DEHN TWISTS AND INVARIANT CLASSES

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Abstract. A degeneration of compact Kähler manifolds gives rise to a monodromy action on Betti moduli space

\[ H^1(X, G) = \text{Hom}(\pi_1(X), G)/G \]

over smooth fibres with a complex algebraic structure group \( G \) being either abelian or reductive. Assume that the singularities of the central fibre is of normal crossing. When \( G = \mathbb{C} \), the invariant cohomology classes arise from the global classes. This is no longer true in general. In this paper, we produce large families of locally invariant classes that do not arise from global ones for reductive \( G \). These examples exist even when \( G \) is abelian, as long as \( G \) contains multiple torsion points. Finally, for general \( G \), we make a new conjecture on local invariant classes and produce some suggestive examples.

1. Introduction

Let \( X \) be a compact Kähler manifold and \( G \) an abelian or reductive complex algebraic group. The adjoint action of \( G \) on the representation variety \( \text{Hom}(\pi_1(X), G) \) gives rise to the categorical quotient \[8, 9]\n
\[ H^1(X, G) = \text{Hom}(\pi_1(X), G)/G. \]

The functor \( H^1(\cdot, G) \) is contravariant and equals the usual first Betti cohomological functor when \( G = \mathbb{C} \). Let

\[ \text{pr} : \text{Hom}(\pi_1(X), G) \to H^1(X, G) \]

be the canonical projection and for a representation \( \rho \in \text{Hom}(\pi_1(X), G) \), let \( [\rho] = \text{pr}(\rho) \).

Let \( f : X \to D \) be a holomorphic map, where \( X \) is a Kähler manifold and \( D \subset \mathbb{C} \) is the unit open disk. Let \( D^* = D \setminus \{0\} \). If \( f \) has maximum rank over all points of \( D^* \) and \( f^{-1}(0) \) has singularities of normal crossing, then \( f \) is called a degeneration of Kähler manifolds (or degeneration for short). Fix \( s \in D^* \) once and for all and let \( X = \ldots \)
This setting gives rise to the Picard-Lefschetz diffeomorphism $T : X \to X$.

Let $X_0 = f^{-1}(0)$. Then $f$ induces a strong deformation retraction $X \to X_0$. This together with the inclusion $X \hookrightarrow X$ give a map $c : X \hookrightarrow X \to X_0$.

Define

\begin{equation}
(1) \quad c^* = H^1(c, G) \quad \text{and} \quad T^* = H^1(T, G).
\end{equation}

The local invariant class theorem implies that if $G = \mathbb{C}$, then

\begin{equation}
(2) \quad \text{im}(c^*) = H^1(X, \mathbb{C})^{T^*},
\end{equation}

where $H^1(X, \mathbb{C})^{T^*}$ denotes the subset of $H^1(X, \mathbb{C})$ fixed by $T^*$ [2, 10, 13, 14]. This is no longer true when $G$ is non-abelian. Isolated examples were found for $G = \text{SL}(n, \mathbb{C})$ for various $n$ [15]. These examples are of the type $[\rho] \in H^1(X, G)$ with finite $\text{im}(\rho)$.

For the rest of the paper, assume $G$ to be reductive; moreover, for a given degeneration $f : X \to D$, always assume $X = f^{-1}(s)$ and that it is a compact Riemann surface with genus $p > 0$ and that $X_0 = f^{-1}(0)$ is singular with normal crossing. Let $e$ be the identity element and $N$ the subset of torsion points of $G$. Since $G$ is reductive, $N \neq \{e\}$. Let

$V_f = H^1(X, G)^{T^*} \setminus \text{im}(c^*)$.

We refer to $V_f$ as the exceptional family of $f$. Theorem 4.1 shows that $V_f$ can be rather large for the simple reason of $N \neq \{e\}$ and this is true even when $G$ is abelian (such as $\mathbb{C}^\times$). From this perspective, one has Equation (2) because $\mathbb{C}$ is unipotent with 0 being its only torsion point.

We introduce the more restrictive notion of simple degeneration (Definition 5.1) and construct more examples with respect to simple degenerations with Theorem 5.5.

These examples motivate us to modify the local invariant class statement to Conjecture 5.7. We define the notion of pseudo-degeneration (Definition 3.2) and produce exceptional families of pseudo-degenerations. The existence of these families sheds light on Conjecture 5.7.

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2. Definitions and constructions

2.1. The fundamental groups and their representations. We begin by recalling our standing assumptions that a degeneration \( f : X \to D \) always has singular central fibre \( X_0 = f^{-1}(0) \) with normal crossing and, for \( s \in D^* \), \( X = f^{-1}(s) \) is a compact Riemann surface of genus \( p > 0 \) throughout the rest of the paper. Moreover,

\[
V_\tau = H^1(X, G)^{T_\tau} \setminus \text{im}(c^*).
\]

Let \( x \in X \) and \( \pi(x) = \pi_1(X, x) \) be the fundamental group of \( X \) with base point \( x \). The definition of \( H^1(X, G) \) depends on \( \pi(x) \), but is independent of \( x \). It is often true in this subject that while the technical calculation in \( \pi(x) \) involves \( x \), the results seldom depend on \( x \). We shorten \( \pi(x) \) to \( \pi \) and largely ignore the issue when the base point is irrelevant. We do not distinguish an element in \( \pi \) and its representative directed loop in \( X \). All loops are assumed to be smooth. A loop in \( X \) is simple if it has no self-intersection; moreover, it is called simple and separating if it is null-homologous in \( H_1(X, \mathbb{Z}) \); otherwise, it is called simple and non-separating.

![Figure 1: The generators of \( \pi \) and Dehn twists along them](image)

To define \( \pi \) explicitly with generators and relations, we begin with the commutator map. Let \( G \) be any group and

\[
C : G \times G \to G, \quad C(g, h) = ghg^{-1}h^{-1}.
\]

For \( 1 \leq i \leq 2p \), let \( A_i \in \pi \) be a simple closed non-separating loop such that the intersection number of \( A_i \) and \( A_j \) equals the Kronecker
\[
\delta_{(i)(j-p)}. \text{ Then there is a presentation (See Figure 1)}
\]

\[
(4) \quad \pi = \langle A_i, 1 \leq i \leq 2p \mid \prod_{i=1}^{p} C(A_i, A_{i+p}) \rangle.
\]

Suppose \( G \) is a group. Denote by \( Z(G) \) and \( \text{Aut}(G) \) the center and the automorphism group of \( G \), respectively. If \( G \) acts on a set \( S \), denote by \( S^G \) the fixed point subset of \( S \). If \( a \in G \), then \( S^a \) means \( S^{\langle a \rangle} \), where \( \langle a \rangle \) is the cyclic group generated by \( a \). The adjoint action is ubiquitous and we reserve the dot notation for it. If an action is not specified, it means the Ad-action.

For the rest of the paper, let \( G \) be a complex reductive group and \( e \) the identity element of \( G \). Let \( d, r \) and \( z \) denote the dimension and rank of \( G \) and the dimension of \( Z(G) \), respectively. Let \( H \) and \( K \) be a Cartan and a maximum compact subgroup of \( G \) with Lie algebras \( \mathfrak{h}, \mathfrak{k} \) and \( \mathfrak{g} \), respectively. Let \( W \) be the Weyl group preserving \( H \) and \( N \) the set of torsion points of \( G \). We adapt the notion of irreducibility from [12]:

**Definition 2.1.** Suppose \( \rho : G \to G \) is a group homomorphism. Then \( \rho \) is irreducible if

\[
g^{\rho(G)} = Z(\mathfrak{g}),
\]

where \( Z(\mathfrak{g}) \) is the center of \( \mathfrak{g} \).

Write \( R(G) \) (resp. \( R_0(G) \)) for the representation variety \( \text{Hom}(\pi, G) \) (resp. \( \text{Hom}(\pi_1(X_0), G) \)). Then

\[
(5) \quad R(G) = \{ a \in G^{2p} : \prod_{i=1}^{p} C(a_i, a_{i+p}) = e \}.
\]

In other words, \( R(G) \) may be identified with a subvariety of \( G^{2p} \). This variety structure on \( R(G) \) defines the variety structure on the quotient \( H^1(X, G) = R(G)/G \).

### 2.2. The mapping class groups and their actions.

Let \( Diff(X) \) be the group of orientation preserving diffeomorphisms of \( X \) and \( Diff(X, x) \) the subgroup fixing the base point \( x \). There is a natural inclusion \( \iota : Diff(X, x) \hookrightarrow Diff(X) \). Let \( \Gamma = \pi_0(Diff(X)) \) and \( \Gamma(x) = \pi_0(Diff(X, x)) \). Since every component of \( Diff(X) \) contains an element that fixes \( x \), the induced map \( \iota_* : \Gamma(x) \to \Gamma \) is onto. The group \( \Gamma(x) \) acts on \( \pi(x) \):

\[
\Gamma(x) \times \pi(x) \to \pi(x), \quad (\gamma, u) \mapsto \gamma(u).
\]
This induces a $\Gamma(x)$-action on $R(G)$:

$$\Gamma(x) \times R(G) \to R(G), \quad (\gamma, \rho) \mapsto \rho \circ \gamma^{-1}. \quad (6)$$

This action factors through $\iota_*$ and induces a $\Gamma$-action on $H^1(X, G)$.

**Definition 2.2.** Suppose $L$ is a simple loop on $X$ such that $x \notin L$. Let $B$ be a small tubular neighborhood of $L$ with $x \notin B$. Cutting along $L$, there are two ways to Dehn twist along $L$ in $B$. Stay consistently on one side of $X$ and call the right turn direction positive and denote by $\tau_L$ the Dehn twist in the negative direction along $L$.

The construction of $\tau_L \in Diff(X, x)$ actually depends on $B$ and the details of the twist; however, its images in $\Gamma(x)$ and $\Gamma$ under $\iota_*$ do not. Denote also by $\tau_L$ its images in $\Gamma(x)$ and $\Gamma$.

**Definition 2.3.** If $C \in \pi$ is simple, then $C'$ denotes a simple loop of a deformation of $C$ in a small tubular neighborhood of $C$ such that $x \notin C'$.

In the cases of $A_i \in \pi$ in the standard presentation (4), deform $A_i$ slightly left and call the resulting loop $A'_i$ for each $1 \leq i \leq 2p$ as in Figure 1.

By a direct calculation [4, 5],

**Proposition 2.4.** Let $\rho \in R(G)$.

1. Let $C \in \pi$ be simple and separating and $X_1, X_2$ the resulting connected components of $X \setminus C'$. Then the Dehn twist $\tau_{C'}$-action on $H^1(X, G)$ is covered by the $\tau_{C'}$-action on $R(G)$ with

$$\tau_{C'}(\rho)(A) = \begin{cases} 
\rho(A) & \text{if } A \in \pi_1(X_1) \\
\rho(BAB^{-1}) & \text{if } A \in \pi_1(X_2) 
\end{cases},$$

where $B = C$ or $B = C^{-1}$.

2. The $\tau_{A'_i}$-action and the $\tau_{A_{p+i}}$-action on $H^1(X, G)$ are covered by

$$\tau_{A'_i}(\rho)(A_{p+i}) = \rho(A_{p+i}A_i^{-1}), \quad \tau_{A_{p+i}}(\rho)(A_i) = \rho(A_iA_{p+i}),$$

for $1 \leq i \leq p$.

3. Dehn twists and degenerations

Consider a degeneration $f : X \to D$. The map $c^*$ is defined as follows. The map

$$c : X \to X_0$$

induces a homomorphism on the fundamental group

$$c_* : \pi \to \pi_1(X_0).$$
This further induces a map

\[ c^* : R_0(G) \to R(G), \quad c^*(\rho) = \rho \circ c_*, \]

which descends to the map \( c^* \) of (1). Hence

**Lemma 3.1.** If \([\rho] \in \text{im}(c^*),\) then \( \rho(C) = e \) for all \( C \in \ker(c_*) \).

**Definition 3.2.** Let \( S = \{C_1, \ldots, C_m\} \subseteq \pi \) be a set of simple loops. Suppose there exists a set of pairwise disjoint simple loops \( \{C'_1, \ldots, C'_m\} \) such that \( C'_i \) is a deformation of \( C_i \) in a small tubular neighborhood of \( C_i \). Then the element

\[ \tau_S = \prod_{i=1}^{m} \tau_{C'_i}^{n_i} \in \Gamma \]

is called a **pseudo-degeneration** for \( S \). We always assume that \( n_i \in \mathbb{Z} \setminus \{0\} \). An element \([\rho] \in H^1(X, G)\) is **exceptional** for \( \tau_S \) if \( \tau_S([\rho]) = [\rho] \) and \( \rho(C_i) \neq e \) for some \( i \).

The construction in [3, Section 2] and [3, Theorem 3] completely determines when a pseudo-degeneration arises from a degeneration:

**Theorem 3.3** (Earle, Sipe). Let \( \tau_S = \prod_{i=1}^{m} \tau_{C'_i}^{n_i} \in \Gamma \) be a pseudo-degeneration for \( S = \{C_1, \ldots, C_m\} \subseteq \pi \). Then \( \tau_S = T_\ast \) for some degeneration \( f : X \to D \) with \( S \subseteq \ker(c_*) \) if and only if \( n_i > 0 \) for all \( i \).

**Remark 3.4.** A degeneration as defined in [3] allows the central fibre to have general isolated singularities. However the actual construction in Section 2 of [3] clearly shows that the central fibre has normal crossings and the degeneration is of the type \( \{uv = t^{n_i}\} \to t \) near the singularity associated with \( C_i \).

By Lemma 3.1

**Corollary 3.5.** Suppose \( f : X \to D \) is a degeneration as in Theorem 3.3 and \( \tau_S = T_\ast \) for \( S = \{C_1, \ldots, C_n\} \). If \([\rho] \in \text{im}(c^*),\) then \( \rho(C_i) = e \) for all \( i \).

4. Exceptional families

In this section, we prove that the exceptional loci \( V_f \) can be rather large. Recall that \( N \neq \{e\} \) is the set of torsion points of \( G \).

**Theorem 4.1.**

1. Suppose \( p = 1 \). Then there exists a degeneration \( f : X \to D \) with \( \dim(V_f) \geq r \).
(2) Suppose $p > 1$. Then here exists a degeneration $f : X \to D$ and $[\rho] \in \mathcal{V}_f$ with $\text{im}(\rho)$ being irreducible and Zariski dense in $G$. Moreover $\dim(\mathcal{V}_f) \geq (2p - 3)d + 2z$.

**Proof.** Suppose that $\lambda \in N \setminus \{e\}$ is an $n$-torsion point and $$V_\lambda = \{ \rho \in R(G) : \rho(A_1) = \lambda \} = \{ a \in R(G) : a_1 = \lambda \},$$ via the identification (5). Let $\rho \in V_\lambda$. By Theorem 3.3 there is a degeneration $f : X \to D$ with $T_s = \tau_S = \tau_{A_1}^\lambda$, where $S = \{ A_1 \} \subseteq \ker(c_s)$. Since $\rho(A_1)^{-n} = \lambda^{-n} = e$, $\tau_S(\rho) = \rho$ by Proposition 2.4(2). Hence $[\rho]$ is a cocycle class invariant under $T^*$. Since $\rho(A_1) = \lambda \neq e$, $[\rho] \notin \text{im}(c^*)$ by Corollary 3.5. Hence $\text{pr}(V_\lambda) \subseteq \mathcal{V}_f$.

Suppose $p = 1$. Then $$V_\lambda \supseteq \{ a_2 \in G : C(\lambda, a_2) = e \} = G^\lambda.$$ Then $\dim(\mathcal{V}_f) \geq r$. This proves part (1).

Suppose $p > 1$. There exists a two generator subgroup $\langle g, h \rangle$ that is irreducible and Zariski dense in $G$. Since $N \cap K$ is dense in $K$ in the usual topology, we may choose $\lambda \in N \cap K$ and $h \in G$ such that $\langle \lambda, h \rangle$ is irreducible and Zariski dense in $G$. Since $p > 1$, there exists $\rho \in V_\lambda$ such that $$\begin{cases} 
\rho(A_1) = \rho(A_{p+2}) = \lambda, & \rho(A_2) = \rho(A_{p+1}) = h \\
\rho(A_i) = \rho(A_{p+i}) = e, & \text{for } 3 \leq i \leq p 
\end{cases}.$$ Then $\rho$ satisfies the required condition. A direct calculation then shows that $$\dim(V_\lambda) \geq (2p - 2)d + z, \quad \dim(\mathcal{V}_f) \geq (2p - 3)d + 2z.$$  

### 5. More simple examples

The previous section shows that the exceptional family can be rather large even when $G$ is abelian. In this section, we give more examples corresponding to a more restrictive class of degenerations. For simplicity, assume that $G$ is semi-simple.

**Definition 5.1.** A degeneration $f : X \to D$ is **simple** if it gives rise to a pseudo-degeneration with $n_i = 1$ for all $i$, where $n_i$ is as in Theorem 3.3.

**Lemma 5.2.** The map $C : K \times K \to K$ is surjective; hence, the center $Z(G)$ is contained in $C(K \times K)$. 


Proof. Since $G$ as well as $K$ are semi-simple, $C(K \times K) = K$ by Remark (2.1.5)]. The Lemma follows because $Z(G)$ is finite and contained in $Z(K)$. □

The following Proposition 5.3 is a direct consequence of [7 Theorem 1.3] as pointed out to me by Jiu-Kang Yu:

**Proposition 5.3.** If $p \geq 1$, then the space

$$\{(a_1, \cdots, a_{2p}) \in G^{2p} : \langle a_1, \cdots, a_{2p} \rangle \text{ is Zariski dense in } G\}$$

is Zariski open in $G^{2p}$.

By identification (5), $R(G)$ is a subvariety of $G^{2p}$. Since the intersection of a subvariety with a Zariski open set is Zariski open in the subvariety, we have

**Corollary 5.4.** If $p \geq 1$, then the space

$$\{\rho \in R(G) : \text{im}(\rho) \text{ is Zariski dense in } G\}$$

is Zariski open in $R(G)$.

**Theorem 5.5.**

(1) If $p > 0$, then there exists simple degeneration $f : X \to D$ such that $V_f \neq \emptyset$.

(2) If $p > 1$ and $G$ contains a closed semi-simple subgroup $G'$ with $Z(G') \neq \{e\}$ and $\dim(G') = d'$, then there exists simple degeneration $f : X \to D$ such that $\dim(V_f) \geq (2p - 2)d' - d$.

(3) If $p > 2$ and $G$ contains a closed semi-simple subgroup $G'$ with $Z(G') \neq \{e\}$ and $\dim(G') = d'$, then there exists simple degeneration $f : X \to D$ with $[p] \in V_f$ such that im($\rho$) is irreducible and Zariski dense in $G'$; moreover, $\dim(V_f) \geq (2p - 3)d'$.

**Proof.** Let $1 \leq n \leq p$ and consider the adjoint action of the Weyl group $W$ on $H$

$$W \times H \to H.$$

Suppose that there exist $w \in W$ and $g, h \in H$ such that $g \neq e$ and

$$\begin{align*}
\begin{cases}
  w.(hg) &= h \\
  C(g, w)^n &= e
\end{cases}
\end{align*}$$

Then

$$h.(g, wg^{-1}) = (g, w).$$
Let $1 \leq i_1 < \cdots < i_n \leq p$ and $\rho \in R(G)$ such that

$$\rho(A_l) = \begin{cases} 
  g & \text{if } l = i_j \text{ for some } j \\
  w & \text{if } l = i_j + p \text{ for some } j \\
  e & \text{otherwise}
\end{cases}$$

(9)

By Proposition 2.4(2) and Theorem 3.3, there exists a degeneration

$$\ker(\tau)$$

Corollary 3.5. Hence $C_G = \mathrm{PGL}(2)$ and find triples $w, g, h$ such that $t > \tau$ via the identification (5). Since $C$ and separating loop $V$ Let $\rho \in G$ calculation shows that $f_T$ is invariant under $T$. This proves part (2).

Hence $\rho \in \mathbf{V}_f$. Hence $\rho = 1$ and $n = 1$). Hence $\rho \in \mathbf{V}_f$ with $A_{ij} \in \ker(c)$. Incidentally, $\im(\rho)$ is abelian in these two cases if $p = 1$, but observe that $h \notin \im(\rho)$. This proves part (1). Notice that in this case $\tau(\rho) \neq \rho$.

For part (2) and (3), let $K'$ be the maximum compact subgroup of $G'$.

Suppose $p > 1$. Let $\lambda \in Z(G') \setminus \{e\}$ and

$$V_\lambda = \{\rho \in R(G') : \rho(C(A_1, A_{p+1})) = \lambda\} = \{a \in R(G') : C(a_1, a_{p+1}) = \lambda\}$$

via the identification (5). Since $p > 1$, by Lemma 5.2, $V_\lambda \neq \emptyset$. A direct calculation shows that

$$\dim(V_\lambda) \geq 2(p - 1)d'.$$

Let $\rho \in V_\lambda$. The element $C(A_1, A_{p+1})$ corresponds to a simple closed and separating loop $C$ on $X$. By Theorem 3.3, there is a degeneration $f : X \to D$ such that $T = \tau_S = \tau_C$ with $S = \{C\}$. Since $\rho(C) = \lambda \in Z(G')$, $\tau_S(\rho) = \rho$ by Proposition 2.4(1). Hence $[\rho]$ is a cocycle class invariant under $T^*$. Since $\rho(C) = \lambda \neq e$, $[\rho] \notin \im(c^*)$ by Corollary 3.5. Hence $\mathbf{pr}(V_\lambda) \subseteq \mathbf{V}_f$. Then

$$\dim(V_\lambda) \geq \dim(V_\lambda) - d \geq 2(p - 1)d' - d.$$

This proves part (2).
Suppose in addition that \( p > 2 \). Let \( g, h \in K' \) such that \( \langle g, h \rangle \) is Zariski dense in \( G' \). Then \( (\lambda C(g, h))^{-1} \in C(K' \times K') \) by Lemma 5.2. Hence there exists \( \rho_0 \in V_\lambda \) such that

\[
\begin{align*}
\rho_0(C(A_1, A_{p+1})) &= \lambda, \\
\rho_0(A_2) &= g, \\
\rho_0(C(A_3, A_{p+3})) &= \lambda^{-1}C(g, h)
\end{align*}
\]

Then \( \langle g, h \rangle \subseteq \text{im}(\rho) \subseteq G' \). Hence \( \text{pr}(V_\lambda) \subseteq V_f \).

The property of \( \text{im}(\rho) \) being irreducible is open. Since \( G' \) is semi-simple, by Corollary 5.4, there exists a Zariski open set \( V_0 \subseteq V_\lambda \) containing \( \rho_0 \) such that \( \rho \in V_0 \) if and only if \( \rho \) is irreducible and with \( \text{im}(\rho) \) being Zariski dense in \( G' \).

Let \( \rho \in V_0 \) and

\[
G_\rho = \{ g \in G : g \cdot \text{im}(\rho) \subseteq G' \}.
\]

Let \( g \in G_\rho \). Since \( \text{im}(\rho) \) is Zariski dense in \( G' \), \( g \cdot G' = G' \), i.e. the \( G_\rho \)-action on \( R(G') \) factors through \( \text{Aut}(G') \). Since \( G' \) is semi-simple, \( \text{dim}(\text{Aut}(G')) = d' \). Hence the \( G_\rho \)-orbit of \( \rho \) is at most \( d' \)-dimensional. Hence

\[
\text{dim}(V_f) \geq \text{dim}(V_0) - d' \geq (2p - 3)d'.
\]

Part (3) now follows.

\[ \square \]

**Remark 5.6.** For an abelian \( G \), the \( \text{Ad}(G) \)-action on itself is trivial. However, here the subgroup of \( G \) preserving the Cartan subgroup \( H \) is larger than \( H \). These extra Weyl group twists on \( H \) allow the construction of the exceptional families in Theorem 5.5(1). When \( G \) is abelian, \( C(G \times G) = \{ e \} \) while almost the opposite is true when \( G \) is semi-simple in the sense that \( C(G \times G) \) is Zariski dense and contains \( K \). This allows one to construct the exceptional families in Theorem 5.5(2, 3).

The examples we have constructed motivate the following conjecture on local invariant classes:

**Conjecture 5.7.** Suppose \( f : X \to D \) is a degeneration. If \([\rho] \in H^1(X, G)^{T^*} \), then \( \rho(C) \in N \) for all simple \( C \in \ker(c_\tau) \).

**Remark 5.8.** Notice that even if \( G \) is reductive and not necessarily semi-simple, it is still the case that \( C(G, G) \cap Z(G) \subseteq N \); otherwise, the construction for Theorem 5.5(2) would have yielded counterexamples to Conjecture 5.7.
6. Pseudo-degenerations

We do not have a counter-example for Conjecture 5.7 (obviously). However we do have the following suggestive example. Assume $G$ is semi-simple in this section.

**Lemma 6.1.** Let $g \in K$. Then

$$C : K \times K^g \to K$$

is onto.

**Proof.** It is sufficient to assume $g$ is regular. The Lemma then follows from [11, Proposition (B.1)].

Let $\text{pr}_1 : W' \to G$ be the projection to the first factor, where

$$W' = \{ (g, h, k) \in G \times G \times G : k.(g, hg^{-1}) = (g, h) \}.$$  

**Proposition 6.2.** The image $\text{pr}_1(W')$ contains $K$.

**Proof.** The equation $k.(hg^{-1}) = h$ is equivalent to $g = C(k^{-1}, h^{-1})$. Let $g \in K$. By Lemma 6.1,

$$\text{pr}_1^{-1}(g) = \{ (h, k) \in G \times G^g : g = C(k^{-1}, h^{-1}) \} \neq \emptyset.$$  

Consider the presentation of $\pi$ as in Figure 1. Let $W \subseteq R(G)$ such that $\rho \in W$ if

$$\begin{cases} 
\rho(A_1) = \rho(A_{p+2}) = g \\
\rho(A_2) = \rho(A_{p+1}) = h \\
\rho(A_i), \rho(A_{p+i}) \in G^k \text{ for } 3 \leq i \leq p
\end{cases},$$

where $(g, h, k) \in W'$. Let

$$S = \{ A_1, A_{p+2} \}, \quad \tau_S = \tau_{A_1}^{-1} \tau_{A_{p+2}}^{-1}.$$  

Let $\rho \in W$. Then by Proposition 2.4(2) (See Figure 1),

$$\tau_S(\rho)(A_2) = \rho(A_2A_{p+2}^{-1}) = hg^{-1} = \rho(A_{p+1}A_1^{-1}) = \tau_S(\rho)(A_{p+1});$$

moreover, $\tau_S(\rho)(A_i) = \rho(A_i)$ for $1 \leq i \leq 2p$ and $i \neq 2, p + 1$. By Proposition 6.2, $\tau_S(\rho) = k.\rho$; hence, $\tau_S([\rho]) = [\rho]$. Moreover, the image of the map

$$W \to G : \rho \to \rho(A_1)$$

contains $K$. Hence the family $\text{pr}(W)$ would have contained counter-examples to Conjecture 5.7 if $\tau_S$ had actually arisen from a degeneration.
For an explicit construction with $G = \text{SL}(2)$, let $s, t \in \mathbb{C}^\times; t^2 = s$ and
\[
g = \begin{bmatrix} s & 0 \\ 0 & 1/s \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}.
\]

Since $\tau_S$ contains a negative power $\tau_{A_p+1}^{-1}$, by Theorem 3.3, the pseudo-degeneration $\tau_S$ does not correspond to a degeneration. However it can be realized by the following family of hyperelliptic curves parameterized by $t$:
\[
y^2 = (z + (t - 1)a_2 - ta_1)(z - a_2)(z - a_3)(z + (\bar{t} - 1)a_5 - \bar{t}a_4) \prod_{i=5}^{2g+2} (z - a_i),
\]
where $a_1, \ldots, a_{2g+2}$ are distinct complex numbers and $\bar{t}$ means the complex conjugate of $t$.

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