A Local Douglas formula for Higher Order Weighted Dirichlet-Type Integrals

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Abstract
We prove a local Douglas formula for higher order weighted Dirichlet-type integrals. With the help of this formula, we study the multiplier algebra of the associated higher order weighted Dirichlet-type spaces $\mathcal{H}_\mu$, induced by an $m$-tuple $\mu = (\mu_1, \ldots, \mu_m)$ of finite non-negative Borel measures on the unit circle. In particular, it is shown that any weighted Dirichlet-type space of order $m$, for $m \geq 3$, forms an algebra under pointwise product. We also prove that every non-zero closed $M$-invariant subspace of $\mathcal{H}_\mu$, has codimension 1 property if $m \geq 3$ or $\mu_2$ is finitely supported. As another application of this local Douglas formula obtained in this article, it is shown that for any $m \geq 2$, weighted Dirichlet-type space of order $m$ does not coincide with any de Branges–Rovnyak space $\mathcal{H}(b)$ with equivalence of norms.

Keywords  Weighted Dirichlet-type integrals · Reproducing kernel Hilbert spaces · De Branges–Rovnyak spaces · Douglas formula

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1 Introduction

The symbols \( \mathbb{Z} \), \( \mathbb{N} \) and \( \mathbb{Z}_{\geq 0} \) will denote the set of integers, positive integers and non-negative integers respectively. We use the symbols \( \mathbb{T} \) and \( \mathbb{D} \) to denote the unit circle and the open unit disc in the complex plane \( \mathbb{C} \) respectively. The notation \( \mathcal{M}_+(\mathbb{T}) \) stands for the set of all finite non-negative Borel measures on \( \mathbb{T} \). Let \( \mathcal{O}(\mathbb{D}) \) denote the space of all complex valued holomorphic functions on \( \mathbb{D} \). For each \( \alpha \in \mathbb{R} \), let \( \mathcal{D}_\alpha \) denote the subspace of \( \mathcal{O}(\mathbb{D}) \) given by

\[
\mathcal{D}_\alpha := \left\{ f = \sum_{k=0}^{\infty} a_k z^k \mid \| f \|_\alpha^2 := \sum_{k=0}^{\infty} (k+1)^\alpha |a_k|^2 < \infty \right\},
\]

see [34] for a detailed study of \( \mathcal{D}_\alpha \) spaces. Many classical function spaces arise as \( \mathcal{D}_\alpha \) spaces. For example \( \mathcal{D}_{-1} \) coincides with the Bergman space, the space \( \mathcal{D}_0 \) coincides with the Hardy space \( \mathcal{H}^2 \) and \( \mathcal{D}_1 \) coincides with the Dirichlet space \( \mathcal{D} \). Recall that the Dirichlet space \( \mathcal{D} \) consists of all functions \( f \) in \( \mathcal{O}(\mathbb{D}) \) for which the Dirichlet integral \( D(f) \), given by

\[
D(f) := \int_{\mathbb{D}} |f'(z)|^2 dA(z),
\]

is finite. Here \( A(z) \) denotes the normalized Lebesgue area measure on \( \mathbb{D} \). It turns out that \( \mathcal{D} \) is contained in the Hardy space \( \mathcal{H}^2 \). In his study of minimal surfaces, Douglas [16] derived and used the following formula (known as the Douglas formula)

\[
D(f) = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f^*(\xi) - f^*(\lambda)|^2}{|\xi - \lambda|^2} d\sigma(\xi) d\sigma(\lambda), \quad f \in \mathcal{D},
\]

where \( \sigma(\xi) \) denotes the normalized arc length measure on \( \mathbb{T} \) and \( f^*(\lambda) \) denotes the non-tangential limit of \( f \) at \( \lambda \in \mathbb{T} \) (which exists a.e. on \( \mathbb{T} \)). This and similar formulae turns out to be very useful in understanding the boundary behaviour of functions in \( \mathcal{D} \), see [6]. Later, Richter and Sundberg (see [28, p. 356]) introduced the notion of the local Dirichlet-type integral \( D_\lambda(f) \) of a function \( f \) in \( \mathcal{H}^2 \) at a point \( \lambda \in \mathbb{T} \) in the following manner:

\[
D_\lambda(f) := \int_{\mathbb{T}} \frac{|f^*(\xi) - f^*(\lambda)|^2}{|\xi - \lambda|^2} d\sigma(\xi),
\]

provided \( f^*(\lambda) \) exists. Otherwise \( D_\lambda(f) \) is defined to be \( +\infty \). They established various remarkable formulae for the local Dirichlet-type integral in [28, Sect. 3]. One of the motivations for studying the local Dirichlet-type integral was to understand the weighted Dirichlet-type integral \( D_\mu(f) \) for a function \( f \in \mathcal{O}(\mathbb{D}) \) and a measure

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\( \mu \in \mathcal{M}_+(\mathbb{T}) \), a notion introduced and studied by Richter in order to describe all cyclic analytic 2-isometries, see [27, Theorem 5.1]. The weighted Dirichlet-type integral \( D_\mu(f) \) turns out to be a key ingredient in describing the structure of the invariant subspaces of the Dirichlet shift, that is, the operator \( M_z \mu \in A \). Local Douglas Formula for Higher Order Page 3 of 30

By weighted Dirichlet-type integral of \( f \) of order \( n \) for a detailed study of \( m \)-isometries, see [1, Theorem 3.23], [30, Theorem 3.1]. Following [30], for a measure \( \mu \in \mathcal{M}_+(\mathbb{T}) \), \( f \in \mathcal{O}(\mathbb{D}) \) and for a positive integer \( n \), we consider the weighted Dirichlet-type integral of \( f \) of order \( n \) by

\[
D_{\mu,n}(f) := \frac{1}{n!(n-1)!} \int_{\mathbb{D}} |f^{(n)}(z)|^2 P_\mu(z)(1 - |z|^2)^{n-1} dA(z).
\]

Here \( f^{(n)}(z) \) denotes the \( n \)-th order derivative of \( f \) at \( z \), and \( P_\mu(z) \) is the Poisson integral of the measure \( \mu \), that is,

\[
P_\mu(z) := \int_{\mathbb{T}} \frac{1 - |z|^2}{|z - \xi|^2} d\mu(\xi), \quad z \in \mathbb{D}.
\]

For a function \( f \in \mathcal{O}(\mathbb{D}) \), it will be also useful for us to consider the weighted Dirichlet-type integral of order 0 defined by

\[
D_{\mu,0}(f) := \lim_{R \to 1^-} \int_{\mathbb{T}} |f(R\xi)|^2 P_\mu(R\xi) d\sigma(\xi),
\]

provided the limit exists. Note that if \( \mu = \sigma \) then the integral \( D_{\sigma,0}(f) \) coincides with the square of the Hardy norm of \( f \). In case of \( n = 1 \), the integral \( D_{\mu,1}(f) \) coincides with the weighted Dirichlet-type integral \( D_\mu(f) \) for every \( \mu \in \mathcal{M}_+(\mathbb{T}) \). In particular if \( \mu = \sigma \) then \( D_{\sigma,1}(f) \) coincides with \( D(f) \). If \( \mu = \delta_{\lambda} \), the Dirac measure at the point \( \lambda \in \mathbb{T} \), we adopt a simpler notation \( D_{\lambda,n}(\cdot) \) in place of \( D_{\delta_{\lambda},n}(\cdot) \) and we refer it as a local Dirichlet-type integral of order \( n \) at \( \lambda \). We would like to mention that a similar notion, namely, local Dirichlet-type integral of order \( n \) of a function at a point \( \lambda \in \mathbb{T} \) was introduced and studied recently in [22, p. 3]. Nevertheless, we will see in Remark 5.4 that this notion differs from that of local Dirichlet-type integral of order \( n \) introduced in this article.

For a measure \( \mu \in \mathcal{M}_+(\mathbb{T}) \setminus \{0\} \) and for each \( n \in \mathbb{Z}_{\geq 0} \), we consider the semi-inner product space \( \mathcal{H}_{\mu,n} \) given by

\[
\mathcal{H}_{\mu,n} := \{ f \in \mathcal{O}(\mathbb{D}) : D_{\mu,n}(f) < \infty \},
\]

associated to the semi-norm \( \sqrt{D_{\mu,n}(\cdot)} \). In case \( \mu = \delta_{\lambda}, \lambda \in \mathbb{T} \), we will use a simpler notation \( \mathcal{H}_{\lambda,n} \) in place of \( \mathcal{H}_{\delta_{\lambda},n} \) and we refer it as the local Dirichlet space of order \( n \) at \( \lambda \). If \( \mu = 0 \), we set \( \mathcal{H}_{0,n} = H^2 \) for every \( n \in \mathbb{Z}_{\geq 0} \). If \( \mu = \sigma \), by a straightforward
computation it follows that for a holomorphic function \( f = \sum_{k=0}^{\infty} a_k z^k \) in \( \mathcal{O}(\mathbb{D}) \), we have

\[
D_{\sigma,n}(f) = \sum_{k=n}^{\infty} \binom{k}{n} |a_k|^2, \quad n \in \mathbb{Z}_{\geq 0},
\]

where \( \binom{k}{n} := \frac{k!}{n!(k-n)!} \) for any \( k \geq n \). It follows that for each \( n \in \mathbb{Z}_{\geq 0} \), the space \( \mathcal{H}_{\sigma,n} \) coincides as a set with the space \( D_n \). The reader is referred to [19] (see also [30]) for several properties of the spaces \( \mathcal{H}_{\mu,n} \) for an arbitrary non-negative measure \( \mu \) and a positive integer \( n \).

Richter and Sundberg had shown that, for any \( \lambda \in \mathbb{T} \), \( D_{\lambda,1}(f) = D_{\lambda}(f) \), for every \( f \) in \( H^2 \), see [28, Proposition 2.2]. In these notations, we note that

\[
D_{\lambda,1}(f) = D_{\sigma,0} \left( \frac{f(z) - f^*(\lambda)}{z - \lambda} \right),
\]

for every \( f \in \mathcal{H}_{\lambda,1} \), see also [31, Proposition 1] for an alternative proof. This formula is often known as the local Douglas formula, see [17, Section 7.2]. In this article, we establish a local Douglas formula for higher order local Dirichlet-type integrals, namely, we prove the following theorem.

**Theorem 1.1** Let \( n \) be a positive integer, \( \lambda \in \mathbb{T} \), and \( f \in \mathcal{O}(\mathbb{D}) \). Then \( f \in \mathcal{H}_{\lambda,n} \) if and only if \( f = \alpha + (z - \lambda)g \) for some \( g \) in \( \mathcal{H}_{\sigma,n-1} \) and \( \alpha \in \mathbb{C} \). Moreover, in this case, the following statements hold:

(i) \( D_{\lambda,n}(f) = D_{\sigma,n-1}(g) \),

(ii) \( f(z) \to \alpha \) as \( z \to \lambda \) in each oricyclic approach region \( |z - \lambda| < \kappa (1 - |z|^2)^{\frac{1}{2}} \), \( \kappa > 0 \). In particular, \( f^*(\lambda) \) exists and is equal to \( \alpha \).

Thus one can rewrite the local Douglas formula for higher order Dirichlet-type integrals in the following form:

\[
D_{\lambda,n}(f) = D_{\sigma,n-1} \left( \frac{f(z) - f^*(\lambda)}{z - \lambda} \right), \quad f \in H^2, \quad \lambda \in \mathbb{T}, \quad n \in \mathbb{N}, \tag{1}
\]

provided \( f^*(\lambda) \) exists. We explore various applications of this generalized local Douglas formula (1) in this article. For \( 0 < r < 1 \), let \( f_r \) be the \( r \)-dilation of a function \( f \in \mathcal{O}(\mathbb{D}) \) given by \( f_r(z) := f(rz), \ z \in \mathbb{D} \). As a first application of the formula (1), we prove in Theorem 2.6 that for each \( n \geq 2 \) and \( \mu \in \mathcal{M}_+(\mathbb{T}) \),

\[
D_{\mu,n}(f_r) \leq \frac{4^{n-1}(2 - r)r^{2n}}{(1 + r)^{2n-2}} D_{\mu,n}(f) \tag{2}
\]

for every \( f \in \mathcal{H}_{\mu,n} \) and \( 0 < r < 1 \). Note that for any \( n \in \mathbb{N} \) and \( 0 < r < 1 \), we have \( \frac{4^{n-1}(2 - r)r^{2n}}{(1 + r)^{2n-2}} \leq 1 \). Thus the inequality (2) is an improvement of [19, Theorem 1.1], where only the contractivity of the map \( f \mapsto f_r \) was shown. Analogous results are known in the case of \( n = 1 \), see [17, 18, 23, 28, 31] and references therein.
In the study of function spaces, it is an important problem to determine the associated multiplier algebra. Let $\mu \in \mathcal{M}_+ (\mathbb{T})$ and $n$ be a non-negative integer. The multiplier algebra of $\mathcal{H}_{\mu,n}$ is defined by

$$\text{Mult}(\mathcal{H}_{\mu,n}) := \{ \varphi \in \mathcal{O}(\mathbb{D}) : \varphi f \in \mathcal{H}_{\mu,n} \text{ for every } f \in \mathcal{H}_{\mu,n} \}.$$ 

Several studies have been done in characterizing the multipliers of the Dirichlet space $\mathcal{D}$, and more generally for the multipliers of the weighted Dirichlet-type space $\mathcal{H}_{\mu,1}$ for any $\mu \in \mathcal{M}_+ (\mathbb{T})$, see [9, 29, 33] and references therein. In Theorem 3.6, we characterize Mult($\mathcal{H}_{\lambda,n}$) for every $\lambda \in \mathbb{T}$ and $n \in \mathbb{N}$. This characterization also helps us to study Mult($\mathcal{H}_{\mu,n}$) for any $\mu \in \mathcal{M}_+ (\mathbb{T})$ and for every $n \in \mathbb{N}$. It is well known that the space $\mathcal{D}_n$ forms an algebra under pointwise product for every $n \geq 2$ and consequently Mult($\mathcal{D}_n$) $= \mathcal{D}_n$ for each $n \geq 2$, see [34, Remark, p. 239]. Even though $\mathcal{D}_1$, the classical Dirichlet space, is not an algebra under pointwise product (see [17, Theorem 1.3.1]), the subspace of bounded functions in it turns out to be an algebra, see [17, Theorem 1.3.2]. More generally, it is well known that $\mathcal{H}_{\mu,1}$ is never an algebra under pointwise product for any $\mu \in \mathcal{M}_+ (\mathbb{T})$, see [17, Sec. 7.1, Exercise 4], but $\mathcal{H}_{\mu,1} \cap H^\infty$ forms an algebra under pointwise product, see [29, p. 170], where $H^\infty$ is the Banach algebra of bounded holomorphic functions on $\mathbb{D}$. We extend this result in the following theorem for higher order weighted Dirichlet-type spaces.

**Theorem 1.2** Let $\mu \in \mathcal{M}_+ (\mathbb{T})$ and $n \in \mathbb{N}$. Then $\mathcal{H}_{\mu,n} \cap \text{Mult}(\mathcal{H}_{\sigma,n-1})$ is an algebra.

As a consequence of this theorem, we find that $\mathcal{H}_{\mu,n}$ also forms an algebra under pointwise product and Mult($\mathcal{H}_{\mu,n}$) $= \mathcal{H}_{\mu,n}$ for every $\mu \in \mathcal{M}_+ (\mathbb{T}) \setminus \{0\}$ and for each $n \geq 3$, see Corollary 3.7. The answer to the question of whether $\mathcal{H}_{\mu,2}$ is an algebra or not, is of mixed nature. We have found that when $\mu = \delta_\lambda$ with $\lambda \in \mathbb{T}$, then $\mathcal{H}_{\lambda,2}$ is not an algebra under pointwise product. On the other hand we have $\mathcal{H}_{\sigma,2} = \mathcal{D}_2$ forms an algebra under pointwise product, see [34, Remark, p. 239].

Let $m \in \mathbb{N}$ and $\mu = (\mu_1, \ldots, \mu_m)$ be an $m$-tuple of non-negative measures in $\mathcal{M}_+(\mathbb{T})$. Let $\mathcal{H}_\mu$ be the linear subspace of $\mathcal{O}(\mathbb{D})$ given by $\mathcal{H}_\mu := \bigcap_{j=1}^m \mathcal{H}_{\mu_{j,j}}$. Note that for each $j = 1, \ldots, m$, the space $\mathcal{H}_{\mu_{j,j}}$ is contained in the Hardy space $H^2$ for any $n \in \mathbb{N}$, see [19, Corollary 2.5]. To $f \in \mathcal{H}_\mu$, we associate the norm $\| f \|_\mu$ given by

$$\| f \|_\mu^2 := \| f \|_{H^2}^2 + \sum_{j=1}^m D_{\mu_{j,j}}(f).$$

The space $\mathcal{H}_\mu$ with respect to the norm $\| \cdot \|_\mu$ turns out to be a reproducing kernel Hilbert space, see [19, p. 13]. In particular, it follows that $\mathcal{H}_{\mu,n}$ is a Hilbert space with respect to the norm $\| \cdot \|_{\mu,n}$, given by

$$\| f \|_{\mu,n}^2 = \| f \|_{H^2}^2 + D_{\mu,n}(f), \quad f \in \mathcal{H}_{\mu,n}.$$ 

The operator $M_z$ of multiplication by the co-ordinate function on $\mathcal{H}_\mu$ turns out to be a cyclic analytic $(m + 1)$-isometry, see [19, Theorem 4.1, Corollary 5.3]. Let
Lat(M_\mu, \mathcal{H}_\mu) denote the lattice of all closed M_\mu-invariant subspaces of \mathcal{H}_\mu. A subspace W in Lat(M_\mu, \mathcal{H}_\mu) is said to have codimension k property if dim(W \ominus (z-\lambda)W) = k, for each \lambda \in \mathbb{D}, see [25, Definition 2.12]. In order to have a Beurling-type description of Lat(M_\mu, \mathcal{H}_\mu), it is expected that W has codimension 1 property for every non-zero W in Lat(M_\mu, \mathcal{H}_\mu), see [25, Proposition 2.13]. In the case of m = 1, that is, when \mu = \mu_1 and \mu_1 \in \mathcal{M}_+(\mathbb{T}), it is well known that any non-zero W in Lat(M_\mu, \mathcal{H}_\mu) has codimension 1 property, see [29, Theorem 3.2]. We find that for any m \geq 3, and any non-zero W in Lat(M_\mu, \mathcal{H}_\mu) has codimension 1 property. For the remaining case m = 2, that is, when \mu = (\mu_1, \mu_2), we show that the result remains true provided \mu_2 is finitely atomic, see Theorem 4.4.

There is an intimate connection between the local Dirichlet spaces \mathcal{H}_\mu and the de Branges–Rovnyak spaces \mathcal{H}(b) (see [15] for basic theory of \mathcal{H}(b) spaces). For any b in the unit ball of H^{\infty}, the de Branges–Rovnyak space \mathcal{H}(b) is the reproducing kernel Hilbert space with the reproducing kernel \kappa_b(z) = \frac{1}{1-\overline{z}w}, z, w \in \mathbb{D}. In [31], Sarason showed that any local Dirichlet space \mathcal{H}_\mu coincides with a de Branges–Rovnyak space \mathcal{H}(b), with equality of norms, where b can be chosen to be b(z) = \frac{(1-r)z}{1-rz}, z \in \mathbb{D}, with r = \frac{3}{2}. In [10, Theorem 3.1], Chevrot, Guillot and Ransford showed that, in fact, if any \mathcal{H}_\mu space is equal to some de Branges–Rovnyak space \mathcal{H}(b), with equality of norms, then \mu has to be a positive multiple of a point mass measure for some \lambda \in \mathbb{T} and further b is identified explicitly. Thereafter, in [12], the problem of characterizing \mu and b such that \mathcal{H}_\mu = \mathcal{H}(b) as a set was addressed by obtaining certain necessary and some sufficient conditions for \mathcal{H}_\mu = \mathcal{H}(b). In this paper, we study the same question for the spaces \mathcal{H}_\mu and \mathcal{H}(b), where \mu = (\mu_1, \ldots, \mu_m), m \geq 2, is an arbitrary m-tuple of non-negative measures in \mathcal{M}_+(\mathbb{T}).

This article is organized as follows. Section 2 is devoted to prove Theorem 1.1. Section 3 deals with a few applications of generalized local Douglas Formula. In this section, we prove Theorem 1.2 and as a corollary it is observed that \mathcal{H}_{\mu,n} is an algebra for any \mu \in \mathcal{M}_+(\mathbb{T}) \setminus \{0\} and n \geq 3, see Corollary 3.7. In Sect. 4, we discuss about the codimension 1 property for any closed M_\mu-invariant subspaces of \mathcal{H}_{\mu,n}. In Sect. 5, it is shown that, for n \geq 2, any weighted Dirichlet-type space of order n does not coincide with de Branges–Rovnyak space H(b), with equivalence of norms, for any b in the unit ball of H^{\infty}.

2 A Generalized Local Douglas Formula

In this section, we shall prove a local Douglas formula for local Dirichlet spaces of order n. For any \lambda \in \mathbb{T} and n \in \mathbb{N}, let \nu_{\lambda,n} denote the positive weighted area measure on \mathbb{D} given by

\[ d\nu_{\lambda,n}(z) = \frac{1}{n!(n-1)!} P_{\delta_{\lambda}}(z)(1 - |z|^2)^{n-1} dA(z), \]

where \( P_{\delta_{\lambda}}(z) = \frac{1 - |z|^2}{|z|^2 - \lambda^2}, z \in \mathbb{D}, \) is the Poisson integral of the Dirac measure \delta_{\lambda}. Let \( A^2(d\nu_{\lambda,n}) \) be the weighted Bergman space corresponding to the measure \( \nu_{\lambda,n} \), that is,
\[ A^2(d\nu_{\lambda,n}) := \{ f \in \mathcal{O}(\mathbb{D}) : \| f \|^2 = \int_{\mathbb{D}} |f(z)|^2 d\nu_{\lambda,n}(z) < \infty \}. \]

**Lemma 2.1** For any \( \lambda \in \mathbb{T} \) and \( n \in \mathbb{N} \), \( A^2(d\nu_{\lambda,n}) \) is a reproducing kernel Hilbert space with the reproducing kernel

\[ (n+1!)(n-1)! \frac{(z-\lambda)(\overline{w}-\overline{\lambda})}{(1-z\overline{w})^{n+2}}, \quad z, w \in \mathbb{D}. \]

**Proof** Let \( A^2((1-|z|^2)^n dA(z)) \) be the weighted Bergman space corresponding to the weighted area measure \((1-|z|^2)^n dA(z)\) on \( \mathbb{D} \), that is,

\[ A^2((1-|z|^2)^n dA(z)) := \left\{ f \in \mathcal{O}(\mathbb{D}) : \| f \|^2 = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^n dA(z) < \infty \right\}. \]

It is evident that the operator \( \sqrt{n!}(n-1)! M_{z-\lambda} \) is a unitary from \( A^2((1-|z|^2)^n dA(z)) \) onto \( A^2(d\nu_{\lambda,n}) \), where \( M_{z-\lambda} \) denotes the operator of multiplication by the function \( (z-\lambda) \). By a direct computation, it is verified that

\[ \left\{ \sqrt{(n+1)} \binom{n+k+1}{k} \frac{z^k}{w^k} : k \in \mathbb{Z}_{\geq 0} \right\} \]

forms an orthonormal basis of \( A^2((1-|z|^2)^n dA(z)) \). Therefore, by [24, Theorem 2.4], \( A^2(d\nu_{\lambda,n}) \) has the reproducing kernel

\[ n!(n-1)! (z-\lambda)(\overline{w}-\overline{\lambda}) \sum_{k=0}^{\infty} \binom{n+k+1}{k} z^k w^k \]

\[ = (n+1)! (n-1)! \frac{(z-\lambda)(\overline{w}-\overline{\lambda})}{(1-z\overline{w})^{n+2}}, \quad z, w \in \mathbb{D}. \]

This completes the proof. \( \square \)

**Lemma 2.2** Let \( \lambda \in \mathbb{T} \), \( n \in \mathbb{N} \), and \( f \in \mathcal{H}_{\sigma,n-1} \). Then \( ((z-\lambda)f)^{(n)}(z) = 0 \) for all \( z \in \mathbb{D} \) if and only if \( D_{\sigma,n-1}(f) = 0 \).

**Proof** Suppose that \( f \in \mathcal{H}_{\sigma,n-1} \) and \( ((z-\lambda)f)^{(n)}(z) = 0 \) for all \( z \in \mathbb{D} \). Then \( (z-\lambda)f \) is a polynomial of degree at most \( n-1 \). Note that \( \mathcal{H}_{\sigma,n-1} \) is contained in the Hardy space \( H^2 \) (see [19, Corollary 2.5], [34, Corollary 1, Theorem 7]). Therefore, in view of [17, Lemma 1.5.4], we find that \( \lim_{|z| \to 1^-} (1-|z|^2)|f(z)|^2 = 0 \). Hence it follows that \( \lim_{r \to 1^-} (r^2-\lambda)f(r\lambda) = 0 \). This gives us that the polynomial \( (z-\lambda)f \) must have a zero at \( \lambda \) and consequently \( f \) is a polynomial of degree at most \( n-2 \). Therefore, \( D_{\sigma,n-1}(f) = 0 \). Conversely, assume that \( D_{\sigma,n-1}(f) = 0 \). This implies \( f \) is a polynomial of degree at most \( n-2 \), and therefore \( ((z-\lambda)f)^{(n)}(z) = 0 \) for all \( z \in \mathbb{D} \). \( \square \)
Let $j \in \mathbb{Z}_{\geq 0}$. Note that, in general $\sqrt{D_{\sigma,j}(\cdot)}$ represents a seminorm on $\mathcal{H}_{\sigma,j}$ (unless $j = 0$). Consider the subspace $\mathcal{U}_{\sigma,j}$ of $\mathcal{H}_{\sigma,j}$ given by

$$\mathcal{U}_{\sigma,j} = \left\{ f \in \mathcal{O}(\mathbb{D}) : f(z) = \sum_{k=j}^{\infty} a_k z^k, D_{\sigma,j}(f) < \infty \right\}.$$ 

It is straightforward to verify that $\sqrt{D_{\sigma,j}(\cdot)}$ represents a norm on $\mathcal{U}_{\sigma,j}$ which is induced by an inner product. Moreover $\mathcal{U}_{\sigma,j}$ is a reproducing kernel Hilbert space in which the monomials given by $\left\{ \binom{k}{j}^{-\frac{1}{2}} z^k : k \geq j \right\}$ forms an orthonormal basis. The reproducing kernel $K_{\sigma,j}$ of $\mathcal{U}_{\sigma,j}$ is given by

$$K_{\sigma,j}(z, w) = \sum_{k=j}^{\infty} \binom{k}{j}^{-1} z^k \overline{w}^k, \quad z, w \in \mathbb{D}. \quad (3)$$

For any $\alpha > 0$ and $j \in \mathbb{Z}_{\geq 0}$, the Pochhammer symbol $(\alpha)_j$ is defined by

$$(\alpha)_j = \begin{cases} \alpha(\alpha + 1) \cdots (\alpha + j - 1), & j \geq 1 \\ 1, & j = 0. \end{cases}$$

**Proposition 2.3** Let $n$ be a positive integer, $\lambda \in \mathbb{T}$, and $T : \mathcal{O}(\mathbb{D}) \to \mathcal{O}(\mathbb{D})$ be the linear map given by

$$(T f)(z) = ((z - \lambda) f)^{(n)}(z), \quad z \in \mathbb{D}, \quad f \in \mathcal{O}(\mathbb{D}).$$

Then $T$ maps the Hilbert space $(\mathcal{U}_{\sigma,n-1}, \sqrt{D_{\sigma,n-1}(\cdot)})$ isometrically onto $A^2(d\nu_{\lambda,n})$.

**Proof** Consider the space $\mathcal{W} := T(\mathcal{U}_{\sigma,n-1})$ endowed with the norm

$$\| T(f) \|_{\mathcal{W}} := \sqrt{D_{\sigma,n-1}(f)}, \quad f \in \mathcal{U}_{\sigma,n-1}.$$ 

By Lemma 2.2, $\| \cdot \|_{\mathcal{W}}$ is a well-defined norm on $\mathcal{W}$. Clearly $T$ maps $\mathcal{U}_{\sigma,n-1}$ isometrically onto $\mathcal{W}$. This gives us that the space $\mathcal{W}$ endowed with the norm $\| \cdot \|_{\mathcal{W}}$ is a Hilbert space. Now we show that $\mathcal{W}$ is a reproducing kernel Hilbert space. This is equivalent to showing that for each $w \in \mathbb{D}$, the evaluation map $E_w : \mathcal{W} \to \mathbb{C}$, defined by $E_w(g) = g(w)$, is continuous. Let $w \in \mathbb{D}$, and $T(f_j)$ converges to $T(f)$ in $\mathcal{W}$, where $f_j, f \in \mathcal{U}_{\sigma,n-1}$. Clearly $f_j$ converges to $f$ in $\mathcal{U}_{\sigma,n-1}$. Since $\mathcal{U}_{\sigma,n-1}$ is a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$, it follows that $f_j$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$ (see [14, Lemma 3.2 (c)]). Therefore $T(f_j)$ also converges to $T(f)$ uniformly on compact subsets of $\mathbb{D}$. This, in particular, shows that $E_w$ is continuous on $\mathcal{W}$, proving that $\mathcal{W}$ is a reproducing kernel Hilbert space.

The proposition will now be established by showing that the reproducing kernels of $\mathcal{W}$ and $A^2(d\nu_{\lambda,n})$ coincide (see [24, Proposition 2.3]). As the operator $T$ is unitary,
it maps the orthonormal basis \( \{ \binom{k}{n-1}^{-\frac{1}{2}} z^k : k \geq n-1 \} \) of \( \mathcal{U}_{\sigma, n-1} \) to an orthonormal basis of \( \mathcal{W} \). Thus the reproducing kernel of \( \mathcal{W} \) is given by

\[
K_{\mathcal{W}}(z, w) = \frac{\partial^{2n}}{\partial z^{n} \partial w^{n}} \left( (z - \lambda)(\overline{w} - \overline{\lambda}) K_{\sigma, n-1}(z, w) \right), \quad z, w \in \mathbb{D}.
\]

(4)

In view of (3), we see that the expression in (4) is equal to

\[
\frac{\partial^{2n}}{\partial z^{n} \partial w^{n}} \left( \sum_{k=n-1}^{\infty} \binom{k}{n-1}^{-1} \left( z^{k+1}w^{k+1} + z^k w^k - \lambda z^{k+1}w^{k+1} - \overline{\lambda} z^k w^k \right) \right) 
= \sum_{k=n-1}^{\infty} \frac{1}{(n-1)!} (k + 2 - n)_n^2 \, \sum_{k=0}^{\infty} \frac{1}{(n-1)!} \left( (k + 1 - n)_n \right)^2 z^{k-n}w^{k-n} 
- \sum_{k=n}^{\infty} \frac{1}{(n-1)!} (k + 2 - n)_n (k + 1 - n)_n \left( \lambda z^{k-n}w^{k-1-n} + \overline{\lambda} z^{k-1-n}w^{k-n} \right) 
= \sum_{k=0}^{\infty} \frac{1}{(n-1)!} \left( (k + 1)_n (2k + n + 1)z^k w^k - \sum_{k=0}^{\infty} (k + 1)_{n+1} \left( \lambda z^{k+1}w^{k+1} + \overline{\lambda} z^k w^k \right) \right) 

\]

Also, by a straightforward calculation, we see that

\[
(z - \lambda)(\overline{w} - \overline{\lambda}) (1 - z\overline{w})^{n+2} 
= \sum_{k=0}^{\infty} \binom{n+k+1}{k} \left( z^{k+1}w^{k+1} + z^k w^k - \lambda z^{k+1}w^{k+1} - \overline{\lambda} z^k w^k \right) 
= 1 + \sum_{k=1}^{\infty} \left( \binom{n+k+1}{k-1} + \binom{n+k+1}{k} \right) z^k w^k - \sum_{k=0}^{\infty} \binom{n+k+1}{k} \left( \lambda z^{k+1}w^{k+1} + \overline{\lambda} z^k w^k \right) 
= 1 + \frac{1}{(n+1)!} \sum_{k=0}^{\infty} (k + 1)_n (2k + n + 1)z^k w^k - \sum_{k=0}^{\infty} (k + 1)_{n+1} \left( \lambda z^{k+1}w^{k+1} + \overline{\lambda} z^k w^k \right) 
= \frac{1}{(n+1)!} \sum_{k=0}^{\infty} (k + 1)_n (2k + n + 1)z^k w^k - \sum_{k=0}^{\infty} (k + 1)_{n+1} \left( \lambda z^{k+1}w^{k+1} + \overline{\lambda} z^k w^k \right) 
= \frac{1}{(n+1)!} \sum_{k=0}^{\infty} (k + 1)_n (2k + n + 1)z^k w^k - \sum_{k=0}^{\infty} (k + 1)_{n+1} \left( \lambda z^{k+1}w^{k+1} + \overline{\lambda} z^k w^k \right) 

\]

Use of Lemma 2.1 now completes the proof. \( \square \)
For any holomorphic function $f$ on the unit disc $\mathbb{D}$ and a complex number $\lambda$ in $\mathbb{T}$, let $f^*(\lambda)$ denote the boundary value of $f$ at $\lambda$, defined by

$$f^*(\lambda) := \lim_{r \to 1^-} f(r\lambda),$$

whenever it exists. As a consequence of Proposition 2.3, we obtain Theorem 1.1 (main theorem of this section), which, in particular, tells us that for any $f$ in $\mathcal{H}_{\lambda,n}$, the boundary value of $f$ at $\lambda$ exists. As another application of Theorem 1.1 and [19, Proposition 2.6], we see that $\mathcal{H}_{\lambda,n}$ is contained in $\mathcal{H}_{\sigma,n-1}$.

**Proof of Theorem 1.1** Assume that $f \in \mathcal{H}_{\lambda,n}$. Thus we have $f^{(n)} \in A^2(d\nu_{\lambda,n})$. Now, by Proposition 2.3, there exists a $\tilde{g}$ in $\mathcal{U}_{\sigma,n-1}$ such that $D\lambda,n(f) = D\sigma,n-1(\tilde{g})$ and $((z - \lambda)\tilde{g})^{(n)} = f^{(n)}$. Therefore,

$$f(z) = (z - \lambda)\tilde{g}(z) + p(z), \quad z \in \mathbb{D},$$

for some polynomial $p$ of degree at most $n - 1$. Thus, with $q(z) = \frac{p(z) - p(\lambda)}{z - \lambda}$, we have

$$f(z) = p(\lambda) + (z - \lambda)g(z), \quad z \in \mathbb{D}$$

where $g = \tilde{g} + q$. Clearly, $D\sigma,n-1(g) = D\sigma,n-1(\tilde{g}) = D\lambda,n(f)$, which is finite. Hence $g \in \mathcal{H}_{\sigma,n-1}$. Conversely, assume that $f = \alpha + (z - \lambda)g$ for some $g$ with $g \in \mathcal{H}_{\sigma,n-1}$. Then it follows from Proposition 2.3 that $f^{(n)} \in A^2(d\nu_{\lambda,n})$, and hence $f \in \mathcal{H}_{\lambda,n}$. This completes the proof of the first part of the theorem. Note that, for $f \in \mathcal{H}_{\lambda,n}$, the equality $D\lambda,n(f) = D\sigma,n-1(g)$ is already established. To complete the proof, suppose that $f = \alpha + (z - \lambda)g$ for some complex number $\alpha$ and $g$ in $\mathcal{H}_{\sigma,n-1}$. Since $\mathcal{H}_{\sigma,n-1}$ is contained in the Hardy space $H^2$, by [17, Lemma 1.5.4] we have

$$\lim_{|z| \to 1} (1 - |z|^2)^{\frac{1}{2}} |g(z)| = 0 \text{ if } |z - \lambda| < \kappa (1 - |z|^2)^{\frac{1}{2}}, \kappa > 0,$$

then

$$|f(z) - \alpha| < \kappa (1 - |z|^2)^{\frac{1}{2}} |g(z)|.$$

Hence it follows that $f(z) \to \alpha$ as $z \to \lambda$ in each oricyclic approach region $|z - \lambda| < \kappa (1 - |z|^2)^{\frac{1}{2}}$. This completes the proof of Theorem 1.1. $\square$

**Corollary 2.4** Let $n$ be a positive integer and $\mu$ be a finite non-negative Borel measure on the unit circle $\mathbb{T}$. Then for any $f$ in $\mathcal{H}_{\mu,n}$, $f^*(\lambda)$ exists for $\mu$-almost every $\lambda \in \mathbb{T}$, and the following holds:

$$D\mu,n(f) = \int_{\mathbb{T}} D\sigma,n-1\left( \frac{f(z) - f^*(\lambda)}{z - \lambda} \right) d\mu(\lambda).$$

**Proof** By Fubini’s theorem,

$$D\mu,n(f) = \frac{1}{n!(n-1)!} \int_{\mathbb{D}} |f^{(n)}(z)|^2 P_\mu(z)(1 - |z|^2)^{n-1} dA(z) = \int_{\mathbb{T}} D\lambda,n(f) d\mu(\lambda).$$

An application of Theorem 1.1 completes the proof of the corollary. $\square$
We finish this section with a few applications of Theorem 1.1. The first application shows that for any \( n > 1 \) the map \( f \mapsto f_r, 0 \leq r < 1 \), is contractive on \( \mathcal{H}_{\mu,n} \). Here for any \( f \in \mathcal{O}(\mathbb{D}) \), the \( r \)th dilation \( f_r \) of \( f \) is defined by \( f_r(z) := f(rz), z \in \mathbb{D} \). We point out that this result is already obtained in [31, Proposition 3] for \( n = 1 \), and [19, Theorem 1.1] for every \( n \geq 1 \). One of the consequences of this result is that \( \lim_{r \to 1^-} D_{\mu,n}(f_r - f) = 0 \) for any \( f \in \mathcal{H}_{\mu,n} \). We start with the following lemma.

**Lemma 2.5** Let \( \mathcal{H} \) be a semi-inner product space of holomorphic functions on the unit disc \( \mathbb{D} \) with the semi-norm \( \| \cdot \| \). Suppose that \( zf \in \mathcal{H} \) whenever \( f \in \mathcal{H} \). If \( \| zf \| \geq \| f \| \) for all \( f \in \mathcal{H} \) then for any \( \lambda \in \mathbb{T}, 0 \leq r < 1 \), we have

\[
\| (z - \lambda)f \| \leq \frac{2}{1 + r} \| (z - r\lambda)f \|, \ f \in \mathcal{H}.
\]

**Proof** Let \( f \in \mathcal{H} \). Since \( \| zf \| \geq \| f \| \), it follows that

\[
|\text{Re} \langle f, zf \rangle| \leq |\langle f, zf \rangle| \leq \| f \| \| zf \| \leq \| zf \|^2,
\]

where for any complex number \( w \), \( \text{Re} w \) denotes the real part of \( w \). Hence

\[
\frac{4}{(1 + r)^2} \| (z - r\lambda)f \|^2 - \| (z - \lambda)f \|^2 = \left( \frac{4}{(1 + r)^2} - 1 \right) \| zf \|^2 - \left( 1 - \frac{4r^2}{(1 + r)^2} \right) \| f \|^2 + 2 \left( 1 - \frac{4r}{(1 + r)^2} \right) \text{Re} \langle f, zf \rangle \\
\geq \left( \frac{4}{(1 + r)^2} - 1 \right) - \left( 1 - \frac{4r^2}{(1 + r)^2} \right) - 2 \left( 1 - \frac{4r}{(1 + r)^2} \right) \| zf \|^2 = 0.
\]

This completes the proof. \( \square \)

**Theorem 2.6** Let \( n \geq 1 \) be an integer, and \( \mu \) be in \( \mathcal{M}_+(\mathbb{T}) \). Then for any \( f \in \mathcal{O}(\mathbb{D}) \), we have

\[
D_{\mu,n}(f_r) \leq \frac{4^{n-1}(2 - r)r^{2n}}{(1 + r)^{2n-2}} D_{\mu,n}(f), \ 0 \leq r < 1.
\]

**Proof** We will establish the proof by induction on \( n \). The base case of \( n = 1 \) follows from [23, Corollary 1.5]. Assume that the statement holds for every \( f \in \mathcal{O}(\mathbb{D}), 0 \leq r < 1 \), and \( n = 1, \ldots, k - 1 \) for \( k \geq 2 \). Let \( f \in \mathcal{O}(\mathbb{D}) \) and \( 0 \leq r < 1 \).

Since \( D_{\mu,k}(f) = \int_{\mathbb{T}} D_{\lambda,k}(f) d\mu(\lambda) \), in order to establish (5) for \( n = k \), it suffices to show that the holds for \( n = k \) and \( \mu = \delta_\lambda \) for each \( \lambda \in \mathbb{T} \). Fix a \( \lambda \) in \( \mathbb{T} \) and assume that \( D_{\lambda,k}(f) < \infty \). Then by Theorem 1.1, we obtain that for any \( z \in \mathbb{D} \), \( f(z) = f_*(\lambda) + (z - \lambda)g(z) \) for some \( g \) in \( \mathcal{H}_{\sigma,k-1} \) and \( D_{\sigma,k}(f) = D_{\sigma,k-1}(g) \). By a straightforward computation, we see that for any \( z \in \mathbb{D}, \)

\[
f_r(z) = f_r^*(\lambda) + (z - \lambda)h_r(z),
\]

where for any complex number \( w \), \( \text{Re} w \) denotes the real part of \( w \). Hence

\[
\frac{4}{(1 + r)^2} \| (z - r\lambda)f \|^2 - \| (z - \lambda)f \|^2 = \left( \frac{4}{(1 + r)^2} - 1 \right) \| zf \|^2 - \left( 1 - \frac{4r^2}{(1 + r)^2} \right) \| f \|^2 + 2 \left( 1 - \frac{4r}{(1 + r)^2} \right) \text{Re} \langle f, zf \rangle \\
\geq \left( \frac{4}{(1 + r)^2} - 1 \right) - \left( 1 - \frac{4r^2}{(1 + r)^2} \right) - 2 \left( 1 - \frac{4r}{(1 + r)^2} \right) \| zf \|^2 = 0.
\]

This completes the proof. \( \square \)
where

\[ h(z) = r \frac{(z - \lambda)(g(z) - g(r\lambda)) + (z - r\lambda)g(r\lambda)}{z - r\lambda}. \]

Using this together with Lemma 2.5 we get that

\[
D_{\sigma,k-1}(h) = r^2 D_{\sigma,k-1} \left( \frac{(z - \lambda)(g(z) - g(r\lambda))}{z - r\lambda} \right)
\leq \frac{4r^2}{(1 + r)^2} D_{\sigma,k-1}(g(z) - g(r\lambda))
= \frac{4r^2}{(1 + r)^2} D_{\sigma,k-1}(g).
\]

By induction hypothesis, we have that

\[
D_{\sigma,k-1}(h_r) \leq \frac{4^{k-2}(2 - r)r^{2k-2}}{(1 + r)^{2k-4}} D_{\sigma,k-1}(h).
\]

Using Theorem 1.1, together with (6), (7) and (8) we get

\[
D_{\lambda,k}(f_r) = D_{\sigma,k-1}(h_r)
\leq \frac{4^{k-2}(2 - r)r^{2k-2}}{(1 + r)^{2k-4}} D_{\sigma,k-1}(h)
\leq \frac{4^{k-1}(2 - r)r^{2k}}{(1 + r)^{2k-2}} D_{\sigma,k-1}(g) = \frac{4^{k-1}(2 - r)r^{2k}}{(1 + r)^{2k-2}} D_{\lambda,k}(f).
\]

This completes the proof.

The following theorem which is a generalization of [8, Theorem 3.1] (see also [12, Lemma 4.2]) describes the weighted Dirichlet-type space \( \mathcal{H}_{\mu,n} \) when \( \mu \) is finitely supported.

**Theorem 2.7** Let \( n, k \) be two positive integers, and \( \mu = \sum_{j=1}^{k} c_j \delta_{\lambda_j} \) for some \( c_j > 0 \) and \( \lambda_j \in \mathbb{T}, j = 1, \ldots, k \). Then \( f \in \mathcal{H}_{\mu,n} \) if and only if there exists a polynomial \( p \) of degree at most \( k - 1 \) and a function \( g \in \mathcal{H}_{\sigma,n-1} \) such that

\[ f(z) = p(z) + g(z) \prod_{j=1}^{k} (z - \lambda_j), \quad z \in \mathbb{D}. \]

**Proof** Note that \( f \in \mathcal{H}_{\mu,n} \) if and only if \( f \in \mathcal{H}_{\lambda_j,n} \) for each \( j = 1, \ldots, k \). Suppose that \( f = p + g \prod_{j=1}^{k} (z - \lambda_j) \) for some polynomial \( p \) of degree at most \( k - 1 \) and a function \( g \in \mathcal{H}_{\sigma,n-1} \). Since \( z \) is a multiplier of \( \mathcal{H}_{\sigma,n-1} \), it follows that for any \( i = 1, \ldots, k, g \prod_{j=1}^{k} (z - \lambda_j) \) is in \( \mathcal{H}_{\sigma,n-1} \). Therefore, by Theorem 1.1, the
function \( g \prod_{j=1}^{k} (z - \lambda_j) \in H_{\lambda_i,n} \) for each \( i = 1, \ldots, k \). Hence \( f \in H_{\mu,n} \). Conversely, suppose that \( f \in H_{\mu,n} \) and assume that, without loss of generality, \( \lambda_1, \ldots, \lambda_k \) are distinct. Then \( f \in H_{\lambda_i,n} \), and applying Theorem 1.1 we see that \( f^\ast(\lambda_j) \) exists for all \( j = 1, \ldots, k \). Let \( p \) be the polynomial of degree \( k - 1 \) such that \( p(\lambda_j) = f^\ast(\lambda_j), j = 1, \ldots, k \). Again, by Theorem 1.1, it follows that \( \frac{f - p}{z - \lambda_j} \) belongs to \( H_{\sigma,n-1} \) for \( j = 1, \ldots, k \). Since \( \lambda_1, \ldots, \lambda_k \) are distinct, using partial fraction formula, we see that

\[
\frac{f - p}{\prod_{j=1}^{k} (z - \lambda_j)} = \sum_{j=1}^{k} \alpha_j \frac{f - p}{z - \lambda_j}
\]

for some \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \). As the right hand side in (9) belongs to \( H_{\sigma,n-1} \), this completes the proof.

\[ \square \]

3 Multipliers of \( H_{\mu,n} \)

In the study of any function space, it is an important problem to characterize the multipliers of the associated function space. Let \( \mu \) be a measure in \( M_+(\mathbb{T}) \) and \( n \) be a positive integer. In this section, we deduce many properties of a multiplier of \( H_{\mu,n} \) and give some descriptions of \( \text{Mult}(H_{\mu,n}) \) in several different ways. Since the co-ordinate function \( z \) is a multiplier of \( H_{\mu,n} \) (see \([19, \text{Proposition 2.6}]\)), it follows that the set of all polynomials is contained in \( \text{Mult}(H_{\mu,n}) \) for every \( n \in \mathbb{N} \). Let \( O(\overline{D}) \) denote the set of functions which are holomorphic in some neighbourhood of the closed unit disc \( \overline{D} \). As an application of the following proposition we show that if \( \varphi \in O(\overline{D}) \) then \( \varphi \in \text{Mult}(H_{\mu,n}) \). Before proving the proposition, we start with the following useful lemma.

**Lemma 3.1** Let \( \mu \in M_+(\mathbb{T}) \) and \( n \in \mathbb{N} \). Then for any \( f \in H_{\mu,n} \) and \( j = 0, \ldots, n \), the integral

\[
I_{\mu,j,n}(f) := \int_{\mathbb{D}} |f^{(j)}(z)|^2 P_\mu(z)(1 - |z|^2)^{n-1} dA(z)
\]

is finite.

**Proof** Let \( A^2(d\rho) \) be the weighted Bergman space with the weighted area measure \( \rho \) on \( \mathbb{D} \) given by \( d\rho(z) = P_\mu(z)(1 - |z|^2)^{n-1} dA(z) \). Note that a function \( f \) in \( O(\mathbb{D}) \) satisfies \( I_{\mu,j,n}(f) < \infty \) if and only if \( f^{(j)} \in A^2(d\rho) \). Let \( f \in H_{\mu,n} \). By definition, \( f^{(n)} \in A^2(d\rho) \). Let \( 0 \leq k \leq n - 1 \). The proof will be established by showing that if \( f^{(j)} \in A^2(d\rho) \) for all \( j = k + 1, \ldots, n \), then \( f^{(k)} \in A^2(d\rho) \). Note that, by the Leibniz’s product rule for differentiation

\[
(z^{n-k} f)^{(n)} = z^{n-k} f^{(n)} + c_0 z^{n-k-1} f^{(n-1)} + \cdots + c_{n-k} z f^{(k+1)}
\]

\[ + \binom{n}{n-k} (n-k)! f^{(k)} \quad (10) \]
for some constants $c_0, \ldots, c_{n-k}$. Since $z$ is a multiplier of $\mathcal{H}_{\mu,n}$ (see [19, Proposition 2.6]), we see that $(z^{n-k}f)^{(n)}(z) \in A^2(d\rho)$. Since $f^{(j)} \in A^2(d\rho)$ for all $j = k+1, \ldots, n$, it follows from (10) that $f^{(k)} \in A^2(d\rho)$.

**Proposition 3.2** Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $n \in \mathbb{N}$. Suppose $\varphi \in \mathcal{O}(\mathbb{D})$ such that $\varphi^{(n)} \in H^\infty$. Then $\varphi \in \text{Mult}(\mathcal{H}_{\mu,n})$.

**Proof** Since $\varphi^{(n)} \in H^\infty$, by repeated applications of the Mean Value Theorem, it follows that $\varphi^{(j)} \in H^\infty$ for each $j = 0, 1, \ldots, n$. By the Leibniz’s product rule for differentiation and the Cauchy–Schwarz’s inequality, note that, for any $f \in \mathcal{H}_{\mu,n}$,

$$|(\varphi f)^{(n)}(z)|^2 \leq \left( \sum_{j=0}^{n} |\varphi^{(n-j)}(z)|^2 |f^{(j)}(z)|^2 \right) \left( \sum_{j=0}^{n} \binom{n}{j}^2 \right), \quad z \in \mathbb{D}.$$ 

Let $C_n = \sum_{j=0}^{n} \binom{n}{j}^2$ and $\|g\|_\infty := \sup\{|g(z)| : z \in \mathbb{D}\}$ for any $g \in H^\infty$. It follows that for any $f \in \mathcal{H}_{\mu,n}$,

$$D_{\mu,n}(\varphi f) \leq \frac{C_n}{n!(n-1)!} \sum_{j=0}^{n} \|\varphi^{(n-j)}\|_\infty^2 \int_{\mathbb{D}} |f^{(j)}(z)|^2 P_\mu(z)(1 - |z|^2)^{n-1}dA(z).$$

(11)

Applying Lemma 3.1, we obtain that $D_{\mu,n}(\varphi f) < \infty$. Hence it follows that $\varphi \in \text{Mult}(\mathcal{H}_{\mu,n})$. □

**Corollary 3.3** Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $n \in \mathbb{N}$. Then $\mathcal{O}(\mathbb{D}) \subset \text{Mult}(\mathcal{H}_{\mu,n})$.

**Proof** Note that for any $\varphi \in \mathcal{O}(\mathbb{D})$, we have $\varphi^{(j)} \in H^\infty$ for every $j \in \mathbb{N}$. An application of Proposition 3.2 now completes the proof. □

Let $\mu = (\mu_1, \ldots, \mu_m)$, where $\mu_j \in \mathcal{M}_+(\mathbb{T})$ for every $j = 1, \ldots, m$. Recall that the linear space $\mathcal{H}_\mu = \cap_{j=1}^{m} \mathcal{H}_{\mu,j}$ is a Hilbert space with respect to the norm $\| \cdot \|_\mu$, given by

$$\|f\|_\mu^2 = \|f\|_{H^2}^2 + \sum_{j=1}^{m} D_{\mu,j}(f), \quad f \in \mathcal{H}_\mu.$$ 

Furthermore, the evaluation functional $ev_z$, defined by $ev_z(f) := f(z)$, $f \in \mathcal{H}_\mu$, is bounded for every $z \in \mathbb{D}$. Thus $\mathcal{H}_\mu$ is a reproducing kernel Hilbert space of holomorphic functions on the unit disc, see [5, 24] for definition and other basic properties of a reproducing kernel Hilbert space. In particular, it follows that for any finite non-negative Borel measure $\mu$ on $\mathbb{T}$ and a positive integer $n$, the linear space $\mathcal{H}_{\mu,n}$ is a reproducing kernel Hilbert space with respect to the norm $\| \cdot \|_{\mu,n}$, given by

$$\|f\|_{\mu,n}^2 = \|f\|_{H^2}^2 + D_{\mu,n}(f), \quad f \in \mathcal{H}_{\mu,n}.$$
Many properties of $\text{Mult}(H_{\mu,n})$ can be derived using the reproducing kernel Hilbert space structure of $H_{\mu,n}$. From now onwards we will often use the notation $(H_{\mu,n}, \| \cdot \|_{\mu,n})$ to denote the Hilbert space $H_{\mu,n}$ equipped with the norm $\| \cdot \|_{\mu,n}$.

In the following part of this section, we consider a general reproducing kernel Hilbert space having some special properties and derive few properties of the associated multiplier space.

Let $K_k$ be a reproducing kernel Hilbert space of $\mathbb{C}$-valued holomorphic functions defined on the unit disc $D$ and let $\kappa : D \times D \to \mathbb{C}$ be the sesqui-analytic reproducing kernel for $K_k$, that is, $\kappa(\cdot, w) \in K_k$ and $\langle f, \kappa(\cdot, w) \rangle_{K_k} = f(w), f \in K_k, w \in D$.

In the remaining part of this section, we shall assume the following properties on $\kappa$:

(A1) $\kappa$ is normalized at the origin, that is, $\kappa(w, 0) = 1$ for every $w \in D$,

(A2) $z$ is a multiplier for $K_k$, that is, $zf \in K_k$ for every $f \in K_k$,

(A3) $\mathbb{C}[z]$, the set of all polynomials, is dense in $K_k$.

We refer to any reproducing kernel Hilbert space $K_k$ satisfying (A1)-(A3) as a functional Hilbert space with property (A).

For example take $\mu = (\mu_1, \ldots, \mu_m)$, where $\mu_j \in M_+ (\mathbb{T})$ for every $j = 1, \ldots, m$ and consider the associated reproducing kernel Hilbert space $H_{\mu}$. Since $(f, 1) = f(0)$ for every $f \in H_{\mu}$, the kernel function associated to $(H_{\mu}, \| \cdot \|_{\mu})$ is normalized at the origin. This together with [19, Theorem 4.1 and Proposition 4.2] gives us that $(H_{\mu}, \| \cdot \|_{\mu})$ is a functional Hilbert space with property (A). In particular, it follows that, for any finite non-negative Borel measure $\mu$ on $\mathbb{T}$ and for any $n \in \mathbb{N}$, the Hilbert space $(H_{\mu,n}, \| \cdot \|_{\mu,n})$ is also a functional Hilbert space with property (A).

For a reproducing kernel Hilbert space $K_k$, which consists of holomorphic functions on the unit disc $D$, the multipliers space of $K_k$, is defined by

$$\text{Mult}(K_k) := \{ \varphi \in K_k : \varphi f \in K_k \text{ for every } f \in K_k \}.$$ 

Note that $\text{Mult}(K_k)$ forms an algebra with respect to the pointwise product. In particular, $\varphi \psi \in \text{Mult}(K_k)$ for every $\varphi, \psi \in \text{Mult}(K_k)$. Moreover one can identify $\text{Mult}(K_k)$ as a closed subalgebra of $B(K_k)$ in the following manner. Given $\varphi \in \text{Mult}(K_k)$, we write $M_\varphi$ for the multiplication operator acting on $K_k$ given by

$$M_\varphi(f) = \varphi f, \ f \in K_k.$$ 

By an application of the Closed Graph Theorem, it follows that for any $\varphi \in \text{Mult}(K_k)$, the operator $M_\varphi$ is bounded from $K_k$ into itself. Consider the norm $\| \cdot \|_{op}$ on $\text{Mult}(K_k)$ defined by

$$\| \varphi \|_{op} := \| M_\varphi \| = \sup\{ \| \varphi f \|_{K_k} : \| f \|_{K_k} = 1 \}, \ \varphi \in \text{Mult}(K_k).$$
It follows that \((\text{Mult}(\mathcal{H}_k), \| \cdot \|_{\text{op}})\) is a commutative Banach algebra with identity. Let \(\mathcal{H}_k\) be a functional Hilbert space with property (A). It follows that \(\text{Mult}(\mathcal{H}_k)\) can be identified as the commutant of the operator \(M_z\) of multiplication by the co-ordinate function on the Hilbert space \(\mathcal{H}_k\). Indeed if \(T \in \mathcal{B}(\mathcal{H}_k)\) satisfying \(TM_z = M_zT\), then a routine argument using the polynomial density of the functional Hilbert space \(\mathcal{H}_k\), it can be shown that \(T\) is given by \(T = M_{\varphi}\) where \(\varphi = T(1)\), see [25, Corollary 2.5] for similar results for multipliers of a class of Banach spaces of analytic functions.

Let \(\mathcal{H}_k\) be a functional Hilbert space with property (A). By assumption (A1), we have \(\langle \kappa(\cdot,0), \kappa(\cdot,w) \rangle = \kappa(w,0) = 1\) for each \(w \in \mathbb{D}\). Therefore \(\kappa(\cdot,w) \neq 0\) for each \(w \in \mathbb{D}\). Hence it follows that the multiplier algebra \(\text{Mult}(\mathcal{H}_k)\) is a subalgebra of \(H_\infty\), the algebra of bounded analytic functions on the unit disc, for every functional Hilbert space \(\mathcal{H}_k\) with property (A), see [20, Problem 68], [24, Corollary 5.22]. Thus, we obtain that \(\text{Mult}(\mathcal{H}_{\mu,n}) \subseteq H_\infty\) for every measure \(\mu\) in \(\mathcal{M}_+(\mathbb{T})\) and \(n \in \mathbb{N}\).

Among all the measures in \(\mathcal{M}_+(\mathbb{T})\), there are two special measures. First one is the Lebesgue measure \(\sigma\) and the second one is the Dirac delta measure \(\delta_\lambda\), \(\lambda \in \mathbb{T}\). As mentioned in the introduction, the space \(\mathcal{H}_{\sigma,n}\) coincides with the weighted Dirichlet-type space \(D_n\) for each \(n \in \mathbb{Z}_{\geq 0}\). The set of multipliers of \(D_n\) is well studied, see for example [33, 34] and references therein. It is known that, for every \(n \geq 2\), the space \(D_n\) forms an algebra, that is, \(fg \in D_n\), for every \(f, g \in D_n\), see [34, Theorem 7]. Consequently it follows that \(\text{Mult}(\mathcal{H}_{\sigma,n}) = \mathcal{H}_{\sigma,n}\) for every \(n \geq 2\). Now we concentrate our study to characterize the multiplier of \(\mathcal{H}_{\sigma,n}\) for any positive integer \(n\). The following lemma will be useful in describing the set \(\text{Mult}(\mathcal{H}_{\lambda,n})\). This lemma can be thought of as a generalization of [28, Lemma 5.3].

**Lemma 3.4** Let \(n\) be a positive integer, \(\lambda \in \mathbb{T}\) and \(f \in \mathcal{H}_{\lambda,n}\) such that \(D_{\lambda,n}(f) \neq 0\). Suppose \(\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n-1})\). Then \(\varphi f \in \mathcal{H}_{\lambda,n}\) if and only if \(\varphi \in \mathcal{H}_{\lambda,n}\) or \(f^*(\lambda) = 0\). Moreover,

\[
D_{\lambda,n}(\varphi f) \leq 2\|\varphi\|^2_{(\sigma,n-1)} D_{\lambda,n}(f) + 2|f^*(\lambda)| D_{\lambda,n}(\varphi) \quad \text{and} \quad |f^*(\lambda)| D_{\lambda,n}(\varphi) \leq 2\|\varphi\|^2_{(\sigma,n-1)} D_{\lambda,n}(f) + 2D_{\lambda,n}(\varphi f),
\]

where

\[
\|\varphi\|^2_{(\sigma,n-1)} := \sup \left\{ \sqrt{D_{\sigma,n-1}(\varphi h)} : D_{\sigma,n-1}(h) = 1 \right\}.
\]

Furthermore, if \(f^*(\lambda) = 0\) then \(D_{\lambda,n}(\varphi f) \leq \|\varphi\|^2_{(\sigma,n-1)} D_{\lambda,n}(f)\).

**Proof** Since \(f \in \mathcal{H}_{\lambda,n}\), the non-tangential limit \(f^*(\lambda)\) of \(f\) at \(\lambda\) exists and \(\frac{f(z) - f^*(\lambda)}{z - \lambda} \in \mathcal{H}_{\sigma,n-1}\), see Theorem 1.1. First let’s consider the case when \(f^*(\lambda) = 0\). Note that \(\varphi \in H_\infty\) as \(\text{Mult}(\mathcal{H}_{\sigma,n-1}) \subseteq H_\infty\) (see [34, p. 233, Corollary 1] or [24, Corollary 5.22]). Thus, the non-tangential limit \((\varphi f)^*(\lambda)\) of \(\varphi f\) at \(\lambda\) exists and is equal to 0. Hence, by Theorem 1.1 along with the assumption that \(\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n-1})\), we have

\[
D_{\lambda,n}(\varphi f) = D_{\sigma,n-1}(\varphi(z) \frac{f(z)}{z - \lambda}) \leq \|\varphi\|^2_{(\sigma,n-1)} D_{\lambda,n}(f).
\]
Thus we conclude that if $f^*(\lambda) = 0$ then $\varphi f \in \mathcal{H}_{\lambda,n}$. For the remaining part of the proof, we assume that $f^*(\lambda) \neq 0$. Consider the identity

$$\frac{(\varphi f)(z) - (\varphi f)^*(\lambda)}{z - \lambda} = \varphi(z) \frac{f(z) - f^*(\lambda)}{z - \lambda} + f^*(\lambda) \frac{\varphi(z) - \varphi^*(\lambda)}{z - \lambda}, \quad z \in \mathbb{D}. \quad (12)$$

Since $\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n-1})$ and $f \in \mathcal{H}_{\lambda,n}$, it follows from Theorem 1.1 that the first term in the R.H.S. of (12) is in $\mathcal{H}_{\sigma,n-1}$. Once again using (12) and Theorem 1.1, it follows that $\varphi f \in \mathcal{H}_{\lambda,n}$ if and only if $\varphi \in \mathcal{H}_{\lambda,n}$. Combining both the above cases, we obtain that $\varphi f \in \mathcal{H}_{\lambda,n}$ if and only if $\varphi \in \mathcal{H}_{\lambda,n}$ or $f^*(\lambda) = 0$. Moreover using the identity (12) along with Theorem 1.1, we find that

$$D_{\lambda,n}(\varphi f) = D_{\sigma,n-1} \left( \frac{(\varphi f)(z) - (\varphi f)^*(\lambda)}{z - \lambda} \right)$$

$$\leq 2D_{\sigma,n-1} \left( \varphi(z) \frac{f(z) - f^*(\lambda)}{z - \lambda} \right) + 2|f^*(\lambda)|^2 D_{\sigma,n-1} \left( \frac{\varphi(z) - \varphi^*(\lambda)}{z - \lambda} \right),$$

$$\leq 2\|\varphi\|^2_{s(\sigma,n-1)} D_{\sigma,n-1} \left( \frac{f(z) - f^*(\lambda)}{z - \lambda} \right)$$

$$+ 2|f^*(\lambda)|^2 D_{\sigma,n-1} \left( \frac{\varphi(z) - \varphi^*(\lambda)}{z - \lambda} \right).$$

Using the identity (12) with Theorem 1.1 one more time, we obtain that

$$|f^*(\lambda)|^2 D_{\lambda,n}(\varphi) = D_{\sigma,n-1} \left( f^*(\lambda) \frac{\varphi(z) - \varphi^*(\lambda)}{z - \lambda} \right),$$

$$= D_{\sigma,n-1} \left( \frac{(\varphi f)(z) - (\varphi f)^*(\lambda)}{z - \lambda} - \varphi(z) \frac{f(z) - f^*(\lambda)}{z - \lambda} \right),$$

$$\leq 2D_{\sigma,n-1} \left( \frac{(\varphi f)(z) - (\varphi f)^*(\lambda)}{z - \lambda} \right)$$

$$+ 2D_{\sigma,n-1} \left( \varphi(z) \frac{f(z) - f^*(\lambda)}{z - \lambda} \right),$$

$$\leq 2D_{\lambda,n}(\varphi f) + 2\|\varphi\|^2_{s(\sigma,n-1)} D_{\sigma,n-1} \left( \frac{f(z) - f^*(\lambda)}{z - \lambda} \right),$$

$$= 2D_{\lambda,n}(\varphi f) + 2\|\varphi\|^2_{s(\sigma,n-1)} D_{\lambda,n}(f).$$

This completes the proof.

\[\square\]

**Remark 3.5** We would like to point out that the condition $D_{\lambda,n}(f) \neq 0$ in Lemma 3.4 can be removed in the following sense: if $D_{\lambda,n}(f)$ is zero then $f$ is a polynomial of degree at most $n - 1$ and therefore $f \in \text{Mult}(\mathcal{H}_{\sigma,n-1})$. It then easily follows that for any $\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n-1})$ with $D_{\sigma,n-1}(\varphi) \neq 0$,

$$D_{\lambda,n}(\varphi f) \leq 2D_{\sigma,n-1}(\varphi)\|g\|^2_{s(\sigma,n-1)} + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi)$$

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\[ |f^*(\lambda)|^2 D_{\lambda,n}(\varphi) \leq 2D_{\sigma,n-1}(\varphi)\|g\|^2_{s(\sigma,n-1)} + 2D_{\lambda,n}(\varphi f), \]

where \(g(z) := \frac{f(z) - f^*(\lambda)}{z - \lambda}, z \in \mathbb{D} \).

The following theorem describes the multiplier algebra of \( \mathcal{H}_{\lambda,n} \) for any \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{T} \).

**Theorem 3.6** Let \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{T} \). Then, \( \text{Mult}(\mathcal{H}_{\lambda,n}) = \mathcal{H}_{\lambda,n} \cap \text{Mult}(\mathcal{H}_{\sigma,n-1}) \).

**Proof** By Lemma 3.4, it follows that \( \varphi \in \text{Mult}(\mathcal{H}_{\lambda,n}) \) whenever \( \varphi \in \mathcal{H}_{\lambda,n} \cap \text{Mult}(\mathcal{H}_{\sigma,n-1}) \). For the converse, let \( \varphi \in \text{Mult}(\mathcal{H}_{\lambda,n}) \). Since the constant function \( 1 \in \mathcal{H}_{\lambda,n} \), it follows that \( \varphi \in \mathcal{H}_{\lambda,n} \). Consider the map \( L_\lambda : \mathcal{H}_{\lambda,n} \to \mathcal{H}_{\sigma,n-1} \) defined by

\[ L_\lambda(f)(z) := \frac{f(z) - f^*(\lambda)}{z - \lambda}, \quad z \in \mathbb{D}. \]

By the generalized local Douglas formula in Theorem 1.1, we have \( L_\lambda \) is an isometry w.r.t. the associated semi-norms, that is, \( D_{\sigma,n-1}(L_\lambda f) = D_{\lambda,n}(f) \) for every \( f \in \mathcal{H}_{\lambda,n} \). Furthermore, it also follows that \( L_\lambda \) is onto. Let \( f \in \mathcal{H}_{\lambda,n} \). An application of the identity (12) gives us

\[
\begin{align*}
D_{\sigma,n-1}(\varphi(f) & \frac{f(z) - f^*(\lambda)}{z - \lambda}) \\
= & D_{\sigma,n-1}\left(\frac{(\varphi f)(z) - (\varphi f)^*(\lambda)}{z - \lambda} - f^*(\lambda)\frac{\varphi(z) - \varphi^*(\lambda)}{z - \lambda}\right), \\
\leq & 2D_{\sigma,n-1}\left(\frac{(\varphi f)(z) - (\varphi f)^*(\lambda)}{z - \lambda}\right) \\
& + 2|f^*(\lambda)|^2 D_{\sigma,n-1}\left(\frac{\varphi(z) - \varphi^*(\lambda)}{z - \lambda}\right), \\
= & 2D_{\lambda,n}(\varphi f) + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi).
\end{align*}
\]

Thus we have \( D_{\sigma,n-1}(\varphi L_\lambda f) < \infty \) for every \( f \in \mathcal{H}_{\lambda,n} \). Since \( L_\lambda \) is onto, we obtain that \( D_{\sigma,n-1}(\varphi g) < \infty \) for every \( g \in \mathcal{H}_{\sigma,n-1} \). Hence \( \varphi \in \text{Mult}(\mathcal{H}_{\sigma,n-1}) \). This completes the proof. \( \Box \)

We now give the proof of Theorem 1.2 which helps us to characterize the \( \text{Mult}(\mathcal{H}_{\mu,n}) \) for any positive measure \( \mu \) in \( \mathcal{M}_+(\mathbb{T}) \) and \( n \geq 3 \), see Corollary 3.7.

**Proof of Theorem 1.2** Let \( \varphi, \psi \in \mathcal{H}_{\mu,n} \cap \text{Mult}(\mathcal{H}_{\sigma,n-1}) \). Since \( \text{Mult}(\mathcal{H}_{\sigma,n-1}) \) is an algebra, it is sufficient to show that \( \varphi \psi \in \mathcal{H}_{\mu,n} \). Note that \( D_{\mu,n}(g) = \int_{\mathbb{T}} D_{\lambda,n}(g) d\mu(\lambda) \), for any \( g \in \mathcal{H}_{\mu,n} \). Hence we obtain that \( \varphi, \psi \in \mathcal{H}_{\lambda,n} \) for \( \mu \text{-a.e. } \lambda \) in \( \mathbb{T} \). By Lemma 3.4 we see that for \( \mu \text{-a.e. } \lambda \) in \( \mathbb{T} \),

\[
\begin{align*}
D_{\lambda,n}(\varphi \psi) & \leq 2\|\varphi\|^2_{s(\sigma,n-1)} D_{\lambda,n}(\psi) + 2|\psi^*(\lambda)|^2 D_{\lambda,n}(\varphi), \\
& \leq 2\|\varphi\|^2_{s(\sigma,n-1)} D_{\lambda,n}(\psi) + 2\|\psi\|^2_{\infty} D_{\lambda,n}(\varphi).
\end{align*}
\]
Here in the last line we used the fact that $\psi \in \text{Mult}(H_{\sigma,n-1}) \subseteq H_\infty$ (see [34, p. 233, Corollary 1] or [24, Corollary 5.22]). Integrating both sides w.r.t $\mu(\lambda)$ we obtain

$$ D_{\mu,n}(\phi\psi) \leq 2\|\psi\|^2_{s(\sigma,n-1)}D_{\mu,n}(\psi) + 2\|\psi\|^2_{D_{\mu,n}}D_{\mu,n}(\phi), \ \lambda \in \mathbb{T}. $$

This completes the proof of the theorem.

\[\Box\]

**Corollary 3.7** Let $n \geq 3$ be a positive integer and $\mu \in M_+(\mathbb{T}) \setminus \{0\}$. Then $H_{\mu,n}$ is an algebra and consequently $\text{Mult}(H_{\mu,n}) = H_{\mu,n}$.

**Proof** Recall that $H_{\sigma,k} = D_k$ for any non-negative integer $k$. Therefore from [34, Theorem 7] we get that $\text{Mult}(H_{\sigma,n-1}) = H_{\sigma,n-1}$ for any $n \geq 3$. Thus applying Theorem 1.2, it follows that $H_{\mu,n} \cap H_{\sigma,n-1}$ is an algebra. By [19, Lemma 2.4], we also have that $H_{\mu,n} \subseteq H_{\sigma,n-1}$. Hence $H_{\mu,n} \cap H_{\sigma,n-1} = H_{\mu,n}$. Thus $H_{\mu,n}$ is an algebra and consequently $\text{Mult}(H_{\mu,n}) = H_{\mu,n}$.

\[\Box\]

**Corollary 3.8** Let $\mu = (\mu_1, \ldots, \mu_m)$ be an $m$-tuple of finite non-negative Borel measures on $\mathbb{T}$ with $\mu_m \neq 0$. Then $H_{\mu} \cap \text{Mult}(H_{\sigma,m-1})$ is an algebra. In particular, $H_{\mu}$ is an algebra for $m \geq 3$.

**Proof** It is well known that for every $n \in \mathbb{N}$, $\text{Mult}(H_{\sigma,n}) \subset \text{Mult}(H_{\sigma,n-1})$. see [34, p. 233, Corollary 1]. So, we have

$$ H_{\mu} \cap \text{Mult}(H_{\sigma,m-1}) = H_{\mu,1} \cap \cdots \cap H_{\mu,m} \cap \text{Mult}(H_{\sigma,m-1}), $$

$$ = (H_{\mu,1} \cap \text{Mult}(H_{\sigma,0})) \cap \cdots \cap (H_{\mu,m} \cap \text{Mult}(H_{\sigma,m-1})). $$

By an application of Theorem 1.2, $H_{\mu,j} \cap \text{Mult}(H_{\sigma,j-1})$ is an algebra for each $j = 1, \ldots, m$. Hence $H_{\mu,j} \cap \text{Mult}(H_{\sigma,m-1})$ is an algebra. Suppose $m \geq 3$. It follows from [34, p. 233, Corollary 1] and [19, Lemma 2.4] that $H_{\mu,m} \cap \text{Mult}(H_{\sigma,m-1}) = H_{\mu,m} \cap H_{\sigma,m-1} = H_{\mu,m}$. Hence we obtain that

$$ H_{\mu} \cap \text{Mult}(H_{\sigma,m-1}) = H_{\mu,1} \cap \cdots \cap H_{\mu,m} \cap \text{Mult}(H_{\sigma,m-1}) = H_{\mu}. $$

The proof is now complete using the first part.

\[\Box\]

Although $H_{\mu,n}$ is an algebra for every $n \geq 3$ and for any $\mu \in M_+(\mathbb{T})$, it is well known that $H_{\mu,1}$ is never an algebra, see [17, Exercise 7.1.4]. So it is natural to ask the following question.

**Question 3.9** For which measures $\mu$ in $M_+(\mathbb{T})$, the spaces $H_{\mu,2}$ form an algebra?

**Example 3.10** We now point out few cases when $H_{\mu,2}$ is an algebra and when it is not.

(i) The space $H_{\lambda,2}$ is not an algebra for any $\lambda \in \mathbb{T}$. If possible assume that $H_{\lambda,2}$ is an algebra for some $\lambda \in \mathbb{T}$. Then we must have $H_{\lambda,2} = \text{Mult}(H_{\lambda,2}) \subset H_\infty$.

We now show that there exists a function in $H_{\lambda,2}$ which is not bounded on $\mathbb{D}$. Choose a $\tilde{\lambda} \neq \lambda \in \mathbb{T}$ and a function $g \in H_{\sigma,1}$, the classical Dirichlet space, such that $\lim_{r \to 1^-} g(r\tilde{\lambda}) = \infty$ (for example, take $g(z) = \sum_{k=2}^{\infty} \frac{z^k}{\lambda^k k \log k}$, $z \in \mathbb{D}$). Then by Theorem 1.1, $(z-\lambda)g \in H_{\lambda,2}$, but $\lim_{r \to 1^-} (r\tilde{\lambda} - \lambda)g(r\tilde{\lambda}) = \infty$. Hence $(z-\lambda)g \notin H_\infty$. 


(ii) Note that $\mathcal{H}_{\sigma,2} = D_2$ and $D_2$ forms an algebra, see [34, Theorem 7]. In fact, if $\mu = (\mu_1, \mu_2 + \sigma)$, where $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{T})$, then $\mathcal{H}_\mu$ is an algebra. To see this, note that $\mathcal{H}_\mu = \mathcal{H}_{\mu_1,1} \cap \mathcal{H}_{\sigma,2} \cap \mathcal{H}_{\mu_2,2}$. Since $\mathcal{H}_{\sigma,2} \subset \text{Mult}(\mathcal{H}_{\sigma,1})$, we obtain that $\mathcal{H}_\mu \subset \text{Mult}(\mathcal{H}_{\sigma,1})$. The conclusion now follows from Corollary 3.8.

The following theorem is a generalization of [17, Theorem 8.3.2]. As a consequence, we obtain a stronger version of Theorem 1.2. We start with a preparatory lemma.

**Lemma 3.11** Let $n \in \mathbb{Z}_{\geq 0}$ and $\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n})$. Then

$$\sup\{\|\varphi_r\|_{s(\sigma,n)} : 0 < r < 1\} < \infty.$$  

**Proof** We will prove this by induction. Since $\mathcal{H}_{\sigma,0}$ is the Hardy space $H^2$, it follows that $\text{Mult}(\mathcal{H}_{\sigma,0})$ is equal to $H^\infty$. Note that

$$D_{\sigma,0}(\varphi_r f) = \|\varphi_r f\|_{H^2}^2 \leq \|\varphi_r\|_{L^2}^2 D_{\sigma,0}(f) \leq \|\varphi\|_{L^\infty}^2 D_{\sigma,0}(f), \quad 0 < r < 1.$$  

This gives us $\|\varphi_r\|_{s(\sigma,0)} \leq \|\varphi\|_{L^\infty}$ for every $0 < r < 1$, proving the induction step for $n = 0$. Now assume that $n \geq 1$, and for any $\psi \in \text{Mult}(\mathcal{H}_{\sigma,n-1})$, $\sup\{\|\psi_r\|_{s(\sigma,n-1)} : 0 < r < 1\} < \infty$. Let $\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n})$ and $f \in \mathcal{H}_{\sigma,n}$ with $D_{\sigma,n}(f) \neq 0$. Since $\text{Mult}(\mathcal{H}_{\sigma,n}) \subset \text{Mult}(\mathcal{H}_{\sigma,n-1})$, see [34, Corollary 1], by induction hypothesis we have

$$\sup\{\|\varphi_r\|_{s(\sigma,n-1)} : 0 < r < 1\} = K < \infty.$$  

Note that for any $g \in \mathcal{H}_{\sigma,n}$, $D_{\sigma,n}(g) = \int_{\mathbb{T}} D_{\lambda,n}(g) d\sigma(\lambda)$. It follows that $f, \varphi \in \mathcal{H}_{\sigma,n}$ for $\sigma$-a.e. $\lambda \in \mathbb{T}$. Let $0 < r < 1$. In view of Lemma 3.4 and [19, Theorem 1.1], we obtain that for $\sigma$-a.e. $\lambda \in \mathbb{T}$, the following holds:

$$D_{\lambda,n}(\varphi_r f) \leq 2 \|\varphi_r\|_{s(\sigma,n-1)} D_{\lambda,n}(f) + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi_r) \\ \leq 2K^2 D_{\lambda,n}(f) + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi), \\ \leq 2K^2 D_{\lambda,n}(f) + 4\|\varphi\|_{s(\sigma,n-1)} D_{\lambda,n}(f) + 4D_{\lambda,n}(\varphi f).$$  

Integrating both sides w.r.t. $d\sigma(\lambda)$, we obtain that

$$D_{\sigma,n}(\varphi_r f) \leq \left(2K^2 + 4\|\varphi\|_{s(\sigma,n-1)}^2\right) D_{\sigma,n}(f) + 4D_{\sigma,n}(\varphi f) \\ \leq \left(2K^2 + 4\|\varphi\|_{s(\sigma,n-1)}^2 + 4\|\varphi\|_{s(\sigma,n)}^2\right) D_{\sigma,n}(f).$$  

This completes the proof. □

**Theorem 3.12** Let $\mu \in \mathcal{M}_+(\mathbb{T}) \setminus \{0\}$, $n \in \mathbb{N}$. Let $\mathcal{M}$ be a closed $M_\sigma$-invariant subspace of $\mathcal{H}_{\sigma,n}$ and $\varphi \in \text{Mult}(\mathcal{H}_{\sigma,n-1})$. Suppose $f \in \mathcal{M}$ is such that $\varphi f \in \mathcal{H}_{\sigma,n}$. Then $\varphi f \in \mathcal{M}$.

**Proof** Since $\mathcal{M}$ is a $M_\sigma$-invariant subspace of $\mathcal{H}_{\sigma,n}$, if $\varphi$ is a polynomial then it is immediate that $\varphi f \in \mathcal{M}$. Therefore, we assume that $\varphi$ is not a polynomial. Fix a $r$ satisfying $0 < r < 1$. Note that the $r$-dilation $\varphi_r \in \mathcal{O}(\overline{\mathbb{D}})$ and hence, by Corollary 3.3, $\varphi_r \in \text{Mult}(\mathcal{H}_{\sigma,n})$. Choose a sequence of polynomials $\{p_{j,r}\}_{j \in \mathbb{N}}$ such that $\|p_{j,r}^{(k)} - \varphi_r^{(k)}\|_{s(\sigma,n)}^2 \to 0$ as $j \to \infty$. Then $\|\varphi_r^{(k)} - \varphi_r\|_{s(\sigma,n)}^2 \to 0$ as $j \to \infty$. Therefore, $\varphi_r f \in \mathcal{M}$.

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that $\phi_r^{(k)} \to 0$ as $j \to \infty$ for each $k = 0, 1, \ldots, n$. In view of (11), it follows that $D_{\mu,n}(p_j, r f - \varphi_r f) \to 0$ as $j \to \infty$. Since $\mathcal{M}$ is a $M_z$-invariant subspace of $\mathcal{H}_{\mu,n}$, we have that $p_j, r f \in \mathcal{M}$ for all $j \in \mathbb{N}$. As $p_j, r f \to \varphi_r f$ in $\mathcal{H}_{\mu,n}$ as $j \to \infty$, it follows that $\varphi_r f \in \mathcal{M}$. Since $r$ is an arbitrary number between 0 and 1, we conclude that $\varphi_r f \in \mathcal{M}$ for every $0 < r < 1$. To show $\varphi f \in \mathcal{M}$, it is enough to show that

$$\sup\{D_{\mu,n}(\varphi f) : 0 < r < 1\} < \infty.$$  

Indeed, if the family $\{\varphi_r f : 0 < r < 1\}$ is norm bounded in $\mathcal{H}_{\mu,n}$ then there exists a sequence $(r_k)_{k \in \mathbb{N}}$ converging to 1 such that $\varphi_{r_k} f$ converges to $\varphi f$ weakly, which in turn implies that $\varphi f \in \mathcal{M}$. In case $D_{\mu,n}(f) \neq 0$, using Lemma 3.4, Lemma 3.11, and [19, Theorem 1.1], we find that for any $0 < r < 1$ and for $\mu$-a.e. $\lambda \in \mathbb{T}$,

$$D_{\lambda,n}(\varphi f) \leq 2\|\varphi_r\|_{\mathcal{S}(\sigma,n-1)}^2 D_{\lambda,n}(f) + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi_r) \leq 2K^2 D_{\lambda,n}(f) + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi) \leq 2K^2 D_{\lambda,n}(f) + 4\|\varphi\|_{\mathcal{S}(\sigma,n-1)}^2 D_{\lambda,n}(f) + 4D_{\lambda,n}(\varphi f),$$

where $K = \sup\{\|\varphi_r\|_{\mathcal{S}(\sigma,n-1)} : 0 < r < 1\}$. Integrating with respect to $\mu$ on both sides of the above inequality, we obtain that

$$D_{\mu,n}(\varphi f) \leq \left(2K^2 + 4\|\varphi\|_{\mathcal{S}(\sigma,n-1)}^2\right) D_{\mu,n}(f) + 4D_{\mu,n}(\varphi f),$$

for every $0 < r < 1$. If $D_{\mu,n}(f) = 0$ then $D_{\lambda,n}(f) = 0$ for all $\lambda \in \mathbb{T}$. From Remark 3.5, it follows that

$$D_{\lambda,n}(\varphi f) \leq 2D_{\sigma,n-1}(\varphi_r)\|g\|_{\mathcal{S}(\sigma,n-1)}^2 + 2|f^*(\lambda)|^2 D_{\lambda,n}(\varphi_r) \leq 6D_{\sigma,n-1}(\varphi)\|g\|_{\mathcal{S}(\sigma,n-1)}^2 + 4D_{\lambda,n}(\varphi f),$$

where $g(z) := \frac{f(z) - f^*(\lambda)}{z - \lambda}$, $z \in \mathbb{D}$. Integrating with respect to $\mu$ on both sides, we get that

$$D_{\mu,n}(\varphi f) \leq 6\mu(\mathbb{D}) D_{\sigma,n-1}(\varphi)\|g\|_{\mathcal{S}(\sigma,n-1)}^2 + 4D_{\mu,n}(\varphi f).$$

This completes the proof. \hfill \Box

**Corollary 3.13** Let $\mu \in \mathcal{M}_+(\mathbb{T})\setminus\{0\}$ and $n \in \mathbb{N}$. If $\mathcal{M}$ is a closed $M_z$-invariant subspace of $\mathcal{H}_{\mu,n}$ then $\mathcal{M} \cap \text{Mult}(\mathcal{H}_{\sigma,n-1})$ is an algebra.

**Proof** Let $\mathcal{M}$ be a closed $M_z$-invariant subspace of $\mathcal{H}_{\mu,n}$ and $\varphi, f \in \mathcal{M} \cap \text{Mult}(\mathcal{H}_{\sigma,n-1})$. From Theorem 1.2, we know that $\varphi f \in \mathcal{H}_{\mu,n}$. Now by Theorem 3.12, we deduce that $\varphi f \in \mathcal{M}$. This completes the proof. \hfill \Box

We have already seen that $\mathcal{H}_{\mu,n}$ is an algebra for every $n \geq 3$ and for any $\mu \in \mathcal{M}_+(\mathbb{T})\setminus\{0\}$. There are many interesting consequences when a reproducing kernel Hilbert space $\mathcal{H}_K$ becomes an algebra under pointwise product, see for example [32,
Proposition 31, pp. 94–95]. In this part of the section, we note down some of these consequences of \( \mathcal{H}_k \) being an algebra. We start with the following preparatory lemma.

**Lemma 3.14** Let \( \mathcal{H}_k \) be a reproducing kernel Hilbert space of \( \mathbb{C} \)-valued holomorphic functions defined on the unit disc \( \mathbb{D} \) such that the constant function 1 is in \( \mathcal{H}_k \). Suppose \( \mathcal{H}_k \) is an algebra under pointwise product, that is, \( \text{Mult}(\mathcal{H}_k) = \mathcal{H}_k \). Then the multiplier norm \( \| \cdot \|_{\text{op}} \) and the Hilbert space norm \( \| \cdot \|_{\mathcal{H}_k} \) on \( \mathcal{H}_k \) are equivalent.

**Proof** Since 1 \( \in \mathcal{H}_k \), it follows that \( \| f \|_{\mathcal{H}_k} \leq \| M_f \|_{\mathcal{H}_k} = \| f \|_{\text{op}} \| 1 \|_{\mathcal{H}_k} \) for every \( f \in \mathcal{H}_k \). This shows that the inclusion map from \( (\text{Mult}(\mathcal{H}_k), \| \cdot \|_{\text{op}}) \) onto \( (\mathcal{H}_k, \| \cdot \|_{\mathcal{H}_k}) \) is continuous. By an application of open mapping theorem it follows that the two norms \( \| \cdot \|_{\text{op}} \) and \( \| \cdot \|_{\mathcal{H}_k} \) on \( \mathcal{H}_k \) are equivalent. \( \square \)

Let \( \mathcal{C}(\overline{\mathbb{D}}) \) denote the algebra of continuous functions on the closed unit disc \( \overline{\mathbb{D}} \). In the following proposition, we first derive that under certain conditions, an abelian Banach algebra contained in \( \mathcal{O}(\mathbb{D}) \) must lie in the disc algebra \( \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}}) \).

**Proposition 3.15** Let \( \mathcal{A} \subseteq \mathcal{O}(\mathbb{D}) \) be an abelian Banach algebra under pointwise product and the set of polynomials be dense in \( \mathcal{A} \). Suppose \( \sigma_{\mathcal{A}}(z) \), the spectrum of the co-ordinate function \( z \) in \( \mathcal{A} \), is equal to the closed unit disc \( \overline{\mathbb{D}} \). Then \( \mathcal{A} \subseteq \mathcal{C}(\overline{\mathbb{D}}) \) and for every \( \lambda \in \overline{\mathbb{D}} \), the evaluation map \( e_{\lambda} : \mathcal{A} \to \mathbb{C} \) defined by \( e_{\lambda}(f) = f(\lambda) \), \( f \in \mathcal{A} \), is continuous. Furthermore \( \sigma_{\mathcal{A}}(f) = f(\overline{\mathbb{D}}) \) holds for every \( f \in \mathcal{A} \).

**Proof** Let \( \Sigma \) be the maximal ideal space of \( \mathcal{A} \), that is, the collection of all non-zero homomorphisms of \( \mathcal{A} \) into \( \mathbb{C} \). Consider \( \Sigma \) with the weak* topology that it inherits as a subset of \( \mathcal{A}^* \). It is well known that \( \Sigma \) is a compact Hausdorff space, see [11, Ch. VII, Theorem 8.6]. Since \( \mathcal{A} \subseteq \mathcal{O}(\mathbb{D}) \), it follows that \( e_{\lambda} \) belongs to \( \Sigma \) for every \( \lambda \in \mathbb{D} \). By our assumption the co-ordinate function \( z \) is a generator for the algebra \( \mathcal{A} \). By [11, Ch. VII, Proposition 8.10], we have that there is a homeomorphism \( \tau : \Sigma \to \overline{\mathbb{D}} \) such that \( \tau(e_{\lambda}) = \lambda \). Furthermore, it follows that the Gelfand transform \( \gamma : \mathcal{A} \to \mathcal{C}(\overline{\mathbb{D}}) \), satisfies the relation \( \gamma(p) = p \) for every polynomial \( p \) in \( \mathbb{C}[z] \). Now let \( f \in \mathcal{A} \). For each \( \lambda \in \overline{\mathbb{D}} \), we have that

\[
f(\lambda) = e_{\lambda}(f) = (\tau^{-1}(\lambda))(f).
\]

So for each \( \lambda \in \mathbb{T} \), it is natural to define \( f(\lambda) := (\tau^{-1}(\lambda))(f) \). Since \( \tau \) is a homeomorphism, it follows that \( f \) extends as a function in \( \mathcal{C}(\overline{\mathbb{D}}) \). Note that \( e_{\lambda} = \tau^{-1}(\lambda) \in \Sigma \) for every \( \lambda \in \overline{\mathbb{D}} \). Hence the continuity of \( e_{\lambda} \), for each \( \lambda \in \overline{\mathbb{D}} \), follows from [11, Ch. VII, Proposition 8.4]. Finally, in view of [11, Ch. VII, Theorem 8.6], it follows that

\[
\sigma_{\mathcal{A}}(f) = \{ h(f) : h \in \Sigma \} = \{ \tau^{-1}(\lambda)(f) : \lambda \in \overline{\mathbb{D}} \} = \{ f(\lambda) : \lambda \in \overline{\mathbb{D}} \}.
\]

This completes the proof. \( \square \)

As an application of the aforementioned proposition, we obtain that every functional Hilbert space \( \mathcal{H}_k \) with property (A), which is also an algebra under pointwise product, must lie in the disc algebra. Furthermore, for such instances, one can describe all the
cyclic vectors in \( \mathcal{H}_k \). Recall that a vector \( f \) in \( \mathcal{H}_k \) is said to be cyclic if \( \mathcal{H}_k \) is the smallest closed \( M_z \)-invariant subspace containing \( f \). The following results can be thought of as a straightforward generalization of the results in [32, Corollary 2, Corollary 3, p. 95].

**Theorem 3.16** Let \( \mathcal{H}_k \) be a functional Hilbert space with property (A). Suppose \( \mathcal{H}_k \) is an algebra under pointwise product and the spectrum \( \sigma(M_z) = \overline{D} \). Then \( \mathcal{H}_k \subseteq \mathcal{C}(\overline{D}) \) and every closed \( M_z \)-invariant subspace of \( \mathcal{H}_k \) is an ideal. Furthermore, for a vector \( f \in \mathcal{H}_k \), the following are equivalent:

(a) \( f \) is cyclic.
(b) \( f \) is invertible in \( \text{Mult}(\mathcal{H}_k) \).
(c) \( f \) has no zero in \( \overline{D} \).

**Proof** By Lemma 3.14, it follows that the two norms \( \| \cdot \|_{\text{op}} \) and \( \| \cdot \|_{\mathcal{H}_k} \) on \( \mathcal{H}_k \) are equivalent. By assumption (A3), it now follows that the set of all polynomials is dense in \( \text{Mult}(\mathcal{H}_k) \), \( \| \cdot \|_{\text{op}} \) as well. Since \( \text{Mult}(\mathcal{H}_k) \) coincides with the commutant of \( M_z \) in \( \mathcal{B}(\mathcal{H}_k) \), it follows that the spectrum of the co-ordinate function \( f \) in the multiplier algebra \( \text{Mult}(\mathcal{H}_k) \) is equal to the spectrum of \( M_z \) in \( \mathcal{B}(\mathcal{H}_k) \). Now applying Proposition 3.15 to the abelian Banach algebra \( \mathcal{A} = (\text{Mult}(\mathcal{H}_k), \| \cdot \|_{\text{op}}) \), we obtain that \( \mathcal{H}_k = \text{Mult}(\mathcal{H}_k) \subseteq \mathcal{C}(\overline{D}) \). For the proof of the second part, consider an element \( f \in \mathcal{H}_k \). Let \([f] \) denote the smallest closed \( M_z \)-invariant subspace of \( \mathcal{H}_k \), containing \( f \), that is,

\[
[f] = \sqrt{\{ z^nf : n \in \mathbb{Z}_{\geq 0} \}}.
\]

To show that every closed \( M_z \)-invariant subspace is an ideal, it is sufficient to show that \([f] \) is an ideal. As the two norms \( \| \cdot \|_{\text{op}} \) and \( \| \cdot \|_{\mathcal{H}_k} \) on \( \mathcal{H}_k \) are equivalent and the set of all polynomials is dense in \( \mathcal{H}_k \), it follows that \([f] \) is an ideal. Since \([f] \) is a closed ideal in the Banach algebra \( \mathcal{A} = (\text{Mult}(\mathcal{H}_k), \| \cdot \|_{\text{op}}) \), then either one of the following two mutually exclusive events must hold.

(i) \([f] = \mathcal{H}_k \),
(ii) \([f] \) is contained in a maximal ideal, that is, \([f] \subseteq \ker(ev_\lambda) \) for some \( \lambda \in \overline{D} \). In particular \( f(\lambda) = 0 \) for some \( \lambda \in \overline{D} \).

Hence we obtain that an element \( f \in \mathcal{H}_k \), is cyclic, that is, \([f] = \mathcal{H}_k \) if and only if \( f \) has no zeros in \( \overline{D} \). Applying Proposition 3.15, we also have that \( \sigma_{\mathcal{A}}(f) = \{ f(\lambda) : \lambda \in \overline{D} \} \). Thus \( f \) has no zeros in \( \overline{D} \) if and only if \( f \) is invertible in \( \text{Mult}(\mathcal{H}_k) \). This completes the proof. \( \square \)

Let \( \mu = (\mu_1, \ldots, \mu_m) \) be an \( m \)-tuple of finite non-negative Borel measures on \( \mathbb{T} \) with \( \mu_m \neq 0 \). Since the operator \( M_z \) on \( \mathcal{H}_\mu \) is an analytic \((m+1)\)-isometry, the spectrum \( \sigma(M_z) \) is equal to the closed unit disc \( \overline{D} \), see [19, Proposition 6.4]. We have also noted that \( \mathcal{H}_\mu \) is a functional Hilbert space with property (A). Thus in view of Corollary 3.8, the following is immediate.

**Corollary 3.17** Let \( \mu = (\mu_1, \ldots, \mu_m) \) be an \( m \)-tuple of finite non-negative Borel measures on \( \mathbb{T} \) with \( \mu_m \neq 0 \). Suppose \( m \geq 3 \). Then \( \mathcal{H}_\mu \subseteq \mathcal{C}(\overline{D}) \) and every closed
4 Codimension $k$ Property of Invariant Subspaces

This section is an attempt to study the codimension $k$ property of non-zero closed $M_z$-invariant subspaces of $\mathcal{H}_\mu$ for any $\mu = (\mu_1, \ldots, \mu_m)$ with $\mu_j \in \mathcal{M}_+(\mathbb{T})$ for $j = 1, \ldots, m$. Since $M_z|_\mathcal{M}$ is an analytic, $(m + 1)$-isometry for every non-zero $\mathcal{M}$ in $\text{Lat}(M_z, \mathcal{H}_\mu)$, it turns out that the dimension of $\ker((M_z|_\mathcal{M})^* - \lambda)$, that is, the dimension of $\mathcal{M} \ominus (z - \lambda)\mathcal{M}$, is a positive integer independent of $\lambda \in \mathbb{D}$, see [19, Proposition 6.4]. Thus any $\mathcal{M}$ in $\text{Lat}(M_z, \mathcal{H}_\mu)$ has codimension $k$ property if and only if $\dim(\mathcal{M} \ominus z\mathcal{M}) = k$. In case of $m = 1$, that is, $\mu = \mu_1$, where $\mu_1 \in \mathcal{M}_+(\mathbb{T})$, it is well known that for any non-zero $\mathcal{M}$ in $\text{Lat}(M_z, \mathcal{H}_\mu)$, $\mathcal{M}$ has codimension $1$ property, see [29, Theorem 3.2]. Here, firstly we show that for any $m \geq 3$ and any non-zero $\mathcal{M}$ in $\text{Lat}(M_z, \mathcal{H}_\mu)$ has codimension $1$ property.

**Theorem 4.1** Let $\mu = (\mu_1, \ldots, \mu_m)$ be an $m$-tuple of finite non-negative Borel measure on $\mathbb{T}$. Suppose $\mathcal{H}_\mu$ is an algebra under pointwise product, that is, $\mathcal{H}_\mu = \text{Mult}(\mathcal{H}_\mu)$. Then for every non-zero $\mathcal{M}$ in $\text{Lat}(M_z, \mathcal{H}_\mu)$, we have $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$. In particular if $m \geq 3$ and $\mu_m \neq 0$, then $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$ for any non-zero $\mathcal{M}$ in $\text{Lat}(M_z, \mathcal{H}_\mu)$.

**Proof** By Lemma 3.14, the multiplier norm $\| \cdot \|_\text{op}$ and the Hilbertian norm $\| \cdot \|_\mu$ on $\mathcal{H}_\mu$ are equivalent. In view of [19, Corollary 5.3], we obtain that the set of all polynomials is dense in $\mathcal{H}_\mu$ w.r.t the multiplier norm $\| \cdot \|_\text{op}$. For any $f \in \mathcal{H}_\mu$, let $[f]$ denotes the smallest closed $M_z$-invariant subspace containing $f$ in $\mathcal{H}_\mu$. Hence it follows that $\varphi[f] \subseteq [f]$ for every $\varphi, f \in \mathcal{H}_\mu$. Applying [7, Proposition 1] to $\mathcal{H}_\mu$, we get that $\mathcal{H}_\mu$ is a cellular-indecomposable space. Now the first part of the theorem follows immediately from [7, Theorem 1]. The last part follows using the first part together with Corollary 3.8. \qed

For the remaining case $m = 2$, that is, $\mu = (\mu_1, \mu_2)$ with $\mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{T})$, we answer this question partially in Theorem 4.4. We start with a basic lemma which will be useful in the proof of Theorem 4.4.

**Lemma 4.2** Let $H$ be a Hilbert space and $V$ be a closed subspace of $H$ with $\dim V^\perp = n$. Let $W$ be another closed subspace of $H$, then $\dim(W \ominus (V \cap W)) \leq n$.

**Proof** Let $P$ denote the orthogonal projection of $H$ onto $V^\perp$. Suppose $X = W \ominus (V \cap W)$. We claim that $P|_X$ is an injective operator. To verify the claim, let $x \in X$ be such that $P|_X(x) = 0$. Since ker $P = V$, $x \in V$. This implies $x = 0$. Thus the claim stands verified and therefore $\dim(W \ominus (V \cap W)) \leq \dim V^\perp = n$. \qed

**Lemma 4.3** Let $\mu$ be a finite non-negative Borel measure on the unit circle $\mathbb{T}$. Let $n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in \mathbb{T}$ be distinct points and $c_1, \ldots, c_n$ be positive real numbers. Let $\mu$ be the $2$-tuple $(\mu, \sum_{j=1}^n c_j \delta_{\lambda_j})$. Then the subspace $\ker(M_z^* M_z^2 - 2M_z^* M_z + I) = \mathcal{V}$ (say) is a closed $M_z$-invariant subspace of $\mathcal{H}_\mu$ and the operator $M_z|_\mathcal{V}$ is a cyclic analytic $2$-isometry. Moreover, $\dim \mathcal{V}^\perp = n$. \hfill \Box

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Proof From [19, Theorem 4.1], we know that $M_z$ on $\mathcal{H}_\mu$ is a cyclic analytic 3-isometry. Thus, by [1, Proposition 1.6], it follows that $\mathcal{V}$ is a closed $M_z$-invariant subspace of $\mathcal{H}_\mu$ and the operator $M_z|_\mathcal{V}$ is a 2-isometry. Since $M_z$ on $\mathcal{H}_\mu$ is analytic, $M_z|_\mathcal{V}$ is also analytic. In view of [26, Theorem 1], to show $M_z|_\mathcal{V}$ is cyclic, it is sufficient to show that $\dim(\mathcal{V} \ominus z\mathcal{V}) = 1$.

Before proving that $\dim(\mathcal{V} \ominus z\mathcal{V}) = 1$, first we show that $\dim \mathcal{V} = n$. In this regard, note that $\mathcal{H}_\mu \subseteq \mathcal{H}_{\lambda_j,2}$ for each $j = 1, \ldots, n$. Thus, by Theorem 1.1, $f^*(\lambda_j)$ exists for all $f \in \mathcal{H}_\mu$ and for every $j = 1, \ldots, n$. Since $\mathcal{H}_\mu$ is a reproducing kernel Hilbert space on $\mathbb{D}$, the linear functional $f \mapsto f(r\lambda_j)$ is bounded on $\mathcal{H}_\mu$ for any $0 < r < 1$ and for each $j = 1, \ldots, n$. An application of uniform boundedness principle gives that $f \mapsto f^*(\lambda_j)$ is a bounded linear functional on $\mathcal{H}_\mu$ for each $j = 1, \ldots, n$. Therefore, by Riesz representation theorem, there exist $h_1, \ldots, h_n \in \mathcal{H}_\mu$ such that

$$f^*(\lambda_j) = \langle f, h_j \rangle, \quad f \in \mathcal{H}_\mu, \quad j = 1, \ldots, n. \tag{13}$$

Since $M_z$ on $\mathcal{H}_\mu$ is a 3-isometry, by [1, Proposition 1.5], the operator $M_z^*M_z - 2M_z^*M_z + I$ is positive. Thus a function $f$ of $\mathcal{H}_\mu$ is in $\mathcal{V}$ if and only if $\|z^2f\|_\mu^2 - 2\|zf\|_\mu^2 + \|f\|_\mu^2 = 0$. By [19, Proposition 2.7], this is equivalent to $D_{\tau,0}(f) = 0$, where $\tau = \sum_{j=1}^n c_j \delta_{\lambda_j}$. Since $D_{\tau,0}(f) = \sum_{j=1}^n c_j |f^*(\lambda_j)|^2$, we see that

$$\mathcal{V} = \{ f \in \mathcal{H}_\mu : f^*(\lambda_j) = 0, \quad 1 \leq j \leq n \}. \tag{14}$$

Hence, by (13), it follows that $\mathcal{V} \perp = \text{span} \{ h_1, \ldots, h_n \}$. Since $\lambda_j$’s are distinct points on $\mathbb{T}$, it follows that $h_1, \ldots, h_n$ are linearly independent. This shows that $\dim \mathcal{V} \perp = n$. To finish the proof, note that $M_z$ on $\mathcal{H}_\mu$ is in the Cowen-Douglas class $B_1(\mathbb{D})$, see [19, Corollary 6.1]. Since $\mathcal{V}$ is a closed $M_z$-invariant subspace of finite co-dimension, it follows from [13, p. 71] that $\dim(\mathcal{V} \ominus z\mathcal{V}) = 1$. This completes the proof. \hfill \Box

Theorem 4.4 Let $\mu$ be a finite non-negative Borel measure on the unit circle $\mathbb{T}$. Let $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{T}$ be distinct points and $c_1, \ldots, c_n$ be positive real numbers. Suppose $\mathcal{M}$ is a non-zero subspace in $\text{Lat}(M_z, \mathcal{H}_\mu)$, where $\mu = (\mu, \sum_{j=1}^n c_j \delta_{\lambda_j})$. Then $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$.

Proof Let $\mathcal{V}$ denote the subspace $\ker(M_z^*M_z - 2M_z^*M_z + I)$. By Lemma 4.3, the operator $T := M_z|_\mathcal{V}$ is a cyclic analytic 2-isometry. Let $\mathcal{N} := \mathcal{V} \cap \mathcal{M}$. Then $\mathcal{N}$ is a closed invariant subspace for $T$. Since $f^*(\lambda_j)$ exists for all $f \in \mathcal{H}_\mu$ and for each $1 \leq i \leq n$ (see the proof of Lemma 4.3), from (14) we see that $(\prod_{i=1}^n (z - \lambda_i))\mathcal{M} \subset \mathcal{N}$. Hence $\mathcal{N} \neq \{0\}$. Thus from [29, Theorem 3.2], it follows that $\dim(\mathcal{N} \ominus z\mathcal{N}) = 1$. Let $v$ be a unit vector in $\mathcal{N} \ominus z\mathcal{N}$. Let $\mathcal{L}$ be the subspace of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$. Then from Lemma 4.2, it follows that $\dim \mathcal{L} \leq n$. Let $P_{\mathcal{M} \ominus z\mathcal{N}}$ denote the orthogonal projection onto $\mathcal{M} \ominus z\mathcal{N} = (\mathcal{N} \ominus z\mathcal{N}) \oplus \mathcal{L}$. Let $\{ u_1, \ldots, u_k \}$ be an orthonormal basis for $\mathcal{L}$. Note that

$$P_{\mathcal{M} \ominus z\mathcal{N}}(zu_i) = c_i v + f_i, \quad \text{for some } c_i \in \mathbb{C}, \ f_i \in \mathcal{L},$$

where $\mathbb{C}$ is the complex field.
for each \( i = 1, \ldots, k \). Since \( N \cap L = \{0\} \), using (14) we further note that \( N \cap zL = \{0\} \). Therefore, \( P_L \) is injective on \( zL \), where \( P_L \) denotes the orthogonal projection onto \( L \). Thus it follows that \( \{f_1, \ldots, f_k\} \) forms a basis for \( L \). Note that \( \mathcal{M} \ominus z\mathcal{M} = (\mathcal{M} \ominus z\mathcal{N}) \cap (\mathcal{M} \ominus zL) \). Let \( x \) be a non-zero vector in \( \mathcal{M} \ominus z\mathcal{M} \). Then \( x = c_x v + g_x \) for some \( g_x \in L \) and \( c_x \in \mathbb{C} \) and \( \langle x, zu_i \rangle = 0 \) for each \( i = 1, \ldots, k \). This will give us
\[
c_x \bar{c}_i + \langle g_x, f_i \rangle = 0, \quad i = 1, \ldots, k.
\]
Note that \( c_x \neq 0 \), otherwise \( g_x \in L \cap L^\perp \) and \( x = 0 \). Let \( y \) be another non-zero vector in \( \mathcal{M} \ominus z\mathcal{M} \). Then \( y = c_y v + g_y \) for some \( g_y \in L \) and \( c_y \in \mathbb{C} \). Arguing as before, we get that \( c_y 
eq 0 \) and \( c_y \bar{c}_i + \langle g_y, f_i \rangle = 0 \), for each \( i = 1, \ldots, k \). This would lead to \( \frac{1}{c_y} g_y = \frac{1}{c_x} g_x \) and consequently \( x \) and \( y \) are linearly dependent. This shows that \( \dim(\mathcal{M} \ominus z\mathcal{M}) = 1 \).

Remark 4.5 Consider \( \mu = (\mu_1, \mu_2 + \sigma) \), where \( \mu_1, \mu_2 \in \mathcal{M}_+(\mathbb{T}) \). By Remark 3.10(ii), the space \( \mathcal{H}_\mu \) is an algebra. Hence in view of Theorem 4.1 we obtain that \( \dim(\mathcal{M} \ominus z\mathcal{M}) = 1 \), for every non-zero \( \mathcal{M} \in \text{Lat}(M_z, \mathcal{H}_\mu) \).

5 Relationship Between \( \mathcal{H}_\mu \) and \( \mathcal{H}(b) \)

In what follows, for two reproducing kernel Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) consisting of holomorphic functions on the unit disc \( \mathbb{D} \), we write \( \mathcal{H}_1 = \mathcal{H}_2 \) to mean that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are equal as a set but the norms are not necessarily same. In case \( \mathcal{H}_1 = \mathcal{H}_2 \), using closed graph theorem, it can be shown that the norms on \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) must be equivalent.

For a non-negative measure \( \mu \) in \( \mathcal{M}_+(\mathbb{T}) \), we set the notation (as in \cite{12})
\[
V_\mu(z) := \int_{\mathbb{T}} \frac{d\mu(\lambda)}{|1 - \lambda z|^2}, \quad z \in \mathbb{D}.
\]
With the help of generalized local Douglas formula, for a fixed \( w \in \mathbb{D} \), the following lemma computes the semi-norm of Szegö kernel \( S(z, w) := \frac{1}{1 - \bar{w} z} \) in \( \mathcal{H}_{\mu, n} \), in terms of \( V_\mu(w) \).

Lemma 5.1 Let \( \mu \) be a finite non-negative Borel measure on \( \mathbb{T} \) and \( n \) be a positive integer. Then
\[
D_{\mu, n} \left( \frac{1}{1 - \bar{w} z} \right) = \frac{|w|^{2n}}{(1 - |w|^2)^n} V_\mu(w), \quad w \in \mathbb{D}.
\]
Proof Let \( \lambda \in \mathbb{T} \). Then by Theorem 1.1, we have
\[
D_{\lambda, n} \left( \frac{1}{1 - \bar{w} z} \right) = D_{\sigma, n-1} \left( \frac{1 - \bar{w} z}{z - \lambda} \right)
\]
\[ D_{\sigma,n-1} \left( \frac{1}{1-z\overline{w}} \right) = \frac{|w|^2}{|1-\lambda\overline{w}|^2} D_{\sigma,n-1} \left( \frac{1}{1-z\overline{w}} \right). \]

To complete the proof for the case \( \mu \) is Dirac delta measure \( \delta_\lambda \) at the point \( \lambda \), we claim that
\[ D_{\sigma,n-1} \left( \frac{1}{1-z\overline{w}} \right) = \frac{|w|^{2n-2}}{(1-|w|^2)^n}. \tag{15} \]

Since \( \frac{1}{1-z\overline{w}} \) is the reproducing kernel for the Hardy space \( H^2 \), it is easy to verify (15) for the case \( n = 1 \). Suppose \( n > 1 \). Note that
\[ D_{\sigma,n-1} \left( \frac{1}{1-z\overline{w}} \right) = (n-1)|w|^{2n-2} \int_{\mathbb{D}} \frac{(1-|z|^2)^{n-2}}{|1-z\overline{w}|^{2n}} dA(z) \]
\[ = \frac{|w|^{2n-2}}{(1-|w|^2)^n}, \]
where the second equality follows from the reproducing property of the weighted Bergman space \( A^2((n-1)(1-|z|^2)^{n-2}dA(z)) \), see [21, Corollary 1.5]. Hence equation (15) stands verified and consequently we have
\[ D_{\lambda,n} \left( \frac{1}{1-z\overline{w}} \right) = \frac{|w|^{2n}}{|1-\lambda\overline{w}|^2(1-|w|^2)^n}. \]

Integrating both the sides with respect to \( \mu \), we get
\[ D_{\mu,n} \left( \frac{1}{1-z\overline{w}} \right) = \int_{\mathbb{D}} D_{\lambda,n} \left( \frac{1}{1-z\overline{w}} \right) d\mu(\lambda) = \frac{|w|^{2n}}{(1-|w|^2)^n} V_\mu(w). \]

This completes the proof of the lemma. \( \square \)

In the following main result of this section, we prove that the spaces \( \mathcal{H}_\mu \) \((m \geq 2)\) and \( \mathcal{H}(b) \) can not be same even as sets.

**Theorem 5.2** Let \( m \geq 2 \) and \( \mu = (\mu_1, \ldots, \mu_m) \) be an \( m \)-tuple of non-negative measures in \( \mathcal{M}_+(\mathbb{T}) \). If \( \mathcal{H}_\mu = \mathcal{H}(b) \) for some \( b \in H^\infty \) with \( \|b\|_\infty \leq 1 \) then
\[ \mu_2 = \cdots = \mu_m = 0. \]

**Proof** Suppose that there exists a \( b \in H^\infty \) with \( \|b\|_\infty \leq 1 \) such that \( \mathcal{H}(b) = \mathcal{H}_\mu \). Recall that \( \mathcal{H}_\mu \) is a Hilbert space with respect to the norm \( \| \cdot \|_\mu \) given by \( \|f\|_\mu^2 = \|f\|_{H^2}^2 + \sum_{j=1}^m D_{\mu,j}(f) \). Then, by an application of closed graph theorem, there exist constants \( \alpha > 0 \) and \( \beta > 0 \) such that
\[ \beta \|f\|_{\mathcal{H}(b)} \leq \|f\|_\mu \leq \alpha \|f\|_{\mathcal{H}(b)}, \quad f \in \mathcal{H}(b). \tag{16} \]
Since $\mathcal{H}_\mu$ contains all functions holomorphic on a neighbourhood of $\overline{D}$, it follows from [12, Theorem 2.1] that $b$ is not an extreme point of the unit ball of $H^\infty$. Hence there exists a unique outer function $a$ with $a(0) > 0$ such that $|a^*(\eta)|^2 + |b^*(\eta)|^2 = 1$ for almost everywhere $\eta \in \mathbb{T}$ (see [31, p. 2136]). Note that the function $\frac{1}{1 - \overline{w}}$ is holomorphic on a neighbourhood of $\overline{D}$ for all $w \in \mathbb{D}$ By [12, Lemma 3.4], we have

$$\| \frac{1 - z}{1 - \overline{w}} \|_{\mathcal{H}(b)}^2 = \frac{1 + |b(w)|^2}{1 - |w|^2}, \ w \in \mathbb{D}. $$

Also by Lemma 5.1 we have

$$\| \frac{1 - z}{1 - \overline{w}} \|_{\mu}^2 \leq \frac{1}{1 - |w|^2} + \sum_{j=1}^{m-1} \frac{|w|^{2j}}{(1 - |w|^2)^j} V_{\mu_j}(w), \ w \in \mathbb{D}. $$

Hence by (16) we obtain

$$\frac{1}{1 - |w|^2} + \sum_{j=1}^{m-1} \frac{|w|^{2j}}{(1 - |w|^2)^j} V_{\mu_j}(w) \leq \alpha \frac{1 + |b(w)|^2}{1 - |w|^2}, \ w \in \mathbb{D}. $$

Thus for $2 \leq j \leq m - 1$, we have

$$\frac{|w|^{2j}}{(1 - |w|^2)^j} V_{\mu_j}(w) \leq \alpha \frac{1 + |b(w)|^2}{1 - |w|^2}. $$

Since $V_{\mu_j}(w) \geq \frac{\mu_j(\mathbb{T})}{4}$ for all $w \in \mathbb{D}$, we get that

$$\mu_j(\mathbb{T}) \leq 4\alpha (1 - |w|^2)^{j-1} \frac{1 + |b(w)|^2}{|w|^{2j}}. $$

Let $\zeta$ be any point in $\mathbb{T}$ such that $\frac{b^*(\zeta)}{a^*(\zeta)}$ exists. Since $j \geq 2$, putting $w = r\zeta$ in the above inequality and taking limit $r \to 1^-$, we get that $\mu_j(\mathbb{T}) = 0$, completing the proof of the theorem. 

**Remark 5.3** Let $B : \mathbb{D} \to (\ell^2)_1$ be an analytic map with $B(z) = (b_i(z))_{i=0}^\infty$, where $(\ell^2)_1$ is the closed unit ball in $\ell^2$. The space $\mathcal{H}(B)$ corresponding to $B$ is defined to be the reproducing kernel Hilbert space with the kernel function $\frac{1 - \overline{z}}{1 - \overline{w}} b_i(z) b_j(w)$, $z, w \in \mathbb{D}$. The space $\mathcal{H}(B)$ is said to be of finite rank if there exists a $N \in \mathbb{N}$ and an analytic map $C : \mathbb{D} \to (\ell^2)_1$, $C(z) = (c_i(z))_{i=0}^\infty$ such that $c_i = 0$ for all $i > N$ and $\mathcal{H}(B) = \mathcal{H}(C)$, with equality of norms. Since the backward shift operator $L$, defined by $(Lf)(z) = \frac{f(z) - f(0)}{z}$, $z \in \mathbb{D}$, is contractive on $\mathcal{H}_\mu$ (see [19, Lemma 2.9]), it follows from [4, Proposition 2.1] that $\mathcal{H}_\mu$ coincides with the space $\mathcal{H}(B)$ for some $B$ with $B(0) = 0$. In the case of finite rank $\mathcal{H}(B)$ spaces with $B(0) = 0$, if the operator $M_\zeta$ is
bounded, then it turns out that \( \overline{\text{ran}}(M_z^m M_z - I) \) is finite dimensional, (see [22, Lemma 5.1]). On the contrary, for the space \( \mathcal{H}_\mu \), where \( \mu = (\mu_1, \ldots, \mu_m) \), \( m \geq 2 \), it is worth noting that the space \( \ker(M_z^m M_z - I) \) is finite dimensional. This follows easily from [19, Proposition 2.7]. Therefore, in this case, \( \overline{\text{ran}}(M_z^m M_z - I) \) is infinite dimensional. Consequently, we conclude that if \( m \geq 2 \), then for any \( \mu = (\mu_1, \ldots, \mu_m) \), \( \mu_m \neq 0 \), \( \mathcal{H}_\mu \) can not be equal to any \( \mathcal{H}(B) \) space of finite rank with equality of norms. It will be interesting to know whether for a fixed \( \mu = (\mu_1, \ldots, \mu_m) \), does there exist a \( \mathcal{H}(B) \) space of finite rank such that \( \mathcal{H}_\mu = \mathcal{H}(B) \) with just equivalence of norms.

**Remark 5.4** The last part of [22, Theorem 1.1] says that if \( M_z \) on \( \mathcal{H}(b) \) is a strict \( 2m \)-isometry for some \( m \in \mathbb{N} \) then there exists a \( \xi \in \mathbb{T} \) such that \( \mathcal{H}(b) \) is equal to \( D_\xi^m \)-space defined in [22, page 3] with equivalence of norms. Theorem 5.2 indicates that the local Dirichlet space of order \( m \) in the present article differs from that in [22].

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**Declarations**

**Conflict of interest** The authors declare that there are no conflict of interest regarding the publication of this paper.

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