Parametric Resonance of a Charged Pendulum with a Suspension Point Oscillating Between Two Vertical Charged Lines

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Received November 02, 2022; revised February 23, 2023; accepted May 13, 2023

Abstract—In this study, we analyze a planar mathematical pendulum with a suspension point that oscillates harmonically in the vertical direction. The bob of the pendulum is electrically charged and is located between two wires with a uniform distribution of electric charges, both equidistant from the suspension point. The dynamics of this phenomenon is investigated. The system has three parameters, and we analyze the parametric stability of the equilibrium points, determining surfaces that separate the regions of stability and instability in the parameter space. In the case where the parameter associated with the charges is equal to zero, we obtain boundary curves that separate the regions of stability and instability for the Mathieu equation.

MSC2010 numbers: 37N05, 70H14, 70J40, 70J25

DOI: 10.1134/S156035472303005X

Keywords: planar charged pendulum, parametric resonance, Hamiltonian systems, Deprit–Hori method

1. INTRODUCTION

In classical mechanics, the planar mathematical pendulum is a system composed of a mass (a massive ball) fixed at the free end of a rigid massless rod that can rotate without friction about the other end, the suspension point, fixed at a point of space, under the action of the gravity field. The variant of the pendulum with a moving suspension point has received considerable attention. In [3, 9, 11, 12, 14, 19], the authors study the pendulum with an oscillating suspension point in the vertical or horizontal direction. Recently the case of a pendulum under the action of additional forces due to a distribution of electric charges has been considered, see [2, 4, 5].

In this paper, we address the case where the suspension point $S$ oscillates harmonically in the vertical direction, the bob of the pendulum has an electric charge $q$ and the suspension point is equidistant from two vertical wires of infinite length with a uniform distribution of electric charges (see Fig. 1). The problem is described by a Hamiltonian function $H(x, y, \tau, \mu, \alpha, \varepsilon)$, see Eq. (2.2).

The system depends on three parameters: a small parameter $\varepsilon$ which is associated with the amplitude of oscillation of the suspension point and the pendulum length, a parameter $\alpha$ which is associated with the natural frequency of oscillation of the pendulum and the pendulum length, and the parameter $\mu$ which is the ratio of the electric charges on the bob and on the vertical wires.

When $\varepsilon = 0$ we have a charged pendulum with a fixed suspension point, while the case $\mu = 0$ represents an uncharged pendulum whose suspension point oscillates vertically in a harmonic way.

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The points \( P_1 = (0, 0) \) and \( P_2 = (\pi, 0) \) are equilibria of the system for all values of the parameters. They are located on the vertical line containing the suspension point \( S \). Our goal is to study the parametric stability of these equilibria using the Krein–Gelfand–Lidskii theorem and the Deprit–Hori normalization process. In the problem at hand, we have a nonautonomous linear system involving parameters. In this context, the Krein–Gelfand–Lidskii theorem plays an important role, enabling the decomposition of the parameter space into stability and instability regions. The Deprit–Hori method, in turn, is significant in the universe of quadratic Hamiltonian functions, aiming to determine a \( 2\pi \)-periodic generating function that allows for a periodic \( 2\pi \)-linear symplectic coordinate change, transforming the time-dependent Hamiltonian function into an autonomous Hamiltonian function. For further historical details and demonstrations of these results, see [6, 15, 22]. This method has been applied in a variety of problems, including celestial mechanics [1, 7, 8, 16–18, 21] and classical mechanics [2, 4, 5, 13].

The structure of the paper is as follows. In Section 2 we describe the problem in its Hamiltonian formulation and find the regions of linear stability of the unperturbed problem \((\varepsilon = 0)\) in the parameters plane \((\alpha, \mu)\). In Section 3, we make some symplectic changes of variables in the linearized Hamiltonian for each equilibrium which significantly reduce the normalization calculations. In Section 4, we normalize the periodic Hamiltonian using the Deprit–Hori method, obtaining an autonomous Hamiltonian that will be used to obtain the boundary surfaces that separate the regions of stability and instability in the parameter space \((\alpha, \mu, \varepsilon)\). We calculate the coefficients of their parameterizations up to fifth order in terms of the parameter \( \mu \). Considering cross-sections obtained by intersecting these regions with planes \( \mu = \text{const} \), we obtain the stability/instability boundary curves in each plane. In the special case of \( \mu = 0 \), we obtain the boundary curves of the Mathieu equation, which correspond to the coefficients obtained in [3].

2. FORMULATION OF THE PROBLEM

We consider a pendulum of length \( \ell \) with a suspension point \( S \) subjected to a vertical harmonic oscillation. The mass \( m \) of the bob is electrically charged, and the pendulum is located between two vertical wires, both of which are charged. Each wire has a homogeneous distribution of positive electric charge with constant linear density \( \sigma \) and total charge \( Q \). Both wires have infinite length and are equidistant from the suspension point \( S \) of the pendulum, with a distance of \( d > \ell \) from point \( S \) to each wire.

The Lagrangian function is given by \( L = T - U \), where \( T \) represents kinetic energy, \( U_g \) is the gravitational potential energy, and \( U_c \) is the electrostatic potential energy, with \( U = U_g + U_c \). We assume that the velocity of the bob is small enough to neglect magnetic force.

If \( \theta \) is the angle that the pendulum makes with the vertical, see Fig. 1, then the pendulum coordinates are \( x = \ell \sin \theta \) and \( y = -(\rho + \ell \cos \theta) \), where \( \rho \) is the distance between the suspension point \( S \) and the point \( O \). The kinetic energy is given by

\[
T = \frac{m}{2} \left[ (\dot{\ell} \cos \theta)^2 + (\dot{\theta} - \ell \dot{\theta} \sin \theta)^2 \right]
\]

and the gravitational potential energy is given by \( U_g = -mg(\rho + \ell \cos \theta) \). Considering that \( d - \ell \sin \theta \) and \( d + \ell \sin \theta \) are, respectively, the distances from the charge \( q \) to the wires on the right and left, the electrostatic potential energy is given by

\[
U_c = -2k_0 q \sigma \left[ \log \left( \frac{d - \ell \sin \theta}{d} \right) + \log \left( \frac{d + \ell \sin \theta}{d} \right) \right],
\]

where \( k_0 \) is the Coulomb constant and the point \( O \) was used as the reference point for the calculation of the electrostatic potential. Thus, the equation of motion of the pendulum is

\[
ml\ddot{\theta} - m\dot{\rho} \sin \theta + mg \sin \theta + 2k_0 q \sigma \left( \frac{1}{d - \ell \sin \theta} - \frac{1}{d + \ell \sin \theta} \right) \cos \theta = 0.
\]

The harmonic law of oscillation at the pendulum suspension point is given by \( \rho = a \cos \nu t \), and it is convenient to use \( \tau = \nu t \) as the independent variable. In this way, we have that \( \ddot{\theta} = \nu^2 \theta'' \)
and $\ddot{\varphi} = \dot{\varphi}^2 \varphi''$, where the primes denote the derivative with respect to $\tau$. By choosing $d = 2\ell$, and defining $\alpha = \frac{g}{\ell \nu^2}$, $\varepsilon = \frac{q}{\ell}$, $\mu = \frac{q}{\ell \sigma}$, and considering $\sigma$ such that $\frac{4k_0\sigma^2}{m\nu^2} = 1$, the equation of motion becomes

$$\ddot{\theta} + \alpha \sin \theta + \varepsilon \cos \tau \sin \theta + \mu \frac{2 \log(7 + \cos^2 \theta)}{7 + \cos 2\theta} \sin 2\theta = 0. \quad (2.1)$$

When $\mu = 0$, this equation describes the motion of a pendulum whose suspension point oscillates harmonically in the vertical direction, as described in [3]. The values $\theta = 0$ and $\theta = \pi$ represent equilibrium points for all parameter values. Our aim is to study the parametric stability of these equilibria.

Setting $x = \theta$ and $y = \dot{\theta}$, we obtain a Hamiltonian system whose Hamiltonian function is given by

$$H(x, y, \tau, \mu, \alpha, \varepsilon) = \frac{1}{2} y^2 - \alpha \cos x - \varepsilon \cos \tau \cos x - \frac{\mu}{2} \log(7 + \cos 2x). \quad (2.2)$$

The points $P_1 = (0, 0)$ and $P_2 = (\pi, 0)$ represent equilibria for all parameter values in this Hamiltonian system. When $\varepsilon = 0$, we have an autonomous system with one degree of freedom that describes the motion of a charged pendulum with a fixed suspension point. In this case, we have the following result on the linear stability of the equilibria.

**Proposition 1.** For $\varepsilon = 0$, the system defined by (2.2) is linearly stable in the regions

$\mu > -4\alpha$ for the equilibrium $P_1$ and $\mu > 4\alpha$ for the equilibrium $P_2$

and unstable in the regions

$\mu < -4\alpha$ for the equilibrium $P_1$ and $\mu < 4\alpha$ for the equilibrium $P_2$.

**Proof.** For both equilibria we have $H_{yy} = 1$ and $H_{xy} = 0$. Computing $H_{xx}$ at the equilibria $P_1$ and $P_2$, we obtain, respectively

$$H_{xx} = \frac{\mu}{4} + \alpha \quad \text{and} \quad H_{xx} = \frac{\mu}{4} - \alpha.$$ 

Thus, the result follows from the Dirichlet theorem [20].
3. PARAMETRIC RESONANCE

We are interested in studying the parametric stability of the linearized system described by (2.2) in the parameter space \((\mu, \alpha, \varepsilon)\). To achieve this goal, we will use the Krein–Gelfand–Lidskii theorem, described in [15]. The theorem will be stated below.

Krein–Gelfand–Lidskii Theorem. Consider a linear Hamiltonian system whose Hamiltonian function is of the form

\[
H = \frac{1}{2} \sum_{k=1}^{n} \sigma_k(x_k^2 + y_k^2) + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots,
\]

where \(H_1, H_2, \ldots\) are quadratic forms in the variables \(x_1, y_1, \ldots, x_n, y_n\) with continuous and \(2\pi\) periodic coefficients in \(t\). For \(\varepsilon > 0\) small enough, the linear system with Hamiltonian (3.1) is stable if and only if the coefficients \(\sigma_k\) are not bound by the relationships

\[
\sigma_k + \sigma_l = N,
\]

with \(k, l = 1, 2, \ldots, n\) and \(N = \pm 1, \pm 2, \ldots\).

The system defined by (2.2) is a Hamiltonian system that varies periodically with time and depends on the parameters \(\mu, \alpha,\) and \(\varepsilon\). We will investigate the stability of the linearized system around the equilibria \(P_1 = (0, 0)\) when \(\mu > -4\alpha\) and \(P_2 = (\pi, 0)\) when \(\mu > 4\alpha\).

Setting \(\xi = x - x_0\) and \(\eta = y,\) with \(x_0 = 0\) for \(P_1\) and \(x_0 = \pi\) for \(P_2,\) the linearized Hamiltonian functions are given by

\[
H(\xi, \eta, \tau, \mu, \alpha, \varepsilon) = \frac{1}{2} \eta^2 + \frac{1}{2} \left[ \varepsilon \cos \tau + \alpha + \frac{\mu}{4} \right] \xi^2 \quad \text{for} \quad P_1,
\]

\[
H(\xi, \eta, \tau, \mu, \alpha, \varepsilon) = \frac{1}{2} \eta^2 - \frac{1}{2} \left[ \varepsilon \cos \tau + \alpha - \frac{\mu}{4} \right] \xi^2 \quad \text{for} \quad P_2.
\]

Applying the symplectic transformation \(\xi, \eta \rightarrow x, y\) given by

\[
\xi = \omega^{-1/2}x, \quad \eta = \omega^{1/2}y,
\]

the Hamiltonians (3.3) and (3.4) take the form

\[
H(x, y, \tau, \mu, \alpha, \varepsilon) = \frac{\omega}{2}(x^2 + y^2) + \frac{x^2 \cos \tau}{2\omega} - \varepsilon,
\]

where \(\omega^2 = \frac{\mu}{4} + \alpha\) for \(P_1\) and \(\omega^2 = \frac{\mu}{4} - \alpha\) for \(P_2\).

According to the Krein–Gelfand–Lidskii theorem, if \(2\omega(\mu, \alpha) = N\) for some integer \(N,\) the unperturbed linear system is unstable. Analyzing the system in the parameter space \((\mu, \alpha, \varepsilon)\), the equation \(2\omega(\mu, \alpha) = N\) defines a curve in the subspace \((\mu, \alpha, 0)\). Therefore, if \(\varepsilon > 0,\) the point \((\mu_0, \alpha_0, \varepsilon)\) may or may not be stable, depending on the point \((\mu_0, \alpha_0, 0)\) on the curve. To separate the regions of stability and instability, boundary surfaces will be constructed in the parameter space \((\mu, \alpha, \varepsilon)\). These surfaces will be defined as graphs in the plane \((\mu, 0, \varepsilon)\) and expressed as power series of \(\varepsilon,\) with coefficients dependent on \(\mu:\)

\[
\alpha = \alpha_0 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \alpha_3 \varepsilon^3 + \alpha_4 \varepsilon^4 + O(\varepsilon^5),
\]

where \(\alpha_j = \alpha_j(\mu) \) for \(j \geq 1\) are functions of \(\mu, \alpha_0 = (N^2 - \mu)/4\) for \(P_1\) and \(\alpha_0 = (\mu - N^2)/4\) for \(P_2\) are curves in the plane \((\mu, \alpha, 0)\) given by the equation \(2\omega(\mu, \alpha) = N\).

The expression of \(\alpha\) in Eq. (3.6) plays a crucial role, as it allows us to identify the boundary surfaces that separate the stability and instability regions in the parameter space. To do this, we will substitute (3.6) into the Hamiltonians (3.3) and (3.4), perform a power series expansion in \(\varepsilon,\) and make a symplectic variable transformation to obtain the Hamiltonian that will be used in the Deprit–Hori method. The next result will treat this.
Proposition 2. The Hamiltonians (3.3) and (3.4) can be transformed through a symplectic variable change to the following Hamiltonian:

$$\mathcal{H}(X, Y, \tau, \mu, \alpha, \varepsilon) = \frac{(-1)^{i+1}}{N} S^2 \cos \tau \varepsilon + \sum_{j \geq 1} \frac{\alpha_j S^2}{4N} \varepsilon^j,$$

where $i = 1$ for $P_1$, $i = 2$ for $P_2$ and $S = X \cos(N\tau/2) + Y \sin(N\tau/2)$.

Proof. By substituting (3.6) into the Hamiltonians (3.3), (3.4) and making the symplectic change of variables given by

$$\xi = \omega_0^{-1/2} \tilde{X}, \quad \eta = \omega_0^{1/2} \tilde{Y},$$

and

$$\tilde{X} = X \cos(\omega_0 \tau) + Y \sin(\omega_0 \tau), \quad \tilde{Y} = -X \sin(\omega_0 \tau) + Y \cos(\omega_0 \tau),$$

we obtain the new Hamiltonian (3.7). \qed

It is this Hamiltonian (3.7) that will be used in the Deprit–Hori method to calculate the coefficients $\alpha_j$ of (3.6). The symplectic rotation performed here eliminates the $\mathcal{H}_0$ term of the Hamiltonian, as described in [4], resulting in a significant reduction in the complexity of the calculations of the coefficients $\alpha_j$. The following section will be dedicated to the calculation of these coefficients.

4. BOUNDARY SURFACES OF THE STABILITY/INSTABILITY REGIONS

In this section, we will apply the Deprit–Hori method [1, 8, 10, 15] to the Hamiltonian (3.7) to find the boundary surfaces that separate the regions of stability and instability. The method allows transforming, through a symplectic change of variables $X, Y \rightarrow p, P$, time-periodic Hamiltonian functions of the form

$$H(X, Y, \nu, \varepsilon) = \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} H_m(X, Y, \nu),$$

into an autonomous Hamiltonian of the form

$$K(p, P) = k_{02}p^2 + k_{11}pP + k_{20}P^2,$$

where $k_{ij} = \sum_{m=1}^{\infty} k_{ij}^{(m)} \varepsilon^m$, with $k_{ij}^{(m)}$ depending on $\alpha_1, \cdots, \alpha_m$.

By applying the Deprit–Hori method to the Hamiltonian (3.7), we obtain a Hamiltonian of the form (4.2), with the $k_{11}$ term identically zero. This implies that the characteristic equation is of the form $\lambda^2 + 4k_{20}k_{02} = 0$. The stability region is determined by the inequality $k_{20}k_{02} > 0$, whose boundary is given by the equation $k_{20}k_{02} = 0$, that is when

$$k_{20} = 0 \quad \text{or} \quad k_{02} = 0.$$

By setting to zero the coefficients of all powers of $\varepsilon$ in the expressions $k_{20}$ and $k_{02}$, we find the coefficients $\alpha_j(\mu)$ in (3.6), which allow us to determine the boundary surfaces in the parameter space $(\mu, \alpha, \varepsilon)$. These surfaces derive from the curve $\alpha_0 = (N^2 - \mu)/4$ for $P_1$ and from $\alpha_0 = (\mu - N^2)/4$ for $P_2$, which are defined by the equation $2\omega(\mu, \alpha) = N$, $N \geq 1$, in the plane $\varepsilon = 0$. 

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4.1. Boundary Surfaces for the Equilibrium $P_1$

In this subsection, we find the boundary surfaces that separate the stability and instability regions for equilibrium $P_1$ using the Deprit–Hori method. This method is applied to the Hamiltonian (3.7) for resonances of the form $2\omega = N$, where $N = 1, 2, 3, \ldots$ and for $\omega_0^2 = \frac{\mu}{4} + \alpha_0$.

Theorem 1. The boundary surfaces separating the stability and instability regions for equilibrium $P_1$ associated with the resonance $2\omega = 1$ are parameterized by

$$\alpha = \frac{1 - \mu}{4} \mp \frac{1}{2} \varepsilon - \frac{1}{8} \varepsilon^2 \pm \frac{1}{32} \varepsilon^3 - \frac{1}{384} \varepsilon^4 \mp \frac{11}{4608} \varepsilon^5 + O(\varepsilon^6).$$

(4.4)

Proof. Applying the Deprit–Hori method to the Hamiltonian (3.7) with $\omega_0^2 = \frac{\mu}{4} + \alpha_0$ and $N = 1$, we obtain a Hamiltonian of the form (4.2) whose coefficient $k_{11}$ is identically equal to zero and the coefficients $k_{20}$ and $k_{02}$ are given by

$$k_{20}^{(1)} = \frac{1}{4} + \frac{\alpha_1}{2},$$
$$k_{20}^{(2)} = -\frac{1}{16}(3 + 12\alpha_1 + 8\alpha_1^2 - 8\alpha_2),$$
$$k_{20}^{(3)} = -\frac{3}{64} + \frac{\alpha_1}{2} + \frac{3\alpha_1^2}{2} + \alpha_1^3 - \frac{3\alpha_2}{4} - \alpha_1\alpha_2 + \frac{\alpha_3}{2},$$

and

$$k_{02}^{(1)} - k_{20}^{(1)} = -\frac{1}{2},$$
$$k_{02}^{(2)} - k_{20}^{(2)} = \frac{3\alpha_1}{8},$$
$$k_{02}^{(3)} - k_{20}^{(3)} = \frac{3}{32} - 3\alpha_1^2 + \frac{3\alpha_2}{2}.$$

From Eqs. (4.3), we obtain two surfaces in the parameter space, which are determined by equating the coefficients in $k_{20}$ and $k_{02}$ to zero. These surfaces are parameterized in Eq. (4.4).

Fig. 2. Left: the surfaces separating stability and instability regions for $P_1$ and $N = 1$. Right: the planar section by the plane $\mu = -\frac{1}{2}$, see Eq. (4.4).

The two surfaces found above define the boundary of the instability region associated with the resonance $2\omega = 1$ for the equilibrium $P_1$. Figure 2 on the left shows the surfaces which bound the instability region, and the figure on the right shows the planar sections of these surfaces by the plane $\mu = -\frac{1}{2}$. 
Theorem 2. The boundary surfaces separating the stability and instability regions for the $P_1$ equilibrium associated with the resonance $2\omega = 2$ are parameterized by
\begin{align*}
\alpha &= \frac{4 - \mu}{4} + \frac{5}{12} \varepsilon^2 - \frac{763}{3456} \varepsilon^4 + \frac{1002401}{4976640} \varepsilon^6 + O(\varepsilon^7), \\
\alpha &= \frac{4 - \mu}{4} - \frac{1}{12} \varepsilon^2 + \frac{5}{3456} \varepsilon^4 + \frac{169249}{4976640} \varepsilon^6 + O(\varepsilon^7). \tag{4.5}
\end{align*}

Proof. In this case, using $\omega_0^2 = \frac{\mu}{4} + \alpha_0$ and $N = 2$, the $k_{20}$ and $k_{02}$ coefficients are given by
\begin{align*}
k_{20}^{(1)} &= \frac{\alpha_1}{4} , \\
k_{20}^{(2)} &= -\frac{1}{48} (5 + 9\alpha_1^2 + 12\alpha_2), \\
k_{20}^{(3)} &= \frac{1}{288} (70\alpha_1 + 45\alpha_1^3 - 108\alpha_1\alpha_2 + 72\alpha_3)
\end{align*}
and
\begin{align*}
k_{02}^{(1)} &= k_{20}^{(1)}, & k_{02}^{(2)} - k_{20}^{(2)} &= \frac{1}{8} + \frac{\alpha_1^2}{4}, & k_{02}^{(3)} - k_{20}^{(3)} &= -\frac{\alpha_1}{48} (13 + 9\alpha_1^2 - 24\alpha_2).
\end{align*}

The equations $k_{20} = 0$ and $k_{02} = 0$ give us the surfaces given by (4.5) and (4.6), respectively. $\square$

Fig. 3. Boundary surfaces separating stability and instability regions for $P_1$ when $N = 1$ and $N = 2$, and a planar section by the plane $\mu = -\frac{1}{4}$, see Eqs. (4.4)–(4.6).

Theorem 3. The boundary surfaces separating the stability and instability regions for equilibrium $P_1$ associated with the resonance $2\omega = 3$ are parameterized by
\begin{align*}
\alpha &= \frac{9 - \mu}{4} + \frac{1}{16} \varepsilon^2 + \frac{1}{32} \varepsilon^3 + \frac{13}{5120} \varepsilon^4 \pm \frac{5}{2048} \varepsilon^5 + O(\varepsilon^6). \tag{4.7}
\end{align*}

Proof. With $\omega_0^2 = \frac{\mu}{4} + \alpha_0$ and $N = 3$, the coefficients of $k_{20}$ and $k_{02}$ are
\begin{align*}
k_{20}^{(1)} &= \frac{\alpha_1}{6} , \\
k_{20}^{(2)} &= -\frac{1}{864} (9 + 24\alpha_1 + 16\alpha_1^2 - 144\alpha_2), \\
k_{20}^{(3)} &= \frac{1}{15552} (-108 + 153\alpha_1 + 384\alpha_1^2 + 64\alpha_1^3 - 576\alpha_1\alpha_2 - 432\alpha_2 + 2592\alpha_3)
\end{align*}
and
\[ k_{01}^{(1)} = k_{20}^{(1)}, \quad k_{02}^{(2)} - k_{20}^{(2)} = \frac{\alpha_1}{18}, \quad k_{02}^{(3)} - k_{20}^{(3)} = -\frac{1}{648}(9 + 32\alpha_1^2 - 36\alpha_2). \]

Setting \( k_{20} = 0 \) and \( k_{02} = 0 \), we obtain the surfaces given in Eq. (4.7) respectively.

Continuing the process for \( N = 4, 5, 6, \ldots \), we obtain for the equilibrium \( P_1 \) a decomposition of the parameter space \((\mu, \alpha, \epsilon)\) into a sequence of alternating regions of stability and regions of instability.

Figure 4 on the left shows the planar sections of these regions for \( N = 1, 2, 3, 4, 5, 6 \) in the plane \( \mu = -\frac{1}{2} \) and on the right an enlargement of the cases \( N = 4, 5, 6 \).

**Fig. 4.** Planar section by the plane \( \mu = -\frac{1}{2} \) for \( P_1 \) when \( N = 1, 2, 3, 4, 5 \) and an amplification of the cases \( N = 4, 5, 6 \), see Eqs. (4.4)–(4.7) and (A.1)–(A.5).

When \( \mu = 0 \), Eq. (2.1) describes the motion of an uncharged pendulum with the suspension point oscillating vertically in a harmonic way, as discussed in [3]. The curves obtained with \( \mu = 0 \) from Eqs. (4.4)–(4.7) and (A.1)–(A.5) are identical to the boundary curves of the Mathieu equation found in [3].

**4.2. Boundary Surfaces for the Equilibrium \( P_2 \)**

In the case of the equilibrium \( P_2 \), we use the Deprit–Hori method on the Hamiltonian function (3.7) for each resonance of the form \( 2\omega = N \), with \( N = 1, 2, 3, \ldots, \omega_0^2 = \frac{\mu}{4} - \alpha_0 \), and using Eqs. (4.3), we obtain the following result:

**Theorem 4.** The boundary surfaces separating the stability and instability regions for the equilibrium \( P_2 \) associated with the resonance \( 2\omega = N \) are given by

- For \( N = 1 \)

\[
\alpha = \frac{\mu}{4} - \frac{1}{2} \epsilon + \frac{1}{8} \epsilon^2 \pm \frac{1}{32} \epsilon^3 + \frac{1}{384} \epsilon^4 \mp \frac{11}{4008} \epsilon^5 + \mathcal{O}(\epsilon^6).
\]
• For $N = 2$

$$
\alpha = \frac{\mu - 4}{4} - \frac{5}{12} \varepsilon^2 + \frac{763}{3456} \varepsilon^4 + \mathcal{O}(\varepsilon^6),
$$

(4.9)

$$
\alpha = \frac{\mu - 4}{4} + \frac{1}{12} \varepsilon^2 - \frac{5}{3456} \varepsilon^4 + \mathcal{O}(\varepsilon^6).
$$

(4.10)

• For $N = 3$

$$
\alpha = \frac{\mu - 9}{4} - \frac{1}{16} \varepsilon^2 + \frac{1}{32} \varepsilon^3 - \frac{13}{5120} \varepsilon^4 \pm \frac{5}{2048} \varepsilon^5 + \mathcal{O}(\varepsilon^6).
$$

(4.11)

Similarly to the process performed for the equilibrium $P_1$, we continue the process for $N = 4, 5, 6, \ldots$ and obtain a decomposition of the parameter space $(\mu, \alpha, \varepsilon)$ into stability and instability regions for the equilibrium $P_2$. Figure 5 presents the planar sections of these regions for $N = 1, 2, 3, 4, 5, 6$ by the plane $\mu = 200$.

![Fig. 5. Planar section by the plane $\mu = 200$ of stability and instability regions for $P_2$ when $N = 1, 2, 3, 4, 5, 6$, see Eqs. (4.8)–(4.11) and (B.1)–(B.5).](image)

5. CONCLUSION

We investigated the parametric stability of a pendulum with a suspension point that oscillates harmonically in the vertical direction, with two vertical wires that have uniformly distributed electric charges, equidistant from the pendulum suspension point. We used the Hamiltonian formulation to determine the stability of the equilibria in the parameter space $(\mu, \alpha, \varepsilon)$. We proved that equilibria $P_1 = (0, 0)$ and $P_2 = (\pi, 0)$ are linearly stable when $\mu > -4\alpha$ and $\mu > 4\alpha$, respectively. Next, we normalized the quadratic part of the Hamiltonian function using the Deprit–Hori method to find the surfaces that separate the stability and instability regions in the parameter space. These surfaces were obtained as graphs over the plane $(\mu, 0, \varepsilon)$, determining the coefficients of their parameterizations up to the fifth order of the parameter $\varepsilon$ and as a function of the parameter $\mu$. When we fixed the value of the parameter $\mu$ we obtained the boundary curves of the stability and instability regions in the plane $(\alpha, \varepsilon)$. In particular, when $\mu = 0$, we obtained the boundary curves of the Mathieu equation found previously in [3].
APPENDIX A. BOUNDARY SURFACES FOR THE EQUILIBRIUM $P_1$ WHEN $N = 4, 5, 6$

Here, we present the boundary surface parameterizations for the $P_1$ equilibrium associated with the resonances $2\omega = N$, where $N = 4, 5, 6$. To do so, we use the Hamiltonian (3.7) with $\omega_0^2 = \frac{\mu}{4} + \alpha_0$.

- For $N = 4$

\[
\alpha = \frac{16 - \mu}{4} + \frac{1}{30} \varepsilon^2 + \frac{433}{216000} \varepsilon^4 - \frac{5701}{170100000} \varepsilon^6 + \mathcal{O}(\varepsilon^7),
\]  
\[
\alpha = \frac{16 - \mu}{4} + \frac{1}{30} \varepsilon^2 - \frac{317}{216000} \varepsilon^4 + \frac{4799}{170100000} \varepsilon^6 + \mathcal{O}(\varepsilon^7).
\]  

- For $N = 5$

\[
\alpha = \frac{25 - \mu}{4} + \frac{1}{48} \varepsilon^2 + \frac{11}{193536} \varepsilon^4 + \frac{1}{18432} \varepsilon^5 + \frac{37}{55738368} \varepsilon^6 + \mathcal{O}(\varepsilon^7).
\]

- For $N = 6$

\[
\alpha = \frac{36 - \mu}{4} + \frac{1}{70} \varepsilon^2 + \frac{187}{1097600} \varepsilon^4 + \frac{6743617}{580849200000} \varepsilon^6 + \mathcal{O}(\varepsilon^7),
\]  
\[
\alpha = \frac{36 - \mu}{4} + \frac{1}{70} \varepsilon^2 - \frac{187}{1097600} \varepsilon^4 - \frac{5861633}{580849200000} \varepsilon^6 + \mathcal{O}(\varepsilon^7).
\]

APPENDIX B. BOUNDARY SURFACES FOR THE EQUILIBRIUM $P_2$ WHEN $N = 4, 5, 6$

Similarly to what was done for the equilibrium $P_1$, we present the boundary surface parameterizations for the equilibrium $P_2$ associated with the resonances $2\omega = N$, where $N = 4, 5, 6$. In this case, we use the Hamiltonian (3.7) with $\omega_0^2 = \frac{\mu}{4} - \alpha_0$.

- $N = 4$

\[
\alpha = \frac{\mu - 16}{4} - \frac{1}{30} \varepsilon^2 - \frac{433}{216000} \varepsilon^4 + \frac{5701}{170100000} \varepsilon^6 + \mathcal{O}(\varepsilon^7),
\]  
\[
\alpha = \frac{\mu - 16}{4} - \frac{1}{30} \varepsilon^2 + \frac{317}{216000} \varepsilon^4 - \frac{4799}{170100000} \varepsilon^6 + \mathcal{O}(\varepsilon^7).
\]  

- $N = 5$

\[
\alpha = \frac{\mu - 25}{4} - \frac{1}{48} \varepsilon^2 - \frac{11}{193536} \varepsilon^4 + \frac{1}{18432} \varepsilon^5 - \frac{37}{55738368} \varepsilon^6 + \mathcal{O}(\varepsilon^7).
\]

- $N = 6$

\[
\alpha = \frac{\mu - 36}{4} - \frac{1}{70} \varepsilon^2 - \frac{187}{1097600} \varepsilon^4 - \frac{6743617}{580849200000} \varepsilon^6 + \mathcal{O}(\varepsilon^7),
\]  
\[
\alpha = \frac{\mu - 36}{4} - \frac{1}{70} \varepsilon^2 + \frac{187}{1097600} \varepsilon^4 + \frac{5861633 \varepsilon^6}{580849200000} + \mathcal{O}(\varepsilon^7).
\]

ACKNOWLEDGMENTS

The authors thank Professor Hildeberto Cabral for useful discussions that contributed to the development of this work. We also thank the anonymous referees whose inquiries and comments greatly contributed to improving the manuscript.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.
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