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ON A NONSYMMETRIC ORNSTEIN-UHLENBECK SEMIGROUP AND ITS GENERATOR*

YONG CHEN

Abstract. If we add a simple rotation term to both the Ornstein-Uhlenbeck semigroup and the H-derivative, then analogue to the classical Malliavin calculus on the real Wiener space [I. Shigekawa, Stochastic analysis, 2004], we get a normal but nonsymmetric Ornstein-Uhlenbeck operator \( L \) on the complex Wiener space. The eigenfunctions of the operator \( L \) are given. In addition, the hypercontractivity for the nonsymmetric Ornstein-Uhlenbeck semigroup is shown.

1. Introduction

In [1], the following stochastic differential equation is considered:

\[
\begin{align*}
\text{d}Z_t &= -\alpha Z_t \text{d}t + \sqrt{2\alpha^2} \text{d}\zeta_t, \quad t \geq 0, \\
Z_0 &= z_0 \in \mathbb{C}^1,
\end{align*}
\]

where \( Z_t = X_1(t) + iX_2(t) \), \( \alpha = ae^{i\theta} = r + i\Omega \) with \( a > 0, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), and \( \zeta_t = B_1(t) + iB_2(t) \) is a complex Brownian motion. Clearly, when \( \Omega \neq 0 \), the generator of the process is a 2-dimensional not symmetric but normal Ornstein-Uhlenbeck (OU) operator

\[
A = \sigma^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (-rx + \Omega y) \frac{\partial}{\partial x} - (\Omega x + ry) \frac{\partial}{\partial y}
\]

\[
= 4\sigma^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \alpha z \frac{\partial}{\partial z} - \bar{\alpha} \frac{\partial}{\partial \bar{z}},
\]

where we denote by \( \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \) the formal derivative of \( f \) at point \( z = x + iy \). Note that \( \Im(\alpha) \neq 0 \) in Eq.(1.1) is the key point for the non-symmetric property. The eigenfunctions of \( A \) are the so called complex Hermite polynomials [2] \(^1\) which can be generated iteratively by the complex creation operator acting on the constant 1. Let \( B = \begin{bmatrix} -r & \Omega \\ -\Omega & -r \end{bmatrix} \) and \( B_0 = \begin{bmatrix} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{bmatrix} \).

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\(^1\)It is called the Hermite-Laguerre-Itô polynomials in [1].
Then $e^{tB} = e^{-rt}B_0(t)$ and the associated OU semigroup of $A$ is
\[
P_t\varphi(z_0) = \int_{\mathbb{R}^2} \varphi(e^{-rt}B_0(t)z_0 + \sqrt{1-e^{-2rt}}z) \mu(dz)
= \int_{\mathbb{C}} \varphi(e^{-at}z_0 + \sqrt{1-e^{-2at}}z) \mu(dz),
\]
where the stationary distribution is $d\mu = \frac{r}{2\pi r^2} \exp \left\{ -\frac{r(x^2+y^2)}{2\pi r^2} \right\} \, dx \, dy$ and we write $\varphi(x, y)$ as the function $\varphi(z)$ of the complex argument $x + iy$ (i.e., we use the complex representation of $\mathbb{R}^2$ in (1.4)). For simplicity, we can choose that $a = 1$ and $r = \sigma^2 = \cos \theta$ then (1.4) becomes
\[
\int_{\mathbb{C}} \varphi(e^{-\cos \theta + i\sin \theta}t z_0 + \sqrt{1-e^{-2t\cos \theta}}z) \mu(dz).
\]
If we let $z_0, z$ be in the infinite dimensional space $(C_0([0, T] \to \mathbb{C}^1))$, we can define the nonsymmetric OU semigroup on $(C_0([0, T] \to \mathbb{C}^1))$ (see Definition 2.1). This idea is similar to the symmetric case [9]. This is the topic of Section 2.

The topic of Section 3 is how to obtain a concrete expression of the generator $L$ of the above OU semigroup with rotation. We extend the definition of the Gateaux derivative and the H-derivative to the function $F : B \to C$ and consider the derivative of the function $F(x + e^{i\theta}ty)$ with $t \in \mathbb{R}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ at $t = 0$ (i.e., here the rotation term in the derivative corresponds to the one in the above OU semigroup). Furthermore, since we consider complex-value functions, we need the conjugate-linear functional. This idea also comes from the symmetric case [9].

In Section 4, we recall the Itô-Wiener chaos decomposition and give all the eigenfunctions of the generator $L$. In addition, we show the hypercontractivity for the above OU semigroup along almost the same lines as the symmetric case.

### 2. The Nonsymmetric OU Semigroup

By the complex representation of $\mathbb{R}^2$, the planar Brownian motion $(B^1, B^2)$ will be written $B = B^1 + iB^2$. Let $H_1$ be the 1-dimensional Cameron-Martin space [9], and denote $H$ the complex Hilbert space $H = H_1 + iH_1$ with the natural inner product
\[
\langle h, k \rangle = \int_0^T \overline{h(s)k(s)} ds.
\]
Clearly, one can choose a c.o.n.s of $H$ to be $\{ \frac{\varphi_m}{\sqrt{m!}} : m = 1, 2, \ldots \}$.

We look the 2-dimensional Wiener space as a complex Wiener space $(C_0([0, T] \to \mathbb{C}^1), \mu)$. The characteristic function of $\mu$ is
\[
\int_B \exp \left\{ \sqrt{-1} \Re(\omega, \varphi) \right\} \, d\mu(\omega) = \exp \left\{ -\frac{1}{2} |\varphi|^2_{H^*} \right\}, \quad \forall \varphi \in B^*.
\]

**Definition 2.1.** Let the above notation prevail. We define transition probability on $B$ as follows. For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $t \geq 0$, $\Omega \in \mathbb{R}$, $x \in B$, $A \in \mathcal{B}(B)$ (the Borel $\sigma$-field generated by all open sets),
\[
P_t(x, A) = \int_B 1_A(e^{-\cos \theta + i\sin \theta}tx + \sqrt{1-e^{-2t\cos \theta}}y) \mu(dy).
\]
The following property about the measure $\mu$ is well known.

**Proposition 2.2.** For any $a \in \mathbb{R}$, the induced measure of $\mu$ under the mapping $x \mapsto e^{ia}x$ is identical to $\mu$, that is to say, $\mu$ is rotation invariant. In addition, for any $t \geq 0$, denote the induced measure of $\mu$ under the mapping $x \mapsto \sqrt{t}x$ by $\mu_t$, then $\mu_t \ast \mu_s = \mu_{t+s}$ ($\ast$ is the convolution operator).

An argument similar to the one used in the real case [9, Proposition 2.2] shows that $P_t(x, A)$ satisfies the Chapman-Kolmogorov equation.

$$
\int_B P_t(x, dy) P_s(y, A) = \int_B P_s(e^{-te^{i\theta}} x + \sqrt{1 - e^{-2t \cos \theta} y}, A) \mu(dy)
$$

$$
= \int_B \int_B A(e^{-se^{i\theta}} x + \sqrt{1 - e^{-2s \cos \theta} y}) + \sqrt{1 - e^{-2s \cos \theta} z) \mu(dz) \mu(dy)
$$

$$
= \int_B \int_B A(e^{-s(1+it)\cos \theta} x + e^{-s \cos \theta} \sqrt{1 - e^{-2t \cos \theta} y} + \sqrt{1 - e^{-2s \cos \theta} z) \mu(dz) \mu(dy)
$$

(by the rotation invariant of the measure $\mu(dy)$)

$$
= \int_B \int_B A(e^{-s(1+it)\cos \theta} x + y) e^{-2s \cos \theta (1-e^{-2t \cos \theta})} \ast \mu_1 e^{-2s \cos \theta} (dy)
$$

$$
= \int_B A(e^{-s(1+it)\cos \theta} x + \sqrt{1 - e^{2(s+it) \cos \theta} y) \mu(dy)
$$

$$
P_{s+t}(x, A).
$$

The associated Markov process to $P_t(x, A)$ is called a complex-valued Ornstein-Uhlenbeck process. Similar to the real case [9, Proposotion 2.2], it follows Kolmogorov’s criterion and the rotation invariance of $\mu$ that the Ornstein-Uhlenbeck process is realized as a measure on $C([0, \infty) \rightarrow B)$.

The associated semigroup $\{T_t, t \geq 0\}$ is defined as follows: for a bounded Borel measurable function $F$,

$$
T_t F(x) = \int_B F(e^{-(\cos \theta + i \sin \theta) t} x + \sqrt{1 - e^{-2t \cos \theta} y) \mu(dy).
$$

(2.4)

An argument similar to the one used in [9, Proposition 2.3, 2.4] shows that

**Proposition 2.3.** $\mu$ is a unique invariant measure, i.e,

$$
\int_B P(t, A) \mu(dx) = \mu(A), \quad \forall A \in \mathcal{B}(B).
$$

And $\{T_t, t \geq 0\}$ is a strongly continuous contraction semigroup in $L^p(B, \mu) (p \geq 1)$.

3. The Ornstein-Uhlenbeck Operator and the Complex H-derivative

The generator of $\{T_t\}$ is called the Ornstein-Uhlenbeck operator, denoted by $L$. We will obtain a concrete expression of $L$ in this section. Since there is a rotation term in the transition probabilities $P_t(x, A)$, to obtain a concrete expression of $L$, we need the complex H-derivative along a direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. 
Definition 3.1. A function $F : B \to \mathbb{C}$ is complex Gateaux differentiable at $x \in B$ along the direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if there exist $\varphi_1, \varphi_2 \in B^*$ such that
\[
\frac{d}{dt} F(x + e^{i\theta}ty)|_{t=0} = \langle y, \varphi_1 \rangle + \langle \bar{y}, \varphi_2 \rangle, \quad \forall y \in B. \tag{3.1}
\]
$(\varphi_1, \varphi_2)$ is called a Gateaux derivative of $F$ at $x$ along the direction $\theta$, denoted by $G_\theta F(x)$.

Remark 3.2. Here we look $\varphi_1$ as a linear functional on $B$, and $\varphi_2$ a conjugate-linear functional. And we inherit the notation in [9] that
\[
B \langle x, G_\theta F(x) \rangle_{B^*} = \langle x, \varphi_1 \rangle + \langle \bar{x}, \varphi_2 \rangle.
\]

Definition 3.3. A function $F : B \to \mathbb{C}$ is complex H-differentiable at $x \in B$ along the direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if there exist $h_1, h_2 \in H$ such that
\[
\frac{d}{dt} F(x + e^{i\theta}th)|_{t=0} = \langle h, h_1 \rangle + \langle h_2, h \rangle, \quad \forall h \in H. \tag{3.2}
\]
$(h_1, h_2)$ is called a complex H-derivative of $F$ at $x$ along the direction $\theta$, denoted by $D_\theta F(x)$. When $\theta = 0$, we denote $D_0 F(x)$ by $DF(x)$ instead.

We can define higher order differentiability. For simplicity, we only present the 2-th case here.

Definition 3.4. $F$ is said to be 2-th H-differentiable along the direction $\theta$ if there exists a mapping $(\Phi_1, \Phi_2, \Phi_3, \Phi_4) : H \times H \to \mathbb{C}^4$ such that $\forall h_1, h_2 \in H,$
\[
\frac{\partial^2}{\partial t_1 \partial t_2} F(x + e^{i\theta}t_1h_1 + t_2h_2)|_{t_1=t_2=0} = \sum_{j=1}^{4} \Phi_j(h_1, h_2) := \Phi(h_1, h_2), \tag{3.3}
\]
where $\Phi_1$ and $\Phi_2$ are the bilinear forms $^2$, and $\Phi_3$ and $\Phi_4$ are the sesquilinear forms$^3$. $\Phi$ is called the 2-th H-derivative of $F$ at $x$ along $\theta$, denoted by $D^2_\theta F(x)$.

Definition 3.5. Let $\Phi$ be as in Definition 3.4. $\Phi$ is said to be of trace class if the supremum
\[
\sup \sum_{n=1}^{\infty} \sum_{i=1}^{4} |\Phi_i(h_n, k_n)|
\]
is finite, where $k_n$ and $h_n$ run over all c.o.n.s of $H$. Furthermore, the trace of $\Phi$ is defined by
\[
\text{tr} \, \Phi = \sum_{n=1}^{\infty} \Phi_1(h_n, \bar{h}_n) + \Phi_2(h_n, \bar{h}_n) + \Phi_3(h_n, h_n) + \Phi_4(h_n, \bar{h}_n). \tag{3.4}
\]
Here $\{h_n\}$ is a c.o.n.s of $H$, and this does not depend on a choice of c.o.n.s.

Remark 3.6. An argument similar to the one used in [7, p44] shows that there exist bounded conjugate-linear operators $A_1, A_2$ such that $\Phi_1(h_1, h_2) = \langle h_1, A_1 h_2 \rangle$ and $\Phi_2(h_1, h_2) = \langle A_2 h_2, h_1 \rangle$.

$^2$Here the bar is used for the conjugate instead of for the closure operator.
$^3$The definition of sesquilinear is that the first argument is linear and the second one is conjugate-linear.
The 2-th H-derivative is given by
\[ F(x) = f((x, \varphi_1), (x, \varphi_2), \ldots, (x, \varphi_n)). \] (3.5)
Here we assume that \( f \) with its derivatives has polynomial growth. If \( F \in S \), then the two derivative are given in the following forms. Let \( z_j = (x, \varphi_j), j = 1, \ldots, n \) and denote
\[ \partial_j f = \frac{\partial}{\partial z_j} f(z_1, \ldots, z_n), \quad \bar{\partial}_j f = \frac{\partial}{\partial \bar{z}_j} f(z_1, \ldots, z_n). \]
If \( \varphi \in B^* \), \( c\varphi \) means that \( (c\varphi)(x) = c\varphi(x) \). Then the Gâteaux derivative is
\[ G_\theta F(x) = \left(e^{i\theta} \sum_{j=1}^n \varphi_j \partial_j f, \quad e^{-i\theta} \sum_{j=1}^n \varphi_j \bar{\partial}_j f \right), \] (3.6)
\[ \beta \langle x, G_\theta F(x) \rangle_{B^*} = \sum_{j=1}^n [e^{i\theta} z_j \partial_j f + e^{-i\theta} \bar{z}_j \bar{\partial}_j f]. \] (3.7)
The H-derivative is given by
\[ D_\theta F(x) = \left(e^{i\theta} \sum_{j=1}^n \varphi_j \partial_j f, \quad e^{-i\theta} \sum_{j=1}^n \varphi_j \bar{\partial}_j f \right), \] (3.8)
\[ D_\theta F(x)(h) = \sum_{j=1}^n [e^{i\theta} \langle h, \varphi_j \rangle \partial_j f + e^{-i\theta} \langle h, \varphi_j \rangle \bar{\partial}_j f], \] (3.9)
where we adopt the convention that \( B^* \) is the subspace of \( H^* \). (3.9) implies that the 2-th H-derivative is given by
\[ \text{tr } DD_\theta F(x)(h_1, h_2) = \sum_{j, k=1}^n [e^{i\theta} \langle h_1, \varphi_j \rangle \langle h_2, \varphi_k \rangle \partial_k \bar{\partial}_j f + \langle \varphi_k, h_2 \rangle \partial_k \partial_j f] \]
\[ + e^{-i\theta} \langle \varphi_j, h_1 \rangle \langle h_2, \varphi_k \rangle \partial_k \bar{\partial}_j f + \langle \varphi_k, h_2 \rangle \bar{\partial}_k \partial_j f]. \]
If, in addition, \( \left\{ \varphi_1, \varphi_2 \right\} \) is an orthonormal system of \( H^* \),
\[ \text{tr } DD_\theta F(x) = 4 \cos \theta \sum_{j=1}^n \partial_j \bar{\partial}_j f. \] (3.10)

**Proposition 3.7.** For \( F \in S \),
\[ LF(x) = \text{tr } DD_\theta F(x) - \beta \langle x, G_\theta F(x) \rangle_{B^*}. \] (3.11)

**Proof.** Suppose that \( F \in S \) is given by (3.5). We may assume that \( \left\{ \varphi_1, \varphi_2 \right\} \) is an orthonormal system of \( H^* \). Thus \( \xi = (\langle x, \varphi_1 \rangle, \ldots, \langle x, \varphi_n \rangle) \in \mathbb{C}^n \) has a \( 2n \)-dimensional standard normal distribution and we have
\[ T_t F(x) = \int_{\mathbb{C}^n} f(e^{-(\cos \theta + i \sin \theta)t} \xi + \sqrt{1 - e^{-2t \cos \theta}}) (2\pi)^{-n} e^{-|\eta|^2 / 2} \, d\eta. \]
When \( t > 0 \),
\[
\frac{d}{dt} T_t F(x) = \frac{d}{dt} \int_{\mathbb{C}^n} f(e^{-(\cos \theta + i \sin \theta)t} \xi + \sqrt{1 - e^{-2t \cos \theta}} (2\pi)^{-n} e^{-|\eta|^2/2} d\eta
\]
\[
= \sum_{j=1}^{n} \int_{\mathbb{C}^n} (-\xi_j e^{i\theta} e^{-e^{-i\theta} t} + \frac{\eta_j \cos \theta e^{-2t \cos \theta}}{\sqrt{1 - e^{-2t \cos \theta}}} \partial_j f(e^{-i\theta} \xi + \sqrt{1 - e^{-2t \cos \theta}}) u(d\eta)
\]
\[
+ \sum_{j=1}^{n} \int_{\mathbb{C}^n} (-\xi_j e^{-i\theta} e^{-e^{-i\theta} t} + \frac{\eta_j \cos \theta e^{-2t \cos \theta}}{\sqrt{1 - e^{-2t \cos \theta}}} \partial_j f(e^{-i\theta} \xi + \sqrt{1 - e^{-2t \cos \theta}}) u(d\eta)
\]
\[
= -e^{i\theta} e^{-e^{-i\theta} t} \sum_{j=1}^{n} \xi_j \int_{\mathbb{C}^n} \partial_j f u(d\eta) - e^{-i\theta} e^{-e^{-i\theta} t} \sum_{j=1}^{n} \xi_j \int_{\mathbb{C}^n} \partial_j \bar{f} u(d\eta)
\]
\[
+ 4 \cos \theta e^{-2t \cos \theta} \sum_{j=1}^{n} \int_{\mathbb{C}^n} \partial_j \bar{\partial}_j f u(d\eta).
\]

The last equation follows from the formula for integration by parts of the complex creation operator (see Lemma 2.3 of [1]). An argument similar to the one used in Proposition 2.7 of [9] shows that the convergence takes place in the topology of \( L^\rho(B) \). Let \( t \to 0 \), we have
\[
LF(x) = -e^{i\theta} \sum_{j=1}^{n} \xi_j \partial_j f - e^{-i\theta} \sum_{j=1}^{n} \xi_j \bar{\partial}_j f + 4 \cos \theta \sum_{j=1}^{n} \partial_j \bar{\partial}_j f,
\]
which is exact (3.11).

\[\Box\]

4. Itô-Wiener Chaos Decomposition, Eigenfunctions and the Hypercontractivity

**Definition 4.1** (Definition of the Hermite-Laguerre-Itô polynomials). Let \( m, n \in \mathbb{N} \) and \( z = x + iy \) with \( x, y \in \mathbb{R} \). We define the sequence on \( \mathbb{C} \)
\[
J_{0,0}(z) = 1,
\]
\[
J_{m,n}(z) = 2^{m+n} (\partial^*)^m (\bar{\partial}^*)^n 1.
\]
(4.1)

We call it the **Hermite-Laguerre-Itô polynomial** in the present paper.

One can show [1] that \( \{m!n!2^{m+n} - \frac{3}{2} \} J_{m,n}(z) : m, n \in \mathbb{N} \} \) is an orthonormal basis of \( L^2(\nu) \) with \( d\nu = \frac{1}{4\pi} e^{-z^2 + \bar{z}^2} dxdy \) and
\[
[e^{i\theta} \frac{\partial}{\partial z} + e^{-i\theta} \frac{\partial}{\partial z} - 4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}}] J_{m,n}(z) = [(m + n) \cos \theta + i(m - n) \sin \theta] J_{m,n}(z).
\]
(4.2)

For a sequence \( \mathbf{m} = \{m_k\}_{k=1}^{\infty} \), write \( |\mathbf{m}| = \sum_k m_k \).

**Definition 4.2.** Take a complete orthonormal system \( \{\varphi_k, \phi_k\} \subseteq B^* \) in \( H^* \) and fix it throughout the section. For two sequences \( \mathbf{m} = \{m_k\}_{k=1}^{\infty}, \mathbf{n} = \{n_k\}_{k=1}^{\infty} \) of
nonnegative integrals with finite sum, define
\[ J_{m,n}(x) := \prod_k \frac{1}{\sqrt{2^{m_k+n_k}m_k!n_k!}} J_{m_k,n_k}(\langle x, \varphi_k \rangle). \] (4.3)

We name it the *Fourier-Hermite-Itô polynomial*. For two \( m, n \in \mathbb{Z}_+ \), the closed subspace spanned by \( \{J_{m,n}(x); |m| = m, |n| = n\} \) in \( L^2_{\mathbb{C}}(B, \mu) \) is called the Itô-Wiener chaos of degree of \((m, n)\) and is denoted by \( \mathcal{H}_{m,n} \).

**Theorem 4.3.** For any fixed integer \( m, n \geq 0 \), the collection of functions
\[ \{J_{m,n}; |m| = m, |n| = n\} \] (4.4)
is an orthogonal basis for the space \( \mathcal{H}_{m,n} \). And if \((m, n)\) varies then the collection of functions
\[ \{J_{m,n}; |m| = m, |n| = n, m, n \geq 0\} \] (4.5)
is an orthogonal basis for the space \( L^2_{\mathbb{C}}(B, \mu) \). And \( L^2_{\mathbb{C}}(B, \mu) \) has the Itô-Wiener expansion in the following way:
\[ L^2_{\mathbb{C}}(B, \mu) = \bigoplus_{m=0}^{\infty} \bigoplus_{n=0}^{\infty} \mathcal{H}_{m,n}. \] (4.6)

The project from \( L^2_{\mathbb{C}}(B, \mu) \) to \( \mathcal{H}_{m,n} \) is denoted by \( J_{m,n} \).

The above theorem is well known, which is exact Example 3.32 of [3, p31] which can be shown from view of the Gaussian Hilbert spaces. The reader can also give an elementary proof using an argument similar to Theorem 9.5.4 and 9.5.7 of [4].

**Theorem 4.4.** Let \( J_{m,n}(x) \) be a Fourier-Hermite-Itô polynomial defined by (4.3). Denote \( m = |m|, n = |n| \). Then
\[ L_{\mathbb{C}}J_{m,n}(x) = -[(m + n) \cos \theta + i(m - n) \sin \theta] J_{m,n}(x), \] (4.7)
\[ T_t J_{m,n}(x) = e^{-(m+n) \cos \theta + i(m-n) \sin \theta} t] J_{m,n}(x). \] (4.8)

**Proof.** Proposition 3.7 and (4.2) imply (4.7) directly. (4.8) follows from (4.7) and the semigroup equation (or say: Kolmogorov’s equation). □

In fact, (4.8) is an alternative procedure for introducing the OU semigroup. Similar to the symmetric OU semigroup (see [6, p54]), we define a nonsymmetric OU semigroup:

**Definition 4.5.** The *nonsymmetric OU semigroup* is the one-parameter semigroup \( \{T_t, t \geq 0\} \) of contraction operators on \( L^2_{\mathbb{C}}(B) \) defined by
\[ T_t F(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-(m+n) \cos \theta + i(m-n) \sin \theta} t] J_{m,n} F \] (4.9)
for any \( F \in L^2_{\mathbb{C}}(B) \).

Finally, along almost the same lines as the proof of [9, Theorem 2.11], we have the hypercontractivity of the OU semigroup.
Proposition 4.6. For the fixed \( t \geq 0 \) and \( p > 1 \), set \( q(t) = e^{2t \cos \theta (p-1)} + 1 \). Then
\[
\|T_t F\|_{q(t)} \leq \|F\|_p, \quad \forall F \in L^p(B, \mu). \quad (4.10)
\]

Proof. Since \( \mathcal{S} \) is dense in \( L^p(B, \mu) \), it is enough to show this when \( B = \mathbb{C}^n \). It is enough to show that for any \( 0 < a < f, g \leq b \) where \( f, g \) are Borel functions on \( \mathbb{C}^n \), the following inequality holds [9, Theorem 2.11].
\[
\int_{\mathbb{C}^n} T_t f(\xi)g(\xi)(2\pi)^{-n}e^{-|\xi|^2/2}d\xi \leq \|f\|_p \|g\|_{q(t)}. \quad (4.11)
\]

Let \( \zeta_t = (\zeta_t^{(1)}, \zeta_t^{(2)}, \ldots, \zeta_t^{(n)})' \), \( 0 \leq t \leq 1 \) be an \( n \)-dimensional standard complex Brownian motion. \( \bar{\zeta}_t \) is the complex conjugate. Let \( \tilde{\zeta}_t \) be an independent copy of \( \zeta_t \). For a given \( 0 < \lambda < 1 \) and \( a \in \mathbb{R} \), define
\[
\tilde{\zeta}_t = \lambda e^{ia\zeta} + \sqrt{1 - \lambda^2} \bar{\zeta}_t. \quad (4.12)
\]
Clearly, \( \tilde{\zeta}_t \) is still a standard complex Brownian motion. Set \( \mathcal{F}_t^\lambda = \sigma(\zeta_s; 0 \leq s \leq t) \), \( \mathcal{F}_t^\lambda = \sigma(\tilde{\zeta}_s; 0 \leq s \leq t) \). Define martingales
\[
M_t = E[f(\zeta_t)|\mathcal{F}_t], \quad 0 \leq t \leq 1, \\
N_t = E[g(\zeta_t)|\mathcal{F}_t], \quad 0 \leq t \leq 1,
\]
where \( \bar{\zeta} \) is the conjugate number of \( \bar{\zeta} \).

It follows from the martingale (on filtrations induced by the complex Brownian motion) representation theorem that
\[
M_t = M_0 + \int_0^t \theta_s d\zeta_s + \int_0^t \theta_s d\bar{\zeta}_s, \quad N_t = N_0 + \int_0^t \phi_s d\zeta_s + \int_0^t \varphi_s d\bar{\zeta}_s.
\]
Since \( M_t, N_t \in \mathbb{R} \), \( \theta_s = \bar{\theta}_s \) and \( \varphi_s = \varphi_s \). It follows from the Itô's table that
\[
dM_t dM_t = 4|\theta_t|^2 dt, \quad dN_t dN_t = 4|\phi_t|^2 dt \quad \text{and} \quad dM_t dN_t = 2\lambda(e^{ia}\theta_t + e^{-ia}\varphi_t)dt.
\]
By Itô's formula, we have
\[
d(M_t^{1/p}N_t^{1/q'}) = \frac{1}{p} M_t^{1/p-1} N_t^{1/q'} dt + \frac{1}{q'} M_t^{1/p} N_t^{1/q'-1} dN_t \\
+ \frac{1}{2p} \frac{1}{q'} (q'-1) M_t^{1/p-1} N_t^{1/q'} dM_t dN_t \\
+ \frac{1}{q'} M_t^{1/p} N_t^{1/q'-1} dM_t dN_t \\
+ \frac{1}{2p} \frac{1}{q'} (q'-1) M_t^{1/p} N_t^{1/q'-2} dN_t dN_t.
\]
Note that $\sqrt{(p-1)(q'-1)} = \lambda$, therefore,

$$E(M_t^{1/p} N_t^{1/q'}) - E(M_0^{1/p} N_0^{1/q'}) = -2E\left[\int_0^t M_t^{1/p-2} N_t^{1/q'} \left(\frac{1}{p} - \frac{1}{q'}\right) N_t^2 |\theta_t|^2 - \frac{1}{p} \frac{1}{q'} \lambda M_t N_t \Re(e^{i\theta_t} \varphi_t) + \frac{1}{q'} (1 - \frac{1}{q'}) M_t^2 |\phi_t|^2 dt\right]$$

$$= -2E\left[\int_0^t M_t^{1/p-2} N_t^{1/q'} - 2 \frac{\sqrt{p-1}}{p} N_t \theta_t - \frac{\sqrt{q'-1}}{q'} e^{i\theta_t} M_t \phi_t\right]^2 dt \leq 0.$$

Let $t = 1$ in the above inequality displayed, we have

$$E(f(\tilde{\zeta}_1) g(\zeta_1)) \leq E[f^{p}(\tilde{\zeta}_1)]^{1/p} E[g^{q'}(\zeta_1)]^{1/q'}.$$

From the definition of $\zeta$ and letting $\lambda = e^{-t \cos \theta}$, $a = -t \sin \theta$, the above inequality displayed is exact (4.11). This ends the proof. \hfill \Box

An argument similar to the one used in [9, Proposition 2.14, 2.15] shows the following boundedness of operator in $L^p(B, \mu) (p > 1)$.

**Corollary 4.7.** $\mathcal{H}_{m,n}$, the Itô-Wiener chaos of degree of $(m, n)$, is a closed subspace in $L^p(B, \mu) (p > 1)$ and its norms $\| \cdot \|_p$ in $L^p(B, \mu) (p > 1)$ are equivalent to each other. In addition, the project operator $J_{m,n}$ is bounded in $L^p(B, \mu) (p > 1)$ and satisfies that

$$J_{m,n} J_{i,j} = J_{i,j} J_{m,n} = \delta_{m,i} \delta_{n,j} J_{m,n} \quad (4.13)$$

$$T_t J_{m,n} = J_{m,n} T_t = e^{-(m+n)t \cos \theta - i(m-n)t \sin \theta} J_{m,n}. \quad (4.14)$$

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