COUPLED NONLINEAR SCHRÖDINGER SYSTEMS WITH POTENTIALS

ALESSIO POMPONIO*

ABSTRACT. Coupled nonlinear Schrödinger systems describe some physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photorefractive media in optics and Bose-Einstein condensates. In this paper, we study the existence of concentrating solutions of a singularly perturbed coupled nonlinear Schrödinger system, in presence of potentials. We show how the location of the concentration points depends strictly on the potentials.

1. INTRODUCTION

Very recently, different authors focused their attention on coupled nonlinear Schrödinger systems which describe physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photorefractive media in optics and Bose-Einstein condensates.

First of all, let us recall that, in the last twenty years, motivated by the study of the propagation of pulse in nonlinear optical fiber, the nonlinear Schrödinger equation,

$$-\Delta u + u = u^3 \quad \text{in} \, \mathbb{R}^3,$$

has been faced by many authors. It has been proved the existence of the least energy solution (ground state solution), which is radial with respect to some point, positive and exponentially decaying with its first derivatives at infinity. Moreover there are also many papers about the semiclassical states for the nonlinear Schrödinger equation with the presence of potentials

$$-\varepsilon^2 \Delta u + V(x)u = u^3 \quad \text{in} \, \mathbb{R}^3,$$

giving sufficient and necessary conditions to the existence of solutions concentrating in some points, and recently, in set with non-zero dimension, (see e.g. [4–8, 13, 15, 16, 20, 21, 27, 28, 30, 31]).

However, by I.P. Kaminow [19], we know that single-mode optical fibers are not really “single-mode”, but actually bimodal due to the presence of birefringence. This birefringence can deeply influence the way in which an optical evolves during the propagation along the fiber. Indeed, it can occur that the linear birefringence makes a pulse split in two, while nonlinear birefringent traps them together against splitting. C.R. Menyuk [25, 26] showed that the evolution of two

* The author was partially supported by the MIUR research project “Metodi Variazionali ed Equazioni Differenziali Nonlineari”.

Keywords: Coupled nonlinear Schrödinger systems, concentrating solutions.

2000 Mathematics Subject Classification: 35B40, 35J50, 35Q55.
orthogonal pulse envelopes in birefringent optical fibers is governed by the following coupled nonlinear Schrödinger system

\[
\begin{aligned}
    i\dot{\phi} + \phi_{xx} + |\phi|^2\phi + \beta |\psi|^2\phi &= 0, \\
    i\dot{\psi} + \psi_{xx} + |\psi|^2\psi + \beta |\phi|^2\psi &= 0,
\end{aligned}
\]

with \(\beta\) positive constant depending on the anisotropy of the fibers. System (1.1) is also important for industrial applications in fiber communications systems [17] and all-optical switching devices [18]. If one seeks for standing wave solutions of (1.1), namely solutions of the form

\[
\phi(x, t) = e^{i\omega_1^2 t}u(x) \quad \text{and} \quad \psi(x, t) = e^{i\omega_2^2 t}v(x),
\]

then (1.1) becomes

\[
\begin{aligned}
    -u_{xx} + u &= |u|^2u + \beta |v|^2u \quad \text{in } \mathbb{R}, \\
    -v_{xx} + w^2v &= |v|^2v + \beta |u|^2v \quad \text{in } \mathbb{R},
\end{aligned}
\]

with \(w^2 = w_2^2/w_1^2\). Finally we want to recall that (1.2) describes also other physical phenomena, such as Kerr-like photorefractive media in optics, (cf. [1, 10]).

Problem (1.2), in a more general situation and also in higher dimension, has been studied by R. Cipolatti & W. Zumpichiatti [11, 12]. By concentration compactness arguments, they prove the existence and the regularity of the ground state solutions \((u, v) \neq (0, 0)\). Later on, in two very recent papers, T.C. Lin & J. Wei [22] and L.A. Maia, E. Montefusco & B. Pellacci [24] deal with problem (1.2), also in the multidimensional case, and, among other results, they prove the existence of least energy solutions of the type \((u, v)\), with \(u, v > 0\). Moreover T.C. Lin & J. Wei [22] prove that, if \(\beta < 0\), then the ground state solution for (1.2) does not exist. We refer to all these papers and to references therein for more complete informations about (1.2).

Another motivation to the study of coupled Schrödinger systems arises from the Hartree-Fock theory for the double condensate, that is a binary mixture of Bose-Einstein condensates in two different hyperfine states \(|1\rangle\) and \(|2\rangle\) (cf. [14]). Indeed these phenomena are governed by the following system:

\[
\begin{aligned}
    -\varepsilon^2 \Delta u + \lambda_1 u &= \mu_1 u^3 + \beta uv^2 \quad \text{in } \Omega, \\
    -\varepsilon^2 \Delta v + \lambda_2 v &= \mu_2 v^3 + \beta u^2 v \quad \text{in } \Omega, \\
    u, v > 0 \quad &\text{in } \Omega, \\
    u = v = 0 \quad &\text{on } \partial \Omega,
\end{aligned}
\]

where \(\Omega\) is a bounded domain of \(\mathbb{R}^3\). Physically, \(u\) and \(v\) represent the corresponding condensate amplitudes, \(\varepsilon^2 = \frac{\hbar^2}{2m}\), with \(\hbar\) the Planck constant and \(m\) the atom mass. Moreover \(\mu_j = -(N_j - 1)U_{jj}\) and \(\beta = -N_2U_{12}\), with \(N_j \geq 1\) a fixed number of atoms in the hyperfine state \(|j\rangle\), and \(U_{ij} = 4\pi \frac{\hbar^2}{m} a_{ij}\), where \(a_{jj}\)’s and \(a_{12}\) are the intraspecies and interspecies scattering lengths. Besides, by E. Timmermans [29], we infer that \(\mu_j = \mu_j(x)\) represents a chemical potential. For more informations about (1.3), see [22, 23] and references therein.
T.C. Lin & J. Wei, in [23], studied problem (1.3) with \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) positive constant and they proved that if \( \beta < \sqrt{\mu_1 \mu_2} \), for \( \varepsilon \) sufficiently small, \( (1.3) \) has a least energy solution \((u_\varepsilon, v_\varepsilon)\). Moreover, they distinguished two cases: the attractive case and the repulsive one. In the attractive case, which occurs whenever \( \beta > 0 \), \( u_\varepsilon \) and \( v_\varepsilon \) concentrate respectively in \( Q_\varepsilon \) and \( Q'_\varepsilon \), with
\[
\frac{|Q_\varepsilon - Q'_\varepsilon|}{\varepsilon} \to 0, \quad \text{as } \varepsilon \to 0.
\]
Precisely they proved that
\[
d(Q_\varepsilon, \partial\Omega) \to \max_{Q \in \Omega} d(Q, \partial\Omega),
\]
\[
d(Q'_\varepsilon, \partial\Omega) \to \max_{Q \in \Omega} d(Q, \partial\Omega).
\]
In the repulsive case, that is when \( \beta < 0 \), the concentration points \( Q_\varepsilon \) and \( Q'_\varepsilon \) satisfy the following condition:
\[
\varphi(Q_\varepsilon, Q'_\varepsilon) \to \max_{(Q, Q') \in \Omega^2} \varphi(Q, Q'),
\]
where
\[
\varphi(Q, Q') = \min\{\sqrt{\lambda_1}|Q - Q'|, \sqrt{\lambda_2}|Q - Q'|, \sqrt{\lambda_1}d(Q, \partial\Omega), \sqrt{\lambda_2}d(Q', \partial\Omega)\}.
\]
In particular
\[
\frac{|Q_\varepsilon - Q'_\varepsilon|}{\varepsilon} \to \infty, \quad \text{as } \varepsilon \to 0.
\]
Motivated by these results and by the fact that we know that \( \mu_j \) may be not constants (cf. [29]), in this paper we study the following problem:

\[
(P_\varepsilon) \quad \begin{cases}
-\varepsilon^2 \Delta u + J_1(x)u = J_2(x)u^3 + \beta uv^2 & \text{in } \Omega, \\
-\varepsilon^2 \Delta v + K_1(x)v = K_2(x)v^3 + \beta u^2v & \text{in } \Omega, \\
u, v > 0 & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

with \( \Omega \subset \mathbb{R}^3 \), possibly unbounded and with smooth boundary, and with \( \beta < 0 \), namely in the repulsive case. We will show that the presence of the potentials change drastically the situation with respect to the case with positive constants for what concerns the location of peaks, but, in some sense, not the repulsive nature of the problem. In fact, with suitable assumptions on the potentials, for \( \varepsilon \) sufficiently small, we will find solutions \((u_\varepsilon, v_\varepsilon)\) of \((P_\varepsilon)\), even if not of least energy, concentrating respectively on \( Q_\varepsilon \) and \( Q'_\varepsilon \) which tend toward the same point, determined by the potentials, as \( \varepsilon \to 0 \), but with the property that the distance between them divided by \( \varepsilon \) diverges (see Remark 1.2).

Up to our knowledge, in this paper we give a first existence result of concentrating solutions for problem \((P_\varepsilon)\), in presence of potentials.

On the potentials \( J_i \) and \( K_i \) we will do the following assumptions:

**(J)** for \( i = 1, 2 \), \( J_i \in C^1(\Omega, \mathbb{R}) \), \( J_i \) and \( DJ_i \) are bounded; moreover,
\[
J_i(x) \geq C > 0 \quad \text{for all } x \in \Omega;
\]
(K) for \( i = 1, 2 \), \( K_i \in C^1(\Omega, \mathbb{R}) \), \( K_i \) and \( DK_i \) are bounded; moreover,
\[
K_i(x) \geq C > 0 \quad \text{for all } x \in \Omega.
\]

Without lost of generality, we can suppose that there exists \( \varepsilon_0 > 0 \), such that \( \Omega_0 := \Omega \cap (\Omega - \varepsilon_0 e_1) \neq \emptyset \), where \( e_1 = (1, 0, 0) \).

Let us introduce an auxiliary function which will play a crucial role in the study of \((P_\varepsilon)\). Let \( \Gamma: \Omega_0 \to \mathbb{R} \) be a function so defined:
\[
(1.4) \quad \Gamma(Q) = J_1(Q)\frac{1}{2}J_2(Q)^{-1} + K_1(Q)\frac{1}{2}K_2(Q)^{-1}.
\]

Let us observe that by (J) and (K), \( \Gamma \) is well defined.

Our main result is:

**Theorem 1.1.** Suppose (J) and (K) and \( \beta < 0 \). Let \( Q_0 \in \Omega_0 \) be an isolated local strict minimum or maximum of \( \Gamma \). There exists \( \bar{\varepsilon} > 0 \) such that if \( 0 < \varepsilon < \bar{\varepsilon} \), then \((P_\varepsilon)\) possesses a solution \((u_\varepsilon, v_\varepsilon)\) such that \( u_\varepsilon \) concentrates in \( Q_\varepsilon \) with \( Q_\varepsilon \to Q_0 \), as \( \varepsilon \to 0 \), and \( v_\varepsilon \) concentrates in \( Q'_\varepsilon \) with \( Q'_\varepsilon \to Q_0 \), as \( \varepsilon \to 0 \).

**Remark 1.2.** Let us observe that, by the proof, it will be clear that, even if
\[
|Q_\varepsilon - Q'_\varepsilon| \to 0, \quad \text{as } \varepsilon \to 0,
\]
we have
\[
\frac{|Q_\varepsilon - Q'_\varepsilon|}{\varepsilon} \to \infty, \quad \text{as } \varepsilon \to 0.
\]

Let us present how Theorem 1.1 becomes in some particular situations. Let \( H: \Omega \to \mathbb{R} \) satisfying the assumption:

**\( (H) \)** \( H \in C^1(\Omega, \mathbb{R}) \), \( H \) and \( DH \) are bounded; moreover,
\[
H(x) \geq C > 0 \quad \text{for all } x \in \Omega.
\]

**Corollary 1.3.** Suppose (H) and \( \beta < 0 \). Suppose, moreover, that we are in one of the following situations:
- all the potentials \( J_i \) and \( K_i \) coincide with \( H \);
- there exists \( i_0 = 1, 2 \) such that \( J_{i_0} \equiv H \) and \( K_{i_0} \equiv H \), for \( i = i_0 \), while \( J_i \) and \( K_i \) are constant for \( i \neq i_0 \);
- all the potentials \( J_i \) and \( K_i \) are constant, except only one, which coincides with \( H \).

Let \( Q_0 \in \Omega_0 \) be an isolated local strict minimum or maximum of \( H \). There exists \( \bar{\varepsilon} > 0 \) such that if \( 0 < \varepsilon < \bar{\varepsilon} \), then \((P_\varepsilon)\) possesses a solution \((u_\varepsilon, v_\varepsilon)\) such that \( u_\varepsilon \) concentrates in \( Q_\varepsilon \) with \( Q_\varepsilon \to Q_0 \), as \( \varepsilon \to 0 \), and \( v_\varepsilon \) concentrates in \( Q'_\varepsilon \) with \( Q'_\varepsilon \to Q_0 \), as \( \varepsilon \to 0 \).

**Remark 1.4.** If, instead of \( \beta \) constant, we consider \( \beta \in C^1(\Omega, \mathbb{R}) \), bounded and bounded above by a negative constant, then we have exactly the same results.

Finally, we want to observe that we can treat also a more general problem than \((P_\varepsilon)\). Let us consider, indeed,
\[
(\tilde{P}_\varepsilon) \quad \begin{cases}
-\varepsilon^2 \Delta u + J_1(x)u = J_2(x)u^{2p-1} + \beta u^{p-1}v^p & \text{in } \Omega, \\
-\varepsilon^2 \Delta v + K_1(x)v = K_2(x)v^{2p-1} + \beta v^{p-1}u^p & \text{in } \Omega, \\
u, v > 0 & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with $\Omega \subset \mathbb{R}^N$, possibly unbounded and with smooth boundary, $N \geq 3$, $2 < 2p < 2N/(N - 2)$ and with $\beta < 0$.

Also in this case, without lost of generality, we can suppose that there exists $\varepsilon_0 > 0$, such that $\Omega_0 := \Omega \cap (\Omega - \varepsilon_0 e_1) \neq \emptyset$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$.

Let us define now $\bar{\Gamma} : \bar{\Omega}_0 \to \mathbb{R}$ be a function so defined:

$$
\bar{\Gamma}(Q) = J_1(Q) - \frac{N}{p+1} J_2(Q) - \frac{1}{p+1} + K_1(Q) - \frac{N}{p+1} K_2(Q) - \frac{1}{p+1}.
$$

In this case, Theorem 1.1 becomes:

**Theorem 1.5.** Let $N \geq 3$ and $2 < 2p < 2N/(N - 2)$. Suppose (J) and (K) and $\beta < 0$. Let $Q_0 \in \bar{\Omega}_0$ be an isolated local strict minimum or maximum of $\bar{\Gamma}$. There exists $\varepsilon > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $\bar{\Omega}_0$ possesses a solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon$ concentrates in $Q_\varepsilon$ with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$, and $v_\varepsilon$ concentrates in $Q'_\varepsilon$ with $Q'_\varepsilon \to Q_0$, as $\varepsilon \to 0$.

**Remark 1.6.** Let us observe that, if $p = 2$ and $N = 3$, then Theorem 1.1 is nothing else than a particular case of Theorem 1.5. Nevertheless, since problem (P) is more natural and more important by a physical point of view, we prefer to present Theorem 1.1 as our main result and to prove it directly, showing how, with slight modifications, the proof of Theorem 1.5 follows.

Theorem 1.1 will be proved as a particular case of a multiplicity result in Section 5 (see Theorem 5.1). The proof of the theorem relies on a finite dimensional reduction, precisely on the perturbation technique developed in [2, 3, 7]. In Section 2 we give some preliminary lemmas and some estimates which will be useful in Section 3 and Section 4, where we perform the Liapunov-Schmidt reduction, making also the asymptotic expansion of the finite dimensional functional. Finally, in Section 5, we give also a short proof of Theorem 1.5.

**Notation**

- We denote $\Omega_0 := \Omega \cap (\Omega - \varepsilon_0 e_1)$, where $e_1 = (1, 0, 0)$ and $\varepsilon_0$ is sufficiently small such that $\Omega_0 \neq \emptyset$.
- If $r > 0$ and $x_0 \in \mathbb{R}^3$, $B_r(x_0) := \{x \in \mathbb{R}^3 : |x - x_0| < r\}$. We denote with $B_r$ the ball of radius $r$ centered in the origin.
- If $u : \mathbb{R}^3 \to \mathbb{R}$ and $P \in \mathbb{R}^3$, we set $u_P := u(\cdot - P)$.
- If $\varepsilon > 0$, we set $\Omega_\varepsilon := \Omega / \varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\}$.
- We denote $H_\varepsilon = H^1_0(\Omega_\varepsilon) \times H^1_0(\Omega_\varepsilon)$.
- If there is no misunderstanding, we denote with $\| \cdot \|$ and with $(\cdot | \cdot)$ respectively the norm and the scalar product both of $H^1_0(\Omega_\varepsilon)$ and of $H_\varepsilon$. While we denote with $\| \cdot \|_{\mathbb{R}^3}$ and with $(\cdot | \cdot)_{\mathbb{R}^3}$ respectively the norm and the scalar product of $H^1(\mathbb{R}^3)$.
- With $C_i$ and $c_i$, we denote generic positive constants, which may also vary from line to line.
2. SOME PRELIMINARY

Performing the change of variable \( x \mapsto \varepsilon x \), problem \( (P) \) becomes:

\[
\begin{cases}
-\Delta u + J_1(\varepsilon x)u = J_2(\varepsilon x)u^3 + \beta uv^2 = 0 & \text{in } \Omega_\varepsilon, \\
-\Delta v + K_1(\varepsilon x)v = K_2(\varepsilon x)v^3 + \beta u^2v = 0 & \text{in } \Omega_\varepsilon, \\
u, v > 0 & \text{in } \Omega_\varepsilon, \\
u = v = 0 & \text{on } \partial \Omega_\varepsilon,
\end{cases}
\]

(2.1)

where \( \Omega_\varepsilon = \varepsilon^{-1}\Omega \). Of course if \((u, v)\) is a solution of (2.1), then \((u(\cdot/\varepsilon), v(\cdot/\varepsilon))\) is a solution of \((P)\).

Solutions of (2.1) will be found in

\[\mathcal{H}_\varepsilon = H^1_0(\Omega_\varepsilon) \times H^1_0(\Omega_\varepsilon),\]

endowed with the following norm:

\[\|(u, v)\|_{\mathcal{H}_\varepsilon}^2 = \|u\|_{H^1_0(\Omega_\varepsilon)}^2 + \|v\|_{H^1_0(\Omega_\varepsilon)}^2, \quad \text{for all } (u, v) \in \mathcal{H}_\varepsilon.\]

If there is no misunderstanding, we denote with \(\| \cdot \|\) and with \((\cdot, \cdot)\) respectively the norm and the scalar product both of \(H^1(\Omega_\varepsilon)\) and of \(\mathcal{H}_\varepsilon\).

Solutions of (2.1) are critical points of the functional \(f_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}\), defined as

\[
f_\varepsilon(u, v) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} J_1(\varepsilon x)u^2 - \frac{1}{4} \int_{\Omega_\varepsilon} J_2(\varepsilon x)u^4 \\
+ \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} K_1(\varepsilon x)v^2 - \frac{1}{4} \int_{\Omega_\varepsilon} K_2(\varepsilon x)v^4 \\
- \frac{\beta}{2} \int_{\Omega_\varepsilon} u^2v^2.
\]

If we define \(f^J_\varepsilon : H^1_0(\Omega_\varepsilon) \to \mathbb{R}\) and \(f^K_\varepsilon : H^1_0(\Omega_\varepsilon) \to \mathbb{R}\) as

\[
f^J_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} J_1(\varepsilon x)u^2 - \frac{1}{4} \int_{\Omega_\varepsilon} J_2(\varepsilon x)u^4, \\
f^K_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla v|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} K_1(\varepsilon x)v^2 - \frac{1}{4} \int_{\Omega_\varepsilon} K_2(\varepsilon x)v^4,
\]

we have

\[
f_\varepsilon(u, v) = f^J_\varepsilon(u) + f^K_\varepsilon(v) - \frac{\beta}{2} \int_{\Omega_\varepsilon} u^2v^2.
\]

Furthermore, for any fixed \(Q \in \Omega\), we define the two functionals \(F^J_Q : H^1(\mathbb{R}^3) \to \mathbb{R}\) and \(F^K_Q : H^1(\mathbb{R}^3) \to \mathbb{R}\), as follows

\[
F^J_Q(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} J_1(Q)u^2 - \frac{1}{4} \int_{\mathbb{R}^3} J_2(Q)u^4, \\
F^K_Q(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^3} K_1(Q)v^2 - \frac{1}{4} \int_{\mathbb{R}^3} K_2(Q)v^4.
\]
The solutions of (2.1) will be found near \((U^Q, V^Q)\), properly truncated, where \(U^Q\) is the unique solution of
\[
\begin{cases}
-\Delta u + J_1(Q)u = J_2(Q)u^3 & \text{in } \mathbb{R}^3, \\
\end{cases}
\]
and \(V_Q\) is the unique solution of
\[
\begin{cases}
-\Delta v + K_1(Q)v = K_2(Q)v^3 & \text{in } \mathbb{R}^3, \\
\end{cases}
\]
for an appropriate choice of \(Q \in \Omega_0\). It is easy to see that
\[
U^Q(x) = \sqrt{J_1(Q)/J_2(Q)} \cdot W \left( \sqrt{J_1(Q)} \cdot x \right),
\]
\[
V^Q(x) = \sqrt{K_1(Q)/K_2(Q)} \cdot W \left( \sqrt{K_1(Q)} \cdot x \right),
\]
where \(W\) is the unique solution of
\[
\begin{cases}
-\Delta z + z = z^3 & \text{in } \mathbb{R}^3, \\
z(0) = \max_{\mathbb{R}^3} z,
\end{cases}
\]
which is radially symmetric and decays exponentially at infinity with its first derivatives (cf. [16, 20]).

For all \(Q \in \Omega_0\), we set \(Q' = Q'(\varepsilon, Q) = Q + \varepsilon e_1 \in \Omega\) and moreover we call \(P = P(\varepsilon, Q) = Q/\varepsilon \in \Omega_x\) and \(P' = P'(\varepsilon, Q) = Q'/\varepsilon \in \Omega_x\). Let us observe that
\[
|P - P'| = \frac{1}{\sqrt{\varepsilon}} \to 0, \quad \text{as } \varepsilon \to 0.
\]

Let \(\chi: \mathbb{R}^3 \to \mathbb{R}\) be a smooth function such that
\[
\begin{align*}
\chi(x) &= 1, & & \text{for } |x| \leq \varepsilon^{-1/4}; \\
\chi(x) &= 0, & & \text{for } |x| \geq 2\varepsilon^{-1/4}; \\
0 &\leq \chi(x) \leq 1, & & \text{for } \varepsilon^{-1/4} \leq |x| \leq 2\varepsilon^{-1/4}; \\
|\nabla \chi(x)| &\leq 2\varepsilon^{1/4}, & & \text{for } \varepsilon^{-1/4} \leq |x| \leq 2\varepsilon^{-1/4}.
\end{align*}
\]

We denote
\[
U_P(x) := \chi(x - P) U^Q(x - P),
\]
\[
V_{P'}(x) := \chi(x - P') V^Q(x - P').
\]

Let us observe that \((U_P, V_{P'}) \in \mathcal{H}_x\). For \(Q\) varying in \(\Omega_0\), \((U_P, V_{P'})\) describes a 3-dimensional manifold, namely,
\[
Z^{\varepsilon} = \{(U_P, V_{P'}) : Q \in \Omega_0\}.
\]

**Remark 2.1.** Of course, if \(\Omega = \mathbb{R}^3\), then \(\Omega_0 = \mathbb{R}^3\) and we do not need to truncate \(U^Q\) and \(V^Q\). In this case, we would have simply \(U_P = U^Q(\cdot - P)\) and \(V_{P'} = V^Q(\cdot - P')\).
First of all let us give the following estimate which will be very useful in the sequel.

**Lemma 2.2.** For all \( Q \in \Omega_0 \) and for all \( \varepsilon \) sufficiently small, if \( Q' = Q + \sqrt{\varepsilon} e_1 \), \( P = Q/\varepsilon \in \Omega_\varepsilon \) and \( P' = Q'/\varepsilon \in \Omega_\varepsilon \), then

\[
\int_{\Omega_\varepsilon} U_P^2 V_{P'}^2 = o(\varepsilon).
\]

**Proof**  Let us start observing that, since

\[
|P - P'| = \varepsilon^{-1/2} > 4\varepsilon^{-1/4},
\]

we infer that

\[
B_{2\varepsilon^{-1/4}}(P) \cup B_{2\varepsilon^{-1/4}}(P') = \emptyset.
\]

Therefore, by the definitions of (2.9) and (2.10) and by the exponential decay of \( U_P \) and \( V_{P'} \), we get

\[
\int_{\Omega_\varepsilon} U_P^2 V_{P'}^2 \leq \int_{B_{2\varepsilon^{-1/4}}(P) \cup B_{2\varepsilon^{-1/4}}(P')} (U_Q)^2 (x - P) (V_Q)^2 (x - P')
\]

\[
\leq c_1 \int_{\mathbb{R}^3 \setminus B_{2\varepsilon^{-1/4}}(P')} (V_Q)^2 (x - P')
\]

\[
+ c_2 \int_{\mathbb{R}^3 \setminus B_{2\varepsilon^{-1/4}}(P)} (U_Q)^2 (x - P) = o(\varepsilon).
\]

This concludes the proof. \( \square \)

In the next lemma we show that the 3-dimensional manifold \( Z_\varepsilon \), defined in (2.11), is actually a manifold of almost critical points of \( f_\varepsilon \).

**Lemma 2.3.** For all \( Q \in \Omega_0 \) and for all \( \varepsilon \) sufficiently small, if \( Q' = Q + \sqrt{\varepsilon} e_1 \), \( P = Q/\varepsilon \in \Omega_\varepsilon \) and \( P' = Q'/\varepsilon \in \Omega_\varepsilon \), then

\[
\|\nabla f_\varepsilon(U_P, V_{P'})\| = O(\varepsilon^{1/2}).
\]

**Proof**  For all \( (u, v) \in \mathcal{H}_\varepsilon \), we have:

\[
(\nabla f_\varepsilon(U_P, V_{P'}) | (u, v)) = \int_{\Omega_\varepsilon} [\nabla U_P \cdot \nabla u + J_1(\varepsilon x) U_P u - J_2(\varepsilon x) U_P^3 u]
\]

\[
+ \int_{\Omega_\varepsilon} [\nabla V_{P'} \cdot \nabla v + K_1(\varepsilon x) V_{P'} v - K_2(\varepsilon x) V_{P'}^3 v]
\]

\[
- \beta \int_{\Omega_\varepsilon} U_P V_{P'}^2 u - \beta \int_{\Omega_\varepsilon} U_P^3 V_{P'} v.
\]
Let us study the first integral of the right hand side of (2.14). By the exponential decay of $U^Q$ and recalling that $U^Q$ is solution of (2.2), we get

$$
\int_{\Omega_\epsilon} \left[ \nabla U_P \cdot \nabla u + J_1(\epsilon x) U_P u - J_2(\epsilon x) U_P^3 u \right]
$$

$$
= \int_{(\Omega - Q)/\epsilon \cap B_{\epsilon^{-1}/4}} \left[ \nabla U^Q \cdot \nabla u_{-P} + J_1(\epsilon x + Q) U^Q u_{-P} \right]
$$

$$
- \int_{(\Omega - Q)/\epsilon \cap B_{\epsilon^{-1}/4}} J_2(\epsilon x + Q) \left( U^Q \right)^3 u_{-P} + o(\epsilon)
$$

$$
= \int_{\mathbb{R}^3} \left[ \nabla U^Q \cdot \nabla u_{-P} + J_1(\epsilon x + Q) U^Q u_{-P} - J_2(\epsilon x + Q) \left( U^Q \right)^3 u_{-P} \right] + o(\epsilon)
$$

$$
= \int_{\mathbb{R}^3} \left[ \nabla U^Q \cdot \nabla u_{-P} + J_1(Q) U^Q u_{-P} - J_2(Q) \left( U^Q \right)^3 u_{-P} \right]
$$

$$
+ \int_{\mathbb{R}^3} \left( J_1(\epsilon x + Q) - J_1(Q) \right) U^Q u_{-P}
$$

$$
- \int_{\mathbb{R}^3} \left( J_2(\epsilon x + Q) - J_2(Q) \right) \left( U^Q \right)^3 u_{-P} + o(\epsilon)
$$

$$
= \int_{\mathbb{R}^3} \left( J_1(\epsilon x + Q) - J_1(Q) \right) U^Q u_{-P}
$$

$$
- \int_{\mathbb{R}^3} \left( J_2(\epsilon x + Q) - J_2(Q) \right) \left( U^Q \right)^3 u_{-P} + o(\epsilon).
$$

Moreover, from the assumption $D J_i$ bounded, we infer that

$$
|J_i(\epsilon x + Q) - J_i(Q)| \leq c_1 |\epsilon x|,
$$

and so,

$$
\int_{\mathbb{R}^3} \left( J_1(\epsilon x + Q) - J_1(Q) \right) U^Q u_{-P} \leq \|u\| \left( \int_{\mathbb{R}^3} |J_1(\epsilon x + Q) - J_1(Q)|^2 |U^Q|^2 \right)^{1/2}
$$

$$
\leq c_1 \|u\| \left( \int_{\mathbb{R}^3} \epsilon^2 |x|^2 |U^Q|^2 \right)^{1/2} = O(\epsilon) \|u\|.
$$

Analogously,

$$
\int_{\mathbb{R}^3} \left( J_2(\epsilon x + Q) - J_2(Q) \right) \left( U^Q \right)^3 u_{-P} = O(\epsilon) \|u\|.
$$

Therefore, by (2.15), (2.16) and (2.17), we infer

$$
\int_{\Omega_\epsilon} \left[ \nabla U_P \cdot \nabla u + J_1(\epsilon x) U_P u - J_2(\epsilon x) U_P^3 u \right] = O(\epsilon) \|u\|.
$$
Similarly, since $V^Q$ is solution of (2.3), we get
\[
\int_{\Omega_{\varepsilon}} \left[ \nabla V_{p'} \cdot \nabla v + K_1(\varepsilon x) V_{p'} v - K_2(\varepsilon x) V_{p'}^3 v \right]
\]
\[
= \int_{\mathbb{R}^3} \left( K_1(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_1(Q) \right) V^Q v_{p'}
\]
\[
- \int_{\mathbb{R}^3} \left( K_2(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_2(Q) \right) (V^Q)^3 v_{p'} + o(\varepsilon).
\]
(2.19)

Therefore, from the assumption $D K_i$ bounded, we infer that
\[
|K_i(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_i(Q)| \leq c_2 \sqrt{\varepsilon} |\sqrt{\varepsilon} x + e_1|,
\]
and so,
\[
\int_{\mathbb{R}^3} \left( K_1(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_1(Q) \right) V^Q v_{p'}
\]
\[
\leq \|v\| \left( \int_{\mathbb{R}^3} |K_1(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_1(Q)|^2 |V^Q|^2 \right)^{1/2}
\]
\[
\leq c_2 \|v\| \left( \int_{\mathbb{R}^3} \varepsilon |\sqrt{\varepsilon} x + e_1|^2 |V^Q|^2 \right)^{1/2} = O(\varepsilon^{1/2}) \|v\|.
\]
(2.20)

Analogously,
\[
\int_{\mathbb{R}^3} \left( K_2(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_2(Q) \right) (V^Q)^3 v_{p'} = O(\varepsilon^{1/2}) \|v\|.
\]
(2.21)

Therefore, by (2.19), (2.20) and (2.21), we infer
\[
\int_{\Omega_{\varepsilon}} \left[ \nabla V_{p'} \cdot \nabla v + K_1(\varepsilon x) V_{p'} v - K_2(\varepsilon x) V_{p'}^3 v \right] = O(\varepsilon^{1/2}) \|v\|.
\]
(2.22)

Let us study the last two terms of (2.14). Arguing as in Lemma 2.2, we get
\[
\int_{\Omega_{\varepsilon}} U_{p} V_{p'}^2 u \leq c_3 \left( \int_{\Omega_{\varepsilon}} U_{p}^{4/3} V_{p'}^{8/3} \right)^{3/4} \|u\| = o(\varepsilon) \|u\|,
\]
(2.23)

and
\[
\int_{\Omega_{\varepsilon}} U_{p}^3 V_{p'} v = o(\varepsilon) \|v\|.
\]
(2.24)

Now the conclusion of the proof easily follows by (2.14), (2.18), (2.22), (2.23) and (2.24). □

3. Invertibility of $D^2 f_{\varepsilon}$ on $(T_{(U_P,V_{p'})} Z_{\varepsilon})^\perp$

In this section we will show that $D^2 f_{\varepsilon}$ is invertible on $(T_{(U_P,V_{p'})} Z_{\varepsilon})^\perp$, where $T_{(U_P,V_{p'})} Z_{\varepsilon}$ denotes the tangent space to $Z_{\varepsilon}$ at the point $(U_P, V_{p'})$.

Let $L_{\varepsilon,Q} : (T_{(U_P,V_{p'})} Z_{\varepsilon})^\perp \to (T_{(U_P,V_{p'})} Z_{\varepsilon})^\perp$ denote the operator defined by setting $(L_{\varepsilon,Q}(h,h'))((k,k')) = D^2 f_{\varepsilon}(U_P,V_{p'})((h,h'),(k,k'))$. 


Lemma 3.1. Given \( \mu > 0 \), there exists \( C > 0 \) such that, for \( \varepsilon \) small enough and for all \( Q \in \Omega_0 \) with \( |Q| \leq \mu \), one has that

\[
\|L_{\varepsilon,Q}(h,h')\| \geq C\|(h,h')\|, \quad \forall (h,h') \in (T(U_P,V_{P'})Z^\varepsilon)'.
\]

Proof First of all, let us observe that, for all \( (h,h'), (k,k') \in \mathcal{H}_\varepsilon \), we have

\[
D^2 f_\varepsilon(u,v) \left[(h,h'), (k,k')\right] = D^2 f_\varepsilon^I(u,h,k) + D^2 f_\varepsilon^K(v,h',k')
\]

\[-\beta \int_{\Omega_\varepsilon} v^2 kh - 2\beta \int_{\Omega_\varepsilon}uvhk' - 2\beta \int_{\Omega_\varepsilon}uv'h - \beta \int_{\Omega_\varepsilon} u^2 h'k'.
\]

By \ref{2.4}, if we set \( a(Q) = \sqrt{J_1(Q)/J_2(Q)} \) and \( b(Q) = \sqrt{J_1(Q)} \), we have that \( U^Q(x) = a(Q)W(b(Q)x) \) and so \( U_P(x) = \chi(x-P)a(\varepsilon P)W(b(\varepsilon P)(x-P)) \). Therefore, we have:

\[
\partial_P U_P(x) = \partial_P \left( \chi(x-P)U^Q(x-P) \right)
\]

\[-U^Q(x-P)\partial_x \chi(x-P) + \chi(x-P)\partial_P U^Q(x-P)
\]

\[-U^Q(x-P)\partial_x \chi(x-P) + \varepsilon \chi(x-P)\partial_P a(\varepsilon P)W(b(\varepsilon P)(x-P))
\]

\[+ \varepsilon \chi(x-P)a(\varepsilon P)\partial_P a(\varepsilon P)\nabla W(b(\varepsilon P)(x-P)) \cdot (x-P)
\]

\[-\chi(x-P)a(\varepsilon P)b(\varepsilon P)(\partial_x W)(b(\varepsilon P)(x-P)).
\]

Hence

\[
\partial_P U_P(x) = -\varepsilon P U_P(x) + O(\varepsilon).
\]

Analogously, we can prove that

\[
\partial_P V_{P'}(x) = \partial_P V_P'(x) = -\varepsilon P V_{P'}(x) + O(\varepsilon).
\]

We recall that

\[
T_{(U_P,V_{P'})}Z^\varepsilon = \operatorname{span}_{\mathcal{H}_\varepsilon} \{ \left( \partial_P U_P, \partial_P V_{P'} \right), \left( \partial_P U_P, \partial_P V_P \right), \left( \partial_P U_P, \partial_P V_{P'} \right) \}.
\]

We set

\[
\mathcal{V}_\varepsilon = \operatorname{span}_{\mathcal{H}_\varepsilon} \{ \left( U_P, V_{P'} \right), \left( \partial_x U_P, \partial_x V_{P'} \right), \left( \partial_x U_P, \partial_x V_P \right), \left( \partial_x U_P, \partial_x V_{P'} \right) \}.
\]

By \ref{3.3} and \ref{3.4}, therefore it suffices to prove equation \ref{3.1} for all \( (h,h') \in \operatorname{span}_{\mathcal{H}_\varepsilon} \{ \left( U_P, V_{P'} \right), (\phi, \phi') \} \), where \( (\phi, \phi') \) is orthogonal to \( \mathcal{V}_\varepsilon \). Precisely we shall prove that there exist \( C_1, C_2 > 0 \) such that, for all \( \varepsilon > 0 \) small enough, one has:

\[
(L_{\varepsilon,Q}(U_P,V_{P'}) \mid (U_P,V_{P'})) \leq -C_1 < 0,
\]

\[
(L_{\varepsilon,Q}(\phi,\phi') \mid (\phi,\phi')) \geq C_2 \|(\phi,\phi')\|^2, \quad \text{for all } (\phi,\phi') \perp \mathcal{V}_\varepsilon.
\]

Proof of \ref{3.5}. By \ref{3.2}, we get:

\[
D^2 f_\varepsilon(U_P,V_{P'})[(U_P,V_{P'}),(U_P,V_{P'})]
\]

\[= D^2 f_\varepsilon^I(U_P,U_P) + D^2 f_\varepsilon^K(V_{P'},V_{P'}) - 6\beta \int_{\Omega_\varepsilon} U_P^2 V_{P'}^2.
\]
Let us study the first term of the right hand side of (3.7).

\[
D^2 f^F_\varepsilon(U_P)[U_P, U_P] = \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \int_{\Omega_\varepsilon} J_1(\varepsilon x) U_P^2 - 3 \int_{\Omega_\varepsilon} J_2(\varepsilon x) U_P^4
\]

\[
= \int_{(\Omega - Q)/\varepsilon \cap B_{\varepsilon^{-1/4}}} \left[ |\nabla U_Q|^2 + J_1(\varepsilon x + Q) (U_Q)^2 - 3J_2(\varepsilon x + Q) (U_Q)^4 \right]
\]

\[
= \int_{\mathbb{R}^3} \left[ |\nabla U_Q|^2 + J_1(Q) (U_Q)^2 - 3J_2(Q) (U_Q)^4 \right]
\]

\[
+ \int_{\mathbb{R}^3} \left( J_1(\varepsilon x + Q) - J_1(Q) \right) (U_Q)^2
\]

\[
- 3 \int_{\mathbb{R}^3} \left( J_2(\varepsilon x + Q) - J_2(Q) \right) (U_Q)^4 + o(\varepsilon)
\]

\[
= -2 \int_{\mathbb{R}^3} J_2(Q) (U_Q)^4 + O(\varepsilon)
\]

\[
= -2J_1(Q)^{3/2}J_2(Q)^{-1} \int_{\mathbb{R}^3} W^4 + O(\varepsilon) \leq -c_1.
\]

In a similar way it is possible to prove that

\[
D^2 f^K_\varepsilon(V_P')[V_P', V_P'] \leq -c_2.
\]

Finally, by Lemma \[2.2\] we know that

\[
\int_{\Omega_\varepsilon} U_P^2 V_P^2 = o(\varepsilon),
\]

and so equation (3.5) is proved.

**Proof of (3.6).** Recalling the definition of \( \chi \) (see (2.8)), we set \( \chi_1 := \chi \) and \( \chi_2 := 1 - \chi_1 \). Given \( \phi, \phi' \perp V_\varepsilon \), let us consider the functions

\[
\phi_i(x) = \chi_i(x - P)\phi(x), \quad i = 1, 2;
\]

\[
\phi'_i(x) = \chi_i(x - P')\phi'(x), \quad i = 1, 2.
\]

With calculations similar to those of [7], we have

\[
\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + 2 \int_{\Omega_\varepsilon} \chi_1 \chi_2 (\phi^2 + |\nabla \phi|^2) + O(\varepsilon^{1/4}) \|\phi\|^2,
\]

\[
= \int_{\Omega_\varepsilon} \chi_1 \chi_2 (\phi^2 + |\nabla \phi|^2) + O(\varepsilon^{1/4}) \|\phi\|^2.
\]

\[
\|\phi'\|^2 = \|\phi'_1\|^2 + \|\phi'_2\|^2 + 2 \int_{\Omega_\varepsilon} \chi_1 \chi_2 (\phi'^2 + |\nabla \phi'|^2) + O(\varepsilon^{1/4}) \|\phi'\|^2.
\]

\[
= \int_{\Omega_\varepsilon} \chi_1 \chi_2 (\phi'^2 + |\nabla \phi'|^2) + O(\varepsilon^{1/4}) \|\phi'\|^2.
\]
We need to evaluate the three terms in the equation below:

\[
(L_{\varepsilon,Q}(\phi, \phi') | (\phi, \phi')) = (L_{\varepsilon,Q}(\phi_1, \phi'_1) | (\phi_1, \phi'_1)) + (L_{\varepsilon,Q}(\phi_2, \phi'_2) | (\phi_2, \phi'_2)) + 2(L_{\varepsilon,Q}(\phi_1, \phi'_1) | (\phi_2, \phi'_2)).
\]

(3.12)

Let us start with \((L_{\varepsilon,Q}(\phi_1, \phi'_1) | (\phi_1, \phi'_1))\). Since \(\beta < 0\), we get

\[
(L_{\varepsilon,Q}(\phi_1, \phi'_1) | (\phi_1, \phi'_1)) = D^2 f^J_{\varepsilon}(U_P)[\phi_1, \phi_1] + D^2 f^K_{\varepsilon}(V_P)[\phi'_1, \phi'_1]
\]

\[
- 4\beta \int_{\Omega_{\varepsilon}} U_P V_P \phi_1 \phi'_1 - \beta \int_{\Omega_{\varepsilon}} U_P^2 \phi'_1^2 - \beta \int_{\Omega_{\varepsilon}} V_P^2 \phi'_1^2
\]

\[
> D^2 f^J_{\varepsilon}(U_P)[\phi_1, \phi_1] + D^2 f^K_{\varepsilon}(V_P)[\phi'_1, \phi'_1]
\]

(3.13)

Arguing as in Lemma 2.2 we know that

\[
\int_{\Omega_{\varepsilon}} U_P V_P \phi_1 \phi'_1 = o(\varepsilon).
\]

(3.14)

Therefore we need only to study the first two terms of the right hand side of (3.13). For simplicity, we can assume that \(Q = \varepsilon P\) is the origin \(O\). In this case, we recall that we denote with \(U^O\) the unique solution of (2.2) whenever \(Q = O\), while we denote with \(U_O\) the truncation of \(U^O\), namely \(U_O = \chi U^O\), where \(\chi\) is defined in (2.8). We have

\[
D^2 f^J_{\varepsilon}(U^O)[\phi_1, \phi_1] = \int_{\Omega_{\varepsilon}} [|\nabla \phi_1|^2 + J_1(\varepsilon x) \phi_1^2 - 3J_2(\varepsilon x) U_O^2 \phi_1^2]
\]

\[
= \int_{\mathbb{R}^3} [|\nabla \phi_1|^2 + J_1(\varepsilon x) \phi_1^2 - 3J_2(\varepsilon x) (U^O)^2 \phi_1^2] + o(\varepsilon) \|\phi\|^2
\]

\[
= D^2 F^J(O)(U^O)[\phi_1, \phi_1]
\]

\[
+ \int_{\mathbb{R}^3} (J_1(\varepsilon x) - J_1(O)) \phi_1^2
\]

\[
- 3 \int_{\mathbb{R}^3} (J_2(\varepsilon x) - J_2(O)) (U^O)^2 \phi_1^2 + o(\varepsilon) \|\phi\|^2
\]

\[
\geq D^2 F^J(O)(U^O)[\phi_1, \phi_1] - c_3 \varepsilon \int_{\mathbb{R}^3} |x| \phi_1^2 + O(\varepsilon) \|\phi\|^2
\]

\[
= D^2 F^J(O)(U^O)[\phi_1, \phi_1] + O(\varepsilon^{3/4}) \|\phi\|^2,
\]

therefore

\[
D^2 f^J_{\varepsilon}(U^O)[\phi_1, \phi_1] \geq D^2 F^J(O)(U^O)[\phi_1, \phi_1] + O(\varepsilon^{3/4}) \|\phi\|^2.
\]

(3.15)

We recall that \(\phi\) is orthogonal to

\[
\mathcal{V}_{\varepsilon}^U = \text{span}_{H^1_0(\Omega_{\varepsilon})}\{U_O, \partial_x U_O, \partial_{xx} U_O, \partial_{x_3} U_O\}.
\]

Moreover by [9], we know that if \(\hat{\phi}\) is orthogonal to \(\mathcal{V}\) with

\[
\mathcal{V}^U = \text{span}_{H^1_0(\mathbb{R}^3)}\{U^O, \partial_x U^O, \partial_{xx} U^O, \partial_{x_3} U^O\},
\]
then the fact that $U^O$ is a Mountain Pass critical point of $F^J(O)$ implies that

\begin{equation}
D^2 F^J(O)(U^O)[\tilde{\phi}, \tilde{\phi}] > c_4 \|\tilde{\phi}\|_{\mathbb{R}^3}^2 \quad \text{for all } \tilde{\phi} \perp \mathcal{V}^U.
\end{equation}

We can write $\phi_1 = \xi + \zeta$, where $\xi \in \mathcal{V}^U$ and $\zeta \perp \mathcal{V}^U$. More precisely

\[
\xi = (\phi_1 | U^O)_{\mathbb{R}^3} \frac{U^O}{\|U^O\|_{\mathbb{R}^3}^2} + \sum_{i=1}^3 (\phi_1 | \partial_{x_i} U^O)_{\mathbb{R}^3} \frac{\partial_{x_i} U^O}{\|\partial_{x_i} U^O\|_{\mathbb{R}^3}}.
\]

Let us calculate $(\phi_1 | U^O)_{\mathbb{R}^3}$. By the exponential decay of $U^O$ and since $\phi \perp \mathcal{V}^U$, we have

\[
(\phi_1 | U^O)_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \nabla \phi_1 \cdot \nabla U^O + \int_{\mathbb{R}^3} \phi_1 U^O
\]

\[
= \int_{\Omega_\varepsilon} \nabla \phi_1 \cdot \nabla U^O + \int_{\Omega_\varepsilon} \phi_1 U^O + o(\varepsilon)\|\phi\|
\]

\[
= \int_{\Omega_\varepsilon} \nabla \phi \cdot \nabla U^O + \int_{\Omega_\varepsilon} \phi U^O + o(\varepsilon)\|\phi\| = o(\varepsilon)\|\phi\|.
\]

In a similar way, we can prove also that $(\phi_1 | \partial_{x_i} U^O)_{\mathbb{R}^3} = o(\varepsilon)\|\phi\|$, and so

\begin{align}
(\phi_1 | U^O)_{\mathbb{R}^3} &= o(\varepsilon)\|\phi\|, \\
(\phi_1 | \partial_{x_i} U^O)_{\mathbb{R}^3} &= o(\varepsilon)\|\phi\|.
\end{align}

Let us estimate $D^2 F^J(O)(U^O)[\phi_1, \phi_1]$. We get:

\begin{align}
D^2 F^J(O)(U^O)[\phi_1, \phi_1] &= D^2 F^J(O)(U^O)[\zeta, \zeta] + 2D^2 F^J(O)(U^O)[\zeta, \xi] \\
&\quad + D^2 F^J(O)(U^O)[\xi, \xi].
\end{align}

By (3.16) and (3.18), since $\zeta \perp \mathcal{V}^U$, we know that

\[
D^2 F^J(O)(U^O)[\zeta, \zeta] > c_3 \|\zeta\|_{\mathbb{R}^3}^2 = c_3 \|\phi_1\|^2 + o(\varepsilon)\|\phi\|^2,
\]

while, by (3.17) and straightforward calculations, we have

\[
D^2 F^J(O)(U^O)[\zeta, \xi] = o(\varepsilon)\|\phi\|^2,
\]

\[
D^2 F^J(O)(U^O)[\xi, \xi] = o(\varepsilon)\|\phi\|^2.
\]

By these last two estimates, (3.19) and (3.15), we can say that

\[
D^2 f^J_\varepsilon(U^O)[\phi_1, \phi_1] > c_4 \|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2.
\]

Hence, in the general case, we infer that, for all $Q \in \Omega_0$ with $|Q| \leq \mu$,

\begin{equation}
D^2 f^J_\varepsilon(U_P)[\phi_1, \phi_1] > c_4 \|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2,
\end{equation}

and, analogously,

\begin{equation}
D^2 f^K_\varepsilon(V_P')[\phi_1', \phi_1'] > c_5 \|\phi_1'\|^2 + O(\varepsilon^{1/2})\|\phi_1'\|^2.
\end{equation}

By (3.13), (3.14), (3.20) and (3.21), we can say that

\begin{equation}
(L_{\varepsilon,Q}(\phi_1, \phi_1') | (\phi_1, \phi_1')) > c_6 \|\phi_1\|^2 + O(\varepsilon^{1/2})\|\phi_1\|^2.
\end{equation}
Let us now evaluate \((L_{\varepsilon,Q}(\phi_2, \phi'_2) \mid (\phi_2, \phi'_2))\). Arguing as in Lemma 2.2, since \(\beta < 0\) and using the definition of \(\chi_i\) and the exponential decay of \(U_{P}\) and of \(V_{P'}\), we easily get:

\[
(L_{\varepsilon,Q}(\phi_2, \phi'_2) \mid (\phi_2, \phi'_2)) = D^2 f_{\varepsilon}^J(U_P)[\phi_2, \phi_2] + D^2 f_{\varepsilon}^K(V_{P'})[\phi'_2, \phi'_2] - 4\beta \int_{\Omega_\varepsilon} U_P V_{P'} \phi_2 \phi'_2 - \beta \int_{\Omega_\varepsilon} U_P^2 \phi_2^2 - \beta \int_{\Omega_\varepsilon} V_{P'}^2 \phi'_2^2 \\
> D^2 f_{\varepsilon}^J(U_P)[\phi_2, \phi_2] + D^2 f_{\varepsilon}^K(V_{P'})[\phi'_2, \phi'_2] + o(\varepsilon)\| (\phi, \phi') \|^2 \\
> c_1 \| (\phi_2, \phi'_2) \|^2 + o(\varepsilon)\| (\phi, \phi') \|^2.
\]

(3.23)

Let us now study \((L_{\varepsilon,Q}(\phi_1, \phi'_1) \mid (\phi_2, \phi'_2))\). Arguing as in Lemma 2.2 we get:

\[
(L_{\varepsilon,Q}(\phi_1, \phi'_1) \mid (\phi_2, \phi'_2)) = D^2 f_{\varepsilon}^J(U_P)[\phi_1, \phi_2] + D^2 f_{\varepsilon}^K(V_{P'})[\phi'_1, \phi'_2] - 2\beta \int_{\Omega_\varepsilon} U_P V_{P'} \phi_1 \phi'_2 - 2\beta \int_{\Omega_\varepsilon} U_P V_{P'} \phi_2 \phi'_1 \\
- \beta \int_{\Omega_\varepsilon} U_P^2 \phi'_1 \phi'_2 - \beta \int_{\Omega_\varepsilon} V_{P'}^2 \phi_1 \phi_2 \\
= D^2 f_{\varepsilon}^J(U_P)[\phi_1, \phi_2] + D^2 f_{\varepsilon}^K(V_{P'})[\phi'_1, \phi'_2] \\
- \beta \int_{\Omega_\varepsilon} U_P^2 \phi'_1 \phi'_2 - \beta \int_{\Omega_\varepsilon} V_{P'}^2 \phi_1 \phi_2 + o(\varepsilon)\| (\phi, \phi') \|^2.
\]

(3.24)

Using the definition of \(\chi_i\) and the exponential decay of \(U_P\) and of \(V_{P'}\), we easily get:

\[
D^2 f_{\varepsilon}^J(U_P)[\phi_1, \phi_2] \geq c_8 I_{\phi} + O(\varepsilon^{1/4})\| \phi \|^2, \tag{3.25}
\]

\[
D^2 f_{\varepsilon}^K(V_{P'})[\phi'_1, \phi'_2] \geq c_9 I_{\phi'} + O(\varepsilon^{1/4})\| \phi' \|^2, \tag{3.26}
\]

where \(I_{\phi}\) and \(I_{\phi'}\) are defined, respectively in \((3.10)\) and \((3.11)\). Moreover, by the definition of \(\chi_i\), (see (2.8)), and by the definitions of \(\phi_i\) and \(\phi'_i\), (see (3.8) and (3.9)),

\[
\phi_1(x) \phi_2(x) = \chi(x - P)(1 - \chi(x - P)) \phi^2(x) \geq 0, \quad \text{for all } x \in \mathbb{R}^3,
\]

and so, also

\[
\phi'_i(x) \phi'_2(x) \geq 0, \quad \text{for all } x \in \mathbb{R}^3.
\]

Therefore

\[
-\beta \int_{\Omega_\varepsilon} U_P^2 \phi'_1 \phi'_2 - \beta \int_{\Omega_\varepsilon} V_{P'}^2 \phi_1 \phi_2 \geq 0. \tag{3.27}
\]

By \((3.24), (3.25), (3.26)\) and \((3.27)\), we infer

\[
(L_{\varepsilon,Q}(\phi_1, \phi'_1) \mid (\phi_2, \phi'_2)) \geq c_{10} (I_{\phi} + I_{\phi'}) + O(\varepsilon^{1/4})\| (\phi, \phi') \|^2. \tag{3.28}
\]

Hence, by \((3.12), (3.22), (3.23), (3.28)\) and recalling \((3.10)\) and \((3.11)\), we get

\[
(L_{\varepsilon,Q}(\phi, \phi') \mid (\phi, \phi')) \geq c_{11} \| (\phi, \phi') \|^2 + O(\varepsilon^{1/4})\| (\phi, \phi') \|^2.
\]

This completes the proof of the lemma. \(\square\)
4. THE FINITE DIMENSIONAL REDUCTION

By means of the Liapunov-Schmidt reduction, the existence of critical points of $f_\varepsilon$ can be reduced to the search of critical points of an auxiliary finite-dimensional functional.

**Lemma 4.1.** Fix $\mu > 0$. For $\varepsilon > 0$ small enough and for all $Q \in \Omega_0$ with $|Q| \leq \mu$, there exists a unique $(w, w') = (w(\varepsilon, Q), w'(\varepsilon, Q)) \in \mathcal{H}_\varepsilon$ of class $C^1$ such that:

1. $(w(\varepsilon, Q), w'(\varepsilon, Q)) \in (T_{(U_P, V_P)} Z^\varepsilon)^\perp$;
2. $\nabla f_\varepsilon(U_P + w, V_P' + w') \in T_{(U_P, V_P)} Z^\varepsilon$.

Moreover, the functional $\mathcal{A}_\varepsilon : \Omega_0 \to \mathbb{R}$, defined as:

$$\mathcal{A}_\varepsilon(Q) := f_\varepsilon(U_{Q/\varepsilon} + w(\varepsilon, Q), V_{(Q + \varepsilon^1/\varepsilon)} + w'(\varepsilon, Q))$$

is of class $C^1$ and satisfies:

$$\nabla \mathcal{A}_\varepsilon(Q_0) = 0 \iff \nabla f_\varepsilon(U_{Q_0/\varepsilon} + w(\varepsilon, Q_0), V_{(Q_0 + \varepsilon^1/\varepsilon)} + w'(\varepsilon, Q_0)) = 0.$$

**Proof** Let $\mathcal{P} = \mathcal{P}_{\varepsilon,Q}$ denote the projection onto $(T_{(U_P, V_P)} Z^\varepsilon)^\perp$. We want to find a solution $(w, w') \in (T_{(U_P, V_P)} Z^\varepsilon)^\perp$ of the equation

$$P \nabla f_\varepsilon(U_P + w, V_P' + w') = 0.$$

One has that

$$\nabla f_\varepsilon(U_P + w, V_P' + w') = \nabla f_\varepsilon(U_P, V_P') + D^2 f_\varepsilon(U_P, V_P')[w, w'] + R(U_P, V_P', w, w')$$

with $\|R(U_P, V_P', w, w')\| = o(\|w, w'\|)$, uniformly with respect to $(U_P, V_P')$. Therefore, our equation is:

$$L_{\varepsilon,Q}(w, w') + \mathcal{P} \nabla f_\varepsilon(U_P, V_P') + \mathcal{P} R(U_P, V_P', w, w') = 0.$$

According to Lemma [3.1], this is equivalent to

$$(w, w') = N_{\varepsilon,Q}(w, w'),$$

where

$$N_{\varepsilon,Q}(w, w') = -(L_{\varepsilon,Q})^{-1} (\mathcal{P} \nabla f_\varepsilon(U_P, V_P') + \mathcal{P} R(U_P, V_P', w, w')).$$

By (2.13) it follows that

$$\|N_{\varepsilon,Q}(w, w')\| = O(\varepsilon^{1/2}) + o(\|w, w'\|).$$

Therefore it is easy to check that $N_{\varepsilon,Q}$ is a contraction on some ball in $(T_{(U_P, V_P)} Z^\varepsilon)^\perp$ provided that $\varepsilon > 0$ is small enough. Then there exists a unique $(w, w')$ such that $(w, w') = N_{\varepsilon,Q}(w, w')$. Let us point out that we cannot use the Implicit Function Theorem to find $(w(\varepsilon, Q), w'(\varepsilon, Q))$, because the map $(\varepsilon, u, v) \mapsto \mathcal{P} \nabla f_\varepsilon(u, v)$ fails to be $C^2$. However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(Q, w, w') \mapsto \mathcal{P} \nabla f_\varepsilon(U_P + w, V_P' + w')$. Then, in particular, the function $(w(\varepsilon, Q), w'(\varepsilon, Q))$ turns out to be of class $C^1$ with respect to $Q$. Finally, it is a standard argument, see [2, 3], to check that the critical points of $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w, V_P' + w')$ give rise to critical points of $f_\varepsilon$. \qed
Remark 4.2. From (4.2) it immediately follows that:

\[
\| (w, w') \| = O(\varepsilon^{1/2}). \tag{4.3}
\]

Let us now make the asymptotic expansion of the finite dimensional functional.

Theorem 4.3. Fix \( \mu > 0 \) and let \( Q \in \Omega_0 \) with \( |Q| \leq \mu, Q' = Q + \sqrt{\varepsilon} e_1, P = Q/\varepsilon \in \Omega_\varepsilon \) and \( P' = Q'/\varepsilon \in \Omega_\varepsilon \). Suppose (J) and (K). Then, for \( \varepsilon \) sufficiently small, we get:

\[
\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w(\varepsilon, Q), V_{P'} + w'(\varepsilon, Q)) = c_0 \Gamma(Q) + o(\varepsilon^{1/4}), \tag{4.4}
\]

where \( \Gamma: \Omega_0 \to \mathbb{R} \) is defined in (1.4), namely

\[
\Gamma(Q) = J_1(Q)^{1/2} J_2(Q)^{-1} + K_1(Q)^{1/2} K_2(Q)^{-1};
\]

and

\[
c_0 := \frac{1}{2} \int_{\mathbb{R}^3} W^4 \tag{4.5}
\]

with \( W \) the unique solution of (2.6).

Proof We have:

\[
\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w(\varepsilon, Q), V_{P'} + w'(\varepsilon, Q))
\]

\[
= \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla(U_P + w)|^2 \mathbf{1} + \frac{1}{2} \int_{\Omega_\varepsilon} J_1(\varepsilon x)(U_P + w)^2 - \frac{1}{4} \int_{\Omega_\varepsilon} J_2(\varepsilon x)(U_P + w)^4 + \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla(V_{P'} + w')|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} K_1(\varepsilon x)(V_{P'} + w')^2 - \frac{1}{4} \int_{\Omega_\varepsilon} K_2(\varepsilon x)(V_{P'} + w')^4 - \frac{\beta}{2} \int_{\Omega_\varepsilon} (U_P + w)^2(V_{P'} + w')^2.
\]

Therefore, by (4.3) and Lemma 2.2

\[
\mathcal{A}_\varepsilon(Q) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} J_1(\varepsilon x) U_P^2 - \frac{1}{4} \int_{\Omega_\varepsilon} J_2(\varepsilon x) U_P^4 + \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla V_{P'}|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} K_1(\varepsilon x) V_{P'}^2 - \frac{1}{4} \int_{\Omega_\varepsilon} K_2(\varepsilon x) V_{P'}^4 + O(\varepsilon^{1/2}) \tag{4.6}
\]

Let us study the first term of the right hand side of (4.6).

\[
f_\varepsilon^{(1)}(U_P) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} J_1(\varepsilon x) U_P^2 - \frac{1}{4} \int_{\Omega_\varepsilon} J_2(\varepsilon x) U_P^4
\]

\[
= \frac{1}{2} \int_{(\Omega - Q)/\varepsilon \cap B_{\varepsilon^{-1/4}}} \left[ |\nabla U|^2 + J_1(\varepsilon x + Q)(U^2) - \frac{1}{2} J_2(\varepsilon x + Q)(U^4) \right] + o(\varepsilon)
\]
\[
\begin{align*}
&= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla U^Q|^2 + \frac{1}{2} \int_{\mathbb{R}^3} J_1(Q) (U^Q)^2 - \frac{1}{4} \int_{\mathbb{R}^3} J_2(Q) (U^Q)^4 \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} (J_1(\varepsilon x + Q) - J_1(Q)) (U^Q)^2 \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3} (J_2(\varepsilon x + Q) - J_2(Q)) (U^Q)^4 + o(\varepsilon) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} J_2(Q) (U^Q)^4 + o(\varepsilon^{1/4}) \\
&= \frac{1}{2} J_1(Q) \frac{1}{2} J_2(Q)^{-1} \int_{\mathbb{R}^3} W^4 + o(\varepsilon^{1/4}).
\end{align*}
\]

Hence

(4.7) \quad f_\varepsilon(J P) = \frac{1}{2} J_1(Q) \frac{1}{2} J_2(Q)^{-1} \int_{\mathbb{R}^3} W^4 + o(\varepsilon^{1/4}).

Analogously,

\[
\begin{align*}
f_\varepsilon^K(V P') &= \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla V_{P'}|^2 + \frac{1}{2} \int_{\Omega_\varepsilon} K_1(\varepsilon x) V_{P'}^2 - \frac{1}{4} \int_{\Omega_\varepsilon} K_2(\varepsilon x) V_{P'}^4 \\
&= \frac{1}{2} \int \left[ |\nabla V^Q|^2 + K_1(\varepsilon x + Q') (V^Q)^2 \right]_{\Omega_\varepsilon \cap \Omega_{\varepsilon/2} \cap B_{\varepsilon/4}} \\
&\quad - \frac{1}{4} \int \left[ K_2(\varepsilon x + Q') (V^Q)^4 + o(\varepsilon) \right]_{\Omega_\varepsilon \cap \Omega_{\varepsilon/2} \cap B_{\varepsilon/4}} \\
&= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla V^Q|^2 + \frac{1}{2} \int_{\mathbb{R}^3} K_1(Q) (V^Q)^2 - \frac{1}{4} \int_{\mathbb{R}^3} K_2(Q) (V^Q)^4 \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} \left( K_1(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_1(Q) \right) (V^Q)^2 \\
&\quad - \frac{1}{4} \int_{\mathbb{R}^3} \left( K_2(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_2(Q) \right) (V^Q)^4 + o(\varepsilon) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} K_2(Q) (V^Q)^4 + o(\varepsilon^{1/4}) \\
&= \frac{1}{2} K_1(Q) \frac{1}{2} K_2(Q)^{-1} \int_{\mathbb{R}^3} W^4 + o(\varepsilon^{1/4}).
\end{align*}
\]

Therefore

(4.8) \quad f_\varepsilon^K(V P') = \frac{1}{2} K_1(Q) \frac{1}{2} K_2(Q)^{-1} \int_{\mathbb{R}^3} W^4 + o(\varepsilon^{1/4}).

Now (4.4) follows immediately by (4.6), (4.7) and (4.8).
5. A MULTIPLECTY RESULT AND PROOFS OF THEOREMS

In this section we give the proofs of our theorems. First of all, let us prove Theorem 1.1 as an easy consequence of the following multiplicity result:

**Theorem 5.1.** Let (J) and (K) hold and suppose \( \Gamma \) has a compact set \( X \subset \Omega_0 \) where \( \Gamma \) achieves a strict local minimum (resp. maximum), in the sense that there exist \( \delta > 0 \) and a \( \delta \)-neighborhood \( X_\delta \subset \Omega_0 \) of \( X \) such that

\[
\begin{align*}
  b &= \inf \{ \Gamma(Q) : Q \in \partial X_\delta \} > a := \Gamma|_X, \quad \text{(resp. sup}\{ \Gamma(Q) : Q \in \partial X_\delta \} < \Gamma|_X. \\
\end{align*}
\]

Then there exists \( \varepsilon > 0 \) such that (P\( \varepsilon \)) has at least \( \text{cat}(X, X_\delta) \) solutions that concentrate near points of \( X_\delta \), provided \( \varepsilon \in (0, \bar{\varepsilon}) \). Here \( \text{cat}(X, X_\delta) \) denotes the Lusternik-Schnirelman category of \( X \) with respect to \( X_\delta \).

**Proof**  First of all, we fix \( \mu > 0 \) in such a way that \( |Q| < \mu \) for all \( Q \in \partial X_\delta \). We will apply the finite dimensional procedure with such \( \mu \) fixed.

We will treat only the case of minima, being the other one similar. We set \( Y = \{ Q \in X_\delta : A_\varepsilon(Q) \leq c_0(a + b)/2 \} \), being \( c_0 \) defined in (4.5). By (4.4) it follows that there exists \( \bar{\varepsilon} > 0 \) such that

\[
\begin{align*}
  X \subset Y \subset X_\delta, \\
\end{align*}
\]

provided \( \varepsilon \in (0, \bar{\varepsilon}) \). Moreover, if \( Q \in \partial X_\delta \) then \( \Gamma(Q) \geq b \) and hence

\[
A_\varepsilon(Q) \geq c_0 \Gamma(Q) + o(\varepsilon^{1/4}) \geq c_0 b + o(\varepsilon^{1/4}).
\]

On the other side, if \( Q \in Y \) then \( A_\varepsilon(Q) \leq c_0(a + b)/2 \). Hence, for \( \varepsilon \) small, \( Y \) cannot meet \( \partial X_\delta \) and this readily implies that \( Y \) is compact. Then \( A_\varepsilon \) possesses at least \( \text{cat}(Y, X_\delta) \) critical points in \( X_\delta \). Using (5.1) and the properties of the category one gets

\[
\text{cat}(Y, Y) \geq \text{cat}(X, X_\delta).
\]

Moreover, by Lemma 4.1 we know that to critical points of \( A_\varepsilon \) there correspond critical points of \( f_\varepsilon \) and so solutions of (2.1). Let \( Q_\varepsilon \in X \) be one of these critical points, if \( Q'_\varepsilon = Q_\varepsilon + \sqrt{\varepsilon} e_1 \), then

\[
(u_\varepsilon^{Q_\varepsilon}, v_\varepsilon^{Q_\varepsilon}) = (U_{Q_\varepsilon/\varepsilon} + w_\varepsilon(Q_\varepsilon), V_{Q_\varepsilon/\varepsilon} + w_\varepsilon'(Q_\varepsilon))
\]

is a solution of (2.1). Therefore

\[
\begin{align*}
  u_\varepsilon^{Q_\varepsilon}(x/\varepsilon) &\simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U_{Q_\varepsilon}(\frac{x - Q_\varepsilon}{\varepsilon}) \\
  v_\varepsilon^{Q_\varepsilon}(x/\varepsilon) &\simeq V_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = V_{Q_\varepsilon}(\frac{x - Q_\varepsilon'}{\varepsilon})
\end{align*}
\]

is a solution of (P\( \varepsilon \)) and also the concentration result follows. \( \square \)

Let us now give a short proof of Theorem 1.5.

**Proof of Theorem 1.5**  We need only to observe that, in this case, the solutions of (P\( \bar{\varepsilon} \)) will be found near \((\bar{U}Q, \bar{V}Q)\), properly truncated, where \( \bar{U}Q \) is the unique
solution of
\[
\begin{aligned}
-\Delta u + J_1(Q)u &= J_2(Q)u^{2p-1} \quad \text{in } \mathbb{R}^N, \\
0 &> u > 0 \quad \text{in } \mathbb{R}^N, \\
u(0) &= \max_{\mathbb{R}^N} u,
\end{aligned}
\]
and \(\bar{V}_Q\) is the unique solution of
\[
\begin{aligned}
-\Delta v + K_1(Q)v &= K_2(Q)v^{2p-1} \quad \text{in } \mathbb{R}^N, \\
0 &> v > 0 \quad \text{in } \mathbb{R}^N, \\
v(0) &= \max_{\mathbb{R}^N} v,
\end{aligned}
\]
for an appropriate choice of \(Q \in \bar{\Omega}_0\). It is easy to see that
\[
\bar{U}_Q(x) = \left(\frac{J_1(Q)}{J_2(Q)}\right)^{1/(2p-2)} \cdot \bar{W} \left(\sqrt{J_1(Q)} \cdot x\right),
\]
\[
\bar{V}_Q(x) = \left(\frac{K_1(Q)}{K_2(Q)}\right)^{1/(2p-2)} \cdot \bar{W} \left(\sqrt{K_1(Q)} \cdot x\right),
\]
where \(\bar{W}\) is the unique solution of
\[
\begin{aligned}
-\Delta z + z &= z^{2p-1} \quad \text{in } \mathbb{R}^N, \\
z &> 0 \quad \text{in } \mathbb{R}^N, \\
z(0) &= \max_{\mathbb{R}^N} z.
\end{aligned}
\]
At this point, we can repeat the previous arguments, with suitable modifications.
\[
\square
\]

**Remark 5.2.** Of course, the analogous of Theorem 5.1 holds also for problem \((\bar{P}_\varepsilon)\).

**References**

[1] N. Akhmediev, A. Ankiewicz, Partially coherent solitons on a finite background, Phys. Rev. Lett., 82, (1999), 2661–2664.
[2] A. Ambrosetti, M. Badiale, Variational perturbative methods and bifurcation of bound states from the essential spectrum, Proc. Roy. Soc. Edinburgh Sect. A, 128, (1998), 1131–1161.
[3] A. Ambrosetti, M. Badiale, S. Cingolani, Semiclassical states of nonlinear Schrödinger equations, Arch. Ration. Mech. Anal., 159, (2001), 253–271.
[4] A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. (JEMS), 7, (2005), 117–144.
[5] A. Ambrosetti, A. Malchiodi, W.M. Ni, Singly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I, Comm. Math. Phys., 235, (2003), 427–466.
[6] A. Ambrosetti, A. Malchiodi, W.M. Ni, Singly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II, Indiana Univ. Math. J., 53, (2004), 297–329.
[7] A. Ambrosetti, A. Malchiodi, S. Secchi, Multiplicity results for some nonlinear Schrödinger equations with potentials, Arch. Ration. Mech. Anal., 159, (2001), 253–271.
[8] H. Berestycki, P.L. Lions, Nonlinear scalar field equations I. Existence of a ground state, Arch. Ration. Mech. Anal., 82, (1983), 313–346.
[9] K.C. Chang, Infinite dimensional Morse theory and multiple solutions problems, Birkhäuser, 1993.
[10] D.N. Christodoulides, T.H. Coskun, M. Mitchell, M. Segev, Theory of incoherent self-focusing in biased photorefractive media, Phys. Rev. Lett., 78, (1997), 646–649.
[11] R. Cipolatti, W. Zumpichiatti, On the existence and regularity of ground states for a nonlinear system of coupled Schrödinger equations, Comput. Appl. Math., 18, (1999), 19–36.
[12] R. Cipolatti, W. Zumpichatti, Orbitally stable standing waves for a system of coupled nonlinear Schrödinger equations, Nonlinear Anal., 42, (2000), 445–461.
[13] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations, 4, (1996), 121–137.
[14] B.D. Esry, C.H. Greene, J.P. Burke, Jr., and J.L. Bohn, Hartree-Fock Theory for Double Condensates, Phys. Rev. Lett., 78, (1997), 3594–3597.
[15] A. Floer, A. Weinstein, Nonscattering wave packets for the cubic Schrödinger equation with a bounded potential, J. Funct. Anal., 69, (1986), 397–408.
[16] B. Gidas, W.N. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in \( \mathbb{R}^n \), Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.
[17] A. Hasegawa, Y. Kodama, Solitons in optical communications, Academic Press, San Diego, 1995.
[18] M.N. Islam, Ultrafast fiber switching devices and systems, Cambridge University Press, New York, 1992.
[19] I.P. Kaminow, Polarization in optical fibers, IEEE J. Quantum Electron., 17, (1981), 15–22.
[20] M.K. Kwong, Uniqueness of positive solutions of \( \Delta u - u + u^p = 0 \) in \( \mathbb{R}^n \), Arch. Ration. Mech. Anal., 105, (1989), 243–266.
[21] Y.Y. Li, On a singularly perturbed elliptic equation, Adv. Differential Equations, 2, (1997), 955–980.
[22] T.C. Lin, J. Wei, Ground State of N Coupled Nonlinear Schrödinger Equations in \( \mathbb{R}^n \), \( n \geq 3 \), Comm. Math. Phys., 255, (2005), 629–653.
[23] T.C. Lin, J. Wei, Spikes in two coupled nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.
[24] L.A. Maia, E. Montefusco, B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, preprint, 2005.
[25] C.R. Menyuk, Nonlinear pulse propagation in birefringence optical fiber, IEEE J. Quantum Electron., 23, (1987), 174–176.
[26] C.R. Menyuk, Pulse propagation in an elliptically birefringent Kerr medium, IEEE J. Quantum Electron., 25, (1989), 2674–2682.
[27] Y.G. Oh, Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of class \( (V)_a \), Comm. Partial Differential Equations, 13, (1988), 1499–1519.
[28] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys., 43, (1992), 270–291.
[29] E. Timmermans, Phase Separation of Bose-Einstein Condensates, Phys. Rev. Lett., 81, (1998), 5718–5721.
[30] X.F. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Comm. Math. Phys., 153, (1993), 229–244.
[31] X.F. Wang, B. Zeng, On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions, SIAM J. Math. Anal., 28, (1997), 633–655.

DIPARTIMENTO DI MATEMATICA
POLITECNICO DI BARI
VIA AMENDOLA 126/B, I-70126 BARI, ITALY
E-mail address: a.pomponio@poliba.it