ABSTRACT. We present a structure theorem for the moduli space $R_7$ of Prym curves of genus 7 as a projective bundle over the moduli space of 7-nodal rational curves. The existence of this parametrization implies the unirationality of $R_7$ and that of the moduli space of Nikulin surfaces of genus 7, as well as the rationality of the moduli space of Nikulin surfaces of genus 7 with a distinguished line. Using the results in genus 7, we then establish that $R_8$ is uniruled.

A polarized Nikulin surface of genus $g$ is a smooth polarized $K3$ surface $(S, c)$, where $c \in \text{Pic}(S)$ with $c^2 = 2g - 2$, equipped with a double cover $f: \tilde{S} \to S$ branched along disjoint rational curves $N_1, \ldots, N_8 \subset S$, such that $c \cdot N_i = 0$ for $i = 1, \ldots, 8$. Denoting by $e \in \text{Pic}(S)$ the class defined by the equality $e \otimes O_S(\sum_{i=1}^{8} N_i) = O_S$, one forms the Nikulin lattice

$$\mathfrak{N} := \langle O_S(N_1), \ldots, O_S(N_8), e \rangle$$

and obtains a primitive embedding $j: \Lambda_g := \mathbb{Z} \cdot [c] \oplus \mathfrak{N} \hookrightarrow \text{Pic}(S)$. Nikulin surfaces of genus $g$ form an irreducible 11-dimensional moduli space $F^\mathfrak{N}_g$ which has been studied in [Do1] and [vGS]. The connection between $F^\mathfrak{N}_g$ and the moduli space $R_g$ of pairs $[C, \eta]$ where $C$ is a curve of genus $g$ and $\eta \in \text{Pic}^0(C)[2]$ is a 2-torsion point, has been pointed out in [FV] and used to describe $R_g$ in small genus. Over $F^\mathfrak{N}_g$ one considers the open set in a tautological $\mathbb{P}^g$-bundle

$$P^\mathfrak{N}_g := \left\{ [S, j : \Lambda_g \hookrightarrow \text{Pic}(S), C] : C \in [c] \text{ is a smooth curve of genus } g \right\},$$

which is endowed with the two projection maps

$$P^\mathfrak{N}_g \rightarrow \mathcal{F}^\mathfrak{N}_g \rightarrow R_g$$

defined by $p_g([S, j, C]) := [S, j]$ and $\chi_g([S, j, C]) := [C, e_C := e \otimes O_C]$ respectively.

Observe that $\dim(P^\mathfrak{N}_g) = \dim(R_7) = 18$. The map $\chi_7: P^\mathfrak{N}_7 \rightarrow R_7$ is a birational isomorphism, precisely $R_7$ is birational to a Zariski locally trivial $\mathbb{P}^7$-bundle over $F^\mathfrak{N}_7$. This is reminiscent of Mukai’s result [Mu]: $M_{11}$ is birational to a projective bundle over the moduli space $F_{11}$ of polarized $K3$ surfaces of genus 11. Note that $M_{11}$ and $R_7$ are the only known examples of moduli spaces of curves admitting a non-trivial fibre bundle structure over a moduli space of polarized $K3$ surfaces. Here we describe the structure of $F^\mathfrak{N}_7$.

Theorem 0.1. The Nikulin moduli space $F^\mathfrak{N}_7$ is unirational. The Prym moduli space $R_7$ is birationally isomorphic to a $\mathbb{P}^7$-bundle over $F^\mathfrak{N}_7$. It follows that $R_7$ is unirational as well.

It is well-known that $R_g$ is unirational for $g \leq 6$, see [Do], [ILS], [V], and even rational for $g \leq 4$, see [Do2], [Cat]. On the other hand, the Deligne-Mumford moduli space $\overline{R}_g$ of
stable Prym curves of genus $g$ is a variety of general type for $g \geq 14$, whereas $\text{kod}(\mathcal{K}_{12}) \geq 0$, see [EL]. Nothing seems to be known about the Kodaira dimension of $\mathcal{K}_g$, for $g = 9, 10, 11$.

We now discuss the structure of $\mathcal{F}^g_i$. For each positive $g$, we denote by

$$\mathfrak{Rat}_g := \overline{\mathcal{M}}_{0,2g}/\mathbb{Z}_2^{\oplus g} \times S_g$$

the moduli space of $g$-nodal stable rational curves of genus $g$. The action of the group $\mathbb{Z}_2^{\oplus g}$ is given by permuting the marked points labeled by $\{1, 2\}, \ldots, \{2g - 1, 2g\}$ respectively, while the symmetric group $S_g$ acts by permuting the 2-cycles $(1, 2), \ldots, (2g - 1, 2g)$ respectively. The variety $\mathfrak{Rat}_g$, viewed as a subvariety of $\overline{\mathcal{M}}_g$, has been studied by Castelnuovo [Cas] at the end of the 19th century in the course of his famous attempt to prove the Brill-Noether Theorem, as well as much more recently, for instance in [GKM].

Unfortunately, in [GKM] the notation $\mathcal{K}_g$ (reserved for the Prym moduli space) is proposed for what we denote in this paper by $\mathfrak{Rat}_g$. We compute that

$$\mathfrak{Rat}_g \cong \text{Hilb}^9(\mathbb{P}^2)/\text{PGL}(2),$$

where $\text{PGL}(2) \subset \text{PGL}(3)$ is regarded as the group of projective automorphisms of $\mathbb{P}^2$ preserving the image of a fixed smooth conic in $\mathbb{P}^5$.

Let us fix once and for all a smooth rational quintic curve $R \subset \mathbb{P}^5$. For general points $x_1, y_1, \ldots, x_7, y_7 \in R$, we note that $[R, (x_1 + y_1) + \cdots + (x_7 + y_7)] \in \mathfrak{Rat}_7$. We denote by

$$N_1 := \langle x_1, y_1 \rangle, \ldots, N_7 := \langle x_7, y_7 \rangle \in G(2, 6),$$

the corresponding bisecant lines to $R$ and observe that $C := R \cup N_1 \cup \ldots \cup N_7$ is a nodal curve of genus 7 and degree 12 in $\mathbb{P}^5$. By writing down the Mayer-Vietoris sequence for $C$, we find the following identifications:

$$H^0(C, \mathcal{O}_C(1)) \cong H^0(\mathcal{O}_R(1)) \quad \text{and} \quad H^0(C, \mathcal{O}_C(2)) \cong H^0(\mathcal{O}_R(2)) \oplus \left( \bigoplus_{i=1}^7 H^0(\mathcal{O}_{N_i}) \right).$$

It can easily be checked that the base locus

$$S := \text{Bs}\left[\mathcal{I}_C/\mathbb{P}^5(2)\right]$$

is a smooth $K3$ surface which is a complete intersection of three quadrics in $\mathbb{P}^5$. Obviously, $S$ is equipped with the seven lines $N_1, \ldots, N_7$. In fact, $S$ carries an eight line as well! If $H \in |\mathcal{O}_S(1)|$ is a hyperplane section, after setting

$$N_8 := 2R + N_1 + \cdots + N_7 - 2H \in \text{Div}(S),$$

we compute that $N_8^2 = -2, N_8 \cdot H = 1$ and $N_8 \cdot N_i = 0$, for $i = 1, \ldots, 7$. Therefore $N_8$ is equivalent to an effective divisor on $S$, which is embedded in $\mathbb{P}^5$ as a line by the linear system $|\mathcal{O}_S(1)|$. Furthermore,

$$N_1 + \cdots + N_8 = 2(R + N_1 + \cdots + N_7 - H) \in \text{Pic}(S),$$

hence by denoting $e := R + N_1 + \cdots + N_7 - H$, we obtain an embedding $\mathcal{R} \hookrightarrow \text{Pic}(S)$. Moreover $C \cdot N_i = 0$ for $i = 1, \ldots, 8$ and we may view $\Lambda_7 \hookrightarrow \text{Pic}(S)$. In this way $S$ becomes a Nikulin surface of genus 7.

We introduce the moduli space $\mathcal{F}^g_i$ of decorated Nikulin surfaces consisting of polarized Nikulin surfaces $[S, f : \Lambda_9 \hookrightarrow \text{Pic}(S)]$ of genus $g$, together with a distinguished line $N_8 \subset S$ viewed as a component of the branch divisor of the double covering $f : \tilde{S} \to S$. There is an

\footnote{Unfortunately, in [GKM] the notation $\mathcal{K}_g$ (reserved for the Prym moduli space) is proposed for what we denote in this paper by $\mathfrak{Rat}_g$.}
obvious forgetful map \( \tilde{\mathcal{M}}^7_g \to \mathcal{F}_7^g \) of degree 8. Having specified \( N_8 \subset S \), we can also specify the divisor \( N_1 + \cdots + N_7 \subset S \) such that \( e_{\mathcal{O}_S}^g = \mathcal{O}_S(N_1 + \cdots + N_7 + N_8) \). We summarize what has been discussed so far:

**Theorem 0.3.** The moduli space \( \hat{\mathcal{M}}^7 \) (see also \([\text{Bo}]\)), we are led to the following result:

Put together Theorems 0.2 and 0.3, we conclude that there exists a dominant rational map \( \varphi : \mathcal{M}_g \to \hat{\mathcal{M}}^7 \) given by

\[
\varphi \left( [R, (x_1 + y_1) + \cdots + (x_7 + y_7)] \right) := [S, \mathcal{O}_S(R + N_1 + \cdots + N_7), N_8]
\]

is a birational isomorphism.

A construction of the inverse map \( \varphi^{-1} \) using the geometry of Prym canonical curves of genus 7 is presented in Section 2. The moduli space \( \mathcal{M}_g \) is related to the configuration space

\[
U^2 := \text{Hilb}^2(\mathbb{P}^2)/\text{PGL}(3)
\]

of \( g \) unordered points in the plane. Using the isomorphism \( \text{PGL}(3)/\text{PGL}(2) \cong \mathbb{P}^5 \), we observe in Section 2 that there exists a (locally trivial) \( \mathbb{P}^5 \)-bundle structure \( \mathcal{M}_g \to U^2_g \). In particular \( \mathcal{M}_g \) is rational whenever \( U^2_g \) is. Since the rationality of \( U^2_7 \) has been established by Katsylo \([\text{Ka}]\) (see also \([\text{Bo}]\)), we are led to the following result:

**Theorem 0.3.** The moduli space \( \hat{\mathcal{M}}^7 \) of marked Nikulin surfaces of genus 7 is rational.

Putting together Theorems 0.2 and 0.3, we conclude that there exists a dominant rational map \( \mathbb{P}^{18} \to \mathcal{R}_7 \) of degree 8. We are not aware of any dominant map from a rational variety to \( \mathcal{R}_7 \) of degree smaller than 8. It would be very interesting to know whether \( \mathcal{R}_7 \) itself is a rational variety. We recall that although \( \mathcal{M}_g \) is known to be rational for \( g \leq 6 \) (see \([\text{Bo}]\) and the references therein), the rationality of \( \mathcal{M}_7 \) is an open problem.

We sum up the construction described above in the following commutative diagram:

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{0,14} & \overset{(2:7).1)}{\longrightarrow} & \mathcal{M}_7 \\
\uparrow & & \uparrow \\
\mathcal{F}_7^g & \overset{8:1}{\longrightarrow} & \mathcal{F}_7^g \\
\downarrow & & \downarrow \\
\mathcal{R}_7 & \overset{\mathbb{P}^5}{\longrightarrow} & U^2_7
\end{array}
\]

The concrete geometry of \( \mathcal{R}_7 \) has direct consequences concerning the Kodaira dimension of \( \overline{\mathcal{R}}_8 \). The projective bundle structure of \( \mathcal{R}_7 \) over \( \mathcal{F}_7^g \) can be lifted to a boundary divisor of \( \overline{\mathcal{R}}_8 \). Denoting by \( \pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g \) the map forgetting the Prym structure, one has the formula

\[
\pi^*(\delta_0) = \delta_0' + \delta_0'' + 2\delta_0^{\text{ram}} \in CH^1(\overline{\mathcal{R}}_g),
\]

where \( \delta_0' := [\Delta_0'] \), \( \delta_0'' := [\Delta_0''], \) and \( \delta_0^{\text{ram}} := [\Delta_0^{\text{ram}}] \) are boundary divisor classes on \( \overline{\mathcal{R}}_g \) whose meaning will be recalled in Section 3. Note that up to a \( \mathbb{Z}_2 \)-factor, a general point of \( \Delta_0' \) corresponds to a 2-pointed Prym curve of genus 7, for which we apply our Theorem 0.4. We establish the following result:

**Theorem 0.4.** The moduli space \( \overline{\mathcal{R}}_8 \) is uniruled.

Using the parametrization of \( \mathcal{R}_7 \) via Nikulin surfaces, we construct a sweeping curve \( \Gamma \) of the boundary divisor \( \Delta_0' \) of \( \overline{\mathcal{R}}_8 \) such that \( \Gamma \cdot \delta_0' > 0 \) and \( \Gamma \cdot K_{\overline{\mathcal{R}}_8} < 0 \). This implies that the canonical class \( K_{\overline{\mathcal{R}}_8} \) cannot be pseudoeffective, hence via \([\text{BDPP}]\), the moduli space \( \overline{\mathcal{R}}_8 \) is uniruled. This way of showing uniruledness of a moduli space, though quite effective, does not lead to an explicit uniruled parametrization of \( \mathcal{R}_8 \). In Section 3, we sketch an alternative,
more geometric way of showing that $R_8$ is uniruled, by embedding a general Prym-curve of genus 8 in a certain canonical surface. A rational curve through a general point of $\overline{R}_8$ is then induced by a pencil on this surface.

1. Polarized Nikulin surfaces

We briefly recall some basics on Nikulin surfaces, while referring to [VGS], [GS] for details. A symplectic involution $\iota$ on a smooth $K3$ surface $Y$ has 8 fixed points and we denote by $Y := Y/\langle \iota \rangle$ the quotient. The surface $Y$ has 8 nodes. Letting $\sigma : \tilde{S} \to Y$ be the blow-up of the fixed points, the involution $\iota$ lifts to an involution $\tilde{\iota} : \tilde{S} \to \tilde{S}$ fixing the eight $(-1)$-curves $E_1, \ldots, E_8 \subseteq \tilde{S}$. Denoting by $f : \tilde{S} \to S$ the quotient map by the involution $\tilde{\iota}$, we obtain a smooth $K3$ surface $S$, together with a primitive embedding of the Nikulin lattice $\Omega \cong E_8(-2) \subseteq \text{Pic}(S)$, where $N_i = f(E_i)$ for $i = 1, \ldots, 8$. In particular, the sum of rational curves $N := N_1 + \cdots + N_8$ is an even divisor on $S$, that is, there exists a class $e \in \text{Pic}(S)$ such that $e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)$. The cover $f : \tilde{S} \to S$ is branched precisely along the curves $N_1, \ldots, N_8$. The following diagram summarizes the notation introduced so far and will be used throughout the paper:

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\alpha} & Y \\
f \downarrow & & \downarrow \\
S & \longrightarrow & \tilde{Y}
\end{array}
$$

(1)

Nikulin [Ni] showed that the possible configurations of even sets of disjoint $(-2)$-curves on a $K3$ surface $S$ are only those consisting of either 8 curves (in which case $S$ is a Nikulin surface as defined in this paper), or of 16 curves, in which case $S$ is a Kummer surface. From this point of view, Nikulin surfaces appear naturally as the Prym analogues of $K3$ surfaces.

**Definition 1.1.** A polarized Nikulin surface of genus $g$ consists of a smooth $K3$ surface and a primitive embedding $j$ of the lattice $\Lambda_g = \mathbb{Z} \cdot e \oplus \Omega \subseteq \text{Pic}(S)$, such that $e^2 = 2g - 2$ and the class $j(e)$ is nef.

Polarized Nikulin surfaces of genus $g$ form an irreducible moduli 11-dimensional moduli space $\mathcal{F}_g^{\text{nl}}$, see for instance [Do1]. Structure theorems for $\mathcal{F}_g^{\text{nl}}$ for genus $g \leq 6$ have been established in [V]. For instance the following result is proven in loc.cit. for Nikulin surfaces of genus $g = 6$. Let $V = \mathbb{C}^5$ and fix a smooth quadric $Q \subseteq \mathbb{P}(V)$. Then one has a birational isomorphism, which, in particular, shows that $\mathcal{F}_6^{\text{nl}}$ is unirational:

$$
\mathcal{F}_6^{\text{nl}} \cong G\left(7, \bigwedge^2 V\right)^{ss} \!/ \text{Aut}(Q).
$$

On the other hand, fundamental facts about $\mathcal{F}_g^{\text{nl}}$ are still not known. For instance, it is not clear whether $\mathcal{F}_g^{\text{nl}}$ is a variety of general type for large $g$. Nikulin surfaces have been recently used decisively in [FK] to prove the Prym-Green Conjecture on syzygies of general Prym-canonical curves of even genus.

For a polarized Nikulin surface $(S, j)$ of genus $g$ as above, we set $C := j(e)$ and then $H \equiv C - e \in \text{Pic}(S)$. It is shown in [GS], that for any Nikulin surface $S$ having minimal Picard lattice $\text{Pic}(S) = \Lambda_g$, the linear system $\mathcal{O}_S(H)$ is very ample for $g \geq 6$. We compute that $H^2 = 2g - 6$ and denote by $\phi_H : S \to \mathbb{P}^{g-2}$ the corresponding embedding. Since $N_i \cdot H = 1$ for $i = 1, \ldots, 8$, it follows that the images $\phi_H(N_i) \subset \mathbb{P}^{g-2}$ are lines. The existence of two closely
linked distinguished polarizations \( \mathcal{O}_S(c) \) and \( \mathcal{O}_S(h) \) of genus \( g \) and \( g - 2 \) respectively on any Nikulin surface is one of the main sources for the rich geometry of the moduli space \( \mathcal{F}^n_g \) for \( g \leq 6 \), see [FV] and [VGS].

Suppose that \( [S,j : \Lambda_7 \hookrightarrow \text{Pic}(S)] \) is a polarized Nikulin surface of genus 7. In this case \( \phi_H : S \hookrightarrow \mathbb{P}^5 \) is a surface of degree 8 which is a complete intersection of three quadrics. For each smooth curve \( C \in |\mathcal{O}_S(j(c))| \), we have that \( [C, \eta := e_C] \in \mathcal{R}_7 \). Since \( \mathcal{O}_C(1) = K_C \otimes \eta \), it follows that the restriction \( \phi_{H[C]} : C \hookrightarrow \mathbb{P}^5 \) is a Prym-canonically embedded curve of genus 7. This assignment gives rise to the map \( \chi_7 : \mathcal{P}^m_7 \rightarrow \mathcal{R}_7 \).

Conversely, to a general Prym curve \( [C, \eta] \in \mathcal{R}_7 \) we associate a unique Nikulin surface of genus 7 as follows. We consider the Prym-canonical embedding \( \phi_{K_C \otimes \eta} : C \hookrightarrow \mathbb{P}^5 \) and observe that \( S := \text{bs}(|\mathcal{I}_{C/P\mathbb{P}^5}(2)) \) is a complete intersection of three quadrics, that is, if smooth, a K3 surfaces of degree 8. In fact, \( S \) is smooth for a general choice of \( [C, \eta] \in \mathcal{R}_7 \), see [FV] Proposition 2.3. We then set \( N \equiv 2(C - H) \in \text{Pic}(S) \) and note that \( N^2 = -16 \) and \( N \cdot H = 8 \). Using the cohomology exact sequence

\[
0 \rightarrow H^0(S, \mathcal{O}_S(N - C)) \rightarrow H^0(S, \mathcal{O}_S(N)) \rightarrow H^0(C, \mathcal{O}_C(N)) \rightarrow 0,
\]

since \( \mathcal{O}_C(N) \) is trivial, we conclude that the divisor \( N \) is effective on \( S \). It is shown in loc.cit. that for a general \([C, \eta] \in \mathcal{R}_7\), we have a splitting \( N = N_1 + \cdots + N_7 \) into a sum of 8 disjoint lines with \( C \cdot N_i = 0 \) for \( i = 1, \ldots, 8 \). This turns \( S \) into a Nikulin surface and explains the birational isomorphisms

\[
\chi_7^{-1} : \mathcal{P}^m_7 \overset{\cong}{\rightarrow} \mathcal{R}_7
\]

referred to in the Introduction.

Suppose now that \([S, \mathcal{O}_S(C), N_8] \in \mathcal{F}^m_7\), that is, we single out a \((-2)\)-curve in the Nikulin lattice. Writing \( e^{\oplus 2} = \mathcal{O}_C(N_1 + \cdots + N_8) \), the choice of \( N_8 \) also determines the sum of the seven remaining lines \( N_1 + \cdots + N_7 \), where \( H \cdot N_i = 1 \), for \( i = 1, \ldots, 8 \). We compute

\[
(C - N_1 - \cdots - N_7)^2 = -2 \quad \text{and} \quad (C - N_1 - \cdots - N_7) \cdot H = 5,
\]

in particular, there exists an effective divisor \( R \) on \( S \), with \( R \equiv C - N_1 - \cdots - N_7 \). Note also that \( R \cdot N_i = 2 \), for \( i = 1, \ldots, 7 \), that is, \( R \subset \mathbb{P}^5 \) comes endowed with seven bisecant lines.

**Proposition 1.2.** For a decorated Nikulin surface \([S, \mathcal{O}_S(C), N_8] \in \mathcal{F}^m_7\) satisfying \( \text{Pic}(S) = \Lambda_7 \), we have that \( H^1(S, \mathcal{O}_S(C - N_1 - \cdots - N_7)) = 0 \). In particular,

\[
R \in |\mathcal{O}_S(C - N_1 - \cdots - N_7)|
\]

is a smooth rational quintic curve on \( S \).

**Proof.** Assume by contradiction that the curve \( R \subset S \) is reducible. In that case, there exists a smooth irreducible \((-2)\)-curve \( Y \subset S \) such that \( Y \cdot R < 0 \) and \( H^0(S, \mathcal{O}_S(R - Y)) \neq 0 \). Assuming \( \text{Pic}(S) \) is generated by \( C, N_1, \ldots, N_8 \) and the class \( e = (N_1 + \cdots + N_8)/2 \), there exist integers \( a, b, c_1, \ldots, c_8 \in \mathbb{Z} \) such that

\[
Y \equiv a \cdot C + \left(c_1 + \frac{b}{2}\right) \cdot N_1 + \cdots + \left(c_8 + \frac{b}{2}\right) \cdot N_8.
\]

Setting \( b_i := c_i + \frac{b}{2} \), the numerical hypotheses on \( Y \) can be rewritten in the following form:

\[
b_1^2 + \cdots + b_8^2 = 6a^2 + 1 \quad \text{and} \quad 6a + b_1 + \cdots + b_8 \leq -1.
\]
Since $Y$ is effective, we find that $a \geq 0$ (use that $C \subset S$ is nef). Applying the same considerations to the effective divisor $R - Y$, we obtain that $a \in \{0, 1\}$.

If $a = 0$, then $Y \equiv b_1 N_1 + \cdots + b_8 N_8 \geq 0$, hence $b_i \geq 0$ for $i = 1, \ldots, 8$, which contradicts the inequality $b_1 + \cdots + b_8 \leq -1$, so this case does not appear.

If $a = 1$, then $R - Y \equiv -(1 + b_1) N_1 - \cdots - (1 + b_7) N_7 - b_8 N_8 \geq 0$, therefore $b_8 \leq 0$ and $b_i \leq -1$ for $i = 1, \ldots, 7$. From (2), we obtain that $b_8 = 0$ and $b_1 = \cdots = b_7 = -1$. Thus $Y \equiv R$, which is a contradiction, for $Y$ was assumed to be a proper irreducible component of $R$. \hfill $\square$

Retaining the notation above, we obtain a map $\psi : \tilde{\mathcal{F}}_7^2 \rightarrow \mathfrak{Rat}_7$, defined by

$$\psi\left([S, \mathcal{O}_S(C), N_8]\right) := [R, N_1 \cdot R + \ldots + N_7 \cdot R],$$

where the cycle $N_i \cdot R \in \text{Sym}^2(R)$ is regarded as an effective divisor of degree 2 on $R$. The map $\psi$ is regular over the dense open subset of $\tilde{\mathcal{F}}_7^2$ consisting of Nikulin surfaces having the minimal Picard lattice $\Lambda_7$. We are going to show that $\psi$ is a birational isomorphism by explicitly constructing its inverse.

We fix a smooth rational quintic curve $R \subset \mathbb{P}^5$ and recall the canonical identification

$$|\mathcal{I}_{R/P^5}(2)| = |\mathcal{O}_{\text{Sym}^2(R)}(3)|$$

between the linear system of quadrics containing $R \subset \mathbb{P}^5$ and that of plane cubics. Here we use the isomorphism $\text{Sym}^2(R) \cong \mathbb{P}^2$, under which to a quadric $Q \in H^0(\mathbb{P}^5, \mathcal{I}_{R/P^5}(2))$ one assigns the symmetric correspondence

$$\Sigma_Q := \{x + y \in \text{Sym}^2(R) : \langle x, y \rangle \subset Q\},$$

which is a cubic curve in $\text{Sym}^2(R)$.

Let $N_1, \ldots, N_7$ general bisecant lines to $R$ and consider the semi-stable curve of genus 7

$$C := R \cup N_1 \cup \ldots \cup N_7 \subset \mathbb{P}^5.$$

**Proposition 1.3.** For a general choice of the bisecants $N_1, \ldots, N_7$ of the curve $R \subset \mathbb{P}^5$, the base locus

$$S := \text{Bs}\left|\mathcal{I}_{C/P^5}(2)\right|$$

is a smooth K3 surface of degree 8.

**Proof.** The bisecant line $N_i$ is determined by the degree 2 divisor $N_i \cdot R \in \text{Sym}^2(R)$. Under the identification (3), the quadrics containing the line $N_i$ are identified with the cubics in $|\mathcal{O}_{\text{Sym}^2(R)}(3)|$ that pass through the point $N_i \cdot R$. It follows that the linear system $|\mathcal{I}_{C/P^5}(2)|$ corresponds to the linear system of cubics in $\text{Sym}^2(R)$ passing through 7 general points. Since the secants $N_i$ (and hence the points $N_i \cdot R \in \text{Sym}^2(R)$) have been chosen to be general, we obtain that $\dim |\mathcal{I}_{C/P^5}(2)| = 2$.

We have showed in Proposition [12] that for a general Nikulin surface $S$,

$$H^1(S, \mathcal{O}_S(H - N_1 - \cdots - N_i - \cdots - N_8)) = 0,$$

for all $i = 1, \ldots, 8$. In particular, the morphism $\psi$ is defined on all components of $\tilde{\mathcal{F}}_7^2$ and the image of each component is an element of $\mathfrak{Rat}_7$. For such a point in $\text{Im}(\psi)$, it follows that $\text{bs}\left|\mathcal{I}_{C/P^5}(2)\right|$ is a smooth surface, in fact a general Nikulin surface of genus 7. \hfill $\square$
Proof of Theorem 0.2. As explained in the Introduction, the map $\varphi : \Rat_7 \longrightarrow \tilde{\mathcal{F}}^{\text{pr}}_7$ is well-defined and clearly the inverse of $\psi$. In particular, it follows that $\tilde{\mathcal{F}}^{\text{pr}}_7$ is also irreducible (and in fact unirational).

2. Configuration spaces of points in the plane

Throughout this section we use the identification $\text{Sym}^2(P^1) \cong P^2$ induced by the map $\rho : P^1 \times P^1 \rightarrow P^2$ obtained by taking the projection of the Segre embedding of $P^1 \times P^1$ to the space of symmetric tensors, that is, $\rho([a_0, a_1], [b_0, b_1]) = [a_0 b_0, a_1 b_1, a_0 b_1 + a_1 b_0]$. We identify the diagonal $\Delta \subset P^1 \times P^1$ with its image $\rho(\Delta)$ in $P^2$. We view $\text{PGL}(2)$ as the subgroup of automorphisms of $P^2$ that preserve the conic $\Delta$. Furthermore, the choice of $\Delta$ induces a canonical identification $PGL(3)/PGL(2) = |O_{P^2}(2)| = P^5$.

For $g \geq 5$, we consider the projection

$$\beta : \Rat_g := \text{Hilb}^g(P^2)\!/\!//SL(2) \rightarrow \text{Hilb}^g(P^2)\!/\!//SL(3) =: U^2_g.$$  

Definition 2.1. If $X$ is a del Pezzo surface of degree 2, a contraction of $X$ is the blow-up $f : X \rightarrow P^2$ of 7 points in general position in $P^2$.

Specifying a pair $(X, f)$ as above, amounts to giving a plane model of the del Pezzo surface, that is, a pair $(X, L)$, where $X$ is a del Pezzo surface with $K_X^2 = 2$ and $L \in \text{Pic}(S)$ is such that $L^2 = 1$ and $K_X \cdot L = -2$. Therefore $U^2_g$ is the GIT moduli space of pairs $(X, f)$ (or equivalently of pairs $(X, L)$) as above.

Proposition 2.2. The morphism $\beta : \text{Hilb}^g(P^2)\!/\!//SL(2) \rightarrow U^2_g$ is a locally trivial $P^5$-fibration.

Proof. Having fixed the conic $\Delta \subset P^2$, we have an identification $P^2 \cong \text{Sym}^2(\Delta) \cong (P^2)^2$, that is, we view points in $\text{Sym}^2(\Delta)$ as lines in $P^2$. A general point $D \in \text{Hilb}^g(P^2)$ corresponds to a union $D = \ell_1 + \cdots + \ell_g$ of $g$ lines in $P^2$, such that $\text{Aut}((\ell_1, \ldots, \ell_g)) = 1$. We consider the rank 6 vector bundle $E$ over $\text{Hilb}^g(P^2)$ with fibre

$$E(\ell_1 + \cdots + \ell_g) := H^0(\mathcal{O}_{\ell_1 + \cdots + \ell_g}(2)).$$

Clearly $E$ descends to a vector bundle $E$ over the quotient $U^2_g$. We then observe that one has a canonical identification $P(E) \cong \text{Hilb}^g(P^2)\!/\!//SL(2)$, or more geometrically, $\Rat_g$ is the moduli space of pairs consisting of an unordered configuration of $g$ lines and a conic in $P^2$. The birational isomorphism $P(E) \rightarrow \text{Hilb}^g(P^2)\!/\!//SL(2)$ is given by the assignment

$$\left(\ell_1 + \cdots + \ell_g, Q\right) \mod SL(3) \mapsto \sigma(\ell_1) + \cdots + \sigma(\ell_g) \mod SL(2),$$

where $\sigma \in SL(3)$ is an automorphism such that $\sigma(Q) = \Delta$. \hfill \Box

Proof of Theorem 0.3. We have established that the moduli space $\tilde{\mathcal{F}}^{\text{pr}}_7$ is birationally isomorphic to the projectivization of a $P^5$-bundle over $U^2_7$. Since $U^2_7$ is rational, cf. [Bo] Theorem 2.2.4.2, we conclude. \hfill \Box

Remark 2.3. In view of Theorem 0.3 it is natural to ask whether there exists a rational modular degree 8 cover $\tilde{\mathcal{R}}_7 \rightarrow \mathcal{R}_7$ which is a locally trivial $P^7$-bundle over the rational variety $\tilde{\mathcal{F}}^{\text{pr}}_7$, such
that the following diagram is commutative:

\[
\begin{array}{c}
\hat{\mathcal{R}}_7 \xrightarrow{?} \hat{\mathcal{F}}_{71}^! \cong \text{Rat}_7 \\
\downarrow 8:1 \quad \downarrow 8:1 \\
\mathcal{R}_7 \xrightarrow{p^7} \mathcal{F}_{71}^!
\end{array}
\]

One candidate for the cover \(\hat{\mathcal{R}}_7\) is the universal singular locus of the Prym-theta divisor,

\[
\hat{\mathcal{R}}_7 := \left\{ [C, \eta, L] \in \mathcal{R}_7 : [C, \eta] \in \mathcal{R}_7 \text{ and } L \in \text{Sing}(\Xi)/\ast \right\},
\]

where \(\text{Sing}(\Xi) = \{ L \in \text{Pic}^{g-2}(\tilde{C}) : \text{Nm}_f(L) = K_C, h^0(C, L) \geq 4, h^0(C, L) \equiv \mod{2} \}\). It is shown in \([\text{De}]\) that for a general point \([C, \eta] \in \mathcal{R}_7\), the locus \(\text{Sing}(\Xi)\) is reduced and consists of 16 points, so indeed \(\text{deg}(\hat{\mathcal{R}}_7/\mathcal{R}_7) = 8\).

### 3. The Uniruledness of \(\overline{\mathcal{R}}_8\)

We now explain how our structure results on \(\mathcal{F}_{71}^!\) and \(\mathcal{R}_7\) lead to an easy proof of the uniruledness of \(\overline{\mathcal{R}}_8\). We begin by reviewing a few facts about the compactification \(\overline{\mathcal{R}}_g\) of \(\mathcal{R}_g\) by means of stable Prym curves, see \([\text{FL}]\) for details. The geometric points of the coarse moduli space \(\overline{\mathcal{R}}_g\) are triples \((X, \eta, \beta)\), where \(X\) is a quasi-stable curve of genus \(g\), \(\eta \in \text{Pic}(X)\) is a line bundle of total degree is 0 such that \(\eta_E = \mathcal{O}_E(1)\) for each smooth rational component \(E \subset X\) with \(|E \cap X - E| = 2\) (such a component is said to be exceptional), and \(\beta : \eta^{\otimes 2} \rightarrow \mathcal{O}_X\) is a sheaf homomorphism whose restriction to any non-exceptional component is an isomorphism. If \(\pi : \overline{\mathcal{R}}_g \rightarrow \overline{\mathcal{M}}_g\) is the map dropping the Prym structure, one has the formula \([\text{FL}]\)

\[
\pi^*(\delta_0) = \delta'_0 + \delta''_0 + 2\delta_{0,\text{ram}} \in CH^1(\overline{\mathcal{R}}_g),
\]

where \(\delta'_0 := [\Delta'_0], \delta''_0 := [\Delta''_0],\) and \(\delta_{0,\text{ram}} := [\Delta_{0,\text{ram}}]\) are irreducible boundary divisor classes on \(\overline{\mathcal{R}}_g\), which we describe by specifying their respective general points.

We choose a general point \([C_{xy}] \in \Delta_0 \subset \overline{\mathcal{M}}_g\) corresponding to a smooth 2-pointed curve \((C, x, y)\) of genus \(g - 1\) and consider the normalization map \(\nu : C \rightarrow C_{xy}\), where \(\nu(x) = \nu(y)\). A general point of \(\Delta_0^\text{ram}\) (respectively of \(\Delta_0'\)) corresponds to a pair \([C_{xy}, \eta]\), where \(\eta \in \text{Pic}^0(C_{xy})\) and \(\nu^*(\eta) \in \text{Pic}^0(C)\) is non-trivial (respectively, \(\nu^*(\eta) = \mathcal{O}_C\)). A general point of \(\Delta_0^\text{ram}\) is a Prym curve of the form \((X, \eta)\), where \(X := C \cup \{x, y\}\) \(\mathbb{P}^1\) is a quasi-stable curve with \(p_a(X) = g\) and \(\eta \in \text{Pic}^0(X)\) is a line bundle such that \(\eta_{p_1} = \mathcal{O}_{p_1}(1)\) and \(\eta_{p_2}^{\otimes 2} = \mathcal{O}_{C}(-x - y)\). In this case, the choice of the homomorphism \(\beta\) is uniquely determined by \(X\) and \(\eta\). Therefore, we drop \(\beta\) from the notation of such a Prym curve. There are similar decompositions of the pull-back \(\pi^*(\Delta_i)\) of the other boundary divisors \(\Delta_i \subset \overline{\mathcal{M}}_g\) for \(1 \leq i \leq \lfloor \frac{g}{2} \rfloor\), see again \([\text{FL}]\) for details.

Via Nikulin surfaces we construct a sweeping curve for the divisor \(\Delta'_0 \subset \overline{\mathcal{R}}_8\). Let us start with a general element of \(\Delta_0\) corresponding to a smooth 2-pointed curve \([C, x, y] \in \mathcal{M}_{7,2}\) and a 2-torsion point \(\eta \in \text{Pic}^0(C_{xy})\) and set \(\eta_C := \nu^*(\eta) \in \text{Pic}^0(C)\). Using \([\text{FV}]\) Theorem 0.2, there exists a Nikulin surface \(f : \tilde{S} \rightarrow S\) branched along 8 rational curves \(N_1, \ldots, N_8 \subset S\) and an embedding \(C \subset S\), such that \(C \cdot N_i = 0\) for \(i = 1, \ldots, 8\) and \(\eta_C = e_C\), where \(e \in \text{Pic}(S)\) is the even class with \(e^{\otimes 2} = \mathcal{O}_S(N_1 + \cdots + N_8)\). We can also assume that \(\text{Pic}(S) = \Lambda_7\). By moving \(C\) in its linear system on \(S\), we may assume that \(x, y \notin N_1 \cup \cdots \cup N_8\), and we set \(\{x_1, x_2\} = f^{-1}(x)\) and \(\{y_1, y_2\} = f^{-1}(y)\).
We pick a Lefschetz pencil $\Lambda := \{ C_t \}_{t \in \mathbb{P}^1}$ consisting of curves on $S$ passing through the points $x$ and $y$. Since the locus $\{ D \in |O_S(C)| : D \supset N_i \}$ is a hyperplane in $|O_S(C)|$, it follows that there are precisely eight distinct values $t_1, \ldots, t_8 \in \mathbb{P}^1$ such that

$$C_{t_i} := C_i = N_i + D_i,$$

where $D_i$ is a smooth curve of genus 6 which contains $x$ and $y$ and intersects $N_i$ transversally at two points. For each $t \in \mathbb{P}^1 - \{ t_1, \ldots, t_8 \}$, we may assume that $C_t$ is a smooth curve and denoting $[ \bar{C}_t := C_t/x \sim y ] \in \overline{M}_8$, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Pic}^0(\bar{C}_t)[2] \longrightarrow \text{Pic}^0(C_t)[2] \longrightarrow 0.$$

In particular, there exist two distinct line bundles $\eta_t', \eta_t'' \in \text{Pic}^0(\bar{C}_t)$ such that

$$\nu_t^*(\eta_t') = \nu_t^*(\eta_t'') = e_{C_t}.$$

Using the Nikulin surfaces, we can consistently distinguish $\eta_t'$ from $\eta_t''$. Precisely, $\eta_t'$ corresponds to the admissible cover

$$f^{-1}(C_t)/x_1 \sim y_1, x_2 \sim y_2 \xrightarrow{2:1} \bar{C}_t$$

whereas $\eta_t''$ corresponds to the admissible cover

$$f^{-1}(C_t)/x_1 \sim y_2, x_2 \sim y_1 \xrightarrow{2:1} \bar{C}_t.$$

First we construct the pencil $R := \{ \bar{C}_t \}_{t \in \mathbb{P}^1} \hookrightarrow \overline{M}_8$. Formally, we have a fibration $u : \text{Bl}_{2g-2}(S) \rightarrow \mathbb{P}^1$ induced by the pencil $\Lambda$ by blowing-up $S$ at its $2g - 2$ base points (two of which being $x$ and $y$ respectively), which comes endowed with sections $E_x$ and $E_y$ given by the corresponding exceptional divisors. The pencil $R$ is obtained from $u$, by identifying inside the surface $\text{Bl}_{2g-2}(S)$ the sections $E_x$ and $E_y$ respectively.

**Lemma 3.1.** The pencil $R \subset \overline{M}_8$ has the following numerical characters:

$$R \cdot \lambda = g + 1 = 8, \quad R \cdot \delta_0 = 6g + 16 = 58, \quad \text{and} \quad R \cdot \delta_j = 0 \quad \text{for} \quad j = 1, \ldots, 4.$$

**Proof.** We observe that $(R \cdot \lambda)_{\overline{M}_8} = (\Lambda \cdot \lambda)_{\overline{M}_8} = g + 1 = 8$ and $(R \cdot \delta_j)_{\overline{M}_8} = (\Lambda \cdot \delta_j)_{\overline{M}_8} = 0$ for $j \geq 1$. Finally, in order to determine the degree of the normal bundle of $\Delta_0$ along $R$, we write:

$$R \cdot \delta_0)_{\overline{M}_8} = (\Lambda \cdot \delta_0)_{\overline{M}_8} + E_x^2 + E_y^2 = 6g + 18 = 58,$$

where we have used the well-known fact that a Lefschetz pencil of curves of genus $g$ on a $K3$ surface possesses $6g + 18$ singular fibres (counted with their multiplicities) and that $E_x^2 = E_y^2 = -1$. \hfill \Box

Next, note that the family $\{ [\bar{C}_t, \eta_t] : \nu_t^*(\eta_t) = e_{C_t} \}_{t \in \mathbb{P}^1} \hookrightarrow \overline{M}_8$ splits into two irreducible components meeting in eight points. We consider one of the irreducible components, say

$$\Gamma := \{ [\bar{C}_t, \eta_t'] \}_{t \in \mathbb{P}^1} \hookrightarrow \overline{M}_8.$$

**Lemma 3.2.** The curve $\Gamma \subset \overline{M}_8$ constructed above has the following numerical features:

$$\Gamma \cdot \lambda = 8, \quad \Gamma \cdot \delta_0 = 42, \quad \Gamma \cdot \delta_0'' = 0 \quad \text{and} \quad \Gamma \cdot \delta^{\text{ram}}_0 = 8.$$

Furthermore, $\Gamma$ is disjoint from all boundary components contained in $\pi^*(\Delta_j)$ for $j = 1, \ldots, 4$. 

Proof. First we observe that $\Gamma$ intersects the divisor $\Delta^{\text{ram}}_t$ transversally at the points corresponding to the values $t_1, \ldots, t_8 \in \mathbb{P}^1$, when the curve $C_t$ acquires the $(-2)$-curve $N_t$ as a component. Indeed, for each of these points $e_{D_t}^{(-2)} = \mathcal{O}_{D_t}(-N_t)$ and $e_{N_t}^{\vee} = \mathcal{O}_{N_t}(1)$, therefore $[C_t, e_{C_t}] \in \Delta^{\text{ram}}_t$. Furthermore, using Lemma 3.1 we write $(\Gamma \cdot \lambda)_{\mathbb{R}^8} = \pi_*(\Gamma) \cdot \lambda = 8$ and

$$\Gamma \cdot (\delta_0 + \delta''_0 + 2\delta^{\text{ram}}_0) = \Gamma \cdot \pi^*(\delta_0) = R \cdot \delta_0 = 58.$$  

Furthermore, for $t \in \mathbb{P}^1 - \{t_1, \ldots, t_8\}$, the curve $f^{-1}(C_t)$ cannot split into two components, else Pic$(S) \not\supset \Lambda_t$. Therefore $\gamma \cdot \delta_0 = 0$ and hence $\Gamma \cdot \delta_0 = 42$. 

Proof of Theorem 0.4. The curve $\Gamma \subset \mathbb{R}^8$ constructed above is a sweeping curve for the irreducible boundary divisor $\Delta^\prime_t$ in particular it intersects non-negatively every irreducible effective divisor $D$ on $\mathbb{R}^8$ which is different from $\Delta^\prime_t$. Since $\Gamma \cdot \delta_0 > 0$, it follows that $D$ intersects non-negatively every pseudoeffective divisor on $\mathbb{R}^8$. Using the formula for the canonical divisor $[\text{FL}]$

$$K_{\mathbb{R}^8} = 13\lambda - 2(\delta'_0 + \delta''_0) - 3\delta^{\text{ram}}_0 - \cdots \in CH^1(\mathbb{R}^8),$$  

applying Lemma 3.2 we obtain that $\Gamma \cdot K_{\mathbb{R}^8} = -4 < 0$, thus $K_{\mathbb{R}^8} \not\in \text{Eff}(\mathbb{R}^8)$. Using [BDPP], we conclude that $\mathbb{R}^8$ is uniruled, in particular its Kodaira dimension is negative.}

3.1. The uniruledness of the universal singular locus of the theta divisor over $\mathbb{R}^8$. In what follows, we sketch a second, more geometric proof of Theorem 0.4 skipping some details. This proof provides a concrete way of constructing a rational curve through a general point of $\mathbb{R}^8$. We fix a general element $[C, \eta] \in \mathbb{R}^8$ and denote by $f : \tilde{C} \to C$ the corresponding unramified double cover and by $\iota : C \to \tilde{C}$ the involution exchanging the sheets of $f$. Following [W], we consider the singular locus of the Prym theta divisor, that is, the locus

$$V^3(C, \eta) = \text{Sing}(\Xi) := \{L \in \text{Pic}^{14}(\tilde{C}) : \text{Nm}_f(L) = K_C, h^0(C, L) \geq 4 \text{ and } h^0(C, L) \equiv 0 \mod 2\}.$$  

It follows from [W], that $V^3(C, \eta)$ is a smooth curve. We pick a line bundle $L \in V^3(C, \eta)$ with $h^0(\tilde{C}, L) = 4$, a general point $\tilde{x} \in \tilde{C}$ and consider the $\iota$-invariant part of the Petri map, that is,

$$\mu^+_{\tilde{x}}(L(-\tilde{x})) : \text{Sym}^2 H^0(\tilde{C}, L(-\tilde{x})) \to H^0(\tilde{C}, K_{\tilde{C}}(-x)), \ s \otimes t + t \otimes s \mapsto s \cdot \iota^*(t) + t \cdot \iota^*(s),$$  

where $x := f(\tilde{x}) \in C$. We set $P^2 := P(H^0(L(-\tilde{x}))^\vee)$, and similarly to [FV] Section 2.2, we consider the map $q : P^2 \times P^2 \to P^5$ obtained from the Segre embedding $P^2 \times P^2 \hookrightarrow P^8$ and then projecting onto the space of symmetric tensors. We have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{C} & \xrightarrow{(L(-\tilde{x}), \iota^*(L(-\tilde{x})))} & P^2 \times P^2 \\
| & \downarrow \mu^+_{\tilde{x}}(L(-\tilde{x})) & \\
C & \xrightarrow{q} & P^5 = P(\text{Sym}^2 H^0(L(-\tilde{x}))^\vee)
\end{array}$$

Let $\Sigma := \text{Im}(q) \subset P^5$ be the determinantal cubic surface; its singular locus is the Veronese surface $V_1$. For a general choice of $[C, \eta] \in \mathbb{R}^8$, $L \in V^3(C, \eta)$ and of $\tilde{x} \in \tilde{C}$, the map
\[ \mu_0^+(L(-\tilde{x})) \] is injective and let \( W \subset H^0(C, K_C(-x)) \) be its 6-dimensional image. Comparing dimensions, we observe that the kernel of the multiplication map

\[ \text{Sym}^2(W) \rightarrow H^0(C, K_C^\otimes 2(-2x)) \]

is at least 2-dimensional. In particular, there exist distinct quadrics \( Q_1, Q_2 \subset \mathbb{P}^5 \) such that

\[ C \subset S := Q_1 \cap Q_2 \cap \Sigma \subset \mathbb{P}^5. \]

Since \( \text{Sing}(\Sigma) = V_4 \), the surface \( S \) is singular at the 16 points of intersection \( Q_1 \cap Q_2 \cap V_4 \). Assuming we can find \( (C, \eta, L, \tilde{x}) \) such that \( \text{Sing}(S) = Q_1 \cap Q_2 \cap V_4 \), we obtain that \( S \) is a canonical surface, that is, \( K_S = \mathcal{O}_S(1) \).

Using the exact sequence \( 0 \rightarrow H^0(S, \mathcal{O}_S) \rightarrow H^0(S, \mathcal{O}_S(C)) \rightarrow H^0(\mathcal{O}_C(C)) \rightarrow 0 \), since \( \mathcal{O}_C(C) = \mathcal{O}_C(x) \), we obtain that \( \dim |\mathcal{O}_S(C)| = 1 \), that is, \( C \) moves on \( S \). Moreover the pencil \( |\mathcal{O}_S(C)| \) has \( x \in S \) as a base point.

We denote by \( \tilde{S} := q^{-1}(S) \subset \mathbb{P}^2 \times \mathbb{P}^2 \). For each curve \( C_t \in |\mathcal{O}_S(C)| \), we denote by \( \tilde{C}_t := q^{-1}(C_t) \subset \tilde{S} \) the corresponding double cover. Furthermore, we define a line bundle \( L_t \in \text{Pic}^{14}(\tilde{C}_t) \), by setting \( \mathcal{O}_{\tilde{C}_t}(1,0) = L_t(-\tilde{x}) \) (in which case, \( \mathcal{O}_{\tilde{C}_t}(0,1) = \nu^*(L_t(-\tilde{x})) \)).

The construction we just explained induces a uniruled parametrization of the universal singular locus of the Prym theta divisor in genus 8 (which dominates \( \mathcal{R}_8 \)). Our result is conditional to a (very plausible) transversality assumption:

**Theorem 3.3.** Assume there exists \( [C, \eta, L, x] \) as above, such that \( S = Q_1 \cap Q_2 \cap \Sigma \subset \mathbb{P}^5 \) is a 16-nodal canonical surface. Then the moduli space

\[ \mathcal{R}_8^3 := \left\{ [C, \eta, L] : [C, \eta] \in \mathcal{R}_8, \ L \in V^3(C, \eta) \right\} \]

is uniruled.

**Proof.** The assignment \( \mathbb{P}^1 \ni t \mapsto [\tilde{C}_t/C_t, L_t] \in \mathcal{R}_8^3 \) described above provides a rational curve passing through a general point of \( \mathcal{R}_8^3 \). \( \square \)

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