Spectral singularities and threshold gain of a slab laser under illumination of a focused Gaussian beam

M Bavaghar*, R Aalipour and K Jamshidi-Ghaleh
Department of physics, Azarbaijan Shahid Madani University, Tabriz 53714-161, Iran

Received: 14 February 2022 / Accepted: 04 November 2022 / Published online: 27 November 2022

Abstract: Spectral singularities of an infinite planar slab containing homogeneous optically active material at the focal plane of a thin lens are investigated. The field distribution of the beam behind the lens is Gaussian that we approximate by a plane wave concentrated near the optical axis at the focal plane of the lens, such that its phase and amplitude are distance-dependent. The consequences of this configuration are explored by determining the threshold gain of the active medium and tuning the resonance frequencies related to spectral singularities. We find that the spectral singularities and the threshold gain, besides that, vary with distance from the center of the Gaussian beam; also they change with the relative aperture of the focusing lens. Numerical studies show that for the central rays of the beam compared to the side rays, the resonances related to the spectral singularities occur at lower wavelengths (higher frequencies) with the highest threshold gain. Also, for the ray at a fixed distance from the center of the beam, increasing the relative aperture of the illuminating system leads to the same results. Numerical results confirm the theoretical findings.

Keywords: Spectral singularity; Threshold gain; Gaussian beam; Transfer matrix; Resonance

1. Introduction

Spectral singularities are specific points of the continuous spectrum of the Schrödinger operator for a complex potential with real scattering energies at which the transmission and reflection amplitudes of the potential diverge. Physically they correspond to scattering states that behave like zero-width resonances [1]. They draw the attention of mathematicians because they are responsible for several mathematical peculiarities that can never arise for Hermitian operators and fascinated physicists because of their optical applications. Spectral singularity was discovered by Naimark and then studied by some mathematicians and physicists [2–5]. Applications of spectral singularities in optical systems were introduced by Mostafazadeh [6]. He shows that a slab involving optical gain material begins emitting purely outgoing coherent waves at resonance frequencies related to the spectral singularities, i.e., it acts as a slab laser. This observation has provided sufficient motivation for continuously studying spectral singularities and their applications [7–24].

Among the notable applications of this approach, those focusing on the structure of incident light and a variety of optical setups, are of interest to us [25–28]. In recent relevant studies, the electromagnetic wave incident on the optically active medium is assumed to be a plane wave, for which the transfer matrix for getting the transmission and reflection spectra of the medium is easily calculated. However, the field distribution of most lasers is Gaussian, so its amplitude and phase are quadratic functions of the ray coordinate. Therefore, it is worthwhile to investigate the spectral singularities of the active medium under the illumination of the Gaussian beam. In this paper, we consider the spectral singularities of a slab laser in the focus of a thin lens under the illumination of a Gaussian beam. The paper is organized as follows: In Sect. 2, we describe the theory and model of the work by exploring the field distribution of a Gaussian beam in the focus of a thin lens, calculating the transmission and reflection coefficients of the Gaussian beam from the laser slab by implementing the transfer matrix method for a ray at a given coordinate on the beam, and getting the spectral singularities and the corresponding threshold gain. Also, we account for the dispersion relations for the equations governing the spectral singularities and the threshold gain in this section. Finally, we report the numerical results in Sect. 3.

*Corresponding author, E-mail: mbavaghar@azaruniv.ac.ir
2. Model and theory

2.1. Propagation of a focused Gaussian beam through an infinite planar slab laser

In Fig. 1, an infinite planar slab of thickness \( L \) is installed in the back focal plane \((x, y)\) of a thin convergent lens. Suppose the slab contains a homogeneous material with the complex refractive index, \( n \). A Gaussian light beam propagating along the \( z \)-axis is focused on the entrance face of the slab through the lens. The transverse \( x \)-component of the electric field describing such an electromagnetic wave can be written as follows [29]:

\[
\tilde{E}(r, t) = \psi(r)e^{i(kz-\omega t)}e_x,
\]

where \( r := (x, y, z) \), \( e_x \) is the unit vector along the positive \( x \)-axis, \( \omega \) is the frequency of the light, \( k := \frac{\omega}{c} \) stands for the wave number, \( c \) is the speed of light in vacuum, and \( \psi(r) \) is a continuously differentiable scalar wave function. Inserting this component of the field in the familiar Helmholtz equation leads to the following optical Schrodinger equation [30]:

\[
-(\nabla + i\kappa e_z)^2 \psi(r) + v(z) \psi(r) = k^2 \psi(r),
\]

with the complex potential \( v(z) = k^2 (1 - \beta^2) \).

\[
\beta := \begin{cases} n & |z| \leq \frac{L}{2} \\ 1 & |z| > \frac{L}{2} \end{cases}
\]

The main characteristic of a Gaussian beam is that it concentrates near the propagation direction \((z\text{-axis})\). The wave does not propagate in the \( x \) or \( y \) direction, and \( \psi(r) \) slowly varies with \( z \). Ignoring \( \frac{\partial^2 \psi}{\partial z^2} \) converts the Schrodinger equation in Eq. (2) to the following paraxial wave equation:

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2i\kappa \frac{\partial \psi}{\partial z} = 0.
\]

Its solution gives the field distribution of the Gaussian beam as follows [31]:

\[
\psi(r) = E_0 \frac{w_0}{w(z)} \exp \left[ -\frac{r^2}{w(z)^2} \right] e^{i\kappa z} \exp \left[ -\frac{i\kappa r^2}{2R(z)} \right],
\]

where \( r^2 = x^2 + y^2 \), \( R(z) = z [1 + (z_0/z)^2] \), \( w(z) = w_0 \sqrt{1 + (z/z_0)^2} \), and \( \phi(z) = \tan^{-1}(z/z_0) \) in which \( w_0 \) and \( z_0 \), respectively, are the minimum beam waist and Rayleigh range, \( z_0 = \frac{nw_0^2}{\lambda} \) [31].

When the Gaussian beam is followed through a lens of diameter \( D \) and focal length \( f \), the minimum beam waist lies to the right of the lens, a distance \( f \) from the lens. The minimum beam waist and the Rayleigh range are given by [31]

\[
w_0 \approx 2\lambda (f/D)
\]

and

\[
z_0 \approx 4\pi\lambda (f/D)^2,
\]

where the ratio \( f/D \) is defined as the relative aperture of the lens [32].

According to Fig. 1, we choose the origin of the coordinates system at the back focal plane of the lens, where the slab is installed. Suppose the thickness of the slab is very small compared to the Rayleigh range, \( L \ll z_0 \). Therefore, we will be allowed to use the approximation \( z \ll z_0 \) for the Gaussian beam parameters as follows:

\[
R(z) \approx \frac{z_0}{z} \quad w(z) \approx w_0 \quad \phi(z) \approx 0.
\]

Substituting these relations in Eq. (5) leads to the following equation for the field distribution of the Gaussian beam in \( z \ll z_0 \) approximation:
The scalar wave function in Eq. (9) describes a plane wave whose phase and amplitude are \( r \)-dependent [33]. The general solution of the Schrödinger equation, Eq. (2), at a given transverse coordinate \( r \) on the beam can be expressed as follows:

\[
\psi(r, z) = \begin{cases} 
A_0^+(r) e^{ik\xi(r)z} + A_0^-(r) e^{-ik\xi(r)z} & \text{for } z < -\frac{L}{2} \\
A_1^+(r) e^{ik\xi(r)z} + A_1^-(r) e^{-ik\xi(r)z} & \text{for } |z| \leq \frac{L}{2} \\
A_2^+(r) e^{ik\xi(r)z} + A_2^-(r) e^{-ik\xi(r)z} & \text{for } z > \frac{L}{2},
\end{cases}
\]

(12)

where \( A_i^+ \) and \( A_i^- \), for \( i = 0, 1, 2 \) are \( r \)-dependent forward and backward wave amplitudes according to Eq. (10).

### 2.2. Reflection and transmission coefficients

The transfer matrix for the system we consider is the \( 2 \times 2 \) matrix \( \mathbf{M} \) satisfying

\[
\begin{pmatrix}
A_2 \\
A_1 \\
A_0 \\
A_3
\end{pmatrix} = \mathbf{M} \begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3
\end{pmatrix},
\]

(13)

We explore the transfer matrix by using the boundary conditions at \( z = -\frac{L}{2} \) and \( z = \frac{L}{2} \) based on the continuity of the tangential components of electric and magnetic fields. The transfer matrix, after some straightforward calculations, is obtained as follows:

\[
\mathbf{M} = \frac{1}{4n} \begin{pmatrix}
|\mathbf{M}| & e^{-ik\xi(r)L}F(n, -\frac{L}{2}) & 2i(n^2 - 1) \sin[kn\xi(r)L] \\
-2i(n^2 - 1) \sin[kn\xi(r)L] & e^{ik\xi(r)L}F(n, \frac{L}{2})
\end{pmatrix},
\]

(14)

where for all \( p, q \in \mathbb{C} \)

\[
F(p, q) = e^{-2ipq}(1 + p)^2 - e^{2ipq}(1 - p)^2.
\]

From Eq. (14), it is clear that the transfer matrix elements depend on the transverse coordinate \( r \). For a case that we consider the center of the beam, i.e., \( r = 0 \), Eq. (14) turns to the expected transfer matrix for the slab laser under the illumination of a plane wave [6].

The left and right reflection and transmission amplitudes are given by [1]

\[
R_l = -\frac{M_{21}}{M_{22}} \quad R_r = \frac{M_{12}}{M_{22}} \quad T_l = \frac{\det M}{M_{22}} \quad T_r = \frac{1}{M_{22}},
\]

(16)

where \( M_{ij} \), \( i, j = 1, 2 \) are the elements of transfer matrix \( M \) and \( \det \) stands for determinant of the matrix, for which \( \det M = 1 \). Inserting transfer matrix elements from Eq. (14) in Eq. (16) yields the following relations:

\[
R_l = R_r = e^{-ik\xi(r)L} \sqrt{\frac{1 - e^{2ik\xi(r)L}}{1 - \text{Re}e^{2ik\xi(r)L}}},
\]

(17)

and

\[
T_l = T_r = e^{-ik\xi(r)L} \frac{(1 - r)e^{ik\xi(r)L}}{1 - \text{Re}e^{2ik\xi(r)L}},
\]

(18)

where \( r := \left(\frac{n - 1}{n + 1}\right)^2 \). Expressing the refractive index of the slab as \( n = \eta + \kappa \), substituting it in Eqs. (17) and (18), and multiplying them by their complex conjugates give the reflection and transmission coefficients as follows:

\[
R = \frac{|1 - G|^2 + 4G \sin^2(\delta/2)}{(1 - G)^2 + 4G |\sin(\delta + \phi)/2|^2},
\]

(19)

\[
T = \frac{|1 - G|^2 G}{(1 - G)^2 + 4G |\sin(\delta + \phi)/2|^2},
\]

(20)

where

\[
|G| := \left(\frac{\eta - 1}{\eta + 1}\right)^2 + \kappa^2, \quad \delta := 2k\xi(r)\eta L, \quad \phi := 2\tan^{-1}\left(\frac{2\kappa}{\eta^2 + \kappa^2 - 1}\right).
\]

(21)

### 2.3. Spectral singularities and threshold gain

The spectral singularities correspond to the real and positive values of the wavenumber \( k \), for which \( M_{22} = 0 \) [1]. This implies that the reflection and transmission amplitudes diverge. So, it occurs when

\[
e^{-2ik\xi(r)L} = r.
\]

(22)

Taking the natural logarithm of both sides of this equation yields

\[
k\xi(r)L = -\frac{1}{2in} \ln r.
\]

(23)

In follow, inserting \( n = \eta + \kappa \) in Eq. (23), leads to

\[
k\xi(r)L = -\frac{1}{2i(\eta + \kappa)} \ln \left(\frac{\eta - 1}{\eta + 1 + i\kappa}\right)^2.
\]

(24)

Because \( k\xi(r)L \) is real, therefore it equates with the real part of the right-hand side of this equation, and so the imaginary part of the right-hand side is set to zero. This gives
\[ k \xi(r)L = \frac{1}{2\kappa} \ln |r| \quad (25) \]

and

\[ \eta \ln |r| + \kappa (\phi - 2m\pi) = 0, \quad (26) \]

where \( m \) is an arbitrary integer.

The gain coefficient of an active medium is defined by [34]

\[
g := -\frac{4\pi \kappa}{\lambda}, \tag{27}
\]

where \( \kappa \) is the imaginary part of the refractive index \( n \) and \( \lambda := 2\pi/k \) is the wavelength. The value of gain coefficient for which the spectral singularity takes place is called threshold gain, which is the lasing threshold condition. For getting the threshold gain of the active medium, we derive \( \kappa \) from Eq. (25) and then substitute in Eq. (27), which yields

\[
g = g_{th} := \frac{1}{2 \left[ 1 + \frac{\kappa^2}{\kappa_0^2} \right]} \ln \left( \frac{(\eta + 1)^2 + \kappa^2}{(\eta - 1)^2 + \kappa^2} \right), \tag{28}
\]

where we have used Eq. (11) for \( \xi(r) \).

### 2.4. Dispersion considerations

For a more realistic situation, we take into account the effect of dispersion, i.e., the refractive index \( n \) and so \( \eta \) and \( \kappa \) are not independent of the frequency. In this regard, suppose the slab contains a doped host medium of refractive index \( n_0 \) that we can model by a two-level atomic system with lower and upper level population densities \( N_l \) and \( N_u \), resonance frequency \( \omega_0 \), damping coefficient \( \gamma \), and the dispersion relation [35]

\[
n^2 = \frac{n_0^2}{1 + \frac{n_0^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2} - i\gamma/\omega}, \tag{29}
\]

where \( \omega_0 \) is the permeability of the vacuum, \( \omega_p^2 := \frac{(N_l-N_u)e^2}{\varepsilon_0 m_e} \), and \( e \) and \( m_e \) are electron’s charge and mass, respectively. Substituting the refractive index of the slab \( n = \eta + i\kappa \) in Eq. (29) and taking \( \kappa = \kappa_0 \) at \( \omega = \omega_0 \), we get the following approximate equations for the real and imaginary parts of the refractive index

\[
\eta \simeq n_0 + \kappa_0 F_1 \quad (30)
\]

and

\[
\kappa \simeq \kappa_0 F_2, \quad (31)
\]

where

\[
F_1 := \frac{(1 - \tilde{\omega}^2)\tilde{\gamma}}{(1 - \tilde{\omega}^2)^2 + \tilde{\omega}^2 \tilde{\gamma}^2}, \quad F_2 := \frac{\tilde{\omega}^2 \tilde{\gamma}}{(1 - \tilde{\omega}^2)^2 + \tilde{\omega}^2 \tilde{\gamma}^2}, \quad (32)
\]

\[
\tilde{\omega} = \frac{\omega}{\omega_0} \quad \text{and} \quad \tilde{\gamma} = \frac{\gamma}{\omega_0}.
\]

Next, by inserting Eqs. (30) and (31) in Eq. (25) and using Eqs. (11) and (7), we find

\[
k_0 \approx \frac{-2F_1}{\kappa_0^2} + \left[ \frac{2\pi(\tilde{\omega})^2}{\kappa_0^2} + \frac{\pi^2}{16\pi(n/D)^4} (\tilde{\omega})^3 LF_2 \right]. \tag{33}
\]

where \( \delta = \left( \frac{m-1}{m+1} \right)^2 \). By this way, threshold gain is obtained from Eq. (27) as follows:

\[
g_{th} := -4\pi n_0 \kappa_0 / \lambda_0, \quad (34)
\]

where \( \lambda_0 := 2\pi c / \omega_0 \), and \( \kappa_0 \) is evaluated from Eq. (33). In deriving Eqs. (30)–(34), the quadratic- and higher-order terms of \( \kappa_0 \) are neglected. Then, extracting \(|r|\) from Eq. (26), substituting in Eq. (25), and using Eqs. (7), (11) and (30)–(32), we find the following mode equation:

\[
\frac{r^2}{16\pi(n/D)^4} \frac{\omega^2}{\lambda_0^2} \frac{\omega^4}{2\gamma^2} + \frac{2\pi n_0 L}{\lambda_0} - \frac{\ln \lambda_0}{2\gamma} \frac{\omega^2}{2\gamma} + \frac{\ln \lambda_0}{2\gamma} - \pi m \omega = 0. \quad (35)
\]

The solutions of this equation are the resonance frequencies that correspond to the spectral singularities. In addition to being dependent on the physical characteristics of the active medium, also they are \( r \)-dependent and change with the relative aperture of the lens.

### 3. Numerical results and discussions

In order to check the numerical results of the theoretical findings in the previous sections, we consider a slab composed of a semiconductor gain medium with the specifications: \( n_0 = 3.4 \), \( L = 300 \mu m \), \( \lambda_0 = 1500 \) nm, and \( \gamma = 0.02 \), in the back focal plane of thin convergent lenses with the relative apertures of \( f/D = 7.5, 8.5, 10, 12, \) and 15 under the illumination of a Gaussian beam. The reflection and transmission coefficients from the slab are calculated using Eqs. (19) and (20). Figure 2 illustrates the logarithmic plots of the reflection (dashed curve) and transmission (full curve) coefficients of a light ray at the center of the focused Gaussian beam \( (r = 0) \) from the specified slab versus the wavelength. The plots show the zero-width resonance frequencies with the free spectral range of the order of \( \delta \lambda \approx 1 \) nm.

To find out how the spectral singularities and the corresponding resonance frequencies depend on the transverse distance \( r \), we calculate the reflection and transmission coefficients at various transverse distances on the beam
spot. Figure 3 shows the logarithmic curves of the calculated transmission coefficient at various transverse distances. Based on these plots, the spectral singularities corresponding resonances are $r$-dependent, and they shift to the higher wavelengths as $r$ increases. In fact, in a Gaussian beam, the outer rays compared to the central ray are obliquely propagating, that is, the incident angle of the rays striking the front face of the slab increases radially, and so the optical path length experienced by the rays in passing through the slab increases radially. Recently, it has been illustrated that the resonance frequencies related to spectral singularities depend strongly on the incident angle [25, 26].

Now, we study the dependence of the threshold gain on the wavelength at different transverse distances from the center of the beam. For this purpose, we calculate the threshold gain using Eq. (34). Figure 4, shows the plots of the calculated threshold gain versus the wavelength at various transverse distances from the center of the beam. Upon these plots, the threshold gain of the slab becomes maximum at the center of the beam and decreases as $r$ increases. This is because the amplitude of the field distribution across the Gaussian beam decreases radially, according to Eq. (5).
In the proposed configuration, the relative aperture of the thin lens $f/D$ is a factor that influences the threshold gain of the active medium and the corresponding spectral singularities, according to Eqs. (33–35). Therefore, to consider this effect, we can use various lenses with different relative apertures. We calculated the transmission coefficient of a Gaussian beam from the specified slab in the focus of the lenses with different relative apertures using Eq. (20). Figure 5 shows the plots of the transmission coefficient. According to these plots, the use of lenses with higher relative apertures leads to shifting the spectral singularities to the lower wavelengths (higher frequencies). One can use this ability to tune the spectral singularities. Also, Fig. 6 shows the plots of the threshold gain calculated using Eq. (34) for the specified slab in the focus of the lenses with different relative apertures. These plots illustrate that by using thin lenses with higher relative apertures, the threshold gain of the slab increases.

In order to look at the results accurately, we solve the mode equation for the mode numbers $1374–1376$ by plotting Eq. (35) versus the frequency, which gives the wavelength of the spectral singularity. And then, by substituting the solution of the mode equation in Eqs. (33) and (34), respectively, the corresponding threshold gain is obtained. The wavelength of spectral singularity and the corresponding threshold gain are given in Tables 1 and 2. At a fixed distance from the center of the beam, by increasing the relative aperture of the lens, the spectral singularity shifts to lower wavelengths, and so the corresponding threshold gain decreases. On the other hand, for a fixed relative aperture of the lens, the spectral singularity and the corresponding threshold gain shift, respectively, to lower and higher wavelengths by distance from the center of the beam.

![Figure 6](image-url)

**Figure 6** The plots of threshold gain of a slab containing a semiconductor gain medium under the illumination of a focused Gaussian beam versus wavelength for various relative apertures of the focusing system. The specifications used are: $n_0 = 3.4$, $L = 300 \mu m$, $\lambda_0 = 1500 nm$, $\gamma = 0.02$, and $r = 0.02 mm$

| $m$ | $f/D = 7.5$ | $f/D = 8.5$ | $f/D = 10$ | $f/D = 12$ |
|-----|-------------|-------------|-------------|-------------|
|     | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) |
| 1374 | 1485.192  | 80.157     | 1485.082  | 80.759     | 1485.010  | 81.155     | 1484.967  | 81.393     |
| 1375 | 1484.127  | 86.114     | 1484.017  | 86.760     | 1483.945  | 87.185     | 1483.902  | 87.440     |
| 1376 | 1483.064  | 92.47      | 1482.954  | 93.169     | 1482.882  | 93.623     | 1482.838  | 93.901     |

The specifications used are: $n_0 = 3.4$, $L = 300 \mu m$, $\lambda_0 = 1500 nm$, $\gamma = 0.02$, and $r = 0.02 mm$

| $m$ | $r = 0$ | $r = 0.02$ (mm) | $r = 0.03$ (mm) | $r = 0.05$ (mm) |
|-----|----------|-----------------|-----------------|-----------------|
|     | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) | $\lambda$ (nm) | $g_{th}$ (cm$^{-1}$) |
| 1374 | 1484.926  | 81.605     | 1485.192  | 80.157     | 1485.524  | 78.368     | 1486.584  | 72.934     |
| 1375 | 1483.861  | 87.683     | 1484.127  | 86.114     | 1484.459  | 84.193     | 1485.520  | 78.332     |
| 1376 | 1482.798  | 94.155     | 1483.064  | 92.479     | 1483.396  | 90.426     | 1484.458  | 84.137     |

The specifications used are: $n_0 = 3.4$, $L = 300 \mu m$, $\lambda_0 = 1500 nm$, $\gamma = 0.02$, and $f/D = 7.5$
4. Conclusions

We studied spectral singularity and threshold gain of a laser slab in the focus of a thin lens under the illumination of a Gaussian beam. Our goal of this study is to extend the recent studies on spectral singularity to Gaussian beams and investigate the effect of the focusing system on these features.

We considered the field distribution of a Gaussian beam in the focus of a thin lens as a plane wave whose phase and amplitude are $r$-dependent, according to Eq. (9). Physically, for Gaussian beam, the outer rays compared to the central ray, obliquely propagate. Therefore, at the focus, it can be considered as a set of plane waves with wave vectors at an angle to the optical axis that increases with $r$. In this way, we calculated the transfer matrix for the active medium under the illumination of Gaussian beam and derived the gain coefficient of the active medium and the corresponding spectral singularities. Because the amplitude of the field distribution across the Gaussian beam decreases radially, according to Eq. (5), therefore the corresponding threshold gain of the medium also decreases radially, as illustrated in Fig. 4 and Table 2. For the Gaussian beam, the rays propagating through the slab experience a longer optical path length by distance from the center of the beam. Therefore, the corresponding spectral singularities occur at higher wavelengths, as shown in Fig. 3. Also, we illustrated that the spectral singularity and threshold gain of the active medium vary with the relative aperture, $f/D$, of the focusing lens, according to Eq. (33). Increasing the relative aperture of the focusing system causes the resonances to shift to lower wavelengths and increases the threshold gain, as illustrated in Figs. 5 and 6, and Table 1. This study can be extended to other features of electromagnetic waves in the active mediums such as coherent perfect absorption and invisibility.

References

[1] A Mostafazadeh Phys. Rev. Lett. 102 220402 (2009)
[2] M A Naimark Am. Math. Soc. Transl. 2 103 (1960)
[3] R R D Kemp Can. J. Math. 10 447 (1958)
[4] J Schwartz Commun. Pure Appl. Math. 13 609 (1960)
[5] S Longhi Phys. Rev. B. 80 165125 (2009)
[6] A Mostafazadeh Phys. Rev. A. 83 045803 (2011)
[7] A Mostafazadeh Journal of Physics A: Mathematical and Theoretical. 45 444024 (2012)
[8] A Sinha and R Roychoudhury J. Math. Phys. 54 112106 (2013)
[9] A Mostafazadeh and M Sarisaman Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 468 3224 (2012)
[10] F Correa and M S Plyushchay Phys. Rev. D. 86 085028 (2012)
[11] A Mostafazadeh and M Sarisaman Phys. Rev. A 88 033810 (2013)
[12] L Chaos-Cador and G García-Calderón Phys. Rev. A. 87 042114 (2013)
[13] A Mostafazadeh Phys. Rev. Lett. 110 260402 (2013)
[14] A Mostafazadeh Phys. Rev. A. 87 063838 (2013)
[15] A Mostafazadeh Studies in Applied Mathematics. 133 353 (2014)
[16] H Ramezani, H-K Li, Y Wang and X Zhang Phys. Rev. Lett. 113 263905 (2014)
[17] G R Li, X Z Zhang and Z Song Annals of Physics. 349 288 (2014)
[18] X Liu, S D Gupta and G S Agarwal Phys. Rev. A. 89 013824 (2014)
[19] K N Reddy and S D Gupta Opt. Lett. 39 4595 (2014)
[20] A Mostafazadeh and M Sarisaman Annals of Physics. 375 265 (2016)
[21] P Wang, L Jin, G Zhang and Z Song Phys. Rev. A. 94 053834 (2016)
[22] C Hang, G Huang and V V Konotop New J. Phys. 18 085003 (2016)
[23] S Pendharker, Y Guo, F Khoosravi and Z Jacob Phys. Rev. A. 95 033817 (2017)
[24] H Ghaemi-Dizicheh, A Mostafazadeh and M Sarisaman Journal of Optics. 19 105601 (2017)
[25] R Aalipour Phys. Rev. A. 90 013820 (2014)
[26] A Mostafazadeh and M Sarisaman Phys. Rev. A. 91 043804 (2015)
[27] K Dogan, A Mostafazadeh and M Sarisaman Annals of Physics. 392 165 (2018)
[28] H Ghaemi-Dizicheh, A Mostafazadeh and M Sarisaman J. Opt. Soc. Am. B. 37 2128 (2020)
[29] M Born and E Wolf Principles of Optics (Cambridge University Press) (1999)
[30] M A M Marte and S Stenholm Phys. Rev. A. 56 2940 (1997)
[31] B D Guenther Modern Optics (Cambridge University Press) (2015)
[32] F L Pedrotti and L S Pedrotti Introduction to Optics (Prentice-Hall International, Inc) (1993)
[33] K Jamshidi-Ghaleh and R Abd-Ghaleh Journal of Nanophotonics. 5 051817 (2011)
[34] W T Silfvast Laser Fundamentals (Prentice-Hall International, Inc) (2004)
[35] A Yariv and P Yen Photonics: Optical Electronics in Modern Communications (Cambridge University Press) (2007)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.