ON FUJITA’S CONJECTURE FOR PSEUDO-EFFECTIVE THRESHOLDS

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Abstract. We show Fujita’s spectrum conjecture for $\epsilon$-log canonical pairs and Fujita’s log spectrum conjecture for log canonical pairs. Then, we generalize the pseudo-effective threshold of a single divisor to multiple divisors and establish the analogous finiteness and the DCC properties.

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1. Introduction

Let $(X, \Delta)$ be a log pair with $X$ a normal projective variety over $\mathbb{C}$. Suppose $D$ is a big $\mathbb{R}$-Cartier divisor, then the pseudo-effective threshold of $D$ with respect to $(X, \Delta)$ is defined by

$$\tau(X, \Delta; D) := \inf\{ t \in \mathbb{R}_{\geq 0} | K_X + \Delta + tD \text{ is pseudo-effective} \}.$$ 

Recall that an $\mathbb{R}$-divisor is called pseudo-effective if it is a limit of effective divisors in $N^1(X)_{\mathbb{R}}$.

Fujita once made the following two conjectures [Fuj92, Fuj96].

Theorem 1.1 (Fujita’s spectrum conjecture, [DC17] Theorem 1.1). Let $n$ be a natural number, $S_n$ be the set of pseudo-effective thresholds $\tau(X, H) := \tau(X, \emptyset; H)$ of an ample divisor $H$ with respect to a smooth projective variety $X$ of dimension $n$. Then $S_n \cap [\epsilon, +\infty)$ is a finite set for any $\epsilon > 0$. 

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Theorem 1.2 (Fujita’s log spectrum conjecture, [DC16] Theorem 1.2). Let $S^{ls}_n$ be the set of pseudo-effective thresholds $\tau(X, \Delta; H)$, where $X$ is a smooth projective variety of dimension $n$, $\Delta$ is a reduced divisor with simple normal crossing support, and $H$ is an ample Cartier divisor on $X$. Then $S^{ls}_n$ satisfies the ACC.

Fujita showed that his spectrum conjecture is a consequence of the minimal model program and the BAB (Borisov-Alexeev-Borisov) conjecture, [Fuj96]. Recently, Di Cerbo studied these two problems. He proved Fujita’s spectrum conjecture by using the special BAB conjecture, [DC17], and proved Fujita’s log spectrum conjecture by using the ACC for the lc thresholds, [DC16].

Recall that for a partially ordered set $(S, \succeq)$, it is said to satisfy the DCC (descending chain condition) if any non-increasing sequence $a_1 \succeq a_2 \succeq \cdots \succeq a_k \succeq \cdots$ in $S$ stabilizes. It is said to satisfy the ACC (ascending chain condition) if any non-decreasing sequence in $S$ stabilizes. When $S$ is a set of real numbers, we consider the usual relation “$\leq$”.

In this paper, we study Fujita’s spectrum conjecture and Fujita’s log spectrum conjecture in a more general setting, namely we allow the pair $(X, \Delta)$ to have singularities, and the coefficients of $\Delta$ and $H$ to vary in some fixed set.

Fix a positive integer $n$, a positive real number $\epsilon$, and a subset $I \subset [0, 1]$. We will consider the following set.

$$T_{n,\epsilon}(I) := \{\tau(X, \Delta; H) \mid \dim X = n, (X, \Delta) is \epsilon$-$lc, $\Delta \in I$, $H$ is a big and nef $\mathbb{Q}$-Cartier Weil divisor\}.

We can now state one of the main results in this paper.

Theorem 1.3. Let $n$ be a natural number, and $\epsilon$ be a positive real number.

1. If $I \subset [0, 1]$ is a finite set, then $T_{n,\epsilon}(I) \cap [\delta, +\infty)$ is a finite set for any $\delta > 0$.

2. If $I \subset [0, 1]$ is a DCC set, then $T_{n,\epsilon}(I)$ satisfies the ACC.

Roughly speaking, we generalize Di Cerbo’s result from a smooth variety with an ample divisor $H$ (Theorem 1.2 and Theorem 1.1) to an $\epsilon$-lc pair with a big and nef divisor. Theorem 1.3(1) was a question asked by Di Cerbo (cf. [DC17] P.244).

Our argument of the above result relies on the minimal model program and the recent progress on the BAB conjecture. In fact, we only need to use the special BAB conjecture to prove Theorem 1.3 (see Theorem 2.3 and the remark below). In some sense, the $\epsilon$-lc condition is the weakest possible condition for Theorem 1.3 to hold. If we relax the singularities from $\epsilon$-lc to klt, Theorem 1.3 is no longer true, see Example 3.1 for more details. However, if $H$ is assumed to be an ample Cartier divisor in Theorem 1.3 (2), then the condition “$(X, \Delta)$ is lc” is enough.
Theorem 1.4. Let $n$ be a natural number and $\mathcal{I}$ a DCC set of nonnegative real numbers. Let $\mathcal{CT}_n(\mathcal{I})$ be the set of pseudo-effective threshold $\tau(X, \Delta; H)$ which satisfies the following conditions.

1. $X$ is a normal projective variety of dimension $n$,
2. $(X, \Delta)$ is lc and the coefficients of $\Delta$ are in $\mathcal{I}$, and
3. $H = \sum \mu_i H_i$, where $H_i$ is a nef Cartier divisor for each $i$, $\mu_i \in \mathcal{I}$, and $H$ is big.

Then $\mathcal{CT}_n(\mathcal{I})$ satisfies the ACC.

The version where $(X, \Delta)$ is log smooth, $\Delta$ is reduced, and $H$ itself is an ample divisor (Theorem 1.2) was proven by Di Cerbo by the ACC for the log canonical thresholds and Global ACC for the lc pairs. Roughly speaking, the main difficulty in our setting comes from the fact that $(X, \Delta + \tau(X, \Delta; H)H)$ may not be log canonical, and we can not apply Global ACC directly. Instead, we use Global ACC for generalized polarized pairs, [BZ16], which generalizes Global ACC for the lc pairs, [HMX14], to the generalized lc pairs. The notation and the theory of generalized polarized pairs were introduced and developed in [BZ16, Bir16a]. Furthermore, we can show a slightly stronger version of Theorem 1.4 by using an effective birationality result for generalized polarized pairs of general type established in [BZ16], see Theorem 3.2.

It is natural to generalize the pseudo-effective threshold of a single divisor to multiple divisors. Let $(X, \Delta)$ be a log pair and $D_1, \ldots, D_m$ be big $\mathbb{R}$-Cartier divisors on $X$. The pseudo-effective polytope (PE-polytope) of $D_1, \ldots, D_m$ with respect to $X$ is defined as

$$P(X, \Delta; D_1, \ldots, D_m) := \{(t_1, \ldots, t_m) \in \mathbb{R}_{\geq 0}^m \mid (K_X + \Delta + \sum_{i=1}^m t_i D_i) \text{ is pseudo-effective}\}.$$  

We will show in Proposition 4.2 that $P(X, \Delta; D_1, \ldots, D_m)$ is indeed an unbounded polytope if $(X, \Delta)$ has klt singularities and $D_i$ is a big and nef $\mathbb{Q}$-Weil divisor for each $i$. For convenience, we include all $(t_1, \ldots, t_m) \in \mathbb{R}_{\geq 0}^m$ such that $K_X + \Delta + \sum_{i=1}^m t_i D_i$ is pseudo-effective in $P(X, \Delta; D_1, \ldots, D_m)$, though the thresholds only happen on the boundary. In particular, by comparing with [1], we see that $P(X, \Delta; D)$ is the interval $[\tau(X, \Delta; D), +\infty)$.

Fix two positive integers $n, m$, a subset $\mathcal{I} \subset [0, 1]$, a positive real number $\epsilon$, and a nonnegative real number $\delta$, we consider the following set of truncated PE-polytopes.

$$\mathcal{P}_{n, m, \mathcal{I}, \epsilon, \delta} := \{P(X, \Delta; H_1, \ldots, H_m) \cap [\delta, +\infty]^m \mid \dim X = n, (X, \Delta) is \epsilon lc,$$

$$\Delta \in \mathcal{I}, H_i is a big and nef \mathbb{Q}-Cartier Weil divisor for each i\}.$$  

For simplicity, if $\delta = 0$, we denote $\mathcal{P}_{n, m, \mathcal{I}, \epsilon} := \mathcal{P}_{n, m, \mathcal{I}, \epsilon, \delta}$. We will show the following results for PE-polytopes.
Theorem 1.5. Let $n, m$ be two natural numbers and $\epsilon, \delta$ be two positive real numbers, and $\mathcal{I} \subseteq [0, 1]$ be a finite set. Then the set of truncated PE-polytope $\mathcal{P}_{n,m,\mathcal{I},\epsilon,\delta}$ is finite.

Letting $m = 1$ in Theorem 1.5, we get Theorem 1.3 (1).

Theorem 1.6. Let $n, m$ be two natural numbers, $\epsilon$ be a positive real number, and $\mathcal{I} \subseteq [0, 1]$ a DCC set. Then the set of PE-polytopes $\mathcal{P}_{n,m,\mathcal{I},\epsilon}$ is a DCC set under the inclusion relation “$\supseteq$”.

Letting $m = 1$ in Theorem 1.5, we get Theorem 1.3 (2).

We note that we can not apply Theorem 1.3 to prove Theorem 1.5 and Theorem 1.6 directly. For example, suppose we consider two testing divisors, and thus $P := P(X, \Delta; H_1, H_2)$ is a two dimensional polytope (suppose it is non-degenerate). It is possible to construct a sequence of strictly decreasing sequence of convex polytopes $\{P_i\}_{i \in \mathbb{N}}$, such that $\{P_i\}_{i \in \mathbb{N}}$ stabilizes along any vertical line, any horizontal line, and any line passing through the origin. For example, in Figure 1, the sequence of polytopes $\{P_i\}_{i \in \mathbb{N}}$ is not stable near the point $\tau$. It is our hope that Theorem 1.5 and Theorem 1.6 would give more information on the testing divisors.

As a corollary of Theorem 1.3 (1), by the same argument of [HJ16], we can improve their main result for big and nef divisors (rather than big and semiample). This corollary was firstly proven by [LTT14] assuming a weak version of the BAB conjecture.

Recall that for smooth variety $X$, and a big $\mathbb{R}$-Cartier divisor $L$ on $X$, the $a$-constant is defined by $a(X, L) := \tau(X, L)$. For a singular projective variety $X$, the $a$-constant is defined by $a(X, L) := a(Y, \pi^*L)$, where $\pi : Y \to X$ is any log resolution of $(X, L)$.

Corollary 1.7 ( [LTT14] Theorem 4.10). Let $X$ be a smooth uniruled projective variety and $L$ a big and nef $\mathbb{Q}$-divisor on $X$. Then there exists a proper closed subset $W \subset X$ such that every subvariety $Y$ satisfying $a(Y, L) > a(X, L)$ is contained in $W$.

Recall that we only use the special BAB conjecture to prove Theorem 1.3 (1), thus yields another proof of the above corollary.
Finally, it is reasonable to propose the following conjecture for PE-polytopes, which was proven in Theorem 1.4 for a single divisor.

**DCC for PE-polytopes.** Let $n, m$ be two natural numbers, $I$ be a DCC set of nonnegative real numbers. Then the set of PE-polytopes $P_{n,m,I} := \{P(X, \Delta; H_1, \ldots, H_m) \mid \dim X = n, (X, \Delta) \text{ is lc,} \Delta \in I, H_i \text{ is a big and nef Cartier divisor for each } i\}$ satisfies the DCC under the inclusion of polytopes.

**Acknowledgements.** J.H. started this work during his visiting at Princeton University from Sep. 2015–Sep. 2016 under the supervision of János Kollár, and with the support of China Scholarship Council (CSC) and “Training, Research and Motion” (TRAM) network. He wishes to thank them all. We thank Chenyang Xu for giving not only the initial inspiration on this research topic, but also many valuable comments on a previous draft. We thank Gabriele Di Cerbo for useful E-mail correspondences, and Chen Jiang for helpful discussions. J.H. would like to thank his advisors Gang Tian and Chenyang Xu in particular for constant support and encouragement. This work is partially supported by NSFC Grant No.11601015 and the Postdoctoral Grant No.2016M591000.

2. Preliminaries

2.1. **Singularities.** For basic definitions of log discrepancies and log canonical (lc), divisorially log terminal (dlt), kawamata log terminal (klt) singularities, we refer to [KM98]. Recall that a log pair $(X, \Delta)$ is $\epsilon$-lc for some $\epsilon \geq 0$, if its minimal log discrepancy is greater or equal to $\epsilon$.

For reader’s convenience, we state the following lemma which is known as dlt modifications.

**Lemma 2.1** (dlt modifications, c.f. [HMX14] Proposition 3.3.1). Let $(X, \Delta)$ be a lc pair. There there is a proper birational morphism $f : X' \to X$ with reduced exceptional divisors $E_i$, such that

1. $(X', \Delta') := (X', f_*^{-1} \Delta + \sum E_i)$ is dlt,
2. $X'$ is $\mathbb{Q}$-factorial,
3. $K_{X'} + \Delta' = f^*(K_X + \Delta)$.

In particular, if $(X, \Delta)$ is klt, then $f$ is small.

We can apply dlt modifications to reduce the study of PE-polytopes from lc (resp. klt) pairs to $\mathbb{Q}$-factorial dlt (resp. klt) pairs.

**Lemma 2.2.** Let $(X, \Delta)$ be a lc pair, and $H_i$ be a big $\mathbb{R}$-Cartier divisor for any $1 \leq i \leq m$. If $f : (X', \Delta') \to (X, \Delta)$ is a dlt modification of $(X, \Delta)$, then

$$P(X, \Delta; H_1, \ldots, H_m) = P(X', \Delta'; H'_1, \ldots, H'_m),$$

where $H'_i = f^*H_i$. 
Proof. On one hand, let \((t_1, \ldots, t_m) \in P(X, \Delta; H_1, \ldots, H_m)\), then \(K_X + \Delta + \sum_{i=1}^m t_iH_i\) is pseudo-effective. We have \(K_{X'} + \Delta' + \sum_{i=1}^m t_iH'_i = f^*(K_X + \Delta + \sum_{i=1}^m t_iH_i)\) is also pseudo-effective.

On the other hand, let \((t_1, \ldots, t_m) \in P(X', \Delta'; H'_1, \ldots, H'_m)\), \(K_{X'} + \Delta' + \sum_{i=1}^m t_iH'_i\) is pseudo-effective. We have \(K_X + \Delta + \sum_{i=1}^m t_iH_i = f_*(K_{X'} + \Delta' + \sum_{i=1}^m t_iH'_i)\) is pseudo-effective. \(\Box\)

2.2. Boundedness of Fano varieties. A set of varieties \(\mathcal{X}\) is said to form a bounded family if there is a projective morphism of schemes \(g : W \rightarrow T\), with \(T\) of finite type, such that for every \(X \in \mathcal{X}\), there is a closed point \(t \in T\) and an isomorphism \(W_t \cong X\), where \(W_t\) is the fibre of \(g\) at \(t\). A variety \(X\) is called Fano if it is lc and \(-K_X\) is ample. The following result is a variant of the conjecture of Borisov-Alexeev-Borisov.

\[\text{Theorem 2.3 (Bir16a Corollary 1.2).}\] Let \(n\) be a natural number and \(\epsilon\) a positive real number. Then the projective varieties \(X\) such that

1. \((X, \Delta)\) is \(\epsilon\)-lc of dimension \(n\) for some boundary \(\Delta\), and
2. \(K_X + \Delta \equiv 0\) and \(\Delta\) is big,

form a bounded family.

Remark 2.4. The special BAB conjecture, which assumes that the coefficients of \(\Delta\) are more than or equal to a positive real number \(\delta\) in Theorem 2.3, was firstly proven in [Bir16a]. In order to show Theorem 1.3 and Theorem 1.5, we only need to apply the special BAB conjecture.

2.3. Generalized polarized pairs. The theory of generalized pair was developed in [BZ16].

\[\text{Definition 2.5 (generalized polarized pairs).}\] A generalized polarized pair consists of a normal variety \(X\) equipped with projective morphisms

\[W \xrightarrow{f} X \rightarrow Z\]

where \(f\) is birational, \(W\) is normal, an \(\mathbb{R}\)-boundary \(\Delta \geq 0\), and an \(\mathbb{R}\)-Cartier divisor \(H_W\) on \(W\) which is nef over \(Z\) such that \(K_X + \Delta + H\) is \(\mathbb{R}\)-Cartier with \(H = f_*H_W\). We call \(\Delta\) the boundary part and \(H_W\) the nef part. We usually refer to the pair by saying \((X, \Delta + H)\) is a generalized pair with data \(W \xrightarrow{f} X \rightarrow Z\) and \(H_W\).

In this paper, we only need to use the case that \(Z\) is a point. Thus, we will drop \(Z\), and say the pair is projective. Note that if \(W' \rightarrow W\) is a projective birational morphism from a normal variety, then there is no harm in replacing \(W\) with \(W'\) and replacing \(H_W\) with its pullback to \(W'\).

\[\text{Definition 2.6 (generalized lc).}\] Let \((X, \Delta + H)\) be a generalized polarized pair, which comes with the data \(W \xrightarrow{f} X \rightarrow Z\), and replacing \(W\), we may assume that \(f\) is a log resolution of \((X, \Delta)\). We can write

\[K_W + \Delta_W + H_W = f^*(K_X + \Delta + H)\]
for some uniquely determined $\Delta_W$. We say $(X, \Delta + H)$ is generalized lc if every coefficient of $\Delta_W$ is less than or equal to 1.

**Remark 2.7** (BZ16 Remark 4.2(6)). Let $(X, \Delta + H)$ be a generalized projective pair with data $W \stackrel{f}{\longrightarrow} X$ and $H_W$. We may assume that $f$ is a log resolution of $(X, \Delta)$. Assume that there is a contraction $X \rightarrow Y$. Let $F$ be a general fibre of $W \rightarrow Y$, $T$ the corresponding fibre of $X \rightarrow Y$, and $g : F \rightarrow T$ the induced morphism. Let

$$\Delta_F = \Delta_W|_F, H_F = H_W|_F, \Delta_T = g_*(\Delta_F), H_T = g_* H_F.$$  

Then $(T, \Delta_T + H_T)$ is a generalized polarized projective pair with the data $F \stackrel{g}{\longrightarrow} T$ and $H_F$. Moreover,

$$K_T + \Delta_T + H_T = (K_X + \Delta + H)|_T.$$  

In addition, $\Delta_T = \Delta|_T$ and $H_T = H|_T$.

For more properties of generalized polarized pairs, we refer to [BZ16, Bir16a]. We will need the following result to prove Theorem 1.4.

**Theorem 2.8** (Global ACC for generalized pairs, BZ16 Theorem 1.6). Let $\mathcal{I}$ be a DCC set of nonnegative real numbers and $n$ a natural number. Then there is a finite subset $\mathcal{I}_0 \subseteq \mathcal{I}$ depending only on $\mathcal{I}, n$ such that

1. $X$ is a normal projective variety of dimension $n$,
2. $(X, \Delta + H)$ is generalized lc with data $W \stackrel{f}{\longrightarrow} X$ and $H_W$,
3. $H_W = \sum \mu_j H_{j,W}$, where $H_{j,W}$ are nef Cartier divisors and $\mu_j \in \mathcal{I}$,
4. $\mu_j = 0$ if $H_{j,W} \equiv 0$,
5. the coefficients of $\Delta$ belong to $\mathcal{I}$, and
6. $K_X + \Delta + H \equiv_R 0$,

then the coefficients of $\Delta$ and $\mu_j$ belong to $\mathcal{I}_0$.

3. Fujita's spectrum conjecture and Fujita's log spectrum conjecture

**Proof of Theorem 1.3.** By Lemma 2.2 we can assume that $X$ is $\mathbb{Q}$-factorial. Let $\tau = \tau(X, \Delta; H)$, then $K_X + \Delta + \tau H$ is pseudo-effective but not big. We can assume that $K_X$ is not pseudo-effective.

Since $H$ is big and nef, there is an effective divisor $E$, and ample $\mathbb{Q}$-divisors $A_k$ such that $H \sim_{\mathbb{Q}} A_k + \frac{E}{k}$, for any $k \gg 1$. Let $N$ be a natural number such that $(X, \Delta + \tau \frac{E}{N})$ is $\frac{\tau}{2}$-lc, and $N'$ a natural numbers such that $N'A_N$ is a very ample divisor. Let $A' \in |N'A_N|$ be a general very ample divisor, and $H' := \frac{1}{N} A' + \frac{E}{N} \sim_{\mathbb{Q}} H$, then

$$(X, \Gamma) := (X, \Delta + \tau H')$$

is still $\frac{\tau}{2}$-lc.
According to [BCHM10], we may run a \((K_X + \Gamma)\)-MMP, \(\phi : X \rightarrow Y\), such that \(\phi\) is a birational map, \(K_Y + \phi_*\Gamma\) is semiample and \((Y, \phi_*\Gamma)\) is still \(\epsilon_2\)-lc. As \(\phi\) is \((K_X + \Gamma)\)-negative, \(K_Y + \phi_*\Gamma\) is not big as well.

Now \(K_Y + \phi_*\Gamma\) defines a contraction \(f : Y \rightarrow Z\). Let \(F\) be a general fiber of \(f\), we have \(\dim(F) > 0\). Restricting to \(F\), we get

\[
K_F + \phi_*\Gamma|_F = K_F + \phi_*\Delta|_F + \tau H_Y|_F \equiv \mathbb{R} 0,
\]

where \(H_Y\) is the strict transform of \(H'\) on \(Y\). We note that \(H_Y\) is big and \(H_Y|_F\) is also big since it is the restriction of a big divisor on a general fiber.

Since \(K_F + \phi_*\Gamma|_F\) is \(\epsilon_2\)-lc, according to Theorem 2.3, \(F\) belongs to a bounded family. We may find a very ample Cartier divisor \(M_F\) on \(F\), so that \(-K_F \cdot M_F^{\dim F - 1}\) is bounded. Besides, as \(H' \sim_{\mathbb{Q}} H\), we have \(\phi_*(H') \equiv_{\mathbb{Q}} \phi_*(H)\), and the intersection number \(d := H_Y|_F \cdot M_F^{\dim F - 1} = \phi_*(H)|_F \cdot M_F^{\dim F - 1}\) is a positive integer. Let \(\phi_*\Delta|_F = \sum_j a_j \Delta_{F,j}\), where \(a_j \in \mathcal{I}\), and \(\Delta_{F,j}\) is a Weil divisor. By intersecting (3) with \(M_F^{\dim F - 1}\), we obtain an equation for \(\tau\),

\[
\tau d + \sum_j a_j b_j = c,
\]

where \(c = -K_F \cdot M_F^{\dim F - 1}\) is a nonnegative integral with only finite possibilities, \(b_j = \Delta_{F,j} \cdot M_F^{\dim F - 1}\) are nonnegative integers.

First, we prove Theorem 1.3 (1). By assumption \(\tau \geq \delta > 0\) and \(\mathcal{I}\) is finite, \(d, b_j\) are bounded above and thus only have finite possibilities. Hence there are only finite possibilities for the \(\tau\) in equation (4).

Next, we prove Theorem 1.3 (2). Recall that \(a_j \in \mathcal{I}\) which is a DCC set, and \(b_j \in \mathbb{N}\), then the finite summations \(\sum_j a_j b_j \in \sum \mathcal{I}\) still form a DCC set. Now

\[
\tau = \frac{1}{d}(c - \sum_j a_j b_j).
\]

The right hand side of the equation (5) belongs to an ACC set, and \(\mathcal{T}_{n, \epsilon}(\mathcal{I})\) satisfies the ACC.

**Proof of Corollary 1.7.** Replacing Theorem 1.1 by Theorem 1.3 in the proof of [HJ16], we get Corollary 1.7. \(\square\)

In the following example, we see that Theorem 1.3 (1) is no longer true even for an ample Cartier divisor if we relax the singularities from \(\epsilon\)-lc to klt, and Theorem 1.3 (2) is no longer true even for an ample \(\mathbb{Q}\)-Cartier Weil divisor if we relax the singularities from \(\epsilon\)-lc to klt. We thank Chen Jiang for providing us this example.
Example 3.1. Let $n$ be a natural number and $\mathbb{P}(1, 1, n)$ be the weighted projective space. It is a toric variety with the lattice $N$ spanned by $\{(1, 0), (\frac{1}{n}, \frac{1}{n})\}$. Let $v_1 = (1, 0), v_2 = (0, 1)$ and $v_3 = (-\frac{1}{n}, -\frac{1}{n})$, then the fan is generated by maximal cones $\sigma_1 = \langle v_1, v_2 \rangle, \sigma_2 = \langle v_2, v_3 \rangle$ and $\sigma_3 = \langle v_1, v_3 \rangle$ (c.f. [Bor97], P.35). Let $D_i$ be the toric invariant divisor corresponding to $v_i$. By choosing $(n, 0)$ and $(0, n)$ in the dual lattice $N^\vee$, we see that $nD_1 \sim D_3$ and $nD_2 \sim D_3$. Thus

$$-K_{\mathbb{P}(1, 1, n)} \sim Q D_1 + D_2 + D_3 \sim Q (1 + \frac{2}{n})D_3.$$ Moreover $D_3$ is an ample Cartier divisor as it is associated to the lattice $(0, 0) \in N^\vee$ for $\sigma_1$, $(n, 0) \in N^\vee$ for $\sigma_2$ and $(0, n) \in N^\vee$ for $\sigma_3$. As $\mathbb{P}(1, 1, n)$ has Picard number one, $(1 + \frac{2}{n})$ is the pseudo-effective threshold of $D_3$. This gives a set of varieties whose pseudo-effective spectrum is infinite away from 0. Notice that the minimal log discrepancy of $\mathbb{P}(1, 1, n)$ is $\frac{2}{n}$ (c.f. [Bor97]), hence it is a counterexample if we replace $\epsilon$-lc by klt in Theorem 1.3 (1).

Besides, $D_1$ is an integral divisor and

$$-K_{\mathbb{P}(1, 1, n)} \sim Q D_1 + D_2 + D_3 \sim Q (n + 2)D_1.$$ Hence the pseudo-effective threshold of $D_1$ is $n + 2$. Thus we get a family whose pseudo-effective thresholds (with respect to $D_1$) are strictly increasing. This gives a counterexample if we replace $\epsilon$-lc by klt in Theorem 1.3 (2).

Proof of Theorem 1.4. Let $\tau = \tau(X, \Delta, H)$. We first prove the theorem for the case that $(X, \Delta)$ is $\mathbb{Q}$-factorial klt.

By assumption, $H$ is big and nef, as in the proof of Theorem 1.3, we can run a $(K_X + \Delta + \tau H)$-MMP, $\phi : X \to Y$, and reach a minimal model $(Y, \phi_*(\Delta + \tau H))$, on which $K_Y + \phi_*(\Delta + \tau H)$ is semiample defining a contraction $Y \to Z$.

Taking a common log resolution $p : W \to (X, \Delta), q : W \to (Y, \phi_* \Delta)$, let $H_W = p^*H$. Then $(X, \Delta + \tau H)$ is generalized lc, as $(X, \Delta)$ is lc. Since $p^*(K_X + \Delta + \tau H) \ge q^*(K_Y + \phi_*(\Delta + \tau H))$, $(Y, \phi_*(\Delta + \tau H))$ is also generalized lc. Let $F$ be a general fiber of $W \to Z$, and $T$ be the corresponding fiber of $Y \to Z$. Again, we have $\dim(T) > 0$. By restricting to the general fiber $T$, $(T, \phi_*(\Delta + \tau H)|_T)$ is generalized lc, and $K_T + \phi_*(\Delta + \tau H)|_T \equiv 0$. Since $H_W$ is big, $H_W|_F$ is not numerically trivial, and there exists some component $q^*(H_1)|_F$ of $H_W|_F$ which is not numerically trivial. If $\{\tau\}$ forms a strictly increasing sequence, then $\{\mu_{j\tau}\}$ belongs to a DCC set. According to Theorem 2.8, $\{\mu_{j\tau}\}$ belongs to a finite set, a contradiction. Therefore, $\{\tau\}$ belongs to an ACC set.

For the general case, according to Lemma 2.2, we may assume that $(X, \Delta)$ is $\mathbb{Q}$-factorial dlt. If the statement were not true, then there exists a sequence of lc pairs $(X^{(i)}, \Delta^{(i)})$, and a big $\mathbb{R}$-Cartier $H^{(i)}$ on $X^{(i)}$ satisfying the assumption of Theorem 1.4, but $\tau_i := \tau(X^{(i)}, \Delta^{(i)}, H^{(i)})$ is strictly increasing. For any $1 \ge \epsilon \ge 0$, let $\tau_{i, \epsilon} := \tau(X^{(i)}, (1 - \epsilon)\Delta^{(i)}; H^{(i)})$. It is clear
that \(\tau_{i,\epsilon} \geq \tau_i\), and there exists a decreasing sequence, \(\epsilon_i \to 0\), such that \(\tau_{i+1} > \tau_{i,\epsilon_i} \geq \tau_i\). Let \(J = \{1-\epsilon_i\}\). Now \((X^{(i)}, (1-\epsilon_i)\Delta^{(i)})\) is \(\mathbb{Q}\)-factorial klt, the coefficients of \((1-\epsilon_i)\Delta^{(i)}\) belong to \(I J\), which is a DCC set. However, the sequence \(\{\tau_{i,\epsilon_i}\}_{i \in \mathbb{N}}\) is strictly increasing. This contradicts to the above \(\mathbb{Q}\)-factorial case. \(\square\)

Inspired by a private communication with Di Cerbo, we can prove a slightly stronger version of Theorem 1.4 by using an effective birationality result for generalized polarized pairs of general type established in [BZ16].

**Theorem 3.2.** Let \(n\) be a natural number and \(I\) a DCC set of nonnegative real numbers. Let \(D_n(I)\) be the set of pseudo-effective threshold \(\tau(X, \Delta; M)\) which satisfies the following conditions.

1. \(X\) is a normal projective variety of dimension \(n\),
2. \((X, \Delta)\) is lc and the coefficients of \(\Delta\) are in \(I\),
3. \(M = \sum \mu_i M_i\), where \(M_i\) is a nef Cartier divisor for each \(i, \mu_i \in I\), and
4. \(K_X + \Delta + M\) is big.

Then \(D_n(I)\) satisfies the ACC.

**Proof.** Otherwise, there exists a sequence \(\{(X^{(i)}, \Delta^{(i)}, M^{(i)})\}\), such that \(\alpha_i = \tau(X^{(i)}, \Delta^{(i)}; M^{(i)})\) is strictly increasing and \(\lim_{i \to +\infty} \alpha_i = \alpha \leq 1\). Then, \(\tau(X^{(i)}, \Delta^{(i)}; \alpha M^{(i)}) = \frac{\alpha}{\alpha} \to 1\), and \(K_X^{(i)} + \Delta^{(i)} + \alpha M^{(i)}\) is big. Since the coefficients of \(\alpha M\) belong to the DCC set \(I \cup \alpha I\), by Theorem 8.2 in [BZ16], there exists a natural number \(m\) depending only on \(n\) and \(I \cup \alpha I\), such that the linear system \(|m(K_X + \Delta)| + \sum |m\alpha j M_j|\) defines a birational map. By Lemma 2.3.4 in [HMX13], \(K_X + \Delta + (2n+1)(m(K_X + \Delta + \alpha M))\) is big. Since

\[
K_X + \Delta + (2n+1)(m(K_X + \Delta + \alpha M)) \sim_R ((2n+1)m+1)(K_X + \Delta + \frac{(2n+1)m\alpha}{(2n+1)m+1} M),
\]

we have \(\alpha_i = \tau(X^{(i)}, \Delta^{(i)}; M^{(i)}) \leq \frac{(2n+1)m}{(2n+1)m+1} \alpha\). This implies that

\[
\lim_{i \to +\infty} \alpha_i \leq \frac{(2n+1)m}{(2n+1)m+1} \alpha < \alpha,
\]
a contradiction. \(\square\)

**Remark 3.3.** In Theorem 1.4, we do not require that \(M\) to be big, and Theorem 3.2 implies Theorem 1.4. In fact, let \(I\) be a DCC set, \((X, \Delta)\) be a projective \(\mathbb{Q}\)-factorial dlt pair of dimension \(n\), \(H = \sum \mu_i H_i\) be a big and nef \(\mathbb{R}\)-Cartier divisor. Set \(\delta = \min\{I^0\}\), where \(\Delta, \mu_j \in I\). One can show that \(K_X + \Delta + \frac{(2n+1)}{\delta} H\) is also big (see, for example, [Bir16a] Lemma 2.30). Let \(M = \frac{(2n+1)}{\delta} H\), Theorem 1.4 follows from Lemma 2.2 and Theorem 3.2.
4. The Finiteness and the DCC property for pseudo-effective polytopes

In this section, we prove Theorem 1.5 and Theorem 1.6, which generalize Theorem 1.3 from a single divisor to multiple divisors. The proofs are similar.

4.1. Pseudo-effective polytopes. Let $X$ be a normal projective variety, and $V$ be a finite dimensional affine subspace of the real vector space $\text{WDiv}_{\mathbb{R}}(X)$ of Weil divisors on $X$. Fix an $\mathbb{R}$-divisor $A \geq 0$ and define (see [BCHM10] Definition 1.1.4)

$E_A(V) = \{ \Delta = A + B \mid B \in V, B \geq 0, K_X + \Delta \text{ is lc and pseudo-effective} \}$.

Notice that $E_A(V)$ is a compact set by the lc requirement. Under this notation, we have the following result.

**Theorem 4.1** ([BCHM10] Corollary 1.1.5). Let $X$ be a normal projective variety, $V$ be a finite dimensional affine subspace of $\text{WDiv}_{\mathbb{R}}(X)$ which is defined over rationals. Suppose there is a divisor $\Delta_0 \in V$ such that $K_X + \Delta_0$ is klt. Let $A$ be a general ample $\mathbb{Q}$-divisor, which has no components in common with any element of $V$. Then $E_A(V)$ is a rational polytope.

We can deduce that PE-polytopes are indeed polytopes under suitable assumptions from Theorem 4.1. Notice that there are no lc restrictions for PE-polytopes. See Figure 2 for a PE-polytope of two divisors.

![Figure 2. A PE-polytope of two divisors](image)

**Proposition 4.2.** Let $(X, \Delta)$ be a klt pair and $H_1, \ldots, H_m$ be big and nef $\mathbb{Q}$-Weil divisors. Then $P(X, \Delta; H_1, \ldots, H_m)$ is an unbounded polytope.

*Proof.* By definition (see (2)), $P(X, \Delta; H_1, \ldots, H_m)$ is convex. If $K_X$ is pseudo-effective, then $P(X, \Delta; H_1, \ldots, H_m) = \mathbb{R}_{\geq 0}^m$, hence we can assume that $K_X$ is not pseudo-effective. Then there exists a rational number $a > 0$ such that $K_X + a(H_1 + \cdots + H_m)$ is also not pseudo-effective.

Because $H_i$ are big divisors, there exists constant $\tau > 0$, such that $K_X + \Delta + \sum_{i=1}^m t_i H_i$ is pseudo-effective whenever $t_i \geq \tau$. In other words, $\mathbb{R}_{\geq 0}^m \setminus [0, \tau]^m$ is contained in $P(X, \Delta; H_1, \ldots, H_m)$. Thus, it suffices to show that $P(X, \Delta; H_1, \ldots, H_m) \cap [0, \tau]^m$ is a polytope.
Recall that a divisor is pseudo-effective if and only if the divisor which is numerical to it is also pseudo-effective, thus,

\[(6) \quad P(X, \Delta; H_1, \ldots, H_m) = P(X, \Delta; H'_1, \ldots, H'_m)\]

for any \(H_i \equiv Q H'_i\). Since \(H_i\) is big and nef, there is an effective divisors \(E_i\), and ample \(Q\)-divisor \(A_{ik}\) such that \(H_i \equiv A_{ik} + \frac{E_i}{N}\), for any \(k \gg 1\). Let \(N\) be a natural number such that \((X, \Delta + \tau \sum_{i=1}^{m} E_i/N)\) is klt, and \(N_i > \tau\) be a natural number such that \(N_i A_{iN}\) is a very ample divisor. Let \(A'_i \in |N_i A_{iN}|\) be a general very ample divisor, and \(H'_i = \frac{1}{N_i} A'_i + \frac{E_i}{N}\), \((X, \Delta + \tau \sum_{i=1}^{m} H'_i)\) is also klt. Now, we only need to show that \(P(X, \Delta; H'_1, \ldots, H'_m) \cap [0, \tau]^m\) is a polytope.

For each \(i\), let \(M'_i \in |A'_i|\) be another general very ample divisor, and let \(M_i = \frac{a}{N_i} M'_i\). Let \(V_i\) be the affine space \(\Delta + \frac{a}{N} E_i + \sum_{i=1}^{m} \tau H'_i\). By Theorem 4.1, \(E_{M_i}(V_i)\) is a rational polytope. Let

\[E_i := \{(t_1, \ldots, t_{m-1}, t_i + a, t_{i+1}, \ldots, t_m) | \Delta + M_i + \frac{a}{N} E_i + \sum_{i=1}^{m} t_i H'_i \in E_{M_i}(V_i)\} .\]

It is clear that \(E_i \cap [0, \tau]^m \subseteq P(X, \Delta; H'_1, \ldots, H'_m) \cap [0, \tau]^m\) for each \(i\). Now if \((t_1, \ldots, t_m) \in P(X, \Delta; H'_1, \ldots, H'_m) \cap [0, \tau]^m\), there exists at least one \(k\) such that \(t_k \geq a\) (recall that we chose \(a\) such that \(K_X + a(H_1 + \cdots + H_m)\) is not pseudo-effective). This implies that \(\Delta + M_k + \frac{a}{N} E_k + \sum_{i=1}^{m} t_i H'_i - a H'_k \in E_{M_k}(V_k)\), and \((t_1, \ldots, t_m) \in E_k\). Thus,

\[\bigcup_{i=1}^{m} E_i \cap [0, \tau]^m = P(X, \Delta; H'_1, \ldots, H'_m) \cap [0, \tau]^m .\]

In particular, \(P(X, \Delta; H'_1, \ldots, H'_m) \cap [0, \tau]^m\) is a polytope by convexity. \(\square\)

**Figure 3.** A truncated PE-polytope

### 4.2. Proofs of Theorem 1.5 and Theorem 1.6

**Proof of Theorem 1.5.** By Lemma 2.2, we can assume that \(X\) is \(Q\)-factorial. Choose arbitrary \(\tau = (\tau_1, \ldots, \tau_m)\) on the boundary of the truncated PE-polytope \(P(X, \Delta; H'_1, \ldots, H'_m) \cap [\delta, +\infty)^m\) and in the interior of \([\delta, +\infty)^m\), then \(K_X + \Delta + \tau_1 H_1 + \cdots + \tau_m H_m\) is pseudo-effective but not big. If \(K_X\) is
pseudo-effective, then \(P(X, \Delta; H_1, \ldots, H_m) \cap [\delta, +\infty)^m = [\delta, +\infty)^m\), we can assume that \(K_X\) is not pseudo-effective.

Since \(H_i\) is big and nef, by the same argument in the proof of Theorem [1.3] we can find \(\mathbb{Q}\)-divisors \(H'_i \sim_q H_i\), such that

\[
(X, \Gamma) := (X, \Delta + \sum_{i=1}^m \tau_i H'_i)
\]

is \(\frac{1}{\tau}\)-lc, and we may run a \((K_X + \Gamma)\)-MMP, \(\phi : X \to Y\), such that \(\phi\) is a birational map, \(K_Y + \phi_* \Gamma\) is semiample and \((Y, \phi, \Gamma)\) is still \(\frac{1}{\tau}\)-lc.

Now \(K_Y + \phi_* \Gamma\) defines a contraction \(f : Y \to Z\). Let \(F\) be a general fiber of \(f\), we have \(\dim(F) > 0\). By restricting to \(F\), we get

\[
K_F + \phi_* \Gamma|_F = K_F + \phi_* \Delta_F + \sum_{i=1}^m \tau_i H_{Y,i}|_F \equiv 0,
\]

where \(H_{Y,i}\) is the strict transform of \(H'_i\) on \(Y\) for each \(i\).

Since \(K_F + \phi_* \Delta + \sum_{i=1}^m \tau_i H_{Y,i}|_F\) is \(\frac{1}{\tau}\)-lc, according to Theorem [2.3] \(F\) belongs to a bounded family. Hence, we may find a very ample Cartier divisor \(M\) on \(F\), so that \(-K_F \cdot M^{\dim F - 1}\) is bounded. Besides, as \(H'_i \sim_q H_i\), we have \(\phi_*(H'_i) = \phi_*(H_i)\), and the intersection numbers \(d_i := H_{Y,i}|_F \cdot M^{\dim F - 1} = \phi_*(H_i)|_F \cdot M^{\dim F - 1}\) is a positive integer. Let \(\phi_* \Delta_F = \sum_j a_j\Delta_{F,j}\), where \(a_j \in \mathbb{Z}\), and \(\Delta_{F,j}\) are Weil divisors. By intersecting (7) with \(M^{\dim F - 1}\), we obtain an equation for \(\tau_i\),

\[
\sum_j a_j b_j + \sum_{i=1}^m \tau_i d_i = c,
\]

where \(c = -K_F \cdot M^{\dim F - 1}\) is a nonnegative integer with only finite possibilities, \(b_j = \Delta_{F,j} \cdot M^{\dim F - 1}\) is a positive integer. Since \(\tau_i \geq \delta\) for all \(i\), and \(\mathbb{Z}\) is finite, \(d_i, b_i\) are bounded above and thus only have finite possibilities.

Hence there are only finite possibilities for the equations (8). In other words, \((\tau_1, \ldots, \tau_m)\) can only lie on finitely many hyperplanes \(L_k \subseteq \mathbb{R}^m, k \in I\). By Proposition [4.2] \(P(X, \Delta; H_1, \ldots, H_m) \cap [\epsilon, +\infty)^m\) is a polytope. If \(\Theta\) is a facet of \(P(X; H_1, \ldots, H_m) \cap [\epsilon, +\infty)^m\) (i.e. an \((m - 1)\)-dimensional face), then

\[
\Theta \subseteq (\cup_{k \in I} L_k) \cup (\cup_{i=1}^m \{t_i = \delta\}).
\]

By irreducibility, \(\Theta \subseteq L_k\) or \(\Theta \subseteq \{t_i = \delta\}\), and hence finite possibilities. This shows that the set \(P_{n,m,I,\epsilon,\delta}\) is finite. \(\square\)

**Proof of Theorem 1.6.** By Lemma [2.2], we can assume that \(X\) is \(\mathbb{Q}\)-factorial. Without loss of generality, we can assume that \(K_X\) is not pseudo-effective.

Choose arbitrary \(\tau = (\tau_1, \ldots, \tau_m)\) on the boundary of the PE-polytope \(P(X, \Delta; H_1, \ldots, H_m)\) which is in the interior of \(\mathbb{R}_{\geq 0}^m\). We then proceed the
same way as in the proof of Theorem [1.3] We can find \( \mathbb{Q} \)-divisors \( H_i' \sim_\mathbb{Q} H_i \), such that
\[
(X, \Gamma) := (X, \Delta + \sum_{i=1}^{m} \tau_i H_i' \sim_\mathbb{Q} H_i)
\]
is \( \frac{1}{2} \)-lc, and we may run a \( (K_X + \Gamma) \)-MMP, \( \phi : X \to Y \), such that \( K_Y + \phi_* \Gamma \) is semiample. \( K_Y + \phi_* \Gamma \) defines a contraction \( f : Y \to Z \). By restricting to a general fiber \( F \), we get a similar equation as (7)
\[
K_F + \phi_* \Gamma|_F = K_F + \sum_j a_j \Delta_{F,j} + \sum_{i=1}^{m} \tau_i H_{Y,i}|_F = 0.
\]
Here we write \( \phi_* \Delta|_F = \sum_j a_j \Delta_{F,j} \), where \( \Delta_{F,j} \) is a Weil divisor. As \( (F, \phi_* \Gamma|_F) \) is \( \frac{1}{2} \)-lc, it belongs to a bounded family by Theorem [2.3]. By taking a very ample divisor \( M \), and intersecting \( M \cdot M^{\dim F - 1} \) with (9), we get
\[
\sum_j a_j b_j + \sum_{i=1}^{m} \tau_i d_i = c,
\]
where \( c = -K_F \cdot M^{\dim F - 1} \) is a nonnegative integer with only finite possibilities, \( b_j = \Delta_{Y,j} \cdot M^{\dim F - 1} \), \( d_i = H_{Y,i}|_F \cdot M^{\dim F - 1} \) are nonnegative integers.

Now, all \( a_j b_j \) form a DCC set, and the finite summations \( \sum_j a_j b_j \in \sum \mathcal{I} \) also form a DCC set.

Set \( P^{(k)} = P(X^{(k)}, \Delta^{(k)}; H_1^{(k)}, \ldots, H_m^{(k)}) \). Suppose \( P^{(1)} \supseteq P^{(2)} \supseteq \cdots \supseteq P^{(k)} \supseteq \cdots \) is a sequence of decreasing polytopes. Then there are \textit{countably} many linear equations \( \{L^{(s)}\}_{s \in \mathbb{N}} \), where
\[
L^{(s)} := \sum_{i=1}^{m} t_i a_i^{(s)} + \sum_j a_j^{(s)} b_j^{(s)} - c^{(s)},
\]
where \( a_i^{(s)}, b_j^{(s)} \in \mathbb{N}, a_j^{(s)} \in \mathcal{I} \), and \( c^{(s)} \) belongs to a finite set. If \( \Theta^{(k)} \) is a facet of \( P^{(k)} \), then \( \Theta^{(k)} \) must lie on the \textit{countably} union of hyperplanes
\[
(\cup_{s \in \mathbb{N}} \{L^{(s)} = 0\}) \cup (\cup_{i=1}^{m} \{t_i = 0\}),
\]
where \( \{ t_i = 0 \} \) is the \( i \)-th coordinate hyperplane. If \( \Theta^{(k)} \) is not contained in any of \( \{ L^{(s)} = 0 \} \) nor \( \{ t_i = 0 \} \), then their intersections on \( \Theta^{(k)} \) are measure zero sets. Thus, their countable unions is still of measure zero. This is impossible and hence \( \Theta^{(k)} \) must be contained in one of \( \{ L^{(s)} = 0 \} \) or \( \{ t_i = 0 \} \).

Since \( \{ P^{(k)} \}_{k \in \mathbb{N}} \) is a strictly decreasing sequence, there must exist a sequence \( \{ \Theta^{(k)} \}_{k \in \mathbb{N}} \) such that \( \Theta^{(k)} \subseteq P^{(k)} \) is a facet, \( \Theta^{(k)} \subseteq \{ t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) = 0 \} \), and

\[
\{ t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) = 0 \}_{k \in \mathbb{N}}
\]

are different sets (c.f. Figure [1]). Using the DCC property, by passing to a subsequence, we can assume that, in (11), \( c^{(k)} = c \) is fixed for all \( k \), the sequence \( \{ \sum_j a_j^{(k)} b_j^{(k)} \}_{k \in \mathbb{N}} \) is non-decreasing, and the sequence \( \{ n_i^{(k)} \}_{k \in \mathbb{N}} \) is non-decreasing for each \( i \).

Now, the set

\[
\{ t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) \geq 0 \}_{k \in \mathbb{N}}
\]

is strictly increasing. However, by assumption, we have

\[
\{ t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) \geq 0 \} \supseteq P^{(k)} \supseteq P^{(k+1)} \supseteq \Theta^{(k+1)}.
\]

This is impossible as there exists \( \theta \in \Theta^{(k+1)} \) lies on \( L^{(k+1)} = 0 \) which is not contained in \( \{ t \in \mathbb{R}^m_{\geq 0} \mid L^{(k)}(t) \geq 0 \} \) by (12).□

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