Relating metric and covariant perturbation theories in $f(R)$ gravity

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Modified theories of gravity have been invoked recently as an alternative to dark energy, in an attempt to explain the apparent accelerated expansion of the universe at the present time. In order to describe inhomogeneities in cosmological models, cosmological perturbation theory is used, of which two formalisms exist: the metric approach and the covariant approach. In this paper I present the relationship between the metric and covariant approaches for modeling $f(R)$ theories of gravity. This provides a useful resource that researchers primarily working with one formalism can use to compare or translate their results to the other formalism.

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I. INTRODUCTION

Current observational evidence indicates that we are living in a universe well described by $\Lambda$CDM cosmology [1]. This is a model based on the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker solution to general relativity, complete with small inhomogeneities, and whose matter content at the present day is dominated by a cosmological constant and cold dark matter.

However, attempts to reconcile particle physics with general relativity, result in the so-called ‘cosmological constant problem’ (see, e.g., Ref. [2]). That is, the observed value of $\Lambda$ differs from that predicted from fundamental theories by over a hundred orders of magnitude. In light of this, there has been much recent work on attempting to pinpoint the nature of this dark energy component of the universe. There are essentially two methods of modeling the present day acceleration of the expansion of the universe, aside from a cosmological constant term. We can either introduce a dark energy fluid with a negative pressure (e.g. Ref. [2]), or we can drop the assumption that Einstein’s gravity is valid on all scales, adding modifications on the largest scales (e.g. Ref. [3]).

One particular theory of modified gravity consists of a modification of the Einstein-Hilbert
action to depend upon a function of the Ricci scalar. This is perhaps the most popular modified
gravity theory of recent years, and is called $f(R)$ gravity \[4–7\]. This is the modified theory of
gravity on which we focus in the present article.

Observational evidence points towards the existence of small inhomogeneities, generated during
the inflationary phase, as the seeds of large scale structure. There are two popular techniques for
modeling these inhomogeneities. The first is inspired by the pioneering early research by Lifshitz \[8\]
and developed by Bardeen \[9\] and is based around considering perturbations to the FLRW metric.
The second method, dubbed the ‘covariant approach’ follows work by Ellis, Bruni and collaborators
\[10\]. To date, the majority of the study of inhomogeneous perturbations of $f(R)$ theories have been
completed in the metric formalism (see Ref. \[7\] for a detailed reference list). However, recently some
authors have used the covariant approach to study the perturbations \[11\]. These approaches each
suffer from their own strengths and weaknesses (see, e.g., Ref. \[12\]) however, they are equivalent,
describing the same physical universe, and thus results obtained in each formalism should be in
agreement.

Previous work has studied the equivalence between the two formalisms for Einstein gravity \[13\].
However, to date, the relationship between the formalisms has not been presented for modified
gravity and, in particular, for $f(R)$ gravity. In this paper we perform such a study, extending
Ref. \[13\] to $f(R)$ gravity. This will enable authors working in one formalisms to compare their
results to the other. The paper is organized as follows: in the next section we review the basics
of $f(R)$ gravity and define our notation. In Section \[\text{III}\] we present perturbations in the metric
formalism, picking two particularly popular gauges before considering the covariant formalism in
Section \[\text{IV}\]. In Section \[\text{V}\] we relate the two approaches, showing how to transform from one to the
other, before concluding in Section \[\text{VI}\].
spacetime range \((0, \ldots, 3)\), and \(i, j, \ldots\) denote spatial indices \((1, \ldots, 3)\).

Alternatively, one can write the field equations as

\[
FG_{ab} = T_M^{ab} + \frac{1}{2} g_{ab}(f - RF) + \nabla_b \nabla_a F - g_{ab} \Box F,
\]

where \(G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R\) is the usual Einstein tensor. This equation can then be written as

\[
G_{ab} = \tilde{T}_M^{ab} + T_R^{ab},
\]

where \(\tilde{T}_M^{ab} = T_M^{ab} / F\) is the rescaled matter energy-momentum tensor, and

\[
T_R^{ab} = \frac{1}{F} \left[ \frac{1}{2} g_{ab}(f - RF) + \nabla_b \nabla_a f - g_{ab} \Box F \right],
\]

is the energy-momentum tensor of the effective ‘curvature fluid’. It is important to be able to write the system as general relativity with effective fluids, since it enables one to apply the covariant approach to perturbation theory to the model.

III. METRIC PERTURBATION THEORY

Metric perturbation theory has been studied by many authors over the past few decades [14,15] building upon the first comprehensive work on gauge invariant linear cosmological perturbations conducted by Bardeen [9]. In the years since, metric perturbation theory has been extended to second order and beyond (see, e.g., Refs. [16–19] and references therein), and recently studies have been developed to encompass modified gravity theories, such as \(f(R)\) (see, e.g., Refs. [20,21]).

In metric perturbation theory we consider small, inhomogeneous perturbations to the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) background spacetime. Doing so gives the perturbed line element

\[
ds^2 = a^2(\eta) \left[ -1(1 + 2\phi) d\eta^2 + 2B_i dx^i d\eta + (\gamma_{ij} + 2C_{ij}) dx^i dx^j \right],
\]

where \(\eta\) is conformal time, \(a(\eta)\) is the scale factor, \(\gamma_{ij}\) is the metric on the spatial 3-hypersurface, and a bar denotes the covariant derivative with respect to this metric. The perturbations can be further split up using the scalar-vector-tensor decomposition [22] as

\[
B_i = B_{|i} - S_i,
\]

\[
C_{ij} = -\psi \gamma_{ij} + E_{|ij} + F_{(ij)} + \frac{1}{2} h_{ij}.
\]
In Appendix A we show the relationship between variables using different notational conventions.

Considering now only scalar perturbations, with a flat spatial metric, results in the line element

\[ ds^2 = a^2(\eta) \left[ - (1 + 2\phi)d\eta^2 + 2B_{ij}d\eta^i d\eta^j + \left( (1 - 2\psi)\delta_{ij} + 2E_{ij} \right) dx^i dx^j \right], \quad (3.4) \]

The background Einstein equations are then

\[ 3F\dot{H}^2 = \frac{a^2}{2}(FR - f) - 3\dot{H}F + \rho_0, \quad (3.5) \]
\[ -2\dot{H}F = \dot{F} - 2\dot{H}F + a^2(\rho_0 + P_0), \quad (3.6) \]

where a dot denotes a derivative with respect to conformal time, \( \dot{H} = \dot{a}/a \) is the Hubble parameter, and the background Ricci scalar is

\[ R = \frac{6}{a^2}(\dot{H}^2 + \dot{\mathcal{H}}). \quad (3.7) \]

The Einstein equations for the linear perturbations then give an equation from the (ADM) energy constraint

\[-\nabla^2\psi + 3\mathcal{H}(\dot{\psi} + \mathcal{H}\phi) - \nabla^2(\dot{E} - B) = \frac{1}{2F} \left[ \dot{F}\nabla^2(\dot{E} - B) - (\nabla^2 + 3\mathcal{H})\delta F + 3\dot{H}\delta F - 3\dot{F}(\dot{\psi} + 2\mathcal{H}\phi) - 8\pi G a^2 \delta \rho \right], \quad (3.8)\]

and from the momentum constraint

\[ \mathcal{H}\phi + \dot{\psi} = \frac{1}{2F} \left[ \delta F - \dot{\mathcal{H}}\phi - \mathcal{H}\delta F - 8\pi G (\rho_0 + P_0) a^2 (v + B) \right], \quad (3.9) \]

where \( \delta F = \frac{\partial F}{\partial \rho} \delta \rho = F\delta R \). From the ADM propagation equation \((\tilde{G}^i_j - \frac{1}{3} \delta^i_j \tilde{G}^k_k \) component\), after applying the operator \( \partial_i \partial^i \),

\[ \ddot{E} - \ddot{B} + 2\mathcal{H}(\dot{E} - B) + \psi - \phi = \frac{1}{F} \left[ \delta F - \dot{\mathcal{H}}(\dot{E} - B) \right]. \quad (3.10) \]

The Raychaudhuri equation \((\tilde{G}^k_k - \tilde{G}^0_0 \) component\) gives the equation

\[ 3\ddot{\psi} + 3\mathcal{H}(\dot{\phi} + \dot{\psi}) - \nabla^2(\dot{E} - \dot{B}) - 2\mathcal{H}\nabla^2(\dot{E} - B) + \left( 6\dot{\mathcal{H}} + 3\mathcal{H}^2 + \nabla^2 + \frac{\dot{F}}{F} \right) \phi \]
\[ + \frac{\dot{F}}{2F} \left[ 3\ddot{\psi} + 3\dot{\phi} - \nabla^2(\dot{E} - B) \right] = \frac{1}{2F} \left[ 3\delta F - (6\mathcal{H}^2 + \nabla^2)\delta F - 8\pi G a^2 (\delta \rho + \delta P) \right], \quad (3.11) \]

and finally, the trace equation \((\tilde{G}^a_a \equiv \tilde{G}^0_0 + \tilde{G}^k_k \) component\) gives the equation

\[ \delta \dot{F} + 2\mathcal{H}\delta F - \left( \frac{R}{3} + \nabla^2 \right) \delta F = \frac{8\pi G}{3} a^2 (3\delta P - \delta \rho) + \dot{F} \left[ 4\mathcal{H}\phi + 3\dot{\psi} - \nabla^2(\dot{E} - B) + \dot{\phi} \right] + 2\ddot{F} \phi - \frac{1}{3} a^2 F \delta R. \quad (3.12) \]

The perturbed Ricci scalar is

\[ \delta R = \frac{2}{a^2} \nabla^2 \left( 2\psi - \phi + \dot{E} - \dot{B} \right) - \frac{6}{a^2} \left[ \mathcal{H}(\phi + 3\dot{\psi}) - \mathcal{H}\nabla^2(\dot{E} - B) + \dot{\psi} + 2(\dot{\mathcal{H}} + \mathcal{H}^2) \phi \right], \quad (3.13) \]

where the spatial Laplacian is \( \nabla^2 \equiv \partial_k \partial^k \).
A. Gauge Choice

When using cosmological perturbation theory, one encounters the ‘problem’ of gauge invariance. As described above, the formalism requires the splitting of the spacetime into a background spacetime and a perturbed spacetime. However, this method of splitting is not a covariant process. That is, one can make a choice of ‘gauge’ which relates points on the background spacetime to points on the perturbed spacetime, but the choice is not unique. Therefore, quantities can change depending on the choice of coordinate correspondence.

One resolution of this issue was proposed by Bardeen in 1980 \cite{9} where he first introduced the idea of looking at solely gauge invariant variables. These are quantities constructed such that they do not change under a gauge transformation. This is equivalent to eliminating the gauge degrees of freedom from the metric from the outset, therefore guaranteeing that one is working with only gauge invariant variables. In the previous section equations were presented without fixing a gauge. Now, we highlight a couple of common gauges, and present the governing equations for $f(R)$ gravity theories in these gauges.

1. Longitudinal gauge

The longitudinal gauge is the gauge in which the shear metric perturbation, $\sigma \equiv \dot{E} - B$, is zero. This gives $B = 0 = E$. The two remaining scalar metric perturbations are then the Bardeen potentials \cite{9} defined as

$$\Phi = \phi - \mathcal{H}(\dot{E} - B) - (\dot{E} - \dot{B}),$$

$$\Psi = \psi + \mathcal{H}(\dot{E} - B),$$

(or, in Bardeen’s notation, $\Phi_A Q^{(0)}$ and $-\Phi_H Q^{(0)}$). The metric then has no off-diagonal terms and is \footnote{Note that this is a gauge which does not exhibit problems when transforming between frames in modified gravity theories. We do not explore this, but see Ref. \cite{23} for details.}

$$ds^2 = a^2(\eta) \left[ -(1 + 2\Phi) d\eta^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j \right].$$

The governing equations for linear perturbations in this gauge are then:

$$- \nabla^2 \Psi + 3\mathcal{H}(\dot{\Psi} + \dot{\Phi}) = \frac{1}{2F} \left[ -(\nabla^2 + 3\dot{\mathcal{H}}) \delta F - 3\mathcal{H} \dot{\delta F} - 3\dot{\mathcal{F}} (\dot{\Psi} + 2\dot{\mathcal{H}} \Phi) - 8\pi G a^2 \delta \rho \right].$$

$$\text{(3.17)}$$
\[ \mathcal{H}\Phi + \Psi = \frac{1}{2F} \left[ \delta F - \dot{F}\Phi - \mathcal{H}\delta F - 8\pi G(\rho_0 + P_0)a^2v_\ell \right], \quad (3.18) \]

\[ \Psi - \Phi = \frac{\delta F}{F}, \quad (3.19) \]

\[ 3\Psi + 3\mathcal{H}(\Phi + \Psi) + \left( 6\dot{\mathcal{H}} + 3\mathcal{H}^2 + \nabla^2 + 3\frac{\ddot{F}}{F}\right)\Phi + \frac{\dot{F}}{2F} \left[ 3\Phi + 3\dot{\Phi} \right] \\
= \frac{1}{2F} \left[ 3\ddot{F} - (6H^2 + \nabla^2)\delta F - 8\pi Ga^2(\delta\rho_\ell + \delta P_\ell) \right], \quad (3.20) \]

\[ \dot{F} + 2\mathcal{H}\delta F - \left( \frac{R}{3} + \nabla^2 \right)\delta F = \frac{8\pi G}{3} a^2 (3\delta P_\ell - \delta\rho_\ell) + \frac{\dot{F}}{2F} \left[ 3\Phi + 3\dot{\Phi} + 2\dot{\Phi} + \frac{1}{3}a^2 F\delta R_\ell \right], \quad (3.21) \]

and the Ricci scalar is

\[ \delta R_\ell = \frac{2}{a^2} \nabla^2 \left( 2\Psi - \Phi \right) - \frac{6}{a^2} \left[ \mathcal{H}(\Phi + 3\Phi) + \ddot{\Psi} + 2(\dot{\mathcal{H}} + \mathcal{H}^2)\Phi \right]. \quad (3.22) \]

In Bardeen’s original work, the chosen gauge invariant matter variable was the density perturbation in the comoving gauge. This can be related to the longitudinal gauge variables used above through

\[ \delta\rho_{\text{com}} = \delta\rho_\ell + \rho_0 v_\ell. \quad (3.23) \]

2. Uniform curvature gauge

The uniform curvature gauge is the one in which \( E = \psi = 0 \), and so the metric tensor is then spatially unperturbed:

\[ ds^2 = a^2(\eta) \left[ - (1 + 2\phi)d\eta^2 + 2B_{ij}d\eta^idx^j + dx^2 \right]. \quad (3.24) \]

The governing equations in this gauge are then

\[ 3\mathcal{H}^2\phi + \nabla^2 B = \frac{1}{2F} \left[ - \dot{F}\nabla^2 B - (\nabla^2 + 3\mathcal{H})\delta F + 3\mathcal{H}\delta F - 6\dot{\mathcal{H}}\Phi - 8\pi Ga^2\delta\rho \right], \quad (3.25) \]

\[ \mathcal{H}\phi = \frac{1}{2F} \left[ \delta F - \dot{F}\phi - \mathcal{H}\delta F - 8\pi G(\rho_0 + P_0)a^2(v + B) \right]. \quad (3.26) \]

\[ \dot{B} + 2\mathcal{H}B + \phi = - \frac{1}{F} \left[ \delta F + \dot{F}B \right], \quad (3.27) \]

\[ 3\mathcal{H}\dot{\phi} + \nabla^2 \dot{B} + 2\mathcal{H}\nabla^2 B + \left( 6\dot{\mathcal{H}} + 3\mathcal{H}^2 + \nabla^2 + 3\frac{\ddot{F}}{F}\right)\phi + \frac{\dot{F}}{2F} \left[ 3\phi + \nabla^2 B \right] \\
= \frac{1}{2F} \left[ 3\ddot{F} - (6H^2 + \nabla^2)\delta F - 8\pi Ga^2(\delta\rho + \delta P) \right], \quad (3.28) \]
\[ \ddot{\delta F} + 2H\dot{\delta F} - \left( \frac{R}{3} + \nabla^2 \right)\delta F = \frac{8\pi G}{3}a^2(3\delta P - \delta \rho) + \dot{\delta} \left[ 4\dot{\phi} + \nabla^2 B + \phi \right] + 2\ddot{\phi} - \frac{1}{3}a^2F\delta R, \] (3.29)

while the perturbed Ricci scalar is

\[ \delta R = -\frac{2}{a^2}\nabla^2 \left( \phi + \dot{B} \right) - \frac{6}{a^2} \left[ \dot{\phi} + H\nabla^2 B + 2(\dot{H} + H^2)\phi \right]. \] (3.30)

### IV. COVARIANT FORMALISM

The starting point for the covariant approach to cosmological perturbations is choosing a suitable frame in which to work. Equivalently, this means making a choice of the four velocity vector, \( u_a \), of an observer in the spacetime. Several different choices can be made, but the most physically motivated choice is the frame associated with standard matter, so \( u_a = u_a^M \). Now following closely Refs. [11, 24], we can derive the kinematic quantities in the standard way. In the following we denote the derivative along the matter fluid flow lines with a dagger, e.g., \( X^\dagger = u_a \nabla^a X \).

The projection tensor is

\[ h_{ab} \equiv g_{ab} + u_a u_b, \] (4.1)

which obeys

\[ h^a_b h^b_c = h^a_c, \ h_{ab} u^b = 0. \] (4.2)

The projected derivative operator orthogonal to \( u^a \) is \( \nabla_a = h^b_a \nabla_b \), and so kinematical quantities are introduced by splitting the covariant derivative of \( u^a \):

\[ \nabla_b u_a = (3)\nabla^b u_a - a^a u_b, \quad (3)\nabla_b u_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab}, \] (4.3)

where \( a_a = u_a^\dagger \) is the acceleration, \( \Theta \) is the expansion, and the shear and vorticity are \( \sigma_{ab} \) and \( \omega_{ab} \), respectively. Further, in the following, angle brackets applied to a vector denote its projection onto tangent 3-spaces

\[ V_{(a)} = h_{ab} V_b. \] (4.4)

When applied to a tensor, they denote the projected, anti-symmetric and trace free part

\[ W_{(ab)} = [h_{(a} c h_{b)} d - \frac{1}{3} h^{cd} h_{ab}] W_{ab}. \] (4.5)

The spatial curl of a variable is

\[ (\text{curl} X)^{ab} = \epsilon^{cd(a} (3) \nabla_c X^{b)} d, \] (4.6)
where $\epsilon_{abc} = u^d \eta_{abcd}$ is the spatial volume.

Finally, we note that, since we treat the additional curvature as a fluid, we can write an energy density and a pressure for this fluid, namely $\rho^R$ and $P^R$ [25].

### A. Linearized equations

Fully non-linear governing equations valid in any spacetime (with suitable choice of $u^a$) can be found in Ref. [11]. In order to study cosmological perturbations, we linearize the equations around a Friedmann-Robertson-Walker background spacetime. The cosmological equations for the background are

\[
\Theta^2 = 3\dot{\rho}^M + 3\rho^R - \frac{(3)^R}{2},
\]

\[
\Theta^\dagger + \frac{1}{3} \Theta^2 + \frac{1}{2} (\dot{\rho}^M + 3\dot{\rho}^M) + \frac{1}{2} (\rho^R + 3P^R) = 0,
\]

\[
\rho^{M^\dagger} + \Theta (\rho^M + P^M) = 0.
\]

Linearization of the propagation and constraint equations gives \[^2\]

\[
\Theta^\dagger + \frac{1}{3} \Theta^2 - (3)^a a_a + \frac{1}{2} (\dot{\rho}^M + 3\dot{\rho}^M) = -\frac{1}{2} (\rho^R + 3P^R),
\]

\[
\omega^a + 2H\omega_a + \frac{1}{2} \text{curl} a_a = 0,
\]

\[
\sigma^a + 2H\sigma_a + E_a = 0.
\]

\[
E_a^\dagger + 3HE_a - \text{curl} E_a + \frac{1}{2} (\dot{\rho}^M + 3\dot{\rho}^M) \sigma_{ab} = -\frac{1}{2} (\rho^R + P^R) \sigma_{ab} - \frac{1}{2} \pi_{ab} - \frac{1}{2} (3)^{a} q^{Rb} - \frac{1}{6} \Theta \pi_{ab}^R,
\]

\[
H_{ab}^\dagger + 3HH_{ab} + \text{curl} E_{ab} = \frac{1}{2} \text{curl} \pi_{ab}^R,
\]

\[
(3)^a \sigma_{ab} - \text{curl} \omega_a - \frac{2}{3} (3)^a \Theta = -q_a^R,
\]

\[
\text{curl} \sigma_{ab} + (3)^{a} \langle a \omega_b \rangle - H_{ab} = 0,
\]

\[
(3)^a E_{ab} - \frac{1}{3} (3)^a \rho^M = -\frac{1}{2} (3)^a \pi_{ab}^R + \frac{1}{3} (3)^a \rho^R - \frac{1}{3} \Theta q_a^R,
\]

\[
(3)^a H_{ab} - (\dot{\rho}^M + \dot{P}^M) \omega_a = -\frac{1}{2} \text{curl} q_a^R + (\rho^R + P^R) \omega_a,
\]

\[^2\] Note in the following that $E_{ab}$ and $H_{ab}$ are the electric and magnetic parts of the Weyl tensor:

\[
E_{ab} = C_{abcd} u^c u^d, \quad H_{ab} = \frac{1}{2} C_{abcd} u^c \eta^{ed} u^f u^g.
\]
\[(3) \nabla^a \omega_a = 0. \quad (4.20)\]

And the linearized conservation equations are
\[\rho^M = -\Theta (\rho^M + P^M), \quad (4.21)\]
\[(3) \nabla^a P^M = -(\rho^M + P^M) u^a, \quad (4.22)\]
\[\rho^R + (3) \nabla^a q^R = -\Theta (\rho^R + P^R) + \rho^M \frac{F'}{F^2} \hat{R}^\dagger, \quad (4.23)\]
\[q^R_{(a)} + (3) \nabla_a P^R + (3) \nabla^b \pi^R_{ab} = -\frac{4}{3} \Theta q^R_a - (\rho^R + P^R) u^a + \rho^M \frac{F'}{F^2} (3) \nabla_a R. \quad (4.24)\]

### B. Scalar equations

In order to study the linearized dynamics, we define the covariant gauge invariant quantities
\[D^M_a = a (\rho^M)^{(3)} \nabla^a \rho^M, \quad Z_a = a (3) \nabla_a \Theta, \quad C_a = a (3) \nabla_a (3) R, \quad (4.25)\]
as well as the gradients describing inhomogeneities in the Ricci scalar
\[R_a = a (3) \nabla_a R, \quad \cal{R}_a = a (3) \nabla_a \hat{R}^\dagger. \quad (4.26)\]

Dynamical and constraint equations for these variables can be found in Ref. [11]. However, since we want to consider scalar perturbations that govern the formation of structure in the universe, we need to use scalar variables. These are obtained by using a local decomposition. The variables of interest are then obtained by applying \((3) \nabla^a\) to those definitions above to give
\[\Delta^M_a = a (3) \nabla^a D^M_a, \quad Z = a (3) \nabla^a Z_a, \quad C = a (3) \nabla^a C_a, \quad \cal{R} = a (3) \nabla^a \cal{R}_a, \quad \cal{R} = a (3) \nabla^a \cal{R}_a. \quad (4.27)\]

Then, assuming the matter content to be well described by a barotropic fluid with equation of state \(P^M = w \rho^M\), the evolution equations for the variables are
\[\Delta^M = w \Theta \Delta - (1 + w) Z, \quad (4.28)\]
\[Z^\dagger = \left[ \frac{R^\dagger F'}{F} - \frac{2\Theta}{3} \right] Z + \left[ \frac{3(w - 1)(3w + 2)}{6(w + 1)} \hat{\rho}^M + \frac{2w\Theta^2 + 3w(\rho^R + 3P^R)}{6(w + 1)} \right] \Delta_M + \Theta \frac{F'}{F} \cal{R}\]
\[+ \left[ \frac{1}{2} - \frac{1}{2} \frac{f F'}{F^2} - \frac{F'}{F} \hat{\rho}^M + R^\dagger \Theta \left( \frac{F'}{F} \right)^2 + R^\dagger \Theta \frac{F'^2}{F} \right] \cal{R} - \frac{w}{w + 1} (3) \nabla^2 \Delta_M - \frac{F'}{F} (3) \nabla^2 \cal{R}. \quad (4.29)\]

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\(^3\) We should note that the term ‘gauge invariant’ is used to mean different things in perturbation theories. In metric perturbation theory, we refer to a quantity as being gauge invariant if it does not change under a gauge transformation. However, a stronger notion of gauge invariance is introduced through the Stewart-Walker lemma [26], that an inhomogeneous linearly perturbed quantity is gauge invariant if the quantity vanishes in the background. Such a quantity is often referred to as identification gauge invariant. See Ref. [12] for more discussion on this point.
\[ R^\dagger = \mathcal{R} - \frac{w}{w+1} R^\dagger \Delta M, \]  
\[ C^\dagger = \langle 3 \rangle \nabla^2 \left[ \frac{4 w a^2 \Theta}{3(w+1)} \Delta M + 2 a^2 \frac{F'}{F} \mathcal{R} - 2 a^2 \frac{\Theta F' - 3 R^\dagger F''}{3 F'} \mathcal{R} \right], \]  
\[ \frac{C}{a^2} + \left( \frac{4}{3} \Theta + 2 R^\dagger \frac{F'}{F} \right) Z - 2 \rho^M \Delta M + \left[ 2 R^\dagger \Theta \frac{F''}{F} - \frac{F'}{F} \left( f - 2 \rho^M + 2 R^\dagger \Theta F' \right) \right] \mathcal{R} - 2 \frac{2 F'}{F} \mathcal{R} = 0. \]

**V. RELATING THE TWO APPROACHES**

In the previous sections we have introduced cosmological perturbation theory using both the metric and covariant formalisms in \( f(R) \) gravity. In this section we show how to relate one to the other focusing on the covariant approach and showing how this maps to the metric approach.

First, the three-dimensional Ricci scalar is defined in the covariant approach as

\[ (3) R = \mathcal{R} + 2 R^\dagger \Delta M. \]  

To compare, we split this into a homogeneous background and a perturbation as usual, \( (3) R = \delta(3) R \). The background is zero, \( (3) \bar{R} = 0 \), for a flat FLRW spacetime. The perturbation, from Ref. \[13\], can be written in terms of metric perturbation variables as

\[ \delta(3) R = 4 a^2 \nabla^2 \left[ \psi - \mathcal{H}(v + B) \right]. \]

Using metric perturbation theory, we can calculate the 3-Ricci scalar to obtain \[27\]

\[ \delta(3) R = \frac{4}{a^2} \nabla^2 \psi. \]

The definition of the curvature perturbation in the comoving gauge in terms of variables in an arbitrary gauge is

\[ \psi_{\text{com}} = \psi - \mathcal{H}(v + B), \]
from which we can see that the curvature quantities in the covariant approach are equivalent to the quantities in the metric approach in the comoving gauge. That is,

$$\delta^{(3)}R = \frac{4}{a^2} \nabla^2 \psi_{\text{com}}, \quad (5.5)$$

This can be written in terms of longitudinal gauge quantities, or Bardeen variables, as

$$\delta^{(3)}R = \frac{4}{a^2} \nabla^2 (\Psi - \mathcal{H} v_\ell). \quad (5.6)$$

The four dimensional Ricci scalar is derived above in Section III and is given in the comoving gauge in terms of Bardeen variables as

$$\delta R_{\text{com}}[\ell] = 2 \frac{a^2}{a^2} \nabla^2 (2\Psi - \Phi) - \frac{6}{a^2} \left[ \mathcal{H}(\dot{\Phi} + 3\dot{\Psi}) + \ddot{\Psi} + 2(\mathcal{H}^2 + \dot{\mathcal{H}})\Phi + (2\mathcal{H}^3 - \ddot{\mathcal{H}})v_\ell \right]. \quad (5.7)$$

Now we consider kinematical quantities, starting with the expansion scalar, in the covariant approach defined as $\Theta = \nabla_a u^a$. This can then be split into a homogeneous background and a linear perturbation as

$$\Theta = \bar{\Theta} + \delta \Theta. \quad (5.8)$$

The background expansion is $\bar{\Theta} = 3\mathcal{H}/a$. Using the definition of $u^M_a$ in terms of metric perturbation theory we arrive at

$$\delta \Theta = -\frac{3}{a} \left[ \mathcal{H} \phi + \dot{\psi} + \frac{1}{3} \nabla^2 (v - \dot{E}) \right]. \quad (5.9)$$

One difference between the two formalisms is in the assumed time-like vector field with which to describe the spacetime. The covariant approach assumes a four-velocity, taken to be comoving with the matter, $u^M_a$, while the metric formalism assumes the FLRW metric as a background. In the latter, the fundamental vector field, $n^a$ is orthogonal to constant $-\eta$ hypersurfaces, and has components

$$n^a = \frac{1}{a} (1 - \phi, -B_i^i + S^i), \quad (5.10)$$

$$n_a = -a (1 + \phi, 0). \quad (5.11)$$

Thus, in metric perturbation theory the expansion scalar is

$$\tilde{\delta} \Theta = -\frac{3}{a} \left[ \mathcal{H} \phi + \dot{\psi} + \frac{1}{3} \nabla^2 (B - \dot{E}) \right]. \quad (5.12)$$

This is not such a problem, it arises simply because in metric perturbation theory we have a choice of unit timelike vector field. If we want to compare the two approaches we can simply evaluate both in the comoving gauge, for which $v = B = 0$. So, Eq. (5.9), becomes

$$\delta \Theta = -\frac{3}{a} \left[ \mathcal{H} \phi_{\text{com}} + \dot{\psi}_{\text{com}} - \frac{1}{3} \nabla^2 \dot{E}_{\text{com}} \right]. \quad (5.13)$$
and on using the relationships between the comoving gauge variables and the longitudinal gauge (or Bardeen) variables,

\[ \phi_{\text{com}} = \Phi + H v_\ell + \dot{v}_\ell, \quad (5.14) \]
\[ \psi_{\text{com}} = \Psi - H v_\ell, \quad (5.15) \]
\[ \dot{E}_{\text{com}} = v_\ell, \quad (5.16) \]

we obtain

\[ \delta \Theta = -\frac{3}{a} \left[ H \Phi + \dot{\Psi} + (H^2 - \dot{H}) v_\ell - \frac{1}{3} \nabla^2 v_\ell \right]. \quad (5.17) \]

Similarly, the acceleration, \( a_a = u_a^{\;\mu} u^\mu_b \) in the comoving gauge is

\[ a_i = \phi_{\text{com},i}, \quad (5.18) \]

which, in terms of Bardeen variables, is

\[ a_i = \Phi_{,i} + H v_\ell_{,i} + \dot{v}_\ell_{,i}. \quad (5.19) \]

### A. Gauge invariant covariant quantities

Now, we show how to relate the gauge invariant gradients defined above to metric perturbation quantities. Again, as above, we work in the comoving gauge. In this gauge, the projected covariant derivative, defined as \( a^{(3)} \nabla_i = h^b_i \nabla_b \) is simply the covariant derivative on the spatial hypersurfaces and, since we are working with a flat background, is simply a partial derivative: \( a^{(3)} \nabla_i = \partial_i \). Thus, we obtain

\[ D_i^M = \delta_{\text{com},i}, \quad (5.20) \]
\[ Z_i = -\frac{3}{a} \left[ H \Phi_{,i} + \dot{\Psi}_{,i} + (H^2 - \dot{H}) v_\ell_{,i} - \frac{1}{3} \nabla^2 v_{\ell,i} \right], \quad (5.21) \]
\[ C_i = \frac{4}{a^2} \nabla^2 (\Psi_{,i} - H v_\ell_{,i}), \quad (5.22) \]
\[ R_i = \partial_i \delta R_{\text{com}}[\ell], \quad (5.23) \]
\[ \mathcal{R}_i = \frac{1}{a} \delta R_{\text{com}}[\ell] - \frac{1}{a} (\Phi_{,i} + H v_\ell_{,i} + \dot{v}_\ell_{,i}) \dot{R}. \quad (5.24) \]

Note that we have left the variables in terms of \( \delta R_{\text{com}}[\ell] \), since we can then simply substitute this into the covariant equations later which will then give us equations comparable to the longitudinal gauge metric equations presented in Section III A 1. The scalar gauge invariant covariant quantities
are then related to metric variables through

\[ \Delta_M = \nabla^2 \delta_{\text{com}}, \] (5.25)

\[ Z = -\frac{3}{a} \nabla^2 \left[ \mathcal{H} \Phi + \dot{\Psi} + (\mathcal{H}^2 - \dot{\mathcal{H}}) v_\ell - \frac{1}{3} \nabla^2 v_\ell \right], \] (5.26)

\[ C = \frac{4}{a^2} \nabla^4 (\Psi - \mathcal{H} v_\ell), \] (5.27)

\[ \mathcal{R} = \nabla^2 \delta R_{\text{com}}[\ell], \] (5.28)

\[ \mathfrak{R} = \frac{1}{a} \nabla^2 \delta R_{\text{com}}[\ell] + \frac{1}{a} \nabla^2 (\Phi + \mathcal{H} v_\ell + \dot{v}_\ell) \dot{R}. \] (5.29)

### B. Equations

Having now presented the gauge invariant covariant variables in terms of metric perturbation variables, we now show how to convert from the equations in the covariant approach to those in the metric approach. We will use the case of general relativity, for which \( f_R = R \), to highlight the procedure. We first note that a dagger derivative applied to a perturbed quantity in the comoving gauge is

\[ x^\dagger = \frac{1}{a} \dot{x}. \] (5.30)

The equivalency is best shown by first performing a harmonic decomposition, such that

\[ (3) \nabla^2 Q = -\frac{k^2}{a^2} Q^{(k)}. \] (5.31)

This removes the \( (3) \nabla^2 \) from the equations in the covariant approach, thus allowing a more direct comparison with the metric approach. Then, the set of equations governing the scalar variables can be written as two, second order differential equations:

\[ \Delta_M^{(k)\dagger\dagger} + \left[ \frac{2}{3} - w \right] \Theta - R^i \frac{F'}{F} \right] \Delta_M^{(k)\dagger} - \left[ \frac{k^2}{a^2} - w(3P_R + \rho^R) - \frac{2wR^i \Theta F'}{F} - \frac{(3w^2 - 1)\rho^M}{F} \right] \Delta_M^{(k)}, \]

\[ = \frac{1}{2} (w + 1) \left[ 2 \frac{k^2}{a^2} F' - 1 + \left( \frac{f}{3} - 2 \rho^M + 2R^i \Theta F' \right) \frac{F'}{F^2} - 2R^i \Theta \frac{F''}{F} \right] \mathcal{R}^{(k)} - \frac{(w + 1) \Theta F'}{F} \mathcal{R}^{(k)\dagger}, \] (5.32)

\[ F' \mathcal{R}^{(k)\dagger\dagger} + \left( \Theta F' + 2R^i \Theta F' \right) \mathcal{R}^{(k)\dagger} - \left[ \frac{k^2}{a^2} F' + \frac{2}{9} \Theta^2 F' - (w + 1) \frac{\rho^M}{2F} F' - \frac{1}{6} \left( \rho^R + 3P_R \right) F' \right] \mathcal{R}^{(k)} = - \left[ \frac{1}{3} (3w - 1) \rho^M \right], \]

\[ - \frac{F}{3} + \frac{f}{6F} F' + \frac{R^i \Theta F'^2}{6F} - R^{i\dagger\dagger} F'' - \Theta F'' R^i - F^{(3)R^i} \right] \mathcal{R}^{(k)} = \frac{w}{w + 1} \left[ \frac{w}{3} R^i \Theta F' + R^i F' \right] \Delta_M^{(k)} - \frac{(w - 1) R^i F'}{w + 1} \Delta_M^{(k)\dagger}. \] (5.33)
On taking the general relativistic limit, the equation governing the evolution of the energy density perturbation for a flat FLRW universe dominated by dust, is

$$\Delta^{(k)\mp}_M + \frac{2}{3} \Theta \Delta^{(k)\mp}_M - \frac{1}{2} \kappa \rho^M \Delta^{(k)}_M = 0.$$ (5.34)

Using the relationships between the covariant and metric perturbation quantities in the comoving gauge, this can be re-written as

$$\ddot{\delta}_{\text{com}} + H \dot{\delta}_{\text{com}} - 4\pi G \rho u^2 \delta_{\text{com}} = 0.$$ (5.35)

This is the usual equation for the evolution of the density contrast in the comoving gauge, which verifies the transformation between the two approaches.

VI. DISCUSSION

In this article we have provided, for the first time, a method for relating the most popular two methods of modeling cosmological perturbations – the covariant and metric approaches – to one another for \(f(R)\) gravity. This builds upon work presented in Ref. [13] for standard Einstein gravity. We started by reviewing \(f(R)\) gravity, and then both the metric and covariant approaches to cosmological perturbations. We presented the governing equations for scalar perturbations in the metric approach in both the longitudinal gauge and the uniform curvature gauge, as well as presenting the equations in Bardeen’s variables (which amounts to using the longitudinal gauge, but with the comoving density contrast). The governing equations in the covariant approach were then presented, again for scalar perturbations, in terms of the covariant gauge invariant quantities.

Then, in Section [V] we presented the relationship between the variables in the covariant and metric approaches. For the curvature variables such as the 3-dimensional Ricci scalar, the covariant variables are essentially already in a form equivalent to the comoving gauge of metric perturbation theory. For kinematic variables, such as the expansion scalar, the covariant variables are equivalent to the metric perturbation theory variables only in the comoving gauge, due to the choice of the velocity four vector \(u^a_M\) as opposed to the unit timelike vector \(n^a\), which depends on metric perturbations. Having presented this relationship, we then outlined the method in which one can transfer from the covariant equations to the metric equations, and vice versa.

It is not surprising that this relationship exists, since the two approaches are complementary methods with which to describe inhomogeneities on top of a homogeneous and isotropic FLRW background, with the covariant approach mapping to the comoving gauge [12]. However, since the
two approaches are different, there will naturally be problems that one or other of the methods are more suitable to solve. This paper allows one to compare calculations done in one of the approaches to the other, thus enabling a deeper understanding of the predictions made by different $f(R)$ cosmological theories.

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**Appendix A: Notation**

In this paper we follow the notation of Ref. [16], however in this Appendix we show how to relate notation to that of Ref. [13], where the perturbed line element is

$$ds^2 = a^2(\eta)\left[-(1 + 2A)d\eta^2 - 2B_\alpha dx^\alpha d\eta + \left((1 + 2H_L)\gamma_{\alpha\beta} + 2H_T|_{\alpha\beta}\right)dx^\alpha dx^\beta\right], \quad (A1)$$

where here Greek indices run over the spatial coordinates. This is also discussed in Ref. [28]. The perturbations are then decomposed as

$$B_\alpha = B|_\alpha + B^S_\alpha, \quad (A2)$$

$$H_{T\alpha\beta} = \nabla_{\alpha\beta}H_T + H^S_{T(\alpha|\beta)} + H^{TT}_{T\alpha\beta}, \quad (A3)$$

where

$$\nabla_{\alpha\beta}\zeta = \zeta_{|\alpha\beta} - \frac{1}{3} \nabla^2 \zeta, \quad (A4)$$
for some scalar, $\zeta$. Thus, we arrive at the equivalences between the scalar perturbations in the conventions of Malik and Wands (left) and Bruni et al (right):

\[ \phi \leftrightarrow A \quad \text{(A5)} \]
\[ B \leftrightarrow -B \quad \text{(A6)} \]
\[ \psi \leftrightarrow \frac{1}{3} \nabla^2 H_T - H_L \quad \text{(A7)} \]
\[ E \leftrightarrow H_T. \quad \text{(A8)} \]

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