Super \((a^*, d^*)-H\)-antimagic total covering of second order of shackle graphs

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**Abstract.** Let \(H\) be a simple and connected graph. A shackle of graph \(H\), denoted by \(G = \text{shack}(H,v,n)\), is a graph \(G\) constructed by non-trivial graphs \(H_1, H_2, \ldots, H_n\) such that, for every \(1 \leq s, t \leq n\), \(H_s\) and \(H_t\) have no a common vertex with \(|s-t| \geq 2\) and for every \(1 \leq i \leq n-1\), \(H_i\) and \(H_{i+1}\) share exactly one common vertex \(v\), called connecting vertex, and those \(k-1\) connecting vertices are all distinct. The graph \(G\) is said to be an \((a^*, d^*)-H\)-antimagic total graph of second order if there exist a bijective function \(f: V(G) \cup E(G) \rightarrow \{1,2,\ldots,|V(G)|+|E(G)|\}\) such that for all subgraphs isomorphic to \(H\), the total \(H\)-weights \(w(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)\) form an arithmetic sequence of second order of \(\{a^*, a^* + d^*, a^* + 3d^*, a^* + 6d^*, \ldots, a^* + (\frac{n^2-n}{2})d^*\}\), where \(a^*\) and \(d^*\) are positive integers and \(n\) is the number of all subgraphs isomorphic to \(H\). An \((a^*, d^*)-H\)-antimagic total labeling of second order \(f\) is called super if the smallest labels appear in the vertices. In this paper, we study a super \((a^*, d^*)-H\) antimagic total labeling of second order of \(G = \text{shack}(H,v,n)\) by using a partition technique of second order.

1. Introduction

All graphs in this study are simple, connected and undirected. A graph \(G\) is said to be an \((a^*, d^*)-H\)-antimagic total graph of second order if there exist a bijective function \(f: V(G) \cup E(G) \rightarrow \{1,2,\ldots,|V(G)|+|E(G)|\}\) such that for all subgraphs of \(G\) isomorphic to \(H\), the total \(H\)-weights \(w(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)\) form an arithmetic sequence of second order \(\{a^*, a^* + d^*, a^* + 3d^*, a^* + 6d^*, \ldots, a^* + (\frac{n^2-n}{2})d^*\}\), where \(a^*\) and \(d^*\) are positive integers and \(n\) is the number of all subgraphs of \(G\) isomorphic to \(H\). If such a function exist then \(f\) is called an \((a^*, d^*)-H\)-antimagic total labeling of second order of \(G\). An \((a^*, d^*)-H\)-antimagic total labeling of second order \(f\) is called super if \(f: V(G) \rightarrow \{1,2,\ldots,|V(G)|\}\). By this notion, the super \((a,d) - H\) antimagic total labeling is classified as the super \((a,d) - H\) antimagic total labeling of first order.

We initiate to study this concept, thus we have not found any relevant results yet. But for the super \((a,d) - H\) antimagic total labeling, we can find many published results, some of them can be cited in [2, 3, 8, 9] and [10, 11, 12, 13, 15]. Inayah et al. in [8] proved that, for \(H\) is a non-trivial connected graph and \(k \geq 2\) is an integer, \(\text{shack}(H,v,k)\) which contains exactly \(k\) subgraphs isomorphic to \(H\) is \(H\)-super antimagic. All these papers only dealt with \(d\) derived...
from the sequence of order one. Our paper attempt to solve a super \((a^*, d^*)\)-H antimagic total labeling of order two of \(G = shack(H, v, n)\).

To show those existence, we will use a special technique, namely an integer set partition technique. We consider the partition \(P^n_{m, d}(i, j)\) of the set \(\{1, 2, \ldots, mn\}\) into \(n\) columns with \(n \geq 2\), \(m\)-rows such that the difference between the sum of the numbers in the \((j + 1)\)th \(m\)-rows and the sum of the numbers in the \(j\)th \(m\)-rows is always equal to the constant \(d\), where \(j = 1, 2, \ldots, n - 1\). We need to establish some lemmas related to the partition \(P^n_{m, d}(i, j)\). Furthermore, the partition will be developed into a second order partition. These lemmas are useful to develop the super \((a^*, d^*)\)-H antimagic total labeling of second order of \(G = shack(H, v, n)\).

Let \(G\) be a shackle of graph denoted by \(G = shack(H, v, n)\). Let \(G\) and \(H\) be a connected graph with \(|V(G)| = p_G, |E(G)| = q_G, |V(H)| = p_H,\) and \(|E(H)| = q_H\). The vertex set and edge set of the graph \(G = shack(H, v, n)\) can be split into following sets: \(V = \{x_{ij}; 1 \leq j \leq n + 1\} \cup \{x_{ij}; 1 \leq i \leq p_H - 2, 1 \leq j \leq n\}\) and \(E = \{e_{ij}; 1 \leq i \leq q_H, 1 \leq j \leq n\}\). Let \(n, i, j\) be positive integers with \(n \geq 2\). Thus \(p_G = |V(G)| = n + 1 + (p_H - 2)n = 1 + np_H + n - 2n = 1 + n(p_H - 1)\) and \(q_G = |E(G)| = nq_H\).

We recall a partition \(P^n_{m, d}(i, j)\) introduced in [4]. We will use the partition for a linear combination in developing a bijection of vertex and edge label of the main theorem.

**Lemma 1.1.** [4] Let \(n\) and \(m\) be positive integers. The sum of \(P^n_{m, d1}(i, j)\) = \{(i - 1)n + j, 1 \leq i \leq m\}\) and \(P^n_{m, d2}(i, j)\) = \{(j - 1)m + i, 1 \leq i \leq m\}\) form a arithmetic sequence of difference \(d_1 = m\) and \(d_2 = m^2\), respectively.

### 2. Main Results

**Lemma 2.1.** Let \(G\) be a simple graph of order \(p\) and size \(q\). If \(G\) is super \((a^*, d^*)\)-H-antimagic total labeling of second order then \(d \leq \frac{(p_G - p_H)q_H + (q_G - q_H)p_H}{n}\). For \(p_G = |V(G)|, q_G = |E(G)|, p_H = |V(H)|, q_H = |E(H)|\), and \(n = |H|\).

**Proof.** Given the function \(f(V) = 1, 2, 3, \ldots, p_G\) and \(f(E) = p_G + 1, p_G + 2, p_G + 3, \ldots, p_G + q_G\). Let \((p_G, q_G)\) admit a super \((a^*, d^*)\)-H antimagic total labeling with the total second order function, \(f(total) = 1, 2, 3, \ldots, p_G + q_G\) then the set of edge weight of a graph is \(\{a^*, a^* + d^*, a^* + 3d^*, a^* + 6d^*, \ldots, a^* + (\frac{n^2 - n}{2}d)\}\) with \(a^*\) is the smallest weight thus:

\[
1 + 2 + \cdots + p_H + (p_G + 1) + (p_G + 2) + \cdots + (p_G + q_H) \leq a^* \\
\frac{p_H}{2}(1 + p_H) + q_Hp_G + \frac{q_H}{2}(1 + q_H) \leq a^* \\
\frac{p_H^2}{2} + \frac{p_H^2}{2} + p_Hq_G + \frac{q_H^2}{2} + \frac{q_H^2}{2} \leq a^*
\]
Lemma 2.4. Let \( a^{*} + \left( \frac{n^{2} - n}{2} \right) d^{*} \leq p_{G} + (p_{G} - 1) + (p_{G} - 2) + \ldots + (p_{G} - (p_{H} - 1)) + (p_{G} + q_{G}) + (p_{G} + q_{G} - 1) + (p_{G} + q_{G} - 2) + \ldots + (p_{G} + q_{G} - (q_{H} - 1)) \)

\[
= p_{H} p_{G} - \frac{p_{H} - 1}{2}(1 + (p_{H} - 1)) + q_{H} p_{G} + q_{H} q_{G} - \frac{q_{H} - 1}{2}(q_{H} - 1) 
\]

\[
= p_{H} p_{G} - \frac{p_{H} - 1}{2} + q_{H} p_{G} + q_{H} q_{G} - \frac{q_{H} - 1}{2} 
\]

\[
\left( \frac{n^{2} - n}{2} \right) d^{*} \leq p_{H} p_{G} - \frac{p_{H} - 1}{2} + q_{H} p_{G} + q_{H} q_{G} - \frac{q_{H} - 1}{2} 
\]

By simple calculation, for \( n \) and \( m \) be positive integers and \( \{\{m,n\}: \text{integer} \geq 1 \leq i \leq m \} \) and \( \{mn+i-mj:1 \leq i \leq m \} \) form an arithmetic sequence of differences \( d_{3} = -m \) and \( d_{4} = -m^{2} \).

Proof. By simple calculation, for \( j = 1, 2, \ldots, n \), it gives \( \sum_{i=1}^{m} P_{m,d_{3}}(i,j) = P_{m,d_{3}}(j) \leftarrow P_{m,d_{3}}(j) = \{ \frac{n}{2}(m^{2} + m) + m - mj \} \leftarrow P_{m,d_{3}}(j) = \{ \frac{n}{2}(m^{2} + m) + m, \frac{n}{2}(m^{2} + m) - m, \frac{n}{2}(m^{2} + m) - 2m, \ldots, \frac{n}{2}(m^{2} + m) - mn \} \) and \( \sum_{i=1}^{m} P_{m,d_{4}}(i,j) = P_{m,d_{4}}(j) \leftarrow P_{m,d_{4}}(j) = \{ \frac{n}{2}(2mn + m + 1) - m^{2}, \frac{n}{2}(2mn + m + 1) - 2m^{2}, \ldots, \frac{n}{2}(2mn + m + 1) - mn \} \). It is easy to see that the differences of those sequences are \( d_{3} = -m, d_{4} = -m^{2} \). It concludes the proof.

Corollary 2.2. If the graph \( G = \text{shack}(H,v,n) \) admits super \( (a^{*},d^{*}) \)-\( H \)-antimagic total labeling of second order for integer \( n \geq 2 \), then \( d \leq \frac{2(p_{H} - p_{H} + q_{H})}{n} \).

The following lemmas are useful for showing the existence of super \( (a^{*},d^{*}) \)-\( H \) antimagic total labeling \( G = \text{shack}(H,v,n) \).

Lemma 2.3. Let \( n \) and \( m \) be positive integers. For \( 1 \leq j \leq n \), the sum of \( P_{m,d_{3}}^{n}(i,j) = \{1 + ni - j:1 \leq i \leq m \} \) and \( P_{m,d_{4}}^{n}(i,j) = \{mn+i-mj:1 \leq i \leq m \} \) form an arithmetic sequence of differences \( d_{3} = -m \) and \( d_{4} = -m^{2} \).

Proof. By simple calculation, for \( j = 1, 2, \ldots, n \), it gives \( \sum_{i=1}^{m} P_{m,d_{3}}^{n}(i,j) = P_{m,d_{3}}^{n}(j) \leftarrow P_{m,d_{3}}^{n}(j) = \{ \frac{n}{2}(m^{2} + m) + m - mj \} \leftarrow P_{m,d_{3}}^{n}(j) = \{ \frac{n}{2}(m^{2} + m), \frac{n}{2}(m^{2} + m) - m, \frac{n}{2}(m^{2} + m) - 2m, \ldots, \frac{n}{2}(m^{2} + m) - mn \} \) and \( \sum_{i=1}^{m} P_{m,d_{4}}^{n}(i,j) = P_{m,d_{4}}^{n}(j) \leftarrow P_{m,d_{4}}^{n}(j) = \{ \frac{n}{2}(2mn + m + 1) - m^{2}, \frac{n}{2}(2mn + m + 1) - 2m^{2}, \ldots, \frac{n}{2}(2mn + m + 1) - mn \} \). It is easy to see that the differences of those sequences are \( d_{3} = -m, d_{4} = -m^{2} \). It concludes the proof.

Lemma 2.4. Let \( n \), \( m \) be positive integers and \( n = m \). The sum of

\[
P_{m,d_{3}}^{n}(i,j) = \begin{cases} 
(2m+2j+1)-(2m-1)-i-j \leq 1 & \text{for } i-j \geq m-1 \cr 
(m^2-m+2j+1) & \text{for } m-1 \geq i-j \cr
\end{cases}
\]
form an arithmetic sequence of second order with common difference \( d_5 = m \).

**Proof.** By simple calculation, for \( j = 1, 2, \ldots, n \), it gives \( \sum_{i=1}^{m} P_{m,d_5}(i,j) = P_{m,d_5}(j) \). It concludes the proof.

**Lemma 2.5.** Let \( n, m \) be positive integers and \( n = m \). The sum of

\[
P_{m,d_5}(i,j) = \left\{ \begin{array}{ll}
2m^2 - 2mi + j^2 + 1 & 1 \leq i, j \leq m + 1 \\
4m^2 + 4m - 4mj - 3i(j + 2j + i + 3j^2 + 1) & 1 \leq i \leq m, m + 1 < i + j
\end{array} \right.
\]

form an arithmetic sequence of second order with common difference \( d_5 = -m \).

**Proof.** By simple calculation, for \( j = 1, 2, \ldots, n \), it gives \( \sum_{i=1}^{m} P_{m,d_5}(i,j) = P_{m,d_5}(j) \). It is easy to see that the differences of those sequences are \( d_5 = -m, d_4 = m^2 \). It concludes the proof.

**Lemma 2.6.** Let \( d^* \) be the common difference of arithmetic sequence of second order and \( d \) be the common difference of arithmetic sequence of first order, the sum of corresponding terms will form an arithmetic sequence of second order with common difference \( d^* \).

**Proof.** An arithmetic sequence of first order is a sequence of the form:

\[ a, a + d, a + 2d, a + 3d, \ldots, a + (n - 1)d \]

where \( a \) is the first term and \( d \) is common difference of the sequence. Whilst an arithmetic sequence of second order is of the form \( a^*, a^* + d^*, a^* + 2d^*, a^* + 3d^*, \ldots, a^* + \left(\frac{n^2 - n}{2}\right)d^* \), where \( a^* \) and \( d^* \) are the first term and common difference of the sequence, respectively. Now, add the corresponding terms of these two expression:

| Sequence | \( a \) | \( a + d \) | \( a + 2d \) | \( a + 3d \) | \( \ldots \) | \( a + (n - 1)d \) |
|----------|--------|--------|--------|--------|----------|--------|
| Sequence | \( a^* \) | \( a^* + d^* \) | \( a^* + 2d^* \) | \( a^* + 3d^* \) | \( \ldots \) | \( a^* + \left(\frac{n^2 - n}{2}\right)d^* \) |
| First order difference | \( d + d^* \) | \( d + 2d^* \) | \( d + 3d^* \) | \( \ldots \) | \( (n - 1)d \) |
| Second order difference | \( d^* \) | \( d^* \) | \( d^* \) | \( \ldots \) | \( d^* \) |

It concludes the proof.

**Theorem 2.7.** Let \( H \) be a connected graph, then the shackle of the connected graph \( G = shack(H, v, n) \) admits super \( (a^*, d^*) - H \) antimagic total labeling.
Proof. Let $G$ be a shackle of graph denoted by $G = shackle(H,v,n)$. Let $G$ and $H$ be a connected graph with $|V(G)| = p_G$, $|E(G)| = q_G$, $|V(H)| = p_H$, and $|E(H)| = q_H$. The vertex set and edge set of the graph $G = shackle(H,v,n)$ can be split into following sets: $V = \{x_j; 1 \leq j \leq n+1\} \cup \{x_{ij}; 1 \leq i \leq p_H-2, 1 \leq j \leq n\}$ and $E = \{e_{ij}; 1 \leq i \leq q_H, 1 \leq j \leq n\}$. Let $n$, $i$, $j$ be positive integers with $n \geq 2$. Thus $p_G = |V(G)| = n + 1 + (p_H - 2)n = 1 + np_H + n - 2n = 1 + n(p_H - 1)$ and $q_G = |E(G)| = nq_H$. Construct a total labeling $f,g : V(shackle(H,v,n)) \cup E(shackle(H,v,n)) \rightarrow \{1, 2, \ldots, 1 + n(p_H + q_H - 1)\}$ constitute the following set:

$$
\begin{align*}
\phi_1(V_{ph}) &= \{j; 1 \leq j \leq n + 1\} \\
\phi_2(V_{ph}) &= \{P_{m_1,d_v}(i,j) \oplus n + 1\} \cup \{P_{m_2,d_v}(i,j) \oplus n(m_1 + 1) + 1\} \\
\phi_3(E_{gh}) &= \{P_{c_1,d_v}(i,j) \oplus (p_H - 1)n + 1\} \cup \{P_{c_2,d_v}(i,j) \oplus (p_H + r_1 - 1)n + 1\}
\end{align*}
$$

where $m_1 + m_2 = p_H - 2$, $c_1 + c_2 = q_H$, $d_v$ and $d_e$ depends on $p_H - 2$ and $q_H$, respectively. Furthermore the weight of the subgraphs $H_i$, $i = 1, 2, \ldots, p_L$ in the following way:

$$
W = \sum_{v \in V(H_1)} f(v) + \sum_{e \in E(H_1)} f(e)
$$

$$
= (2j + 1) + \sum_{i=1}^{m_1} (P_{m_1,d_v}(j) \oplus n + 1) + \sum_{i=1}^{m_2} (P_{m_2,d_v}(j) \oplus n(m_1 + 1) + 1)
$$

$$
+ (c_1) \sum_{i=1}^{c_1} (P_{c_1,d_v}(j) \oplus (p_H - 1)n + 1) + (c_2) \sum_{i=1}^{c_2} (P_{c_2,d_v}(j) \oplus (p_H + r_1 - 1)n + 1)
$$

$$
= [2j + 1] + [C_{m_1,d_v}^{n} + d_v^c (j^2 - j + 2 / 2) m_1(n + 1)] + [C_{m_2,d_v}^{n} + d_v c j + m_2(n(m_1 + 1) + 1)]
$$

$$
+ [C_{c_1,d_v}^{n} + d_v c (j^2 - j + 2 / 2) c_1((p_H - 1)n + 1)]
$$

$$
+ [C_{c_2,d_v}^{n} + d_v c + c_2(n(p_H + r_1 - 1) + 1)]
$$

based on Lemma 2.6 we obtained:

$$
= 1 + C_{m_1,d_v}^{n} + C_{m_2,d_v}^{n} + C_{c_1,d_v}^{n} + C_{c_2,d_v}^{n} + m_1(n + 1) + m_2(n(m_1 + 1) + 1)
$$

$$
+ c_1((p_H - 1)n + 1) + c_2(n(p_H + r_1 - 1) + 1) + [d_v^c + d_v c + 2j]
$$

3. Special Families

**Theorem 3.1.** Suppose $G = shackle(C_m,v,n)$, with $s \geq 3$ dan $2s \geq n+1$, graph $G$ admits super $(a^*, d^*)$-H-anti-magic total covering of second order with $a = 3 + \left[\frac{2m_1 + 4m_2 + 3m_3 - 3m_1}{6}\right] + m_1(n + 1) + \left[\frac{2m_1 - m_2}{2} + m_2 \right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right] + \left[\frac{m_1 - m_2}{2} + m_2(n(m_1 + 1) + 1)\right]
$$

and $d = m_1 + c_1$.

**Proof.** The graph $G = shackle(C_m,v,n)$ have vertex set $V = \{x_j; 1 \leq j \leq n + 1\} \cup \{x_{ij}; 1 \leq i \leq m - 2, 1 \leq j \leq n\}$ and edge set $E = \{e_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$. Thus $p_G = |V(G)| = mn - n + 1$ and $q_G = |E(G)| = mn$ where $p_H = m - 2$ and $q_H = m$ respectively are the cardinality of the vertex and edge on one cover $H$. We can define the vertex labeling
\[ f_1 : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p_G + q_G\} \] by using the linear combination of \( P_{m,m}^n \), \( P_{m,m}^{n,m} \), \( P_{m,m}^{m,n} \), \( P_{m,m}^{n,m} \) and \( P_{m,m}^{m,n} \). By Lemma 1.1.2.3.2.4 and 2.5 we use \( m_1 \) and \( c_1 \) for the partition \( P_{m,m}^{n,m}(i,j) \), \( m_2 \) and \( c_2 \) for the partition \( P_{m,m}^{m,n}(i,j) \), \( m_3 \) and \( c_3 \) for the partition \( P_{m,m}^{n,m}(i,j) \), \( m_4 \) and \( c_4 \) for the partition \( P_{m,m}^{m,n}(i,j) \), \( m_5 \) and \( c_5 \) for the partition \( P_{m,m}^{n,m}(i,j) \) and we use \( m_6 \) and \( c_6 \) for the partition \( P_{m,m}^{m,n}(i,j) \). For \( i = 1, 2, \ldots, m \), \( l = 1, 2, \ldots, c \) and \( j = 1, 2, \ldots, n \), the total labels can be expressed as follows

\[
\begin{align*}
f_1(V_{PH}) &= \{ j ; 1 \leq j \leq n + 1 \} \\
f_2(V_{PH}) &= \{ P_{m_1,d_1,c_1}^{n,m}(i,j) \oplus n + 1 \} \cup \{ P_{m_2,d_2,c_2}^{n,m}(i,j) \oplus n(m_1 + 1) + 1 \} \\
f(E_{PH}) &= \{ P_{c_1,d_1}^{n}(i,j) \oplus (p_H - 1)n + 1 \} \cup \{ P_{c_2,d_2}^{n}(i,j) \oplus (p_H + r_1 - 1)n + 1 \}
\end{align*}
\]

The total vertex and edge-weights of \( G = shack(C_m, v, n) \) under the labeling \( f_1 \), for \( 1 \leq j \leq n \), constitute the following sets:

\[
W = \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)
\]

\[
= (2j + 1) + \left( \sum_{i=1}^{m_1} (P_{m_1,d_1,c_1}^{n,m}(j) \oplus n + 1) \right) + \left( \sum_{i=1}^{m_2} (P_{m_2,d_2,c_2}^{n,m}(j) \oplus n(m_1 + 1) + 1) \right) + \left( \sum_{i=1}^{c_1} (P_{c_1,d_1}^{n}(j) \oplus (p_H - 1)n + 1) \right) + \left( \sum_{i=1}^{c_2} (P_{c_2,d_2}^{n}(j) \oplus (p_H + r_1 - 1)n + 1) \right)
\]

\[
= [2j + 1] + [C_{m_1,d_1,c_1}^{n,m} + d_1 j (\frac{j^2 - j + 2}{2}) + m_1 (n + 1)] + [C_{m_2,d_2,c_2}^{n,m} + d_2 j + m_2 (n + 1)] + [C_{c_1,d_1}^{n} + d_1 j (\frac{j^2 - j + 2}{2}) + c_1 (p - 1)n + 1] + [C_{c_2,d_2}^{n} + d_2 j + c_2 (p_H + r_1 - 1) + 1]
\]

The total weights of \( G = shack(C_m, v, n) \) constitute the following sets:

\[
W_{f_1} = w_{f_1}^1 + w_{f_1}^2 + w_{f_1}^3
\]

\[
= [2j + 1] + w_{f_1}^2 + w_{f_1}^3
\]

\[
= C^* + C + 1 + [2 + \frac{3m_1 - 3m_1}{2} + m_2 + m_3 - m_4 - m_5 - \frac{3c_1 - 3c_1}{2} + c_2 + c_3 - c_4 - c_4 j ; 1 \leq j \leq n]
\]

By simple calculation, for \( j = 1, 2, \ldots, n \), it gives \( W_{f_1} = C^* + C + 1 + [2 + \frac{3m_1 - 3m_1}{2} + m_2 + m_3 - m_4 - m_5 - \frac{3c_1 - 3c_1}{2} + c_2 + c_3 - c_4 - c_4 j ; 1 \leq j \leq n] \) \( W_{f_1} \).
m_2(n(m_1 + 1) + 1)] + \left[ \frac{m_3 - m_2^2}{2} + m_3^2 + m_3(n(m_1 + m_2 + 1) + 1) \right] + \left[ \frac{m_4}{2} (m_3^2 + m_4) + m_4 - m_4 + m_4 \left( \sum_{i=1}^{3} m_i + 1 \right) + 1 \right] + \left[ \frac{m_5}{2} (2m_5^n + m_5 + 1) - m_5^2 + m_5 \left( \sum_{i=1}^{4} m_i + 1 \right) + 1 \right] \left[ \frac{2c_1^3 + 4c_1 + 3c_1 - 3c_1}{6} \right] +
\left[ \frac{m_5}{2} (2m_5^n + m_5 + 1) + c_5 \left( \sum_{i=1}^{4} c_i + 1 \right) + 1 \right] + \left[ \frac{m_5}{2} (2m_5^n + m_5 + 1) + c_5 \left( \sum_{i=1}^{4} c_i + 1 \right) + 1 \right] \left[ \frac{2c_5^3 + 4c_5 + 3c_5 - 3c_5}{6} \right] +
\text{when the total edge weights at } j = 1 \text{ and the difference } d = [2 + m_1^2 + c_1]. \text{ It concludes the proof. }$

\square

4. Concluding Remarks
We have shown the existence of super antimagic labeling of second order for graph operation $G = sh\text{ack}(H, v, n)$. We have found that $G = sh\text{ack}(H, v, n)$ admits a super($a^*, d^*$)-$H$ antimagic labeling of second order for all differences $d = 2 + d_1^* + d_2^*$ where $d_1^*$ and $d_2^*$ are respectively feasible difference of second order of integer set partition. We have not found the result for another graph operations. Thus, we propose the following open problems.

**Open Problem 4.1.** Analyse the existence of super ($a^*, d^*$)-$H$ antimagic total labeling of second order of other graph operations.

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