From 3 nucleons to 3 quarks

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Abstract

Some short history of few-body methods originated from the famous Skornyakov-Ter-Martirosyan equation is given, including latest development of Faddeev formalism and Efimov states. The 3q system is shown to require an alternative, which is provided by the hyperspherical method (K-harmonics) which is highly successful for baryons.

1 Introduction

The Skornyakov and Ter-Martirosyan paper [1] which appeared in 1956 marked the beginning of a new era in few-body physics, when a somewhat neglected part of nuclear physics was promoted to the successful domain of theoretical physics. As a result the few-body science has become a field accumulating fast developing methods: L.D.Faddeev generalized the Skornyakov-Ter-Martirosyan Equation (STME) [2] and has given a rigorous mathematical foundation for the theory of 3 particles [3], many numerical methods have been introduced, for a review and references see [4]. As an immediate consequence of STME, a new effect was found in 1971, called the Efimov effect [5], which is studied till now with respect to possible experimental consequences [6].

The STME and Faddeev technic is most useful when particles are nearly on-shell, so that e.g. 3-body results do not depend much on the potential shape, but rather described by the on-shell two-body t-matrix as it is for the quartet n – d scattering. The bound states of tritium and $^3He$ provide another example, where the interaction at small distances (far off-shell) is important. To treat such systems an alternative method - the Hyperspherical Formalism (HF) (or K-harmonics method) was developed and the system of
the Schroedinger-like equations was written [7]. Its development was marked with many successful applications both in nuclear and atomic physical see e.g. [8, 9]. Recently it was understood that HF is probably the best suitable for systems with confinement such as 3 quarks, where the interaction is a three-body one, and confining so that the $t$-matrix formalism cannot be applied. Accuracy of HF as applied to the 3q system was found to be remarkably good [10, 11] allowing for the 1% bias in the baryon mass [12].

This talk is intended to demonstrate the physics of the 3-body system, and a qualitative analysis of two alternative approaches discussed above.

2 The STME and Faddeev approach

In the system of 3 equal-mass particles with arbitrary numeration one can introduce the total kinetic energy $E$ and the momentum $k$ in the pair $k_{23}$ and the relative momentum $p$ of particle 1, namely $p = \frac{k_2 + k_3}{2} - \frac{2}{3}k_1$, $k = \frac{k_2 - k_3}{2}$.

The symmetric function of the ground state $\Psi_{symm}$ is expressed through partial w.v.

$$
\Psi_{symm} = \psi(k_{23}, p_1) + \psi(k_{31}, p_2) + \psi(k_{12}, p_3)
$$

(1)

with the normalization condition

$$
\int |\Psi_{symm}|^2 d^3k d^3p = 1.
$$

(2)

It is convenient to extract the free 3-body Green’s function, introducing

$$
\psi(k, p) = \frac{\chi(k, p)}{k^2 + \frac{2}{7}p^2 - mE}
$$

(3)

and the 3-body rescattering equation, equivalent to the summing the ”bridge” Feynman diagrams (nonrelativistic) is [2]

$$
\chi(k, p) = \chi_0(k, p) - 2 \int m \frac{t(k, \frac{p}{2} + p', E - \frac{3}{7}p^2)\chi(|p + p'|, p')dp'}{p^2 + pp' + p'^2 - mE}.
$$

(4)

Here $t(k, k', \epsilon)$ is the 2-body $t$-matrix, representing the ”knot” in a bridge diagram, and

$$
\chi_0(k, p) = -2m \frac{t(k, \frac{p}{2} + p_0, E - \frac{3}{7}p^2)\varphi_0(p + \frac{1}{7}p_0)}{p^2 + p_0^2 + pp_0 - mE},
$$

(5)
where $\varphi_\alpha$ is the 2-body bound state, while $p_0$ is the momentum of incident particle.

Near the bound-state pole $t$-matrix can be written as

$$t(k, k', \varepsilon) = g(k \varepsilon) g(k', \varepsilon) \frac{(2\pi)^2}{m(\alpha + i\sqrt{2m\varepsilon})} + O(r_0)$$

where $\alpha = 1/a$, $a$ is the scattering length and $g(k, \varepsilon)$ formfactor, $g(0, 0) = 1$ and $g(k, \varepsilon)$ fast decreases when $k \sim 1/r_0$ and $\varepsilon \sim \frac{1}{mr_0}$. 

Let us assume now that the range of integration in (4) is small $p, p' \ll 1/r_0$. Then one can insert (6) in (4) with $g \sim 1$, and one gets- for the 3-body bound-state w.f.

$$\chi(k, p) + 8\pi \int \frac{dp'}{(2\pi)^3} \frac{\chi(|p + p'|, p')}{p^2 + pp' + p'^2 - mE} = 0\,.$$ (7)

This is the STME for a 3-body bound state. The off-shell generalization of STME is the Faddeev equation (4). As it was correctly stated in [1], the bound-state equation (7) cannot be used for tritium and $^3$He, since it has no lower bound for energy due to the Thomas theorem [13]. This can be easily understood rewriting (7) in the form $\chi = \int K \chi dp'$, and calculating the norm of $K$, $\|K\| = \int dp dp' (K(p, p'))^2$, which diverges logarithmically at large momenta, implying that there are formally infinitely many bound states. The physical situation corresponds to the cut-off form-factors $g(k, \varepsilon)$ present in $K$, which leads to the finite result for the norm $\|K\|$.

A specific situation occurs when the 2-body scattering length $a$ is large, $a \gg r_0$. Then the number of bound states lying between $\left(-\frac{1}{ma^2}\right)$ and $\left(-\frac{1}{mr_0^2}\right)$ is approximately equal to

$$N \sim \frac{1}{\pi} \ln \frac{|a|}{r_0}$$

and when $|a|$ is increasing, $|a| \rightarrow \infty$, there appears an accumulation point of bound states (the Efimov effect). For 3 nucleons however $N < 1$ and the effect is absent, but for three $^4$He atoms $a = 104\text{A}, r_0 \approx 7\text{A}$ and the effect is theoretically possible [4].

Since the Efimov states are almost on-shell, it is convenient to calculate them using the 3-body unitarity and the $N/D$ method [14]. Numerical results obtained (see Fig. 7 of [14]) support the estimate (8) and yield the explicit position of levels near the energy threshold.
To conclude with the bound state equation \[ (7) \] it is interesting to study the properties of the bound wave function, e.g. the size of the bound system. Here one encounters an important difference between 2- and 3-body systems \[ (13) \]. Namely the 2-body loosely bound system with a small binding energy \( \varepsilon \), \( m \varepsilon r_0^2 \ll 1 \) has a radius of the order of \( r_2 = \frac{1}{\sqrt{m \varepsilon}} \), \( r_2 \gg r_0 \). For the 3-body system the situation may be twofold. In case when a bound 2-body system exists as a subsystem and the 3-body bound state is close to the 2+1 threshold, one has a quasi-two-body situation, whereas when 2-body bound subsystems are absent the size of the 3-body bound state is always \( r_0 \) however small binding energy is \[ (14) \]. Modern calculations of 3-body bound states in the framework of STME and its development – Faddeev equations are done for \( ^3H \) and \( ^3He \) using systems of around 30 equations and exploiting realistic potentials describing \( NN \) scattering and bound states in the large energy interval (0-350 MeV), see e.g. the review in \[ (17) \]. Unfortunately results of calculations yield significant underbinding of around 10-15%, may be due to three-body forces, which are not exactly known, hence final results are model dependent, see \[ (15) \] for an example and references.

We go now to the \( dN \) scattering which was also the topic of the primary paper \[ (4) \]. The corresponding equations look like

\[
\begin{align*}
\frac{(\sqrt{3k^2/4-ME-\alpha_1})}{k^2-k_0^2} a_{3/2}(k, k_0) &= \frac{-1}{k_0^2+k^2+kk_0-ME} - \\
&- \int \frac{4\pi a_{3/2}(k', k_0)}{(k_0^2+k^2+kk_0-ME)(k^2-k_0^2)} \frac{dk'}{(2\pi)^3} ; \\
\frac{(\sqrt{3k^2/4-ME-\alpha_1})}{k^2-k_0^2} a_{3/2}(k, k_0) &= \frac{1}{k_0^2+k^2+kk_0-ME} + \\
+ \int \frac{4\pi (1/2a_{1/2}(k', k_0)+3/2b_{1/2}(k', k_0))}{(k_0^2+k^2+kk_0-ME)(k^2-k_0^2)} \frac{dk'}{(2\pi)^3} ; \\
\frac{(\sqrt{3k^2/4-ME-\alpha_1})}{k^2-k_0^2} b_{1/2}(k, k_0) &= \frac{3/2}{k_0^2+k^2+kk_0-ME} + \\
+ \int \frac{4\pi (3/2a_{1/2}(k', k_0)+1/2b_{1/2}(k', k_0))}{(k_0^2+k^2+kk_0-ME)(k^2-k_0^2)} \frac{dk'}{(2\pi)^3} .
\end{align*}
\]

Here \( a_{3/2} \) is the \( Nd \) quartet \( (S = 3/2) \) scattering amplitude, while \( b_{1/2} \) and \( a_{1/2} \) are doublet \( (S = \frac{1}{2}) \) amplitudes corresponding to the singlet and triplet last \( NN \) interaction respectively. It is seen that the kernel for \( S = 3/2 \) is mostly negative and allows for a faster convergence, in contrast to the doublet \( (S = 1/2) \) case. Numerical result for quartet scattering length \( a_{3/2} = 5.1 \text{fm} \) obtained in \[ (4) \] is not far from experimental value \[ (19) \], whereas doublet scattering requires full off-shell calculation \[ (4) \].
3 Hyperspherical Method

Heretofore the basic dynamics was assumed to be quasi-two-body (however the Faddeev technic allows for the full off-shell description), in the sense that typical distance $R$ between an interacting pair and a third spectator particle is large, $R \gg r_0$. However this situation is an exclusion, and not the rule, which can be understood from the representation of the w.f. through the 3 body Green’s function ($\xi, \eta$ are Jacobi coordinates)

$$\psi(\xi, \eta) = \psi_0(\xi, \eta) + \int G(\xi - \xi', \eta - \eta')V_3(\xi', \eta')\psi(\xi', \eta')d\xi'd\eta'. \quad (10)$$

Here $G(\xi, \eta) = \frac{\kappa^2(\kappa \rho)}{\rho^5}$, $\kappa = \sqrt{2m|E|}$, $\rho = \sqrt{\xi^2 + \eta^2}$, and $V_3$ includes all interaction terms. The asymptotics of $\psi$ is given by $G$ and is equal to

$$\psi(\xi, \eta) \sim \frac{1}{\rho^4}, \quad \rho \rightarrow \infty. \quad (11)$$

Hence the 3-body kynematics tends to concentrate all 3-body w.f. inside the interaction region of all 3 particles which generates small radius of w.f. even for barely bound 3-body states. (This is also true for $N$-body systems $N \geq 3$). In this situation any pair angular momentum $l_{ij}$ contributes to the total energy of the system an amount $\Delta E \sim \frac{l_{ij}(l_{ij}+1)}{2mr_0^2}$ which for the 3 nucleon system with $r_0 \sim 1$ fm and for $l_{ij} = 1$ is of the order of $\Delta E \sim 50$ MeV, while for the 3q system with $m = m_q \sim 0.3$ GeV and $r_0 \sim 0.5$ fm, $\Delta E_q \sim 600$ MeV.

Therefore it is advantageous to have a wave function with the minimal number of pair internal angular momenta for the given total momentum $L$. This basis is provided by hyperspherical functions ($K$-harmonics) due to the following properties:

i) the solution of the condition $\hat{l}_{ij}\Psi = 0, i \neq j = 1, ... N$ is given by the representation $\Psi_{K=0} = \Psi_0(\rho)$,

$$\rho^2 = \frac{1}{N} \sum_{i<j=1}^n (r_i - r_j)^2 \quad (12)$$

where all particles are assumed to have the same mass.

ii) The function $\Psi_K(r_1, ... r_N) = u_K(\Omega)\chi_K(\rho)$, where $u_K(\Omega) = \frac{P_K(r_1, ... r_N)}{\rho^K}$ and $P_K$ - harmonic polynomial contains excited angular momenta $l_1, ... l_{N-1}$ the arithmetic sum of which is equal to $K$. 

5
Therefore the basis $\Psi_K$ corresponds to the minimal excitation of angular momenta and is advantageous for compact $N$-body systems. Since as was explained the majority of such systems are compact, the Hyperspherical Expansion Approach (HEA) \[7\] formulated as a system of coupled integral or differential equations has proved to be very successful both for few-nucleon systems \[7,8\], where short-range correlations can be taken into account in the hyperspherical correlated basis (last ref. in \[18\]), and atomic physics \[9\]. It was understood afterwards \[10,11\] that the HEA works even better for $3q$ systems, since interaction there contains no repulsive core and confinement excludes two-body channels.

Therefore already the lowest approximation with $K = 0$ yields the 1% accuracy for the baryon energy \[10,11,12\].

In this case the baryon state is characterized by the grand angular momentum $K$ and radial quantum number $n = 0, 1, 2$, which counts number of zeros of the w.f. in the $\rho$-space. A typical calculation was done in \[12\], and the result depends on only two input parameters: string tension $\sigma = 0.15$ GeV$^2$ and $\alpha_s = 0.4$, while current masses of light quarks have been put to zero. The spin-averaged masses $\frac{1}{2}(M_n + M_\Delta)$ have been computed to eliminate effect of hyperfine splitting.

To illustrate the simplicity of the method, let us quote the the equation for the for the dominant hyperspherical harmonics $\psi_K(\rho) = \frac{\chi_K(\rho)}{\sqrt{\rho}}$,

$$-\frac{1}{2\mu} \frac{d^2\psi_K}{d\rho^2} + W_{KK}(\rho)\psi_K(\rho) = E_K\psi_K(\rho)$$

(13)

where $W_{KK}(\rho)$ is the sum of kinetic (angular) and potential energies, and $\mu$ is a constituent quark mass to be found below dynamically. The nonrelativistic appearance of this equation contains nevertheless the full relativistic dynamics, since $\mu$ is the einbein field needed to get rid of square roots in the relativistic quark action.

The explicit expression for $W_{KK}$ is

$$W_{KK}(\rho) = \frac{d}{2\mu \rho^2} + V_{KK}(\rho), \quad d = (K + \frac{3}{2})(K + \frac{5}{2})$$

(14)

while $V_{KK}(\rho) = (u_K^+(\Omega)\hat{V}u_K(\Omega))$, is the total potential $\hat{V}$, including 2-body and 3-body parts, averaged over hyperspherical harmonics, which is done analytically. E.g. for the $Y$-type $3q$ confining potential one has $V_{KK}(\rho) = \ldots$
1.58σρ. It is remarkable that to find the eigenvalues $E_K$ with the 1% accuracy one does not need to solve equation (13), but instead is approximating $W_{KK}(ρ)$ near the minimum point $ρ_0$ by the oscillator well:

$$W_{KK}(ρ) = W_{KK}(ρ_0) + \frac{1}{2}(ρ - ρ_0)^2W''_{KK}(ρ_0), \quad \frac{dW_{KK}}{dρ}|_{ρ=ρ_0} = 0. \quad (15)$$

The resulting eigenvalues are found immediately:

$$E_{Kn} \simeq W_{KK}(ρ_0) + \omega(n + \frac{1}{2}), \quad \omega^2 = W''_{KK}/μ. \quad (16)$$

The total baryon mass is calculated as $M_{Kn}(μ) = \frac{3}{2}μ + E_{Kn}(μ)$, and finally $μ = μ_0$ is to be found from the stationary point condition $\frac{∂M_{Kn}(μ)}{∂μ}|_{μ=μ_0} = 0$. This gives the constituent quark mass $μ_0 = 0.957\sqrt{σ} = 0.37 \text{ GeV}$ and finally the baryon mass is $M_{Kn}(μ_0)$. The masses $\frac{1}{2}(M_N + M_Δ)$ computed in this way are shown below in Table 1.

| State     | $M_{Kn} + \langle ΔH \rangle_{self}$ | $⟨ΔH⟩_{coul}$ | $M_{tot}^{Kn}$ | $M_{tot}^{exp}$ |
|-----------|------------------------------------|---------------|---------------|----------------|
| $K = 0, n = 0$ | 1.36                              | -0.274        | 1.08          | 1.08           |
| $K = 0, n = 1$ | 2.19                              | -0.274        | 1.91          | ?              |
| $K = 0, n = 2$ | 2.9                               | -0.274        | 2.62          | ?              |
| $K = L = 1, n = 0$ | 1.85                              | -0.217        | 1.63          | 1.6            |
| $K = 2, n = 0$ | 2.23                              | -0.186        | 2.04          | ?              |

As it seen from the Table the calculated spin-averaged mass $\frac{1}{2}(M_N + M_Δ)$ agrees well with the experimental average, the same is also true for lowest negative parity states with $K = L = 1$, which should be compared with $\frac{1}{2}, \frac{3}{2}$ states of $N$ and $Δ$ respectively.

We also notice that breathing modes ($n > 0$) have excitation energy around 0.8 GeV while orbital excitations $K = L = 1$ have energy interval around 0.5 GeV.

One of important advantages of HEA is that in the lowest approximation there is no need for numerical computations – as demonstrated above the result for the mass can be obtained analytically with 1% accuracy as can be checked by comparison with exact calculations, see [10]-[12].
To conclude, the on-shell approach of STME (and its Faddeev generalization) and the HEA are two alternatives which describe opposite physical situations. Their coexistence has played a very important stimulating role for the development of the few-body physics in the last four decades.

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