OPTIMAL CONTROL PROBLEMS FOR SOME ORDINARY DIFFERENTIAL EQUATIONS WITH BEHAVIOR OF BLOWUP OR QUENCHING

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ABSTRACT. This paper is concerned with some optimal control problems for equations with blowup or quenching property. We first study the existence and Pontryagin’s maximum principle for optimal controls which have the minimal energy among all the controls whose corresponding solutions blow up at the right-hand time end-point of a given functional. Then, the same problem for quenching case is discussed. Finally, we establish Pontryagin’s maximum principle for optimal controls of extended problems after quenching.

1. Introduction. It is well known that solutions to some evolution equations have the behavior of blowup or quenching. Such equations can describe a large class of phenomena in applied science. For instance, equations with the property of blowup can describe the dramatic increase in temperature leading to the ignition of a chemical reaction, and those with the property of quenching can be used to represent the potential differences of the polarization field in ionic conductors achieving the balance. Roughly speaking, blowup is a conception which means that a solution is unbounded in finite time. Quenching of a solution means that the derivative in time $t$ of the solution goes to infinity in finite time, while it keeps bounded itself. In the past decades, such equations have attracted many researchers’ attention (see, for instance, [2], [3], [5]-[11]). In recent years, optimal control problems for minimal or maximal blowup/quenching time have been studied by several authors (see [4], [13]-[16], [18]-[20]).

A natural interesting problem is to find the control minimizing a given cost functional among all the controls whose corresponding solutions blow up or quench at the right-hand time end-point of the functional. It is one purpose of this paper. Optimal control problems for some partial differential equations with the property of blowup have been studied in few papers (see, for instance, [1], [17]). However, the intention of these papers was to find an optimal control with the minimal energy.

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from all the controls whose corresponding solutions exist in the closed interval where
the cost functional is defined. On the other hand, Lin [14] studied the extendabil-
ity and an optimal control problem after quenching for some ordinary differential
equations. It was shown in [14] that a solution, which quenches for the first time at
finite time, may hold different properties: not extendable after quenching, extend-
able uniquely or having at least two extended solutions. The extended solutions
may quench for the second time. Thus, we can also consider the following optimal
control problem: for a given extendable solution which quenches for the first time
at finite time, one seeks the control with minimal energy among all the extended
controls that can make the corresponding extended solution quenches for the second
time at the right-hand time end-point of a cost functional. This is another purpose
of the paper.

Quenching problem for ordinary differential equations is more complex than
blowup one. If the set of quenching points for an ordinary differential equation
contains only one point (or finite isolated points), then theoretically, quenching
problem could be converted to blowup one by suitable variable substitution. How-
ever, if there exists a quenching point which has a neighborhood holding infinite
quenching points, quenching problem may in general not be transformed into blowup
one. Furthermore, a solution for an ordinary differential equation blowing up means
that the Euclidean norm of it goes to infinity at finite time. We need not consider
the behavior of this solution after blowup. While, as we have stated above, the
solution may be extendable after quenching.

In this paper, we consider the following controlled differential equation:

\[
\begin{aligned}
\frac{dy(t)}{dt} &= f(t, y(t), u(t)), \quad t > 0, \\
y(0) &= y_0.
\end{aligned}
\]

In the above, \(y_0\) is an initial data in \(\mathbb{R}^n\) and \(f(\cdot, \cdot, \cdot) : [0, +\infty) \times \mathbb{R}^n \times U \to \mathbb{R}^n\) is
a given map, where \(U\) is a nonempty set in \(\mathbb{R}^m\). \(y(\cdot)\) is called the state trajectory
taking values in \(\mathbb{R}^n\), and \(u(\cdot)\) is called the control taking values in \(U\). We say
\(y(\cdot) \in C([0, T]; \mathbb{R}^n)\) is a solution of (1) on \([0, T]\), if \(y(\cdot)\) satisfies

\[
y(t) = y_0 + \int_0^t f(s, y(s), u(s)) \, ds, \quad \forall \, t \in [0, T].
\]

We shall first study the existence and Pontryagin’s maximum principle for optimal
controls which have the minimal energy among all the controls whose corre-
sponding solutions blow up at the right-hand time end-point of a given functional.
Then, the same problem for quenching case will be discussed. Finally, among all
the extended controls whose corresponding extended solutions quench for the second
time at the right-hand time end-point of a cost functional, Pontryagin’s maximum
principle will be established for optimal controls which have the minimal energy. It
is different with the problem considered in [14], where every admissible state does
not quench for the second time at any point of the closed interval in which the cost
functional is defined.

As far as we know, since equations with blowup or quenching property will show
singularities at blowup or quenching time, the key to getting Pontryagin’s maximum
principle for optimal control problems related to blowup or quenching is to obtain
so-called “the initial period optimality”. For instance, in order to obtain Pontrya-
gin’s maximum principle, Lin and Wang [15] pointed out the fact in the proof of
Corollary 1.3 that the optimal controls of the minimal blowup time hold the properties of both the initial period optimality and the terminal period optimality for a special autonomous system; for general controlled autonomous systems, Lou and Wang [19] established the initial period optimality for the optimal controls of the minimal and the maximal blowup/quenching time; meanwhile, for non-autonomous case with blowup property, Lou and Wang [18] introduced a class of monotone systems and deduced the initial period optimality based on the monotonicity. However, for non-autonomous and non-monotone systems with blowup property, and for non-autonomous systems with quenching property and those having not only one quenching point (even if they are monotone), “the initial period optimality” may not hold for blowup/quenching time optimal control problems. Fortunately, we can establish “the initial period optimality” for our optimal control problems and by means of the results of classical optimal control problems with terminal constraints, the corresponding Pontryagin’s maximum principle for general systems without autonomous and monotone assumptions could be established in this paper.

The rest of the paper is organized as follows. In Section 2, we will obtain the existence result of optimal controls. In order to establish the existence result of optimal controls, we will introduce some notations. Denote

$$\mathcal{U} = \left\{ u(\cdot) : [0, +\infty) \to U \mid u(\cdot) \text{ is measurable} \right\},$$

$$\mathcal{U}_T = \left\{ u(\cdot) : [0, T] \to U \mid u(\cdot) \text{ is measurable} \right\},$$

$$\mathcal{P}_T = \left\{ (T, y(\cdot), u(\cdot)) \in [0, +\infty) \times C([0, T]; \mathbb{R}^n) \times \mathcal{U}_T \mid (1) \text{ holds on } [0, T] \right\}. \quad (2)$$

Let $T_1 > 0$ and let $f^0(\cdot, \cdot, \cdot) : [0, +\infty) \times \mathbb{R}^n \times U \to \mathbb{R}$ be a given function. Define

$$\mathcal{P}_{ad}^B = \left\{ (y(\cdot), u(\cdot)) \mid (T_1, y(\cdot), u(\cdot)) \in \mathcal{P}_{T_1}, \lim_{t \to T_1^+} |y(t)| = +\infty \text{ and } f^0(\cdot, y(\cdot), u(\cdot)) \in L^1(0, T_1) \right\}.$$

Then, we study the following optimal control problem.

**Problem (PB).** to find a pair $(y_*, u_*) \in \mathcal{P}_{ad}^B$, such that

$$J(y_*(\cdot), u_*(\cdot)) = \inf_{(y(\cdot), u(\cdot)) \in \mathcal{P}_{ad}^B} J(y(\cdot), u(\cdot)),$$

where $J(y(\cdot), u(\cdot)) = \int_0^{T_1} f^0(t, y(t), u(t)) \, dt$.

In order to establish the existence result of optimal controls, we will introduce relaxed control, which can also be used in the next section for quenching case. Precisely, define

$$\mathcal{M}_+^1(U) = \left\{ \theta \mid \theta \text{ is a Randon probability measure on } U \right\},$$

$$\mathcal{P}(U) = \left\{ \sigma(\cdot) : [0, +\infty) \to \mathcal{M}_+^1(U) \mid t \mapsto \int_U h(u) \sigma(t)(du) \text{ is measurable for any } h(\cdot) \in C(U) \right\}. \quad (3)$$
\( \mathcal{R}_T(U) = \left\{ \sigma(\cdot) : [0, T] \rightarrow \mathcal{M}^1(U) \mid t \mapsto \int_U h(u) \sigma(t)(du) \right\} \text{ measurable for any } h(\cdot) \in C(U) \right\}. \\
We call \( \mathcal{R}_T(U) \) the relaxed control set on \([0, T]\). Also, we say \\
\[ \sigma_k(\cdot) \rightarrow \sigma(\cdot) \text{ in } \mathcal{R}_T(U), \quad \text{as } k \to +\infty, \]
if it holds that for any \( g(\cdot, \cdot) \in L^1([0, T]; C(U)) \),
\[ \int_0^T dt \int_U g(t, u) \sigma_k(t)(du) \to \int_0^T dt \int_U g(t, u) \sigma(t)(du), \quad \text{as } k \to +\infty. \]

The following lemma is an important property of relaxed controls.

**Lemma 2.1.** (See [21]) Suppose that \( U \subseteq \mathbb{R}^m \) is compact. Then, for any \( T > 0 \), the relaxed control set \( \mathcal{R}_T(U) \) is convex and sequentially compact.

Consider the following relaxed system
\[
\begin{align*}
\frac{dy(t)}{dt} &= \int_U f(t, y(t), u) \sigma(t)(du), \quad t > 0, \\
y(0) &= y_0.
\end{align*}
\]

Denote
\[
\mathcal{R}_\mathcal{B}_\text{ad} = \left\{ (y(\cdot), \sigma(\cdot)) \in C([0, T_1]; \mathbb{R}^n) \times \mathcal{R}^1(U) \mid (4) \text{ holds on } [0, T_1], \quad \lim_{t \to T_1^-} |y(t)| = +\infty \text{ and } \int U f^0(\cdot, y(\cdot), u) \sigma(\cdot)(du) \in L^1(0, T_1) \right\}.
\]

Correspondingly, the optimal relaxed control problem can be stated as

**Problem (RPB).** to find a pair \( (y_\ast(\cdot), \sigma_\ast(\cdot)) \in \mathcal{R}_\mathcal{B}_\text{ad} \) such that
\[ J(y_\ast(\cdot), \sigma_\ast(\cdot)) = \inf_{(y(\cdot), \sigma(\cdot)) \in \mathcal{R}_\mathcal{B}_\text{ad}} J(y(\cdot), \sigma(\cdot)), \]

where \( J(y(\cdot), \sigma(\cdot)) = \int_0^{T_1} dt \int_U f^0(t, y(t), u) \sigma(t)(du) \).

To establish the existence result of Problem (RPB), we need the following assumptions.

(A1) The set \( U \subseteq \mathbb{R}^m \) is compact.

(A2) For any \( (y, u) \in \mathbb{R}^n \times U \), the function \( f(\cdot, y, u) : [0, +\infty) \rightarrow \mathbb{R}^n \) and \( f^0(\cdot, y, u) : [0, +\infty) \rightarrow \mathbb{R} \) are measurable; for any \( t \in [0, +\infty) \), \( f(t, \cdot, \cdot) : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n \) and \( f^0(t, \cdot, \cdot) : \mathbb{R}^n \times U \rightarrow \mathbb{R} \) are continuous; for any \( R > 0 \), there exists an \( L_R > 0 \), such that
\[ |f(t, y, u)| + |f^0(t, y, u)| \leq L_R, \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, |y| \leq R. \]

(A3) For any \( (t, u) \in [0, +\infty) \times U \), the function \( f(t, \cdot, u) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( f^0(t, \cdot, u) : \mathbb{R}^n \rightarrow \mathbb{R} \) are continuously differentiable; for any \( R > 0 \), there exists an \( \tilde{L}_R > 0 \), such that
\[ |f_y(t, y, u)| + |f^0_y(t, y, u)| \leq \tilde{L}_R, \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, |y| \leq R. \]

(A4) There exist a constant \( R_0 > 0 \) and a continuous positive function \( \xi(\cdot) : [R_0, +\infty) \rightarrow \mathbb{R}^+, \) such that
\[ \left\langle \frac{y}{|y|}, f(t, y, u) \right\rangle \geq \xi(|y|), \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, |y| \geq R_0, \]
and
\[ \int_{R_0}^{+\infty} \frac{1}{\xi(r)} \, dr < +\infty. \]

(A5) There is a function \( \eta(t) \in L^1([0, T_1]; C(U)) \), such that
\[ |f^0(t, y, u)| \leq \eta(t, u), \quad \forall (t, y, u) \in [0, T_1] \times \mathbb{R}^n \times U. \]

Remark 1. The assumption (A4) implies that for any \( u(\cdot) \in \mathcal{U} \) (or \( \sigma(\cdot) \in \mathcal{R}(U) \)), the solution \( y(\cdot; u(\cdot)) \) of (1) (or the solution \( y(\cdot; \sigma(\cdot)) \) of (4)) blows up at finite time when the norm of the initial value \( y_0 \) is large enough.

To make preparations for the existence result of Problem (RPB), we need the following lemma.

Lemma 2.2. Suppose (A2) and (A3) hold. Let \( \{(y_k(\cdot), \sigma_k(\cdot))\}_{k=1}^{+\infty} \subseteq \mathcal{R}^{B}_{ad} \) and \( (y(\cdot), \sigma(\cdot)) \in \mathcal{P}^{B}_{ad} \). For any \( T \in (0, T_1) \), if
\[ \sigma_k(\cdot) \to \sigma(\cdot) \quad \text{in} \quad \mathcal{R}_T(U), \quad \text{as} \ k \to +\infty, \quad (5) \]
then it holds that
\[ y_k(\cdot) \to y(\cdot) \quad \text{in} \quad C([0, T]; \mathbb{R}^n), \quad \text{as} \ k \to +\infty. \quad (6) \]

Proof. Let \( M = \max_{t \in [0, T]} |y(t)| \). By (A2), for any \( 0 < \varepsilon < e^{-\tilde{L}_{M+1} T} \), there is a constant \( \delta_\varepsilon > 0 \), such that for any \( k \in \mathbb{N} \),
\[ \int_t^{(t+\delta)\wedge T} ds \int_U f(s, y(s), u) (\sigma_k(s) - \sigma(s))(du) | < \frac{\varepsilon^2}{4}, \quad \forall 0 < \delta < \delta_\varepsilon, \ t \in [0, T], \quad (7) \]
where \( \tilde{L}_{M+1} \) is defined in (A3). And (5) implies that for some \( K_\varepsilon > 0 \),
\[ \int_0^{\frac{j \delta T}{N}} ds \int_U f(s, y(s), u) (\sigma_k(s) - \sigma(s))(du) | < \frac{\varepsilon^2}{4}, \quad \forall j = 1, 2, \ldots, N, \ k \geq K_\varepsilon, \quad (8) \]
where \( N = \lceil \frac{T}{\delta_\varepsilon} \rceil + 1 \). For any \( t \in [0, T] \), we can find a \( j_\varepsilon \in \{0, 1, \ldots, N-1\} \) satisfying
\[ \frac{j_\varepsilon T}{N} \leq t \leq \frac{(j_\varepsilon + 1) T}{N}. \quad (9) \]
Combining (7) with (8) and (9), we have
\[ \begin{aligned}
\left| \int_0^t ds \int_U f(s, y(s), u) (\sigma_k(s) - \sigma(s))(du) \right| &
\leq \left| \int_0^{\frac{j_\varepsilon T}{N}} ds \int_U f(s, y(s), u) (\sigma_k(s) - \sigma(s))(du) \right| \\
&\quad + \left| \int_{\frac{j_\varepsilon T}{N}}^t ds \int_U f(s, y(s), u) (\sigma_k(s) - \sigma(s))(du) \right| \\
&< \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4} = \frac{\varepsilon^2}{2}, \quad \forall t \in [0, T], \ k \geq K_\varepsilon. \quad (10)
\end{aligned} \]
Further, for each \( k \geq K_\varepsilon \), let
\[ t_\varepsilon^k = \sup \left\{ t \in [0, T] \mid |y_k(t) - y(t)| < \varepsilon \right\}. \]
It remains to prove \( t_\varepsilon^k = T \). Obviously, \( t_\varepsilon^k > 0 \). By contradiction, we suppose that \( t_\varepsilon^k < T \). Then, by (A3) and (10), it follows that
\[ |y_k(t) - y(t)| \]
Suppose (A2)—(A4) hold. Let \( t^*_k \) be such that

\[
\int_0^{t^*_k} \left| f(s, y_k(s), u) - f(s, y(s), u) \right| \sigma_k(s)(du) + \left| f(s, y(s), u)(\sigma_k(s) - \sigma(s))(du) \right| \leq \int_0^{t^*_k} L_{M+1} |y_k(s) - y(s)| ds + \frac{\varepsilon^2}{2}, \quad \forall \ t \in [0, t^*_k].
\]

By Grönwall’s inequality, we have

\[
|y_k(t) - y(t)| \leq \frac{\varepsilon^2}{2} e^{L_{M+1}t} \leq \frac{\varepsilon}{2}, \quad \forall \ t \in [0, t^*_k],
\]

which contradicts to the definition of \( t^*_k \). Thus, \( t^*_k = T \). Combining with the continuity of \( y(\cdot) \) and \( y_k(\cdot) \) for all \( k \in \mathbb{N} \), we deduce that (6) holds.

We have completed the proof. \( \square \)

By Lemma 2.2, we can obtain the following corollary.

**Corollary 1.** Suppose (A2)—(A4) hold. Let \( \{(y_k(\cdot), \sigma_k(\cdot))\}_{k=1}^{+\infty} \subseteq \mathcal{R}_T \mathcal{P}_B \) and \( \sigma(\cdot) \in \mathcal{R}_T (U) \). If

\[
\sigma_k(\cdot) \rightarrow \sigma(\cdot) \quad \text{in} \quad \mathcal{R}_T (U), \quad \text{as} \quad k \rightarrow +\infty,
\]

then, it holds that \( y(\cdot) \) blows up at \( T_1 \), where \( y(\cdot) = y(\cdot; \sigma(\cdot)) \).

**Proof.** Suppose that \( y(\cdot) \) does not blow up at \( T_1 \). There are two cases to discuss.

**Case One.** \( y(\cdot) \) exists globally or blows up at some finite time \( S > T_1 \).

In this case, by repeating the same strategies in Lemma 2.2, we have that \( y_k(\cdot) \) exists on \([0, T_1]\) when \( k \) is large enough, which contradicts to the fact that \( \{(y_k(\cdot), \sigma_k(\cdot))\}_{k=1}^{+\infty} \subseteq \mathcal{R}_T \mathcal{P}_B \).

**Case Two.** \( y(\cdot) \) blows up at some finite time \( S < T_1 \).

By (A4), there exists an \( R_1 > R_0 \) such that

\[
\int_{R_1}^{+\infty} \frac{1}{\xi(r)} dr < \frac{T_1 - S}{2}, \quad (11)
\]

where \( R_0 \) is defined in (A4). On the other hand, since \( y(\cdot) \) blows up at \( S \), we easily obtain a \( T \in (0, S) \), such that

\[
|y(T)| \geq R_1 + 1.
\]

By the similar argument in the proof of Lemma 2.2, we can find a \( K > 0 \) such that

\[
|y_k(T)| \geq R_1, \quad \forall \ k \geq K.
\]

Then, by (A4), we can use the similar argument in the proof of Lemma 2.3 in [15] to obtain

\[
|y_k(t)| \geq R_1, \quad \forall \ t \in [T, S), \ k \geq K. \quad (12)
\]

Then, it follows by (A4) that

\[
\frac{d|y_k(t)|}{dt} = \int_U \left< y_k(t), f(t, y_k(t), u) \right> \sigma_k(t)(du) \geq \xi(|y_k(t)|), \quad \text{a.e.} \ t \in [T, S), \forall \ k \geq K. \quad (13)
\]
Combining (11), (12) with (13), we obtain that for any $k \geq K$,
\[
\frac{T_1 - S}{2} > \int_{R_1}^{+\infty} \frac{1}{\xi(r)} \, dr \geq \int_{|y_k(T)|}^{+\infty} \frac{1}{\xi(r)} \, dr
\]
\[
= \int_T^{T_1} \frac{1}{\xi(|y_k(t)|)} \frac{d|y_k(t)|}{dt} \, dt \geq T_1 - T > T_1 - S,
\]
which is a contradiction to the fact that $S < T_1$.

Thus, we have proved that $y(\cdot)$ blows up at $T_1$.

Based on the above convergence results, we can obtain the existence result of Problem (RPB).

**Theorem 2.3.** (Existence of Problem (RPB)) Suppose (A1)—(A5) hold and the set $\mathcal{R}P_d \neq \emptyset$. Then, Problem (RPB) admits at least one optimal pair.

**Proof.** Take a minimizing sequence $\{y_k(\cdot), \sigma_k(\cdot)\}_{k=1}^{+\infty} \subseteq \mathcal{R}P_d$. From Lemma 2.1, we have a relaxed control $\sigma_*(\cdot) \in \mathcal{R}_T(U)$, such that, at least along a subsequence,
\[
\sigma_k(\cdot) \to \sigma_*(\cdot) \text{ in } \mathcal{R}_T(U), \quad \text{as } k \to +\infty.
\] (14)

Denote $y_*(\cdot) = y(\cdot; \sigma_*(\cdot))$. Then, Lemma 2.2, Corollary 1 and (A5) ensure that $(y_k(\cdot), \sigma_k(\cdot)) \in \mathcal{R}P_d$.

For any $\varepsilon > 0$, (A5) and (14) imply that for some $K_0 > 0$,
\[
\left| \int_0^{T_1} dt \int_U f^0(t, y_k(t), u) \sigma_k(t) - \sigma_*(t) \right| < \frac{\varepsilon}{3} \quad \forall k \geq K_0.
\] (15)

By the absolute continuity of integral, we can choose a time $T \in (0, T_1)$ satisfying
\[
\int_T^{T_1} \max_{u \in U} |\eta(t, u)| \, dt < \frac{\varepsilon}{6}.
\] (16)

Based on Lemma 2.2, it holds that
\[
y_k(\cdot) \to y_*(\cdot) \text{ in } C([0, T]; \mathbb{R}^n), \quad \text{as } k \to +\infty,
\]
which means that for some $K_* > K_0$,
\[
\max_{t \in [0, T]} |y_k(t) - y_*(t)| < \min \left\{1, \frac{\varepsilon}{3L_{M_* + 1} T_1} \right\}, \quad \forall k \geq K_*,
\] (17)

where $M_* = \max_{t \in [0, T]} |y_*(t)|$ and $\hat{L}_{M_* + 1}$ is defined in (A3). Then, combining (A3), (15), (16) with (17), we deduce that
\[
\left| \int_0^{T_1} dt \int_U f^0(t, y_k(t), u) \sigma_k(t)(du) - \int_0^{T_1} dt \int_U f^0(t, y_*(t), u) \sigma_*(t)(du) \right|
\]
\[
\leq \int_0^{T_1} dt \int_U |f^0(t, y_k(t), u) - f^0(t, y_*(t), u)| \sigma_k(t)(du)
\]
\[
+ \int_0^{T_1} dt \int_U |f^0(t, y_*(t), u) \sigma_k(t) - \sigma_*(t)(du)|
\]
\[
= \int_0^{T_1} dt \int_U |f^0(t, y_k(t), u) - f^0(t, y_*(t), u)| \sigma_k(t)(du)
\]
\[
+ \int_0^{T_1} dt \int_U |f^0(t, y_*(t), u) - f^0(t, y_*(t), u)| \sigma_k(t)(du)
\]
and that by (18), we can use Filippov’s Lemma to get that there exists a $u^*$, which implies that

$$
\int_0^{T_1} dt \int_U f^0(t, y^*(t), u) \sigma_*(t)(du) = \inf_{(y^*(\cdot), \sigma_*(\cdot)) \in \mathcal{R}^{ad}_{\mathcal{B}}} \int_0^{T_1} dt \int_U f^0(t, y(t), u) \sigma(t)(du).
$$

Thus, we have proved that $(y^*(\cdot), \sigma_*(\cdot))$ is an optimal pair of Problem (RPB), which completes the proof.

To obtain the existence of Problem (PB), we need an extra assumption.

(A6) For almost any $t \in [0, T_1]$, the set $\mathcal{E}(t, y)$ is closed and convex for any $y \in \mathbb{R}^n$, where

$$
\mathcal{E}(t, y) = \left\{ (z, z^0) \in \mathbb{R}^n \times \mathbb{R} \mid z = f(t, y, u), \ z^0 \geq f^0(t, y, u), \ u \in U \right\}.
$$

**Theorem 2.4.** (Existence of Problem (PB)) Suppose (A1)–(A6) hold and the set $\mathcal{R}^{ad}_{\mathcal{B}} \neq \emptyset$. Then, Problem (PB) admits at least one optimal pair.

**Proof.** The set $\mathcal{R}^{ad}_{\mathcal{B}} \neq \emptyset$ implies that $\mathcal{R}^{ad}_{\mathcal{B}} \neq \emptyset$. Then, based on Theorem 2.3, we already have that $(y^*(\cdot), \sigma_*(\cdot))$ is an optimal pair of Problem (PRB). By (3) and (A6), it is easy to check that

$$
\left( \int_U f(t, y^*(t), u) \sigma_*(t)(du), \int_U f^0(t, y^*(t), u) \sigma_*(t)(du) \right) \in \overline{\mathcal{E}}(t, y^*(t)) = \mathcal{E}(t, y^*(t)), \text{ a.e. } t \in [0, T_1].
$$

(18)

By (18), we can use Filippov’s Lemma to get that there exists a $u^*(\cdot) \in \mathcal{U}_T$, such that

$$
f(t, y^*(t), u^*(t)) = \int_U f(t, y^*(t), u) \sigma_*(t)(du)
$$

and

$$
f^0(t, y^*(t), u^*(t)) \leq \int_U f^0(t, y(t), u) \sigma_*(t)(du),
$$

which ensures that $(y^*(\cdot), u^*(\cdot))$ is an optimal pair of Problem (PB). 

Now, we aim to obtain Pontryagin’s maximum principles of Problem (PB) by establishing the initial period optimality. For this reason, we introduce a class of auxiliary problems. Precisely, for any $T \in (0, T_1)$, denote

$$
\tilde{\mathcal{R}}^{B}_{T} = \left\{ (y(\cdot), u(\cdot)) \in \mathcal{R}^{ad}_{\mathcal{B}} \mid y(T) = y^*(T) \right\},
$$

where $(y^*(\cdot), u^*(\cdot))$ is an optimal pair of Problem (PB).

Define the optimal control problems as follows.

**Problem (PB$_T$).** to find a $(y^*_T(\cdot), u^*_T(\cdot)) \in \tilde{\mathcal{R}}^{B}_{T}$, such that

$$
J_T(y^*_T(\cdot), u^*_T(\cdot)) = \inf_{(y(\cdot), u(\cdot)) \in \tilde{\mathcal{R}}^{B}_{T}} J_T(y(\cdot), u(\cdot)),
$$

where $J_T(y(\cdot), u(\cdot)) = \int_0^T f^0(t, y(t), u(t)) dt.$
The following lemma shows the relation between Problem (PB₁) and Problem (PB).

**Lemma 2.5.** (Initial Period Optimality) Let \((y_*(\cdot), u_*(\cdot)) \in \mathcal{P}_{ad}^B\) be an optimal pair of Problem (PB). Then, for any \(T \in (0, T_1]\), the pair \((y_*(\cdot), u_*(\cdot))\) is also an optimal pair of Problem (PB₁).

**Proof.** Obviously, \((y_*(\cdot), u_*(\cdot)) \in \mathcal{P}^B\). If \((y_*(\cdot), u_*(\cdot))\) is not the optimal pair of Problem (PB₁), then there exists a pair \((\tilde{y}(\cdot), \tilde{u}(\cdot)) \in \mathcal{P}^B\) such that

\[
J_T(\tilde{y}(\cdot), \tilde{u}(\cdot)) = \int_0^T f^0(t, \tilde{y}(t), \tilde{u}(t)) \, dt < \int_0^T f^0(t, y_*(t), u_*(t)) \, dt = J_T(y_*(\cdot), u_*(\cdot)).
\]

Let

\[
u_{**}(t) = \begin{cases} \tilde{u}(t), & 0 \leq t \leq T, \\ u_*(t), & T < t \leq T_1, \end{cases}
\]

and \(y_{**}(\cdot) = y(\cdot, u_{**}()\)). It is easy to see that \(y_{**}(\cdot)\) blows up at \(T_1\). Then, we have

\[
J(y_{**}(\cdot), u_{**}(\cdot)) = \int_0^{T_1} f^0(t, y_{**}(t), u_{**}(t)) \, dt = \int_0^T f^0(t, y_*(t), u_*(t)) \, dt + \int_{T_1}^T f^0(t, y_{**}(t), u_{**}(t)) \, dt < J_T(y_*(\cdot), u_*(\cdot)),
\]

which contradicts to the optimality of \((y_*(\cdot), u_*(\cdot))\) for Problem (PB). Thus, we have proved that the pair \((y_*(\cdot), u_*(\cdot))\) is also optimal for Problem (PB₁). \(\square\)

On the basis of Lemma 2.5, we can establish Pontryagin’s maximum principle of Problem (PB). We need an extra assumption.

(A7) There is a constant \(R_0 > 0\), such that for any \(\varphi \in \mathbb{R}^n\),

\[
\langle \varphi, f_y(t, y, u) \varphi \rangle \geq 0, \quad \forall \ (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, \ |y| \geq R_0.
\]

**Remark 2.** The assumption (A7) is reasonable for some commonly used systems in the analysis of blowup phenomenon. For instance, let

\[
f(t, y, u) = |y|^{p-1}y + \ell(t, u), \quad \forall \ (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U,
\]

or

\[
f(t, y, u) = e^{|y|}y + \ell(t, u), \quad \forall \ (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U,
\]

where \(p > 1\) and \(\ell(\cdot, \cdot) : [0, +\infty) \times U \rightarrow \mathbb{R}^n\). It can be easily verified that the assumption (A7) holds in these two cases.

**Theorem 2.6.** (Pontryagin’s Maximum Principle of Problem (PB)) Suppose \((A2), (A3), (A4)\) and \((A7)\) hold. Let \((y_*(\cdot), u_*(\cdot))\) be an optimal pair of Problem (PB).
and the function $f^0_\ast(\cdot,y_\ast(\cdot),u_\ast(\cdot)) \in L^1([0,T_1];\mathbb{R}^n)$. Then, there exist $\varphi^0_\ast \leq 0$ and $\varphi_\ast(\cdot) \in C([0,T_1];\mathbb{R}^n)$, such that
\[ |\varphi_\ast(t)|^2 + |\varphi^0_\ast|^2 > 0, \quad \forall \ t \in [0,T_1), \]
and for almost any $t \in [0,T_1)$,
\[ \frac{d\varphi_\ast(t)}{dt} = -f_y(t,y_\ast(t),u_\ast(t))\varphi_\ast(t) - \varphi^0_\ast f^0_y(t,y_\ast(t),u_\ast(t)), \tag{19} \]
\[ H(t,y_\ast(t),u_\ast(t),\varphi_\ast(t),\varphi^0_\ast) = \max_{u \in U} H(t,y_\ast(t),u,\varphi_\ast(t),\varphi^0_\ast), \tag{20} \]
where $H(t,y,u,\varphi,\varphi^0) = \langle \varphi, f(t,y,u) \rangle + \varphi^0 f^0(t,y,u)$.

**Proof.** By Lemma 2.5, we have that for any $0 < \varepsilon < T_1$, $(y_\ast(\cdot),u_\ast(\cdot))$ is also an optimal pair of Problem $\text{(PB}_{T_1-\varepsilon})$. We introduce the Ekeland distance $d^\ast$ over the set $\mathcal{W}_{T_1}$ by setting
\[ d^\ast(u(\cdot),v(\cdot)) = \text{meas}\{ \{t \in [0,T_1] \mid u(t) \neq v(t)\} \} \] for all $u(\cdot),v(\cdot) \in \mathcal{W}_{T_1}$, where $\text{meas}(E)$ stands for the Lebesgue measure of a measurable set $E$ in $\mathbb{R}$. Then, $(\mathcal{W}_{T_1},d^\ast)$ forms a complete metric space (see [12, p. 145]). Denote
\[ \tilde{\mathcal{W}}_{T_1} = \left\{ u(\cdot) \in \mathcal{W}_{T_1} \mid (T_1,y(\cdot;u(\cdot)),u(\cdot)) \in \mathcal{P}_{T_1}, \lim_{t \to T_1^-} |y(t;u(\cdot))| = +\infty \right\}. \]

We claim that $(\tilde{\mathcal{W}}_{T_1},d^\ast)$ is a complete metric sub-space of $(\mathcal{W}_{T_1},d^\ast)$.

Indeed, it is clear that $(\tilde{\mathcal{W}}_{T_1},d^\ast)$ is a metric sub-space of $(\mathcal{W}_{T_1},d^\ast)$. Let $\{u_k(\cdot)\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{W}_{T_1}$ and
\[ d^\ast(u_k(\cdot),u(\cdot)) \to 0, \quad \text{as } k \to \infty, \]
where $u(\cdot) \in \mathcal{W}_{T_1}$. Then, $(T_1,y(\cdot;u_k(\cdot)),u_k(\cdot)) \in \mathcal{P}_{T_1}$ and $\lim_{t \to T_1^-} |y(t;u_k(\cdot))| = +\infty$. By adopting the similar argument in the proof of Corollary 1, we can get $(T_1,y(\cdot;u(\cdot)),u(\cdot)) \in \mathcal{P}_{T_1}$ and $\lim_{t \to T_1^-} |y(t;u(\cdot))| = +\infty$, which implies $u(\cdot) \in \mathcal{W}_{T_1}$ and $(\tilde{\mathcal{W}}_{T_1},d^\ast)$ is a complete metric sub-space of $(\mathcal{W}_{T_1},d^\ast)$.

For any $\varepsilon > 0$, any $u(\cdot) \in \mathcal{W}_{T_1}$ and $y(\cdot) = y(\cdot;u(\cdot))$, set
\[ J_\varepsilon(y(\cdot),u(\cdot)) = \left\{ \left| y(T_1-\varepsilon) - y_\ast(T_1-\varepsilon) \right|^2 \right. \\
+ \left( \max_{t} \left( \int_0^{T_1-\varepsilon} f^0(t,y(t),u(t)) dt - \int_0^{T_1-\varepsilon} f^0(t,y_\ast(t),u_\ast(t)) dt + \varepsilon \right) \right)^2 \right\}^{1/2}. \]

Then, we can use Ekeland’s variational principle and the classical techniques to obtain Pontryagin’s maximum principle of Problem $\text{(PB}_{T_1-\varepsilon})$. That means there exist $\varphi^0_{T_1-\varepsilon} \leq 0$ and $\varphi_{T_1-\varepsilon}(\cdot) \in C([0,T_1-\varepsilon];\mathbb{R}^n)$, such that
\[ |\varphi_{T_1-\varepsilon}(t)|^2 + |\varphi^0_{T_1-\varepsilon}|^2 > 0, \quad \forall \ t \in [0,T_1-\varepsilon), \]
\[ |\varphi_{T_1-\varepsilon}(T_1-\varepsilon)|^2 + |\varphi^0_{T_1-\varepsilon}|^2 = 1, \tag{21} \]
and for almost any $t \in [0,T_1-\varepsilon)$,
\[ \frac{d\varphi_{T_1-\varepsilon}(t)}{dt} = -f_y(t,y_\ast(t),u_\ast(t))\varphi_{T_1-\varepsilon}(t) - \varphi^0_{T_1-\varepsilon} f^0_y(t,y_\ast(t),u_\ast(t)), \tag{22} \]
\[ H(t,y_\ast(t),u_\ast(t),\varphi_{T_1-\varepsilon}(t),\varphi^0_{T_1-\varepsilon}) = \max_{u \in U} H(t,y_\ast(t),u,\varphi_{T_1-\varepsilon}(t),\varphi^0_{T_1-\varepsilon}). \tag{23} \]
Now, we take a sequence \( \{\delta_m\}_{m=1}^{+\infty} \) of numbers from the interval \((0, T_1)\), such that

\[
\text{(i) } \lim_{m \to +\infty} \delta_m = 0 \quad \text{and} \quad \text{(ii) } \delta_1 > \delta_2 > \cdots > \delta_m > \cdots .
\]

Corresponding to the number \( \delta_1 \), we take a sequence \( \{\varepsilon_n\}_{n=1}^{+\infty} \) of numbers from the interval \((0, T_1)\), such that

\[
\text{(i) } \lim_{n \to +\infty} \varepsilon_n = 0 \quad \text{and} \quad \text{(ii) } [0, T_1 - \delta_1] \subseteq [0, T_1 - \varepsilon_n], \quad \forall \ n = 1, 2, \cdots .
\]

For any \( n = 1, 2, \cdots \), denote

\[
\varphi_{T_1-\varepsilon_n}(t) = \begin{cases} 
\tilde{\varphi}_{T_1-\varepsilon_n}(t), & \text{if } |\tilde{\varphi}_{T_1-\varepsilon_n}(0)| \leq 1, \\
\frac{\varphi_{T_1-\varepsilon_n}(t)}{|\varphi_{T_1-\varepsilon_n}(0)|}, & \text{if } |\tilde{\varphi}_{T_1-\varepsilon_n}(0)| > 1,
\end{cases}
\]

and

\[
\varphi_{T_1}^0 - \varepsilon_n = \begin{cases} 
\tilde{\varphi}_{T_1}^0\varepsilon_n, & \text{if } |\tilde{\varphi}_{T_1-\varepsilon_n}(0)| \leq 1, \\
\frac{\varphi_{T_1}^0\varepsilon_n}{|\varphi_{T_1-\varepsilon_n}(0)|}, & \text{if } |\tilde{\varphi}_{T_1-\varepsilon_n}(0)| > 1.
\end{cases}
\]

It is clear that \( (\varphi_{T_1-\varepsilon_n}(\cdot), \varphi_{T_1}^0 - \varepsilon_n) \) also satisfies (22) and (23).

We claim that the sequence \( \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \) is uniformly bounded and equicontinuous on \([0, T_1 - \delta_1]\).

Indeed, it holds that for any \( t \in [0, T_1 - \delta_1] \),

\[
\varphi_{T_1-\varepsilon_n}(t) = \varphi_{T_1-\varepsilon_n}(0) - \int_0^t \left( f_p(s, y_s(s), u_s(s))\varphi_{T_1-\varepsilon_n}(s) + \varphi_{T_1-\varepsilon_n}^0 f_p^0(s, y_s(s), u_s(s)) \right) ds,
\]

which implies

\[
|\varphi_{T_1-\varepsilon_n}(t)| \leq |\varphi_{T_1-\varepsilon_n}(0)| + \int_0^t L_{M_{\delta_1}} (|\varphi_{T_1-\varepsilon_n}(s)| + |\varphi_{T_1-\varepsilon_n}^0|) ds, \quad \forall \ t \in [0, T_1 - \delta_1],
\]

where \( M_{\delta_1} = \max_{t \in [0, T_1 - \delta_1]} |y_t(t)| \) and \( \tilde{L}_{M_{\delta_1}} \) is defined in (A3). By (21), (24), (25) and Grönwall’s inequality, we have

\[
|\varphi_{T_1-\varepsilon_n}(t)| \leq C \triangleq e^{\tilde{L}_{M_{\delta_1}} T_1} (\tilde{L}_{M_{\delta_1}} T_1 + 1), \quad \forall \ t \in [0, T_1 - \delta_1].
\]

On the other hand, it holds that

\[
|\varphi_{T_1-\varepsilon_n}(t_1) - \varphi_{T_1-\varepsilon_n}(t_2)| = \left| \int_{t_1}^{t_2} \left( f_p(s, y_s(s), u_s(s))\varphi_{T_1-\varepsilon_n}(s) + \varphi_{T_1-\varepsilon_n}^0 f_p^0(s, y_s(s), u_s(s)) \right) ds \right|
\]

\[
\leq \tilde{L}_{M_{\delta_1}} (C + 1) |t_1 - t_2|, \quad \forall \ t_1, t_2 \in [0, T_1 - \delta_1],
\]

which implies that \( \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \) is equicontinuous on \([0, T_1 - \delta_1]\).

According to Arzelà–Ascoli’s lemma, we can take a subsequence \( \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \) from the sequence \( \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \), such that it is uniformly convergent on the interval \([0, T_1 - \delta_1]\). In general, for each number \( \delta_m \), we can take a subsequence \( \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \) holding the following properties:

\[
\text{(i) } \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \subseteq \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{j+1}, \quad \forall \ j = 2, \cdots, m,
\]

and

\[
\text{(ii) } \{\varphi_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{+\infty} \text{ is uniformly convergent on } [0, T_1 - \delta_m].
\]
Now, by the standard diagonal argument, we see the subsequence \( \{\varphi^n_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{\infty} \) is uniformly convergent on \([0, T_1-\delta]\) for each \( \delta \in (0, T_1) \).

Denote \( \varphi_*(t) = \lim_{n \to +\infty} \varphi^n_{T_1-\varepsilon_n}(t) \) for any \( t \in [0, T_1) \). It follows that for any number \( \delta \in (0, T_1) \),
\[
\varphi^n_{T_1-\varepsilon_n}(\cdot) \to \varphi_*(\cdot) \text{ in } C([0, T_1-\delta]; \mathbb{R}^n), \quad \text{as } n \to \infty. \tag{26}
\]
Then, by (21) and (25), we may as well assume
\[
\varphi^n_{T_1-\varepsilon_n} \to \varphi_*, \quad \text{as } n \to \infty. \tag{27}
\]
Combining with (22), (23), (26) and (27), we deduce that (19) and (20) hold.

It remains to prove the non-triviality of \((\varphi_*(\cdot), \varphi_*^0)\). Actually, we may suppose \(\varphi_*^0 = 0\).

**Case One.** \( \{\varphi^n_{T_1-\varepsilon_n}(0)\}_{n=1}^{\infty} \) has infinite terms satisfying \( |\varphi^n_{T_1-\varepsilon_n}(0)| > 1 \).

By (25), we assume that \( |\varphi^n_{T_1-\varepsilon_n}(0)| = 1 \) for any \( n = 1, 2, \cdots \) without loss of generality. Combining with (26), it implies that \( |\varphi_*(0)| = 1 \). Hence,
\[
\varphi_*(t) \neq 0, \quad \forall t \in [0, T_1).
\]

**Case Two.** \( \{\varphi^n_{T_1-\varepsilon_n}(0)\}_{n=1}^{\infty} \) has at most finite terms satisfying \( |\varphi^n_{T_1-\varepsilon_n}(0)| > 1 \).

We take a subsequence from \( \{\varphi^n_{T_1-\varepsilon_n}(\cdot)\}_{n=1}^{\infty} \), still denoted in the same way, such that \( |\varphi^n_{T_1-\varepsilon_n}(0)| \leq 1 \) for any \( n = 1, 2, \cdots \). Then, (21), (24) and (25) ensure that
\[
|\varphi^n_{T_1-\varepsilon_n}(T_1-\varepsilon_n)|^2 + |\varphi^n_{T_1-\varepsilon_n}|^2 = 1. \tag{28}
\]
Then, since \( y_* \) blows up at \( T_1 \), there exists a \( \delta_0 \in (0, T_1) \), such that
\[
|y_*(t)| \geq R_0, \quad \forall t \in [T_1-\delta_0, T_1), \tag{29}
\]
where \( R_0 \) is defined in (A7). Meanwhile, based on (27) and (28), we can find a natural number \( N \), such that
\[
-\frac{1}{4M_0} < \frac{\varphi^n_{T_1-\varepsilon_n}}{\varepsilon_n} \leq 0, \quad |\varphi^n_{T_1-\varepsilon_n}(T_1-\varepsilon_n)| > \frac{1}{2} \quad \text{and} \quad \varepsilon_n < \delta_0, \quad \forall n \geq N. \tag{30}
\]
where
\[
M_0 = \int_0^{T_1} |f^0(t, y_*(t), u_*(t))| \, dt + 1.
\]

By (22), we have that for each \( n \geq N \) and for almost any \( t \in [0, T_1-\varepsilon_n] \),
\[
\frac{d |\varphi^n_{T_1-\varepsilon_n}(t)|}{dt} = - \left\langle \frac{\varphi^n_{T_1-\varepsilon_n}(t)}{|\varphi^n_{T_1-\varepsilon_n}(t)|}, f^0(t, y_*(t), u_*(t))\varphi^n_{T_1-\varepsilon_n}(t) \right\rangle - \left\langle \frac{\varphi^n_{T_1-\varepsilon_n}(t)}{|\varphi^n_{T_1-\varepsilon_n}(t)|}, \varphi^n_{T_1-\varepsilon_n} f^0(t, y_*(t), u_*(t)) \right\rangle.
\]
Combining with (29), (30) and (A7), it holds that for each \( n \geq N \),
\[
|\varphi^n_{T_1-\varepsilon_n}(T_1-\delta_0)| = |\varphi^n_{T_1-\varepsilon_n}(T_1-\varepsilon_n)| + \int_{T_1-\delta_0}^{T_1-\varepsilon_n} \left\langle \frac{\varphi^n_{T_1-\varepsilon_n}(t)}{|\varphi^n_{T_1-\varepsilon_n}(t)|}, f^0(t, y_*(t), u_*(t))\varphi^n_{T_1-\varepsilon_n}(t) \right\rangle \, dt
\]
\[
+ \int_{T_1-\delta_0}^{T_1-\varepsilon_n} \left\langle \frac{\varphi^n_{T_1-\varepsilon_n}(t)}{|\varphi^n_{T_1-\varepsilon_n}(t)|}, \varphi^n_{T_1-\varepsilon_n} f^0(t, y_*(t), u_*(t)) \right\rangle \, dt
\]
\[
> \frac{1}{2} + \int_{T_1-\delta_0}^{T_1-\varepsilon_n} \left\langle \frac{\varphi^n_{T_1-\varepsilon_n}(t)}{|\varphi^n_{T_1-\varepsilon_n}(t)|}, \varphi^n_{T_1-\varepsilon_n} f^0(t, y_*(t), u_*(t)) \right\rangle \, dt.
\]
3. Quenching case. In this section, we aim to establish the existence and Pontryagin’s maximum principles for optimal control problems governed by the systems with quenching phenomenon. Precisely, for a given nonempty set $\Omega$, we consider the case that the function $f \in (1)$ may be unbounded near $\Omega$. For this reason, we need similar assumptions with the blowup case. We restate these assumptions as follows except (A1), (A5) and (A6).

(H1) $\Omega$ is a nonempty, closed and convex set in $\mathbb{R}^n$. Also, $y_0 \notin \Omega$.
(H2) For any $(y, u) \in (\mathbb{R}^n \setminus \Omega) \times U$, the function $f(\cdot, y, u) : [0, +\infty) \to \mathbb{R}^n$ and $f^0(\cdot, y, u) : [0, +\infty) \to \mathbb{R}$ are measurable; for any $t \in [0, +\infty)$, $f(t, \cdot, \cdot) : (\mathbb{R}^n \setminus \Omega) \times U \to \mathbb{R}^n$ and $f^0(t, \cdot, \cdot) : (\mathbb{R}^n \setminus \Omega) \times U \to \mathbb{R}$ are continuous; for any $r > 0$, there is an $L_r > 0$, such that

$$|f(t, y, u)| + |f^0(t, y, u)| \leq L_r, \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, \quad d\Omega(y) \geq r.$$  
(H3) For any $(t, u) \in [0, +\infty) \times U$, the function $f(t, \cdot, u) : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^n$ and $f^0(t, \cdot, u) : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ are continuously differentiable; for any $r > 0$, there is an $L_r > 0$, such that

$$|f_y(t, y, u)| + |f^0_y(t, y, u)| \leq L_r, \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, \quad d\Omega(y) \geq r.$$  
(H4) There exist a constant $r_0 > 0$ and a positive function $\zeta(\cdot) : [0, r_0] \to \mathbb{R}^+$, such that

$$\langle \nabla d\Omega(y), f(t, y, u) \rangle \leq -\zeta(d\Omega(y)), \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, \quad 0 < d\Omega(y) \leq r_0,$$

and

$$\int_0^{r_0} \frac{1}{\zeta(r)} \, dr < +\infty.$$  
(H5) There is a constant $r_0 > 0$, such that for any $\varphi \in \mathbb{R}^n$,

$$\langle \varphi, f_\Omega(t, y, u) \varphi \rangle \geq 0, \quad \forall (t, y, u) \in [0, +\infty) \times \mathbb{R}^n \times U, \quad 0 < d\Omega(y) \leq r_0.$$

Remark 3. For convenience, we assume the set $\Omega$ is convex in (H1) to ensure the differentiability of $d\Omega(\cdot)$ on $(0, +\infty)$ in (H4). In the following part of the paper, we only need the differentiability of $d\Omega(\cdot)$.

Let $T_1 > 0$ and let $f^0(\cdot, \cdot, \cdot) : [0, +\infty) \times \mathbb{R}^n \times U \to \mathbb{R}$ be a given function. We denote

$$\mathcal{P}_{ad}^Q = \left\{ (y(\cdot), u(\cdot)) \in C([0, T_1]; \mathbb{R}^n \setminus \Omega) \times \mathcal{W}_{T_1} \, \bigg| \, (1) \text{ holds on } [0, T_1), \right\}.$$

Here, $\mathcal{W}_{T_1}$ is defined in (2).
Similarly, we study the following optimal control problem.

**Problem (PQ).** To find a pair \((y_{**}(\cdot), u_{**}(\cdot))\) in \(\mathcal{P}_{ad}^Q\), such that

\[
J(y_{**}(\cdot), u_{**}(\cdot)) = \inf_{(y(\cdot), u(\cdot)) \in \mathcal{P}_{ad}} J(y(\cdot), u(\cdot)),
\]

where \(J(y(\cdot), u(\cdot)) = \int_0^{T_1} f^0(t, y(t), u(t)) \, dt\).

Now, we give the definition of quenching.

**Definition 3.1.** Let the control \(u(\cdot) \in \mathcal{U} \). We say that the solution \(y(\cdot; u(\cdot))\) in \(C([0, T); \mathbb{R}^n \setminus \Omega)\) of (1) quenches at finite time \(T > 0\) (for the first time), if

\[
y(t; u(\cdot)) = y_0 + \int_0^t f(s, y(s; u(\cdot)), u(s)) \, ds, \quad \forall \ t \in [0, T),
\]

\[
\lim_{t \to T^-} y(t; u(\cdot)) \text{ exists, and for each } T_0 \in (0, T),
\]

the set \(\{f(t, y(t; u(\cdot)), u(t)) \in \mathbb{R}^n \mid t \in (T_0, T)\}\) is unbounded.

**Remark 4.** It is obvious that Problem (PQ) can contain some optimal control problems whose purposes are to find the control minimizing a given cost functional among all the controls leading to the corresponding solutions’ quenching at the right-hand time end-point of the functional.

**Remark 5.** The assumption (H5) is reasonable for some commonly used systems in the analysis of quenching phenomenon. For instance, let

\[
f(t, y, u) = \frac{1}{1 - y} + \ell(t, u), \quad \forall \ t, y, u \in [0, +\infty) \times \mathbb{R} \times U, \ y \neq 1,
\]

where \(\ell(\cdot, \cdot) : [0, +\infty) \times U \to \mathbb{R}\). It follows that the assumption (H5) holds.

Similarly to the optimal control problem including blowup phenomenon, we introduce the relaxed control for Problem (PQ). For convenience, we restate the relaxed problem. Consider the relaxed system,

\[
\begin{cases}
\frac{dy(t)}{dt} = \int_U f(t, y(t), u) \sigma(t)(du), \quad t > 0, \\
y(0) = y_0,
\end{cases}
\]

and the relaxed admissible set

\[
\mathcal{R}_{ad}^Q = \left\{(y(\cdot), \sigma(\cdot)) \in C([0, T_1] ; \mathbb{R}^n \setminus \Omega) \times \mathcal{R}_{T_1} (U) \mid (32) \text{ holds on } [0, T_1), \right. \\
\left. \lim_{t \to T_1^-} d_{\Omega}(y(t)) = 0 \text{ and } \int_U f^0(\cdot, y(\cdot), u) \sigma(\cdot)(du) \in L^1(0, T_1) \right\}.
\]

Here, \(\mathcal{R}_{T_1} (U)\) is defined in (3).

The corresponding optimal relaxed control problem can be stated as

**Problem (RPQ).** To find a pair \((y_{**}(\cdot), \sigma_{**}(\cdot))\) in \(\mathcal{R}_{ad}^Q\), such that

\[
J(y_{**}(\cdot), \sigma_{**}(\cdot)) = \inf_{(y(\cdot), \sigma(\cdot)) \in \mathcal{R}_{ad}^Q} J(y(\cdot), \sigma(\cdot)),
\]

where \(J(y(\cdot), \sigma(\cdot)) = \int_0^{T_1} d\int_U f^0(t, y(t), u) \sigma(t)(du)\).

The following convergence result can be established by adopting the similar strategies of Lemma 2.2.
Lemma 3.2. Suppose (H1)–(H3) hold. Let \( \{(y_{k}(\cdot), \sigma_{k}(\cdot))\}_{k=1}^{\infty} \subseteq \mathcal{R}_{ad}^{Q} \) and \( (y(\cdot), \sigma(\cdot)) \in \mathcal{R}_{ad}^{Q} \). For any \( T \in (0, T_1) \), if
\[
\sigma_{k}(\cdot) \rightarrow \sigma(\cdot) \quad \text{in} \quad \mathcal{R}_{T}(U), \quad \text{as} \quad k \rightarrow +\infty,
\]
then it holds that
\[
y_{k}(\cdot) \rightarrow y(\cdot) \quad \text{in} \quad C([0, T]; \mathbb{R}^{n}), \quad \text{as} \quad k \rightarrow +\infty.
\]

Corollary 2. Suppose (H1)–(H4) hold. Let \( \{(y_{k}(\cdot), \sigma_{k}(\cdot))\}_{k=1}^{\infty} \subseteq \mathcal{R}_{ad}^{Q} \) and \( \sigma(\cdot) \in \mathcal{R}_{T_{1}}(U) \). If
\[
\sigma_{k}(\cdot) \rightarrow \sigma(\cdot) \quad \text{in} \quad \mathcal{R}_{T_{1}}(U), \quad \text{as} \quad k \rightarrow +\infty,
\]
then, it holds that \( y(\cdot) \) quenches at \( T_{1} \), where \( y(\cdot) = y(\cdot; \sigma(\cdot)) \).

Proof. Suppose that \( y(\cdot) \) does not quench at \( T_{1} \). There are two cases to discuss.

Case One. There exists an \( S \geq T_{1} \), such that
\[
y(\cdot) \in C([0, S]; \mathbb{R}^{n} \setminus \Omega) \quad \text{and} \quad (32) \quad \text{holds on} \quad [0, S].
\]

In this case, by the similar argument in Lemma 2.2, we have that \( y_{k}(\cdot) \in C([0, S]; \mathbb{R}^{n} \setminus \Omega) \) when \( k \) is large enough, which contradicts to the fact that the sequence \( \{(y_{k}(\cdot), \sigma_{k}(\cdot))\}_{k=1}^{\infty} \subseteq \mathcal{R}_{ad}^{Q} \).

Case Two. There exists an \( S < T_{1} \), such that
\[
y(\cdot) \in C([0, S]; \mathbb{R}^{n} \setminus \Omega), \quad (32) \quad \text{holds on} \quad [0, S] \quad \text{and} \quad \lim_{t \to S^{-} d_{Q}(y(t))} = 0. \quad (34)
\]

By (H4), there exists a \( 0 < r_{1} < r_{0} \), such that
\[
\int_{0}^{r_{1}} \frac{1}{\zeta(r)} \, dr < \frac{T_{1} - S}{2}, \quad (35)
\]
where \( r_{0} \) is defined in (H4). Then, by (34), we can obtain a \( T \in (0, S) \), such that
\[
d_{Q}(y(T)) < 2r_{1}.
\]

By the similar argument in the proof of Lemma 2.2, we can find a \( K > 0 \) such that
\[
d_{Q}(y_{k}(T)) \leq r_{1}, \quad \forall \quad k \geq K.
\]

Then, by (H4), we can use the similar techniques in the proof of Lemma 2.3 in [15] to get
\[
d_{Q}(y_{k}(t)) \leq r_{1}, \quad \forall \quad t \in [T, S], \quad k \geq K. \quad (36)
\]

Hence, it follows by (H4) that
\[
\frac{d}{dt} d_{Q}(y_{k}(t)) = \int_{U} \nabla d_{Q}(y_{k}(t)), f(t, y_{k}(t), u)) \sigma_{k}(t)(du)
\leq -\zeta(d_{Q}(y_{k}(t))), \quad \text{a.e.} \quad t \in [T, S], \quad \forall \quad k \geq K. \quad (37)
\]

Combining (35), (36) with (37), we obtain that for any \( k \geq K \),
\[
\frac{T_{1} - S}{2} > \int_{0}^{r_{1}} \frac{1}{\zeta(r)} \, dr \geq \int_{0}^{d_{Q}(y(T))} \frac{1}{\zeta(r)} \, dr
= \int_{T}^{T_{1}} \frac{1}{\zeta(d_{Q}(y_{k}(t)))} \frac{d}{dt} d_{Q}(y_{k}(t)) \, dt
= -\int_{T}^{T_{1}} \frac{1}{\zeta(d_{Q}(y_{k}(t)))} \frac{d}{dt} d_{Q}(y_{k}(t)) \, dt \geq T_{1} - T > T_{1} - S, \quad (38)
\]

which is a contradiction to the fact that \( S < T_{1} \).
Thus, we have proved this corollary. \qed

Then, we can easily deduce the existence of Problem (RPQ).

**Theorem 3.3.** (Existence of Problem (RPQ)) Suppose (H1)—(H4), (A1), (A5) hold and \( R^Q \neq \emptyset \). Then, Problem (RPQ) admits at least one optimal pair.

Based on the above results and the assumption (A6), we have the existence of Problem (PQ).

**Theorem 3.4.** (Existence of Problem (PQ)) Suppose (H1)—(H4), (A1), (A5), (A6) hold and \( P^n_{ad} \neq \emptyset \). Then, Problem (PQ) admits at least one optimal pair.

Further, we introduce the initial period optimality of Problem (PQ). Auxiliary optimal control problems are considered. Precisely, for any \( T \in (0,T_1) \), denote

\[
\mathcal{P}_{ad}^Q = \left\{ (y(\cdot), u(\cdot)) \in \mathcal{P}_{ad}^Q \mid y(T) = y_*(T) \right\},
\]

where \((y_*(\cdot), u_*(\cdot))\) is an optimal pair of Problem (PQ).

Define the auxiliary optimal control problems as follows.

**Problem (PQ_T).** Find \((y^*_T(\cdot), u^*_T(\cdot)) \in \mathcal{P}^Q_T\), such that

\[
J_T(y^*_T(\cdot), u^*_T(\cdot)) = \inf_{(y(\cdot), u(\cdot)) \in \mathcal{P}^Q_T} J_T(y(\cdot), u(\cdot)),
\]

where \( J_T(y(\cdot), u(\cdot)) = \int_0^T f_0(t, y(t), u(t)) dt \).

**Lemma 3.5.** (Initial Period Optimality) Let \((y_*(\cdot), u_*(\cdot)) \in \mathcal{P}_{ad}^Q\) be an optimal pair of Problem (PQ). Then, for any \( T \in (0,T_1) \), the pair \((y_*(\cdot), u_*(\cdot))\) is also an optimal pair of Problem (PQ_T).

On the basis of the initial period optimality, Pontryagin’s maximum principle of Problem (PQ) can be stated as follows.

**Theorem 3.6.** (Pontryagin’s Maximum Principle of Problem (PQ)) Suppose (H1)—(H5) hold. Let \((y_*(\cdot), u_*(\cdot))\) be an optimal pair of Problem (PQ) and the function \( f_0^0(\cdot, y_*(\cdot), u_*(\cdot)) \in L^1([0,T_1]; \mathbb{R}^n) \). Then, there exist \( \varphi_{**}^0 \leq 0 \) and \( \varphi_{**}(\cdot) \in C([0,T_1]; \mathbb{R}^n) \), such that

\[
|\varphi_{**}(t)|^2 + |\varphi_{**}^0|^2 > 0, \quad \forall t \in [0,T_1),
\]

and for almost any \( t \in [0,T_1) \),

\[
d\varphi_{**}(t) = -f_0(t, y_*(t), u_*(t)) \varphi_{**}(t) - \varphi_{**}^0 f_0^0(t, y_*(t), u_*(t)),
\]

\[
H(t, y_*(t), u_*(t), \varphi_{**}(t), \varphi_{**}^0) = \max_{u \in U} H(t, y_*(t), u, \varphi_{**}(t), \varphi_{**}^0),
\]

where \( H(t, y, u, \varphi, \varphi^0) = \langle \varphi, f(t, y, u) \rangle + \varphi^0 f_0(t, y, u) \).

**Remark 6.** The existence result and Pontryagin’s maximum principle of Problem (PQ) can be proved by adopting the same strategies of Problem (PB). Thus, we omit the details of the proof.
4. Case of extendable control problem after quenching. In this section, we focus on the optimal control problems governed by an extendable equation after quenching. For convenience, denote
\[
\mathcal{U}|_I = \left\{ u(\cdot) : I \to U \mid u(\cdot) \text{ is measurable} \right\},
\]
where \( I \) is an arbitrary subset of \( \mathbb{R} \). Then, we give the following definition (see [14]) on the extendability of a solution of (1) after quenching.

**Definition 4.1.** Let \( y_0 \notin \Omega \) and \( u(\cdot) \in \mathcal{U} \). Suppose that the solution \( y(\cdot; u(\cdot)) \) of equation (1) quenches at finite time \( T > 0 \) for the first time, and \( \lim_{t \to T^-} y(t; u(\cdot)) = \hat{y} \). We say that \( y(\cdot; u(\cdot)) \) is extendable after quenching, if

(i) the function \( f(\cdot, y(\cdot; u(\cdot)), u(\cdot)) \in L^1([0,T] ; \mathbb{R}^n) \).

(ii) there are a number \( \delta > 0 \), a control \( \tilde{u}(\cdot) \in \mathcal{U}|_{[T,T+\delta]} \) and a function \( \tilde{Y}(\cdot) \in C((T,T+\delta) ; \mathbb{R}^n \setminus \Omega) \), such that

\[
\lim_{t \to T^+} \tilde{Y}(t) = \hat{y}, \quad f(\cdot, \tilde{Y}(\cdot), \tilde{u}(\cdot)) \text{ is integrable over } (T, T+\delta),
\]

and the function \( Y(t) \triangleq \begin{cases} y(t, u(\cdot)), & t \in [0,T), \\ \hat{y}, & t = T, \\ \tilde{Y}(t), & t \in (T, T+\delta) \end{cases} \) satisfies

\[
Y(t) = y_0 + \int_0^T f(s, Y(s), u(s)) \, ds + \int_t^T f(s, Y(s), \tilde{u}(s)) \, ds, \quad \forall \ t \in [T, T+\delta).
\]

**Remark 7.** We call \( \tilde{u}(\cdot) \) and \( \tilde{Y}(\cdot) \) with \( \tilde{Y}(T) = \hat{y} \) in Definition 4.1 an extended control of \( u(\cdot) \), and a corresponding extended solution of \( y(\cdot; u(\cdot)) \) over the interval \([T, T+\delta]\), respectively. While, (1) is called an extendable equation.

Let \( y_0 \notin \Omega \) and \( u(\cdot) \in \mathcal{U} \). Suppose that the solution \( y(\cdot; u(\cdot)) \) of (1) quenches at \( T_1 > 0 \) for the first time, \( \lim_{t \to T_1^-} y(t; u(\cdot)) = \hat{y} \), and it is extendable after quenching. Let \( T_2 > T_1 \). We set

\[
\mathcal{U}|_{ad}^E = \left\{ \tilde{u}(\cdot) \in \mathcal{U}|_{[T_1,T_2]} \mid \tilde{u}(\cdot) \text{ is an extended control of } u(\cdot) \text{ over } [T_1,T_2] \right\},
\]

and

\[
\mathcal{D}|_{ad}^E = \left\{ (\tilde{Y}(\cdot), \tilde{u}(\cdot)) \in C((T_1,T_2) ; \mathbb{R}^n \setminus \Omega) \times \mathcal{U}|_{ad}^E \mid \tilde{Y}(\cdot) \text{ is an extended solution of } y(\cdot; u(\cdot)) \text{ with respect to } \tilde{u}(\cdot), \right. \\
\left. \lim_{t \to T_2^-} d_\Omega(\tilde{Y}(t)) = 0 \right\}
\]

\[
\text{and } f^0(\cdot, \tilde{Y}(\cdot), \tilde{u}(\cdot)) \in L^1(T_1, T_2)
\]

We introduce the following problem:

**Problem (PE).** to find \((\tilde{Y}_*(\cdot), \tilde{u}_*(\cdot)) \in \mathcal{D}|_{ad}^E \), such that

\[
\tilde{J}(\tilde{Y}_*(\cdot), \tilde{u}_*(\cdot)) = \inf_{(\tilde{Y}(\cdot), \tilde{u}(\cdot)) \in \mathcal{D}|_{ad}^E} \tilde{J}(\tilde{Y}(\cdot), \tilde{u}(\cdot)),
\]

where \( \tilde{J}(\tilde{Y}(\cdot), \tilde{u}(\cdot)) = \int_{T_1}^{T_2} f^0(t, \tilde{Y}(t), \tilde{u}(t)) \, dt \).

The following Pontryagin’s maximum principle holds for Problem (PE).
Theorem 4.2. (Pontryagin’s Maximum Principle of Problem (PE)) Suppose (H1)—(H5) hold. Let \((\hat{Y}_\epsilon(\cdot), \hat{u}_\epsilon(\cdot))\) be an optimal pair of Problem (PE) and the function \(f_\epsilon^0(\cdot, \hat{Y}_\epsilon(\cdot), \hat{u}_\epsilon(\cdot)) \in L^1([T_1, T_2]; \mathbb{R}^n)\). Then, there exist \(\bar{\varphi}_\epsilon \leq 0\) and \(\hat{\varphi}_\epsilon(\cdot) \in C((T_1, T_2); \mathbb{R}^n)\), such that

\[
|\varphi_\epsilon(t)|^2 + |\varphi_\epsilon^0|^2 > 0, \quad \forall t \in (T_1, T_2),
\]

and for almost any \(t \in (T_1, T_2),\)

\[
\frac{d\varphi_\epsilon}{dt} = -f_\epsilon(t, \hat{Y}_\epsilon(t), \hat{u}_\epsilon(t)) \varphi_\epsilon(t) - \varphi_\epsilon^0 f_\epsilon^0(t, \hat{Y}_\epsilon(t), \hat{u}_\epsilon(t)),
\]

\[
H(t, \hat{Y}_\epsilon(t), \hat{u}_\epsilon(t), \varphi_\epsilon(t), \varphi_\epsilon^0) = \max_{u \in \mathcal{U}} H(t, \hat{Y}_\epsilon(t), u, \varphi_\epsilon(t), \varphi_\epsilon^0),
\]

where \(H(t, y, u, \varphi, \varphi^0) = \langle \varphi, f(t, y, u) \rangle + \varphi^0 f^0(t, y, u)\).

Proof. For each \(\epsilon \in (0, \frac{T_2 - T_1}{2})\), and for each \(\tilde{u}(\cdot) \in \mathcal{U}|_{[T_1 + \epsilon, T_2]}\), consider the following equation:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{dy}{dt} = f(t, y(t), \tilde{u}(t)), \quad t \in [T_1 + \epsilon, T_2), \\
y(T_1 + \epsilon) = \hat{Y}_\epsilon(T_1 + \epsilon).
\end{array} \right.
\tag{39}
\end{aligned}
\]

Since \(\hat{Y}_\epsilon(T_1 + \epsilon) \notin \Omega\), it is easy to see that the local solution to (39) is unique. Denote

\[
\mathcal{P}_{ad}^\epsilon = \left\{ (\tilde{y}(\cdot), \tilde{u}(\cdot)) \in C([T_1 + \epsilon, T_2]; \mathbb{R}^n \setminus \Omega) \times \mathcal{U}|_{[T_1 + \epsilon, T_2]} \left| (39) \right. \text{ holds} \right. \}
\]

on \([T_1 + \epsilon, T_2], \left. \lim_{t \to T_2^-} d\Omega(\tilde{y}(t)) = 0 \right. \text{ and } \tilde{y}(T_2 - \epsilon) = \hat{Y}_\epsilon(T_2 - \epsilon) \right. \}

Similarly as in the proof of Theorem 3.1 in [14], we can construct the following approximate control problem for Problem (PE).

Problem (PE\(^\epsilon\)). to find \((\tilde{y}_\epsilon^\epsilon(\cdot), \tilde{u}_\epsilon^\epsilon(\cdot)) \in \mathcal{P}_{ad}^\epsilon, \text{ such that}\)

\[
J^\epsilon(\tilde{y}_\epsilon^\epsilon(\cdot), \tilde{u}_\epsilon^\epsilon(\cdot)) = \inf_{(\tilde{y}(\cdot), \tilde{u}(\cdot)) \in \mathcal{P}_{ad}^\epsilon} J^\epsilon(\tilde{y}(\cdot), \tilde{u}(\cdot)),
\]

where \(J^\epsilon(\tilde{y}(\cdot), \tilde{u}(\cdot)) = \int_{T_1 + \epsilon}^{T_2 - \epsilon} f^0(t, \tilde{y}(t), \tilde{u}(t)) dt\).

Then, by adopting the similar techniques in the proof of Lemma 2.5 and Theorem 2.6, we can easily check that \((\hat{Y}_\epsilon^\epsilon(\cdot), \hat{u}_\epsilon^\epsilon(\cdot))\) is an optimal control of Problem (PE\(^\epsilon\)). Moreover, there exist \(\bar{\varphi}_\epsilon^\epsilon \leq 0\) and \(\hat{\varphi}_\epsilon^\epsilon(\cdot) \in C([T_1 + \epsilon, T_2 - \epsilon]; \mathbb{R}^n)\), such that

\[
|\tilde{\varphi}_\epsilon(t)|^2 + |\tilde{\varphi}_\epsilon^0|^2 > 0, \quad \forall t \in [T_1 + \epsilon, T_2 - \epsilon],
\]

\[
|\tilde{\varphi}_\epsilon(T_2 - \epsilon)|^2 + |\tilde{\varphi}_\epsilon^0|^2 = 1,
\]

and for almost any \(t \in [T_1 + \epsilon, T_2 - \epsilon],\)

\[
\frac{d\tilde{\varphi}_\epsilon}{dt} = -f_\epsilon(t, \hat{Y}_\epsilon(t), \hat{u}_\epsilon(t)) \tilde{\varphi}_\epsilon(t) - \tilde{\varphi}_\epsilon^0 f_\epsilon^0(t, \hat{Y}_\epsilon(t), \hat{u}_\epsilon(t)),
\]

\[
H(t, \hat{Y}_\epsilon(t), \hat{u}_\epsilon(t), \varphi_\epsilon(t), \varphi_\epsilon^0) = \max_{u \in \mathcal{U}} H(t, \hat{Y}_\epsilon(t), u, \varphi_\epsilon(t), \varphi_\epsilon^0).
\]

Now, we take a sequence \(\{\delta_m\}_{m=1}^{+\infty}\) of numbers from the interval \((0, \frac{T_2 - T_1}{2})\), such that

(i) \(\lim_{m \to +\infty} \delta_m = 0\) and (ii) \(\delta_1 > \delta_2 > \cdots > \delta_m > \cdots\).
Corresponding to the number $\delta_1$, we can take a sequence $\{\varepsilon_n\}^{+\infty}_{n=1} \subseteq (0, \frac{T_2-T_1}{2})$, such that

(i) $\lim_{n \to +\infty} \varepsilon_n = 0$ and (ii) $[T_1 + \delta_1, T_2 - \delta_1] \subseteq [T_1 + \varepsilon_n, T_2 - \varepsilon_n]$, $\forall n = 1, 2, \ldots$.

Let

$$\varphi_{\varepsilon_n}(\cdot) = \begin{cases} \tilde{\varphi}_{\varepsilon_n}(\cdot), & \text{if } |\tilde{\varphi}_{\varepsilon_n}(\frac{T_2-T_1}{2})| \leq 1, \\ \frac{\tilde{\varphi}_{\varepsilon_n}(\cdot)}{|\tilde{\varphi}_{\varepsilon_n}(\frac{T_2-T_1}{2})|}, & \text{if } |\tilde{\varphi}_{\varepsilon_n}(\frac{T_2-T_1}{2})| > 1, \end{cases}$$

and

$$\varphi_{\varepsilon_n}^0 = \begin{cases} \varphi_{\varepsilon_n}^0, & \text{if } |\tilde{\varphi}_{\varepsilon_n}(\frac{T_2-T_1}{2})| \leq 1, \\ \frac{\varphi_{\varepsilon_n}^0}{|\tilde{\varphi}_{\varepsilon_n}(\frac{T_2-T_1}{2})|}, & \text{if } |\tilde{\varphi}_{\varepsilon_n}(\frac{T_2-T_1}{2})| > 1. \end{cases}$$

It is clear that $(\varphi_{\varepsilon_n}(\cdot), \varphi_{\varepsilon_n}^0)$ also satisfies (40), (41) and (42).

Then, we can use the similar argument in the proof of Theorem 2.6 to complete this theorem.

5. Perspectives. For technical reasons, we proved the existence of optimal controls for Problem (PB) and (PQ) under a strong assumption (A5). It is an interesting problem to get the same existence results for Problem (PB) and (PQ) without this assumption.

Meanwhile, Pontryagin’s maximum principles for optimal controls of Problems (PB), (PQ) and (PE) were proved under the assumption that the function $f_0^0 \in L^1$ corresponding to optimal state-control pair in the interval, where the cost functional is defined. These are also technical conditions. How to obtain Pontryagin’s maximum principles in three cases without such conditions remains to be explored.

On the other hand, the transversality conditions for Problems (PB), (PQ) and (PE) need more complex techniques to obtain.

Moreover, since the extended solutions after quenching may be multiple for the same extended control, the existence of optimal controls for problem (PE) is difficult. This problem deserves to be studied.

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