The CMO-Dirichlet Problem for the Schrödinger Equation in the Upper Half-Space and Characterizations of CMO

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Abstract
Let \( L \) be a Schrödinger operator of the form \( L = -\Delta + V \) acting on \( L^2(\mathbb{R}^n) \) where the non-negative potential \( V \) belongs to the reverse Hölder class \( RH_q \) for some \( q \geq (n + 1)/2 \). Let \( CMO_L(\mathbb{R}^n) \) denote the function space of vanishing mean oscillation associated to \( L \). In this article, we will show that a function \( f \) of \( CMO_L(\mathbb{R}^n) \) is the trace of the solution to \( Lu = -u_{tt} + Lu = 0, u(x, 0) = f(x) \), if and only if, \( u \) satisfies a Carleson condition

\[
\sup_{B: \text{balls}} C_{u, B} := \sup_{B(x_B, r_B): \text{balls}} r_B^{-n} \int_0^{r_B} \int_{B(x_B, r_B)} \left| t \nabla u(x, t) \right|^2 \frac{dx \, dt}{t} < \infty,
\]

and

\[
\lim_{a \to 0} \sup_{B: r_B \leq a} C_{u, B} = \lim_{a \to \infty} \sup_{B: r_B \geq a} C_{u, B} = \lim_{a \to \infty} \sup_{B: B \subseteq (B(0, a))^c} C_{u, B} = 0.
\]

This continues the lines of the previous characterizations by Duong et al. (J Funct Anal 266(4):2053–2085, 2014) and Jiang and Li (ArXiv:2006.05248v1) for the BMO\(_L\) spaces, which were founded by Fabes et al. (Indiana Univ Math J 25:159–170, 1976) for the classical BMO space. For this purpose, we will prove two new characterizations of the \( CMO_L(\mathbb{R}^n) \) space, in terms of mean oscillation and the theory of tent spaces, respectively.

Keywords CMO · Schrödinger operators · Dirichlet problem · BMO · Tent spaces

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1 Introduction

The space of bounded mean oscillation (BMO) was introduced by John and Nirenberg [1]. A locally integrable function $f$ on $\mathbb{R}^n$ is said to be in $\text{BMO}(\mathbb{R}^n)$, if

$$
\|f\|_{\text{BMO}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|} \int_B |f(y) - f_B| \, dy < \infty,
$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^n$ and $f_B := \frac{1}{|B|} \int_B f(x) \, dx$.

A celebrated theorem of Fefferman and Stein [2] states that a function $f$ of BMO is the trace of the solution to

$$
\begin{cases}
\partial_{tt} u(x, t) + \Delta u(x, t) = 0, & (x, t) \in \mathbb{R}^n_+; \\
u(x, 0) = f(x), & x \in \mathbb{R}^n
\end{cases}
$$

where $u$ satisfies

$$
\sup_{xB, rB} r_B^{-n} \int_0^{r_B} \int_{B(xB, rB)} |t \nabla u(x, t)|^2 \frac{dx \, dt}{t} < \infty, \quad (1.1)
$$

where $\nabla = (\nabla_x, \partial_t)$. Expanding on this result, Fabes et al. [3] showed that the condition (1.1) characterizes all the harmonic functions whose traces are in $\text{BMO}(\mathbb{R}^n)$. We refer the reader to [4] for the earlier study of the $H^p$ traces, and to [5–8] for further results on this topic.

In the last two decades, the theory of BMO spaces associated to differential operators attracted lots of attentions. See for example, [9–13]. Especially, consider the Schrödinger operator

$$
\mathcal{L} = -\Delta + V(x) \quad \text{on} \quad L^2(\mathbb{R}^n), \quad n \geq 3,
$$

where the non-negative potential $V$ is not identically zero, and $V \in \text{RH}_q$ for some $q > n/2$. Recall that $V \in \text{RH}_q$ means that $V \in L^q_{\text{loc}}(\mathbb{R}^n)$, $V \geq 0$, and there exists a constant $C > 0$ such that the reverse Hölder inequality

$$
\left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq \frac{C}{|B|} \int_B V(y) \, dy \quad (1.3)
$$

holds for all balls $B$ in $\mathbb{R}^n$. Recall that $f$ belongs to $\text{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ ([9]) if $f$ is a locally integrable function and satisfies

$$
\|f\|_{\text{BMO}_{\mathcal{L}}(\mathbb{R}^n)} := \sup_{B=B(xB, rB): rB<\rho(xB)} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy
$$
Here, the function $\rho(x)$, introduced by Shen [14, 15], is defined by

$$
\rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \ dy \leq 1 \right\}. \tag{1.5}
$$

Note that this $\text{BMO}_L(\mathbb{R}^n)$ space is a proper subspace of the classical BMO space and when $V \equiv 1$, $\text{BMO}_{-\Delta+1}$ is just the bmo space introduced by Goldberg [16]. It is known that there is an alternative characterization of $\text{BMO}_L(\mathbb{R}^n)$ that $f \in \text{BMO}_L(\mathbb{R}^n)$, if and only if, $f \in L^2(\mathbb{R}^n, (1 + |x|)^{-(n+\beta)}dx)$ for some $\beta > 0$ and

$$
\| f \|_{\text{BMO}_L(\mathbb{R}^n)} := \sup_B \left( \frac{1}{|B|} \int_B \left| f(x) - e^{-r_B \sqrt{\langle r \rangle}} f(x) \right|^2 \ dx \right)^{1/2} < \infty, \tag{1.6}
$$

where the supremum is taken over all balls in $\mathbb{R}^n$. Moreover, $\| f \|_{\text{BMO}_L(\mathbb{R}^n)} \approx \| f \|_{\text{BMO}_L(\mathbb{R}^n)}$. See, for example, [17,Proposition 6.11].

Recently, Duong et al. [18] extended the study by Fabes et al [3] to the Dirichlet problem for the Schrödinger equation with BMO$^1$ traces, and established the following characterization: whenever $V \in RH_q$ with $q \geq n$, a solution $u$ to the equation

$$
\begin{aligned}
\partial_t u(x, t) - \mathcal{L} u(x, t) &= 0, \quad (x, t) \in \mathbb{R}^{n+1}; \\
u(x, 0) &= f(x), \quad x \in \mathbb{R}^n
\end{aligned} \tag{1.7}
$$

satisfies (1.1), if and only if, $u$ can be represented as $u = e^{-t\sqrt{\mathcal{L}}} f$, where $f$ is in BMO$^1$. Very recently, the condition $V \in RH_q$ with $q \geq n$ in [18], was improved by Jiang and Li [19] to $q \geq (n + 1)/2$.

On the other hand, it came to our attention that Martell et al [20] established the well-posedness of the Dirichlet problem for any homogeneous, second-order, constant complex coefficient elliptic system in the upper half-space, with boundary data in the VMO$(\mathbb{R}^n)$ space of Sarason [21]. Here VMO is the BMO-closure of $\text{UC} \cap \text{BMO}$, where $\text{UC}$ denotes the class of all uniformly continuous functions. There is yet another significant space of functions of vanishing mean oscillations, $\text{CMO}(\mathbb{R}^n)$, which is defined by the closure in the BMO norm of $C^\infty_c(\mathbb{R}^n)$, the space of smooth functions with compact support. Obviously $\text{CMO}(\mathbb{R}^n)$ is a proper subspace of VMO$(\mathbb{R}^n)$. One well-known fact is that the Hardy space $H^1(\mathbb{R}^n)$ is the dual space of $\text{CMO}(\mathbb{R}^n)$; see [22,Theorem 4.1]. Notably, in a subsequent paper [23] by Martell et al, the authors posed an open question to formulate and prove a well-posedness result for the Dirichlet problem in the upper half-space, for the elliptic system as in [20] with CMO traces.

The main aim of this article is to continue the lines of [18, 19] to process the study on the Dirichlet problem for the Schrödinger equation with boundary value in $\text{CMO}_L(\mathbb{R}^n)$. The $\text{CMO}_L(\mathbb{R}^n)$ is the space of functions of vanishing mean oscillation

\footnote{CMO is also called VMO in [22].}
Theorem 1.2 Suppose \( V \in \mathcal{L} \), introduced in [24] by Deng, Duong, Tan, Yan and the first named author of this article under a more general setting.

**Definition 1.1** [24] We say that a function \( f \in \text{CMO}_L(\mathbb{R}^n) \) if \( f \) is in \( \text{BMO}_L(\mathbb{R}^n) \) and satisfies the limiting conditions \( \gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0 \), where

\[
\gamma_1(f) = \lim_{a \to 0} \sup_{B : r_B \leq a} \left( r_B^{-n} \int_B \left| f(x) - e^{-r_B} \mathcal{L} f(x) \right|^2 dx \right)^{1/2};
\]

\[
\gamma_2(f) = \lim_{a \to \infty} \sup_{B : r_B \geq a} \left( r_B^{-n} \int_B \left| f(x) - e^{-r_B} \mathcal{L} f(x) \right|^2 dx \right)^{1/2};
\]

\[
\gamma_3(f) = \lim_{a \to \infty} \sup_{B \subseteq (B(0, a))^c} \left( r_B^{-n} \int_B \left| f(x) - e^{-r_B} \mathcal{L} f(x) \right|^2 dx \right)^{1/2}.
\]

We endow \( \text{CMO}_L(\mathbb{R}^n) \) with the norm of \( \text{BMO}_L(\mathbb{R}^n) \).

Note that, whenever \( \mathcal{L} = -\Delta \), i.e., \( V \equiv 0 \), the space \( \text{CMO}_{-\Delta}(\mathbb{R}^n) \) coincides with \( \text{CMO}(\mathbb{R}^n) \); see [24, Proposition 3.6]. Besides, the following results hold.

**Theorem 1.2** Suppose \( V \in \mathcal{R}H_q \) for some \( q > n/2 \).

(i) [24] \( \text{CMO}_L(\mathbb{R}^n) \) is the pre-dual space of \( H^1_L(\mathbb{R}^n) \).

(ii) [25] \( \text{CMO}_L(\mathbb{R}^n) \) is the closure of \( C_c^\infty(\mathbb{R}^n) \) in the \( \text{BMO}_L(\mathbb{R}^n) \) norm.

The reader is referred to Sect. 2 for the definition of \( H^1_L(\mathbb{R}^n) \), the Hardy space associated to \( \mathcal{L} \). We say that \( u \in W^{1,2}(\mathbb{R}^n_{++1}) \) is an \( \mathbb{L} \)-harmonic function in \( \mathbb{R}^n_{++1} \), if \( u \) is a weak solution of \( \mathbb{L} u := -u_{tt} + \mathcal{L} u = 0 \), that is,

\[
\int_{\mathbb{R}^n_{++1}} \int_{\mathbb{R}^n_{++1}} \partial_t u \partial_t \phi \, dx \, dt + \int_{\mathbb{R}^n_{++1}} \int_{\mathbb{R}^n_{++1}} (\nabla_x u, \nabla_x \phi) \, dx \, dt + \int_{\mathbb{R}^n_{++1}} V u \phi \, dx \, dt = 0
\]

holds for all Lipschitz functions \( \phi \) with compact support in \( \mathbb{R}^n_{++1} \). The space \( \text{HMO}_L(\mathbb{R}^n_{++1}) \) is defined as the class of all \( \mathbb{L} \)-harmonic functions \( u \), that satisfies

\[
\|u\|_{\text{HMO}_L(\mathbb{R}^n_{++1})} := \sup_{x_B, r_B} \left( r_B^{-n} \int_0^{r_B} \int_{B(x_B, r_B)} |t \nabla u(x, t)|^2 \frac{dx \, dt}{t} \right)^{1/2} < \infty,
\]

where \( \nabla := (\partial_t, \nabla_x) \).

**Definition 1.3** We say that \( u \) belongs to \( \text{HCMO}_L(\mathbb{R}^n_{++1}) \) if \( u \in \text{HMO}_L(\mathbb{R}^n_{++1}) \), and satisfies the limiting conditions \( \beta_1(u) = \beta_2(u) = \beta_3(u) = 0 \), where

\[
\beta_1(u) = \lim_{a \to 0} \sup_{B : r_B \leq a} \left( r_B^{-n} \int_0^{r_B} \int_B |t \nabla u(x, t)|^2 \frac{dx \, dt}{t} \right)^{1/2};
\]

\[
\beta_2(u) = \lim_{a \to \infty} \sup_{B : r_B \geq a} \left( r_B^{-n} \int_0^{r_B} \int_B |t \nabla u(x, t)|^2 \frac{dx \, dt}{t} \right)^{1/2};
\]
\[
\beta_3(u) = \lim_{a \to \infty} \sup_{B: B \subseteq (B(0, a))^c} \left( r^{-n} \int_0^{r_B} \int_B |t \nabla u(x, t)|^2 \frac{dx \, dt}{t} \right)^{1/2},
\]

We endow \( \text{HCMO}_L(\mathbb{R}^{n+1}_+) \) with the norm of \( \text{HMO}_L(\mathbb{R}^{n+1}_+) \).

The main result of this paper is the following characterization.

**Theorem A** Suppose \( V \in \text{RH}_q \) for some \( q \geq (n + 1)/2 \) and let \( \mathcal{L} = -\Delta + V \).

(i) If \( u \in \text{HCMO}_L(\mathbb{R}^{n+1}_+) \), then there exists a function \( f \in \text{CMO}_L(\mathbb{R}^n) \) such that

\[
\begin{align*}
&u(x, t) = e^{-\frac{t}{\sqrt{\mathcal{L}}}} f(x), \\
&\text{there exists a constant } C > 1, \text{ independent of } u, \text{ such that}
\end{align*}
\]

\[
\| f \|_{\text{BMO}_L(\mathbb{R}^n)} \leq C \| u \|_{\text{HMO}_L(\mathbb{R}^{n+1}_+)}.
\]

(ii) If \( f \in \text{CMO}_L(\mathbb{R}^n) \), then \( u(x, t) = e^{-\frac{t}{\sqrt{\mathcal{L}}}} f(x) \in \text{HCMO}_L(\mathbb{R}^{n+1}_+) \), and there exists a constant \( C > 1 \), independent of \( f \), such that

\[
\| u \|_{\text{HMO}_L(\mathbb{R}^{n+1}_+)} \leq C \| f \|_{\text{BMO}_L(\mathbb{R}^n)}.
\]

Based on the previous works \([18, 19]\) on this Dirichlet problem with \( \text{BMO}_L(\mathbb{R}^n) \) traces, the main difficulty of proving Theorem A is to reveal the connections between limiting conditions equipped by solutions and traces, respectively. In order to show (i) of Theorem A, we will establish an equivalent characterization of the space \( \text{CMO}_L(\mathbb{R}^n) \) in terms of tent spaces.

Let \( T^p_2, 0 < p \leq \infty \), be the classical tent spaces introduced by Coifman et al in \([26, 27]\) (see Sect. 2 for precise definitions). Let \( T^2_2, \) denote the set of all \( f \in T^2_2 \) with compact support in \( \mathbb{R}^{n+1}_+ \). Denote by \( T^\infty_2, \) the closure of the set \( T^2_2, \) in \( T^\infty_2 \), and we endow \( T^\infty_2, \) with the norm of \( T^\infty_2 \). The following result is a special case of Proposition 3.3 in \([24]\), by taking the operator therein to be the Schrödinger operator \( \mathcal{L} \).

**Proposition 1.4** Suppose \( V \in \text{RH}_q \) for some \( q > n/2 \) and let \( \mathcal{L} = -\Delta + V \). Then \( f \in \text{CMO}_L \) if and only if \( f \in L^2(\mathbb{R}^n, (1 + |x|)^{(n+\beta)} \, dx) \) for some \( \beta > 0 \) and \( t^{\sqrt{L}} e^{-t^{\sqrt{L}}} (1 - e^{-t^{\sqrt{L}}}) f \in T^\infty_2, \) with

\[
\| f \|_{\text{CMO}_L} \approx \left\| t^{\sqrt{L}} e^{-t^{\sqrt{L}}} \left( I - e^{-t^{\sqrt{L}}} \right) f \right\|_{T^\infty_2}.
\]

However, the above proposition can not be used directly to show (i) of Theorem A. As a result, we have to establish a revised version of Proposition 1.4, Theorem B below.

**Theorem B** Suppose \( V \in \text{RH}_q \) for some \( q > n/2 \). Then \( f \in \text{CMO}_L \) if and only if \( f \in L^2(\mathbb{R}^n, (1 + |x|)^{(n+\beta)} \, dx) \) for some \( \beta > 0 \) and \( t^{\sqrt{L}} e^{-t^{\sqrt{L}}} f \in T^\infty_2, \) with

\[
\| f \|_{\text{CMO}_L} \approx \left\| t^{\sqrt{L}} e^{-t^{\sqrt{L}}} f \right\|_{T^\infty_2}.
\]
We note that analogous versions of Theorem B have been established for second-order divergence form elliptic operators in [28, 29]. The argument in the proof of Theorem B is based on a modification of techniques in [13, 28].

On the other hand, to prove (ii) of Theorem A, we will give another new characterization of $\text{CMO}_L(\mathbb{R}^n)$ in terms of limiting behaviors of mean oscillation.

**Theorem C** Suppose $V \in RH_q$ for some $q > n/2$ and let $\mathcal{L} = -\Delta + V$. The following statements are equivalent.

(a) $f$ is in $\text{CMO}_L(\mathbb{R}^n)$.
(b) $f$ is in the closure of $C_0(\mathbb{R}^n)$ in $\text{BMO}_L(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n)$ is the space of all continuous functions on $\mathbb{R}^n$ which vanish at infinity.
(c) $f$ is in $\mathcal{B}_L$, where $\mathcal{B}_L$ is the subspace of $\text{BMO}_L(\mathbb{R}^n)$ satisfying $\tilde{\gamma}_i(f) = 0$ for $1 \leq i \leq 5$, where

$$
\begin{align*}
\tilde{\gamma}_1(f) &= \lim_{a \to 0} \sup_{B: r_B \leq a} \left( |B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2}; \\
\tilde{\gamma}_2(f) &= \lim_{a \to \infty} \sup_{B: r_B \geq a} \left( |B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2}; \\
\tilde{\gamma}_3(f) &= \lim_{a \to \infty} \sup_{B: B \subseteq (B(0,a))^c} \left( |B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2}; \\
\tilde{\gamma}_4(f) &= \lim_{a \to \infty} \sup_{B: r_B \geq \max\{a, \rho(x_B)\}} \left( |B|^{-1} \int_B |f(x)|^2 \, dx \right)^{1/2}; \\
\tilde{\gamma}_5(f) &= \lim_{a \to \infty} \sup_{B \subseteq (B(0,a))^c \atop r_B \geq \rho(x_B)} \left( |B|^{-1} \int_B |f(x)|^2 \, dx \right)^{1/2}.
\end{align*}
$$

Here $x_B$ denotes the center of $B$, and the function $\rho$ is defined in (1.5).

(d) $f$ is in $\text{BMO}_L(\mathbb{R}^n)$ and satisfies $\tilde{\gamma}_1(f) = \tilde{\gamma}_2(f) = \tilde{\gamma}_3(f) = 0$.

Recall that Uchiyama [30] proved that $f \in \text{CMO}(\mathbb{R}^n)$ if and only if $f \in B$, where $B$ is the subspace of $\text{BMO}(\mathbb{R}^n)$ satisfying

$$
\begin{align*}
\lim_{a \to 0} \sup_{B: r_B \leq a} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx &= 0; \\ 
\lim_{a \to \infty} \sup_{B: r_B \geq a} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx &= 0; \\ 
\lim_{a \to \infty} \sup_{B: B \subseteq (B(0,a))^c} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx &= 0.
\end{align*}
$$

It should be pointed out that the above result was first announced by Neri [31] without proof and the three limiting conditions above are mutually independent (see [32,p. 49] for some examples). Theorem C may be seen as a generalization of the Neri–Uchiyama theorem from the CMO–$\Delta$ space to the CMO$_L$ space. Indeed, when $V \equiv 0$,
the auxiliary function \( \rho(x) \equiv \infty \) for each \( x \in \mathbb{R}^n \), then the last two requirements in (c) of Theorem C, \( \gamma_4(f) = 0 \) and \( \gamma_5(f) = 0 \), are trivial.

In the case of \( V \equiv 1 \), it is well known that \( \text{BMO}_{-\Delta+1} = \text{bmo} \), which is the dual of the local Hardy space \( h^1 \) ([16]). The pre-dual space of \( h^1 \) is a local version of \( \text{CMO} \), which can also be regarded as our \( \text{CMO}_{-\Delta+1}(\mathbb{R}^n) \). It was proved by Dafni in [32] that \( f \in \text{CMO}_{-\Delta+1} \) is equivalent to \( f \in \text{BMO}_{-\Delta+1} \) and \( \gamma_1(f) = \gamma_5(f) = 0 \). However, for general \( V \in \text{RH}_q, q > n/2 \), the situation is different. Theorem C states that \( f \in \text{CMO}_L \) is equivalent to \( f \in \text{BMO}_L \) and \( \gamma_1(f) = \gamma_3(f) = \gamma_5(f) = 0 \). Also, we will construct an example at the end of Sect. 4, which satisfies \( \gamma_1(f) = \gamma_5(f) = 0 \), while \( \gamma_3(f) \neq 0 \). This implies that \( \gamma_3(f) = 0 \) can not be deduced by \( \gamma_1(f) = \gamma_5(f) = 0 \).

The main difficulty of showing Theorem C arises from the implicit function \( \rho(x) \) occurring in \( \gamma_4(f) \) and \( \gamma_5(f) \). Concretely, even though it is known that \( \rho(x) \) is a slowly varying function (see Lemma 2.3), there is no a uniformly positive bound (from above or below) for \( \rho(x) \). For this reason, to verify the averaged behaviors of functions on balls in the case of \( r_B \geq \rho(x_B) \) becomes more subtle. For clarity, we will begin by showing the standard modifier is not sufficient to approximate a given function in \( B_L \) directly (see (4.3b) in Lemma 4.1), although such an approach has been successfully applied in [32, Theorem 6] to character the \( \text{CMO}_{-\Delta+1} \) space. The difficulty will be overcome in this article by combining a modified Uchiyama’s construction (see Lemma 4.3) and the standard modifier, relied heavily on properties of \( \rho(x) \).

The layout of the article is as follows. In Sect. 2, we recall some preliminary results, including the theory of tent spaces and the kernel estimates of the heat and Poisson semigroups of \( L \). Sect. 3 is mainly devoted to showing Theorem B, by combining ideas of [13, 24, 28]. Our purpose in Sect. 4 is to prove Theorem C, based on two auxiliary estimates following from the standard approximation to the identity and a modified Uchiyama’s construction, respectively. In Sect. 5, Theorem A is proved by applying Theorem B and Theorem C.

Throughout this article, the letter “C” or “c” will denote (possibly different) constants that are independent of the essential variables. By \( A \approx B \) (resp. \( A \lesssim B \)), we mean that there exists a positive constant \( C \) such that \( C^{-1}A \leq B \leq CA \) (resp. \( A \leq CB \)).

### 2 Preliminaries and Auxiliary Results Associated to Operators

In this section, we recall some basic definitions and properties of tent spaces and the critical radii function \( \rho(x) \).

#### 2.1 Tent Spaces

Let \( \Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\} \) be the standard cone (of aperture 1) with vertex \( x \in \mathbb{R}^n \). For any closed subset \( F \subseteq \mathbb{R}^n \), \( \mathcal{R}(F) := \bigcup_{x \in F} \Gamma(x) \). If \( O \) is an open subset of \( \mathbb{R}^n \), then the “tent” over \( O \), denoted by \( \mathcal{O} \), is given as \( \mathcal{O} = [\mathcal{R}(O^c)]^c \).
For any function $F(y, t)$ defined on $\mathbb{R}^{n+1}$, we will denote

$$
\mathcal{A}(F)(x) = \left( \int \int_{\Gamma(x)} |F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}
$$

and

$$
\mathcal{C}(F)(x) = \sup_{x \in B} \left( r_B^{-n} \int \int_B |F(y, t)|^2 \frac{dy dt}{t} \right)^{1/2}.
$$

As in [27], the tent space $T^2_p$ is defined as the space of functions $F$ such that $\mathcal{A}(F) \in L^p(\mathbb{R}^n)$ when $p < \infty$. The resulting equivalence classes are then equipped with the norm $\|F\|_{T^2_p} = \|\mathcal{A}(F)\|_p$. When $p = \infty$, the space $T^\infty_2$ is the class of functions $F$ for which $\mathcal{C}(F) \in L^\infty(\mathbb{R}^n)$ and the norm $\|F\|_{T^\infty_2} = \|\mathcal{C}(F)\|_\infty$. Let $T^p_{2,c}$ be the set of all $f \in T^p_2$ with compact support in $\mathbb{R}^{n+1}$. We denote by $T^\infty_{2,C}$ the closure of the set $T^2_{2,c}$ in $T^\infty_2$, and we endow $T^\infty_{2,C}$ with the norm of $T^\infty_2$.

Let $\mathcal{H}$ be the set of all $f \in T^\infty_2$ satisfying the following three conditions:

(i) $\eta_1(F) := \lim_{a \to 0} \sup_{B:B \subseteq (B(x, a))} \left( r_B^{-n} \int \int_B |F(y, t)|^2 \frac{dy dt}{t} \right)^{1/2} = 0$;

(ii) $\eta_2(F) := \lim_{a \to \infty} \sup_{B:B \subseteq (B(x, a))} \left( r_B^{-n} \int \int_B |F(y, t)|^2 \frac{dy dt}{t} \right)^{1/2} = 0$;

(iii) $\eta_3(F) := \lim_{a \to \infty} \sup_{B:B \subseteq (B(x, a))} \left( r_B^{-n} \int \int_B |F(y, t)|^2 \frac{dy dt}{t} \right)^{1/2} = 0$.

It can be verified that $\mathcal{H}$ is a closed linear subspace of $T^\infty_2$.

**Lemma 2.1** (a) $(T^\infty_{2,C})^* = T^1_2$, i.e., $T^1_2$ is the dual space of $T^\infty_{2,C}$.

(b) $f \in T^\infty_{2,C}$ if and only if $f \in \mathcal{H}$.

**Proof** (a) was proved in [33,Theorem 1.7]. (b) was proved in [24,Lemma 3.2].

2.2 Basic Properties of the Critical Radii Function $\rho(x)$

In this subsection, we recall some basic properties of the critical radii function $\rho(x)$ defined in (1.5), which were first proved by Z.W. Shen in [15].

**Lemma 2.2** ([15,Lemma 1.2]) Suppose $V \in RH_q$ for $q > 1$. There exists $C > 0$ such that, for $0 < r < R < \infty$,

$$
\frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq C \left( \frac{R}{r} \right)^{\frac{n-2}{q-2}} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y) dy.
$$
Lemma 2.3 ([15, Lemma 1.4].) Suppose \( V \in RH_q \) for some \( q > n/2 \). There exist \( c > 1 \) and \( k_0 \geq 1 \) such that for all \( x, y \in \mathbb{R}^n \),

\[
c^{-1} \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k_0} \rho(x) \leq \rho(y) \leq c \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{k_0} \rho(x). \tag{2.1}
\]

In particular, \( \rho(x) \approx \rho(y) \) when \( y \in B(x, r) \) and \( r \lesssim \rho(x) \).

Noting that \( \rho(x) > 0 \) for each \( x \in \mathbb{R}^n \), Lemma 2.3 implies that the implicit function \( \rho \) is locally bounded from above and below. This fact will be used frequently in the sequel.

2.3 Basic Properties of the Heat and Poisson Semigroups of Schrödinger Operators

Let \( \{ e^{-tL} \}_{t > 0} \) be the heat semigroup associated to \( L \):

\[
e^{-tL} f(x) = \int_{\mathbb{R}^n} K_t(x, y) f(y) \, dy, \quad f \in L^2(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad t > 0. \tag{2.2}
\]

Lemma 2.4 (see [9].) Suppose \( V \in RH_q \) for some \( q > n/2 \). For every \( N > 0 \) there exist constants \( C_N \) and \( c \) such that for \( x, y \in \mathbb{R}^n, \ t > 0, \)

(i) \( 0 \leq K_t(x, y) \leq C_N t^{-n/2} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \)

(ii) \( |\partial_t K_t(x, y)| \leq C_N t^{-n/2} \exp \left( -\frac{|x - y|^2}{ct} \right) \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \).

Denote by \( h_t(x) \) the kernel of the classical heat semigroup \( \{ e^{t\Delta} \}_{t > 0} \) on \( \mathbb{R}^n \). Then the following result is valid.

Lemma 2.5 (See [34, Proposition 2.16].) Suppose \( V \in RH_q \) for some \( q > n/2 \). There exists a non-negative Schwartz function \( \phi \) on \( \mathbb{R}^n \) such that

\[
|h_t(x - y) - K_t(x, y)| \leq \left( \frac{\sqrt{t}}{\rho(x)} \right)^{2-n/q} \varphi_t(x - y), \quad x, y \in \mathbb{R}^n, \ t > 0,
\]

where \( \varphi_t(x) = t^{-n/2} \varphi \left( x / \sqrt{t} \right) \).

The Poisson semigroup associated to \( L \) can be obtained from the heat semigroup through Bochner's subordination formula:

\[
e^{-t\sqrt{L}} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} \sqrt{u} e^{-u \frac{t}{4u} L} f(x) \, du. \tag{2.3}
\]

From (2.3), the semigroup kernels \( P_t(x, y) \), associated to \( e^{-t\sqrt{L}} \), satisfy the following estimates.
Lemma 2.6 ([35, Proposition 3.6]) Suppose \( V \in \text{RH}_q \) for some \( q > n/2 \). For any \( 0 < \delta < \min \left\{ 1, 2 - \frac{n}{q} \right\} \) and every \( N > 0 \), there exists a constant \( C = C_N \) such that

(i)

\[
|\mathcal{P}_t(x, y)| \leq C \frac{t}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}} \\
\left( 1 + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(x)} \right)^N + \left( \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(y)} \right)^{-N} ;
\]

(ii) For every \( m \in \mathbb{N} = \{1, 2, 3, \ldots\} \),

\[
|t^m \partial_t^m \mathcal{P}_t(x, y)| \leq C \frac{t^m}{(t^2 + |x - y|^2)^{\frac{n+2m}{2}}} \\
\left( 1 + \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(x)} \right)^N + \left( \frac{(t^2 + |x - y|^2)^{1/2}}{\rho(y)} \right)^{-N} ;
\]

(iii) For every \( m \in \mathbb{N} \),

\[
|t^m \partial_t^m e^{-t\sqrt{\lambda}}(1)(x)| \leq C \left( \frac{t}{\rho(x)} \right)^{\delta} \left( 1 + \frac{t}{\rho(x)} \right)^{-N} .
\]

Moreover, combining Lemma 2.5 and (2.3), it’s easy to verify, for each \( x \in \mathbb{R}^n \),

\[
|e^{-t\sqrt{\lambda}}(1)(x) - 1| = |e^{-t\sqrt{\lambda}}(1)(x) - e^{-t\sqrt{-\Delta}}(1)(x)| \leq C \left( \frac{t}{\rho(x)} \right)^{2-n/q} . \tag{2.4}
\]

For \( s > 0 \), we define

\[
\mathbb{F}(s) := \left\{ \psi : \mathbb{C} \to \mathbb{C} \text{ measurable} : |\psi(z)| \leq C \frac{|z|^s}{(1 + |z|^{2s})} \right\} ;
\]

Then for any non-zero function \( \psi \in \mathbb{F}(s) \), we have that \( \left( \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \). Denote \( \psi_t(z) := \psi(tz) \) for \( t > 0 \). It follows from the spectral theory in [36] that for any \( f \in L^2(\mathbb{R}^n) \),

\[
\left\{ \int_0^\infty \|\psi(t\sqrt{\lambda}) f\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right\}^{1/2} = \left\{ \int_0^\infty \left( \overline{\psi(t\sqrt{\lambda})} \psi(t\sqrt{\lambda}) f, f \right) \frac{dt}{t} \right\}^{1/2} \\
= \left\{ \left\{ \int_0^\infty |\psi|^2(t\sqrt{\lambda}) \frac{dt}{t} f, f \right\}^{1/2} \right\}^{1/2} \leq \kappa \|f\|_{L^2(\mathbb{R}^n)} ; \tag{2.5}
\]

where \( \kappa = \left\{ \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right\}^{1/2} \). The estimate will be often used in this article.
2.4 BMO Spaces

The following characterization theorem for $\text{BMO}_\mathcal{L}(\mathbb{R}^n)$ was proved in [9].

**Theorem 2.7** Let $V \neq 0$ be a non-negative potential in $\text{RH}_q$, for some $q > n/2$. The following statements are equivalent.

(i) $f$ is a function in $\text{BMO}_\mathcal{L}(\mathbb{R}^n)$;

(ii) $f \in L^2(\mathbb{R}^n, (1 + |x|)^{-n+\beta} \, dx)$ for some $\beta > 0$, and $\|t \sqrt{\mathcal{L}} e^{-t \sqrt{\mathcal{L}}} f(x)\|_{T^2_\infty} < \infty$;

(iii) $f \in L^p_{\text{Loc}}(\mathbb{R}^n)$ and $\|f\|_{\text{BMO}_\mathcal{L}} < \infty$, where $1 < p < \infty$ and

$$\|f\|_{\text{BMO}_\mathcal{L}} := \sup_B \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p \, dy \right)^{\frac{1}{p}} + \sup_{B: r_B \geq \rho(x_B)} \left( \frac{1}{|B(x_B, r_B)|} \int_{B(x_B, r_B)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

(iv) $f$ is in the dual space of $\mathcal{H}^1_{\mathcal{L}}(\mathbb{R}^n)$. Here, $\mathcal{H}^1_{\mathcal{L}}(\mathbb{R}^n)$ is defined by

$$\mathcal{H}^1_{\mathcal{L}}(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : \|f\|_{\mathcal{H}^1_{\mathcal{L}}} := \sup_{t > 0} \left| e^{-t \sqrt{\mathcal{L}}} f(x) \right|_{L^1} < \infty \right\}.$$

Moreover, the norms in above cases are equivalent:

$$\|f\|_{\text{BMO}_\mathcal{L}} \approx \|t \sqrt{\mathcal{L}} e^{-t \sqrt{\mathcal{L}}} f\|_{T^2_\infty} \approx \|f\|_{\text{BMO}_\mathcal{L}} \approx \|f\|_{(\mathcal{H}^1_{\mathcal{L}})^*}.$$

The following fact is used often below, which can be found in [9, Lemma 2].

**Lemma 2.8** There exists $C > 0$ such that, for any function $f \in \text{BMO}_\mathcal{L}$ and any ball $B(x, r)$ of $\mathbb{R}^n$ with $r < \rho(x)$, then

$$\left| \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy \right| \leq C \left( 1 + \log \frac{\rho(x)}{r} \right) \|f\|_{\text{BMO}_\mathcal{L}}.$$

**Remark 2.9** For further theory of BMO and CMO spaces associated to differential operators, we refer the reader to [10–13, 17, 25, 29, 37–41] and the references therein.

3 Proof of Theorem B

In this section, we will show an equivalent characterization for $\text{CMO}_\mathcal{L}(\mathbb{R}^n)$ by using tent spaces.

**Proof of Theorem B** Suppose $f \in \text{CMO}_\mathcal{L}(\mathbb{R}^n)$, then $f \in \text{BMO}_\mathcal{L}(\mathbb{R}^n)$. By Theorem 2.7, we have that $t \sqrt{\mathcal{L}} e^{-t \sqrt{\mathcal{L}}} f(x) \in T^2_\infty$ and $\|t \sqrt{\mathcal{L}} e^{-t \sqrt{\mathcal{L}}} f\|_{T^2_\infty} \approx \|f\|_{\text{BMO}_\mathcal{L}}$. Springer
We will prove \( \eta_1(t \sqrt{L} e^{-t \sqrt{L}} f) = \eta_2(t \sqrt{L} e^{-t \sqrt{L}} f) = \eta_3(t \sqrt{L} e^{-t \sqrt{L}} f) = 0 \), where 
\[
\left\{ \eta_i(t \sqrt{L} e^{-t \sqrt{L}} f) \right\}_{i=1}^3
\]
are defined in Sect. 2.

To this end, we will prove that there exists a positive constant \( c > 0 \) such that, for any ball \( B = B(x_B, r_B) \),
\[
\left( \frac{1}{|B|} \int_B \left| t \sqrt{L} e^{-t \sqrt{L}} f(x) \right|^2 \frac{dx \, dt}{t} \right)^{1/2} \leq c \sum_{k=0}^{\infty} 2^{-k} \delta_k(f, B), \tag{3.1}
\]
where
\[
\delta_k(f, B) = \sup_{B' : B' \subseteq 2^{k+2} B, r_{B'} \in [2^{-1} r_B, 2 r_B]} \left( \frac{1}{|B'|} \int_{B'} \left| (1 - e^{-r' \sqrt{L}}) f(x) \right|^2 \, dx \right)^{1/2}. \tag{3.2}
\]

Once the estimate (3.1) is proved, \( F := t \sqrt{L} e^{-t \sqrt{L}} f \in \mathcal{T}_{2,L}^\infty \) follows readily. Concretely, it is clear that for any \( k = 0, 1, \ldots, \), there holds \( \delta_k(f, B) \leq \| f \|_{\text{BMO}_L(\mathbb{R}^n)} \leq \| f \|_{\text{BMO}_L(\mathbb{R}^n)} \approx \| f \|_{\text{BMO}_L(\mathbb{R}^n)} \), where the \( \text{BMO}_L \)-norm is given in (1.6). Moreover, one may apply the condition \( \gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0 \) to obtain that for any given \( k \),
\[
\lim_{a \rightarrow 0} \sup_{B : r_B \leq a} \delta_k(f, B) = \lim_{a \rightarrow +\infty} \sup_{B : r_B \geq a} \delta_k(f, B) = \lim_{a \rightarrow +\infty} \sup_{B : B \subseteq (B(0,a))^c} \delta_k(f, B) = 0. \tag{3.3}
\]
It follows from (3.1) that
\[
\left( \frac{1}{|B|} \int_B \left| t \sqrt{L} e^{-t \sqrt{L}} f(x) \right|^2 \frac{dx \, dt}{t} \right)^{1/2} \leq c \sum_{k=0}^{k_0} 2^{-k} \delta_k(f, B) \\
+ c \sum_{k=k_0+1}^{\infty} 2^{-k} \| f \|_{\text{BMO}_L} \\
\leq c \sum_{k=0}^{k_0} 2^{-k} \delta_k(f, B) + c 2^{-k_0} \| f \|_{\text{BMO}_L}.
\]

Note that if \( k_0 \) is large enough, then the quantity \( 2^{-k_0} \| f \|_{\text{BMO}_L} \) is sufficiently small. Fix a \( k_0 \), we then use the property (3.3) to obtain \( \eta_1(F) = \eta_2(F) = \eta_3(F) = 0 \), as desired.

It suffices to prove estimate (3.1). As observed in [13], we rewrite
\[
f = \frac{1}{r_B} \int_{r_B}^{2r_B} (I - e^{-s \sqrt{L}}) f \, ds + \frac{1}{r_B} \int_{r_B}^{2r_B} e^{-s \sqrt{L}} f \, ds \\
= \frac{1}{r_B} \int_{r_B}^{2r_B} (I - e^{-s \sqrt{L}}) f \, ds + (r_B \sqrt{L})^{-1} e^{-r_B \sqrt{L}} (I - e^{-r_B \sqrt{L}}) f.
\]
for any $B \subset \mathbb{R}^n$. Then

$$\text{LHS of (3.1)} \leq \sup_{s \in [r_B, 2r_B]} \left( \frac{1}{|B|} \int_{B} \left| t \sqrt{L} e^{-t \sqrt{L}} \left( I - e^{-s \sqrt{L}} \right) f(x) \right|^2 \frac{dx dt}{t} \right)^{1/2}$$

$$+ \left( \int_{0}^{r_B} \frac{t^2}{r_B^2} \left\| e^{-(t+r_B) \sqrt{L}} \left( I - e^{-r_B \sqrt{L}} \right) f \right\|_{L^\infty(B)}^2 \frac{dt}{t} \right)^{1/2}.$$

For any given $s \in [r_B, 2r_B]$, let

$$F_{s,0}(y) := \chi_{2B}(y) \left( I - e^{-s \sqrt{L}} \right) f(y) \quad \text{and}$$

$$F_{s,k}(y) := \chi_{2^{k+1}B \setminus 2^kB}(y) \left( I - e^{-s \sqrt{L}} \right) f(y) \quad \text{for } k \geq 1 \text{ and } y \in \mathbb{R}^n.$$

Then

$$\text{LHS of (3.1)} \leq \sup_{s \in [r_B, 2r_B]} \left( \frac{1}{|B|} \int_{\mathbb{R}^{n+1}} \left| t \sqrt{L} e^{-t \sqrt{L}} F_{s,0}(x) \right|^2 \frac{dx dt}{t} \right)^{1/2}$$

$$+ \sup_{s \in [r_B, 2r_B]} \sum_{k=1}^{\infty} \left( \int_{0}^{r_B} \left\| t \sqrt{L} e^{-t \sqrt{L}} F_{s,k} \right\|_{L^\infty(B)}^2 \frac{dt}{t} \right)^{1/2}$$

$$+ \sum_{k=0}^{\infty} \left( \int_{0}^{r_B} \frac{t^2}{r_B^2} \left\| e^{-(t+r_B) \sqrt{L}} F_{r_B,k} \right\|_{L^\infty(B)}^2 \frac{dt}{t} \right)^{1/2}$$

$$=: I_0 + \sum_{k=1}^{\infty} I_k + \sum_{k=0}^{\infty} I I_k. \quad (3.4)$$

By (2.5), we have

$$I_0 \leq \sup_{s \in [r_B, 2r_B]} \left( \frac{1}{|B|} \int_{2B} \left| \left( I - e^{-s \sqrt{L}} \right) f(x) \right|^2 \frac{dx}{2} \right)^{1/2}.$$

Note that there exists a positive constant $N_0 = N_0(n)$ such that for every fixed $s \in [r_B, 2r_B]$, the ball $2B$ can be covered by finite-overlapped balls $\{B(x_i, s)\}_{i=1}^{N_0}$, where each $B(x_i, s) \subseteq 4B$. Hence,

$$I_0 \leq C \delta_0(f, B).$$

For any $s \in [r_B, 2r_B]$, $x \in B$ and $k \geq 1$, it follows from (ii) of Lemma 2.6 that

$$\left| t \sqrt{L} e^{-t \sqrt{L}} F_{s,k}(x) \right| \leq C \frac{t}{2^{k(n+1)}r_B} \frac{1}{|B|} \int_{2^{k+1}B} \left| \left( I - e^{-s \sqrt{L}} \right) f(y) \right| dy.$$

Moreover, it can be verified that for any ball $B(x_B, 2^{k+1}r_B)$, there exists a corresponding collection of balls $B_1^{(k)}$, $B_2^{(k)}$, ..., $B_{N_k}^{(k)}$ such that
(a) each ball $B_i^{(k)}$ is of the radius $s$ and $B_i^{(k)} \subseteq B(x_B, (2+2^{k+1})r_B) \subseteq B(x_B, 2^{k+2}r_B)$;
(b) $B(x_B, 2^{k+1}r_B) \subseteq \bigcup_{i=1}^{N_k} B_i^{(k)}$;
(c) there exists a constant $c > 0$ independent of $k$ such that $N_k \leq c 2^{kn}$;
(d) $\sum_{i=1}^{N_k} \chi_{B_i^{(k)}}(x) \leq K$ for each $x \in B(x_B, 2^{k+1}r_B)$, where $K$ is independent of $k$.

From the properties (a) – (d) above, we may apply Hölder’s inequality to obtain

$$\left| t \sqrt{L} e^{-t \sqrt{L}} F_{s,k}(x) \right| \leq \frac{t}{2^{k(n+1)}r_B} \sum_{i=1}^{N_k} \left( \frac{1}{|B_k|} \int_{B_k} \left| (I - e^{-t \sqrt{L}}) f(y) \right|^2 dy \right)^{1/2} \leq C \frac{t}{2^k r_B^2} \delta_k(f, B),$$

which gives

$$I_k \leq C \left( \int_0^{r_B} \frac{t^2}{r_B^2} \frac{dt}{t} \right)^{1/2} 2^{-k} \delta_k(f, B) \leq C 2^{-k} \delta_k(f, B).$$

A similar argument can be used to show $II_k \leq C 2^{-k} \delta_k(f, B)$ for $k \geq 0$, by noting that the kernel $\mathcal{P}_{t+r_B}(x, y)$ of $e^{-(t+r_B)\sqrt{L}}$, where $0 \leq t \leq r_B$, satisfies

$$\left| \mathcal{P}_{t+r_B}(x, y) \right| \leq \begin{cases} Cr_B^{-n}, & \text{if } |x-y| < 2r_B, \\ Cr_B |x-y|^{-(n+1)}, & \text{otherwise,} \end{cases}$$

which follows from (i) of Lemma 2.6.

Plugging all estimates of terms $I_k$ and $II_k$ ($k \geq 0$) into (3.4), we deduce (3.1), and then get $t \sqrt{L} e^{-t \sqrt{L}} f(x) \in T^{\infty}_{2, L}$.

Conversely, suppose that $f \in L^2(\mathbb{R}^n, (1 + |x|)^{-(n+\beta)} dx)$ for some $\beta > 0$ and $t \sqrt{L} e^{-t \sqrt{L}} f(x) \in T^{\infty}_{2, L}$. Let us prove $f \in \text{CMO}_L$. In fact, by applying an argument similar to that of [17, Proposition 5.1], one can prove that the following identity

$$\int_{\mathbb{R}^n} f(x) g(x) \, dx = 4 \int_{\mathbb{R}^n+1} \left( t \sqrt{L} e^{-t \sqrt{L}} f \right)(x) \left( t \sqrt{L} e^{-t \sqrt{L}} g \right)(x) \frac{dx \, dt}{t} \quad (3.5)$$

holds for any $f \in \text{BMO}_L$ and $g \in H^1_L \cap L^2$. Then the aimed result of $f \in \text{CMO}_L$ easily follows by a simple modification of [24, Proposition 3.3] in which the representation formula (3.17) is replaced by (3.5). We have completed the proof of Theorem B. \hfill \Box

### 4 Proof of Theorem C

Due to (ii) of Theorem 1.2, the main difficulty of showing Theorem C is to prove the implication (c) $\Rightarrow$ (a) of Theorem C. That is, for any given $f \in B^c_L$, one needs to approximate it in BMO norm by using $C^\infty_c(\mathbb{R}^n)$ functions. To this end, we first

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make an attempt to show Theorem C by using a standard mollifier. Such an approach has been successfully applied to character a local version of $\text{CMO}(\mathbb{R}^n)$, which can be regarded as $\text{CMO}_{-\Delta+1}$; see [32] for details. However, Lemma 4.1 below tells us this approach is not completely effective to approximate a given function in $\mathcal{B}_\mathcal{L}$ directly, which in turn reveals a certain difference between $\text{CMO}_{-\Delta+1}$ and $\text{CMO}_{-\Delta+\nu}$, due to the lack of uniform bounds for the variable function $\rho(x)$ defined by $V$.

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a radial bump function satisfying:

$$\text{supp } \phi \subseteq B(0, 1), \quad 0 \leq \phi \leq 1 \quad \text{and} \quad \int \phi(x) \, dx = 1. \quad (4.1)$$

Let $\phi_t(x) := t^{-n} \phi(x/t)$ for every $x \in \mathbb{R}^n$ and $t > 0$. For any given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define

$$A_t(f)(x) := \phi_t * f(x) = \int_{\mathbb{R}^n} \phi_t(x - y) f(y) \, dy, \quad x \in \mathbb{R}^n. \quad (4.2)$$

It’s clear that $A_t(f) \in C^\infty(\mathbb{R}^n)$. For $z \in \mathbb{R}^n$, denote $\tau_z(B) := \{x - z : x \in B\}$. We have the following lemma.

**Lemma 4.1** Suppose $V \in \text{RH}_q$ for some $q > n/2$. Let $f \in \mathcal{B}_\mathcal{L}$, where $\mathcal{B}_\mathcal{L}$ is the space defined in Theorem C. Let $\phi$ and $A_t(f)$ be given in (4.1) and (4.2), respectively.

(i) $A_t(f)$ is uniformly continuous on $\mathbb{R}^n$ and $\tilde{\gamma}_1(A_t(f)) = \tilde{\gamma}_2(A_t(f)) = \tilde{\gamma}_3(A_t(f)) = \tilde{\gamma}_4(A_t(f)) = 0$ for each $0 < t < 1$.

(ii) For any $\varepsilon > 0$, there exist positive constants $R > > 1$ and $t_0 << 1$ such that for all $0 < t < t_0$,

$$\|A_t(f) - f\|_{\text{BMO}} < \varepsilon, \quad \text{and}$$

$$\|A_t(f) - f\|_{\text{BMO}_\mathcal{L}(\mathbb{R}^n)} \leq \varepsilon + \sup_{B(x_B, r_B) \subseteq (B(0, (2c+2)R))^c} \sup_{\rho(x_B) < R, \rho(x) < t} \left( \frac{1}{\left| \tau_z(B) \right|} \int_{\tau_z(B)} |f(x)|^2 \, dx \right)^{1/2}, \quad (4.3b)$$

where $c$ is the constant in Lemma 2.3.

(iii) If $f \in \mathcal{B}_\mathcal{L}$ with compact support, then $A_t(f) \in C_c^\infty(\mathbb{R}^n)$ and so $A_t(f) \in \mathcal{B}_\mathcal{L}$. Also,

$$\lim_{t \to 0} \|A_t(f) - f\|_{\text{BMO}_\mathcal{L}} = 0.$$

**Proof** (i). The uniform continuity of $A_t(f)$ for $f \in \text{BMO}$ was first proved by Dafni in [32]. Of course, it holds for $f \in \mathcal{B}_\mathcal{L}$ since $\mathcal{B}_\mathcal{L} \subseteq \text{BMO}$.

For any $t > 0$ and ball $B := B(c_0, r_0) \subseteq \mathbb{R}^n$, One may apply Minkowski’s inequality and the fact $\int \phi_t(x) \, dx = 1$ to obtain
\[
\left( \frac{1}{|B|} \int_B \left| A_t(f)(x) - (A_t(f))_B \right|^2 \, dx \right)^{1/2} \\
= \left( \frac{1}{|B|} \int_B \left| \int_{\mathbb{R}^n} \left( f(x - z) - \frac{1}{|B|} \int_B f(y - z) \, dy \right) \phi_t(z) \right|^2 \, dx \right)^{1/2} \\
\leq \int_{\mathbb{R}^n} \left( \frac{1}{|B|} \int_B \left| f(x - z) - \frac{1}{|B|} \int_B f(y - z) \, dy \right|^2 \, dx \right)^{1/2} \phi_t(z) \, dz \\
= \int_{\mathbb{R}^n} \left( \frac{1}{|\tau_z(B)|} \int_{\tau_z(B)} \left| f(x) - f_{\tau_z(B)} \right|^2 \, dx \right)^{1/2} \phi_t(z) \, dz \\
\leq \sup_{|z| \leq 1} \left( \frac{1}{|\tau_z(B)|} \int_{\tau_z(B)} \left| f(x) - f_{\tau_z(B)} \right|^2 \, dx \right)^{1/2} . \tag{4.4}
\]

Similarly,
\[
| (A_t(f))_B | \leq \left[ \left( \left| A_t(f) \right|^2 \right)_B \right]^{1/2} \leq \sup_{|z| \leq 1} \left( \frac{1}{|\tau_z(B)|} \int_{\tau_z(B)} |f(x)|^2 \, dx \right)^{1/2} . \tag{4.5}
\]

Combining these two estimates and \( f \in \mathcal{B}_C \), we can verify directly \( \widetilde{\gamma}_1(A_t(f)) = \widetilde{\gamma}_2(A_t(f)) = \widetilde{\gamma}_3(A_t(f)) = 0 \). If \( r_0 \geq \max\{a, \rho(c_0)\} \) and \( a >> 1 \), then \( \tau_z(B) \subseteq 2B \) for any \( |z| < 1 \). Thus, for any \( 0 < t < 1 \), one has
\[
\widetilde{\gamma}_4(A_t(f)) \leq \lim_{a \to \infty} \sup_{r_0 \geq \max\{a, \rho(c_0)\}} \sup_{|z| \leq 1} \left( \frac{1}{|\tau_z(B)|} \int_{\tau_z(B)} |f(x)|^2 \, dx \right)^{1/2} \\
\leq 2^{n/2} \lim_{a \to \infty} \sup_{r_0 \geq \max\{a, \rho(c_0)\}} \left( \frac{1}{|2B|} \int_{2B} |f(x)|^2 \, dx \right)^{1/2} = 0.
\]

However, we may not obtain \( \widetilde{\gamma}_5(A_t(f)) = 0 \).

(ii). Now we start to prove estimates (4.3a) and (4.3b). Note that for any \( \varepsilon > 0 \), it follows from \( \widetilde{\gamma}_i(f) = 0 \) for \( 1 \leq i \leq 5 \) that there exist positive constants \( \delta << 1 \) and \( R >> 1 \) such that
\[
\sup_{B: r_B \leq \delta} \left( \frac{1}{|B|} \int_B \left| f(x) - f_B \right|^2 \, dx \right)^{1/2} < \varepsilon , \tag{4.6a}
\]
\[
\sup_{B: r_B \geq R} \left( \frac{1}{|B|} \int_B \left| f(x) - f_B \right|^2 \, dx \right)^{1/2} < \varepsilon , \tag{4.6b}
\]
\[
\sup_{B: B \subseteq (B(0,R))^c} \left( \frac{1}{|B|} \int_B \left| f(x) - f_B \right|^2 \, dx \right)^{1/2} < \varepsilon , \tag{4.6c}
\]
\[
\sup_{B=B(x_B, r_B): r_B \geq \max\{R, \rho(x_B)\}} \left( \frac{1}{|B|} \int_B \left| f(x) \right|^2 \, dx \right)^{1/2} < \varepsilon . \tag{4.6d}
\]
and

\[
\sup_{B=\overline{B}(x_B,r_B): B \subseteq \overline{B}(B(0,r))} \left( \frac{1}{|B|} \int_B |f(x)|^2 \, dx \right)^{1/2} < \varepsilon. \tag{4.6e}
\]

We start by proving (4.3a). For any fixed ball \( B_0 := B(x_0, r_0) \), let us consider the following cases.

**Case 1.** \( r_0 \leq \delta \). In this case, one may apply (4.4) and (4.6a) that for any \( 0 < t < 1 \),

\[
\left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x) - (A_t(f) - f)_{B_0}|^2 \, dx \right)^{1/2} \leq 2 \sup_{|z| \leq t} \left( \frac{1}{|\tau_z(B_0)|} \int_{\tau_z(B_0)} |f(x) - f_{\tau_z(B_0)}|^2 \, dx \right)^{1/2} < 2\varepsilon.
\]

**Case 2.** \( r_0 > \delta \) and \( B_0 \cap B(0, 2R) \neq \emptyset \). In this case, we just need to consider two subcases as follows.

**Subcase 2-1.** \( \delta < r_0 < R \). In this subcase, \( B_0 \subseteq B(0, 4R) \) and so

\[
\left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x) - (A_t(f) - f)_{B_0}|^2 \, dx \right)^{1/2} \leq 2 \left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x)|^2 \, dx \right)^{1/2} \leq 2 \delta^n \| A_t(f) - f \|_{L^2(B(0, 4R))}.
\]

Notice that \( f \in L^2_{\text{loc}}(\mathbb{R}^n) \) and \( \{ \phi_t \}_{0 < t < 1} \) is an approximate identity as \( t \to 0 \), there exists a constant \( t_\varepsilon > 0 \) small enough such that

\[
\| A_t(f) - f \|_{L^2(B(0, 4R))} < \frac{\delta^n \varepsilon}{2} \quad \text{for} \quad 0 < t < t_\varepsilon. \tag{4.7}
\]

From the above, we have

\[
\left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x) - (A_t(f) - f)_{B_0}|^2 \, dx \right)^{1/2} < \varepsilon \quad \text{for} \quad 0 < t < t_\varepsilon.
\]

**Subcase 2-2.** \( r_0 > R \). It follows from (4.4) and (4.6b) that

\[
\left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x) - (A_t(f) - f)_{B_0}|^2 \, dx \right)^{1/2} \leq 2 \sup_{|z| \leq t} \left( \frac{1}{|\tau_z(B_0)|} \int_{\tau_z(B_0)} |f(x) - f_{\tau_z(B_0)}|^2 \, dx \right)^{1/2}.
\]
\[
\leq 2 \sup_{B : r_B \geq R} \left( \frac{1}{|B|} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2} < 2\varepsilon.
\]

**Case 3.** \( r_0 > \delta \) and \( B_0 \cap B(0, 2R) = \emptyset. \) In this case, it’s clear \( \tau_z(B_0) \subseteq (B(0, R))^c \) for \( 0 < t < 1 \) since \( R > 0 \) is sufficiently large. This, combined with (4.4) and (4.6c), deduces that

\[
\left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x) - (A_t(f) - f)_{B_0}|^2 \, dx \right)^{1/2} \\
\leq 2 \sup_{|z| \leq t} \left( \frac{1}{|\tau_z(B_0)|} \int_{\tau_z(B_0)} |f(x) - f_{\tau_z(B_0)}|^2 \, dx \right)^{1/2} \\
\leq 2 \sup_{B : B \subseteq (B(0, R))^c} \left( \frac{1}{|B|} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2} < 2\varepsilon.
\]

Combining estimates in Cases 1-3, we obtain (4.3a), as desired.

It remains to verify the estimate (4.3b). Let \( \varepsilon, \) \( R \) be constants in (4.6a) – (4.6e). For any \( B_0 = B(x_0, r_0) \) satisfying \( r_0 \geq \rho(x_0), \) consider the following cases.

**Case I.** \( r_0 \geq R, \) i.e., \( r_0 \geq \max\{R, \rho(x_0)\}. \) In this case, \( \tau_z(B_0) \subseteq 2B_0 \) for any \( |z| < 1. \) This, together with (4.5) and (4.6d), deduces that

\[
\left( \frac{1}{|B_0|} \int_{B_0} |A_t(f)(x) - f(x)|^2 \, dx \right)^{1/2} \leq 2 \sup_{|z| \leq 1} \left( \frac{1}{|\tau_z(B_0)|} \int_{\tau_z(B_0)} |f(x)|^2 \, dx \right)^{1/2} \\
\leq 2^{n+1} \sup_{|z| \leq 1} \left( \frac{1}{|2B_0|} \int_{2B_0} |f(x)|^2 \, dx \right)^{1/2} \\
\leq 2^{n+1} \sup_{B(x_B, r_B) : r_B \geq \max\{R, \rho(x_B)\}} \left( |B|^{-1} \int_B |f(x)|^2 \, dx \right)^{1/2} \\
\lesssim \varepsilon.
\]

**Case II.** \( \rho(x_0) \leq r_0 < R. \) We need to consider the position of \( B_0. \)

**Subcase II-1.** \( B(x_0, r_0) \cap B(0, 2(c + 1)R) \neq \emptyset. \) Then \( B_0 \subseteq B(0, 2(c + 2)R) \) due to \( r_0 \leq R. \) Besides, by Lemma 2.3,

\[
r_0 \geq \rho(x_0) \geq c^{-1} \left\{ 1 + \frac{|x_0|}{\rho(0)} \right\}^{-k_0} \rho(0) \geq sc^{-1} \left\{ 1 + \frac{(2c + 4)R}{\rho(0)} \right\}^{-k_0} \rho(0) := C_R, \rho(0).
\]

Note that there exists a constant \( \tilde{t}_\varepsilon > 0 \) small enough such that

\[
\|A_t(f) - f\|_{L^2(B(0,(2c+4)R))} < (C_R, \rho(0))^{n} \varepsilon \quad \text{for} \quad 0 < t < \tilde{t}_\varepsilon. \quad (4.8)
\]
Hence,

\[
\left(\frac{1}{|B_0|} \int_{B_0} |A_r(f)(x) - f(x)|^2 \, dx\right)^{1/2} \leq \frac{1}{r_0^n} \|A_r(f) - f\|_{L^2(B(0,(2c+4)R))} < \varepsilon \quad \text{for} \quad 0 < t < \tilde{\varepsilon}.
\]

**Subcase II-2.** \(B(x_0, r_0) \cap B(0, 2(c + 1)R) = \emptyset\).

- If \(\rho(x_0) \geq t\), then it follows from Lemma 2.3 again to see that for any \(|z| \leq t\),
  \[
  \rho(x_0 - z) \leq c \left(1 + \frac{|z|}{\rho(x_0)}\right)^{k_0/(k_0 + 1)} \rho(x_0) \leq c (t + \rho(x_0)) \leq 2cr_0.
  \]

This, combined with the fact \(2c \cdot \tau_z(B_0) \subseteq (B(0, R))^c\), allows us to apply (4.5) and (4.6e) to obtain

\[
\left(\frac{1}{|B_0|} \int_{B_0} |A_r(f)(x) - f(x)|^2 \, dx\right)^{1/2} \leq 2 \sup_{|z| \leq t} \left(\frac{1}{|\tau_z(B_0)|} \int_{2c \cdot \tau_z(B_0)} |f(x)|^2 \, dx\right)^{1/2} \\
\leq 2(2c)^n \sup_{B(x_B,r_B) : B \subseteq (B(0, R))^c, r_B \geq \rho(x_B)} \left(\frac{1}{|B|} \int_B |f(x)|^2 \, dx\right)^{1/2} \lesssim \varepsilon.
\]

- Consider \(\rho(x_0) < t\). For any \(|z| < t\), it holds \(\rho(x_0 - z) \leq c \rho(x_0) \frac{1}{\rho(x_0) + t} (\rho(x_0) + t)^{k_0+1}\). Compared to \(t\), \(\rho(x_0)\) may be much smaller. So it fails to bound \(\rho(x_0 - z)\) for \(|z| \leq t\) by \(C \cdot \rho(x_0)\). This is why the second term of RHS of (4.3b) appears.

From the above, (ii) is proved.

(iii). It is a direct corollary of (i) and (ii). \(\square\)

**Remark 4.2** Consider \(\mathcal{L} = -\Delta + 1\). In this case \(\rho(x)\) is constant. By (4.4) and (4.5), it can be seen that \(f \in B_{\mathcal{L}}\) implies \(A_r(f) \in B_{\mathcal{L}}\). Moreover, \(\lim_{t \to 0} \|A_r(f) - f\|_{\text{BMO}_{\mathcal{L}}} = 0\) follows from (4.3b). However, for \(\mathcal{L} = -\Delta + V(x)\), in order to obtain the same result, the additional condition that \(f\) has compact support is needed.

Lemma 4.1 hints that before smoothing \(f \in B_{\mathcal{L}}\), some data pre-processing should be considered. Recall that Uchiyama [30, pp. 166-167] gave an explicit construction to approximate a function in \(B\) satisfying (1.8a) – (1.8c) by step functions. We will use a modified Uchiyama’s construction to approximate a given function in \(B_{\mathcal{L}}\) by step functions with compact supports, which relies heavily on the properties of the function \(\rho\). This result is useful for proving the aimed Theorem C.

Let \(Q := Q(c_Q, \ell(Q)) \subseteq \mathbb{R}^n\) be a cube of center \(c_Q\) and sidelength \(\ell(Q)\). For any constant \(c > 0\), denote \(cQ := Q(c_Q, c\ell(Q))\). Observe the following facts: for
each ball \( B \subseteq \mathbb{R}^n \), there exists a cube \( Q \subseteq \mathbb{R}^n \) satisfying \( B \supseteq Q \) and \( B \subseteq \sqrt{n}Q \). Furthermore, there exists a constant \( C = C(n) > 1 \) independent of \( B \) and \( f \) such that

\[
\frac{1}{C} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} \leq \left( \frac{1}{|B|} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2} \leq C \left( \frac{1}{|\sqrt{n}Q|} \int_{\sqrt{n}Q} |f(x) - f_{\sqrt{n}Q}|^2 \, dx \right)^{1/2}.
\]

Hence we can substitute cubes for balls (simultaneously replacing \( r_B \) and \( f_B \) by \( \ell(Q) \) and \( f_Q \), respectively) in the definitions of \( \tilde{\gamma}_i(f) \) for \( f \in B_L \), where \( 1 \leq i \leq 5 \).

Therefore, for any given \( f \in B_L \) and \( \epsilon > 0 \), it follows from \( \tilde{\gamma}_1(f) = 0 \) and \( \tilde{\gamma}_i(f) = 0 \) \((2 \leq i \leq 5)\), respectively, to see there exist two integers \( I_\epsilon >> 1 \) and \( J_\epsilon >> 1 \) such that

\[
\sup_{Q: \ell(Q) \leq 2^{l_\epsilon}} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} < \frac{\epsilon}{5 \cdot 4^n}, \quad (4.9a)
\]

\[
\sup_{Q: \ell(Q) \geq 2^{l_\epsilon}} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} < \frac{\epsilon}{5 \cdot 4^n}, \quad (4.9b)
\]

\[
\sup_{Q: \ell(Q) \geq \max\{2^{l_\epsilon}, \rho(c_Q)\}} \left( \frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} < \frac{\epsilon}{2}, \quad (4.9c)
\]

and

\[
\sup_{Q \subseteq (Q(0, 2^{l_\epsilon+1}))^c, \ell(Q) \geq \rho(c_Q)} \left( \frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} < \frac{\epsilon}{2}. \quad (4.9d)
\]

Note that

\[
R_k := Q(0, 2^{k+1}) = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq 2^k \text{ for } 1 \leq i \leq n \right\} \quad (4.10)
\]

is a union of \( 2^n \) dyadic cubes of sidelength \( 2^k \) for each \( k \in \mathbb{Z} \). Besides, the set \( R_{k+1} \setminus R_k \) can be divided into mutually disjoint dyadic cubes with sidelength \( 2^l \) for any fixed \( l \leq k \). These, combined with properties (4.9a)–(4.9e), motivate us to give the following construction which is partly adapted from [30]: throughout this proof, whenever we mention \( Q_x \) for \( x \in \mathbb{R}^n \), it always denotes the unique dyadic cube that contains \( x \) as follows.

- for \( x \in R_{l_\epsilon} \), let \( Q_x \) be the dyadic cube of sidelength \( 2^{l_\epsilon-2} \) that contains \( x \);
• for $x \in \mathcal{R}_{m+1} \setminus \mathcal{R}_m$ whenever the integer $m \geq J_\varepsilon$, let $Q_x$ be the dyadic cube of sidelength $2^{m-J_\varepsilon-J_{\varepsilon}-1}$ that contains $x$.

Now define
\[ A_\varepsilon(f)(x) := \frac{1}{|Q_x|} \int_{Q_x} f(y) \, dy, \quad x \in \mathbb{R}^n. \quad (4.11) \]
which, together with (4.9a) and (4.9c), implies that
\[ \left( \frac{1}{|Q_x|} \int_{Q_x} |f(y) - A_\varepsilon(f)(y)|^2 \, dy \right)^{1/2} \leq \frac{\varepsilon}{5 \cdot 4^n} \text{ for all } Q_x \subseteq \mathbb{R}^n. \quad (4.12) \]

**Lemma 4.3** Suppose $V \in \mathcal{RH}_q$ for some $q > n/2$. Let $f \in \mathcal{B}_L$, where $\mathcal{B}_L$ is the space defined in Theorem C. For any $\varepsilon > 0$, let $I_\varepsilon$ and $J_\varepsilon$ be given in (4.9a) – (4.9e), and let $A_\varepsilon(f)$ be the function defined in (4.11). Then
\[ \| f - A_\varepsilon(f) \|_{\text{BMO}} \lesssim \varepsilon. \quad (4.13) \]
In addition, assume that $I_\varepsilon$ is sufficiently large such that $2^{-I_\varepsilon-1} \leq \inf_{x \in \mathcal{R}_{J_\varepsilon+2}} \rho(x)$. We have
\[ \| f - A_\varepsilon(f) \|_{\text{BMO}_L} \lesssim \varepsilon. \quad (4.14) \]

**Remark 4.4** We note that the assumption $2^{-I_\varepsilon-1} \leq \inf_{x \in \mathcal{R}_{J_\varepsilon+2}} \rho(x)$ is workable and non-contradictory. Indeed, using Lemma 2.3, it holds
\[ \rho(x) \geq c^{-1} \left( 1 + \frac{|x|}{\rho(0)} \right)^{-k_0} \rho(0) \geq c^{-1} \left( 1 + \frac{\sqrt{n} J_{\varepsilon+2}}{\rho(0)} \right)^{-k_0} \rho(0), \quad \text{for any } x \in \mathcal{R}_{J_\varepsilon+2}. \]
Taking $2^{-I_\varepsilon-1} \leq c^{-1} \left( 1 + \frac{\sqrt{n} J_{\varepsilon+2}}{\rho(0)} \right)^{-k_0} \rho(0)$, yields the desired assumption.

**Proof of Lemma 4.3** Firstly, we claim that $A_\varepsilon(f)$ has the following two properties: there exists a positive integer $M_\varepsilon \gtrsim I_\varepsilon + J_\varepsilon$ such that
\[ \sup_{x \in \mathbb{R}^n \setminus \mathcal{R}_{M_\varepsilon}} |A_\varepsilon(f)(x)| < \varepsilon/2, \quad \text{(P1)} \]
and
\[ \sup \left\{ |A_\varepsilon(f)(x) - A_\varepsilon(f)(y)| : Q_x \cap Q_y \neq \emptyset \right\} < \varepsilon, \quad \text{(P2)} \]
where $\overline{Q}$ is the closure of $Q$ in $\mathbb{R}^n$. 

\[ \text{ Springer} \]
Let us prove the above claim. It follows from Lemma 2.3 that for any \( x \in \mathcal{R}_{m+1} \setminus \mathcal{R}_m \) with \( m \geq J_\varepsilon \),
\[
\rho(x) \leq C \left\{ 1 + \frac{|x|}{\rho(0)} \right\}^{\frac{k_0}{2^{k_0+1}}} \rho(0) \leq C \cdot 2^{\frac{k_0}{2^{k_0+1}}},
\]
where \( C \) is a constant dependent on \( \rho(0) \). Meanwhile, \( \ell(Q_x) = 2^{-m-J_\varepsilon - J_\varepsilon - 1} \), so one has
\[
\rho(y) \leq \ell(Q_x) \quad \text{for all} \quad y \in Q_x, \quad \text{if} \quad m \geq (k_0 + 1) \left( \log_2 C + J_\varepsilon + J_\varepsilon + 1 \right) =: M_\varepsilon.
\]

In particular, denote the center of \( Q_x \) by \( c_{Q_x} \), then \( \rho(c_{Q_x}) \leq \ell(Q_x) \). Hence, (P1) is a straightforward consequence of (4.9e).

Then we turn to (P2). Suppose \( Q_x \cap Q_y \neq \emptyset \). Let \( Q_{x,y} \) be the smallest cube that contains \( Q_x \) and \( Q_y \), and we remind that \( Q_{x,y} \) may not be a dyadic cube. Assume \( \ell(Q_x) \leq \ell(Q_y) \), then it follows from the definition of dyadic cubes \( \{Q_z\}_{z \in \mathbb{R}^n} \) that \( \ell(Q_x) = \ell(Q_y)/2 \) if \( \ell(Q_x) \neq \ell(Q_y) \) and \( |Q_{x,y}| \leq 2^n |Q_x| \). Note that
\[
|A_\varepsilon(f)(x) - A_\varepsilon(f)(y)| \leq |f_{Q_x} - f_{Q_y}| + |f_{Q_y} - f_{Q_{x,y}}| \leq 2 \cdot 3^n \int_{Q_{x,y}} |f(z) - f_{Q_{x,y}}| dz.
\]

It suffices to show (P2) in the following two cases. In the case of \( x, y \in \mathcal{R}_{J_\varepsilon+2} \), we have that \( \ell(Q_x) \leq \ell(Q_y) \leq 2^{-J_\varepsilon} \) and \( \ell(Q_{x,y}) \leq 2^{-J_\varepsilon + 1} \). Then (P2) follows from (4.9a). In the case of \( x, y \notin \mathcal{R}_{J_\varepsilon+1} \), we have \( Q_{x,y} \subseteq (\mathcal{R}_{J_\varepsilon})^c = (Q(0,2^{J_\varepsilon+1}))^c \). Then (P2) follows from (4.9c).

With (P1) and (P2) at our disposal, we now show (4.13) and (4.14).

Denote by \( Q := (c_{Q_x}, \ell(Q)) \) the cube in \( \mathbb{R}^n \). Let’s prove (4.13) by considering the following cases.

**Case I.** \( \ell(Q) < \frac{1}{8} \max \{ \ell(Q_x) : Q_x \cap Q \neq \emptyset \} \).

By the construction of \( \{Q_z\}_{z \in \mathbb{R}^n} \), it is not difficult to show the fact: if \( Q_x \cap Q \neq \emptyset \), \( Q_y \cap Q \neq \emptyset \) and \( \ell(Q) < \frac{1}{8} \max \{ \ell(Q_x) : Q_x \cap Q \neq \emptyset \} \), then
\[
\ell(Q_x)/\ell(Q_y) \in \left\{ \frac{1}{2}, 1, 2 \right\} \quad \text{and} \quad \overline{Q_x} \cap \overline{Q_y} \neq \emptyset.
\] (4.15)

One can compute
\[
\left( \frac{1}{|Q|} \int_Q |f(x) - A_\varepsilon(f)(x) - (f - A_\varepsilon(f))_Q|^2 \, dx \right)^{1/2} \leq \left( \frac{1}{|Q|} \int_Q |A_\varepsilon(f)(x) - (A_\varepsilon(f))_Q|^2 \, dx \right)^{1/2} + \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2}.
\] (4.16)
Note that \(x, y \in Q\) implies \(Q_x, Q_y \in \{Q_z : Q_z \cap Q \neq \emptyset\}\) and by (4.15) we get \(Q_x \cap Q_y \neq \emptyset\). Then one may apply (P1) and (P2) to obtain

\[
\left( \frac{1}{|Q|} \int_Q \left| A_\varepsilon(f)(x) - (A_\varepsilon(f))_Q \right|^2 \, dx \right)^{1/2} \leq \left( \frac{1}{|Q|^2} \int_Q \int_Q \left| A_\varepsilon(f)(x) - A_\varepsilon(f)(y) \right|^2 \, dy \, dx \right)^{1/2} \leq \varepsilon. \tag{4.17}
\]

Consider \(\left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2}\).

- if \(Q \cap R_{J_\varepsilon} \neq \emptyset\), then \(\ell(Q) < \frac{1}{8} \max\{\ell(Q_x) : Q_x \cap Q \neq \emptyset\}\) \(\leq 2^{-L^2/2}\). This allows us to apply (4.9a) to obtain

\[
\left( \frac{1}{|Q|} \int_Q \left| f(x) - f_Q \right|^2 \, dx \right)^{1/2} < \frac{\varepsilon}{5 \cdot 4^n};
\]

- if \(Q \cap R_{J_\varepsilon} = \emptyset\), then it follows from (4.9c) that

\[
\left( \frac{1}{|Q|} \int_Q \left| f(x) - f_Q \right|^2 \, dx \right)^{1/2} < \frac{\varepsilon}{5 \cdot 4^n}.
\]

This, combined with (4.17), implies

\[
\left( \frac{1}{|Q|} \int_Q \left| f(x) - A_\varepsilon(f)(x) - (f(x) - A_\varepsilon(f))_Q \right|^2 \, dx \right)^{1/2} < 2\varepsilon.
\]

**Case II.** \(\ell(Q) \geq \frac{1}{8} \max\{\ell(Q_x) : Q_x \cap Q \neq \emptyset\}\).

In this case \(\bigcup_{Q_x \cap Q \neq \emptyset} Q_x \subseteq 20Q\). By (4.12), one can write

\[
\left( \frac{1}{|Q|} \int_Q \left| f - A_\varepsilon(f) - (f - A_\varepsilon(f))_Q \right|^2 \, dx \right)^{1/2} \leq 2 \left( \int_Q \left| f(y) - A_\varepsilon(f)(y) \right|^2 \, dy \right)^{1/2} \leq 2 \left( \frac{1}{|Q|} \int_{Q_x} \left| Q_x \right| \left| f(y) - A_\varepsilon(f)(y) \right|^2 \, dy \right)^{1/2} \leq 2 \left( \frac{\varepsilon}{5 \cdot 4^n} \left( \frac{\left| \bigcup_{Q_x \cap Q \neq \emptyset} Q_x \right|}{|Q|} \right)^{1/2} \leq \frac{20^{n/2}}{4^n \varepsilon}, \tag{4.18}
\]

as desired.

Combining the two cases above, we obtain that \(\|f - A_\varepsilon(f)\|_{BMO} \lesssim \varepsilon\).
Lastly, we prove (4.14). It suffices to prove

$$\sup_{Q: \ell(Q) \geq \rho(c_Q)} \left( \frac{1}{|Q(c_Q, \ell(Q))|} \int_{Q(c_Q, \ell(Q))} |f - A_\epsilon(f)|^2 \, dx \right)^{1/2} \lesssim \varepsilon. \quad (4.19)$$

If \( \ell(Q) \geq \frac{1}{8} \max \{ \ell(Q_x) : Q_x \cap Q \neq \emptyset \} \), we may use (4.18) to obtain

$$\left( \frac{1}{|Q(c_Q, \ell(Q))|} \int_{Q(c_Q, \ell(Q))} |f(x) - A_\epsilon(f)(x)|^2 \, dx \right)^{1/2} \lesssim \varepsilon.$$

It remains to consider the case of \( \ell(Q) < \frac{1}{8} \max \{ \ell(Q_x) : Q_x \cap Q \neq \emptyset \} \) and \( \ell(Q) \geq \rho(c_Q) \). It follows from (4.15) that \( \ell(Q) \leq \ell(Q_x) \) whenever \( Q_x \cap Q \neq \emptyset \).

We claim that there holds

$$Q \cap R_{J_{k+1}} = \emptyset.$$ 

In fact, if \( Q \cap R_{J_{k+1}} \neq \emptyset \), then \( \ell(Q) \leq 2^{-I_{k+2}} \) and \( Q \subseteq R_{J_{k+2}} \). The assumption that \( I_{k+1} \) is sufficiently large such that \( 2^{-I_{k+1}} \leq \min_{x \in R_{J_{k+2}}} \rho(x) \), gives \( \ell(Q) < \rho(c_Q) \), which contradicts our condition.

If \( Q_x \cap Q \neq \emptyset \), then

$$|c_Q - c_{Q_x}| \leq \sqrt{n} (\ell(Q) + \ell(Q_x)) / 2 \leq \sqrt{n} \ell(Q_x).$$

By using Lemma 2.3 and \( \rho(c_Q) \leq \ell(Q) \leq \ell(Q_x) \), one can compute

$$\rho(c_{Q_x}) \leq c \left( \frac{\rho(c_Q) + \sqrt{n} \ell(Q_x)}{\rho(c_Q)} \right)^{k_0/(k_0+1)} \rho(c_Q) \leq c \left( \sqrt{n} + 1 \right) \ell(Q_x).$$

Denote \( C_1 := c \left( \sqrt{n} + 1 \right) \) and \( Q_x^* := Q(c_{Q_x}, C_1 \ell(Q_x)) \). Clearly, \( \rho(c_{Q_x}^*) \leq \ell(Q_x^*) \).

The fact \( Q \cap R_{J_{k+1}} = \emptyset \) implies that \( Q_x^* \subseteq (R_{J_{k+2}})^c \) if \( J_{k+1} \) is chosen large enough. For any \( x \in Q \), we have

$$|A_\epsilon(f)(x)| = |f_{Q_x^*}| \leq (C_1)^{n/2} \left( \frac{1}{|Q_x^*|} \int_{Q_x^*} |f(y)|^2 \, dy \right)^{1/2} \lesssim \varepsilon,$$

where in the last inequality we used (4.9e). Hence \( \|A_\epsilon(f)\|_{L^\infty(Q)} \lesssim \varepsilon \). This, combined with (4.9e) and the fact \( Q \subseteq (R_{J_{k+2}})^c \) and \( \ell(Q) \geq \rho(Q) \), gives that

$$\left( \frac{1}{|Q|} \int_Q |f(x) - A_\epsilon(f)(x)|^2 \, dx \right)^{1/2} \leq \left( \frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} + \|A_\epsilon(f)\|_{L^\infty(Q)} \leq \varepsilon.$$ 

The proof of Lemma 4.3 is completed. \( \square \)
We are now in a position to show Theorem C, by combining Lemma 4.1 and Lemma 4.3.

**Proof of Theorem C**  The proof follows from the sequence of implications

\[(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) \quad \text{and} \quad (c) \Leftrightarrow (d).\]

The implication \((a) \Rightarrow (b)\) follows directly from (ii) of Theorem 1.2 and the fact \(C^\infty_c(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)\).

**Proof of “(b) \Rightarrow (c)”**. We first show that \(f \in C_0(\mathbb{R}^n) \Rightarrow f \in B_L\).

Observe that \(f \in C_0(\mathbb{R}^n)\) implies \(f \in \text{BMO}_L(\mathbb{R}^n)\) by the simple fact \(C_0 \subseteq L^\infty \subseteq \text{BMO}_L\).

Note that

\[
\lim_{a \to 0} \sup_{B: r_B \leq a} \left( |B|^{-1} \int_B |f(x) - f_B|^2 \, dx \right)^{1/2} \leq \lim_{a \to 0} \sup_{x, y \in B: r_B \leq a} |f(x) - f(y)|,
\]

which, together with the uniform continuity of \(f \in C_0\), gives \(\tilde{\gamma}_1(f) = 0\).

Since \(f \in C_0\), then for any given \(\varepsilon > 0\), there exists a constant \(N_\varepsilon > 0\) such that \(|f(x)| \leq \varepsilon\) whenever \(|x| \geq N_\varepsilon\). For every ball \(B\) with \(r_B > a\), where \(a > 0\) is sufficiently large, one has

\[
\left( |B|^{-1} \int_B |f(x)|^2 \, dx \right)^{1/2} \leq \frac{\|f\|_{L^2(B(0, N_\varepsilon) \cap B)}}{|B|^{1/2}} + \frac{\|f\|_{L^\infty(B(0, N_\varepsilon))}}{|B|^{1/2}} \frac{|B \setminus B(0, N_\varepsilon)|^{1/2}}{|B|^{1/2}} \lesssim \frac{\|f\|_{L^\infty N_\varepsilon}}{a^{n/2}} + \varepsilon.
\]

This says that for any given \(\varepsilon > 0\), there exists \(a = a(f, \varepsilon)\) sufficiently large, such that

\[
\sup_{B: r_B \geq a} \left( \frac{1}{|B|} \int_B |f(x)|^2 \, dx \right)^{1/2} < 2\varepsilon,
\]

which yields

\[
\tilde{\gamma}_2(f)(x) = 0 \quad \text{and} \quad \tilde{\gamma}_4(f) = 0.
\]

By using \(\lim_{|x| \to \infty} f(x) = 0\), we have

\[
\lim_{a \to \infty} \sup_{B: B \subset B(0, a)^c} \left( \frac{1}{|B|} \int_B |f(x)|^2 \, dx \right)^{1/2} \leq \lim_{a \to \infty} \sup_{|x| \geq a} |f(x)| = 0.
\]
which gives
\[ \tilde{\gamma}_3(f) = 0 \quad \text{and} \quad \tilde{\gamma}_5(f) = 0. \]

Thus, we have shown \( C_0 \subseteq B_{\mathcal{L}} \), as desired.

To complete the proof of \( C_0 \subseteq B_{\mathcal{L}} \), it suffices to show that \( B_{\mathcal{L}} \) is closed in \( \text{BMO}_{\mathcal{L}} \). Suppose that \( f \in \text{BMO}_{\mathcal{L}} \) and \( f_k \in B_{\mathcal{L}}, \ k \in \mathbb{N}, \) satisfying \( \lim_{k \to \infty} \| f_k - f \|_{\text{BMO}_{\mathcal{L}}} = 0 \). We will prove \( f \in B_{\mathcal{L}} \).

For any ball \( B \subseteq \mathbb{R}^n \) and \( k \in \mathbb{N} \), it follows from Theorem 2.7 that
\[
\left( \frac{1}{|B|} \int_B \left| f(x) - f_B \right|^2 \, dx \right)^{1/2} \lesssim \left( \frac{1}{|B|} \int_B \left| f(x) - f_B \right| \, dx \right)^{1/2} \]
and
\[
\left( \frac{1}{|B|} \int_B \left| f(x) \right| \, dx \right)^{1/2} \lesssim \left( \frac{1}{|B|} \int_B \left| f(x) \right| \, dx \right)^{1/2}.
\]

Hence, it follows from \( f_k \in B_{\mathcal{L}} \) that \( \tilde{\gamma}_j(f) = 0 \) for \( 1 \leq j \leq 5 \). It implies \( f \in B_{\mathcal{L}} \).

We completed the proof of “(b) \Rightarrow (c)”.

**Proof of “(c) \Rightarrow (a)”**. Let \( f \in B_{\mathcal{L}} \). We will show that for any given \( \varepsilon > 0 \), there exists a function \( F_\varepsilon \in C_\infty^\infty(\mathbb{R}^n) \) such that
\[ \| f - F_\varepsilon \|_{\text{BMO}_{\mathcal{L}}} \lesssim \varepsilon. \] (4.21)

Firstly, let \( I_\varepsilon, J_\varepsilon, M_\varepsilon, R_m, Q_x \) and \( A_\varepsilon(f) \) be as in the proof of Lemma 4.3. By Remark 4.4, we can assume that \( 2^{-l_\varepsilon-1} \leq \min_{x \in R_{M_\varepsilon+2}} \rho(x) \). Thus it follows from Lemma 4.3 that
\[ \| f - A_\varepsilon(f) \|_{\text{BMO}_{\mathcal{L}}} \lesssim \varepsilon. \] (4.22)

This, together with (P1), gives
\[
\| f - A_\varepsilon(f) \chi_{R_{M_\varepsilon+2}} \|_{\text{BMO}_{\mathcal{L}}} \leq \| f - A_\varepsilon(f) \|_{\text{BMO}_{\mathcal{L}}} + \| A_\varepsilon(f) \chi_{(R_{M_\varepsilon+2})^c} \|_{\text{BMO}_{\mathcal{L}}} \lesssim \varepsilon + \| A_\varepsilon(f) \chi_{(R_{M_\varepsilon+2})^c} \|_{L^\infty} \lesssim \varepsilon.
\] (4.22)

By Theorem 2.7 again,
\[
\left( \frac{1}{|B|} \int_B \left| A_\varepsilon(f) \chi_{R_{M_\varepsilon+2}}(x) - (A_\varepsilon(f) \chi_{R_{M_\varepsilon+2}})_B \right|^2 \, dx \right)^{1/2} \lesssim \| f - A_\varepsilon(f) \chi_{R_{M_\varepsilon+2}} \|_{\text{BMO}_{\mathcal{L}}} + \left( \frac{1}{|B|} \int_B \left| f(x) - (f)_B \right|^2 \, dx \right)^{1/2}
\]
and
\[
\left( \frac{1}{|B|} \int_B |A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2}(x)|^2 \, dx \right)^{1/2} \lesssim \|f - A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2}\|_{BMO_\mathcal{C}} + \left( \frac{1}{|B|} \int_B |f(x)|^2 \, dx \right)^{1/2}.
\]

These two estimates, together with (4.22), ensure that the estimates (4.6a)-(4.6e) still hold with the parameter \( R > 2^{M_{\varepsilon} + 4} \) whenever replacing \( f \) by \( A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2} \) (the constants therein should be changed accordingly). Recalling that \( A_t \) is defined in (4.2), it is clear that \( A_t (A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2}) \in C_\varepsilon^\infty (\mathcal{R}_{M_{\varepsilon} + 3}) \) for any \( t \leq 1 \). It follows from the proof of (ii) of Lemma 4.1 that there exists \( t_\varepsilon < 1 \) sufficiently small such that
\[
\|A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2} - A_{t_\varepsilon} (A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2})\|_{BMO_\mathcal{C}} \lesssim \varepsilon + \sup_{B : B \subseteq B(0,R)^c, |z| \leq 1} \sup_{B(\varepsilon)} \left( \frac{1}{|\tau_\varepsilon(B)|} \int_{\tau_\varepsilon(B)} |A_\varepsilon(f)(x) \chi_{R_{M_{\varepsilon}} + 2}(x)|^2 \, dx \right)^{1/2} \lesssim \varepsilon.
\]

Therefore, we obtain (4.21) by taking \( F_\varepsilon = A_{t_\varepsilon} (A_\varepsilon(f) \chi_{R_{M_{\varepsilon}} + 2}) \). The proof of “(c) ⇒ (a)” is completed.

The implication “(c) ⇒ (d)” is obvious.

Proof of “(d) ⇒ (c)”. We first show \( \tilde{\gamma}_4(f) = 0 \) can be deduced by \( \tilde{\gamma}_5(f) = 0 \). To this end, we use the notation in Lemma 4.3, then
\[
\tilde{\gamma}_4(f) \leq \lim_{a \to 2^{M_{\varepsilon} + 2}} \sup_{Q \cap \mathcal{R}_{M_{\varepsilon}} = \emptyset : \ell(Q) \geq 2 \rho(Q)} \left( \frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} + \lim_{a \to 2^{M_{\varepsilon} + 2}} \sup_{Q \cap \mathcal{R}_{M_{\varepsilon}} \neq \emptyset : \ell(Q) \geq a} \left( \frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} =: I(f) + II(f).
\]

Observe that \( I(f) < \varepsilon/2 \) by (4.9e) which is a consequence of \( \tilde{\gamma}_5(f) = 0 \). Besides, for any given \( Q \) involved in the term \( II(f) \), it’s clear that \( Q \cap \mathcal{R}_{M_{\varepsilon}} \neq \emptyset \) and \( Q \cap (\mathcal{R}_{M_{\varepsilon} + 1})^c \neq \emptyset \), due to \( \ell(Q) > 2^{M_{\varepsilon} + 2} \) is assumed therein. Therefore, there exists a positive integer \( \kappa_Q \) such that
\[
Q \subseteq \mathcal{R}_{\kappa_Q} \quad \text{and} \quad |\mathcal{R}_{\kappa_Q}| \leq 2^{3n} |Q|.
\]

Hence, for such \( Q \),
\[
\frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \leq \frac{1}{|Q|} \int_{\mathcal{R}_{\kappa_Q}} |f(x)|^2 \, dx \leq \frac{\|f\|_{L^2(\mathcal{R}_{M_{\varepsilon}})}^2}{|Q|} + \frac{2^{3n}}{|\mathcal{R}_{\kappa_Q}|} \sum_{Q_s \subseteq \mathcal{R}_{\kappa_Q} \setminus \mathcal{R}_{M_{\varepsilon}}} \int_{Q_s} |f(x)|^2 \, dx
\]

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\[
\leq \frac{\|f\|^2_{L^2(\mathcal{R}_M)}}{|Q|} + 2^{3n} \cdot \left(\frac{\varepsilon}{2}\right)^2,
\]

where the last inequality we used (P1), which is a consequence of \(\hat{\gamma}_5(f) = 0\). Meanwhile, \(\|f\|_{L^2(\mathcal{R}_M)}\) is bounded and independent of \(Q\), by noticing \(f \in L^2_{loc}(\mathbb{R}^n)\). Combining these estimates above, we obtain

\[
\tilde{\gamma}_4(f) \lesssim \varepsilon
\]

for arbitrary given \(\varepsilon > 0\).

Next, we will show that \(\tilde{\gamma}_2(f) = 0\) can be deduced by \(\tilde{\gamma}_3(f) = 0\) and \(\tilde{\gamma}_5(f) = 0\). Similarly to the argument in (i) above,

\[
\tilde{\gamma}_2(f) \leq \lim_{a > 2M + 2, a \to \infty} \sup_{Q \subseteq \mathcal{R}_M, \ell(Q) \geq a} \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} + 2 \lim_{a > 2M + 2, a \to \infty} \sup_{Q \subseteq \mathcal{R}_M, \ell(Q) \geq a} \left( \frac{1}{|Q|} \int_Q |f(x)|^2 \, dx \right)^{1/2} =: I'(f) + 2 \cdot II(f),
\]

where \(II(f)\) is the second term occurred in (i) above, and we have shown that \(II(f) \lesssim \varepsilon\) holds by \(\tilde{\gamma}_5(f) = 0\). Observe that \(I'(f) < \varepsilon / (5 \cdot 2^n)\) by (4.9c) which is a consequence of \(\tilde{\gamma}_3(f) = 0\). Hence we obtain \(\tilde{\gamma}_2(f) = 0\).

The proof of Theorem C is completed. \(\square\)

**Remark 4.5** Assume that \(\sup_{x \in \mathbb{R}^n} \rho(x) < +\infty\), \(\tilde{\gamma}_3(f) = 0\) is a consequence of \(\tilde{\gamma}_1(f) = \tilde{\gamma}_5(f) = 0\). We refer to [32] in the case of \(\rho \equiv 1\). However, in general, we could not deduce \(\tilde{\gamma}_3(f) = 0\) by combining \(\tilde{\gamma}_1(f) = 0\) and \(\tilde{\gamma}_5(f) = 0\). In fact, one can construct a function \(f \in \text{BMO}_L\) satisfying \(\tilde{\gamma}_1(f) = \tilde{\gamma}_5(f) = 0\), while \(\tilde{\gamma}_3(f) \neq 0\).

To clarify this fact, consider the potential

\[
V(x) = \frac{1}{|x|^{2-\varepsilon}}, \quad \text{where} \quad \varepsilon = 2 - (n/q_0)
\]

(4.23)

for any given \(q_0 > n/2\). Then \(V \in RH_q\) for any \(q < q_0\). See [15, p. 545].

We first observe that

\[
\rho(x) \approx |x|^{1-\frac{\varepsilon}{2}} \quad \text{for} \quad |x| >> 1.
\]

(4.24)

In fact, if \(|x| >> 1\), \(y \in B(x, r)\) and \(r > |x|/2\), we have \(|y| \leq |x| + r \leq 3r\). Then

\[
I_r(x) := \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy = \frac{1}{r^{n-2}} \int_{B(x,r)} \frac{1}{|y|^{2-\varepsilon}} \, dy \geq \frac{v_n}{2^{2-\varepsilon}} r^\varepsilon \geq \frac{v_n}{3^{2-\varepsilon}} \left( \frac{|x|}{2} \right)^\varepsilon > 1,
\]

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where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$. So, the conditions $|x| >> 1$ and $I_r(x) \leq 1$ imply $r \leq |x|/2$. Furthermore, it yields $|y| \approx |x|$ whenever $y \in B(x, r)$. One has

$$I_r(x) \approx \frac{r^2}{|x|^{2-\varepsilon}},$$

and therefore $I_r(x) \approx 1$ is equivalent to $r \approx |x|^{1-\varepsilon/2}$. By the definition (1.5), we showed (4.24).

Next, let’s choose a function $\varphi \in C_c(\mathbb{R}^n)$ satisfying $\text{supp } \varphi \subseteq B(0, 1)$ and $\|\varphi\|_{L^\infty} \approx \|\varphi\|_{BMO} \approx 1$. Denote $x_k := (3^k, 0, \cdots, 0) \in \mathbb{R}^n$ for $k \in \mathbb{N}$. We define

$$f(x) = \sum_{k=1}^{\infty} \varphi(x - x_k), \ x \in \mathbb{R}^n.$$

Notice that the support sets of $\{\varphi(\cdot - x_k)\}_{k=1}^{\infty}$ are mutually disjoint. Then we have that $f$ is uniformly continuous and bounded on $\mathbb{R}^n$. So $f \in BMO\mathcal{L}$ and $\tilde{\gamma}_1(f) = 0$. Besides, $\tilde{\gamma}_3(f) \geq \|\varphi\|_{BMO} \approx 1$.

Lastly, let us estimate $\tilde{\gamma}_5(f)$. Suppose $a >> 1$ and $B := B(x_B, r_B) \subseteq B(0, a)^c$ with $r_B \geq \rho(x_B)$. Noting that $|x_B| \geq a + r_B$, it follows from (4.24) that $r_B \geq |x_B|^{1-\varepsilon/2} \geq (a + r_B)^{1-\varepsilon/2}$.

$$|B|^{-1} \int_B \sum_{k=1}^{\infty} \left| \varphi(x - x_k) \right|^2 dx = |B|^{-1} \sum_{k : B \cap B(x_k, 1) \neq \emptyset} \int_B \left| \varphi(x - x_k) \right|^2 dx \lesssim \frac{\log_3 r_B}{(a + r_B)^{(1-\varepsilon)n}} \lesssim a^{-\frac{q}{2}(1-\varepsilon)}, \quad (4.25)$$

where in the first inequality above we used one observation $\# \{k : B \cap B(x_k, 1) \neq \emptyset\} \lesssim \log_3 r_B$. This gives

$$\tilde{\gamma}_5(f) \lesssim \lim_{a \to \infty} a^{-\frac{q}{2}(1-\varepsilon)} = 0.$$

5 Proof of Theorem A

With Theorem B and Theorem C at our disposal, we now prove Theorem A. Let us begin by introducing the following key estimates on the space derivative of the Poisson kernel of $e^{-t\sqrt{L}}$, which were first proved by Jiang and Li in [19].

**Lemma 5.1** ([19, Proposition 5.2].) Let $V \in RH_q$ for some $q > n/2$. Suppose $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+t}} dx < \infty$. Then there exists a constant $C > 0$ such that for any ball $B = B(x_B, r_B)$, it holds

$$\int_0^{r_B} \int_B \left| t \nabla_x e^{-t\sqrt{L}} f \right|^2 \frac{dx dt}{t} \leq C \int_0^{2r_B} \int_{2B} \left( t^2 \partial^2_t e^{-t\sqrt{L}} f \right) \left| e^{-t\sqrt{L}} f \right|$$
Moreover, for any constant \( c_0 \neq 0 \), it holds

\[
\int_0^T \int_B |t \nabla_x e^{-t \sqrt{L}} f(x)|^2 \frac{dx}{t} \leq C \int_0^T \int_{2B} \left( |t^2 \partial_t^2 e^{-t \sqrt{L}} f| e^{-t \sqrt{L}} f - c_0 \right) dx \frac{dt}{t} + \frac{t^2}{r_B^2} \left| e^{-t \sqrt{L}} f - c_0 \right|^2 \frac{dx}{t}
+ C \int_0^T \int_{2B} t |e^{-t \sqrt{L}} f| \left| e^{-t \sqrt{L}} f - c_0 \right| V dx \frac{dt}{t}.
\]

We now prove the main result of this article, Theorem A.

**Proof of Theorem A**

(i). If \( u \in \text{HCMO}_L^q(\mathbb{R}_+^{n+1}) \), then \( u \in \text{HMO}_L^q(\mathbb{R}_+^{n+1}) \). By

Theorem 1.1 in [18] (or Theorem 1.1 in [19]), there exists a function \( f \in \text{BMO}_L(\mathbb{R}^n) \) such that

\[
u(x, t) = e^{-t \sqrt{L}} f(x)
\]

and \( \| f \|_{\text{BMO}_L(\mathbb{R}^n)} \leq C \| u \|_{\text{HMO}_L(\mathbb{R}_+^{n+1})} \). It follows from

the definition of \( u = e^{-t \sqrt{L}} f \in \text{HCMO}_L^q(\mathbb{R}^n) \) that

\[\sqrt{t} e^{-t \sqrt{L}} f \in T^\infty_{2C}.\]

Applying Theorem B, we have \( f \in \text{CMO}_L(\mathbb{R}^n) \) as desired.

(ii). If \( f \in \text{CMO}_L(\mathbb{R}^n) \), then \( f \in \text{BMO}_L(\mathbb{R}^n) \). By noting that \( V \in \text{RH}_q \) for some

\( q \geq (n + 1)/2 \), it follows from Theorem 1.1 in [19] that \( u(x, t) := e^{-t \sqrt{L}} f(x) \in \text{HMO}_L(\mathbb{R}_+^{n+1}) \) and \( \| u \|_{\text{HMO}_L(\mathbb{R}_+^{n+1})} \leq C \| f \|_{\text{BMO}_L(\mathbb{R}^n)} \). Moreover, using Theorem B, we know

\( t \partial_t u(x, t) = \sqrt{t} e^{-t \sqrt{L}} f \in T^\infty_{2C} \). Thus, to prove \( u \in \text{HCMO}_L^q \), it remains to prove

\( \tilde{\beta}_1(f) = \tilde{\beta}_2(f) = \tilde{\beta}_3(f) = 0 \), where

\[
\tilde{\beta}_1(f) = \lim_{a \to 0} \sup_{B : r_B \leq a} \left( r_B^{-n} \int_0^{r_B} \int_B |t \nabla_x e^{-t \sqrt{L}} f(x)|^2 \frac{dx}{t} \right)^{1/2},
\]

\[
\tilde{\beta}_2(f) = \lim_{a \to \infty} \sup_{B : r_B \geq a} \left( r_B^{-n} \int_0^{r_B} \int_B |t \nabla_x e^{-t \sqrt{L}} f(x)|^2 \frac{dx}{t} \right)^{1/2},
\]

\[
\tilde{\beta}_3(f) = \lim_{a \to \infty} \sup_{B \subset (B(0, a))^c} \left( r_B^{-n} \int_0^{r_B} \int_B |t \nabla_x e^{-t \sqrt{L}} f(x)|^2 \frac{dx}{t} \right)^{1/2}.
\]

To this end, for any given ball \( B = B(x_B, r_B) \subseteq \mathbb{R}^n \), split the function \( f \) into three

parts as follows

\[f = (f - f_{4B}) \chi_{4B} + (f - f_{4B}) \chi_{(4B)^c} + f_{4B} =: f_1 + f_2 + f_3,
\]

where \( 4B := B(x_B, 4r_B) \). For \( i = 1, 2, 3 \), we denote

\[J_{B,i} := \left( r_B^{-n} \int_0^{r_B} \int_B |t \nabla_x e^{-t \sqrt{L}} f_i(x)|^2 \frac{dx}{t} \right)^{1/2}.
\]

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Then
\[
\left( r^{-n} \int_0^{r_B} \int_B \left| t \nabla x e^{-t \sqrt{L}} f(x) \right|^2 \frac{dx \, dt}{t} \right)^{1/2} \leq \sum_{i=1}^{3} J_{B,i},
\]

Let us first estimate \( J_{B,1} \). By the well-known fact that the Riesz transform \( \nabla x L^{-1/2} \) is bounded on \( L^2(\mathbb{R}^n) \), one may obtain
\[
J_{B,1} = \left( r^{-n} \int_0^{r_B} \int_B \left| \nabla x L^{-1/2} t \sqrt{L} e^{-t \sqrt{L}} f_1(x) \right|^2 \frac{dx \, dt}{t} \right)^{1/2}
\leq C \left( r^{-n} \int_0^{\infty} \int_{\mathbb{R}^n} \left| t \sqrt{L} e^{-t \sqrt{L}} f_1(x) \right|^2 \frac{dx \, dt}{t} \right)^{1/2}
\leq C \left( r^{-n} \int_{4B} \left| f - f_{4B} \right|^2 dx \right)^{1/2},
\]
where we used (2.5) in the last inequality above.

Consider the second term \( J_{B,2} \). One may apply (5.1) to obtain
\[
J_{B,2} \leq C \left( r^{-n} \int_0^{2r_B} \int_{2B} \left( t^2 \partial_t^2 e^{-t \sqrt{L}} f_2(x) \right) \left| e^{-t \sqrt{L}} f_2(x) \right| \, dx \, dt \right)^{1/2}
\leq C \left( \frac{t^2}{r_B^2} \left| e^{-t \sqrt{L}} f_2(x) \right|^2 \right) \, dx \, dt^{1/2}
\]
Then for any \( x \in 2B \) and \( t < 2r_B \), it follows from (i) and (ii) of Lemma 2.6 to see that for \( m = 0, 2, \)
\[
\left| t^m \partial_t^m e^{-t \sqrt{L}} f_2(x) \right| \leq C \int_{(4B)^c} \frac{t}{|y - x_B|^{n+1}} \left| f(y) - f_{4B} \right| \, dy
\leq C \sum_{k=1}^{\infty} \int_{4^{k+1}B \setminus 4^k B} \left| f(y) - f_{4B} \right| \, dy
\leq C \left( \frac{t}{r_B} \right) \sum_{k=1}^{\infty} \frac{1}{4^{-k}} \int_{4^{k+1}B} \left| f(y) - f_{4B} \right| \, dy.
\]

Note that
\[
\frac{1}{|4^{k+1}B|} \int_{4^{k+1}B} \left| f(y) - f_{4B} \right| \, dy \leq \frac{1}{|4^{k+1}B|} \int_{4^{k+1}B} \left| f(y) - f_{4^{k+1}B} \right| \, dy
+ \left| f_{4B} - f_{4^{k+1}B} \right|
\leq \frac{1}{|4^{k+1}B|} \int_{4^{k+1}B} \left| f(y) - f_{4^{k+1}B} \right| \, dy
\]
\[ + \sum_{j=1}^{k} |f_{4j B} - f_{4j+1 B}| \]
\[ \leq (4^n k + 1) \sup_{1 \leq j \leq k} \frac{1}{|4j+1 B|} \]
\[ \times \int_{4j+1 B} |f(y) - f_{4j+1 B}| \, dy. \]

Then for any \( x \in 2B, t < 2r_B \) and \( m = 0, 2, \)
\[ \left| t^m \partial_t^m e^{-t \sqrt{E}} f_2(x) \right| \leq C \left( \frac{t}{r_B} \right)^{\infty} \sum_{k=1}^{2^{-k}} \sigma_k(f, B), \]
where
\[ \sigma_k(f, B) := \sup_{0 \leq j \leq k} \frac{1}{|4j+1 B|} \int_{4j+1 B} |f(y) - f_{4j+1 B}| \, dy. \]

This gives
\[ J_{B,2} \leq C \sum_{k=1}^{\infty} 2^{-k} \sigma_k(f, B). \] (5.4)

Consider the term \( J_{B,3} \). By applying (i) and (iii) of Lemma 2.6, we have
\[ \left| e^{-t \sqrt{E}} (f_{4B}) (x) \right| \leq C |f_{4B}| \text{ and } \left| t^2 \partial_t^2 e^{-t \sqrt{E}} (f_{4B}) (x) \right| \leq C \left( \frac{t}{\rho(x)} \right)^{\delta} \left( 1 + \frac{t}{\rho(x)} \right)^{-N} |f_{4B}| \] for each \((x, t) \in \mathbb{R}_+^{n+1}\), where \( \delta > 0 \) is the parameter in Lemma 2.6. We consider the following two cases.

**Case 1.** \( 2r_B \geq \rho(x_B) \). In this case, it follows from Corollary 1 in [9] that we can select a finite family of critical balls \( \{ B(x_i, \rho(x_i)) \} \) such that
\[ 2B \subseteq \bigcup_i B(x_i, \rho(x_i)) \text{ and } \sum_i |B(x_i, \rho(x_i))| \leq c |B|, \]
where \( c = c(\rho) < \infty \) independent of \( B \). Hence, \( \rho(x) \approx \rho(x_i) \) for each \( x \in B(x_i, \rho(x_i)) \), and it follows from (5.1) and (5.5) to see
\[ (J_{B,3})^2 \leq Cr_B^{-n} \sum_i \int_0^{2r_B} \int_{B(x_i, \rho(x_i))} \left( \frac{t}{\rho(x)} \right)^{\delta} \left( 1 + \frac{t}{\rho(x)} \right)^{-N} |f_{4B}|^2 + \frac{t^2}{r_B^2} |f_{4B}|^2 \, dx \, dt \]
\[ \leq C |f_{4B}|^2 r_B^{-n} \sum_i \int_0^{\rho(x)} \int_{B(x_i, \rho(x_i))} \left( \frac{t}{\rho(x)} \right)^{\delta} \, dx \, dt \]
\[ + \int_{\rho(x)}^{\infty} \int_{B(x_i, \rho(x_i))} \left( \frac{t}{\rho(x)} \right)^{-N} \, dx \, dt + |B(x_i, \rho(x_i))| \]
\[ \leq C |f_{4B}|^2 r_B^{-n} \sum_i |B(x_i, \rho(x_i))| \leq C |f_{4B}|^2. \]

**Case 2.** $2r_B < \rho(x_B)$. For any $x \in 2B$, we get $|x - x_B| < 2r_B < \rho(x_B)$. It follows from Lemma 2.3 that $\rho(x) \approx \rho(x_B)$, for any $x \in 2B$. We then apply (5.2) by taking the parameter $c_0 := f_{4B}$, and combine (5.5) and (2.4) to obtain

\[ (J_{B,3})^2 \leq C r_B^{-n} \int_0^{2r_B} \int_2 B \left[ |f_{4B}|^2 \left( \frac{t}{\rho(x)} \right)^{\delta + 2 - n/q} + \frac{t^2}{r_B^2} |f_{4B}|^2 \left( \frac{t}{\rho(x)} \right)^{2(2-n/q)} \right] \frac{dx dt}{t} \]

\[ + C r_B^{-n} \int_0^{2r_B} \int_2 B t |f_{4B}|^2 \left( \frac{t}{\rho(x)} \right)^{2-n/q} V(x) dx dt. \]

By Lemma 2.2, we have the fact $\int_2 B V(x) dx \leq C (2r_B)^{n-2}$ for $2r_B < \rho(x_B)$. Then, one can get

\[ (J_{B,3})^2 \leq C |f_{4B}|^2 \left( \frac{r_B}{\rho(x_B)} \right)^{\delta} \]

since $0 < \delta < 2 - n/q$.

Combining Case 1 and Case 2 above, we obtain

\[ J_{B,3} \leq C |f_{4B}| \min \left\{ \left( \frac{2r_B}{\rho(x_B)} \right)^{\delta/2}, 1 \right\}. \tag{5.6} \]

We are now in a position to prove the aimed $\tilde{\beta}_1(f) = \tilde{\beta}_2(f) = \tilde{\beta}_3(f) = 0$. By (5.3) and (5.4),

\[ J_{B,1} + J_{B,2} \leq C \sum_{k=1}^{\infty} 2^{-k} \sigma_k(f, B). \tag{5.7} \]

Since $f \in \text{CMO}_C$, it follows from Theorem C that $\tilde{\gamma}_j(f) = 0$, $j = 1, \ldots, 5$. Then one can show that for any $k \in \mathbb{N}$

\[ \lim_{a \to 0} \sup_{B : r_B \leq a} \sigma_k(f, B) = \lim_{a \to \infty} \sup_{B : r_B \geq a} \sigma_k(f, B) = \lim_{a \to \infty} \sup_{B : B \subseteq (B(0, a))^c} \sigma_k(f, B) = 0. \tag{5.8} \]

In fact, the first two terms in (5.8) vanish due to $\tilde{\gamma}_1(f) = 0$ and $\tilde{\gamma}_2(f) = 0$, respectively. To estimate the third term in (5.8), for any given large positive number $a$, we classify balls $B \subseteq (B(0, a))^c$ by the size of $B$. Then we can use $\tilde{\gamma}_2(f) = 0$ (in the case of $r(B) \geq R_0$, where $R_0 >> 1$) and $\tilde{\gamma}_3(f) = 0$ (in the case of $r_B < R_0$) to prove

\[ \lim_{a \to \infty} \sup_{B \subseteq (B(0, a))^c} \sigma_k(f, B) = 0. \]
For any $\varepsilon > 0$, there exists $N > 0$, such that $\sum_{k=N}^{\infty} 2^{-k} < \varepsilon$. By noting that $\sigma_k(f, B) \leq \|f\|_{\text{BMO}} \leq \|f\|_{\text{BMO}_L}$ for any $k \in \mathbb{N}$, we then have

$$
\sum_{k=1}^{\infty} 2^{-k} \sigma_k(f, B) \leq \sum_{k=1}^{N} 2^{-k} \sigma_k(f, B) + \sum_{k=N+1}^{\infty} 2^{-k} \|f\|_{\text{BMO}_L} \leq \sum_{k=1}^{N} 2^{-k} \sigma_k(f, B) + \varepsilon \|f\|_{\text{BMO}_L},
$$

which, together with (5.8), gives

$$
\lim_{a \to 0} \sup_{B : r_B \leq a} (J_{B,1} + J_{B_2}) = \lim_{a \to \infty} \sup_{B : r_B \geq a} (J_{B,1} + J_{B_2}) = \lim_{a \to \infty} \sup_{B \subseteq (B(0,a))^c} (J_{B,1} + J_{B_2}) = 0. \tag{5.9}
$$

In the end, we are concerned with the behavior of $J_{B,3}$ as $B$ is small, or large, or far away from the origin. Note that when $r_B < \rho(x_B)/4$, one can apply Lemma 2.8 to obtain

$$
|f_{4B}| \leq C \min \left\{ |f|_{B(x_B, \rho(x_B))} \left( \frac{\rho(x_B)}{r_B} \right)^n, \|f\|_{\text{BMO}_L} \left( 1 + \log \frac{\rho(x_B)}{r_B} \right) \right\}
\leq C \left( |f|_{B(x_B, \rho(x_B))} \left( \frac{\rho(x_B)}{r_B} \right)^n \right)^{1-\theta} \left( \|f\|_{\text{BMO}_L} \left( 1 + \log \frac{\rho(x_B)}{r_B} \right) \right)^{\theta},
$$

for any $\theta \in [0, 1]$. It follows from (5.6) and (5.10) that

$$
\lim_{a \to \infty} \sup_{B : r_B \geq a, r_B < \rho(x_B)/4} J_{B,3} \lesssim \lim_{a \to \infty} \sup_{B : r_B \geq a, r_B < \rho(x_B)/4} |f_{4B}| \left( \frac{r_B}{\rho(x_B)} \right)^{\delta/2}
\lesssim \lim_{a \to \infty} \sup_{B : r_B \geq a, r_B < \rho(x_B)/4} |f|_{B(x_B, \rho(x_B))}^{1-\theta} \left( \frac{\rho(x_B)}{r_B} \right)^{(n-\delta/2)(1-\theta) - \delta \theta/4} \|f\|_{\text{BMO}_L}^{\theta}
$$

for any $\theta \in (0, 1)$. Take $\theta^* \in (0, 1)$ sufficiently close to 1, such that $(n - \delta/2)(1 - \theta^*) - \delta \theta^*/4 < 0$. Then it follows from $\tilde{\gamma}_4(f) = 0$ that

$$
\lim_{a \to \infty} \sup_{B : r_B \geq a, r_B < \rho(x_B)/4} J_{B,3} \lesssim \lim_{a \to \infty} \sup_{B : r_B \geq a, r_B < \rho(x_B)/4} |f|_{B(x_B, \rho(x_B))}^{1-\theta^*} \|f\|_{\text{BMO}_L}^{\theta^*}
\lesssim \|f\|_{\text{BMO}_L} \left( \tilde{\gamma}_4(f) \right)^{1-\theta^*} = 0,
$$
which, together with the fact \( \lim_{a \to \infty} \sup_{B : r_B \geq \max[a, \rho(x_B)/4]} |f_{4B}| \lesssim \tilde{\gamma}_4(f) \), implies that
\[
\lim_{a \to \infty} \sup_{B : r_B \geq a} J_{B,3} = 0.
\]

Similarly, one may apply \( \tilde{\gamma}_4(f) = \tilde{\gamma}_5(f) = 0 \) to obtain
\[
\lim_{a \to \infty} \sup_{B : B \subseteq (B(0,a))^c, r_B \geq \rho(x_B)/4} J_{B,3} \lesssim \lim_{a \to \infty} \sup_{B : B \subseteq (B(0,a))^c, r_B \geq \rho(x_B)/4} |f_{4B}|
\lesssim \tilde{\gamma}_5(f) = 0
\]
and
\[
\lim_{a \to \infty} \sup_{B : B \subseteq (B(0,a))^c, r_B < \rho(x_B)/4} J_{B,3}
\lesssim \|f\|_{\text{BMO}_C} \sup_{B : B \subseteq (B(0,a))^c, r_B < \rho(x_B)/4} |f|^{1 - \theta^*}_{B(x_B, \rho(x_B))}
\lesssim \|f\|_{\text{BMO}_C} \lim_{a \to \infty} \left\{ \sup_{B : B \subseteq (B(0,a))^c, r_B < \rho(x_B)/4, a/8} |f|^{1 - \theta^*}_{B(x_B, \rho(x_B))} \right. \\
+ \sup_{B \subseteq (B(0,a))^c : \max[r_B, a/8] < \rho(x_B)/4} |f|^{1 - \theta^*}_{B(x_B, \rho(x_B))} \right\} \lesssim \|f\|_{\text{BMO}_C} \left\{ (\tilde{\gamma}_5(f))^{1 - \theta^*} + (\tilde{\gamma}_4(f))^{1 - \theta^*} \right\} = 0.
\]
These give that
\[
\lim_{a \to \infty} \sup_{B : B \subseteq (B(0,a))^c} J_{B,3} = 0. \tag{5.11}
\]

It remains to prove \( \lim_{a \to 0} \sup_{B : r_B \leq a} J_{B,3} = 0 \). Due to (5.11), for any \( \varepsilon > 0 \), there exists \( R_\varepsilon >> 1 \) such that \( \sup_{B \subseteq B(0,R_\varepsilon)} J_{B,3} < \varepsilon \). One can write
\[
\lim_{a \to 0} \sup_{B : r_B \leq a} J_{B,3} \leq \lim_{a \to 0} \sup_{B : B \subseteq B(0,R_\varepsilon+1), r_B \leq a} J_{B,3} + \lim_{a \to 0} \sup_{B : B \subseteq B(0,R_\varepsilon)^c} J_{B,3}
\leq \lim_{a \to 0} \sup_{B : B \subseteq B(0,R_\varepsilon+1), r_B \leq a} J_{B,3} + \varepsilon. \tag{5.12}
\]

It follows from Lemma 2.3 that \( \inf_{x \in B(0,R_\varepsilon+1)} \rho(x) > 0 \) (we denote the infimum by \( m_\varepsilon \)). So, if \( a < m_\varepsilon / 4 \), one can apply (5.6) and Lemma 2.8 to get
\[
\sup_{B : B \subseteq B(0,R_\varepsilon+1), r_B \leq a} J_{B,3} \leq C \sup_{B : B \subseteq B(0,R_\varepsilon+1), r_B \leq a} |f_{4B}| \left( \frac{r_B}{\rho(x_B)} \right)^{\delta/2}
\leq C \sup_{B : B \subseteq B(0,R_\varepsilon+1), r_B \leq a} \left( 1 + \log \frac{\rho(x_B)}{4r_B} \right) \|f\|_{\text{BMO}_C} \left( \frac{r_B}{\rho(x_B)} \right)^{\delta/2}.
\]
\[ \leq C \| f \|_{\text{BMO}} \frac{a^{3/4}}{m^{1/4}} \to 0, \quad \text{as } a \to 0. \]

This, together with (5.12), implies \( \lim_{a \to 0} \sup_{B : r_B \leq a} J_{B,3} = 0 \), as desired. We finish the proof of (ii) of Theorem A. \( \square \)

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