Instanton Moduli and Topological Soliton Dynamics

PAUL M. SUTCLIFFE

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Cambridge CB3 9EW, England
email p.m.sutcliffe@amtp.cam.ac.uk

ABSTRACT

It has been proposed by Atiyah and Manton [1] that the dynamics of Skyrmions in $\mathbb{R}^{3+1}$ may be approximated by motion on a finite dimensional manifold obtained from the moduli space of SU(2) Yang-Mills instantons in $\mathbb{R}^4$. Motivated by this work we describe how similar results exist for other soliton and instanton systems. We describe in detail two examples for the approximation of the infinite dimensional dynamics of sine-Gordon solitons in $\mathbb{R}^{1+1}$ by finite dimensional dynamics on a manifold obtained from instanton moduli. In the first example we use the moduli space of $\mathbb{CP}^1$ instantons in $\mathbb{R}^2$ and in the second example we use the moduli space of SU(2) Yang-Mills instantons in $\mathbb{R}^4$. The metric and potential functions on these manifolds are constructed and the resulting dynamics is compared with the explicit exact soliton solutions of the sine-Gordon theory.

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1. Introduction

There are numerous examples in theoretical physics of theories which possess topological solitons, which are extended structure solutions with a particle-like interpretation. In (1+1)-dimensions there are models which are integrable at both the classical and quantum level, thereby allowing an explicit and exact study of dynamical soliton interactions. However, in (d + 1)-dimensions there are no known integrable models for d > 1 if one restricts to lagrangian theories with the physically desirable property of Lorentz covariance. We are therefore led to the study of non-integrable nonlinear field and gauge theories, which even at the classical level is a difficult task. Invariably analytical and/or numerical approximations are required. For d > 1 an investigation of classical soliton dynamics through a numerical simulation of the field equations requires the use of a large amount of computing power for even a modest study. Furthermore, it is unclear how to address the quantum aspects of these theories from a consideration of the full field theory; which are in fact often non-renormalizable. It is therefore evident that there is sufficient motivation for attempting to find analytical approximation schemes. This paper is concerned with the construction, implementation and accuracy of such schemes for the situation in which there are forces between static solitons.

For theories of Bogomolny type there are no forces between static solitons and the second order nonlinear partial differential equation which determines the static soliton solutions can be reduced to a first order equation. For such theories there is an analytical approximation scheme which works remarkably well; namely the geodesic approximation [2]. Here one assumes that the n-soliton configuration at any fixed time may be well approximated by a static n-soliton solution. The only time dependence allowed is therefore in the dynamics of the finite dimensional n-soliton moduli space $\mathcal{M}_n$ (effectively the parameter space of static n-soliton solutions). A lagrangian on $\mathcal{M}_n$ is inherited from the field theory lagrangian, but since all elements of $\mathcal{M}_n$ have the same potential energy the kinetic part of the action completely determines the dynamics. It determines a metric on $\mathcal{M}_n$ and the dynamics is given by geodesic motion on $\mathcal{M}_n$ with respect to this metric. The geodesic approximation has been applied to the study of soliton dynamics in many theories of Bogomolny type including BPS monopoles [3], vortices at critical coupling [4] and $\sigma$-model lumps [5]. For some of these systems numerical simulations of the field theory equations of motion have been performed and demonstrate the validity of the geodesic approximation. Quantum soliton interactions can also be studied by quantization of the finite dimensional system, which has been studied in great detail for example in the case of BPS monopoles [6].

Turning now to the situation in which forces exist between static solitons (ie non-Bogomolny theories) it is clear that if a truncation to a finite dimensional manifold is to be performed then a substitute for the moduli space must be found. In the special case that the theory may be considered as a perturbation of a Bogomolny theory then one may use the moduli space of this
associated theory. The difference now is that there will be both a metric and a potential function on \( \mathcal{M}_n \) so that the dynamics will no longer be given by geodesic motion on \( \mathcal{M}_n \) but by a more complicated dynamics determined from the lagrangian on \( \mathcal{M}_n \). Such a scheme was applied to a planar Skyrme-like model [7], considered as a perturbation of the O(3) \( \sigma \)-model (a Bogomolny theory), and found to produce accurate results when compared with numerical simulations of the field equations. Other theories which are natural perturbations of Bogomolny theories are, of course, non-BPS monopoles and vortices which are not at critical coupling. Very recently this scheme has been applied to the study of vortex scattering at near critical coupling [8].

In the general case it is more difficult to find a substitute for the moduli space. One approach is to use the parameter space of an \( n \)-soliton configuration obtained by patching together \( n \) copies of a single soliton. This will certainly be an adequate manifold to describe solitons which are well separated when inter-soliton forces are weak. However, it may not provide an adequate description of solitons which are close together, when the solitons become greatly distorted and lose their individual identities. In particular this patching method is inadequate for the Skyrme model, where the minimal energy 2-soliton cannot be described by a patched configuration of two single solitons. One proposal for the Skyrme model is to use the union of gradient flow curves descending from a suitable family of static saddle point solutions [2]. Recently Atiyah and Manton [1] have shown that an apparently very similar manifold can be obtained from the moduli space of SU(2) Yang-Mills instantons in \( \mathbb{R}^4 \) by calculating holonomies. They have shown that it provides a good truncation of the 1-soliton sector and have investigated the differential and topological structure in the 2-soliton sector. This connection between instantons and solitons is still somewhat mysterious, and a clear explanation of the underlying reason for its existence is still lacking. There are some technical difficulties which together with the complexity of the calculations are responsible for the lack of a result at present on the metric and potential functions on this manifold.

In this paper we describe in detail two simpler examples of an Atiyah-Manton-like procedure as applied to a truncation of the sine-Gordon model in \((1+1)\)-dimensions. We show how to obtain suitable finite dimensional manifolds from both \( \mathcal{F}\mathbb{P}^1 \) instanton moduli in \( \mathbb{R}^2 \) and SU(2) Yang-Mills instanton moduli in \( \mathbb{R}^4 \). By calculating the metric and potential functions on these manifolds we obtain an approximation to the dynamics of solitons in the one and two soliton sectors. These results are compared with the exact sine-Gordon soliton solutions. We find that the method works well; however, an unexpected discovery is that a constraint must be placed on the orientation of the instantons in order to avoid a spurious bound state in the two soliton sector.

The plan of the paper is as follows. In section 2 we briefly review the Skyrme model and describe how we view the sine-Gordon theory as its lower dimensional analogue. In section 3 we compute sine-Gordon kink fields from \( \mathcal{F}\mathbb{P}^1 \) instantons and discuss the finite dimensional manifold
this produces. Section 4 describes an alternative possibility to the previous section where we use Yang-Mills instantons. Section 5 is devoted to an analysis of the dynamics generated on the finite dimensional manifolds obtained in the previous two sections. Finally in section 6 we make some concluding remarks and observations.

2. Topological Solitons in the Skyrme and Sine-Gordon theories

The Skyrme model is a nonlinear field theory which provides an effective description of low energy hadron physics. Originally introduced by Skyrme [9] this model has been the subject of a great deal of recent study following the observations of Witten [10]. The field of the model is an SU(2) matrix $U$ with lagrangian density given by

$$L = -\frac{1}{2} \text{Tr}(\partial_\mu U \partial^\mu U^{-1}) + \frac{1}{16} \text{Tr}([U^{-1} \partial_\mu U, U^{-1} \partial_\nu U][U^{-1} \partial^\mu U, U^{-1} \partial^\nu U]) + \frac{m^2}{2} \text{Tr}(U + U^\dagger - 2) \quad (2.1)$$

where $x^\mu = (t, x)$, $\mu = 0, 1, 2, 3$ are the spacetime coordinates in Minkowski space with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and Tr denotes trace. We have scaled away some phenomenological constants by a suitable choice of energy and length units, and $m$ is a small constant equal to the pion mass. For the purposes of our discussion we can neglect the pion mass term and will therefore consider the Skyrme model with $m = 0$ unless otherwise stated.

In order to consider configurations with finite energy the boundary condition

$$U(x) \to 1 \quad \text{as} \quad |x| \to \infty \quad (2.2)$$

is imposed. This effectively compactifies space from $\mathbb{R}^3$ to $S^3$, so that at a fixed time, $U$ is a map $U(x) : S^3 \to SU(2)$. Since $S^3$ is the group manifold of SU(2) the relevant identity is the homotopy group relation

$$\Pi_3(S^3) = \mathbb{Z} \quad (2.3)$$

which implies that to each configuration there may be associated an integer $B$, which is conserved and represents the winding number of the field as a map from space to the target manifold. It is known as the topological charge and counts the number of solitons in the configuration. It is physically identified as the baryon number so that a soliton is interpreted as a baryon. Explicitly $B$ is given by

$$B = \frac{1}{24\pi^2} \int \epsilon_{ijk} \text{Tr}((U^{-1} \partial_i U)(U^{-1} \partial_j U)(U^{-1} \partial_k U)) d^3x \quad (2.4)$$

where indices range over the values 1,2,3. There is a lower bound on the energy of a given configuration in terms of the soliton number i.e $E \geq 12\pi^2|B|$. 
The static one-soliton solution \((\text{ie } B = 1)\) is known as the Skyrmion and has the hedgehog form

$$U = \exp(i f(r)\hat{x}) \quad \text{where} \quad \hat{x} = \frac{x}{r}, \quad r = |x| \quad (2.5)$$

and denote the Pauli matrices. \(f(r)\) is a profile function, with boundary conditions \(f(0) = \pi\), \(f(\infty) = 0\), which has to be determined numerically [11]. This Skyrmion is situated at the origin but may be moved to any position by a translation. The orientation may also be changed by conjugation of \(U\) by a fixed element of \(SU(2)\). The Skyrmion has energy \(E_1 = 1.23 \times 12\pi^2\) and so does not saturate the topological lower bound on the energy. This allows the possibility of a bound state of two solitons in the \(B = 2\) sector, and indeed a bound state exists, although it is not of the hedgehog form. The complicated nature of the 2-soliton sector and the forces between individual solitons make the study of soliton dynamics in the Skyrme model a difficult problem. This is the reason that the approximation scheme proposed in [1] is both useful and interesting.

We now describe the sine-Gordon model in a way that illustrates its role in this paper, as a lower dimensional analogue of the Skyrme model. In fact Skyrme himself [12] proposed the sine-Gordon model as a toy model for a nonlinear meson field theory with the baryon interpretation of solitons \(\text{ie the forerunner of the Skyrme model. Let us consider the Skyrme lagrangian (2.1) (including the pion mass term) not in (3+1)-dimensions with an } SU(2)-\text{valued field, but in (1+1)-dimensions with a } U(1)-\text{valued field. We can then write}

$$U = e^{i\phi} \quad (2.6)$$

where \(\phi\) is a real-valued field, upon which (2.1) becomes (after dividing through by the constant 4 to obtain the standard normalization)

$$\mathcal{L}_{sg} = \frac{1}{8} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{m^2}{4} (\cos \phi - 1) \quad (2.7)$$

where now \(x^\mu = (t, x), \mu = 0, 1\). This is of course the sine-Gordon lagrangian. Note that in three dimensional space it is not necessary to include the potential term (pion mass term) in order to have a finite size soliton solution, although the term quadratic in derivatives (the Skyrme term) is required. In contrast in one dimensional space the Skyrme term is not required (the Skyrme term contribution has vanished in (2.7)) but the potential term must be present. This can easily be seen as a consequence of the behaviour of the sigma model term (the first term) under a scale transformation of the space coordinates. In the intermediate case of two dimensional space both a Skyrme term and a potential term are required, and this produces the planar Skyrme-like model which we mentioned briefly in the introduction. Since we require \(m \neq 0\) in the sine-Gordon lagrangian we can choose length units to set \(m = 1\) in (2.7) without loss of generality.
The sine-Gordon equation which follows from (2.7) is

\[ \partial_\mu \phi \partial^\mu \phi + \sin \phi = 0. \] (2.8)

The soliton solutions of the sine-Gordon equation (which are known as kinks) have finite energy so that the field \( \phi(x) \) at fixed time must tend to an integer multiple of \( 2\pi \) as \( x \to \pm\infty \). As with the Skyrme model there is therefore an integer-valued conserved topological charge \( B_{sg} \), given by

\[ B_{sg} = \frac{1}{2\pi} \lim_{-\infty}^{+\infty} \int \partial_x \phi \, dx. \] (2.9)

Again this counts the number of solitons (kinks) in a given configuration and we regard it as the analogue of the baryon number \( B \) of the Skyrme model. There is also a lower bound on the energy of a configuration in terms of the soliton number \( \text{ie. } E \geq 2\pi |B_{sg}| \).

The static one-kink solution is given by

\[ \phi = 4\text{arctan}(e^{x-a}) \] (2.10)

where \( a \) is the position of the kink. This solution saturates the energy bound so that it has energy \( E = 2 \). Note that this is an important difference between the sine-Gordon model and the Skyrme model since it implies that there can be no soliton bound states in the sine-Gordon theory (although there are soliton-antisoliton bound states known as breathers). Another difference between the two models, which will be important in the next section, is that there is no internal phase or orientation for a kink, in contrast to the already mentioned orientation of the Skyrmion.

There are no static two-kink solutions to the sine-Gordon equation, but the integrability of the theory allows the explicit construction of the exact two-kink solution where both kinks have arbitrary velocity (but at least one non-zero). The solution

\[ \phi = 4\text{arctan}\left( \frac{u \sinh(\gamma x)}{\cosh(\gamma ut)} \right) \] (2.11)

where \( \gamma = (1 - u^2)^{-\frac{1}{2}}, 0 < u < 1 \), was first found by Perring and Skyrme [13] and describes two kinks which approach along the \( x \)-axis with velocity \( u \), scatter elastically at \( t = 0 \), and emerge from the interaction with a phase shift given by

\[ \delta = \frac{2 \log(u^{-1})}{\gamma}. \] (2.12)

This 2-kink solution has energy \( E = 4\gamma \).

In the following section we briefly review the principal idea of the Atiyah-Manton approach to a finite dimensional truncation of the Skyrme theory, and then we construct in detail an analogous finite dimensional truncation of the sine-Gordon theory.
3. Kinks from $\mathbb{CP}^1$ Instantons

Let $A_\mu$ be the $su(2)$-valued Yang-Mills gauge potential for an instanton in four-dimensional euclidean space with coordinates $x_\mu = (x, x_4)$, $\mu = 1, 2, 3, 4$. Note that the euclidean time $x_4$ in the instanton theory is quite distinct from (and not to be confused with) the “physical time” $t$ of the soliton theories introduced in the previous section. The main idea of the Atiyah-Manton scheme [1] is to construct the holonomy

$$U(x) = \mathcal{P}\exp\int_{-\infty}^{+\infty} -A_4(x, x_4)dx_4$$

(3.1)

where $\mathcal{P}$ denotes path ordering.

The fundamental topological result is that if we consider this holonomy as a Skyrme field then it has baryon number $B = n$, where $n$ is the topological charge (ie instanton number) of the gauge potential $A_\mu$. The moduli space of charge $n$ SU(2) Yang-Mills instantons has dimension $8n - 3$, and this generates a manifold of Skyrme fields with dimension $8n - 1$ (after including a choice of gauge at infinity). It turns out that this manifold appears to be a good truncation of the Skyrme model. Here we will only mention the example for $n = 1$, where a 1-instanton positioned at the origin with scale $\lambda$ generates a Skyrmion of the hedgehog form (2.5) with a profile function

$$f(r) = \pi(1 - \frac{\lambda r}{\sqrt{1 + \lambda^2 r^2}}).$$

(3.2)

This Skyrme field has minimum energy when $\lambda = 0.69$ at which it takes the value $\tilde{E}_1 = 1.24 \times 12\pi^2$, which is within 1% of the numerically determined value $E_1$.

There is a lower dimensional analogue of this procedure which was briefly introduced by the present author in [14]. Here we describe this result in more detail and use it to construct finite dimensional manifolds which we use in section 5 to study soliton dynamics.

We use the gauge field formulation of the $\mathbb{CP}^1$ $\sigma$-model in two-dimensional euclidean space. It is defined in terms of a 2-component column vector $Z$, which is a function of the euclidean spacetime coordinates $x^\mu = (x^1, x^2)$, and is constrained to satisfy the condition

$$Z^\dagger Z = 1.$$ 

(3.3)

The action density has a U(1) gauge symmetry and is given by

$$\mathcal{L} = \text{Tr}(D_\mu Z)^\dagger(D^\mu Z)$$

(3.4)
where $\text{Tr}$ denotes trace and $D_\mu$ are the covariant derivatives

$$D_\mu = \partial_\mu - A_\mu$$  \hspace{1cm} (3.5)

with the composite gauge fields being purely imaginary and defined by

$$A_\mu = Z^\dagger \partial_\mu Z.$$  \hspace{1cm} (3.6)

The equation of motion resulting from (3.4) is

$$[\partial_\mu \partial^\mu, \mathbf{P}] = 0$$  \hspace{1cm} (3.7)

where $\mathbf{P}$ is the one-dimensional hermitian projector

$$\mathbf{P} = ZZ^\dagger.$$  \hspace{1cm} (3.8)

Instantons are finite action solutions to (3.7) and are most easily written using the parametrization

$$Z = \frac{1}{\sqrt{1 + |W|^2}} \begin{pmatrix} 1 \\ W \end{pmatrix}$$  \hspace{1cm} (3.9)

where they are given by $W$ a rational function of $x_+ = x_1 + ix_2$.

The target manifold is $\mathbb{C}\mathbb{P}^1$, which is isometric to $S^2$, so that due to the homotopy relation

$$\Pi_2(S^2) = \mathbb{Z}$$  \hspace{1cm} (3.10)

each finite action field configuration has an associated integer winding number. This winding number is the topological charge (instanton number) of the configuration and is given by

$$n = \frac{1}{\pi} \int \text{Im}(\partial_1 W \partial_2 \bar{W}) \frac{d^2x}{(1 + |W|^2)^2}$$  \hspace{1cm} (3.11)

where Im denotes the imaginary part. For instanton fields the degree of the rational function $W(x_+)$ is equal to the instanton number $n$.

Following (3.1) we construct the holonomy

$$U(x_1) = (-1)^n \exp \left( \int_{-\infty}^{+\infty} A_2(x_1, x_2) dx_2 \right)$$  \hspace{1cm} (3.12)

which is a $\text{U}(1)$-valued field. Note that there is no path ordering required here since we are dealing with an abelian gauge theory. The prefactor in (3.12) may be interpreted (see below) as the extra holonomy along a large semicircle at infinity in the $(x_1, x_2)$ plane which connects the points $x_2 = +\infty$ and $x_2 = -\infty$ and arises from the fact that the instanton may be in a singular gauge at infinity. In fact the holonomy (3.1) also requires a prefactor of -1 if the instanton is in a singular gauge at infinity [1].
We now make use of our interpretation of the sine-Gordon model as a U(1) Skyrme model through the relation (2.6) ie from the holonomy (3.12) we construct the sine-Gordon field through the definition

\[ e^{i\phi(x)} = U(x). \] (3.13)

The fundamental topological result is that the sine-Gordon field defined in this way has a kink number equal to the topological charge of the instanton gauge potential \( A_\mu \) ie \( B_{sg} = n \). This can be proved using the following steps. Combining (3.12) with (3.13) we have that

\[ \phi(x) = n(\text{mod}2)\pi - i \int_{-\infty}^{+\infty} A_2 dx_2 \bigg|_{x_1 = x} \] (3.14)

Let \( L \) denote the line which joins the point \((x_1, -\infty)\) to the point \((x_1, +\infty)\), and \( C \) be the large semicircle at infinity which joins these points with opposite sense of direction. Then we can write (3.14) as

\[ \phi(x) = n(\text{mod}2)\pi + i \int_{C} A_\mu dx_\mu \bigg|_{x_1 = x} - i \int_{L+C} A_\mu dx_\mu \bigg|_{x_1 = x} \] (3.15)

The second term may be evaluated to give \(-n\pi\) which cancels exactly the first term (since \( \phi \) takes values in \( S^1 \) we can freely add integer multiples of \( 2\pi \)). The remaining term is an integral over a closed contour so we can use Stokes’ theorem to write it as an area integral. In terms of \( W \) the result is

\[ \phi(x) = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \frac{2\text{Im}(\partial_1 W \partial_2 \bar{W})}{(1 + |W|^2)^2}. \] (3.16)

Comparing with equation (3.11) we see that \( \phi(x) \) is exactly \( 2\pi \) times the instanton topological charge contained in the region of the \((x_1, x_2)\) plane given by \( x_1 < x \). Hence \( \phi(-\infty) = 0 \) and \( \phi(\infty) = 2\pi n \), and we have the final result

\[ B_{sg} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \partial_x \phi \ dx = \frac{1}{2\pi} (\phi(\infty) - \phi(-\infty)) = n \] (3.17)

The moduli space of charge \( n \mathbb{D}^1 \) instantons has dimension \( 4n - 1 \). Since the holonomy (3.12) is invariant under translations of \( x_2 \) the above procedure produces kink fields which depend on a manifold, which we shall call \( \mathcal{M}^{np}_n \), that has dimension \( 4n - 2 \).
Let us consider the case \( n = 1 \). Then we may take \( W \) to be given by

\[
W = \lambda (x_+ - a)
\]  

(3.18)

where \( \lambda \) is real and positive and is the scale of the instanton, and \( a \) may be taken to be real due to the above mentioned invariance. This generates the 1-kink field

\[
\phi = \phi_{1}^{cp}(x, \lambda, a) = \pi \left( 1 + \frac{\lambda(x-a)}{\sqrt{1 + \lambda^2 (x-a)^2}} \right).
\]  

(3.19)

We therefore have that \( \mathcal{M}_{1}^{cp} = \mathbb{R}^+ \times \mathbb{R} \), with the interpretation of the coordinates as a scale and position for the kink. The energy of (3.19) is obviously independent of \( a \), and is minimized when \( \lambda = \lambda_{cp} = 0.695 \) at which it takes the value \( E = M_{cp} = 2 \times 1.010 \). This is the mass of the \( \Phi \mathbb{R}^1 \) generated approximate kink, which is within 1% of the energy of the exact 1-kink solution (2.10). Let us note that not only is the procedure we have described very similar to the Atiyah-Manton procedure but the results in the 1-soliton sector have a remarkable similarity also. Compare the form of the approximate kink (3.19) with the approximate profile function (3.2), the values of the instanton scales at which the approximate soliton energies are minimized, and the percentage error by which the energy of each approximate soliton exceeds that of the exact solution.

Now consider the case \( n = 2 \). In order to make the interpretation of each coordinate as clear as possible we write \( W \) in the form

\[
W^{-1} = \frac{\lambda_1^{-1}}{(x_+ - a_1)} + \frac{e^{i\theta} \lambda_2^{-1}}{(x_+ - a_2 + i c)}
\]  

(3.20)

where \( \lambda_1, \lambda_2 \) are the positive scales of each instanton and \( a_1, a_2, c \) are real and give the position on the \( x_1 \) axis of each instanton and their separation in \( x_2 \) space respectively, and \( \theta \) gives the relative internal phase between the two instantons.

We have that \( \mathcal{M}_{2}^{cp} = S^1 \times \mathbb{R}^3 \times (\mathbb{R}^+)^2 \). However, as we shall demonstrate shortly, the manifold \( \mathcal{M}_{2}^{cp} \) is not a good truncation of the 2-kink sector, unless we restrict to a certain submanifold. Before we begin this discussion let us make a few simplifications by considering only two-kink configurations in which the kinks are placed symmetrically about the origin and have momenta of equal magnitude but opposite sign \( ie \) we are considering 2-kink solutions that are given by the exact solution (2.11). This corresponds to setting \( \lambda_1 = \lambda_2 = \lambda \) and \( a_1 = -a_2 = a \); we also choose to set \( c = 0 \) for simplicity. Our first argument is no more than an intuitive observation as to why the value of \( \theta \) should be taken to be zero. We then make an explicit computation to show that the manifold obtained by allowing \( \theta \) to take arbitrary values results in an incorrect approximation to the sine-Gordon model, since it predicts a spurious 2-kink bound state.
As noted earlier an important difference between the Skyrme theory and the sine-Gordon theory is that the solitons of the latter have no internal degrees of freedom ie no orientation or phase. We therefore conjecture that to approximate sine-Gordon solitons one should, in contrast to the Skyrme case, consider only instantons which are all in phase. For \( n = 2 \) this means setting \( \theta = 0 \). We have now restricted to a two dimensional submanifold of \( \mathcal{M}_2^{cp} \) which we denote by \( \tilde{\mathcal{M}}_2^{cp} \). In fact \( \tilde{\mathcal{M}}_2^{cp} = \mathbb{R}^+ \times \mathbb{R}^+ \), where the coordinates are \( a \) and \( \lambda \). Note that \( a \) is no longer allowed to be equal to zero since the two-instanton solution would then degenerate into a one-instanton solution. Explicitly from \( \tilde{\mathcal{M}}_2^{cp} \) we obtain (after calculating the integral in (3.12) by contour integration) the 2-kink field

\[
\phi_2^{cp}(x, \lambda, a) = 2\pi x (1 + \frac{x^2 - a^2 + 2\lambda^{-2}}{\sqrt{(x^2 - a^2)^2 + 4x^2\lambda^{-2}}}) \times \frac{1}{\sqrt{x^2 + a^2 + 2\lambda^{-2} + 2\sqrt{a^2(x^2 + \lambda^{-2}) + \lambda^{-4}} + \sqrt{x^2 + a^2 + 2\lambda^{-2} - 2\sqrt{a^2(x^2 + \lambda^{-2}) + \lambda^{-4}}}}}
\]

(3.21)

If we compute numerically the energy of this configuration as a function of \( a \) and \( \lambda \) we find that it is minimized in the limit \( a \to \infty \), in which (3.21) becomes asymptotically two approximate kinks of the form (3.19) with positions \( a \) and \(-a\). The energy in this limit is therefore minimized when \( \lambda = \lambda_{cp} \) at which it takes the value \( E = 2M_{cp} \). \( \tilde{\mathcal{M}}_2^{cp} \) therefore produces the correct qualitative sine-Gordon behaviour of no bound states in the two-soliton sector. Note that this feature is not automatic for our approximate 2-soliton sector since the approximate 1-soliton does not attain the topological lower bound ie \( M_{cp} > 2 \). We will now demonstrate that it is precisely this qualitative “no bound states” requirement which is violated by \( M_2^{cp} \) as a finite dimensional truncation. In the same way that we defined the submanifold \( \tilde{\mathcal{M}}_2^{cp} \) we define the similar two dimensional submanifold \( \hat{\mathcal{M}}_2^{cp} \) but where we set \( \theta = \pi \) rather than the previous case of \( \theta = 0 \).

We are now considering the case in which the two instantons are exactly out of phase. Note that \( \hat{\mathcal{M}}_2^{cp} = \mathbb{R} \times \mathbb{R}^+ \) since the position coordinate \( a \) is now allowed to take the value zero, since this 2-instanton solution no longer degenerates into a 1-instanton solution in the \( a \to 0 \) limit now that the instantons are not identical (they have opposite orientation). Explicitly from \( \hat{\mathcal{M}}_2^{cp} \) we obtain the two kink field (where for notational convenience we define \( \zeta = \lambda^2x^2 - 1 \))
\[ \Phi_{2}^{cp}(x, \lambda, a) = \begin{cases} 
\pi x \sqrt{\frac{\lambda}{\zeta}} \left( \frac{\sqrt{\zeta + \lambda a}}{\sqrt{\lambda(x^2 + a^2) + 2a\sqrt{\zeta}}} + \frac{\sqrt{\zeta - \lambda a}}{\sqrt{\lambda(x^2 + a^2) - 2a\sqrt{\zeta}}} \right) & \text{if } \zeta \geq 0 \\
\frac{\pi x a \sqrt{2\lambda}}{\zeta \sqrt{\lambda^2(x^2 + a^2)^2 + 4a^2\zeta^2}} \times \left\{ \zeta \sqrt{\lambda^2(x^2 + a^2)^2 + 4a^2\zeta^2 + \lambda(x^2 + a^2)} + \lambda a \sqrt{\lambda^2(x^2 + a^2)^2 + 4a^2\zeta^2 - \lambda(x^2 + a^2)} \right\} & \text{if } \zeta < 0. 
\end{cases} \] (3.22)

Of course in the limit \( a \to \infty \) this solution again becomes two approximate 1-kinks as before, with the energy minimized at \( \lambda = \lambda_{cp} \) and taking the value \( 2M_{cp} \). However, the crucial point is that the energy of (3.22) is not minimized by this limit of \( a \). Numerically calculating the energy of (3.22) we find it is minimized at \( a = 3.16 \) and \( \lambda = 0.55 \), at which it has the value \( E = 4 \times 1.003 < 2M_{cp} \). The exact values of these constants are unimportant, the point being that the energy is minimized at a finite value of \( a \). This means that \( \tilde{\mathcal{M}}_{2} \) does not provide a qualitatively good truncation of the 2-soliton sector since it describes a theory with a 2-kink bound state. This observation is a warning that care must be taken when applying the instanton method to the study of solitons. If we had been studying a non-integrable soliton theory, with no exact results to guide us, then the instanton method may have led us to the incorrect conclusion that a 2-kink bound state existed in the full theory. It would appear that the only way to attempt to avoid making such errors in non-integrable theories is the kind of intuitive argument given above, based upon relating the instanton orientation to the soliton orientation (or lack of it).

It is in principal possible to calculate explicitly the 2-kink field corresponding to the manifold \( \mathcal{M}_{2} \), since using contour integration methods it depends on the zeros of a quartic polynomial, for which a closed form expression exists. However, we have already demonstrated that a submanifold of \( \mathcal{M}_{2} \), namely \( \tilde{\mathcal{M}}_{2} \), does not provide a good truncation and this implies that neither does \( \mathcal{M}_{2} \). This supports our conjecture that the correct manifold to use is \( \tilde{\mathcal{M}}_{2} \) obtained from instantons with the same orientation. In the rest of this paper we will restrict to this manifold when discussing the 2-soliton sector generated from \( \Phi\mathbb{P}^1 \) instantons.
4. Kinks from Yang-Mills Instantons

As briefly described earlier the Atiyah-Manton procedure involves taking SU(2) Yang-Mills instantons in $\mathbb{R}^4$ and computing a holonomy. In the previous section we introduced an analogue of this procedure which again involves computing holonomies but this time of $\mathbb{CP}^1$ instantons in $\mathbb{R}^2$. There is yet another analogue, which we shall now discuss, where we begin with the same instantons used in the Atiyah-Manton scheme.

It is obvious that to obtain sine-Gordon kinks from SU(2) Yang-Mills instantons in $\mathbb{R}^4$ one has to do something other than simply compute a holonomy. The problem is how to construct a real-valued field in $\mathbb{R}$ from such an instanton. The solution is suggested by equation (3.16) of the previous section. There we saw that computing the holonomy (3.12) was equivalent, in that case, to computing the integral of the topological charge density over a certain region of space. For other instanton systems there will be no such equivalence but it is clear that the second interpretation can be taken as a definition to define the procedure. In reference [15] the authors considered such a procedure by defining

$$\phi(x) = 2\pi \int_{-\infty}^{x} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots \int_{-\infty}^{+\infty} dx_d q_d(x_1, \ldots, x_d)$$

where $q_d$ denotes the topological charge density of any instanton theory in $d$-dimensional euclidean space. Clearly by construction if we consider an $n$-instanton solution to the gauge theory in question then $\phi(\infty) = 2\pi n$ and $\phi(0) = 0$ so that $B_{xy} = n$. This procedure becomes the holonomy procedure of the previous section if we choose $d = 2$ and the instanton theory to be the $\mathbb{CP}^1$ $\sigma$-model. The analysis in [15] was concerned only with the 1-soliton sector where it was shown that, for the theories they considered, the best approximate 1-kink was obtained from four-dimensional SU(2) Yang-Mills theory. In this section we will therefore concentrate on this particular instanton theory.

Let $A_\mu$ be the $\mathfrak{su}(2)$-valued Yang-Mills gauge potential for an instanton in four-dimensional euclidean space with coordinates $x_\mu, \mu = 1, 2, 3, 4$. The gauge field is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. In terms of these quantities the formula (4.1) is

$$\phi(x) = \frac{-1}{16\pi} \int_{-\infty}^{x} dx_1 \int_{-\infty}^{+\infty} dx_2 dx_3 dx_4 \epsilon_{\mu\nu\alpha\beta} \text{Tr}(F^{\mu\nu} F^{\alpha\beta})$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric symbol on four indices.

The moduli space of charge $n$ SU(2) Yang-Mills instantons has dimension $8n - 3$, but since the above integral is invariant under shifts of $x_j$ for $j = 2, 3, 4$, this procedure produces kink fields which depend on a manifold, which we call $\mathcal{M}^{\mu\nu}_n$, which has dimension $8n - 6$. 

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One could now simply substitute an $n$-instanton solution into (4.2) and attempt to perform the integration, as done in [15] for the $n = 1$ case. However, let us first comment that, as mentioned above, (4.2) is to be thought of as a similar expression to (3.16). It would be interesting, and certainly extremely useful from a computational point of view, if another form of the relation (4.2) could be found which was closer in spirit to the form (3.12); in the sense that an integral over $x_1$ did not have to be performed. In the following we restrict to instantons obtained from a superpotential [16, 17] (‘t Hooft or the more general JNR instantons) and show that such a relation exists. Note that for the cases $n = 1, 2$ such a restriction results in no loss of generality since all instantons are of this form. It is also possible to produce a similar result in the general instanton case, using the ADHM [18] construction and an expression for the topological charge density as a double laplacian [19], but we shall not describe this here.

The instantons we consider are therefore given by

$$A_\mu = \frac{i}{2} \sigma_\mu \partial^\nu \log \rho$$

(4.3)

where $\sigma_{\mu\nu} = (\sigma_4 \delta_{\mu\nu} - \delta_{\mu4} \delta_{\nu4}) \sigma_4$. The real-valued superpotential satisfying the wave equation in euclidean four-space $\Box \rho = 0$. Then (4.2) becomes

$$\phi(x) = -\frac{1}{8\pi} \int_{-\infty}^{x} dx_1 \int_{-\infty}^{+\infty} dx_2 dx_3 dx_4 \, \Box \partial_1 \log \rho$$

(4.4)

where, as in [17], this integral must be interpreted as excluding infinitesimal regions of integration containing the singularities of $\rho$; this is equivalent to defining $\Box \partial_1 \log (x_\mu - X_\mu)^2$ to be zero for any constant $X_\mu$ in $\mathbb{R}^4$. By converting total derivative terms to boundary terms we write (4.4) in its final form

$$\phi(x) = n(\mod 2) \pi - \frac{1}{8\pi} \int_{-\infty}^{+\infty} dx_2 dx_3 dx_4 \partial_1^3 \log \rho \bigg|_{x_1=x}$$

(4.5)

where we have succeeded in removing the integral over $x_1$.

Take the case $n = 1$, where we may write (using the above mentioned shift invariances)

$$\rho = 1 + \frac{\lambda^{-2}}{(x_\mu - a \delta_{1\mu})^2}$$

(4.6)

Here $\lambda$ is the positive scale of the instanton and $a$ is real, giving the position of the instanton on the $x_1$ axis. $M_1^m = \mathbb{R}^+ \times \mathbb{R}$ and corresponds to the explicit 1-kink field (obtained from
integrating (4.5))

$$
\phi^{ym}_{1}(x, \lambda, a) = \pi + \frac{\pi}{2} \partial_x^2 \{ (x-a) \sqrt{\lambda^{-2} + (x-a)^2} \} = \pi \left( 1 + \frac{\lambda(x-a)}{\sqrt{1 + \lambda^2(x-a)^2}} + \frac{\frac{1}{2} \lambda(x-a)}{(1 + \lambda^2(x-a)^2)^{2}} \right)
$$

(4.7)

The final expression is the form in which it was obtained in [15] by calculating explicitly the integral (4.2). The energy of (4.7), independent of the kink position $a$, is minimized when $\lambda = \lambda_{ym} = 0.419$ at which it takes the value $E = M_{ym} = 2 \times 1.0002$. This is the mass of the Yang-Mills generated approximate kink, which exceeds that of the exact solution (2.10) by around 0.02%. It is therefore a remarkably accurate approximation.

We now consider the $n = 2$ sector, and following the remarks of the previous section, consider instantons which have no relative orientation. We have not performed an explicit calculation to see whether the “no bound states” requirement is violated for Yang-Mills instantons with a relative orientation. Note that since $(M_{ym} - 2)$ is two orders of magnitude smaller than $(M_{cp} - 2)$ the window for such a possible violation is greatly reduced. Accurate numerical work is required to settle this issue. We also make the same symmetric kink simplifications that we made in the previous section, and consider the submanifold $\tilde{M}_{2}^{ym} = \mathbb{R}^+ \times \mathbb{R}^+$ corresponding to the instanton superpotential

$$
\rho = 1 + \frac{\lambda^{-2}}{(x_{\mu} - a \delta_{1\mu})^2} + \frac{\lambda^{-2}}{(x_{\mu} + a \delta_{1\mu})^2}
$$

(4.8)

This gives the 2-kink field

$$
\phi^{ym}_{2}(x, \lambda, a) = \pi \partial_x^2 \left\{ \frac{x (x^2 + 3a^2 + \lambda^{-2} + \sqrt{(x^2 - a^2)^2 + 2\lambda^{-2}(x^2 + a^2)})}{\sqrt{x^2 + a^2 + \lambda^{-2} + \sqrt{4x^2a^2 + \lambda^{-2}} + \sqrt{x^2 + a^2 + \lambda^{-2} - \sqrt{4x^2a^2 + \lambda^{-2}}}}} \right\}
$$

(4.9)

Note that upon performing the double differential in the above expression the result will be quite lengthy and would have been tedious to calculate from the expression (4.2) directly.

The energy of (4.9) is minimized in the limit $a \to \infty$, when the field becomes two approximate kinks of the form (4.7) with positions $a$ and $-a$, so that the energy is minimized when $\lambda = \lambda_{ym}$ at which it is equal to $2M_{ym}$.
5. Dynamics on $\mathcal{M}_{\text{cp}}^n$ and $\mathcal{M}_{\text{ym}}^n$

In this section we compute the approximate soliton dynamics, in the one and two-soliton sectors, obtained from the finite dimensional truncations constructed in the previous two sections.

We first consider $\mathcal{M}_{\text{cp}}^1$, with parameters $a$ and $\lambda$. As described in the introduction, time dependence is introduced by allowing these parameters to be functions of the “physical time” $t$ i.e $a(t)$ and $\lambda(t)$. A lagrangian for this system is then obtained by restricting the field theory lagrangian to the configuration space described by $\mathcal{M}_{\text{cp}}^1$ i.e the lagrangian is obtained by substituting the ansatz (3.19) into the lagrangian density (2.7) and integrating over $x$. Before we give the result it is convenient to introduce the normalized scale $\mu = \lambda/\lambda_{\text{cp}}$, so that the minimum energy of a static configuration is obtained at the value $\mu = 1$. The result for the lagrangian is

$$L_{\text{cp}}^1 = \frac{M_{\text{cp}}}{2} (\mu \dot{a}^2 + k_{\text{cp}} \frac{\dot{\mu}^2}{\mu^3} - \mu - \frac{1}{\mu})$$

(5.1)

where dot denotes differentiation with respect to time $t$ and we have introduced the constant $k_{\text{cp}} = \frac{1}{3 \lambda_{\text{cp}}} \approx 0.69$. It is easy to see that the equations which follow from (5.1) are

$$\ddot{a} \mu + \dot{a} \dot{\mu} = 0$$

$$2k_{\text{cp}} \ddot{\mu} - 3k_{\text{cp}} \dot{\mu}^2 + \mu^4 (1 - \dot{a}^2) - \mu^2 = 0$$

(5.2)

and have a one parameter family of solutions (independent of $k_{\text{cp}}$) given by

$$a = ut, \quad \mu = \gamma$$

(5.3)

for $-1 < u < 1$ constant, where $\gamma$ is the Lorentz factor $(1 - u^2)^{-\frac{1}{2}}$. The energy of such a solution is $\gamma M_{\text{cp}}$. Hence we obtain a simple Lorentz boost of the static approximate kink. Note that to be able to obtain an exact Lorentz boost from the instanton scale is a feature unique to the approximation of solitons in one dimensional space.

If we now consider $\mathcal{M}_{\text{ym}}^1$, by performing the same procedure with the ansatz (4.7) we obtain the same results as above after a labelling replacement $\text{cp} \rightarrow \text{ym}$, where the only quantity we have yet to define is $k_{\text{ym}} = \frac{1}{\lambda_{\text{ym}}} \approx 0.81$. It is interesting to compare these two results with that obtained by taking the exact kink solution (2.10) as an ansatz and introducing a scale through the replacement $x \rightarrow \mu x$. We then get a similar result with the obvious notation $M_{\text{exact}} = 2$ and $k_{\text{exact}} = \frac{\pi^2}{12} \approx 0.82$. This demonstrates that the coupling of translational and scale degrees of freedom is very similar for the approximate kinks as it is for the exact kink; compare $k_{\text{cp}}$ with $k_{\text{exact}}$ and particularly $k_{\text{ym}}$ with $k_{\text{exact}}$. 

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Let us now turn to the 2-soliton sector and consider $\tilde{M}^{cp}_2$. We are going to make one further simplification (it turns out that we obtain almost identical results with or without this simplification). In the limit of large $a$ (ie well-separated kinks) a 2-kink configuration becomes two 1-kink configurations so that the lagrangian becomes $L^{cp}_2 \approx 2L^{cp}_1$. The solution in this region will therefore be two free, Lorentz boosted, kinks with $\dot{a} = -u$, $\lambda = \gamma \lambda_{cp}$, corresponding to studying kink scattering with kink velocity $0 < u < 1$; compare the exact solution (2.11). Our simplification is to assume that the scale changes little during the evolution so that we can fix it at this value ie we set $\dot{\lambda} = 0$, and truncate to a system with only one degree of freedom. (This is in fact a good approximation. We have studied the two-dimensional system, which arises by allowing $\lambda(t)$, numerically using a Runge-Kutta evolution and find that the variation of $\lambda$ from the constant value $\lambda = \gamma \lambda_{cp}$ is so small as to have a negligible effect on our results).

Substituting the ansatz $\phi^{cp}_2(x, \gamma \lambda_{cp}, a(t))$ into (2.7) and integrating over $x$ gives

$$L^{cp}_2 = \frac{1}{2}g_{cp}(a)\dot{a}^2 - V_{cp}(a)$$

(5.4)

where the metric $g_{cp}$ and the potential $V_{cp}$ are given by

$$g_{cp}(a) = \frac{1}{2} \int_0^\infty dx \left( \frac{\partial \phi^{cp}_2(x, a, \gamma \lambda_{cp})}{\partial a} \right)^2$$

$$V_{cp}(a) = \int_0^\infty dx \frac{1}{4} \left( \frac{\partial \phi^{cp}_2(x, a, \gamma \lambda_{cp})}{\partial x} \right)^2 + \frac{1}{2} (1 - \cos \phi^{cp}_2(x, a, \gamma \lambda_{cp})).$$

(5.5)

Of course, in the Yang-Mills case we have exactly the same formalism, again simply replacing $cp$ by $ym$ so that we are then using the function $\phi^{ym}_2(x, \gamma \lambda_{ym}, a(t))$ etc. In Fig. 1. we plot the metric and potential functions $g_{cp}, V_{cp}, g_{ym}, V_{ym}$ obtained by integrating the above expressions numerically.
Fig 1: The metric and potential functions $g_{cp}, V_{cp}$ (continuous curves) and $g_{ym}, V_{ym}$ (dashed curves). In each case the potential curves are the monotonic decreasing functions of $a$.

To produce the plots of Fig 1. we have set $\gamma = 1$. The range of $a$ reflects the values which are relevant in the scattering processes we shall discuss below. It is easy to see that since $\phi_{cp}^2$ is a symmetric function of $a$ then $g_{cp} \to 0$ as $a \to 0$. Also it can be shown that the slope of $\phi_{cp}^2$ at the origin behaves like $\frac{1}{a^2}$, so that the potential grows without bound as $a \to 0$. Similar remarks apply for $\phi_{ym}^2$. From (5.4) the equations of motion are

$$g_{cp}\ddot{a} + \frac{1}{2} \frac{dg_{cp}}{da} \dot{a}^2 + \frac{dV_{cp}}{da} = 0 \quad (5.6)$$

with initial conditions $a(t = 0) = a_0$, $\dot{a}(t = 0) = -u$.

The solution can be obtained implicitly by quadrature as

$$t(a) = \int_{a_0}^{a} \sqrt{\frac{g_{cp}(\alpha)}{2(E - V_{cp}(\alpha))}} \, d\alpha \quad (5.7)$$

where $E = \frac{1}{2} g_{cp}(a_0) u^2 + V_{cp}(a_0)$ is the total energy. This solution is valid upto $t = t_1$, where $t_1$ is the turning time $t_1 = t(a_1)$ and $a_1$ is the turning point $V_{cp}(a_1) = E$. The position for $t > t_1$ can
be determined from the symmetric property of the motion \(a(t-t_1) = a(t_1-t)\). The approximate 2-kink phase shift is given by

\[ \delta_{cp} = 2(a_0 - ut_1) \]  

(5.8)

in the limit in which \(a_0 \to \infty\) with \(t_1\) determined as above.

We now compare the results of our finite dimensional truncations with the exact results given in section 2. First note that to compare with the exact 2-kink solution (2.11) it is necessary to shift the time coordinate \(t\) in this solution to \(t - \frac{1}{u} \left( \log(u^{-1}) - a_0 \right) \), so that the kinks have positions \( \pm a_0 \) at \( t = 0 \). In Fig. 2a, we plot the function \( \phi_{cp}^2(x, \gamma \lambda_{cp}, a(t)) \) (continuous curve) and the exact solution (2.11) (dashed curve) for a velocity \( u = 0.5 \), at various times. We have taken \( a_0 = 10 \) and all required integrals are performed numerically. Fig. 2b is as Fig. 2a, but for the Yang-Mills function \( \phi_{ym}^2(x, \gamma \lambda_{ym}, a(t)) \) with the time evolution of \( a \) determined by \( L_{ym}^2 \).

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Fig 2a: $\phi_{2}^{p}$ (continuous curve) and the exact solution (dashed curve), at increasing times.
Fig 2b : As Fig 2a. but for the approximation $\phi_2^{ym}$. 

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Fig 2a. (1) shows the initial \( t = 0 \) configuration, and it can be seen that it is very close to the initial exact solution. The kinks then collide, and Fig 2a. (2) shows the configurations at the time for which the separation \( a \) is a minimum. Note that since we are considering the scattering of identical kinks it is impossible to distinguish forward from backward scattering. Our instanton moduli coordinates make backward scattering the most natural interpretation (since \( a \in \mathbb{R}^+ \)) but we interpret the results in the more usual forward scattering manner in order to compare with the exact results. We can see that even when the kinks have lost their individual identities the approximation is still quite good, although it is clear that there are two regions in which the error is now appreciable. Fig 2a. (3) corresponds to the time for which the position \( a \) has returned to its initial value. The error here is quite clear, with the approximation giving a phase shift in excess of the exact result.

We now turn to Fig 2b. These figures correspond to those of Fig 2a. but now for the Yang-Mills generated approximation. Again two curves are graphed in each window, but the approximation is so close to the exact solution that it appears there is only a single graph! This approximation is clearly extremely accurate and much the superior of the two schemes. This is to be expected since the one-kink field \( \phi_{ym} \) is a much better approximation than the one-kink field \( \phi_{cp} \).

In Fig. 3, we plot the exact phase shift \( \delta \) (continuous curve), the \( \mathcal{CP}^1 \) generated approximation \( \delta_{cp} \) (stars) and the Yang-Mills generated approximation \( \delta_{ym} \) (circles) for velocities in the range \( 0.1 \leq u \leq 0.9 \). This figure displays the main result of this paper, showing how accurately the instanton truncations approximate the infinite dimensional field theory dynamics. Although the \( cp \) approximation gives qualitatively correct results the errors are quite large. This may seem surprising when one considers the accuracy of the one-kink field \( \phi_{cp} \); recall its energy is only 1% larger than that of the exact one-kink. However, an important point to bear in mind is that the approximation decays power-like, as opposed to the exponential decay of the exact one-kink. This would therefore suggest that we would be over optimistic to hope to obtain results of too great an accuracy. This is where the \( ym \) approximation yields surprising results. It too generates kinks with a power-like tail behaviour, but from Fig. 3, we see that the accuracy is remarkable, for the whole range of velocities. So it would seem that the exponential localization of sine-Gordon solitons is not a major factor in influencing their interaction. This is an interesting point which requires further investigation and is of relevance to higher dimensional systems where solitons are often localized in a power-like way eg lumps in planar sigma models.
Figure 3: The exact phase shift $\delta$ (continuous curve), and the approximations $\delta_{cp}$ (stars), and $\delta_{ym}$ (circles) for velocities $0.1 \leq u \leq 0.9$. 
6. Conclusion

This paper has dealt with some simpler analogues of the Atiyah-Manton construction of Skyrmions from instantons. We have shown how their procedure may be adapted to study other topological solitons, and furthermore have demonstrated the validity of the scheme in a case where exact results are available for comparison. The discovery that constraints may have to be imposed upon the instanton orientations, in order to correctly reproduce the qualitative bound states behaviour of the soliton theory, is unexpected. It is clearly an important issue to consider when applying the scheme to non-integrable soliton theories. For the particular case of the Skyrme model the intuitive argument described earlier suggests that no constraint is required, since the orientation of the Yang-Mills instanton produces the required isospin of the Skyrmion.

The results in this paper are encouraging for the use of the instanton method in approximating the dynamics of Skyrmions. As mentioned earlier, there are as yet no complete results on the instanton truncation of Skyrmion dynamics, mainly due to the heavy computations this involves, and some technical difficulties in computing the non-abelian holonomy. Using numerical methods [20] there are some results regarding the potential function on the finite dimensional two-Skyrmion manifold, and perhaps a full numerical scheme could also be used to construct the metric function and solve the resulting ordinary differential equations which result.

In this paper we have dealt with only the classical dynamics of solitons. It is also possible to study quantum scattering by a quantization of the finite dimensional system obtained from instanton moduli. A finite dimensional truncation of the sine-Gordon model (obtained from patched kinks) was quantized in [21] and found to compare well with the known exact results in the weak coupling limit. A similar analysis carries over to the finite dimensional manifolds constructed in this paper and results of a similar quality can be obtained.

It appears that the soliton from instanton construction is a useful approximation technique, the wide applicability of which is only just beginning to emerge. It is already known that this scheme can be applied to other kink systems, such as the double sine-Gordon equation [15], to produce accurate approximate static kinks, and it produces excellent results when applied to $\phi^{2n}$ field theory [22]. No doubt the method will prove to be a useful tool as other areas of application come to light.

Finally, let us make a few remarks about Skyrmions which arise from the work in this paper. One point to note is that our earlier comments about the power-like decay properties of the approximate kinks, as compared to the exponential decay of the exact kink, also have relevance in the Skyrme context. The instanton generated profile function (3.2) decays like $\frac{1}{r^2}$ for large $r$, as does the (numerical) exact solution. However, if a pion mass term is added then the exact solution now has an exponential decay. The results of this paper would seem to suggest that this is not a serious defect if one wished to use the Atiyah-Manton scheme to study Skyrmion
dynamics even in the massive pion case. A further observation is based on the similarity of the Skyrmion profile functions (3.2) and the kink function (3.19). In fact, if one considers a $\mathbb{RP}^1$ anti-instanton (rather than an instanton) positioned at the origin and makes the replacement $x \to r$ then one obtains exactly the profile function (3.2). This identification prompted the use of a sine-Gordon anti-kink field as a trial function for the Skyrme profile function [23] and slightly reduced the energy as compared to (3.2). Now note that we can use a similar identification with a Yang-Mills generated anti-kink, to propose a modification of the Atiyah-Manton profile function (3.2) to

$$f(r) = \pi(1 - \frac{\lambda r}{\sqrt{1 + \lambda^2 r^2}} - \frac{\frac{1}{2} \lambda r}{(1 + \lambda^2 r^2)^{\frac{3}{2}}}).$$

(6.1)

This function is the same as (3.2) but with an additional term added which does not affect the boundary conditions. As described earlier (3.2) gives a Skyrme field with energy about 1% above the (numerical) exact solution, whereas (6.1) reduces this excess to around $\frac{1}{3}$%. If one replaces the $\frac{1}{2}$ coefficient of the final term in (6.1) with an arbitrary constant $c$, then the energy is minimized for $c \approx 0.4$ at which the excess energy is less than $\frac{1}{100}$%. By introducing independent scales for the two terms in (6.1) one could perhaps reduce the energy still further.

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