On 3-regular 4-ordered graphs

Karola Mészáros
Massachusetts Institute of Technology
karola@math.mit.edu

Abstract

A simple graph $G$ is $k$-ordered (respectively, $k$-ordered hamiltonian), if for any sequence of $k$ distinct vertices $v_1, \ldots, v_k$ of $G$ there exists a cycle (respectively, hamiltonian cycle) in $G$ containing these $k$ vertices in the specified order. In 1997 Ng and Schultz introduced these concepts of cycle orderability and posed the question of the existence of 3-regular 4-ordered (hamiltonian) graphs other than $K_4$ and $K_{3,3}$. Ng and Schultz observed that a 3-regular 4-ordered graph on more than 4 vertices is triangle free. We prove that a 3-regular 4-ordered graph $G$ on more than 6 vertices is square free, and we show that the smallest graph that is triangle and square free, namely the Petersen graph, is 4-ordered. Furthermore, we prove that the smallest graph after $K_4$ and $K_{3,3}$ that is 3-regular 4-ordered hamiltonian is the Heawood graph, and we exhibit forbidden subgraphs for 3-regular 4-ordered hamiltonian graphs on more than 10 vertices. Finally, we construct an infinite family of 3-regular 4-ordered graphs.

1 Introduction

The concept of $k$-ordered graphs was introduced in 1997 by Ng and Schultz [8]. A simple graph $G$ is a graph without loops or multiple edges, and it is called hamiltonian if there exists a cycle that contains all vertices of $G$. In this paper we consider only connected finite simple graphs. A simple graph $G$ is called $k$-ordered (respectively, $k$-ordered hamiltonian), if for any sequence of $k$ distinct vertices $v_1, \ldots, v_k$ of $G$ there exists a cycle (respectively, hamiltonian cycle) in $G$ containing these $k$ vertices in the specified order. Previous results concerning cycle orderability focus on minimum degree and forbidden subgraph conditions that imply $k$-orderedness or $k$-ordered hamiltonicity [2, 4, 5]. A comprehensive survey of results can be found in [8].

Any hamiltonian graph is necessarily 3-ordered hamiltonian as well as 3-ordered, thus we study $k$-orderedness for $k \geq 4$. Indeed, it is easy to see that hamiltonicity is equivalent to 3-ordered hamiltonicity, and 3-cyclability to 3-orderedness (a graph is said to be 3-cyclable, if for any three vertices of the graph there exists a cycle containing them). If $G$ is a hamiltonian graph of order $n \geq 3$, then $G$ being $k$-ordered hamiltonian implies that $G$ is $(k-1)$-connected (see [8]). The arguments made in [8] hold in case of $k$-orderedness as well, namely, if $G$ is a graph of order $n \geq 3$, then $G$ being $k$-ordered implies that $G$ is $(k-1)$-connected. In particular, this implies that $\delta(G)$, the minimum degree of any vertex in a $k$-ordered graph $G$ is at least $k-1$.

In [8], a search for small degree 4-ordered hamiltonian graphs was conducted and an infinite family of 4-regular 4-ordered hamiltonian graphs was exhibited. However, the only two 3-regular 4-ordered hamiltonian graphs found were $K_4$ and $K_{3,3}$. In this paper we analyze the class of all 3-regular graphs with the aim of determining whether or not there exist other 4-ordered (and 4-ordered hamiltonian) 3-regular graphs. In Section 2 we prove that a 3-regular 4-ordered graph on
more than 6 vertices is not only triangle free, but it is also square free and we show that the smallest (by the number of vertices) 3-regular triangle and square free graph, namely the Petersen graph, is 4-ordered. We also consider a common family of graphs generalizing the Petersen graph, and we show that the Petersen graph itself is the only member of this family that is 4-ordered. In Section 3, we exhibit a 4-ordered hamiltonian graph on 14 vertices, the Heawood graph, and show that it is the smallest graph after $K_4$ and $K_{3,3}$ that is 3-regular and 4-ordered hamiltonian.

In Section 3 we also exhibit forbidden subgraphs for 3-regular and 4-ordered hamiltonian graphs. Finally, in Section 4, we conclude our paper by constructing an infinite family of 3-regular 4-ordered graphs that we call torus-graphs (torus-graphs can be embedded on the torus without crossing edges). Since 3-regular graphs have the lowest possible degree for 4-ordered graphs, the construction of torus-graphs answers the question of whether there are low degree 4-ordered graphs.

2 Forbidden subgraphs and the Petersen graph

It is easy to see that no 3-regular 4-ordered hamiltonian graph on more than 4 vertices contains a triangle (see [8]). Also, 3-regular 4-ordered graphs must be triangle free, by an analogous argument.

**Theorem 2.1.** A 3-regular 4-ordered graph on more than 6 vertices does not contain a square.

**Proof.** Suppose $G$ is a 4-ordered graph on more than 6 vertices and it contains a square. By 4-orderedness, $G$ is triangle free, as noted before. If there exists a square, say with vertices $A$, $B$, $C$, and $D$ (in order) such that some pair of edges incident to opposite vertices of the square $ABCD$ do not share a vertex, then we can show that $G$ is not 4-ordered. Indeed, suppose without loss of generality, that the third edge incident to $A$ is $AE$, that the third edge incident to $C$ is $CF$, and that $E \neq F$. In this case, there can be no cycle in $G$ containing the vertices $F$, $E$, $C$, $A$ in this order because $CF$ and $AE$ cannot be edges in this cycle, which implies that $AB$, $BC$, $CD$, $DA$ are all edges in this cycle, which contradicts the existence of a cycle containing vertices $F$, $E$, $C$, $A$ in that order.

We now show that $K_{3,3}$ is the only 3-regular triangle free graph containing a square such that the edges incident to opposite vertices of any square it contains do share a vertex. Indeed, suppose $H$ is a 3-regular triangle free graph containing a square $ABCD$, such that the edges incident to opposite vertices of any square it contains do share a vertex. Then, there exists a vertex $E$ and a vertex $F$ such that such that $BE$, $DE$, $AF$, $CF$ are edges. As $H$ is triangle free, it follows that $E \neq F$. Consider the square $ADCF$ in $H$, and its opposite vertices $D$ and $F$. By assumption, the edges incident to opposite vertices of any square in $H$ share a vertex, and as the degree of $D$ is already 3, it follows that $EF$ is an edge in $H$. As all of the vertices $A, B, C, D, E, F$ already have degree 3 it follows that $H$ is the graph on these six vertices with edges as described. It is easy to see that $H = K_{3,3}$.

**Corollary 2.2.** If $G$ is a 3-regular 4-ordered graph on more than 6 vertices, then every vertex has exactly 6 vertices at distance 2.

**Proof.** Note that in a 3-regular graph $G$ any vertex has 2, 3, 4, 5, or 6 vertices at distance 2. By Theorem 2.1, in order for graph $G$ on more than 6 vertices to be 4-ordered, it has to be square free. Observe that if there is a vertex $v$ that has 2, 3, 4, or 5 vertices at distance 2, then $v$ is a vertex of...
a square in $G$. Thus, if $G$ is a 3-regular 4-ordered graph on more than 6 vertices, then each vertex of $G$ has exactly 6 vertices at distance 2.

The next two lemmas will be used to prove Theorem 2.5, in which we show that the smallest 3-regular graph that is triangle and square free is 4-ordered. This graph is the well-known Petersen graph, (see Figure 1).

A walk is a sequence of (not necessarily distinct) vertices $x_1, x_2, \ldots, x_n$ such that $x_i$ is adjacent to $x_{i+1}$ for all $1 \leq i \leq n - 1$. The length of a walk is the number of edges in the walk. Following [6], an $n$-route is a vertex disjoint walk of length $n$ with specified initial point. A graph $G$ is $n$-transitive, $n \geq 1$, if it has an $n$-route and if there is always an automorphism of $G$ sending each $n$-route into any other $n$-route.

**Lemma 2.3.** ([6], p.175) The Petersen graph is 3-transitive.

The following well-known fact follows from Lemma 2.3:

**Lemma 2.4.** Given any two 5-cycles in the Petersen graph, there exists an automorphism that takes one of the 5-cycles into the other.

**Theorem 2.5.** The Petersen graph is 4-ordered.

**Proof.** Consider the Petersen graph as shown in Figure 1 and consider 4 vertices $v_1, v_2, v_3, v_4$ specified in order in the Petersen graph. We consider three cases depending on how the 4 vertices are distributed: either all 4 specified vertices are among vertices $a, b, c, d, e$, or 3 of them are from $a, b, c, d, e$, or 2 of them are from $a, b, c, d, e$. Call the cycle containing the vertices $a, b, c, d, e$ the outer cycle, and the cycle containing the vertices $f, h, j, g, i$ the inner cycle of the Petersen graph.

Consider the case when there are 3 vertices specified on the outer cycle, and 1 vertex specified on the inner cycle. Without loss of generality, the vertex on the inner cycle can be specified to be the first vertex, $v_1$, and the 3 vertices specified on the outer cycle the second, third, and forth, $v_2, v_3, v_4$. We now show that in this case regardless of exactly which 4 vertices $v_1, v_2, v_3, v_4$ are, there is a cycle containing them in this order.

Let $x_2$ and $x_4$ be the vertices on the inner cycle that are adjacent to $v_2$ and $v_4$, respectively. Go from $v_1$ on the inner cycle $f-h-j-g-i$ until $x_2$, without meeting $x_4$. Then go from $x_2$ to $v_2$, and from
v_2 \text{ go to } v_3 \text{ and then to } v_4 \text{ on the outer cycle } a-b-c-d-e. \text{ From } v_4 \text{ go to } x_4 \text{ and then to } v_1 \text{ without meeting } x_2. \text{ This completes the cycle that contains } v_1, v_2, v_3, v_4 \text{ in this order. }

Thus, by Lemma 2.4, if there are exactly 3 of the 4 specified vertices on any 5-cycle in the Petersen graph, then we have a cycle containing the 4 vertices in the specified order.

This observation makes it unnecessary to check the case of all 4 vertices being among vertices a, b, c, d, e, as in this case there is a 5-cycle containing exactly 3 of the specified vertices. Furthermore, in the case that 2 of the specified vertices are from a, b, c, d, e, and 2 from f, g, h, i, j it suffices to consider the case when these 4 vertices are in relative positions as a, c, i, j since in all other cases there is a 5-cycle containing 3 of the specified vertices. For these remaining cases, one can easily find a cycle containing the vertices no matter how we specify their order.

We now consider a common family of graphs generalizing the Petersen graph and show that the Petersen graph itself is the only member of this family that is 4-ordered, although they are all 3-ordered hamiltonian with the exception of the Petersen graph ([7], p.136).

A **star graph** $S_{n,k}$, where $n, k$ are positive integers, is a graph on vertices $A_1, A_2, \ldots, A_n$ such that $A_i$ and $A_j$ are adjacent if the indices $i$ and $j$ differ by $k$ modulo $n$. The **generalized Petersen graph** $P_{n,\lfloor n-1\rfloor}$, $n \geq 5$, is a graph consisting of an cycle of length $n$ on the vertices $B_1, B_2, \ldots, B_n$ (in order) and a star graph $S_{n,\lfloor n-1\rfloor}$ on vertices $A_1, A_2, \ldots, A_n$, such that $A_i$ and $B_i$ are adjacent for all $i = 1, 2, \ldots, n$. One can imagine the vertices $B_1, B_2, \ldots, B_n$ to be drawn on an outer cycle and the vertices $A_1, A_2, \ldots, A_n$ on an inner cycle. Note that $P_{5,2}$ is the standard Petersen graph.

**Proposition 2.6.** The generalized Petersen graph $P_{n,\lfloor n-1\rfloor}$ is not 4-ordered for $n > 5$.

**Proof.** If $n = 6$, $P_{6,2}$ is 3-regular and contains a triangle, thus it is not 4-ordered. In Figure 2 we depicted some edges of $P_{n,\lfloor n-1\rfloor}$ in the case $n > 6$. The thick lines denote edges in the graph, and
the thin line denotes the circle upon which the star graph is drawn. Note that if \( n = 7 \), then \( b = c \) and \( g = l \) in Figure 2; however, this has no effect on the following arguments.

In Figure 2 we also marked vertices 1, 2, 3, 4 and we now show that there is no cycle containing these vertices in this specified order. Indeed, suppose that there is such a cycle \( C \). As 13 and 24 are edges that cannot occur in \( C \), it follows that \( C \) must contain edges \( j1, 1a, 2h, i4, 4b, g3, \) and \( 3c \); and since \( 1-a-2 \) is a path in \( C \) this also shows that the edges \( j1, 1a, 2h \) are oriented from their first vertex to the second (i.e., \( j1 \) is oriented from \( j \) to 1, etc.). As \( hk \) and \( hj \) are the remaining edges from \( h \), and \( j1 \) is an edge in \( C \), it follows that \( C \) contains \( hk \) with this orientation. Also, as \( il \) and \( ik \) are the remaining edges from \( i \), and \( k \) has been visited when we were going from 2 to 3, it follows that \( C \) contains the edge \( il \). As \( hk, ki \), and \( ke \) are the edges from \( k \), and edge \( hk \) has been used, and \( ki \) cannot be used as it would create a path in \( C \) directly from 2 to 4, it follows that \( C \) contains the edge \( ke \) with this orientation. Because edge \( hj \) cannot be in \( C \), the edge \( dj \) must be in \( C \), with this orientation. If the edge \( cd \) were in \( C \), then there would be a closed cycle from 1 to 2 and back to 1 in \( C \), which is also impossible, giving us a contradiction.

3 The Heawood Graph

In this section we show that the Heawood graph, the smallest 3-regular graph that is triangle, square, and pentagon free ([1], p. 42), is 4-ordered hamiltonian (Figure 3).

**Lemma 3.1.** ([6], p.174) The Heawood graph is 4-transitive.

**Corollary 3.2.** The diameter of the Heawood graph is 3. Furthermore, if two vertices \( X \) and \( Y \) of the Heawood graph are at distance 3, then there are two disjoint paths of length 3 between \( X \) and \( Y \).

**Proof.** Take any vertex \( v \) in the Heawood graph. Observe that in order to prove that the diameter is 3, it suffices to check that all vertices are at distance less than 4 from \( v \) and there is a vertex at distance 3 from \( v \), which is easily done. In order to show that if two vertices \( X \) and \( Y \) of the Heawood graph are at distance 3, then there are two disjoint paths of length 3 between \( X \) and \( Y \), it suffices to find two vertices \( X \) and \( Y \) at distance 3 with two disjoint paths between them, say \( P_1 \) and \( P_2 \), which is easily done. Then, by applying Lemma 3.1 the claim follows for any two vertices \( X' \) and \( Y' \) at distance 3.

**Theorem 3.3.** The Heawood graph is 4-ordered hamiltonian.

**Proof.** We will consider cases depending on the distances between the 4 specified vertices \( v_1, v_2, v_3, v_4 \). We denote the distance between vertices \( X \) and \( Y \) by \( d(X,Y) \).

**Case 1.** Some 2 of the 4 specified vertices are at distance 1. Without loss of generality these are either \( v_1 \) and \( v_2 \) or \( v_1 \) and \( v_3 \).

**Case 1.1.1.** \( d(v_1,v_2) = 1, d(v_2,v_3) = 1 \). By Lemma 3.1 we can suppose without loss of generality that \( v_1, v_2, v_3 \) are respectively \( A, B, C \). In this case wherever \( v_4 \) is, it is clear that \( A-B-C-D-E-F-G-H-I-J-K-L-M-N-A \) is the desired hamiltonian cycle.

**Case 1.1.2.** \( d(v_1,v_2) = 1, d(v_2,v_3) = 2 \), and there is a length 2 path from \( v_2 \) to \( v_3 \) not containing \( v_1 \). By Lemma 3.1 we can suppose without loss of generality that \( v_1, v_2, v_3 \) are respectively \( A, B \),
D. In the case \( v_4 \neq C \), it is clear that \( A-B-C-D-E-F-G-H-I-J-K-L-M-N-A \) is the desired hamiltonian cycle. In case \( v_4 = C \), the desired hamiltonian cycle is \( A-B-G-H-I-J-K-F-E-D-C-L-M-N-A \).

**Case 1.1.3.** \( d(v_1, v_2) = 1, d(v_2, v_3) = 2 \), and there is no length 2 path from \( v_2 \) to \( v_3 \) not containing \( v_1 \). We can suppose that \( v_1, v_2, v_3 \) are respectively \( A, B, N \). In the case \( v_4 \neq C, M, L \), the desired hamiltonian cycle is \( A-B-C-L-M-N-E-D-I-H-G-F-K-J-A \). In the case \( v_4 = C \) or \( v_4 = M \) or \( v_4 = L \), the desired hamiltonian cycle is \( A-B-G-F-E-N-M-H-I-D-C-L-K-J-A \).

**Case 1.2.1.** \( d(v_1, v_2) = 1, d(v_2, v_3) = 3 \). By Corollary 3.2 there is a length 3 path from \( v_2 \) to \( v_3 \) not containing \( v_1 \). Thus, by Lemma 3.1 we can suppose that \( v_1, v_2, v_3 \) are respectively \( A, B, E \).

In the case \( v_4 \neq F, G, H, I, J, K \), the desired hamiltonian cycle is \( A-B-G-H-I-J-K-F-E-D-C-L-M-N-A \). Clearly, in the case \( v_4 = F, G, H, I, J, \) or \( K \) the desired hamiltonian cycle is \( A-B-C-D-E-F-G-H-I-J-K-L-M-N-A \).

**Case 1.2.2.** \( d(v_1, v_3) = 1, d(v_3, v_4) = 2 \), and there is a length 2 path from \( v_3 \) to \( v_4 \) not containing \( v_1 \). We can suppose that \( v_1, v_3, v_4 \) are respectively \( A, B, D \). In the case \( v_2 = C \), the desired hamiltonian cycle is \( A-J-K-C-L-B-G-F-E-D-I-H-M-N-A \). In the case \( v_2 \neq C, I, J \), the desired hamiltonian cycle is \( A-N-E-F-K-L-M-H-G-B-C-D-I-J-A \). In the case \( v_2 = I \) or \( v_2 = J \) the desired hamiltonian cycle is \( A-J-I-H-M-L-K-F-G-B-C-D-E-N-A \).

**Case 1.2.3.** \( d(v_1, v_3) = 1, d(v_3, v_4) = 2 \), and every length 2 path from \( v_2 \) to \( v_3 \) contains \( v_1 \). We can suppose that \( v_1, v_3, v_4 \) are respectively \( A, B, N \). In the case \( v_2 \neq E, F, G \), the desired hamiltonian cycle is \( A-J-K-L-M-H-I-D-C-B-G-F-E-N-A \). In the case \( v_2 = E \) or \( v_2 = F \) or \( v_2 = G \), the desired hamiltonian cycle is \( A-J-K-F-E-D-I-H-G-B-C-L-M-N-A \).

**Case 1.2.4.** \( d(v_1, v_3) = 1, d(v_3, v_4) = 3 \). By Corollary 3.2 there is a path from \( v_3 \) to \( v_4 \) not containing \( v_1 \). Thus, we can suppose that \( v_1, v_3, v_4 \) are respectively \( A, B, E \). Applying Lemma 3.1 to 3-routes \( B-A-N-E \) and \( A-B-C-D \), we can identify this case with \( v_3, v_1, v_4 \) being \( A, B, D \) respectively. If \( v_2 = C, L, M, \) or \( N \), the desired hamiltonian cycle is \( B-C-L-M-N-A-J-K-F-E-D-I-H-G-B \). If \( v_2 =
\[ F, G, J, \text{ or } K \] the desired hamiltonian cycle is \( B-G-F-K-J-A-N-E-D-I-H-M-L-C-B \). The remaining cases that we have to consider are when \( v_2 = E, H, \text{ or } I \). As \( A-B-C-D-E \) and \( A-B-C-D-I \) are both 4-routes, by Lemma 3.1 it suffices to consider only the cases when \( v_2 = H \) or \( v_2 = I \). In this case the desired hamiltonian cycle is \( B-G-H-I-J-A-N-M-L-K-F-E-D-C-B \).

Case 2. No 2 of the 4 specified vertices are at distance 1. As the diameter is 3, the possible distances are 2 and 3.

Case 2.1. \( d(v_1, v_2) = d(v_1, v_3) = d(v_1, v_4) = 2 \). Without loss of generality we can assume that either \( v_4, v_1, v_2 \) are \( N, B, D \) or \( v_4, v_1, v_2 \) are \( L, B, D \).

In the case \( v_4, v_1, v_2 \) are \( N, B, D \) as no two of the four specified vertices are at distance 1, \( v_3 \neq A, C \); and thus the desired hamiltonian cycle is \( B-C-D-E-F-G-H-I-J-K-L-M-N-A-B \).

In the case \( v_4, v_1, v_2 \) are \( L, B, D \) as no two of the four specified vertices are at distance 1, \( v_3 \neq A, C, M \); and thus if \( v_3 \neq N \) the desired hamiltonian cycle is \( B-C-D-E-F-G-H-I-J-K-L-M-N-A-B \). If \( v_3 = N \), then the desired hamiltonian cycle is \( B-G-F-E-D-I-H-M-N-A-J-K-L-C-B \).

Case 2.2. Some 2 vertices are at distance 2 from \( v_1 \). In the case \( v_2 \) and \( v_4 \) are at distance 2 from \( v_1 \) can be solved analogously to Case 2.1. Consider the case \( d(v_1, v_2) = d(v_1, v_3) = 2 \) and \( d(v_1, v_4) = 3 \). Without loss of generality, either \( v_4, v_1, v_2 \) are \( N, B, D \) or \( v_4, v_1, v_2 \) are \( L, B, D \). In the case \( v_4, v_1, v_2 \) are \( N, B, D \) the only possibility for \( v_4 \) is \( K \), and the desired hamiltonian cycle is \( B-C-D-E-N-M-L-K-F-G-H-I-J-A-B \). In the case \( v_4, v_1, v_2 \) are \( L, B, D \), then the all the points that are distance 3 from \( v_1 \) would be at distance 1 from some of the vertices \( v_2 \) or \( v_3 \), contradicting our assumption.

Case 2.3. Some 2 vertices are at distance 3 from \( v_1 \) and the remaining vertex is at distance 2. We can suppose without loss of generality that either \( d(v_1, v_2) = 2 \), or \( d(v_1, v_3) = 2 \).

If \( d(v_1, v_2) = 2 \), suppose \( v_1, v_2 \) are \( B, D \). As the 4 vertices at distance 3 from \( v_1 \) are \( E, I, K, M \) and \( d(v_2, E) = 1, d(v_2, I) = 1 \), we have that \( \{v_3, v_4\} = \{K, M\} \). If \( v_4 = K \) and \( v_3 = M \), then the desired hamiltonian cycle is \( B-C-D-E-N-M-L-K-F-G-H-I-J-A-B \), whereas if \( v_3 = K \) and \( v_4 = M \), then the desired hamiltonian cycle is \( B-C-D-E-N-M-L-K-F-G-H-I-J-A-B \).

If \( d(v_1, v_3) = 2 \), suppose \( v_1, v_3 \) are \( B, D \). By an analogous argument as above \( \{v_2, v_4\} = \{K, M\} \). If \( v_2 = K \) and \( v_4 = M \), then the desired hamiltonian cycle is \( B-A-N-M-H-G-F-E-D-I-J-K-L-C-B \).

Case 2.4. Suppose that all 3 points are at distance 3 from \( v_1 \). Without loss of generality, let \( v_1 = B \). The distance 3 vertices from \( v_1 \) are \( E, I, K, M \) and regardless of which 3 of these we choose for \( v_2, v_3, v_4 \), there will be two at distance 2. However, as the choice of \( v_1 \) was without loss of generality in all previous cases, this case cannot occur.

\[ \square \]

**Theorem 3.4.** The Heawood graph is the graph on the fewest vertices, after \( K_4 \) and \( K_{3,3} \), that is 3-regular 4-ordered hamiltonian.

**Proof.** The only 3-regular 4-ordered hamiltonian graphs on fewer than 7 vertices are \( K_4 \) and \( K_{3,3} \), \( K_5 \). By \( K_5 \) and Theorem 2.1, all 3-regular 4-ordered hamiltonian graphs on more than 6 vertices are both triangle and square free. The smallest such 3-regular graph is the Petersen graph on 10 vertices, and it is not hard to see that it is the only 3-regular triangle- and square-free graph on 10 vertices. As it is not hamiltonian, it is not 4-ordered hamiltonian either. Since a 3-regular graph has even number of vertices, the only possibility for a 3-regular 4-ordered hamiltonian graph on fewer than 14 vertices is a 3-regular graph on 12 vertices that is triangle and square free.
As square-freeness of a graph $G$ is equivalent to each vertex of $G$ having 6 vertices at distance 2, it follows given any triangle- and square-free graph $G$ on at least 10 vertices and an arbitrary vertex $v$ of $G$, the induced subgraph of $G$ on $v$ and the vertices that are at distance 1 and at distance 2 from $v$ contains the graph in Figure 4 as a subgraph. Since the graph in Figure 4 has 10 vertices, only two vertices have to be added to obtain a 3-regular triangle- and square-free graph on 12 vertices. It is not hard to see that there are only two non-isomorphic 3-regular graphs on 12 vertices that are triangle free and square free (Figure 5).

We now show that neither of the two triangle- and square-free 3-regular graphs on 12 vertices are 4-ordered hamiltonian.

Consider Figure 6. We show that there is no hamiltonian cycle containing vertices $A, E, D, J$ in this order. Suppose for sake of contradiction that there is a hamiltonian cycle $C$ containing vertices $A, E, D, J$ in this order. Then edges $AD$ and $EJ$ are not in the cycle, thus edges $CA, AB, BE, EK, ID, DJ, JL$ are in the cycle, and it is not hard to see that they are oriented in this way (i.e., we are going from $C$ to $A$, from $A$ to $B$, etc.). Note that the only way to get from $E$ to $D$ without meeting $C$ and $L$ (vertices that are on $C$ after meeting all of $A, E, D, J$) is to take the edge $KI$ from $K$ to $I$. However, there is no path from $L$ to $C$ exactly including vertices $G$ and $H$, thus there is no hamiltonian cycle $C$ containing vertices $A, E, D, J$ in this order.

Consider Figure 7. We show that there is no hamiltonian cycle containing vertices $A, E, C, G$ in this order. Suppose for sake of contradiction that there is a hamiltonian cycle $C$ contain-
Figure 6: There is no hamiltonian cycle containing vertices $A, E, D, J$ in this order.

Figure 7: There is no hamiltonian cycle containing vertices $A, E, C, G$ in this order.
Proposition 3.5. A 4-ordered hamiltonian graph $G$ on more than 10 vertices contains neither of the graphs in Figure 8 as subgraphs.

Proof. The proof follows the arguments in the proof of Theorem 3.4, where $v_1, v_2, v_3, v_4$ are the four vertices for which there is no hamiltonian cycle containing them in this order. See Figure 8.

Proposition 3.6. There is no 4-ordered hamiltonian graph $G$ that contains the graph in Figure 9 as a subgraph.

Proof. Suppose the graph in Figure 9 is a subgraph of a 4-ordered hamiltonian graph $G$. Then there exists a hamiltonian cycle $\mathcal{C}$ in $G$ containing vertices $D, E, G, H$ in this order. Since edges $DG$ and $EH$ cannot be in $\mathcal{C}$, edges $CD, DB, BE, EF, IG, GH$ and $HJ$ are in $\mathcal{C}$. As $\mathcal{C}$ is a hamiltonian cycle, and since edge $BA$ cannot be in $\mathcal{C}$ as $DB$ and $BE$ are, it follows that edges $IA$ and $AJ$ must be in $\mathcal{C}$. However, edges $AI, IG, GH, HJ$ and $JA$ form a cycle, which contradicts the existence of $\mathcal{C}$.

Figure 8: There is no 4-ordered hamiltonian graph $G$ on more than 10 vertices that contains these graphs as subgraphs.
Figure 9: There is no 4-ordered hamiltonian graph $G$ that contains this graph as a subgraph.

Figure 10: The torus-graphs. Note that the vertices with the same names are identified, and consequently $l_3$ is both the bottom and the top line.

4 A Family of 4-ordered 3-regular graphs

In this section we introduce an infinite family of 3-regular 4-ordered graphs that answers the question of the existence of an infinite family of low degree 4-ordered graphs. These graphs, which we will refer to as torus-graphs, are similar to the Heawood graph in that they have girth 6, and just as the Heawood graph can be embedded on the torus, they too can be embedded in the torus. A general torus-graph is shown in Figure 10. The dashed lined signify that there are more hexagons in each of the three rows, and we will always assume we have sufficiently many hexagons in each row. The labelings $a, b, c, \ldots, k, l$ show which vertices are identified.

Theorem 4.1. The torus-graphs presented in Figure 10 are 3-regular 4-ordered graphs for long enough rows.

Proof. We consider the graphs as drawn in the plane, and we shall refer to left and right movements, as well as up and down movements, when referring to moves on the horizontal and vertical lines respectively. Furthermore, we refer to the horizontal lines in Figure 10 as lines $l_1, l_2, l_3$, where the
top and the bottom lines are $l_3$, and $l_1$ and $l_2$ are, respectively, the second and third horizontal lines from the top.

We analyze all possible ways of specifying the 4 points, $v_1, v_2, v_3, v_4$, and in each case we exhibit a cycle containing them in the specified order.

We consider cases depending on how the 4 points are distributed among the lines.

Case 1. All 4 points are on the same line. Without loss of generality, say the line is $l_2$.

Case 1.1. If beginning from $v_1$ the 4 points are in the order $v_1, v_2, v_3, v_4$ along the line, either going to the right on the line or to the left starting at $v_1$, then just trace along $l_2$ to obtain the desired cycle.

Case 1.2. If the 4 points are not in order $v_1, v_2, v_3, v_4$, we can suppose without loss of generality that they are in order $v_1, v_2, v_4, v_3$ when going to the right from $v_1$. In this case we consider two subcases.

Case 1.2.1. $v_1$ and $v_3$ are not adjacent. Then, do the following:

- go right from $v_1$ to $v_2$ on line $l_2$
- go down or up from $v_2$ (depending whether the vertical edge starting at $v_2$ is down or up) to the nearest line; suppose without loss of generality that the edge we took was down
- go right until reaching either the vertical edge that has $v_3$ as its end or the vertical edge with an end that is one step to the right of $v_3$, and traverse this vertical edge, returning to the line $l_2$
- go left, hitting $v_3$ and $v_4$ and go until the vertex next to $v_2$, and from this vertex take the edge up
- if the vertical edge from $v_1$ is up, continue on this line ($l_1$) until hitting that edge and take it to $v_1$; if the vertical edge from $v_1$ is down, go one step left, go up, and then continue left along $l_3$ until hitting the vertical edge of $v_1$ and take it to $v_1$

Case 1.2.2. $v_1$ and $v_3$ are adjacent. We can suppose without loss of generality that the vertical edge of $v_1$ is down, and consequently that the vertical edge of $v_3$ is up.

If the vertical edge of $v_2$ is down,

- go right from $v_1$ to $v_2$
- take the vertical edge from $v_2$ down to $l_3$
- go right on $l_3$ until the vertex that is next to the vertical edge from $v_1$
- go down to $l_1$
- take a step left, and then step down to $l_2$, hitting $v_3$
- go left, hitting $v_4$, until the vertex next to $v_2$, and from this vertex take the edge up
- go one step left, go up, and then go left until hitting the vertical edge of $v_1$ and take it to $v_1$

If the vertical edge of $v_2$ is up,

- go right from $v_1$ to $v_2$
• take the vertical edge from $v_2$ up to $l_1$
• go right until hitting the vertical edge from $v_3$
• go down to $l_1$ hitting $v_3$
• go left, hitting $v_4$, until the vertex that is next to $v_2$, and from this vertex take the edge down
• go left until hitting the vertical edge of $v_1$ and take it to $v_1$

Case 2. There are 3 points on one line (say on line $l_1$) and 1 on another. Without loss of generality we can suppose that the 3 points on line $l_1$ are $v_1, v_2, v_3$, and that this is the order as we go from $v_1$ to the right. Also, we can suppose without loss of generality that $v_4$ is on line $l_2$.

Case 2.1. Both $v_1$ and $v_3$ have vertical edges down.

• go from $v_1$ to the right to $v_2$ and then to $v_3$
• go down, go in the direction so as to hit $v_4$ and then the vertical edge of $v_1$
• go up to $v_1$

Case 2.2. Both $v_1$ and $v_3$ have vertical edges up. Analysis can be done similarly as before.

Case 2.3. If the vertical edge from $v_1$ is up, from $v_3$ is down, $v_1$ and $v_3$ are non-adjacent,

• go from $v_1$ to the right to $v_2$ and then to $v_3$
• go down, go in the direction so as to hit $v_4$ and then the vertical edge to the left of $v_1$
• go up to $l_1$ and go one step right to $v_1$

If the vertical edge from $v_1$ is up, from $v_3$ is down, $v_1$ and $v_3$ are adjacent,

• go from $v_1$ to the right to $v_2$ and then to $v_3$
• go down, go in either direction so as to hit $v_4$ and then take the first vertical edge down to $l_3$
• go on $l_3$ so as to hit the vertical edge of $v_1$, and then take it and go to $v_1$

Case 2.4. If the vertical edge from $v_1$ is down, and the vertical edge from $v_3$ is up, $v_1$ and $v_3$ are non-adjacent, or if the vertical edge from $v_1$ is down, and the vertical edge from $v_3$ is up, $v_1$ and $v_3$ are adjacent, an analysis similar as before gives the desired cycle.

Case 3. There is a line with exactly 2 vertices on it. Without loss of generality these are either $v_1$ and $v_2$ or $v_1$ and $v_3$.

Case 3.1. $v_1$ and $v_2$ are on line $l_1$. The construction of a cycle is easy in this case, considering two cases depending on whether $v_3$ and $v_4$ are on the same line (without loss of generality on $l_2$) or on two different lines.

Case 3.2. $v_1$ and $v_3$ are on the same line $l_1$ and they are non-adjacent.

Case 3.2.1. $v_2$ and $v_4$ are on the same line, without loss of generality on $l_2$

If $v_2$ and $v_4$ are non-adjacent,

• go from $v_1$ down if there is a vertical edge, or if not go to one of its neighbors, so that when taking the edge down from the neighbor, the path is shorter to $v_2$ (without hitting $v_4$)
• go to $v_2$, and then go up to $l_1$ immediately or after the next step

• go along $l_1$ until hitting $v_3$ (without hitting $v_1$) and go down either from $v_3$ or from the next vertex

• go until hitting $v_4$ (without hitting $v_2$) and then go down to $l_3$ either from $v_4$ or from the next vertex

• go along $l_3$ until hitting the vertical from $v_1$, or the vertical from one of the neighbors of $v_1$ that was not yet used

• take the vertical edge to $l_1$ and go to $v_1$

If $v_2$ and $v_4$ are adjacent, a similar analysis can be performed, also taking into account whether the vertical edges from $v_1, v_2, v_3,$ and $v_4$ are up or down.

Case 3.2.2 $v_2$ and $v_4$ are on different lines, without loss of generality $v_2$ on $l_2$ and $v_4$ on $l_3$.

• either go down from $v_1$, or take one step left and go down to line $l_2$

• go in one of the directions along $l_2$ until hitting $v_2$ so that either it is possible to go up from $v_2$, or it is possible to continue one more step and go up to $l_1$

• go until hitting $v_3$ (in the direction along $l_1$ so that $v_1$ is not hit again), and go up either at $v_3$ or continue one more step and go up to $l_3$

• go in the direction along $l_3$ so as to hit $v_4$ and then hit either the vertical edge leading to $v_1$ or the vertical edge leading to one of the neighbors of $v_1$; take this vertical edge and go to $v_1$

Case 3.3. $v_1$ and $v_3$ are on the same line $l_1$ and they are adjacent.

Case 3.3.1 $v_2$ and $v_4$ are on the same line.

If $v_2$ and $v_4$ are non-adjacent, the analysis can be carried out in a way similar to Case 3.2.1.

If $v_2$ and $v_4$ are adjacent, the analysis can be carried out depending on whether the vertical edges from $v_1, v_2, v_3,$ and $v_4$ are up or down, taking into account the relative position of $v_1, v_3$ and $v_2, v_4$.

3.3.2. $v_2$ and $v_4$ are on different lines. Without loss of generality $v_2$ is on $l_2$ and $v_4$ on line $l_3$

If the vertical edge from $v_1$ is down and the vertical edge from $v_3$ is up,

• go from $v_1$ down to $l_2$

• go to $v_2$ and take the vertical edge from $v_2$ or from a neighbor up to $l_1$

• go to $v_3$ and take the vertical edge up to $l_3$

• go in one of the directions along $l_3$ so as to hit first $v_4$ and then the vertical edge that is one step to the left of $v_1$

• take the vertical edge to $l_1$ and take one step right, ending the cycle at $v_1$

If the vertical edge from $v_1$ is up and the vertical edge from $v_3$ is down, the analysis can be carried out in a similar fashion.

□
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