CONSTRUCTIBLE CHARACTERS AND $b$-INVARIENT

by

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Let $(W,S)$ be a finite Coxeter system and let $\varphi : S \to \mathbb{R}_{>0}$ be a weight function that is, a map such that $\varphi(s) = \varphi(t)$ whenever $s$ and $t$ are conjugate in $W$. Associated with this datum, G. Lusztig has defined [Lu3, §22] a notion of constructible characters of $W$: it is conjectured that a character is constructible if and only if it is the character afforded by a Kazhdan-Lusztig left cell (defined using the weight function $\varphi$). These constructible characters depend heavily on $\varphi$ so we will call them the $\varphi$-constructible characters of $W$: the set of $\varphi$-constructible characters will be denoted by $\text{Cons}^\text{Lus}_\varphi(W)$. We shall also define a graph $\mathcal{G}_{W,\varphi}$ as follows: the vertices of $\mathcal{G}_{W,\varphi}$ are the irreducible characters and two irreducible characters $\chi$ and $\chi'$ are joined in this graph if there exists a $\varphi$-constructible character $\gamma$ of $W$ such that $\chi$ and $\chi'$ both occur as constituents of $\gamma$. The connected components of $\mathcal{G}_{W,\varphi}$ (viewed as subsets of $\text{Irr}(W)$) will be called the Lusztig $\varphi$-families: the set of Lusztig $\varphi$-families will be denoted by $\text{Fam}^\text{Lus}_\varphi(W)$. If $\mathcal{F} \in \text{Fam}^\text{Lus}_\varphi(W)$, we denote by $\text{Cons}^\text{Lus}_\varphi(\mathcal{F})$ the set of $\varphi$-constructible characters of $W$ all of whose irreducible components belong to $\mathcal{F}$.

On the other hand, using the theory of rational Cherednik algebras at $t = 0$ and the geometry of the Calogero-Moser space associated with $(W,\varphi)$, R. Rouquier and the author (see [BoRo1] and [BoRo2]) have defined a notion of Calogero-Moser $\varphi$-cells of $W$, a notion of Calogero-Moser $\varphi$-cellular characters of $W$ (whose set is denoted by $\text{Cell}^\text{CM}_\varphi(W)$) and a notion of Calogero-Moser $\varphi$-families (whose set is denoted by $\text{Fam}^\text{CM}_\varphi(W)$).

Conjecture (see [BoRo1], [BoRo2] and [GoMa]). With the above notation,

$$\text{Cons}^\text{Lus}_\varphi(W) = \text{Cell}^\text{CM}_\varphi(W) \quad \text{and} \quad \text{Fam}^\text{Lus}_\varphi(W) = \text{Fam}^\text{CM}_\varphi(W)$$

for every weight function $\varphi : S \to \mathbb{R}_{>0}$.

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The statement about families in this conjecture holds for classical Weyl groups thanks to a case-by-case analysis relying on [Lu3, §22] (for the computation of Lusztig \(\varphi\)-families), [GoMa] (for the computation of Calogero-Moser \(\varphi\)-families in type A and B) and [Be2] (for the computation of the Calogero-Moser \(\varphi\)-families in type D). It also holds whenever \(|S| = 2\) (see [Lu3, §17 and Lemma 22.2] and [Be1, §6.10]).

The statement about constructible characters is much more difficult to establish, as the computation of Calogero-Moser \(\varphi\)-cellular characters is at that time out of reach: it has been proved whenever the Caloger-Moser space associated with \((W, S, \varphi)\) is smooth [BoRo2, Theorem 14.4.1] (it has also been checked if \(W\) is of type \(B_2\))...

Our aim in this paper is to show that this conjecture is compatible with properties of the \(b\)-invariant (as defined below). With each irreducible character \(\chi\) of \(W\) is associated its fake degree \(f_\chi(t)\), using the invariant theory of \(W\) (see for instance [BoRo2, Definition 1.5.7]). Let us denote by \(b_\chi\) the valuation of \(f_\chi(t)\): \(b_\chi\) is called the \(b\)-invariant of \(\chi\). For instance, \(b_1 = 0\) and \(b_\varepsilon\) is the number of reflections of \(W\) (here, \(\varepsilon : W \to \{1, -1\}\) denotes the sign character). Also, \(b_\chi = 1\) if and only if \(\chi\) is an irreducible constituent of the canonical reflection representation of \(W\). The following result is proved in [BoRo2, Theorems 9.6.1 and 12.3.14]:

**Theorem CM.** Let \(\varphi : S \to \mathbb{R}_{>0}\) be a weight function. Then:

(a) If \(\mathcal{F} \in \text{Fam}^{\text{CM}}_{\varphi}(W)\), then there exists a unique \(\chi \in \mathcal{F}\) with minimal \(b\)-invariant.

(b) If \(\gamma \in \text{Cons}^{\text{CM}}_{\varphi}(W)\), then there exists a unique irreducible constituent \(\chi\) of \(\gamma\) with minimal \(b\)-invariant.

The next theorem is proved in [Lu2, Theorem 5.25 and its proof] (see also [Lu1] for the first occurrence of the special representations):

**Theorem (Lusztig).** Assume that \(\varphi\) is constant. Then:

(a) If \(\mathcal{F} \in \text{Fam}^{\text{Lus}}_{\varphi}(W)\), then there exists a unique \(\chi_\mathcal{F} \in \mathcal{F}\) with minimal \(b\)-invariant (\(\chi_\mathcal{F}\) is called the special character of \(\mathcal{F}\)).

(b) If \(\gamma \in \text{Cons}^{\text{Lus}}_{\varphi}(\mathcal{F})\), then \(\chi_\mathcal{F}\) is an irreducible constituent of \(\gamma\) (and, of course, among the irreducible constituents of \(\gamma\), \(\chi_\mathcal{F}\) is the unique one with minimal \(b\)-invariant).

It turns out that, for general \(\varphi\), there might exist Lusztig \(\varphi\)-families \(\mathcal{F}\) such that no element of \(\mathcal{F}\) occurs as an irreducible constituent of all the \(\varphi\)-constructible characters in \(\text{Cons}^{\text{Lus}}_{\varphi}(\mathcal{F})\) (this already occurs in type \(B_2\), and the reader can also check this fact in type \(E_6\), using the tables given by Geck [Ge, Table 2]). Nevertheless, we will prove in this paper the following result, which is compatible with the above conjecture and the above theorems:
**Theorem L.** Let $\varphi : S \to \mathbb{R}_{>0}$ be a weight function. Then:

(a) If $\mathcal{F} \in \text{Fam}^\text{Lus}_\varphi(W)$, then there exists a unique $\chi \in \mathcal{F}$ with minimal $b$-invariant.

(b) If $\gamma \in \text{Cons}^\text{Lus}_\varphi(W)$, then there exists a unique irreducible constituent $\chi$ of $\gamma$ with minimal $b$-invariant.

The proof of Theorem CM is general and conceptual, while our proof of Theorem L goes through a case-by-case analysis, based on Lusztig’s description of $\varphi$-constructible characters and Lusztig $\varphi$-families [Lu3, §22].

**Remark 0.** Let $\gamma_\chi$ denote the coefficient of $t^b$ in $F_\chi(t)$. Then it has been noticed by Lusztig [Lu1, §2, Page 325] that $\gamma_\chi = 1$ whenever $\chi$ is special.

As the only irreducible Coxeter systems affording possibly unequal parameters are of type $I_2(2m)$, $F_4$ or $B_n$, and as $\gamma_\chi = 1$ for any character $\chi$ in these groups, we can conclude that, in general (equal or unequal parameters), $\gamma_\chi = 1$ for all the characters $\chi$ with minimal $b$-invariant constructed in Theorem L (for both (a) and (b)).

The same property holds for the characters $\chi$ with minimal $b$-invariant constructed in Theorem CM (in this case, the proof is again general and conceptual [BoRo2]).

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1. **Proof of Theorem L**

1.A. **Reduction.** — It is easily seen that the proof of Theorem L may be reduced to the case where $(W,S)$ is irreducible. If $W$ is of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, $H_3$ or $H_4$, then $\varphi$ is necessarily constant and Theorem L follows immediately from Lusztig’s Theorem. If $W$ is dihedral, then Theorem L is easily checked using [Lu3, §17 and Lemma 22.2]. If $W$ is of type $F_4$, then Theorem L follows from inspection of [Ge, Table 2]. Therefore, this shows that we may, and we will, assume that $W$ is of type $B_n$, with $n \geq 2$. Write $S = \{t, s_1, s_2, \ldots, s_{n-1}\}$ in such a way that the Dynkin diagram of $(W,S)$ is

```
(1)   t   s_1   s_2   \cdots   s_{n-1}
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Write $b = \varphi(t)$ and $a = \varphi(s_1) = \varphi(s_2) = \cdots = \varphi(s_{n-1})$. If $b \not\in a\mathbb{N}^*$, then $\text{Cons}^\text{Lus}_\varphi(W) = \text{Irr}(W)$ (see [Lu3, Proposition 22.25]) and Theorem L becomes obvious. So we may assume that $b = ra$ with $r \in \mathbb{N}^*$. Therefore:
Hypothesis. From now on, and until the end of this section, we will assume that $(W, S)$ is of type $B_n$, with $n \geq 2$, that $S = \{t, s_1, s_2, \ldots, s_{n-1}\}$ is such that the Dynkin diagram of $(W, S)$ is given by $(\#)$, that $\varphi(t) = r\varphi(s_1) = r\varphi(s_2) = \cdots = r\varphi(s_{n-1}) = 1$ with $r \in \mathbb{N}^*$. 

1.B. Admissible involutions. — Let $l \geq 0$ and let $Z$ be a totally ordered set of size $2l + r$. We shall define by induction on $l$ what is an $r$-admissible involution of $Z$. Let $\iota : Z \to Z$ be an involution. Then $\iota$ is said $r$-admissible if it has $r$ fixed points and, if $l \geq 1$, there exist two consecutive elements $b$ and $c$ of $Z$ such that $\iota(b) = c$ and the restriction of $\iota$ to $Z \setminus \{b, c\}$ is $r$-admissible.

Note that, if $\iota$ is an $r$-admissible involution and if $\iota(b) = c > b$ and $\iota(z) = z$, then $z < b$ or $z > c$ (this is easily proved by induction on $|Z|$).

1.C. Symbols. — We shall denote by $\text{Sym}_k(r)$ the set of symbols $\Lambda = \left(\begin{array}{c} \beta \\ \gamma \end{array}\right)$ where $\beta = (\beta_1 < \beta_2 < \cdots < \beta_{k+r})$ and $\gamma = (\gamma_1 < \gamma_2 < \cdots < \gamma_k)$ are increasing sequences of non-zero natural numbers. We set

$$|\Lambda| = \sum_{i=1}^{k+r} (\beta_i - i) + \sum_{j=1}^{k} (\gamma_j - j)$$

and

$$b(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)(\beta_i - i) + \sum_{j=1}^{k} (2k + 1 - 2j)(\gamma_j - j).$$

The number $b(\Lambda)$ will be called the $b$-invariant of $\Lambda$. For simplifying our arguments, we shall define

$$\nabla_{k,r} = \sum_{i=1}^{k+r} (2k + 2r - 2i)i + \sum_{j=1}^{k} (2k + 1 - 2j)j$$

so that

$$b(\Lambda) = \sum_{i=1}^{k+r} (2k + 2r - 2i)\beta_i + \sum_{j=1}^{k} (2k + 1 - 2j)\gamma_j - \nabla_{k,r}.$$
so that

\[
\mathbf{b}(\Lambda) = \sum_{i=1}^{r} (2k + 2r - 2i)z_i'(\Lambda) + \sum_{i=r+1}^{2k+r} (2k + r - i)z_i'(\Lambda) - \nabla_{k,r}
\]

\[
\bigstar = \sum_{i=1}^{r} (r - i)z_i'(\Lambda) + \sum_{i=1}^{2k+r} (2k + r - i)z_i'(\Lambda) - \nabla_{k,r}
\]

\[
\sum_{i=1}^{r} \left( \sum_{j=1}^{i} z_j'(\Lambda) \right) + \sum_{i=1}^{2k+r} \left( \sum_{j=1}^{i} z_j'(\Lambda) \right) - \nabla_{k,r}.
\]

1.D. Families of symbols. — We denote by \( z(\Lambda) \) the sequence \( z_1 \leq z_2 \leq \cdots \leq z_{2k+r} \) obtained after rewriting the sequence \( (\beta_1, \beta_2, \ldots, \beta_{k+r}, \gamma_1, \gamma_2, \ldots, \gamma_k) \) in non-decreasing order.

Remark 1 - Note that the sequence \( z'(\Lambda) \) determines the symbol \( \Lambda \), contrarily to the sequence \( z(\Lambda) \). However, \( z(\Lambda) \) determines completely \( |\Lambda| \) thanks to the formula

\[ |\Lambda| = \sum_{z \in \mathcal{E}(\Lambda)} z - r(r+1)/2 - (k+r)(k+r+1)/2. \]

We say that two symbols \( \Lambda = \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \) and \( \Lambda' = \left( \begin{array}{c} \beta' \\ \gamma' \end{array} \right) \) in \( \mathbf{Sym}_k(r) \) are in the same family if \( z(\Lambda) = z(\Lambda') \). Note that this is equivalent to say that \( \beta \cap \gamma = \beta' \cap \gamma' \) and \( \beta \cup \gamma = \beta' \cup \gamma' \).

If \( \mathcal{F} \) is the family of \( \Lambda \), we set \( X_{\mathcal{F}} = \beta \cap \gamma \) and \( Z_{\mathcal{F}} = \beta + \gamma \); note that \( X_{\mathcal{F}} \) and \( Z_{\mathcal{F}} \) depend only on \( \mathcal{F} \) (and not on the particular choice of \( \Lambda \in \mathcal{F} \)).

If \( \iota \) is an \( r \)-admissible involution of \( Z_{\mathcal{F}} \), we denote by \( \mathcal{F}_\iota \) the set of symbols \( \Lambda = \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \) in \( \mathcal{F} \) such that \( |\beta \cap \omega| = 1 \) for all \( \iota \)-orbits \( \omega \).

1.E. Lusztig families, constructible characters. — Let \( \Lambda \in \mathbf{Sym}_k(r) \) be such that \( |\Lambda| = n \). Let \( \mathbf{Bip}(n) \) be the set of bipartitions of \( n \). We set

\[
\lambda_1(\Lambda) = (\beta_{k+r} - (k+r) \geq \cdots \geq \beta_2 - 2 \geq \beta_1 - 1),
\]

\[
\lambda_2(\Lambda) = (\gamma_k - k \geq \cdots \geq \gamma_2 - 2 \geq \gamma_1 - 1)
\]

and

\[
\lambda(\Lambda) = (\lambda_1(\Lambda), \lambda_2(\Lambda)).
\]

Then \( \lambda(\Lambda) \) is a bipartition of \( n \). We denote by \( \chi_\lambda \) the irreducible character of \( W \) denoted by \( \chi_{\lambda(\Lambda)} \) in [Lu3, §22] or in [GePf, §5.5.3]. Then [GePf, §5.5.3]

\[
\boxed{b_{\chi_\lambda} = \mathbf{b}(\Lambda).}
\]

With these notation, Lusztig described the \( \varphi \)-constructible characters in [Lu3, Proposition 22.24], from which the description of Lusztig \( \varphi \)-families follow by using [Lu3, Lemma 22.22]:
Theorem 2 (Lusztig). Let $\mathcal{F}_{\text{Lus}}$ be a Lusztig $\varphi$-family and let $\gamma \in \text{Cons}^{\text{Lus}}(\mathcal{F}_{\text{Lus}})$. If we choose $k$ sufficiently large, then:

(a) There exists a family $\mathcal{F}$ of symbols in $\text{Sym}_k(r)$ such that
$$\mathcal{F}_{\text{Lus}} = \{ \chi_{\Lambda} \mid \Lambda \in \mathcal{F} \}.$$ 

(b) There exists an $r$-admissible involution $\iota$ of $\mathbb{Z}$ such that
$$\gamma = \sum_{\Lambda \in \mathcal{F}} \chi_{\Lambda}.$$ 

If $\Lambda = \left( \begin{array}{c} \beta \\ \gamma \end{array} \right)$, we set $\Lambda^s = \left( \begin{array}{c} \beta \setminus (\beta \cap \gamma) \\ \gamma \setminus (\beta \cap \gamma) \end{array} \right)$.

Definition 3. The symbol $\Lambda$ is said special if $z(\Lambda^s) = z'(\Lambda^s)$.

Remark 4 - According to Remark 1, there is a unique special symbol in each family. It will be denoted by $\Lambda_{\mathcal{F}}$. Finally, note that, if $\Lambda, \Lambda'$ belong to the same family, then $|\Lambda| = |\Lambda'|$. □

Now, Theorem L follows from Theorem 2, Formula (♦) and the following next Theorem:

Theorem 5. Let $\mathcal{F}$ be a family of symbols in $\text{Sym}_k(r)$, let $\iota$ be an $r$-admissible involution of $\mathbb{Z}$ and let $\Lambda \in \mathcal{F}$. Then:

(a) $b(\Lambda) \geq b(\Lambda_{\mathcal{F}})$ with equality if and only if $\Lambda = \Lambda_{\mathcal{F}}$.

(b) There is a unique symbol $\Lambda_{\mathcal{F},i}$ in $\mathcal{F}_i$ such that, if $\Lambda \in \mathcal{F}_i$, then $b(\Lambda) \geq b(\Lambda_{\mathcal{F},i})$, with equality if and only if $\Lambda = \Lambda_{\mathcal{F},i}$.

The rest of this section is devoted to the proof of Theorem 5.

1.F. Reduction for the proof of Theorem 5. — First, assume that $X_{\mathcal{F}} \neq \emptyset$. Let $b \in X_{\mathcal{F}}$ and let $\Lambda = \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \in \mathcal{F}$. Then $b \in \beta \cap \gamma = X_{\mathcal{F}}$ and we denote by $\beta[b]$ the sequence obtained by removing $b$ to $\beta$. Similarly, let $\Lambda[b] = \left( \begin{array}{c} \beta[b] \\ \gamma[b] \end{array} \right)$.

Then $\Lambda[b] \in \text{Sym}_{k-1}(r)$ and

$$b(\Lambda) = b(\Lambda[b]) + \nabla_{k,r} - \nabla_{k-1,r} + b\left( 4k + 2r + 1 - \sum_{z \in \mathcal{L}(\Lambda) \atop z \leq b} 2 \right) + 2 \sum_{z \in \mathcal{L}(\Lambda) \atop z < b} z.$$
Proof of (帨). Let $i_0$ and $j_0$ be such that $\beta_{i_0} = b$ and $\gamma_{j_0} = b$. Then

$$b(\Lambda) - b(\Lambda[b]) = \nabla_{k,r} - \nabla_{k-1,r} + (2k + 2r - 2i_0)b + \sum_{i=1}^{i_0-1} 2\beta_i + (2k + 1 - 2j_0)b + \sum_{j=1}^{j_0-1} 2\gamma_j.$$ 

But the numbers $\beta_1, \beta_2, \ldots, \beta_{i_0}, \gamma_1, \gamma_2, \ldots, \gamma_{j_0}$ are exactly the elements of the sequence $z(\Lambda)$ which are $\leq b$. So

$$i_0 + j_0 = \sum_{z \in z(\Lambda)} 1$$

and

$$\sum_{i=1}^{i_0-1} \beta_i + \sum_{j=1}^{j_0-1} \gamma_j = \sum_{z \in z(\Lambda)} z.$$ 

This shows (帨). ■

Now, the family of $\Lambda[b]$ depends only on the family of $\Lambda$ (and not on $\Lambda$ itself): indeed, $z(\Lambda[b])$ is obtained from $z(\Lambda)$ by removing the two entries equal to $b$. We will denote by $\mathcal{F}[b]$ the family of $\Lambda[b]$. Moreover, $Z_{\mathcal{F}[b]} = Z_{\mathcal{F}}$ and the map $\Lambda \mapsto \Lambda[b]$ induces a bijection between $\mathcal{F}$ and $\mathcal{F}[b]$, and also induces a bijection between $\mathcal{F}_i$ and $\mathcal{F}[b]_i$.

On the other hand, the formula (帨) shows that the difference between $b(\Lambda)$ and $b(\Lambda[b])$ depends only on $b$ and $\mathcal{F}$, so proving Theorem 5 for the pair $(\mathcal{F}, \tau)$ is equivalent to proving Theorem 5 for the pair $(\mathcal{F}[b], \tau)$. By applying several times this principle if necessary, this means that we may, and we will, assume that

$$X_{\mathcal{F}} = \emptyset.$$ 

1.G. Proof of Theorem 5(a). — First, note that $z(\Lambda) = z(\Lambda_{\mathcal{F}}) = z'(\Lambda_{\mathcal{F}})$ (the last equality follows from the fact that $\Lambda_{\mathcal{F}}$ is special and $X_{\mathcal{F}} = \emptyset$). As $z'(\Lambda)$ is a permutation of the non-decreasing sequence $z'(\Lambda_{\mathcal{F}})$, we have

$$\sum_{j=1}^{i} z_j'(\Lambda) \geq \sum_{j=1}^{i} z_j'(\Lambda_{\mathcal{F}})$$

for all $i \in \{1, 2, \ldots, 2k + r\}$. So, it follows from (_sat) that

$$b(\Lambda) - b(\Lambda_{\mathcal{F}}) = \sum_{i=1}^{r-1} \left( \sum_{j=1}^{i} (z_j'(\Lambda) - z_j'(\Lambda_{\mathcal{F}})) \right) + \sum_{i=1}^{2k + r-1} \left( \sum_{j=1}^{i} (z_j'(\Lambda) - z_j'(\Lambda_{\mathcal{F}})) \right).$$

So $b(\Lambda) \geq b(\Lambda_{\mathcal{F}})$ with equality only whenever $\sum_{j=1}^{i} z_j'(\Lambda) = \sum_{j=1}^{i} z_j'(\Lambda_{\mathcal{F}})$ for all $i \in \{1, 2, \ldots, 2k + r\}$. The proof of Theorem 5(a) is complete.
1.H. Proof of Theorem 5(b). — We denote by \( f_r < \cdots < f_1 \) the elements of \( Z_f \) which are fixed by \( \iota \). We also set \( f_{r+1} = 0 \) and \( f_0 = \infty \). As \( \iota \) is \( r \)-admissible, the set \( Z^{(d)}_f = \{ z \in Z_f \mid f_{d+1} < z < f_d \} \) is \( r \)-stable and contains no \( r \)-fixed point (for \( d \in \{0, 1, \ldots, r\} \)). Let \( k_d = |Z^{(d)}_f|/2 \) and let \( \iota_d \) be the restriction of \( \iota \) to \( Z^{(d)}_f \). Then \( \iota_d \) is a 0-admissible involution of \( Z^{(d)}_f \).

If \( \Lambda = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathcal{F}_r \), we set \( \beta^{(d)} = \beta \cap Z^{(d)}_f \), \( \gamma^{(d)} = \gamma \cap Z^{(d)}_f \) and \( \Lambda^{(d)} = \begin{pmatrix} \beta^{(d)} \\ \gamma^{(d)} \end{pmatrix} \). Then \( \Lambda^{(d)} \in \text{Sym}_{k_d}(0) \) and, if \( \mathcal{F}^{(d)} \) denotes the family of \( \Lambda^{(d)} \), then \( \Lambda^{(d)} \in \mathcal{F}^{(d)}_{\iota_d} \).

Now, if \( \Lambda' = \begin{pmatrix} \beta' \\ \gamma' \end{pmatrix} \in \text{Sym}_{k_d}(0) \), we set

\[
\mathbf{b}_d(\Lambda') = \sum_{i=1}^{k'} (2k' + 2d - 2i)\beta'_i + \sum_{j=1}^{k'} (2k' + 1 - 2j)\gamma'_j.
\]

The number \( \mathbf{b}_d(\Lambda') \) is called the \( \mathbf{b}_d \)-invariant of \( \Lambda' \). It then follows from the definition of \( \mathbf{b} \) and \( \nabla_{k,r} \) that

\[
(\spadesuit) \quad \mathbf{b}(\Lambda) = \sum_{d=0}^{r} \mathbf{b}_d(\Lambda^{(d)}) - \nabla_{k,r} + \sum_{d=1}^{r} 2(k_0 + k_1 + \cdots + k_{d-1}) \left( f_d + \sum_{z \in Z^{(d)}} z \right).
\]

Since the map

\[
\mathcal{F}_r \quad \xrightarrow{\Lambda} \quad \prod_{d=0}^{r} \mathcal{F}^{(d)}_{\iota_d} \quad \mapsto \quad (\Lambda^{(0)}, \Lambda^{(1)}, \ldots, \Lambda^{(d)})
\]

is bijective and since \( \mathbf{b}(\Lambda) - \sum_{d=0}^{r} \mathbf{b}_d(\Lambda^{(d)}) \) depends only on \( (\mathcal{F}, \iota) \) and not on \( \Lambda \) (as shown by the formula (\spadesuit)), Theorem 5(b) will follow from the following lemma:

**Lemma 6.** There exists a unique symbol in \( \mathcal{F}^{(d)}_{\iota_d} \) with minimal \( \mathbf{b}_d \)-invariant.

The proof of Lemma 6 will be given in the next section.

2. Minimal \( \mathbf{b}_d \)-invariant

For simplifying notation, we set \( Z = Z^{(d)}_f \), \( l = k_d \), \( \mathcal{G} = \mathcal{F}^{(d)} \) and \( j = \iota_d \). Let us write \( Z = \{ z_1, z_2, \ldots, z_{2l} \} \) with \( z_1 < z_2 < \cdots < z_{2l} \). Recall from the previous section that \( j \) is a 0-admissible involution of \( Z \).
2.A. Construction. — We shall define by induction on $l \geq 0$ a symbol $\Lambda_j^{(d)}(Z) \in \mathcal{Q}_j$. If $l = 0$, then $\Lambda_j^{(d)}(Z)$ is obviously empty. So assume now that, for any set of non-zero integers $Z'$ of order $2(l-1)$, for any 0-admissible involution $j'$ of $Z'$ and any $d' \geq 0$, we have defined a symbol $\Lambda_j^{(d')}(Z')$. Then $\Lambda_j^{(d)}(Z) = \left( \Lambda_j^{(d')}(Z) \right)$ is defined as follows: let $Z' = Z \setminus \{ z_1, \iota(z_1) \}$, $j'$ the restriction of $j$ to $Z'$ and let

$$d' = \begin{cases} 
    d-1 & \text{if } d \geq 1, \\
    1 & \text{if } d = 0. 
\end{cases}$$

Then $|Z'| = 2(l-1)$ and $j'$ is 0-admissible. So $\Lambda_j^{(d')}(Z') = \left( \beta_j^{(d')}(Z') \right)$ is well-defined by the induction hypothesis. We then set

$$\beta_j^{(d)}(Z) = \begin{cases} 
    \beta_j^{(d')}(Z') \cup \{ z_1 \} & \text{if } d \geq 1, \\
    \beta_j^{(d')}(Z') \cup \{ \iota(z_1) \} & \text{if } d = 0, 
\end{cases}$$

and

$$\gamma_j^{(d)}(Z) = \begin{cases} 
    \gamma_j^{(d')}(Z') \cup \{ j(z_1) \} & \text{if } d \geq 1, \\
    \gamma_j^{(d')}(Z') \cup \{ z_1 \} & \text{if } d = 0. 
\end{cases}$$

Then Lemma 6 is implied by the next lemma:

**Lemma 6**. Let $\Lambda \in \mathcal{Q}_j$. Then $b_d(\Lambda) \geq b_d(\Lambda_j^{(d)}(Z))$ with equality if and only if $\Lambda = \Lambda_j^{(d)}(Z)$.

The rest of this section is devoted to the proof of Lemma 6. We will first prove Lemma 6 whenever $d \in \{0, 1\}$ using Lusztig’s Theorem. We will then turn to the general case, which will be handled by induction on $l = |Z|/2$. We fix $\Lambda = \left( \beta, \gamma \right) \in \mathcal{Q}_j$.

2.B. Proof of Lemma 6 whenever $d = 1$. — Let $z$ be a natural number strictly bigger than all the elements of $Z$. Let $\tilde{\Lambda} = \left( \beta \cup \{ z \} \right) \in \mathcal{Q}_j$. Then $b_1(\Lambda) = b(\tilde{\Lambda}) + C$, where $C$ depends only on $Z$. Let $\tilde{\Lambda}_0 = \left( \gamma, z_1, z_3, \ldots, z_{2l-1}, z \right) \in \mathcal{Q}_j$. Since $j$ is 0-admissible, it is easily seen that, if $j(z_i) = z_j$, then $j - i$ is odd. So $\tilde{\Lambda}_0 \in \mathcal{Q}_j$. But, by [Lus], §5, $b(\tilde{\Lambda}) \geq b(\tilde{\Lambda}_0)$ with equality if and only if $\tilde{\Lambda} = \tilde{\Lambda}_0$. So it is sufficient to notice that $\tilde{\Lambda}_0^{(1)}(Z) = \tilde{\Lambda}_0$, which is easily checked.
2.C. Proof of Lemma 6\(^+\) whenever \(d = 0\). — Assume in this subsection, and only in this subsection, that \(d = 0\) or 1. We denote by \(\Lambda^{\text{op}} = \left(\begin{array}{c}\gamma \\ \beta \end{array}\right) \in \mathscr{G}_i\). It is readily seen from the construction that \(\Lambda_j^{(0)}(Z)^{\text{op}} = \Lambda_j^{(1)}(Z)\) and that
\[
\mathbf{b}_1(\Lambda) = \mathbf{b}_0(\Lambda^{\text{op}}) + \sum_{z \in Z} z.
\]
So Lemma 6\(^+\) for \(d = 0\) follows from Lemma 6\(^+\) for \(d = 1\).

2.D. Proof of Lemma 6\(^+\) whenever \(d \geq 2\). — Assume now, and until the end of this section, that \(d \geq 2\). We shall prove Lemma 6\(^+\) by induction on \(l = |Z|/2\). The result is obvious if \(l = 0\), as well as if \(l = 1\). So we assume that \(l \geq 2\) and that Lemma 6\(^+\) holds for \(l' \leq l - 1\). Write \(j(z_1) = z_{2m}\), where \(m \leq l\) (note that \(j(z_1) \not\in \{z_1, z_3, z_5, \ldots, z_{2l-1}\}\) since \(j\) is 0-admissible).

Assume first that \(m < l\). Then \(Z\) can be written as the union \(Z = Z^+ \cup Z^-\), where \(Z^+ = \{z_1, z_2, \ldots, z_{2m}\}\) and \(Z^- = \{z_{2m+1}, z_{2m+2}, \ldots, z_{2l}\}\) are \(j\)-stable (since \(j\) is 0-admissible). If \(\varepsilon \in \{+,-\}\), let \(j^\varepsilon\) denote the restriction of \(j\) to \(Z^\varepsilon\), let \(\beta^\varepsilon = \beta \cap Z^\varepsilon\), \(\gamma^\varepsilon = \gamma \cap Z^\varepsilon\) and \(\Lambda^\varepsilon = \left(\begin{array}{c}\beta^\varepsilon \\ \gamma^\varepsilon \end{array}\right)\), and let \(\mathscr{G}^\varepsilon\) denote the family of \(\Lambda^\varepsilon\). Then it is easily seen that \(\Lambda^\varepsilon \in \mathscr{G}^\varepsilon\), that \(\mathbf{b}_d(\Lambda) = (\mathbf{b}_d(\Lambda^+) + \mathbf{b}_d(\Lambda^-))\) depends only on \((\mathscr{G}, j)\) and that \(\Lambda_j^{(d)}(Z^\varepsilon) = \Lambda_j^{(d)}(Z^\varepsilon)\). By the induction hypothesis, \(\mathbf{b}_d(\Lambda^\varepsilon) \geq \mathbf{b}_d(\Lambda_j^{(d)}(Z^\varepsilon))\) with equality if and only if \(\Lambda^\varepsilon = \Lambda_j^{(d)}(Z^\varepsilon)\). So the result follows in this case. This means that we may, and we will, work under the following hypothesis:

**Hypothesis.** From now on, and until the end of this section, we assume that \(j(z_1) = z_{2l}\).

As in the construction of \(\Lambda_j^{(d)}(Z)\), let \(Z' = Z \setminus \{z_1, z_{2l}\} = \{z_2, z_3, \ldots, z_{2l-1}\}\), let \(j'\) denote the restriction of \(j\) to \(Z'\) and let
\[
d' = \begin{cases} 
  d - 1 & \text{if } d \geq 1, \\
  1 & \text{if } d = 0.
\end{cases}
\]
Then \(|Z'| = 2(l - 1)\) and \(j'\) is 0-admissible. Let \(\Lambda' = \left(\begin{array}{c}\beta' \\ \gamma' \end{array}\right)\) where \(\beta' = \beta \setminus \{z_1, z_{2l}\}\) and \(\gamma' = \gamma \setminus \{z_1, z_{2l}\}\). Since \(d \geq 2\), we have \(z_1 \not\in \beta_j^{(d)}(Z)\) and \(z_{2l} \not\in \gamma_j^{(d)}(Z)\). This implies that
\[
\mathbf{b}_d(\Lambda_j^{(d)}(Z)) = \mathbf{b}_{d-1}(\Lambda_j^{(d-1)}(Z')) + z_{2l} + 2(l + d)z_1 + 2\sum_{z \in Z'} z.
\]

Thus, the result follows for \(d = 2\), and the induction is complete.
If \( z_1 \in \beta \), then \( \Lambda = \Lambda^{(d)}_j(Z) \) if and only if \( \Lambda' = \Lambda^{(d)}_{j'}(Z') \) and again
\[
b_d(\Lambda) = b_{d-1}(\Lambda') + z_{2l} + 2(l + d)z_1 + 2 \sum_{z \in Z'} z.
\]

So the result follows from (\( \bigstar \)) and from the induction hypothesis.

This means that we may, and we will, assume that \( z_1 \in \gamma \). In this case,
\[
b_d(\Lambda) = b_{d+1}(\Lambda') + 2d z_{2l} + (2l + 1)z_1.
\]

Then it follows from (\( \bigstar \)) that
\[
b_d(\Lambda) - b_d(\Lambda^{(d)}_j(Z)) = b_{d+1}(\Lambda') - b_{d-1}(\Lambda^{(d-1)}_{j'}(Z')) + (2d - 1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.
\]

So, by the induction hypothesis,
\[
b_d(\Lambda) - b_d(\Lambda^{(d)}_j(Z)) \geq b_{d+1}(\Lambda^{(d+1)}_{j'}(Z')) - b_{d-1}(\Lambda^{(d-1)}_{j'}(Z')) + (2d - 1)(z_{2l} - z_1) - 2 \sum_{z \in Z'} z.
\]

Since \( z_{2l} - z_1 > z_{2l-1} - z_2 \), it is sufficient to show that
\[
(?) \quad b_{d+1}(\Lambda^{(d+1)}_{j'}(Z')) - b_{d-1}(\Lambda^{(d-1)}_{j'}(Z')) \geq -(2d - 1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z.
\]

This will be proved by induction on the size of \( Z' \). First, if \( j(z_2) < z_{2l} \), then we can separate \( Z' \) into two \( j' \)-stable subsets and a similar argument as before allows to conclude thanks to the induction hypothesis.

So we assume that \( j'(z_2) = z_{2l-1} \). Let \( Z'' = Z' \setminus \{z_2, z_{2l-1}\} \) and let \( j'' \) denote the restriction of \( j' \) to \( Z'' \). Since \( z_2 \in \beta_{j'}^{(d+1)}(Z') \), we can apply (\( \bigstar \)) one step further to get
\[
b_{d+1}(\Lambda^{(d+1)}_{j'}(Z')) - b_{d-1}(\Lambda^{(d-1)}_{j'}(Z')) = b_d(\Lambda^{(d)}_{j''}(Z'')) + z_{2l-1} + 2(l + d)z_2 + 2 \sum_{z \in Z''} z
\]
\[
-(b_{d-2}(\Lambda^{(d-2)}_{j''}(Z'')) + z_{2l-1} + 2(l + d - 2)z_2 + 2 \sum_{z \in Z''} z)
\]
\[
= b_d(\Lambda^{(d)}_{j''}(Z'')) - b_{d-2}(\Lambda^{(d-2)}_{j''}(Z'')) + 4z_2.
\]

So, by the induction hypothesis,
\[
b_{d+1}(\Lambda^{(d+1)}_{j'}(Z')) - b_{d-1}(\Lambda^{(d-1)}_{j'}(Z')) \geq -(2d - 3)(z_{2l-2} - z_3) + 2 \sum_{z \in Z'} z + 4z_2
\]
\[
\geq -(2d - 3)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z + 2z_2 - 2z_{2l-1}
\]
\[
= -(2d - 1)(z_{2l-1} - z_2) + 2 \sum_{z \in Z'} z,
\]

as desired. This shows (?) and completes the proof of Lemma 6â. 
3. Complex reflection groups

If \( \mathcal{W} \) is a complex reflection group, then R. Rouquier and the author have also defined Calogero-Moser cellular characters and Calogero-Moser families (see [BoRo1] or [BoRo2]). If \( \mathcal{W} \) is of type \( G(l, 1, n) \) (in Shephard-Todd classification), then Leclerc and Miyachi [LeMi, §6.3] proposed, in link with canonical bases of \( U_v(\mathfrak{sl}_\infty) \)-modules, a family of characters that could be good analogue of constructible characters: let us call them the Leclerc-Miyachi constructible characters of \( G(l, 1, n) \). If \( l = 2 \), then they coincide with constructible characters [LeMi, Theorem 10].

Of course, it would be interesting to know if Calogero-Moser cellular characters coincide with the Leclerc-Miyachi ones: this seems rather complicated but it should be at least possible to check if the Leclerc-Miyachi constructible characters satisfy the analogous properties with respect to the \( b \)-invariant.

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