CONFINEMENT

Yu.A.Simonov
Institute of Theoretical and Experimental Physics
117259, Moscow, B.Cheremushkinskaya 25, Russia

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Abstract
Numerous aspects and mechanisms of color confinement in QCD are surveyed. After a gauge–invariant definition of order parameters, the phenomenon is formulated in the language of field correlators, to select a particular correlator responsible for confinement. In terms of effective Lagrangians confinement is viewed upon as a dual Meissner effect and a quantitative correspondence is established via the popular abelian projection method, which is explained in detail. To determine the field configurations possibly responsible for confinement, the search is made among the classical solutions, and selection criterium is introduced. Finally all facets of confinement are illustrated by a simple example of string formation for a quark moving in the field of heavy antiquark.

1. Introduction.
2. Definition of confinement and order parameters.
3. Field correlators and confinement.
4. Confinement and superconductivity. Dual Meissner effect.
5. Abelian projection method.
6. Search for classical solutions. Monopoles, multiinstantons and dyons.
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1 Introduction

Ten years ago the author has delivered lectures on confinement at the XX LINP Winter School [1]. Different models of confinement were considered, and the idea was stressed that behind the phenomenon of confinement is a disorder or stochasticity of vacuum at large distances.

This phenomenon becomes especially beautiful when one discovers that the disorder is due to topologically nontrivial configurations [2], and the latter could be quantum or classical. In that case one should look for appropriate classical solutions: instantons, multistatons, dyons etc.

The tedious work of many theorists during last ten years has given an additional support for the stochastic mechanism, and has thrown away some models popular in the past, e.g. the dielectric vacuum model and the $Z_2$ flux model. Those are reported in [1] (see also refs. cited in [1]). At the same time a new and deeper understanding of confinement has grown.

It was found that the stochastic mechanism reveals itself on the phenomenological level as a dual Meissner effect, which was suggested as a confinement mechanism 20 years ago [3]. During last years there appeared a lot of lattice data in favour of the dual Meissner effect and the so-called method of abelian projection was suggested [4], which helps to quantify the analogy between the QCD string and the Abrikosov-Nilsen-Olesen (ANO) string.

At the same time the explicit form of confining configuration – whether it is some classical solutions or quantum fluctuations – is still unclear, some candidates are being considered and active work is going on.

The present review is based on numerous data obtained from lattice calculations, phenomenology of strong interaction and theoretical studies.

The structure of the review is the following. After formulation of criteria for confining mechanisms in the next chapter, we describe in Section 3 the general method of vacuum correlators (MVC) [5] to characterize confinement as a property of field correlators. A detailed comparison of superconductivity and confinement in the language of MVC and in the effective Lagrangian formalism of Ginzburg–Landau type is given in Section 4.

The Abelian Projection (AP) method is introduced in Section 5 and lattice data are used to establish the similarity of the QCD vacuum and the dual superconductor medium. At the same time the AP method reveals the topological properties of confining configurations.

In Section 6 the most probable candidates for such configurations are
looked for among classical solutions. A new principle is introduced to select solutions which are able to confine when a dilute gas is formed of those.

An interconnection of topology and stochastic properties of the QCD vacuum is discussed in Section 7.

In conclusion the search for confining configurations is recapitulated and a simple picture of the confining vacuum and string formation is given as seen from different points of view. A possible temperature deconfinement scenario is shortly discussed.

2 What is confinement?

By confinement it is understood the phenomenon of absence in physical spectrum those particles (fields) which are present in the fundamental Lagrangian. In the case of QCD it means that quarks and gluons and in general all colored objects cannot exist as separate asymptotic objects.

To be more exact let us note, that sometimes the massless particles entering with zero bare masses in the Lagrangian, e.g. photons or gluons, can effectively acquire mass – this is the phenomenon of screening in plasma or in QCD vacuum above the deconfinement temperature. Due to the definition of confinement given above, the screened asymptotic color states cannot exist in the confining phase.

On the other hand in the deconfined phase quarks and gluons can evolve in the Euclidean (or Minkowskian) time separately, not connected by strings. From this fact it follows that the free energy for large temperature $t$ is proportional to $t^4$ as it usual for the Stefan–Boltzmann gas, and it is natural to call this phase the quark–gluon plasma (QGP). It turns out, however, that the dynamics of QGP is very far from the ideal gas of quarks and gluons, and a new characteristic interaction appears there, called the magnetic confinement – this will be discussed in more detail in conclusions, but now we come back to the confinement phase.

It is very important to discuss confinement in gauge–invariant terms. Then the physical contents of the confinement phenomenon can be best understood comparing a gauge–invariant system of an electron and positron $(+e, -e)$ in QED and a system of quark antiquark $(q\bar{q})$ in QCD when no other charges are present. When distance between $+e$ and $-e$ is large the electromagnetic interaction becomes negligible and the wave function (w.f.) of $(+e, -e)$
factorizes into a product of individual w.f. (the same is true for a gauge-invariant Green’s function of \((+e, -e)\). Therefore the notion of isolated e.m. charge and its individual dynamics makes sense. In contrast to that in the modern picture of confinement in agreement with experiment and lattice data (see below), quark and antiquark attract each other with the force of approximately 14 ton. Therefore \(q\) and \(\bar{q}\) cannot separate and individual dynamics, individual w.f. and Green’s function of a quark (antiquark) has in principle no sense: quark (or antiquark) is confined by its partner. This statement can be generalized to include all nonzero color changes, e.g. two gluons are confined and never escape each other (later we shall discuss what happens when additional color charges come into play).

It is also clear now that the phenomenon of confinement is connected to the formation of a string between color charges – the string gives a constant force at large distances; and the dynamics of color charges there without the string is inadequate.

This picture is nicely illustrated by many lattice measurements of potential between static quarks, Fig. 1.

One can see in Fig. 1 clearly the linear growth of potential \(V(r) = \sigma r\) at large distance \(r; r \geq 0.25\) fm.

We shall formulate this as a first property of confinement.

I. The linear interaction between colored objects.

To give an exact meaning to this statement, making it possible to check in lattice calculations, it is convenient to introduce the so-called Wilson loop [6], and to define through it the potential between color changes. This is done as follows. Take a heavy quark \(Q\) and a heavy antiquark \(\bar{Q}\) and consider a process, where the pair \(QQ\) is created at some point \(x\), then the pair separates at distance \(r\), and after a period of time \(T\) the pair annihilates at the point \(y\).

It is clear that trajectories of \(Q\) and \(\bar{Q}\) form a loop \(C\) (starting at the point \(x\) and passing through the point \(y\) and finishing at the point \(x\)). The amplitude (the Green’s function) of such a process according to Quantum Mechanics, is proportional to the phase (Schwinger) factor \(W \sim \exp ig \int A_\mu j_\mu d^4x\), where \(A_\mu\) is the total color vector potential of quarks and vacuum fields, and \(j_\mu\) – the current of the pair \(QQ\), depicting its motion along the loop \(C\). Taking into account that \(A_\mu\) is a matrix in color space \((A_\mu = A_\mu^a T^a, \ a = 1, \ldots N_c^2 - 1, \ and \ T^a = \frac{1}{2} \sigma^a \ for \ SU(2) \ and \ T^a = \frac{1}{2} \lambda^a \ for \ SU(3), \ where \ N_c \ number \ of \ colors \ \sigma^a, \ \lambda^a \ – \ are \ Pauli \ and \ Gell–Mann \ matrices \ respectively)\) one should insert an ordering operator \(P\) which orders these matrices along the loop, and the
trace operator in color indices $tr$, since quarks were created and annihilated in a white state of the object $Q\bar{Q}$. Hence one obtains the Wilson operator for a given field distribution $W(C) = tr P \exp i g \int_C A_\mu dx_\mu$ where we have used the point-like structure of quarks, $j_\mu(x) \sim \delta^{(3)}(x - x(t)) \frac{dx_\mu}{dt}$.

Now one must take into account, that the vacuum fields $A_\mu(x)$ form a stochastic ensemble, and one should average over it. This is a necessary consequence of the vector character of $A_\mu(x)$ and of the Lorentz–invariance of the vacuum, otherwise for any fixed function $A_\mu(x)$ this invariance would be violated contrary to the experiments.

Finally, in field theory in general and dealing with field vacuum in particular it is convenient to use the Euclidean space–time for the path–integral representation of partition functions, and also for the Monte-Carlo calculations on the lattice. There are at least two reasons for that: a technical reason; Euclidean path integrals have a real (and positive–definite) measure, $exp(-S_E)$, where $S_E$ is the Euclidean action, and the convergence of the path integrals is better founded (but not strictly proved in the continuum).

Another, and possibly a deeper reason: Till now all nontrivial classical solutions in QCD (instantons, dyons etc.) have Euclidean nature, i.e. in the usual Minkowskian spacetime they describe some tunneling processes. As for the covalent bonds between atoms due to the electron tunneling from one atom to another, also the Euclidian configurations in ghudynamics and QCD yield attraction, i.e. the lowering of the vacuum energy, and therefore are advantageous for the nonperturbative vacuum reconstruction (cf discussion in Conclusion). Therefore everywhere in the review we shall use the Euclidian space–time, i.e. we go over from $x_0, A_0$ to the real Euclidean components $x_4 = ix_0, A_4 = iA_0$.

Now we come back to the Wilson loop and relate it to the potential. To this end we recall that $W$ is an amplitude, or the Green’s function of the $Q\bar{Q}$ system, and therefore it can be expressed through the Hamiltonian $H$, namely $W = \exp(-HT)$, where $T$ is the Euclidean time. For heavy quarks the kinetic part of $H$ vanishes and only the $Q\bar{Q}$ potential $V(r)$ is left, so that we finally obtain for the averaged (over all vacuum field configurations denoted by angular brackets) Wilson loop

$$< W(C) >= < tr P \exp i g \int_C A_\mu dx_\mu >= \exp(-V(r)T) \quad (1)$$

where the loop $C$ can be conveniently chosen as a rectangular $r \times T$, and
confinement corresponds to the linear dependence of the potential $V(r)$, $V(r) = \sigma r$, where $\sigma$ is called the string tension.

Having said this, one should add necessary elaborations. First, what happens when other charges are present, e.g. $q\bar{q}$ pairs are created from the vacuum. In both cases screening occurs, but whereas for $e\bar{e}$ system nothing crucial happens at large distances, the $q\bar{q}$ system can be split by an additional $q_1\bar{q}_1$ pair into two neutral systems of $q\bar{q}_1$ and $q_1\bar{q}$ which can now separate – the string breaks into two pieces.

The same is always true for a pair of gluons which can be screened by gluon pairs from the vacuum.

In practice (i.e. in lattice computations) the pair creation from the vacuum is suppressed even for gluons, [7], one of suppression factors for quark pairs is $1/N_c$ [8], another one is numerical and not yet understood, for $N_c = 3$ the overall factor is around 0.1, as can be seen e.g. in ratio of resonance widths over their masses $\Gamma/M$.

This circumstance allows one to see on the lattice the almost constant force between static $q$ and $\bar{q}$ up to the distance of around 1 fm or larger. This property can be seen in lattice calculations made with the account of dynamical fermions (i.e. additional quark pairs), e.g. on Fig.2 one can see the persistence of linear confinement in all measured region with accuracy of 10%. Therefore we formulate the second property of confinement in QCD, which should be obeyed by all realistic theoretical models.

II. Linear confinement between static quarks persists also in presence of $q\bar{q}$ pairs in the physical region $(0.3\, fm \leq r \leq 1.5\, fm)$. Another important comment concerns quark (or gluon) dynamics at small distances. Perturbative interaction dominates there because it is singular, which can be seen in the one–gluon–exchange force of $\alpha_s r^2$. Comparing this with the confinement force mentioned above ($\sigma = 0.2 GeV^2 \approx 14 ton$), one can see that perturbative dynamics dominates for $r < 0.25\, fm$.

At these distances quarks and color charges in general can be considered independent, with essentially perturbative dynamics, which is supported by many successes of perturbative QCD.

Till now only color charges in fundamental representation of $SU(N_c)$ (quarks) have been discussed.

Very surprising results have been obtained for interaction of static charges in other representations. For example adjoint charges, which can easily be screened by gluons from the vacuum, in lattice calculations are linearly con-
fined in the physical region \( r \leq 1.5\,fm \), Fig. 3.

One can partly understand this property from the point of view of large \( N_c \): the screening part of potential \( V_S \sim \frac{1}{r} e^{-\mu r} \) is suppressed by the factor \( 1/N_c^2 \) as compared with linear part, but wins at large distances \[7\]

\[
< W_{adj}(C) > = C_1 e^{-\sigma_{adj} r T} + \frac{C_2}{N_c^2} e^{-V_s(r) T} \tag{2}
\]

The same property holds for other charge representations \[10\], and moreover the string tension \( \sigma(j) \) for a given representation \( j \) satisfies an approximate relation \[7\]

\[
\frac{\sigma(j)}{\sigma(fund)} = \frac{C_2(j)}{C_2(fund)} \tag{3}
\]

where \( C_2(j) \) is the quadratic Casimir operator,

\[
C_2(adj, N_c) = N_c, \quad C_2(fund, N_c) = \frac{N_c^2 - 1}{2N_c}
\]

Correspondingly we formulate the third property of confinement.

III. Adjoint and other charges are effectively confined in the physical region \( r \leq 1.5\,fm \) with string tension satisfying (2).

We conclude this section with two remarks, one concerning interrelation of confinement and gauge invariance, another about many erroneous attempts to define confinement through some specific form of nonperturbative gluon and quark propagator.

Firstly, gauge invariance is absolutely necessary to study properly the mechanism of confinement. It requires that any gauge–noninvariant quantity, like quark or gluon propagator, 3–gluon vertex etc. vanish when averaged over all gauge copies. One can fix the gauge only for the gauge–invariant amplitude, otherwise one lacks important part of dynamics, that of confinement.

As one popular example one may consider the "nonperturbative" quark propagator, which is suggested in a form without poles at real masses. As was told above, this propagator has no sense, since on formal level it is gauge–noninvariant and vanish upon averaging, and on physical grounds, propagator of a colored object cannot be considered separately from other colored partner(s), since it is connected by a string to it, and this string (confinement) dynamics dominates at large distances. The same can be told
about the "nonperturbative" gluon propagator behaving like \(1/q^4\) at small \(q\), which in addition imposes wrong singular nonanalytic behaviour of two–gluon–glueball Green’s function.

Likewise the well–known Dyson–Schwinger equations (DSE) for one–particle Green’s functions cannot be used for QCD in the confined phase, because again they are i) gauge–noninvariant ii) there should be a string connected to any of propagators of DSE, omitted there.

More subtle formally, but basically the same is situation with Bethe–Salpeter equation (BSE) on the fundamental level, i.e, when the kernel of BSE contains colored gluon or quark exchanges: any finite approximation (e.g. ladder–type) of the kernel violates gauge invariance and looses confinement. When averaging over all vacuum fields is made in the amplitude on the other hand, the resulting effective interaction can be treated approximately in the framework of BSE. Their value if any might lie in phenomenological applications and not in fundamental understanding of the confinement, which is the primary purpose of the present review.

Let us turn now to the order parameters defining the confining phase. To distinguish between confined and deconfined phase several order parameters are used in absence of dynamical (sea) quarks. One is the introduced above Wilson loop (1). Confinement is defined as the phase where the area law (linear potential) is valid for large contours, whereas in the deconfining phase the perimeter law appears.

Another, and sometimes practically more convenient for nonzero temperature \(t\), is the order parameter called the Polyakov line:

\[
< L(x) > = \frac{1}{n} < \mathrm{tr} \ P \ exp \ i g \int_0^\beta A_4 dx_4 >, \quad \beta = 1/t
\]  

Since \(< L \) is connected to the free energy \(F\) of isolated color charge, \(< L > = exp(-F/T)\), vanishing of \(< L >\) at \(T < T_C\) means that \(F\) is infinite in the confined phase. On the other hand, vanishing of \(< L >\) is connected to the \(Z(N_c)\) symmetry, which is respected in the confining vacuum and broken in the deconfined phase. As for the Wilson loops, Polyakov lines can be defined both for fundamental and adjoint charges; the first in absence of dynamical quarks vanish rigorously in confined phase, while adjoint lines are very small there. When dynamical quarks are admitted in the vacuum, both Wilson loop and Polyakov line are not order parameters, strictly speaking, there is no area law for large enough Wilson loops and \(< L >\) does not
vanish in the confined phase. However, for large $N_c$, and practically even for $N_c = 2, 3$ these quantities can be considered as approximate and useful order parameters even in presence of dynamical quarks, as will be seen in next sections.

From dynamical point of view the (approximate) validity of area law (linear potential) for any color charges even in presence of dynamical quarks at distances $r \leq 1.5 \text{fm}$ means that strings are formed at these distances and string dynamics defines the behaviour of color charges in the most physically interesting region.

### 3 Field correlators and confinement picture

As can be seen in the area law (1) the phenomenon of confinement necessarily implies appearance of a new mass parameter in the theory – the string tension $\sigma$ has dimension $[\text{mass}]^2$.

Since perturbative QCD depends on the mass scale only due to the renormalization through $\Lambda_{\text{QCD}}$, and using to the asymptotic freedom (and renormalization group properties) one can express the coupling constant $g(\Lambda)$ at the scale $\Lambda$ through $\Lambda_{\text{QCD}}$ as $g^2(\Lambda) = \frac{(4\pi)^2}{\beta_0 \ln \frac{\Lambda^2}{\Lambda_{\text{QCD}}^2}}$, $\beta_0 = \frac{11}{3} N_C - \frac{2}{3} n_f$, one can write

$$\sigma = \text{const} \Lambda_{\text{QCD}}^2 \sim \Lambda^2 \exp\left(-\frac{16\pi^2}{\beta_0 g^2(\Lambda)}\right)$$

where $\Lambda$ is the cut–off momentum. It is clear from (5) that $\sigma$ cannot be obtained from the perturbation series, hence the source of $\sigma$ and of the whole confinement phenomenon is purely nonperturbative. Correspondingly one should admit in the QCD vacuum the nonperturbative component and split the total gluonic vector potential $A_\mu$ as

$$A_\mu = B_\mu + a_\mu$$

where $B_\mu$ is nonperturbative and $a_\mu$ – perturbative part. As for $B_\mu$, it can be

a) quasiclassical, i.e. consisting of a superposition of classical solutions like instantons, multiinstantons, dyons etc. This possibility will be discussed below in Section 6.
b) purely quantum (but nonperturbative). The picture of Gaussian stochastic vacuum gives an example, which is discussed below.

It is important to stress at this point, that the formalism of field correlators, given below in this chapter, is of general character and allows to discuss both situations a) and b), quasiclassical and stochastic. In the first case, however, some modifications are necessary which will be introduced at the end of the chapter. As it was mentioned above, confinement implies the string formation between color charges. To understand how string is related to field correlators, consider a simple example of a nonrelativistic quark moving at a distance $r$ from a heavy antiquark fixed at the origin. As it is known from quantum mechanics, the quark Green’s function is proportional to the phase integral along its trajectory

$$G(r, t) \sim \exp \left( \int \mathcal{A}_\mu(\vec{r}(t'), t') dz_\mu \right)$$

where the averaging is over all vacuum configurations. It is convenient to express $A_\mu$ though the field strength $F_{\mu\nu}$, since the latter would be the basic stochastic quantities, and this can be done e.g. using the Fock–Schwinger gauge

$$A_\mu(x) = \int_0^x F_{\nu\mu}(u)\alpha(u)du_\nu, \quad \alpha(u) = \frac{u}{x}$$

Hence in the lowest order one obtains

$$G(r, t) \sim 1 - \frac{g^2}{2} \int d\sigma_{\nu\mu}(u)d\sigma_{\nu'\mu'}(u') < F_{\nu\mu}(u)F_{\nu'\mu'}(u') > + ...$$

where notations are used such that $d\sigma_{\nu\mu} = \alpha(u)dx_\mu du_\nu$.

On the other hand one introduce the potential $V(r)$ acting on the quark

$$G \sim \exp(- \int V(r, t')dt') \sim 1 - \int V(r, t')dt'$$

the string formation implies that $V(r, t)$ is proportional to $r$ and this depends, as we see, on the field correlators $<FF>$. 

For the exact Lorentz–invariant treatment let us introduce gauge–invariant field correlators (FC) and express the Wilson loop average through FC. This is done using the nonabelian Stokes theorem [11]

$$< W(C) >= < \frac{1}{N_C} Tr \ exp \ ig \int_c A_\mu dx_\mu >=$$

(7)
\[ \frac{1}{N_C} \langle P \text{ tr } \exp ig \int_S d\sigma F_{\mu\nu}(u,z_0) \rangle \]

where we have defined

\[ F_{\mu\nu}(u,z_0) = \Phi(z_0,u)F_{\mu\nu}(u)\Phi(u,z_0), \Phi(x,y) = P \exp ig \int_x^y A_\mu dz_\mu \]  

and integration in (7) is over the surface \( S \) inside the contour \( C \), while \( z_0 \) is an arbitrary point, on which \( \langle W(C) \rangle \) evidently does not depend. In the Abelian case the parallel transporters \( \Phi(z_0,u) \) and \( \Phi(u,z_0) \) cancel and one obtains the usual Stokes theorem.

Note that the nonabelian Stokes theorem, eq. (7), is gauge invariant even before averaging over all vacuum configurations – the latter is implied by the angular brackets in (7).

One can now use the cluster expansion theorem [12] to express the r.h.s. of (7) in terms of \( FC \), namely [5]

\[ \langle W(C) \rangle = \frac{tr}{N_C} \exp \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \int d\sigma(1)d\sigma(2)\ldots d\sigma(n) \ll F(1)\ldots F(n) \gg \]  

where lower indices of \( d\sigma_{\mu\nu} \) and \( F_{\mu\nu} \) are suppressed and \( F(k) \equiv F_{\mu_1\nu_1}(u^{(k)},z_0) \).

Note an important simplification – the averages \( \ll F(1)\ldots F(n) \gg \) in the color symmetric vacuum are proportional to the unit matrix in color space, and the ordinary operator \( P \) is not needed any more.

Eq. (9) expresses Wilson loop in terms of gauge invariant \( FC \), also called cumulants [12], defined in terms of \( FC \) as follows:

\[ \ll F(1)F(2) \gg = \langle F(1)F(2) \rangle - \langle F(1) \rangle \langle F(2) \rangle \]  

\[ \ll F(1)F(2)F(3) \gg = \langle F(1)F(2)F(3) \rangle - \langle F(1)F(2) \rangle \langle F(3) \rangle - \langle F(1) \rangle \langle F(2) \rangle \langle F(3) \rangle \]

Let us have a look at the lowest cumulant

\[ \langle F(x)\Phi(x,z_0)\Phi(z_0,y)F(y)\Phi(y,z_0)\Phi(z_0,x) \rangle \]  

It depends not only on \( x, y \) but also on the arbitrary point \( z_0 \). In case when a classical solution (a dyon or instanton) is present it is convenient to place \( z_0 \) at its center, and then \( z_0 \) acquires a clear physical meaning. We shall investigate
this case in detail in chapter 6, but now we consider the limit of stochastic
vacuum, when the expansion (9) is particularly useful. To this end consider
parameters on which a generic cumulant \( \ll F(1)\ldots F(n) \) depends. When
all coordinates \( u^k \) coincide with \( z_0 \), one obtains condensate
\( \ll (F_{\mu \nu}(0))^n \), to which we assign an order of magnitude \( F^n \). The coordinate dependence
can be characterized by the gluon correlation length \( T_g \), which is assumed to
be of the same order of magnitude for all cumulants. Then the series in (9)
has the following estimate

\[
< W(C) > = \frac{tr}{N_c} \exp \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} F^n T_g^{2(n-1)} S
\]

where \( S \) is the area of the surface inside the contour \( C \). To obtain the result
(12) we have taken into account that in each cumulant

\[
\ll F(x^{(1)}, z_0) F(x^{(2)}, z_0) \ldots F(x^{(n)}, z_0) \]

whenever \( x \) and \( y \) are close to each other,

\[
| x - y | \ll | x - z_0 |, \quad | y - z_0 |
\]

the dependence on \( z_0 \) drops out, therefore in (13) in a generic situation where
all distances \( | x^{(i)} - x^{(j)} | \sim T_g \ll | x^{(i)} - z_0 |, \quad | x^{(j)} - z_0 | \) one can omit
dependence on \( z_0 \).

The expansion in (12) is in powers of \( FT_g^2 \), and when this parameter is
small,

\[
FT_g^2 \ll 1 \quad (14)
\]

one gets the limit of Gaussian stochastic ensemble where the lowest (quadratic
in \( F_{\mu \nu} \)) cumulant is dominant.

In the same approximation (e.g. \( T_g \to 0 \) while \( < F_{\mu \nu}^2 > \) is kept fixed)
one can neglect in this cumulant the \( z_0 \) dependence, using the equivalent
(effective) form of (11)

\[
D_{\mu \nu \lambda \sigma} = \frac{1}{N_c} tr < F_{\mu \nu}(x) \Phi(x, y) F_{\lambda \sigma}(y) \Phi(y, x) > \quad (15)
\]

The form (15) has a general decomposition in terms of two Lorentz scalar
functions \( D(x - y) \) and \( D_1(x - y) \) [5]

\[
D_{\mu \nu \lambda \sigma} = (\delta_{\mu \lambda} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \lambda})D(x - y) + \
\frac{1}{2} \partial_{\mu} \{(h_{\lambda \sigma} - h_{\lambda \delta} \sigma_{\nu \delta}) + \ldots]D_1(x - y)\}
\]

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Here the ellipsis implies terms obtained by permutation of indices. It is important that the second term on the r.h.s. of (16) is a full derivative by construction.

Insertion of (16) into (9) yields the area law of Wilson loop with the string tension \( \sigma \)

\[
< W(C) > = \exp(-\sigma S_{\min})
\]

\[
\sigma = \int D(x) d^2x (1 + O(FT_g^2))
\]

where \( O(FT_g^2) \) stands for the contribution of higher cumulants, and \( S_{\min} \) is the minimal area for contour \( C \).

Note that \( D_1 \) does not enter \( \sigma \), but gives rise to the perimeter term and higher order curvature terms. On the other hand the lowest order perturbative QCD contributes to \( D_1 \) and not to \( D \), namely the one–gluon–exchange contribution is

\[
D_{pert}^{1}(x) = \frac{16\alpha_s}{3\pi x^4}
\]

Nonperturbative parts of \( D(x) \) and \( D_1(x) \) have been computed on the lattice [13] using the looking method, which suppresses perturbative fluctuations, and are shown in Fig.4. As one can see in Fig.4, both functions are well described by an exponent in the measured region, and \( D_1(x) \sim \frac{1}{3}D(x) \sim \exp(-x/T_g) \) , where \( T_g \sim 0.2 \text{fm} \). The smallness of \( T_g \) as compared to hadron size confirms the approximations made before, in particular the stochasticity condition (14). One should also take into account that \( F \) in (14) is an effective field in cumulants, which vanish when vacuum insertion is made and therefore can be small as compared with \( F \) from the gluonic condensate.

The representation (16) is valid both for abelian and nonabelian theories, and it is interesting whether the area law and nonzero string tension obtained in (17) could be valid also for QED (or \( \mathbb{U}(1) \) in lattice version). To check it let us apply the operator \( \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \frac{\partial}{\partial x_\alpha} \) to both sides of (15-16) [].

In the abelian case, when \( \Phi \) cancel in (15), one obtains (the term with \( D_1 \) drops out)

\[
\partial_\alpha < \tilde{F}_{\alpha\beta}(x) F_{\lambda\sigma}(y) > = \varepsilon_{\lambda\sigma\gamma\beta} \partial_\gamma D(x - y)
\]

where \( \tilde{F}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} F_{\mu\nu} \).

If magnetic monopoles are present in Abelian theory (e.g. Dirac monopoles) with the current \( \tilde{j}_\mu \), one has

\[
\partial_\alpha \tilde{F}_{\alpha\beta}(x) = \tilde{j}_\beta(x)
\]
In absence of magnetic monopoles (for pure QED) the abelian Bianchi identity (the second pair of Maxwell equation) requires that

$$\partial_\alpha \tilde{F}_{\alpha\beta} \equiv 0$$

Thus for QED (without magnetic monopoles) the function $D(x)$ vanishes due to (19) and hence confinement is absent, as observed in nature.

In the lattice version of $U(1)$ magnetic monopoles are present (as lattice artefacts) and the lattice formulation of our method would predict the confinement regime with nonzero string tension, as it is observed in Monte–Carlo calculations [14].

The latter can be connected through $D(x)$ to the correlator of magnetic monopole currents. Indeed multiplying both sides of (19) with $\frac{1}{2} \varepsilon_{\lambda\sigma\gamma\delta} \frac{\partial}{\partial y_\gamma}$ one obtains

$$< \tilde{j}_\beta(x) \tilde{j}_\delta(y) > = (\frac{\partial}{\partial x_\alpha} \cdot \frac{\partial}{\partial y_\alpha} \cdot \delta_{\beta\delta} - \frac{\partial}{\partial x_\delta} \frac{\partial}{\partial y_\beta}) D(x - y) \quad (22)$$

The form of Eq.(22) identically satisfies monopole current conservation: applying $\frac{\partial}{\partial x_\beta}$ or $\frac{\partial}{\partial y_\delta}$ on both sides of (22) gives zero.

It is interesting to note, that confinement (nonzero $\sigma$ and $D$, see Eq. (17)) in $U(1)$ theory with monopole currents occurs not due to average monopole density $< \tilde{j}_4(x) >$, but rather due to a more subtle feature – the correlator of monopole currents (22), which can be nonzero for the configuration where $< \tilde{j}_4(x) > = 0$. The latter is fulfilled for the system with equal number of monopoles and antimonopoles.

Note that our analysis here is strictly speaking applicable when stochasticity condition (14) is fulfilled. Therefore the case of monopoles with Dirac quantization condition is out of the region of (14) and needs some elaboration to be discussed later in this chapter.

Let us turn now to the nonabelian case, again assuming stochasticity condition (14), so that one can keep only the lowest cumulant (15-16).

Applying as in the Abelian case the operator $\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \frac{\partial}{\partial x_\nu}$ to the r.h.s. of (15-16), one obtains [15]

$$< D_\alpha \tilde{F}_{\alpha\beta}(x) \Phi(x,y) F_{\lambda\sigma}(y) \Phi(y,x) > + \Delta_\beta\lambda\sigma(x,y) = \quad (23)$$

$$= \varepsilon_{\alpha\beta\lambda\sigma} \frac{\partial}{\partial x_\alpha} D(x - y)$$
The term with $D_1$ in (23) drops out as in the Abelian case; the first term on the l.h.s. of (23) now contains the nonabelian Bianchi identity term which should vanish also in presence of dyons (magnetic monopoles) – classical solutions of Yang-Mills theory, i.e. one has

$$D_\alpha \tilde{F}_{\alpha\beta}(x) = 0. \quad (24)$$

It is another question, whether or not in lattice formulation one can violate (24) in the definition of lattice artefact monopoles, similarly to the Abelian case. We shall discuss this topic when studying lattice results on Abelian projected monopoles in Section 5. To conclude discussion of (23) one should define $\Delta^\beta_\delta_\sigma$; the latter appears only in the nonabelian case due to the shift of the straight-line contour $(x, y)$ of the correlator (15) into the position $(x + \delta x, y)$, which is implied by differentiation $\frac{\partial}{\partial x_\alpha}$. This ”contour differentiation” is well known in literature [16] and leads to the answer

$$\Delta^\beta_\lambda_\sigma(x, y) = ig \int_y^x dz_\rho \alpha(z) < tr[\tilde{F}_{\alpha\beta}(x, z) \phi(x, z) F_{\lambda\rho}(y) \phi(y, x)\phi(z, y)] \quad (25)$$

Especially simple form of (23) occurs when using (23) and tending $x$ to $y$; one obtains [15]

$$\frac{dD(z)}{dz^2} \bigg|_{z=0} = \frac{g}{8} f^{abc} < F_{\alpha\beta}(0) F_{\beta\gamma}(0) F_{\gamma\alpha}(0) >. \quad (26)$$

Thus confinement (nonzero $\sigma$ due to nonzero $D$) occurs in nonabelian case due to the purely nonabelian correlator

$$< trF_{\alpha\beta}F_{\beta\gamma}F_{\gamma\alpha} >= 3 < trE_iE_jB_k > \varepsilon_{ijk}.$$

To see the physical meaning of this correlator, one can visualize magnetic and electric field strength lines (FSL) in the space. Each magnetic monopole is a source of FSL, whether it is a real object (classical solution or external object like Dirac monopole) or lattice artefact.

In nonabelian theory these lines may form branches, and e.g. electric FSL may emit a magnetic FSL at some point, playing the role of magnetic monopole at this point. This is what exactly nonzero triple correlator

$$< \text{TrFFF} >$$

implies. Note that this could be a purely quantum effect, and no real magnetic monopoles are necessary for this mechanism of confinement.
Till now we have discussed confinement in terms of the lowest cumulant \( D(x) \), which is justified when stochasticity condition (14) is fulfilled and \( D(x) \) gives a dominant contribution. Let us now turn on other terms in the cluster expansion (9). It is clear that the general structure of higher cumulants is much more complicated than (16), but there is always present a Kronecker-type term \( D(x_1, x_2, ..., x_n) \prod \delta_{\mu\nu \kappa} \) similar to \( D(x - y) \) in (16) and other terms containing derivatives and coordinate differences like \( D_1 \).

The term \( D(x_1, x_2, ..., x_n) \) contributes to string tension, and application of the same operator \( \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} \frac{\partial}{\partial x_\alpha} \) again reveals the nonabelian Bianchi identity term (24) and an analog of \( \Delta_{\beta\lambda\sigma} \) in (23). This means that the string tension in the general case is a sum

\[
\sigma = \sum_{n=2}^{\infty} \sigma^{(n)} \quad \sigma^{(n)} = g^{n} \int \ll F(1)F(2)...F(n) \gg d\sigma(2)...d\sigma(n) \quad (27)
\]

When the stochasticity condition (14) holds, the lowest term, \( \sigma^{(2)} \), dominates in the sum; in the general case all terms in the sum (27) are important. The most important case of such a situation is the discussed above the case of quasiclassical vacuum, which we now shortly discuss, shifting the detailed discussion to the Section 6.

For a dilute gas of classical solutions the role of vacuum correlation length \( T_g \) is played by the size of the solution \( \rho \). For the correlator \( \langle F(x_1)...F(x_n) \rangle \) the essential nonzero result occurs when all \( x_1, ...x_n \) are inside the radius \( \rho \) of the solution (e.g. dyon or instanton).

On the other side the typical \( F \) of the solution, e.g.: \( F = F_{\mu\nu}(0) \), is connected to \( \rho \) by the value of topological charge, e.g. for instanton

\[
Q = \frac{g^2}{32\pi^2} \int d^4xF_{\mu\nu}^2 \quad (28)
\]

Since \( Q \) is an integer one immediately obtains that

\[
(gF \cdot \rho^2)^2 \sim n, \quad n = 1, 2...
\]

The same estimate holds for the dyon, solution which is discussed in Section 6.

Hence the series (12) and (27) has a parameter of expansion \((gFT_g^2)\) of order of unity and may not converge. A more detailed analysis of the gas of
instantons and magnetic monopoles shows that the string tension series (27) for instantons (for simplicity centers of instantons and monopoles were taken on the plane (12) of the Wilson loop) looks like [17]

\[
\sigma = \rho_0 (1 - \langle \cos \beta \rangle) ; \quad \beta = g \int F_{12}(z) d^2 z = 2\pi, \quad \sigma = 0
\]

while for magnetic monopoles

\[
\beta = \pi, \quad \sigma = 2\rho_0, \quad (30)
\]

and \( \rho_0 \) is the surface density of instantons (monopoles).

Note that \( \sigma^{(2)} = \rho_0 \langle \frac{\beta^2}{2} \rangle \) in both cases is positive, and for instantons the total sum for \( \sigma \) vanishes while for monopoles \( \sigma \) is nonzero. As we shall see below this is in agreement with the statement in chapter 6: all topological charges (29) yield flux through Wilson loop equal to \( 2\pi Q \), and for (multi)instantons with \( Q = n = 1, 2, ... \), Wilson loop is \( W = \exp 2\pi Qi = 1 \), and no confinement results for the dilute gas of such solutions. For magnetic monopoles (dyons) topology is different and elementary flux is equal to \( \pi \), bringing confinement for the dilute gas of monopoles in agreement with (31).

Thus the lowest cumulant \( < F(x)\Phi F(y)\Phi > \) might give a misleading result in case of quasiclassical vacuum, and one should sum up all the series to get the correct answer as in (30-31). Therefore to treat the vacuum containing topological charges one should separate the latter and write their contribution explicitly, while the rest – quantum fluctuations with \( FT_g^2 \ll 1 \) – can be considered via the lowest cumulants. An example of such vacuum with instanton gas and confining configurations was studied in [18] to obtain chiral symmetry breaking; this work demonstrates the usefulness of such an approach.

We conclude this chapter with discussion of confinement for charges in higher representations. As it was stated in the previous chapter, our definition of confinement based on lattice data, requires the linear potential between static charges in any representation, with string tension proportional to the quadratic Casimir operator.

Consider therefore the Wilson loop (1) for the charge in some representation; the latter was not specified above in all eqs. leading to (27). One can write in general

\[
A_\mu(x) = A_\mu^a T^a, \quad tr(T^a T^b) = \frac{1}{2} \delta^{ab}
\]

(32)
Similarly to (9) one has for the representation $j = (m_1, m_2, \ldots)$ of the group $SU(N)$ with dimension $N(j)$

$$< W(C) = \frac{1}{N(j)} tr_{j} e^{x p} \sum_{n=1}^{\infty} \frac{(ig)^n}{n!} \int d\sigma(1) \ldots d\sigma(n) \ll F(1) \ldots F(n) \gg \quad (33)$$

and by the usual arguments one has Eq.(27).

Due to the color neutrality of the vacuum each cumulant is proportional to the unit matrix in the color space, e.g. for the lowest cumulant one has

$$< F^{e}(1) F^{e}(2) > = \frac{1}{N_c^2 - 1} T^{c}_{an} T^{c}_{nb} = \Lambda^{(2)} C_2(j) \cdot \hat{1}_{ab}, \quad (34)$$

where we have used the definition

$$T^{c} T^{c} = C_2(j) \hat{1} \quad (35)$$

and introduced a constant not depending on representation,

$$\Lambda^{(2)} \equiv \frac{1}{N_C^2 - 1} < F^{e}(1) F^{e}(2) >, \quad (36)$$

and also used the color neutrality of the vacuum,

$$< F^{c}(1) F^{d}(2) > = \delta_{cd} < F^{e}(1) F^{e}(2) > \frac{1}{N^2_C - 1} \quad (37)$$

For the next – quartic cumulant one has

$$< F^{a_1}(1) F^{a_2}(2) F^{a_3}(3) F^{a_4}(4) > \ll \times \quad (38)$$

$$\times T^{a_1}_{\alpha \beta} T^{a_2}_{\beta \gamma} T^{a_3}_{\gamma \delta} T^{a_4}_{\delta \epsilon} = \Lambda^{(4)}_1 (C_2(j))^2 \delta_{\alpha \epsilon} + \Lambda^{(4)}_2 (T^{a_1}_{\alpha \beta} T^{a_2}_{\beta \gamma} T^{a_3}_{\gamma \delta} T^{a_4}_{\delta \epsilon})_{\alpha \epsilon}$$

Thus one can see in the quartic cumulant a higher order of quadratic Casimir and higher Casimir operators.

The string tension for the representation $j$ is the coefficient of the diagonal element in (34) and (38)

$$\sigma(j) = C_2(j) \int \frac{g^2 \Lambda^{(2)}}{42} d^2 x + O(C^2_2(j)) \quad (39)$$
where the term $O(C_2^2(j))$ contains higher degrees of $C_2(j)$ and higher Casimir operators.

Comparing our result (39) with lattice data [10] and Fig.3 one can see that the first quadratic cumulant should be dominant as it ensures proportionality of $\sigma(j)$ to the quadratic Casimir operator.

Another interesting and important check of the dominance of bilocal correlator (of the Gaussian stochasticity) is the calculation of the QCD string profile, done in [19]. Here the string profile means the distribution $\rho_{11}$ of the longitudinal component of colorelectric field as a function of distance $x_\perp$ to the string axis. This distribution can be expressed through integral of functions $D(x)$, $D_1(x)$ [18], and with measured values of $D$, $D_1$ from [13] one can compute $\rho_{11}(x_\perp)$ and compare it with independent measurements. This comparison was done in [19] and shown in Fig. 5. One can see a good agreement of the computed $\rho_{11}(x_\perp)$ (broken line) with ”experimental” values. Thus MVC gives a good description of data in the simplest (bilocal) approximation, even for such delicate characteristics as field distributions inside the string.

4 Dual Meissner mechanism, confinement and superconductivity

The physical essence of the confinement phenomenon is the formation of the string between the probing charges introduced into the vacuum, which in turn means that the electric field distribution is drastically changes from the usual dipole picture (for the empty vacuum) and is focused instead into a string picture in the confining vacuum. From the point of view of macroscopic electrodynamics of media, this effect can be described introducing dielectric function $\varepsilon(x)$, and electric induction $\vec{D}(x)$ together with electric field $\vec{E}(x)$, $\vec{D}(x) = \varepsilon(x)\vec{E}(x)$.

One can then adjust $\varepsilon(x)$, or better $\varepsilon(D)$ or $\varepsilon(E)$ to obtain the string formation. Another possibility is to choose the effective action as a function of $F_{\mu\nu}^2$ in such a way, as to reproduce string-type distribution.

This direction was reviewed in [1], the main conclusion reached long ago [20] was that the physical vacuum of QCD considered as a medium could be called a pure dielectric, i.e. $\varepsilon = 0$ far from probing charges [21].
It is also shown, that it is possible to choose \( \varepsilon(E) \) in such a way as to obtain a string of constant radius [21] or with radius slowly dependent on length. We shall not follow these results below, as well as results of the so-called dielectric model [1,21], referring the reader to the mentioned literature.

Instead we focus in this and following chapter on another approach, which has proved to be fruitful during last years - confinement as dual Meissner effect [3] and ideologically connected to it abelian projection method [4].

The physical idea used by 'tHooft and Mandelstam [3] is the analogy between the Abrikosov string formation in type II– superconductor between magnetic poles and proposed string formation between color electric charges in QCD.

We shall study this analogy here from several different point of view:
1) energetics of the vacuum – minimal free energy of the vacuum
2) classical equations of motion -Maxwell and London equations
3) condensate formation and symmetry breaking
4) vacuum correlation functions of fields and currents.

We consider in this chapter the 4d generalization of the Ginzburg-Landau model of superconductivity, which is called the Abelian Higgs model with the Lagrangian [22]

\[
\mathcal{L} = \frac{1}{4} F_{\mu\nu}^2 - |D_\mu \varphi|^2 - \frac{\lambda}{4} (|\varphi|^2 - \varphi_0^2), \quad D_\mu = \partial_\mu - ieA_\mu. \tag{40}
\]

This model is known to possess classical solutions–Nielsen–Olesen strings, which are 4d generalization of the Abrikosov strings, occurring in type II superconductor. The latter can be described by the Ginzburg–Landau Lagrangian, when coupling constants are chosen correspondingly [23].

The Lagrangian (40) combines two fields: the electromagnetic field \( (A_\mu, F_{\mu\nu}) \) – which will be analogue of gluonic field of QCD, and the complex Higgs field \( \varphi(x) \), which describes the amplitude of the Cooper–paired electrons in a superconductor. When \( \lambda \to \infty \) the wave functional \( \Psi\{A_\mu, \varphi(x)\} \) has a strong maximum around \( \varphi(x) = \varphi_0 \), which means formation of the condensate of Cooper pairs of amplitude \( \varphi_0 \).

Let us look more closely at the model (40) discussing points 1)–4) successively.

1) The energy density corresponding to the (40) is

\[
\varepsilon = \frac{\vec{E}^2 + \vec{B}^2}{2} + |D\varphi|^2 + |D_0\varphi|^2 + \frac{\lambda}{4} (|\varphi|^2 - \varphi_0^2)^2 \tag{41}
\]
From (41) it is clear that in absence of external sources and for large \( \lambda \) the lowest (vacuum) state corresponds to \( \varphi = \varphi_0 = \text{const.} \), and \( \vec{E} = \vec{B} = 0 \). For future comparison with QCD it is worthwhile to stress that formation of the condensate \( \langle F^2_{\mu\nu} \rangle \) is not advantageous for large \( \lambda \), since the mixed term \( |A_{\mu}\varphi|^2 \) will give very large positive contribution: condensate of electric charges \( \varphi \) repels and suppresses everywhere electromagnetic field \( F_{\mu\nu} \). The same situation occurs in the nonabelian Higgs – the Georgi – Glashow model: there appear two phases depending on values of \( \lambda, g \) and for large \( \lambda \) the deconfined phase with \( \varphi = \varphi_0 \) persists. Here the word deconfined means that external (color) electric charges are not confined, while magnetic monopoles can be confined. For smaller values of \( \lambda \) one can reach a region where it is advantageous to form the condensate \( \langle F^2_{\mu\nu} \rangle \) and then possibly also nonzero \( D(x) \). As we discussed in Section 3 then there is a possible confinement of colorelectric charges in this phase. But let us come back to the Abelian Higgs model and the limit of large \( \lambda \), which is of primary interest to us.

In this case it is advantageous to form the condensate of electric charges \( \langle \varphi(x) = \varphi_0 \rangle \), condensate of electromagnetic field is suppressed, and one obtains Abrikosov–Nielsen–Olesen (ANO) strings connecting magnetic poles – this is the confinement of magnetic monopoles and mass generation of e.m. field – leading to the deconfinement of electric charges.

In the dual picture one would assume that condensate of magnetic charges (monopoles) would help to create strings of electric field, connecting (color) electric charges, yielding confinement of the latter. Our tentative guess is that the condensate of magnetic monopoles (dyons) in gluodynamics is associated with the gluonic condensate \( \langle F^2_{\mu\nu} \rangle \).

2) We now discuss the structure of the ANO strings from the point of view of classical equations of motion. The Maxwell equations are

\[
\text{rot}\vec{B} = \vec{j}
\]

(42)

where \( \vec{j} \) is the microscopic current of electric charges, including the condensate of Cooper pairs.

To obtain a closed equation for \( \vec{B} \) one needs the specific feature of superconductor in the form of Londons equation

\[
\text{rot}\vec{j} = \delta^{-2}\vec{B}
\]

(43)

The latter can be derived from the Ginzburg–Landau–type Lagrangian (40). Indeed writing the current for the Lagrangian (40) in the usual way,
we have
\[ j_\mu = ie(\phi^+ \partial_\mu \phi - \partial_\mu \phi^+) - 2e^2 A_\mu |\phi|^2 \] (44)

Assume that there exists a domain where \( \phi \) is already constant, and \( A_\mu \) is still nonzero (we shall define this region later in a better way), and apply the rot operation to both sides of (44). Then one obtains the London equations (43) and \( \delta \) is defined by
\[ \delta^{-2} = 2e^2 \phi_0^2 \] (45)

Insertion of (43) into (42) yields equation for \( \vec{B} \)
\[ \Delta \vec{B} - \delta^{-2} \vec{B} = 0 \] (46)

Solving (46) one obtains the exponential fall–off of \( \vec{B} \) away from the center of the ANO string,
\[ B(r) = \text{const} K_0(r/\delta), \quad B(r) \sim \exp(-r/\delta), \quad r \gg \delta \] (47)

From (47) it is clear that \( \delta^{-1} \) is the photon mass, generated by the Higgs mechanism. On the other hand the field \( \phi \) has its own correlation length \( \xi \), connected to the mass of quanta of the field \( \phi \) (the "Higgs mass")
\[ \xi = 1/m_\phi, \quad m_\phi^2 = 2\lambda \phi_0^2 \] (48)

As it is known [23] the London’s limit for the superconductor of the second kind, corresponds to the relations
\[ \delta \gg \xi, \quad e \ll \lambda \] (49)

One can also calculate the string energy per unit length–the string tension – for the string of the minimal magnetic flux \( 2\pi \). Using (41) and (47-48) one obtains [23]
\[ \sigma_{ANO} = \frac{\pi}{\delta^2 \ln \frac{\delta}{\xi}}, \quad \delta/\xi \gg 1 \] (50)

Note that the main contribution to \( \sigma_{ANO} \) (50) comes from the term \( |\vec{D}\phi|^2 \) in (41).

Thus the physical picture of the ANO string in the London’s limit implies the mass generation of the magnetic quanta, \( m = 1/\delta \), which are much less than the Higgs mass \( m_\phi = 1/\xi \).
It is instructive to see how the screening of the magnetic field (mass generation) occurs:

First magnetic field creates around its flux (field–strength line) the circle of the current $\vec{j}$ from the superconducting medium, as described by London's equations (43). Then Maxwell equation (42) tells that around the induced current $\vec{j}$ there appears a circulating magnetic field, directed opposite to the original magnetic field $\vec{B}$ and proportional to it. As a result magnetic field is partly screened in the middle and finally completely screened far from the center of the magnetic flux (which is the ANO string).

The profile of the string, i.e. $B_{11}(r)$ as a function of the distance of the string axis is shown in Fig.6, as obtained from Eqs. (47).

It is interesting to compare this ANO–string profile with the corresponding profile obtained in the gluodynamics. As was discussed in the previous chapter, the distribution of the parallel component of the colorelectric field was measured in [24] and is also exponentially decreasing far off axis, as can be seen in Fig.6, and both profiles of the ANO string and of the QCD string in Fig.6 are very much similar, thus supporting the idea of dual Meissner mechanism.

To look more closely at the similarity of classical equations (4.9-4.10) with the corresponding equations, obtained in lattice Monte–Carlo simulations, one needs first an instrument to recognize the similarity of effective Lagrangians in the model (40) and in QCD. This is discussed in the next chapter.

We conclude point 2) of this chapter discussing parameters $\delta_{QCD}$ and $\xi_{QCD}$, parameters dual–analogical to $\delta$ and $\xi$ of (44-45). This is done via the Abelian projection method in [25] with the result (see Fig. 7)

$$\delta_{QCD} \sim \xi_{QCD} \sim 0.2 \text{fm}$$  \hspace{1cm} (51)

Thus in QCD situation is somewhat in the middle between the type I and type II dual superconductor. The calculation of the effective potential $V(\phi)$ in SU(2) gluodynamics, made recently in [26], shows a two-well structure, but with a rather shallow well, hence the effective $\lambda_{QCD}$ in the Lagrangian (40) is not large, again in agreement with (51). A very interesting discussion of properties of dual monopoles and their measurement on the lattice is contained in [27].

3) Condensate formation and symmetry breaking
The phenomenon of superconductivity is usually associated with the formation of the condensate of Cooper pairs (although it is not necessary).

The notion of condensate is most clearly understood for the noninteracting Bose–Einstein gas at almost zero temperature, where the phenomenon of the Bose–Einstein condensation takes place. Ideally in quantum–mechanical systems condensate can be considered as a coherent state.

When interaction is taken into account, the meaning of the condensate is less clear [28].

In quantum field theory one associates condensate with the properties of the wave functional and/or Fock columns. Again in the case of no interaction one can construct the coherent state in the second–quantized formalism (see e.g. the definition of wave operators in the superfluidity case in [29]).

One of the properties of such a state is the fixed phase of the wave functional, which means that the $U(1)$ symmetry is violated.

The simplest example is given by the Ginzburg–Landau theory (40), where in the approximation when $\lambda \to \infty$, the wave functional $\Psi\{\varphi\}$ can be approximated by the classical solution $\varphi = \varphi_0$, $\Psi\{\varphi\} \to \Psi\{\varphi_0\}$.

The solution $\varphi = \varphi_0$, where $\varphi_0$ has a fixed phase, violates $U(1)$ symmetry of the Lagrangian (40), and one has the phenomenon of spontaneous symmetry breaking (SSB) [30]. The easiest way to exemplify the SSB is the double–well Higgs potential like in (40). Therefore if one looks for the dual Meissner mechanism in QCD (gluodynamics) one may identify the magnetic monopole condensate $\tilde{\varphi}$, dual to the Cooper pair condensate $\varphi = \varphi_0$ and find the effective potential $V(\tilde{\varphi})$, demonstrating that it has a typical double–well shape. Such an analysis was performed in [26] using the Abelian projection method and will be discussed in the next chapter.

4) Finally in this chapter we shall look at the dual Meissner mechanism from the point of view of field and current correlators [31]. This will enable us to formulate the mechanism in the most general form, valid both in (quasi) classical and quantum vacuum.

One can use the correlator (16) to study confinement of both magnetic and electric charges (the latter was discussed in chapter 3 – Eq. (19) and subsequent text). Let us rewrite (16) for correlators of electric and magnetic fields separately

$$< E_i(x)E_j(y) > = \delta_{ij}(D^E + D^E_1 + \hbar^2 \frac{\partial D^E_1}{\partial \hbar^2}) + \hbar_i \hbar_j \frac{\partial d^E}{\partial \hbar^2}$$ (52)
\begin{equation}
< H_i(x)H_j(y) > = \delta_{ij}(D^H + D^H_1 + h^2 \frac{\partial D^H_1}{\partial h^2}) - h_i h_j \frac{\partial d^H_1}{\partial h^2}
\end{equation}

where \( h_\mu = x_\mu - y_\mu \), \( h^2 = h_\mu h_\mu \).

In (52)-(53) we have specified correlators \( D, D_1 \) for electric and magnetic fields separately, since in Lorentz-invariant vacuum \( D^E = D^H, D^E_1 = D^H_1 \), otherwise, e.g. in the Ginzburg-Landau model, electric and magnetic correlators may differ, as also in any theory for nonzero temperature.

Now let us compare the Wilson loop averages for electric and magnetic charges. In case of electric charges the result is the area law (17) with the function \( D \to D^E \) responsible for confinement.

Now consider a magnetic charge in the contour \( C \) in the 14 plane, the corresponding Wilson loop is

\begin{equation}
< \tilde{\mathcal{W}}(C) > = < \exp(ig \int \tilde{F}_{\mu\nu} d\sigma_{\mu\nu}) > = \exp(-\sigma^* S_{min})
\end{equation}

Here \( \tilde{F}_{\mu\nu} \) is the dual field, \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \), and \( \tilde{F}_{14} = H_1 \). From (17) one obtains

\begin{equation}
\sigma^* = \frac{g^2}{2} \int d^2 x D^H_1(x)(1 + O(T^2 g H))
\end{equation}

Eq. (55) is kind of surprise. For electric charges \( D^E \) yields confinement and is nonperturbative, supported by magnetic monopoles, cf. Eq. (22), while \( D^E \) contributes perimeter correction to the area law and contains also perturbative contributions like Coulomb term.

The same electric charges in the (23) – plane Wilson loop bring about again the area law (17) with \( D \to D^H \). Also \( D^H \) is coupled to magnetic monopoles in the vacuum, indeed taking divergence from both sides of (53) one gets

\begin{equation}
< \text{div} H(x), \text{div} H(y) > = -\partial^2 G^H(x - y)
\end{equation}

Instead confinement of external magnetic charges, Eq. (4.16), is coupled to the function \( D^H_1 \) (or \( D^E_1 \) if one takes \( < \tilde{\mathcal{W}}(C) > \) for the loop in the (23) –plane).

Thus duality of electric – magnetic external charges requires interchange \( D^H, D^E \leftrightarrow D^H_1, D^E_1 \).

At this point one should be careful and separate out perturbative interaction which is contained in \( D_1 \), namely one should replace in (55) \( D^H_1 \) by
\[ \tilde{D}_1^H \] where
\[ \tilde{D}_1^H = D_1^H - \frac{4e^2}{x^4} \]  

(57)

It is important to stress again that it is the nonperturbative contents of correlators which may create a new mass parameter like \( \sigma^* \) and which should enter therefore in correlators (the perturbative term contributes the usual Coulomb–like interaction which is technically easier consider not as a part of \( D_1^H \), but to separate at earlier stage, see e.g. [32]).

This mass creation can be visualized in the Ginzburg–Landau model, where \( D_1 \) can be computed explicitly from (40)
\[ D_1^{LG}(x - y) = (e^2|\phi|^2 - \partial^2)^{-1}_{xy} \approx e^{-m|x-y|} \]  

(58)

where in the asymptotic region \(|\phi| = |\phi_0|\) and \( m = e|\phi_0| = 1/\delta \).

From the point of view of duality \( D_1^H(x) \) Eq.(49) should be compared with the behaviour of \( D^E(x) \), which was discussed above in Section 3 and measured in [13].
\[ D(x) = D^E(x) = e^{-\mu|x|}, \quad \mu \approx 1\text{GeV} \]  

(59)

Eqs. (58-59) demonstrate validity of dual Meissner mechanism on the level of field correlators.

5 Abelian projection method

In the previous section it was shown how the string is formed between magnetic sources in the vacuum described by the Abelian Higgs model.

In the dual Meissner mechanism of confinement [3] it is assumed that the condensate of magnetic monopoles occurs in QCD, which creates strings between color electric charges. There are two possible ways to proceed from this point. In the first one may assume the form of Abelian or nonabelian Higgs model for dual gluonic field and scalar field of magnetic monopoles. This type of approach was pursued in [33] and phenomenologically is quite successful: the linear confinement appears naturally and even spin–dependent forces are predicted in reasonable agreement with experiment [34]. We shall not go into details of this very interesting approach, referring the reader to the cited papers, since here lacks the most fundamental part of the problem
– the derivation of this dual Meissner model from the first principles – the QCD Lagrangian.

Instead we turn to another direction which was pursued very intensively the last 8-10 years – the Abelian projection method dating from the seminal paper by ’tHooft [4].

The main problem – how to recognize configurations with properties of magnetic monopoles, which are responsible for confinement. The ’tHooft’s suggestion [4] is to chose a specific gauge, where monopole degrees of freedom hidden in a given configuration become evident. The corresponding procedure was elaborated both in continuum and on lattice [35] and most subsequent efforts have been devoted to practical separation (abelian projection) of lattice configurations and study of the separated degrees of freedom, and construction of the effective Lagrangian for them. We start with the formal procedure in continuum for SU(N) gluodynamics, following [4,35].

For any composite field $X$ transforming as an adjoint representation, like $F_{\mu\nu}$, e.g.,

$$X \rightarrow X' = VXV^{-1}$$  \hspace{1cm} (60)

let us find the specific unitary matrix $V$ (the gauge), where $X$ is diagonal

$$X' = VXV^{-1} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_N)$$  \hspace{1cm} (61)

For $X$ from the Lie algebra of SU(N), one can choose $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_N$. It is clear that $V$ is determined up to the left multiplication by a diagonal SU(N) matrix.

This matrix belongs to the Cartan or largest Abelian subgroup of SU(N), $V(1)^{N-1} \subset SU(N)$.

Now we transform $A_\mu$ to the gauge (61)

$$\tilde{A}_\mu = V(A_\mu + \frac{i}{g} \partial_\mu)V^{-1}$$  \hspace{1cm} (62)

and consider how components of $\tilde{A}_\mu$ transform under $U(1)^{N-1}$. The diagonal ones

$$a_\mu^i \equiv (\tilde{A}_\mu)_{ii}$$  \hspace{1cm} (63)

transform as ”photons”:

$$a_\mu^i \rightarrow a_\mu^i = a_\mu^i + \frac{1}{g} \partial_\mu \alpha_i$$  \hspace{1cm} (64)
while nondiagonal, $c_{ij}^i \equiv (A^i_{ij})_{\mu}$, transform as charged fields.

$$C'_{ij}^i = \exp[i(\alpha_i - \alpha_j)]C_{ij}^i \quad (65)$$

But as 'tHooft remarks [4], this is not the whole story – there appear singularities due to a possible coincidence of two or more eigenvalues $\lambda_i$ and those bear properties of magnetic monopoles. To make it explicit consider as in [35] the ”photon” field strength

$$f^i_{\mu\nu} = \partial_\mu a^i_\nu - \partial_\nu a^i_\mu = V F^i_{\mu\nu}V^{-1} + ig[V(A_\mu + \frac{i}{g}\partial_\mu)V^{-1}, V(A_\nu + \frac{i}{g}\partial_\nu)V^{-1}] \quad (66)$$

and define the monopole current

$$K^i_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\rho\sigma} \partial_\nu f^i_{\rho\sigma}, \partial_\mu K^i_\mu = 0. \quad (67)$$

Since $F^i_{\mu\nu}$ is regular, the only singularity giving rise to $K^i_\mu$ is the commutator term in (66), otherwise the smooth part of $a^i_\mu$ does not contribute to $K^i_\mu$ because of the antisymmetric tensor.

Hence one can define the magnetic charge $m^i(\Omega)$ in the 3d region $\Omega$,

$$m^i(\Omega) = \int_\Omega d^3\sigma K^i_\mu = \frac{1}{8\pi} \int_{\partial\Omega} d^2\sigma \varepsilon_{\mu\nu\rho\sigma} f^i_{\rho\sigma} \quad (68)$$

Consider now the situation when two eigenvalues of (61) coincide, e.g. $\lambda_1 = \lambda_2$. This may happen at one 3d point in $\Omega$, $x^{(1)}$, i.e. on the line in the 4d, which one can visualize as the magnetic monopole world line. The contribution to $m^i(\Omega)$ comes only from the infinitesimal neighborhood $B_\varepsilon$ of $x^{(1)}$

$$m^i(B_\varepsilon(x^{(1)})) = \frac{i}{4\pi} \int_{S_\varepsilon} d^2\sigma \varepsilon_{\mu\nu\rho\sigma} [V\partial_\rho V^{-1}, V\partial_\sigma V^{-1}]_{ii} =$$

$$= -\frac{i}{4\pi} \int d^2\sigma \varepsilon_{\mu\nu\rho\sigma} \partial_\rho [V\partial_\sigma V^{-1}]_{ii} \quad (69)$$

The term $V\partial_\sigma V^{-1}$ is singular and should be treated with care. To make it explicit, one can write.

$$V = W \begin{pmatrix} \cos \frac{1}{2} \theta + i\vec{e}_\phi \sin \frac{1}{2} \theta, & 0 \\ 0, & 1 \end{pmatrix} \quad (70)$$

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where $W$ is a smooth SU(N) function near $x^{(1)}$. Inserting it in (69) one obtains

$$\mathbf{m}^i(B_\xi(x^{(1)})) = \frac{1}{8\pi} \int_{S_\xi} d^2\sigma_{\mu\nu}\epsilon_{\mu\nu\rho\sigma}\partial_\rho(1 - \cos\theta)\partial_\sigma\phi[\sigma_3]_{ii}$$  (71)

where $\phi$ and $\theta$ are azimuthal and polar anges.

The integrand in (71) is a Jacobian displaying a mapping from $S^2_\xi(x^{(1)})$ to $(\theta, \phi) \sim SU(2)/U(1)$.

Since

$$\Pi_2(SU(2)/U(1)) = Z$$  (72)

the magnetic charge is $m^i = 0, \pm 1/2, \pm 1, \ldots$

From the derivation above it is clear, that the point $x = x^{(1)}$, where $\lambda_1(x^{(1)}) = \lambda_2(x^{(1)})$, is a singular point of the gauge transformed $\tilde{A}_\mu$ and $a^i_\mu$, and the latter behaves near $x = x^{(1)}$ as $0(1\left|x-x^{(1)}\right|)$, and abelian projected field strength $f^i_{\mu\nu}$ is $0(1\left|x-x^{(1)}\right|^2)$, like the field of a point–like magnetic monopole.

However several points should be stressed now

i) the original vector potential $A_\mu$ and $F_{\mu\nu}$ are smooth and do not show any singular behaviour

ii) at large distances $f^i_{\mu\nu}$ is not, generally speaking, monopole–like, i.e. does not decrease as $1\left|x-x^{(1)}\right|^2$. So that similarity to the magnetic monopole can be seen only in topological properties in the vicinity of the singular point $x^{(1)}$.

iii) the fields $A_\mu, a^i_\mu$ have in general nothing to do with classical solutions, and may be quantum fluctuations. Actually almost any field distribution in the vacuum may be abelian projected into $a^i_\mu, f^i_{\mu\nu}$ and magnetic monopoles then can be detected.

Examples of this statement will be given below, but before doing that we must say several words about the choice of the field $X$ in (60) and more generally about the explicit gauge choice.

By now the most popular choices for the adjoint operator $X$, (60) is for nonzero temperature the fundamental Polyakov line

$$L_{ij} = (Pexpig\int_0^\beta dx_4 A_4(x_4, \bar{x}))_{ij}$$  (73)

where $\beta = 1/T$, and $T$ is the temperature, $A_\mu(x_4, \bar{x})$ is required to be periodic in $x_4$ and $i, j$ are fundamental color indices.
Another widely used gauge is the so-called maximal abelian gauge (MAG) [35], which in lattice notations can be expressed as a gauge where the following quantity is maximized

\[
R = \sum_{s,\mu} Tr(\sigma_3 \tilde{U}_\mu(s) \sigma_3 \tilde{U}_\mu^+(s))
\] (74)

Here the link matrix \( U_\mu \sim \exp igA_\mu^0\Delta z_\mu \), and \( \tilde{U} \) is gauge transformed with the help of \( V \)

\[
\tilde{U}_\mu(s) = V(s)U_\mu(s)V^{-1}(s + \mu)
\] (75)

In the continuum MAG is characterized by the condition, which in the SU(2) case looks most simply

\[
(\partial_\mu \mp igA_\mu^3)A_\mu^\pm = 0, \quad A_\mu^\pm = A_\mu^1 \pm iA_\mu^2
\] (76)

As will be seen the choice of gauge in the abelian projection method is crucial: e.g. for the minimal abelian gauge, corresponding to the minimum of \( R \) (74), Abelian projected monopoles have no influence on confinement [36].

Since the total of papers on abelian projection is now enormous, let us discuss shortly the main ideas and results. The most part of results are obtained doing abelian projection on the lattice. (A short introduction and discussion of lattice technic is given in [27]). The separation of monopole degrees of freedom was done as follows. For an abelian projected link (75) one can define the \( U(1) \) angle \( \theta_\mu \)

\[
\tilde{U}_\mu = \exp(i\theta_\mu \sigma_3), \quad -\pi \leq \theta_\mu \leq \pi
\] (77)

Then for the plaquette \( \tilde{U}_{\mu\nu} \sim \tilde{U}_\mu \tilde{U}_\nu \sum \exp(i\theta_{\mu\nu} \sigma_3) \) one can write \(-4\pi \leq \theta_{\mu\nu} \leq 4\pi\) and define the ”Coulomb part” of \( \theta_{\mu\nu}, \bar{\theta}_{\mu\nu} = \text{mod}_{2\pi}\{\theta_{\mu\nu}\}, \) so that

\[
\bar{\theta}_{\mu\nu} = \theta_{\mu\nu} + 2\pi n_{\mu\nu}
\] (78)

and \( n_{\mu\nu} \) counts the number of Dirac strings across the plaquette \( (\mu\nu) \).

Now one can calculate different observables in AP and find the contribution to them separately of the ”Coulomb part” \( \bar{\theta}_{\mu\nu} \) and the monopole part \( m_{\mu\nu} \cong 2\pi n_{\mu\nu} \). This was done in a series of papers and the monopole dominance was demonstrated i) for the string tension [38] ii) for fermion propagators and hadron masses [39] iii) for the topological susceptibility [40].
In Fig. 8 a comparison is made of the total AP contribution to the string tension and of the monopole part, which is seen to dominate as compared to the Coulomb part.

Another line of activity in AP is the derivation of effective Lagrangians for AP degrees of freedom. (See e.g. in [41]). The resulting Lagrangian however does not confine AP neutral objects like ”photons” and the latter should contribute to the spectrum in contradiction to the experiment. Therefore the Lagrangian is not very useful both phenomenologically and fundamentally. We shall not discuss these Lagrangians referring the reader to the cited literature. Let us come back to the investigation of the confinement mechanism with the help of the AP method.

A direct check of the dual Meissner mechanism of confinement should contain at least two elements: a detection of dual London current and a check of magnetic monopole condensation. The first was done in [42]. The dual of the London equation (43) is

\[ \vec{E} = \delta^2 \text{rot} j_M, \quad \delta = 1/m \]  

(79)

For the ideal type -II superconducting picture one needs that \( \delta \gg \xi \), where \( \xi = m^{-1}_\phi \) is the Higgs mass (corresponding to the magnetic monopole condensation). In practice in [42] it was found that \( \delta \sim \xi \), and therefore the condensate is ”soft”, implying that exact solutions of ANO string are necessary to compare with. Such analysis was done in [43] for \( SU(2) \) and \( SU(3) \) and recently in [44].

The string profile \( \rho(x_\perp) \) was also studied using AP, and the resulting \( \rho(x_\perp) \) [45] is similar to the one, obtained in the full lattice simulation [46]. Thus the whole picture of currents and density is compatible with the dual superconductivity.

Now we come to the second check – of monopole condensation. Of special interest is the problem of definition of monopoles. In the first lattice studies [35] the AP monopoles in the maximally Abelian gauge have been identified through magnetic currents on the links of the dual lattice, and the perimeter density of the currents was measured below and above transition temperature \( T_c \), showing a strong decrease of this density at \( T > T_c \).

Later it was realized [47] that monopole density defined in this way may not be a good characteristic of dual superconductivity and monopole condensate, and for the latter one needs the creation operator of the magnetic
monopole. In this way one can define the dual analog of the Higgs field \( \varphi \), and settle the question of condensation. The general mathematical construction of the monopole creation operator was given before in [48], and several papers [49] used this U(1) construction for the AP monopoles.

Another construction was used for the U(1) theory in [50] and as in [49] the condensation of monopoles was also demonstrated. In the SU(2) case the analysis was performed in [51] and [26].

In the last paper the important step was done in finding an effective potential \( V(\varphi) \) for the monopole creation operator \( \varphi \) (defined similarly to Froelich–Marchetti [48], the exact equivalence to [48] is still not proved in [26]).

If one believes in the dual Meissner picture of confinement, then one should expect the two–well structure of \( V(\varphi) \), symmetric with respect to the change \( \varphi \to -\varphi \). One can see in Fig.9a the r.h.s. of \( V(\phi) \) (for positive \( \phi \)) which indeed has a minimum at \( \phi = \phi_c \), shifted to the right from \( \phi = 0 \), for values of \( \beta = \frac{1}{g^2} \) in the region of confinement. In Fig.9b the same quantity \( V(\phi) \) is measured in the deconfinement phase, and as one can see, the minimum \( \varphi = \varphi_c \) is at zero, \( \phi_c = 0 \). In this way the analysis of [26] gives an evidence of AP monopole condensation. Note, however, that strictly speaking the condensation should be proved in the London limit \( (\lambda \to \infty) \), otherwise quantum fluctuations of \( \varphi \) might prevail for a shallow well like that in Fig.9a (we remind that the system has a finite number of degrees of freedom in a finite lattice volume). Till now we have said nothing about the nature of configurations which due to AP disclose the magnetic monopole structure and ensure confinement. They could be classical configurations or quantum fluctuations (the last possibility is preferred by the majority of researches).

Recently an interesting AP analysis of classical configurations has been done [52-54]. We shall mostly pay attention to the first paper [52], created a wave of further activity. The authors of [52] make AP analytically of an isolated instanton, multiinstanton and Prasad–Sommerfield monopole and demonstrate in all cases the appearance of a straight–line monopole current. In the first case the current is concentrated at the centre of instanton and its direction depends on the parametrization chosen. Later on in [53] the same type of analysis was made numerically with the instanton–antiinstanton gas, and the appearance of monopole–current loops was demonstrated, of the size
of instanton radius and stable with respect to quantum fluctuations.

Hence everything looks as if there is a hidden magnetic monopole inside an instanton. This result is extremely surprising from several points of view. First of all the flux of magnetic monopole through the Wilson loop is equal to \( \pi \) (this will be shown in the next Section), whereas the same flux of instanton is equal to zero (modulo \( 2\pi \)), this fact helps to understand why the monopole gas may ensure confinement, while the instanton gas does not. Therefore the identification of monopoles and instantons is not possible. Secondly, confinement in the instanton gas was shown to be absent in several independent calculations \[55\], and the appearance of rather large monopole loops in this gas \[53\] looks suspiciously. To understand what happens in the AP method, and whether it can generate monopoles where they were originally absent, we turn again to the case of one instanton \[52\] and take into account that AP contains a singular gauge transformation, which transforms an originally smooth field \( F_{\mu\nu} \) into a singular one, namely \[52\]

\[
F_{\mu\nu}(x) = VF_{\mu\nu}(x)V^{-1} + F_{\mu\nu}^{\text{sing}},
\]

where

\[
F_{\mu}^{\text{sing}}(x) = \frac{i}{g}V(x)[\partial_\mu \partial_\nu - \partial_\nu \partial_\mu]V^{-1}(x)
\]

The matrix \( V(x) \) is singular and \( F_{\mu\nu}^{\text{sing}} \) therefore does not vanish. Hence the correlator function \( D(x) \), of the fields (15) and currents (22) acquires due to this singular gauge transformation a new term

\[
D(x) \rightarrow D(x) + D^{\text{sing}}(x),
\]

where

\[
D^{\text{sing}}(x - y) \sim < F^{\text{sing}}(x)F^{\text{sing}}(y) >
\]

The same \( F^{\text{sing}}_{\mu\nu} \) causes the appearance of a magnetic monopole current which passes through the center of an isolated instanton \[52\]. Therefore the AP method indeed inserts a singularity of magnetic monopole type into practically any configuration and therefore cannot be a reliable method of separation of confining configurations. On the other side the use of AP for lattice configurations reproduces well the confinement observables, e.g. the string tension \[38\], which means that confining contributions pass the AP test very well.
From this point of view it is interesting to consider another example – the dyonic (Prasad–Sommerfield) solution, also studied in [52]. It appears that the AP monopole current passes exactly through the multistanton centers and coincides with the trajectory of the physical dyon, which can be calculated independently, i.e. the AP magnetic monopole current coincides with the total magnetic monopole current. This again supports the idea that the genuine confining configurations are well projected by the AP method.

6 Classical solutions as a possible source of confinement

In the last Section we have seen, that the AP method gives no answer to the question what are the confining configurations, they could be both classical fields and quantum fluctuations. Nothing about it can be said till now from the field correlators, however lattice measurements of correlators yield some information on the possible profile of confining configurations.

At the same time an interesting information can be obtained from the lattice calculations using the so-called cooling method [56], where at each step of cooling quantum fluctuations are suppressed more and more, and configurations evolve in the direction of decreasing of the action. Roughly these results demonstrate that out of tens of thousands of original configurations (which are mostly quantum noise) at some step of cooling only few (sometimes 15-25) are retained, which ensure the same string tension, as the original – ”hot” vacuum. With more cooling the number of configurations drops to several units (they are mostly (anti)instantons) and confinement disappears.

Thus one can think that confining configurations in the vacuum differ from usual quantum fluctuations, and their action is probably larger than instantonic action or they are less stable.

Therefore it is interesting to look carefully into all existing classical solutions and check whether one can build up the confining vacuum out of those solutions.

In this chapter we shall study several classical solutions: instantons, dyons and lattice periodic instantons, and give a short discussion of some other objects, like torons.
After disposing the individual properties of those we specifically concentrate on the contribution of each of the object to the Wilson loop—what we call the elementary flux of the object—and argue that flux proportional to π of dyons and twisted instantons with \( Q = 1/2 \) may yield confinement, in contrast to the case of instantons with flux equal to \( 2\pi \).

To prove this one should construct the dilute gas of objects, which we do in the most nontrivial example of dyons.

6.1. Classical solutions i.e. solutions of the equation

\[
D_\mu F_{\mu\nu} = 0
\]

(84)

can be written in the (anti)selfdual case in the form of the so-called 'tHooft ansatz [57] or in the most general form of Atiya–Drinfeld–Hitchin–Manin (ADHM) [58].

In the first (simpler and less general) case one has

\[
A^a_\mu = -\frac{1}{g} \bar{\eta}^a_{\mu\nu} \partial_\nu \ln W, \tag{85}
\]

where \( \bar{\eta}^a_{\mu\nu} \) is the 'tHooft symbol,

\[
\bar{\eta}^a_{\mu\nu} = \epsilon_{\mu\nu}, \mu, \nu = 1, 2, 3 \quad \text{and} \quad \delta_{a\nu}, \mu = 4 \quad \text{or} \quad -\delta_{a\mu}, \nu = 4, \tag{86}
\]

and \( W \) satisfies equation \( \partial^2 W = 0 \), with a particular solution

\[
W = 1 + \sum_{n=1}^{N} \frac{\rho_n^2}{(x - x^{(n)})^2}. \tag{87}
\]

Here \( \rho_n, x^{(n)}_\mu, n = 1, \ldots N \) are real parameters. In the simplest case, \( N = 1 \), one has the instanton solution [59] with size \( \rho_n = \rho \) and position \( x^{(1)}_\mu \). For \( \rho_n \) finite and \( N \) arbitrary (85),(87) gives a multiinstanton solution with topological charge \( Q = N \). In particular for \( N \to \infty, x^{(n)}_4 = nb, \ x^{(n)}_i = r_i \), one gets the Harrington–Shepard periodic instanton [60], for which \( A^a_\mu \) periodically depends on (Euclidean) time \( x_4 \equiv t \).

A specific feature of instantons is their finite size: fields \( F_{\mu\nu} \) fall off at large distances from the center as \( x^{-4} \). This is very different from the case of magnetic monopole, where fields decay as \( x^{-2} \).
Another class of solutions with these latter properties can be obtained from (85), (87) in the limit \( \rho_n = \rho \to \infty \).

The case of \( N = 1 \), when \( W = \frac{1}{(x-x^{(0)})^2} \), yields no solution, since it is a pure gauge.

The next case, \( N = 2 \), is gauge equivalent to the (anti)instanton with the position \( \frac{1}{2} (x^{(1)} + x^{(2)}) \) and size \( \frac{1}{2} |x^{(1)} - x^{(2)}| \).

We shall be interested in the case when \( N \to \infty, \rho_i = \rho \to \infty, x^{(n)}_4 = nb, \vec{x}^{(n)} \equiv \vec{R} \) and call this solution a dyon, since, as we demonstrate below, it has both electric and magnetic long-distance field.

In this case \( W \) can be written as

\[
W \equiv \sum_{n=-\infty}^{\infty} \frac{1}{(\vec{x} - \vec{R})^2 + (x_4 - nb)^2},
\]  
(88)

and one can rewrite (88) using variables

\[
\gamma|\vec{x} - \vec{R}| \equiv r, \quad x_4 \gamma \equiv t, \quad \gamma = 2\pi/b
\]  
(89)

as

\[
W = \frac{1}{2r} \frac{shr}{chr - cost}.
\]  
(90)

From (85) one obtains vector–potentials

\[
A_{ia} = \epsilon_{aik} n_k (\frac{1}{r} - cthr + \frac{shr}{chr - cost}) - \frac{\delta_{ia}sint}{chr - cost},
\]  
(91)

\[
A_{4a} = n_a (\frac{1}{r} - cthr + \frac{shr}{chr - cost}),
\]  
(92)

with

\[
\vec{n} = (\vec{x} - \vec{R})/|\vec{x} - \vec{R}|.
\]

One can notice, that the dyonic field in this (singular, or ’tHooft) gauge is now long ranged in spatial coordinates

\[
A_{\mu a} \sim \frac{1}{r}, \quad F_{\mu \nu} \sim \frac{1}{r^2}
\]  
(93)

and periodic in ”time” \( t \).
Even more similarity to the magnetic monopole field can be seen, when one makes the (singular) gauge transformation [61]

\[ \tilde{A}_\mu = U^+(A_\mu + \frac{i}{g} \partial_\mu)U, \quad U = \exp\left(\frac{i \tau^\mu n}{2} \theta\right), \quad \text{(94)} \]

with

\[ \tan \theta = W_4 \left[ \frac{W_r + W_r}{r} \right]^{-1}, \quad W_\mu \equiv \partial_\mu W. \]

In this gauge, sometimes called Rossi gauge, \( \tilde{A}_\mu \) looks like Prasad–Sommerfield solution [62]:

\[ \tilde{A}_{ia} = f(r)e_{iba}n_b, \quad f(r) = \frac{1}{gr}(1 - \frac{r}{shr}), \quad \text{(95)} \]

\[ \tilde{A}_{4a} = \varphi(r)n_a, \quad \varphi(r) = \frac{1}{gr}(rch - 1). \quad \text{(96)} \]

Note that \( \tilde{A}_{\mu a} \) does not depend on time, it describes a static dyonic solution, since it has both (color)electric and (color)magnetic field:

\[ E_{ka} = B_{ka} = \delta_{ak}(-f' - f/r) + n_a n_k(f' - f/r + gf^2). \quad \text{(97)} \]

One may further gauge rotate \( E_{ka}, B_{ka} \) to the quasiabelian gauge, where the only long–ranged component is along 3 axis:

\[ E'_{k3} = B'_{k3}(r \to \infty) \sim -\frac{1}{gr^2}n_k. \quad \text{(98)} \]

Eq.(98) justifies our use of the name dyon for the solution and demonstrates its clear similarity to the magnetic monopole. Note also, that \( \tilde{A}_{4a} \) (96) is tending to a constant at spacial infinity, like the Higgs field component of the 'tHooft–Polyakov monopole [63].

The total action of the dyon is calculated from (95), (96) or (91), (92) to be

\[ S = \frac{1}{2} \int d^3r \int_0^T dt (B_{ak}^2 + E_{ak}^2) = \frac{8\pi^2}{g^2b} T, \quad \text{(99)} \]

where \( T \) is the length of the ”dyonic string”, in terms of the number \( N \) of centers in (87) it is \( T = b(N - 1) \). For the given \( N \) one also has

\[ S(N) = \frac{8\pi^2}{g^2} Q(N), \quad Q(N) = N - 1. \quad \text{(100)} \]
6.2. This short section is devoted to another type of classical solutions – those depending on boundary conditions and defined in finite volume. Here we consider torons and instantons on torus [64], which obey the twisted boundary conditions (b.c.) in the box \( 0 \leq x_\mu \leq a_\mu \). Periodic b.c. are imposed modulo gauge transformation (twisted b.c.)

\[
A_\lambda(x_\mu = a_\mu) = \Omega_\mu \left[ A_\lambda(x_\mu = 0) - i \frac{\partial}{\partial x_\lambda} \right] \Omega_\mu^+, \tag{101}
\]

To ensure selfconsistency of \( A_\lambda \) on the lines, four functions \( \Omega_\mu (\mu = 1, ..., 4) \) should satisfy conditions

\[
\Omega_1(x_2 = a_2)\Omega_2(x_1 = 0) = \Omega_2(x_1 = a_1)\Omega_1(x_2 = 0)Z_{12}, \tag{102}
\]

and analogous conditions for \( 1, 2 \to i, j \), where \( Z_{12} \in \mathbb{Z}(N) \) is the center of the group \( SU(N) \),

\[
Z_{\mu\nu} = \exp\left( 2\pi i \frac{n_{\mu\nu}}{N} \right), \quad n_{\mu\nu} = -n_{\nu\mu}. \tag{103}
\]

Here \( n_{\mu\nu} \) are integers not depending on coordinates \( x_\mu \).

The twisted solutions \( A_\mu \) (101) contribute to the topological charge

\[
\frac{g^2}{16\pi^2} \int_{|x_\mu| \leq a_\mu} Tr(F_{\mu\nu} \tilde{F}_{\mu\nu}) d^4 x = \nu - \frac{\chi}{N}, \tag{104}
\]

where \( \nu \) – an integer and \( \chi = \frac{1}{4} n_{\mu\nu} \tilde{n}_{\mu\nu} = n_{12}n_{34} + n_{13}n_{42} + n_{14}n_{23} \).

The action in the box is bounded from below

\[
\frac{1}{2} \int Tr(F_{\mu\nu} F_{\mu\nu}) d^4 x \geq \frac{8\pi^2}{g^2} |\nu - \frac{\chi}{N}|. \tag{105}
\]

Consider e.g. the case of \( n_{34} = -n_{12} = 1 \), all other \( n_{\mu\nu} = 0, \chi = 1 \). Then one has

\[
A_\lambda(x) = -\frac{\omega}{g} \sum_{\mu} \frac{\alpha_{\mu\lambda} x_\mu}{a_\mu a_\lambda}, \quad \alpha_{\mu\lambda} = -\alpha_{\lambda\mu}, \tag{106}
\]

where \( \alpha_{12} = \frac{1}{2Nk}, \quad \alpha_{34} = \frac{1}{2Nl}, \quad k + l = N, \)

\[
\omega = 2\pi \cdot diag(l, ..., l, -k, ..., -k), \tag{107}
\]

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the matrix $\omega$ has $k$ elements equal to $l$ and $l$ elements equal to $(-k)$, and a condition is imposed $a_1a_2(a_3a_4)^{-1} = \frac{l}{k} = \frac{N-k}{k}$. As a simple example take SU(2) and cubic box, then $k = l = 1$, $\omega = 2\pi\tau_3$ and

$$A_\lambda(x) = -\frac{\tau_3}{a^2} \frac{\pi}{2g} \sum_\mu \bar{\alpha}_{\mu\lambda} x_\mu, \quad \bar{\alpha}_{12} = \bar{\alpha}_{34} = 1.$$  

(108)

This solution is selfdual and the following relation holds

$$Tr F_{\mu\nu} F_{\mu\nu} = Tr F_{\mu\nu} \tilde{F}_{\mu\nu} = \frac{16\pi^2}{\prod_{\mu} a_\mu N g^2}.$$  

(109)

From (109) one can see that toron (108) is a particular case of a selfdual solution with constant field $\tilde{F}_{\mu\nu}$

$$A_\mu(x) = \tilde{F}_{\mu\nu} x_\nu \frac{\tau_3}{2},$$  

(110)

where the amplitude of constant field $\tilde{F}_{\mu\nu}$ is quantized. For constant (anti)selfdual field the analysis of Leutwyler [65] tells that such solutions are stable with respect to quantum fluctuations.

The flux through the Wilson loop for the solution (108) is the planes (12) or (34) is

$$P \exp(i g \int_C A_\mu dx_\mu) = \exp(-i\pi \frac{S}{a^2 \tau_3}),$$  

(111)

where $S$ is the area, bounded by the contour $C$.

As we shall see in the next section the flux equal to $\pi$ for $S = a^2$ is a property very important for confinement. Another interesting property of torons, not shared by any other solutions, that its action is proportional to $1/N_c g^2$ and therefore stays constant for large $N_c$, where $g^2 = g_0^2/N_c$. We come back to this property in conclusions.

Another type of twisted solutions are twisted instantons [64]. These are solutions with topological charge $Q$ (104) and $\nu$ integer non zero.

Those solutions have been seen on the lattice [66], and the profile (distribution of $tr F_{\mu\nu}^2(x)$) is very close to that of the usual instanton.

Unfortunately the analytic form of twisted instanton is still unknown; the top charge was found to be $1/2$ [66], and the extrapolated string tension is probably nonzero. These two facts are not accidental – in the next section we show that halfinteger top. charge ensures a flux of $\pi$ and that in turn
may lead to confinement.

6.3. In this subsection we compute elementary flux inside Wilson loop for (multi)instanton, dyon and twisted instanton and connect properties of elementary flux to confinement in the gas of classical solutions [67].

Consider a circular Wilson loop in the plane (12) and take $A_\mu$ in the form of the 'tHooft ansatz (5.2), where $N$ is fixed and $x_i^{(n)} = 0, x_4^{(n)} = nb$. In this way one can study the case of instanton ($N = 1$), periodic Harrington–Shepard instanton ($N \to \infty, \rho_i = \rho$ fixed), multiinstanton ($N$ finite, $\rho_i$ finite) and dyon ($\rho_i = \rho \to \infty, N \to \infty$).

When the radius of the loop $R$ is much larger than the core of the solution (i.e. $R \gg \rho$ for (multi)instantons or $R \gg b/2\pi$ for dyon), the Wilson loop is

$$W(C_R) = \exp\left( i\tau_3 \pi \frac{R W_r}{W} \right) \equiv \exp(i\tau_3 \cdot \text{flux}), \quad (112)$$

where $W_r = \frac{\partial W}{\partial |\vec{x}|=R}$, $|\vec{x}| \equiv r$.

Now for (multi)instanton one has for $R \gg \rho$

$$\frac{R W_r}{W} \big|_{r=R} = - \sum \frac{\rho_i^2 2R}{1 + \sum \frac{\rho_i^2}{R^2 + (x_4 - nb)^2}} \to 0. \quad (113)$$

In nonsingular gauge one would obtain for (multi)instanton the flux $2\pi$ [68], in all gauges one has, that

$$W(C_R) = 1, \quad \text{(multi)instantons.} \quad (114)$$

Consider now the case of dyons, which amounts to tending $\rho_n \to \infty$ in $W_r$ in (112).

One can use the form (90) to obtain for dyon

$$\frac{R W_r}{W} = -1; \quad \text{flux} = -\pi, \quad W(C_R) = -1. \quad (115)$$

It is amusing to consider also the intermediate case of so–called $\tau$–monopoles [69], when $\rho_n \to \infty$, but $N$ is fixed, so the length of the chain $L = Nb$ is finite. One can use (88) to find two limiting cases:

$$R \gg L, \quad \frac{R W_r}{W} = -2, \quad \text{flux} = -2\pi; \quad W(C_R) = 1; \quad (116)$$

$$R \gg \rho, \quad \frac{R W_r}{W} \to 0, \quad \text{flux} \to 0; \quad W(C_R) \to 1. \quad (117)$$

$$R \gg b/2\pi, \quad \frac{R W_r}{W} \to 0, \quad \text{flux} \to 0; \quad W(C_R) \to 1. \quad (118)$$

$$R \gg \rho, \quad \frac{R W_r}{W} \to 0, \quad \text{flux} \to 0; \quad W(C_R) \to 1. \quad (119)$$
\[ R \ll L, \quad \frac{R W}{W} = -1, \quad \text{flux} = -\pi; \quad W(C_R) = -1. \quad (117) \]

Thus only \( \tau \)-monopoles long enough, i.e. almost dyons, may ensure non-trivial Wilson loop, \( W(C_R) \neq 1 \).

To connect flux values (114)-(117) to confinement one can use model consideration of stochastic distribution of fluxes in the dilute gas, as was done in [68, 69]. More generally the picture of stochastic fluxes was formulated in the model of stochastic confinement [70] and checked on the lattice in [71].

We shall come back to the model of stochastic confinement in the next Section, and now shall use simple arguments from [68, 69].

Indeed, consider a thin 3d layer above and below the Wilson loop, of thickness \( l \ll R \), and assume that it is filled with gas of (multi)instantons or dyons. If 3d density of the gas is \( \nu \), so that average number of objects is \( \bar{n} = \nu S l \), then Poisson distribution gives probability of having \( n \) objects in the layer around the plane of Wilson loop is

\[ w(n) = e^{-\bar{n}} \frac{(\bar{n})^n}{n!}. \quad (118) \]

If contribution to the Wilson loop of one object is \( \lambda \), \( \lambda = +1 \) and \( -1 \) for instantons and dyons respectively, then the total contribution is

\[ < W(C_R) > = \sum_n e^{-\bar{n}} \frac{\lambda^n \bar{n}^n}{n!} = e^{-\bar{n}(1-\lambda)} = e^{-\sigma S}, \quad (119) \]

where

\[ \sigma = (1 - \lambda) \nu l. \quad (120) \]

Thus for instantons (\( \lambda = 1 \)) one obtains zero string tension in agreement with other calculations [55], while for dyons (\( \lambda = -1 \)), confinement is present according to the model. Let us stress, which features are important for this conclusion of confinement:

i) the flux = \( \pi \), so that \( W = -1 \) for one dyon,

ii) stochastic distribution of fluxes, which enables us to use Poisson (or similar) distribution,

iii) existence of finite thickness, i.e. of finite screening length, so that objects more distant than \( l \) completely screen each other and do not contribute to the Wilson loop.

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Notice that point iii) is necessary for area law, otherwise (for \( l = R \)) one obtains \( \sigma \) growing with \( R \), i.e. superconfinement.

The same reasoning is applicable to torons and twisted instantons [66]. Indeed from (111) one can see, that their elementary flux is equal to \((-\pi)\). Therefore if one divides all volume into a set of twisted cubic cells, and ensures stochasticity of fluxes in the cells one should have the same result (119) with confinement present.

Torons [72] and twisted instantons [66] have been studied from the point of view of confinement both analytically [72] and on the lattice [66]. For torons the requirement of stochasticity is difficult to implement, since boundary conditions of adjacent cells should ensure continuity of \( A_\mu(x) \), and this introduces ordering in the fluxes, and confinement may be lost. In case of twisted instantons with \( Q = 1/2 \) [66] the field is essentially nonzero around the center of instanton, and b.c. are much less essential. The authors of [66] note a possibility of nonzero extrapolated value for the string tension when the box size is increased beyond 1.2. \( \text{fm} \). It is still unclear what would be result when the twisted b.c. are imposed only on the internal boundaries.

**6.4.** Below we concentrate on dyons, as most probable candidates for classical confining configurations. One must study properties of dyon gas and show that interaction in this gas is weak enough to ensure validity of the dilute gas approximation. As it is usual, one assumes the superposition ansatz

\[
A_\mu = \sum_{i=1}^{N_+} A_\mu^{\pm(i)}(x) + \sum_{i=1}^{N_-} A_\mu^{-(i)}(x),
\]

where \( N_+, N_- \) are numbers of dyons and antidyons respectively. To make the QCD vacuum \( O(4) \) invariant, one should take any direction of the dyonic line, characterized by the unit vector \( \omega^{(i)}_\mu \) and position vector \( R^{(i)}_\mu \), so that

\[
A^{(i)}_\mu(x) = \Omega_i^+ (LA)_\mu(r, t)\Omega_i.
\]

Here \( \Omega_i \) is the color orientation matrix and \( L \) is the \( O(4) \) (Lorentz) rotation matrix, while \( r \) and \( t \) are

\[
r = [(x - R^{(i)})^2 - ((x - R^{(i)})_\mu \omega^{(i)}_\mu)^2]^{1/2},
\]

\[
t = (x - R^{(i)})_\mu \omega^{(i)}_\mu.
\]
It is now nontrivial, in which gauge take solution $A_\mu$ in (122), e.g. one may take singular gauge solution (91) or time–independent one (95),(96). The sum (121) is not obtained by gauge transformation from one case to another. Indeed it appears, that the form (95),(96) is not suitable, since the action for the sum (121) in this case diverges (see [67] for details).

The form (91) is adoptable in this sense and we consider it in more detail.

Since solutions fall off fast enough, cf. Eq. (93), the interaction between dyons, defined as $S_{int}$,

$$S(A) = \sum_{i=1}^{N_+ + N_-} S_i(A^{(i)}) + S_{int}$$

is Coulomb–like at large distances, e.g. for parallel dyon lines one has for two dyons

$$S_{int}(R^{(1)}, R^{(2)}) = \frac{\text{const} \cdot T}{|R^{(1)} - R^{(2)}|},$$

where $T = Nb$ is the length of dyon lines. A similar estimate can be obtained for nonparallel lines.

As was discussed in the previous section, the crucial point for the appearing of the area law of Wilson loop is the phenomenon of screening. To check this, let us consider the field of tightly correlated pair $d\bar{d}$. When distance between $d$ and $\bar{d}$ is zero, the resulting vector potential is obtained using the superposition ansatz (121) and $d$ vector potentials (91)-(92) for $d$ and the corresponding one for $\bar{d}$, which differs from (91) by the sign of the last term and the total sign in (92):

$$A_{ia}(d\bar{d}) = 2 e^{aik} n_k \left( \frac{1}{r} - cthr + \frac{shr}{chr - cost} \right),$$

$$A_{4a}(d\bar{d}) = 0.$$  \hspace{1cm} (127)

At long range one has

$$A_{ia}(d\bar{d}) = 2e^{aik} \frac{1}{gr} + O(e^{-r}).$$  \hspace{1cm} (128)

Calculation of $B_{ka}$ amounts to insertion in (98) $f = \frac{2}{gr}$, which immediately yields:

$$B_{ia}(d\bar{d}) = O(e^{-r}), \quad E_{ia} \equiv O(e^{-r}).$$  \hspace{1cm} (130)
Thus fields of $d$ and $\bar{d}$ completely screen each other at large distances; note, that this is purely nonabelian effect, since the cancellation is due to the quadratic term in $f$ in (98).

Now take distance between $d$ and $\bar{d}$ equal to $\vec{\rho} = \vec{R}^{(1)} - \vec{R}^{(2)}$ and distance between observation point $\vec{x}$ and center of $d\bar{d}$ equal to $\vec{r} = \vec{x} - \frac{\vec{R}^{(1)} + \vec{R}^{(2)}}{2}$; assume that $r \gg \rho$. Then the field of $d\bar{d}$ (averaged over direction of $\vec{\rho}$) is of the order

$$B_k, E_k = O(\rho^2/r^4).$$

Hence contribution of the distant correlated pair of $d\bar{d}$ is unessential, and indeed in calculation of the Wilson loop one can take into account the distances to the plane of the loop smaller than the correlation length $l$, which is actually the screening length.

From the dimensional arguments – we have the only parameter in our Coulomb–like system – the average distance between the nearest neighbors $\nu^{-1/3}$, therefore one has

$$l = c\nu^{-1/3},$$

where $c$ is some numerical constant, and $\nu$ is the 3d density of the dyon gas.

Hence one expects the string tension in the dyonic gas to be of the order

$$\sigma = c\nu^{2/3}.$$ 

Numerical calculations of $< W(C) >$ for the dyonic gas have been done in [73], but the density used was still much below that which is necessary for the observation of the screening; work is now in progress.

Summarizing this Section, let us discuss perspectives of the classical solutions reported above as candidates for confining configurations. Only two solutions, dyons and twisted instantons, yield the suitable flux, equal to $\pi$, through the Wilson loop, therefore we discuss them separately. Dyons can be represented as a coherent chain of instantons of large radius and correlated orientation of color field. When one goes from the instanton gas to these coherent chains, the action changes a little, but the entropy decreases significantly, and possibly confinement occurs. To estimate the advantage or disadvantage of dyonic configurations from the point of view of the minimum of the vacuum free energy, one must perform complicated computations, which are planned in the nearest future.
As to the twisted instantons, they require an internal lattice structure in the vacuum, which may violate the Lorentz invariance in some field correlators.

After all, the problem is solved, as in dyonic case, by the calculation of the free energy of the vacuum: in the nature there should come into existence that vacuum structure, which ensures the minimal free energy. Lattice Monte-Carlo calculations satisfy this principle of minimal free energy (up to the finite size effects) and predict the confining vacuum with special nonperturbative configurations responsible for confinement.

It is possible that those configurations are dyons, it is probable that they are not at all classical, but it is not excluded, that there exist unknown classical solutions, which ensure confinement after all.

7 Topology and stochasticity of confinement

In the previous chapter we have used the stochasticity of fluxes to obtain area law for dyons (magnetic monopoles). We start this chapter giving more rigorous treatment of this stochasticity and comparing it to lattice data.

For abelian theory magnetic flux through the loop $C$ is defined unambiguously through the Wilson loop

$$W(C) = \exp i e \oint_C A_\mu dx_\mu = \exp i e \int S \vec{H} d\vec{\sigma},$$

and the magnetic flux is

$$\mu = e \int S \vec{H} d\vec{\sigma}.$$  \hfill (135)

For SU($N$) theory the flux can be defined analogously [70] (we omit the word "magnetic", since it depends on the orientation of the loop).

Consider eigenvalues of the Wilson operator (note absence of trace in its definition):

$$U(C) = P \exp i g \oint_C A_\mu dx_\mu \equiv \exp i \hat{\alpha}(C).$$

The eigenvalues of the unitary operator $U(C)$ are equal to $\exp i \hat{\alpha}(C)$, where $\hat{\alpha}(C)$ is a diagonal matrix $N \times N$, depending on $A_\mu$.

In electrodynamics $\alpha_n(C_{12})$ are additive for the contour $C_{12}$, consisting of two closed contours $C_1$ and $C_2$:

$$\alpha_n(C_{12}) = \alpha_n(C_1) + \alpha_n(C_2).$$ \hfill (137)
In SU($N$) theory this is not so in general. Consider now the spectral density $\rho_c(\alpha)$, i.e. averaged with the weight $\exp(-S_0(A))$ the probability of the flux $\alpha(C)$:

$$\rho_c(\alpha) = \int \frac{DA_\mu e^{-S_0(A)}}{\int DA_\mu e^{-S(A)}} \frac{1}{N} \sum_{m=1}^{N} \delta_{2\pi}(\alpha - \alpha_m(A_\mu, C)),$$  \hspace{1cm} (138)

where $S_0(A)$ is the standard action of the $SU(N_c)$ theory.

Now any averaged Wilson operator over contour $C$, and also those for contour $C^n$, i.e. contour $C$, followed $n$ times, can be calculated with the help of $\rho_c(\alpha)$:

$$< W(C^n) > = \int_{-\pi}^{\pi} d\alpha e^{i\alpha \rho_c(\alpha)}. \hspace{1cm} (139)$$

Assume, that there is confinement in the system, i.e. the area law holds both for contour $C$ and $C^n$:

$$< W(C^n) > = \exp(-k_n S). \hspace{1cm} (140)$$

Then the following equality holds [70,71]:

$$\rho^C_S(\alpha) = \int_{-\pi}^{\pi} d\alpha_1... \int_{-\pi}^{\pi} d\alpha_n \rho^C_{S_1}(\alpha_1)...\rho^C_{S_n}(\alpha_n) \delta_{2\pi}(\alpha - \alpha_1 - ... - \alpha_n), \hspace{1cm} (141)$$

where the contour $C$ with the area $S$ is made of contours $C_i$ splitting the area $S$ into pieces $S_i$. The proof [70,71] can be done in both directions: from (139),(140) to (141) and back. Sometimes this statement is formulated as a theorem [70]: The necessary and sufficient condition of confinement is the additivity of random fluxes.

The randomness is seen in (141), which has the form of convolution as it should be for the product of probabilities for independent events. The additivity is evident from the argument of the $\delta$-function in (141).

The density $\rho_c(\alpha)$ was measured in lattice calculations [71], and it was found, that $\rho_c(\alpha)$ indeed satisfies (141) and approximately coincides with the $\rho^{d=2}(\alpha)$ – density for $d = 2$ chromodynamics, if one renormalizes properly the charge. For the $d = 2$ case $\rho_c(\alpha)$ is also known explicitly and satisfies (141) exactly [71]. In that case confinement exists for trivial reasons.

Let us now consider the nonabelian Stokes theorem [11], which for the operator (136) looks like

$$U(C) = P \exp ig \int_S d\sigma_{\mu\nu}(u) F_{\mu\nu}(u, x_0), \hspace{1cm} (142)$$
and take into account, that under gauge transformation $V(x)$ it transforms as

$$U(C) \rightarrow V^+(x_0)U(C)V(x_0), \quad F_{\mu\nu}(u, x_0) \rightarrow V^+(x_0)F_{\mu\nu}V(x_0). \quad (143)$$

Since $U(C)$ can be brought to the form $U(C) = \exp i\hat{\alpha}$ with $\hat{\alpha}$ diagonal by some unitary transformation, one can deduce, that this is some gauge transformation $V(x_0)$, and, moreover, this is the same, which makes $\int d\sigma\mu\nu F_{\mu\nu}(u, x_0)$ diagonal. Thus one can define the flux $\mu$ similarly to (135):

$$\hat{\mu} = \text{diag}\left\{igV^+(x_0) \int_{S} d\sigma\mu\nu(u)F_{\mu\nu}(u, x_0)V(x_0)\right\}. \quad (144)$$

Note, that dependence on $x_0$ presents in $U(C)$, cancels in $W(C) = trU(C)$.

The additivity of fluxes is seen in (144) explicitly.

Now consider the statistical independence of fluxes, which obtain, when one divides the surface $S$ into pieces $S_1, ..., S_n$. Using the cluster expansion theorem [12] and discussion in Section 3, one can conclude, that necessary and sufficient condition for this is the finite correlation length $T_g$, which appears in correlation functions (cumulants) $\ll F(1) ... F(n) \gg$. In this case, when pieces $S_k, k = 1, ..., n$, are all much larger in size than $T_g$, then different pieces become statistical independent. Thus our consideration in the framework of the field correlators in Section 3 is in clear agreement with the idea of stochastic confinement [70,71]. The MVC in addition contains the quantitative method to calculate all observables in terms of given local correlators, which is absent in the stochastic confinement model [70,71].

The appearance of new physical quantity – $T_g$ and cumulants is a further development of the idea of stochastic vacuum, which gives an exact quantitative characteristics of randomness.

When size of contours is of the order of $T_g$, the fluxes are no more random, and area law at such distances disappears – there is no area law at small distances, as is explained in Section 3.

The lattice measurements in [71] show that $\rho_c(\alpha)$ is strongly peaked around $\alpha = 0$ for small contours, when there is no area law, which corresponds to the perturbative regime.

Instead for large contours the measured $\rho_c(\alpha)$ are rather isotropic.

This fact one can compare with our result for fluxes for instantons and dyons. From our definition of fluxes (136) dyon in the plane of contour $C$
corresponds to (cf. Eq. (112))

\[ \hat{\alpha} = \begin{pmatrix} \pi & 0 \\ 0 & -\pi \end{pmatrix}, \tag{145} \]

and dyon with the center off the plane has smaller eigenvalues \( \alpha_m \). It is clear, that instantons with flux zero cannot bring about isotropic distribution of fluxes, while having maximal flux (\( \pm \pi \)) are most effective in creating the isotropic \( \rho_c(\alpha) \), when one integrates over all dyons in the layer above and below the plane.

It is instructive now to study the question of fluxes for adjoint Wilson loop and in general for Wilson loops of higher representations.

One can keep the definition (136) also in this case, but \( A_\mu \) and \( \hat{\alpha}(C) \) should be expressed through generators of given representation:

\[ A_\mu = \sum_a A_{\mu a} T^a, \quad tr T^a T^b = \frac{1}{2} \delta_{ab}. \tag{146} \]

Thus \( \hat{\alpha}(C) \) for the Wilson loop in adjoint representation is a matrix \((N_c^2 - 1) \times (N_c^2 - 1)\). E.g. for SU(2) \((T^a)_{bc} = \frac{i}{2} \epsilon_{abc}\). To understand how stochastic vacuum model works for the adjoint representation, let us take as an example the flux of one dyon and calculate \( \hat{\alpha}_{adj}(C) \). Repeating our discussion, preceding eq. (112) for large loops in the (12) plane, one concludes, that again only color index \( a = 3 \) contribute, and one has

\[ \hat{\alpha}_{adj}(C) = \pi \cdot diag(T^3) \lim \left( \frac{RW_r}{W} \right). \tag{147} \]

Since the last factor for dyon is \( \lim \frac{RW_r}{W} = -1 \), and \( diag(T^3) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \),

one has finally

\[ \hat{\alpha}_{adj}(C) = \begin{pmatrix} -\pi & \pi \\ \pi & -\pi \end{pmatrix}. \tag{148} \]

Thus our conclusion of the elementary flux equal to \( \pi \) holds true also in the adjoint representation (and all higher representations), which gives an argument for confinement of adjoint charges on the same ground as for fundamental charges.
One has
\[
< W_{adj} >= \frac{1}{\mathcal{N}_C^2-1} \text{tr}_{adj} e^{i\hat{\alpha}_{adj}(C)} = -1,
\] (149)
as well as \(< \hat{W}_{fund} >= -1\).

So far we have discussed stochasticity of the vacuum from the point of view of fluxes and conclude, that it shows up as random distribution of fluxes. In Section 3 the vacuum stochasticity was formulated in the language of field correlators. Through the AP method one can connect the latter with the distribution of AP magnetic monopole currents. (in the $U(1)$ theory an exact connection holds even without AP). One may wonder, why magnetic monopoles or dyons are needed to maintain the stochastic picture of the vacuum?

To answer this question we start with the Abelian theory. Without magnetic monopoles Bianchi identities $\text{div} \vec{H} = 0$ are operating, requiring, that all magnetic field strength lines are closed.

This introduces strong ordering in the distribution of magnetic field, and no stochastic picture emerges.

As a result, confinement is not present in the system, as can be seen from eq.(22). In presence of magnetic monopoles the magnetic lines can start and end at any place, where monopole is present, and one can have a really stochastic distribution. As we discussed it in Section 3, the nonabelian dynamics can mimick the effect of monopoles due to triple correlators $< E_i E_j B_k >$ and ensure in this way the stochastic distribution of fields.

So magnetic monopoles in Abelian theory, dyons in gluodynamics create disorder in the system.

The same situation occurs in other spin and lattice systems, e.g. in the planar Heisenberg model the Berezinsky–Costerlitz–Thouless vortices create disorder and and master the phase transition into the high-temperature phase [74] (for details see also [1]).

All that is a manifestation of a general principle [2]:

**Topologically nontrivial field configurations are responsible for creation of disorder and they drive the phase transition order–disorder.**

From the QCD point of view the "ordered phase" is the perturbative vacuum of QCD with long distance correlations ($D_1(x) \sim \frac{1}{x}$) and flux distribution $\rho_c(\alpha)$ centered at zero, while the "disordered phase" is the real QCD vacuum with short correlation length $T_g$ and with random fluxes; there is no real
phase transition in continuum: the phases coexist on two different scales of distances (or moments). In the lattice version of $U(1)$ theory there are indeed two phases: weak coupling phase, corresponding to the usual QED, and strong coupling phase with magnetic monopoles – lattice artefacts – driving the phase transition.

The dyon is a continuum example of topologically nontrivial configuration. In singular (‘tHooft’s) gauge dyon has multiinstanton topological number, proportional to its length.

Dyon saturates the triple correlator $< E_i E_j B_k >$ and may be a source of randomness of field distribution.

However the ultimate answer to the question about the nature of confining configurations is still missing. The analysis of the dyonic vacuum as a model of the QCD vacuum is not yet completed, and it is possible that the topologically nontrivial configurations responsible for confinement are dyons, or some other not known solutions or else purely quantum fluctuations.

We conclude this Section with the discussion of a possible connection between confinement and the Anderson localization [75].

At the base of the similarity between these two phenomena lies the field stochasticity in the vacuum (medium), where quark (electron) propagates. The similarity however ends up just here.

Namely, for an electron one can discuss its individual Green’s function, which always (for any density of defects) decays exponentially with distance. [There exists, however, a special correlator, e.g. the direct current conductivity $\sigma_{dc}$, which vanishes for localized states (for large density of localized defects [76]) and is nonzero for the delocalized states].

In the case of a quark in the confining vacuum its Green’s function (more precise: the gauge–invariant Green’s function of the $q\bar{q}$ system averaged over vacuum configurations) corresponds to the linear potential, i.e. it behaves as $G(r) \sim exp(-r^{3/2})$, where $r$ is the distance between $q$ and $\bar{q}$. Thus the quark Green’s function decays faster than any exponent, in contrast to the Green’s function of an electron in the medium, always decaying exponentially. This property of vacuum Green’s functions was coined by the author [77] the superlocalization. If the average potential $\bar{V}$, acting on a quark, had been finite, then quark at some high energy could be freed and get to the detector.

The essence of the superlocalization is exactly the fact, that the averaged potential $\bar{V}$ grows with distance without limits and therefore quarks are confined at any energy –this is the absolute confinement.
It is interesting to follow the mechanism, how the unbounded growth of $\bar{V}$ occurs. To this end consider, as we did at the beginning of Section 3, a nonrelativistic quark, moving in the $x, t$ plane, while the heavy antiquark is fixed at the origin.

According to the quantum mechanical textbooks [78] the quark Green’s function is proportional to the phase integral, and using the Fock–Schwinger gauge, one can write

$$G(X, T) = \langle \exp ig \int_{T_0}^T A_4(x, t) dt \rangle \approx$$

$$\approx 1 - \frac{g^2}{2} \int_{T_0}^T dt \int_{T_0}^T dt' \int_{X_0}^X du \int_{X_0}^X du' < E_1(u, t)E_1(u', t') > \approx$$

$$\approx 1 - \bar{V} T \quad (150)$$

Stochasticity of vacuum fields implies the finite correlation length $T_g$ for the correlator $< E_1(u, t)E_1(u', t') >$, i.e. according to (16) one has

$$< E_1(u, t)E_1(u', t') > = D(u - u', t - t') + ... \quad (151)$$

and for large $T$ and $X$ we obtain

$$\bar{V} \approx \text{const}|X|, \quad |X| \to \infty \quad (152)$$

Thus the linear growth of $\bar{V}$ is a consequence of random distribution of field strength $\vec{E}(u, t)$ and of the fact, that $\bar{V}$ is a result of the averaging of vector-potentials $A_\mu$, which are connected to $F_{\mu\nu}$ by an additional integral. This extra integration causes the linear growth of $\bar{V}$, and from the physical point of view this means the accumulation of fluctuations of the field $F_{\mu\nu}$ on the whole distance $X$ from the quark to the antiquark.

This is the essence of the superlocalization phenomenon, which still has no analogue in the physics of condensed matter.

8 Conclusions

In this review we have looked at confinement from different sides and described the mechanism of this phenomenon in the language of field correlators, using more phenomenological language of dual superconductivity –
effective classical equations of the Ginzburg–Landau type, in the language of the stochastic flux distributions and finally we have studied classical configurations which may be responsible for confinement.

All the way long we have stressed that confinement is the string formation between color charges, the string mostly consisting of the longitudinal color electric field.

Let us try now to combine different descriptions of confinement, given above in the review, and show in a simple example how the string looks like.

To this end we use the simple picture of the nonrelativistic quark and the heavy antiquark at the distance $X$ between them, discussed at the end of the last Section. From the point of view of field correlators, confinement – the string formation – is the consequence of the fact, that there exists the correlation length $T_g$, such that fields inside this length are coherent and those outside of this length are random. This is shown in Fig.10.1 (left part), where the strip of size $T_g$ is indicated in the plane (14) (one could take instead any other plane, i.e. the plane (12) or (13)). Inside this strip the field is directed mostly in the same way (i.e. as on the string axis), but outside of the strip directions are random. The correlation length $T_g$ characterizes the string thickness (if the plane (12) or (14) is chosen). Thus $T_g$ plays the double role, it gives the coherence length, where the string is created, and beyond which – stochastic vacuum field, existing also before quark and antiquark have been inserted in the vacuum.

Let us look at the same construction from the point of view of the dual superconductivity. Then one obtains the picture shown in Fig.10.2. Here an arrow depicts the monopole current $\tilde{j}_\mu$, which is caused by the color electric field $E_x$ of the string in accordance with the dual London’s equation $\text{rot} \tilde{j} = m^2 \tilde{E}$. The effect of this current is the squeezing of the string field, which prevents the field flux lines from diverging into the space, and as a result $E_x$ decays exponentially away from the string axis $Ox$ like $exp(-m\sqrt{y^2 + z^2})$. Thus $m$ defines the string thickness and one can conclude that $m \sim 1/T_g$. And indeed Eq.(22) confirms this conclusion.

Let us turn now to the flux distribution and to the stochastic vacuum model, Fig. 10.3. In this case the strip, corresponding to the string in Fig. 10.1 can be divided into pieces $S_1, S_2, ...$ of the size $d^2$, such that fluxes inside each piece are coherent and equal to e.g. $\pm \pi$ for the case of dyons, but two neighboring pieces are noncoherent – their fluxes are mutually random. The string thickness is now built up due to the size $d$ of the piece $S_n$, containing
a coherent flux. If the surface $S_n$ is penetrated by a monopole or a dyon, then $d$ coincides with the monopole or dyon size.

To clarify this point let us find the minimal size of the loop $R$ where the dyon flux is equal to the asymptotic value of $(-\pi)$. To this end we use (115) and insert there (90), and obtain the result, that for $R \gg b/2\pi \equiv \gamma^{-1}$ the flux is equal to $(-\pi)$ with exponential accuracy; hence the size of the dyon flux is equal to $\gamma^{-1}$ and this should be the size $d$ of the piece $S_n$.

From the point of view of field correlators $d$ should coincide with $T_g$, therefore the string thickness is of the order of the typical dyon (or monopole) size (or an other classical solutions).

Hence, all our pictures represented in Fig. 10.1- 10.3 can be combined in one generalized mechanism of the string formation, which is based on the existence of coherent field domains of the size $T_g$, and beyond that size the fields are independent and random.

A question arises: who manages this structure of QCD vacuum and why in the case of QCD and gluodynamics the vacuum is made up this way, while in the case of QED and the Weinberg–Salam theory, the nonperturbative configurations are probably suppressed and the vacuum structure is different. To be able to answer this question is also to answer the question of phase transition mechanism for the temperature deconfinement, which was observed on the lattice [79]. This topic requires a separate review paper, since the amount of information accumulated here by now is very large. We shall confine ourselves to only few remarks on this point.

Firstly, the density of the (nonperturbative) vacuum energy one can connect with the help of the scale anomaly theorem with the magnitude of the nonperturbative gluonic condensate [8]

$$
\varepsilon = \frac{\beta(\alpha_s)}{16\alpha_s} < F_{\mu\nu}^a(0) F_{\mu\nu}^a(0) >
$$

(153)

For small $\alpha_s$ the function $\beta(\alpha_s)$ is negative – in contrast to QED, and if it keeps the sign in all effective region of $\alpha_s$, then one can deduce that the nonperturbative vacuum shift (153) is advantageous, since it diminishes the vacuum energy (and also the free energy for small temperatures).

This conclusion can be considered as an intuitive idea why the nonperturbative vacuum in QCD is advantageous and comes into existence, while QCD – not advantageous and is not realized.

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Secondly, let us briefly discuss the phase transition with increasing temperature in QCD, referring the reader to lattice calculations [79] and original papers [81] for details. The main criterium which defines the vacuum structure preferred at a given temperature, is the criterium of the minimum of the free energy (which is a corollary of the second law of thermodynamics). In the confining phase for $T > 0$ the free energy consists of the term (153) and of the contribution of hadronic excitations (glueballs, mesons and baryons), which slowly grows up to $T \sim 150 MeV$. Note that the gluonic condensate contains both colorelectric and colormagnetic fields, but only the first ones have to do with confinement in the proper sense of this term.

The deconfinement phase, realized at $T > T_c$, usually was identified as the phase with perturbative vacuum, where quarks and gluons in the lowest order in $g$ are free [82]. However, from the point of view of the minimum of free energy it is advantageous to keep in the vacuum colormagnetic fields and the corresponding part of the condensate (153) since quarks in this case stay practically free, and one significantly gains in energy – around one half of the amount in (153). This is the ”magnetic confinement” phase [81]. Calculations in [81] yield $T_c$ in good agreement with lattice data for number of flavours $n_f = 0, 2, 4$.

The main prediction of the ”magnetic confinement” is the area law for $T > T_c$ of the spacial Wilson loops [83], and the phenomenon of the ”hadronic screening lengths”, i.e. the existence of hadronic spectra for $T > T_c$ in the Green’s functions with evolution along space directions [81], which agrees well with the lattice measurements [79].

Hence the picture of the phase transition [81] into the ”magnetic confinement” phase, supported by computations, seems to be well founded. In this picture at $T > T_c$ disappear colorelectric correlators, more explicitly, the correlators of the type of of $D(x)$, contributing to the string tension.

What happens then with effective or real magnetic monopoles and dyons? In the AP method at $T > T_c$ the monopole density strongly decreases [35], which can be understood as an active annihilation or a close pairing of monopoles and antimonopoles. The same can be said about pairs of dyons and antidyons. Thus the deconfinement phase of color charges can be associated with the confinement phase of monopoles (or dyons). However, the ”magnetic confinement” phenomenon imposes definite requirements on the vacuum structure at $T > T_c$. E.g. there should exist magnetic monopole (dyonic) currents along the 4-th (euclidean time) axis –i.e. static or periodic

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monopoles (dyons). Such currents can confine in spacial planes (what is observed on lattices [83]), but do not participate in the usual confinement (i.e. in the temporal planes \((i4), i = 1, 2, 3\)). These points will be elucidated in separate publications.

We also had no space to discuss an important question of the connection between confinement and spontaneous breaking of chiral symmetry and \(U_A(1)\), where magnetic monopoles (dyons) can play an important role [84].

We always held above that confinement is the property not only of QCD (with quarks present in the vacuum), but also of gluodynamics (without quarks). This conclusion follows from numerous lattice data (see e.g. [10]), and also from computations supporting the dual Meissner effect as a basis of confinement discussed in the review, where quarks do not play an important role.

On the other hand there are not any calculation or experimental data which support a key role of quarks in the confinement mechanism. For this reason we have not discussed above the model of V.N.Gribov (interesting by itself) and the reader is referred to original papers [85].

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Figure captions

Fig. 1. Potential between static quarks in the triplet representation of SU(3), computed in [10] on the lattice $32^4$. The solid line – the fit of the form $C/R + \sigma R + \text{const.}$ The potential and distance $R$ are measured in lattice units $a$, equal to 0.055 fm for $\beta = 6.9$. Dynamical quarks are absent.

Fig. 2. The same potential as in Fig. 1, but with dynamical quarks taken into account in two versions (staggered fermions – upper part, $a \simeq 0.11$ fm, Wilson fermions – lower part, $a \simeq 0.16$ fm); calculations of Heller et al (second entry of [10]).

Fig. 3. The same potential as in Fig. 1, but for quarks in the sextet (a) and octet (b) representation. Broken line – the triplet potential of Fig. 1 multiplied by the ratio of Casimir operators, equal to 2.25 for the octet and 2.5 for the sextet.

Fig. 4. Correlators $D_{11}(x) = D + D_1 + x^2 \frac{\partial D_1}{\partial x}$ (lower set of points) and $D_{\perp}(x) = D + D_1$ (upper set of points) as functions of distance $x$. Crosses correspond to $\beta = 5.8$, diamonds – to $\beta = 5.9$ and squares – to $\beta = 6.0$. Solid lines are best fits in the form of independent exponents for $D(x)$ and $D_1(x)$. Computations from [13].

Fig. 5. Distribution of the parallel colorelectric field $E_{11}$ as a function of distance $x_{\perp}$ to the string axis. Measurements from [19] are made at different distances $x_{11}$ from the string end: filled circle - $x_{11} = 3a$, diamond – $x_{11} = 5a$ and triangle – $x_{11} = 7a$ ($a$ is the lattice unit). Solid line - Gaussian fit, dashed line – calculation in [19] with the help of $D(x)$ taken from [13].

Fig. 6. The same distribution $E_{11}$, as in Fig. 5, for SU(2) gluodynamics measured in [24] on the lattice $24^4$ for $\beta = 2.7$, $x_{11} = 5a$, (the length of the string is 10a) – empty circles, compared to the $B_{11}$ distribution in the Abrikosov string – solid line. The growth of the solid line at small $x_1$, is unphysical and is due to violation of approximations made in (47).

Fig. 7. Lattice measurements [25] of the penetration length $\delta \equiv \lambda$ and the coherence length $\xi$ as functions $\beta = 2N_c/g^2$ for AP configurations in
SU(2) gluodynamics (upper two figures) and for SU(3) (the lower figure). The values of $\delta$ and $\xi$ are defined by comparison of field distribution and AP monopole currents with the solution of the Ginzburg–Landau equations.

Fig. 8. The string tension measured in [43] for all AP configurations (squares) and separately for AP monopoles (empty circles) and ”photons” (filled circles) as functions of $\beta = 4/g^2$ in SU(2) gluodynamics.

Fig. 9. Effective potential of the AP monopole field $\varphi$, defined according to [48], measured in [26] for two values of $\beta$ in SU(2) gluodynamics, corresponding to confinement (Fig.9a) and deconfinement (Fig.9b).

Fig. 10. The picture of string formation between a nonrelativistic quark and a heavy antiquark, is illustrated in three different approaches, discussed in the text.

(1) – in the formalism of field correlators

(2) – in the formalism of dual superconductivity

(3) – in the picture of the stochastic flux distribution.
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