A method for solve integrable $A_2$ spin chains combining different representations

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Abstract
A non homogeneous spin chain in the representations $\{3\}$ and $\{3^*\}$ of $A_2$ is analyzed. We find that the naive nested Bethe ansatz is not applicable to this case. A method inspired in the nested Bethe ansatz, that can be applied to more general cases, is developed for that chain. The solution for the eigenvalues of the trace of the monodromy matrix is given as two coupled Bethe equations different from that for a homogeneous chain. A conjecture about the form of the solutions for more general chains is presented.
Integrable magnetic chains are interesting physical systems with a rich mathematical structure. Homogeneous integrable spin chains with spin 1/2 and higher have been found and solved [1]–[5]. Other interesting systems are non-homogenous chains combining two different kinds of spin states in the sites. De Vega and Wojnarovich [6] pioneered the work in these systems, whose thermodynamic limit was obtained by de Vega, L. Mezincescu and R.I. Nepomechie [7] (see too [8]–[10]). These authors considered an alternating chain with spin 1/2 and spin 1 associated to a $su(2)$ algebra. Following the quantum inverse scattering method (QISM) and using the Yang Baxter equation (YBE) [11], they found integrable hamiltonians that present terms coupling pairs of neighboring sites and others coupling three neighboring spin sites. The method introduces a monodromy matrix on an auxiliary space whose elements are tensors on the state space of the chain, tensorial product of the site spaces. The spectrum and eigenstates of the trace of the monodromy matrix are obtained by using the Bethe ansatz (BA) method and solving the set of equations, known as Bethe equations, derived by this method.

In a homogeneous system with the associated algebra of higher rank ([12]–[15]), the solutions are obtained by applying the BA method successively $(d - 1)$-times, $d$ being the dimension of the spin. This method is known as nested Bethe ansatz (NBA) [16].

In this paper, we are going to solve a non-homogeneous chain with spin associated to the algebra $su(3)$ and states of the sites alternating between the elementary representations $\{3\}$ and $\{3^*\}$. In order to obtain hamiltonians associated to alternating chains, we have made an extension of the method used in ref. [6] for systems where $P$ and $T$ symmetry are not conserved. The diagonalization of our hamiltonians is not possible by the usual NBA method. The reason why is that we have not a reference state, which would be taken as pseudovacuum annihilated by the elements under the diagonal of the monodromy matrix, to build the eigenstates of the diagonal elements of the monodromy matrix whose trace is related to the hamiltonian.

In the homogeneous case, the eigenstates are built by the action of the elements above the diagonal of the monodromy matrix on the pseudovacuum state; this fact makes the charges of $su(3)$ be related to the number of applied operators, which can be considered as a particle number. In the non-homogeneous case this does not happen anymore, since there are site states with charges of different sign which belong to different representations and therefore there are states of the chain with the same total charge and different number of particles. This makes the usual NBA method non-applicable.
The method we are going to present is inspired in NBA one and in the method given for solutions of $O(2N)$ symmetric theories in ref. [17], but it has important differences with them. In a first step, we look for a subspace of states of the chain that are eigenstates of the element $T_{1,1}$ of the monodromy matrix and are annihilate by the elements $T_{j,1}$ ($j > 1$), the whole subspace being invariant under application of $T_{i,j}$ ($i, j > 1$). On this subspace, we build the eigenstates of the trace of $T$ by the application of operators $T_{1,j}$ ($j > 1$) and obtain the eigenvalues in terms of the restriction of $T$ to one less dimension. The procedure should be repeated to end with a monodromy matrix of dimension one. As we see, our method can be considered as an extension of the NBA one.

In this paper we start by showing the general features of a non-homogeneous chain [8]. Then develop our method in an alternating chain based in the representations $\{3\}$ and $\{3^*\}$ of $su(3)$, but the method has a general application that we will applied in a forthcoming paper.

The YBE is usually written

$$R(\theta - \theta') \cdot [t(\theta) \otimes t(\theta')] = [t(\theta') \otimes t(\theta)] \cdot R(\theta - \theta'),$$

(1)

with

$$[t_{a,b}(\theta)]_{c,d} = R_{c,a}^{b,d}(\theta)$$

(2)

and the product $\otimes$ being understood, as usual, in the indices $c$ and $d$ which lie in the site space and the $\cdot$ product in the space tensorial product of the two auxiliary spaces in which lie the indices $a$ and $b$ of $t$ and all indices of $R$. Both operators are represented in fig. 1(a).
There is another solution to (1) with the same $R$-matrix and a $t$-matrix, with the site space in other representation, that we call $\tilde{t}$ and is represented in fig. 1(c) with self-explanatory notation. The YBE (1) for these operators is represented in fig. 2.

As said before, the key of the method is to build the monodromy operator as a product of $t$ and $\tilde{t}$ matrices, one per site, in the indices of the auxiliary space. The method could
be applied to any distribution of the spin representations in the chain. Here, we are considering an alternating chain. So, we have for the monodromy operator $T^{(alt)}$

$$T^{(alt)}_{a,b} (\theta, \alpha) = t_{a,a_1}^{(1)} (\theta) t_{a_1,a_2}^{(2)} (\theta + \alpha) \ldots t_{a_{2N-2},a_{2N-1}}^{(2N-1)} (\theta) \tilde{t}_{a_{2N-1},b}^{(2N)} (\theta + \alpha),$$  \hspace{1cm} (3)

is graphically expressed in fig. 3(a)

![Graphical expression of $T^{(alt)}_{a,b} (\theta, \alpha)$](image)

(a)

![Graphical expression of $\tilde{T}^{(alt)}_{\alpha,\beta} (\theta, \sigma)$](image)

(b)

This operator, due to (3), fulfills the YBE.

In the same way we have obtained this system, we can considered an analogous system by interchanging the representations $\{3\}$ and $\{3^*\}$. The corresponding $R^*$, $t^*$ and $\tilde{t}$ operators, represented in fig. 1(b) and (d), satisfy

$$R^* (\theta - \theta') \cdot [t^* (\theta) \otimes t^* (\theta')] = [t^* (\theta') \otimes t^* (\theta)] \cdot R^* (\theta - \theta'),$$  \hspace{1cm} (4)

with

$$[t^*_{a,b}(\theta)]_{c,d} = R^{*b,c}_{a,d}(\theta).$$  \hspace{1cm} (5)

The corresponding monodromy matrix, $\tilde{T}^{(alt)}$, is represented in fig. 3(b).

In the most general case, the $R$-matrix has the following properties

i) It is $PT$-symmetric

$$R^{c,d}_{a,b}(\theta) = R^{a,b}_{c,d}(\theta),$$  \hspace{1cm} (6a)

$$R^{*\gamma,\delta}_{\alpha,\beta}(\theta) = R^{*\alpha,\beta}_{\gamma,\delta}(\theta).$$  \hspace{1cm} (6b)
ii) It is unitary

\[ R^{c,d}_{a,b}(\theta) R^{e,f}_{c,d}(-\theta) = \rho(\theta) \delta_{a,e} \delta_{b,f}, \]  
\[ R^{*\gamma,\delta}_{\alpha,\beta}(\theta) R^{*\mu,v}_{\gamma,d}(-\theta) = \rho^*(\theta) \delta_{\alpha,\mu} \delta_{\beta,v}. \]  

(7a) (7b)

iii) The values for the spectral parameter \( \theta = 0 \) are \( R(0) = c_0 I \) and \( R^*(0) = c_0^* I \).

iv) A matrix \( M \) and a constant \( \eta \) exist such that

\[ R^{c,d}_{a,b}(\theta) M_{b,e} R^{g,e}_{f,d}(-\theta - 2\eta) M_{f,h}^{-1} \propto \delta_{a,g} \delta_{c,h}. \]  

(8)

v) The \( t \)-matrices verify

\[ [\tilde{t}_{a,b}(\theta)]_{\alpha,\beta} [\tilde{t}_{\beta,\gamma}(-\theta)]_{b,c} = \tilde{\rho}(\theta) \delta_{a,c} \delta_{\alpha,\gamma}. \]  

(9)

The hamiltonian is obtained by taking the trace of \( T^{(alt)} \) in the auxiliary space

\[ \tau^{(alt)}(\theta, \alpha) = T^{(alt)}_{a,a}(\theta, \alpha), \]  

(10)

and doing

\[ H = \frac{d}{d\theta} \ln \tau^{(alt)}(\theta, \alpha) \bigg|_{\theta=0}. \]  

(11)

The successive derivatives of \( \tau^{(alt)} \) give a complete set of commuting operators that make the system integrable.

Unlike in the homogeneous case, the hamiltonian has pieces that involve two and three neighboring sites

\[ H = \frac{1}{\rho(\alpha)} \sum_{i=1}^{2N-1} h^{(1)}_{i,i+1} + \frac{1}{c_0 \rho(\alpha)} \sum_{i=1}^{2N-2} h^{(2)}_{i,i+1,i+2}, \]  

(12)

with

\[ (h^{(1)}_{i,i+1})_{a,\beta;b,\gamma} = [\tilde{t}_{a,c}(\alpha)]_{\beta,\delta} [\tilde{t}_{\delta,\gamma}(-\alpha)]_{c,b} \]  

(13)

and

\[ (h^{(2)}_{i,i+1,i+2})_{a,\beta;c,b;\gamma,d} = [\tilde{t}_{a,e}(\alpha)]_{\beta,\delta} [\tilde{t}_{e,d}(0)]_{c,f} [\tilde{t}_{\delta,\gamma}(-\alpha)]_{f,b} \]  

(14)

where \( \dot{t} \) means \( dt/d\theta \).

In the same way, we can obtain another system by interchanging the representations \( \{3\} \) and \( \{3^*\} \). The hamiltonian \( \tilde{H} \) so obtained commutes with \( H \) and both of them can be simultaneously diagonalized.
The chain we are considering, the simplest one in a higher rank algebra, combines alternating the two elementary representations of $su(3)$. In fig. 1, we associate the solid line to the $\{3\}$ representation and the wavy line to the $\{3^*\}$. The $t$ matrix is

$$t(\theta) = \sinh \left( \frac{3}{2} \theta + \gamma \right) \sum_{i=1}^{3} e_{i,i} \otimes e_{i,i}^s + \sinh \left( \frac{3}{2} \theta \right) \sum_{i,j=1 \atop i \neq j}^{3} e_{i,i} \otimes e_{j,j}^s$$

$$+ \sinh (\gamma) \sum_{i,j=1 \atop i \neq j}^{3} \exp \left( (i - j - \frac{n}{2} \text{sign}(i - j)) \theta \right) e_{i,j} \otimes e_{j,i}^s,$$

where the superindex $s$ means site space and, in the $\{3\}$ representation, the $e_{i,j}$ are the matrices $(e_{i,j})_{l,m} = \delta_{i,l} \delta_{j,m}$.

The $t$ operator can be written in matrix form

$$t(\theta) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & a \end{pmatrix},$$

with

$$a(\theta) = \sinh \left( \frac{3}{2} \theta + \gamma \right),$$

$$b(\theta) = \sinh \left( \frac{3}{2} \theta \right),$$

$$c(\theta) = \sinh (\gamma) e^{\frac{\theta}{2}},$$

$$d(\theta) = \sinh (\gamma) e^{-\frac{\theta}{2}}.$$

In the same way, we can write

$$\tilde{t}(\theta) = \begin{pmatrix} \bar{a} & 0 & 0 & \bar{c} & 0 & 0 & 0 & 0 & \bar{d} \\ 0 & \bar{b} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\ 0 & 0 & \bar{b} & 0 & 0 & 0 & 0 & 0 & \bar{d} \\ 0 & 0 & 0 & \bar{b} & 0 & 0 & 0 & 0 & \bar{d} \\ \bar{d} & 0 & 0 & 0 & \bar{a} & 0 & 0 & 0 & \bar{c} \\ 0 & 0 & 0 & 0 & \bar{b} & 0 & 0 & 0 & \bar{c} \\ 0 & 0 & 0 & 0 & 0 & \bar{b} & 0 & 0 & \bar{c} \\ 0 & 0 & 0 & 0 & 0 & \bar{b} & 0 & 0 & \bar{c} \\ \bar{c} & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & \bar{a} \end{pmatrix},$$

where

$$\bar{a} = \text{sinh} \left( \frac{3}{2} \theta + \gamma \right),$$

$$\bar{b} = \text{sinh} \left( \frac{3}{2} \theta \right),$$

$$\bar{c} = \text{sinh} (\gamma) e^{\frac{\theta}{2}},$$

$$\bar{d} = \text{sinh} (\gamma) e^{-\frac{\theta}{2}}.$$
with
\[ \bar{a}(\theta) = \sinh \left(\frac{3}{2} \theta + \frac{\gamma}{2}\right), \tag{19a} \]
\[ \bar{b}(\theta) = \sinh \left(\frac{3}{2} (\theta + \gamma)\right), \tag{19b} \]
\[ \bar{c}(\theta) = -\sinh (\gamma)e^{\frac{\theta + \gamma}{2}}, \tag{19c} \]
\[ \bar{d}(\theta) = -\sinh (\gamma)e^{-\frac{\theta + \gamma}{2}}. \tag{19d} \]

The monodromy matrix \( T_{\text{alt}} \) can be written as a matrix in the auxiliary space,
\[
T_{\text{alt}}(\theta) = \begin{pmatrix}
A(\theta) & B_2(\theta) & B_3(\theta) \\
C_2(\theta) & D_{2,2}(\theta) & D_{2,3}(\theta) \\
C_3(\theta) & D_{3,2}(\theta) & D_{3,3}(\theta)
\end{pmatrix},
\tag{20}
\]
whose elements are operators in the tensorial product of the site spaces.

The YBE for \( T_{\text{alt}} \), can be written in terms of its components
\[
B(\theta) \otimes B(\theta') = R^{(2)}(\theta - \theta') \cdot (B(\theta') \otimes B(\theta)) = (B(\theta') \otimes B(\theta)) \cdot R^{(2)}(\theta - \theta'), \tag{21a} \\
A(\theta)B(\theta') = g(\theta' - \theta)B(\theta')A(\theta) - B(\theta)A(\theta') \cdot \tilde{r}^{(2)}(\theta' - \theta), \tag{21b} \\
D(\theta) \otimes B(\theta') = g(\theta - \theta')(B(v) \otimes D(\theta)) \cdot R^{(2)}(\theta - \theta') - B(\theta) \otimes (r^{(2)}(\theta - \theta') \cdot D(\theta')), \tag{21c}
\]
where
\[
R^{(2)}(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & d & b & 0 \\
0 & b & c & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad r^{(2)}(\theta) = \begin{pmatrix}
h_+ & 0 \\
0 & h_-
\end{pmatrix}, \quad \tilde{r}^{(2)}(\theta) = \begin{pmatrix}
h_+ & 0 \\
0 & h_-
\end{pmatrix}, \tag{22}
\]
and
\[
g(\theta) = \frac{1}{b(\theta)}, \quad h_+(\theta) = \frac{c(\theta)}{b(\theta)}, \quad h_- (\theta) = \frac{d(\theta)}{b(\theta)}. \tag{23}
\]

For the site states, we use the notation
\[
u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{s} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{24}
\]

Inspired in the NBA method, we look for an eigenstate of \( A \) that serves as a pseudovacuum. Firstly we group two neighbor sites and call \( w_i \) the subspace generated by the
vectors $| u, \bar{s} >$ and $| u, \bar{d} >$ in the two site space formed with the $2i$-th and $(2i+1)$-th sites. Then, we build the subspace

$$\Omega = w_1 \otimes w_2 \otimes \cdots \otimes w_N$$

in the total space of states of a chain with $2N$ sites.

In a nonhomogeneous chain, we have not a state $| s >$ such that $D_{i,j} | s > \propto \delta_{i,j} | s >$. For this reason, the NBA method can not be used. Our method, instead, starts with a state $| 1 > \in \Omega$ verifying

$$A(\theta) | 1 > = [a(\theta)]^{N_3} [\bar{b}(\theta)]^{N_3^*} | 1 >,$$ (26a)

$$B_i | 1 > \neq 0, \quad i = 2, 3,$$ (26b)

$$C_i | 1 > = 0, \quad i = 2, 3,$$ (26c)

$$D_{i,j} | 1 > \in \Omega, \quad i, j = 2, 3,$$ (26d)

$N_3 (N_3^*)$ being the number of sites in the representation $\{3\} (\{3^*\})$. In our case $N_3 = N_3^* = N$.

Following the steps inspired in the NBA, we apply $r$-times the $B$ operators to $| 1 >$ and build the state

$$\Psi(\vec{\mu}) \equiv \Psi(\mu_1, \cdots, \mu_r) = B_{i_1}(\mu_1) \otimes \cdots \otimes B_{i_r}(\mu_r) X_{i_1, \cdots, i_r} | 1 >,$$ (27)

$X_{i_1, \cdots, i_r}$ being a $r$-tensor that, together with the values of the spectral parameters $\mu_1, \cdots, \mu_r$, will be determined at the end.

The action of $A(\mu)$ and $D_{i,j}(\mu)$ on $\Psi$ is found by pushing them to the right through the $B_{i_j}(\mu_j)$’s using the commutations rules (21b,c). Two types of terms arise when $A$ and $D_{i,j}$ pass through $B$’s, the wanted and unwanted terms, similar to the obtained in the NBA method. The first one comes from the first terms of (21b,c). In this type of terms the $A$ or $D_{i,i}$ and the $B$’s keep their original arguments and give a state proportional to $\Psi$. The terms coming from the second terms in (21b,c) are called unwanted since they contain $B_{i}(\mu)$ and so they never give a state proportional to $\Psi$; so, they must cancel each other out when we sum the trace of $T^{alt}$.

The wanted term obtained by application of $A$ is

$$[a(\mu)]^{N_3} [\bar{b}(\mu)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - \mu) B_{i_1}(\mu_1) \otimes \cdots \otimes B_{i_r}(\mu_r) X_{i_1, \cdots, i_r} | 1 >,$$ (28)
and the $k$-th unwanted term

$$-[a(\mu_k)]^{N_3}[\tilde{b}(\mu_k)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - \mu_k) \left( B(u) \tilde{\tau}(2)(\mu_k - u) \right) \otimes B(\mu_{k+1}) \otimes \cdots$$

$$\cdots \otimes B(\mu_r) \otimes B(\mu_1) \otimes B(\mu_{k-1}) M^{(k-1)} X \| 1 >,$$

where $M$ being the operator arising by repeated application of (21),

$$B(\mu_1) \otimes \cdots \otimes B(\mu_r) = B(\mu_{k+1}) \otimes \cdots B(\mu_r) \otimes B(\mu_1) \cdots \otimes B(\mu_{k-1}) M^{(k-1)}.\quad (30)$$

The wanted term results

$$[D_{k,j}(\mu) B_{i_1}(\mu_1) \otimes \cdots \otimes B_{i_r}(\mu_r) X_{i_1, \ldots, i_r} \| 1 >]_{\text{wanted}} = \prod_{i=1}^{r} g(\mu - \mu_i)$$

$$R^{(2)}_{j_r, a_r} (\mu - \mu_r) \cdots R^{(2)}_{j_2, a_2} (\mu - \mu_2) \cdot \tau^{(2)}_{j_1, a_1} (\mu - \mu_1) D_{k, a_r} X_{i_1, \ldots, i_r} \| 1 >,$$

where the $R^{(2)}$'s product is taken in the auxiliary space and has the form

$$\Phi(\mu, \tilde{\mu})_{a_r, j} \equiv R^{(2)}_{j_r, a_r, j} \cdots R^{(2)}_{j_2, a_2} \cdot \tau^{(2)}_{j_1, a_1} = \left( \begin{matrix} 1 & 2 \\ \alpha(\mu, \tilde{\mu}) & \beta(\mu, \tilde{\mu}) \\ \gamma(\mu, \tilde{\mu}) & \delta(\mu, \tilde{\mu}) \end{matrix} \right). \quad (32)$$

The action of $D_{k,j}$ with $k \neq j$ on $\| 1 >$ is not zero. This is the main difference with the models that can be solved by NBA. Then, we try to diagonalize the matrix product

$$F(\mu, \tilde{\mu}) = D(\mu) \cdot \Phi(\mu, \tilde{\mu}) = \left( \begin{matrix} A^{(2)}(\mu, \tilde{\mu}) & B^{(2)}(\mu, \tilde{\mu}) \\ C^{(2)}(\mu, \tilde{\mu}) & D^{(2)}(\mu, \tilde{\mu}) \end{matrix} \right). \quad (33)$$

By taking the terms in (31) with $k = j$ and adding them for $k = 2$ and $3$, we obtain the wanted term

$$\prod_{j=1}^{r} g(\mu - \mu_j) B_{i_1}(\mu_1) \otimes \cdots \otimes B_{i_r}(\mu_r) \tau^{(2)}(\mu, \tilde{\mu}) X_{i_1, \ldots, i_r} \| 1 >,$$

where

$$\tau^{(2)}(\mu, \tilde{\mu}) = \text{trace}(F) = A^{(2)}(\mu, \tilde{\mu}) + D^{(2)}(\mu, \tilde{\mu}). \quad (35)$$

In the same form, the $k$-th unwanted term results

$$- \prod_{j=1}^{r} g(\mu_k - \mu_j) \left( B(\mu) \tilde{\tau}(2)(\mu - \mu_k) \right) \otimes B(\mu_{k+1}) \otimes \cdots$$

$$\cdots \otimes B(\mu_r) \otimes B(\mu_1) \otimes B(\mu_{k-1}) M^{(k-1)} \tau^{(2)}(\mu_k, \tilde{\mu}) X \| 1 >.$$

9
The sum of the wanted terms and the cancelation of the unwanted terms give us the relations

\[ \tau_{(2)}(\mu, \vec{\mu})X \parallel 1 > = \Lambda_{(2)}(\mu, \vec{\mu})X \parallel 1 > \] (37)

and

\[ \Lambda_{(2)}(\mu_k, \vec{\mu}) = \left[a(\mu_k)\right]^{N_3}[\bar{b}(\mu_k)]^{N_3^*} \prod_{j=1, j \neq k}^{r} \frac{g(\mu_j - \mu_k)}{g(\mu_k - \mu_j)}. \] (38)

We must now diagonalize (37).

The tensor \( X_{i_1, \ldots, i_r}, (i_j = 2, 3) \) lies in a space with \( 2^r \) dimensions and \( \| 1 > \in \Omega \). Then, the vector \( X \parallel 1 > \) yields in a space \( \Omega^{(2)} \) with \( 2^r + N \) dimensions. In this space, we take the element

\[ \| 1 >^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |u\bar{s} >_1 \otimes \cdots \otimes |u\bar{s} >_N, \] (39)

which is annihilated by \( C^{(2)}(\mu, \vec{\mu}) \). (Note that the operators \( \alpha, \beta, \gamma \) and \( \delta \) of (32) act on first part of \( \| 1 >^{(2)} \) and the operators \( D_{i,j} \) on the second part). The application of the operators \( A^{(2)} \) and \( D^{(2)} \) gives

\[ A^{(2)}(\mu, \vec{\mu}) \| 1 >^{(2)} = [\bar{b}(\mu_k)]^{N_3}[\bar{b}(\mu_k)]^{N_3^*} \| 1 >^{(2)}, \] (40a)

\[ D^{(2)}(\mu, \vec{\mu}) \| 1 >^{(2)} = \prod_{i=1}^{r} \frac{1}{g(\mu - \mu_i)}[a(\mu_k)]^{N_3}[\bar{b}(\mu_k)]^{N_3^*} \| 1 >^{(2)}, \] (40b)

The important fact is that \( F(\mu, \vec{\mu}) \) verifies the YBE with the \( R^{(2)} \) matrix given in (24),

\[ R^{(2)}(\mu - \mu') F(\mu, \vec{\mu}) \otimes F(\mu', \vec{\mu}) \] = \[ F(\mu', \vec{\mu}) \otimes F(\mu, \vec{\mu}) \] \( R^{(2)}(\mu - \mu') \), (41)

which, in a second step, permits us to solve the system. From this equation, we obtain the commutation rules

\[ A^{(2)}(\mu) \cdot B^{(2)}(\mu') = g(\mu' - \mu)B^{(2)}(\mu') \cdot A^{(2)}(\mu) - h_+(\mu' - \mu)B^{(2)}(\mu) \cdot A^{(2)}(\mu'), \] (42a)

\[ D^{(2)}(\mu) \cdot B^{(2)}(\mu') = g(\mu - \mu')B^{(2)}(\mu') \cdot D^{(2)}(\mu) - h_+(\mu - \mu')B^{(2)}(\mu) \cdot D^{(2)}(\mu'). \] (42b)

In this second step, we build the vector

\[ \Psi^{(2)}(\vec{\lambda}, \vec{\mu}) = B^{(2)}(\lambda_1, \vec{\mu}) \cdots B^{(2)}(\lambda_s, \vec{\mu}) \parallel 1 >^{(2)}. \] (43)
The action of $A^{(2)}(\lambda, \bar{\mu})$ on $\Psi^{(2)}$ gives the wanted term

$$[b(\lambda)]^{N_3} \bar{b}(\lambda)]^{N_3} \prod_{i=1}^{s} g(\lambda - \lambda_i) B^{(2)}(\lambda_1, \bar{\mu}) \ldots B^{(2)}(\lambda_s, \bar{\mu}) \parallel 1 >^{(2)},$$

and the $k$-th unwanted term

$$-h_+ (\lambda - \lambda) [b(\lambda)]^{N_3} \bar{b}(\lambda)]^{N_3} \prod_{i=1}^{s} g(\lambda - \lambda_i) B^{(2)}(\lambda, \bar{\mu}) B^{(2)}(\lambda_{k+1}, \bar{\mu}) \ldots$$

$$\ldots B^{(2)}(\lambda_{k-1}, \bar{\mu}) \parallel 1 >^{(2)}.$$

In the same form, the action of $D^{(2)}(\lambda, \bar{\mu})$ on $\Psi^{(2)}$ gives the wanted term

$$[b(\lambda)]^{N_3} [\bar{a}(\lambda)]^{N_3} \prod_{i=1}^{s} g(\lambda - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\lambda_k - \mu_j)} B^{(2)}(\lambda_1, \bar{\mu}) \ldots B^{(2)}(\lambda_s, \bar{\mu}) \parallel 1 >^{(2)},$$

and the $k$-th unwanted term

$$-h_- (\lambda - \lambda_k) [b(\lambda)]^{N_3} [\bar{a}(\lambda)]^{N_3} \prod_{i=1}^{s} g(\lambda - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\lambda_k - \mu_j)} B^{(2)}(\lambda, \bar{\mu}) B^{(2)}(\lambda_{k+1}, \bar{\mu}) \ldots$$

$$\ldots B^{(2)}(\lambda_{k-1}, \bar{\mu}) \parallel 1 >^{(2)}.$$

The cancelation of the unwanted terms and the sum of the wanted terms, give us the equations

$$\left[ \frac{\bar{a}(\lambda_k)}{b(\lambda_k)} \right]^{N_3} \prod_{j=1}^{r} \frac{1}{g(\lambda_k - \mu_j)} = \prod_{i=1}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} \quad k = 1, \ldots, s$$

and

$$\Lambda^{(2)}(\mu_k, \bar{\mu}) = \prod_{i=1}^{s} g(\lambda_i - \mu_j) [b(\mu_k)]^{N_3} [\bar{a}(\mu_k)]^{N_3}.$$

Then, by comparing equations (38) and (49) and calling $\bar{g}(\theta) = \bar{a}(\theta)/\bar{b}(\theta)$, we obtain the coupled Bethe equations

$$[\bar{g}(\lambda_k)]^{N_3} = \prod_{j=1}^{r} \frac{g(\lambda_k - \mu_j)}{g(\lambda_k - \lambda_i)} \prod_{i=1}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)},$$

$$[g(\mu_k)]^{N_3} = \prod_{j=1}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_k - \mu_j)} \prod_{i=1}^{s} \frac{g(\lambda_i - \mu_k)}{g(\lambda_i - \mu_k)}. $$
and the eigenvalue of the trace of $T^{(alt)}$

$$
\Lambda(\mu) = [a(\mu)]^N_3 [\bar{b}(\mu)]^N_3 \prod_{j=1}^{r} g(\mu_j - \mu) + \\
[b(\mu)]^N_3 \prod_{j=1}^{r} g(\mu - \mu_j) \left[ [\bar{b}(\mu)]^N_3 \prod_{i=1}^{s} g(\lambda_i - \mu) + [a(\mu)]^N_3 \prod_{i=1}^{s} g(\mu - \lambda_i) \prod_{j=1}^{r} \frac{1}{g(\mu - \mu_j)} \right]
$$

(51)

that is the solution to our problem.

The solution we have obtained, permits us to conjecture the form of the solution for a non homogeneous chain combining the two elementary representations of an algebra $A_n$. Each elementary representation introduces a function $g$ and $\bar{g}$ (that we can call source functions). Such solution will have $n$ sets of Bethe equations (the same number of basic representations of $A_n$). The first and last sets will have in the first member their respective source functions powered to the number of sites of each representation and in the second member a product of source functions similar to those in the right hand side of eqns. (50a, b). The left hand sides of the rest of the sets of equations are the unity, since there is not source function. For other representations, we expect a similar scheme with source functions related to the source functions of the elementary representations [18]. Since our method has a wide applicability, we will study these points in a forthcoming paper.

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References

[1] H. M. Babujian, Phys. Lett. A 90 (1982) 479.
[2] L.A. Takhtajan, Phys. Lett. A 87 (1982) 479.
[3] A. N. Kirillov and N. Y. Reshetikhin, J. Phys. A 20 (1987) 1565.
[4] H. M. Babujian and A. M. Tsvelick, Nucl. Phys. B 265 (1986) 24.
[5] A. B. Zamolodchikov and V. A. Fateev, Sov. J. Nucl. Phys. 32 (1980) 298.
[6] H.J. de Vega and F. Woynarovich, J. Phys. A 25 (1992) 4499.
[7] H.J. de Vega, L. Mezincescu and R.I. Nepomechie, Phys. Rev. B 49 (1994) 13223.
[8] S. R. Aladin and M. J. Martins, J. Phys. A 26 (1993), 1529 and J. Phys. A 26 (1993) 7287.
[9] M. J. Martins, J. Phys. A 26 (1993) 7301.
[10] H.J. de Vega, L. Mezincescu and R.I. Nepomechie, J. Mod. Phys. B 8 (1994) 3473.
[11] L. D. Fadeev, Sov. Sc. Rev. Math Phys. C1 (1981) 107;
    V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, Quantum Inverse Scattering Method and Correlation Functions. Cambridge University Press (1991).
[12] M. Jimbo, Commun. Math. Phys. 102 (1986) 537.
[13] V. V. Bazhanov, Phys. Lett B 159 (1985), 321 and Commun. Math. Phys. 113 (1987) 471.
[14] M. Jimbo, T. Miwa and M. Okado, Mod. Phys. Lett. B1 (1987) 73.
[15] H. J. de Vega and E. Lopez, Phys. Rev. Lett. 67 (1981) 489; Nucl. Phys. B 362 (1991) 261; Preprint LPTHE 91/29.
[16] H.J. de Vega, J. Mod. Phys. A4 (1989) 2371;
    H.J. de Vega, Nucl. Phys. B (Proc. nad Suppl.) 18A (1990) 229;
    J. Abad and M. Rios, Univ. of Zaragoza preprint DFTUZ 94-11 (1994).
[17] H.J. de Vega and M. Karowski, Nucl Phys. B 280 (1987) 225.
[18] Discussions on those points with H. de Vega are acknowledge.