On kernel estimators of density for reversible Markov chains

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Abstract

In this paper we investigate the kernel estimator of the density for a stationary reversible Markov chain. The proofs are based on a new central limit theorem for a triangular array of reversible Markov chains obtained under conditions imposed to covariances, which has interest in itself.

1 Introduction and main results

For estimating the marginal density for dependent sequences the dependence structure plays an important role. One possible estimator is the kernel estimator introduced by Rosenblatt (1956a). In general the dependence is imposed in terms of mixing conditions (Bradley, 1993, Bosq et al, 1999 among many others), in terms of coupling coefficients for functions of i.i.d. (Wu et al, 2010) or positive association of random variables (Lin, 2003).

In this paper we study the kernel estimator for reversible Markov chains. It is well known that for strictly stationary reversible Markov chains the covariances can be viewed as a measure of dependence (see Kipnis and Varadhan, 1986). When estimating the density via kernel estimators we introduce a triangular array of random variables which is only row-wise stationary. This makes it difficult for studying the kernel density of the marginal distribution for reversible Markov chains without imposing recurrence conditions. As a matter of fact results on the kernel estimators for marginal density of reversible Markov chains are very rare. We noticed only the paper by Lei (2006) dealing with large deviations results for the integrated error of the kernel density estimators for reversible Markov chains. The class they considered is of reversible irreducible Markov chains with the transitions satisfying a uniform integrability condition in square mean. However their result cannot be applied when studying the density at a point or several points. In this paper we develop tools that make this study possible.

Let \((X_n)_{n \in \mathbb{Z}}\) be a stationary reversible Markov chain with marginal distribution \(\pi(A) = P(X_n \in A)\), for all Borel sets \(A\). For a stationary Markov chain the reversibility means that the

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distribution of \((X_0, X_1)\) is the same as of \((X_1, X_0)\). Assume that \(\pi\) has a marginal density \(f(x)\), continuous at \(x\). We shall consider in this paper the Rosenblatt (1956a) estimator of density defined by

\[
\hat{f}_n(x) = \frac{1}{nb_n} \sum_{k=1}^{n} K\left(\frac{x - X_k}{b_n}\right),
\]

where \(b_n\) is a bandwidth converging to 0 and \(K\) is a kernel, a known density function.

The problem considered in this paper is the consistency and the speed of convergence of the kernel density estimator of the density at several points \((x_j)_{1 \leq j \leq m}\) which will be given via a multivariate CLT.

To treat the problem we shall use the following notation

\[
H_k(u, v) = P(X_0 > u, X_k > v) - P(X_0 > u)P(X_k > v)
\]

and we shall denote

\[
\eta_k = \iint |H_k(u, v)| dudv.
\]

The condition we shall impose to \(\eta_k\) is

\[
\eta_k \leq \frac{1}{k^4 l(k)},
\]

where \(l(x)\) is a function increasing to infinity such that for any positive \(k\), \(\lim_{x \to \infty} l(kx)/l(x) = 1\) (slowly varying at infinite).

The following condition is imposed to the joint density of the vector \((X_0, X_2)\) and a family of points of interest \((x_j)_{1 \leq j \leq m}\): there exists the joint density \(f_2(x, y)\) of \((X_0, X_2)\) which is locally bounded around any pair \((x_i, x_j)\) in the sense that there exists a constant \(M\) and a constant \(C_M\) (both depending on \((x_i, x_j)\)) such that

\[
\sup_{|a| < M} |f_2(x_i + a, x_j + a)| < C_M.
\]

This condition is weaker than the condition which is usually imposed in the dependent cases which requires that all the densities of vectors \((X_0, X_j)\) are uniformly bounded on \(\mathbb{R}^2\) (see condition in Bosq, 1998, or in Bosq et al, 1999). Local conditions can be found for instance in papers by Liebscher (1999) and Dedecker and Merlevède (2002).

All along the paper, we assume that the kernel \(K\) satisfies the Condition C below:

(C1) \(K\) is symmetric decreasing on \((0, \infty)\) and \(\int K(u)du = 1\).
(C2) \(x^2K(x) \to 0\) as \(x \to \infty\).
(C3) \(K\) is differentiable with \(K'(x)\) bounded.

Note that a normal kernel will satisfy all these conditions. The convergence in distribution will be denoted by \(\Rightarrow\), and \(\overset{P}{\to}\) denotes the convergence in probability.

The main result of this paper is the following:

**Theorem 1** Let \((X_j)_{j \in \mathbb{Z}}\) be a stationary reversible Markov chain with marginal density function \(f(x)\) satisfying condition (4). Assume that the bandwidth \(b_n\) in the estimator (1) satisfies \(nb_n^4 \to \infty\) and the kernel \(K\) satisfies Condition C. Then, at any points \(x_1, \ldots, x_m\) where \(f(x)\) is
continuous, different of 0 and the joint densities satisfy condition (5), we have

\[ \sqrt{nb_n} \left( \frac{\hat{f}_n(x_j) - \mathbb{E}\hat{f}_n(x_j)}{\left(\hat{f}_n(x_j) \int K^2(u)du\right)^{1/2}}, 1 \leq j \leq m \right) \Rightarrow N(0, I_m), \]

where \( I_m \) is the identity matrix.

It is well known that if the density is twice continuously differentiable at \( x_j \) then the bias is of order (see Härdle 1991, relation (2.3.2))

\[ \mathbb{E}(\hat{f}_n(x_j)) - f(x_j) = \frac{b_n^2}{2} f''(x_j) + o(b_n^2) \text{ as } b_n \to 0. \]

By combining this result with Theorem 1 we get the following corollary:

**Corollary 2** In addition to the conditions of Theorem 1, assume that \( f \) is twice continuously differentiable at \( (x_j)_{1 \leq j \leq m} \) and \( nb_n^5 \to 0 \). Then

\[ \sqrt{nb_n} \left( \frac{\hat{f}_n(x_j) - f(x_j)}{\left(\hat{f}_n(x_j) \int K^2(u)du\right)^{1/2}}, 1 \leq j \leq m \right) \Rightarrow N(0, I_m). \]

Let us comment about the dependence coefficient used in our results defined in (3).

If we have positive dependence, in the sense that \( H_k(x, y) \geq 0 \) for all \( (x, y) \in \mathbb{R}^2 \), then by the Lehmann Lemma (see Newman, 1980) we have

\[ \eta_k = \text{cov}(X_0, X_k). \]

This coefficient can also be controlled by a pairwise mixing condition which is weaker than strong mixing coefficient introduced by Rosenblatt (1956b). As in Rio (2000) relation (1.8a), define

\[ \bar{\alpha}_k = \bar{\alpha}_k(X_0, X_k) = 2 \sup_{(x, y) \in \mathbb{R}^2} |H_k(x, y)| \]

Theorem 1.1 in Rio (2000) states an estimate of the covariance between \( X_0 \) and \( X_k \) in terms of \( \alpha_k \). However, in the proof, the author actually estimated \( \eta_k \). Therefore we have

\[ \eta_k \leq 2 \int_{0}^{\bar{\alpha}_k} Q_{|X_0|}(u)du, \]

where \( Q_{|X_0|} \) is the quantile function of \( |X_0| \), i.e. the generalized inverse of the function \( \mathbb{P}(|X_0| > t) \).

In particular, for \( \delta > 0 \)

\[ \eta_k \leq 2 \bar{\alpha}_k^{\delta/2} |X_0|^{2+\delta}, \]

and if \( ||X_0||_{\infty} \leq 1 \) then \( \eta_k \leq 2 \bar{\alpha}_k \).

Recall that, in the stationary setting, the Rosenblatt pairwise strong mixing coefficient, for
an integer \( k > 0 \), is defined by

\[
\alpha_k = \alpha_k(X_0, X_k) = \sup_{A, B \in \mathcal{B}} |\mathbb{P}(X_0 \in A, X_k \in B) - \mathbb{P}(X_0 \in A)(X_k \in B)|,
\]

where \( \mathcal{B} \) denotes the Borel sigma algebra on the line. Clearly \( \bar{\alpha}_k \leq 2\alpha_k \). In terms of these mixing coefficients we make the following remark.

**Remark 3** Theorem 1 also holds if we replace condition (4) by \( \sum_{k \geq 1} k\alpha_k < \infty \).

It should be noted that this strong mixing rate was already pointed out in Bosq et al. (1999), without assuming reversibility. The advantage here is that we can have the asymptotic normality of the kernel estimators by only requiring a condition on the joint density of the vector \((X_0, X_2)\) and not on all the joint densities.

We finish this section by mentioning a few notations which will be used in this paper. The largest integer smaller or equal to \( x \) will be denoted by \( \lfloor x \rfloor \). By \( c_n \ll d_n \) we understand that \( c_n \leq Cd_n \) for some \( C > 0 \) and all \( n \); we denote by \( a \lor b \) the maximum between \( a \) and \( b \).

### 2 Technical results

The proof relies on the properties of reversible Markov chains. We shall point first some monotonicity conditions for integrals of functions of reversible Markov chains. The regular conditional probability of \( X_1 \) given \( X_0 \) will be denoted by \( Q(x, A) = \mathbb{P}(X_1 \in A | X_0 = x) \). Let \( Q \) also denote the Markov operator acting via \((Qf)(x) = \int_S f(s)Q(x, ds)\). Next, let \( L^0_2(\pi) \) be the set of measurable functions such that \( \int g^2d\pi < \infty \) and \( \int gd\pi = 0 \). In operator terms the Markov chain is called reversible if \( Q = Q^* \), where \( Q^* \) is the adjoint operator of \( Q \).

For some function \( g \in L^0_2(\pi) \), let \( Y_i = g(X_i) \). Denote \( \mathcal{F}_n = \sigma(\cdots X_{n-1}, X_n) \); \( \mathbb{E}_k Y = \mathbb{E}(Y | \mathcal{F}_k) \). It is well known that, by the definition of Markov chains, for \( Y \in \sigma(X_i, i \geq 1) \) we have \( \mathbb{E}_0 Y = \mathbb{E}(Y | X_0) = \mathbb{E}_{X_0}(Y) \). From the spectral theory of self-adjoint operators on Hilbert spaces (see for instance Rudin, 1991), it is well known that for every \( g \in L^0_2(\pi) \) there is a unique \textit{transition spectral measure} \( \nu \) supported on the spectrum of the operator \([-1, 1] \), such that

\[
\mathbb{E}(\mathbb{E}_0(Y_i)\mathbb{E}_0(Y_j)) = \int_{-1}^{1} s^{i+j}\nu(ds). \quad (6)
\]

By using this representation we give the following lemma which relates the conditional expectation with the covariances for functions of reversible Markov chains. It also points out several monotonicity conditions for the covariances.

**Lemma 4** For stationary reversible Markov chains and every positive integers \( k, j \) we have

\[
\mathbb{E}(\mathbb{E}_0 Y_k\mathbb{E}_0 Y_j) = \mathbb{E}(Y_0 Y_{k+j}). \quad (7)
\]

For any integer \( k \geq 0 \)

\[
\mathbb{E}(Y_0(Y_{2k} + Y_{2k+1})) \geq 0, \quad (8)
\]
also, for any integers \( j \) and \( k \) such that \( 0 \leq j \leq k \)
\[
\mathbb{E}(Y_0Y_{2j}) \geq \mathbb{E}(Y_0Y_{2k}) \geq 0,
\]
and for integers \( k \geq 2 \)
\[
\mathbb{E}(Y_0Y_k) \leq \mathbb{E}(Y_0Y_2).
\]
For any positive integer \( \ell \) and any \( j \geq 2\ell \)
\[
\sum_{k=2\ell}^{j} \mathbb{E}(Y_0Y_k) \geq 0.
\]
For any positive integer \( \ell \) and any \( n \geq 2\ell \)
\[
\max_{2\ell \leq j \leq n} \sum_{k=2\ell}^{j} \mathbb{E}(Y_0Y_k) \leq \sum_{k=2\ell}^{n} \mathbb{E}(Y_0Y_k) + \mathbb{E}(Y_0Y_{2\ell}).
\]

**Proof.** To prove relation (7) just note that \( \mathbb{E}(Y_0Y_{k+j}) = \mathbb{E}(Y_0\mathbb{E}Y_{k+j}) \) and apply relation (6). Then, note that
\[
\mathbb{E}(Y_0(Y_{2k} + Y_{2k+1})) = \int_{-1}^{1} (s^{2k} + s^{2k+1})\nu(ds),
\]
and \( s^{2k}(1 + s) \geq 0 \) for all \(-1 \leq s \leq 1\); so relation (8) holds by (6). Relation (9) is clearly true since for \( 0 \leq j \leq k \), we have \( s^{2j} \geq s^{2k} \geq 0 \) for all \( s \). Finally in order to show relation (10) just note that for all \(-1 \leq s \leq 1\) and any integer \( k \geq 2 \) we have \( s^k \leq s^2 \), which we combine with (6).

Relations (11) and (12) are obtained via a blocking argument. Now, if \( j \) is odd, say \( j = 2m + 1 \), we can write
\[
\sum_{k=2\ell}^{j+1} \mathbb{E}(Y_0Y_k) = \sum_{k=2\ell}^{m} (\mathbb{E}(Y_0Y_{2k}) + \mathbb{E}(Y_0Y_{2k+1})) \geq 0,
\]
since, by relation (8) in the right hand side we have a sum of positive terms.

On the other hand, if \( j \) is even, say \( j = 2(m + 1) \)
\[
\sum_{k=2\ell}^{j+2} \mathbb{E}(Y_0Y_k) = \mathbb{E}(Y_0Y_{2(m + 1)}) + \sum_{k=2\ell}^{2m+1} \mathbb{E}(Y_0Y_k).
\]
By relation (9), \( \mathbb{E}(Y_0Y_{2(m + 1)}) \geq 0 \). Therefore (11) is true for all \( j \geq 2\ell \).

Now, if \( n \geq 2\ell \), by combining the latter considerations with (9) and (10)
\[
\max_{2\ell \leq j \leq n} \sum_{k=2\ell}^{j} \mathbb{E}(Y_0Y_k) \leq \max_{2\ell \leq j \leq n} \sum_{k=2\ell}^{2j} \mathbb{E}(Y_0Y_k) \vee \max_{2\ell \leq 2m+1 \leq n} \sum_{k=2\ell}^{2m+1} \mathbb{E}(Y_0Y_k)
\]
\[
\leq \max_{2\ell \leq 2m+1 \leq n} \sum_{k=2\ell}^{2m+1} \mathbb{E}(Y_0Y_k) + \max_{2\ell \leq j \leq n} \mathbb{E}(Y_0Y_{2j}) \leq \mathbb{E}(Y_0Y_{2\ell}) + \sum_{k=2\ell}^{m} \mathbb{E}(Y_0Y_k).
\]
where \( m_n \) is the largest odd integer smaller than \( n \). If \( n \) is odd \( n = m_n \). If \( n \) is even, \( n > 2\ell \), then \( m_n = n - 1 \). By taking into account relation (9) we can add in this case a positive term, \( \mathbb{E}(Y_0 Y_n) \), and obtain overall relation (12). □

Next, we give a CLT for a triangular array of row-wise stationary reversible Markov chains. The conditions for the CLT are imposed to the covariances of both the variables and their squares.

**Theorem 5** Let \((X_i)_{i \in \mathbb{Z}}\) be a stationary reversible Markov chain. For real functions \( f_n \), define

\[
X_{n,k} = f_n(X_k). \tag{13}
\]

Assume that

\[
\mathbb{E}X_{n,k}^4 < \infty; \quad \mathbb{E}X_{n,k} = 0 \quad \text{and} \quad \mathbb{E}(X_{n,0}^2) \to \sigma^2, \tag{14}
\]

\[
\text{cov}(X_{n,0}, X_{n,2}) + \sum_{k=2}^{n} \text{cov}(X_{n,0}, X_{n,k}) \to 0, \tag{15}
\]

and

\[
\frac{1}{n}(\text{var}(X_{n,0}^2) + \sum_{u=0}^{n} \text{cov}(X_{n,0}^2, X_{n,u}^2)) \to 0. \tag{16}
\]

Then

\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_{n,k} \Rightarrow N(0, \sigma^2).
\]

**Proof.** We start from a standard martingale decomposition by using projections:

\[
S_n = \sum_{k=1}^{n} (X_{n,k} - \mathbb{E}_{k-1}X_{n,k}) + \sum_{k=1}^{n} \mathbb{E}_{k-1}X_{n,k}
\]

\[
= \sum_{k=1}^{n} D_{n,k} + \sum_{k=1}^{n} \mathbb{E}_{k-1}X_{n,k}.
\]

Note that \( D_{n,k} = X_{n,k} - \mathbb{E}_{k-1}X_{n,k} \) are martingale differences adapted to \((\mathcal{F}_k)_{k \geq 1}\).

We show first that the second term divided by \( \sqrt{n} \) is negligible for the convergence in distribution. To show this we estimate \( \text{var}(\sum_{k=1}^{n} \mathbb{E}_{k-1}X_{n,k}) \). By using the properties of conditional expectation, stationarity and relation (7) in Lemma 4, for \( k \leq j \) we obtain

\[
\mathbb{E}(\mathbb{E}_{k-1}X_{n,k}\mathbb{E}_{j-1}X_{n,j}) = \mathbb{E}(X_{n,j-1}X_{n,k}) = \mathbb{E}(X_{n,j-k+1}X_{0}X_{n,1})
\]

\[
= \mathbb{E}(X_{0}X_{n,1}X_{0}X_{n,j-k+1}) = \mathbb{E}(X_{n,0}X_{n,j-k+2})
\]
and so,
\[
\mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}X_{n,k})^2 = n \cdot \text{cov}(X_{n,0}, X_{n,2}) + 2 \sum_{j=2}^{n} \sum_{k=1}^{j-1} \mathbb{E}(X_{n,0}X_{n,j-k+2})
\]
\[
\leq 2n \max_{2 \leq j \leq n} \sum_{k=2}^{j} \mathbb{E}(X_{n,0}X_{n,k}).
\]

By applying now relation (12) of Lemma 4
\[
\frac{1}{n} \mathbb{E}(\sum_{k=1}^{n} \mathbb{E}_{k-1}X_{n,k})^2 \leq 2\mathbb{E}(X_{n,0}X_{n,2}) + 2 \sum_{k=2}^{n} \mathbb{E}(X_{n,0}X_{n,k}),
\]
which converges to 0 by (15).

Now we analyze the martingale differences via Theorem 6 given in Appendix. To show that
\[
\max_{1 \leq k \leq n} |D_{n,k}| / \sqrt{n}
\]
is uniformly integrable we show that \(\mathbb{E}(\max_{1 \leq k \leq n} D_{n,k}^2) \leq Cn\), for all \(n\) and some constant \(C > 0\). Indeed, since \(\mathbb{E}(D_{n,0}^2) \leq \mathbb{E}(X_{n,0}^2)\), by (14) we note that there is a positive constant \(C\) such that \(\mathbb{E}(D_{n,0}^2) \leq C\) and therefore, by stationarity
\[
\mathbb{E}(\max_{1 \leq k \leq n} D_{n,k}^2) \leq \sum_{k=1}^{n} \mathbb{E}(D_{n,k}^2) \leq Cn.
\]

It remains to verify
\[
\frac{1}{n} \sum_{k=1}^{[nt]} D_{n,k}^2 \overset{P}{\to} \sigma^2.
\]
We note that
\[
D_{n,k}^2 = X_{n,k}^2 + (\mathbb{E}_{k-1}X_{n,k})^2 - 2X_{n,k}(\mathbb{E}_{k-1}X_{n,k}) = X_{n,k}^2 + I_{n,k}.
\]
Furthermore, by the Cauchy-Schwartz inequality, (14), Lemma 4 and stationarity
\[
\frac{1}{n} \mathbb{E} \left( \sum_{k=1}^{n} |I_{n,k}| \right) = \frac{1}{n} \mathbb{E} \left( \sum_{k=1}^{n} |(\mathbb{E}_{k-1}X_{n,k})^2 - 2X_{n,k}(\mathbb{E}_{k-1}X_{n,k})| \right)
\]
\[
\leq \mathbb{E}((\mathbb{E}_{-1}X_{n,0})^2) + 2 \sum_{k=1}^{n} ||X_{n,k}||_2 ||\mathbb{E}_{k-1}X_{n,k}||_2
\]
\[
\leq \text{cov}(X_{n,0}, X_{n,2}) + C \sqrt{\text{cov}(X_{n,0}, X_{n,2})}.
\]
We see that the last quantity converges to 0 by condition (15) combined with relation (11) in Lemma 4. We also note that by stationarity and (14),
\[
\frac{1}{n} \mathbb{E} \left( \sum_{k=1}^{[nt]} X_{n,k}^2 \right) = \frac{[nt]}{n} \mathbb{E}(X_{n,0}^2) \to \sigma^2 t.
\]
So, it remains to show that
\[ \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} (X_{n,k}^2 - \mathbb{E}(X_{n,0}^2)) \overset{P}{\rightarrow} 0, \]
which will be implied by
\[ \text{var} \left( \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} X_{n,k}^2 \right) \rightarrow 0. \]
We estimate now this variance. Note that by relation (12) in Lemma 4,
\[ \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \text{var}(X_{n,k}^2) + \frac{2}{n^2} \sum_{k=2}^{\lfloor nt \rfloor} \sum_{u=1}^{k-1} \text{cov}(X_{n,k}^2, X_{n,u}^2) \leq \frac{2}{n^2} \left( \max_{0 \leq k \leq n} \sum_{u=0}^{k} \text{cov}(X_{n,0}^2, X_{n,u}^2) \right) \leq 2 t \frac{n}{t} \left( \text{var}(X_{n,0}^2) + \sum_{u=0}^{n} \text{cov}(X_{n,0}^2, X_{n,u}^2) \right). \]
The result follows by condition (16). □

3 Proof of Theorem 1

By the consistency of \( \hat{f}_n(x_j) \) due to the continuity of \( f(x) \) at \( x_j \) and the assumptions we made on the bandwidth and kernel we only need to show that
\[ \sqrt{nb_n} \left( \frac{\hat{f}_n(x_j) - \mathbb{E}\hat{f}_n(x_j)}{(f(x_j) \int K^2(u)du)^{1/2}}, 1 \leq j \leq m \right) \Rightarrow N(0, I_m). \]
By the Cramer-Wold device, it suffices to prove that
\[ \sqrt{nb_n} \sum_{j=1}^{m} \lambda_j \frac{\hat{f}_n(x_j) - \mathbb{E}\hat{f}_n(x_j)}{(f(x_j) \int K^2(u)du)^{1/2}} \Rightarrow \sum_{j=1}^{m} \lambda_j Z_j \]
for arbitrary fixed \( \lambda_1, \cdots, \lambda_m \). Here \( Z_j, 1 \leq j \leq m \) are i.i.d. standard normal random variables.
Let \( S_n = \sum_{i=1}^{n} Y_{n,i} \) where
\[ Y_{n,i} = \frac{1}{\sqrt{b_n}} \sum_{j=1}^{m} \frac{\lambda_j}{(f(x_j) \int K^2(u)du)^{1/2}} (K(x_j - X_i) - \mathbb{E}K(x_j - X_i)). \]
We shall verify the conditions of Theorem 5. We verify first (14).
\[ \text{var}(Y_{n,i}) = \frac{1}{b_n} \sum_{j=1}^{m} \frac{\lambda_j^2}{(f(x_j) \int K^2(u)du)^{1/2}} \text{var}(K(x_j - X_0)), \]
\[ + \frac{2}{b_n} \sum_{j=1}^{m} \sum_{p=1}^{m} \frac{\lambda_j \lambda_p}{(f(x_j) f(x_p))^{1/2} \int K^2(u)du} \text{cov}(K(x_j - X_0), K(x_p - X_0)) = I_n + II_n. \]
Now, by Bochner’s lemma (see Parzen, 1962, or Bosq, 1998) and the fact $b_n \to 0$,
\[
\lim_{n \to \infty} \text{var} \left( \frac{1}{\sqrt{b_n}} K \left( \frac{x_j - X_0}{b_n} \right) \right) = f(x_j) \int K^2(u) du.
\]
Therefore
\[
I_n \to \sum_{j=1}^m \lambda_j^2.
\]
On the other hand by simple calculus computations involving the symmetry of $K$, for $j \neq p$ we have
\[
\frac{1}{b_n^2} \text{cov} \left( K \left( \frac{x_j - X_0}{b_n} \right), K \left( \frac{x_p - X_0}{b_n} \right) \right)
= \frac{1}{b_n^2} \int K \left( \frac{x_j - u}{b_n} \right) K \left( \frac{x_p - u}{b_n} \right) f(u) du - \frac{1}{b_n} \int K \left( \frac{x_j - u}{b_n} \right) f(u) du \int K \left( \frac{x_p - u}{b_n} \right) f(u) du
= \int K(v) K \left( v + \frac{x_p - x_j}{b_n} \right) f(x_j - b_n v) dv - b_n \int K(v) f(x_j - b_n v) dv \int K(v) f(x_p - b_n v) dv.
\]
Clearly the second term is convergent to 0 by Bochner’s lemma and the fact that $b_n \to 0$. For the first term we cannot apply directly the Bochner lemma, but by using the same arguments as in its proof presented in (Parzen, 1962) along with the Lebesgue dominated convergence theorem, under our conditions we deduce that this term is also negligible. Hence $\text{Var}(Y_{n,i}) \to \sum_{j=1}^m \lambda_j^2$.

To verify condition (16) we introduce the function
\[
\tilde{g}(u) = \left( \sum_{j=1}^m \frac{\lambda_j}{f(x_j)} \int K^2(u) du \right)^{1/2} \left( K \left( \frac{x_j - u}{b_n} \right) - \mathbb{E} K \left( \frac{x_j - X_0}{b_n} \right) \right)^2.
\]
Since by our conditions on $K$, the function $\tilde{g}(u)$ has bounded derivative, by Newman extension of Hoeffding lemma (see relation (22) in Newman, 1980),
\[
\text{cov}(Y_{n,0}^2, Y_{n,k}^2) = \frac{1}{b_n^2} \int \int \tilde{g}'(u) \tilde{g}'(v) H_k(u, v) du dv.
\]
Therefore, with $C = (\sum_{j=1}^m \lambda_j) (f(x_j) \int K^2(u) du)^{1/2}$ we obtain
\[
\frac{1}{n} \sum_{k=1}^n |\text{cov}(Y_{n,0}^2, Y_{n,k}^2)| \leq \frac{16C}{n b_n^4} \|KK'\|_2^2 \sum_{k=1}^n \eta_k,
\]
which converges to 0 by taking into account our conditions on $b_n$ and $\eta_k$.

We have also to treat $\text{var}(Y_{n,0}^2)/n$. We shall apply first H"older inequality to obtain
\[
\text{var}(Y_{n,0}^2) \leq \mathbb{E}(Y_{n,0}^4) \leq \frac{8 m^3}{b_n^2} \sum_{j=1}^m \frac{\lambda_j^4}{(f(x_j) \int K^2(u) du)^2} (K^4 \left( \frac{x_j - X_0}{b_n} \right) + (\mathbb{E} K \left( \frac{x_j - X_0}{b_n} \right))^4).
\]
Note that for any $p \geq 1$

$$\mathbb{E}K^p(\frac{x - X_0}{b_n}) = b_n \int K^p(u)f(x + b_n u)du.$$ 

So by the Bochner’s lemma and the fact that the kernel is bounded

$$\frac{1}{n} \var(Y_{n,0}^2) \leq \max_{1 \leq j \leq m} \frac{C_m}{nb_n} \int K^4(u)f(x_j + b_n u)du + \frac{4}{n} b_n^2 (\int K(u)f(x_j + b_n u)du)^4 \to 0$$

provided $nb_n \to \infty$. Therefore (16) is satisfied.

We turn now to verify condition (15). Since

$$\text{cov}(Y_{n,0}, Y_{n,k}) = \frac{1}{b_n} \sum_{j,p=1}^m \frac{\lambda_j \lambda_p}{(f(x_j)f(x_p))^{1/2}} \int K^2(u)du \text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_k}{b_n})), $$

it is enough to show that for any $j$ and $p$ fixed

$$\frac{1}{b_n} \text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_k}{b_n})) \to 0$$

(18)

and

$$\sum_{k=2}^n \frac{1}{b_n} \text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_k}{b_n})) \to 0.$$  

(19)

We shall estimate $\text{cov}(X_{n,0}, X_{n,k})$ in two different ways and take the minimum of these estimates.

By Lemma 4, for $k \geq 2$

$$A = \text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_k}{b_n})) \leq \text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_2}{b_n}))$$

$$= \frac{1}{b_n} \int \int K(\frac{x_j - u}{b_n})K(\frac{x_p - v}{b_n})(f_2(u,v) - f(u)f(v))dudv.$$

By changing the variable

$$A = b_n \int \int K(u)K(v)(f_2(x_j + ub_n, x_p + vb_n) - f(x_j + ub_n)f(x_p + vb_n))dudv$$  

(20)

$$\leq b_n \int \int K(u)K(v)f_2(x_j + ub_n, x_p + vb_n)dudv.$$

To analyze this term we divide the integral in (20) on 4 sets

$$(|u - x_j| \leq M) \times (|v - x_p| \leq M)$$

$$(|u - x_j| > M) \times (|v - x_p| \leq M)$$

$$(|u - x_j| \leq M) \times (|v - x_p| > M)$$

$$(|u - x_j| \geq M) \times (|v - x_p| > M).$$
On the first set, \(|u - x_j| \leq M \times |v - x_p| \leq M\), we change the variable and obtain

\[
 b_n \int_{-\frac{M}{b_n}}^{\frac{M}{b_n}} \int_{-\frac{M}{b_n}}^{\frac{M}{b_n}} K(u)K(v)(f_2(x_j - b_n u, x_p - b_n v)\,du\,dv \leq 
\leq b_n \sup_{|a| < M} |f_2(x_j + a, x_p + a)| \int_{-\frac{M}{b_n}}^{\frac{M}{b_n}} \int_{-\frac{M}{b_n}}^{\frac{M}{b_n}} K(u)K(v)\,du\,dv.
\]

By our assumptions this term is smaller than \(b_n C_M\). On the set \(|u - x_j| > M \times |v - x_p| \leq M\) we have

\[
 \frac{1}{b_n}E(K\left(\frac{x_j - X_0}{b_n}\right)K\left(\frac{x_p - X_2}{b_n}\right)I(|X_0 - x_j| > M)I(|X_2 - x_p| \leq M) \leq ||K||_{\infty} \frac{1}{b_n}K\left(\frac{M}{b_n}\right).
\]

A similar estimate is obtained on the set \(|u - x_j| \leq M \times (|v - x_p| > M\). On \(|u - x_j| > M \times (|v - x_p| > M\) we estimate in the following way

\[
 \frac{1}{b_n}E(K\left(\frac{x_j - X_0}{b_n}\right)K\left(\frac{x_p - X_2}{b_n}\right)I(|X_0 - x_j| > M)I(|X_2 - x_p| > M) \leq \frac{1}{b_n}K^2\left(\frac{M}{b_n}\right).
\]

So, for \(n\) sufficiently large

\[
 \frac{1}{b_n} \text{cov}(K\left(\frac{x_j - X_0}{b_n}\right), K\left(\frac{x_p - X_2}{b_n}\right)) \leq b_n C(x_j, x_p) \tag{21}
\]

where

\[
 C(x_j, x_p) = C_M + 2||K||_{\infty} \frac{1}{b_n^2}K\left(\frac{M}{b_n}\right) + \frac{1}{b_n^2}K^2\left(\frac{M}{b_n}\right).
\]

By our conditions on \(K\) for \(n\) sufficiently large \(C(x_j, x_p)\) is bounded.

So clearly by (21) condition (18) is satisfied.

On the other hand,

\[
 \frac{1}{b_n^3} \int \int K'\left(\frac{x_j - u}{b_n}\right)K'\left(\frac{x_p - v}{b_n}\right)H_k(u, v)\,du\,dv \leq ||K'||_{\infty}^2 \frac{1}{b_n^3} \eta_k.
\]

So by combining the estimates in (21) and (22) we have proven that

\[
 \frac{1}{b_n} \text{cov}(K\left(\frac{x_j - X_0}{b_n}\right), K\left(\frac{x_p - X_2}{b_n}\right)) \tag{23}
\]

\[
 \leq \min(b_n C(x_j, x_p), c \frac{1}{b_n^2} \text{cov}(X_0, X_2)) \ll \min(b_n, \frac{1}{b_n^3} \eta_k).
\]

To continue we use the estimate from (23) to bound the sum in the right hand side of (19). We shall divide the sum in two, up to \(m_n\) and after \(m_n\). This sequence of positive integers \(m_n\) will be selected later. On the first part of the sum we use the bound of order \(b_n\) and on the second
part of the sum we use the bound $\eta_k/b_n^3$. So, by the properties of slowly varying function $l$,

$$
\frac{1}{b_n^3} \sum_{k=2}^{n} \text{cov}(K(x_j - X_0), K(x_p - X_k)) \ll \sum_{k=2}^{m_n} b_n + \sum_{k=m_n+1}^{n} \frac{1}{b_n^3} \eta_k \\
\ll m_n b_n + \frac{1}{b_n^3 m_n^3 l(m_n)}.
$$

To optimize the sum we take

$$m_n = \lceil \max\left( \frac{1}{\sqrt{b_n}}, \frac{1}{b_n l^{1/6}(1/b_n^{1/2})} \right) \rceil + 1.
$$

Clearly $m_n b_n \to 0$ (since $b_n/\sqrt{b_n} \to 0$ and $1/l^{1/6}(1/b_n^{1/2}) \to 0$).

Since $m_n > b_n^{-1/2}$ and $l$ is increasing we have $l(m_n) > l(b_n^{-1/2})$ and so, since $m_n > (b_n l^{1/6}(1/b_n^{1/2}))^{-1}$,

$$
\frac{1}{b_n^3 m_n^3 l(m_n)} \leq \frac{1}{b_n^3 m_n^3 l(b_n^{-1/2})} \leq \frac{l^{1/2}(1/b_n^{1/2})}{l(b_n^{-1/2})} \leq \frac{1}{\sqrt{l(1/b_n^{1/2})}} \to 0 \text{ as } n \to \infty.
$$

By taking now into account Lemma 4, it follows that the sum in (19) is positive and then (19) follows. Now by (18) and (19) we conclude that condition (15) is satisfied and the result follows. □

**Proof of Remark 3.** To prove this remark we have to replace in the proof of Theorem 1 relation (22) by relation (3.12) in Bosq et al. (1999), namely

$$
\frac{1}{b_n} |\text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_k}{b_n}))| \leq \frac{4}{b_n} ||K||_\infty^2 \alpha_k.
$$

Then, we replace relation (23) by

$$
\frac{1}{b_n} \text{cov}(K(\frac{x_j - X_0}{b_n}), K(\frac{x_p - X_k}{b_n})) \ll \min(b_n, \frac{1}{b_n} \alpha_k),
$$

and follow the proof from the page 88 in Bosq et al. (1999), to obtain the result of this remark.

4 Appendix

**Martingale limit theorem** (Gänsler and Häusler, 1986, pages 315-317).

**Theorem 6** Assume $(D_{n,k})_{1 \leq k \leq n}$ is a triangular array of martingales adapted to an increasing in $k$ filtration $\mathcal{F}_{n,k}$. Assume $\sum_{k=1}^{[nt]} D_{n,k}^2 \overset{P}{\to} \tau^2$ and

$$
\max_{1 \leq k \leq n} |D_{n,k}| \text{ is uniformly integrable} \quad (24)
$$

(as before, by $[x]$ we denote as usual the integer part of $x$). Then $S_{[nt]} \Rightarrow \sigma W(t)$ where $W(t)$ is a standard Brownian measure. In particular $S_n \Rightarrow N(0,\sigma^2)$. 

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References

[1] Bradley, R. C. (1993). Asymptotic normality of some kernel-type estimators of probability density, *Statist. Probab. Lett.* 1 295-300.

[2] Bosq, D. (1998). *Nonparametric statistics for stochastic processes*, Second edition, Springer.

[3] Bosq, D., Merlevède, F. and Peligrad, M. (1999). Asymptotic normality for density kernel estimators in discrete and continuous time, *J. Multivariate Anal.* 68 78-95.

[4] Dedecker J. and Merlevède, F. (2002). Necessary and sufficient conditions for the conditional central limit theorem. *Ann. Probab.* 30 1044-1081.

[5] Gänssler, P. and Häusler, E. (1986). On martingale central limit theory. *Dependence in Probability and Statistics, Progress in Probability and statistics*, Vol 11, Birkhauser 303-335.

[6] Härdle, W. (1991). *Smoothing techniques with implementation in S*, Springer Series in Statistics.

[7] Kipnis, C. and Varadhan, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov chains, *Comm. Math. Phys.* 104 1-19.

[8] Lei, L. (2006). Large deviations of the kernel density estimator in $L_1(\mathbb{R}^d)$ for reversible Markov processes, *Bernoulli* 12 65-83.

[9] Liebscher, E. (1999). Asymptotic normality of nonparametric estimators under $\alpha$-mixing condition. *Statist. Probab. Lett.* 43 243-250.

[10] Lin, Z. (2003). Asymptotic normality of kernel estimates of a density function under association dependence, *Acta Math. Sci. Ser. B Engl. Ed.* 23 345-350.

[11] Newman, C. M. (1980). Normal fluctuations and the FKG inequalities, *Comm. Math. Phys.* 74 119-128.

[12] Parzen, E. (1962). On estimation of a probability function and mode, *Ann. Math. Statist.* 33 1065-1076.

[13] Rio, E. (2000). *Théorie asymptotique des processus aléatoires faiblement dépendants*. Ed. J.M. Ghidaglia et X. Guyon. Mathématiques et Applications 31. Springer.

[14] Rudin, W. (1991). *Functional analysis*, Second edition, McGraw-Hill, Inc., New York.

[15] Rosenblatt, M. (1956a). Remarks on some nonparametric estimates of a density function, *Ann. Math. Statist.* 27 832-837.

[16] Rosenblatt, M. (1956b). A central limit theorem and a strong mixing condition, *Proc. Natl. Acad. Sci. U.S.A.* 42 43-47.
[17] Wu, W. B., Huang, Y. and Huang, Y. (2010). Kernel estimation for time series: An asymptotic theory, *Stochastic Process. Appl.* **120** 2412-2431.