KIPPENHAN VARIETIES AND THE WEYL CALCULUS FOR SEVERAL MATRICES I

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Abstract. The paper reviews properties of the Weyl functional calculus for several operators and its relation to the generalised numerical range of n hermitian matrices. The support and singular support of the Weyl functional calculus for n hermitian matrices are determined by Kippenhahn varieties in algebraic geometry.

Dedicated to the memory of Alan McIntosh.

1. Introduction

Let \( n \) be an integer and \( A = (A_1, \ldots, A_n) \) an \( n \)-tuple of linear operators densely defined in a complex Banach space \( X \) whose joint spectra satisfy the spectral reality condition

\[
\sigma(\langle A, \xi \rangle) \subset \mathbb{R}, \quad \xi \in \mathbb{R}^n,
\]

with \( \langle A, \xi \rangle = A_1\xi_1 + \cdots + A_n\xi_n \). Such a system \( A \) is called hyperbolic.

The term comes from the case when \( A \) consists of \( N \times N \) matrices with \( N = 2, 3, \ldots \).

With the characteristic polynomial of a matrix \( B \) defined by \( p_B(z) = \det(B - zI) \) for \( z \in \mathbb{C} \), the \( n \)-tuple \( A \) is hyperbolic exactly when the only solutions \( z \) of the equation \( p(\langle A, \xi \rangle) = 0 \) are real for any \( \xi \in \mathbb{R}^n, \xi \neq 0 \), or that \( P^A : \xi \mapsto \det(\xi_0I + \langle A, \xi \rangle), \xi \in \mathbb{R}^{n+1}, \) is a homogeneous hyperbolic polynomial with respect to the standard basis \((e_0, \ldots, e_n)\) of \( \mathbb{R}^{n+1} \) and the same notation is used for \( \zeta \in \mathbb{C}^{n+1} \).

General hyperbolic polynomials \( p : \mathbb{R}^{n+1} \to \mathbb{R} \) were studied by L. Gårding [16] in relation to hyperbolic partial differential equations in order that the equation \( p(\tau, \mathcal{D}) = \delta_0 \) be well-posed in the sense of distributions with respect to the differential operators

\[
\tau = \frac{1}{i} \frac{\partial}{\partial t}, \quad \mathcal{D} = \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_n} \right).
\]

The unique distributional solution \( F_p \) of \( p(\tau, \mathcal{D}) = \delta_0 \) [20, Theorem 12.5.1] may then be expressed by a formula of Herglotz-Petrovsky-Leray systematically studied in [3],[4] and [20, Chapter 12]. The associated hyperbolic system

\[
I \frac{\partial}{\partial t} + \langle A, \nabla \rangle = i\delta_0 I
\]

then has a unique solution \( F_A = \Phi(A)F_{pA} \) for a matrix differential operator \( \Phi(A) \) of order \( N - 1 \) in \( (\tau, \mathcal{D}) \) formed by multiplying the Fourier transform of \( F_{pA} \) by the \( (N - 1) \) matrix minors of \( \xi_0I + \langle A, \xi \rangle, \xi = \xi_0e_0 + \xi \in \mathbb{R}^{n+1} \).

The Fourier transform of the Schwartz function \( f \in \mathcal{S}(\mathbb{R}^n) \) is taken to be

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i(x,\xi)} f(x) \, dx, \quad \xi \in \mathbb{R}^n.
\]
On the other hand, taking the Fourier transform of equation (2) in the space variables only for \( t > 0 \) and solving for the initial value problem, we get the distribution \( \mathcal{W}_{tA} \), \( t \in \mathbb{R} \). At \( t = 1 \), the operator valued distribution \( \mathcal{W}_A \) given by

\[
\langle \mathcal{W}_A, f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(A, \xi)} \hat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),
\]

is the Weyl functional calculus for \( A \) studied since the late 60’s by E. Nelson [28], M. Taylor [35] and R.F.V. Anderson [1, 2]. This expression has the advantage that it does not involve determinants so \( \mathcal{W}_A \) makes sense for any system \( A \) of collectively densely defined linear operators which satisfy an exponential estimate of the form

\[
\|e^{i(A, \xi)}\| \leq C(1 + |\xi|)^s, \quad \xi \in \mathbb{R}^n,
\]

for some \( C, s > 0 \), for example, selfadjoint operators defined in a Hilbert space with \( s = 0 \). A single bounded linear operator satisfying the estimate (4) is a generalised scalar operator [13, Theorem 5.4.5]. More precisely, if \( A = (A_1, \ldots, A_n) \) are densely defined selfadjoint operators such that every real linear combination has a selfadjoint closure \( \langle A, \xi \rangle \) on the intersection of the relevant domains, then (4) follows with \( C = 1 \) and \( s = 0 \) from successive applications of the Lie-Kato-Trotter product formula (see [14, Chapter III, Corollary 5.8]).

The prime candidate for the Weyl functional calculus is the system \( (X, D) \) of densely defined operators. Here \( X = (X_1, \ldots, X_n) \) with \( X_j \) the operator of multiplication by the variable \( x_j \), \( j = 1, \ldots, n \), so that \( \mathcal{W}_{(X,D)} \) is a \( \mathcal{L}(L^2(\mathbb{R}^n)) \)-valued Schwartz distribution on \( \mathbb{R}^{2n} \). This is the original Weyl functional calculus and much studied as a pseudodifferential operator [18],[36, VII §14]. The more general definition of \( \mathcal{W}_A \) for selfadjoint operators \( A \) formulated by E. Nelson [28] was inspired by ideas of R. Feynman on a general operator calculus and “disentangling” procedure [15],[24]. In the case that \( A \) are selfadjoint elements of a von Neumann algebra with a given trace \( \rho \), then the scalar distribution \( \rho \circ \mathcal{W}_A \) is the Wigner transform [12, 33].

In a similar fashion, the Kohn-Nirenberg functional calculus \( \sigma \mapsto \sigma(X, D)_{KN} \) defined for symbols \( \sigma \in \mathcal{S}(\mathbb{R}^{2n}) \) by

\[
\sigma(X, D)_{KN} = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{i(X, q)} e^{i(D, p)} \hat{\sigma}(q, p) \, dq \, dp
\]

is an important tool in the analysis of partial differential equations [36, VII (1.10)].

If \( A \) are bounded linear operators on a Banach space \( X \) satisfying the exponential bound (4) then by the Paley-Wiener theorem \( \mathcal{W}_A \) has compact support and there exist \( r > 0 \) such that

\[
\|e^{i(A, \xi)}\| \leq C'(1 + |\Re \zeta|)^{s'} e^{r|\Im \zeta|}, \quad \zeta \in \mathbb{C}^n,
\]

for some \( C', s' > 0 \). The operator valued distribution \( \mathcal{W}_A \) has support in the ball \( B_r \) of radius \( r \) centred at zero. Any number \( r \) strictly greater than \( \|A\| = (\|A_1\|^2 + \cdots + \|A_n\|^2)^{1/2} \) will do. All such linear operators satisfy the spectral reality condition (1).

Back to hyperbolic \((N \times N)\) matrices \( A \), the solution of (2) has an expression by the Herglotz-Petrovsky-Leray formula so the bound (5) is satisfied with \( s' \) at most \( N - 1 \). In the infinite dimensional situation this may no longer be the case. The paper of Nelson [28] also contains a formula for hermitian matrices \( A \) acting on \( \mathbb{C}^N \) expressing \( \mathcal{W}_A \) as a
differential operator of order \((N-1)\) acting on the measure \(\mu \circ n^{-1}_A\). Here \(\mu\) is the unitarily invariant probability measure on the unit sphere \(S(\mathbb{C}^N)\) in \(\mathbb{C}^N\) centred at zero and
\[
n_A : h \mapsto (\langle A_1 h, h \rangle, \ldots, \langle A_n h, h \rangle), \quad h \in S(\mathbb{C}^N),
\]
is the joint numerical range map. The Payle-Wiener theorem shows that \(\mathcal{W}_A\) is supported by the convex hull of the joint numerical range \(N_A = n_A(S(\mathbb{C}^N))\), even for unbounded selfadjoint operators (with a common dense domain so that \(\langle A, \xi \rangle\) has a selfadjoint closure for all \(\xi \in \mathbb{R}^n\)).

The joint numerical range \(N_A\) of an \(n\)-tuple \(A\) of hermitian matrices has attracted recent interest because it arises in many areas such as optimisation theory, quantum optics, quantum statistical mechanics, Wigner transforms and quantum error correction, see [30] for an overview. In the Herglotz-Petrovsky-Leray formula the joint numerical range \(N_A\) manifests as the trace of the propagation cone \(K(P^A)\) at time \(t = 1\) and the fundamental formula
\[
\mu \circ n^{-1}_A = (-i)^N (N-1)! F_{PA}(1, \cdot)
\]
holds as distributions linking the Nelson and Herglotz-Petrovsky-Leray representations for systems of hermitian matrices. It is not immediately obvious why \(F_p(1, \cdot)\) should be a distribution of order zero in the case that \(p = P^A\) is a hyperbolic determinantal polynomial for a system \(A\) of hermitian matrices, see Theorem 5.2 below.

For the case \(n = 2\) of two hermitian matrices \(A = (A_1, A_2)\), the set \(N_A\) may be identified with the usual numerical range in \(\mathbb{C}\) of the single matrix \(A = A_1 + iA_2\) under the identification \(j : (x, y) \mapsto x + iy, x, y \in \mathbb{R}\). Then \(jN_A\) is the convex hull of an algebraic curve \(C(A) = jC(A)\) studied by R. Kippenhahn [26]. In this case M. Atiyah, R. Bott and L. Gårding show that the singular support \(\text{ss}(\mathcal{W}_A)\) of the Weyl calculus for two hermitian matrices \(A\) is exactly \(C(A)\) [4, Theorem 14.20]. That \(\text{ss}(\mathcal{W}_A) \subset C(A)\) was shown by J. Bazer and D. Yen [6]. None of these authors reference the numerical range studies of Kippenhahn [26]. Bazer and Yen manage to avoid discussing hyperbolic polynomials by invoking a plane wave decomposition of the distribution \(\mathcal{W}_{tA}\) whose density with respect to Lebesgue measure is the “Riemann matrix” for equation (2).

At this stage it is prudent to introduce some modern developments. The Cauchy transform of a Schwartz distribution \(T \in \mathcal{S}'(\mathbb{R})\) is the function \(\tilde{T} : z \mapsto \langle T, g_z \rangle\) with
\[
g_z(x) = \frac{1}{2\pi i} \frac{1}{x-z}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad x \in \mathbb{R} \ [10, \S 5.1].
\]
Then
\[
T = \lim_{\epsilon \to 0^+} \tilde{T}(x + i\epsilon) - \tilde{T}(x + i\epsilon)
\]
in the sense of distribution [10, \S 5.6]. If we want a similar formula for distributions on \(\mathbb{R}^n\) like the Weyl functional calculus \(\mathcal{W}_A\), then we need an analogue of the normalised Cauchy kernel \(\frac{1}{2\pi i} \frac{1}{z} \), \(z \in \mathbb{C}, z \neq 0\), in higher dimensions. Then we may obtain a representation for the density of \(\mathcal{W}_A\) with respect to Lebesgue measure like the Riemann matrix of Bazer and Yen [5, 6]. This is the point where Clifford analysis enters as a functional calculus technique.

The Clifford algebra \(\mathbb{C}_{(n)}\) over the field \(\mathbb{C}\) is generated by standard basis vectors
\((e_0, e_1, \ldots, e_n)\) in \(\mathbb{R}^{n+1}\) with multiplication so that \(e_0\) is the unit and the formula \(x^2 = -|x|^2\) holds for \(x = \sum_{j=1}^n x_j e_j, x_j \in \mathbb{R}\). Then we obtain \(e_j e_k = -e_k e_j\) for \(j \neq k, j, k = 1, \ldots, n\).
and \( e_j^2 = -1 \) for \( j = 1, \ldots, n \). The Kelvin inverse of \( x \in \mathbb{R}^{n+1} \) is the vector \( x^{-1} = \frac{x}{|x|^2}, x \neq 0 \).

The Cauchy-Riemann operator is \( D = \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j} \) and for \( \Sigma_n = \frac{2\pi}{\Gamma(n+1)} \) the function

\[
E(x) = \frac{1}{\sum_n |x|^{n+1}}, \quad x \in \mathbb{R}^{n+1}, \ x \neq 0,
\]

is the corresponding Cauchy kernel with \( DE = 0 \) on \( \mathbb{R}^{n+1} \setminus \{0\} \). The involution is given by \( \overline{x} = x_0 - x \) for \( x = x_0 e_0 + x \) with \( x_0 \in \mathbb{R} \) and \( x \in \mathbb{R}^n \equiv \{0\} \times \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \). For product vectors \( \overline{e}_j e_k = \mathcal{E}_k \mathcal{E}_j, j \neq k, j, k \neq 0 \), and so on. It is easily checked that \( E = \overline{D} \Gamma_{n+1} \) for the fundamental solution \( \Gamma_{n+1} \) of the Laplacian operator \( \Delta e_0 = D \overline{D} \) in \( \mathbb{R}^{n+1} \) so that \( \Gamma_{n+1} = \frac{1}{\Sigma_n |x|^{n+1}} \Gamma_0 \) and \( \Delta \Gamma_{n+1} = \delta_0 \) for \( n \geq 2 \). Setting \( G_\omega(x) = E(\omega - x), \ x \in \mathbb{R}^n \), for any Schwartz distribution \( T \in \mathcal{S}'(\mathbb{R}^n) \) we have

\[
(9) \quad T = \lim_{\epsilon \to 0+} \overline{T}(x + \epsilon e_0) - \overline{T}(x - \epsilon e_0)
\]

in the sense of distributions for the Cauchy transform \( \overline{T}(\omega) = \langle T, G_\omega \rangle, \ \omega \in \mathbb{R}^{n+1}, \ \omega_0 \neq 0 \) [9, Theorem 27.7]. When the Weyl functional calculus \( \mathcal{W}_A \) exists for operators \( A \), the operator valued function \( \omega \mapsto G_\omega(A) = \mathcal{W}_A(\omega) \) defined for all \( \omega \in \mathbb{R}^{n+1} \) with \( \omega_0 \neq 0 \) is called the Cauchy kernel for \( A \). For any bounded linear operators \( A \) on a Banach space \( X \), the Cauchy kernel \( G_\omega(A) \) can be defined by a series expansion for \( \omega \in \mathbb{R}^{n+1} \) with \( |\omega| \) large enough but this is not very useful. For \( n = 1 \), \( G_\omega(A) = \frac{1}{2\pi}(j \omega - A)^{-1} \) in \( \mathcal{L}(X) \) if \( j \omega \notin \sigma(A) \).

Finally, if \( A \) is a hyperbolic system of bounded linear operators (or unbounded with a common dense domain and uniform resolvent bounds) then \( G_\omega(A) \) may be defined by a plane wave decomposition which agrees with the definition in case \( A \) satisfies the exponential bounds (5) and so \( \mathcal{W}_A \) exists. This central idea is due to Alan McIntosh in 1988 out of which the present investigation grew.

When \( n \) is even, the plane wave formula and equation (9) tell us that the distribution \( \mathcal{W}_A \) is a constant times the limit

\[
(10) \quad \lim_{\epsilon \to 0+} \int_{S^{n-1}} \left( \langle x I - A, s \rangle - \epsilon s I \right)^{-n} + \left( \langle x I - A, s \rangle + \epsilon s I \right)^{-n} ds
\]

in the sense of distributions for \( x \in \mathbb{R}^n \). Here \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). Even for matrices this is a useful observation because (10) is similar to the plane wave formula employed by Bazer and Yen [5] in their study of the Riemann matrix but with the pleasing advantage that it also works for \( n \geq 2 \). The formula for odd \( n \) is

\[
(11) \quad \lim_{\epsilon \to 0+} \int_{S^{n-1}} s \left( \langle x I - A, s \rangle + \epsilon s I \right)^{-n} - s \left( \langle x I - A, s \rangle - \epsilon s I \right)^{-n} ds.
\]

By perturbing the integrals over \( S^{n-1} \) into the complex domain and observing that we have a closed matrix valued differential form by homogeneity, the limits can be evaluated pointwise and converge to the density of the distribution \( \mathcal{W}_A \) with respect to Lebesgue measure. This is essentially the argument of Bazer and Yen in the case \( n = 2 \) for two hermitian matrices. The method just described is limited to finite dimensional operators.
but works for all hyperbolic $n$-tuples $A$ of matrices and uses the full force of the Herglotz-Petrovsky-Leray representation devised by Atiyah, Bott and Gårding.

In addition, we need the analogue $C(A)$ of the Kippenhahn curves in dimensions $n = 3, 4, \ldots$ for which $N_A = \text{co}(C(A))$. These semi-algebraic sets have only recently been determined by Plaumann, Sinn and Weis [30] and promise to have applications to the many areas where joint numerical range is a central concept. After settling the many issues involved, the position of the singular support $\text{ss}(W_A)$ of the Weyl functional calculus with respect to the boundary generating set $C(A)$ or Kippenhahn variety in $\mathbb{R}^n$ is examined.

2. The Kippenhahn Curves

Let $A = (A_1, A_2)$ be a pair of $(N \times N)$ hermitian matrices. Set $A = A_1 + iA_2$. As mentioned in the Introduction, an application of the Paley-Wiener Theorem yields that the convex hull of the support $\text{supp}(W_A)$ of the associated Weyl distribution $W_A$ coincides with the numerical range $K(A) := jN_A$

$$K(A) := \{(Ax, x) \mid x \in \mathbb{C}^N, |x| = 1\}$$

of the matrix $A$. For more precise information on the location of $\text{supp}(W_A)$ within the numerical range of $A$, we need to have a closer look at the fine structure of $K(A)$.

Of particular interest are certain plane algebraic curves associated with $A$ that were first investigated by R. Kippenhahn [26] in 1952. We briefly recall the concepts involved.

Let $F = \mathbb{R}$ or $\mathbb{C}$. For $0 \leq k \leq 3$, the Grassmannian $G_{3,k}F$, defined as the set of all $k$-dimensional $F$-subspaces of $F^3$, is a compact analytic $F$-manifold of dimension $k(3 - k)$. It has a natural topology, induced by the differential structure of the manifold, which is determined, for example, by the metric $h$ on $G_{3,k}F$ with

$$h(U, V) = \sup_{v \in V, |v| = 1} \inf_{u \in U, |u| = 1} \|u - v\|$$

for all $U, V \in G_{3,k}F$.

The projective plane $\text{PG}(F^3)$ over $F$ is given by

$$\text{PG}(F^3) = \bigcup_{0 \leq k \leq 3} G_{3,k}F.$$ 

The one and two dimensional subspaces of $F^3$ are usually called the points and lines in $\text{PG}(F^3)$, respectively.

By common abuse of notation we introduce homogeneous coordinates for the points in $\text{PG}(F^3)$ as $(u_1 : u_2 : u_3) = F(u_1, u_2, u_3)$. The coordinates of a vector in $F^3$ are expressed with respect to the standard basis for $F^3$.

A polarity of $\text{PG}(F^3)$ is a bijection on $\text{PG}(F^3)$ which reverses the inclusion of subspaces and the square of which equals the identity mapping. The standard polarity $\pi$ is characterised by

$$u^\pi = \{v \in F^3 \mid \sum_{j=1}^3 u_jv_j = 0\}$$

for all $u \in G_{3,1}F$,

which gives $u^\pi \in G_{3,2}F$. Using the polarity $\pi$, we can also introduce homogeneous coordinates for the lines in $\text{PG}(F^3)$ by setting $[v_1 : v_2 : v_3] = (v_1 : v_2 : v_3)^\pi$. 

A nonempty subset $C$ of $G_{3,1}\mathbb{F}$ is called a plane $\mathbb{F}$-algebraic curve if it is the zero locus of a homogeneous 3-variate polynomial over $\mathbb{F}$. The defining polynomial of $C$ is not uniquely determined: if $f$ defines the curve, then so does, for example, $f^k$ for any $k \geq 1$. However, every curve $C$ has a defining polynomial of minimal degree which is unique up to a constant factor. A curve is said to be irreducible if it has an irreducible defining polynomial. Since a polynomial ring over a field is a unique factorisation domain, each algebraic curve $C$ is the union of finitely many irreducible curves. If $C_1, \ldots, C_k$ are the irreducible components of $C$ with irreducible defining polynomials $f_1, \ldots, f_k$, then $f = f_1 \cdots f_k$ is a defining polynomial of $C$ of minimal degree. We call $f$ a minimal polynomial of $C$. Note that an irreducible real algebraic curve is not necessarily connected.

Let $f$ be a minimal polynomial of the algebraic curve

$$C = \{ u \in G_{3,1}\mathbb{F} \mid f(u) = 0 \}.$$  

A point $u \in C$ is called singular or a singularity of $C$ if $(\partial f/\partial u_j)(u) = 0$ for $j = 1, 2, 3$. Observe that $C$ has at most finitely many singular points. These are the singular points of the irreducible components of $C$ together with the points of intersection of any two of these components. A nonsingular point $u \in C$ is called a simple point of $C$. The curve $C$ is the topological closure of its simple points. Also, to every simple point $u \in C$, there exists a neighbourhood of $u$ in which $C$ admits a smooth parametrization.

Let $C$ be an irreducible plane algebraic curve with minimal polynomial $f$. At each simple point $u \in C$, we have a unique tangent line to $C$ which is given by

$$\mathcal{T}_u C = \left[ \frac{\partial f}{\partial u_1}(u) : \frac{\partial f}{\partial u_2}(u) : \frac{\partial f}{\partial u_3}(u) \right].$$

If $C$ is not a projective line or a point, then it is well-known that the set

$$\{(\mathcal{T}_u C)^\pi \mid u \in C \text{ simple}\}$$

is contained in a unique irreducible algebraic curve $C^*$, the so-called dual curve of $C$. In fact, since an algebraic curve has at most finitely many singularities, the dual curve is the topological closure of the set $\{(\mathcal{T}_u C)^\pi \mid u \in C \text{ simple}\}$. We have $C^{**} = C$. If $C$ is a projective line, then $\{(\mathcal{T}_u C)^\pi \mid u \in C \}$ consists of a single point $u$ in $\text{PG}(\mathbb{F}^3)$. In this case, we set $C^* = \{u\}$ and define $C^{**}$ to be the image under $\pi$ of the set of all lines in $\text{PG}(\mathbb{F}^3)$ which pass through $u$. This again yields $C^{**} = C$. The dual curve of a general plane algebraic curve $C$ is the union of the dual curves of its irreducible components. In particular, $C$ and $C^*$ have the same number of irreducible components.

In general, it is difficult to derive an explicit equation for the dual curve $C^*$ from the given equation of a curve $C$. However, from the above we obtain the following criterion for a point in $\text{PG}(\mathbb{F}^3)$ to belong to $C^*$.

**Lemma 2.1.** Let $(x_1 : x_2 : 1) \in G_{3,1}\mathbb{F}$. If there exists a smooth local parametrization $\zeta \mapsto (c(\zeta) : s(\zeta) : \mu(\zeta))$ of $C$, for $\zeta$ in an open set $U \subseteq \mathbb{F}$, and a point $z \in U$ such that $x_1 c(z) + x_2 s(z) + \mu(z) = 0$ and $x_1 c'(z) + x_2 s'(z) + \mu'(z) = 0$, then the point $(x_1 : x_2 : 1)$ belongs to $C^*$.

**Proof.** The two points $(c(z) : s(z) : \mu(z))$ and $(c'(z) : s'(z) : \mu'(z))$ span the tangent line $\mathcal{T}_{(c(z), s(z), \mu(z))} C$ to $C$ at $(c(z) : s(z) : \mu(z))$. The equations $x_1 c(z) + x_2 s(z) + \mu(z) = 0$ and
The details and further information on complex algebraic curves can be found, for example, in [32]. The literature for the real case is somewhat less easy to access. As a general reference to the theory of real algebraic geometry, see [8].

Let $A = A_1 + iA_2 \in L(C^N)$. Following R. Kippenhahn [26], we define the complex algebraic curve $C_C(A)$ in the complex projective plane $PG(C^3)$ by setting its dual curve to be

$$D(A) = \{(c : d : \mu) \in G_{3,1}C \mid \det(cA_1 + dA_2 + \mu I) = 0\}.$$ 

In [26], Kippenhahn showed that the real part $C_R(A)$ of the curve $C_C(A) = D(A)^*$ is contained in the affine subplane $F = \{(\alpha_1 : \alpha_2 : 1) \mid (\alpha_1, \alpha_2) \in \mathbb{R}^2\}$ of $PG(\mathbb{R}^3)$ and, identifying $F$ with $\mathbb{R}^2$ in the canonical way, that the convex hull $co(C_R(A))$ of $C_R(A)$ is precisely the numerical range of $A$.

The curve $C_R(A)$ considered as a real algebraic curve in $PG(\mathbb{R}^3)$ is the dual curve of the real part of $D(A)$ given by

$$D_R(A) = \{(c : d : \mu) \in G_{3,1}\mathbb{R} \mid \det(cA_1 + dA_2 + \mu I) = 0\}.$$ 

Every point $u \in D_R(A)$ has a representation $(\cos \theta : \sin \theta : \mu)$ for some $\theta \in [0, \pi)$ and $\mu \in \mathbb{R}$. As $u$ is a zero of $\det(cA_1 + dA_2 + \mu I)$, it follows that $-\mu$ is an eigenvalue of the operator $A(\theta) = \cos \theta A_1 + \sin \theta A_2$.

Note that the points in $D_R(A)$ are in one-to-one correspondence with the lines $L_{y,t}$ in $\mathbb{R}^2$ satisfying $\langle x, t \rangle \in \sigma(\langle A, t \rangle)$ for all $x \in L_{y,t}$. For $u = (\cos \theta : \sin \theta : \mu) \in D_R(A)$, take $t = (\cos \theta, \sin \theta) \in \mathbb{T}$ and $y \in \mathbb{R}^2$ such that $\langle y, t \rangle = -\mu$. Then $u^\pi$ is the two dimensional subspace

$$\bigcup \{(x_1 : x_2 : 1) \mid (x_1, x_2) \in L_{y,t}\}$$

of $\mathbb{R}^3$, that is, $L_{y,t} \times \{1\}$ is the line in which the plane $u^\pi$ normal to $u$ in $\mathbb{R}^3$ cuts the plane $\{x_3 = 1\}$.

3. Clifford Analysis

The basic idea of forming a Clifford algebra $\mathcal{A}$ with $n$ generators is to take the smallest real or complex algebra $\mathcal{A}$ with an identity element $e_0$ such that $\mathbb{R} \oplus \mathbb{R}^n$ is embedded in $\mathcal{A}$ via the identification of $(x_0, x) \in \mathbb{R} \oplus \mathbb{R}^n$ with $x_0 e_0 + x \in \mathcal{A}$ and the identity

$$x^2 = -|x|^2 e_0 = -(x_1^2 + x_2^2 + \cdots + x_n^2) e_0$$

holds for all $x \in \mathbb{R}^n$. Then we arrive at the following definition.

Let $\mathbb{F}$ be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The *Clifford algebra* $\mathbb{F}(n)$ over $\mathbb{F}$ is a $2^n$-dimensional algebra with unit defined as follows. Given the standard basis vectors $e_0, e_1, \ldots, e_n$ of the vector space $\mathbb{F}^{n+1}$, the basis vectors $e_S$ of $\mathbb{F}(n)$ are indexed by all finite subsets $S$ of $\{1, 2, \ldots, n\}$. The basis vectors are determined by the following rules for multiplication on $\mathbb{F}(n)$:

- $e_0 = 1$,
- $e_j^2 = -1$, for $1 \leq j \leq n$,
- $e_j e_k = -e_k e_j = e_{(j,k)}$, for $1 \leq j < k \leq n$.
\[ e_{j_1} e_{j_2} \cdots e_{j_s} = e_S, \quad \text{if} \quad 1 \leq j_1 < j_2 < \cdots < j_s \leq n \]
\[ \quad \text{and} \quad S = \{ j_1, \ldots, j_s \}. \]

Here the identifications \( e_0 = e_0 \) and \( e_j = e_{(j)} \) for \( 1 \leq j \leq n \) have been made.

Suppose that \( m \leq n \) are positive integers. The vector space \( \mathbb{R}^m \) is identified with a subspace of \( \mathbb{F}_n \) by virtue of the embedding \( (x_1, \ldots, x_m) \mapsto \sum_{j=1}^m x_j e_j \). On writing the coordinates of \( x \in \mathbb{R}^{n+1} \) as \( x = (x_0, x_1, \ldots, x_n) \), the space \( \mathbb{R}^{n+1} \) is identified with a subspace of \( \mathbb{F}_n \) with the embedding \( (x_0, x_1, \ldots, x_n) \mapsto \sum_{j=0}^n x_j e_j \).

The product of two elements \( u = \sum_S u_u e_S \) and \( v = \sum_S v_v e_S, v_S \in \mathbb{F} \) with coefficients \( u_S \in \mathbb{F} \) and \( v_S \in \mathbb{F} \) is \( uv = \sum_{S,R} u_S v_R e_{SR} \). According to the rules for multiplication, \( e_S e_R \) is \( \pm 1 \) times a basis vector of \( \mathbb{F}_n \). The scalar part of \( u = \sum_S u_u e_S \), \( u_S \in \mathbb{F} \) is the term \( u_0 \), also denoted as \( u_0 \).

The Clifford algebras \( \mathbb{R}(0), \mathbb{R}(1) \) and \( \mathbb{R}(2) \) are the real, complex numbers and the quaternions, respectively. In the case of \( \mathbb{R}(1) \), the vector \( e_1 \) is identified with \( i \) and for \( \mathbb{R}(2) \), the basis vectors \( e_1, e_2, e_1 e_2 \) are identified with \( i, j, k \) respectively.

The conjugate \( \overline{e_S} \) of a basis element \( e_S \) is defined so that \( e_S \overline{e_S} = \overline{e_S} e_S = 1 \). Denote the complex conjugate of a number \( c \in \mathbb{F} \) by \( \overline{c} \). Then the operation of conjugation \( u \mapsto \overline{u} \)

\[ \overline{u} = \sum_S u_S \overline{e_S} \]

for every \( u = \sum_S u_u e_S, u_S \in \mathbb{F} \) is an involution of the Clifford algebra \( \mathbb{F}_n \) and \( \overline{uv} = \overline{v} \overline{u} \) for all elements \( u \) and \( v \) of \( \mathbb{F}_n \). Because \( e_j^2 = -1 \), the conjugate \( \overline{e_j} \) of \( e_j \) is \( -e_j \). An inner product is defined on \( \mathbb{F}_n \) by the formula \( \langle u, v \rangle = \langle u \overline{v} \rangle_0 = \sum_S u_S \overline{v}_S \overline{S} \)

for every \( u = \sum_S u_S e_S \) and \( v = \sum_S v_S e_S \) belonging to \( \mathbb{F}_n \). The corresponding norm is written as \( | \cdot | \).

For a Banach space \( X \), the tensor product \( X^{(n)} := X \otimes \mathbb{C}^{(n)} \) denotes the module of all sums \( u = \sum_S u_S e_S \) with coefficients \( u_S \in X \) endowed with the norm

\[ \| u \|_{X^{(n)}} = \left( \sum_S \| u_S \|^2_X \right)^{\frac{1}{2}}. \]

The left product \( \lambda u \) and right product \( u \lambda \) are defined in the obvious way for all \( \lambda \in \mathbb{C}^{(n)} \).

The space \( L^{(n)}(X^{(n)}) \) of right-module homomorphisms is identified with \( L(X)^{(n)} \) by writing

\[ \left( \sum_S T_S e_S \right) \left( \sum_{S'} u_{S'} e_{S'} \right) = \sum_S (T_S u_{S'}) e_S e_{S'}, \]

so that \( T(\lambda u) = (Tu) \lambda \) for all \( u \in X^{(n)}, \lambda \in \mathbb{C}^{(n)} \) and \( T = L(X)^{(n)} \). The linear subspace \( \{ T_{e_0} : T \in L(X) \} \) of \( L(X)^{(n)} \) \( \equiv L(X)^{(n)}(X^{(n)}) \) is identified with \( L(X) \). The norm induced on \( L^{(n)}(X^{(n)}) \equiv L(X)^{(n)} \) by the norm of \( X^{(n)} \) is given by

\[ \| \sum_S T_S e_S \|_{L^{(n)}(X^{(n)})} = \left( \sum_S \| T_S \|^2_{L(X)} \right)^{\frac{1}{2}}. \]

For an \( n \)-tuple \( A \) of bounded linear operators on \( X \), it is convenient to write \( | A | \) for the norm \( \| A_1 e_1 + \cdots + A_n e_n \|_{L^{(n)}(X^{(n)})} = \left( \sum_j \| A_j \|^2_{L(X)} \right)^{\frac{1}{2}} \).

Because \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \) is identified with the element \( \sum_{j=0}^n x_j e_j \) of \( \mathbb{R}_n \), the conjugate \( \overline{x} \) of \( x \) in \( \mathbb{R}^{n+1} \) is \( x_0 e_0 - x_1 e_1 - \cdots - x_n e_n \). A useful feature of Clifford algebras is that a nonzero vector \( x \in \mathbb{R}^{n+1} \) has an inverse \( x^{-1} \) in the algebra \( \mathbb{R}_n \) (the Kelvin inverse).
given by
\[ x^{-1} = \frac{\bar{\chi}}{|x|^2} = \frac{x_0 e_0 - x_1 e_1 - \cdots - x_n e_n}{x_0^2 + x_1^2 + \cdots + x_n^2}. \]

The vector \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \) will often be written as \( x = x_0 e_0 + x \) with \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).

Because \( \mathbb{C}_m \) is an algebra over the complex numbers, the spectrum \( \sigma(ix) \) of the vector \( ix \) with \( x \in \mathbb{R}^n \) is the set of all \( \lambda \in \mathbb{C} \) for which \( \lambda e_0 - i x \) is not invertible in \( \mathbb{C}_m \). The formula
\[ (\lambda e_0 - i x)^{-1} = \frac{\lambda e_0 + i x}{\lambda^2 - |x|^2}, \quad \lambda \neq \pm |x|, \]
ensures that \( \sigma(ix) = \{ \pm |x| \} \) if \( x \neq 0 \) and \( \sigma(0) = \{ 0 \} \). For \( x \neq 0 \), the spectral representaion
\[ ix = |x| \chi_+(x) + (-|x|) \chi_-(x) \]
holds with respect to the spectral idempotents
\[ \chi_{\pm}(x) = \frac{1}{2} \left( e_0 \pm \frac{x}{|x|} \right). \]

The vector \( ix \) is actually selfadjoint with respect to the inner product of \( \mathbb{C}_m \) defined above. For every function \( f : \{ \pm |x| \} \to \mathbb{C} \) there is an element
\[ f(ix) = f(|x|) \chi_+(x) + f(-|x|) \chi_-(x) \]
of \( \mathbb{C}_m \) associated with \( f \) by the functional calculus for selfadjoint operators. For a polynomial \( p(z) = a_0 + a_1 z + \cdots + a_k z^k \), the expression
\[ p(ix) = a_0 e_0 + a_1 (ix) + \cdots + a_k (ix)^k \]
expected in \( \mathbb{C}_m \) is obtained. The identities \( \chi_{\mathbb{R}_+}(ix) = \chi_{\pm}(x) \) hold for the characteristic functions \( \chi_{\mathbb{R}_+} \) of the half lines \( \mathbb{R}_+ = \{ t > 0 \} \) and \( \mathbb{R}_- = \{ t < 0 \} \).

Given an \( n \)-tuple of operators \( A = (A_1, \ldots, A_n) \) acting on a Hilbert space \( \mathcal{H} \), the expression \( A = \sum_{j=1}^n e_j A_j \) acts on \( \mathcal{H}(n) := \mathcal{H} \otimes \mathbb{C}_m \) via the formula
\[ Au = \sum_{j=1}^n \sum_S (e_j e_S)(A_j u_S), \quad u = \sum_S u_S e_S. \]
The coefficients \( u_S \) are elements of the Hilbert space \( \mathcal{H} \) and \( u_S \otimes e_S \) is written simply as \( u_S e_S \) for all \( S \subseteq \{ 1, \ldots, n \} \).

Using Fourier theory, the functional calculus for the selfadjoint differential operator
\[ D = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j} \]
acting in the Hilbert space \( L^2_n(\mathbb{R}^n) := L^2(\mathbb{R}^n) \otimes \mathbb{C}_m \) can be calculated explicitly.

If \( \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i(x,\xi)} u(x) \, dx \) denotes the Fourier transform of \( u \in L^1(\mathbb{R}^n) \), then according to the Fourier-Plancherel Theorem, the linear map \( u \mapsto (2\pi)^{-n/2} \hat{u} \), \( u \in L^1 \cap L^2(\mathbb{R}^n) \), extends to an isometry of \( L^2(\mathbb{R}^n) \). For each \( j = 1, \ldots, n \), the selfadjoint operator \( \frac{1}{i} \frac{\partial}{\partial x_j} \)
defined in \( L^2(\mathbb{R}^n) \) satisfies
\[ \left( \frac{1}{i} \frac{\partial}{\partial x_j} \right) \hat{u}(\xi) = \xi_j \hat{u}(\xi). \]
almost everywhere for each \( u \in L^2(\mathbb{R}^n) \) in its domain. Furthermore, for any bounded measurable function \( \varphi \) defined on \( \mathbb{R}^n \), the operator \( \varphi \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_n} \right) \) satisfies

\[
\left( \varphi \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_n} \right) u \right)(\xi) = \varphi(\xi) \hat{u}(\xi)
\]

almost everywhere for \( u \in L^2(\mathbb{R}^n) \). Similarly,

\[
(f(D)u)(\xi) = f(i\xi) \hat{u}(\xi), \quad u \in L^2_{(n)}(\mathbb{R}^n), \ \xi \in \mathbb{R}^n,
\]

is valid for any bounded measurable function \( f : \mathbb{R} \to \mathbb{C} \) with the understanding that for each \( \xi \in \mathbb{R}^n \), the vector \( f(i\xi) \) is given by the functional calculus (12) of the selfadjoint element \( i\xi \) of the Clifford algebra \( \mathbb{C}(n) \).

What is usually called \textit{Clifford analysis} is the study of functions of finitely many real variables, which take values in a Clifford algebra, and which satisfy higher dimensional analogues of the Cauchy-Riemann equations.

It is worthwhile to spell out the direction this analogy takes. The Cauchy-Riemann equations for a complex valued function \( f \) defined in an open subset of the complex plane may be represented as \( \overline{\partial}f = 0 \) for the operator

\[
\overline{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \quad z = x + iy \in \mathbb{C}.
\]

The fundamental solution \( E \) of the operator \( \overline{\partial} \) is the solution in the sense of Schwartz distributions of the equation \( \overline{\partial}E = \delta_0 \) for the unit point mass \( \delta_0 \) at zero. Then

\[
E(z) = \frac{1}{2\pi} \frac{1}{z} = \frac{1}{2\pi} \frac{\overline{z}}{|z|^2}, \quad \text{for } z = x + iy \in \mathbb{C} \setminus \{0\}.
\]

A function \( f \) satisfying \( \overline{\partial}f = 0 \) in a neighbourhood of a simple closed contour \( C \) together with its interior can be represented as

\[
f(z) = \frac{1}{i} \int_C E(\zeta - z)f(\zeta) \, d\zeta = \int_C E(\zeta - z)n(\zeta)f(\zeta) \, d|\zeta|
\]

at all points \( z \) inside \( C \). Here \( n(\zeta) \) is the outward unit normal at \( \zeta \in \mathbb{C} \), \( d|\zeta| \) is arclength measure so that \( in(\zeta)d|\zeta| = d\zeta \). The higher dimensional analogue for functions taking values in a Clifford algebra is as follows.

A function \( f : U \to \mathbb{F}(n) \) defined in an open subset \( U \) of \( \mathbb{R}^{n+1} \) has a unique representation \( f = \sum_S f_S e_S \) in terms of \( \mathbb{F} \)-valued functions \( f_S, S \subseteq \{1, \ldots, n\} \) in the sense that \( f(x) = \sum_S f_S(x)e_S \) for all \( x \in U \). Then \( f \) is continuous, differentiable and so on, in the normed space \( \mathbb{F}(n) \), if and only if for all finite subsets \( S \) of \( \{1, \ldots, n\} \), its scalar component functions \( f_S \) have the corresponding property. Let \( \partial_j \) be the operator of differentiation of a scalar function in the \( j \)-th coordinate in \( \mathbb{R}^{n+1} \) – the coordinates of \( x \in \mathbb{R}^{n+1} \) are written as \( x = (x_0, x_1, \ldots, x_n) \). For a continuously differentiable function \( f : U \to \mathbb{F}(n) \) with \( f = \sum_S f_S e_S \) defined in an open subset \( U \) of \( \mathbb{R}^{n+1} \), the functions \( Df \) and \( fD \) are defined by

\[
Df = \sum_S \left( (\partial_0 f_S)e_S + \sum_{j=1}^n (\partial_j f_S)e_j e_S \right)
\]

\[
fD = \sum_S \left( (\partial_0 f_S)e_S + \sum_{j=1}^n (\partial_j f_S)e_s e_j \right)
\]
Similarly, the conjugate operator is defined by
\[ \mathcal{D} = e_0 \partial_0 - \sum_{j=1}^{n} e_j \partial_j \]
so that the Laplace operator \( \Delta = \partial_0^2 + \cdots + \partial_n^2 \) in \( \mathbb{R}^{n+1} \) has factorizations
\[ \Delta = D\overline{D} = \overline{D}D. \]

Now suppose that \( f \) is an \( \mathbb{F}_n \)-valued, continuously differentiable function defined in an open subset \( U \) of \( \mathbb{R}^{n+1} \). Then \( f \) is said to be left monogenic in \( U \) if \( Df(x) = 0 \) for all \( x \in U \) and right monogenic in \( U \) if \( fD(x) = 0 \) for all \( x \in U \).

The following result connects \( \mathbb{R}^{n+1} \)-valued monogenic functions with systems of conjugate harmonic functions.

**Proposition 3.1.** Let \( F = u_0e_0 - \sum_{j=1}^{n} u_je_j \) be an \( \mathbb{R}^{n+1} \)-valued function defined on an open subset \( \Omega \) of \( \mathbb{R}^{n+1} \). Conditions (1)-(4) below are equivalent in \( \Omega \).

1. The \((n+1)\)-tuple \( U = (u_j)_{j=0}^{n} \) is a system of conjugate harmonic functions in \( \Omega \), that is, \( U \) satisfies the generalised Cauchy-Riemann equations \( \text{div}U = 0 \) and \( \text{curl}U = 0 \).
2. \( F \) is left monogenic.
3. \( F \) is right monogenic.
4. The 1-form \( \omega := u_0dx_0 - u_1dx_1 - \cdots - u_ndx_n \) satisfies \( d\omega = 0 \) and \( d^*\omega = 0 \), where \( d \) and \( d^* \) are the exterior differential operator and its formal transpose, respectively.
5. In the case that \( \Omega \) is simply connected, then the above conditions are equivalent to the existence of a real valued harmonic function \( v \) in \( \Omega \) such that \( U = \text{grad}v \), so that \( F = \overline{D}v \).

For each \( x \in \mathbb{R}^{n+1} \), the function \( G(\cdot, x) \) defined by
\[ G(\omega, x) = G_\omega(x) = \frac{1}{\Sigma_n} \frac{\omega - x}{|\omega - x|^{n+1}} \]
for every \( \omega \neq x \) is both left and right monogenic as a function of \( \omega \). Here the volume \( 2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right) \) of the unit \( n \)-sphere in \( \mathbb{R}^{n+1} \) has been denoted by \( \Sigma_n \) and we have used the identification of \( \mathbb{R}^{n+1} \) with a subspace of \( \mathbb{R}(n) \) mentioned earlier.

The function \( G(\cdot, x) \), \( x \in \mathbb{R}^{n+1} \) plays the role in Clifford analysis of a Cauchy kernel. Rewriting \( G(\omega, x) \) as \( E(\omega - x) \) for all \( \omega \neq x \) in \( \mathbb{R}^n \), it follows that the \( \mathbb{R}^{n+1} \)-valued function
\[ E(x) = \frac{1}{\Sigma_n} \frac{x}{|x|^{n+1}} \]
defined for all \( x \neq 0 \) belonging to \( \mathbb{R}^{n+1} \) is the fundamental solution of the operator \( D \), that is, \( DE = \delta_0e_0 \) in the sense of Schwartz distributions, because \( E = \overline{D}\Gamma_{n+1} = \Gamma_{n+1}\overline{D} \) for the fundamental solution
\[ \Gamma_{n+1}(x) = \begin{cases} 
-\frac{1}{(n-1)\Sigma_n} \frac{1}{|x|^{n-1}}, & x \neq 0, \ n \geq 2 \\
\frac{1}{2\pi} \log |x|, & x \neq 0, \ n = 1,
\end{cases} \]
of the Laplace operator \( \Delta \) in \( \mathbb{R}^{n+1} \). Then a function satisfying \( Df = 0 \) in an open set can be retrieved from a surface integral involving \( E \).
Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set with smooth boundary $\partial \Omega$ and exterior unit normal $n(\omega)$ defined for all $\omega \in \partial \Omega$. For any left monogenic function $f$ defined in a neighbourhood $U$ of $\overline{\Omega}$, the Cauchy integral formula

$$\int_{\partial \Omega} G(\omega, x)n(\omega)f(\omega)\, d\mu(\omega) = \left\{ \begin{array}{ll} f(x), & \text{if } x \in \Omega; \\ 0, & \text{if } x \in U \setminus \overline{\Omega}. \end{array} \right.$$  

is valid. Here $\mu$ is the surface measure of $\partial \Omega$. The result is proved in [9, Corollary 9.6] by appealing to by Stoke’s theorem. If $g$ is right monogenic in $U$ then $\int_{\partial \Omega} g(\omega)n(\omega)f(\omega)\, d\mu(\omega) = 0$ [9, Corollary 9.3]. The manifold $\partial \Omega$ integral (14) can then be deformed across any set not containing $x$.

In terms of differential forms, the $\mathbb{F}^{n+1}$-valued $n$-form

$$\eta = \sum_{j=0}^{n} (-1)^j e_j \, dx_0 \wedge \cdots \wedge \widehat{dx}_j \wedge \cdots \wedge dx_n \quad (\widehat{\cdot} \equiv \text{omitted}),$$

is defined on $\mathbb{R}^{n+1}$ and its pullback to the orientable $n$-dimensional manifold $\partial \Omega$ is denoted by the same symbol. Then for any continuous $\mathbb{F}^{(n)}$-valued functions $u, v$ on $\partial \Omega$, the equality

$$\int_{\partial \Omega} u\eta v = \int_{\partial \Omega} u(\omega)n(\omega)v(\omega)\, d\mu(\omega)$$

holds. Now suppose that $f, g$ are any continuously differentiable $\mathbb{F}^{(n)}$-valued functions defined on $U$. The differential of $g\eta f$ is equal to $((gD)f + g(Df))\, dx_0 \wedge \cdots \wedge dx_n$, so Stokes’ Theorem gives

$$\int_{\Omega} ((gD)f + g(Df))\, d\lambda = \int_{\partial \Omega} g\eta f$$

with respect to Lebesgue measure $\lambda$ on $\mathbb{R}^{n+1}$. The Cauchy integral formula (14) follows by shrinking $\partial \Omega$ to a sphere about $x \in \Omega$, see [9, Corollary 9.6].

**Example 3.2.** For the case $n = 1$, the Clifford algebra $\mathbb{R}_{(1)}$ is identified with $\mathbb{C}$. A continuously differentiable function $f : U \to \mathbb{R}_{(1)}$ defined in an open subset $U$ of $\mathbb{R}^2$ satisfies $Df = 0$ in $U$ if and only if it satisfies the Cauchy-Riemann equations $\overline{\partial}f = 0$ in $U$. For each $x, \omega \in \mathbb{R}^2, x \neq \omega$, the formula

$$G(\omega, x) = \frac{1}{2\pi} \frac{1}{\omega - x}$$

holds. The inverse is taken in $\mathbb{C}$. As indicated above, the tangent at the point $\zeta(t)$ of the portion $\{\zeta(s) : a < s < b\}$ of a positively oriented rectifiable curve $C$ is $i$ times the normal $n(\zeta(t))$ at $\zeta(t)$, so the equality $d\zeta = i \, n(\zeta)\, d|\zeta|$ shows that (14) is the Cauchy integral formula for a simple closed contour $C$ bounding a region $\Omega$.

4. The Cauchy kernel

Armed with the Cauchy integral formula (14) for monogenic functions, formula (25) is established for the $n$-tuple $A = (A_1, \ldots, A_n)$ of bounded linear operators on a Banach space $X$ by substituting the $n$-tuple $A$ for the vector $y \in \mathbb{R}^n$. If $n$ is odd and $A$ is a commutative $n$-tuple, that is, $A_jA_k = A_kA_j$ for $j, k = 1, \ldots, n$, and each operator $A_j$ has real spectrum $\sigma(A_j) \subset \mathbb{R}$ for $j = 1, \ldots, n$, then for suitable $\omega \in \mathbb{R}^{n+1}$, the expression

$$G_\omega(A) = \frac{1}{\sum_{\omega}} |\omega I - A|^{-n-1}(\omega I - A)$$

is valid. Here $\mu$ is the surface measure of $\partial \Omega$. The result is proved in [9, Corollary 9.6] by appealing to by Stoke’s theorem. If $g$ is right monogenic in $U$ then $\int_{\partial \Omega} g(\omega)n(\omega)f(\omega)\, d\mu(\omega) = 0$ [9, Corollary 9.3]. The manifold $\partial \Omega$ integral (14) can then be deformed across any set not containing $x$.

In terms of differential forms, the $\mathbb{F}^{n+1}$-valued $n$-form

$$\eta = \sum_{j=0}^{n} (-1)^j e_j \, dx_0 \wedge \cdots \wedge \widehat{dx}_j \wedge \cdots \wedge dx_n \quad (\widehat{\cdot} \equiv \text{omitted}),$$

is defined on $\mathbb{R}^{n+1}$ and its pullback to the orientable $n$-dimensional manifold $\partial \Omega$ is denoted by the same symbol. Then for any continuous $\mathbb{F}^{(n)}$-valued functions $u, v$ on $\partial \Omega$, the equality

$$\int_{\partial \Omega} u\eta v = \int_{\partial \Omega} u(\omega)n(\omega)v(\omega)\, d\mu(\omega)$$

holds. Now suppose that $f, g$ are any continuously differentiable $\mathbb{F}^{(n)}$-valued functions defined on $U$. The differential of $g\eta f$ is equal to $((gD)f + g(Df))\, dx_0 \wedge \cdots \wedge dx_n$, so Stokes’ Theorem gives

$$\int_{\Omega} ((gD)f + g(Df))\, d\lambda = \int_{\partial \Omega} g\eta f$$

with respect to Lebesgue measure $\lambda$ on $\mathbb{R}^{n+1}$. The Cauchy integral formula (14) follows by shrinking $\partial \Omega$ to a sphere about $x \in \Omega$, see [9, Corollary 9.6].

**Example 3.2.** For the case $n = 1$, the Clifford algebra $\mathbb{R}_{(1)}$ is identified with $\mathbb{C}$. A continuously differentiable function $f : U \to \mathbb{R}_{(1)}$ defined in an open subset $U$ of $\mathbb{R}^2$ satisfies $Df = 0$ in $U$ if and only if it satisfies the Cauchy-Riemann equations $\overline{\partial}f = 0$ in $U$. For each $x, \omega \in \mathbb{R}^2, x \neq \omega$, the formula

$$G(\omega, x) = \frac{1}{2\pi} \frac{1}{\omega - x}$$

holds. The inverse is taken in $\mathbb{C}$. As indicated above, the tangent at the point $\zeta(t)$ of the portion $\{\zeta(s) : a < s < b\}$ of a positively oriented rectifiable curve $C$ is $i$ times the normal $n(\zeta(t))$ at $\zeta(t)$, so the equality $d\zeta = i \, n(\zeta)\, d|\zeta|$ shows that (14) is the Cauchy integral formula for a simple closed contour $C$ bounding a region $\Omega$.
makes sense as an element of \( \mathcal{L}_{(n)}(X_{(n)}) \). For an even integer \( m \)

\[
|\omega I - A|^{-m} = \left( \frac{\omega^2}{\omega_0^2} I + \sum_{j=1}^{n} (\omega_j I - A_j)^2 \right)^{m/2}
\]

and \( \omega I - A = \omega_0 I - \sum_{j=1}^{n} (\omega_j I - A_j)^2 \omega_j \) for \( \omega = \omega_0 e_0 + \omega \in \mathbb{R}^{n+1}, \omega_j = \sum_{j=1}^{n} \omega_j e_j \).

An appeal to the Spectral Mapping Theorem shows that the operator

\[
\omega_0^2 I + \sum_{j=1}^{n} (\omega_j I - A_j)^2
\]

is invertible in \( \mathcal{L}(X) \) for each \( \omega_0 \neq 0 \), so the \( \mathcal{L}_{(n)}(X_{(n)}) \)-valued function \( \omega \mapsto G_\omega(A) \) is defined on the set \( \mathbb{R}^{n+1} \setminus \{0\} \times \gamma(A) \) with

\[
\gamma(A) = \left\{ (\omega_1, \ldots, \omega_n) : \sum_{j=1}^{n} (\omega_j I - A_j)^2 \text{ is not invertible in } \mathcal{L}(X) \right\}.
\]

Off \( \gamma(A) \), the function \( \omega \mapsto G_\omega(A) \) is left and right monogenic and formula (25) defines a functional calculus \( \tilde{f} \mapsto f(A) \) which coincides with Taylor's functional calculus \( \tilde{f} \mapsto \tilde{f}(A) \). Any left monogenic function \( f \) defined in a neighbourhood \( U \) of \( \gamma(A) \) in \( \mathbb{R}^{n+1} \) has a holomorphic counterpart \( \tilde{f} \) defined in a neighborhood \( \tilde{U} \) of \( \gamma(A) \) in \( \mathbb{C}^n \) by taking the power series expansion about points of \( U \cap \{0\} \times \mathbb{R}^n \) [9]. The left monogenic function \( f \) is referred to as the Cauchy-Kowaleski extension of \( f \rvert (\tilde{U} \cap \mathbb{R}^n) \) to \( \mathbb{R}^{n+1} \) [9].

In the case of even \( n = 2, 4, \ldots \), the operator \( |\omega I - A|^{-n-1} \) needs to be defined suitably. The direct formulation employs Taylor's functional calculus, but by using the plane wave decomposition of the Cauchy kernel (see Section 4.2), the case of even \( n \) and noncommuting operators can be treated simultaneously.

For an \( n \)-tuple \( A = (A_1, \ldots, A_n) \) of commuting bounded linear operators on a Banach space \( X \) with real spectra, the nonempty compact subset \( \gamma(A) \) of \( \mathbb{R}^n \) coincides with Taylor's joint spectrum defined in terms of the Koszul complex [37, Definition III.6.4].

In general, the symbol \( \gamma(A) \) is used to denote the set of points of \( \mathbb{R}^{n+1} \) in the complement of the domain where the Cauchy kernel \( \omega \mapsto G_\omega(A) \) is defined and monogenic in the Banach module \( \mathcal{L}_{(n)}(X_{(n)}) \). For a single operator \( A \), its spectrum \( \sigma(A) \) is precisely the set of singularities of the Cauchy kernel or resolvent \( \lambda \mapsto (\lambda I - A)^{-1} \) for \( \lambda \in \mathbb{C} \), that is, the set of \( \lambda \in \mathbb{C} \) for which \( \lambda I - A \) is not invertible in \( \mathcal{L}(X) \).

A commuting \( n \)-tuple \( A = (A_1, \ldots, A_n) \) of bounded selfadjoint operators on a Hilbert space \( H \) has a ready-made functional calculus given by formula (25). The support of the joint spectral measure \( P_A \) is naturally interpreted as the joint spectrum \( \sigma(A) \) of \( A \). The observation that \( \sigma(A) \) is actually the Gelfand spectrum of the commutative \( C^* \)-algebra generated by \( A \) lends credence to the interpretation of \( \sigma(A) \) as the joint spectrum. Setting \( G_\omega(A) = \int_{\sigma(A)} G_\omega(\lambda) \, dP_A(\lambda) \) for all \( \omega \in \mathbb{R}^{n+1} \setminus \{0\} \times \sigma(A) \), it is easy to check by the vector valued version of Fubini's Theorem, that with the assumptions of formula (25)
below for the left monogenic function $f : U \to \mathbb{C}_n$ and the open set $\Omega$, the equalities

$$
\int_{\partial \Omega} G_\omega(A)n(\omega)f(\omega) \, d\mu(\omega) = \int_{\partial \Omega} \left( \int_{\sigma(A)} G_\omega(\lambda) \, dP(A)(\lambda) \right) n(\omega)f(\omega) \, d\mu(\omega)
$$

$$
= \int_{\sigma(A)} \left( \int_{\partial \Omega} G_\omega(\lambda)n(\omega)f(\omega) \, d\mu(\omega) \right) \, dP(A)(\lambda)
$$

$$
= \int_{\sigma(A)} f(\lambda) \, dP(A)(\lambda)
$$

$$
= f(A)
$$

hold. Furthermore $\gamma(A) = \sigma(A)$.

4.1. The Weyl calculus. There is an operator valued distribution $W_A$ on $\mathbb{R}^n$ associated with any $n$-tuple $A = (A_1, \ldots, A_n)$ of selfadjoint operators on a Hilbert space $H$. Now the operators $A_1, \ldots, A_n$ need not commute with each other. The distribution $W_A$ is a substitute for the joint spectral measure $P_A$. If $A$ is a commuting $n$-tuple of bounded selfadjoint operators, then $W_A : f \mapsto \int_{\sigma(A)} f \, dP_A$ for all smooth functions defined in a neighbourhood of $\sigma(A)$ in $\mathbb{R}^n$.

Suppose that $T : C^\infty(\mathbb{R}^n) \to \mathcal{L}(X)$ is an operator valued distribution with compact support $\text{supp}(T)$ acting on a Banach space $X$. For any smooth $\mathbb{C}_n$-valued function function $\check{f} = \sum_S f_S e_S$ defined in a neighborhood of $\text{supp}(T)$ in $\mathbb{R}^n$, the element $Tf$ of $\mathcal{L}_n(X(n))$ is defined by $Tf = \sum_S e_S T(f_S)$

The Cauchy integral formula (14) may be viewed as an equality

$$
f = \int_{\partial \Omega} G_\omega n(\omega) f(\omega) \, d\mu(\omega)
$$

between smooth $\mathbb{C}_n$-valued functions defined in a neighbourhood $U \cap \mathbb{R}^n$ of the support of $T$ when $f$ is left monogenic on $U$, $\text{supp}(T) \subset \Omega$ and $\overline{\Omega} \subset U$, so that

$$
Tf = T \int_{\partial \Omega} G_\omega n(\omega) f(\omega) \, d\mu(\omega)
$$

$$
= \int_{\partial \Omega} T(G_\omega) n(\omega) f(\omega) \, d\mu(\omega).
$$

The last inequality is a property of the Bochner integral for functions with values in the Fréchet space $C^\infty_n(\mathbb{R}^n)$. The $\mathcal{L}_n(X(n))$-valued function $\omega \mapsto T(G_\omega)$ is left and right monogenic in $\mathbb{R}^{n+1}$ away from $\text{supp}(T)$. If $\varphi$ is a smooth function with compact support in a neighbourhood of $\text{supp}(T)$, $x \in X$ and $\xi \in X'$, then according to [9, Theorem 27.7],

$$
\langle (T\varphi)x, \xi \rangle = \lim_{t \to 0^+} \int_{\mathbb{R}^n} \langle (T(G_{u+te_0}) - T(G_{u-te_0}))x, \xi \rangle \varphi(u) \, du,
$$

which may be compared with equation (8).

For any $n$-tuple $A = (A_1, \ldots, A_n)$ of bounded selfadjoint operators on a Hilbert space $H$, the Weyl functional calculus is the $\mathcal{L}(H)$-valued distribution

$$
W_A = \frac{1}{(2\pi)^n} (e^{i\xi \cdot A})^*.
$$
The operator $\langle \xi, A \rangle = \langle A, \xi \rangle = \sum_{j=1}^{n} \xi_j A_j$ is selfadjoint for each $\xi \in \mathbb{R}^n$ and the Fourier transform is taken with respect $\xi$ in the sense of distributions. If $A$ is a commuting system, then the distribution $\mathcal{W}_A$ is integration with respect to the joint spectral measure $P_A$.

Setting $G_\omega(A) = \mathcal{W}_A(G_\omega \upharpoonright \mathbb{R}^n)$ for $\omega \in \mathbb{R}^{n+1} \setminus \{0\} \times \text{supp}(\mathcal{W}_A)$, the equality $f(A) = \mathcal{W}_A(f \upharpoonright \mathbb{R}^n)$ holds for the element $f(A)$ of $L(\mathcal{H}_{(n)})$ defined by formula (15) below and $\gamma(A) = \text{supp}(\mathcal{W}_A)$. E. Nelson [28] has identified $\gamma(A)$ with the Gelfand spectrum of a certain commutative Banach algebra.

**Example 4.1.** Let $n = 3$ and consider the simplest noncommuting example of the Pauli matrices,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

viewed as linear transformations acting on $H = \mathbb{C}^2$. Set $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Then $\langle \xi, \sigma \rangle^2 = |\xi|^2 I$, so the exponential series gives

$$e^{i\langle \xi, \sigma \rangle} = \cos(t|\xi|)I + i\langle \xi, \sigma \rangle \frac{\sin(t|\xi|)}{|\xi|}, \quad t \in \mathbb{R}.
$$

Because $(2\pi)^{-3} (\sin(t|\xi|)/|\xi|)^3 = t \mu_t$, with $\mu_t$ the unit surface measure on the sphere $tS^2$ of radius $t > 0$ centred at zero in $\mathbb{R}^3$ and

$$
\cos(t|\xi|) = \frac{d}{ds} \frac{\sin(s|\xi|)}{|\xi|} \bigg|_{s=t},
$$

for each $f \in C^\infty(\mathbb{R}^3)$, the matrix $\mathcal{W}_{\iota\sigma}(f)$ is given by

$$
\mathcal{W}_{\iota\sigma}(f) = I \int_{tS^2} (f + \mathbf{n} \cdot \nabla f) \, d\mu_t + t \int_{tS^2} \sigma \cdot \nabla f \, d\mu_t, \quad t > 0.
$$

Here $\mathbf{n}(x)$ is the outward unit normal at $x \in S^2$. Thus, $\text{supp}(\mathcal{W}_\sigma) = S^2$.

For all $\omega = \omega_0 + \omega \in \mathbb{R}^4$ such that $\omega \notin S^2 \subset \mathbb{R}^3$, the Cauchy kernel $G_\omega(\sigma) \in L(\mathbb{C}^2)_{(3)}$ is given by

$$
G_\omega(\sigma) = \mathcal{W}_\sigma(G_\omega) = I \int_{S^2} (G_\omega + \mathbf{n} \cdot \nabla G_\omega) \, d\mu + \int_{S^2} \sigma \cdot \nabla G_\omega \, d\mu.
$$

By the operator valued version of the Paley-Wiener Theorem (see [31, Theorem 7.23] for the scalar version), the distribution (16) exists for $n$-tuple $A = (A_1, \ldots, A_n)$ of bounded linear operators on a Banach space $X$ provided that the exponential growth estimate

$$
\|e^{i\langle \zeta, A \rangle}\|_{L(X)} \leq C(1 + |\xi|)^s e^{\eta |\eta|}, \quad \zeta = \xi + i\eta, \ \xi, \eta \in \mathbb{R}^n,
$$

for $\langle \zeta, A \rangle = \sum_{j=1}^{n} \zeta_j A_j$ holds, for some positive numbers $C, s, r$ independent of $\zeta \in \mathbb{C}^n$. Then $\text{supp}(\mathcal{W}_A)$ is contained in the ball of radius $r > 0$ centred at zero in $\mathbb{R}^n$. The estimate (19) holds if $A$ is finite system of simultaneously triangularisable matrices with real eigenvalues [21, Theorem 5.10].

The Weyl functional calculus $\mathcal{W}_A$ has the property that operator products are *symmetrically* ordered. For an $n$-tuple $A = (A_1, \ldots, A_n)$ of bounded selfadjoint operators on a Hilbert space $H$, other choices of operator ordering define an operator valued distribution $\mathcal{F}_{A, \mu}$. The weighting for operator products is determined by an $n$-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of continuous Borel probability measures on $[0, 1]$, see [21, Chapter 7].
The collection of single operators $A$ satisfying the bound (19) is called the class of \textit{generalized scalar operators} and these have been extensively studied [13].

4.2. Plane wave decomposition of the Cauchy kernel. The exponential growth estimate (19) leads to a $C^\infty$-functional calculus $W_A$, so it is desirable to have a condition weaker than (19) for which the Cauchy kernel can be defined in a way that agrees with the preceding definition.

It turns out that it suffices for $A$ to be \textit{hyperbolic} to make sense of the Cauchy kernel $G_\omega(A)$ in the Cauchy integral formula (25). For matrices, this is equivalent to the bound (19), because as noted in the Introduction it says that

$$I \frac{\partial}{\partial t} + \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}$$

is a \textit{hyperbolic differential operator} on $\mathbb{R}^{n+1}$.

The key to constructing the Cauchy kernel $G_\omega(A)$ for a hyperbolic $n$-tuple $A$ of bounded linear operators is the plane wave decomposition of the fundamental solution

$$E : x \mapsto \frac{1}{\sum_{n} |x|^{n+1}}, \quad x \in \mathbb{R}^{n+1} \setminus \{0\},$$

of the generalised Cauchy-Riemann operator $D = \sum_{j=0}^{n} e_j \partial_j$. The plane wave decomposition of $E$ was first given by F. Sommen [34] and is most simply realised with the proof of Li, McIntosh, Qian [27] using Fourier analysis. The unit hypersphere $S^{n-1}$ in $\mathbb{R}^n$ is the set \{ $s \in \mathbb{R}^n : |s| = 1$ \}.

The Fourier transform of $u \in L^1 \cap L^2(\mathbb{R}^n)$ is $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) \, dx$ and the inverse map $u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \hat{u}(\xi) \, d\xi$ recovers $u$ from $\hat{u}$ when $\hat{u} \in L^1 \cap L^2(\mathbb{R}^n)$. For any $\xi \in \mathbb{R}^n$, the linear function $x \mapsto i\langle x, \xi \rangle$, $x \in \mathbb{R}^n$, defined in $\mathbb{R}^n$ extends monogenically to $\mathbb{R}^{n+1}$ to the function

$$x \mapsto i\langle x, \xi \rangle e_0 - i\xi x_0, \quad x = x_0 e + x, x_0 \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

According to the functional calculus for the selfadjoint element $i\xi$ of $\mathbb{C}(n)$, the unique monogenic extension of the function $x \mapsto e^{i\langle x, \xi \rangle}$, $x \in \mathbb{R}^n$, is given by

$$\exp(i\langle x, \xi \rangle e_0 - i\xi x_0) = e^{i\langle x, \xi \rangle - |x|x_0} \, \chi_+(x) + e^{i\langle x, \xi \rangle + |x|x_0} \, \chi_-(x)$$

for $x = x_0 e + x$ with $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

\textbf{Theorem 4.2.} Let $x = x_0 e + x$ be an element of $\mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n$. If $x_0 > 0$, then

$$E(x) = \frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n \int_{S^{n-1}} (e_0 + is)(\langle x, s \rangle - x_0 s)^{-n} \, ds.$$  \hspace{1cm} (20)

\textit{If $x_0 < 0$, then}

$$E(x) = (-1)^{n+1} \frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n \int_{S^{n-1}} (e_0 + is)(\langle x, s \rangle - x_0 s)^{-n} \, ds.$$  \hspace{1cm} (21)
Now suppose that $A$ is a hyperbolic $n$-tuple of bounded linear operators on a Banach space $X$. Then $\langle A, s \rangle - x_0 s I$ is an element of the space $L_n(X_n)$ of module homomorphisms for each $s \in S^{n-1}$. When $x_0 \neq 0$ and $a \in \mathbb{R}$, the inverse of $(aI - \langle A, s \rangle)e_0 - x_0 s I$ in the Clifford module $L_n(X_n)$ is given by

$$
((aI - \langle A, s \rangle)e_0 - x_0 s I)^{-1} = ((aI - \langle A, s \rangle) + x_0 s I)((aI - \langle A, s \rangle)^2 + x_0^2 I)^{-1}.
$$

Because $\sigma(\xi, A) \subset \mathbb{R}$, the Spectral Mapping Theorem ensures that the bounded linear operator $(aI - \langle A, s \rangle)^2 + x_0^2 I$ is invertible and

$$
\sigma((aI - \langle A, s \rangle)^2 + x_0^2 I) = \varphi_a(\sigma(\langle s, A \rangle))
$$

for the function $\varphi_a : t \mapsto (a - t)^2 + x_0^2$, $t \in \mathbb{R}$. Moreover,

$$
((aI - \langle A, s \rangle)e_0 - x_0 s I)^{-n} = ((aI - \langle A, s \rangle)e_0 + x_0 s I)^{n}((aI - \langle A, s \rangle)^2 + x_0^2 I)^{-n}
$$

in $L_n(X_n)$. To define $G_x(A)$, the promised substitution $y \rightarrow A$ in the Cauchy kernel

$$
G_x(A) = E(x - y)
$$

is now made by setting

$$
(22) \quad G_x(A) = \frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n \int_{S^{n-1}} (e_0 + is)(((x, s) - x_0 s I - \langle A, s \rangle)^{-n} ds
$$

for $x = x_0 e_0 + x$ with $x \in \mathbb{R}^n$ and $x_0 > 0$, and

$$
(23) \quad G_x(A) = (-1)^{n+1} \frac{(n-1)!}{2} \left( \frac{i}{2\pi} \right)^n \int_{S^{n-1}} (e_0 + is)(((x, s) - x_0 s I - \langle A, s \rangle)^{-n} ds
$$

for $x_0 < 0$. The expression $\langle (x, s) - x_0 s I - \langle A, s \rangle^{-n}$ is left and right monogenic in $L_n(X_n)$ for the variable $x = x_0 e_0 + x$ with $x_0 \neq 0$, so differentiating under the integral sign shows that $x \mapsto G_x(A)$ is itself two sided monogenic for $x_0 \neq 0$.

Note that for $n$ even, symmetry in the integral gives

$$
\int_{S^{n-1}} (e_0 + is)(((x, s) - x_0 s I - \langle A, s \rangle)^{-n} ds = e_0 \int_{S^{n-1}} (((x, s) - x_0 s I - \langle A, s \rangle)^{-n} ds
$$

for $n$ even and

$$
\int_{S^{n-1}} (e_0 + is)(((x, s) - x_0 s I - \langle A, s \rangle)^{-n} ds = i \int_{S^{n-1}} s (((x, s) - x_0 s I - \langle A, s \rangle)^{-n} ds
$$

for $n$ odd.

4.3. The McIntosh functional calculus. For a general matrix or operator $A$, the Riesz-Dunford formula

$$
(24) \quad f(A) = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} f(\lambda) d\lambda
$$

is valid for all functions $f$ holomorphic in a neighbourhood of the spectrum $\sigma(A)$ in the complex plane. The simple closed contour $C$ surrounds $\sigma(A)$ and is contained in the domain of $f$. The higher-dimensional analogue of the Riesz-Dunford formula (24) is then

$$
(25) \quad f(A) = \int_{\partial \Omega} G_x(A) n(x) f(x) d\mu(x)
$$
for the hyperbolic $n$-tuple $A = (A_1, \ldots, A_n)$ of bounded linear operators on a Banach space $X$, simply by substituting the $n$-tuple $A$ for the vector $x \in \{0\} \times \mathbb{R}^n$ in the Cauchy integral formula (14). The price paid is that Clifford regular functions have values in the Clifford algebra $\mathbb{C}(n)$, which is noncommutative for $n = 2, 3, \ldots$ and the correspondence between Clifford regular functions and their holomorphic counterparts needs to be investigated. The algebra $\mathbb{R}_{(1)}$ is isomorphic to $\mathbb{C}$ and formulas (25) and (24) coincide in the case $n = 1$.

Let $A$ be a hyperbolic $n$-tuple of bounded linear operators on a Banach space $X$ so that $G(A)$ is defined by the plane wave decomposition. If $x \mapsto G_x(A)$ has a continuous extension to a neighborhood in $\mathbb{R}^{n+1}$ of a point $a \in \mathbb{R}^n$, then $G(A)$ is actually monogenic in a neighborhood of $a$ in $\mathbb{R}^{n+1}$ by the monogenic analogue of Painlevé’s Theorem [9, Theorem 10.6 p. 64].

The joint monogenic spectrum $\gamma(A)$ of the $n$-tuple $A$ is the subset of $\mathbb{R}^n$ for which \{0\} $\times \gamma(A)$ is the set of singularities of the Cauchy kernel $G(A)$. In the case that the $n$-tuple $A$ satisfies the exponential growth estimates (19), the joint spectrum $\gamma(A)$ coincides with $\text{supp}(\mathcal{W}_A)$ [21, Theorem 4.8]. In the case that the $n$-tuple $A$ consists of bounded selfadjoint operators, $\gamma(A)$ equals the Gelfand spectrum of a certain commutative Banach algebra (operators) associated with $A$ [28]. The monogenic analogue of Liouville’s Theorem ensures that the joint spectrum $\gamma(A)$ is nonempty and compact [21, Theorem 4.16].

The joint spectral radius $r(A) = \sup \{|x| : x \in \gamma(A)\}$ of the hyperbolic $n$-tuple $A$ is the radius of the smallest ball centred at zero containing $\gamma(A)$.

The joint spectrum $\gamma(A)$ is the analogue of the spectrum $\sigma(A)$ of a single operator $A$ in the sense that it is the set of singularities of the Cauchy kernel $G(A)$. In the case that the $n$-tuple $A$ is a polynomial in $x$, $\gamma(A)$ equals the Gelfand spectrum of a certain commutative Banach algebra (operators) associated with $A$ [28]. The monogenic analogue of Liouville’s Theorem ensures that the joint spectrum $\gamma(A)$ is nonempty and compact [21, Theorem 4.16].

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Given a left monogenic function $f$ defined in a neighbourhood $U$ of the joint spectrum $\gamma(A)$, the element $f(A)$ of $\mathcal{L}(\mathbb{C}(n),X(\mathbb{C}(n)))$ is defined by formula (25) independently of the oriented $n$-manifold $\partial \Omega$ such that $\gamma(A) \subset \Omega$ and $\overline{\Omega} \subset U$.

A real analytic function $f : V \to \mathbb{C}$ defined in a neighbourhood $V$ of $\gamma(A)$ in $\mathbb{R}^n$ has a unique two-side monogenic extension $\tilde{f}$ (the Cauchy-Kowaleski extension) to a neighbourhood $U$ of $\{0\} \times \gamma(A)$ in $\mathbb{R}^{n+1}$. The extension is provided by an expansion in a series of monogenic polynomials [9]. Then the definition $f(A) := \tilde{f}(A)$ makes sense and does not depend on the domain $U$ of $\tilde{f}$ containing $\{0\} \times \gamma(A)$.

It is important to know that $f(A) \in \mathcal{L}(X)$ (where $\mathcal{L}(X) \equiv \mathcal{L}(X)e_0$) and what the bounded linear operator $p(A) \in \mathcal{L}(X)$ is in the case that $p$ is a polynomial in $n$ real variables. The following results are taken from [21, §4.3].

**Theorem 4.3.** Let $A$ be a hyperbolic $n$-tuple of bounded operators acting on a Banach space $X$.

(i) Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial and $\zeta \in \mathbb{C}^n$. Set $f(z) = p(\langle z, \zeta \rangle)$, for all $z \in \mathbb{C}^n$. Then $f(A) = p(\langle A, \zeta \rangle)$.

(ii) Suppose that $k_1, \ldots, k_n = 0, 1, 2, \ldots, k = k_1 + \cdots + k_n$ and $f(x) = x_1^{k_1} \cdots x_n^{k_n}$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then

$$f(A) = \frac{k_1! \cdots k_n!}{k!} \sum_{\pi} A_{\pi(1)} \cdots A_{\pi(k)},$$
where the sum is taken over every map $\pi$ of the set $\{1, \ldots, k\}$ into $\{1, \ldots, n\}$ which assumes the value $j$ exactly $k_j$ times, for each $j = 1, \ldots, n$.

(iii) Let $\Omega$ be an open set in $\mathbb{R}^{n+1}$ containing $\gamma(A)$ with a smooth boundary $\partial \Omega$. Then for all $\omega \in \overline{\Omega}$, $G_\omega(A) = \int_{\partial \Omega} G_\zeta(A) n(\zeta) G_\omega(\zeta) d\mu(\zeta)$.

(iv) Suppose that $U$ is an open neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$ and $f : U \to \mathbb{C}$ is an analytic function. Then $f(A) \in \mathcal{L}(X)$.

For a commuting $n$-tuple $A$ of bounded operators with real spectra, the McIntosh functional calculus $f \mapsto f(A)$ given by formula (25) coincides with Taylor's functional calculus $\tilde{f} \mapsto \tilde{f}(A)$ for the holomorphic counterpart $\tilde{f} : \tilde{U} \to \mathbb{C}$ of the monogenic function $f : U \to \mathbb{R}^{n+1}$, that is, $\tilde{U}$ is an open subset of $\mathbb{C}^n$ containing $\gamma(A)$ and $U$ is an open subset of $\mathbb{R}^{n+1}$ containing $\{0\} \times \gamma(A)$ such that $\tilde{f}(x) = f(x)$ for every element $x$ of the open subset $(\tilde{U} \cap \mathbb{R}^n) \cap \pi_{\mathbb{R}^n}(U \cap \{(0) \times \mathbb{R}^n\})$ of $\mathbb{R}^n$. Here $\pi_{\mathbb{R}^n}$ is the projection $\pi_{\mathbb{R}^n}(x_0, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in \mathbb{R}$.

**Theorem 4.4.** Let $A$ be a commuting $n$-tuple of bounded operators acting on a Banach space $X$ such that $\sigma(A_j) \subset \mathbb{R}$ for all $j = 1, \ldots, n$.

Then $\gamma(A)$ is the complement in $\mathbb{R}^n$ of the set of all $\lambda \in \mathbb{R}^n$ for which the operator $\sum_{j=1}^n (\lambda_j I - A_j)^2$ is invertible in $\mathcal{L}(X)$.

Moreover, $\gamma(A)$ is the Taylor spectrum of $A$. If the complex valued function $f$ is real analytic in a neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$, then the operator $f(A) \in \mathcal{L}(X)$ coincides with the operator obtained from Taylor’s functional calculus [37].

In the noncommuting case, there is no homomorphism properties for the McIntosh functional calculus, but it does enjoy symmetry properties similar to those of the Weyl calculus $\mathcal{W}_A$ when it exists, that is, when the exponential estimate (19) obtains.

The following general properties of the Weyl functional calculus [1, Theorem 2.9], suitably interpreted, are also enjoyed by the McIntosh functional calculus.

**Theorem 4.5.** Let $A$ be an hyperbolic $n$-tuple of bounded operators acting on a Banach space $X$.

(i) **Affine covariance:** if $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, then $\gamma(LA) \subseteq L\gamma(A)$ and for any function $f$ analytic in a neighbourhood in $\mathbb{R}^m$ of $L\gamma(A)$, the equality $f(LA) = (f \circ L)(A)$ holds.

(ii) **Consistency with the one-dimensional calculus:** if $g : \mathbb{R} \to \mathbb{C}$ is analytic in a neighbourhood of the projection $\pi_1 \gamma(A)$ of $\gamma(A)$ onto the first ordinate, and $f = g \circ \pi_1$, then $f(A) = g(A_1)$. We also have consistency with the $k$-dimensional calculus, $1 < k < n$.

(iii) **Continuity:** The mapping $(T, f) \mapsto f(T)$ is continuous for $T = \sum_{j=1}^n T_j e_j$ from $\mathcal{L}(n)(X(n)) \times M(\mathbb{R}^{n+1}, \mathbb{C}_{n+1})$ to $\mathcal{L}(n)(X(n))$ and from $\mathcal{L}(X) \times H_M(\mathbb{R}^n)$ to $\mathcal{L}(X)$.

(iv) **Covariance of the Range:** If $T$ is an invertible continuous linear map on $X$ and $TAT^{-1}$ denotes the $n$-tuple with entries $TA_jT^{-1}$ for $j = 1, \ldots, n$, then $\gamma(TAT^{-1}) = \gamma(A)$ and $f(TAT^{-1}) = Tf(A)T^{-1}$ for all functions $f$ analytic in a neighbourhood of $\gamma(A)$ in $\mathbb{R}^n$. 
It is time to make the first sentence of the Introduction precise by specifying the term *hyperbolic* for a system \( A \) of unbounded operators in a Banach space \( X \). The object is to ensure that the Cauchy kernel \( G_{x_0,0}(A) \) is sensibly defined for all \( x_0 \neq 0 \) and \( x \in \mathbb{R}^n \).

The set of \( s \in S^{n-1} \) with nonzero coordinates \( s_j \) for every \( j = 1, \ldots, n \) is denoted by \( S_0^{n-1} \). Then \( S_0^{n-1} \) is a dense open subset of \( S^{n-1} \) with full surface measure.

The notation \( \mathcal{D}(f) \) is used for the domain of a function \( f \).

**Definition 4.6.** An \( n \)-tuple \( A = (A_1, \ldots, A_n) \) of densely defined operators in a Banach space \( X \) is called *hyperbolic* provided that

(a) \( \cap_{j=1}^{n} \mathcal{D}(A_j) \) is dense in \( X \),
(b) \( \langle \cdot, s \cdot \rangle \) is closable on \( \cap_{j=1}^{n} \mathcal{D}(A_j) \) for every \( s \in S_0^{n-1} \) and
(c) \( \sigma(\langle \cdot, s \cdot \rangle) \subset \mathbb{R} \) for every \( s \in S_0^{n-1} \).

The operator \( \left( (\langle x, s \rangle I - \langle A, s \rangle)^2 + x_0^2 I \right)^{-n} \) makes sense for \( s \in S_0^{n-1} \) because we can interpret it as the bounded linear operator

\[
\left( (\langle x, s \rangle + i|x_0|)I - \langle A, s \rangle \right)^{-n} \left( (\langle x, s \rangle - i|x_0|)I - \langle A, s \rangle \right)^{-n}
\]

and also

\[
((\langle x, s \rangle + x_0 s)I - \langle A, s \rangle)^n \left( (\langle x, s \rangle + i|x_0|)I - \langle \overline{A}, \overline{s} \rangle \right)^{-n}
\]

is a bounded linear operator for every \( x_0 \neq 0 \) and \( x \in \mathbb{R}^n \).

Because \( S_0^{n-1} \) is a set of full measure \( x \mapsto G_x(A) \) is defined off \( \{0\} \times \mathbb{R}^n \) by formulas (22) and (23). Differentiation under the integral sign ensures that \( x \mapsto G_x(A) \) is two-sided monogenic in \( \mathcal{L}_n(X_n) \) so that the joint spectrum \( \gamma(A) \) is a closed and nonempty subset of \( \mathbb{R}^n \). The same argument works if \( S_0^{n-1} \) is replaced by a set of full Hausdorff measure. If \( X \) is a Hilbert space and elements of \( A \) are selfadjoint, then (b) implies (c) in the Definition 4.6.

The next step is to verify that (25) produces the right result for elementary functions. For bounded operators the case of polynomials is treated in Theorem 4.3. For unbounded operators we should check functions like \( x \mapsto p(x, \xi)(\lambda - \langle x, \xi \rangle)^{-k}, \; x \in \mathbb{R}^n \), for \( \lambda \in \mathbb{C}, \exists \lambda \neq 0 \) with \( p \) a polynomial of degree less than or equal to \( k = 1, 2, \ldots \). As we are concerned here only with matrices we won’t go any further into the case of unbounded operators.

5. **Systems of Matrices**

Let \( A = (A_1, \ldots, A_n) \) be a hyperbolic \( n \)-tuple of \( N \times N \) matrices. If \( A_1, \ldots, A_n \) are hermitian then the exponential bound (19) follows from the Lie-Kato-Trotter product formula with \( C = 1, s = 0 \) and \( r = \|A\| \) [36, Theorem 1].

In general, the exponential bound (19) follows from the properties of hyperbolic polynomials considered below. It is instructive to see this directly. The **characteristic polynomial** of a square matrix \( B \) is \( p_B(z) = \det(B - zI), \; z \in \mathbb{C} \).

**Proposition 5.1.** Let \( A \) be a hyperbolic \( n \)-tuple of \( N \times N \) matrices. Then for each \( r > \|A\|, \) there exists \( C > 0 \) such that the exponential bound (19) holds with \( s = N - 1 \).
Proof. Following [23, V §2], to see directly that \( A \) satisfies the bound (19) we can take the Fourier transform of (2), estimate the factor associated with a differential operator of order \( s = N - 1 \) and look at the function

\[
Z(\xi, t) = \frac{1}{2\pi} \int_{C(\xi)} e^{itz} \frac{dz}{p(\xi, \zeta)}
\]

for a suitable contour \( C(\xi) \) with \( \xi \in \mathbb{R}^n \). Let \( D_r \) be the closed disk of radius \( r \geq 0 \) centred at zero in \( \mathbb{C} (D_0 = \{0\}) \). If we know that \( \sigma((A, \xi)) \subset D_r(\xi) \) with \( r(\xi) \geq 0 \) for \( \xi \in \mathbb{R}^n \) and

\[
r = \sup_{|\xi|=1} r(\xi) < \infty,
\]

then we can take \( C(\xi) \) to be that part of the circle of radius \(|\xi|(r + 1)\) centred at zero in the half-plane \( \{ z : \Re z > -1 \} \) joined with a segment of the line \( \{ z : \Im z = -1 \} \), oriented clockwise. Then dist\((C(\xi), \sigma((A, \xi))) \geq 1 \) for each \( \xi \in \mathbb{R}^n \), so

\[
|Z(\xi, t)| \leq \frac{e^r}{2\pi} \int_{C(\xi)} \frac{|dz|}{|p(\xi, \zeta)|} \leq e^r|\xi|(r + 1), \quad \xi \in \mathbb{R}^n.
\]

With \( t = 1 \), an appeal to the Payley-Weiner Theorem shows that \( A \) is of type \((s, r)\), the Weyl functional calculus \( W_A \) exists and \( \gamma(A) = \text{supp}(W_A) \). It suffices to take

\[
r(\xi) = \|\langle A, \xi \rangle\| \leq \|A\| |\xi|.
\]

The number \( r > 0 \) in the bound (19) may be taken to be any number greater than the spectral radius \( r(A) = \sup\{|x| : x \in \gamma(A)\} \) and \( s = N - 1 \). \( \square \)

The speed of propagation is equal to the joint spectral radius \( r(A) \) [23, p. 131]. The inclusion

\[
\sigma((A, \xi)) \subset r(\xi) D_{|\xi|}, \quad \xi \in \mathbb{C}^n
\]

with the \( r(\xi) \) given by equation

\[
r(\xi) = \sup\{|x| : x \in \gamma((\langle A, \xi \rangle, \langle A, \eta \rangle))\}, \quad \xi = \xi + i\eta, \, \xi, \eta \in \mathbb{R}^n,
\]

is a general fact about bounded hyperbolic operators on a Banach space [21, Theorem 5.7]. The relevant result for hyperbolic polynomials is [20, Theorem 12.5.1].

5.1. The Fundamental Solution. Let

\[
P^A(\xi_0, \xi_1, \ldots, \xi_n) = \det(\xi_0 I + \xi_1 A_1 + \cdots + \xi_n A_n)
\]

for all \( \xi \in \mathbb{C}^{n+1} \) with the representation \( \xi = \xi_0 e_0 + \xi, \xi \in \mathbb{C}^n \).

Let \( \mathbb{R}P^n \) be real \( n \)-dimensional projective space. Then

\[
\Xi(A) = \{ (\xi_0 : \xi_1 : \cdots : \xi_n) \in \mathbb{R}P^n \mid P^A(\xi_0, \xi_1, \cdots, \xi_n) = 0 \}
\]

is an algebraic hypersurface. Identifying elements of \( \mathbb{R}P^n \) with lines \([b] \) in \( \mathbb{R}^{n+1} \), \( b \in \mathbb{R}^{n+1} \setminus \{0\} \), let \( \Gamma(P^A) \) denote the open connected component of \( \mathbb{R}^{n+1} \setminus \Xi(A) \) containing \( e_0 \). The convex cone \( \Gamma(P^A) \) in \( \mathbb{R}^{n+1} \) is called the hyperbolicity cone of \( A \). The trace of the dual cone \( K(P^A) \) of \( \Gamma(A) \) on the set \( x_0 = 1 \), referred to as the propagation set of \( A \), is given by

\[
K(A) = \{ x \in \mathbb{R}^n \mid \langle e_0 + x, \xi \rangle \geq 0, \forall \xi \in \Gamma(A) \}.
\]
In the case that $n = 2$ and $A = (A_1, A_2)$ is a pair of hermitian matrices, then the result of Kippenhahn mentioned in Section 2 ensures that the set $K(A)$ can be identified with the numerical range of the matrix $A = A_1 + iA_2$. Writing $F\phi = \hat{\phi}$, $\phi \in \mathcal{S}(\mathbb{R}^{n+1})$ for the Fourier transform, its inverse map is

$$( F^{-1}\psi)(x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{i(x,\xi)}\psi(\xi) \, d\xi, \quad \psi \in \mathcal{S}(\mathbb{R}^{n+1}), \; x \in \mathbb{R}^{n+1}$$

and by a common abuse of notation the dual linear maps on $\mathcal{S}'(\mathbb{R}^{n+1})$ are denoted by the same symbols.

The distribution

$$( F^{-1}\left( \frac{1}{P_A(\xi - i0)} \right) ) (te_0 + x)$$

is the fundamental solution of $P_A(\tau e_0 + D) = \delta_0$ in the sense that

$$( F^{-1}\left( \frac{1}{P_A(\xi - i\epsilon e_0)} \right) ) (te_0 + x) = 0, \quad t < 0.$$

Furthermore

$$\frac{1}{P_A(\xi - i0)} = \lim_{\epsilon \to 0^+} \frac{1}{P_A(\xi - i\epsilon e_0)}$$

as the limit of distributions in $\mathcal{S}'(\mathbb{R}^n)$ [3, Theorem 4.1]. Similarly, $F^{-1}\left( \frac{1}{P_A(\xi + i0)} \right)$ is supported in $-K(P_A)$. The equality (28) also establishes Proposition 5.1.

5.2. The Numerical Range Distribution. Let $n_A$ be the joint numerical range map (6) for the $n$-tuple $A$ of $(N \times N)$ hermitian matrices and $\mu$ the unitarily invariant probability on $\mathcal{S}(\mathbb{C}^N)$. Then the Borel probability measure $\nu_A = \mu \circ n_A^{-1}$ is supported by the joint numerical range $N_A = n_A(S(\mathbb{C}^N))$ of $A$. The probability measure $\nu_A$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called the numerical range distribution of the $n$-tuple $A$. For general hyperbolic matrices $A$, the measure $\nu_A$ lives on $(\mathbb{C}^n, \mathcal{B}(\mathbb{C}^n))$.

Let $P_A(\xi) = \det(\xi_0 I + \langle A, \xi \rangle)$, $\xi \in \mathbb{R}^{n+1}$. Nelson’s representation (31) suggests that the numerical range distribution $\nu_A$ and the fundamental solution of $F_{P_A}$ of $P_A(\tau, D) = \delta_0$ must be related. The relationship is curiously the basis of the generalisation of the Cauchy integral formula considered in the paper [22] which intentionally does not mention the fundamental solution $F_{P_A}$.

The adjugate of an invertible square matrix $B$ is the matrix $\text{adj}B = (\det B)B^{-1}$, that is, the transpose of the matrix of signed minor determinants of $B$. The same rules apply if $B$ is a matrix of differential operators acting on $\mathcal{S}'(\mathbb{R}^m)$, $m = 1, 2, \ldots$. 
Theorem 5.2. Let \( A \) be an \( n \)-tuple of \((N \times N)\) hermitian matrices. The equality
\[
\hat{\nu}_A(t) = (-i)^N (N-1)! \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} \right) (te_0 + \cdot), \quad t > 0.
\]
holds in the sense of distributions on \((0, \infty) \times \mathbb{R}^n\) and
\[
\nu_A = (-i)^N (N-1)! \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} \right) (e_0 + \cdot)
\]
as distributions on \( \mathbb{R}^n \). Furthermore
\[
\mathcal{W}_t A = i\text{adj}(\partial_t I + \langle A, \nabla \rangle) \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} \right) (te_0 + x)
\]
\[
= i^N \text{adj}(\tau I + \langle A, D \rangle) \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} \right) (te_0 + x), \quad t > 0.
\]

Proof. According to [21, equation (5.7)] we have
\[
\frac{1}{2\pi i} \int_{(\mathbb{R}-i\eta)^+} \frac{e^{iz}}{p(A, \xi)(z)} dz = -\frac{(-i)^{N-1}(2\pi)^n}{(N-1)!} \hat{\nu}_A(\xi)
\]
for \( \xi \in \mathbb{R}^n \) and a simple closed contour \( C(\xi) \) about the real eigenvalues \( \sigma(\langle A, \xi \rangle) \).

This equation arises in a proof of Nelson’s representation for the Weyl calculus [21, Proposition 5.4] (a forward slash is obviously missing in equations (5.4) and (5.5) there) but was not mentioned by Nelson himself. For \( N \geq 2 \), the characteristic polynomial has only real roots so that the left hand side is
\[
\frac{1}{2\pi i} \int_{(\mathbb{R}-i\eta)^+} \frac{e^{iz}}{p(A, \xi)(z)} dz
\]
for any \( \eta > 0 \), with the understanding that \( \mathbb{R} \) is oriented from \(-\infty\) to \( \infty \).

Taking a Fourier transform and substituting \( z = -\xi \) yields
\[
\frac{1}{2\pi i} \int_{(\mathbb{R}-i\eta)^+} \int_{\mathbb{R}^n} \frac{e^{-i\xi} \hat{\phi}(\xi)}{p(A, \xi)(-\xi)} d\xi d\zeta = \frac{(-i)^{N-1}(2\pi)^n}{(N-1)!} \int_{\mathbb{R}^n} \hat{\phi}(\xi) \hat{\nu}_A(\xi) d\xi
\]
\[
= \frac{1}{(2\pi)^{n+1}} \int_{(\mathbb{R}-i\eta)^+} \int_{\mathbb{R}^n} \frac{e^{-itz} \hat{\phi}(\xi)}{P(A)(z\xi_0 + \xi)} d\xi dz = \frac{(-i)^N t^{N-1}}{(N-1)!} \int_{\mathbb{R}^n} \hat{\phi}(\xi) \hat{\nu}_A(\xi) d\xi, \quad t > 0,
\]
and all \( \phi \in \mathcal{S}(\mathbb{R}^n) \).

\[
\hat{\nu}_A(t) = (-i)^N (N-1)! \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} - \frac{1}{P(A)(\xi + i0)} \right) (-te_0 - x)
\]
\[
= i^N (N-1)! \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi + i0)} \right) (-te_0 - x)
\]
\[
= (-i)^N (N-1)! \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} \right) (te_0 + x),
\]
as distributions. The distribution \( \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi - i0)} \right) \) is supported in \( K(P^A) \) while \(-te_0 - x \in -K(P^A)\) and \( \mathcal{F}^{-1} \left( \frac{1}{P(A)(\xi + i0)} \right) \) is supported in \(-K(P^A)\) as mentioned above. Convolution
with $\delta_1 \otimes \delta_0$ gives the second equality. Using similar reasoning
\[
e^{it\langle A, \xi \rangle} = \frac{1}{2\pi i} \int C(\xi) e^{i\lambda (\lambda - \langle A, \xi \rangle)^{-1}} d\lambda = -\frac{1}{2\pi i} \int_{(\mathbb{R} - in)+(\mathbb{R} + in)} e^{i\lambda \text{adj}(-\lambda I + \langle A, \xi \rangle)} \frac{1}{PA(-\lambda e_0 + \xi)}
\]
from which formula (29) follows by taking the Fourier transform. \qed

**Example 5.3.** If $\sigma$ are the Pauli matrices, then $P^\sigma(\xi) = \xi_0^2 - |\xi|^2$ for $\xi \in \mathbb{R}^{n+1}$. According to [17, p. 204], the solution of the wave equation in $\mathbb{R}^4$ is
\[
\delta(r - t)/(4\pi t) = -\mathcal{F}^{-1}\left(\frac{1}{P\sigma(\xi - i0)}\right) (te_0 + x).
\]
The negative sign in the formula is because the fundamental solution of the wave equation satisfies $p(\tau, D)E = \delta_0$ for the hyperbolic polynomial $p(\xi) = -\xi_0^2 + |\xi|^2$. The convention is required when considering nonhomogeneous hyperbolic polynomials $p$—these are not needed in the present context of hyperbolic systems $A$.

Computing $\nu_{t\sigma}$ requires the evaluation of an integral in polar coordinates for $S(\mathbb{R}^4)$ but we can simply read it off formula (18) and Nelson’s formula [21, Theorem 5.1] given by
\[
\mathcal{W}_{t\sigma} = \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{k+m} \binom{j}{m} \frac{1}{(1-j+m)!} \times \langle t\sigma, \nabla \rangle^k \tilde{\phi}_{n-j-k} \langle (t\sigma, \nabla) \rangle (\nabla \cdot id)^m \nu_{t\sigma}
\]
\[
= \sum_{j=0}^{N-1} \sum_{m=0}^{j} (-1)^{m} \binom{j}{m} \frac{1}{(1-j+m)!} \times \phi_{n-j} \langle (t\sigma, \nabla) \rangle (\nabla \cdot id)^m \nu_{t\sigma} - \langle (t\sigma, \nabla) \rangle \nu_{t\sigma}
\]
\[
= \nu_{t\sigma} - (\nabla \cdot id) \nu_{t\sigma} - \langle (t\sigma, \nabla) \rangle \nu_{t\sigma}.
\]
to verify
\[
t \nu_{t\sigma} = t\delta(r - t)/(4\pi t^2) = -\mathcal{F}^{-1}\left(\frac{1}{P\sigma(\xi - i0)}\right) (te_0 + x).
\]
Further identities follow from the Nelson formula
\[
\mathcal{W}_{tA} = \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \sum_{m=0}^{j} (-1)^{k+m} \binom{j}{m} \frac{1}{(N-1-j+m)!} \times \langle tA, \nabla \rangle^k \tilde{\phi}_{n-j-k-1} \langle (tA, \nabla) \rangle (\nabla \cdot id)^m \nu_{tA}
\]
by equating coefficients in powers of $t$ with $i\text{adj}(\partial_t I + \langle A, \nabla \rangle)\mathcal{F}^{-1}\left(\frac{1}{PA(\xi - i\eta)}\right) (te_0 + x)$.

6. **LACUNAS**

The Herglotz-Petrovsky-Leray formula for the fundamental solution $E$ of the hyperbolic differential operator $P^A(\tau, D)$ for $P^A(\zeta) = \det(\zeta_0 I + \langle A, \zeta \rangle)$, $\zeta \in \mathbb{R}^{n+1}$ is derived from the distribution
\[
\mathcal{F}^{-1}\left(\frac{1}{PA(\xi - i0)}\right)
\]
by the explicit integration of the radial variable. In particular for the hermitian case of $N \geq n + 1$, it turns out that $d\nu_A/d\lambda$ is a polynomial of degree $N - n - 1$ in regions where the Petrovsky cycle vanishes. Such regions are called lacunas because they make up the connected components of $\text{co}(N_A)^o \setminus \text{supp}(W_A)$.

In the hermitian cases $n = 2, N \geq 2$ and $n = 3, N \geq 3$, the joint numerical range $N_A$ is convex and the set $(N_A)^o \setminus \text{supp}(W_A)$ consists of the gaps between the joint numerical range $N_A$ and the support $\text{supp}(W_A)$ of the Weyl functional calculus. The trivial gaps are eliminated by restricting the elements of $A$ to the subspaces $X$ of $\mathbb{C}^N$ on which $A$ has no further nontrivial joint invariant subspaces of $X$. By spectral theory and induction $\mathbb{C}^N$ can be written as the orthogonal sum of subspaces on which $\text{supp}(W_A)$ has no further decomposition.

The linearised equations of magnetohydrodynamics correspond to the case $n = 2$ and $N = 7$ for hermitian matrices. A beautiful instrument drawing before the era of consumer computer graphics appears in Figures 1a, 1b of the fundamental paper of J. Bazer and D. Yen [5]. The corresponding symmetric matrices have no joint eigenvalues or nontrivial joint invariant subspaces.

Let $n$ be an even integer and $A = (A_1, \ldots, A_n)$ a hyperbolic $n$-tuple of $(N \times N)$ matrices. Modifications required for the case of odd $n$ are indicated later. The purpose of this section is to outline a general method using Clifford analysis for establishing that a point $x \in \mathbb{R}^n$ belongs to the joint spectrum $\gamma(A)$ or not and to determine if $\text{supp}(W_A) = K(A)$ when $A$ has no nontrivial joint invariant subspaces. Because the propagation set $K(A)$ is convex by construction and $\text{supp}(W_A) \subseteq K(A)$ [3, Theorem 4.1], the set $K(A) \setminus \text{supp}(W_A)$ consists of genuine lacunas for the Weyl functional calculus.

Roughly speaking, the approach of Atiyah, Bott and Gårding [3] is interpreted in the present matrix setting [21, Section 5.3] and we see that the detailed explanation given in [21] for the fundamental case $n = 2$ may be generalised by using the appropriate tools from algebraic topology. The presentation of this section is based on the summary of the Herglotz-Petrovsky-Leray formulas [3] given by Y. Berest in [7]. Another brief account is given in [20, Section 12.6].

A general element $x = (x_0, x_1, \ldots, x_n)$ of $\mathbb{R}^{n+1}$ will be written as $x = x + x_0 e_0$ with $x = \sum_{j=1}^n x_j e_j$. Because $n$ is assumed to be an even integer,

$$\int_{S^{n-1}} s \left( \langle xI - A, s \rangle - x_0 s \right)^{-n} ds = 0$$

and the plane wave decomposition (22), (23) for the Cauchy kernel is

$$G_x(A) = W_A(G_x)$$

$$= \frac{(n - 1)!}{2} \left( \frac{i}{2\pi} \right)^n \text{sgn}(x_0) \int_{S^{n-1}} \left( \langle xI - A, s \rangle - x_0 s \right)^{-n} ds,$$

for $x \in \mathbb{R}^{n+1}$ with $x_0 \neq 0$. For ease of notation, an element $xI$ of $L_{(n)}(\mathbb{C}^N)$ for $x \in \mathbb{C}_{(n)}$ will often be written as $x$. Because $x \mapsto G_x(A)$ is actually the Cauchy transform of the Weyl calculus $W_A$ off $\mathbb{R}^n$, we have

$$W_A = \lim_{\epsilon \to 0^+} G_{x + \epsilon e_0}(A) - G_{x - \epsilon e_0}(A)$$
in the sense of distributions. Consequently, if the limit on the right hand side of equation (32) exists uniformly for all \( x \) in a neighbourhood of a point \( a \in \mathbb{R}^n \) and is zero there, then \( a \) lies outside the support of the matrix valued distribution \( \mathcal{W}_A \), that is, \( a \in \gamma(A)^c \). We shall seek conditions which guarantee that the limit

\[
\lim_{\epsilon \to 0^+} \int_{S^{n-1}} ((xI - A, s) - \epsilon s)^{-n} + ((xI - A, s) + \epsilon s)^{-n} \, ds
\]

exists uniformly for all elements \( x \) of an open subset of \( \mathbb{R}^n \). In any case \( \mathcal{W}_A \) is \((n-1)! (\frac{1}{2\pi})^n\) times the limit (33) in the distributional sense.

For the case \( n = 2 \) considered in [21, Section 5.3], the integral (33) was calculated in an elementary manner by converting it into a contour integral and actually computing the residues associated with the spectral representation of the hermitian matrix \( \langle A, s \rangle \) following the analysis of Bazer and Yen [5].

6.1. Hyperbolic polynomials. In this subsection properties and concepts of hyperbolic polynomials are stated in the context of the determinantal polynomial

\[
P^A(\zeta) = \det(\zeta_0 I + \langle A, \zeta \rangle), \quad \zeta = \zeta_0 + \zeta \in \mathbb{R}^{n+1}
\]

associated with a hyperbolic \( n \)-tuple \( A \) of matrices. Most phenomena are already exhibited in the classes of simultaneously triangularisable matrices with real spectra, hermitian matrices and their direct sums.

A localisation \( P^A_{\xi} \) of \( P^A \) at \( \xi \in \mathbb{R}^{n+1} \), is the lowest nonzero term of the polynomial

\[
t \mapsto P^A(\xi + t\zeta) = t^{\mu_\xi} P^A_{\xi}(\zeta) + O(t^{\mu_\xi+1}), \quad \mu_\xi = \deg P^A_{\xi}.
\]

Let \( A \) and \( \xi \in \mathbb{R}^{n+1} \) be fixed. Consider the localisation \( P^A_{\xi} \) of \( P^A \) at \( \xi \). The local hyperbolicity cone and the local propagation set of \( P^A \) at \( \xi \) are defined by setting, respectively,

\[
\Gamma_{\xi}(A) := \Gamma(P^A_{\xi}), \quad K_{\xi}(A) := \{ x \in \mathbb{R}^n : [c_0 + x] \in K(P^A_{\xi}) \}.
\]

Here the polynomial \( P^A \) has been replaced by \( P^A_{\xi} \) in the definitions (26) and (27). A similar notation is used for the real lineality \( \Lambda(P^A_{\xi}) \) of the polynomial \( P^A_{\xi} \).

Clearly, \( \Gamma_{\xi}(A) \supseteq \Gamma(A) \) and, hence, \( K_{\xi}(A) \subseteq K(A) \) for all \( \xi \in \mathbb{R}^{n+1} \). More precisely, the mapping \( (\xi, A) \mapsto \Gamma_{\xi}(A) \) (and \( (\xi, A) \mapsto K_{\xi}(A) \)) is inner (resp., outer) continuous in the sense that \( \Gamma_{\xi}(A) \cap \Gamma_{\tilde{\xi}}(\tilde{A}) \) (resp., \( K_{\xi}(A) \cup K_{\tilde{\xi}}(\tilde{A}) \)) is close to \( \Gamma(\xi)(A) \) (resp., \( K(\xi)(A) \)) when \( (\xi, A) \) is close to \( (\xi, A) \) with \( \xi, \tilde{\xi} \in \mathbb{R}^{n+1} \) and \( A, \tilde{A} \) hyperbolic.

**Example 6.1.** Suppose that \( A \) consists of two hermitian matrices. Then \( \xi \in D_\mathbb{R}(A) \) is a simple point if and only if \( P^A_{\xi}(\zeta) = \langle b, \zeta \rangle \) where \( b_0 \neq 0 \) and \( b \) is the tangent vector at \( \xi \), that is

\[
[b] = \left[ \frac{\partial P^A}{\partial \xi_0}(\xi) : \frac{\partial P^A}{\partial \xi_1}(\xi) : \frac{\partial P^A}{\partial \xi_2}(\xi) \right].
\]

Then \( K_{\xi}(A) = \{ x : [c_0 + x] = [b] \} \) and

\[
C(A) = \bigcup_{\mu_\xi(A) = 1} K_{\xi}(A).
\]

[4, Theorem 14.20], shows that \( ss(\mathcal{W}_A) = C(A) \).
If \( \xi \in D_\mathbb{R}(A) \) and \( \deg P_\xi^A = 2 \) (the multiplicity of tangent vectors to \( D_\mathbb{R}(A) \) at \( \xi \)) and
\[
P_\xi^A(\zeta) = a(\zeta_0 + \langle b_1, \zeta \rangle)(\zeta_0 + \langle b_2, \zeta \rangle)
\]
for some \( a \in \mathbb{R} \), \( b_1, b_2 \in \mathbb{R}^2 \), then \( b_1, b_2 \in C(A) \) and
\[
\Gamma(P_\xi^A) = \{ \zeta_0 + \langle b_1, \zeta \rangle > 0, \zeta_0 + \langle b_2, \zeta \rangle > 0 \}, \quad K_\xi^A(\mathbb{R}) = \text{co}\{b_1, b_2\}.
\]
Hence \( K_\xi^A(\mathbb{R}) \) is the line segment joining \( b_1 \) and \( b_2 \) lying on the “double tangent” corresponding to \( \xi \in \mathbb{R}^3 \).

At this stage, we need to take into account that the homogeneous polynomial \( P^A \) may not depend on all variables in \( \mathbb{C}^{n+1} \). For example, one of the matrices \( A_j \) could be the zero matrix.

The real linearity \( \Lambda(A) \) of \( A \), is the maximal linear subspace of \( \mathbb{R}^{n+1} \) such that the restriction of \( P^A \) the quotient \( \mathbb{R}^{n+1}/\Lambda(A) \) is again a polynomial. Then \( \Lambda(A) \) coincides with the edge of the hyperbolicity cone \( \Gamma(A) \), so that \( \Gamma + \Lambda = \Gamma \), and \( K(A) \) spans the intersection of its orthogonal complement \( \Lambda^+(A) \) in \( \mathbb{R}^{n+1} \) with the plane \( x_0 = 1 \).

The system \( A \) is called complete if \( A \) has a trivial linearity. In this case, \( P_\xi^A(\zeta) \equiv P^A(\zeta) \) implies \( \xi = 0 \), the cone \( \Gamma(A) \) is proper (peaked) in the sense that \( \overline{\Gamma(A)} \) does not contain any straight lines, and then \( K(A) \) has a non-empty interior \( K^+(A) \) in \( \mathbb{R}^n \).

The wave front surface \( W(A) \) of the system \( A \) of matrices is generated by the union of local propagation cones:
\[
(34) \quad W(A) := \bigcup_{0 \neq \xi \in \mathbb{R}^{n+1}} K_\xi^A(\mathbb{R}).
\]

**Theorem 6.2** (Joswig and Straub [25]). Let \( A \) be two \((N \times N)\) hermitian matrices. The set of critical points of the numerical range map \( n_A : S(\mathbb{C}^N) \to \mathbb{R}^2 \) is \( n_A^{-1}(W(A)) \).

By the implicit function theorem, the numerical range distribution \( \nu_A \) has a real analytic density with respect to Lebesgue measure outside the wave front set \( W(A) \). Although the distribution \( \mathcal{W}_A \) consists of a differential operator of order \((N - 1)\) acting on the numerical range distribution \( \nu_A \) as may be seen from Nelson’s representation (31), the equality \( \text{ss}(\mathcal{W}_A) = C(A) \) proved in [4, Theorem 14.20], shows that \( \mathcal{W}_A \) does not see the double tangents mentioned in Example 6.1 in the hermitian case—a fact already apparent from the simplest of examples.

For an ordered set \( a = (a_1, \ldots, a_N) \) of real numbers, \( \text{diag} \ a \) denotes the \((N \times N)\) diagonal matrix with entries \( a_1, \ldots, a_N \) down the diagonal.

**Example 6.3.**

(a) Let \( A_1 = \text{diag}(1,0), \ A_2 = \text{diag}(0,1) \) and \( A = (A_1, A_2) \). Then
\[
\mathcal{W}_A = \delta_{(1,0)}P_1 + \delta_{(0,1)}P_2, \quad P_1 : x \mapsto \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \quad P_2 : x \mapsto \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \quad x \in \mathbb{R}^2,
\]
but \( W(A) = K_{(1,1)}(\mathbb{R}) = \text{co}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \).

(b) Let \( A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Then as in [21, Example 5.31], we have
\[
\mathcal{W}_A = \begin{pmatrix} 0 & \delta_0 \otimes \chi_{[0,1]} \\ 0 & \delta_0 \otimes \delta_1 \end{pmatrix} + \begin{pmatrix} 0 & \delta_0 \otimes (\delta_0 - \delta_1) \\ 0 & \delta_0 \otimes \delta_1 \end{pmatrix}
\]
so \( \text{supp}(\mathcal{W}_A) = \{0\} \times [0, 1] = K(A) = W(A) \). The polynomial \( P^A \) is not complete [21, Example 5.36]. The spectral projections associated with \( \langle A, \zeta \rangle \) for \( \zeta \in \mathbb{C}^2 \) have poles on the unit circle centred at zero—a phenomenon forbidden by Rellich’s lemma in the hermitian case [25].

(c) Not so obvious is that the operator valued distribution \( \mathcal{W}_A \) vanishes across the double tangent of the Kippenhahn curve \( C(A) \) associated with

\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\]

The curve \( C(A) \) is the third type in Kippenhahn’s classification [26, Theorem 26] of numerical range of \((3 \times 3)\) matrices, see [25, Figure X]. The argument goes as follows. The couple \( A = A_1 + iA_2 \) has no joint invariant subspaces otherwise \( \text{supp}(\mathcal{W}_A) \) would be an elliptical region or an elliptical region together with an outside point—the joint eigenvalue, or just finitely many points corresponding to the case when \( A_1 + iA_2 \) is a normal matrix, which it is not.

The elliptical region for two \((3 \times 3)\) matrices \( B \) derives from a two dimensional joint invariant subspace \( X \), should one exist, on which \( B \) is represented by \((2 \times 2)\) hermitian matrices. Denote the restriction of \( B \) to the two dimensional joint invariant subspace \( X \) by \( B_X \). The Pauli matrices \( \sigma \) and the identity \( I_2 \) form a linear basis of the \((2 \times 2)\) hermitian matrices, so there exists an affine transformation \( L : x \mapsto Tx+a, x \in \mathbb{R}^3 \) with rank two linear \( T \) and \( a \in \mathbb{R}^2 \) such that \( B_X = T\sigma + aI_2 \).

By Theorem 4.5 (i), \( \mathcal{W}_{B_X} \) is an affine transformation of \( \mathcal{W}_\sigma \) and \( \gamma(B_X) = LS(\mathbb{R}^3) \) is an elliptical region in \( \mathbb{R}^2 \). However, \( C(A) \) is a cardioid so the set \( (N_A)^c \setminus \text{supp}(\mathcal{W}_A) \) is a nontrivial lacuna of \( \mathcal{W}_A \).

To check the double tangent, observe that

\[
\zeta_0 I + \langle A, \zeta \rangle = \begin{pmatrix}
\zeta_0 + \z_1 & 0 & \zeta_2 \\
0 & \zeta_0 - \z_1 & \zeta_2 \\
\zeta_2 & \zeta_2 & \zeta_0 - \z_1
\end{pmatrix}
\]

and

\[
P^A(\zeta) = (\zeta_0 + \z_1)((\zeta_0 - \z_1)^2 - \zeta_2^2) - (\zeta_0 - \z_1)\zeta_2^2 \\
= (\zeta_0^2 - \z^2)(\zeta_0 - \z_1) - 2\zeta_0\zeta_2^2.
\]

Then \( P^A((1, 1, 0)) = 0 \) and \( P^A_{(1,1,0)}(\zeta) = 2((\zeta_0 - \z_1)^2 - \zeta_2^2) \), so a line parallel to \( \{x_1 = 0\} \) is tangential to \( C(A) \) at two points satisfying \( x_2 = \pm(x_1 - 1) \) in \( \mathbb{R}^2 \).

We finish Part I of this paper with an explicit calculation of \( E_A = F^{-1}\left(\frac{1}{P^A(\zeta-\alpha)}\right) \) for two \((3 \times 3)\) hermitian matrices \( A \) with a joint eigenvalue. Here \( E_A \) has a Petrovsky lacuna on which \( \mathcal{W}_A \) vanishes. In this example, it is easy to write down the Weyl functional calculus:

\[
\mathcal{W}_A = \delta_a P_a \oplus \mathcal{W}_{(\sigma_1, \sigma_2)} P
\]

for selfadjoint projections \( P_a, P \) with \( P_a + P = I \) on \( \mathbb{C}^3 \) and as mentioned above \( \mathcal{W}_{(\sigma_1, \sigma_2)} \) is the affine image of \( \mathcal{W}_\sigma \) so \( \text{supp}(\mathcal{W}_A) \) is the union of \( \{a\} \) and the closed unit disk centred
at zero. The numerical range \( N_A \) is the convex hull of \( \text{supp}(\mathcal{W}_A) \). On the other hand calculating the fundamental solution \( E_A \) takes more effort.

**Example 6.4.** Let \( A = (a_1 \oplus \sigma_1, a_2 \oplus \sigma_2) \) for the Pauli matrices (17) and \( a \in \mathbb{R}^2 \). Then

\[
P_A(\xi) = \det(\xi_0 I + \langle A, \xi \rangle) = (\xi_0 + \langle a, \xi \rangle)(\xi_0^2 - |\xi|^2),
\]

The Kippenhahn curve \( C(A) \) is the union of the singleton \( \{a\} \) and the unit circle centred at 0 and \( N_A \) is the convex hull of \( C(A) \).

Then by \([7, \text{equation (4.1)}]\) we have

\[
F^{-1} \left( \frac{1}{P(\xi - i0)} \right) = \frac{i}{2\pi^\frac{3}{2} \Gamma(\frac{1}{2})} H(x_0 + \langle a, x \rangle) \delta(x - x_0a) \ast H(x_0)(x_0^2 - |x|^2)^{\frac{1}{2}}
\]

\[
= \frac{i}{2\pi^\frac{3}{2} \Gamma(\frac{1}{2})} \int_0^\infty H(x_0 - y_0)(y_0^2 - |x - (x_0 - y_0)a|^2)^{\frac{1}{2}} dy_0
\]

\[
= \frac{i}{2\pi^\frac{3}{2} \Gamma(\frac{1}{2})} \int_0^{x_0} (y_0^2 - |x - (x_0 - y_0)a|^2)^{\frac{1}{2}} dy_0,
\]

where \( H \) is the Heaviside function and \( \delta \) is the unit point mass at zero, both interpreted as distributions. The notation means that we are integrating the function

\[
y_0 \longmapsto (y_0^2 - |x - (x_0 - y_0)a|^2)^{\frac{1}{2}}
\]

with values in the space of distributions over the interval \([0, x_0]\).

Suppose \( x_0 = 1, 1 < |x| < |a| \). Then \( \langle a, x - a \rangle < 0 \) and \( |\langle a, x - a \rangle| > \sqrt{|a|^2 - 1}|x - a| \) because the angle \( \theta \) between \( a \) and \( x - a \) satisfies \( \cos \theta > \sqrt{|a|^2 - 1}/|a| \).

\[
y_0^2 - |x - (1 - y_0)a|^2 = -(|x - a|^2 - 2\langle a, x - a \rangle y_0 + (|a|^2 - 1)y_0^2) := p(y_0).
\]

\[-(|a|^2 - 1)(y_0 + \langle a, x - a \rangle/|a|^2 - 1)) - (\langle a, x - a \rangle^2/(|a|^2 - 1) - |x - a|^2/(|a|^2 - 1)) \]

If \( |\langle a, x - a \rangle| < \sqrt{|a|^2 - 1}|x - a| \), then the expression is negative and the integral is zero. Suppose that \( |\langle a, x - a \rangle| > \sqrt{|a|^2 - 1}|x - a| \).

\[
y_0 = \left(-\langle a, x - a \rangle \pm \sqrt{\langle a, x - a \rangle^2 - (|a|^2 - 1)|x - a|^2} \right) \vee 0/(|a|^2 - 1)
\]

\[
\langle a, x - a \rangle^2 - (|a|^2 - 1)|x - a|^2 = (|a|^2 \cos \theta - (|a|^2 - 1))|x - a|^2 > 0.
\]

If \( |x| > 1 \), then the limits of integration are

\[
(y_0 + \langle a, x - a \rangle/|a|^2 - 1)) = \pm \sqrt{\langle a, x - a \rangle^2 - (|a|^2 - 1)|x - a|^2/|a|^2 - 1}.
\]

If \( |x| < 1 \), then \( -\langle a, x - a \rangle + \sqrt{\langle a, x - a \rangle^2 - (|a|^2 - 1)|x - a|^2/|a|^2 - 1} > (|a|^2 - 1) \) and

\[
0 < -\langle a, x - a \rangle - \sqrt{\langle a, x - a \rangle^2 - (|a|^2 - 1)|x - a|^2} < (|a|^2 - 1)
\]

because the polynomial \( p \) is equal to \( 1 - |x|^2 \) at \( y_0 = 1 \), so it has a zero to the right of 1.

If \( |x| > 1 \) and \( |\langle a, x - a \rangle| > |a|^2 - 1 \), that is, on the other side of the unit circle to \( a \), then the polynomial \( p \) has zeros on the positive axis because it has the values \(-|x - a|^2\) at \( y_0 = 0, 1 - |x|^2 \) at \( y_0 = 1 \) and the maximum value is at

\[
y_0 = |\langle a, x - a \rangle|/(|a|^2 - 1) > 1.
\]

Hence the least zero of \( p \) satisfies the inequality

\[
1 < \frac{-\langle a, x - a \rangle - \sqrt{\langle a, x - a \rangle^2 - (|a|^2 - 1)|x - a|^2}}{|a|^2 - 1} < \frac{|\langle a, x - a \rangle|}{|a|^2 - 1}
\]
and \( p(y_0) = y_0^2 - |x - (1 - y_0)a|^2 < 0 \) for all \( 0 \leq y_0 \leq 1 \). The polynomial \( p \) is increasing on \([0, 1]\) where \((y_0^2 - |x - (x_0 - y_0)a|^2)_+ \) vanishes.

Let \( u(x) = \left( \sqrt{(a, x - a)^2 - (|a|^2 - 1)|x - a|^2} \right) / (|a|^2 - 1) \).
\[
\int_0^1 \frac{1}{\sqrt{b^2 - x^2}} \, dx = \sin^{-1}(x/b)
\]
Then for \(|x| > 1\), \( \cos \theta > \sqrt{|a|^2 - 1/|a|} \) and \( (a, x - a) < 0 \), the integral
\[
\int_0^1 (y_0^2 - |x - (x_0 - y_0)a|^2)^{1/2} \, dy_0
\]
equals
\[
\left[ \sin^{-1} \left( t(|a|^2 - 1)/\sqrt{(a, x - a)^2 - (|a|^2 - 1)|x - a|^2} \right) \right]_0^{t = u(x)} = \pi.
\]
For \(|x| \leq 1\), \( \int_0^1 (y_0^2 - |x - (x_0 - y_0)a|^2)^{1/2} \, dy_0 \) equals
\[
\left[ \sin^{-1} \left( t(|a|^2 - 1)/\sqrt{(a, x - a)^2 - (|a|^2 - 1)|x - a|^2} \right) \right]_{t = -u(x)}^{t = 1 + (a, x - a)/(|a|^2 - 1)}
\]
\[
= \pi/2 + \sin^{-1} \left( (a, x - 1)/\sqrt{(a, x - a)^2 - (|a|^2 - 1)|x - a|^2} \right).
\]
Also \( W_A = \delta_a P_a \oplus W_{(\sigma_1, \sigma_2)} P_2 \)

To calculate the wave front set, we note that
\[
(\xi_0 + t\zeta_0^2 - |\xi + t\zeta|^2 = \xi_0^2 - |\xi|^2 + 2t(\xi_0 \zeta_0 - (\xi, \zeta)) + t^2(\zeta_0^2 - |\zeta|^2)).
\]
For \( \xi_0^2 - |\xi|^2 = 0\), \( \eta = \pm |\xi| e_0 - \xi\), we have \( \Gamma_\eta = \{ \pm \eta, (\xi, \zeta) > 0 \} \), \( K_\eta = \{ \pm \eta |t > 0 \} \). \( \cup_{\eta \neq 0} K_\eta \) is a circular cone in \( \mathbb{R}^3 \) and
\[
\begin{align*}
P(\xi) &= (\xi_0 + (a, \xi))(\xi_0^2 - |\xi|^2) \\
\Gamma(P) &= \{ \xi_0 + (a, \xi) > 0, \xi_0 - |\xi| > 0, \xi_0 + |\xi| > 0 \} \\
K(P) &= \text{co}\{ t(e_0 + a), t(e_0 + \xi) : |\xi| = 1, \xi \in \mathbb{R}^n, t > 0 \}
\end{align*}
\]
\[
P(\xi + t\zeta) = (\xi_0^2 - |\xi|^2)(\xi_0 + (a, \xi)) + 2t(1/2)(\xi_0^2 - |\xi|^2)(\xi_0 + (a, \xi)) + (\xi_0 \zeta_0 - (\xi, \zeta))(\xi_0 + (a, \xi))
\]
\[
+ t^2((\zeta_0^2 - |\zeta|^2)(\xi_0 + (a, \xi)) + 2(\xi_0 \zeta_0 - (\xi, \zeta))(\xi_0 + (a, \zeta)) + t^3(\zeta_0^2 - |\zeta|^2)(\xi_0 + (a, \zeta))
\]
\[
\xi_0 + (a, \xi) = 0, \eta = e_0 + a.
\]
\[
|a| > 1, \quad \xi_0^3 = |\xi|^2, (a, \xi) = -\xi_0.
\]
\[
P_\xi(\zeta) = 2(\xi_0 \zeta_0 - (\xi, \zeta))(\xi_0 + (a, \zeta))
\]
\[
\Xi_\xi(\zeta) = \{ \zeta : (\xi', \zeta_0 - (\xi', \zeta) = 0 \} \cup \{ \zeta_0 + (a, \xi) = 0 \}, \ \xi_0 = |\xi|'.
\]
Because \( -\xi' = \xi \) for one of the two solutions of \( (a, \xi) = |\xi| \), this set is equal to
\[
\Xi_\xi(\zeta) = \{ \zeta : (\xi_0 + (a, \zeta) = 0 \} \cup \{ \zeta_0 + (a, \xi) = 0 \}, \ \xi_0 = -|\xi| \}
\]
Now choose \( |\xi| = 1 \). Then
\[
\Gamma_\xi = \{ \zeta : \zeta_0 + (\xi, \zeta) > 0, \zeta_0 + (a, \zeta) > 0 \},
\]
\[
K_\xi = \text{co}\{ \xi, a \}, \ \xi_0 = -1,
\]
for each of the two solutions of \( (a, \xi) = 1 \). The wave front set \( W(\mathcal{A}) \) is the unit circle plus \{a\} as well as segments tangential to the circle meeting at \{a\}.
References

[1] R.F.V. Anderson, The Weyl functional calculus, *J. Funct. Anal.* **4** (1969), 240–267.

[2] ———, On the Weyl functional calculus, *J. Funct. Anal.* **6** (1970), 110–115.

[3] M. Atiyah, R. Bott, L. Gårding, Lacunas for hyperbolic differential operators with constant coefficients I, *Acta Math.* **124** (1970), 109–189.

[4] ———, Lacunas for hyperbolic differential operators with constant coefficients II, *Acta Math.* **131** (1973), 145–206.

[5] J. Bazer and D.H.Y. Yen, The Riemann matrix of (2+1)-dimensional symmetric hyperbolic systems, *Comm. Pure Appl. Math.* **20** (1967), 329–363.

[6] ———, Lacunas of the Riemann matrix of symmetric-hyperbolic systems in two space variables, *Comm. Pure Appl. Math.* **22** (1969), 279–333.

[7] Y. Berest, The problem of lacunas and analysis on root systems, *Trans. Amer. Math. Soc.* **352** (2000), 3743–3776.

[8] J. Bochnak, M. Coste, and M.-F. Roy, *Géométrie Algébrique Réelle*, Springer, New York – Berlin, 1987.

[9] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Research Notes in Mathematics 76, Pitman, Boston/London/Melbourne, 1982.

[10] H. Bremermann, *Distributions, complex variables, and Fourier transforms*, Addison-Wesley, 1965.

[11] C. Canzi and G. Guerra, A simple counterexample related to the Lie-Trotter product formula. *Semigroup Forum* **84** (2012), 499-504, https://doi.org/10.1007/s00233-011-9326-6.

[12] W. Case, Wigner functions and Weyl transforms for pedestrians, *American Journal of Physics* **76** (2008), 937–946; https://doi.org/10.1119/1.2957889.

[13] I. Colojoara and C. Foias, *Theory of Generalized Spectral Operators*, Gordon and Breach, 1968.

[14] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics **194**, SpringerVerlag, New York, 2000.

[15] R. Feynman, An operator calculus having applications in quantum electrodynamics, *Phys. Rev.* **84** (1951), 108–128.

[16] L. Gårding, An inequality for hyperbolic polynomials, *J. Math. Mech.* **8** (1959) 957–965.

[17] I. M. Gel’fand, N. Ya. Vilenkin, *Generalized Functions*, Volume 4: Applications of Harmonic Analysis, AMS Chelsea Publishing **380**, 1964.

[18] L. Hörmander, The Weyl calculus of pseudodifferential operators, *Comm. Pure Appl. Math.* **32** (1979), 359–443.

[19] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol I, Springer, 1985.

[20] ———, *The Analysis of Linear Partial Differential Operators*, vol II, Springer, 1985.

[21] B. Jefferies, *Spectral Properties of Noncommuting Operators*, Lecture Notes in Mathematics **1843**, Springer 2004.

[22] ———, A Generalisation of the Cauchy Integral Formula for Normal Matrices. *Complex Anal. Oper. Theory* **6** (2012), 1037-1046, https://doi.org/10.1007/s11785-012-0226-x

[23] F. John, *Partial Differential Equations* (4th ed.), Springer, 1982.

[24] G.W. Johnson, M. Lapidus and L. Nielsen, *Feynman’s Operational Calculus and Beyond: Noncommutativity and Time-Ordering*, OUP Oxford, 2015.

[25] M. Joswig and B. Straub On the numerical range map, *J. Austral. Math. Soc. (Series A)* **65** (1998), 267–283.

[26] R. Kippenhahn, Über den Wertevorrat einer Matrix, *Math. Nachr.* **6** (1951), 193–228. (Transl. by Paul F. Zachlin & Michiel E. Hochstenbach (2008), On the numerical range of a matrix, *Linear and Multilinear Algebra*, 56:1-2, 185-225).
[27] C. Li, A. McIntosh, T. Qian, Clifford algebras, Fourier transforms and singular convolution operators on Lipschitz surfaces, *Rev. Mat. Iberoamericana* **10** (1994), 665–721.

[28] E. Nelson, Operants: A functional calculus for non-commuting operators, in: *Functional analysis and related fields, Proceedings of a conference in honour of Professor Marshal Stone*, Univ. of Chicago, May 1968 (F.E. Browder, ed.), Springer-Verlag, Berlin/Heidelberg/New York, 1970, pp. 172–187.

[29] I. Petrovsky On the diffusion of waves and lacunas for hyperbolic equations, *Mat. Sbornik* **17** (1945), 289–368.

[30] D. Plaumann, R. Sinn, S. Weis, Kippenhahn’s Theorem for joint numerical ranges and quantum states, *J. Appl. Algebra Geometry* **5** (2021), 86–113.

[31] W. Rudin, *Real and Complex Analysis*, McGraw-Hill 2nd Ed., New York, 1987.

[32] I.R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, New York/Berlin, 1977.

[33] R. Schwonnek and R. Werner, The Wigner distribution of $n$ arbitrary observables, Journal of Mathematical Physics **61** (2020), 082103; https://doi.org/10.1063/1.5140632

[34] F. Sommen Plane wave decompositions of monogenic functions, *Annales Pol. Math.* **49** (1988), 101–114.

[35] M.E. Taylor, Functions of several self-adjoint operators, *Proc. Amer. Math. Soc.* **19** (1968), 91–98.

[36] ______, *Pseudodifferential Operators*, Princeton U.P., Princeton, 1981.

[37] F.-H. Vasilescu, *Analytic functional calculus and spectral decompositions*, Mathematics and its Applications (East European Series), D. Reidel Publishing Co., Dordrecht, 1982.

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