ASYMPTOTIC GEOMETRY OF THE MODULI SPACE OF PARABOLIC SL(2, C)-HIGGS BUNDLES

LAURA FREDRICKSON1, RAFe MAZZEO2, JAN SWOBODA3, AND HARTMUT WEISS4

ABSTRACT. Given a generic stable strongly parabolic SL(2, C)-Higgs bundle \((\mathcal{E}, \varphi)\), we describe the family of harmonic metrics \(h_t\) for the ray of Higgs bundles \((\mathcal{E}, t\varphi)\) for \(t \gg 0\) by perturbing from an explicitly constructed family of approximate solutions \(h_t^{\text{app}}\). We then describe the natural hyperKähler metric on \(\mathcal{M}\) by comparing it to a simpler “semiflat” hyperKähler metric. We prove that \(g_{L^2} - g_{sf} = O(e^{-\gamma t})\) along a generic ray, proving a version of Gaiotto-Moore-Neitzke’s conjecture. Our results extend to weakly parabolic SL(2, C)-Higgs bundles as well.

A centerpiece of this paper is our explicit description of the moduli space and its \(L^2\) metric for the case of the four-punctured sphere. We prove that the hyperKähler metric in this case is ALG and that its rate of exponential decay to the semiflat metric is the conjectured optimal one, \(\gamma = 4L\), where \(L\) is the length of the shortest geodesic on the base curve measured in the singular flat metric \(|\det \varphi|\).

1. INTRODUCTION

Various conjectures and results over the past decade have illuminated the large-scale asymptotic structure of the SU\((N)\) Hitchin moduli space associated to a Riemann surface \(C\), and in particular, its natural hyperKähler metric \(g_{L^2}\). A conjectural picture due to Gaiotto, Moore and Neitzke [GMN13] has provided stimulus for much of this work, and some part of that has now been established rigorously through the sharp asymptotic results obtained by one of us [Fre20], cf. also [DN19], following closely related work by the other three authors together with Witt [MSWW16, MSWW19]. The goal of the present paper is to extend these results to the parabolic setting, and to closely analyze the simplest special case: the moduli space of rank 2 bundles over the four-punctured sphere, sometimes called the ‘toy model’, which we show is a four-dimensional ALG gravitational instanton.

1DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403-1222, USA
2DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305-2125, USA
3MATHEMISCHES INSTITUT, RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG , IM NEUENHEIMER FELD 205, 69120 HEIDELBERG, GERMANY
4MATHEMATISCHES SEMINAR, CHRISTIAN-ALBRECHTS-UNIVERSITÄT KIEL, HEINRICH-HECHT-PLATZ 6, 24118 KIEL, GERMANY

E-mail addresses: 1lfredric@uoregon.edu, 2rmazzeo@stanford.edu, 3swoboda@mathi.uni-heidelberg.de, 4weiss@math.uni-kiel.de.
Using the formalism of spectral networks, Gaiotto, Moore and Neitzke [GMN13] introduced a new hyperKähler metric $g_{GMN}$ which they conjectured is the same as the Hitchin metric $g_{L^2}$. There is a simpler ‘semiflat’ metric $g_{sf}$ on the part of the Hitchin moduli space above the complement of the discriminant locus in the Hitchin base, and they predicted that $g_{GMN} - g_{sf}$ decays exponentially (with respect to the distance from a fixed compact set and in conical sectors away from the preimage of the discriminant locus). Their conjecture includes a far more detailed description of the asymptotic development of this difference of metrics as a series of terms with increasing exponential rates given by geodesic lengths of an underlying family of flat conic metrics on the curve $C$ and with coefficients given in terms of associated BPS states, which are Donaldson-Thomas invariants in this setting. (See Figure 1.1.)

While the full scope of this conjectural picture remains out of reach, there has been substantial progress. As a first step, the paper [MSWW16] uses gluing methods to construct ‘large’ elements of the SU(2) Hitchin moduli space away from the discriminant locus; this was extended to the SU($N$) case in [Fre18]. Unlike Hitchin’s original proof of existence of solutions, this new construction gives a precise description, up to exponentially small errors, of the actual fields $(\bar{\partial}_E, \varphi, h)$ which solve the Hitchin equations, and hence a parametrization of the ends of the Hitchin moduli space. This was then used in [MSWW19] to show that the difference $g_{L^2} - g_{sf}$ has an asymptotic development in a series of polynomially decaying terms. As a parametrization of the moduli space and its tangent bundle, these polynomial terms seem to be actually present, but a miraculous cancellation, first observed by Dumas and Neitzke [DN19] in a special case and later proved by one of us in full generality [Fre20], shows that this difference does in fact decay exponentially.

All of this was carried out for connections and Higgs fields without singularities. However, in the applications to mathematical physics it is necessary to consider fields admitting simple (or higher order) poles. Our first goal in the present work is to extend the results of [MSWW16, MSWW19, Fre20] to the setting of parabolic Higgs bundles, i.e., for fields with simple poles. The analysis proceeds in broad outline much as in the smooth case, but several new technical challenges must be faced. However, having carried this out, we are able to treat, in particular, a special case where the Hitchin moduli space is only four-dimensional, and where the discriminant locus lies in a compact set of the Hitchin base. This is the case where $C$ is the Riemann sphere $\mathbb{CP}^1$ with four punctures. It is possible here to write out elements of the Hitchin base explicitly, for example. The Hitchin moduli space in this case has been already examined in detail in [Hau01, Kon93, Nak96, Bla15] since so much can be done explicitly here. (This has also
sometimes been called the ‘toy model’.) It was conjectured in these papers, and independently also by Sergey Cherkis, that the moduli space in this setting is an ALG gravitational instanton. We learned of this conjecture in a lecture of Nigel Hitchin at the Newton Institute in August 2015. We prove this here, and also discuss how the family of ALG metrics obtained through this construction by varying the parabolic data fit into the recent classification of ALG metrics by Chen-Chen [CC21]. In particular, using an alternate description available in this four-dimensional case, we show that for strongly parabolic data, the exponential rate $g_{L^2} - g_{sf}$ equals the rate predicted in [GMN13].

§2 reviews general background material about parabolic Higgs bundles. We introduce the two main building blocks, namely limiting configurations and fiducial solutions, in §3. For each of these, we review the construction near simple zeros and then describe the generalization to strongly and weakly parabolic data successively. This leads immediately to the family of approximate solutions. Analysis of the linearized Hitchin equations is carried out first locally in §4, then globally in §5. The key new feature here, over what was done in [MSWW16], is the incorporation of ‘curvature bubbling’ at the parabolic points. (In addition, in distinction to [MSWW16], the analysis here is all done at the level of Hermitian metrics.) The deformation to exact solutions is in §6 and the proof of exponential decay of $g_{L^2} - g_{sf}$ is in §7. Finally, §8 contains the detailed and explicit analysis in the case of the sphere punctured at four points. The precise statements of results will be given inter alia.

Acknowledgments: We received substantial advice throughout this project and we wish to thank in particular Sergey Cherkis, Lorenzo Foscolo, Andy Neitzke (BPS predictions), Nigel Hitchin, André Oliveira (Hitchin section) and Claudio Meneses (chamber structure).

Each of the four of us were supported at various times by the GEAR network NSF grant DMS 1107452, 1107263, 1107367 “RNMS: Geometric Structures and Representation Varieties”; RM was supported by NSF DMS-1608223, and JS and HW were supported by DFG SPP 2026. JS was supported by a Heisenberg grant of the DFG and within the DFG RTG 2229 “Asymptotic invariants and limits of groups and spaces”. This work is supported by DFG under Germany’s Excellence Strategy EXC-2181/1 – 390900948 (the Heidelberg STRUCTURES Cluster of Excellence). This paper is based upon work supported by the National Science Foundation under Grant No. DMS-1440140 while we were all in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2019 semester.
2. Background

Fix a compact Riemann surface $C$ of genus $\geq 2$, with metric $g_C$, complex structure $I$, and Kähler form $\omega_C$. Let $K_C$ be the canonical line bundle. Fix also a complex vector bundle $E \to C$ of rank $N$ and degree $d$. Its determinant line bundle is denoted $\text{Det} E$. Let $\text{SL}(E)$ be the bundle of automorphisms of $E$ inducing the identity map on $\text{Det} E$ and $\text{sl}(E) = \text{End}_0 E$ the bundle of tracefree endomorphisms. Then $G_C = \Gamma(\text{SL}(E))$ is the group of complex gauge transformations with Lie algebra $\Gamma(\text{sl}(E))$.

We also choose on the complex line bundle $\text{Det} E$ a holomorphic structure $\partial$ and denote by $h_{\text{Det} E}$ the associated Hermitian-Einstein metric, i.e.,

$$F_D = -\sqrt{-1} \deg E \cdot \frac{2\pi \omega_C}{\text{vol}_{g_C}(C)} \text{Id}_{\text{Det} E},$$

where $D = D(\overline{\partial}_{\text{Det} E}, h_{\text{Det} E})$ is the Chern connection on this holomorphic Hermitian line bundle.

We now recall two different representations of the moduli space $\mathcal{M}$ of $\text{SL}(N, C)$-Higgs bundles associated to this fixed data. The central objects are stable Higgs pairs.

A Higgs pair (or more simply, a Higgs bundle) $(\overline{\partial} E, \varphi)$ consists of a holomorphic structure on $E$ and a section $\varphi \in \Omega^{1,0}(C, \text{End}_0 E)$ satisfying $\overline{\partial}_E \varphi = 0$, where $\overline{\partial}_E$ induces $\overline{\partial}_{\text{Det} E}$; this pair is called stable if any proper holomorphic subbundle $E' \subset E$ which satisfies $\varphi(E') \subset E' \otimes K_C$ satisfies $\mu(E') < \mu(E)$, where the slope $\mu(F)$ of any holomorphic bundle $F$ is defined to equal the quotient $\text{deg} F / \text{rk} F$. A pair is called polystable if it is a direct sum of stable pairs of lower rank and equal slope. The complex gauge group acts on such pairs by

$$g \cdot (\overline{\partial} E, \varphi) = (g^{-1} \circ \overline{\partial} E \circ g, g^{-1} \varphi g) \text{ for any } g \in G_C. \tag{2.1}$$

The Dolbeault representation of the Higgs bundle moduli space consists of the space of (polystable) pairs modulo the complex gauge action.

We shall be concerned, however, with the alternate representation of this moduli space as consisting of triples $(\overline{\partial} E, \varphi, h)$ which solve Hitchin’s equations up to unitary gauge equivalence. Here $(\overline{\partial} E, \varphi)$ is as before, and $h$ is a Hermitian metric on $E$ which induces
We write $D(\bar{\partial}_E,h)$ for the Chern connection associated to $\bar{\partial}_E$ and $h$ and $\varphi^* h \in \Omega^{0,1}(C, \text{End}_0 E)$ for the $h$-Hermitian adjoint. In addition, denote by $F^\perp_D$ the trace-free part of the curvature of $D$,

$$F^\perp_D(\bar{\partial}_E,h) = F_D(\bar{\partial}_E,h) + \sqrt{-1} \mu(E) \frac{2\pi \omega_C}{\text{vol}_C(C)} \text{Id}_E.$$ 

The triple $(\bar{\partial}_E, \varphi, h)$ satisfies Hitchin’s equations for $G = SU(N)$ if

$$F^\perp_D(\bar{\partial}_E,h) + [\varphi, \varphi^* h] = 0, \quad \text{and} \quad \bar{\partial}_E \varphi = 0.$$ 

The first of these two equations may be regarded in two ways: either as an equation for $\bar{\partial}_E$ (and hence the full connection $D$) and $\varphi$ if the metric $h$ is fixed, or else as an equation for the metric $h$ once the Higgs pair $(\bar{\partial}_E, \varphi)$ is fixed. In the latter case, $h$ is called the harmonic metric associated to $(\bar{\partial}_E, \varphi)$; we shall write out the equation for $h$ more explicitly below. One of the first main results in the theory, due in this formulation to Simpson, is that a Higgs bundle $(\bar{\partial}_E, \varphi)$ admits a harmonic metric if and only if $(\bar{\partial}_E, \varphi)$ is polystable.

It will also be convenient to refer to the first formulation. For this we fix a Hermitian metric $h_0$ on $E$ and consider pairs $(A, \Phi)$ where $A$ is an $h_0$-unitary connection and $\Phi \in \Omega^{1,0}(C, \text{End}_0 E)$. The Hitchin equations are now that $\bar{\partial}_A \Phi = 0$ and $F^\perp_A + [\Phi, \Phi^* h_0] = 0$.

The Hitchin moduli space $\mathcal{M}$ for $G = SU(N)$ consists of the space of pairs which solve these equations modulo $h_0$-unitary gauge equivalence. We will write $\mathcal{G} = \Gamma(SU(E))$ for the unitary gauge group with Lie algebra $\Gamma(\mathfrak{su}(E))$.

It is not difficult to pass back and forth between these two formulations. Indeed, given a triple $(\bar{\partial}_E, \varphi, h)$ where $h$ is the harmonic metric for this Higgs bundle, there is an $\text{SL}(E)$-valued $h_0$-Hermitian section $H$ such that $h(v,w) = h_0(Hv,w)$. Take the complex gauge transformation $g = H^{-1/2}$. Observe that in general, $(g \cdot h)(v,w) = h_0((g^{*h_0} H g)v,w)$; consequently, for our choice of gauge transformation $g = H^{-1/2}$, indeed $(g \cdot h) = h_0$. Then, for the complex gauge action in (2.1), $g \cdot (\bar{\partial}_E, \varphi, h) = (H^{1/2} \circ \bar{\partial}_E \circ H^{-1/2}, H^{1/2} \varphi H^{-1/2}, h_0)$. Consequently, the associated pair $(A, \Phi)$ is defined by $\bar{\partial}_A = H^{1/2} \circ \bar{\partial}_E \circ H^{-1/2}$ and $\Phi = H^{1/2} \varphi H^{-1/2}$.

2.1. **Parabolic Higgs bundles.** We next recall the salient facts about parabolic Higgs bundles, and in particular the moduli spaces of (weakly or strongly) parabolic Higgs bundles, following [BY96]. For simplicity, we restrict this discussion to the setting of rank 2 bundles.

A parabolic Higgs bundle consists of a Higgs bundle where the fields have simple poles at a given divisor $D = p_1 + \ldots + p_n$, and where extra algebraic data is specified at each $p_j$. This extra data consists of a weighted flag, which amounts to specifying a boundary
condition for the harmonic metric at that puncture, and in the weakly parabolic case a prescription of the eigenvalues of the residues.

We now explain this in more detail. Fix a divisor $D$ and at each $p \in D$, fix also a weight vector $\bar{\alpha}(p) = (\alpha_1(p), \alpha_2(p)) \in [0,1)^2$ and in the weakly parabolic case $\sigma(p) \in \mathbb{C}$. In the weakly parabolic case, we split $D = D_s \cup D_w$ where the strongly parabolic divisor is $D_s = \{ p \in D : \sigma(p) = 0 \}$ and the weakly parabolic divisor is $D_w = \{ p \in D : \sigma(p) \neq 0 \}$. Thus a strongly parabolic Higgs bundle is a weakly parabolic Higgs bundle with $D_w = \emptyset$.

A parabolic $\text{SL}(2,\mathbb{C})$-Higgs bundle over $(C, D)$ is then a rank 2 complex vector bundle over $C$ and a triple $(\tilde{\partial}_E, \{ \mathcal{F}(p) \}_{p \in D}, \varphi)$, where $\tilde{\partial}_E$ is a holomorphic structure on $E$ inducing a fixed holomorphic structure on $\text{Det} E$, $\varphi$ is a holomorphic map $\mathcal{E} \to \mathcal{E} \otimes K(D)$ (i.e., $\varphi$ has simple poles at each $p$) which is traceless and for each $p \in D$, $\mathcal{F}(p) = F_{*}(p)$ is a complete flag in the fiber $E_p$:

$$E_p = F_1(p) \supset F_2(p) \supset 0 \quad 0 \leq \alpha_1(p) < \alpha_2(p) < 1.$$  

This triple is called strongly parabolic at $p$ if $\varphi(p) : F_i(p) \to F_i(1+p) \otimes K(D)_p$ and weakly parabolic if $\varphi(p) : F_i(p) \to F_i(1+p) \otimes K(D)_p$ for each $i$. Thus in the strongly parabolic case, the residue of the Higgs field is nilpotent with respect to the flag, while in the weakly parabolic case, this residue preserves the flag. Furthermore, in the $\text{SL}(2,\mathbb{C})$-case we require $\alpha_1(p) + \alpha_2(p) = 1$ for all $p \in D$ and, additionally, for weakly parabolic points the residue to have eigenvalues $\sigma(p)$ and $-\sigma(p)$. In either setting we will say that it is compatible with the flag. For simplicity of notation, we write $\mathcal{E}$ for the holomorphic bundle $(E, \tilde{\partial}_E)$ together with the flag $\{ \mathcal{F}(p) \}_{p \in D}$; thus a parabolic Higgs bundle is again simply a pair $(\mathcal{E}, \varphi)$.

Stability in this setting depends on the weight vector $\bar{\alpha}(p)$. Define the parabolic degree of the parabolic bundle $\mathcal{E} = (E, \tilde{\partial}_E, \{ \mathcal{F}(p) \}_{p \in D})$ by

$$\text{pdeg}_{\bar{\alpha}} \mathcal{E} = \deg \mathcal{E} + \sum_{p \in D} (\alpha_1(p) + \alpha_2(p)),$$

which reduces to $\deg \mathcal{E} + |D|$ in the $\text{SL}(2,\mathbb{C})$-case. Note also that the parabolic structure on $\mathcal{E}$ induces parabolic structures on its holomorphic subbundles. We say that $\mathcal{E}$ is $\bar{\alpha}$-stable if

$$\frac{\text{pdeg}_{\bar{\alpha}} \mathcal{E}}{\text{rank} \mathcal{E}} > \text{pdeg}_{\bar{\alpha}} \mathcal{L}$$

for every holomorphic line subbundle $\mathcal{L}$ preserved by $\varphi$. The parabolic weights of $\mathcal{L}$ at $p$ are defined to be $\alpha_2(p)$ if the fiber $L_p = F_2(p)$ and $\alpha_1(p)$ otherwise.

We are now in a position to define the moduli space $\mathcal{M}_{\text{Higgs}}$ as the set of isomorphism classes of $\bar{\alpha}$-stable parabolic $\text{SL}(2,\mathbb{C})$-Higgs bundles (whose residues have eigenvalues $\pm \sigma(p)$ at $p \in D$ in the weakly parabolic case), where isomorphism in this category means
holomorphic bundle isomorphism commuting with the Higgs fields and preserving the flag structure. Notice that the weight data but not the flags are fixed. This only defines \( \mathcal{M}_{\text{Higgs}} \) as a set; a quotient in the algebraic category has been constructed by [Yok93].

There is a differential-geometric definition of \( \mathcal{M}_{\text{Higgs}} \) which clarifies the role of the weights \( \vec{\alpha} \). For this we fix the smooth complex bundle \( E \), divisor \( D \), and flag \( F(p) \) on each fiber \( E_p \), \( p \in D \). Thus here we fix the flag, but not yet the weights. Let \( \text{ParEnd}(E) \) be the bundle of endomorphisms of \( E \) which preserve the flag \( F(p) \) for each \( p \in D \) and \( G_C = \Gamma(\text{SL}(E) \cap \text{ParEnd}(E)) \) the complex gauge group.

Letting \( A_0 \) denote the affine space of all holomorphic structures on \( E \) inducing the fixed holomorphic structure \( \bar{\partial} \) on the determinant line bundle, define the space

\[
\mathcal{H} = \left\{ (\bar{\partial}_E, \varphi) \in A_0 \times \Omega^{1,0}(C, \text{End}_0 E) \mid \varphi \text{ meromorphic with respect to } \bar{\partial}_E \text{ with simple pole at each } p \in D \text{ such that } \text{Res}_p(\varphi) \text{ is compatible with the flag} \right\}.
\]

At this point, fix the weight vector \( \vec{\alpha}(p) \) at each \( p \), and assume that the weights are generic in the sense that any \( \vec{\alpha} \)-semistable bundle is in fact \( \vec{\alpha} \)-stable. Now let \( \mathcal{H}_{\vec{\alpha}} \subset \mathcal{H} \) be the subspace of pairs \( (\bar{\partial}_E, \varphi) \) which are \( \vec{\alpha} \)-stable and define \( \mathcal{M}_{\text{Higgs}} = \mathcal{H}_{\vec{\alpha}} / G_C \), where \( G_C \) is the complex gauge group as above. We recall that

\[
\dim_{\mathbb{C}} \mathcal{M}_{\text{Higgs}} = 6(g - 1) + 2n,
\]

where \( g = \text{genus}(C) \) and \( n \) is the number of parabolic points [BY96].

2.2. Non-abelian Hodge correspondence. The usual non-abelian Hodge correspondence extends to this parabolic case, and to a stable parabolic Higgs bundles we can associate a harmonic metric adapted to the parabolic structure. To describe what it means for these metrics to be adapted, let us restrict attention for simplicity to the case of rank 2 where the flags are full.

As described carefully in [Moc06, §3.5], both a parabolic structure and a Hermitian structure determine a filtration of the sections of \( E \otimes \mathcal{O}_C(*D) \) near any \( p \), where \( \mathcal{O}_C(*D) \) is the sheaf of algebras of rational functions with poles at \( D \). The filtration associated to the Hermitian metric is by the order of growth of holomorphic sections: \( |s(x)|_h \sim \text{dist}(x, p)^{\vec{\alpha}} \). The Hermitian metric \( h \) is said to be adapted to the parabolic structure if these two filtrations coincide.

Let us write this down in a model situation, cf. [BGP97, §2]. Consider a parabolic vector bundle \( (\mathcal{E}, \{F(p)\}_{p \in D}, \{\vec{\alpha}\}_{p \in D}) \) of rank 2 with a complete flag and parabolic weights \( \vec{\alpha}(p) = (\alpha_1(p), \alpha_2(p)) \) at every \( p \in D \). Choose a local holomorphic coordinate \( z \) and a holomorphic splitting of \( \mathcal{E} \) compatible with the filtration near \( p \), i.e. a holomorphic basis
of sections \(e_1, e_2\) with \(e_i(p) \in F_i(p)\). Then the metric
\[
h_{\tilde{\alpha}} = \begin{pmatrix}
|z|^{2\alpha_1(p)} & 0 \\
0 & |z|^{2\alpha_2(p)}
\end{pmatrix}
\]
is adapted to the parabolic structure. A unitary frame is provided by
\[
\tilde{e}_1 = |z|^{-\alpha_1(p)}e_1, \quad \tilde{e}_2 = |z|^{-\alpha_2(p)}e_2
\]
with respect to which the Chern connection of \(h_{\tilde{\alpha}}\) is given by
\[
d + \begin{pmatrix}
 i\alpha_1 & 0 \\
0 & i\alpha_2
\end{pmatrix} d\theta
\]
for \(z = re^{i\theta}\). In the \(\text{SL}(2, \mathbb{C})\)-case, where \(\alpha_1(p) + \alpha_2(p) = 1\), we see that the local monodromy around \(p\) lies in \(\text{SU}(2)\).

For a stable parabolic \(\text{SL}(2, \mathbb{C})\)-Higgs bundle \((E, \varphi)\) we can uniquely solve Hitchin’s equation
\[
F_{\text{D}(\varphi, h)}^\perp + [\varphi, \varphi^*h] = 0
\]
in the class of Hermitian metrics \(h\) adapted to the parabolic structure [Sim90]. Here,
\[
F_{\text{D}(\varphi, h)}^\perp = F_{\text{D}(\varphi, h)} + \sqrt{-1}\mu(E)\frac{2\pi\omega_C}{\text{vol}_G(C)}\text{Id}_E,
\]
where \(\mu(E)\) is the parabolic slope of \(E\). This gives rise to (part of) the nonabelian Hodge correspondence, namely the diffeomorphism
\[
\text{NAHC} : \mathcal{M}_{\text{Higgs}} \to \mathcal{M}
\]
obeained by mapping \((E, \varphi)\) to the triple \((E, \varphi, h)\).

Let us finally discuss the other part of the non-abelian Hodge correspondence. If we assume for simplicity that \(\text{pdeg}_{\tilde{\alpha}} E = 0\), then for a solution \(h\) of Hitchin’s equation the \(\text{SL}(2, \mathbb{C})\)-connection \(D(\varphi, h) + \varphi + \varphi^*h\) is flat and hence gives rise to representation \(\rho : \pi_1(C \setminus D) \to \text{SL}(2, \mathbb{C})\). The local monodromies around points in \(D\) are determined up to conjugacy by the parabolic weights and the eigenvalues of the residues of the Higgs field. They lie in \(\text{SL}(2, \mathbb{C})\) rather than \(\text{GL}(2, \mathbb{C})\) precisely because \(\alpha_1(p) + \alpha_2(p) = 1\).

**Remark 2.1.** Note that the complexities involved in considering \(\text{deg} E \neq 0\) are similar to the complexities from including parabolic weights.

The simplest case is the case of ordinary Higgs bundles where the fixed data \(\text{Det} E = (\text{Det} E, \tilde{\varphi}_{\text{Det} E})\) is the trivial holomorphic bundle with trivialization \(s\). The Hermitian-Einstein metric \(h_{\text{Det} E}(s, s)\) is constant, and we can renormalize \(s\) so that \(h_{\text{Det} E}(s, s) = 1\). Then locally, we can represent \(h\) by a matrix of determinant 1 in a basis of sections \(e_1, \cdots, e_n\) such that \(e_1 \wedge \cdots \wedge e_n = s\).

In the case of ordinary Higgs bundles where the fixed data \(\text{Det} E\) is not the trivial holomorphic bundle, there exists a local holomorphic section \(s\) of \(\text{Det} E\); \(h_{\text{Det} E}(s, s)\) need not
be constant. We can then represent \( h \) by a matrix which has determinant \( h_{\text{Det } E}(s, s) \) in a basis of sections \( e_1, \cdots, e_n \) for which \( e_1 \wedge \cdots \wedge e_n = s \), i.e.

\[
h = (h_{\text{Det } E}(s, s))^{1/n} h^\circ,
\]

where \( h^\circ \) is represented by a matrix of determinant 1. We emphasize that this prefactor \( (h_{\text{Det } E}(s, s))^{1/n} \) is completely determined by \( \text{Det } E \) and the choice of section \( s \) and is not a source of any additional freedom in the problem. Note that in the basis \( e_1, \cdots, e_n, \varphi \) and \([\varphi, \varphi^*] \) are still represented by traceless matrices, but \( F_{D(\delta_E, h)} \) is not represented by a traceless matrix.

Finally, in the setting of parabolic Higgs bundles, the additional complexity is that the metric \( h \) is adapted to the parabolic structure. This means that \( h_{\text{Det } E}(s, s) \) (hence \( \text{Det } h \)) has a singularity of the form \( |z|^{2 \sum a_i(p)} \) in a local holomorphic coordinate \( z \) centered at \( p \in D \).

### 2.3. Hyperkähler metric.

In this section we review the construction of a hyperKähler metric on the moduli space of stable parabolic \( \text{SL}(2, \mathbb{C}) \)-Higgs bundles. In the strongly parabolic case this was carried out by [Kon93] and later generalized to the weakly parabolic case by [Nak96].

Fix \( h_0 \) adapted to the parabolic structure (which for now we may assume to coincide with the model metric \( h_{\alpha} \) near \( p \in D \)) and a parabolic Higgs bundle \( (\bar{\partial}_E, \varphi) \). Let \( A_0 = D(\bar{\partial}_E, h_0) \) be the Chern connection and \( \Phi_0 = \varphi \). The pair \( (A_0, \Phi_0) \) will serve as a basepoint in the following construction. We consider pairs \( (A, \Phi) \) on \( \mathbb{C} \setminus D \) such that \( \bar{\partial}_A - \bar{\partial}_{A_0} \) and \( \Phi - \Phi_0 \) lie in certain function spaces encoding decay properties near the punctures. This decay is measured using weighted Sobolev spaces in [Kon93] and [Nak96], and in weighted \( b \)-Hölder spaces here.

In either approach, we impose that in the unitary frame \( \tilde{e}_1 = |z|^{-a_1(p)} e_1, \tilde{e}_2 = |z|^{-a_2(p)} e_2, \)

\[
\bar{\partial}_A - \bar{\partial}_{A_0} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} dz \quad \text{and} \quad \Phi - \Phi_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} dz,
\]

where \( \alpha, a = O(1) \) and \( \beta, \gamma, b, c = O(r^\epsilon) \) for some \( \epsilon > 0 \) (the decay being measured in an \( L^2 \) sense in the first approach and an \( L^\infty \) sense in the second approach). This infinite dimensional affine space of fields is acted on by the unitary gauge group \( \mathcal{G} = \{ g \in G_\mathbb{C} \mid g h_0\text{-unitary} \} \). The moment maps for this action are

\[
\mu_I(A, \Phi) = F_{A}^\perp + [\Phi, \Phi^*]
\]

\[
(\mu_I + \mu_K)(A, \Phi) = \bar{\partial}_A \Phi,
\]
corresponding to complex structures
\[ I(\hat{A}^{0,1}, \Phi) = (i\hat{A}^{0,1}, i\Phi), \quad J(\hat{A}^{0,1}, \Phi) = (i\Phi^*, -i(\hat{A}^{0,1})^*), \quad K(\hat{A}^{0,1}, \Phi) = (-\Phi^*, (\hat{A}^{0,1})^*), \]
where \( * = *_{h_0} \). The natural \( L^2 \)-metric on the configuration space is hyperKähler and descends to a hyperKähler metric on the moduli space of solutions to Hitchin’s equation
\[ \mathcal{M} = \mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0) / \mathcal{G} \]
via the process of hyperKähler reduction. By the non-abelian Hodge correspondence NAHC : \( \mathcal{M}_{\text{Higgs}} \to \mathcal{M} \) the hyperKähler metric can be thought to live on \( \mathcal{M}_{\text{Higgs}} \).

2.4. Spectral data. We next review briefly the spectral data associated to a parabolic Higgs bundles, cf. [LM10, §2.3] for a more thorough treatment.

The Hitchin fibration is given by the usual map
\[ \text{Hit} : (\mathcal{E}, \mathcal{F}(p), \varphi) \mapsto \text{char}_\varphi(\lambda) \]
where \( \text{char}_\varphi(\lambda) \) is the characteristic polynomial of \( \varphi \); this does not depend on the parabolic structure. The holomorphicity of \( \varphi : \mathcal{E} \to \mathcal{E} \otimes K_C(D) \) identifies the Hitchin base \( \mathcal{B} \) with the vector space of coefficients of \( \text{char}_\varphi(\lambda) \); since \( \text{tr} \varphi = 0 \), in this rank 2 case the only remaining coefficient is the determinant \( q = \det \varphi \). For \( p \in D_s \), \( \varphi \) has a nilpotent residue at \( p \), hence this is a quadratic differential with at most simple poles at \( D_s \). For \( p \in D_w \), \( q \) is a quadratic differential with at most a double pole like \( -\varphi^2 \frac{dz^2}{z^2} \) near \( p \).

Each point in \( \mathcal{B} \) can be regarded as a spectral curve \( \Sigma \) in the total space of the bundle \( K_C(D) \to \mathbb{C} \). The map \( \pi : \Sigma \to \mathbb{C} \) is a \( 2 : 1 \) ramified cover, and each sheet represents a different eigenvalue of the Higgs field. The genus of \( \Sigma \) is
\[ g(\Sigma) = 4(g - 1) + n + 1, \]
see [LM10, Eq. 6].

Let \( \mathcal{B}' \) be the subset of points in \( \mathcal{B} \) for which the spectral covers are smooth and \( \mathcal{M}' = \text{Hit}^{-1}(\mathcal{B}') \) the regular locus. To any \( (\mathcal{E}, \varphi) \in \mathcal{M}' \) we can associate the spectral data \( \mathcal{L} \to \Sigma \), where \( \mathcal{L} \) is a line bundle on the spectral cover \( \Sigma \subset \text{Tot}(K_C(D)) \). Away from ramification points, the fiber of \( \mathcal{L} \to \Sigma \) is the corresponding eigenspace of \( \varphi \) in \( \pi^* \mathcal{E} \to \Sigma \). The degree of \( \mathcal{L} \) is [LM10, p. 10]
\[ \deg \mathcal{L} = \deg \mathcal{E} + 2(g - 1) + n. \quad (2.3) \]

We can reverse this process and recover the parabolic Higgs bundle \( (\mathcal{E}, \varphi) \) from the spectral data \( (\Sigma, \mathcal{L}) \). Indeed, as a holomorphic bundle, \( \mathcal{E} = \pi_* \mathcal{L} \), while the Higgs field is recovered by pushing down multiplication by the tautological section \( \lambda \) of \( \pi^* K_C(D) \to \Sigma \). We also need to construct the flags at each of the marked points. In general, there is some
finite ambiguity at this step, see Logares-Martens [LM10, Proposition 2.2], but in this rank 2 case, the flags are full and there is no ambiguity.

3. Asymptotic Profiles and Approximate Solutions

Fix a stable parabolic SL(2, C)-Higgs bundle (E, φ) in $\mathcal{M}'$, so that det $\varphi = q$ has only simple zeros. Our eventual goal is to construct the harmonic metrics $h_t$ associated to the one-parameter family of Higgs bundles $(E, t\varphi)$ when $t \gg 1$. Equivalently, we fix $(E, \varphi)$ and view $h_t$ as the unique solution of the $t$-rescaled Hitchin equations

$$F^\perp_{D,h_t} + t^2[\varphi, \varphi^*_{h_t}] = 0. \quad (3.1)$$

In this section we introduce the two main ingredients used to construct this family: the limiting configurations and the model solutions on $\mathbb{C}$, also known as the fiducial solutions. The metrics $h_t$ are constructed by desingularizing limiting configurations using fiducial solutions. This may also be understood in reverse, by considering the possible limiting behavior of $h_t$ as $t \to \infty$. The ansatz made in [MSWW16] is that the fields $A_t$ and $t\Phi_t$ asymptotically decouple, and in the limit, on compact sets away from the zeros and poles of $q$, converge to solutions of the decoupled equations

$$F^\perp_{D_{\Lambda_\infty}} = 0, \quad \text{and} \quad [\Phi_{h_\infty}, \Phi^*_{h_\infty}] = 0. \quad (3.2)$$

This was later vindicated by Mochizuki [Moc16]:

**Theorem 3.1.** [Moc16, Theorem 2.7] For any compact set $K \subset \mathbb{C} \setminus (Z \cup D)$, there exist positive constants $c_0$ and $\epsilon$ such that the family of solutions of Hitchin’s equations $(E, t\varphi, h_t)$ satisfy

$$|[\varphi, \varphi^*_{h_t}]|_{h_t, \mathbb{C}} \leq c_0 \exp(-\epsilon t)$$

in $K$.

His analysis is local, and thus applies also to the parabolic setting.

The fiducial solutions, by contrast, are the limits of rescalings of the fields near the zeros and poles; in other words, they are the ‘bubbles’ in this theory.

3.1. Limiting configurations. We now describe the limiting configurations more carefully, emphasizing the Hermitian metrics rather than the fields $(A, \Phi)$. In particular, we describe normal forms for these limiting configurations near the zeros and poles of $\varphi$.

Let $(E, \varphi) \in \mathcal{M}'$ be a parabolic SL(2, C)-Higgs bundle. We now construct a singular Hermitian metric $h_\infty$ on $E$ as the pushforward of a singular metric on the associated spectral data $\mathcal{L} \to \Sigma$, and which solves the decoupled Hitchin equations. This metric can be chosen to induce a fixed Hermitian metric on Det $E$.

First denote by $Z \subset \mathbb{C}$ the set of zeros of det $\varphi \in H^0(\mathbb{C}, K_\mathbb{C}^2)$. Let $R = Z \cup D_\varphi \subset \mathbb{C}$ be the set of zeros of det $\varphi \in H^0(\mathbb{C}, K_\mathbb{C}(D)^{\otimes 2})$. (Recall that the zeros and simple poles of det $\varphi$,
regarded as a section of $H^0(C, K_C^2)$, are zeros of det $\varphi$ regarded as a section of $K_C(D)^{\otimes 2}$.

Let $\mathcal{L} \to \Sigma$ be the spectral data associated to det $\varphi$. The projection $\pi : \Sigma \to C$ is ramified at $R$ but not at $D_w$. At points $\tilde{p} \in \tilde{D}_s = \pi^{-1}(D_s)$ coming from simple poles of $q$, define $\tilde{\alpha}_{\tilde{p}} = \frac{1}{2}$; at points $\tilde{p} \in \tilde{Z}$, define $\tilde{\alpha}_{\tilde{p}} = -\frac{1}{2}$. Over each point in $D_w$ there are exactly two points $\tilde{p} \in \tilde{D}_w$. At one of these the naturally associated weight is $\alpha_1(\pi(\tilde{p}))$ and at the other the weight is $\alpha_2(\pi(\tilde{p}))$. Define $\tilde{\alpha}_{\tilde{p}}$ to be this naturally associated value.

Now equip $\mathcal{L}$ with a parabolic structure by setting the parabolic weights at $\tilde{p} \in \tilde{R} \cup \tilde{D}_w$ to be equal $\alpha_{\tilde{p}}$.

**Remark 3.2.** The number of ramification points is $|R| = \deg K_C(D)^{\otimes 2} = 4(g - 1) + 2|D|$. From (2.3) we see that

$$\deg \mathcal{L} = \deg \mathcal{E} + 2(g - 1) + |D| = \deg \mathcal{E} + \frac{1}{2}|R|.$$ 

The sum of the parabolic weights on $\mathcal{L}$ is

$$\sum_{\tilde{p} \in \tilde{R} \cup \tilde{D}_w} \alpha_{\tilde{p}} = -\frac{1}{2}|R| + |D|$$

Consequently, $\text{pdeg}_{\tilde{R}} \mathcal{L} = \text{pdeg}_{\tilde{R}} \mathcal{E}$.

By [Biq97, Sim90], there exists a Hermitian-Einstein metric $h_{\mathcal{L}}$ on the parabolic line bundle $\mathcal{L}$ adapted to the parabolic structure. This metric is unique up to a constant factor and solves $F_{(\tilde{\partial}_E, h_{\mathcal{L}})} = 0$.

Finally, define $h_\infty$ on $\mathcal{E}|_{C \setminus R}$ as the (orthogonal) pushforward of $h_{\mathcal{L}}$. In other words, we let $h_\infty$ be the metric on $\mathcal{E}$ such that the eigenspaces of $\varphi$ are orthogonal, and which equals the metric induced by $h_{\mathcal{L}}$ on each eigenspace. The metric $h_\infty$ solves $F_{(\tilde{\partial}_E, h_\infty)} = 0$, where the constant appearing in the trace-free part is defined in (2.2), precisely because $\text{pdeg}_{\tilde{R}} \mathcal{L} = \text{pdeg}_{\tilde{R}} \mathcal{E}$. Note that $h_\infty$ is adapted to the parabolic structure at points in $D_w$ but not at points in $D_s$.

**Corollary 3.3.** There exists a Hermitian metric $h_\infty$ which solves the decoupled Hitchin’s equations (3.2) and induces the fixed Hermitian structure on $\text{Det}\mathcal{E}$.

**Proof.** The metric $h_\infty$ constructed above induces a Hermitian-Einstein metric on $\text{Det}\mathcal{E}$. Since Hermitian-Einstein metrics are unique up to scale, we can rescale $h_{\mathcal{L}}$ so that this metric on $\text{Det} E$ agrees with the given one. $\square$

### 3.2. Normal forms

We now exhibit the normal form for $(\tilde{\partial}_E, \varphi, h_\infty)$ around simple zeros, and simple and double poles of det $\varphi$. Here and in several places below, we treat these three different settings in sequence; in most of these cases, for simple zeros we simply record the appropriate results from elsewhere, making the necessary modifications to the $\text{Det}\mathcal{E} \neq \mathcal{O}$ setting, as described in Remark 2.1.
Proposition 3.4. [MSWW16, Lemma 4.2] Fix $(E, \varphi) \in \mathcal{M}'$ and suppose that $\det \varphi$ has a simple zero at $p$. Then there is a holomorphic coordinate $z$ centered at $p$ such that $-\det \varphi = z d z^2$. In addition, in some holomorphic gauge,

$\partial E = \overline{\partial}, \quad \varphi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} d z, \quad h_\infty = Q h_\infty^w \quad \text{where} \quad h_\infty^w = \begin{pmatrix} |z|^{1/2} & 0 \\ 0 & |z|^{-1/2} \end{pmatrix}.$

Here, $Q$ is a locally-defined function completely determined by $h_{\det E}$ and the choice of holomorphic section of $\det E$.

Turning now to the parabolic setting, first note that if $(E, \varphi)$ is a parabolic Higgs bundle with weights $0 \leq \alpha_1(p) < \alpha_2(p) < 1$, then $\det E$ inherits a parabolic structure with weight $\alpha_1(p) + \alpha_2(p) = 1$. Hence in the same local holomorphic coordinate $z$ as above, and using the holomorphic frame $s_1 \wedge s_2$, the metric $|z|^2 = |z|^{2(\alpha_1(p) + \alpha_2(p))}$ is adapted to the induced parabolic structure on $\det E$.

Proposition 3.5. Fix $p \in D_s$. Then for any $(E, \varphi) \in \mathcal{M}'$, $\det \varphi$ has a simple pole at $p$, and there is a holomorphic coordinate $z$ centered at $p$ such that $-\det \varphi = z^{-1} d z^2$, and in some holomorphic gauge,

$\partial E = \overline{\partial}, \quad \varphi = \begin{pmatrix} 0 & 1 \\ \frac{1}{z} & 0 \end{pmatrix} d z, \quad h_\infty = Q h_\infty^w \quad \text{where} \quad h_\infty^w = |z| \begin{pmatrix} |z|^{-1/2} & 0 \\ 0 & |z|^{1/2} \end{pmatrix}.$

Here, $Q$ is a locally-defined function completely determined by $h_{\det E}$, the choice of holomorphic section of $\det E$, the unique coordinate $z$, and the parabolic weights.

Proof. The proof is a small modification of [MSWW16, Lemma 4.2]. Choose a holomorphic coordinate $z$ around $p$ so that $-\det \varphi = z^{-1} d z^2$, and fix some local holomorphic frame. Since $p$ is a simple pole of $\det \varphi$, $z \varphi |_{z=0}$ is nilpotent, but not identically zero. Applying a constant gauge transformation, we may assume that

$$z \varphi = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} d z \quad z \varphi |_{z=0} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Since $c(0) = 1$, $\sqrt{c(z)}$ is well-defined and holomorphic near 0, so if we define

$$g(z) = \frac{1}{\sqrt{c(z)}} \begin{pmatrix} 1 & a(z) \\ 0 & c(z) \end{pmatrix},$$

then

$$g^{-1} \varphi g = \begin{pmatrix} 0 & 1 \\ \frac{1}{z} & 0 \end{pmatrix} d z,$$

as needed. The metric $h_\infty$ defined here satisfies $F_{D(\overline{\partial} E, h_\infty)}^\perp = 0$, normalizes the Higgs field and induces the correct metric on $\det E$. \qed
In unitary gauge,
\[
d_A = d + \frac{1}{2} \left( \frac{\partial Q}{Q} - \frac{\bar{Q}}{Q} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \left( i \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right) d\theta - \frac{1}{4} \left( i \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix} \right) d\theta,
\]
\[
\Phi = \begin{pmatrix} 0 & |z|^{-1/2} \\ |z|^{1/2}/z & 0 \end{pmatrix} dz.
\]

**Remark 3.6.** Note that if \((E, \varphi) \in M^{\text{sing}} = M \setminus M'\), then there need not be a frame near \(p \in D_s\) where \((\bar{\partial}_E, \varphi, h_\infty)\) has local form in Proposition 3.9. In the extreme case, in any moduli space of strongly parabolic Higgs bundles, there is a distinguished point \(q \equiv 0 \in B\); for any Higgs bundle in the nilpotent cone and any choice of \(p \in D_s\), there is no holomorphic coordinate \(z\) such that \(-\det \varphi = z^{-1} dz^2\) near \(p \in D_s\). More generally, as one moves in \(B\), the zeros of \(q\) wander. They can coincide with each other or with a point of \(D_w\), but never with \(D_w\) because of the fixed non-zero residue \(\sigma\). In either of these two cases of coincidence, \(q\) no longer has all simple zeros, viewed as a section of \(K_C(D)^{\otimes 2}\).

**Proposition 3.7.** Fix \(p \in D_w\). Then for any \((E, \varphi) \in M\), \(-\det \varphi\) has a double pole at \(p\), and there is a holomorphic coordinate \(z\) centered at \(p\) such that \(-\det \varphi = \sigma^2 z^{-2} dz^2\) for some \(\sigma \in \mathbb{C}^*\) and a holomorphic gauge such that
\[
\bar{\partial}_E = \bar{\partial}, \quad \varphi = \sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dz, \quad \text{and} \quad h_\infty = Q h_\infty^\circ \quad \text{where} \quad h_\infty^\circ = \begin{pmatrix} |z|^{2\alpha_1(p)} & 0 \\ 0 & |z|^{2\alpha_2(p)} \end{pmatrix}.
\]

Here \(Q\) is a locally-defined function completely determined by \(h_{\text{Det} E}\), the choice of holomorphic section of \(\text{Det} E\), the coordinate \(z\), and the parabolic weights.

The constant \(\sigma\) is an invariant since the residue of \(\sigma dz/z\) is independent under holomorphic change of variables.

**Proof.** Choose \(z\) so that \(-\det \varphi = \sigma^2 z^{-2} dz^2\), so \(z\varphi\) is regular at \(p\) and \(-\det(z\varphi) = \sigma^2 dz^2\). Hence there exists a local holomorphic frame such that
\[
z\varphi = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} dz, \quad \text{i.e.,} \quad \varphi = \sigma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dz.
\]

As before, the metric \(h_\infty\) satisfies \(\frac{1}{2} F_{D(\bar{\partial}_E, h_\infty)} = 0\), normalizes the Higgs field and induces the correct metric on \(\text{Det} E\). \(\square\)

In unitary gauge,
\[
d_A = d + \frac{1}{2} \left( \frac{\partial Q}{Q} - \frac{\bar{Q}}{Q} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left( F_{\alpha_1} 0 \right) d\theta,
\]
\[ \Phi = \frac{\sigma}{z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, dz. \]

In this case, \( h_\infty \) is already adapted to the parabolic structure, hence requires no further desingularization.

### 3.3. Fiducial solutions

We next display the model solutions on the complex plane, known as the fiducial solutions, which we shall use to ‘smooth out’ the limiting configurations. These model solutions were used for this purpose in [MSWW16], but also arose many years earlier in the physics literature.

**Proposition 3.8.** [MSWW16] There exists a family \( (\partial E, t\varphi, h_t) \) of smooth solutions of Hitchin’s equations on \( \mathbb{C} \) with

\[
\partial E = \overline{\partial}, \quad \varphi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \, dz, \quad h_t^{\text{model}} = \begin{pmatrix} r^{1/2}e^{\ell_t(r)} & 0 \\ 0 & r^{-1/2}e^{-\ell_t(r)} \end{pmatrix},
\]

where, here and below, \( r = |z| \). The function \( \ell_t \) is the solution of the Painlevé equation

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ell_t = 8t^2r \sinh(2\ell_t)
\]

with asymptotics

\[
\ell_t(r) \sim \frac{1}{\pi} K_0 \left( \frac{8}{3}tr^2 \right) \quad \text{as} \; r \to \infty, \quad \ell_t(r) \sim -\frac{1}{2} \log r \quad \text{as} \; r \to 0.
\]

In unitary gauge, this solution takes the form

\[
d_A = d + \frac{1}{4} \left( \frac{1}{2} + r \frac{d\ell_t}{dr} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right)
\]

\[
\Phi = \begin{pmatrix} 0 & r^{1/2}e^{\ell_t(r)} \\ zr^{-1/2}e^{-\ell_t(r)} & 0 \end{pmatrix} \, dz.
\]

Note that \( 2i d\theta = dz/z - d\bar{z}/\bar{z} \).

Near a simple pole there is a similar radial solution.

**Proposition 3.9.** For parabolic weights \( 0 \leq \alpha_1 < \alpha_2 < 1 \), there exists a family \( (\partial E, t\varphi, h_t) \) of solutions of Hitchin’s equations which are smooth on \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \), with

\[
\partial E = \overline{\partial}, \quad \varphi = \begin{pmatrix} 0 & 1 \\ \frac{1}{z} & 0 \end{pmatrix} \, dz, \quad h_t^{\text{model}} = r \begin{pmatrix} r^{-1/2}e^{m_t(r)} & 0 \\ 0 & r^{1/2}e^{-m_t(r)} \end{pmatrix}.
\]

Here \( m_t \) solves

\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) m_t = 8t^2r^{-1} \sinh(2m_t)
\]

(3.4)
and satisfies
\[ m_t \sim \frac{1}{rt} K_0(8tr^{1/2}) \text{ as } r \to \infty, \quad m_t \sim \left( \frac{1}{2} + \alpha_1 - \alpha_2 \right) \log r \text{ as } r \to 0, \]
so that
\[ r \cdot r^{-\frac{1}{2}} e^{m_t(r)} \sim r^{2\alpha_1}, \quad r \cdot r^{\frac{1}{2}} e^{-m_t(r)} \sim r^{2\alpha_2}. \]

The corresponding filtration at \( z = 0 \) is
\[ F_1 = C^2 \supset F_2 = \left\langle \begin{pmatrix} 0 & 1 \end{pmatrix} \right\rangle \supset 0 \]
with weights \( 0 \leq \alpha_1 < \alpha_2 < 1 \).

In unitary gauge
\[ dA_t = d + \left( \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + F^p_t(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \]
\[ \Phi_t = \begin{pmatrix} 0 & r^{-1/2} e^{m_t(r)} \\ r^{-1/2} e^{-m_t(r)} & 0 \end{pmatrix} \]
\[ F^p_t(r) = \frac{1}{4} \left( -\frac{1}{2} + rm_t'(r) \right). \]

**Proof.** The existence of the family \( h^\text{model}_t \) may be understood in two ways. First, by the change of variables \( \rho = 8tr^{1/2} \), we have
\[ \left( \frac{d^2}{^2} + \frac{1}{\rho} \frac{d}{d\rho} \right) m_t = \frac{1}{2} \sinh(2m_t), \]
McCoy-Tracy-Wu [MTW77] prove the existence and uniqueness of solutions of this ODE on \((0, \infty)\) with the prescribed asymptotic behavior.

Alternately, we may also follow [FN17, §1.10] where Proposition 3.8 is derived at simple zeros. In the terminology of [Moc13], \((\mathcal{E}, t\phi)\) is a family of “good filtered Higgs bundle” on \( \mathbb{C}P^1 \) with marked points 0 and \( \infty \). The singularity at 0 is regular and the one at \( \infty \) irregular. Consequently, by [BB04, Moc11], there is an associated family of adapted harmonic metrics \( h^\text{model}_t \). These good filtered Higgs bundle are fixed by the \( C^\infty \)-action that rotates the base and simultaneously rescales the Higgs field. Hence \( h^\text{model}_t \) has the claimed shape and radial symmetry. \( \square \)

### 3.4. Approximate solutions

We now assemble the two pieces above to construct a family of adapted Hermitian metrics \( h^\text{app}_t \) which approximately solve the \( t \)-rescaled Hitchin’s equations in (3.1). In fact, we show in Proposition 3.10 below that
\[ F^\perp_{D(\mathcal{E}, h^\text{app}_t)} + t^2 [\phi, \phi^* h^\text{app}_t] \]
decays exponentially in $t$. The further step of perturbing $h_t^{\text{app}}$ to an exact harmonic metric requires a closer study of the linearized operator, which is carried out in the next section.

Choose a smooth nonnegative cutoff function $\chi$ on $\mathbb{R}^+$ taking values in $[0,1]$, with $\chi(r) = 1$ for $r \leq 1/2$ and $\chi(r) = 0$ for $r \geq 1$. Now define $h_t^{\text{app}}$ on $\mathcal{E}$ as follows:

- in a holomorphic coordinate and gauge near zeros of $\det \varphi$, where $\bar{\sigma}_E$ and $\varphi$ are in normal form, set
  \[
  h_t^{\text{app}} := Q \begin{pmatrix}
  r^{1/2}e^{\ell_t(r)}\chi(r) & 0 \\
  0 & r^{-1/2}e^{-\ell_t(r)}\chi(r)
  \end{pmatrix};
  \]

- in a holomorphic coordinate and gauge near simple poles of $\det \varphi$ where $\bar{\sigma}_E$ and $\varphi$ are in normal form, set
  \[
  h_t^{\text{app}} = Qr \begin{pmatrix}
  r^{-1/2}e^{m_t(r)}\chi(r) & 0 \\
  0 & r^{1/2}e^{-m_t(r)}\chi(r)
  \end{pmatrix};
  \]

- elsewhere on $C$, set $h_t^{\text{app}} = h_\infty$.

Here $Q$ is the same locally-defined function appearing in Proposition 3.4 and 3.5; consequently $h_t^{\text{app}}$ induces the metric $h_{\text{Det} E}$ on $\text{Det} E$. Recall that the function $Q$ is completely determined by $h_{\text{Det} E}$, the choice of holomorphic section of $\text{Det} \mathcal{E}$, the coordinate $z$, and parabolic weights. For the analysis, it will be important to note that $Q$ is smooth.

**Proposition 3.10.** For $t_0 \gg 1$, there exist positive constants $c$, $\mu$, such that for any $t > t_0$,

\[
\| F_{D(\bar{\sigma}_E,h_t^{\text{app}})} + t^2 \left[ \varphi, \varphi^{* h_t^{\text{app}}} \right] \| \leq ce^{-\mu t}.
\]

The norm here can be taken either in $L^2$ or in a Hölder norm with respect to $h_t^{\text{app}}$ and the fixed Riemannian metric on $C$.

**Proof.** Clearly $h_t^{\text{app}}$ solves Hitchin’s equations exactly away from the annuli where $\chi' \neq 0$. The exponential decay rates of $\ell_t$ and $r\ell_t'$, cf. [MSWW16, Lemma 3.3], as well as those of $m_t$ and $rm_t'$ stated above, then imply the stated bounds. \qed

We now convert $h_t^{\text{app}}$ into the corresponding family of connections and Higgs fields. Fixing any compatible metric $h_0$, then given a triple $(\bar{\sigma}_E, \varphi, h)$, there is an $\text{End} E$-valued $h_0$-Hermitian section $H$ such that $h(w_1, w_2) = h_0(Hw_1, w_2)$. The complex gauge transformation $g = H^{-1/2}$ satisfies $g \cdot h = h_0$ since in general $(g \cdot h)(w_1, w_1) = h(gw_1, gw_2)$. The associated pair $(d_A, \Phi)$ equals $\bar{\sigma}_A = H^{1/2} \circ \bar{\sigma}_E \circ H^{-1/2}$, $\Phi = H^{1/2} \varphi H^{-1/2}$. Using these
formulae, we record the forms of these fields in unitary gauge:

$$A_t = d + \frac{1}{2} \left( \frac{\partial Q}{Q} - \frac{\overline{\partial Q}}{Q} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \left( \frac{1}{2} + r \frac{d(\ell_i \chi)}{dr} \right) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta$$

$$\Phi_t = \begin{pmatrix} 0 & r^{1/2}e^{\ell_i \chi} \\ zr^{-1/2}e^{-\ell_i \chi} & 0 \end{pmatrix} dz$$
on disks where $$- \text{det} \Phi = zdz^2$$,

$$A_t = d + \frac{1}{2} \left( \frac{\partial Q}{Q} - \frac{\overline{\partial Q}}{Q} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} d\theta + \frac{1}{2} \left( -\frac{1}{2} + r \frac{d(m_i \chi)}{dr} \right) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta,$$

$$\Phi_t = \begin{pmatrix} 0 & r^{1/2}e^{m_i \chi} \\ z^{-1}r^{1/2}e^{-m_i \chi} & 0 \end{pmatrix} dz$$
on disks where $$- \text{det} \Phi = z^{-1}dz^2$$, and

$$A = \frac{1}{2} \left( \frac{\partial Q}{Q} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} i\chi & 0 \\ 0 & i\chi \end{pmatrix} d\theta, \quad \Phi = \frac{\sigma}{z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dz$$
on disks where $$- \text{det} \Phi = \sigma^2z^{-2}dz^2$$.

### 4. The Linearization

Define the nonlinear operator

$$F_t(\overline{\sigma}_E, \phi, h) := H^{1/2} \left( F_{D(\overline{\sigma}_E, h)} + t^2[\phi, \phi*] \right) H^{-1/2},$$

where $$H^{1/2}$$ is the End($E$)-valued $$h_0$$-Hermitian section satisfying $$h(v, w) = h_0(H^{1/2}v, H^{1/2}w)$$, as discussed at the end of the introduction of §2. By doing this, the output $$F_t(\overline{\sigma}_E, \phi, h)$$ is an $$h_0$$-skew-Hermitian section of $$\Omega^{1,1}(C, End E)$$. Or equivalently in the unitary formulation of Hitchin’s equations, this operator $$F_t$$ is equal to $$F_t(A_t, t\Phi_t) = F_{A_t}^\perp + t^2[\Phi_t, \Phi_t*]$$.) The output of the operator $$F_t$$ measures the failure of $$(\overline{\sigma}_E, \phi, h)$$ from being a solution of the $$t$$-rescaled Hitchin equations. We fix the underlying Higgs bundle $$(\overline{\sigma}_E, \phi)$$ and regard $$F_t$$ as an operator acting on Hermitian metrics, which we assume are perturbations of the approximate solution $$h_{\text{app}}^t$$. In unitary gauge, the Higgs bundle data becomes

$$\Phi_t = (H_{\text{app}}^t)^{1/2} \circ \phi \circ (H_{\text{app}}^t)^{-1/2}, \quad A_{t0}^{0,1} = (H_{\text{app}}^t)^{1/2} \circ \overline{\sigma}_E \circ (H_{\text{app}}^t)^{-1/2},$$

where $$H_{\text{app}}^t$$ is the End $$E$$-valued $$h_0$$-Hermitian section such that

$$h_{\text{app}}^t(v, w) = h_0((H_{\text{app}}^t)^{1/2}v, (H_{\text{app}}^t)^{1/2}w). \quad (4.1)$$

Local expressions for these fields near the zeros and poles of $$\text{det} \phi$$ are recorded above. Finally, write $$h_t(w_1, w_2) = h_{\text{app}}^t(e^{\gamma}w_1, e^{\gamma}w_2)$$, or equivalently $$h_t^{1/2} = (H_{\text{app}}^t)^{1/2}e^{\gamma}$$, and
break unitary invariance by assuming that \( \gamma_t \) is \( h_0 \)-Hermitian. Thus we focus on the operator
\[
\mathcal{F}_t(\gamma) := F_{A_{\exp(\gamma)}}^t + t^2 [e^{-\gamma} \Phi_t \Phi_t^* + e^{\gamma} \Phi_t^* h_0^t e^{-\gamma}].
\]
Note that this operator is computed relative to the background approximate solution fields. The error term \( \mathcal{F}_t(0) \), which is supported on a union of annuli around the zeros and poles of \( \det \varphi \), was estimated in the last section.

Up to a simple isomorphism, the linearization of \( \mathcal{F}_t \) at 0 equals
\[
\mathcal{L}_t \gamma := -i \star D \mathcal{F}_t(0)[\gamma] = -i \frac{d}{de} \bigg|_{e=0} \mathcal{F}_t(e \gamma) = \Delta_{A_t} \gamma - i \ast t^2 M_{\Phi_t} \gamma,
\]
where
\[
\Delta_{A_t} := d_{A_t}^* d_{A_t} \gamma, \quad \text{and} \quad M_{\Phi_t} \gamma := [\Phi_t^* \wedge [\Phi_t, \gamma]] - [\Phi_t \wedge [\Phi_t^*, \gamma]].
\]

The set of exceptional points in the analysis below is a union \( Z \cup D \), \( D = D_s \cup D_w \), where \( Z \) is the set of zeros of \( \det \varphi \) (all simple, by assumption), and \( D_s \) and \( D_w \) are the sets of strongly and weakly parabolic points, respectively. Near any \( p \in D \), choose a holomorphic frame \((e_1, e_2)\) compatible with the flag at \( p \) and the model metric
\[
h_{\bar{\alpha}} = \begin{pmatrix} |z|^{2\alpha_1} & 0 \\ 0 & |z|^{2\alpha_2} \end{pmatrix}.
\]
In this holomorphic frame, the Chern connection \( A_{\bar{\alpha}} \) of \( h_{\bar{\alpha}} \) equals
\[
d_{A_{\bar{\alpha}}} = \bar{\partial} + \partial^{h_{\bar{\alpha}}} = d + \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \frac{dz}{z},
\]
while in the unitary frame \((\tilde{e}_1, \tilde{e}_2)\), where \( \tilde{e}_i = e_i / |z|^{\alpha_i} \), it equals
\[
d_{A_{\bar{\alpha}}} = d + \begin{pmatrix} i\alpha_1 & 0 \\ 0 & i\alpha_2 \end{pmatrix} \, d\theta.
\]

We see from these expressions that near each \( p \in D \), \( A_{\bar{\alpha}} \) has a simple pole, \( A_t \) differs from \( A_{\bar{\alpha}} \) only by lower order terms, and the pole of \( \Phi_t \) is simple. This means that the operator \( \mathcal{L}_t \) is an elliptic operator of conic type, with singularities at points of \( D \). Of course, the coefficients of \( \mathcal{L}_t \) are smooth near points of \( Z \), but this operator develops conic singularities as \( t \to \infty \) and we must keep track of this ‘emergent’ behavior.

We now describe the precise local expressions for \( \mathcal{L}_t \) near each of these three types of points. In preparation for analyzing the mapping properties of \( \mathcal{L}_t \), both for each fixed \( t \) and uniformly as \( t \to \infty \), we also describe the indicial roots of \( \mathcal{L}_t \). By definition, a number \( \nu \) is called an indicial root of a conic operator \( L \) if there exists \( \psi \in \mathcal{C}^\infty(S^1) \) such that \( L(r^\nu \psi) = O(r^{\nu-1}) \). The exponent on the right should normally be \( \nu - 2 \), so this condition entails a leading order cancellation. In fact, the coefficient of \( r^{\nu-2} \) is a type of
eigenvalue equation for \( \psi \). The indicial roots play a significant role in determining the mapping properties of \( L \) and regularity properties of its solutions.

**Remark 4.1.** In the local analysis, we assume \( Q = 1 \). Because \( Q \) is smooth and everywhere positive, this simplification does not change the mapping properties of the operator.

### 4.1. \( L_t \) near simple zeros

Recall from Proposition 3.8 that

\[
d_{A_t} = d + F_t(0) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \, d\theta, \quad \Phi_t = \begin{pmatrix} 0 & |z|^{1/2} e^{\ell_t} \\ z|z|^{-1/2} e^{-\ell_t} & 0 \end{pmatrix} \, dz,
\]

where \( \ell_t \) solves (3.3) and \( F_t(0) = \frac{1}{2}(\frac{1}{2} + r \partial_r \ell_t) \). Writing \( \gamma = \begin{pmatrix} u_0 & u_1 \\ \bar{u}_1 & -u_0 \end{pmatrix} \), then we calculate that

\[
L_t \gamma = (\Delta_{A_t} - i \ast t^2 M_{\Phi_t}) \begin{pmatrix} u_0 & u_1 \\ \bar{u}_1 & -u_0 \end{pmatrix}
= \begin{pmatrix} \Delta_0 u_0 + 4 \partial_\theta \bar{u}_1 - \Delta_0 u_0 & 4 \partial_\theta \bar{u}_1 + 4(\partial_1^2)^2 \bar{u}_1 \\ \Delta_0 \bar{u}_1 + 4 \partial_\theta \bar{u}_1 + 4(\partial_1^2)^2 \bar{u}_1 & -\Delta_0 \bar{u}_1 \end{pmatrix}
+ 8t^2 \begin{pmatrix} 2 \cosh(2\ell_t) u_0 & \cosh(2\ell_t)u_1 - e^{-i\theta} \bar{u}_1 \\ \cosh(2\ell_t)\bar{u}_1 - e^{i\theta} u_1 & -2 \cosh(2\ell_t) u_0 \end{pmatrix}.
\]

Note that \( F_t(0) \) vanishes and \( r \cosh(2\ell_t) \) is regular at \( r = 0 \); in fact \( L_t \) has only polar coordinate singularities at \( r = 0 \). Thus its indicial roots are simply those of the scalar Laplacian \( \Delta_0 \), namely \( \mathbb{Z} \).

The Hermitian operator \(-i \ast M_{\Phi_t}\) is positive definite for every \( t > 0 \), with eigenvalues

\[
\lambda_0 = 16r \cosh 2\ell_t, \quad \lambda_1 = 8r(\cosh 2\ell_t - 1), \quad \lambda_2 = 8r(\cosh 2\ell_t + 1).
\]

The eigenvector for \( \lambda_0 \) has \( \gamma \) diagonal, i.e., \( u_1 \equiv 0 \), while the other two eigenspaces are spanned by vectors with \( u_0 \equiv 0 \).

We also consider the limits of the two summands \( \Delta_{A_t} \) and \( M_{\Phi_t} \) as \( t \to \infty \). Near any point of \( \Sigma \),

\[
d_{A_\infty} = d + \frac{1}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \, d\theta, \quad \Phi_\infty = \begin{pmatrix} 0 & r^{1/2} \\ zr^{-1/2} & 0 \end{pmatrix} \, dz,
\]

and hence

\[
\Delta_{A_\infty} \gamma = \begin{pmatrix} \Delta_0 u_0 & \frac{\Delta_1}{2} u_1 \\ \frac{\Delta_1}{2} u_1 & -\Delta_0 u_0 \end{pmatrix}, \quad -i \ast M_{\Phi_\infty} \gamma = 8r \begin{pmatrix} 2u_0 & u_1 - e^{-i\theta} \bar{u}_1 \\ \bar{u}_1 - e^{i\theta} u_1 & -2u_0 \end{pmatrix},
\]

where

\[
\Delta_{1/2} := -\partial_r^2 u_1 - \frac{1}{r} \partial_r u_1 - \frac{1}{r^2} \left( \partial_\theta + \frac{i}{2} \right)^2.
\]
The limiting operator $-i \star M_{\Phi}$ is only semi-definite, with eigenvalues

$$\lambda_0 = 16r, \quad \lambda_1 = 0, \quad \lambda_2 = 16r.$$ 

Later on we shall consider the conic operator $L_c^0 = \Delta_{A_c} - i \star M_{\Phi_c}$. This has indicial roots consisting of the integers, for the diagonal terms, and $\mathbb{Z} + \frac{1}{2}$ for the off-diagonal terms.

### 4.2. $L_t$ near strongly parabolic points.

Following (3.5), the model solution at a strongly parabolic point, in unitary gauge, is

$$d_{A_t} = d + \left( \begin{array}{cc} i & 0 \\ 0 & i \end{array} \right) + 2iF_t^p(r) \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) d\theta$$

$$\Phi_t = \left( \begin{array}{cc} 0 & e^{\sigma_t(r)} \\ z^{-1}e^{-\sigma_t(r)} \end{array} \right) dz,$$

where $m_t$ and $F_t^p$ are as in (3.4) and (3.6), and $\sigma_t(r) = m_t(r) - \frac{1}{2} \log r$. We then obtain

$$L_t \gamma = \begin{pmatrix} \Delta u_0 \\ \Delta \tilde{u}_1 + \frac{2i}{r^2} (4F_t^p) \partial_\theta \tilde{u}_1 + \frac{(4F_t^p)^2}{r^2} \tilde{u}_1 \\ -\Delta u_0 + \frac{8t^2}{r} \left( \begin{array}{cc} 2\cosh(2m_t)u_0 & -\tilde{u}_1 e^{i\theta} + \cosh(2m_t)u_1 \\ -\tilde{u}_1 e^{-i\theta} + \cosh(2m_t)\tilde{u}_1 & -2\cosh(2m_t)u_0 \end{array} \right) \end{pmatrix}.$$ 

The matrix $-i \star M_{\Phi_t}$ is Hermitian and strictly positive, with eigenvalues

$$\lambda_0 = \frac{16}{r} \cosh(2m_t), \quad \lambda_1 = \frac{8}{r} (\cosh(2m_t) - 1), \quad \lambda_2 = \frac{8}{r} (\cosh(2m_t) + 1).$$

As before, the eigenvector corresponding to $\lambda_0$ occurs when $\gamma$ is diagonal.

Now, $m_t(r) \sim \left( \frac{1}{2} + \alpha_1 - \alpha_2 \right) \log r$, $\sigma_t(r) \sim (\alpha_1 - \alpha_2) \log r$ and $F_t^p(r) \sim \frac{\alpha_1 - \alpha_2}{4}$ as $r \to 0$, so rewriting

$$d_{A_t} = d + \left( \begin{array}{cc} i\alpha_1 & 0 \\ 0 & i\alpha_2 \end{array} \right) d\theta + 2 \left( F_t^p - \frac{\alpha_1 - \alpha_2}{4} \right) \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right) d\theta,$$

then the final term vanishes in the limit as $r \to 0$, and hence the leading part of $d_{A_t}$ is the Chern connection $d_{A_{\tilde{z}}}$. Consequently, to leading order as $r \to 0$,

$$\Delta_{A_t} \gamma \sim \Delta_{A_{\tilde{z}}+} \gamma = \begin{pmatrix} \Delta u_0 \\ \Delta \tilde{u}_1 + \frac{2i}{r^2} (\alpha_1 - \alpha_2) \partial_\theta \tilde{u}_1 + \frac{(\alpha_1 - \alpha_2)^2}{r^2} \tilde{u}_1 \\ -\Delta u_0 + \frac{8t^2}{r} (\alpha_1 - \alpha_2) \partial_\theta u_1 + \frac{(\alpha_1 - \alpha_2)^2}{r^2} u_1 \end{pmatrix}.$$ 

Furthermore, $\cosh 2m_t \sim r^{1+2(\alpha_1 - \alpha_2)}$, and since $|\alpha_1 - \alpha_2| < 1$, $| -i \star M_{\Phi_t} | \leq Cr^{-2+\delta}$ for some $\delta > 0$. Hence the indicial roots of $L_t$ are the same as those for $\Delta_{A_{\tilde{z}}+}$. The indicial
roots corresponding to $\gamma$ diagonal are the integers, just as before. The induced operator for the off-diagonal part is

$$u_1 \mapsto \left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} (\partial_\theta + i(\alpha_1 - \alpha_2))^2\right) u_1.$$  

Writing $- (\partial_\theta + i(\alpha_1 - \alpha_2))^2 = T$, then $\nu$ is an indicial root if and only if $\nu^2$ is an eigenvalue of $T$. But $(T - \nu^2) \zeta = 0$ has a nontrivial solution if and only if $\nu^2 = (\ell + \alpha_1 - \alpha_2)^2$ for some $\ell \in \mathbb{Z}$, and hence

$$\Gamma(L_t) = \Gamma(\Delta_{A_t}) = \mathbb{Z} \cup \{\pm(\ell + \alpha_1 - \alpha_2) \mid \ell \in \mathbb{Z}\}.$$

For later reference, $\Gamma(\Delta_{A_t}) \cap (-1,1) = \{\pm(\alpha_2 - \alpha_1)\}$ and $\pm(1 - (\alpha_2 - \alpha_1))\}$.

As $t \to \infty$, the fields converge to

$$d_{A_\infty} = d + \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} d\theta - \frac{1}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} d\theta, \quad \Phi_\infty = \begin{pmatrix} 0 & r^{-\frac{1}{2}} \\ z^{-1} r^{\frac{1}{2}} & 0 \end{pmatrix} dz,$$

and hence, denoting

$$\Delta_{-1/2} := -\partial^2_r u_1 - \frac{1}{r} \partial_r u_1 - \frac{1}{r^2} \left(\partial_\theta - i \frac{1}{2}\right)^2,$$

the operators are

$$\Delta_{A_\infty} \gamma = \begin{pmatrix} \Delta_0 u_0 & \Delta_{-1/2} u_1 \\ \Delta_{-1/2} u_1 & -\Delta_0 u_0 \end{pmatrix}, \quad -i \ast M\Phi_\infty \gamma = \frac{8}{r} \begin{pmatrix} 2u_0 & u_1 - e^{i\theta} \bar{u}_1 \\ \bar{u}_1 - e^{-i\theta} u_1 & -2u_0 \end{pmatrix}.$$

Thus, here too $M\Phi_\infty$ has no effect on the indicial roots, and to leading order, $L_\infty^0 := \Delta_{A_\infty} - i \ast M\Phi_\infty \sim \Delta_{A_\infty}$ so $\Gamma(L_\infty^0) = \mathbb{Z} \cup \{\mathbb{Z} + \frac{1}{2}\}$.

4.3. $L_t$ near weakly parabolic points. Finally, if $p \in D_w$, then near $p$, the model solution is the $t$-independent pair of fields

$$d_A = d + \begin{pmatrix} i\alpha_1 & 0 \\ 0 & i\alpha_2 \end{pmatrix} d\theta, \quad \Phi = \frac{\sigma}{z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dz.$$

Thus $d_A = d_{A_\tilde{\alpha}}$ is the Chern connection of the model metric and $\Delta_A = \Delta_{A_\tilde{\alpha}}$. In addition,

$$-i \ast M\Phi \gamma = 16 \frac{r}{|\sigma|^2} \begin{pmatrix} 0 & u_1 \\ \bar{u}_1 & 0 \end{pmatrix};$$

this transformation has eigenvalues

$$\lambda_0 = 0, \quad \lambda_1 = 16 \frac{|\sigma|^2}{r^2}, \quad \lambda_2 = 16 \frac{|\sigma|^2}{r^2},$$

hence is nonnegative with nullspace consisting of the diagonal matrices. This is a ‘Hardy-type’ potential term and the indicial roots depend on $\sigma$. 

ASYMPTOTIC GEOMETRY OF THE MODULI SPACE OF PARABOLIC SL(2, C)-HIGGS BUNDLES

The action of $\Delta_A$ on diagonal $\gamma$ is simply $\Delta_0$, so the indicial roots for this part are the integers $\mathbb{Z}$. For the off-diagonal terms of $L_t$, the indicial roots are

$$\{ \pm [ (\ell + \alpha_1 - \alpha_2)^2 + 16|\sigma|^2 ]^{1/2} \}.$$

5. MAPPING PROPERTIES

The operator $L_t$ acts on various natural function spaces. To accommodate its conic structure, the most common are weighted $b$-Sobolev or $b$-Hölder spaces. (There are also closely related conic versions of these spaces which we do not introduce for simplicity.) The former are more convenient when using duality and other Hilbert space arguments, while the latter are more convenient for the nonlinear aspects of the problem. The geometric microlocal approach makes it possible to switch back and forth between the two.

We remind the reader that in this section we take $Q = 1$ for notational simplicity, as discussed in Remark 4.1.

5.1. Fredholm map and weighted spaces. We begin by defining the basic $b$-Hölder space. Let $P$ be a finite set of points on $C$, which we take below to be either $D$ or $Z \cup D$. For any $0 < \alpha < 1$, define

$$C^0_b(C, P) = \left\{ u \in C^0(C \setminus P) : \sup_{z \neq z'} \frac{|u(z) - u(z')|(r + r')^\alpha}{\text{dist}_g(z, z')^\alpha} := [u]_{b, 0, \alpha} < \infty \right\},$$

where we use polar coordinates $z = re^{i\theta}$ near each point of $P$. The space is defined in the standard way elsewhere on $C$. Note that $\text{dist}_g(z, z')^\alpha$ is comparable in a neighborhood of each such $p$ to $|r - r'|^\alpha + (r + r')^\alpha |\theta - \theta'|^\alpha$. Observe also that functions with this $b$-Hölder regularity need not be continuous at $P$.

It is useful in defining the higher regularity $b$-Hölder (and Sobolev) spaces to pass from the surface $C$ to its blowup at the points of $P$. This entails replacing each $p \in P$ by its circle of unit normal vectors; thus $C_P$ is a manifold with boundary equal to a union of circles and with smooth structure determined by the lifts of the smooth functions on $C$ and polar coordinates $(r, \theta)$. Now consider the space $V_b$, the span over $C^\infty(C_P)$ of the vector fields $r \partial_r$ and $\partial_\theta$. Invariantly, $V_b$ consists of all smooth vector fields on $C_P$ which are tangent to all boundaries. In terms of these, the higher regularity spaces $C^{0, \alpha}_b(C, P; i\text{su}(E))$ consist of those sections $u$ for which $V_1 \ldots V_j u \in C^{0, \alpha}_b(C, P; i\text{su}(E))$ for any $j \leq \ell$ and where each $V_i \in V_b$. We also define their weighted versions $r^\nu C^{0, \alpha}_b(C, P; i\text{su}(E))$ for any $\nu \in \mathbb{R}$.

The weighted $b$-Sobolev spaces are defined by

$$r^\delta H^\ell_b = \{ u = r^\ell v : V_1 \ldots V_j u \in L^2 \forall j \leq \ell \text{ and } V_i \in V_b \}.$$

All of these definitions adapt immediately to sections of vector bundles over $C$. 
There is a useful ‘comparison’ between the $b$-Sobolev and $b$-Hölder spaces. Functions in $r^{\nu+1}L^2$ just fail to lie in $r^\nu C^{0,\alpha}_b$, and conversely, $r^{\nu+\epsilon} C^{0,\alpha}_b$ is contained in $r^{\nu+1}L^2$ for any $\epsilon > 0$ but does not lie in this weighted $L^2$ space if $\epsilon \leq 0$. We say that the two spaces $r^{\nu} C^{0,\alpha}_b$ and $r^{\nu+1}L^2$ are commensurate.

We now refocus on the case where $P = D$. It is obvious from the definitions that for every $\nu \in \mathbb{R}$ and $\ell = 0, 1, 2, \ldots$,

$$L_t : r^\nu C^{\ell+2,\alpha}_b(C, D; i\text{ su}(E)) \to r^{\nu-2} C^{\ell,\alpha}_b(C, D; i\text{ su}(E))$$

is bounded. The first main result explains when this map is Fredholm.

**Proposition 5.1.** Writing $\Gamma(L_t)$ as the set of indicial roots of $L_t$, fix $\nu \in \mathbb{R} \setminus \Gamma(L_t)$.

(i) For any $\ell \geq 0$, the operator

$$L_t : r^\nu C^{\ell+2,\alpha}_b(C, P; i\text{ su}(E)) \to r^{\nu-2} C^{\ell,\alpha}_b(C, P; i\text{ su}(E))$$

is Fredholm, with index and nullspace both constant as $\nu$ varies over each connected component of $\mathbb{R} \setminus \Gamma(L_t)$.

(ii) If $L_t \gamma = \eta \in r^{\nu-2} C^{\ell,\alpha}_b(C, P; i\text{ su}(E))$ and $\gamma \in r^{\nu+1} L^2(C, P; i\text{ su}(E))$ for some $\nu' < \nu$, then $\gamma = \sum \gamma_j r^{\nu_j} + \tilde{\gamma}$, where the finite sum is over all indicial roots with $\nu_j \in [\nu', \nu)$ and $\tilde{\gamma} \in r^\nu C^{\ell+2,\alpha}_b$. (If $0 \in (\nu', \nu)$, one must allow an additional term $\tilde{\gamma}_0 \log r$ in this sum.)

(iii) If $L_t \gamma = \eta \in r^{\nu-2} C^{\ell,\alpha}_b$ and $\gamma \in r^{\nu+2} L^2$, then $\gamma \in r^\nu C^{\ell+2,\alpha}_b$.

(iv) If $\eta$ is polyhomogeneous and $\gamma \in r^\nu C^{2,\alpha}_b$ for some $\nu$, $\gamma$ is also polyhomogeneous, with exponents in its expansion determined by those in the expansion for $\eta$ together with the indicial roots of $L_t$ lying in $[\nu, \infty)$. In particular, any element of the nullspace of $L_t$ is polyhomogeneous.

**Remark 5.2.** As a clarification of part iv) of this Proposition, a function (or section) $u$ is said to be polyhomogeneous means if it is smooth away from the punctures and admits a classical expansion of the form

$$u \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^{N_j} u_{j\ell}(\theta) r^{\sigma_j}(\log r)^\ell,$$

where $\sigma_j$ is a sequence of possibly complex numbers with real parts tending to infinity, and where each $u_{j\ell}(\theta)$ is smooth. This expansion is to be interpreted in the sense that the difference between $u$ and any partial sum of the expression on the right vanishes like the next term in the asymptotic series, and that similar estimates hold for any derivative of $u$ and the corresponding term-by-term derivative of the formal series. In our specific situation, and more generally for solutions of conic elliptic operators, the exponents which
occur are closely related to the indicial roots of the problem. Polyhomogeneity is a reasonable substitute for smoothness, but one which is broad enough to contain solutions of conic elliptic equations.

5.2. **Pseudo-Friedrichs extensions.** We now adapt and modify a concept from $L^2$ theory: the Friedrichs extension of $L_t$. Classically, this is a canonical realization of $L_t$ as a self-adjoint operator, using only its semiboundedness and associated quadratic form, and starting from the core domain $C_0^\infty(C \setminus P)$. Briefly, for any $\gamma$ in this core domain we have that

$$\langle L_t \gamma, \gamma \rangle_{L^2} = \|d_A \gamma\|^2_{L^2} + 2r^2 \|\Phi_t \gamma\|^2_{L^2} \geq 0, \quad (5.2)$$

cf. [MSWW16, Proposition 5.1]. Then there is a unique self-adjoint extension of $L_t$, defined on its Friedrichs domain $D^2_{Fr}(L_t)$, with the property that (5.2) remains valid and for which the domain of this quadratic form is minimal amongst all such extensions. It is well-known that, despite the singularities of $L_t$, $D^2_{Fr}(L_t)$ is compactly included in $L^2$ and hence $(L_t, D^2_{Fr}(L_t))$ has discrete spectrum; by (5.2), all eigenvalues lie in $[0, \infty)$.

We now consider a Hölder space analogue which includes some additional flexibility in the range space. Fix any $\nu \geq 0$ and define

$$D^\ell, \alpha_{Fr}(L_t)(\nu) = \{ \gamma \in C^\ell, \alpha_b : L_t \gamma \in r^{-2}C^\ell + 2, \alpha_b \}. \quad \text{Since } L_t \gamma \in r^{-2}C^\ell + 2, \alpha_b \text{ is guaranteed only when } \gamma \in r^\nu C^\ell + 2, \alpha_b, \text{ the a priori hypothesis in this definition that } \gamma \in C^\ell, \alpha_b \implies \text{that } \gamma \text{ must have some special regularity properties in order that } L_t \gamma \text{ lies in the correct space. Indeed, there is a regularity theorem for conic elliptic operators which states that if } \gamma \in D^\ell, \alpha_{Fr}(L_t)(\nu) \text{ for some } \nu > 0, \text{ then necessarily } \gamma \text{ has a finite expansion involving the indicial roots of } L_t \text{ in } [0, \nu) \text{ plus a remainder in } r^\nu C^\ell + 2, \alpha_b, \text{ i.e.,}

$$D^\ell, \alpha_{Fr}(L_t)(\nu) = \left\{ \gamma = \sum_{0 \leq v_j < \nu} \gamma_j r^{v_j} + \tilde{\gamma} : \gamma_j \in C^\infty(S^1), \tilde{\gamma} \in r^\nu C^\ell + 2, \alpha_b \right\}. \quad \text{The coefficients } \gamma_j \text{ here are eigenfunctions of the induced operator on } S^1 \text{ with eigenvalue } -v_j^2. \text{ Since } \gamma \text{ is assumed in this definition to be bounded, this partial expansion omits the unbounded term } \log r, \text{ which also corresponds to the indicial root } 0. \text{ Note that we are making the simplifying assumption here that } \nu < 1; \text{ in general, when } \nu \text{ is larger, this partial expansion will also involve terms of the form } r^{v_j + i}, i \in \mathbb{N}_0, \text{ so long as } v_j + i < \nu. \text{ These extra terms are needed to account for error terms caused by the higher order terms in the Taylor expansion of the coefficients of } L_t; \text{ the guiding principle is that } L_t \text{ must annihilate all terms up to order } v - 2.
Proposition 5.3. The graph of $\mathcal{L}_t$ over $D^t_{Fr}(\mathcal{L}_t)(v)$ is a closed subspace of $C^{\ell,\alpha}_b \oplus r^{\nu-2}C^{\ell,\alpha}_b$, and the induced graph norm is equivalent to
\[
||\gamma||'_{D^t_{Fr}(\mathcal{L}_t)(v)} = \sum_{0 \leq \nu_j < \nu} \sup \gamma_j + ||\tilde{\gamma}||_{r^{\nu}C^\ell,2.\alpha}.
\]

Proof. The span $V$ of the functions $\gamma_j(\theta)r^{\nu_j}$ is finite dimensional, and has only trivial intersection with $r^{\nu}C^\ell,2.\alpha$. Furthermore, by virtue of the a priori estimate
\[
||\gamma||_{r^{\nu}C^\ell,2.\alpha} \leq C(||\mathcal{L}_t\gamma||_{r^{\nu-2}C^\ell,\alpha} + ||\gamma||_{C^\ell,\alpha}),
\]
$r^{\nu}C^\ell,2.\alpha$ is a closed subspace in the graph norm, so the natural map $V \oplus r^{\nu}C^\ell,2.\alpha \to D^t_{Fr}(\mathcal{L}_t)(v)$ is continuous and bijective, hence by the open mapping theorem is a topological isomorphism. This shows that $D^t_{Fr}(\mathcal{L}_t)(v)$ is a closed subspace in the graph norm, and furthermore, since the norms on these two spaces are the norms we wish to compare, they must be equivalent.

Remark 5.4. Near any point $p \in D$, the leading term $\gamma_0$ in the expansion of any $\gamma \in D^0_{Fr}(v)$ is a constant diagonal Hermitian matrix. Indeed, the indicial root 0 for $\mathcal{L}_t$ is associated with the $0^{th}$ eigenvalue of the Laplacian on $S^1$, and occurs only for the indicial equation on the diagonal part of $\gamma$.

Lemma 5.5. If $\gamma \in D^0_{Fr}(v)$, then (5.2) holds whenever $\nu > 0$.

Proof. The boundary term in the integration by parts leading to (5.2) is
\[
\lim_{\nu \searrow 0} \int_{r=e} \langle \partial_r \gamma, \gamma \rangle r d\theta.
\]
The leading term $\gamma_0$ is annihilated by $\partial_r$, so the leading term in the pointwise inner product is $O(r^{\mu-1})$, where $\mu$ is the minimum of $\nu$ and the first positive indicial root of $\mathcal{L}_t$ for the diagonal operator. Multiplying by $r^\mu$, this boundary term vanishes since $\mu > 0$. We note also that both $\langle M_{\Phi}, \gamma, \gamma \rangle$ and $||[\Phi_t, \gamma]||^2$ are integrable. Indeed, in the strongly parabolic case, these are each bounded pointwise by $C r^{2(a_1-\alpha_2)}$, while in the weakly parabolic case, the leading part of $\Phi_t$ and its adjoint are both diagonal, hence commute with $\gamma_0$, so these two terms are bounded by $C r^{2\mu-2}$ for some $\mu > 0$.

Lemma 5.6. The map $\mathcal{L}_t : D^2_{Fr} \to L^2$ is invertible provided $Z \cup D_s \neq \emptyset$.

Remark 5.7. The sum of the orders of the zeros and poles of any meromorphic quadratic differential on $C$ equals $4g-4$. If there are no zeros, then there can be at most two poles of order 2, and in fact the only cases where $Z \cup D_s = \emptyset$ is when $C = S^2$ and $q = c^2 dz^2/z^2$, or when $C = T^2$ and $q$ has no zeros or poles. We henceforth exclude these two trivial cases.
Proof. If $\gamma$ is in the classical $L^2$ Friedrichs domain, then (5.2) holds, and hence if $L_t\gamma = 0$, then $d_{A_t}\gamma = 0$ and $[\Phi_t, \gamma] = 0$. The first equality implies that $\gamma$ is parallel, so $|\gamma|^2$ is constant on $C \setminus D$. If $D_s$ is nonempty, then in a standard holomorphic coordinate near a strongly parabolic point, the equation $[\Phi, \gamma] = 0$ yields

$$\gamma = \begin{pmatrix} 0 & u_1 \\ \bar{a}_1 & 0 \end{pmatrix}. $$

Taking into account the indicial root set for this off-diagonal term, we must have $u_1 = O(r^{\nu_0})$, where $\nu_0$ is the first positive indicial root for the off-diagonal operator. Hence in this case we obtain that $\gamma \equiv 0$. Similarly, near any point of $Z$, $\gamma$ is also off-diagonal and in this case $[\Phi, \gamma] = 0$ yields that $u_1 z = re^{2\ell t}a_1$. A simple computation shows that $u_1 \equiv 0$ in the region where $\ell_t > 0$, hence $\gamma \equiv 0$ once again. Since $L_t$ on this Friedrichs domain is self-adjoint and injective, it is bijective and hence an isomorphism.

\[\Box\]

**Lemma 5.8.** For $v > 0$ and $\gamma \in D_{Fr}^{0,\alpha}(L_t)(v)$, if $L_t\gamma = 0$, then $d_{A_t}\gamma = 0$ and $[\Phi_t, \gamma] = 0$. Hence if $Z \cup D_s \neq \emptyset$, then $\gamma = 0$. In this case, $L_t : D_{Fr}^{0,\alpha}(L_t)(v) \to r^{-v-2}C_{b}^{0,\alpha}$ is an isomorphism.

**Proof.** Since $\gamma = \gamma_0 + \tilde{\gamma}$, the boundary term $\int_{r=\epsilon} \langle \partial_r \tilde{\gamma}, r^0 \rangle r d\theta$ vanishes as $\epsilon \to 0$, so $d_{A_t}\gamma = 0$ and $[\Phi_t, \gamma] = 0$ as in the previous Lemma. Thus repeating the same arguments, except for the trivial case where $Z \cup D_s$ is empty, $L_t$ is injective.

Now fix any $\eta \in r^{-v-2}C_{b}^{0,\alpha}$. If $v > 1$, then $r^{-v-2}C_{b}^{0,\alpha} \subset L^2$, so there exists some $\gamma \in D_{Fr}^{0,\alpha}$ such that $L_t\gamma = \eta$. By parts (iii) and (iv) of Proposition 5.1, $\gamma$ decomposes as a finite sum $\sum r^{v_j} \gamma_j$, with each indicial root $v_j \in [-1, v)$ and a remainder term $\tilde{\gamma} \in r^{v}C_{b}^{2,\alpha}$. However, no term with $r^{v_j}$ for $v_j < 0$ or log $r$ can lie in the $L^2$ Friedrichs domain, so the sum is actually only over nonnegative $v_j$. Thus $\gamma$ lies in $D_{Fr}^{0,\alpha}(v)$. By the open mapping theorem, $L_t : D_{Fr}^{0,\alpha} \to C_{b}^{0,\alpha}$ is an isomorphism.

If $0 < v \leq 1$, we must argue differently since we cannot obtain the solution $\gamma$ from the $L^2$ theory. For this we note that, extending part (i) of Proposition 5.1, $L_t : r^{-v'}C_{b}^{2,\alpha} \to r^{-v'-2}C_{b}^{0,\alpha}$ is surjective when $v' > 0$. This can be deduced from a duality statement on weighted $L^2$ spaces and some version of part (iii) of this same Proposition. We refer to [MW17] for details. In any case, now apply part (ii) of this Proposition to obtain that $\gamma$ is a finite sum of terms $\gamma_j r^{v_j}$ with $v_j \in [-v', v)$, a term $\tilde{\gamma}_0 \log r$ since $0 \in (-v', v)$, and a ‘remainder’ term $\tilde{\gamma} \in r^{v}C_{b}^{2,\alpha}$. We can take $v'$ arbitrarily small, so the only potentially problematic term in this expansion is $\tilde{\gamma}_0 \log r$.

At this point we have obtained a solution $\gamma$ to $L_t\gamma = \eta$ which has the property that $\gamma \sim \tilde{\gamma}_0(p) \log r + \gamma_0(p) + \ldots$ at each $p \in D$. We claim that there is another solution to this same equation which is bounded, i.e., does not have these logarithmic terms at any of the singular points of $q$. To prove this, we show that there exists an element $\tilde{\gamma}$ in the nullspace of $L_t$ which has the same asymptotics $\tilde{\gamma} \sim \tilde{\gamma}_0(p) \log r + O(1)$ at each $p \in D$. 
Equivalently, we claim that the map
\[ \ker (L_t) \cap r^{-\nu} C^2_b \ni \gamma \mapsto (\tilde{\gamma}_0(p))_{p \in D} \in \mathbb{R}^{|D|} \]
is a bijection. (Here \(|D|\) is the number of points in \(D\).)

To prove this claim, we quote two further facts, both discussed in [MW17]. First, consider the two Fredholm mappings \( L_t : r^{\pm \nu} C^2_b \to r^{\pm \nu - 2} C^0_b \). There is a relative index theorem which states that the difference between the indices of these two maps is the algebraic multiplicity of the indicial root 0, which in this case equals \(2|D|\). (This is because at each \(p \in D\) the multiplicity of the indicial root 0 for the one-dimensional diagonal part of \(L_t\) is 2, while 0 is not an indicial root for the nondiagonal part). Next using that \(L_t\) is injective on positively weighted spaces and surjective on negatively weighted spaces, we see that this difference of indices is in fact equal to twice the dimension of the nullspace of \(L_t\) on \(r^{-\nu} C^2_b\). Finally, the map from this nullspace to the vector \((\tilde{\gamma}_0(p))_{p \in D} \in \mathbb{R}^{|D|}\) is injective, since any nullspace element with no log terms lies in \(D^0_{Fr}(\nu')\), and hence must be trivial. This proves that the map is bijective, and hence we can find a nullspace element \(\tilde{\gamma}\) with precisely the correct log \(r\) coefficients at each \(p \in D\). This proves the claim. □

We have now established the existence of an inverse \(L_t^{-1}\) both in this Hölder setting, and we denote this by \(G_t\). It is a conic pseudodifferential operator of order \(-2\), and the structure of its Schwartz kernel will play a role below.

5.3. Uniform mapping properties. In this section we estimate the growth of the norm of \(G_t = L_t^{-1} : C^0_b \to D^0_{Fr}\) as \(t \to \infty\). We begin with \(L^2\) and Sobolev estimates.

**Proposition 5.9.** The norm of the inverse \(G_t : L^2 \to L^2\) is uniformly bounded as \(t \to \infty\).

**Proof.** Let \(\lambda_t\) denote the smallest eigenvalue of \(L_t\). We have already shown that \(\lambda_t > 0\) for every \(t\), and this proposition is equivalent to the claim that there exists \(\kappa > 0\) such that \(\lambda_t \geq \kappa\) for every \(t \geq 1\). Define \(L_t^0 = \Delta_{A_t} - i * M_{\Phi_t}\). Since \(-i * M_{\Phi_t} \geq 0\), it follows that \(L_t \geq L_t^0\) for \(t \geq 1\), and thus it suffices to show that the smallest eigenvalue \(\lambda_t^0\) of \(L_t^0\) is bounded below by some \(\kappa^0 > 0\).

A special role is played in this argument by points in the sets \(Z\) and \(D_s\) since the structure of \(L_t\) changes as \(t \to \infty\) at these points (but not at points in \(D_w\)). Choose a local holomorphic coordinate \(z\) centered at each point \(p \in Z \cup D_s\) and a disc \(\mathbb{D}_p\) around each such \(p\). We do not label these coordinate patches separately, and for simplicity, tacitly assume that \(\mathbb{D}_p = \{|z| \leq 3/4\}\). Define the family of smooth positive weight functions \(\mu_t\)
on $C$ such that

$$\mu_t(z) = \begin{cases} 
(t^{-\frac{4}{5}} + |z|^2)^{\frac{1}{2}} & \text{on each } D_p, \ p \in Z, \\
(t^{-4} + |z|^2)^{\frac{1}{2}} & \text{on each } D_p, \ p \in D, \\
1 & \text{on } C \setminus \bigcup_{p \in Z \cup D} D_p.
\end{cases}$$

For each $t \geq 1$, denote by $\psi_t$ the eigenfunction of $L^0_t$ with eigenvalue $\lambda^0_t$, normalized so that

$$\sup_{C \setminus D} \mu^\delta t|\psi_t| = 1,$$

where $\delta > 0$ is a fixed constant specified below. We assume by way of contradiction that $\lambda^0_t \to 0$, at least for some sequence of values $t_j \to \infty$. Write $\mu_j = \mu_{t_j}$ and $\psi_j = \psi_{t_j}$, and for each $j$, choose a point $q_j$ such that $\mu^\delta_j(q_j)|\psi_j(q_j)| = 1$.

**Case 1.** Suppose that (at least some subsequence) of the $q_j$ does not converge to $Z \cup D$. By the local boundedness of the $\mu_j$, elliptic regularity, and a diagonalization argument, we may choose a subsequence of the $\psi_j$ (labeled again as $\psi_j$) which converges in $C^\infty$ on any compact subset of $C^\times := C \setminus (Z \cup D)$ to a limit $\psi_\infty$. Since $\lambda^0_j \to 0$, this limiting function satisfies $L^0_\infty \psi_\infty = 0$ and $|\psi_\infty| \leq |z|^{-\delta}$. Furthermore, $\psi_\infty$ is not identically 0 since it is nonvanishing at the point $q_j$.

As described earlier, $L^0_\infty$ is a conic differential operator. By local conic elliptic regularity theory, $\psi_\infty$ has a complete asymptotic expansion near each $p \in Z \cup D$ in powers $r^{\nu_i}$ where the $\nu_i$ are indicial roots of $L^0_\infty$, and possibly also including the term $\log r$. Choosing $\delta$ smaller than the absolute value of the first nonzero indicial root at any point of $Z \cup D$, each $\nu_i$ must be nonnegative, but a priori the $\log r$ term may still be present. To rule this out, integrate by parts when $t$ is large but finite to get

$$\lambda^0_j \|\psi_j\|^2 = \langle L^0_j \psi_j, \psi_j \rangle = \int |dA_j \psi_j|^2 + |[\Phi_j, \psi_j]|^2.$$

Since $|\psi_j|^2 \leq C \mu_j^{-2\delta}$ is uniformly $L^1$, the left hand side tends to 0 and we conclude that $dA_\infty \psi_\infty = 0$, $[\Phi_\infty, \psi_\infty] = 0$. The latter equation implies that $\psi_\infty$ is a multiple of $\Phi_\infty$, hence purely off-diagonal near $Z \cup D$, and hence must vanish at least like $r^{1/2}$. The first equation implies that $|\psi_\infty|$ is constant, and hence $\psi_\infty \equiv 0$. This is a contradiction and hence this case cannot occur.

**Case 2.** Next suppose that $q_j$ converges to a point $q_\infty \in Z$, and that $\mu^\delta_j|\psi_j|$ converges to zero on any compact subset of $C^\times$. The point $q_j$ corresponds to $z_j$ in a fixed holomorphic coordinate $z$ around $q_\infty$, and we must distinguish two cases, depending on the rate at which $z_j \to 0$.

To understand this, we first digress and consider the rescaled problem. Near any point of $Z$, write $\rho = t^{2/3}r$ (note the difference with the change of variables $\rho = \frac{8}{3} tr^{3/2}$ used
two cases based on whether or not $t \to t^2$. Since $A_{\text{app}}$ equals $A_{\text{model}}$ in this neighborhood, its coefficients depend only on $\varrho$ here. A brief calculation shows that $\Delta_{A_t} = t^{4/3} \Delta_{\varrho}$, the connection Laplacian associated to $A_{\text{model}}^t$. Recall that the difference $\Delta_{\varrho} - \Delta_{A_{\text{app}}}^t$ decays exponentially as $\varrho \to \infty$. The other term $-i \ast t^2 M_{\varrho}$ behaves similarly: the matrix entries of $\Phi_t$ and $\Phi_t^r \ast \varrho$ equal $r^{1/2}$ times functions of $t^{2/3} r = \varrho$, so $t^2 M_t = t^{4/3} M_{\varrho}$, where $M_{\varrho}$ is an endomorphism with coefficients depending only on $(\varrho, \theta)$. Altogether, in this neighborhood,

$$L_t = t^{4/3} (\Delta_{\varrho} - i \ast M_{\varrho}), \quad L_t^0 = t^{4/3} (\Delta_{\varrho} - i \ast t^{-4/3} M_{\varrho}).$$

Return now to the argument at hand. Set $w = t_j^{2/3} z$ and $w_j = t_j^{2/3} z_j$. We must distinguish two cases based on whether or not $w_j$ remains bounded.

Suppose that $|w_j| \leq C$ for all $j$. Set $\psi_j(w) = t^{-4\delta/3} \psi_j(t_j^{-2/3} w)$ so that $|\psi_j(z)| \leq \mu_j^{-\delta}$ with equality at $z_j$ is the same as $|\tilde{\psi}_j(w)| \leq (1 + |w|^2)^{-\delta/2}$ with equality at $w_j$. This function solves

$$(\Delta_{\varrho} - i \ast t_j^{-4/3} M_{\varrho}) \tilde{\psi}_j(w) = t_j^{-4/3} \lambda_j^{\delta} \tilde{\psi}_j(w).$$

Taking the limit, we obtain a nontrivial $\tilde{\psi}_\infty$ defined on all of $\mathbb{C}$, which decays like $|w|^{-\delta}$ as $w \to \infty$, and which solves $\Delta_{\varrho} \tilde{\psi}_\infty = 0$. The splitting of $\tilde{\psi}_\infty$ into its diagonal and off-diagonal parts is now global on $\mathbb{C}$; we call these parts $\psi'_\infty$ and $\psi''_\infty$, respectively. The operator induced by $\Delta_{\varrho}$ on $\psi'_\infty$ is simply the scalar Laplacian $\Delta_{\varrho}$, and hence by standard conic theory, $\psi'_\infty$ decays like $\varrho^{-1}$. This is sufficient to justify the integration by parts $\langle \Delta_{\varrho} \psi'_\infty, \psi'_\infty \rangle = |d_{A_{\varrho}} \psi'_\infty|^2 = 0$. Hence $\psi'_\infty$ has has constant norm, but also decays at infinity, so it must vanish identically. On the other hand, the induced operator on the off-diagonal part is $\Delta_0 - F_1^r (k + 4 (F_1^r)^2)$. Expanding $\psi''_\infty$ into Fourier series, then the $k$th Fourier component satisfies $(-r^{-2} (r \partial_r)^2 - (i \partial_\theta - 2 F_1^r)^2) \psi''_{\infty,k} = 0$. As $\varrho \to \infty$, this operator converges to $-r^{-2} (r \partial_r)^2 - (k - 1/2)^2$, and by standard ODE theory, any bounded solution must decay exponentially. Hence by the same argument, this term too must vanish identically.

The other possibility is that $\sigma_j := |w_j| \to \infty$. We now rescale further, letting $\bar{w} = w/\sigma_j$ (or altogether, $\bar{w} = t_j^{-2/3} \sigma_j^{-1} z$). Defining $\psi_j(\bar{w}) = \sigma_j^{-\delta} \tilde{\psi}_j(\sigma_j \bar{w})$, then

$$\sigma_j^{-\delta} |\tilde{\psi}_j(\sigma_j \bar{w})| \leq \sigma_j^{-\delta} (1 + \sigma_j^2 |\bar{w}|^2)^{-\delta/2} \Rightarrow |\tilde{\psi}_j(\bar{w})| \leq (\sigma_j^{-2} + |\bar{w}|^2)^{-\delta/2},$$

with equality at some point $\bar{w}_j$ with $|\bar{w}_j| = 1$.

The limit $\tilde{\psi}_\infty$ satisfies $|\tilde{\psi}_\infty| \leq |w|^{-\delta}$ and $\Delta_{\varrho} \tilde{\psi}_\infty = 0$. This operator is conic at both 0 and $\infty$, and in fact is homogeneous of degree $-2$. This means that if we expand $\tilde{\psi}_\infty$ into Fourier series in $\theta$, then each coefficient is a sum of at most two monomials $r^{\nu_j}$; the Fourier mode at energy 0 has coefficient $r^0$ or $\log r$. However, none of these terms are bounded by $r^{-\delta}$ at both 0 and $\infty$, which is a contradiction. Hence this case cannot occur.
**Case 3.** The next scenario is that $q_j \rightarrow q\infty \in D_s \cup D_w$. As before, we distinguish between two cases, depending on the rate at which the points $z_j$ tend to 0. Near strongly parabolic points, the appropriate scaling factor is $t^2$ so we define $	ilde{\psi}_j(w) = t_j^{-2\delta}\psi_j(w/t_j)$ which gives $|\tilde{\psi}_j(w)| \leq (1 + |w|^2)^{-\delta/2}$, with equality at some point $w_j$. Note that since $\psi_j \in D_{Fr}^2$, it is bounded but does not necessarily extend smoothly across the origin.

Assume first that $w_j$ remains bounded. We can repeat the arguments for the previous case almost verbatim and obtain a nontrivial limit $\tilde{\psi}_\infty(w)$ on $\mathbb{C} \setminus \{0\}$ which satisfies $\Delta_0 \tilde{\psi}_\infty = 0$, where now $\Delta_0$ is the connection Laplacian associated to the fiducial solution around a strongly parabolic point if $q\infty \in D_s$ and the $t$-independent fiducial solution when $q\infty \in D_w$. In addition $|\tilde{\psi}_\infty(w)| \leq (1 + |w|^2)^{-\delta/2}$ with equality at some point $w\infty$. By conic elliptic theory, $\tilde{\psi}_\infty$ has a complete asymptotic expansion at $w = 0$ and another as $w \rightarrow \infty$. Since $\tilde{\psi}_\infty$ is bounded near $w = 0$, its expansion there has only nonnegative exponents and no log $r$ term. Choosing $0 < \delta < \min\{a_2 - a_1, 1 - (a_2 - a_1)\}$, then $\tilde{\psi}_\infty$ decays fast enough to justify the usual integration by parts, which then implies as before that $|\tilde{\psi}_\infty|$ is constant. It cannot vanish at infinity unless it is identically zero.

If $w_j = t_j^2z_j$ is unbounded, then proceeding as in the second part of Case 2, we arrive at a nontrivial solution $\tilde{\psi}_\infty$ which satisfies $\Delta_{A,\infty} \tilde{\psi}_\infty = 0$ and $|\tilde{\psi}_\infty| \leq |\tilde{\omega}|^{-\delta}$, where $\tilde{\omega} = w/|w_j|$. This case is ruled out as before. □

We have now proved that the unbounded operator $L_t : L^2 \rightarrow L^2$ is invertible and has inverse $G_t = L_t^{-1}$ which has norm bounded independently of $t$ for $t \geq 1$. We now determine the behavior of the norm of $G_t$ mapping between other spaces.

First consider $G_t : L^2 \rightarrow H^2_{Fr}$, which is well-defined since $D_{Fr}^2 \subset H^2_{Fr}$. Fix $\eta \in L^2$ and rewrite $L_t\eta = \eta$ as $L_t^*\eta = \eta + (t^2 - 1)i \ast M_\Phi \gamma$.

Since $\|\gamma\|_{L^2} \leq C\|\eta\|_{L^2}$ with $C$ independent of $t$, consider the right hand side as a function $\tilde{\eta}$ with $\|\tilde{\eta}\|_{L^2} \leq Ct^2\|\eta\|_{L^2}$. The $H^2$ bound for $\gamma$ on any compact set disjoint from $Z \cup D$ follows from standard elliptic theory and the $L^2$ estimate for $\gamma$. We consider this estimate near each of the different types of points of $Z \cup D$ in turn.

The easiest is in fact the estimate near $q \in D_w$ for the simple reason that the operator is $t$-independent there. Thus $L_t^*\eta = \tilde{\eta}$ is a fixed conic operator in a neighborhood around such a $q$ and the estimate follows from standard conic theory, cf. [MW17].

If $q \in D_s$, then $L_t^2$ is an elliptic conic operator for all $t$ including $t = \infty$, but the indicial root structure changes in the limit as $t \rightarrow \infty$.

It is possible to construct a uniformly $t$-dependent family of parametrices $G_{p,t}$ for $L_t^2$ in the ‘calculus with bounds’. This implies that any combination of up to two $b$-derivatives involving $r\partial_r$ and $\partial_\rho$ applied to $G_{p,t}$ is bounded on $L^2$ with norm independent of $t$, which is what we require. However, we can derive this in a more elementary way using the
Mellin transform. Recall that if \( f(r, \theta) \in L^2(rdrd\theta) \) is supported in \( r \leq 1 \), then
\[
f_M(\zeta, \theta) = \int_0^\infty f(r, \theta) r^{i\zeta} \frac{dr}{r}
\]
is holomorphic in the lower half-plane \( \text{Im} \, \zeta < 1 \) with \( L^2 \) norm on each line \( \text{Im} \, \zeta = -\epsilon \) uniformly bounded as \( \epsilon \searrow 0^- \). If \( f \in \mathcal{L}^2(rdrd\theta) \) then \( f_M \) is holomorphic in the lower half-plane \( \text{Im} \, \zeta < v - 1 \). Now write \( L_1 \gamma = \bar{\eta} \) as \( r^2 \Delta A_1 \gamma = i \ast r^2 M_{\Phi_1} \gamma + r^2 \bar{\eta} \). The operator on the left is \( -(r \partial_r)^2 - \partial_\theta^2 \) for the diagonal part and \( -(r \partial_r)^2 + (i \theta - 2 F_i^p(r))^2 \) for the off-diagonal part.

Now, \( i \ast r^2 M_{\Phi_1} \gamma \) is uniformly \( \in \mathcal{L}^2(rdrd\theta) \) as \( t \leq \infty \) for some \( \delta > 0 \). For the diagonal part, pass to the Mellin transform and write the equation as \( (\zeta^2 + \partial_\theta^2) M_{\gamma M} = F_M \) where \( F \) is the sum of the two terms on the right and \( F_M \) its Mellin transform. Thus \( F_M \) is holomorphic in the half-plane \( \text{Im} \, \zeta < 1 + \delta \). The resolvent \( (\zeta^2 + \partial_\theta^2)^{-1} \) has poles at the points \( ik, k \in \mathbb{Z} \), hence applying it to both sides we see that \( \gamma_M \) is meromorphic in \( \text{Im} \, \zeta < 1 + \delta \) with poles possibly at \( ki, k = 1, 0, -1, \ldots \) However, since \( \gamma \in L^2 \), then a priori \( \gamma_M \) is holomorphic in \( \text{Im} \, \zeta < 1 \) with uniform \( L^2 \) bounds on each horizontal line; this prohibits any of the possible poles of \( \gamma_M \). The expression \( \gamma_M = (\zeta^2 + \partial_\theta^2)^{-1} F_M \) shows that in fact \( \gamma_M(\zeta, \theta) \) takes values in \( H^2(S^1) \) and both \( \zeta \gamma_M \) and \( \zeta^2 \gamma_M \) satisfy uniform \( L^2 \) estimates on each horizontal line \( \text{Im} \, \zeta < 1 \). Taking the inverse Mellin transform, we find that \( \gamma \in H_b^2 \).

The argument for the off-diagonal part has one additional step. Indeed, we can write the equation as
\[
-(r \partial_r)^2 + \partial_\theta^2 \gamma = i \ast r^2 M_{\Phi_1} \gamma + r^2 \bar{\eta} + 4i F_i^p \partial_\theta \gamma - 4(F_i^p)^2 \gamma.
\]

The right hand side lies in \( L^2 \) as a function of \( r \), uniformly in \( t \), with values in \( H^{-1}(S^1) \). Now take the Mellin transform and apply the resolvent \( (\zeta^2 + \partial_\theta^2)^{-1} \); this shows that \( \gamma_M, \zeta \gamma_M \) and \( \zeta^2 \gamma_M \) are holomorphic in \( \text{Im} \, \zeta < 1 \) with values in \( H^1(S^1) \). We can recycle this information back into the initial equation, so that the right hand side is now \( L^2 \) in both \( r \) and \( \theta \) so we deduce that \( \gamma \in H_b^2 \) uniformly in \( t \) as claimed.

The analysis near points of \( Z \) is essentially the same, and we leave the details to the reader. In summary, we have proved the

**Proposition 5.10.** The norm of the inverse \( L_t^{-1} : L^2 \rightarrow H_b^2 \) is bounded by \( Ct^2 \) for some \( C > 0 \).

We may now prove bounds for the norm of this inverse acting between \( \text{b-Hölder spaces} \). We show first that \( L_t^{-1} : C_b^{0,\alpha} \rightarrow C_b^{0,\alpha} \) has norm bounded by \( Ct^2 \). This is not an optimal estimate, but is sufficient for our purposes. The key to this is Sobolev embedding and the scale-invariant nature of the \( \text{b-Hölder} \) and \( \text{b-Sobolev} \) norms. More specifically, using the embedding \( H^2 \subset C^{0,\alpha} \) on any compact set of \( \mathbb{C} \setminus (Z \cup D) \), it suffices to establish the norm estimate in a neighborhood \( \mathbb{D}_p \) of each \( p \in Z \cup D \). Using cutoff functions, we may as well assume that \( \gamma \) is supported in such a neighborhood. Decompose \( \mathbb{D}_p \setminus \{ p \} \) into a
countable union of annuli $A_j = \{2^{-j-1} \leq r \leq 2^{-j}\}$. Denote by $A_j' = \{2^{-j-2} \leq r \leq 2^{-j+1}\}$ a slight enlargement of $A_j$. Let $k_j : A_1 \to A_j$ be the dilation $(r, \theta) \to (2^{-j} r, \theta)$, and let $\{\chi_j\}$ be a partition of unity relative to this cover. Then, by definition of the $b$-norms
\[
\|\eta\|_{C^0_b} \simeq \sup_j \|\chi_j \eta\|_{C^0_b} = \sup_j \|k_j^* (\chi_j \eta)\|_{C^0_b(A_1)}
\]
and
\[
\|\gamma\|_{H^2_b} \simeq \sum_{j=0}^\infty \|\chi_j \gamma\|_{H^2_b} = \sum_{j=0}^\infty \|k_j^* (\chi_j \gamma)\|_{H^2_b(A_1)}^2.
\]
Now using that $H^2(A_1) \subset C^{0,\alpha}(A_1) \subset L^2(A_1)$, we have
\[
\|\gamma\|_{C^0_b} \leq C \sup_j \|\chi_j \gamma\|_{C^0_b} \leq C \sup_j \|\chi_j \gamma\|_{H^2_b}^2 \\
\leq C \sum_j \|\chi_j \gamma\|_{H^2_b}^2 \leq C \|\gamma\|_{H^2_b} \leq Ct^2 \|\eta\|_{L^2} \leq Ct^2 \|\eta\|_{C^0_b}.
\]
The second inequality is the result of dilating by $k_j^*$, applying the ordinary Sobolev embedding bound, then dilating back by $(k_j^{-1})^*$. This establishes that

**Proposition 5.11.** The norm of the inverse $L_i^{-1} : C^{0,\alpha}_b \to C^{0,\alpha}_b$ is bounded by $Ct^2$ for some $C > 0$. 

Finally, write $L_i \gamma = \tilde{\eta} := \eta + i \ast t^2 M_{\Phi_t} \gamma$ again, and observe that $\|\tilde{\eta}\|_{C^0_b} \leq Ct^2 \|\eta\|_{C^0_b}$. Localizing to each annulus $A_j$ and applying ordinary Schauder estimates to the rescalings of $\gamma$ and $\tilde{\eta}$ there, we obtain
\[
\|\gamma|_{A_j}\|_{C^{2,\alpha}_b} \leq C \left( \|\tilde{\eta}|_{A_j'}\|_{C^{0,\alpha}_b(A_j')} + \|\gamma|_{A_j'}\|_{C^{0,\alpha}_b(A_j')} \right),
\]
so we now conclude that

**Corollary 5.12.** The norm of the inverse $L_i^{-1} : C^{0,\alpha}_b \to C^{2,\alpha}_b$ is bounded by $Ct^4$ for some $C > 0$. The image of this mapping is the Hölder Friedrichs domain $D^{0,\alpha}_{Fr}$. 

We now come to the main estimate.

**Proposition 5.13.** If $v > 0$ is less than the smallest positive indicial root of $L_i$, then the norm of the inverse $L_i^{-1} : r^{v-2} C^{0,\alpha}_b \to D^{0,\alpha}_{Fr}(v)$ is bounded by $Ct^4$ for some $C > 0$.

**Proof.** Fix any $\eta \in r^{v-2} C^{0,\alpha}_b$ and write $\gamma = L_i^{-1} \eta \in D^{0,\alpha}_{Fr}(v)$.

For each $p \in Z \cup D$, choose a nonnegative smooth cutoff function $\chi_p$ supported in the unit disk $D_p(1)$ and equaling 1 in $D_p(1/2)$, and define $\gamma_p = \chi_p G^\text{model}_i(\chi_p \eta)$. Now, $L_i \gamma_p = \eta$ in $D_p(1/2)$ since $L_i G^\text{model}_i = 1$ in this neighborhood. Thus if $\zeta = \gamma - \sum_{p \in Z \cup D} \gamma_p$, then $L_i \zeta = \tilde{\eta}$, where $\tilde{\eta}$ vanishes near each $p \in D$ and $\tilde{\eta} = \eta$ outside $\cup_p D_p(1)$. Clearly $\tilde{\eta} \in C^{0,\alpha}_b$, so by Corollary 5.12, $\|\zeta\|_{C^{2,\alpha}_b} \leq Ct^4 \|	ilde{\eta}\|_{C^{0,\alpha}_b}$.
It remains to estimate each $\| \gamma_p \|_{C_b^{2,\alpha}}$ in terms of $\| \eta_p \|_{\nu^{-2}C_b^{0,\alpha}}$, and then to estimate $\| \bar{\eta} \|_{C_b^{0,\alpha}}$ in terms of $\sum \| \eta \|_{\nu^{-2}C_b^{0,\alpha}}$.

First let $p \in \mathbb{Z}$. Then $G_t^{\text{model}}(z, \bar{z}) = G_\rho(t^{2/3}z, t^{2/3}\bar{z})$, see [MSWW19, §5.1] for details. Writing $(\eta_p)_\lambda(z) = \eta_p(\lambda z)$, we calculate that

$$\left\| \int G_t^{\text{model}}(z, \bar{z}) \eta_p(\bar{z}) \, d\bar{z} \right\|_{C_b^{2,\alpha}} = t^{-4/3} \left\| \int G_\rho(t^{2/3}z, \bar{w})(\eta_p)(t^{-2/3}(\bar{w})) \, d\bar{w} \right\|_{C_b^{2,\alpha}}$$

$$\leq C t^{-4/3} \left\| \int G_\rho(w, \bar{w}) \eta_p(\bar{w}) \right\|_{C_b^{2,\alpha}} \leq C t^{-4/3} \| \eta_p \|_{C_b^{0,\alpha}}$$

by dilation invariance of the $b$-Hölder norms.

Next, if $p \in D_s$ then $G_t^{\text{model}}(z, \bar{z}) = G_\rho(t^2z, t^2\bar{z}) \, d\bar{z}$. Rescaling in both $z$ and $\bar{z}$, we obtain

$$\left\| \int G_t^{\text{model}}(z, \bar{z}) \eta_p(\bar{z}) \, d\bar{z} \right\|_{C_b^{2,\alpha}} = t^{-4} \left\| \int G_\rho(t^2z, \bar{w})(\eta_p)(t^{-2}\bar{w}) \, d\bar{w} \right\|_{C_b^{2,\alpha}}$$

$$\leq C t^{-4} \left\| \int G_\rho(w, \bar{w}) \eta_p(\bar{w}) \right\|_{C_b^{2,\alpha}} .$$

Now write

$$\int G_\rho(w, \bar{w}) \eta_p(\bar{w}) = \int G_\rho(w, \bar{w}) \left| \bar{w} \right|^{\nu - 2} \left( \left| \bar{w} \right|^{2 - \nu} \eta_p(\bar{w}) \right).$$

We must estimate the $C_b^{2,\alpha}$ norm of this integral in terms of the $C_b^{0,\alpha}$ norm of $\left| \bar{w} \right|^{2 - \nu} \eta_p(\bar{w})$. At least for $|w| \leq 10$, this follows from the general mapping properties of $G_\rho$ as a $b$-pseudodifferential operator of order $-2$, proved in [Maz91].

The hypothesis that needs to be checked is that the integration makes sense. In other words, in the region where $\bar{w} \to 0$ and $w \neq 0$, the Schwartz kernel $G_\rho(w, \bar{w}) \left| \bar{w} \right|^{\nu - 2}$ is asymptotic to $\nu - 2$ since the smallest nonnegative indicial root of $L_\rho$ is zero, and this is integrable with respect to the area form on $\mathbb{R}^2$. On the other hand, as $w \to \infty$, the sup norm, and a fortiori the $C_b^{2,\alpha}$ norm of this integral is uniformly bounded. This proves that

$$\left\| \int G_t^{\text{model}}(z, \bar{z}) \eta_p(\bar{z}) \, d\bar{z} \right\|_{C_b^{2,\alpha}} \leq C t^{-4} \| \eta_p \|_{\nu^{-2}C_b^{0,\alpha}} .$$

Finally, if $p \in D_w$, then there is no longer any scaling, and we must simply check that the $C_b^{2,\alpha}$ norm of $\int G \eta_p$ is bounded by the $\nu^{-2}C_b^{0,\alpha}$ norm of $\eta_p$. For this, note that the expansions of $G$ at each of the boundaries involve only the nonnegative indicial roots of $L_t$, and hence the integral is well-defined and one can use the same pseudodifferential boundedness theorem.

The final estimate is the trivial observation that

$$\| \bar{\eta} \|_{C_b^{0,\alpha}} \leq \| \eta \|_{C_b^{0,\alpha}} + \sum \| \eta_p \|_{\nu^{-2}C_b^{0,\alpha}} \leq C \| \eta \|_{C_b^{0,\alpha}} .$$

This completes the proof. \qed
6. Correcting to an Exact Solution

We now come to the final step, to modify the approximate solution $h_i^{\text{app}}$ to obtain an exact solution. As we have explained earlier, this amounts to solving the equation $F_i(\gamma) = 0$ for some $\gamma \in \mathcal{D}^{0,a}_{Fr}(\delta)$.

Recall that if $g$ is any Hermitian gauge transformation and $A$ any connection, then

$$F_{As} = g^{-1} \left( F_A + \overline{\delta}_A(g^2 \partial_AG^{-2}) \right) g.$$

Thus, keeping in mind that $A_t$ and $\Phi_t$ refer to the to the background approximate solution fields, the equation $F_i(\gamma) = 0$ is equivalent to

$$F_{Ai} + \overline{\delta}_A(e^{2\gamma} \partial_A e^{-2\gamma}) + e^\gamma \left[ e^{-\gamma} \Phi_t e^\gamma \wedge e^\gamma \Phi^*_t e^{-\gamma} \right] e^{-\gamma} = 0.$$

Recall also the general formula

$$\nabla e^\gamma = \nabla \gamma E(\gamma) e^\gamma = e^\gamma E(\gamma) \nabla \gamma,$$

where

$$E(\gamma) = \frac{\exp(\text{ad} \gamma) - 1}{\text{ad} \gamma}.$$

Putting all of these identities together, the equation becomes

$$F_i(\gamma) := F_{Ai} - 2\overline{\delta}_A(e^{-2\gamma} \partial_A e^{2\gamma}) + \left( \Phi_t e^{2\gamma} \wedge \Phi^*_t e^{-2\gamma} + e^{2\gamma} \Phi_t \wedge e^{-2\gamma} \Phi_t \right) = 0.$$

**Proposition 6.1.** The map $F_i: \mathcal{D}^{0,a}_{Fr}(\delta) \longrightarrow r^{\delta-2}C^0_b$ is smooth.

**Proof.** The expression for $F_i$ above is a polynomial in $\overline{\delta}_A, \partial_A, \gamma, \overline{\partial}_A, \gamma$ and $\partial_A, \gamma$ with coefficients $E(-2\gamma), D E|_{-2\gamma}$ and $e^{\pm 2\gamma}$. Schematically,

$$F_i(\gamma) = B_1(\gamma) \overline{\delta}_A \partial_A \gamma + B_2(\gamma) \partial_A \gamma \overline{\partial}_A \gamma + B_3(\gamma) (\Phi_t, \Phi_t^*),$$

where $B_1$ and $B_2$ are smooth functions of $\gamma$ and $B_3(\gamma)$ is a bilinear form in $\Phi_t$ and $\Phi_t^*$ which coefficients smooth in $\gamma$. Furthermore, each $\gamma \mapsto B_i(\gamma)$ is smooth with respect to the norm on $\mathcal{D}^{0,a}_{Fr}(\delta)$. Now recall that $\gamma = \gamma_0 + \tilde{\gamma}$ where $\gamma_0$ is independent of $r$ and $\theta$ and diagonal, and $\tilde{\gamma} \in r^{\delta-1}C^1_b$. This implies that $\nabla A_t, \gamma = \nabla A_t + \tilde{\gamma}[A_t, \tilde{\gamma}]$ since $[A_t, \gamma_0] = 0$ because $A_t$ is also diagonal. Hence $\nabla A_t, \gamma \in r^{\delta-1}C^1_b$. Finally, the products of terms involving entries of $\Phi_t$ and $\Phi_t^*$ are polyhomogeneous and lie in $r^{2(a_1-a_2)}C^{0,a}_b \subset r^{\delta-2}C^0_b$.

The structure of the equation as written above clearly implies that each summand lies in $r^{\delta-2}C^0_b$, and it is also clear that $F_i$ is a smooth map. \hfill $\Box$

We now expand $F_i$ in a Taylor series around $\gamma = 0$, writing

$$F_i(\gamma) = \eta_i + \mathcal{L}_i \gamma + \mathcal{Q}_i(\gamma).$$
The nonlinear error term $Q_t$ is smooth in $\gamma$ and by inspecting (6.1), we see that
\[
\|Q_t(\gamma)\|_{p^\gamma} \leq C\|\gamma\|_{C^2_b} \left(\|\gamma\|_{C^1_b} + \|\nabla\gamma\|_{p^\gamma} + \|\nabla^2\gamma\|_{p^\gamma}\right).
\]
The covariant derivative $\nabla$ here is any reference connection which is smooth across the points of $Z$ and $D$. Furthermore,
\[
\|Q_t(\gamma_1) - Q_t(\gamma_2)\|_{p^\gamma} \leq C\|\gamma\|_{C^2_b} \left(\|\gamma_1 - \gamma_2\|_{C^1_b} + \|\nabla(\gamma_1 - \gamma_2)\|_{p^\gamma} + \|\nabla^2(\gamma_1 - \gamma_2)\|_{p^\gamma}\right).
\]
The constants $C$ which appear in the two preceding inequalities are independent of $t$.

**Theorem 6.2.** There exists $t_0 > 1$ such that for every $t > t_0$ there exists $\gamma_t \in D^0_t(\delta)$ which satisfies $\mathcal{F}_t(\gamma_t) = 0$. Furthermore, this $\gamma_t$ is unique amongst Hermitian endomorphisms of small norm and satisfies an estimate $\|\gamma_t\|_{C^2_b} \leq e^{-(\mu/2)t}$ for some $\mu > 0$. Hence by Proposition 5.3, $\gamma_t = \sum \gamma_{j,t}(\theta) r^j + \tilde{\gamma}_t$ where $\tilde{\gamma}_t \in r^2 C^2_b$, and
\[
\sum_j \sup |\gamma_{j,t}| + \|\tilde{\gamma}_t\|_{r^2 C^2_b} \leq Ce^{-(\mu/2)t}.
\]

**Proof.** Write the equation to be solved as
\[
\gamma_t = -G_t(\eta_t + Q_t(\gamma_t))
\]
where $G_t$ is the inverse described above. Now recall that $\|\eta_t\| \leq Ce^{-\mu t}$ for some $\mu > 0$. The norm estimate of $G_t$ and a straightforward contraction mapping argument, using the estimates above for $Q_t$, show that $\|\gamma_t\|_{C^2_b} \leq e^{-(\mu/2)t}$. To estimate $\|\mathcal{L}_t \gamma_t\|_{p^\gamma}$ similarly, write $\mathcal{L}_t \gamma_t = -\eta_t - Q_t(\gamma_t)$ and use the estimates for these quantities noted above.

## 7. Asymptotic Geometry of $\mathcal{M}'$

In this section we establish that for parabolic Higgs bundle moduli spaces, the difference between $g_{t^2}$ and $g_{sf}$ decays exponentially in $t$ along rays in the portion of the moduli space $\mathcal{M}'$ away from the preimage of the discriminant locus. There are some slight differences between the strongly and weakly parabolic settings, and we comment on these as we go along. The statement of the result in the strongly parabolic setting is directly in line with what we have discussed before:

**Theorem 7.1.** Let $\mathcal{M}$ be a moduli space of strongly parabolic $SL(2,\mathbb{C})$ Higgs bundles and $(\bar{\partial}_E, \varphi) \in \mathcal{M}'$ any stable Higgs bundle. Suppose that $\psi = (\eta, \varphi)$ is an infinitesimal variation of the Higgs bundle moduli space. Consider the family of tangent vectors $\psi_t = (\eta_t, t\varphi) \in T_t(\bar{\partial}_E, t\varphi) \mathcal{M}$
over the ray \((\partial E, t\varphi, h_t)\). Then as \(t \to \infty\),
\[
\|(\eta, t\varphi, \dot{v}_t)\|_{g_{L^2}}^2 - \|(\eta, \varphi^t, \dot{v}_t)\|_{g_{sf}}^2 = O(e^{-\varepsilon t})
\]
for some \(\varepsilon > 0\).

This extends the analogous result for the Hitchin moduli space over \(SU(n)\), with holomorphic (rather than meromorphic) data, in [Fre20], building on [MSWW19] and [DN19]. The proof here is a relatively straightforward adaption of the one in [Fre20].

The feature here which does not extend to the weakly parabolic case is the fact that \(R^+\) (and in fact \(C^*\)) acts on the moduli space only in the strongly parabolic case. The correct way to extend this theory then is to consider the family of moduli spaces \(M_t\), for each \(t > 0\), defined as follows. Given \([([\mathcal{E}, \varphi]) \in M_{\text{Higgs}}\), define the \(t\)-Hitchin moduli space
\[
M_t = \{([[\mathcal{E}, \varphi, h_t]) : ([\mathcal{E}, \varphi]) \in M_{\text{Higgs}}\}
\]
where \(h_t\) solves the rescaled Hitchin equations
\[
F_D(\partial_E, h_t) + t^2[\varphi, \varphi^* h_t] = 0 \tag{7.1}
\]
and is adapted to the parabolic structure.

The space \(M_t\) has a natural \(L^2\)-metric \(g_{L^2,t}\), defined in the obvious way, which is hyperkähler as before. The goal now is to compare \(g_{L^2,t}\) to the \(t\)-rescaled semiflat metric \(g_{sf,t}\).

**Theorem 7.2.** Fix \(M_{\text{Higgs}}\) a moduli space of (weakly or strongly) parabolic \(SL(2, \mathbb{C})\) Higgs bundles. Let \((\partial E, \varphi) \in M_{\text{Higgs}}\) be any stable Higgs bundle and \((\eta, \varphi)\) an infinitesimal variation of the Higgs bundle moduli space. Identifying \((\eta, \varphi)\) with its image in \(T_{(\partial E, \varphi, h_t)} M_t\), as \(t \to \infty\),
\[
\|(\eta, \varphi, \dot{v}_t)\|_{g_{L^2}}^2 - \|(\eta, \varphi, \dot{v}_\infty)\|_{g_{sf,t}}^2 = O(e^{-\varepsilon t})
\]
for some \(\varepsilon > 0\).

The proofs involve comparing the \(L^2\)-metrics on the three moduli spaces
\[
M' = \{(\partial E, \varphi, h)\} / \sim, \quad M'_\infty = \{(\partial E, \varphi, h_\infty)\} / \sim, \quad \text{and} \quad M'_{\text{app}} = \{(\partial E, \varphi, h_{\text{app}})\} / \sim
\]
(or their \(t\)-rescaled versions). We begin by reviewing the gauged infinitesimal deformations and expressions for \(g_{L^2}\). In §7.3 we show that the semiflat metric is naturally identified with the \(L^2\)-metric on \(M'_\infty\), the regular locus of the moduli space of limiting configurations. In §7.4 and §7.5 we establish that \(g_{sf} - g_{\text{app}}\) and \(g_{\text{app}} - g_{L^2}\) decay exponentially; the proofs of Theorems 7.1 and 7.2 follow from this.
7.1. The tangent space to $\mathcal{M}_{\text{Higgs}}$. We first describe elements of $T\mathcal{M}_{\text{Higgs}}$ using the differential-geometric setup as in §2.1. Fix the smooth complex vector bundle $E$ over $C$ and complete flag $\mathcal{F}(p)$ and weight vector $\tilde{a}(p)$ at each $p \in D$. In terms of these, $\mathcal{M}$ is the quotient space $\mathcal{M} = \mathcal{H}_{\tilde{a}} / \mathcal{G}_C$, where $\mathcal{H}_{\tilde{a}} \subset \mathcal{H}$ is the space of $\tilde{a}$-stable Higgs bundle structures $(\tilde{\partial}_E, \varphi)$ on $(E, \mathcal{F})$ and $\mathcal{G}_C$ is the group of complex gauge transformations preserving $\mathcal{F}(p)$ at each $p \in D$. Recall that $\tilde{\partial}_E$ is a nonsingular holomorphic structure on $E$, $\varphi$ has only simple poles at $D$ and $\tilde{\partial}_E \varphi = 0$ on $C \setminus D$. The residue of $\varphi$ at $p$ is nilpotent with respect to the flag in the strongly parabolic case, but only preserves it in the weakly parabolic case with eigenvalues $\sigma(p)$ and $-\sigma(p)$. In either setting we say that it is compatible with the flag.

Differentiating the equality $(\tilde{\partial}_E)_\epsilon \varphi_\epsilon = 0$ for a family of parabolic Higgs bundle

\[ (\tilde{\partial}_E)_\epsilon = \tilde{\partial}_E + \epsilon \tilde{\eta} + O(\epsilon^2), \quad \varphi_\epsilon = \varphi + \epsilon \tilde{\phi} + O(\epsilon^2), \]

yields

\[ \tilde{\partial}_E \varphi + [\tilde{\eta}, \varphi] = 0 \]
on $C \setminus D$. Since the singular structure of $(\tilde{\partial}_E)_\epsilon$ is fixed, $\tilde{\eta} \in \Omega^{0,1}(\text{End}_0 E)$ is nonsingular and represents an infinitesimal deformation of the holomorphic structure; on the other hand, $\tilde{\phi} \in \Omega^{1,0}(\text{End}_0 C)$ can have at most simple poles at $p \in D$, with residues compatible with $\mathcal{F}(p)$. More precisely, choose a local family of $(\tilde{\partial}_E)_\epsilon$-holomorphic frames and a holomorphic coordinate $z$ on the disk $D_p$, and write $\varphi_\epsilon = f_\epsilon \, dz$; then there exists a smooth $\mathfrak{sl}(2, \mathbb{C})$-valued function $\dot{\gamma}$ which preserves the flag and a meromorphic $\dot{f}$ with simple pole at each $p$ and residue compatible with $\mathcal{F}(p)$ such that

\[ \dot{\eta} = \tilde{\partial} \dot{\gamma} \quad \text{and} \quad \dot{\phi} = \dot{f} \, dz + [\varphi, \dot{\gamma}]. \quad (7.2) \]

Infinitesimal gauge transformations act on the space of infinitesimal deformations by

\[ (\dot{\eta}, \dot{\phi}) \mapsto (\dot{\eta}, \dot{\phi}) + (\tilde{\partial}_E \dot{\gamma}, [\dot{\phi}, \dot{\gamma}]). \quad (7.3) \]

We then set

\[ T_{(\tilde{\partial}_E, \varphi)} \mathcal{M}'_{\text{Higgs}} = \{ (\dot{\eta}, \dot{\phi}) | \tilde{\partial}_E \dot{\phi} + [\dot{\eta}, \dot{\phi}] = 0 \} / \sim, \]

where $\sim$ is the equivalence (7.3).

**Lemma 7.3.** Suppose that $q$ has a simple pole at $p \in C$, and that $z$ is a holomorphic coordinate such that $q = z^{-1} \, dz^2 = - \det \varphi$ near $p$. Fix a stable Higgs bundle $(\tilde{\partial}_E, \varphi) \in \pi^{-1}(q)$ and an infinitesimal Higgs bundle deformation $[(\eta_1, \dot{\phi}_1)] \in T_{(\tilde{\partial}_E, \varphi)} \mathcal{M}'_{\text{Higgs}}$. In the flat metric $|dz|^2$ and the holomorphic gauge where

\[ \tilde{\partial}_E = \partial, \quad \varphi = \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} \, dz, \]
there is a unique \((\eta_2, \phi_2) \sim (\eta_1, \phi_1)\) on this disk such that

\[ \dot{\eta}_2 = 0, \quad \dot{\phi}_2 = \begin{pmatrix} 0 & 0 \\ \frac{\dot{P}}{z} & 0 \end{pmatrix} \, dz, \quad \dot{P} \text{ holomorphic.} \]

If \(q\) has a double pole, and the holomorphic gauge is chosen so that

\[ \tilde{\partial}_E = \tilde{\partial}, \quad \varphi = \frac{\sigma}{z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, dz, \]

then there exists a unique infinitesimal Higgs bundle deformation \((\dot{\eta}_2, \dot{\phi}_2)\) in the equivalence class such that

\[ \dot{\eta}_2 = 0, \quad \dot{\phi}_2 = \frac{\dot{P}}{z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, dz, \quad \dot{P} \text{ holomorphic.} \]

Proof. By (7.2) and the Poincaré lemma, there exists a representative such that \(\dot{\eta}_2 = 0\). We then adjust by a constant matrix so that the flag at each \(p\) is preserved. In the strongly parabolic case, the residue of the Higgs field is nilpotent, hence at this stage

\[ \dot{\phi} = \begin{pmatrix} \dot{P}_1 & \dot{P}_2 \\ \frac{\dot{P}_3}{z} & -\dot{P}_1 \end{pmatrix} \, dz, \]

where the \(\dot{P}_j\) are holomorphic. Now set

\[ \dot{\gamma} = \begin{pmatrix} \frac{\dot{P}_2}{z} \\ z(\dot{P}_1 + f) \\ f - \frac{\dot{P}_2}{z} \end{pmatrix}, \]

where \(f\) is an arbitrary holomorphic function; this too respects the flag structure at \(z = 0\). Then

\[ \dot{\phi} + [\varphi, \dot{\gamma}] = \begin{pmatrix} 0 & 0 \\ \frac{\dot{P}_3 + \dot{P}_3}{z} & 0 \end{pmatrix} \, dz; \]

setting \(\dot{P} = \dot{P}_2 + \dot{P}_3\), this has the desired form.

In the weakly parabolic setting, the residue is not nilpotent, so at first we only have

\[ \dot{\phi} = \begin{pmatrix} \frac{\dot{P}_1}{z} & \frac{\dot{P}_2}{z} \\ \frac{\dot{P}_3}{z} & -\frac{\dot{P}_1}{z} \end{pmatrix} \]

where \(\dot{P}_1, \dot{P}_2, \dot{P}_3\) are all holomorphic. Taking

\[ \dot{\gamma} = \begin{pmatrix} f & -\frac{\dot{P}_3}{z} \\ \frac{\dot{P}_3}{z} & f \end{pmatrix}, \]

which respects the flag structure at \(z = 0\), we obtain

\[ \phi = \phi_1 + [\varphi, \dot{\gamma}] = \frac{\dot{P}_1}{z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, dz \]
7.2. Hyperkähler metric. The usual definition of $g_{L^2}$ involves the unitary formulation of the Hitchin equations, cf. §2. However, following [Fre20], it is also possible to define this metric in terms of Higgs bundle deformations $(\eta, \phi)$ and one-parameter deformations of Hermitian metrics expressed in terms of an $\mathfrak{sl}(E)$-valued section $\nu$ as

$$h_\epsilon(v, w) = h(e^\epsilon v, e^\epsilon w).$$

Note that $h_\epsilon(v, w) = h(v, w) + \epsilon (h(\nu v, w) + h(v, \nu w)) + O(\epsilon^2)$. In the $h_0$-unitary formulation of Hitchin’s equations, the corresponding deformation is

$$\hat{A}^{0,1} = H^{1/2} \left( \eta - \bar{\partial}_E \check{v} \right) H^{-1/2}, \quad \Phi = H^{1/2} (\phi + [\check{v}, \varphi]) H^{-1/2},$$

where $H$ is an $\text{End } E$-valued $h_0$-Hermitian section such that $h(v, w) = h_0(Hv, w)$. By [Fre20, Proposition 2.2], the linearized Hitchin equation and Coulomb gauge condition are satisfied if and only if

$$\partial^h_E \bar{\partial}_E \check{v} - \partial^h_E \eta - [\varphi^h, \phi + [\check{v}, \varphi]] = 0,$$

or equivalently,

$$\mathcal{P} \check{v} := \partial^h_E \bar{\partial}_E \check{v} - [\varphi^h, [\check{v}, \varphi]] = \partial^h_E \eta + [\varphi^h, \phi].$$

Here the Coulomb gauge condition demands the pair $(\hat{A}, \Phi)$ to be $L^2$-orthogonal to the unitary gauge orbit at $(A, \Phi)$, i.e. $L^2$-orthogonal to all pairs $(d_A \gamma, [\Phi, \gamma])$ for infinitesimal unitary gauge transformations $\gamma$.

In the compact case it is easy to see that if $(\bar{\partial}_E, \varphi)$ is stable, then for a given Higgs bundle deformation $(\eta, \phi)$ there is a unique $\check{v}$ solving (7.4). Indeed, the index of $\mathcal{P}$ is 0, so we need only prove that this operator is injective, which can be checked by taking the inner product with $\check{v}$, integrating by parts and using that a stable Higgs bundle is simple, cf. [Fre20, Corollary 2.5]. Having determined $\check{v}$ in this way, we then define the $L^2$-metric on the Hitchin moduli space by

$$\| (\eta, \phi, \check{v}) \|_{g_{L^2}}^2 = 2 \int_C \left( |\eta - \bar{\partial}_E \check{v}|_h^2 + |\phi + [\check{v}, \varphi]|_h^2 \right)$$

$$= 2 \int_C \langle \eta - \bar{\partial}_E \check{v}, \eta \rangle_h + \langle \phi + [\check{v}, \varphi], \varphi \rangle_h.$$
The coefficients in this matrix are all bounded by $\|\|_{h_0}$ in unitary frame, and hence indicial roots of $P$ are the same as those for the operator associated to the flat model metric $h$. We computed earlier as

$$\| (\tilde{A}^0, \Phi) \|_{g_{L^2}}^2 = 2 \int C \| \tilde{A}^{0,1} \|_{h_0}^2 + |\Phi|^2_{h_0} \quad = 2 \int C \left| H^{1/2} \left( \eta - \overline{\partial_E} \nu \right) \right|^2_{h_0} + \left| H^{1/2} (\tilde{\phi} + [\tilde{\nu}, \nu]) \right|^2_{h_0} \quad (7.5)$$

In the parabolic setting, $\varphi$, $\tilde{\varphi}$ and $h$ have singularities, so we must discuss the solvability of (7.4) in suitable function spaces. Regularity of $\nu$ is needed to justify the integration by parts in (7.5). To that end, first recall the Kähler identities, cf. [MSWW16, §5.1],

$$2\partial_E^h \bar{E}^h \nu = F_{A_h} + i \Delta_{A_h}$$

Now, $P$ is a conic operator and acts on weighted $b$-Hölder spaces as before. Its indicial roots are the same as those for the operator associated to the flat model metric $h_g$, i.e., to $2\partial_E^h \bar{E}^h \nu = i \Delta_{A_{h_g}}$. We compute the other term in $P$. Write $\nu$ in the holomorphic frame $e_1, e_2$ and in the unitary frame $\tilde{e}_1 = e_1/|z|^{a_1}$, $\tilde{e}_2 = e_2/|z|^{a_2}$ are

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \tilde{a} & b \\ \tilde{c} & -\tilde{a} \end{pmatrix} = \begin{pmatrix} a & |z|^{a_1-a_2} b \\ |z|^{a_2-a_1} c & -a \end{pmatrix},$$

respectively. The expression for $\varphi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \, \text{dz}$ in holomorphic frame becomes

$$\varphi = \begin{pmatrix} 0 & |z|^{a_2-a_1} \\ \frac{1}{z} & 0 \end{pmatrix} \, \text{dz} \quad \text{and} \quad \varphi^{*_{h_g}} = \begin{pmatrix} 0 & \frac{|z|^{a_2-a_1}}{z} \\ |z|^{a_1-a_2} & 0 \end{pmatrix} \, \text{d\bar{z}}$$

in unitary frame, and hence

$$[\varphi^{*_{h_g}}, [\tilde{\nu}, \nu]] = \begin{pmatrix} -2\tilde{a}(|z|^{2(a_2-a_1-1)} + |z|^{2(a_1-a_2)}) & 2\tilde{c} - 2\tilde{b}|z|^{2(a_2-a_1-1)} \\ 2\tilde{b} - 2\tilde{c}|z|^{2(a_1-a_2)} & 2\tilde{a}(|z|^{2(a_2-a_1-1)} + |z|^{2(a_1-a_2)}) \end{pmatrix} \, \text{d\bar{z}} \wedge \text{dz}.$$
In the weakly parabolic setting, the expression \( \varphi = \frac{1}{\bar{z}} \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} dz \) leads to the equality

\[
[\varphi^{*}_E, [\bar{\nu}, \varphi]] = \frac{1}{|z|^2} \begin{pmatrix} 0 & -4|\sigma|^2\bar{b} \\ -4|\sigma|^2\bar{c} & 0 \end{pmatrix} dz \wedge dz,
\]

in unitary frame; this is now of order \(|z|^{-2}\). The indicial root set for the off-diagonal part changes to

\[
\left\{ \pm \left( (\ell + \alpha_1 - \alpha_2)^2 + 16|\sigma|^2 \right)^{1/2} | \ell \in \mathbb{Z} \right\}.
\]

The key point is that there is a gap around 0 for the off-diagonal part in both the strongly and weakly parabolic cases.

We now consider the issue of solvability.

**Proposition 7.4.** Given any infinitesimal Higgs bundle deformation \((\bar{\eta}, \varphi)\) of the parabolic Higgs bundle \((\bar{\partial}_E, \varphi)\), there is a unique \(\bar{\nu} \in \mathcal{D}_{0}^{Fr}(\mathcal{P})(\mu)\), where \(\mu > 0\) is sufficiently small, which solves

\[
\bar{\partial}_E^h \bar{\nu} - \partial_E^h \bar{\eta} - [\varphi^{*}_E, \varphi + [\bar{\nu}, \varphi]] = 0.
\]

**Proof.** If \(\bar{\nu} \in \mathcal{D}_{0}^{Fr}(\mu)\) and \(\mathcal{P}\bar{\nu} = 0\), then

\[
0 = \int_C \left\langle \bar{\partial}_E^h \bar{\partial}_E \bar{\nu}, \bar{\nu} \right\rangle_h - \left\langle [\varphi^{*}_E, [\bar{\nu}, \varphi]], \bar{\nu} \right\rangle_h = \int_C d \left\langle \bar{\partial}_E \bar{\nu}, \bar{\nu} \right\rangle_h + \int_C \left| \bar{\partial}_E \bar{\nu} \right|^2_h + \left| [\bar{\nu}, \varphi] \right|^2_h.
\]

By Stokes’ theorem,

\[
\int_C d \left\langle \bar{\partial}_E \bar{\nu}, \bar{\nu} \right\rangle_h = \lim_{\delta \to 0} \int_{C_\delta} \left\langle \bar{\partial}_E \bar{\nu}, \bar{\nu} \right\rangle_h,
\]

where \(C_\delta = C \setminus \cup_{p \in D} D_p(\delta)\). In any one of these disks we have

\[
\bar{\nu} = \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} \quad \text{and} \quad \bar{\partial}_E = \bar{\partial} - \frac{1}{2} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \frac{d\bar{z}}{\bar{z}}
\]

in the unitary frame \((\bar{e}_1, \bar{e}_2)\), and so

\[
\left\langle \bar{\partial}_E \bar{\nu}, \bar{\nu} \right\rangle_h = \left\langle \begin{pmatrix} \bar{z} \bar{a} \\ \bar{z} \bar{b} - \frac{1}{\bar{z}} (\alpha_1 - \alpha_2) \bar{c} \\ -\bar{z} \bar{a} \end{pmatrix}, \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} \right\rangle dz.
\]

Now, \(\bar{a} = \bar{a}_0 + \bar{a}_1\) where \(\bar{a}_0\) is constant and \(\bar{a}_1, \bar{b}, \bar{c} = O(r^\mu)\), so \(\int_{C_\delta} \left\langle \bar{\partial}_E \bar{\nu}, \bar{\nu} \right\rangle_h = O(\delta^\mu)\). This shows that \(\bar{\partial}_E \bar{\nu} = [\bar{\nu}, \varphi] = 0\). In particular, the entries of

\[
\bar{\nu} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} \bar{a} \\ |z|^{\alpha_2 - \alpha_1} \bar{b} \end{pmatrix}.
\]

in the holomorphic frame \((e_1, e_2)\) are holomorphic functions. Now, this equality and the boundedness of \(\bar{a}, \bar{b}, \bar{c}\) show that \(a\) is bounded, \(|c| \leq C|z|^{-1+\varepsilon}\) and \(|b| \leq C|z|^{\varepsilon}\), hence \(|c| = O(1)\) and \(|b| = O(|z|)\), so in particular \(\bar{\nu}\) must extend holomorphically across the puncture.
and preserve the flag. The stable parabolic Higgs bundle \((\delta_E, \varphi)\) is simple, therefore \(\dot{v}\) must be a constant multiple of the identity, and since \(\text{tr} \, \dot{v} = 0\), we see finally that \(\dot{v} = 0\). As in the proof of Lemma 5.8 this proves that 
\[
P : D_{Fr}^{0, \alpha}(P)(\mu) \rightarrow r^{-2} C_{b}^{0, \alpha}
\]
is an isomorphism. Finally, for a Higgs bundle deformation \((\dot{\eta}, \dot{\phi})\) we compute using Lemma 7.3 that 
\[
\partial^h_E \dot{\eta} + [\varphi^* h, \dot{\phi}] \in r^{-2} C_{b}^{0, \alpha}
\]
for some small \(\mu > 0\). Altogether we have shown that there exists a unique \(\dot{v} \in D_{Fr}^{0, \alpha}(P)(\mu)\) solving 
\[
P \dot{v} = \partial^h_E \dot{\eta} + [\varphi^* h, \dot{\phi}].
\]

Now suppose that \(\dot{v} \in D_{Fr}^{0, \alpha}(P)(\mu)\) solves \(\partial^h_E \delta_E \dot{v} - \partial^h_E \eta - [\varphi^* h, \dot{\phi} + [\dot{v}, \varphi]] = 0\). Following the proof of [Fre20, Proposition 2.6], we justify the equalities in (7.5) by checking that 
\[
\lim_{\delta \to 0} \int_{\partial C_\delta} \langle \dot{\eta} - \delta_E \dot{v}, \dot{v} \rangle_h = 0.
\]
However, \(\dot{\eta}\) is bounded and \(|\delta_E \dot{v}| \leq C|z|^{-1+\epsilon}\), so this follows as above.

*The \(L^2\)-metrics on \(M'_\infty\) and \(M'_{app}\):* The \(L^2\)-metrics on \(M'_\infty\) and \(M'_{app}\) are defined similarly. The metric deformation \(\dot{v}_\infty\) of \(h_\infty\) satisfies the decoupled equations 
\[
\partial^h_{E} \overline{\delta}_E \dot{v}_\infty = \partial^h_{E} \eta, \quad [\varphi^* h_\infty, \dot{\phi} + [\dot{v}_\infty, \varphi]] = 0,
\]
so the \(L^2\)-metric on \(M'_\infty\) is 
\[
\| (\dot{\eta}, \dot{\phi}, \dot{v}_\infty) \|^2_{\dot{g}_{\infty}} = 2 \int_C \langle \dot{\eta} - \overline{\delta}_E \dot{v}_\infty, \dot{\eta} \rangle_{h_\infty} + \langle \dot{\phi} + [\dot{v}_\infty, \varphi], \dot{\phi} \rangle_{h_\infty}.
\]
Likewise, the metric deformation \(\dot{v}_{app}\) of \(h_{app}\) satisfies 
\[
\partial^h_E \overline{\delta}_E \dot{v}_{app} = \partial^h_E \eta - [\varphi^* h_{app}, \dot{\phi} + [\dot{v}_{app}, \varphi]] = 0,
\]
so the \(L^2\)-metric on \(M'_{app}\) is 
\[
\| (\dot{\eta}, \dot{\phi}, \dot{v}_{app}) \|^2_{\dot{g}_{app}} = 2 \int_C \langle \dot{\eta} - \overline{\delta}_E \dot{v}_{app}, \dot{\eta} \rangle_{h_{app}} + \langle \dot{\phi} + [\dot{v}_{app}, \varphi], \dot{\phi} \rangle_{h_{app}}.
\]
The integration by parts in both cases is justified by a similar argument to what was done above.

*The \(t\)-rescaled metric \(g_{L^2,t}\):* For moduli spaces of strongly parabolic Higgs bundles, we can state the main theorem purely in terms of the metric \(g_{L^2}\) defined above. More generally, for moduli spaces of weakly parabolic Higgs bundles, we must define a \(t\)-rescaled \(L^2\)-metric \(g_{L^2,t}\). As will be discussed in §7.6, we need a \(t\)-rescaled \(L^2\)-metric precisely because \(\mathcal{M}_{\text{Higgs}}\) is not preserved by \(\mathbb{C}_z\)-action \((\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \zeta \varphi)\).
Given \((\mathcal{E}, \varphi) \in \mathcal{M}_{\text{Higgs}}\), define the \(t\)-Hitchin moduli space
\[
\mathcal{M}_t = \{((\mathcal{E}, \varphi, h_t)) : ((\mathcal{E}, \varphi)) \in \mathcal{M}_{\text{Higgs}}\}
\]
where \(h_t\) solves the \(t\)-rescaled Hitchin’s equations
\[
F_{D(\mathcal{E}, h_t)} + t^2 [\varphi, \varphi^{* h_t}] = 0
\]
and is adapted to the parabolic structure.

The non-abelian Hodge correspondence gives a diffeomorphism
\[
\text{NAHC}_t : \mathcal{M}_{\text{Higgs}} \rightarrow \mathcal{M}_t
\]
\[
[(\mathcal{E}, \varphi)] \mapsto [(\mathcal{E}, \varphi, h_t)].
\]

Given a Higgs bundle deformation \([(\dot{\eta}, \dot{\varphi})] \in T_{[(\mathcal{E}, \varphi)]} \mathcal{M}_{\text{Higgs}}\), there is a corresponding map of tangent spaces
\[
d\text{NAHC}_t : T \mathcal{M}_{\text{Higgs}} \rightarrow T \mathcal{M}_t
\]
\[
[(\dot{\eta}, \dot{\varphi})] \mapsto [(\dot{\eta}, \dot{\varphi}, \dot{v}_t)],
\]
where \(\dot{v}_t\) is determined by
\[
\partial_E^h \partial_E \dot{v}_t - \partial_E^h \dot{\eta} - t^2 [\varphi^{* h_t}, \dot{\varphi} + [\dot{v}_t, \varphi]] = 0. \tag{7.7}
\]

Given \([(\dot{\eta}, \dot{\varphi}, \dot{v}_t)] \in T_{[(\dot{\mathcal{E}}, \dot{\varphi}, \dot{h}_t)]} \mathcal{M}_t\), the \(L^2\)-metric on the moduli space \(\mathcal{M}_t\) then takes the form
\[
\|(\dot{\eta}, \dot{\varphi}, \dot{v}_t)\|_{g_{L^2}, t}^2 = 2 \int_C \langle \dot{\eta} - \partial_E \dot{v}_t, \dot{\eta} \rangle_{h_t} + t^2 \langle \dot{\varphi} + [\dot{v}_t, \varphi], \dot{\varphi} \rangle_{h_t}. \tag{7.8}
\]

Remark 7.5. When \(\mathcal{M}_{\text{Higgs}}\) is preserved by the \(C_2^X\)-action \((\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \zeta \varphi)\), the \(g_{L^2, t}\) norm of \([(\dot{\eta}, \dot{\varphi}, \dot{v}_t)] \in T_{[(\dot{\mathcal{E}}, \dot{\varphi}, \dot{h}_t)]} \mathcal{M}_t\) equals the usual \(g_{L^2}\) norm of \([(\dot{\eta}, \dot{\varphi}, \dot{v}_t)] \in T_{[(\dot{\mathcal{E}}, t \dot{\varphi}, h_t)]} \mathcal{M}.

The \(t\)-rescaled \(L^2\)-metrics on \(\mathcal{M}_t^\prime\) and \(\mathcal{M}_{\text{app}}^\prime\). We similarly define \(t\)-rescaled \(L^2\)-metrics on \(\mathcal{M}_t^\prime\) and \(\mathcal{M}_{\text{app}}^\prime\). The metric deformation \(\dot{v}_\infty\) of \(h_\infty\) is independent of \(t\) and still satisfies the decoupled equations
\[
\partial_E^{h_\infty} \partial_E \dot{v}_\infty - \partial_E^{h_\infty} \dot{\eta} = 0, \quad [\varphi^{* h_\infty}, \dot{\varphi} + [\dot{v}_\infty, \varphi]] = 0.
\]

In particular, it solves
\[
\partial_E^{h_\infty} \partial_E \dot{v}_\infty - \partial_E^{h_\infty} \dot{\eta} + t^2 [\varphi^{* h_\infty}, \dot{\varphi} + [\dot{v}_\infty, \varphi]] = 0.
\]

for any value of \(t\). The \(t\)-rescaled \(L^2\)-metric on \(\mathcal{M}_\infty^\prime\) is
\[
\|(\dot{\eta}, \dot{\varphi}, \dot{v}_\infty)\|_{g_{L^2, t}}^2 = 2 \int_C \langle \dot{\eta} - \partial_E \dot{v}_\infty, \dot{\eta} \rangle_{h_\infty} + t^2 \langle \dot{\varphi} + [\dot{v}_\infty, \varphi], \dot{\varphi} \rangle_{h_\infty}.
\]
Likewise, the metric deformation $\dot{v}_t^{\text{app}}$ of $h_t^{\text{app}}$ satisfies
\[
\partial_E^{\text{app}} \delta_E \dot{v}_t^{\text{app}} - \partial_E^{\text{app}} \dot{\eta} - t^2 [\varphi^* h_t^{\text{app}}, \dot{\varphi} + [\dot{v}_t^{\text{app}}, \varphi]] = 0,
\]
so the $t$-rescaled $L^2$-metric on $M'_{\text{app}}$ is
\[
\| (\dot{\eta}, \dot{v}_t^{\text{app}}) \|_{g_{\text{app},t}}^2 = 2 \int_C \langle \dot{\eta} - \delta_E \dot{v}_t^{\text{app}}, \dot{\eta} \rangle_{h_t^{\text{app}}} + t^2 \langle \dot{\varphi} + [\dot{v}_t^{\text{app}}, \varphi], \dot{\varphi} \rangle_{h_t^{\text{app}}}.
\]

The justifications for all these expressions follow by arguments similar to what was done above.

### 7.3. The semiflat metric on $M'_{\infty}$

The portion of the moduli space $M'$ not lying over the discriminant locus has, in addition to the $L^2$-metric, another natural hyperkähler metric $g_{\text{sf}}$, called the semiflat metric. This is an artifact of the complex integrable system structure [Fre99, Theorem 3.8]. For smooth rank 2 Higgs data, [MSWW19, Proposition 3.7, Proposition 3.11, Lemma 3.12] proves that the this semiflat metric actually equals the $L^2$-metric on the moduli space of limiting configurations $M'_{\infty}$. A different proof of this is given in [Fre20], which holds also in the $SU(n)$ case with smooth data. That uses the description of limiting configurations as pairs consisting of a spectral curve $\Sigma$ and a line bundle $L$ over $\Sigma$ with parabolic structure at the ramification points of the covering $\Sigma \to C$. We recall this latter perspective and adapt it to the present setting.

The first point is to parametrize variations of limiting configurations. Such a variation consists of a family of embeddings of a fixed smooth curve $\Sigma_{\text{top}}$ into $\text{Tot}(K_C)$ and a variation of the associated holomorphic line bundle. The same complex line bundle $L \to \Sigma_{\text{top}}$ underlies all nearby Higgs bundles, so we can view spectral data as pair $(\overline{\partial}_L, \tau)$ where $\tau$ is the tautological eigenvalue of $\pi^* \varphi$ which gives the embedding of $\Sigma_{\text{top}}$ and $\overline{\partial}_L$ is a holomorphic structure on $L$. This gives an isomorphism
\[
T_{(\overline{\partial}_E, \varphi)} M' \to H^0(\Sigma_b, K_{\Sigma_b})^* \oplus H^0(\Sigma_b, K_{\Sigma_b})_{\text{odd}}
\]
\[
(\dot{\eta}, \dot{\varphi}) \mapsto (\dot{\xi}, \dot{\tau}),
\]
corresponding to the family $\overline{\partial}_L + c \dot{\xi}$ and $\tau + c \dot{\tau}$. Here $H^0(\Sigma_b, K_{\Sigma_b})_{\text{odd}}$ is the subspace of sections which are anti-invariant under the involution of $\Sigma_b$. Variations of the holomorphic structure on $L$ alone are called vertical variations and involve only $\dot{\xi}$, while horizontal variations involve only $\dot{\tau}$. The limiting Hermitian metric $h_{\infty}$ on $E \to C$ is the ‘orthogonal push-forward’ of the unique Hermitian-Einstein metric $h_L$ on $L \to \Sigma$, cf. §3.1 and [Fre18]. Given a deformation $(\dot{\xi}, \dot{\tau})$ of the spectral data, the associated deformation $\dot{v}_L$ of the Hermitian metric $h_L$ is determined by the equation
\[
\partial_L^{h_L} \overline{\partial}_L \dot{v}_L - \partial_L^{h_L} \dot{\xi} = 0.
\]
Due to the obvious homogeneities of the data, there is a one-parameter family of semiflat metrics:

**Proposition 7.6.** [Fre20, Proposition 2.14] For each \( t > 0 \) there is a semiflat metric \( g_{sf,t} \) (at ‘scale \( t \)’) characterized by the following three properties:

1. On horizontal deformations, the semiflat metric is \( t^2 \int_{\Sigma} 2 |\dot{\tau}|^2 \);
2. On vertical deformations, the semiflat metric is \( \int_{\Sigma} 2 |\dot{\xi} - \overline{\partial} L \dot{v}_L|^2 \);
3. Horizontal and vertical deformations are orthogonal.

**Remark 7.7.** We use a slightly different convention than in [Fre20] and [Fre99] where horizontal and vertical deformations are weighted by \( t \) and \( t^{-1} \), respectively. The convention adopted here is more useful in §7.6.

The semiflat metric \( g_{sf,1} \) is called the semiflat metric, and denoted simply \( g_{sf} \).

**Remark 7.8.** The characterization of \( g_{sf} \) in [Fre20] and here is independent of the Hitchin section, but instead is phrased in terms of the horizontal subspaces with respect to the Gauss-Manin connection.

**Corollary 7.9.** The semiflat metric \( g_{sf,t} \) is the natural \( t \)-rescaled \( L^2 \)-metric on the moduli space of limiting configurations \( M'_{\infty} \), for deformations in formal Coulomb gauge, and moreover,\[
\| (\dot{\eta}, \dot{\phi}, \dot{v}_\infty) \|^2_{g_{sf,t}} = 2 \int_C \langle \dot{\eta} - \overline{\partial} E \dot{v}_\infty, \dot{\eta} \rangle_{h_\infty} + t^2 \langle \dot{\phi} + [\dot{v}_\infty, \phi], \dot{\phi} \rangle_{h_\infty}. \tag{7.10}
\]

**Proof.** The corresponding fact in the setting without poles was proved in [MSWW19]; the proof here is a small modification of [Fre20, Theorem 2.15]. The \( t \)-rescaled \( L^2 \)-metric on \( M'_{\infty} \) equals\[
\| (\dot{\eta}, \dot{\phi}, \dot{v}_\infty) \|^2_{g_{sf,t}} = \lim_{\delta \to 0} \left( 2 \int_{C_\delta} \langle \dot{\eta} - \overline{\partial} E \dot{v}_1, \dot{\eta} \rangle_{h_\infty} + t^2 \langle \dot{\phi} + [\dot{v}_\infty, \phi], \dot{\phi} \rangle_{h_\infty} - 2 \text{Re} \int_{\partial C_\delta} \langle \dot{\eta} - \overline{\partial} E \dot{v}_\infty, \dot{v}_\infty \rangle_{h_\infty} \right), \tag{7.11}
\]
where \( C_\delta = C - \cup_{p \in Z \cup D} |D_p(\delta) \). Since \( h_\infty \) is singular at \( Z \cup D \), \( \dot{v}_\infty \) may also be singular at this same collection of points. We show that the boundary terms vanish.

In the setting without poles, if \( p \in Z \), the monodromy of the associated abelian Chern connection is fixed to be \((-1)^{\text{ord}(p)}\), hence its infinitesimal variation vanishes. A short argument [Fre20] shows that the boundary terms vanish.

Similarly, if \( p \in D \), the monodromy is determined by the fixed parabolic weights, hence its infinitesimal variation also vanishes. Indeed, the \( h_L \)-unitary deformation of the Chern connection is \( \dot{D} = \dot{\xi} - \overline{\partial} L \dot{v}_L - (\dot{\xi} - \overline{\partial} L \dot{v}_L)^* h_L \) (with apologies for the different use of \( D \) here), and the variation of monodromy is the integral of \( \dot{D} \) around a small loop surrounding \( p \). It follows from (7.6) that \( \dot{D} \) is a harmonic 1-form. The fact that the boundary integrals vanish
for each loop implies that the corresponding cohomology class in $H^1(S^1, \mathbb{R})$ vanishes, thus $\tilde{D} \equiv 0$ and so $\int_{\partial C_s} \langle \tilde{\xi} - \tilde{\partial}_L \hat{v}_L, \hat{v}_L \rangle = 0$ as well. This proves that the $L^2$-metric is as in (7.10).

We now pull back the expression for $g_{\infty,t}$ in (7.11) to the spectral cover. Because everything diagonalizes on $\Sigma_b$, 

$$
\langle (\hat{\eta}_1, \phi_1, \hat{v}_{1,\infty}), (\hat{\eta}_2, \phi_2, \hat{v}_{2,\infty}) \rangle_{g_{\infty,t}} 
= 2 \text{Re} \int_{\Sigma_b} \langle \hat{\eta}_1 - \hat{\partial}_L \hat{v}_{1,\infty}, \hat{\eta}_2 \rangle_{h_{\infty}} + t^2 \langle \phi_1 + [\hat{v}_{1,\infty}, \phi], \phi_2 \rangle_{h_{\infty}} 
= \text{Re} \int_{\Sigma_b} \langle \pi^* \hat{\eta}_1 - \pi^* \hat{\partial}_L \pi^* \hat{v}_{1,\infty}, \pi^* \hat{\eta}_2 \rangle_{\pi^* h_{\infty}} + t^2 \langle \pi^* \phi_1 + [\pi^* \hat{v}_{1,\infty}, \pi^* \phi], \pi^* \phi_2 \rangle_{\pi^* h_{\infty}} 
= 2 \text{Re} \int_{\Sigma_b} \langle \hat{\xi}_1 - \hat{\partial}_L \hat{v}_{1,L}, \hat{\xi}_2 \rangle_{h_L} + t^2 \langle \hat{t}_1 + [\hat{v}_{1,L}, \lambda], \hat{t}_2 \rangle_{h_L} 
= 2 \text{Re} \int_{\Sigma_b} \langle \hat{\xi}_1 - \hat{\partial}_L \hat{v}_{1,L}, \hat{\xi}_2 - \hat{\partial}_L \hat{v}_{2,L} \rangle + t^2 \langle \hat{t}_1, \hat{t}_2 \rangle
$$

The last line uses integration by parts and (7.9).

This expression shows that the natural $t$-rescaled $L^2$-metric $g_{\infty,t}$ on $\mathcal{M}_\infty$ satisfies the three properties of the semiflat metric $g_{sf,t}$ in the statement of the Proposition since $\hat{\xi}_1 - \hat{\partial}_L \hat{v}_{1,L} = 0$ for the horizontal deformations and $\hat{t}_2 = 0$ for the vertical deformations. This proves the result. □

7.4. **Comparing $g_{sf,t}$ and $g_{app,t}$**. Extending to the parabolic setting an argument due to Dumas and Neitzke [DN19] and Fredrickson [Fre20], we now show that $g_{app,t} - g_{sf,t}$ decays exponentially in $t$. Suppose that $p \in D_\chi$ and fix $(\mathcal{E}, \varphi) \in \mathcal{M}'$, so $q = -\det \varphi = z^{-1} dz^2$ for some holomorphic coordinate $z$. Fix the usual flat (non-singular) metric on $\mathcal{D}_p$, and a holomorphic frame for which 

$$
\tilde{\partial}_E = \partial, \quad \varphi = \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} dz, \quad h_{\infty} = Q |z| \begin{pmatrix} |z|^{-1/2} & 0 \\ 0 & |z|^{1/2} \end{pmatrix}.
$$

The harmonic metric $h_L$ is not necessarily diagonal in this frame, but as in (3.7), we define the approximate harmonic metric 

$$
h^{\text{app}}_L = Q |z| \begin{pmatrix} |z|^{-1/2} e^{m_{\lambda}(|z|)} \chi(|z|) & 0 \\ 0 & |z|^{1/2} e^{-m_{\lambda}(|z|)} \chi(|z|) \end{pmatrix},
$$

where $Q$ is determined by the frame and the cutoff function $\chi$ is the same as in that earlier setting. Recall also from [MSWW19, Def. 5.1] that an exponential packet in the region $\{\chi(|z|) = 1\}$ is a (possibly matrix-valued) function of the form 

$$
\rho_t(z) = t^2 \rho(t^2 z),
$$

where $\rho \in C^\infty(\mathbb{C})$ converges exponentially to 0 as $|z| \to \infty$. 

Lemma 7.10. Let \( [(\eta, \phi)] \in T_{(\bar{\eta}, \phi)} M' \) be an infinitesimal Higgs bundle deformation. There is a unique representative in this equivalence class which has the form, in some holomorphic frame on \( D_p \),

\[
\eta = 0, \quad \phi = \begin{pmatrix} 0 & 0 \\ \bar{\eta} & 0 \end{pmatrix} \, dz, \quad \dot{\nu}_\infty = \frac{\dot{p}}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(7.12)

where \( \dot{p} \) is holomorphic. The difference \( \dot{\nu}_\text{app} - \dot{\nu}_\infty \) is an exponential packet, as defined above.

Proof. The lemma follows from the following two claims.

Claim 1: There is a unique representative in the equivalence class of \( [(\eta, \phi)] \) taking the form (7.12).

\( \triangleright \) The existence of a representative in the class \( [(\eta, \phi)] \) of this form is proved in Lemma 7.3. The deformation \( \dot{\nu}_\infty \) solves \( [\phi^* h_\infty, \phi + [\nu_\infty, \phi]] = 0 \) and \( \partial E \bar{\nu}_\infty - \bar{\partial} h_\infty \dot{\eta} = 0 \). Using the posited form of \( [(\eta, \phi)] \), the equation \( [\phi^* h_\infty, \phi + [\nu_\infty, \phi]] = 0 \) implies that

\[
\dot{\nu}_\infty = \frac{\dot{p}}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + F \phi,
\]

for some function \( F \). The second equation implies that \( \bar{\partial} F = 0 \). Since \( \dot{\nu}_\infty \) is bounded, we have \( F(0) = 0 \). As in the proof of Lemma 7.3, applying an infinitesimal gauge transformation of the form

\[
\gamma = \begin{pmatrix} 0 & f z \\ f & 0 \end{pmatrix} =zf \phi, \quad \text{where } f \text{ is any holomorphic function},
\]

does not alter the representative \( (\eta, \phi) \) and only changes \( \dot{\nu}_\infty \) to \( \dot{\nu}_\infty + \gamma \). Thus setting \( f = F/z \) (which is holomorphic) yields \( \dot{\nu}_\infty \) as in (7.12). \( \triangleright \)

Claim 2: The difference \( \dot{\nu}_\text{app} - \dot{\nu}_\infty \) is an exponential packet.

\( \triangleright \) We use \( \dot{\nu}_\infty \) as an approximate solution for the equation determining \( \dot{\nu}_\text{app} \) from \( (\eta, \phi) \) for \( h_\text{app} \). Since \( h_\text{app} = h_\infty \) outside the disks \( D_p \), the defect

\[
\rho_t := \partial E h_\text{app} \bar{\nu}_\infty - \bar{\partial} h_\text{app} \dot{\eta} - \left[ \phi^* h_\text{app}, \phi + [\nu_\infty, \phi] \right]
\]

which measures how far \( \dot{\nu}_\infty \) is from solving the equation is supported inside these disks. More precisely, computing in a frame as above and using Claim 1 we get \( \partial E \bar{h}_\text{app} \bar{\nu}_\infty = 0 \) (since \( \bar{\partial} \nu_\infty = 0 \)), \(-\bar{\partial} h_\text{app} \dot{\eta} = 0 \) (since \( \dot{\eta} = 0 \)) and finally

\[
\rho_t = - \left[ \phi^* h_\text{app}, \phi + [\nu_\infty, \phi] \right] = \frac{\dot{p}}{|z|} \sinh (2m_t(|z|)\chi(|z|)) \sigma_3.
\]

Now \( \rho_t = O(r^{-2+\epsilon}) \) for some \( \epsilon > 0 \) and

\[
\mathcal{P}_t : D^0_{Fr}(\mathcal{P}_t)(\mu) \to r^{\nu-2} C^0_{Fr}, \quad \nu \mapsto \mathcal{P}_t \dot{\nu} := \partial E h_\text{app} \bar{\nu}_\infty - \left[ \phi^* h_\text{app}, [\nu_\infty, \phi] \right].
\]
is an isomorphism. Hence we can find a unique \(\mu_t \in \mathcal{D}_{Fr}^{0,\alpha}(\mathcal{P}_t)(\mu)\) such that \(\mathcal{P}_t \mu_t = \rho_t\). Now by definition of \(\rho_t\),

\[\mathcal{P}_t \hat{v}_\infty = \rho_t + \left[\phi^*_{h^\text{app}_t}, \hat{\phi}\right],\]

so that with \(\mu_t\) as above,

\[\mathcal{P}_t (\hat{v}_\infty - \mu_t) = \left[\phi^*_{h^\text{app}_t}, \hat{\phi}\right],\]

e.i. \(\hat{v}^\text{app}_t = \hat{v}_\infty - \mu_t\) solves the desired equation. Consequently, by a straightforward adap-
tion of [MSWW19, Prop. 5.3] to the present case, the inverse \(\mu_t = \mathcal{P}_t^{-1}\rho_t\) is the product of
an exponential packet supported in \(D_p\) with an extra factor \(t^\sigma\) for some \(\sigma\). In particular,
its restriction to \(\partial D_p\) is of order \(O(e^{-\gamma_t^*})\). Hence \(\hat{v}^\text{app}_t\) differs from the diagonal \(\hat{v}_\infty\) by this
exponentially small term. Thus \(\hat{v}^\text{app}_t\) is a solution on \(D_p\) of the boundary value problem
for the equation

\[\mathcal{P}_t \hat{v}^\text{app}_t = \left[\phi^*_{h^\text{app}_t}, \hat{\phi}\right]\]

boundary conditions which are diagonal up to an exponentially small perturbation. We note also that the right-hand side is exactly diagonal. Since \(\mathcal{P}_t\) decouples on \(D_p\) into diagonal and off-diagonal components, it follows by a standard maximum principle argument that the off-diagonal component of \(\hat{v}^\text{app}_t\) is of order \(O(e^{-\epsilon t})\) everywhere on \(D_p\). 

**Proposition 7.11.** Fix \((\hat{\eta}, \hat{\phi}) \in T_{(\mathcal{P}_E, \mathcal{P}_F)} \mathcal{M}'\). Then for some \(\epsilon > 0\),

\[\| (\hat{\eta}, \hat{\phi}, \hat{v}^\text{app}_t) \|^2_{g^\text{app}, t} - \| (\hat{\eta}, \hat{\phi}, \hat{v}_\infty) \|^2_{g^\text{app}, t} = O(e^{-\epsilon t}).\]

**Proof.** We first observe that the integrand localizes on disks \(D = D_p(\frac{1}{2})\) of radius \(\frac{1}{2}\)
around the zeros and strongly parabolic points, i.e. is exponentially small on their complement. On these disks (where \(h^\text{app}_t = h^\text{model}_t\)), we choose the unique representative
\((\hat{\eta}, \hat{\phi})\) of the equivalence class \([\hat{\eta}, \hat{\phi}]\) and a holomorphic frame from Lemma 7.10.

For convenience, since our representative of \(\hat{\eta}\) vanishes on the disk, we introduce the notation

\[\delta\left((0,t\phi), (h_1, \hat{v}_1), (h_2, \hat{v}_2)\right) = 2t^2 \langle \hat{\phi} + [\hat{v}_1, \hat{\phi}], \hat{\phi} \rangle_{h_1} - 2t^2 \langle \hat{\phi} + [\hat{v}_2, \hat{\phi}], \hat{\phi} \rangle_{h_2}.\]

In order to prove the exponential decay of

\[\| (0, \hat{\phi}, \hat{v}^\text{app}_t) \|^2_{g^\text{app}, t(D)} - \| (0, \hat{\phi}, \hat{v}_\infty) \|^2_{g^\text{app}, t(D)} = 2t^2 \int_D \langle \hat{\phi} + [\hat{v}^\text{app}_t, \hat{\phi}], \hat{\phi} \rangle_{h^\text{app}_t} - \langle \hat{\phi} + [\hat{v}_\infty, \hat{\phi}], \hat{\phi} \rangle_{h^\infty_t} = \int_D \delta\left((0,t\phi), (h^\text{app}_t, \hat{v}^\text{app}_t), (h^\infty_t, \hat{v}_\infty)\right),\]
on a disk of radius $\frac{1}{2}$ around a zero or strongly parabolic point, we break the integrand into two terms as
\[
\delta \left( (0, t\phi), (h^\text{app}_t, \dot{v}^\text{app}_t), (h_\infty, v_\infty) \right) = \delta \left( (0, t\phi), (h^\text{model}_t, \dot{v}^X_t), (h_\infty, v_\infty) \right) + \delta \left( (0, t\phi), (h^\text{model}_t, \dot{v}^\text{app}_t), (h^\text{model}_t, \dot{v}^X_t) \right)
\]
and consider each term separately. Here, $\dot{v}^X_t$ is defined using a well-chosen holomorphic variation, as follows. If $\dot{P} = \sum_{n=0}^{\infty} a_n z^n$, then following Dumas-Neitzke [DN19, Eq. 10.12], we set
\[
\mathcal{X} = \sum_{n=0}^{\infty} \frac{a_n}{4n-2} z^{n+1}
\]
and check that $z\dot{P} + 2\mathcal{X} - 4z\mathcal{X}' = 0$. This yields a holomorphic vector field $X = \mathcal{X} \frac{\partial}{\partial z}$ generating the holomorphic deformation adapted to $\dot{q} = \frac{\dot{P}}{z} dz^2$. Now define
\[
F^X_t = \partial_z \mathcal{X} + 2\mathcal{X} \partial_z \left( -\frac{1}{2} \log |z| + m_t \right), \quad \text{so } F^X_\infty = \frac{1}{4} \dot{P}.
\]
Then $\dot{v}^X_t = F^X_t \sigma_3$ satisfies the complex variation equation (7.4) for $(\partial_E, t\phi, h^\text{model}_t)$ and Higgs bundle variation $(0, t\phi)$ as in (7.12). The fact that $\dot{v}^X_t$ satisfies the complex variation equation reduces to the equality
\[
\left( \partial_z \partial_{\bar{z}} - 4t^2 |z|^{-1} \cosh(2m_t) \right) F^X_t + t^2 e^{-2m_t} |z|^{-1} \partial_{\bar{z}} \dot{P} = 0. \quad (7.13)
\]
The assertion of the proposition then follows from the following two claims.

**Claim 1:** There is a positive constant $\varepsilon > 0$ such that
\[
\int_{D_\varepsilon(1/2)} \delta \left( (0, t\phi), (h^\text{model}_t, \dot{v}^X_t), (h_\infty, v_\infty) \right) = O(e^{-\varepsilon t}). \quad (7.14)
\]

The contribution of the $L^2$-metric to the integrand is
\[
\langle \phi_t, \phi_t \rangle_{h^\text{model}_t} + \langle [\phi_t, \dot{v}_t], \phi_t \rangle_{h^\text{model}_t} = \frac{e^{-2m_t} |\dot{P}|^2}{|z|} - \frac{2e^{-2m_t} F^X_t \overline{\dot{P}}}{|z|}. \quad (7.15)
\]
Using $z\dot{P} + 2\mathcal{X} - 4z\mathcal{X}' = 0$, we rewrite
\[
F^X_t = \partial_z \mathcal{X} + 2\mathcal{X} \partial_z \left( -\frac{1}{2} \log |z| + m_t \right) = \partial_z \mathcal{X} - \frac{\mathcal{X}}{2z} + 2\mathcal{X} \partial_z m_t = \frac{\dot{P}}{4} + 2\mathcal{X} \partial_z m_t. \quad (7.16)
\]
Then (7.15) becomes
\[
\langle \phi_t, \phi_t \rangle_{h^\text{model}_t} + \langle [\phi_t, \dot{v}_t], \phi_t \rangle_{h^\text{model}_t} = \frac{e^{-2m_t} |\dot{P}|^2}{2|z|} - \frac{4e^{-2m_t} F^X_t \overline{\dot{P}}}{|z|} \partial_z m_t.
\]
The contribution of the semiflat metric to the integrand is $-\frac{|\dot{P}|^2}{2}$, so the integrand in (7.14) is

$$I_t = t^2 |\dot{P}|^2 (e^{-2m_t} - 1) - 8e^{-2m_t} \bar{P} \chi \partial_z m_t \frac{dz}{|z|} \text{d}\bar{z}.$$ 

A brief calculation using the identities $z\dot{P} + 2\chi - 4z\chi' = 0$ and $\partial_z \bar{P} = 0$ shows that $d\beta_t = I_t$, where

$$\beta_t = 4t^2 (e^{-2m_t} - 1) \bar{P} \chi \frac{dz}{|z|} \text{d}\bar{z}.$$ 

By Stokes' theorem,

$$\int_{D_p(1/2)} I_t = \int_{\partial D_p(1/2)} \beta_t. \quad (7.17)$$

This identity is justified by the fact that the integrand $I_t$ blows up only like $|z|^{-2+\delta}$ for some $\delta > 0$; indeed, since $\chi(z) \sim z$, we have

$$e^{-2m_t} \partial_z m_t \sim |z|^{-1-2(\alpha_1-\alpha_2)} |z|^{-1} = |z|^{-2+2(\alpha_2-\alpha_1)}.$$ 

Since $\beta_t$ is exponentially small on $\partial D_p(1/2)$, the claim follows from (7.17). $\lhd$

CLAIM 2: There is a positive constant $\epsilon > 0$ such that

$$\int_{D_p(1/2)} \delta \left( (0, t\phi), (\dot{h}_t^\text{model}, \dot{\psi}_t^\text{app}), (\dot{h}_t^\text{model}, \dot{\psi}_t^X) \right) = O(e^{-\epsilon t}). \quad (7.18)$$

$\lhd$ Since the difference $\dot{\psi}_t^\text{app} - \dot{\psi}_t^X$ is an exponential packet, $\dot{\psi}_t^\text{app}$ is exponentially close to

$$\dot{\psi}_\infty = \frac{\bar{P}}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on the boundary of $D_p(1/2)$. This is also true for $\dot{\psi}_t^X$ as follows from the right-hand side of (7.16). Hence

$$\left. \left( \dot{\psi}_t^\text{app} - \dot{\psi}_t^X \right) \right|_{\partial D_{1/2}(p)} = O(e^{-\epsilon t})$$

(and similarly for all derivatives of that difference). Both are solutions of the equation

$$\partial_{\dot{h}_t^\text{model}}^E \bar{\partial}_E \dot{\psi} - [\phi^* h_t^\text{model}, \dot{\psi} + [\dot{\psi}, \phi]] = 0.$$ 

hence their difference $\mu_t := \dot{\psi}_t^\text{app} - \dot{\psi}_t^X$ satisfies the homogeneous equation

$$\partial_{\dot{h}_t^\text{model}}^E \bar{\partial}_E \mu_t - [\phi^* h_t^\text{model}, [\mu_t, \phi]] = 0. \quad (7.19)$$

Now consider the function $|\mu_t|_{\dot{h}_t^\text{model}}^2$; we calculate that

$$\frac{1}{2} \Delta |\mu_t|_{\dot{h}_t^\text{model}}^2 = -|\nabla_{\dot{h}_t^\text{model}} \mu_t|_{\dot{h}_t^\text{model}}^2 + \langle \Delta_{\dot{h}_t^\text{model}} \mu_t, \mu_t \rangle_{\dot{h}_t^\text{model}} \leq 0,$$
where $\Delta = \ast d$ and $D^\text{model}$ is the Chern connection with respect to $\overline{\partial}_E$ and $h^\text{model}_t$. To show that $\langle \Delta D^\text{model}_t \mu_t, \mu_t \rangle_{h^\text{model}_t} \leq 0$, substitute

$$\Delta D^\text{model}_t \mu_t = -2 i \ast \left[ \varphi^* h^\text{model}_t, [\mu_t, \varphi] \right] + i \ast [F_{D^\text{model}_t}, \mu_t]$$

using (7.19) and the general identity $2\overline{\partial}_E^2 = F_{D^\text{model}_t} + i \ast \Delta D^\text{model}_t$. We see that $(\overline{\partial}_E, \varphi, h^\text{model}_t)$ is a solution of the self-duality equations, whence

$$i \ast [F_{D^\text{model}_t}, \mu_t] = i \ast [F_{D^\text{model}_t}, \mu_t] = -i \ast \left[ [\varphi, \varphi^* h^\text{model}_t], \mu_t \right].$$

Now use the Jacobi identity to compute

$$\langle \Delta D^\text{model}_t \mu_t, \mu_t \rangle_{h^\text{model}_t} = \langle -2 i \ast \left[ \varphi^* h^\text{model}_t, [\mu_t, \varphi] \right], \mu_t \rangle_{h^\text{model}_t} + \langle -i \ast \left[ [\varphi, \varphi^* h^\text{model}_t], \mu_t \right], \mu_t \rangle_{h^\text{model}_t}$$

$$= -2 \left| [\varphi, \mu_t] \right|^2_{h^\text{model}_t} - \left| [\varphi^* h^\text{model}_t, \mu_t] \right|^2_{h^\text{model}_t} + \left| [\varphi, \mu_t] \right|^2_{h^\text{model}_t} = -\left| [\varphi, \mu_t] \right|^2_{h^\text{model}_t} - \left| [\varphi^* h^\text{model}_t, \mu_t] \right|^2_{h^\text{model}_t} \leq 0.$$

This shows that $|\mu_t|^2_{h^\text{model}_t}$ is subharmonic and its restriction to $\partial D(1/2)$ is $O(e^{-ct})$. By the maximum principle, $\mu_t$ itself is of order $O(e^{-ct})$, and this implies (7.18). □

7.5. Proof of Theorem 7.2.

**Proposition 7.12.** Fix the moduli space $\mathcal{M}_{\text{Higgs}}$ of (either weakly or strongly) parabolic $SL(2, \mathbb{C})$ Higgs bundles. Let $(\overline{\partial}_E, \varphi) \in \mathcal{M}'_{\text{Higgs}}$ be any stable Higgs bundle and $(\hat{\eta}, \hat{\phi})$ an infinitesimal variation. Identify $(\hat{\eta}, \hat{\phi})$ with its image in $T_{(\overline{\partial}_E, \varphi, h_1)} \mathcal{M}_1$. As $t \to \infty$, the difference between $g_{1, t}$ and $g_{\text{app}, t}$ decays exponentially in $t$:

$$\|(\hat{\eta}, \hat{\phi}, \psi_t)\|^2_{g_{1, t}} - \|(\hat{\eta}, \hat{\phi}, \psi^\text{app}_t)\|^2_{g_{\text{app}, t}} = O(e^{-ct}).$$

**Proof.** We use that $h_t(w_1, w_2) = h^\text{app}_t(e^{\gamma} w_1, e^{\gamma} w_2)$ for any $h_0$-Hermitian $\gamma_t \in \mathcal{D}^{0,A}_{\text{Fr}}(\delta)$ satisfying $\|\gamma_t\|^2_{c_b^*} \leq e^{-ct}$, cf. Theorem 6.2, as well as the observation that $\psi_t - \psi^\text{app}_t$ also decays exponentially (which follows since these two quantities satisfy equations whose coefficients and inhomogeneous terms differ by exponentially small amounts). □

**Proof of Theorem 7.2.** The theorem follows directly from Proposition 7.11, Corollary 7.9 and Proposition 7.12. □

7.6. Review: The conjecture of Gaiotto, Moore and Neitzke. We review Gaiotto-Moore-Neitzke’s conjecture for a general (strongly or weakly) parabolic Higgs bundles. In (7.8),
we defined the following metric on $\mathcal{M}_t$:
\[
\| (\dot{\eta}, \dot{\phi}, \dot{\nu}_t) \|^2_{g_{L^2,t}} = 2 \int_C \langle \dot{\eta} - \mathcal{D}_E \dot{\nu}_t, \eta \rangle_{h_t} + t^2 \langle \dot{\phi} + [\dot{\nu}_t, \phi], \phi \rangle_{h_t},
\]
where $\dot{\nu}_t$ is the unique solution to
\[
\partial^h_{E} \mathcal{D}_E \dot{\nu}_t - \partial^h_{E} \dot{\eta} - t^2 [\phi^* h_t, \dot{\phi} + [\dot{\nu}_t, \phi]] = 0.
\]

In §7.3, we defined the $t$-rescaled semiflat metric
\[
\| (\dot{\eta}, \dot{\phi}, \dot{\nu}_\infty) \|^2_{g_{sf,t}} = 2 \int_C \langle \dot{\eta} - \mathcal{D}_E \dot{\nu}_\infty, \eta \rangle_{h_\infty} + t^2 \langle \dot{\phi} + [\dot{\nu}_\infty, \phi], \phi \rangle_{h_\infty},
\]
where the $t$-independent section $\dot{\nu}_\infty$ solves the decoupled equations
\[
\partial^h_{E} \mathcal{D}_E \dot{\nu}_\infty - \partial^h_{E} \dot{\nu}_\infty = 0, \quad [\phi^* h_\infty, \dot{\phi} + [\dot{\nu}_\infty, \phi]] = 0.
\]

Fixing the Higgs bundle $(E, \phi)$ and deformation $(\dot{\eta}, \dot{\phi})$, we consider the difference
\[
\| (\dot{\eta}, \dot{\phi}, \dot{\nu}_t) \|_{g_{L^2,t}} - \| (\dot{\eta}, \dot{\phi}, \dot{\nu}_\infty) \|_{g_{sf,t}}.
\]
Gaiotto-Moore-Neitzke give a beautiful conjectural description of the hyperkähler metric on the Hitchin moduli space; at the coarsest level, it states

**Conjecture 7.13** (Weak form of Gaiotto-Moore-Neitzke’s conjecture for $\mathcal{M}_{SU(2)}$). Fix a Higgs bundle $(\mathcal{D}_E, \phi)$ in the regular locus $\mathcal{M}_t'_{\text{Higgs}}$. Then, as $t \to \infty$
\[
\| (\dot{\eta}, \dot{\phi}, \dot{\nu}_t) \|^2_{g_{L^2,t}} - \| (\dot{\eta}, \dot{\phi}, \dot{\nu}_\infty) \|^2_{g_{sf,t}} = O(e^{-2Mt})
\]
where $M$ is the length of the shortest geodesic on the punctured spectral curve $\Sigma_b - \pi^{-1}(D)$ for $b = \text{Hit}(E, \phi)$ which is not a loop around $\pi^{-1}(D)$ (see Figure 7.1), as measured in the singular flat metric $\pi^* \det \phi$.

![Figure 7.1](image-url)
Note that although Theorem 7.2 proves some degree of exponential decay, the constant is not the sharp one conjectured by Gaiotto-Moore-Neitzke. A closer examination of our the methods in this paper (which we do not explain carefully here) can improve our result to $O(e^{-(M-\varepsilon)t})$ for any $\varepsilon > 0$. (Note, however, that in a special case considered in the next section we are able to get the exact predicted rate.)

There is a significantly stronger version of this conjecture which we explain below. The hyperkähler metric $g_{L^2}$ on $\mathcal{M}$ is completely determined by and determines a twisted fiber-wise holomorphic symplectic structure on the twistor space $Z = \mathcal{M} \times \mathbb{CP}^1$. In [GMN13, GMN10], the authors conjecture that certain holomorphic Darboux coordinates $\mathcal{X}_\gamma$ on the twistor space $Z = \mathcal{M} \times \mathbb{CP}^1$ solve a certain integral equation, see [Nei14, Eq 4.8]. Implicit in the solution to this equation is the hyperkähler metric, but it is difficult to pass between the twistor space formulation of the conjecture and what it says about $g_{L^2}$.

We now describe a slightly stronger version of Conjecture 7.13 that highlights the ingredients in this integral equation. This holds only on the tangent space to the Hitchin section. One may attempt to solve the integral equation by a Picard iteration starting from the initial hyperkähler metric $g_{sf,t}$. The successive iterates should approach $g_{L^2,t}$. The first iterate yields the following expression for the difference of the two metrics over a ray $(E, \varphi, h_t) \in \mathcal{M}_t$:

$$g_{L^2,t} = g_{sf,t} - \frac{2}{\pi} \sum_{\gamma \in \Gamma_b} \Omega(\gamma; b) K_0(2|Z_{\gamma,t}|)(d|Z_{\gamma,t}|)^2 + \ldots$$  \hspace{1cm} (7.20)

Here

• $\Sigma_b \xrightarrow{\pi} \mathbb{C}$ is the spectral cover;
• $\Gamma_b$ is the fiber of a certain local system of lattices (known as the charge lattice, see [Nei18]) $\Gamma \to B'$ over $b$, fitting into the exact sequence

$$0 \to \Gamma^{\text{flavor}} \to \Gamma \to \Gamma^{\text{gauge}} \to 0.$$  

The other two lattices $\Gamma^{\text{flavor}}$ and $\Gamma^{\text{gauge}}$ are determined in the following way. If $\Sigma_b \subset \text{Tot}(K(D))$ is the compactified spectral curve, then the fiber over $b \in B$ of the gauge lattice is

$$\Gamma^{\text{gauge}}_b = \ker \left( H_1(\Sigma_b, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \right).$$  

On the other hand, the fiber $\Gamma^{\text{flavor}}_b$ of the flavor lattice is, for generic divisors $D_s \cup D_w$, a free $\mathbb{Z}$-module generated by loops $\gamma$ around the points of $\pi^{-1}(D)$ in the punctured spectral curve $\Sigma_b - \pi^{-1}(D)$, see [Nei18]. Hence, for generic $D$,

$$\Gamma_b = \ker \left( H_1(\Sigma_b - \pi^{-1}(D), \mathbb{Z}) \to H_1(C, \mathbb{Z}) \right).$$

Note that this involves the homology of the punctured spectral curve.
Finally, $Z_{\ast,t}$ is the period map

$$Z_{\ast,t} : \Gamma_b \rightarrow \mathbb{C} \quad Z_{\gamma,t} = t \oint_{\gamma} \lambda,$$

where $\lambda$ is the tautological (Liouville) 1-form on Tot($K_C$).

Remark 7.14. For loops $\gamma$ in the image of $\Gamma^\text{flavor}$, $Z_{\gamma,t} : \mathcal{B}' \rightarrow \mathbb{C}$ is a constant function. For example, for $G = \text{SU}(2)$, if $\gamma$ is a small counter-clockwise oriented loop around $p \in D_w$ with residue $\frac{\text{Re}z}{z^2}$, then $Z_{\gamma,t} \equiv 2\pi|\sigma|$.

- $K_0$ is the modified Bessel function of the second kind.
- $\Omega(\gamma;b)$ is an integer-valued generalized Donaldson-Thomas invariant $^1$ (see [KS08, JS12] and the discussion in [Nei14]).

The first correction in (7.20) is from the smallest value $2|Z_{\gamma_0}|$ for which $\Omega(\gamma_0;b) \neq 0$ and $\gamma_0 \notin \Gamma^\text{flavor}$. This is why, in Conjecture 7.13, we only consider the length of the shortest geodesic which is not a loop around $\pi^{-1}(D)$. Since $K_0(x) \sim \sqrt{\frac{\pi}{x}}e^{-x}$, the first correction $K_0(2t|Z_{\gamma_0}|) = O\left(e^{-2|Z_{\gamma_0}|t}\right)$. The cross-terms in (7.20) are of order $O\left(e^{-4|Z_{\gamma_0}|t}\right)$, see [Nei18, Eq. 5.3].

The constant of exponential decay conjectured for the whole Hitchin moduli space in Conjecture 7.13 is equal to this smallest allowable exponent $e^{-2|Z_{\gamma_0}|}$. For $G = \text{SU}(2)$, $|Z_{\gamma_0}|$ is the length of a geodesic in the class; indeed, $t \oint_{\gamma} |\lambda|$ is the length of $\gamma$ with respect to the singular flat metric $t^2\pi^*|\det \varphi|$ on the spectral cover $\Sigma_b$. Note that $|\oint_{\gamma} \lambda| \leq \oint_{\gamma} |\lambda|$, with equality if and only if $\gamma$ is a geodesic. In particular, $Z_{\gamma_0} = M$ is the length of the shortest geodesic on $\Sigma_b$ not surrounding a double pole in the singular flat metric $\pi^*|\det \varphi|$, cf. Figure 7.1.

In the strongly parabolic setting, the Hitchin moduli space admits a $\mathbb{C}_\zeta^\times$-action, $(\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \zeta \varphi)$. We can then phrase the conjecture in terms of the decay of the difference of the Hitchin and semiflat metrics along an $\mathbb{R}^+_t$-ray $[(\mathcal{E}, t\varphi, h_t)]$ rather than comparing metrics on the different moduli spaces $\mathcal{M}_t$ and $\mathcal{M}_\infty$. There is no $\mathbb{C}_\zeta^\times$-action in the weakly parabolic case because at the double poles, multiplication of the Higgs field by $\zeta$ induces $\frac{dz^2}{z^2} \mapsto \frac{\zeta^2 dz^2}{z^2}$. In the strongly parabolic setting, the $\mathbb{C}_\zeta^\times$-action identifies $\mathcal{M}_t$ (with metric $g_{L^2,t}$) with $\mathcal{M}_1 = \mathcal{M}$ (with metric $g_{L^2}$) isometrically via $(\bar{\partial}_E, \varphi, h_t) \mapsto (\bar{\partial}_E, t\varphi, h_t)$. On the tangent spaces this becomes $(\eta, \varphi, \bar{v}_t) \mapsto (\eta, t\varphi, \bar{v}_t)$.

$^1$If $\Gamma \rightarrow \mathcal{B}'$ is the local system with fiber $\Gamma_b$, then $\Omega : \Gamma \rightarrow \mathbb{Z}$. Given a section $\gamma$ of $\Gamma$, the function $\Omega(\gamma; \cdot) : \mathcal{B}' \rightarrow \mathbb{Z}$ is typically not continuous, but jumps at (real) codimension 1 walls in $\mathcal{B}'$ and satisfies the Kontsevich-Soibelman wall-crossing formula.
56 ASYMPTOTIC GEOMETRY OF THE MODULI SPACE OF PARABOLIC SL(2,C)-HIGGS BUNDLES

**Conjecture 7.15** (Conjecture 7.13 when \( \mathcal{M}_{\text{Higgs}} \) admits a \( \mathbb{C}^\times \)-action). Fix a ray of Higgs bundles \( (\mathcal{E}, t\varphi) \in \mathcal{M} \) and a corresponding family of tangent vectors \( (\dot{\eta}, t\dot{\varphi}) \in T_{(\mathcal{E}, t\varphi)} \mathcal{M} \); then

\[
\| (\dot{\eta}, t\dot{\varphi}, \dot{\nu}) \|_{\mathcal{g}_{L^2}}^2 - \| (\dot{\eta}, t\dot{\varphi}, \dot{\nu}_\infty) \|_{\mathcal{g}_{\text{sf}}}^2 = O(e^{-2Mt}),
\]

where \( M \) is the length of a shortest geodesic with respect to the singular flat metric \( \pi^*| \det \varphi | \) on the punctured spectral curve \( \Sigma_b - \pi^{-1}(D) \) which is not a loop around \( \pi^{-1}(D) \).

8. THE ASYMPTOTIC GEOMETRY OF THE MODULI SPACE OF STRONGLY-PARABOLIC HIGGS BUNDLES ON THE FOUR-PUNCTURED SPHERE

In this section we specialize the entire previous discussion so as to describe the asymptotic geometry of the moduli space of SL\((2, \mathbb{C})\)-Higgs bundles on the 4-punctured sphere. This is sometimes called the toy model because the moduli space in this case is four-dimensional, the lowest dimension possible. For simplicity we restrict to the strongly parabolic setting here so there is a \( \mathbb{C}^\times \) action. This action identifies the various torus fibers \( \pi^{-1}(q), q \neq 0 \), with one another. The only singular fiber is the nilpotent cone \( \pi^{-1}(0) \). Thus the discriminant locus is compact and \( \mathcal{M}' \) contains an entire neighborhood of infinity. Because of this we can sharpen our results considerably to obtain the optimal rate of exponential decay for \( \mathcal{g}_{L^2} - \mathcal{g}_{\text{sf}} \). We use this to show that the moduli space is a gravitational instanton of type ALG, decaying exponentially to the flat model metric in the sense of [CC21]. This exponential behavior is exceptional in the class of ALG-metrics.

We begin, in §8.1, with an explicit description of the elements of the moduli space of Higgs bundles and the associated spectral data in this 4-punctured sphere setting. The special Kähler metric on the base and semiflat metric are both quite simple §8.2. We then turn in §8.3 to a discussion of the predictions by GMN in this particular case. The optimal rate of exponential decay and attendant curvature decay are derived in §8.4. Finally, §8.5 contains a description of how this case fits into the Chen-Chen classification of ALG metrics [CC21].

8.1. The moduli space \( \mathcal{M} \). We now describe the moduli space of strongly parabolic SL\((2, \mathbb{C})\)-Higgs bundles on the four-punctured sphere in detail. Take \( C = \mathbb{CP}^1 \) and choose any divisor \( D \) with four distinct points of multiplicity 1. The Higgs bundle moduli space depends on the complex structure of \( (\mathbb{C}, D) \), and without loss of generality, we may use a Möbius transformation to arrange that the points of \( D \) are \( 0, 1, \infty, p_0 \). As before we consider the case where the parabolic bundle \( \mathcal{E} \) has full flags and that \( \alpha_i(p) \in [0, 1) \) with \( \alpha_1(p) + \alpha_2(p) = 1 \) for each \( p \in D \), i.e., \( \bar{\alpha}(p) = (\alpha_p, 1 - \alpha_p) \) for \( \alpha_p \in (0, \frac{1}{2}) \). We assume also that \( \text{pdeg}_{\mathbb{R}} \mathcal{E} = 0 \). In summary, the moduli space \( \mathcal{M} \) of strongly parabolic SL\((2, \mathbb{C})\)-Higgs bundles depends on the choice of fourth point \( p_0 \notin \{0, 1, \infty\} \), the complex vector bundle \( E \to \mathbb{CP}^1 \) of rank 2 and degree \(-4\), and the (real) parabolic
weight vector \( \vec{a}(p), p \in D \), with \( a_0(p) \in (0, \frac{1}{2}) \), \( a_1(p) = 1 - a_0(p) \). For generic weight vector, the moduli space \( M \) is a noncompact complex manifold of complex dimension 2 [BY96, Theorem 4.2]. As an algebraic surface it is the blowup of \( \mathbb{C} \times T^2_\tau / \pm 1 \) at four singular points, where \( T^2_\tau \) is an elliptic curve [Hau01]. We show later that the modulus \( \tau \in \mathfrak{h}/\text{SL}(2, \mathbb{Z}) \) is determined by the choice of \( p_0 \). By [Kon93], \( M \) carries a complete hyperkähler metric. The space of weights \( (0, \frac{1}{2})^4 \) is partitioned into open chambers by semistability walls, and the weight vector is called generic if it lies in one of these chambers. If the weight vector lies on a wall of semistability, the moduli space is singular. Details about this chamber structure and the associated wall-crossing phenomena appear in work of Meneses [Men20]. We see below that the structure of the regular part \( M' \) as a complex manifold is the same for all weight vectors, generic or not.

Note finally that for \( \text{SL}(2, \mathbb{C}) \)-Higgs bundles, we fix the underlying holomorphic and Hermitian structure on \( \text{Det} E \). However, on \( \mathbb{C}P^1 \), these are determined by the degree \( \text{deg} \text{Det} E = -4 \). Indeed, \( \text{Det} \mathcal{E} \simeq \mathcal{O}(-4) \) and we fix the Hermitian metric

\[
h_{\text{Det}E} = |z|^2|z-1|^2|z-p_0|^2
\]

which is adapted to the induced parabolic weights on \( \text{Det} E \rightarrow \mathbb{C}P^1 \).

8.1.1. Hitchin fibration. The moduli space \( M \) fibers over the space of meromorphic quadratic differentials with simple poles at the points of \( D \) by the map

\[
\text{Hit} : M \twoheadrightarrow \mathcal{B}, \quad (\vec{a}_E, \varphi) \mapsto \det \varphi.
\]

Since \( \text{deg} \ K^{2}_{\mathbb{C}P^1} = -4 \) equals the number of zeros minus the number of poles (counted with multiplicity) for any section of \( K^{\otimes 2}_{\mathbb{C}P^1} \), we see that meromorphic quadratic differentials in \( \mathcal{B} \) have no zeros. Concretely, fix the usual holomorphic coordinate \( z \) on \( \mathbb{C} = \mathbb{C}P^1 - \{\infty\} \). The Hitchin base is then

\[
\mathcal{B} = \left\{ q = \frac{B}{z(z-1)(z-p_0)}dz^2 \big| B \in \mathbb{C} \right\} \simeq \mathbb{C}_B.
\]

The regular locus is \( \mathcal{B}' \simeq \mathbb{C}_B^\times \).

8.1.2. Explicit description of Higgs bundles in \( M' \).
Proposition 8.1. With all notation as above, the regular locus $M'$ is stratified by the underlying holomorphic bundle type of $E$, which is either $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-3) \oplus \mathcal{O}(-1)$. The large stratum is parametrized by triples $(B, u, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ solving the cubic

$$Bx(x-1)(x-p_0) + u^2 = 0.$$  \hfill (8.1)

The corresponding Higgs bundle is

$$E \cong \mathcal{O}(-2) \oplus \mathcal{O}(-2) \quad \varphi_{B,u,x} = \left( \begin{array}{cc} u & -\frac{Bz(z-1)(z-p_0)+u^2}{z-x} \\ z-x & -u \end{array} \right) \frac{dz}{z(z-1)(z-p_0)}.$$  

The small stratum consists of pairs

$$E \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3) \quad \varphi = \left( \begin{array}{cc} 0 & 1 \\ -\frac{B}{z(z-1)(z-p_0)} & 0 \end{array} \right) dz.$$  

Consequently, the fiber over $\frac{B}{z(z-1)(z-p_0)}dz^2 \in B$ is the elliptic curve in (8.1), compactified by adding the relevant point in the small stratum.

Remark 8.2. The elliptic curve $Bx(x-1)(x-p_0) + u^2 = 0$ is isomorphic to the complex torus $\mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$ where $p_0$ is related to $\tau$ by $p_0 = \lambda(\tau)$; here $\lambda$ is the elliptic modular lambda function.

Proof. Because $p\deg_{\overline{\mathbb{R}}} E = 0$ and the sum of the parabolic weights is 4, $\deg E = -4$. Thus using Grothendieck's theorem for vector bundles over $\mathbb{CP}^1$, we have $E \cong \mathcal{O}(m) \oplus \mathcal{O}(-4-m)$ $m \geq -2$. Note that for any choice of $m$,

$$\text{End } E \cong \left( \begin{array}{cc} \mathcal{O} & \mathcal{O}(4+2m) \\ \mathcal{O}(-4-2m) & \mathcal{O} \end{array} \right).$$

In $M'$, the flags are determined from the Higgs field. Indeed, since $\varphi$ has at most simple poles at 0, 1, $p_0$, we can write

$$\varphi = \frac{dz}{z(z-1)(z-p_0)} \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix}$$

where $a(z), b(z), c(z) \in \mathbb{C}[z]$.

Passing to the coordinate $w = z^{-1}$ using the transition function $w \mapsto w^k$ for $\mathcal{O}(k)$,

$$\varphi = -\frac{dw}{w(1-w)(1-wp_0)} \cdot w^2 \begin{pmatrix} w^0a(w^{-1}) & w^{4+2m}b(w^{-1}) \\ w^{-4-2m}c(w^{-1}) & -w^0a(w^{-1}) \end{pmatrix},$$

and since $\varphi$ has at most a simple pole at $\infty$ ($w = 0$), we have

$$\deg a \leq 2, \quad \deg b \leq 6 + 2m, \quad \deg c \leq -2 - 2m.$$
Additionally, \( \det \varphi = \frac{B}{z(z-1)(z-p_0)} \mathrm{d}z^2 \) for some \( B \in \mathbb{C}^\times \), i.e.,
\[
a(z)^2 + b(z)c(z) = -B(z-1)(z-p_0). \tag{8.2}
\]
We can rule out the cases \( m \geq 0 \) as follows since in these cases, \( \deg c < 0 \) so \( c = 0 \). Thus the left side of (8.2) has even degree while the right side is nonvanishing and has odd degree; this is a contradiction. The only two possible bundle types which remain are when \( m = -2 \) and \( E \cong \mathcal{O}(-2) \oplus \mathcal{O}(-2) \) or \( m = -1 \) and \( E \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3) \).

Suppose that \( E = \mathcal{O}(-2) \oplus \mathcal{O}(-2) \). In this case \( \deg a, \deg b, \deg c \leq 2 \), so
\[
\varphi = \frac{\mathrm{d}z}{z(z-1)(z-p_0)} \begin{pmatrix} a_2z^2 + a_1z + a_0 & b_2z^2 + b_1z + b_0z \\ c_2z^2 + c_1z + c_0 & -a_2z^2 - a_1z - a_0 \end{pmatrix}.
\]
The complex gauge group is the group of constant \( \text{SL}(2, \mathbb{C}) \) gauge transformations. We use this gauge group to write down an explicit representative in each equivalence class. From the condition in (8.2), \( a_2^2 + b_2c_2 = 0 \), so the highest degree part is nilpotent. Using the complex gauge group, we can arrange that \( \ker \begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix} \supset \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Thus each gauge orbit contains a representative of the form
\[
\varphi = \frac{\mathrm{d}z}{z(z-1)(z-p_0)} \begin{pmatrix} a_1z + a_0 & b_2z^2 + b_1z + b_0z \\ c_1z + c_0 & -a_1z - a_0 \end{pmatrix}.
\]
Note that from (8.2), \( b_2 \neq 0 \) and \( c_1 \neq 0 \). By the diagonal gauge transformation
\[
g = \begin{pmatrix} c_1^{-1/2} & 0 \\ 0 & c_1^{1/2} \end{pmatrix},
\]
we can make \( c(z) \) monic. Thus, each gauge orbit contains a representative
\[
\varphi = \frac{\mathrm{d}z}{z(z-1)(z-p_0)} \begin{pmatrix} a_1z + a_0 & b_2z^2 + b_1z + b_0z \\ z + c_0 & -a_1z - a_0 \end{pmatrix}.
\]
Now use
\[
g = \begin{pmatrix} 1 & -a_1 \\ 0 & 1 \end{pmatrix},
\]
to make \( a(z) \) constant:
\[
\varphi = \frac{\mathrm{d}z}{z(z-1)(z-p_0)} \begin{pmatrix} a_0 & b_2z^2 + b_1z + b_0z \\ z + c_0 & -a_0 \end{pmatrix}.
\]
For notational simplicity, write $x = -c_0 \in \mathbb{C}$ and $u = a_0 \in \mathbb{C}$. Imposing (8.2), we find in each equivalence class a representative
\[
\varphi = \frac{dz}{z(z-1)(z-p_0)} \left( \begin{array}{cc}
u & -\frac{B(z-1)(z-p_0)+u^2}{z-x} \\ z-x & -u \end{array} \right),
\]
where $B, x$ and $u$ satisfy (8.1). (Note that when this elliptic equation holds, the upper right entry is a polynomial.) There is no further gauge freedom.

If $\mathcal{E} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-3)$, then
\[
\text{End } \mathcal{E} \cong \left( \begin{array}{cc} \mathcal{O} & \mathcal{O}(2) \\ \mathcal{O}(-2) & \mathcal{O} \end{array} \right)
\]
and the entries of the Higgs field
\[
\varphi = \frac{dz}{z(z-1)(z-p_0)} \left( \begin{array}{cc} a(z) & b(z) \\ c(z) & -a(z) \end{array} \right),
\]
are polynomials with $\deg a \leq 2$, $\deg b \leq 4$ and $\deg c = 0$, satisfying (8.2). From this equation, $c \neq 0$, otherwise the left side of (8.2) has even degree while the right is nonvanishing of odd degree. Now make the gauge transformation
\[
g = \left( \begin{array}{cc} -i & \frac{a(z)}{\zeta^{1/2}B^{1/2}} \\ \zeta^{1/2}B^{1/2} & 0 \end{array} \right).
\]
to make $\varphi$ off-diagonal with $c(z) = -B$. Thus each gauge orbit contains a representative of the form
\[
\varphi = \frac{dz}{z(z-1)(z-p_0)} \left( \begin{array}{cc} 0 & z(z-1)(z-p_0) \\ -B & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ \frac{B}{z(z-1)(z-p_0)} & 0 \end{array} \right) dz.
\]

\[
8.1.3. \mathbb{C}^\times\text{-action. } \text{The moduli space } \mathcal{M} \text{ admits } \mathbb{C}^\times\text{-action } (\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \zeta \varphi). \text{ This action preserves the decomposition } \mathcal{M}' = \mathcal{M}'_{\text{big}} \sqcup \mathcal{M}'_{\text{small}}. \text{ A coordinate for the Higgs bundles in } \mathcal{M}'_{\text{small}} \text{ is } B \in \mathbb{C}^\times, \text{ and in terms of this, the } \mathbb{C}^\times \text{ action is } B \mapsto \zeta^2 B. \text{ We have}
\]
\[
\zeta \varphi_B = g_\zeta \varphi \zeta B g_\zeta^{-1}, \quad g_\zeta = \left( \begin{array}{cc} \zeta^{-1/2} & 0 \\ \zeta^{1/2} & 0 \end{array} \right).
\]
The Higgs bundles in $\mathcal{M}'_{\text{big}}$ are labeled by $(B, u, x) \in \mathbb{C}^\times \times \mathbb{C} \times \mathbb{C}$, and in these coordinates, the $\mathbb{C}^\times$ action is $(B, u, x) \mapsto (\zeta^2 B, \zeta u, x)$. Now,
\[
\zeta \varphi_{(B,u,x)} = g_\zeta \varphi (\zeta^2 B, \zeta u, x) g_\zeta^{-1}, \quad g_\zeta = \left( \begin{array}{cc} \zeta^{-1/2} & 0 \\ 0 & \zeta^{1/2} \end{array} \right).
\]
8.2. **Semiflat metric.** We now determine the semiflat metric: Proposition 8.3 shows that in rescaled polar coordinates on the Hitchin base $\mathbb{C}_B$ the special Kähler metric is the flat conic metric of cone angle $\pi$

$$\frac{dr^2 + r^2d\theta^2}{r}.$$ 

Proposition 8.4 shows that the flat torus fibers in the semiflat metric all equal $T^2_\tau$, normalized so that the area of $T^2_\tau$ is $4\pi^2$.

8.2.1. **Special Kähler metric.**

**Proposition 8.3.** The special Kähler metric on $B'$ is

$$g_{sK} = \frac{dr^2 + r^2d\theta^2}{r}, \quad csK = \int_{\mathbb{CP}^1} \frac{i\,dz \wedge d\bar{z}}{|z(z-1)(z-p_0)|'},$$

in rescaled polar coordinates on $\mathbb{C}_B$ given by $re^{i\theta} = csKB$. This has a conic singularity of cone angle $\pi$ at $B = 0$.

**Proof.** Fix $q = \frac{B}{z(z-1)(z-p_0)}dz^2$, and consider the variation $\dot{q} = \frac{\dot{B}}{z(z-1)(z-p_0)}dz^2 \in T_qB'$. From the definition of the special Kähler metric in Proposition 7.6 and $\dot{\tau} = \frac{q}{z\sqrt{q}}$, the special Kähler metric is

$$||\dot{q}||_{sK}^2 = \frac{1}{2} \int_{\Sigma_B} |\dot{q}|^2 = csK |\dot{B}|^2 |B|', \quad \text{where } csK = \int_{\mathbb{CP}^1} \frac{i\,dz \wedge d\bar{z}}{|z(z-1)(z-p_0)|}.$$ (8.3)

This yields the stated metric. $\square$

8.2.2. **Area of fibers.** In complex analytic terms, the nonsingular fibers are all $T^2_\tau$, and since the semiflat metric is flat on these fibers, it induces a constant multiple of the Euclidean metric on $T^2_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. A priori, we do not know this constant, or equivalently the area of each fiber.

**Proposition 8.4.** For $B \neq 0$, the area of $\text{Hit}^{-1}(B) \simeq T^2_\tau$ in the semiflat metric equals $4\pi^2$, so $T^2_\tau = C_w/c_{fib}(\mathbb{Z} \oplus \tau\mathbb{Z})$ with Euclidean metric $g_{Euc} = dwd\bar{w} = dx^2 + dy^2$ and [New value of $c_{fib}$!]

$$c_{fib} = \frac{2\pi}{\sqrt{\text{Im } \tau}}.$$ (8.4)

**Proof.** Recall from §3.1 that the fibers of Hit are canonically torsors over $\text{Prym}(\Sigma_b, \mathbb{C})$. The identification of $\text{Hit}^{-1}(B)$ with $\text{Prym}(\Sigma_B, \mathbb{CP}^1)$ arises by tensoring by a holomorphic line bundle with fixed Hermitian metric, so the deformation spaces of $\text{Hit}^{-1}(B)$ and $\text{Prym}(\Sigma_B, \mathbb{CP}^1)$ can be identified.

Since $\text{Jac}(\mathbb{CP}^1)$ is a point, $\text{Prym}(\Sigma_B, \mathbb{CP}^1) = \text{Jac}(\Sigma_B)$. To describe the holomorphic line bundles in $\text{Jac}(\Sigma_B)$ explicitly, let $L \to T^2_\tau = \mathbb{C}_z/(\mathbb{Z} \oplus \tau\mathbb{Z})$ be the trivial complex line bundle. The space of all holomorphic line bundles on $T^2_\tau$ is given by $L_\psi = \left(L, \overline{\partial}_L, \overline{\partial}_L + \psi \partial_\bar{z}\right)$
where

\[ \psi \in \mathbb{C} / \frac{\pi}{\text{Im } \tau} (\mathbb{Z} \oplus \tau \mathbb{Z}) \]

is constant. (To see that \( L_\psi \) is trivial if and only if \( \psi \in \frac{\pi}{\text{Im } \tau} (\mathbb{Z} \oplus \tau \mathbb{Z}) \), consider the space of global holomorphic sections of \( L_\psi \). If \( \partial L_\psi f = 0 \), then \( f(z, \bar{z}) = h(z)e^{2i\psi \text{Im } z} \) for \( h(z) \) some holomorphic function \( h(z) \). Because \( f(z, \bar{z}) \) is \( \mathbb{Z} \oplus \tau \mathbb{Z} \)-periodic, it follows that \( h(z + 1) = h(z) \) and \( h(z + \tau) = h(z)e^{-2i\psi \text{Im } \tau} \). One possible solution is \( h(z) = \lambda e^{2\pi imz} \) for \( m \in \mathbb{Z} \) provided there is some \( n \in \mathbb{Z} \) such that \( \pi(m\tau + n) = -\psi \text{Im } \tau \). Thus there is a basis \( \{ h(z) = \lambda e^{2\pi imz} \}_{\lambda \in \mathbb{C}} \) of global holomorphic sections of \( L_\psi \) trivializing \( L_\psi \) if and only if \( \psi \) lies in the claimed lattice.)

The tangent space to \( \text{Jac}(\Sigma_B) \) at \( (\lambda, \bar{\partial}_L) \) is

\[
T_{(\lambda, \bar{\partial}_L)} \text{Hit}^{-1}(B) = \{ (\lambda = 0, \bar{\xi} = \psi d\bar{z}) \} \simeq \mathbb{C}_\psi.
\]

For each deformation of the Higgs bundle spectral data, the associated deformation of the Hermitian-Einstein metric is trivial. The semiflat metric is characterized in Proposition 7.6 by the property that on vertical deformations the semiflat metric is

\[
\int_{\Sigma_B} 2 \left\| \bar{\xi} - \partial_L v_L \right\|^2.
\]

Hence

\[
\left\| (0, \psi d\bar{z}) \right\|^2_{\text{sf}} = \int_{T^2_{\tau}} 2 \left\| \psi d\bar{z} \right\|^2 = 4 \text{Im } \tau \left\| \psi \right\|^2.
\]

It follows that the metric on the torus \( \mathbb{C}_\psi / \left( \frac{\pi}{\text{Im } \tau} \mathbb{Z} \oplus \frac{\pi}{\text{Im } \tau} \tau \mathbb{Z} \right) \) equals \( g = 4 \text{Im } \tau |d\psi|^2 \) and this has area

\[
\text{Vol}_{\text{sf}}(T^2_{\tau}) = (\text{Im } \tau) \left( \frac{\pi}{\text{Im } \tau} \right)^2 (4 \text{Im } \tau) = 4\pi^2.
\]

\[\Box\]

**Figure 8.2.** (LEFT) The four-punctured sphere with flat metric with cone points of angle \( \pi \) at each marked point, from identifying the edges of a triangle. (RIGHT) The induced flat metric on \( \Sigma \), the double cover, from identifying opposite sides of the parallelogram. The map \( \pi : \Sigma \to \mathbb{C}\mathbb{P}^1 \) is given by quotienting by the \( \mathbb{Z}/2 \)-action generated by rotating 180° around \( \infty \).

**8.3. Gaiotto-Moore-Neitzke’s conjecture on the four-punctured sphere.** Fix a holomorphic quadratic differential \( q = \frac{B}{z(z-1)(z-p_0)}dz^2 \in B' \), and consider the ray of holomorphic quadratic differentials \( t^2q \). The optimal coefficient of exponential decay is \( -2M_B t \) where \( M_B \) is the shortest geodesic on \( \Sigma_B \), as measured in the flat metric \( \pi^* |q| \). We can easily
compute the length of this geodesic because with respect to this metric, the spectral cover is the flat torus \( T_2^\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}) \) of area \( |B|c_{sK} \), with \( c_{sK} \) as in (8.3). Hence the correct scaling for this flat metric is

\[
\frac{|B|c_{sK}}{\Im \tau} (dx^2 + dy^2).
\]

Since \( \tau \) lies in the usual fundamental domain \( \{|\tau| \geq 1, -\frac{1}{2} < \Re \tau < \frac{1}{2}\} \) for \( \text{PSL}(2, \mathbb{Z}) \), the length of the shortest geodesic in \( T_2^\tau \) with unscaled metric \((dx^2 + dy^2)\) is 1, and the area of \( T_2^\tau \) is \( \Im \tau \). Thus with respect to the correctly scaled metric above, the shortest geodesic has length

\[
M_B = \sqrt{\frac{|B|c_{sK}}{\Im \tau}}.
\]

The full Gaiotto-Moore-Neitzke conjecture also specifies coefficients of the exponentially decaying terms given by BPS indices. The following BPS counts for the four-punctured sphere were communicated to the authors by Andy Neitzke. Since \( \text{Jac}(\mathbb{CP}^1) \) is a point and \( D_w = \emptyset \), \( \Gamma_B = H_1(\Sigma_B, \mathbb{Z}) \). Given a saddle connection \( \gamma_{\text{saddle}} \) connecting the punctures \( D \subset \mathbb{CP}^1 \), the corresponding lift \( \gamma_{\text{saddle}} \) is a primitive class in \( \Gamma_B = H_1(\Sigma_B, \mathbb{Z}) \) and contributes \( \Omega(\gamma_{\text{saddle}}) = 8 \) to the BPS count. Consider a path \( \gamma_{\text{pair}} \) which separates the 4 marked points into 2 + 2, and let \( \gamma_{\text{pair}} \) be the corresponding lift. (Note that there are infinitely many distinct possible homotopy classes for \( \gamma_{\text{pair}} \).) The BPS invariants receive contributions from an annulus of closed loops homotopic to (as shown in Figure 8.3): these give \( \Omega(\gamma_{\text{pair}}) = -2 \). Note that in \( H_1(\Sigma_B, \mathbb{Z}) \), there is a correspondence between

\[
\gamma_{\text{saddle}} \quad \text{and} \quad \gamma_{\text{pair}} \quad \text{such that} \quad 2[\gamma_{\text{saddle}}] = [\gamma_{\text{pair}}], \quad \text{as illustrated in Figure 8.3.}
\]

Altogether, for \( \gamma \in \Gamma = H_1(\Sigma_B, \mathbb{Z}) \) primitive, we have

\[
\Omega(n\gamma) = \begin{cases} 
8 & \text{for } n = 1 \\
-2 & \text{for } n = 2 \\
0 & \text{for } n > 2.
\end{cases}
\]

\[\text{Figure 8.3. (Left) Paths on } \mathbb{CP}^1 \text{ and (Right) corresponding paths on spectral torus.}\]
Incorporating these BPS indices, we recall the version (7.20) of the conjecture restricted to the Hitchin section, and supposing that the flat spectral torus has a unique shortest geodesic, i.e., \( \text{Im } \tau \neq 1 \). (This includes both the square torus corresponding to \( p_0 = \frac{1}{2}, \tau = i \) and the double cover of the equilateral triangle corresponding to \( p_0 = \tau = e^{i \frac{\pi}{3}} \).)

**Conjecture 8.5** (Weak version of conjecture on Hitchin section). If \( \text{Im } \tau = \text{Im } \lambda(p_0) \neq 1 \), then on the Hitchin section, and in polar coordinates \( re^{i\theta} = c_\mathcal{sK}B \in \mathbb{C} \):

\[
g_{L^2} - g_{sf} = - \frac{2}{\pi} \cdot 8 \cdot K_0(2\sqrt{r/\text{Im } \tau}) \frac{dr^2 + r^2d\theta^2}{2r \text{Im } \tau} + O(e^{-\eta\sqrt{r}}),
\]

where \( \eta > 2\sqrt{2}/\text{Im } \tau \). Since \( d|Z|^2 = \frac{dr^2 + r^2d\theta^2}{2r \text{Im } \tau} \), this agrees with (7.20).

This formulation (which does not feature the parameter \( t \)) is of course sharper than one which only gives decay rates along radial paths \( t^2q \in B' \). We prove this weaker form of the conjecture almost entirely, though unfortunately our methods do not yield the coefficient of this leading exponential term.

### 8.4. Proof of optimal rate of exponential decay.

Theorem 7.2 asserts that the difference between the Hitchin and semiflat metrics decays exponentially along rays in the base, but the decay rate in Theorem 7.2 is far from optimal. We now establish the sharp predicted decay rate on the four-punctured sphere in the strongly parabolic case. The key is that in this special case, \( \dim_{\mathbb{R}} \mathcal{M} = 4 \) and the natural \( \mathbb{C}^\times \)-action on Higgs bundles restricts to a circle action preserving the Hitchin metric and one of the complex structures, but rotating the other two. This symmetry makes it possible to write the Hitchin metric in terms of a single scalar function \( u : T^2 \times \mathbb{R}^+ \rightarrow \mathbb{R} \), at least away from the fiber over the discriminant locus. This reduction appears in LeBrun’s paper [LeB91]. The function \( u \) satisfies the first of the equations in (8.5) below. We now explain how to use this equation to deduce the optimal decay rate starting from any \( \textit{a priori} \) exponential decay.

As noted above, the Gaiotto-Moore-Neitzke conjecture predicts the coefficient of the leading exponential term as a BPS index. We hope to return to a determination of this coefficient using global analytic techniques elsewhere.

**Figure 8.4.** Schematic of \( g_{sf} \) (shown in gray) versus \( g_{L^2} \) (shown in black).
Background: Reduction to a PDE. Given a principal $U(1)$ bundle $M$ over an open set $U \subset \mathbb{R}^3$ and a positive harmonic function $w : U \to \mathbb{R}^+$, the Gibbons-Hawking ansatz produces a hyperkähler metric $g$ on $M$ as follows: define a connection $\omega$ on $M$ with curvature $F = \star_{\mathbb{R}^3} dw$ and then set

$$g = w g_{\mathbb{R}^3} + w^{-1}\omega^2.$$  

LeBrun’s formula generalizes this by describing all Ricci-flat Kähler metrics in two complex dimensions with a holomorphic circle action in terms of solutions of an elliptic equation.

**Proposition 8.6.** [LeB91, Proposition 1] Let $w > 0$ and $u$ be smooth real-valued functions on an open set $U \subset \mathbb{R}^3$ which satisfy

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0$$   \hspace{1cm} (8.5)

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0.$$

Suppose also that the deRham class of the closed 2-form

$$\frac{1}{2\pi} F = \frac{1}{2\pi} (w_x dy \wedge dz + w_y dz \wedge dx + (we^u)_z dx \wedge dy)$$

is integral, i.e., lies in the image of $H^2(U, \mathbb{Z}) \to H^2(U, \mathbb{R})$. Fix an $S^1$ bundle $M \to U$ such that $[c_1(M)]^\mathbb{R} = [\frac{1}{2\pi} F]$, and let $\omega$ be a connection 1-form on $M$ with curvature $F$. Then

$$g = e^u w(dx^2 + dy^2) + w dz^2 + w^{-1}\omega^2$$

is a scalar-flat Kähler metric on $M$; conversely, every scalar-flat Kähler surface with $S^1$-symmetry arises this way.

The metric $g$ is Ricci-flat if, and only, if $u_z = cw$ for some constant $c$.

This reduces to the usual Gibbons-Hawking formula when $u$ and hence $c$ both vanish.

Since $SU(2) = Sp(1)$, hyperkähler 4-manifolds are Calabi-Yau and vice versa. Hence, this result describes all hyperkähler metrics with a holomorphic circle action

**Corollary 8.7.** Write $u_z = cw$. If $c \neq 0$, then the pair $(u, w)$ solves (8.5) if and only if $u$ solves

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0.$$ 

Proof. When $c \neq 0$, the equation for $w$ is obtained by applying $c^{-1}\partial_z$ to the equation for $u$. Clearly $c^{-1}(u_{xx})_z = w_{xx}$, $c^{-1}(u_{yy})_z = w_{yy}$ and $c^{-1}(e^u)_{zzz} = (we^u)_{zzz}$. □

The PDE for the four-punctured sphere. In the strongly parabolic setting, both the semiflat metric $g_{sf}$ and the Hitchin metric $g_{L^2}$ are hyperkähler with circle action lifted from the Hitchin base which fixes one of the complex structures, and hence, using this formalism, we may write these metrics in terms of functions $u_{sf}, w_{sf}$ and $u, w$. One subtlety is that
the coordinate $z$ in LeBrun’s formalism is the Hamiltonian vector field associated to the generator $\partial_\theta$ for the $S^1$ action with respect to the given metric, i.e. $dz = -\partial_\theta J \Omega$. Since we are dealing with two different metrics, there are two radial coordinates, which we denote by $r$ and $\hat{r}$; these are associated to $g_{sf}$ and $g_{L^2}$, respectively. In fact, we have two distinct coordinate systems, $(r, \theta, x, y)$ and $(\hat{r}, \hat{\theta}, \hat{x}, \hat{y})$, with $\hat{r}(r, x, y)$ determined as above (independent of $\theta$ because of the $S^1$ symmetry) and $\hat{\theta} = \theta, \hat{x} = x, \hat{y} = y$.

The coordinates and functions associated to $g_{sf}$ are particularly simple. Indeed, by Propositions 8.3 and 8.4, we can write $T^2_\tau = C_w / (c_{\fib}(Z \oplus \tau Z))$, and $r = c_{sK}|B| \in \mathbb{R}^+$. (In other words, the radial coordinate $r$ is simply the obvious one on the Hitchin base.) We can also choose $\omega_{sf} = d\theta, \theta = \Arg(B)$. We then have

$$u_{sf}(x, y, r) = \log r, \quad w_{sf}(x, y, r) = r^{-1}.$$  

The estimate $g_{L^2} - g_{sf} = O(e^{-\epsilon \sqrt{r}})$ and the definitions of $r$ and $\hat{r}$ above show that, possibly normalizing $\hat{r}$ by an additive constant,

$$\hat{r} - r = O(e^{-\epsilon \sqrt{r}}). \quad (8.6)$$

We now obtain asymptotic estimates for the functions $u$ and $w$ corresponding to the Hitchin metric $g_{L^2}$ in terms of the function $\hat{r}$.

**Proposition 8.8.** If $u : T^2_{x, y} \times \mathbb{R}^+_\hat{r}$ satisfies

$$\Delta_{T^2} u + \partial_\hat{r}^2 e^u = 0.$$  

and $v = u - \log \hat{r}$ satisfies $|v| \leq C e^{-\epsilon \sqrt{r}}$ as $\hat{r} \to \infty$, then

$$v(x, y, \hat{r}) = \hat{r}^{-1/2} K_1(2\lambda_T \sqrt{\hat{r}})(A_1 \cos(2\pi (x, y) \cdot \mu_0) + A_2 \sin(2\pi (x, y) \cdot \mu_0)) + O(e^{-\eta \sqrt{\hat{r}}})$$

for some $\eta > 2\lambda_T$ as $\hat{r} \to \infty$; here $K_1$ is the Bessel function of imaginary argument, $\lambda_T^2$ is the smallest positive eigenvalue of $-\Delta_{T^2}$ and $\cos(2\pi (x, y) \cdot \mu_0)$ and $\sin(2\pi (x, y) \cdot \mu_0)$ are the corresponding eigenfunctions. The precise growth rate of this leading term is $\hat{r}^{-3/4} e^{-2\lambda_T \sqrt{\hat{r}}}$. 

**Proof.** Substitute $u = \log \hat{r} + v$ into the equation to get

$$\Delta_T (\log \hat{r} + v) + \partial_\hat{r}^2 e^{v+\log \hat{r}} = \Delta_T v + e^v (\hat{r} \partial_\hat{r}^2 v + \hat{r} (\partial_\hat{r} v)^2 + 2\partial_\hat{r} v) = 0.$$  

We first transform this by multiplying both sides by $\hat{r}$ and setting $\hat{r} = \rho^2$. Since $\hat{r} \partial_{\hat{r}} = \frac{1}{2} \rho \partial_\rho$, this becomes, after some simplification,

$$e^v (\rho^2 \partial_\rho^2 v + 3\rho \partial_\rho v + (\rho \partial_\rho v)^2) + 16\rho^2 \Delta_T v = 0,$$

which we write more simply as

$$Lv := \rho^2 \partial_\rho^2 v + 3\rho \partial_\rho v + 16\rho^2 \Delta_T v = Q(v, \rho \partial_\rho v, \rho^2 \partial_\rho^2 v), \quad (8.7)$$

where $Q = (1 - e^v)(\rho^2 \partial_\rho^2 v + 3\rho \partial_\rho v + (\rho \partial_\rho v)^2)$. 


We first observe that if $|f(\rho, x, y)| \leq C_0$ in the region $\rho \geq \rho_0$, then for any $N > 0$, there exists a constant $C_N$ such that
\[
\sup_{i+j+k \leq N} |(\rho \partial_\rho)^i(\rho \partial_x)^j(\rho \partial_y)^k v| \leq C_0 C_N.
\] (8.8)
To prove this, note that $\rho \partial_\rho, \rho \partial_x, \rho \partial_y$ are invariant with respect to the dilation $(\rho, x, y) \mapsto (\lambda \rho, \lambda x, \lambda y)$ for any $\lambda > 0$, and hence so is the entire equation (8.7). Thus we may estimate these scale-invariant derivatives in some region $2^{-k-1} < \rho < 2^{-k+1}$ and $|x - x_0| + |y - y_0| < 2^{-k}$ for some $(x_0, y_0) \in T^2$ by dilating by the factor $\lambda = 2^k$ and then invoking standard Schauder theory in $1/2 \leq \rho \leq 2$, where (8.7) is uniformly elliptic. In other words, the a priori estimate when $\rho$ is small reduces to one in a region where $\rho \approx 1$. This argument reflects the fact that the operator is of ‘uniformly degenerate type’, and this sort of rescaling argument is standard in that context, see [Maz91].

We next obtain bounds for the solution to the inhomogeneous linear problem $Lv = f$. For this, decompose $v$ and $f$ into eigenfunctions on the torus. Write $T^2 = \mathbb{R}^2/\Lambda$ and denote by $\Lambda^\vee$ the dual lattice. The exponentials $e^{2\pi i x \cdot \mu}/\mu \in \Lambda^\vee$, give a complete basis of $L^2(T^2)$ by eigenfunctions of $\Lambda_T$ with associated eigenvalues $4\pi^2|\mu|^2$. (Of course, $v$ is real-valued so we really should be working with the real and imaginary parts of these eigenfunctions.) This reduces the problem to the family of equations $L_\mu v_\mu = f_\mu$ where $v_\mu$ and $f_\mu$ are the eigencomponents of $v$ and $f$, and
\[
L_\mu = \rho^2 \partial_\rho^2 + 3\rho \partial_\rho - 16\pi^2|\mu|^2 \rho^2.
\]
This is essentially the Bessel equation. There is a unique (up to constant multiple) solution which decays exponentially as $\rho \to \infty$, namely $\varphi_\mu(\rho) = |\mu|^{1/2}\rho^{-1}K_1(4\pi|\mu|\rho)$. This satisfies $\varphi_\mu(\rho) \sim C\rho^{-3/2}e^{-4\pi|\mu|\rho}$, where $C$ is independent of $\mu$. When $\mu = 0$, the unique (up to constant multiple) solution decaying at infinity is $\varphi_0(\rho) = \rho^{-2}$. In terms of these we can write a particular solution to the inhomogeneous equation as
\[
v_\mu(\rho) = -\varphi_\mu(\rho) \int_a^\rho \varphi_\mu(s)^{-2}s^{-3} \int_s^\infty \varphi_\mu(\sigma)f_\mu(\sigma)\sigma^3 \, d\sigma \, ds,
\]
where we may take $a = \infty$ if the outer integral converges. The general solution is the sum of this $v_\mu$ and a multiple of $\varphi_\mu$.

Now suppose that $f \in C^\infty$ and for every $N \geq 0$, $|\partial^N f| \leq C_N e^{-\eta|\mu|}$ for some $\eta > 0$. Here (and below) $\partial^N$ denotes any monomial of order $N$ in the vector fields $\rho \partial_\rho, \rho \partial_x, \rho \partial_y$. For every $N \geq 0$, each eigencomponent of $f$ satisfies
\[
|f_\mu| \leq C_{2N}(1 + |\mu|^2)^{-N} e^{-\eta|\mu|}.
\]
We next make some elementary estimates: first, for $\mu \neq 0$,
\[ \int_s^\infty e^{3/2}e^{-4\pi |\mu|s}e^{-\eta s}e^s \, ds \leq C s^{3/2}e^{-(4\pi |\mu|+\eta)s}, \]
next, if $\eta < 4\pi |\mu|$, we must take $a$ to be some finite number and find that
\[ \int_a^\infty e^{8\pi |\mu|s} e^{-4\pi |\mu|s} e^{-4\pi \eta s} \, ds \leq C e^{4\pi |\mu|\eta s} + C, \]
so finally, taking the product of this with $\varphi_\mu$, we conclude that
\[ |v_\mu(\rho)| \leq \sup |v_\mu| |\eta|^{-3/2} e^{-4\pi |\mu|\rho} (Ce^{4\pi |\mu|\eta \rho} + C) \leq C \sup |v_\mu| e^{-\eta \rho}. \]
If $\eta > 4\pi |\mu|$, then we take $a = \infty$, and deduce once again that $|v_\mu(\rho)| \leq C \sup |v_\mu| e^{-\eta \rho}$. (We may exclude the case that $\eta = 4\pi |\mu|$ for any $\mu$.) The final constant $C$ is independent of $\mu$. We conclude from all of this that for any $N > 0$ and $\mu \neq 0$,
\[ |v_\mu(\rho)| \leq C C_{2N} (1 + |\mu|^2)^{-N} e^{-\eta \rho}, \]
along with corresponding estimates for any derivative $(\rho \partial_\rho)^j v_\mu$. When $\mu = 0$ we can obtain the same estimate for $|v_0(\rho)|$ by a slightly more elementary calculation, taking $a = \infty$ in the outer integral. Reassembling these components, we obtain that if $\lambda_T = \min_{\mu \neq 0} 2\pi |\mu| = 2\pi |\mu_0|$, so that $2\lambda_T = 4\pi |\mu_0|$, then since the actual solution $v$ must be the sum of these particular solutions and some homogeneous solution, we have proved that
\[ |v(\rho, x, y)| \leq Ce^{-\eta \rho} \quad \text{if} \quad \eta < 2\lambda_T \]
and
\[ v(\rho, x, y) = A \varphi_\mu(\rho) e^{2\pi i \rho \cdot \mu_0} + O(e^{-\eta \rho}) \quad \text{if} \quad \eta > 2\lambda_T. \]
Note that
\[ \varphi_\mu(\rho) = \rho^{-3/2} \left( \sum_{j=0}^{\infty} a_j \rho^j \right) e^{-2\lambda_T \rho}, \]
where the sum is convergent, and the remainder term decays at a higher exponential rate than the sum of this series. There is a corresponding statement for all derivatives.

Finally let us return to the nonlinear equation (8.7). We start with the $C^0$ estimate $|v| \leq Ce^{-\eta \rho}$. The Schauder estimates imply that $|\partial^N v| \leq C_N e^{-\eta \rho}$, and hence $|Q(v, \rho \partial_\rho v, \rho^2 \partial_\rho^2 v)| \leq C e^{-2\rho}$. Regarding $Q$ as the inhomogeneous term $f$ and applying the argument above, we obtain that $|v| \leq Ce^{-2\rho}$ so long as $2\epsilon < 2\lambda_T$. Iterating this argument a finite number of times, and recalling at last that $\rho = \sqrt{\tau}$, we conclude finally that the decomposition in the statement of this Proposition holds.

Lemma 8.9. The smallest nonzero eigenvalue of $-\Delta_{T^2}$ on the flat torus $T^2_\tau$ in the semiflat metric is $\lambda^2 = 1/\text{Im} \tau$. Hence, $u - \log \tau \sim T(x, y) \tau^{-5/4} e^{-\frac{2}{\sqrt{\text{Im} \tau}} \tau^{1/2}} + O(e^{\eta \sqrt{\tau}})$ for some $\eta > 2\sqrt{2}/\text{Im} \tau$ as $\tau \to \infty$. 

\[ \square \]
Proof. For a 2-torus $\mathbb{T}^2 = \mathbb{C}/(\alpha \mathbb{Z} \oplus \beta \mathbb{Z})$, the dual lattice is given by $\Lambda^\vee = \hat{\alpha} \mathbb{Z} \oplus \hat{\beta} \mathbb{Z}$ where

$$\hat{\alpha} = \frac{i \beta}{\text{Im}(\alpha \beta)}, \quad \hat{\beta} = -\frac{i \alpha}{\text{Im}(\alpha \beta)}.$$

(If the lattice has basis $B$ the dual lattice has basis $(B^T)^{-1}$.) In our setting, $T_2^2 = \mathbb{C}/(c_{\text{fib}} \mathbb{Z} \oplus c_{\text{fib}} \tau \mathbb{Z})$, hence $\hat{\alpha} = -\frac{r T}{c_{\text{fib}} \text{Im} \tau}, \hat{\beta} = \frac{i}{c_{\text{fib}} \text{Im} \tau}$. The smallest value of $|\mu|$ for $\mu \in \Lambda^\vee$ is $\frac{1}{c_{\text{fib}} \text{Im} \tau}$, hence the smallest eigenvalue of $-\Delta_{T^2}$ is $\lambda_T^2 = (\frac{2 \pi}{c_{\text{fib}} \text{Im} \tau})^2$. Using the value $c_{\text{fib}} = \frac{2 \pi}{\sqrt{\text{Im} \tau}}$ computed in (8.4), $\lambda_T = 1/\sqrt{\text{Im} \tau}$.

The constant of exponential decay of $u - \log \hat{r}$ (and hence of $w - \hat{r}$) matches exactly the predicted one in Conjecture 8.5 for $g_{L^2} - g_{sf}$. However, this is not the end of the story, since $\hat{r}$ is not the correct radial coordinate and we have not yet taken care of the dependence of $g_{L^2}$ on the connection form $\omega$.

To conclude the estimate, we must therefore pass from $\hat{r}$ to $r$ in such a way as to preserve this sharp estimate and obtain some control on the difference between $\omega$ and $d\theta$.

**Proposition 8.10.** There is a constant $A$ such that, restricted to the tangent bundle of the Hitchin section,

$$g_{L^2} - g_{sf} = A K_0(2\sqrt{2/\text{Im} \tau}) \frac{dr^2 + r^2 d\theta^2}{r} + O(e^{-\eta \sqrt{\hat{r}}}),$$

for some $\eta > 2\sqrt{2}/\text{Im} \tau$. In fact, there is a slightly more complicated expression for the difference $g_{L^2} - g_{sf}$ even away from the Hitchin section, which is the sum of terms involving the Bessel functions $K_0(2\lambda_T \sqrt{\hat{r}})$ and $K_1(2\lambda_T \sqrt{\hat{r}})$, which each decay like $r^{-1/4}e^{-2\lambda_T \sqrt{\hat{r}}}$, and a remainder which decays at a faster exponential rate. This expression will be recorded at the end of the proof.

Proof. First note that, as $\hat{r} \to \infty$,

$$v = \hat{r}^{-1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T(x, y) + O(e^{-\eta \sqrt{\hat{r}}})$$

for some $\eta > 2\lambda_T$, where $T(x, y)$ denotes the appropriate trigonometric factor. Therefore

$$v_{\hat{r}} = \left(\lambda_T \hat{r}^{-1} K_1'(2\lambda_T \sqrt{\hat{r}}) - \frac{1}{2} \hat{r}^{-3/2} K_1(2\lambda_T \sqrt{\hat{r}})\right) T(x, y) + O(e^{-\eta \sqrt{\hat{r}}})$$

Using the identity $zK_1'(z) - K_1(z) = -zK_2(z)$, this is the same as

$$v_{\hat{r}} = -\lambda_T \hat{r}^{-1} K_1(2\lambda_T \sqrt{\hat{r}}) T(x, y) + O(e^{-\eta \sqrt{\hat{r}}}) \quad (8.9)$$

The principal difficulty is to find some sharp comparison of the functions $r$ and $\hat{r}$. To accomplish this, we use the fact that these two metrics are Kähler for the same complex structure. This complex structure, which we denote by $\mathcal{I}$, satisfies (and is determined by):

$$\mathcal{I}(dr) = rd\theta, \mathcal{I}(d\hat{r}) = w^{-1} \omega, \text{ and } \mathcal{I}(dx) = dy, \mathcal{I}(dy) = -dx.$$
Let us write
\[ d\hat{r} = a_1 dr + a_2 dx + a_3 dy, \]
\[ dr = b_1 d\hat{r} + b_2 dx + b_3 dy, \] and
\[ \omega = d\theta + c_1 d\hat{r} + c_2 dx + c_3 dy. \]
Note that \( d\theta \) does not appear in the first two expressions and none of the \( a_i, b_i \) or \( c_i \) depend on \( \theta \) because of the \( S^1 \) symmetry. Then
\[ I d\hat{r} = w^{-1} \omega = w^{-1} (d\theta + c_1 d\hat{r} + c_2 dx + c_3 dy) \]
\[ = a_1 r d\theta + a_2 dy - a_3 dx, \]
from which we conclude that \( a_1 = \frac{1}{rw} \) as well as \( c_1 = 0, c_2 = -wa_3, c_3 = wa_2. \) Altogether,
\[ d\hat{r} = (rw)^{-1} dr + a_2 dx + a_3 dy, \]
\[ dr = (rw)d\hat{r} - rwa_2 dx - rwa_3 dy, \] and
\[ \omega = d\theta - wa_3 dx + wa_2 dy. \]

The Hitchin section is \( \{(x,y) = (0,0)\} \), so \( dx = dy = 0 \) on its tangent bundle. This follows, since the image of the Hitchin section is horizontal with respect to the Gauss-Manin connection \( \nabla^{GM} \) on the Hitchin fibration and \( \nabla^{GM} dx = \nabla^{GM} dy = 0 \), cf. [MSWW19, §3.2]. Therefore, restricted to the Hitchin section,
\[ g_{L^2} = wd\hat{r}^2 + w^{-1} \omega^2 = w((rw)^{-1} dr)^2 + w^{-1} d\theta^2 = \frac{1}{rw} \left( \frac{dr^2}{r} + r d\theta^2 \right). \]
Hence,
\[ g_{L^2} - g_{st} = \left( \frac{1}{rw} - 1 \right) \left( \frac{dr^2}{r} + r d\theta^2 \right). \]

We now conclude with a remarkable identity. On the Hitchin section,
\[ d\hat{r} = \frac{dr}{rw} \implies wd\hat{r} = \left( \frac{1}{\hat{r}} + v_r \right) d\hat{r} = \frac{dr}{r}, \]
and hence by integration and choosing the constant of integration appropriately, \( \log \hat{r} + v = \log r \), or equivalently, \( r = \hat{r} e^v = \hat{r} (1 + v + O(v^2)) \). Inserting the asymptotic expression for \( v \) we find that
\[ r = \hat{r} (1 + \hat{r}^{-1/2} K_1(2\lambda T \sqrt{\hat{r}}) T + \ldots) = \hat{r} + \hat{r}^{1/2} K_1(2\lambda T \sqrt{\hat{r}}) T + O(e^{-\eta \sqrt{\hat{r}}}) \]
and so
\[ rw = \left( \hat{r} + \hat{r}^{1/2} K_1(2\lambda T \sqrt{\hat{r}}) T + \ldots \right) \left( \frac{1}{\hat{r}} - \frac{\lambda T}{\hat{r}} K_2(2\lambda T \sqrt{\hat{r}}) T + \ldots \right) \]
\[ = 1 + 2\lambda T \left( \frac{1}{2\lambda T \sqrt{\hat{r}}} K_1(2\lambda T \sqrt{\hat{r}}) - \frac{1}{2} K_2(2\lambda T \sqrt{\hat{r}}) \right) T + \ldots \]
We now avail ourselves of the classical formula $K_0(z) - K_2(z) = -(2/z) K_1(z)$ and replace $\hat{r}$ by $r$ on the right hand side to see that
\[ rw = 1 - \lambda_T K_0(2\lambda_T \sqrt{r}) T + O(e^{-\eta \sqrt{r}}), \]
and so, at long last,
\[ \frac{1}{rw} - 1 = \lambda_T K_0(2\lambda_T \sqrt{r}) T + O(e^{-\eta \sqrt{r}}) \]
for some $\eta > 2\lambda_T$. Substituting $\lambda_T = 1/\sqrt{\Im \hat{r}}$, this is precisely the claim.

Let us proceed to find (or at least estimate) the metric on the entire end of the moduli space. For this we need to know a bit more about the coefficients $a_2$ and $a_3$, which we can learn from the curvature equation $d\omega = F$. This yields the three equalities
\[ w_x = -\partial_{\hat{r}}(wa_2), \quad w_y = -\partial_{\hat{r}}(wa_3), \quad \partial_x(wa_2) + \partial_y(wa_3) = \partial_{\hat{r}}(we^v). \]

Now
\[ w_x = -\lambda_T \hat{r}^{-1} K_2(2\lambda_T \sqrt{\hat{r}}) T_x + \ldots, \quad w_y = -\lambda_T \hat{r}^{-1} K_2(2\lambda_T \sqrt{\hat{r}}) T_y + \ldots, \]
so integrating in from infinity, we obtain that $wa_2$ and $wa_3$ are of the form $\lambda_T \int \hat{r}^{-1} K_2(2\lambda_T \sqrt{\hat{r}}) d\hat{r} = -K_1(2\lambda_T \sqrt{\hat{r}}) \hat{r}^{-1/2}$ multiplied by $T_x$ or $T_y$, respectively, plus a faster exponential error. The third curvature equation does not give additional information.

We can now integrate the equation for $d\hat{r}$ along the $r$-radial rays to obtain that
\[ r = \hat{r} + \hat{r}^{1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T + O(e^{-\eta \sqrt{\hat{r}}}) \quad (8.10) \]
even away from the Hitchin section. With this, we can finally show that $g_{L^2} - g_{sf}$ has the predicted rate of exponential decay of $g_{L^2} - g_{sf}$ where
\[ g_{L^2} = e^u w(dx^2 + dy^2) + \omega \hat{r}^2 + w^{-1} \omega^2 \]
\[ g_{sf} = (dx^2 + dy^2) + r^{-1} dr^2 + r d\theta^2. \]

Here, $w = u_{\hat{r}}$ and
\[ v = u - \log \hat{r} = \hat{r}^{-1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T + O(e^{-\eta \sqrt{\hat{r}}}) \]
From this alone, we see
\[ e^u u_{\hat{r}} = e^v (1 + v \hat{r}) = \left(1 + \hat{r}^{1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T \right) \left(1 - \lambda_T K_2(2\lambda_T \sqrt{\hat{r}}) T \right) + O(e^{\eta \sqrt{\hat{r}}}) \quad (8.11) \]
\[ = 1 - \lambda_T K_0(2\lambda_T \sqrt{\hat{r}}) T + O(e^{\eta \sqrt{\hat{r}}}) \]
\[ = 1 - \lambda_T K_0(2\lambda_T \sqrt{\hat{r}}) T + O(e^{\eta \sqrt{\hat{r}}}). \]
In the last line we used the relation between $r$ and $\hat{r}$ in (8.10). Similarly, we see that

\[(w \hat{r}^2 + w^{-1} \omega^2) = w \left( \frac{1}{r \hat{r}} \, dr + a_2 \, dx + a_3 \, dy \right)^2 + w^{-1} \left( d\theta - wa_3 \, dx + wa_2 \, dy \right)^2\]

\[= \frac{1}{\hat{r}} \left( 1 - \lambda_T K_2(2\sqrt{\hat{r}} \lambda_T) T \right) \cdot \left( \frac{\hat{r}}{r} \left( 1 + \lambda_T K_2(2\sqrt{\hat{r}} \lambda_T) \right) T \, dr - \hat{r}^{1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T_x \, dx - \hat{r}^{1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T_y \, dy \right)^2 + \hat{r} \left( 1 + \lambda_T K_2(2\lambda_T \sqrt{\hat{r}}) \right) \left( d\theta + \hat{r}^{-1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T_y \, dx - \hat{r}^{-1/2} K_1(2\lambda_T \sqrt{\hat{r}}) T_x \, dy \right)^2 + O(e^{-\eta \sqrt{\hat{r}}})\]

To make these estimates in $r$ rather than $\hat{r}$, we use the relation between $r$ and $\hat{r}$ in (8.10). In particular, note that $K_i(2\lambda_T \sqrt{\hat{r}}) = K_i(2\lambda_T \sqrt{\hat{r}}) + O(e^{-\eta \sqrt{\hat{r}}})$. It follows that

\[(w \hat{r}^2 + w^{-1} \omega^2) = \left( 1 + \lambda_T K_0(2\lambda_T \sqrt{\hat{r}}) T \right) \frac{dr^2 + r^2 d\theta^2}{r} \quad (8.12)\]

\[+ 2r^{-1/2} K_1(2\lambda_T \sqrt{\hat{r}}) \left( -dr \cdot (T_x \, dx + T_y \, dy) + r d\theta \cdot (T_y \, dx - T_x \, dy) \right) + O(e^{-\eta \sqrt{\hat{r}}}).\]

At long last, looking at (8.11) and (8.12), we see every term appearing in the difference $g_{L^2} - g_{sf}$ is of order $O(e^{-2\lambda \sqrt{\hat{r}}})$. This completes the proof.

\[\square\]

8.5. ALG gravitational instantons. Chen-Chen have given a classification of noncompact complete connected hyperkähler manifold $X$ of real dimension 4 with faster than quadratic curvature decay [CC15]. They prove that any connected complete gravitational instanton with this curvature decay must be asymptotic to one element in a short list of standard models which are torus bundles over the flat cone $\mathbb{C}_\beta$ of cone angle $2\pi \beta \in (0, 2\pi]$. The list of possible torus bundles $E \to \mathbb{C}_\beta$ is quite restricted.

**Theorem 8.11.** [CC15, Theorem 3.11] [CC21, Theorem 3.2] Suppose $\beta \in (0, 1]$ and $\tau \in \mathbb{H} = \{ \tau | \text{Im} \, \tau > 0 \}$ are parameters in the following table:

| $\beta$ | $\tau$ | $D$ | $\text{Regular}$ | $I^*_0$ | $I^*_I$ | $II^*_I$ | $III^*_I$ | $IV^*_I$ |
|---------|--------|-----|-----------------|---------|---------|---------|---------|---------|
| $1$     | $\mathbb{H}$ | $1$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{5}{6}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{3}$ |
| $2\pi i/3$ | $e^{2\pi i/3}$ | $e^{2\pi i/3}$ | $i$ | $i$ | $e^{2\pi i/3}$ | $e^{2\pi i/3}$ |

Suppose $\ell > 0$ is some scaling parameter. Let $E$ be the manifold obtained by identifying $(u, v) \simeq (e^{2\pi i \beta} u, e^{2\pi i \beta} v)$ in the space

\[\{ u \in \mathbb{C} : \text{Arg}(u) \in [0, 2\pi \beta] \& \, |u| \geq R \} \times \mathbb{C}_v / (\mathbb{Z} \ell + \mathbb{Z} \ell \tau).\]
Every ALG gravitational instanton $X$ is asymptotic to one of these standard models $(E, g_{\text{model}})$. Moreover, if $\beta = 1$, then $X$ is the standard flat gravitational instanton $C \times T^2$.

For this four-dimensional family of Hitchin moduli spaces on $\mathbb{C}P^1$, the semiflat metric is the model metric for $\beta = \frac{1}{2}$ and $\tau = \lambda^{-1}(p_0)$. From the bounds on $g_{L^2}$ and all derivatives in (8.8), it follows that each component of the Riemann curvature tensor is also exponentially decaying as one approaches the ends, hence the curvature decay hypothesis is satisfied. Consequently, these spaces fit into the Chen-Chen classification of ALG gravitational instantons.

Let $\mathcal{N}_{\beta,\tau}$ denote the moduli space of ALG gravitational instantons with faster than quadratic curvature decay which are asymptotic to the $(\beta, \tau)$ standard model. Generic hyperkähler metrics $g \in \mathcal{N}_{\beta,\tau}$ decay at a polynomial rate to the model metric $g_{\text{model}}$. However, the hyperkähler metrics from the strongly parabolic Hitchin moduli spaces on the four-punctured sphere decay exponentially to the model metric. The four-dimensional family parameterized by parabolic data at $0, 1, p_0, \infty$ lies in a distinguished locus in $\mathcal{N}_{\beta,\tau}$ consisting of those metrics with exponential decay. We shall investigate this further in a future paper.

**References**

[BB04] O. Biquard and P. Boalch, “Wild nonabelian Hodge theory on curves,” Compositio Math. 140 (2004) 179–204, arXiv:math/0111098.

[BGP97] O. Biquard and O. García-Prada, “Parabolic vortex equations and instantons of infinite energy,” Journal of Geometry and Physics 21 (1997) 238–254.

[Biq97] O. Biquard, “Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse),” Ann. Sci. Éc Norm 30 (1997) 41–96.

[Bla15] J. Blaavand, The Dirac-Higgs bundle. PhD thesis, University of Oxford, 2015. http://people.maths.ox.ac.uk/hitchin/hitchinstudents/blaavand.pdf.

[BY96] H. Boden and K. Yokogawa, “Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves: I,” Internat. J. Math. 7 (1996) 573–598, arXiv:alg-geom/9610014.

[CC15] G. Chen and X. Chen, “Gravitational instantons with faster than quadratic decay (I),”(2015) arXiv:1505.01790.

[CC21] G. Chen and X. Chen, “Gravitational instantons with faster than quadratic decay (III),” Math. Ann. 380 no. 1-2, (2021) 687–717, arXiv:1603.08465.

[DN19] D. Dumas and A. Neitzke, “Asymptotics of Hitchin’s metric on the Hitchin section,” Comm. Math. Phys. 367 no. 1, (2019) 127–150, arXiv:1802.07200.

[FN17] L. Fredrickson and A. Neitzke, “From $S^1$-fixed points to $\mathcal{W}$-algebra representations,” (2017) arXiv:1709.06142.

[Fre99] D. Freed, “Special Kähler manifolds,” Comm. Math. Phys. 203 no. 1, (1999) 31–52.
L. Fredrickson, “Generic ends of the moduli space of $SL(n, \mathbb{C})$-Higgs bundles,” (2018) arXiv:1810.01556.

L. Fredrickson, “Exponential decay for the asymptotic geometry of the Hitchin metric,” Comm. Math. Phys. 375 no. 2, (2020) 1393–1426, arXiv:1810.01554.

D. Gaiotto, G. W. Moore, and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” Comm. Math. Phys. 299 no. 1, (2010) 163–224, arXiv:0807.4723.

D. Gaiotto, G. W. Moore, and A. Neitzke, “Wall-crossing, Hitchin systems, and the WKB approximation,” Adv. Math. 234 (2013) 239–403, arXiv:0907.3987.

T. Hausel, “Geometry of the moduli space of Higgs bundles,” (2001) arXiv:math/0107040.

D. Joyce and Y. Song, “A theory of generalized Donaldson-Thomas invariants,” Memoirs of the AMS 217 (2012), arXiv:0810.5645.

T. Hausel, “Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface,” J. Math. Soc. Japan 45 no. 2, (1993) 253–276.

M. Kontsevich and Y. Soibelman, “Stability structures, motivic Donaldson-Thomas invariants and cluster transformations,” (2008) arXiv:0811.2435.

C. LeBrun, “Explicit self-dual metrics on $\mathbb{CP}^2 \# \cdots \mathbb{CP}^2$,” Journal of Differential Geometry 34 (1991) 223–253.

M. Logares and J. Martens, “Moduli spaces of parabolic Higgs bundles and Atiyah algebroids,” J. reine angew. Math. 649 (2010) 89–116, arXiv:0811.0817.

R. Mazzeo, “Elliptic theory of differential edge operators. I,” Comm. Partial Differential Equations 16 no. 10, (1991) 1615–1664.

C. Meneses, “Geometric models and variation of weights on moduli of parabolic Higgs bundles over the Riemann sphere: a case study,” (2020) arXiv:2012.13389.

T. Mochizuki, “Kobayashi-Hitchin Correspondence for tame harmonic bundles and an application,” Astérisque no. 309, (2006) viii+117, arXiv:math/0411300.

T. Mochizuki, “Wild harmonic bundles and wild pure twistor $D$-modules,” Astérisque no. 340, (2011) x+607, arXiv:0803.1344.

T. Mochizuki, “Harmonic bundles and Toda lattices with opposite sign,” (2013) arXiv:1301.1718.

T. Mochizuki, “Asymptotic behavior of certain families of harmonic bundles on Riemann surfaces,” J. Topol. 9 no. 4, (2016) 1021–1073, arXiv:1508.05997.

R. Mazzeo, J. Swoboda, H. Weiss, and F. Witt, “Ends of the moduli space of Higgs bundles,” Duke Math. J. 165 no. 12, (2016) 2227–2271, arXiv:1405.5765.

R. Mazzeo, J. Swoboda, H. Weiss, and F. Witt, “Asymptotic geometry of the Hitchin metric,” Comm. Math. Phys. 367 no. 1, (2019) 151–191, arXiv:1709.03433.

B. McCoy, C. Tracy, and T. T. Wu, “Painlevé functions of the third kind,” J. Mathematical Phys. 18 no. 5, (1977) 1058–1092.

R. Mazzeo and H. Weiss, “Teichmüller theory for conic surfaces,” in Geometry, analysis and probability, vol. 310 of Progr. Math., pp. 127–164. Birkhäuser/Springer, Cham, 2017.

H. Nakajima, “Hyper-Kähler structures on moduli spaces of parabolic Higgs bundles on Riemann surfaces,” in Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), vol. 179 of Lecture Notes in Pure and Appl. Math., pp. 199–208. Dekker, New York, 1996.
A. Neitzke, “Notes on a new construction of hyperkähler metrics,” in *Homological mirror symmetry and tropical geometry*, vol. 15 of *Lect. Notes Unione Mat. Ital.*, pp. 351–375. Springer, Cham, 2014. arXiv:1308.2198.

A. Neitzke, “Metric on the moduli space of Higgs bundles,” (2018). https://www.ma.utexas.edu/users/neitzke/expos/higgs-metric.pdf.

C. T. Simpson, “Harmonic bundles on noncompact curves,” *Journal of the American Mathematical Society* 3 no. 3, (1990) 713–770.

K. Yokogawa, “Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves,” *J. Math. Kyoto Univ.* 33 no. 2, (1993) 451–504.