DONALDSON FUNCTIONAL IN TEICHMÜLLER THEORY

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Abstract. In this paper we define a Donaldson type functional whose Euler-Lagrange equations are a system of differential equations which corresponds to Hitchin’s self-duality equations for a suitable choice of Higgs bundle on closed Riemann surfaces. The main challenge of this functional is its lack of regularity and lack of compactness when defined in its natural domain of definition. Though a standard variational approach cannot directly be applied, we provide the appropriate analytical tools that make Donaldson functional treatable by a variational viewpoint. We prove that this functional admits a unique critical point corresponding to its global minimum. As an immediate consequence, we find that this system of self-duality equations admits a unique solution. Among the applications in geometry of this fact, we obtain a parametrization of closed constant mean curvature immersions in hyperbolic manifolds (possibly incomplete), and their moduli spaces.

1. Introduction

In his paper we present a variational approach to construct minimal immersions and minimal Lagrangian immersions of closed surfaces in hyperbolic three-manifolds and complex hyperbolic 2-manifolds. In this way we obtain useful information about the representations of the fundamental group of the surface into PSL(2, C) and PU(2, 1).

Throughout this paper, we let S be a smooth, closed, oriented surface of genus \( g \geq 2 \) and \( \pi_1(S) \) be its fundamental group. The Teichmüller space \( T_g(S) \) is the space of conformal structures on \( S \), modulo biholomorphisms in the homotopy class of the identity. Uhlenbeck ([Uhl83]) initiated a study of moduli spaces of minimal immersions of a closed surface into a three-manifold of constant sectional curvature \(-1\). Typically in this context, the three-manifold is hyperbolic, homeomorphic to \( S \times \mathbb{R} \) and possibly incomplete. These minimal immersions naturally induce representations of \( \pi_1(S) \) into the group PSL(2, \( C \)), the (orientation preserving) isometry group of \( \mathbb{H}^3 \). She considered the possibility of characterizing such class of irreducible representations, by fixing a conformal class \( X \) on \( S \) and a holomorphic quadratic differential \( q(z)dz^2 \) on \( X \). She proved a range of results for minimal immersions in quasi-Fuchsian manifolds by using a bifurcation analysis based on the implicit function theorems. More recently, it was shown ([HL12, HLT21]) that for a given data \( (X, q(z)dz^2) \), a minimal immersion may not exist. When it exists, one obtains

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a second solution in addition to the stable one constructed by Uhlenbeck. A similar multiplicity result was pointed out in [HLL13] for prescribed holomorphic cubic differentials in the construction of minimal Lagrangian immersions. It is natural then to ask if one can parametrize the space of minimal immersions in another way.

In this spirit, Gonçalves-Uhlenbeck ([GU07]) proposed to parametrize the space of immersions of constant mean curvature surfaces in hyperbolic three-manifolds (homeomorphic to $S \times \mathbb{R}$ and possibly incomplete) by elements of the tangent bundle of the Teichmüller space. Such tangent bundle is identified as the collection of pairs $(X, [\beta])$, where $X$ is a conformal structure on the surface and $[\beta]$ is a cohomology class of $(0,1)$-forms valued in $T^{1,0}_X$, the holomorphic tangent bundle over $X$. We can trace back an analogous point of view to the Higgs bundle approach introduced by Hitchin in [Hit87]. For example, we see that, for given a pair $(X, [\beta])$ one can obtain a minimal immersion of $X$ in a hyperbolic three-manifold by solving the Gauss-Codazzi equations:

$$\begin{cases}
\partial \bar{\partial} \log(h) - h^2 \|\beta\|^2 - h^2 = 0 , \\
\bar{\partial} (*_h \beta) = 0
\end{cases}$$

expressed in terms of an Hermitian metric $h$ defined on the line bundle $K^{-\frac{3}{2}}$, where $K = (T^{1,0}_X)^{-1}$ is the canonical bundle of $X$, and a suitable representative in the class $[\beta]$ which, abusing our notations, is still denoted by $\beta$ in (1.1). As usual we use the Hermitian extension of the hyperbolic metric on $X$ to define $\|\beta\|^2$, while $*_h$ denotes the Hodge dual operator with respect to the metric $h$. From a solution of (1.1) we obtain the pullback metric $g$ of the immersion from the Hermitian metric $h^2$ on $T^{1,0}_X$, and also we find that $\beta$ is harmonic with respect to $g$. Furthermore, $4 *_h \beta$ defines a holomorphic quadratic differential on $X$ whose real part identifies the second fundamental form of the immersion.

We can show the equivalence between (1.1) and Hitchin’s self-duality equations for a suitable nilpotent SL(2, $\mathbb{C}$)-Higgs bundle $(\mathcal{E}, \phi)$ given as follows. We let the rank two bundle $\mathcal{E} = K^{-\frac{3}{2}} \oplus K^\frac{1}{2}$ equipped with the holomorphic structure $\bar{\partial}_{\mathcal{E}} = \bar{\partial} + \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$ and Higgs field $\phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then as explicitly derived in [ALS21], the pair $(h, \beta)$ satisfies (1.1) if and only if the Hermitian metric $H = \begin{pmatrix} h & 0 \\ 0 & \frac{1}{h} \end{pmatrix}$ on $\mathcal{E}$ is the unique solution to Hitchin’s self-duality equation:

$$F_{\nabla_H} + [\phi, \phi^{*H}] = 0,$$

where $F_{\nabla_H}$ is the curvature form of the Chern connection $\nabla_H$, and $\phi^{*H}$ is the Hermitian adjoint of $\phi$ with respect to $H$. In particular, by the general results of [Hit87, Don87], the given Higgs bundle $(\mathcal{E}, \phi)$ is stable. For full details of the one-to-one correspondence between the system (1.1) and Hitchin’s self-duality equation (1.2), we refer to [ALS21]. See also recent survey articles ([Wen16, Li19]) on Higgs bundles in relation to harmonic maps and other topics.
It turns out that problem (1.1) has a natural variational structure and the associated functional is referred by [GU07] as the Donaldson functional, inspired by those introduced in Kähler geometry in ([Don85, Sim88]) for the construction of Hermitian-Einstein metrics in holomorphic vector bundles.

We aim to pursue such a variational approach proposed in [GU07], and analyze the corresponding functional $D$ (see definition 1.5) for more general cohomology classes $[\beta]$ (dual to holomorphic $k$-differentials, for any $k \geq 2$), and prove a general uniqueness result showing that the global minimum is the only critical point of $D$.

Actually from ([LM13]), we know that when $k = 3$ there is a close relation between holomorphic cubic differentials on $X$ and equivariant minimal Lagrangian immersions from $\mathbb{D}$ into $CH^2$. Our results provide an alternative proof for the one-to-one correspondence established in [LM19a] by means of Higgs bundles theory.

On the other hand, for $k \geq 4$, a Higgs bundles theory approach to the existence and uniqueness issue is not available at present. It will be interesting and useful to interpret our solution in terms of a Hermitian metric solving Hitchin’s equation for an appropriate choice of a “stable” Higgs bundle.

More importantly, the variational approach provides us with the analytical framework to investigate the asymptotic behavior of minima corresponding to the data $(X, [\beta])$ as $t \to \infty$, and this will be the main objective of our future work.

In the present paper we consider a Donaldson type functional defined in terms of the data $(X, [\beta])$, where $X$ is a conformal structure on the surface and $[\beta]$ is a cohomology class of $(0,1)$-forms valued in the bundle $E = \bigotimes^{k-1} T^{1,0}_X$ ($k \geq 2$), namely $[\beta] \in \mathcal{H}^{0,1}(X, E)$. To this purpose, let $\overline{\partial}$ be the induced holomorphic structure on $E$, and $g_X$ be the unique hyperbolic metric on $X$, with Hermitian extension $h_X$. We let $\beta_0$ in $[\beta]$ be the unique harmonic element with respect to the hyperbolic metric, so that $[\beta] = [\beta_0 + \overline{\partial}\eta]$ for some section $\eta$ of $E$.

As usual, to remain in the given conformal class, we let $h = e^u h_X$ so that $g = e^{2u} g_X$. In this way, system (1.1) can be formulated in terms of the unknowns $(u, \eta)$ as follows:

$$
\begin{cases}
\Delta u + \frac{1}{4} - e^{2u} - \|\beta_0 + \overline{\partial}\eta\|^2 e^{2(k-1)u} = 0 & \text{on } X, \\
\overline{\partial} \left( e^{2(k-1)u} *_E (\beta_0 + \overline{\partial}\eta) \right) = 0,
\end{cases}
$$

where the Laplacian $\Delta$, the Hodge dual $*_E$, and the norm $\|\cdot\|$ are taken with respect to the background hyperbolic metric $g_X$ and corresponding Hermitian extension $h_X$.

To simplify notations it is convenient to operate the following change of variables: $u \mapsto 2(u + \ln 2)$, $[\beta] \mapsto [\beta e^\beta / (2^{k-1} \sqrt{k-1})]$, so that system (1.3) takes the form:

$$
\begin{cases}
\Delta u + 2 - 2e^u - 8(k-1)\|\beta_0 + \overline{\partial}\eta\|^2 e^{(k-1)u} = 0 & \text{on } X, \\
\overline{\partial} \left( e^{(k-1)u} *_E (\beta_0 + \overline{\partial}\eta) \right) = 0.
\end{cases}
$$
Interestingly, solutions to system (1.4) correspond to critical points of the following functional:

\[(1.5)\quad \mathcal{D}(u, \eta) = \int_X \left\{ \frac{1}{4} |\nabla u|^2 - u + e^u + 4 \| \beta_0 + \mathcal{F}_\eta \|^2 e^{(k-1)u} \right\} dA,\]

considered in its natural domain

\[(1.6)\quad \mathcal{W} = \{(u, \eta) \in H^1(X) \times W^{1,2}(X, E) : \int_X e^{(k-1)u} \| \beta_0 + \mathcal{F}_\eta \|^2 dA < \infty\},\]

where \(H^1(X)\) and \(W^{1,2}(X, E)\) are the usual Sobolev spaces (see section 3 for details). In view of the connection with Hitchin’s self-duality equations, as in [GU07], the functional \(\mathcal{D}\) is referred to as a “Donaldson functional”.

For \(k = 2\), the authors in [GU07] observed through a formal computation that the second variation of \(\mathcal{D}\) at each possible critical point is positive definite. By this observation they claimed uniqueness of the critical point, as typically it would follow by standard global bifurcation arguments. However, \(\mathcal{D}\) is not continuous, or even weakly lower semicontinuous in \(\mathcal{W}\), and therefore it needs particular care in order to be tackled by nonlinear techniques, as far as “regularity” and “compactness” issues are concerned.

Thus, to gain some “regularity” for \(\mathcal{D}\), we work in the stronger Banach space \(\mathcal{V} = H^1(X) \times W^{1,p}(X, E)\) with a fixed \(p > 2\). However, while \(\mathcal{D}\) is differentiable of any order in \(\mathcal{V}\) (see Theorem 3.2), now we face a serious problem when verifying any sort of “compactness” property for \(\mathcal{D}\) in \(\mathcal{V}\), as for example the well known Palais-Smale condition. For these reasons, the available variational approaches developed for nonsmooth functionals (e.g. the “approximation” approach proposed by Struwe ([Str08])) fail to apply to \(\mathcal{D}\). In fact, without “compactness”, even the information (we have obtained) that all critical points of \(\mathcal{D}\) in \(\mathcal{V}\) are strict local minima (see Proposition 4.8) isn’t strong enough to imply “uniqueness”. Indeed we could run into a situation similar to the function: \(f(z) = |e^z - 1|^2\), \(z \in \mathbb{C}\), whose critical points are infinitely many strict local minima exactly located at \(z = 2\pi ni\), \(n \in \mathbb{Z}\). Clearly the main difficulty is to gain control on the component \(\eta\). However, such component is well behaved along critical points, by the holomorphic condition in (1.4). We exploit exactly this fact, and by using Ekeland \(\epsilon\)-variational principle, we succeed to construct pre-compact Palais-Smale sequences. In this way we carry out a variational approach and show that indeed \(\mathcal{D}\) admits a global minimum in \(\mathcal{W}\), which is its unique critical point.

**Theorem A.** Let \(X \in \mathcal{T}_g(S)\) be a closed Riemann surface, and \(E = k^{-1} \otimes T_X^{1,0}\) be the tensor product of its holomorphic tangent bundles. Then for each cohomology class \([\beta]\) of \((0, 1)\)-forms valued in \(E\), the Donaldson functional (1.5) admits a unique critical point \((u, \eta) \in \mathcal{V}\) corresponding to a global minimum. Furthermore, \((u, \eta)\) is smooth and it is the only solution to the system (1.4).

There are several applications of Theorem A. For instance, in the case \(k = 2\) and \(E = T_X^{1,0}\), the minimal immersion provided by Theorem A can be lifted to a
minimal immersion from the universal cover $D$ into $H^3$ which is equivariant with respect to the associated representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ (see Section 5 of [Uhl83]). On the other hand, it is always possible to recover the pair of data $(X, [\beta])$ out of an equivariant minimal immersion from $D$ into $H^3$ with respect to some representation. Recalling that a representation is irreducible if and only if the minimal immersion is not totally geodesic ([LM19b]), by virtue of Theorem A, we conclude:

**Corollary 1.** The moduli space of minimal immersions of $D$ into $H^3$ which are equivariant with respect to an irreducible representation of the fundamental group $\pi_1(S)$ into the group $PSL(2, \mathbb{C})$ can be identified with the space $\mathcal{T}_g(S) \times H^{0,1}(X, E)\{0\}$.

Corollary 1 was recently proved in ([LM19b]) via a Higgs bundle approach (and they also attributed this to [DEL97] from the point of view of birational algebraic geometry).

In case $k = 3$, we have $E = T_{X}^{1,0} \otimes T_{X}^{1,0}$ and by Serre duality, $H^{0,1}(X, E)$ is isomorphic to the space $\mathfrak{C}_3(X)$ of holomorphic cubic differentials on $X$. As before, these are used to parametrize the space of equivariant minimal Lagrangian immersions from $D$ into complex hyperbolic plane $CH^2$ (see for instance [LM13, HLL13]). Our main theorem implies the following characterization of all such equivariant minimal Lagrangian immersions, as seen in [LM19a]:

**Corollary 2.** The minimal Lagrangian immersions of $D$ into $CH^2$ which are equivariant with respect to an irreducible representation of $\pi_1(S)$ into the group $PU(2, 1)$ are in one-to-one correspondence with the pairs $(X, [\beta]) \in \mathcal{T}_g(S) \times H^{0,1}(X, E)\{0\}$.

Another application of Theorem A concerns the parametrization of the moduli space of constant mean curvature immersions of $S$ into germs of hyperbolic three-manifolds (see [Tau04]). This problem can be reduced to study the Donaldson functional (with $k = 2$) after a change of variable (see details in Section 6), and in view of Theorem A, it holds:

**Corollary 3.** For each given constant $c$ (with $c^2 < 1$), there is a one-to-one correspondence between the space of constant mean curvature $c$ immersions in a germ of hyperbolic three-manifolds and the space $\mathcal{T}_g(S) \times H^{0,1}(X, E)$.

The paper is organized as follows. In §2, after introducing the necessary notation, we focus on proving a Poincaré inequality with a sharp constant which is crucial to show that the Hessian of the functional $\mathcal{D}$ at a critical point is positive definite.

We will break down the main result to the existence and uniqueness parts. In §3, we prove the Donaldson functional is well defined, bounded from below and smooth in the Banach space $\mathcal{V}$ defined above. We manage to establish appropriate “compactness” properties for $\mathcal{D}$ in $\mathcal{V}$ and construct a convergent minimizing sequence yielding to the desired critical point. Furthermore, this minimum is regular by standard elliptic estimates.
Our strategy for proving the uniqueness part is done in two steps, contained in §4 and §5. We prove first that the quadratic form associated to the second variation of $\mathcal{D}$ at a critical point $(u, \eta)$ is coercive in the space $H^1(X) \times W^{1,2}(X, E)$. Combining this fact with the holomorphic property of $\eta$ in (1.4) and the convexity of the functional with respect to such variable, we succeed in establishing that actually the critical point $(u, \eta)$ is a strict local minimum for $\mathcal{D}$ even with respect to the stronger norm of the space $V$ (Proposition 4.8). To conclude uniqueness we argue by contradiction, and by assuming that $\mathcal{D}$ admits two distinct critical points, by a “mountain-pass” construction and a suitable use of Ekeland’s $\epsilon$-variational principle, we produce an extra critical point for $\mathcal{D}$ which can not be a strict local minimum (Theorem 5.5). This provides a full proof of our main result Theorem A.

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2. Background Material and Poincaré inequalities

2.1. Holomorphic Differentials and First Dolbeault Group. In this subsection, we present the duality between the space of holomorphic $k$-differentials and the $(0,1)$-Dolbeault cohomology group. For this purpose, we collect some notations.

1. If $(x, y)$ is a real local coordinate where $z = x + iy$, then $\frac{\partial}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial z} - i \frac{\partial}{\partial \bar{z}})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial z} + i \frac{\partial}{\partial \bar{z}})$, also $dz = dx + idy$ and $d\bar{z} = dx - idy$;

2. $X \in T_g(S)$ is a conformal structure on $S$. It has a unique hyperbolic metric denoted by $g_X$, whose volume form in a holomorphic coordinate $\{z\}$ can be written as $dA = \frac{1}{2}g_X(z)dz \wedge d\bar{z}$, where $dz \wedge d\bar{z} = \frac{1}{4}(dz \otimes d\bar{z} - d\bar{z} \otimes dz)$;

3. Throughout this paper, for Laplace operator, inner products, norms and volume elements, we always use the hyperbolic metric as the background metric unless we specify a lower index such as $\Delta_g$, $\langle \cdot, \cdot \rangle_g$, $\| \cdot \|_g$ or $dA_g$.

4. $T^{1,0}_X$: the holomorphic tangent bundle over $X$. Since the complex dimension of $X$ is 1, the dual of $T^{1,0}_X$ coincides with the canonical bundle $K_X = K$.

5. We always assume $k \geq 2$, and denote the tensor product $E = E_k = \bigotimes_{i=1}^{k-1} T^{1,0}_X$, and $E^*$ the dual bundle of $E$;

6. $A^0(E) = \{\eta\}$ is the space of smooth sections of $E$; we denote $A^{0,1}(X, \mathbb{C})$ the space of complex-valued $(0,1)$-forms on $X$, and $A^{0,1}(X, E) = \{\beta\}$ the space of $(0,1)$-forms on $X$ valued in $E$, i.e., $A^{0,1}(X, E) = A^{0,1}(X, \mathbb{C}) \otimes E$;

7. $\mathcal{C}_k(X)$: the space of holomorphic $k$-differentials on $X$, or equivalently the space of holomorphic sections of the bundle $\bigotimes T^{1,0}_X$, often also denoted by $H^0(X, \bigotimes T^{1,0}_X)$. Any such differential is locally of the form $q(z)dz^k$ on $X$. 6
where $q(z)$ is holomorphic. As a consequence of the Riemann-Roch Theorem, the complex dimension of $C_k(X)$ is $(2k - 1)(g - 1)$.

(8) The $(0, 1)$-Dolbeault cohomology group $\mathcal{H}^{0,1}(X, E)$ is defined as the quotient space $A^{0,1}(X, E) / \overline{\partial}(A^0(E))$, where $\overline{\partial} : A^0(E) \to A^{0,1}(X, E)$ is the $\bar{\partial}$-operator. By Hodge Theory, there is a natural isomorphism between $\mathcal{H}^{0,1}(X, E)$ and the space of harmonic $(0, 1)$-forms valued in $E$. Therefore $\forall [\beta] \in \mathcal{H}^{0,1}(X, E)$, there is a unique harmonic element $\beta_0$ with respect to the hyperbolic metric such that $\beta = \beta_0 + \overline{\partial} \eta$, with $\eta \in A^0(E)$.

Given a Riemannian metric $g$ on $X$, it induces a Hermitian metric $h$ on $X$ with $h(v, w) = g_{C}(v, \overline{w})$ where $g_{C}$ is the complex extension of $g$ as a bilinear form. We can extend it to obtain a Hermitian metric $\langle \cdot, \cdot \rangle_g$ on sections and forms valued in $E$.

Let $\alpha$ be a differential form on $X$ valued in $\mathbb{C}$, then we define Hodge star $\ast \alpha$ as $\psi \wedge \ast \alpha = \langle \psi, \alpha \rangle dA$ for every $\psi$. In a local holomorphic coordinate $\{z\}$, we have $\ast dz = id\overline{z}$ and $\ast d\overline{z} = -idz$. This can be extended to differential forms valued in $E$ as follows. We define $\bar{\partial}$ operator to be the identification of $E$ and $E^\ast$, i.e. $\bar{\partial} = \{ \beta_0 \in A^{0,1}(X, \mathbb{C}) \text{ and } e \in E \}$. We can also define wedge product for forms valued in vector bundles. Particularly, if $\alpha = \alpha_0 \otimes e \in A^{1,0}(X, E^\ast)$ and $\beta = \beta_0 \otimes e \in A^{0,1}(X, E)$, with $\alpha_0 \in A^{1,0}(X, \mathbb{C})$ and $\beta_0 \in A^{0,1}(X, \mathbb{C})$, $e \in A^0(E)$ and $e^\ast \in A^0(E^\ast)$, then

$$\alpha \wedge \beta = e^\ast(e)\alpha_0 \wedge \beta_0.$$ 

With this and the definition of $\ast E$ on forms valued in bundles, for $\beta_1, \beta_2 \in A^{0,1}(X, E)$, we note:

$$\ast E \beta_1 \wedge \beta_2 = \langle \beta_1, \beta_2 \rangle dA. \tag{2.1}$$

We have the following natural bilinear form:

$$A^{1,0}(X, E^\ast) \times A^{0,1}(X, E) \to \mathbb{C}, \ (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta.$$ 

Since we have the identification $A^{1,0}(X, E^\ast) \cong A^0(X, E^* \otimes K_X)$, and noting that Stoke’s Theorem gives $\int_X \alpha \wedge \overline{\partial} \eta = 0$ for all $\alpha \in A^{1,0}(X, E^\ast)$ which are holomorphic, and all $\eta \in A^0(E)$, we obtain the following bilinear form:

$$H^0(X, E^* \otimes K_X) \times \mathcal{H}^{0,1}(X, E) \to \mathbb{C}, \ (\alpha, [\beta]) \mapsto \int_X \alpha \wedge \beta.$$ 

This is a nondegenerate bilinear form and therefore it induces an injective homomorphism between $H^0(X, E^* \otimes K_X)$ and $\mathcal{H}^{0,1}(X, E)^\ast$, which by Serre’s duality Theorem (see [Vo07]), is an isomorphism. As a consequence, given $[\beta_0 + \overline{\partial} \eta] \in \mathcal{H}^{0,1}(X, E)$, where $\beta_0$ is harmonic, by considering the following linear map

$$\mathcal{H}^{0,1}(X, E) \to \mathbb{C}, \ [\xi] \mapsto \int_X \langle \xi, \beta_0 \rangle dA$$
(well defined since $\int_X \langle \bar{\partial} \eta, \beta_0 \rangle dA = 0$), there exists a unique $\tilde{\alpha} \in H^0(X, E^* \otimes K_X)$ such that $\int_X \tilde{\alpha} \wedge \xi = \int_X \langle \xi, \beta_0 \rangle dA$, i.e., $\tilde{\alpha} = \ast_E \beta_0$. Since in our notation, $C^k(X) = H^0(X, E^* \otimes K_X)$, we obtain the following isomorphism:

$$H^0,1(X, E) \rightarrow C^k(X), \; [\beta] \mapsto \ast_E \beta_0.$$  

2.2. Poincaré type inequalities. In this subsection, we will present two Poincaré type inequalities in Proposition 2.1 and Proposition 2.2. We suspect inequalities of this type are standard, and we only include the proofs here for the sake of completeness.

Proposition 2.1. Let $X$ be a closed Riemann surface with the Riemannian metric $g$, and $K(g)$ be its Gaussian curvature. Then

$$\int_X \langle \bar{\partial} E \psi, \bar{\partial} E \psi \rangle_g dA_g \geq -(k-1) \int_X K(g) \langle \psi, \psi \rangle_g dA_g,$$

holds for every section $\psi \in W^{1,2}(X, E)$. In particular if we use the hyperbolic metric $g_X$ and $dA$ its volume form, we have

$$\int_X \langle \bar{\partial} E \psi, \bar{\partial} E \psi \rangle dA \geq (k-1) \int_X \langle \psi, \psi \rangle dA.$$

Proof. By density of smooth sections in $W^{1,2}(X, E)$, it is enough to prove this for smooth section $\psi$. In the conformal coordinate $\{z\}$ of $X$, we write the metric $g = e^{2\phi} |dz|^2$ and let $\omega = \frac{i}{2} e^{2\phi} dz \wedge d\bar{z}$ be its Kähler form. Then we have $*\omega = 1$ and $*1 = \omega$.

The exterior differentiation $d$ does not extend to vector-valued forms, we will work with Chern connection on holomorphic bundles here. For a holomorphic vector bundle $E$ and a local holomorphic frame $F$, the Chern connection $D_E = D' + D''$ can be characterized with respect to $F$ as follows:

$$D' = \partial_E + \theta(F), \quad D'' = \bar{\partial}_E,$$

where $\theta(F)$ is the connection matrix with respect to $F$.

Writing $\Delta'' = D'' \delta'' + \delta'' D''$ and $\Delta' = D' \delta' + \delta' D'$, where $\delta'$ and $\delta''$ are the formal adjoint operators of $D'$ and $D''$, respectively, we have the Chern curvature form $\Theta(E) = d\theta + \theta \wedge \theta$. We define the operators

$$0 \xrightarrow{L} A^0(X, \mathbb{C}) \xrightarrow{L} A^{1,1}(X, \mathbb{C}),$$

where the second arrow is defined as $\eta \mapsto \omega \wedge \eta = \eta \omega$, and its adjoint

$$\Lambda : A^{1,1}(X, \mathbb{C}) \rightarrow A^0(X, \mathbb{C}), \; \Lambda = \ast^{-1} L \ast.$$  

We extend them on forms valued in the bundle $E$ by tensor product with the identity map on $E$.

Using the Akizuki-Nakano identity ([AN54]): $\Delta'' = \Delta' + [i \Theta(E), \Lambda]$, we obtain:

$$\int \langle \Delta'' \psi, \psi \rangle_g = \int \langle \Delta' \psi, \psi \rangle_g + \int \langle [i \Theta(E), \Lambda] \psi, \psi \rangle_g.$$
For a smooth section $\psi$ of $E$, we have $\delta'(\psi) = \delta''(\psi) = 0$, and $D'' = \overline{\partial}E$, therefore,
\[
\int \langle \overline{\partial}E\psi, \overline{\partial}E\psi \rangle_g = \int \langle D'\psi, D'\psi \rangle_g + \int \langle [i\Theta(E), \Lambda]\psi, \psi \rangle_g \\
\geq \int \langle [i\Theta(E), \Lambda]\psi, \psi \rangle_g.
\]

Let us calculate the term $[i\Theta(E), \Lambda]$ for our case where $E = \bigotimes_{k=1}^{k-1} T_X^{1,0}$. Since the frame $F$ is locally $\bigotimes_{k=1}^{k-1} \frac{\partial}{\partial z}$, we have $\theta(F) = \|F\|^{-2k-1}\partial\|F\|^{2(k-1)} = 2(k-1)\partial\phi$, and $\Theta(E) = d\theta + \theta \wedge \theta = 2(k-1)\overline{\partial}\partial\phi$, given that the complex dimension of the surface is one. Therefore, for each $\psi \in A^0(E)$, we get
\[
\int \frac{1}{2(k-1)} \Theta(E)(\psi) = (\overline{\partial}\partial\phi)(\psi) = \phi_{\bar{z}z}\psi d\bar{z} \wedge dz = -\phi_{\bar{z}z}\psi dz \wedge d\bar{z}.
\]

Thus,
\[
\int \frac{1}{2(k-1)} i(\Lambda \circ \Theta(E))(\psi) = -i\Lambda(\phi_{\bar{z}z}dz \wedge dz) \otimes \psi = -2\Lambda(\omega) \otimes (e^{-2\phi}\phi_{\bar{z}z})\psi.
\]

Since $\Lambda\psi = 0$, and $\Lambda(\omega) = *^{-1}L * \omega = *^{-1}L1 = *^{-1}\omega = 1$, we have
\[
[i\Theta(E), \Lambda]\psi = i\Theta(E)\Lambda\psi - i\Lambda\Theta(E)\psi = 4(k-1)e^{-2\phi}\phi_{\bar{z}z}\psi.
\]

Using $K(g) = -4e^{-2\phi}\phi_{\bar{z}z}$, we find:
\[
\int \langle \overline{\partial}E\psi, \overline{\partial}E\psi \rangle_g \geq \int 4(k-1)e^{-2\phi}\phi_{\bar{z}z}\langle \psi, \psi \rangle_g \\
= -(k-1) \int K(g)\langle \psi, \psi \rangle_g.
\]

\[\square\]

We end this section with the following $L^p$-version of the Poincaré inequality, and we state it for a general holomorphic vector bundle $\tilde{E}$ over $X$:

**Proposition 2.2.** Let $X$ be a closed Riemann surface with the hyperbolic metric $g_X$, and $\tilde{E}$ be a holomorphic vector bundle over $X$, and $h$ be an Hermitian metric on $\tilde{E}$. Suppose the only global holomorphic section of $\tilde{E}$ is zero. Then for $1 < p < \infty$, there exists a positive constant $C = C(X, \tilde{E}, h, p)$, such that, for any section $\psi$ of $W^{1,2}(X, \tilde{E})$, there holds:

\[
(2.4) \quad \int_X \|\overline{\partial}E\psi\|^p dA \geq C \int_X \|\psi\|^p dA.
\]

**Proof.** We prove this by contradiction. If not, there exists a sequence of smooth sections $\psi_j$, such that
\[
(2.5) \quad \|\psi_j\|_{L^p} = 1 \text{ and } \|\overline{\partial}E\psi_j\|_{L^p} < \frac{1}{j}, \text{ for } j = 1, 2, \cdots.
\]

We claim that, regarding $\overline{\partial}E$ as a real operator, it is elliptic. Let $e_\alpha$ be a local holomorphic frame of $\tilde{E}$, then locally, we can write $\psi = \psi^\alpha e_\alpha$, where $\psi^\alpha$ is a complex valued function.
Let $\psi^\alpha = u^\alpha + iv^\alpha$ and $z = x + iy$ be a local holomorphic coordinate system on $X$. Then $d\bar{z} \otimes e_\alpha$ is a local frame for $(T^1_X)^* \otimes \bar{E}$, and locally, we have

$$\overline{\partial}_E \psi = \frac{1}{2}(\partial_x(u^\alpha + iv^\alpha) - i\partial_y(u^\alpha + iv^\alpha))d\bar{z} \otimes e_\alpha = \frac{1}{2}((\partial_x u^\alpha + \partial_y v^\alpha) + i(-\partial_y u^\alpha + \partial_x v^\alpha))d\bar{z} \otimes e_\alpha.$$

Considering now $\overline{\partial}_E$ as a real differential operator, we have

$$\overline{\partial}_E : (u^\alpha v^\alpha) \mapsto \frac{1}{2} \begin{pmatrix} \xi_x & \xi_y \\ -\xi_y & \xi_x \end{pmatrix} \begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix} d\bar{z} \otimes e_\alpha.$$ 

Since the matrix symbol associated to each component of the operator is given by $\frac{1}{2} \begin{pmatrix} \xi_x & \xi_y \\ -\xi_y & \xi_x \end{pmatrix}$, and it has nonzero determinant, we deduce that $\overline{\partial}_E$ is elliptic. (Notice that this is not true in general for high dimensional base manifolds). Then by standard elliptic estimates ([DN55]), there exists a positive constant $C$, independent of $j$, such that

$$\|\psi_j\|_{W^{1,p}} \leq C(\|\overline{\partial}_E \psi_j\|_{L^p} + \|\psi_j\|_{L^p}).$$

Then by (2.5), we see that the $W^{1,p}$-norm of $\psi_j$ is uniformly bounded. Then we can choose a subsequence which converges weakly in $W^{1,p}$. We denote the weak limit by $\psi_0$. Then from the compact embedding theorem, $\psi_j$ converges strongly in $L^p$ to $\psi_0$. This implies $\|\psi_0\|_{L^p} = 1$. On the other hand, since the functional $\int_X \|\overline{\partial}_E \psi\|^p dA$ is convex, then by weak lower semi-continuity,

$$\int_X \|\overline{\partial}_E \psi_0\|^p dA \leq \liminf \int_X \|\overline{\partial}_E \psi_j\|^p dA = 0.$$

Hence $\overline{\partial}_E \psi_0 = 0$, almost everywhere. From the elliptic regularity (see for instance [Nar92]), $\psi_0$ is smooth, then holomorphic. From the assumption that there is no global holomorphic section, $\psi_0$ must be 0, which contradicts the assumption that $\|\psi_0\|_{L^p} = 1$. This completes the proof.

Remark 2.3. Proposition 2.1 implies that the only global holomorphic section for the negative holomorphic line bundle $E$ is zero. This fact can also be seen from standard Kodaira-vanishing theorems in complex geometry (see [Chapter VI, Theorem 2.4(b), [Wel08]]). Therefore the assumptions in Proposition 2.2 hold for the bundle $E$. In the case of $L^2$-version of the inequality, the constant is explicit and universal, while the constant for general $L^p$-version is less explicit. We later will take advantage of both properties.

3. Existence of Critical Point for the Donaldson Functional

Recall that the Sobolev space $H^1(X)$ is the closure of $C^1$-functions on Riemann surface $X$, with respect to the $H^1$-norm defined as:

$$\|u\|_{H^1} = \left\{ \int_X (|u|^2 + |\nabla u|^2) dA \right\}^{\frac{1}{2}}.$$
Let us also define some Sobolev spaces for differential forms valued in $E$. The Hermitian form $\langle \cdot , \cdot \rangle$ defined fiberwise on each space $A^{0,j}(X, E)$ ($j = 0, 1$) induces a norm denoted by $\| \cdot \|$. In local coordinates, if $\eta \in A^{0,0}(X, E) = A^0(E)$, and $\beta \in A^{0,1}(X, E)$, then

$$
\|\eta\| = |\eta(z)|(g_X)^{\frac{1}{2}} , \quad \|\beta\| = |\beta(z)|(g_X)^{\frac{1}{2}}.
$$

Given $q \geq 1$, a $(0,j)$-form $\alpha$ valued in $E$ is said to belong to $L^q(A^{0,j}(X, E))$ if $\int_X \|\alpha\|^q dA < \infty$. We also say a section $\eta$ of the bundle $E$ belongs to $W^{1,q}(X, E)$ if $\int_X (\|\eta\|^q + \|\bar{\partial}\eta\|^q) dA < \infty$, and its $W^{1,q}$-norm is given as follows:

$$
\|\eta\|_{W^{1,q}(X, E)} = \left\{ \int_X (\|\eta\|^q + \|\bar{\partial}\eta\|^q) dA \right\}^{\frac{1}{q}}.
$$

We analyze the functional $D$ in the space $\mathcal{V} = H^1(X) \times W^{1,p}(X, E)$, for fixed $p > 2$, endowed with the norm $\| \cdot \|_\mathcal{V}$ given as follows

$$
\|(u, \eta)\|_\mathcal{V} = \sqrt{\|u\|^2_{H^1} + \|\eta\|^2_{W^{1,p}(X, E)}}.
$$

We observe that $(\mathcal{V}, \| \cdot \|_\mathcal{V})$ is a uniformly convex Banach space ([Cla36]), a property we explore in section §5. We will prove the existence and regularity of a critical point for $D$.

The fact that $D$ is well defined is a consequence of the Moser-Trudinger inequality, which states that on any closed Riemannian surface $(X, g_X)$, there exists some constant $C_X > 0$ such that (see Theorem 2.50 in [Aub98] or [Fon93]):

$$
\int_X e^{4\pi \sqrt{\|\bar{\partial}\eta\|_{L^2}}^2} dA \leq C_X , \; \forall u \in H^1(X) \text{ with } \int_X u dA = 0.
$$

This inequality has the following consequence:

$$
\int_X e^v dA \leq C_X e^{\int_X v dA} e^{\frac{\|\bar{\partial}\eta\|^2_{L^2}}{16\pi}}, \; \forall v \in H^1(X).
$$

Since $|e^v - e^{v_0}| \leq e^{v_0} |e^{v-v_0}| |v - v_0|$ (by mean value theorem) for any $v, v_0 \in H^1(X)$, using (3.4), we have:

$$
\int_X |e^v - e^{v_0}| dA \leq (C_X \int_X e^{4v_0} dA) \left\{ \frac{e^{4v_0} - e^{v_0}}{16\pi} + \frac{\|\bar{\partial}\eta\|^2_{L^2}}{4\pi} \right\} \|v - v_0\|_{L^2}.
$$

In particular, these inequalities imply that, for every $q \geq 1$, the map $H^1(X) \to L^q(X)$, $v \mapsto e^v$ is well defined, $C^\infty$, and compact.

3.1 Differentiability of the Donaldson functional. In this subsection we show that $D$ is smooth in the space $\mathcal{V}$. This justifies why we choose to work with such space, even though it does introduce some difficulties when dealing with “compactness” issues. To this purpose, we first notice that the Donaldson functional

$$
D(u, \eta) = \int_X \left\{ \frac{1}{4} |\nabla u|^2 - u + e^u + 4\|\bar{\partial}\eta\|^2 e^{(k-1)u} \right\} dA
$$

can be written as $D(u, \eta) = A(u) + 4\mathcal{B}(u, \eta)$, where

$$
A(u) = \int_X \left\{ \frac{1}{4} |\nabla u|^2 - u + e^u \right\} dA,
$$

\[\text{(3.6)}\]
Lemma 3.1. The expression of $A$ is the space of continuous bilinear maps from $Y$ to $Z$, and $\beta$ is a composition of the map $\tilde{\tau}$ with vanishing higher derivatives; and

$\tilde{\tau}'(\beta, 1, \zeta) = 2\left( v_1 \Re \langle \beta, \zeta \rangle + v_2 \Re \langle \beta, \zeta \rangle + g \Re \langle \beta, \zeta \rangle \right)$.

Proof. (i) The expressions of $\tau_F'[\zeta]$ and $\tau_F''[\zeta_1, \zeta_2]$ are obtained by explicit calculations. Furthermore, setting $Y = L^p(\mathcal{A}^{0,1}(X, E))$ and $Z = \tilde{H}(X)$, and using Cauchy-Schwarz inequality, we check the continuity of the following maps

$\begin{align*}
Y &\to L(Y, Z), & Y &\to \text{Bil}(Y \times Y, Z), \\
F &\to \tau_F', & F &\to \tau_F''
\end{align*}$

where $\mathcal{L}(Y, Z)$ is the space of continuous linear maps from $Y$ to $Z$, and $\text{Bil}(Y \times Y, Z)$ is the space of continuous bilinear maps from $Y \times Y$ to $Z$, both endowed with the sup-norm.

(ii) The map $\tilde{\tau}$ is a composition of the map $\tilde{\tau}' \times id : L^p(\mathcal{A}^{0,1}(X, E)) \times L^{\frac{p}{p-2}}(X) \to L^{\tilde{\tau}'}(X) \times L^{\tilde{\tau}''}(X)$

$(F, f) \mapsto (\langle F, F \rangle, f)$

with the map $L^{\tilde{\tau}'}(X) \times L^{\tilde{\tau}''}(X) \to L^1(X)$, $(g_1, g_2) \mapsto g_1 g_2$. \hfill $\square$

Theorem 3.2. The functional $B \in C^\infty(\mathcal{V})$, and consequently $D$ is smooth in $\mathcal{V}$. Moreover, let $\beta = \beta_0 + \beta \eta$, then for every $(u, \eta) \in \mathcal{V}$, we have

$B'_\eta[u, \eta](v, \ell) = \int_X \left( c^{k-1}u ((k-1)\|\beta\|^2v + 2\Re \langle \beta, \beta \ell \rangle) \right) \, dA$, \quad \forall \, (v, \ell) \in \mathcal{V},$
and

\[ B''(u, \eta)[(v_1, \ell_1), (v_2, \ell_2)] = \int_X e^{(k-1)u} \left\{ (k-1)^2 \| \beta \|^2 v_1v_2 + 2\langle \overline{\partial} \ell_1, \overline{\partial} \ell_2 \rangle + 2(k-1)(v_1 \Re(\beta, \overline{\partial} \ell_2) + v_2 \Re(\beta, \overline{\partial} \ell_1)) \right\}, \tag{3.11} \]

for all \((v_1, \ell_1), (v_2, \ell_2) \in \mathcal{V}.

**Proof.** From (3.5), we see that for each \(q \geq 1\), the map \(u \mapsto e^{(k-1)u}\) from \(H^1(X)\) to \(L^q(X)\) is smooth, and its derivative can be easily calculated. At this point we write \(B(u, \eta) = \int_X \bar{\nabla} (\beta_0 + \overline{\eta}, e^{(k-1)u}) dA\). Now formulas (3.10) and (3.11) follow easily via direct calculations. \(\Box\)

### 3.2. Characterization of critical points.

From now on, we use \(\ast\) to denote the Hodge star \(\ast_E\) for forms valued in \(E\). The main result in this subsection is to characterize the critical points of the Donaldson functional:

**Theorem 3.3.** The pair \((u, \eta) \in \mathcal{V}\) is a critical point for \(\mathcal{D}(u, \eta)\) if and only if \((u, \eta)\) is a smooth solution for the system (1.4).

**Proof.** For any family of \((u_t, \eta_t)\) with \(\beta_t = \beta_0 + \eta_t\) and \(u_t = u + tv, \eta_t = \eta + t\ell\), where \((v, \ell) \in \mathcal{V}\), we can readily compute the first variation of \(\mathcal{D}\) from (3.8) and (3.10) as follows:

\[
d/dt|_{t=0} \mathcal{D}(u_t, \beta_t) = \int_X \left\{ \frac{1}{2} \nabla u \nabla v + (1 + e^u + 4(k-1)\| \beta_0 + \overline{\eta} \|^2 e^{(k-1)u}) v \right\} dA,
\]

Hence we find that \((u, \eta)\) is a critical point of \(\mathcal{D}\) if and only if \(u\) is a weak solution in \(H^1(X)\) of the equation

\[ \Delta u + 2e^u - 8(k-1)\| \beta_0 + \overline{\eta} \|^2 e^{(k-1)u} = 0, \tag{3.12} \]

and this is the first equation in the system (1.4), and furthermore:

\[ \int_X \Re(\beta_0 + \overline{\eta}, \overline{\partial} \ell) e^{(k-1)u} = 0, \quad \text{for any } \ell \in A^0(E). \tag{3.13} \]

By taking \(\sqrt{-1} \ell\) instead of \(\ell\) in (3.13), we find more generally that, at a critical point \((u, \eta)\) of \(\mathcal{D}\),

\[ \int_X \langle \beta_0 + \overline{\eta}, \overline{\partial} \ell \rangle e^{(k-1)u} dA = 0, \quad \text{for any } \ell \in A^0(E). \tag{3.14} \]

We need to show that (3.14) is equivalent to the condition that \(e^{(k-1)u} \ast (\beta_0 + \overline{\eta})\) is a holomorphic \(k\)-differential, as stated by the second equation in the system (1.4).

This is obtained as follows:

\[ \int_X \langle \beta_0 + \overline{\eta}, \overline{\partial} \ell \rangle e^{(k-1)u} dA = 0, \quad \text{for any } \ell \in A^0(E) \]

\[ \iff \int_X \langle \overline{\partial} \ell, \beta_0 + \overline{\eta} \rangle e^{(k-1)u} dA = 0, \quad \text{for any } \ell \in A^0(E). \]
\[ \int_X e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta) \wedge (\overline{\partial} \ell) dA = 0, \quad \text{for any } \ell \in A^0(E) \]
\[ \int_X \overline{\partial}(e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta)) dA = \int_X e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta) dA, \quad \forall \ell \in A^0(E) \]
\[ \int_X \partial(e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta)) dA = 0, \quad \forall \ell \in A^0(E) \]
\[ \overline{\partial}(e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta)) = 0. \]

Now we are left to show the regularity for the critical point \((u, \eta) \in V\). We start with the following:

**Claim:** If \( \eta \) satisfies \((3.14)\) with \( u \in H^1(X) \), then \( \eta \in W^{1,q}(X, E) \) for any \( q \geq 1 \).

**Proof of the Claim:** To establish this, we consider the bundle \( E \otimes K_X \) over \( X \), which has a Hermitian inner product that arises from the Hermitian product defined on \( X \). Since the surface is closed, we have that the space of holomorphic sections over \( X \) is a finite dimensional vector space over \( \mathbb{C} \) (see for instance Finiteness Theorems in [Nar92]). Let us choose a basis \( \{ s_1, \cdots, s_N \} \) on this space of holomorphic sections such that
\[ \int_X \langle * s_i, * s_j \rangle = \delta_{ij}. \]

Since \((3.14)\) holds, by Weyl’s regularity Lemma, we find that \( e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta) \) is a holomorphic section (or a holomorphic \( k \)-differential on \( X \)). Hence by the finiteness property of the space of holomorphic sections, for some \( \alpha^i \in \mathbb{C} \), we have
\[ e^{(k-1)u} \ast (\beta_0 + \overline{\partial} \eta) = \sum_{i=1}^{N} \alpha^i s_i. \]

We denote the inverse map of the Hodge star \(*\) by \(*^{-1}\), then
\[ e^{(k-1)u} (\beta_0 + \overline{\partial} \eta) = \sum_{i=1}^{N} \alpha^i (*^{-1} s_i). \]

From this we have
\[ \overline{\partial} \eta = -\beta_0 + e^{-(k-1)u} \sum_{i=1}^{N} \alpha^i (*^{-1} s_i). \]

Since \( e^u \in L^q(X) \) for all \( q \geq 1 \), and \( \overline{\partial} \) is an elliptic operator, using trivialization and standard elliptic estimates, we deduce that \( \eta \in W^{1,q}(X, E) \) for all \( q \geq 1 \). The Claim is proved.

Therefore we can apply this information in \((3.12)\) and by elliptic regularity obtain that \( u \in W^{2,q}(X) \) for all \( q \geq 1 \). In particular, we see that \( u \in C^{1,b}(X) \) for some \( b \in (0,1) \). Now elliptic regularity theory applied to equations \((3.12)\) and \((3.16)\) combined with a bootstrapping argument allow us to obtain all the desired regularity for \((u, \eta)\).
3.3. A Priori Estimates. Clearly the Donaldson functional is bounded from below by the value $4\pi(g-1)$. To analyze a minimizing sequence, we first provide some elementary estimates.

Lemma 3.4. For each $C > 0$, consider the sublevel set

$$D^C := \{(u, \eta) \in V : D(u, \eta) \leq C\}.$$

Then we have:

(i) The set \( \{u : (u, \eta) \in D^C\} \) is bounded in \( H^1(X) \), and

(ii) For any \( a \in [1, 2) \), there exists a constant \( C_a > 0 \) such that:

$$\hat{\mathcal{X}} \|\beta_0 + \partial \eta\|^a dA \leq C_a, \quad \forall (u, \eta) \in D^C.$$

Proof. We write \( \beta = \beta_0 + \partial \eta \). Since the function \( e^x - x \) is always positive, and by assumption we have \( D(u, \eta) \leq C \), then we find

$$\hat{\mathcal{X}} \|\nabla u\|^2 dA \leq C,$$

and

$$\hat{\mathcal{X}} \|\partial \eta\|^2 e^{(k-1)u} dA \leq C,$$

and

$$\hat{\mathcal{X}} \{e^u - u\} dA \leq C.$$

Furthermore we observe that for some positive constant \( C_1 \), we have:

$$e^u - u \geq |u| - C_1, \quad \forall u \in \mathbb{R}.$$

Thus, we conclude that,

$$\hat{\mathcal{X}} |u| < C_2,$$

for some suitable \( C_2 > 0 \). Hence by means of (3.18) and Poincaré inequality we obtain the desired \( H^1 \)-estimate.

To show (3.17), let \( a \in [1, 2) \), \( q = \frac{2}{a} \), and \( b = \frac{(k-1)u}{2} \), we have

$$\hat{\mathcal{X}} \|\beta\|^a dA = \hat{\mathcal{X}} \|\beta\|^a e^{bu} e^{-bu} dA$$

$$\leq \left( \hat{\mathcal{X}} \|\beta\|^{qa} e^{bu} \right)^{\frac{1}{q}} \left( \hat{\mathcal{X}} e^{-q'bu} \right)^{\frac{1}{q'}}$$

$$= \left( \hat{\mathcal{X}} \|\beta\|^2 e^{(k-1)u} \right)^{\frac{a}{2}} \left( \hat{\mathcal{X}} e^{-q'bu} \right)^{\frac{2-a}{2}},$$

with \( q' = \frac{2}{2-a} \).

For \((u, \eta)\) in the set \( D^C \), we know that the first term on the right hand side is bounded by (3.19), and also the second term is bounded by part (i) and Moser-Trudinger inequality.

We prove the following important “compactness” result:
Proposition 3.5. Let \((u_n, \eta_n) \in \mathcal{V}\) be a sequence satisfying
\[(3.20) \quad \int_X e^{(k-1)u_n} \langle \beta_0 + \overline{\eta}_n, \overline{\ell} \rangle = 0, \text{ for any } \ell \in A^0(E).\]

Then if \(u_n \rightharpoonup u\) weakly in \(H^1(X)\), we have, along a subsequence, \(\eta_n \to \eta\) in \(W^{1,p}(X, E)\), with \(\eta\) satisfying (3.14), namely,
\[\int_X e^{(k-1)u} \langle \beta_0 + \overline{\eta}, \overline{\ell} \rangle = 0, \text{ for any } \ell \in A^0(E).\]

Proof. Since \(u_n \rightharpoonup u\), we know that \(u_n\) is uniformly bounded in \(H^1(X)\), and along a subsequence there holds,
\[e^{(k-1)u_n} \xrightarrow{L_q} e^{(k-1)u}, \text{ for any } q \geq 1.\]

By choosing \(\ell = \eta_n\) in (3.20), we have:
\[\int_X e^{(k-1)u_n} \|\overline{\eta}_n\|^2 \, dA = - \int_X e^{(k-1)u_n} \langle \beta_0, e^{(k-1)u_n} \overline{\eta}_n \rangle \, dA \leq C_0 \left( \int_X e^{(k-1)u_n} \right) \frac{1}{2} \left( \int_X e^{(k-1)u_n} \|\overline{\eta}_n\|^2 \right)^{\frac{1}{2}},\]
for some constant \(C_0 > 0\). This implies
\[(3.21) \quad \int_X e^{(k-1)u_n} \|\overline{\eta}_n\|^2 \, dA \leq C,\]
and so we have \(\mathcal{D}(u_n, \eta_n) \leq C\) for some suitable \(C > 0\). Therefore for some \(a \in (1, 2)\), by Lemma 3.4, we find a constant \(C_a > 0\) such that \(\int_X \|\overline{\eta}_n\|^a \leq C_a\).

Hence by using Proposition 2.2, we conclude that \(\eta_n\) is bounded in \(W^{1,a}(X, E)\), and therefore, along a subsequence, \(\eta_n \to \eta\) in \(W^{1,a}(X, E)\). In particular, for \(a' = \frac{a}{a-1}\) the dual exponent of \(a\), and for each fixed \(\xi_0 \in L^{a'}(A^0(E))\), we have
\[(3.22) \quad \int_X \langle \overline{\eta}(\eta - \eta_n), \xi_0 \rangle \, dA \to 0, \text{ as } n \to \infty,\]

since the map \(\xi \mapsto \int_X \langle \overline{\xi}, \xi_0 \rangle\) is a continuous linear map (by Hölder inequality).

Thus, for any smooth \(\ell \in A^0(E)\), using (3.20), we find:
\[
\left| \int_X e^{(k-1)u_n} \langle \beta_0 + \overline{\eta}_n, \overline{\ell} \rangle \right| = \left| \int_X e^{(k-1)u_n} \langle \beta_0 + \overline{\eta}_n, \overline{\ell} \rangle - e^{(k-1)u_n} \langle \beta_0 + \overline{\eta}_n, \overline{\ell} \rangle \right| \\
\leq \left| \int_X (e^{(k-1)u_n} - e^{(k-1)u_n}) \langle \beta_0 + \overline{\eta}_n, \overline{\ell} \rangle \right| \\
\quad + \left| \int X e^{(k-1)u_n} \langle \overline{\eta}(\eta - \eta_n), \overline{\ell} \rangle \right| \\
\leq \left| \int X e^{(k-1)u_n} - e^{(k-1)u_n} \right| \|\beta_0 + \overline{\eta}_n\| \|\overline{\ell}\|_\infty \\
\quad + \left| \int \langle \overline{\eta}(\eta - \eta_n), e^{(k-1)u_n} \overline{\ell} \rangle \right| \\
\leq \|e^{(k-1)u_n} - e^{(k-1)u_n}\|_{L^{a'}} \|\beta_0 + \overline{\eta}_n\|_{L^a} \|\overline{\ell}\|_\infty.
\]
Indeed, the first term in the last inequality goes to zero by the strong convergence of \( e^{(k-1)u_n} \) to \( e^{(k-1)u} \) in \( L^a \), and \( \beta_0 + \overline{\partial} \eta_n \) is bounded in \( L^a \). The second term also goes to zero as a consequence of (3.22) with \( \xi_0 = e^{(k-1)u} \overline{\partial} \ell \). Now we have \((u, \eta) \in V \) and (3.14) holds.

Now we are left to prove \( \eta_n \to \eta \) in \( W^{1,q}(X,E) \), for each \( q > 1 \). To this end, we use (3.15) to write

\[
\alpha^i = \int_X (e^{(k-1)u} (\beta_0 + \overline{\partial} \eta), (s \overline{\partial}^{n-i}))
\]

and

\[
\alpha^i_n = \int_X (e^{(k-1)u} (\beta_0 + \overline{\partial} \eta_n), (s \overline{\partial}^{n-i})).
\]

Furthermore, for \( 1 < a < 2 \), we have:

\[
|\alpha^i - \alpha^i_n| = \left| \int_X (e^{(k-1)u} (\beta_0 + \overline{\partial} \eta) - e^{(k-1)u_n} (\beta_0 + \overline{\partial} \eta_n), (s \overline{\partial}^{n-i}) \right|
\]

\[
\leq \|e^{(k-1)u} - e^{(k-1)u_n}\|_{L^a} \|\beta_0 + \overline{\partial} \eta_n\|_{L^s} \|s \overline{\partial}^{n-i}\|_{\infty}
\]

\[
+ \left| \int_X (\overline{\partial}(\eta - \eta_n), e^{(k-1)u} (s \overline{\partial}^{n-i})) \right|
\]

and as before we conclude that, \( \alpha^i_n \to \alpha^i \), as \( n \to \infty \). On the other hand,

\[
\overline{\partial}(\eta_n - \eta) = e^{-(k-1)u_n} \sum_{i=1}^{N} \alpha^i_n (s \overline{\partial}^{n-i}) - e^{-(k-1)u} \sum_{i=1}^{N} \alpha^i (s \overline{\partial}^{n-i})
\]

\[
= (e^{-(k-1)u_n} - e^{-(k-1)u}) \sum_{i=1}^{N} \alpha^i_n (s \overline{\partial}^{n-i})
\]

\[
+ e^{-(k-1)u} \sum_{i=1}^{N} (\alpha^i_n - \alpha^i) (s \overline{\partial}^{n-i}),
\]

and readily we derive that \( \eta_n \xrightarrow{W^{1,q}} \eta \), for any \( q \geq 1 \). This concludes the proof. \( \square \)

### 3.4. The partial map.

**Definition 3.6.** For each fixed \( u \in H^1(X) \), let us consider the following map:

\( D_u : W^{1,p}(X,E) \to \mathbb{R} \) with \( \eta \mapsto D(u, \eta) \).

This map will be very important in our strategy of proving the existence of a minimizer for the Donaldson functional in the space \( V = H^1(X) \times W^{1,p}(A^0(E)) \), with \( p > 2 \). We first show this map has a unique minimum in \( W^{1,p}(X,E) \).

**Theorem 3.7.** For each \( u \in H^1(X) \), the map above \( D_u \) admits a minimizer \( \eta(u) \) which is its unique critical point in \( W^{1,p}(X,E) \). Furthermore, \( \eta(u) \) lies in \( W^{1,q}(X,E) \), for all \( q \geq 1 \).
Proof. Let \( \eta_n \in W^{1,p}(X,E) \) be a minimizing sequence for the map \( D_u \). By Lemma 3.4, for each \( a \in (1,2) \), we have a constant \( C_a > 0 \) such that

\[
\int_X \| \eta_n \|^a \, dA \leq C_a.
\]

Therefore, for some \( \eta \in W^{1,a}(X,E) \), we have \( \eta_n \xrightarrow{W^{1,a}} \eta \). In addition, as in the proof of Proposition 3.5, we find, for all smooth \( \ell \in A^0(E) \),

\[
\int_X e^{(k-1)u}(\beta_0 + \overline{\eta}, \overline{\eta}) = 0,
\]

and by regularity \( \eta \in W^{1,q}(X,E) \), for all \( q \geq 1 \). Furthermore, from

\[
\int_X e^{(k-1)u}(\overline{\eta}(\eta_n - \eta), \overline{\eta}(\eta_n - \eta)) \geq 0,
\]

we get

\[
\int_X e^{(k-1)u}(\overline{\eta}(\eta_n - \eta), \overline{\eta}(\eta_n - \eta)) \geq \int_X \left\{ (\overline{\eta}(\eta_n - \eta), e^{(k-1)u}\overline{\eta}) + e^{(k-1)u}(\overline{\eta}, \overline{\eta}(\eta_n - \eta)) \right\}
\]

\[
+ \int_X e^{(k-1)u}(\overline{\eta}(\eta_n - \eta), \overline{\eta}).
\]

Since \( \overline{\eta}(\eta_n - \eta) \xrightarrow{L^a} 0 \), and \( e^{(k-1)u}\overline{\eta} \in L^a \), so the first integral in the right hand side of the above inequality goes to zero as \( n \to \infty \). Therefore we have

\[
\inf_{\eta \in W^{1,p}(X,E)} D_u(\eta) = \lim_{n \to \infty} \int_X e^{(k-1)u}\| \beta_0 + \overline{\eta}\eta_n \|^2 \geq \int_X e^{(k-1)u}\| \beta_0 + \overline{\eta}\|^2,
\]

and therefore \( \eta \) is a minimum for the map \( D_u \).

We now observe that the map \( D_u \) is strictly convex in \( \eta \), and so \( \eta \) is the unique critical point of \( D_u \) in \( W^{1,p}(X,E) \).

Clearly, as in Theorem 3.3, we see that, for each \( u \in H^1(X) \), the unique critical point \( \eta(u) \) of the map \( D_u \) given by Theorem 3.7 satisfies:

\[
\int_X \langle e^{(k-1)u}(\beta_0 + \overline{\eta}(u)), \overline{\ell} \rangle = 0, \quad \text{for any } \ell \in A^0(E).
\]

Furthermore by the uniqueness, we deduce:

\[
(3.25) \text{if } (u, \eta) \in \mathcal{V}, \text{and } \int_X \langle e^{(k-1)u}(\beta_0 + \overline{\eta}), \overline{\ell} \rangle = 0, \forall \ell \in A^0(E), \text{then } \eta = \eta(u).
\]

As a direct consequence of Proposition 3.5, we deduce

**Proposition 3.8.** If \( u_n \rightharpoonup u \) in \( H^1(X) \), then \( \eta(u_n) \rightharpoonup \eta(u) \) in \( W^{1,q}(X,E) \), for all \( q \geq 1 \).

We can now show:

**Theorem 3.9.** The Donaldson functional \( \mathcal{D} \) admits a global minimum \((u_0, \eta_0)\) in \( \mathcal{V} = H^1(X) \times W^{1,p}(X,E) \).
Proof. Clearly the functional $D$ is bounded from below by the value $4\pi(g-1)$, and so we may consider a minimizing sequence $(u_n, \eta_n) \in \mathcal{V}$, such that $D(u_n, \eta_n) \to \inf \{D(u, \eta)\}$. By the definition of the map $D_u$ and Theorem 3.7, we have

$$D(u_n, \eta_n) \geq D(u_n, \eta(u_n)),$$

where $\eta(u_n)$ is the unique minimum for the map $D_{u_n}$. Therefore $(u_n, \eta(u_n))$ is also a minimizing sequence for the Donaldson functional. By part (i) of Lemma 3.4, we can further assume that, $u_n \rightharpoonup u$ in $H^1(X)$, and therefore by Proposition 3.8, $\eta(u_n)$ converges to $\eta(u)$ in $W^{1,p}(X, E)$. Therefore, we find

$$\inf D = \lim_{n \to \infty} D(u_n, \eta(u_n)) \geq D(u, \eta(u)),$$

and so $(u, \eta(u))$ is a minimum for the Donaldson functional.

$$\square$$

4. Second Variation of the Donaldson functional

In this section, we study the second variation of the Donaldson functional. The main result in §4.1 is that, at a critical point, the Hessian is positive definite in the space $H^1(X) \times W^{1,2}(X, E)$ (Theorem 4.3). By additional estimates, we will prove in §4.2 that indeed every critical point is a strict local minimum for $D$ in the stronger space $\mathcal{V}$.

4.1. The Second Variation. Recall from (1.5) that the functional $D(u, \eta)$ is defined as:

$$D(u, \eta) = \int_X \left\{ \frac{1}{4} |\nabla u|^2 - u + e^u \right\} dA + 4 \int_X \| \beta_0 + \phi \eta \|^2 e^{(k-1)u} dA,$$

and at a critical point $(u, \eta) \in \mathcal{V}$ we have:

$$\Delta u = 2 - 2e^u - 8(k-1)\| \beta \|^2 e^{(k-1)u} = 0,$$

for $\beta = \beta_0 + \phi \eta$, and $\mathcal{O}(e^{(k-1)u} * (\beta_0 + \phi \eta)) = 0$, that is

$$\int_X (\beta_0 + \phi \eta, \phi \ell) e^{(k-1)u} dA = 0,$$

for all $\ell \in A^{0,1}(X, E)$.

We first observe the following consequence of the Proposition 2.1:

Lemma 4.1. Let $E = \bigotimes^{k-1} T^1_X$, and $\phi$ be a smooth function on $X$, then

$$\int_X \| \mathcal{O} \ell \|^2 e^{4(k-1)\phi} dA \geq (k-1) \int_X (\Delta \phi + 1)\| \ell \|^2 e^{4(k-1)\phi} dA$$

holds for any $\ell \in W^{1,2}(X, E)$.

Proof. We use the metric $g = e^{2\phi}g_X$ conformal to the hyperbolic metric $g_X$. Then its Gaussian curvature can be calculated according to

$$K(g) = e^{-2\phi}(-\Delta \phi - 1),$$

where $\Delta$ is used with respect to the hyperbolic metric $g_X$ which has constant sectional curvature $-1$.  

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Now using this Riemannian metric $g$, we have from (2.3)\[\int_X \langle \overline{\partial \ell}, \overline{\partial \ell} \rangle_g dA_g \geq -(k-1) \int_X K(g)(\ell, \ell)_g dA_g.\]We change these inner products to be in terms of the hyperbolic metric $g_X$. Since $E$ is the tensor product of $k-1$ copies of $T^{0,1}X$, and $\ell \in W^{1,2}(X,E)$, we have\[\langle \ell, \ell \rangle_g = e^{4(k-1)\phi}(\ell, \ell)\]
and since $\overline{\partial \ell}$ is a $(0,1)$-form valued in $E$, we also have\[\langle \overline{\partial \ell}, \overline{\partial \ell} \rangle_g = e^{4(k-1)\phi} e^{-2\phi} \langle \overline{\partial \ell}, \overline{\partial \ell} \rangle.
Finally we use the volume form $dA_g = e^{2\phi} dA$ to conclude the proof. \[\square\]

As a corollary, by taking $\phi = \frac{\alpha}{4}$, we have

**Corollary 4.2.** Let $E = k^{-1} \bigotimes T_X^{1,0}$, and $(u, \eta)$ be a solution of the system of equations (4.1) and (4.2), then\[(4.4) \quad \int_X \| \overline{\partial \ell} \|^2 e^{(k-1)u} \geq 2(k-1)^2 \int_X \| \beta \|^2 \| \ell \|^2 e^{(k-1)u} + \frac{(k-1)}{2} \int_X \| \ell \|^2 e^{(k-1)u}\]
holds for any $\ell \in W^{1,2}(X,E)$.

**Proof.** Since $u$ satisfies\[\Delta u + 2 - 2 e^n - 8(k-1)\| \beta \|^2 e^{(k-1)u} = 0,\]we choose $\phi = \frac{\alpha}{4}$ in Lemma 4.1 to find:\[\int_X \| \overline{\partial \ell} \|^2 e^{(k-1)u} \geq \frac{(k-1)}{2} \int_X \left\{ e^u + 1 + 4(k-1)\| \beta \|^2 e^{(k-1)u} \right\} \| \ell \|^2 e^{(k-1)u} \]
\[\geq 2(k-1)^2 \int_X \| \beta \|^2 \| \ell \|^2 e^{(k-1)u} + \frac{(k-1)}{2} \int_X \| \ell \|^2 e^{(k-1)u}.\]

\[\square\]

Our main result in this subsection is the following:

**Theorem 4.3.** At any critical point $(u, \eta)$, setting $\beta = \beta_0 + \overline{\partial \eta}$, the second variation of the Donaldson functional $D'(u, \eta)$ is strictly positive definite. More specifically we have, for all $v \in H^1(X)$ and all $0 \neq \ell \in W^{1,1}(X,E)$,

\[D''_{(u, \eta)}(v, \ell) = \int_X \left\{ e^u v^2 + 4\| (k-1)v \| \beta + \| \overline{\partial \ell} \|^2 e^{(k-1)u} \right\} dA + R_{(u, \eta)}(v, \ell),\]

(4.5)\[+ 2 \int_X \left\{ \| 2(k-1)e^{(k-1)u} \| \ell \| \beta - \overline{\partial \ell} \| \ell \|^2 \right\} dA + R_{(u, \eta)}(v, \ell),\]

where

(4.6)\[R_{(u, \eta)}(v, \ell) = 4 \int_X \left\{ \| \overline{\partial \ell} \|^2 e^{(k-1)u} - 2(k-1)^2\| \beta \|^2 e^{2(k-1)u} \| \ell \|^2 \right\} dA \]
\[\geq 2(k-1) \int_X e^{(k-1)u} \| \ell \|^2 dA.\]
Proof. At a critical point \((u, \eta)\), we have for any tangent vector \((v, \ell)\), with \(v \in H^1(X)\) and \(\ell \in W^{1,p}(X, E)\), the second variation can be computed as follows:

\[
\mathcal{D}''(v, \ell) = \int_X \left\{ \frac{\nabla v^2}{2} + e^u v^2 \right\}
+ 4 \int_X \left\{ (k - 1)^2 \|\beta\|^2 v^2 + 4(k - 1)\Re(\beta, \overline{\Theta} \ell) v + 2\|\overline{\Theta} \ell\|^2 \right\} e^{(k - 1)u}
= A + B + \int_X \left\{ e^u v^2 \right\},
\]

where we write

\[
A = 4 \int_X \left\{ (k - 1)^2 \|\beta\|^2 v^2 + 2(k - 1)\Re(\beta, \overline{\Theta} \ell) v + \|\overline{\Theta} \ell\|^2 \right\} e^{(k - 1)u},
\]

and

\[
B = \int_X \left\{ \frac{\nabla v^2}{2} \right\} + 4 \int_X \left\{ 2(k - 1)\Re(\beta, \overline{\Theta} \ell) v + \|\overline{\Theta} \ell\|^2 \right\} e^{(k - 1)u}.
\]

Clearly we have:

\[
A = 4 \int_X ((k - 1)\beta + \overline{\Theta} \ell, (k - 1)\beta + \overline{\Theta} \ell) e^{(k - 1)u} \\
\geq 0.
\]

Now let us consider the term \(B\). Since \((u, \eta)\) is a critical point, we have:

\[
\int_X \langle \beta, \overline{\Theta} (v \ell) \rangle e^{(k - 1)u} = 0.
\]

Then by Leibniz’s rule for the operator \(\overline{\Theta} E = \overline{\Theta}\) (see for instance [Wel08]), we have

\[
\overline{\Theta} E(v \ell) = (\overline{\Theta} v) \otimes \ell + v \overline{\Theta} E \ell,
\]

where \(\overline{\Theta} v = \overline{\delta}(v) d\overline{\xi} \in A^{0,1}(X)\) for a function \(v\). Therefore

\[
\int_X \langle \beta, v (\overline{\Theta} \ell) \rangle e^{(k - 1)u} = - \int_X \langle \beta, (\overline{\Theta} v) \otimes \ell \rangle e^{(k - 1)u}.
\]

Noting that

\[
\|\overline{\Theta} v\|^2 = \frac{1}{4} \left\| \partial_x v + \sqrt{-1} \partial_y v \right\|^2 \|d\overline{\xi}\|^2 = \frac{1}{4} |\nabla v|^2,
\]

we can express \(B\) as follows:

\[
B = 2 \int_X \left\{ \|\overline{\Theta} v\|^2 - 4(k - 1)\Re(\beta, \overline{\Theta} v \otimes \ell) e^{(k - 1)u} + 2\|\overline{\Theta} \ell\|^2 e^{(k - 1)u} \right\}.
\]

Since \(\|\overline{\Theta} v\|^2 = \|\overline{\Theta} v \otimes \ell\|^2\), the above equality is equivalent to

\[
B = 2 \int_X \left\{ \|\overline{\Theta} v\|^2 - 2(k - 1) e^{(k - 1)u} \|\beta\|^2 \right\} dA + \mathcal{R}_{(u, \eta)}(v, \ell),
\]

with the remainder term

\[
\mathcal{R}_{(u, \eta)}(v, \ell) = 4 \int_X \left\{ \|\overline{\Theta} \ell\|^2 e^{(k - 1)u} - 2(k - 1)^2 \|\beta\|^2 e^{2(k - 1)u} \right\} dA.
\]

By (4.4), we have established

\[
\int_X \|\overline{\Theta} \ell\|^2 e^{(k - 1)u} \geq 2(k - 1)^2 \int_X \|\beta\|^2 e^{2(k - 1)u} + \frac{(k - 1)}{2} \int_X e^{(k - 1)u} \|\ell\|^2.
\]

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Therefore we have
\[ R_{(u,\eta)}(v,\ell) \geq 2(k-1) \int_X e^{(k-1)u}\|\ell\|^2 dA. \]
This completes the proof. \(\Box\)

4.2. Every critical point is a strict local minimum. Unfortunately the estimates in Theorem 4.3 do not directly imply that every critical point is a strict local minimum of \(\mathcal{D}\) in the stronger norm of \(\mathcal{V}\). As we shall see in Proposition 4.5, these estimates ensures only that this is true in the weaker (but natural) space \(H^1(X) \times W^{1,2}(X,E)\).

**Lemma 4.4.** Let \(\tilde{\beta}\) be a given continuous \((0,1)\)-form valued in \(E\). Then there exists \(\sigma = \sigma(\tilde{\beta}) > 0\) such that \(\forall (f,F) \in L^2(X) \times L^2(A^{0,1}(X,E))\),
\[ \int_X f^2 dA + \int_X \|f\tilde{\beta} + F\|^2 dA \geq \sigma \int_X (f^2 + \|F\|^2) dA. \]  

**Proof.** It is equivalent to prove
\[ \inf_{\|f\|_{L^2}^2 + \|F\|_{L^2}^2 = 1} \left\{ \int_X f^2 + \|f\tilde{\beta} + F\|^2 dA \right\} > 0. \]  
We argue by contradiction. Let \((f_n,F_n) \in L^2(X) \times L^2(A^{0,1}(X,E))\) be a sequence with \(\|f_n\|_{L^2}^2 + \|F_n\|_{L^2}^2 = 1\) such that \(\int_X (f_n^2 + \|f_n\tilde{\beta} + F_n\|^2) dA \to 0\), as \(n \to \infty\). Thus we have \(\int_X f_n^2 dA \to 0\) and \(\int_X \|f_n\tilde{\beta} + F_n\|^2 dA \to 0\). Therefore,
\[ \|F_n\|^2 = \|F_n + f_n\tilde{\beta} - f_n\tilde{\beta}\|^2 \leq \|F_n + f_n\tilde{\beta}\|^2 + \|f_n\tilde{\beta}\|^2 \to 0, \]
and this contradicts the assumption: \(\|f_n\|_{L^2}^2 + \|F_n\|_{L^2}^2 = 1\). \(\Box\)

As a consequence, we also obtain:

**Proposition 4.5.** For every critical point \((u,\eta)\) of \(\mathcal{D}\), there exists \(\sigma = \sigma(u,\eta) > 0\) such that, \(\forall (v,\ell) \in H^1(X) \times W^{1,2}(X,E)\),
\[ \mathcal{D}''_{(u,\eta)}(v,\ell) \geq \sigma \left( \|v\|_{H^1(X)}^2 + \|\ell\|_{W^{1,2}(A^0(\eta))}^2 \right). \]  

**Proof.** The second variation \(\mathcal{D}''_{(u,\eta)}(v,\ell)\) at a critical point \((u,\eta)\) is given in the expression (4.5). Since \((u,\eta)\) is smooth, we apply Lemma 4.4 with \(\tilde{\beta} = (k-1)\beta = (k-1)(\beta_0 + \bar{\eta})\) fixed, and \((f,F) = (v,\overline{\partial}v) \in L^2(X) \times L^2(A^{0,1}(X,E))\), to find
\[ \int_X \left\{ e^u v^2 + 4\|(k-1)v\beta + \overline{\partial}v\|^2 e^{(k-1)u} \right\} dA \geq \sigma_1 \int_X \left\{ v^2 + \|(k-1)v\beta + \overline{\partial}v\|^2 \right\} dA \]
\[ \geq \sigma_1 \int_X \left\{ v^2 + \|\overline{\partial}v\|^2 \right\} dA. \]
Now we inspect the last two terms in (4.5). From (4.6), and the fact that \(e^{(k-1)u}\) is smooth, we have
\[ R_{(u,\eta)}(v,\ell) \geq C \int_X \|\ell\|^2 dA. \]
Next we apply Lemma 4.4 again, with $\beta = 2(k-1)e^{(k-1)u} \beta$ and $(f, F) = (\|\ell\|, \bar{\partial}v \otimes \ell)$, to get

$$
2 \int_X \left\{ 2(k-1)e^{(k-1)u}\|\ell\| \beta - \bar{\partial}v \otimes \ell \right\}^2 dA + R_{(u,0)}(v, \ell) \geq \sigma_2 \int_X (\|\bar{\partial}v\|^2 + \|\ell\|^2).
$$

The proof is complete. \hfill \Box

Clearly Proposition 4.5 does not suffice to show that a critical point of $D$ is a strict local minimum in $\mathcal{V}$. To this purpose, we establish the following estimate:

**Lemma 4.6.** Let $u_0 \in H^1(X), \eta_0 := \eta(u_0)$, and $b > 0$. Then there exists a constant $C_0 = C_0(b, \|u_0\|_{H^1(X)}) > 0$ such that:

$$
\|\bar{\partial}\eta(u) - \bar{\partial}\eta_0\|_{L^p}^2 \leq C_0 \left\{ \|\bar{\partial}\eta(u) - \bar{\partial}\eta_0\|_{L^2}^2 + \|u - u_0\|_{L^2(X)}^2 \right\}
$$

holds for all $u \in H^1(X)$ with $\|u\|_{H^1(X)} < b$.

**Proof.** Recall that the space

$$
C_k(X) = \{ \alpha \in A^{1,0}(X, E^*) : \bar{\partial}\alpha = 0 \}
$$

is the space of holomorphic $k$-differentials on $X$, and it is a finite dimensional vector space. Therefore all norms on $C_k(X)$ are equivalent. Hence for any $q \geq 1$, there is a constant $C_q > 0$ such that

$$
\|\alpha\|_{L^q} \leq C_q \|\alpha\|_{L^1}, \quad \forall \alpha \in C_k(X),
$$

and we note $e^{(k-1)u} * (\beta_0 + \bar{\partial}\eta(u)) \in C_k(X)$. Furthermore, by (3.5), for each $q \geq 1$, there is a suitable constant $B_q > 0$, depending only on $\|u_0\|_{H^1(X)}, q$ and $b$, such that

$$
\|e^{(k-1)u} - e^{(k-1)u_0}\|_{L^q} \leq B_q \|u - u_0\|_{L^2(X)}.
$$

We use

$$
\bar{\partial}\eta(u) - \bar{\partial}\eta_0 = e^{-(k-1)u} \left\{ e^{(k-1)u}(\beta_0 + \bar{\partial}\eta(u)) - e^{(k-1)u_0}(\beta_0 + \bar{\partial}\eta_0) \right\}
$$

$$
\quad \quad + (e^{-(k-1)u} - e^{-(k-1)u_0})e^{(k-1)u}(\beta_0 + \bar{\partial}\eta_0)
$$

to estimate:

$$
\|\bar{\partial}\eta(u) - \bar{\partial}\eta_0\|_{L^p} \leq \|e^{-(k-1)u} \left\{ e^{(k-1)u}(\beta_0 + \bar{\partial}\eta(u)) - e^{(k-1)u_0}(\beta_0 + \bar{\partial}\eta_0) \right\}\|_{L^p}
$$

$$
\quad \quad + \|(e^{-(k-1)u} - e^{-(k-1)u_0})e^{(k-1)u}(\beta_0 + \bar{\partial}\eta_0)\|_{L^p}.
$$

Using Hölder inequality and the fact that the operator $* : A^{0,1}(X, E) \to A^{1,0}(X, E^*)$ is an isometry, and setting $\alpha_0 = *e^{(k-1)u_0}(\beta_0 + \bar{\partial}\eta_0)$ and $\alpha = *e^{(k-1)u}(\beta_0 + \bar{\partial}\eta(u))$, we have:

$$
\|\bar{\partial}\eta(u) - \bar{\partial}\eta_0\|_{L^p} \leq \|e^{-(k-1)u}\|_{L^{2(1/p)}(X)} \|\alpha - \alpha_0\|_{L^2(X)}
$$

$$
\quad \quad + \|(e^{-(k-1)u} - e^{-(k-1)u_0})e^{(k-1)u}(\beta_0 + \bar{\partial}\eta_0)\|_{L^p}.
$$

Therefore, by means of (3.4), the estimates (4.12) and (4.13), there exists a constant $C_1 > 0$ (depending only on $b, p, \|u_0\|_{H^1}$) such that

$$
\|\bar{\partial}\eta(u) - \bar{\partial}\eta_0\|_{L^p} \leq C_1 (\|u - u_0\|_{L^2} + \|\alpha - \alpha_0\|_{L^1}).
$$
Writing \[ \alpha - \alpha_0 = \star \left\{ e^{(k-1)u} (\partial_\eta(u) - \partial_\eta_0) + (e^{(k-1)u} - e^{(k-1)u_0})(\beta_0 + \partial_\eta_0) \right\}, \] we deduce
\[ (4.15) \quad \| \alpha - \alpha_0 \|_{L^1} \leq \| e^{(k-1)u} \|_{L^2} \| \partial_\eta(u) - \partial_\eta_0 \|_{L^2} + C_2 \| u - u_0 \|_{L^2}, \]
where, again, \( C_2 = C_2(b, p, \| u_0 \|_{H^1}) > 0 \). Thus, from (4.14) and (4.15), we obtain a constant \( C_0 = C_0(b, p, \| u_0 \|_{H^1}) > 0 \), such that
\[ \| \partial_\eta(u) - \partial_\eta_0 \|_{L^p} \leq C_0 \{ \| \partial_\eta(u) - \partial_\eta_0 \|_{L^2} + \| u - u_0 \|_{L^2(X)} \} \]
for all \( u \in H^1(X) \) with \( \| u \|_{H^1(X)} < b \), and (4.11) is established. \( \Box \)

Now we are ready to prove that each critical point of \( D \) in \( V \) is a strict local minimum. To start, we show the following:

**Theorem 4.7.** Let \( p_0 = (u_0, \eta_0) \) be a critical point for \( D \). Then there exist \( \delta_0 > 0 \) and \( t_0 > 0 \) such that \( \| u - u_0 \|_{H^1} < \delta_0 \) holds whenever \( \| u - u_0 \|_{H^1(X)} < \delta_0 \).

**Proof.** Given \( r > 0 \), let \( B_r(p_0) \) in \( (V, \| \cdot \|_V) \) be the ball centered at \( p_0 \) of radius \( r \).

By the continuity of the map \( u \rightarrow (u, \eta(u)) \) from \( H^1(X) \) to \( V \), there exists some \( \delta_r > 0 \), such that \( (u, \eta(u)) \in B_r(p_0) \), whenever \( \| u - u_0 \|_{H^1} < \delta_r \). In particular, \( 0 < \delta_r < r \).

We apply Taylor expansion in the fixed ball \( B_r(p_0) \), and for \( (u, \eta) \in B_r(p_0) \), in virtue of Proposition 4.5 (with \( \sigma_0 = \sigma(u_0, \eta_0) \)), we can write
\[ D(u, \eta) = D(p_0) + \frac{D''(p_0)(u - u_0, \eta - \eta_0)}{2} + O((\| u - u_0 \|_{V} + \| \eta - \eta_0 \|_{V^*})^2), \]

On the other hand, when \( \eta = \eta(u) \) then by Lemma 4.6 and Poincaré inequality, we obtain
\[ \| u - u_0 \|_{H^1}^2 + \| \eta(u) - \eta_0 \|_{W^{1,2}}^2 \geq \frac{1}{2} \| u - u_0 \|_{H^1}^2 + \frac{1}{2 \sigma_0} \| \partial_\eta(u) - \partial_\eta_0 \|_{L^2}^2 \]

with a suitable \( \alpha_p > 0 \) for any \( u \in H^1(X) : \| u - u_0 \|_{H^1} < \delta_r \), and for any \( r > 0 \) sufficiently small. As a consequence, we have:
\[ D(u, \eta(u)) \geq D(p_0) + \left( \frac{\sigma_0 \alpha_p}{2} + o(1) \right) \| (u, \eta(u)) - p_0 \|^2_V, \]

Thus, by choosing \( t_0 = \frac{\sigma_0 \alpha_p}{2} \), we find \( r_0 > 0 \) sufficiently small and corresponding \( \delta_0 = \delta_{r_0} > 0 \), such that, for \( \| u - u_0 \|_{H^1} < \delta_0 \), we have:
\[ D(u, \eta(u)) \geq D(u_0, \eta_0) + t_0 \| (u, \eta(u)) - p_0 \|^2_V, \]

as claimed. \( \Box \)

Consequently, we deduce:
Corollary 4.8. Let \( p_0 = (u_0, \eta_0) \) be a critical point of the Donaldson functional \( D \). Then \( p_0 \) is a strict local minimum for \( D \) in \( V \). More precisely, for suitable \( \delta_0 > 0 \) sufficiently small it holds:

\[
D(u, \eta) > D(u_0, \eta_0), \quad \forall (u, \eta) \in V \quad \text{with} \quad \|u - u_0\|_{H^1} < \delta_0, \quad \text{and} \quad (u, \eta) \neq (u_0, \eta_0).
\]

Proof. For any given critical point \( p_0 = (u_0, \eta_0) \) of \( D \), let \( \delta_0 > 0 \) and \( t_0 > 0 \) be as given in Theorem 4.7, so that

\[
D(u, \eta) \geq D(u, \eta(u)) \geq D(u_0, \eta_0) + t_0 \|(u, \eta(u)) - p_0\|^2_V
\]

holds, whenever \( \|u - u_0\|_{H^1} < \delta_0 \).

Now we assume that \( (u, \eta) \neq p_0 \), then there are two cases to consider. In the case \( u \neq u_0 \), then from (4.16), we deduce that \( D(u, \eta) > D(u_0, \eta_0) \), as claimed. In the other case where \( u = u_0 \), then necessarily we have \( \eta \neq \eta_0 = \eta(u_0) \). Since \( \eta_0 \) is the unique strict minimum of the partial map \( D_{u_0} \), in this case we have

\[
D(u, \eta) = D(u_0, \eta) = D_{u_0}(\eta) > D_{u_0}(\eta_0) = D(u_0, \eta_0).
\]

In conclusion (4.16) holds and the proof is completed.

5. A “Weaker” Palais-Smale Condition, Ekeland Principle and Uniqueness of the Critical Point

As pointed out in the introduction, a functional may have many critical points which are all strict local minima. Thus to prove that \( D \) admits a unique critical point (i.e. its global minimum), our approach is to assume (by contradiction) the existence of more strict local minima for \( D \) then arrive to a contradiction by a “mountain-pass” construction ([AR73]) that yields to an additional critical point of \( D \), which however is not a local minimum. But to successfully carry out such program, we need the Donaldson functional to satisfy the following (well known) Palais-Smale condition: if a sequence \( (u_n, \eta_n) \in V \) satisfies:

\[
D(u_n, \eta_n) \to c, \quad \text{and} \quad \|D'(u_n, \eta_n)\| \to 0, \quad \text{as} \quad n \to \infty,
\]

then up to a subsequence, \( (u_n, \eta_n) \) (called a Palais-Smale sequence) converges strongly in \( V \). Unfortunately, it is not obvious to check such property, since (5.1) does not provide any reasonable control of the component \( \eta_n \) in the space \( W^{1,p}(X, E) \) when \( p > 2 \). In fact as seen above, the best we can hope for is a uniform estimate in \( W^{1,a}(X, E) \), with \( 1 < a < 2 \).

On the other hand, if we assume a priori that \( \{\eta_n\} \) is bounded in \( W^{1-p}(X, E) \), a “weaker” form of the Palais-Smale condition holds (see Lemma 5.1), which suffices for our purpose. Indeed, by means of the Ekeland Principle (Theorem 5.3 below) we are able to obtain an “ad-hoc” Palais-Smale sequence satisfying such additional uniform bound for the component \( \eta_n \).
5.1. A “weaker” Palais-Smale condition. We start with the following lemma:

**Lemma 5.1.** Let \((u_n, \eta_n) \in V\) be a Palais-Smale sequence satisfying (5.1). If \(\{\eta_n\}\) is uniformly bounded in \(W^{1,p}(X, E)\), then along a subsequence, as \(n \to \infty\), we have

(i) \(u_n \to u\) strongly in \(H^1(X)\), and \(\eta_n \to \eta\) strongly in \(W^{1,2}(X,E)\);
(ii) \(D(u_n, \eta_n) \to D(u,\eta)\) and \(D'(u,\eta) = 0\).

In particular, \(c\) is a critical value for \(D\) with corresponding critical point \((u,\eta)\).

**Proof.** Since by assumption we have \(D(u_n, \eta_n) \leq C\) and \(\|\eta_n\|_{W^{1,p}(X,E)} \leq C\), then, along a subsequence, we can assume:

\[ u_n \xrightarrow{H^1} u, \quad \eta_n \xrightarrow{W^{1,p}(X,E)} \eta; \]

and in addition:

\[ u_n \xrightarrow{L^\alpha} u, \quad e^{u_n} \xrightarrow{L^\alpha} e^u, \quad \forall \alpha \geq 1, \]

as \(n \to \infty\).

Furthermore, by assumption we have:

\[ |D'(u_n,\eta_n)(\xi,\ell)| = o(1)\|\xi,\ell\|_V \to 0, \text{ as } n \to \infty, \quad \forall (\xi,\ell) \in V. \]

Thus, by arguing as in Proposition 3.5, we immediately derive that \(D'(u_n,\eta_n) = 0\), and so \((u,\eta)\) is a (smooth) critical point of \(D\). Moreover, by using this information together with (5.3) with \(\xi = 0\) and \(\ell = \eta_n - \eta\) (uniformly bounded in \(W^{1,p}(X,E)\)), we obtain:

\[ \int_X e^{(k-1)u_n} \|
\]

and consequently,

\[ \int_X e^{(k-1)u} \|
\]

Therefore, (by Poincaré inequality), \(\eta_n \to \eta\) strongly in \(W^{1,2}(X,E)\), and

\[ \int_X e^{(k-1)u_n} \|
\]

Next we choose \(\xi_n = u_n - u\) and \(\ell = 0\) in (5.3) to find

\[ |D'(u_n,\eta_n)(u_n - u,0)| = o(1), \text{ as } n \to \infty. \]

This means,

\[ \int_X \nabla u_n \nabla (u_n - u) + (e^{u_n} - 1)(u_n - u) + 4(k-1)\|
\]

as \(n \to +\infty\). Thus, we have

\[ \int_X \frac{1}{2} |\nabla (u_n - u)|^2 = \int_X (u_n - u) - \int_X e^{u_n} (u_n - u) \]

\[ -4(k-1) \int_X \|
\]

\[ 26 \]
and the right hand side goes to 0 as \( n \to +\infty \). Since by (5.2) we know that
\[
\int_X e^{u_n(u_n - u)} \to 0, \text{ in } L^q(X), \forall q \geq 1,
\]
and so by using Hölder inequality, and by recalling that \( \eta_n \) is uniformly bounded in \( W^{1,p}(A^0(E)) \),
\[
\int_X \| \beta_n + \Omega \eta_n \|^2 e^{(k-1)u_n(u_n - u)} \leq \left( \int_X \| \beta_n + \Omega \eta_n \|^p \right)^{\frac{2}{p}} \| e^{(k-1)u_n(u_n - u)} \|_{\infty} \to 0, \text{ as } n \to +\infty.
\]

Hence, \( u_n \to u \) strongly in \( H^1(X) \). Consequently, \( D(u_n, \eta_n) \to D(u, \eta) = c \), and the proof is complete. \( \blacksquare \)

**Remark 5.2.** We note that, even under the stronger assumption of Lemma 5.1, we do not know whether or not the sequence \( (u_n, \eta_n) \) converges in \( V \).

### 5.2. The Ekeland principle

Let us first recall the Ekeland’s \( \epsilon \)-variational principle as follows:

**Theorem 5.3. ([AE84])** Let \( (Y, d) \) be a complete metric space, and \( F : Y \to \mathbb{R} \) a nonnegative and lower semi-continuous functional. Let there be given \( \epsilon > 0 \), and \( \gamma_\epsilon^0 \in Y \), such that \( F(\gamma_\epsilon^0) \leq \epsilon + \inf F \). Then there is some point \( \gamma_\epsilon \in Y \) such that
\[
F(\gamma_\epsilon) \leq F(\gamma_\epsilon^0),
\]
\[
d(\gamma_\epsilon, \gamma_\epsilon^0) \leq \sqrt{\epsilon},
\]
and
\[
F(\gamma) \geq F(\gamma_\epsilon) - \sqrt{\epsilon}d(\gamma, \gamma_\epsilon), \text{ for all } \gamma \in Y.
\]

We will use the Ekeland principle to prove an important lemma in a more general form than we need but may be of independent interest. To this end, we consider two distinct points \( P_1 \) and \( P_2 \) in some Banach space \((V, \| \cdot \|)\), and set
\[
\mathcal{P} = \{ \gamma \in C^0([0,1], V) : \gamma(0) = P_1, \text{ and } \gamma(1) = P_2 \}.
\]
Clearly \( \mathcal{P} \) is not empty, as it contains the path: \( \gamma(t) = (1 - t)P_1 + tP_2 \), and \( (\mathcal{P}, d) \) is a complete metric space equipped with the metric \( d(\gamma_1, \gamma_2) = \max_{t \in [0,1]} \| \gamma_1(t) - \gamma_2(t) \| \).

**Lemma 5.4.** Let \((V, \| \cdot \|)\) be a Banach space which is uniformly convex and \( J : V \to \mathbb{R} \) be a \( C^1 \)-function on \( V \). Suppose there exist \( \epsilon > 0 \) and \( \gamma_\epsilon \in \mathcal{P} \) such that
\[
\max_{t \in [0,1]} J(\gamma(t)) \geq \max_{t \in [0,1]} J(\gamma_\epsilon(t)) - \sqrt{\epsilon}d(\gamma, \gamma_\epsilon)
\]
holds for all \( \gamma \in \mathcal{P} \). Set
\[
T_\epsilon = \{ \tilde{t} \in [0,1] : J(\gamma_\epsilon(\tilde{t})) = \max_{t \in [0,1]} J(\gamma_\epsilon(t)) \}.
\]
If \( T_\epsilon \subset (0,1) \), i.e., compactly contained in \((0,1)\), then there is \( t_\epsilon \in T_\epsilon \) such that
\[
\| J'_{\gamma_\epsilon(t_\epsilon)} \|_* \leq \sqrt{\epsilon}.
\]
Proof: We define $F(\gamma) = \max_{t \in [0,1]} J(\gamma(t))$. Let $\rho_\epsilon : [0,1] \to [0,1]$ be a continuous function such that $\rho_\epsilon(0) = \rho_\epsilon(1) = 0$, and $\rho_\epsilon(t) \equiv 1$, $\forall t \in T_\epsilon$. This cut-off function exists since $T_\epsilon \subset [0,1]$.

Since $V$ is uniformly convex, it is reflexive by a theorem of Milman-Pettis ([Bre83]), and its bi-dual $V^{**}$ is also uniformly convex (since the canonical map $V \to V^{**}$ is an isometry). In particular, given $f \in V^*$, there exists a unique $\hat{f} \in V^{**}$ satisfying $\hat{f}(f) = \|f\|_2^2$ and $\|\hat{f}\| = \|f\|_*$, where $\| \cdot \|_*$ is the norm on $V^*$. This gives a well-defined “duality map” $V^* \to V^{**} \cong V : f \to \hat{f}$ which is continuous (see Proposition 32.22 in [Zei90]). Hence for each $t \in [0,1]$, there exists $\psi_t(t) \in V$ such that

$$J'_{\gamma_t(t)}[\psi_t(t)] = \|J'_{\gamma_t(t)}\|_*^2, \quad \text{and} \quad \|\psi_t(t)\| = \|J'_{\gamma_t(t)}\|_*.$$

Since the map $t \to J'_{\gamma_t(t)}$ and the duality map are both continuous, we have the map $t \to \psi_t(t)$ also continuous.

For $h > 0$, we consider the path

$$\gamma_h(t) = \gamma_\epsilon(t) - h\rho_\epsilon(t)\psi_t(t) \in P,$$

and let $t_h \in [0,1]$ be such that

$$J(\gamma_h(t_h)) = \max_{t \in [0,1]} J(\gamma_h(t)) = F(\gamma_h). \quad (5.12)$$

On the one hand, by assumption (5.8) we find:

$$F(\gamma_h) \geq F(\gamma_\epsilon) - \sqrt{\epsilon}d(\gamma_\epsilon, \gamma_h) \geq J(\gamma_\epsilon(t_h)) - \sqrt{\epsilon}h\|\psi_t(t_h)\|. \quad (5.13)$$

On the other hand, we have

$$F(\gamma_h) = J(\gamma_\epsilon(t_h) - h\rho_\epsilon(t_h)\psi_t(t_h))$$

$$= J(\gamma_\epsilon(t_h)) - h\rho_\epsilon(t_h)J'_{\gamma_\epsilon(t_h)}[\psi_t(t_h)] + o(h). \quad (5.14)$$

Therefore from (5.13) and (5.14), we find

$$\rho_\epsilon(t_h)J'_{\gamma_\epsilon(t_h)}[\psi_t(t_h)] \leq \sqrt{\epsilon}\|\psi_t(t_h)\| + o(1), \quad \text{as} \quad h \to 0^+.$$

This allows us to pass to the limit along a sequence $h_n \to 0^+$ with $t_{h_n} \to t_0 \in [0,1]$. We claim that $t_0 \in T_\epsilon$. Indeed, this follows from a general fact that if a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ converges uniformly to a function $f$ (as $n \to \infty$), and if $f_n(t_n) = \max_{t \in [0,1]} f_n(t)$, then along a subsequence, $t_n \to t_0$ with $f(t_0) = \max_{t \in [0,1]} f(t)$.

And we conclude $J'_{\gamma_\epsilon(t_0)}[\psi_t(t_0)] \leq \sqrt{\epsilon}\|\psi_t(t_0)\|$, and this gives

$$\|J'_{\gamma_\epsilon(t_0)}\|_*^2 \leq \sqrt{\epsilon}\|J'_{\gamma_\epsilon(t_0)}\|_*^*.$$

Now the proof is complete. □
5.3. **Uniqueness.** In this subsection, we prove our main result:

**Theorem 5.5.** The Donaldson functional $D$ admits a unique critical point corresponding to its global minimum.

**Proof.** We assume by contradiction there are two distinct critical points $P_1 = (u_1, \eta_1)$ and $P_2 = (u_2, \eta_2)$ for $D$ in $V$, as in (5.7), we consider,

\[ P = \{ \gamma \in C^0([0,1], V) : \gamma(0) = P_1, \text{ and } \gamma(1) = P_2 \}, \]

which defines a nonempty complete metric space, with

\[ d(\gamma_1, \gamma_2) = \max_{t \in [0,1]} ||\gamma_1(t) - \gamma_2(t)||_V. \]

We will use Theorem 5.3 and Lemma 5.4 on $(V, \| \cdot \|_V)$, which is a uniformly convex Banach space ([Cla36]). We take the functional $J = D$, and $F(\gamma) = \max_{t \in [0,1]} D(\gamma(t))$.

Set

\[ c_0 = \max_{t \in [0,1]} \{ D(u_1, \eta_1), D(u_2, \eta_2) \}, \]

so that,

\[ \max_{t \in [0,1]} D(\gamma(t)) \geq c_0, \forall \gamma \in P. \]

Therefore, it is well defined:

\[ c = \inf_{\gamma \in P} \max_{t \in [0,1]} D(\gamma(t)) \geq c_0. \]

Note that, if the path $\gamma(t) = (u(t), \eta(t))$ lies in the space $P$, then so does the path $\tilde{\gamma}(t) = (u(t), \eta(u(t)))$, where $\eta(u) \in W^{1,p}(X, E)$ is the map defined by Theorem 3.7.

Indeed, since the map $\eta(u)$ is continuous, we see that $\tilde{\gamma}$ is also continuous, and we also check that $\tilde{\gamma}(0) = P_1$ and $\tilde{\gamma}(1) = P_2$, since by (3.25) we have $\eta(u(0)) = \eta(u_1) = \eta_1$ and $\eta(u(1)) = \eta(u_2) = \eta_2$. In addition, from Theorem 3.7 we have:

\[ D(\gamma(t)) \geq \mathcal{D}(\tilde{\gamma}(t)), \forall t \in [0,1]. \]

We also emphasize that by applying Theorem 4.7 to the critical points $P_1$ and $P_2$, there exist $\delta_0 > 0$ and $\epsilon_0 > 0$, such that

\[ D(u, \eta(u)) \geq D(P_j) + \epsilon_0, \quad \forall (u, \eta(u)) \in \partial B_{\frac{\delta_0}{2}}(P_j), \quad j = 1, 2; \]

and without loss of generality, we can assume further that $0 < \delta_0 \leq \| P_1 - P_2 \|_V$. Therefore for every $\gamma(t) = (u(t), \eta(t)) \in P$, by continuity, we find $t_1, t_2 \in [0,1]$ such that for $\tilde{\gamma}(t) = (u(t), \eta(u(t))) \in P$, there holds:

\[ \| \tilde{\gamma}(t_j) - P_j \|_V = \frac{\delta_0}{2}, \quad j = 1, 2. \]

So by (5.19), we have

\[ \max_{t \in [0,1]} D(\gamma(t)) \geq \max_{t \in [0,1]} D(\tilde{\gamma}(t)) \geq D(P_j) + \epsilon_0, \quad j = 1, 2. \]

Therefore, for any $\gamma \in P$, we have $\max_{t \in [0,1]} D(\gamma(t)) \geq c_0 + \epsilon_0$ with suitable $\epsilon_0 > 0$, and so the set

\[ T = \{ \tilde{t} \in [0,1] : \max_{t \in [0,1]} D(\gamma(t)) = D(\gamma(\tilde{t})) \} \subset (0,1). \]
Furthermore the value $c$ in (5.17) satisfies: $c \geq c_0 + \epsilon_0$.

In view of (5.17) and (5.18), for given $\epsilon > 0$, we find a continuous map $u^0_\epsilon : [0,1] \to H^1(X)$ such that

\[ u^0_\epsilon(0) = u_1, \quad u^0_\epsilon(1) = u_2, \]

and a path

\[ \gamma^0_\epsilon(t) = (u^0_\epsilon(t), \eta(u^0_\epsilon(t))) \in \mathcal{P} \]

satisfying:

\[ c \leq \max_{t \in [0,1]} D(\gamma^0_\epsilon(t)) < c + \epsilon. \]

At this point, we are in position to apply Theorem 5.3 with the lower continuous map $F : \mathcal{P} \to [0, \infty)$, with $F(\gamma) = \max_{t \in [0,1]} D(\gamma(t))$, with respect to the path $\gamma^0_\epsilon$ in (5.22). Thus, we obtain another path $\gamma_\epsilon(t)$ in $\mathcal{P}$, which in turn satisfies the assumptions of Lemma 5.4 with the functional $J = D$. Thus, we obtain $t_\epsilon \in (0,1)$, such that

\[ c \leq D(\gamma_\epsilon(t_\epsilon)) = \max_{t \in [0,1]} D(\gamma_\epsilon(t)) < c + \epsilon, \]

and

\[ d(\gamma_\epsilon(t_\epsilon), \gamma^0_\epsilon(t_\epsilon)) \leq \sqrt{\epsilon}, \]

and

\[ \|D'(\gamma_\epsilon(t_\epsilon))\|_\epsilon \leq \sqrt{\epsilon}. \]

Hence, as $\epsilon \to 0$, by (5.24) and (5.25), we have

\[ D(\gamma_\epsilon(t_\epsilon)) \to c, \quad \|\gamma_\epsilon(t_\epsilon) - \gamma^0_\epsilon(t_\epsilon)\|_V \to 0, \quad D'(\gamma_\epsilon(t_\epsilon)) \to 0, \]

with $\gamma^0_\epsilon$ in (5.22).

Therefore, along a subsequence $\epsilon_n \to 0$, applying Lemma 3.4, Proposition 3.8, and (5.26), we can find sequences $(u_n, \eta_n) := \gamma_\epsilon_n(t_{\epsilon_n})$ and $(\tilde{u}_n, \tilde{\eta}_n) := \gamma^0_\epsilon(t_{\epsilon_n})$ such that

\[ u_n \xrightarrow{H^1} u, \quad \tilde{u}_n \xrightarrow{H^1} u, \quad \text{as } n \to +\infty; \]

also

\[ \tilde{\eta}_n = \eta(\tilde{u}_n) \xrightarrow{W^{1,p}(X,E)} \eta(u) = \eta, \quad \eta_n \xrightarrow{W^{1,p}(X,E)} \eta, \quad \text{as } n \to +\infty. \]

In other words, $(u_n, \eta_n)$ defines a Palais-Smale sequence for $D$, to which Lemma 5.1 applies. Hence, we conclude (along a subsequence) that, $u_n \to u$ strongly in $H^1(X)$. In summarizing, we have established that, $(u_n, \eta_n) \to (u, \eta)$ strongly in $V$ with $D(u, \eta) = c > c_0$ ($c_0$ in (5.16)), and $D'(u, \eta) = 0$. Therefore $P_3 = (u, \eta)$ defines a critical point for $D$ different from $P_1$ and $P_2$.

We can apply Theorem 4.7 to $P_3$ to find suitable $\delta_0 > 0$, which we can always choose to satisfy: $0 < \delta_0 < \min\{\|P_j - P_3\|, j = 1, 2\}$, and $\epsilon_0 > 0$, such that

\[ D(u, \eta(u)) \geq D(P_3) + \epsilon_0, \quad \forall (u, \eta(u)) \in \partial B_{\delta_0}(P_3). \]
Since $\gamma_0^n(t_{\tau_n}) = (\bar{u}_n, \bar{\eta}_n) \to P_3$, as $n \to \infty$, while $\gamma_0^n(0) = P_1 \neq P_3$ and $\gamma_0^n(1) = P_2 \neq P_3$, by the continuity of $\gamma_0^n$, for $n$ sufficiently large, we find $\bar{t}_n \in (0, 1)$ such that $\gamma_0^n(\bar{t}_n) \in \partial \mathcal{B}_{\frac{2}{\delta}}(P_3)$, and so (by (5.29)) we have: $D(\gamma_0^n(\bar{t}_n)) \geq c + \epsilon_0$. While in view of (5.23), we also have:

$$D(\gamma_0^n(\bar{t}_n)) \geq D(\gamma_0^n(t_n)) \to c, \text{ as } n \to \infty,$$

and we arrive at the desired contradiction.

6. Final Remarks

As discussed in [Uhl83] a minimal immersion of $X$ into (a germ of) hyperbolic three-manifold $M \cong S \times \mathbb{R}$ (not necessarily complete) with a pullback metric $g = e^{2u} g_X$ and second fundamental form $\Pi_g = \Re(q)$ is governed by the Gauss-Codazzi equations:

$$\begin{cases}
\frac{\Delta u + 1}{4} - \frac{\|q\|^2}{16} e^{-2u} - e^{2u} = 0 & \text{on } X, \\
\mathcal{J}(q) = 0,
\end{cases}
$$

expressing the Gauss consistency condition between intrinsic and extrinsic curvatures and the fact that $q$ defines a holomorphic quadratic differential on $X$, namely $q \in \mathcal{C}_2(X)$.

More generally, if we concern with constant mean curvature (CMC) immersions of mean curvature $c$, then the Gauss-Codazzi equations are modified accordingly as follows:

$$\begin{cases}
\frac{\Delta u + 1}{4} - \frac{\|q\|^2}{16} e^{-2u} - (1 - c^2)e^{2u} = 0 & \text{on } X, \\
\mathcal{J}(q) = 0.
\end{cases}
$$

If we attack (6.2) by considering $q \in \mathcal{C}_2(X)$ fixed, it has been pointed out ([HLT21]) that, by setting $\lambda = 1 - c^2$, then the first equation in (6.2) admits a solution if and only if $0 \leq \lambda \leq \lambda_0$ for a suitable $\lambda_0 = \lambda_0(q)$. Moreover we have multiple solutions when $0 < \lambda < \lambda_0$. Such a failure of one-to-one correspondence also prompted [GU07] to take a different viewpoint inspired in some sense by the Higgs bundle approach pioneered by Hitchin ([Hit87]). Hence, in [GU07] the authors proposed to fix a class $[\beta] \in \mathcal{H}^{0,1}(X, E)$ with $E = T^1_X$ (i.e. $k = 2$ in the notations above) and let $[\beta] = [\beta_0 + \partial \eta]$ where $\beta_0$ is the harmonic representative. For $\lambda = 1 - c^2 > 0$ and $\eta$ a section of $T^1_X$ as above, now we aim to solve the system:

$$\begin{cases}
\frac{\Delta u + 1}{4} - \lambda e^{2u} - \|\beta_0 + \partial \eta\|^2 e^{2u} = 0 & \text{on } X, \\
\mathcal{J}(e^{2u} \star_E (\beta_0 + \partial \eta)) = 0,
\end{cases}
$$

and obtain a posteriori the holomorphic quadratic differential $q = 4 \star_E (\beta_0 + \partial \eta)e^{2u}$, conveniently devised together with the metric $g = e^{2u} g_X$, and such that its real part identifies the traceless part of the second fundamental form of the CMC immersion.
As before, a suitable change of variables reduces system (6.3) to the Euler-Lagrange equations for the functional:

$$D_\lambda(u, \eta) = \int_X \left\{ \frac{1}{4} |\nabla u|^2 - u + \lambda e^u + 4\|\beta_0 + \overline{\eta}\|_2 e^u \right\} dA.$$  

Obviously, for $\lambda > 0$, the functional $D_\lambda$ enjoys exactly the same properties of $D$ in (1.5). So it admits a unique critical point $(u_\lambda, \eta_\lambda)$, and

$$D_\lambda(u_\lambda, \eta_\lambda) = \min_{\mathcal{W}} D_\lambda(u, \eta),$$

where $\mathcal{W}$ is defined in (1.6) as the natural domain for the functional. As a consequence, we establish Corollary 3 as stated in the Introduction.

We conclude by pointing out that another advantage of our variational approach is that now we can aim to construct CMC immersions with $|\mathbf{c}| = 1$. Namely, we can try to see whether $(u_\lambda, \eta_\lambda)$ survives the passage to the limit as $\lambda \searrow 0$, in order to obtain a solution for problem (6.3) when $\lambda = 0$. This is a nontrivial task, since for $\lambda = 0$, the functional $D_0 = D_{\lambda=0}$ may not even be bounded from below. This occurs when $[\beta] = 0$ where we easily check that,

$$\min_{\mathcal{W}} D_\lambda = 4\pi(g - 1) \ln(\lambda) \to -\infty, \quad \text{as} \quad \lambda \searrow 0,$$

recall that $g$ is the genus of the surface. Hence a first interesting question is to understand the pair of data $(X, [\beta])$ (if any) that yields to a functional $D_0$ bounded from below in $\mathcal{W}$. But, since $D_0$ is no longer coercive, in this case it is important to understand whether the minimum of $D_0$ is attained. To this purpose, it will be relevant to provide a detailed description about the asymptotic behavior of $(u_\lambda, \eta_\lambda)$ as $\lambda \searrow 0$, a task that becomes particularly delicate in case $D_0$ is unbounded from below in $\mathcal{W}$. To this purpose, one needs to elaborate a proper blow up analysis for sequence of solutions to (6.3) as $\lambda \searrow 0$. This line of investigation will be expanded in future work.

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