Superluminal behavior and the Minkowski space-time

Daniela Mugnai

Nello Carrara Institute of Applied Physics, CNR Florence Research Area,
Via Madonna del Piano 10, 50019 Sesto Fiorentino (FI), Italy

Bessel X-waves, or Bessel beams, have been extensively studied in last years, especially with regard to the topic of superluminality in the propagation of a signal. However, in spite of many efforts devoted to this subject, no definite answer has been found, mainly for lack of an exact definition of signal velocity. The purpose of the present work is to investigate the field of existence of Bessel beams in order to overcome the specific question related to the definition of signal velocity. Quite surprisingly, this field of existence can be represented in the Minkowski space-time by a Super-Light Cone which wraps itself around the well-known Light Cone.

The propagation of Bessel X-waves has been extensively analyzed in last years, especially with regard to the topic of superluminality in connection to the signal propagation. Many contributions were devoted to this topic, both from a theoretical and experimental point of view.

Bessel X-waves, which are also known as Bessel beams, belong to the class of localized waves. The peculiarity of this type of waves is that they are well localized in space, unlike a “usual” wave which occupies the entire space. As is well known, a $u_B$ Bessel beam is the result of superimposing an infinite number of plane waves, each of them with a direction of propagation tilted by the same angle $\theta_0$ with respect to a given axis, say $z$. In cylindrical coordinates $(\rho, z, \psi)$, the beam is given by

$$u_B(\rho, z, t) = J_0(k_0 \rho \sin \theta_0) \exp[ik_0 z \cos \theta_0] \exp(-i\omega t)$$  \hspace{1cm} (1)

where $k_0 = \omega/c$ is the wavenumber in the vacuum, and $\omega$ is the frequency of the beam. Function $J_0$ denotes the zero-order Bessel function of first kind, which, apart from inessential factors, can be written as

$$J_0(x) = \int_0^\pi \exp(ix \cos \varphi) \, d\varphi.$$  \hspace{1cm} (2)

The characteristic features of a Bessel beam are that it supplies well-localized energy, that propagates along the $z$-axis with no deformation in its amplitude, and that both phase and group velocities are greater than the light velocity $c$.

A $U_B$ Bessel pulse limited in time, which is the theoretical definition of signal, can be obtained by superimposing an infinite number of frequencies. After integration of Eq. (1) over $d\omega$, and by substituting the Bessel function $J_0$ with its integral form, we obtain

$$U_B(\rho, z, t) = \int_0^\pi \delta \left( \frac{\rho}{c} \sin \theta_0 \cos \varphi + \frac{z}{c} \sin \theta_0 - t \right) \, d\varphi,$$  \hspace{1cm} (3)

which is different from zero only if

$$t \leq \left| \frac{1}{c} (z \cos \theta_0 + \rho \sin \theta_0) \right|,$$  \hspace{1cm} (4)

where $0 \leq \theta_0 < \theta_{max}$, $\theta_{max} \ll \pi/2$ depends on the experimental set-up.

Thus, the time interval in which the beam is different from zero is

$$t_{min}(\theta_0 = \theta_{max}) \leq t < t_{max}(\theta_0 = 0).$$  \hspace{1cm} (5)
FIG. 1: Bessel beam velocities for three different values of parameter θ₀, in the z – t plane, for ρ = 0 (Euclidean space). The red line indicates light velocity c (for θ₀ = 0) taken here as equal to 1. Green, pink, and blue lines represent the beam velocities for θ₀ = 20°, 30°, and 40°, respectively. For ρ ≠ 0, the intersection point changes its position with no modification in the line behavior.

FIG. 2: Schematic representation of the Super-Light Cone in the Minkowski space-time (pseudo-Euclidean space). The green zone represents the Light-Cone, while the blue zone around it is the field of existence of the Bessel beam. Quantity v_b is the beam velocity for a given axicon angle, θ₀. For θ₀ = 0, the beam is reduced to a plane wave, and its velocity then becomes equal to c. In this situation, the field of existence of the beam goes to zero, the Super-Light Cone narrows and becomes equal to the Light-Cone.

Since the Bessel pulse propagates along the z-axis, we can deduce that the motion of the beam in the z – t plane (see Fig. 1) is within a conical surface similar to the Light Cone, where light velocity c is replaced by velocity v_b = c/ cos θ₀, and t is a real quantity: we can say that the propagation of a Bessel pulse in the Euclidean-space corresponds to a Super-Light Cone in the pseudo-Euclidean space-time of Minkowski. In other words, by introducing a second spatial coordinate, for a given value of θ₀, we obtain a Super-Light Cone like the one of Fig. 2, where straight line v_b, which depends on θ₀, is the beam velocity.

For θ₀ = θ_max, v_b represents the border line which determines the existence of the field: the Bessel beam exists only in the blue zone. Inside this cone of existence, the past Super-Light Cone, t < 0, represents the time interval prior to generation of the beam. The beam originates at t = 0, and for t > 0 (future Super-Light Cone) propagates along the z axis with velocity v_b (blue line, in Fig. 2). For θ₀ = 0 the beam reduces to a plane wave, its velocity becomes equal to c (green line, in Fig. 2), and the Super-Light Cone becomes the Light Cone (green cone in Fig. 2).

Now, since Bessel beams are real quantities (they have been experimentally generated and measured), and since Eq. 1 is capable of describing the scalar field of the beam as being due to a specific experimental set-up [10], we can
conclude that the Super-Light Cone places a new upper speed limit for all objects. Massless particles can travel not only along the Light Cone, but also along the Super-Light Cone in the region between the Super-Cone and the Cone, while the world-lines remain confined within the Light-Cone. In substance, we can think that $c$ is the velocity of light in its simplest manifestation (wave), while more complex electromagnetic phenomena, such as the interference among an infinite number of waves, may originate different velocities. The maximum value $\theta_{\text{max}}$ of axicon angle $\theta_0$ sets the maximum value of the beam velocity. Since the filed depth, that is, the spatial range in which the beam exists, is proportional to $\tan^{-1}\theta_0$, $\theta_0$ can never reach the value of $\pi/2$. If it were possible to obtain values of $\theta_0$ close to $\pi/2$, we should have almost immediate propagation in a nearly-zero space, rather like an ultra fast shot destined to slow down immediately.

The change in the upper limit of the light velocity (the Bessel beam is “light”) does not modify the fundamental principles of relativity and the principle of causality, as demonstrated by recent theory dealing with new geometrical structure of space-time \[1\]. The principle that “the speed of light is the same for all inertial observers, regardless of the motion of the source”, remains unchanged, provided that the substitution $c \rightarrow v_b (= c/\cos \theta_0)$ is made in the Lorentz transformations. In this way, the direction of the beam-light does not depend on the motion of the source, and all observers measure the same speed ($v_b$) in all directions, independently of their motions.

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