A CONSTRUCTION OF SOME OBJECTS IN MANY BASE CASES OF AN AUSONI-ROGNES CONJECTURE

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ABSTRACT. Let $p$ be a prime, $n \geq 1$, $K(n)$ the $n$th Morava $K$-theory spectrum, $\mathbb{G}_n$ the extended Morava stabilizer group, and $K(A)$ the algebraic $K$-theory spectrum of a commutative $S$-algebra $A$. For a type $n+1$ complex $V_n$, Ausoni and Rognes conjectured that the unit map $i_n: L_{K(n)}(S^0) \to E_n$ from the $K(n)$-local sphere to the Lubin-Tate spectrum induces a weak equivalence

$$K(L_{K(n)}(S^0)) \wedge v_{n+1}^{-1} V_n \to (K(E_n))^{h\mathbb{G}_n} \wedge v_{n+1}^{-1} V_n,$$

where $\pi_*(-)$ of the target is the abutment of a homotopy fixed point spectral sequence. Since $\mathbb{G}_n$ is profinite, $(K(E_n))^{h\mathbb{G}_n}$ denotes a continuous homotopy fixed point spectrum, and for every $n$ and $p$, there are no known constructions of it, the above target, or the spectral sequence. For $n = 1, p \geq 5$, and $V_1 = V(1)$, we give a way to realize the last two objects by proving that $i_1$ induces a map

$$K(L_{K(1)}(S^0)) \wedge v_2^{-1} V_1 \to (K(E_1)) \wedge v_2^{-1} V_1)^{h\mathbb{G}_1} =: K_1,$$

where $K_1$ is a continuous homotopy fixed point spectrum, with the expected spectral sequence associated to it. Though we do not construct $(K(E_1))^{h\mathbb{G}_1}$, we prove that $K_1 \simeq (K(E_1))^{h\mathbb{G}_1} \wedge v_2^{-1} V_1$, where $(K(E_1))^{h\mathbb{G}_1}$ is the homotopy fixed points with $\mathbb{G}_1$ regarded as a discrete group.

1. INTRODUCTION

1.1. An overview of an Ausoni-Rognes conjecture and statements of our main theorems. Let $n \geq 1$ and let $p$ be a prime. Let $E_n$ be the Lubin-Tate spectrum with $\pi_*(E_n) = W(F_{p^n})[[u_1, ... , u_{n-1}][u^\pm 1]]$, where $W(F_{p^n})$ is the ring of Witt vectors of the field $F_{p^n}$ (with $p^n$ elements), the complete power series ring is in degree zero, and $|u| = 2$, and let $\mathbb{G}_n$ be the $n$th extended Morava stabilizer group. By [19, 21], $E_n$ is a commutative $S$–algebra and the group $\mathbb{G}_n$ acts on $E_n$ by maps of commutative $S$–algebras. Given a commutative $S$–algebra $A$, the algebraic $K$–theory spectrum of $A$, $K(A)$, is a commutative $S$–algebra. Thus, $K(E_n)$ is a commutative $S$–algebra, and by the functoriality of $K(-)$, $\mathbb{G}_n$ acts on $K(E_n)$ by maps of commutative $S$–algebras.

Let $L_{K(n)}(S^0)$ denote the Bousfield localization of the sphere spectrum with respect to $K(n)$, the $n$th Morava $K$–theory spectrum. The group $\mathbb{G}_n$ is profinite, and by [31, 7], the $K(n)$–local unit map

$$L_{K(n)}(S^0) \to E_n$$

is a consistent profaithful $K(n)$–local profinite $\mathbb{G}_n$–Galois extension.

Now let $V_n$ be a finite $p$–local complex of type $n+1$ and let $v: \Sigma^d V_n \to V_n$ be a $v_{n+1}$–self-map, where $d$ is some positive integer (see [23, Theorem 9]). The map $v$ induces a sequence

$$V_n \to \Sigma^{-d} V_n \to \Sigma^{-2d} V_n \to \cdots$$
of maps of spectra, and we let
\[ v_{n+1}^{-1}V_n = \operatorname{colim}_{j \geq 0} \Sigma^{-jd}V_n, \]
the colimit of the above sequence, denote the mapping telescope associated to the \( v_{n+1} \)-self-map \( v \). As hinted at by the notation, the mapping telescope \( v_{n+1}^{-1}V_n \) is independent of the choice of self-map \( v \).

In [3, Conjecture 4.2], [2, page 46; Remark 10.8], and [3, paragraph containing (0.1)], Christian Ausoni and John Rognes conjectured that the \( \mathbb{G}_n \)-Galois extension \( L_{K(n)}(S^0) \to E_n \) induces a map
\[ (1.2) \quad K(L_{K(n)}(S^0)) \land v_{n+1}^{-1}V_n \to (K(E_n))^{h\mathbb{G}_n} \land v_{n+1}^{-1}V_n \]
that is a weak equivalence, and associated with the target of this weak equivalence, there exists a homotopy fixed point spectral sequence that has the form
\[ E_2^{s,t} = H^s_c(\mathbb{G}_n; (V_n)_i(K(E_n))[v_{n+1}^{-1}]) \implies (V_n)_{t-s}((K(E_n))^{h\mathbb{G}_n})[v_{n+1}^{-1}], \]
where the \( E_2 \)-term is given by continuous cohomology and the conjectural spectrum \( (K(E_n))^{h\mathbb{G}_n} \) is a continuous homotopy fixed point spectrum. This conjecture is an extension of the Lichtenbaum-Quillen conjectures (for example, see [35, (0.1), Theorem 4.1]), which can be viewed as corresponding to \( n = 0 \) versions of the above (see [4, 2, Section 10]). More generally, the conjecture is related to trying to understand \( \acute{e} \text{tale} \) descent for the algebraic \( K \)-theory of commutative \( S \)-algebras; for more details about this, see [3, Introduction] and [32, Section 4].

Remark 1.3. The above two conjectural statements are just a piece of an important family of conjectures – which include the chromatic redshift conjecture – made by Ausoni and Rognes; we only state the part that we focus on in this paper. For more information about these conjectures, see [2, 3, 4, 5, 22].

Notice that for every integer \( t \), there is an isomorphism
\[ (V_n)_i(K(E_n))[v_{n+1}^{-1}] \cong \pi_t(K(E_n) \land v_{n+1}^{-1}V_n). \]
Thus, when the above homotopy fixed point spectral sequence exists, since its abutment should be \( \pi_t(-) \) of a homotopy fixed point spectrum, there should also be an equivalence
\[ (1.4) \quad (K(E_n))^{h\mathbb{G}_n} \land v_{n+1}^{-1}V_n \simeq (K(E_n) \land v_{n+1}^{-1}V_n)^{h\mathbb{G}_n}, \]
where the right-hand side is a continuous homotopy fixed point spectrum. Obtaining equivalence (1.4) and a homotopy fixed point spectral sequence
\[ E_2^{s,t} = H^s_c(\mathbb{G}_n; \pi_t(K(E_n) \land v_{n+1}^{-1}V_n)) \implies \pi_{t-s}((K(E_n) \land v_{n+1}^{-1}V_n)^{h\mathbb{G}_n}) \]
immediately implies the existence of the conjectural spectral sequence above.

To make progress on the above two-part conjecture, one obstacle that must be overcome is that currently, there are no known constructions of the continuous homotopy fixed point spectra
\[ (K(E_n))^{h\mathbb{G}_n}, \quad (K(E_n) \land v_{n+1}^{-1}V_n)^{h\mathbb{G}_n} \]
for any \( n \) and \( p \). Directly related to this issue is the fact that there are also no known constructions of the above two descent spectral sequences (here and elsewhere, we use the term “descent spectral sequence” in place of “homotopy fixed point spectral sequence”).

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In this paper, we remedy part of the situation just described in certain base cases: for \( n = 1, \ p \geq 5, \) and \( V_1 = V(1) \), the type 2 Smith-Toda complex \( S^0/(p, V_1) \), we give an elementary construction of
\[
(K(E_1) \wedge v_2^{-1}V_1)^{hG_1}
\]
and we obtain the desired descent spectral sequence
\[
E_2^{s,t} = H^s_c(G_1; \pi_t(K(E_1) \wedge v_2^{-1}V_1)) \Rightarrow \pi_{t-s}((K(E_1) \wedge v_2^{-1}V_1)^{hG_1}).
\]

**Remark 1.5.** Our work addresses aspects of an Ausoni-Rognes conjecture involving a certain Galois extension, where the relevant group, \( G_n \), is infinite and profinite. For \( K(n)\)-local \( G \)-Galois extensions \( A \to B \), where \( G \) is a finite group, Ausoni and Rognes have made a conjecture similar to the one encapsulated above in \([1,2,4, \text{Conjecture 4.2}]\), and in these cases, since \( G \) is naturally discrete, it is well-known that \( (K(B))^{hG} \) always exists, and so there is no issue with the statement of the conjecture. For these cases, progress on the conjecture has been made by \([13]\).

Given our hypotheses – \( n = 1, \ p \geq 5, \) and \( V_1 = V(1) \), we can be a little more concrete about some of the main actors in the scenario that we focus on:
\[
E_1 = KU_p,
\]
p-completed complex \( K \)-theory;
\[
G_1 = \mathbb{Z}^\times_p,
\]
the group of units in the \( p \)-adic integers \( \mathbb{Z}_p \); and
\[
v_2^{-1}V(1) = \lim_{j \to 0} \Sigma^{-jd}V(1).
\]

Then our first result is actually an extension of the aforementioned new \( n = 1 \) constructions to all closed subgroups of \( \mathbb{Z}^\times_p \).

**Theorem 1.6.** Let \( p \geq 5 \). Given any closed subgroup \( K \) of \( \mathbb{Z}^\times_p \), there is a strongly convergent descent spectral sequence
\[
E_2^{s,t} = H^s_c(G_1; \pi_t(K(KU_p) \wedge V(1))^{[v_2^{-1}])} \Rightarrow \pi_{t-s}((K(KU_p) \wedge v_2^{-1}V(1))^{hK}),
\]
with \( E_2^{s,t} = 0 \), for all \( s \geq 2 \) and any \( t \in \mathbb{Z} \). Also, there is an equivalence of spectra
\[
(K(KU_p) \wedge v_2^{-1}V(1))^{hK} \simeq \lim_{j \to 0} (K(KU_p) \wedge \Sigma^{-jd}V(1))^{hK}.
\]

In the above result, the subgroup \( K \) is a profinite group and each application of \((-)^{hK}\) denotes a continuous homotopy fixed point spectrum (as in \([7]\)), formed in the setting of symmetric spectra of simplicial sets.

Our next two results are about \( (K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times} \).

**Theorem 1.7.** When \( p \geq 5 \), there is a canonical map
\[
K(L_{K(1)}(S^0)) \wedge v_2^{-1}V(1) \to (K(KU_p) \wedge v_2^{-1}V(1))^{h\mathbb{Z}_p^\times},
\]
induced by the \( K(1)\)-local unit map \( L_{K(1)}(S^0) \to KU_p \), in the category of symmetric spectra.
For \( n = 1, \ p \geq 5, \) and \( V_1 = V(1), \) if \([1,4]\) were valid, then Theorem \([1.7]\) would yield the conjectural map in \([1,2]\), as a map in the stable homotopy category. However, in this paper, we do not show that the morphism in Theorem \([1.7]\) is a weak equivalence. But we hope that the spectral sequence of Theorem \([1.6]\) when \( K = \mathbb{Z}_p^\times \) is a useful computational tool for this problem.

Though this paper does not construct \((K(KU_p))^h\mathbb{Z}_p^\times\) for any \( p \) (so that \([1,4]\) remains conjectural in all \( n = 1 \) cases), we do have a related result. Before stating this result, we recall that if \( G \) is any profinite group and \( X \) is a (naive) \( G \)-spectrum, then \( G \) can be regarded as a discrete group and one can always form the “discrete homotopy fixed point spectrum”

\[
X^\tilde{h}G = \text{Map}_G(EG_+, X)
\]

(the usual notation for \( X^\tilde{h}G \) omits the “\( \sim \)” but we use it here to distinguish \((-)^\tilde{h}G\) from the continuous \((-)^{hG}\).

**Theorem 1.8.** When \( p \geq 5, \) there is an equivalence of spectra

\[
(K(KU_p) \wedge v_2^{-1}V(1))^\mathbb{Z}_p^\times \simeq (K(KU_p))^\mathbb{Z}_p^\times \wedge v_2^{-1}V(1).
\]

**Remark 1.9.** It is worth pointing out that in proving Theorem \([1.8]\) we show that (for \( p \geq 5 \)) there is a map

\[
\text{colim}_{j \geq 0}(K(KU_p) \wedge \Sigma^{-jd}V(1))^\mathbb{Z}_p^\times \xrightarrow{\sim} \text{colim}_{j \geq 0}(K(KU_p) \wedge \Sigma^{-jd}V(1))^\mathbb{Z}_p^\times
\]

that is a weak equivalence.

In \([1,4]\), when \( n = 1, \) if \((K(E_1))^hG_1 = (K(KU_p))^\mathbb{Z}_p^\times\) is changed to \((K(KU_p))^\mathbb{Z}_p^\times\), then Theorem \([1.8]\) is an instance of this “modified \([1,4]\).” But we do not take this observation as evidence that \((K(KU_p))^\mathbb{Z}_p^\times\) should be the definition of \((K(KU_p))^\mathbb{Z}_p^\times\) for some \( p \), and thus, as noted earlier, we believe that for all \( p \), the proper construction of \((K(KU_p))^\mathbb{Z}_p^\times\) is still an open problem.

The proofs of Theorems \([1.6, 1.7, 1.8]\) are given in the first part of Section \([8]\) that section’s second part, and Part \([9]\) respectively.

### 1.2. The construction of the continuous homotopy fixed point spectra in Theorem \([1.6]\)

We now explain our work in more detail. Let \( G \) be a profinite group and let \( X \) be a \( G \)-spectrum. Then there is \( X^{hG}\) and one can always form the associated descent spectral sequence

\[
E_2^{s,t} = H^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG}),
\]

with \( E_2 \)-term given by (non-continuous) group cohomology. However, it is not \((K(E_n) \wedge v_n^{-1}V_n)^{hG_n}\) that the conjecture of Ausoni and Rognes is concerned with. Since \( G_n \) is profinite and the \( E_2 \)-term of the conjectured spectral sequence is given by continuous cohomology, one wants a continuous homotopy fixed point spectrum \((K(E_n) \wedge v_n^{-1}V_n)^{hG_n}\) that takes the profinite topology of \( G_n \) into account; that is, we would like to know that \((K(E_n) \wedge v_n^{-1}V_n)\) is a continuous \( G_n \)-spectrum in some sense, and that \((K(E_n) \wedge v_n^{-1}V_n)^{hG_n}\) can be formed with respect to the continuous action.

To address this problem in the \( n = 1, \ p \geq 5 \) case, given a profinite group \( G \), we work with discrete \( G \)-spectra (as in \([7]\)) within the framework of symmetric spectra
of simplicial sets (for more detail, see the end of the introduction). For the moment, let \( X \) be a discrete \( G \)-spectrum. Then for all \( k, l \geq 0 \), the \( l \)-simplices of the \( k \)th pointed simplicial set of \( X \), \( X_{k,l} \), is a discrete \( G \)-set. Also, the homotopy fixed point spectrum \( X^{hG} \) is defined (in \([7]\), as recalled at the end of the introduction) in a way that respects the profinite topology of \( G \). Throughout this paper, we use \((-)^{hG}\) for these “continuous” homotopy fixed points.

The following convention and terminology (from \([10]\)) will be helpful to us.

**Definition 1.10.** Let \( X \) be a spectrum (that is, a symmetric spectrum). By “\( \pi_*(X) \),” we always mean the homotopy groups \( \pi_t(X) := [S^t, X] \), \( t \in \mathbb{Z} \), of morphisms \( S^t \to X \) in the homotopy category of symmetric spectra, where here, \( S^t \) denotes a fixed cofibrant and fibrant model for the \( t \)-th suspension of the sphere spectrum.

**Definition 1.11** ([10, page 5]). A spectrum \( X \) is an \( f \)-spectrum if \( \pi_t(X) \) is finite for every integer \( t \).

Recall that a profinite group is strongly complete if every subgroup of finite index is open. Let \( p \) be any prime: since \( \mathbb{Z}_p \) is strongly complete, it follows that the profinite group \( \mathbb{Z}_p \times H \), where \( H \) is any finite discrete group and \( \mathbb{Z}_p \times H \) is equipped with the product topology, is strongly complete. Thus (see Remark 3.2), if \( M \) is any \( (\mathbb{Z}_p \times H) \)-module that is finite, then \( M \) is a discrete \( (\mathbb{Z}_p \times H) \)-module. Then, as an immediate consequence of Theorem 3.6 – the proof of which uses [18] in a key way – and our central result, Theorem 4.9, we have the following.

**Theorem 1.12.** Let \( p \) be any prime and let \( H \) be any finite discrete group. If \( X \) is a \( (\mathbb{Z}_p \times H) \)-spectrum and an \( f \)-spectrum, then \( X \) is a discrete \( (\mathbb{Z}_p \times H) \)-spectrum.

We state the conclusion of the above result more precisely: under the hypotheses of Theorem 1.12, there is a zigzag

\[
X \xrightarrow{\simeq} X' \xleftarrow{\simeq} X^\text{dis}_X
\]

of \( (\mathbb{Z}_p \times H) \)-spectra and \( (\mathbb{Z}_p \times H) \)-equivariant maps that are weak equivalences of symmetric spectra, and \( X^\text{dis}_X \) is a discrete \( (\mathbb{Z}_p \times H) \)-spectrum. Thus, as in Definition 6.2, it is natural to identify \( X \) with the discrete \( (\mathbb{Z}_p \times H) \)-spectrum \( X^\text{dis}_X \) and to define

\[
X^{h(\mathbb{Z}_p \times H)} = (X^\text{dis}_X)^{h(\mathbb{Z}_p \times H)}.
\]

To go further, we need to introduce some notation and make a few comments. Let \( \Sigma \text{Sp} \) denote the model category of symmetric spectra (as in \([24]\), Theorem 3.4.4). We use

\[
(-)_f : \Sigma \text{Sp} \to \Sigma \text{Sp}, \quad Z \mapsto Z_f
\]

to denote a fibrant replacement functor, so that given the spectrum \( Z \), there is a natural map \( Z \to Z_f \) that is a trivial cofibration, with \( Z_f \) fibrant. It is useful to note that if \( X \) is a \( G \)-spectrum, then \( X_f \) is also a \( G \)-spectrum and the trivial cofibration \( X \to X_f \) is \( G \)-equivariant. Similarly, if \( p : X \to Y \) is a map of \( G \)-spectra (thus, \( p \) is \( G \)-equivariant), then \( p_f : X_f \to Y_f \) is a map of \( G \)-spectra.
We want to highlight the fact that in zigzag (1.13), the construction of \( X^\text{dis}_N \) is elementary: by Definition 4.4,

\[
X^\text{dis}_N = \operatorname{colim}_{m \geq 0} \operatorname{holim}_{(n+1) \text{ times}} \left( \prod_{[n] \in \Delta} \left( \operatorname{Sets}(\mathbb{Z}_p \times H, \cdots, \operatorname{Sets}(\mathbb{Z}_p \times H, X_f) \cdots) \right) \right),
\]

where each \((\mathbb{Z}_p \times H) \times \{e\}\) is an (open normal) subgroup of \( \mathbb{Z}_p \times H \) and \( p^m \mathbb{Z}_p \) has its usual meaning. We would like the reader to see how basic the construction of \( X^\text{dis}_N \) is, and thus, in this introduction, we do not think it is necessary to give any further explanation of (1.14). It turns out that for a \((\mathbb{Z}_p \times H)\)–spectrum \( X \) that is an \( f \)–spectrum,

\[
X^h(\mathbb{Z}_p \times H) \simeq \left( \operatorname{holim}_{(n+1) \text{ times}} \left( \prod_{[n] \in \Delta} \left( \operatorname{Sets}(\mathbb{Z}_p \times H, \cdots, \operatorname{Sets}(\mathbb{Z}_p \times H, X_f) \cdots) \right) \right) \right),
\]

by Theorem 6.3. We are confident that without any additional explanation, the reader has at least an almost complete understanding of the meaning of the expression in (1.15); later reading about it (and (1.14)) will mostly just confirm the reader’s “native conclusions.”

We now explain our application of Theorem 1.12 to the conjecture of Ausoni and Rognes. Let \( p \geq 5 \). Then

\[
\mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1),
\]

and as discussed earlier, \( K(KU_p) \) is a \( \mathbb{Z}_p^\times \)–spectrum. By giving \( V(1) \) the trivial \( \mathbb{Z}_p^\times \)–action, \( K(KU_p) \wedge V(1) \) is a \( \mathbb{Z}_p^\times \)–spectrum under the diagonal action.

Let \( ku_p \) be the \( p \)–completed connective complex \( K \)–theory spectrum, with coefficients \( \pi_*(ku_p) = \mathbb{Z}_p[u] \), where \( |u| = 2 \), as before. In \cite{BlumbergMandell2011}, Andrew Blumberg and Michael Mandell proved a conjecture of Rognes that there is a localization cofiber sequence

\[
K(\mathbb{Z}_p) \to K(ku_p) \to K(KU_p) \to \Sigma K(\mathbb{Z}_p),
\]

and hence, there is a cofiber sequence

\[
K(\mathbb{Z}_p) \wedge V(1) \to K(ku_p) \wedge V(1) \to K(KU_p) \wedge V(1) \to \Sigma K(\mathbb{Z}_p) \wedge V(1)).
\]

By \cite{Rognes2008}, it is known that \( K(\mathbb{Z}_p) \wedge V(1) \) is an \( f \)–spectrum (see also \cite{Ausoni2016} pages 663–664 for a helpful discussion about \( V(1), K(\mathbb{Z}_p) \)). Also, Ausoni \cite[Theorems 1.1, 8.1]{Ausoni2016} showed that there exists an element \( b \in V(1)_{2p+2}K(ku_p) \) such that if

\[
F_p[b] \subset V(1)_*K(ku_p)
\]

denotes the polynomial \( F_p \)–subalgebra generated by \( b \), then there is a short exact sequence of graded \( F_p[b] \)–modules

\[
0 \to \Sigma^{2p-3}F_p[b] \to V(1)_*K(ku_p) \to F \to 0,
\]

where \( F \) is a free \( F_p[b] \)–module on \( 4p+4 \) generators. (Work of Rognes with Ausoni played a role in the Ausoni result: for example, see \cite[Section 8]{Rognes2008}. Also, \cite[Theorems 1.1, 8.1]{Ausoni2016} was, in some sense, anticipated by \cite[discussion of Lemma 6.6]{Rognes2008}, as explained in \cite[discussion of Proposition 1.4]{Ausoni2016}.)

It follows from the last result that \( K(ku_p) \wedge V(1) \) is an \( f \)–spectrum, and hence, cofiber sequence (1.16) implies that \( K(KU_p) \wedge V(1) \) is an \( f \)–spectrum. Therefore, by setting \( H = \mathbb{Z}/(p-1) \) in Theorem 1.12, we obtain that \( K(KU_p) \wedge V(1) \) is (in the sense of zigzag (1.13)) a discrete \( \mathbb{Z}_p^\times \)–spectrum.
Our next step is to note that there is an equivalence
\[ K(KU_p) \wedge v_2^{-1}V(1) = K(KU_p) \wedge (\colim_{j \geq 0} \Sigma^{-jd}V(1)) \]
\[ \simeq \colim_{j \geq 0} (K(KU_p) \wedge \Sigma^{-jd}V(1))_f, \]
where \( \{(K(KU_p) \wedge \Sigma^{-jd}V(1))_f\}_{j \geq 0} \) is a diagram of \( \mathbb{Z}_p^\times \)-spectra and \( \mathbb{Z}_p^\times \)-equivariant maps (as in the case of \( V(1) \), each spectrum \( \Sigma^{-jd}V(1) \) is given the trivial \( \mathbb{Z}_p^\times \)-action). Since \( K(KU_p) \wedge V(1) \) is an \( f \)-spectrum, it is immediate that for each \( j \geq 0 \), \( (K(KU_p) \wedge \Sigma^{-jd}V(1))_f \) is an \( f \)-spectrum, and hence, Theorem 1.12 implies that each \((K(KU_p) \wedge \Sigma^{-jd}V(1))_f\) can be regarded as a discrete \( \mathbb{Z}_p^\times \)-spectrum.

**Remark 1.17.** As above, we let \( p \geq 5 \). To aid the reader in making connections between the theory developed in this paper and the application of it that is discussed in this introduction, we use the terminology that is set up in later sections to express our main conclusions above. Let \( \mathcal{N} \) denote the collection of open normal subgroups of \( \mathbb{Z}_p^\times \) that corresponds to the family \( \{(p^n \mathbb{Z}_p) \times \{e\}\}_{n \geq 0} \) of subgroups of \( \mathbb{Z}_p \times \mathbb{Z}/(p-1) \). Then \( \mathbb{Z}_p^\times \) has a good filtration (see Definition 3.3), and we have shown that \((\mathbb{Z}_p^\times, K(KU_p) \wedge V(1), \mathcal{N})\) is a suitably finite triple (see Definition 6.1) and
\[ (\mathbb{Z}_p^\times, \{(K(KU_p) \wedge \Sigma^{-jd}V(1))_f\}_{j \geq 0}, \mathcal{N}) \]
is a suitably filtered triple (Definition 7.1).

Let \( \mathcal{N} \) be as defined in Remark 1.17. As explained (in greater generality) in the discussion centered around (7.2), there is a zigzag of \( \mathbb{Z}_p^\times \)-equivariant maps
\[ C_p := \colim_{j \geq 0} (K(KU_p) \wedge \Sigma^{-jd}V(1))_f \xrightarrow{\sim} \colim_{j \geq 0} (K(KU_p) \wedge \Sigma^{-jd}V(1))_f \]
\[ \simeq \colim_{j \geq 0} ((K(KU_p) \wedge \Sigma^{-jd}V(1))_f)_{\mathcal{N}^d}, \]
with each map a weak equivalence of symmetric spectra, and \( C_p^{\text{dis}} \) is a discrete \( \mathbb{Z}_p^\times \)-spectrum. The above zigzag is obtained by taking a colimit of the zigzags that are obtained from (1.13) by setting \( X \) (in (1.13)) equal to \((K(KU_p) \wedge \Sigma^{-jd}V(1))_f\), for each \( j \geq 0 \).

Let us now put the various equivalences above together. Following Definition 7.2, we identify the \( \mathbb{Z}_p^\times \)-spectrum \( C_p \) with the discrete \( \mathbb{Z}_p^\times \)-spectrum \( C_p^{\text{dis}} \) and we make the concomitant definition
\[ (C_p)^{\mathbb{Z}_p^\times} = (C_p^{\text{dis}})^{\mathbb{Z}_p^\times}. \]
Similarly, it is natural to identify the \( \mathbb{Z}_p^\times \)-spectrum \( K(KU_p) \wedge v_2^{-1}V(1) \) with \( C_p \), and hence, with the discrete \( \mathbb{Z}_p^\times \)-spectrum \( C_p^{\text{dis}} \) (the mapping telescope \( v_2^{-1}V(1) \) has the trivial \( \mathbb{Z}_p^\times \)-action). Thus, we define
\[ (K(KU_p) \wedge v_2^{-1}V(1))^{\mathbb{Z}_p^\times} = (C_p^{\text{dis}})^{\mathbb{Z}_p^\times}. \]
More explicitly, we have
\[ (K(KU_p) \wedge v_2^{-1}V(1))^{\mathbb{Z}_p^\times} = \left( \colim_{j \geq 0} ((K(KU_p) \wedge \Sigma^{-jd}V(1))_f)_{\mathcal{N}^d} \right)^{\mathbb{Z}_p^\times}. \]
Now let $K$ be an arbitrary closed subgroup of $\mathbb{Z}_p^\times$. By the identification above of $K(KU_p) \wedge v^{-1}_1 V(1)$ with $C^{\text{dis}}_p$ in the world of $\mathbb{Z}_p^\times$--spectra and as in Definition 7.3 it follows that the $K$--spectrum $K(KU_p) \wedge v^{-1}_1 V(1)$ can be regarded as the discrete $K$--spectrum $C^{\text{dis}}_p$, and hence, it is natural to define

$$(K(KU_p) \wedge v^{-1}_1 V(1))^{hK} = \left(\colim_{j \geq 0} ((K(KU_p) \wedge \Sigma^{-jd} V(1))^{\text{dis}}_p)^{hK}\right).$$

Similarly (and easier; see the discussion just above [8,5]), for each $j \geq 0$, it is natural to define

$$(K(KU_p) \wedge \Sigma^{-jd} V(1))^{hK} = ((K(KU_p) \wedge \Sigma^{-jd} V(1))^{\text{dis}}_p)^{hK}.$$  

This completes the construction of the continuous homotopy fixed point spectra that appear in Theorem 1.6.

1.3. Concluding introductory remarks: our underlying framework, terminology, etc. In work in preparation, we use the theory developed in this paper to study $(KU_p)^{h\mathbb{Z}_p^\times}$, and more generally, $E^{hG}_n$, for $G$ a closed subgroup of $\mathbb{G}_n$.

We work in the framework of symmetric spectra in this paper because it is a symmetric monoidal category and such a category is important for studying the algebraic $K$--theory of commutative $S$--algebras. For example, in symmetric spectra, the role of commutative $S$--algebras is played by commutative symmetric ring spectra, and their properties are essential in the statement that $\mathbb{Z}_p^\times$ acts on $K(KU_p)$ by morphisms of commutative symmetric ring spectra. Furthermore, use of the framework of symmetric spectra makes available for future work the model category $\text{Alg}_{A,G}$ of discrete commutative $G$--$A$--algebras, where $G$ is any profinite group and $A$ is a commutative symmetric ring spectrum (see [7, Section 5.2]). Since the $\mathbb{G}_n$--action on $K(E_n)$ is by maps of commutative symmetric ring spectra, the model category $\text{Alg}_{K((KU_p)(\mathbb{P}^n)),\mathbb{G}_n}$ (or $\text{Alg}_{\mathbb{G}_n}$) might play a role in understanding the conjectural continuous homotopy fixed point spectrum $(K(E_n))^{h\mathbb{G}_n}$.

We conclude this introduction with some preparatory comments for the upcoming work. For the rest of the paper, “spectrum” means symmetric spectrum of simplicial sets (except for a few instances in which the exception is clearly noted). It is useful to recall that given any collection $\{X_\gamma\}_{\gamma \in \Gamma}$ of fibrant spectra, there is an isomorphism $\pi_k(\prod_{\gamma \in \Gamma} X_\gamma) \cong \prod_{\gamma \in \Gamma} \pi_k(X_\gamma)$ of abelian groups, where $k$ is any integer, for the product of spectra $\prod_{\gamma \in \Gamma} X_\gamma$. Also, it is helpful to note that if a map $f$ of spectra is, when regarded as a map of Bousfield-Friedlander spectra, a weak equivalence (in the usual stable model structure on Bousfield-Friedlander spectra), then the map $f$ is a weak equivalence of spectra, by [24, Theorem 3.1.11]. We use holim to denote the homotopy limit for $\Sigma\text{Sp}$, as defined in [22, Definition 18.1.8].

Let $G$ be any profinite group. A “discrete $G$--spectrum” is a discrete symmetric $G$--spectrum, as defined in [7, Section 2.3] (see also [14 Section 3]); these objects, together with the $G$--equivariant maps (see [7] for the precise definition), constitute the category $\Sigma\text{Sp}_G$ of discrete $G$--spectra. By [7, Theorem 2.3.2], there is a model category structure on $\Sigma\text{Sp}_G$ in which a morphism $f$ in $\Sigma\text{Sp}_G$ is a weak equivalence (cofibration) if and only if $f$ is a weak equivalence (cofibration) in $\Sigma\text{Sp}$. Given a fibrant replacement functor

$$(-)_f: \Sigma\text{Sp}_G \to \Sigma\text{Sp}_G, \quad X \mapsto X_f$$
(thus, $X_{fg}$ is fibrant in $\Sigma\Sp_G$), such that there is a natural trivial cofibration $\eta: X \to X_{fg}$ in $\Sigma\Sp_G$, there is the induced map

$$\eta^G: X^G \to (X_{fg})^G = X^{hG}.$$ 

By [4, Section 3.1], the target of $\eta^G$, the homotopy fixed point spectrum $X^{hG}$, is the output of the right derived functor of fixed points.

Given any profinite group $G$, a “$G$–spectrum” is a naive symmetric $G$–spectrum and not a genuine equivariant symmetric $G$–spectrum. Thus, when $G$ is finite, a $G$–spectrum need not be an equivariant symmetric $G$–spectrum in the sense of [20] (defined by using the spheres $S(G) = \Lambda G S^1$ in the bonding maps).

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## 2. SOME PRELIMINARIES

In this section, we explain some constructions and a result (Lemma 2.1) that will be useful for our main work later. As in the introduction, we let $G$ be any profinite group.

Given a set $S$, let $\text{Sets}(G, S)$ be the $G$–set of all functions $f: G \to S$, with $G$–action defined by

$$(g \cdot f)(g') = f(g' g), \quad g, g' \in G.$$ 

Let $U$ be the forgetful functor from the category of $G$–sets to the category of sets. Then it is easy to see that $\text{Sets}(G, -)$ is the right adjoint of $U$. By analogy with a standard construction in group cohomology, $\text{Sets}(G, S)$ can be thought of as the “coinduced $G$-set on $S$.”

The construction $\text{Sets}(G, S)$ prolongs to the category of $G$–spectra and the forgetful functor $U_G$ from the category of $G$–spectra to $\Sigma\Sp$ has a right adjoint that is given by the prolongation $\text{Sets}(G, -)$, so that, given a spectrum $Z$ and any $k, l \geq 0$, the set of $l$-simplices of the pointed simplicial set $\text{Sets}(G, Z)_k$ is defined by

$$\text{Sets}(G, Z)_{k,l} = \text{Sets}(G, Z_{k,l}).$$

Thus, for any $Z \in \Sigma\Sp$, there is an isomorphism

$$\text{Sets}(G, Z) \cong \prod_G Z$$

in $\Sigma\Sp$, where the right-hand side of the isomorphism is the product of $|G|$ copies of $Z$. Since the functors $U_G$ and $\text{Sets}(G, -)$ are an adjoint pair, there is the associated triple (e.g., see [37 8.6.2]), and, for any $G$-spectrum $X$, we let

$$\text{Sets}(G^{\star+1}, X)$$

denote the cosimplicial $G$–spectrum that is given in the usual way by the triple (for more detail, see [37 8.6.4]).

For any $m \geq 0$, we use $G^m$ to denote the Cartesian product of $m$ copies of $G$, with $G^0 = *$, the point. Then it is not hard to see that, for any $G$–spectrum $X$ and any $m \geq 0$, the “$G$–spectrum of $m$–cosimplices” of the cosimplicial $G$–spectrum $\text{Sets}(G^{\star+1}, X)$ satisfies the $G$–equivariant isomorphism

$$\text{Sets}(G^{\star+1}, X)^m \cong \text{Sets}(G, \text{Sets}(G^m, X)),$$

where, as before, $\text{Sets}(G^m, X)$ is the spectrum defined on the level of sets by $\text{Sets}(G^m, X)_{k,l} = \text{Sets}(G^m, X_{k,l})$, for every $k, l \geq 0$. 


We make no claim of originality for Lemma 2.1 below; for example, it is a variation on the fact that if \( L \) is a discrete group, \( Z \) an \( L \)-spectrum that is fibrant in \( 
abla \Sp \), and \( P \) a subgroup of \( L \), then the descent spectral sequence
\[
E_{2}^{s,t} = \pi_{t-s}(\Map_{P}(EL_{+}Z)) \cong \pi_{t-s}(Z^{hP})
\]
has an \( E_{2} \)-term that satisfies
\[
E_{2}^{s,t} = H^{s}(P; \pi_{t}(Z)),
\]
the (non-continuous) group cohomology of \( P \) with coefficients in the \( P \)-module \( \pi_{t}(Z) \). Also, the result below is a “discrete version” of [20] page 210 and the proof of Lemma 5.4 and [14, proof of Lemma 7.12]. But, since Lemma 2.1 is a useful tool for our work later, we give a complete proof.

**Lemma 2.1.** Let \( G \) be a profinite group. If \( X \) is a \( G \)-spectrum and \( K \) is a subgroup of \( G \), then, for every \( s \geq 0 \) and any \( t \in \mathbb{Z} \), there is an isomorphism
\[
\lim_{\Delta}^{s} \pi_{t}(\Sets(G^{s+1},X_{f})^{K}) \cong H^{s}(K; \pi_{t}(X)).
\]

**Remark 2.2.** To avoid any confusion, we note that in the statement of Lemma 2.1 \( K \) is any subgroup of \( G \) (thus, for example, \( K \) does not have to be a closed subgroup of \( G \)).

**Proof of Lemma 2.1.** If \( A \) is an abelian group and \( P \) is a profinite group, let \( \Sets(P,A) \) be the abelian group of functions \( P \to A \); in fact, \( \Sets(P,A) \) is a \( P \)-module, with its \( P \)-action defined by \((p \cdot f)(p') = f(p'p)\). Then there is an isomorphism
\[
\lim_{\Delta}^{s} \pi_{t}(\Sets(G^{s+1},X_{f})^{K}) \cong H^{s}\left[\Sets(G^{s+1},\pi_{t}(X))^{K}\right],
\]
where \( \Sets(G^{s+1},\pi_{t}(X))^{K} \) is the cochain complex obtained by applying, for each \( m \geq 0 \), the chain of isomorphisms
\[
\pi_{t}\left((\Sets(G^{s+1},X_{f})^{K})^{m}\right) \cong \pi_{t}(\Sets(G,\Sets(G^{m},X_{f}))^{K})
\]
\[
\cong \pi_{t}\left(\prod_{G/K} \prod_{G^{m}} X_{f}\right)
\]
\[
\cong \prod_{G/K} \prod_{G^{m}} \pi_{t}(X_{f})
\]
\[
\cong \Sets(G,\Sets(G^{m},\pi_{t}(X)))^{K}
\]
\[
\cong \Sets(G^{m+1},\pi_{t}(X))^{K}.
\]
Above, for \( m \geq 1 \), \( \Sets(G^{m},\pi_{t}(X)) \) is the \( K \)-module of functions \( G^{m} \to \pi_{t}(X) \) whose \( K \)-action is given by
\[
(k \cdot p)(g_{1},g_{2},g_{3},...,g_{m}) = p(g_{1}k,g_{2},g_{3},...,g_{m}),
\]
for \( k \in K, p \in \Sets(G^{m},\pi_{t}(X)) \), and \( g_{1},g_{2},...,g_{m} \in G \). (In the preceding sentence, since \( m \geq 1 \), it goes without saying that this sentence also defines the \( K \)-action on the \( K \)-module \( \Sets(G^{m+1},\pi_{t}(X)) \) that appears in the last expression in the above chain of isomorphisms.)

Notice that there is a \( G \)-equivariant monomorphism
\[
\pi_{t}(X) \xrightarrow{\eta} \Sets(G,\pi_{t}(X)), \quad [f] \mapsto (g \mapsto g \cdot [f])
\]
and a homomorphism
\[
\text{ev}_{i} : \Sets(G,\pi_{t}(X)) \to \pi_{t}(X), \quad p \mapsto p(1),
\]
such that $ev \circ \eta = \text{id}_{\pi_t(X)}$. Then, since the cochain complex $\text{Sets}(G^{*+1}, \pi_t(X))$ originally comes from a triple, there is an exact sequence

(2.3) \[ 0 \rightarrow \pi_t(X) \xrightarrow{\eta} \text{Sets}(G^{*+1}, \pi_t(X)) \]
of $K$–modules (for example, see the dual of [37 Corollary 8.6.9]).

There is a chain

\[
\text{Sets}(G, \text{Sets}(G^m, \pi_t(X))) \cong \prod_K \prod_{G/K} \text{Sets}(G^m, \pi_t(X))
\]
\[
\cong \text{Hom}_{\mathbb{Z}}(\bigoplus_K \mathbb{Z}, \prod_{G/K} \text{Sets}(G^m, \pi_t(X)))
\]
of isomorphisms of $K$–modules, where $\text{Hom}_{\mathbb{Z}}(\bigoplus_K \mathbb{Z}, \prod_{G/K} \text{Sets}(G^m, \pi_t(X)))$ is a coinduced $K$–module, and hence, Shapiro’s Lemma implies that

\[
H^s(K; \text{Sets}(G, \text{Sets}(G^m, \pi_t(X)))) 
\cong H^s(K; \text{Hom}_{\mathbb{Z}}(\bigoplus_K \mathbb{Z}, \prod_{G/K} \text{Sets}(G^m, \pi_t(X)))) 
\]
\[
= 0,
\]
whenever $s > 0$, for all $m \geq 0$.

Our last conclusion above implies that exact sequence (2.3) is a resolution of the $K$–module $\pi_t(X)$ by ($-)^K$–acyclic $K$–modules, and therefore, $H^s\left[\text{Sets}(G^{*+1}, \pi_t(X))^K\right] \cong H^s(K; \pi_t(X))$, as desired.

\[ \square \]

3. PROFINITE GROUPS THAT HAVE A GOOD FILTRATION

As usual, let $G$ be a profinite group. In this section, after explaining the notion of a good filtration for $G$ and making several comments about it, we show that $\mathbb{Z}_p \times H$, where $p$ is any prime and $H$ is a finite discrete group, has a good filtration.

**Definition 3.1.** Given a discrete $G$–module $M$, let

\[ \lambda_M^s : H^s(G; M) \rightarrow H^s(G; M) \]

be the natural homomorphism between continuous cohomology and non-continuous cohomology that is obtained by regarding each group $\text{Map}_c(G^m, M)$ of continuous cochains as a subgroup of the corresponding group $\text{Sets}(G^m, M)$ of all cochains. Then, in this paper (see Remark 3.2 below), we say that $G$ is **good** if $\lambda_M^s$ is an isomorphism for all $s \geq 0$ and every finite discrete $G$–module $M$.

**Remark 3.2.** The above definition is taken from [34 page 13, Exercise 2]: if $G$ is strongly complete, so that $G \cong \hat{G}$, where $\hat{G}$ is the profinite completion of $G$, and $\lambda_M^s$ is an isomorphism for all $s \geq 0$ and every finite $G$–module $M$ (a finite $G$–module consists of finite orbits, so that every stabilizer subgroup of $G$ has finite index, and hence, is an open subgroup (since $G$ is strongly complete), so that a finite $G$–module is automatically a discrete $G$–module), then, following Serre, $G$ is “bon.” In general, since $G$ and $\hat{G}$ need not be the same, our definition of “good” is different from the usual one (that is, the aforementioned “bon”) in group theory. However, our use of “good” in this paper should cause no confusion, since, throughout this paper, we only use “good” in the sense of Definition 3.1.
We say that $G$ has **finite cohomological dimension** (“finite c.d.”) if there exists some positive integer $r$ such that the continuous cohomology $H^*_c(G; M) = 0$, for all discrete $G$–modules $M$, whenever $s > r$.

**Definition 3.3.** A profinite group $G$ has a **good filtration** if

(a) there exists a directed poset $\Lambda$ such that there is an inverse system

$$\mathcal{N} = \{ N_\alpha \}_{\alpha \in \Lambda}$$

of open normal subgroups of $G$, with the maps in the diagram given by the inclusions (that is, $\alpha_1 \leq \alpha_2$ in $\Lambda$ if and only if $N_{\alpha_2}$ is a subgroup of $N_{\alpha_1}$);

(b) the intersection $\bigcap_{\alpha \in \Lambda} N_\alpha$ is the trivial group $\{e\}$;

(c) each $N_\alpha$ is a good profinite group, in the sense of Definition 3.1; and

(d) the collection $\{N_\alpha\}_{\alpha \in \Lambda}$ has uniformly bounded finite c.d.; that is, there exists a fixed natural number $r_G$, such that $H^*_c(N_\alpha; M) = 0$, for all $s > r_G$, whenever $\alpha \in \Lambda$ and $M$ is any discrete $N_\alpha$–module.

**Remark 3.4.** Let $G$ be a profinite group with a good filtration and let $\mathcal{N} = \{ N_\alpha \}_{\alpha \in \Lambda}$ satisfy (a)–(d) in Definition 3.3. It follows from (a) and (b) that $\mathcal{N}$ is a cofinal subcollection of the family of all open normal subgroups of $G$, and hence, the canonical homomorphism $G \rightarrow \lim_{\alpha \in \Lambda} G/N_\alpha$ is a homeomorphism. Now choose any $\alpha \in \Lambda$, so that $N_\alpha$ is good, by (c) above. We give an argument that is suggested by [33, page 14, Exercise 2, (c)] (for instances of Serre’s argument that are closely related to the version given here, see [18, proof of Theorem 2.10] and [33, proof of Proposition 3.1]). Since

$$\lambda^*_M : H^*_c(N_\alpha; M) \rightarrow H^*(N_\alpha; M)$$

is an isomorphism in each degree for any finite discrete $G$–module $M$, the $E_2$–term of the Lyndon-Hochschild-Serre spectral sequence

$$E^{p,q}_{2} = H^p(G/N_\alpha; H^q(N_\alpha; M)) \Rightarrow H^{p+q}(G; M)$$

for continuous group cohomology (since $G/N_\alpha$ is a finite discrete group, the $E_2$–term is given by just group cohomology) is isomorphic to the $E_2$–term of the corresponding Lyndon-Hochschild-Serre spectral sequence

$$H^p(G/N_\alpha; H^q(N_\alpha; M)) \Rightarrow H^{p+q}(G; M)$$

for group cohomology, and hence, by comparison of spectral sequences, the map

$$\lambda^*_M : H^*_c(G; M) \cong H^*(G; M)$$

is an isomorphism, for all $s \geq 0$ and any finite discrete $G$–module $M$.

**Remark 3.5.** Let $G$ be a profinite group that has finite c.d. and let $\{ N_\alpha \}_{\alpha \in \Lambda}$ be an inverse system of open normal subgroups of $G$ that satisfies (a)–(c) in Definition 3.3. Then the inverse system also satisfies (d), so that $G$ has a good filtration. This conclusion follows from the fact that for $r$ as in our definition of finite c.d. above (just before Definition 3.3), Shapiro’s Lemma implies that whenever $s > r$, given any $\alpha \in \Lambda$,

$$H^*_c(N_\alpha; M) \cong H^*_c(G; \text{Coind}^G_{N_\alpha}(M)) = 0,$$

for all discrete $N_\alpha$–modules $M$ (above, Coind$^G_{N_\alpha}(M)$ is the coinduced module of continuous functions $G \rightarrow M$ that are $N_\alpha$–equivariant).

**Theorem 3.6.** Let $p$ be any prime and let $G = \mathbb{Z}_p \times H$, where $H$ is a finite discrete group and $G$ is equipped with the product topology. Then $G$ has a good filtration.
Proof. Recall that there is a descending chain
\[ Z_p = U_0 \geq U_1 \geq \cdots \geq U_m \geq \cdots \]
of open normal subgroups of \( Z_p \), with \( U_m = p^m Z_p \) for each \( m \geq 0 \) and \( \bigcap_{m \geq 0} U_m = \{ e \} \). For each \( m \geq 0 \), we set \( N_m = U_m \times \{ e \} \), a subgroup of \( G \). We will show that \( \{ N_m \}_{m \geq 0} \) satisfies conditions (a)–(d) in Definition 3.3.

It is easy to see that \( \{ N_m \}_{m \geq 0} \) satisfies (a) and (b). By [13, Theorem 2.9], \( Z_p \) is a good profinite group and, for each \( \{ e \} \), \( \{ e \} \) is a subgroup of \( G \). We will show that \( \{ N_m \}_{m \geq 0} \) satisfies conditions (a)–(d) in Definition 3.3.

Finally, since the pro-\( p \)-group \( Z_p \) has cohomological \( p \)-dimension equal to one, it follows that \( Z_p \) has finite c.d. This fact, coupled with another application of the isomorphisms \( N_m \cong Z_p \) for all \( m \geq 0 \), shows that (d) holds. \( \square \)

4. An \( r \)-\( Z_p \)-spectrum is a discrete \( Z_p \)-spectrum

In this section, we prove one of the key results of this paper, Theorem 4.9, the title above illustrates a special case of this result, and the unfamiliar term in the title is defined below.

Definition 4.1. Let \( G \) be a profinite group and \( X \) a \( G \)-spectrum. If \( \pi_t(X) \) is a finite discrete \( G \)-module for every \( t \in \mathbb{Z} \), then we say that \( X \) is an \( r \)-\( G \)-spectrum.

Remark 4.2. Since an \( r \)-\( G \)-spectrum is both a \( G \)-spectrum and an \( f \)-spectrum, our first thought was to use the term “\( f \)-\( G \)-spectrum” for such an object, but this term is already used (often) by [19] (see [ibid., Definition 3.1]), and so we removed the horizontal stroke in the term’s “\( f \)” to “obtain” the “\( r \)” in “\( r \)-\( G \)-spectrum” (also, “restricted” is, roughly speaking, a synonym of “finite”). If \( G \) is strongly complete, then every \( r \)-\( G \)-spectrum \( X \) has an \( f \)-\( G \)-spectrum associated to it in the following way: \( X_f \) is a \( G \)-spectrum and since it is a fibrant spectrum, for each integer \( t \), there is an isomorphism
\[ \pi_t(X_f) \cong \operatorname{colim}_k \pi_{t+k}(X_k) = \pi_t(U(X_f)) \]
of finite abelian groups, where the last expression in (4.3) refers to the \( t \)-th (classical) stable homotopy group of the Bousfield-Friedlander spectrum \( U(X_f) \) that underlies \( X_f \), and hence, by an application of [29, Theorem 5.15], there is a \( G \)-equivariant map and weak equivalence \( U(X_f) \cong F^g_G(U(X_f)) \) of Bousfield-Friedlander spectra, with \( F^g_G(U(X_f)) \) an \( f \)-\( G \)-spectrum.

For the remainder of this section (with the exception of Lemma 4.7), \( G \) denotes a profinite group that has a good filtration. Thus, we let \( \mathcal{N} = \{ N_\alpha \}_{\alpha \in \Lambda} \) be an inverse system of open normal subgroups of \( G \) that satisfies the requirements of Definition 3.3.

Definition 4.4. Let \( X \) be a \( G \)-spectrum. We set
\[ X^\text{dis}_{\mathcal{N}} = \operatorname{colim}_{\alpha \in \Lambda} \operatorname{holim}_{\Delta} \operatorname{Sets}(G^{\bullet+1}, X_f)^{N_\alpha}, \]
where the colimit is formed in \( \Sigma \text{Sp} \).

Since each \( N_\alpha \) is an open normal subgroup of \( G \), with \( G/N_\alpha \) a finite discrete group, \( \operatorname{Sets}(G^{\bullet+1}, X_f)^{N_\alpha} \) is a cosimplicial \( G/N_\alpha \)-spectrum. Thus, the spectrum \( \operatorname{holim}_{\Delta} \operatorname{Sets}(G^{\bullet+1}, X_f)^{N_\alpha} \) is a \( G/N_\alpha \)-spectrum, and hence, a discrete \( G \)-spectrum.
(via the canonical projection \( G \to G/N_a \)). By [7 Section 3.4], colimits in \( \Sigma Sp_G \) are formed in \( \Sigma Sp \), and hence, we have the following observation.

**Lemma 4.5.** If \( X \) is a \( G \)-spectrum, where \( G \) is a profinite group that has a good filtration, then \( X^\text{dis}_N \) is a discrete \( G \)-spectrum.

**Remark 4.6.** Let \( X \) be a \( G \)-spectrum. Since \( \mathcal{N} \) is cofinal in the collection of all open normal subgroups of \( G \), there is an isomorphism
\[
X^\text{dis}_\mathcal{N} \cong \colim_{N \triangleleft_G G} \lim_{\Delta} \text{Sets}(G^\ast+1, X_f)^N
\]
of discrete \( G \)-spectra, where above, \( N \triangleleft G \) means that \( N \) is an open normal subgroup of \( G \). Similarly, if \( \mathcal{N}' = \{N_{a'}\}_{a' \in A'} \) is another inverse system of open normal subgroups of \( G \) that satisfies Definition [3], there is an isomorphism
\[
X^\text{dis}_{\mathcal{N}'} \cong \colim_{\alpha' \in A'} \lim_{\Delta} \text{Sets}(G^\ast+1, X_f)^{N_{a'}} \cong \colim_{N \triangleleft_G G} \lim_{\Delta} \text{Sets}(G^\ast+1, X_f)^N
\]
in \( \Sigma Sp_G \), and hence, there is an isomorphism \( X^\text{dis}_\mathcal{N} \cong X^\text{dis}_{\mathcal{N}'} \) in \( \Sigma Sp_G \). It follows that the definition of \( X^\text{dis}_\mathcal{N} \) is independent of the choice of inverse system \( \mathcal{N} \) up to isomorphism.

Now we are ready to prove the central result of this paper: its conclusion can be abbreviated by saying that if \( X \) is an \( r \)-\( G \)-spectrum (as in Definition [11]), then \( X \) is a discrete \( G \)-spectrum. We break up our work for this result into two pieces. The first piece, Lemma [4.7] below, can be regarded as a special case of [17 Proposition 6.4], in the setting of \( G \)-spectra.

**Lemma 4.7.** If \( G \) is any profinite group and \( X \) is any \( G \)-spectrum, then there is a \( G \)-equivariant map
\[
i_X : X \xrightarrow{\sim} \lim_{\Delta} \text{Sets}(G^\ast+1, X_f)
\]
that is a weak equivalence in \( \Sigma Sp \).

**Proof.** Given a spectrum \( Z \), let \( cc^\ast(Z) \) denote the constant cosimplicial spectrum on \( Z \). Then the \( G \)-equivariant map \( i_X \) is defined to be the composition
\[
i_X : X \xrightarrow{\sim} X_f \xrightarrow{\text{cc}^\ast} \lim_{\Delta} \text{Sets}(X_f) \xrightarrow{\text{cc}^\ast} \lim_{\Delta} \text{Sets}(G^\ast+1, X_f),
\]
where the last (rightmost) map is induced by repeated use of the \( G \)-equivariant monomorphism \( i : Y \to \text{Sets}(G, Y) \) of \( G \)-spectra, that is defined on the level of sets, for any \( G \)-spectrum \( Y \), by the maps
\[
y_{k,l} \to \text{Sets}(G, Y_{k,l}), \quad y \mapsto (g \mapsto g \cdot y).
\]

Notice that for each \( m \geq 0 \), the spectrum of \( m \)-cosimplicies of \( \text{Sets}(G^\ast+1, X_f) \),
\[
\left(\text{Sets}(G^\ast+1, X_f)\right)^m \cong \prod_{G^\ast+1} X_f,
\]
is fibrant, so that \( \text{Sets}(G^\ast+1, X_f) \) is a cosimplicial fibrant spectrum. Thus, there is a homotopy spectral sequence
\[
E_2^{s,t} \cong H^s[\pi_t(\text{Sets}(G^\ast+1, X_f))] \Rightarrow \pi_{t-s}(\lim_{\Delta} \text{Sets}(G^\ast+1, X_f)).
\]
By Lemma [2.1] we have
\[
E_2^{s,t} \cong H^s(\{e\}; \pi_t(X)) = \begin{cases} \pi_t(X), & s = 0; \\ 0, & s > 0, \end{cases}
\]
and hence, spectral sequence $^lE_{r}^*$ of (4.8) collapses, showing that $i_X$ is a weak equivalence.

**Theorem 4.9.** Let $G$ be a profinite group that has a good filtration and let $N$ be a diagram of subgroups of $G$ that satisfies Definition 3.3. If $X$ is an $r$–$G$–spectrum, then there is a zigzag of $G$–equivariant maps

$$X \xrightarrow{\cong} \text{holim}_{\mathcal{D}} \text{Sets}(G^{\bullet+1}, X_f) \rightleftarrows X_{\text{dis}}^N$$

that are weak equivalences in $\Sigma \text{Sp}$.

**Remark 4.11.** As stated just before Lemma 4.7, the above theorem says that (given a suitable profinite group $G$) an $r$–$G$–spectrum can be regarded as a discrete $G$–spectrum (in a canonical way): the “$G$–equivariant zigzag” of weak equivalences in (4.10) makes this statement precise.

**Proof of Theorem 4.9.** By Lemma 4.7, it suffices to construct a $G$–equivariant map

$$\phi_X : X_{\text{dis}}^N = \lim_{\rightarrow} \text{holim}_{\mathcal{D}} \text{Sets}(G^{\bullet+1}, X_f)^{N_\alpha} \rightarrow \text{holim}_{\mathcal{D}} \text{Sets}(G^{\bullet+1}, X_f)$$

and then show that it is a weak equivalence of spectra. The $G$–equivariant map $\phi_X$ is defined to be the composition

$$\lim_{\rightarrow} \text{holim}_{\mathcal{D}} \text{Ens}(G^{\bullet+1}, X_f)^{N_\alpha} \xrightarrow{\phi^1_X} \lim_{\rightarrow} \text{holim}_{\mathcal{D}} \text{Ens}(G^{\bullet+1}, X_f)^{N_\alpha} \xrightarrow{\phi^2_X} \text{holim}_{\mathcal{D}} \text{Ens}(G^{\bullet+1}, X_f)$$

of canonical maps, where, here (and below), to conserve space, we (sometimes) use the notation “$\lim_{\rightarrow}$” to denote “$\text{colim}_{\mathcal{D}}$”, and “$\text{Ens}$” in place of “$\text{Sets}$.”

The definition of the map $\phi^2_X$ is given explicitly as follows: the collection of inclusions $\text{Sets}(G^{\bullet+1}, X_f)^{N_\alpha} \hookrightarrow \text{Sets}(G^{\bullet+1}, X_f)$ induces the morphism

$$\phi^2_X : \lim_{\rightarrow} \text{Sets}(G^{\bullet+1}, X_f)^{N_\alpha} \rightarrow \text{Sets}(G^{\bullet+1}, X_f)$$

of cosimplicial $G$–spectra, and $\phi^2_X = \text{holim}_{\alpha \in \Lambda} \phi^2_X$. The morphism $\phi^2_X$ also induces a map

$$E_r^{*,*}(\overline{\phi^2_X}) : \text{II}E_r^{*,*} \rightarrow \text{I}E_r^{*,*},$$

from the homotopy spectral sequence

$$\text{II}E_{2}^{s,t} = H^s \left[ \pi_t \left( \lim_{\rightarrow} \text{Ens}(G^{\bullet+1}, X_f)^{N_\alpha} \right) \right] \Rightarrow \pi_{t-s} \left( \lim_{\rightarrow} \text{Ens}(G^{\bullet+1}, X_f)^{N_\alpha} \right)$$

to spectral sequence (4.8). We point out that the construction of spectral sequence (4.12) uses the fact that for each $m \geq 0$, the spectrum of $m$–cosimplices of $\lim_{\rightarrow} \text{Ens}(G^{\bullet+1}, X_f)^{N_\alpha}$ satisfies

$$(\lim_{\rightarrow} \text{Sets}(G^{\bullet+1}, X_f)^{N_\alpha})^m \cong \text{colim}_{\alpha \in \Lambda} \left( \prod_{G/N_\alpha} \prod_{G^{m}X_f} \right),$$

which is a fibrant spectrum, since products and filtered colimits of fibrant spectra are again fibrant (the second fact is justified, for example, in [12, Section 5]), so that $\lim_{\rightarrow} \text{Ens}(G^{\bullet+1}, X_f)^{N_\alpha}$ is a cosimplicial fibrant spectrum.
Notice that for spectral sequence \( \text{II} E^r_{s,t} \), there is the chain of isomorphisms
\[
\text{II} E^s_{t} \cong \text{colim}_{\alpha \in \Lambda} H^s(N_{\alpha}; \pi_t(X)) \\
\cong \text{colim}_{\alpha \in \Lambda} H^s_c(N_{\alpha}; \pi_t(X)) \\
\cong H^s_c(\bigcap_{\alpha \in \Lambda} N_{\alpha}; \pi_t(X)) \\
= H^s((\{e\}; \pi_t(X)),
\]
where the first isomorphism uses Lemma 2.1 and the fact that filtered colimits of fibrant spectra commute with \([S^t, -] \); the second isomorphism applies the assumption that each \( N_{\alpha} \) is a good profinite group; and the last step (involving the equality) is due to property (b) of Definition 3.3. Therefore, there is an isomorphism \( \text{II} E^s_{t} \cong I E^s_{t} \), for all \( s \) and \( t \), so that the map \( E^r_{s,t}( \phi^2_X ) \) of spectral sequences is an isomorphism from the \( E_2 \)-terms onward. Hence, the map \( \pi_s( \phi^2_X ) = [S^s, \phi^2_X] \) between the abutments of these conditionally convergent spectral sequences is an isomorphism, so that \( \phi^2_X \) is a weak equivalence.

As in (4.13), there are isomorphisms
\[
H^s \left[ \pi_t(\text{Sets}(G^{s+1}, X_f)^{N_{\alpha}}) \right] \cong H^s(N_{\alpha}; \pi_t(X)) \cong H^s_c(N_{\alpha}; \pi_t(X))
\]
for each \( \alpha \), and hence, condition (d) of Definition 3.3 implies that
\[
H^s \left[ \pi_t(\text{Sets}(G^{s+1}, X_f)^{N_{\alpha}}) \right] = 0, \text{ for all } s > r_G, \text{ every } t \in \mathbb{Z}, \text{ and each } \alpha.
\]
Therefore, the map \( \phi^1_X \) is a weak equivalence, by [27, Proposition 3.4].

Finally, we can conclude that \( \phi_X \) is a weak equivalence, since \( \phi^1_X \) and \( \phi^2_X \) are weak equivalences. □

5. An extension of the central result, Theorem 4.9

In this section, we show – in Theorem 5.1 – that the hypotheses of Theorem 4.9 can be slightly loosened. We give this result in this later section so that Theorem 4.9 (and Section 4) is ready-made for the intended applications. Suppose that \( X \) is a \( G \)-spectrum with homotopy groups that are always torsion discrete \( G \)-modules: as explained in the second part of this section, the homotopy groups of such a \( G \)-spectrum are closely related to those of \( r \)-\( G \)-spectra. However, we explain why our proof of Theorem 4.9 does not extend to this more general “torsion case.” We conclude this section by noting that if \( G \) is of type \( FP_{\infty} \) as an abstract group (for background on this notion, we refer to [11]), then the proof of Theorem 4.9 does extend to the torsion case.

For the rest of this section, we suppose that \( G \) is an arbitrary profinite group and \( X \) is any \( G \)-spectrum. Given this context, it is easy to see that Definition 4.4 and Lemma 4.5 depend only on condition (a) of Definition 3.3 and hence, under only the additional assumption of condition (a), the spectrum \( X^{(G)}_{\rho} \) is defined and is a discrete \( G \)-spectrum. Also, the proof of Theorem 4.9 depends only on

(i) condition (a);
(ii) the assumption that the \( G \)-module \( \pi_t(X) \) is a discrete \( G \)-module, for every \( t \in \mathbb{Z} \); and
(iii) part (b) of Definition 3.3 \( \bigcap_{\alpha \in \Lambda} N_{\alpha} = \{e\} \),
except in three spots:

- in the second isomorphisms of (4.13) and (4.14), in addition to (i) and (ii) above, the proof of Theorem 4.9 uses both the assumption that \( \pi_t(X) \) is finite for every integer \( t \) and part (c) of Definition 3.3 and
- in (4.15), besides (i) and (ii) above, the proof uses part (d) of Definition 3.3.

These observations imply the following result.

**Theorem 5.1.** Let \( G \) be a profinite group, with \( \mathcal{N} = \{ N_\alpha \}_{\alpha \in \Lambda} \) an inverse system of open normal subgroups of \( G \) that satisfies properties (a) and (b) of Definition 3.3, and let \( X \) be a \( G \)-spectrum such that condition (ii) above holds. Also, suppose that the map

\[
\lambda^s_{\pi_t(X)}: H^s_c(N_\alpha; \pi_t(X)) \to H^s(N_\alpha; \pi_t(X))
\]

is an isomorphism for all \( s \geq 0 \), every integer \( t \), and each \( \alpha \in \Lambda \). If

- there exists a natural number \( r \), such that for all integers \( t \) and every \( \alpha \in \Lambda \), \( H^s_c(N_\alpha; \pi_t(X)) = 0 \), for all \( s > r \); or
- there exists some fixed integer \( l \), such that \( \pi_t(X) = 0 \), for all \( t > l \),

then there is a zigzag of \( G \)-equivariant maps

\[
X \xrightarrow{\sim} \text{holim}_{\Delta} \text{Sets}(G^{m+1}, X_f) \xleftarrow{\sim} X^\text{dis}_N
\]

that are weak equivalences in \( \Sigma \text{Sp} \), with \( X^\text{dis}_N \) (defined as in Definition 4.4) a discrete \( G \)-spectrum.

**Proof.** The only part of the theorem that is not justified by the remarks preceding it is the following. In our proof of Theorem 4.9, in (4.15), we assumed condition (d) of Definition 3.3, but by [27, Proposition 3.4], an alternative to assuming condition (d) is to require that there exists some fixed integer \( l \), such that for each \( m \geq 0 \) and every \( \alpha \in \Lambda \),

\[
\pi_t(\text{Sets}(G^{m+1}, X_f)^{N_\alpha}) \cong \prod_{G/N_\alpha \times G^{m}} \pi_t(X) = 0, \text{ for all } t > l,
\]

which is equivalent to assuming that \( \pi_t(X) = 0 \), for all \( t > l \). \( \square \)

We conclude this section by explaining why the proof of Theorem 4.9 fails to extend to the case when \( X \) is a \( G \)-spectrum with each homotopy group a (possibly infinite) discrete \( G \)-module that is also a torsion abelian group. With \( G \) as in Theorem 4.9, our assumptions imply that for each \( t \in \mathbb{Z} \),

\[
\pi_t(X) = \bigcup_\beta M_{t, \beta}
\]

is the union of its finite \( G \)-submodules \( M_{t, \beta} \), each of which is automatically a discrete \( G \)-module.

As discussed at the beginning of this section, in the second isomorphisms in (4.13) and (4.14), we need to know that for each \( \alpha \) and every integer \( t \), the natural map

\[
\lambda^s_{\pi_t(X)}: H^s_c(N_\alpha; \pi_t(X)) \to H^s(N_\alpha; \pi_t(X))
\]
is an isomorphism, for all $s \geq 0$. Since each $N_\alpha$ is a good profinite group, there are isomorphisms
\[
H^s(N_\alpha; \pi_t(X)) \cong \lim_{\beta} H^s(N_\alpha; M_{t, \beta}) \\
\cong \lim_{\beta} H^s(N_\alpha; M_t) \\
\cong H^s \left( \lim_{\beta} \text{Sets}(N_\alpha, M_t) \right),
\]
where, here, given an $N_\alpha$–module $M$, Sets$(N_\alpha, M)$ denotes the usual cochain complex such that $H^s \left( \text{Sets}(N_\alpha, M) \right) = H^s(N_\alpha; M)$, for each $s \geq 0$, with the abelian group of $m$–cochains equal to
\[
\text{Sets}(N_\alpha, M)^m = \prod_{N_\alpha} M,
\]
for each $m \geq 0$.

It follows that the map $\lambda^s_{\pi_t(X)}$ is an isomorphism if and only if the canonical map
\[
h^{s,t} : H^s \left( \lim_{\beta} \text{Sets}(N_\alpha, M_t) \right) \to H^s \left( \text{Sets}(N_\alpha, \bigcup_{\beta} M_t) \right) = H^s(N_\alpha; \pi_t(X))
\]
is an isomorphism.

Since filtered colimits and infinite products do not commute in general, the map $h^{s,t}$ above need not be an isomorphism, so that $\lambda^s_{\pi_t(X)}$ need not be an isomorphism: this situation is the crux of what prevents the proof of Theorem 4.9 from going through in the case when each $\pi_t(X)$ is a torsion discrete $G$–module.

Remark 5.2. Let $G$ be as in Theorem 4.9 and suppose that $X$ is a $G$–spectrum such that $\pi_t(X)$ is a discrete $G$–module and torsion abelian group, for every integer $t$. Then it is clear from the above discussion that if $G$, as an abstract group, is of type $F P_\infty$, then $H^*(G; -) \cong \text{Ext}^*_{Z[G]}(Z, -)$ commutes with direct limits, and hence, the conclusion of Theorem 4.9 is still valid. However, we do not know of any examples of infinite profinite groups that are of type $F P_\infty$ as abstract groups.

6. The spectrum $X_N^{dis}$, fibrancy, and homotopy fixed points

In this section, we let $G$ be any profinite group and $X$ any $G$–spectrum.

Definition 6.1. If $G$, $X$, and $N$ (an inverse system of open normal subgroups of $G$) satisfy the hypotheses of Theorem 4.9 or Theorem 5.1, then we say that the triple $(G, X, N)$ is suitably finite. (In the preceding sentence, by satisfying the hypotheses of Theorem 5.1, we mean that $G$, $X$, and $N$ satisfy the conditions of the first two sentences of Theorem 5.1 and at least one of the two “itemized conditions” (that is, the conditions marked by a “•” listed in the third sentence of Theorem 5.1).) Notice that if $(G, X, N)$ is a suitably finite triple, then there is a zigzag of $G$–equivariant maps
\[
X \xrightarrow{\sim} \text{holim}_{\Delta} \text{Sets}(G^{\star+1}, X_f) \xleftarrow{\sim} X_N^{\text{dis}}
\]
that are weak equivalences in $\Sigma\text{Sp}$.

Definition 6.2. If $(G, X, N)$ is a suitably finite triple, then because of the above zigzag of equivalences between $X$ and $X_N^{\text{dis}}$, it is natural to identify $X$ with the discrete $G$–spectrum $X_N^{\text{dis}}$, and hence, to define
\[
X^{hG} = (X_N^{\text{dis}})^{hG}.
\]
Remark 6.3. Let \((G, X, \mathcal{N})\) be a suitably finite triple, with the inverse system \(\mathcal{N}\) written as \(\{N_\alpha\}_{\alpha \in \Lambda}\), and suppose that \(X\) is a discrete \(G\)-spectrum (that is, before the identification of Definition 6.2 \(X \in \Sigma \text{Sp}_G\)). In this case, after following Definition 6.2 \(X^{hG}\) can mean \((X_{fG})^G\) or \((X^{\text{dis}})^G\). Since \(X \in \Sigma \text{Sp}_G\), the weak equivalence \(i_X: X \xrightarrow{\simeq} \text{holim}_{\Delta}(G^{\bullet+1}, X_f)\) factors into the map \(\delta: X \to X^{\text{dis}}\), which is defined to be the composition

\[
X \xrightarrow{\simeq} \text{colim}_{\alpha \in \Lambda} X^{N_\alpha} \overset{\text{colim}(i_X)^{N_\alpha}}{\longrightarrow} \text{colim}_{\Delta} (\text{Sets}(G^{\bullet+1}, X_f))^{N_\alpha} \overset{\cong}{\longrightarrow} X^{\text{dis}}
\]

(the first isomorphism in the composition is due to the fact that, since \(\mathcal{N}\) satisfies (a) and (b) in Definition 6.2 \(\mathcal{N}\) is a cofinal subcollection of \(\{N \mid N \leq_a G\}\)), followed by the weak equivalence \(X^{\text{dis}} \xrightarrow{\simeq} \text{holim}_{\Delta} \text{Sets}(G^{\bullet+1}, X_f)\), and hence, the map \(\delta\) is a weak equivalence of spectra. It follows that \(\delta\) is a weak equivalence in \(\Sigma \text{Sp}_G\); therefore, \(\delta\) induces a weak equivalence \((X_{fG})^G \xrightarrow{\simeq} (X^{\text{dis}})^G\), showing that the two possible interpretations of \(X^{hG}\) are equivalent to each other.

Several interesting consequences of Definition 6.2 are stated in Theorem 6.3 below. Before giving this result, we need to give some background material for its proof.

Let \(G-\Sigma \text{Sp}\) be the category of \(G\)-spectra (as defined at the end of the introduction): \(G-\Sigma \text{Sp}\) has a model category structure in which a morphism \(f\) is a weak equivalence (cofibration) if and only if \(f\) is a weak equivalence (cofibration) when regarded as a morphism in \(\Sigma \text{Sp}\). The existence of this model structure follows, for example, from the fact that \(G-\Sigma \text{Sp}\) is isomorphic to \(\Sigma \text{Sp}^{(\ast_G)}\), the category of functors \(\{\ast_G\} \to \Sigma \text{Sp}\), where \(\{\ast_G\}\) is the one-object groupoid associated to \(G\), and this diagram category can be equipped with an injective model structure, by [25 Proposition A.2.8.2], since \(\Sigma \text{Sp}\) is a combinatorial model category.

Since the forgetful functor \(U_G: G-\Sigma \text{Sp} \to \Sigma \text{Sp}\) preserves weak equivalences and cofibrations, the adjoint functors \((U_G, \text{Sets}(G, -))\) are a Quillen pair. Also, it will be helpful to recall the standard fact that if \(Y\) is fibrant in \(G-\Sigma \text{Sp}\), then \(Y\) is fibrant in \(\Sigma \text{Sp}\) (since, for example, an injective fibrant object in \(\Sigma \text{Sp}^{(\ast_G)}\) is projective fibrant in \(\Sigma \text{Sp}^{(\ast_G)}\) (one reference for this is [25 Remark A.2.8.5]: \(\Sigma \text{Sp}^{(\ast_G)}\) has a projective model structure by [ibid., Proposition A.2.8.2])).

The left adjoint functor \(\Sigma \text{Sp} \to G-\Sigma \text{Sp}\) that sends a spectrum to itself, but now regarded as a \(G\)-spectrum with trivial \(G\)-action, preserves weak equivalences and cofibrations. It follows that the right adjoint, the \(G\)-fixed points functor \((-)^G: G-\Sigma \text{Sp} \to \Sigma \text{Sp}\), is a right Quillen functor, and if \(Y \to Y_{\text{fib}}\) is a trivial cofibration to a fibrant object, in \(G-\Sigma \text{Sp}\), then

\[
Y^{hG} = (Y_{\text{fib}})^G.
\]

As in [25 Example 1.1.5.8], the category \(\{\ast_G\}\) can be regarded as a simplicial category by defining the simplicial set \(\text{Map}_{\{\ast_G\}}(\{\ast_G\}, \{\ast_G\})\) to be the constant simplicial set on \(\text{Hom}_{\{\ast_G\}}(\{\ast_G\}, \{\ast_G\})\). With \(S\) equal to the category of simplicial sets, it is easy to see that the category of \(S\)-enriched functors from \(\{\ast_G\}\) to the simplicial category \(\Sigma \text{Sp}\), with morphisms the \(S\)-enriched natural transformations, is identical to the usual functor category \(\Sigma \text{Sp}^{\{\ast_G\}}\). Since \(\Sigma \text{Sp}\) is a simplicial model category, it follows from [25 Proposition A.3.3.2, Remark A.3.3.4] that the injective model structure on \(\Sigma \text{Sp}^{\{\ast_G\}}\) is simplicial, and hence, the model category \(G-\Sigma \text{Sp}\) is simplicial.
Let $\text{holim}^G$ denote the homotopy limit for $G\Sigma\text{Sp}$, as defined in [22] Definition 18.1.8 (this definition uses the fact that $G\Sigma\text{Sp}$ is a simplicial model category). Since the forgetful functor $U_{\Sigma}$ is a right adjoint (its left adjoint is given by the functor $\Sigma\text{Sp} \to G\Sigma\text{Sp}$ that sends a spectrum $Z$ to the $G$–spectrum $\bigvee_G Z$, where $G$ acts only on the indexing set of the coproduct), limits in $G\Sigma\text{Sp}$ are formed in $\Sigma\text{Sp}$. Also, it is a standard fact that the cotensor $Y \otimes_S^G$ in $G\Sigma\text{Sp}$, where $Y$ is a $G$–spectrum and $S_*$ is a simplicial set, is equal to the corresponding cotensor $Y \otimes_S^G$ in $\Sigma\text{Sp}$ equipped with the natural $G$–action. Since $\text{holim}^G$ is defined as the equalizer of maps between products of cotensors, it follows that $\text{holim}^G$ is formed in $\Sigma\text{Sp}$: if $\{Y_i\}_{i \in I}$ is a small diagram of $G$–spectra, then $\text{holim}^G \{Y_i\}_{i \in I}$ is equal to the spectrum $\text{holim}_I \{Y_i\}_{i \in I}$ equipped with the induced $G$–action.

Now we recall [15, Theorem 4.3], but we rewrite it for symmetric spectra ([loc. cit.] is written in the world of Bousfield-Friedlander spectra, but the argument is the same when using symmetric spectra). The forgetful functor $U : \Sigma\text{Sp}_G \to G\Sigma\text{Sp}$ has a right adjoint, the discretization functor $(-)_d : G\Sigma\text{Sp} \to \Sigma\text{Sp}_G$, $Y \mapsto (Y)_d = \colim_{N \triangleleft G} Y^N$.

Since $U$ preserves weak equivalences and cofibrations, the functors $(U, (-)_d)$ are a Quillen pair.

**Theorem 6.4.** If $(G, X, N)$ is a suitably finite triple, then

$$X^{hG} \simeq (X_{\Delta}^{\text{dis}})^G \cong \left(\text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f)\right)^G \simeq X^{\tilde{h}G}.$$  

**Proof.** Since $X_f$ is fibrant in $\Sigma\text{Sp}$, $\text{Sets}(G, X_f)$ is fibrant in $G\Sigma\text{Sp}$, and hence, it is fibrant in $\Sigma\text{Sp}$. By iterating this argument, we obtain that $\text{Sets}(G^{*+1}, X_f)$ is a cosimplicial fibrant $G$–spectrum (that is, for each $m \geq 0$, the $m$–cosimplices are a fibrant $G$–spectrum). It follows that $\text{holim}^G_{\Delta} \text{Sets}(G^{*+1}, X_f)$ is a fibrant $G$–spectrum. Since $\text{holim}^G_{\Delta} \text{Sets}(G^{*+1}, X_f)$ is equal to the $G$–spectrum $\text{holim}^G_{\Delta} \text{Sets}(G^{*+1}, X_f)$, we write the latter instead of the former.

Let $X \to X_{\text{fib}}$ be a trivial cofibration to a fibrant object, in $G\Sigma\text{Sp}$, and notice that the equivalence $X \xrightarrow{\simeq} \text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f)$ (in Definition 6.1) is a weak equivalence with fibrant target, in $G\Sigma\text{Sp}$. Then there exists a weak equivalence $X_{\text{fib}} \xrightarrow{\simeq} \text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f)$ in $G\Sigma\text{Sp}$, and since $(-)^G : G\Sigma\text{Sp} \to \Sigma\text{Sp}$ is a right Quillen functor, the induced map

$$X^{\tilde{h}G} = (X_{\text{fib}})^G \xrightarrow{\simeq} \left(\text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f)\right)^G$$

is a weak equivalence.

Since $N$ satisfies conditions (a) and (b) in Definition 3.3, there is an isomorphism

$$X^{\text{dis}}_N \cong \colim_{N \triangleleft G} \left(\text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f)\right)^N \cong \left(\text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f)\right)_d$$

of discrete $G$–spectra, as noted in Remark 4.6 and since the functor $(-)_d$ is a right Quillen functor, $(\text{holim}_{\Delta} \text{Sets}(G^{*+1}, X_f))_d$ is a fibrant discrete $G$–spectrum, and hence, so is $X^{\text{dis}}_N$. Thus, applying the right Quillen functor $(-)^G : \Sigma\text{Sp}_G \to \Sigma\text{Sp}$ to the fibrant replacement map $X^{\text{dis}}_N \to (X^{\text{dis}}_N)^G$, which is a trivial cofibration between fibrant objects in $\Sigma\text{Sp}_G$, yields the weak equivalence

$$(X^{\text{dis}}_N)^G \xrightarrow{\simeq} \left((X^{\text{dis}}_N)^G\right)^G = (X^{\text{dis}}_N)^{hG} = X^{hG}.$$
The final step is to note that

\[
(X^\text{dis}_N)^G \cong \left( \colim_{N \leq_G} \text{holim Sets}(G^{\bullet+1}, X_f) \right)^N \Delta^G \cong \text{holim} \left( \text{Sets}(G^{\bullet+1}, X_f) \right)^G,
\]
as desired. \(\square\)

**Remark 6.5.** Let \((G, X, \mathcal{N})\) be a suitably finite triple. In light of the proof of Theorem 6.4, we reexamine the \(G\)-equivariant zigzag

\[
X \xrightarrow{\cong} \text{holim Sets}(G^{\bullet+1}, X_f) \xleftarrow{\cong} X^\text{dis}_N
\]
of equivalences: the first map is taking an explicit fibrant replacement of \(X\) — call it \(X'\) — in the model category of \(G\)-spectra (here, we do not require the fibrant replacement map to be a cofibration) and the second map is the inclusion into \(X'\) from its largest discrete \(G\)-subspectrum \(X^\text{dis}_N \cong (X')_d\) (this description of the output of the functor \((-)_d\) is used in \([36, \text{page } 861]\) and it is meant to be taken literally: if \(X''\) is a discrete \(G\)-subspectrum of \(X'\), then the isomorphism

\[
X'' \cong \colim_{N \leq_G} (X'')^N
\]
shows that \(X''\) is a \(G\)-subspectrum of

\[
\colim_{N \leq_G} (X')^N = (X')_d \cong X^\text{dis}_N.
\]

Therefore, we can think of the above zigzag as saying that \(X\) is equivalent to an explicit model — \(X^\text{dis}_N\) — for \((\tilde{R}(-)_d)(X)\), the output of the total right derived functor \(\tilde{R}(-)_d\) of \((-)_d\) (recall from the proof of Theorem 6.4 that there is a weak equivalence \(X^\text{fib}_d \xrightarrow{\cong} X'\) between fibrant objects in \(G\Sigma\text{Sp}\), and hence, there is a weak equivalence

\[
(\tilde{R}(-)_d)(X) = (X^\text{fib}_d)_d \xrightarrow{\cong} (X')_d \xrightarrow{\cong} X^\text{dis}_N
\]
of discrete \(G\)-spectra).

We can use Theorem 6.4 to build a descent spectral sequence, as follows.

**Corollary 6.6.** If \((G, X, \mathcal{N})\) is a suitably finite triple, then there is a conditionally convergent descent spectral sequence that has the form

\[
E_2^{s,t} \cong H^s(G; \pi_t(X)) \cong H^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(\text{holim } X^\text{h}_G) \cong \pi_{t-s}(X^hG).
\]

**Proof.** At the beginning of the proof of Theorem 6.4, we noted that \(\text{Sets}(G^{\bullet+1}, X_f)\) is a cosimplicial fibrant \(G\)-spectrum, and hence, \(\text{Sets}(G^{\bullet+1}, X_f)^G\) is a cosimplicial fibrant spectrum. Thus, there is a homotopy spectral sequence

\[
E_2^{s,t} = H^s[\pi_t(\text{Sets}(G^{\bullet+1}, X_f)^G)] \Rightarrow \pi_{t-s}(\text{holim } \text{Sets}(G^{\bullet+1}, X_f)^G).
\]

This spectral sequence is the descent spectral sequence described in the corollary, and the isomorphism that occurs in the abutment of the descent spectral sequence follows immediately from applying Theorem 6.4 to the abutment of the above homotopy spectral sequence.

Lemma yields the isomorphism \(E_2^{s,t} \cong H^s(G; \pi_t(X))\), for all \(s \geq 0\) and any integer \(t\). If the triple \((G, X, \mathcal{N})\) satisfies the hypotheses of Theorem 6.4 then there is an isomorphism \(H^s(G; \pi_t(X)) \cong H^s(G; \pi_t(X))\), for all \(s \geq 0\) and any \(t \in \mathbb{Z}\), by Remark 3.3. If the triple \((G, X, \mathcal{N})\) satisfies the hypotheses of Theorem 5.1 then this same isomorphism is obtained by applying the spectral sequence argument of Remark 3.3 to the case where the "\(M\)" in the remark is changed to \(\pi_t(X)\). \(\square\)
To illustrate the previous result, we have the following special case for $G = \mathbb{Z}_p$.

**Corollary 6.7.** Let $p$ be any prime. If $X$ is a $\mathbb{Z}_p$–spectrum and an $f$–spectrum, then there is a strongly convergent descent spectral sequence

$$E^{s,t}_2 = H^s_\ast(\mathbb{Z}_p; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{h\mathbb{Z}_p}),$$

with $E^{s,t}_2 = 0$, whenever $s \geq 2$ and $t$ is any integer.

**Proof.** By Theorem 3.4, $\mathbb{Z}_p$ has a good filtration, with $\mathcal{N} = \{p^m \mathbb{Z}_p\}_{m \geq 0}$. Any subgroup of finite index in $\mathbb{Z}_p$ is open in $\mathbb{Z}_p$ and $\pi_s(X)$ is finite in each degree, so that $\pi_t(X)$ is a discrete $\mathbb{Z}_p$–module (see Remark 3.2), for every integer $t$. It follows that $X$ is an $r$–$G$–spectrum, and hence, $(\mathbb{Z}_p, X, \{p^m \mathbb{Z}_p\}_{m \geq 0})$ is a suitably finite triple, $X$ can be identified with the discrete $\mathbb{Z}_p$–spectrum $X^{dis}_\mathcal{N}$, $X^{h\mathbb{Z}_p}$ is defined, and Corollary 6.6 gives the conditionally convergent spectral sequence described above.

Since each $\pi_t(X)$ is finite and $\mathbb{Z}_p$ has cohomological $p$–dimension one, $E^{s,t}_2 = H^s_\ast(\mathbb{Z}_p; \pi_t(X)) = 0$, whenever $s \geq 2$, for all integers $t$ (this fact is well-known; as a reference for the argument, see, for example, [18, proof of Theorem 2.9]), and this vanishing result implies that the spectral sequence is strongly convergent, by [35, Lemma 5.48].

\[ \square \]

7. **Filtered diagrams of suitably finite triples and their colimits.**

In this section, we extend Definitions 6.1 and 6.2 to the case of a filtered diagram of $G$–spectra.

**Definition 7.1.** Let $G$ be a profinite group with $\mathcal{N}$ a fixed inverse system of open normal subgroups of $G$, and let $\{X_\mu\}_\mu$ be a filtered diagram of $G$–spectra (thus, the morphisms in the diagram are $G$–equivariant), such that for each $\mu$, $(G, X_\mu, \mathcal{N})$ is a suitably finite triple and $X_\mu$ is a fibrant spectrum. We refer to $(G, \{X_\mu\}_\mu, \mathcal{N})$ as a suitably filtered triple.

Let $(G, \{X_\mu\}_\mu, \mathcal{N})$ be a suitably filtered triple. Since the colimit of a filtered diagram of weak equivalences between fibrant spectra is a weak equivalence, there is a zigzag of $G$–equivariant maps

$$\text{colim}_{\mu} X_\mu \xrightarrow{\sim} \text{colim}_{\mu} \text{holim}_{\Delta} \text{Sets}(G^{\bullet+1}, (X_\mu)_f) \xleftarrow{\sim} \text{colim}_{\mu} (X_\mu)^{dis}_\mathcal{N}$$

that are weak equivalences in $\Sigma \text{Sp}$ (since each $\text{Sets}(G^{\bullet+1}, (X_\mu)_f)$ is a cosimplicial fibrant spectrum, each $\text{holim}_{\Delta} \text{Sets}(G^{\bullet+1}, (X_\mu)_f)$ is a fibrant spectrum; also, by the proof of Theorem 6.4 each $(X_\mu)^{dis}_N$ is a fibrant discrete $G$–spectrum, and thus, by [7, Corollary 5.3.3], each $(X_\mu)^{dis}_N$ is a fibrant spectrum). The right end of zigzag (7.2) satisfies

$$\text{colim}_{\mu} (X_\mu)^{dis}_N = \text{colim}_{\mu} \text{colim}_{\alpha \in \Lambda} \text{Sets}(G^{\bullet+1}, (X_\mu)_f)^{N_\alpha}$$

and $\text{colim}_{\mu} (X_\mu)^{dis}_N$ is a discrete $G$–spectrum. (In zigzag (7.2), since each $X_\mu$ is a fibrant spectrum, the fibrant replacement in each $(X_\mu)_f$ is not necessary. However, we believe that by leaving the $(-)_f$ in each $(X_\mu)_f$ and by continuing to use the maps $1_{X_\mu}$ as previously defined (in the proof of Lemma 6.7), our presentation is less cumbersome.) Notice that for every integer $t$, our hypotheses on the triple and zigzag (7.2) imply that the composition

$$\text{colim}_{\mu} \pi_t(X_\mu) \xrightarrow{\sim} \pi_t(\text{colim}_{\mu} X_\mu) \xrightarrow{\sim} \pi_t(\text{colim}_{\mu} (X_\mu)^{dis}_N) \xrightarrow{\sim} \text{colim}_{\mu} \pi_t((X_\mu)^{dis}_N)$$

implies...
consists of three isomorphisms in the category of discrete $G$-modules (in particular, each of the four abelian groups above is a discrete $G$-module).

**Definition 7.4.** Given a suitably filtered triple $(G, \{X_\mu\}_{\mu}, N)$, the weak equivalences in zigzag (7.2) imply that the $G$–spectrum $\text{colim}_\mu X_\mu$ can be identified with the discrete $G$–spectrum $\text{colim}_\mu (X_\mu)^{\text{dis}}$. Thus, it is natural to define

$$(\text{colim}_\mu X_\mu)^{hG} = (\text{colim}_\mu (X_\mu)^{\text{dis}})^{hG}.$$ 

We can extend this definition to an arbitrary closed subgroup $K$ in $G$: since the $K$–spectrum $\text{colim}_\mu X_\mu$ can be regarded as the discrete $K$–spectrum $\text{colim}_\mu (X_\mu)^{\text{dis}}$, we define

$$(\text{colim}_\mu X_\mu)^{hK} = (\text{colim}_\mu (X_\mu)^{\text{dis}})^{hK}.$$ 

**Remark 7.5.** Let $(G, \{X_\mu\}_{\mu}, N)$ be a suitably filtered triple and let $K$ be a closed subgroup of $G$. Suppose that $K$ is finite, so that its topology is both profinite and discrete. It follows that any $K$–spectrum can itself be regarded as a discrete $K$–spectrum, whenever desired. Thus, the notation $(\text{colim}_\mu X_\mu)^{hK}$ can mean $((\text{colim}_\mu X_\mu)_{fK})^K$ or it can mean $(\text{colim}_\mu (X_\mu)^{\text{dis}})^{hK}$. In the remainder of this remark, to avoid any ambiguity, we take $(\text{colim}_\mu X_\mu)^{hK}$ to have the latter meaning, $(\text{colim}_\mu (X_\mu)^{\text{dis}})^{hK}$, and for the former meaning, $((\text{colim}_\mu X_\mu)_{fK})^K$, we just write it out as needed. Since (7.2) can be regarded as a zigzag of weak equivalences in the category of discrete $K$–spectra, there is a zigzag of weak equivalences

$$((\text{colim}_\mu X_\mu)_{fK})^K \simeq \Delta \rightarrow (\text{colim}_\mu \text{holim Sets}(G^{*+1}, (X_\mu)_f))^K \simeq \rightarrow (\text{colim}_\mu (X_\mu)^{\text{dis}})^{hK}.$$ 

Also, given an arbitrary $K$–spectrum $Y$, let $Y \rightarrow Y_{\text{fib}}$ be a trivial cofibration to a fibrant object, in $K$-$\Sigma Sp$, the category of $K$–spectra. Then we have

$$Y^{hK} = (Y_{\text{fib}})^K \simeq (Y_{fK})^K,$$

where the last equivalence follows from the fact that $Y_{\text{fib}}$ is fibrant in $\Sigma Sp_K$ (and this fibrancy assertion is true because the functor $(-)_d: K$-$\Sigma Sp \rightarrow \Sigma Sp_K$ preserves fibrant objects and $(Y_{\text{fib}})_d \cong Y_{\text{fib}}$ is an isomorphism in $\Sigma Sp_K$). We conclude that when $K$ is finite, there are equivalences

$$(\text{colim}_\mu X_\mu)^{hK} \simeq ((\text{colim}_\mu X_\mu)_{fK})^K \simeq (\text{colim}_\mu (X_\mu)^{hK},$$

as one would expect.

We say that a profinite group $G$ has *finite virtual cohomological dimension* (“finite v.c.d.”) if $G$ contains an open subgroup that has finite c.d. Under the assumption that $G$ has this property, the following result gives a descent spectral sequence for the situation described by Definition 7.1.

**Theorem 7.6.** Let $G$ be a profinite group with finite v.c.d. If $(G, \{X_\mu\}_{\mu}, N)$ is a suitably filtered triple and $K$ is a closed subgroup of $G$, then there is a conditionally convergent descent spectral sequence $E_{s,t}^{*,*}(K)$ that has the form

$$E_2^{s,t}(K) = H^s(K; \pi_t(\text{colim}_\mu X_\mu)) \implies \pi_{t-s}((\text{colim}_\mu X_\mu)^{hK}).$$

**Remark 7.8.** If $G$ has a good filtration, then condition (d) of Definition 3.3 implies that $G$ has finite v.c.d. Thus, if $(G, \{X_\mu\}_{\mu}, N)$ is a suitably filtered triple such that there is some $\mu_0 \in \{\mu\}_\mu$ for which the triple $(G, X_{\mu_0}, N)$ satisfies the hypotheses of
Theorem 7.9 then $G$ has finite v.c.d. and the first sentence of Theorem 7.6 can be omitted.

Proof of Theorem 7.6. Let $U$ be an open subgroup of $G$ that has finite c.d. Then $U \cap K$ is an open subgroup of $K$, and since $U$ has finite c.d. and $U \cap K$ is closed in $U$, there exists some $r$ such that for any discrete $(U \cap K)$–module $M$,

$$H^s_\ast(U \cap K; M) \cong H^s_\ast(U; \text{Coind}_{U \cap K}^U(M)) = 0, \quad \text{whenever } s > r,$$

by Shapiro’s Lemma. This shows that $K$ has finite v.c.d. Therefore, [7] proofs of Theorem 3.2.1, Proposition 3.5.3, and [14] proof of Theorem 7.9] yield the conditionally convergent spectral sequence

$$E_2^{s,t} = H^s_\ast(K; \pi_t(\text{colim}_\mu (X_\mu)_N^{\text{dis}})) \Longrightarrow \pi_{t-s} \left( (\text{colim}_\mu (X_\mu)_N^{\text{dis}})^{hK} \right),$$

and this is the desired spectral sequence, since the middle map in composition (7.3) is an isomorphism of discrete $K$–modules.

We provide some more detail (based on the above two references) because it will be useful to us later. Since $K$ has finite v.c.d.,

$$(\text{colim}_\mu (X_\mu)_N^{\text{dis}})^{hK} \cong \text{holim}_\Delta \text{colim}_\mu (X_\mu)_N^{\text{dis}},$$

and for each $m \geq 0$, the $m$-cosimplices of cosimplicial spectrum $\Gamma_K \text{colim}_\mu (X_\mu)_N^{\text{dis}}$ satisfy the isomorphism

$$(\Gamma_K \text{colim}_\mu (X_\mu)_N^{\text{dis}})^m \cong \text{colim}_{V \leq K^m} \prod_{K^m/V} \text{colim}_\mu (X_\mu)_N^{\text{dis}},$$

where $K^m$ is the $m$-fold Cartesian product of $K$ ($K^0$ is the trivial group $\{e\}$, equipped with the discrete topology). (For more detail about this, we refer the reader to [2, Sections 2.4, 3.2].)

The above spectral sequence is the homotopy spectral sequence for the spectrum $\text{holim}_\Delta \Gamma_K \text{colim}_\mu (X_\mu)_N^{\text{dis}}$. Based on [7] proof of Theorem 3.2.1 and [14] proof of Theorem 7.9], the reader might expect us to instead form the homotopy spectral sequence for $\text{holim}_\Delta \Gamma_K (\text{colim}_\mu (X_\mu)_N^{\text{dis}})^{fK}$. But since each $(X_\mu)_N^{\text{dis}}$ is a fibrant spectrum, $\text{colim}_\mu (X_\mu)_N^{\text{dis}}$ is already a fibrant spectrum, so that we do not need to apply $(-)^{fK}$ to it (so that we are taking the homotopy limit of a cosimplicial fibrant spectrum).

\[\] Notice that if $(G, \{X_\mu\}_{\mu}, N)$ is a suitably filtered triple, then for each $\mu' \in \{\mu\}_\mu$, $(G, \{X_\mu\}_{\mu' \in \{\mu\}_\mu}, N)$ is a suitably filtered triple, so that Definition 3.4 gives

$$(X_{\mu'})^{hK} = ((X_{\mu'})^N)^{hK},$$

for any closed subgroup $K$ of $G$.

Theorem 7.10. Let $G$ be a profinite group with finite v.c.d., let $(G, \{X_\mu\}_{\mu}, N)$ be a suitably filtered triple such that $\{\mu\}_\mu$ is a directed poset, and let $K$ be a closed subgroup of $G$. If there exists a nonnegative integer $r$ such that for all $t \in \mathbb{Z}$ and each $\mu$, $H^s_\ast(K; \pi_t(X_\mu)) = 0$ whenever $s > r$, then descent spectral sequence $E_2^{s,t}(K)$ in (7.7) is strongly convergent and there is an equivalence of spectra

$$\text{colim}_\mu X_\mu^{hK} \cong \text{colim}_\mu (X_\mu)^{hK}.$$
Proof. For all $t \in \mathbb{Z}$, when $s > r$, we have
\[
E_2^{s,t}(K) = H_c^s(K; \pi_t(\text{colim } X_{\mu})) \cong \text{colim } H_c^s(K; \pi_t(X_{\mu})) = 0,
\]
so that the spectral sequence is strongly convergent, by [35, Lemma 5.48].

If $V$ is an open normal subgroup of $K^n$, where $m \geq 0$, then $K^n/V$ is finite, and hence, isomorphism (7.9) implies that
\[
(\Gamma_K^• \text{colim}(X_{\mu})_{\mathcal{N}})^m \cong \text{colim } \bigcup_{V \triangleleft K^n} \prod_{p \leq v} \Gamma_K(X_{\mu})_{\mathcal{N}} \cong \text{colim } (\Gamma_K^•(X_{\mu})_{\mathcal{N}})^m,
\]
so that there is an isomorphism
\[
\Gamma_K^• \text{colim}(X_{\mu})_{\mathcal{N}} \cong \text{colim } \Gamma_K^•(X_{\mu})_{\mathcal{N}}
\]
of cosimplicial spectra. Therefore, we have
\[
\left(\text{colim } (X_{\mu})_{\mathcal{N}}\right)^{hK} \cong \text{holim } \Gamma_K^• \text{colim}(X_{\mu})_{\mathcal{N}} \cong \text{holim } \Gamma_K^•(X_{\mu})_{\mathcal{N}},
\]
which gives
\[
\left(\text{colim } X_{\mu}\right)^{hK} \cong \text{holim } \Gamma_K^•(X_{\mu})_{\mathcal{N}} \leftarrow \text{colim } \Gamma_K^•(X_{\mu})_{\mathcal{N}}
\]
and the canonical colim/holim exchange map above is a weak equivalence if there exists a nonnegative integer $r$ such that for every $t$ and all $\mu$, 
\[
H^s[\pi_t(\Gamma_K^•(X_{\mu})_{\mathcal{N}})] = 0, \quad \text{when } s > r,
\]
by [27, Proposition 3.4]. The proof is completed by noting that there are isomorphisms
\[
H^s[\pi_t(\Gamma_K^•(X_{\mu})_{\mathcal{N}})] \cong H^s_c(K; \pi_t((X_{\mu})_{\mathcal{N}})) \cong H^s_c(K; \pi_t(X_{\mu})),
\]
for all $s \geq 0$. \hfill $\square$

8. The proofs of Theorems 1.6 and 1.7

After proving Theorem 1.6 a task which ends with (8.6), we prove Theorem 1.7.

Let $p \geq 5$ and let $K$ be any closed subgroup of $\mathbb{Z}_p^\times$. As noted in the proof of Theorem 5.6, $\mathbb{Z}_p$ has finite c.d., and since it is open in $\mathbb{Z}_p^\times$, $\mathbb{Z}_p^\times$ has finite v.c.d. Also, in the introduction (see Remark 1.17), we showed that
\[
(\mathbb{Z}_p^\times, \{ (K(KU_p) \wedge \Sigma^{-jd}V(1))_{j \geq 0}, \mathcal{N} \}),
\]
where $\mathcal{N}$ is as defined in Remark 1.17 is a suitably filtered triple. Therefore, by Theorem 7.6 there is a conditionally convergent descent spectral sequence that has the form
\[
E_2^{s,t} \Rightarrow \pi_{t-s}(\left( K(KU_p) \wedge v_2^{-1}V(1) \right)^{hK}),
\]
where
\[
E_2^{s,t} = H_c^s(K; \pi_t(\text{colim } (K(KU_p) \wedge \Sigma^{-jd}V(1))_{j \geq 0}))
\cong H_c^s(K; \pi_t(K(KU_p) \wedge V(1))[v_2^{-1}]),
\]
as desired.

Since $p \geq 5$, $V(1)$ is a homotopy commutative and homotopy associative ring spectrum [28], so that $\pi_*(K(KU_p) \wedge V(1))$ is a graded right $\pi_*(V(1))$–module, and
hence, \( \pi_*(K(U_p) \land V(1)) \) is a unitary \( \mathbb{F}_p \)-module. It follows that for every integer \( t \), the finite abelian group \( \pi_*(K(U_p) \land V(1)) \) is a \( p \)-torsion group (that is, \( pm = 0 \), for all \( m \in \pi_*(K(U_p) \land V(1)) \)).

Given any profinite group \( G \), we use \( cd_p(G) \) to denote its cohomological \( p \)-dimension. Since \( K \) is closed in \( \mathbb{Z}_p^\times \),

\[
\text{cd}_p(K) \leq \text{cd}_p(\mathbb{Z}_p^\times) = \text{cd}_p(\mathbb{Z}_p) = 1,
\]

where the first equality is due to the fact that \( \mathbb{Z}_p \) is the \( p \)-Sylow subgroup of \( \mathbb{Z}_p^\times \), and hence,

\[
H_s^c(K; M) = 0, \quad \text{for all } s \geq 2,
\]

whenever \( M \) is a discrete \( K \)-module that is also \( p \)-torsion. Now choose any \( j \geq 0 \).

For each \( t \in \mathbb{Z} \) and all \( s \geq 0 \), there is an isomorphism

\[
H_s^c(K; \pi_1((K(U_p) \land \Sigma^{-jd}V(1)))_f) \cong H_s^c(K; \pi_{t+jd}(K(U_p) \land V(1))).
\]

Then \( \text{S}2 \) implies that for every integer \( t \),

\[
H_s^c(K; \pi_1((K(U_p) \land \Sigma^{-jd}V(1)))_f) = 0, \quad \text{for all } s \geq 2,
\]

since the discrete \( K \)-module \( \pi_{t+jd}(K(U_p) \land V(1)) \) is \( p \)-torsion.

We have now verified the hypotheses of Theorem \( \text{S}10 \) so that descent spectral sequence \( \text{S}1 \) is strongly convergent, \( E_\infty^{a,t} = 0 \) for all integers \( t \) whenever \( s \geq 2 \) (see the first sentence of the proof of Theorem \( \text{S}10 \)), and there is the equivalence

\[
(K(U_p) \land v_2^{-1}V(1))^{hK} \cong \text{colim}_{j \geq 0} \left( (K(U_p) \land \Sigma^{-jd}V(1))_f \right)^{hK}.
\]

Let \( G \) be any profinite group and let \( X_1 \) and \( X_2 \) be arbitrary \( G \)-spectra, such that \( (G, X_1, \mathcal{U}) \) and \( (G, X_2, \mathcal{U}) \) are suitably finite triples (the inverse system \( \mathcal{U} \) is the same in each triple) and there is a weak equivalence \( w: X_1 \to X_2 \) in \( G \)-S\( \Sigma \)Sp. The equivalence \( w \) induces the commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{w} & \text{holim\,Sets}(G^{*+1}, (X_1)_f) \\
\Delta \downarrow & & \Delta \\
X_2 & \xrightarrow{w} & \text{holim\,Sets}(G^{*+1}, (X_2)_f)
\end{array}
\]

in which each "\( \simeq \)" denotes a weak equivalence in \( G \)-S\( \Sigma \)Sp. From the left commutative square, it follows that the middle vertical map in the diagram is a weak equivalence in \( G \)-S\( \Sigma \)Sp, and hence, the right commutative square implies that the \( G \)-equivariant map \( w_{\mathcal{U}}^{\text{dis}} \) is a weak equivalence of spectra, which allows us to conclude that \( w_{\mathcal{U}}^{\text{dis}} \) is a weak equivalence in \( \Sigma \text{Sp}_G \).

As in Definition \( \text{S}4 \), for any suitably finite triple \( (G, X, \mathcal{U}) \) and any closed subgroup \( P \) of \( G \), it is natural to define

\[
X^{hP} = (X_{\mathcal{U}}^{\text{dis}})^{hP}
\]

(this extends Definition \( \text{S}2 \)). For any \( P \), since \( w_{\mathcal{U}}^{\text{dis}} \) is a weak equivalence in the category of discrete \( P \)-spectra, it follows that the induced map

\[
(X_1)^{hP} = ((X_1)_{\mathcal{U}}^{\text{dis}})^{hP} \xrightarrow{\simeq} ((X_2)_{\mathcal{U}}^{\text{dis}})^{hP} = (X_2)^{hP}
\]

is a weak equivalence.
For each \( j \geq 0 \), the triples
\[
(\mathbb{Z}_p^\infty, K(KU_p) \land \Sigma^{-jd}V(1), N) \quad \text{and} \quad (\mathbb{Z}_p^\infty, (K(KU_p) \land \Sigma^{-jd}V(1))_f, N)
\]
are suitably finite, the natural fibrant replacement map
\[
K(KU_p) \land \Sigma^{-jd}V(1) \xrightarrow{\simeq} (K(KU_p) \land \Sigma^{-jd}V(1))_f
\]
is a weak equivalence in the category of \( \mathbb{Z}_p^\infty \)-spectra, and \( K(KU_p) \land \Sigma^{-jd}V(1) \)
can be identified with the discrete \( \mathbb{Z}_p^\infty \)-spectrum \( (K(KU_p) \land \Sigma^{-jd}V(1))_{\text{dis}} \), as in Definition 5.2. It follows from the above discussion that for each \( j \geq 0 \) and each closed subgroup \( K \), there is the definition
\[
(K(KU_p) \land \Sigma^{-jd}V(1))^{hK} = ((K(KU_p) \land \Sigma^{-jd}V(1))_f)^{hK}
\]
and there is a weak equivalence
\[
(K(KU_p) \land \Sigma^{-jd}V(1))^{hK} \xrightarrow{\simeq} \lim_{j \geq 0} \colim_{j \geq 0} \colim_{(K(KU_p) \land \Sigma^{-jd}V(1))_f}^{hK}
\]
between fibrant spectra, giving a weak equivalence
\[
\colim_{j \geq 0} \colim_{(K(KU_p) \land \Sigma^{-jd}V(1))_f}^{hK} \xrightarrow{\simeq} \colim_{j \geq 0} \colim_{(K(KU_p) \land \Sigma^{-jd}V(1))_f}^{hK}.
\]

From (8.4) and (8.5), we obtain an equivalence
\[
(K(KU_p) \land \Sigma^{-jd}V(1))^{hK} \simeq \colim_{j \geq 0} (K(KU_p) \land \Sigma^{-jd}V(1))^{hK}.
\]

Proof of Theorem 1.7. Setting \( n = 1 \) in (1.1) gives the \( (K(1) \)–local profinite \( \mathbb{Z}_p^\infty \)–Galois extension \( L_{K(1)}(S^0) \rightarrow KU_p \), and this map yields a \( \mathbb{Z}_p^\infty \)-equivariant map \( K(L_{K(1)}(S^0)) \rightarrow K(KU_p) \), with \( \mathbb{Z}_p^\infty \) acting trivially on \( K(L_{K(1)}(S^0)) \). Thus, for each \( j \geq 0 \), the induced map
\[
K(L_{K(1)}(S^0)) \land \Sigma^{-jd}V(1) \rightarrow K(KU_p) \land \Sigma^{-jd}V(1) \xrightarrow{\simeq} (K(KU_p) \land \Sigma^{-jd}V(1))_f
\]
is \( \mathbb{Z}_p^\infty \)-equivariant, giving the canonical map to the fixed points,
\[
K(L_{K(1)}(S^0)) \land \Sigma^{-jd}V(1) \rightarrow \lim_{j \geq 0} \colim_{j \geq 0} (K(KU_p) \land \Sigma^{-jd}V(1))_f^{\mathbb{Z}_p^\infty}.
\]

It follows that there is the map
\[
K(L_{K(1)}(S^0)) \land \Sigma^{-jd}V(1) \rightarrow \colim_{j \geq 0} (K(KU_p) \land \Sigma^{-jd}V(1))_f^{\mathbb{Z}_p^\infty},
\]
which is defined to be the composition
\[
K(L_{K(1)}(S^0)) \land \Sigma^{-jd}V(1) \xrightarrow{\simeq} \colim_{j \geq 0} \colim_{(K(KU_p) \land \Sigma^{-jd}V(1))_f^{\mathbb{Z}_p^\infty}} \rightarrow \colim_{j \geq 0} (KV_j)^{\mathbb{Z}_p^\infty},
\]
where here and below, we use the notation
\[
KV_j := (K(KU_p) \land \Sigma^{-jd}V(1))_f, \quad \text{for} \quad j \geq 0,
\]
to keep certain expressions from being too long (and the second map in the composition is obtained by taking a colimit of the maps given by (8.7)).

For the diagram of \( \mathbb{Z}_p^\infty \)-equivariant maps
\[
\{(i_{KV_j} : KV_j \xrightarrow{\simeq} \lim_{\Delta} \text{Sets}((\mathbb{Z}_p^\infty)^{\bullet+1}, (KV_j)_f)\}_{j \geq 0},
\]
taking fixed points and then the colimit gives the canonical map
\[
\colim_{j \geq 0} (KV_j)^{\mathbb{Z}_p^\infty} \rightarrow \colim_{j \geq 0} \lim_{\Delta} \text{Sets}((\mathbb{Z}_p^\infty)^{\bullet+1}, (KV_j)_f)^{\mathbb{Z}_p^\infty}.
\]
Also, for each $j \geq 0$ (and with $\mathcal{N}$ as defined in Remark 1.17), there are natural isomorphisms

\[
\text{colim}_{\Delta} \left( \text{holim}_{\mathcal{N}} (\mathbb{Z}_p^\times)^{\bullet+1}, (K(V_j)_f) \right)^{Z_p^\times} \cong \left( \text{colim}_{\Delta} \left( \text{holim}_{\mathcal{N}} (\mathbb{Z}_p^\times)^{\bullet+1}, (K(V_j)_f) \right)^{\mathbb{Z}_p^\times} \right)^{Z_p^\times} \\
\cong \left( (K(V_j)_N^\delta)^{\mathbb{Z}_p^\times} \right)^{Z_p^\times},
\]

where the last step is due to the isomorphism

\[
\text{colim}_{\mathcal{N}} \text{holim}_{\mathcal{N}} (\mathbb{Z}_p^\times)^{\bullet+1}, (K(V_j)_f) \cong (K(V_j)_N^\delta)^{Z_p^\times},
\]

in the category of discrete $\mathbb{Z}_p^\times$-spectra (which itself is valid by Remark 3.4 (see its first two sentences)), and hence, there is the isomorphism

\[
(8.10) \quad \text{colim}_{\Delta} \left( \text{holim}_{\mathcal{N}} (\mathbb{Z}_p^\times)^{\bullet+1}, (K(V_j)_f) \right)^{Z_p^\times} \cong \text{colim}_{\Delta} \left( (K(V_j)_N^\delta)^{\mathbb{Z}_p^\times} \right)^{Z_p^\times}.
\]

Finally, there is the composition of canonical maps

\[
(8.11) \quad \text{colim}_{\Delta} \left( (K(V_j)_N^\delta)^{\mathbb{Z}_p^\times} \right) \rightarrow \left( \text{colim}_{\Delta} (K(V_j)_N^\delta)^{\mathbb{Z}_p^\times} \right) \Rightarrow \left( C_p^{\text{dis}} \right)^{Z_p^\times} \Rightarrow \left( (C_p^{\text{dis}})_{\mathbb{Z}_p^\times} \right)^{Z_p^\times},
\]

where the first map is due to the universal property of the colimit and the second map is obtained by applying fixed points to the fibrant replacement map. The target of map (8.11) is equal to \((K(KU_p) \wedge v_2^{-1}V(1))^h\mathbb{Z}_p^\times\), and the composition of maps (8.8), (8.9), (8.10), and (8.11) (that is, after omitting the source and target from each map, the composition \((8.8) \Rightarrow (8.9) \Rightarrow (8.10) \Rightarrow (8.11)\) defines the desired map

\[
K(L_{K(1)}(S^0)) \wedge v_2^{-1}V(1) \rightarrow (K(KU_p) \wedge v_2^{-1}V(1))^h\mathbb{Z}_p^\times.
\]

9. The proof of Theorem 1.8

As in the preceding section, we continue with letting $p \geq 5$. By Theorem 1.6 (in particular, see (8.6)), there is an equivalence

\[
(K(KU_p) \wedge v_2^{-1}V(1))^h\mathbb{Z}_p^\times \cong \text{colim}_{j \geq 0} (K(KU_p) \wedge \Sigma^{-j}V(1))^{h\mathbb{Z}_p^\times},
\]

and for each $j \geq 0$, \((\mathbb{Z}_p^\times, K(KU_p) \wedge \Sigma^{-j}V(1), \mathcal{N})\) (with $\mathcal{N}$ as defined in Remark 1.17) is a suitably finite triple. Then by the proof of Theorem 6.3 (the spectrum \((K(KU_p) \wedge \Sigma^{-j}V(1))^{\text{dis}}_N\) is a fibrant discrete $\mathbb{Z}_p^\times$-spectrum, for each $j$), there are weak equivalences

\[
\text{colim}_{j \geq 0} (K(KU_p) \wedge \Sigma^{-j}V(1))^{h\mathbb{Z}_p^\times} = \text{colim}_{j \geq 0} \left( \left( (K(KU_p) \wedge \Sigma^{-j}V(1))^{\text{dis}}_N \right)_{f\mathbb{Z}_p^\times} \right)^{\mathbb{Z}_p^\times} \cong \text{colim}_{j \geq 0} \left( (K(KU_p) \wedge \Sigma^{-j}V(1))^{\text{dis}}_N \right)^{\mathbb{Z}_p^\times}.
\]

The last weak equivalence above requires a little more justification. Let $J$ denote the indexing category \(\{j \geq 0\}\) for the above colimits. For any profinite group $G$, the model structure on $G-\Sigma\text{Sp}$ is combinatorial, by [25, Proposition A.2.8.2], and hence, $(G-\Sigma\text{Sp})^J$, the category of $J$-shaped diagrams in $G-\Sigma\text{Sp}$, can be equipped
Putting all of the equivalences above together yields

\[ \{ (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0} \xrightarrow{\simeq} \{ \text{holim } \text{Sets}((Z_p^\times)^{\bullet+1}, (K(U_p) \land \Sigma^{-jd}V(1))_f) \}_{j \geq 0} \]

be a trivial cofibration to a fibrant object, in \((Z_p^\times-\Sigma \text{Sp})^J\). It follows from our last equivalence (for example, see [31, Lemma 6.2.6]; the key point here is that a homotopy limit \(J\) for all \(j \geq 0\) of Theorem 6.4, the morphism

\[ \{ (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0} \xrightarrow{\simeq} \{ \text{holim } \text{Sets}((Z_p^\times)^{\bullet+1}, (K(U_p) \land \Sigma^{-jd}V(1))_f) \}_{j \geq 0} \]

is a weak equivalence to a fibrant object, in \((Z_p^\times-\Sigma \text{Sp})^J\), and hence, there is a morphism

\[ \{ (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0} \xrightarrow{\simeq} \{ \text{holim } \text{Sets}((Z_p^\times)^{\bullet+1}, (K(U_p) \land \Sigma^{-jd}V(1))_f) \}_{j \geq 0} \]

that is a weak equivalence between fibrant objects, in \((Z_p^\times-\Sigma \text{Sp})^J\). As in the proof of Theorem 6.4, the application of the right Quillen functor

\[ (-)^{Z_p^\times} : Z_p^\times-\Sigma \text{Sp} \to \Sigma \text{Sp} \]

to the last morphism induces compositions

\[ \{ (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0} \xrightarrow{\simeq} \{ \text{holim } \text{Sets}((Z_p^\times)^{\bullet+1}, (K(U_p) \land \Sigma^{-jd}V(1))_f) \}_{j \geq 0} \]

for all \(j \geq 0\), with each composition a weak equivalence between fibrant spectra. Taking the colimit over \(J\) of these weak equivalences yields the weak equivalence

\[ \omega : \text{colim}_{j \geq 0} ((K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0}^{Z_p^\times} \xrightarrow{\simeq} \text{colim}_{j \geq 0} ((K(U_p) \land \Sigma^{-jd}V(1))^{\text{di}}_{\Delta}^{Z_p^\times}) \]

By [25] Remark A.2.8.5, every projective cofibration is an injective cofibration, so that for each \(j\), the map

\[ (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0} \xrightarrow{\simeq} (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0} \]

is a trivial cofibration to a fibrant object in \(Z_p^\times-\Sigma \text{Sp}\). It follows from this that the source of weak equivalence \(\omega\) satisfies the equality

\[ \text{colim}_{j \geq 0} ((K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0}^{Z_p^\times} = \text{colim}_{j \geq 0} (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0}^{hZ_p^\times} \]

and thus, \(\omega\) is the weak equivalence that we set out in this paragraph to obtain.

Fix \(j \geq 0\). Since \(V(1)\) is a finite spectrum, \(\Sigma^{-jd}V(1)\) is too, and hence, there is an equivalence

\[ (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0}^{hZ_p^\times} \simeq (K(U_p))_{j \geq 0}^{hZ_p^\times} \land \Sigma^{-jd}V(1) \]

(for example, see [31] Lemma 6.2.6]; the key point here is that a homotopy limit commutes with smashing with a finite spectrum). It follows from our last equivalence that

\[ \text{colim}_{j \geq 0} (K(U_p) \land \Sigma^{-jd}V(1))_{j \geq 0}^{hZ_p^\times} \simeq \text{colim}_{j \geq 0} (K(U_p))_{j \geq 0}^{hZ_p^\times} \land \Sigma^{-jd}V(1)) \]

\[ \simeq (K(U_p))_{j \geq 0}^{hZ_p^\times} \land v^{-1}_2 V(1). \]

Putting all of the equivalences above together yields

\[ (K(U_p) \land v^{-1}_2 V(1))_{j \geq 0}^{hZ_p^\times} \simeq (K(U_p))_{j \geq 0}^{hZ_p^\times} \land v^{-1}_2 V(1), \]
which is the desired equivalence.

REFERENCES

[1] Christian Ausoni. On the algebraic $K$-theory of the complex $K$-theory spectrum. *Invent. Math.*, 180(3):611–668, 2010.
[2] Christian Ausoni and John Rognes. Algebraic $K$-theory of the fraction field of topological $K$-theory. 54 pages, arXiv:0911.4781.
[3] Christian Ausoni and John Rognes. Algebraic $K$-theory of topological $K$-theory. *Acta Math.*, 188(1):1–39, 2002.
[4] Christian Ausoni and John Rognes. The chromatic red-shift in algebraic $K$-theory. In *Guido’s Book of Conjectures*, Monographie de L’Enseignement Mathématique, volume 40, pages 13–15, 2008.
[5] Christian Ausoni and John Rognes. Algebraic $K$-theory of the first Morava $K$-theory. *J. Eur. Math. Soc. (JEMS)*, 14(4):1041–1079, 2012.
[6] Nils A. Baas, Bjørn Ian Dundas, and John Rognes. Two-vector bundles and forms of elliptic cohomology. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 18–45. Cambridge Univ. Press, Cambridge, 2004.
[7] Mark Behrens and Daniel G. Davis. The homotopy fixed point spectra of profinite Galois extensions. *Trans. Amer. Math. Soc.*, 362(9):4983–5042, 2010.
[8] Andrew J. Blumberg and Michael A. Mandell. The localization sequence for the algebraic $K$-theory of topological $K$-theory. *Acta Math.*, 200(2):155–179, 2008.
[9] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. *Astérisque*, (226):7–8, 57–143, 1994. $K$-theory (Strasbourg, 1992).
[10] Edgar H. Brown, Jr. and Michael Comenetz. Pontrjagin duality for generalized homology and cohomology theories. *Amer. J. Math.*, 98(1):1–27, 1976.
[11] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.
[12] J. Daniel Christensen and Daniel C. Isaksen. Duality and pro-spectra. *Algebr. Geom. Topol.*, 4:781–812 (electronic), 2004.
[13] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent in algebraic $K$-theory and a conjecture of Ausoni–Rognes. *J. Eur. Math. Soc. (JEMS)*, 22(4):1149–1200, 2020.
[14] Daniel G. Davis. Homotopy fixed points for $L_{K(n)}(E_n \wedge X)$ using the continuous action. *J. Pure Appl. Algebra*, 206(3):322–354, 2006.
[15] Daniel G. Davis. Homotopy fixed points for profinite groups emulate homotopy fixed points for discrete groups. *New York J. Math.*, 19:909–924, 2013.
[16] Daniel G. Davis and Gereon Quick. Profinite and discrete $G$-spectra and iterated homotopy fixed points. *Algebr. Geom. Topol.*, 16(4):2257–2303, 2016.
[17] William Dwyer, Haynes Miller, and Joseph Neisendorfer. Fibrewise completion and unstable Adams spectral sequences. *Israel J. Math.*, 66(1-3):160–178, 1989.
[18] G. A. Fernández-Alcober, I. V. Kazachkov, V. N. Remeslennikov, and P. Symonds. Comparison of the discrete and continuous cohomology groups of a pro-$p$ group. *Algebra i Analiz*, 19(6):126–142, 2007.
[19] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.
[20] Paul G. Goerss. Homotopy fixed points for Galois groups. In *The Čech centennial (Boston, MA, 1993)*, pages 187–224. Amer. Math. Soc., Providence, RI, 1995.
[21] Paul G. Goerss and Michael J. Hopkins. André-Quillen (co)-homology for simplicial algebras over simplicial operads. In *Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps* (Arolla, 1999), pages 41–85. Amer. Math. Soc., Providence, RI, 2000.
[22] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
[23] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math.* (2), 148(1):1–49, 1998.
[24] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.
[25] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
[26] Michael A. Mandell. Equivariant symmetric spectra. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 399–452. Amer. Math. Soc., Providence, RI, 2004.
[27] Stephen A. Mitchell. Hypercohomology spectra and Thomason’s descent theorem. In *Algebraic K-theory (Toronto, ON, 1996)*, pages 221–277. Amer. Math. Soc., Providence, RI, 1997.
[28] Shichirô Oka. Multiplicative structure of finite ring spectra and stable homotopy of spheres. In *Algebraic topology, Aarhus 1982 (Aarhus, 1982)*, volume 1051 of *Lecture Notes in Math.*, pages 418–441. Springer, Berlin, 1984.
[29] Gereon Quick. Profinite $G$-spectra. *Homology Homotopy Appl.*, 15(1):151–189, 2013.
[30] John Rognes. Arithmetic of some brave new rings. Notes, 10 pages, March 29th, 2006. Available from the author’s homepage.
[31] John Rognes. Galois extensions of structured ring spectra. In *Galois extensions of structured ring spectra/Stably dualizable groups*, in *Mem. Amer. Math. Soc.*, vol. 192 (898), pages 1–97, 2008.
[32] John Rognes. Algebraic K-theory of strict ring spectra. In *Proceedings of the International Congress of Mathematicians, Seoul 2014, Volume II*, pages 1259–1283. Kyung Moon Sa, Seoul, Korea, 2014.
[33] Stefan Schröer. Topological methods for complex-analytic Brauer groups. *Topology*, 44(5):875–894, 2005.
[34] Jean-Pierre Serre. *Cohomologie Galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.
[35] R. W. Thomason. Algebraic $K$-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.
[36] Takeshi Torii. Discrete $G$-spectra and embeddings of module spectra. *J. Homotopy Relat. Struct.*, 12(4):853–899, 2017.
[37] Charles A. Weibel. *An introduction to homological algebra*. Cambridge University Press, Cambridge, 1994.