VIRTUAL CALCULUS — PART I

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Abstract

An alternative organization for Differential and Integral Calculus, based on an extension of real numbers that include infinitesimal and infinite quantities, is presented. Only Elementary Set Theory is used, without reference to methods or results from Mathematical Logic.

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I. Introduction

The title of this work refers to an alternative organization for Differential and Integral Calculus, based on an extension of the real numbers set, not on the traditional concept of “limit”. We discuss the relevancy of such alternative below.

The notion of limit, traditionally defined by manipulation of logical quantifiers, has the historical merit for having risen the foundations of Calculus to the level of formal rigour required by contemporary mathematics. However, the use of that notion as a base of Calculus can be criticized for many reasons:

(i) The formal definition of limit is “non-intuitive”, i.e., there is a large distance between the rigorous definition and the corresponding heuristic concept. The use of a suggestive notation, which introduces a “notion of time” on the strictly symbolic plane, does not seem to solve this problem satisfactorily.

(ii) The formal demonstrations of results involving the concept of limit are clumsy and artificial, due to the simultaneous use of many logic quantifiers in a same proposition. We know that “technicalities” are sometimes unavoidable in formal mathematics. Nevertheless, the amount present in the manipulation of limits is not reasonable, obscuring even the proof of the most elementary results.

(iii) The notation and operational rules about limits are not comfortable for “calculations”, and they exclude many important notions which are kept undefined from the formal viewpoint. Perhaps the best example is the concept of “order of an infinitesimal” which is essential for comprehension of many important mathematical facts.

The main motivation for this work was to approximate the intuitive ideas of Calculus to its formalizations. Having done so, we expect that definitions and statements become simpler and clearer, without loss of rigour which characterizes contemporary mathematical methods. Moreover, we hope to approximate formal proofs to the intuitive ideas behind the results.

To accomplish that, we will apply the process of “virtual extension” described in Ref. 1 to the real numbers set. We do not intend to show here a complete and detailed formulation of Calculus based on this extension, since it would be too long. Our aim is just to indicate how it can be done, and offer enough elements so that this Calculus alternative organization can be compared to the traditional one.
II. Virtual Numbers

We will call virtual numbers, or simply virtuals, the elements of the virtual extension $\mathbb{R}$ of the set $\mathbb{R}$ of “real numbers”. According to the identification $\mathbb{R} = K(\mathbb{R})$, we will consider $\mathbb{R} \subset \mathbb{R}$, so that every real number is a virtual number. (The reciprocal statement clearly does not hold.)

The real numbers will be represented by low-case Latin letters ($x, y, a, b$...), and generic virtual numbers by low-case Greek letters ($\xi, \alpha, \beta, \lambda, \omega$...). The letter ‘$\pi$’ is an exception, and it will keep its usual mathematical meaning: $\pi \in \mathbb{R}$ is the constant ratio between the circumference and the diameter of a circle. With these conventions, we can omit the “bar” which distinguishes a relation or function from its virtual extension, for its argument indicates which is the case. Since we are considering $\mathbb{R} \subset \mathbb{R}$, we will also write $0 \in \mathbb{R}, 1 \in \mathbb{R}, -3 \in \mathbb{R}$ or $e \in \mathbb{R}$ (Neper’s number), instead of $\overline{0}, \overline{1}, \overline{-3}$ or $\overline{e}$. Furthermore, all syntactic and notational conventions universally accepted for real numbers will be maintained in $\mathbb{R}$. For example:

$$\alpha \beta^3 + \gamma = [\alpha(\beta^3)] + \gamma,$$

$$\sin^2 \xi = (\sin \xi)^2 \quad \text{and} \quad \sin \xi^2 = \sin(\xi^2).$$

The Virtual Extension Theorem (VET, Ref. 1) guarantees the consistence of identifications and notations above, provided that we observe relational symbols which have the underlying logical connectives ‘not’ and ‘or’. In this case, we will maintain the “bar” which distinguishes a relation in $\mathbb{R}$ from its virtual extension in $\mathbb{R}$, to avoid ambiguity. For instance, the expression

$$\alpha \neq \beta$$

means:

$$\alpha = \beta \quad \text{is false},$$

whereas the virtual extension of the relation $\neq \subset \mathbb{R}^2$ is represented by ‘$\overline{\neq}$’. According to item (ii) of the VET, the condition $\alpha \overline{\neq} \beta$ is sufficient but not necessary for the validity of $\alpha \neq \beta$, i.e.:

$$\alpha \overline{\neq} \beta \Rightarrow \alpha \neq \beta.$$

Analogously, the expression

$$\alpha \leq \beta$$
means
\[ \alpha < \beta \text{ or } \alpha = \beta, \]
whereas the virtual extension of the relation \( \leq \subset \mathbb{R}^2 \) is represented by ‘\( \leq^\ast \)’. By item (iv) of the VET, the condition \( \alpha \leq \beta \) is sufficient but not necessary for the validity of \( \alpha \leq^\ast \beta \), i.e.:
\[ \alpha \leq \beta \Rightarrow \alpha \leq^\ast \beta. \]

To avoid ambiguities, we will not write

\[ \alpha \leq^\ast \beta, \]
or any other combination of symbols which at the same time involves the bar that indicates virtual extension and the logical connectives ‘not’ and ‘or’. However, there is no risk of misunderstandings when writing:
\[ \alpha \not< \beta, \]
which simply means
\[ \alpha < \beta \text{ is false,} \]
or while combining many simultaneous relations on the same variable:
\[ \alpha \neq \beta \leq \gamma, \]
which means simply
\[ \alpha \neq \beta \text{ and } \beta \leq \gamma. \]

The multiplicative inverse of a virtual number \( \alpha \in \overline{\mathbb{R}} \) will be indicated by \( \alpha^{-1} \) or
\[ \frac{1}{\alpha}. \]
It is important to note that those symbols are meaningful just when \( \alpha \neq 0 \), since only virtuals satisfying this condition admit multiplicative inverse. Otherwise, the fraction above will be treated exactly like
\[ \frac{1}{0}. \]
By the VET, we have that if \( \alpha > 0 \) or \( \alpha < 0 \) then \( \alpha \neq 0 \), so \( \alpha \) is reversible.
We will introduce now a special notation for some “non-real virtuals”: the class of the sequence $(1, 2, 3, \ldots)$ will be represented by $\infty \in \overline{\mathbb{R}}$, and the multiplicative inverse of this virtual number will be represented by $\partial \in \overline{\mathbb{R}}$, i.e.:

$$\partial = (1, \frac{1}{2}, \frac{1}{3}, \ldots) \in \overline{\mathbb{R}}.$$ 

Besides, for any $\alpha = (a_1, a_2, a_3, \ldots) \in \overline{\mathbb{R}}$, we define:

$$\pm \alpha = (-a_1, +a_2, -a_3, \ldots) \in \overline{\mathbb{R}}$$

and

$$\mp \alpha = (+a_1, -a_2, +a_3, \ldots) \in \overline{\mathbb{R}}.$$

Those definitions are clearly independent of the representative sequence of the class $\alpha \in \overline{\mathbb{R}}$.

We will often use the following ternary relation: a real number $b$ is between two other reals $a$ and $c$ when $a \leq b \leq c$ or $c \leq b \leq a$. This ternary relation defined on $\mathbb{R}$ extends, as any other $n$-ary relation, to the set of virtual numbers $\overline{\mathbb{R}}$. We will say $\beta$ is between $\alpha$ and $\gamma$ when the triple $(\alpha, \beta, \gamma)$ satisfies that extension. Thus, if $\alpha \leq \beta \leq \gamma$ or $\gamma \leq \beta \leq \alpha$ then $\beta$ is between $\alpha$ and $\gamma$, but the reciprocal statement does not hold: zero is between $\pm 1$ and $\mp 1$, despite $\pm 1 \leq 0 \leq \mp 1$ and $\mp 1 \leq 0 \leq \pm 1$ being both false.

The aim of all conventions and notations stated in this section is to make the manipulation of virtual numbers as simple and intuitive as possible. Those conventions, along with the VET, allow us to work with virtual numbers “as if they were real”, without any risk of ambiguities or contradictions. For instance, the following statements in $\overline{\mathbb{R}}$ immediately come from the VET:

$$\lambda > 0 \Rightarrow \mu + \lambda > \mu,$$

$$\xi > 0 \Rightarrow -\xi < 0,$$

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2,$$

and

$$\sin^2 \omega + \cos^2 \omega = 1.$$

Therefore:

$$\infty + 1 > \infty, \quad 2 - \partial < 2, \quad (\infty + \partial)^2 = \infty^2 + 2 + \partial^2,$$
and
\[ \sin^2 \infty + \cos^2 \infty = 1. \]

As another example of application of the VET, we also have:

\[ \sin(\nu \pi) = 0, \text{ for every virtual integer } \nu \in \mathbb{Z}. \]

Hence:

\[ \sin(\infty \pi) = 0, \]

for \( \infty \in \mathbb{Z} \subset \mathbb{R} \).

**III. Absolute Finitude**

We will say that a virtual number is *infinitesimal* when its absolute value is less than each real positive number. In other words, \( \varepsilon \in \mathbb{R} \) is infinitesimal when \( |\varepsilon| < x \), for every real number \( x > 0 \).

As an illustration, the virtuals \( \partial \), \( \pm \partial \) and \( \mp \partial \) are all infinitesimal. We also have that \( \sin \partial \) is infinitesimal, since \( |\sin \partial| < |\partial| \). It is clear that we obtain an equivalent definition substituting “\( |\varepsilon| < x \)” by “\( |\varepsilon| \leq x \)” or “\( |\varepsilon| \leq x \)” in the condition above.

A virtual number will be called *finite* when its absolute value is less than some real number, i.e., \( \lambda \in \mathbb{R} \) is finite if there exists a real \( x \) such that \( |\lambda| < x \).

For instance, the virtual \( \pm 1 \) is finite, since \( |\pm 1| = 1 < 2 \). The virtuals \( \cos \alpha \) and \( \sin \alpha \) are also finite, for every \( \alpha \in \mathbb{R} \), since the VET guarantees that:

\[ |\cos \alpha| < 2 \quad \text{and} \quad |\sin \alpha| < 2. \]

In particular, we have that \( \cos \partial \) and \( \sin \infty \) are both finite.

According to the definitions above, every infinitesimal is finite, but the reciprocal statement does not hold, as the following fact shows:

*The unique infinitesimal real number is the zero, but all real numbers are finite.*

**Proof:** For every non-zero real number \( x \), it holds that:

\[ |x| > \frac{|x|}{2}, \]

hence \( x \neq 0 \) is not infinitesimal.

On the other hand, for any real \( y \), we have:

\[ |y| < |y| + 1, \]
so $y$ is finite. ■

We will say that a virtual is \textit{infinite} when it is not finite, i.e., $\omega \in \overline{\mathbb{R}}$ is infinite when $|\omega| < x$, for every real $x$. Moreover, we will say that $\omega > \mathbb{R}$ when $\omega > x$ for each $x \in \mathbb{R}$; and that $\omega < \mathbb{R}$ when $\omega < x$ for each $x \in \mathbb{R}$.

With those definitions, if $\omega > \mathbb{R}$ or $\omega < \mathbb{R}$ then $\omega$ is infinite, but there exists infinite virtuals which are neither more nor less than $\mathbb{R}$. In addition, if ‘$\infty$’ has been read as “infinity”, then we should clearly distinguish this noun which denotes one particular virtual number from the adjective “infinite”, which merely means “not finite”, and that can be applied to many virtual numbers different from $\infty \in \mathbb{R}$.

Exemplifying: the virtuals $\infty$, $-\infty$, $\pm \infty$ and $\mp \infty$ are all infinite. Besides, we have $\infty > \mathbb{R}$ and $-\infty < \mathbb{R}$, but $\pm \infty \not< \mathbb{R}$ and $\pm \infty \not> \mathbb{R}$.

We present below some basic relations between finitude and the operations of addition and multiplication, whose proofs can be compared to the traditional techniques to formal manipulation of limits.

\textit{The sum of two infinitesimals is also infinitesimal.}
\textit{Proof:} We have $|a + b| \leq |a| + |b|$, for any $a, b \in \mathbb{R}$. Thus the VET guarantees that:

$$|\varepsilon + \delta| \leq |\varepsilon| + |\delta|,$$

for any virtuals $\varepsilon$ and $\delta$. If $\varepsilon$ and $\delta$ are infinitesimals then $|\varepsilon| < x/2$ and $|\delta| < x/2$, for every real $x > 0$. Hence, $|\varepsilon + \delta|$ is less than each positive real $x$. ■

\textit{The product of an infinitesimal and a finite virtual is infinitesimal.}
\textit{Proof:} By the VET, he have $|\varepsilon \lambda| = |\varepsilon| |\lambda|$, for any virtuals $\varepsilon$ and $\lambda$. If $\lambda$ is finite then there exists a positive real $y$ such that $|\lambda| < y$. If $\varepsilon$ is infinitesimal then $|\varepsilon| < x/y$, for each $x > 0$. Therefore $|\varepsilon \lambda|$ is less than every positive real $x$. ■

This result shows that the \textit{inverse of an infinitesimal, if it exists, is necessarily infinite}. Otherwise, we would conclude that $1 \in \mathbb{R}$ is infinitesimal.

\textit{The sum and the product of two finite virtuals are both finite.}
\textit{Proof:} If $\lambda$ and $\mu$ are finite then there exist reals $x$ and $y$ such that $|\lambda| < x$ and $|\mu| < y$. Thus we have:

$$|\lambda + \mu| < x + y \quad \text{and} \quad |\lambda \mu| < xy,$$

hence $\lambda + \mu$ and $\lambda \mu$ are both finite. ■

As a consequence, we have that \textit{the sum of an infinite virtual $\omega$ and a finite virtual $\lambda$ is necessarily infinite}, for if $\omega + \lambda$ were finite then $\omega = (\omega + \lambda) - \lambda$ would also be.
However, it is not true that the product of an infinite virtual and a finite virtual is necessarily infinite, since the finite factor might be infinitesimal. Nevertheless, we have:

**The product of an infinite virtual and a non-zero real is also infinite.**

*Proof:* Let \( \omega \) be an infinite virtual and \( a \) a real number different from zero. If \( a\omega \) were finite then \(|a\omega| < x\), for some real \( x \). In this case, we would have \(|\omega| < x/|a|\), and so \( \omega \) would not be infinite. 

If \( \omega > R \) or \( \omega < R \) then \( \omega \) is inversible, and its inverse is infinitesimal.

*Proof:* If \( \omega > R \) then \( \omega > 0 \), hence \( \omega \) is inversible. Furthermore, we have that \( \omega > (1/x) \), for every positive \( x \), so
\[
0 < \frac{1}{\omega} < x,
\]
for every positive \( x \). The case \( \omega < R \) is analogous.

Many other results involving infinitely big virtuals are easily obtained. For instance:

- If \( \omega > R \) and \( \psi > R \) then \( \omega + \psi > R \) and \( \omega\psi > R \).
- If \( \omega < R \) and \( \psi < R \) then \( \omega + \psi < R \) and \( \omega\psi > R \).
- If \( \omega > R \) and \( \psi < R \) then \( \omega\psi < R \).

**IV. Proximity**

We will now introduce an equivalence relation on the set \( \mathbb{R} \) of virtual numbers: we will say that \( \alpha \) is infinitely close to \( \beta \), or simply that \( \alpha \) is near \( \beta \), when the difference between them is infinitesimal. In addition, we will write \( \alpha \approx \beta \) to indicate that \( \alpha \in \mathbb{R} \) is near \( \beta \in \mathbb{R} \). In other words, \( \alpha \approx \beta \) when \( \alpha - \beta \) is infinitesimal.

It is easily seen that \( \approx \) is a symmetric and reflexive relation. To verify its transitivity, it is enough to note that:
\[
\alpha - \gamma = (\alpha - \beta) + (\beta - \gamma),
\]
hence \( \alpha \approx \beta \) and \( \beta \approx g \) implies \( \alpha \approx g \), since the sum of two infinitesimals is also infinitesimal. So we have that proximity is in fact an equivalence relation on \( \mathbb{R} \).

According to the definition above, infinitesimals are exactly the virtuals infinitely close to zero. The following proposition shows that the proximity relation preserves the finitude attributes:

If \( \alpha \approx \beta \) then:

- (i) \( \alpha \) is infinitesimal if and only if \( \beta \) is infinitesimal;
- (ii) \( \alpha \) is finite if and only if \( \beta \) is finite;

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(iii) $\alpha$ is infinite if and only if $\beta$ is infinite.

Proof: The two first statements follow from $\alpha = (\alpha - \beta) + \beta$, since the sum of two infinitesimals is also infinitesimal, and the sum of two finites is also finite. The third statement is an immediate consequence of the second. ■

Two distinct real numbers cannot be near each other, since the difference between them, being a non-zero real, cannot be infinitesimal. In other words:

$$\text{if } x \approx y \text{ then } x = y.$$  

As a consequence, we have that:

$$\text{if } x \approx \alpha \approx y \text{ then } x = y.$$  

Which means: none of the virtuals can be infinitely close to two distinct reals at the same time.

Not every virtual is near some real, i.e., there exists $\xi \in \mathbb{R}$ such that $\xi \not\approx x$, for each $x \in \mathbb{R}$. For example, $\infty$ and $\pm 1$ are not infinitely close to any real number.

It is also not true that $\omega > \mathbb{R}$ and $\psi > \mathbb{R}$ implies $\omega \approx \psi$. To illustrate, $\infty > \mathbb{R}$ and $\infty^2 > \mathbb{R}$, but $\infty \not\approx \infty^2$, since $\infty^2 - \infty = \infty(\infty - 1) > \mathbb{R}$ is not infinitesimal.

The following proposition corresponds to the “Confront Theorem” of the traditional formalization:

If $\beta$ is between $\alpha$ and $\gamma$ and $\alpha \approx \gamma$ then $\alpha \approx \beta \approx \gamma$.

Proof: In $\mathbb{R}$ we have that if $b$ is between $a$ and $c$, then:

$$|b - a| \leq |c - a|.$$  

Thus, the VET guarantees that if $\beta$ is between $\alpha$ and $\gamma$ then:

$$|\beta - \alpha| \leq |\gamma - \alpha|.$$  

Therefore $\gamma \approx \alpha$ implies $\beta \approx \alpha$. ■

Addition preserves the proximity between virtual numbers:

If $\alpha_1 \approx \alpha_2$ and $\beta_1 \approx \beta_2$ then $\alpha_1 + \beta_1 \approx \alpha_2 + \beta_2$.

Proof: It is enough to note that:

$$(\alpha_1 + \beta_1) - (\alpha_2 + \beta_2) = (\alpha_1 - \alpha_2) + (\beta_1 - \beta_2),$$  

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and remember that the sum of two infinitesimals is also infinitesimal.

In particular, we have that \( \alpha \approx x \) and \( \beta \approx y \) implies \( \alpha + \beta \approx x + y \). This corresponds to the statement “The limit of the sum equals the sum of the limits” from traditional Calculus.

About the relation between multiplication and proximity, we have:

If \( \lambda_1 \approx \lambda_2 \) and \( \mu_1 \approx \mu_2 \), with \( \lambda_1 \) and \( \mu_1 \) finite, then \( \lambda_1 \mu_1 \approx \lambda_2 \mu_2 \).

Proof: For every quadruple of virtuals \( \lambda_1, \mu_1, \lambda_2 \) and \( \mu_2 \), it holds that:

\[
\lambda_1 \mu_1 - \lambda_2 \mu_2 = \lambda_1 \mu_1 - \lambda_1 \mu_2 - \mu_1 \mu_2 + \lambda_1 \mu_2 = \lambda_1 (\mu_1 - \mu_2) + (\lambda_1 - \lambda_2) \mu_2.
\]

Since \( \lambda_1 \) and \( \mu_2 \) are finite, and \((\mu_1 - \mu_2)\) and \((\lambda_1 - \lambda_2)\) are infinitesimals, we conclude that \( \lambda_1 \mu_1 - \lambda_2 \mu_2 \) is also infinitesimal.

A counterexample having one of the factors infinite: let \( \lambda_1 = 1 \), \( \lambda_2 = 1 + \partial \), and \( \mu_1 = \mu_2 = \infty \). In this case:

\[
\lambda_1 \mu_1 - \lambda_2 \mu_2 = \infty - (1 + \partial) \infty = -\partial \infty = -1,
\]

so \( \lambda_1 \mu_1 \not\approx \lambda_2 \mu_2 \).

Particularizing the result above, we obtain that \( \alpha \approx x \) and \( \beta \approx y \) implies \( \alpha \beta \approx xy \). (The limit of the product equals the product of the limits.)

For multiplicative inversibility we have:

If \( \alpha \approx x \not= 0 \) then \( \alpha \) is invertible and \( 1/\alpha \approx 1/x \).

Proof: If \( x > 0 \) then \( x/2 < \alpha < 2x \), and if \( x < 0 \) then \( 2x < \alpha < x/2 \). In both cases \( \alpha \) is invertible with \( 1/\alpha \) finite. Since

\[
\frac{1}{\alpha} - \frac{1}{x} = \frac{1}{\alpha x} (x - \alpha),
\]

we have \( 1/\alpha \approx 1/x \), for \( 1/(\alpha x) = (1/\alpha)(1/x) \) is finite and \( (x - \alpha) \) is infinitesimal.

Therefore \( \alpha \approx x \not= 0 \) and \( \beta \approx y \) implies:

\[
\frac{\beta}{\alpha} \approx \frac{y}{x}.
\]

We present below some examples that can be compared to the corresponding “computations” in the traditional formalization:

\[
\frac{(5 + \partial)^2 - 25}{\partial} = 25 + 10\partial + \partial^2 - 25 = \frac{10\partial + \partial^2}{\partial} = 10 + \partial \approx 10.
\]
\[
\frac{2\infty^3 + 4\infty^2 - 1}{\infty^3 - 5} = \frac{\infty^3 (2 + \frac{4}{\infty} - \frac{1}{\infty^2})}{\infty^3 (1 - \frac{5}{\infty^2})} = \frac{2 + 4\partial - \partial^3}{1 - 5\partial^3} \approx 2.
\]

\[
\infty - \sqrt{\infty^2 + 1} = \left(\infty - \sqrt{\infty^2 + 1}\right) \frac{\infty + \sqrt{\infty^2 + 1}}{\infty + \sqrt{\infty^2 + 1}}
\]

\[
= \frac{\infty^2 - (\infty^2 + 1)}{(\infty + \sqrt{\infty^2 + 1})}
\]

\[
= \frac{-1}{(\infty + \sqrt{\infty^2 + 1})}
\]

\[
\approx 0,
\]

hence \(\infty \approx \sqrt{\infty^2 + 1}\).

V. Continuity

A function \(f: \mathbb{D} \to \mathbb{C}\) with \(\mathbb{D} \subset \mathbb{R}\) and \(\mathbb{C} \subset \mathbb{R}\) is continuous when its virtual extension:

\[
\overline{f}: \overline{\mathbb{D}} \to \overline{\mathbb{C}}
\]

preserves the proximity relation on \(\overline{\mathbb{R}}\). As established in Sec. II, we will omit the bar which distinguishes \(f\) from its virtual extension, in order to simplify notation.

We will say that \(f\) is continuous at \(x \in \mathbb{D}\) when

\[
\forall \alpha \in \overline{\mathbb{D}}, \alpha \approx x \Rightarrow f(\alpha) \approx f(x).
\]

This condition is equivalent to the usual definition of continuity of \(f\) at a point of its domain. We will say that \(f\) is continuous when it is continuous at each \(x \in \mathbb{D}\). This definition is equivalent to the “pointwise” continuity of traditional Calculus.

If \(f\) and \(g\) are real functions defined on the same domain \(\mathbb{D} \subset \mathbb{R}\) and continuous at \(x \in \mathbb{D}\) then \(f + g\) is also continuous at \(x\).

Proof: Since \((f + g)(a) = f(a) + g(a)\), for every \(a \in \mathbb{D}\), The VET guarantees that \((f + g)(\alpha) = f(\alpha) + g(\alpha)\), for every \(\alpha \in \overline{\mathbb{D}}\). Now, it is enough to remember that \(f(\alpha) \approx f(x)\) and \(g(\alpha) \approx g(x)\) implies \(f(\alpha) + g(\alpha) \approx f(x) + g(x)\).

Therefore, the sum of two continuous functions defined on the same domain is also continuous.

Analogously one verifies that:

If \(f\) and \(g\) are real functions defined on the same domain \(\mathbb{D} \subset \mathbb{R}\) and continuous at \(x \in \mathbb{D}\) then \(fg\) is also continuous at \(x\). Besides, if \(g(x) \neq 0\) then \(f/g\) is also continuous at \(x\).
Consequently, the product and the quotient of two continuous functions defined on the same domain are both continuous.

The intuitive argument for the following result and its formal demonstration are identical:

*If two real functions \( f \) and \( g \) are continuous and make a chain then the composite function \((g \circ f)\) is also continuous.*

**Proof:** If \( x \in D \) and \( \alpha \in \overline{D} \) with \( \alpha \approx x \) then \( f(\alpha) \approx f(x) \), for \( f \) is continuous. Since \( g \) is also continuous, we have \( g[f(\alpha)] \approx g[f(x)] \). Therefore the composite \((g \circ f)\) is continuous.

Using virtual numbers, we can state a more restrictive condition of continuity than the one used so far: we say that \( f:D \to \mathbb{C} \) is *uniformly continuous* on \( A \subset D \) when

\[
\forall \alpha, \beta \in \overline{A}, \ \alpha \approx \beta \Rightarrow f(\alpha) \approx f(\beta).
\]

This condition is equivalent to the uniform continuity of traditional Mathematical Analysis. We can demonstrate this fact by the usual methods, keeping in mind that virtual numbers are classes of sequences of real numbers. However, we will present a proof based on the VET, which illustrates how it can substitute those traditional techniques.

*If \( f:D \to \mathbb{C} \) and \( A \subset D \), then the following conditions are equivalent:*  

(i) \( \forall \alpha, \beta \in \overline{A}, \ \alpha \approx \beta \Rightarrow f(\alpha) \approx f(\beta) \).

(ii) \( \forall r > 0, \exists s > 0, \forall a, b \in A, \ |a - b| < s \Rightarrow |f(a) - f(b)| < r \).

**Proof:** If condition (ii) holds, then, by the VET, we have:

\[
\forall r > 0, \exists s > 0, \forall \alpha, \beta \in \overline{A}, \ |\alpha - \beta| < s \Rightarrow |f(\alpha) - f(\beta)| < r.
\]

Since \( \alpha \approx \beta \) implies \( |\alpha - \beta| < s \), for every \( s > 0 \), we conclude that \( \alpha \approx \beta \) implies

\[
\forall r > 0, \ |f(\alpha) - f(\beta)| < r.
\]

That means, \( \alpha \approx \beta \) implies \( f(\alpha) \approx f(\beta) \). Therefore (i) follows from (ii).

On the other hand, if condition (ii) does not hold, then:

\[
\exists r > 0, \forall s > 0, \exists a, b \in A, \ |a - b| < s \text{ and } |f(a) - f(b)| > r.
\]

By the VET, we obtain:

\[
\exists r > 0, \forall \sigma > 0, \exists \alpha, \beta \in \overline{A}, \ |\alpha - \beta| < \sigma \text{ and } |f(\alpha) - f(\beta)| > r.
\]
Now taking \( \sigma \approx 0 \) (or \( \partial \) for example), we get:

\[
\exists r > 0, \exists \alpha, \beta \in \overline{A}, \ \alpha \approx \beta \text{ and } |f(\alpha) - f(\beta)| > r,
\]

i.e., there exist \( \alpha, \beta \in \overline{A} \) with \( \alpha \approx \beta \) and \( f(\alpha) \not\approx f(\beta) \), which is the negation of (i).

We can verify the uniform continuity of a real function directly by the proposed definition, i.e., by condition (i) above. For example, the function cosine is uniformly continuous on \( \mathbb{R} \). To prove this, we note that:

\[
|\cos x - \cos y| \leq |x - y|, \]

for any reals \( x \) and \( y \). Therefore, by the VET, we have

\[
|\cos \xi - \cos \upsilon| \leq |\xi - \upsilon|, \]

for any virtuals \( \xi \) and \( \upsilon \). Hence, \( \xi \approx \upsilon \) implies \( \cos \xi \approx \cos \upsilon \).

The notion of pointwise continuity and that of uniform continuity can both be defined using only the proximity relation between virtual numbers. Many other topological notions depend just on that relation, which motivates the following definition:

A virtual topology on a generic set \( A \) is an equivalence relation \( \approx \) on its virtual extension \( \overline{A} \). We will say that \( \approx \) is separable when there exists at the most one element of \( A \subset \overline{A} \) in each equivalence class, i.e., when \( x \approx y \) with \( x, y \in A \) implies \( x = y \). As we have seen, the proximity between virtual numbers defined in the previous section is a separable virtual topology on \( \mathbb{R} \).

The notions of metric space and topological space are generalizations of the traditional definition of continuity. A virtual topology allows us to introduce topological notions on any set in a simpler, more direct and intuitive way. In addition, we can define uniform continuity using just a virtual topology (as we have seen above), which cannot be done by using just the traditional topological structure.

We define below two familiar topological concepts on any virtual topological space. They will be used further for the particular case of real line.

We will say that \( x \in A \) is an interior point of a set \( B \subset A \) when \( \alpha \approx x \) implies \( \alpha \in B \), and we will say that \( B \subset A \) is an open set when each \( x \in B \) is an interior point of \( B \).
VI. Derivation

To simplify notation, we will now define another relation between virtual numbers: we will say that $\alpha$ and $\beta$ are neighbours when $\alpha \approx \beta$ and $\alpha \neq \beta$. Moreover, we will write $\alpha \sim \beta$ to indicate that $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are neighbours. This neighbourliness relation is symmetric, but it is neither reflexive nor associative, and so it is not an equivalence relation on $\mathbb{R}$. According to this definition, the inversible infinitesimals are exactly the neighbours of zero.

The following proposition illustrates the use of the neighbourliness relation:

*If $\varepsilon \in \mathbb{R}$ is an inversible infinitesimal, i.e., if $\varepsilon \sim 0$, then:*

$$\frac{\sin \varepsilon}{\varepsilon} \approx 1.$$ 

*Proof:* It holds in $\mathbb{R}$ that:

$$0 \neq |x| < \frac{\pi}{2} \Rightarrow \cos x < \frac{\sin x}{x} < 1.$$ 

Therefore, by the VET, we get in $\mathbb{R}$:

$$0 \neq |\xi| < \frac{\pi}{2} \Rightarrow \cos \xi < \frac{\sin \xi}{\xi} < 1.$$ 

Thus we have, for any $\varepsilon \sim 0$:

$$\cos \varepsilon < \frac{\sin \varepsilon}{\varepsilon} < 1,$$

so $(\sin \varepsilon)/\varepsilon$ is between $\cos \varepsilon$ and 1. Since the cosine function is continuous, we conclude that $\cos \varepsilon \approx \cos 0 = 1$, hence:

$$\frac{\sin \varepsilon}{\varepsilon} \approx 1. \quad \blacksquare$$

Let now $f: \mathbb{D} \to \mathbb{C}$ be a function with $\mathbb{D} \subset \mathbb{R}$ and $\mathbb{C} \subset \mathbb{R}$. We will say that $f$ is **derivable at** $x \in \mathbb{D}$ when there exists $m \in \mathbb{R}$ such that

$$\forall \alpha \in \overline{\mathbb{D}}, \; \alpha \sim x \Rightarrow \frac{f(\alpha) - f(x)}{\alpha - x} \approx m.$$ 

That condition is equivalent to the traditional definition of derivability at $x \in \mathbb{D}$, not presuming that $x$ is an interior point of the domain of $f$. We will say simply that $f$ is **derivable** when it is derivable at each point of its domain.
If \( f: D \to C \) is a derivable function then there exists exactly one \( m_x \in \mathbb{R} \) which satisfies the derivability condition for each \( x \in D \). Thus we define the derivative \( f' \) of \( f \) by:

\[
f': D \to \mathbb{R} \\
x \mapsto m_x.
\]

Let \( y \) be the real variable depending on \( x \) by a function \( f \), i.e.:

\[
y = f(x).
\]

If \( x \) is an interior point of the domain of \( f \), we can consider an infinitesimal variation of \( x \) as a virtual number \( dx \sim 0 \). The corresponding variation of \( y \) is:

\[
dy = f(x + dx) - f(x).
\]

So, it is clear that \( f \) is derivable at \( x \) when, for any virtual variation \( dx \sim 0 \), the quotient of \( dy \) and \( dx \) is infinitely close to the same real number. In this case, we have:

\[
\frac{dy}{dx} \approx f'(x),
\]

which can be seen as a formalization of the “manipulations of infinitesimal variations” presented in the classical notation due to Leibniz.

Some examples:

(i) If \( f(x) = k \), for every \( x \in \mathbb{R} \), then:

\[
\frac{dy}{dx} = \frac{f(x + dx) - f(x)}{dx} = \frac{k - k}{dx} = 0,
\]

hence, any constant function is derivable, and its derivative is the constant zero function.

(ii) If \( f \) is the identity function, i.e., if \( y = x \), then:

\[
\frac{dy}{dx} = \frac{(x + dx) - x}{dx} = \frac{dx}{dx} = 1,
\]

so the identity function is derivable and \( f'(x) = 1 \), for every \( x \in \mathbb{R} \).

(iii) Let \( y = \sin x \). By the VET, we have:

\[
\sin \alpha - \sin \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2},
\]
for any virtuals $\alpha$ and $\beta$. Hence:

\[
\frac{dy}{dx} = \frac{\sin(x + dx) - \sin x}{dx} \\
= \frac{2 \sin \frac{dx}{2} \cos \frac{2x + dx}{2}}{dx} \\
= \frac{\sin \frac{dx}{2}}{dx} \cos \left( x + \frac{dx}{2} \right) \\
\approx \cos x.
\]

Therefore the sine function is derivable and its derivative is the cosine function.

We also have the usual derivation rules:

If $f$ and $g$ are real functions derivable at an interior point $x$ of both domains, then $(f + g)$ is also derivable at $x$, and:

\[
(f + g)'(x) = f'(x) + g'(x).
\]

Proof: For every $dx \sim 0$ we have:

\[
\frac{(f + g)(x + dx) - (f + g)(x)}{dx} = \frac{f(x + dx) + g(x + dx) - f(x) - g(x)}{dx} \\
= \frac{f(x + dx) - f(x)}{dx} + \frac{g(x + dx) - g(x)}{dx} \\
\approx f'(x) + g'(x). \quad \blacksquare
\]

If $f$ and $g$ are real functions derivable at an interior point $x$ of both domains, then $(fg)$ is also derivable at $x$, and:

\[
(fg)'(x) = f'(x)g(x) + f(x)g'(x).
\]

Proof: For every $dx \sim 0$ we have:

\[
\frac{(fg)(x + dx) - (fg)(x)}{dx} = \\
= \frac{f(x + dx)g(x + dx) - f(x)g(x + dx) + f(x)g(x + dx) - f(x)g(x)}{dx} \\
= \frac{f(x + dx) - f(x)}{dx} g(x + dx) + \frac{f(x)g(x + dx) - g(x)}{dx} f(x) \\
\approx f'(x)g(x) + f(x)g'(x). \quad \blacksquare
\]
If \( g: D \to \mathbb{R} \) is a real function derivable at an interior point \( x \) of its domain and \( g(x) \neq 0 \) then \( (1/g) \) is derivable at \( x \in D \), and:

\[
\left( \frac{1}{g} \right)'(x) = -\frac{g'(x)}{[g(x)]^2}.
\]

**Proof:** Let \( dx \sim 0 \). We have that \( g(x + dx) \) is inversive, since \( g \) continuous at \( x \) implies \( g(x + dx) \approx g(x) \neq 0 \). Furthermore:

\[
\frac{1}{g(x + dx)} \frac{1}{g(x)} dx = -\frac{g(x + dx) - g(x)}{dx} \frac{1}{g(x + dx)g(x)} \approx -\frac{g'(x)}{[g(x)]^2}.
\]

There is a more restrictive condition of derivability than the one used so far: we will say that \( f: D \to \mathbb{R} \) is differentiable at \( x \in D \) when there exists \( m \in \mathbb{R} \) such that:

\[
\forall \alpha, \beta \in D, \alpha \sim \beta \approx x \Rightarrow \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \approx m.
\]

It is clear that differentiability implies derivability (as defined in this section). The following theorem explains the relation between these two concepts:

Let \( f: D \to \mathbb{R} \) be a derivable function defined on an open set \( D \subset \mathbb{R} \). Its derivative \( f': D \to \mathbb{R} \) is continuous at \( x \in D \) if and only if \( f \) is differentiable at this point \( x \).

To prove this theorem, we will use the following lemma:

If \( f: D \to \mathbb{C} \) is a derivable function on an open set \( D \subset \mathbb{R} \) then, for every \( \alpha \in \overline{D} \), there exists \( \beta \in \overline{D} \) with \( \beta \sim \alpha \), and

\[
\frac{f(\alpha) - f(\beta)}{\alpha - \beta} \approx f'(\alpha).
\]

**Proof:** We will show initially that if \( f \) is derivable on \( D \) then:

\[
\forall a \in D, \forall r > 0, \exists b \neq a, |b - a| < r \text{ and } \left| \frac{f(a) - f(b)}{a - b} - f'(a) \right| < r.
\]

If this statement were false we would have:

\[
\exists a \in D, \exists r > 0, \forall b \neq a, |b - a| < r \Rightarrow \left| \frac{f(a) - f(b)}{a - b} - f'(a) \right| > r.
\]

Thus, we would conclude, by the VET, that:

\[
\exists a \in D, \exists r > 0, \forall \beta \neq a, |\beta - a| < r \Rightarrow \left| \frac{f(a) - f(\beta)}{a - \beta} - f'(a) \right| > r,
\]
and so:

\[ \exists a \in D, \exists r > 0, \forall \beta \sim a, \left| \frac{f(a) - f(\beta)}{a - \beta} - f'(a) \right| > r, \]

which means \( f \) would not be derivable at \( a \in D \).

Applying the VET to the statement proved above we get:

\[ \forall \alpha \in \overline{D}, \forall \rho > 0, \exists \beta \neq \alpha, |\beta - \alpha| < \rho \text{ and } \left| \frac{f(\alpha) - f(\beta)}{\alpha - \beta} - f'(\alpha) \right| < \rho. \]

Now taking \( \rho \) infinitesimal (\( \partial \) for example), we conclude that, for every \( \alpha \in \overline{D} \), there exists \( \beta \sim \alpha \) such that:

\[ \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \approx f'(\alpha). \]

To prove the theorem, let \( f: D \to C \) be a function derivable on an open set \( D \subset R \). If \( f' \) is not continuous at \( x \in D \), then there exists \( \alpha \approx x \) such that \( f'(\alpha) \neq f'(x) \).

Thus, by the lemma above, we conclude that there exists \( \beta \sim \alpha \approx x \) such that

\[ \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \approx f'(\alpha) \neq f'(x), \]

which reads: the function \( f \) is not differentiable at \( x \).

On the other hand, the Mean Value Theorem of traditional Calculus guarantees that for any pair \( a, b \in D \) with \( a \neq b \), there exists \( c \) between \( a \) and \( b \) such that:

\[ \frac{f(a) - f(b)}{a - b} \approx f'(c). \]

Thus, by the VET, we have that for any pair \( \alpha, \beta \in \overline{D} \) with \( a \neq \beta \) there exists \( \gamma \) between \( \alpha \) and \( \beta \) such that:

\[ \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \approx f'(\gamma). \]

If the derivative of \( f \) is a function continuous at \( x \in D \) then, for any pair \( \alpha, \beta \in \overline{D} \) with \( \alpha \sim \beta \approx x \), there exists \( \gamma \approx x \) such that

\[ \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \approx f'(\gamma) \approx f'(x). \]

Therefore \( f \) is differentiable at \( x \).
VII. Relative Finitude

Let $\alpha \in \overline{\mathbb{R}}$ be an inversible virtual, i.e., $\alpha \neq 0$. We will say that $\beta \in \overline{\mathbb{R}}$ is of order $\alpha$ when the quotient $\beta/\alpha$ is finite. In addition, we will represent the set of all virtual numbers of order $\alpha$ by $O(\alpha) \subset \overline{\mathbb{R}}$. That means:

$$O(\alpha) = \left\{ \beta \in \overline{\mathbb{R}} \mid \frac{\beta}{\alpha} \text{ is finite} \right\}.$$ 

For instance, $\sin \varepsilon \in O(\varepsilon)$, for any inversible infinitesimal $\varepsilon \sim 0$.

We will say that $\gamma \in \overline{\mathbb{R}}$ is negligible when compared to $\alpha$, if the quotient $\gamma/\alpha$ is infinitesimal, i.e., if

$$\frac{\gamma}{\alpha} \approx 0.$$ 

We will indicate that by writing:

$$\gamma \ll \alpha.$$ 

Exemplifying: the square of any inversible infinitesimal $\varepsilon \sim 0$ is negligible when compared to $\varepsilon$ itself:

$$\varepsilon \sim 0 \Rightarrow \varepsilon^2 \ll \varepsilon.$$ 

According to those definitions, it is obvious that if $\gamma$ is negligible when compared to $\alpha$, then $\gamma$ is of order $\alpha$:

$$\gamma \ll \alpha \Rightarrow \gamma \in O(\alpha),$$

and that the reciprocal statement does not hold.

Using results about absolute finitude and proximity (like those presented in Secs. III and IV, for instance) we can establish many relations between those concepts and the notions of relative finitude introduced above. As an illustration, it is easily shown that any infinitesimal is negligible when compared to any non-zero real number, and that any finite virtual is negligible when compared to $\infty$.

The following proposition states the Taylor’s Formula of a real function as it is actually used by most scientists. Our notation will be like this:

$$\alpha = \beta + O(\gamma),$$

which merely means:

$$\alpha - \beta \in O(\gamma).$$
If $x$ is an interior point of the domain of a function $f$ which is differentiable $n + 1$ times ($n \in \mathbb{N}$), then for any $\varepsilon \sim 0$ we have:

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + \cdots + \frac{f^{(n)}(x)}{n!}\varepsilon^n + \mathcal{O}(\varepsilon^{n+1}).$$

**Proof:** Let $I$ be an open interval inside the domain of $f$ such that $x \in I$. Taylor’s Formula with Lagrange’s Remainder guarantees that, for each $a \in I$, there exists $b$ between $x$ and $a$ such that:

$$f(a) = f(x) + f'(x)(a - x) + \cdots + \frac{f^{(n)}(x)}{n!}(a - x)^n + \frac{f^{(n+1)}(b)}{(n + 1)!}(a - x)^{n+1}.$$  

Thus, we conclude, by the VET, that for each $\alpha \in I$ there exists $\beta$ between $x$ and $\alpha$ such that:

$$f(\alpha) = f(x) + f'(x)(\alpha - x) + \cdots + \frac{f^{(n)}(x)}{n!}(\alpha - x)^n + \frac{f^{(n+1)}(\beta)}{(n + 1)!}(\alpha - x)^{n+1}.$$  

Now taking $\alpha = x + \varepsilon$, it follows that there exists $\beta \approx x$ such that:

$$f(x + \varepsilon) = f(x) + f'(x)\varepsilon + \cdots + \frac{f^{(n)}(x)}{n!}\varepsilon^n + \frac{f^{(n+1)}(\beta)}{(n + 1)!}\varepsilon^{n+1}.$$  

Since $f^{(n+1)}$ is continuous, we have $f^{(n+1)}(\beta) \approx f^{(n+1)}(x) \in \mathbb{R}$. Hence:

$$\frac{f^{(n+1)}(\beta)}{(n + 1)!}\varepsilon^{n+1} \in \mathcal{O}(\varepsilon^{n+1}).$$

According to the definitions introduced in this section, we can consider the **First Rule of L’Hospital** as a criterion for deciding the relative magnitude between two infinitesimals $\varepsilon$ and $\delta$ of the kind:

$$\varepsilon = f(\alpha) \approx 0 \quad \text{and} \quad \delta = g(\alpha) \sim 0.$$  

Analogously, the **Second Rule of L’Hospital** might be seen as a criterion to match the magnitude of two infinitely big virtuals, i.e., which are more than $\mathbb{R}$ or less than $\mathbb{R}$.

**VIII. Integration**

Our aim in this section is to illustrate the usefulness of relative finitude notions introduced in the last section. We will begin with a definition of integral which uses the
language of virtual numbers instead of the idea of limit. However, it is essentially the
construction due to Riemann.

There is another way to define the integral as a sum of infinitely many infinitesimal
terms, using the concept of “virtual sequence”. We will not go this way here because it
would be premature to introduce that notion now.

Let \( f \) be a function defined on an closed interval with extremes \( a, b \in \mathbb{R} \). We define
the following relation \( P \) between pairs \((m, s)\) of real numbers:

(i) if \( a = b \) then \( P(m, s) \) if and only if \( m > 0 \) and \( s = 0 \);

(ii) if \( a < b \) then \( P(m, s) \) if and only if there exists an extended partition of interval
\([a, b]\):

\[
a = x_0 < z_1 < x_1 < z_2 < x_2 < \cdots < x_n = b,
\]

with norm less then \( m \):

\[
m > x_i - x_{i-1} \quad (i = 1, \ldots, n),
\]

whose Riemann sum is equal to \( s \):

\[
s = \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1});
\]

(iii) if \( b < a \) then \( P(m, s) \) if and only if there exists an extended partition of the
interval \([b, a]\), with norm less than \( m \), whose Riemann sum equals \(-s\).

We will use the virtual extension \( \mathcal{P} \) of the relation \( P \) above to define the integrability
of the function \( f \). According to the convention adopted in Sec. II, we will omit the bar
which distinguishes \( P \) from its virtual extension.

We say that \( f \) is integrable on the interval of extremes \( a, b \in \mathbb{R} \) when there exists
\( s \in \mathbb{R} \) such that \( P(\mu, \sigma) \) and \( \mu \approx 0 \) implies \( \sigma \approx s \). Informally: \( f \) is integrable when
any extended partition infinitely fine has the Riemann sum near the same real number.
Obviously, if \( f \) is integrable then there exists a unique real number \( s \) which satisfies that
condition. This number \( s \) will be called integral of \( f \) between \( a \) and \( b \).

It is not difficult to see that the integrability condition above and the corresponding
definition of integral are equivalent to those due to Riemann. Therefore, any continuous
function is integrable in this sense.

We can think of the integral as a “finite sum of an infinite quantity of infinitesimal
elements”. The notation universally adopted for the integral of \( f \) between \( a \) and \( b \)
suggests this idea:

\[ s = \int_{a}^{b} f(x) \, dx. \]

It is important to note that this notation is founded on the following fact: the infinitesimal elements which we “add” during the integration process can be approximated by the product \( f(x) \, dx \), without altering the final result. This is, essentially, the Fundamental Theorem of Calculus.

To explain this assertion better, let \( f \) be a real continuous function, defined on an open interval, and \( a \) a generic point in this interval. Since \( f \) is integrable, we can define:

\[ s = g(x) = \int_{a}^{x} f(t) \, dt. \]

Thus, for each infinitesimal increment \( dx \) of variable \( x \), we have a corresponding infinitesimal variation of \( s \):

\[ ds = g(x + dx) - g(x). \]

Those are the elements \( ds \) which we “add” during the integration process.

The Fundamental Theorem of Calculus states that \( g \) is derivable and its derivative equals the function \( f \):

\[ g'(x) = f(x). \]

This statement is equivalent to

\[ \frac{ds}{dx} \approx f(x), \]

which can be rewritten as:

\[ \left[ \frac{ds}{dx} - f(x) \right] \approx 0, \]

or:

\[ \left[ \frac{ds - f(x) \, dx}{dx} \right] \approx 0. \]

That means, the Fundamental Theorem of Calculus is equivalent to the following statement:

*If \( f \) is a continuous function, then, for any \( dx \sim 0 \), the error in the approximation: \( ds \approx f(x) \, dx \) is negligible when compared to \( dx \), i.e.:

\[ [ds - f(x) \, dx] \ll dx. \]
This assertion admits a quite intuitive graphic interpretation.

Every time we calculate a finite quantity by integration, we use an approximation for the corresponding infinitesimal element. In order not to change the final result, it is necessary that the approximation error be negligible when compared to the infinitesimal element to which we are approximating. For instance, let us consider the calculation of lengths, areas, and volumes of some simple geometric objects:

If \( f: [a, b] \rightarrow \mathbb{R} \) is a strictly positive derivable function then:

(i) The area of the plan region under the graph of \( f \) is given by:

\[
A = \int da = \int_a^b f(x) \, dx.
\]

(ii) The volume of the solid obtained by the revolution of the region under the graph of \( f \) around the \( x \)-axis is given by:

\[
V = \int dv = \int_a^b \pi [f(x)]^2 \, dx.
\]

(iii) The length of the graph of \( f \) is given by:

\[
L = \int d\ell = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]

(iv) The area of the surface obtained by the revolution of the graph of \( f \) around the \( x \)-axis is given by:

\[
S = \int ds = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx.
\]

The derivative of the function \( f \) is not present in the integrands of the two first formulae above, but it is indispensable in the last two.

In the first formula, we can approximate the element of area \( da \) by \( f(x) \, dx \) because the approximation error is negligible when compared to \( da \) itself:

\[
da = f(x) \, dx + \varepsilon, \quad \text{with} \quad \varepsilon \ll da.
\]

Analogously, in the second formula, if we approximate:

\[
dv \approx \pi [f(x)]^2 \, dx,
\]

we make a negligible error when compared to the volume element \( dv \) itself, which does not alter the final result \( V \).
On the other hand, if we approximate the element of length $d\ell$ simply by $dx$, the error would not be negligible opposed to $d\ell$ itself, and the integral formula thus obtained:

$$L = \int_{a}^{b} dx = b - a$$

would not hold. Taking

$$d\ell \approx \sqrt{1 + [f'(x)]^2} dx,$$

we commit a negligible error when compared to $d\ell$, hence it does not alter the length $L$ obtained from formula (iii).

The same way, approximating the area $ds$ of the surface element merely by

$$2\pi f(x) dx,$$

the error is not negligible opposed to $d\ell$ itself, and the corresponding integral formula:

$$S = 2\pi \int_{a}^{b} f(x) dx$$

is wrong. Formula (iv) is right because the error in the approximation:

$$ds \approx 2\pi f(x) d\ell \approx 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

is negligible when compared to $ds$ itself.

Reference

1 S. F. Cortizo: “Virtual Extensions”, to appear (1995).