New Puncture Initial Data for Black-Hole Binaries: High Spins and High Boosts

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We solve the Hamiltonian and momentum constraints of General Relativity for two black-holes with nearly extremal spins or ultra-relativistic boosts in the puncture formalism. We use a non-conformally-flat ansatz with an attenuated superposition of two conformally Kerr or Lorentz-boosted-Schwarzschild 3-metrics and their corresponding extrinsic curvatures. We compare evolutions of these data with the standard Bowen-York conformally-flat ansatz (technically limited to intrinsic spins $S/M_{\text{ADM}} = 0.928$ and boosts $P/M_{\text{ADM}} = 0.897$), finding an order of magnitude smaller burst of spurious radiation. As a case study, we evolve two equal-mass black-holes from rest with an initial separation of $d = 12M$ and spins $S_i/m_i^2 = 0.99$, compute the waveforms produced by the collision, the energy radiated, and the recoil of the final remnant black-hole. We find that the black-hole trajectories curve at closer separation, which leads to the radiation of angular momentum. We also study orbiting (nonspinning) black-hole binaries and binaries with the two black-holes boosted towards each other at relativistic speeds. These non-spinning data also show a substantial reduction in the non-physical initial burst of radiation which leads to cleaner waveforms. Finally, we study different choices of the initial lapse and lapse evolution equation in the moving punctures approach to improve the accuracy and efficiency of the simulations.

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I. INTRODUCTION

General Relativity is central to the modern understanding of much of astrophysics from, cosmological evolutions down to the end-state of large stars. Crucial to this is the correctness of the theory itself and the elucidation of its predictions. While much can be done using analytic techniques, one of the most interesting regimes, the merge-phase of compact-object binaries, requires the use of large-scale numerical relativity (NR) simulations. However, in order to evolve these systems one needs appropriate initial data that allow for the simulation of binaries with astrophysically realistic parameters. Perhaps most important of all is the inclusion of large spins.

Highly-spinning black-holes (BHs) are thought to be common. For example, supermassive BHs with high intrinsic spins are fundamental to the contemporary understanding of active galaxies and galactic evolution, in general.

In units with $c = 1$ and $G = 1$, a BH’s spin $S$ (i.e., intrinsic angular momentum) is bounded by its mass $M$, where the maximum spin is given by $S = M^2$. While it is actually hard to have an accurate measure of astrophysical BH spins, in a few cases the spins have been measured [1] and were found to be near the maximal value. Since galactic mergers are expected to lead to mergers of highly-spinning BHs, it is important to be able to simulate black-hole binaries (BHBs) with high spins in order to model the dynamics of these ubiquitous objects.

Spin can greatly affect the dynamics of a merging BHB. Important spin-based effects include the hangup mechanism [2], which delays or prompts the merger of the binary according to the sign of the spin-orbit coupling; the $\text{flip-flop}$ of spins [3] due to a spin-spin coupling effect that is capable of completely reversing the sign of individual spins; and finally, highly spinning binaries may recoil at thousands of km/s [4, 5] due to asymmetrical emission of gravitational radiation induced by the BH spins [6, 7]. These effects are maximized for highly spinning BHs.

No less important is the modeling of gravitational waveforms from compact-object mergers. Unlike low-spin binaries, highly spinning binaries can radiate more than 11% of their rest mass [8, 9], the majority of which emanates during the last moments of merger, down to the formation of a final single spinning BH. Efforts to interpret gravitational wave signals from such systems require accurate model gravitational waveforms [10–14].

The moving punctures approach [15, 16] has proven to be very effective in evolving BHBs with similar masses and relatively small initial separations, as well as small mass ratios [17] and large separations [18]. It is also effective for more general multiple BH systems [19], hybrid BH-neutron-star binaries [20], and gravitational collapse [21]. However, numerical simulations of highly spinning BHs have proven to be very challenging. The most commonly used initial data to evolve those binaries, which are based on the Bowen-York (BY) ansatz [22], use a conformally flat 3-metric. This method has a fundamental spin upper limit of $S/M^2 = 0.928$ [23, 24]. Even when relaxing the BY ansatz (while retaining conformal flatness), the spin is still bounded above by $S/M^2 = 0.932$ [25].

In order to exceed this limit and approach maximally spinning BHs with $S/M^2 = 1$, one has to allow for a more general 3-metric that captures the non-conformally flat aspects of the Kerr geometry. Dain showed [26] that it is possible to find solutions to the initial value problem rep-
representing a pair of Kerr-like BHs. This proposal was implemented for the case of thin-sandwich initial data (with excision of the BH interiors) and used to produce stable evolutions of orbiting BHBs with $S/M^2 \sim 0.97$ \cite{27}, and more recently $S/M^2 \sim 0.98$ \cite{28}. On the other hand, for the moving punctures approach, which does not use excision, Dain’s method was tested for the case of head-on collisions (from rest) and compared to the BY data for spinning BHs up to $S/M^2 = 0.96$ \cite{29}.

In this paper we revisit the question of finding solutions to the puncture initial value problem of two near-extremal-spin BHBs and the subsequent evolution using the moving punctures approach—the most widespread method to evolve BHBs and which is implemented in the open source EinsteinToolKit \cite{30,33}.

To solve for these new data, we combine a superposition of two conformally Kerr 3-metrics with the corresponding superposition of Kerr extrinsic curvatures. To regularize the problem, the superposition is such that very close to each BH, the metric and extrinsic curvature are exactly Kerr \cite{34}. We then simultaneously solve the Hamiltonian and momentum constraints for an overall conformal factor for the metric and nonsingular correction to the extrinsic curvature using a modification of the TwoPunctures \cite{35} spectral initial data solver. We refer to this extension as Highly SPinning Initial Data (HiSpID, pronounced “high speed”).

We evolve these data sets and find that the spurious initial radiation is significantly reduced compared to BY initial data for $S/M^2 \leq 0.9$. We also perform accurate evolutions of highly spinning BHs with $S/M^2 = 0.99$, which has not been possible before for moving punctures codes.

In addition, we consider Lorentz boosted Schwarzschild BHs in a quasicircular orbital configuration. It is important to minimize the spurious radiation content of the initial data in order to achieve a very accurate gravitational waveform computation. We present here the results of using superposed boosted BHs instead of the traditional BY solution for the puncture initial data approach. BY solutions are also limited in the maximum boost BHs can achieve (up to $P/M_{ADM} = 0.897$, see Ref. \cite{36}), while our new data do not suffer this restriction since it does not use the conformally flat 3-metric ansatz.

This paper is organized as follows. In Sec. II we present the formalism to solve for the initial data. We choose the standard thin-sandwich version since it provides the simplest set of equations and allows one to achieve both a reduction of the spurious initial radiation content and overcome the technical limits of the conformally flat initial data reaching highly spinning BHs and highly relativistic velocities. We describe the explicit conformal decomposition and attenuation functions used to regularize the superposition of conformal Kerr and Lorentz boosted BHs in the puncture approach.

In Sec. III we describe the numerical techniques to solve for the initial data as an extension of those used to solve the Hamiltonian constraint with the TwoPunctures code. We also provide a summary of the evolution techniques used in the regime of parameters previously unexplored with the moving punctures approach.

In Sec. IV we exhibit the convergence with spectral collocation points of the spinning initial data to levels of accuracy acceptable for evolution. We compare waveforms from the new HiSpID initial data to those of the standard spinning BY solution for $S_i/m_i^2 = 0.90$. We then evolve highly spinning BHs with $S_i/m_i^2 = 0.99$ from rest and discuss the results for the radiated energy and momenta as well as the horizon measures of mass and spin for the individual and final BHs.

In Sec. V we show the convergence with spectral collocation points of the initial data for nonspinning Lorentz boosted BHs. We compare waveforms for our new initial data with the standard boosted BY solution in quasicircular orbit to highlight the benefits of lower initial spurious radiation of our data.

In Sec. VI we study how the initial choice of the lapse and its subsequent gauge evolution \cite{37} affects the accuracy of the simulation at the typical marginal resolutions used to evolve highly spinning BHs, orbiting BHs, and in high energy head-on BH collisions.

In the Discussion Sec. VII we summarize the results and consider the next series of developments for initial data and evolution of BHBs.

In the Appendix we test the idea of finding alternative coordinate representations of the superposed 3-metrics, for the Kerr solution \cite{38} in order to improve the subsequent evolution with a given grid point structure. Finally, we give the explicit definitions used to compute the ADM mass and momenta of our initial data.

II. INITIAL DATA

In this section, we summarize the initial data formalism used to describe BHBs with spin magnitudes up to near maximal. First, we review the conformal decomposition of General Relativity’s field equations into a set of constraint and evolution equations. Then, we discuss a method for extracting the singularities at the punctures from the constraints, allowing us to find solutions using numerical pseudo-spectral methods. Finally, we construct initial data out of the superposition of two Kerr-like spinning or Lorentz boosted Schwarzschild BH backgrounds.

A. Constraints

A four-dimensional pseudo-Riemannian manifold is foliated into three-dimensional spatial hypersurfaces $\Sigma_t$, parametrized as surfaces of constant time function $t$. Vacuum solutions to General Relativity’s field equations
on the initial slice Σ₀ = Σₑₜ₀ must satisfy \[ \mathcal{H} \equiv \mathcal{R} + K^2 - K_{ij} K^{ij} = 0 \tag{1} \]

\[ \mathcal{M}^i \equiv \nabla_j (K^{ij} - \gamma^{ij} K) = 0 \tag{2} \]

known, respectively, as the Hamiltonian and momentum constraints. Latin indices represent spatial degrees of freedom. Here, \( \gamma_{ij} \) is the induced spatial metric tensor on \( \Sigma_0 \) with the associated covariant derivative \( \nabla_i \), and \( R = \gamma^{ij} R_{ij} \) is the trace of the spatial Ricci tensor \( R_{ij} \). The extrinsic curvature tensor of \( \Sigma_0 \) and its trace (the mean curvature) are denoted by \( K_{ij} \) and \( K = \gamma^{ij} K_{ij} \), respectively.

In the conformal thin-sandwich (CTS) formalism \[40-43\], the constraints \( (1) \) and \( (2) \) become a set of elliptic differential equations through a conformal transformation

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} . \]

We call \( \tilde{\gamma}_{ij} \) the conformally related metric tensor. All quantities with a tilde are associated with \( \tilde{\gamma}_{ij} \). The conformal factor \( \psi \) is a scalar function that is everywhere positive. The extrinsic curvature tensor is split into trace and trace-free parts

\[ K_{ij} = A_{ij} + \frac{1}{3} \gamma_{ij} K . \tag{3} \]

It is convenient to adopt the conformal rescaling

\[ A_{ij} = \psi^{-2} \tilde{A}_{ij} , \tag{4} \]

while leaving the mean curvature conformally invariant, \( K = \tilde{K} \). CTS also defines \( \tilde{u}_{ij} = \partial_t \tilde{\gamma}_{ij} \) and the lapse function \( N = \psi^6 \tilde{N} \). The evolution equation for \( \tilde{\gamma}_{ij} \) is rearranged into

\[ \tilde{A}^{ij} = \frac{1}{2N} \left[ (\tilde{\nabla} \beta)^{ij} - \tilde{u}^{ij} \right] , \tag{5} \]

where \( \beta^i \) are the components of the shift vector and

\[ (\tilde{\nabla} \beta)^{ij} \equiv 2\tilde{\nabla}^{(i} \beta^{j)} - \frac{2}{3} \gamma^{ij} \tilde{\nabla}^k \beta_k \]

defines the longitudinal vector operator. Together, these constitute the 3 + 1 invariant spacetime line element

\[ \text{d}s^2 = -N^2 \text{d}t^2 + \gamma_{ij} \left( \text{d}x^i + \beta^i \text{d}t \right) \left( \text{d}x^j + \beta^j \text{d}t \right) . \]

In the CTS formalism, the freely specifiable degrees of freedom are contained in \( \tilde{\gamma}_{ij}, \tilde{u}_{ij}, K, \) and \( \tilde{N} \). For quasi-stationary initial data, we let \( \tilde{u}_{ij} = 0 \) \[13\]-\[46\]. In the case with no boost, the Kerr metric admits the maximal slicing condition \( K = 0 \) \[26\]. In the boosted case, we will adopt a non-trivial \( K \) (see Section 11C). For simplicity, we set \( \tilde{N} = 1 \) everywhere. Inspired by \[4\], we write our ansatz for the extrinsic curvature as

\[ \tilde{A}_{ij} = \tilde{A}^{(1)}_{ij} + \tilde{A}^{(2)}_{ij} + (\tilde{L} b)_{ij} , \tag{6} \]

where the first two terms are the known singular, trace-free extrinsic curvature tensors for individual Kerr or Lorentz boosted Schwarzschild BHs. The sub- and superscripts “(1)” and “(2)” designate the individual BHs. This turns the momentum constraint into a differential equation for the correction to the shift vector \( b^i \). With these choices, the constraints \( (1) \) and \( (2) \) become

\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^8 \tilde{K}^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 , \tag{7} \]

\[ \tilde{\Delta}_L b^i + \tilde{\nabla}_j \left( \tilde{A}^{(1)ij} + \tilde{A}^{(2)ij} \right) - \frac{2}{3} \psi \tilde{\gamma}^{ij} \tilde{\nabla}^k K = 0 , \tag{8} \]

where the vector Laplacian is defined as

\[ \tilde{\Delta}_L b^i = \tilde{\nabla}_j (\tilde{L} b)^{ij} = \tilde{\nabla}^2 b^i + \frac{1}{3} \tilde{\nabla}^i (\tilde{\nabla}_j b^j) + \tilde{R}_{j} b^j . \]

B. Conformal Kerr

In spherical quasi-isotropic coordinates, the Kerr conformal spatial line element is \[26\]-\[47\]

\[ \text{d}s^2 = \tilde{\gamma}_{ij} \text{d}x^i \text{d}x^j = \text{d}r^2 + r^2 \text{d}\Omega^2 + a^2 \text{d}r \text{d}\varphi \text{d}\varphi^2 , \]

where \( m \) is the puncture mass, \( a \) is the angular momentum per unit mass, \( r \) is the quasi-isotropic radial coordinate and \[29\]

\[ R = r + m + \frac{m^2 - a^2}{4r} , \]

\[ \Sigma = R^2 + a^2 \cos^2 \theta , \]

\[ \sigma = \frac{2mR}{\Sigma} , \]

\[ h = \frac{1 + \sigma}{\Sigma r^2} . \]

The non-vanishing component of the shift vector is

\[ \beta^\varphi = -\frac{2amr}{(r^2 + a^2)^2 - a^2(r^2 - 2ar + a^2) \sin^2 \theta} . \]

The non-vanishing components of the conformal extrinsic curvature associated with this metric are given by \[26\]-\[48\]

\[ \tilde{A}_{r\varphi} = \frac{H_E \sin^2 \theta}{r} , \]

\[ \tilde{A}_{\theta\varphi} = \frac{H_F \sin \theta}{r} , \]

with the definitions

\[ e^{-2q} = \frac{\Sigma}{R^2 + a^2 \left[ 1 + \sigma \sin^2 \theta \right]} , \]

\[ H_E = e^{-q} \frac{am}{\Sigma^2} \left[ (R^2 - a^2) \Sigma + 2R^2(R^2 + a^2) \right] , \]

\[ H_F = -e^{-q} \frac{a^3 m R}{2\Sigma^2} \left( 4r^2 - m^2 + a^2 \right) \cos \theta \sin^2 \theta . \]
The quasi-isotropic Kerr conformal factor is
\[ \psi_{QI} = \left( \frac{\Sigma}{r^2} \right)^{1/4} . \]

Ultimately, only the asymptotic behavior is important, so sometimes just the lowest order terms of \( \psi_{QI} \) are used [29].

\[ \psi_{QI} \approx 1 + \frac{\sqrt{m^2 - a^2}}{2r} . \]

However, we find that HiSPID initial data implementing \( \psi_{QI} \) converges to the desired tolerance in approximately \( 1/3 \) as many iterations as compared to the approximation. Therefore, we utilize the exact conformal factor \( \psi_{QI} \).

All fields are transformed to a Cartesian basis, with coordinates related by
\[
x = r \sin(\theta) \cos(\varphi) \\
y = r \sin(\theta) \sin(\varphi) \\
z = r \cos(\theta) .
\]

At this point, the spin is parallel to the \( z \)-axis. The specified local angular momentum vector has Cartesian components \( S^i \), so that \( a = \sqrt{S^i S_i} / m \). All of the fields are then rotated such that they are oriented in the desired direction, spin aligned with \( S^i \).

C. Conformal Lorentz Boosted Schwarzschild

To describe a BH with arbitrary linear momentum \( P^i \), we begin with the Schwarzschild line element in isotropic Cartesian coordinates \( (t_0, x_0, y_0, z_0) \):
\[
ds^2 = -N_0^2 \, dt_0^2 + \psi_0^2 \left( dx_0^2 + dy_0^2 + dz_0^2 \right) ,
\]
where
\[
N_0 = \frac{1 - \frac{m}{2r}}{1 - \frac{m}{2r_0}}
\]
is the lapse,
\[
\psi_0 = 1 + \frac{m}{2r_0}
\]
is the puncture conformal factor, and \( r_0 = \sqrt{x_0^2 + y_0^2 + z_0^2} \). Next, we perform a Lorentz transformation in the \( y_0 \)-direction, with the associated change of variables
\[
t_0 = \gamma(t - vy) \\
x_0 = x \\
y_0 = \gamma(y - vt) \\
z_0 = z ,
\]
where \( (t, x, y, z) \) are the coordinates of the boosted reference frame, \( v \) is the magnitude of the local velocity vector
\[
v^j = \frac{P^i}{\sqrt{m^2 + P^i P_i}} ,
\]
and \( \gamma = (1 - v^2)^{-1/2} \). Afterwards, all of the fields are rotated such that they are oriented in the desired direction, momentum aligned with \( P^i \).

From the boosted spacetime metric, we extract the lapse function, shift vector, and spatial metric. The only non-vanishing component of the shift is
\[
\beta^y = -mv(m^2 + 6mr + 16r^2)(m^3 + 6m^2r + 8mr^2 + 16r^3) / B^2 ,
\]
with
\[
B = \sqrt{(m + 2r)^6 - 16(m - 2r)^2 r^4 v^2} .
\]

On the \( t_0 = 0 \) hypersurface, \( r_0 \to r = \sqrt{x^2 + y^2 \gamma^2 + z^2} \) and the conformal factor is
\[
\psi_B = 1 + \frac{m}{2r}.
\]

The conformal spatial line element on \( \Sigma_0 \) is
\[
d\tilde{\ell}^2 = dx^2 + \gamma^2 \left[ 1 - \frac{16(m - 2r)^2 r^4 v^2}{(m + 2r)^6} \right] dy^2 + dz^2 .
\]

The evolution equation for the spatial metric gives us an expression for the extrinsic curvature
\[
K_{ij} = \frac{1}{2N} \left( \nabla_i \beta_j + \nabla_j \beta_i - \partial_t \gamma_{ij} \right) .
\]

The mean curvature is
\[
K = \frac{32\gamma mv [(m + 2r)^2 - 32(m - 2r)^2 (m - r) r^4 v^2] r^2 y}{(m + 2r)^3 B^3} .
\]

The non-vanishing components of the trace-free, longitudinal, conformal extrinsic curvature tensor are
\[
\tilde{A}_{xx} = \tilde{A}_{zz} = \frac{\gamma mv (m - 4r)(m + 2r)^3 BCy}{3r^4 D} \\
\tilde{A}_{xy} = \frac{-\gamma mv (m - 4r)(m + 2r)^3 x}{2r^4 B} \\
\tilde{A}_{yy} = \frac{-2\gamma^3 mv (m - 4r) Cy}{3r^2 (m + 2r)^3 B} \\
\tilde{A}_{yz} = \frac{-\gamma mv (m - 4r)(m + 2r)^3 z}{2r^4 B} ,
\]
with
\[
C = (m + 2r)^6 - 8(m - 2r)^2 r^4 v^2 ,
\]
\[
D = (m + 2r)^{12} - 32(m - 2r)^2 r^4 (m + 2r)^6 v^2 + 256(m - 2r)^4 r^8 v^4 .
\]
D. Punctures

In the puncture approach, we re-write the conformal factor as singular parts plus a finite correction:

$$\psi = \psi_1 + \psi_2 - 1 + u,$$

where $\psi_1$ and $\psi_2$ are the conformal factors associated with the individual, isolated BHs with metric tensors $\tilde{g}_{ij}^{(1)}$ and $\tilde{g}_{ij}^{(2)}$, respectively. The factor of $-1$ ensures $\psi \to 1$ as $r \to \infty$. The property $u \to 0$ as $r \to \infty$ is taken to be a boundary condition. This turns (7) into an elliptic differential equation for $u$,

$$\tilde{\nabla}^2 u - \frac{1}{8} R - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} + \tilde{\nabla}^2 (\psi_1 + \psi_2) = 0,$$

which is regular. For the conformally Kerr case, $\psi_1 = \psi_1^{QI}$. For the conformally boosted Schwarzschild case, $\psi_1 = \psi_1^{B}$.

E. Attenuation and Superposition

In an effort to tame the singularities in the source term of the momentum constraint (5), we introduce attenuation factors to the conformal metric. We seek fields which have the following properties: at the location of puncture 1, the contribution from puncture 2 falls to zero sufficiently fast, and vice versa; and towards spatial infinity, the metric becomes flat. We achieve this with a simple superposition

$$\tilde{\gamma}_{ij} = \delta_{ij} + f_1 \left( \tilde{\gamma}_{ij}^{(1)} - \delta_{ij} \right) + f_2 \left( \tilde{\gamma}_{ij}^{(2)} - \delta_{ij} \right).$$

We attenuate the mean curvature in a similar fashion:

$$K = f_1 K_1 + f_2 K_2.$$  

(9)

We pick the attenuation function

$$f_1 = 1 - e^{-(r_2/\omega_1)p},$$

where $r_2$ is the coordinate distance from the origin to the location of puncture 2. We make the analogous definition for $f_2$. The parameters $\omega_1$ and $\omega_2$ are weights that control the steepness of the attenuation. We take the smallest possible power index $p = 4$.

Even with attenuation, round-off error makes the divergence in the source term in (8) difficult to calculate numerically near either of the punctures. We alleviate this by rewriting the source with the divergence removed explicitly:

$$\tilde{\nabla}_j \tilde{A}_{ij}^{(1)} = \left( \tilde{\Gamma}_{jk}^{(1)} - \tilde{\Gamma}_{jk} \right) \tilde{A}_{ij}^{(1)} + \left( \tilde{\Gamma}_{jk}^{(1)} - \tilde{\Gamma}_{jk} \right) \tilde{A}_{ij}^{(1)} - \frac{2}{3} \psi_1^{QI} \tilde{\gamma}_{ij} \partial_j K_1,$$

with the analogous definition for puncture 2. We define $\tilde{\Gamma}_{jk}^{(1)}$ as the Christoffel symbols associated with the isolated, unattenuated conformal metric $\tilde{\gamma}_{ij}^{(1)}$, and use the same prescription for puncture 2. This has the additional benefit that no derivatives of the trace-free extrinsic curvature tensor need be calculated.

Within a small coordinate distance $r \lesssim 0.2$ of one of the punctures, that puncture's contribution is removed analytically from the constraint equations, so that there remains is a finite contribution from the other (distant) puncture.

III. NUMERICAL TECHNIQUES

A. Initial data solver with Spectral Methods

The TwoPunctures thorn \cite{2005JCoPh.201...48S} generates conformally flat $\tilde{\gamma}_{ij} = \delta_{ij}$ initial data via a spectral expansion of the Hamiltonian constraint on a collocation point grid. The residual values of the constraint are minimized at the collocation points, yielding a series solution that is interpolated onto a Carpet grid \cite{1994PhDT........23C} to be evolved. The momentum constraint for conformally flat initial data is satisfied analytically by the BY solutions for linear and angular momentum \cite{1979ApJ...231...25B,1978ApJ...225...88B}.

Conformally Kerr initial data requires that all four constraint equations be solved numerically. This is achieved by extending TwoPunctures to solve for $u$ and $b^i$ at each collocation point. The solver handles the nonlinearities in the constraint equations by using a linearized iterative method. The linearized constraints are

$$dH = \tilde{\nabla}^2 du - \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} du - \frac{1}{8} \tilde{R} du - \frac{5}{12} \psi^4 K^2 du + \frac{1}{4} \psi^{-7} \tilde{A}_{ij} \tilde{\nabla}_i \tilde{\nabla}_j du,$$

$$dM^i = \tilde{\nabla}_i db^i - 4\psi^5 \tilde{\gamma}^{ij} \tilde{\nabla}_j K du,$$

where $du$ and $db^i$ represent small changes in $u$ and $b^i$, respectively.

B. Evolution

We use the extended TwoPunctures thorn to generate puncture initial data \cite{2011PhRvD..83d4048S} for the BHB simulations. These data are characterized by mass parameters $m_p$ (which are not the horizon masses), as well as the momentum and spin of each BH, and their initial coordinate separation. We evolve these BHB data sets using the LAZEv \cite{2001PhRvD..64h4038S} implementation of the moving punctures approach with the conformal function $W = \sqrt{\lambda} = \exp(-2\varphi)$ suggested by Ref. \cite{2001PhRvD..64h4038S}. For the runs presented here, we use centered, eighth-order finite differencing in space \cite{2011PhRvD..83d4048S} and a fourth-order Runge Kutta time integrator. (Note that we do not upwind the advection terms.) Our code uses the CACTUS/EINSTEIN/TOOLKIT \cite{2011PhRvD..83d4048S}.
We locate the apparent horizons using the AHFINDERDIRECT code [54] and measure the horizon spin using the isolated horizon (IH) algorithm detailed in [55].

For the computation of the radiated angular momentum components, we use formulas based on “flux-linkages” [56] and explicitly written in terms of $\psi_4$ in [57, 58]. We obtain accurate, convergent waveforms and horizon parameters by evolving this system in conjunction with a modified 1+log lapse and a modified Gamma-driver shift condition [13, 59, 60]. The lapse and shift are evolved with

\[
(\partial_t - \beta^i \partial_i)\alpha = -\alpha^2 f(\alpha) K, \quad (10a) \\
\partial_i \beta^a = 3/4\Gamma^a - \eta \beta^a. \quad (10b)
\]

In the original moving punctures approach we used $\alpha(t) = 2/\alpha$ and an initial lapse $\alpha(t = 0) = \psi_{BL}^2$ [15] or $\alpha(t = 0) = 2/(1 + \psi_{BL}^4)$ [2], where $\psi_{BL} = 1 - m_1/(2r_1) - m_2/(2r_2)$. In Secs. V and VI we also use $\alpha(t = 0) = 1/(2\psi_{BL}^2 - 1)$ which seems better suited for the highly spinning or highly boosted BH evolutions.

There, we also explore other gauge conditions for the lapse in the form of $f(\alpha) = 1/\alpha$ (gauge speed = 1) and $f(\alpha) = 8/(3\alpha(2 - \alpha))$ (shock avoiding) [37] which prove to be more convenient when dealing with highly boosted moving punctures.

IV. EVOLUTION OF HIGHLY SPINNING BLACK-HOLES

To assess the characteristics of this superposed Kerr-like BH initial data we compare with the corresponding BY-type. For given binary separations and spin parameters, the horizon masses and spins are not identical, as shown in Fig. 1 since the initial radiation content and distortions are not the same. However, we consider them close enough for comparisons of physical quantities such as the gravitational waveforms.

We study a few test cases of BHB initial configurations with equal-mass BHs starting from rest and spins aligned (UU) or counter aligned (UD) with each other and perpendicular to the line joining the BHs. We evolve both BHs with the HiSpID data and the standard BY choice (for spins within the BY limit). We also evolve BHs with near-maximal spin, $S_+/m_+^2 = 0.99$, a regime unreachable for BY initial data. Table I gives the initial data parameters of these BHB configurations.

Fig. 2 shows a comparison of waveforms $r\psi_{4}$ extracted at an observer location $r = 75M$. We clearly see that the initial radiation content (located around $t \sim 80M$) of the BY data for equal mass spinning BHs with $S_+/m_+^2 = 0.9$ has an amplitude comparable to that of the physical merger. On the other hand, our Kerr-like initial data has much lower initial radiation content (one order of magnitude smaller). Although not apparent in these plots, the much lower initial radiation content is not only more physical, but also leads to more accurate computations of waveforms. This initial pulse reflects from the refinement boundaries (since they are not perfectly transmissive) leading to high frequency errors and convergence issues when looking at much finer details of the waveform phase [61, 62].

One of our main motivations to study a new set of initial data is to be able to simulate highly spinning BHs, beyond the BY (or conformally flat) limit, $S_+/m_+^2 \approx 0.93$ [23, 25, 36]. In Fig. 3 we show the level of satisfaction of the constraints for our new initial data for spinning BHBs with equal masses and spin parameters $S_+/m_+^2 = 0.99$. We observe an exponential convergence of the $L_2$ norm of the constraint violations with spectral collocation points down to a level of $\approx 10^{-8}$, which is accurate enough to

| Configuration | $x/M$ | $m_p$ | $S/M^2$ | $a/m_H$ | $m_H$ | $M_{ADM}/M$ |
|---------------|-------|-------|---------|----------|-------|------------|
| BY90UU       | 6     | 0.191475 | 0.225 | 0.8977 | 0.500702 | 0.982362 |
| HS90UU       | 6     | 0.5 | 0.225 | 0.8958 | 0.501287 | 0.982353 |
| BY90UD       | 6     | 0.191475 | 0.225 | 0.8977 | 0.500702 | 0.982396 |
| HS90UD       | 6     | 0.5 | 0.225 | 0.8955 | 0.501208 | 0.982388 |
| HS99UU       | 6     | 0.5 | 0.2475 | 0.9896 | 0.500162 | 0.980124 |
| HS99UD       | 6     | 0.5 | 0.2475 | 0.9887 | 0.499981 | 0.980163 |
start BHB evolutions. If one requires greater satisfaction of the constraints, one can fine-tune the attenuation functions to that end.

The evolution of BHs with $S_i/m_i^2 = 0.99$ requires high resolution, particularly during the first 10$M$ of evolution, but otherwise proceeds with the standard moving punctures set up [15]. Even if the initial data had no initial orbital angular momentum, the case of a BHB from rest in the UU configuration radiates angular momentum due to mutual frame dragging effects in the opposite direction, as observed in [63]. On the other hand, the UD configuration leads to a recoil velocity, that can be modeled as [7]

$$V_{\text{recoil}} = \sum_{j=1,3,5...} k_j \Delta^j$$

where $\Delta = (S_2 - S_1)/(m_1 + m_2)^2$ and the $k_j$ are fitting constants (this form applies only to equal mass binaries with vanishing total spin). A summary of the properties of the final merger remnant BH are listed in Table II.

Fig. 2 shows the waveforms of the UU and UD cases for highly spinning BHs. It confirms that the initial radiation content has a much smaller amplitude than the merger waveform—even in the head-on case—significantly reducing contamination of the physical signals by unresolved high-frequency reflections.

### V. ORBITING BLACK-HOLES

One of the most astrophysically important applications of NR is the evolution of BHBs in quasicircular orbits. Fig. 5 shows the convergence rate of the initial data solution as nearly exponential with the number of collocation points for a typical set of orbital parameters. Hamiltonian and momentum constraint residuals reach levels below $10^{-7}$ above 80 collocation points. This is accurate enough for most applications.

In order to evaluate these initial data we perform a numerical evolution of a binary in the merger regime and compare our Lorentz boost data with the traditional BY solution. We chose initial parameters that lower the eccentricity for each set of data, as given by Table III. The BHs orbit nearly five times before merging (see Fig. 6), and at $t \sim 700M$, merge to a spinning remnant BH with the properties given in Table IV.

Fig. 7 shows the waveforms of a binary with the Lorentz boost and BY initial data for the modes $(\ell, m) = (2, 2)$ and $(\ell, m) = (4, 4)$. While most of the waveforms superpose, the most notable difference lies in the initial

![Graph](https://via.placeholder.com/150)

**FIG. 2:** $\ell = 2, m = 2$ mode of $\Psi_4$ at $r = 75M$ (above). $(2,0)$ mode of $\Psi_4$ at $r = 75M$ (below) for spinning binaries with $S_i/m_i^2 = 0.9$. BY data in blue (showing much larger initial burst), HiSpID in red.

**FIG. 3:** Convergence of the residuals of the Hamiltonian and momentum constraints versus number of collocation points $N$ for BHBs with $\chi = 0.99$ in the UU configuration.

**TABLE II:** The final mass, final remnant spin, and recoil velocity for each configuration.

| Configuration | $M_{\text{rem}}/M$ | $\alpha_{\text{rem}}$ | $V$  |
|---------------|--------------------|-----------------------|------|
| BY90UU        | 0.98053            | 0.46554               | 0    |
| HS90UU        | 0.98162            | 0.46483               | 0    |
| BY90UD        | 0.98073            | 0                     | 35.90|
| HS90UD        | 0.98181            | 0                     | 36.01|
| HS99UU        | 0.97971            | 0.52501               | 0    |
| HS99UD        | 0.97900            | 0                     | 38.40|
Note the small amplitude of the initial data radiation content relative to the Lorentz boosted data for the initial burst of radiation (located at around $t = 75M$), compared to the merger signal after $t = 130M$.

The original moving punctures breakthrough formulation [15] remains widely used, and produces reliable BHB evolutions, as well as multi-BH systems [19]. It also functions in the presence of matter, as in neutron star mergers [54–60]. The choice of the gauges, i.e., Eq. (10a) plays a crucial role in stabilizing the numerical evolutions. There is still a range of possibilities for choosing the specific form of the gauges. While preserving the numerical stability properties one would like to improve the accuracy of the simulation for a given resolution and grid structure.

Some questions about the accuracy and convergence of the moving punctures method have been raised in [61]. Recently, Etienne et al. [62] studied how modifications to the lapse evolution can ameliorate the numerical errors that lead to poor waveform convergence.

Here we study other choices for the initial lapse and its time evolution to control and improve the accuracy of the numerical results for highly spinning BHs and relativistic collision of BHs generating large amplitude gauge waves.

The Bona-Massò gauge condition for the lapse evolution to control and improve the accuracy of the simulation for a given resolution and grid structure.

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VI. GAUGE CONDITIONS

The Bona-Massò gauge condition for the lapse evolution to control and improve the accuracy of the numerical results for highly spinning BHs and relativistic collision of BHs generating large amplitude gauge waves.

The Bona-Massò gauge condition for the lapse evolution to control and improve the accuracy of the numerical results for highly spinning BHs and relativistic collision of BHs generating large amplitude gauge waves.
Notably, the simple choice of the initial lapse $\alpha_0 = 1/(2\psi_{BL} - 1)$ has advantages over the other choices studied for the entire evolution by providing increased accuracy and computational efficiency. Here, we display the results of the evolution from rest of BHBs with intrinsic spin $S_i/m_i^2 = 0.99$. Fig. 8 shows that we can achieve comparably accurate results with many fewer grid points. The curves follow closely to each other, but with the new lapse we use 80 points per dimension compared to the 125 needed with the original initial lapse. This provides a speed up factor of $(125/80)^4 \sim 6$.

The improvement of the new initial lapse also translates into a more accurate description of the final remnant BH, as shown in Fig. 9. Note that even at lower resolutions we observe a similar gain.

We interpret these results as indicating that a better choice of the initial lapse leads to a better coordinate evolution, reducing the initial gauge waves, thus allowing a better distribution of grid points, resulting in a more efficient numerical computation. See for instance the horizon coordinate radius evolution in Figs. 8 and 9.

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Fig. 10 displays the effects of the initial lapse on the waveform. We see the notable reduction of the unphysical oscillations pre-merger while reproducing accurately the physical merger waveform for the dominant modes $(\ell, m) = (2, 0)$ and $(\ell, m) = (2, 2)$. Note that this reduction of the errors due to improved gauge choices is in addition to and independent from the reduction of the initial burst of radiation (with respect to BY data) that has a physical content, despite being an undesirable effect.

2. Quasicircular orbits

Here, we study the effects of lapse evolution choices on the case study of equal mass, nonspinning, orbiting BHBs. The Lorentz boost initial data has a lower radiation content than the boost BY data and allows us to see more clearly the effects of the initial choice and evolution of the lapse.

Fig. 11 displays the effects of gauge versus resolution on physical quantities like the horizon mass (left column) and horizon radius (right column). We expect the horizon mass to be essentially conserved during the orbital period up to merger. We can see that this physical observable varies very little with different gauge choices. On the other hand, we observe that the coordinate radius varies with the evolution of the lapse choice, but not as much with the initial lapse. After a sudden growth, typical of a gauge settling, the horizon radius reaches a constant value. The original moving punctures choice, $f(\alpha) = 2/\alpha$, keeps the value of the horizon coordinate closer to its original value which could be beneficial for setting up the initial mesh refinement levels.

Fig. 12 displays the waveform as seen by an observer
at \( r = 90M \) from the sources for different evolution functions \( f(\alpha) \) for the lapse. The initial lapse here is \( \alpha_0 = 2/(1 + \psi^4_{\text{full}}) \). While physical quantities like the waveform and its amplitude are essentially independent of the gauge choices, numerical errors, which produce the high frequency noise, are not. The bottom panel of the figure shows a close-up view of the amplitude during the post initial pulse period. We observe that overall the choice \( f(\alpha) = 2/\alpha \) produces a lower amplitude of this high frequency noise.

Figs. 13 and 14 display a similar behavior for the waveforms, but their close-up view of the noise shows a smaller amplitude, which suggests that the choice of the initial lapses \( \alpha_0 = 1/\psi^2_{\text{BL}} \) or \( \alpha_0 = 1/(2\psi_{\text{BL}} - 1) \) lead to smaller amplitude gauge waves.

Since the moving punctures approach is a free evolution of the general relativistic field equations, a very important method to monitor its accuracy is to verify the satisfaction of the Hamiltonian and momentum constraints. We also monitor the BSSN constraints, which
are on the order of $10^{-7}$ throughout the duration of the evolution.

Figs. 15, 17 display the $L_2$ norm of the nonvanishing values of the Hamiltonian and momentum components of the constraints. We observe that the propagation of errors travel at different speeds, associated with the gauge velocities $\sqrt{2}$, $\sqrt[4]{3}$, and $1$ for $f(\alpha) = 2/\alpha$, $8/(3\alpha(3-\alpha))$, and $1/\alpha$, respectively. We also observe slightly larger violations for the choice $f(\alpha) = 1/\alpha$, and $\alpha_0 = 2/(1+\psi_{BL}^2)$.

We thus conclude that while all three evolution choices for the lapse are viable to evolve typical BHB simulations, the original moving punctures choice $f(\alpha) = 2/\alpha$ and initial lapse $\alpha_0 = 1/\psi_{BL}^2$, or $\alpha_0 = 1/(2\psi_{BL} - 1)$ are somewhat preferred. This study suggests there might be even more optimal choices of $\alpha_0$ and $f(\alpha)$, as well as shift evolution gauge conditions. We also note that in the independent study of Ref. 62, a higher gauge velocity is preferred for the early stage of evolution.

3. Relativistic head-on collisions

Since we observe a notable benefit on using the initial lapse $\alpha_0 = 1/(2\psi_{BL} - 1)$ in evolutions of highly spinning BHs, we would like to explore their effect on another extreme configuration: high energy relativistic collisions of BHs. The collisions were studied in Refs. 69, 72 with regard to potential applications to collider-generated mini BHs. Here we will consider them as test case for comparing different gauge conditions.

In Fig. 18 we use physical observables such as the individual horizon masses and the gravitational radiation waveforms as indicators of the numerical accuracy.
of the evolutions. We observe that the initial lapse \( \alpha_0 = 1/(2\psi_{\text{BL}} - 1) \) gives the best behavior for the horizons mass (i.e., most constant) and a waveform with reduced noise.

The preferred behavior of the initial lapse \( \alpha_0 = 1/(2\psi_{\text{BL}} - 1) \) is also confirmed with regards to the constraint preservation as shown in Fig. 19 closely followed by the choice \( \alpha_0 = 1/(\psi_{\text{full}})^2 \).

In these evolutions we have taken the standard choice for the moving punctures evolution of the lapse, \( f(\alpha) = 2/\alpha \) in Eq. (11). It is also worthwhile to explore alternative evolutions of \( f(\alpha) = 1/\alpha \), with gauge speed equal to 1, and \( f(\alpha) = 8/(3\alpha(2-\alpha)) \), with approximate shock avoiding properties \( ^{3M} \). The results of such evolutions are displayed in Figs. 20 and 21 where we have taken an initial separation of the binary \( d = 66M \), \( P_x/m_H = \pm 2 \), and used the initial lapse \( \alpha_0 = 1/(2\psi_{\text{BL}} - 1) \).

We first observe that the results of Figs. 20 and 21 indicate that with our numerical setup the evolution \( f(\alpha) = 1/\alpha \) fails to complete (i.e., crashes) generating large errors, while the form \( f(\alpha) = 8/(3\alpha(2-\alpha)) \) is stable, but less accurate than the standard \( f(\alpha) = 2/\alpha \).

However, we found that for larger initial \( P_x/m_H \) values the lapse evolution equation characterized by \( f(\alpha) = 2/\alpha \) fails to complete the evolution while the (approximate) shock avoiding form \( f(\alpha) = 8/(3\alpha(2-\alpha)) \) always succeeds. In these cases, a large amplitude gauge wave is generated by the high energy collision initial data which leads to an inability for the numerics to resolve the waves.

FIG. 13: Waveforms extracted at an observer location \( r = 90M \). Real part of \( \psi_4 \) (upper left) and the amplitude of those waveforms (upper right). The bottom panel shows a zoom-in of the amplitude oscillations for different evolution functions \( f(\alpha) \) for the lapse. The initial lapse implemented here is \( \alpha_0 = 1/\psi_{\text{BL}} \).

FIG. 14: Waveforms extracted at an observer location \( R = 90M \). Real part of \( \psi_4 \) (upper left) and the amplitude of those waveforms (upper right). On the lower panel a zoom-in of the amplitude oscillations for different evolution functions \( f(\alpha) \) for the lapse. Initial lapse here is \( \alpha_0 = 1/(2\psi_{\text{BL}} - 1) \).

FIG. 15: \( L_2 \)-norms of the violations of the Hamiltonian and three components of the momentum constraints versus time for different evolution functions \( f(\alpha) \) for the lapse. Initial lapse here is \( \alpha_0 = 2/(1 + \psi_{\text{full}}) \).

FIG. 16: \( L_2 \)-norms of the violations of the Hamiltonian and three components of the momentum constraints versus time for different evolution functions \( f(\alpha) \) for the lapse. Initial lapse here is \( \alpha_0 = 1/\psi_{\text{BL}}^2 \).
and stabilize the system. While one can try to fine tune parameters of the evolution or change the evolution equations (for instance to a Z4-type \cite{23}) the form \( f(\alpha) = 8/(3\alpha(2-\alpha)) \) represents a valid alternative to the standard \( f(\alpha) = 2/\alpha \) evolution (which can still be used by starting collisions further apart or slightly grazing).

\section{Conclusions and Discussion}

In this paper we have been able to implement puncture initial data for highly spinning and highly boosted BHBs by attenuated superposition of conformal Kerr and Lorentz boosted Schwarzschild metrics. We verified the validity of the data by showing convergence of the Hamiltonian and momentum constraint residuals with the number of collocation points of the spectral solver. We then showed, by evolving this data, that the radiation content of these initial data was much lower than the standard conformally flat choice. This produced a more accurate and realistic computation of gravitational radiation waveforms. These cleaner initial data allowed us to explore different choices of the moving punctures gauge (initial lapse and its evolution).

The natural extension of this work is to superpose Lorentz boosted Kerr metrics into the initial data formalism we developed, and to implement them into HiSpID, our extended spectral solver. This will allow for the simulation of extremely boosted and orbiting highly spinning BHBs in a regime of astrophysical interest, and to
explore the corners of the BHB parameter space. Revisiting some of the most interesting spin dynamic effects in BHBs, such as the hangup [2], flip-flops [3], and large recoils [74], is of particular interest. Another venue that the current studies opened up is the search for an improvement of the moving punctures gauges (for both the lapse and the shift) in order to better deal with the near extreme cases and to improve the accuracy and efficiency of the numerical simulations.

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Appendix A: UIUC Coordinates

While it is possible to evolve BHB initial data in initially quasi-isotropic coordinates, Liu et al. [38] found that it was more efficient to perform a fisheye [39] transformation to the background Kerr metric to open up the throats of the holes in the numerical coordinates. The radial quasi-isotropic coordinate suffers from the fact that the BH horizon at \( r = \sqrt{m^2 - a^2/2} \) shrinks to zero in the \( a \to m \) limit. This is remedied by the use of the coordinate \( \tilde{r} \), defined implicitly by [76]

\[
\tilde{r} \left( 1 + \frac{R_+}{4\tilde{r}} \right)^2 = R ,
\]

where \( R_\pm = m \pm \sqrt{m^2 - a^2} \) are the Boyer-Lindquist radii of the inner (–) and outer (+) horizons of the BH.

The conformal spatial line element in these UIUC coordinates is

\[
d\tilde{\ell}^2 = \frac{\left( \tilde{r} + \frac{R_+}{4\tilde{r}} \right)^2}{\tilde{r}(R - R_-)} d\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \frac{A}{\Sigma^2} \tilde{r}^2 \sin^2(\theta) d\varphi^2 ,
\]

with

\[
A = \left( R^2 + a^2 \right)^2 - \left( R^2 - 2mR + a^2 \right) a^2 \sin^2(\theta) .
\]

This choice of conformal metric utilizes \( \psi_{\text{UIUC}} = \psi_{\text{QI}} \). The non-vanishing components of the trace-free, conformal extrinsic curvature tensor are

\[
\tilde{A}_{\tilde{r}\tilde{\varphi}} = \frac{ma \sin^2(\theta)}{\tilde{r} \Sigma \sqrt{\tilde{r}A(R - R_-)}} \left[ 3R^4 + 2a^2 R^2 - a^4 - a^2(R^2 - a^2) \sin^2(\theta) \right] \left( 1 + \frac{R_+}{4\tilde{r}} \right) ,
\]

\[
\tilde{A}_{\tilde{\theta}\tilde{\varphi}} = -\frac{2a^3 m R \cos(\theta) \sin^3(\theta)}{\tilde{r} \Sigma \sqrt{\tilde{r}A}} \left( \tilde{r} - \frac{R_+}{4} \right) \sqrt{R - R_-} .
\]

We have implemented these coordinates in our initial data numerical solver and reproduced the results of Ref. [70] for a single BH. While we are able to produce a convergent set of initial data (see Fig. [22]), there appears to be no obvious gain over the quasi-isotropic implementation for the BHB cases studied in this paper.

Appendix B: Calculating the ADM mass

For the sake of completeness we give here the explicit form of the Arnowitt-Deser-Misner (ADM) mass, linear and angular momenta used in the identification of the initial data parameters.

In an asymptotically flat spacetime, in asymptotically Cartesian coordinates, the ADM mass is given by [77]

\[
E[h_{ab}] = \frac{1}{16\pi} \lim_{r \to \infty} \sum_{a,b=1}^{3} \oint (h_{ab,a} - h_{aa,b}) \frac{2}{r} r^2 d\Omega \sin \theta d\varphi d\phi ,
\]

where \( h_{ab} = \delta_{ab} + c_{ab}(\theta, \phi)/r + O(1/r^2) \) is the 3 metric, \( x^a = (x, y, z) \) are Cartesian coordinates (at spatial infinity) and \( (r, \theta, \phi) \) are the usual spherical coordinates. The integral is over an \( r = \text{const} \) sphere, and \( d\Omega = \sin \theta d\theta d\phi \). Only the \( O(1/r) \) terms in the metric contribute to the ADM mass.
For the case of superimposed boosted Schwarzschild BHs, we have

$$h_{ab} = \left(1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2} + u\right)^4 \left(\tilde{S}_{ab}^{(1)} + \tilde{S}_{ab}^{(2)} - \delta_{ab}\right),$$

where

$$\left(1 + \frac{m_1}{2r_1}\right)^4 \tilde{S}_{ab}^{(1)} = S_{ab}^{(1)},$$

$$\left(1 + \frac{m_2}{2r_2}\right)^4 \tilde{S}_{ab}^{(2)} = S_{ab}^{(2)},$$

$m_1$ and $m_2$ are the mass parameters of the two Schwarzschild BHs, $r_1$ and $r_2$ are $O(r)$ with angular dependence, and $S_{ab}^{(1)}$ and $S_{ab}^{(2)}$ are Schwarzschild metrics in boosted coordinates. Since $\tilde{S}_{ab} = \delta_{ab} + O(1/r)$, the ADM mass takes the form

$$E[h_{ab}] = E[\psi^4 \tilde{S}_{ab}^{(1)}] + E[\psi^4 \tilde{S}_{ab}^{(2)}] - E[\psi^4 \delta_{ab}],$$

$$= E\left[\left(1 + \frac{2m_1}{r_1}\right) S_{ab}^{(1)} + \left(1 + \frac{2m_2}{r_2}\right) S_{ab}^{(2)}\right] + E\left[\left(\frac{2m_2}{r_2} + \frac{4\dot{u}}{r}\right) \delta_{ab}\right]$$

$$- E\left[\left(\frac{2m_1}{r_1} + \frac{2m_2}{r_2} + \frac{4\dot{u}}{r}\right) \delta_{ab}\right],$$

$$(B1)$$

where $u = \dot{u}(\theta, \phi)/r + O(1/r^2)$. The first two terms on Eq. (B1) are the ADM masses of boosted Schwarzschild BHs and are thus equal to $\gamma_1 m_1$ and $\gamma_2 m_2$, respectively. Finally, $E[4\dot{u}/r \delta_{ab}] = \frac{1}{8\pi} \oint \delta_{ab} d\Omega$.

The ADM momentum is given by

$$P_a[K_{ab}] = \frac{1}{8\pi} \lim_{r \to \infty} \sum_{b=1}^{3} \oint (K_{ab} - \delta_{ab} K) \frac{x^b}{r} r^2 d\Omega,$$

with $K = \sum_{a=1}^{3} K^a_a$. Using Eqs. (3), (4), (6), and (9) in the asymptotic region, the integrand becomes

$$K_{ab} - \delta_{ab} K = K_{ab}^{(1)} - \delta_{ab} K^{(1)} + K_{ab}^{(2)} - \delta_{ab} K^{(2)} + (\bar{L}b)_{ab}$$

where $K_{ab}^{(1)}$ and $K_{ab}^{(2)}$ are the extrinsic curvature tensors for isolated BHs. Given the momentum parameters $P_a^{(1)}$, the corresponding ADM momentum for an isolated BH is $P_a[K_{ab}] = P_a^{(1)}$. Therefore, using the linearity of the integral, the ADM momentum for a BHB is

$$P_a[K_{ab}] = P_a^{(1)} + P_a^{(2)} + P_a[(\bar{L}b)_{ab}].$$

The ADM angular momentum is given by the first moment of the ADM momentum:

$$J^a[K_{ab}] = \frac{\epsilon_{abc}}{8\pi} \lim_{r \to \infty} \sum_{b,c,d=1}^{3} \oint x_b (K_{cd} - \delta_{cd} K) \frac{x^d}{r} r^2 d\Omega.$$

Using the same linearity and asymptotic properties, we can write this as

$$J^a[K_{ab}] = J_a^{(1)} + J_a^{(2)} + J_a[(\bar{L}b)_{ab}].$$

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