Cohomology of $\text{aff}(1)$ and $\text{aff}(1|1)$ acting on the space of $n$-ary differential operators on the superspace $\mathbb{R}^{1|1}$

Mabrouk Ben Ammar    Maha Boujelben    Amina Jabeur    Rabeb Sidaoui *

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Abstract

We consider the $\mu$-densities spaces $\mathcal{F}_\mu$ with $\mu \in \mathbb{R}$, we compute the space $H^1_{\text{diff}}(\text{aff}(1), D_{\lambda,\mu})$ where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $D_{\lambda,\mu}$ is the space of $n$-ary differential operators from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$. We also compute the super analog space $H^1_{\text{diff}}(\text{aff}(1|1), D_{\lambda,\mu})$.

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1 Introduction

The space of weighted densities of weight $\mu$ on $\mathbb{R}$ (or $\mu$-densities for short), denoted by:

$$\mathcal{F}_\mu = \{ f dx^\mu, \ f \in C^\infty(\mathbb{R}) \}, \ \mu \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^\otimes \mu$. The Lie algebra $\text{Vect}(\mathbb{R})$ of vector fields $X_h = h \frac{d}{dx}$, where $h \in C^\infty(\mathbb{R})$, acts by the Lie derivative. Alternatively, this action can be written as follows:

$$X_h \cdot (f dx^\mu) = L^\mu_{X_h} (f) dx^\mu \quad \text{with} \quad L^\mu_{X_h} (f) = hf' + \mu h' f, \quad (1.1)$$

where $f'$, $h'$ are $\frac{df}{dx}$, $\frac{dh}{dx}$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ we denote by $D_{\lambda,\mu}$ the space of $n$-ary differential operators $A$ from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$. The Lie algebra $\text{Vect}(\mathbb{R})$ acts on the space $D_{\lambda,\mu}$ of these differential operators by:

$$X_h \cdot A = L^\mu_{X_h} \circ A - A \circ L^\lambda_{X_h} \quad (1.2)$$

where $L^\lambda_{X_h}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule. If we restrict ourselves to the Lie algebra $\text{aff}(1)$ which is isomorphic to the Lie subalgebra of $\text{Vect}(\mathbb{R})$ spanned by

$$\{X_1, X_x\},$$

we have a family of infinite dimensional $\text{aff}(1)$-modules still denoted by $D_{\lambda,\mu}$.

*Université de Sfax, Faculté des Sciences, Département de Mathématiques, Laboratoire d’Algèbre, Géométrie et Théorie Spectrale (AGTS) LR11ES53, BP 802, 3038 Sfax, Tunisie. E.mail: mabrouk.benammar@fss.rnu.tn
According to Nijenhuis-Richardson [10], the space $H^1(\mathfrak{g}; \text{End}(V))$ classifies the infinitesimal deformations of a $\mathfrak{g}$-module $V$ and the obstructions to integrability of a given infinitesimal deformation of $V$ are elements of $H^2(\mathfrak{g}; \text{End}(V))$. While the spaces $H^1(\mathfrak{g}; L(\otimes^k_s V, V))$ appear naturally in the problem of normalization of nonlinear representations of $\mathfrak{g}$ in $V$. To be more precise, let

$$T : \mathfrak{g} \to \bigoplus_{k \geq 0} L(\otimes^k_s V, V), \; X \mapsto T_X = \sum T^k_X,$$

be a nonlinear representation of $\mathfrak{g}$ in $V$, that is, $T_{[X,Y]} = [T_X, T_Y]$. In [2], it is proved that the representation $T$ is normalized if $T^k_X$ is in a supplementary of $B^1(\mathfrak{g}; L(\otimes^k_s V, V))$ in $Z^1(\mathfrak{g}; L(\otimes^k_s V, V))$.

In fact if $A$ is a differential operator on the line, $A$ can be viewed as an homomorphism from $F^\lambda$ to $F^\mu$. If $A$ is with order $n$, we can define its symbol as an element in $S^n_\delta = \bigoplus_{j=0}^n F^\delta - j$ for $\delta = \mu - \lambda$. If $n$ goes to $+\infty$, the space $S_\delta = \bigoplus_{j \geq 0} F^\delta - j$ appears as the space of symbols for all differential operators. The space $H^1(\mathfrak{g}; L(\otimes^2_s S_\delta, S_\delta))$ can be decomposed as a sum of spaces $H^1(\mathfrak{g}, D_{\lambda,\mu})$ with $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Thus, the computation of the spaces $H^1(\mathfrak{g}, D_{\lambda,\mu})$ is the first step to normalize any nonlinear representation of $\mathfrak{g}$ in $S_\delta$.

For $\lambda \in \mathbb{R}$ the spaces $H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\mu})$ are computed by Gargoubi [7] and Lecomte [9] and the spaces $H^1_{\text{diff}}(\mathfrak{osp}(1|2), D_{\lambda,\mu})$ are computed by Basdouri and Ben Ammar [3], where $H^1_{\text{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. For $\lambda \in \mathbb{R}^2$ the spaces $H^1_{\text{diff}}(\mathfrak{sl}(2), D_{\lambda,\mu})$ are computed by Bouarroudj [5] and the spaces $H^1_{\text{diff}}(\mathfrak{osp}(1|2), D_{\lambda,\mu})$ are computed by Ben Ammar et al [4], while we are interested in this paper in the spaces $H^1_{\text{diff}}(\mathfrak{aff}(1), D_{\lambda,\mu})$ and $H^1_{\text{diff}}(\mathfrak{aff}(1|1), D_{\lambda,\mu})$ where $\lambda \in \mathbb{R}^n$.

## 2 Definitions and Notations

### 2.1 The Lie superalgebra of contact vector fields on $\mathbb{R}^{1|1}$

We define the superspace $\mathbb{R}^{1|1}$ in terms of its superalgebra of functions, denoted by $C^\infty(\mathbb{R}^{1|1})$ and consisting of elements of the form:

$$F(x, \theta) = f_0(x) + f_1(x)\theta,$$

where $x$ is the even variable, $\theta$ is the odd variable ($\theta^2 = 0$) and $f_0(x), f_1(x) \in C^\infty(\mathbb{R})$. Even elements in $C^\infty(\mathbb{R}^{1|1})$ are the functions $F(x, \theta) = f_0(x)$, the functions $F(x, \theta) = \theta f_1(x)$ are odd elements. We consider the contact bracket on $C^\infty(\mathbb{R}^{1|1})$ defined on $C^\infty(\mathbb{R}^{1|1})$ by:

$$\{F, G\} = FG' - F'G + \frac{1}{2}D(F)\overline{D}(G),$$

where $D = \frac{\partial}{\partial x} + \theta \frac{\partial}{\partial \theta}$ and $\overline{D} = \frac{\partial}{\partial x} - \theta \frac{\partial}{\partial \theta}$. The superspace $\mathbb{R}^{1|1}$ is equipped with the standard contact structure given by the following 1-form:

$$\alpha = dx + \theta d\theta.$$

Let $\text{Vect}(\mathbb{R}^{1|1})$ be the superspace of vector fields on $\mathbb{R}^{1|1}$:

$$\text{Vect}(\mathbb{R}^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_i \in C^\infty(\mathbb{R}^{1|1}) \right\}.$$
where $\partial_\theta$ stands for $\frac{\partial}{\partial \theta}$ and $\partial_x$ stands for $\frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(1)$ of contact vector fields on $\mathbb{R}^{1\mid1}$. That is, $\mathcal{K}(1)$ is the superspace of vector fields on $\mathbb{R}^{1\mid1}$ preserving the distribution $\langle D \rangle$:

$$\mathcal{K}(1) = \{ X \in \text{Vect}(\mathbb{R}^{1\mid1}) \mid [X, \overline{D}] = F_X D \text{ for some } F_X \in C^\infty(\mathbb{R}^{1\mid1}) \}. $$

Any contact vector field on $\mathbb{R}^{1\mid1}$ has the following explicit form:

$$X_F = F \partial_x + \frac{1}{2} D(F) \overline{D}, \text{ where } F \in C^\infty(\mathbb{R}^{1\mid1}).$$

The bracket on $\mathcal{K}(1)$ is given by

$$[X_F, X_G] = X_{\{F,G\}}.$$

Thus, the map $F \mapsto X_F$ is a Lie superalgebra isomorphism from $\mathcal{K}(1)$ to $C^\infty(\mathbb{R}^{1\mid1})$. Of course, the set $\{ X_f = f \partial_x + \frac{1}{2} f' \theta \theta, f \in C^\infty(\mathbb{R}) \}$ is a subalgebra of $\mathcal{K}(1)$ isomorphic to $\text{Vect}(\mathbb{R})$.

### 2.2 The subalgebra $\mathfrak{aff}(1\mid1)$

The Lie algebra $\mathfrak{aff}(1)$ is realized as subalgebra of the Lie algebra $\text{Vect}(\mathbb{R})$:

$$\mathfrak{aff}(1) = \text{Span}(X_1, X_x).$$

Similarly, we now consider the Lie superalgebra $\mathfrak{aff}(1\mid1)$ as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{aff}(1\mid1) = \text{Span}(X_1, X_x, X_\theta).$$

The space of even elements is isomorphic to $\mathfrak{aff}(1)$, while the space of odd elements is one dimensional:

$$(\mathfrak{aff}(1\mid1))_1 = \text{Span}(X_\theta).$$

The commutation relations are

$$[X_x, X_\theta] = -\frac{1}{2} X_\theta, \quad [X_x, X_1] = -X_1, \quad [X_\theta, X_\theta] = \frac{1}{2} X_1.$$

### 2.3 The space of weighted densities on $\mathbb{R}^{1\mid1}$

We have analogous definition of weighted densities in super setting (see [1]) with $dx$ replaced by $\alpha$. The elements of these spaces are indeed (weighted) densities since all spaces of generalized tensor fields have just one parameter relative $\mathcal{K}(1)$ — the value of $X_x$ on the lowest weight vector (the one annihilated by $X_\theta$). From this point of view the volume element (roughly speaking, “$dx \frac{\partial}{\partial \theta}$”) is indistinguishable from $\alpha^\frac{1}{2}$. We denote by $\mathfrak{F}_\mu$ the space of all weighted densities on $\mathbb{R}^{1\mid1}$ of weight $\mu \in \mathbb{R}$:

$$\mathfrak{F}_\mu = \left\{ F(x, \theta) \alpha^\mu \mid F(x, \theta) \in C^\infty(\mathbb{R}^{1\mid1}) \right\}.$$

As a vector space, $\mathfrak{F}_\mu$ is isomorphic to $C^\infty(\mathbb{R}^{1\mid1})$, but the Lie derivative of the density $G \alpha^\mu$ along the vector field $X_F$ in $\mathcal{K}(1)$ is now:

$$\mathcal{L}_{X_F}(G \alpha^\mu) = \mathcal{L}_{X_F}^\mu(G) \alpha^\mu, \quad \text{with} \quad \mathcal{L}_{X_F}^\mu(G) = \mathcal{L}_{X_F}(G) + \mu F'G.$$
Especially, if \( f \in C^\infty(\mathbb{R}) \) and \( G(x, \theta) = g_0(x) + g_1(x)\theta \), then we easily check that

\[
\mathcal{L}^\mu_{X_f}(G) = L^\mu_{X_f}(g_0) + \left( L^{\mu+\frac{1}{2}}_{X_f}(g_1) \right) \theta.
\]

(2.1)

Of course, for all \( \mu \), \( \mathfrak{g}_\mu \) is a \( \mathcal{K}(1) \)-module:

\[
\left[ \mathcal{L}^\mu_{X_f}, \mathcal{L}^\mu_{X_G} \right] = \mathcal{L}^\mu_{[X_F, X_G]}.
\]

2.4 Differential operators on weighted densities

A differential operator on \( \mathbb{R}^{1|1} \) is an operator on \( C^\infty(\mathbb{R}^{1|1}) \) of the following form:

\[
A = \sum_{i=0}^\ell a_i(x, \theta) \partial_x^i + \sum_{i=0}^\ell b_i(x, \theta) \partial_x^i \partial_\theta.
\]

In [8], it is proved that any local operator \( A \) on \( \mathbb{R}^{1|1} \) is in fact a differential operator.

Of course, any differential operator defines a linear mapping from \( \mathfrak{f}_\lambda \) to \( \mathfrak{f}_\mu \) for any \( \lambda, \mu \in \mathbb{R} \): \( F^\lambda \alpha \mapsto A(F)\alpha^\mu \). Similarly, if \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \) and \( \mu \in \mathbb{R} \), we consider a family of \( \mathcal{K}(1) \) and \( \mathfrak{aff}(1|1) \) modules denoted \( \mathfrak{D}_\lambda, \mathfrak{D}_\mu \), for the natural action:

\[
X_F \cdot A = \mathcal{L}^\mu_{X_F} \circ A - (-1)^{AF} A \circ \mathcal{L}^\lambda_{X_F}
\]

where \( \mathcal{L}^\lambda_{X_F} \) is the Lie derivative on \( \mathfrak{f}_\lambda \otimes \cdots \mathfrak{f}_\lambda \) defined by the Leibnitz rule.

3 The space \( H^1_{\text{diff}}(\mathfrak{aff}(1|1), \mathfrak{D}_\lambda, \mathfrak{D}_\mu) \)

3.1 Cohomology

We will compute the first cohomology space of \( \mathfrak{aff}(1|1) \) with coefficients in \( \mathfrak{D}_\lambda, \mathfrak{D}_\mu \) where \( \lambda \in \mathbb{R}^n \) and \( \mu \in \mathbb{R} \). Let us first recall some fundamental concepts from cohomology theory (see, e.g., [8]). Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra acting on a superspace \( V = V_0 \oplus V_1 \) and let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \). (If \( \mathfrak{h} \) is omitted it is assumed to be \( \{0\} \)). The space of \( \mathfrak{h} \)-relative \( n \)-cochains of \( \mathfrak{g} \) with values in \( V \) is the \( \mathfrak{g} \)-module

\[
C^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_\mathfrak{h}(\Lambda^n(\mathfrak{g}/\mathfrak{h}); V).
\]

The coboundary operator \( \partial^n : C^n(\mathfrak{g}, \mathfrak{h}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}; V) \) is a \( \mathfrak{g} \)-map satisfying \( \partial^n \circ \partial^{n-1} = 0 \). The kernel of \( \partial^n \), denoted \( Z^n(\mathfrak{g}, \mathfrak{h}; V) \), is the space of \( \mathfrak{h} \)-relative \( n \)-cocycles, among them, the elements in the range of \( \partial^{n-1} \) are called \( \mathfrak{h} \)-relative \( n \)-coboundaries. We denote \( B^n(\mathfrak{g}, \mathfrak{h}; V) \) the space of \( n \)-coboundaries.

By definition, the \( n \)-th \( \mathfrak{h} \)-relative cohomology space is the quotient space

\[
H^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V)/B^n(\mathfrak{g}, \mathfrak{h}; V).
\]

We will only need the formula of \( \partial^n \) (which will be simply denoted \( \partial \)) in degrees 0 and 1. For \( v \in C^0(\mathfrak{g}, \mathfrak{h}; V) = V^h \),

\[
\partial v(g) := (-1)^{gv} g \cdot v,
\]

where \( V^h \) is the subspace of \( \mathfrak{h} \)-invariant elements of \( V \). For \( \Omega \in C^1(\mathfrak{g}, \mathfrak{h}; V) \) and \( g, h \in \mathfrak{g} \),

\[
\partial(\Omega)(g, h) := (-1)^{g}\partial g \cdot \Omega(h) - (-1)^{h}(g+\Omega) \cdot \Omega(g) - \Upsilon([g, h]).
\]
Proposition 3.1. 1) Let $\Omega \in Z^1(\mathfrak{g}; V)$ and $a \in \mathfrak{g}$. If $\Omega(a) = 0$ then $a \cdot \Omega = 0$. Thus, if $\Omega|_{\mathfrak{h}} = 0$ then $\Omega$ is $\mathfrak{h}$-invariant: $\Omega \in Z^1(\mathfrak{g}, \mathfrak{h}; V)$. Moreover, $H^1(\mathfrak{g}, \mathfrak{h}; V) \subset H^1(\mathfrak{g}; V)$.

2) If $H^1(\mathfrak{g}, \mathfrak{h}; V) = 0$ then $\Omega \in B^1(\mathfrak{g}; V)$ if and only if $\Omega|_{\mathfrak{h}} \in B^1(\mathfrak{h}; V)$. Thus, in this case, the space $H^1(\mathfrak{g}; V)$ is characterized by the space $H^1(\mathfrak{h}; V)$.

Proof. 1) Indeed, for any $x \in \mathfrak{g}$, we have

$$(-1)^{a_0}a \cdot \Omega(x) - (-1)^{x(a + \Omega)}x \cdot \Omega(a) = \Omega([a, x]) = 0$$

Moreover, for $\Omega \in Z^1(\mathfrak{g}, \mathfrak{h}; V)$ we denote by $\overline{\Omega}$ (respectively $\tilde{\Omega}$) the class of $\Omega$ up to $B^1(\mathfrak{g}, \mathfrak{h}; V)$ (respectively $B^1(\mathfrak{g}; V)$). We easily check that the map $\overline{\Omega} \mapsto \tilde{\Omega}$, from $H^1(\mathfrak{g}, \mathfrak{h}; V)$ to $H^1(\mathfrak{g}; V)$, is injective: if $\Omega = \partial v$ with $v \in V^\mathfrak{h}$ then $v \in V^\mathfrak{h}$ since $\Omega|_{\mathfrak{h}} = 0$.

2) Obviously if $\Omega \in B^1(\mathfrak{g}; V)$ then $\Omega|_{\mathfrak{h}} \in B^1(\mathfrak{h}; V)$. Inversely, if $\Omega|_{\mathfrak{h}} \in B^1(\mathfrak{h}; V)$ then, up to a coboundary, $\Omega$ vanishes on $\mathfrak{h}$, therefore, $\Omega \in Z^1(\mathfrak{g}, \mathfrak{h}; V)$. But $H^1(\mathfrak{g}, \mathfrak{h}; V) = \{0\} \subset H^1(\mathfrak{g}; V)$, so, $\Omega \in B^1(\mathfrak{g}; V)$. $\square$

3.2 The space $H^1_{\text{diff}}(\text{aff}(1), D_{\lambda, \mu})$

Let $\mu \in \mathbb{R}, \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, we consider $\delta = \mu - \sum_{i=1}^n \lambda_i$ and $|\alpha| = \sum \alpha_i$. For $F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_\lambda \otimes \cdots \otimes \mathcal{F}_\lambda$, we define

$$F^{(\alpha)} := f_1^{(\alpha_1)} \cdots f_n^{(\alpha_n)}.$$  

Recall that the space $\mathcal{F}_\lambda \otimes \cdots \otimes \mathcal{F}_\lambda$ is an $\text{aff}(1)$-module:

$$X_h \cdot F := L^\lambda_{X_h}(F) = \sum_{i=1}^n f_1 \otimes \cdots \otimes L^\lambda_{X_h}(f_i) \otimes \cdots \otimes f_n.$$  

The following lemma gives the general form of any 1-cocycle.

Lemma 3.1. Up to a coboundary, any 1-cocycle $\Omega \in Z^1_{\text{diff}}(\text{aff}(1), D_{\lambda, \mu})$ can be expressed as follows. For all $F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ and for all $X_h \in \text{aff}(1)$:

$$\Omega(X_h, F) = \sum_{\alpha} B_{\alpha} h^\prime F^{(\alpha)},$$  

where the $B_{\alpha}$ are constants.

Proof. Any 1-cocycle on $\text{aff}(1)$ should retains the following general form:

$$\Omega(X_h, F) = \sum_{\alpha} N_{\alpha} h F^{(\alpha)} + \sum_{\alpha} M_{\alpha} h^\prime F^{(\alpha)},$$  

where $N_{\alpha}$ and $M_{\alpha}$ are, a priori, functions. First, we prove that the terms in $h$ can be annihilated by adding a coboundary. Let $b : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \to \mathcal{F}_\mu$ be a $n$-ary differential operator defined by

$$b(F) = \sum_{\alpha} D_{\alpha} F^{(\alpha)},$$  

where
We have
\[ \partial b(X_h, F) = h(b(F))' + \mu h' b(F) - b(X_h \cdot F) \]
\[ = \sum_{\alpha} D'_{\alpha} h F^{(\alpha)} + \sum_{\alpha} (\delta - |\alpha|) D_{\alpha} h' F^{(\alpha)} \]  
(3.2)

Thus, if \( D'_{\alpha} = N_{\alpha} \) then \( \Omega - \partial b \) does not contain terms in \( h \). So, we can replace \( \Omega \) by \( \Omega - \partial b \). That is, up to a coboundary, any 1-cocycle on \( \text{aff}(1) \) can be expressed as follows:
\[ \Omega(X_h, F) = \sum_{\alpha} B_{\alpha} h' F^{(\alpha)} . \]

Now, consider the 1-cocycle condition:
\[ \Omega([X_{h_1}, X_{h_2}], F) - X_{h_1} \cdot \Omega(X_{h_2}, F) + X_{h_2} \cdot \Omega(X_{h_1}, F) = 0, \]
where \( X_{h_1}, X_{h_2} \in \text{aff}(1) \). That is,
\[ \sum_{\alpha} B'_{\alpha} (h_1 h_2' - h_1' h_2) F^{(\alpha)} = 0. \]

So, for all \( \alpha \), we have \( B'_{\alpha} = 0 \).
\[ \square \]

**Theorem 3.2.** 1) If \( \delta \not\in \mathbb{N} \) then \( H_{\text{diff}}^1(\text{aff}(1); D_{\lambda,\mu}) = 0. \)

2) If \( \delta \in \mathbb{N} \) then, up to a coboundary, any 1-cocycle \( c \in Z_{\text{diff}}^1(\text{aff}(1); D_{\lambda,\mu}) \) can be expressed as follows. For all \( F = f_1 \otimes \cdots \otimes f_n \in \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n} \) and for all \( X_h \in \text{aff}(1) \):
\[ \Omega(X_h, F) = \sum_{|\alpha| = \delta} B_{\alpha} h' F^{(\alpha)}, \]
(3.3)

Proof. 1) Indeed, according to Lemma 3.1, we can easily show the 1-cocycle \( \Omega \) defined by (3.1) is nothing but the operator \( \partial b \) where
\[ b(F) = \sum_{\alpha} \frac{B_{\alpha}}{\delta - |\alpha|} F^{(\alpha)}, \]

2) Consider the 1-cocycle \( \Omega \) defined by (3.1) and consider the operator \( \partial b \) where
\[ b(F) = \sum_{|\alpha| \not\equiv \delta} \frac{B_{\alpha}}{\delta - |\alpha|} F^{(\alpha)}. \]

We easily show that
\[ (\Omega - \partial b)(X_h, F) = \sum_{|\alpha| = \delta} B_{\alpha} h' F^{(\alpha)}. \]
\[ \square \]

**Theorem 3.3.** If \( \delta = k \in \mathbb{N} \) then
\[ \dim H_{\text{diff}}^1(\text{aff}(1); D_{\lambda,\mu}) = \binom{n + k - 1}{k}. \]

Proof. According to Theorem 3.2 and to the formula (3.2), the classes of 1-cocycles \( \Omega^\alpha \) defined by \( \Omega^\alpha(X_h, F) = h' F^{(\alpha)} \), where \( |\alpha| = k \), constitute a basis of \( H_{\text{diff}}^1(\text{aff}(1); D_{\lambda,\mu}) \). Thus, \( \dim H_{\text{diff}}^1(\text{aff}(1); D_{\lambda,\mu}) \) is the cardinal of the set \( \{ \alpha \in \mathbb{N}^n, |\alpha| = k \} \).
4 Relationship between $H^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda,\mu})$ and $H^1_{\text{diff}}(\text{aff}(1), D_{\lambda,\mu})$

We need to present here some results illustrating the analogy between the cohomology spaces in super and classical settings. We consider

$$E = \{ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n), \varepsilon_i = 0, \frac{1}{2}, \text{ and } |\varepsilon| \in \mathbb{N} \},$$

$$O = \{ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n), \varepsilon_i = 0, \frac{1}{2}, \text{ and } |\varepsilon| \notin \mathbb{N} \}.$$

**Proposition 4.1.**

1) As a $\text{aff}(1)$-module, we have

$$\mathfrak{A}_\mu \simeq F_\mu \oplus \Pi(F_{\mu+\frac{1}{2}}) \quad \text{and} \quad \text{aff}(1|1) \simeq \text{aff}(1) \oplus \Pi(\mathbb{R} dx^{-\frac{1}{2}}),$$

where $\mu \in \mathbb{R}$ and $\Pi$ is the change of parity.

2) For $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, as a $\text{aff}(1)$-module, we have, for the homogeneous relative parity components:

$$(\mathcal{D}_{\lambda,\mu})_0 \simeq \bigoplus_{\varepsilon \in E} D_{(\lambda+\varepsilon,\mu)} \oplus \bigoplus_{\varepsilon \in O} D_{(\lambda+\varepsilon,\mu+\frac{1}{2})} \quad \text{and} \quad (\mathcal{D}_{\lambda,\mu})_1 \simeq \Pi \left( \bigoplus_{\varepsilon \in O} D_{(\lambda+\varepsilon,\mu)} \oplus \bigoplus_{\varepsilon \in E} D_{(\lambda+\varepsilon,\mu+\frac{1}{2})} \right).$$

(4.1)

Proof. The first statement is immediately deduced from (2.1) and from the fact that $K(1) \simeq F_{-\frac{1}{2}}$. The second statement can be deduced from the first one (see [1]).

Now, in order to describe $H^1(\text{aff}(1|1), \mathcal{D}_{\lambda,\mu})$, we need first to describe the $\text{aff}(1)$-relative cohomology space $H^1_{\text{diff}}(\text{aff}(1|1), \text{aff}(1); \mathcal{D}_{\lambda,\mu})$. So, we shall need the following description of some $\text{aff}(1)$-invariant mappings.

**Lemma 4.1.** Let

$$A : \mathbb{R} dx^{-\frac{1}{2}} \otimes F_{\lambda_1} \otimes \cdots \otimes F_{\lambda_n} \rightarrow F_\mu,$$

$$(adx^{-\frac{1}{2}}, f_1 dx^{\lambda_1}, \ldots, f_n dx^{\lambda_n}) \rightarrow A(a, f_1, \ldots, f_n)(dx)^\mu$$

be an $n+1$-ary differential operator and let $\delta = \mu - \sum i \lambda_i$. If $A$ is a nontrivial $\text{aff}(1)$-invariant operator then

$$\delta + \frac{1}{2} \in \mathbb{N}.$$

The corresponding operator $A$ is of the form:

$$A(a, f_1, \ldots, f_n) = \sum_{|\alpha| = \delta + \frac{1}{2}} c_\alpha F^{(\alpha)},$$

where the $c_\alpha$ are constants.

Proof. The invariance with respect the vector field $X_1 = \partial_x$ yields that $A$ must be expressed with constant coefficients. Thus, the operator $A$ can be expressed as follows:

$$A(a, f_1, \ldots, f_n) = \sum_\alpha c_\alpha F^{(\alpha)},$$

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The equations (4.4) and (4.5) express the $\text{aff}_X$ to the decomposition (4.1) the map $\Omega:\Omega$ of $c$ where the $\text{aff}_X$ to the decomposition (4.1) the map $\Omega:\Omega$ of $c$

The invariance property of $A$ with respect the vector fields $X_{\varepsilon}$ reads:

$$0 = x(A(f_1,\ldots,f_n))' + (\mu + \frac{1}{2})A(f_1,\ldots,f_n) - A(xf_1' + \lambda_1f_1, f_2,\ldots,f_n)
$$

$$- A(f_1,xf_2' + \lambda_2f_2, f_3,\ldots,f_n) - \cdots - A(f_1, f_2,\ldots,xf_n' + \lambda_nf_n).$$

(4.3)

Consider any non vanishing coefficient $c_\alpha$ and consider terms in $F^{(\alpha)}$ in (4.3), we get

$$\delta + \frac{1}{2} = |\alpha|.$$ 

\[ \square \]

**Proposition 4.2.** The $\text{aff}(1)$-relative cohomology spaces $H^1_{\text{diff}}(\text{aff}(1)|1), \text{aff}(1); \mathcal{D}_{\lambda,\mu})$ are all trivial. That is, any 1-cocycle $\Omega$ is a coboundary over $\text{aff}(1)|1$ if and only if its restriction to $\text{aff}(1)$ is a coboundary over $\text{aff}(1)$.

Proof. First, it is well known that the space $H^1_{\text{diff}}(\text{aff}(1)|1), \text{aff}(1); \mathcal{D}_{\lambda,\mu})$ is nothing but the space of cohomology classes of 1-cocycles vanishing on $\text{aff}(1)$ [4].

Let $\Omega$ be a 1-cocycle vanishing on $\text{aff}(1)$, then, by the 1-cocycle condition, we have:

$$X_1 \cdot \Omega(X_{\theta}) - \Omega([X_1,X_{\theta}]) = 0,$$

(4.4)

$$X_x \cdot \Omega(X_{\theta}) - \Omega([X_x,X_{\theta}]) = 0,$$

(4.5)

$$X_{\theta} \cdot \Omega(X_{\theta}) = 0.$$ 

(4.6)

The equations (4.4) and (4.5) express the $\text{aff}(1)$-invariance property of the map $\Omega$. According to the decomposition (4.1) the map $\Omega(X_{\theta})$ is decomposed into some $\text{aff}(1)$-invariant elements $\Omega$ of $\bigoplus_{\varepsilon \in \mathcal{E}} D_{\lambda+\varepsilon,\mu} \oplus \bigoplus_{\varepsilon \in \mathcal{E}} D_{\lambda+\varepsilon,\mu + \frac{1}{2}}$ or $\Pi \left( \bigoplus_{\varepsilon \in \mathcal{E}} D_{\lambda+\varepsilon,\mu} \oplus \bigoplus_{\varepsilon \in \mathcal{E}} D_{\lambda+\varepsilon,\mu + \frac{1}{2}} \right)$, according to that $\Omega$ is odd or even. Therefore, the expressions of these maps are given by Lemma 4.1 in fact, the change of parity functor $\Pi$ commutes with the the $\text{aff}(1)$-action.

So, if $\Omega$ is an even 1-cocycle then we must have $\delta \in \mathbb{N}$ and if $\Omega$ is an odd 1-cocycle then we must have $\delta + \frac{1}{2} \in \mathbb{N}$. Otherwise, the operator $\Omega$ is identically the zero map.

If $\Omega$ is an even 1-cocycle then $\Omega(X_{\theta})$ is odd, so

$$\Omega(X_{\theta}) = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_{\varepsilon} + \frac{1}{2}} C_\alpha^{\varepsilon} \Omega_\alpha^{\varepsilon} + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_{\varepsilon} + 1} D_\alpha^{\varepsilon} \Omega_\alpha^{\varepsilon},$$

where, if $F_i = f_i + \theta g_i$, then

$$\Omega_\varepsilon^{\alpha}(F_1,\ldots,F_n) = H_\varepsilon^{(\alpha)}$$

with

$$H_\varepsilon^{(\alpha)} = h_1^{(\alpha_1)} \cdots h_n^{(\alpha_n)}$$

where $h_i = \begin{cases} f_i & \text{if } \varepsilon_i = 0 \\ g_i & \text{if } \varepsilon_i = \frac{1}{2} \end{cases}$

**Lemma 4.2.** For any $X_h \in \text{aff}(1)$ we have

$$\partial \theta \Omega_\varepsilon^{\alpha}(X_h) = (\delta_{\varepsilon} + \frac{1}{2} - |\alpha|)\theta \Omega_\varepsilon^{\alpha},$$

where $\Omega_\varepsilon^{\alpha}$ is viewed as element of $\mathcal{D}_{\lambda,\mu}$.
Proof. By a straightforward computation. We can also see $\Omega^\alpha_\varepsilon$ as element of $D_{\lambda+\varepsilon,\mu+\frac{1}{2}}$, so, we deduce the result from (3.2).

Now, since $X_\theta = \frac{1}{2} D$, then by the equation (4.6) we have

$$D(\Omega(X_\theta, F_1, \ldots, F_n)) = -\sum_{i=1}^{n} \xi_i \Omega(X_\theta, F_1, \ldots, D(F_i), \ldots, F_n)$$

(4.7)

where

$$\xi_i = (-1)^{\sum_{j=1}^{i-1} F_j}.$$

For $\varepsilon \in \mathcal{E} \cup \mathcal{O}$ and $i \in \{1, \ldots, n\}$ we consider the element $\varepsilon^i = \eta \in \mathcal{E} \cup \mathcal{O}$ defined by

$$\eta_i \neq \varepsilon_i \text{ and } \eta_j = \varepsilon_j \text{ for } j \neq i$$

and we define $\xi^i_\varepsilon$ by

$$\xi^i_\varepsilon = (-1)^{2 \sum_{j<i} \varepsilon_j}.$$

Note that

$$\xi^{i}_\varepsilon = \xi^{i}_\varepsilon^i.$$  (4.8)

For $\alpha \in \mathbb{N}^n$, we define

$$\alpha^i = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n) \text{ and } \alpha^{\overline{i}} = (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n).$$

Note that

$$(\varepsilon^i)^i = \varepsilon, \quad (\alpha^i)^i = (\alpha^{\overline{i}})^i = \alpha \text{ and } \varepsilon \in \mathcal{E} \iff \varepsilon^i \in \mathcal{O}.$$  (4.9)

If $F = f + \theta g$ then $D(F) = g + \theta f'$. Thus,

$$\Omega^\alpha_\varepsilon(F_1, \ldots, D(F_i), \ldots, F_n) = \Omega^{\alpha^i}_\varepsilon(F_1, \ldots, \ldots, F_n).$$

where

$$\alpha^i_\varepsilon = \left\{ \begin{array}{ll} \alpha & \text{if } \varepsilon_i = 0 \\ \alpha^i & \text{if } \varepsilon_i = \frac{1}{2} \end{array} \right..$$

Define $\alpha^{\overline{i}}_\varepsilon$ so that

$$(\alpha^{\overline{i}}_\varepsilon)^i = \alpha,$$

(4.10)

that is

$$\alpha^{\overline{i}}_\varepsilon = \left\{ \begin{array}{ll} \alpha & \text{if } \varepsilon_i = \frac{1}{2} \\ \alpha^{\overline{i}} & \text{if } \varepsilon_i = 0 \end{array} \right..$$

Thus, the equation (4.7) becomes

$$\sum_{\varepsilon \in \mathcal{E}, \alpha} D^\varepsilon_\alpha \Omega^\alpha_\varepsilon + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} C^\varepsilon_\alpha \sum_{i=1}^{n} \Omega^{\alpha^i_i}_{\varepsilon_i} = -\sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_\alpha \sum_{i=1}^{n} \xi^i_\varepsilon C^{\overline{i}}_{\varepsilon_i} \Omega^{\alpha^i_i}_{\varepsilon_i} - \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} D^\varepsilon_\alpha \sum_{i=1}^{n} \xi^i_\varepsilon C^{\overline{i}}_{\varepsilon_i} \Omega^{\alpha^i_i}_{\varepsilon_i}$$

(4.11)

or equivalently (according to (4.9), (4.8) and (4.10))

$$\sum_{\varepsilon \in \mathcal{E}, \alpha} D^\varepsilon_\alpha \Omega^\alpha_\varepsilon + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^{n} C^\varepsilon_{\alpha^i} \right) \Omega^\alpha_\varepsilon = -\sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^{n} \xi^i_\varepsilon C^{\overline{i}}_{\varepsilon_i} \right) \Omega^\alpha_\varepsilon - \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^{n} \xi^i_\varepsilon D^{\overline{i}}_{\varepsilon_i} \right) \Omega^\alpha_\varepsilon$$

(4.12)
Thus, the coefficients $C_{\alpha}^\varepsilon$ must satisfy the following conditions
\[
\begin{align*}
D_{\alpha}^\varepsilon &= -\sum_{i=1}^{n} \xi_i C_{\alpha}^{\varepsilon i} \quad \text{for} \quad \varepsilon \in \mathcal{E} \\
\sum_{i=1}^{n} C_{\alpha}^{\varepsilon i} &= -\sum_{i=1}^{n} \xi_i D_{\alpha}^{\varepsilon i} \quad \text{for} \quad \varepsilon \in \mathcal{O}.
\end{align*}
\] (4.13)

Thus,
\[
\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, \alpha} C_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha - \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^{n} \xi_i C_{\alpha}^{\varepsilon i} \right) \Omega_\varepsilon^\alpha = \sum_{\varepsilon \in \mathcal{O}, \alpha} C_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha - \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \sum_{i=1}^{n} \xi_i \Omega_\varepsilon^{\alpha i}
\]

with
\[
\sum_{i=1}^{n} C_{\alpha}^\varepsilon = \sum_{i,j} \xi_i \xi_j C_{\alpha}^{ij},
\] (4.14)

where $\alpha_\varepsilon = (\alpha_{\varepsilon i})_{i=1}^n$. But, it is easy to see that
\[
\xi_i \xi_j = -\xi_j \xi_i, \quad \varepsilon^{ii} = \varepsilon \quad \text{and} \quad \alpha_{\varepsilon i} = \alpha_i.
\]

Therefore the relation (4.14) is trivial. Of course, we agree that if $\alpha_\varepsilon \notin \mathbb{N}^n$ or $\alpha_i \notin \mathbb{N}^n$ then the correspondent coefficients $C_{\alpha}^{\varepsilon i}$ or $C_{\alpha}^{\varepsilon i}$ are zero.

Now, let $B \in (\mathcal{Q}, \theta)\bar{a}$, according to the decomposition (4.11), we can write
\[
B = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + 1} B_{\varepsilon, \alpha}^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + 1+\frac{1}{2}} R_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha.
\]

\[
\partial B(X_\theta) = \frac{1}{2} \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + 1+\frac{1}{2}} \left( R_{\alpha}^\varepsilon - \sum_{i=1}^{n} \xi_i B_{\alpha}^{\varepsilon i} \right) \Omega_\varepsilon^\alpha + \frac{1}{2} \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + 1} \left( \sum_{i=1}^{n} R_{\alpha}^{\varepsilon i} - \sum_{i=1}^{n} \xi_i R_{\alpha}^{\varepsilon i} \right) \Omega_\varepsilon^\alpha.
\] (4.15)

Thus, according to Lemma 4.2 we have
\[
\Omega = \partial B
\]

where
\[
B = 2\theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + 1+\frac{1}{2}} C_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha.
\]

Now, if $\Omega$ is an odd 1-cocycle then $\delta + \frac{1}{2} \in \mathbb{N}$ and $\Omega(X_\theta)$ is even, so
\[
\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + 1+\frac{1}{2}} C_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + 1} D_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha.
\]

By the equation (4.10) we have
\[
D(\Omega(X_\theta, F_1, \ldots, F_n)) = \sum_{i=1}^{n} \Omega(X_\theta, F_1, \ldots, D(F_i), \ldots, F_n).
\] (4.16)

Thus, the equation (4.17) becomes
\[
\sum_{\varepsilon \in \mathcal{O}, \alpha} D_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \sum_{i=1}^{n} C_{\alpha}^\varepsilon \Omega_\varepsilon^{\alpha i} = \sum_{\varepsilon \in \mathcal{E}, \alpha} C_{\alpha}^\varepsilon \sum_{i=1}^{n} \xi_i \Omega_\varepsilon^{\alpha i} + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \sum_{i=1}^{n} \xi_i \Omega_\varepsilon^{\alpha i}.
\] (4.17)
or equivalently (according to (1.9) and (1.10))

\[
\sum_{\varepsilon \in \mathcal{O}, \alpha} D^\varepsilon_{\alpha} \Omega^\varepsilon + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^{n} C^\varepsilon_{\alpha} \right) \Omega^\varepsilon = \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^{n} \xi^i_{\varepsilon} C^\varepsilon_{\alpha} \right) \Omega^\varepsilon + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^{n} \xi^i_{\varepsilon} D^\varepsilon_{\alpha} \right) \Omega^\varepsilon \quad (4.18)
\]

Thus, the coefficients \( C^\varepsilon_{\alpha} \) must satisfy the following conditions

\[
D^\varepsilon_{\alpha} = \sum_{i=1}^{n} \xi^i_{\varepsilon} C^\varepsilon_{\alpha} \quad \text{for} \quad \varepsilon \in \mathcal{O}. \tag{4.19}
\]

Thus,

\[
\Omega(X_{\theta}) = \sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_{\alpha} \Omega^\varepsilon + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( \sum_{i=1}^{n} \xi^i_{\varepsilon} C^\varepsilon_{\alpha} \right) \Omega^\varepsilon = \sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_{\alpha} \Omega^\varepsilon + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_{\alpha} \sum_{i=1}^{n} \xi^i_{\varepsilon} \Omega^\varepsilon^i \Omega^\varepsilon.
\]

Now, let \( B \in (\mathcal{D}_{\lambda, \mu})_1 \), according to the decomposition (4.1), we can write

\[
B = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + 1} B^\varepsilon_{\alpha} \Omega^\varepsilon + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} R^\varepsilon_{\alpha} \Omega^\varepsilon.
\]

\[
\partial B(X_{\theta}) = \frac{1}{2} \sum_{\varepsilon \in \mathcal{O}, \alpha} \left( R^\varepsilon_{\alpha} + \sum_{i=1}^{n} \xi^i_{\varepsilon} B^\varepsilon_{\alpha} \right) \Omega^\varepsilon + \frac{1}{2} \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} \left( \sum_{i=1}^{n} B^\varepsilon_{\alpha} + \sum_{i=1}^{n} \xi^i_{\varepsilon} R^\varepsilon_{\alpha} \right) \Omega^\varepsilon. \quad (4.20)
\]

Thus, according to Lemma 4.2 we have

\[
\Omega = \partial B
\]

where

\[
B = 2\theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} C^\varepsilon_{\alpha} \Omega^\varepsilon.
\]

**Corollary 4.3.** Any 1-cocycle \( \Omega \in Z^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu}) \) of \( \text{aff}(1|1) \) is a coboundary if and only if its restriction to \( \text{aff}(1) \) is a coboundary.

## 5 The space \( H^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu}) \)

According to Theorem 3.2 the restriction of any 1-cocycle \( \Omega \in Z^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu}) \) to \( \text{aff}(1) \) has the following structure:

**Proposition 5.1.** 1) If \( 2\delta \notin \mathbb{N} \) then \( H^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu}) = 0 \).

2) If \( \delta \in \mathbb{N} \) then, up to a coboundary, any 1-cocycle \( \Omega \in Z^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu}) \) is even and its restriction to \( \text{aff}(1) \) has the following form:

\[
\Omega(X_{h}) = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon} C^\varepsilon_{\alpha} h^\varepsilon \Omega^\varepsilon + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} D^\varepsilon_{\alpha} h^\varepsilon \Omega^\varepsilon
\]

where \( \delta_\varepsilon = \mu - |\lambda + \varepsilon| \).
3) If $\delta \in \mathbb{N} + \frac{1}{2}$ then, up to a coboundary, any 1-cocycle $\Omega \in Z^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu})$ is odd and its restriction to $\text{aff}(1)$ has the following form:

$$
\Omega(X_h) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon} C^\varepsilon_\alpha h^\alpha \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} D^\varepsilon_\alpha h^\alpha \Omega^\varepsilon_\alpha.
$$

**Theorem 5.1.**  
1) If $2\delta \notin \mathbb{N}$ then $H^1_{\text{diff}}(\text{aff}(1|1), \mathcal{D}_{\lambda, \mu}) = 0$.

2) If $\delta = k \in \mathbb{N}$ then

$$
\dim H^1(\text{aff}(1|1); \mathcal{D}_{\lambda, \mu}) = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2r} \binom{n+k-r-1}{k-r}.
$$

3) If $\delta = k + \frac{1}{2} \in \mathbb{N} + \frac{1}{2}$ then

$$
\dim H^1(\text{aff}(1|1); \mathcal{D}_{\lambda, \mu}) = \sum_{r=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2r+1} \binom{n+k-r-1}{k-r}.
$$

A basis of $H^1(\text{aff}(1|1); \mathcal{D}_{\lambda, \mu})$ is given by the family $\Gamma^\alpha_\varepsilon$ defined on $\text{aff}(1)$ by

$$
\Gamma^\alpha_\varepsilon(X_h) = h^\alpha \Omega^\varepsilon_\alpha + \theta \sum_{i=1}^{n} \xi_i h^\alpha \Omega^\xi_i^{\alpha_i}
$$

where $\varepsilon \in \mathcal{E}$ and $|\alpha| = \delta_\varepsilon$ if $\delta \in \mathbb{N}$ and $\varepsilon \in \mathcal{O}$ and $|\alpha| = \delta_\varepsilon + \frac{1}{2}$ if $\delta \in \mathbb{N} + \frac{1}{2}$.

**Proof.** If $\delta \in \mathbb{N}$ then, according to Proposition 5.1, we have for any $X_h \in \text{aff}(1)$

$$
\Omega(X_h) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_\varepsilon} C^\varepsilon_\alpha h^\alpha \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_\varepsilon + \frac{1}{2}} D^\varepsilon_\alpha h^\alpha \Omega^\varepsilon_\alpha.
$$

In this case, $X_\theta$ is an odd operator, so, it is of the following form:

$$
\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, \alpha} B^\varepsilon_\alpha \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} R^\varepsilon_\alpha \Omega^\varepsilon_\alpha.
$$

Using the 1-cocycle conditions we seek to establish relationships that must be satisfied by the coefficients $B^\varepsilon_\alpha, R^\varepsilon_\alpha, C^\varepsilon_\alpha, \ldots$.

The 1-cocycle condition reads:

$$
X_x \cdot \Omega(X_\theta) - X_\theta \cdot \Omega(X_x) + \frac{1}{2} \Omega(X_\theta) = 0, \tag{5.1}
$$

where

- $X_x \cdot \Omega(X_\theta) = \Omega^\mu_{X_x} \circ \Omega(X_\theta) - \Omega(X_\theta) \circ \Omega^\lambda_{X_x}$,
- $\Omega^\varepsilon_\alpha \circ \Omega^\lambda_{X_x} = \sum_{i=1}^{n} \left( x \Omega^\alpha_i + (\alpha_i + \lambda_i + \varepsilon_i) \Omega^\varepsilon_i \right)$,
- $\Omega^\mu_{X_x} \circ \Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, \alpha} B^\varepsilon_\alpha L^\mu_{X_x} \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} R^\varepsilon_\alpha L^\mu_{X_x} \Omega^\varepsilon_\alpha$,
- $L^\mu_{X_x} \Omega^\varepsilon_\alpha = \sum_{i=1}^{n} \Omega^\varepsilon_i + \mu \Omega^\varepsilon_i$,
- $2X_\theta \cdot \Omega(X_x) = \sum_{\varepsilon \in \mathcal{O}, \alpha} D^\varepsilon_\alpha \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_\alpha \sum_{i=1}^{n} \Omega^\varepsilon_i$.

Using the 1-cocycle conditions we seek to establish relationships that must be satisfied by the coefficients $B^\varepsilon_\alpha, R^\varepsilon_\alpha, C^\varepsilon_\alpha, \ldots$.

The 1-cocycle condition reads:

$$
X_x \cdot \Omega(X_\theta) - X_\theta \cdot \Omega(X_x) + \frac{1}{2} \Omega(X_\theta) = 0, \tag{5.1}
$$

where

- $X_x \cdot \Omega(X_\theta) = \Omega^\mu_{X_x} \circ \Omega(X_\theta) - \Omega(X_\theta) \circ \Omega^\lambda_{X_x}$,
- $\Omega^\varepsilon_\alpha \circ \Omega^\lambda_{X_x} = \sum_{i=1}^{n} \left( x \Omega^\alpha_i + (\alpha_i + \lambda_i + \varepsilon_i) \Omega^\varepsilon_i \right)$,
- $\Omega^\mu_{X_x} \circ \Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, \alpha} B^\varepsilon_\alpha L^\mu_{X_x} \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} R^\varepsilon_\alpha L^\mu_{X_x} \Omega^\varepsilon_\alpha$,
- $L^\mu_{X_x} \Omega^\varepsilon_\alpha = \sum_{i=1}^{n} \Omega^\varepsilon_i + \mu \Omega^\varepsilon_i$,
- $2X_\theta \cdot \Omega(X_x) = \sum_{\varepsilon \in \mathcal{O}, \alpha} D^\varepsilon_\alpha \Omega^\varepsilon_\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_\alpha \sum_{i=1}^{n} \Omega^\varepsilon_i - \sum_{\varepsilon \in \mathcal{E}, \alpha} C^\varepsilon_\alpha \sum_{i=1}^{n} \Omega^\varepsilon_i$.
Therefore, the relation (5.1) is satisfied if and only if

\[
\begin{cases}
(\delta_x - |\alpha| + \frac{1}{2})B_\alpha^\varepsilon - \frac{1}{2}D_\alpha^\varepsilon + \frac{1}{2} \left( \sum_{i=1}^{n} \xi_i C_{\alpha^i}^\varepsilon \right) = 0 \\
(\delta_x - |\alpha| + 1)R_\alpha^\varepsilon - \frac{1}{2} \sum_{i=1}^{n} C_{\alpha^i}^\varepsilon + \frac{1}{2} \sum_{i=1}^{n} \xi_i D_{\alpha^i}^\varepsilon = 0.
\end{cases}
\]

Thus, for \(|\alpha| = \delta_x + \frac{1}{2}\) we have

\[D_\alpha^\varepsilon = \sum_{i=1}^{n} \xi_i C_{\alpha^i}^\varepsilon\]

and for \(|\alpha| \neq \delta_x + \frac{1}{2}\) we have \(|\alpha| = \delta_x + \frac{1}{2}\) and \(C_{\alpha^i}^\varepsilon\) appear only for \(|\alpha| = \delta_x\). Therefore, \(B_\alpha^\varepsilon = 0\) for \(|\alpha| \neq \delta_x + \frac{1}{2}\).

For \(|\alpha| = \delta_x + 1\) we have

\[\sum_{i=1}^{n} C_{\alpha^i}^\varepsilon = \sum_{i=1}^{n} \xi_i D_{\alpha^i}^\varepsilon\]

and for \(|\alpha| \neq \delta_x + 1\) we have \(C_{\alpha^i}^\varepsilon = D_{\alpha^i}^\varepsilon = 0\), therefore \(R_\alpha^\varepsilon = 0\).

Thus,

\[\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_x + \frac{1}{2}} B_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_x + 1} R_\alpha^\varepsilon \Omega_\varepsilon^\alpha\]

and

\[\Omega(X_h) = \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_x} C_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_x + \frac{1}{2}} \sum_{i=1}^{n} \xi_i C_{\alpha^i}^\varepsilon \Omega_\varepsilon^\alpha\]

From Corollary 4.3 that the dimension of \(H^1(\text{aff}(1|1); \mathfrak{D}_{\lambda,\mu})\) is equal to the number of parameters \(C_{\alpha}^\varepsilon\). That is,

\[
\text{dim}H^1(\text{aff}(1|1); D_{\lambda,\mu}) = \sum_{r=0}^{[\frac{n}{2}]} \binom{n}{2r} \binom{n + k - r - 1}{k - r},
\]

where \(\binom{n}{2r}\) is the number of \(\varepsilon\) with \(|\varepsilon| = r\) and \(\binom{n + k - r - 1}{k - r}\) is the number of \(\alpha\) with \(|\alpha| = \delta_x = k - r\).

Now, If \(\delta \in \mathbb{N} + \frac{1}{2}\) then we have

\[\Omega(X_h) = \sum_{\varepsilon \in \mathcal{O}, |\alpha| = \delta_x} C_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{E}, |\alpha| = \delta_x + \frac{1}{2}} D_{\alpha}^\varepsilon \Omega_\varepsilon^\alpha.
\]

In this case, \(X_\theta\) is an even operator, so, it is of the following form:

\[\Omega(X_\theta) = \sum_{\varepsilon \in \mathcal{E}, \alpha} B_\alpha^\varepsilon \Omega_\varepsilon^\alpha + \theta \sum_{\varepsilon \in \mathcal{O}, \alpha} R_\alpha^\varepsilon \Omega_\varepsilon^\alpha.
\]

As before we prove that

\[
\begin{cases}
(\delta_x - |\alpha| + \frac{1}{2})B_\alpha^\varepsilon - \frac{1}{2}D_\alpha^\varepsilon + \frac{1}{2} \left( \sum_{i=1}^{n} \xi_i C_{\alpha^i}^\varepsilon \right) = 0 \\
(\delta_x - |\alpha| + 1)R_\alpha^\varepsilon - \frac{1}{2} \sum_{i=1}^{n} C_{\alpha^i}^\varepsilon + \frac{1}{2} \sum_{i=1}^{n} \xi_i D_{\alpha^i}^\varepsilon = 0.
\end{cases}
\]
Thus,
\[
\dim H^1(\text{aff}(1|1); \mathfrak{D}_{\lambda,\mu}) = \sum_{r=0}^{\lfloor n/2 \rfloor - 1} \binom{n}{2r+1} \binom{n+k-r-1}{k-r},
\]
where \( \binom{n}{2r+1} \) is the number of \( \varepsilon \) with \( |\varepsilon| = r + \frac{1}{2} \) and \( \binom{n+k-r-1}{k-r} \) is the number of \( \alpha \) with \( |\alpha| = k-r \).

For \( n = 2 \), \( \dim H^1(\text{aff}(1|1); \mathfrak{D}_{\lambda,\mu}) = 2\delta + 1 \).

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