STRONG ASYMPTOTICS FOR CHRISTOFFEL FUNCTIONS OF PLANAR MEASURES

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Abstract. We prove a version of strong asymptotics of Christoffel functions with varying weights for a general class of sets $E$ and measures $\mu$ in the complex plane $\mathbb{C}$. This class includes all regular measures $\mu$ in the sense of Stahl-Totik [14] on regular compact sets $E$ in $\mathbb{C}$ and even allows varying weights. Our main theorems cover some known results for $E \subset \mathbb{R}$, the real line; in particular, we recover information in the case of $E = \mathbb{R}$ with Lebesgue measure $dx$ and weight $w(x) = \exp(-Q(x))$ where $Q(x)$ is a nonnegative, even degree polynomial having positive leading coefficient.

1. Introduction.

Let $w(x)$ be a positive uppersemicontinuous (usc) function on the interval $[-1, 1]$. Let $\{q_j\}_{j=1,2,...}$ be a sequence of orthonormal polynomials with respect to the measure $d\nu(x) = w(x)dx$ where $\text{deg} q_j = j - 1$. Form the sequence of Christoffel functions $K_n(z) := \sum_{j=1}^{n+1} |q_j(z)|^2$. It is straightforward to see that

$$\frac{1}{2n} \log K_n(z) \to \log |z + \sqrt{z^2 - 1}|$$

uniformly on $\mathbb{C}$ as $n \to \infty$ and hence

$$(1.1) \quad d\mu_n(x) := [\Delta(\frac{1}{2n} \log K_n(x))]dx \to \frac{1}{\pi \sqrt{1 - x^2}}dx$$

weak-* where $\Delta$ is the Laplacian. More generally, let $E$ be a compact subset of $\mathbb{C}$, $w$ an admissible weight function on $E$, and $\mu$ a positive Borel measure on $E$ such that the triple $(E, w, \mu)$ satisfies a weighted Bernstein-Markov inequality (see [2,3]). If we take, for each $n = 1, 2, ..., a$ set of orthonormal polynomials $q_1^{(a)}, ..., q_{n+1}^{(a)}$ with respect to the varying measures $w(z)^{2n}d\mu(z)$ where $\text{deg} q_j^{(a)} = j - 1$ and form the sequence of Christoffel functions $K_n(z) :=$...
\[
\sum_{j=1}^{n+1} |q_j^{(n)}(z)|^2, \text{ then the functions } \frac{1}{2n} \log K_n(z) \text{ converge uniformly on } \mathbb{C} \text{ and}
\]

\[
d\mu_n := \Delta \left( \frac{1}{2n} \log K_n \right) \to d\mu_{eq}^w
\]

weak-* where \( \mu_{eq}^w \) is the potential-theoretic weighted equilibrium measure (cf., [4], Lemma 2.3).

A deeper result than (1.1) for the interval \([-1, 1]\) is a stronger asymptotic for \(K_n\):

\[
\lim_{n \to \infty} K_n(x) = \pi \sqrt{1 - x^2}/w(x)
\]
a.e. on \((-1, 1)\). In [16], Totik generalized (1.3) to certain “regular” measures \(d\nu(x)\) on the real line having compact support \(E\) where \(\mathbb{C} \setminus E\) is regular for the Dirichlet problem (he later observed that the regularity assumption on \(E\) was unnecessary; cf., [17], section 8). Here, the arcsine measure \(\frac{1}{\pi \sqrt{1 - x^2}} \, dx\) is replaced by the (unweighted) equilibrium measure \(d\mu_{eq}(x)\) for \(E\) and the right-hand-side of (1.3) is replaced by the appropriate Radon-Nikodym derivative. An earlier result of Totik (in [13]) gives an analogous generalization of (1.2) in the special case where \(E\) is a finite union of intervals in the real line, \(\mu\) is normalized Lebesgue measure \(dx\) on \(E\), and \(w\) is a positive continuous function on \(E\). That is, forming the sequence of Christoffel functions \(K_n(x) := \sum_{j=1}^{n+1} |q_j^{(n)}(x)|^2\), in this case, the asymptotic relation

\[
\frac{1}{n+1} K_n(x) w(x)^2 dx \to d\mu_{eq}^w(x) \text{ weak-*}
\]

holds. We will refer to any result similar to (1.4) as strong asymptotics of Christoffel functions with varying weights (this is not standard terminology).

Motivated by the study of statistical quantities related to distributions of eigenvalues of random matrices, Johansson [8] and others (cf., Pastur [11] and related articles) studied strong asymptotics of Christoffel functions with varying weights in the situation where \(E\) is the whole real line \(\mathbb{R}\) and \(w(x) = \exp(-Q(x))\) where \(Q(x)\) is an even degree polynomial with positive leading coefficient and \(Q(x) \geq 0\) on \(\mathbb{R}\); e.g., \(Q(x) = x^{2m}, \quad m = 1, 2, \ldots\). We recommend Chapter 6 of Deift’s elegant book [6]; for the reader’s convenience we include a statement (Corollary 2.1) and proof of a result in this case. Recently there has been a flurry of activity in universality limits involving orthogonal polynomials on subsets of the real line – roughly speaking, the same (strong) asymptotics occur for a wide class of measures supported on the same set \(E \subset \mathbb{R}\). Very precise limiting behavior involving the reproducing kernels

\[
K_n(\mu; x, y) := \sum_{j=1}^{n+1} q_j(x) q_j(y)
\]
where \( \{q_j\}_{j=1,2,...} \) is a sequence of orthonormal polynomials (with positive leading coefficients) with respect to a measure \( d\mu(x) \) on \( E \subset \mathbb{R} \) have been studied for certain \( E \) and \( \mu \) by Lubinsky, Simon, Totik and others (cf., \[10\], \[17\] and \[13\]). We utilize very different techniques, motivated from several complex variables, to prove a general version of (1.4) for nonpolar compact sets \( E \) in the complex plane \( \mathbb{C} \) and triples \( (E, w, \mu) \) satisfying a weighted Bernstein-Markov inequality (Theorem 2.2). Our main tool is the correspondence between weighted potential theory in \( \mathbb{C} \) and pluripotential theory (the study of plurisubharmonic functions) in \( \mathbb{C}^2 \) as developed in \[3\]. We also make use of some generalizations of certain statistical quantities described in \[6\]. A crucial difference between our approach and that of Johansson is that we use potential theoretic consequences of measures satisfying a Bernstein-Markov inequality to prove the existence of “free energy” (Theorem 2.1). Our “large deviation” estimate (Proposition 4.2) is then a straightforward consequence of this result and the existence of the limit in (2.1); this latter item is a standard fact in potential theory. An alternate approach to studying strong asymptotics of Christoffel functions with varying weights in \( \mathbb{C}^N \), \( N > 1 \) has been developed by Berman (cf., \[1\], \[2\]).

To keep the article self-contained, we include some brief background material on weighted potential theory in \( \mathbb{C} \) and pluripotential theory in \( \mathbb{C}^2 \). We refer the reader to \[3\] for details of stated results. For more on general univariate weighted potential theory, we refer the reader to \[12\]; for more on general pluripotential theory, we refer the reader to \[9\].

In the next section, we give the relevant definitions and state our main results, Theorems 2.1 and 2.2. We prove Theorem 2.1 in section 3 and Theorem 2.2 in section 4. Corollary 2.1 is proved in section 5.

We would like to thank Vilmos Totik for his valuable comments; in particular, for kindly pointing out several recent papers on universality.

2. Main results.

In this paper, we let \( E \subset \mathbb{C} \) be a closed set with an admissible weight function \( w \): \( w \) is a nonnegative, uppersemicontinuous function on \( E \) with \( \{z \in E : w(z) > 0\} \) nonpolar (in particular, \( E \) is nonpolar); if \( E \) is unbounded we require also the growth condition \( |z| w(z) \to 0 \) as \( |z| \to \infty \). It turns out that \( S_w \) is always bounded (Theorem 1.3, p. 27 of \[12\]).

The limit

\[
(2.1) \quad \lim_{n \to \infty} \left[ \max_{\lambda_i \in E} |VDM(\lambda_0, \ldots, \lambda_n)| w(\lambda_0)^n \cdots w(\lambda_n)^n \right]^{2/n^2} := \delta^w(E)
\]

exists and is called the weighted transfinite diameter of \( E \) (with respect to \( w \)). Here \( VDM(\zeta_1, \ldots, \zeta_n) = \det[\zeta_i^{j-1}]_{i,j=1,...,n} = \prod_{j<k} (\zeta_j - \zeta_k) \) is the
classical Vandermonde determinant. Points $\lambda_0, \ldots, \lambda_n \in E$ for which

$$|VDM(\lambda_0, \ldots, \lambda_n)|w(\lambda_0)^n \cdots w(\lambda_n)^n = |\det \begin{bmatrix} 1 & \lambda_0 & \cdots & \lambda_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^n \end{bmatrix}|w(\lambda_0)^n \cdots w(\lambda_n)^n$$

is maximal are called weighted Fekete points of order $n$. The quantity $\delta^w(E)$ in (2.1) comes from a discrete version of a weighted energy minimization problem: for a probability measure $\tau$ on $E$, consider the weighted energy

$$I^w(\tau) := \int_E \int_E \log \frac{1}{|z - t|w(z)w(t)} d\tau(t) d\tau(z).$$

Then

$$\inf_\tau I^w(\tau) = - \log \delta^w(E)$$

where the infimum is taken over all probability measures $\tau$ on $E$. Moreover, the infimum is attained by a unique measure $\mu^w_{eq}$. If $w \equiv 1$ we are in the classical (unweighted) case and we simply write $\mu_{eq}$. We remark that there exists $\eta > 0$ such that the support $S_w$ of $\mu^w_{eq}$ is contained in $\{z \in E : w(z) \geq \eta\}$ (Remark 1.4, p. 27 of [12]).

A weighted polynomial on $E$ is a function of the form $w(z)^n p_n(z)$ where $p_n$ is a holomorphic polynomial of degree at most $n$. Let $\mu$ be a measure with support in $E$ such that $(E, w, \mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials (referred to as a weighted B-M inequality in [3]): given $\epsilon > 0$, there exists a constant $M = M(\epsilon)$ such that for all weighted polynomials $w^n p_n$

$$||w^n p_n||_E \leq M(1 + \epsilon)^n||w^n p_n||_{L^2(\mu)}.$$

In this setting, we will restrict our attention to compact sets $E$.

For a compact set $K \subset \mathbb{C}^N$ and a measure $\nu$ on $K$, we say that the pair $(K, \nu)$ satisfies the Bernstein-Markov inequality for holomorphic polynomials in $\mathbb{C}^N$ if, given $\epsilon > 0$, there exists a constant $\tilde{M} = \tilde{M}(\epsilon)$ such that for all such polynomials $Q_n$

$$||Q_n||_K \leq \tilde{M}(1 + \epsilon)^n||Q_n||_{L^2(\nu)}.$$

The terminology “Bernstein-Markov” in this context is standard in several complex variables. Our main results are as follows.

**Theorem 2.1.** Let $E$ be compact and let $(E, w, \mu)$ satisfy a Bernstein-Markov inequality for weighted polynomials. Then

$$\lim_{n \to \infty} Z_n^{1/n^2} = \delta^w(E)$$
where
\[
Z_n = Z_n(E, w, \mu) := 
\int_{E^{n+1}} |VDM(\lambda_0, \ldots, \lambda_n)|^2 w(\lambda_0)^{2n} \cdots w(\lambda_n)^{2n} d\mu(\lambda_0) \cdots d\mu(\lambda_n).
\]

\textbf{Theorem 2.2.} Let \( E \) be compact and let \( (E, w, \mu) \) satisfy a Bernstein-Markov inequality for weighted polynomials. Define the probability measures
\[
d\mu_n(z) := \frac{1}{Z_n} R_1^{(n)}(z) w(z)^{2n} d\mu(z)
\]
where
\[
R_1^{(n)}(z) := 
\int_{E^n} |VDM(\lambda_0, \ldots, \lambda_{n-1}, z)|^2 w(\lambda_0)^{2n} \cdots w(\lambda_{n-1})^{2n} d\mu(\lambda_0) \cdots d\mu(\lambda_{n-1}).
\]
Then \( d\mu_n(z) \to d\mu_w^{eq}(z) \) weak-*. 

Utilizing similar arguments as in the proof of Theorem 2.1, we obtain another proof (cf., [6], Chapter 6) of the following.

\textbf{Corollary 2.1.} Let \( w(x) = \exp(-Q(x)) \) where \( Q(x) \) is a nonnegative, even degree polynomial on the real line \( \mathbb{R} \) having positive leading coefficient. Then
\[
\lim_{n \to \infty} Z_n^{1/n^2} = \delta^w(\mathbb{R})
\]
where
\[
Z_n = Z_n(\mathbb{R}, w, dx) := 
\int_{\mathbb{R}^{n+1}} |VDM(\lambda_0, \ldots, \lambda_n)|^2 w(\lambda_0)^{2n} \cdots w(\lambda_n)^{2n} d\lambda_0 \cdots d\lambda_n.
\]

\textbf{Remark 2.1.} We observe that with the notation in (2.6) and (2.5)
\[
R_1^{(n)}(z) = \frac{1}{Z_n} \sum_{j=1}^{n+1} |q_j^{(n)}(z)|^2
\]
where \( q_1^{(n)}, \ldots, q_{n+1}^{(n)} \) are orthonormal polynomials of degree 0, 1, \ldots, \( n \) with respect to the measure \( w(z)^{2n} d\mu(z) \); hence Theorem 2.2 generalizes (1.4). To verify (2.8), if we apply Gram-Schmidt in \( L^2(w^{2n} \mu) \) to the monomials \( 1, z, \ldots, z^n \) to obtain orthogonal polynomials \( p_1^{(n)}(z) \equiv 1, \ldots, p_{n+1}^{(n)}(z) \), we have, upon using elementary row operations on \( VDM(\lambda_0, \ldots, \lambda_{n-1}, z) \) and expanding the integrand in (2.6),
\[
R_1^{(n)}(z) = 
\sum_{I,S} \sigma(I) \cdot \sigma(S) \int_{E^n} p_{I_0}^{(n)}(\lambda_0) \cdots p_{I_n}^{(n)}(z) p_{S_0}^{(n)}(\lambda_0) \cdots p_{S_n}^{(n)}(z) w(\lambda_0)^{2n} \cdots
\]
\[ \ldots w(\lambda_{n-1})^{2n} d\mu(\lambda_0) \cdots d\mu(\lambda_{n-1}) \]

\[ = \sum_{I,S} \sigma(I) \cdot \sigma(S) \left[ \int_{E} p_{i_0}^{(n)}(\lambda_0)p_{s_0}^{(n)}(\lambda_0) w(\lambda_0)^{2n} d\mu(\lambda_0) \cdots \right. \]

\[ \ldots \int_{E} p_{i_{n-1}}^{(n)}(\lambda_{n-1}) p_{s_{n-1}}^{(n)}(\lambda_{n-1}) w(\lambda_{n-1})^{2n} d\mu(\lambda_{n-1}) \left] p_{i_n}^{(n)}(z) p_{s_n}^{(n)}(z) \right. \]

\[ (2.9) \]

\[ = n! \sum_{j=1}^{n+1} \left( \prod_{i \neq j} \|p_i^{(n)}\|_{L^2(w^{2n}\mu)} \right) \int_{E} |p_j^{(n)}(z)|^2 w(z)^{2n} d\mu(z) \]

\[ (2.10) \]

Here \( I = (i_0, \ldots, i_n) \) and \( S = (s_0, \ldots, s_n) \) are permutations of \((0, 1, \ldots, n)\) and \( \sigma(I) \) is the sign of \( I \) (+1 if \( I \) is even; −1 if \( I \) is odd). Then

\[ Z_n = \int_{E} R_1^{(n)}(z) w(z)^{2n} d\mu(z) \]

\[ = n! \sum_{j=1}^{n+1} \left( \prod_{i \neq j} \|p_i^{(n)}\|_{L^2(w^{2n}\mu)} \right) \int_{E} |p_j^{(n)}(z)|^2 w(z)^{2n} d\mu(z) \]

Dividing (2.9) by (2.10) yields (2.8) since \( |q_j^{(n)}(z)| = |p_j^{(n)}(z)|/\|p_j^{(n)}\|_{L^2(w^{2n}\mu)} \).

In particular, then, if we take \( E \) to be a finite union of intervals on the real line, \( d\mu(x) = dx = \text{Lebesgue measure} \) on \( E \), and \( w \) positive and continuous, then \((E, w, dx)\) satisfies a Bernstein-Markov inequality for weighted polynomials (see Remark 2.2 below). Theorem 2.2 gives

\[ \left[ \frac{1}{n+1} \sum_{j=1}^{n+1} |q_j^{(n)}(x)|^2 \right] \cdot w(x)^{2n} d\mu(x) \rightarrow \rho(x) d\mu(x) \]

weak-* where \( \rho(x)dx = d\mu_{\omega w}(x) \). For such \( E \) and \( w \), under the additional hypotheses that \( \rho \) be continuous on an interval \( J \) in \( \text{Int}E \) – this occurs if \( J \subset \text{Int}(S_w) \) and \( w \) is \( C^{1+\epsilon} \) on a neighborhood of \( J \) – Totik [13] proved that

\[ \left[ \frac{1}{n+1} \sum_{j=1}^{n+1} |q_j^{(n)}(x)|^2 \right] \cdot w(x)^{2n} \rightarrow \rho(x) \]

uniformly on \( J \).

For more general subsets \( E \subset \mathbb{R} \) and measures \( \mu \) such that \((E, \mu)\) satisfies a Bernstein-Markov inequality, if \( w = 1 \), Theorem 2.2 implies

\[ \left[ \frac{1}{n+1} \sum_{j=1}^{n+1} |q_j(x)|^2 \right] d\mu(x) \rightarrow d\mu_{\omega_0}(x). \]
In [16], Totik proves this result for a regular compact subset $E$ of $\mathbb{R}$ ($C \setminus E$ is regular for the Dirichlet problem) with a “regular” measure $\mu$. According to Theorem 3.2.3 of [14], for a regular compact set $E$, regularity of $\mu$ is equivalent to $(E, \mu)$ satisfying a Bernstein-Markov inequality.

**Remark 2.2.** If $E$ is a regular compact set in $\mathbb{C}$, then $(E, \mu_{eq})$ satisfies the Bernstein-Markov inequality. More generally, for such $E$, if $w$ is an admissible continuous weight function, then the triple $(E, w, \mu_{w_{eq}})$ satisfies a Bernstein-Markov inequality for weighted polynomials (cf., Corollary 3.1 [3]). We mention that Theorem 4.2.3 of [14] provides a sufficient condition for the pair $(E, \mu)$ to satisfy a Bernstein-Markov inequality when $E = \text{supp} \mu \subset \mathbb{C}$ is a regular compact set. For instance, a finite union $E$ of intervals is regular, and $(E, dx)$ satisfies this sufficient condition where $dx$ is Lebesgue measure on $E$. If $w$ is a positive, continuous weight on $E$, appealing to Theorem 3.2.3 (vi) of [14] with $g_n = w^{2n}$, it follows that $(E, w, dx)$ satisfies a Bernstein-Markov inequality for weighted polynomials.

**Remark 2.3.** In the setting of [6], where $E = \mathbb{R}$ and $w(x) = \exp(-Q(x))$ with $Q(x)$ an even degree polynomial having positive leading coefficient and $Q(x) \geq 0$ on $\mathbb{R}$, Corollary 2.1 is referred to as the existence of the free energy (Corollary 6.90 in [6]). Note that $w$ is an admissible weight.

3. **Proof of Theorem 2.1.**

We let $E \subset \mathbb{C}$ be a nonpolar compact set with an admissible weight function $w$. Following [3], we define the circled set

$$F = F(E, w) := \{(t, z) = (t, \lambda t) : \lambda \in E, |t| = w(\lambda)\}.$$ 

We first relate weighted univariate Vandermonde determinants for $E$ with homogeneous bivariate Vandermonde determinants for $F$. To this end, for each positive integer $n$, choose $n+1$ points $\{(t_i, z_i)\}_{i=0,...,n}$ in $F$ and form the $n$–homogeneous Vandermonde determinant

$$VDMH_n((t_0, z_0), ..., (t_n, z_n)) := \det \left[ t_i^{n-j} z_j \right]_{i,j=0,...,n}.$$ 

Note that we evaluate the $n+1$ homogeneous monomials

$$t^n, t^{n-1}z, ..., tz^{n-1}, z^n$$

at the $n+1$ points $\{(t_i, z_i)\}_{i=0,...,n}$. Factoring $t_i^n$ out of the $i$–th row, we obtain

$$VDMH_n((t_0, z_0), ..., (t_n, z_n)) = t_0^n \cdots t_n^n \cdot VDM(\lambda_0, ..., \lambda_n);$$
i.e.,

\[
\begin{bmatrix}
  t_0^n & t_0^{n-1}z_0 & \cdots & z_0^n \\
  \vdots & \vdots & \ddots & \vdots \\
  t_n^n & t_n^{n-1}z_n & \cdots & z_n^n \\
\end{bmatrix}
= \begin{bmatrix}
  \lambda_0 & \cdots & \lambda_0^n \\
  \vdots & \ddots & \vdots \\
  \lambda_n & \cdots & \lambda_n^n \\
\end{bmatrix},
\]

where \( \lambda_j = z_j/t_j \) provided \( t_j \neq 0 \) and \( VDM(\lambda_0, \ldots, \lambda_n) = \prod_{j<k}(\lambda_j - \lambda_k) \) is a standard (univariate) Vandermonde determinant. By definition of \( F \), since \( (t_i, z_i) \in F \), we have \( |t_i| = w(\lambda_i) \) so that from (3.1)

\[ |VDMH_n((t_0, z_0), \ldots, (t_n, z_n))| = |VDM(\lambda_0, \ldots, \lambda_n)|w(\lambda_0)^n \cdots w(\lambda_n)^n. \]

Thus

\[
\max_{(t_i, z_i) \in F} |VDMH_n((t_0, z_0), \ldots, (t_n, z_n))| = \max_{\lambda_i \in E} |VDM(\lambda_0, \ldots, \lambda_n)|w(\lambda_0)^n \cdots w(\lambda_n)^n.
\]

Note that the maximum will occur when all \( |t_j| = w(\lambda_j) > 0 \). Now the limit

\[
\lim_{n \to \infty} \left[ \max_{\lambda_i \in E} |VDM(\lambda_0, \ldots, \lambda_n)|w(\lambda_0)^n \cdots w(\lambda_n)^n \right]^{1/n^2} =: D^H(F)
\]

exists (the homogeneous bivariate transfinite diameter \( D^H(F) \) of \( F \); cf., [7]); also, as previously mentioned, the limit

\[
\lim_{n \to \infty} \left[ \max_{\lambda_i \in E} |VDM(\lambda_0, \ldots, \lambda_n)|w(\lambda_0)^n \cdots w(\lambda_n)^n \right]^{2/n^2} =: \delta^w(E)
\]

exists (the weighted transfinite diameter of \( E \) with respect to \( w \)) and thus we have

(3.2) \[
\delta^w(E) = D^H(F)^2.
\]

Let \( \mu \) be a measure with support in \( E \) such that \( (E, w, \mu) \) satisfies a Bernstein-Markov inequality for weighted polynomials. Note that the integrand

\[ |VDM(\lambda_0, \ldots, \lambda_n)|^2w(\lambda_0)^{2n} \cdots w(\lambda_n)^{2n} \]

in the definition of \( Z_n \) in (2.5) thus has a maximal value on \( E^{n+1} \) whose \( 1/n^2 \) root tends to \( \delta^w(E) \). To show that the integrals themselves have the same property, we proceed as follows. On the set \( F \subset \mathbb{C}^2 \), there exists a measure \( \nu \) associated to \( \mu \) such that \( (F, \nu) \) satisfies the Bernstein-Markov property for holomorphic polynomials in \( \mathbb{C}^2 \); i.e., (2.4) holds. Indeed, take

\[
\nu := m_\lambda \otimes \mu, \ \lambda \in E
\]

where \( m_\lambda \) is normalized Lebesgue measure on the circle \( |t| = w(\lambda) \) in the complex \( t \)-plane given by

\[
C_\lambda := \{(t, t\lambda) \in \mathbb{C}^2 : t \in \mathbb{C} \}.
\]
That is, if $\phi$ is continuous on $F$,
\[ \int_F \phi(t, z) d\nu(t, z) = \int_E \left[ \int_{C_\lambda} \phi(t, t\lambda) dm_\lambda(t) \right] d\mu(\lambda). \]

Equivalently, if $\pi: \mathbb{C}^2 \to \mathbb{C}$ via $\pi(t, z) = z/t := \lambda$, then $\pi_*(\nu) = \mu$. Moreover, if $p_1(t, z)$ and $p_2(t, z)$ are two homogeneous polynomials in $\mathbb{C}^2$ of degree $n$, say, and we write
\[ p_j(t, z) = p_j(t, \lambda t) = t^n p_j(1, \lambda) =: t^n G_j(\lambda), \quad j = 1, 2 \]
for univariate $G_j$, then it is straightforward to see that
\[ (3.3) \quad \int_F p_1(t, z) p_2(t, z) d\nu(t, z) = \int_E G_1(\lambda) G_2(\lambda) w(\lambda) 2^n d\mu(\lambda) \]
(cf., [3], Lemma 3.1 and its proof). Note that if $p(t, z) = t^i z^{n-i}$ for $i = 0, ..., n$, then
\[ p(t, z) = t^n \left( z/t \right)^{n-i} = t^n G(\lambda) \]
where $G(\lambda) = \lambda^{n-i}$.

**Proposition 3.1.** Let
\[ \tilde{Z}_n := \int_{F^{n+1}} |VDMH_n((t_0, z_0), ..., (t_n, z_n))|^2 d\nu(t_0, z_0) \cdots d\nu(t_n, z_n). \]

Then $\tilde{Z}_n = Z_n$.

**Proof.** Using the notation from the proof of (2.3), expanding the homogeneous Vandermonde determinant in $\tilde{Z}_n$ gives
\[ \tilde{Z}_n = \sum_{I, S} \sigma(I) \cdot \sigma(S) \left[ \int_F t_0^{i_0} s_0^{n-i_0} z_0^{n-s_0} d\nu(t_0, z_0) \cdots \right. \]
\[ \left. \cdots \int_F t_n^{i_n} s_n^{n-i_n} z_n^{n-s_n} d\nu(t_n, z_n) \right]. \]

Expanding the ordinary Vandermonde determinant in $Z_n$ gives
\[ Z_n = \sum_{I, S} \sigma(I) \cdot \sigma(S) \left[ \int_E \lambda_0^{n-i_0} \lambda_0^{n-s_0} w(\lambda_0) 2^n d\mu(\lambda_0) \cdots \right. \]
\[ \left. \cdots \int_E \lambda_n^{n-i_n} \lambda_n^{n-s_n} w(\lambda_n) 2^n d\mu(\lambda_n) \right]. \]

Since $|t_j| = w(\lambda_j)$, using (3.3) completes the proof. \qed

We need the following result, Theorem 5.9 of [7], which is the homogeneous analogue of Theorem 3.3 in [5].
Proposition 3.2. Let $F \subset \mathbb{C}^2$ be a circled set and let $\nu$ be a measure with support in $F$ such that $(F, \nu)$ satisfies the Bernstein-Markov property. Then

$$D^H(F) = \lim_{n \to \infty} \frac{G_n^{1/2n^2}}{n^2},$$

where

$$G_n := \det \left[ \int_F t^{n-j} z^i \overline{z^j} d\nu(t, z) \right]_{i,j=0,\ldots,n}$$

is the $n$-th homogeneous Gram determinant associated with $(F, \nu)$.

The last step is to work in $\mathbb{C}^2$ with the $\tilde{Z}_n$ integrals and verify the following.

Proposition 3.3. We have

$$\lim_{n \to \infty} \frac{\tilde{Z}_n^{1/2n^2}}{n^2} = D^H(F).$$

Proof. Fix $n$ and consider the monomials

$$t^n, t^{n-1}z, \ldots, tz^{n-1}, z^n$$

utilized in $VDMH_n((t_0, z_0), \ldots, (t_n, z_n))$ and in the computation of the Gram determinant $G_n$ associated to $(F, \nu)$. Use Gram-Schmidt in $L^2(\nu)$ to obtain orthogonal polynomials

$$p_0(t, z) = t^n, \quad p_1(t, z) = t^{n-1}z + \cdots, \quad p_n(t, z) = z^n + \cdots.$$ 

Then

$$VDMH_n((t_0, z_0), \ldots, (t_n, z_n)) = \det [p_i(t_j, z_j)]_{i,j=0,\ldots,n}.$$ 

By orthogonality, as in the calculation of $Z_n$ in proving (2.10) in the introduction, we obtain

$$Z_n = (n+1)! \|p_0\|^2_{L^2(\nu)} \cdots \|p_n\|^2_{L^2(\nu)}.$$ 

On the other hand, using the orthogonal polynomials diagonalizes the $n$-th Gram matrix while preserving its determinant; hence

$$G_n = \|p_0\|^2_{L^2(\nu)} \cdots \|p_n\|^2_{L^2(\nu)}.$$ 

From Proposition 3.2, we have

$$\lim_{n \to \infty} \left( \|p_0\|^2_{L^2(\nu)} \cdots \|p_n\|^2_{L^2(\nu)} \right)^{1/2n^2} = D^H(F)$$

and the result follows. \qed

Combining Propositions 3.1 and 3.3 with equation (3.2) completes the proof of Theorem 2.1. \qed
Remark 3.1. A version of Theorem 2.1 is valid in $\mathbb{C}^N$ for $N > 1$. Here, in the definition of $Z_n$ in (2.5), we replace $VDM(\lambda_1,...,\lambda_n)$ by the generalized Vandermonde determinant

$$VDM(\lambda_1,...,\lambda_{hn}) := |\det[e_i(\lambda_j)]_{i,j=1,...,hn}|$$

where $e_1(z),...,e_i(z),...,e_{hn}(z)$ is a listing of the monomials in $\mathbb{C}^N$ of degree at most $n$. If $\ell_n = \sum_{i=1}^{hn} \deg e_i$, the result is then that

$$\lim_{n \to \infty} Z_n^{1/2\ell_n} = \delta^w(E)$$

where in (2.1) we use

$$\delta^w(E) := \lim_{n \to \infty} \left[ \max_{\lambda_i \in E} |VDM(\lambda_1,...,\lambda_{hn})| w(\lambda_1)^n \cdots w(\lambda_{hn})^n \right]^{1/\ell_n}.$$

Details of this and related results will be given in a future work.

4. Proof of Theorem 2.2.

As in the previous section, we let $E \subset \mathbb{C}$ be a nonpolar compact set with an admissible weight function $w$. Let $\mu$ be a measure with support in $E$ such that $(E,w,\mu)$ satisfies a Bernstein-Markov inequality for weighted polynomials. We will need the following fact, which is claimed in Remark 1.4 on p. 147 of [12]. This says that for any doubly indexed array of points $\{z_{k}^{(n_j)}\}_{k=1,\ldots,n_j}$ in $E$ which satisfies asymptotically the relation (2.1), the limiting measures

$$(4.1) \quad d\mu_{n_j} := \frac{1}{n_j} \sum_{k=1}^{n_j} \delta_{z_k^{(n_j)}}$$

have the same weak-* limit, the weighted equilibrium measure $d\mu^w_{eq}$.

Proposition 4.1. Let $E \subset \mathbb{C}$ be compact and let $w$ be an admissible weight on $E$. If, for a subsequence of positive integers $\{n_j\}$ with $n_j \uparrow \infty$, the points $z_1^{(n_j)},...,z_{n_j}^{(n_j)} \in E$ are chosen so that

$$\lim_{j \to \infty} \left[ |VDM(z_1^{(n_j)},...,z_{n_j}^{(n_j)})|^2 w(z_1^{(n_j)})^{2n_j} \cdots w(z_{n_j}^{(n_j)})^{2n_j} \right]^{1/n_j^2} = \delta^w(E),$$

then $d\mu_{n_j} \to d\mu^w_{eq}$ weak-* where $d\mu_{n_j}$ is defined in (4.1).

We recall that the support $S_w$ of $\mu^w_{eq}$ is contained in $E_\eta := \{z \in E : w(z) \geq \eta\}$ for some $\eta > 0$ and we mention that the proof on p. 146 of [12] for weighted Fekete points works verbatim under the assumption that the points $z_1^{(n_j)},...,z_{n_j}^{(n_j)}$ lie in $E_\tilde{\eta}$ for some $\tilde{\eta} > 0$. For the reader’s convenience, and since we do not make this assumption, we include a proof of Proposition 4.1.
Proof. Take a subsequence of the measures \{\mu_{n_j}\} which converges weak-* to a probability measure \sigma on \(E\). We use the same notation for the subsequence and the original sequence. We show that \(I^w(\sigma) = -\log \delta^w\); by uniqueness of the weighted energy minimizing measure (2.2) we will then have \(\sigma = \mu^w_{eq}\).

First of all, choose continuous admissible weight function \(w_m\) with \(w_m \downarrow w\) and \(w_m \geq \alpha_m > 0\) on \(E\) and for a real number \(M\) let

\[
h_{M,m}(z,t) := \min[M, \log \frac{1}{|z-t|w_m(z)w_m(t)}] \leq \log \frac{1}{|z-t|w_m(z)w_m(t)}
\]

and

\[
h_M(z,t) := \min[M, \log \frac{1}{|z-t|w(z)w(t)}] \leq \log \frac{1}{|z-t|w(z)w(t)}.
\]

Then \(h_{M,m} \leq h_M\). By the Stone-Weierstrass theorem, every continuous function on \(E \times E\) can be uniformly approximated by finite sums of the form \(\sum_j f_j(z)g_j(t)\) where \(f_j, g_j\) are continuous on \(E\); hence \(\mu_{n_j} \times \mu_{n_j} \rightarrow \sigma \times \sigma\) and we have

\[
I^w(\sigma) = \lim_{M \rightarrow \infty} \lim_{m \rightarrow \infty} \int_E \int_E h_{M,m}(z,t) d\sigma(z) d\sigma(t)
\]

\[
= \lim_{M \rightarrow \infty} \lim_{m \rightarrow \infty} \int_E \int_E h_{M,m}(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t)
\]

\[
\leq \lim_{M \rightarrow \infty} \limsup_{j \rightarrow \infty} \int_E \int_E h_M(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t)
\]

since \(h_{M,m} \leq h_M\). Now

\[
h_M(z_k^{(n_j)}, z_l^{(n_j)}) \leq \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}|w(z_k^{(n_j)})w(z_l^{(n_j)})}
\]

if \(k \neq l\) and hence

\[
\int_E \int_E h_M(z,t) d\mu_{n_j}(z) d\mu_{n_j}(t) \leq
\]

\[
\frac{1}{n_j} M + \left(\frac{1}{n_j^2 - n_j}\right) \left[\sum_{k \neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}|w(z_k^{(n_j)})w(z_l^{(n_j)})}\right].
\]

By assumption, given \(\epsilon > 0\),

\[
\left(\frac{1}{n_j^2 - n_j}\right) \left[\sum_{k \neq l} \log \frac{1}{|z_k^{(n_j)} - z_l^{(n_j)}|w(z_k^{(n_j)})w(z_l^{(n_j)})}\right] \leq -\log[\delta^w(E) - \epsilon]
\]

for \(j \geq j(\epsilon)\); in particular, \(w(z_k^{(n_j)}) > 0\) for such \(j\) and hence

\[
I^w(\sigma) \leq \lim_{M \rightarrow \infty} \limsup_{j \rightarrow \infty} \frac{1}{n_j} M - \log[\delta^w(E) - \epsilon] = -\log[\delta^w(E) - \epsilon]
\]

for all \(\epsilon > 0\); i.e., \(I^w(\sigma) = -\log \delta^w(E)\). \(\square\)
We also need a “large deviation” result, which follows easily from Theorem 2.1. Define a probability measure $P_n$ on $E_n^{n+1}$ via, for a Borel set $A \subset E_n^{n+1}$,

$$P_n(A) := \frac{1}{Z_n} \int_A \left| VDM(z_0, \ldots, z_n) \right|^2 w(z_0)^{2n} \cdots w(z_n)^{2n} d\mu(z_0) \cdots d\mu(z_n).$$

**Proposition 4.2.** Given $\eta > 0$, define

$$A_{n, \eta} := \{ (z_0, \ldots, z_n) \in E_n^{n+1} : |VDM(z_0, \ldots, z_n)|^2 w(z_0)^{2n} \cdots w(z_n)^{2n} \geq (\delta_w(E) - \eta)^n \}.$$

Then there exists $n^* = n^*(\eta)$ such that for all $n > n^*$,

$$P_n(E_n^{n+1} \setminus A_{n, \eta}) \leq (1 - \frac{\eta}{2\delta_w(E)})^n.$$

**Proof.** From Theorem 2.1, given $\epsilon > 0$,

$$Z_n \geq [\delta_w(E) - \epsilon]^n$$

for $n \geq n(\epsilon)$. Thus

$$P_n(E_n^{n+1} \setminus A_{n, \eta}) = \frac{1}{Z_n} \int_{E_n^{n+1} \setminus A_{n, \eta}} \left| VDM(z_0, \ldots, z_n) \right|^2 w(z_0)^{2n} \cdots w(z_n)^{2n} d\mu(z_0) \cdots d\mu(z_n)$$

$$\leq \frac{[\delta_w(E) - \eta]^n}{[\delta_w(E) - \epsilon]^n}$$

if $n \geq n(\epsilon)$. Choosing $\epsilon < \eta/2$ and $n^* = n(\epsilon)$ gives the result. \qed

To prove Theorem 2.2, we fix $\phi \in C(E)$. Recalling that

$$d\mu_n(z) := \frac{1}{Z_n} R_1^{(n)}(z) w(z)^{2n} d\mu(z),$$

for each $n$ we have

$$\int_E \phi(z) d\mu_n(z)$$

$$= \frac{1}{Z_n} \int_E \phi(z) \left( \int_{E_0} \left| VDM(z_0, \ldots, z_{n-1}, z) \right|^2 w(z_0)^{2n} \cdots w(z_{n-1})^{2n} \right. d\mu(z_0) \cdots d\mu(z_{n-1}) \left. \right) w(z)^{2n} d\mu(z)$$

$$= \frac{1}{Z_n} \int_{E_n^{n+1}} \phi(z_n) \left| VDM(z_0, \ldots, z_n) \right|^2 w(z_0)^{2n} \cdots w(z_n)^{2n} d\mu(z_0) \cdots d\mu(z_n)$$

$$= \frac{1}{Z_n} \int_{E_n^{n+1}} \sum_{j=0}^{n} \frac{\phi(z_j)}{n+1} \left| VDM(z_0, \ldots, z_n) \right|^2 w(z_0)^{2n} \cdots w(z_n)^{2n} d\mu(z_0) \cdots d\mu(z_n)$$

$$= \int_{E_n^{n+1}} \psi_n(z_0, \ldots, z_n) dP_n(z_0, \ldots, z_n)$$

where $\psi_n(z_0, \ldots, z_n) := \frac{\sum_{j=0}^{n} \phi(z_j)}{n+1}$. 

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Now take a sequence \( \{ \eta_j \} \) with \( \eta_j \downarrow 0 \) and a corresponding sequence \( \{ n_j \} \) with \( n_j \geq n^*(\eta_j) \) from Proposition 4.2 and \( n_j \uparrow \infty \). By choosing \( n_j \) larger if necessary we may assume that

\[
(4.2) \quad n_j^2 \eta_j \uparrow \infty \quad \text{so that} \quad \left( 1 - \frac{\eta_j}{2 \delta w(E) \eta_j} \right)^{n_j^2} \to 0.
\]

Choose points \( z_0^{(n_j)}, \ldots, z_{n_j}^{(n_j)} \in A_{n_j, \eta_j} \) with

\[
\psi_{n_j}(z_0^{(n_j)}, \ldots, z_{n_j}^{(n_j)}) = \max_{(w_0, \ldots, w_{n_j}) \in A_{n_j, \eta_j}} \psi_{n_j}(w_0, \ldots, w_{n_j}).
\]

If \( |\phi| \leq M \) on \( E \), then \( |\psi_{n_j}| \leq M \) on \( E_{n_j+1} \); using the large deviation result, Proposition 4.2, and (4.2),

\[
\limsup_{j \to \infty} \int_E \phi(z) d\mu_{n_j}(z) = \limsup_{j \to \infty} \int_{E_{n_j+1}} \psi_{n_j} dP_{n_j}
\]

\[
= \limsup_{j \to \infty} \int_{A_{n_j, \eta_j}} \psi_{n_j} dP_{n_j} + \int_{E_{n_j+1} \setminus A_{n_j, \eta_j}} \psi_{n_j} dP_{n_j}
\]

\[
\leq \limsup_{j \to \infty} \left( \frac{1}{n_j + 1} \sum_{k=0}^{n_j} \phi(z_k^{(n_j)}) + M \left( 1 - \frac{\eta_j}{2 \delta w(E) \eta_j} \right)^{n_j^2} \right)
\]

\[
= \limsup_{j \to \infty} \frac{1}{n_j + 1} \sum_{k=0}^{n_j} \phi(z_k^{(n_j)}).
\]

Now since \( z_0^{(n_j)}, \ldots, z_{n_j}^{(n_j)} \in A_{n_j, \eta_j} \),

\[
|VDM(z_0^{(n_j)}, \ldots, z_{n_j}^{(n_j)})|^2 w(z_0^{(n_j)})^{2n_j} \ldots w(z_{n_j}^{(n_j)})^{2n_j} \geq (\delta w(E) - \eta_j)^{n_j^2}
\]

so that

\[
\lim_{j \to \infty} \left( |VDM(z_0^{(n_j)}, \ldots, z_{n_j}^{(n_j)})|^2 w(z_0^{(n_j)})^{2n_j} \ldots w(z_{n_j}^{(n_j)})^{2n_j} \right)^{1/n_j^2} = \delta w(E).
\]

By Proposition 4.1,

\[
\frac{1}{n_j + 1} \sum_{k=0}^{n_j} \delta_{z_k^{(n_j)}} \to d\mu_{eq}^w.
\]

Thus

\[
\frac{1}{n_j + 1} \sum_{k=0}^{n_j} \phi(z_k^{(n_j)}) \to \int_E \phi(z) d\mu_{eq}^w(z)
\]

and hence

\[
(4.3) \quad \limsup_{j \to \infty} \int_E \phi(z) d\mu_{n_j}(z) \leq \int_E \phi(z) d\mu_{eq}^w(z).
\]
Applying (4.3) to $-\phi$ we obtain
\[
\limsup_{j \to \infty} \int_E (-\phi(z))d\mu_{n_j}(z) \leq \int_E (-\phi(z))d\mu^{w}(z);
\]
i.e.,
\[
\liminf_{j \to \infty} \int_E \phi(z)d\mu_{n_j}(z) \geq \int_E \phi(z)d\mu^{w}(z),
\]
so that
\[
(4.4) \quad \lim_{j \to \infty} \int_E \phi(z)d\mu_{n_j}(z) = \int_E \phi(z)d\mu^{w}(z).
\]
Thus for any sequence of positive integers increasing to infinity we can choose a subsequence $\{n_j\}$ satisfying (4.2) for some $\eta_j \downarrow 0$ so that (4.4) holds; hence $d\mu_{n}(z) \to d\mu^{w}(z)$ weak-*.

**Remark 4.1.** More generally, if we consider, for any positive integer $m \geq 1$, the generalized $m$-point correlation functions $R^{(n)}_{m}(z_1, \ldots, z_m)$ defined as
\[
R^{(n)}_{m}(z_1, \ldots, z_m) := \int_{E^n - m + 1} |VDM(\lambda_0, \ldots, \lambda_{n-m}, z_1, \ldots, z_m)|^2 \cdot w(\lambda_0)^{2n} \cdots w(\lambda_{n-m})^{2n}d\mu(\lambda_0) \cdots d\mu(\lambda_{n-m}),
\]
then one may verify that
\[
\frac{1}{Z_n} R^{(n)}_{m}(z_1, \ldots, z_m)w(z_1)^{2n} \cdots w(z_m)^{2n}d\mu(z_1) \cdots d\mu(z_m)
\]
converge weak-* as $n \to \infty$ to $d\mu^{w}(z_1) \cdots d\mu^{w}(z_m)$. See [6] for the case $E = \mathbb{R}$ and $w(x) = \exp(-Q(x))$ where $Q(x)$ is an even degree polynomial with positive leading coefficient and $Q(x) \geq 0$ on $\mathbb{R}$. In our setting, one may prove the analogue of Lemma 6.77 of [6] with slight modifications and then the proof of the analogues of Corollary 6.94 and Theorem 6.96 follow word-for-word.

5. Proof of Corollary 2.1.

We indicate the modifications needed to prove Corollary 2.1. We have $E = \mathbb{R}$, $\mu = dx$ and $w(x) = \exp(-Q(x))$ with $Q(x)$ an even degree polynomial having positive leading coefficient and $Q(x) \geq 0$ on $\mathbb{R}$. As mentioned in section 2, it is known that for unbounded sets and admissible measures, the support $S_{\mu}$ of the weighted energy minimizing measure $\mu^{w}_{eq}$ is compact. Related to this is the observation that the $L^2$-norms of our weighted polynomials essentially “live” on a compact subset of $\mathbb{R}$ (cf., Theorem III.6.1 of [12]). To be precise, we can take $\tilde{E}$ to be a large enough compact interval
in \(\mathbb{R}\) so that \(S_w \subset \tilde{E}\) and such that there exist positive constants \(a\) and \(b\) independent of \(n\) and \(p_n\) so that if \(p_n\) is a polynomial of degree at most \(n\),

\[
(5.1) \quad \int_{\mathbb{R}} |p_n(x)|^2 w(x)^{2n} dx \leq (1 + ae^{-bn}) \int_{\tilde{E}} |p_n(x)|^2 w(x)^{2n} dx.
\]

We apply Gram-Schmidt in \(L^2(w^{2n}dx)\) to the monomials \(1, x, \ldots, x^n\) to obtain orthogonal polynomials \(p_1^{(n)}(x) \equiv 1, \ldots, p_{n+1}^{(n)}(x)\); and we apply Gram-Schmidt in \(L^2(w^{2n}dx|\tilde{E})\) to the monomials \(1, x, \ldots, x^n\) to obtain orthogonal polynomials \(q_1^{(n)}(x) \equiv 1, \ldots, q_{n+1}^{(n)}(x)\). From (5.1),

\[
(5.2) \quad \|p_j^{(n)}\|_{L^2(w^{2n}dx)}^2 \leq (1 + ae^{-b(j-1)})\|p_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})}^2 \quad \text{and} \quad \|q_j^{(n)}\|_{L^2(w^{2n}dx)}^2 \leq (1 + ae^{-b(j-1)})\|q_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})}^2.
\]

Also, from the definitions of the orthogonal polynomials, we have

\[
(5.3) \quad \|q_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})} \leq \|p_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})} \leq \|p_j^{(n)}\|_{L^2(w^{2n}dx)} \leq \|q_j^{(n)}\|_{L^2(w^{2n}dx)}.
\]

so that, combining (5.2) and (5.3),

\[
(5.4) \quad \|q_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})} \leq \|p_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})} \leq (1 + ae^{-b(j-1)})^{1/2}\|q_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})}.
\]

As in Remark 2.1,

\[
\mathcal{Z}_n = \mathcal{Z}_n(\mathbb{R}, w, dx) := \int_{\mathbb{R}^{n+1}} |VDM(\lambda_0, \ldots, \lambda_n)|^2 w(\lambda_0)^{2n} \cdots w(\lambda_n)^{2n} d\lambda_0 \cdots d\lambda_n
\]

can be written as (see (2.10))

\[
\mathcal{Z}_n = (n + 1)! \prod_{i=1}^{n+1} \|p_i^{(n)}\|_{L^2(w^{2n}dx)}^2.
\]

Note the \(L^2\)-norms are finite because of the decay as \(|x| \to \infty\) of \(w(x)\). Using (5.2),

\[
\|p_j^{(n)}\|_{L^2(w^{2n}dx)}^2 \leq (1 + ae^{-b(j-1)})\|p_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})}^2 \leq (1 + ae^{-b(j-1)})\|p_j^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})}^2;
\]

multiplying these inequalities for \(j = 1, \ldots, n+1\) and taking \(n^2\)-roots, we see that

\[
(5.5) \quad \lim_{n \to \infty} \mathcal{Z}_n^{1/n^2} = \lim_{n \to \infty} \left( (n + 1)! \prod_{i=1}^{n+1} \|p_i^{(n)}\|_{L^2(w^{2n}dx|\tilde{E})}^2 \right)^{1/n^2}
\]

provided this limit exists.
On the other hand, applying Theorem 2.1 and (2.10) to $(\tilde{E}, w, dx|_{\tilde{E}})$ (recall Remark 2.2), we have

$$\lim_{n \to \infty} \left[ (n+1)! \prod_{i=1}^{n+1} \left| q_i^{(n)} \right|^2 \right]^{1/n^2} = \delta^w(\tilde{E}).$$

But since $\mu_{eq}$ has support in $\tilde{E}$, from (2.2),

$$\delta^w(\tilde{E}) = \delta^w(\mathbb{R}).$$

The proof of (2.7), including the existence of the limit, now follows from (5.5), (5.6) and (5.7) by applying (5.4).

\[\square\]

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