The Capacity of Random Ad hoc Networks under a
Realistic Link Layer Model

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Abstract— The problem of determining asymptotic bounds on the
capacity of a random ad hoc network is considered. Previous
approaches assumed a threshold-based link layer model in which a
packet transmission is successful if the SINR at the receiver
is greater than a fixed threshold. In reality, the mapping from
SINR to packet success probability is continuous. Hence, over
each hop, for every finite SINR, there is a non-zero probability
of packet loss. With this more realistic link model, it is shown
that for a broad class of routing and scheduling schemes, a fixed
fraction of hops on each route have a fixed non-zero packet loss
probability. In a large network, a packet travels an asymptotically
large number of hops from source to destination. Consequently, it
is shown that the cumulative effect of per-hop packet loss results
in a per-node throughput of only \( O(\frac{1}{n}) \) (instead of \( O(\sqrt{\frac{1}{n \log n}}) \) as shown previously for the threshold-based link model).

A scheduling scheme is then proposed to counter this effect. The proposed scheme improves the link SINR by using conservative
spatial reuse, and improves the per-node throughput to \( O(\frac{1}{K_n \sqrt{\log n}}) \), where each cell gets a transmission opportunity
at least once every \( K_n \) slots, and \( K_n \to \infty \) as \( n \to \infty \).

Keywords
Ad hoc networks, capacity, SINR, interference, packet error,
scheduling

I. INTRODUCTION

Recently, there has been considerable interest in the field of
ad-hoc networks due to the vast range of applications that
they offer in several areas such as home networking, disaster
recovery networks, military, etc. The important question of
the capacity of such ad-hoc networks was first systematically
analyzed by Gupta and Kumar in [1]. This work has motivated
several works under additional assumptions such as node
mobility [6], presence of limited infrastructure [7], capacity-
delay trade-offs [8], [11], network information theoretic view
of network capacity [5], etc.

In [1], Gupta and Kumar derive asymptotic bounds on the
capacity of a random ad hoc network. In a random ad hoc
network, nodes are deployed randomly and uniformly over the
surface of a sphere of unit area. Each node picks a random
node as its destination node, and sends packets to that node by
using multi-hop communication. All the nodes use a common
transmit power level. To derive their results, the authors
assume that the region is covered with a tessellation of cells.
The cells are assumed to have certain uniformness properties in
terms of size and shape. Such a uniform tessellation simplifies
the capacity analysis, since it is easier to account for routing
and scheduling when the cells have uniform size and shape.

The authors then demonstrate that under certain simplifying
link layer assumptions for successful packet reception, each
node can achieve a throughput of \( \Theta\left(\frac{1}{\sqrt{n \log n}}\right) \) packets per
second. The authors also provide a scheduling and routing
strategy in order to achieve this throughput. The authors
assume a link layer model in which, if the Signal to Interference
and Noise Ratio (SINR) at the receiver is greater than a certain
threshold \( \beta \), then the packet is received successfully by the
receiver with probability one.

In reality, for a given modulation and coding scheme, as
long as there is some noise and interference, i.e., as long as the
SINR is finite, there is always a non-zero probability of packet
error, and this probability of error approaches zero as the SINR
approaches infinity [3], [4]. In other words, over each hop,
the mapping from SINR to packet success probability is a
continuous function that approaches one only asymptotically
as the SINR approaches infinity. While the threshold-based
packet reception model used in [1] is a reasonable choice
for successful packet reception in a single hop network such
as a cellular network, we argue that it needs to be refined
when applied to a multi-hop network. In a cellular network,
if the SINR threshold \( \beta \) is chosen to be sufficiently high,
then the packet error rate between the mobile and the base
station (and vice versa) can be made very small for a given
choice of modulation and coding. However, in the context
of an ad hoc network, we know that each packet traverses
multiple hops. The links of these hops receive interference
from other ongoing transmissions which could potentially
corrupt the packet transmission over the given link. When a
packet is relayed over a large number of links, each of which is
likely to drop the packet with a certain probability, the end-
to-end throughput depends on the probability of end-to-end
packet delivery. In [1], the capacity determining constraint is
the number of source-destination paths that pass through a
cell, i.e., the relaying burden of a cell. However we believe
that other factors such as interference, and the end-to-end
packet error probability due to interference on each individual
hop, should also be taken into account when determining the
network capacity.

Thus, our work is motivated by the fact that in a large ad-
hoc network, a packet has to travel over an asymptotically
large number of links, each of which is unreliable with a
certain probability. We show that the existence of a \( p < 1 \)
that uniformly upper bounds the packet success probability of
at least a fixed fraction of hops on each source-destination path is sufficient to considerably degrade the network throughput. We then show the existence of such a $p < 1$ for a broad range of routing and scheduling schemes (including the one proposed in [1]), and this results in an achievable per-node throughput of just $O\left(\frac{1}{n^{\frac{1}{2}}}\right)$ instead of $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$. Note that proving the existence of such a $p$ is not easy because, due to the random node distribution it is possible that for certain hops, the transmitter and the receiver nodes may be arbitrarily close to each other. This results in a very high SINR, and consequently a probability of packet success that is arbitrarily close to one.

To counter the throughput reduction due to this cumulative packet error effect, we also show that by choosing a more conservative scheduling policy, we can reduce the interference (and thereby improve the packet success probability) by trading off spatial reuse. This improves the per-node throughput to $O\left(\frac{1}{K_n \sqrt{n \log n}}\right)$. Here $K_n$ is the length of the schedule, i.e., each cell gets a transmission opportunity at least once every $K_n$ slots, and $K_n$ is chosen such that $K_n \to \infty$ as $n \to \infty$.

Since our arguments are mainly based on the impact of interference (via scheduling) and packet error probability over long paths (asymptotically large number of hops), it is easy to see that the results are not restricted to a specific communication paradigm such as the any-to-any communication model of [1]. The key insight is that if, in a network, nodes use long multi-hop paths to reach their destination, and scheduling is used for exploiting spatial reuse, then interference from other ongoing transmissions, and end-to-end error probability may degrade the network throughput. Hence a trade-off between spatial reuse and cumulative packet error probability has to be studied.

The rest of the paper is organized as follows. We motivate the problem, and outline our approach in Section II. We discuss some of the related work in Section III. In Section IV we formulate and solve the capacity problem. Finally, we present our conclusion and future directions in Section V.

II. Motivation and Approach

The arguments in this section are only meant to provide intuitive insights about the problem and our approach. We provide precisely proved results for all the arguments in Section IV. In [1], the authors study the problem of determining the capacity of random ad hoc networks. They propose a routing and scheduling scheme to achieve a per-node throughput of $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$. It can be shown that for this scheme a packet traverses $\Theta\left(\sqrt{\frac{\log n}{n}}\right)$ intermediate hops on its way from a source node to its destination node. Thus asymptotically, the number of hops that a packet has to travel from source to destination goes to infinity as $n$ scales. The authors assume that if the SINR over a hop is at least $\beta$, the packet transmission is successful. Let us call this the ideal link model. The authors then use a scheduling scheme which ensures an SINR of at least $\beta$ over all the hops. Thus, under the ideal link model, and the proposed scheduling policy, every packet transmission is successful.

Now consider another link model in which a link is reliable with a probability $p < 1$, i.e., each packet transmitted on the link is received successfully at the receiver with a probability of $p$. Let us call this the probabilistic lossy link model. For simplicity, assume that over each hop, if a packet is not received successfully by the receiver, the transmitter does not retransmit the packet, and the packet is lost. Also assume that we use the same routing and scheduling policy as used above for the ideal link model. Since there are $\Theta\left(\sqrt{\frac{n}{\log n}}\right)$ hops from source to destination, the probability that the packet reaches its destination scales as $p^{\sqrt{\frac{n}{\log n}}}$. It is easy to show that this quantity is $O\left(\sqrt{\frac{\log n}{n}}\right)$, since $p^m$ is $O\left(\frac{1}{m}\right)$ as $m$ tends to infinity. Let us consider two identical networks, one with the ideal link model, and the other with the probabilistic lossy link model, and assume that for both of them, we use the same routing and scheduling algorithm as outlined in [1]. The only difference between the two networks is that while in the former case, all the packets injected by the source into the network reach the destination, in the latter case, only a fraction of the injected packets reach the destination. Hence, under the probabilistic lossy link model, the achievable throughput is the product of the achievable throughput of the ideal link model, $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$, and an end-to-end packet delivery probability term that is $O\left(\sqrt{\frac{\log n}{n}}\right)$. This results in an end-to-end throughput of $O\left(\frac{1}{m}\right)$ for the probabilistic lossy link model.

A realistic link model lies somewhere between the ideal link model and the probabilistic lossy link model. In a realistic link model, the probability of link reliability, $p$, is a continuous function of the SINR of the link. The SINR in turn depends on factors such as transmitter-receiver separation, power of interference from simultaneous transmissions, and noise power. These factors are different for different links along a path. For example, since the nodes are randomly placed, it is possible that for a given hop, the two nodes could be arbitrarily close to each other resulting in a very high SINR for that link. This in turn means that the packet success probability for that link could be arbitrarily close to one. Thus, there is no single $p < 1$ which uniformly upper bounds the probability of packet success for all the links. However, if for at least a fixed fraction of links along every source-destination path, there is a $p < 1$ which upper bounds the probability of successful packet reception, then the per-node throughput bound of $O\left(\frac{1}{\sqrt{n \log n}}\right)$ which holds for the probabilistic link model also holds for the realistic link model. To get around this problem we must design scheduling (and/or retransmission strategies) so that no such $p$ exists.

The above arguments are made more precise in Section IV.

III. Related Work

Throughout this paper, we refer to the work of Gupta and Kumar on the capacity of random ad hoc networks [1]. In this work, the authors assume a simplified link layer model in which each packet reception is successful if the receiver has an SINR of at least $\beta$. The authors assume that each
packet is decoded at every hop along the path from source to destination. No co-operative communication strategy is used, and interference signal from other simultaneous transmissions is treated just like noise. For this communication model, the authors propose a routing and scheduling strategy, and show that a per-node throughput of $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$ can be achieved.

In [5], the authors discuss the limitations of the work in [1], by taking a network information theoretic approach. The authors discuss how several co-operative strategies such as interference cancellation, network coding, etc. could be used to improve the throughput. However these tools cannot be exploited fully with the current technology which relies on point-to-point coding, and treats all forms of interference as noise. The authors also discuss how the problem of determining the network capacity from an information theoretic viewpoint is a difficult problem, since even the capacity of a three node relay network is unknown. In Theorem 3.6 in [5], the authors determine the same bound on the capacity of a random network as obtained in [1].

In [1], the authors consider another model of ad hoc networks called arbitrary networks. In an arbitrary network, nodes are placed at arbitrary locations in a region of fixed area, source-destination pairs are chosen arbitrarily, and each node can transmit at an arbitrary power level. This is unlike a random ad hoc network where node are deployed randomly and uniformly, source-destination pairs are chosen at random, and all the nodes transmit at the same power level\(^1\). For the arbitrary network model, the authors show that a per-node throughput of $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$ is achievable. However, for random networks, the routing and scheduling scheme proposed in [1] only achieves a throughput of $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$. This gap between the arbitrary networks and the random networks is closed in [9]. In [9], the authors note that the requirement of connectivity with high probability in random networks as required in [1], requires higher transmission power at all the nodes, and results in excessive interference. This in turn lowers the throughput of random networks from $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$ to $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$. Motivated by this, the authors in [9] propose using a backbone-based relaying scheme in which instead of ensuring connectivity with probability one, they allow for a small fraction of nodes to be disconnected from the backbone. The nodes in the backbone are densely connected, and can communicate over each hop at a constant rate. Such a backbone traverses up to $\sqrt{n}$ hops. The nodes that are not a part of the backbone, send their packets to the backbone using single hop communication. The authors then show that the interference caused by these long range transmissions does not impact the traffic carrying capacity of the backbone nodes. Finally, the authors show that the relaying load of the backbone determines the per-node throughput of such a scheme, and this results in an achievable per-node throughput of $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$. The above approach of allowing a few disconnected nodes in the network has also been used in [10].

However, just as in [1], all the above mentioned works assume that over each link a certain non-zero rate can be achieved. They do not take into account the fact that in reality, such a rate is achieved with a probability of bit error arbitrarily close (but not equal) to one. Once the coding and modulation scheme is fixed, the function corresponding to the probability of bit error is also fixed.

IV. Problem Formulation and Solution
We use the same terminology as used by Gupta and Kumar in [1].

- $f_n = \Theta(g_n)$, if there exist constants $a_1$ and $a_2$ such that, for $n$ large enough, $a_1 g_n \leq f_n \leq a_2 g_n$.
- $f_n = \Omega(g_n)$, if there exists a constant $a_1$ such that, for $n$ large enough, $a_1 g_n \leq f_n$.
- $f_n = O(g_n)$, if there exists a constant $a_2$ such that, for $n$ large enough, $f_n \leq a_2 g_n$.
- $f_n = \omega(g_n)$, if there exists a constant $a_1$ such that, for $n$ large enough, $a_1 g_n < f_n$.
- $f_n = o(g_n)$, if there exists a constant $a_2$ such that, for $n$ large enough, $f_n < a_2 g_n$.

Consider a sphere of unit area, say $S^2$, over which $n$ nodes are deployed randomly and uniformly. Each node picks a random node which is its destination node, and sends packets to this destination node. Each node has a common transmit power level $P$, and it uses intermediate nodes as relays to reach the destination node. The surface of the sphere is covered by a Voronoi tessellation in such a way that each Voronoi cell $V$ can be enclosed inside a circle of radius $2\rho_n$, and each circle encloses a circle of radius $\rho_n$ (all the distances are measured along the surface of $S^2$). We use such a tessellation to ensure uniform cell size. The cells can be made arbitrarily small in a uniform way by choosing $\rho_n$ small. In [2], Gupta and Kumar have derived necessary and sufficient conditions for asymptotic connectivity of a random ad-hoc network. By using results from [2] in [1], the authors choose $\rho_n$ to be the radius of a circle of area $\frac{100 \log n}{n}$ on $S^2$. With this choice of $\rho_n$, and with the communication radius of each node chosen to be $8\rho_n$, the authors show that all the nodes in the network are connected with probability approaching one as $n$ approaches infinity.

The authors in [1] use the following criterion for successful packet reception. If the SINR at a receiver is greater than a certain fixed threshold $\beta$, then the packet is successfully decoded by the receiver. In other words, a transmission from node $i$ to node $j$ is successful if:

$$N + \sum_{k \in T, k \neq i} \frac{P}{|X_k - X_j|^2} \geq \beta,$$

where $T$ is the set of all the nodes that are transmitting simultaneously with node $i$, and $\alpha > 2$ is the propagation loss exponent.

In our analysis, to begin we make the following assumptions:

A1 The region $S^2$ is partitioned into a Voronoi tessellation such that each cell contains a circle of radius $\rho_n$, and each cell is contained inside a circle of radius $2\rho_n$. We assume that $\rho_n$ is at least $\Omega \left( \frac{\log n}{n} \right)$ to

\(^1\)The definitions of random networks and arbitrary networks have been taken from [1].
ensure network connectivity with high probability. To ensure that the cell sizes shrink as \( n \) scales, \( \rho_n \to 0 \) as \( n \) approaches infinity. Each node transmits at a fixed power level \( P \).

A2 For a given modulation and coding scheme, the probability of a successful packet reception is a continuous increasing function of SINR that approaches one as the SINR approaches infinity. Hence, for every finite SINR value, we assume that there is a non-zero probability of packet loss. Even as \( n \) scales, the modulation and coding scheme is fixed. We also assume that over each hop, if a packet is not received successfully by the receiver, the transmitter does not retransmit the packet, and the packet is lost.

A3 A scheduling algorithm that guarantees each cell a transmission opportunity at least once every \( K \) slots, where \( K \) is a finite number that is independent of \( n \). In other words, the length of the schedule is bounded even as \( n \) scales. In [1], it was shown that such a scheduling strategy guarantees an SINR of at least \( \beta \) for all the scheduled transmissions.

A4 A routing scheme in which packets are routed along straight line paths between source-destination pairs, i.e., every cell that intersects the straight line joining a source-destination pair, relays the packets of that pair.

The key assumption that distinguishes our work from the previous approaches is assumption A2. The remaining three assumptions are also used by Gupta and Kumar in [1]. We later on relax assumptions A3 and A4, and also comment on relaxing assumption A2. Note that even with A3 and A4, the assumptions above are general enough to encompass a vast array of routing and scheduling policies that could be used for multi-hop communication.

### A. Per-node Capacity under Assumptions A1 to A4

Consider Fig. 1. Let \( L_i \) be the line segment along the surface of \( S^2 \) that connects the \( i \)th source-destination pair (henceforth referred to as the \( i \)th connection). We also use \( L_i \) to denote the length of the line segment joining the \( i \)th source-destination pair. As per A4, the packets of the \( i \)th connection are relayed hop-by-hop by every cell which intersects line \( L_i \). Over each hop, any node in the relaying cell may forward the packet. The scheduling algorithm, and the uniform cell sizes ensure that communication between any two nodes in the neighboring cells is possible by guaranteeing that the SINR at the receiving node is greater than or equal to \( \beta \) (see [1] for more details).

We will now prove that for each source-destination pair, there are at least \( \Theta \left( \frac{L_i}{\rho_n} \right) \) hops over which the SINR is upper bounded by a fixed constant. For this, we need the next three lemmas.

**Lemma 1:** For the routing scheme in A4, the number of hops \( H_i \) for connection \( i \) is \( \Theta \left( \frac{L_i}{\rho_n} \right) \). More precisely,

\[
\frac{1}{8} \frac{L_i}{\rho_n} \leq H_i \leq \frac{16}{\pi} \frac{L_i}{\rho_n}.
\]

**Proof:** Since, each Voronoi cell is contained in a circle of radius \( 2\rho_n \), the maximum distance between a point in a given cell, and a point in its neighboring cell is \( 8\rho_n \) (see Fig. 1). Thus, over each hop, the maximum distance that a packet can cover is upper bounded by \( 8\rho_n \). Hence the lower bound.

For the upper bound, if we look at a strip that is \( 4\rho_n \) wide on either sides of \( L_i \) (see Fig. 1), we observe that if a cell is used as a relay cell, then it must lie entirely within this strip. This is because each cell is completely contained inside a circle of diameter \( 4\rho_n \). The area of this strip is \( 8\rho_n L_i \). Each cell contains a circle of radius \( \rho_n \), and hence the area of each cell is at least \( \pi\rho_n^2 / 2 \) (see Lemma 9 in Appendix). Thus, the maximum number of cells that lie entirely within this strip is upper bounded by \( 16L_i / \pi\rho_n \). Hence the upper bound.

The lower bound is actually \( \max \left( \frac{L_i}{8\rho_n}, 1 \right) \), but with \( L_i = \Theta(1) \), \( \rho_n \) is small for \( n \) large, and we can ignore one. In any case, we require the upper bound, and not so much the lower bound in our subsequent analysis. Also note, that for a more exact upper bound, we should consider a strip of length \( L_i + 4\rho_n + 4\rho_n \) instead of \( L_i \), since the source and destination nodes could be located at the peripheries of their respective cells. However, since \( L_i = \Theta(1) \), and \( \rho_n \) approaches zero for \( n \) large, we can safely ignore the two terms corresponding to the sizes of the source and destination cells.

**Lemma 2:** Fix \( t \) such that \( 0 < t < 1 \). For connection \( i \), out of \( H_i \) total hops, let \( h_i \) hops be such that each of these hops covers a distance of less than \( t\rho_n \). Then

\[
H_i - h_i \geq \frac{L_i}{\rho_n} \left( 1 - \frac{16}{\pi} \frac{8 - t}{8 - t} \right)
\]

Thus, for the above \( H_i - h_i \) hops, the signal received at the receiver is at the most \( P(t\rho_n)^{-\alpha} \), where \( P \) is the transmit power common to all the nodes, and \( \alpha \) is the propagation loss exponent (\( \alpha > 2 \)).

**Proof:** Since \( h_i \) hops each cover less than \( t\rho_n \) distance, the leftover distance, which is at least \( L_i - h_i t\rho_n \) has to be covered by the rest of the \( H_i - h_i \) hops. Each of these \( H_i - h_i \)
hops can cover a distance of at the most $8\rho_n$. Hence, we have

$$ (H_i - h_i) 8\rho_n \geq L_i - h_i(t\rho_n) $$

$$ \Rightarrow h_i \leq H_i \left( \frac{8 - t}{8 - t} \right) - \frac{L_i}{\rho_n} \left( \frac{1}{8 - t} \right) $$

$$ \Rightarrow H_i - h_i \geq H_i \left( \frac{-t}{8 - t} \right) + \frac{L_i}{\rho_n} \left( \frac{1}{8 - t} \right) $$

$$ \geq \frac{L_i}{\rho_n} \left( \frac{1 + c_1}{\pi} \right) \left( \frac{-t}{8 - t} \right) + \frac{L_i}{\rho_n} \left( \frac{1}{8 - t} \right), $$

where we have used Lemma 4 to upper bound $H_i$ on the right hand side in the last step. Thus,

$$ H_i - h_i \geq \frac{L_i}{\rho_n} \left( \frac{1 - 4\pi c_1}{8 - t} \right). $$

In [1], it was shown that there exists a scheduling policy that ensures an SINR of at least $\beta$ at the receiver of every scheduled transmission. This scheduling policy corresponds to a graph coloring problem, and it was shown in [1] that the maximum number of colors required to color all the cells is upper bounded by $1 + c_1$, where $c_1$ is a fixed constant that is independent of $n$ (see Lemma 4.4 in [1]). Using this scheduling scheme, each cell gets a transmission opportunity at least once every $1 + c_1$ time slots. Thus, with respect to A3, $K = 1 + c_1$.

The objective of the following lemma is to show that except for a small fraction of hops, all the remaining hops of a connection receive a certain minimum amount of interference from other simultaneous transmissions.

**Lemma 3:** Fix $M > 9$. Let $N_i$ be the number of hops of connection $i$ such that there is no simultaneous interfering transmission within a circle of radius $(M+8)\rho_n$ around the receivers of those hops. To avoid tedious boundary conditions, let us not count the hops containing the source and the destination nodes in $N_i$. Then,

$$ N_i \leq \frac{L_i}{\rho_n} (\frac{2(1 + c_1)}{M}), $$

where $c_1$ is the constant from Lemma 4.4 in [1].

**Proof:** There are at the most $1 + c_1$ colors, i.e., each cell gets a transmission opportunity at least once every $1 + c_1$ time slots. Let us index these $1 + c_1$ colors, and consider any one of these colors, say color $j$. We would like to determine the number of color-$j$ cells of connection $i$ that are at least $M\rho_n$ distance away from other color-$j$ cells.

For this, let $U^j_i$ be the set of all the cells of color $j$ that a packet of connection $i$ traverses. We refer to a circle as a “surrounding circle of a cell” if its center lies within the cell. Consider all the surrounding circles of radius $M\rho_n$ around each of the cells in set $U^j_i$. For each cell, there are many such circles, since the only condition that the center of the circle lie within the cell. A subset $V^j_i$ of $U^j_i$ is then formed as follows. A cell from set $U^j_i$ is added to set $V^j_i$ if none of its surrounding circles of radius $M\rho_n$ contains any other color-$j$ cell either partially or fully. Thus every cell in set $V^j_i$ is at least $M\rho_n$ distance away from any other color-$j$ cell.

Hence, any surrounding circle of radius $\frac{M\rho_n}{2}$ around a cell in set $V^j_i$ does not overlap with any surrounding circle of radius $\frac{M\rho_n}{2}$ around another cell in set $V^j_i$. Let there be $N^j_i$ cells in set $V^j_i$. The total length of the path is $L_i$ (referring to Fig. 2). We would like to determine the maximum number of non-overlapping surrounding circles of radius $\frac{M\rho_n}{2}$ around color-$j$ cells that can be accommodated along $L_i$. This number is an upper bound for $N^j_i$. Each cell along $L_i$ can be enclosed inside a circle of diameter $4\rho_n$. Referring to Fig. 2, maximal packing of the non-overlapping circles of radius $\frac{M\rho_n}{2}$ occurs when the chord formed by the intersection of $L_i$ with these circles is of minimal length. This occurs when the chord is shifted from its diametrical position by $4\rho_n$. This is the maximum distance that the chord can be shifted while still ensuring that $L_i$ intersects the cell, and the center of the circle of radius $\frac{M\rho_n}{2}$ surrounding the cell lies inside the cell. The length of each chord formed by the intersection of $L_i$ and the surrounding circle of radius $\frac{M\rho_n}{2}$ is at least $2\left(\frac{(M\rho_n)^2}{2} - (4\rho_n)^2\right)$, which in turn is at least $\frac{M\rho_n}{2}$ if $M > 9$. Since $N^j_i$ such chords are to be accommodated along the length of the path,

$$ N^j_i \leq \frac{L_i}{\rho_n} \left( \frac{2(1 + c_1)}{M} \right), $$

Therefore for all the colors,

$$ N_i = \sum_{j=1}^{1+c_1} N^j_i \leq \frac{L_i}{\rho_n} \left( \frac{2(1 + c_1)}{M} \right). $$

Thus except for these $N_i$ hops, each of the hops of connection $i$ has at least one cell that has a simultaneous ongoing transmission within a radius of $M\rho_n$ of the given cell. Or equivalently, within a radius of $(M+4+4)\rho_n = (M+8)\rho_n$ of the receivers of these hops, there is at least one more node that is transmitting simultaneously. This is because the maximum size of a cell is $4\rho_n$. And this proves the result.

Let $A_i$ be the set of hops of connection $i$ for which the received signal is at the most $P(t\rho_n)^{-\epsilon_1}$. Then, given $\epsilon_1 > 0$, we can find $t > 0$ small enough so that using Lemma 2

$$ |A_i| \geq \frac{L_i}{\rho_n} \left( \frac{1}{8 - \epsilon_1} \right) $$

Let $B_i$ be the set of hops of connection $i$ for which there is no simultaneous transmissions within a distance of $(M+8)\rho_n$.
of the receiver. Using Lemma 3, \( |B_i| = N_i \), and given \( \epsilon_2 > 0 \), we can choose \( M \) large enough so that
\[
|B_i| \leq \frac{L_i}{\rho_n} \left( \frac{2(1 + \epsilon_1)}{M} \right)
\]
\[
\leq \frac{L_i}{\rho_n} \epsilon_2
\]
Thus using 2 and 3,
\[
|A_i \cap B_i^c| \geq |A_i| - |B_i|
\]
\[
\geq \frac{L_i}{\rho_n} \left( \frac{1}{8 - \epsilon_1} \right) - \frac{L_i}{\rho_n} \epsilon_2
\]
\[
= \frac{L_i}{\rho_n} \left( \frac{1}{8 - \epsilon_1 - \epsilon_2} \right)
\]
If we pick \( \epsilon_1 = \epsilon_2 = 1/32 \), and choose \( t = t_0 \) and \( M = M_0 \) corresponding to this choice of \( \epsilon_1, \epsilon_2 \), then
\[
|A_i \cap B_i^c| \geq \frac{L_i}{16\rho_n}
\] (4)
Note that \( A_i \cap B_i^c \) is the set of hops over which the received signal is no more than \( P(t_0\rho_n)^{\alpha} \), and there is at least one simultaneous transmission within a distance of \( (M_0 + 8)\rho_n \) of the receiver. This in turn means that for these hops, the SINR is upper bounded by
\[
SINR \leq \frac{P(t_0\rho_n)^{-\alpha}}{P((M_0 + 8)\rho_n)^{-\alpha}}
\]
\[
= \left( \frac{M_0 + 8}{t_0} \right)^{\alpha}
\] (5)
\[
= \beta_0
\] (6)
Thus we have proved the following Proposition.

**Proposition 1:** There exist fixed constants \( t_0 \) and \( M_0 \), that do not depend on \( n \), such that for at least \( L_i/16\rho_n \) hops of connection \( i \), the SINR is less than a fixed constant \( \beta_0 \) given by
\[
\beta_0 = \left( \frac{M_0 + 8}{t_0} \right)^{\alpha}
\]
As per A2, since the SINR is upper bounded by a fixed constant \( \beta_0 \), the probability of successful packet reception is also upper bounded by a fixed constant \( \phi(\beta_0) < 1 \).

Let us assume that time is slotted such that each slot is long enough to transmit a single packet of fixed length. Let \( \lambda_n(i) \) be the rate in packets/slot at which source node \( i \) injects packets in the network. Although the source node injects packets at a rate of \( \lambda_n(i) \) packets per slot, not all the packets make it to the destination node. This is because, at each hop, the SINR is finite, and hence there is a non-zero probability of the packet getting dropped. The actual end-to-end throughput of connection \( i \), denoted by \( \Lambda_n(i) \), is given by,
\[
\Lambda_n(i) = \lambda_n(i) \text{ Prob } \{ \text{ packet is received successfully over all the hops of connection } i \}
\]
\[
= \lambda_n(i) \prod_{j=1}^{L_i} \text{ Prob } \{ \text{ packet is received successfully over the } j^\text{th} \text{ hop of connection } i \},
\]
where we assume that the interference signal, and noise observed by a packet at each of the hops are independent. We know from Proposition 1 that, among the \( H_i \) hops of connection \( i \), at least \( L_i/16\rho_n \) hops have a probability of packet success of no more than \( \phi(\beta_0) \). Thus,
\[
\Lambda_n(i) \leq \lambda_n(i) \{ \phi(\beta_0) \}^{L_i/16\rho_n}
\] (7)
Note that \( L_i \) are i.i.d. random variables. Hence if we remove the conditioning on \( L_i \) by taking the expectation with respect to \( L_i \), the end-to-end throughput \( \Lambda_n \), we take expectations at two levels; once to take into account the randomness due to the possibility of packet error on each link, and once to take into account the randomness due to the locations of the source and destination nodes. Also note that \( 0 < \delta < 1 \). Since \( L_i \) is a line connecting two points picked at random on the surface of \( S^2 \), we can show that (See Lemma 3 in Appendix).
\[
E_L[\delta^{L_i}] = \frac{2\pi (1 + \delta^{2})}{4\pi + (\log \delta)^2}
\] (11)
Using 10 and 11,
\[
\Lambda_n \leq \lambda_n(i) \frac{2\pi (1 + \delta^{2})}{(4\pi + (\log \delta)^2)}
\]
\[
= \lambda_n(i) \left( \frac{2\pi}{4\pi + \left( \frac{1}{16\rho_n \log \phi(\beta_0)} \right)^2} \right)
\]
\[
= \lambda_n(i) \left( \frac{512\pi \rho_n^2}{1 + \left( \frac{1}{16\rho_n \log \phi(\beta_0)} \right)^2} \right)
\]
\[
< \lambda_n(i) \frac{1024\pi \rho_n^2}{(1024\pi \rho_n^2 + (\log \phi(\beta_0))^2)}, \text{ since } \phi(\beta_0) < 1.
\]
\[
< \lambda_n(i) \frac{1024\pi \rho_n^2}{(1024\pi \rho_n^2 + (\log \phi(\beta_0))^2)}
\] (12)
So far, i.e., in proving Lemmas 1, 2 and 3, Proposition 1 and 12, we have not made any assumptions about the exact form of \( \rho_n \). We have just assumed A1 to A4. Let us now study the per-node throughput bound for the choice of \( \rho_n \) in [1], i.e., when \( \rho_n \) is chosen to be the radius of a disk of area \( \frac{100 \log n}{n} \). In [1], it was shown through Lemma 4.8, that for this choice of \( \rho_n \) each Voronoi cell contains at least one node with high probability. More precisely, there exists a sequence \( \delta_n \to 0 \), such that
\[
\text{Prob } \{ \text{ Number of nodes in cell } V \geq 50 \log n, \text{ for every cell } V \text{ in the tessellation } \} > 1 - \delta_n
\]
Thus we have the following Lemma for the above choice of $\rho_n$.

**Lemma 4:** If $\lambda_n(i)$ is the rate in packets/slot at which every source node injects packets in the network, then with high probability,

$$\lambda_n(i) \leq \frac{1}{50 \log n}$$

**Proof:** The scheduling algorithm proposed in [1] guarantees each cell a time slot at least once every $1 + c_1$ slots. However, since each cell contains at least $50 \log n$ nodes with high probability, even if each node were to transmit only its own packets, and not relay packets, it would still get no more than one transmission opportunity every $50 \log n$ slots. Hence, the rate at which a node injects its own packets into the network can never be more than $1/50 \log n$.

Using the above Lemma to bound $\lambda_n(i)$ in (12) we get

$$\Lambda_n < \frac{1}{1024 \pi \rho_n^2} \frac{1}{(\log \phi(\beta_0))^2}$$

Also, since $\rho_n$ is the radius of a circle of area $\frac{100 \log n}{n}$, using Lemma 2 in Appendix we have,

$$\frac{\pi \rho_n^2}{2} < \frac{100 \log n}{n}$$

Substituting the above in (13),

$$\Lambda_n = \frac{2^{12}}{(\log \phi(\beta_0))^2} \frac{1}{n}$$

Thus we have proved the following important Proposition.

**Proposition 2:** Under assumptions A1 to A4, and with $\rho_n$ chosen to be the radius of a disk of area $\frac{100 \log n}{n}$ (as in [1]), the per-node throughput that can be achieved, $\Lambda_n$, is $O \left( \frac{1}{\sqrt{n \log n}} \right)$ instead of $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$. More precisely,

$$\Lambda_n \leq \frac{c_0}{n}$$

where $c_0 = \frac{2^{12}}{(\log \phi(\beta_0))^2}$ is a constant that does not depend on $n$.

Note that for keeping all the calculations simple, we have tried to ensure that the constant in the asymptotic expression is a power of two. However, a tighter bound with a smaller asymptotic constant can be easily obtained.

The upper bound of $O \left( \frac{1}{\sqrt{n \log n}} \right)$ on per-node throughput can also be extended to any other choice of $\rho_n$ that satisfies assumption A1.

**Lemma 5:** Consider a tessellation that satisfies A1 with a certain $\rho_n$. Then there exists a sequence $\delta_n \rightarrow 0$, such that

$$\Pr \left\{ \text{Number of nodes in cell } V \geq \frac{\pi \rho_n^2}{4}, \text{ for every cell } V \text{ in the tessellation} \right\} > 1 - \delta_n,$$

i.e., with high probability each cell contains at least $\frac{\pi \rho_n^2}{4}$ nodes.

**Proof:** Let $a_n$ be the area of a disk of radius $\rho_n$. We can use similar arguments as in Lemma 4.8 in [1]. However instead of inscribed disks of area $\frac{100 \log(n)}{n}$, we have inscribed disks of area $a_n$. Since $\rho_n \rightarrow 0$ by assumption A1, all the arguments in Lemma 4.8 of [1] go through. Hence

$$\Pr \{ \text{Number of nodes in cell } V \geq \frac{\pi \rho_n^2}{4}, \text{ for every cell } V \text{ in the tessellation} \} > 1 - \delta_n,$$

Also note that the area of a disk of radius $\rho_n$ is lower bounded by $\frac{\pi \rho_n^2}{4}$, and hence the result follows.

Thus with high probability, each node can inject packets into the network at a rate no greater than $\frac{4}{\pi \rho_n^2}$. Thus,

$$\lambda_n(i) \leq \frac{4}{\pi \rho_n^2}$$

Substituting the above in (12) results in the same $O \left( \frac{1}{n} \right)$ upper bound on the per-node throughput. Thus we have the following generalized version of Proposition 2 that holds for any $\rho_n$ that satisfies A1.

**Proposition 3:** For any choice of $\rho_n$ that satisfies assumptions A1 to A4, the per-node throughput that can be achieved is $O \left( \frac{1}{\sqrt{n \log n}} \right)$.

Note that in obtaining the above result, we have assumed that each node transmits a packet just once. If the packet is not received successfully at the receiver, no retransmission attempt is made. It is easy to see that the conclusion in Proposition 3 holds even if a fixed number of retransmissions are allowed, and this number does not scale with $n$. Thus, under a realistic link layer model (assumption A2), if we do not scale the number of retransmission attempts with $n$, and also keep the schedule length $K$ fixed, then the Gupta-Kumar result of per-node throughput of $\Theta \left( \frac{1}{\sqrt{n \log n}} \right)$ does not hold.

**B. Per-node Capacity: Assumption A4 relaxed**

In the previous subsection, we saw that when straight line routing (as proposed in [1]) is used, the per-node throughput is $O \left( \frac{1}{\sqrt{n \log n}} \right)$. In this sub-section, we show that this bound on the throughput also holds for a broad class of routing schemes, i.e., Proposition 3 holds even when assumption A4 is relaxed. In A4 we assume a straight line routing path between the source and the destination nodes. However, we would like to know if a higher per-node throughput can be achieved if we choose routing paths intelligently, i.e., not necessarily straight line paths between source-destination pairs. We only consider those routing paths in which multi-hop communication takes place through packet-relaying between adjacent cells, and there are no loops in the routing path. The first requirement implies that communication is not possible between two nodes if they do not belong to adjacent cells, while the second requirement implies that a packet does not visit a cell more than once on its path from source to destination. Note that even with this restriction, we can account for a vast range of routing schemes.

Let us assume that a packet of connection $i$ takes a certain path (not necessarily straight line path) from the source node to the destination node. Let $L_i$ be the length of this path, i.e., the actual distance traveled by the packet along the surface of $S^2$. Then we have the following Lemma which is essentially a generalization of Lemma 2. In this Lemma, we show that at
least a fixed fraction of hops for any routing scheme are over a distance of \( tp_n \) or longer for a fixed \( t > 0 \).

**Lemma 6**: Fix \( W = 40 \), and \( t = 0.01 \). Assume any arbitrary routing path such that

- R1: Each hop is between nodes of two adjacent cells,
- R2: There are no loops in the routing path.

Then, under assumptions A1 to A3, R1 and R2, we cannot have \( W \) consecutive hops of distance \( tp_n \) or less. In other words, for any routing scheme that satisfies A1 to A3, R1 and R2, at least one out of every \( W \) consecutive hops is at least \( tp_n \) long.

**Proof**: By way of contradiction, assume that there exists some routing scheme that satisfies constraints A1 to A3, R1 and R2, and has \( W \) consecutive hops, each of length \( tp_n \) or less. Referring to Fig. 3, let point \( P \) be the initial location of the packet, and assume that the packet traverses \( W \) hops from this position. Since each of these hops is less than \( tp_n \), the final location of the packet after traversing these \( W \) hops is inside a circle of radius \( Wtp_n \) centered at \( P \). Let \( C_1 \) be this circle. Clearly, every cell that the packet traversed during these \( W \) hops must intersect circle \( C_1 \). Otherwise, the packet could not have traversed that cell. We can easily find an upper bound on the number of cells that intersect \( C_1 \) as follows.

Consider another circle \( C_2 \) of radius \( (Wt + 4)\rho_n \) centered at \( P \). If a cell intersects \( C_1 \), it should lie completely inside circle \( C_2 \). This is because the distance between any two points of a cell cannot exceed \( 4\rho_n \), and if a cell has at least one point inside \( C_1 \), its other point cannot be more than \( 4\rho_n \) farther from the circumference of circle \( C_1 \). The number of cells that lie completely inside \( C_2 \), denoted by \( n_2 \) is upper bounded as follows

\[
n_2 \leq \frac{\pi ((Wt + 4)\rho_n)^2}{\pi \rho_n^2} = 2(Wt + 4)^2,
\]

where we have used Lemma 6 in the first step. For \( W = 40 \) and \( t = 0.01 \), we have \( n_2 \leq 38.72 \). Since 40 hops require at least 41 unique cells, and there are only up to 38 unique cells that the packet can visit, we clearly have a contradiction. Thus, there does not exist a routing scheme that satisfies A1 to A3, R1 and R2, and that has \( W \) consecutive hops of \( tp_n \) or less, where \( W = 40 \), and \( t = 0.01 \).

Note that the arguments in the above Lemma are easier to understand in case of simple tessellations such as triangular, square, hexagonal, etc. For example, consider a square tessellation (square-shaped cells), such that each side of the cell is \( 2\rho_n \) long. Thus, each cell contains a circle of radius \( \rho_n \), and is contained in a circle of radius \( 2\rho_n \). In this case, it is easy to see that circle \( C_1 \) defined in Lemma 6 cannot intersect more than 4 circles, and this happens in the neighborhood of one of the vertices of the tessellation. Consequently, for a square tessellation, we cannot have 4 consecutive hops, each of length 0.01\( \rho_n \) or less. In fact, the upper bound could be made tighter by noting that there cannot be more than 4 consecutive hops that are strictly less than 0.5\( \rho_n \). Similarly, for a hexagonal tessellation such that each cell has sides of length \( 2\rho_n \), we cannot have 3 consecutive hops, each of length strictly less than 0.5\( \rho_n \). Thus the value of \( W \) is much smaller, and the value of \( t \) is much larger for simple tessellations. However when we consider general tessellations that satisfy assumption A1, we note that the tessellation need not be as well-behaved as square or hexagonal. Yet, we can obtain an upper bound on the maximum number of successive hops of arbitrarily small lengths. To do this, we need to choose a fairly large (but finite and fixed) \( W \), and a fairly small (but positive and fixed) \( t \).

In the next Lemma, we show that except for a small fraction of hops, all other hops have an interfering transmitter within a distance of \( (M + 8)\rho_n \) of the receiver, and this fraction can be made arbitrarily small by choosing \( M \) large enough.

**Lemma 7**: Fix \( M > 16 \). Consider an arbitrary routing path between source and destination nodes such that R1 and R2 are satisfied. Let \( N_i \) be the number of hops of connection \( i \) such that there is no simultaneous interfering transmission within a circle of radius \( (M + 8)\rho_n \) around the receivers of those hops. To avoid tedious boundary conditions, let us not count the hops containing the source and the destination nodes in \( N_i \). Then,

\[
N_i \leq \frac{\tilde{L}_i}{\rho_n} \left( \frac{2(1 + c_1)}{M} \right),
\]

where \( \tilde{L}_i \) is the length of the routing path and \( c_1 \) is the constant from Lemma 4.4 in [1].

**Proof**: We use an approach identical to the one used to prove Lemma 6. Let \( U^j_i \) be the set of all the cells of color \( j \) that a packet of connection \( i \) traverses. Consider all the surrounding circles of radius \( M\rho_n \) around each of the cells in set \( U^j_i \). For each cell, there are many such circles, since the only requirement is that the center of the circle lie within the cell. A subset \( V^j_i \) of \( U^j_i \) is then formed as follows. A cell from set \( U^j_i \) is added to set \( V^j_i \) if none of its surrounding circles of radius \( M\rho_n \) contains any other color-\( j \) cell either partially or fully. Thus every cell in set \( V^j_i \) is at least \( M\rho_n \) distance away.
from any other color-$j$ cell. Hence, any surrounding circle of radius $\frac{M\rho_n}{2}$ around a cell in set $V^j_i$ does not overlap with any surrounding circle of radius $\frac{M\rho_n}{2}$ around another cell in set $V^j_i$. Let there be $N^j_i$ cells in set $V^j_i$, and let $N_i$ be the sum of the number of cells in $V^j_i$ for all $j$. We would like to determine an upper bound on $N_i$.

Referring to Fig. 4, consider the path taken by a packet of connection $i$ (solid line in the figure). The path is piece-wise linear, with the locations of the relaying node of each hop as points of discontinuity. Let $\hat{L}_i$ be the length of this routing path. Let us rearrange the cells in set $V^j_i$ in the order in which a packet of connection $i$ traverses them, and join them in this order (dotted lines in Fig. 4). Let $l_{i,k}(j)$ be the length of the $k$th dotted line segment. Define $\hat{L}_i(j)$ to be the total length of the dotted path. Then we have

$$\hat{L}_i(j) = \sum_k l_{i,k}(j) \leq \hat{L}_i$$  \hfill (16)

Consider a typical cell in set $V^j_i$ as shown in Fig. 5. As in the proof of Lemma 4 we first determine the length of the minimum possible section of the routing path (in this case the dotted path) that lies inside the surrounding circle. Since we know the total length of the dotted path, this gives us an upper bound on the number of surrounding circles of radius $\frac{M\rho_n}{2}$ that can be accommodated along the dotted path.

Referring to Fig. 5 the dotted path has its smallest section inside the surrounding circle of radius $\frac{M\rho_n}{2}$ when the relaying node of the cell, i.e., the point of discontinuity of the dotted path is located as close as possible to the circumference of the circle. However, since this point belongs to the cell, it cannot be farther than $4\rho_n$ from any other point in the cell. But the center of the surrounding circle must also lie inside the cell. Hence the point of discontinuity can be at the most $4\rho_n$ away from the center of the circle. Hence the each portion of the dotted path that lies within the circle can be no shorter than $\frac{M\rho_n}{2} - 4\rho_n$. Thus, for each cell in $V^j_i$, at least $(M - 8)\rho_n$ portion of the dotted path must lie inside the surrounding circle. Let $M - 8 > \frac{M\rho_n}{2}$, i.e., $M > 16$. Then the maximum number of such surrounding circles that can be accommodated along the dotted path, i.e., the maximum number of cells in set $V^j_i$ is:

$$N_i \frac{M\rho_n}{2} \leq \hat{L}_i \implies N_i \leq \frac{2\hat{L}_i}{M\rho_n}$$

Therefore for all the colors,

$$N_i = \sum_{j=1}^{1+c_1} N^j_i$$

$$\leq \frac{\hat{L}_i}{\rho_n} \frac{(2(1+c_1))}{M}$$

$$\leq \frac{\hat{L}_i}{\rho_n} \frac{(2(1+c_1))}{M}$$

Thus except for these $N_i$ hops, each of the hops of connection $i$ has at least one cell that has a simultaneous on-going transmission within a radius of $M\rho_n$ of the given cell. Or equivalently, within a radius of $(M+4+4)\rho_n = (M+8)\rho_n$ of the receivers of these hops, there is at least one more node that is transmitting simultaneously. This is because the maximum size of a cell is $4\rho_n$. And this proves the result.

For any routing path, the number of hops is lower bounded by $\frac{\hat{L}_i}{\rho_n}$, since each hop can traverse a distance of at the most $8\rho_n$. Lemma 4 shows that at least $\frac{1}{40}$th of these hops, i.e., at least $\frac{\hat{L}_i}{320\rho_n}$ hops of connection $i$ have transmitter-receiver pair separated by $0.01\rho_n$ or more. As in Subsection IV-A, let $A_i$ be the set of hops of connection $i$ for which the received signal is at the most $P(0.01\rho_n)^{-\alpha}$. Then,

$$|A_i| \geq \frac{\hat{L}_i}{320\rho_n}$$  \hfill (17)

Let $B_i$ be the set of hops of connection $i$ for which there is no simultaneous transmissions within a distance of $(M+8)\rho_n$ of the receiver. Using Lemma 7 given $\epsilon > 0$, we can choose $M$ large enough so that

$$|B_i| \leq \frac{\hat{L}_i}{\rho_n} \frac{(2(1+c_1))}{M}$$

$$\leq \frac{\hat{L}_i}{\rho_n} \epsilon$$  \hfill (18)

Then using (17) and (18),

$$|A_i \cap B_i^c| \geq |A_i| - |B_i|$$

$$\geq \frac{\hat{L}_i}{320\rho_n} - \frac{\hat{L}_i}{\rho_n} \epsilon$$

$$= \frac{\hat{L}_i}{\rho_n} \frac{1}{320} - \epsilon$$

If we pick $\epsilon = 1/640$, and choose $M = M_0$ corresponding to this choice of $\epsilon$,

$$|A_i \cap B_i^c| \geq \frac{\hat{L}_i}{640\rho_n}$$  \hfill (19)

Note that $A_i \cap B_i^c$ is the set of hops over which the received signal is no more than $P(0.01\rho_n)^{-\alpha}$, and there is at least one simultaneous transmission within a distance of $(M_0+8)\rho_n$ of
the receiver. This in turn means that for these hops, the SINR is upper bounded by

$$\text{SINR} \leq \frac{P(0.01\rho_n)^{-\alpha}}{P((M_0 + 8)\rho_n)^{-\alpha}} = 100^\alpha (M_0 + 8)^\alpha = \beta_1$$

The above upper bound on SINR may seem large due to $100^\alpha$ term. However, this term appears because we picked up a conservative hop length of 0.01$\rho_n$ in proving Lemma 2 to account for all possible tessellations. For simple tessellations such as square and hexagonal, we can use a minimum hop length of $\rho_n$, and obtain a tighter bound on SINR. However, as far as the asymptotic nature of the solution is concerned, the values of these constants do not matter. Thus we have proved the following Proposition which is a generalization of Proposition 4 when A4 is relaxed.

**Proposition 4:** There exists a fixed constant $M_0$, that does not depend on $n$, such that for at least $\hat{L}_i/64\rho_n$ hops of connection $i$, the SINR is less than a fixed constant $\beta_1$ given by

$$\beta_1 = 100^\alpha (M_0 + 8)^\alpha$$

As per assumption A2, since the SINR is upper bounded by a fixed constant $\beta_1$, the probability of successful packet reception is also upper bounded by a fixed constant $\phi(\beta_1) < 1$.

We can now use the above Proposition, and proceed as in Subsection IV-A to determine the end-to-end throughput of a connection. For this, we note that at least $\hat{L}_i/64\rho_n$ hops of connection $i$ have an SINR that is upper bounded by a constant $\beta_1$, i.e., a packet success probability that is upper bounded by a constant $\phi(\beta_1)$. Hence, replacing $\hat{L}_i$ by $\hat{L}_i$ in (20) to (10) we obtain the throughput a connection, $\Lambda_n$ as

$$\Lambda_n \leq \lambda_n(i) E_L [\delta^{\hat{L}_i}]$$

Recall that $\hat{L}_i$ denotes the length of the straight line joining the source-destination node pair of connection $i$, and hence $\hat{L}_i \geq L_i$. Also, since $\delta \leq 1$, we get

$$\Lambda_n \leq \lambda_n(i) E_L [\delta^{\hat{L}_i}]$$

After this stage, we use (11), and proceed along the same lines as Subsection IV-A and obtain the same asymptotic upper bound of $O\left(\frac{1}{n}\right)$ as in Propositions 2 and 4.

**Proposition 5:** For any choice of $\rho_n$ that satisfies assumptions A1 to A3, and for any routing scheme that satisfies R1 and R2, the per-node throughput that can be achieved is $O\left(\frac{1}{n}\right)$.

Thus the asymptotic per-node throughput is $O\left(\frac{1}{n}\right)$ even with arbitrary routing, i.e., when assumption A4 is relaxed.

**C. Per-node Capacity: Assumption A3 relaxed**

In the previous subsection, we saw that even when assumption A4 was relaxed, for a broad range of routing schemes the achievable per-node throughput is $O\left(\frac{1}{n}\right)$. To prove this, we showed that at least a fixed fraction of hops have the transmitter-receiver separation of no more than $\hat{t}\rho_n$ for some fixed $\hat{t} > 0$, and there exists an interfering transmitter within a distance of $(M + 8)\rho_n$ of receivers of these hops for some fixed, and finite $M$. This enabled us to upper bound the SINR (and hence the packet success probability) of these hops, and this in turn led to the $O\left(\frac{1}{n}\right)$ per-node throughput. In this subsection, we show that if we reduce the extent of spatial reuse via scheduling, then we can get better per-node throughput. Reduced spatial reuse results in lower interference from other ongoing transmissions, and this in turn improves SINR.

Recall that in proving Lemma 7, we showed that there exists an interfering transmitter within a distance of $(M + 8)\rho_n$ of at least a certain fraction of hops of connection $i$, and this fraction can be made as close to one as desired by choosing $M$ appropriately large. To prove this, we considered one color at a time, and determined the maximum number of cells of that color that belong to connection $i$ such that these cells do not have a simultaneous transmission within a radius $M\rho_n$ of themselves. We then summed up over all the colors to determine the maximum number of cells of connection $i$ with this property. Due to assumption A3, the number of colors used during scheduling, i.e., the length of the schedule is $1 + c_1$, and $c_1$ is a constant.

To relax assumption A3, let us assume that the total number of colors that are used during scheduling is $K_n$, i.e., the schedule length is not constant, but it scales with $n$. Also assume that $K_n$ goes to infinity as $n$ scales goes to infinity as $n$ scales. This implies that as the number of nodes in the network increases, the schedule become progressively longer, or equivalently, the extent of spatial reuse decreases with increasing network size. Due to this choice of $K_n$, i.e., due to reduced spatial reuse, we cannot use Lemma 4 to upper bound $N_i$. One way to bound $N_i$ would be to consider surrounding circles of radius $MK_n\rho_n$ instead of $M\rho_n$, so that $N_i$ can be made arbitrarily small by choosing $M$ large as before. However, the problem with this approach is that it guarantees that the fraction of hops of connection $i$ with an interfering transmitter within a distance of $(MK_n + 8)\rho_n$ can be made as close to one as desired by choosing $M$ large. However, our ultimate goal is to upper bound the SINR, and this was done in two parts; first by showing that at least a certain fraction of hops have transmitter-receiver separated by $f_n$, and that for except for a small fraction of hops, all the hops have an interferer within a distance of $g_n$. By showing that $f_n = \Theta(g_n)$, we were able to show that the SINR is upper bounded by a constant. Hence in addition to showing that there is an interfering node within $(MK_n + 8)\rho_n$ of the receiver, we must also show that at least a fixed fraction of hops have transmitter and receiver separated by no more than $tK_n\rho_n$ for some fixed positive $t$. Then we can claim an upper bound on the SINR of these hops. However there is no analog of Lemma 6 in this case. The reason being that we cannot claim that there are at the most $W$ consecutive hops such that each of them is shorter than $tK_n\rho_n$ because $K_n \to \infty$. Consequently, we can not no longer claim that the per-node capacity of the network is $O\left(\frac{1}{n}\right)$.

Thus by choosing a more conservative scheduling policy, i.e., by choosing a schedule of length $K_n$ such that $K_n$ goes to infinity as $n$ scales, we reduce the interference on the links
and improve the link SINR. However note that a reduced spatial reuse has an impact on the per-node throughput. We know from Theorem 4.1 in [1], that the straight line routing scheme achieves a per-node throughput of $\frac{c}{n \rho_n (1+c_1)}$, where $c$ is a constant. In [1], this throughput was shown to be achievable by determining the number of connections that pass though a given cell, and using the fact that each cell gets a transmission opportunity once every $1 + c_1$ slots, i.e., the schedule length is $1 + c_1$. For a general schedule length $K_n$, the throughput is given by $\frac{c}{n \rho_n K_n}$. To ensure connectivity, $\rho_n$ cannot decay any faster than $\sqrt{\frac{\log n}{n}}$, and hence the per node throughput is upper bounded by $\frac{c}{n \rho_n K_n \sqrt{n \log n}}$. Thus, to overcome the cumulative packet error probability over an asymptotically long connection, we must compromise on the spatial reuse in the network. By employing a schedule of asymptotically long length, we can reduce the extent of interference received by a link from other ongoing transmissions. This improves the link SINR. The effect of reduced spatial reuse is that now the per-node throughput is $O\left(\frac{1}{K_n \sqrt{n \log n}}\right)$ instead of $O\left(\frac{1}{n}\right)$. Note that for this to be true, $K_n$ has to be $\omega(1)$.

We have not shown that the above throughput can be achieved, i.e., with the above choice of $K_n$, and under assumptions A1 and A2, either the straight-line routing scheme in [1] or any other routing scheme achieves a throughput of $\Theta \left(\frac{1}{K_n \sqrt{n \log n}}\right)$. This is an interesting problem, and will be addressed in a separate work.

D. A Remark about Retransmission Strategy

In our analysis so far, we have assumed that over each hop, a single attempt is made to transmit the packet. If the transmission fails, then the node does not make a retransmission attempt. In reality, a multi-hop link layer employs a hop-by-hop acknowledgment and retransmission strategy in which the transmitter retransmits the packet if its previous transmission fails. If we assume that an infinite number of retransmissions are allowed for each packet, then the packet will eventually be successfully relayed over each hop. However, most practical link layer protocols have a fixed upper bound on the number of allowable retransmissions to avoid unbounded delays. If no more than $R$ retransmissions are allowed over each hop, and if $p$ is the probability of packet success over each transmission, then the probability that the packet will be transmitted successfully within $R$ attempts is simply given by $1 - (1 - p)^R$. This quantity is strictly less than one for finite $R$. Thus, there is an upper bound on the packet success probability which is strictly less than one. Note that for $R = 1$, this probability reduces to $p$, which is the case we have studied. With a constant $R$ that does not scale with $n$, all our previous arguments still go through with slight modifications. We can now think of replacing $p$ as the packet success probability by $1 - (1 - p)^R$, which is still a constant independent of $n$, and strictly less than one. As a result, the per-node throughput still remains $O\left(\frac{1}{n}\right)$.

Note that asymptotically, allowing infinite number of retransmissions at the link level is equivalent to assuming that the packet will be successfully received in a single transmission if the SINR is greater than $\beta$. However a realistic link layer protocol allows only a fixed bounded number of retransmission attempts. Besides, using a more conservative scheduling policy, we could also scale the number of allowable retransmissions so that the probability of transmission success over each link approaches one asymptotically. The impact of a retransmission scheme that scales with $n$, on the network capacity is an interesting problem that will be addressed in an extension of this work.

V. CONCLUSION AND EXTENSIONS

The key observation in this paper is that for a broad range of routing and scheduling schemes (including the one proposed in [1]), the number of hops of each connection scales to infinity with $n$. In a realistic link layer model, for a given modulation and coding scheme, the probability of packet loss over any given link is non-zero for finite SINR values. With the above link layer model, we show that for a broad class of routing and scheduling policies we cannot achieve a per-node throughput of more than $O\left(\frac{1}{n}\right)$ due to the cumulative probability of packet loss over all the hops of the connection. However, it is possible to improve the link SINR by using a more conservative scheduling policy with a lower spatial reuse. In this case the per-node throughput is $O\left(\frac{1}{K_n \sqrt{n \log n}}\right)$, where $K_n$ is the length of the schedule, i.e., each cell gets a transmission opportunity once every $K_n$ slots, and $K_n$ goes to infinity as $n$ scales. Thus the capacity of random ad-hoc networks scales even more pessimistically than what was previously thought.

Although we have shown that with such a choice of $K_n$, the per-node throughput is $O\left(\frac{1}{K_n \sqrt{n \log n}}\right)$, whether this throughput can indeed be achieved by a routing and scheduling scheme needs to be studied, and will be addressed in an extension of this work. We would also like to study how the throughput scales when we allow the number of retransmissions to scale with $n$. The work in [1] assumes that nodes pick source destination pairs at random, and then use multi-hop communication. The authors then note that each cell has to relay packets of other connections, and it is this relaying load that determines how fast a node can inject its own packets into the network. Thus the results are strongly dependent on the any-to-any communication paradigm, i.e., a network architecture in which any node may wish to communicate with any other node. On the other hand, our results are mainly based on the impact of interference and cumulative packet error probability on the network throughput. We note that the scheduling strategy and the number of hops are the key parameters that may determine the network throughput, and these parameters do not require any additional assumptions on the communication paradigm (any-to-any, many-to-one, arbitrary). As the network size increases, paths are likely to have an asymptotically large number of hops with high probability. Thus our results can be generalized to more generic network architectures such as [9], [10], and this will also be addressed in an extension of this work.

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Along the surface of the sphere. We have, sphere, which is one. Note that all the distances are measured by, Therefore the distribution function of \( L \) is given by, \[ F_L(l) = \frac{1}{2} (1 - \cos (2\sqrt{\pi}l)). \] Note that \( 0 \leq L \leq \sqrt{\pi}/2 \). The density function of \( L \) is given by, \[ f_L(l) = \sqrt{\pi} \sin (2\sqrt{\pi}l). \] Now we can compute \( E[\delta^L] \) as follows. \[
E[\delta^L] = \int_{0}^{\sqrt{\pi}} \delta^L \sqrt{\pi} \sin (2\sqrt{\pi}l) dl = \frac{1}{2} \int_{0}^{\pi} \gamma^x \sin x dx, \text{ where } \gamma = \delta^2 \sqrt{\pi}
= \frac{1}{2} l
\] (22)

Solving for \( I \) using integration by parts,
\[
I = \frac{1 + \gamma^n}{1 + (\log \gamma)^2}
\]
Replacing \( \gamma \) by \( \delta^2 \sqrt{\pi} \), and substituting the above in (22),
\[
E[\delta^L] = \frac{2\pi (1 + \delta^2 \sqrt{\pi})}{4\pi + (\log \delta)^2} \quad \text{ (23) }
\]

Using (21) from the above Lemma, we also have the following lemma for the area of a disk of radius \( \rho_n \) on \( S^2 \).

**Lemma 9:** If \( \rho_n \) is the radius of a disk (measured along the surface of \( S^2 \)), and the area of this disk is \( a_n = 100 \log n/n \), then
\[
\frac{\pi \rho_n^2}{2} \leq a_n \leq \pi \rho_n^2.
\] (24)

**Proof:** Using (21) with \( l = \rho_n \) gives
\[
a_n = \frac{1}{2} (1 - \cos (2\sqrt{\pi} \rho_n)) = \sin^2 (\sqrt{\pi} \rho_n)
\]
For \( \theta \leq \pi/4 \),
\[
\frac{\theta}{\sqrt{2}} \leq \sin \theta \leq \theta
\]
From this, (24) follows if \( n \) is large enough so that \( \rho_n \leq \sqrt{\pi}/4 \). The latter can always be ensured since
\[
a_n = \frac{100 \log n}{n} = \sin (\sqrt{\pi} \rho_n)
\]

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