A trajectory-based framework for data-driven system analysis and control

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Abstract—The vector space of all input-output trajectories of a discrete-time linear time-invariant (LTI) system is spanned by time-shifts of a single measured trajectory, given that the respective input signal is persistently exciting. This fact, which was proven in the behavioral control framework, shows that a single measured trajectory can capture the full behavior of an LTI system and might therefore be directly used for system analysis and controller design, without explicitly identifying a model. In this paper, we translate the result from the behavioral context to the classical state-space control framework and we extend it to certain classes of nonlinear systems. Further, we sketch links to several potential applications, including a framework for data-driven model predictive control as well as the verification of dissipativity properties from a single input-output trajectory. In this way, we argue that the presented approach may serve as a foundation for providing rigorous guarantees in data-based system analysis and controller design.

I. INTRODUCTION

Finding rigorous and efficient ways to integrate data into control theory has been a problem of great interest for many decades. Since most of the classical contributions in control theory rely on model knowledge, the problem of finding such a model from measured data, i.e., system identification, has become a mature research field [1]. More recently, learning controllers directly from data has received increasing interest, not least due to many successful practical applications of reinforcement learning techniques [2]. However, as is thoroughly evaluated in [3], such methods typically require large amounts of data, they are often not reproducible, and their analysis rarely addresses rigorous guarantees on e.g. stability of the closed loop.

Also in the control community, several approaches for the direct design of controllers from data have been proposed. Established methods include the Virtual Reference Feedback Tuning paradigm [4] or Iterative Feedback Tuning [5]. However, fundamental problems such as the direct data-driven design of linear quadratic optimal controllers with guarantees from finite data are still not fully solved. The recent contribution [6] uses novel statistical estimation techniques in combination with robust control to give guarantees as well as sample complexity bounds on this problem, using only finitely many data samples. A stochastic, and hence less conservative, approach towards the same problem has been made in [7]. Although these contributions are important steps towards data-driven control with rigorous guarantees, they have opened up many new questions, which have not yet been fully answered.

It is the goal of this paper to present an alternative, unifying framework for data-based control, which allows for the development of various system analysis and controller design methods based directly on measured data. This framework relies on the characterization of all input-output trajectories of an unknown dynamical system using a single input-output trajectory, which may be obtained via simulation or an experiment. The latter problem has been considered and solved in the context of behavioral systems theory for discrete-time linear time-invariant (LTI) systems in [8]. In the behavioral approach, a system is not defined via a differential or difference equation with inputs and outputs, but rather as the space of all system trajectories [9], [10]. Thus, it seems naturally well-suited for the development of purely data-based approaches to system analysis and control. With the behavioral theory, an alternative view on control has been developed, providing us also with a deeper understanding of the classical control framework. However, despite the theoretical maturity of the behavioral approach, a large majority of today’s control community prefers the classical state-space framework, put forward by Kalman [11] and others. Therefore, it is one of our main goals to translate the work from [8] to the classical control language.

The remainder of this paper is structured as follows. In Section III we phrase the main theorem of [8], which uses measured data to characterize all system trajectories, in the classical control setting. Further, we show how this result can be improved by weaving multiple such trajectories together. In Section IV we extend the idea to classes of nonlinear systems, which are linear in suitably chosen and known nonlinear coordinates. Building on these results, links to three applications are presented: The data-driven simulation problem (Section V-A), a data-based model predictive control (MPC) framework (Section V-B), as well as the inference of dissipativity properties from data (Section V-C).

II. SETTING

We denote the set of integers in the interval \([k_0, k_1]\) by \(I_{[k_0, k_1]}\). The Kronecker product is written as \(\otimes\). For a...
sequence \( \{x_k\}_{k=0}^{N-1} \), we define the Hankel matrix

\[
H_L(x) := \begin{bmatrix}
x_0 & x_1 & \ldots & x_{N-L} \\
x_1 & x_2 & \ldots & x_{N-L+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{L-1} & x_L & \ldots & x_{N-1}
\end{bmatrix}.
\]

For a stacked window of the sequence, we write

\[
x_{[k_0,k_1]} = \begin{bmatrix} x_{k_0} \\ \vdots \\ x_{k_1} \end{bmatrix}.
\]

Further, \( x \) will denote either the sequence itself or the stacked vector \( x_{[0,N-1]} \) containing all of its components. A key assumption for our results will be persistence of excitation of the input signal, as captured in the following standard definition.

**Definition 1.** We say that a signal \( \{x_k\}_{k=0}^{N-1} \) with \( x_k \in \mathbb{R}^n \) is persistently exciting of order \( L \) if \( \operatorname{rank}(H_L(x)) = nL \).

Note that the above definition implies \( N \geq (n+1)L - 1 \). This means that, for a signal to be persistently exciting, it is not sufficient that its time-shifts are linearly independent, but the signal must also be long enough. A large part of this paper deals with discrete-time multi-input multi-output LTI systems, i.e., systems of the form

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k, \quad x_0 = \bar{x}, \\
y_k &= Cx_k + Du_k,
\end{align*}
\]

where the matrices \( A, B, C, D \) as well as the initial condition \( \bar{x} \) are unknown and only input-output data \( \{u_k, y_k\}_{k=0}^{N-1} \), which may be obtained from (I) via simulation or an experiment, is available. Throughout this paper, \( n \) will denote the order of the unknown system and it is only assumed to be known in terms of a potentially rough upper bound. Further, we denote the input and output dimension by \( m \) and \( p \), respectively.

We are interested in using a single trajectory to characterize all other trajectories, which might be produced from the system (I), i.e., which satisfy the following definition.

**Definition 2.** We say that an input-output sequence \( \{u_k, y_k\}_{k=0}^{N-1} \) is a trajectory of an LTI system \( G \), if there exists an initial condition \( \bar{x} \in \mathbb{R}^n \) as well as a state sequence \( \{x_k\}_{k=0}^{N} \) such that

\[
\begin{align*}
x_{k+1} &= Ax_k + Bu_k, \quad x_0 = \bar{x}, \\
y_k &= Cx_k + Du_k,
\end{align*}
\]

for \( k = 0, \ldots, N-1 \), where \( (A, B, C, D) \) is a minimal realization of \( G \).

It follows directly from linearity that the set of all trajectories of an LTI system in the sense of Definition 2 forms a vector space. As we will see in Section III a basis for this vector space is formed by time-shifts of a single measured trajectory, given that the respective input signal is persistently exciting.

Throughout this paper, we make extensive use of the well-known fact that any LTI system admits a controllable and observable minimal realization with the same input-output behavior. The particular choice of a specific minimal realization is however not relevant. Further, we exploit that any fixed window of an input-output trajectory \( \{u_k, y_k\}_{k=k_0}^{k_1} \) induces a unique state trajectory \( \{x_k\}_{k=k_0}^{k_1} \) (in a given minimal realization), whenever \( k_1 - k_0 \geq n - 1 \).

### III. Trajectory-Based Representation of Linear Systems

In this section, we translate the main result of [8], which characterizes the trajectory space of an unknown system from measured data, to the classical state-space control framework. While the behavioral theory is naturally well-suited for such a result, we illustrate that it can also be formulated in the classical framework in an elegant way. Further, as a novel contribution, we show how a required persistence of excitation assumption can be relaxed by weaving multiple trajectories together to achieve an overall larger time horizon.

The following result is the correspondence of [8, Theorem 1] in the classical control setting and it will serve as the basis for the remainder of this paper.

**Theorem 3.** Suppose \( \{u_k, y_k\}_{k=0}^{N-1} \) is a trajectory of an LTI system \( G \), where \( u \) is persistently exciting of order \( L + n \). Then, \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) is a trajectory of \( G \) if and only if there exists \( \alpha \in \mathbb{R}^{N-L+1} \) such that

\[
\begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} \alpha = \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}.
\]

**Proof.** This is a direct application of [8, Theorem 1] to the special case of controllable state-space systems. \( \square \)

Note that (2) is equivalent to

\[
\begin{align*}
\bar{u}_{[0,L-1]} &= \sum_{i=0}^{N-L} \alpha_i u_{[i,L-1+i]}, \\
\bar{y}_{[0,L-1]} &= \sum_{i=0}^{N-L} \alpha_i y_{[i,L-1+i]},
\end{align*}
\]

i.e., the trajectory space is spanned by time-shifts of the measured trajectory. Thus, the "if"-direction in Theorem 3 follows directly from the fact that \( G \) is LTI, without adhering to the persistence of excitation assumption. The intuition about the "only if"-direction is sketched in the following. Take any trajectory \( \{\hat{u}_k, \hat{y}_k\}_{k=0}^{L-1} \) of \( G \). Clearly, we need \( L \) degrees of freedom in the input to choose \( \alpha \in \mathbb{R}^{N-L+1} \) such that (4) holds. Additional \( n \) degrees of freedom can then be used to attain the initial internal condition \( \bar{x}_0 \). Since \( \{y_k\}_{k=0}^{L-1} \) is a linear combination of \( \{u_k\}_{k=0}^{L-1} \) and \( \bar{x}_0 \), this is enough to find an \( \alpha \) which satisfies both (3) and (4), and thus (2). Therefore, persistence of excitation of order \( L + n \) is required for the equivalence in Theorem 3.

Theorem 3 shows that all trajectories of an unknown LTI system can be constructed from a single persistently exciting trajectory. Equivalently, the vector space of all system trajectories is equal to the range of a data-dependent
Hankel matrix. Thus, in a way, the measured input-output trajectory serves as a system representation on its own, without using it to identify a model. Prior knowledge of the unknown system’s order is only implicitly needed in Theorem 3 through the condition that \( u \) has to be persistently exciting of order \( L + n \). Hence, if the amount of available data \( N \) is significantly larger than \( n \) and the input is persistently exciting of a sufficiently high order, only a rough upper bound on \( n \) is required.

As described above, persistence of excitation is essential for establishing the equivalence in Theorem 3. Note however that it also sets a fundamental limit on the application of Theorem 3. In order to span the space of all trajectories of length \( L \), Theorem 3 requires \( N \geq (m + 1)(L + n) - 1 \) or, equivalently, \( L \leq \frac{N + 1}{m} - n \). Loosely speaking, if \( m = 1 \), \( L \) can only be half as long as \( N \) and, with increasing input dimension \( m \), the maximum length \( L \) decreases by a factor of \( \frac{1}{m} \).

An intuitive solution to overcome this limitation would be to weave several, say \( \xi \in \mathbb{N} \), trajectories of length \( L \) together to construct an overall trajectory of length \( \xi L \). This is however not trivial, since the internal states of the separate trajectories have to align at the intersections. In [12, Lemma 3], it is shown that two distinct input-output trajectories can be woven together if they align over a sufficiently long window at their intersection. The following result is an extension of [12, Lemma 3] to more than two trajectories.

**Proposition 4.** Suppose \( \{u_k, y_k\}_{k=0}^{N-1} \) is a trajectory of an LTI system \( G \), where \( u \) is persistently exciting of order \( L + n \).

Then, \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) with \( L = \xi L + (1 - \xi)n \), \( \xi \in \mathbb{N} \), is a trajectory of \( G \) if and only if there exist \( \alpha^i \in \mathbb{R}^{N-L+1}, i \in [1, \xi] \), such that

\[
\begin{bmatrix}
H_L(u) \\
H_L(y)
\end{bmatrix}
\begin{bmatrix}
I_{\xi-1} \otimes H_{L-n}(u_{[n,N-1]}) \\
I_{\xi-1} \otimes H_{L-n}(y_{[n,N-1]})
\end{bmatrix}
\begin{bmatrix}
\alpha^1 \\
\vdots \\
\alpha^\xi
\end{bmatrix} =
\begin{bmatrix}
\bar{u}_{[0,L-1]} \\
\bar{y}_{[0,L-1]}
\end{bmatrix},
\]

(5)

\[
H_n(u_{[L-n,N-1]}) \alpha^i = H_n(u_{[0,N-L+n-1]}) \alpha^{i+1},
\]

(6)

\[
H_n(y_{[L-n,N-1]}) \alpha^i = H_n(y_{[0,N-L+n-1]}) \alpha^{i+1},
\]

(7)

\( i \in [1, \xi-1] \).

**Proof.** Define \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) via

\[
\begin{bmatrix}
\bar{u}_{[0,L-1]} \\
\bar{y}_{[0,L-1]}
\end{bmatrix} =
\begin{bmatrix}
H_L(u) \\
H_L(y)
\end{bmatrix}
\begin{bmatrix}
I_{\xi-1} \otimes H_{L-n}(u_{[n,N-1]}) \\
I_{\xi-1} \otimes H_{L-n}(y_{[n,N-1]})
\end{bmatrix}
\begin{bmatrix}
\alpha^1 \\
\vdots \\
\alpha^\xi
\end{bmatrix},
\]

and note that (5) means that \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) is a stacked version of the sequences \( \{\tilde{u}_k, \tilde{y}_k\}_{k=0}^{L-1} \) in the sense that

\[
\bar{u}_{[0,L-1]} =
\begin{bmatrix}
\tilde{u}^1_{[0,L-1]} \\
\vdots \\
\tilde{u}^\xi_{[0,L-1]} \\
\tilde{u}^1_{[n,L-1]} \\
\vdots \\
\tilde{u}^\xi_{[n,L-1]}
\end{bmatrix},
\]

\[
\bar{y}_{[0,L-1]} =
\begin{bmatrix}
\tilde{y}^1_{[0,L-1]} \\
\vdots \\
\tilde{y}^\xi_{[0,L-1]} \\
\tilde{y}^1_{[n,L-1]} \\
\vdots \\
\tilde{y}^\xi_{[n,L-1]}
\end{bmatrix}.
\]

(8)

According to Theorem 3, the sequences \( \{\tilde{u}_k, \tilde{y}_k\}_{k=0}^{L-1} \) are trajectories of \( G \). Further, (6) and (7) imply that, at the transitions between the separate trajectories, they align over windows of length \( n \), i.e.,

\[
\tilde{u}_{i[L-n,L-1]} = \tilde{u}_{i+1[n,n-1]}, \quad i \in [1,\xi-1],
\]

(9)

\[
\tilde{y}_{i[L-n,L-1]} = \tilde{y}_{i+1[n,n-1]}, \quad i \in [1,\xi-1].
\]

(10)

Denote by \( \{\bar{x}_i\}_{k=0}^{L-1} \) the state trajectory corresponding to \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) in some minimal realization of \( G \). The conditions (9) and (10) imply that, at the transitions between the separate trajectories, the internal states align, i.e., \( \bar{x}_L = \bar{x}_{L+1} \), and thus, \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) is a trajectory of \( G \).

**Only If.** Suppose \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) is a trajectory of \( G \). Define \( \{\tilde{u}_k, \tilde{y}_k\}_{k=0}^{L-1} \), \( i \in [1,\xi] \), according to (8) and note that any of these sequences is itself a trajectory of \( G \). Hence, it follows directly from Theorem 3 that there exist \( \alpha^i \in \mathbb{R}^{N-L+1}, i \in [1,\xi] \), such that (5)-(7) hold.

Proposition 4 weaves multiple trajectories \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \) together to form a single, longer sequence \( \{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1} \). To make this sequence a trajectory of \( G \) in the sense of Definition 2 it only needs to be ensured that the shorter trajectories align over at least \( n \) steps at their intersections. Note that the number of trajectories \( \xi \) can be chosen arbitrarily large and thus, Proposition 4 can be used to construct trajectories of arbitrary length, using a single measured trajectory of finite length.

Although we assumed for notational simplicity that all trajectories contributing to the overall trajectory are of equal length, the same idea can be applied to weave trajectories of different lengths together. Further, one can straightforwardly employ measurements from multiple experiments of possibly different time horizons.

**IV. TRAJECTORY-BASED REPRESENTATION OF NONLINEAR SYSTEMS**

In this section, we extend Theorem 3 to certain classes of nonlinear systems. First, the special cases of Hammerstein and Wiener systems are considered, for which an extension of Theorem 3 is immediate. Thereafter, we combine these results and consider nonlinear systems, which are linear in suitably chosen and known coordinates. Trajectories of such systems are defined as in Definition 2 i.e., as trajectories of an LTI system in the lifted coordinates. During the last decades, there have been many contributions to identify Hammerstein and Wiener systems from data [13, 14]. In this context, our results can be seen as an alternative to the identification of such systems, using a single measured trajectory to represent them.

**A. Hammerstein systems**

A Hammerstein system is a nonlinear system, composed of a static nonlinearity followed by an LTI system, i.e., it is of the form

\[
x_{k+1} = Ax_k + B\psi(u_k), \quad x_0 = \bar{x},
\]

\[
y_k = Cx_k + D\psi(u_k),
\]

(11)
with a nonlinear function $\psi : \mathbb{R}^m \to \mathbb{R}^m$. In the following, we deal only with the case $\tilde{m} = 1$ for notational simplicity, although the same ideas can be employed for $\tilde{m} > 1$. We assume that $\psi$ can be written as $\psi(u) = \sum_{i=1}^r a_i \psi_i(u)$ for $r$ known basis functions $\psi_i$. Further, we define the auxiliary input trajectory $\{v_k\}_{k=0}^{N-1}$ with components

$$v_k = \begin{bmatrix} \psi_1(u_k) \\ \vdots \\ \psi_r(u_k) \end{bmatrix}. \quad (12)$$

Note that a Hammerstein system can also be viewed as a linear map from $v$ to $y$. This insight is used in the following result, which is an immediate extension of Theorem 3 to Hammerstein systems.

**Proposition 5.** Suppose $\{u_k, y_k\}_{k=0}^{N-1}$ is a trajectory of a Hammerstein system (11), where $v$ from (12) is persistently exciting of order $L + n$. Then, $\{\tilde{u}_k, \tilde{y}_k\}_{k=0}^{L-1}$ is a trajectory of (11) if and only if there exists $\alpha \in \mathbb{R}^{N-L+1}$ such that

$$\begin{bmatrix} H_L(v) \\ H_L(y) \end{bmatrix} \alpha = \begin{bmatrix} \tilde{v} \\ \tilde{y} \end{bmatrix}, \quad (13)$$

where $\{\tilde{v}_k\}_{k=0}^{L-1}$ is the trajectory with components

$$\tilde{v}_k = \begin{bmatrix} \psi_1(\tilde{u}_k) \\ \vdots \\ \psi_r(\tilde{u}_k) \end{bmatrix}. \quad (12)$$

**Proof.** Define the LTI system

$$x_{k+1} = Ax + Bu, \quad y_k = Cx + Du,$$

with input $v$ and output $y$, with $A^T = [a_1 \ldots a_r]$, $B = Ba^T$, $D = Da^T$, and $A, B, C, D$ from (11).

Clearly, a sequence $\{u_k, \tilde{y}_k\}_{k=0}^{L-1}$ is a trajectory of (11) if and only if $\{\tilde{v}_k, \tilde{y}_k\}_{k=0}^{L-1}$ is a trajectory of (14). Further, using that $v$ is persistently exciting, it follows from Theorem 3 that $\{\tilde{v}_k, \tilde{y}_k\}_{k=0}^{L-1}$ is a trajectory of (14) if and only if there exists $\alpha \in \mathbb{R}^{N-L+1}$ such that (13) holds, which was to be shown.

For the application of Proposition 5 the basis functions $\psi_i$ of $\psi$ have to be known. In practice, it may be adequate to simply choose sufficiently many basis functions, thereby approximating the true ones. Note however that the number of basis functions $r$ enters into the persistence of excitation assumption on the auxiliary input $v$. To be more precise, for $v$ to be persistently exciting of order $L + n$, it is necessary that $N \geq (r + 1)(L + n) - 1$ and hence, one cannot choose arbitrarily many basis functions.

**B. Wiener systems**

Wiener systems are in a sense the dual of Hammerstein systems. A Wiener system consists of an LTI system followed by a static nonlinearity, i.e., it is of the form

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 = \bar{x}$$

$$y_k = \phi(Cx_k + Du_k), \quad (15)$$

with a nonlinear function $\phi : \mathbb{R}^q \to \mathbb{R}^r$. Similar to Section IV-A we consider in the following only the case $\tilde{p} = 1$. To apply the same reasoning as for Hammerstein systems, we assume that $\phi$ is invertible and that its inverse admits a basis function decomposition as $\phi^{-1}(y) = \sum_{i=1}^{q} b_i \hat{\psi}_i(y)$ with $q$ known basis functions $\hat{\psi}_i$. We define an auxiliary output trajectory $\{z_k\}_{k=0}^{N-1}$ with components

$$z_k = \begin{bmatrix} \hat{\phi}_1(y_k) \\ \vdots \\ \hat{\phi}_q(y_k) \end{bmatrix},$$

which will serve as the output of an equivalent LTI system. The following result is the correspondence of Proposition 5 for the Wiener system case.

**Proposition 6.** Suppose $\{u_k, y_k\}_{k=0}^{N-1}$ is a trajectory of a Wiener system (15), where $u$ is persistently exciting of order $L + n$. Then, $\{\bar{u}_k, \bar{y}_k\}_{k=0}^{L-1}$ is a trajectory of (15) if and only if there exists $\alpha \in \mathbb{R}^{N-L+1}$ such that

$$\begin{bmatrix} H_L(u) \\ H_L(z) \end{bmatrix} \alpha = \begin{bmatrix} \bar{u}_{0:L-1} \\ \bar{z}_{0:L-1} \end{bmatrix}, \quad (16)$$

where $\{\bar{z}_k\}_{k=0}^{L-1}$ is the trajectory with components

$$\bar{z}_k = \begin{bmatrix} \hat{\phi}_1(\bar{y}_k) \\ \vdots \\ \hat{\phi}_q(\bar{y}_k) \end{bmatrix}.$$

**Proof.** The proof proceeds in the same way as the proof of Proposition 5 and is therefore omitted.

Note that, contrary to the Hammerstein case, the above result does not pose any limit on the maximal number of basis functions we may choose. However, they represent the inverse of $\phi$ and are thus more difficult to select in a practical scenario.

**C. Nonlinear systems in lifted spaces**

From the perspective of Koopman operator theory, there has recently been a renewed interest in viewing nonlinear systems as linear systems in lifted state coordinates [15]. In a similar fashion, Propositions 5 and 6 can be directly combined to provide trajectory-based representations of nonlinear systems, which are linear in suitable higher-dimensional input-output coordinates. Even if such coordinates do not exist or are not known, one may in practice simply choose sufficiently many basis functions to approximate the unknown nonlinear system. In this way, the result of Theorem 3 can be extended to relatively general system classes [16]. It remains an open issue for future research to analyze the influence of not exactly known basis functions in practice.

**V. APPLICATIONS**

In this section, we illustrate how the proposed trajectory-based framework may serve as a foundation for the development of mathematically sound data-based system analysis and control methods. We provide a concise, but certainly not exhaustive, list of potential application areas, where
Theorem 3 leads to simple and elegant solutions: data-driven simulation, a framework for data-based MPC, and the data-driven inference of dissipativity properties.

A. Data-driven simulation

The data-driven simulation problem is concerned with the computation of an unknown system’s output resulting from the application of a given input, using no model but only a previously measured input-output trajectory. Its solution is described in the behavioral context in [17]. Loosely speaking, the idea is to fix $\hat{u}$ in (2) to first solve $\hat{u} = H_L(u)\alpha$ for $\alpha$, in order to then compute the new predicted output $\hat{y} = H_L(y)\alpha$.

To fix a unique such output, initial conditions have to be specified [17]. Since a state-space model is not available, we consider an initial input-output trajectory over a length of at least $n$, since this induces a unique initial state in some minimal realization. The following is the main result of [17].

**Proposition 7.** Suppose $\{u_k, y_k\}_{k=0}^{N-1}$ is a trajectory of a discrete-time LTI system $G$, where $u$ is persistently exciting of order $L + n$. Let $\{\hat{u}_k, \hat{y}_k\}_{k=0}^{L-1}$ be an arbitrary trajectory of $G$. If $\nu \geq n$, then there is a unique solution $\alpha \in \mathbb{R}^{N-L+1}$ to

$$
\begin{bmatrix}
H_L(u) \\
H_\nu(y_{0,N-L+\nu-1})
\end{bmatrix}
\alpha = \begin{bmatrix}
\hat{u} \\
\hat{y}_{[0,\nu-1]}
\end{bmatrix}.
$$

Further, it holds that $\hat{y} = H_L(y)\alpha$.

**Proof.** This follows directly from the corresponding result in the behavioral framework [17, Proposition 1].

The main idea of Proposition 7 is that, as described above, the new input $\{\hat{u}_k\}_{k=0}^{L-1}$ together with the initial trajectory $\{\hat{u}_k, \hat{y}_k\}_{k=0}^{\nu-1}$ fixes a unique $\alpha$, which can then be used to predict the remaining elements of $\hat{y}$. This condition $\nu \geq n$ means that $\nu$ is an upper bound on the order $n$ of $G$ and implies that $\{\hat{u}_k, \hat{y}_k\}_{k=0}^{L-1}$ specifies a unique initial condition for the internal state.

**Algorithm 8. Data-driven simulation**

**Given:** Data $\{u_k, y_k\}_{k=0}^{N-1}$, a new input $\{\hat{u}_k\}_{k=0}^{L-1}$, initial conditions $\{\hat{u}_k, \hat{y}_k\}_{k=0}^{\nu-1}$.

1) Solve (17) for $\alpha$.

2) Compute the remaining elements of $\hat{y}$ as $\hat{y} = H_L(y)\alpha$.

The practical application of Proposition 7 is illustrated in Algorithm 8. Although the classical simulation problem is commonly approached using a model, it can be solved in the proposed trajectory-based framework using a single measured input-output trajectory. Several extensions of Proposition 7 have been suggested to account for noise [18], to simulate systems in closed loop [19], and to compute feedback controllers [17]. Note that the results of Section IV apply also to the data-driven simulation problem, i.e., they can be used to simulate nonlinear systems, which are linear in suitable known coordinates. An interesting open issue is the comparison of the nonlinear data-driven simulation approach to existing methods for system identification of Hammerstein-Wiener systems.

B. A framework for data-based model predictive control

In MPC, a model is used to predict future trajectories and to optimize over them. After optimization, the first predicted input is applied, the resulting new state is measured, and the whole procedure is repeated [20]. In this section, we show how MPC schemes can be naturally formulated in the trajectory-based framework using Theorem 3.

If a persistently exciting trajectory $\{u_k, y_k\}_{k=0}^{N-1}$ of an unknown LTI system is available, it can be used to replace the dynamics in the standard MPC optimization problem via Theorem 3. This leads us to the following optimal control problem, which has to be solved online.

$$
\begin{aligned}
\minimize_{u \in \mathbb{R}^{mL}, y \in \mathbb{R}^{pL}} & \sum_{k=0}^{L-1} \begin{bmatrix} \hat{u}_k^T & Q \end{bmatrix} \begin{bmatrix} S & R \end{bmatrix} \begin{bmatrix} \hat{u}_k \\ \hat{y}_k \end{bmatrix} \\
\text{s.t.} & \begin{bmatrix} H_L(u) \\ H_\nu(y) \end{bmatrix} \alpha = \begin{bmatrix} \hat{u} \\ \hat{y}_{[0,n-1]} \end{bmatrix} = \begin{bmatrix} v \\ z \end{bmatrix}, \\
& \hat{u}_k \in \mathbb{U}, \hat{y}_k \in \mathbb{Y}, \ k = 0, \ldots, L-1.
\end{aligned}
$$

As in Section V-A, we have initial input and output trajectories $v \in \mathbb{R}^{mn}$ and $z \in \mathbb{R}^{pl}$, respectively, which would in practice correspond to measurements from past iterations. Note the simplicity of the above MPC scheme, which does not require any identification or learning step, but is directly based on measured data. Moreover, for convex quadratic or affine constraints $U$, $Y$, and convex cost, the above problem is a convex quadratic program, which can be solved efficiently.

The above MPC scheme was proposed in [21], [22] using the behavioral results from [8], [17]. Although, in [21], the formal equivalence to a standard model-based MPC scheme was proven, stability guarantees on the closed loop could not be given. Nevertheless, the authors of [21] successfully applied the scheme to a nonlinear stochastic quadcopter system, using suitable regularization techniques to account for the noise and the nonlinearity. Thus, the approach seems very promising, and addressing the lack of guarantees provides an interesting challenge for future research.

C. Data-driven dissipativity

It is well-known that knowledge of the $L_2$-gain, passivity properties or, more generally, dissipation inequalities can give useful insights into a system and can even be exploited for controller design [23], [24]. Therefore, there have been various approaches to learn dissipativity properties directly from data, without prior system identification [25], [26], [27]. As we briefly sketch in the following, Theorem 3 can be employed to identify dissipativity properties from a single measured trajectory in a simple and efficient way. This idea was first proposed in the behavioral context in [28], where it led to an indefinite quadratic matrix inequality. An efficient reformulation was presented in [29] and will be sketched in the following. Since we have only finite data, we consider the relaxed version of finite-horizon- or $L$-dissipativity.
Definition 9. We say that an LTI system $G$ is $L$-dissipative w.r.t. the supply rate $\Pi$ if
\[
\sum_{k=0}^{r} \begin{bmatrix} \bar{u}_k \\ \bar{y}_k \end{bmatrix}^\top \Pi \begin{bmatrix} \bar{u}_k \\ \bar{y}_k \end{bmatrix} \geq 0, \quad \forall r \in [0, L-1] \quad (18)
\]
for all trajectories $\{ \bar{u}_k, \bar{y}_k \}_{k=0}^{L-1}$ of $G$ with initial condition $\bar{x}_0 = 0$.

It is not difficult to show that (13) only needs to be verified for $r = L - 1$ and is then equivalent to
\[
\begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}^\top \bar{\Pi} \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix} \geq 0
\]
for a suitably stacked matrix $\bar{\Pi}$. Using Theorem 5 $\bar{u}$ and $\bar{y}$ can be replaced by $H_L(u)\alpha$ and $H_L(y)\alpha$, respectively. Then, after an application of Finsler’s Lemma (cf. [30]) to eliminate the initial condition, one arrives at an equivalent positive semidefiniteness condition on a single, fixed matrix, which depends on the measured data $\{ u_k, y_k \}_{k=0}^{N-1}$. Hence, dissipativity in the sense of Definition 9 can be directly verified from data, without identifying a model. While this approach works very well in practice (cf. the examples in [29]), there are many open questions regarding the implications on standard (infinite-horizon) dissipativity as well as the rigorous treatment of noise.

VI. CONCLUSION

In this paper, we proposed a novel framework for rigorous data-based system analysis and control. All trajectories of an unknown system can be constructed from a single measured trajectory and thus, this trajectory captures all the required information needed for analysis and controller design, without explicit identification of a model. After translating this result from the behavioral context to the classical control framework, we extended it to certain classes of nonlinear systems and we provided links to several promising applications, including data-driven MPC as well as the verification of dissipativity properties from data. Future research should include further exploration of the applications sketched in Section V as well as a systematic treatment of noisy measurements.

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