YANG-MILLS FOR QUANTUM HEISENBERG MANIFOLDS

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Abstract. We consider the Yang-Mills problem for a quantum Heisenberg manifold, which is a $C^*$-algebra defined by the (strict) deformation quantization of the ordinary Heisenberg manifold, in the setting of non-commutative differential geometry following Connes and Rieffel [Co] [Co1].

1. Preliminaries

Classical Yang-Mills theory is concerned with the set of connections (i.e. gauge potentials) on a vector bundle of a smooth manifold. The Yang-Mills functional $YM$ measures the “strength” of the curvature of a connection. The Yang-Mills problem is determining the nature of the set of connections where $YM$ attains its minimum, or more generally the nature of the set of critical points for $YM$. Since there is a well-developed non-commutative analogue of this setting, we can define a non-commutative Yang-Mills problem as follows [CoRie]: Let $(A,G,\alpha)$ be a $C^*$-dynamical system, where $G$ is a Lie group. It is said that $x \in A$ is $C^\infty$-vector if and only if $g \mapsto \alpha_g(x)$ from $G$ to the normed space is of $C^\infty$. Then $A^\infty = \{ x \in A | x \text{ is of } C^\infty \}$ is norm dense in $A$. In this case we call $A^\infty$ the smooth dense subalgebra of $A$. Since a $C^*$-algebra with a smooth dense subalgebra is an analogue of a smooth manifold, finitely generated projective $A^\infty$-modules are the appropriate generalizations of vector bundles over the manifold. By Connes [Co1], a finitely generated projective $A^\infty$-module $\Xi^\infty$ always exists if there is a finitely generated projective $A$-module $\Xi$ under $G$-invariant maps. In addition, an hermitian structure on $\Xi^\infty$ is given by a $A^\infty$-valued positive definite inner product $\langle \xi,\eta \rangle \in A^\infty$ for $\xi,\eta \in \Xi^\infty$. Let $L$ be the Lie-algebra of (unbounded) derivations of $A^\infty$ given by the representation $\delta$ from the Lie-algebra $\mathfrak{g}$ of $G$ where $\delta_X(x) = \lim_{t \to 0} \frac{1}{t} (\alpha_{\exp(tX)}(x) - x)$ for $X \in \mathfrak{g}$.

Definition 1.1. Given $\Xi^\infty$, a connection on $\Xi^\infty$ is a linear map $\nabla : \Xi^\infty \to \Xi^\infty \otimes L^*$ such that, for all $X \in \mathfrak{g}$, $\xi \in \Xi^\infty$ and $x \in A^\infty$ one has

$$\nabla_X (x \cdot \xi) = x \cdot \nabla_X (\xi) + \delta_X (x) \cdot \xi.$$ 

We shall say that $\nabla$ is compatible with the hermitian metric if and only if

$$\langle \nabla_X (\xi),\eta \rangle + \langle \xi,\nabla_X (\eta) \rangle = \delta_X (\langle \xi,\eta \rangle)$$

for all $\xi,\eta \in \Xi^\infty$, $X \in \mathfrak{g}$.

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Definition 1.2. Let $\nabla$ be a connection on $\Xi^\infty$, the curvature of $\nabla$ is the element $\Theta$ of $\text{End}_{A^\infty}(\Xi^\infty) \otimes \Lambda^2(\mathfrak{g})^*$ given by
$$\Theta_{\nabla}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$
for all $X,Y \in \mathfrak{g}$.

If $\nabla$ is compatible with the Hermitian metric, then the values of $\Theta$ are the self-adjoint elements of $E = \text{End}_{A^\infty}(\Xi^\infty)$. Since $L$ is playing the role of the tangent space of $A^\infty$, the analogue of a Riemannian metric on a manifold will be just an ordinary positive inner product on $L$. With the curvature form in mind, we need the bilinear form on the space of alternating 2-forms with values in $E$. Then given alternating $E$-valued 2-forms $\Phi$ and $\Psi$ we let
$$\{\Phi, \Psi\}_E = \sum_{i<j} \Phi(Z_i, Z_j) \Psi(Z_i, Z_j),$$
which is an element of $E$ where $Z_1, \cdots, Z_n$ is an orthonormal basis of $L$. Finally, we need an analogue of integration over a manifold, and we need this to be $G$-invariant. Thus it is appropriate to assume that $A^\infty$ is given a faithful trace $\tau$ on $A^\infty$ which is invariant under the action of $L$ i.e. $\delta$-invariant so that $\tau(\delta_X(x)) = 0$ for all $X \in \mathfrak{g}$ and $x \in A^\infty$. One can define an $E$-valued inner product, $\langle, \rangle_E$, by
$$\langle \xi, \eta \rangle_E(\zeta) = \xi \cdot <\eta, \zeta>$$
Then every element of $E$ will be a finite linear combination of terms of form $\langle \xi, \eta \rangle_E$ so that we can define $\tau_E$ by
$$\tau_E(\langle \xi, \eta \rangle_E) = \tau(\langle \xi, \eta \rangle_{A^\infty})$$

Definition 1.3. The Yang-Mills functional $YM$ is defined for a compatible connection $\nabla$ by
$$YM(\nabla) = -\tau_E(\{\Theta_{\nabla}, \Theta_{\nabla}\})$$
It is not hard to show that the set of compatible connection $CC(\Xi^\infty)$ is closed under conjugation of a unitary element of $E$. In fact, if we define $(\gamma_u(\nabla))_X \xi = u(\nabla_X(u^*(\xi)))$ for $u \in UE$, it is easily verified that $\gamma_u(\nabla) \in CC(\Xi^\infty)$. Also, it is verified that
$$\Theta_{\gamma_u(\nabla)}(X,Y) = u \Theta_{\nabla}(X,Y) u^*$$
for $X,Y \in \mathfrak{g}$, and that
$$\{\Theta_{\gamma_u(\nabla)}, \Theta_{\gamma_u(\nabla)}\} = u \{\Theta_{\nabla}, \Theta_{\nabla}\} u^*.$$ 
It follows that
$$YM(\gamma_u(\nabla)) = YM(\nabla)$$
for every $u \in UE$ and $\nabla \in CC(\Xi^\infty)$. Thus YM is a well-defined functional on the quotient space $CC(\Xi^\infty)/UE$. If $MC(\Xi^\infty)$ denotes the set of compatible connections where YM attains its minimum, we call $MC(\Xi^\infty)/UE$ the moduli space for $\Xi^\infty$, or more generally the set of critical points}/UE the moduli space.

Connes and Rieffel [CoRie] studied Yang-Mills for the irrational rotation $C^*$-algebras which is non-commutative analogue of 2-tori or non-commutative 2-tori and Rieffel extended $YM$ for the higher dimensional non-commutative tori [Rie]. In view of deformation quantization, the higher dimensional non-commutative torus is the example of deformation quantization of $d$-dimensional torus $\mathbb{T}^d$. A further aspect of this special deformation quantization is that
the ordinary torus acts on the non-commutative tori as a group of symmetries which is a Lie-

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non-commutative Heisenberg manifold can be found in arXiv: 0904.4291.

2. Main results

For any positive integer \(c\) let \(S^c\) be the space of \(C^\infty\) functions \(\phi\) on \(\mathbb{R} \times \mathbb{T} \times \mathbb{Z}\) which satisfy

(a) \(\phi(x+k, y, p) = e^{ckp}\phi(x, y, z)\) for all \(k \in \mathbb{Z}\)

(b) \(\sup_K \|p^m \partial_x^m \partial_y^n \phi(x, y, p)\| < \infty\) for all \(k, m, n \in \mathbb{N}\) and any compact set \(K\) of \(\mathbb{R} \times \mathbb{T}\)

We can give \(S^c\) a \(C^*\)-algebra structure for each \(\hbar \in \mathbb{R}\) as follows;

(c) \(\phi \ast \psi(x, y, p) = \sum q \phi(x - \hbar(q-p)\mu, y - \hbar(q-p)\nu, q)\psi(x - \hbar\mu, y - \hbar\nu, p-q)\)

(d) \(\phi^*(x, y, p) = \overline{\phi(x, y, -p)}\)

with the norm coming from the representation on \(L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})\) defined by

\[
\phi(f)(x, y, p) = \sum_q \phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)f(x, y, p-q)
\]

where \(\mu, \nu\) are non-zero real numbers. \(D^{c,\hbar}_{\mu,\nu}\) will denote the norm completion of \(S^c\). Let \(G\) be the Heisenberg group given by

\[
(r, s, t) \leftrightarrow \begin{pmatrix} 0 & s & t/c \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix}
\]

so that when it is identified with \(\mathbb{R}^3\) the product is given by \((r, s, t)(r', s', t') = (r + r', s + s', t + t' + csr')\). Then we have a canonical action of \(G\) on \(D^{c,\hbar}_{\mu,\nu}\) by

\[
\alpha_{(r,s,t)}(\phi)(x, y, p) = e(p(t + cs(x-r)))\phi(x-r, y-s, p)
\]

which comes from a left action of \(G\) on the Heisenberg manifold and \((D^{c,\hbar}_{\mu,\nu}, G, \alpha)\) is a \(C^*\)-dynamical system [Rie1, p557].

From now on, we only consider the case \(\hbar = 1\) and thus \(D^{c}_{\mu,\nu}\) will denote the corresponding \(C^*\)-algebra named by quantum Heisenberg manifold. Thanks to Abadie, we have a different description of \(D^{c}_{\mu,\nu}\).

Theorem 2.1. [Ab1, p17] \(D^{c}_{\mu,\nu}\) is the closure in the multiplier algebra of \(C_0(\mathbb{R} \times \mathbb{T}) \times_\lambda \mathbb{Z}\) of the \(*\)-subalgebra \(D_0\) consisting of functions \(\phi\) in \(C(\mathbb{R} \times \mathbb{T} \times \mathbb{Z})\) which have compact support on \(\mathbb{Z}\) and satisfy

\[
\phi(x+k, y, p) = e(-ckp(y-p\nu))\phi(x, y, p)
\]

for all \(k, p \in \mathbb{Z}\), and \((x, y) \in \mathbb{R} \times \mathbb{T}\) where \(\lambda(x, y) = (x + 2\mu, y + 2\nu)\)
There is a faithful trace $\tau_D$ on $D^c_{\mu,\nu}$ defined for $\phi \in D^c_{\mu,\nu}$, by

$$
(1) \quad \tau_D(\phi) = \int_{T^2} \phi(x, y, 0)dxdy
$$

The virtue of this alternative definition is the much simpler Morita equivalence picture than the original Morita equivalence found by Rieffel (for the latter, see [Rie2, Theorem 5.5]).

**Theorem 2.2.** [Ab1, Theorem 2.12] Let $E^c_{\mu,\nu}$ be the closure in the multiplier algebra of $C(\mathbb{R} \times T) \times_\sigma \mathbb{Z}$ of the $*$-subalgebra $E_0$ consisting of functions $\psi$ in $C(\mathbb{R} \times T \times \mathbb{Z})$ with compact support on $\mathbb{Z}$ and satisfy

$$
\psi(x - 2p\mu, y - 2p\nu, k) = e(ckp(y - p\nu))\psi(x, y, k)
$$

for all $k, p \in \mathbb{Z}$, and $(x, y) \in \mathbb{R} \times T$ where $\sigma(x, y) = (x - 1, y)$. Then $E^c_{\mu,\nu}$ and $D^c_{\mu,\nu}$ are strong-Morita equivalent via the bimodule $\Xi = C(\mathbb{R} \times T)$.

**Corollary 2.3.** $\Xi$ is a finitely generated $D^c_{\mu,\nu}$-module and $\text{End}_{D^c_{\mu,\nu}}(\Xi)$ is isomorphic to $E^c_{\mu,\nu}$ via the map $f \to f \cdot \psi$.

**Proof.** Note that both $E^c_{\mu,\nu}$ and $D^c_{\mu,\nu}$ have identity elements [Ab2, p309]. From the strong Morita equivalence, it is well known that $\Xi$ is a finitely generated and the endomorphism ring $\text{End}_{D^c_{\mu,\nu}}(\Xi)$ is equal to $E^c_{\mu,\nu}$ (see [Rie2, Proposition 2.1]). □

For our purpose, we need to recall the action of $D^c_{\mu,\nu}$ on $\Xi$ by

$$
(\phi \cdot f)(x, y) = \sum_p \phi(x, y, p)f(x - 2p\mu, y - 2p\nu) = \sum_p \phi(x, y, p)(\lambda_{2p}f)(x, y)
$$

for $\phi \in D^c_{\mu,\nu}$ and $f \in \Xi$ and $D^c_{\mu,\nu}$-valued inner product by

$$
< f, g >_{D^c_{\mu,\nu}}(x, y, p) = \sum_k e(ckp(y - p\nu))f(x + k, y)g(x - 2p\mu + k, y - 2p\nu)
$$

for $f, g \in \Xi$.

Accordingly, $G$ act on $D^c_{\mu,\nu}$ by

$$
(2) \quad \alpha_{(r, s, t)}(\phi)(x, y, p) = e(p(t + cs(x - p\mu - r)))\phi(x - r, y - s, p)
$$

for $\phi \in D^c_{\mu,\nu}$ and $(D^c_{\mu,\nu}, G, \alpha)$ becomes a $C^*$-dynamical system. Also, we can check that $\tau$ is $\delta$-invariant using (1). Since we never work with $D^c_{\mu,\nu}$ and $\Xi$, but only with $C^\infty$ versions, we shall denote the latter by $D^c_{\mu,\nu}$ and $\Xi$. It is easy to see that the Lie algebra of the (parametrized) Heisenberg group has an orthonormal basis consisting of

$$
X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1/c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Then we have the derivations corresponding to this basis.
Proposition 2.4. The representation $\delta$ of $L$ as a Lie algebra of derivations on $D_{\mu \nu}^c$ is given by

$$
\begin{align*}
\delta_X(\phi)(x, y, p) &= -\frac{\partial}{\partial x}\phi(x, y, p), \\
\delta_Y(\phi)(x, y, p) &= -\frac{\partial}{\partial y}\phi(x, y, p) + 2\pi ip(x - p\mu)\phi(x, y, p), \\
\delta_Z(\phi)(x, y, p) &= 2\pi ip\phi(x, y, p)
\end{align*}
$$

Proof. In our case, $\exp(A) = I + \sum_{n=1}A^n/n!$. Using the action of $G$ defined as (2), we can compute $\delta_X(\phi) = \lim_{t \to 0} \frac{1}{t}(\alpha_{\exp(tX)}(\phi) - \phi)$ for each $X, Y, Z \in \mathfrak{g}$ and $\phi \in D_{\mu \nu}^c$.

Thus the smooth dense subalgebra of $D_{\mu \nu}^c$ is the Schwartz space of complex valued functions on the $\mathbb{R} \times \mathbb{T} \times \mathbb{Z}$ satisfying (2.1) and the corresponding finitely generated module is the Schwartz space of complex valued functions on the $\mathbb{R} \times \mathbb{T}$. If we let $\{Z_i\}$ be the basis of a Lie algebra $\mathfrak{g}$, a linear map $\hat{\nabla}^*$ which takes $E_i$-valued 2 forms to 1-forms is defined by

$$(\hat{\nabla}^*\Omega)(Z_i) = \sum_j [\nabla_{Z_j}, \Omega(Z_i \wedge Z_j)] - \sum_{j < k} c_{j,k}^i \Omega(Z_j \wedge Z_k)$$

where $c_{j,k}^i$ are the structure constants of $\mathfrak{g}$ for the basis $Z_j$.

Theorem 2.5. [Rie3, Theorem 1.1] A compatible connection $\nabla$ is a critical point of YM exactly when it satisfies the Yang-Mills equation $\hat{\nabla}^*(\Theta_\nabla) = 0$

Now we define a linear map $\nabla$ on $\Xi$ by

$$(\nabla_X f)(x, y) = -\frac{\partial}{\partial x}f(x, y), \quad (\nabla_Y f)(x, y) = -\frac{\partial}{\partial y}f(x, y) + \frac{\pi ci}{2\mu}x^2f(x, y), \quad (\nabla_Z f)(x, y) = \frac{\pi ix}{\mu}f(x, y)$$

Proposition 2.6. $\nabla$ is a compatible connection.

Proof. We need to verify that $\nabla$ satisfy the Definition [1,1]. It is enough to show that

$$\nabla_W(\phi \cdot f) = \phi \cdot \nabla_W(f) + \delta_W(\phi) \cdot f$$

for $W = X, Y, Z$.

The first case is obvious.

We note that

$$(\phi \cdot (\nabla_Y f))(x, y) = \sum_p \phi(x, y, p)\frac{\partial}{\partial y}(\lambda_{2p}f)(x, y) + \sum_p \phi(x, y, p)\frac{\pi ci}{2\mu}(x - 2p\mu)^2f(x - 2p\mu, y - 2p\nu),$$

$$(\delta_Y(\phi) \cdot f)(x, y) = \sum_p \frac{\partial}{\partial y}\phi(x, y, p)(\lambda_{2p}f)(x, y) + \sum_p 2\pi i cp(x - p\mu)\phi(x, y, p)f(x - 2p\mu, y - 2p\nu).$$

Thus $\phi \cdot (\nabla_Y f) + \delta_Y(\phi) \cdot f = \frac{\partial}{\partial y}(\phi \cdot f) + \frac{\pi ci}{2\mu}x^2(\phi \cdot f) = \nabla_Y(\phi \cdot f)$.

Similarly,

$$(\phi \cdot (\nabla_Z f))(x, y) = \sum_p \phi(x, y, p)\frac{\pi i(x - 2p\mu)}{\mu}(\lambda_{2p}f)(x, y),$$

and

The proof is completed.
\[(\delta_Z(\phi) \cdot f)(x, y) = \sum_p 2\pi i p \phi(x, y, p)(\lambda_{2p} f)(x, y)\]

Thus
\[
(\phi \cdot (\nabla_Z f) + \delta_Z(\phi) \cdot f)(x, y) = \sum_p \left( \frac{\pi i (x - 2p\mu)}{\mu} + 2\pi i p \right) \phi(x, y, p)(\lambda_{2p} f)(x, y)
\]
\[
= \frac{\pi i x}{\mu} \sum_p \phi(x, y, p)(\lambda_{2p} f)(x, y)
\]
\[
= (\nabla_Z(\phi \cdot f))(x, y)
\]

We also need to check the compatibility. By the product rule of differentiation, it follows that

\[\delta_X(<f, g>_D_{\mu\nu}) = <\nabla_X f, g>_D_{\mu\nu} + <f, \nabla_X g>_D_{\mu\nu}\]

Now
\[\delta_X(<f, g>)(x, y, p) = \frac{\partial}{\partial y}(<f, g>_D_{\mu\nu})(x, y, p) + 2\pi i cp(x - p\mu)(<f, g>_D_{\mu\nu})(x, y, p)\]
\[= \sum_k ((2\pi i cp) + 2\pi i c(x - 2p\mu)) e(cp(y - p\nu)(\sigma_k f)(x, y)(\overline{\lambda_{2p} g})(\sigma_k(x, y))\]
\[+ \sum_k e(cp(y - p\nu))\left(\frac{\partial}{\partial y}(\lambda_{2p} g)(\sigma_k(x, y))\right)\]
\[+ \sum_k e(cp(y - p\nu))\left(\sigma_k f)(x, y)\frac{\partial}{\partial y}(\lambda_{2p} g)(\sigma_k(x, y))\right)\]

Note that for each \(k\),
\[
\frac{\pi ci}{2\mu} (x + k)^2 + \frac{\pi ci}{2\mu} (x - 2p\mu + k)^2 = \frac{\pi ci}{2\mu}((x + k)^2 - (x + k - 2p\mu)^2)
\]
\[= \frac{\pi ci}{2\mu}(2p\mu)(2x - 2p\mu + 2k)
\]
\[= 2\pi i cp(x - p\mu) + 2\pi i cpk\]

Thus
\[<\nabla_Y f, g>(x, y, p) + <f, \nabla_Y g>(x, y, p) = \sum_k e(cp(y - p\nu))\frac{\partial}{\partial y}(\sigma_k f)(x, y)(\overline{\lambda_{2p} g})(\sigma_k(x, y))\]
\[+ \sum_k \frac{\pi ci}{2\mu}(x + k)^2 e(cp(y - p\nu))(\sigma_k f)(x, y)(\overline{\lambda_{2p} g})(\sigma_k(x, y)) + \sum_k \frac{\pi ci}{2\mu}(x - 2p\mu + k)^2(\sigma_k f)(x, y)(\overline{\lambda_{2p} g})(\sigma_k(x, y))\]
\[+ \sum_k e(cp(y - p\nu))(\sigma_k f)(x, y)\frac{\partial}{\partial y}(\lambda_{2p} g)(\sigma_k(x, y))\]
\[+ \sum_k e(cp(y - p\nu))(\sigma_k f)(x, y)\frac{\partial}{\partial y}(\lambda_{2p} g)(\sigma_k(x, y))\]
\[= \sum_k (2\pi i cp(x - p\mu) + 2\pi i cpk)e(cp(y - p\nu))(\sigma_k f)(x, y)(\overline{\lambda_{2p} g})(\sigma_k(x, y)) + \sum_k e(cp(y - p\nu))(\sigma_k f)(x, y)(\overline{\lambda_{2p} g})(\sigma_k(x, y))\]
\[= \delta_Y(<f, g>).\]
Finally,
\[
\langle \nabla_Z(f), g \rangle (x, y, p) + \langle f, \nabla_Z(g) \rangle (x, y, p) = \sum_k e(ckp(y - p\mu)) \frac{\pi i(x + k)}{\mu}(\sigma_k f)(x, y)
\]
\[
\overline{\lambda_{2g}}(\sigma_k(x, y)) + \sum_k e(ckp(y - p\mu))\frac{\pi i(x - 2p\mu + k)}{\mu}(\lambda_{2g})(\sigma_k(x, y))
\]
\[
= \sum_k \frac{\pi i(x + k)}{\mu} + \frac{\pi i(x - 2p\mu + k)}{\mu}e(ckp(y - p\mu))\frac{\pi i(x - 2p\mu + k)}{\mu}(\lambda_{2g})(\sigma_k(x, y))
\]
\[
= \sum_k 2\pi ip e(ckp(y - p\mu))(\sigma_k f)(x, y)\frac{\pi i(x - 2p\mu + k)}{\mu}(\lambda_{2g})(\sigma_k(x, y)) = \delta_Z(\langle f, g \rangle)(x, y, p).
\]

**Theorem 2.7.** The connection \( \nabla : \Xi^\infty \to \Xi^\infty \otimes L^* \) satisfying (3) is a critical point of Yang-Mills equation for the quantum Heisenberg manifold \( D_{\mu\nu}^c \).

**Proof.** First, we note that \([X, Y] = -cZ, [Y, Z] = 0, [Z, X] = 0\). Therefore, \( C_{12}^3 = -c \) and all other structure constants are zero. We must show that the Yang-Mills equation \( (\nabla^*(\Theta_{\nabla})) = 0 \) for our choice \( \nabla \). Thus it is enough to show that
\[
(\nabla^*(\Theta_{\nabla}))(Z_i) = \sum_j [\nabla_{Z_j}, \Theta_{\nabla}(Z_i \wedge Z_j)] - \sum_{j < k} c_{jk}^i \Theta_{\nabla}(Z_j \wedge Z_k) = 0
\]
for each \( i \).

Let \( Z_1 = Z, Z_2 = Y, Z_3 = Z \). We can easily see that \( \Theta_{\nabla}(X, Y) = (-\frac{\pi ci}{\mu} + \frac{\pi ci}{\mu})I_E = 0 \), \( \Theta_{\nabla}(Y, Z) = [\nabla_Y, \nabla_Z] = 0 \), \( \Theta_{\nabla}(Z, X) = \frac{\pi i}{\mu} I_E \) where \( I_E \) is the identity element of \( E = \text{End}_{D_{\mu\nu}^c}(\Xi) \).

Thus
\[
(\nabla^*\Theta_{\nabla})(X) = [\nabla_Y, \Theta_{\nabla}(X, Y)] + [\nabla_Z, \Theta_{\nabla}(X, Z)] = 0
\]
\[
(\nabla^*\Theta_{\nabla})(Y) = [\nabla_X, \Theta_{\nabla}(Y, X)] = 0
\]
\[
(\nabla^*\Theta_{\nabla})(Z) = [\nabla_X, \Theta_{\nabla}(Z, X)] - c_{12}^3 \Theta_{\nabla}(X, Y) = 0
\]

\( \square \)

**Corollary 2.8.** All other critical points of Yang-Mills equation are of the form \( \nabla + \gamma \) where \( \gamma(Z_i) \in E_{\mu\nu}^c \) such that \( \gamma(Z_i) \) is imaginary valued function for each \( Z_i \). In addition, \( (\nabla_X + \gamma_X)(f) = \nabla_X(f) + f \cdot \gamma(X) \) where the latter is defined by the action of \( E_{\mu\nu}^c \) on \( \Xi \).

**Proof.** Suppose \( \nabla' \) be another critical point. Then \( \nabla'_X - \nabla_X \) is a skew-adjoint element of \( E = \text{End}_{D_{\mu\nu}^c}(\Xi) \) for each \( X \in g \). If let \( \gamma(X) = \nabla'_X - \nabla_X, \nabla'_X = \nabla_X + \gamma(X) \) and \( \gamma(X) \) is pure-imaginary valued by Corollary 2.3.

\( \square \)

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