Fuzzy spacetime from a null-surface version of GR

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Abstract. The null-surface formulation of general relativity – recently introduced – provides novel tools for describing the gravitational field, as well as a fresh physical way of viewing it. The new formulation provides “local” observables corresponding to the coordinates of points — which constitute the spacetime manifold — in a geometrically defined chart, as well as non-local observables corresponding to lightcone cuts and lightcones. In the quantum theory, the spacetime point observables become operators and the spacetime manifold itself becomes “quantized”, or “fuzzy”. This novel view may shed light on some of the interpretational problems of a quantum theory of gravity. Indeed, as we discuss briefly, the null-surface formulation of general relativity provides (local) geometrical quantities — the spacetime point observables — which are candidates for the long-sought physical operators of the quantum theory.

1. Introduction

The issue of the relationship or unification of the general theory of relativity with quantum theory was raised early on, and remains a subject of active research [1]. Two basic types of problems have been at the core of the work: 1. In a purely formal manner, can (or how can) a quantum theory of general relativity (GR) be constructed in a mathematically consistent fashion? 2. If, in fact, a theory can be so constructed, can (or how can) it be given a reasonable physical interpretation, i.e., what relationship does it have to physical reality? In particular, what possible physical meaning could be given to a “quantum spacetime”?

According to some these problems can be solved by simply applying the ideas of quantum theory to GR or to one of its extensions [2, 3]. There are others who feel that both the mathematical and conceptual problems are so overwhelming that new physical ideas (perhaps radical) must be introduced to unify GR with quantum theory - that there must be a true unification or amalgamation of the ideas of GR with the ideas of quantum theory, rather than the “simple” subjugation of GR to the machinery of quantum theory [4].

Without strongly taking sides in the debate (the authors among themselves have differences of opinion) we want to point out that by viewing the classical Einstein equations in an unorthodox manner and following a “natural” trail towards the “quantization” of this unorthodox version of GR, we appear to have been led to a new point of view towards the subject. The new view essentially states that the spacetime points themselves must be “quantized”, i.e., turned into operators with commutation relations, etc. It is not the fields on a manifold that must be quantized (metric, etc.), but...
is (in some sense) the manifold idea itself that must be changed. This is not simply an empty conjecture; there is in principle (in full theory) and in practice (in linear theory) a means to calculate the commutators between individual spacetime points. We are not claiming anything profound nor are we even advocating an approach to the two major questions posed above. We, however, feel that the new point of view has the potential for shedding light on the problem of the “meaning of quantum gravity” and thereby being possibly of importance.

In Sec. 2 we give a brief introduction to a new (classical) point of view to GR, referred to as a null-surface version of GR (where the basic variable has been shifted from the metric, connection, or other – field –, to families of characteristic surfaces) while, in Secs. 3 and 4, we discuss what happens when we try to turn the new point of view into a quantum theory. In Sec. 5 we discuss possible meanings and ramifications of these ideas.

This is a report of work very much in progress and as such we will mainly “wave our hands”, describe what we have in mind and give a minimum number of equations and no proofs.

2. Null-Surface View of GR

During the past few years a new formulation of classical general relativity (specially suited for asymptotically flat spacetimes) has become available [5]. In this formulation, the emphasis has shifted from the usual type of field variable (metric, connection, holonomy, curvature, etc.) to, instead, families of three-dimensional surfaces. On a manifold with no other structure, equations are given for the determination of these surfaces. From the surfaces themselves, by differentiation, a (conformal) metric can be obtained; the surfaces themselves are then automatically characteristic surfaces of this conformal metric. The equations simultaneously determine a choice of conformal factor such that the metric automatically satisfies the vacuum Einstein equations. In other words, the vacuum Einstein equations are formulated as equations for families of surfaces and a single scalar conformal factor. In our present discussion we will be primarily interested in only the characteristic surfaces, though, in the actual formalism, the conformal factor plays an essential role in the determination of the surfaces. Our point of view will be that we have already integrated the equations for both – the surfaces and the conformal factor – and we are interested now in information that we can retrieve from knowledge of the surfaces.

Consider asymptotically flat spacetimes with (future) null infinity $I^+ = S^2 \times \mathbb{R}$. Let $\zeta$ be a complex stereographic coordinate† on $S^2$ which labels the generators of $I^+$,

† In this paper, when we write a function of complex variables, we make no assumption of it being holomorphic. Consequently, functions that are of the form $f(\zeta, \bar{\zeta})$ will be written simply as $f(\zeta)$.
and let \( u \in \mathbb{R} \) be an appropriately normalized parameter along the generators. Thus \((u, \zeta)\) are the Bondi coordinates on \( I^+ \). The free data for Einstein’s equations is then associated with a connection (the Bondi shear) on \( I^+ \), and can be specified by the choice of a complex spin-weight-2 field \( \sigma(u, \zeta) \) on \( I^+ \). The space of all such fields \((\sigma)\), together with the appropriate symplectic structure, constitutes the reduced phase space of general relativity [6].

The characteristic surfaces of the spacetime are described as follows: We obtain a function (our fundamental variable) \( Z(x^a, \zeta; [\sigma]) \) as a solution of certain differential equations involving the given data \([\sigma]\). (The square brackets \([\ ]\) denote functional dependence.) The \( x^a \) appearing in the argument of \( Z \) are local coordinates on the spacetime manifold in an arbitrary chart. They appear as (integration) parameters in the solutions of the differential equations that determine \( Z \). For fixed values of \((u, \zeta)\), the equation

\[
Z(x^a, \zeta; [\sigma]) = u
\]

(1)

describes a characteristic (or null) surface in terms of the given local chart, \( x^a \), on our manifold. In fact, the null surface is the past lightcone of the point \((u, \zeta)\) on \( I^+ \). As the value of \( u \) varies (for fixed \( \zeta \)) we have a one-parameter foliation (of a local region) by the characteristic surfaces. The \( \zeta \) then labels a sphere’s worth of these null foliations; equivalently, for each point \( x^a \), as the \( \zeta \) varies, we obtain a sphere’s worth of characteristic surfaces through \( x^a \). An alternate interpretation of the function \( u = Z(x^a, \zeta; [\sigma]) \) is that, for fixed point \( x^a \), it describes the lightcone cut of \( x^a \); i.e., the intersection of the future lightcone of \( x^a \) with \( I^+ \).

Assuming that the \( Z \) satisfies our differential equations, one can then, in a prescribed fashion, express a conformal Einstein metric in terms of derivatives of \( Z \) [5]. For simplicity, a special member of the conformal class can be chosen (in a natural† fashion), yielding an explicit metric in terms of \( Z \). This natural representative of the conformal class does not satisfy the Einstein equations. Nevertheless, once the conformal representative is fixed, the conformal factor needed to transform it to an Einstein metric can be determined as well. In what follows, we assume that the function \( Z \) is always implicitly associated with an appropriate conformal factor that guarantees an Einstein metric. Notice, first, that all the conformal information about the spacetime is contained in the knowledge of \( Z(x^a, \zeta; [\sigma]) \) and, second, that the (Einstein) conformal factor itself depends on the data. (For the sake of simplicity of presentation, we have slightly simplified the discussion. See [5] for the details.)

Since, for each fixed value of \( \zeta \), as \( u \) varies, the \( Z \) describes a foliation by null Consistently, we will write \( \zeta \) instead of \((\zeta, \bar{\zeta})\), except when confusion could arise.

† A special member of the conformal class is chosen by requiring that the coordinate \( R \) introduced in (2d) becomes an affine parameter with respect to that representative of the conformal class [5].
surfaces, we can associate a null coordinate system with each such foliation. Though it is not at all obvious (the proof is given in [3]) a characteristic coordinate system is easily explicitly obtained from the $Z$ — by taking several $(\zeta, \bar{\zeta})$ derivatives — in the following fashion:

$$ u = Z(x^a, \zeta, [\sigma]), $$

$$ \omega = \bar{\partial}Z(x^a, \zeta, [\sigma]) $$

$$ \bar{\omega} = \bar{\partial}Z(x^a, \zeta, [\sigma]), $$

$$ R = \bar{\partial}\bar{\partial}Z(x^a, \zeta, [\sigma]), $$

where $\partial$ and $\bar{\partial}$ are (essentially) the $\zeta$ and $\bar{\zeta}$ derivatives respectively [4]. The geometrical meaning of the coordinates $(u, \omega, \bar{\omega}, R)$ is extensively discussed below. Note that we automatically have a sphere’s worth of these characteristic coordinate systems (a single coordinate system for every value of $\zeta$).

With the notation

$$ \theta^i = (\theta^0, \theta^+, \theta^-, \theta^1) = (u, \omega, \bar{\omega}, R) $$

we have that

$$ \theta^i = \theta^i(x^a, \zeta; [\sigma]) $$

is a coordinate transformation from the “old” coordinates $x^a$ to a sphere’s worth of null coordinate systems $\theta^i$. It should however be stressed that (4) is much more than a set of coordinate transformations. The $\theta^i$ contain the full information about the solutions of the conformal Einstein equations, through their dependence on the data $\sigma$.

Eq. (4) can be algebraically inverted to express the local coordinates $x^a$ in terms of the $\theta^i$ and $\zeta$

$$ x^a = x^a(\theta^i, \zeta; [\sigma]) \equiv x^a(u, \omega, R, \zeta; [\sigma]). $$

The inversion shows that, expressed in the chart $\theta^i$, the $x^a$ depend on the data. Note that, like (4), (5) also contains the full information about the solutions of the conformal Einstein equations; i.e., from (5) a metric conformal to an Einstein metric can be obtained. Though this feature is basic, (4) encodes other information which is, at the moment, of more direct interest to us. Before we proceed to the quantum theory, it is worthwhile to discuss the meaning and the dual role of equations (2) and (5). We present three related and complementary interpretations, which focus on three different sets of geometric entities. For the purposes of the following discussion, we fix the data $\sigma$ to some arbitrary value, thus fixing an Einstein 4-metric.

### 2.1. Lightcone cuts

Consider first Eq. (5). We assume that $Z$ is known, as a solution of our equations (equivalent to the Einstein equations). Everything we say follows from the $Z$. 

On the one hand, Eq. (1) has been introduced as describing null surfaces of the interior spacetime in terms of the “old” coordinates $x^a$, since, viewed as a function of the $x^a$, (1) defines the past lightcone of the point $(u, \zeta)$ on $I^+$: spacetime points $x^a$ which solve $Z(x^a, \zeta) = u$ lie on this lightcone.

However, there is another meaning (mentioned earlier), which extends to the other $\theta^i$. We can view $Z(x^a, \zeta)$, for fixed $x^a$, as describing a cut on $I^+$ (the lightcone cut of $x^a$); i.e., $u$ as a function of $\zeta$. Consider a fixed $x^a$. For every fixed value of $\zeta$ there is (in general) a single null geodesic emanating from $x^a$ that reaches $I^+$. This null geodesic can be characterized by geometric information given on $I^+$; three real parameters labeling the point of intersection $(u, \zeta)$ and two parameters specifying a direction from $I^+$ (complex $\omega$, geometrically a 2-blade). These parameters correspond to the five real parameters $(u, \omega, \zeta)$ given by Eqs. (2). The parameter $R$ (Eq. (2d)) parametrizes the geodesic labeled by $(u, \omega, \zeta)$ and specifies the interior points that lie on it. Eqs. (2) fix a one-to-one correspondence between interior points and geometric structures on $I^+$. For the moment, this is the viewpoint we will take for the meaning of (2):

For a fixed spacetime point $x^a$, (2a) yields its lightcone cut on $I^+$. (2b) yields the angle of intersection with the generator $\zeta$ of $I^+$ of the null geodesic between $x^a$ and the generator $\zeta$, and (2d) yields the corresponding non-affine geodesic distance from $x^a$ to $I^+$. (Note that $R$ also has a geometric meaning on $I^+$; it is the curvature of the lightcone cut at $\zeta$.)

2.2. Spacetime points

Alternatively to (4), we can think of its inverse, namely (5), as locating interior spacetime points by following null geodesics inward, from $I^+$, as follows. The five-dimensional space of null geodesics (of the spacetime specified by $\sigma$) is coordinatized by $(u, \omega, \zeta)$: $(u, \zeta)$ label the point on $I^+$ and $\omega$ labels the generator of the past lightcone; whereas $R$ parametrizes the individual geodesics. For fixed values of $(u, \omega, R, \zeta)$, (5) yields an interior spacetime point.

Given a fixed “observation point” $(u, \zeta) \in I^+$, (5) gives the coordinates (in the local chart) of the point which lies a parameter distance $R$ along the generator $\omega$ of the past lightcone of $(u, \zeta)$.

Observe that a particular point $x^a$ can be reached by coming in along many different null geodesics; i.e., there are many different sets of parameters $(u, \omega, \zeta)$ that will focus $x^a$.†

† Exactly $S^2$ worth, see (3).
the null geodesics onto the same point $x^a$. Spacetime points are thus identified by the set of null geodesics that focus onto them.

As an aside we also point out that (5) written as $x^a = x^a(R, u, \omega, \zeta, [\sigma])$ for arbitrary but fixed $(u, \omega, \zeta)$, and variable $R$, is the parametric form for all null geodesics.

2.3. Lightcones

Finally, there is a third structure that is hidden in the $Z$-function. We are interested in describing the lightcone of a specific point in the interior of the spacetime. Fix a spacetime point $x^a_0$, and consider the $S^2$ generators of its lightcone (see (3)), specified by:

$$
u[x_0](\zeta) = Z(x_0^a, \zeta, [\sigma]),$$
$$\omega[x_0](\zeta) = \bar{\omega}Z(x_0^a, \zeta, [\sigma]),$$
$$\bar{\omega}[x_0](\zeta) = \bar{\nu}Z(x_0^a, \zeta, [\sigma])$$

and the null geodesic parameter distance from $x_0^a$ to the $\zeta$ generator of $I^+$:

$$R[x_0](\zeta) = \bar{\nu}\bar{\omega}Z(x_0^a, \zeta, [\sigma]).$$

Now, in (5), which describes all null geodesics, at each value of $\zeta$ we fix the null geodesic on which $x_0^a$ lies, by substituting (3) and (5) into (5). This gives us, in parametric form, the full lightcone of $x_0^a$:

$$x^a[x_0](r, \zeta; [\sigma]) = x^a(u[x_0], \omega[x_0], R[x_0] - r, \zeta; [\sigma]),$$

where the $\theta^i[x_0]$ depend on the initial data $\sigma$ and on $\zeta$ through (3) and (5). Note that we have set $R = R[x_0] - r$, so that, at $r = 0$, $x^a = x_0^a$. Thus:

As the three parameters $(r, \zeta)$ vary, we obtain the full lightcone of the point $x_0^a$ in the conformal spacetime that is specified by $[\sigma]$.

We will return to these equations and ideas later, when we describe the quantization of linear gravity.

3. Vacuum Electrodynamics

Classical vacuum electrodynamics and classical linearized GR can be given a parallel development in terms of characteristic data (D'Adhamar formulation) both of which we briefly review. Though our primary interest is in gravity, in this section, we use source-free Maxwell theory as a model in which to introduce various ideas. As we are only interested in giving an overview, we refrain from giving the details [8].

Consider Minkowski space with coordinates $x^a$ and future null infinity $I^+ = S^2 \times \mathbb{R}$. The characteristic data on null infinity for the free Maxwell field is specified by the
arbitrary complex spin-wt-1 function, \( A(u, \zeta) \) where, as before, \( (u, \zeta) \) are the Bondi coordinates on \( \mathcal{I}^+ \). The future lightcone of an interior point \( x^a \) intersects \( \mathcal{I}^+ \) on a (topological) sphere \( S^2(x^a) \), called the lightcone cut of \( x^a \), which is described by

\[
u = Z_M(x^a, \zeta) = x^a \ell_a(\zeta) = x^a \eta_{ab} \ell^b,
\]

where \( \ell^b(\zeta) \) is the null vector with components \( 1/\sqrt{2(1+\zeta \bar{\zeta})} \) \( (1 + \zeta \bar{\zeta}, \zeta + \bar{\zeta}, i(\bar{\zeta} - \zeta), -1 + \zeta \bar{\zeta}) \) defining the null cone at any point, and \( Z_M \) is the \( Z \)-function for Minkowski space.

The data \( A \) restricted to the lightcone cut of \( x^a \), becomes \( A_R(x^a, \zeta) = A(Z_M(x^a, \zeta), \zeta) \). The vacuum Maxwell equations can be written in terms of a real scalar “potential”, \( F(x^a, \zeta) \), which satisfies \([1]\) the equation

\[
\delta \delta F = \delta A_R(x^a, \zeta) + \delta A_R(x^a, \zeta) \equiv D_M(x^a, \zeta)[A].
\]

This is easily solved in the form

\[
F(x^a, \zeta; [A]) = \int_{S^2} d^2 S_\eta G_M(\zeta, \eta)D_M(x^a, \eta)[A]
\]

where \( G_M(\zeta; \eta) \) is a known Green’s function \([10]\), and \( d^2 S_\eta = -2i \eta \wedge d\bar{\eta}/(1 + \eta \bar{\eta})^2 \) is the area form for \( S^2 \). \([A]\) indicates the functional dependence of the solution on the free data. From \( F \), satisfying \([10]\), it is easy to construct \([8]\); by differentiation, a vector potential \( \gamma_a(x^a, [A]) \) and a Maxwell field \( F_{ab}(x^a; [A]) \) that automatically satisfies the vacuum Maxwell equations; i.e., from \( F \) we have

\[
F(x^a, \zeta, [A]) \Rightarrow \gamma_a(x^a, [A]) \Rightarrow F_{ab}(x^a, [A]).
\]

The quantization of the Maxwell field is accomplished by constructing operators corresponding to the data \( A \) which satisfy the commutation relations

\[
[\hat{A}(u, \zeta), \hat{A}(u', \zeta')] = i\Delta(u - u')\delta^2(\zeta - \zeta')\hat{1},
\]

where \( \Delta(u) = \text{sgn}(u)/2 \) is the skew step function, and we have set \( \hbar = c = 1 \). Eq. \([11]\) can be used to define

\[
\hat{F}(x^a, \zeta) \equiv F(x^a, \zeta, [\hat{A}]) = \int_{S^2} d^2 S_\eta G_M(\zeta, \eta)D_M(x^a, \eta)[\hat{A}]
\]

and it is straightforward to find an integral representation (which can be evaluated in closed form with considerable effort \([5]\)) of the commutation relations

\[
[\hat{F}(x^a, \zeta), \hat{F}(x'^a, \zeta')]
\]

as well as

\[
[\hat{\gamma}_a(x^a), \hat{\gamma}_b(x'^a)] \quad \text{and} \quad [\hat{F}_{ab}(x^a), \hat{F}_{cd}(x'^a)].
\]

The last two \([13]\) turn out to be, respectively, the standard commutation relations of the vector potential in the Coulomb gauge and covariant commutation relations for the
We note that, within the formalism, the vector potential has been automatically chosen to be in the Coulomb gauge. Alternate choices for the Green’s function \( G_M \) yield all other gauge choices.

We point out that the above asymptotic quantization is not merely heuristic, but has been constructed explicitly elsewhere \( \text{[8]} \), and a representation isomorphic to the asymptotic Fock representation \( \text{[6]} \) has been obtained, which in turn is isomorphic to the usual Fock representation, through D’Adhamar integrals.

4. Linearized GR: Quantization and interpretation

The linearized Einstein equations (off Minkowski space) can be considered as being analogous to the Maxwell equations but now for a spin-2 field rather than for a spin-1 field. For the moment we will adopt this point of view and describe the linearized Einstein equations in a fashion completely analogous to the Maxwell description, Eqs. (10)-(16). This version of the linearized equations arises as the linearization of the null-surface theory of GR described in Sec.3.

The data is given by a complex-valued spin-weight-2 function on \( \mathcal{I}^+ \), namely \( \sigma(u, \zeta) \) which can be given freely. The data restricted to the lightcone cut of \( x^a \) becomes \( \sigma_R(x^a, \zeta) = \sigma(Z_M(x^a, \zeta), \zeta) \). The variable analogous to the potential \( F(x^a, \zeta) \) is \( Z(x^a, \zeta) \) with the generalization of (10) to this case being \( \text{[10]} \).

\[
\sigma^2 \partial^2 Z = \sigma^2 \sigma_R(x^a, \zeta) + \partial^2 \sigma_R(x^a, \zeta) := D_{GR}(x^a, \zeta)[\sigma].
\]

(17)

Though it is far from obvious, Eq. (17) is completely equivalent to the linearized vacuum Einstein equations. The solution is easily obtained in integral form:

\[
Z(x^a, \zeta, [\sigma]) = \int_{S^2} d^2 \eta G_{GR}(\zeta, \eta) D_{GR}(x^a, \eta)[\sigma],
\]

(18)

where \( G_{GR} \) is a known Green’s function \( \text{[10]} \) and the spacetime points \( x^a \) again enter as parameters in the solution \( \text{[18]} \). From this, one can construct a linearized metric by taking various derivatives of \( Z \). (See Section \( \text{[2]} \)) This metric automatically satisfies the spin-2 equations — but in a particular gauge, namely a generalized version of the Coulomb gauge. Analogous to the Maxwell case, we have that from \( Z \) we can calculate \( h_{ab}(x^a, [\sigma]) = g_{ab} - \eta_{ab} \), the deviation of the metric from the Minkowski metric, which automatically satisfies the linearized Einstein equations, and the linearized Weyl tensor \( C_{abcd}(x^a, [\sigma]) \); i.e.,

\[
Z(x^a, \zeta, [\sigma]) \Rightarrow h_{ab}(x^a, [\sigma]) \Rightarrow C_{abcd}(x^a, [\sigma]).
\]

(19)

The formal quantization of the spin-2 fields can be carried out by representing the data as operators that satisfy the commutation relations

\[
[\hat{\sigma}(u, \zeta), \hat{\sigma}(u', \zeta')] = i \Delta(u - u') \delta^2(\zeta - \zeta') \hat{1}.
\]

(20)
From (18), one can define
\[ \hat{Z}(x^a, \zeta) \equiv Z(x^a, \zeta, [\hat{\sigma}]) = \int_{S^2} d^2 \eta \ G_{GR}(\zeta, \eta) D_{GR}(x^a, \eta)[\hat{\sigma}], \] 
(21)
and then from (20) one can obtain commutation relations for the \( \hat{Z} \); i.e.,
\[ [\hat{Z}(x^a, \zeta), \hat{Z}(x^a, \zeta')] \] 
(22)
From these one can obtain the algebra of the \( h_{ab} \) and \( C_{abcd} \); i.e., the commutation relations for
\[ [\hat{h}_{ab}(x^a), \hat{h}_{cd}(x'^a)] \quad \text{and} \quad [\hat{C}_{abcd}(x^a), \hat{C}_{efgh}(x'^a)]. \] 
(23)
(At the present, we have an integral representation for these commutation relations but have not evaluated the integrals in closed form.)

Up to this point there is little difference between the Maxwell and linearized GR theory other than the spin-1 versus spin-2. However, GR is a theory of the geometry of spacetime. The geometric structure can be seen in the meaning of the function, \( Z(x^a, \zeta, [\sigma]) \); though, so far, in this section it has only played the role of a potential for the metric (and Weyl tensor), it is an important geometric quantity. \( Z \) is in fact the linearization of the \( Z(x^a, \zeta, [\sigma]) \) of equation (1) of the full theory, and as such describes the characteristic surfaces of the linearized metric. We are now presented with a potential problem or conundrum when trying to understand the quantization: From the point of view of a spin-2 theory, \( Z \) was a field (or potential) that could be promoted to an operator – but from the point of view of the geometry, it describes a (characteristic) surface and making a “surface” into an operator certainly raises issues of meaning. At the very least, it is not clear \textit{a priori} what this may mean.

A hint as to a possible meaning of a quantum operator corresponding to a classical definition of a surface is given by the following \textit{analogy}. In ordinary quantum mechanics, consider the trajectory of a particle. This is described by the function \( x^i(t), i = 1, 2, 3 \); geometrically, this is a curve in spacetime. In the quantum theory, this curve becomes a “quantum trajectory” \( \hat{x}^i(t) \). That is, we have three 1-parameter families of operators which correspond to the position of the particle at any given value of the parameter, the Newtonian time \( t \). The common eigenstate of the operators at time \( t \), with eigenvalues \( x^i \), is the localized state \( | x^i; t \rangle \). For each value of the \textit{classical parameter} \( t \), the projection of a quantum state onto the eigenstates of \( \hat{x}^i(t) \) allows us to compute the \textit{probability} distribution of the position of the particle at the time \( t \). Note that \( t \) is a classical parameter which specifies the “experimental” situation, and the probability is a density on \( \{ x^i \} \), the spectrum of \( \hat{x}^i(t) \). The interpretation of quantum gravity that we attempt in this paper is an extension of this very idea to the three classical geometrical entities described in Sec. 2.1, 2.2, 2.3.
4.1. Fuzzy lightcone cuts

Though the situation at hand is very different from, and much more complicated than, the Newtonian particle considered above – we are not quantizing particle motion but geometric properties of spacetimes – let us first attempt a similar interpretation of \( \hat{Z}(x^a, \zeta; \sigma) \equiv Z(x^a, \zeta; [\sigma]) \). \( \hat{Z}(x^a, \zeta) \) is a 6-parameter family of operators. Like \( t \), \((x^a, \zeta)\) are classical parameters which help us define the specific “experimental” situation, and like the set \( \{x^i\} \), the eigenvalues of \( \hat{Z} \) constitute the space of possible outcomes \( \{u\} \).

Classically, (1) defined the lightcone cut of \( x^a \), or the value of \( u \) at which the future lightcone from \( x^a \) intersects \( I^+ \) on the generator \( \zeta \). Now, in the quantum theory, we will only have a probabilistic interpretation. Fix a spacetime point \( x^a \), a generator of \( I^+ \) labeled by \( \zeta \) and a physical state \( \psi \) of “quantum gravity”. Let \( |u; (x^a, \zeta)\rangle \) denote an eigenstate of \( \hat{Z} \) with eigenvalue \( u \). Then we expect that \( |\langle u; (x^a, \zeta) | \psi \rangle|^2 \) is the probability distribution (in \( u \)) that the future lightcone of \( x^a \) intersects \( I^+ \) (at \( u \)) on the generator \( \zeta \).

In this preliminary interpretation, it is the points, i.e. values of \( u \), along the generators of \( I^+ \) which are “fuzzy”. Thus the lightcone cut of the point \( x^a \) appears to be fuzzy.

We now develop this interpretation further. We used \( Z \) to construct a null coordinate system \((u, \omega, R)\) (in fact a sphere’s worth of them). From Sec. 2.1, let us recall the geometrical meaning of these coordinates: \((u, \omega)\) label a null geodesic from the generator \( \zeta \) of \( I^+ \), and \( R \) is the geodesic distance between \( I^+ \) and a point \( x^a \) on the geodesic. In the quantum theory, since they depend on the data through \( Z \), these coordinates will in turn be operators:

\[
\begin{align*}
\hat{u}(x^a, \zeta) &= Z(x^a, \zeta; [\sigma]), \\
\hat{\omega}(x^a, \zeta) &= \partial Z(x^a, \zeta; [\sigma]), \\
\hat{\omega}(x^a, \zeta) &= \partial Z(x^a, \zeta; [\sigma]), \\
\hat{R}(x^a, \zeta) &= \partial \partial Z(x^a; \zeta; [\sigma]).
\end{align*}
\]

Let us collectively denote them as \( \hat{\theta}^i \). A number of problems now arise. What are the commutation relations between these four operators? How do the commutators depend on \( \zeta \)? If we fix \( \zeta \), is there a set of spacetime points such that the four geodesic operators form a commuting set? (Note that these questions can be answered in the linear theory [8].) What is the relationship between these points? What is clear, at such a preliminary stage, is that a generic state of quantum gravity will not correspond to well-defined values of \( \theta^i \), since they are operators subject to nontrivial commutation relations; in general, one can only associate probabilities with various allowed sets of their eigenvalues. In this sense, the lightcone cut, the angle of emittance of the null geodesic at \( I^+ \) and the curvature of the lightcone cut corresponding to a fixed spacetime
point are “fuzzy”.

### 4.2. Fuzzy spacetime points

In this subsection, we construct the “dual” formulation (see Sec. 2.2), in which the spacetime points themselves are fuzzy. Recall that the equations for the null coordinates of a spacetime point $x^a$, Eq. (22)

$$\theta^i = \theta^i(x^a, \zeta, [\sigma])$$

(25)

can be inverted (order-by-order in a perturbative approach, or to first order in linear theory) to obtain the coordinates in the local chart:

$$x^a = x^a(\theta^i; \zeta; [\sigma]).$$

(26)

In the quantization process (for the linear theory) the data, $\sigma$, was made into operators which implied that the $\theta^i$ were all operator functions of the $c$–numbers $(x^a, \zeta)$. Alternatively, in the dual picture corresponding to (26), we can treat the inversion equations (26) as a set of equations for the operators $\hat{x}^a$ as operator-valued functions of the operator data and the $c$–numbers $(\theta^i, \zeta)$:

$$\hat{x}^a = x^a(\theta^i; \zeta; [\sigma]).$$

(27)

From this point of view the (coordinates of the) spacetime points themselves become operators. Let us explore the possible significance of this.

What is the analog of a point in the quantum theory? A candidate could be a common eigenstate $|x^a; (u, \omega, R), \zeta\rangle$ of the four operators $\hat{x}^a$; this would correspond to well-defined values of all four coordinates, and thus a well-defined “spacetime point”. Let us fix a specific null-coordinate system, by fixing $\zeta$ corresponding to an asymptotic observer†. There now appear to be three levels at which one could fail to have well-defined points in the quantum theory:

- Consider the four coordinate operators for the same spacetime point; i.e., fix also $(u, \omega, R)$, and consider $[\hat{x}^a, \hat{x}^b]$. It is possible that the four operators $\hat{x}^a$, corresponding to the same spacetime point, do not commute amongst themselves. Thus, common eigenstates $|x^a; (u, \omega, R), \zeta\rangle$ would fail to exist and, in this sense, there would be no spacetime points in the quantum theory. However, preliminary calculations suggest that they do commute.
- What if the coordinates of the same spacetime point do commute amongst themselves? Then, we have common eigenstates $|x^a; (u, \omega, R), \zeta\rangle$, and the existence of the quantum analog of a spacetime point. However, a generic state of quantum

† Recall that there many ways ($S^2$ of them) in which to reach an interior point from $I^+$ along null geodesics.
gravity will not be such an eigenstate, and at most we will obtain a probability
distribution (in $x^a$).

- We expect that the sets of coordinates, $\hat{x}^a$ and $\hat{x}'^a$, of two separate spacetime points
will not commute. Thus, even if we do find a spacetime point eigenstate, all the
other points in the manifold are generically fuzzed out.

Note that, here one obtains probability densities in $x^a$, the eigenvalues of the spacetime
point operators, whereas, in contrast to the situation in the previous subsection, it is
$(u, \omega, R)$, together with $\zeta$ that are classical parameters which define the “experiment”
or the measurement situation. Thus, in this picture, while one loses the interior of
the spacetime as a distinct classical manifold, the manifold $I^+$ always remains free of
quantum fluctuations, and is the classical scaffolding from which quantum measurements
can be made. What is the rôle of the (6 real) parameters $(u, \omega, R, \zeta)$? In this picture, we
interpret $(u, \zeta)$ as the location of the classical, asymptotic observer; the remaining three
coordinates $(\omega, R)$ attempt to locate a specific point on this observer’s past lightcone.

4.3. Fuzzy lightcones

Recall from the classical theory in Sec. 2 that we have a third interpretation (see Sec.
2.3), one in which the content of the Einstein equations is expressed in the equations
for the lightcones of spacetime points. In parametric form, the lightcone of the point $x^a_0$
is given by (8), and through the dependence on $\sigma$, is a function on the reduced phase
space. In the quantum theory, the data, which are coordinates on the reduced phase
space, are operators. The parametric form for the lightcone of the $c$-number $x_0$ becomes
four 3-parameter families of operators:

$$\hat{x}^a[x_0](r, \zeta) = x^a[x_0](r, \zeta; [\sigma]).$$

(28)

Again, without belaboring points we have raised before, there are interesting issues
which arise from the commutation relations between the $\hat{x}^a[x_0](r, \zeta)$. If we fix a
spacetime point $x_0$, are there “lightcone eigenstates”? What physical significance can
we attribute to nontrivial commutation relations between lightcones of different points?

Since in the above formulation the spacetime points $x^a_0$ play the role of $c$-number
classical parameters, it is natural to interpret the uncertainties in the coordinates of the
lightcones as indicating a fuzzing out of the lightcones themselves.

5. Discussion

We summarize the steps we have taken to get from the hypersurface formulation of
classical general relativity to the quantum theory in which we have fuzzy spacetime
points and other fuzzy geometric objects. In Sec. 2 we mentioned that the physical
phase space of general relativity can be coordinatized by the characteristic data $\sigma$ given
on $I^+$, and comes equipped with a symplectic structure. Einstein’s equations are coded in the function $Z$ — a 6-parameter family of functionals on the physical phase space $\{\sigma\}$ — which eventually determines a (conformal) Einstein spacetime metric. $Z$ is an element of a set of four geometrical variables on $I^+$, namely $\{\theta^i\}$. The equations for these four geometric quantities can be inverted to yield the local coordinates of interior spacetime points $\{\tilde{x}^a\}$, which are also four 6-parameter families of functionals on the phase space. Next, in Sec. 4, we outlined the quantum theory by promoting the characteristic data to operators. We then constructed operators corresponding to the four geometric variables $\theta^i$ (in section 4.1) and the spacetime points $x^a$ (in section 4.2), by promoting the classical functions to operator valued functions of the operators corresponding to the characteristic data. We then tried to interpret these operators. We discussed some consequences of the fact that a generic physical state of “quantum gravity” is not a common eigenstate of the spacetime point operators. This discussion led us to the conclusion that spacetime points themselves must be quantized, and to a relatively precise formulation of the notion of a quantized or fuzzy point. Thus the spacetime manifold ceases to exist as a well-defined entity in the quantum theory.

We close with some comments.

(i) We have an immediate conceptual problem: In our version of “quantum gravity”, though we can construct the operators $\hat{\theta}^i$ and $\hat{x}^a$ in linear theory (and, perturbatively, to higher orders), we do not yet understand how to do this construction (even formally) for the full non-perturbative theory.

(ii) The quantization of the null-surface formulation can be completed in the case of Maxwell and in the case of linearized general relativity. In the later case, the Fock representation itself has been constructed, and work is well under way to compute all the required commutation relations. These calculations, which are lengthy but not at all conceptually difficult, are being carried out at present.

In principle, at least perturbatively, the same can be done in full theory, the important case. However, major technical problem are in the way, for examples, the factor ordering of the operators, their algebraic complexity, and the need to control the infinities. We have not addressed these problems here. Rather, we have discussed a possible interpretation of the operators one hopes to be able to define in the full theory.

(iii) The quantities $x^a(u, \omega, R, \zeta, [\sigma])$ are functionals on the phase space of characteristic data for the theory. Now, the phase space of characteristic data is essentially equivalent to the reduced phase space of the canonical theory. Therefore, the quantities $x^a(u, \omega, R, \zeta, [\sigma])$, considered as families of functions on the reduced phase space, are concrete examples of the “evolving constants of motion” discussed in Ref. [12]. In Ref. [12] it was argued that the “evolving constants of motion” are the
quantities that describe evolution in a diffeomorphism-invariant manner (because they are defined on the reduced phase space, and thus are diffeomorphism-invariant, but they code information about the evolution via their dependence on parameters), and therefore they are the quantities that must be promoted to operators in the quantum theory. However, no such quantity was known in pure GR. Here, we point out that “evolving constants of motion” in pure GR do in fact exist, and that they are realized by the quantities \(x^a(u, \omega, R, \zeta, [\sigma])\).

(iv) The issue of changing the coordinates by means of a gauge transformation is not a problem, at least perturbatively, since the formalism has already chosen a gauge; if the gauge were changed it would entail an associated change in the commutation relations. This is analogous to the situation in Maxwell theory where the vector potential can be quantized in the Coulomb gauge —with specific commutation relations, which then are changed by a gauge transformation.

(v) The picture of quantum spacetime that emerges is still very vague and tentative. Perhaps this view can be seen as complementary to the discrete quantum geometry that is emerging from the loop representation of quantum gravity [13], but the connection is certainly obscure at the moment. We emphasize the fact that the picture that we have sketched is not based on operators which are spacetime fields (or operator valued spacetime distributions). In this sense, the picture is a very radical departure from quantum field theory: physical spacetime is ill-defined. On the other hand, we have kept ourselves rather on the conservative side as far as quantum mechanics is concerned, retaining the basis of its operatorial/probabilistic interpretation. The reason we did this is not so much blind trust in quantum mechanics (which some of us do not hold), but rather lack of viable alternatives. Still, we expect that the quantities \(x^a\), derived from the null surfaces, i.e. Eq. (5), could represent a key to handle the mystery of a physical description of a quantum fluctuating geometry, free of matter.

(vi) Many years ago it was pointed out by Peter Bergmann, among others, that since the classical theory of GR predicts the (pondermotive) equations of motion of its own sources, its quantization then should predict, or at least be closely associated with, the quantization of the particle orbits. Since the geodesic equations are limiting cases of the pondermotive equations, we appear to have made some contact with this idea: namely, there is no need to quantize the motion of particles separately: perhaps the quantization of our spacetime points already contains the quantization of classical particle trajectories.

(vii) Notice that the approach considered here has remarkable similarities with some of the ideas in the twistor approach [14] to quantum gravity in both spirit and
technique. Twistor theory too emphasizes the quantum theoretic but not field theoretic viewpoint, and gives great importance to null surfaces and null geodesics — as opposed to gravitational fields — as descriptors of Einstein geometries.

Work by Doplicher et al. [15] has recently come to our attention, in which, motivated by semiclassical arguments, they postulate certain commutation relations on the coordinates of spacetime points and take preliminary steps towards a quantum field theory on such a quantum spacetime. Connes [16] has proposed and developed the idea of doing quantum field theory on a noncommutative manifold. Thus the idea of a quantum spacetime itself is not new. The difference is that we propose to derive the commutation relations on spacetime points from a quantization of (linearized) vacuum GR, without recourse to semiclassical ideas.

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