THE INDEX OF FLOER MODULI PROBLEMS FOR PARAMETRIZED ACTION FUNCTIONALS

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ABSTRACT. We define an index for the critical points of parametrized Hamiltonian action functionals. The expected dimension of moduli spaces of parametrized Floer trajectories equals the difference of indices of the asymptotes.

1. MAIN DEFINITION AND MAIN THEOREM

1.1. The parametrized action functional. Let $\Lambda$ be a manifold of dimension $m$, $(W, \omega)$ a symplectic manifold of dimension $2n$, and $H : S^1 \times W \times \Lambda \to \mathbb{R}$, $H(\theta, x, \lambda) = H_\lambda(\theta, x)$ a smooth family of Hamiltonians defined on $W$. Let $\mathcal{L}W$ denote the space of loops in $W$ and assume for simplicity that $\omega = d\alpha$ is exact. We are interested in the parametrized Hamiltonian action functional $A_H : \mathcal{L}W \times \Lambda \to \mathbb{R}$, $(\gamma, \lambda) \mapsto -\int_{S^1} \alpha - \int_{S^1} H_\lambda(\theta, \gamma(\theta))\,d\theta$.

Such functionals appear in a variety of settings (Appendix A), and we analyzed in [3] their Fredholm theory and their transversality theory.

1.2. Critical points. A pair $(\gamma, \lambda) \in \mathcal{L}W \times \Lambda$ is a critical point of $A_H$ iff it solves the system

$$\dot{\gamma}(\theta) = X_{H_\lambda}(\theta, \gamma(\theta)), \ \theta \in S^1 \text{ and } \int_{S^1} \frac{\partial H}{\partial \lambda}(\theta, \gamma(\theta), \lambda)\,d\theta = 0.$$  

Our convention for the definition of $X_{H_\lambda}$ is $\omega(X_{H_\lambda}, \cdot) = dH_\lambda$. We say that the critical point $(\gamma, \lambda)$ is nondegenerate if the Hessian $d^2A_H(\gamma, \lambda)$ is injective. If the critical points of $A_H$ are all nondegenerate (which is a generic assumption), they can be used to define a Floer chain complex whose differential is expressed as a count of rigid $L^2$-gradient trajectories [2]. The purpose of the present paper is to associate an index to each critical point of $A_H$, in such a way that the dimension of the moduli space of connecting Floer trajectories is expressed as the difference of the indices at the endpoints.

Equation (1.1) can be interpreted as follows. Every loop $\gamma : S^1 \to W$ determines a function

$$F_\gamma : \Lambda \to \mathbb{R}, \ \lambda \mapsto \int_{S^1} H(\theta, \gamma(\theta), \lambda)\,d\theta.$$  

A pair $(\gamma, \lambda)$ belongs to $\text{Crit}(A_H)$ iff $\gamma$ is a 1-periodic orbit of $X_{H_\lambda}$ and $\lambda$ is a critical point of $F_\gamma$. However, the nondegeneracy of $(\gamma, \lambda)$ does not imply that $\gamma$ is a nondegenerate orbit of $H_\lambda$, nor that $\lambda$ is a nondegenerate critical point of $F_\gamma$. This situation is already present in Morse theory, as the following example shows.
Example 1. Consider the Morse function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, \lambda) \mapsto x\lambda \). Then \((x_0, \lambda_0) = (0, 0)\) is a nondegenerate critical point, but \(f\) is constant along \(\mathbb{R} \times \{0\}\) and \(\{0\} \times \mathbb{R}\), hence \(x_0 = 0\) and \(\lambda_0 = 0\) are degenerate critical points.

It is thus not \textit{a priori} clear how to define the index of a critical point \((\gamma, \lambda)\), unless the Hamiltonian \(H\) is \textit{split}, i.e. of the form \(H(\theta, x, \lambda) = K(\theta, x) + f(\lambda)\), in which case the system \((1.1)\) is uncoupled. Our discovery is that one can define the index using a parametrized version of the Robbin-Salamon index which we now explain. Our method works in general and our approach is fundamentally different from other attempts dealing with particular cases \([11, 4]\).

1.3. The parametrized Robbin-Salamon index. Given a Hamiltonian \(H : S^1 \times W \times \Lambda \to \mathbb{R}\), we extend it to \(\widetilde{H} : S^1 \times W \times T^*\Lambda \to \mathbb{R}\) by the formula

\[
\widetilde{H}(\theta, x, (\lambda, p)) := H(\theta, x, \lambda) = H_\Lambda(\theta, x),
\]

so that

\[
X_{\widetilde{H}} = X_{H_\Lambda} - \frac{\partial H}{\partial \lambda} \frac{\partial}{\partial p}.
\]

(We use the symplectic form \(d\lambda \wedge dp\) on \(T^*\Lambda\).) A 1-periodic orbit \(\widetilde{\gamma}\) of \(X_{\widetilde{H}}\) is of the form \(\widetilde{\gamma} = (\gamma(\cdot), \lambda, p(\cdot))\), with \(\gamma\) a 1-periodic orbit of \(X_{H_\Lambda}\) and \(p(\theta) = p(0) - \int_0^\theta \frac{\partial H}{\partial \lambda}(\tau, \gamma(\tau), \lambda) d\tau\). The closing condition \(p(1) = p(0)\) is equivalent to \(\int_0^1 \frac{\partial H}{\partial \lambda}(\tau, \gamma(\tau), \lambda) d\tau = 0\), while \(p(0) \in T^*_\Lambda\) can be chosen arbitrarily. Thus critical points of \(A_H\) are in one-to-one bijective correspondence with families of 1-periodic orbits of \(X_{\widetilde{H}}\), of dimension \(\dim T^*_\Lambda = \dim \Lambda\).

We assume in this paper that

\[
\langle c_1(W), \pi_2(W) \rangle = 0
\]

and we consider only critical points \((\gamma, \lambda)\) such that \(\gamma\) is contractible in \(W\). These restrictions are only meant to focus the discussion and are by no means essential. The associated periodic orbits \(\widetilde{\gamma}\) are then contractible in \(W \times T^*\Lambda\), and we have \(\langle c_1(W \times T^*\Lambda), \pi_2(W \times T^*\Lambda) \rangle = 0\). In this situation we can associate without ambiguity to the periodic orbit \(\widetilde{\gamma}\) a half-integer called the Robbin-Salamon index. This index is defined as the Maslov index \([7]\) of the path of symplectic matrices obtained by linearizing the Hamiltonian flow of \(\widetilde{H}\) along \(\widetilde{\gamma}\) and by trivializing \(T(W \times T^*\Lambda)\) over a disc bounded by \(\widetilde{\gamma}\).

Main Definition. The parametrized Robbin-Salamon index \(\mu(\gamma, \lambda)\) of a critical point of \(A_H\) is the Robbin-Salamon index of one of the corresponding 1-periodic orbits \((\gamma(\cdot), \lambda, p(\cdot))\) of \(\widetilde{H}\).

1.4. The parametrized Floer equation. Let \(J = (J^0_\lambda), \lambda \in \Lambda, \theta \in S^1\) be a family of compatible almost complex structures on \(W\). This induces a \(\Lambda\)-family of \(L^2\)-metrics on \(C^\infty(S^1, W)\), defined by

\[
\langle \zeta, \eta \rangle_\lambda := \int_{S^1} \omega(\zeta(\theta), J^0_\lambda \eta(\theta))d\theta, \quad \zeta, \eta \in T_\gamma C^\infty(S^1, W) = \Gamma(\gamma^*TW).
\]

Such a metric can be coupled with any metric \(g\) on \(\Lambda\) and gives rise to a metric on \(C^\infty(S^1, W) \times \Lambda\) acting at a point \((\gamma, \lambda)\) by

\[
\langle (\zeta, \ell), (\eta, k) \rangle_{J, g} := \langle \zeta, \eta \rangle_\lambda + g(\ell, k), \quad (\zeta, \ell), (\eta, k) \in \Gamma(\gamma^*TW) \oplus T_\Lambda\Lambda.
\]
The parametrized Floer equation is the negative gradient equation for \( A_H \) with respect to such a metric \( \langle \cdot, \cdot \rangle_{J,g} \). More precisely, given \((\gamma, \lambda), (\gamma, \lambda) \in \text{Crit}(A_H)\) we denote by
\[
\mathcal{M}((\gamma, \lambda), (\gamma, \lambda); H, J, g)
\]
the space of parametrized Floer trajectories, consisting of pairs \((u, \lambda)\) with
\[
u : \mathbb{R} \times S^1 \to W, \quad \lambda : \mathbb{R} \to \Lambda,
\]
satisfying
\[
(1.3) \quad \partial_s u + J_{\lambda(s)}^\theta((\partial_\theta u - X_{H_{\lambda(s)}}^\theta)(u)) = 0,
\]
\[
(1.4) \quad \dot{\lambda}(s) - \int_{S^1} \nabla_\lambda H(\theta, u(s, \theta), \lambda(s))d\theta = 0,
\]
and
\[
(1.5) \quad \lim_{s \to -\infty} (u(s, \cdot), \lambda(s)) = (\gamma, \lambda), \quad \lim_{s \to +\infty} (u(s, \cdot), \lambda(s)) = (\gamma, \lambda).
\]
Here and in the sequel we use the notation \( \nabla \) for a gradient vector field, whereas \( \nabla \) will denote a covariant derivative.

1.5. The index theorem for the linearized operator. Let us fix \( p > 1 \). By linearizing equations (1.3-1.4) we obtain the operator
\[
D_{(u, \lambda)} : W^{1,p}(u^*TW) \oplus W^{1,p}(\lambda^*TA) \to L^p(u^*TW) \oplus L^p(\lambda^*TA),
\]
\[
(1.6) \quad D_{(u, \lambda)}(\zeta, \ell) := \left( D_u \zeta + (D_\lambda J \cdot \ell)((\partial_\theta u - X_{H_\lambda}(u)) - J_\lambda(D_\lambda X_{H_\lambda} \cdot \ell) \right),
\]
where
\[
D_u : W^{1,p}(u^*TW) \to L^p(u^*TW)
\]
is the usual Floer-Gromov operator
\[
D_u \zeta := \nabla_\zeta - J_\lambda \nabla_{\theta \zeta} - J_\lambda \nabla_\zeta X_{H_\lambda} + \nabla_\zeta J_\lambda(\partial_\theta u - X_{H_\lambda}).
\]
Let us denote
\[
W^{1,p} := W^{1,p}(\mathbb{R} \times S^1, u^*TW) \oplus W^{1,p}(\mathbb{R}, \lambda^*TA),
\]
\[
L^p := L^p(\mathbb{R} \times S^1, u^*TW) \oplus L^p(\mathbb{R}, \lambda^*TA).
\]
We proved in [3, Theorem 2.6] that, given \((\gamma, \lambda), (\gamma, \lambda) \in \text{Crit}(A_H)\) which are nondegenerate, and given \((u, \lambda) \in \mathcal{M}((\gamma, \lambda), (\gamma, \lambda); H, J, g)\), the operator
\[
D_{(u, \lambda)} : W^{1,p} \to L^p
\]
is Fredholm for \( 1 < p < \infty \). Moreover, for a generic choice of the triple \((H, J, g)\), the space of Floer trajectories \( \mathcal{M}((\gamma, \lambda), (\gamma, \lambda); H, J, g)\) is a smooth manifold whose local dimension at \((u, \lambda)\) is equal to \( \text{ind} D_{(u, \lambda)} \) [3, Theorem 4.1].

Main Theorem. Assume \((\gamma, \lambda), (\gamma, \lambda) \in \text{Crit}(A_H)\) are nondegenerate and fix \( 1 < p < \infty \). For any \((u, \lambda) \in \mathcal{M}((\gamma, \lambda), (\gamma, \lambda); H, J, g)\) the index of the Fredholm operator
\[
D_{(u, \lambda)} : W^{1,p} \to L^p
\]
is
\[
\text{ind} D_{(u, \lambda)} = -\mu(\gamma, \lambda) + \mu(\gamma, \lambda).
\]

\[
\mu(\gamma, \lambda) := \mu(\gamma, \lambda) := \frac{1}{2\pi} \int_{S^1} H\big|_{\gamma}(\theta) - H\big|_{\lambda}(\theta)\,d\theta.
\]
2. Proof of the main theorem

2.1. A subgroup of $\text{Sp}(2n+2m)$. Let $n, m \geq 1$ be integers and define the subgroup $\mathcal{S}_{n,m} \subset \text{Sp}(2n+2m)$ to consist of matrices of the form

$$M = M(\Psi, X, E) = \begin{pmatrix} \Psi & \Psi X & 0 \\ 0 & \mathbb{I} & 0 \\ X^T J_0 & E + \frac{1}{2}X^T J_0 X & \mathbb{I} \end{pmatrix},$$

with $\Psi \in \text{Sp}(2n)$, $X \in \text{Mat}_{2n,m}(\mathbb{R})$, and $E \in \text{Mat}_m(\mathbb{R})$ symmetric. Here we have denoted $J_0 := \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ the standard complex structure on $\mathbb{R}^{2n}$, and the elements $\Psi \in \text{Sp}(2n)$ are characterized by the condition $\Psi^T J_0 \Psi = J_0$. Similarly, we denote the standard complex structure on $\mathbb{R}^{2n} \times \mathbb{R}^{2m}$ by $\tilde{J}_0 := \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & -\mathbb{I} \\ 0 & \mathbb{I} & 0 \end{pmatrix}$, and the elements $\tilde{\Psi} \in \text{Sp}(2n+2m)$ are characterized by the condition $\tilde{\Psi}^T \tilde{J}_0 \tilde{\Psi} = \tilde{J}_0$. We have that $\mathcal{S}_{n,m}$ is a subgroup (but we shall not use this fact). The subgroup property follows from the relations

$$M(\Psi_1, X_1, E_1) \cdot M(\Psi_2, X_2, E_2) = M(\Psi_1 \Psi_2, X_2 + \Psi_2^{-1}X_1, E_1 + E_2 + \text{Sym}(X_1^T J_0 \Psi_2 X_2))$$

and

$$M(\Psi, X, E)^{-1} = M(\Psi^{-1}, -\Psi X, -E).$$

Here we have used the notation

$$\text{Sym}(P) := (P + P^T)/2$$

for the symmetric part of a square matrix $P$.

The form of the elements of $\mathcal{S}_{n,m}$ may seem less artificial in view of the following Lemma. Elements of the form (2.3) arise naturally in the next section.

**Lemma 2.** Let

$$M = \begin{pmatrix} \Psi & A & 0 \\ 0 & \mathbb{I} & 0 \\ B & C & \mathbb{I} \end{pmatrix}$$

be a square $(2n+2m)$-matrix, such that $\Psi$ is a square $2n$-matrix and $\mathbb{I}$ is the identity $m$-matrix. Then $M$ is symplectic if and only if $\Psi$ is symplectic and there exists a matrix $X$ and a symmetric matrix $E$ such that $M = M(\Psi, X, E)$.

**Proof.** The proof is a straightforward computation using block matrices and the condition $M^T \tilde{J}_0 M = \tilde{J}_0$. \qed

We refer to Appendix B for a summary of the properties of the Robbin-Salamon index of paths with values in $\mathcal{S}_{n,m}$. 
2.2. The linearized flow of $\tilde{H}$. Recall from §1.3 the Hamiltonian

$$\tilde{H} : S^1 \times W \times T^* \Lambda \to \mathbb{R}, \quad \tilde{H}(\theta, x, (\lambda, p)) := H(\theta, x, \lambda),$$

whose flow is given by

$$\varphi_{\tilde{H}}^\theta(x, \lambda, p) = \left( \varphi_{H_\lambda}^\theta(x), \lambda, p - \int_0^\theta \frac{\partial H}{\partial \lambda}(\tau, \varphi_{\tilde{H}_\lambda}^\theta(x), \lambda) \, d\tau \right).$$

Let $(\gamma, \lambda) \in \text{Crit}(A_H)$ be a critical point and $\tilde{g} = (\gamma(\cdot), \lambda, p(\cdot))$ be an associated 1-periodic orbit of $X_{\tilde{H}}$. We fix a unitary trivialization of $\gamma^*TW$ coming from a spanning disc and we fix an isometry $T_{\Lambda} \equiv \mathbb{R}^m$, and these together determine a unitary trivialization of $\tilde{g}^*T(W \times T^* \Lambda)$. The linearized flow $d\varphi_{\tilde{H}}^\theta$ read in such a trivialization determines a path $M(\theta, \theta) \in [0,1]$ of symplectic matrices of the form (2.3), and this path takes values in $S_{n, m}$ by Lemma 2. By definition, the index $\mu(\gamma, \lambda)$ is equal to the Robbin-Salamon index of the path $M$.

The matrices $\Psi(\theta), X(\theta)$, and $E(\theta)$ that determine $M(\theta) = M(\Psi(\theta), X(\theta), E(\theta))$ are expressed as follows. We denote $\Psi$ and $A$ the components of the linearization of the flow $\varphi_{H_\lambda}^\theta$ in the given trivializations of $\gamma^*TW$ and of $T_{\Lambda}$, and set $X := \Psi^{-1}A$. Thus the linearized flow acts as

$$T_{(\gamma(0), \lambda)}(W \times \Lambda) \to T_{\gamma(0)}W,$$

(2.5)

$$(\zeta_0, \ell) \mapsto \Psi(\theta)\zeta_0 + \Psi(\theta)X(\theta)\ell.$$

The matrix $E(\theta)$ is the symmetric part of the endomorphism

$$T_{\Lambda} \to T_{\Lambda},$$

(2.6)

$$\ell \mapsto -\frac{d}{d\lambda} \int_0^\theta \nabla_\lambda H(\tau, \Psi^\tau(\gamma(0), \lambda), \lambda) \, d\tau \cdot \ell.$$

2.3. The spectral flow of the linearized operator $D_{(u, \lambda)}$. Let us fix a connecting trajectory $(u, \lambda) \in \mathcal{M}((\gamma, \lambda); H, J, g)$ between two nondegenerate critical points of $A_H$. We recall here from [3, Lemma 2.3] that the nondegeneracy of a critical point $(\gamma, \lambda)$ is equivalent to the bijectivity of the \textit{asymptotic operator}

$$D_{(\gamma, \lambda)} : H^1(S^1, \gamma^*TW) \times T_{\Lambda} \to L^2(S^1, \gamma^*TW) \times T_{\Lambda},$$

(2.7)

$$D_{(\gamma, \lambda)}(\zeta, \ell) = \left( J_{\lambda}(\nabla_{\theta} \zeta - \nabla_{\ell} X_{H_\lambda}) - (D_{\lambda}X_{H_\lambda}) \cdot \ell \right) - \int_{S^1} \nabla_{\ell} \frac{\partial H}{\partial \theta} \, d\theta - \int_{S^1} \nabla_{\theta} \frac{\partial H}{\partial \theta} \, d\theta.$$

The operator $D_{(\gamma, \lambda)}$ is formally obtained from the linearized operator $D_{(u, \lambda)}$ in equation (1.6) by setting $(u(s, \theta), \lambda(s)) \equiv (\gamma(\theta), \lambda)$ and $(\zeta(s, \theta), \ell(s)) \equiv (\zeta(\theta), \ell)$.

Given a unitary trivialization of $u^*TW$ and an orthogonal trivialization of $\lambda^*T\Lambda$, the operator $D_{(u, \lambda)}$ defined by (1.6) can be written for $p = 2$ as

$$D_{(u, \lambda)} : H^1(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times H^1(\mathbb{R}, \mathbb{R}^m) \to L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times L^2(\mathbb{R}, \mathbb{R}^m),$$

$$D_{(u, \lambda)}(\zeta, \ell) = \left( \frac{\partial s}{\partial \ell} \zeta + A(s) \left( \begin{array}{c} \zeta \\ \ell \end{array} \right) \right).$$

Here $A(s) : H^1(S^1, \mathbb{R}^{2n}) \times \mathbb{R}^m \to L^2(S^1, \mathbb{R}^{2n}) \times \mathbb{R}^m$ has the property that $A(s) \to A^\pm$, $s \to \pm \infty$ and $A^\pm$ coincide through the given trivializations with the asymptotic operators $D_{(\gamma, \lambda)}$ and $D_{(\gamma, \lambda)}^\pm$, which are bijective in view of our nondegeneracy assumption. The operators $A(s)$ are of order one and their principal part is self-adjoint. Thus, up to an order zero (and hence compact) perturbation, we can...
assume for the purpose of computing the index that $A(s)$ is self-adjoint for all $s \in \mathbb{R}$. In this situation, the Fredholm index of the operator $D_{(u,\lambda)}$ is equal to the spectral flow of the family of self-adjoint operators $A(s)$, $s \in \mathbb{R}$ [8, Theorem A].

The spectral flow is described as follows. Let us call $s \in \mathbb{R}$ a crossing if $\ker A(s) \neq 0$, and define the crossing form $\Gamma(A, s) : \ker A(s) \to \mathbb{R}$ by $\Gamma(A, s)\xi = \langle \xi, \frac{d}{ds} A(s) \xi \rangle$. A crossing $s \in \mathbb{R}$ is called regular if the crossing form $\Gamma(A, s)$ is nondegenerate. Such crossings are isolated. If all crossings are nondegenerate, the spectral flow is given by the sum over all crossings of the signature of the crossing form $\Gamma(A, s)$, which is the number of positive minus the number of negative eigenvalues. Heuristically, the spectral flow measures the net difference between the number of eigenvalues of $A(s)$ which cross from $-$ to $+$ and those which cross from $+$ to $-$. Up to a compact perturbation we can always assume that all the crossings of $A(s)$ are regular.

In view of (2.7), the operators can be written in the given trivializations of $TW$ and $TA$ along $u$ and as

$$A(s)(\zeta, \ell) = \left( J_0 \partial_0 \zeta(\theta) + S(s, \theta) \zeta(\theta) + C(s, \theta)^T \ell \int_{S^1} C(s, \theta) \zeta(\theta) d\theta + \int_{S^1} D(s, \theta) d\theta \ell \right),$$

where $S(s, \theta) = S(s, \theta)^T$ and $D(s, \theta) = D(s, \theta)^T$ are symmetric matrices.

### 2.3.1. Computation of $\ker A(s)$, $s \in \mathbb{R}$

We define $\Psi : \mathbb{R} \times [0, 1] \to \text{Sp}(2n)$ by $\hat{\Psi}(s, \theta) = J_0 S(s, \theta) \Psi(s, \theta)$ and $\Psi(s, 0) = 1$, so that

$$\lim_{s \to -\infty} \Psi(s, \cdot) = \Psi(\cdot), \quad \lim_{s \to \infty} \Psi(s, \cdot) = \Psi(\cdot).$$

For $(\zeta, \ell) \in \ker A(s)$, we write $\zeta(\theta) = \Psi(s, \theta) \eta(\theta)$ for some smooth function $\eta : [0, 1] \to \mathbb{R}^{2n}$. Substituting this in the first component of $A(s)(\zeta, \ell)$, we obtain

$$\eta(\theta) = \Psi(s, \theta)^{-1} J_0 C(s, \theta)^T \ell.$$ 

We define $X : \mathbb{R} \times [0, 1] \to \text{Mat}_{2n, m}(\mathbb{R})$ by

$$\hat{X}(s, \theta) = \Psi(s, \theta)^{-1} J_0 C(s, \theta)^T$$

and $X(s, 0) = 0$. The solution of (2.9) is then $\eta(\theta) = X(s, \theta) \ell + \eta(0)$, so that

$$\zeta(\theta) = \Psi(s, \theta) \zeta_0 + \Psi(s, \theta) X(s, \theta) \ell,$$

with $\zeta_0 = \zeta(0) = \eta(0)$. Comparing (2.11) with (2.5) we see that

$$\lim_{s \to -\infty} X(s, \cdot) = \bar{X}(\cdot), \quad \lim_{s \to \infty} X(s, \cdot) = \bar{X}(\cdot).$$

The solution $\zeta(\theta)$ given by (2.11) descends to $S^1 = \mathbb{R}/\mathbb{Z}$ if and only if

$$\zeta_0 = \Psi(s, 1) \zeta_0 + \Psi(s, 1) X(s, 1) \ell.$$ 

Substituting the expression (2.11) for $\zeta(\theta)$ in the second component of $A(s)(\zeta, \ell)$, we obtain

$$\int_0^1 C(s, \theta) \Psi(s, \theta) d\theta \zeta_0 + \int_0^1 (C(s, \theta) \Psi(s, \theta) X(s, \theta) + D(s, \theta)) d\theta \ell = 0.$$ 

We now notice that we have

$$C(s, \theta) \Psi(s, \theta) = \hat{X}(s, \theta)^T J_0,$$
which implies in particular

\[(2.15) \quad \int_0^\theta C(s, \tau) \Psi(s, \tau) d\tau = \int_0^\theta \dot{X}(s, \tau) J_0 d\tau = X(\theta)^T J_0.\]

We define

\[E : \mathbb{R} \times [0, 1] \rightarrow \text{Mat}_m(\mathbb{R})\]

by

\[(2.16) \quad E(s, \theta) = \int_0^\theta \left( C(s, \tau) \Psi(s, \tau) X(s, \tau) + D(s, \tau) \right) d\tau - \frac{1}{2} X(s, \theta)^T J_0 X(s, \theta).\]

We claim that the matrix \(\frac{1}{\theta} X(s, \theta)^T J_0 X(s, \theta)\) is the anti-symmetric part of the matrix \(\int_0^\theta C(s, \tau) \Psi(s, \tau) X(s, \tau) d\tau\), so that \(E(s, \theta)\) is symmetric. Omitting the \(s\)-variable for clarity and using that \(C(\tau) \Psi(\tau) = \dot{X}(\tau)^T J_0\) we obtain

\[
\int_0^\theta C(\tau) \Psi(\tau) X(\tau) d\tau - \int_0^\theta X(\tau)^T \Psi(\tau)^T C(\tau)^T d\tau = \int_0^\theta \dot{X}(\tau)^T J_0 X(\tau) d\tau + \int_0^\theta X(\tau)^T J_0 \dot{X}(\tau) d\tau = X(\theta)^T J_0 X(\theta).
\]

It follows that \(E(s, \theta)\) is the symmetric part of \(\int_0^\theta (C\Psi X + D)(s, \tau) d\tau\). Comparing this with (2.6), it follows that

\[
\lim_{s \to -\infty} E(s, \cdot) = \mathcal{E}(\cdot), \quad \lim_{s \to \infty} E(s, \cdot) = E(\cdot).
\]

With these notations in place, we see that (2.12) and (2.13) are equivalent to the \((2n + m) \times (2n + m)\) system of linear equations

\[(2.17) \quad \begin{pmatrix} \Psi(s, 1) - \mathbb{I} & \Psi(s, 1) X(s, 1) \\ X(s, 1)^T J_0 & E(s, 1) + \frac{1}{2} X(s, 1)^T J_0 X(s, 1) \end{pmatrix} \begin{pmatrix} \zeta_0 \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

The solutions of the system (2.17) are in bijective correspondence with the elements \((\zeta, \ell) \in \ker A(s)\) through equation (2.11). On the other hand, it follows from the definition of \(S_{n,m}\) that solutions of (2.17) are in bijective correspondence with elements

\[(\zeta_0, \ell, 0) \in \ker (M(\Psi(s, 1), X(s, 1), E(s, 1)) - \mathbb{I}).\]

Since \((0, 0, v) \in \ker (M(\Psi(s, 1), X(s, 1), E(s, 1)) - \mathbb{I})\) for all \(v \in \mathbb{R}^m\), we infer that \(\ker A(s) \neq 0\) if and only if

\[(2.18) \quad \dim \ker (M(\Psi(s, 1), X(s, 1), E(s, 1)) - \mathbb{I}) > m.
\]

\textbf{Remark 3.} We associated to each operator \(A(s)\) of the form (2.8) a path of matrices \(M : [0, 1] \rightarrow S_{n,m}, M(\theta) = M(\Psi(\theta), X(\theta), E(\theta))\) such that \(M(0) = \mathbb{I}\). Conversely, any such path \(M\) determines a unique operator \(A(s)\) of the form (2.8) by the formulas

\[
S(\theta) = -J_0 \dot{\Psi}(\theta) \Psi(\theta)^{-1} \\
C(\theta) = \dot{X}(\theta)^T \Psi(\theta)^T J_0 \\
D(\theta) = \dot{E}(\theta) + \text{Sym} \left( X(\theta)^T J_0 \dot{X}(\theta) \right).
\]
2.3.2. Computation of the crossing form $\Gamma(A, s)$ on $\ker A(s)$. We have

$$
\frac{d}{ds} A(s)(\zeta, \ell) = \left( \begin{array}{c}
\partial_s S(s, \theta) \zeta(\theta) + \partial_s C(s, \theta)^T \ell \\
\int_{S^1} \partial_s C(s, \theta) \zeta(\theta) d\theta + \int_{S^1} \partial_s D(s, \theta) d\theta \ell
\end{array} \right).
$$

Since $(\zeta, \ell) \in \ker A(s)$, we have $\zeta(\theta) = \Psi(s, \theta) \zeta_0 + \Psi(s, \theta) X(s, \theta) \ell$. We obtain

$$
\Gamma(A, s)(\zeta, \ell) = (\langle \zeta, \ell \rangle, \frac{d}{ds} A(s)(\zeta, \ell))
= \int_{S^1} \langle \zeta(\theta), \partial_s S(s, \theta) \zeta(\theta) + \partial_s C(s, \theta)^T \ell \rangle d\theta
+ \langle \ell, \int_{S^1} \partial_s C(s, \theta) \zeta(\theta) d\theta + \int_{S^1} \partial_s D(s, \theta) d\theta \ell \rangle
= \int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \Psi(s, \theta)^T \partial_s S(s, \theta) \Psi(s, \theta)(\zeta_0 + X(s, \theta) \ell) d\theta
+ \int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \Psi(s, \theta)^T \partial_s C(s, \theta)^T \ell d\theta
+ \ell^T \int_0^1 \partial_s C(s, \theta) \Psi(s, \theta)(\zeta_0 + X(s, \theta) \ell) d\theta
+ \ell^T \int_{S^1} \partial_s D(s, \theta) d\theta \ell.
$$

(2.19)

Let us define symmetric matrices $\tilde{S}(s, \theta)$ by $\partial_s \Psi(s, \theta) = J_0 \tilde{S}(s, \theta) \Psi(s, \theta)$. The condition $\Psi(s, 0) = I$ implies $\tilde{S}(s, 0) = 0$. We claim that (see also [9, proof of Lemma 2.6])

$$
\Psi(s, \theta)^T \partial_s S(s, \theta) \Psi(s, \theta) = \partial_0 \left( \Psi(s, \theta)^T \tilde{S}(s, \theta) \Psi(s, \theta) \right).
$$

(2.20)

Dropping the $(s, \theta)$ variables for clarity, we have [9]

$$
\partial_0 \left( \Psi^T \tilde{S} \Psi \right) = \Psi^T S^T (-J_0) \tilde{S} \Psi + \Psi^T \partial_0 (\tilde{S} \Psi)
= -\Psi^T S \partial_s \Psi + \Psi^T \partial_0 (-J_0 \partial_s \Psi)
= -\Psi^T S \partial_s \Psi - \Psi^T J_0 \partial_s \partial_0 \Psi
= -\Psi^T S \partial_s \Psi - \Psi^T J_0 \partial_s (J_0 S \Psi)
= \Psi^T \partial_s S \Psi.
$$
Using (2.21), the term (2.19) becomes

\[
\int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \partial_0 \left( \Psi(s, \theta)^T \tilde{S}(s, \theta) \Psi(s, \theta) \right) \left( \zeta_0 + X(s, \theta) \ell \right) d\theta
\]

\[
= \left( \zeta_0 + X(s, \theta) \ell \right)^T \Psi(s, \theta)^T \tilde{S}(s, \theta) \Psi(s, \theta) \left( \zeta_0 + X(s, \theta) \ell \right) d\theta
\]

\[
- \ell^T \int_0^1 \partial_0 X(s, \theta)^T \Psi(s, \theta)^T \tilde{S}(s, \theta) \Psi(s, \theta) \left( \zeta_0 + X(s, \theta) \ell \right) d\theta
\]

\[
- \int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \Psi(s, \theta)^T \tilde{S}(s, \theta) \Psi(s, \theta) \partial_0 X(s, \theta) d\theta \ell
\]

\[
= \zeta_0^T \tilde{S}(s, 1) \zeta_0 + \ell^T \int_0^1 C(s, \theta) J_0 \tilde{S}(s, \theta) \Psi(s, \theta) (\zeta_0 + X(s, \theta) \ell) d\theta
\]

\[
- \int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \Psi(s, \theta)^T \tilde{S}(s, \theta) J_0 C(s, \theta)^T d\theta \ell
\]

\[
= \zeta_0^T \tilde{S}(s, 1) \zeta_0 + \ell^T \int_0^1 C(s, \theta) \partial_s \Psi(s, \theta) (\zeta_0 + X(s, \theta) \ell) d\theta
\]

\[
+ \int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \partial_s \Psi(s, \theta)^T C(s, \theta)^T d\theta \ell.
\]

The second equality uses (2.12) and (2.10). Thus, equation (2.20) becomes

\[
\Gamma(A, s)(\zeta, \ell) = \zeta_0^T \tilde{S}(s, 1) \zeta_0 + \ell^T \int_0^1 \partial_s \left( C(s, \theta) \Psi(s, \theta) \right) (\zeta_0 + X(s, \theta) \ell) d\theta
\]

\[
+ \int_0^1 (\zeta_0 + X(s, \theta) \ell)^T \partial_s \left( \Psi(s, \theta)^T C(s, \theta)^T \right) d\theta \ell
\]

\[
+ \ell^T \int_{S^1} \partial_s D(s, \theta) d\theta \ell
\]

\[
= \zeta_0^T \tilde{S}(s, 1) \zeta_0 + \ell^T \left( \partial_s X(s, 1)^T J_0 \right) \zeta_0 + \zeta_0^T \left( - J_0 \partial_s X(s, 1) \right) \ell
\]

\[
+ \ell^T \int_0^1 \left( \partial_s (C \Psi) X + X^T \partial_s (\Psi^T C^T) + \partial_s D \right) \ell.
\]

(2.22)

We used (2.14) in the second equality. We claim that the matrix of the quadratic form \( \Gamma(A, s) \) acting on the space of elements \((\zeta_0, \ell)\) satisfying (2.12) is given by

\[
(2.23) \quad \begin{pmatrix} \tilde{S}(s, 1) & -J_0 \partial_s X(s, 1) \\ \partial_s X(s, 1)^T J_0 & \partial_s E(s, 1) - \text{Sym}(X^T(s, 1) J_0 \partial_s X(s, 1)) \end{pmatrix}.
\]

This amounts to proving the identity

\[
(2.24) \quad \partial_s E(s, 1) - \text{Sym}(X^T(s, 1) J_0 \partial_s X(s, 1)) = \int_0^1 \partial_s (C \Psi) X + X^T \partial_s (\Psi^T C^T) + \partial_s D
\]

\[
= \int_0^1 \partial_s (C \Psi) X + X^T \partial_s (\Psi^T C^T) + \partial_s D
\]
for the term in the lower right corner. This is seen by a direct computation:
\[
\int_0^1 \partial_s(C\Psi)X + X^T \partial_s(\Psi^T C^T) + \partial_s D
\]
\[
= \partial_s \text{Sym} \int_0^1 (C\Psi X + D) + \text{Sym} \int_0^1 \partial_s(C\Psi)X - \text{Sym} \int_0^1 C\Psi \partial_s X
\]
\[
= \partial_s E(s, 1) + \text{Sym} \int_0^1 \partial_s(C\Psi)X
\]
\[
- \text{Sym} \left( X(s, 1)^T J_0 \partial_s X(s, 1) \right) + \text{Sym} \int_0^1 X^T J_0 \partial_s X
\]
\[
= \partial_s E(s, 1) - \text{Sym} \left( X(s, 1)^T J_0 \partial_s X(s, 1) \right).
\]

The second equality uses the definition of \( E \), the identity \( C\Psi = \bar{X}^T J_0 \) from (2.14), and integration by parts. The third equality uses that \( X^T J_0 \partial_s \bar{X} = -\left( \partial_s(C\Psi)X \right)^T \), which is a consequence of \( C\Psi = \bar{X}^T J_0 \).

2.4. Proof of the Main Theorem. Let us compute the crossing form \( \Gamma(M, s) \) for the Robbin-Salamon index of the path
\[
s \mapsto M(s, 1) = M(\Psi(s, 1), X(s, 1), E(s, 1)).
\]

By definition, the crossing form is \( \Gamma(M, s)(\zeta_0, \ell, v) = \langle (\zeta_0, \ell, v), Q(s)(\zeta_0, \ell, v) \rangle \), with \( Q(s) := -J_0 \partial_s M(s, 1) M(s, 1)^{-1} \). Using (2.2) and the definition of \( \bar{S}(s, 1) = -J_0 \partial_s \Psi(s, 1) \Psi(s, 1)^{-1} \) from §2.3.2, a straightforward computation shows that \( Q(s) \) is given by
\[
\begin{pmatrix}
\bar{S}(s, 1) & -J_0 \Psi(s, 1) \partial_s X(s, 1) & 0 \\
0 & \partial_s E(s, 1) + \text{Sym} \left( X(s, 1)^T J_0 \partial_s X(s, 1) \right) & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The key observation now is that, for any \( (\zeta_0, \ell, 0) \in \ker (M(s, 1) - \mathbb{1}) \), we have
\[
\Gamma(M, s)(\zeta_0, \ell, 0) = \Gamma(A, s)(\zeta, \ell),
\]
with \( \zeta(\theta) = \Psi(s, \theta) \zeta_0 + \Psi(s, \theta) X(s, 1) \ell \). This is seen by a direct computation, substituting \( \zeta_0 = \Psi(s, 1) \zeta_0 + \Psi(s, 1) X(s, 1) \ell \) in the non-diagonal terms of \( \Gamma(M, s)(\zeta_0, \ell, 0) \):
\[
\Gamma(M, s)(\zeta_0, \ell, 0)
\]
\[
= \zeta_0^T \bar{S} \zeta_0 + \ell^T (\partial_s E + \text{Sym} (X^T J_0 \partial_s X)) \ell + \ell^T \partial_s X^T \Psi^T J_0 \zeta_0 + \zeta_0^T (-J_0 \Psi \partial_s X) \ell
\]
\[
= \zeta_0^T \bar{S} \zeta_0 + \ell^T (\partial_s E + \text{Sym} (X^T J_0 \partial_s X)) \ell
\]
\[
+ \ell^T \partial_s X^T J_0 \zeta_0 + \ell^T \partial_s X^T J_0 \ell + \zeta_0^T (-J_0 \partial_s X) \ell + \ell^T X^T (-J_0 \partial_s X) \ell
\]
\[
= \zeta_0^T \bar{S} \zeta_0 + \ell^T (\partial_s E + \text{Sym} (X^T J_0 \partial_s X)) \ell
\]
\[
+ \ell^T \partial_s X^T J_0 \zeta_0 + \ell^T (-J_0 \partial_s X) \ell - 2 \ell^T \text{Sym}(X^T J_0 \partial_s X) \ell.
\]

This last expression is equal to \( \Gamma(A, s)(\zeta, \ell) \) in view of (2.23).

By Proposition 6 in Appendix B (applied with \( E(s) = \{0\} \oplus \{0\} \oplus \mathbb{R}^m \)), it follows that the spectral flow of \( A(s) \) coincides with the Robbin-Salamon index of the degenerate path \( s \mapsto M(s, 1) \). Thus
\[
\text{ind } D_{(u, \lambda)} = \mu_{RS} (M(\Psi(s, 1), X(s, 1), E(s, 1)), s \in \mathbb{R}).
\]
By the (Homotopy) and (Catentation) axioms for the Robbin-Salamon index [7], and using that $\lim_{s \to -\infty} M(s, \theta) = \overline{M}(\theta)$ and $\lim_{s \to \infty} M(s, \theta) = \underline{M}(\theta)$, we obtain

$$\text{ind} \, D_{(u, \lambda)} = \mu_{RS} (M(\overline{\Psi}(\theta), \overline{X}(\theta)), \theta \in [0, 1]) - \mu_{RS} (M(\overline{\Psi}(\theta), \overline{X}(\theta)), \theta \in [0, 1]) = \mu(\overline{\gamma}, \overline{\lambda}) - \mu(\overline{\tau}, \overline{\lambda}).$$

\[\square\]

**Appendix A. Examples**

We explain in this appendix several examples in which parametrized Hamiltonian action functionals appear naturally.

A.1. **$S^1$-equivariant Floer homology** [11, 2]. One takes $\Lambda = ES^1$ (or rather a finite-dimensional approximation of it), where $ES^1$ is up to equivariant homotopy the unique contractible $S^1$-space carrying a free action. The $S^1$-equivariant Floer homology groups are defined using Hamiltonians which are invariant

$$H(\theta + \tau, x, \lambda \rho) = H(\theta, x, \lambda), \quad \tau \in S^1,$$

Compared to the classical, non-equivariant Floer homology groups, these carry refined information coming from the $S^1$-action on $\mathcal{L}W$ given by reparametrization at the source.

A.2. **Rabinowitz-Floer homology** [4]. One takes $\Lambda = \mathbb{R}$ and

$$H(\theta, x, \lambda) = \lambda K(x),$$

with $K : W \to \mathbb{R}$ an autonomous Hamiltonian. The critical points of $A_H$ solve the equations $\dot{\gamma} = \lambda X_K$ and $\int_{S^1} K(\gamma(\theta)) \, d\theta = 0$, which are equivalent to $\dot{\gamma} = \lambda X_K$ and $\text{im} \, \gamma \subset K^{-1}(0)$. Thus critical points of $A_H$ correspond to closed characteristics on the fixed energy level $K^{-1}(0)$.

A.3. **Rabinowitz-Floer homology for leafwise intersections of hypersurfaces** [1]. One takes again $\Lambda = \mathbb{R}$ but

$$H(\theta, x, \lambda) = \lambda \rho(\theta) K(x) + F(\theta, x).$$

Here $\rho : S^1 \to \mathbb{R}$ is such that $\text{supp}(\rho) \subset [0, \frac{1}{2}]$ and $\int_{S^1} \rho(\theta) \, d\theta = 1$, while $F(\theta, \cdot) = 0$ for $\theta \in [0, \frac{1}{2}]$. The equations for a critical point $(\gamma, \lambda)$ are equivalent to $x := \gamma(\frac{1}{2}) \in K^{-1}(0)$ and $\gamma(0) = \gamma(1) = \phi^t_{\rho}(x) \in L_x$, with $L_x$ the orbit of the characteristic flow passing through $x$. One calls $x$ a leafwise intersection of the flow $\phi^t_{\rho}$.

A.4. **Rabinowitz-Floer homology for leafwise coisotropic intersections** [6]. Let $\mathcal{K} = (K_1, \ldots, K_k) : W \to \mathbb{R}^k$ be a system of autonomous Poisson-commuting Hamiltonians. The preimage $\mathcal{K}^{-1}(a)$ of a regular value $a \in \mathbb{R}^k$ is then a coisotropic submanifold which is foliated by isotropic leaves that are tangent to the span of the Hamiltonian vector fields $X_{K_1}, \ldots, X_{K_k}$. Let $(\cdot, \cdot)$ be the Euclidean scalar product on $\mathbb{R}^k$, take $\Lambda = \mathbb{R}^k$ and define

$$H(\theta, x, \lambda) = \rho(\theta) \langle \lambda, K(x) \rangle + F(\theta, x)$$

with $\rho$ and $F$ as above. The equations for a critical point $(\gamma, \lambda)$ are equivalent to $x := \gamma(\frac{1}{2}) \in K^{-1}(0)$ and $\gamma(0) = \gamma(1) = \phi^t_{\rho}(x) \in L_x$, with $L_x$ the isotropic leaf through $x$. One calls $x$ a leafwise coisotropic intersection of the flow $\phi^t_{\rho}$. 
A.5. Floer homology for families [5]. This construction generalizes the setup that we consider in this paper to nontrivial fibrations. Assume \( \pi : E \to \Lambda \) is a symplectic fibration endowed with an exact 2-form \( \Omega = d\Theta \in \Omega^2(E;\mathbb{R}) \) which restricts to a symplectic form in the fibers (this is called a coupling form). Let \( H : S^1 \times E \to \mathbb{R} \) be a Hamiltonian and let \( \mathcal{L}_\Lambda E \) denote the space of loops \( \gamma \) in \( E \) such that \( \pi \circ \gamma \) is constant. One considers the action functional

\[
A_H : \mathcal{L}_\Lambda E \to \mathbb{R}, \quad \gamma \mapsto -\int_S \Theta - \int_{S^1} H(\theta, \gamma(\theta)) \, d\theta.
\]

The critical points of \( A_H \) are the basis for the Floer homology groups of the family \((E, \Lambda)\) [5]. Since the fibration is locally trivial and the critical points of \( A_H \) are contained in a fiber, the definition of the index that we give in this paper applies also to this more general setup.

Appendix B. The parametrized Robbin-Salamon index

We summarize in this appendix the properties of the Robbin-Salamon index on paths with values in the subgroup \( S_{n,m} \subset \text{Sp}(2n + 2m) \) defined in §1.3. We also prove a result (Proposition 6) which is used in the proof of our Main Theorem.

We recall that \( S_{n,m} \) consists of matrices of the form

\[
M = M(\Psi, X, E) = \begin{pmatrix}
\Psi & \Psi X & 0 \\
0 & \Psi X & 0 \\
X^T J_0 & E + \frac{k}{2} X^T J_0 X & \Psi \\
\end{pmatrix},
\]

with \( \Psi \in \text{Sp}(2n), X \in \text{Mat}_{2n,m}(\mathbb{R}), \) and \( E \in \text{Mat}_m(\mathbb{R}) \) symmetric. We have denoted by \( J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \) the standard complex structure on \( \mathbb{R}^{2n}, \) so that \( \Psi \in \text{Sp}(2n) \) if and only if \( \Psi^T J_0 \Psi = J_0. \) The standard complex structure on \( \mathbb{R}^{2n} \times \mathbb{R}^{2m} \) is

\[
\tilde{J}_0 := \begin{pmatrix} J_0 & 0 & 0 \\
0 & 0 & -\mathbb{1} \\
0 & \mathbb{1} & 0 \end{pmatrix},
\]

and we have \( M^T \tilde{J}_0 M = \tilde{J}_0. \) Note that we have natural embeddings (which respect the group structure)

\[
S_{n,m} \times S_{n',m'} \hookrightarrow S_{n+n',m+m'}
\]

which associate to \( M = M(\Psi, X, E) \in S_{n,m} \) and \( M' = M(\Psi', X', E') \in S_{n',m'} \) the matrix

\[
M \oplus M' := M(\Psi \oplus \Psi', X \oplus X', E \oplus E') \in S_{n+n',m+m'}.
\]

The space \( S_{n,m} \) is stratified as \( \bigsqcup_{k=0}^{2n+m} S^k_{n,m}, \) with

\[
S^k_{n,m} := \{ M \in S_{n,m} : \dim \ker (M - \mathbb{1}) = m + k \}.
\]

The following are general properties of the Robbin-Salamon index \( \mu = \mu_{\text{RS}} \) defined on paths with values in \( \text{Sp}(2n + 2m) \) [7, Theorem 4.1 and Theorem 4.7].

(Homotopy): If \( M, M' : [a, b] \to S_{n,m} \) are homotopic with fixed endpoints then

\[
\mu(M) = \mu(M');
\]

(Catenation): For any \( c \in [a, b] \) we have

\[
\mu(M) = \mu(M|_{[a,c]}) + \mu(M|_{[c,b]});
\]
(Naturality): For any path \( P : [a, b] \rightarrow \text{Sp}(2n) \times \text{Sp}(2m) \) of the form
\[
P(\theta) = \begin{pmatrix}
\Phi(\theta) & 0 & 0 \\
0 & A(\theta) & 0 \\
0 & 0 & A(\theta)
\end{pmatrix}
\]
(with \( \Phi(\theta) \in \text{Sp}(2n) \) and \( A(\theta) \in O(m) \), hence \( PMP^{-1} \in \mathcal{S}_{n,m} \)), we have
\[
\mu(PMP^{-1}) = \mu(M);
\]
(Product): For any \( M \in \mathcal{S}_{n,m} \) and \( M' \in \mathcal{S}_{n',m'} \) we have
\[
\mu(M \oplus M') = \mu(M) + \mu(M');
\]
(Zero): For any path \( M : [a, b] \rightarrow \mathcal{S}_{n,m}^{k} \) we have
\[
\mu(M) = 0;
\]
(Integrality): Given a path \( M : [a, b] \rightarrow \mathcal{S}_{n,m}^{k_a} \) with \( M(a) \in \mathcal{S}_{n,m}^{k_a}, M(b) \in \mathcal{S}_{n,m}^{k_b} \), we have
\[
\mu(M) + \frac{k_a - k_b}{2} \in \mathbb{Z};
\]
The next statement summarizes properties that are specific to the index function restricted to paths with values in \( \mathcal{S}_{n,m} \).

**Proposition 4.** The Robbin-Salamon index \( \mu = \mu_{RS} \) defined on paths \( M : [a, b] \rightarrow \mathcal{S}_{n,m} \), \( M(\theta) = M(\Psi(\theta), X(\theta), E(\theta)) \) has the following properties.

(Loop): For any loop \( P : [a, b] \rightarrow \text{Sp}(2n) \times \text{Sp}(2m) \) of the form (B.1), we have
\[
\mu(PM) = \mu(M) + 2\mu(\Phi);
\]
(Splitting): Given \( M = M(\Psi, 0, E) : [a, b] \rightarrow \mathcal{S}_{n,m} \), we have
\[
\mu(M) = \mu(\Psi) + \frac{1}{2} \text{sign } E(b) - \frac{1}{2} \text{sign } E(a);
\]
(Signature): Given symmetric matrices \( E \in \mathbb{R}^{m \times m} \) and \( S \in \mathbb{R}^{2n \times 2n} \) with \( \|S\| < 2\pi \), we have
\[
\mu\{M(\exp(J_0St), 0, tE), t \in [0, 1]\} = \frac{1}{2} \text{sign } (S) + \frac{1}{2} \text{sign } (E);
\]
(Determinant): Given a path \( M = M(\Psi, X, E) : [a, b] \rightarrow \mathcal{S}_{n,m} \) with \( M(a) = 1 \) and \( M(b) \in \mathcal{S}_{n,m}^{0} \), we have
\[
(-1)^{n+\frac{1}{2} - \mu(M)} = \text{sign } \det \begin{pmatrix}
\Psi - \frac{1}{4} X^T J_0 X & \frac{1}{4} X^T J_0 X
\end{pmatrix};
\]
We have denoted for simplicity \( \Psi = \Psi(b), X = X(b), E = E(b) \).

(Involution): For any \( M = M(\Psi, X, E) : [a, b] \rightarrow \mathcal{S}_{n,m} \), we have
\[
\mu(M(\Psi, X, E)) = \mu(M(\Psi, -X, E))
\]
and
\[
\mu(M(\Psi^{-1}, X, -E)) = \mu(M(\Psi^T, J_0 \Psi X, -E)) = -\mu(M(\Psi, X, E)).
\]
Proof. To prove the (Loop) property we use the equality

\[ \mu(PM) = \mu(M) + 2\mu(P) = \mu(M) + 2\mu(\Phi) + 2\mu \left( \begin{array}{cc} A & 0 \\ 0 & A \end{array} \right). \]

Since \( \pi_1(O(m)) = \mathbb{Z}/2\mathbb{Z} \) and \( \pi_1(\text{Sp}(2m)) = \mathbb{Z} \), the last term vanishes.

The (Splitting) property follows from the (Product) property and the normalization axiom for the Robbin-Salamon index of a symplectic shear.

The (Signature) property follows from the (Splitting) property and from the identity \( \mu_{RS}(\exp(J_0St)) = \frac{1}{2}\text{sign}(S) \) \cite[Theorem 3.3.(iv)].

We prove the (Involution) property. The first identity \( \mu(M(\Psi^{-1}, Y, -E)) = \mu(M(\Psi^T, J_0Y, -E)) \) follows from the (Naturality) axiom by conjugating with the constant path \( J_0 \oplus \mathbb{I}_{2m} \). The identity \( \mu(M(\Psi, X, E)) = \mu(M(\Psi, X, E^{-1})) \) follows by conjugating twice with \( J_0 \oplus \mathbb{I}_{2m} \). Finally, using (2.2) we obtain \( -\mu(M(\Psi, X, E)) = \mu(M(\Psi^{-1}, -\Psi X, -E)) = \mu(M(\Psi^{-1}, \Psi X, E)) \).

It remains to prove the (Determinant) property. Given a path \( N : [0, 1] \to \text{Sp}(2n + 2m) \) satisfying \( N(0) = \mathbb{I} \) and \( \det (N(1) - \mathbb{I}) \neq 0 \), we have \cite[Theorem 3.3.(iii)]{10}

\[ (-1)^{n+m-\mu_{RS}(N)} = \text{sign} \det (N(1) - \mathbb{I}). \]

We construct such a path \( N : [a, b + \varepsilon] \to \text{Sp}(2n + 2m) \) by catenating \( M = M(\Psi(\theta), X(\theta), E(\theta)) \) with the path \( M' : [b, b + \varepsilon] \to \text{Sp}(2n + 2m) \) given by

\[ M'(b + \theta) := \left( \begin{array}{ccc} \Psi & \Psi X & \theta \Psi X \\ 0 & \mathbb{I} & \theta \mathbb{I} \\ X^T J_0 & E + \frac{1}{2}X^T J_0 X & \mathbb{I} + \theta (E + \frac{1}{2}X^T J_0 X) \end{array} \right). \]

We have denoted for simplicity \( \Psi := \Psi(b), X := X(b), E := E(b) \). Since \( M(b) \in S_{n,m}, \) the path \( M' \) has a single crossing at \( b \) and the kernel of \( M'(b) - \mathbb{I} = M(b) - \mathbb{I} \) is \( \{0\} \oplus \{0\} \oplus \mathbb{R}^m \). The crossing form at \( b \) is \( -\mathbb{I}_m \), so that \( \mu_{RS}(M') = -\frac{m}{2} \). Thus \( \mu_{RS}(N) = \mu(M) - \frac{m}{2} \). On the other hand

\[ \det (N(b + \varepsilon) - \mathbb{I}) = \varepsilon^m (-1)^m \det \left( \begin{array}{cc} \Psi - \mathbb{I} & \Psi X \\ X^T J_0 & E + \frac{1}{2}X^T J_0 X \end{array} \right) . \]

This implies the desired statement. \( \square \)

Example 5. The index \( \mu(M(\Psi, X, E)) \) depends in an essential way on \( X \), as the following example shows. Given \( a, b \in \mathbb{R} \), let

\[ \Psi := \left( \begin{array}{cc} 2 & 0 \\ 0 & \frac{1}{2} \end{array} \right), \quad X_{a,b} := \left( \begin{array}{c} a \\ b \end{array} \right), \quad E := 1. \]

We denote \( M_{a,b} := M(\Psi, X_{a,b}, E) \in S_{1,1} \). It follows from the (Determinant) property that a path in \( S_{1,1} \) starting at \( \mathbb{I} \) and ending at \( M_{0,0} \) has an index in \( \frac{1}{2} + 2\mathbb{Z} \), whereas a path in \( S_{1,1} \) starting at \( \mathbb{I} \) and ending at \( M_{1,1} \) has an index in \( \frac{1}{2} + 2\mathbb{Z} + 1 \) (the value of the relevant determinant is \( -\frac{1}{2} + \frac{3}{2}ab \)).

For the rest of this Appendix we place ourselves in \( \mathbb{R}^{2N} \) equipped with the standard symplectic form \( \omega_0 \) and the standard complex structure \( J_0 \). The next Proposition is relevant for the parametrized Robbin-Salamon index when applied with \( N = n + m \) and \( E(t) \equiv \{0\} \oplus \{0\} \oplus \mathbb{R}^m \). We recall that, given a path of symplectic matrices \( M : [0, 1] \to \text{Sp}(2N) \), the crossing form at a point \( t \in [0, 1] \) is the quadratic form \( \Gamma(M, t) \) on \( \ker (M(t) - \mathbb{I}) \) given by \( \Gamma(M, t)(v) = \langle v, -J_0 \dot{M}(t)M(t)^{-1}v \rangle \).
Proposition 6. Let $M : [0, 1] \rightarrow \text{Sp}(2N)$ be a $C^1$-path of symplectic matrices with the following property: there exists a continuous family of vector spaces $t \mapsto E(t) \subset \mathbb{R}^{2N}$ such that $E(t) \subset \ker (M(t) - \mathbb{I})$ and the crossing form $\Gamma(M, t)$ induces a nondegenerate quadratic form on $\ker (M(t) - \mathbb{I}) / E(t)$. Assume $\omega_0$ has constant rank on $E(t)$. Then

$$\mu_{RS}(M) = \frac{1}{2} \text{sign} \Gamma(M, 0) + \sum_{t : \dim \ker (M(t) - \mathbb{I}) / E(t) > 0} \text{sign} \Gamma(M, t) + \frac{1}{2} \text{sign} \Gamma(M, 1).$$

Proof. Let us first assume that the rank of $\omega_0$ is constant equal to 0 on $E(t)$, i.e. $E(t)$ is isotropic. Let us decompose $\mathbb{R}^{2N} = E(t) \oplus J_0E(t) \oplus F(t)$, where $F(t)$ is the symplectic orthogonal of $E(t) \oplus J_0E(t)$. Given $\varepsilon > 0$ we denote by $\beta_\varepsilon : [0, 1] \rightarrow [0, \varepsilon]$ a smoothing of the function

$$t \mapsto \left\{ \begin{array}{ll}
\varepsilon, & 0 \leq t \leq \varepsilon, \\
\varepsilon - 1, & 1 - \varepsilon \leq t \leq 1.
\end{array} \right.$$

We define an element $\Phi_\varepsilon^0(t) \in \text{Sp}(2N)$ which has the following matrix form with respect to the splitting $E(t) \oplus J_0E(t) \oplus F(t)$:

$$\Phi_\varepsilon^0(t) = \begin{pmatrix}
\mathbb{I} & 0 & 0 \\
\beta_\varepsilon(t) & \mathbb{I} & 0 \\
0 & 0 & \mathbb{I}
\end{pmatrix}.$$

We define $\tilde{M}(t) := M(t)\Phi_\varepsilon^0(t)$, and we have $\mu_{RS}(\tilde{M}) = \mu_{RS}(M)$ since these paths are homotopic with fixed endpoints. We claim that the following equality holds for all $t \in \mathbb{R}$:

$$(B.2) \quad \ker (\tilde{M}(t) - \mathbb{I}) = \ker (M(t) - \mathbb{I}) \cap (J_0E(t) \oplus F(t)).$$

That $\ker (M(t) - \mathbb{I}) \cap (J_0E(t) \oplus F(t)) \subset \ker (\tilde{M}(t) - \mathbb{I})$ follows from the fact that $\Phi_\varepsilon^0(t)$ acts by the identity on $J_0E(t) \oplus F(t)$. Conversely, let $v = v_1 + v_2 \in \ker (\tilde{M}(t) - \mathbb{I})$, with $v_1 \in E(t)$ and $v_2 \in J_0E(t) \oplus F(t)$. The identity $\tilde{M}(t)v = v$ is equivalent to $M(t)(v_1 + \beta_\varepsilon(t)J_0v_1 + v_2) = v_1 + v_2$, hence to $(M(t) - \mathbb{I})v_2 = -\beta_\varepsilon(t)M(t)J_0v_1$. Using that $M(t)v_1 = v_1$ we obtain

$$0 = \omega_0(v_1, (M(t) - \mathbb{I})v_2) = -\beta_\varepsilon(t)\omega_0(v_1, M(t)J_0v_1) = -\beta_\varepsilon(t)\omega_0(v_1, J_0v_1).$$

Since $\beta_\varepsilon(t) \neq 0$, this implies $v_1 = 0$, so that $v = v_2 \in J_0E(t) \oplus F(t)$ and $(M(t) - \mathbb{I})v_2 = (\tilde{M}(t) - \mathbb{I})v_2 = 0$, as desired.

Since the restrictions of $M(t)$ and $\tilde{M}(t)$ to $J_0E(t) \oplus F(t)$ are the same, it follows that the crossing form $\tilde{\Gamma}(\tilde{M}, t)$ coincides with $\Gamma(M, t)$ on $\ker (\tilde{M}(t) - \mathbb{I})$ for $t \in [0, 1]$. On the other hand, a straightforward computation shows that

$$(B.3) \quad \text{sign} \Gamma(\tilde{M}, 0) = \text{sign} \Gamma(M, 0) + \dim E(0),$$

$$(B.3) \quad \text{sign} \Gamma(\tilde{M}, 1) = \text{sign} \Gamma(M, 1) - \dim E(1).$$

Thus, the contributions at the endpoints compensate each other, and the conclusion follows using the definition of the Robbin-Salamon index via crossing forms.

We now assume that the rank of $\omega_0$ on $E(t)$ is equal to $\dim E(t)$, i.e. $E(t)$ symplectic. Let us decompose $\mathbb{R}^{2N} = E(t) \oplus F(t)$, where $F(t)$ is the symplectic orthogonal of $E(t)$. Let $J(t)$ be a continuous family of complex structures on $E(t)$.
which are compatible with $\omega_0$. For $\varepsilon > 0$ we define a path $\Phi^1_\varepsilon : [0, 1] \to \text{Sp}(2N)$ whose matrix with respect to the decomposition $E(t) \oplus F(t)$ is

$$\Phi^1_\varepsilon(t) := \begin{pmatrix} \exp(J(t)\beta_\varepsilon(t)) & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote $\widetilde{M}(t) := M(t)\Phi^1_\varepsilon(t)$, so that we have $\mu_{RS}(\widetilde{M}) = \mu_{RS}(M)$. We claim that

\begin{equation}
\text{ker}(\widetilde{M}(t) - 1) = \text{ker}(M(t) - 1) \cap F(t)
\end{equation}

for all $t \in [0, 1]$, whenever $0 < \varepsilon < \pi$. That $\text{ker}(M(t) - 1) \cap F(t) \subseteq \text{ker}(\widetilde{M}(t) - 1)$ follows from the fact that $\Phi^1_\varepsilon(t)$ acts as the identity on $F(t)$. Conversely, let $v = v_1 + v_2 \in \text{ker}(\widetilde{M}(t) - 1)$ such that $v_1 \in E(t)$ and $v_2 \in F(t)$. The relation $\widetilde{M}(t)v = v$ is equivalent to $(M(t) - 1)v_2 = (1 - \exp(J(t)\beta_\varepsilon(t)))v_1$. Then

$$0 = \omega_0(v_1, (M(t) - 1)v_2) = \omega_0(v_1, (1 - \exp(J(t)\beta_\varepsilon(t)))v_1) = -\sin(\beta_\varepsilon(t))\omega_0(v_1, J(t)v_1).$$

Since $\sin(\beta_\varepsilon(t)) \neq 0$, we obtain $v_1 = 0$ and the claim follows.

Since the restrictions of $M(t)$ and $\widetilde{M}(t)$ to $F(t)$ are the same, it follows that the crossing form $\Gamma(\widetilde{M}, t)$ coincides with $\Gamma(M, t)$ on $\ker(\widetilde{M}(t) - 1)$ for $t \in [0, 1]$. On the other hand, a straightforward computation shows that equations (B.3) still hold, and the conclusion follows.

Finally, we assume that the rank of $\omega_0$ on $E(t)$ lies strictly between 0 and $\dim E(t)$. We choose a continuous splitting $E(t) = E_1(t) \oplus E_0(t)$ with $E_0(t) := E(t) \cap E(t)^{\omega_0}$ isotropic and $E_1(t) = E_0(t)^{\perp}$ symplectic. Here $E(t)^{\omega_0}$ denotes the symplectic orthogonal of $E(t)$, and $E_0(t)^{\perp}$ denotes the Euclidean orthogonal of $E_0(t)$ in $E(t)$. We decompose $\mathbb{R}^{2N} = E_1(t) \oplus E_0(t) \oplus J_0E_0(t) \oplus F(t)$, such that $F(t)$ is the symplectic orthogonal of $E_1(t) \oplus E_0(t) \oplus J_0E_0(t)$. Given $0 < \varepsilon < \pi$ we define as above two paths $\Phi^0_\varepsilon(t)$ acting as the identity on $E_1(t) \oplus F(t)$, and $\Phi^1_\varepsilon(t)$ acting as the identity on $E_0(t) \oplus J_0E_0(t) \oplus F(t)$. We denote $\widehat{M}(t) := M(t)\Phi^0_\varepsilon(t)\Phi^1_\varepsilon(t)$, so that $\mu_{RS}(\widehat{M}) = \mu_{RS}(M)$. One proves as above that the crossings of $\widehat{M}$ and $M$ on $[0, 1]$ are the same, with the same crossing forms on $\ker(\widehat{M}(t) - 1)$, and moreover equations (B.3) still hold. This finishes the proof. 

\textbf{Remark 7}. The crossing form $\Gamma(M, t)$ vanishes identically on $E(t)$. Indeed, given a path $v(t) \in E(t)$ we have $M(t)v(t) = v(t)$ and $\dot{M}(t)v(t) + M(t)\dot{v}(t) = \dot{v}(t)$. Dropping the $t$-variable for clarity, we have

$$\Gamma(M, t)(v(t)) = \langle v, -J_0\dot{M}M^{-1}v \rangle = \langle v, -J_0(\dot{v} - M\dot{v}) \rangle = -\langle v, J_0\dot{v} \rangle + \langle v, (M^{-1})^TJ_0\dot{v} \rangle = -\langle v, J_0\dot{v} \rangle + \langle M^{-1}v, J_0\dot{v} \rangle = 0.$$

\textbf{Appendix C. Grading in Rabinowitz-Floer homology}

We give in this section a sample computation of the index within the setup of \S A.2. The index of a critical point of $A_H$ is defined as the Robbin-Salamon
index of a corresponding 1-periodic orbit for the Hamiltonian $\tilde{H} : W \times T^*\mathbb{R} \to \mathbb{R}$, $\tilde{H}(x, \lambda, p) = \lambda K(x)$. The flow of the latter is

$$\varphi^0_H(x, \lambda, p) = (\varphi^0_K(x), \lambda, p - \theta K(x)),$$

and its linearization is

$$d\varphi^0_H(x, \lambda, p) = \begin{pmatrix}
    \begin{array}{ccc}
d\varphi^0_K(x) & \theta X_K(\varphi^0_K(x)) & 0 \\
0 & 1 & 0 \\
-\theta dK(x) & 0 & 1
\end{array}
\end{pmatrix}.$$

We shall compute the index under the following simplifying assumptions:

- the level set $\Sigma := K^{-1}(0)$ is regular and of restricted contact type. This means that the restriction to $\Sigma$ of the primitive of the symplectic form is a contact form, which we denote $\pi$. There exists then a neighborhood $\mathcal{W}$ of $\Sigma$ and a diffeomorphism $\mathcal{W} \simeq [1 - \varepsilon, 1 + \varepsilon] \times \Sigma$, $\varepsilon > 0$ which transforms the symplectic form into $d(\pi \xi)$, $r \in [1 - \varepsilon, 1 + \varepsilon]$.
- the Hamiltonian $K$ has the form $K(r, \bar{x}) = k(r)$ on $\mathcal{W}$, with $\bar{x}$ denoting a point on $\Sigma$ and $k(1) = 0$, $k'(1) \neq 0$. Then $X_K(r, \bar{x}) = -k'(r)R(\bar{x})$, with $R$ the Reeb vector field on $\Sigma$ defined by $d(\pi R, \cdot) = 0$ and $\pi(R) = 1$. Thus

$$\varphi^0_R(r, \bar{x}, \lambda, p) = \left(r, \varphi^0_R(\bar{x}), \lambda, p - \theta k(r)\right).$$

Let $(\gamma, \lambda) \in \text{Crit}(A_H)$ and choose a symplectic trivialization $\gamma^*\mathcal{W} \simeq \mathbb{R}^{2n} = \mathbb{R}^{2n-2} \oplus \mathbb{R} \oplus \mathbb{R}$ which maps the contact distribution $\xi$ to $\mathbb{R}^{2n-2}$, the vector field $\partial/\partial r$ to the constant vector $(0, 1, 0)$, and the Reeb vector field to the constant vector $(0, 0, 1)$. The tangent bundle $T(T^*\mathbb{R})$ is in turn naturally trivialized as $T^*\mathbb{R} \times (\mathbb{R} \oplus \mathbb{R})$. When read in these trivializations, the linearization $d\varphi^0_R$, $\theta \in [0, 1]$ determines a path of symplectic matrices of the form $\begin{pmatrix} \Phi(\theta) & 0 \\ 0 & M(\theta) \end{pmatrix}$, where $\Phi(\theta) \in \text{Sp}(2n-2)$ corresponds to $d\varphi^{-\theta k'(1)}_R(\gamma(0))|_\xi$ and $M(\theta) \in S_{1,1} \subset \text{Sp}(4)$ has the form

$$M(\theta) = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
\theta T & 1 & \theta A & 0 \\
0 & 0 & 1 & 0 \\
\theta A & 0 & 0 & 1
\end{pmatrix}, \quad T = -\lambda k''(1), \quad A = -k'(1).$$

The matrix that represents the crossing form is

$$-\tilde{J}_0 \dot{M}(\theta) M(\theta)^{-1} = \begin{pmatrix}
    T & 0 & A & 0 \\
0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

Using that $A \neq 0$ we obtain that $\ker(M(\theta) - \mathbb{I})$ is equal to $\mathbb{R}^4$ if $\theta = 0$, respectively to $0 \oplus \mathbb{R} \oplus 0 \oplus \mathbb{R}$ if $\theta > 0$. We can use Proposition 6 to compute the Robbin-Salamon index of the path $M(\theta)$, $\theta \in [0, 1]$ and we find

$$\mu_{RS}(M) = \frac{1}{2} \text{sign} \left( \begin{array}{cc} T & A \\ A & 0 \end{array} \right) = 0.$$
Given \((\gamma, \lambda) \in \text{Crit}(A_H)\), we denote by \(\gamma(\theta) = \varphi^\theta_{\lambda^k(1)}(\gamma(0))\) the positively parametrized closed Reeb orbit that underlies \(\gamma\), and denote \(\mu(\gamma)\) its index. Then

\[
\mu(\gamma, \lambda) = \mu_{RS}(\Phi(\theta), \theta \in [0, 1]) = \begin{cases} 
\text{sign}(-k'(1))\mu(\gamma), & \lambda > 0, \\
0, & \lambda = 0, \\
-\text{sign}(-k'(1))\mu(\gamma), & \lambda < 0.
\end{cases}
\]

This agrees with the Rabinowitz-Floer homology grading in [4] up to a global shift of \(\frac{1}{2}\).

**Acknowledgements.** F.B.: Partially supported by ERC Starting Grant StG-239781-ContactMath. A.O.: This material is based upon work supported by the National Science Foundation under agreement No. DMS-0635607. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation. A.O. was also partially supported by ANR project “Floer Power” ANR-08-BLAN-0291-03 and ERC Starting Grant StG-259118-STEIN. A.O. is grateful to the organizers of the GESTA 2011 conference in Castro Urdiales for having given him the opportunity to lecture on \(S^1\)-equivariant symplectic homology.

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