A SYNTACTIC APPROACH TO THE MACNEILLE COMPLETION OF $\Lambda^*$, THE FREE MONOID OVER AN ORDERED ALPHABET $\Lambda$.

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Abstract. Let $\Lambda^*$ be the free monoid of (finite) words over a not necessarily finite alphabet $\Lambda$, which is equipped with some (partial) order. This ordering lifts to $\Lambda^*$, where it extends the divisibility ordering of words. The MacNeille completion of $\Lambda^*$ constitutes a complete lattice ordered monoid and is realized by the system of "closed" lower sets in $\Lambda^*$ (ordered by inclusion) or its isomorphic copy formed of the "closed" upper sets (ordered by reverse inclusion). Under some additional hypothesis on $\Lambda$, one can easily identify the closed lower sets as the finitely generated ones, whereas it is more complicated to determine the closed upper sets. For a fairly large class of ordered sets $\Lambda$ (including complete lattices as well as antichains) one can generate the closure of any upper set of words by means of binary operations ("syntactic rules") thus obtaining an efficient procedure to test closedness. Closed upper set of words are involved in an embedding theorem for valuated oriented graphs. In fact, generalized paths (so-called "zigzags") are encoded by words over an alphabet $\Lambda$. Then the valuated oriented graphs which are "isometrically" embeddable in a product of zigzags have the characteristic property that the words corresponding to the zigzags between any pair of vertices form a closed upper set in $\Lambda^*$.

1. Introduction

Our motivation for studying the MacNeille completion stems from distance-preserving embeddings of graphs into products of path-like graphs. Here is an outline of the embedding question. For undirected graphs there is a universal embedding theorem, due independently to Quilliot [13, 14] and Nowakowski and Rival [11]:

Every undirected graph $G$ isometrically embeds in a product of paths.

"Isometry" and "product" certainly need a word of explanation. Isometry requires preservation of the shortest path distance, that is, any two vertices of $G$ are sent to two vertices at the same distance in the product. The product in question is the strong product, which is the canonical product in the category of undirected graphs with loops; namely, two vertices in such a product are adjacent if and only if all pairs of their corresponding coordinates form edges (which may be loops). Now, for directed graphs (binary relations with edges) the embedding question is more
intricate. The strong product one considers here is, of course, the direct product for reflexive relations, but the potential factors are not just those directed graphs whose symmetric closures constitute undirected paths (see Kabil and Pouzet [7]). Distance and thus isometry have a natural meaning here, too – but one needs to measure distances by sets of words over a two or three letter alphabet rather than numbers. The appropriate notion of distance for directed graphs was introduced by Quilliot [13] and is subsumed in the general approach taken by Jawhari, Misane and Pouzet [5] and further developed by Pouzet and Rosenberg [12]. Let us focus on the case of oriented graphs (with loops), i.e., reflexive antisymmetric binary relations as in this case a two letter alphabet (distinguishing 'forward' and 'backward') will do: an oriented graph is a directed graph in which every pair of vertices is linked by at most one arc. The oriented analogues of undirected complete graphs, for instance, are the tournaments. The oriented versions of undirected paths are called zigzags, see Figure 1.

\[ \text{Figure 1. A zigzag from } a \text{ to } b \]

The distance from the initial to the terminal vertex of a zigzag is not just a number (counting the edges) as in the undirected case but rather the isomorphy class of its homomorphic zigzag pre-images, thus a subset of what we call the ordered monoid of zigzags; this set consists of all zigzags (up to isomorphism) ordered as follows: \( P \preceq Q \) if and only if there is an arc-preserving mapping from the zigzag \( Q \) onto the zigzag \( P \) (which may collapse vertices because of the ubiquity of loops). The multiplication is simply the concatenation of zigzags. The singleton zigzag, i.e. the loop, is the least element as well as the neutral element. Coding forward arcs by ‘+’ and backward arcs by ‘−’ we can identify zigzags with words (i.e., finite sequences) over the alphabet \( \{+,−\} \). For example, the zigzag from \( a \) to \( b \) shown in Figure 1 receives the code \( +++−−+−− \) and the reverse zigzag from \( b \) to \( a \) is coded by \( +−−+++−− \).

Hence the ordered monoid of zigzags is nothing else but the 2-generated free monoid \( \{+,−\}^* \). The ordering of \( \{+,−\}^* \) is the subword or divisibility ordering. Now,
the distance \(d(a,b)\) from vertex \(a\) to vertex \(b\) in any oriented graph \(G\) is the "upper" subset of \(\{+,-\}^*\) consisting of all words coding zigzags which map homomorphically to a subzigzag of \(G\) from \(a\) to \(b\). Every upper set \(Z\) of \(\{+,-\}^*\) (such that every word above some element of \(Z\) belongs to \(Z\)) may occur as a distance in some oriented graph except the set of all nonempty words. This exception simply reflects the hypothesis of antisymmetry, for, if \(+\) as well as \(-\) belong to \(d(a,b)\), then we would get a double arc between \(a\) and \(b\) unless \(a = b\). The set of all words (being the upper set generated by the empty word) then constitutes the "zero" distance.

An isometry (or isometric embedding) of \(G\) into another oriented graph \(H\) is a mapping \(f\) from \(G\) to \(H\) preserving distances, i.e. \(d(f(a), f(b)) = d(a,b)\), and is necessarily injective and preserves arcs. The isometric embedding in products of zigzags is governed by the Galois connection induced by the ordering between zigzags (or words), viz., the MacNeille completion of \(\{+,-\}^*\). First observe that the "lower cone" \(d(a,b)^\top\) formed by the words below all words in a distance \(d(a,b)\) of \(G\) has an obvious interpretation: it consists of the words coding zigzags \(P\) for which there are arc-preserving mappings from \(G\) onto \(P\) sending \(a\) and \(b\) to the initial and terminal vertices of \(P\), respectively. So, if \(a := (a_i)\) and \(b := (b_i)\) are vertices in a product of zigzags, then the words coding the zigzags from the \(a_i\)'s to the \(b_i\)'s in the factors are exactly the members of the lower cone of the distance from \(a\) to \(b\). This merely rephrases the universal property of products in our category. Hence the distance from \(a\) to \(b\) in a product is a "closed" upper set, and therefore every oriented graph \(G\) which isometrically embeds in such a product has MacNeille closed distances.

That the converse is also true, is affirmed by a (more general) result of Jawhari, Misane and Pouzet \[5\], Proposition IV-4.1. This characterization is, however, not yet completely satisfactory because of its partially extrinsic nature: how would we check that a distance is closed other than by computing the lower cone, which consists of the words coding potential zigzag factors? Fortunately, there is an intrinsic way to verify closedness: assume \(y + z\) and \(y - z\) are any two words in a distance \(d(a,b)\) with common circumfix \(y \ldots z\) but different infix letters; then the common subword \(yz\) must belong to \(d(a,b)\) whenever this distance is closed. We refer to this checking procedure as the "cancellation rule". To give an example, assume that a closed upper set \(Z\) of words contains \(++\) and \(--\). Then, as \(Z\) is an upper set, we have both \(++-+\) and \(--++\) in \(Z\), whence \(+--+\) belongs to \(Z\) by the cancellation rule. Further, as \(++--\) is in \(Z\), the rule applied to the latter two words returns \(+--\), and since also \(+++\) is in \(Z\), so must be the common subword \(++\). Interchanging the roles of \(+\) and \(\) we infer that \(Z\) contains \(--\). Repeating essentially the same argument for the words \(++\) and \(--\) proves that \(+\) and \(\) are words in \(Z\), whence \(Z\) contains the empty word and thus is all of \(\{+,-\}^*\).

The category of oriented graphs is not the only one where embeddability in products of certain paths or zigzags can be characterized by closedness of distances. Suitable coding schemes are then necessary in order to capture the kind of adjacency relation for pairs of vertices. Consider, for instance, undirected multigraphs (with loops of unbounded multiplicity) and mappings that do not decrease the multiplicity of edges. Adjacency is coded by the multiplicity of the corresponding edges,
and thus the alphabet $\Lambda$ is linearly ordered here (as the negative integers). If we want to deal with arbitrary directed graphs (with loops), then the appropriate alphabet consists of three letters $+, -, \#$ coding for backward, forward, and two-way arcs, respectively. This alphabet is necessarily ordered so that the letter $\#$ is below $+$ and $-$, which exactly expresses the fact that a two-way arc entails a forward and a backward arc; see Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ordered_alphabets}
\caption{Ordered alphabets for undirected multigraphs and directed graphs, respectively}
\end{figure}

This motivates the use of ordered alphabets. The ordering of letters extends (freely) to an ordering of words, viz., the Higman ordering of $\Lambda^*$ (refining the divisibility ordering). Then, with the right notions of product and path/zigzag, we obtain analogous embedding theorems for multigraphs and directed graphs.

The next section provides the necessary details on the free ordered monoid $\Lambda^*$ (over an ordered alphabet $\Lambda$) and its MacNeille completion. Under some additional assumption on $\Lambda$ the closed lower sets of $\Lambda^*$ other than $\Lambda^*$ are exactly the lower sets generated by finitely many words, see Section 3. A syntactical description of the upper closure (in the MacNeille completion of $\Lambda^*$) is established for particular classes of ordered sets $\Lambda$ in Section 4 (see Theorem 4.4). This applies to the ordered alphabets coding arcs in oriented graphs or multigraphs with bounded multiplicity of edges, respectively, but not to the ordered alphabets displayed in Figure 2.

2. THE FREE ORDERED MONOID AND ITS COMPLETION

An alphabet is a not necessarily finite, ordered set $\Lambda$. Its elements are letters and denoted by small Greek letters $\alpha, \beta, \gamma, \delta, \lambda$ etc. A finite sequence $(\alpha_1, \ldots, \alpha_m)$ of letters is a word of length $m$ and is written as $\alpha_1 \alpha_2 \ldots \alpha_m$. The word of length 0 is the empty word, denoted by $\Box$. The words of length 1 are identified with the corresponding letters. The concatenation of two words $x := \alpha_1 \alpha_2 \ldots \alpha_m$ and $y := \beta_1 \beta_2 \ldots \beta_n$ is the word $xy$ given by

$$xy := \alpha_1 \alpha_2 \ldots \alpha_m \beta_1 \beta_2 \ldots \beta_n.$$
For each \(i\) with \(1 \leq i \leq m\), the word \(\alpha_1 \ldots \alpha_i\) is a **prefix** of \(x = \alpha_1 \ldots \alpha_m\) while the word \(\alpha_i \ldots \alpha_m\) is a **suffix** of \(x\).

The set \(\Lambda^*\) of all words is a **monoid** with respect to concatenation, where the empty word is the neutral element. The order relation of \(\Lambda\), denoted by \(\leq\), extends to \(\Lambda^*\) in the following way:

\[
x = \alpha_1 \alpha_2 \ldots \alpha_m \leq y = \beta_1 \beta_2 \ldots \beta_n
\]

if and only if

\[
\alpha_j \leq \beta_{i_j}\text{ for all } j = 1, \ldots m \text{ with some } 1 \leq i_1 < \ldots i_m \leq n.
\]

That is, \(x\) is below \(y\) in \(\Lambda^*\) exactly when there exists a subword \(\beta_{i_1} \beta_{i_2} \ldots \beta_{i_m}\) of \(y\) which is letter-wise above \(x\) (in the ordering of \(\Lambda^*\)). Then \(\Lambda^*\) becomes an ordered monoid (i.e., \(x \leq y\) implies \(xz \leq yz\) and \(zx \leq zy\)) in which the empty word is the least element. The ordered monoid \(\Lambda^*\) is freely generated by the ordered set \(\Lambda\), see [2], that is, \(\Lambda^*\) is the free object in the category of ordered monoids (whose neutral elements are also the least elements) and order-preserving homomorphisms.

The ordered monoid \(\Lambda^*\) can be extended to a complete lattice ordered monoid by applying the MacNeille completion. The necessary notation (cf. Skornjakow [15]) is introduced next. Let \(X\) be a subset of \(\Lambda^*\); then

\[
\uparrow X := \{ y \in \Lambda^* : x \leq y \text{ for some } x \in X \}
\]

is the **upper set** generated by \(X\) and

\[
\downarrow X := \{ x \in \Lambda^* : x \leq y \text{ for some } y \in X \}
\]

is the **lower set** generated by \(X\). Upper sets and lower sets are **finitely generated** if they are of the form \(\uparrow X\), resp. \(\downarrow X\) for some finite set \(X\). For a singleton \(X = \{x\}\), we omit the set brackets and call \(\uparrow x\) and \(\downarrow x\) a **principal upper set** and a **principal lower set**, respectively. Then

\[
X^\Delta := \bigcap_{x \in X} \uparrow x
\]

and

\[
X^\vee := \bigcap_{x \in X} \downarrow x
\]

are the **upper cone** and the **lower cone** respectively, generated by \(X\).

The pair \((\Delta, \nabla)\) of mappings on \(\mathcal{P}(\Lambda^*)\), the power set lattice of \(\Lambda^*\), constitutes a Galois connection, yieildings the **MacNeille completion** of \(\Lambda^*\). This completion is realized as the complete lattice

\[
\{ W \subseteq \Lambda^* : W = W^\Delta \nabla \}
\]

ordered by inclusion or its isomorphic copy

\[
\{ Y \subseteq \Lambda^* : Y = Y^\nabla \Delta \}
\]

ordered by reverse inclusion. The members of those two sets are said to be (MacNeille) **closed**. The set \(\Lambda^*\) embeds into the former set via \(x \mapsto \downarrow x\) and into the latter via \(x \mapsto \uparrow x\) \((x \in \Lambda^*)\).
To give an example, consider the alphabet $\Lambda := \{+, -\}$ where $+$ and $-$ are incomparable letters. Then:

$$\{+-+--+, ++--+, +++, +++, ++-\} \uparrow = \{+++, +++, ++-\}$$

and

$$\{+-+--+, ++--+, +++, +++, ++-\} \downarrow = \{+-+--+, ++--+, +++, +++, ++-\},$$

showing that the latter upper set $\uparrow \{+-+--+, ++--+, +++, +++, ++-\}$ is closed. In contrast, $\uparrow \{++, -\}$ is not closed since $\{++, -\} \uparrow \Lambda^* = \Lambda^*$.

The completion of $\Lambda^*$ inherits its monoid structure from the power set, where the multiplication is given by

$$XY := \{xy : x \in X, y \in Y\}$$

for any subsets $X$ and $Y$ of $\Lambda^*$. The cone operators preserve this multiplication as the following lemma confirms.

**Lemma 2.1.** For any subsets $X, Y$ of $\Lambda^*$,

$$(XY)^\uparrow = X^\uparrow Y^\uparrow,$$

and $$(XY)^\downarrow = X^\downarrow Y^\downarrow$$

whence $$(XY)^\downarrow = X^\downarrow Y^\downarrow$$

and $$(XY)^\uparrow = X^\uparrow Y^\uparrow$$

**Proof.** First, observe that $\emptyset^\uparrow = \emptyset^\downarrow = \Lambda^*$ and $\Lambda^* \uparrow = \emptyset$, while $\emptyset^\uparrow$ consists of the empty word. Further, $\emptyset Z = Z \emptyset = \emptyset$ for every subset $Z$ of $\Lambda^*$. The inclusions $X^\uparrow Y^\uparrow \subseteq (XY)^\uparrow$ and $X^\downarrow Y^\downarrow \subseteq (XY)^\downarrow$ are then immediate.

Suppose that there exists a word $w$ in $(XY)^\uparrow$ that does not belong to $X^\uparrow Y^\downarrow$. Then let $u$ be the longest prefix of $w$ from $X^\uparrow$, and let $v$ be the longest suffix of $w$ from $Y^\downarrow$ so that $w$ is of the form

$$w = u\alpha_1 \ldots \alpha_k v$$

for some letters $\alpha_1, \ldots, \alpha_k$, where $k \geq 1$. By the choice of $u$ and $v$, there are words $x \in X$ and $y \in Y$ such that

$$u\alpha_1 \not\in x \text{ and } \alpha_k v \not\in y.$$ 

This, however, is in conflict with

$$w = u\alpha_1 \ldots \alpha_k v \leq xy.$$ 

Therefore $(XY)^\uparrow$ equals $X^\uparrow Y^\downarrow$. Finally, suppose $z$ is a word in $(XY)^\downarrow$ which does not belong to $X^\downarrow Y^\uparrow$, where $X$ and $Y$ are nonempty. Then the shortest prefix of $z$ from $X^\downarrow$ and the shortest suffix of $z$ from $Y^\uparrow$ intersect in a nonempty subword

$$w := \alpha_1 \ldots \alpha_k$$

so that $z$ can be written as

$$z = uwv \text{ with } uw \in X^\downarrow \text{ and } vw \in Y^\downarrow.$$ 

By the choice of the words $u$ and $v$, we can find words $x \in X$ and $y \in Y$ with

$$x \not\in u\alpha_1 \ldots \alpha_{k-1} \text{ and } y \not\in v.$$
This contradicts the hypothesis that

\[ xy \leq z = u\alpha_1 \ldots \alpha_{k-1}\alpha_k v. \]

We conclude that \((XY)^\Delta = X^\Delta Y^\Delta\), completing the proof.

The completion of \(\Lambda^*\), realized by the upper closed sets, is a complete lattice in which suprema are set-theoretic intersections, whereas infima are the closures of set-theoretic unions. The closed union of a family \(Z_i (i \in I)\) of upper sets in \(\Lambda^*\) is given by:

\[
\bigcup_{i \in I} Z_i = (\bigcup_{i \in I} Z_i)^\Delta.
\]

The following result entails that the completion of \(\Lambda^*\) is a complete lattice ordered monoid (in the sense of Birkhoff [1]).

**Proposition 2.2.** For any ordered alphabet \(\Lambda\), the collection of all closed upper sets of words over \(\Lambda\) is a monoid and complete lattice such that the multiplication distributes over intersection and closed unions, that is

\[
Y(\bigcap_{i \in I} Z_i) = \bigcap_{i \in I} YZ_i \quad \text{and} \quad (\bigcap_{i \in I} Z_i)Y = \bigcap_{i \in I} Z_i Y,
\]

\[
Y(\bigcup_{i \in I} Z_i) = \bigcup_{i \in I} YZ_i \quad \text{and} \quad (\bigcup_{i \in I} Z_i)Y = \bigcup_{i \in I} Z_i Y
\]

for any index set \(I\) and all closed upper sets \(Y, Z_i (i \in I)\).

**Proof.** Since \(Y\) and all \(Z_i\) are closed and \((\Delta, \triangledown)\) is a Galois connection, we have \(Y = Y^\triangledown\Delta\) and \(\bigcup_{i \in I} Z_i = (\bigcup_{i \in I} Z_i)^\triangledown\Delta = (\bigcap_{i \in I} Z_i^\triangledown)^\Delta\). Analogous formulae hold for \(Y^\triangledown\) and \(Z_i^\triangledown (i \in I)\). Hence by Lemma 2.1

\[
Y(\bigcup_{i \in I} Z_i) = Y^\triangledown\Delta(\bigcup_{i \in I} Z_i)^\triangledown\Delta = (Y(\bigcup_{i \in I} Z_i))^\triangledown\Delta =
\]

\[
= (\bigcup_{i \in I} YZ_i)^\triangledown\Delta = \bigcup_{i \in I}(YZ_i)^\triangledown\Delta = \bigcup_{i \in I}Y^\triangledown\Delta Z_i^\triangledown\Delta = \bigcup_{i \in I}YZ_i.
\]

Further

\[
Y(\bigcap_{i \in I} Z_i) = Y^\triangledown\Delta(\bigcap_{i \in I} Z_i)^\triangledown\Delta = (Y^\triangledown \bigcup_{i \in I} Z_i^\triangledown)^\Delta = (\bigcup_{i \in I} Y^\triangledown Z_i^\triangledown)^\Delta = \bigcap_{i \in I} Y^\triangledown Z_i^\triangledown = \bigcap_{i \in I} YZ_i.
\]

This settles left distributivity; the proof of right distributivity is analogous.

Note that the collection of all closed upper sets of words over \(\Lambda\) is in fact a free monoid, see [8].
3. Closed lower sets

In this section, we describe the closed lower sets. The characterization of the closed upper sets is more involved and shall occupy us for the rest of the next section.

Finiteness assumptions on the alphabet $\Lambda$ allow to argue by induction or to obtain finite generation. $\Lambda$ is said to be well-founded (or to satisfy the descending chain condition, DCC for short, [1]) if $\Lambda$ does not contain any infinite decreasing chain $\lambda_0 > \lambda_1 > \ldots$. Then $\Lambda$ is called well-quasi-ordered if $\Lambda$ is well-founded and has no infinite antichain (that is, contains no infinite subset of pairwise incomparable elements). We recall a fundamental result.

**Theorem 3.1.** (Higman [3]) If $\Lambda$ is well-quasi-ordered then $\Lambda^*$ is well-quasi-ordered too, whence the complete lattice of all lower sets of $\Lambda^*$ is well-founded, which means that every upper set in $\Lambda^*$ is finitely generated.

For a proof see also Nash-Williams [10] or Cohn [2]. For more information on well-quasi-ordered sets, see the survey paper of Milner [9]. If $\Lambda$ is well-quasi-ordered, then by virtue of Higman’s Theorem the MacNeille completion of $\Lambda^*$ as realized within the complete lattice of all lower sets is necessarily well-founded. However, this completion can be well-founded even when $\Lambda$ contains infinite antichains. Well-founded dual forests constitute pertinent examples, as will be seen next.

An ordered set $\Lambda$ is called an ordered tree if every principal lower set of $\Lambda$ is a chain and $\Lambda$ is down-directed (that is, any two elements of $\Lambda$ are bounded below). An ordered forest is a disjoint union of ordered trees. The dual (alias opposite) of an ordered set is obtained by reversing the order relation. Observe that the dual of an ordered forest (a dual forest, for short) is just an ordered set in which any two incomparable elements are incompatible, i.e., not bounded below (Figure 3).

![Figure 3. A (finite) dual forest](image)

Consider an ordered set $\Lambda$ that is well-quasi-ordered and the dual of an ordered forest. Since $\Lambda$ is well-founded, every element is above some minimal element. Let $K$ be the subset of $\Lambda$ consisting of all existing joins of minimal elements. Then, as $\Lambda$ has no infinite antichain, $K$ is a dual finite forest. It is not difficult to see that $\Lambda$ is the lexicographic sum of a family of ordinals indexed by $K$. Note that adding a least element to $\Lambda$ (if necessary) results in a complete lattice.
It will turn out (see Theorem 3.4 below) that the finitely generated lower sets of $\Lambda^*$ together with $\Lambda^*$ are exactly the closed lower sets in any well-founded dual forest $\Lambda$. Two lemmas are needed to establish this.

**Lemma 3.2.** Let $\Lambda$ be an ordered set. Then all (MacNeille) closed lower sets of $\Lambda^*$ different from $\Lambda^*$ are finitely generated lower sets if and only if $\Lambda$ is well-founded and the intersection of any two principal lower sets of $\Lambda$ is a finitely generated lower set. Hence, in this case, the MacNeille completion of $\Lambda^*$ is necessarily well-founded.

**Proof.** $\Lambda$ can be regarded as a lower set of $\Lambda^* \setminus \{\boxempty\}$. Therefore, if $\Lambda^*$ is well-founded, so is $\Lambda$. The intersection of any two principal lower sets in $\Lambda$ is closed, whence this is a finitely generated lower set under the assumption that all closed lower sets in $\Lambda^*$ other than $\Lambda^*$ be finitely generated. Restricting this intersection to $\Lambda$ amounts to removing the empty word. This establishes necessity of the conditions on $\Lambda$.

To prove the converse, assume that $\Lambda$ is well-founded such that any two principal lower sets of $\Lambda$ intersect in a finitely generated lower set. We extend the original alphabet $\Lambda$ to the set $\overline{\Lambda} := \Lambda \cup \{\boxempty\}$, where $\boxempty$ becomes the last element of the extended alphabet. The set $\overline{\Lambda}^*$ of words over $\overline{\Lambda}$ is the union of $\overline{\Lambda}^n$ for all $n \geq 0$ (here the empty word is distinct from $\boxempty$). There is a canonical map $\varphi$ from $\overline{\Lambda}^*$ onto $\Lambda^*$ which "forgets" the empty letter $\boxempty$, viz., $\varphi$ maps the empty tuple to $\boxempty$ and a nonempty tuple $x$ from $\overline{\Lambda}^n$ $(n > 0)$ to the concatenation of its coordinates with respect to the indexing order. For instance, both tuples $(\boxempty, +, \boxempty, -)$ and $(+, -, \boxempty)$ from $\{+, -, \boxempty\}^*$ are mapped to $-+$ under $\varphi$. Thus, the pre-images under $\varphi$ of a fixed word differ only in the number and positions of the empty letter $\boxempty$. The map $\varphi$ obviously is a monoid homomorphism such that for any two words $w$ and $x$ in $\Lambda^*$ we have $w < x$ exactly when for each tuple $x'$ in the pre-image of $x$ under $\varphi$ there exist a tuple $w'$ in the pre-image of $w$ such that $w' < x'$.

First, we claim that $\Lambda^*$ is well-founded because $\Lambda$ is. Trivially, $\overline{\Lambda}$ is well-founded and hence so any of its finite Cartesian powers $\overline{\Lambda}^n$ $(n > 0)$ by virtue of the pigeonhole principle. If there was an infinite descending chain $x_0 > x_1 > x_2 > \ldots$ in $\Lambda^*$ starting with some word $x_0$ of length $n$, then we could lift this chain to $\overline{\Lambda}^n$ by selecting $x_0' = x_0$ and successively choosing tuples in $\overline{\Lambda}^n$ with $x_1' < x_0', x_2' < x_1', \ldots$, contrary to the observation that $\overline{\Lambda}^n$ is well-founded.

Second, we assert that any two principal lower sets $\downarrow w$ and $\downarrow x$ of $\Lambda^*$ intersect in a finitely generated lower set. This is true in the particular case that $w$ and $x$ belong to $\overline{\Lambda}$ because of the corresponding property assumed for the alphabet $\Lambda$. If $w' := (w_1, \ldots, w_n)$ and $x' := (x_1, \ldots, x_n)$ belong to $\overline{\Lambda}^n$ $(n > 0)$, then $\downarrow w' \cap \downarrow x'$ is simply the Cartesian product of $\downarrow w_i \cap \downarrow x_i$ for $i = 1, \ldots, n$, whence as a product of finitely generated lower sets of $\overline{\Lambda}$ it is a finitely generated lower set of $\overline{\Lambda}^n$. Now, if $w$ and $x$ are words of length at most $n$, then we can take corresponding tuples $w'$ and $x'$ in $\overline{\Lambda}^n$ which are mapped to $w$ and $x$ by $\varphi$. Since $\varphi$ maps lower sets onto lower sets, we infer that $\downarrow w \cap \downarrow x$ is a finitely generated lower set of $\Lambda^*$.
Third, we claim that for every finite subset $Z$ of $\overline{\Lambda}^*$, the lower cone $Z^\vee$ is a finitely generated lower set. If $Z$ has cardinality at most 2, this has just been established. Now, by an induction hypothesis, for any $y$ in $Z$, there is a finite antichain $X$ in $\Lambda^*$ such that $(Z \setminus \{y\})^\vee = \downarrow X$. Then $\downarrow X \cap \downarrow y$ equals the union of all $\downarrow x \cap \downarrow y$ for $x$ from $X$ and thus is a finitely generated lower set, as required.

Fourth, a result of Birkhoff [1], Theorem 2, p. 182, states that the set of finitely generated lower sets of any well-founded ordered set $P$ is well-founded. Hence, the set of finitely generated initial segments of $\Lambda^*$ is well-founded. From this and the well-foundedness of $\Lambda^*$ we derive that every closed lower set $X$ other than $\Lambda^*$ is finitely generated. Consider the collection of all lower cones of the form $Z^\vee$ where $Z$ is a finite subset of the upper cone $\Delta\downarrow X$ (so that $X \subseteq Z^\vee$). This collection is nonempty because $X \neq \Lambda^*$, and it contains some minimal member $Y^\vee$. Suppose we could find $w \in Y^\vee \setminus X$. Then $w \not\leq z$ for some $z \in X^\Delta$ because $X$ is closed. Now, by minimality of $Y^\vee$ we have

$$w \in Y^\vee = (Y \cup \{z\})^\vee \subseteq \downarrow z,$$

giving a contradiction. This completes the proof. \qed

Note that every closed upper set of $\Lambda^*$ is of the form $Y^\vee\Delta$ for some finite subset $Y$ whenever the MacNeille completion of $\Lambda^*$ is well-founded. Observe that $Y^\vee\Delta$ need not be a finitely generated upper set. Lemma 3.2 applies, in particular, to a well-founded conditional lattice $\Lambda$ (such as a well-founded dual forest), yielding the finiteness conditions for the MacNeille completion of $\Lambda^*$. Here we say that an ordered set $\Lambda$ is a conditional lattice if it is obtained from a bounded lattice by removing the bounds. In other words, $\Lambda$ is a conditional lattice if and only if every pair of elements bounded below has a meet and every pair of element bounded above has a join.

**Lemma 3.3.** Let $\Lambda$ be an ordered set. Then every finitely generated lower set in $\Lambda^*$ is (MacNeille) closed if and only if each pair of letters from $\Lambda$ that is bounded below is also bounded above.

**Proof.** Let $\alpha, \beta, \lambda$ be letters such that $\lambda < \alpha, \lambda < \beta$, but $\alpha, \beta$ do not have an upper bound. Consider the lower set $W := \downarrow \{\alpha, \beta\}$ in $\Lambda^*$. Since $\{\alpha, \beta\}$ is not bounded above, every word above $\alpha$ and $\beta$ is above $\alpha \beta$ or $\beta \alpha$. Hence $W^\Delta = \uparrow \{\alpha \beta, \beta \alpha\}$. Then the word $\lambda \lambda$ belongs to $W^\Delta^\vee$ but not to $W$, showing that $W$ is not closed.

Conversely assume that $\Lambda$ satisfies the condition of the lemma. Let $w, x, y$ be words in $\Lambda^*$ such that $w$ does not belong to $\downarrow \{x, y\}$, that is,

$$w \not\in x \text{ and } w \not\in y.$$

We claim that $w \not\in \{x, y\}\Delta^\vee$, that is, there exists a word $z$ such that

$$x \leq z, y \leq z, \text{ and } w \not\leq z.$$

Assume that $w = \alpha_1 \ldots \alpha_n$ with $\alpha_i \in \Lambda$. Let $x_n$ be the (possibly empty) largest suffix of $x$ consisting only of letters not above $\alpha_n$. If $\alpha_n \not\leq x$, then $x_n = x$. Otherwise,
there exists some $\beta_n \in \Lambda$ such that $\alpha_n \leq \beta_n$ and $x$ is of the form $x = u\beta_n x_n$ for some (possibly empty) word $u$. Then $n \geq 2$ and $\alpha_1 \ldots \alpha_{n-1} \notin u$. We continue as before, so that we eventually obtain a representation of $x$ as

$$x = x_i \beta_{i+1} x_{i+1} \ldots x_{n-1} \beta_n x_n,$$

where $1 \leq i \leq n$, and

$$\alpha_k \notin x_k \text{ for } i \leq k \leq n,$$
$$\alpha_k \leq \beta_k \text{ for } i < k \leq n.$$

Then $\alpha_{i+1} \ldots \alpha_n$ is the largest suffix of $w$ that is below some subword $\beta_{i+1} \ldots \beta_n$ of $x$, where in addition $\beta_{i+1} \ldots \beta_n$ is the right-most subword of $x$ with this property. Similarly, we have a representation

$$y = y_j \gamma_{j+1} y_{j+1} \ldots y_{n-1} \gamma_n y_n,$$

where $1 \leq j \leq n$, and

$$\alpha_k \notin y_k \text{ for } j \leq k \leq n,$$
$$\alpha_k \leq \gamma_k \text{ for } j < k \leq n.$$

We may assume that $i \leq j$. Now, by the condition on $\Lambda$, we can find a letter $\lambda_k$ such that

$$\beta_k \leq \lambda_k \text{ and } \gamma_k \leq \lambda_k \text{ whenever } j \leq k \leq n.$$

Put

$$z = x_i \beta_{i+1} \ldots x_{j-1} \beta_j x_j \lambda_{j+1} \ldots x_{n-1} y_{n-1} \lambda_n x_n y_n.$$

Then $x \leq z$ and $y \leq z$, but $w \notin z$ by the choice of the $x_k$ and $y_k$. This proves the claim. Now, by a trivial induction we get that for any words $w, x_1, \ldots, x_m$ with $w \notin x_k$ for all $k$ there exists a word $z$ such that $w \notin z$ and $x_k \leq z$ for all $k$. So, if $X = \downarrow \{x_1, \ldots, x_m\}$ is some finitely generated set and $w \in \Lambda^* \setminus X$, then there exists $z \in \{x_1, \ldots, x_m\}^\Delta$ such that $w \notin \downarrow z$, that is, $w \notin X^\Delta$. This proves that $X^\Delta \subseteq X$, whence $X$ is closed.

The preceding lemma covers the result of Jullien \[4\] for unordered finite alphabets (i.e., in the case that $\Lambda$ is a finite antichain); see also Kabil and Pouzet \[6\], Proposition 2.2.

Recall that a pair of elements $\alpha, \beta \in \Lambda$ is compatible if these elements have a common lower bound.

**Theorem 3.4.** Let $\Lambda$ be an ordered set. Then the (MacNeille) closed lower sets of $\Lambda^*$ form a well-founded lattice which exactly comprises $\Lambda^*$ and all finitely generated lower sets if and only if $\Lambda$ is well-founded and every compatible pair of (incomparable) elements $\alpha, \beta \in \Lambda$ is bounded above and the common lower bounds of $\alpha, \beta$ form a finitely generated lower set. In particular, the ordered set $\Lambda$ obtained from some disjoint union of well-founded lattices by removing the antichain of minimal elements is of this kind.
Proof. From Lemma 3.2 we infer that every closed lower set \( W \neq \Lambda^* \) in \( \Lambda^* \) is finitely generated. Conversely, a finitely generated lower set of \( \Lambda^* \) is closed by Lemma 3.3 since \( \Lambda \) satisfies the hypothesis of this Lemma.

4. Closed upper sets

Given a set \( Y \) of words over an ordered set \( \Lambda \), we wish to build up its closure \( Y^{\nabla \Delta} \) by successively applying a few (partial) binary operations and taking upper sets (which, of course, is governed by a family of unary operations indexed by \( \Lambda^* \)). Certainly, one cannot circumvent some finiteness condition on \( Y \) as the MacNeille completion is inherently infinitary. Since we reserve the name ”closed upper set” for members of this completion, we say that \( Z \subseteq \Lambda^* \) is stable with respect to a partial operation \( f \) defined on \( D(f) \subseteq \Lambda^* \times \Lambda^* \) if

\[
(x, x') \in D(f) \cap (Z \times Z) \implies f(x, x') \in Z.
\]

Lemma 4.1. Let \( \Lambda \) be an ordered set. Then every closed upper set \( Z \) in \( \Lambda^* \) is stable with respect to the four partial binary operations ”cancellation”, ”reduction”, ”permutation”, and ”meet”:

(cancellation rule) if \( y\alpha z \in Z \) and \( y\beta z \in Z \) where \( \alpha, \beta \) are incompatible letters (that is, not bounded below) and \( y, z \in \Lambda^* \), then \( yz \in Z \);

(reduction rule) if \( y\alpha\alpha z \in Z \) and \( y\gamma z \in Z \) for \( \alpha < \gamma \) in \( \Lambda \) and \( y, z \in \Lambda \), then \( yaz \in Z \);

(permutation rule) if \( y\alpha\beta z \in Z \) and \( y\gamma z \in Z \) where \( \alpha, \beta, \gamma \in \Lambda \) and \( y, z \in \Lambda^* \) such that \( \alpha, \beta \) are incomparable and below \( \gamma \), then \( y\beta\alpha z \in Z \);

(meet rule) if \( y\alpha z \in Z \) and \( y\beta z \in Z \) such that \( \alpha, \beta \in \Lambda \) are incomparable letters with meet \( \alpha \land \beta \) in \( \Lambda \) and \( y, z \in \Lambda^* \), then \( y(\alpha \land \beta)z \in Z \).

Proof. Let \( u, v, y, z \in \Lambda^* \) such that \( yuz, yvz \in Z \). Then, according to Lemma 2.1, we obtain

\[
\{yuz, yvz\}^{\nabla} = (y\{u, v\}z)^{\nabla} = (\downarrow y)\{u, v\}^{\nabla}(\downarrow z)
\]

and hence

\[
\{yuz, yvz\}^{\nabla \Delta} = (\uparrow y)\{u, v\}^{\nabla \Delta}(\uparrow z).
\]

Since \( Z \) is a closed upper set, the preceding upper cone is included in \( Z \), that is,

\[
ywz \in Z \text{ for all } w \in \{u, v\}^{\nabla \Delta}.
\]

This applies to each of the four asserted rules. In each case the closure of \( \{u, v\} \) is readily determined: if \( \alpha \) and \( \beta \) are incompatible, then \( \{\alpha, \beta\}^{\nabla \Delta} = \Lambda^* \); and if \( \alpha \land \beta \) exists then \( \{\alpha, \beta\}^{\nabla \Delta} = \uparrow (\alpha \land \beta) \). For \( \alpha < \beta \) we get \( \{\alpha\alpha, \beta\}^{\nabla \Delta} = \uparrow \alpha \). Finally, if \( \alpha \) and \( \beta \) are incomparable such that \( \alpha, \beta < \gamma \), then

\[
\{\alpha\beta, \gamma\}^{\nabla \Delta} = (\downarrow \alpha \cup \downarrow \beta)^{\Delta} = \{\alpha, \beta\}^{\Delta} = (\uparrow \alpha \cap \uparrow \beta \cap \Lambda) \cup \{\alpha\beta, \beta\alpha\},
\]
which equals \( \uparrow \{ \alpha \beta, \beta \alpha, \alpha \lor \beta \} \) whenever the join \( \alpha \lor \beta \) exists. This completes the proof of the lemma.

The final argument in the preceding proof actually yields an extension of the permutation rule that also entails the reduction rule, viz.

\((\text{permuto-reduction rule})\) if \( y\alpha \beta z \in Z \) and \( y\gamma z \in Z \) where \( \alpha, \beta, \gamma \in \Lambda \) and \( y, z \in \Lambda^* \) such that \( \alpha, \beta \) are incomparable and below \( \gamma \), then \( y\beta \alpha z \in Z \) and \( y\delta z \in Z \) for all \( \delta \in \Lambda \) with \( \alpha, \beta < \delta \).

One can also derive the second assertion in this rule from the reduction rule: if \( \alpha, \beta < \delta < \gamma \) such that \( y\alpha \beta z \in Z \) and \( y\gamma z \in Z \) then \( y\delta \beta z \in Z \) and hence \( y\delta z \in Z \) by the reduction rule.

The cancellation rule and the meet-rule can be regarded as a single rule with respect to the meet in \( \Lambda^* \):

\((\text{extended meet rule})\) if \( yaz \in Z \) and \( y\beta z \in Z \) such that \( \alpha, \beta \in \Lambda \) are incomparable letters such that their meet \( w \) in \( \Lambda^* \) exists, then \( ywz \in Z \).

In fact, this meet exists exactly when \( \alpha \) and \( \beta \) either are incompatible (so that \( w = \square \)) or have a meet \( w = \alpha \land \beta \) in \( \Lambda \). Hence, any two incomparable letters have a meet in \( \Lambda^* \) if and only if \( \Lambda \) is a \textit{conditional meet-semilattice}, that is, every pair of compatible elements (i.e., bounded below) has a meet.

**Lemma 4.2.** Let \( \Lambda \) be a conditional meet-semilattice. An upper set \( Z \) in \( \Lambda^* \) is stable with respect to cancellation, reduction, permutation, and meet precisely when \( Z \) obeys the following "compound" rule: if \( y\alpha_1 \ldots \alpha_n z \in Z(n \geq 1) \) and \( y\beta z \in Z \) such that \( y, z \in \Lambda^* \) and \( \alpha_i, \beta \in \Lambda \) with \( \beta \notin \alpha_i \) for all \( i \), then \( ytz \in Z \) where \( t \) is a word (possibly empty) formed by the maximal elements of \( \{ \alpha_i \land \beta : i = 1, \ldots, n \text{ such that } \alpha_i \land \beta \text{ exists} \} \) in any order.

**Proof.** Evidently the rules described in Lemma 4.1 are particular instances of the compound rule. To prove the converse, assume first that there is some letter \( \alpha_i \) incompatible with \( \beta \). Then, as \( Z \) is an upper set containing \( y\beta z \), the word \( y\alpha_1 \ldots \alpha_{i-1} \beta \alpha_{i+1} \ldots \alpha_n z \) belongs to \( Z \), whence so does \( y\alpha_1 \ldots \alpha_{i-1} \alpha_{i+1} \ldots \alpha_n z \) by virtue of the cancellation rule. Continuing this way we can eliminate all letters \( \alpha_i \) from the subword \( \alpha_1 \ldots \alpha_n \) in \( y\alpha_1 \ldots \alpha_n z \) that are incompatible with \( \beta \), thus resulting in \( y\lambda_1 \ldots \lambda_k z \in Z \) where \( \lambda_1 \ldots \lambda_k \) is a subword of \( \alpha_1 \ldots \alpha_n \). Since \( y\lambda_1 \ldots \lambda_k \beta z \in Z \), the meet rule gives \( y\lambda_1 \ldots \lambda_{k-1} (\lambda_k \land \beta) z \in Z \). Iterating this argument yields \( y\mu_1 \ldots \mu_k z \in Z \) with \( \mu_i = \lambda_i \land \beta \) for all \( i \). In a similar way we successively apply the reduction and permutation rules: as every permutation of a word is the composition of transpositions interchanging two consecutive letters, it suffices to manipulate the letters \( \mu_i, \mu_{i+1} \) for \( i = 1, \ldots, k - 1 \). If \( \mu_i \leq \mu_{i+1} \), then both \( y\mu_1 \ldots \mu_{i-1} \mu_{i+1} \mu_{i+2} \ldots \mu_k z \) and \( y\mu_1 \ldots \mu_{i-1} \beta \mu_{i+2} \ldots \mu_k z \) belong to \( Z \), whence \( y\mu_1 \ldots \mu_{i-1} \mu_{i+1} \ldots \mu_k z \) by the reduction rule. If \( \mu_i \notin \mu_{i+1} \), then the permutation rule
guarantees $y\mu_1 \ldots \mu_{i-1}\mu_{i+1}\mu_{i+2} \ldots \mu_k z \in Z$. This finally, shows that $ytz$ is in $Z$. □

The next lemma we need is the analogue of Lemma 2.1 for stable sets.

**Lemma 4.3.** If $U$ and $V$ are two upper sets that are stable with respect to cancellation, reduction, permutation, and meet, then the concatenation $UV$ is stable as well.

**Proof.** Let $s, y, z \in \Lambda^*$ and $\lambda \in \Lambda$ such that $ysz$ and $y\lambda z$ are words for which the compound rule, say, would return the word $ytz$. Assume $ysz, y\lambda z \in UV$. We wish to show that $ytz \in UV$. Since $U, V$ are upper sets, we infer from $y\lambda z \in UV$ either $y \in U$ or $z \in V$; say, the latter holds. If $y \in U$, then $yt \in U$ and hence $ytz \in UV$. So assume that $y$ does not belong to $U$. Let $v$ be the shortest suffix of $z$ belonging to $V$ such that $z$ is of the form $xv$ with $y\lambda x \in U$. Then, by the minimal choice of $v$, the word $ysx$ belongs to $U$ as well (because $U$ is an upper set) and hence $yt \in U$ as $U$ is stable. We conclude that $ytz = ytxv \in UV$, as required. □

We are now in position to prove the main theorem. For every subset $Y$ of $\Lambda^*$ let $[Y]$ denote the smallest upper set of words which contains $Y$ and is stable with respect to cancellation, reduction, permutation, and meet (as described in Lemma 4.1). Then by this lemma we have $[Y] \subseteq Y^{\land \Delta}$.

**Theorem 4.4.** Let $\Lambda$ be an ordered set in which any two elements bounded below have a meet and an upper bound. Then for every finite nonempty subset $Y$ of $\Lambda^*$ the smallest closed upper set and the smallest stable upper set containing $Y$ coincide: $Y^{\land \Delta} = [Y]$. If, in addition, $\Lambda$ is well-founded, then the closed upper sets of $\Lambda^*$ are exactly the stable upper sets.

**Proof.** We will show that $[Y]$ is closed for all finite nonempty sets $Y \subseteq \Lambda^*$ by induction. To this end, define the total length $||Y||$ of $Y$ as the sum of the lengths of the words in $Y$. If $||Y|| = 0$, that is, $Y$ consists only of the empty word, then we get $[Y] = \Lambda^*$. So let $n =||Y|| \geq 1$, and assume that $[X]$ is closed for all nonempty sets $X \subseteq \Lambda^*$ with $||X|| < n$. If $Y$ is not an antichain, then $\uparrow Y = \uparrow X$ for some proper subset $X$ of $Y$ (giving $||X|| < ||Y||$), whence $[Y] = [X]$ is closed by the induction hypothesis. Therefore we can assume that $Y$ is an antichain. We aim at representing $Z := [Y]$ as an intersection of concatenations of stable upper sets to which the induction hypothesis applies.

Consider the set $K$ of front letters (i.e., prefixes of length 1) of the words in $Y$. For $\delta \in K$ and any $W \subseteq \Lambda^*$ let $W_\delta$ be the set of words obtained from $W$ by cancelling all front letters $\delta$, that is, $x \in W_\delta$ if and only if either $\delta x \in W$, or $x \in W$ and $\delta$ is not a prefix of $x$. In case that $W$ is an upper set we simply have $W_\delta = \{ x \in \Lambda^* : \delta x \in W \}$. It is easy to see (by putting $\delta$ in front of all words in question) that each $W_\delta$ is a stable upper set whenever $W$ is a stable upper set. In particular, $Z_\delta$ is a stable upper set containing $Y_\delta$ (for $\delta \in K$). Therefore, as $Z$ is an upper set and $\uparrow Y \subseteq \uparrow Y_\delta$ holds, we obtain the following inclusions

$$(\uparrow \delta)Z_\delta \subseteq Z = [Y] \subseteq [Y_\delta] \subseteq Z_\delta \text{ for all } \delta \in K.$$
Furthermore, $\|Y_\delta\| < \|Y\|$, and consequently $[Y_\delta]$ is closed by virtue of the induction hypothesis. These facts will be used in each of the subsequent cases (without explicit mention).

**Case 1.** $K$ is a singleton $\{\delta\}$.
This means that all words in $Y$ have the letter $\delta$ in front, that is, $Y = \delta Y_\delta$. Since the concatenation of stable upper sets is a stable upper set by Lemma 4.3, it follows

$$(\uparrow \delta)Z_\delta \subseteq Z = [\delta Y_\delta] \subseteq (\uparrow \delta)[Y_\delta] \subseteq (\uparrow \delta)Z_\delta,$$

and thus equality holds throughout. Then $Z = (\uparrow \delta)[Y_\delta]$ is closed by Lemma 2.1.

**Case 2.** $K$ is not bounded below in $\Lambda$.
Let $\lambda$ be the meet of a maximal subset of $K$. Then there is a letter $\mu$ in $K$ incompatible with $\lambda$. Now if $x \in \bigcap_{\delta \in K} Z_\delta$, then $\delta x \in Z$ for all $\delta \in K$. Applying the meet rule several times, we eventually get $\alpha x \in Z$. Since $\mu x \in Z$, the cancellation rule returns $x \in Z$, thus proving that $Z$ contains the intersection of all $Z_\delta$ ($\delta \in K$). On the other hand, we already know that $Z \subseteq (\uparrow \alpha)\bigcup_{\delta \in K} Y_\delta$. Therefore $Z$ equals the intersection of all $[Y_\delta]$ ($\delta \in K$) and hence is closed.

**Case 3.** $K$ is not a singleton, but bounded below.
Then the meet $\alpha$ of $K$ in $\Lambda$ exists, and the hypothesis on $\Lambda$ guarantees an upper bound $\beta$ of $K$. Necessarily, $\alpha < \beta$. Remove the front letters from all words in $Y$, which results in the set

$$Y_K = \{x \in \Lambda^* : \delta x \in Y \text{ for some } \delta \in K\} = \bigcup_{\delta \in K} Y_\delta.$$

Since $\|Y_\delta\| < \|Y\|$, the stable set $[Y_K]$ must be closed. Note that

$$Y \subseteq (\uparrow \alpha)Y_K \text{ and } Y_K \subseteq Z_\beta$$

as $\alpha \leq \delta \leq \beta$ for all $\delta \in K$. Hence, by Lemma 4.3

$$Z \subseteq [(\uparrow \alpha)Y_K] \subseteq (\uparrow \alpha)[Y_K] \subseteq (\uparrow \alpha)Z_\beta.$$

Now, applying the meet rule successively, we get $\alpha x \in Z$ whenever $\delta x \in Z$ for all $\delta \in K$ (where $x \in \Lambda^*$). Therefore

$$Z \subseteq \bigcap_{\delta \in K} [Y_\delta] \subseteq \bigcap_{\delta \in K} Z_\delta \subseteq Z_\alpha.$$

Combining both chains of inclusions yields

$$Z \subseteq \bigcap_{\delta \in K} [Y_\delta] \cap ((\uparrow \alpha)[Y_K]) \subseteq Z_\alpha \cap ((\uparrow \alpha)Z_\beta).$$

To prove the converse inclusion, assume $x \in Z_\alpha \cap ((\uparrow \alpha)Z_\beta)$. Then $\alpha x \in Z$ and $x = wy$ for some word $w \geq \alpha$ and $y \in Z_\beta$. If $w \geq \beta$, then $x \in Z$ follows immediately. So, let $\beta \notin w$. Writing $w = u\gamma v$ with $u, v \in \Lambda^*$ and a letter $\gamma \geq \alpha$, we have $\alpha u\gamma vy \in Z$. On the other hand, we have $\gamma \in K$. If $\gamma \notin \beta$, then $\gamma \in Z_\delta$ (for some $\delta \in K$) and $\alpha u\gamma vy \in Z_\delta \subseteq Z_\beta$, which is a contradiction. Therefore, $\gamma = \beta$ and $\beta \notin w$. Writing $w = uv$, we have

$$\alpha uv \in Z_\beta \subseteq Z_\delta \subseteq Z_\alpha \subseteq Z_\alpha \cap ((\uparrow \alpha)Z_\beta),$$

which proves that $Z_\alpha \cap ((\uparrow \alpha)Z_\beta)$ is closed. Therefore, $Z = (\uparrow \alpha)[Y_K]$ is closed by Lemma 2.1.
and $\beta y \in Z$. Now we can apply the compound rule (according to Lemma 4.2) and thus obtain $u'((\beta \land \gamma)v'y \in Z$, where $u'$ and $v'$ are some words below $u$ resp. $v$ (and only comprising letters below $\beta$) and $\alpha$ got removed because $\alpha \leq \beta \land \gamma$. Hence $x = u'yv' \in Z$, as required. Hence $Z$ is equal to the intersection of all $[Y_{\delta}]$ with $(\uparrow \alpha)[Y_K]$. Since the latter set is closed by Lemma 4.3, so is $Z$.

This completes the induction and thus establishes the equality of $[Y]$ and $Y^{\dagger \Delta}$ for all finite nonempty sets of words. These sets $Y^{\dagger \Delta}$ exhaust all closed upper sets when $\Lambda$ is well-founded. Indeed, in that case, the MacNeille completion of $\Lambda^*$ is well-founded by Lemma 3.2.

Some of the four rules for generating the smallest stable upper set may become redundant in the case of particular alphabets. Specifically, we have the following consequence of Theorem 4.4.

**Corollary 4.5.** Let $\Lambda$ be a well-founded ordered set in which every (finite) subset bounded below has its meet and join in $\Lambda$. Then

(a) $\Lambda$ is a lattice,
(b) $\Lambda$ is a dual forest,
(c) $\Lambda$ is a disjoint union of chains,
(d) $\Lambda$ is a chain, or
(e) $\Lambda$ is an antichain, respectively,

if and only if the closed upper sets of $\Lambda^*$ are exactly the upper sets obeying the

(a) reduction, permutation, and meet rules,
(b) cancellation, reduction, and permutation rules,
(c) cancellation and reduction rules,
(d) reduction rule, or
(e) cancellation rule, respectively.

**Proof.** Necessity is clear. As to sufficiency, consider sets of the form

(a) $\uparrow \{\alpha, \beta\}$ where $\alpha, \beta$ are incompatible,
(b) $\uparrow \{\alpha, \beta\}$ where $\alpha, \beta$ are incomparable, but bounded,
(c) $\uparrow \{\alpha \lor \beta, \alpha \beta\}$ where $\alpha, \beta$ are incomparable,
(d) $\uparrow \{\alpha\alpha, \beta\}$ for $\alpha < \beta$, respectively.

In each case, the upper set as described is not closed, but obeys the corresponding subset of rules. Finally, (d) follows from (a) and (c).

**Conjecture 4.6.** Let $\Lambda$ be a well-founded conditional lattice. Then an upper set $Z$ of $\Lambda^*$ is closed if and only if it satisfies the four rules.

We do not even have a proof of this assertion in the simplest case of a 3-letter alphabet $\Lambda = \{\lambda, \mu, \nu\}$ with $\nu < \lambda$ and $\nu < \mu$ ($\lambda, \mu$ being incomparable) so that $\nu = \lambda \land \mu$.

Theorem 4.4 does not apply to the ordered alphabets displayed in Figure 2. So, we have not yet achieved a thorough understanding of the MacNeille completion of all free ordered monoids. It would also be interesting to characterize the closed
upper sets of $\Lambda^*$ without imposing any condition on $\Lambda$; then, of course, finitary rules are no longer sufficient:

**Problem 4.7.** Characterize the closed upper sets of $\Lambda^*$ for an arbitrary ordered set $\Lambda$.

5. **Final remark**

Originally, the main motivation for the description of upper sets belonging to the MacNeille completion by means of syntactic rules was to characterize absolute retracts among oriented graphs. The difficulty of a characterization is due to the fact that, in general, not every oriented graph (e.g., an oriented cycle) is isometrically embeddable in an absolute retract in the category of oriented graphs (that is, a graph which is a retract of all its isometric extensions). Our main result, Theorem 4.4 entails that on a two-letter alphabet $\Lambda := \{+, -\}$, closed sets are characterized by the satisfaction of the cancellation rule. This allows to characterize among oriented graphs those which are absolute retracts in the category of oriented graphs. Indeed, it turns out that these graphs are simply the retracts of products of oriented zigzags. This result, as well as others in the same vein, obtained in collaboration with F. Saïdane, will be developed in a forthcoming paper.

**Acknowledgement**

The second author thanks the members of the Mathematical Department of the University of Hamburg for their hospitality in February 2016, and the first author is grateful to the Camille Jordan Institute for a visit in August 2017.

The authors are pleased to thank the referee for his very careful reading.
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