HOM-ENTWINING STRUCTURES AND HOM-HOPF-TYPE MODULES

SERKAN KARAÇUHA

Abstract. The notions of Hom-coring, Hom-entwining structure and associated entwined Hom-module are introduced. A theorem regarding base ring extension of a Hom-coring is proven and then is used to acquire the Hom-version of Sweedler coring. Motivated by [4], a Hom-coring associated to an entwining Hom-structure is constructed and an identification of entwined Hom-modules with Hom-comodules of this Hom-coring is shown. The dual algebra of this Hom-coring is proven to be a $\psi$-twisted convolution algebra. By a construction, it is shown that a Hom-Doi-Koppinen datum comes from a Hom-entwining structure and that the Doi-Koppinen Hom-Hopf modules are the same as the associated entwined Hom-modules, following [3]. A similar construction regarding an alternative Hom-Doi-Koppinen datum is also given. A collection of Hom-Hopf-type modules are gathered as special examples of Hom-entwining structures and corresponding entwined Hom-modules, and structures of all relevant Hom-corings are also considered.

1. Introduction

Motivated by the study of symmetry properties of noncommutative principal bundles, entwining structures (over a commutative ring $k$ with unit) were introduced in [5] as a triple $(A, C)_\psi$ consisting of a $k$-algebra $A$, a $k$-coalgebra $C$ and a $k$-module map $\psi : C \otimes A \rightarrow A \otimes C$ satisfying, for all $a, a' \in A$ and $c \in C$,

\[(aa')_\kappa \otimes c^\kappa = a_\kappa a'_\kappa \otimes c^{\kappa} \lambda, \quad 1_\kappa \otimes c^{\kappa} = 1 \otimes c,\]
\[a_\kappa \otimes c_1^{\kappa} \otimes c_2^{\kappa} = a_{\kappa\lambda} \otimes c_1^{\kappa} \otimes c_2^\lambda, \quad a_\kappa \varepsilon(c^{\kappa}) = a \varepsilon(c),\]

where the notation $\psi(c \otimes a) = a_\kappa \otimes c^{\kappa}$ (summation over $\kappa$ is understood) is used. Given an entwining structure $(A, C)_\psi$, the notion of $(A, C)_\psi$-entwined module $M$ was first defined in [3] as a right $A$-module with action $m \otimes a \mapsto m \cdot a$ and a right $C$-comodule with coaction $\rho^M : m \mapsto m(0) \otimes m(1)$ (summation understood) such that the following compatibility condition holds:

\[\rho^M(m \cdot a) = m(0) \cdot a_\kappa \otimes m(1)^{\kappa} \cdot a \varepsilon(c), \quad \forall a \in A, m \in M.\]

Hopf-type modules are typically the objects with an action of an algebra and a coaction of a coalgebra which satisfy some compatibility condition. The family of Hopf-type modules includes well-known examples such as Hopf modules of Sweedler [35], relative Hopf modules of Doi and Takeuchi [13], [37], Long dimodules [27], Yetter-Drinfeld modules [33], [42], Doi-Koppinen Hopf modules [15], [25] and alternative Doi-Koppinen Hopf modules of Schauenburg [34]. All these modules except alternative Doi-Koppinen modules are special cases of Doi-Koppinen modules. As newer special cases of them, the family of Hopf-type modules also includes anti-Yetter-Drinfeld modules which were obtained as coefficients for the cyclic cohomology of Hopf algebras [19], [20], [22], and their generalizations termed $(\alpha, \beta)$-Yetter-Drinfeld modules [32] (also called $(\alpha, \beta)$-equivariant $C$-comodules in [24]). Basically, entwining structures and modules associated to them enable us to unify several categories of Hopf modules in the sense that the compatibility conditions for all of them can be restated in the form of the above condition required for entwined modules. One can refer to [6] and [9] for more information on the relationship between entwining structures and Hopf-type modules.

Hom-type algebras have been introduced in the form of Hom-Lie algebras in [21], where the Jacobi identity was twisted along a linear endomorphism. Meanwhile, Hom-associative algebras have been suggested in [28] to give rise to a Hom-Lie algebra using the commutator bracket.

Key words and phrases. Hom-coring, Hom-entwining structure, entwined Hom-module, (alternative) Hom-Doi-Koppinen data, unifying Hom-Hopf module, generalized Hom-Yetter-Drinfeld module.
Other Hom-type structures such as Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras and their properties have been considered in [29, 30, 38]. The so-called twisting principle has been introduced in [39] to provide Hom-type generalization of algebras, and it has been used to obtain many more properties of Hom-bialgebras and Hom-Hopf algebras; for instance see [1, 16, 17, 31, 40, 41]. The authors of [8] investigated the counterparts of Hom-bialgebras and Hom-Hopf algebras in the context of tensor categories, and termed them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras with slight variations in their definitions. Further properties of monoidal Hom-Hopf algebras and many structures on them have been lately studied [10, 11, 12, 13, 18, 23, 20].

Entwining structures have been generalized to weak entwining structures in [7] to define Doi-Koppinen data for a weak Hopf algebra, motivated by [2]. Thereafter, it has been proven in [4] that both entwined modules and weak entwined modules are comodules of certain type of corings which built on a tensor product of an algebra and a coalgebra, and shown that various properties of entwined modules can be obtained from properties of comodules of a coring. Here we recall from [35] that for an associative algebra $A$ with unit, an $A$-coring is an $A$-bimodule $C$ with $A$-bilinear maps $\Delta_C : C \to C \otimes A C$, $c \mapsto c_1 \otimes c_2$ called coproduct and $\varepsilon_C : C \to A$ called counit, such that

$$\Delta_C(c_1) \otimes c_2 = c_1 \otimes \Delta_C(c_2), \quad \varepsilon_C(c_1) c_2 = c = c_1 \varepsilon_C(c_2), \quad \forall c \in C.$$ 

Given an $A$-coring $C$, a right $C$-comodule is a right $A$-module $M$ with a right $A$-linear map $\rho^M : M \to M \otimes C$, $m \mapsto m_{(0)} \otimes m_{(1)}$, called coaction, such that

$$\rho^M(m_{(0)}) \otimes m_{(1)} = m_{(0)} \otimes \Delta_C(m_{(1)}), \quad m = m_{(0)} \varepsilon_C(m_{(1)}), \quad \forall m \in M.$$ 

The main aim of the present paper is to generalize the entwining structures, entwined modules and the associated corings within the framework of monoidal Hom-structures and then to study Hopf-type modules in the Hom-setting. The idea is to replace algebra and coalgebra in a classical entwining structure with a monoidal Hom-algebra and a monoidal Hom-coalgebra to make a definition of Hom-entwining structures and associated entwined Hom-modules. Following [4], these entwined Hom-modules are identified with Hom-comodules of the associated Hom-coring. The dual algebra of this Hom-entwining is proven to be the Koppinen smash. Furthermore, we give a construction regarding Hom-Doi-Koppinen datum and Doi-Koppinen Hom-Hopf modules as special cases of Hom-entwining structures and associated entwined Hom-modules. Besides, we introduce alternative Hom-Doi-Koppinen datum. By using these constructions, we get Hom-versions of the aforementioned Hopf-type modules as special cases of entwined Hom-modules, and give examples of Hom-corings in addition to trivial Hom-coring and canonical Hom-coring.

Throughout the paper $k$ will be a commutative ring with unit. Unadorned tensor product is over $k$.

### 2. Preliminaries

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the monoidal category of $k$-modules. We associate to $\mathcal{M}_k$ a new monoidal category $\mathcal{H}(\mathcal{M}_k)$ whose objects are ordered pairs $(M, \mu)$, with $M \in \mathcal{M}_k$ and $\mu \in \text{Aut}_k(M)$, and morphisms $f : (M, \mu) \to (N, \nu)$ are morphisms $f : A \to B$ in $\mathcal{M}_k$ satisfying $\nu \circ f = f \circ \mu$. The monoidal structure is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$ and $(k, 1)$.

If we speak briefly, all monoidal Hom-structures are objects in the tensor category $\mathcal{H}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, id), \tilde{a}, \tilde{l}, \tilde{r})$ introduced in [8], with the associativity constraint $\tilde{a}$ defined by

$$(2.1) \quad \tilde{a}_{A,B,C} = a_{A,B,C} \circ ((\alpha \otimes id) \otimes \gamma^{-1}) = (\alpha \otimes (id \otimes \gamma^{-1})) \circ a_{A,B,C},$$

for $(A, \alpha), (B, \beta), (C, \gamma) \in \mathcal{H}(\mathcal{M}_k)$, and the right and left unit constraints $\tilde{r}, \tilde{l}$ given by

$$(2.2) \quad \tilde{r}_A = \alpha \circ r_A = r_A \circ (\alpha \otimes id); \quad \tilde{l}_A = \alpha \circ l_A = l_A \circ (id \otimes \alpha),$$

which we write elementwise:

$$\tilde{a}_{A,B,C}((a \otimes b) \otimes c) = \alpha(a) \otimes (b \otimes \gamma^{-1}(c)),$$

$$\tilde{l}_A(x \otimes a) = x\alpha(a) = \tilde{r}_A(a \otimes x).$$
The category $\tilde{H}(\mathcal{M}_k)$ is termed Hom-category associated to $\mathcal{M}_k$, where a $k$-submodule $N \subset M$ is called a subobject of $(M, \mu)$ if $(N, \mu_{|N}) \in \tilde{H}(\mathcal{M}_k)$, that is $\mu$ restricts to an automorphism of $N$.

We now recall some definitions of monoidal Hom-structures.

**Definition 2.1.** An algebra in $\tilde{H}(\mathcal{M}_k)$ is called a monoidal Hom-algebra and a coalgebra in $\tilde{H}(\mathcal{M}_k)$ is termed a monoidal Hom-coalgebra, that is, respectively,

1. A monoidal Hom-algebra is an object $(A, \alpha) \in \tilde{H}(\mathcal{M}_k)$ together with a $k$-linear map $m : A \otimes A \to A$, $a \otimes b \mapsto ab$ and an element $1_A \in A$ such that

   \begin{align}
   \alpha(a)(bc) &= (ab)\alpha(c) \ ; \ a1_A = 1_A a = \alpha(a), \\
   \alpha(ab) &= \alpha(a)\alpha(b) \ ; \ \alpha(1_A) = 1_A,
   \end{align}

   for all $a, b, c \in A$.

2. A monoidal Hom-coalgebra is an object $(C, \gamma) \in \tilde{H}(\mathcal{M}_k)$ together with $k$-linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \to k$ such that

   \begin{align}
   \gamma^{-1}(c_1) \otimes c_21 \otimes c_22 &= c_1 \otimes c_2 \otimes \gamma^{-1}(c_2) \ ; \ c_1 \varepsilon(c_2) &= \gamma^{-1}(c) = \varepsilon(c_1)c_2, \\
   \Delta(\gamma(c)) &= \gamma(c_1) \otimes \gamma(c_2) \ ; \ \varepsilon(\gamma(c)) = \varepsilon(c),
   \end{align}

   for all $c \in C$.

**Definition 2.2.** Now we consider modules and comodules over a Hom-algebra and a Hom-coalgebra, respectively.

1. A right $(A, \alpha)$-Hom-module consists of an object $(M, \mu) \in \tilde{H}(\mathcal{M}_k)$ together with a $k$-linear map $\psi : M \otimes A \to M$, $\psi(m \otimes a) = ma$, in $\tilde{H}(\mathcal{M}_k)$ satisfying the following

   \begin{align}
   \mu(m)(ab) &= (ma)\alpha(b) \ ; \ m1_A = \mu(m),
   \end{align}

   for all $m \in M$ and $a, b \in A$. $\psi$ being a morphism in $\tilde{H}(\mathcal{M}_k)$ means that

   \begin{align}
   \mu(ma) &= \mu(m)\alpha(a).
   \end{align}

   $\psi$ is termed a right Hom-action of $(A, \alpha)$ on $(M, \mu)$. Let $(M, \mu)$ and $(N, \nu)$ be two right $(A, \alpha)$-Hom-modules. We call a morphism $f : M \to N$ right $(A, \alpha)$-linear if $f \circ \mu = \nu \circ f$ and $f(ma) = f(m)a$ for all $m \in M$ and $a \in A$.

2. A right $(C, \gamma)$-Hom-comodule consists of an object $(M, \mu) \in \tilde{H}(\mathcal{M}_k)$ together with a $k$-linear map $\rho : M \to M \otimes C$, $\rho(m) = m_{[0]} \otimes m_{[1]}$, in $\tilde{H}(\mathcal{M}_k)$ such that

   \begin{align}
   \mu^{-1}(m_{[0]}) \otimes m_{[1]1} \otimes m_{[12]} &= m_{[0]0} \otimes m_{[0][1]} \otimes \gamma^{-1}(m_{[1]}1) : m_{[0]} \varepsilon(m_{[1]}) = \mu^{-1}(m)
   \end{align}

   for all $m \in M$. $\rho$ being a morphism in $\tilde{H}(\mathcal{M}_k)$ stands for

   \begin{align}
   \rho(\mu(m)) &= \mu(m)_{[0]} \otimes \mu(m)_{[1]} = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).
   \end{align}

   $\rho$ is called a right Hom-coaction of $(C, \gamma)$ on $(M, \mu)$. Let $(M, \mu)$ and $(N, \nu)$ be two right $(C, \gamma)$-Hom-comodules, then we call a morphism $f : M \to N$ right $(C, \gamma)$-colinear if $f \circ \mu = \nu \circ f$ and $f(m_{[0]}) \otimes m_{[1]} = f(m_{[0]}) \otimes f(m_{[1]})$ for all $m \in M$.

**Definition 2.3.** A bialgebra in $\tilde{H}(\mathcal{M}_k)$ is called a monoidal Hom-bialgebra and a Hopf algebra in $\tilde{H}(\mathcal{M}_k)$ is called a monoidal Hom-Hopf algebra, in other words
(1) A monoidal Hom-bialgebra \((H, \alpha)\) is a sextuple \((H, \alpha, m, \eta, \Delta, \varepsilon)\) where \((H, \alpha, m, \eta)\) is a monoidal Hom-algebra and \((H, \alpha, \Delta, \varepsilon)\) is a monoidal Hom-coalgebra such that

\[
\Delta(hh') = \Delta(h)\Delta(h') : \Delta(1_H) = 1_H \otimes 1_H,
\]

(2.11)
\[
\varepsilon(hh') = \varepsilon(h)\varepsilon(h') : \varepsilon(1_H) = 1,
\]

(2.12)
for any \(h, h' \in H\).

(2) A monoidal Hom-Hopf algebra \((H, \alpha)\) is a septuple \((H, \alpha, m, \eta, \Delta, \varepsilon, S)\) where \((H, \alpha, m, \eta, \Delta, \varepsilon)\) is a monoidal Hom-bialgebra and \(S : H \rightarrow H\) is a morphism in \(\mathcal{H}(\mathcal{M}_H)\) such that \(S \ast \text{id}_H = \text{id}_H \ast S = \eta \circ \varepsilon\).

\(S\) is called antipode and it has the following properties

\[
S(gh) = S(h)S(g) : S(1_H) = 1_H ;
\]

\[
\Delta(S(h)) = S(h_2) \otimes S(h_1) ; \varepsilon \circ S = \varepsilon,
\]

for any elements \(g, h\) of the monoidal Hom-Hopf algebra \(H\).

**Definition 2.4.** Let \((B, \beta)\) be a monoidal Hom-bialgebra. A right \((B, \beta)\)-Hom-module algebra \((A, \alpha)\) is a monoidal Hom-algebra and a right \((B, \beta)\)-Hom-module with a Hom-action \(\rho_A : A \otimes B \rightarrow A\), \(a \otimes b \mapsto a \cdot b\) such that, for any \(a, a' \in A\) and \(b \in B\)

\[
b \cdot (aa') = (b_1 \cdot a)(b_2 \cdot a'), \quad b \cdot 1_A = \varepsilon(b)1_A,
\]

\[
\rho_A \circ (\alpha \otimes \beta) = \alpha \circ \rho_A.
\]

**Definition 2.5.** Let \((B, \beta)\) be a monoidal Hom-bialgebra. A right \((B, \beta)\)-Hom-comodule algebra \((A, \alpha)\) is a monoidal Hom-algebra and a right \((B, \beta)\)-Hom-comodule with a Hom-coaction \(\rho^A : A \rightarrow A \otimes B, \ a \mapsto a_0(0) \otimes a(1)\) such that \(\rho^A\) is a Hom-algebra morphism, i.e., for any \(a, a' \in A\)

\[
(aa')_0 \otimes (aa')_1 = a_0 a'_0 \otimes a_1 a'_1, \quad \rho^A(1_A) = 1_A \otimes 1_B,
\]

\[
\rho^A \circ \alpha = (\alpha \otimes \beta) \circ \rho^A.
\]

**Definition 2.6.** Let \((A, \alpha)\) and \((B, \beta)\) be two monoidal Hom-algebras. A left \((A, \alpha)\), right \((B, \beta)\) Hom-bimodule consists of an object \((M, \mu) \in \mathcal{H}(\mathcal{M}_H)\) together with a left \((A, \alpha)\)-Hom-action \(\phi : A \otimes M \rightarrow M, \ \phi(a \otimes m) = am\) and a right \((B, \beta)\)-Hom-action \(\varphi : M \otimes B \rightarrow M, \ \varphi(m \otimes b) = mb\) fulfilling the compatibility condition, for all \(a \in A, b \in B\) and \(m \in M\),

\[
(\alpha m)(\beta b) = \alpha(\alpha)(\beta b).
\]

We call a left \((A, \alpha)\), right \((B, \beta)\) Hom-bimodule a \([\alpha, B, \beta]\)-Hom-bimodule. Let \((M, \mu)\) and \((N, \nu)\) be two \([\alpha, \beta]\)-Hom-bimodules. A morphism \(f : M \rightarrow N\) is called a morphism of \([\alpha, \beta]\)-Hom-bimodules if it is both left \((A, \alpha)\)-linear and right \((B, \beta)\)-linear, and satisfies the following property

\[
(af(m))\beta(b) = \alpha(a)(f(m)b),
\]

for all \(a \in A, b \in B\) and \(m \in M\).

**Definition 2.7.** Let \((M, \mu)\) be a right \((A, \alpha)\)-Hom-module and \((N, \nu)\) be a left \((A, \alpha)\)-Hom-module. The tensor product \((M \otimes_A N, \mu \otimes \nu)\) of \((M, \mu)\) and \((N, \nu)\) over \((A, \alpha)\) is the coequalizer of \(\rho \otimes \text{id}_N, (\text{id}_M \otimes \rho) \circ \tilde{a}_{M,A,N} : (M \otimes A) \otimes N \rightarrow M \otimes N\), where \(\rho\) and \(\rho\) are the right and left Hom-actions of \((A, \alpha)\) on \((M, \mu)\) and \((N, \nu)\) respectively. That is,

\[
m \otimes_A n = \{m \otimes n \in M \otimes N \mid m a \otimes n = \mu(m) \otimes \nu^{-1}(n), \forall a \in A\}.
\]
3. Hom-corings and Hom-Entwining structures

Definition 3.1. (1) Let \((A, \alpha)\) be a monoidal Hom-algebra. An \((A, \alpha)\)-Hom-coring consists of an \((A, \alpha)\)-Hom-bimodule \((C, \chi)\) together with \(A\)-bilinear maps \(\Delta_C : C \to C \otimes_A C, c \mapsto c_1 \otimes c_2\) and \(\varepsilon_C : C \to A\) called comultiplication and counit such that

\[
\Delta_C(c_1) \otimes \Delta_C(c_2) = c_{11} \otimes (c_{12} \otimes \chi^{-1}(c_2)); \quad \varepsilon_C(c_1)c_2 = c = c_1\varepsilon_C(c_2),
\]

\[
\Delta_C(\chi(c)) = \chi(c_1) \otimes \chi(c_2); \quad \varepsilon_C(\chi(c)) = \alpha(\varepsilon_C(c)).
\]

(2) A right \((C, \chi)\)-Hom-comodule \((M, \mu)\) is defined as a right \((A, \alpha)\)-Hom-module with a right \(A\)-linear map \(\rho : M \to M \otimes_A C, m \mapsto m(0) \otimes m(1)\) satisfying

\[
\mu^{-1}(m(0)) \otimes \Delta_C(m(1)) = m(0) \otimes (m(1) \otimes \chi^{-1}(m(1)));
\]

\[
\mu(m(0)) \otimes \mu(m(1)) = \mu(m(0)) \otimes \chi(m(1)).
\]

Theorem 3.2. Let \(\phi : (A, \alpha) \to (B, \beta)\) be a morphism of monoidal Hom-algebras. Then, for an \((A, \alpha)\)-Hom-coring \((C, \chi)\), \((BC)B = ((B \otimes_A C) \otimes_A B, (\beta \otimes \chi) \otimes \beta)\) is a \((B, \beta)\)-Hom-coring with a comultiplication and a counit,

\[
\Delta_{(BC)B}((b \otimes_A c) \otimes_A b') = ((\beta^{-1}(b) \otimes_A c_1) \otimes_A 1_B) \otimes_B ((1_B \otimes_A c_2) \otimes_A \beta^{-1}(b'))
\]

\[
\varepsilon_{(BC)B}((b \otimes_A c) \otimes_A b') = (b\phi(\varepsilon_C(c)))b'.
\]

Proof. For \(b, b', b'' \in B\) and \(c \in C\),

\[
\Delta_{(BC)B}(((b' \otimes_A c) \otimes_A b'')b) = \Delta_{(BC)B}((\beta(b') \otimes_A \chi(c)) \otimes_A b'' \beta^{-1}(b))
\]

\[
= ((b' \otimes_A \chi(c_1)) \otimes_A 1_B) \otimes_B ((1_B \otimes_A \chi(c_2)) \otimes_A \beta^{-1}(b'') \beta^{-1}(b)) \tag{3.19}
\]

\[
= ((b' \otimes_A \chi(c_1)) \otimes_A 1_B) \otimes_B ((1_B \otimes_A \chi(c_2)) \otimes_A \beta^{-1}(b'') \beta^{-1}(b))
\]

\[
= ((b' \otimes_A \chi(c_1) \otimes_A 1_B) \otimes_B ((1_B \otimes_A \chi(c_2)) \otimes_A \beta^{-1}(b'') \beta^{-1}(b))
\]

\[
= \Delta_{(BC)B}((b' \otimes_A c) \otimes_A b'').
\]

It can also be shown that \(\Delta_{(BC)B} \circ \tilde{\chi} = (\tilde{\chi} \otimes \tilde{\chi}) \circ \Delta_{(BC)B}\), where \(\tilde{\chi} = (\beta \otimes \chi) \otimes \beta\), which proves the right \(A\)-linearity of \(\Delta_{(BC)B}\). In a similar way, the left \(A\)-linearity of \(\Delta_{(BC)B}\) and the fact that it preserves the compatibility condition between the left and right \((B, \beta)\)-Hom-actions on \((BC)B\) are checked, that is,

\[
\Delta_{(BC)B}(b((b' \otimes_A c) \otimes_A b'')) = \Delta_{(BC)B}((b' \otimes_A c) \otimes_A b'')b,
\]

\[
(b\Delta_{(BC)B}((b' \otimes_A c) \otimes_A b''))\beta(b') = \beta(b)(\Delta_{(BC)B}((b'' \otimes_A c) \otimes_A b'')b').
\]

Next we prove the Hom-coassociativity of \(\Delta_{(BC)B}\):

\[
\Delta_{(BC)B}(((b \otimes_A c) \otimes_A b')_1) = ((b \otimes_A c) \otimes_A b')_1 \otimes_B ((b \otimes_A c) \otimes_A b')_2
\]

\[
= \Delta_{(BC)B}((\beta^{-1}(b) \otimes_A c_1) \otimes_A 1_B)
\]

\[
= ((\beta^{-1}(b) \otimes_A c_1) \otimes_A A \otimes_B ((1_B \otimes_A c_2) \otimes_A \beta^{-1}(b''))\beta^{-1}(b))
\]

\[
= \Delta_{(BC)B}((b' \otimes_A c) \otimes_A b'').
\]
which helps us to compute

\[
((\beta^{-1} \otimes \chi^{-1}) \otimes \beta^{-1})(((b \otimes_A c) \otimes_A b')_1) \otimes_B \Delta_{(BC)B}(((b \otimes_A c) \otimes_A b')_2)
\]

\[
= ((\beta^{-2}(b \otimes_A \chi^{-1}(c_1)) \otimes_A 1_B) \otimes_B (((1_B \otimes_A c_2) \otimes_A 1_B)
\otimes_B ((1_B \otimes_A c_2) \otimes_A \beta^{-2}(b')))
\]

\[
= ((\beta^{-2}(b \otimes_A c_11) \otimes_A 1_B) \otimes_B (((1_B \otimes_A c_12) \otimes_A 1_B)
\otimes_B ((1_B \otimes_A \chi^{-1}(c_2)) \otimes_A \beta^{-2}(b')))
\]

\[
= ((b \otimes_A c) \otimes_A b'11 \otimes_B (((b \otimes_A c) \otimes_A b')_{12}
\otimes_B ((\beta^{-1} \otimes \chi^{-1}) \otimes \beta^{-1})((b \otimes_A c) \otimes_A b')_2).
\]

Now we demonstrate that \(\varepsilon_{(BC)_B}\) is left \(A\)-linear:

\[
\varepsilon_{(BC)_B}(b((b' \otimes_A c) \otimes_A b')) = \varepsilon_{(BC)_B}((\beta^{-2}(b)\otimes_A \chi(c)) \otimes_A \beta(b'))
\]

\[
= ((\beta^{-2}(b)\otimes_A \phi(\varepsilon_C(\chi(c)))) \otimes_A \beta(b')) = ((\beta^{-2}(b)\otimes_A \phi(\varepsilon_C(\chi(c)))) \otimes_A \beta(b'))
\]

\[
= b((b' \otimes_A c) \otimes_A b''),
\]

where \(\phi \circ \alpha = \beta \circ \phi\) was used in the fifth equality. Additionally, we have

\[
(\varepsilon_{(BC)_B} \otimes I)((b \otimes_A c) \otimes_A b') = (\beta(b)\phi(\varepsilon_C(\chi(c)))) \otimes_A \beta(b')
\]

\[
= \beta((b\phi(\varepsilon_C(\chi(c)))) \otimes_A b') = (\beta \circ \varepsilon_C)((b \otimes_A c) \otimes_A b'),
\]

meaning \(\varepsilon_{(BC)_B} \in \tilde{H}(\mathcal{M}_k)\). In the same manner, what is more one can show

\[
\varepsilon_{(BC)_B}((\beta^{-1}((b' \otimes_A c) \otimes_A b'))_1) \otimes_B \varepsilon_{(BC)_B}((\beta^{-1}((b' \otimes_A c) \otimes_A b'))_2)
\]

\[
= ((\beta^{-1}(b) \otimes_A c_1) \otimes_A 1_B)\varepsilon_{(BC)_B}((1_B \otimes_A c_2) \otimes_A \beta^{-1}(b'))
\]

\[
= ((\beta^{-1}(b) \otimes_A c_1) \otimes_A 1_B)((1_B \otimes_A \chi(\varepsilon_C(c_2))) \otimes_A \beta^{-1}(b'))
\]

\[
= (b \otimes_A \chi(c_1)) \otimes_A \beta(\phi(\varepsilon_C(c_2))) \otimes_A \beta^{-1}(b')
\]

\[
= b((b' \otimes_A c) \otimes_A \phi(\alpha(\varepsilon_C(c_2)))) \otimes_A \beta^{-1}(b')
\]

\[
= b((b' \otimes_A c) \otimes_A \varepsilon_C(c_2)) \otimes_A b'
\]

\[
= b((b' \otimes_A c) \otimes_A \varepsilon_C(c_2)) \otimes_A b'
\]

\[
= b((b' \otimes_A c) \otimes_A \varepsilon_C(c_2)) \otimes_A b'
\]

\[
= b((b' \otimes_A c) \otimes_A \varepsilon_C(c_2)) \otimes_A b'
\]

which completes the proof that given a morphism of monoidal Hom-algebras \(\phi : (A, \alpha) \rightarrow (B, \beta)\), \((B \otimes_A C) \otimes_A B, (\beta \otimes \chi) \otimes \beta)\) is a \((B, \beta)\)-Hom-coring, called a base ring extension of the \((A, \alpha)\)-Hom-coring \((C, \chi)\).\(\Box\)
Example 3.3. A monoidal Hom-algebra \((A, \alpha)\) has a natural \((A, \alpha)\)-Hom-bimodule structure with its Hom-multiplication. \((A, \alpha)\) is an \((A, \alpha)\)-Hom-coring by the canonical isomorphism \(A \to A \otimes_A A\), \(a \mapsto \alpha^{-1}(a) \otimes 1_A\), in \(\mathcal{M}_k\), as a comultiplication and the identity \(A \to A\) as a counit. This Hom-coring is called a trivial \((A, \alpha)\)-Hom-coring.

Example 3.4. Let \(\phi : (B, \beta) \to (A, \alpha)\) be a morphism of monoidal Hom-algebras. Then \((C, \chi) = (A \otimes_B A, \alpha \otimes \alpha)\) is an \((A, \alpha)\)-Hom-coring with comultiplication

\[\Delta_C(a \otimes_B a') = (\alpha^{-1}(a) \otimes_B 1_A) \otimes_A (1_A \otimes_B \alpha^{-1}(a')) = (\alpha^{-1}(a) \otimes_B 1_A) \otimes_B a',\]

and counit

\[\varepsilon_C(a \otimes_B a') = aa'.\]

Proof. By Proposition 3.2, for \(\phi : (B, \beta) \to (A, \alpha)\) and the trivial \((B, \beta)\)-Hom-coring \((B, \beta)\) with \(\Delta_B(b) = \beta^{-1}(b) \otimes_B 1_B\) and \(\varepsilon_B(b) = b\), we have the base ring extension of the trivial Hom-coring \((B, \beta)\) to \((A, \alpha)\)-Hom-coring \((AB)A = ((A \otimes_B B) \otimes_B A, (\alpha \otimes \beta) \otimes (\alpha \otimes \alpha))\) with

\[\Delta_{(AB)A}((a \otimes_B b) \otimes_B a') = ((\alpha^{-1}(a) \otimes_B \beta^{-1}(b)) \otimes_B 1_A) \otimes_A ((1_A \otimes_B 1_B) \otimes_B \alpha^{-1}(a')),\]

\[\varepsilon_{(AB)A}((a \otimes_B b) \otimes_B a') = (a \phi(b))a'.\]

On the other hand we have the isomorphism \(\varphi : A \to A \otimes_B B, \ a \mapsto \alpha^{-1}(a) \otimes_B 1_B\), in \(\mathcal{M}_k\), with the inverse \(\psi : A \otimes_B B \to A, \ a \otimes_B b \mapsto a \phi(b)\): Indeed,

\[\psi(\varphi(A)) = \alpha^{-1}(a)\phi(1_B) = \alpha^{-1}(a)1_A = a,\]

\[\varphi(\psi(a \otimes_B b)) = \varphi(a \phi(b)) = \alpha^{-1}(a \phi(b)) \otimes_B 1_B = \alpha^{-1}(a)\phi(\beta^{-1}(b)) \otimes_B 1_B = \alpha^{-1}(a) \otimes_B 1_B = a \otimes_B 1_B = a \otimes_B b,\]

in addition one can check that \(\alpha \circ \psi = \psi \circ (\alpha \otimes \beta)\) and \((\alpha \otimes \beta) \circ \varphi = \varphi \circ \alpha\). Thus, \((AB)A \cong A \otimes_B A = C\)

and

\[\Delta_C(a \otimes_B b) = ((\psi \otimes id) \circ (\varphi \otimes id))(a \otimes_B b) = (\alpha^{-1}(a) \otimes_B 1_A) \otimes_A (1_A \otimes_B \alpha^{-1}(a')) = \varepsilon_C(a \otimes_B a') = \varepsilon_{(AB)A} \circ (\varphi \otimes id)(a \otimes_B a') = aa'.\]

\((A \otimes_B A, \alpha \otimes \alpha)\) is called the Sweedler or canonical \((A, \alpha)\)-Hom-coring associated to a monoidal Hom-algebra extension \(\phi : (B, \beta) \to (A, \alpha)\).

For the monoidal Hom-algebra \((A, \alpha)\) and the \((A, \alpha)\)-Hom-coring \((C, \chi)\), let us put \(*C = \mathcal{A}Hom^H(C, A)\), consisting of left \((A, \alpha)\)-linear morphisms \(f : (C, \chi) \to (A, \alpha)\), that is, \(f(ac) = af(c)\) for \(a \in A, \ c \in C\) and \(f \circ \chi = \alpha \circ f\). Similarly, \(*C^* = \mathcal{A}Hom^H(C, A)\) consist of right \((A, \alpha)\)-Hom-module maps and \((A, \alpha)\)-Hom-bimodule maps, respectively. Now we prove that these modules of \((A, \alpha)\)-linear morphisms \(C \to A\) have ring structures.

Proposition 3.5. (1) \(*C\) is an associative algebra with unit \(\varepsilon_C\) and multiplication \((f \ast g)(C) = f(c_1)g(c_2), \ f, g \in *C, \ c \in C\).

(2) \(*C^*\) is an associative algebra with unit \(\varepsilon_C\) and multiplication \((f \ast^r g)(C) = g(f(c_1)c_2), \ f, g \in *C, \ c \in C\).

(3) \(*C^*\) is an associative algebra with unit \(\varepsilon_C\) and multiplication \((f \ast g)(C) = f(c_1)g(c_2), \ f, g \in *C, \ c \in C\).
Proof. (1) For \( f, g, h \in \mathcal{C} \) and \( c \in \mathcal{C} \),

\[
(f \ast g) \ast h)(c) = f((c_1 h(c_2)) g((c_1 h(c_2))_2)) = f(\chi(c_1) g(c_1 h(c_2)))
\]

where the second equality comes from the fact that \( \Delta_C \) is right \((A, \alpha)\)-linear, i.e., \( \Delta_C(\alpha a) = (\alpha a)_1 \otimes (\alpha a)_2 = \Delta_C(\alpha)a = (\alpha_1 \otimes \alpha_2)a = \chi(c_1) \otimes \alpha_2^{-1}(a), \forall c \in \mathcal{C}, a \in A. \)

\[
(f \ast \varepsilon_C)(c) = f(c_1 \varepsilon_C(c_2)) = f(c), \quad (\varepsilon_C \ast f)(c) = \varepsilon_C(c_1 f(c_2)) = \varepsilon_C(c_1) f(c_2) = f(\varepsilon_C(c_1) c_2) = f(c).
\]

By similar computations one can prove (2) and (3).

\[ \square \]

**Definition 3.6.** A (right-right) Hom-entwining structure is a triple \([([A, \alpha], (C, \gamma)])_\psi\) consisting of a monoidal Hom-algebra \((A, \alpha)\), a monoidal Hom-coalgebra \((C, \gamma)\) and a \(k\)-linear map \(\psi : C \otimes A \rightarrow A \otimes C\) in \(\mathcal{H}(\mathcal{M}_k)\) satisfying the following conditions for all \(a, a' \in A, c \in C\):

\[
\begin{align*}
(aa')_\kappa \otimes \gamma(c)^\kappa &= a_\kappa a'_\lambda \otimes \gamma(c^{\kappa \lambda}), \\
\alpha^{-1}(a_\kappa) \otimes c_\gamma \otimes c_2^{\kappa_2} &= \alpha^{-1}(a)_\kappa \otimes c_1^{\lambda} \otimes c_2^{\kappa}, \\
1_\kappa \otimes c^{\kappa} &= 1 \otimes c, \\
a_\kappa \varepsilon(c^{\kappa}) &= a \varepsilon(c),
\end{align*}
\]

where we have used the notation \(\psi(c \otimes a) = a_\kappa \otimes c^{\kappa}, a \in A, c \in C,\) for the so-called entwining map \(\psi\). It is said that \((C, \gamma)\) and \((A, \alpha)\) are entwined by \(\psi\). \(\psi \in \mathcal{H}(\mathcal{M}_k)\) means that the relation

\[
\alpha(a)_\kappa \otimes \gamma(c)^\kappa = \alpha(a_\kappa) \otimes \gamma(c^{\kappa})
\]

holds.

**Definition 3.7.** A \([([A, \alpha], (C, \gamma)])_\psi\)-entwined Hom-module is an object \((M, \mu) \in \mathcal{H}(\mathcal{M}_k)\) which is a right \((A, \alpha)\)-Hom-module with action \(\rho_M : M \otimes A \rightarrow M, m \otimes a \mapsto ma\) and a right \((C, \gamma)\)-Hom-comodule with coaction \(\rho^M : M \rightarrow M \otimes C, m \mapsto m_{(0)} \otimes m_{(1)}\) fulfilling the condition, for all \(m \in M, a \in A,\)

\[
\rho^M(ma) = m_{(0)} \alpha^{-1}(a)_\kappa \otimes \gamma(m_{(1)}^{\kappa}.
\]

By \(\mathcal{H}_\mathcal{M}_k(\psi)\), we denote the category of \([([A, \alpha], (C, \gamma)])_\psi\)-entwined Hom-modules together with the morphisms in which are both right \((A, \alpha)\)-linear and right \((C, \gamma)\)-colinear.

With the following theorem, we construct a Hom-coring associated to an entwining Hom-structure and show an identification of entwined Hom-modules with Hom-comodules of this Hom-coring.

**Theorem 3.8.** Let \((A, \alpha)\) be a monoidal Hom-algebra and \((C, \gamma)\) be a monoidal Hom-coalgebra.

1. For a Hom-entwining structure \([([A, \alpha], (C, \gamma)])_\psi\), \((A \otimes C, \alpha \otimes \gamma)\) is an \((A, \alpha)\)-Hom-bimodule with a left Hom-module structure \(a(a' \otimes c) = \alpha^{-1}(a)a' \otimes \gamma(c)\) and a right Hom-module structure \((a' \otimes c)a = a'\alpha^{-1}(a)_\kappa \otimes \gamma(c^{\kappa})\), for all \(a, a' \in A, c \in C\). Furthermore, \((C, \chi) = (A \otimes C, \alpha \otimes \gamma)\) is an \((A, \alpha)\)-Hom-coring with the comultiplication and counit

\[
\Delta_C : C \rightarrow C \otimes A \mathcal{C}, \quad a \otimes c \mapsto (\alpha^{-1}(a) \otimes c_1) \otimes A (1 \otimes c_2),
\]

\[
\varepsilon_C : C \rightarrow A, \quad a \otimes c \mapsto \alpha(a) \varepsilon(c).
\]
(2) If $\mathcal{C} = (A \otimes C, \alpha \otimes \gamma)$ is an $(A, \alpha)$-Hom-coring with the comultiplication and counit given above, then $[(A, \alpha), (C, \gamma)]_{\psi}$ is a Hom-entwining structure, where
\[
\psi : C \otimes A \to A \otimes C, \quad c \otimes a \mapsto (1 \otimes \gamma^{-1}(c))a.
\]

(3) Let $(\mathcal{C}, \chi) = (A \otimes C, \alpha \otimes \gamma)$ be the $(A, \alpha)$-Hom-coring associated to $[(A, \alpha), (C, \gamma)]_{\psi}$ as in (1). Then the category of $[(A, \alpha), (C, \gamma)]_{\psi}$-entwined Hom-modules is isomorphic to the category of right $(\mathcal{C}, \chi)$-Hom-comodules.

**Proof.**

(1) We first show that the right Hom-action of $(A, \alpha)$ on $(A \otimes C, \alpha \otimes \gamma)$ is Hom-associative and Hom-unital (or weakly unital), for all $a, d, e \in A$ and $c \in C$:

\[
(\alpha(a) \otimes \gamma(c))(de) = \alpha(a)a^{-1}(de)_{\kappa} \otimes \gamma(\gamma(c)^{\kappa}) = \alpha(a)(\alpha^{-1}(d)\alpha^{-1}(e))_{\kappa} \otimes \gamma(\gamma(c)^{\kappa})
\]

proves the compatibility condition between left and right $(A, \alpha)$-Hom-actions, and the compatibility condition between them as follows:

\[
(a \otimes c)1 = \alpha^{-1}(1)_{\kappa} \otimes \gamma(c^{\kappa}) = a1_{\kappa} \otimes \gamma(c^{\kappa}) \quad \overset{\text{Hom-associativity}}{=} \quad \alpha^{-1}(\alpha(a))1_{\kappa} \otimes \gamma(c^{\kappa}) = \alpha(a)(1 \otimes c^{\kappa})
\]

One can also show that the left Hom-action, too, satisfies the Hom-associativity and Hom-unity. For any $a, b, d \in A$ and $c \in C$,

\[
(b(a \otimes c))\alpha(d) = (\alpha^{-1}(b)a \otimes \gamma(c))\alpha(d) = (\alpha^{-1}(b)a)\alpha^{-1}(\alpha(d))_{\kappa} \otimes \gamma(\gamma(c)^{\kappa}) = (\alpha^{-1}(b)a)\alpha^{-1}(d)_{\kappa} \otimes \gamma(\gamma(c)^{\kappa}) = b(aa^{-1}(d))_{\kappa} \otimes \gamma^{2}(c^{\kappa}) = \alpha^{-1}(\alpha(b))(aa^{-1}(d))_{\kappa} \otimes \gamma(\gamma(c^{\kappa})) = \alpha(b)(aa^{-1}(d))_{\kappa} \otimes \gamma(c^{\kappa}) = \alpha(b)((a \otimes c)d),
\]

proves the compatibility condition between left and right $(A, \alpha)$-Hom-actions is fulfilled.

First, it can easily be proven that the morphisms

\[
A \otimes (C \otimes A C) \to C \otimes A C, \quad a \otimes ((a' \otimes c) \otimes A (a'' \otimes c')) \mapsto \alpha^{-1}(a)(a' \otimes c) \otimes A (a'' \otimes c'),
\]

\[
(C \otimes A C) \otimes A \to C \otimes A C, \quad ((a' \otimes c) \otimes A (a'' \otimes c'))a \mapsto (\alpha(a') \otimes \gamma(c)) \otimes A (a'' \otimes c')^{\alpha^{-1}(a)}
\]

define a left Hom-action and a right Hom-action of $(A, \alpha)$ on $(C \otimes A C, \chi \otimes \chi)$, respectively. Next it is shown that the comultiplication $\Delta_C$ is $A$-bilinear, that is, $\Delta_C$ preserves the left and right $(A, \alpha)$-Hom-actions, and the compatibility condition between them as follows: Let $a, a', b, d \in A$ and $c \in C$, then we have the following computations
$$\Delta_C(a' \otimes c) = \alpha^{-1}(\alpha^{-1}(a') \otimes \gamma(c_1)) \otimes_A (1 \otimes \gamma(c_2))$$

$$2.6 \quad (\alpha^{-2}(a)\alpha^{-1}(a') \otimes \gamma(c_1)) \otimes_A (1 \otimes \gamma(c_2))$$

$$= \alpha^{-1}(a)(\alpha^{-1}(a') \otimes c_1) \otimes_A (\alpha(1) \otimes \gamma(c_2))$$

$$3.32 \quad a((\alpha^{-1}(a') \otimes c_1) \otimes_A (1 \otimes c_2))$$

$$= a\Delta_C(a' \otimes c),$$

$$\Delta_C((a' \otimes c)a) = \Delta_C(a'\alpha^{-1}(a) \otimes \gamma(c^s))$$

$$= (\alpha^{-1}(a')\alpha^{-1}(a) \otimes \gamma(c_1^s)) \otimes_A (1 \otimes \gamma(c_2^s))$$

$$2.6 \quad (\alpha^{-2}(a')\alpha^{-2}(a) \otimes \gamma(c_1^s)) \otimes_A (1 \otimes \gamma(c_2^s))$$

$$= (\alpha^{-1}(a') \otimes c_1)(\alpha(\alpha^{-2}(a)) \otimes \gamma(c_2^s))$$

$$= (a' \otimes \gamma(c_1)) \otimes_A (1 \otimes \gamma(c_2^s))$$

$$= (a' \otimes \gamma(c_1)) \otimes_A (\alpha^{-1}(a)(1 \otimes c_2)\alpha^{-1}(a))$$

$$= ((\alpha^{-1}(a') \otimes c_1) \otimes_A (1 \otimes c_2))a$$

$$= \Delta_C(a' \otimes c)a,$$

$$\alpha(b)(\Delta_C(a \otimes c)d) = \alpha(b)(((\alpha^{-1}(a) \otimes c_1) \otimes_A (1 \otimes c_2))d)$$

$$= \alpha(b)((a \otimes c_1) \otimes_A (1 \otimes c_2)\alpha^{-1}(d))$$

$$3.33 \quad \alpha(b)((a \otimes \gamma(c_1)) \otimes_A \alpha(\alpha^{-2}(d) \otimes \gamma(c_2^s)))$$

$$= \alpha(b)((a \otimes \gamma(c_1)) \otimes_A (\alpha(\alpha^{-2}(d)) \otimes \gamma(c_2^s)))$$

$$= b(a \otimes \gamma(c_1)) \otimes_A (\alpha^2(\alpha^{-2}(d)) \otimes \gamma^2(c_2^s))$$

$$= (\alpha^{-1}(b)a \otimes \gamma^2(c_1)) \otimes_A \alpha^2(\alpha^{-2}(d) \otimes \gamma(c_2^s))$$

$$2.6 \quad (\alpha^{-1}(b)a \otimes \gamma(c_1)) \alpha^2(\alpha^{-2}(d)) \otimes \gamma^2(c_2^s)) \otimes_A (1 \otimes \gamma^2(c_2^s))$$

$$= ((\alpha^{-2}(b)\alpha^{-1}(a)) \alpha(\alpha^{-2}(d)) \otimes \gamma^2(c_2^s)) \otimes_A (1 \otimes \gamma^2(c_2^s))$$

$$3.33 \quad (((\alpha^{-2}(b)\alpha^{-1}(a)) \alpha(\alpha^{-2}(d))) \otimes \gamma(c_1^s)) \otimes_A (1 \otimes \gamma^2(c_2^s))$$

$$= (\alpha^{-1}(b)a \otimes \gamma^2(c_1)) \otimes_A (\alpha^{-1}(d) \otimes \gamma^2(c_2^s))$$

$$3.33 \quad (\alpha^{-1}(b)a \otimes \gamma^2(c_1)) \otimes_A (\alpha^{-1}(d) \otimes \gamma^2(c_2^s)) \otimes_A (1 \otimes \gamma^2(c_2^s))$$

$$2.6 \quad (\alpha^{-1}(b)a \otimes \gamma^2(c_1)) \otimes_A (1 \otimes \gamma^2(c_2^s))$$

$$= \Delta_C((\alpha^{-1}(b)a \otimes \gamma^2(c_1)) \otimes_A (1 \otimes \gamma^2(c_2^s)))$$

$$= \Delta_C(b(a \otimes c))\alpha(d).$$

One easily checks that the counit $\varepsilon_C$ is both left and right $A$-linear. Take any $a, b, d \in A$ and $c \in C$ and compute...
\[
\varepsilon_C((b(a \otimes c))a(d)) = \varepsilon_C(b(\alpha^{-1}(d))) \otimes \gamma^2(c^*)
\]
\[
= \alpha(b(\alpha^{-1}(d)))\varepsilon(\gamma^2(c^*))
\]

\[\tag{2.10}\]
\[
\alpha(b)(\alpha(a)\alpha^{-1}(d))\varepsilon(c^*)
\]
\[
= \alpha(b)(\alpha(a)\alpha^{-1}(d))\varepsilon(c^*)
\]

\[\tag{3.27}\]
\[
\alpha(b)(\alpha(a)\alpha^{-1}(d))\varepsilon(c))
\]
\[
= \alpha(b)(\alpha(a)\varepsilon(c)d)
\]
\[
= \alpha(b)(\varepsilon_C(a \otimes c)d).
\]

This finishes the proof that \(\varepsilon_C\) is \(A\)-bilinear. By using the equalities
\[
\Delta_C(a \otimes c) = (a \otimes c)_1 \otimes_A (a \otimes c)_2 = (\alpha^{-1}(a \otimes c))_1 \otimes_A (1 \otimes c_2),
\]
\[
\Delta_C((a \otimes c)_1) = (a \otimes c)_{11} \otimes_A (a \otimes c)_{12} = (\alpha^{-2}(a \otimes c_{11})) \otimes_A (1 \otimes c_{12}),
\]
we get the following
\[
(\alpha^{-1} \otimes \gamma^{-1})((a \otimes c)_1) \otimes_A \Delta_C((a \otimes c)_2) = \alpha^{-2}(a) \otimes \gamma^{-1}(c) \otimes_A (1 \otimes c_{21}) \otimes_A (1 \otimes c_{22})
\]
\[
= \alpha^{-2}(a) \otimes (c_{11}) \otimes_A (1 \otimes c_{12}) \otimes_A (1 \otimes \gamma^{-1}(c))
\]
\[
= (a \otimes c)_{11} \otimes_A (a \otimes c)_{12} \otimes_A (\alpha^{-1} \otimes \gamma^{-1})((a \otimes c)_2),
\]

where in the second step the Hom-coassociativity of \((C, \gamma)\) is used.
\[
\varepsilon_C((a \otimes c)_1)(a \otimes c)_2 = \varepsilon_C((\alpha^{-1}(a \otimes c)))(1 \otimes c_2)
\]
\[
= \alpha^{-1}(a)\varepsilon(c)(1 \otimes c_2) = a(1 \otimes \varepsilon(c_1)c_2)
\]
\[
= a(1 \otimes \gamma^{-1}(c)) = a \otimes c,
\]
on the other hand we have
\[
(a \otimes c)_1\varepsilon_C((a \otimes c)_2) = (\alpha^{-1}(a) \otimes c_1)\alpha(1)\varepsilon(c_2)
\]
\[
= (\alpha^{-1}(a) \otimes c_1\varepsilon(c_2))1
\]
\[
= (\alpha^{-1}(a) \otimes \gamma^{-1}(c))1
\]
\[
= a \otimes c.
\]

We also show that the following relations
\[
\Delta_C(\alpha(a) \otimes \gamma(c)) = (\alpha^{-1}(\alpha(a)) \otimes \gamma(c))_1 \otimes_A (1 \otimes \gamma(c)_2)
\]
\[
= (\alpha^{-1}(\alpha(a)) \otimes \gamma(c)) \otimes_A (\alpha(1) \otimes \gamma(c)_2)
\]
\[
= ((\alpha \otimes \gamma) \otimes (\alpha \otimes \gamma))(\Delta_C(a \otimes c)),
\]
\[
\varepsilon_C(\alpha(a) \otimes \gamma(c)) = \alpha(\alpha(a))\varepsilon(\gamma(c))
\]
\[
= \alpha(\alpha(a))\varepsilon(c)
\]
\[
= \varepsilon_C(a \otimes c)
\]

hold, which completes the proof that \((A \otimes C, \alpha \otimes \gamma)\) is an \((A, \alpha)\)-Hom-coring.

(2) Let us denote \(\psi(c \otimes a) = (1 \otimes \gamma^{-1}(c))a = a_\kappa \otimes c^e\). \(\psi\) is in \(\tilde{H}(M_k)\):
\[
(\alpha \otimes \gamma)(\psi(c \otimes a)) = \alpha(a_\kappa) \otimes \gamma(c^e) = (\alpha \otimes \gamma)((1 \otimes \gamma^{-1}(c))a)
\]
\[
= ((\alpha(1) \otimes \gamma(\gamma^{-1}(c)))\alpha(a)) = (1 \otimes c)\alpha(a)
\]
\[
= (1 \otimes \gamma^{-1}(\gamma(c)))\alpha(a) = \alpha(a) \otimes \gamma(c)^e
\]
\[
= \psi(\gamma(c) \otimes a(a),
\]
where in the third equality the fact that the right Hom-action of \((A, \alpha)\) on \((A \otimes C, \alpha \otimes \gamma)\) is a morphism in \(\mathcal{H}(M_k)\) was used. Now, let \(a, a' \in A\) and \(c \in C\), then

\[
\psi(c \otimes aa') = (aa')_\kappa \otimes c^\kappa = (1 \otimes \gamma^{-1}(c))(aa')
\]

\[
= ((\alpha^{-1}(1) \otimes \gamma^{-1}(\gamma^{-1}(c)))a)\alpha(a') = ((1 \otimes \gamma^{-1}(\gamma^{-1}(c)))a)\alpha(a')
\]

\[
= (a_\kappa \otimes \gamma^{-1}(\gamma^{-1}(c))a' \otimes \gamma^{-1}(\gamma^{-1}(c)))(a_\kappa) = (a_\kappa \otimes \gamma^{-1}(\gamma^{-1}(c))a_\kappa \otimes \gamma^{-1}(\gamma^{-1}(c))a_\kappa)
\]

\[
= a_\kappa a_\kappa' \otimes \gamma(\gamma^{-1}(c))a_\kappa.
\]

In the above equality, if we replace \(c\) by \(\gamma(c)\) we obtain \((aa')_\kappa \otimes \gamma(c)^\kappa = a_\kappa a_\kappa' \otimes \gamma(c^\kappa)\).

Next, by using the right \(A\)-linearity of \(\Delta_C\) we prove the following

\[
\alpha^{-1}(a_\kappa \otimes c_1^\kappa \otimes c_2^\kappa) = \psi(c_1 \otimes \alpha^{-1}(a_\kappa) \otimes c_2^\kappa
\]

\[
= (1 \otimes \gamma^{-1}(c_1))\alpha^{-1}(a_\kappa) \otimes c_2^\kappa
\]

\[
= (1 \otimes \gamma^{-1}(c_1))\alpha^{-1}(a_\kappa) \otimes c_2^\kappa
\]

\[
= (id_{A \otimes C} \otimes id_A \otimes \gamma^{-1})((1 \otimes c_1) \otimes_A \psi(c_2 \otimes \alpha^{-1}(a)))
\]

\[
= (id_{A \otimes C} \otimes id_A \otimes \gamma^{-1})((1 \otimes c_1) \otimes_A (1 \otimes \gamma^{-1}(c_2))\alpha^{-1}(a))
\]

\[
= (id_{A \otimes C} \otimes id_A \otimes \gamma^{-1})(((\alpha^{-1}(1) \otimes \gamma^{-1}(c_1)) \otimes_A (1 \otimes \gamma^{-1}(c_2)))\alpha^{-1}(a))
\]

\[
\alpha^{-1}(a_\kappa) \otimes c_1^\kappa \otimes c_2^\kappa
\]

We also find

\[
\psi(c \otimes 1) = 1_\kappa \otimes c^\kappa = (1 \otimes \gamma^{-1}(c))1 = 1 \otimes c.
\]

Finally, the fact of \(\varepsilon_C\) being right \(A\)-linear gives

\[
\alpha(a_\kappa)\varepsilon(c^\kappa) = \varepsilon_C(a_\kappa \otimes c^\kappa) = \varepsilon_C((1 \otimes \gamma^{-1}(c))a)
\]

\[
= \varepsilon_C(1 \otimes \gamma^{-1}(c))a = \alpha(1)\varepsilon(\gamma^{-1}(c))a = 1a\varepsilon(c)
\]

\[
= \alpha(a)\varepsilon(c),
\]

which means that \(a_\kappa \varepsilon(c^\kappa) = a\varepsilon(c)\). Therefore \([(A, \alpha), (C, \gamma)]_\psi\) is a Hom-entwining structure.

(3) The essential point is that if \((M, \mu)\) is a right \((A, \alpha)\)-Hom-module, then \((M \otimes C, \mu \otimes \gamma)\) is a right \((A, \alpha)\)-Hom-module with the Hom-action \(\rho_{M \otimes C} : (M \otimes C) \otimes A \to M \otimes C, (m \otimes c) \otimes a \mapsto (m \otimes c)a = ma^{-1}(a_\kappa \otimes \gamma(c^\kappa), \rho_{M \otimes C}\) indeed satisfies Hom-associativity and Hom-unity as follows. For all \(m \in M, a, a' \in A\) and \(c \in C\),
\[(\mu(m) \otimes \gamma(c))(aa') = \mu(m) \alpha^{-1}(aa')_\kappa \otimes \gamma(\gamma(c)_\lambda) \quad \text{\textbf{3.24}}\]

\[
\begin{align*}
\mu(m)(\alpha^{-1}(a)_\kappa & \alpha^{-1}(a')_\lambda) \otimes \gamma(\gamma(c)_{e\lambda}) \\
= (\lambda \alpha^{-1}(a)_\kappa & \alpha^{-1}(a')_\lambda) \otimes \gamma(\gamma(c)_{e\lambda}) \\
= (\lambda \alpha^{-1}(a)_\kappa & \alpha(a')_\lambda) \\
= (m \otimes c & \alpha(a')),
\end{align*}
\]

\[(m \otimes c)1 = m\alpha^{-1}(1)_\kappa \otimes \gamma(c_\epsilon) = m1_\kappa \otimes \gamma(c) \quad \text{\textbf{3.26}}\]

\[
\equiv m1 \otimes \gamma(c) = \mu(m) \otimes \gamma(c).
\]

With respect to this Hom-action of \((A, \alpha)\) on \((M \otimes C, \mu \otimes \gamma)\), becoming an \([[A, \alpha], (C, \gamma)]_\psi\)-entwined Hom-module is equivalent to the fact that the Hom-coaction of \((C, \gamma)\) on \((M, \mu)\) is right \(A\)-linear.

Let \((M, \mu) \in \mathcal{M}_\psi^C(\psi)\) with the right \((C, \gamma)\)-Hom-comodule structure \(m \mapsto m_{(0)} \otimes m_{(1)}\). Then \((M, \mu) \in \mathcal{M}_\psi^C\) with the Hom-coaction \(\rho^M : M \rightarrow M \otimes_A C, m \mapsto m_{(0)} \otimes_A (1 \otimes \gamma^{-1}(m_{(1)})\)), which actually is

\[\rho^M(m) = m_{(0)} \otimes_A (1 \otimes \gamma^{-1}(m_{(1)})\) = \mu^{-1}(m)1 \otimes \gamma(\gamma^{-1}(m_{(1)})) = m_{(0)} \otimes m_{(1)},\]

where in the second equality we have used the canonical identification

\[
\phi : M \otimes_A (A \otimes C) \simeq M \otimes C, \quad m \otimes_A (a \otimes c) \mapsto \mu^{-1}(m)a \otimes \gamma(c),
\]

and \(\rho^M\) is \(A\)-linear since

\[\rho^M(ma) = (ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)} \alpha^{-1}(a)_\kappa \otimes \gamma(m_{(1)}\kappa) = (m_{(0)} \otimes m_{(1)})a.\]

Conversely, if \((M, \mu)\) is a right \((A \otimes C, \alpha \otimes \gamma)\)-Hom-comodule with the coaction \(\rho^M : M \rightarrow M \otimes_A (A \otimes C)\), by using the canonical identification above, one gets the \((C, \gamma)\)-Hom-comodule structure \(\hat{\rho}^M = \phi \circ \rho^M : M \rightarrow M \otimes C\) on \((M, \mu)\). One can also check that \(\phi\) is right \(A\)-linear once the following \((A, \alpha)\)-Hom-module structure on \(M \otimes_A C\) is given:

\[\rho_{M \otimes_A C} : (M \otimes_A C) \otimes A \rightarrow M \otimes A \otimes_C, \quad (m \otimes_A (a \otimes c)) \otimes a' \mapsto \mu(m) \otimes_A (a \otimes c) \alpha^{-1}(a'),\]

thus \(\hat{\rho}^M\) is \(A\)-linear since by definition \(\rho^M\) is \(A\)-linear. Therefore \((M, \mu)\) has an \([[A, \alpha], (C, \gamma)]_\psi\)-entwined Hom-module structure.

\[\square\]

**Theorem 3.9.** Let \([[A, \alpha], (C, \gamma)]_\psi\) be an entwining Hom-structure and \((C, \chi) = (A \otimes C, \alpha \otimes \gamma)\) be the associated \((A, \alpha)\)-Hom-coring. Then the so-called Koppinen smash or \(\psi\)-twisted convolution algebra \(\text{Hom}_\psi^H(C, A) = (\text{Hom}_\psi^H(C, A), *_{\psi}, \eta_\alpha \circ \varepsilon_C)\), where \((f \ast_{\psi} g)(c) = f(c_2)g(c_1^\kappa)\) for any \(f, g \in \text{Hom}_\psi^H(C, A)\), is anti-isomorphic to the algebra \((\varepsilon_C, \varepsilon_C, \varepsilon_C)\) in Proposition 3.9.

**Proof.** For \(f, g, h \in \text{Hom}_\psi^H(C, A)\) and \(c \in C\),
\[(f * \psi g) * \psi h)(c)\]
\[= (f * \psi g)(c_2)h(c_1^\kappa) = (f(c_{22})\lambda g(c_{21}^\lambda))h(c_1^\kappa)\]
\[= (f(c_{22})\lambda g(c_{21}^\lambda)h(\gamma^{-1}(c_1)^{\kappa\sigma})) = (f(c_{22})\lambda g(c_{21}^\lambda)\alpha(h(\gamma^{-1}(c_1)^{\kappa\sigma})))\]
\[= \alpha(f(c_{22})\kappa g(c_{21}^\lambda)h(\gamma^{-1}(c_1)^{\kappa\sigma})) = \alpha(f(c_{22})\kappa g(c_{21}^\lambda)h(\gamma^{-1}(c_1)^{\kappa\sigma}))\]
\[= f(c_2)\kappa(g(c_{12}^\lambda)\sigma h(c_{11}^\kappa)) = \alpha^{-1}(f(c_2))\kappa(g(c_{12}^\lambda)\sigma h(c_{11}^\kappa))\]
\[= f(c_2)\kappa(g(c_{12}^\lambda)\sigma h(c_{11}^\kappa))\]
\[= f(c_2)\kappa(g * \psi h)(c_1^\kappa)\]
\[= (f * \psi (g * \psi h))(c),\]
proving that *\psi is associative. Now we show that \(\eta\) is the unit for *\psi:

\[(\eta * \psi f)(c) = \eta(c_2)\kappa f(c_1^\kappa) = \varepsilon(c_2)(1_\kappa f(c_1^\kappa)) = 1_\kappa f(\gamma^{-1}(c)) = f(c)\]

The map \(\phi : \mathcal{C} = \mathcal{A}\text{Hom}^\mathcal{H}(A \otimes C, A) \rightarrow \text{Hom}^\mathcal{H}(C, A)\) given by

\[(3.34) \quad \phi(\xi)(c) = \xi(1 \otimes \gamma^{-1}(c))\]

for any \(\xi \in \mathcal{C}\) and \(c \in C\), is a \(k\)-module isomorphism with the inverse \(\varphi : \text{Hom}^\mathcal{H}(C, A) \rightarrow \mathcal{C}\) given by \(\varphi(f)(a \otimes c) = af(c)\) for all \(f \in \text{Hom}^\mathcal{H}(C, A)\) and \(a \otimes c \in A \otimes C\): Let \(a \in A, a' \otimes c \in A \otimes C\) and \(f \in \text{Hom}^\mathcal{H}(C, A)\). Then

\[\varphi(f)(a(a' \otimes c)) = \varphi(f)(a^{-1}(a)a' \otimes \gamma(c)) = (a^{-1}(a)a')f(\gamma(c))\]
\[= (a^{-1}(a)a')\alpha(f(c)) = a(a'f(c)) = a\varphi(f)(a' \otimes c)\]

and

\[\varphi(f)(a(a) \otimes \gamma(c)) = \alpha(a)f(\gamma(c)) = \alpha(af(c)) = \alpha(\varphi(f)(a) \otimes c),\]

showing that \(\varphi(f)\) is \((A, \alpha)\)-linear. On the other hand,

\[\varphi(\phi(\xi))(a \otimes c) = a\phi(\xi)(c) = a\xi(1 \otimes \gamma^{-1}(c)) = \xi(a(1 \otimes \gamma^{-1}(c))) = \xi(a \otimes c),\]

\[\phi(\varphi(f))(c) = \varphi(f)(1 \otimes \gamma^{-1}(c)) = 1f(\gamma^{-1}(c)) = f(c).\]

Now if we put \(\phi(\xi) = f\) and \(\phi(\xi') = f'\), we have \(f(c) = \xi(1 \otimes \gamma^{-1}(c))\), \(f'(c) = \xi'(1 \otimes \gamma^{-1}(c))\) for \(c \in C\), and then

\[(\xi * \xi')(a \otimes c) = \xi((a \otimes c)\xi'((a \otimes c)c))\]
\[= \xi((a \otimes a) \otimes c)(\xi(1 \otimes c))\]
\[= \xi((a^{-1}(a) \otimes c)\xi'(1 \otimes c))\]

which induces the following
\[
\phi(\xi \ast \xi')(c) = (\xi \ast \xi')(1 \otimes \gamma^{-1}(c)) = 1(f \ast f)(\gamma^{-1}(c))
= \alpha((f \ast f)(\gamma^{-1}(c))) = (f \ast f)(\gamma(\gamma^{-1}(c)))
= (f \ast f)(c) = (\phi(\xi') \ast \phi(\xi))(c).
\]
Moreover, \(\phi(\varepsilon_C)(c) = \varepsilon_C(1 \otimes \gamma^{-1}(c)) = \alpha(1)\varepsilon(\gamma^{-1}(c)) = \eta\varepsilon(c)\). Therefore \(\phi\) is the anti-isomorphism of the algebras \(^*C\) and \(\text{Hom}^H_C(C, A)\).

4. ENTWININGS AND HOM-HOPF-TYPE MODULES

**Definition 4.1.** Let \((B, \beta)\) be a monoidal Hom-bialgebra. A right \((B, \beta)\)-Hom-module coalgebra \((C, \gamma)\) is a monoidal Hom-coalgebra and a right \((B, \beta)\)-Hom-module with the \(\text{Hom}\)-action \(\rho: C \otimes B \to C, c \otimes b \mapsto cb\) such that \(\rho_C\) is a Hom-coalgebra morphism, that is, for any \(c \in C\) and \(b \in B\)

\[
(c b)_1 \otimes (c b)_2 = c_1 b_1 \otimes c_2 b_2, \quad \varepsilon_C(cb) = \varepsilon_C(c)\varepsilon_B(b),
\]

\[
\rho_C \circ (\gamma \otimes \beta) = \gamma \circ \rho_C.
\]

By the following construction, we show that a Hom-Doi-Koppenen datum comes from a Hom-entwining structure and that the Doi-Koppenen Hom-Hopf modules are the same as the associated entwined Hom-modules, and give the structure of Hom-coring corresponding to the relevant Hom-entwining structure.

**Proposition 4.2.** Let \((B, \beta)\) be a monoidal Hom-bialgebra. Let \((A, \alpha)\) be a right \((B, \beta)\)-Hom-comodule algebra with Hom-coaction \(\rho_A: A \to A \otimes B, a \mapsto a(0) \otimes a(1)\) and \((C, \gamma)\) be a right \((B, \beta)\)-Hom-module coalgebra with Hom-action \(\rho_C: C \otimes B \to C, c \otimes b \mapsto cb\). Define the morphism

\[
\psi: C \otimes A \to A \otimes C, \quad c \otimes a \mapsto \alpha(a(0)) \otimes \gamma^{-1}(c)a(1) = a_c \otimes e^c.
\]

Then the following assertions hold.

1. \([(A, \alpha), (C, \gamma)]_{\psi}\) is an Hom-entwining structure.
2. \((M, \mu)\) is an \([(A, \alpha), (C, \gamma)]_{\psi}\)-entwined Hom-module iff it is a right \((A, \alpha)\)-Hom-module with \(\rho_M: M \otimes A \to M, m \otimes a \mapsto ma\) and a right \((C, \gamma)\)-Hom-comodule with \(\rho^M: M \to M \otimes C, m \mapsto m(0) \otimes m(1)\) such that

\[
\rho^M(ma) = m(0)a(0) \otimes m(1)a(1)
\]

for any \(m \in M\) and \(a \in A\).
3. \((C, \chi) = (A \otimes C, \alpha \otimes \gamma)\) is an \((A, \alpha)\)-Hom-coring with comultiplication and counit given by \(\text{Eq.}2.13\) and \(\text{Eq.}2.11\), respectively, and it has the \((A, \alpha)\)-Hom-bimodule structure \(a(a' \otimes c) = \alpha^{-1}(a)a' \otimes \gamma(c), (a' \otimes c)a = a'a(0) \otimes ca(1)\) for \(a, a' \in A\) and \(c \in C\).
4. \(\text{Hom}^A(C, A)\) is an associative algebra with the unit \(\eta\varepsilon\) and the multiplication \(\ast\) defined by

\[
(f \ast g)(c) = \alpha(f(c(2))(0))g(\gamma^{-1}(c)f(c(2))(1)) = \alpha(f(c(2))(0))\alpha^{-1}(g(c(1)\alpha(f(c(2))(1)))),
\]

for all \(f, g \in \text{Hom}^A(C, A)\) and \(c \in C\).

**Proof.**
1. By \(\text{Eq.}4.36\) we have \(a_\kappa \otimes \gamma(c)^\kappa = \alpha(a(0)) \otimes ca(1)\), and thus

\[
(aa')_\kappa \otimes \gamma(c)^\kappa = \alpha((aa')(0)) \otimes c((aa')(1))
= \alpha(a(0)a'(0)) \otimes c(a(1)a'(1)) = \alpha(a(0))\alpha(a'(0)) \otimes (\gamma^{-1}(c)a(1))\beta(a'(1))
= a_\kappa a(0) \otimes c^\kappa \beta(a'(1))
= a_\kappa a(0)' \otimes \gamma(\gamma^{-1}(c)^\kappa)a(1')
= a_\kappa a(0)' \otimes \gamma(c^\kappa)a(1'),
\]

which shows that \( \psi \) satisfies (3.24). To prove that \( \psi \) fulfills (3.25) we have the computation

\[
\alpha^{-1}(a_\kappa) \otimes c_1^\kappa \otimes c_2^\kappa = \alpha^{-1}(\alpha(a_0)) \otimes (\gamma^{-1}(c)a_{1(0)})_1 \otimes (\gamma^{-1}(c)a_{1(0)})_2
\]

\[
= a_0 \otimes \gamma^{-1}(c_1)a_{1(0)} \otimes \gamma^{-1}(c_2)a_{1(0)} = a_0 \otimes \gamma^{-1}(c_1)a_{1(0)} \otimes \gamma^{-1}(c_2)a_{1(0)}
\]

\[
= \alpha(a_0) \otimes \gamma^{-1}(c_1)a_{0(0)} \otimes \gamma^{-1}(c_2)a_{0(0)}
\]

\[
= \alpha(a_0) \otimes c_1^\kappa \otimes \gamma^{-1}(c_2)a_{1(0)}
\]

\[
= \alpha^{-1}(a_{1(0)}) \otimes c_1^\kappa \otimes \gamma^{-1}(c_2)a_{1(0)}
\]

\[
= \alpha^{-1}(a_{1(0)}) \otimes c_1^\kappa \otimes c_2^\lambda.
\]

To finish the proof of (1) we finally verify that \( \psi \) satisfies (3.26) and (3.27) as follows,

\[
1_\kappa \otimes c^\kappa = \alpha(1_{0(0)}) \otimes \gamma^{-1}(c)1_{1(1)} = \alpha(1_A) \otimes \gamma^{-1}(c)1_B = 1 \otimes c,
\]

\[
a_\kappa \varepsilon(c^\kappa) = \alpha(a_0) \varepsilon(\gamma^{-1}(c)a_{1(0)}) = \alpha(a_0) \varepsilon(\gamma^{-1}(c_1)a_{1(0)}))
\]

\[
= \alpha(a_0) \varepsilon(c_1a_{1(0)}) \otimes c_1^\kappa \otimes \gamma^{-1}(c_2)a_{1(0)}
\]

\[
= \alpha(a_0) \varepsilon_B(a_{1(0)}) \varepsilon(c) = \alpha(a_0) \varepsilon_B(a_{1(0)}) \varepsilon(c)
\]

\[
= a \varepsilon(c).
\]

(2) We see that the condition for entwined Hom-modules, i.e., \( \rho^M(ma) = m_0(\alpha^{-1}(a_\kappa) \otimes \gamma(m_{1(\kappa)}) \right. \) and the condition in (3.37) are equivalent by the following, for \( m \in M \) and \( a \in A, \)

\[
m_{0(0)} \alpha^{-1}(a_\kappa) \otimes \gamma(m_{1(\kappa)}) = m_{0(0)} \alpha^{-1}(a_0(0)) \otimes \gamma^{-1}(m_{1(1)}) \alpha^{-1}(a_{1(1)})
\]

\[
= m_{0(0)} \alpha^{-1}(a_0(0)) \otimes \gamma^{-1}(m_{1(1)}) \alpha^{-1}(a_{1(1)})
\]

\[
= m_{0(0)} \alpha^{-1}(a_0(0)) \otimes \gamma^{-1}(m_{1(1)}) \alpha^{-1}(a_{1(1)})
\]

\[
= m_{0(0)} \alpha^{-1}(a_0(0)) \otimes \gamma^{-1}(m_{1(1)}) \alpha^{-1}(a_{1(1)})
\]

(3) We only prove that the right \((A, \alpha)\)-Hom-module structure holds as is given in the assertion. The rest of the structure of the corresponding Hom-coring can be seen at once from Theorem (3.88). For \( a, a' \in A \) and \( c \in C, \)

\[
(a' \otimes c)a = a' \alpha^{-1}(a) \kappa \otimes \gamma(c^\kappa)
\]

\[
= a' \alpha^{-1}(a_0(0)) \otimes \gamma^{-1}(c_1)a_{1(0)} = a' a_0(0) \otimes \gamma^{-1}(c_1)a_{1(0)}
\]

\[
= a' a_0(0) \otimes c a_{1(0)}.
\]

(4) By the definition of product \( * \) given in Theorem (3.39) and the definition of \( \psi \) given in (3.36) we have, for \( f, g \in Hom^R(C, A) \) and \( c \in C, \)

\[
(f * g)(c) = f(c_2) \gamma(c_1^\kappa)
\]

\[
= \alpha(f(c_2)) \gamma(c_1^\kappa) = \alpha(f(c_2)) \gamma(c_1^\kappa)
\]

\[
= \alpha(f(c_2)) \alpha^{-1}(g(c_1^\kappa))(f(c_2))_{(1)} = \alpha(f(c_2)) \alpha^{-1}(g(c_1^\kappa))(f(c_2))_{(1)}
\]

\[
= \alpha(f(c_2))_{(0)} \alpha^{-1}(g(c_1^\kappa))(f(c_2))_{(1)}
\]

\[
= \alpha(f(c_2))_{(0)} \alpha^{-1}(g(c_1^\kappa))(f(c_2))_{(1)}.
\]

\[\boxed{}\]
Definition 4.3. A triple \([\mathcal{(A,\alpha),(B,\beta),(C,\gamma)}]\) is called a \((\text{right-right})\) Hom-Doi-Koppinen datum if it satisfies the conditions of Proposition \([4.2]\), that is, if \((A,\alpha)\) is a right \((B,\beta)\)-Hom-comodule algebra and \((C,\gamma)\) is a right \((B,\beta)\)-Hom-module coalgebra for a monoidal Hom-bialgebra \((B,\beta)\).

\([\mathcal{(A,\alpha),(C,\gamma)}]\) in Proposition \([4.2]\) is called a Hom-entwining structure associated to a Hom-Doi-Koppinen datum.

A Doi-Koppinen Hom-Hopf module or a unifying Hom-Hopf module is a Hom-module satisfying the condition \([4.3]\).

Now we give the following collection of examples. Each of them is a special case of the construction given above.

Example 4.4. Hom-bialgebra entwinings and Hom-Hopf modules Let \((B,\beta)\) be a monoidal Hom-bialgebra with Hom-multiplication \(m_B: B \otimes B \rightarrow B\), \(b \otimes b' \mapsto bb'\) and Hom-commultiplication \(\Delta_B: B \rightarrow B \otimes B\), \(b \mapsto b_1 \otimes b_2\).

1. \(((B,\beta),(B,\beta))\)\(_{\psi}\), with \(\psi: B \otimes B \rightarrow B \otimes B\), \(b \otimes b' \mapsto \beta(b_1) \otimes \beta^{-1}(b')b_2\), is an Hom-entwining structure.

2. \(((M,\mu))\) is an \(((B,\beta),(B,\beta))\)\(_{\psi}\)-entwined Hom-module iff it is a right \((B,\beta)\)-Hom-module with \(\rho_M: M \otimes B \rightarrow M\), \(m \otimes b \mapsto mb\) and a right \((B,\beta)\)-Hom-comodule with \(\rho^M: M \rightarrow M \otimes B, m \mapsto m_{(0)} \otimes m_{(1)}\) such that

\[
\rho^M(mb) = m_{(0)}b_1 \otimes m_{(1)}b_2
\]

for all \(m \in M\) and \(b \in B\). Such Hom-modules are called Hom-Hopf modules.

3. \(((C,\chi)) = ((B\otimes B,\beta)\otimes\beta)\) is a \((B,\beta)\)-Hom-coring with comultiplication \(\Delta_C(b \otimes b') = (\beta^{-1}(b) \otimes b'_1) \otimes_B (1_B \otimes b'_2)\) and counit \(\varepsilon_C(b \otimes b') = \beta(b)\varepsilon_B(b')\), and \((B,\beta)\)-Hom-bimodule structure

\[
b(b' \otimes b'') = \beta^{-1}(b'b' \otimes \beta(b'), (b' \otimes b'')b = b'b_1 \otimes b'b_2
\]

for all \(b, b', b'' \in B\).

Proof. Since \(\Delta_B\) is a Hom-algebra morphism, \((B,\beta)\) is a right \((B,\beta)\)-Hom-comodule algebra with Hom-coaction

\[
\rho^B = \Delta_B: B \rightarrow B \otimes B, b \mapsto b_1 \otimes b_2,
\]

and since \(m_B\) is a Hom-coalgebra morphism, \((B,\beta)\) is a right \((B,\beta)\)-Hom-module coalgebra with Hom-action \(\rho_B = m_B: B \otimes B \rightarrow B, b \otimes b' \mapsto bb'\). So, we have the triple \(\mathcal{[(B,\beta),(B,\beta),(B,\beta)]}\) as Hom-Doi-Koppinen datum, and the associated Hom-entwining structure is \(\mathcal{[(B,\beta),(B,\beta)]}_{\psi}\), where \(\psi(b' \otimes b) = \beta(b_{(0)}) \otimes \beta^{-1}(b')b_{(1)} = \beta(b_1) \otimes \beta^{-1}(b')b_2\). The rest of the assertions are immediately obtained by the above proposition.

Example 4.5. Relative entwinings and relative Hom-Hopf modules Let \((B,\beta)\) be a monoidal Hom-bialgebra and let \((A,\alpha)\) be a \((B,\beta)\)-Hom-comodule algebra with Hom-coaction \(\rho^A: A \rightarrow A \otimes B, a \mapsto a_{(0)} \otimes a_{(1)}\).

1. \(\mathcal{[(A,\alpha),(B,\beta)]}_{\psi}\), with \(\psi: B \otimes A \rightarrow A \otimes B, b \otimes a \mapsto \alpha(a_{(0)}) \otimes \beta^{-1}(b)a_{(1)}\), is an Hom-entwining structure.

2. \(\mathcal{[(M,\mu)]}_{\psi}\)-entwined Hom-module iff it is a right \((A,\alpha)\)-Hom-module with \(\rho_M: M \otimes A \rightarrow M\), \(m \otimes a \mapsto ma\) and a right \((B,\beta)\)-Hom-comodule with \(\rho^M: M \rightarrow M \otimes A, m \mapsto m_{(0)} \otimes m_{(1)}\) such that

\[
\rho^M(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}
\]

for all \(m \in M\) and \(a \in A\). Hom-modules fulfilling the above condition are called relative Hom-Hopf modules.

3. \(\mathcal{[(C,\chi)]} = ((A \otimes B,\alpha)\otimes\beta)\) is a \((A,\alpha)\)-Hom-coring with comultiplication \(\Delta_C(a \otimes b) = (\alpha^{-1}(a) \otimes b_1) \otimes_A (1_A \otimes b_2)\) and counit \(\varepsilon_C(a \otimes b) = \alpha(a)\varepsilon_B(b)\), and \((A,\alpha)\)-Hom-bimodule structure

\[
a(a' \otimes b) = \alpha^{-1}(a)a' \otimes \beta(b), (a' \otimes b)a = a'a_{(0)} \otimes ba_{(1)}
\]

for all \(a, a' \in A\) and \(b \in B\).
Proof. The relevant Hom-Doi-Koppinen datum is $[(A, \alpha), (B, \beta), (B, \beta)]$, where the first object $(A, \alpha)$ is assumed to be a right $(B, \beta)$-Hom-comodule algebra with the Hom-coaction $\rho^A : a \mapsto a_{(0)} \otimes a_{(1)}$ and the third object $(B, \beta)$ is a right $(B, \beta)$-Hom-module coalgebra with Hom-action given by its Hom-multiplication. Hence, $[(A, \alpha), (B, \beta)]_\psi$ is the associated Hom-entwining structure, where $\psi(b \otimes a) = \alpha(a_{(0)}) \otimes \beta^{-1}(b)a_{(1)}$. Assertions (2) and (3) can be seen at once from Proposition 4.2. □

Remark 4.6. $(A, \alpha)$ itself is a relative Hom-Hopf-module by its Hom-multiplication and the $(B, \beta)$-Hom-coaction $\rho^B$.

Example 4.7. Dual-relative entwinings and $[(C, \gamma), (A, \alpha)]$-Hom-Hopf modules Let $(A, \alpha)$ be a monoidal Hom-bialgebra and let $(C, \gamma)$ be a right $(A, \alpha)$-Hom-module coalgebra with Hom-action $\rho_C : C \otimes A \to C, c \otimes a \mapsto ca$.

(1) $[(A, \alpha), (C, \gamma)]_\psi$, with $\psi : C \otimes A \to A \otimes C, c \otimes a \mapsto \alpha(a_1) \otimes \beta^{-1}(c)a_2$, is an Hom-entwining structure.

(2) $(M, \mu)$ is an $[(A, \alpha), (C, \gamma)]_\psi$-entwined Hom-module iff it is a right $(A, \alpha)$-Hom-module with $\rho_M : M \otimes A \to M, m \otimes a \mapsto ma$ and a right $(C, \gamma)$-Hom-comodule with $\rho^M : M \to M \otimes B, m \mapsto m_{(0)} \otimes m_{(1)}$ such that

$$\rho^M(ma) = m_{(0)}a_1 \otimes m_{(1)}a_2$$

for all $m \in M$ and $a \in A$. Such a Hom-module is called $[(C, \gamma), (A, \alpha)]$-Hom-Hopf module.

(3) $(C, \chi) = (A \otimes C, \alpha \otimes \gamma)$ is a $(A, \alpha)$-Hom-coring with comultiplication $\Delta_C(a \otimes c) = (a^{-1}(a) \otimes c_1) \otimes (1_a \otimes c_2)$ and counit $\varepsilon_C(a \otimes b) = \alpha(a)\varepsilon_C(c)$, and $(A, \alpha)$-Hom-bimodule structure

$$a(a' \otimes b) = \alpha(a)\alpha(a') \otimes \gamma(c), \quad (a' \otimes c)a = a'a_1 \otimes ca_2$$

for all $a, a' \in A$ and $c \in C$.

Proof. $(A, \alpha)$ is a right $(A, \alpha)$-Hom-comodule algebra with Hom-coaction given by the Hom-comultiplication

$$\rho^A = \Delta_A : A \to A \otimes A, \quad a \mapsto a_{(0)} \otimes a_{(1)} = a_1 \otimes a_2,$$

since $\Delta_A$ is a Hom-algebra morphism. Besides $(C, \gamma)$ is assumed to be a right $(A, \alpha)$-Hom-module coalgebra with Hom-action $\rho_C(c \otimes a) = ca$. Thus, the related Hom-Doi-Koppinen datum is $[(A, \alpha), (A, \alpha), (C, \gamma)]$. Then $[(A, \alpha), (C, \gamma)]_\psi$ is the Hom-entwining structure associated to the datum, where

$$\psi(c \otimes a) = \alpha(a_{(0)}) \otimes \gamma^{-1}(c)a_{(1)} = \alpha(a_1) \otimes \gamma^{-1}(c)a_2.$$

The assertions (2) and (3) are also immediate by Proposition 4.2. □

Remark 4.8. $(C, \gamma)$ itself is a $[(C, \gamma), (A, \alpha)]$-Hom-Hopf-module by the $(A, \alpha)$-Hom-action $\rho_C$ and its Hom-comultiplication.

Example 4.9. Generalized Yetter-Drinfeld entwinings and $\phi, \varphi$-Hom-Yetter-Drinfeld modules Let $(H, \alpha)$ be a monoidal Hom-Hopf algebra and let $\phi, \varphi : H \to H$ be two monoidal Hom-Hopf algebra automorphisms. Define the map, for all $h, g \in H$

$$\psi : H \otimes H \to H \otimes H, \quad g \otimes h \mapsto \alpha^2(h_{(2)}) \otimes \varphi(S(h_1))((\alpha^{-2}(g)\phi(h_{(2)}),$$

where $S$ is the antipode of $H$.

(1) $[(H, \alpha), (H, \alpha)]_\psi$ is an Hom-entwining structure.

(2) $(M, \mu)$ is an $[(H, \alpha), (H, \alpha)]_\psi$-entwined Hom-module iff it is a right $(H, \alpha)$-Hom-module with $\rho_M : M \otimes H \to M, m \otimes h \mapsto mh$ and a right $(H, \alpha)$-Hom-comodule with $\rho^M : M \to M \otimes H, m \mapsto m_{(0)} \otimes m_{(1)}$ such that

$$\rho^M(mh) = m_{(0)}\alpha(h_{(2)}) \otimes \varphi(S(h_1))((\alpha^{-1}(m_{(1)})\phi(h_{(2)})$$

for all $m \in M$ and $h \in H$. A Hom-module $(M, \mu)$ satisfying this condition is called $(\phi, \varphi)$-Hom-Yetter-Drinfeld module.
(3) \((C, \chi) = (H \otimes H, \alpha \otimes \alpha)\) is an \((H, \alpha)\)-Hom-coring with comultiplication \(\Delta_C(h \otimes h') = (\alpha^{-1}(h) \otimes h') \otimes_H (1_H \otimes h')\) and counit \(\varepsilon_C(h \otimes h') = \alpha(h)\varepsilon_H(h')\), and \((H, \alpha)\)-Hom-bimodule structure
\[
g(h \otimes h') = \alpha^{-1}(g)h \otimes \alpha(h'), \quad (h \otimes h')g = h \alpha(g_{21}) \otimes \varphi(S(g_1))(\alpha^{-1}(h')\phi(g_{22}))
\]
for all \(h, h', g \in H\).

**Proof.** In the first place, we prove that the map
\[
\rho^H : H \to H \otimes (H^{op} \otimes H), \quad h \mapsto h_{(0)} \otimes h_{(1)} := \alpha(h_{21}) \otimes (\alpha^{-1}(\varphi(S(h_1)))) \otimes h_{22}
\]
defines a \((H^{op} \otimes H, \alpha \otimes \alpha)\)-Hom-comodule algebra structure on \((H, \alpha)\). Let us put \((H^{op} \otimes H, \alpha \otimes \alpha) = (\tilde{H}, \tilde{\alpha})\). Then

\[
h_{(0)(0)} \otimes (h_{(0)(1)}) \otimes \tilde{\alpha}^{-1}(h_{(1)})
\]
\[
= \alpha(\alpha(h_{21})) \otimes ((\alpha^{-1}(\varphi(S(h_{21})))) \otimes \alpha(h_{22})) \otimes (\alpha^{-1}(\varphi(S(h_1)))) \otimes \alpha^{-1}(h_{22}))
\]
\[
= \alpha^2(h_{2121}) \otimes ((\alpha^{-1}(\varphi(S(h_{211})))) \otimes \alpha(h_{2122})) \otimes (\alpha^{-1}(\varphi(S(h_1)))) \otimes \alpha^{-1}(h_{22}))
\]
\[
= \alpha^2(h_{2121}) \otimes ((\varphi(S(h_{211})))) \otimes \alpha(h_{2122})) \otimes (\alpha^{-1}(\varphi(S(h_1)))) \otimes \alpha^{-1}(h_{22}))
\]
\[
= h_{21} \otimes ((\alpha^{-1}(\varphi(S(h_{12})))) \otimes h_{221}) \otimes (\alpha^{-1}(\varphi(S(h_{11})))) \otimes h_{222})
\]
\[
= h_{21} \otimes ((\alpha^{-1}(\varphi(S(h_{11})))) \otimes (\alpha^{-1}(\varphi(S(h_{11})))) \otimes h_{221}) \otimes (\alpha^{-1}(\varphi(S(h_{12})))) \otimes h_{222})
\]
\[
= \alpha^{-1}(h_{(0)}) \otimes \Delta_{\tilde{H}}(h_{(1)}),
\]
where in the fourth step we used
\[
\alpha(h_{11}) \otimes \alpha^{-1}(h_{12}) \otimes \alpha^{-1}(h_{21}) \otimes \alpha^{-1}(h_{22}) = h_1 \otimes h_{211} \otimes h_{212} \otimes h_{222},
\]
which can be obtained by applying the Hom-coassociativity of \(\Delta_H\) three times. We also have

\[
h_{(0)}\varepsilon_{\tilde{H}}(h_{(0)}) = \alpha(h_{21})\varepsilon(\alpha^{-1}(\varphi(S(h_1))))\varepsilon(h_{22})
\]
\[
= \alpha(h_{21})\varepsilon(h_{22})\varepsilon(\alpha^{-1}(\varphi(S(h_1)))) = \alpha(\alpha^{-1}(h_2))\varepsilon(h_1)
\]
\[
= \alpha^{-1}(h),
\]
where in the third equality we used the relations \(\varepsilon \circ \alpha^{-1} = \varepsilon, \varepsilon \circ \varphi = \varepsilon\) and \(\varepsilon \circ S = \varepsilon\). One can easily check that the relations \(\rho^H \circ \alpha = (\alpha \otimes \tilde{\alpha}) \circ \rho^H\) and \(\rho^H(1_H) = 1_H \otimes 1_{\tilde{H}}\) hold. For \(g, h \in H\),

\[
\rho^H(g)\rho^H(h) = (\alpha(g_{21}) \otimes (\alpha^{-1}(\varphi(S(g_1)))) \otimes g_{22}))\alpha(h_{21}) \otimes (\alpha^{-1}(\varphi(S(h_1)))) \otimes h_{22}))
\]
\[
= \alpha(g_{21})\alpha(h_{21}) \otimes (\alpha^{-1}(\varphi(S(h_1))))\alpha^{-1}(\varphi(S(g_1)))) \otimes g_{22}h_{22})
\]
\[
= \alpha(g_{21}h_{21}) \otimes (\alpha^{-1}(\varphi(S(h_1)S(g_{11})))) \otimes g_{22}h_{22})
\]
\[
= \alpha((gh)_{21}) \otimes (\alpha^{-1}(\varphi(S((gh)_{11})))) \otimes (gh)_{22})
\]
\[
= \rho^H((gh),
\]
which completes the proof of the statement that \(\rho^H\) makes \((H, \alpha)\) an \((\tilde{H}, \tilde{\alpha})\)-Hom-comodule algebra. We next consider the map, for all \(g, h, k \in \tilde{H}\)

\[
\rho_H : H \otimes \tilde{H} \to H, \quad g \cdot (h \otimes k) := (h\alpha^{-1}(g))\phi(\alpha(k))
\]
and we claim that it defines an \((\tilde{H}, \tilde{\alpha})\)-Hom-module coalgebra structure on \((H, \alpha)\): Indeed,
\[(g \cdot (h \otimes k)) \cdot (\alpha(h') \otimes \alpha(k')) = ((h\omega^{-1}(g))\phi(\alpha(k))) \cdot (\alpha(h') \otimes \alpha(k')) \]
\[= (\alpha(h'))((\alpha^{-1}(h)\omega^{-2}(g))\alpha^{-1}(\phi(\alpha(k))))\phi(\alpha^2(k')) \]
\[= (\alpha(h'))((\alpha^{-1}(h)\omega^{-2}(g))\phi(\alpha^2(k'))) \]
\[= (h'\omega^{-2}(h)\alpha^{-1}(g))\phi(\alpha^2(k')) \]
\[= (\alpha^{-1}(h')\omega^{-1}(g))\phi(\alpha^2(k')) \]
\[= (\alpha^{-1}(h')\omega^{-1}(g))((\alpha^2(k')) = ((h'\omega^{-1}(h)\phi(\alpha(k)))\phi(\alpha(k'))) \]
\[= (h'\omega^{-1}(h)\phi(\alpha((kk')) = ((h'\omega^{-1}(h)\phi(\alpha((kk')) \)
\[= \alpha(g) \cdot (h'h \otimes kk') = \alpha(g) \cdot ((h \otimes k)(h' \otimes k')), \]
\[h \cdot (1_H \otimes 1_H) = (1_H\omega^{-1}(h))\phi(\alpha(1_H)) = \alpha(h), \]

\[
\varepsilon(g \cdot (h \otimes k)) = \varepsilon((h\omega^{-1}(g))\phi(\alpha(k))) = \varepsilon(h)\varepsilon(\alpha^{-1}(g))\varepsilon(\phi(\alpha(k))) = \varepsilon(h)\varepsilon(g)\varepsilon(k) = \varepsilon(h)\varepsilon_H(g \otimes k),
\]
proving that \((H, \alpha)\) is an \((\hat{H}, \hat{\alpha})\)-Hom-module coalgebra with the Hom-action \(\rho_H\). Hence, the Hom-Doi-Koppinen datum is given by \([H, \alpha), (H^{\text{op}} \otimes H, \alpha \otimes \alpha), (H, \alpha)\] to which the Hom-entwining structure \([H, \alpha), (H, \alpha)]_{\psi}\) is associated, where we have the entwining map \(\psi : H \otimes H \to H \otimes H\) as

\[
\psi(g \otimes h) = \alpha(h(0))\alpha^{-1}(g) \cdot h(1) = \alpha(h(21)) \otimes \alpha^{-1}(g) \cdot (\alpha^{-1}(\varphi(S(h_1))) \otimes h_{22}) \]
\[= \alpha^2(h_{21}) \otimes (\alpha^{-1}(\varphi(S(h_1)))\alpha^{-2}(g))\phi(\alpha(h_{22})) \]
\[= \alpha^2(h_{21}) \otimes \varphi(S(h_1))((\alpha^{-2}(g)\phi(h_{22})).
\]

For \(m \in M\) and \(h \in H\), we have the condition (4.43)

\[
\rho^M(mh) = m(0)h(0) \otimes m(1) \cdot h(1) \]
\[= m(0)\alpha(h(21)) \otimes m(1) \cdot (\alpha^{-1}(\varphi(S(h_1))) \otimes h_{22}) \]
\[= m(0)\alpha(h_{21}) \otimes \alpha^{-1}(\varphi(S(h_1)))m(1)\phi(\alpha(h_{22})) \]
\[= m(0)\alpha(h_{21}) \otimes \varphi(S(h_1))((\alpha^{-1}(m(1))\phi(h_{22})).
\]
By the above proposition, the \((H, \alpha)\)-Hom-coring structure of \((H \otimes H, \alpha \otimes \alpha)\) is immediate. Here we only write down the right Hom-module condition

\[
(h \otimes h')g = hg(0) \otimes h' \cdot g(1) \]
\[= \alpha(0g(21)) \otimes h' \cdot (\alpha^{-1}(\varphi(g(1))) \otimes g_{22}) \]
\[= \alpha(g(21)) \otimes \varphi(S(g(1))((\alpha^{-1}(h')\phi(g_{22})),
\]
completing the proof. \(\square\)

**Remark 4.10.** (1) By putting \(\phi = id_H = \varphi\) in the compatibility condition (4.43) we get the usual condition for (right-right) Hom-Yetter-Drinfeld modules, which is

\[\rho^M(mh) = m(0)\alpha(h(21)) \otimes S(h_1)(\alpha^{-1}(m(1))h_{22}).\]
Proposition 4.11. Assume that (4.46) holds, then
\[ \rho^M(mh) = m_{(0)}\alpha(h_{21}) \otimes S^{-1}(h_1)(\alpha^{-1}(m_{(1)})h_{22}). \]

We get an equivalent condition for the generalized Hom-Yetter-Drinfeld modules by the following

**Proposition 4.11.** The compatibility condition (4.43) for (\(\phi, \varphi\))-Hom-Yetter-Drinfeld modules is equivalent to the equation
\[ m_{(0)}\alpha^{-1}(h_1) \otimes m_{(1)}\phi(\alpha^{-1}(h_2)) = (mh_2)_{(0)} \otimes \alpha^{-1}(\varphi(h_1)(mh_2)_{(1)}). \]

**Proof.** Assume that (4.46) holds, then
\[
\begin{align*}
m_{(0)}\alpha(h_{21}) \otimes \varphi(S(h_1))&(\alpha^{-1}(m_{(1)})\phi(h_{22})) \\
&= m_{(0)}\alpha^{-1}(\alpha^2(h_{21})) \otimes \varphi(S(h_1))(\alpha^{-1}(m_{(1)})\alpha^{-2}(\phi(\alpha^2(h_{22})))) \\
&= m_{(0)}\alpha^{-1}(\alpha^2(h_{21})_1) \otimes \varphi(S(h_1))\alpha^{-1}(m_{(1)})\alpha^{-2}(\phi(\alpha^2(h_{22})_2))) \\
&= (m\alpha^2(h_{21}))(\alpha^{-2}(\alpha^{-2}(\alpha^2(h_{21})))) \otimes \varphi(S(h_1))(\alpha^{-2}(m\alpha^2(h_{22}))_{(1)})) \\
&= (m\alpha^2(h_{21}))(\alpha^{-2}(\alpha^{-2}(\alpha^2(h_{21})))) \otimes \varphi(S(h_1))(\alpha^{-2}(m\alpha^2(h_{22}))_{(1)})) \\
&= \varepsilon(h_1)(m\alpha^2(h_{21}))(\alpha^{-2}(m\alpha^2(h_{22}))_{(1)}) \\
&= \varepsilon(h_1)\rho^M(m\alpha^2(h_{21})) = \rho^M(mh),
\end{align*}
\]

which gives us (4.43). One can easily show that by applying the Hom-coassociativity condition twice we have
\[ \alpha^{-1}(h_1) \otimes h_{21} \otimes \alpha(h_{221}) \otimes \alpha(h_{222}) = h_{11} \otimes h_{12} \otimes h_{21} \otimes h_{22}, \]
which is used in the below computation. Thus, if we suppose that (4.48) holds, then
\[ (mh_2)_{(0)} \otimes \alpha^{-1}(\varphi(h_1)(mh_2)_{(1)}), \]

finishing the proof.

**Remark 4.12.** The above result implies that the equations (4.36) and (4.43) are equivalent to
\[ m_{(0)}\alpha^{-1}(h_1) \otimes m_{(1)}\alpha^{-1}(h_2) = (mh_2)_{(0)} \otimes \alpha^{-1}(h_1(mh_2)_{(1)}) \]

and
\[ m_{(0)}\alpha^{-1}(h_1) \otimes m_{(1)}\alpha^{-1}(h_2) = (mh_2)_{(0)} \otimes \alpha^{-1}(S^{-2}(h_1)(mh_2)_{(1)}), \]

respectively.

**Example 4.13.** The flip and Hom-Long dimodule Let \((H, \alpha)\) be a monoidal Hom-bialgebra. Then:

1. \( [(H, \alpha), (H, \alpha)]_{\psi}, \) where \(\psi : H \otimes H \to H \otimes H, g \otimes h \mapsto h \otimes g,\) is an Hom-entwining structure.
(2) \((M, \mu)\) is an \([H, \alpha], (H, \alpha)\)_\(\psi\)-entwined Hom-module iff it is a right \((H, \alpha)\)-Hom-module with \(\rho_M : M \otimes H \to M, m \otimes h \mapsto mh\) and a right \((H, \alpha)\)-Hom-comodule with \(\rho^M : M \to M \otimes H, m \mapsto m_{(0)} \otimes m_{(1)}\) such that

\[
\rho^M(mh) = m_{(0)} a^{-1}(h) \otimes \alpha(m_{(1)})
\]

for all \(m \in M\) and \(h \in H\). Such Hom-modules \((M, \mu)\) are called (right-right) \((H, \alpha)\)-Hom-Long dimodules.

(3) \((C, \chi) = (H \otimes H, \alpha \otimes \alpha)\) is an \((H, \alpha)\)-Hom-coring with comultiplication \(\Delta_C(h \otimes h') = (\alpha^{-1}(h)h_1) \otimes_H (1_H \otimes h_2')\) and counit \(\varepsilon_C(h \otimes h') = \alpha(h) \varepsilon_H(h')\), and \((H, \alpha)\)-Hom-bimodule structure

\[
g(h \otimes h') = \alpha^{-1}(g)h \otimes \alpha(h'), \ (h \otimes h')g = h \alpha^{-1}(g) \otimes \alpha(h')
\]

for all \(h, h', g \in H\).

**Proof.** \((H, \alpha)\) itself is a right \((H, \alpha)\)-Hom-comodule algebra with Hom-coaction \(\rho^H = \Delta_H : H \to H \otimes H, h \mapsto h_{(0)} \otimes h_{(1)} = h_1 \otimes h_2\). In addition, \((H, \alpha)\) becomes a right \((H, \alpha)\)-Hom-module coalgebra with the trivial Hom-Hom-action \(\rho_H : H \otimes H \to H, g \otimes h \mapsto g \cdot h = \alpha(g) \varepsilon(h)\). Hence we have \([H, \alpha], (H, \alpha), (H, \alpha)\] as Hom-Doi-Koppinen datum with the associated Hom-entwining structure \([\psi(\rho), (H, \alpha)\] \(\psi\), where \(\psi(h') \otimes h) = \alpha(h_{(0)}) \otimes \alpha^{-1}(h') \cdot h_{(1)} = \alpha(h_1) \otimes \alpha^{-1}(h') \cdot h_2 = \alpha(h_1) \otimes (\alpha^{-1}(h') \varepsilon(h_2)) \varepsilon(h_2) = h \otimes h'\). \(\square\)

**Definition 4.14.** Let \((B, \beta)\) be a monoidal Hom-bialgebra. A left \((B, \beta)\)-Hom-module coalgebra \((C, \gamma)\) is a monoidal Hom-coalgebra and a left \((B, \beta)\)-Hom-comodule with a Hom-coaction \(\rho : C \to B \otimes C, c \mapsto c_{(-1)} \otimes c_{(0)}\) such that, for any \(c \in C\)

\[
c_{(-1)} \otimes c_{(0)} = c_{(-1)} \otimes c_{(0)}, c_{(-1)} \otimes c_{(0)} = c_{(-1)} \varepsilon_C(c_{(0)}) = 1_B \varepsilon_C(c),
\]

\[
\rho \circ \gamma = (\beta \otimes \gamma) \circ \rho.
\]

We lastly introduce the below construction regarding the Hom-version of the so-called alternative Doi-Koppinen datum.

**Proposition 4.15.** Let \((B, \beta)\) be a monoidal Hom-bialgebra. Let \((A, \alpha)\) be a left \((B, \beta)\)-Hom-module algebra with Hom-action \(A \otimes \alpha : B \otimes A \to A, b \otimes a \mapsto b \cdot a\) and \((C, \gamma)\) be a left \((B, \beta)\)-Hom-comodule coalgebra with Hom-coaction \(\gamma^C : C \to B \otimes C, c \mapsto c_{(-1)} \otimes c_{(0)}\). Define the map

\[
\psi : C \otimes A \to A \otimes C, c \otimes a \mapsto c_{(-1)} \cdot \alpha^{-1}(a) \otimes \gamma(c_{(0)})
\]

Then the following statements hold.

1. \([[(A, \alpha), (C, \gamma)]]\) is an Hom-entwining structure.
2. \((M, \mu)\) is an \([A, \alpha], (C, \gamma)\)_\(\psi\)-entwined Hom-module iff it is a right \((A, \alpha)\)-Hom-module with \(\rho_M : M \otimes A \to M, m \otimes a \mapsto ma\) and a right \((C, \gamma)\)-Hom-comodule with \(\rho^M : M \to M \otimes C, m \mapsto m_{(0)} \otimes m_{(1)}\) such that

\[
\rho^M(ma) = (ma)_{[0]} \otimes (ma)_{[1]} = m_{[0]}(m_{[1]}(\cdot) \cdot \alpha^{-2}(a)) \otimes \gamma^2(m_{[1]}(a))
\]

for any \(m \in M\) and \(a \in A\).
3. \((C, \chi) = (A \otimes C, \alpha \otimes \gamma)\) is an \((A, \alpha)\)-Hom-coring with comultiplication and counit given by \([3, \beta, \gamma]\) and \([3, \beta, \gamma]\), respectively, and the \((A, \alpha)\)-Hom-bimodule structure \(a(a' \otimes c) = \alpha^{-1}(a) a' \otimes \gamma(c), (a' \otimes c)a = a' (c_{(-1)} \cdot \alpha^{-2}(a)) \otimes \gamma^2(c_{(0)})\) for \(a, a' \in A\) and \(c \in C\).

**Proof.** The first two conditions for Hom-entwining structures will be checked and the rest of the proof can be completed by performing similar computations as in Proposition 4.12. For \(a, a' \in A\) and \(c \in C\),
\[(aa')_κ \otimes γ(c) κ = γ(c)_{(-1)} \cdot α^{-1}(aa') \otimes γ(γ(c)_{(0)})
= β(c_{(-1)}) \cdot (α^{-1}(a)α^{-1}(a')) \otimes γ^2(c_{(0)})
= (β(c_{(-1)})_1 \cdot α^{-1}(a))(β(c_{(-1)})_2 \cdot α^{-1}(a')) \otimes γ^2(c_{(0)})
= (β(β^{-1}(c_{(-1)})) \cdot α^{-1}(a))(β(c_{(0)}(0)) \cdot α^{-1}(a')) \otimes γ^2(γ(c_{(0)})_{(0)})
= (c_{(-1)} \cdot α^{-1}(a))(γ(c_{(0)})_{(-1)} \cdot α^{-1}(a')) \otimes γ^2(γ(c_{(0)})_{(0)})
= (c_{(-1)} \cdot α^{-1}(a))a_κ' \otimes γ(γ(c_{(0)})^λ)
= a_κ a_κ' \otimes γ(κ^λ),
\]

\[α^{-1}(a_κ) \otimes c_γ^1 \otimes c_2^\gamma = α^{-1}(c_{(-1)} \cdot α^{-1}(a)) \otimes γ(c_{(0)})_{(1)} \otimes γ(c_{(0)})_{(2)}
= β^{-1}(c_{(-1)}) \cdot α^{-2}(a) \otimes γ(c_{(0)})_{(1)} \otimes γ(c_{(0)})_{(2)}
= β^{-1}(c_{(-1)}c_{(2)}^{-1}) \cdot α^{-2}(a) \otimes γ(c_{(0)}) \otimes γ(c_{(2)})
= (β^{-1}(c_{(-1)})β^{-1}(c_{(2)}^{-1})) \cdot α^{-2}(a) \otimes γ(c_{(0)}) \otimes γ(c_{(2)})
= c_{(-1)} \cdot β^{-1}(c_{(-1)}c_{(2)}^{-1}) \cdot α^{-3}(a) \otimes γ(c_{(0)}) \otimes γ(c_{(2)})
= c_{(-1)} \cdot α^{-1}(c_{(2)}^{-1} \cdot α^{-2}(a)) \otimes γ(c_{(0)}) \otimes γ(c_{(2)})
= (c_{(2)}^{-1} \cdot α^{-1}(a))a_κ \otimes c_1^κ \otimes γ(c_{(2)})
= α^{-1}(a)λ_κ \otimes c_1^κ \otimes c_2^λ.
\]

**Definition 4.16.** A triple \([(A, α), (B, β), (C, γ)]\) satisfying the assumptions of Proposition 4.15 is called an alternative Hom-Doi-Koppinen datum.

5. Acknowledgments

The author would like to thank Professor Christian Lomp for his valuable suggestions. This research was funded by the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through the FCT-Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0144/2013. The author was supported by the grant SFRH/BD/51171/2010.

**References**

[1] F. Ammar, A. Makhlof, *Hom-Lie superalgebras and Hom-Lie admissible superalgebras*, J. Algebra 324 (2010), 1513-1528.
[2] G. Böhm, *Dori-Hopf modules over weak Hopf algebras*, Comm. Algebra 28 (2000), 4687-4698.
[3] T. Brzeziński, *On modules associated to coalgebra-Galois extensions*, J. Algebra 215 (1999), 290-317.
[4] T. Brzeziński, *The structure of corings: Induction functors, Masche-type theorem, and Frobenius and Galois-type properties*, Algebr. Represent. Theory 5 (2002), 21-35.
[5] T. Brzeziński, S. Majid, *Coalgebra bundles*, Comm. Math. Phys. 191 (1998), 467-492.
[6] T. Brzeziński, R. Wisbauer, *Corings and Comodules*, Cambridge University Press, Cambridge, 2003.
[7] S. Caenepeel, E. De Groot, *Modules over weak entwining structures*, Contemp. Math. 267 (2000), 31-54.
[8] S. Caenepeel, I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra 39 (2011), 2216-2240.
[9] S. Caenepeel, G. Militaru, S. Zhu, *Frobenius and Separable Functors for Generalized Hopf Modules and Non-linear Equations*, LNM 1787, Springer, Berlin, 2002.
[10] Y. Y. Chen, Z. W. Wang, L. Y. Zhang, *Integrals for monoidal Hom-Hopf algebras and their applications*, J. Math. Phys. 54 (2013), 073515.
[11] Y. Y. Chen, Z. W. Wang, L. Y. Zhang, *The FRT-type theorem for the Hom-Lang equation*, Comm. Algebra 41 (2013), 3931-3948.
[12] Y. Y. Chen, L. Y. Zhang, *The category of Yetter-Drinfel’d Hom-modules and the quantum Hom-Yang-Baxter equation*, J. Math. Phys. 55 (2014), 031702.
[13] Y. Y. Chen, L. Y. Zhang, *Hopf-Galois extensions for monoidal Hom-Hopf algebras*, arXiv:1405.5184.
[14] Y. Doi, *On the structure of relative Hopf modules*, Comm. Algebra 11 (1983), 243-253.
[15] Y. Doi, *Unifying Hopf modules*, J. Algebra 153 (1992), 373-385.
[16] Y. Frégier, A. Gohr, S. D. Silvestrov, Unital algebras of Hom-associative type and surjective or injective twistings, J. Gen. Lie Theory Appl. 3(4) (2009), 285-295.
[17] A. Gohr, On Hom-algebras with surjective twisting, J. Algebra 324 (2010), 1483-1491.
[18] S. J. Guo, X. L. Chen, A Maschke-type theorem for relative Hom-Hopf modules, [arXiv:1411.7204] (to appear Czech. Math. J.).
[19] P. M. Hajac, M. Khalkhali, R. Rangipour, Y. Sommerhauser, Stable anti-Yetter-Drinfeld modules, C. R. Math. Acad. Sci. Paris 338 (2004), 587-590.
[20] P. M. Hajac, M. Khalkhali, R. Rangipour, Y. Sommerhauser, Hopf-cyclic homology and cohomology with coefficients, C. R. Math. Acad. Sci. Paris 338 (2004), 667-672.
[21] J. T. Hartwig, D. Larsson, S. D. Silvestrov, Deformations of Lie algebras using $\sigma$-derivations, J. Algebra 295 (2006), 314-361.
[22] P. Jara, D. Ştefan, Cyclic homology of Hopf-Galois extensions and Hopf algebras, Proc. London Math. Soc. 93 (2006), 138-174.
[23] S. Karaçuha, Covariant bimodules over monoidal Hom-Hopf algebras, [arXiv:1404.1296].
[24] A. Kaygun, M. Khalkhali, Hopf modules and noncommutative differential geometry, Lett. Math. Phys. 76 (2006), 77-91.
[25] M. Koppinen, Variations on the smash product with applications to group-graded rings, J. Pure Appl. Alg. 104 (1994), 61-80.
[26] L. Liu, B. L. Shen, Radford’s biproducts and Yetter-Drinfel’d modules for monoidal Hom-Hopf algebras, J. Math. Phys. 55 (2014), 031701.
[27] F. W. Long, The Brauer group of dimodule algebras, J. Algebra 31 (1974), 559-601.
[28] A. Makhlof, S. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl. 2(2) (2008), 51-64.
[29] A. Makhlof, S. Silvestrov, Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras, Ch. 17, pp. 189-206, in "Generalized Lie Theory in Mathematics, Physics and Beyond" (Eds. S. Silvestrov, E. Paal, V. Abramov, A. Stolin), Springer-Verlag, Berlin, 2008.
[30] A. Makhlof, S. Silvestrov, Hom-algebras and Hom-coalgebras, J. Algebra Appl. 9(4) (2010), 553-589.
[31] A. Makhlof, F. Panaite, Yetter-Drinfeld modules for Hom-bialgebras, J. Math. Phys. 55 (2014), 013501.
[32] F. Panaite, M. D. Staic, Generalized (anti) Yetter-Drinfel’d modules as components of a braided $T$-category, Isr. J. Math. 158(1) (2007), 349-365.
[33] D. E. Radford, J. Towber, Yetter-Drinfel’d categories associated to an arbitrary bialgebra, J. Pure Appl. Algebra 87 (1993), 259-279.
[34] P. Schauenburg, Dov-Koppinen Hopf modules versus entwined modules, New York J. Math 6 (2000), 325-329.
[35] M. E. Sweedler, Integrals for Hopf algebras, Ann. Math. 89 (1969), 323-335.
[36] M. E. Sweedler, The preduel theorem to the Jacobson-Bourbaki theorem, Amer. Math. Soc. 213 (1975), 391-406.
[37] M. Takeuchi, Relative Hopf modules-equivalence and freeness criteria, J. Algebra 60 (1979), 452-471.
[38] D. Yau, Hom-bialgebras and comodule Hom-algebras, Int. E. J. Algebra, 8 (2010), 45-64.
[39] D. Yau, Hom-algebras and homology, J. Lie Theory, 19 (2009), 409-421.
[40] D. Yau, Hom-quantum groups I: Quasitriangular Hom-bialgebras, J. Phys. A, 45 (2012), 065203.
[41] D. Yau, Hom-Yang-Baxter-equation, Hom-Lie algebras and quasitriangular bialgebras, J. Phys. A, 42 (2009), 165202.
[42] D. N. Yetter, Quantum groups and representations of monoidal categories, Math. Proc. Cambridge Philos. Soc. 108 (1990), 261-290.

Department of Mathematics, FCUP, University of Porto, Rua Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: s.karacuha3@gmail.com