STRUCTURE THEORY FOR ONE CLASS OF
LOCALLY FINITE LIE ALGEBRAS.

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Abstract

In this paper I consider locally finite Lie algebras of characteristic zero satisfying the condition that for every finite number of elements $x_1, x_2, \ldots, x_k$ of such an algebra $L$ there is finite-dimensional subalgebra $A$ which contains these elements and $L(\text{ad}A)^n \subset A$ for some integer $n$. For such algebras I prove several structure theorems that can be regarded as generalizations of the classical structure theorems of the finite-dimensional Lie algebras theory.

INTRODUCTION. The subject of this article is similar to that of Refs.[1, 2, 3]. I consider locally finite Lie algebras of characteristic zero. A Lie algebra is called locally finite if every its finite subset is contained in a finite-dimensional subalgebra. We will study representations $R = (M, L)$ of such algebras. A representation $R = (M, L)$ is a homomorphism of a Lie algebra $L$ into the algebra of linear transformations of a linear space $M$. We will assume that $R = (M, L)$ satisfies the following condition:

(1) for every $x \in L$ the linear transformation $x^R$ has the Fitting null component of a finite co-dimension. Here $x^R$ is the linear transformation that corresponds to $x \in L$ in the representation $R$.

For such linear transformation the following is true:

LEMMA 1. Let $A$ be a linear transformation of a linear space $M$ of an infinite dimension, $M$ be the Fitting null component of $A$, that is, the set of all $x \in M$ with $xA^m = 0$ for some integer $m$. If $M/M_0$ has finite dimension, then $M$ is a direct sum $M_0 \oplus M_1$, where $M_1$ is a finite-dimensional space invariant under $A$, and the transformation induced by $A$ in $M_1$ is an automorphism.

Lemma 1 allows us to determine the trace $trA$ of $A$ as a trace of the linear transformation, which is induced by $A$ in the finite-dimensional space.

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Besides, given a representation \( R = (M, L) \) satisfying condition (1), this Lemma allows us to construct a decomposition
\[
M = M_\rho \oplus M_\sigma \oplus \ldots \oplus M_\tau \oplus M_0
\]
into weight spaces relative to a nilpotent subalgebra \( H \) of \( L \). This decomposition has the same properties as in the finite-dimensional case and is used in the proof of

**THEOREM 3.** Let \( L \) be a locally finite Lie algebra of characteristic zero. Suppose \( L \) has a representation \( R = (M, L) \) in a vector space \( M \) satisfying the condition (1) and the associative span \( A \) of \( L \) in the representation \( R = (M, L) \) is locally finite. If the kernel of the representation \( R \) is locally solvable and \( tr(x^R)^2 = 0 \) for every \( x \in L' \), then \( L \) is locally solvable.

Theorem 3 can be considered as a generalization of the Cartan’s criterion for solvability.

Subsequent results are obtained for Lie algebras and their representations that satisfy the following conditions:

(2) representations \( R = (M, L) \): for every finite-dimensional subalgebra \( A \) of \( L \) there is such an integer \( n \) that \( MA^n \) has finite dimension.

(3) Lie algebra \( L \): for every finite set \( x_1, x_2, \ldots, x_k \in L \) there is a finite-dimensional subalgebra \( A \) which contains these elements and for which \( L(adA)^n \subset A \) for some integer \( n \).

**COROLLARY.** An algebra \( L \) satisfying the condition (3) is locally solvable if and only if \( tr(adx)^2 = 0 \) for every \( x \in L' \).

Since the intersection of a finite number of finite-codimensional subspaces has finite codimension, the trace can be defined simultaneously for any finite number of transformations for algebras that satisfy condition (2). All properties of the usual finite-dimensional trace are true in this case. Therefore, for the representation \( R = (M, L) \) satisfying the condition (2) we can define a trace form
\[
f(a, b) = tr a^R b^R, \ a, b \in L.
\]
In particular, if \( L \) satisfies the condition (3), we obtain the form
\[
K(a, b) = tr(ada)(adb)
\]
It is natural to name this form the Killing form of \( L \).

Lie algebra \( L \) is said to be semi-simple if its locally solvable ideal equals to 0, that is, if \( L \) has no non-zero locally solvable ideals.
THEOREM 4. Let $L$ be a locally finite semi-simple Lie algebra of characteristic 0, and $R = (M, L)$ be an arbitrary faithful representation satisfying the condition (2). Then the trace form $f(a, b)$ is non-degenerate.

If $L$ satisfies the condition (3) and the Killing form $K(a, b)$ is non-degenerate, then $L$ is semi-simple.

This theorem can be regarded as a generalization of the Cartan’s criterion for semi-simplicity. Finally, the following theorem generalizes the structure theorem.

THEOREM 6 (Structure Theorem). Let $L$ be a semi-simple Lie algebra that satisfies condition (3). Then $L$ is a subdirect sum of a set of finite-dimensional simple algebras.

1. WEIGHT SPACES. PROOF OF LEMMA 1. Let $N$ be a finite-dimensional subspace of $M$ such that $M = N \oplus M_0$ and let $e_1, e_2, \ldots, e_n$ be a basis of the $N$. We have $e_iA = \sum_{j=1}^{n} a_{ij}e_j + m_i$, where $m_i \in M_0$. For any $m_i$ there is an integer number $n_i$ such that $m_iA^{n_i} = 0$. Let $P$ denote the linear subspace spanned by $m_i, m_iA, \ldots, m_iA^{n_i-1}, i = 1, 2, \ldots, n$. It is clear that $P$ is invariant under $A$ and has a finite dimension. From the equality $e_iA = \sum_{j=1}^{n} a_{ij}e_j + m_i$ it follows that $NA \subseteq P + N$. Hence $K = P + N$ is invariant under $A$. We have $M = K + M_0$. Since $K$ is finite-dimensional, it can be represented as $K = K_1 \oplus K_0$, where $K_1$ and $K_0$ are, respectively, the Fitting one and the Fitting null components of $K$ relative to the transformation induced by $A$ in $K$. Then $M = K_1 + K_0 + M_0$. Since $K_0 \subseteq M_0$, then $M = K_1 + M_0$. We shall show now that this sum is direct. Since $K$ is finite-dimensional there is $t$ such that $yA^t = 0$ for any $y \in K \cap M_0$. On the other hand, there exists $s$ such that $K_1 = KA^s = KA^{s+1} = \cdots$. Let $r = \max(s, t)$. Then $K_1 = KA^r$ and $yA^r = 0$ for any $y \in K \cap M_0$. Now let $x \in K_1 \cap M_0$. Then from the equality $K_1 = KA^r$ it follows that $x = yA^r$ for some $y \in K$. On the other hand, since $x \in K_1 \cap M_0$, it holds $xA^r = 0$. But then $0 = xA^r = yA^{2r}$. Hence $y \in K \cap M_0$ and, consequently, $yA^r = 0$. But then $x = yA^r$ is equal to zero and $K_1 \cap M_0 = 0$. From the construction of $K_1$ it follows, that $A$ is a linear transformation acting in $K_1$ as an automorphism. Therefore, the only possibility is to put $M_1 = K_1$.

This lemma may be considered as a generalization of the well-known Fitting’s lemma. Let $M_0$ and $M_1$ are the Fitting null and the Fitting one.
components of $M$ relative to $A$, respectively.

As in [4], a linear transformation $A$ will be called algebraic, if every vector $x \in M$ is contained in a finite-dimensional subspace that is invariant under $A$.

LEMMA 2. If the Fitting null component $M_0$ of $M$ relative to $A$ has finite codimension, then $A$ is an algebraic linear transformation.

PROOF. By Lemma 1, $M = M_1 \oplus M_0$, where $M_1$ has finite dimension and $A$ acts in $M_1$ as an automorphism. Let $x \in M$. We shall show that the dimension of the smallest subspace that contains $x$ and is invariant relative to $A$ is finite. Let $x = y + z$, where $y \in M_1$ and $z \in M_0$. Since $M_1$ is invariant under $A$, $yA \in M_1$. On the other hand, $zA^m = 0$ for some integer $m$. Let $N$ be the subspace, which is generated by $M_1$, $z, zA, \ldots, zA^{m-1}$. It is clear that $x \in N$, $N$ is invariant under $A$, and $N$ has finite dimension. The lemma is proved.

Let the characteristic roots of $A$ be in the base field $\Phi$ and let $M_1 = M_\alpha \oplus M_\beta \oplus \cdots \oplus M_\gamma$ be the decomposition of $M_1$ into the weight spaces relative to $A$. Then $M = M_0 \oplus (\bigoplus \alpha \neq 0 M_\alpha)$. Also we have that all $M_\alpha$ with $\alpha \neq 0$ are of finite dimension. It is worth recalling that by definition $x \in M_\alpha$, if and only if $x(A - \alpha E)^m = 0$ for some integer $m$.

We will require some known results which I outline here for completeness. Let $A$ be an associative algebra, and $a \in A$. Let us consider the inner derivation $D_a : x \rightarrow x' = [x, a]$ in $A$. If we denote $x^{(k)} = (x^{(k-1)})', x^{(0)} = x$, then the following formulas hold:

$$xa^k = a^kx + \binom{k}{1} a^{k-1}x' + \binom{k}{2} a^{k-2}x'' + \cdots + x^{(k)}$$

$$a^kx = a^kx - \binom{k}{1} x'a^{k-1} + \binom{k}{2} x''a^{k-2} + \cdots + (-1)^k x^{(k)}$$

$$x\phi(a) = \phi(a)x + \phi_1(a)x' + \phi_2(a)x'' + \cdots + x^{(r)},$$

where $\phi(\lambda)$ is a polynomial of degree $r$ and $\phi_k(\lambda) = \phi^{(k)}(\lambda)/k!$.

LEMMA 3 [6]. Let $A, B$ be linear transformations in a vector space $M$ satisfying $B(adA)^u = 0$ for some integer $u$. Let $\mu(\lambda)$ be a polynomial and let $M_{\mu A} = \{x | x\mu(A)^m = 0 \text{ for some integer } m\}$. Then $M_{\mu A}$ is invariant under $B$.

PROOF. Let $x \in M_{\mu A}$ and suppose that $x\mu(A)^m = 0$. Putting $\phi(\lambda) = \mu(\lambda)^m$, we obtain $B\phi(A) = \phi(A)B + \phi_1(A)B' + \cdots + \phi_{u-1}(A)B^{(u-1)}$. Since
\[ \phi_0(\lambda) = \phi(\lambda), \phi_1(\lambda), \ldots, \phi_{u-1}(\lambda) \] are divisible by \( \mu(\lambda)^m \), \( x\phi_j(A) = 0, 0 \leq j \leq u-1 \). Therefore \( xB\phi(A) = 0 \) and \( xB \in M_{\mu A} \).

**COROLLARY.** If \( B(adA)^n = 0 \) then the weight spaces \( M_\alpha \) are invariant under \( B \).

Let \( R = (M, L) \) be a representation of a Lie algebra \( L \) in a vector space \( M \) of infinite dimension, satisfying the condition (1) and let characteristic roots of every \( A \in L \), lie in the base field.

We shall also assume that for every finite-dimensional subalgebra \( H \) of \( L \) there is such an integer \( m \) that \( (adH)^m = 0 \).

**THEOREM 1.** If \( H \) is a finite-dimensional subalgebra of \( L \), then \( M \) can be decomposed as \( \bigoplus_{\alpha \neq 0} M_\alpha \oplus M_0 \) where \( M_\alpha, \alpha \neq 0 \), are finite-dimensional weight spaces relative to \( L \) with the weights \( \alpha \), and \( M_0 \) is a weight space relative to \( H \) with the weight \( \alpha = 0 \). The dimension of \( M_0 \) is infinite.

**PROOF.** First let's show that for every \( x \in M \) the smallest subspace \( N \) that is invariant under \( H \) and contains \( x \) has finite dimension. Indeed (see also [4]), if \( B_1, B_2, \ldots, B_r \) is a basis of \( H \), then \( N \) is the linear span of the set of all elements of the form \( xB_1^{m_1}B_2^{m_2} \ldots B_r^{m_r}, i_1 \leq i_2 \leq \ldots \leq i_r \). Since all \( B_i \) are algebraic, there is only a finite set of linearly independent elements of a given form and all of them may be found among the elements \( xB_1^{m_1}B_2^{m_2} \ldots B_r^{m_r} \) for which \( m_j \leq s_j \), where \( s_j \) are integers and \( j = 1, 2, \ldots, k \). Suppose now that every element \( A \) of a finite subset \( F \subset H \) is locally nilpotent, that is, for \( A \) the following condition is satisfied: for any \( x \in M \) there exists \( m \) such that \( xA^m = 0 \). We shall show that this condition holds for every element of the subalgebra \( \{F\} \) generated by the set \( F \). Indeed, since \( H \) is nilpotent, \( F \) is contained in Jacobson radical of representation \((N, H)\) [5]. Hence \( \{F\} \) is contained in it. But this means that \( \{F\} \) consists of nilpotent relative to \( N \) transformations and hence the given condition holds for the elements of \( \{F\} \).

Since \( H \) is nilpotent, there exists a chain of ideals \( 0 \subset H_1 \subset H_2 \subset \ldots H_{n-1} \subset H_n = H \) such that \( \dim H_{i+1}/H_i = 1 \). Take an arbitrary element \( A \in H_1 \) and let \( M = \bigoplus_\alpha M_\alpha \) be a decomposition of \( M \) into the weight spaces relative to \( A^R \). From Lemma 3 it follows that all \( M_\alpha \) are invariant under \( L \). Besides there is just a finite number of the subspaces \( M_\alpha \), and \( \dim M_\alpha < \infty \) if \( \alpha \neq 0 \). Therefore \( M_1 = \bigoplus_{\alpha \neq 0} M_\alpha \) can be decomposed into a direct sum of a finite number of weight spaces relative to \( L \) (see [6, p.
43]). We recall that a map \( \alpha : A \rightarrow \alpha(A) \) of \( L \) into the base field \( \Phi \) is called the weight of \( M \) relative to \( L \) if there exists a nonzero element \( x \in M \) such that \( x(A^R - a(A))^m = 0 \) for all \( A \in L \). Here \( m \) is an integer which depends on \( x \) and \( A \). The set of elements (zero included) satisfying this condition forms the subspace that is called the weight subspace. It should be recalled that \( M_{0A} \) also is invariant under \( L \). Let \( B \) be an element of \( H_2 \setminus H_1 \) and \( M_{0A} = \bigoplus M_{\alpha B} \) be a decomposition of \( M_{0A} \) into the weight spaces relative to \( B^R \). The subspace \( M'_{1B} = \bigoplus_{\alpha B \neq 0} M_{\alpha B} \) has a finite dimension, is invariant under \( L \) and can be decomposed into a direct sum of a finite number of weight spaces relative to \( L \). If we add this decomposition to decomposition of \( M_{1A} \), we obtain that \( M = (\bigoplus M_{\alpha}) \oplus M'_{0B} \). \( A^R \) and \( B^R \) act in \( M'_{0B} \) as locally nilpotent transformations. Therefore, the subalgebra \( \{A, B\} \) consists of the locally nilpotent in \( M'_{0B} \) transformations [7]. Continuing in this way we obtain - by virtue of finite dimensionality of \( \mathcal{H} \) - the statement of the theorem.

If \( L \) is nilpotent we may combine Lie’s theorem with Theorem 1 to obtain the following

**THEOREM 2.** If \( L \) is finite-dimensional, then \( M \) is a direct sum of weight spaces \( M_\alpha \), and the matrices in the weight space \( M_\alpha, \alpha \neq 0 \) can be taken simultaneously in the form

\[
A_\alpha = \begin{pmatrix}
\alpha(A) & 0 & \cdots & 0 \\
* & \alpha(A) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
* & * & \cdots & \alpha(A)
\end{pmatrix}
\]

This theorem is proved in exactly the same way as in Ref.[6]. In a similar fashion we obtain

**COROLLARY.** The weights \( \alpha : A \rightarrow \alpha(A) \) are linear functions on \( L \) which vanish on \( L' \).

2. **CARTAN’S CRITERION.** Let \( L \) be a finite-dimensional Lie algebra, \( H \) be a nilpotent subalgebra of \( L \), \( R = (M, L) \) be a representation of \( L \) in a vector space \( M \) satisfying the condition (1).

**PROPOSITION 1.** Let

\[
M = M_\rho \oplus M_\sigma \cdots M_\tau \oplus M_0
\]
be the decompositions of $M$ and $L$ into weight spaces relative to $H$. (The existence of the first decomposition was proved in the previous section). Then $M_\rho L_\alpha \subset M_{\rho+\alpha}$ if $\rho + \alpha$ is the weight of $M$ relative to $H$; otherwise $M_\rho L_\alpha = 0$.

**PROOF.** For every $x \in M$ and $A, B \in L$ we have the equality $xA(B - \rho I - \alpha I) = x(B - \rho I)A + x(A(adB - \alpha I))$. If $x(B - \rho I)^m = 0$ and $A(adB - \alpha I)^n = 0$ then by repeating this equality we obtain $xA(B - \rho I - \alpha I)^{m+n+1} = 0$. Here $\rho = \rho(B)$, $\alpha = \alpha(B)$, and $I$ is an identity operator of $M$.

It is also true that $[L_\alpha, L_\beta] \subset \mathcal{L}_{\alpha + \beta}$ if $\alpha + \beta$ is a root of $L$ and $[L_\alpha, L_\beta] = 0$ otherwise (see [6, p. 64]).

Suppose now that $H$ is a Cartan subalgebra. Then $H = L_0$, the root module corresponding to the root 0. Also, we have $L' = [L, L] = \sum_{\alpha, \beta} [L_\alpha, L_\beta]$, where the sum is taken over all roots $\alpha, \beta$, and $L' \cap H = \sum_{\alpha} [L_\alpha, L_{-\alpha}]$, where the summation is taken over all $\alpha$ such that $-\alpha$ is also a root (see [6, p. 67]).

Let $A$ be a linear transformation of $M$ with the Fitting null component $M_0A$ of a finite codimension. Then, as noted in Introduction, $trA$ can be defined as the trace of the linear transformation induced by $A$ in the quotient space $M/M_0A$. It is easy to see that $trA$ is equal to the trace of the linear transformation induced by $A$ in the quotient space of $M$ by any invariant under $A$ subspace of a finite codimension contained in $M_0A$.

**LEMMA 4.** Let $\Phi$ be algebraically closed of characteristic 0. Under the assumptions of this section let $H$ be a Cartan subalgebra of $L$ and let $\alpha$ be a root such that $-\alpha$ is also a root. Let $e_\alpha \in L_\alpha$, $e_{-\alpha} \in L_{-\alpha}$, $h_\alpha = [e_\alpha, e_{-\alpha}]$. Then $r(h_\alpha)$ is a rational multiple of $\alpha(h_\alpha)$ for every weight $\rho$ of $H$ in $M$.

**PROOF.** Let $M_0^\rho = M_0 + \sum_i M_{i\alpha}$, $i = 0, \pm 1, \pm 2, \ldots$. Let us turn to the quotient space $\overline{M} = M/M_0^\rho$. If $M_0^\rho$ is invariant under $x \in L$, then the operator induced by $x$ in $\overline{M}$ is denoted as $x\overline{\rho}$. Consider functions of the form $\rho(h) + i\alpha(h), i = 0, \pm 1, \pm 2, \ldots$, which are weights, and form the subspace $N = \sum_i \overline{M}_{\rho+i\alpha}$ where $\overline{M}_{\rho+i\alpha} = M_{\rho+i\alpha} + M_0^\rho / M_0^\rho$ and the sum is taken over the corresponding weight spaces of the representation $R = (M, L)$. $N$ is invariant relative to $H$ and, by Proposition 1, it is also invariant relative to the linear transformations $e_{\overline{\alpha}}$ and $e_{-\overline{\alpha}}$. Thus, if $tr_N$ denotes the trace of an induced mapping in $N$, then $tr_N h_\overline{\alpha} = tr_N [e_{\overline{\alpha}}, e_{-\overline{\alpha}}] = 0$. On the other hand, the restriction of $h_\overline{\alpha}$ to $\overline{M}_\sigma = \sigma + M_0^\rho / M_0^\rho$ has the single characteristic root.
\[ \sigma(h) \]. Hence \( 0 = tr_N h_{h}^R = \sum n_{\rho + i\alpha}(\rho + i\alpha)(h) \) where \( n_{\rho + i\alpha} = \dim M_{\rho + i\alpha} \).

Thus we have \( (\sum n_{\rho + i\alpha})\rho(h) + (\sum i n_{\rho + i\alpha})\alpha(h) = 0 \). Since \( \sum n_{\rho + i\alpha} \) is a positive integer, this shows that \( \rho(h) \) is a rational multiple of \( \alpha(h) \).

**PROOF OF THEOREM 3.** Assume first that the base field \( \Phi \) is algebraically closed. It suffices to prove that \( C' \subset C \) for every finite-dimensional subalgebra \( C \) of the algebra \( L \). Hence we shall have that \( C \supset C' \supset C'' \supset C^{(k)} = 0 \). We therefore suppose that there exists a finite-dimensional subalgebra \( C \) such that \( C' = C \). Let \( H \) be a Cartan subalgebra of \( C \) and let

\[
M = M_{\rho} \oplus M_{\alpha} \oplus \cdots \oplus M_{r} \oplus M_{0}
\]

\[
C = C_{\alpha} \oplus C_{\beta} \oplus \cdots \oplus C_{\gamma} \oplus C_{0}.
\]

be the decomposition of \( M \) and \( C \) into weight spaces relative to \( H \). Then the formula \( H \cap C' = \sum [C_{\alpha}, C_{-\alpha}] \) implies that \( H = \sum [C_{\alpha}, C_{-\alpha}] \) summed on \( \alpha \) such that \(-\alpha \) is also a root. Choose such \( \alpha \), let \( e_{\alpha} \in C_{\alpha}, e_{-\alpha} \in C_{-\alpha} \), and consider the element \( h_{\alpha} = [e_{\alpha}, e_{-\alpha}] \). The formula \( H = \sum [C_{\alpha}, C_{-\alpha}] \) implies that every element of \( H \) is a sum of terms of the form \([e_{\alpha}, e_{-\alpha}] \). Let us turn to the quotient space \( M/M_0 \) and denote by \( h_{\alpha}^R \) the operator induced by \( h_{\alpha} \) in \( M/M_0 \). The restriction of \( h_{\alpha}^R \) to \( M_0 = M_{\rho} + M_0/M_0 \) has the single characteristic root \( \rho(h_{\alpha}) \). Hence the restriction of \( (h_{\alpha}^R)^2 \) has the single characteristic root \( \rho(h_{\alpha}) \). Let \( n_{\rho} \) be the dimension of \( M_{\rho} \). Then we have \( tr(h_{\alpha}^R)^2 = \sum n_{\rho}(\rho(h_{\alpha}))^2 \). On the other hand, \( tr(h_{\alpha}^R) = tr(h_{\alpha}^R) \). Thus \( \sum n_{\rho}(\rho(h_{\alpha}))^2 = 0 \) since \( tr(h_{\alpha}^R)^2 = 0 \). By the Lemma 4, \( \rho(h_{\alpha}) = r_{\rho} \alpha(h_{\alpha}) \), where \( r_{\rho} \) is rational. Hence \( \alpha(h_{\alpha})^2(\sum n_{\rho}r_{\rho}^2) = 0 \). Since \( n_{\rho} \) are positive integers, this implies that \( \alpha(h_{\alpha}) = 0 \) and \( \rho(h_{\alpha}) = 0 \). Since \( \rho \) are linear functions and every \( h \in H \) is a sum of elements of the form \( h_{\alpha}, h_{\beta}, \ldots, \), we see that \( \rho(h) = 0 \). Thus 0 is the only weight for \( M \), that is, we have \( M = M_0 \). If \( \alpha \) is a root, then the condition

\[
M_{\rho}C_{\alpha} = \begin{cases} 
0 & \text{if } \rho + \alpha \text{ is not a weight of } M \\
\subset M_{\rho+\alpha} & \text{if } \rho + \alpha \text{ is a weight}
\end{cases}
\]

implies that \( MC_{\alpha} = 0 \) for every \( \alpha \neq 0 \). Hence \( C_{\alpha} \oplus C_{\beta} \oplus \cdots \oplus C_{\gamma}, \alpha, \beta, \ldots, \gamma \neq 0 \) is contained in the kernel \( K \) of representation \((M,C)\). Hence \( C/K \) is a
homomorphic image of $H$. It follows that the $C/K$ is nilpotent. According to our assumptions the kernel $K$ is solvable, and it follows that $C$ is solvable which contradicts $C' = C$.

If the base field is not algebraically closed, then let $\Omega$ be its algebraic closure. Then $(M_\Omega, L_\Omega) = R_\Omega$ is the representation of $L_\Omega$ in $M_\Omega$ and $K_\Omega$ is the kernel of this representation if $K$ is the kernel of $R = (M, L)$. Since $K$ is locally solvable, $K_\Omega$ is locally solvable. Next we note that $tr(x^R)^2 = 0$ and $tr x^R y^R = tr y^R x^R$ imply that $tr x^R y^R = (1/2) tr(x^R y^R + y^R x^R) = (1/2)(tr(x^R + y^R)^2 - tr(x^R)^2 - tr(y^R)^2) = 0$. Hence if $x_i \in L$ and $\omega_i \in \Omega$, then $tr(\sum_i \omega_i x_i^R)^2 = \sum_i \omega_i x_i^R x_i^R = 0$. To prove that the condition (1) holds we use the fact that associative span $A$ of $L$ in the representation $R = (M, L)$ is locally finite. We need to show that $(\sum_{i=1}^m \omega_i x_i)^R$ has the Fitting null component of finite codimension for every $x_1, x_2, \ldots, x_m \in L$ and $\omega_1, \omega_2, \ldots, \omega_m \in \Omega$. Since $A$ is locally finite, the subalgebra $A$ in $A$ generated by $x_1, x_2, \ldots, x_m$ has a finite dimension. Therefore there exists an integer $u$ such that $y(x^R)^u = 0$. Here $x$ is an arbitrary linear combination of $x_1, x_2, \ldots, x_m$ with coefficients from the base field $\Phi$, and $y$ is an arbitrary element from the Fitting null component $M_{0X}$ of the linear transformation $x^R$.

Take $(u+1)^m$ elements of the form

$$k_{i_1} x_1 + m_{i_2} x_2 + n_{i_3} x_3 + \cdots + s_{i_m} x_m$$

where $i_1, i_2, i_3, \ldots, i_m$ receive their values $1, 2, 3, \ldots, u+1$ independently, and $k_1, k_2, \ldots, k_{u+1}, m_1, m_2, \ldots, m_{u+1}, n_1, n_2, \ldots, n_{u+1}, s_1, s_2, \ldots, s_{u+1}$ are pairwise different nonzero integers. The intersection $N$ of the Fitting null components of these elements has a finite codimension. Let us prove that for any $y \in N$ the equality $y(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_m x_m)^u = 0$ takes place for every $\omega_1, \omega_2, \ldots, \omega_m \in \Omega$. We have

$$(k_j x_1 + m_{i_2} x_2 + n_{i_3} x_3 + \cdots + s_{i_m} x_m)^u = 0, \quad j = 1, 2, \ldots, u + 1$$

for any $m_{i_2}, n_{i_3}, \ldots, s_{i_m}$ which are taken from the set of integers shown above. It follows immediately that

$$y P_0 + k_j y P_1 + k_j^2 y P_2 + \cdots + k_j^u y P_u = 0, \quad j = 1, 2, \ldots, u + 1$$

where $P_i = P_i(x_1, m_{i_2} x_2, n_{i_3} x_3, \ldots, s_{i_m} x_m)$ is the homogeneous component of $i$-th degree relative to $x_1$ of $(k_j x_1 + m_{i_2} x_2 + n_{i_3} x_3 + \cdots + s_{i_m} x_m)^u$. Since the
determinant of this system is Vandermonde’s determinant, it follows that $yP_0 = yP_1 = yP_2 = \cdots = yP_u = 0$. Next for $P_i$, $i = 0, 1, 2, \ldots, u$, we have $yP_i(x_1, m_1x_2, n_i x_3, \ldots, s_i x_m) = 0$, $l = 1, 2, \ldots, u - i + 1$. It immediately follows that

$$yP_0 + m_i yP_1 + m_i^2 yP_2 + \cdots + m_i^{u-i} yP_{i(u-i)} = 0, l = 1, 2, \ldots, u - i + 1$$

where $P_{ij}(x_1, x_2, n_i x_3, \ldots, s_i x_m)$ is a component of $P$, which is homogeneous of degree $i$ relative to $x_1$ and of degree $j$ relative to $x_2$. Since the determinant of this system is Vandermonde’s determinant, it follows that $yP_0 = yP_1 = \cdots = yP_{i(u-i)} = 0$.

Continuing in this way we obtain that $yP_{i_1i_2\cdots i_m} = 0$, $i_1, i_2, \ldots, i_m = 1, 2, \ldots, u$, $i_1 + i_2 + \cdots + i_m = u$, where $P_{i_1i_2\cdots i_m} = P_{i_1i_2\cdots i_m}(x_1, x_2, \ldots, x_m)$ are homogeneous polynomials of degree $i_1$ relative to $x_1$, of degree $i_2$ relative to $x_2$ and so on, and finally, of degree $i_m$ relative to $x_m$, which arise in computing of the power $(x_1 + x_2 + \cdots + x_m)^u$, and which are its multihomogeneous components. On the other hand,

$$y(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_m x_m)^u = \sum \omega_1^{i_1} \omega_2^{i_2} \cdots \omega_m^{i_m} yP_{i_1i_2\cdots i_m}(x_1, x_2, \ldots, x_m).$$

Hence $y(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_m x_m)^u = 0$.

Thus we have proved that the conditions of the theorem hold in $R_{\Omega} = (M_{\Omega}, L_{\Omega})$. The first part of the proof, therefore, implies that $L_{\Omega}$ is locally solvable.

**LEMMA 5.** Let $R = (M, L)$ and $A$ is a finite-dimensional subalgebra of $L$ such that $MA^n$ has finite dimension for some integer $n$. Then a subspace of all $x \in L$ for which $x A^n = 0$ has finite codimension.

**PROOF.** Let $z_1, z_2, \ldots, z_k$ is a basis of $A$. Then any product of $n$ elements of the basis transfers $M$ into finite-dimensional subspace of $M$. Thus the kernel of this product has finite codimension. Since there is only a finite number of different product of $n$ elements of the basis, the intersection of all kernels of such products has a finite codimension as well. The lemma is proved.

From Lemma 5 it follows that the associative algebra generated by $L$ in $R = (M, L)$ is locally finite. Therefore we can apply the results of [5] about the Jacobson radical of Lie algebra. Thus we obtain the following

**COROLLARY.** Let $L$ be a locally finite Lie algebra over a field of characteristic $0$. Suppose $L$ satisfies the condition (3). Then $L$ is locally solvable if and only if $tr(ada)^2 = 0$ for every $a \in L'$. 

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PROOF. The sufficiency of the condition is a consequence of Theorem 3, since the kernel of the adjoint representation is the centre. Conversely, assume $L$ is locally solvable. Then from [5] it follows that $ada, a \in L'$, belongs to radical $J_{adL^*}(adL)$. Hence $ada$ is a nilpotent linear transformation and $tr(ada)^2 = 0$.

3. A TRACE FORM. Let $R = (M, L)$ be a representation of a Lie algebra $L$ in a vector space $M$ which satisfies condition (2). Then we can define a trace form $f(a, b) = tr a^R b^R, a, b \in L$. The function $f(a, b)$ is evidently a symmetric bilinear form on $M$ with values in the base field $\Phi$. In particular, if $L$ satisfies (3), we obtain the Killing form $K(a, b) = tr(ada)(adb)$.

If $f(a, b)$ is the trace form defined by the representation $R = (M, L)$, then $f([a, c], b) + f([a, [b, c]]) = tr([a, c]^R b^R + a^R [b, c]^R) = tr([a^R, c^R] b^R + a^R[b^R, c^R]) = tr([a^R b^R, c^R]) = 0$.

As noted in the Introduction, we can calculate the trace simultaneously for every finite number of elements of $L$, since these elements can be considered as linear transformations acting on a common quotient space of $M$ of a finite dimension. Therefore, the last chain of equalities is correct.

A bilinear form $f(a, b)$ on $L$ that satisfies the condition $f([a, c], b) + f([a, [b, c]]) = 0$ is called an invariant form on $L$. Hence the trace form is invariant. We note next that if $f(a, b)$ is any symmetric invariant form on $L$, then the radical $L^\perp$ of the form - that is, the set of elements $z$ such that $f(a, z) = 0$ for all $a \in L$ - is an ideal. This is clear since $f(a, [z, b]) = -f([a, b], z) = 0$.

PROOF OF THEOREM 4. Let $f(a, b)$ be a trace form of $R = (M, L)$. Then $L^\perp$ is an ideal of $L$ and $f(a, a) = tr(a^R)^2 = 0$ for every $a \in L^\perp$. Hence $L^\perp$ is locally solvable by Theorem 3. Since $L$ is semi-simple, $L^\perp = 0$, and $f(a, b)$ is non-degenerate. Next suppose that the Killing form is non-degenerate. If $R(L)' \neq 0$ then by the Theorem 7 from [5] $R(L)' \subset J(L)$, Jacobson radical of $L$. This implies that for every $a \in R(L)'$, it is true that $ada \in J(adL^*)$. Hence for every $b \in L$ we have $ada \cdot adb \in J(adL^*)$. Since by Levitzki theorem $J(adL^*)$ is locally nilpotent ideal, then $ada \cdot adb$ is a nilpotent linear transformation and therefore $tr(ada \cdot adb) = 0$. This contradicts our assumption that the trace form $tr(ada \cdot adb)$ is non-degenerate. Therefore, it is required of $R(L)$ that $R(L)' = 0$.

Let $a \in R(L), b \in L$ and $N$ is a subspace of $L$ such that $L/N$ is finite-dimensional and $ada$ and $adb$ act in $N$ as nil transformations. Let us denote
$L = L/N$ and $R(L) = R(L) + N/N$ and choose a basis for $L$ such that the first vectors form a basis for $R(L)$. The matrices of linear transformations induced by $ada$ and $adb$ in $L$, respectively, are of the forms
\[
\begin{pmatrix} 0 & 0 \\ \ast & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ast & 0 \\ \ast & \ast \end{pmatrix}.
\]
This implies that $tr(ada)(adb) = 0$. Hence $R(L) \subset L^\perp$, and the Killing form is degenerate.

Using Killing form the following characterization of the locally solvable radical in the characteristic 0 case (cf. [6, p. 73]) can be obtained.

THEOREM 5. If a locally finite Lie algebra $L$ over a field of characteristic 0 satisfies the condition (3), then the locally solvable radical $R(L)$ of $L$ is the orthogonal complement $L^\perp$ of $L'$ relative to the Killing form $K(a,b)$.

PROOF. The algebra $B = L^\perp$ is an ideal. Further, if $b \in B'$, then $tr(adb)^2 = K(b,b) = 0$. The kernel of the representation $a \to ada$, $a \in B$, is abelian. Hence $B$ is locally solvable, by Corollary of theorem 2. Hence $B \subset R(L)$. Next, let $x \in R(L), a, b \in L$. Then $K([x,[a,b]]) = K([x,a],b)$. By [5] the element $[x,a]$ belongs to the Jacobson radical $J(L)$ of $L$. Consequently, $ad[x,a]$ belongs to $J(adL^*)$ and $ad[x,a] \cdot adb$ is nilpotent for every $b$. Hence $K([x,[a,b]]) = tr(ad[x,a] \cdot adb) = 0$. Thus $K([x,[a,b]]) = 0$ and $x \in L'^\perp$. Thus $R(L) \subset L'^\perp$ and so $R(L) = L'^\perp$.

Let $\Omega$ be an extension of the base field $\Phi$ of $L$. Then the Killing form $f_{\Omega}$ of $L_{\Omega}$ is obtained from the Killing form $f$ of $L$ by the extension. Therefore, $f_{\Omega}$ is non-degenerate if and only if $f$ is non-degenerate [8]. Consequently, $L_{\Omega}$ is semi-simple if and only if $L$ is semi-simple (see also [9]).

4. STRUCTURE OF SEMI-SIMPLE ALGEBRAS. We continue the consideration of locally finite Lie algebras $L$ which satisfy the condition (3).

LEMMA 6 (cf. [6, p. 29]). Let $A$ be a finite-dimensional subalgebra of $L$ such that $L(adA)^n \subset A$. Then $A^\omega = \bigcap_{k=1}^{\infty} A^k$ is an ideal of $L$.

PROOF. If $A^\omega = 0$ then the assertion of Lemma is trivial. Let $A^\omega \neq 0$. We have $[L, A^n] \subset L(adA)^n$. $A$ is finite-dimensional. Therefore, $A^\omega = A^m$ for some integer $m$. Thus $A^\omega = A^{n+m-1}$. Now we have $[L, A^\omega] = [L, A^{n+m-1}] \subset L(adA)^{n+m-1} \subset A(adA)^{m-1} = A^m = A^\omega$, which completes the proof.
Let $f(a, b)$ be any symmetric invariant bilinear form on $L$ and let $A$ be a subspace of $L$. Denote by $A^\perp$ the subspace of $L$ that consists of all elements $b \in L$ such that $f(a, b) = 0$ for all $a \in A$.

**Lemma 7.** If $A$ is a finite-dimensional subspace of $L$ such that $A \cap A^\perp = 0$, then $L = A \oplus A^\perp$.

**Proof.** Since $A \cap A^\perp = 0$, $A$ is a non-degenerate subspace. Let us show that $L = A + A^\perp$. Take any basis $a_1, a_2, \ldots, a_m$ in $A$ and let $c$ be any element of $L$. Find a decomposition $c = a + a^\perp$, where $a \in A$ and $a^\perp \in A^\perp$.

We will look for $a$ in the form $a = x_1a_1 + x_2a_2 + \cdots + x_m a_m$. Then $c$ will look like: $c = x_1a_1 + x_2a_2 + \cdots + x_m a_m + a^\perp$. From $f(a_i, a^\perp) = 0$ it follows that $f(a_i, c) = \sum_{k=1}^{m} x_k f(a_i, a_k), i = 1, 2, \ldots, m$. This system of equations has exactly one solution, since its determinant is the Gram’s determinant of the system $a_1, a_2, \ldots, a_m$. Since $A$ is a non-degenerate subspace, this determinant is non-zero. The vector $a = x_1a_1 + x_2a_2 + \cdots + x_m a_m$, where $x_k$ were just found, satisfies the conditions $f(a, c-a) = 0$. Indeed, $f(a_i, c-a) = f(a_i, c) - \sum_{k=1}^{m} x_k f(a_i, a_k) = 0$. From the equalities $f(a_i, c-a) = 0$, it follows that $c-a \in A^\perp$. To complete the proof it remains to put $a^\perp = c-a$.

**Proof of Theorem 6.** Let $A$ be a finite-dimensional subalgebra for which $L(adA)^n \subset A$. By Lemma 6, subalgebra $A^\omega$ is an ideal of $L$. By Theorem 4, the Killing form $K(a, b)$ is non-degenerate. Since $K(a, b)$ is invariant, $(A^\omega)^\perp$ is an ideal of $L$. Indeed, for every $a \in A^\omega$, every $b \in (A^\omega)^\perp$ and every $c \in L$ we have $K(a, [b, c]) = -K([a, c], b) = 0$. Let us prove that $A^\omega \cap (A^\omega)^\perp = 0$. If, on the contrary, $b_1, b_2 \in A^\omega \cap (A^\omega)^\perp$ and $a$ is any element of $L$, then $K([b_1, b_2], a) = -K(b_1, [a, b_2]) = 0$. Since $K(a, b)$ is non-degenerate, $[b_1, b_2] = 0$. Hence $A^\omega \cap (A^\omega)^\perp$ is an abelian ideal of $L$.

Since $L$ is semi-simple, $A^\omega \cap (A^\omega)^\perp = 0$. By Lemma 7 this implies that $L = A^\omega \oplus (A^\omega)^\perp$. Since $A^\omega$ and $(A^\omega)^\perp$ are direct summands, every ideal of $A^\omega$ or $(A^\omega)^\perp$ is an ideal of $L$. Therefore $A^\omega$ and $(A^\omega)^\perp$ are semi-simple subalgebras. Moreover, since $A^\omega$ is finite-dimensional, $A^\omega$ is a direct sum of ideals which are simple. Let us denote by $\Pi$ the set of all finite-dimensional simple ideals of $L$. This set is not empty; otherwise we have that $A^\omega = 0$ for every $x_1, x_2, \ldots, x_k \in L$. The previous implies that $L$ is a locally nilpotent algebra. But by condition of the Theorem, $L$ is semi-simple. Since $M^\omega = M$ for every finite-dimensional simple ideal $M$, $L = M \oplus M^\perp$. Now denote by $N$ the intersection of all $M^\perp$ where $M \in \Pi$. Let us prove that $N = 0$. Suppose
not, and let \( x_1, x_2, \cdots, x_k \) be a non-zero element of \( N \). Then by condition of the Theorem, there exists a finite-dimensional subalgebra \( A \), such that \( x_1, x_2, \cdots, x_k \in A \) and \( L(adA)^n \subset A \). Let \( B = A \cap N \). Then \( L(adB)^n \subset A \) and \( L(adB)^n \subset N \), since \( N \) is an ideal of \( L \). Consequently, \( L(adB)^n \subset B \) and \( B^\omega \) is ideal of \( L \). If \( B^\omega \neq 0 \), then \( B^\omega \) is a semi-simple ideal contained in \( N \). Next, \( B^\omega \) is a direct sum of simple ideals contained in \( N \) and, consequently, these ideals do not belong to \( \Pi \). But this contradicts the definition of \( \Pi \). Hence \( B^\omega = 0 \) for every finite subset \( x_1, x_2, \cdots, x_k \in N \) and \( N \) is a locally nilpotent ideal of \( L \). But \( L \) is a semi-simple Lie algebra. Hence \( N = 0 \). Then by Remac’s theorem \( L \) is a subdirect sum of simple finite-dimensional ideals \( M \cong L/M^\perp \). The proof of the theorem is complete.

It is well known that every derivation of a semi-simple finite-dimensional Lie algebra is inner. What about the infinite-dimensional case? Let \( L = \bigoplus \alpha L_\alpha \), where \( L_\alpha \) are finite-dimensional simple ideals of \( L \). In this case Stewart [10] proved the theorem that we give here for completeness.

**THEOREM 7.** Let \( L \) be a semi-simple Lie algebra and let \( L \) is a direct sum, \( L = \bigoplus \alpha L_\alpha \), where \( L_\alpha \) are finite-dimensional simple ideals of \( L \). Then

1) every derivation of \( L \) is an element of the complete direct sum \( \tilde{L} \) of \( L_\alpha \) and the algebra of all derivations \( D(L) \) of \( L \) is isomorphic to \( \tilde{L} \).

2) a derivation \( D \) is inner if and only if \( L_\alpha D = 0 \) for every \( L_\alpha \) with the exception of a finite number of it

3) \( L \) has outer derivations

4) every derivation of \( L \) is locally finite.

It follows from Theorem 7 that any semi-simple Lie algebra satisfying condition (3) is a subalgebra of the algebra of all derivations of such an algebra which is a direct sum of its simple finite dimensional ideals.

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