The large $N_c$ limit of four-point functions in $N = 4$ super-Yang-Mills theory from anti-de Sitter Supergravity

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Abstract

We compute the imaginary part of scalar four-point functions in the AdS/CFT correspondence relevant to $N = 4$ super Yang-Mills theory. Unitarity of the AdS supergravity demands that the imaginary parts of the correlation functions factorize into products of lower-point functions. We include the exchange diagrams for scalars as well as gravitons and find explicit expressions for the imaginary parts of these correlators. In momentum space these expressions contain only rational functions and logarithms of the kinematic invariants, in such a manner that the correlator is not a free-field result. The simplicity of these results, however, indicate the possibility of additional symmetry structures in $N = 4$ super Yang-Mills theory in the large $N_c$ limit at strong effective coupling. The complete expressions may be computed from the integral results derived here.

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1 Introduction

In the past few months much work has focused on the correspondence between superconformal field theory and string (M-)theory in an anti-de Sitter background \cite{1}. In particular, the low energy supergravity regime of the compactified AdS$_{p+1}$ string theory should correspond to the large $N_c$ limit of a $p$-dimensional conformal field theory at strong 't Hooft coupling $\lambda = g^2_{YM} N_c$. A holographic relation between conformal field theory correlators and those in the gauged supergravity is given in \cite{2,3}: correlations in the CFT are generated via,

$$
\left. \prod_{j=1}^k \left( \frac{\delta}{\delta \phi_{0,j}(\vec{z}_j)} \right) e^{i S_{\text{sugra}}[\phi(\phi_0)]} \right|_{\phi_{0,j}=0} = \langle \prod_{j=1}^k \mathcal{O}^j(\vec{z}_j) \rangle_{\text{CFT}}. \tag{1.1}
$$

$S_{\text{sugra}}$ is the bulk action of the compactified string theory considered as a functional of the boundary values of the fields, $\phi_{0,j}$, and $\mathcal{O}$ are composite (gauge invariant) operators of the conformal Yang-Mills theory. These operators are dual to the boundary values of the supergravity fields in the sense that the latter act as sources for the former, as can be seen from (1.1).

A specific example is the correspondence between $d = 4$ $N = 4$ super-Yang-Mills theory in the large $N_c$ limit at large $\lambda$ and type IIB supergravity on AdS$_5 \times S^5$ with radii $R^2_{\text{AdS}} = R^2_{S^5} = \alpha' \sqrt{4\pi g_{\text{st}} N_c}$, where $g^2_{YM} = 4\pi g_{\text{st}}$. It is clear that testing the conjecture is not an easy task as the accessible regimes for computations, $\lambda \ll 1$ for super-Yang-Mills theory and $\lambda \gg 1$ for supergravity, do not overlap. Nevertheless two- and three-point correlators of certain chiral primary operators and their descendants have been calculated and the supergravity results were found to agree with the free-field results on the super-Yang-Mills side \cite{4,5,6,7,8}. For two-point functions this was expected due to the known existence of a non-renormalization theorem. For certain three-point correlators a similar theorem was conjectured for all three-point functions of chiral operators from explicit computation. Such a non-renormalization theorem should follow from the stringent covariance constraints of the $N = 4$ superconformal algebra. Recent evidence has been given in \cite{9}.

Superconformal constraints are not expected to protect the four-point or higher order correlation functions from renormalization. In particular, it is not clear that free-field Yang-Mills result will be reproduced by the supergravity calculation at the four-point level. Evidence for this claim is the fact that certain four-point functions have non-trivial $\lambda$-dependence. They are affected by $\alpha'$ string corrections to the classical supergravity action, which correspond to $\lambda^{-1/2}$ terms in the strong ('t Hooft) coupling expansion of the CFT \cite{10,11,12}. Thus, an explicit comparison at $n \geq 4$ order is a non-trivial test of the dynamics of $N = 4$ super Yang-Mills theory. In view of the above, any indications of further non-renormalization theorems for four-point functions of particular operators would be very interesting on the field.
In this article we will calculate the imaginary part of four-point functions in AdS supergravity using the unitarity constraints on Green’s functions. Unitary quantum field theories obey the identity,

$$2 \text{Im} T = T^\dagger T$$

(1.2)

where $T$ is the transition matrix $iT = S - \mathbb{1}$. These unitarity relations hold also for general Green’s functions and not just the case where the external lines are on-shell. Classical supergravity, as well as finite-$N_c$ super Yang-Mills theory, is certainly unitary and we will use this to relate the imaginary parts of four-point functions to the product of three-point boundary-boundary-bulk functions, whose explicit expressions can be computed. Explicit computations of four-point functions are technically challenging; however, following the unitary cutting approach we avoid the technical complications associated with integrating over the bulk-bulk propagator. Though we limit ourselves to the imaginary parts in particular channels our approach enables us to make predictions regarding the infinite summation of planar graphs. Specifically, we may compare with the generically complicated logarithmic dependence of the perturbative series in field theory. (The complete correlators may also be found from our computations by performing an additional one-parameter integral and adding diagrams containing four-point vertices.)

This work is organized as follows. In section 2 we formulate the holographic Feynman rules in Lorentzian signature AdS used to generate the boundary correlators. We find the bulk-bulk and bulk-boundary kernels for scalars and gravitons used in the following sections. The bulk vertices are also found. We then discuss unitarity principles in section 3 and explain the cutting rules. In section 4 we derive the imaginary parts of the correlator of four axions in IIB supergravity on $\text{AdS}_5 \times S_5$ and show how the calculation reduces to the square of three-point functions; in section 5 we discuss how other scalar correlators may be computed. In section 6 we interpret the result within the AdS/CFT correspondence for the $N = 4$ super-Yang-Mills theory.

Previous to this work contributions to four-point functions involving four-point vertices were given in [13]. Scalar exchange contributions were discussed in [14, 15], and recently the expression for the contribution due to the exchange of a vector boson in a covariant gauge was obtained [16].

## 2 Kernels in $\text{AdS}_{d+1}$

We need to formulate the supergravity theory in a Lorentzian signature version of anti-de Sitter space in order to derive the imaginary parts within a given channel. Most of the work relating to the AdS/CFT correspondence has been performed...
using the Euclidean version described in \[3\]. Our prescription will be to naively Wick rotate from Euclidean AdS to a Lorentzian form. Some care has to be taken in Lorentzian signature as one has to confront the subtlety that there are additional homogenous solutions to the field equations \[18\]; the bulk supergravity modes are no longer uniquely determined from their boundary values. We will comment on this below.

In Euclidean space the bulk-bulk propagator for a given supergravity field may be written as

$$
\Delta(x, y) = -\sum_n \frac{\varphi_n^\ast(y) \varphi_n(x)}{\lambda_n^2}
$$

(2.1)

where the $\varphi_n(x)$ span a complete set of eigenfunctions of the kinetic operator that obey the correct boundary conditions. The summation extends over all quantum numbers, denoted by \{\{n\}\}, necessary to describe the solution and the $\lambda_n^2$ are the eigenvalues associated with $\varphi_n(x)$. In Wick rotating to Lorentzian signature one needs to provide an $i\epsilon$ prescription, $\lambda_n^2 \rightarrow \lambda_n^2 \pm i\epsilon$, in the denominator of (2.1). The presence (as well as the sign) of the $i\epsilon$ term is determined by the physical requirement that at large timelike separations the positive frequency modes dominate.

We shall consider the halfspace Poincaré coordinate system for $\text{AdS}_{d+1}$ with metric

$$
ds^2 = \frac{A^2}{x_0^2} \left( dx_0^2 + dx^i dx^j \eta_{ij} \right), \quad i = 1, \ldots, d
$$

(2.2)

where $x_0 \geq 0$. The metric $\eta_{ij}$ is Minkowski with mostly plus signature and $\eta_{dd} = -1$, and the analytical continuation of the metric to its Euclidean form is obvious. The boundary of AdS in these coordinates is the Minkowski space $\mathbb{R}^{3,1}$ at $x_0 = 0$ and the single point $x_0 = \infty$. This metric has a timelike Killing vector $\partial/\partial x^d$ whose conserved charge we may interpret as the energy and according to which an $i\epsilon$ prescription may be given \[19\], \[20\].

This section is devoted to constructing the eigenfunctions $\varphi_n(x)$ for the scalar and graviton fields in the AdS background. In this article we will consider a particular set of boundary conditions (Dirichlet) for the bulk AdS modes. This uniquely selects a complete set of modes $\varphi_n(x)$ for the Euclidean kernel. After the Wick rotation some of these may become zero modes but all have to be retained in the kernel to preserve completeness. Besides such zero modes there are further ones which alter the form of the correlators and their boundary behaviour when added to the propagator \[18\]. These are necessary to describe the AdS theory with another choice of boundary conditions for the bulk fields. In view of unitarity and the dependence in the imaginary parts this is interesting in that there may be choices where particular cuts vanish. Here we will take the conventional route by not adding these solutions, which is analogous to a vacuum choice \[18\], \[21\].

\footnote{There is a family of AdS quantum vacua with Euclidean signature preserving the isometry structure \[3\].}
We will concern ourselves with scalar four-point functions in the dilaton-axion sector of IIB supergravity on $\text{AdS}_5 \times S_5$. It was shown in [14] that at the level of four-point functions this sector of the compactified supergravity theory does not receive contributions from any other fields. The CFT operators which correspond to these fluctuations are $\text{Tr} F^2$ and $\text{Tr} F \tilde{F}$ for the dilaton and axion respectively. The relevant part of the IIB action is given by

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \sqrt{g} \left[ -(R - \Lambda) - \frac{1}{2} g^\mu\nu \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} e^{2\phi} g^\mu\nu \partial_\mu C \partial_\nu C \right],$$

(2.3)

where $d = 4$ and $1/2\kappa_5^2 = \Omega_5 N_c^2/(2\pi)^5 = N_c^2/15\pi^3$ in terms of the $N = 4$ super-Yang-Mills variables. The proportionality of the supergravity boundary values to the CFT operators may be fixed by a two-point function calculation. As in (2.3) the correspondence between supergravity and super-Yang-Mills theory is usually formulated in the Einstein frame. The exact relation between the five-dimensional supergravity fields and the conformal operators is not precisely clear because of subtleties in the consistent truncation from the Kaluza-Klein modes. The cosmological constant $\Lambda$ is related to the scale $A$ of the Poincaré metric as

$$\Lambda = \frac{d(d-1)}{A^2}.$$  

(2.4)

Below we will set the scale $A$ to unity, in which case the scalar curvature associated with the background AdS metric equals $R = d(d+1)$.

## 2.1 Scalars

We first recall the derivation of the propagator for a scalar field on anti-de Sitter space. Along the way we find the correct $i\epsilon$ prescription needed in Lorentzian signature.

The covariant kinetic operator for massive scalars in a curved background is

$$\hat{K}\Phi(x) = \left( \frac{1}{\sqrt{g}} \left( \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \right) - m^2 \right) \Phi(x)$$

$$= \left( x_0^{d+1} \partial_0 \left( \frac{1}{x_0^{d-1}} \partial_0 \right) + x_0^2 \partial^i \partial_i - m^2 \right) \Phi(x).$$

(2.5)

The differential operator in (2.3) has the eigenfunctions [14]

$$\varphi_\lambda(x) = x_0^{d/2} e^{i \vec{k} \cdot \vec{x}} J_\lambda(\lambda x_0), \quad \vec{k} \cdot \vec{x} \equiv \sum_{i=1}^d k_i x_i$$

$$\hat{K}\varphi_\lambda(x) = -(\lambda^2 + \vec{k}^2) x_0^2 \varphi_\lambda(x),$$

(2.6)

\footnote{Our conventions for the Riemann and the Ricci tensor are that $R^{\mu\nu\rho\sigma} = \partial_\mu \Gamma^{\rho\sigma}_{\nu\rho} + \Gamma^{\rho\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho} - (\mu \leftrightarrow \nu)$ and $R_{\nu\rho} = R^{\mu\nu\rho\nu}$.}
where \( \nu = \sqrt{m^2 + d^2/4} > 0 \). The eigenfunctions \( \varphi_\lambda(x) \) are labelled by the four-vector \( \vec{k} \), the conserved momentum along the boundary, and a continuous eigenvalue \( \lambda \). The other Bessel functions that are possible solutions of (2.3) do not obey the Dirichlet boundary condition at \( x_0 = 0 \) or are not well-behaved in the interior of AdS.

The Bessel functions \( J_\nu(\lambda x_0) \) and the Fourier modes are complete in the sense that

\[
\int_0^\infty d\lambda \lambda J_\nu(\lambda x_0) J_\nu(\lambda y_0) = \frac{\delta(x_0 - y_0)}{\sqrt{x_0 y_0}},
\]

and

\[
\int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} = \delta^d(\vec{x} - \vec{y}) .
\]

The Euclidean bulk-bulk propagator for scalars is thus given by

\[
G_\Phi(x, y) \equiv \langle \Phi(x) \Phi(y) \rangle = \int_0^\infty d\lambda \lambda \int \frac{d^d k}{(2\pi)^d} \frac{\varphi_\lambda^*(x) \varphi_\lambda(y)}{\lambda^2 + \vec{k}^2} ,
\]

and obeys

\[
\hat{K} \langle \Phi(x) \Phi(y) \rangle = -x_0^{d+1} \delta^{d+1}(x - y) = -\frac{\delta^{d+1}(x - y)}{\sqrt{g}} .
\]

To obtain the Lorentzian signature version of the propagator we analytically continue the \( d \)th coordinate. Requiring that the positive frequency modes \( k_{dLor}^2 = -\omega(\vec{k}) < 0 \) dominate at large times we find that we should effectively replace \((k_d^E)^2\) with \(-(k_{dLor}^2)^2 - i\epsilon\) in (2.9).

In order to compute the conformal field theory correlation functions from the supergravity diagrams we also need the bulk-boundary kernel for the scalars:

\[
\Delta(\vec{x}, y) = \sqrt{h} n^\mu \partial_\mu G_\Phi(x, y)|_{x_0 = 0} , \\
\phi_{bulk}(y) = \int d^d x \Delta(\vec{x}, y) \phi_{bound}(\vec{x}) ,
\]

where \( h \) is the determinant of the induced metric on the boundary. These are attached as the external legs of the supergravity Green’s function, according to the prescription of [4, 3]. Care has to be taken in defining the bulk-boundary propagator, since formally the Poincaré metric blows up at the boundary [3, 4]. One way to regularise is to impose the Dirichlet conditions at \( x_0 = \zeta \) and let \( \zeta \to 0 \) at the end of the calculation. The accordingly modified Green’s function is [4, 13]

\[
G^\zeta_\Phi(x, y) = G_\Phi(x, y) - (x_0 y_0)^{d/2} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} K_\nu(k x_0) K_\nu(k y_0) \frac{I_\nu(k \zeta)}{K_\nu(k \zeta)} ,
\]

where \( k = \sqrt{\vec{k} \cdot \vec{k}} \). Taking the normal derivative then yields

\[
\Delta^\zeta(\vec{x}, y) = \left( \frac{y_0}{\zeta} \right)^{d/2} \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \frac{K_\nu(k y_0)}{K_\nu(k \zeta)} .
\]
We have not examined the $i\epsilon$ procedure of the bulk-boundary kernels in order to determine cuts in single-particle channels here, although one may certainly do so.

For massless scalars the index of the Bessel function $J_\nu(\lambda x_0)$ equals $\nu = d/2$ and the limit of $\zeta \to 0$ can be taken without ambiguity. Hence for this special case the bulk-boundary propagator is regularisation independent and equals

$$\Delta(\vec{x}, y)_{\nu=d/2} = \frac{2}{\Gamma(d/2)} \int \frac{d^dk}{(2\pi)^d} \left( \frac{k y_0}{2} \right)^{d/2} K_{d/2}(k y_0) e^{i\vec{k} \cdot (\vec{x} - \vec{y})}. \quad (2.14)$$

We will not apply our methods in this work to the evaluation of boundary correlators for massive scalars; however, the techniques presented below are certainly available for this case as well.

2.2 Gravitons in $h_{\mu0} = 0$ Gauge

For simplicity we have chosen to work in the $h_{\mu0} = 0$ gauge. Of course this choice breaks the background isometries of the AdS space and is therefore not preferred above a covariant gauge such as $g^{\mu\nu} D_\mu h_{\nu\rho} = 0$. However, the kinetic operators in the latter gauges are not readily inverted due to the presence of the nontrivial background curvature (see e.g. [22]). The external supergravity fields satisfy field equations near the boundary of the AdS space. This indicates that final results for correlator expressions should in principle be independent of the gauge-choice, although this has not yet been examined in detail.

Expanding the action for the graviton, $\hat{g}_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, in the presence of a source term $T_{\mu\nu}$,

$$S = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \left( \sqrt{\hat{g}(x)} \left[ -(R - \Lambda) \right] + \sqrt{g(x)} h_{\mu\nu} T^{\mu\nu} \right). \quad (2.15)$$

we find the effective field equations for $h_{\mu\nu}$,

$$\frac{1}{2} \left\{ g^{\alpha\sigma} g^{\nu\rho} (D_\rho D_\alpha h_{\tau\sigma} - D_\rho D_\tau h_{\alpha\sigma} - D_\rho D_\sigma h_{\alpha\tau}) + g^{\mu\nu} g^{\sigma\tau} D_\rho D_\sigma h_{\alpha\tau} + g^{\mu\nu} g^{\sigma\tau} D_\tau D_\sigma h - g^{\mu\nu} g^{\sigma\alpha} D_\rho D_\sigma h \right\} + \frac{1}{2} g^{\mu\nu} g^{\sigma\rho} h_{\tau\sigma} (R - \Lambda) + \frac{1}{4} g^{\mu\rho} R^{\rho\sigma} h_{\sigma\alpha} + \frac{1}{4} R^{\mu\rho} h = -T^{\mu\nu} + O(h_{\mu\nu}^2). \quad (2.16)$$

The indices are raised with the background AdS metric $g^{AdS}_{\mu\nu}$ and the derivatives $D_\alpha$ are covariant with respect to this metric.

Setting $h_{\mu0} = 0$, we find the field equation

$$\partial_0^2 (x_0^2 h_{mn} - \eta_{mn}(x_0^2 h)) - \frac{(d - 1)}{x_0} \partial_0 (x_0^2 h_{mn} - \eta_{mn}(x_0^2 h)) + x_0^2 \left( \triangle h_{mn} - 2 \partial_m \partial_n h + \eta_{mn} \eta_{ij} \partial_i \partial_j h_{jk} \right) = -\frac{2}{x_0^4} T_{mn}, \quad (2.17)$$
where \( \tilde{h}_{mn} = h_{mn} - \frac{1}{2} \eta_{mn} h \) and \( h = \eta^{mn} h_{mn} \). The box is defined as \( \Box = \partial_i^2 \) with flat \( d \)-dimensional indices. The field equations for \( h_{0\mu} \) generate constraints,

\[
\partial_0 (\partial_m (x_0^2 h) - \partial_i (x_0^2 h_{im})) = -\frac{2}{x_0^4} T^0_m \, ,
\]

(2.18)

and,

\[
\eta^{ij} \partial_i (x_0^2 h_{jk}) - \Box (x_0^2 h) + \frac{(d - 1)}{x_0} \partial_0 (x_0^2 h) = -\frac{2}{x_0^4} T^{00} \, .
\]

(2.19)

In equations (2.17) through (2.19) boundary indices have been raised and lowered with \( \eta_{ij} \). Following \[14\] we decompose \( h_{mn} \) into,

\[
h_{mn} = h^\perp_{mn} + \partial_m V^\perp_n + \partial_n V^\perp_m + \partial_m \partial_n S + \frac{1}{(d - 1)} (\delta_{mn} - \frac{\partial_m \partial_n}{\Box}) h' \, ,
\]

(2.20)

Projecting the field equation (2.17) onto the boundary transverse traceless part we obtain

\[
\left( x_0^2 \delta_0^2 + x_0^2 \Box - (d - 5)x_0 \partial_0 - 2(d - 2) \right) h^\perp_{mn} = -\frac{2}{x_0^4} t_{mn} \, , \quad t_{mn} = P_{mnij} T_{ij} \, ,
\]

(2.21)

where \( P_{mnij} \) is the transverse traceless projector on the boundary flat Minkowski space. The propagator for the fluctuation \( x_0^2 h^\perp_{mn} \) is then the same as that of a massless scalar.

The remaining unphysical components are constrained by (2.18) and (2.19),

\[
\partial_0 (x_0^2 V^\perp_i) = -\frac{2}{x_0^4} \frac{1}{\Box} \left( \delta_{ij} - \frac{\partial_i \partial_j}{\Box} \right) T^0_j \, ,
\]

(2.22)

\[
\partial_0 (x_0^2 h') = -\frac{2}{x_0^4} \frac{1}{\Box} \partial_j T^0_j \, ,
\]

(2.23)

\[
\partial_0 (x_0^2 S) = \frac{x_0}{(d - 1)} \left( -\frac{2}{x_0^4} \frac{1}{\Box} T^{00} + x_0^2 h' \right) + \frac{2}{x_0^4} \frac{1}{\Box} \partial_j T^0_j \, .
\]

(2.24)

We then use the fact that \( T^{\mu\nu} \) is conserved,

\[
D^\mu T^{\mu\nu} = 0 \rightarrow \begin{cases} \partial_i T^{ij} = -x_0^{d+3} \partial_0 x_0^{-d-3} T^{0j} \\ \partial_i T^{0i} = -x_0^{d+2} \partial_0 x_0^{-d-2} T^{00} = -\frac{1}{x_0} T^{jj} \end{cases}
\]

(2.25)

and use these relations in the action. Integrating twice by parts we find the source action (up to boundary terms which do not contribute to bulk propagation) \[14\],

\[
S = \frac{1}{2 \kappa^2_{d+1}} \int d^{d+1} y \sqrt{g(y)} \, d^{d+1} z \sqrt{g(z)} \, t_{ijkl}^2 \eta_{ik} \eta_{jkl} g_{h^\perp}(y, z) t^{kl}(z) \bigg|_{T^{0i} = 0} \frac{4}{x_0^4} \frac{1}{\Box} T^{00} = -\frac{1}{x_0^4} \frac{1}{\Box} \partial_j T^{0j} \bigg|_{x_0^4} \frac{(d - 2)}{(d - 1)x_0^4} \frac{1}{\Box} \partial_j T^{0i} \bigg| .
\]

(2.26)
Explicitly the Green’s function for the physical modes $h_{ij}^\perp$ is:

\[ G_{h^\perp}(x, y) = \frac{1}{(x_0 y_0)^2} G_{\Phi, m^2=0}(x, y) \]

\[ = (x_0 y_0)^{d/2-2} \int d\lambda \int \frac{d^dk}{(2\pi)^d} \frac{J_{d/2}(\lambda x_0) J_{d/2}(\lambda y_0)}{\lambda^2 + \vec{k}^2 - i\epsilon} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}. \tag{2.27} \]

One may also find the propagator for vectors in the $A_0 = 0$ gauge in a similar way. In that case one obtains the source action \[14\].

\[ S = \frac{1}{2} \int d^{d+1}y \sqrt{g(y)} \int d^{d+1}z \sqrt{g(z)} \mathcal{J}^\perp_{\perp, ij} G_{A}(y, z) \mathcal{J}^\perp_{\perp, j} \]

\[ - \frac{1}{2} \int d^{d+1}x \sqrt{g(x)} \mathcal{J}_0 \Box \mathcal{J}_0, \tag{2.28} \]

The correlator of physical polarizations is proportional to the propagator of a massive field with mass $m^2 = 1 - d$

\[ G_{A}(x, y) = \frac{1}{(x_0 y_0)^{d/2-1}} G_{\Phi, m^2=1-d}(x, y) \]

\[ = (x_0 y_0)^{d/2-1} \int d\lambda \int \frac{d^dk}{(2\pi)^d} \frac{J_{d}(\lambda x_0) J_{d}(\lambda y_0)}{\lambda^2 + k^2 - i\epsilon} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}. \tag{2.30} \]

where $\rho^2 = (1 - d) + d^2/4$ (or $\rho^2 = 1$ when $d = 4$).

A brief comment is in order about the apparent additional poles in the $A_0 = 0$ and $h_{0\mu} = 0$ gauge fixed form of the bulk action. (The effective action will in general be gauge dependent but gauge invariant when the external sources do not themselves satisfy field equations, for example in the background field method.) Related to these gauge fixed actions is the light-cone form of Yang-Mills theory, in the spinor notated form,

\[ \mathcal{L} = \text{Tr} \quad \bar{A} \Box A + \partial_+ \bar{A} \left[ \left( \partial^a + \frac{1}{\partial_+} A \right) \left( \partial_a + \frac{1}{\partial_+} A \right) \right] + \text{c.c.} + \mathcal{O}(A^2, \bar{A}^2), \tag{2.31} \]

which has a similar structure as the gauge fixed actions above, and is readily available to compute gauge invariant S-matrices as well as effective actions (including its $N = 4$ extension). Time derivatives are $\partial_+$. In gauge invariant expressions the potential poles are spurious.

Within the imaginary parts of the correlator involving non-covariantly gauge fixed intermediate states, the poles in the gauge fixed action in the latter terms of (2.26) do contribute. They lead to contributions in the holographic Feynman diagrams with a $\Box - i\epsilon$ in the denominator and contribute to a cut only when $(\vec{k}_i + \vec{k}_j)^2 = 0$. Presumably, these contact interactions exist to cancel spuriously introduced infra-red divergent terms.
3 Unitarity and Cutting

We shall adopt the unitarity cutting procedure to the supergravity theory in order to compute the four-point (and higher) correlators. In flat space the imaginary part in a particular channel is read off by complex conjugation through the identity,

\[
\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} = 2\pi i \delta(x) \tag{3.1}
\]

Holding, for example, in the momentum space expression the external momenta \(k_i^2 > 0\) and analytically continuing \(s_{12}\) to negative values will pick out the unitarity cut in this channel. If we denote the generic four-point correlator, \(C(k_j)\) and its complex conjugate \(C^*(k_j)\); the imaginary part \(C - C^*\), after imposing \(s_{12} < 0\) and \(k_i^2 > 0\) receives a contribution only from cutting the intermediate propagator in the \(s_{12}\) channel via the relation in (3.1).

To see the unitarity relations in its simplest form consider a scalar field theory in Minkowski space with a cubic interaction. The propagator for this field is given by the usual

\[
\langle \varphi(x)\varphi(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \tag{3.2}
\]

The tree level \(2 \rightarrow 2\) scattering diagram is thus given by two vertices sandwiching a propagator. The imaginary part of this graph in the two-particle channel is found through the use of (3.1); we immediately see that the graph splits in the product of two three point graphs where the intermediate state is on shell. Note, however, that the external lines are not required to be on shell, but only have to satisfy momentum conservation.

The supergravity result should yield the leading strong effective coupling term in the planar diagram sector of the theory. In gauge theory at one-loop the reduced tensor integrations in supersymmetric theories enable one to find the complete correlator (and effective actions) from explicit information of only the imaginary parts in all channels [25, 26]. The extraction of the imaginary part requires performing the phase space integral,

\[
\text{Im}|_{s} A_{n;1}(k_1, \ldots, k_n) = \sum_{\lambda_1, \lambda_2} \int d\phi_2 A_{n;1}^{\text{tree}}(k_1, \ldots, p_1^{\lambda_1}, \ldots, p_2^{\lambda_2}, \ldots, k_n) \nonumber
\]

\[
\times A_{n;1}^{\text{tree}}(k_1, \ldots, p_1^{-\lambda_1}, \ldots, p_2^{-\lambda_2}, \ldots, k_n), \tag{3.3}
\]

where \(d\phi_2\) is the two-particle phase space measure, \(\delta(p_1^2)\delta(p_2^2)\Theta(p_1^0)\Theta(p_2^0)\), and we have suppressed color structures. Extracting the full amplitude follows from the logarithmic orthogonality of the one-loop integral reduction formulae [25, 26].

At two or more loops the cuts in a particle channel require higher than two-particle cuts; for example, one may take a cut of a double box through a two-particle cut. This is even the case for non-supersymmetric theories, provided one keeps the full form of the dimensionally regulated integral functions.
or three-particle channel. Such generalizations of the program to finding complete correlators (and S-matrix elements) from the imaginary parts, as at one-loop, have only been partially developed [29].

In the previous section we introduced the necessary \( i \epsilon \) terms in the Lorentzian formulation of the AdS supergravity theory. The delta-function identity \( (3.1) \) then relates the imaginary part of the four-point function in the two-particle channels to two three-point boundary-boundary-bulk functions, the functional form of which may be explicitly calculated. Different from conventional field theory is that the prescription of [2, 3] instructs us to work with Green’s functions with bulk-boundary kernels attached to the external legs.

### 4 Factorization

We shall illustrate our technique with a calculation involving four scalar fields, axions, of IIB supergravity on \( \text{AdS}_5 \times S_5 \). The general four-point scalar function involves graphs built with a bulk four-point vertex, which do not contribute to the imaginary parts in the \( s_{ij} \) channels, and graphs with two three-point vertices joined by an intermediate scalar or graviton line. We shall here consider in detail the \( \langle C(\vec{x}_1)C(\vec{x}_2)C(\vec{x}_3)C(\vec{x}_4) \rangle \) correlator dual to the the \( N = 4 \) SYM four-point function \( \langle \prod_{i=1}^4 \text{Tr} F\tilde{F}(\vec{x}_i) \rangle \) as it is the simplest to analyze. In this case there are no four-point vertices and only six holographic diagrams of the second type described above, each with an internal line associated with a scalar or graviton in the \( s, t, \) or \( u \) channel.

#### 4.1 Scalar Exchange

We first examine the contribution to this correlator coming from the dilaton exchange. The \( \phi(x)C(x)C(x) \) vertex is,

\[
\mathcal{L}_{\phi CC} = -\frac{1}{2\kappa^2} \sqrt{g} g^{\mu\nu} \phi \partial_\mu C \partial_\nu C ,
\]

and hence the expression for the correlator arising from an internal dilaton (\( \phi \)) line is,

\[
A^{\phi, s}_{CCCC}(\vec{x}_i) = \frac{4 \cdot 2}{2!} \frac{1}{2\kappa^2} \int d^{d+1}z_1 \sqrt{g(z_1)} \int d^{d+1}z_2 \sqrt{g(z_2)}
\]

\[
\times \left[ g^{\mu\nu}(z_1) \partial_\mu \Delta(\vec{x}_1, z_1) \partial_\nu \Delta(\vec{x}_2, z_1) \right] G_{\Phi, m^2=0}(z_1, z_2)
\]

\[
\left[ g^{\alpha\beta}(z_2) \partial_\alpha \Delta(\vec{x}_3, z_2) \partial_\beta \Delta(\vec{x}_4, z_2) \right] ,
\]

(4.2)

together with the \( t \)- and \( u \)-channel diagrams. After Fourier transforming with respect to the directions parallel to the boundary

\[
A^{\phi, s}_{CCCC}(\vec{k}_i) = \left( \prod_{j=1}^4 \int d^d x_i e^{i\vec{k}_j \cdot \vec{x}_j} \right) A^{\phi, s}_{CCCC}(x_1, x_2, x_3, x_4) .
\]

(4.3)
we find the momentum space expression for the \(s\)-channel contribution to the correlator:

\[
A^{\phi,s}_{CCCC}(\vec{k}_1) = \frac{2}{\kappa^2} \delta^{(d)}(k_1 + k_2 + k_3 + k_4) \int d\nu d\mu \, \frac{\lambda}{\lambda^2 + (\vec{k}_0 + \vec{\kappa})^2} \int dz_0 \, \frac{\nu}{\nu^2 - \mu^2} J_{d/2}(\lambda y_0) J_{d/2}(\lambda z_0),
\]

where

\[
I(\vec{k}_1, \vec{k}_2, y_0) = -\vec{k}_1 \cdot \vec{k}_2 \Delta(k_1, y_0) \Delta(k_2, y_0) + \partial_{y_0} \Delta(k_1, y_0) \partial_{y_0} \Delta(k_2, y_0). \tag{4.5}
\]

The imaginary part of the expression in (4.4), when holding \(k_0^2 > 0\) and \(s = (\vec{k}_1 + \vec{k}_2)^2 < 0\), receives a contribution equal to the effective replacement of the bulk-bulk propagator with

\[
-\pi \int d\lambda \lambda \delta(\lambda^2 + \vec{k}^2) (y_0 z_0)^{d/2} J_{d/2}(\lambda y_0) J_{d/2}(\lambda z_0). \tag{4.6}
\]

The \(\lambda\) integral may be performed and we shuffle the factor \(y_0^{d/2} J_{d/2}(\lambda y_0)\) to one side of the cut and the \(z_0\)-term to the other side (the additional factor of \(\lambda\) is cancelled by the delta function). We also have \(\lambda = \pm i \sqrt{k^2}\), with \(\vec{k} = \vec{k}_1 + \vec{k}_2\).

In general then, for any massive intermediate field (or gauge field for that matter), the cut of the four-point holographic Feynman diagram reduces to the product of two boundary-boundary-bulk three-point functions. In the case above, in (4.4), we have the example of an internal scalar and hence

\[
\text{Im} \ A^{\phi,s}_{CCCC}(\vec{k}_j) = -\frac{\pi}{\kappa^2} \delta^{(d)}(k_1 + k_2 + k_3 + k_4) M(\vec{k}_1, \vec{k}_2, \lambda) M(\vec{k}_3, \vec{k}_4, \lambda) |_{\lambda^2 = -\vec{k}^2}, \tag{4.7}
\]

where \(\vec{k} = \vec{k}_1 + \vec{k}_2\) and

\[
M(\vec{k}_1, \vec{k}_2, \lambda) = \frac{2^{2-d}}{\Gamma^2(d/2)} \int d\nu d\mu \int dz_0 \frac{\lambda}{\lambda^2 + (\vec{k}_0 + \vec{\kappa})^2} \int dz_0 \frac{\nu}{\nu^2 - \mu^2} J_{d/2}(\lambda y_0) J_{d/2}(\lambda z_0)
\]

\[
\times \left[ -\vec{k}_1 \cdot \vec{k}_2 (y_0 k_1)^{d/2} (y_0 k_2)^{d/2} K_{d/2}(y_0 k_1) K_{d/2}(y_0 k_2) + \partial_{y_0} \left\{ (y_0 k_1)^{d/2} K_{d/2}(y_0 k_1) \right\} \partial_{y_0} \left\{ (y_0 k_2)^{d/2} K_{d/2}(y_0 k_2) \right\} \right]. \tag{4.8}
\]

The second integral in (4.8) may be simplified by noting that

\[
\partial_{y_0} \left\{ (y_0 k_1)^{d/2} K_{d/2}(y_0 k_1) \right\} = k_1 \frac{\partial}{\partial k_1} \left\{ \frac{1}{y_0} (y_0 k_1)^{d/2} K_{d/2}(y_0 k_1) \right\}, \tag{4.9}
\]

\[\text{Note that since the imaginary part is nonzero the above diagram does not reduce solely to that of an effective four-point vertex. Previous articles [14, 15] used partial integration of the bulk derivatives to relate scalar exchange diagrams to an effective four-point vertex. The integration, however, produces boundary terms that contribute to cuts.}\]
and then extracting the momentum derivatives outside the integration.

To proceed further, we need the expression for a triple convolution of Bessel functions (related integrals are given in [27]). In order to evaluate integrals of the general form,

$$B_{\nu_1,\nu_2,\nu_3}^t = \int_0^\infty dy \, y^{t+1}K_{\nu_1}(k_1y)K_{\nu_2}(k_2y)J_{\nu_3}(\lambda y) , \quad (4.10)$$

we use the following integral expression for the modified Bessel function $K_\nu(x)$:

$$K_\nu(\lambda x) = \frac{1}{2} \left( \frac{2x}{\lambda} \right)^\nu \int_0^\infty d\tau \, \tau^{\nu-1} e^{-x^2/4\tau - \lambda^2/4\tau} . \quad (4.11)$$

We would also like to point out that in three-point correlation function calculations of bilinear operators, the $\tau$ parameters of the bulk-boundary kernels when expressed through (4.11) become Schwinger parameters of triangle integrals [6] (offering a partial explanation that these correlations are derived in field theory at one-loop).

Using (4.11) the triple convolution in (4.10) may be expressed as

$$B_{\nu_1,\nu_2,\nu_3}^t = \frac{1}{4} \left( \frac{4}{k_1^2} \right)^{\nu/2} \left( \frac{4}{k_2^2} \right)^{\nu/2} \int \prod_{j=1}^2 d\tau_j \, \tau_j^{\nu_j - 1} e^{-k_j^2/4\tau_j} \int_0^\infty dy \, y^{t+1+\nu_1+\nu_2} e^{-(\tau_1+\tau_2)y^2} J_{\nu_3}(\lambda y) . \quad (4.12)$$

The $y$-integration may now be done, and it gives the formal expression [28],

$$\int_0^\infty dy \, y^{t+1+\nu_1+\nu_2} e^{-(\tau_1+\tau_2)y^2} J_{\nu_3}(\lambda y) = \frac{1}{2} \frac{\Gamma(\frac{1}{2}[\nu_1 + \nu_2 + \nu_3 + t] + 1)}{\Gamma(\nu_3 + 1)} \times \left[ \frac{\lambda^2}{4(\tau_1 + \tau_2)} \right]^{\nu_3/2} _1F_1 \left[ 1 + \frac{1}{2}(\nu_1 + \nu_2 + \nu_3 + t); \nu_3 + 1; -\frac{\lambda^2}{4(\tau_1 + \tau_2)} \right] , \quad (4.13)$$

where $_1F_1$ is a confluent hypergeometric function. Its defining series representation,

$$_1F_1[a; b; x] = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b + n)} \frac{x^n}{n!} , \quad (4.14)$$

with the condition that $a \geq b$, permits us to write it in the form,

$$_1F_1(a; b; x) = \frac{\Gamma(b)}{\Gamma(a)} P_{a-b}(x) e^x . \quad (4.15)$$

The $P_{a-b}(x)$ is a polynomial of degree $a - b$, related to the associated Laguerre polynomials $L_n^k(x)$, with leading unit coefficient,

$$P_{a-b}(x) = x^{a-b} + \gamma_0 x^{a-b-1} + \ldots + \gamma_{a-b}$$

$$= \Gamma(a - b + 1)e^x L_{a-b}^{b-1}(-x) . \quad (4.16)$$
For \( a - b \in \mathbb{Z}^+ \), the operator \( P_{a-b}(\frac{\partial}{\partial \mu}) \) has the simple representation

\[
P_{a-b}(\frac{\partial}{\partial \mu})f(\mu) = \left( \frac{\partial}{\partial \mu} \right)^{a-b} \left[ \mu^{a-1} f(\mu) \right].
\]

(4.17)

In the case \( a - b \) is non-integral we may formally regard the operator as an infinite series expansion. However, all the cases analyzed in detail here will pertain to integral values of \( a - b \).

The re-expression of the confluent function \( _1F_1 \) in (4.15) allows us to complete the \( \tau_i \) integrals. The integral \( B_{\nu_1,\nu_2,\nu_3}^t \) after extracting the polynomial derivative \( P_{a-b} \) is,

\[
B_{\nu_1,\nu_2,\nu_3}^t = \frac{1}{8} \left( \frac{4}{k^2_1} \right)^{\nu_1/2} \left( \frac{4}{k^2_2} \right)^{\nu_2/2} \left( \frac{\lambda^2}{4} \right)^{\nu_3/2} P_{\frac{1}{2}((\nu_1+\nu_2+t-\nu_3)}(\frac{\partial}{\partial \mu})
\]

\[
\times \int \prod_{j=1}^{2} d\tau_j \tau_j^{\nu_j-1} e^{-k^2_j/4\tau_j} (\tau_1 + \tau_2)^{-\frac{1}{2}(\nu_1+\nu_2+\nu_3+t+2)} e^{-\mu \lambda^2/4(\tau_1+\tau_2)}.
\]

(4.18)

Now switching integration variables first to,

\[
\tau_i \rightarrow \frac{1}{\tau_i} \quad d\tau_i \rightarrow -\frac{1}{\tau_i^2} d\tau_i,
\]

(4.19)

and then to

\[
\tau_1 = \alpha \tau \quad \tau_2 = (1-\alpha) \tau \quad d\tau_1 d\tau_2 = \tau d\tau d\alpha,
\]

(4.20)

we may also complete the integral over \( \tau \),

\[
B_{\nu_1,\nu_2,\nu_3}^t = \frac{2^{t-1} \lambda^{\nu_3}}{k_1^{\nu_1} k_2^{\nu_2}} \Gamma(a - \nu_1 - \nu_2) \left( \frac{\partial}{\partial \mu} \right)^{a-b} \mu^{a-1}
\]

\[
\times \int_0^1 d\alpha \left[ \frac{\alpha^{a-\nu_2-1}(1-\alpha)^{a-\nu_1-1}}{(k_1^2(1-\alpha) + k_2^2 \alpha + \mu \lambda^2 \alpha(1-\alpha))^{a-\nu_1-\nu_2}} \right]_{\mu=1},
\]

(4.21)

with

\[
2a = \nu_1 + \nu_2 + \nu_3 + t + 2, \quad b = \nu_3 + 1.
\]

(4.22)

In the case of massless scalar exchange, we have \( \nu_1 = \nu_2 = \nu_3 = t = d/2 \) and the remaining \( \alpha \) integration is finite; we set \( d = 4 \) as well. Substituting these values the last integral may be performed, and yields

\[
B_{2,2,2}^2 = -\frac{2}{k_1^2 k_2^2} \left( \frac{\partial}{\partial \mu} \right)^2 \mu^3 \left\{ \frac{1}{3} - \frac{3}{2} q_+ + \frac{3}{2} q_- + (q_+^2 + q_+ q_- + q_-^2)
\right.
\]

\[
+ \frac{q_+^2(1-q_-)^2}{(q_+ - q_-)} \ln \left[ \frac{1 - q_+}{q_+} \right] - \frac{q_-^2(1-q_+)^2}{(q_+ - q_-)} \ln \left[ \frac{1 - q_-}{q_-} \right] \right\}_{\mu=1}
\]

(4.23)

where \( q_\pm \) are the roots of the quadratic,

\[
\alpha^2 - \frac{k_1^2 - k_2^2 + \mu \lambda^2}{\mu \lambda^2} \alpha - \frac{k_2^2}{\mu \lambda^2} = 0,
\]

(4.24)
or,

$$q_{\pm} = \frac{(k_{1}^2 - k_{2}^2 + \mu \lambda^2)}{2\mu \lambda^2} \pm \sqrt{(k_{1}^2 - k_{2}^2 + \mu \lambda^2)^2 + 4\mu k_{1}^2 \lambda^2} \cdot \frac{2}{2\mu \lambda^2}. \quad (4.25)$$

It is noteworthy that the functional form of (4.23) contains only rational functions in the external momenta and logarithms.

Similarly, the second integral in (4.8) has \( \nu_1 = \nu_2 = \nu_3 = d/2 \) and \( t = d/2 - 2 \). This integral is also finite and may be evaluated to:

$$k_1 \frac{\partial}{\partial k_1} k_2 \frac{\partial}{\partial k_2} B^{d/2-2}_{d/2,d/2,d/2} = 32 k_1^2 k_2^2 \lambda^2 P_1 \left( \frac{\partial}{\partial \mu} \right) \left[ \frac{\partial}{\partial (\mu \lambda^2)} \right]^2 \times \left\{ \ln(-\mu k_3^2) + \sum_{\pm} \ln(1 - q_{\pm}) + q_{\pm} \ln(-\frac{q_{\pm}}{1 - q_{\pm}}) \right\} \bigg|_{\mu = 1}, \quad (4.26)$$

with \( d = 4 \). Alternatively, one may use the recursion relation

$$\frac{1}{x^\nu} \partial_x (x^\nu K_\nu(\lambda x)) = -\lambda K_{\nu - 1}(\lambda x), \quad (4.27)$$

instead of (4.9). The second contribution to \( M(\vec{k}_1, \vec{k}_2, \lambda) \) is then proportional to,

$$B_{1,1,2}^2 = \frac{2}{\lambda^2 \sqrt{k_1^2 k_2^2}} \left( \frac{\partial}{\partial \mu} \right) \left( \frac{\mu}{(q_+ - q_-)^2} \right) \left\{ \frac{1}{q_+(q_+ - 1)} + \frac{1}{q_-(1 - q_-)} \right\} \times \left[ \frac{1 - q_+}{q_+} \right] + \frac{2}{(q_+ - q_-)} \ln \left( \frac{1 - q_+}{q_+} \right) \right\} \bigg|_{\mu = 1}. \quad (4.28)$$

The cut in the s-channel of the \( A^0_{CCCC} \)-correlation function arising from the intermediate dilaton exchange is then,

$$\text{Im} \ A^{\phi,4}_{CCCC}(\vec{k}_j) = -\frac{\pi}{k^2} \delta^4(k_1 + k_2 + k_3 + k_4) M(\vec{k}_1, \vec{k}_2, \lambda) M(\vec{k}_3, \vec{k}_4, \lambda) \big|_{\lambda^2 = -\vec{k}^2}, \quad (4.29)$$

with

$$M(\vec{k}_1, \vec{k}_2, \lambda) = \left[ -k_1 \cdot k_2 B^2_{2,2,2} + k_1 \frac{\partial}{\partial k_1} k_2 \frac{\partial}{\partial k_2} B^0_{2,2,2} \right] \left[ -k_1 \cdot k_2 B^2_{2,2,2} + \sqrt{k_1^2 k_2^2} B^2_{1,1,2} \right]. \quad (4.30)$$

Similar manipulations may be used to find the contribution from massive states as well.

We end this section by noting how the entire exchange diagram with the intermediate scalar may be found. Effectively in the above we have integrated completely the fifth coordinates \( y_0 \) and \( z_0 \) laying at the ends of the bulk-bulk propagator. We

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may alternatively use these integrals to reconstruct the full diagram through one more additional integral (i.e. the one over \( \lambda \)),

\[
A^{\phi,s}_{CCCC}(\vec{k}_j) = \frac{2}{\kappa^2} \delta^d(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \int_0^\infty d\lambda \frac{\lambda}{\lambda^2 + (\vec{k}_1 + \vec{k}_2)^2} M(\vec{k}_3, \vec{k}_4, \lambda). 
\]

(4.31)

The \( \lambda \) integration is over a product of logarithms and rational functions, and it is reasonable to suspect that this last integration may be done generally and explicitly.

### 4.2 Graviton Exchange in \( h_{\mu 0} = 0 \) Gauge

In this gauge the contribution of the intermediate graviton is also easily computed. The bulk-bulk Green’s function for the physical polarizations is that of a massless scalar, and the calculation is nearly the same as in the previous section.

The coupling of the axion fields to the graviton is derived by expanding the quadratic term

\[
L_{hCC} = -\frac{1}{4\kappa^2} \sqrt{g} g^{\mu\nu} \partial_\mu C \partial_\nu C + \frac{1}{2\kappa^2} \sqrt{g} h_{\mu\nu} T^{\mu\nu}_C
\]

(4.32)

about the anti-de Sitter background \( g_{\mu\nu} \). This defines the energy-momentum tensor induced from the scalar matter,

\[
2T^{\mu\nu}_C(z) = \partial^\mu C(z) \partial^\nu C(z) - \frac{1}{2} g^{\mu\nu} \partial^\alpha C(z) \partial_\alpha C(z).
\]

(4.33)

Using the induced action in (2.26) we may derive the \( s \)-cut in the graviton exchange diagram.

The projection onto the transverse traceless components \( t_{ij} \) gives

\[
t^{ij}(z) = \frac{x^4_0}{2} P^{ij, mn} \left[ \partial_m C \partial_n C - \frac{1}{2} g_{mn} \partial^\alpha C \partial_\alpha C \right]
\]

(4.34)

with which we may compute the exchange diagram coming from the physical components of the graviton. Recall that \( P^{ij, mn} \) is the transverse traceless projector on the \( d \)-dimensional boundary.

The first term in (2.26) then gives the contribution,

\[
A^{h_{\mu 0} = 0}_{CCCC}(x_i) = \frac{4 \cdot 2}{2!} \frac{1}{(2\kappa^2)} \int d^{d+1} y \sqrt{g(y)} \int d^{d+1} z \sqrt{g(z)} \left[ \partial_m \Delta(x_1, y) \partial_n \Delta(x_2, z) \right]
\]

\[
\times P^{mn, rs}(y_0 z_0)^4 G_{h_{\mu 0} = 0}(y, z) \left[ \partial_{\alpha} \Delta(x_3, z) \partial_\alpha \Delta(x_4, z) \right]
\]

(4.35)

Hence we find the contribution in the \( s \)-channel,

\[
\text{Im} A^{h_{\mu 0} = 0}_{CCCC} = -\frac{\pi}{\kappa^2} \delta^{(d)}(k_1 + k_2 + k_3 + k_4) M^{ab}(\vec{k}_1, \vec{k}_2, \lambda) M_{ab}(\vec{k}_3, \vec{k}_4, \lambda) \bigg|_{\lambda^2 = -\vec{k}^2}.
\]

(4.36)
Immediate application of the preceding techniques gives us,

\[ M^{ab}(\vec{k}_1, \vec{k}_2, \lambda) = -P^{ab, mn}k_1^m k_2^n B^{d/2}_{d/2,d/2,d/2} \] (4.37)

In \( d = 4 \) the integral is finite and has been done before; we only quote the result,

\[ B^{2,2,2} = -\frac{2}{k_1^2 k_2^2} \left( \frac{\partial}{\partial \mu} \right)^2 \mu^3 \left\{ \frac{1}{3} - \frac{3}{2}(q_+ + q_-) + (q_+^2 + q_+ q_- + q_-^2) \right. 
+ \left. \frac{q_+^2 (1 - q_+)^2}{(q_+ - q_-)} \ln \left[ \frac{1 - q_+}{q_+} \right] - \frac{q_-^2 (1 - q_-)^2}{(q_+ - q_-)} \ln \left[ \frac{1 - q_-}{q_-} \right] \right\} \bigg|_{\mu=1}. \] (4.38)

Again the final form is such that there are only logarithms and rational functions contributing.

The remaining diagrams are generated by interactions resembling four-point vertices; however, from the \( i\epsilon \) contained in the transverse flat box, \( \Box - i\epsilon \), there are unitarity cuts only when the on-shell condition \( (\vec{k}_i + \vec{k}_j)^2 = 0 \) is satisfied. Although we shall not consider these terms in any more detail, these integrals may be evaluated and are necessary in the construction of the full correlator.

5 Other Correlator examples

Although we have examined a few of the diagrams involved in the general correlator, the methods (and momentum space integration techniques) generalize to other correlator expressions.

Let us consider for example the correlator of two dilatons and two axions, i.e. \( \langle \phi(\vec{k}_1) C(\vec{k}_2) \phi(\vec{k}_3) C(\vec{k}_4) \rangle \). The main difference between this example and those in preceding sections involves the derivative structure acting on the intermediate axion line joined between two three-point vertices:

\[ A^{C,s}_{\phi C,\phi C}(\vec{x}_j) = \frac{2}{(2\pi)^2} \int \sqrt{g(z_1)} dz_1 \int \sqrt{g(z_2)} dz_2 \]
\[ g^{\alpha\beta}(z_1) \left[ \Delta(\vec{x}_1, z_1) \partial^{(1)}_\beta \Delta(\vec{x}_2, z_1) \right] g^{\mu\nu}(z_2) \left[ \Delta(\vec{x}_3, z_2) \partial^{(2)}_\mu \Delta(\vec{x}_4, z_2) \right] \]
\[ \times \partial^{(1)}_\alpha \partial^{(2)}_\nu G_{\Phi, m^2=0}(z_1, z_2). \] (5.1)

The derivative with respect to \( y_0 \) on the (bulk-)Bessel function \( J_\nu(\lambda z_i) \) can be evaluated using either the recursion relation in (4.13) or (4.27). Alternatively one may partially integrate this derivative away onto the bulk-boundary propagator carrying no derivative. The boundary term vanishes due to the Dirichlet condition on the bulk to bulk Green’s function and the remaining terms produced by the partial integration conspire to

\[ \frac{1}{\sqrt{g(z_i)}} \left( \partial^{(i)}_\mu \sqrt{g(z_i)} g^{\mu\nu}(z_i) \partial^{(i)}_\nu \right) \Delta(\vec{x}, z_i) = 0. \] (5.2)

\(^5\)This is a special case; in general boundary terms do not vanish.\]
The integral is then the same as in (4.4).

We see that again the imaginary part of these expressions factorizes into two identical functions.

\[ \text{Im} \ A_{\phi C \phi C}^{C,s}(\vec{k}_j) = -\frac{\pi}{4\kappa^2}\delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \ M(\vec{k}_1, \vec{k}_2, \lambda)M(\vec{k}_3, \vec{k}_4, \lambda) \bigg|_{\lambda^2 = -\vec{k}_2} \ . \tag{5.3} \]

This factorization occurs for all diagrams because the correlation functions are generated through supergravity tree diagrams. In principle, one may explicitly compute this way the contributions from the exchange of massive scalars and other tensor fields, once the propagators are known. The general structure of the imaginary part of any correlator in momentum space, following from the generic structure of intermediate propagators in (2.1), is thus

\[ \text{Im} \ C(\vec{x}_j) = \sum_{I} \delta^{(d)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \ M^{I}(\vec{k}_1, \vec{k}_2, \lambda)M^{I}(\vec{k}_3, \vec{k}_4, \lambda) \bigg|_{\lambda^2 = -\vec{k}_2} \ . \tag{5.4} \]

This factorization at strong coupling is a notable feature in field theory, given that in general, unitarity cuts in perturbation theory introduce multi-particle phase space integrals over intermediate states (momenta, helicity, spin); these integrations generically do not result in sums of factorized products of functions.

\section{Implications for $N = 4$ SYM}

We would like to point out how the logarithmic dependence in the $\langle \prod_{j=1}^{4} \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}(k_i) \rangle$ correlator might arise. First, generically in perturbation theory one encounters $\text{Li}_{t+1}(x)$ functions, i.e. polylogarithms,

\[ \text{Li}_k(x) = -\int_0^x dt \frac{\text{Li}_{k-1}(t)}{t} \ , \quad \text{Li}_0(x) = \frac{x}{1-x} \ , \tag{6.1} \]

at each $l$-loop order. For example, in $d$-dimensions at one-loop all $n$-point functions may be algebraically reduced onto a basis of integral functions present in $p$-point functions with $p \leq d$ \[30\]. In four dimensions all of these integral functions have been computed and contain only rational functions, logarithms, and dilogarithms. At higher-loop certain classes of Feynman diagrams have been computed; however, one may understand the logarithmic dependence by examining the cut structure. For example, consider taking a two-particle cut of a double box function in the $s$-channel, which separates the double box into a product of a four-point tree and four-point loop. The phase space integral of the (off-shell, i.e. $k_i^2 \neq 0$) cut double-box involves a two-body phase space integration over a rational function times a function involving dilogarithms. Such integrations, in principle, produce $\text{Li}_3(x)$ functions. One may approximately understand the complicated cut structure of higher loop Feynman diagrams in this manner.

\footnote{We would like to thank Zvi Bern for numerous discussions on this point.}
The fact that at large coupling, the imaginary part in a two-particle channel involves only a product of functions possessing at most logarithms has the immediate prediction that the result may not arise at one-loop in field theory. This is because there are no one-loop integral functions possessing a $s_{ij}$-channel cut that can produce squares of logarithms (as they do not cancel in preceding sections). From the above computations it is clear that the presence of logarithm squared terms is generic; it appears then that AdS four-point correlators of fields dual to bilinear (chiral primary and descendent) operators do not reflect free-field computations from the CFT-side given this property.

The interpretation of the strong coupling result within the field theory context requires that either: the degree $Li_k$ functions conspire to cancel at every order in perturbation theory, or that the resummation of the large number of loops add in a dramatic fashion (after resumming and expanding in the inverse ’t Hooft coupling constant $\lambda = g^2_{YM}N_c$). An explicit demonstration of either of these possibilities is difficult to present at the moment and deserves further study.

7 Conclusion

In this work we have computed all of the integrals necessary for a complete evaluation of the imaginary parts of the general $N = 4$ current correlator at strong coupling in the ’t Hooft limit. In the Lorentzian signature formulation of the correspondence, an $i\epsilon$ prescription is provided that permits the cutting of the scalar and graviton propagators. In this work the graviton is analyzed in the non-covariant $h_{\mu0} = 0$ gauge; however, other gauge choices lead to similar results.

Explicit expressions are given here for the imaginary parts in the two-particle channels in the boundary theory momentum space. There are two noteworthy aspects of these results. First, the imaginary part for the $\langle \prod_{j=1}^{4} \text{Tr} F_{\mu
u} \tilde{F}_{\mu\nu}(\vec{k}_j) \rangle$ correlator, as well as others, at strong ’t Hooft coupling factorizes into a product of identical functions. Generically (and in any gauge choice) any correlator will factorize at strong effective coupling into a sum of products of functions due to the correspondence with the classical supergravity formulation. Second, after explicitly computing the integrals we see that the final expressions contain only logarithms and rational functions in the kinematic invariants. This feature is difficult to see in pure $N = 4$ super-Yang-Mills theory, as generically polylogarithms of any degree are expected. The expressions derived in this work indicate large orders of cancellations of such logarithmic functions.

The simplicity of the final expressions suggest that an additional non-trivial structure besides the superconformal constraints might be appearing in the $N = 4$ super Yang-Mills theory at strong coupling (e.g. [31]). It would be interesting to find out if there were any further symmetries of the classical field equations of gauged
supergravity on anti-de Sitter space; the associated conserved charges would induce those on the boundary theory and give rise to further Ward identities (possibly explaining the simple momentum space integral results obtained in this work). Such symmetries are known to exist in several sets of field theory equations of motion: in the context of recursively generated conserved currents of self-dual Yang-Mills field equations, and Liouville theory which is classically equivalent to a free field theory.

Further work in field theory requires in the least higher order calculations associated with the four-point correlators. A calculation of the \( \langle \prod_{j=1}^{4} \text{Tr} F_{\mu \nu} \tilde{F}_{\mu \nu}(k_j) \rangle \) correlator in \( N = 4 \) field theory would shed light, for example, on how our results do not agree with free-field \( N = 4 \) super-Yang-Mills theory. Free-field expressions agree at the three-point function level for correlators of chiral primary operators (and their descendants); however, conformal invariance constrains these functions tightly (up to a finite number of constants). Our results indicate that generically for any four-point AdS boundary correlator the imaginary part will contain squares of logarithms (with specific examples presented in this work). Such terms cannot be produced in field theory at one-loop and thus the four-point functions indeed carry non-trivial information about the dynamics of \( N = 4 \) super-Yang-Mills theory, where the kinematic structure is not constrained by conformal invariance alone. This would be evident from a correlation function of bilinear operators where the free-field result is one-loop. We should still emphasize that within the AdS/CFT correspondence these functions are relatively simple and factorize in the two-particle channels at strong coupling. Similar features are not expected in \( 1/N_c \) corrections arising from string-loop effects in the AdS background and require intermediate phase space integrations.

A complete evaluation of the correlator is also accessible with the techniques we have generated in this work. Most of the necessary integrals have been computed and only the ones arising from four-point vertices need to be added. The latter integrals have been computed in position space. Lastly, possible connections to the Regge limit of \( N = 4 \) super-Yang-Mills theory would be worth exploring considering the simple results \[32\].

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Note added: That the large \( N_c \) limit of four-point functions of gauge invariant
operators in $N = 4$ SYM (at finite $\lambda$) is indeed not given by free-field theory has consequently been shown in [33, 34].

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