MINIMUM DISTANCE FUNCTIONS OF GRADED IDEALS AND REED-MULLER-TYPE CODES

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ABSTRACT. We introduce and study the minimum distance function of a graded ideal in a polynomial ring with coefficients in a field, and show that it generalizes the minimum distance of projective Reed-Muller-type codes over finite fields. This gives an algebraic formulation of the minimum distance of a projective Reed-Muller-type code in terms of the algebraic invariants and structure of the underlying vanishing ideal. Then we give a method, based on Gröbner bases and Hilbert functions, to find lower bounds for the minimum distance of certain Reed-Muller-type codes. Finally we show explicit upper bounds for the number of zeros of polynomials in a projective nested cartesian set and give some support to a conjecture of Carvalho, Lopez-Neumann and López.

1. Introduction

Let \( S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^\infty S_d \) be a polynomial ring over a field \( K \) with the standard grading and let \( I \neq (0) \) be a graded ideal of \( S \) of Krull dimension \( k \). The Hilbert function of \( S/I \) is:
\[
H_I(d) := \dim_K(S_d/I_d), \quad d = 0, 1, 2, \ldots,
\]
where \( I_d = I \cap S_d \). By a theorem of Hilbert, there is a unique polynomial \( h_I(x) \in \mathbb{Q}[x] \) of degree \( k-1 \) such that \( H_I(d) = h_I(d) \) for \( d \gg 0 \). The degree of the zero polynomial is \(-1\).

The degree or multiplicity of \( S/I \) is the positive integer
\[
\deg(S/I) := \begin{cases} 
(k-1)! \lim_{d \to \infty} H_I(d)/d^{k-1} & \text{if } k \geq 1, \\
\dim_K(S/I) & \text{if } k = 0.
\end{cases}
\]

Let \( F_d \) be the set of all zero-divisors of \( S/I \) not in \( I \) of degree \( d \geq 0 \):
\[
F_d := \{ f \in S_d \mid f \notin I, (I : f) \neq I \},
\]
where \( (I : f) = \{ h \in S \mid hf \in I \} \) is a quotient ideal. Notice that \( F_0 = \emptyset \).

The main object of study here is the function \( \delta_I : \mathbb{N} \to \mathbb{Z} \) given by
\[
\delta_I(d) := \begin{cases} 
\deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in F_d\} & \text{if } F_d \neq \emptyset, \\
\deg(S/I) & \text{if } F_d = \emptyset.
\end{cases}
\]

We call \( \delta_I \) the minimum distance function of \( I \). If \( I \) is a prime ideal, then \( F_d = \emptyset \) for all \( d \geq 0 \) and \( \delta_I(d) = \deg(S/I) \). We show that \( \delta_I \) generalizes the minimum distance function of projective Reed-Muller-type codes over finite fields (Theorem 4.7). This abstract algebraic formulation of the minimum distance gives a new tool to study these type of linear codes.

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To compute $\delta_I(d)$ is a difficult problem. For certain family of ideals we will give lower bounds for $\delta_I(d)$ which are easier to compute.

Fix a monomial order $<$ on $S$. Let $\Delta_<(I)$ be the footprint of $S/I$ consisting of all the standard monomials of $S/I$, with respect to $<$, and let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis of $I$. Then $\Delta_<(I)$ is the set of all monomials of $S$ that are not a multiple of any of the leading monomials of $g_1, \ldots, g_r$ (Lemma 2.7). A polynomial $f$ is called standard if $f \neq 0$ and $f$ is a $K$-linear combination of standard monomials. If $\Delta_<(I) \cap S_d = \{t^{a_1}, \ldots, t^{a_n}\}$ and $F_{<,d} = \{f = \sum_i \lambda_i t^{a_i} \mid f \neq 0, \lambda_i \in K, (I : f) \neq I\}$, then using the division algorithm [4, Theorem 3, p. 63] we can write:

$$
\delta_I(d) = \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in F_d\} = \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in F_{<,d}\}.
$$

Notice that $F_d \neq \emptyset$ if and only if $F_{<,d} \neq \emptyset$. If $K = \mathbb{F}_q$ is a finite field, then the number of standard polynomials of degree $d$ is $n^d - 1$, where $n$ is the number of standard monomials of degree $d$. Hence, we can compute $\delta_I(d)$ for small values of $n$ and $q$ (Examples 7.1 and 7.2).

Upper bounds for $\delta_I(d)$ can be obtained by fixing a subset $F'_{<,d}$ of $F_{<,d}$ and computing

$$
\delta'_I(d) = \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in F'_{<,d}\} \geq \delta_I(d).
$$

Typically one uses $F'_{<,d} = \{f = \sum_i \lambda_i t^{a_i} \mid f \neq 0, \lambda_i \in \{0,1\}, (I : f) \neq I\}$ or a subset of it.

Lower bounds for $\delta_I(d)$ are harder to find. Thus, we seek to estimate $\delta_I(d)$ from below. So, with this in mind, we introduce the footprint function of $I$:

$$
fp_I(d) = \begin{cases} 
\deg(S/I) - \max\{\deg(S/(\text{in}_<(I), t^{a})) \mid t^a \in \Delta_<(I)_d\} & \text{if } \Delta_<(I)_d \neq \emptyset, \\
\deg(S/I) & \text{if } \Delta_<(I)_d = \emptyset,
\end{cases}
$$

where $\text{in}_<(I) = (\text{in}_<(g_1), \ldots, \text{in}_<(g_r))$ is the initial ideal of $I$, $\text{in}_<(g_i)$ is the initial monomial of $g_i$ for $i = 1, \ldots, s$, and $\Delta_<(I)_d = \Delta_<(I) \cap S_d$.

The contents of this paper are as follows. In Section 2 we present some of the results and terminology that will be needed throughout the paper.

Some of our results rely on a degree formula to compute the number of zeros that a homogeneous polynomial has in any given finite set of points in a projective space (Lemma 6.2).

In Section 3 we study $\delta_I$ and present an alternative formula for $\delta_I$, pointed out to us by Vasconcelos, valid for unmixed graded ideals (Theorem 4.1). If $F_d \neq \emptyset$ for $d \geq 1$ and $I$ is unmixed, then, by Lemma 4.1 $fp_I(d)$ is a lower bound of $\delta_I(d)$, but even in this case $fp_I(d)$ could be negative (Example 7.3). For this reason we will sometimes make extra assumptions requiring that $\dim(S/I) \geq 1$ and that $t_i$ is a zero-divisor of $S/I$ for $i = 1, \ldots, s$ (cf. Lemma 2.7).

One of our first main results gives some general properties of $\delta_I$ for an interesting class of graded ideals and show its relation to $fp_I$:

**Theorem 4.5.** Let $I \subset S$ be an unmixed graded ideal of dimension $\geq 1$ such that $t_i$ is a zero-divisor of $S/I$ for $i = 1, \ldots, s$. The following hold.

(i) $F_d \neq \emptyset$ for $d \geq 1$.

(ii) $\delta_I(d) \geq fp_I(d)$ for $d \geq 1$.

(iii) $\deg(S/(I, t^a)) \leq \deg(S/(\text{in}_<(I), t^a)) \leq \deg(S/I)$ for any $t^a \in \Delta_<(I) \cap S_d$.

(iv) $fp_I(d) \geq 0$.

(v) $\delta_I(d) \geq \delta_I(d+1) \geq 0$ for $d \geq 1$.

(vi) If $I$ is a radical ideal and its associated primes are generated by linear forms, then there is $r \geq 1$ such that $\delta_I(1) > \cdots > \delta_I(r) = \delta_I(d) = 1$ for $d \geq r$. 


The study of $\delta_1$ was motivated by the notion of minimum distance of linear codes in coding theory. For convenience we recall this notion. Let $K = F_q$ be a finite field. A linear code is a linear subspace of $K^m$ for some $m$. The basic parameters of a linear code $C$ are length: $m$, dimension: $\dim_K(C)$, and minimum distance:

\[ \delta(C) := \min\{ \|v\| : 0 \neq v \in C \}, \]

where $\|v\|$ is the number of non-zero entries of $v$.

The minimum distance of affine Reed-Muller-type codes has been studied using Gröbner bases techniques; see [2] [8] [9] and the references therein. Of particular interest to us is the footprint technique introduced by Geil [8] to bound from below the minimum distance. In this work we extend this technique to projective Reed-Muller-type codes, a special type of linear codes that generalizes affine Reed-Muller-type codes [15]. These projective codes are constructed as follows.

Let $K = F_q$ be a finite field with $q$ elements, let $P^{s-1}$ be a projective space over $K$, and let $X$ be a subset of $P^{s-1}$. The vanishing ideal of $X$, denoted $I(X)$, is the ideal of $S$ generated by the homogeneous polynomials that vanish at all points of $X$. In this case the Hilbert function of $S/I(X)$ is denoted by $H_X(d)$. We can write $X = \{ [P_1], \ldots, [P_m] \} \subset P^{s-1}$ with $m = |X|$.

Fix a degree $d \geq 0$. For each $i$ there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$. There is a $K$-linear map given by

\[ \text{ev}_d: S_d \to K^m, \quad f \mapsto \left( \frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_m)}{f_m(P_m)} \right). \]

The image of $S_d$ under $\text{ev}_d$, denoted by $C_X(d)$, is called a projective Reed-Muller-type code of degree $d$ on $X$ [5] [12]. The basic parameters of the linear code $C_X(d)$ are:

(a) length: $|X|$,  
(b) dimension: $\dim_K C_X(d)$,  
(c) minimum distance: $\delta_X(d) := \delta(C_X(d))$.

The regularity of $S/I(X)$, denoted $\text{reg}(S/I(X))$, is the least integer $r \geq 0$ such that $H_X(d)$ is equal to $h_{I(X)}(d)$ for $d \geq r$. As is seen below, the knowledge of the regularity of $S/I(X)$ is important for applications to coding theory. According to [10] and Proposition 2.15 there are integers $r \geq 0$ and $r_1 \geq 1$ such that

\[ 1 = H_X(0) < H_X(1) < \cdots < H_X(r-1) < H_X(d) = |X| \]

for $d \geq r = \text{reg}(S/I(X))$, and

\[ |X| = \delta_X(0) > \delta_X(1) > \cdots > \delta_X(r-1) > \delta_X(r_1) = \delta_X(d) = 1 \text{ for } d \geq r_1, \]

respectively. The integer $r_1$ is called the minimum distance regularity of $S/I(X)$. In general $r_1 \leq r$ (see the discussion below). Using the methods of [14] [23], the regularity of $S/I(X)$ can be effectively computed when $X$ is parameterized by monomials, but $r_1$ is very difficult to compute.

The Hilbert function and the minimum distance are related by the Singleton bound:

\[ 1 \leq \delta_X(d) \leq |X| - H_X(d) + 1. \]

In particular, if $d \geq \text{reg}(S/I(X)) \geq 1$, then $\delta_X(d) = 1$. The converse is not true (Example [7, 21]).

Thus, potentially good Reed-Muller-type codes $C_X(d)$ can occur only if $1 \leq d < \text{reg}(S/I(X))$. There are some families where $d \geq \text{reg}(S/I(X)) \geq 1$ if and only if $\delta_X(d) = 1$ [14] [20] [21], but we do not know of any set $X$ parameterized by monomials where this fails. If $X$ is parameterized by monomials we say that $C_X(d)$ is a projective parameterized code [19] [23].

A main problem in Reed-Muller-type codes is the following. If $X$ has nice algebraic or combinatorial structure, find formulas in terms of $s, q, d$, and the structure of $X$, for the basic parameters
of $C_X(d)$: $H_X(d)$, $\deg(S/I(X))$, $\delta_X(d)$, and $\reg(S/I(X))$. Our main results can be used to study this problem, especially when $X$ is parameterized by monomials or when $X$ is a projective nested cartesian set (see Definition 6.1).

The basic parameters of projective Reed-Muller-type codes have been computed in some cases. If $X = \mathbb{P}^{s-1}$, $C_X(d)$ is the classical projective Reed–Muller code. Formulas for its basic parameters were given in [21, Theorem 1]. If $X$ is a projective torus (see Definition 2.11), $C_X(d)$ is the generalized projective Reed–Solomon code. Formulas for its basic parameters were given in [20, Theorem 3.5]. If $X$ is the image of a cartesian product of subsets of $K$, under the map $K^{s-1} \to \mathbb{P}^{s-1}$, $x \to [x, 1]$, then $C_X(d)$ is an affine cartesian code and formulas for its basic parameters were given in [9, 14].

We give a formula, in terms of the degree, for the number of zeros in $X \subset \mathbb{P}^{s-1}$ of any homogeneous polynomial (Lemma 3.2). As a consequence we derive our second main result:

**Theorem 4.7.** If $|X| \geq 2$, then $\delta_X(d) = \delta_{1(X)}(d) \geq 1$ for $d \geq 1$.

In particular, if $<$ is a monomial order on $S$, then

$$
\delta_X(d) = \deg(S/I) - \max\{\deg(S/(I, f)) | f \in \mathcal{F}_{<d}\}.
$$

This description allows us to compute the minimum distance of Reed-Muller-type codes for small values of $q$ and $s$ (Corollary 4.3) and it gives an algebraic formulation of the minimum distance in terms of the algebraic properties and invariants of the vanishing ideal. The formula of Eq. (1.1) is more interesting from the theoretical point of view than from a computational perspective. Indeed, if $t_i$ is a zero-divisor of $S/I(X)$ for all $i$, then one has:

$$
\delta_X(d) \geq \delta_{1(X)}(d) \geq 0 \quad \text{for} \quad d \geq 1.
$$

This inequality gives a lower bound for the minimum distance of any Reed-Muller-type code over a set $X$ parameterized by relatively prime monomials because in this case $t_i$ is a zero-divisor of $S/I(X)$ for $i = 1, \ldots , s$ (Corollary 4.9). Using SAGE [15] and a generator matrix of $C_X(d)$ one can compute the minimum distance of $C_X(d)$ in a more efficient way than by using our formula at least in the case that $C_X(d)$ arises from an affine Reed-Muller type code [15].

Let $d_1, \ldots , d_s$ be a non-decreasing sequence of positive integers with $d_1 \geq 2$ and $s \geq 2$, and let $L$ be the ideal of $S$ generated by the set of all $t_i t_j^{d_j}$ such that $1 \leq i < j \leq s$. It turns out that the ideal $L$ is the initial ideal of the vanishing ideal of a projective nested cartesian set (see the discussion below). In Section 5 we study the ideal $L$ and show some degree equalities as a preparation to show some applications.

Projective nested cartesian codes were introduced and studied in [8] (see Definition 6.1). This type of evaluation codes generalize the classical projective Reed–Muller codes [21]. As an application we will give some support for the following interesting conjecture.

**Conjecture 6.2.** (Carvalho, Lopez-Neumann, and López [8]) Let $A_1, \ldots , A_s$ be subsets of $K$ and let $C_X(d)$ be the $d$-th projective nested cartesian code on the set $X = [A_1 \times \cdots \times A_s]$ with $d_i = |A_i|$ for $i = 1, \ldots , s$. Then its minimum distance is given by

$$
\delta_X(d) = \begin{cases} 
(d_{k+2} - \ell + 1) d_{k+3} \cdots d_s, & \text{if } d \leq \sum_{i=2}^{s} (d_i - 1) , \\
1, & \text{if } d \geq \sum_{i=2}^{s} (d_i - 1) + 1 ,
\end{cases}
$$

where $0 \leq k \leq s - 2$ and $\ell$ are integers such that $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$. 


Let \(<\) be the lexicographical order on \(S\) with \(t_1 < \cdots < t_s\). Carvalho et al. found a Gröbner basis for \(I(\mathcal{X})\) whose initial ideal is \(L\), and obtained formulas for the regularity and the degree of the coordinate ring \(S/I(\mathcal{X})\) \([3]\) (Proposition 6.3).

They showed the conjecture when the \(A_i\)'s are subfields of \(\mathbb{F}_q\), and essentially showed that their conjecture can be reduced to:

**Conjecture 6.4.** (Carvalho, Lopez-Neumann, and López \([3]\)) If \(f \in S_q\) is a standard polynomial such that \((I(\mathcal{X}) : f) \neq I(\mathcal{X})\), \(1 \leq d \leq \sum_{i=2}^s (d_i - 1)\), and \(V_X(f)\) is the zero set of \(f\) in \(\mathcal{X}\), then \(|V_X(f)| \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1) d_{k+3} \cdots d_s\), where \(0 \leq k \leq s - 2\) and \(\ell\) are integers such that \(d = \sum_{i=2}^{k+1} (d_i - 1) + \ell\) and \(1 \leq \ell \leq d_{k+2} - 1\).

Let \(f \neq 0\) be a standard polynomial, with respect to \(<\), of degree \(d \geq 1\) with \(d \leq \sum_{i=2}^s (d_i - 1)\) and \((I(\mathcal{X}) : f) \neq I(\mathcal{X})\), and let \(\in_X(f) = t^a\) be its initial monomial. We can write

\[
t^a = t_r^a_r \cdots t_s^a_s,
\]

with \(1 \leq r \leq s\), \(a_r \geq 1\), \(0 \leq a_i \leq d_i - 1\) for \(i > r\).

We show an explicit upper bound for the number of zeros of \(f\) in \(\mathcal{X}\):

**Theorem 6.5.** \(|V_X(f)| \leq \deg(S/\in_X(I(\mathcal{X})), t^a)\) =

\[
\begin{align*}
\deg(S/I(\mathcal{X})) - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) & \quad \text{if } a_r \leq d_r, \\
\deg(S/I(\mathcal{X})) - (d_{r+1} - a_{r+1}) \cdots (d_s - a_s) & \quad \text{if } a_r \geq d_r + 1.
\end{align*}
\]

Then we use Theorem 6.5 to give some support for Conjecture 6.4.

**Theorem 6.6.** If \(t_1\) divides \(t^a = \in_X(f)\), then

\[
|V_X(f)| \leq \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1) d_{k+3} \cdots d_s,
\]

where \(0 \leq k \leq s - 2\) and \(\ell\) are integers such that \(d = \sum_{i=2}^{k+1} (d_i - 1) + \ell\) and \(1 \leq \ell \leq d_{k+2} - 1\).

As a consequence we show that the minimum distance of \(C_X(d)\) proposed in Conjecture 6.2 is in fact the minimum distance of certain evaluation linear code (Corollary 6.9). Finally in Section 7 we show some examples that illustrate how some of our results can be used in practice.

For all unexplained terminology and additional information, we refer to \([1, 4, 8]\) (for the theory of Gröbner bases, commutative algebra, and Hilbert functions), and \([16, 26]\) (for the theory of error-correcting codes and linear codes).

2. Preliminaries

In this section, we present some of the results that will be needed throughout the paper and introduce some more notation. All results of this section are well-known. To avoid repetitions, we continue to employ the notations and definitions used in Section 1.

Let \(S = K[t_1, \ldots, t_s] = \bigoplus_{d=0}^\infty S_d\) be a graded polynomial ring over a field \(K\), with the standard grading, and let \((0) \neq I \subset S\) be a graded ideal. We will use the following multi-index notation: for \(a = (a_1, \ldots, a_s) \in \mathbb{N}^s\), set \(t^a := t_1^{a_1} \cdots t_s^{a_s}\). The multiplicative group of \(K\) is denoted by \(K^*\). As usual, \(m\) will denote the maximal ideal of \(S\) generated by \(t_1, \ldots, t_s\), and \(\text{ht}(I)\) will denote the height of the ideal \(I\). By the dimension of \(I\) (resp. \(S/I\)) we mean the Krull dimension of \(S/I\). The Krull dimension of \(S/I\) is denoted by \(\dim(S/I)\).

One of the most useful and well-known facts about the degree is its additivity:
Proposition 2.1. (Additivity of the degree [14 Proposition 2.5]) If \( I \) is an ideal of \( S \) and \( I = q_1 \cap \cdots \cap q_m \) is an irredundant primary decomposition, then
\[
\deg(S/I) = \sum_{\text{ht}(q_i) = \text{ht}(I)} \deg(S/q_i).
\]

Theorem 2.2. (Hilbert [11 Theorem 4.1.3]) Let \( I \subset S \) be a graded ideal of dimension \( k \). Then there is a polynomial \( h_I(x) \in \mathbb{Q}[x] \) of degree \( k \) such that \( H_I(d) = h_I(d) \) for \( d \gg 0 \).

If \( f \in S \), the quotient ideal of \( I \) with respect to \( f \) is given by \( (I : f) = \{ h \in S \mid hf \in I \} \). The element \( f \) is called a zero-divisor of \( S/I \) if there is \( \mathfrak{a} \neq \mathfrak{p} \in S/I \) such that \( f\mathfrak{p} = \mathfrak{a} \), and \( f \) is called regular on \( S/I \) if \( f \) is not a zero-divisor. Notice that \( f \) is a zero-divisor if and only if \( (I : f) \neq I \).

An associated prime of \( I \) is a prime ideal \( \mathfrak{p} \) of \( S \) of the form \( \mathfrak{p} = (I : f) \) for some \( f \in S \).

Theorem 2.3. [27 Lemma 2.1.19, Corollary 2.1.30] If \( I \) is an ideal of \( S \) and \( I = q_1 \cap \cdots \cap q_m \) is an irredundant primary decomposition with \( \text{rad}(q_i) = p_i \), then the set of zero-divisors \( Z(S/I) \) of \( S/I \) is given by
\[
Z(S/I) = \bigcup_{i=1}^{m} p_i,
\]
and \( p_1, \ldots, p_m \) are the associated primes of \( I \).

In the introduction we defined the regularity of the coordinate ring of a finite set in a projective space. This notion is defined for any graded ideal.

Definition 2.4. The regularity of \( S/I \), denoted \( \text{reg}(S/I) \), is the least integer \( r \geq 0 \) such that \( H_I(d) \) is equal to \( h_I(d) \) for \( d \geq r \).

If \( I \subset S \) is Cohen-Macaulay and \( \dim(S/I) = 1 \), then \( \text{reg}(S/I) \) is the Castelnuovo-Mumford regularity of \( S/I \) in the sense of [7]. If \( \dim(S/I) = 0 \), then \( \text{reg}(S/I) \) is the least positive integer \( d \) such that \( m^d \subset I \).

The footprint of an ideal. Let \( \prec \) be a monomial order on \( S \) and let \( (0) \neq I \subset S \) be an ideal. If \( f \) is a non-zero polynomial in \( S \), then one can write
\[
f = \lambda_1 t^{\alpha_1} + \cdots + \lambda_r t^{\alpha_r},
\]
with \( \lambda_i \in K^* \) for all \( i \) and \( t^{\alpha_1} > \cdots > t^{\alpha_r} \). The leading monomial \( t^{\alpha_1} \) of \( f \) is denoted by \( \text{in}_{\prec}(f) \).

The initial ideal of \( I \), denoted by \( \text{in}_{\prec}(I) \), is the monomial ideal given by
\[
\text{in}_{\prec}(I) = \{ \text{in}_{\prec}(f) \mid f \in I \}.
\]

A monomial \( t^a \) is called a standard monomial of \( S/I \), with respect to \( \prec \), if \( t^a \) is not the leading monomial of any polynomial in \( I \). The set of standard monomials, denoted \( \Delta_{\prec}(I) \), is called the footprint of \( S/I \). The image of the standard monomials of degree \( d \), under the canonical map \( S \to S/I, x \to x \), is equal to \( S_d/I_d \), and the image of \( \Delta_{\prec}(I) \) is a basis of \( S/I \) as a \( K \)-vector space (see [27 Proposition 3.3.13]). In particular, if \( I \) is graded, then \( H_I(d) \) is the number of standard monomials of degree \( d \).

A subset \( \mathcal{G} = \{ g_1, \ldots, g_r \} \) of \( I \) is called a Gröbner basis of \( I \) if
\[
\text{in}_{\prec}(I) = \{ \text{in}_{\prec}(g_1), \ldots, \text{in}_{\prec}(g_r) \}.
\]

Lemma 2.5. [2 p. 2] Let \( I \subset S \) be an ideal generated by \( \mathcal{G} = \{ g_1, \ldots, g_r \} \), then
\[
\Delta_{\prec}(I) \subset \Delta_{\prec}(\text{in}_{\prec}(g_1), \ldots, \text{in}_{\prec}(g_r)),
\]
with equality if \( \mathcal{G} \) is a Gröbner basis.
Proposition 2.10. Let \( f \) be a point in \( \mathbb{P}^s \), then \( f \) can be written as

\[
  f = a_1 g_1 + \cdots + a_r g_r + h,
\]

where \( a_i, h \in S \) and either \( h = 0 \) or \( h \neq 0 \) and no term of \( h \) is divisible by one of the initial monomials \( \langle g_1 \rangle, \ldots, \langle g_r \rangle \). Furthermore if \( a_i g_i \neq 0 \), then \( \langle f \rangle \geq \langle a_i g_i \rangle \).

Lemma 2.7. Let \( G = \{ g_1, \ldots, g_r \} \) be a Gröbner basis of \( I \). If for some \( i \), the variable \( t_i \) does not divide \( \langle g_j \rangle \) for all \( j \), then \( t_i \) is a regular element on \( S/I \).

Proof. Assume that \( t_i f \in I \). By the division algorithm we can write \( f = g + h \), where \( g \in I \) and \( h \) is 0 or a standard polynomial. It suffices to show that \( h = 0 \). If \( h \neq 0 \), then \( t_i \langle h \rangle \in \langle I \rangle \), hence, using our hypothesis on \( t_i \), we get \( \langle h \rangle \in \langle I \rangle \), a contradiction.

This lemma tells us that if \( t_i \) is a zero-divisor of \( S/I \) for all \( i \), then any variable \( t_i \) must occur in an initial monomial \( \langle g_j \rangle \) for some \( j \).

Theorem 2.8. (Macaulay [27, Corollary 3.3.15]) If \( I \) is a graded ideal of \( S \), then \( S/I \) and \( S/\langle \langle I \rangle \rangle \) have the same Hilbert function and the same degree and regularity.

Vanishing ideals of finite sets. The projective space of dimension \( s - 1 \) over the field \( K \), denoted \( \mathbb{P}_K^{s-1} \), or simply \( \mathbb{P}^{s-1} \), is the quotient space

\[
  (K^s \setminus \{0\})/\sim
\]

where two points \( \alpha, \beta \) in \( K^s \setminus \{0\} \) are equivalent under \( \sim \) if \( \alpha = c \beta \) for some \( c \in K^* \). It is usual to denote the equivalence class of \( \alpha \) by \([\alpha]\).

For a given a subset \( X \subset \mathbb{P}^{s-1} \) define \( I(X) \), the vanishing ideal of \( X \), as the ideal generated by the homogeneous polynomials in \( S \) that vanish at all points of \( X \), and given a graded ideal \( I \subset S \) define its zero set relative to \( X \) as

\[
  V_X(I) = \{ [\alpha] \in X | f(\alpha) = 0, \forall f \in I \ \text{homogeneous} \}.
\]

In particular, if \( f \in S \) is homogeneous, the zero set \( V_X(f) \) of \( f \) is the set of all \([\alpha] \in X \) such that \( f(\alpha) = 0 \), that is \( V_X(f) \) is the set of zeros of \( f \) in \( X \).

Lemma 2.9. Let \( X \) be a finite subset of \( \mathbb{P}^{s-1} \), let \([\alpha]\) be a point in \( X \), with \( \alpha = (\alpha_1, \ldots, \alpha_s) \) and \( \alpha_k \neq 0 \) for some \( k \), and let \( I_{[\alpha]} \) be the vanishing ideal of \([\alpha]\). Then \( I_{[\alpha]} \) is a prime ideal,

\[
  I_{[\alpha]} = \langle \{ \alpha_k t_i - \alpha_i t_k | k \neq i \in \{1, \ldots, s\} \} \rangle, \quad \deg(S/I_{[\alpha]}) = 1,
\]

\[
  \text{ht}(I_{[\alpha]}) = s - 1, \quad \text{and} \quad I(X) = \bigcap_{[\beta] \in X} I_{[\beta]} \quad \text{is the primary decomposition of} \ I(X).
\]

If \( X \) is a subset of \( \mathbb{P}^{s-1} \) it is usual to denote the Hilbert function of \( S/I(X) \) by \( H_X \).

Proposition 2.10. [10] If \( X \subset \mathbb{P}^{s-1} \) is a finite set, then

\[
  1 = H_X(0) < H_X(1) < \cdots < H_X(r - 1) < H_X(d) = |X|
\]

for \( d \geq r = \text{reg}(S/I(X)) \).

Definition 2.11. The set \( T = \{ [(x_1, \ldots, x_s)] \in \mathbb{P}^{s-1} | x_i \in K^* \forall i \} \) is called a projective torus.
Projective Reed-Muller-type codes. Let $K = \mathbb{F}_q$ be a finite field, let $\mathcal{X}$ be a subset of $\mathbb{P}^{s-1}$, and let $P_1, \ldots, P_m$ be a set of representatives for the points of $\mathcal{X}$ with $m = |\mathcal{X}|$. In this paragraph all results are valid if we assume that $K$ is any field and that $\mathcal{X}$ is a finite subset of $\mathbb{P}^{s-1}$ instead of assuming that $K$ is finite. However the interesting case for coding theory is when $K$ is finite.

Fix a degree $d \geq 0$. For each $i$ there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$. Indeed suppose $P_i = [(a_1, \ldots, a_s)]$, there is at least one $k$ in $\{1, \ldots, s\}$ such that $a_k \neq 0$. Setting $f_i(t_1, \ldots, t_s) = t_k^d$ one has that $f_i \in S_d$ and $f_i(P_i) \neq 0$. There is a $K$-linear map:

$$
evd: S_d = K[t_1, \ldots, t_s]_d \rightarrow K^{[\mathcal{X}]}, \quad f \mapsto \left( \frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_m)}{f_m(P_m)} \right).$$

The map $\nevd$ is called an evaluation map. The image of $S_d$ under $\nevd$, denoted by $C_{\mathcal{X}}(d)$, is called a projective Reed-Muller-type code of degree $d$ over $\mathcal{X}$ [11]. It is also called an evaluation code associated to $\mathcal{X}$ [11].

**Definition 2.12.** The basic parameters of the linear code $C_{\mathcal{X}}(d)$ are its length $|\mathcal{X}|$, dimension $\dim_K C_{\mathcal{X}}(d)$, and minimum distance

$$\delta_{\mathcal{X}}(d) := \min\{\|v\| : 0 \neq v \in C_{\mathcal{X}}(d)\},$$

where $\|v\|$ is the number of non-zero entries of $v$.

**Lemma 2.13.** (a) The map $\nevd$ is well-defined, i.e., it is independent of the set of representatives that we choose for the points of $\mathcal{X}$. (b) The basic parameters of the Reed-Muller-type code $C_{\mathcal{X}}(d)$ are independent of $f_1, \ldots, f_m$.

**Proof.** (a): If $P'_1, \ldots, P'_m$ is another set of representatives, there are $\lambda_1, \ldots, \lambda_m$ in $K^*$ such that $P'_i = \lambda_i P_i$ for all $i$. Thus, $f(P'_i)/f_i(P'_i) = f(P_i)/f_i(P_i)$ for $f \in S_d$ and $1 \leq i \leq m$.

(b): Let $f'_1, \ldots, f'_m$ be homogeneous polynomials of $S$ of degree $d$ such that $f'_i(P_i) \neq 0$ for $i = 1, \ldots, m$, and let

$$\nevd': S_d \rightarrow K^{[\mathcal{X}]}, \quad f \mapsto \left( \frac{f(P_1)}{f_1(P_1)}, \ldots, \frac{f(P_m)}{f_m(P_m)} \right)$$

be the evaluation map relative to $f'_1, \ldots, f'_m$. Then $\ker(\nevd) = \ker(\nevd')$ and $\|\nevd(f)\| = \|\nevd'(f)\|$ for $f \in S_d$. It follows that the basic parameters of $\nevd(S_d)$ and $\nevd'(S_d)$ are the same. \hfill $\square$

**Lemma 2.14.** Let $\mathcal{Y} = \{[\alpha], [\beta]\}$ be a subset of $\mathbb{P}^{s-1}$ with two elements. The following hold.

(i) $\reg(S/I(\mathcal{Y})) = 1$.

(ii) There is $h \in S_1$, a form of degree 1, such that $h(\alpha) \neq 0$ and $h(\beta) = 0$.

(iii) For each $d \geq 1$, there is $f \in S_d$, a form of degree $d$, such that $f(\alpha) \neq 0$ and $f(\beta) = 0$.

(iv) If $\mathcal{X}$ is a subset of $\mathbb{P}^{s-1}$ with at least two elements and $d \geq 1$, then there is $f \in S_d$ such $f \notin I(\mathcal{X})$ and $(I(\mathcal{X}) : f) \neq I(\mathcal{X})$.

**Proof.** (i): As $H_{\mathcal{X}}(0) = 1$ and $|\mathcal{Y}| = 2$, by Proposition 2.10 we get that $H_{\mathcal{X}}(1) = |\mathcal{Y}| = 2$. Thus $S/I(\mathcal{Y})$ has regularity equal to 1.

(ii): Consider the evaluation map

$$\nev_1: S_1 \rightarrow K^2, \quad f \mapsto (f(\alpha)/f_1(\alpha), f(\beta)/f_2(\beta)).$$

By part (i) this map is onto. Thus $(1, 0)$ is in the image of $\nev_1$ and the result follows.

(iii): It follows from part (ii) by setting $f = h^d$. 

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(iv): By part (iii), there are distinct $[\alpha], [\beta]$ in $X$ and $f \in S_d$ such that $f(\alpha) \neq 0, f(\beta) = 0$. Then $f \notin I(X)$. Notice that $f(\beta) = 0$ if and only if $f \in I_{[\beta]}$. Hence, by Theorem 2.3 and Lemma 2.9, $f$ is a zero-divisor of $S/I(X)$, that is, $(I(X): f) \neq I(X)$. □

The next result was shown in [19, Proposition 5.2] and [24, Proposition 2.1] for some special types of Reed-Muller-type codes (cf. Theorem 4.5(vi)). In [19] (resp. [24]) it is assumed that $X$ is contained in a projective torus (resp. $X$ is not contained in a hyperplane and that there is $f \in S_d$ not vanishing at any point of $X$).

**Proposition 2.15.** There is an integer $r \geq 0$ such that

$$|X| = \delta_X(0) > \delta_X(1) > \cdots > \delta_X(d) = \delta_X(r) = 1 \text{ for } d \geq r.$$  

**Proof.** Assume that $\delta_X(d) > 1$, it suffices to show that $\delta_X(d) > \delta_X(d + 1)$. Pick $g \in S_d$ such that $g \notin I(X)$ and

$$|V_X(g)| = \max\{|V_X(f)| : ev_d(f) \neq 0; f \in S_d\}.$$

Then $\delta_X(d) = |X| - |V_X(g)| \geq 2$. Thus there are distinct points $[\alpha], [\beta]$ in $X$ such that $g(\alpha) \neq 0$ and $g(\beta) \neq 0$. By Lemma 2.14 there is a linear form $h \in S_1$ such that $h(\alpha) \neq 0$ and $h(\beta) = 0$. Hence the polynomial $hg$ is not in $I(X)$, has degree $d + 1$, and has at least $|V_X(g)| + 1$ zeros. Thus $\delta_X(d) > \delta_X(d + 1)$, as required. □

The following summarizes the well-known relation between projective Reed-Muller-type codes and the theory of Hilbert functions.

**Proposition 2.16.** ([12], [19]) The following hold.

(i) $H_X(d) = \dim_K C_X(d)$ for $d \geq 0$.
(ii) $\deg(S/I(X)) = |X|$.
(iii) $\delta_X(d) = 1$ for $d \geq \reg(S/I(X))$.
(iv) $S/I(X)$ is a Cohen–Macaulay reduced graded ring of dimension 1.
(v) $C_X(d) \neq (0)$ for $d \geq 1$.

3. Computing the number of zeros using the degree

In this section we give a degree formula to compute the number of zeros that a homogeneous polynomial has in any given finite set of points in a projective space over any field.

An ideal $I \subset S$ is called unmixed if all its associated primes have the same height and $I$ is called radical if $I$ is equal to its radical. The radical of $I$ is denoted by $\text{rad}(I)$.

**Lemma 3.1.** Let $I \subset S$ be a radical unmixed graded ideal. If $f \in S$ is homogeneous, $(I : f) \neq I$, and $A$ is the set of all associated primes of $S/I$ that contain $f$, then $\ht(I) = \ht(I, f)$ and

$$\deg(S/(I, f)) = \sum_{p \in A} \deg(S/p).$$

**Proof.** As $f$ is a zero-divisor of $S/I$ and $I$ is unmixed, there is an associated prime ideal $p$ of $S/I$ of height $\ht(I)$ such that $f \in p$. Thus $I \subset (I, f) \subset p$, and consequently $\ht(I) = \ht(I, f)$. Therefore the set of associated primes of $(I, f)$ of height equal to $\ht(I)$ is not empty and is equal to $A$. There is an irredundant primary decomposition

$$\deg(S/(I, f)) = \sum_{p \in A} \deg(S/p) = \sum_{p \in A} \deg(S/p).$$

(3.1) $(I, f) = q_1 \cap \cdots \cap q_r \cap q_{r+1} \cap \cdots \cap q'_l,$
where \( \text{rad}(q_i) = p_i, A = \{p_1, \ldots, p_r\} \), and \( \text{ht}(q_i) > \text{ht}(I) \) for \( i > r \). We may assume that the associated primes of \( S/I \) are \( p_1, \ldots, p_m \). Since \( I \) is a radical ideal, we get that \( I = \cap_{i=1}^m p_i \). Next we show the following equality:

\[
\tag{3.2}
p_1 \cap \cdots \cap p_m = q_1 \cap \cdots \cap q_r \cap q_{r+1} \cap \cdots \cap q_{r+\ell} \cap p_{r+1} \cap \cdots \cap p_m.
\]

The inclusion \( \supset \) is clear because \( q_i \subset p_i \) for \( i = 1, \ldots, r \). The inclusion \( \subset \) follows by noticing that the right hand side of Eq. (3.2) is equal to \( (I, f) \cap p_{r+1} \cap \cdots \cap p_m \), and consequently it contains \( I = \cap_{i=1}^m p_i \). Notice that \( \text{rad}(q_i^j) = p_i^j \) for all \( i, j \) and \( \text{p} \not\subset p_i \) for \( i \neq j \). Hence localizing Eq. (3.2) at the prime ideal \( p_i \) for \( i = 1, \ldots, r \), we get that \( p_i = I, f \cap (q_i)p_i \cap S = q_i \) for \( i = 1, \ldots, r \). Using Eq. (3.1) and the additivity of the degree the required equality follows. \( \square \)

**Lemma 3.2.** Let \( \subseteq \) be a finite subset of \( \mathbb{P}^{n-1} \) over a field \( \mathbb{K} \) and let \( I(\subseteq) \subset S \) be its graded vanishing ideal. If \( 0 \neq f \in S \) is homogeneous, then the number of zeros of \( f \) in \( \subseteq \) is given by

\[
|V_\subseteq(f)| = \begin{cases} 
\deg S/(I(\subseteq), f) & \text{if } (I(\subseteq): f) \neq I(\subseteq), \\
0 & \text{if } (I(\subseteq): f) = I(\subseteq). 
\end{cases}
\]

**Proof.** Let \( [P_1], \ldots, [P_m] \) be the points of \( \subseteq \) with \( m = |\subseteq| \), and let \( [P] \) be a point in \( \subseteq, \) with \( P = (\alpha_1, \ldots, \alpha_s) \) and \( \alpha_k \neq 0 \) for some \( k \). Then the vanishing ideal \( I_{[P]} \) of \( [P] \) is a prime ideal of height \( s-1 \),

\[
I_{[P]} = (\{\alpha_k t_i - \alpha_i t_k | k \neq i \in \{1, \ldots, s\}\}, \deg(S/I_{[P]}) = 1,
\]

and \( I(\subseteq) = \cap_{i=1}^m I_{[P]} \) is a primary decomposition (see Lemma [2.3]). In particular \( I(\subseteq) \) is an unmixed radical ideal of dimension 1.

Assume that \( (I(\subseteq): f) \neq I(\subseteq) \). Let \( \mathcal{A} \) be the set of all \( I_{[P]} \) that contain the polynomial \( f \). Then \( f(P_i) = 0 \) if and only if \( I_{[P]} \) is in \( \mathcal{A} \). Hence, by Lemma [3.1] we get

\[
|V_\subseteq(f)| = \sum_{[P_i] \in V_\subseteq(f)} \deg(S/I_{[P_i]}) = \sum_{f \in I_{[P_i]}} \deg(S/I_{[P_i]} = \deg S/(I(\subseteq), f).
\]

If \( (I(\subseteq): f) = I(\subseteq) \), then \( f \) is a regular element of \( S/I(\subseteq) \). This means that \( f \) is not in any of the associated primes of \( I(\subseteq) \), that is, \( f \not\in I_{[P]} \) for all \( i \). Thus \( V_\subseteq(f) = 0 \) and \( |V_\subseteq(f)| = 0 \). \( \square \)

4. The minimum distance function of a graded ideal

In this section we study the minimum distance function \( \delta_f \) of a graded ideal \( I \) and show that it generalizes the minimum distance function of a projective Reed-Muller-type code. To avoid repetitions, we continue to employ the notations and definitions used in Sections [1] and [2].

The next result will be used to bound the number of zeros of polynomials over finite fields (see Corollary [4.3]) and to study the general properties of \( \delta_f \).

**Lemma 4.1.** Let \( I \subset S \) be an unmixed graded ideal and let \( \prec \) be a monomial order. If \( f \in S \) is homogeneous and \( (I: f) \neq I \), then

\[
\deg(S/(I, f)) \leq \deg(S/(\text{in}_\prec(I), \text{in}_\prec(f))) \leq \deg(S/I),
\]

and \( \deg(S/(I, f)) < \deg(S/I) \) if \( I \) is an unmixed radical ideal and \( f \not\in I \).

**Proof.** To simplify notation we set \( J = (I, f) \) and \( L = (\text{in}_\prec(I), \text{in}_\prec(f)) \). We denote the Krull dimension of \( S/I \) by \( \dim(S/I) \). Recall that \( \dim(S/I) = \dim(S) - \text{ht}(I) \). First we show that \( S/J \) and \( S/L \) have Krull dimension equal to \( \dim(S/I) \). As \( f \) is a zero-divisor of \( S/I \) and \( I \) is unmixed, there is an associated prime ideal \( p \) of \( S/I \) such that \( f \in p \) and \( \dim(S/I) = \dim(S/p) \).
Since $I \subset J \subset \mathfrak{p}$, we get that $\dim(S/J) = \dim(S/I)$. Since $S/I$ and $S/\text{in}_<(I)$ have the same Hilbert function, and so does $S/\mathfrak{p}$ and $S/\text{in}_<(\mathfrak{p})$, we obtain
\[ \dim(S/\text{in}_<(I)) = \dim(S/I) = \dim(S/\mathfrak{p}) = \dim(S/\text{in}_<(\mathfrak{p})). \]

Hence, taking heights in the inclusions $\text{in}_<(I) \subset L \subset \text{in}_<(\mathfrak{p})$, we obtain $\text{ht}(I) = \text{ht}(L)$.

Pick a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_r\}$ of $I$. Then $J$ is generated by $\mathcal{G} \cup \{f\}$ and by Lemma 2.5 one has the inclusions
\[ \Delta_<(J) = \Delta_<(I, f) \subset \Delta_<(\text{in}_<(g_1), \ldots, \text{in}_<(g_r), \text{in}_<(f)) = \Delta_<(\text{in}_<(I), \text{in}_<(f)) = \Delta_<(L) \subset \Delta_<(\text{in}_<(g_1), \ldots, \text{in}_<(g_r)) = \Delta_<(I). \]

Thus $\Delta_<(J) \subset \Delta_<(L) \subset \Delta_<(I)$. Recall that $H_J(d)$, the Hilbert function of $I$ at $d$, is the number of standard monomials of degree $d$. Hence $H_J(d) = H_L(d) \leq H_I(d)$ for $d \geq 0$. If $\dim(S/I)$ is equal to 0, then
\[ \deg(S/J) = \sum_{d \geq 0} H_J(d) \leq \deg(S/L) = \sum_{d \geq 0} H_L(d) \leq \deg(S/I) = \sum_{d \geq 0} H_I(d). \]

Assume now that $\dim(S/I) \geq 1$. By the Hilbert theorem, $H_J$, $H_L$, $H_I$ are polynomial functions of degree equal to $k = \dim(S/I) - 1$. Thus
\[ k! \lim_{d \to \infty} H_J(d)/d^k \leq k! \lim_{d \to \infty} H_L(d)/d^k \leq k! \lim_{d \to \infty} H_I(d)/d^k, \]
that is $\deg(S/J) \leq \deg(S/L) \leq \deg(S/I)$.

If $I$ is an unmixed radical ideal and $f \notin I$, then there is at least one minimal prime that does not contains $f$. Hence, by Lemma 3.1 it follows that $\deg(S/(I, f)) < \deg(S/I)$. \hfill \Box

**Remark 4.2.** Let $I \subset S$ be an unmixed graded ideal of dimension 1. If $f \in S_d$, then $(I : f) = I$ if and only if $\dim(S/(I, f)) = 0$. In this case $\deg(S/(I, f))$ could be greater than $\deg(S/I)$.

**Corollary 4.3.** Let $\mathcal{X}$ be a finite subset of $\mathbb{P}^{s-1}$, let $I(\mathcal{X}) \subset S$ be its vanishing ideal, and let $<$ be a monomial order. If $0 \neq f \in S$ is homogeneous and $(I(\mathcal{X}); f) \neq I(\mathcal{X})$, then
\[ |V_\mathcal{X}(f)| = \deg(S/(I(\mathcal{X}), f)) \leq \deg(S/(\text{in}_<(I(\mathcal{X})), \text{in}_<(f))) \leq \deg(S/I(\mathcal{X})), \]
and $\deg(S/(I(\mathcal{X}), f)) < \deg(S/I(\mathcal{X}))$ if $f \notin I(\mathcal{X})$.

**Proof.** It follows from Lemma 3.2 and Lemma 4.1. \hfill \Box

The next alternative formula for the minimum distance function is valid for unmixed graded ideals. It was pointed out to us by Vasconcelos.

**Theorem 4.4.** Let $I \subset S$ be an unmixed graded ideal and let $<$ be a monomial order on $S$. If $\Delta_<(I)_d^p$ is the set of homogeneous standard polynomials of degree $d$ and $\mathfrak{m}^d \notin I$, then
\[ \delta_I(d) = \min\{\deg(S/(I : f)) | f \in S_d \setminus I\} = \min\{\deg(S/(I : f)) | f \in \Delta_<(I)_d^p\}. \]

**Proof.** The second equality is clear because by the division algorithm any $f \in S_d \setminus I$ can be written as $f = g + h$, where $g \in I$ and $h \in \Delta_<(I)_d^p$, and $(I : f) = (I : h)$. Next we show the first equality. If $\mathcal{F}_d = \emptyset$, $\delta_I(d) = \deg(S/I)$ and for any $f \in S_d \setminus I$, one has that $(I : f)$ is equal to $I$.
Thus equality holds. Assume that \( F_d \neq \emptyset \). Take \( f \in F_d \). Using that \( I \) is unmixed, it is not hard to see that \( S/I, S/(I : f) \), and \( S/(I, f) \) have the same dimension. There are exact sequences
\[
\begin{align*}
0 & \to (I : f)/I \to S/I \to S/(I : f) \to 0, \\
0 & \to (I : f)/I \to (S/I)[−d] \xrightarrow{f} S/I \to S/(I, f) \to 0.
\end{align*}
\]

Hence, by the additivity of Hilbert functions, we get
\[
H_I(i) - H_{(I : f)}(i) = H_I(i - d) - H_I(i) + H_{(I,f)}(i) \quad \text{for} \quad i \geq 0.
\]

By definition of \( \delta_I(d) \) it suffices to show the following equality
\[
\text{deg}(S/(I : f)) = \text{deg}(S/I) - \text{deg}(S/(I, f)).
\]

If \( \dim S/I = 0 \), then using Eq. (4.1) one has
\[
\sum_{i \geq 0} H_I(i) - \sum_{i \geq 0} H_{(I : f)}(i) = \sum_{i \geq 0} H_I(i - d) - \sum_{i \geq 0} H_I(i) + \sum_{i \geq 0} H_{(I,f)}(i).
\]

Hence, using the definition of degree, the equality of Eq. (4.2) follows. If \( k = \dim S/I - 1 \geq 0 \), by the Hilbert theorem, \( H_I, H_{(I,f)} \), and \( H_{(I,f)} \) are polynomial functions of degree \( k \). Then dividing Eq. (4.1) by \( i^k \) and taking limits as \( i \) goes to infinity, the equality of Eq. (4.2) holds. \( \square \)

We come to one of our main results.

**Theorem 4.5.** Let \( \prec \) be a monomial order and let \( I \subset S \) be an unmixed ideal of dimension \( \geq 1 \) such that \( t_i \) is a zero-divisor of \( S/I \) for \( i = 1, \ldots, s \). The following hold.

(i) The set \( F_d = \{ f \in S_d : f \not\in I, (I : f) \neq I \} \) is not empty for \( d \geq 1 \).

(ii) \( \delta_I(d) \geq \text{fp}_I(d) \) for \( d \geq 1 \).

(iii) \( \text{deg}(S/(I, t^a)) \leq \text{deg}(S/(\text{in}_{\prec}(I), t^a)) \leq \text{deg} S/I \) for any \( t^a \in \text{in}_{\prec}(I) \cap S_d \).

(iv) \( \text{fp}_I(d) \geq 0 \).

(v) \( \delta_I(d) \geq \delta_I(d + 1) \geq 0 \) for \( d \geq 1 \).

(vi) If \( I \) is a radical ideal and its associated primes are generated by linear forms, then there is an integer \( r \geq 1 \) such that
\[
\delta_I(1) > \cdots > \delta_I(r) = \delta_I(d) = 1 \quad \text{for} \quad d \geq r.
\]

**Proof.** (i): Since \( \dim(S/I) \geq 1 \), there is \( 1 \leq \ell \leq s \) such that \( t_{d}^{\ell} \) is not in \( I \), and \( (I : t_{d}^{\ell}) \neq I \) because \( t_{d}^{\ell} \) is a zero-divisor of \( S/I \). Thus \( t_{d}^{\ell} \) is in \( F_d \).

(ii): The set \( F_d \) is not empty for \( d \geq 1 \) by part (i). Pick a polynomial \( F \) of degree \( d \) such that \( \delta_I(d) = \text{deg}(S/I) - \text{deg}(S/(I,F)) \), \( F \not\in I \), and \( (I : F) \neq I \). We may assume that \( F \) is a sum of standard monomials of \( S/I \) with respect to \( \prec \) (this follows using a Gröbner basis of \( I \) and the division algorithm). Then, by Lemma 4.1 we get
\[
\text{deg} S/(I, F) \leq \text{deg} S/(\text{in}_{\prec}(I), \text{in}_{\prec}(F)) \leq \text{deg} S/I,
\]
where \( \text{in}_{\prec}(F) \) is a standard monomial of \( S/I \). Therefore \( \delta_I(d) \geq \text{fp}_I(d) \) for \( d \geq 1 \).

(iii), (iv): Since any standard monomial of degree \( d \) is a zero-divisor, by Lemma 4.1 we get the inequalities in item (iii). Part (iv) follows at once from part (iii).

(v): The set \( F_d \) is not empty for \( d \geq 1 \) by part (i). Then, by parts (ii) and (iv), \( \delta_I(d) \geq 0 \). Pick \( F \in S_d \) such that \( F \not\in I \), \( (I : F) \neq I \) and
\[
\text{deg}(S/(I, F)) = \max\{\text{deg}(S/(I, f)) : f \not\in I, f \in S_d, (I : f) \neq I\}.
\]
There is $h \in S_1$ such that $hF \notin I$, because otherwise $m = (t_1, \ldots, t_s)$ is an associated prime of $S/I$, a contradiction to the assumption that $I$ is unmixed of dimension $\ge 1$. As $F$ is a zero-divisor of $S/I$, so is $hF$. The ideals $(I, F)$ and $(I, hF)$ have height equal to $\text{ht}(I)$. Therefore taking Hilbert functions in the exact sequence

$$0 \to (I, F)/(I, hF) \to S/(I, hF) \to S/(I, F) \to 0$$

it follows that $\deg(S/(I, hF)) \le \deg(S/(I, F))$. This proves that $\delta_{t}(d) \ge \delta_{t}(d + 1)$.

(vi): By Lemma 3.1, $\delta_{t}(d) \ge 1$ for $d \ge 1$. Assume that $\delta_{t}(d) > 1$. By part (v) it suffices to show that $\delta_{t}(d) > \delta_{t}(d + 1)$. Pick a polynomial $F$ as in part (v). Let $p_1, \ldots, p_m$ be the associated primes of $I$. Then, by Lemma 3.1 one has

$$\delta_{t}(d) = \deg(S/I) - \deg(S/(I, F)) = \sum_{i=1}^{m} \deg(S/p_i) - \sum_{F \in p_i} \deg(S/p_i) \ge 2.$$ 

Hence there are $p_k \neq p_j$ such that $F$ is not in $p_k \cup p_j$. Pick a linear form $h$ in $p_k \setminus p_j$ which exists because $I$ is unmixed and $p_k$ is generated by linear forms. Then $hF \notin I$ because $hF \notin p_j$, and $hF$ is a zero-divisor of $S/I$ because $(I : F) \neq I$. Noticing that $F \notin p_k$ and $hF \in p_k$, by Lemma 3.1, we get

$$\deg(S/(I, F)) = \sum_{F \in p_i} \deg(S/p_i) < \sum_{F \in p_i} \deg(S/p_i) = \deg(S/(I, hF)).$$

Therefore $\delta_{t}(d) > \delta_{t}(d + 1)$.

**Corollary 4.6.** If $I \subset S$ is a Cohen-Macaulay square-free monomial ideal, then there is an integer $r \ge 1$ such that

$$\delta_{t}(1) > \cdots > \delta_{t}(r) = \delta_{t}(d) = 1$$

for $d \ge r$.

**Proof.** If $I$ is prime, then $I$ is generated by a subset of $\{t_1, \ldots, t_s\}$, $\deg(S/I) = 1$, and $\mathcal{F}_d = \emptyset$ for all $d$. Hence $\delta_{t}(d) = 1$ for $d \ge 1$. Thus we may assume that $I$ has at least two associated primes. Any Cohen-Macaulay ideal is unmixed [27]. Thus the degree of $S/I$ is the number of associated primes of $I$. Hence, we may assume that all variables are zero-divisors of $S/I$ and the result follows from Theorem 4.5(vi). \hfill \Box

The next result gives an algebraic formulation of the minimum distance of a projective Reed-Muller-type code in terms of the degree and the structure of the underlying vanishing ideal.

**Theorem 4.7.** Let $K$ be a field and let $\mathbb{X}$ be a finite subset of $\mathbb{P}^{n-1}$. If $|\mathbb{X}| \ge 2$, then

$$\delta_{\mathbb{X}}(d) = \delta_{I(\mathbb{X})}(d) \ge 1$$

for $d \ge 1$.

**Proof.** Setting $I = I(\mathbb{X})$, by Lemma 2.14 the set $\mathcal{F}_d := \{f \in S_d : f \notin I, (I : f) \neq I\}$ is not empty for $d \ge 1$. Hence, using the formula for $V_{\mathbb{X}}(f)$ of Lemma 3.2 we obtain

$$\max\{|V_{\mathbb{X}}(f)| : \text{ev}_d(f) \neq 0, f \in S_d\} = \max\{|\text{deg}(S/(I, f))| : f \in \mathcal{F}_d\}.$$ 

Therefore, using that $\deg(S/I) = |\mathbb{X}|$, we get

$$\delta_{\mathbb{X}}(d) = \min\{|\text{ev}_d(f)| : \text{ev}_d(f) \neq 0, f \in S_d\} = |\mathbb{X}| - \max\{|V_{\mathbb{X}}(f)| : \text{ev}_d(f) \neq 0, f \in S_d\} = \deg(S/I) - \max\{\text{deg}(S/(I, f))| : f \in \mathcal{F}_d\} = \delta_{t}(d),$$

where $|\text{ev}_d(f)|$ is the number of non-zero entries of $\text{ev}_d(f)$.

\hfill \Box
If \( I \) is a graded ideal and \( \Delta_{\prec}(I) \cap S_d = \{t^{a_1}, \ldots, t^{a_s}\} \), recall that \( \mathcal{F}_{\prec,d} \) is the set of homogeneous standard polynomials of \( S/I \) of degree \( d \) which are zero-divisors of \( S/I \):

\[
\mathcal{F}_{\prec,d} := \{f = \sum \lambda_i t^{a_i} \mid f \not= 0, \, \lambda_i \in K, \, (I: f) \not= I\}.
\]

The next result gives a description of the minimum distance which is suitable for computing this number using a computer algebra system such as Macaulay2 \[13\].

**Corollary 4.8.** If \( K = \mathbb{F}_q \), \(|X| \geq 2\), \( I = I(X) \), and \( \prec \) a monomial order, then

\[
\delta_X(d) = \deg S/I - \max\{\deg(S/(I,f)) \mid f \in \mathcal{F}_{\prec,d}\} \geq 1 \text{ for } d \geq 1.
\]

**Proof.** It follows from Theorem 4.7 because by the division algorithm any polynomial \( f \in S_d \) can be written as \( f = g + h \), where \( g \) is in \( I_d \) and \( h \) is a \( K \)-linear combination of standard monomials of degree \( d \). Notice that \((I: f) = (I: h)\). \qed

The expression for \( \delta_X(d) \) of Corollary 4.8 gives and algorithm that can be implemented in Macaulay2 \[13\] to compute \( \delta_X(d) \) (see Example 7.2). However, in practice, we can only find the minimum distance for small values of \( q \) and \( d \). Indeed, if \( n = |\Delta_{\prec}(I) \cap S_d| \), to compute \( \delta_{I(X)} \) requires to test the inequality \((I(X): f) \not= I(X)\) and compute the corresponding degree of \( S/(I(X), f) \) for the \( n^q - 1 \) standard polynomials of \( S/I \).

**Corollary 4.9.** Let \( K \) be a field, let \( \prec \) be a monomial order, and let \( X \) be a finite subset of \( \mathbb{P}^{s-1} \). If \( t_i \) is a zero-divisor of \( S/I(X) \) for \( i = 1, \ldots, s \), then \( \delta_X(d) \geq \text{fp}_{I(X)}(d) \geq 0 \) for \( d \geq 1 \).

**Proof.** The inequalities \( \delta_X(d) \geq \text{fp}_{I(X)}(d) \geq 0 \) follow from Theorems 4.5 and 4.7. \qed

One can use Corollary 4.9 to estimate the minimum distance of any Reed-Muller-type code over a set \( X \) parameterized by a set of relatively prime monomials because in this case \( t_i \) is a zero-divisor of \( S/I(X) \) for \( i = 1, \ldots, s \) and one has the following result that can be used to compute the vanishing ideal of \( X \) using Gröbner bases and elimination theory.

**Theorem 4.10.** \[23\] Let \( K = \mathbb{F}_q \) be a finite field. If \( X \) is a subset of \( \mathbb{P}^{s-1} \) parameterized by monomials \( y_1^{v_1}, \ldots, y_n^{v_s} \) in the variables \( y_1, \ldots, y_n \), then

\[
I(X) = (\{t_i - y_1^{v_i} z_i \}_{i=1}^s \cup \{y_j^q - y_1 \}_{i=1}^n) \cap S,
\]

and \( I(X) \) is a binomial ideal.

As an application, Corollary 4.9 will be used to study the minimum distance of projective nested cartesian codes \[3\] over a set \( X \). In this case \( t_i \) is a zero-divisor of \( S/I(X) \) for \( i = 1, \ldots, s \) and one has a Gröbner basis for \( I(X) \) \[3\] (see Section 6).

5. Degree Formulas and Some Inequalities

Let \( S = K[t_1, \ldots, t_s] \) be a polynomial ring over a field \( K \), let \( d_1, \ldots, d_s \) be a non-decreasing sequence of positive integers with \( d_1 \geq 2 \) and \( s \geq 2 \), and let \( L \) be the ideal of \( S \) generated by the set of all \( t_i^d_j \) such that \( 1 \leq i < j \leq s \). In this section we show a formula for the degree of \( S/(L, t^a) \) for any standard monomial \( t^a \) of \( S/L \).

**Lemma 5.1.** The ideal \( L \) is Cohen-Macaulay of height \( s - 1 \), has a unique irredundant primary decomposition given by

\[
L = q_1 \cap \cdots \cap q_s,
\]

where \( q_i = (t_1, \ldots, t_{i-1}, t^d_{i+1}, \ldots, t_s^d) \) for \( 1 \leq i \leq s \), and \( \deg(S/L) = 1 + \sum_{i=2}^s d_i \cdots d_s \).
Proof. Using induction on $s$ and the depth lemma (see [22, Lemma 2.3.9]) it is seen that $L$ is Cohen-Macaulay. In particular $L$ is unmixed. Since the radical of $L$ is generated by all $t_it_j$ with $i < j$, the minimal primes of $L$ are $p_1, \ldots, p_s$, where $p_i$ is generated by $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_s$. The $p_i$-primary component of $L$ is uniquely determined and is given by $LS_{p_i} \cap S$. Inverting the variable $t_i$ in $LS_{p_i}$ it follows that $LS_{p_i} = q_iS_{p_i}$. As $q_i$ is an irreducible ideal, it is $p_i$-primary and one has the equality $LS_{p_i} \cap S = q_i$. By the additivity of the degree we obtain the required formula for the degree of $S/L$.

**Proposition 5.2.** [22] Propositions 3.1.33 and 5.1.11] Let $A = R_1/I_1$, $B = R_2/I_2$ be two standard graded algebras over a field $K$, where $R_1 = K[x]$, $R_2 = K[y]$ are polynomial rings in disjoint sets of variables and $I_1$ is an ideal of $R_1$. If $R = K[x, y]$ and $I = I_1R + I_2R$, then

$$(R_1/I_1) \otimes_K (R_2/I_2) \simeq R/I$$

and $F(A \otimes_K B, x) = F(A, x)F(B, x)$, where $F(A, x)$ and $F(B, x)$ are the Hilbert series of $A$ and $B$, respectively.

**Proposition 5.3.** Let $t^a = t_r^a \cdots t_s^a$ be a standard monomial of $S/L$ with respect to a monomial order $\prec$. If $a_r \geq 1$, $a_i = 0$ for $i < r$, and $1 \leq r \leq s$, then $0 \leq a_i \leq d_i - 1$ for $i > r$ and

$$\deg S/(L, t^a) = \begin{cases} 
\deg S/L - \sum_{i=2}^{s} (d_i - a_i) \cdots (d_s - a_s) - 1 & \text{if } r = s, a_s \leq d_s, \\
\deg S/L - 1 & \text{if } r = s, a_s \geq d_s + 1, \\
\deg S/L - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) & \text{if } r < s, a_r \leq d_r, \\
\deg S/L - (d_r+1 - a_{r+1}) \cdots (d_s - a_s) & \text{if } r < s, a_r \geq d_r + 1.
\end{cases}$$

Proof. As $f = t^a$ is not a multiple of $t_it_j$ for $i < j$, we get that $0 \leq a_i \leq d_i - 1$ for $i > r$. To show the formula for the degree we proceed by induction on $s \geq 2$. In what follows we will use the theory of Hilbert functions of graded algebras, as introduced in a classical paper of Stanley [22]. In particular, we will freely use the additivity of Hilbert series, a formula for the Hilbert series of a complete intersection [22, Corollary 3.3], the formula of Lemma 5.1 for the degree of $S/L$, and the fact that any monomial is a zero-divisor of $S/L$ (this follows from Lemma 5.1). We split the proof of the case $s = 2$ in three easy cases.

Case (1): Assume $s = 2$, $r = 1$. This case is independent of whether $a_1 \leq d_1$ or $a_1 \geq d_1 + 1$ because the two possible values of $\deg S/(L, f)$ coincide. There are exact sequences

$$0 \to S/(t_1)[-d_2] \overset{t_2}{\to} S/(L, f) \to S/(t_2^2, f) \to 0,$$

$$0 \to S/(t_2^2)[-a_1] \overset{t_1^a}{\to} S/(t_2^2, f) \to S/(t_2^2, t_1^a) \to 0.$$

Taking Hilbert series we get

$$F(S/(L, f), x) = \frac{x^2}{1-x} + \frac{x^a(1 + x + \cdots + x^{a-1})}{(1-x)} + \left(\sum_{i=0}^{d_1-1} x^i\right) \left(\sum_{i=0}^{a_1-1} x^i\right).$$

Writing $F(S/(L, f), x) = h(x)/(1-x)$ with $h(x) \in \mathbb{Z}[x]$ and $h(1) > 0$, and noticing that $h(1)$ is the degree of $S/(L, f)$, we get

$$\deg S/(L, f) = 1 + a_2 = (d_2 + 1) - (d_2 - a_2) = \deg(S/L) - (d_2 - a_2).$$

Case (2): Assume $s = 2$, $r = 2$, $a_2 \leq d_2$. In this case $(L, f)$ is equal to $(t_2^2)$. Thus

$$\deg S/(L, f) = a_2 = (1 + d_2) - (d_2 - a_2) - 1 = \deg S/L - (d_2 - a_2) - 1.$$
Case (3): Assume $s = 2$, $r = 2$, $a_2 \geq d_2 + 1$. Taking Hilbert series in the exact sequence

$$0 \longrightarrow S/(t_1, t_2^{d_2 - d_2})[-d_2] \xrightarrow{t_2^{d_2}} S/(L, f) \longrightarrow S/(t_2^{d_2}) \longrightarrow 0,$$

we obtain

$$F(S/(L, f), x) = x^{d_2}(1 + x + \cdots + x^{a_2 - d_2 - 1}) + \frac{(1 + x + \cdots + x^{d_2 - 1})}{1 - x}. $$

Thus we may proceed as in Case (1) to get $\deg(S/(L, f)) = d_2 = \deg(S/L) - 1$.

This completes the initial induction step. We may now assume that $s \geq 3$ and split the proof in three cases.

Case (I): Assume $r = s \geq 3$ and $a_s \leq d_s$. Thus $f = t_1^{a_s}$ and $a_i = 0$ for $i < s$. Setting $L'$ equal to the ideal generated by the set of all $t_it_j^{d_i}$ such that $2 \leq i < j \leq s$, there is an exact sequence

$$0 \longrightarrow S/(t_2^{d_2}, \ldots, t_{s-1}^{d_{s-1}}, t_s^{a_s})[-1] \xrightarrow{t_1} S/(L, t_s^{a_s}) \longrightarrow S/(L', t_s^{a_s}, t_1) \longrightarrow 0.$$ 

Taking Hilbert series one has

$$F(S/(L, t_s^{a_s}), x) = xF(S/(t_2^{d_2}, \ldots, t_{s-1}^{d_{s-1}}, t_s^{a_s}), x) + F(S/(L', t_s^{a_s}, t_1), x).$$

Hence, setting $S' = K[t_2, \ldots, t_s]$, from the induction hypothesis applied to $S'/(L', t_s^{a_s})$, and using that $\deg S'/L' = \deg S/L - d_2 \cdots d_{s-1}d_s$ (see Lemma 5.1), we obtain

$$\deg(S/(L, f)) = d_2 \cdots d_s - a_s + \deg(S'/L') - \sum_{i=3}^{s} d_i \cdots d_{s-1}(d_s - a_s) - 1$$

$$= \deg(S/L) - \sum_{i=2}^{s} d_i \cdots d_{s-1}(d_s - a_s) - 1.$$ 

Case (II): Assume $r = s \geq 3$ and $a_s \geq d_s + 1$. Using the exact sequence

$$0 \longrightarrow S/(t_2^{d_2}, \ldots, t_{s-1}^{d_{s-1}}, t_s^{d_s})[-1] \xrightarrow{t_1} S/(L, t_s^{a_s}) \longrightarrow S/(L', t_s^{a_s}, t_1) \longrightarrow 0,$$

we can proceed as in Case (1) to get $\deg(S/(L, f)) = \deg(S/L) - 1$.

Case (III): Assume $r < s$. Then, by assumption, $a_s < d_s$. Let $L'$ be the ideal generated by the set of all $t_it_j^{d_i}$ such that $1 \leq i < j \leq s - 1$. Setting $f' = t_1^{a_1} \cdots t_{s-1}^{a_{s-1}}$ and $S' = K[t_1, \ldots, t_{s-1}]$, there are exact sequences

$$0 \longrightarrow S/(t_1, \ldots, t_{s-1})[-d_s] \xrightarrow{t_s^{d_s}} S/(L, f) \longrightarrow S/(L', f, t_s^{d_s}) \longrightarrow 0,$$

$$0 \longrightarrow S/(L', f', t_s^{d_s-a_s})[-a_s] \xrightarrow{t_s^{a_s}} S/(L', f, t_s^{d_s}) \longrightarrow S/(L', t_s^{a_s}) \longrightarrow 0.$$ 

Hence taking Hilbert series, and applying Proposition 5.2 we get

$$F(S/(L, f), x) = \frac{x^{d_s}}{1 - x} + F(S'/(L', f'), x)F(K[t_s]/(t_s^{d_s-a_s}), x) + F(S'/L', x)F(K[t_s]/(t_s^{a_s}), x).$$

Writing $F(S/(L, f), x) = h(x)/(1 - x)$ with $h(x) \in \mathbb{Z}[x]$ and $h(1) > 0$, and noticing that $h(1)$ is the degree of $S/(L, f)$, the induction hypothesis applied to $S'/(L', f')$ yields the equality

$$\deg S/(L, f) = 1 + \left( \deg S'/L' - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_{s-1} - a_{s-1}) \right) (d_s - a_s) + \deg S'/L')a_s$$

$$= 1 + (\deg S'/L')d_s - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) \text{ if } a_r \leq d_r.$$
or the equality
\[
\deg S/(L, f) = 1 + (\deg S'/L' - (d_{r+1} - a_{r+1}) \cdots (d_{s-1} - a_{s-1})) (d_s - a_s) + \deg(S'/L') a_s
\]

To complete the proof it suffices to notice that \(\deg(S/L) = 1 + \deg(S'/L') d_s\). This equality follows readily from Lemma 5.1.

\[\square\]

**Remark 5.4.** Cases (1), (2), and (3) can also be shown using Hilbert functions instead of Hilbert series, but case (III) is easier to handle using Hilbert series.

**Lemma 5.5.** Let \(a_1, \ldots, a_r, a, b, e\) be positive integers with \(e \geq a\). Then

(a) \(a_1 \cdots a_r \geq (a_1 + \cdots + a_r) - (r - 1)\), and
(b) \(a(e - b) \geq (a - b)e\).

\[\text{Proof.} \text{ Part (a) follows by induction on } r, \text{ and part (b) is straightforward.} \quad \square\]

The next inequality is a generalization of part (a).

**Lemma 5.6.** Let \(1 \leq e_1 \leq \cdots \leq e_m\) and \(0 \leq b_i \leq e_i - 1\) for \(i = 1, \ldots, m\) be integers. Then

\[
(5.1) \prod_{i=1}^{m} (e_i - b_i) \geq \left(\sum_{i=1}^{k} (e_i - b_i) - (k - 1) - \sum_{i=k+1}^{m} b_i\right) e_{k+1} \cdots e_m
\]

for \(k = 1, \ldots, m\), where \(e_{k+1} \cdots e_m = 1\) and \(\sum_{i=k+1}^{m} b_i = 0\) if \(k = m\).

**Proof.** Fix \(m\) and \(1 \leq k \leq m\). We will proceed by induction on \(\sigma = \sum_{i=1}^{k} (e_i - b_i - 1)\). If \(\sigma = 0\), then \(e_i - b_i - 1 = 0\) for \(i = 1, \ldots, k\). Thus either \(1 - \sum_{i=k+1}^{m} b_i < 0\) or \(1 - \sum_{i=k+1}^{m} b_i \geq 1\). In the first case the inequality is clear because the left hand side of Eq. (5.1) is positive and in the second case one has \(b_i = 0\) for \(i = k + 1, \ldots, m\) and equality holds in Eq. (5.1). Assume that \(\sigma > 0\). If \(k = m\) or \(b_i = 0\) for \(i = k + 1, \ldots, m\), the inequality follows at once from Lemma 5.5(a). Thus, we may assume \(k < m\) and \(b_j > 0\) for some \(k + 1 \leq j \leq m\). To simplify notation, and without loss of generality, we may assume that \(j = m\), that is, \(b_m > 0\). If the right hand side of Eq. (5.1) is negative or zero, the inequality holds. Thus we may also assume that

\[
(5.2) \sum_{i=1}^{k} (e_i - b_i) - \sum_{i=k+1}^{m} b_i \geq k.
\]

Hence there is \(1 \leq \ell \leq k\) such that \(e_\ell - b_\ell \geq 2\).

Case (1): Assume \(e_\ell - b_\ell - b_m \geq 1\). Setting \(a = e_\ell - b_\ell, e = e_m, \text{ and } b = b_m\) in Lemma 5.5(b), we get

\[
(5.3) (e_\ell - b_\ell)(e_m - b_m) \geq (e_\ell - (b_\ell + b_m)) e_m.
\]

Therefore using Eq. (5.3), and then applying the induction hypothesis to the two sequences of integers

\[e_1, \ldots, e_{\ell-1}, e_\ell, e_{\ell+1}, \ldots, e_{m-1}, e_m; b_1, \ldots, b_{\ell-1}, b_\ell + b_m, b_{\ell+1}, \ldots, b_{m-1}, 0,\]
we get the inequalities

\[
\prod_{i=1}^{m} (e_i - b_i) = \left( \prod_{i \not\in \ell, m} (e_i - b_i) \right) (e_\ell - b_\ell)(e_m - b_m)
\geq \left( \prod_{i \not\in \ell, m} (e_i - b_i) \right) (e_\ell - (b_\ell + b_m))e_m
\geq \left( \sum_{\ell \neq i=1}^{k} (e_i - b_i) + (e_\ell - (b_\ell + b_m)) - (k - 1) - \sum_{i=k+1}^{m-1} b_i \right) e_{k+1} \cdots e_m
= \left( \sum_{i=1}^{k} (e_i - b_i) - (k - 1) - \sum_{i=k+1}^{m} b_i \right) e_{k+1} \cdots e_m.
\]

Case (2): Assume \( e_\ell - b_\ell - b_m < 1 \). Setting \( r_\ell = e_\ell - b_\ell - 1 \geq 1 \), one has

\[
b_\ell + r_\ell = e_\ell - 1 \geq 1, \quad b_m - r_\ell \geq 1, \quad e_\ell - (b_\ell + r_\ell) = 1.
\]

On the other hand, by Lemma 5.5(a), one has

\[
(e_\ell - b_\ell)(e_m - b_m) \geq (e_\ell - b_\ell) + (e_m - b_m) - 1 = (e_\ell - (b_\ell + r_\ell))(e_m - (b_m - r_\ell)).
\]

Therefore using Eq. (5.3), and then applying the induction hypothesis to the two sequences of integers

\[ e_1, \ldots, e_\ell-1, e_\ell, e_{\ell+1}, \ldots, e_{m-1}, e_m; \quad b_1, \ldots, b_{\ell-1}, b_\ell + r_\ell, b_{\ell+1}, \ldots, b_{m-1}, b_m - r_\ell, \]

we get the inequalities

\[
\prod_{i=1}^{m} (e_i - b_i) = \left( \prod_{i \not\in \ell, m} (e_i - b_i) \right) (e_\ell - b_\ell)(e_m - b_m)
\geq \left( \prod_{i \not\in \ell, m} (e_i - b_i) \right) (e_\ell - (b_\ell + r_\ell))(e_m - (b_m - r_\ell))
\geq \left( \sum_{\ell \neq i=1}^{k} (e_i - b_i) + (e_\ell - (b_\ell + r_\ell)) - (k - 1) - \sum_{i=k+1}^{m-1} b_i - (b_m - r_\ell) \right) e_{k+1} \cdots e_m
= \left( \sum_{i=1}^{k} (e_i - b_i) - (k - 1) - \sum_{i=k+1}^{m} b_i \right) e_{k+1} \cdots e_m. \quad \square
\]

**Proposition 5.7.** Let \( 1 \leq e_1 \leq \cdots \leq e_m \) and \( 0 \leq b_i \leq e_i - 1 \) for \( i = 1, \ldots, m \) be integers. If \( b_0 \geq 1 \), then

\[
\prod_{i=1}^{m} (e_i - b_i) \geq \left( \sum_{i=1}^{k+1} (e_i - b_i) - (k - 1) - b_0 - \sum_{i=k+2}^{m} b_i \right) e_{k+2} \cdots e_m
\]

for \( k = 0, \ldots, m - 1 \), where \( e_{k+2} \cdots e_m = 1 \) and \( \sum_{i=k+2}^{m} b_i = 0 \) if \( k = m - 1 \).
Proposition 6.3. Let \( S \) be the lexicographical order on \( F_1 \) such that \( d \) is its corresponding nested cartesian code on the set \( X \) (Carvalho, Lopez-Neumann, and López [3]).

Conjecture 6.2. Let \( X \) be the image of \( A_1, \ldots, A_s \) be a collection of subsets of \( K \), and let

\[ \mathcal{X} = [A_1 \times \cdots \times A_s] \]

be the image of \( A_1 \times \cdots \times A_s \setminus \{0\} \) under the map \( K^s \setminus \{0\} \to \mathbb{P}^{s-1}, x \to [x] \).

Definition 6.1. [3] The set \( \mathcal{X} \) is called a projective nested cartesian set if

(i) \( \{0, 1\} \subset A_i \) for \( i = 1, \ldots, s \),
(ii) \( a/b \in A_j \) for \( 1 \leq i < j \leq s \), \( a \in A_j \), \( 0 \neq b \in A_i \), and
(iii) \( d_1 \leq \cdots \leq d_s \), where \( d_i = |A_i| \) for \( i = 1, \ldots, s \).

If \( \mathcal{X} \) is a projective nested cartesian set, we call \( C_{\mathcal{X}}(d) \) a projective nested cartesian code.

Conjecture 6.2. (Carvalho, Lopez-Neumann, and López [3]) Let \( C_{\mathcal{X}}(d) \) be the \( d \)-th projective nested cartesian code on the set \( \mathcal{X} = [A_1 \times \cdots \times A_s] \) with \( d_i = |A_i| \) for \( i = 1, \ldots, s \). Then its minimum distance is given by

\[ \delta_{\mathcal{X}}(d) = \begin{cases} 
(d_{k+2} - \ell + 1) d_{k+3} \cdots d_s & \text{if } d \leq \sum_{i=2}^{s} (d_i - 1), \\
1 & \text{if } d \geq \sum_{i=2}^{s} (d_i - 1) + 1,
\end{cases} \]

where \( 0 \leq k \leq s - 2 \) and \( \ell \) are the unique integers such that \( d = \sum_{i=2}^{k+1} (d_i - 1) + \ell \) and \( 1 \leq \ell \leq d_{k+2} - 1 \).

In what follows \( \mathcal{X} = [A_1 \times \cdots \times A_s] \) denotes a projective nested cartesian set and \( C_{\mathcal{X}}(d) \) is its corresponding \( d \)-th projective Reed-Muller-type code. Throughout this section \( \prec \) is the lexicographical order on \( S \) with \( t_1 \prec \cdots \prec t_s \) and \( \text{in}_{\prec}(I(\mathcal{X})) \) is the initial ideal of \( I(\mathcal{X}) \).

Proposition 6.3. [3] The initial ideal \( \text{in}_{\prec}(I(\mathcal{X})) \) is generated by the set of all monomials \( t_i t_j^{d_j} \) such that \( 1 \leq i < j \leq s \),

\[ \deg(S/I(\mathcal{X})) = 1 + \sum_{i=2}^{s} d_i \cdots d_s, \quad \text{and} \quad \text{reg}(S/I(\mathcal{X})) = 1 + \sum_{i=2}^{s} (d_i - 1). \]

Carvalho, Lopez-Neumann and López, showed the conjecture when the \( A_i \)'s are subfields of \( \mathbb{F}_q \). They also showed that the conjecture can be reduced to:
Conjecture 6.4. (Carvalho, López-Neumann, and López) If $0 \neq f \in S_d$ is a standard polynomial, with respect to $\prec$, such that $(I(X) : f) \neq I(X)$ and $1 \leq d \leq \sum_{i=2}^{s}(d_i - 1)$, then
$$|V_X(f)| \leq \deg(S/I(X)) - (d_{k+2} - \ell + 1) d_{k+3} \cdots d_s,$$
where $0 \leq k \leq s - 2$ and $\ell$ are integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

We show a degree formula and use this to give an upper bound for $|V_X(f)|$.

Theorem 6.5. Let $\prec$ be the lexicographical order on $S$ with $t_1 < \cdots < t_s$ and let $f \neq 0$ be a standard polynomial with $\inf_\prec(f) = t_{a_1} \cdots t_{a_s}$ and $a_r \geq 1$. Then $0 \leq a_i \leq d_i - 1$ for $i > r$ and
$$|V_X(f)| \leq \deg(S/\inf_\prec(I(X)), \inf_\prec(f)) =
\begin{cases} 
\deg(S/I(X)) - \sum_{i=2}^{r+1}(d_i - a_i) \cdots (d_s - a_s) & \text{if } a_r \leq d_r, \\
\deg(S/I(X)) - (d_{r+1} - a_{r+1}) \cdots (d_s - a_s) & \text{if } a_r \geq d_r + 1,
\end{cases}$$
where $(d_i - a_i) \cdots (d_s - a_s) = 1$ if $i > s$ and $a_i = 0$ for $i < r$.

Proof. By Proposition 6.3 the initial ideal of $I(X)$ is generated by the set of all $t_i t_j^{d_j}$ such that $1 \leq i < j \leq s$ and the degree of $S/\inf_\prec(I(X))$ is equal to the degree of $S/I(X)$. As $\inf_\prec(f)$ is a standard monomial, it follows that $0 \leq a_i \leq d_i - 1$ for $i > r$. Notice that if $f$ is not a zero-divisor of $S/I(X)$, then $V_X(f) = \emptyset$. Thus the inequality follows at once from Corollary 4.3 and the equality follows from Proposition 5.3.

The next result gives some support for Conjecture 6.4.

Theorem 6.6. Let $\prec$ be the lexicographical order on $S$ with $t_1 < \cdots < t_s$. If $0 \neq f \in S_d$ is a standard polynomial such that $1 \leq d \leq \sum_{i=2}^{s}(d_i - 1)$ and $t_1$ divides $\inf_\prec(f)$, then
$$|V_X(f)| \leq \deg(S/I(X)) - (d_{k+2} - \ell + 1) d_{k+3} \cdots d_s,$$
where $0 \leq k \leq s - 2$ and $\ell$ are integers such that $d = \sum_{i=2}^{k+1}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+2} - 1$.

Proof. By Lemma 3.3 we may assume that $(I(X) : f) \neq I(X)$. Let $t^a = \inf_\prec(f)$ be the initial monomial of $f$. By Proposition 6.3 we can write
$$t^a = t_{a_1}^{a_1} \cdots t_{a_s}^{a_s},$$
with $a_1 \geq 1$, $0 \leq a_i \leq d_i - 1$ for $i > 1$. By Lemmas 5.2 and 4.1 it suffices to show that the following inequality holds
$$\deg(S/\inf_\prec(I(X)), t^a)) \leq \deg(S/I(X)) - (d_{k+2} - \ell + 1) d_{k+3} \cdots d_s.$$

If we substitute $-\ell = \sum_{i=2}^{k+1}(d_i - 1) - \sum_{i=1}^{s}a_i$ in Eq. (6.1), and use the formula for the degree of $S/\inf_\prec(I(X)), t^a)$ given in Theorem 6.5 we need only show that the following inequalities hold for $r = 1$:

$$\sum_{i=2}^{r+1}(d_i - a_i) \cdots (d_s - a_s) \geq \left(\sum_{i=2}^{k+2}(d_i - a_i) - (k - 1) - a_1 - \sum_{i=k+3}^{s}a_i\right) d_{k+3} \cdots d_s \text{ if } a_r \leq d_r,$$

$$\prod_{i=r+1}^{s}(d_i - a_i) \geq \left(\sum_{i=2}^{k+2}(d_i - a_i) - (k - 1) - a_1 - \sum_{i=k+3}^{s}a_i\right) d_{k+3} \cdots d_s \text{ if } a_r \geq d_r + 1,$$
for $0 \leq k \leq s - 2$, where $(d_i - a_i) \cdots (d_s - a_s) = 1$ if $i > s$ and $a_i = 0$ for $i < r$.

Assume $r = 1$. Then Eqs. (6.2) and (6.3) are the same. Thus we need only show the inequality
\[ \prod_{i=2}^{s} (d_i - a_i) \geq \left( \sum_{i=2}^{k+2} (d_i - a_i) - (k - 1) - a_1 - \sum_{i=k+3}^{s} a_i \right) d_{k+3} \cdots d_s, \]
for $0 \leq k \leq s - 2$. This inequality follows making $m = s - 1$, $e_i = d_{i+1}$, $b_i = a_{i+1}$ for $i = 1, \ldots, m$, and $b_0 = a_1$ in Proposition 5.7. \qed

The following resembles Conjecture 6.2 when $d = 1$:

**Conjecture 6.7.** [25] Conjecture 4.9] Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^{s-1}$. If $I(\mathbb{X})$ is a complete intersection generated by $f_1, \ldots, f_{s-1}$, with $e_i = \deg(f_i)$ for $i = 1, \ldots, s-1$, and $2 \leq e_i \leq e_{i+1}$ for all $i$, then $\delta\mathbb{X}(1) \geq (e_1 - 1)e_2 \cdots e_{s-1}$.

Next we show that the corresponding conjecture is true for projective nested cartesian codes by proving that Conjecture 6.2 is true for $d = 1$.

**Proposition 6.8.** $\delta\mathbb{X}(1) = d_2 \cdots d_s$.

**Proof.** It suffices to show that Conjecture 6.4 is true for $d = 1$, that is, if $0 \neq f \in S_1$ is a standard polynomial such that $(I(\mathcal{X}) : f) \neq I(\mathcal{X})$, then we must show the inequality
\[ |V_{\mathbb{X}}(f)| \leq \deg(S/I(\mathcal{X})) - d_2 \cdots d_s. \]

As in $u(f) = t_r$ for some $1 \leq r \leq s$, we can write in $u(f) = t_1^{a_1} \cdots t_r^{a_r} \cdots t_s^{a_s}$, where $a_r = 1$, $a_i = 0$ for $i \neq r$, and $2 \leq d_2 \leq \cdots \leq d_s$. Therefore, by Theorem 6.5 we get
\[ |V_{\mathbb{X}}(f)| \leq \deg(S/I(\mathcal{X})) - \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s), \]
where $(d_i - a_i) \cdots (d_s - a_s) = 1$ if $i > s$. Hence the proof reduces to showing the inequality
\[ \sum_{i=2}^{r+1} (d_i - a_i) \cdots (d_s - a_s) \geq d_2 \cdots d_s. \]

If $r = s$, this inequality follows readily by induction on $s \geq 2$. If $r < s$ the inequality also follows by induction on $s$ by noticing that, in this case, both sides of the inequality have $d_s$ as a common factor because $d_s$ appears in all terms of the summation of the left hand side. \qed

Let $\mathcal{L}_d$ be the $K$-vector space generated by all $t^a \in S_d$ such that $t_1$ divides $t^a$ and let $C_d$ be the image of $\mathcal{L}_d$ under the evaluation map $ev_d$. From the next result it follows that the minimum distance of $C_\mathbb{X}(d)$ proposed in Conjecture 6.2 is in fact the minimum distance of the evaluation linear code $C_d$.

**Corollary 6.9.** Let $\mathcal{L}_d$ be the $K$-vector space generated by all $t^a \in S_d$ such that $t^a$ contains $t_1$. If $1 \leq d \leq \sum_{i=2}^{s} (d_i - 1)$, then
\[ \max \{|V_{\mathbb{X}}(f)| : f \notin I(\mathcal{X}), f \in \mathcal{L}_d\} = \deg(S/I(\mathcal{X})) - (d_{k+2} - \ell + 1) d_{k+3} \cdots d_s, \]
where $0 \leq k \leq s - 2$ and $\ell$ are integers, $d = \sum_{i=2}^{k+1} (d_i - 1) + \ell$, and $1 \leq \ell \leq d_{k+2} - 1$. 
Proof. Take \( f \in \mathcal{L}_d \setminus \mathcal{I}(\mathcal{X}) \). Let \( \prec \) be the lexicographical order with \( t_1 \prec \cdots \prec t_s \) and let \( \mathcal{G} \) be the Gröbner basis of \( \mathcal{I}(\mathcal{X}) \) given in [3, Proposition 2.14]. By the division algorithm, we can write \( f = \sum_{i=1}^{s} a_i g_i + g \), where \( g_i \in \mathcal{G} \) for all \( i \) and \( g \) is a standard polynomial of degree \( d \).

The polynomial \( g \) is again in \( \mathcal{L}_d \setminus \mathcal{I}(\mathcal{X}) \). Indeed if \( g \notin \mathcal{L}_d \setminus \mathcal{I}(\mathcal{X}) \), there is at least one monomial of \( g \) that do not contain \( t_1 \), then making \( t_1 = 0 \) in the last equality, we get an equality of the form \( 0 = \sum_{i=1}^{s} b_i g_i + h \), where \( h \) is a non-zero standard polynomial of \( \mathcal{I}(\mathcal{X}) \), a contradiction. Hence, by Theorem 6.6, the inequality \( \leq \) follows because \( |V_X(f)| = |V_X(g)| \). To show equality notice that according to the proof of [3, Lemma 3.1], there is a polynomial \( f \) of degree \( d \) in \( \mathcal{L}_d \setminus \mathcal{I}(\mathcal{X}) \) whose number of zeros in \( \mathcal{X} \) is equal to the right hand side of the required equality. \( \square \)

7. Examples

In this section we show some examples that illustrates how some of our results can be used in practice.

Example 7.1. Let \( K \) be the field \( \mathbb{F}_3 \), let \( \mathcal{X} \) be the subset of \( \mathbb{P}^3 \) given by

\[
\mathcal{X} = \{ [e_1], [e_2], [e_3], [e_4], [(1, -1, -1, 1)], [(1, 1, 1, 1)], [(-1, -1, 1, 1)], [(-1, 1, -1, 1)] \},
\]

where \( e_i \) is the \( i \)-th unit vector, and let \( \mathcal{I} = \mathcal{I}(\mathcal{X}) \) be the vanishing ideal of \( \mathcal{X} \). Using Lemma 2.9 and Macaulay2 [13], we get that \( \mathcal{I} \) is the ideal of \( \mathbb{S} = \mathbb{K}[t_1, t_2, t_3, t_4] \) generated by the binomials \( t_1 t_2 - t_3 t_4, t_1 t_3 - t_2 t_4, t_2 t_3 - t_1 t_4 \). Hence, using Theorem 4.4 and the procedure below for Macaulay2 [13], we get:

\[
\begin{array}{c|cccc}
   & 1 & 2 & 3 & \cdots \\
\hline
\deg(S/\mathcal{I}) & 8 & 8 & 8 & \cdots \\
H(\mathcal{I}(d)) & 4 & 7 & 8 & \cdots \\
\delta(\mathcal{I}(d)) & 4 & 2 & 1 & \cdots \\
\end{array}
\]

Example 7.2. Let \( \mathcal{X} \) be the set parameterized by \( y_1 y_2, y_2 y_3, y_3 y_4, y_1 y_4 \) over the field \( \mathbb{F}_3 \). Using Corollary 4.8, Theorem 4.10, and the following procedure for Macaulay2 [13] we get:

\[
\begin{array}{c|cccc}
   & 1 & 2 & 3 & \cdots \\
\hline
|\mathcal{X}| & 16 & 16 & 16 & \cdots \\
H(\mathcal{X}(d)) & 4 & 9 & 16 & \cdots \\
\delta(\mathcal{X}(d)) & 9 & 4 & 1 & \cdots \\
\mathbb{P}(\mathcal{I}(\mathcal{X})(d)) & 6 & 3 & 1 & \cdots \\
\end{array}
\]

Q=3

\[
\begin{align*}
R=\mathbb{Z}/q[y_1, y_2, y_3, y_4, z, t_1, t_2, t_3, t_4, \text{MonomialOrder}=>\text{Eliminate 5}] ; \\
f1=y_1*y_2, f2=y_2*y_3, f3=y_3*y_4, f4=y_4*y_1 \\
J=\text{ideal}(y_1^q-y_1, y_2^q-y_2, y_3^q-y_3, y_4^q-y_4, t_1-f_1*z, t_2-f_2*z, t_3-f_3*z, t_4-f_4*z)
\end{align*}
\]
MINIMUM DISTANCE FUNCTIONS AND REED-MULLER-TYPE CODES

\begin{verbatim}
C4=ideal selectInSubring(1,gens gb J)
S=ZZ/q[t1,t2,t3,t4];
I=sub(C4,S)
M=coker gens gb I
h=(d)->degree M - max apply(apply(apply(apply(
toList (set(0..q-1))^**(hilbertFunction(d,M))-
(set{0})^**(hilbertFunction(d,M)), toList),x->basis(d,M)*vector x),
z->ideal(flatten entries z)),x-> if not
quotient(I,x)==I then degree ideal(I,x) else 0)--The function h(d)
--gives the minimum distance in degree d
h(1), h(2)
f=(x1) -> degree ideal(x1,leadTerm gens gb I)
fp=(d)->degree M - max apply(flatten entries basis(d,M),f)--The
--function fp(d) gives the footprint in degree d
L=toList(1..regularity M)
apply(L,fp)
\end{verbatim}

Example 7.3. Let $X$ be a projective torus in $\mathbb{P}^2$ over the field $K = \mathbb{F}_3$. The vanishing ideal $I = I(X)$ is generated by $t_1^2 - t_2^2$, $t_2^2 - t_3^2$. The polynomial $F = (t_1 - t_2)^d$ is a zero-divisor of $S/I$ because $(t_1 - t_3, t_2 - t_3)$ is an associated prime of $S/I$ and $F \notin I$ because $F$ does not vanish at $[(1, -1)]$. Hence, $\mathcal{F}_d \neq \emptyset$. If $<$ is the lexicographical order $t_1 > t_2 > t_3$, then $t_3^d$ is a standard monomial which is not a zero-divisor of $S/I$ and $S/(\mathcal{F}_d(I), t_3^d)$ to be greater than $\deg(S/I)$. Using Macaulay2 [13] we obtain

\begin{verbatim}
| d | 1 | 2 | \ldots |
|---|---|---|--------|
| | 4 | 4 | \ldots |
| $H_I(d)$ | 3 | 4 | \ldots |
| $\delta_I(d)$ | 2 | 1 | \ldots |
| fpI(d) | 0 | -4 | \ldots |
\end{verbatim}

Let $C_X(d)$ be a projective Reed-Muller-type code. If $d \geq \text{reg}(S/I(X))$, then $\delta_X(d) = 1$. The converse is not true as the next example shows.

Example 7.4. Let $X = \{(1, 1, 1), (1, -1, 0), (1, 0, -1), (0, 1, -1), (1, 0, 0)\}$ and let $I$ be its vanishing ideal over the finite field $\mathbb{F}_3$. Using Macaulay2 [13] we obtain that $\text{reg}(S/I) = 3$. Notice that $\delta_X(1) = 1$ because the polynomial $t_1 + t_2 + t_3$ vanishes at all points of $X \setminus \{(1, 0, 0)\}$.

The next example shows that $\delta_I$ is not in general non-increasing. This is why we often require that the dimension of $I$ be at least 1 or that $I$ is unmixed with at least 2 minimal primes.

Example 7.5. Let $I$ be the ideal of $\mathbb{F}_5[t_1, t_2]$ generated by $t_1^7$, $t_2^5$, $t_1^2 t_2$, $t_1 t_2^3$. Using Corollary 4.8 and Macaulay2 [13] we get that the regularity of $S/I$ is 7, that is, $H_I(d) = 0$ for $d \geq 7$, and

\begin{verbatim}
| d | 1 | 2 | 3 | 4 | 5 | 6 | \ldots |
|---|---|---|---|---|---|---|--------|
| deg(S/I) | 13 | 13 | 13 | 13 | 13 | 13 | \ldots |
| $H_I(d)$ | 2 | 3 | 3 | 2 | 1 | 1 | \ldots |
| $\delta_I(d)$ | 6 | 2 | 1 | 1 | 2 | 1 | \ldots |
\end{verbatim}

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