STABLE CLT FOR DETERMINISTIC SYSTEMS

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Abstract. We show that for every ergodic and aperiodic probability preserving transformation and \( \alpha \in (0, 2) \) there exists a function whose associated time series is in the standard domain of attraction of a non-degenerate symmetric \( \alpha \)-stable distribution.

1. Introduction

A random variable \( Y \) is stable if there exists a sequence \( Z_1, Z_2, \ldots \) of i.i.d. random variables and sequences \( a_n, b_n \) such that

\[
\frac{\sum_{k=1}^{n} Z_k - a_n}{b_n},
\]

converges in distribution to \( Y \), as \( n \to \infty \).

In other words, \( Y \) arises as a distributional limit of a central limit theorem, see \[3\]. Furthermore in this case, \( b_n \) is regularly varying of index \( \frac{1}{\alpha} \) which implies that \( b_n = n^{1/\alpha} L(n) \) where \( L(n) \) is a slowly varying function. The Normal and the Cauchy distribution are stable distributions and one can parametrize the class of stable distribution via their characteristic functions (Fourier transform). Namely a random variable is \( \alpha \)-stable, \( 0 < \alpha \leq 2 \), if there exists \( \sigma > 0, \beta \in [-1, 1] \) and \( \mu \in \mathbb{R} \) such that for all \( \theta \in \mathbb{R} \).

\[
\mathbb{E}(\exp(i\theta Y)) = \begin{cases} 
\exp \left(-\sigma^\alpha |\theta|^\alpha (1 - i\beta (\text{sign}(\theta) \tan(\frac{\pi \alpha}{2}) + i\mu \theta))\right), & \alpha \neq 1, \\
\exp \left(-\sigma^\alpha |\theta|^\alpha (1 + i\beta (\text{sign}(\theta) \ln(\theta) + i\mu \theta))\right), & \alpha = 1.
\end{cases}
\]

The constant \( \sigma > 0 \) is the dispersion parameter and \( \beta \) is the skewness parameter. In this case we will say that \( Y \) is a \( S_\alpha(\sigma, \beta, \mu) \) random variable. If \( \mu = \beta = 0 \) and \( \sigma > 0 \) then the random variable is symmetric \( \alpha \) stable and we will abbreviate \( Y \) is \( S_\alpha S(\sigma) \). See \[5\] for a detailed account of infinite variance (\( \alpha \neq 2 \)) stable processes and its appearance in various fields of mathematics and science.

A probability preserving dynamical system is a quadruplet \((\mathcal{X}, \mathcal{B}, m, T)\) where \((\mathcal{X}, \mathcal{B}, m)\) is a standard probability space and \( T : \)

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\( \mathcal{X} \to \mathcal{X} \) is measurable and \( m \circ T^{-1} = m \). The system is \textbf{aperiodic} if the collection of all periodic points is a null set. It is \textbf{ergodic} if every \( T \)-invariant set is either a null or a co-null set.

A function \( f : \mathcal{X} \to \mathbb{R} \) generates a stationary process \((f \circ T^n)_{n=1}^\infty\) and \( S_n(f) = \sum_{k=0}^{n-1} f \circ T^k \) is its corresponding \textbf{sum process}. Given \( Y \) a \( \mathcal{S} \alpha \mathcal{S}(\sigma) \) random variable, a function \( f \) is a \( Y \)-\textbf{CLT function} if there exists \( b_n \to \infty \) and \( a_n \) such that \( \frac{S_n(f) - a_n}{b_n} \) converges in distribution to \( Y \). Since the distribution of \( Y \) is non-atomic, this is equivalent to: for all \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} m \left( \frac{S_n(f) - a_n}{b_n} \leq t \right) = \mathbb{P}(Y \leq t)
\]

If in addition \( a_n = 0 \) and \( b_n = n^{1/\alpha} \) then the time series generated by \( f \) is in the \textbf{standard domain of attraction} of \( Y \).

It seems that general methods of proof of the central limit theorem in the dynamical systems setting work only in the case of positive entropy systems. For example, if \((f \circ T^n)_{n=0}^\infty\) is a martingale difference sequence and \( T \) has zero entropy then \( f \equiv 0 \). Consequently martingale approximation can hardly be used. It was a natural open problem whether every aperiodic dynamical system admits a function which satisfies the CLT with a nondegenerate normal distribution as a limit.

In 1986, Burton and Denker \cite{Bur86} answered this question in the affirmative by showing that for every aperiodic dynamical system, even very deterministic ones such as irrational rotations, there exists a CLT function \( f \) for \( Y \), a standard normal distribution.

For the moment suppose that \( Y \) is a standard normal random variable. By \( L^2_0 \) we denote the space of \( L^2 \) functions with zero mean. One can notice that the functions found by Burton and Denker are from \( L^2_0 \). As remarked in \cite{Bur86}, because coboundaries are dense in \( L^2_0 \), the set of \( Y \)-CLT functions is dense in \( L^2_0 \). As shown in \cite{Kos94} for any sequence \( b_n \to \infty \), \( b_n = o(n) \), there exists a dense \( G_\delta \) subset of \( f \in L^2_0 \) such that every probability law is a weak limit of the distributions of \((1/b_n)S_n(f)\).

The set of \( Y \)-CLT functions is therefore meagre.

Burton and Denker asked whether there exists a function \( f \) which satisfies the Weak Invariance Principle (WIP), meaning that the partial sums process \( W_n : X \times [0,1], W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{[nt]} f \circ T^k \), when viewed as a random process with values in the space of C\(à\)dl\(à\)g functions converges in distribution to a Brownian motion. This question was resolved in the affirmative by the second author in \cite{Vol89} and recently we showed in \cite{Kos99} that when \( T \) is ergodic and aperiodic there exists a function \( f : X \to \mathbb{Z} \) for which the lattice local central limit theorem holds.
Weiss and Thouvenot showed in [6] that for every free probability preserving system and random variable $Y$ there exists a function $f$ such that $\frac{1}{n}S_n(f)$ converges in distribution to $Y$. See also [1] where a refined result for positive valued processes is obtained with normalizing constants of the form $\frac{1}{c_n}$ with $c_n$ a 1-regularly varying sequence.

In this work we show the existence of CLT functions for the whole range of symmetric $\alpha$ stable distributions with the scaling $b_n = n^{1/\alpha}$. This normalization corresponds to that for iid sequences, unlike the others mentioned above.

**Theorem 1.** Let $(\mathcal{X}, \mathcal{B}, m, T)$ be an ergodic, aperiodic probability preserving system, For every $\alpha \in (0, 2)$, $\sigma > 0$, there exists $f : \mathcal{X} \to \mathbb{R}$ such that $\frac{1}{n^{1/\alpha}}S_n(f)$ converges in distribution to a $S\alpha S(\sigma)$ random variable.

We remark that a considerable part of the statement is that the scaling is of the form $n^{1/\alpha}$. One reason for interest in this scaling is that if a stationary process satisfies a WIP with a non-degenerate $S\alpha S$ Lévy motion as a limit then $b_n$ must be $1/\alpha$ regularly varying. Furthermore, by Fact 3 when $\alpha \in (0, 1)$, this scaling is the largest possible growth rate of the dispersion parameter for the sum process of a stationary $S\alpha S$ process.

1.1. **Organisation of the paper.** In Section 2 we introduce a carefully chosen triangular array and use a Proposition 2 from [4] to embed it in a given aperiodic, ergodic probability preserving system. We then construct, using the functions from the embedding, the function which satisfies the $\alpha$-stable CLT.

Section 3 is concerned with the proof of the CLT for the function from Section 2. The last section is a short appendix containing some standard properties of $S\alpha S$ random variables which are used in Section 3.

1.1.1. **Notations.** In what follows we will write for $f, g$ two positive valued functions (or sequences), $f(t) \sim g(t)$ if $\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1$. We will denote by $f(t) \preceq g(t)$ if there exists $C > 0$ such that $f(t) \leq C g(t)$ for all large $t$ and $f \asymp g$ if $f(t) \preceq g(t)$ and $g(t) \preceq f(t)$.

In addition when $f$ and $g$ are real valued functions with $2 \leq f(t) \leq g(t)$, we write $\sum_{k=f(t)}^{g(t)} a_k$ for the sum $\sum_{k=\lceil f(t) \rceil}^{\lceil g(t) \rceil} a_k$ where $\lceil x \rceil$ is the floor function of $x$.

Given a sequence $(Y_n)_{n=1}^{\infty}$ of random variables and a random variable $Y$, $Y_n \Rightarrow^d Y$ denotes $Y_n$ converges in distribution to $Y$, $X =^d Z$.
means $X$ and $Y$ are equally distributed and $Y \sim^d S\alpha S(\sigma)$ means $Y$ is distributed $S\alpha S(\sigma)$.

For a sequence of random variables $Y(1), Y(2), \ldots$ and $n \in \mathbb{N}$, we write $S_n(Y) = \sum_{j=1}^{n} Y(j)$.

2. Stable laws and a CLT for a target process

2.1. Target triangular array. The first step is to describe a triangular array, consisting of finite valued random variables, which we will be able to embed in subsection 2.2 in every aperiodic, ergodic, probability preserving system.

Let $d_k := \left\lfloor \frac{2^{2k}}{2^{2k} - \alpha} \right\rfloor$.

Consider the following triangular array of random variables:

(a) For each $k \in \mathbb{N}$, $\{X_k(i) : 0 \leq i \leq 2d_k\}$ are i.i.d, $S\alpha S\left( k^{-1/\alpha} \right)$ random variables.

(b) For each $k \in \mathbb{N}$, $\{X_k(i) : 0 \leq i \leq 2d_k\}$ is independent of $\{X_j(i) : 1 \leq j < k, 0 \leq i \leq 2d_j\}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space on which all these random variables are defined.

We now define a sequence of finite valued random variables as follows; First set

\[ Y_k(j) := X_k(j)1_{\left[ 2^k \leq |X_k(j)| \leq 2^{k+2} \right]} \]

Now let $(t^{(k)}_l)_{l=0}^{L_k} \subset \left[ 2^k, 2^{k+2} \right]$, satisfying:

- $t^{(k)}_0 = 2^k, t^{(k)}_{L_k} = 2^{k+2}$. Here $L_k + 1 \in \mathbb{N}$ is the number of points in the partition.
- For all $0 \leq l < L_k$, $0 < t^{(k)}_{l+1} - t^{(k)}_l \leq \frac{1}{d_k}$.

Now for all $1 \leq j \leq 2d_k$, let

\[ Z_k(j) = \begin{cases} \text{sign}(Y_k(j)) t^{(k)}_l, & \exists 0 \leq l < L_k, \ t^{(k)}_l \leq |Y_k(j)| < t^{(k)}_{l+1}, \\ 0, & |Y_k(j)| \notin \left[ 2^k, 2^{k+2} \right] \end{cases} \]

The following claim follows easily from the definition.

**Fact 1.** The sequence $(Z_k(j))_{k \in \mathbb{N}, 1 \leq j \leq 2d_k}$ is a triangular array of random variables so that for every $k \in \mathbb{N}$, $(Z_k(j))_{j=1}^{2d_k}$ are finite-valued, i.i.d. random variables.

We summarise several key properties of the sequences defined above which will be used in the sequel.

**Lemma 2.** For every $k \in \mathbb{N}$, $1 \leq j \leq 2d_k$:
(a) \(|Z_k(j) - Y_k(j)| \leq \frac{1}{dk}\).
(b) \(\mathbb{P}(Z_k(j) \neq 0) \leq \mathbb{P}(|X_k(j)| \geq 2^k) \leq C_\alpha 2^{-\alpha k}\). Here \(C_\alpha\) is a global constant independent of \(k\) and \(j\).

**Proof.** Part (a) and the first inequality in part (b) are immediate consequences of the definitions of \(Y_k\) and \(Z_k\) as functions of \(X_k\). By Proposition 3 we find that the array \(\{Z_k\}_{k \in \mathbb{N}, 1 \leq j \leq 2d_k}\) is i.i.d. distributed as the finite-valued random variable \(X\). Let \((f \circ T^j)_{j=0}^{n-1}\) be distributed as \((U_j)_{j=1}^n\) and \((f \circ T^j)_{j=0}^{n-1}\) is independent of \(\mathcal{P}\).

\[\mathbb{P}(|X_k(j)| \geq 2^k) \leq \frac{C_\alpha}{k} 2^{-\alpha k}.\]

\[\square\]

2.2. Embedding the array in the dynamical system. Let \((\mathcal{X}, \mathcal{B}, m)\) be a standard probability space. A finite partition of \(\mathcal{X}\) is measurable if all of its pieces (atoms) are Borel-measurable. Recall that a finite sequence of random variables \(X_1, \ldots, X_n : \mathcal{X} \to \mathbb{R}\), each taking finitely many of values, is independent of a finite partition \(\mathcal{P} = (P)_{P \in \mathcal{P}}\) if for all \(s \in \mathbb{R}^n\) and \(P \in \mathcal{P}\),

\[m\left(\left(X_j\right)_{j=1}^n = s | P\right) = m\left(\left(X_j\right)_{j=1}^n = s\right).\]

We will embed the triangular array using the following key proposition.

**Proposition 3.** [4, Proposition 2] Let \((\mathcal{X}, \mathcal{B}, m, T)\) be an aperiodic, ergodic, probability preserving transformation and \(\mathcal{P}\) a finite-measurable partition of \(\mathcal{X}\). For every finite set \(A\) and \(U_1, U_2, \ldots, U_n\) an i.i.d. sequence of \(A\) valued random variables, there exists \(f : \mathcal{X} \to \mathcal{X}\) such that \((f \circ T^j)_{j=0}^{n-1}\) is distributed as \((U_j)_{j=1}^n\) and \((f \circ T^j)_{j=0}^{n-1}\) is independent of \(\mathcal{P}\).

An easy corollary of this proposition and Fact 4 is the following.

**Corollary 4.** Let \((\mathcal{X}, \mathcal{B}, m, T)\) be an aperiodic, ergodic, probability preserving transformation and \((Z_k(j))_{k \in \mathbb{N}, 1 \leq j \leq 2d_k}\) be the triangular array from subsection 2.1. There exist functions \(f_k : \mathcal{X} \to \mathbb{R}\) such that \((f_k \circ T^{j-1})_{k \in \mathbb{N}, 1 \leq j \leq 2d_k}\) is distributed as \((Z_k(j))_{k \in \mathbb{N}, 1 \leq j \leq 2d_k}\).

**Proof.** Starting with \(\mathcal{P} = \{\mathcal{X}\}\), the trivial partition, and applying Proposition 3 we find \(f_1 : \mathcal{X} \to \mathbb{R}\) such that \(f_1, f_1 \circ T, \ldots, f \circ T^{2d-1}\) are i.i.d. distributed as the finite-valued random variable \(Z(1)\).

In the inductive step we are given \(f_k : \mathcal{X} \to \mathbb{R}, 1 \leq k \leq m\), such that the array \(\{f_k \circ T^{j-1} : 1 \leq k \leq m, 1 \leq j \leq 2d_k\}\) is distributed as \(\{Z_k(j) : 1 \leq k \leq m, 1 \leq j \leq 2d_k\}\).

Let \(\mathcal{P}_m\) be the finite partition of \(\mathcal{X}\) according to the values of the (finite valued) random vector \(V := (f_k \circ T^{j-1} : 1 \leq k \leq m, 1 \leq j \leq 2d_k)\).
Apply Proposition 3 and obtain a function $f_{m+1} : \mathcal{X} \to \mathbb{R}$ such that \( \{f_{m+1} \circ T^{j-1} : 1 \leq j \leq 2d_{m+1}\} \) is an i.i.d. sequence distributed as \( \{Z_{m+1}(j) : 1 \leq j \leq 2d_{m+1}\} \) and independent of \( \mathcal{P}_m \). Since being independent of \( \mathcal{P}_m \) is equivalent to being independent of \( \mathcal{V} \), we see that the array \( \{f_k \circ T^{j-1} : 1 \leq k \leq m+1, 1 \leq j \leq 2d_k\} \) is distributed as \( \{Z_k(j) : 1 \leq k \leq m+1, 1 \leq j \leq 2d_k\} \). □

2.3. Definition of the function. Let \((\mathcal{X}, \mathcal{B}, m, T)\) be an aperiodic, ergodic, probability preserving system and \((f_k)_{k=1}^{\infty}\) the functions from Corollary 4.

**Lemma 5.** For \( m \) almost every \( \omega \in \mathcal{X} \), there exists \( K(\omega) \in \mathbb{N} \) such that for all \( k > K(\omega) \), \( f_k(\omega) - f_k \circ T^{d_k}(\omega) = 0 \).

**Proof.** By the definition of the functions, we have for all \( k \in \mathbb{N} \),
\[
m(f_k \neq 0) = P(Z_k(0) \neq 0).
\]
By Lemma 2(b), there exists \( C_\alpha \) such that for all \( k \),
\[
P(Z_k(0) \neq 0) \leq \frac{C_\alpha}{k^{2\alpha}}.
\]
Consequently, as \( T \) is \( m \) preserving,
\[
\sum_{k=1}^{\infty} m(f_k - f_k \circ T^{d_k} \neq 0) \leq 2 \sum_{k=1}^{\infty} m(f_k \neq 0) \leq 2 \sum_{k=1}^{\infty} P(Z_k(0) \neq 0) < \infty.
\]
The conclusion follows from the Borel-Cantelli lemma. □

Set
\[
f := \sum_{k=1}^{\infty} (f_k - f_k \circ T^{d_k}).
\]
This function is well defined as it is almost surely a sum of finitely many values. The following theorem implies Theorem 4. In what follows, \( \log(x) \) denotes the logarithm of \( x \) in base 2 and \( \ln(x) \) is the natural logarithm of \( x \).

**Theorem 1.b.** \( \frac{S_n(f)}{n^{\frac{1}{\alpha}}} \Rightarrow^d S_{\alpha S}(\sigma) \) with \( \sigma^\alpha = 2 \ln \left(\frac{2}{2-\alpha}\right) \).
3. Proof of Theorem 1.b

For a measurable function \( g : \mathcal{X} \to \mathbb{R} \) and \( k \in \mathbb{N} \), we write \( U^k g = g \circ T^k \). The proof of Theorem 1.b begins by writing

\[
S_n(f) = S_n^{(S)}(f) + S_n^{(M)}(f) + S_n^{(L)}(f)
\]

where

\[
S_n^{(M)}(f) := \sum_{k=(\frac{1}{\alpha} - \frac{1}{2}) \log(n)+1}^{\frac{1}{\alpha} \log(n)} (S_n(f_k) - U^d_k S_n(f_k))
\]

\[
S_n^{(S)}(f) := \sum_{k=1}^{(\frac{1}{\alpha} - \frac{1}{2}) \log(n)} (S_n(f_k) - U^d_k S_n(f_k))
\]

\[
S_n^{(L)}(f) := \sum_{k=\frac{1}{\alpha} \log(n)+1}^{\infty} (S_n(f_k) - U^d_k S_n(f_k))
\]

Theorem 1.b follows from the following proposition, (1) and the converging together lemma (also known as Slutsky’s Theorem).

Proposition 6.

(a) \( \frac{1}{n^{1/\alpha}} S_n^{(S)}(f) \to 0 \) in probability.

(b) \( \frac{1}{n^{1/\alpha}} S_n^{(M)}(f) \Rightarrow d S_{\alpha} S \left( \sqrt{2 \log \left( \frac{2}{2-\alpha} \right)} \right) \).

(c) \( \lim_{n \to \infty} m \left( S_n^{(L)}(f) \neq 0 \right) = 0 \).

We first prove the simplest part.

Proof of Proposition 6(c). By Corollary 4 and Lemma 2(b) for every \( k > \frac{1}{\alpha} \log(n) \),

\[
m(f_k \neq 0) = \mathbb{P} (Z_k(1) \neq 0) \leq C_{\alpha} \frac{2^{-\alpha k}}{k}.
\]

We have for all \( k > \frac{1}{\alpha} \log(n) \),

\[
m \left( S_n(f_k) - U^d_k S_n(f_k) \neq 0 \right) \leq m \left( \exists j \in [0, n] \cup [d_k, d_k + n], f_k \circ T^j \neq 0 \right)
\]

\[
\leq \sum_{j=0}^{n-1} \left[ m \left( f_k \circ T^j \neq 0 \right) + m \left( f_k \circ T^{d_k+j} \neq 0 \right) \right]
\]

\[
\leq 2n \cdot m(f_k \neq 0) \leq 2n C_{\alpha} \frac{2^{-\alpha k}}{k}.
\]
Here the third inequality is the union bound. A similar argument using the union bound gives
\[
\Pr \left( S_n^{(L)}(f) \neq 0 \right) \leq \sum_{k=\frac{1}{\alpha} \log(n)+1}^{\infty} m \left( S_n(f_k) - U^{dk} S_n(f_k) \neq 0 \right)
\]
\[
\leq 2nC_{\alpha} \sum_{k=\frac{1}{\alpha} \log(n)+1}^{\infty} \frac{2^{-k\alpha}}{k}
\]
\[
\leq \frac{2nC_{\alpha}}{\alpha \log(n)} \sum_{k=\frac{1}{\alpha} \log(n)+1}^{\infty} 2^{-k\alpha} \lesssim \frac{1}{\log(n)} \to 0 \text{ as } n \to \infty.
\]

3.1. Proving Proposition 6(b). For \( W \in \{X, Y, Z\} \), write
\[
S_n^{(M)}(W) = \sum_{k=(\frac{1}{\alpha} - \frac{1}{2}) \log(n)+1}^{\frac{1}{\alpha} \log(n)} \sum_{j=1}^{n} (W_k(j) - W_k(j + d_k))
\]
Proposition 6(b) follows from the following two Lemmas.

**Lemma 7.** For all large \( n \),
\[
S_n^{(M)}(Z) n^{1/\alpha} \rightarrow d S_{\alpha S} \left( \sqrt{2 \ln \left( \frac{2}{2 - \alpha} \right)} \right)
\]

**Proof of Proposition 6(b).** By Lemma 7, it suffices to show convergence of \( S_n^{(M)}(Z) n^{1/\alpha} \). To that end, write
\[
\frac{S_n^{(M)}(Z)}{n^{1/\alpha}} = \frac{S_n^{(M)}(Z) - S_n^{(M)}(Y)}{n^{1/\alpha}} + \frac{S_n^{(M)}(Y)}{n^{1/\alpha}}.
\]
It follows from Lemma 8 and the convergence together lemma that
\[
\frac{S_n^{(M)}(Z)}{n^{1/\alpha}} \rightarrow d S_{\alpha S} \left( \sqrt{2 \ln \left( \frac{2}{2 - \alpha} \right)} \right).
\]

**Proof of Lemma 7 and Lemma 8(a).** Note that if \( k \geq (\frac{1}{\alpha} - \frac{1}{2}) \log(n) \) then \( d_k = \left[ \frac{2^{2k - \alpha}}{2} \right] \geq n \). Consequently \( n + d_k \leq 2d_k \) and
\[
S_n^{(M)}(f) = d G_n \left( f_k \circ T^{j-1} : k \in \mathbb{N}, 1 \leq j \leq 2d_k \right)
\]
where
\[
G_n(x_{k,j} : k \in \mathbb{N}, 1 \leq j \leq 2d_k) = \sum_{k=(\frac{1}{\alpha} - \frac{1}{2}) \log(n) + 1}^{\frac{1}{\alpha} \log(n)} \sum_{j=1}^{n} (x_{k,j} - x_{k,j+d_k})
\]

Since \(G_n\) is continuous and \((f_k \circ T^j)\) is distributed as \((Z_k(j))\), we see that for all large \(n\),
\[
S_n^{(M)}(f) = G_n(Z_k(j) : k \in \mathbb{N}, 1 \leq j \leq 2d_k) = S_n^{(M)}(Z),
\]
concluding the proof of Lemma 7.

Now by Lemma 2(a), if \(k \geq (\frac{1}{\alpha} - \frac{1}{2}) \log(n)\), then
\[
\sum_{j=1}^{n} (|Z_k(j) - Y_k(j)| + |Z_k(j + d_k) - Y_k(j + d_k)|) \leq \frac{n}{d_k} \leq 1.
\]

Lemma 8(a) readily follows from this as for all large \(n\)
\[
|S_n^{(M)}(Z) - S_n^{(M)}(Y)| \leq \frac{1}{\alpha} \log(n).
\]

The proof of Lemma 8(b) is more involved and is done in two stages. The first stage, which is Lemma 9, is to interchange the \(Y_k\) random variables with \(X_k\)’s. The second, Lemma 10, is to show the distributional convergence of \(n^{-1/\alpha} S_n^{(M)}(X)\).

\section*{Lemma 9.} \(\frac{1}{n^{1/\alpha}} \left( S_n^{(M)}(Y) - S_n^{(M)}(X) \right) \) converges to 0 in probability.

\section*{Lemma 10.} \(\frac{1}{n^{1/\alpha}} S_n^{(M)}(X) \Rightarrow S_{\alpha S} \left( \sqrt{2 \ln \left( \frac{2}{1-\alpha} \right)} \right)\).

\textit{Proof of Lemma 10.} For every \(k > (\frac{1}{\alpha} - \frac{1}{2}) \log(n)\), \(d_k > n\) and \(d_k + n \leq 2d_k\).

Therefore, for all \(n\) and \(k > (\frac{1}{\alpha} - \frac{1}{2}) \log(n)\), \((X_k(j) - X_k(j + d_k))_{k \in \mathbb{N}, 1 \leq j \leq n}\) is a sequence of i.i.d. \(S_{\alpha S} \left( \sqrt{\frac{2}{k}} \right)\) random variables. It follows that
\[
\sum_{j=1}^{n} (X_k(j) - X_k(j + d_k)) \sim S_{\alpha S} \left( \sqrt{\frac{2n}{k}} \right) \left( \frac{2}{1-\alpha} \right)^{1/\alpha}.
\]

Secondly, since \(\{X_k(j) : k \in \mathbb{N}, 1 \leq j \leq 2d_k\}\) are independent, we see that
\[
\left\{ \sum_{j=1}^{n} (X_k(j) - X_k(j + d_k)) : k > (\frac{1}{\alpha} - \frac{1}{2}) \log(n) \right\}
\]
are independent $S\alpha S$ random variables. As a result, $\frac{1}{n^{1/\alpha}}S^{(M)}_{n}(X)$ is $S\alpha S(\sigma_{n})$ distributed with

$$\sigma_{n}^{\alpha} = \frac{1}{n} \sum_{k=\left(\frac{1}{\alpha} - \frac{1}{2}\right) \log(n) + 1}^{\log(n)} \frac{2n}{k} \sim 2 \ln \left( \frac{2}{2 - \alpha} \right), \text{ as } n \to \infty.$$  

We conclude from this and Fact 2 that

$$\frac{1}{n^{1/\alpha}}S^{(M)}_{n}(X) \Rightarrow \mathcal{D} S\alpha S \left( \alpha \sqrt{2 \ln \left( \frac{2}{2 - \alpha} \right)} \right).$$

□

The rest of this subsection is concerned with the proof of Lemma 9. Observe that

$$S^{(M)}_{n}(X) - S^{(M)}_{n}(Y) = V^{(M)}_{n} + V^{(M)}_{n}$$

where

$$V^{(M)}_{n} = \frac{1}{n} \sum_{k=\left(\frac{1}{\alpha} - \frac{1}{2}\right) \log(n) + 1}^{\log(n)} \sum_{j=1}^{n} \left( X_{k}(j)1[|X_{k}(j)| \geq 2^{k^{2}}] - X_{k}(j + d_{k})1[|X_{k}(j + d_{k})| \geq 2^{k^{2}}] \right)$$

$$V^{(M)}_{n} = \frac{1}{n} \sum_{k=\left(\frac{1}{\alpha} - \frac{1}{2}\right) \log(n) + 1}^{\log(n)} \sum_{j=1}^{n} \left( X_{k}(j)1[|X_{k}(j)| \leq 2^{k}] - X_{k}(j + d_{k})1[|X_{k}(j + d_{k})| \leq 2^{k}] \right).$$

**Lemma 11.** $\frac{1}{n^{1/\alpha}}V^{(M)}_{n} \to 0$ in probability.

**Proof.** We write for all $k, j \in \mathbb{N}$, $\hat{X}_{k}(j) = X_{k}1[|X_{k}(j)| \geq 2^{k^{2}}]$ so that for every $n \in \mathbb{N}$,

$$V^{(M)}_{n} = \frac{1}{n} \sum_{k=\left(\frac{1}{\alpha} - \frac{1}{2}\right) \log(n) + 1}^{\log(n)} \sum_{j=1}^{n} \left( \hat{X}_{k}(j) - \hat{X}_{k}(j + d_{k}) \right).$$

For $k \in \mathbb{N}$, let $A_{k}$ be the event

$$\left\{ \exists j \in [1, 2d_{k}], \hat{X}_{k}(j) \neq 0 \right\}.$$

Similarly to the proof of Proposition 3(c), there exists $C_{\alpha} > 0$ such that for all but finitely many $k \in \mathbb{N}$,

$$\mathbb{P}(A_{k}) \leq 2d_{k}\mathbb{P} \left( |X_{k}(1)| \geq 2^{k^{2}} \right) \leq 2C_{\alpha}d_{k}2^{-\alpha k^{2}}.$$
The right hand side being summable, the Borel-Cantelli lemma implies that \( P \)- almost surely, \( A_k \) happens only for finitely many \( k \)'s. We now deduce the claim from this fact.

For all \( \left( \frac{1}{\alpha} - \frac{1}{2} \right) \log(n) \leq k \leq \frac{1}{\alpha} \log(n) \), \( n \leq d_k \) and

\[
\left[ \sum_{j=1}^{n} \left( \tilde{X}_k(j) - \tilde{X}_k(j + d_k) \right) \neq 0 \right] \subset A_k.
\]

Since \( \log(n) \to \infty \) and almost surely \( A_k \) happens finitely often we have \( \lim_{n \to \infty} \frac{1}{n^{1/\alpha}} V_M^{(M)} = 0 \) almost surely.

\( \square \)

**Lemma 12.** \( \frac{1}{n^{1/\alpha}} V_M^{(M)} \to 0 \) in probability.

For the proof of Lemma 12 we need the following variance bound.

**Proof of Lemma 12.** Write \( \tilde{X}_k(j) = X_k(j) 1_{|X_k(j)| \leq 2^k} \). Fix \( k > \left( \frac{1}{\alpha} - \frac{1}{2} \right) \log(n) \) so that \( n \leq d_k \). The sequence \( \{ \tilde{X}_k(j) : 1 \leq j \leq n + d_k \} \) is an i.i.d sequence of symmetric random variables. We have,

\[
\mathbb{E} \left( \left( \sum_{j=1}^{n} \left( \tilde{X}_k(j) - \tilde{X}_k(j + d_k) \right) \right)^2 \right) = \sum_{j=1}^{n} \left( \text{Var} \left( \tilde{X}_k(j) \right) + \text{Var} \left( \tilde{X}_k(j + d_k) \right) \right)
= 2n \text{Var} \left( \tilde{X}_k(1) \right).
\]

Since \( X_k(1) \) is \( S_{\alpha}S \left( k^{-1/\alpha} \right) \) distributed, it follows from Lemma 17 with \( K = 2^k \) that,

\[
\mathbb{E} \left( \left( \sum_{j=1}^{n} \left( \tilde{X}_k(j) - \tilde{X}_k(j + d_k) \right) \right)^2 \right) \leq 2Cn \frac{2^{(2-\alpha)k}}{k}.
\]

Now by properties (a) and (b) of the array \( (X_k(j))_{k,j \in \mathbb{N}} \),

\[
\left\{ \sum_{j=1}^{n} \left( \tilde{X}_k(j) - \tilde{X}_k(j + d_k) \right) : \left( \frac{1}{\alpha} - \frac{1}{2} \right) \log(n) < k \leq \frac{1}{\alpha} \log(n) \right\}
\]
are independent, zero mean random variables, therefore

\[
\mathbb{E} \left( \left( n^{-1/\alpha} V_n(M) \right)^2 \right) = n^{-2/\alpha} \sum_{k=\left(\frac{1}{n} - \frac{1}{2}\right) \log(n)+1}^{\frac{1}{2} \log(n)} \mathbb{E} \left( \left( \sum_{j=1}^{n} (\tilde{X}_k(j) - \tilde{X}_k(j + d_k)) \right)^2 \right)
\]

\[
\leq 2C n^{-2/\alpha} \sum_{k=\left(\frac{1}{n} - \frac{1}{2}\right) \log(n)+1}^{\frac{1}{2} \log(n)} \frac{n2(2-\alpha)k}{k}
\]

\[
\lesssim n^{-\frac{2}{\alpha}+1} \frac{2(2-\alpha) \log(n)}{\log(n)} = \frac{1}{\log(n)} \to 0.
\]

A routine application of Markov’s inequality shows that \( n^{-1/\alpha} V_n(M) \) tends to 0 in probability.

\[\square\]

Proof of Lemma 6. The result readily follows from Lemmas 11 and 12 and equation (2).

\[\square\]

We have now concluded the proof of Proposition 6.(b).

3.2. Proving Proposition 6.(a). Write

\[
G_n(f) = \sum_{k=1}^{(\frac{1}{\alpha} - \frac{1}{2}) \log(n)} S_{d_k}(f_k)
\]

Proposition 13.

(a) For all \( n \in \mathbb{N} \), \( S_n^{(S)}(f) = G_n(f) - U^n (G_n(f)) \).

(b) \( \frac{1}{n^{1/\alpha}} G_n(f) \to 0 \) in probability.

Proof of Proposition 13.(a). For all \( k \leq (\frac{1}{\alpha} - \frac{1}{2}) \log(n), d_k \leq n \). Consequently,

\[
S_n(f_k) - U^{d_k} S_n(f_k) = S_{d_k}(f_k) - U^n S_{d_k}(f_k).
\]

Identity (a) follows from summing these identities over all \( 1 \leq k \leq (\frac{1}{\alpha} - \frac{1}{2}) \log(n) \).

\[\square\]

The proof of part(b) in 13 is longer and goes along identical lines as in Subsection 3.1. Recall the notation

\[
\tilde{X}_k(j) = X_k(j)1_{[|X_k(j)| > 2^k]} \text{ and } \tilde{X}_k(j) = X_k(j)1_{[|X_k(j)| < 2^k]}.
\]
For \( W \in \{ X, \tilde{X}, \hat{X}, Y, Z \} \), write
\[
G_n(W) = \frac{1}{n^{1/\alpha}} \sum_{k=1}^{d_k} \sum_{j=1}^{(\frac{1}{\alpha} - \frac{1}{2}) \log(n)} W_k(j)
\]

Proposition 13(b) follows directly from the following lemma.

**Lemma 14.**

(a) For every \( n \in \mathbb{N} \), \( G_n(f) = d \cdot G_n(Z) \).

(b) \( \frac{G_n(Z) - G_n(Y)}{n^{1/\alpha}} \xrightarrow{n \to \infty} 0 \) pointwise.

(c) \( \frac{G_n(Y) - G_n(X)}{n^{1/\alpha}} \xrightarrow{n \to \infty} 0 \) in probability.

(d) \( \frac{G_n(X)}{n^{1/\alpha}} \xrightarrow{n \to \infty} 0 \) in probability.

**Proof.** Fix \( n \in \mathbb{N} \) and note that \( G_n(f) \) is a continuous function of \( F_n := (f_k \circ T^j : 1 \leq k \leq (\frac{1}{\alpha} - \frac{1}{2}) \log(n), 0 \leq j < d_k) \). Since \( F_n \) and \((Z_k(j) : 1 \leq k \leq (\frac{1}{\alpha} - \frac{1}{2}) \log(n), 1 \leq j \leq d_k)\) are equally distributed we see that part (a) holds.

Similarly as in the proof of Lemma 8(a), for all \( n \in \mathbb{N} \),
\[
|G_n(Z) - G_n(Y)| \leq \sum_{k=1}^{(\frac{1}{\alpha} - \frac{1}{2}) \log(n)} \sum_{j=1}^{d_k} |Z_k(j) - Y_k(j)| \leq \sum_{k=1}^{(\frac{1}{\alpha} - \frac{1}{2}) \log(n)} \sum_{j=1}^{d_k} \frac{1}{d_k} = o \left(n^{1/\alpha}\right),
\]

concluding the proof of part (b).

Now for all \( n \in \mathbb{N} \),
\[
G_n(X) - G_n(Y) = G_n(\hat{X}) - G_n(\tilde{X}).
\]

Part (c) follows from Lemma 15.

As in the proof of Lemma 11, \( \sum_{j=1}^{d_k} X_k(j) \sim d \cdot S \alpha S \left( \sqrt{\frac{d_k}{k}} \right) \) as sum of \( d_k \) i.i.d. \( S \alpha S \left( \sqrt{\frac{1}{k}} \right) \) random variables. By the triangular array property, \( \{ \sum_{j=1}^{d_k} X_k(j) : k \in \mathbb{N} \} \) are independent and consequently,
\[
G_n(X) = \frac{1}{n^{1/\alpha}} \sum_{k=1}^{d_k} \left( \sum_{j=1}^{d_k} X_k(j) \right) \sim d \cdot S \alpha S \left( \sigma(n) \right),
\]
where
\[
\sigma(n)^\alpha = \frac{1}{n} \sum_{k=1}^{d_k} \frac{d_k}{k} \lesssim \frac{1}{\log(n)} \rightarrow 0.
\]

Part (d) now follows from Fact 2. \qed

Lemma 15.

- Almost surely, \(\lim_{n \to \infty} G_n(\hat{X}) = \sum_{k=1}^{\infty} \sum_{j=1}^{d_k} \hat{X}_k(j) \in \mathbb{R}\).
- \(n^{-1/\alpha} G_n(\bar{X}) \rightarrow 0\) in probability.

Proof. The proof of the first claim goes along similar lines to the proof of Lemma 11. Write \(A_k := \{\exists j \in [1, d_k], \hat{X}_k(j) \neq 0\}\). By the union bound and Proposition 16,
\[
P(A_k) \leq d_k P(|X_k(1)| > 2^k) \leq C\alpha d_k k^{-\alpha} k^2.
\]
Since the right hand side is summable, it follows from the Borel-Cantelli lemma that almost surely, \(A_k\) holds for only finitely many \(k\). This implies that almost surely
\[
\# \{(k, j) \in \mathbb{N}^2 : X_k(j) \neq 0\} < \infty.
\]
Consequently \(\sum_{k=1}^{\infty} \sum_{j=1}^{d_k} \hat{X}_k(j)\) is almost surely a sum of finitely many terms. This concludes the proof of the first part.

For the second part, note that by independence of \((X_k(j))_{j=1}^{d_k}\) and Lemma 17, there exists \(C > 0\) so that
\[
\text{Var} \left( \sum_{j=1}^{d_k} \bar{X}_k(j) \right) = \sum_{j=1}^{d_k} \text{Var} \left( \bar{X}_k(j) \right) \leq C \frac{d_k 2^{(2-\alpha)k}}{k}.
\]
As \(\{\sum_{j=1}^{d_k} \tilde{X}_k(j) : k \in \mathbb{N}\}\) are independent, centred and square integrable random variables, writing \(\kappa_n = \left( \frac{1}{\alpha} - \frac{1}{2} \right) \log(n)\), we have

\(^1\)recall that for \(k \leq \left( \frac{1}{\alpha} - \frac{1}{2} \right) \log(n)\), \(d_k \leq n\).
\[
\mathbb{E} \left( \left( n^{-1/\alpha} G_n(\tilde{X}) \right)^2 \right) = n^{-2/\alpha} \sum_{k=1}^{\kappa n} \text{Var} \left( \sum_{j=1}^{d_k} \tilde{X}_k(j) \right) \\
\leq C n^{-2/\alpha} \sum_{k=1}^{\kappa n} \frac{d_k 2^{(2-\alpha)k}}{k} \\
\leq n^{-2/\alpha} d_{\kappa n} \frac{2^{(2-\alpha)\kappa n}}{\kappa n} \\
\leq n^{1-\frac{2}{\alpha} 2^{(2-\alpha)2} \log(n)} = n^{1-\frac{2}{\alpha} 2^{(2-\alpha)2} \kappa n} \xrightarrow{n \to \infty} 0,
\]
since for \( \alpha \in (0, 2) \),
\[1 - \frac{2}{\alpha} + \frac{(2 - \alpha)^2}{2\alpha} = \frac{\alpha^2 - 2\alpha}{2\alpha} < 0.\]
The second part follows from a routine application of Markov’s inequality.

We can now conclude the proof of Proposition 6.(a).

**Proof of Proposition 6.(a).** Since \( G_n(f) = d U_n^m (G_n(f)) \), it follows from Proposition 13(b) that \( \frac{1}{n^{1/\alpha}} G_n(f) \) and \( \frac{1}{n^{1/\alpha}} U_n^m (G_n(f)) \) converge to 0 in probability. By Proposition 13 we see that \( \frac{1}{n^{1/\alpha}} S_n(f) \) converges to 0 in probability. \( \square \)

4. APPENDIX: GROWTH OF DISPERSION FOR STATIONARY \( S\alpha S \) PROCESSES

As \( S\alpha S \) random variables are defined by their characteristic functions, Lévy’s continuity theorem implies the following fact.

**Fact 2.** If for all \( n \in \mathbb{N} \), \( Z_n \) is \( S\alpha S(\sigma_n) \) distributed and \( \lim_{n \to \infty} \sigma_n^\alpha = A^\alpha \), then \( Z_n \Rightarrow^d S\alpha S(A) \). In addition, If \( \sigma_n \to 0 \) then \( Z_n \Rightarrow^d 0 \).

The following tail bound is used extensively in this work.

**Proposition 16.** There exists \( C_\alpha > 0 \) such that for all \( 0 < \sigma \leq 1 \), if \( X \) is an \( S\alpha S(\sigma) \) random variable and \( t \geq 1 \) then,
\[\mathbb{P} (|X| \geq t) \leq C_\alpha \sigma^\alpha t^{-\alpha}\]

**Proof.** By Proposition 1.2.15 in [5], there exists \( c_\alpha > 0 \) such that if \( X \sim^d S\alpha S(1) \), then
\[\mathbb{P} (|X| \geq t) \sim c_\alpha t^{-\alpha}, \quad \text{as } t \to \infty.\]
We deduce that
\[ C_\alpha := \sup_{t \geq 1} \frac{\mathbb{P}(|X| \geq t)}{t^{-\alpha}} < \infty. \]

Finally if \( X \sim^d S\alpha S(\sigma) \) with \( \sigma \leq 1 \) and \( t \geq 1 \), we have
\[ \mathbb{P}(|X| \geq t) = \mathbb{P}\left(\frac{|X|}{\sigma} \geq \frac{t}{\sigma}\right) \leq C_\alpha \left(\frac{t}{\sigma}\right)^{-\alpha}. \]

The tail bound implies the following inequality for the variance.

**Lemma 17.** There exists \( c = c(\alpha) > 0 \) such that for all \( K \geq 1 \), \( 0 < \sigma \leq 1 \) and \( m \in \mathbb{N} \), if \( X \) is a \( S\alpha S(\sigma) \) random variable, then
\[ \text{Var}(X_{1|[X| \leq K]}) \leq cK^{2-\alpha}\sigma^\alpha. \]

**Proof.** As \( X \) is symmetric the random variable \( X_{1|[X| \leq K]} \) has zero mean. By Proposition [16] there exists \( C_\alpha > 0 \) such that,
\[
\text{Var}(X_{1|[X| \leq K]}) = \mathbb{E}((X_{1|[X| \leq K]})^2) = \int x^2 \mathbb{P}(|X| \geq K > x) dx \\
\leq 1 + \int_1^K x \mathbb{P}(|X| > x) dx \\
\leq [1 + o_{K \to \infty}(1)]C_\alpha \sigma^\alpha \int_1^K x^{1-\alpha} dx \\
= C_\alpha \cdot \sigma^\alpha K^{2-\alpha} \left[1 + o_{K \to \infty}(1)\right].
\]

We conclude that there exists \( C \) depending only on \( \alpha \) such that for all \( K \geq 1 \),
\[ \text{Var}(X_{1|[X| \leq K]}) \leq C\sigma^\alpha K^{2-\alpha}. \]

In our construction of CLT functions we used a triangular array of random variables \( Y_k \) which are not \( S\alpha S \) distributed but are in the domain of attraction of an \( S\alpha S \) distribution. A main reason for this choice lies in the fact that the dispersion of a stationary \( S\alpha S \) process does not go fast enough for the methods of [13] to work.

A real valued stationary process \((X_n)_{n=1}^\infty \) is a \( S\alpha S \) process if every \( Z \) in the linear span of \( \{X_n : n \in \mathbb{N}\} \) is \( S\alpha S \) distributed. In that case the function
\[ \|X\|_\alpha = \left(-\log \mathbb{E}(e^{iX})\right)^{1/\alpha}, \]
is a quasi-norm from \( \text{Lin}(X) := \text{span}\{X_n : n \in \mathbb{N}\} \) to \([0, \infty)\) and for all \( Z \in \text{Lin}(X), \|Z\|_\alpha \) equals the dispersion parameter of \( Z \). The following is a well known fact on stationary \( S\alpha S \) processes.

**Fact 3.** If \( 0 < \alpha < 1 \) and \( (X_n)_{n=1}^\infty \) is a stationary \( S\alpha S \) process, then for every \( N \in \mathbb{N} \),

\[
\left\| \sum_{j=1}^N X_j \right\|_\alpha \leq N \|X_1\|_\alpha^\alpha.
\]

**Proof.** By [5, Property 2.10.5], if \( X, Y \) are \( S\alpha S \) random variables with \( 0 < \alpha < 1 \), then

\[
\|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha - \|X + Y\|_\alpha^\alpha \geq 0.
\]

A straightforward inductive procedure gives the claim. \( \square \)

**Remark 18.** One can show using stochastic integrals that there is equality if and only if \( X_1, \ldots, X_N \) are independent.

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