A novel MOND effect in isolated high-acceleration systems

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ABSTRACT
We discuss a novel MOND effect that entails a correction to the dynamics of isolated mass systems even when they are deep in the Newtonian regime: systems whose extent $R \ll r_M$, where $r_M \equiv (GM_\odot/a_0)^{1/2}$ is the MOND radius and $M_\odot$ is the total mass. Interestingly, even if the MOND equations approach Newtonian dynamics arbitrarily fast at high accelerations, this correction decreases only as a power of $R/r_M$. The effect appears in formulations of MOND as modified gravity, governed by generalizations of the Poisson equation. The MOND correction to the potential is a quadrupole field $\phi_a \approx G \hat{Q}_{ij} r^2 r_j / r^2$, where $r$ is the radius from the centre of mass. In quasilinear MOND (QUMOND), $\hat{Q}_{ij} = -\alpha Q_{ij} r^2_M$, where $Q_{ij}$ is the quadrupole moment of the system and $\alpha > 0$ is a numerical factor that depends on the interpolating function. For example, the correction to the Newtonian force between two masses, $m$ and $M$, a distance $\ell$ apart ($\ell \ll r_M$) is $F_a = 2\alpha (\ell/r_M)^3 (mM)^5 M^{-3} a_0$ (attractive). Its strength relative to the Newtonian force is $2\alpha (mM/M^5)^5 (a_0/g_N)^{5/2}$ ($g_N \equiv GM/\ell^2$). For generic MOND theories, which approach Newtonian dynamics quickly for accelerations beyond $a_0$, the predicted strength of the effect in the Solar system is rather much below present testing capabilities. In MOND theories that become Newtonian only beyond $\kappa a_0$, the effect is enhanced by $\kappa^2$.

Key words: galaxies; kinematics and dynamics.

1 INTRODUCTION
One would suppose (as we had for many years) that MOND dynamics always approach Newtonian dynamics, in regions of high accelerations, as precipitously as its field equations approach those of Newtonian dynamics. (For a recent review of MOND, see Famaey & McGaugh 2012.) One would surmise that if the interpolating function that characterizes the theory, $\mu(x)$, approaches unity, very quickly, at high $x$, then the field, the forces and the dynamics in general approach their Newtonian values as quickly, when the system accelerations become high compared with the MOND constant $a_0$. This may be so in some formulations (e.g. in ‘modified inertia’ ones), and it is so for spherical systems, in general. However, we show here that this is not the general rule in formulations of MOND as an extension of the Poisson equation: the non-linear Poisson formulation of Bekenstein & Milgrom (1984) and the quasi-linear\(^1\) formulation (QUMOND) of Milgrom (2010a). No matter how vanishingly small $1 - \mu$ is for high $x$ – namely how close the field equations themselves approach the Poisson equation at high accelerations – in aspherical systems, $g/g_N - 1$ remains finite, and is of the order of $(a_0/g)^{5/2}$; here $g$ and $g_N$ are the MOND and Newtonian accelerations, respectively.

Previously noted MOND corrections in systems with $g \gg a_0$ are of several distinct types.

(1) Corrections stemming from remaining departure of $\mu(x)$ from unity even for high accelerations ($x \gg 1$). Such corrections have been discussed extensively since the early days of MOND (e.g. Milgrom 1983; Sereno & Jetzer 2006; Milgrom 2009a). We do not discuss these here. To isolate away such effects, we shall assume here that $\mu(\infty) - 1 \to 0$ fast enough as $x \to \infty$ (how fast will become clear below). We can then put $\mu = 1$ for $x \gg 1$, to the desired accuracy.

(2) Effects stemming from a MONDian background field in which the system is embedded, such as the Galactic field in the context of a stellar system. This effect and its consequences for the Solar system have been considered in detail in Milgrom (2009a) and Blanchet & Novak (2011), and these too are not considered in this paper, where we assume that the system is isolated.

(3) Even in systems that are of high acceleration almost everywhere, there are, generically, small regions where the accelerations are smaller than $a_0$. These include, for example, the very centres of stars and the regions around the points of zero gravity and in many-body systems. Such near-zero-gravity regions in the Solar system have been discussed as possible sites for testing MOND using test particle probes (Bekenstein & Magueijo 2006; Galianni et al. 2011; Magueijo & Mozaffari 2012). Here we are not interested in the

\(^1\) Since the term ‘quasi-linear’ partial differential equations is used by mathematicians in a different sense from here, we are now referring to such theories as ‘practically linear’.

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direct probing of such regions, but, rather, in the effect their presence has on the dynamics of the mass sources themselves. We shall show that the effects of these zero-gravity points are, generically, much smaller than the new effect we discuss here.

In Section 2, we describe the new effect qualitatively. In Section 3, we calculate it in the framework of QUMOND. In Section 4, we comment on the effect in the non-linear Poisson formulation. Section 5 treats the zero-gravity points in the system. In Section 6, we mainly discuss the effect in theories that restore Newtonian dynamics only at accelerations much larger than \( \mu_0 \).

\section{Qualitative Explanation}

Consider an isolated distribution of non-relativistic masses, \( \rho_0(x) \), of total mass, \( M_0 \), contained within a region of radius, \( R \), much smaller than the MOND radius of the system, \( r_M = (M_0 G a_0)^{1/2} \). Also assume that the system is quasi-static, in the sense that it varies only on time-scales much longer than \( r_\text{sd}/c \). Apart for small regions around zero-gravity point – which we consider separately in Section 5, and which have negligible effects – the accelerations in the system are everywhere much larger than \( a_0 \), and hence we may put \( \mu = 1 \) everywhere within \( R \). In such regions, the MOND equations coincide with the standard Poisson equation, and so the MOND potential satisfies the linear Poisson equation. However, it is not the standard, Newtonian solution, which assumes that the Poisson equation is valid everywhere to infinity. The fact that the field equation departs from Poisson beyond \( r_M \) imposes a different solution even at much smaller radii.

It is easy to understand the effect with the aid of the so-called ‘phantom’ mass (PM) distribution of the system, defined as (Milgrom 1986b)

\[
\rho_\text{PM} = \frac{1}{4\pi G} \left( \Delta \phi - \Delta \phi_0 \right),
\]

where \( \phi \) and \( \phi_0 \) are the MOND and Newtonian potentials of the system, respectively. In other words, the MOND correction to the Newtonian field is simply the Newtonian field of the PM. This concept is very useful in MOND generally, where it helps us bring to bear our long experience with Newtonian gravity, especially in cases where we have some prior knowledge of the PM distribution. The concept of PM was used effectively, for example, in Milgrom (1986b), Milgrom & Sanders (2008), Wu et al. (2008) and Zhao & Famaey (2010). It is especially useful in the context of QUMOND, where \( \rho_\text{PM} \) can be easily calculated at the outset.

Quasi-staticity, in the sense defined above, has to hold for the auxiliary of PM to be useful, since we assume that its distribution is uniquely determined by the instantaneous distribution of \( \rho \). Our non-relativistic approximation breaks down if the system varies on time-scales that are not much longer than \( r_\text{sd}/c \).

Under the above conditions, the MOND dynamics of any high acceleration, spherical system are exactly Newtonian. This follows from applying the Gauss theorem to spherical volumes concentric with the system. The fact that asymptotically the field crosses to the Newtonian field at, and beyond, \( r_M \), the MOND acceleration field is traceless. It is \( \tilde{Q}_{ij} \) that we want to calculate.

To recapitulate, much inside \( r_M \) the MOND acceleration field is very high; therefore, there \( \mu = 1 \) apart from the above-mentioned, inconsequential, MOND islands. Thus, \( \phi \) is, there, a solution of the Poisson equation. However, it is not the standard solution, \( \phi_0 \), which is oblivious of the MOND limit of the theory beyond \( r_M \). Instead, it is \( \phi \approx \phi_0 + \phi_\text{PM} \), while at larger radii, \( r \gtrsim r_M \), \( \phi \) becomes the full MOND solution of the problem. Because of the assumption of quasi-staticity, we can assume that even if the system changes over time, the distribution of PM ‘adjusts itself’ continuously to the instantaneous configuration.

\subsection{Quasi-static approximation}

Even in deeply non-relativistic systems, there are aspects that require relativistic treatment. Such is the case, for example, when we consider influences over distances for which the light travel time is longer than time-scales over which the influences vary. Such may be the case in the present context. The origin of the anomaly here is the behaviour of the field at distances \( \gtrsim r_M \), which, in turn, is determined by the matter distribution near the origin. If the latter changes on a time-scale \( \tau \) that is not much longer than \( 2r_\text{sd}/c \), the field at \( r_\text{sd} \) cannot be assumed to adjust to the instantaneous configuration, and instantaneously influences back the dynamics within \( \rho \). A relativistic treatment is then needed, and our treatment below is not valid.

In a system, such as a binary, whose \( \rho \) varies on a dynamical time-scale \( \tau \sim R/c \sim (R^2/MG)^{1/2} \), the condition for quasi-staticity can be written as \( 4G \rho_\text{sd}/(v/c)^2 \ll 1 \). This can also be written as \( R \gg R_\text{s} \equiv (2R^2 r_\text{M}^3)^{1/3} \sim (2\pi R_\text{s}^2 v_\text{H})^{1/3} \sim (1/(M/M_c))^{2/3} \) au (3) \((R_c \) is the Schwarzschild radius of \( M \) and \( v_\text{H} \) the Hubble radius).

When the opposite of inequality (3) holds, one might be tempted to simply time average the distribution of the PM; however, this has to be justified via a relativistic treatment.

\section{Calculation in QUMOND}

QUMOND (Milgrom 2010a) is a practically linear formulation of MOND derived from an action. It is the non-relativistic limit of a certain formulation of bimetric MOND (BIMOND) (Milgrom 2009b). The field equation for the gravitational potential is

\[
\Delta \phi = \nabla \cdot \left[ \nu \left( \frac{\nabla \phi_0}{a_0} \right) \nabla \phi_0 \right], \quad \text{where} \quad \Delta \phi_0 = 4\pi G \rho_0, \tag{4}
\]

with \( \nabla \phi \to 0 \) at infinity. Here, \( \nu(y \to \infty) \to 1 \) and \( \nu(y \to 0) \to y^{-1/2} \). It is related to the usual MOND interpolating function, \( \mu(x) \),

\[\text{Since } R \ll r_M, \text{ the MOND field at, and beyond, } r_M \text{ is spherical to the lowest order, with a correction of quadrupolar angular distribution.} \]

\[\text{Then the PM is also spherical to the lowest order, plus a small aspherical contribution. The spherical part has no effect on the dynamics of the system. Since the aspherical PM lies beyond } r_M, \text{ its added potential much inside } r_M \text{ is of the quadrupole type (the dipole will be shown to vanish):}
\]

\[\phi_\text{PM} \approx \frac{G}{M} \tilde{Q}_{ij} r^i r^j. \tag{2}\]

This is a vacuum solution of the Poisson equation since \( \tilde{Q}_{ij} \) is traceless. It is \( \tilde{Q}_{ij} \) that we want to calculate.

\[\text{To recapitulate, much inside } r_M \text{ the MOND acceleration field is very high; therefore, there } \mu = 1 \text{ apart from the above-mentioned, } \text{MOND islands. Thus, } \phi \text{ is, there, a solution of the Poisson equation. However, it is not the standard solution, } \phi_0, \text{ which is oblivious of the MOND limit of the theory beyond } r_M. \text{ Instead, it is } \phi \approx \phi_0 + \phi_\text{PM} \text{, while at larger radii, } r \gtrsim r_M, \phi \text{ becomes the full MOND solution of the problem. Because of the assumption of quasi-staticity, we can assume that even if the system changes over time, the distribution of PM ‘adjusts itself’ continuously to the instantaneous configuration.}

\[\text{2 We assume that the quadrupole of the system does not vanish. Otherwise, the dominant correction is of a higher multipole and of a higher order in } R/r_M.\]

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by \( v(\mu(x)) = 1/\mu(x) \). QUMOND thus requires solving (twice) the linear Poisson equation (and not a non-linear differential equation). It has already been put to good use for predicting and calculating MOND effects in the Solar system (Milgrom 2009a; Gaianni et al. 2011), for calculating MOND fields of galaxies (e.g. Angus et al. 2012), and for structure formation in MOND (Llinares 2011) (see also Zhao & Famaey 2010). Its relation to the non-linear Poisson formulation is elaborated on in Milgrom (2012). For example, it is shown there that the two theories are equivalent for one-dimensional systems when the non-linear Lagrangians of the two theories are the Legendre transforms of each other, leading to the above relation between \( \mu(x) \) and \( v(y) \).

In regions where \( \rho = 0 \), the phantom density is given by

\[
\rho_p = \frac{1}{4\pi G} \nabla \cdot \left[ v \left( \frac{\nabla \phi_N}{a_0} \right) \right],
\]

and is easily calculated from the Newtonian potential. Since \( \rho_p \) is non-zero only at distances much larger than \( R \), we can take only the lowest contributing harmonic to the Newtonian potential. Taking the origin at the centre of mass of the system, the dipole moment vanishes, and we keep only the monopole and quadrupole contributions, writing as

\[
\phi_N \approx -\frac{GM}{r} + \eta,
\]

where

\[
\eta = Gr^{-5/2} Q_{ij}, \quad Q_{ij} = \frac{1}{2} \int \rho(r)(r^2 \delta_{ij} - 3r_i r_j) \, d^3r.
\]

Using equations (6)–(7) in expression (5), keeping only terms up to first order in \( \eta \), we get

\[
4\pi G \rho_p \approx -2yv'(y) \frac{GM}{r^3} + 18v(y)^2 + 6y^2 v''(y) \frac{\eta}{r^2},
\]

where \( y = r_0/R \). The first term is spherical. At large distances, where \( y \ll 1 \), so \( v' \rightarrow -y^{-3/2} \), it gives the asymptotic phantom matter ‘isothermal holo’: \( \rho_p \propto M/r_0 r^2 \). At short distances, \( v' \rightarrow 0 \), and, as stated above, we assume that it does so fast enough that we can take it as zero within \( R \). The first term thus describes a spherical distribution with a spherical cavity; therefore, it has no effect on the dynamics of \( \rho \). In the second term, call it \( 4\pi G \rho_p \), all the factors are spherical, except \( \eta \), which is a quadrupole. This part of \( \rho_p \) also has an empty cavity for \( r \ll r_m \), but it does produce a field inside it. Because \( \rho_p \) is non-vanishing only at distances \( r \gg R \), we can keep only its lowest multipole contribution, which is the quadrupole, since \( \eta \) is reflection symmetric. Its field within \( R \) can be written to the dominant power in \( r/r_m \) as

\[
\phi_q \approx Gr^{-1} \hat{Q}_{ij}, \quad \hat{Q}_{ij} = \frac{1}{2} \int \rho_p(r)(r^2 \delta_{ij} - 3r_i r_j) \, d^3r.
\]

Inserting expression (8) for \( \rho_p \), and expression (7) for \( \eta \), we get

\[
\hat{Q}_{ij} = \frac{9}{8\pi r_m^3} Q_{ij} \int_0^\infty dy y^{5/2} \left[ v(y) + \frac{y}{3} v''(y) \right] \times \int \delta^{(2)}(\delta_{ij} - 3n_i n_j) n_i n_j,
\]

where \( r = r/\ell \). For the radial integral to converge at \( r \rightarrow 0 \), it is necessary that \( v'(y) \) vanishes at large \( y \) faster than \( y^{-7/2} \). This is, indeed, what is quantitatively meant when we say repeatedly that \( v \) is assumed to approach 1 fast enough. For slower vanishing, the MOND correction of type 1 discussed in Section 1 can be shown to dominate the present effect, and our approximation here is not valid.

After performing the angular integral, and integrating the \( y \) integrals by parts, we get

\[
\hat{Q}_{ij} = \alpha \frac{\partial}{\partial \mu} Q_{ij}, \quad \alpha = \frac{3}{4\pi} \int_0^\infty dy y^{5/2} [v(y) - 1].
\]

Thus, \( \alpha \) is a numerical factor that depends on the interpolating function.\(^4\)

A generic choice of \( v(y) \) in MOND would be one for which \( v(y) \rightarrow 1 \approx y^{-1/2} \) for \( y \ll 1 \), and can be made to vanish quickly for \( y \gg 1 \). Such a choice would give \( \alpha \approx 1 \), to within an order of magnitude roughly.\(^5\) For example, for the very sharply transiting, limiting form of \( \mu, \mu(x) = x \) for \( x \ll 1 \) and \( \mu(x) = 1 \) for \( x \gg 1 \) [for which \( v(y) = 1/y \) and \( v(y) = 1 \) for \( y \gg 1 \)], we get \( \alpha = 3/40 \). For \( v(y) = (1 - e^{-y^{1/2}})^{-1} \) one gets \( \alpha \approx 0.6 \). For the slowly transiting \( v(y) = (1 - e^{-y^{1/2}})^{-1} \), which was used successfully in Famaey & McGaugh (2012) for rotation curve predictions, we have \( \alpha \approx 37 \).

There may also be interest in interpolating functions for which very-near-Newtonian behaviour is reached, not beyond \( g_m \approx a_0 \), but beyond \( g_\infty \approx \kappa a_0 \), with, possibly, \( \kappa \gg 1 \) (see Section 6). For these, \( v - 1 \) is made to vanish quickly only for \( y \gg \kappa \), while \( v \sim 1 \). We can write some such functions in the form

\[
v = v_0(y) = 1 + \kappa^{-1/2} [v(y/\kappa) - 1],
\]

where \( v \) is of the generic type defined above. Substituting expression (12) in equation (11), we see that for such a function \( \alpha_0 = \kappa^2 \alpha - 1 \), where \( \alpha_{\kappa = 1} \) is roughly of order 1.

For a highly aspherical system, for which \( Q \sim M R^2 \) (such as a system of two comparable masses), the anomalous acceleration is \( g_\infty \sim GM/R^3 \), and its strength relative to the Newtonian acceleration, \( g_m \sim GM/M R^2 \), is \( g_m/g_\infty \sim (R/r_m)^3 \). This is so, we re-emphasize, even if \( v_0(g_m) - 1 \) approaches zero arbitrarily fast with increasing \( g_m/a_0 \). In almost spherical systems, the anomaly is further reduced. If \( m \ll M \), the ‘aspherical’ mass (e.g. the mass of planets in a planetary system), then the anomalous MOND acceleration in the system is \( g_\infty \sim (m/M)(R/r_m)^3 a_0 \).

As an example, consider the MOND correction to the Newtonian, two-body force between masses \( m \) and \( M \), a distance \( \ell \) apart, for which condition (3) for quasi-staticity holds. Here, \( Q_{ij} = -\ell m M/\ell^2 \), and \( \kappa_{ij} = Q_{ij} = -Q_{ij}/2 \), where the masses lie on the z-axis. The MOND correction to the Newtonian force is then attractive, and equals

\[
F_z = 2\alpha(\ell/r_m)^3 (m M)^2 \ell^{-3} a_0.
\]

\(^3\) All the first-order terms are scalars that are linear in \( Q_{ij} \), so they must be of the form \( G(\nu(r)/v)^2 \hat{Q}_{ij} \phi = f(\nu)^2 \left( \nabla \nu \right) (Q_{ij} = 0) \). For example, use is made of \( r - \nabla \eta = -3\eta \) and \( r^i \nabla^n \eta = 12\eta \).

\(^4\) Note that \( \hat{Q}_{ij} \) is the ‘internal’ quadrupole of the PM. It is finite and picks up its main contribution from radii around \( r_m \). The expression for the ‘external’ quadrupole moment of the PM, defined as in equation (7), diverges linearly at large radii (small \( y \)) for isolated systems, just as the total PM mass does. This does not lead to divergences in the fields and is, anyhow, cut off by external fields. In some systems, there are also departures from quasi-staticity to be reckoned with at large radii.

\(^5\) Of course, \( \alpha \) can be made arbitrarily large if \( v - 1 \) vanishes nearly as \( y^{-3/2} \).
4 THE NON-LINEAR POISSON THEORY

In this theory, the MOND potential for an isolated system is the (unique) solution of the equation (Bekenstein & Milgrom 1984)

\[ \nabla \cdot \left( \mu \left( \frac{\nabla \phi}{a_0} \right) \nabla \phi \right) = 4\pi G \rho, \tag{14} \]

with \( \nabla \phi \to 0 \) at infinity. This is the non-relativistic limit of Einstein aether theories (Zlosnik, Ferreira & Starkman 2007), and it is also part of the non-relativistic limit of tensor-vector-scalar gravity (TeVeS) (Bekenstein 2004) and of generic formulations of BI-MOND (Milgrom 2009b, 2010b).

If \( \mu \) is related to \( \phi \) of QUMOND as described above, the two theories coincide for spherical systems, and the (radial) dominant asymptotic behaviour is the same in the two theories. Again, in the phantom-density approach, we need the dominant, asymptotic, aspherical MOND potential. However, in this theory we cannot calculate \( \rho \) before a full solution of the problem is known.

In Milgrom (1986a), we showed that the aspherical, far field still has, generically, a quadrupolar angular dependence, but its radial dependence is somewhat different from that in QUMOND. The asymptotic potential, beyond \( R_M \), is of the form

\[ \phi \approx (G_0 M) \sqrt{2} \xi \ln(r) + G \frac{S_0 r^4}{r^{2+\gamma}}, \tag{15} \]

where \( S_0 \) is a symmetric, traceless (constant) matrix; it depends on the mass distribution \( r \), but we do not know how. The phantom mass density outside \( \rho \) is \( 4\pi G \rho = \Delta \phi \), so its asymptotic form is

\[ 4\pi G \rho \sim G M \frac{S_0 r^4}{r^{2+\gamma}}, \tag{16} \]

where \( \gamma = (\sqrt{3} + 2) / (\sqrt{3} - 3) \). Here too, in high-acceleration regions, roughly within \( R_M \), \( \rho \sim \rho_0 \) vanishes. This is because there is \( \Delta \phi = \Delta \phi_0 \). The first term in the expression for the asymptotic \( \rho \) is spherical and does not affect the dynamics in the system. The second term gives rise again to a quadrupole field:

\[ \phi_\nu \propto G \frac{S_0 r^4}{r^{3+\gamma}}. \tag{17} \]

It vanishes for a spherical system, and also in the limit \( a_0 \to 0 \). Therefore, we may write it as \( S \sim M R^{3+\gamma} \) \((R/r_0)\), where \( R \) is the characteristic extent of the system. The dimensionless function \( \xi(u) \) is unknown, however, and depends on the various dimensionless parameters of the system (mass ratios, ratios of distances, etc.), and has to vanish for \( u = 0 \), which corresponds to \( a_0 = 0 \). Numerical calculations are needed to say more on this.

5 CONTRIBUTION OF THE SMALL MOND DOMAINS NEAR ZERO-GRAVITY POINTS

Clearly, in every mass system there are critical points of the potential, where its gradient vanishes.\(^6\) This follows from topological considerations, but is otherwise obvious. There are, for example, the points at (or very near) the centres of spherical stars or planets. Also, in a system of many compact objects, there are zero-gravity points, around which there is a MONDian region where \( |\nabla \phi| < a_0 \). These regions are surrounded by Newtonian regions, where we assume that \( \mu = v = 1 \) to the desired accuracy. This implies that the total PM within each such MONDian region vanishes. Applying the Gauss theorem to the volume \( V \), within a surface \( \Sigma \) that is wholly in the Newtonian region, we have

\[ \int_\Sigma \nabla \cdot \phi dV \propto \int_\Sigma (\nabla \phi - \nabla \phi_0) \cdot dS = \int_\Sigma |\mu| (\nabla \phi/|a_0|) \nabla \phi - \nabla \phi_0 | dS = \int_\Sigma |\mu| |\nabla \phi/|a_0| \nabla \phi - \nabla \phi_0 | dV = 0, \]

where we used the fact that on \( \Sigma \), \( \mu = 1 \). In a similar way, it is seen that this is also true in QUMOND.

The effects of such regions on the dynamics of \( \rho \) can be calculated if we know the PM distribution in them. Since the typical acceleration within such a region is, by definition, \( |\nabla \phi| \sim a_0 \) (we restrict ourselves to MOND theories that approach Newtonian dynamics quickly beyond \( a_0 \)), the characteristic size, \( L \), and phantom density are related by \( 4\pi G \rho L \sim a_0 \).

The MONDian regions near the centres of spherical objects are, themselves, spherical (they may be slightly aspherical if the general geometry is strongly aspherical, but we neglect this); therefore, since they have vanishing total PM, they have no outside effect.\(^7\)

Other zero-gravity points typically occur in vacuum, so they cannot be extrema of the potential; they are thus saddle points. Take, for example, a system of a few comparable masses with intermass distances \( \sim R \). If \( g(r) \) is the gravitational acceleration field in the system, \( g \sim GM/R^2 \), then at a zero-gravity point, \( r_0 \), we have \( g(r_0) = 0 \). The size of the region around this point, within which \( g \leq a_0 \) is \( L \sim a_0 (d g/d r) ^{-1} \sim (a_0/g) R \sim R^2 / R_M ^2 \). The total PM in this region vanishes. To lowest order in \( R / R_M \) the region is reflection symmetric; therefore, its dipole moment vanishes to this order (it would otherwise scale as \( \rho_0 L^3 \)). The dipole moment is thus of the higher order; \( d \sim \rho_0 L^4 / R \). The phantom quadrupole moment is \( Q \sim \rho L^4 \). The acceleration field of the dipole at a characteristic distance \( R \) is \( g_d \sim G d R^2 \sim G \rho_0 L^2 \), and that of the quadrupole is \( g_Q \sim G Q R^4 \sim G \rho_0 L^2 \). They thus scale in the same way. Since \( G \rho_0 \sim a_0 \), we find that the anomalous MOND correction to the acceleration in the system is \( g_a \sim a_0 (L/R)^4 \sim (a_0 / g) R \sim R^4 / R_M ^4 \). This estimate can also be made for the case of a small mass \( m \) in a larger Newtonian system whose characteristic size is \( R \), and whose total mass is \( M \gg m \). Again, one estimates that the MOND correction to the acceleration on \( m \) is \( g_a \sim a_0 (R / R_M )^4 \). All this is confirmed by explicit calculations in QUMOND, which we omit here.

The accelerations due to this effect thus scale as \( (R / R_M )^4 \), compared with \( (R / R_M )^3 \) for the correction discussed above; however, for this effect the anomalous acceleration on a small mass does not vanish in the limit \( m \to 0 \).

6 DISCUSSION

Intuitively, one would expect that the MONDian, two-body force becomes exactly Newtonian at high accelerations, if the interpolating function does so. We have shown that this is not the case, and that, more generally, there are MOND effects that linger even in high-acceleration systems.

For theories with \( \alpha \sim 1 \), we cannot think, at present, of a precision experiment to test this effect: in the Solar system, the effect is

\( ^7 \) For a star of central density \( \rho \), the characteristic phantom density in the region is also \( \sim \rho \); therefore, the size of this region is \( L \sim a_\odot / G \rho \), which for \( \rho \sim 1 \) g cm\(^{-3} \), is a fraction of a centimetre.

\( ^8 \) If some masses are much smaller than others, then the zero-gravity points are much nearer such masses than \( R \); say a distance \( R_m \). Then \( L \sim (a_0/g) R_m \); however, our result below remains valid.

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A novel high-acceleration MOND effect

Thus, in all these theories $\kappa \sim 1$ – which is also preferred by galaxy dynamics – is admissible. As already mentioned, the original versions of TeVeS are forced to adopt $\kappa \gg 1$ because they are not GR compatible, a fact that puts them in danger of conflicts with tests in the high-acceleration regime, unless $k \ll 1$. Babichev, Deffayet & Esposito-Farèse (2011) have recently devised a theory that might be viewed as a short-distance modification of TeVeS (introducing an additional scale length besides $a_0$), and that avoids conflicts with observations, while admitting $\kappa \sim 1$. We feel that a theory with $\kappa \gg 1$ and all the consequences that follow from it are not generic MOND results.

The QUMOND versions of theories with $\kappa \gg 1$ would involve an interpolating function as given in equation (12). Then, as we saw in Section 3, the coefficient $\alpha$ defined in equation (11) is $\alpha = \kappa_0 = k^2 \alpha_{\kappa_0}$, where $\alpha_{\kappa_0} = 1$. In particular, this means that $a_0$ that appears in the mass–asymptotic speed relation, $M G a_0 = V_a^2$, also marks the boundary of the Newtonian regime, namely that beyond $\sim a_0$, Newtonian behaviour is reached quickly. This underlies our conclusion above regarding the Solar system, since it implies that $\alpha \sim 1$ within roughly an order of magnitude.

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