ASYMPTOTIC FORMULAS FOR THE GAMMA FUNCTION CONSTRUCTED BY BIVARIATE MEANS

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ABSTRACT. Let $K, M, N$ denote three bivariate means. In the paper, the author proves the asymptotic formulas for the gamma function have the form

$$\Gamma(x + 1) \sim \sqrt{2\pi M} (x + \theta, x + 1 - \theta)^{K(x + \epsilon, x + 1 - \epsilon)} e^{-N(x + \sigma, x + 1 - \sigma)}$$
or

$$\Gamma(x + 1) \sim \sqrt{2\pi M} (x + \theta, x + \sigma)^{K(x + \epsilon, x + 1 - \epsilon)} e^{-M(x + \theta, x + \sigma)}$$
as $x \to \infty$, where $\epsilon, \theta, \sigma$ are fixed real numbers. This idea can be extended to the psi and polygamma functions. As examples, some new asymptotic formulas for the gamma function are presented.

1. Introduction

The Stirling’s formula

$$n! \sim \sqrt{2\pi nn} e^{-n} := s_n$$

has important applications in statistical physics, probability theory and number theory. Due to its practical importance, it has attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements.

Burnside’s formula [1]

$$n! \sim \sqrt{2\pi (n + 1/2)} e^{-(n + 1/2)} := b_n$$

slightly improves (1.1). Gosper [2] replaced $\sqrt{2\pi n}$ by $\sqrt{2\pi (n + 1/6)}$ in (1.1) to get

$$n! \sim \sqrt{2\pi (n + 1/6)} \left(\frac{n}{e}\right)^n := g_n,$$

which is better than (1.1) and (1.2). In the recent paper [3], N. Batir obtained an asymptotic formula similar to (1.3):

$$n! \sim \frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n - 1/6}} := b_n'$$

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which is stronger than (1.1) and (1.2). A more accurate approximation for the factorial function

\[(1.5)\quad n! \sim \sqrt{2\pi} \left( \frac{n^2 + n + 1/6}{e^2} \right)^{n/2+1/4} := m_n\]

was presented in [4] by Mortici.

The classical Euler’s gamma function \(\Gamma\) may be defined by

\[(1.6)\quad \Gamma (x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad \text{for} \quad x > 0,\]

and its logarithmic derivative \(\psi (x) = \Gamma'(x) / \Gamma (x)\) is known as the psi or digamma function, while \(\psi', \psi''\), ... are called polygamma functions (see [5]).

The gamma function is closely related to the Stirling’s formula, since \(\Gamma(n + 1) = n!\) for all \(n \in \mathbb{N}\). This inspires some authors to also pay attention to find better approximations for the gamma function. For example, Ramanujan’s [6, P. 339] double inequality for the gamma function:

\[(1.7)\quad \sqrt{\frac{\pi}{x}} \left( \frac{x e}{8x^3 + 4x^2 + x + 1/100} \right)^{1/6} < \Gamma (x+1) < \sqrt{\frac{\pi}{x}} \left( \frac{x e}{8x^3 + 4x^2 + x + 1/30} \right)^{1/6}\]

for \(x \geq 1\). Batir [7] showed that for \(x > 0\),

\[(1.8)\quad \sqrt{2e^{4/9}} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{2}} \exp \left( -\frac{1}{6(x+3/8)} \right) \quad < \quad \Gamma (x+1) < \sqrt{2\pi} \left( \frac{x}{e} \right)^x \sqrt{x + \frac{1}{2}} \exp \left( -\frac{1}{6(x+3/8)} \right) .\]

Mortici [8] proved that for \(x \geq 0\),

\[(1.9)\quad \sqrt{2\pi e^{-\omega}} \left( \frac{x + \omega}{e} \right)^{x+1/2} \quad < \quad \Gamma (x+1) \leq \alpha \sqrt{2\pi e^{-\omega}} \left( \frac{x + \omega}{e} \right)^{x+1/2} ,\]

\[(1.10)\beta \sqrt{2\pi e^{-\varsigma}} \left( \frac{x + \varsigma}{e} \right)^{x+1/2} \quad < \quad \Gamma (x+1) \leq \beta \sqrt{2\pi e^{-\varsigma}} \left( \frac{x + \varsigma}{e} \right)^{x+1/2} ,\]

where \(\omega = (3 - \sqrt{3}) / 6, \alpha = 1.072042464...\) and \(\varsigma = (3 + \sqrt{3}) / 6, \beta = 0.988503589...\).

More results involving the asymptotic formulas for the factorial or gamma functions can consult [9], [10], [11], [12], [13], [14], [15], [16] and the references cited therein.

Mortici [17] presented an idea that by replacing an under-approximation and an upper-approximation of the factorial function by one of their geometric mean to improve certain approximation formula of the factorial. In fact, by observing and analyzing these asymptotic formulas for factorial or gamma function, we find out that they have the common form of

\[(1.11)\quad \ln \Gamma (x+1) \sim \frac{1}{2} \ln 2\pi + P_1 (x) \ln P_2 (x) - P_3 (x) + P_4 (x) ,\]
where \( P_1(x) \), \( P_2(x) \) and \( P_3(x) \) are all means of \( x \) and \((x+1)\), while \( P_4(x) \) satisfies \( P_4(\infty) = 0 \). For example, (1.1)–(1.5) can be written as

\[
\ln n! \sim \frac{1}{2} \ln 2\pi + \left( n + \frac{1}{2} \right) \ln n - n, \\
\ln n! \sim \frac{1}{2} \ln 2\pi + \left( n + \frac{1}{2} \right) \ln \left( n + \frac{1}{2} \right) - \left( n + \frac{1}{2} \right), \\
\ln n! \sim \frac{1}{2} \ln 2\pi + \left( n + \frac{1}{2} \right) \ln n - n + \frac{1}{2} \ln \left( 1 + \frac{1}{6n} \right), \\
\ln n! \sim \frac{1}{2} \ln 2\pi + \left( n + \frac{1}{2} \right) \ln n - n - \frac{1}{2} \ln \left( 1 - \frac{1}{6n} \right), \\
\ln n! \sim \frac{1}{2} \ln 2\pi + \left( n + \frac{1}{2} \right) \ln \sqrt{\frac{n^2 + 4n(n+1)+ (n+1)^2}{6}} - \left( n + \frac{1}{2} \right).
\]

Inequalities (1.7)–(1.10) imply that

\[
\ln \Gamma (x+1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln x - x + \frac{1}{6} \ln \left( 1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^5} \right), \\
\ln \Gamma (x+1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln x - x + \frac{1}{2} \ln \left( 1 + \frac{1}{2x} \right) - \frac{1}{6(x+3/8)}, \\
\ln \Gamma (x+1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln ((1-a)x + a(x+1)) - ((1-a)x + a(x+1)),
\]

where \( a = \omega = (3 - \sqrt{3})/6, \varsigma = (3 + \sqrt{3})/6 \).

The aim of this paper is to prove the validity of the form (1.11) which offers such a new way to construct asymptotic formulas for Euler gamma function in terms of bivariate means. Our main results are included in Section 2. Some new examples are presented in the last section.

2. Main results

Before stating and proving our main results, we recall some knowledge on means. Let \( I \) be an interval on \( \mathbb{R} \). A bivariate real valued function \( M : I^2 \to \mathbb{R} \) is said to be a bivariate mean if

\[
\min (a, b) \leq M(a, b) \leq \max (a, b)
\]

for all \( a, b \in I \). Clearly, each bivariate mean \( M \) is reflexive, that is,

\[
M(a, a) = a
\]

for any \( a \in I \). \( M \) is symmetric if

\[
M(a, b) = M(b, a)
\]

for all \( a, b \in I \), and \( M \) is said to be homogeneous (of degree one) if

\[
M(ta, tb) = tM(a, b)
\]

for any \( a, b \in I \) and \( t > 0 \).

The lemma is crucial to prove our results.
Lemma 1 ([3] Theorem 1, 2, 3]). If \( M : I^2 \rightarrow \mathbb{R} \) is a differentiable mean, then for \( c \in I \),

\[
M'_a(c,c), M'_b(c,c) \in (0,1) \quad \text{and} \quad M'_a(c,c) + M'_b(c,c) = 1.
\]

In particular, if \( M \) is symmetric, then

\[
M'_a(c,c) = M'_b(c,c) = 1/2.
\]

Now we are in a position to state and prove main results.

Theorem 1. Let \( M : (0, \infty) \times (0, \infty) \rightarrow (0, \infty) \) and \( N : (-\infty, \infty) \times (-\infty, \infty) \rightarrow (-\infty, \infty) \) be two symmetric, homogeneous and differentiable means and let \( r \) be defined on \((0, \infty)\) satisfying \( \lim_{x \to \infty} r(x) = 0 \). Then for fixed real numbers \( \theta, \theta^*, \sigma, \sigma^* \) with \( \theta + \theta^* = \sigma + \sigma^* = 1 \) such that \( x > -\min(1, \theta, \theta^*) \), we have

\[
\Gamma(x+1) \sim \sqrt{2\pi} M(x+\theta, x+\theta^*) x^{1/2} e^{-N(x+\sigma, x+\sigma^*)} e^{r(x)}, \quad \text{as} \quad x \to \infty.
\]

Proof. Since \( \lim_{x \to \infty} r(x) = 0 \), the desired result is equivalent to

\[
\lim_{x \to \infty} \left( \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln M(x+\theta, x+\theta^*) + N(x+\sigma, x+\sigma^*) \right) = 0.
\]

Due to \( \lim_{x \to \infty} r(x) = 0 \) and the known relation

\[
\lim_{x \to \infty} \left( \ln \Gamma(x+1) - \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) + \left( x + \frac{1}{2} \right) \right) = \frac{1}{2} \ln 2\pi,
\]

it suffices to prove that

\[
D_1 : = \lim_{x \to \infty} \left( x + \frac{1}{2} \right) \ln \frac{M(x+\theta, x+\theta^*)}{x+1/2} = 0,
\]

\[
D_2 : = \lim_{x \to \infty} \left( N(x+\sigma, x+\sigma^*) - \left( x + \frac{1}{2} \right) \right) = 0.
\]

Letting \( x = 1/t \), using the homogeneity of \( M \), that is, \( (2.1) \), and utilizing L’Hospital rule give

\[
D_1 = \lim_{t \to 0^+} \frac{1 + t/2}{t} \ln \left( \frac{M(1+\theta t, 1+\theta^* t)}{1+t/2} \right) = \lim_{t \to 0^+} \ln \left( M(1+\theta t, 1+\theta^* t) - \ln(1+t/2) \right) = \lim_{t \to 0^+} \left( \frac{\theta M_x(1+\theta t, 1+\theta^* t) + \theta^* M_y(1+\theta t, 1+\theta^* t)}{M(1+\theta t, 1+\theta^* t)} - \frac{1}{2+t} \right)
\]

\[
= \frac{\theta M_x(1,1) + \theta^* M_y(1,1)}{M(1,1)} - \frac{1}{2} = 0,
\]

where the last equality holds due to Lemma [1].
Similarly, we have
\[ D_2 = \lim_{x \to \infty} \left( N(x + \sigma, x + \sigma^*) - \left( x + \frac{1}{2} \right) \right) \]
\[ = \lim_{t \to 0^+} \frac{N(1 + \sigma t, 1 + \sigma^* t) - (1 + t/2)}{t} \]
\[ = \sigma \sigma^* - \frac{1}{2} = 0, \]
which proves the desired result.

\[ \Box \]

**Theorem 2.** Let \( M : (0, \infty) \times (0, \infty) \to (0, \infty) \) be a mean and let \( r \) be defined on \((0, \infty)\) satisfying \( \lim_{x \to \infty} r(x) = 0 \). Then for fixed real numbers \( \theta, \sigma \) such that \( x > -\min (1, \theta, \sigma) \), we have
\[ \Gamma(x + 1) \sim \sqrt{2\pi M(x + \theta, x + \sigma)}^{x + 1/2} e^{-M(x + \theta, x + \sigma)} e^{r(x)}, \text{ as } x \to \infty. \]

**Proof.** Since \( \lim_{x \to \infty} r(x) = 0 \), the desired result is equivalent to
\[ \lim_{x \to \infty} \left( \ln \Gamma(x + 1) - \ln \sqrt{2\pi} - \left( x + \frac{1}{2} \right) \ln M(x + \theta, x + \sigma) + M(x + \theta, x + \sigma) \right) = 0. \]

Similarly, it suffices to prove that
\[ D_3 := \lim_{x \to \infty} \left( x + \frac{1}{2} \right) \ln M(x + \theta, x + \sigma) - M(x + \theta, x + \sigma) - \left( x + \frac{1}{2} \right) \right) \]
\[ = \lim_{x \to \infty} \left( M(x + \theta, x + \sigma) - \left( x + \frac{1}{2} \right) \right) \times \left( \frac{1}{L(y, 1)} - 1 \right) = 0, \]
where \( L(a, b) \) is the logarithmic mean of positive \( a \) and \( b \), \( y = M(x + \theta, x + \sigma)/(x + 1/2) \).

Now we first show that
\[ D_4 := M(x + \theta, x + \sigma) - \left( x + \frac{1}{2} \right) \]
is bounded. In fact, by the property of mean we see that
\[ x + \min (\theta, \sigma) - \left( x + \frac{1}{2} \right) < D_4 < x + \max (\theta, \sigma) - \left( x + \frac{1}{2} \right) \]
that is,
\[ \min (\theta, \sigma) - \frac{1}{2} < D_4 < \max (\theta, \sigma) - \frac{1}{2}. \]

It remains to prove that
\[ \lim_{x \to \infty} D_5 := \lim_{x \to \infty} \left( \frac{1}{L(y, 1)} - 1 \right) = 0. \]

Since
\[ \frac{x + \min (\theta, \sigma)}{x + 1/2} < y = \frac{M(x + \theta, x + \sigma)}{x + 1/2} < \frac{x + \max (\theta, \sigma)}{x + 1/2}, \]
so we have \( \lim_{x \to \infty} y = 1 \). This together with
\[ \min (y, 1) \leq L(y, 1) \leq \max (y, 1) \]
yields \( \lim_{x \to \infty} L(y, 1) = 1 \), and therefore, \( \lim_{x \to \infty} D_5 = 0. \)

This completes the proof. \( \Box \)
Corollary 1. Suppose that while the second one is clearly equal to zero.

\[ K \text{ symmetric, homogeneous, and differentiable means; } \]

\[ x > \] such that

\[ \lim_{x \to \infty} r(x) = 0. \]

Proof. Due to \( \lim_{x \to \infty} r(x) = 0 \), the result in question is equivalent to

\[ \lim_{x \to \infty} \left( \ln \Gamma(x + 1) - \ln \sqrt{2\pi} - K(x + \epsilon, x + \epsilon^*) \ln \left( x + \frac{1}{2} \right) + \left( x + \frac{1}{2} \right) \right) = 0. \]

Clearly, we only need to prove that

\[ D_6 := \lim_{x \to \infty} \left( K(x + \epsilon, x + \epsilon^*) - \left( x + \frac{1}{2} \right) \right) \ln \left( x + \frac{1}{2} \right) = 0. \]

By the homogeneity of \( K \), we get

\[ D_6 \overset{1/x \to t}{=} \lim_{t \to 0^+} \frac{K(1 + ct, 1 + \epsilon t) - (1 + t/2)}{t} \left( \ln \left( 1 + \frac{t}{2} \right) - \ln t \right) = \lim_{t \to 0^+} \frac{K(1 + ct, 1 + \epsilon t) - (1 + t/2)}{t^2} \lim_{t \to 0^+} \left( t \ln \left( 1 + \frac{t}{2} \right) - t \ln t \right) = 0, \]

where the first limit, by L’Hospital’s rule, is equal to

\[ \lim_{t \to 0^+} \frac{2t}{2t} = \lim_{t \to 0^+} \frac{2t}{2t} = \frac{2}{2} \]

while the second one is clearly equal to zero.

The proof ends.

By the above three theorems, the following assertion is immediate.

Corollary 1. Suppose that

(i) the function \( K : \mathbb{R}^2 \to \mathbb{R} \) is a symmetric, homogeneous and twice differentiable mean;

(ii) the functions \( M : (0, \infty) \times (0, \infty) \to (0, \infty) \) and \( N : \mathbb{R}^2 \to \mathbb{R} \) are two symmetric, homogeneous, and differentiable means;

(iii) the function \( r : (0, \infty) \to (-\infty, \infty) \) satisfies \( \lim_{x \to \infty} r(x) = 0 \).

Then for fixed real numbers \( \epsilon, \epsilon^*, \theta, \theta^* \), \( \sigma, \sigma^* \) with \( \epsilon + \epsilon^* = \theta + \theta^* = \sigma + \sigma^* = 1 \) such that \( x > -\min(1, \theta, \theta^*) \), we have

\[ \Gamma(x + 1) \sim \sqrt{2\pi} M(x + \theta, x + \theta^*) K^{(x + \epsilon, x + \epsilon^*)} e^{-N(x + \sigma, x + \sigma^*)} e^{r(x)}, \text{ as } x \to \infty. \]

Corollary 2. Suppose that

(i) the function \( K : (-\infty, \infty)^2 \to (-\infty, \infty) \) is a symmetric, homogeneous and twice differentiable mean;

(ii) the functions \( M, N : (0, \infty)^2 \to (0, \infty) \) are two means;

(iii) the function \( r : (0, \infty) \to (-\infty, \infty) \) satisfies \( \lim_{x \to \infty} r(x) = 0 \).
Then for fixed real numbers $\epsilon, \epsilon^*, \theta, \sigma$ with $\epsilon + \epsilon^* = 1$ such that $x > -\min (1, \theta, \sigma)$, we have
\[
\Gamma (x + 1) \sim \sqrt{2\pi} M (x + \theta, x + \sigma) e^{K(x+\epsilon,x+\epsilon^*)} e^{-M(x+\theta,x+\sigma)} e^{r(x)}, \quad as \ x \to \infty.
\]

Further, it is obvious that our ideas constructing asymptotic formulas for the gamma function in terms of bivariate means can be extended to the psi and polygamma functions.

**Theorem 4.** Let $M : (0, \infty)^2 \to (0, \infty)$ be a mean and let $r$ be defined on $(0, \infty)$ satisfying $\lim_{x \to \infty} r(x) = 0$. Then for fixed real numbers $\theta, \sigma$ such that $x > -\min (1, \theta, \sigma)$, the asymptotic formula for the psi function
\[
\psi (x + 1) \sim \ln M (x + \theta, x + \sigma) + r(x)
\]
holds as $x \to \infty$.

**Proof.** It suffices to prove
\[
\lim_{x \to \infty} (\psi (x + 1) - \ln M (x + \theta, x + \sigma)) = 0.
\]
Since $M$ is a mean, we have $x + \min (\theta, \sigma) \leq M (x + \theta, x + \sigma) \leq x + \max (\theta, \sigma)$, and so
\[
\psi (x + 1) - \ln (x + \max (\theta, \sigma)) < \psi (x + 1) - \ln M (x + \theta, x + \sigma) < \psi (x + 1) - \ln (x + \min (\theta, \sigma)),
\]
which yields the inquired result due to
\[
\lim_{x \to \infty} (\psi (x + 1) - \ln (x + \max (\theta, \sigma))) = \lim_{x \to \infty} (\psi (x + 1) - \ln (x + \min (\theta, \sigma))) = 0.
\]

**Theorem 5.** Let $M : (0, \infty)^2 \to (0, \infty)$ be a mean and let $r$ be defined on $(0, \infty)$ satisfying $\lim_{x \to \infty} r(x) = 0$. Then for fixed real numbers $\theta, \sigma$ such that $x > -\min (1, \theta, \sigma)$, the asymptotic formula for the polygamma function
\[
\psi^{(n)} (x + 1) \sim \frac{(-1)^{n-1} (n-1)!}{M^n (x + \theta, x + \sigma)} + r(x)
\]
holds as $x \to \infty$.

**Proof.** It suffices to show
\[
\lim_{x \to \infty} \left( (-1)^{n-1} \psi^{(n)} (x + 1) - \frac{(n-1)!}{M^n (x + \theta, x + \sigma)} \right) = 0.
\]
For this purpose, we utilize a known double inequality that for $k \in \mathbb{N}$
\[
\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)} (x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}
\]
holds on $(0, \infty)$ proved by Guo and Qi in [19] Lemma 3 to get
\[
\frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)} (x) - \frac{(k-1)!}{x^k} < \frac{k!}{x^{k+1}}.
\]
This implies that
\[
\lim_{x \to \infty} \left( (-1)^{k-1} \psi^{(k)} (x) - \frac{(k-1)!}{x^k} \right) = 0.
\]
On the other hand, without loss of generality, we assume that \( \theta \leq \sigma \). By the property of mean, we see that
\[
x + \theta \leq \bar{M}(x + \theta, x + \sigma) \leq x + \sigma,
\]
and so
\[
(-1)^{n-1} \psi^{(n)}(x + 1) - \frac{(n-1)!}{(x+\theta)^{n}} < (-1)^{n-1} \psi^{(n)}(x + 1) - \frac{(n-1)!}{M^{n}(x + \theta, x + \sigma)} < (-1)^{n-1} \psi^{(n)}(x + 1) - \frac{1}{(x + \sigma)^{n}}.
\]
Then, by (2.2), for \( a = \theta, \sigma \), we get
\[
(-1)^{n-1} \psi^{(n)}(x + 1) - \frac{(n-1)!}{(x+a)^{n}}
= \left((-1)^{n-1} \psi^{(n)}(x + 1) - \frac{(n-1)!}{(x+1)^{n}}\right) + \left(\frac{(n-1)!}{(x+1)^{n}} - \frac{(n-1)!}{(x+a)^{n}}\right)
\to 0 + 0 = 0, \text{ as } x \to \infty,
\]
which gives the desired result.
Thus we complete the proof. \( \square \)

3. Examples

In this section, we will list some examples to illustrate applications of Theorems [1] and [2]. To this end, we first recall the arithmetic mean \( A \), geometric mean \( G \), and identric (exponential) mean \( I \) of two positive numbers \( a \) and \( b \) defined by
\[
A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab},
\]
\[
I(a,b) = (b^a/a^b)^{1/(b-a)}/e \text{ if } a \neq b \text{ and } I(a,a) = a,
\]
(see [20], [21]). Clearly, these means are symmetric and homogeneous. Another possible mean is defined by
\[
H^{n,n-1}_{p_k,q_k}(a,b) = \frac{\sum_{k=0}^{n} p_k a^k b^{n-k}}{\sum_{k=0}^{n-1} q_k a^k b^{n-1-k}},
\]
(3.1)
where
\[
\sum_{k=0}^{n} p_k = \sum_{k=0}^{n-1} q_k = 1.
\]
(3.2)
It is clear that \( H^{n,n-1}_{p_k,q_k}(a,b) \) is homogeneous and satisfies \( H^{n,n-1}_{p_k,q_k}(a,a) = a \).
When \( p_k = p_{n-k} \) and \( q_k = q_{n-1-k} \), we denote \( H^{n,n-1}_{p_k,q_k}(a,b) \) by \( S^{n,n-1}_{p_k,q_k}(a,b) \), which can be expressed as
\[
S^{n,n-1}_{p_k,q_k}(a,b) = \frac{\sum_{k=0}^{[n/2]} p_k (ab)^k (a^{n-2k} + b^{n-2k})}{\sum_{k=0}^{([n-1]/2]} q_k (ab)^k (a^{n-1-2k} + b^{n-1-2k})},
\]
(3.3)
where \( p_k \) and \( q_k \) satisfy
\[
\sum_{k=0}^{[n/2]} (2p_k) = \sum_{k=0}^{([n-1]/2]} (2q_k) = 1,
\]
(3.4)
\[ x \] denotes the integer part of real number \( x \). Evidently, \( S_{n,n-1}^{p,q} \) is symmetric and homogeneous, and \( S_{n,n-1}^{p,q} (a,a) = a \). But \( H_{n,n-1}^{p,q} (a,b) \) and \( S_{n,n-1}^{p,q} (a,b) \) are not always means of \( a \) and \( b \). For instance, when \( p = 2/3 \),

\[
S_{p,1/2}^{2,1} (a,b) = \frac{pa^2 + pb^2 + (1 - 2p) ab}{(a+b)/2} = \frac{2}{3} \frac{2a^2 + 2b^2 - ab}{a+b} > \max(a,b)
\]

in the case of \( \max(a,b) > 4 \min(a,b) \). Indeed, it is easy to prove that \( S_{p,1/2}^{2,1} (a,b) \) is a mean if and only if \( p \in [0, 1/2) \).

Secondly, we recall the so-called completely monotone functions. A function \( f \) is said to be completely monotonic on an interval \( I \), if \( f \) has derivatives of all orders on \( I \) and satisfies

\[
(-1)^n f^{(n)}(x) \geq 0 \text{ for all } x \in I \text{ and } n = 0, 1, 2, ....
\]

If the inequality \((3.5)\) is strict, then \( f \) is said to be strictly completely monotonic on \( I \). It is known (Bernstein’s Theorem) that \( f \) is completely monotonic on \((0, \infty)\) if and only if

\[
f(x) = \int_0^\infty e^{-xt} d\mu(t),
\]

where \( \mu \) is a nonnegative measure on \([0, \infty)\) such that the integral converges for all \( x > 0 \), see [22] p. 161.

**Example 1.** Let

\[
K (a,b) = N (a,b) = A (a,b) = \frac{a+b}{2},
\]

\[
M (a,b) = A^{2/3} (a,b) G^{1/3} (a,b) = \left( \frac{a+b}{2} \right)^{2/3} \left( \sqrt{ab} \right)^{1/3}
\]

and \( \theta = \sigma = 0 \) in Theorem 2. Then we can obtain an asymptotic formulas for the gamma function as follows.

\[
\ln \Gamma(x+1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln \left( \left( x + \frac{1}{2} \right)^{2/3} \left( \sqrt{x(x+1)} \right)^{1/3} \right) - \left( x + \frac{1}{2} \right)
\]

\[
= \frac{1}{2} \ln 2\pi + \frac{2}{3} \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) + \frac{1}{6} \left( x + \frac{1}{2} \right) \ln x
\]

\[
+ \frac{1}{6} \left( x + \frac{1}{2} \right) \ln (x+1) - \left( x + \frac{1}{2} \right), \text{ as } x \to \infty.
\]

Further, we can prove

**Proposition 1.** For \( x > 0 \), the function

\[
f_1(x) = \ln \Gamma(x+1) - \frac{1}{2} \ln 2\pi - \frac{2}{3} \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) - \frac{1}{6} \left( x + \frac{1}{2} \right) \ln x
\]

\[
- \frac{1}{6} \left( x + \frac{1}{2} \right) \ln (x+1) + \left( x + \frac{1}{2} \right)
\]

is a completely monotone function.
Proof. Differentiating and utilizing the relations
\begin{equation}
\psi(x) = \int_0^\infty \left( \frac{e^{-t} - e^{-xt}}{t} \right) dt \quad \text{and} \quad \ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt
\end{equation}
yield
\[
f'_1(x) = \psi(x+1) - \frac{1}{6} \ln(x+1) - \frac{1}{6} \ln x - \frac{2}{3} \ln \left( x + \frac{1}{2} \right) + \frac{1}{12} (x + 1) - \frac{1}{12x}
\]
\[
= \int_0^\infty \left( \frac{e^{-t} - e^{-(x+1)t}}{t} \right) dt - \int_0^\infty \frac{e^{-t} - e^{-xt}}{6t} dt - \int_0^\infty \frac{e^{-t} - e^{-(x+1)t}}{6t} dt
\]
\[
- \int_0^\infty 2 \left( \frac{e^{-t} - e^{-(x+1/2)t}}{3t} \right) dt + \frac{1}{12} \int_0^\infty e^{-(x+1)t} dt - \frac{1}{12} \int_0^\infty e^{-xt} dt
\]
\[
= \int_0^\infty e^{-xt} \left( \frac{1}{6t} + \frac{e^{-t}}{6t} + \frac{2e^{-t/2}}{3t} - \frac{e^{-t/2}}{1 - e^{-t}} + \frac{1}{12} (e^{-t} - 1) \right) dt
\]
\[
= \int_0^\infty e^{-xt} e^{-t/2} \left( \frac{\cosh (t/2)}{3t} + \frac{2}{3t} - \frac{1}{2 \sinh (t/2)} - \frac{1}{6} \sinh \left( \frac{t}{2} \right) \right) dt
\]
\[
: = \int_0^\infty e^{-xt} e^{-t/2} u \left( \frac{t}{2} \right) dt,
\]
where
\[
u(t) = \frac{\cosh t}{6t} + \frac{1}{3t} - \frac{1}{2 \sinh t} - \frac{1}{6} \sinh t.
\]
Factoring and expanding in power series lead to
\[
u(t) = -\frac{t \cosh 2t - \sinh 2t - 4 \sinh t}{12t \sinh t}
\]
\[
= -\sum_{n=1}^\infty \frac{2^{2n-2}2^{2n-1}}{(2n-2)!} t^{2n-2} - \sum_{n=1}^\infty \frac{2^{2n-1}2^{2n-1}}{(2n-1)!} - 4 \sum_{n=1}^\infty \frac{2^{2n-1}}{(2n-1)!} + 5t
\]
\[
= -\frac{\sum_{n=3}^\infty \frac{(2n-3)2^{2n-2}4^{2n-1}}{(2n-3)!}}{12t \sinh t} < 0
\]
for \( t > 0 \). This reveals that \(-f'_1\) is a completely monotone function, which together with \( f_1(x) > \lim_{x \to \infty} f_1(x) = 0 \) leads us to the desired result.
\hfill \Box

Using the decreasing property of \( f_1 \) on \((0, \infty)\) and notice that
\[
f_1(1) = \ln \frac{2^{3/4}e^{3/2}}{3\sqrt{2\pi}} \quad \text{and} \quad f_1(\infty) = 0
\]
we immediately get

**Corollary 3.** For \( n \in \mathbb{N} \), it is true that
\[
\sqrt{2\pi} \left( \frac{(n+1/2)^4}{e^{6}} \right) \frac{(n+1/2)^{n+1/2}}{n!} \leq n < \frac{2^{3/4}e^{3/2}}{3} \left( \frac{(n+1/2)^4}{e^{6}} \right) \frac{(n+1/2)^{n+1/2}}{n!}
\]
with the optimal constants \( \sqrt{2\pi} \approx 2.5066 \) and \( 2^{3/4}e^{3/2}/3 \approx 2.5124 \).

**Example 2.** Let
\[
K(a, b) = N(a, b) = A(a, b) = \frac{a + b}{2},
\]
\[
M(a, b) = \mathcal{I}(a, b) = (b^{a^2})^{1/(b-a)} / e \quad \text{if} \; a \neq b \; \text{and} \; I(a, a) = a
\]
and \( \theta = 0 \) in Theorem [4]. Then we get the asymptotic formulas:
\[
\ln \Gamma(x + 1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \left( (x + 1) \ln(x + 1) - x \ln x - 1 \right) - \left( x + \frac{1}{2} \right),
\]
as \( x \to \infty \).

And, we have

**Proposition 2.** For \( x > 0 \), the function
\[
f_2(x) = \ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - \left( x + \frac{1}{2} \right) \left( (x + 1) \ln(x + 1) - x \ln x - 1 \right) + x + \frac{1}{2}
\]
is a completely monotone function.

**Proof.** Differentiation gives
\[
f_2'(x) = \psi(x + 1) - \left( 2x + \frac{3}{2} \right) \ln(x + 1) + \left( 2x + \frac{1}{2} \right) \ln x + 2,
\]
\[
f_2''(x) = \psi'(x + 1) - 2 \ln(x + 1) + 2 \ln x + \frac{1}{2(x + 1)} + \frac{1}{2x}.
\]

Application of the relations (3.6), \( f_2''(x) \) can be expressed as
\[
f_2''(x) = \int_0^\infty e^{-(x+1)t} \frac{1}{1-e^{-t}} dt - 2 \int_0^\infty e^{-xt} - e^{-(x+1)t} \frac{1}{t} dt + \frac{1}{2} \int_0^\infty (e^{-(x+1)t} + e^{-xt}) \frac{1}{t} dt
\]
\[
= \int_0^\infty e^{-xt} \left( \frac{te^{-t}}{1-e^{-t}} - 2 \frac{1-e^{-t}}{t} + \frac{1}{2} (e^{-t} + 1) \right) dt
\]
\[
= \int_0^\infty e^{-xt} e^{-t/2} \left( \frac{t}{2 \sinh(t/2)} - 4 \frac{\sinh(t/2)}{t} + \cosh(t/2) \right) dt
\]
\[
= \int_0^\infty e^{-xt} e^{-t/2} v(t) dt,
\]
where
\[
v(t) = \frac{t}{\sinh t} - 2 \frac{\sinh t}{t} + \cosh t.
\]

Employing hyperbolic version of Wilker inequality proved in [23] (also see [24], [25])
\[
\left( \frac{t}{\sinh t} \right)^2 + \frac{t}{\tanh t} > 2,
\]
we get
\[
\frac{\sinh t}{t} v(t) = \left( \frac{t}{\sinh t} \right)^2 + \frac{t}{\tanh t} - 2 > 0,
\]
and so \( f_2''(x) \) is complete monotone for \( x > 0 \). Hence, \( f_2'(x) \leq \lim_{x \to \infty} f_2'(x) = 0 \),
and then, \( f_2(x) \geq \lim_{x \to \infty} f_2(x) = 0 \), which indicate that \( f_2 \) is complete monotone
for \( x > 0 \).

This completes the proof. \( \square \)

The decreasing property of \( f_2 \) on \((0, \infty)\) and the facts that
\[
f_2(0^+) = \ln \frac{e}{\sqrt{2 \pi}}, \quad f_2(1) = \ln \frac{e}{8}, \quad f_2(\infty) = 0
\]
give the following
Corollary 4. For $x > 0$, the sharp double inequality
\[
\sqrt{2\pi}e^{-2x-1} \frac{(x+1)^{x+1}(x+1/2)}{x^{x+1/2}} < \Gamma(x+1) < e^{-2x} \frac{(x+1)^{(x+1)(x+1/2)}}{x^{x(x+1/2)}}
\]
holds.
For $n \in \mathbb{N}$, it holds that
\[
\sqrt{2\pi}e^{-2n-1} \frac{(n+1)^{(n+1)(n+1/2)}}{n^{n(n+1/2)}} < n! < \frac{e^3}{8} e^{-2n-1} \frac{(n+1)^{(n+1)(n+1/2)}}{n^{n(n+1/2)}}
\]
with the best constants $\sqrt{2\pi} \approx 2.5066$ and $e^3/8 \approx 2.5107$.

Example 3. Let
\[
K(a, b) = N(a, b) = A(a, b) = \frac{a + b}{2},
\]
\[
M(a, b) = M_{p,q}^{3,2}(a, b) = \frac{p a^3 + pb^3 + (1/2 - p) a^2 b + (1/2 - p) ab^2}{qa^2 + qb^2 + (1 - 2q) ab} = \frac{a + b}{2} \frac{2pa^2 + 2pb^2 + (1 - 4p) ab}{qa^2 + qb^2 + (1 - 2q) ab}
\]
and $\theta = 0$ in Theorem 7, where $p$ and $q$ are parameters to be determined. Then, we have
\[
K(x, x + 1) = N(x, x + 1) = x + \frac{1}{2},
\]
\[
M(x, x + 1) = S_{p,q}^{3,2}(x, x + 1) = (x + 1/2) \frac{x^2 + x + 2p}{x^2 + x + q}.
\]

Straightforward computations give
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x+1) - \ln \sqrt{2\pi} \frac{(x+1)^{(x+1)(x+1/2)}}{x^{x+1/2}}}{x} = q - 2p - \frac{1}{24},
\]
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x+1) - \ln \sqrt{2\pi} \frac{(x+1/2)^{(x+1/2)(x+1/2)}}{x^{x+1/2}}}{x} = \frac{160}{1920} \left( p - \frac{23}{160} \right),
\]
and solving the equation set
\[
q - 2p - \frac{1}{24} = 0 \quad \text{and} \quad \frac{160}{1920} \left( p - \frac{23}{160} \right) = 0
\]
leads to
\[
p = \frac{23}{160}, \quad q = \frac{79}{240}.
\]

And then,
\[
M(x, x + 1) = \left( x + \frac{1}{2} \right) \frac{x^2 + x + 23}{x^2 + x + 24}
\]
It is easy to check that $S_{p,q}^{3,2}(a, b)$ is a symmetric and homogeneous mean of positive numbers $a$ and $b$ for $p = 23/160$, $q = 79/240$. Hence, by Theorem 7 we have the optimal asymptotic formula for the gamma function
\[
\ln \Gamma(x+1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln \frac{x(x+1/2)(x^2+x+23/80)}{x^2+x+79/240} - \left( x + \frac{1}{2} \right),
\]
as $x \to \infty$, and
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x+1) - \ln \sqrt{2\pi} \frac{(x+1/2)^{(x+1/2)(x+1/2)}}{x^{x+1/2}}}{x} = \frac{18.029}{29.030400}.
\]
Also, this asymptotic formula have a well property.

**Proposition 3.** For \( x > -1/2 \), the function \( f_3 \) defined by
\[
(3.7) \quad f_3(x) = \ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - \left( x + \frac{1}{2} \right) \ln \left( \frac{(x+1/2)(x^2+x+23/80)}{x^2+x+79/240} \right) + \left( x + \frac{1}{2} \right),
\]
is increasing and concave.

**Proof.** Differentiation gives
\[
f'_3(x) = \psi(x+1) + \ln \left( \frac{x^2 + x + \frac{79}{240}}{x^2 + x + \frac{23}{80}} \right) - \ln \left( \frac{x^2 + x + \frac{23}{80}}{x^2 + x + \frac{79}{240}} \right) - 2 \frac{(x+1/2)^2}{x^2 + x + 23/80} + 2 \frac{(x+1/2)^2}{x^2 + x + 79/240},
\]
\[
f''_3(x) = \psi'(x+1) + 6 \frac{x+1/2}{x^2 + x + 79/240} - 6 \frac{x+1/2}{x^2 + x + 23/80} - \frac{1}{x+1/2} + 4 \frac{(x+1/2)^3}{(x^2 + x + 23/80)^2} - 4 \frac{(x+1/2)^3}{(x^2 + x + 79/240)^2}.
\]
Denote by \( x + 1/2 = t \) and make use of recursive relation
\[
(3.8) \quad \psi^{(n)}(x+1) - \psi^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}
\]
yield
\[
f''_3(t + \frac{1}{2}) - f''_3(t - \frac{1}{2}) = - \frac{1}{(t+1/2)^2} + 6 \frac{t+1}{(t+1)^2 + t + (3/80)} - 6 \frac{t+1}{(t+1)^2 + (3/80)} - \frac{1}{t+1} + 4 \frac{(t+1)^3}{(t+1)^2 + (3/80)}
\]
\[= -4 \frac{(t+1)^3}{(t+1)^2 + 19/240} - 6 \frac{t+1}{(t+1)^2 + 3/80} - \frac{1}{t+1} + 4 \frac{t^3}{(t+1)^2 + (3/80)^2} - 4 \frac{t^3}{(t+1)^2 + 19/240}.
\]
\[= \frac{f_{31}(t)}{t(t+1) \left( t + \frac{1}{2} \right)^2 \left( t^2 + 2t + 83/80 \right)^2 \left( t^2 + 3/80 \right)^2 \left( t^2 + 2t + 259/240 \right)^2 \left( t^2 + 19/240 \right)^2}, \]
where
\[
f_{31}(t) = \frac{1812801}{138240 t^{12}} + \frac{1812801}{23040 t^{11}} + \frac{83674657}{41472000 t^{10}} + \frac{24184837}{8294400 t^9} + \frac{333658121}{13270400000 t^8} + \frac{489465167}{3377600000 t^7}
\]
\[+ \frac{73296657213}{132787000000 t^6} + \frac{20147292749}{2147483648 t^5} + \frac{29709203547}{94371840000 t^4} + \frac{66777391051}{1435577600000 t^3}
\]
\[+ \frac{295012866563}{5562318000000000 t^2} + \frac{3972595981}{188436800000000000 t} + \frac{166825684249}{60397597600000000}
\]
\[> 0 \text{ for } t = x + 1/2 > 0.
\]
This shows that \( f''_3(t + \frac{1}{2}) - f''_3(t - \frac{1}{2}) > 0 \), that is, \( f''_3(x+1) - f''_3(x) > 0 \), and so
\[
f''_3(x) < f''_3(x+1) < f''_3(x+2) < ... < f''_3(\infty) = 0.
\]
It reveals that shows \( f_3 \) is concave on \((-1/2, \infty)\), and we conclude that, \( f_3(x) \geq \lim_{x \to \infty} f_3(x) = 0 \), which proves the desired result. \( \square \)

As a consequence of the above proposition, we have
Corollary 5. For \( x > 0 \), the double inequality
\[
\sqrt{\frac{158e}{69}} \left( \frac{x + 1/2}{e} \right)^{x+1/2} < x^{x+1/2} \Gamma(x + 1) < \sqrt[2]{2\pi} \left( \frac{x + 1/2}{e} \right)^{x+1/2}
\]
holds true, where \( \sqrt{158e/69} \approx 2.4949 \) and and \( \sqrt[2]{2\pi} \approx 2.5066 \) are the best.
For \( n \in \mathbb{N} \), it is true that
\[
\left( \frac{1118}{1647} \right)^{3/2} \left( \frac{n+1/2}{e} \right)^{n+1/2} < n! < \sqrt[2]{2\pi} \left( \frac{n+1/2}{e} \right)^{n+1/2}
\]
holds true with the best constants \( (1118/1647)^{3/2} \approx 2.5065 \) and \( \sqrt[2]{2\pi} \approx 2.5066 \).

Example 4. Let
\[
K(a, b) = M(a, b) = A(a, b) = \frac{a + b}{2},
\]
\[
N(a, b) = S_{p,q}^{3,2}(a, b) = \frac{pa^3 + pb^3 + (1/2 - p) ab^2 + (1/2 - p) a^2 b}{qa^2 + qb^2 + (1 - 2q) ab}
\]
and \( \sigma = 0 \) in Theorem 4 where \( p \) and \( q \) are parameters to be determined. Direct computations give
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - (x + 1/2) \ln (x + 1/2) + \left( \frac{x^2 + x + 2p}{x^2 + x + 2} \right)}{x}\]
\[
= 2p - q - \frac{1}{24},
\]
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - (x + 1/2) \ln (x + 1/2) + \left( \frac{x^2 + x + 2p}{x^2 + x + 2} \right)}{x}\]
\[
= \frac{7}{480} - \frac{1}{12} p.
\]
Solving the simultaneous equations
\[
2p - q - \frac{1}{24} = 0,
\]
\[
\frac{7}{480} - \frac{1}{12} p = 0
\]
leads to \( p = 7/40 \), \( q = 37/120 \). And then,
\[
N(x, x + 1) = (x + 1/2) \frac{x^2 + x + 7/20}{x^2 + x + 37/120}.
\]
An easy verification shows that \( S_{p,q}^{3,2}(a, b) \) is a symmetric and homogeneous mean of positive numbers \( a \) and \( b \) for \( p = 7/40 \), \( q = 37/120 \). Hence, by Theorem 4 we get the best asymptotic formula for the gamma function
\[
\ln \Gamma(x + 1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) - \left( x + \frac{1}{2} \right) \frac{x^2 + x + 7/20}{x^2 + x + 37/120},
\]
as \( x \to \infty \). And we have
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - (x + 1/2) \ln (x + 1/2) + \left( \frac{x^2 + x + 7/20}{x^2 + x + 37/120} \right)}{x}
\]
\[
= -\frac{1517}{2419200}.
\]
Now we prove the following assertion related to this asymptotic formula.

Proposition 4. Let the function \( f_4 \) be defined on \((-1/2, \infty)\) by
\[
f_4(x) = \ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - x + \frac{1}{2} \ln (x + \frac{1}{2}) + \frac{1}{2} \frac{x^2 + x + 7/20}{x^2 + x + 37/120}.
\]
Then \( f_4 \) is increasing and convex on \((-1/2, \infty)\).
Proof. Differentiation gives
\[ f'_4(x) = \psi'(x + 1) - \ln \left( x + \frac{1}{2} \right) + \frac{1}{24} \frac{1}{x^2 + x + 37/120}, \]
\[ f''_4(x) = \psi''(x + 1) - \frac{1}{x + 1/2} - \frac{1}{4} \frac{x + 1/2}{(x^2 + x + 37/120)^2} + \frac{1}{3} \frac{(x + \frac{1}{2})^3}{(x^2 + x + 37/120)^3}. \]

Denote by \( x + 1/2 = t \) and make use of recursive relation (3.3) yield
\[
\begin{align*}
&f''_4(t + \frac{1}{2}) - f''_4(t - \frac{1}{2}) \\
&= -\frac{1}{(t+1/2)^2} - \frac{1}{t+1} - \frac{1}{4} \frac{t + 1}{(t + 1)^2 + 7/120}^2 + \frac{1}{3} \frac{(t + 1)^3}{(t + 1)^2 + 7/120}^3 \\
&\quad - \left( -\frac{1}{t} - \frac{1}{4} \frac{t}{(t^2 + 7/120)^2} + \frac{1}{3} \frac{t^3}{(t^2 + 7/120)^2} \right) \\
&= f_{41}(t) \quad t > 0 \text{ for } t = x + 1/2 > 0.
\end{align*}
\]

This implies that \( f''_4(t + \frac{1}{2}) - f''_4(t - \frac{1}{2}) > 0 \), that is, \( f''_4(x + 1) - f''_4(x) > 0 \), and so \( f''_4(x) < f''_4(x + 1) < f''_4(x + 2) < ... < f''_4(\infty) = 0 \).

It reveals that shows \( f_4 \) is concave on \((-1/2, \infty)\), and therefore, \( f'_4(x) > \lim_{x \to \infty} f'_4(x) = 0 \), which proves the desired result. \( \square \)

By the increasing property of \( f_4 \) on \((-1/2, \infty)\) and the facts
\[ f_4(0) = \ln \frac{\varepsilon^{21/37}}{\sqrt{\pi}}, \quad f_4(1) = \ln \frac{2e^{423/277}}{3\sqrt{3\pi}}, \quad f_4(\infty) = 0, \]
we have

**Corollary 6.** For \( x > 0 \), the double inequality
\[ e^{21/37} \sqrt{2} \left( \frac{x+1/2}{\exp \left( x + 1/2 + 37/120 \right)} \right)^{x+1/2} < \Gamma(x + 1) < \sqrt{2\pi} \left( \frac{x+1/2}{\exp \left( x + 1/2 + 37/120 \right)} \right)^{x+1/2}, \]
holds, where \( e^{21/37} \sqrt{2} \approx 2.4946 \) and \( \sqrt{2\pi} \approx 2.5066 \) are the best.

For \( n \in \mathbb{N} \), the double inequality
\[ e^{423/277} \frac{n+1/2}{\sqrt{n}} \varepsilon^{n+1/2} \exp \left( -\frac{1}{24} \frac{n+1/2}{n + 37/120} \right) > n! < \sqrt{2\pi} \left( \frac{n+1/2}{e} \right)^n \exp \left( -\frac{1}{24} \frac{n+1/2}{n + 37/120} \right) \]
holds true with the best constants \( 2\varepsilon e^{423/277} \approx 2.5065 \) and \( \sqrt{2\pi} \approx 2.5066 \).
Example 5. Let

\[ K(a, b) = M(a, b) = A(a, b) = x + 1/2, \]

\[ N(a, b) = S_{p, q, r}^{4, 3}(a, b) = \frac{pa^4 + pb^4 + qa^3b + qab^3 + (1 - 2p - 2q)a^2b^2}{ra^3 + rb^3 + (1/2 - r)a^2b + (1/2 - r)ab^2} \]

and \( \sigma = 0 \) in Theorem 1. In a similar way, we can determine that the best parameters satisfy

\[ r = 2p + \frac{1}{2} q - \frac{7}{48}, \quad p = \frac{21}{40} - \frac{7}{4} q, \quad q = \frac{7303}{35280}, \]

which imply

\[ p = \frac{3281}{20160}, \quad q = \frac{7303}{35280}, \quad r = \frac{111}{392}. \]

Then,

\[ (3.9) \quad N(x, x + 1) = x + \frac{1}{2} + \frac{1517}{44640} x + \frac{1}{2} + \frac{343}{44640} \frac{x + 1/2}{x^2 + x + 111/196} := N_{4/3}(x, x + 1), \]

In this case, we easily check that \( S_{p, q, r}^{4, 3}(a, b) \) is a mean of \( a \) and \( b \). Consequently, from Theorem 1 the following best asymptotic formula for the gamma function

\[ \ln \Gamma(x + 1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln(x + 1/2) - N_{4/3}(x, x + 1) \]

holds true as \( x \to \infty \). And, we have

\[ \lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - (x + 1/2) \ln(x + 1/2) + N_{4/3}(x, x + 1)}{x} = \frac{10981}{31610880}. \]

We now present the monotonicity and convexity involving this asymptotic formula.

Proposition 5. Let \( f_5 \) defined on \((-1/2, \infty)\) by

\[ f_5(x) = \ln \Gamma(x + 1) - \frac{1}{2} \ln 2\pi - (x + 1/2) \ln(x + 1/2) + N_{4/3}(x, x + 1), \]

where \( N_{4/3}(x, x + 1) \) is defined by (3.9). Then \( f_5 \) is decreasing and convex on \((-1/2, \infty)\).

Proof. Differentiation gives

\[ f'_5(x) = \psi(x + 1) - \ln \left( x + \frac{1}{2} \right) - \frac{1517}{44640} (x + 1/2)^2 + \frac{343}{44640} (x^2 + x + 111/196) - \frac{343}{22320} (x + 1/2)^2, \]

\[ f''_5(x) = \psi'(x + 1) - \frac{1}{x + 1/2} + \frac{1517}{22320} (x + 1/2)^3 - \frac{343}{7440} (x^2 + x + 111/196)^2 + \frac{343}{5580} (x + 1/2)^3. \]
Denote by $x + 1/2 = t$ and make use of recursive relation (3.5) yield

\[
\begin{align*}
    f''_5(t + \frac{1}{2}) - f''_5(t - \frac{1}{2}) \\
    &= -\frac{1}{(t+1/2)^2} - \frac{1}{(t+1)^2} + \frac{1517}{22320(t+1)^3} - \frac{1517}{22320(t+31/96)^3} + \frac{343}{7440 (t+1/2)^{31/96}} + \frac{343}{7440 (t+1/2)^{31/96}} \\
    &= -\left( \frac{1}{t^2} + \frac{1517}{22320t^2} - \frac{343}{7440 (t+2)^{31/96}} + \frac{343}{5580 (t+2)^{31/96}} \right) \\
    &= -\frac{f_5(t)}{80 (t + 1/2)^2 t^3 (t + 1)^3 (t^2 + 2t + 129/98)^4 (t^2 + 31/98)^5},
\end{align*}
\]

where

\[
    f_5(t) = \frac{10981}{784} t^{10} + \frac{54905}{784} t^9 + \frac{21028}{134} t^8 + \frac{27614}{134} t^7 + \frac{294820}{134} t^6 + \frac{739}{22} \cdot 471 t^5 + \frac{138}{22} \cdot 266 \cdot 105 \cdot 451 t^4 + \frac{251}{22} \cdot 15 \cdot 604 \cdot 49 t^3 + \frac{2728}{129} \cdot 31 \cdot 421 t^2 + \frac{506}{129} \cdot 195 \cdot 646 t + \frac{337}{129} \cdot 724 \cdot 79977 + \frac{54}{129} \cdot 369 \cdot 523165 < 0
\]

for $t = x + 1/2 > 0$.

This implies that $f''_5(t + \frac{1}{2}) - f''_5(t - \frac{1}{2}) < 0$, that is, $f''_5(x + 1) - f''_5(x) < 0$, and so

\[
    f''_5(x) > f''_5(x + 1) > f''_5(x + 2) > \ldots > f''_5(\infty) = 0.
\]

It reveals that shows $f_5$ is convex on $(-1/2, \infty)$, and therefore, $f_5'(x) < \lim_{x \to \infty} f_5'(x) = 0$, which proves the desired statement.

**Corollary 7.** For $x > 0$, the double inequality

\[
    \sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp\left( -\frac{1517}{41640} \frac{x+1/2}{x+2} \right) < \Gamma(x+1) < e^{2987/3960} \sqrt{2\pi} \left( \frac{x+1/2}{e} \right)^{x+1/2} \exp\left( -\frac{1517}{41640} \frac{x+1/2}{x+2} \right)
\]

holds, where $\sqrt{2\pi} \approx 2.5066$ and $e^{2987/3960} \sqrt{2\pi} \approx 2.5126$ are the best constants.

For $n \in \mathbb{N}$, it holds that

\[
    \sqrt{2\pi} \left( \frac{n+1/2}{e} \right)^{n+1/2} \exp\left( -\frac{1517}{41640} \frac{n+1/2}{n+2} \right) < n! < \frac{2\sqrt{6}}{9} \exp\left( \frac{839}{543} \frac{n+1/2}{e} \right) \exp\left( -\frac{1517}{41640} \frac{n+1/2}{n+2} \right)
\]

with the best constants $\sqrt{2\pi} \approx 2.5066$ and $2\sqrt{6} \exp(\frac{839}{543}) / 9 \approx 2.5067$.

Lastly, we give an application example of Theorem 2.

**Example 6.** Let

\[
    M(a, b) = H_{p,q,r}^2(a, b) = \frac{pb^2 + qa^2 + (1 - p - q)ab}{rb + (1 - r)a}
\]

and $\theta = 0, \sigma = 1$ in Theorem 3. Then by the same method previously, we can derive two best arrays

\[
    (p_1, q_1, r_1) = \left( \frac{129 + 59\sqrt{3}}{360}, \frac{129 + 59\sqrt{3}}{360}, \frac{90 - 29\sqrt{3}}{180} \right),
\]

\[
    (p_2, q_2, r_2) = \left( \frac{129 + 59\sqrt{3}}{360}, \frac{129 - 59\sqrt{3}}{360}, \frac{90 + 29\sqrt{3}}{180} \right).
\]
Then,
\[
(3.10) \ H_{p_1,q_1;r_1}^{2,1}(x, x+1) = \frac{x^2 + \frac{180-59\sqrt{3}}{180} x + \frac{129-59\sqrt{3}}{360}}{x + \frac{90-29\sqrt{3}}{180}} := M_1(x, x+1),
\]
\[
(3.11) \ H_{p_2,q_2;r_2}^{2,1}(x, x+1) = \frac{x^2 + \frac{180+59\sqrt{3}}{180} x + \frac{129+59\sqrt{3}}{360}}{x + \frac{90+29\sqrt{3}}{180}} := M_2(x, x+1)
\]

It is easy to check that \(M(a,b)\) are means of \(a\) and \(b\) for \((p,q,r) = (p_1,q_1,r_1)\) and \((p_2,q_2,r_2)\). Thus, application of Theorem 2 implies that both the following two asymptotic formulas
\[
\ln \Gamma(x+1) \sim \frac{1}{2} \ln 2\pi + (x+1/2) \ln M_i(x, x+1) - M_i(x, x+1), \quad i = 1, 2
\]
are valid as \(x \to \infty\). And, we have
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x+1) - \frac{1}{2} \ln 2\pi - (x+1/2) \ln M_1(x, x+1) + M_1(x, x+1)}{x-1} = -\frac{1481\sqrt{3}}{2332800},
\]
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x+1) - \frac{1}{2} \ln 2\pi - (x+1/2) \ln M_2(x, x+1) + M_2(x, x+1)}{x-1} = \frac{1481\sqrt{3}}{2332800}.
\]

The above two asymptotic formulas also have well properties.

**Proposition 6.** Let \(f_6, f_7\) be defined on \((0, \infty)\) by
\[
f_6(x) = \ln \Gamma(x+1) - \frac{1}{2} \ln 2\pi - (x+1/2) \ln M_1(x, x+1) + M_1(x, x+1),
\]
\[
f_7(x) = \ln \Gamma(x+1) - \frac{1}{2} \ln 2\pi - (x+1/2) \ln M_2(x, x+1) + M_2(x, x+1),
\]
where \(M_1\) and \(M_2\) are defined by \((3.10)\) and \((3.11)\), respectively. Then \(f_6\) is increasing and concave on \((0, \infty)\), while \(f_7\) is decreasing and convex on \((0, \infty)\).

**Proof.** Differentiation gives
\[
f_6'(x) = \psi(x+1) - \ln \frac{x^2 + \frac{180-59\sqrt{3}}{180} x + \frac{129-59\sqrt{3}}{360}}{x + \frac{90-29\sqrt{3}}{180}} - \left( \frac{x + \frac{1}{2}}{x + \frac{90-29\sqrt{3}}{180}} \right)^2 - \frac{x^2 + \frac{180+59\sqrt{3}}{180} x + \frac{129+59\sqrt{3}}{360}}{x + \frac{90+29\sqrt{3}}{180}},
\]
\[
f_6''(x) = \psi'(x+1) - \frac{2x + \frac{180-59\sqrt{3}}{180}}{x^2 + \frac{180-59\sqrt{3}}{180}} + \frac{1}{x + \frac{90-29\sqrt{3}}{180}} + \frac{59\sqrt{3}}{180} \left( x^2 + \frac{129-59\sqrt{3}}{360} \right)^2 - \frac{7\sqrt{3}}{45} \left( x + \frac{90-29\sqrt{3}}{180} \right)^2.
\]

Employing the recursive relation \((3.5)\) and factoring reveal that
\[
f_6''(x+1) - f_6''(x) = \frac{1481\sqrt{3}}{19440} f_6''(x),
\]
where

\[ f_{61}(x) = x^9 + \left( 9 - \frac{4471.134 \sqrt{3}}{26459.859} \right) x^8 + \left( \frac{2291.07423}{20.058.090} - \frac{474.306 \sqrt{3}}{66.615} \right) x^7 \\
+ \left( \frac{2349.907.961}{20.058.090} - \frac{169.081.132.727 \sqrt{3}}{1.997.129.900} \right) x^6 + \left( \frac{4333.292.090.469}{28.9.041.000} - \frac{55.797.724.727 \sqrt{3}}{799.140.000} \right) x^5 \\
+ \left( \frac{956.621.902.709}{575.8.128.000} - \frac{148.442.768.304.491 \sqrt{3}}{1727.438.400.000} \right) x^4 \\
+ \left( \frac{229.288.958.388.788.929}{1.869.633.427.000.000} - \frac{3.145.014.047.29 \sqrt{3}}{431.839.600.000} \right) x^3 \\
+ \left( \frac{36.305.075.316.164.929}{621.871.524.000.000} - \frac{55.416.599.045.055.111.861 \sqrt{3}}{1679.070.124.800.000.000} \right) x^2 \\
+ \left( \frac{179.958.706.278.746.628.611}{11.193.800.832.000.000.000} - \frac{773.145.289.282.423.861 \sqrt{3}}{8.59.506.062.400.000.000} \right) x \\
+ \left( \frac{21.862.691.463.638.689.611}{11.193.800.832.000.000.000} - \frac{5.58.677.417.732.710.687 \sqrt{3}}{497.522.592.000.000.000.000} \right), \]

\[ f_{62}(x) = (x + 1)^2 \left( x^2 + \frac{180 - 59 \sqrt{3}}{180} x + \frac{129 - 59 \sqrt{3}}{180} \right)^2 \left( x^2 + \frac{540 - 59 \sqrt{3}}{180} x + \frac{283 - 59 \sqrt{3}}{180} \right)^2 \\
\times \left( x + \frac{270 - 29 \sqrt{3}}{180} \right)^3 \left( x + \frac{90 - 29 \sqrt{3}}{180} \right)^3. \]

By direct verifications we see that all coefficients of \( f_{61} \) and \( f_{62} \) are positive, so \( f_{61}(x), f_{62}(x) > 0 \) for \( x > 0 \). Therefore, we get \( f_6''(x + 1) - f_6''(x) > 0 \), which yields

\[ f_6''(x) < f_6''(x + 1) < f_6''(x + 2) < \ldots < f_6''(\infty) = 0. \]

It shows that \( f_6 \) is concave on \((0, \infty)\), and therefore, \( f_6(x) > \lim_{x \to \infty} f_6(x) = 0 \), which proves the monotonicity and concavity of \( f_6 \).

In the same way, we can prove the monotonicity and convexity of \( f_7 \) on \((0, \infty)\), whose details are omitted. \( \square \)

As direct consequences of the previous proposition, we have

**Corollary 8.** For \( x > 0 \), the double inequality

\[ \delta_0 \sqrt{2 \pi} \left( \frac{x^2 + \frac{180 - 59 \sqrt{3}}{180} x + \frac{129 - 59 \sqrt{3}}{180}}{x + \frac{90 - 29 \sqrt{3}}{180}} \right)^{x+1/2} \exp \left( - \frac{x^2 + \frac{180 - 59 \sqrt{3}}{180} x + \frac{129 - 59 \sqrt{3}}{180}}{x + \frac{90 - 29 \sqrt{3}}{180}} \right) < \Gamma(x + 1) < \sqrt{2 \pi} \left( \frac{x^2 + \frac{180 - 59 \sqrt{3}}{180} x + \frac{129 - 59 \sqrt{3}}{180}}{x + \frac{90 - 29 \sqrt{3}}{180}} \right)^{x+1/2} \exp \left( - \frac{x^2 + \frac{180 - 59 \sqrt{3}}{180} x + \frac{129 - 59 \sqrt{3}}{180}}{x + \frac{90 - 29 \sqrt{3}}{180}} \right) \]

holds, where \( \delta_0 = \exp f_6(0) \approx 0.96259 \) and 1 are the best constants.

For \( n \in \mathbb{N} \), it holds that

\[ \delta_2 \sqrt{2 \pi} \left( \frac{n^2 + \frac{180 - 59 \sqrt{3}}{180} n + \frac{129 - 59 \sqrt{3}}{180}}{n + \frac{90 - 29 \sqrt{3}}{180}} \right)^{n+1/2} \exp \left( - \frac{n^2 + \frac{180 - 59 \sqrt{3}}{180} n + \frac{129 - 59 \sqrt{3}}{180}}{n + \frac{90 - 29 \sqrt{3}}{180}} \right) < n! < \sqrt{2 \pi} \left( \frac{n^2 + \frac{180 - 59 \sqrt{3}}{180} n + \frac{129 - 59 \sqrt{3}}{180}}{n + \frac{90 - 29 \sqrt{3}}{180}} \right)^{n+1/2} \exp \left( - \frac{n^2 + \frac{180 - 59 \sqrt{3}}{180} n + \frac{129 - 59 \sqrt{3}}{180}}{n + \frac{90 - 29 \sqrt{3}}{180}} \right) \]

with the best constants \( \delta_1 = \exp f_6(1) \approx 0.99965 \) and 1.
Corollary 9. For $x > 0$, the double inequality
\[
\sqrt{2\pi} \left( \frac{x^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{x + \frac{90 + 29 \sqrt{\pi}}{180}} \right)^{x+1/2} \exp \left( - \frac{x^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{x + \frac{90 + 29 \sqrt{\pi}}{180}} \right) < \Gamma(x + 1) < \tau_0 \sqrt{2\pi} \left( \frac{x^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{x + \frac{90 + 29 \sqrt{\pi}}{180}} \right)^{x+1/2} \exp \left( - \frac{x^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{x + \frac{90 + 29 \sqrt{\pi}}{180}} \right)
\]
holds, where $\tau_0 = \exp f_7(0) \approx 1.0020$ and 1 are the best constants.

For $n \in \mathbb{N}$, it holds that
\[
\sqrt{2\pi} \left( \frac{n^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{n + \frac{90 + 29 \sqrt{\pi}}{180}} \right)^{n+1/2} \exp \left( - \frac{n^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{n + \frac{90 + 29 \sqrt{\pi}}{180}} \right) < n! < \sqrt{2\pi} \left( \frac{n^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{n + \frac{90 + 29 \sqrt{\pi}}{180}} \right)^{n+1} \exp \left( - \frac{n^2 + 180 + 59 \sqrt{\pi} \tau + 129 + 59 \sqrt{\pi}}{n + \frac{90 + 29 \sqrt{\pi}}{180}} \right)
\]
with the best constants $\delta_1 = \exp f_7(1) \approx 1.0001$ and 1.

4. Open problems

Inspired by Examples 3–5, we propose the following problems.

Problem 1. Let $S_{p_k/q_k}^{n,n-1}(a,b)$ be defined by (3.3). Finding $p_k$ and $q_k$ such that the asymptotic formula for the gamma function
\[
\ln \Gamma(x + 1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln S_{p_k/q_k}^{n,n-1}(x, x + 1) - \left( x + \frac{1}{2} \right) := F_1(x)
\]
holds as $x \to \infty$ with
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - F_1(x)}{x^{-2n+1}} = c_1 \neq 0, \pm \infty.
\]

Problem 2. Let $S_{p_k/q_k}^{n,n-1}(a,b)$ be defined by (3.3). Finding $p_k$ and $q_k$ such that the asymptotic formula for the gamma function
\[
\ln \Gamma(x + 1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln \left( x + \frac{1}{2} \right) - S_{p_k/q_k}^{n,n-1}(x, x + 1) := F_2(x)
\]
holds as $x \to \infty$ with
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - F_2(x)}{x^{-2n+1}} = c_2 \neq 0, \pm \infty.
\]

Problem 3. Let $H_{p_k/q_k}^{n,n-1}(a,b)$ be defined by (3.7). Finding $p_k$ and $q_k$ such that the asymptotic formula for the gamma function
\[
\ln \Gamma(x + 1) \sim \frac{1}{2} \ln 2\pi + \left( x + \frac{1}{2} \right) \ln H_{p_k/q_k}^{n,n-1}(x, x + 1) - H_{p_k/q_k}^{n,n-1}(x, x + 1) := F_3(x)
\]
holds as $x \to \infty$ with
\[
\lim_{x \to \infty} \frac{\ln \Gamma(x + 1) - F_3(x)}{x^{-2n+1}} = c_3 \neq 0, \pm \infty.
\]
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