Non-Abelian Vortices
without Dynamical Abelianization

Daniele DORIGONI (1), Kenichi KONISHI (1,2), Keisuke OHASHI (3)

Dipartimento di Fisica “E. Fermi” – Università di Pisa (1),
Istituto Nazionale di Fisica Nucleare – Sezione di Pisa (2),
Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy (1,2)
Department of Mathematics and Theoretical Physics (DAMTP) (3)
University of Cambridge, Cambridge, UK

Abstract: Vortices carrying truly non-Abelian flux moduli, which do not dynamically reduce to Abelian vortices, are found in the context of softly-broken $\mathcal{N} = 2$ supersymmetric chromodynamics (SQCD). By tuning the bare quark masses appropriately we identify the vacuum in which the underlying $SU(N)$ gauge group is partially broken to $SU(n) \times SU(r) \times U(1)/\mathbb{Z}_K$, where $K$ is the least common multiple of $(n, r)$, and with $N_{f^{su(n)}} = n$ and $N_{f^{su(r)}} = r$ flavors of light quark multiplets. At much lower energies the gauge group is broken completely by the squark VEVs, and vortices develop which carry non-Abelian flux moduli $\mathbb{C}P^{n-1} \times \mathbb{C}P^{r-1}$. For $n > r$ we argue that the $SU(n)$ fluctuations become strongly coupled and Abelianize, while leaving weakly fluctuating $SU(r)$ flux moduli. This allows us to recognize the semi-classical origin of the light non-Abelian monopoles found earlier in the fully quantum-mechanical treatment of 4D SQCD.

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1. Introduction

Attempts to understand better the mechanism of confinement of non-Abelian variety, which is probably the case for the realistic world of QCD, has eventually led to the discovery of vortices with non-Abelian continuous flux moduli [1],[2], triggering a remarkable development of research activity in related problems [3]-[25]. A typical system considered is a $U(n)$ theory with $N_f = n$ scalar quark flavors, whose vacuum expectation value (VEV) breaks the gauge symmetry completely, leaving however the color-flavor diagonal $SU(n)_{C+F}$ symmetry unbroken (color-flavor locking). Vortices in such a system develop a continuous zeromodes (moduli) parametrizing

$$SU(n)/SU(n-1) \times U(1) \sim CP^{n-1},$$

where the divisor represents the symmetry respected by individual vortices. When the vortex orientation is allowed to fluctuate along $z$ (the direction of the vortex length) and in time $t$, the dynamics of such fluctuations is described by a two dimensional $CP^{n-1}$ sigma model [2, 4, 5]. If the original system is the bosonic sector of a $\mathcal{N} = 2$ supersymmetric model, the sigma model has $(2,2)$ supersymmetry, as half of the supersymmetry is broken by the vortex. In the infrared limit, the sigma model becomes strongly coupled, and the 2D system reproduces exactly [4, 5] the dynamics of the corresponding 4D gauge theory in Coulomb phase, encoded by Seiberg-Witten curves [26, 27, 28], realizing thus the idea of duality between two-dimensional sigma model and a four-dimensional gauge theory discussed earlier by Dorey [29].

Beautiful as it may be, the very result of the analysis shows that the vortices considered in [2, 4, 5] dynamically Abelianize to Abrikosov-Nielsen-Olesen (ANO) vortices (see the next section). This fact can be seen both in two and four dimensions. In the sigma model analysis, the fluctuations inside the vortex become strongly coupled and generates the mass scale, $\Lambda$; there are $n$ degenerate ground states [7] (Witten-CFIV index [30, 31]). Monopoles appear as kinks (domain walls) connecting two adjacent vortex ground states. Each monopole is confined by two vortices carrying the “adjacent” $U(1)$ fluxes, a typical situation for a monopole arising from the breaking of $SU(2) \subset U(n)$ to $U(1)$. The global $SU(N_f) = SU(n)$ flavor symmetry
is not spontaneously broken by the vortex dynamics\[1\]; this however does not contradict the fact that the monopoles in the infrared carry only Abelian magnetic $U(1)^n$ charges.

In four dimensions, the model considered can be seen as the (bosonic part of the) low-energy effective action of $\mathcal{N} = 2$ supersymmetric $SU(N)$, with $N = n + 1$ and with $N_f = n$ flavors. The gauge group is broken by the adjoint scalar VEV,

$$\langle \phi \rangle = \text{diag} \left( m_1, m_2, \ldots, m_n, -\sum_{j=1}^{n} m_j \right), \quad m_i \rightarrow m,$$

(1.1)

to $SU(n) \times U(1)/\mathbb{Z}_n \sim U(n)$. The light monopoles and the magnetic gauge quantum numbers of these, in the limit of small $m_i$ and $\mu$, can be read off from the singularities of the Seiberg-Witten curves \[36, 32\]. Semi-classically (large $m_i$), instead, the vacua of this theory are classified according to the number of quark flavors which remain massless due to the cancellation between the bare quark mass and the adjoint scalar VEV in the superpotential,

$$\tilde{Q} \left( \sqrt{2} \Phi + M \right) Q.$$

The model considered in \[2, 4, 5\], as can be seen from the VEV of the adjoint scalar, corresponds to the $r = n = N_f$ vacuum of the above theory. The light monopoles in Table 1 correspond to the limit $m_i \rightarrow m \rightarrow 0$, and we need to know to which quantum vacuum each semi-classical vacuum corresponds. This problem of matching the semi-classical and fully quantum mechanical vacua one by one, has been solved by using the vacuum counting and by symmetry considerations. The classical $r$ vacua, $r = 0, 1, \ldots, N_f$ found in the semiclassical regime $|m_i| \gg |\mu| \gg \Lambda$ are found to correspond \[32, 33, 34\] to the quantum $r$ vacua, $r = 0, 1, \ldots, N_f/2$, as

$$\{r, N_f - r\} \iff r, \quad r = 0, 1, \ldots \leq N_f/2.$$

(1.2)

where the left hand side stands for the classical vacuum classification. Note that the quantum $r$ vacua (with $SU(r)$ non-Abelian magnetic gauge symmetry) exist only up to $r \leq N_f/2$ for dynamical reasons \[35\]. Therefore the model considered in \[2, 4, 5\] must correspond to the $r = 0$ quantum vacuum. The latter is characterized by the fact that all monopoles are Abelian (see Table 1); furthermore none of them carries any flavor $SU(N_f)$ quantum numbers. The condensation of the light monopoles (which occurs when the adjoint scalar masses $\mu \Phi^2$ are added in the theory) does not break $SU(N_f)$ symmetry, consistently with the finding from the vortex dynamics\[2\].

\[1\] Of course this is consistent with Coleman’s theorem.

\[2\] The authors thank R. Auzzi and G. Marmorini for discussions on this point.
On the other hand, one knows that in four dimensional $\mathcal{N} = 2$ supersymmetric QCD there appear light monopoles carrying non-Abelian charges ($r$ vacua with $2 \leq r \leq N_f/2$ in Table 1), and one wonders whether such truly non-Abelian vortices which do not Abelianize dynamically can be found in some appropriate regime, through which one can identify a semi-classical origin of the non-Abelian monopoles and the associated vortices.

We shall show below that such a system can indeed be found. The underlying model is the same as the one discussed in [2, 32]: an $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with $N_f = N$ flavors. But the gauge group is broken partially down to $SU(n) \times SU(r) \times U(1)$ gauge symmetry ($N = n + r$) by the adjoint scalar VEV.

| $r$ | Deg. Freed. | Eff. Gauge Group | Phase | Global Symmetry |
|-----|-------------|-----------------|-------|-----------------|
| 0   | monopoles   | $U(1)^{N-1}$    | Confinement | $U(n_f)$       |
| 1   | monopoles   | $U(1)^{N-1}$    | Confinement | $U(N_f - 1) \times U(1)$ |
| $2, \ldots, \lfloor N_f/2 \rfloor$ | NA monopoles | $SU(r) \times U(1)^{N-r}$ | Confinement | $U(N_f - r) \times U(r)$ |
| $N_f/2$ | rel. nonloc. | -               | -     | Almost SCFT $U(N_f/2) \times U(N_f/2)$ |

Table 1: Confining vacua of $SU(N)$ gauge theory with $N_f$ flavors. In the superconformal $r = N_f/2$ vacuum, relatively nonlocal monopoles and dyons appear both as the low-energy effective degrees of freedom. “Almost SCFT” means that the theory is a non-trivial superconformal theory when $\mu = 0$ but confines upon $\mu \neq 0$ perturbation. In the theory with $N_f = N$ considered here, the vacua at the “baryonic root”, in free magnetic phase, are absent. They appear only for $N_f > N$, with an effective gauge group, $SU(N_f - N)$.

2. Dynamical Abelianization

As the question of dynamical Abelianization is central to this work, and as this point might be somewhat misleading, let us add a few clarifying remarks before proceeding, even risking the vice of over-repetition. Dynamical Abelianization, as normally understood, concerns the gauge symmetry. It means by definition that a non-Abelian gauge symmetry of a given theory reduces at low energies by quantum effects to an Abelian (dual or not) gauge theory. (Related concepts are dynamical Higgs mechanism, or tumbling [37]). Example of the theories in which this is known

\[\text{We thank the referee of the first version of this paper for urging us to do so.}\]
to occur are the pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theories \cite{26,28} which reduce to Abelian gauge theories at low energies, and the $SU(2)$ $\mathcal{N} = 2$ theories with $N_f = 1, 2, 3$ matter hypermultiplets \cite{27}. But as has been emphasized repeatedly and in Introduction above, $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD (with $N \geq 3$) \textit{with quark multiplets}, do not Abelianize in general \cite{36,32,35}. Whether or not the standard QCD with light quarks Abelianizes is not known. The ’t Hooft-Mandelstam scenario implies a sort of dynamical Abelianization, as it assumes the Abelian $U(1)^2$ monopoles to be the dominant degrees of freedom at some relevant scales, but this has not been proven.

As the vortex orientation fluctuation modes turn out to be intimately connected to the way \textit{dual} gauge symmetry emerges at low-energies \cite{19,35} and below), it is perfectly reasonable to use the same terminology for the vortex modes.

Nevertheless, one could \textit{define} – and in this paper we shall use it in this sense – the concept of \textit{non-Abelian or Abelian vortices}, independently of the usual meaning attributed to it in relation to a gauge symmetry. A vortex is non-Abelian, if it carries a non-trivial, internal non-Abelian moduli, which can fluctuate along its length and in time. We exclude from this consideration other vortex moduli associated with their (transverse) positions, shapes or sizes (in the case of higher-winding \cite{5,12,17} or semi-local vortices \cite{10,16,23}). Otherwise, a vortex is Abelian. The standard ANO vortex is Abelian, as it possesses no-continuous moduli. The vortices found in the context of $U(N)$ models \cite{1,2} \textit{are} indeed non-Abelian in this sense.

But just as a non-Abelian gauge theory may or may not Abelianize depending on dynamical details, a non-Abelian vortex may or may not dynamically Abelianize. In the very papers in which these vortices have been discovered \cite{2,5} and in those which followed \cite{4}, it was shown that they dynamically reduced to Abelian, ANO like vortices at long distances. The orientational moduli fluctuate strongly and at long distances they effectively lose their orientation. A recent observation \cite{45} nicely exhibits this aspect through the Lüscher term of the string tension. It is quite sensible therefore to call those vortices in the $U(N)$, $N_f = N$ models as \textit{elementary} non-Abelian vortices \cite{46}.

In what follows, it will be shown that this fate is not unavoidable. Semi-classical non-Abelian vortices which remain so at low-energies do exist; they can be found in appropriate vacua, selected by a careful tuning of the bare quark masses. This is quite similar to the situation in $\mathcal{N} = 1$ supersymmetric QCD, where a vacuum with
a prescribed chiral symmetry breaking pattern can be selected out of the degenerate set of vacua by appropriately tuning the bare quark mass ratios, before sending them to zero. The symmetry breaking pattern in those theories is aligned with the bare quark masses, as is well-known [44].

And this finding closes the gap in matching the results in the 4D theories at fully quantum regimes (where all bare mass parameters are small) and in semi-classical regimes where the vortices can be reliably studied. In other words the work which follows allows us to identify the semi-classical origin of the quantum non-Abelian monopoles found in [36, 32].

3. Non-Abelian vortices which do not dynamically reduce to ANO vortices

The model on which we shall base our consideration is the softly broken $\mathcal{N} = 2$ supersymmetric QCD with $SU(N)$ and $N_f = N$ flavors of quark multiplets,

$$\mathcal{L} = \frac{1}{8\pi} \text{Im} \tau_{cl} \left[ \int d^4 \theta \text{Tr} (\Phi^\dagger e^V \Phi e^{-V}) + \int d^2 \theta \frac{1}{2} \text{Tr} (WW) \right] + \mathcal{L}^{(\text{quarks})} + \int d^2 \theta \mu \text{Tr} \Phi^2;$$

$$\mathcal{L}^{(\text{quarks})} = \sum_i \left[ \int d^4 \theta (Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger) + \int d^2 \theta (\sqrt{2} \tilde{Q}_i \Phi Q_i^i + m_i \tilde{Q}_i Q_i) \right];$$

where $\tau_{cl} \equiv \theta_0/\pi + 8\pi i/g_0^2$ contains the coupling constant and the theta parameter, $\mu$ is the adjoint scalar mass, breaking softly $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. We tune the bare quark masses as

$$m_1 = \ldots = m_n = m^{(1)}; \quad m_{n+1} = m_{n+2} = \ldots = m_{n+r} = m^{(2)}, \quad N = n + r;$$

$$n \, m^{(1)} + r \, m^{(2)} = 0 , \quad (3.3)$$

or

$$m^{(1)} = \frac{r \, m_0}{\sqrt{r^2 + n^2}} , \quad m^{(2)} = -\frac{n \, m_0}{\sqrt{r^2 + n^2}}, \quad (3.4)$$

and their magnitude is taken as

$$|m_0| \gg |\mu| \gg \Lambda.$$  

(3.5)
The adjoint scalar VEV can be taken to be
\[ \langle \Phi \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m^{(1)} & 0 \\ 0 & m^{(2)} \end{pmatrix} \] (3.6)

Below the mass scale \( v_1 \sim |m_i| \) the system thus reduces to a gauge theory with gauge group
\[ G = \frac{SU(n) \times SU(r)}{\mathbb{Z}_K}, \quad K = \text{LCM}\{n, r\} \] (3.7)
where \( K \) is the least common multiple of \( n \) and \( r \). The higher \( n \) color components of the first \( n \) flavors (with the bare mass \( m^{(1)} \)) remain massless, as well as the lower \( r \) color components of the last \( r \) flavors (with the bare mass \( m^{(2)} \)); they will be denoted as \( q^{(1)} \) and \( q^{(2)} \), respectively. They carry the charges \( \lambda_1, -\lambda_2 \),
\[ \lambda_1 \equiv \frac{r}{\sqrt{2nr(r+n)}}, \quad \lambda_2 \equiv \frac{n}{\sqrt{2nr(r+n)}}. \] (3.8)
with respect to the \( U(1) \) gauge symmetry generated by
\[ t^{(0)} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \quad \text{Tr} t^{(0)2} = \frac{1}{2}. \] (3.9)
Non-Abelian gauge groups are generated by the standard \( SU \) generators
\[ t^a_{su(n)} = \begin{pmatrix} \delta_{aa'} & 0 \\ 0 & \delta_{aa'} \end{pmatrix}; \quad t^b_{su(r)} = \begin{pmatrix} \delta_{bb'} & 0 \\ 0 & \delta_{bb'} \end{pmatrix}, \] (3.10)
a = 1, 2, \ldots, n^2 - 1; \quad b = 1, 2, \ldots, r^2 - 1, \quad \text{with the normalization}
\[ \text{Tr}_n (t^a t^{a'}) = \frac{\delta_{aa'}}{2}, \quad \text{Tr}_r (t^b t^{b'}) = \frac{\delta_{bb'}}{2}. \]

Our model for studying the vortices then is:
\[ \mathcal{L} = -\frac{1}{4g^2_0} F_{\mu\nu}^0 \frac{1}{4g^2_n} F_{\mu\nu}^{(n)} \frac{1}{4g^2_r} F_{\mu\nu}^{(r)} + \frac{1}{g^2_0} |D_\mu \Phi^{(0)}|^2 + \frac{1}{g^2_n} |D_\mu \Phi^{(n)}|^2 + \frac{1}{g^2_r} |D_\mu \Phi^{(r)}|^2 
+ |D_\mu q^{(1)}|^2 + |D_\mu q^{(1)}|^2 + |D_\mu q^{(2)}|^2 + |D_\mu q^{(2)}|^2 - V_D - V_F, \] (3.11)

One could very well start with a model of this sort directly. The squark VEVs can be induced by a Fayet-Iliopoulos term introduced by hand. By an \( SU_R(2) \) rotation, which rotates \((q, \tilde{q}^\dagger)\) as a doublet, such a model can be seen to be equivalent to the one being considered here.
plus fermionic terms, where $V_D$ and $V_F$ are the $D$-term and $F$-term potentials. The $D$–term potential $V_D$ has the form,

$$V_D = \frac{1}{8} \sum_A \left( \Tr t^A \left[ \frac{2}{g^2} [\Phi, \Phi^\dagger] + \sum_i (Q_i Q_i^\dagger - \tilde{Q}_i \tilde{Q}_i^\dagger) \right] \right)^2,$$  \hspace{1cm} (3.12)

where the generators $A$ takes the values 0 for $U(1)$, $a = 1, 2, \ldots, n^2 - 1$ for $SU(n)$ and $b = 1, 2, \ldots r^2 - 1$ for $SU(r)$. $V_F$ is of the form

$$g_0^2 [\mu \Phi^{(a)} + \sqrt{2} \tilde{Q} t^{(a)} Q]^2 + g_n^2 [\mu \Phi^{(b)} + \sqrt{2} \tilde{Q} t^{(b)}_{su(n)} Q]^2 + g_r^2 [\mu \Phi^{(b)} + \sqrt{2} \tilde{Q} t^{(b)}_{su(r)} Q]^2 + \tilde{Q} [M + \sqrt{2} \Phi] [M + \sqrt{2} \Phi^\dagger] \tilde{Q} + Q [M + \sqrt{2} \Phi] [M + \sqrt{2} \Phi^\dagger] Q,$$  \hspace{1cm} (3.13)

where

$$M = \begin{pmatrix} m^{(1)} & 0 \\ 0 & m^{(2)} \end{pmatrix}$$

is the mass matrix and the (massless) squark fields have the form,

$$Q(x) = \begin{pmatrix} q^{(1)}(x) & 0 \\ 0 & q^{(2)}(x) \end{pmatrix}, \hspace{1cm} \tilde{Q}(x) = \begin{pmatrix} \tilde{q}^{(1)}(x) & 0 \\ 0 & \tilde{q}^{(2)}(x) \end{pmatrix},$$  \hspace{1cm} (3.14)

if written in a color-flavor mixed matrix notation. The light squarks (supersymmetric partners of the left-handed quarks in supersymmetric model) are summarized in Table 2.

Table 2:

| fields | $U(1)$ | $SU(n)$ | $SU(r)$ |
|--------|--------|---------|---------|
| $q^{(1)}$ | $\lambda_1$ | $\frac{n}{1}$ | $\frac{1}{1}$ |
| $\tilde{q}^{(1)}$ | $-\lambda_1$ | $\frac{n^*}{1}$ | $\frac{1}{1}$ |
| $q^{(2)}$ | $-\lambda_2$ | $\frac{1}{r}$ | $\frac{1}{r^2}$ |
| $\tilde{q}^{(2)}$ | $\lambda_2$ | $\frac{1}{r}$ | $\frac{1}{r^2}$ |

We set $V_D$ to zero identically, in the vacuum and in the vortex configurations, by keeping

$$\tilde{q}^{(1)} = (q^{(1)})^\dagger, \hspace{1cm} q^{(2)} = - (\tilde{q}^{(2)})^\dagger;$$  \hspace{1cm} (3.15)

the redefinition

$$q^{(1)} \rightarrow \frac{1}{\sqrt{2}} q^{(1)}, \hspace{1cm} \tilde{q}^{(2)} \rightarrow \frac{1}{\sqrt{2}} \tilde{q}^{(2)}$$  \hspace{1cm} (3.16)
brings the kinetic terms for these fields back to the original form.

The VEVs of the adjoint scalars are given by

\[ \langle \Phi^{(0)} \rangle = -m_0, \quad \langle \Phi^{(a)} \rangle = \langle \Phi^{(b)} \rangle = 0, \quad (3.17) \]

while the squark VEVs are given (from the vanishing of the first line of Eq. (3.13)) by

\[ \langle Q \rangle = \left( \begin{array}{cc} v^{(1)} \, \mathbb{1}_{n \times n} & 0 \\ 0 & -v^{(2)} \, \mathbb{1}_{r \times r} \end{array} \right), \quad \langle \tilde{Q} \rangle = \left( \begin{array}{cc} v^{(1)} \, \mathbb{1}_{n \times n} & 0 \\ 0 & v^{(2)} \, \mathbb{1}_{r \times r} \end{array} \right), \quad (3.18) \]

with

\[ |v^{(1)}|^2 + |v^{(2)}|^2 = \sqrt{\frac{n + r}{n \, r \, \mu \, m_0}}. \quad (3.19) \]

There is a continuous vacuum degeneracy; we assume that

\[ v^{(1)} \neq 0; \quad v^{(2)} \neq 0, \]

in the following. The presence of the flat direction implies the existence of the so-called semi-local vortex moduli; but we shall not be concerned with these here.

“Non-Abelian” vortices exist in this theory as the vacuum breaks the gauge group $G$ (Eq. (3.7)) completely, leaving at the same time a color-flavor diagonal symmetry

\[ [SU(n) \times SU(r) \times U(1)]_{C+F} \quad (3.20) \]

unbroken. The full global symmetry, including the overall global $U(1)$ is given by

\[ U(n) \times U(r). \quad (3.21) \]

The minimal vortex in this system corresponds to the smallest nontrivial loop in the $G$ group space, Eq. (3.7). It is the path in the $U(1)$ space

\[ \left( \begin{array}{cc} e^{i \alpha r} \, \mathbb{1}_{n \times n} & 0 \\ 0 & e^{i \alpha n} \, \mathbb{1}_{r \times r} \end{array} \right), \quad \alpha : 0 \to \frac{2\pi}{n \, r}, \quad (3.22) \]

that is,

\[ \mathbb{1}_{N \times N} \to \mathbb{Y}, \quad \mathbb{Y} = \left( \begin{array}{cc} e^{2\pi i / n} \, \mathbb{1}_{n \times n} & 0 \\ 0 & e^{2\pi i / r} \, \mathbb{1}_{r \times r} \end{array} \right), \quad (3.23) \]

followed by a path in the $SU(n) \times SU(r)$ manifold

\[ \mathbb{1}_{n \times n} \to \mathbb{Z}_n = e^{-2\pi i / n} \, \mathbb{1}_{n \times n}; \quad \mathbb{1}_{r \times r} \to \mathbb{Z}_r = e^{-2\pi i / r} \, \mathbb{1}_{r \times r}; \quad (3.24) \]
Figure 1: Numerical result for the profile functions $f_{1,2}, g_{1,2}$ as functions of the radius $\rho$, for $SU(3) \times SU(2) \times U(1)$ theory. The coupling constants and the ratio of the VEVs are taken to be $g_0 = 0.1, g_3 = 10, g_2 = 1, |v_2|/|v_1| = 3$.

back to the unit element. For instance one may choose $(\beta : 0 \rightarrow 2\pi; \gamma : 0 \rightarrow 2\pi)$

$$
\begin{pmatrix}
    e^{i\beta(n-1)/n} & 0 \\
    0 & e^{-i\beta/n} \mathbb{1}_{(n-1) \times (n-1)}
\end{pmatrix};
\begin{pmatrix}
    e^{i\gamma(r-1)/r} & 0 \\
    0 & e^{-i\gamma/r} \mathbb{1}_{(r-1) \times (r-1)}
\end{pmatrix}.
$$

As

$$
\mathbb{Y}^K = \mathbb{1}_{N \times N}, \quad K = \text{LCM}\{n, r\}
$$

it follows that the tension (and the winding) with respect to the $U(1)$ is $\frac{1}{K}$ of that in the standard ANO vortex.

The squark fields trace such a path asymptotically, i.e., far from the vortex core, as one goes around the vortex; at finite radius the vortex has, for instance, the form,

$$
q^{(1)} = \begin{pmatrix}
    e^{i\phi} f_1(\rho) & 0 \\
    0 & f_2(\rho) \mathbb{1}_{(n-1) \times (n-1)}
\end{pmatrix},
q^{(2)} = \begin{pmatrix}
    e^{i\phi} g_1(\rho) & 0 \\
    0 & g_2(\rho) \mathbb{1}_{(r-1) \times (r-1)}
\end{pmatrix},
$$

where $\rho$ and $\phi$ stand for the polar coordinates in the plane perpendicular to the vortex axis, $f_{1,2}, g_{1,2}$ are profile functions. The adjoint scalar fields $\Phi$ are taken to be equal to their VEVs, Eq. (3.17). They are accompanied by the appropriate gauge fields so that the tension is finite. The BPS equations for the squark and gauge fields, and the properties of their solutions are discussed in Appendix A. The behavior of numerically integrated vortex profile functions $f_{1,2}, g_{1,2}$ is illustrated in Fig. 1.

We note here only that the necessary boundary conditions on the squark profile functions have the form,

$$
f_1(\infty) = f_2(\infty) = v^{(1)}, \quad g_1(\infty) = g_2(\infty) = v^{(2)},
$$
while at the vortex core,

\[ f_1(0) = 0, \quad g_1(0) = 0, \quad f_2(0) \neq 0, \quad g_2(0) \neq 0, \]  

(3.27)

The most important fact about these minimum vortices is that one of the \( q^{(1)} \) and one of the \( \tilde{q}^{(2)} \) fields must necessarily wind at infinity, simultaneously. As the individual vortex breaks the (global) symmetry of the vacuum as

\[ [SU(n) \times SU(r) \times U(1)]_{C+F} \rightarrow SU(n-1) \times SU(r-1) \times U(1)^3, \]  

(3.28)

the vortex acquires Nambu-Goldstone modes parametrizing

\[ CP^{n-1} \times CP^{r-1} : \]  

(3.29)

they transform under the exact color-flavor symmetry \( SU(n) \times SU(r) \) as the bifundamental representation, \((n,r)\). Allowing the vortex orientation to fluctuate along the vortex length and in time, we get a \( CP^{n-1} \times CP^{r-1} \) two-dimensional sigma model as an effective Lagrangian describing them. The details have been worked out in \([4,5]\) and need not be repeated here.

The main idea of the present paper is this. Let us assume without losing generality that \( n > r \), excluding the special case of \( r = n \) for the moment. As has been shown in \([4,5]\) the coupling constant of the \( CP^{n-1} \) sigma models grows precisely as the coupling constant of the 4D \( SU(n) \) gauge theory. At the point the \( CP^{n-1} \) vortex moduli fluctuations become strong and the dynamical scale \( \Lambda \) gets generated, with vortex kinks (Abelian monopoles) acquiring mass of the order of \( \Lambda \), the vortex still carries the unbroken \( SU(r) \) fluctuation modes \((CP^{r-1})\), as the \( SU(r) \) interactions are still weak. See Fig. 2. Such a vortex will carry one of the \( U(1) \) flux arising from the dynamical breaking of \( SU(n) \times U(1) \rightarrow U(1)^n \), as well as an \( SU(r) \) flux. As these vortices end at a massive monopole (arising from the high-energy gauge symmetry breaking, Eq. (3.6)), the latter necessarily carries a non-Abelian continuous moduli, whose points transform as in the fundamental representation of \( SU(r) \). This can be interpreted as the (electric description of) dual gauge \( SU(r) \) system observed in the infrared limit of the 4D SQCD \([36,32]\).

The special case \( r = 1 \) corresponds to the \( U(N) \) model \([2,4,5,15]\), mentioned in the Introduction, and in this case the vortices dynamically Abelianize. This is not in contradiction with the claim made above, after Eq. (1.2), that the \( U(n) \) models considered in those papers corresponded to the quantum \( r = 0 \) vacuum of the \( SU(n+\)
Figure 2:

1) model, with $N_f = n$. The point is that here we start with the underlying theory with $SU(N)$, $N_f = N$, where $N = n + r$; the classical-quantum vacuum matching condition (Eq. (1.2)) implies that the $U(n)$ models studied earlier, if embedded in our general scheme, correspond to the $r = 1$, rather than $r = 0$, vacua. The symmetry breaking pattern Eq. (3.21) also perfectly matches the full quantum result in Table II as it does for generic $r$.

There is no difficulty in generalizing our construction and finding vortices with fluctuations corresponding to more than two non-Abelian factors,

$$SU(n) \times SU(r_1) \times SU(r_2) \times \ldots,$$

as long as we remain in the semi-classical region with $|m_i|, |\mu| \gg \Lambda$. However, the main aim of this paper is to identify the semi-classical origin of the non-Abelian monopoles seen in the fully quantum effective low-energy action of the theory at $m_i \to 0$, $\mu \sim \Lambda$. In such a limit, the breaking of the gauge symmetry is a dynamical question; the result of the analysis of the 4$D$ theory (Table II) suggests that in that limit the surviving non-Abelian dual group $SU(r_1) \times SU(r_2) \times \ldots$ gets enhanced to a single factor $SU(r)$. In order for gauge groups with more than one non-Abelian factors to survive dynamically, a nontrivial potential in the adjoint scalar field $\Phi$ needs to be present in the underlying theory [33].
4. Vortex moduli, kinks and monopoles in 4D theory

It is somewhat a puzzle why the exact 2D-4D correspondence holds. A particularly intriguing point is that the two-dimensional vortex sigma-model dynamics in the Higgs phase of the four dimensional theory reproduces exactly the 4D gauge dynamics in the Coulomb phase. One might be tempted to argue that the reason for such a correspondence is that in the vortex core the full gauge symmetry is restored, as in the case of an instanton. Actually, it is not. A glance at Eq. (3.26) and Eq. (3.27) shows that the gauge symmetry at the vortex core is only partially restored, to $U(1) \times U(1)$. The global symmetry in the vortex core, on the other hand, is smaller than that outside the vortex (Eq. (3.28)). This difference in the global symmetries means that there are certain Nambu-Goldstone excitations (and their superpartners) which can propagate only inside the vortex. In the vacuum exterior to the vortex these modes become massive and cannot propagate. They correspond to the various broken $SU(n)_{C+F} \times SU(r)_{C+F}$ generators,

$$
\begin{pmatrix}
0 & \mathbb{b}^\dagger & 0 & 0 \\
\mathbb{b} & 0_{(n-1)\times(n-1)} & 0 & 0 \\
0 & 0 & 0 & e^\dagger \\
0 & 0 & e & 0_{(r-1)\times(r-1)}
\end{pmatrix},
$$

with complex $n-1$ component vector $\mathbb{b}$ and $r-1$ component vector $e$. They are precisely the inhomogeneous coordinates of $CP_{n-1}$ and $CP_{r-1}$, respectively, which are the non-Abelian vortex flux orientation moduli.

In our opinion, the true reason for the exact 2D-4D correspondence is in the consistency of being able to consider the model for the vortex, such as Eq. (3.11) or similar models with $U(n)$ gauge symmetry, as a low-energy approximation of (e.g.) an $SU(N)$ gauge theory, $N > n$. The fact that $\Pi_2(SU(N)) = \mathbb{1}$ means that any regular ’t Hooft-Polyakov monopoles arising from a partial breaking such as

---

5In the strictly low-energy approximation, Eq. (3.11), where small terms arising from the symmetry breaking at high energies are neglected, the vortices are BPS saturated: their moduli space turns out to be considerably larger and shows a richer structure. Here we restrict ourselves to the vortex moduli arising form the global symmetry alone. The latter is an exact symmetry of the system, valid in the full theory, while most of the moduli in the BPS approximation will be absent in the exact theory. As emphasized in [40, 19] the fact that the high-energy monopoles and low-energy vortices are both approximately BPS but not exactly so, is fundamental in the monopole-vortex matching argument.
Figure 3: Vortex carrying non-Abelian flux moduli can convert to a monopole anywhere and at any time.

$SU(N) \rightarrow SU(n) \times SU(r) \times U(1)$ at an intermediate mass scale, must eventually all disappear from the spectrum, confined by the vortices developing at the lower energies, when much smaller squark VEVs are taken into account. Vice versa, no vortex appearing in the low-energy $SU(n) \times SU(r) \times U(1)$ theory in Higgs phase can be there in the underlying $SU(N)$ theory: they are meta-stable and must end at the massive monopoles. Consistency requires that the vortex with each orientation must have its counterpart – a monopole with the corresponding orientation. Symmetry-based vortex moduli space implies an associated monopole moduli space.

Note that the color-flavor diagonal symmetry $U(n) \times U(r)$ is an exact symmetry of the full system (Eq. (3.18)). When the low-energy vortex orientation is rotated in the quotient space $CP^{n-1} \times CP^{r-1}$, a corresponding rotation must be performed on the monopole at the extrema, to keep the energy of the configuration invariant. Although the origin of such fluctuation modes is color-flavor global symmetry, the vortex can end (or originate) anywhere and at any instant of time into (from) a monopole. (Fig. 3). This could be the reason why these fluctuation modes, dynamically broken or not, manifest themselves as a dual local gauge group. The latter is realized however in a confining phase, as the original, electric gauge group is in Higgs phase. The vortex of the electric theory is the confining string of the dual theory.

6Basically, this is not very different from what happens in the two-dimensional Ising model at the critical temperature, although there the kinks in the spin chain manifest themselves as massless unconfined fermionic particles in the dual picture. (See for instance, Kogut [38] for a review.)
5. Matching to the 4D theory

There remains the task of matching the light magnetic degrees of freedom found in the $r$ vacua of the underlying $SU(N)$, $N_f = N$ SQCD, in the $m_i \to 0$, $\mu \sim \Lambda$ limit, see Table 3 to the vortices and their endpoints seen in the low-energy model (in the region $|m_i| \gg |\mu| \gg \Lambda$). The vortex carrying $SU(n) \times SU(r) \times U(1)$ quantum numbers, in which $SU(n)$ Abelianizes dynamically to $U(1)^{n-1}$, so that the monopoles at which these vortices end carry the quantum numbers of $SU(r) \times U(1)$, is an excellent candidate to explain the appearance of the non-Abelian monopoles in the infrared in 4D theory [36, 32]. The fact that both in 4D and in 2D these solitons exist only for $r \leq N_f/2$ is a strong indication that such an identification is indeed correct.

The fact that the monopoles carrying the $SU(r)$ charge appear $N_f$ times and represent the global $SU(N_f)$ symmetry group (see Table 3), is important for the 4D low-energy effective action to possess the correct global symmetry group of the underlying theory [32, 34]. From the semi-classical point of view, this can be understood as due to the Jackiw-Rebbi effect [41, 42]. Note that due to this effect, the dual $SU(r)$ group of the fully quantum mechanical regime, $|m_i|, |\mu| \ll \Lambda$, is infrared free.

In the semi-classical region, $|m_i| \gg |\mu| \gg \Lambda$, where we study the vortices, the Jackiw-Rebbi effect is due to the quark clouds (normalizable fermion zeromodes in the quantization around the background semiclassical monopoles), of the size of $\sim 1/|m| \ll 1/\Lambda$. We claim that these are effects distinct from the color-flavor symmetry breaking effect, which involves a much larger length scale of the order of $1/\sqrt{\mu m}$, and which, we believe, explains the origin of the dual gauge group.

| $SU(r)$ | $U(1)_0$ | $U(1)_1$ | $\ldots$ | $U(1)_{n-1}$ | $U(1)_B$ |
|---------|---------|---------|---------|-------------|----------|
| $n_f \times q$ | $\mathbb{1}$ | 1 | 0 | $\ldots$ | 0 | 0 |
| $e_1$ | $\mathbb{1}$ | 0 | 1 | $\ldots$ | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ |
| $e_{n-1}$ | $\mathbb{1}$ | 0 | 0 | $\ldots$ | 1 | 0 |

Table 3: The effective low-energy degrees of freedom and their quantum numbers at the confining vacuum characterized by a magnetic dual $SU(r)$ gauge group.

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7This is precisely the reason why such monopoles carrying non-Abelian charges can appear as the low-energy degrees of freedom.
The Abelian monopoles seen as kinks in the low-energy vortex theory might be identified with the Abelian monopoles in Table 3. Note that our argument (the monopoles should not be present in the full theory) applies to these monopoles as well. Though these monopoles are stable in 2D theory, with the vortex extending along a fixed (e.g., in z) direction (the first figure in Fig. 4), they are not stable when such a system is embedded in 4D theory: they are confined by a pair of vortices (the second picture of Fig. 4). That each of the vortices on both sides of the kink transforms as \( r \) of \( SU(r)_{C+F} \) group is not in contradiction with the claim that these (kink) monopoles are singlets of \( SU(r)_{C+F} \). Composite vortices transform as in a product representation [17], which in our case is:

\[ r \otimes r^* = 1 \oplus r^2 - 1; \]

it can very well be that the lower-tension double-vortex belongs to the singlet.

It is interesting to consider the case, \( r = N_f/2 \). In four dimensional \( \mathcal{N} = 2 \) SQCD this is a special vacuum, it is a (strongly-coupled) non-trivial superconformal theory. The infrared degrees of freedom include relatively nonlocal monopoles and dyons, and no effective Lagrangian description is available there. Nevertheless, it has been argued [43, 34] that these represented an interesting type of confining theory (with \( \mu \neq 0 \)) in which confinement is induced by the condensation of monopole composites, caused by the strong interactions. The symmetry breaking pattern reflects such a mechanism. In two dimensional vortex effective theory, this particular case deserves indeed further study.

In order to really sew things up, one must answer the following question: do not \( CP^{r-1} \) fluctuations also eventually become strongly coupled, generating still another, hierarchically small, mass scale \( \Lambda' \), and Abelianize? If the \( SU(n) \times SU(r) \times U(1) \) theory were considered in its own right, without referring to a 4D theory, then the answer would be obviously: yes. The new scale at which \( SU(r) \) fluctuations become strongly coupled, \( \Lambda' \), however, would depend on the coupling constants \( g_r \) at the ultraviolet cutoff, which is an arbitrary parameter.

Actually, as our \( SU(n) \times SU(r) \times U(1) \) theory is a low-energy approximation of the underlying 4D \( SU(N) \) theory, the above argument does not hold. We assume that our 2D system corresponds to the quantum \( r \) vacua, with \( r < N_f/2 \). Such an identification is justified, apart from the fact that the condition \( r < N_f/2 \) is needed for

\[ ^8\text{When higher quantum effects are taken into account the two multiplets are expected to split.} \]
for both of them, by the unbroken global symmetry $U(n) \times U(r)$, common to both of the systems. The vortex carrying a quantum $CP^{r-1}$ modulation, being unstable, ends at a monopole before the new scale $\Lambda'$ is generated by the strong $CP^{r-1}$ interactions.

In the 2D – 4D matching, a subtle role is played by the adjoint mass $\mu$. In our vortex study the Fayet-Iliopoulos term of the low-energy model (see Eq. (A.1)) is given by the mass $\xi \sim \sqrt{\mu \Lambda}$ which should be taken much larger than $\Lambda$ to analyse the vortices semi-classically. On the other hand, in the fully quantum regime where 4D theory is analyzed by use of the Seiberg-Witten solutions it is necessary to choose $\mu \ll \Lambda$ so that the dual Higgs phenomenon (for $\tilde{H} = SU(r) \times U(1)^{N-r}$) occurring at the mass scale $\Lambda'' \sim \sqrt{\mu \Lambda}$ can be reliably studied [32] in an effective low-energy action defined at scales lower than $\Lambda$. It is not known whether a more quantitative 2D – 4D matching procedure eventually allows us to identify the two small scales $\Lambda'$ (the scale at which $CP^{r-1}$ becomes strongly coupled in the 2D theory) and $\Lambda''$ (in the 4D theory). The question is rather subtle, as we are really talking about two different kinematical regions, semi-classical ($m_i, \mu \gg \Lambda$) and fully quantum ($m_i, \mu \sim \Lambda$), of the underlying 4D theory.

In any case, both in 2D and 4D theories, the $SU(r)$ group disappears at scales lower than $\Lambda'$ or $\Lambda''$. Of course, the emergence of a non-Abelian dual group concerns the mass scales higher than these scales ($\Lambda'$ in the 2D theory or $\Lambda''$ in the 4D theory). As dual $SU(r)$ gauge interactions correctly describe the monopole interactions at scales higher than $\Lambda''$ in 4D theory, there must be some range of mass scales at which vortex modulation modes in $CP^{r-1}$ survive, at mass scale higher than $\Lambda'$ but much lower than the scale of gauge symmetry breaking, $SU(n) \times SU(r) \times U(1) \rightarrow SU(r) \times U(1)^n$. This is indeed what we have found.

6. Conclusion

In this note we have constructed vortices having non-Abelian moduli, which do not dynamically Abelianize. Semi-classically, they are simply vortices carrying the $SU(n) \times SU(r) \times U(1)$ color-flavor flux. More precisely, they carry the Nambu-Goldstone modes

$$CP^{n-1} \times CP^{r-1},$$

resulting from the partial breaking of the $SU(n) \times SU(r) \times U(1)$ global symmetry to $SU(n-1) \times SU(r-1) \times U(1)^3$ by the vortex. For $n > r$, $CP^{n-1}$ field fluctuations
Figure 4: Our vortex has $CP^{n-1} \times CP^{r-1}$ orientational modes which can fluctuate along the vortex length and in time (top figure). At low energies $CP^{n-1}$ orientational modes fluctuate strongly and Abelianize, leaving the weakly fluctuating $CP^{r-1}$ modes (middle figure). The vortex ends at a monopole which, absorbing the $CP^{r-1}$ fluctuations, turns into a non-Abelian monopole. The latter transforms according to the fundamental representation of the dual $SU(r)$ group (bottom picture). The kink monopoles are Abelian.
propagating along the vortex length become strongly coupled in the infrared, the $SU(n) \times U(1)$ part dynamically Abelianizes; the vortex however still carries weakly-fluctuating $SU(r)$ flux modulations. In our theory where $SU(n) \times SU(r) \times U(1)$ model emerges as the low-energy approximation of an underlying $SU(N)$ theory, such a vortex is not stable. When the vortex ends at a monopole, its $CP^{r-1}$ orientational modes are turned into the dual $SU(r)$ color modulations of the monopole.

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Appendix A: Vortex configurations

To be complete we present here the vortex equations and their solutions of the model Eq. (3.11), in the vacuum Eq. (3.17), Eq. (3.18). The action of our model, after setting $\Phi$ to its VEV (Eq. (3.17)), after making the Ansätze-reduction on the squark fields Eqs. (3.15), (3.16), takes the form,

$$S = \int d^4x \left[ \frac{1}{4g_n^2} (F_{\mu\nu})^2 + \frac{1}{4g_r^2} (F_{\mu\nu}^b)^2 + \frac{1}{4g_0^2} (F_{\mu\nu}^{(0)})^2 + |\nabla_{\mu} q^{(1)}|^2 + |\nabla_{\mu} \tilde{q}^{(2)}|^2 
+ \frac{g_n^2}{2} (q^{(1)} \dagger t^a q^{(1)})^2 + \frac{g_r^2}{2} (\tilde{q}^{(2)} t^b \tilde{q}^{(2)} \dagger)^2 + \frac{g_0^2}{2} (\lambda_1 q^{(1)} \dagger q^{(1)} + \lambda_2 \tilde{q}^{(2)} \tilde{q}^{(2)} \dagger - \xi)^2 \right],$$

(A.1)
\[ \xi = \sqrt{2} \mu m_0 . \] (A.2)

The tension can be written completing the squares à la Bogomolny [39], as:

\[
T = \int d^2x \left( \sum_{a=1}^{n^2-1} \left[ \frac{1}{2g_n} F_{ij}^{(a)} + \frac{g_n}{2} (q^{(1)} \dagger t^a q^{(1)}) \right]^2 + \sum_{b=1}^{r^2-1} \left[ \frac{1}{2g_r} F_{ij}^{(b)} + \frac{g_r}{2} (\bar{q}^{(2)} t^b \bar{q}^{(2)}) \right]^2 - \frac{2}{2g_r} \right) \]

where \( B^{(0)} \equiv \frac{1}{2} \epsilon_{ij} F_{ij}^{(0)} \) is the magnetic flux density along the \( z \) direction. The first-order Bogomolny equations are obtained by setting to zero all square bracket terms in Eq. (A.3), that is, all terms except the last, topological invariant, winding-number term. Their solutions can be elegantly expressed in terms of the moduli matrices \( z \equiv x + iy \)

\[
q^{(1)} = S_n^{-1}(z, \bar{z}) e^{-\lambda_1 \psi(z, \bar{z})} H_0^{(n)}(z); \quad \bar{q}^{(2)} = S_r^{-1}(z, \bar{z}) e^{-\lambda_2 \psi(z, \bar{z})} H_0^{(r)}(z); \quad \] (A.4)

where \( H_0^{(n)}(z) \) and \( H_0^{(r)}(z) \) are \( n \times n \) and \( r \times r \) matrices holomorphic in \( z \), while \( S_n \) (\( S_r \)) is a regular \( SL(n, C) \) (\( SL(r, C) \)) matrix; \( \psi(z, \bar{z}) \) is a complex function, which can be chosen real by an appropriate choice of gauge.

\[
\lambda_1 = \frac{r}{\sqrt{2}nr(n+r)}, \quad \lambda_2 = \frac{n}{\sqrt{2}nr(n+r)} \quad (A.5)
\]

are the \( U(1) \) charges of the \( q^{(1)} \) and \( \bar{q}^{(2)} \) fields, respectively, see Eq. (3.9). \( S_n \) (\( S_r \)) corresponds to the complexified \( SU(n) \) (\( SU(r) \)) transformations. Note that \( H_0^{(n)} \)'s and \( S_n \)'s are defined up to transformations of the form

\[
H_0^{(n)}(z) \rightarrow V_n(z) H_0^{(n)}(z); \quad S_n(z, \bar{z}) \rightarrow V_n(z) S_n(z, \bar{z}),
\]

where \( V_n(z) \) is an arbitrary regular, holomorphic \( n \times n \) (\( vis-\`a-vis \), \( r \times r \) for \( H_0^{(r)}(z) \), \( S_r \)) matrix of determinant one. \( H_0^{(n)}(z) \) and \( H_0^{(r)}(z) \), called moduli matrices, contain all the moduli parameters [15]. \( SU(n) \), \( SU(r) \), \( U(1) \) gauge fields are given by \( \bar{\partial} \equiv \partial/\partial \bar{z} \)

\[
A_1^{(n)} + i A_2^{(n)} = -2i S_n^{-1}(z, \bar{z}) \bar{\partial} S_n(z, \bar{z}); \quad A_1^{(r)} + i A_2^{(r)} = -2i S_r^{-1}(z, \bar{z}) \bar{\partial} S_r(z, \bar{z});
\]

\[
A_1^{(0)} + i A_2^{(0)} = -2i \bar{\partial} \psi . \quad (A.6)
\]
These Ansätze solve the matter part of the Bogomolnyi equations

$$(\mathcal{D}_1 + i\mathcal{D}_2) q^{(1)} = (\mathcal{D}_1 + i\mathcal{D}_2) \bar{q}^{(2)} = 0,$$  \hspace{1cm} (A.7)$$

automatically (they reduce to \(\bar{\partial}H_0 = 0\)). In order to simplify the (linearized) gauge field equations let us introduce

$$\Omega_n = S_n S_n^\dagger, \quad \Omega_r = S_r S_r^\dagger;$$

the (Bogomolnyi) gauge field equations (sometimes called master equations) are\endnote{For instance, the \(SU(n)\) gauge field components can be written from Eq. (A.6) as

\[ A_1 = -i(S^{-1}\bar{\partial}S + S^\dagger\partial(S^\dagger)^{-1}); \quad A_2 = -(S^{-1}\bar{\partial}S - S^\dagger\partial(S^\dagger)^{-1}). \]

By a straightforward algebra one finds then \((F_{12} = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2]):\)

\[ (S^\dagger)^{-1} F_{12} S^\dagger = -2\partial(\Omega^{-1}\partial \Omega), \quad \Omega = S S^\dagger. \]}

\begin{align*}
\partial(\Omega_n^{-1}\partial \Omega_n) &= \frac{g_n^2}{4} e^{-2\lambda_1 n} \left[ \Omega_n^{-1} H_0^{(n)} H_0^{(n)}\dagger - \frac{1}{n} \text{Tr}_n (\Omega_n^{-1} H_0^{(n)} H_0^{(n)}\dagger) 1_{n \times n} \right]; \\
\partial(\Omega_r^{-1}\partial \Omega_r) &= \frac{g_r^2}{4} e^{-2\lambda_2 r} \left[ \Omega_r^{-1} H_0^{(r)} H_0^{(r)}\dagger - \frac{1}{r} \text{Tr}_r (\Omega_r^{-1} H_0^{(r)} H_0^{(r)}\dagger) 1_{r \times r} \right]; \\
\bar{\partial}\bar{\partial} &= \frac{g_0^2}{4} \left[ \lambda_1 e^{-2\lambda_1 n} \text{Tr}_n (\Omega_n^{-1} H_0^{(n)} H_0^{(n)}\dagger) + \lambda_2 e^{-2\lambda_2 r} \text{Tr}_r (\Omega_r^{-1} H_0^{(r)} H_0^{(r)}\dagger) - \xi \right].
\end{align*}

Since \(SU(n), SU(r)\) and \(U(1)\) all commute with each other, the above construction is basically just a straightforward generalization of the formulas given in the case of \(U(n) \sim SU(n) \times U(1)\) theory, see e.g., [1], except for one point. As there is just one \(U(1)\) gauge group factor but two non-Abelian groups \(SU(n)\) and \(SU(r)\), the moduli matrices are subject to a constraint. In fact, from Eq. (A.4) and the fact that \(S_n (S_r)\) belongs to \(SL(n, C) (SL(r, C))\) it follows that

\[ e^{-2\lambda_1 n} \psi \det H_0^{(n)} H_0^{(n)}\dagger = \det(q^{(1)} q^{(1)}\dagger); \]

\[ e^{-2\lambda_2 r} \psi \det H_0^{(r)} H_0^{(r)}\dagger = \det(\bar{q}^{(2)} \bar{q}^{(2)}\dagger). \]

As \(\lambda_1 n = \lambda_2 r\) (see Eq. (A.5)), these are consistent with the asymptotic behavior,

\[ q^{(1)} q^{(1)}\dagger \sim |v_1|^2 1_{n \times n}, \quad \bar{q}^{(2)} \bar{q}^{(2)}\dagger \sim |v_2|^2 1_{r \times r}. \]
if a constraint
\[
\frac{\det H^{(n)}_0 H^{(n)\dagger}_0}{\det H^{(r)}_0 H^{(r)\dagger}_0} \sim \frac{|v_1|^{2n}}{|v_2|^{2r}}
\]  
(A.8)
is satisfied at large $|z|$. So for a vortex of winding number $k$,
\[
\det H^{(n)}_0 H^{(n)\dagger}_0 \propto \det H^{(r)}_0 H^{(r)\dagger}_0 \sim |z|^{2k},
\]
i.e., the same winding in $q$ and $\tilde{q}$ fields, but with the condition, Eq. (A.8).

The tension for the minimum vortex ($k = 1$) can be worked out easily as follows. A typical such vortex has the form, Eq. (A.4), where the moduli matrices can be brought to the form locally, e.g.,
\[
H^{(n)}_0(z) = \begin{pmatrix} c_1 z & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix}; \quad H^{(r)}_0(z) = \begin{pmatrix} c_2 z & 0 \\ 0 & I_{(r-1)\times(r-1)} \end{pmatrix},
\]
with
\[
\frac{c_1}{c_2} = \frac{v_1^n}{v_2^n}. \tag{A.9}
\]
Note that one of $c_1$ and $c_2$, for instance $c_1$, can be set to unity by an appropriate choice of $S_n$ and $\psi$. The other is then fixed uniquely. In order for the behavior (by setting $c_1 = 1$)
\[
H^{(n)}_0(z) H^{(n)\dagger}_0(z) = \begin{pmatrix} \rho^2 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix}
\]
to be consistent with $q^{(1)}q^{(1)\dagger} \sim |v_1|^2 I_{n \times n}$, the large $\rho$ behavior of $\psi$ and $S_n$ must be such that
\[
e^{-2\lambda_1\psi} S^{-1}_n (S^\dagger_n)^{-1} \sim \begin{pmatrix} 1/\rho^2 & 0 \\ 0 & I_{(n-1)\times(n-1)} \end{pmatrix};
\]
and this is possible if
\[
S_n \sim \begin{pmatrix} e^{(n-1)\lambda_1\psi} & 0 \\ 0 & e^{-\lambda_1\psi} I_{(n-1)\times(n-1)} \end{pmatrix}
\]
and
\[
e^{-2n\lambda_1\psi} \sim 1/\rho^2, \quad \therefore \quad \psi \sim \sqrt{\frac{n+r}{2n\: r}} \log \rho^2.
\]
Of course, the same conclusion for $\psi$ is reached by considering the asymptotic behavior of $\tilde{q}^{(2)}$ and $S_r$. As $F^{(0)}_{12} = -4 \bar{\partial} \partial \psi$
\[
T = \xi \int d^2x F^{(0)}_{12} = \xi \int d^2x \nabla^2 \psi = 4\pi \sqrt{\frac{n+r}{2n\: r}} \xi = 4\pi \sqrt{\frac{n+r}{n\: r}} \mu m_0,
\]
24
that is
\[ T = 4\pi (|v(1)|^2 + |v(2)|^2). \]

An (axially symmetric) vortex of generic \( SU(n) \times SU(r) \) orientations can be represented by the moduli matrix of the form,

\[
H^{(n)}_0(z) = \begin{pmatrix} c_1z & 0 \\ \mathbb{b} & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix} ; \quad H^{(r)}_0(z) = \begin{pmatrix} c_2z & 0 \\ \mathbb{c} & \mathbb{1}_{(r-1)\times(r-1)} \end{pmatrix}.
\]

where \( \mathbb{b} (\mathbb{c}) \) is an \( (n-1) \)-component \( (r-1) \)-component complex vector, representing the inhomogeneous coordinates of \( CP^{n-1} \) \( (CP^{r-1}) \). Under the color-flavor \( SU(n) \) \( (SU(r)) \) symmetry group they transform as in the fundamental representation of \( SU(n) \) \( (SU(r)) \). This is the content of some of the claims made in the main text.

The BPS equations actually allow more general kinds of vortex solutions. The moduli space, for general winding numbers and with more general position and orientation parameters, shows a very rich and interesting spectrum. This and other questions will be discussed elsewhere.