Coalgebraic modal logic and games for coalgebras with side effects

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Abstract
We study coalgebraic modal logic and games to characterise behavioural equivalence in the presence of side effects, i.e., when coalgebras live in a (co)Kleisli or an Eilenberg-Moore category. Our aim is to develop a general framework based on indexed categories/fibrations that is common, at least, to the aforementioned categories. In particular, we show how the coalgebraic notion of behavioural equivalence arises from a relation lifting (a special kind of indexed morphism) and we give a general recipe to construct such liftings in the above three cases. Lastly, we apply this framework to derive games and logical characterisations for (weighted) language equivalence and conditional bisimilarity.

1 Introduction
Coalgebra [29] offers a categorical framework for specifying and reasoning about state-based transition systems in a generic way. In particular, new types of transition systems, behavioural equivalences (or distances), modal logics and games can be obtained by suitably instantiating the theory of coalgebras. While many types of transition systems can already be studied in the category \( \text{Set} \), systems with side effects – leading to a notion of trace equivalence or conditional bisimilarity – usually require to move to a setting beyond \( \text{Set} \), using Kleisli, coKleisli or Eilenberg-Moore categories, where the (co)monad specifies the side-effects.

Behavioural equivalences for such scenarios have already been studied extensively (see e.g. [8]). Modal logics on the other hand, have been considered to a lesser extent [17, 19], while we are not aware of any work describing coalgebraic behavioural equivalence games for trace equivalences. The aim of the present paper is to work in an indexed category framework – strongly related to the notion of fibrations [10] — that provides the means to lift predicates and relations and to give a fixpoint characterisation of behavioural equivalence. Concerning modal logics, we apply the dual adjunction setup for fibrations by Kupke and Rot [25] to derive logical characterisations for coalgebras with side effects.

Hermida and Jacobs in [10] provided a systematic account (based on the theory of fibrations [11]) of how to capture behavioural equivalences (including bisimilarity) at the level of coalgebras. Since then, fibrations have increasingly appeared in the coalgebraic literature [5, 6, 9, 13, 18, 20, 25, 32]. The general idea is as follows and is illustrated above in (1).

First, a system is modelled as a coalgebra \( X \xrightarrow{\alpha} FX \) for some endofunctor \( \text{Set} \xrightarrow{\lambda} \text{Set} \). Second a fibration \( E \) of binary relations on the working category of sets is realised, whose fibres are all the relations on the underlying state space. This approach can also be used to capture bisimulation metrics [5, 20, 25]. Third, a mechanism \( \mathcal{P}(X \times X) \xrightarrow{\Lambda} \mathcal{P}(FX \times FX) \)
Predicate and relation liftings (aka relation lifting) to lift a relation on $X$ to a relation on $FX$ is defined, which amounts to the lifting of $F$ to an endofunctor $F_\lambda$ on $E$. Now one can study the coalgebras induced by $F_\lambda$ and, more importantly, this category $\text{Coalg}_E(F_\lambda)$ can be again arranged (see (1)) as a fibration on $\text{Coalg}_{\text{Set}}(F)$. Lastly, the applicability is shown by characterising bisimulation relations on $X$ as coalgebras of a certain endofunctor living in the fibre above $(X, \alpha)$.

One of the objectives of this paper is to extend this ‘categorical’ picture w.r.t. coalgebraic notion of behavioural equivalence for dynamical systems having side effects, i.e., those systems that can be modelled as coalgebras living in a (co)Kleisli or Eilenberg-Moore categories for some (co)monad on $\text{Set}$. Typical examples, in the context of this paper, are the following: nondeterministic (linear weighted) automata modelled as coalgebras in the Kleisli category for the powerset (multiset) monad (see Sections 6 and 3); conditional transition systems (which facilitate formal modelling of software product lines) modelled as coalgebras in the coKleisli category for the writer comonad $K \times -$ (see Section 7).

We explore the general conditions on relation liftings (technically they are called indexed morphisms in the paper like the ones indicated by $\lambda$ in (1)) to ensure that behavioural equivalences can be viewed as coalgebras living in the fibre above a given coalgebra $(X, \alpha)$ with side effects. Another contribution is a recipe for obtaining relation and predicate liftings (a special type of indexed morphisms) whose definition and correctness proof are otherwise (at least in the Kleisli case) quite cumbersome to establish. Predicate liftings are instrumental in providing interpretation to various modalities for coalgebras in (co)Kleisli or Eilenberg-Moore categories, just like in the case of $\text{Set}$ (cf. [13, 27, 30]).

Our study focus is on lifting an indexed morphism for a given endofunctor $F$ on $\text{Set}$ to an indexed morphism for a (co)Kleisli extension/Eilenberg-Moore lifting $\overline{F}$ of $F$. And to the best of our knowledge, this question is open at least for coalgebras with side effects. As soon as we have established predicate liftings in Kleisli categories, we have assembled the necessary requirements for defining a spoiler-duplicator game that characterises behavioural equivalence for coalgebras living in Kleisli and Eilenberg-Moore categories, while the issue of games is still open for coKleisli categories.

Once we have captured behavioural equivalence in a fibration, we can then apply the Kupke-Rot setup [25] based on dual adjunctions (see the survey [24] on coalgebraic modal logic) to establish the logical characterisation of behavioural equivalence. In particular, we first construct the Kupke-Rot setup for behavioural equivalences and show that the sufficient conditions for adequacy (i.e., behavioural equivalence is contained in logical equivalence) and expressivity (i.e., converse of adequacy) given in [25] are satisfied. This setup is later used to derive the logical characterisation for (weighted) language equivalence and conditional bisimilarity; note that these notions were not studied in [25].

While several ingredients (especially encompassing fibrations/indexed category) used in this paper are already known, our paper contains the following original contributions:

▷ We capture behavioural equivalences on arbitrary coalgebras as a fibred notion and we study behavioural equivalences as special types of coalgebras beyond $\text{Set}$.
▷ We give concrete recipes for defining predicate and relation liftings (which is both tedious and error-prone) in (co)Kleisli and Eilenberg-Moore categories.
▷ We extend the dual adjunction framework for fibrations by Kupke and Rot to side effects, in particular to Kleisli categories. Here we need a mechanism to factor the state space of a coalgebra by behavioural equivalence, which is difficult if the category has no coequalisers. We provide a technique based on reflective subcategories to circumvent this issue.
▷ To the best of our knowledge, we are the first to describe coalgebraic games that characterise linear equivalences like trace, (weighted) language, failure, and ready equivalences. The latter
two are captured as behavioural equivalence in an Eilenberg-Moore category (cf. Appendix B).

This paper is organised as follows. Section 2 sets the relevant categorical preliminaries required for this paper. It is assumed that the reader is already familiar with basic category theory, particularly, how a Kleisli or an Eilenberg-Moore category is induced by a monad. Section 3 introduces the assumptions that ensure behavioural equivalence is a fibred notion (in the sense of [1]). Section 4 is devoted to coalgebraic modal logic where general adequacy and expressivity results for behavioural equivalence are derived from [25]. Section 5 gives the recipe to construct relation/predicate liftings in Kleisli categories and introduces the game. In the next sections, the results of this paper are applied in the context of nondeterministic automata (see Sections 6 and 7 for generalised Moore machines and weighted automata respectively) and conditional transition systems. Section 8 concludes this paper with some discussions on future research. Note that proofs (Section A) as well as additional material (Sections B–E) can be found in the appendix.

2 Preliminaries

Coalgebraic preliminaries \[14, 29\]

Let \( \mathcal{C} \) be a category and let \( \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C} \) be an endofunctor modelling the branching type of the system of interest. Then the behaviour of a state-based system will be modelled as an \( \mathcal{F} \)-coalgebra (or, simply coalgebra), i.e. as a morphism \( X \xrightarrow{\alpha} \mathcal{F}X \) in the category \( \mathcal{C} \).

\[ \text{Definition 1.} \quad \text{A coalgebra homomorphism } f \text{ between } X \xrightarrow{\alpha} \mathcal{F}X \text{ and } Y \xrightarrow{\beta} \mathcal{F}Y \in \mathcal{C} \text{ is a morphism } X \xrightarrow{f} Y \in \mathcal{C} \text{ satisfying } \mathcal{F}f \circ \alpha = \beta \circ f. \text{ The collection of coalgebras and their homomorphisms forms a category denoted } \text{Coalg}_\mathcal{C}(\mathcal{F}). \]

Moreover, one can define behavioural equivalence on the ‘states’ of a coalgebra under the assumption that there is a functor \( \mathcal{C} \xrightarrow{\iota} \text{Set} \). Note that there is a functor \( \iota \) left adjoint to \( \_ \xrightarrow{\_} \text{Set} \) when \( \mathcal{C} = \text{Kl}(T) \) or \( \mathcal{C} = \text{EM}(T) \).

\[ \text{Definition 2.} \quad \text{Two states } x, x' \in |X| \text{ of a coalgebra } X \xrightarrow{\alpha} \mathcal{F}X \text{ are behaviourally equivalent iff there is a coalgebra homomorphism } f \text{ such that } |f| x = |f| x'. \]

\[ \text{Example 3.} \quad \text{An interesting example of coalgebras living in Kleisli categories is non-deterministic automata (NDA). By following [8] we model an NDA as a coalgebra living in } \mathcal{C} = \text{Rel} = \text{Kl}(\mathcal{P}). \text{ Recall that a Kleisli extension } \text{Rel} \xrightarrow{\bar{\mathcal{F}}} \text{Rel of } \mathcal{C} \text{ (i.e. } \bar{\mathcal{F}} \circ \iota = \iota \circ \mathcal{F}) \text{ is in correspondence (see [26, Theorem 2.2] for a general statement) with a distributive law } \mathcal{F}T \xrightarrow{\vartheta} \mathcal{F}T \text{ such that the following diagrams commute in } \text{Set}. \]

\[ \begin{array}{ccc}
FX & \xrightarrow{\mathcal{F}\eta_X} & FX \\
F\eta_X & \xrightarrow{\mathcal{F}\eta_X} & FX \\
FTX & \xrightarrow{\varphi_X} & TFX \\
F\varphi_X & \xrightarrow{F\varphi_X} & FTX \\
F\mu_X & \xrightarrow{F\mu_X} & FX \\
FTX & \xrightarrow{\theta_X} & TFX \\
F\theta_X & \xrightarrow{F\theta_X} & FTX \\
FTX & \xrightarrow{\varphi_X} & TFX \\
T\varphi_X & \xrightarrow{T\varphi_X} & TFX \\
\end{array} \]  

Consider \( F\_ = A \times \_ + 1 \) (where \( 1 = \{\bullet\} \)) with the following distributive law [12, Section 4]:

\[ \text{Act} \times \mathcal{P}X + 1 \xrightarrow{\varphi_X} \mathcal{P}(\text{Act} \times X + 1) \quad (a, U) \mapsto \{a\} \times U, \bullet \mapsto \{\bullet\}. \]

This induces a functor \( \text{Rel} \xrightarrow{\bar{\mathcal{F}}} \text{Rel} \) which acts on a relation \( X \xrightarrow{f} Y \), seen as a Kleisli arrow \( X \xrightarrow{f} \mathcal{P}Y \), as follows: \( \bar{\mathcal{F}} f = \vartheta_Y \circ \mathcal{F} f' \). Notice that \( \mathcal{F} \)-coalgebras model implicit nondeterminism (i.e. this side-effect is hidden to an outside observer) [8], thus behavioural equivalence typically coincides with language equivalence (instead of bisimilarity) in this case.
Predicate and relation liftings

Predicate liftings as indexed morphisms

Predicates and their liftings are quite common within the literature on (coalgebraic) modal logic. In particular, a predicate is used as the semantics of a logical formula [27], or as a relation on the state space of a coalgebra [10]. In the basic setting, when \( C = \mathbf{Set} \), the predicates on a set \( X \) are given by the subsets of \( X \). Now given a function \( f: X \rightarrow Y \) then a predicate \( V \) on \( X \) (i.e. \( V \subseteq Y \)) can be transformed into a predicate on \( X \) by the pullback operation \( f^{-1}V \subseteq X \) in \( \mathbf{Set} \). Note that this operation is functorial in nature; thus this ‘logical’ structure can be organised as a functor \( \mathbf{Set}^{\mathbf{op}} \rightrightarrows \mathbf{Cat} \) [13], where \( \hat{P}X \) is the poset \((P, \subseteq)\) viewed as a category. So as noted by Jacobs in [13], predicate logic on a category is given by an indexed category and predicate liftings are (endo)morphisms of indexed categories.

Definition 4. An indexed category is a \( \mathbf{Cat} \)-valued presheaf, i.e., a contravariant functor \( \Phi \) from \( C \) to \( \mathbf{Cat} \). In addition, a morphism between two indexed categories \( \mathbf{C}^{\mathbf{op}} \rightrightarrows \mathbf{Cat} \) and \( \mathbf{D}^{\mathbf{op}} \rightrightarrows \mathbf{Cat} \) is a pair of a functor \( C \rightrightarrows \mathbf{D} \) and a natural transformation \( \Phi \rightrightarrows \Psi \circ \mathbf{G}^{\mathbf{op}} \).

Notation 1. Often the application of \( \Phi \) on an arrow \( f \in C \) is denoted as \( f^* = \Phi f \). We also omit the use of superscript ‘\( \cdot \)’ on functors by writing them as contravariant functors. Moreover, we will use the phrases ‘indexed morphism’ and ‘predicate lifting’ interchangeably.

Remark 5. Another, but equivalent, way to organise logic is by specifying the fibrations of predicates over a category [11]. The transformation of a fibration over \( \mathbf{Cat} \) into a contravariant pseudofunctor \( \mathbf{C} \rightrightarrows \mathbf{D} \) is given by taking the fibres at each object in \( C \). Conversely one has to invoke the so-called Grothendieck construction to get a fibration, which glues all the fibres \( \Phi(X) \) into a total category of predicates \( \mathbb{E}(\Phi) \) defined as follows.

\[
\begin{align*}
X \in C \land U \subseteq \Phi X \quad \Rightarrow \quad (X, U) \in \mathbb{E}(\Phi) \\
(X, U) \rightrightarrows (Y, V) \in \mathbb{E}(\Phi)
\end{align*}
\]

Moreover, there is an obvious ‘forgetful’ functor \( \mathbb{E}(\Phi) \rightrightarrows C \) given by \( (X, U) \mapsto X \) that induces a (split) fibration on \( C \) [11] [13]. Often, in applications, the fibres \( \Phi(X, \subseteq) \) (at each \( X \in C \) form a poset (rather than a full-fledged category); we label such a indexed category/fibration as thin. We restrict ourselves to thin fibrations in this paper.

Example 6. The contravariant powerset functor \( \mathbf{Set}^{\mathbf{op}} \rightrightarrows \mathbf{Cat} \) is an example of an indexed category such that \( \mathbb{E}(\hat{P}) \rightrightarrows \mathbf{Set} \) is a bifibration [11]. This is because the reindexing functor \( f^{-1} \) (for any function \( f \)) has a left adjoint given by the direct image functor \( f_! \). Moreover, as an example of a predicate lifting, consider \( F = \mathcal{P} \) over \( C = \mathbf{Set} \) (which describes the branching type of unlabelled transition systems) with \( E \rightrightarrows \hat{P}FX \) given by \( U \mapsto \mathcal{P}U \). It is well known that the above predicate lifting encodes the box modality \( \square \) from logic [13].

3 Behavioural equivalence through indexed morphisms

Indexed morphisms not only induce modalities of interest in Computer Science; but, as we show now, they can also be used to characterise behavioural equivalence. The original idea (due to [10]) is to work with an indexed category (fibration) \( \mathbf{Set}^{\mathbf{op}} \rightrightarrows \mathbf{Cat} \) of binary relations, i.e., \( \Psi \) is given by the composition \( \mathbf{Set}^{\mathbf{op}} \rightrightarrows \mathbf{Set}^{\mathbf{op}} \rightrightarrows \mathbf{Cat} \). In particular, \( \Psi X \) is the set of all relations on \( X \). Then, for a relation lifting \( \Psi X \rightrightarrows \Psi FX \) and a given coalgebra \( X \rightrightarrows \alpha \rightrightarrows FX \in \mathbf{Set} \), bisimilarity can be seen as the largest fixpoint of the functional:

\[
\Psi X \rightrightarrows \Psi FX \rightrightarrows \Psi X.
\]
Unfortunately, this idea immediately does not generalise to coalgebras with side effects; e.g., in the case of conditional transition systems (CTSs) viewed as coalgebras living in the coKleisli category of the writer comonad $\text{Set} \xrightarrow{\times} \text{Set}$ (see Section 2). In particular, the problem lies in the definition of fibres by requiring it to be the set of all binary relations on the state space. There are situations (as in CTSs) where the fibres will only be some subset of all the relations on the state space. As a result, we impose the following restrictions:

A1 there is a functor $C \xrightarrow{\lambda} \text{Set}$ and a collection $\Psi X \xleftarrow{\Phi} \hat{\mathcal{D}}(|X| \times |X|)$ (for each $X \in C$) such that this collection is closed under $(|f| \times |f|)^{-1}$, i.e., for any $S \in \Psi Y$ and an arrow $X \xrightarrow{f} Y \in C$ we have $(|f| \times |f|)^{-1}S \in \Psi X$.

A2 the syntactic equality is expressible in $\Psi$, i.e., for every object $X \in C$ we have $\equiv_{|X|} \in \Psi X$.

Proposition 7. Under the assumptions A1 and A2, we have the following results.

1. the indexed collection $\Psi$ induces an indexed category $\text{Set}^{\text{op}} \xrightarrow{\lambda} \text{Cat}$.
2. there is a functor $C \xrightarrow{\text{Eq}} \mathcal{E}(\Psi)$ (henceforth called equality functor) that maps an object $X$ to the underlying syntactic equality $\equiv_{|X|} \in \Psi X$.

The next proposition (original from [13]) is a general result on indexed categories which is useful in lifting an endofunctor on $C$ to an endofunctor on the given fibration $E(\Phi)$. Moreover the category of coalgebras of the lifted endofunctor forms a fibration on the given category of coalgebras in which our original system of interest is modelled.

Proposition 8. The following statements hold for a given $C \xrightarrow{F} C$ and $\Phi \xrightarrow{\lambda} \Phi F$.

- The map $\lambda$ induces a map $E(\Phi) \xrightarrow{F_X} E(\Phi)$ of fibrations given by $(X, U) \mapsto (FX, \lambda_X U)$.
- The category of coalgebras induced by $F_X$ forms a fibration on $\text{Coalg}_{C}(F)$ (see the above diagram), where $\text{Coalg}_{C}(F)^{\text{op}} \xrightarrow{\Phi^F} \text{Cat}$ is the mapping: $(X, \alpha) \mapsto \text{Coalg}_{\Phi X}(\alpha^* \circ \lambda_X)$.

Now recall [11] and $\Psi$ as indexed category of relations on $\text{Set}$, an arbitrary bisimulation relation $R$ on a coalgebra $X \xrightarrow{\alpha} FX \in \text{Set}$ is the relation $R \in \Psi X$ satisfying $R \subseteq \alpha^* \lambda_X R$. In other words, bisimulation relations on the state space $X$ are again coalgebras of the functor $\alpha^* \circ \lambda_X$ living in the fibre $\Psi X$. Next we show that the same holds for behavioural equivalence in general, however, under the following assumptions:

- A3 the given morphism $\Psi \xrightarrow{\lambda} \Psi F$ preserves equalities, i.e., $\lambda_X(\equiv_{|X|}) = \equiv_{|FX|}$, for $X \in C$.
- A4 the functor $\text{Eq}$ has a left adjoint $Q$.

Remark 9. Assumption A4 already to model quotient types in the context of type theory. However, our usage is in the unit $\kappa$ of $Q \dashv \text{Eq}$ to construct a witnessing coalgebra homomorphism in Theorem 11. In the setting when $C = \text{Set}$, $Q$ maps an relation $R$ on $X$ to the quotient generated by the smallest equivalence containing $R$; the unit $\kappa_X$ (for any set $X$) is the usual quotient function mapping an element to its equivalence class.

Theorem 10. Let $\Psi \xrightarrow{\lambda} \Psi F$ be an indexed morphism. Under Assumptions A1, A2 and A3, the behavioural equivalence induced by a coalgebra homomorphism $f \in \text{Coalg}_{C}(F)$ on a coalgebra $(X, \alpha) \in \text{Coalg}_{C}(F)$ is a $\alpha^* \circ \lambda$-coalgebra living in the fibre $\Psi X$. 

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Theorem 11. Under Assumptions \( \mathbf{A1} \), \( \mathbf{A2} \), \( \mathbf{A3} \) and \( \mathbf{A4} \) for every \( \alpha^* \circ \lambda \)-coalgebra \( R \) there is a coalgebra map \( f \in \text{Coalg}_{\Psi}(F) \) such that \( R \preceq f^*(\text{cod}(f)) \), where \( \text{cod}(f) \) denotes the codomain of \( f \). Moreover, \( R = f^*(\text{cod}(f)) \) when the unit of \( \mathcal{Q} \vdash \mathcal{E} \) is Cartesian.

4 Coalgebraic modal logic

The ‘partial’ characterisation of behavioural equivalence as a fibred notion (cf. Theorems \( \mathbf{10} \) and \( \mathbf{11} \)) enable us to use the dual (contravariant) adjunction framework of \( \mathbf{23} \) (depicted in \( \mathbf{6} \)) to develop a logical characterisation of behavioural equivalence. It should be noted that, although, this framework can handle behavioural preorders and distances, but we prove our results only for behavioural equivalence (i.e. in the context of Assumptions \( \mathbf{A1} \) and \( \mathbf{A2} \)).

Lemma 12. Suppose \( \mathcal{C}^{\text{op}} \xrightarrow{\Psi} \text{Cat} \) has indexed final objects (i.e., the final object exists in each fibre \( \Phi X \) for \( X \in \mathcal{C} \)) and the reindexing functor \( f^* \) preserves these final objects. Then there is a functor \( \mathcal{C} \xrightarrow{1} \mathcal{E}(\Phi) \) that is right adjoint to \( p \).

Example 13. Consider the indexed category \( \Psi \) induced by binary relations on sets and a labelled transition system modelled as a coalgebra \( \xrightarrow{\alpha^* \circ \lambda} (PX)^{\text{Act}} \), i.e., our \( \mathcal{C} = \mathbf{Set}, F = (\mathbf{P})^{\text{Act}} \). Consider the function \( \Psi X \xrightarrow{\lambda_X} \Psi FX \) (below \( q, q' \in (\mathbf{P})^{\text{Act}} \)):

\[
q \lambda_X R q' \iff \forall a, x \exists x' (x \in qa \implies x' \in q'a) \land \forall a, x' \exists x (x' \in q'a \implies x \in qa).
\]

Then, as first observed in \( \mathbf{10} \), a bisimulation relation \( R \) on \( X \) is a coalgebra \( \xrightarrow{\alpha^* \lambda_X} \Psi X \). Moreover, bisimilarity \( \equiv_X \) (the largest bisimulation relation) on \((X, \alpha)\) then corresponds to the terminal object living in the fibre \( \Psi_X^*(X, \alpha) \), i.e., in particular \( \equiv_X \subseteq (\alpha \times \alpha)^{-1} \lambda_X \equiv_X \).

As for the dual adjunction \( \mathcal{S} \vdash \mathcal{T} \) in \( \mathbf{6} \), it provides a connection (cf. \( \mathbf{23} \)) between states and theories (the formulae satisfied by a state). The syntax of logic is given by a functor \( \mathcal{A} \xrightarrow{L} \mathcal{A} \) and it

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\(^1\) The meaning of \( \mathcal{T} \) will be clarified later.
is assumed that the initial algebra exists \( L A \xrightarrow{h} A \in A \) for \( L \), which models the typical Lindenbaum algebra induced by the term algebra. Lastly, the natural transformation \( \delta \) gives the one-step interpretation to the formulae which can be given its mate \( \theta \) as described below (such a result is probably folklore cf. [16, Proposition 2]).

**Proposition 14.** Given \( E \xrightarrow{F} C, A \xrightarrow{L} A \), and \( C \xrightarrow{\tau} A^{op} \) with \( S \dashv T \), there is a correspondence between the natural transformations of types: \( F T \xrightarrow{\delta} T L \) and \( S F \xrightarrow{\theta} L S \).

Now given a coalgebra \( X \xrightarrow{\alpha} F X \in C \), the semantics \( A \xrightarrow{\_ \_} S X \) of logic \((L, \delta)\) is given by the universal property of the initial algebra \( L A \xrightarrow{h} A \). In particular, it is the unique arrow in \( A \) that makes the following diagram (drawn on the left) commutative.

\[
\begin{array}{ccc}
L A & \xrightarrow{\underline{\_ \_}} & LS X \\
\downarrow h & & \downarrow \theta X \\
A & \xrightarrow{\_ \_} & SX
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{\underline{\_ \_}} & T A \\
\downarrow \alpha & & \downarrow T h \\
F X & \xrightarrow{\underline{\_ \_}} & F T A
\end{array}
\]

And the transpose of the semantics map \( \underline{\_ \_} \) under \( S \dashv T \) gives a ‘theory’ map \( X \xrightarrow{\underline{\_ \_}} Q A \); it is the unique arrow in \( C \) that makes the above diagram on the right commutative.

Once these niceties are set up, one can argue when a logic \((L, \delta)\) is adequate and expressive. The formulation below is a straightforward generalisation of adequacy and expressivity given in [25] to the case of non-thin indexed categories.

**Definition 15.** Suppose the behavioural conformance functor \( \Lambda^\lambda \) exists (for some \( \lambda \)) with \( A^{op} \xrightarrow{\_ \_} E(\Psi) \) such that \( p \circ \_ = \_ \_ \). Then a logic \((L, \delta)\) is adequate (resp. expressive) w.r.t. \( \_ \_ \) if \( (X, 1_X) \xrightarrow{\underline{\_ \_}} \_ \_ \) is a (resp. Cartesian) map in \( E(\Psi) \), for every coalgebra \( X \xrightarrow{\alpha} F X \in C \).

The role of \( \_ \_ \) is to encode a relationship between the theories of any two states (cf. [25]); so we let \( \_ \_ = \text{Eq} \circ \_ \_ \) in the context of behavioural equivalence (i.e. when our working indexed category is \( \Psi \) due to Assumption \( A1 \)). Next we state the main result of this section, which is a refinement of adequacy and expressivity results given in [25].

**Theorem 16.** Under the assumptions of Theorem 10, if \( \_ \_ \) has a left adjoint \( \_ \_ \) (see below on the left), the logic \((L, \delta)\) is adequate. Moreover it is expressive if \( |\alpha| \) is injective.

In short, the Kupke-Rot logical setup for behavioural equivalence can be summarised as drawn below on the left. Now if our indexed category \( \Psi \) satisfies \( A4 \) (like in the case of coKleisli and Eilenberg-Moore categories), then \( \_ \_ = \_ \_ \circ \_ \_ \) as indicated below on the right.

However, in Kleisli categories we will construct \( \_ \_ \) under some restrictions (cf. Theorem 19).

\[
\begin{array}{ccc}
\text{E}(\Psi) & \xrightarrow{\_ \_} & A^{op} \\
\downarrow \text{Eq} \quad \quad \quad \downarrow \_ \_ \\
C & \xrightarrow{\_ \_} & A^{op}
\end{array}
\]

\[
\begin{array}{ccc}
\text{E}(\Psi) & \xrightarrow{\_ \_} & A^{op} \\
\downarrow \text{Eq} \quad \quad \quad \downarrow \_ \_ \\
\_ \_ & \xrightarrow{\_ \_} & A^{op}
\end{array}
\]
underlying base category \( \text{Set} \) (see the next paragraph). In a similar spirit, our aim is to construct indexed morphisms on a Kleisli category by lifting an indexed morphism on \( \text{Set} \). Another aim of this section is to prove the existence of \( \hat{S} \) (which is a sufficient condition for both adequacy and expressivity in Kupke-Rot setup) when \( C \) is a Kleisli category.

So given a monad \( \text{Set} \xrightarrow{T} \text{Set} \), a Kleisli extension \( \text{Kl}(T) \xrightarrow{\Phi} \text{Kl}(T) \) of \( \text{Set} \xrightarrow{F} \text{Set} \) (i.e., \( \bar{F} \circ \iota = \iota \circ F \)) induced by a distributive law \( FT \xrightarrow{\sigma} TF \), then our indexed categories modelling predicates and relations, denoted by \( \Phi \) and \( \Psi \), are given as follows: \( \Phi = \bar{\Phi} \circ | \_ | \) and \( \Psi \) is the composition \( \text{Kl}(T) \xrightarrow{\Phi} \text{Set} \xrightarrow{\times} \text{Set} \) \( \xrightarrow{\Phi} \text{Cat} \). Clearly \( \Psi \) satisfies Assumptions \( A1 \) and \( A2 \) thus there is an equality functor \( \text{Kl}(T) \xrightarrow{E_{\Phi}} \text{E}(\Psi) \).

Unfortunately, arbitrary (co)limits in general do not exist in a Kleisli category. For instance, one of our working categories \( \text{Kl}(P) \cong \text{Rel} \) (the category of sets and relations) does not have all coequalisers, but \( \text{Rel} \) has a reflective subcategory \( \text{Set} \) that does. The presence of these coequalisers in the reflective subcategory will then be used to construct \( \hat{S} \).

**Definition 17.** A subcategory \( B \xrightarrow{j} C \) is reflective when the inclusion functor \( j \) has a left adjoint \( \iota \) (often called as reflector).

**Theorem 18** ([1]). Let \( B \xrightarrow{j} C \) be a reflective subcategory of \( C \). If \( C \xrightarrow{F} \text{Cat} \) preserves \( B \), i.e., \( \forall_{B,F \in B} (FB \in B \land Fj \in B) \) and \( Fj = j \circ F \), then we have a diagram \( \text{Coalg}_{B}(F) \xrightarrow{\iota} \text{Coalg}_{C}(F) \) with \( \bar{\iota} \dashv j \). Here, \( j \) is the obvious inclusion.

The reflector \( \bar{\iota} \) typically results in a form of (on-the-fly) determinisation (cf. Example 26). Moreover, in our case studies, these reflective subcategories will also take the place of algebras in \([5]\), and if these reflective subcategories has coequalisers then we can construct \( \hat{S} \) in general. So let \( B = A^{\text{op}}, S = \iota, T = j, \) and \( (X, R) \in \text{E}(\Psi) \). Then the idea is to use the following series of transformations (depicted below on the left) to construct \( \hat{S} \) as the equaliser of the parallel arrows \( S\bar{p}'_1, S\bar{p}'_2 \in A \). Below \( p_i \) (for \( i \in \{1, 2\} \)) are the obvious projection functions and each \( \bar{p}'_i \) is the transpose of \( p_i \) under the free-forgetful adjunction \( \iota \dashv | \_ | \).

\[
\begin{array}{ccc}
R \xrightarrow{p_1} |X| \in \text{Set} & \xrightarrow{\iota R} X \in \text{Kl}(T) & \xrightarrow{\bar{S}(X, R)} \hat{S}(X, R) \\
\xrightarrow{p_2} & \xrightarrow{X \in \text{Kl}(T)} & \xrightarrow{\bar{S}(X, R)} \hat{S}(X, R) \\
\iota R \xrightarrow{\bar{p}'_1} X \in \text{Kl}(T) & \xrightarrow{\bar{S}(X, R)} \hat{S}(X, R) & \xrightarrow{\bar{S}(Y, S)} \hat{S}(Y, S) \\
\xrightarrow{\bar{p}'_2} S\bar{p}'_2 & \xrightarrow{\bar{S}(X, R)} \hat{S}(X, R) & \xrightarrow{\bar{S}(Y, S)} \hat{S}(Y, S)
\end{array}
\]

Let \( (\bar{S}(X, R), e) \) be the equaliser of \( \bar{p}'_1 \) in \( A \). Now \( (X, R) \xrightarrow{f} (Y, S) \in \text{E}(\Psi) \) means that \( X \xrightarrow{f} Y \in \text{Kl}(T) \) and \( R \subseteq ([f] \times [f])^{-1} S \). So there is a function \( \bar{R} \xrightarrow{\bar{f}} S \) such that \( [f] \circ p_1 = q_1 \circ g \) with \( p_i, q_i \) being the obvious projections when the relations \( R, S \) are viewed as spans in \( \text{Set} \). Moreover \( f \circ \bar{p}'_1 = q'_1 \circ \bar{g} \) due to the naturality of the counit of \( \iota \dashv | \_ | \). So the two squares in \([6]\) commute and the universal property of equalisers gives the unique \( \bar{S}f \).

**Theorem 19.** Let \( A^{\text{op}} \) be a reflective subcategory of \( \text{Kl}(T) \) having coequalisers. Then the above defined map \( \text{E}(\Psi) \xrightarrow{\bar{S}} A^{\text{op}} \) is a functor and left adjoint to \( T = \text{Eq} \circ \iota \).

**Lifting of indexed morphisms in Kleisli categories**

Next we give a recipe to construct a predicate lifting of type \( \Phi \xrightarrow{\lambda} \Phi \bar{F} \); note \( \Phi = \bar{\Phi} \circ | \_ | \). In particular, we need an endofunctor \( \text{Set} \xrightarrow{G} \text{Set} \), a predicate lifting \( \bar{\Phi} \xrightarrow{\sigma} \bar{\Phi}G \), and a natural transformation \( \gamma \) as indicated left on the next page.
Then a natural transformation (commutes) induces a distributive law as follows from the current game instance (i.e., each \( \Lambda = \Psi \) is the indexed category of all binary relations on the underlying set, i.e., \( \Lambda = \Psi \)).

A straightforward calculation using the naturality of \( \sigma \) and \( \gamma \) gives the following result.

**Theorem 20.** The above mapping \( \lambda \) is an indexed morphism.

Note that this technique (depicted in the square drawn above on the left) also works in the case of coKleisli (cf. Section 7] and Eilenberg-Moore (cf. Section 13] categories. We simply let \( G = F \) and \( \tilde{F} \) be a coKleisli extension/Eilenberg-Moore lifting of \( F \), which results in a distributive law of type \( |\tilde{F}| \rightarrow G |\_|; \) in the case of Eilenberg-Moore categories, such natural transformations are also known as EM-law [15].

In the case of Kleisli categories, the situation is slightly complicated. This is because the distributive law \( FT \rightarrow T \tilde{F} \) (which induces a Kleisli extension \( \tilde{F} \) of \( F \)) results in a natural transformation in the ‘wrong’ direction \( F |\_| \rightarrow |\tilde{F}| \). However, in various applications, \( G \) is typically associated with the branching type of a determinstic version of the corresponding system of interest (such as \( G = A \times X \) in the case of NDA) exists. The next result helps in finding such a distributive law \( \gamma \) for a given \( G \) in a more elementary way.

**Lemma 21.** Let \( \tilde{F} \) be a Kleisli extension of \( F \) induced by a distributive law \( FT \rightarrow T \tilde{F} \). Then a natural transformation \( \tilde{F} \rightarrow GT \tilde{F} \) compatible with \( \vartheta \) and \( \mu \) (i.e., Square 10 commutes) induces a distributive law \( |\_| \circ \tilde{F} \rightarrow G |\_|. \) Moreover, the converse also holds.

**Remark 22.** Note that the compatibility property in the previous lemma already appeared in [15] as part of an ‘extension’ natural transformation. In short, the properties of an extension natural transformation are more stringent than of Lemma 21.

Lastly, this technique can also be used to construct a relation lifting of type \( \Psi \rightarrow \Psi \tilde{F} \) since \( \Psi \) is the indexed category of all binary relations on the underlying set, i.e., \( \Psi X = \tilde{P}([X] \times [X]) \).

So given a relation lifting \( \tilde{P}(X \times X) \rightarrow \tilde{P}(GX \times GX) \) of \( G \), then we can create \( \Psi \rightarrow \Psi \tilde{F} \) by extending the following:

\[
\Psi X = \tilde{P}(([X] \times [X])) \rightarrow \tilde{P}(G([X] \times [X])) \rightarrow \tilde{P}(([\tilde{F}X] \times [\tilde{F}X])) = \Psi \tilde{F}X.
\]  

**Coalgebraic games in Kleisli categories**

Yet another application of indexed morphisms (predicate liftings) is in the game-theoretic characterisation of behavioural equivalence. In this section, we will adopt the two player game of [24] designed for coalgebras living in \( \text{Set} \) to \( \text{Kl}(T) \) (for some functor \( \text{Set} \rightarrow \text{Set} \)). Recall that \( \Phi = \tilde{P} |\_| \) just like in the previous subsections.

**Definition 23.** Given a coalgebra \( X \rightarrow \tilde{F}X \) in \( \text{Kl}(T) \), a set of chosen predicate liftings \( \Lambda \) (i.e., each \( \lambda \in \Lambda \) is of type \( \Phi \rightarrow \Phi \tilde{F} \)), and two states \( x, x' \in [X] \), then the game works as follows from the current game instance \((x,x')\):
1. Spoiler chooses a state $s \in \{x, x'\}$ and a predicate $U \in \Phi X$. 
2. Duplicator picks the remaining state $t$ (i.e., $t = x$ if $s = x'$ or $t = x'$ if $s = x$) and a predicate $U' \in \Phi X$ such that: $\alpha(s) \in \lambda_X U \Rightarrow \alpha(t) \in \lambda_X U'$ for all $\lambda_X \in \Lambda$.
3. Spoiler chooses a predicate $\bar{U} \in \{U, U'\}$ and a state $y \in |C|$ such that $y \in \bar{U}$.
4. Duplicator chooses a state $y' \in |C|$ with the remaining predicate $\bar{U}'$ such that $y' \in \bar{U}'$.

The game instance changes to $(y, y')$. $D$ wins the game if the game continues forever or if $S$ cannot perform Step 3. $S$ wins the game whenever $D$ has no moves at Step 2 or Step 4.

It is not hard to show that the defender has a winning strategy from every pair of behaviourally equivalent states (cf. Theorem 25). For the converse, we argue with the condition of separability found in coalgebraic modal logic [27, 30] and the ‘game’ equivalence relation $\equiv \subseteq |X| \times |X|$ defined as: $x \equiv x'$ if $D$ has a winning strategy from the instance $(x, x')$. Now we reuse the coequaliser construction (used to define $\bar{S}$ in Theorem 19) to find a coalgebra homomorphism from the game equivalence (instead of logical equivalence).

**Definition 24.** A given set $\Lambda$ of predicate liftings is separable w.r.t. $\bar{F}$ if, and only if, for any $x, x' \in |\bar{F}X|$ we have $\forall U \in \Phi X, \lambda \in \Lambda \ (t \in \lambda_X U \iff t' \in \lambda_X U) \Rightarrow t = t'$.

In Appendix A we prove the following statement in more general terms (cf. Theorem 12), which allow us to derive the game characterisation even for coalgebras in Eilenberg-Moore categories. Nevertheless, it should be noted that this general theorem is not applicable for coKleisli categories and as such the game characterisation is left open for the future.

**Theorem 25.** Given a reflective subcategory $A^{op} \subseteq \bar{Kl}(T)$ having coequalisers (as in Theorem 19) and $\bar{F}$ is separable w.r.t. $\Lambda$, then behaviourally equivalent states are exactly those from which the defender has a winning strategy.

# 6 Nondeterministic automata (NDA)

Recall the necessary parameters from Example 3 for coalgebraic modelling of an NDA, i.e., $T = \mathcal{P}$, $C = Kl(\mathcal{P}) \cong \mathbf{Rel}$, $F = \text{Act} \times \underline{+} + 1$, the distributive law $\vartheta$ given in [3], and the free-forgetful adjunction $\iota \dashv \underline{\_}$ associated with any Kleisli category. To apply Theorem 19 we first recall the reflective subcategory $\mathbf{Set}^{op}$ of $\mathbf{Rel}$ from [1].

As an example, consider the functors $j$ and $\tau$ (below $X \xrightarrow{f} Y \in \mathbf{Set}$ and $X \xrightarrow{g} Y \in \mathbf{Rel}$):

1. $jX = X$ $\quad Y \xrightarrow{f} X \in \mathbf{Rel} \quad \Rightarrow \quad y \ j \ f \ x \iff f \ x = y$
2. $\tau X = \mathcal{P} \ X$ $\quad \tau Y \in \mathbf{Set}^{op}$ $\quad \Rightarrow \quad \tau g (V) = \{x \mid \exists y \in V \ x \ y \}$.

Next we illustrate the definition of $\bar{S}$ and how the unit of $\bar{S} \dashv \bar{T}$ maps an NDA to the largest subautomaton (respecting language equivalence) obtained after backward determinisation of the given NDA.

**Example 26.** Consider the NDA drawn above on the right with the accepting state $z$ as a coalgebra $X \xrightarrow{\alpha} \text{Act} \times + 1 \subseteq \mathbf{Kl}(\mathcal{P})$. Logical equivalence $\equiv$ is the least equivalence that equates $\{x, y\}$ with $\{y\}$ (both accept the language $\{a, b\}$) and $\{x, y, z\}$ with $\{y, z\}$ (both accept the language $\{a, b, \epsilon\}$). Also, for any $U \cap U'$ we have $(U, U') p_1' x \iff x \in U$ and $(U, U') p_2' x \iff x \in U'$. So the coequaliser $\bar{S}(X, \equiv)$ of $\tau f'$ can be computed as follows:

$$\bar{S}(X, \equiv) = \{W \in \mathcal{P} \ X \mid \bar{\tau} p'_1 W = \bar{\tau} p'_2 W\} = \{W \mid \forall_{U, V} (U \cap W \neq \emptyset \land U \equiv V) \Rightarrow V \cap W \neq \emptyset\}.$$
The arrow $\tilde{S}(X, \equiv) \xrightarrow{\beta} F\tilde{S}(X, \equiv) \in \text{Set}^{\text{op}}$ is defined by the following (depicted on the left) universal property of equaliser in $\text{Set}$. Here, $\tilde{\omega}$ is the backward determinisation of the given coalgebra (as described, e.g. in [1] as a deterministic automaton accepting the reverse language), i.e. it maps $(a, U) \mapsto \{x \mid \exists x' \in U (a, x') \in \alpha(x)\}$ and $\bullet \mapsto \{x \mid \bullet \in \alpha(x)\}$. Thus, in essence, $\beta$ acts like $\tilde{\omega}$ on the elements of $\tilde{S}(X, \equiv)$.

\[
\begin{align*}
\tilde{S}(X, \equiv) & \xrightarrow{e_X} \mathcal{P}X \xrightarrow{\xi_1} \mathcal{P}X \\
& \xrightarrow{\beta} F\tilde{S}(X, \equiv) \xrightarrow{F\xi_1} F\mathcal{P}X
\end{align*}
\]

In this example, we obtain as $\beta$ the automaton drawn above on the right with six states. The relation $\kappa_X$ indicated by dotted line is the transpose of $e_X$ w.r.t. $\tau \dashv \eta$; concretely, $x \kappa_X U \iff x \in U$. Furthermore $\kappa_X \in \text{Rel}$ is a witnessing coalgebra homomorphism because $\tilde{F}(\kappa_X) \circ \alpha = \beta \circ \kappa_X$. Note that $[\kappa_X]$ maps both $\{x, y\}, \{y\}$ to $\{\{x, y\}, \{y\}, \{y, z\}, \{x, y, z\}\}$, witnessing the fact that they are language equivalent. Hence, the coequaliser gives us the largest sub-automaton of the backwards determinisation that respects logical equivalence (removing $\emptyset$, $\{y, z\}$, and $\{x, y, z\}$ will result in smallest such sub-automaton).

**Predicate liftings for NDAs**

To apply techniques from Section 5, we set $G = \_\text{Act} \times 2$ and define $\gamma$ as follows:

\[
\mathcal{P}(\text{Act} \times X + 1) \xrightarrow{\gamma_X} (\mathcal{P}X)^{\text{Act} \times 2} \xrightarrow{\hat{U} \mapsto (\gamma_X^{\text{Act}} \hat{U}, \gamma_X^2 \hat{U})},
\]

where $\gamma_X^{\text{Act}} \hat{U}(a) = \{x \mid (a, x) \in \hat{U}\}$ and $\gamma_X^2 \hat{U} = 1 \iff \bullet \in \hat{U}$.

Moreover, from [15] we know that $\gamma$ is compatible with $\theta$ and $\bigcup$ in the sense of Lemma 21. Next consider the family of liftings $\hat{\mathcal{P}} X \xrightarrow{\sigma^\_\text{Act}} \hat{\mathcal{P}}(\mathcal{P}X)^{\text{Act} \times 2}$ (for each $a \in \text{Act}$) and $\hat{\mathcal{P}} X \xrightarrow{\sigma^\_\text{Act}} \hat{\mathcal{P}}(\mathcal{P}X)^{\text{Act} \times 2}$: $U \mapsto \{(p, b) \in X^{\text{Act} \times 2} \mid p(a) \in U\}$ and $U \mapsto \{(p, 1) \mid p \in X^{\text{Act}}\}$, respectively.

**Lemma 27.** The above mappings $\sigma^\_\text{Act}$ (for $a \in \text{Act}$) and $\sigma^\_\text{Act}$ are indexed morphisms.

Thanks to Theorem 20 $\lambda^a = \gamma^{-1} \circ \sigma^a, \lambda^1 = \gamma^{-1} \circ \sigma^1$ are valid predicate liftings for the functor $\text{Rel} \xrightarrow{\text{Act} \times X + 1} \text{Rel}$. Moreover, for any $U \subseteq \mathcal{P}X$ we find (below $\hat{U} \subseteq X^{\text{Act} \times X + 1}$):

\[
\begin{align*}
\lambda^X(\hat{U}) &= \gamma^{-1} \sigma^X(\hat{U}) \\
&= \gamma_X^{-1}(\{p, b\} \in (\mathcal{P}X)^{\text{Act} \times 2} \mid p(a) \in \hat{U}\) \\
&= \{\hat{U} \mid \{x \mid (a, x) \in \hat{U}\} \subseteq \hat{U}\}
\end{align*}
\]

The above indexed morphisms $\lambda^a, \lambda^1$ induce modalities that can be interpreted on determinised automata. Given an NDA $(X, \text{Act}, \rightarrow, \downarrow)$ with $\downarrow \subseteq X$ being the termination predicate (or $X \xrightarrow{\alpha} \mathcal{P}(\text{Act} \times X + 1) \in \text{Set}$), then the determinised automaton has state space $\mathcal{P}X$ with dynamics given by the SOS rules or abstractly by the composition $\gamma_X \circ \bigcup \circ \mathcal{P}\alpha$:

\[
\begin{align*}
U \subseteq X & \quad U_{\alpha} = \{x' \mid \exists x \in U \ x \xrightarrow{\alpha} x'\} \\
U \xrightarrow{\alpha} U_{\alpha} & \quad U \subseteq X & \quad \exists x \in U \ x \downarrow
\]
\]
In turn, we can now rewrite the two modalities to a simpler form:
\[
|\alpha|^{-1} \lambda^\alpha_X U = |\alpha|^{-1} \{ \tilde{U} \mid \{ x \mid (a, x) \in \tilde{U} \} \in \mathbb{U} \} = \{ U \mid U \xrightarrow{\alpha} U_{\alpha} \implies U_{\alpha} \in \mathbb{U} \} = \{ U \mid U \downarrow \}. 
\]
For NDAs the predicate liftings that we derived are indeed separating.

\textbf{Lemma 28.} The set $\Lambda = \{ \lambda^a \mid a \in \text{Act} \} \cup \{ \lambda^1 \}$ is separating w.r.t. $A \times X + 1$.

\textbf{Example 29.} Here we illustrate how the coalgebraic game based on predicate liftings works by recalling the automaton given in Example 26. Consider the initial situation $\{ \{ x \}, \{ x, y \} \}$ (notice the concrete state space is the set $\mathcal{P}\{ x, y, z \}$) and $S$ chooses $\{ x, y \}$ with $U = \{ \{ z \} \}$, so we have $|\alpha|\{ x, y \} = \{ (a, z), (b, z) \} \in \lambda^X_\mathcal{X}(\{ \{ z \} \})$. To answer this move $D$ chooses $\{ x \}$ and $U' = \{ \emptyset, \{ z \} \}$ with $|\alpha|\{ x \} = \{ (a, z) \} \in \lambda^X_\mathcal{X}(\{ \emptyset, \{ z \} \})$. Note here, that the essential component of $U'$ is the $\emptyset$, since no $b$-transition is observable from $\{ x \}$. Next $S$ chooses $\emptyset, U'$ and so $D$ can only go on with $\{ z \}$. Therefore, in Step 2 of the next round $S$ selects $\{ z \}$ with $|\alpha|\{ z \} = \{ \bullet \}$ and wins with any predicate due to $\bullet$. $D$ is unable to find a valid predicate $U'$ for $|\alpha|\emptyset = \emptyset$ satisfying $\emptyset \in \lambda^X_\mathcal{X}(U')$.

\textbf{Language equivalence through relation lifting}

Just like in the case of predicate lifting, we can create a relation lifting of $\tilde{F}$ from the following relation lifting $\hat{\mathcal{P}}(X \times X) \xrightarrow{\hat{\delta}_X} \hat{\mathcal{P}}(GX \times GX)$ of $G$ (below $R \subseteq X \times X$):
\[
(p, b) \hat{\delta}_X R (p', b') \iff b = b' \land \forall a \in \text{Act} \ R \ p'a.
\]

\textbf{Lemma 30.} The mapping $\hat{\lambda}^X$ defined above is an indexed morphism.

So the mapping $\lambda^X \xrightarrow{\hat{\lambda}^X} \Psi \tilde{F}$ given in (12) is an indexed morphism. In more concrete terms, $\lambda^X$ maps a relation $R \subseteq \mathcal{P}X \times \mathcal{P}X$ to a relation $\lambda^X_R \subseteq \mathcal{P}FX \times \mathcal{P}FX$ defined as:
\[
\hat{U} \lambda^X_R \hat{U}' \iff (\bullet \in \hat{U} \iff \bullet \in \hat{U}') \land \left( \forall a \in \text{Act} \ R \{ x \mid (a, x) \in \hat{U} \} \right).
\]

\textbf{Lemma 31.} The indexed morphism $\hat{\lambda}$ preserves arbitrary intersections at each component; therefore, so does the predicate lifting $\hat{\lambda}$. Moreover, $\hat{\lambda}$ satisfies Assumption A3.

\textbf{Theorem 32.} Let $X \xrightarrow{\alpha} FX \in \text{Rel}$ be an NDA. Then language equivalence $\equiv_X \subseteq \mathcal{P}X \times \mathcal{P}X$ on the determinised system is a $\Psi(\alpha) \circ \hat{\lambda}^X$-coalgebra, i.e., $\equiv_X \subseteq (|\alpha| \times |\alpha|)^{-1}(\hat{\lambda}^X_X \equiv_X)$.

Moreover, $\equiv_X = (|f| \times |f|)^{-1} \equiv_Y$ for any coalgebra homomorphism $(X, \alpha) \xrightarrow{f} (Y, \beta)$; thus, there is a behavioural conformance functor $\text{Coalg}_{\text{Rel}}(\tilde{F}) \xrightarrow{\lambda^X} \text{Coalg}_{\text{E}}(\hat{F}_\lambda)$.

\textbf{Logical characterisation of language equivalence}

Recall the adjoint situation $\tau \dashv j$ that witnesses $\text{Set}^{op}$ is a reflective subcategory of $\text{Rel}$. We use this dual adjunction to model our logic because (intuitively) conjunction is not needed to characterise language equivalence. Thus we fix $A = \text{Set}$, $S = \tau$, and $T = j$. Moreover, a left adjoint $\hat{S}$ of $\tilde{F}$ exists due to Theorem 19.

Since to establish language equivalence one needs to ascertain whether a word in $\text{Act}^*$ is accepting or not, so we take our syntax functor $L = \text{Act} \times _+ 1$. Note that the initial algebra of $L$ exists and is given by $A = \text{Act}^*$. As for the one-step semantics given by a natural transformation $\delta$, we are going to define it (indirectly) by defining its mate $LSX = \text{Act} \times \mathcal{P}X + 1 \xrightarrow{\delta_X} \mathcal{P}(\text{Act} \times X + 1) = SFX \in \text{Set}$ that acts on objects like the distributive law $\delta_X$ (see (3)). Note that, however, they differ in their naturality conditions.
We next consider conditional transition systems (CTSs) [1], strongly related to the featured work (see Section C in the appendix).

The algebra Corollary 34. for CTSs without upgrades in the following way (below

Now given a coalgebra $\sigma$ indexed morphism once we have fixed the predicate lifting is given by the composition in (11) and, moreover, Theorem 20 ensures that the state to the set of traces accepted by it (below $\Rightarrow$ is the reachability relation):

$$\langle x \rangle = \{ w \in \text{Act}^* \mid \exists x', x \xrightarrow{w} x' \wedge x' \downarrow \} \quad \text{(for each } x \in X \} .$$

Corollary 34. The above defined mapping $\theta$ is a natural transformation.

The algebra $\text{Act} \times \text{Act}^* + 1 \xrightarrow{h} \text{Act}^*$ is given by the unary concatenation of words and the constant $\varepsilon$ (i.e., $h(a, w) = aw$ and $h \cdot = \varepsilon$). Consider the map $X \xrightarrow{j} \mathcal{A} \in \text{Rel}$ that maps a state to the set of traces accepted by it (below $\Rightarrow$ is the reachability relation):

$$\langle x \rangle = \{ w \in \text{Act}^* \mid \exists x', x \xrightarrow{w} x' \wedge x' \downarrow \} \quad \text{(for each } x \in X \} .$$

An analogous approach can be chosen to handle linear weighted automata in this framework (see Section C in the appendix).

### 7 Conditional transition systems: an application in coKleisli categories

We next consider conditional transition systems (CTSs) [1], strongly related to the featured transition systems used for modelling software product lines [7]. Here we consider the simpler case of conditional transition systems without upgrades and action labels; their full treatment [2, 3] is left for the future. In our earlier work, CTSs were coalgebras living in the Kleisli category induced by the reader monad [2] and, moreover, Theorem 20 ensures that it is both adequate and expressive for language equivalence on determinised systems.

Conditional transition systems used for modelling software product lines [7]. Here we consider the simpler case of conditional transition systems without upgrades and action labels; their full treatment [2, 3] is left for the future. In our earlier work, CTSs were coalgebras living in the Kleisli category induced by the reader monad [2] and, moreover, Theorem 20 ensures that it is both adequate and expressive for language equivalence on determinised systems.

Consider the coKleisli category $\mathbf{coKl}(\mathcal{G})$ whose objects, just like in any Kleisli category, are sets; an arrow $X \xrightarrow{f} Y$ corresponds to a function $GX \xrightarrow{Ff} FY$. Now there is a forgetful functor $\mathbf{coKl}(\mathcal{G}) \xrightarrow{\mathcal{L}} \text{Set}$; however, in contrast to the Kleisli setting, it is now left adjoint to the inclusion $\text{Set} \xrightarrow{\mathcal{J}} \mathbf{coKl}(\mathcal{G})$. Concretely, this forgetful functor maps an object $X \xrightarrow{\mathcal{L}} GX$ and an arrow $X \xrightarrow{f} Y \in \mathbf{coKl}(\mathcal{G})$ to a function $[f]$ mapping $(k, x) \mapsto (k, f(k, x))$.

Next to model the branching type of CTSs, take $\lambda \in \mathbf{coKl}(\mathcal{G})$ whose objects, just like in any Kleisli category, are sets; an arrow $X \xrightarrow{\lambda} Y$ corresponds to a function $GX \xrightarrow{\lambda} FY$. Now there is a forgetful functor $\mathbf{coKl}(\mathcal{G}) \xrightarrow{\mathcal{L}} \text{Set}$; however, in contrast to the Kleisli setting, it is now left adjoint to the inclusion $\text{Set} \xrightarrow{\mathcal{J}} \mathbf{coKl}(\mathcal{G})$. Concretely, this forgetful functor maps an object $X \xrightarrow{\mathcal{L}} GX$ and an arrow $X \xrightarrow{\lambda} Y \in \mathbf{coKl}(\mathcal{G})$ to a function $[\lambda]$ mapping $(k, x) \mapsto (k, f(k, x))$.

As mentioned earlier in Section 5, it is easier to lift a predicate lifting $\bar{\mathcal{P}} \xrightarrow{\sigma} \bar{\mathcal{P}}F$ of $F$ than in the Kleisli case to define a predicate lifting $\bar{\mathcal{P}}|_{\mathcal{K} \times X} \xrightarrow{\lambda} \bar{\mathcal{P}}|_{\mathcal{K} \times X}F$ of $F$. In particular, $\lambda$ is given by the composition in (11) and, moreover, Theorem 20 ensures that $\lambda$ is indeed an indexed morphism once we have fixed the predicate lifting $\sigma$ of $F$. To this end, we simply take $\sigma$ that corresponds to box modality (cf. Example 5). To answer whether these definitions give the right kind of ‘box’ modality for CTSs, let us first instantiate $\lambda_X$ for any $U \subseteq \mathcal{K} \times X$:

$$\lambda_X U = \gamma_X^{-1} U = \gamma_X^{-1} \{ U' \mid U' \subseteq U \} = \{(k, U') \mid \{ k \} \times U' \subseteq U \} .$$

Now given a coalgebra $\mathcal{K} \times X \xrightarrow{\alpha} \mathcal{P}X \in \text{Set}$, we derive the interpretation of box modality for CTSs without upgrades in the following way (below $x \xrightarrow{\alpha} x' \iff x' \in \alpha xk$):

$$|\alpha|^{-1} \lambda_X U = \{(k, x) \mid \forall x' \ x \xrightarrow{\alpha} x' \Rightarrow (k, x') \in U \} .$$
Conditional bisimilarity through relation lifting

Definition 35. Given a CTS $X \xrightarrow{\rho} \hat{\mathcal{P}}X \in \text{coKl}(G)$, a conditional bisimulation is a relation $\mathcal{R} \subseteq (\mathbb{K} \times X) \times (\mathbb{K} \times X)$ satisfying the following transfer conditions:

1. $\forall_{k,k' \in \mathbb{K}, x,x' \in X} (k, x) \mathcal{R} (k', x') \implies k = k'$.
2. $\forall_{x_1,x_2,x_3 \in \mathbb{K}, x \in \mathbb{K} \times X} (x_1 \xrightarrow{k} x_3 \land (k, x_1) \mathcal{R} (k, x_2)) \implies \exists_{x \in \mathbb{K} \times X} (x_2 \xrightarrow{k} x_4 \land (k, x_3) \mathcal{R} (k, x_4))$.

Two states $x, x' \in X$ are conditional bisimilar under $\mathcal{R}$ such that $(k, x) \mathcal{R} (k, x')$. Moreover, two states $x, x'$ are conditional bisimilar, denoted $x \equiv_X x'$, if $x$ and $x'$ are conditional bisimilar under every condition $k \in \mathbb{K}$.

In order to capture conditional bisimilarity, we first need a fibration $\Psi$ of binary relations on the state space. The first choice for $\Psi$ is to consider the set of all binary relations on the underlying state space, i.e., $\Psi = \hat{\mathcal{P}}([\cdot] \times [\cdot])$. Assumption [A2] is satisfied by this indexed category, since the equality functor $\text{Eq}$ maps a set $X \in \text{coKl}(G)$ to the pair $(X, \equiv_{GX})$. Nevertheless, Assumption [A4] fails to hold which we explain next.

Remark 36. We argue that $\text{Eq}$ cannot have a left adjoint since it does not preserve finite limits (in particular, terminal objects). Clearly, $1 = \{\bullet\}$ is the terminal object in $\text{coKl}(G)$ because $1$ is the right adjoint of $[\cdot]$. Now suppose $\text{Eq}1 = (1, \equiv_{1 \times 1})$ is the terminal object in $\mathbb{E}(\Psi)$. Then, for any $(X, \mathcal{R})$, there is a unique arrow $(X, \mathcal{R}) \xrightarrow{\lambda_X} \text{Eq}1$, i.e., $X \xrightarrow{\lambda_X} 1 \in \text{coKl}(G)$ and $\mathcal{R} \subseteq ([|X|] \times [|X|])^{-1} = 1 \times 1$. But we argue that $\lambda_X$ is not a map in $\mathbb{E}(\Psi)$ for $|\mathbb{K}| \geq 2$. To see this, let $k, k' \in \mathbb{K}$ with $k \neq k'$ and let $\mathcal{R} \subseteq (\mathbb{K} \times X) \times (\mathbb{K} \times X)$ be an equivalence relation such that $(k, x) \mathcal{R} (k', x)$. Then we find a contradiction

$$R \subseteq ([|X|] \times [|X|])^{-1} \implies (k, !X(k, x)) = (k', !X(k', x)) \implies k = k'.$$

So we restrict ourselves to relations satisfying the first property of conditional bisimulation (see Definition 35). Let $\Psi X \xhookrightarrow{\lambda_X} \hat{\mathcal{P}}([|X|] \times [|X|])$ be the set of relations on $|X|$ satisfying Definition 35(1). Note $\Psi X$ is well defined because Definition 35(1) is closed under the reindexing of $\hat{\mathcal{P}} \circ [\cdot]$, i.e., if $S \in \Psi Y$ and $\mathbb{K} \times X \xrightarrow{f} Y \in \text{Set}$ then $([f] \times [f])^{-1} S \in \Psi X$. Clearly, the equality relation on $GX$ satisfies Definition 35(1); thus, $\Psi$ satisfies [A2]. For [A4] we use coequalisers in $\text{Set}$ to construct $\mathcal{Q}$ (see the proof of the next proposition).

Proposition 37. The indexed category $\Psi$ defined above satisfies [A1], [A2] and [A4].

Next, we construct a relation lifting of type $\Psi \xrightarrow{\lambda} \Psi \hat{\mathcal{P}}$ from the predicate lifting $\lambda$ corresponding to box modality for CTSs. First we note that binary products $\otimes$ exist in $\text{coKl}(G)$. Second we relate the fibres of relations $\Psi$ and predicates $\Phi$ by using the natural transformations $\Psi X \xrightarrow{\rho_X} \hat{\mathcal{P}}[|X|]$ by letting $\rho_X$ and $\rho'_X$ be the inverse and direct image induced by the function $\mathcal{F} X \xrightarrow{(|p_1|, |p_2|)} |X| \times |X|$, i.e., $\rho_X = (|p_1|, |p_2|)^{-1}$ and $\rho'_X = (|p_1|, |p_2|)$. Finally, consider $\lambda$ given by the following composition:

$$\Psi X \xrightarrow{\lambda_X} \Phi(X \otimes X) \xrightarrow{\lambda_X^{X \otimes X}} \Phi(\hat{\mathcal{F}}X \otimes \hat{\mathcal{F}}X) \xrightarrow{\rho_X^{X \otimes X}} \Psi F X,$$

where $\hat{\mathcal{F}}(X \otimes X) \xrightarrow{\pi_X} \hat{\mathcal{F}}X \otimes \hat{\mathcal{F}}X$ is the unique arrow $\hat{\mathcal{F}}p_1 \otimes \hat{\mathcal{F}}p_2$ due to products in $\text{coKl}(G)$.

Lemma 38. The naturality square induced by $|\pi_X|$ (for each $X \in \text{Set}$) is a weak pullback square in $\text{Set}$. As a result, $\lambda$ is an indexed morphism.
Theorem 39. Alternatively, the indexed morphism $\Psi \xrightarrow{\lambda} \Psi \mathcal{P}$ can be defined as follows: $(k, U) \xrightarrow{\lambda} R (k', U') \iff k = k' \land \forall x \in U, \exists x' \in U' (k, x) R (k, x') \land \forall x \in U, \exists x' \in U' (k, x) R (k, x')$. Moreover, $\lambda$ satisfies Assumption A.3.

Corollary 40. Let $X \xrightarrow{\alpha} \mathcal{P}X \in \text{coKl}(G)$ be a CTS. A relation $R$ on $\mathbb{K} \times X$ is a conditional bisimulation iff $R$ is an $\alpha^* \circ \lambda$-coalgebra in $\Psi X$, i.e., $R \subseteq (|\alpha| \times |\alpha|)^{-1} \lambda_X R$.

Moreover, for any $(X, \alpha) \xrightarrow{\tilde{f}} (Y, \beta) \in \text{coKl}(G)$ we have $\tilde{\varepsilon}_X = (|f| \times |f|)^{-1} \varepsilon_Y$; thus, there is a behavioural conformance functor $\text{Coalg}_{\text{coKl}(G)}(\mathcal{P}) \xrightarrow{1_X^k} \text{Coalg}_{\Psi}(\mathcal{P}_\lambda)$.

Modal characterisation of conditional bisimilarity

$$\text{coKl}(\mathbb{K} \times _) \xrightarrow{|-|} \text{Set} \xleftarrow{\iota} \mathcal{T} \xrightarrow{S} \text{BA}^{\text{op}}$$

Consider the above adjoint situations where the adjoint situation on the right is the well known duality (see, for instance [24]) between $\text{Set}$ and the opposite category of Boolean algebras $\text{BA}$; $\mathcal{S}$ is the contravariant powerset functor $\mathcal{P}$ and $\mathcal{T}$ maps a Boolean algebra to its set of ultrafilters. We will follow [16] and use the proposed syntax functor $\text{BA} \xrightarrow{L} \text{BA}$ and the interpretation $\mathcal{P} \mathcal{T} \xrightarrow{\delta} \mathcal{T} L$ induced by the box modality on (unlabelled) transition systems. Note that this inline with our construction of relation lifting $\lambda$ which was based on box modality for CTSs (see [14]). Since $\mathcal{P}$ is a coKleisli extension of $\mathcal{P}$, i.e., $\mathcal{P} \circ \iota = \iota \circ \mathcal{P}$, we consider the following logical interpretation for CTSs: $\mathcal{P} \circ \iota \circ T = \iota \circ \mathcal{P} \circ T \xrightarrow{\delta} \iota \circ T \circ L$.

Corollary 41. Since the relation lifting $\lambda$ preserves equalities, the logic $(L, \lambda \delta)$ defined above is adequate for conditional bisimilarity. Moreover, since $\delta$ is injective in each component (cf. [16]), the function $|\delta_\mathcal{A}|$ is injective; thus $(L, \lambda \delta)$ is expressive.

8 Conclusions

To recapitulate, we gave a systematic way to construct both predicate and relation liftings in (co)Kleisli categories and Eilenberg-Moore categories. Predicate liftings form the basis of defining moves in our 2-player coagbebraic game; while relation liftings form the basis to define behavioural equivalence as a coagebra of certain lifted endofunctor in the fibre of relations, although in some cases (such as CTSs) such fibres can be subtle to define.

Once behavioural equivalence is captured as a fibred notion, the Kupke and Rot setup becomes applicable to obtain its corresponding logical characterisation. In particular, we gave a recipe to find the left adjoint $\mathcal{S}$ of $\mathcal{T}$ which is a sufficient condition for both adequacy and expressivity. For coKleisli and Eilenberg-Moore categories, the construction [8] of $\mathcal{S}$ is based on the existence of coequalisers in the underlying categories, while in the Kleisli case one has to resort to a reflective subcategory having coequalisers (cf. Theorem [19]).

In the future, we plan to extend our 2-player game for coalgebras living in coKleisli categories. One possibility is to apply the codensity games developed in [20] and also establish a precise relationship between the proposed game for Kleisli and Eilenberg-Moore categories. Lastly, we also plan to investigate whether the given recipe of constructing predicate/relation liftings can also be extended to more general monads (like the ones on pseudometric spaces). This should help in developing quantitative modal logics for coalgebras with side effects; thus providing a pertinent litmus test for the categorical unification of quantitative expressivity as claimed in the recent work [21].
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Appendix

The appendix is organized as follows: we will start by providing the omitted proofs for the main body of the paper in Section A. Then we will provide some additional material, which we could not fit into the main part due to space constraints. This additional material is an add-on and is not required to read or understand the main results. In particular we instantiate our framework to Eilenberg-Moore categories (Section B) and give the corresponding proofs (Section D). Furthermore we show the range of our approach by working out trace semantics for linear weighted automata (Section C) together with the proofs (Section E).

A Proofs

Theorem 10. Let \( \Psi \xrightarrow{\lambda} \Psi \circ F \) be an indexed morphism. Under Assumptions A1, A2, and A3, the behavioural equivalence induced by a coalgebra homomorphism \( f \in \text{Coalg}_C(F) \) on a coalgebra \( (X, \alpha) \in \text{Coalg}_C(F) \) is a \( \alpha^* \circ \lambda \)-coalgebra living in the fibre \( \Psi X \).

Proof. Let \( (X, \alpha) \xrightarrow{f} (Y, \beta) \in \text{Coalg}_C(F) \) be a coalgebra homomorphism and consider the following commutative diagram.

Then using Assumption A2 and \( \lambda \) preserves syntactical equalities we derive

\[
\begin{align*}
\alpha^* = |Y| & \geq \beta^* \lambda_Y = |Y| \quad \Rightarrow \quad f^* = |Y| \geq f^* \beta^* \lambda_Y = |Y| \quad \Rightarrow \quad f^* = |Y| \geq \alpha^* \lambda_X f^* = |Y| .
\end{align*}
\]

Theorem 11. Under Assumptions A1, A2, A3, and A4, for every \( \alpha^* \circ \lambda \)-coalgebra \( R \) there is a coalgebra map \( f \in \text{Coalg}_C(F) \) such that \( R \preceq f^* (= |\text{cod}(f)|) \), where \( \text{cod}(f) \) denotes the codomain of \( f \). Moreover, \( R = f^* (= |\text{cod}(f)|) \) when the unit of \( Q \dashv \text{Eq} \) is Cartesian.

Proof. Let \( R \in \Psi X \) such that \( R \preceq \alpha^* \lambda_X R \) and let \( \kappa \) be the unit of adjunction \( Q \dashv \text{Eq} \). For a given coalgebra \( X \xrightarrow{\alpha} FX \), we are going to first construct a coalgebra of the form \( \Psi(\kappa_X) \xrightarrow{\alpha} F\Psi(\kappa_X) \) such that \( \text{pr}_X \) is the witnessing coalgebra homomorphism. To define \( \alpha_R \), consider the following diagram in \( \mathbb{E}(\Psi) \) without the dashed arrow.

Note that by the assumption we have \( \lambda_X(=|X|) = |FX| \) and the unit \( \kappa_X \) gives \( R \preceq \kappa_X^* \). Thus,

\[
\begin{align*}
\lambda_X R & \preceq \lambda_X \kappa_X^* (=|Q(X,R)|) \quad \Rightarrow \quad \lambda_X R \preceq (F\kappa_X)^* \lambda_{Q(X,R)} = |Q(X,R)| \\
& \quad \Rightarrow \quad \lambda_X R \preceq (F\kappa_X)^* = |FQ(X,R)| .
\end{align*}
\]
As a result, the map $F \kappa_X$ drawn in (16) is a valid arrow in $\mathbb{E}(\Psi)$. Furthermore, the universal property of the unit $\kappa_X$ gives a unique arrow $Q(X, R) \xrightarrow{\alpha_R} FQ(X, R)$, i.e., a coalgebraic structure on $Q(X, R)$, such that Square (16) commutes. Clearly, applying the forgetful functor $p$ ensures that $p \kappa_X$ is a coalgebra homomorphism.

**Lemma 12.** Suppose $\mathbf{C}^{op} \xrightarrow{\Phi} \mathbf{Cat}$ has indexed final objects (i.e., the final object exists in each fibre $\Phi X$ for $X \in \mathbf{C}$) and the reindexing functor $f^*$ preserves these final objects. Then there is a functor $\mathbf{C} \xrightarrow{\mathbb{I}} \mathbb{E}(\Phi)$ that is right adjoint to $p$.

**Proof.** Define $\mathbb{I}X = (X, 1_X)$ where $1_X \in \Phi X$ is the final object. Moreover, for any $X \xrightarrow{f} Y \in \mathbf{C}$, we have $1_X = 1_Y$, $1_X = f^* 1_Y$ since the reindexing functor preserves the indexed final objects. Thus, we let $\mathbb{I}f = (f, \text{id}_{1_X})$. Next we show that $p \circ \mathbb{I}$.

\[
\begin{pmatrix} (X, U) \xrightarrow{(g, \text{id}_{1_Y})} (Y, \text{id}_{1_Y}) \in \mathbb{E}(\Phi) \\
(X, U) \xrightarrow{(g, \text{id}_{1_Y})} 1_Y \in \mathbb{E}(\Phi) \\
p(X, U) \xrightarrow{f} Y \in \mathbf{C} \\
X \xrightarrow{f} Y \in \mathbf{C}
\end{pmatrix}
\]

\[\uparrow \text{Let } X \xrightarrow{f} Y \in \mathbf{C}. \text{ Then take } g = f \text{ and let } \bar{g} \text{ be the unique arrow } U \xrightarrow{\bar{g}} 1_X \in \Phi X \text{ to the final object } 1_X = f^* 1_Y.\]

\[\downarrow \text{Take } f = g \text{ and the implication follows directly from the construction of } \mathbb{E}(\Phi).\]

**Theorem 16.** Under the assumptions of Theorem 16, if $\bar{T}$ has a left adjoint $\bar{S}$ (see below on the left), the logic $(L, \delta)$ is adequate. Moreover it is expressive if $|\delta_A|$ is injective.

**Proof.** For adequacy, we show the conditions of [25] Theorem 18 are satisfied. It only remains to show that there is a natural transformation $(FTA =_{FTA}) = F \delta A \bar{T} A \xrightarrow{\delta_A} \bar{T} A = (TLA =_{TLA}).$ We can simply let $\bar{\delta} = \delta$ because for any function $X \xrightarrow{f} Y$ we have $=_{X} \subseteq (f \times f)^{-1} X$. For expressivity, we show that the conditions of [25] Theorem 19 are satisfied, i.e., the arrow $(FTA =_{FTA}) \xrightarrow{\delta_A} (TLA =_{TLA}) \in \mathbb{E}(\Psi)$ is Cartesian. But this follows directly from the injectivity of $|\delta_A|$.}

**Theorem 19.** Let $\mathbf{A}^{op}$ be a reflective subcategory of $\mathbf{KL}(T)$ having coequalisers. Then the above defined map $\mathbb{E}(\Psi) \xrightarrow{\bar{S}} \mathbf{A}^{op}$ is a functor and left adjoint to $\bar{T} = \text{Eq} \circ \bar{T}$.

**Proof.** We show that $\bar{S} \dashv \bar{T}$, i.e., the following correspondence holds

\[
\begin{pmatrix} X \xrightarrow{f} jY \in \mathbf{KL}(T) \wedge R \subseteq ([|f| \times |f|])^{-1} =_{|Y|} \\
(X, R) \xrightarrow{f} (jY, =_{|Y|}) \in \mathbb{E}(\Psi) \\
\bar{S}(X, R) \xrightarrow{g} Y \in \mathbf{A}^{op} \\
Y \xrightarrow{g} \bar{S}(X, R) \in \mathbf{A}
\end{pmatrix}
\]

\[\uparrow \text{Let } X \xrightarrow{f} jY \in \mathbf{KL}(T) \text{ such that } R \subseteq ([|f| \times |f|])^{-1} =_{|Y|}. \text{ Then we find a unique } Y \xrightarrow{g} \tau X \in \mathbf{A} \text{ as the transpose of } f \text{ under } \tau \dashv j. \text{ Moreover, using the counit of } \tau \dashv || \text{ one can show that if } |f| \circ p_1 = |f| \circ p_2 \implies f \circ p_1 = f \circ p_2. \text{ Thus, } \tau p_1 \circ g = \tau p_2 \circ g. \text{ And by the universal property of equalisers we find a unique } Y \xrightarrow{u_g} \bar{S}(X, R).\]
Let $Y \xrightarrow{g} \tilde{S}(X, R) \in \mathcal{A}$. Then take $f$ as the transpose of $e \circ g$ under $\tau \dashv j$. To show that $j \circ p'_1 = f \circ p'_2$ consider the following commutative diagram in $\text{KL}(T)$:

\[
\begin{array}{c}
j \circ R \xrightarrow{j \circ p'_1} j \circ X \xrightarrow{j(e \circ g)} jY \\
\eta \circ R \xrightarrow{\eta \circ p'_1} X \xrightarrow{\eta X} X
\end{array}
\]

where $\eta$ is the unit of $\tau \dashv j$.

\[\nabla\]

\[\nabla\]

Lemma 21. Let $\bar{F}$ be a Kleisli extension of $F$ induced by a distributive law $FT \xrightarrow{\varphi} TF$. Then a natural transformation $TF \xrightarrow{\gamma} GT$ compatible with $\varphi$ and $\mu$ (i.e., Square 10 commutes) induces a distributive law $\bar{\mu} \circ \bar{F} \xrightarrow{\bar{\gamma}} G \circ \bar{\mu}$. Moreover, the converse also holds.

Proof. Let $C \xrightarrow{f} D \in \text{KL}(T)$. Then we need to show that the following square on the left commutes. But this follows immediately by the commutative diagram drawn on the right, where the top square commutes due to the naturality of $\gamma$.

\[
\begin{array}{c}
|fC| \xrightarrow{\gamma C} G|C| \\
|\bar{F}f| \xrightarrow{\gamma D} G|fD| \\
|\bar{F}D| \xrightarrow{\gamma D} G|D|
\end{array}
\]

For the converse, take $f = \text{id}_{fC}$ and view it as a Kleisli arrow $fC \xrightarrow{\bar{\gamma}} C$.

\[\nabla\]

Theorem 42 (Generalisation of Theorem 25). Under the following assumptions:

1. a category $C$ with a free-forgetful adjunction $C \xrightarrow{\dashv} \text{Set}$.
2. the indexed categories $\Phi$ and $\Psi$ of predicates and relations are defined like in the Kleisli case, i.e., $\Phi = \hat{\mathcal{P}}|\_|$ and $\Psi = \hat{\mathcal{P}}(\_| \times \_|)$.
3. a reflective subcategory $\mathcal{A}^\text{op} \subset \tilde{C}$ having all coequalisers.
4. an endofunctor $C \xrightarrow{\bar{F}} C$ is separable w.r.t. $\Lambda$ and it also preserves the subcategory $\mathcal{A}^\text{op}$.

Then behaviourally equivalent states are exactly those from which the defender has a winning strategy.

Proof. Let $x, x' \in |X|$ be behaviourally equivalent states, i.e., there is some coalgebra homomorphism $(X, \alpha) \xrightarrow{f} (Y, \beta)$ such that $|f|x = |f|x'$. Suppose the spoiler chooses the state $x$ (the symmetric case when the spoiler chooses $x'$ is similar) and a predicate $U \in \Phi X$. Since $\mathbb{E}(\Phi) \xrightarrow{\dashv} C$ has bifibration property, there is an adjunction between the fibre categories:

\[
\begin{array}{c}
\exists fU \xrightarrow{\dashv} V \in \Phi Y \\
U \xrightarrow{\dashv} f^*V \in \Phi X
\end{array}
\]

$(U \in \Phi X, V \in \Phi Y)$. 


Thus, from the unit of this adjunction, we can find $U \subseteq f^* \exists f U$. So, we let the duplicator choose the state $x'$ and fix $U' = f^* \exists f U$. Clearly, $\lambda_X U \subseteq \lambda_X U'$, for any $\lambda \in \Lambda$.

Consider the diagram in (17) and note that its commutativity follows from the naturality of $\lambda$ and the fact that $f$ is a coalgebra homomorphism.

$$
\begin{array}{ccc}
\Phi X & \xrightarrow{\lambda_X} & \Phi X \\
|\alpha|^{-1} & \Downarrow & |\alpha|^{-1} \\
\Phi Y & \xrightarrow{\lambda_Y} & \Phi Y
\end{array}
\quad
\begin{array}{ccc}
f^* & \Downarrow & f^* \\
(Ff)^* & \Downarrow & (Ff)^*
\end{array}
\tag{17}
$$

Moreover, we find

$$
|\alpha| x \in \lambda_X f^* \exists f U \iff x \in |\alpha|^{-1} \lambda_X f^* (\exists f U)
$$

(17) $\downarrow$

$$
|\beta| (f|\alpha|) x \in \lambda_Y (\exists f U)
$$

$$
|\beta| ((f|\alpha|) x') \in \lambda_Y (\exists f U)
$$

(19) $\uparrow$

$$
|\alpha| x' \in \lambda_X f^* (\exists f U).
$$

Now, without loss of generality, suppose Spoiler chooses a predicate $\bar{U} \in \{U, U'\}$. Then we distinguish the following cases:

1. Let $\bar{U} = U$. By construction we have $U \subseteq U'$. Moreover, $\lambda_X U \subseteq \lambda_X U'$, for any $\lambda \in \Lambda$.

So duplicator can choose $\gamma' = \gamma$ if Spoiler chooses $y \in U$ and, clearly, we have $\gamma' \in U'$.

2. Let $\bar{U} = U'$. Now, suppose spoiler chooses a state $y' \in |X|$ such that $y' \in U'$. Then, $y' \in f^* \exists f U$ and $|y'| \in |f| U$. Thus, we find some $y \in U$ such that $|f| y = |f| y'$.

$$
\xrightarrow{\text{Let } R \xrightarrow{p_1} |X| \text{ be the relation consisting those pairs of states from which Duplicator}}
\xrightarrow{\text{has a winning strategy. Recall } \bar{S} \models \text{Eq} \circ \bar{J} \text{ from Theorem 19}} \xrightarrow{\text{We will show that the underlying}}
\xrightarrow{\text{map } X \xrightarrow{\bar{g}} \bar{S}(X, R) \text{ (obtained by applying the forgetful functor on the unit of this}}
\xrightarrow{\text{adjunction } (X, R) \xrightarrow{\lambda_X} \text{ Eq} \circ \bar{J} \circ \bar{S}(X, R)) \text{ is going to be a coalgebra homomorphism. To complete}}
\xrightarrow{\text{the proof, it suffices to show that } |Fg||\alpha| p_1 = |Fg||\alpha| p_2 \text{. This is because the}}
\xrightarrow{\text{universal property of coequaliser then gives a coalgebraic structure on } \bar{S}(X, R)}
\xrightarrow{\text{For simplicity, let } B = S(X, R). \text{ Moreover since the given set } \Lambda \text{ of predicate liftings is}}
\xrightarrow{\text{separable w.r.t } \bar{F} \text{ then it suffices to show that}}
\xrightarrow{\forall x, x' \in |X| \ x R x' \Rightarrow \forall \lambda \in \Lambda, U \in \Phi_B \left(|Fg||\alpha|x \in \lambda_J B U \iff |Fg||\alpha|x' \in \lambda_J B U\right)}
\xrightarrow{\forall \lambda \in \Lambda, U \in \Phi_B \left(|\alpha|x \in \lambda_X g^* U \iff |\alpha|x \in \lambda_X g^* U\right)}
\xrightarrow{\forall x, x' \in |X| \ x R x' \Rightarrow \forall \lambda \in \Lambda, U \in \Phi_B \left(|\alpha|x \in \lambda_X g^* U \iff |\alpha|x' \in \lambda_X g^* U\right)}
\xrightarrow{\text{So let } x R x' \text{ and suppose Spoiler picks } x \text{ and the predicate } g^* U \text{ (the case when Spoiler picks}}
\xrightarrow{\text{x' is symmetric). Then Duplicator answers with a predicate } U' \text{ such that}}
\xrightarrow{|\alpha|x \in \lambda_X g^* U \iff |\alpha|x' \in \lambda_X U'}}
\xrightarrow{(18)}
\xrightarrow{\text{Next we claim that } \exists g U' = \exists g g^* U \text{ (recall } \exists g = |g|) \text{ for which we identify the following cases.}}
\xrightarrow{1. \text{ Suppose Spoiler picks } y \in |X| \text{ and } U' \text{ such that } y \in U' \text{ (i.e., } |g|y \in \exists^2 g U'). \text{ Then}}
\xrightarrow{\text{Duplicator answers this move by finding some } y' \in |X| \text{ such that } y' \in g^* U. \text{ Since the}}
\xrightarrow{\text{game instance moves to } (y, y') \text{ and Duplicator has a winning strategy from } (y, y') \text{ we}}
\xrightarrow{\text{know that } |g|y = |g|y' \text{ (since } g \circ p_1 = g \circ p_2). \text{ Therefore, } |g|y \in \exists^2 g g^* U.}}
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2. Suppose Spoiler picks \( y \in |X| \) and \( g^*U \) such that \( y \in g^*U \) (i.e., \( |g|y \in \exists_y g^*U \)). Similar to the previous case.

This proves the claim and we conclude that
\[
\exists_y U' \subseteq \exists_y g^*U \iff U' \subseteq g^*U \iff U' \subseteq g^*U.
\]

Thus, we find \( \lambda_X U' \subseteq \lambda_X g^*U \) and the implication in (18) can be refined to get the desired result: \( |\alpha| x \in \lambda_X g^*U \implies |\alpha| x' \in \lambda_X g^*U \). ▶

Lemma 27. The above mappings \( \sigma^a_X \) (for \( a \in \text{Act} \)) and \( \sigma^1_X \) are indexed morphisms.

Proof. Let \( V \subseteq Y \) and \( X \xrightarrow{f} Y \) be a function. Then,
\[
(f^{\text{Act}} \times 2)^{-1} \sigma^a_X V = (f^{\text{Act}} \times 2)^{-1} \{(q, b) \in Y^{\text{Act}} \times 2 \mid qa \in V\}
= \{(p, b) \in X^{\text{Act}} \times 2 \mid (f^{\text{Act}} \times 2)(p, b) \in \{(q, b) \mid qa \in V\}\}
= \{(p, b) \mid f(p) \in V\}
= \{f^{-1}V \mid p \in X \}.
\]

For the mapping associated with termination, we derive
\[
(f^{\text{Act}} \times 2)^{-1} \sigma^1_X \bar{V} = (f^{\text{Act}} \times 2)^{-1} \{(q, 1) \mid q \in Y^{\text{Act}}\}
= \{(p, b) \in X^{\text{Act}} \times 2 \mid (f^{\text{Act}} \times 2)(p, b) \in \{(q, 1) \mid q \in Y^{\text{Act}}\}\}
= \{(p, 1) \in X^{\text{Act}} \times 2 \mid f \circ p \in Y^{\text{Act}}\}
= \sigma^1_X f^{-1}V. \quad \blacktriangleleft
\]

Lemma 28. The set \( \Lambda = \{\lambda^a \mid a \in \text{Act}\} \cup \{\lambda^1\} \) is separating w.r.t. \( A \times X + 1 \).

Proof. Let \( t_1, t_2 \in \mathcal{P}(A \times X + 1) \) s.t. \( t_1 \neq t_2 \). Hence we have to consider only two cases:

1. For some \( (a, x) \) it holds that \( (a, x) \in t_1 \) but \( (a, x) \notin t_2 \): Here we consider the predicate \( U = \{(x)\} \) together with the lifting \( \lambda_X^a \). The lifting then yields all such \( U \subseteq A \times X + 1 \) containing at least \( (a, x) \), hence \( t_2 \notin \lambda_X^a(U) \). Obviously \( t_1 \in \lambda_X^a(U) \).

2. For \( 1 \) it holds that \( 1 \in t_1 \) but \( 1 \notin t_2 \): Here it does not matter, which predicate we choose as long as we work with \( \lambda_X^1 \). Based on this lifting, we know that \( t_2 \notin \lambda_X^1(U) \) for all \( U \in \mathcal{P}\mathcal{P}X \), hence all elements in the lifting contain the set with termination. ▶

Lemma 30. The mapping \( \bar{\sigma} \) defined above is an indexed morphism.

Proof. Let \( R, R' \) be two binary relations on \( X \) such that \( R \subseteq R' \). We first show that \( \bar{\sigma}' \bar{X} R \subseteq \bar{\sigma}' \bar{X} R' \). Let \( (p, b) \bar{\sigma}' \bar{X} R (p', b') \). Then we find \( \forall a \in \text{Act} \) \( (pa R p'a \implies pa R' p'a) \). Thus, \( (p, b) \bar{\sigma}' \bar{X} R (p', b') \).

Lastly, we need to show that the equation \( \bar{\sigma} \bar{X} \circ (f \times f)^{-1} = (Gf \times Gf)^{-1} \circ \bar{\sigma} Y \) holds. Below \( S \) is a relation on \( Y \).
\[
(p, b) \bar{\sigma} \bar{X} \circ (f \times f)^{-1} S (p', b') \iff b = b' \land \forall a \ pa (f \times f)^{-1} p'a
\]
\[
\iff b = b' \land \forall a \ fpa R fpa
\]
\[
\iff (f \circ p, b) \bar{\sigma} Y S (f \circ p', b')
\]
\[
\iff Gf(p, b) \bar{\sigma} Y S Gf(p', b')
\]
\[
\iff (p, b) (Gf \times Gf)^{-1} \circ \bar{\sigma} Y S (p', b'). \quad \blacktriangleleft
\]
Lemma 31. The indexed morphism $\bar{\sigma}$ preserves arbitrary intersections at each component; therefore, so does the predicate lifting $\lambda$. Moreover, $\lambda$ satisfies Assumption $\mathcal{A}_3$

Proof. Let $R_i$ (for $i \in \mathbb{N}$) be relations on $X$. Then

$$(p,b) \left( \bigcap_{i \in \mathbb{N}} R_i \right) (p',b') \iff b = b' \land \forall a \in \text{Act} \; pa \left( \bigcap_{i \in \mathbb{N}} R_i \right) p'a$$

$$\iff b = b' \land \forall a \in \text{Act}, \; \forall a \in \text{Act} \; pa \; R_i \; p'a$$

$$\iff \forall i \in \mathbb{N} \; (b = b' \land \forall a \in \text{Act} \; R_i \; p'a)$$

$$\iff \forall i \in \mathbb{N} \; (p,b) \left( \bigcap_{i \in \mathbb{N}} \sigma_X^G R_i \right) (p',b')$$

$$\iff (p,b) \left( \bigcap_{i \in \mathbb{N}} \sigma_X^G R_i \right) (p',b').$$

Note that $\lambda$ is the composition $(\gamma_X \times \gamma_X)^{-1} \circ \sigma^G$ and since inverse functions preserve arbitrary intersections, we can conclude that $\lambda$ also preserves intersections at each component.

Lastly, to show that $\lambda$ preserves equalities, consider $R$ to be equality relation on $\mathcal{P}X$, i.e., $R = = \mathcal{P}X$. Let $U, U' \subseteq \text{Act} \times X + 1$ such that $U \lambda_X R U'$. Clearly, $\bullet \in U \iff \bullet \in U'$. So consider $(a,x) \in U$. Then $\bar{x} \in \{x \mid (a,x) \in U\}$ and since the two sets $\{x \mid (a,x) \in U\}, \{x \mid (a,x) \in U'\}$ are the same, we find that $(a,x) \in U'$. Likewise, we can show the other direction and we conclude that $U = U'$.

Theorem 32. Let $X \xrightarrow{\alpha} FX \in \text{Rel}$ be an NDA. Then language equivalence $\equiv_X \subseteq \mathcal{P}X \times \mathcal{P}X$ on the determinised system is a $\Psi(\alpha) \circ \lambda_X$-coalgebra, i.e., $\equiv_X \subseteq (|\alpha| \times |\alpha|)^{-1}(\lambda_X \equiv_X)$.

Moreover, $\equiv_X = (|f| \times |f|)^{-1} \equiv_Y$ for any coalgebra homomorphism $(X, \alpha) \xrightarrow{f} (Y, \beta)$; thus, there is a behavioural conformance functor $\text{Coalg}_\text{Rel}(F) \xrightarrow{1^\lambda} \text{Coalg}_\text{Rel}(\Psi(\bar{F})\lambda)$.

Proof. Let $X \xrightarrow{\alpha} \text{Act} \times X + 1 \in \text{Rel}$ be an NDA. Since the relation lifting $\lambda$ preserves intersections, so from Kleene fixed point theorem, the largest fixpoint $\sim \in \Psi X$ of $\alpha^* \circ \lambda$ exists. That is, the following sequence stabilises (below $1_X \in \Psi X$ is the total relation on $\mathcal{P}X$):

$$1_X \supseteq (|\alpha| \times |\alpha|)^{-1} \circ \lambda_X (\top_X) \supseteq ((|\alpha| \times |\alpha|)^{-1} \circ \lambda_X)^2(\top_X) \supseteq \cdots$$

Let us denote the above projective sequence as a diagram $\xrightarrow{D} \Psi X$, i.e., $Di = (|\alpha|^{-1} \circ \lambda_X)^i \top_X$ (note that the zeroth iteration of a function is assumed to return the full relation $\top_X$ and $N$ has the set $N$ all natural numbers as objects and arrows induced by the less-than-equal-to relation). So $\sim = \bigcap_i Di$ and note that by definition $\sim$ is $\alpha^* \circ \lambda$-coalgebra.

Let $U_w$ is the unique state reached in the determinised system from the state $U$ after performing the trace $w$. Note that for any $i > 1$ we have:

$$U \xrightarrow{Di} U' \iff (U \downarrow \iff U' \downarrow) \land \forall a \in \text{Act} \; U_a \xrightarrow{Di} U'_a.$$ 

Moreover, it is not hard to show by induction on $w$ that we have

$$U \equiv U' \iff \forall w \in \text{Act}^* \; U_w \downarrow \iff U'_w \downarrow.$$ 

Next we show that $\equiv \subseteq \sim$. To show that $\equiv \subseteq \sim$, it suffices to show that $\equiv$ is an $\alpha^* \circ \lambda_X$-coalgebra, i.e., for any $U, U' \subseteq X$, if $U \equiv U'$ then $|U| \lambda_X \equiv |U'|$. So let $U \equiv U'$, clearly, $U \downarrow \iff U' \downarrow$. Moreover, $U_a \equiv U'_a$ (for any $a$) because $U \equiv U'$. So $\equiv \subseteq \sim$.

For the other direction, let $U \sim U'$. Then for any $w \in \text{Act}^*$ we have $U \xrightarrow{D(|w| + 1)} U'$, where $|w|$ is the length of the word $w$. Thus, $U_w \downarrow \iff U'_w \downarrow$. Therefore, $U \equiv U'$. 

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Lastly, let \((X, \alpha) \xrightarrow{f} (Y, \beta) \in \text{Coalg}_{\text{Rel}}(\bar{F})\), i.e., \(X \xrightarrow{f} Y\) is a relation satisfying the following two transfer properties:

\[
\forall_{x,y',a} \left( \exists_y (x \; f \; y \wedge y \; \xrightarrow{a} y') \iff \exists_{x'} (x' \; f \; y' \wedge x' \; \xrightarrow{a} x') \right).
\]

\(\forall_{x} \left( x \downarrow \iff \exists_y (x \; f \; y \wedge y \downarrow) \right).

Now the first property says that \(|f|(U_a) = (|f|U)_a\); so by induction we have \(|f|(U_w) = (|f|U)_w\). And the second property says that \(\left( |f|U \right) \downarrow = U \downarrow\). Thus, for any \(w\) we have \((U_w \downarrow \iff U'_w \downarrow) \iff ((|f|U)_w \downarrow \iff (|f|U)_w \downarrow)\). Hence \(\exists_X = (|f| \times |f|)^{-1} \equiv Y\).

\[\textbf{Proposition 33.} \text{ The above defined mapping } \theta \text{ is a natural transformation.}\]

\[\textbf{Proof.} \text{ We need to show that the following square commutes for any } X \xrightarrow{f} Y \in \text{Rel}.\]

\[
\begin{array}{ccc}
\text{Act} \times \mathcal{P}X + 1 & \xrightarrow{\theta_X} & \mathcal{P}(\text{Act} \times X + 1) \\
\downarrow \text{Lt}_f & & \downarrow \mathcal{F}\bar{f} \\
\text{Act} \times \mathcal{P}Y + 1 & \xrightarrow{\theta_Y} & \mathcal{P}(\text{Act} \times Y + 1)
\end{array}
\]

Let \(a \in \text{Act}, V \subseteq Y\). Then we find

\[
\tau(\bar{F}f)\theta_Y(a,V) = \tau(\bar{F}f)(\{a\} \times V)
\]

\[
= \{ (a', x) \mid \exists_{y \in V} \, (a', x) \; Ff \; (a, y) \}
\]

\[
= \{ (a, x) \mid \exists_{y \in V} \, x \; f \; y \}
\]

\[
= \{a\} \times \{ x \mid \exists_{y \in V} \, x \; f \; y \}
\]

\[
= \{a\} \times \tau f(V)
\]

\[
= \theta_X(a, \tau f(V)) = \theta_X \circ L(\tau s f)(a, V).
\]

Likewise, the case for \(\bullet \in 1\) can be verified; thus, \(LS \xrightarrow{\theta} SF\) is a natural transformation.

\[\textbf{Corollary 34.} \text{ The above map } \lfloor \_ \rfloor \text{ is indeed the theory map for a given NDA. So the logic } (L, \delta) \text{ is both adequate and expressive for language equivalence on determinised systems.}\]

\[\textbf{Proof.} \text{ First observe that the function } \text{Act}^* \xrightarrow{\lfloor \_ \rfloor} \bar{t} X \in \text{Set} \text{ is the transpose of the theory map under adjunction } \tau \dashv j. \text{ In other words, } \lfloor \_ \rfloor X \text{ is given by the composition:}\]

\[\text{Act}^* \xrightarrow{\tau \circ \alpha} \bar{t} \text{Act}^* \xrightarrow{\bar{t} \circ j} \bar{t} X \in \text{Set}.\]

The counit \(\epsilon\) is the function that maps every element to the singleton set containing that element. Thus, we compute \(\lfloor w \rfloor_X = \lfloor \_ \rfloor X \circ \epsilon_{\text{Act}^*}(w) = \lfloor \_ \rfloor X \{ w \} = \{ x \in X \mid w \in \lfloor \_ \rfloor X \}\), for any \(w \in \text{Act}^*\). Now it remains to show that the diagram in \([7]\) commutes because the uniqueness follow from the initiality of \((\text{Act}^*, h)\). Let \(a \in \text{Act}, w \in \text{Act}^*\). Then,

\[
\lfloor h(a, w) \rfloor_X = [aw]_X = \{ x \mid \exists_{x', x} x \xrightarrow{aw} x' \wedge x' \downarrow \}. \]

Furthermore,

\[
\tau \alpha \circ \theta_X \circ (\text{Act} \times \lfloor \_ \rfloor X + 1)(a, w) = \tau \alpha \circ \theta_X(a, [w]_X)
\]

\[
= \tau \alpha([a] \times [w]_X)
\]

\[
= \{ x \mid \exists_{x', x} x \xrightarrow{a} x' \wedge x' \in [a] \times [w]_X \}
\]

\[
= \{ x \mid \exists_{x', x''} x \xrightarrow{aw} x' \wedge x' \xrightarrow{w} x'' \wedge x'' \downarrow \}
\]

\[
= \{ x \mid \exists_{x', x''} x \xrightarrow{aw} x' \wedge x' \downarrow \} = \lfloor h(aw) \rfloor_X. \quad \square
\]
From Theorem 32 we know that behavioural conformance functor \( \mathbb{I}^{\Delta} \) which maps an NDA to its language equivalence on the determinised systems. Clearly, Assumptions \( A_1, A_2 \) and \( A_3 \) are satisfied; moreover, Theorem 19 ensures that Theorem 16 becomes applicable. However, to investigate expressivity, we need a concrete definition of \( \delta \) by unfolding its mate \( \theta \) (cf. Proposition 14) as follows (below \( \eta, \epsilon \) are the unit and counit of \( \tau \cdot j \)):

\[
\text{Act} \times X + 1 \xrightarrow{\eta X} \nu(\text{Act} \times X + 1) \xrightarrow{\theta X} \text{Act} \times \mathcal{P}X + 1 \xrightarrow{\ell X} \text{Act} \times X + 1.
\]

We find that the above composition corresponds to the identity relation on \( \text{Act} \times X + 1 \).

\[
(a, x) \xrightarrow{j \ell X} \eta X \circ \theta X \circ \eta_{FX} (a', x') \iff \exists \alpha', V \ (a, x) \xrightarrow{j \theta X} \eta_{FX} (a'', V) \land (a'', V) \xrightarrow{j \ell X} (a', x') \iff \exists \bar{V} (a, x) \bar{\in} \bar{U} \land \bar{\theta} X (a', \{x'\}) \iff a = a' \land x = x'.
\]

So, in particular, \( |\delta_{Act}| \) is injective.

**Proposition 37.** The indexed category \( \Psi \) defined above satisfies \( A_1, A_2 \) and \( A_4 \).

**Proof.** We focus on the existence of left adjoint of Eq. Let \( G = I \times _\_ \) and let \( (X, R) \in \mathcal{E}(\Psi) \), i.e., \( X \in \text{Set} \) and \( R \) is a relation on \( GX \) satisfying the filtering property. This means we have the following diagram in \( \text{Set} \):

\[
R \xrightarrow{p_1} GX \xrightarrow{q} GX/R,
\]

where \( GX/R \) is the coequaliser of the parallel arrows \( p_1, p_2 \). So we define \( Q(X, R) \) to be the object \( \iota(GX/R) \). In other words, \( Q \) can be defined by the composition:

\[
\mathcal{E}(\Psi) \xlongleftarrow{} \mathcal{E}(\Psi') \xrightarrow{\iota'} \text{Set} \xrightarrow{\iota} \text{coKl}(G),
\]

where \( \Psi' \) is the standard indexed category of relations on sets and \( \iota' \) is the usual quotient functor as described in Remark 9. So \( Q \) is a functor.

Next we show that \( Q \dashv \iota \), i.e., we establish the following correspondence:

\[
\begin{align*}
\iota(GX/R) & \xrightarrow{\ell} Y \in \text{coKl}(G) \\
(X, R) & \xrightarrow{g} \text{Eq}Y = (Y, =_{GY}) \in \mathcal{E}(\Psi) \\
X & \xrightarrow{g} Y \in \text{coKl}(G) \land R \subseteq (|g| \times |g|)^{-1} =_{GY}
\end{align*}
\]

\[\overset{\uparrow}{\text{Let } X \xrightarrow{g} Y \in \text{coKl}(G) \text{ (i.e., } GX \xrightarrow{g} Y \in \text{Set} \text{) and } R \subseteq (|g| \times |g|)^{-1} =_{GY}. \text{ We claim that } g \circ p_1 = g \circ p_2. \text{ So let } (k, x) R (k', x'). \text{ Then we find } k = k' \text{ and } g(k, x) = g(k, x') \text{ because of the inequality } R \subseteq (|g| \times |g|)^{-1} =_{GY}. \text{ This proves the claim. And from the universal property of coequalisers in } \text{Set}, \text{ we find a unique function } GX/R \xrightarrow{g'} Y \text{ satisfying } g' \circ g = g. \text{ So fix } f = ig' \text{ and clearly } f \text{ is uniquely determined by } g.\]

\[\overset{\uparrow}{\text{Let } \iota(GX/R) \xrightarrow{\iota} Y \in \text{coKl}(G). \text{ Recall the unit } X \xrightarrow{\eta X} \iota[X] \in \text{coKl}(G) \text{ of the adjunction } \ulcorner \_ \urcorner \dashv \iota. \text{ Then, let } g \text{ be the following composition:}}
\]

\[
X \xrightarrow{\eta X} \iota[X] \xrightarrow{\iota} Q(X, R) \xrightarrow{f} Y \in \text{coKl}(G).
\]
Clearly, $g$ is uniquely determined by $f$. Now it remains to show that for any pairs $(k, x), (k', x') \in GX$, if $(k, x) R (k', x')$ then $|g|(k, x) = (k, g(k, x)) = (k', g(k', x')) = |g|(k', x')$. So suppose $(k, x) R (k', x')$. Then we find $k = k'$ since $R$ satisfies the filtering property by construction of $\Psi$. Thus it remains to show that $g(k, x) = g(k, x')$.

To this end, we first evaluate $g$. Note that $\eta$ is given by the identity arrow and using the definition of coKleisli composition we find that

$$(k, x) \xrightarrow{\Delta_X} (k, k, x) \xrightarrow{Gq_X} (k, k, x) \xrightarrow{\eta_X} q(k, x).$$

Thus, $\eta q \circ \eta_X = q$. Furthermore, computing the composition $X \xrightarrow{q} Qu(X, R) \xrightarrow{f} Y \in \mathbf{coKl}(G)$ results in the evaluation of $g$. In particular,

$$g(k, x) = f(Gq(\Delta_X(k, x))) = f(k, q(k, x)).$$

So if $(k, x) R (k', x')$ then $k = k'$ and $q(k, x) = q(k, x')$. Thus, $g(k, x) = f(k, q(k, x)) = f(k, q(k, x')) = g(k, x')$.

\[\shortintertext{\textbf{Lemma 38.} The naturality square induced by $|\pi_X|$ (for each $X \in \mathbf{Set}$) is a weak pullback square in $\mathbf{Set}$. As a result, $\lambda$ is an indexed morphism.}\]

\[\textbf{Proof.} \text{Recall that } \pi_X \text{ is defined as follows: } \pi_X(k, R) = (\{x \mid \exists y \ x R x', \{x' \mid \exists y \ x R x'\}\}) \text{, for every } k \in \mathbb{K} \text{ and } R \subseteq X \times X. \text{ We show that the following square is a weak pullback in } \mathbf{Set}.\]

\[
\begin{array}{ccc}
|\mathcal{P}(X \times X)| & \xrightarrow{|\pi_X|} & |\mathcal{P}X \times \mathcal{P}X| \\
\downarrow |\mathcal{P}(f \times f)| & & \downarrow |\mathcal{P}f \times \mathcal{P}f| \\
|\mathcal{P}(Y \times Y)| & \xrightarrow{|\pi_Y|} & |\mathcal{P}Y \times \mathcal{P}Y|
\end{array}
\]

Let $k \in \mathbb{K}, S \subseteq Y \times Y, U, U' \subseteq X$ such that $\pi_Y(k, S) = (\mathcal{P}f \times \mathcal{P}f)(k, U, U')$, i.e.,

$$\{y \mid \exists y' y S y'\} = \{f(k, x) \mid x \in U\} \text{ and } \{y' \mid \exists y y S y'\} = \{f(k, x') \mid x' \in U'\}.$$

So let $R = U \cup U'$. Clearly, $R \subseteq X \times X$. Moreover, we find that

- $\pi_X(k, R) = (U, U')$ for any $k \in \mathbb{K}$; and
- $\mathcal{P}(f \times f)(k, R) = \{(f \times f)(k, (x, x')) \mid x R x'\} = \{(f(k, x), f(k, x')) \mid x \in U \land x' \in U\}$.

Clearly, $\mathcal{P}(f \times f)(k, R) = S$ because $\pi_Y(k, S) = (\mathcal{P}f \times \mathcal{P}f)(k, U, U')$.

Thus the above square is a weak pullback square in $\mathbf{Set}$. As a result, the Beck-Chevalley condition on $\mathcal{P}$ becomes applicable; thus, $|\pi_X| \circ (\mathcal{P}f \circ f)^{-1} = (\mathcal{P}f \circ f)^{-1} \circ |\pi_Y|$. This is instrumental in showing the naturality of $\lambda$.

\[\shortintertext{\textbf{Theorem 39.} Alternatively, the indexed morphism } \Psi \xrightarrow{\lambda} \Psi \mathcal{P} \text{ can be defined as follows: } (k, U) \lambda_X R (k', U') \iff k = k' \land \forall x \in U \exists x' \in U' \ (k, x) R (k, x') \land \forall x \in U \exists x' \in U' \ (k, x) R (k, x'). \text{ Moreover, } \lambda \text{ satisfies Assumption } A3.\]

\[\textbf{Proof.} \text{Recall from } [13], \text{ the map } \lambda \text{ is given by the composition:}\]

$$\Psi X \xrightarrow{((p_1, |p_2|)^{-1})} \mathcal{P}|\Pi X| \xrightarrow{\lambda_X} \mathcal{P}|\mathcal{P}|\Pi X| \xrightarrow{|\pi_X|} \mathcal{P}|\Pi \mathcal{P} X| \xrightarrow{|(q_1, |q_2|)|} \Psi \mathcal{P} X,$$
where $\hat{P}X \times \hat{P}X \xrightarrow{\text{def}} \hat{P}X \in \text{coKl}(G)$ given by $q_1(k, U, U') = U$ and $q_2(k, U, U') = U'$. Let $R \subseteq GX \times GX$, where $G = K \times \_$. Then we compute

$$((|p_1|, |p_2|)^{-1}R = \{(k, x, x') \mid |p_1|(k, x, x') R |p_2|(k, x, x')\} = \{(k, x, x') \mid (k, x) R (k, x')\}$$

$$\lambda_{HX}((|p_1|, |p_2|)^{-1}R = \{(k, R') \in K \times X \times X \mid \{k\} \times R' \subseteq \{(k, x, x') \mid (k, x) R (k, x')\}\} = \{(k, R') \mid \forall_{x,x'} x R' x' \implies (k, x) R (k, x')\}$$

$$|\pi_X|\lambda_{HX}((|p_1|, |p_2|)^{-1}R = \{(|\pi_X|(k, R') \mid \forall_{x,x'} x R' x' \implies (k, x) R (k, x')\} = \{(|\pi_X|(k, k, R'), |\pi_X|(k, k, R')) \mid \forall_{x,x'} x R' x' \implies (k, x) R (k, x')\}$$

Let us abbreviate the set derived in the last step by $\bar{R}$. We show that $\bar{\lambda}_X R = \bar{R}$.

- Let $(k, U) \bar{\lambda}_X R (k, U')$. Then, consider the relation $R' \subseteq X \times X$ given by: $x R' x' \iff (k, x) R (k, x') \land x \in U \land x' \in U'$. We claim that $U = \{x \mid \exists_{x'} x R' x'\}$. Clearly, the inclusion $\subseteq$ holds by the construction of $R'$. Let $x \in U$. Then we find some $x' \in U$ such that $(k, x) R (k, x')$, i.e., $x R' x'$. Let $(k, U) \bar{R} (k, U')$ and let $x \in U$. Then there is some $x' \in U'$ such that $x R' x'$. Moreover, by the construction of $\bar{R}$ we get $(k, x) R (k, x')$.

Furthermore, let $(k, U) \bar{\lambda}_X \equiv_{|X|} (k, U')$. Let $x \in U$. Then we find some $x' \in U'$ such that $x = x'$. So $U \subseteq U'$; likewise, we can show that $U' \subseteq U$.

### B Eilenberg-Moore categories

Although, in this paper, there are no case studies worked out in the setting of Eilenberg-Moore categories, it is still worthwhile to report that Eilenberg-Moore categories (at least when $T$ is finitary) are more well-behaved than Kleisli categories in satisfying the assumptions of this paper. Consider $\Phi$ and $\Psi$ as in the Kleisli case, i.e., $\Phi = \hat{P}(\_ \times \_)$ and $\Psi = \hat{P}(\_ \times \_ \times \_)$. First it is straightforward to show that every predicate lifting $\hat{P} \longrightarrow \hat{P}S$ of $S$ induces an indexed morphism $\Phi \longrightarrow \Phi S$.

- **Proposition 43.** Let $\bar{F}$ be a lifting of $F$ to $\text{EM}(T)$ with $C \xrightarrow{T} C$. Then,

  1. every indexed category on $C$ induces an indexed category $\text{EM}(T)$ by composing with $\_ \times \_$.  
  2. every indexed morphism $\Phi \longrightarrow \Phi S$ induces an indexed morphism of type $\Phi \_ \times \_ \longrightarrow \Phi \_ \times \_ S$.

Second, the adjunction $Q \dashv \text{Eq}$ does exist when $T$ is finitary, which we state next without proof. Intuitively, $Q$ creates quotients w.r.t. the smallest congruence relation generated by a relation on the underlying algebra. Formally, for an algebra $(X, h) \in \text{EM}(T)$, let $\text{Cong}(X, h)$ denote the poset of congruences on $X$ ordered by $\subseteq$. Next we claim that arbitrary meets exists in $\text{Cong}(X, h)$ since $\text{EM}(T)$ is complete. Therefore, we can construct

$$C(R) = \bigwedge\{(R', h') \mid (R', h') \in \text{Cong}(X, h) \land R \subseteq R'\} \,.$$  

Moreover, the algebraic structure on $C(R)$, i.e., a function $T^*C(R) \longrightarrow C(R)$ exists due to the universal property of limits, which we denote simply $h_R$. Now we let $Q((X, h), R)$ be the coequaliser of the diagram: $(C(R), h_R) \xrightarrow{p} (X, h)$ in $\text{EM}(T)$.
Predicate liftings for generalised Moore machines

In [3][31], the authors captured failure/ready/trace equivalences as behavioural equivalence for coalgebras living in the Eilenberg-Moore category introduced by a finitary power set monad \( P \). Interestingly, these linear equivalences can be seen as an instance of behavioural equivalence induced by a common ‘generalised Moore’ endofunctor \( F_S = S \times \_^{\text{Act}} \) [4] Section 3], where \( S \in \text{EM}(P) \) is an arbitrary semi-lattice.

Given a map \( X \xrightarrow{\alpha} S \in \text{EM}(P) \), then the generalised determinisation [5] of an LTS \( X \xrightarrow{\alpha} (P\omega X)^{\text{Act}} \) is a coalgebra \( P\omega X \xrightarrow{(\alpha',\alpha')} F_S(P\omega X) \in \text{EM}(P) \), where the two functions are given as follows (cf. [4]):

\[
d'(U) = \bigvee_{x \in U} o(x) \quad \text{and} \quad \alpha'(U)(a) = \bigcup_{x \in U} o(x)(a) \quad \text{for } U \subseteq X, a \in \text{Act}.
\]

In other words, the generalised determinisation induces a transition relation on \( P\omega X \) and a predicate on \( P\omega X \times S \) defined by the following SOS rules:

\[
\begin{align*}
U \subseteq X &\quad U_{\alpha} = \{ x' \mid \exists x \in U \cdot x \xrightarrow{a} x' \} && U \subseteq X &\quad d'(U) = \bigvee_{x \in U} o(x) \quad U \downarrow d'(U).
\end{align*}
\]

The next theorem is recalled from [31] and captures the aforementioned linear equivalences as instances of behavioural equivalence.

**Theorem 44.** Two states \( U,U' \subseteq X \) in the induced coalgebra \( P\omega X \xrightarrow{(\alpha',\alpha')} F_S(P\omega X) \in \text{EM}(P) \) are behaviourally equivalent if, and only if, they get mapped to a common point in the final coalgebra (which exists for the Moore functor \( F_S \)). Moreover,

1. For \( S = 2 \) and \( o \) as constant 1, behavioural equivalence coincides with trace equivalence.
2. For \( S = P\omega \) and \( o(x) \) (for each \( x \in X \)) is the set of sets of actions that are refused by the state \( x \), behavioural coincides with failure equivalence.
3. For \( S = P\omega \) and \( o(x) \) (for each \( x \in X \)) is the singleton containing the set of actions enabled at \( x \), behavioural coincides with ready equivalence.

Next we define two families of indexed morphisms \( \hat{P} \xrightarrow{\lambda^x} \hat{P} \circ F_S \) (for each \( a \in \text{Act} \)) and \( \hat{P} \xrightarrow{\lambda^s} \hat{P} \circ F_S \) (for each \( s \in S \)).

\[
\lambda^x_{\gamma} U = \{ (s',p) \in S \times X^{\text{Act}} \mid p(a) \in U \} \quad \text{and} \quad \lambda^s_{\gamma} U = \{ (s,p) \in X^{\text{Act}} \} \quad \text{for } U \subseteq X.
\]

**Proposition 45.** The above mappings \( \hat{P} \xrightarrow{\lambda^x,\lambda^s} \hat{P} \circ F_S \) are indexed morphisms. Moreover, the family \( \Lambda = \{ \lambda_{\gamma} \mid \gamma \in \text{Act} \cup S \} \) is separating w.r.t \( \hat{F}_S \).

As a result, thanks to Proposition [3] the mappings \( \lambda^x,\lambda^s \) are also indexed morphisms of type \( \hat{P} \xrightarrow{\lambda^x,\lambda^s} \hat{P} \circ F_S \). Recall the determinised systems \( P\omega X \xrightarrow{(\alpha',\alpha')} F_S(P\omega X) \in \text{EM}(P) \) (which is induced by an LTS \( X \xrightarrow{\alpha} (P\omega X)^{\text{Act}} \)). We are now ready to derive the action and observation modalities on the determinised system.

**Theorem 46.** Consider the compositions: \( \hat{P} \xrightarrow{\lambda^x,\lambda^s} \hat{P} \circ F_S \xrightarrow{|(\alpha',\alpha')|^{-1}} \hat{P} \circ F_S \xrightarrow{\hat{F}_S \circ P\omega X} \hat{P} \circ P\omega X \) for \( \diamond \in \{ a,s \} \). Let \( U \) be a predicate on \( P\omega X \). Then, \( |(\alpha',\alpha')|^{-1} \lambda^x_{\gamma} P\omega X U = \{ U \subseteq X \mid U \xrightarrow{\alpha'} \hat{U} \implies U_{\alpha} \subseteq U \} \quad \text{and} \quad |(\alpha',\alpha')|^{-1} \lambda^s_{\gamma} P\omega X U = \{ U \subseteq X \mid U \downarrow s \}. \)
Furthermore, a well-known application of the Linton theorem when the monad \( T \) is finitary is that the category of Eilenberg-Moore algebras is cocomplete. So, in particular, the finite powerset monad \( P_{\omega} \) is finitary, thus, \( EM(P_{\omega}) \) has all coequalisers. Therefore, we take \( EM(P_{\omega}) \) as the reflective subcategory of itself; hence, Theorem 42 pertaining to game characterisation of behavioural equivalence becomes applicable.

C Linear Weighted Automaton (LWA)

We consider (linear) weighted automata (LWA) as coalgebras as studied in [15]. LWA are modelled as coalgebras of the endofunctor \( M_{F}(1 + Act \times _{-}) \), where \( F \) is a field and \( M_{F} \) is the multiset monad. The set of functions \( F \times \) having finite support is \( M_{F}X \) and on arrows it is given by \( M_{F}(\tau)(y) = \sum_{x \in f^{-1}(y)} \tau(x) \) (for \( X \xrightarrow{f} Y \in \text{Set} \)). We write \( x \xrightarrow{a} x' \) whenever \( \alpha(x)(\bullet) = s \) and \( \alpha(x)(a,x') = s' \) for a given LWA \( X \xrightarrow{\alpha} M_{F}(Act \times X + 1) \).

Recall the language of a given LWA \( \alpha \) starting from a state \( x \in X \) is an inductively defined function \( Act^{\ast} \xrightarrow{tr} F \) described below, where \( a \in Act \), \( w \in Act^{\ast} \), and \( \varepsilon \) is the empty word.

\[
tr(x)(\varepsilon) = \alpha(x)(\bullet), \quad tr(x)(aw) = \sum \{ s \cdot tr(x')(w) \mid x \xrightarrow{a,s} x' \}.
\]

Two states \( x, x' \in X \) are (weighted) language equivalent iff \( tr(x) = tr(x') \). This coincides with coalgebraic behavioural equivalence in \( KI(M_{F}) \) (see [15, 22]). Note that probabilistic automata can be encoded by letting \( F = \mathbb{R} \) and restricting the weights to the interval \([0, 1]\).

Predicate lifting for weighted automata

To apply techniques of Section 5 we fix \( F = Act \times _{-} + 1, G = _{-}^{Act} \times F \), and recall from [15] Section 7.3 the distributive laws \( FM_{F}X \xrightarrow{\vartheta_{X}} M_{F}Fx \) and \( M_{F}Fx \xrightarrow{\gamma_{X}} GM_{F}X \):

\[
\vartheta_{X}(\bullet)(\varnothing) = \begin{cases} 1, & \text{if } \varnothing = \bullet \\ 0, & \text{otherwise.} \end{cases} \quad \vartheta_{X}(a,\tau)(\varnothing) = \begin{cases} \tau(x), & \text{if } \varnothing = (a, x), \text{ for some } x \in X \\ 0, & \text{otherwise.} \end{cases}
\]

\[
\gamma_{X}(\bar{p}) = (\gamma_{X}^{Act}\bar{p}, \bar{p}(\bullet)), \text{ where } \gamma_{X}^{Act}\bar{p}(a)(x) = \bar{p}(a, x) \text{ (for } x \in X, \bar{p} \in M_{F}Fx). \]

We know (from [15]) that \( \gamma \) is compatible with \( \theta \) and \( \mu \) (the multiplication of the monad \( M_{F} \)) in the sense of Lemma 21. Similar to NDAs, consider the following predicate liftings \( \hat{\mathcal{P}} \xrightarrow{\sigma_{X}^{\lambda}} \hat{\mathcal{P}}(X^{Act} \times F) \) (for \( a \in Act \) and \( s \in F \)):

\[
U \mapsto \{ (p, s) \in X^{Act} \times F \mid p(a) \in U \} \quad \text{and} \quad U \mapsto \{ (p, s) \mid p \in X^{Act} \}, \text{ respectively.}
\]

The proof of the following lemma is similar to the proof of Lemma 27. And thanks to Theorem 20 we know that \( \gamma^{-1} \circ \sigma^{a} \) and \( \gamma^{-1} \circ \sigma^{s} \) are valid predicate liftings.

\[\text{Lemma 47.} \quad \text{The above mappings } \sigma_{X}^{\lambda}, \lambda_{X}^{\lambda} \text{ (for } a \in Act, s \in F) \text{ are indexed morphisms.}\]

\[\text{Lemma 48.} \quad \text{For any } U \subseteq M_{F}X \text{ we find that } \lambda_{X}^{\lambda}(U) = \{ \bar{p} \in M_{F}(Act \times X + 1) \mid \gamma_{X}^{Act}\bar{p}(a) \in U \} \text{ and } \lambda_{X}^{\lambda}(U) = \{ \bar{p} \in M_{F}(Act \times X + 1) \mid \bar{p}(\bullet) = s \}.\]

Just like in our running example, the determinisation of an LWA \( \alpha \) is the composition:

\[
M_{F}X \xrightarrow{\alpha} M_{F}(Act \times X + 1) \xrightarrow{\mu_{Act \times X + 1}} M_{F}(Act \times X + 1) \xrightarrow{\gamma_{X}} (M_{F}X)^{Act} \times F.
\]
More concretely, it maps a \( p \in \mathcal{M}_F X \) to a pair \((\bar{p}, s)\), where \( \bar{p}(a)(x') = \sum_{x \in X} p(x) \cdot \alpha(x)(a) \) and \( s = \sum_{x \in X} p(x) \cdot \alpha(x)(\bullet) \). In terms of SOS rules, determination is given as follows:

\[
\begin{align*}
\text{Lemma 49.} & \quad \text{For any } \mathbb{U} \subseteq \mathcal{M}_F X \text{ and } X \xrightarrow{\alpha} \mathcal{M}_F(\text{Act} \times X + 1), \text{ we have } |\alpha|^{-1} \lambda^\alpha \mathbb{U} = \{ p \in \mathcal{M}_F X \mid p \overset{\alpha}{\Rightarrow} \bar{p}(a) \iff \bar{p}(a) \in \mathbb{U} \} \text{ and } |\alpha|^{-1} \lambda^\alpha \mathbb{U} = \{ p \in \mathcal{M}_F X \mid s = \sum_{x \in X} p(x) \cdot \alpha(x)(\bullet) \}. \\
\text{Moreover, the set } \Lambda = \{ \lambda^a \mid a \in \text{Act} \} \cup \{ \lambda^s \mid s \in \mathbb{F} \} \text{ is separating w.r.t. } A \times X + 1.
\end{align*}
\]

**Weighted language equivalence through relation lifting**

Consider the relation lifting \( \tau \) of \( G \) that maps a relation \( R \) on \( X \) to \( \tau_X R \) defined as:

\[
(\bar{p}, s) \overset{(\bar{p}', s')}{\Rightarrow} \iff s = s' \land \forall a \in \text{Act} pa \ 	op \ R p'a. \text{ So } (12) \text{ gives a relation lifting } \bar{\lambda} \text{ for } F. \text{ Concretely, } \bar{\lambda} \text{ maps a relation } R \text{ on } \mathcal{M}_F X \text{ to a relation } \lambda_X R \text{ on } \mathcal{M}_F \text{FX} \text{ given as: }
\]

\[
\bar{p} \lambda_X R \bar{p}' \iff \bar{p}(\bullet) = \bar{p}'(\bullet) \land \forall a \in \text{Act} \ 	op \ ar{\gamma} \lambda^\alpha \bar{p}(a) R \bar{\gamma} \lambda^\alpha \bar{p}'(a).
\]

**Theorem 50.** Let \( X \xrightarrow{\alpha} F X \in \text{Kl}(\mathcal{M}_F) \) be an LWA. Then weighted language equivalence \( \equiv_X \subseteq \mathcal{M}_F X \times \mathcal{M}_F X \) on the determinised system is a \( \Psi(\alpha) \circ \lambda_X \)-coalgebra.

Moreover, \( \equiv_X = ([\mathbb{F}] \times [\mathbb{F}])^{-1} \equiv_Y \) for any coalgebra homomorphism \( (X, \alpha) \xrightarrow{f} (Y, \beta); \) thus, there is a behavioural conformance functor \( \text{Coal}\text{Kl}(\mathcal{M}_F)(\hat{F}) \xrightarrow{\equiv_X} \text{Coal}\text{Kl}(\mathcal{M}_F)(\hat{F}_Y) \).

**Logical characterisation of weighted language equivalence**

Just like language equivalence, its weighted variant is a ‘linear’ notion of behavioural equivalence; thus, we set \( \mathcal{A} = \text{Set} \) and \( L = F \). As for the dual adjunction, first recall that \( \text{Kl}(\mathcal{M}_F) \) is equivalent to the Eilenberg-Moore category \( EM(\mathcal{M}_F) = \text{Vect}_F \) \[15\], which is due to the fact that every vector space has a basis. In particular, the comparison functor \( \text{Kl}(\mathcal{M}_F) \xrightarrow{\mathcal{K}} \text{EM}(\mathcal{M}_F) \) is fully faithful and essentially surjective. As a result, \( \mathcal{K} \) has an inverse \( \mathcal{K}' \) that maps every vector space to its basis which can be arranged as \( \mathcal{K} \dashv \mathcal{K}' \).

Moreover, the category of vector spaces is related with the category of sets by a dual adjunction \[15\] as indicated below.

\[
\begin{align*}
\text{Kl}(\mathcal{M}_F) & \xleftarrow{\mathcal{K}} \text{EM}(\mathcal{M}_F) & \text{hom}_{\text{Set}}(\cdot, \mathbb{F}) & \xrightarrow{\text{Set}^{op}} \text{Set}^{op}.
\end{align*}
\]

So \( \mathcal{S} = \text{hom}(\mathcal{K}_{op}, \mathbb{F}) \) and \( \mathcal{T} = \mathcal{K}'\mathbb{F} - \). Note that for any set \( X \) we have \( \mathcal{S}X \cong \mathbb{F}X \) and \( \mathcal{K}'(\mathbb{F}X) \cong X \). The former is true because \( \text{hom}(\mathcal{M}_F X, \mathbb{F}) \cong \text{Set}(X, \mathbb{F}) \), while the latter is because the basis of a vector space generated by a set is isomorphic to the set itself. So \( \text{Set}^{op} \) is a subcategory of \( \text{Kl}(\mathcal{M}_F) \) and Theorem \[19\] ensures that \( \text{Eq} \circ \mathcal{T} \) has a left adjoint \( \mathcal{S} \). Finally, \( \delta_X \) (for any set \( X \)) should be a natural transformation of type \( FTX = FX \xrightarrow{\delta_X} FX = TLX \). Therefore, we simply take \( \delta_X \) to be an identity arrow on \( FX \) in \( \text{Kl}(\mathcal{M}_F) \). Lastly, the theory map \( X \xrightarrow{1_X} \mathcal{T}\text{Act}^* \in \text{Kl}(\mathcal{M}_F) \) is given by the trace function \( \text{tr} \).

**Theorem 51.** The above defined map \( \llbracket \_ \rrbracket \) is a theory map for a given LWA. As a result, the logic \( (L, \delta) \) is both adequate and expressive for weighted language equivalence.
The family $\Lambda = \{\lambda_\diamond \mid \diamond \in \text{Act} \cup S\}$ is separating w.r.t $\tilde{F}_S$.

**Proposition 45.** The above mappings $\tilde{P} \xrightarrow{\lambda_X} \tilde{P} \circ F_S$ are indexed morphisms. Moreover, $\Lambda$ is a separating property.

**Proof.** Let $t \subseteq Y$ and $X \xrightarrow{\bar{f}} Y \in \text{Set}$. Then we derive

$$\lambda_X f^{-1}V = \{(s, p) \mid p(a) \in f^{-1}V\}$$

$$= \{(s, p) \mid f(pa) \in V\}$$

$$= \{(s, p) \mid (F_S f)(s, p) \in \{(s, q) \mid qa \in V\}\}$$

$$= (F_S f)^{-1} \lambda_X^V.$$

Moreover, $\lambda_X$ preserves arbitrary meets since

$$\lambda_X \bigcap_{i \in I} U_i = \{s \mid X \in \text{Act}\} \subseteq \bigcap_{i \in I} \{s \mid X \in \text{Act}\} = \bigcap_{i \in I} \lambda_X U_i.$$

To establish separating property, let $t, t' \subseteq \tilde{F}_S X = S \times X \text{Act}$ such that $t \neq t'$, for some $X \in \text{EM}(\mathcal{P}_\omega)$. Then we distinguish the following two cases:

1. Let $t = (s, p)$ and $t = (s', p')$ with $s \neq s'$. We need to find a predicate lifting $\lambda \in \Lambda$ that distinguishes $t$ and $t'$. Clearly, $t \in \lambda_X 0$, but $t' \notin \lambda_X 0$.

2. Let $t = (s, p)$ and $t = (s', p')$ with $p \neq p'$. Since $p \neq p'$, we find some $a \in \text{Act}$ such that $pa \neq p'a$. Let $U = \{pa\}$. Then we find $t \in \lambda_X U$ and $t' \notin \lambda_X U$.

**Theorem 46.** Consider the compositions: $\tilde{P}|_{\mathcal{P}_\omega X} \xrightarrow{\lambda_{\mathcal{P}_\omega X}^0} \tilde{P}|_{\tilde{F}_S \mathcal{P}_\omega X} \xrightarrow{\lambda_{\mathcal{P}_\omega X}^\diamond} \tilde{P}|_{\mathcal{P}_\omega X}$ for $\diamond \in \{a, s\}$. Let $U \subseteq \mathcal{P}_\omega X$. Then, $\lambda_{\mathcal{P}_\omega X}^\diamond U = \mathcal{P}_\omega X \subseteq \{U \subseteq X \mid U \xrightarrow{\alpha} U_\alpha \Rightarrow U_\alpha \in \mathcal{U}\}$ and $\lambda_{\mathcal{P}_\omega X}^\diamond U = \{U \subseteq X \mid U \downarrow s\}$.  

**Proof.** Let $U \subseteq \mathcal{P}_\omega X$. Then we derive the modality for action transition:

$$\lambda_{\mathcal{P}_\omega X}^\diamond U = \alpha'(s, p) \in \mathcal{P}_\omega X \text{Act} \mid p(a) \in U\}$$

$$= \{U \subseteq X \mid (s, p) \in \mathcal{P}_\omega X \text{Act} \mid p(a) \in \mathcal{U}\}$$

$$= \{U \subseteq X \mid \alpha(U) \in \mathcal{U}\}$$

$$= \{U \subseteq X \mid U \xrightarrow{\alpha} U_\alpha \Rightarrow U_\alpha \in \mathcal{U}\}.$$

Likewise, we derive the modalities for observations living in $S$:

$$\lambda_{\mathcal{P}_\omega X}^\diamond U = \alpha'(s, p) \in \mathcal{P}_\omega X \text{Act} \mid p(a) \in U\}$$

$$= \{U \subseteq X \mid (s, p) \in \mathcal{P}_\omega X \text{Act} \mid p(a) \in \mathcal{U}\}$$

$$= \{U \subseteq X \mid \alpha(U) = s\}$$

$$= \{U \subseteq X \mid U \downarrow s\}.$$
Proofs of the results in Appendix C

Lemma 48. For any \( U \subseteq M_{T}X \) we find that \( \lambda^a_X(U) = \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \gamma_X^{Act} \bar{p}(a) \in U \} \) and \( \lambda^a_X(U) = \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \bar{p}(\bullet) = s \} \).

Proof. For any \( U \subseteq M_{T}X \) we find that
\[
\lambda^a_X(U) = \gamma_X^{-1} \sigma^a_{M_{T}X}(U)
= \gamma_X^{-1} \{ (p, s) \in (M_{T}X)^{Act} \times F \mid p(a) \in U \}
= \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \gamma_X \bar{p} \in \{ (p, s) \in (M_{T}X)^{Act} \times F \mid p(a) \in U \} \}
= \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \gamma_X^{Act} \bar{p}(a) \in U \}.
\]
Similarly, in the context of termination, we find (for each \( s \in F \)):
\[
\lambda^a_X(U) = \gamma_X^{-1} \sigma^a_{M_{T}X}(U)
= \gamma_X^{-1} \{ (p, s) \mid p \in (M_{T}X)^{Act} \}
= \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \gamma_X \bar{p} \in \{ (p, s) \mid p \in (M_{T}X)^{Act} \} \}
= \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \bar{p}(\bullet) = s \}.
\]
The proof that \( \lambda^a_X, \lambda^a \) preserves finite meet is similar to the Boolean case (cf. Lemma 27).

Lemma 49. For any \( U \subseteq M_{T}X \) and \( X \xrightarrow{\alpha} M_{T}(Act \times X + 1) \), we have \( |\alpha|^{-1} \lambda^a_X(U) = \{ p \in M_{T}X \mid p \overset{\alpha}{\rightarrow} \bar{p}(a) \Rightarrow \bar{p}(a) \in U \} \) and \( |\alpha|^{-1} \lambda^a_X = \{ p \in M_{T}X \mid s = \sum_{x \in X} p(x) \cdot \alpha(x)(\bullet) \} \).

Moreover, the set \( \Lambda = \{ \lambda^a \mid a \in Act \} \cup \{ \lambda^a \mid s \in F \} \) is separating w.r.t. \( Act \times X + 1 \).

Proof. Let \( U \subseteq M_{T}X \) and a given LWA \( X \xrightarrow{\alpha} M_{T}X \). Then we derive
\[
|\alpha|^{-1} \lambda^a_X(U) = |\alpha|^{-1} \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \gamma_X^{Act} \bar{p}(a) \in U \}
= \{ p \in M_{T}X \mid |\alpha| \bar{p} \in D(Act \times X + 1) \mid \gamma_X^{Act} \bar{p}(a) \in U \}
= \{ p \in M_{T}X \mid \gamma_X^{Act}(\mu_{Act \times X + 1} M_{T} \alpha(\bullet)) a \in U \}
= \{ p \in M_{T}X \mid p \overset{\alpha}{\rightarrow} \bar{p}(a) \Rightarrow \bar{p}(a) \in U \}.
\]
Similarly, we have a modality to handle termination that can be derived as follows:
\[
|\alpha|^{-1} \lambda^a_X(U) = |\alpha|^{-1} \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \bar{p}(\bullet) = s \}
= \{ p \in M_{T}X \mid |\alpha| \bar{p} \in \{ \bar{p} \in M_{T}(Act \times X + 1) \mid \bar{p}(\bullet) = s \} \}
= \{ p \in M_{T}X \mid s = \sum_{x \in X} p(x) \cdot \alpha(x)(\bullet) \}.
\]
Now it remains to show that \( \Lambda \) is separating w.r.t. \( Act \times X + 1 \). So let \( \bar{p}, \bar{p}' \in M_{T}(Act \times X + 1) \) with \( \bar{p} \neq \bar{p}' \). Then we identify the following two cases:

1. Let \( \bar{p}(\bullet) \neq \bar{p}'(\bullet) \). Consider an empty predicate \( \emptyset \) with \( r = \bar{p}(\bullet) \). Then we find \( \bar{p} \in \lambda_X^a \emptyset \); however, \( \bar{p}' \notin \lambda_X^a \emptyset \) (for any \( U \subseteq M_{T}X \)) since the definition of \( \lambda^a \) is independent of \( U \) and, moreover, \( \bar{p}'(\bullet) \neq r \).

2. Let \( \bar{p}(a, x) \neq \bar{p}'(a, x) \), for some \( a \in Act, x \in X \). Consider the function
\[
p(x') = \begin{cases} p(a, x), & \text{if } x' = x \\ 0, & \text{otherwise} \end{cases}
\]
together with a predicate \( U = \{ p \} \). Clearly, \( \bar{p} \in \lambda_X^a U \); however, \( \bar{p}' \notin \lambda_X^a U \) because \( p(x) = \bar{p}(a, x) \neq \bar{p}'(a, x) \).
Theorem 50. Let $X \xrightarrow{\alpha} FX \in \text{KI}(\mathcal{M}_\mathcal{F})$ be an LWA. Then weighted language equivalence $\equiv_X \subseteq \mathcal{M}_\mathcal{F}X \times \mathcal{M}_\mathcal{F}X$ on the determinised system is a $\Psi(\alpha) \circ \lambda_X$-coalgebra.

Moreover, $\equiv_X = (|f| \times |f|)^{-1} \equiv_Y$ for any coalgebra homomorphism $(X, \alpha) \xrightarrow{f} (Y, \beta)$; thus, there is a behavioural conformance functor $\text{Coalg}_\text{KI}(\mathcal{M}_\mathcal{F})(\bar{F}) \xrightarrow{1^\lambda} \text{Coalg}_\Psi(\bar{F}_X)$.

Proof. Let $X \xrightarrow{\alpha} FX \in \text{KI}(\mathcal{M}_\mathcal{F})$ be an LWA. Just like in the proof of Theorem [32], we will show that weighted language equivalence $\equiv_X$ coincides with the relation $\sim$ defined as the limit of the diagram $N \xrightarrow{D} \Psi X$ defined in the proof of Theorem [32]. Recall the determinisation of an LWA in terms of SOS rules:

\[
p \in \mathcal{M}_\mathcal{F}X \xrightarrow{p} p_a \quad p \in \mathcal{M}_\mathcal{F}X \quad s = \sum_{x \in X} p(x) \cdot \alpha(x) \cdot (\cdot)
\]

where $p_a(x') = \sum_{x \in X} p(x) \cdot \alpha(x)(a, x')$. We can extend this notation to words as follows: $p_c = p$ and $p_{aw} = (p_a)_w$. Thus, $p \equiv_X p'$ if and only if $\forall_{s \in \mathbb{F}} p_{aw} \downarrow s \iff p'_{aw} \downarrow s$.

Note that for any $i > 1$ and $p, p' \in \mathcal{M}_\mathcal{F}X$ we have

\[
p D(i + 1) p' \iff \forall_{s \in \mathbb{F}} (p \downarrow s \iff p' \downarrow s) \land p_a p'_{aw}.
\]

Clearly, $\equiv_X \subseteq \sim$ because $\equiv_X \subseteq \alpha^* \lambda_X \equiv_X$. For the other direction, suppose $p \sim p'$. Then for any word $w \in \text{Act}^*$ we have $p D(\#w + 1) p'$ which results $p_w \downarrow s \iff p'_w \downarrow s$, for any $s \in \mathbb{F}$. Since $w$ was chosen arbitrarily, we find that $p \equiv_X p'$.

Theorem 51. The above defined map $\llbracket \_ \rrbracket$ is a theory map for a given LWA. As a result, the logic $(L, \delta)$ is both adequate and expressive for weighted language equivalence.

Proof. We show the square associated with theory map is commutative, i.e., $\mathcal{T}h \circ \llbracket \_ \rrbracket_X = \delta_{\text{Act}^*} F(\llbracket \_ \rrbracket_X \circ \alpha)$; the uniqueness follows from structural recursion. Since $\delta_{\text{Act}^*}$ is identity arrow, so it suffices to show that $\llbracket \_ \rrbracket_X$ is a coalgebra homomorphism; $\mathcal{T}h \circ \llbracket \_ \rrbracket_X = F(\llbracket \_ \rrbracket_X \circ \alpha)$.

Note that we follow [22] by using matrix multiplication to denote the composition of arrows in $\text{KI}(\mathcal{M}_\mathcal{F})$. Also, for a function $Y \xrightarrow{f} X$, $\mathcal{K}(\mathcal{F}f)$ corresponds to unit matrix; namely, the cell $(\delta_x, \delta_y)$ in $\mathbb{F}^\mathbb{F}$ corresponds to 1 if $\mathcal{F}f(\delta_x) = \delta_y \circ f = \delta_y$; 0 otherwise. So concretely, $\mathcal{T}f$ is a function defined as follows: $\mathcal{T}fxy = 1$ if $fy = x$; 0, otherwise.

\[
F(\llbracket \_ \rrbracket_X \circ \alpha(x, (a, w))) = \sum_{(a', x') \in \text{supp}(\alpha x)} \alpha(x, (a', x')) \cdot \bar{F}(\llbracket \_ \rrbracket_X((a', x'), (a, w)))
\]

\[
= \sum_{(a', x') \in \text{supp}(\alpha x)} \alpha(x, (a, x')) \cdot [x'](w)
\]

\[
= [x]_X(aw)
\]

\[
= \sum_{w' = aw} [x]_Xw' \cdot \mathcal{T}h(w', (a, w))
\]

\[
= \mathcal{T}h \circ \alpha(x, (a, w)).
\]

Adequacy follows from Theorem [16]. Furthermore, expressivity follows since $|\delta_{\text{Act}^*}|$ is injective.