Non-linear critical current thermal response of an asymmetric Josephson tunnel junction

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(Dated: July 10, 2018)

We theoretically investigate the critical current of a thermally-biased SIS Josephson junction formed by electrodes made by different BCS superconductors. The response of the device is analyzed as a function of the asymmetry parameter, \( r = T_{c1}/T_{c2} \). We highlight the appearance of discontinuities in the critical current of an asymmetric junction, namely, when \( r \neq 1 \). In fact, in such a case at temperatures at which the BCS superconducting gaps coincide, the critical current suddenly increases or decreases. In particular, we thoroughly discuss the counterintuitively behaviour of the critical current, which increases by enhancing the temperature of one lead, instead of monotonically reducing. In this case, we found that the largest jump of the critical current is obtained for moderate asymmetries, \( r \approx 3 \). In view of these results, the discussed behavior can be speculatively proposed as a temperature-based threshold single-photon detector, which operates non-linearly in the non-dissipative channel.

I. INTRODUCTION

More than 50 years after its discovery, the Josephson’s effect \(^1\text{[2]}\) is still a province able to provide intriguing, even unexpected, physical phenomena, from which novel devices are continuously conceived. This is the case of the plethora of works descending only recently \(^3\text{[6]}\), from the earlier intuition that a temperature bias imposed across a Josephson junction (JJ) produces a phase-dependent heat flow through the device \(^7\text{[7]}\). We are dealing with the phase-coherent caloritronics \(^3\text{[6]}\), namely, an emerging research field from which fascinating Josephson-based devices, such as heat diodes \(^10\text{[10]}\), thermal transistors \(^11\text{[11]}\), solid-state memories \(^12\text{[12]}\), microwave refrigerators \(^13\text{[13]}\), thermal routers \(^16\text{[16]}\), heat amplifier \(^18\text{[18]}\), and heat oscillator \(^19\text{[19]}\), were recently designed and actualized. Even the critical current \(I_c\) of a Josephson tunnel junction, namely, the maximum dissipationless current that can flow through the device, deviates from the well-known Ambegaokar-Baratoff relation \(^20\text{[20]}\) in the presence of a thermal bias imposed across the junction, namely, as the superconducting electrodes reside at different temperatures, as portrayed in Fig. 1.

In this work we explore peculiar features of the critical current of a thermally-biased asymmetric tunnel JJ. We theoretically demonstrate that the critical current \(I_c\) of a junction formed by different superconductors behaves discontinuously and it is asymmetric in the temperature switch. Specifically, we show that the critical current suddenly jumps at specific temperatures at which the BCS superconducting gaps \(^21\text{[21]}\) \(\Delta_1, \Delta_2\) are equal. The discontinuities are due to the matching in the singularities of the anomalous Green functions in the two superconductors \(^21\text{[21]}\). This feature is the non-dissipative counterpart of the discontinuities discussed in the quasiparticle current flowing through a voltage-biased S1IS2 junction \(^21\text{[21]}\) and the heat current flowing through a temperature biased junction \(^11\text{[11]}\), both stemming from the alignment of the singularities of the BCS DOSs in the superconductors \(^21\text{[21]}\).

We observe that discontinuities in the critical current were already noted, but not extensively discussed so far \(^11\text{[11]}\). Additionally, for appropriate parameters values we will show that the critical current counterintuitively behaves, since it increases by enhancing the temperature, instead of decreasing. Furthermore, we study the asymmetry of the critical current with respect to the switching of the temperatures, through the definition of a suitable temperature-switching asymmetry parameter. We also discuss the linear regime in response to a thermal gradient, by studying the first-order coefficients of

FIG. 1. Schematic illustration of a temperature-biased SIS Josephson tunnel junction formed by the superconducting leads \(S_1, S_2\), with critical temperatures \(T_{c1}, T_{c2}\), and residing at temperatures \(T_1, T_2\). The junction is enclosed in a superconducting ring pierced by a magnetic flux \(\Phi\) which allows phase biasing of the weak link. The ring is supposed to be made by a third superconductor \(S_3\) with energy gap \(\Delta_3 \gg \Delta_1, \Delta_2\) so to suppress the heat losses.
the critical current expansion as a function of the average temperature, at a few values of the Dynes parameter.

Finally, according to the step-like behavior of the critical current, we suggest the application of this device as a non-dissipative threshold single-photon detector, based on the sudden increase of $I_c$ due to a photon-induced heating of one of the electrodes of the junction.

The paper is organized as follows. In Sec. II we study the behavior of the critical current by varying non-linearly the temperatures of the device and the ratio between the critical temperatures of the two superconductors. In Sec. III we address the linear approximation in the temperature gradient. We discuss in Sec. IV some possible applications of the discussed effects, and also their implication in view of recent researches. In Sec. V the conclusions are drawn.

II. THE CRITICAL CURRENT

Here, we explore how the critical current of a temperature-biased SIS JJ depends on the superconductors composing the device. Indeed, we consider a junction formed by different BCS superconductors, so that we can define an asymmetry parameter

$$ r = \frac{T_{c_1}}{T_{c_2}} = \frac{\Delta_{10}}{\Delta_{20}}, \quad (1) $$

where $T_{c_j}$ is the critical temperature and $\Delta_{j0} = 1.764k_BT_{c_j}$ is the zero-temperature superconducting BCS gap of the $j$-th superconductor (with $k_B$ being the Boltzmann constant).

A Josephson tunnel junction formed by two superconducting leads $S_1$ and $S_2$ with energy gaps $\Delta_1$ and $\Delta_2$ residing at temperatures $T_1$ and $T_2$, see Fig. 1, can support a non-dissipative Josephson current \[ I_c(T_1, T_2) \]

$$ I_c(T_1, T_2) = I_c(T_1, T_2) \sin \varphi, \quad (2) $$

with $\varphi$ being the macroscopic quantum phase difference between the superconductors across the junction, and $I_c(T_1, T_2)$ being the critical current, which reads

$$ I_c(T_1, T_2) = \frac{1}{2eR} \int_{-\infty}^{\infty} \left\{ f(\varepsilon, T_1) \text{Re}[\mathcal{F}_1(\varepsilon, T_1)] \text{Im}[\mathcal{F}_2(\varepsilon, T_2)] + f(\varepsilon, T_2) \text{Re}[\mathcal{F}_2(\varepsilon, T_2)] \text{Im}[\mathcal{F}_1(\varepsilon, T_1)] \right\} d\varepsilon. \quad (3) $$

Here, $R$ is the normal-state resistance of the junction, $e$ is the electron charge, $f(\varepsilon, T_j) = \tanh(\varepsilon / 2k_BT_j)$, and

$$ \mathcal{F}_j(\varepsilon, T_j) = \frac{\Delta_j(T_j)}{\sqrt{\varepsilon + \Gamma_j^2}} \quad (4) $$

is the anomalous Green’s function of the $j$-th superconductor \[ 1 \], with $\Gamma_j = \gamma_j \Delta_{j0}$ being the Dynes parameter \[ 31 \]. The so-called Dynes model \[ 31, 32 \] is based on an expression of the BCS DOS including a lifetime broadening. It allows to take into account the smearing of the $IV$ characteristics of a JJ, that is the persistence of a small subgap current at low voltages. In fact, a non-vanishing $\gamma_j$ introduces effectively states within the gap region, $|\varepsilon| < \Delta_j$, as opposed to the ideal BCS DOS obtained at $\gamma_j = 0$, which instead results in a vanishing DOS within the gap \[ 25, 24 \]. Unless otherwise stated, hereafter we assume $\gamma_1 = \gamma_2 = \gamma = 10^{-4}$, namely, a value often used to describe realistic superconducting tunnel junctions \[ 10, 14, 33 \].

Fig. 1 shows a possible experimental realization of the discussed setup where we clearly indicate how to master the phase difference across the device. The thermally-biased junction is enclosed, through clean contacts, within a superconducting ring pierced by a control magnetic flux $\Phi$. In this way, we achieve the phase-biasing via this external flux, which allows us to thoroughly play with the macroscopic phase difference across the JJ. In fact, neglecting the ring inductance, the phase-flux relation is given by $\varphi = 2\pi \Phi / \Phi_0$ \[ 26 \] ($\Phi_0 = h/2e \simeq 2 \times 10^{-15}$ Wb is the magnetic flux quantum, with $h$ being the Planck constant). Accordingly, the phase drop across the junction can vary within the whole phase space, i.e., $-\pi \leq \varphi \leq \pi$. The ring is supposed to be made by a third superconductor $S_3$ with energy gap $\Delta_3 \gg \Delta_1, \Delta_2$ so to suppress the heat losses thanks to Andreev reflection heat mirroring effect \[ 27 \].

A. Non-linear temperature behavior of $I_c$

We first study the critical current of the device by choosing the asymmetry parameter $r$, and changing the temperature of $S_2$ at fixed $T_1$ for non-linear regimes of temperatures.

The behavior of the critical current $I_c$, in units of $\sqrt{\Delta_{10}\Delta_{20}}/(2eR)$, as a function of the normalized temperature $T_2/T_{c_2}$ at a few values of the normalized temperature $T_1/T_{c_1}$, for $r = \{0.5, 1, 2\}$ is shown in Fig. 2. We note that the critical current generally reduces by increasing $T_1$, an effect that may be naively interpreted as the usual detrimental effect of the temperature on the critical current. Anyway, we will see that for $r \neq 1$, the temperature may affect the critical current in an unexpected way. In fact, we observe that the critical current as a function of $T_2$ may present discontinuous behaviors. Specifically, for $r \neq 1$, i.e., the asymmetric junction case, curves may show jump discontinuities, see Figs. 2(a) and (c), whereas for a symmetric junction, namely, $r = 1$, curves present only a change of slope, see Fig. 2(b). These discontinuous behaviors stem from the alignment of the singularities in the Green’s functions $\mathcal{F}_j$ at $\varepsilon = \Delta_j$ when

$$ \Delta_1(T_1) = \Delta_2(T_2). \quad (5) $$

In order to correctly interpret this phenomenology, we discuss first the critical currents for $r < 1$, i.e., $r = 0.5$ shown in Fig. 2(a). In this case, the superconducting gap
\( \Delta_1 \) is smaller than \( \Delta_2 \), namely, \( \Delta_1(T) \leq \Delta_2(T) \ \forall T \in [0 - T_c] \), see the inset of Fig. 2(a). So that for each temperature \( T_1 \), there exists a temperature \( T_2 \) satisfying Eq. 6. However, this condition is fulfilled only when \( T_2 \) is higher than the threshold value \( T_2^h \) at which \( \Delta_2(T_2^h) = \Delta_{10} \) (where \( \Delta_{10} = \Delta_1(T_1 = 0) \)). Specifically, for \( r = 0.5 \), one obtains \( T_2^b \approx 0.91 T_c \), see dashed lines in the inset of Fig. 2(a). Therefore, the sharp jumps in the current critical emerge at \( T_2 > T_2^h \), namely, at the \( T_2 \)'s values within the shaded region in the inset of Fig. 2(a). We note that the height of the jumps reduces by increasing \( T_1 \).\( ^{38} \)

In the symmetric case, namely, \( r = 1 \), shown Fig. 2(b), the condition \( 6 \) can be satisfied only at \( T_1 = T_2 \). In this case, there is no jump, so that the curves have a change of slope, in the place of a jump, at \( T_1 = T_2 \).

Finally, for \( r > 1 \), i.e., \( r = 2 \) in Fig. 2(c), \( \Delta_1(T) > \Delta_2(T) \ \forall T \in [0 - T_c] \), see the inset of Fig. 2(c), so that the condition \( 6 \) is fulfilled only at temperatures \( T_1 \) higher than the \( T_1^h \) at which \( \Delta_1(T_1^h) = \Delta_{20} \) (where \( \Delta_{20} = \Delta_2(T_2 = 0) \)), see the shaded region in the inset of Fig. 2(c). Specifically, for \( r = 2 \), one obtains \( T_1^h \approx 0.91 T_c \). Indeed, among those shown in Fig. 2(c), only the curve at \( T_1 = 0.94 T_c \) shows a jump. Interestingly, in this case the critical current \( I_c \) behaves counterintuitively, since by raising the temperature it sharply increases undergoing a discontinuous jump, instead of decreasing monotonically. Moreover, this positive jump becomes higher at a temperature \( T_1 \) just above \( T_1^h \) and reduces by further increasing it. This odd behaviour of the critical current can be anticipated also by further inspecting Fig. 2(a), since the point where the jump is located, i.e. \( T_2^h \), shifts towards higher temperatures by increasing \( T_1 \). So, by inverting the role of \( T_1 \) and \( T_2 \) the jumps showed in Fig. 2(a) would necessarily imply the behaviour shown in Fig. 2(c).

We note that, both in \( r > 1 \) and \( r < 1 \) cases, the temperature ranges in which the discontinuous jumps in \( I_c \) appear can be enlarged by reducing the temperatures \( T_1^h \) and \( T_2^h \), namely, by considering junctions less and less asymmetric, i.e., \( r \to 1 \). Nonetheless, in this case the height of the jumps tends to reduce, up to vanish just for \( r = 1 \). Conversely, by increasing the asymmetry between the gaps, namely, for \( r \gg 1 \) (or \( r \ll 1 \)), we are suppressing one superconducting gap with respect to the other. In these cases, \( T_2^h \to T_1 \) (or \( T_2^h \to T_2 \)), and the ranges of temperature in which the \( I_c \) jumps appear get narrower. Accordingly, since \( I_c \to 0 \), we expect that, also in these regimes, the height of the \( I_c \) jumps will tend to diminish.

In light of this, we investigate the dependence of the height of the critical current jump, \( \Delta I_c'(T_1) \), on the asymmetry parameter \( r \) by varying the temperature \( T_1 \). Specifically, we explore the cases for \( r > 1 \), namely, the cases giving positive jumps of \( I_c \), as already discussed in Fig. 2(c). In fact, for \( r > 1 \), at a temperature \( T_2 = T_2^j \) satisfying Eq. 6, the critical current \( I_c(T_1 > T_1^h, T_2^j) \) sudden increases. In this case, we additionally observe that \( I_c \) has a minimum just before the jump, i.e., for \( T_2 < T_2^j \), and a maximum just after the jump, i.e., for \( T_2 > T_2^j \). Therefore, we define the height of the critical current jump as the difference between these maximum and minimum \( I_c \) values, namely,

\[
\Delta I_c'(T_1) = \max_{T_2} I_c(T_1, T_2 > T_2^j) - \min_{T_2} I_c(T_1, T_2 < T_2^j),
\]

where \( T_2^j \) is the temperature \( T_2 \) at which the jump occurs, \( T_1 > T_1^h, \) and \( r > 1 \). The behavior of \( \Delta I_c'(T_1) \) as a function of \( T_1 \) at a few values of \( r \) is shown in Fig. 3. The vertical dashed-dotted lines indicate the threshold temperatures \( T_1^h/T_c \), above which the discontinuous jumps of \( I_c \) appear, calculated at the values of \( r \) used in the figure. We observe that, at a given \( r \), \( \Delta I_c'(T_1) \) is maximal for a \( T_1 \) just above \( T_1^h \) and then it reduced linearly by increasing \( T_1 \) up to vanishes for \( T_1 \to T_{c1} \). Interestingly, we observe that the maximum value of \( \Delta I_c'(T_1) \), calculated as \( \Delta I_c^{\text{max}} = \max_{T_1} \Delta I_c'(T_1) \), behaves non-monotonically by increasing \( r > 1 \), approaching zero for \( r \to 1 \) and \( r \gg 1 \) and reaching a maximum for \( r \approx 3 \), as shown in the inset of Fig. 3. Accordingly, the highest \( I_c \) jump is...
obtained for $T_{c_1} \approx 3T_{c_2}$.

To have an idea of the situation in which the present effect can be observed, we assume a JJ with a barrier resistance of $R = 100$ $\Omega$ between, for instance, $\text{Nb}$ ($T_{c_1} = 9.2$ K) and $\text{Ta}$ ($T_{c_2} = 4.4$ K), corresponding to an asymmetry parameter of $r \approx 2$, one finds a jump of $\Delta I_c = 2 \frac{\Delta I_{c_0}}{2eR}$, as a function of $r$, when the maximal critical current at low temperatures is $I_{c,max}^{r=2} \approx 3.05 \frac{\Delta I_{c_0}}{2eR} = 14.5$ $\mu$A, see Fig. 2(c).

Nonetheless, we observe that in this case the range of $T_1$ at which the discontinuous behaviour of $I_c$ emerges is very nearby to the critical temperature.

In the previous discussion we analyzed the jump for $r > 1$, although one can easily generalize the previous results also to the $r < 1$ case, due to the discussed symmetry between the $r < 1$ and $r > 1$ cases by exchanging the role of the temperatures $T_1$ and $T_2$. In particular, for $r < 1$ the jump height will be maximum for $r \approx 1/3$.

In this case, the value of $T_2$ at which the jump appears is really nearby the critical temperature $T_{c_2}$, as can be easily seen by comparing Fig. 2(a) with Fig. 2(c).

The impact of the Dynes parameter, $\gamma$, on the critical current is highlighted in Fig. 4. In this figure, the behavior of $I_c$, in units of $\sqrt{\Delta_{10} \Delta_{20}}/(2eR)$, as a function of $T_2/T_{c_2}$, at a few values of $\gamma$, for $r = 0.5$ and $T_1/T_{c_1} = 0.1$, is shown. Specifically, we evidence how the critical current changes by varying $\gamma$ in a neighborhood of a jump. We observe that the higher the $\gamma$ value, the smoother the $I_c$.

In Sec. IV we will discuss some possible applications of this device, but certainly the sharpness of the jump is an important figure of merit, which is potentially connected to the sensitivity of the junction to small temperature variations around the operating point $T_2^{\ast}$. Higher sensitivities in temperature can be obtained by maximizing the jump sharpness, i.e., by increasing the current jump height $\Delta I_c$ and/or by minimizing the Dynes parameter $\gamma$.

As discussed so far, the critical current strongly depends on the asymmetry parameter $r$. In this regard, in Fig. 5(a) we illustrates the behavior of the critical current $I_c$, in units of $\sqrt{\Delta_{10} \Delta_{20}}/(2eR)$, as a function of $r$, at a few values of the normalized temperature $T_1/T_{c_1}$ and $T_2/T_{c_2} = 0.8$. We observe that also these curves may show a jump, except for the curve at $T_1/T_{c_1} = T_2/T_{c_2}$. In the latter case, $I_c$ shows a cusp in $r = 1$, since its slope suddenly changes from negative to positive around $r = 1$, and it is symmetric, in a semi-log plot, with respect to this point. The position $r_j$ of the discontinuous jump of $I_c$ changes with the temperature $T_1/T_{c_1}$ and can be estimated through Eq. (6). In Fig. 5(b), we display the jump position $r_j$ as a function of $T_1/T_{c_1}$, for $T_2/T_{c_2} = 0.8$. Additionally, the height of the $I_c$ jumps, $|\Delta I_c(r_j)|$, as a function of $T_1/T_{c_1}$ is shown in Fig. 5(c) for $T_2/T_{c_2} = 0.8$. We observe that $\Delta I_c(r_j)$ has a plateau at low $T_1$'s and it decreases by increasing $T_1$, up to vanish at $T_1 = T_2$, whereupon it raises again.

Finally, with the aim to quantify the asymmetry of the critical current with respect to the switch of the temperatures $T_1$ and $T_2$, keeping fixed the structural asymmetry $r$, we define the temperature-switching asymmetry parameter $\mathcal{R}$,

$$\mathcal{R}(\%) = \frac{I_c(T_1, T_2) - I_c(T_2, T_1)}{I_c(T_2, T_1)} \times 100. \quad (7)$$

This parameter synthetically describes how the struc-
tural asymmetry \( r \) induces a strong asymmetrical behavior on the non-dissipative branch represented by an asymmetry of the critical current with the exchange of the temperatures of the superconducting leads. In the density plot shown in Fig. 6(a) we display the behavior of \( R \) as a function of \( T_1/T_{c_1} \) and \( T_2/T_{c_2} \), for \( r = 0.5 \). We observe that also \( R \) shows discontinuities, just in correspondence of the \( I_c \) jumps previously discussed in Fig. 3. Furthermore, the sign of \( R \) switches in correspondence of a jump. If \(|R| \) is maximum, it means that the variation of \( I_c \) by switching the temperatures is maximal too. Conversely, if \( R = 0 \) the critical current is symmetric with respect to a temperature switch, although the system is intrinsically asymmetric, since \( r \neq 1 \). Three selected profiles of \( R \) as a function of \( T_2/T_{c_2} \) for different \( T_1/T_{c_1} \)'s are shown as well in Fig. 6(b). The situations plotted in this figure correspond to the colored dashed lines in Fig. 5(a). For \( T_1 < T_1^{th} \), by varying \( T_2/T_{c_2} \) we note that \( R \) undergoes to only one jump at a temperature \( T_2 > T_2^{th} \), see curves at \( T_1/T_{c_1} = 0.2 \) and 0.7 in Fig. 6(b). In these cases, \( R \) monotonically increases before the jump, whereas it becomes negative and monotonically decreases after the jump. Moreover, the height of these jump reduces by increasing \( T_1 \). Conversely, at a temperature \( T_1 > T_1^{th} \), we observe two jumps in \( R \), see the curve at \( T_1/T_{c_1} = 0.92 \) in Fig. 5(b), since both \( I_c(T_1,T_2) \) and \( I_c(T_2,T_1) \) behaves discontinuously at some values of \( T_2 \). Also in this case \( R \) becomes negative after a discontinuous jump.

The behavior of \( R \) as a function of \( T_2/T_{c_2} \) at a few values of the asymmetric parameter \( r < 1 \) is shown in Fig. 5(c), at \( T_1/T_{c_1} = 0.2 \). We note that the lower the value of \( r \), the higher are both the temperature at which \( R \) discontinuously changes and the height of its jump. Conversely, in the symmetric case, \( r = 1 \), the critical current is symmetric in the temperatures switch, namely, \( I_c(T_1,T_2) = I_c(T_2,T_1) \), so that \( R = 0 \ \forall T_1,T_2 \).

III. LINEAR RESPONSE APPROXIMATION

In this section we analyze the variation of the critical current for small temperature differences between the two superconductors. Our aim is to quantify how small temperature differences will affect the non-dissipative regime in the presence of a structural asymmetry, \( r \neq 0 \), in the junction. We assume that \( T_1 > T_2 \), so that we can define \( T \) and \( \delta T \) such that \( T_1 = T + \delta T/2 \) and \( T_2 = T - \delta T/2 \), and we can investigate the linear response approximation by imposing \( \delta T = T_1 - T_2 \ll T = (T_1 + T_2)/2 \).

The critical current, see Eq. (3), depends on the lead temperatures through both the statistical factors \( f_j \equiv f(\varepsilon,T_j) \) and the self-consistent superconducting gap \( \Delta_j \equiv \Delta_j(T_j) \) (with \( j = 1,2 \)). The linear behaviour in \( \delta T \) of the critical current \( I_c = \int_{-\infty}^{+\infty} d\varepsilon J(\varepsilon) \) can be easily written as

\[
\delta I_c \sim \int_{-\infty}^{+\infty} d\varepsilon \left( \sum_i \frac{\delta f_i}{f_i} \frac{\partial f_i}{\partial \Delta_i} \frac{\delta f_i}{f_i} \frac{\partial \Delta_i}{\Delta_i} \right) I_c(\varepsilon,T),
\]

where in the first term \( \alpha_1 \) we consider only the temperature variation of the statistical weights \( f_i \) and in the second \( \alpha_2 \) the temperature variation of the gaps \( \Delta_i \). Finally, the critical current can be written as

\[
I_c(T,\delta T) \sim I_c(T,0) + \alpha_1(T)\delta T + \alpha_2(T)\delta T,
\]

where

\[
I_c(T,0) = \frac{1}{2eR} \int_{-\infty}^{+\infty} f(\varepsilon,T) \text{Im} \left[ \tilde{G}_1(\varepsilon,T)\tilde{G}_2(\varepsilon,T) \right] \, d\varepsilon
\]

coincides exactly with the well known Ambegaokar-Baratoff relation [21]. Therefore, the linear contribution to the critical current can be seen as a correction to the usual relation, Eq. (10), due to the junction asymmetry.
and the temperature gradient. This contribution is determined by two different terms, \( \alpha_1 \) and \( \alpha_2 \), see Eq. (5). The former is associated to the variation of the electron distribution assuming temperature-independent gaps. Instead, the latter, \( \alpha_2 \), is computed by considering only temperature variations of the superconducting gaps included in the anomalous Green’s functions \( \tilde{\gamma}_j \), see Eq. (4).

According to the modulus in Eq. (3), if we recast the critical current as \( I_c = |\mathcal{E}| \), its derivative can be written as \( \frac{\partial I_c}{\partial T} = \text{sgn}(\mathcal{E}) \frac{\partial \mathcal{E}}{\partial T} \). Then the coefficient \( \alpha_1 \) reads

\[
\alpha_1(T) = \frac{\text{sgn}(\mathcal{E})}{8eR_k B T^2} \int_{-\infty}^{\infty} d\varepsilon \frac{\text{Im}[\tilde{\gamma}_1(\varepsilon, T)\tilde{\gamma}_2^*(\varepsilon, T)]}{\cosh^2(\varepsilon/2k_BT)},
\]

where it is easy to recognise the derivative contribution of \( f_i \), as taken directly from Ambegaokar-Baratoff, Eq. (10). Instead, by expanding the anomalous terms in Eq. (3) to the first order in \( \delta T \), the coefficient \( \alpha_2 \) can be expressed as

\[
\alpha_2(T) = \frac{\text{sgn}(\mathcal{E})}{4eR} \int_{-\infty}^{\infty} d\varepsilon f(\varepsilon, T) \left( \sum_i (-1)^{i-1} \frac{\Delta_i'(T)}{\Delta_i(T)} \beta_i(\varepsilon, T) \right),
\]

where \( \Delta_i'(T) \) is the derivative with respect to \( T \) of the \( i \)-th superconducting gap, and \( \beta_j(\varepsilon, T) = \text{Im}(\tilde{\gamma}_1\tilde{\gamma}_2^*) - \frac{1}{2} \Re(\tilde{\gamma}_1\tilde{\gamma}_2^*) \tilde{\gamma}_j^2 \), with \( \Theta_j(\varepsilon, T) = \frac{\varepsilon + i\Gamma_j}{\sqrt{\varepsilon^2 + \Gamma_j^2}} - \Delta_j^*(T) \). We see that the gaps affect the linear coefficient \( \alpha_2 \) via their logarithmic derivatives \( \Delta_i'(T)/\Delta_i(T) \) only.

We note that both \( \alpha_1 \) and \( \alpha_2 \) are linear coefficients of the dissipationless regime so they can be defined only for \( T \leq \min\{T_{c1}, T_{c2}\} \). In order to efficiently represents these terms for different structural asymmetries \( r \), it is convenient to normalize the temperature with respect to \( \sqrt{T_{c1}T_{c2}} \). So, one can easily verify that the linear coefficients are defined only for \( \frac{T}{\sqrt{T_{c1}T_{c2}}} \leq \min\{\sqrt{r}, \frac{1}{\sqrt{r}}\} \).

The behaviors of the coefficients \( \alpha_1 \) and \( \alpha_2 \) as a function of the normalized temperature \( T/\sqrt{T_{c1}T_{c2}} \) for \( r \in [0.2 \pm 5] \) are shown in Figs. (a) and (b), respectively. Hereafter, we will assume a Nb (\( T_{c1} = 9.2 \text{ K} \)) electrode \( S_1 \) and we suppose to be able to set the gap of \( S_2 \) at will, in order to get the appropriate value of the asymmetry parameter \( r \). The barrier resistance is set to \( R = 100 \text{ \Omega} \), which results in a junction that, for the symmetric case \( r = 1 \), has a low-temperatures critical current approximatively of \( 22 \mu\text{A} \).

First of all, we observe that both \( \alpha_1 \) and \( \alpha_2 \) vanish for \( r = 1 \), see Figs. (a) and (b), respectively, namely, there is no linear contribution with the temperature gradient to the critical current in the symmetric case. Conversely, both coefficients are positive for \( r > 1 \) and negative for \( r < 1 \). This remark can be rationalized by observing that the critical current roughly scales according to the geometric mean of the superconducting gaps \( \sqrt{\Delta_1(T_1)\Delta_2(T_2)} \), see Eq. (3). Interestingly, the \( \delta T \) derivative of this quantity is positive for \( r > 1 \) and negative for \( r < 1 \). This shows that the sign of \( \delta I_c/\delta T \) in a JJ under a small temperature gradient \( \delta T \) directly reflects on the structural asymmetry \( r \) in the junction.

We observe that \( \alpha_1 \) behaves non-monotonically, see Fig. (a), since, for \( r < 1 \) it starts from zero, reaches...
Specifically, it rapidly vanishes at $T = T_{c1}$, that is at $T = T_{c1}$. Similarly, for $r > 1$ it starts from zero, reaches a maximum and finally vanishes at $\frac{T}{\sqrt{T_{c1}T_{c2}}} = 1/r$, that is at $T = T_{c2}$. For low temperatures, the behavior of $\alpha_1$ is ruled by the exponential suppression of the hyperbolic contribution for $T \to 0$. Instead, for $T \to 0$ we observe that the maximum value of $|\alpha_1|$ increases if $r \to 1$. This apparently odd result is consistent with the fact that when $T_1 \approx T_2$ the critical current is not-analytical in the asymmetry parameter $r$, as implied by the cusp shown in Fig. 3(a) for $r = 1$ and $T_1 = T_2$.

Conversely, $\alpha_2$ behaves monotonically, see Fig. 3(b). Specifically, it rapidly vanishes at $T \to 0$ and diverges at $\frac{T}{\sqrt{T_{c1}T_{c2}}} \to \operatorname{Min}\{\sqrt{r}, \sqrt{1/r}\}$. The low-temperatures behavior of $\alpha_2$ is mainly governed by the gap logarithmic derivatives, being the superconducting gap roughly constant at $T \lesssim T_{c1}/4$ so that $\Delta_j(T) \to 0$ at $T \to 0$. Instead, for $T \to \operatorname{Min}\{T_{c1}, T_{c2}\}$, although $\Delta_j(T) \to 0$, we observe that the logarithmic derivative diverges making $\alpha_2$ also diverging.

Interestingly, we observe that the coefficients $\alpha_1$ and $\alpha_2$ behave quite differently by varying the Dynes parameter $\gamma$, as it is clearly shown in Figs. 7(c) and (d) for a few values of $\gamma \in [10^{-1} \div 10^{-1}]$, and $r = 1.05$. We observe that $\alpha_1$ is strongly affected by $\gamma$, since it significantly reduces by increasing $\gamma$, up to become even five times lower passing from $\gamma = 10^{-5}$ to $\gamma = 10^{-1}$, see Fig. 7(c). Conversely, the coefficient $\alpha_2$ is practically independent of $\gamma$, as it is shown in Fig. 7(d). Interestingly, we observe that to appreciate concrete variations in $\alpha_2$ we should consider quite higher, unrealistic values of $\gamma$, see the curve shown in the inset in Fig. 7(d) obtained at $\frac{T}{\sqrt{T_{c1}T_{c2}}} = 0.97$.

IV. DISCUSSION

The physical effect described so far could promptly find an application in several contexts. For instance, according to the possible lowering of $I_c$ upon temperature bias reversal one can envisage the use of this device as the Josephson-counterpart of a thermal current recti-
fier. Interestingly, several examples of thermal rectifiers, namely, structures allowing high heat conduction in one direction but suppressing thermal transport upon temperature switch, based on Josephson junctions [10–11], phononic devices [12–11], and quantum dot [15], were also recently conceived.

Alternatively, a single-photon or bolometric detector [16–12] based on a temperature-biased asymmetric Josephson tunnel junction might be conceived. Superconducting sensors operating at cryogenic temperatures are increasingly attractive, since superconductivity assures highly suppressed heat leakage [10–13], a drastic resistance drop at the critical temperature [14], and negligible Johnson noise when operating in dissipationless regime [55].

In our device concept, the measurable abrupt increase of the critical current could be exploited to sense radiation. In such a setup, a small superconducting electrode of the junction can be heated by the absorption of an incident photon, so that, if the temperature of the electrode is close enough to the threshold value giving an \( I_c \) jump, namely, at \( T_2 \lesssim T_c^2 \), the critical current suddenly increases (or reduces). We note that during this part of the sensor response, the detector is not sensitive to a subsequent photon, since a further temperature increase would not induce an abrupt \( I_c \) variation. Anyway, after the first absorption, due to the thermal contact with a phononic bath, the electrode recovers its initial steady temperature with a time constant determined by the heat capacity of electrode, and both the electron-phonon and electron thermal conductances of the junction [17]. The above setup resembles a superconducting tunnel junction (STJ) detector where a tunnel junction between two superconductors is exploited in the dissipative regime [56–58]. Conversely, in our proposal we operate the tunnel junction in the dissipationless regime without any quasiparticle charge current involved [59]. Reading of the photon-induced \( I_c \) variation could be performed by conventional well-established techniques, for instance, via a Josephson sensor [55] based on the modifications of the kinetic inductance, \( L_k \propto 1/I_c \) [21–22], of the junction working in the dissipationless regime and inductively coupled to a superconducting quantum interference device (SQUID).

Alternatively, the variation of the Josephson kinetic inductance of the junction can be performed dispersively through an LC tank circuit inductively coupled to the JJ [60–61]. As a matter of fact, in this readout scheme the modifications of the inductance can be measured through a shift, or a broadening, of the circuit’s transmission or reflection resonance [62]. The detector based on dispersive detection have a huge potential in fast detection and quantum limited energy-resolution [61]. Those platforms combined with the dissipationless configuration of our tunnel junction promise minimal low-noise performances with reduced dark-counts and, consequently, high energy sensitivity.

Before concluding, we wish to remark that our observation of jump in the critical current, in the presence of both an asymmetry of the junction and a temperature bias, is purely based on a conventional BCS mechanism, i.e., gaps matching. This means that for all those experiments where jumps in the critical current are indeed discussed as a smoking-gun proof of more elaborate mechanisms, such as, for instance, topological transitions [63–65], one need to deserve extra care, in order to be sure that a structural asymmetry, in the presence of an uncontrollable thermal gradient evolution, could eventually provide a simpler explanation.

V. CONCLUSIONS

In conclusion we discuss in this paper the behavior of the critical current, \( I_c \), of a Josephson tunnel junction formed by different superconductors. We analyze in detail the behavior of \( I_c \) by changing both the temperatures of the electrodes and the ratio, \( r \), between the critical temperatures of the superconductors. We observe that the critical current is asymmetric in the temperatures switch and that it behaves discontinuously at specific temperatures, namely, at the temperatures at which the BCS superconducting gaps coincides. Specifically, in these conditions the critical current of an asymmetric junction, i.e., \( r \neq 1 \), suddenly jumps. We observe also an unexpected behavior, since, for \( r > 1 \), by enhancing the temperature the critical current in correspondence of a jump increases.

Studying the height of the \( I_c \) jump, we observe a non-monotonic behavior, according to which we found that an optimal \( r \) value, giving a maximum increase of the critical current upon temperature variations, exists. We also discuss how Dynes parameters in the superconductors affect the sharpness of the \( I_c \) transition. Finally we discuss in detail the behavior of the critical current for a small thermal gradient along the junction as a function of the average temperature and the Dynes parameters.

The peculiar temperature-dependence of the critical current of an asymmetric Josephson junction can be relevant to conceive intriguing applications. For instance, the step-like variation with the temperature of the critical current will allow us to design a single-photon threshold detector in which the absorption of a photon produces a temperature enhancement, that can correspond to a measurable critical current variation. This system operating in the non-dissipative branch is likely to provide very-high energy sensitivity which deserve further investigation.

ACKNOWLEDGMENTS

C.G., A.B., and F.G. acknowledge the European Research Council under the European Union’s Seventh Framework Program (FP7/2007-2013)/ERC Grant agreement No. 615187-COMANCHE and the Tuscany Region under the FARFAS 2014 project SCIADRO for
partial financial support. P.S. and A.B. have received funding from the European Union FP7/2007-2013 under REA Grant agreement No. 630925 – COHEAT. A.B. acknowledges the CNR-CONICET cooperation programme “Energy conversion in quantum nanoscale hybrid devices” and the Royal Society though the International Exchanges between the UK and Italy (grant IES R3 170054).

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