BOUNDARY TERMS IN COMPLEX GENERAL RELATIVITY

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Abstract. A recent analysis of real general relativity based on multisymplectic techniques has shown that boundary terms may occur in the constraint equations, unless some boundary conditions are imposed. This paper studies the corresponding form of such boundary terms in complex general relativity, where space-time is a four-complex-dimensional complex-Riemannian manifold. A complex Ricci-flat space-time is recovered providing some boundary conditions are imposed on two-complex-dimensional surfaces. One then finds that the holomorphic multimomenta should vanish on an arbitrary three-complex-dimensional surface, to avoid having restrictions at this surface on the spinor fields which express the invariance of the theory under holomorphic coordinate transformations. The Hamiltonian constraint of real general relativity is then replaced by a geometric structure linear in the holomorphic multimomenta, and a link with twistor theory is found. Moreover, a deep relation emerges between complex space-times which are not anti-self-dual and two-complex-dimensional surfaces which are not totally null.

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1. Introduction

Among the various approaches to the quantization of the gravitational field, much insight has been gained by the use of twistor theory and Hamiltonian techniques (see [1-5] and references therein). For example, it is by now well-known how to reconstruct an anti-self-dual space-time out of deformations of flat projective twistor space, and the various definitions of twistors in curved space-time enable one to obtain relevant information about complex space-time geometry within a holomorphic, conformally invariant framework. Moreover, the recent approaches to canonical gravity described in [3] have led to many exact solutions of the quantum constraint equations of general relativity, although their physical relevance for the quantization programme remains unclear. A basic difference between the Penrose formalism [1-2,5] and the Ashtekar formalism [3] is as follows. The twistor programme refers to a four-complex-dimensional complex-Riemannian manifold with holomorphic metric, holomorphic connection and holomorphic curvature tensor, where the complex Einstein equations are imposed. By contrast, in the recent approaches to canonical gravity, one studies complex tetrads on a four-real-dimensional Lorentzian manifold, and real general relativity may be recovered providing one is able to impose suitable reality conditions.

The aim of this paper is to describe a new property of complex general relativity within the holomorphic framework relevant for twistor theory, whose derivation results from recent attempts to obtain a manifestly covariant formulation of Ashtekar’s programme [6]. For this purpose, section 2 studies boundary conditions relevant for the multisymplectic
description of Lorentzian space-times, whilst their holomorphic counterpart appears in section 3. The multisymplectic form of complex general relativity, with the corresponding equations, which are linear in the holomorphic multimomenta, is studied in section 4. Open problems are presented in section 5.

2. Boundary conditions in the Lorentzian theory

It has been recently shown in [6] that the constraint analysis of general relativity may be performed by using multisymplectic techniques, without relying on a 3+1 split of the space-time four-geometry. The constraint equations (cf section 4) have been derived while paying attention to boundary terms, and the Hamiltonian constraint turns out to be linear in the multimomenta. Whilst the latter property is more relevant for the (as yet unknown) quantum theory of gravitation, the former result on boundary terms deserves further thinking already at the classical level, and is the object of our investigation.

We here write the Lorentzian space-time 4-metric as

\[ g_{ab} = e_a^\hat{a} \ e_b^\hat{b} \ \eta_{\hat{a}\hat{b}} \]  

(2.1)

where \( e_a^\hat{a} \) is the cotetrad and \( \eta \) is the Minkowski metric. In first-order theory, the tetrad \( e_a^\hat{a} \) and the connection 1-form \( \omega_a^\hat{b}\hat{c} \) are regarded as independent variables. In [6] it has been shown that, on using jet-bundle formalism and covariant multimomentum maps (see appendix), the constraint equations of real general relativity hold on an arbitrary three-real-dimensional hypersurface \( \Sigma \) providing one of the following three conditions holds:
Boundary terms in complex general relativity

(i) $\Sigma$ has no boundary;

(ii) the multimomenta $\tilde{p}^{ab}_{\hat{c}\hat{d}} \equiv e\left(e^a_{\hat{c}} e^b_{\hat{d}} - e^b_{\hat{c}} e^a_{\hat{d}}\right)$ vanish at $\partial \Sigma$, $e$ being the determinant of the tetrad;

(iii) an element of the algebra $o(3,1)$ corresponding to the gauge group, represented by the antisymmetric $\lambda^{\hat{a}\hat{b}}$, vanishes at $\partial \Sigma$, and the connection 1-form $\omega_a^{\hat{b}\hat{c}}$ or $\xi^b$ vanishes at $\partial \Sigma$, $\xi$ being a vector field describing diffeomorphisms on the base-space.

In other words, boundary terms may occur in the constraint equations of real general relativity, and they result from the total divergences of [6]

$$\sigma^{ab} \equiv \tilde{p}^{ab}_{\hat{c}\hat{d}} \lambda^{\hat{c}\hat{d}}$$  \hspace{1cm} (2.2)

$$\rho^{ab} \equiv \tilde{p}^{ab}_{\hat{c}\hat{d}} \omega_{f}^{\hat{c}\hat{d}} \xi^{f}$$  \hspace{1cm} (2.3)

integrated over $\Sigma$.

In two-component spinor language, denoting by $\tau_{\hat{a}BB'}$ the Infeld-van der Waerden symbols, the two-spinor version of the tetrad reads

$$e^{a}_{BB'} \equiv e^a_{\hat{a}} \tau^{\hat{a}}_{BB'}$$  \hspace{1cm} (2.4)

which implies that $\sigma^{ab}$ in (2.2) takes the form

$$\sigma^{ab} = e\left(e^{a}_{CC'} e^{b}_{DD'} - e^{a}_{DD'} e^{b}_{CC'}\right) \tau_{\hat{a}CC'} \tau^{\hat{b}DD'} \lambda^{\hat{a}\hat{b}}.$$  \hspace{1cm} (2.5)

Thus, on defining the spinor field

$$\lambda^{CC'CD'} \equiv \tau_{\hat{a}CC'} \tau^{\hat{b}DD'} \lambda^{\hat{a}\hat{b}} \equiv \Lambda^{(CD)}_{1} \epsilon^{C'D'} + \Lambda^{(C'D')}_{2} \epsilon^{CD}$$  \hspace{1cm} (2.6)
the first of the boundary conditions in (iii) is satisfied providing $\Lambda_1^{(CD)} = 0$ at $\partial \Sigma$ in real general relativity, since then $\Lambda_2^{(C'D')}$ is obtained by complex conjugation of $\Lambda_1^{(CD)}$, and hence the condition $\Lambda_2^{(C'D')} = 0$ at $\partial \Sigma$ leads to no further information.

3. Boundary conditions in the holomorphic framework

In the holomorphic framework, no complex conjugation relating primed to unprimed spin-space can be defined, since such a map is not invariant under holomorphic coordinate transformations [1,5]. Hence spinor fields belonging to unprimed or primed spin-space are *totally independent*, and the first of the boundary conditions in (iii) reads

$$\Lambda^{(CD)} = 0 \quad \text{at } \partial \Sigma_c$$

(3.1)

$$\tilde{\Lambda}^{(C'D')} = 0 \quad \text{at } \partial \Sigma_c$$

(3.2)

where $\partial \Sigma_c$ is a two-complex-dimensional complex surface, bounding the three-complex-dimensional surface $\Sigma_c$, and the tilde is used to denote *independent* spinor fields [1,5], not related by any conjugation.

Similarly, $\rho^{ab}$ in (2.3) takes the form

$$\rho^{ab} = e\left(e^a_{CC'} e^b_{DD'} - e^a_{DD'} e^b_{CC'}\right)\left(\Omega_f^{(CD)} e^{C'D'} + \tilde{\Omega}_f^{(C'D')} e^{C'D}\right)\xi^f$$

(3.3)

and hence the second of the boundary conditions in (iii) leads to the independent boundary conditions

$$\Omega_f^{(CD)} = 0 \quad \text{at } \partial \Sigma_c$$

(3.4)
Boundary terms in complex general relativity

\[ \tilde{\Omega}_f^{(C'D')} = 0 \quad \text{at } \partial \Sigma_c \]  

(3.5)

in complex general relativity. The equations (3.4)-(3.5) may be replaced by the condition

\[ u^{AA'} = 0 \quad \text{at } \partial \Sigma_c \]  

(3.6)

where \( u \) is a holomorphic vector field describing holomorphic coordinate transformations on the base-space, i.e. on complex space-time.

4. Multisymplectic form of complex general relativity

The picture of complex general relativity resulting from sections 2-3, and from the analysis in [6], is highly non-trivial. One starts from a one-jet bundle \( J^1 \) which, in local coordinates, is described by a holomorphic coordinate system, with holomorphic tetrad, holomorphic connection 1-form \( \omega_a \mapsto \hat{c} \), multivelocities corresponding to the tetrad and multivelocities corresponding to \( \omega_a \mapsto \hat{b} \), both of holomorphic nature. The intrinsic form of the field equations, which is a generalization of a mathematical structure already existing in classical mechanics, leads to the complex vacuum Einstein equations \( R_{ab} = 0 \), and to a condition on the covariant divergence of the multimomenta. Moreover, the covariant multimomentum map (see appendix and [6]), evaluated on a section of \( J^1 \) and integrated on an arbitrary three-complex-dimensional surface \( \Sigma_c \), reflects the invariance of complex general relativity under all holomorphic coordinate transformations. Since space-time is now a complex manifold, one deals with holomorphic coordinates which are all on the same footing, and hence no time coordinate can be defined. Thus, the constraints result from the holomorphic version
Boundary terms in complex general relativity

of the covariant multimomentum map, but cannot be related to a Cauchy problem as in the Lorentzian theory (cf [7] and references therein). In particular, the Hamiltonian constraint of Lorentzian general relativity is replaced by a geometric structure which is linear in the holomorphic multimomenta, providing two boundary terms can be set to zero (of course, our multimomenta are holomorphic by construction, since in complex general relativity the tetrad is holomorphic). For this purpose, one of the following three conditions should hold:

(i) \( \Sigma_c \) has no boundary;

(ii) the holomorphic multimomenta vanish at \( \partial \Sigma_c \);

(iii) the equations (3.1)-(3.2) hold at \( \partial \Sigma_c \), and the equations (3.4)-(3.5), or (3.6), hold at \( \partial \Sigma_c \).

Before imposing the boundary conditions (i), or (ii), or (iii), the constraint equations (see previous remarks) of complex general relativity read (cf (2.2)-(2.3))

\[
\int_{\Sigma_c} \partial_a \sigma^{ab} \, d^3x_b - \int_{\Sigma_c} \lambda^{\hat{c}\hat{d}} \left( D_a \tilde{\rho}^{ab} \right)_{\hat{c}\hat{d}} \, d^3x_b = 0
\]

\[\text{(4.1)}\]

\[
\int_{\Sigma_c} \partial_a \rho^{ab} \, d^3x_b - \int_{\Sigma_c} \text{Tr} \left[ \tilde{\rho}^{af} \Omega_{ad} - \frac{1}{2} \tilde{\rho}^{ab} \Omega_{ab} \delta^f_d \right] u^d \, d^3x_f = 0.
\]

\[\text{(4.2)}\]

With our notation, \( \Omega_{a\hat{c}\hat{d}} \) is the holomorphic curvature of the holomorphic connection 1-form \( \omega_a \hat{c}\hat{d} \). Moreover, \( D \) is a connection which annihilates the internal-space metric \( \eta_{\hat{a}\hat{b}} \) (cf [6]). On imposing the boundary conditions studied so far, the first term on the left-hand side of (4.1)-(4.2) vanishes, and the preservation of constraints yields the contracted Bianchi
Boundary terms in complex general relativity

identities. Thus, the full set of field equations linear in the holomorphic multimomenta takes the form (cf [6])

\[
\text{Tr} \left[ \tilde{p}^{ij} \Omega_{ij} \right] = 0 \quad (4.3)
\]

\[
\text{Tr} \left[ \tilde{p}^{i0} \Omega_{ij} \right] = 0 \quad (4.4)
\]

\[
(D_a \tilde{p}^{a0})_{\hat{c}\hat{d}} = 0. \quad (4.5)
\]

We omit the details to avoid repeating the analysis appearing in [6]. However, we should emphasize that (4.3)-(4.5) are obtained by fixing the holomorphic coordinate $z^0$, which does not have a distinguished role with respect to $z^1, z^2, z^3$. Hence the interpretation of our particular coordinate system is quite different from the Lorentzian case. In other words, the equations (4.3)-(4.5) correspond to the Hamiltonian, momentum and Gauss constraints of the Lorentzian theory, respectively, but they should not be regarded as describing a 3+1 split of the four-complex-dimensional geometry.

Note that it is not a priori obvious that the three-complex-dimensional surface $\Sigma_c$ has no boundary. Hence one really has to consider the boundary conditions (ii) or (iii) in the holomorphic framework. They imply that the holomorphic multimomenta have to vanish everywhere on $\Sigma_c$ (by virtue of a well-known result in complex analysis), or the elements of $\mathfrak{o}(4, C)$ have to vanish everywhere on $\Sigma_c$, jointly with the self-dual and anti-self-dual parts of the connection 1-form. The latter of these conditions may be replaced by the vanishing of the holomorphic vector field $u$ on $\Sigma_c$. In other words, if $\Sigma_c$ has a boundary, unless the holomorphic multimomenta vanish on the whole of $\Sigma_c$, there are restrictions at $\Sigma_c$ on the spinor fields expressing the holomorphic nature of the theory and its invariance under all
holomorphic coordinate transformations. Indeed, already in real Lorentzian four-manifolds one faces a choice between boundary conditions on the multimomenta and restrictions on the invariance group resulting from boundary effects. We choose the former and emphasize their role in complex general relativity. Of course, the spinor fields involved in the boundary conditions are instead non-vanishing on the four-complex-dimensional space-time.

Remarkably, to ensure that the holomorphic multimomenta \( \tilde{p}^{ab}_{\hat{c}\hat{d}} \) vanish at \( \partial \Sigma_c \), and hence on \( \Sigma_c \) as well, the determinant \( e \) of the tetrad should vanish at \( \partial \Sigma_c \), or \( e^{-1} \tilde{p}^{ab}_{\hat{c}\hat{d}} \) should vanish at \( \partial \Sigma_c \). The former case admits as a subset the totally null two-complex-dimensional surfaces known as \( \alpha \)-surfaces and \( \beta \)-surfaces [1-2,5]. Since the integrability condition for \( \alpha \)-surfaces is expressed by the vanishing of the self-dual Weyl spinor, our formalism enables one to recover the anti-self-dual (also called right-flat) space-time relevant for twistor theory, where both the Ricci spinor \( R_{AA'B'B'} \) and the self-dual Weyl spinor \( \tilde{\psi}_{AA'B'C'D'} \) vanish. However, if \( \partial \Sigma_c \) is not totally null, the resulting theory does not correspond to twistor theory. The latter case implies that the tetrad vectors are turned into holomorphic vectors \( u_1, u_2, u_3, u_4 \), say, such that one of the following conditions holds at \( \partial \Sigma_c \), and hence on \( \Sigma_c \) as well: (i) \( u_1 = u_2 = u_3 = u_4 = 0 \); (ii) \( u_1 = u_2 = u_3 = 0, u_4 \neq 0 \); (iii) \( u_1 = u_2 = 0, u_3 = \gamma u_4, \gamma \in \mathcal{C} \); (iv) \( u_1 = 0, \gamma_2 u_2 = \gamma_3 u_3 = \gamma_4 u_4, \gamma_i \in \mathcal{C}, i = 2, 3, 4 \); (v) \( \gamma_1 u_1 = \gamma_2 u_2 = \gamma_3 u_3 = \gamma_4 u_4, \gamma_i \in \mathcal{C}, i = 1, 2, 3, 4 \).
5. Open problems

It now appears essential to understand the relation between complex general relativity derived from jet-bundle theory and complex general relativity as in the Penrose twistor programme. For this purpose, one needs to study the topology and the geometry of the space of two-complex-dimensional surfaces $\partial \Sigma_c$ in the generic case. This leads to a deep link between complex space-times which are not anti-self-dual and two-complex-dimensional surfaces which are not totally null. In other words, on going beyond twistor theory, one finds that the analysis of two-complex-dimensional surfaces still plays a key role. Last, but not least, one has to solve equations (cf (4.3)-(4.5)) which are now linear in the holomorphic multimomenta, both in classical and in quantum gravity (these equations correspond to the constraint equations of the Lorentzian theory). Hence we hope that our paper may provide the first step towards a new synthesis in relativistic theories of gravitation.

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Appendix

To help the readers who are not familiar with multisymplectic geometry, we present a very brief outline of jet bundles and momentum maps.

In section 4, the notation $J^1$ means what follows. Let $X$ be a manifold, and let $Y$ be a fibre bundle having $X$ as its base space, with projection map $\pi_{XY}$. Moreover, let $\gamma : T_xX \rightarrow T_yY$ be a linear map between the tangent space to $X$ at $x$ and the tangent space to $Y$ at $y \in \pi_{XY}^{-1}(x)$. Given a point $y$ belonging to the fibre $Y_x$ through $x \in X$, one considers all $\gamma$ maps relative to $y \in Y_x$. This leads to a fibre bundle $J^1(Y)$ having the fibre bundle $Y$ as its base space and fibres given by the $\gamma$ maps. Such a $J^1(Y)$ is called the one-jet bundle on $Y$.

A familiar property of classical mechanics and classical field theory is that, if the Lagrangian is invariant under the action of a group, by virtue of Noether’s theorem there exist functions which are constant along solutions of the equations of motion. The constraints of a field theory result from Noether’s theorem through the action of the gauge group or the group of space-time diffeomorphisms. The covariant multimomentum map is the mathematical tool which enables one to describe these properties of classical fields. In section 4, the covariant multimomentum map $\tilde{J}(u)$ reads

$$
\tilde{J}(u) \equiv \left[ e \frac{\partial L}{\partial \omega_a} \left( u_a \hat{e} \hat{a} - \omega_a \hat{e} \hat{d} , u^b \right) + \frac{e}{2} e^a \hat{c} e^b \hat{d} \Omega_{ab} \hat{c} \hat{d} u^f \right] d^3x_f
$$

(A1)
Boundary terms in complex general relativity

where \( L \equiv \frac{e}{2} e^a \hat{c} e^b \hat{d} \Omega_{ab} \hat{c} \hat{d} \) is the Lagrangian. With our notation, \( u_a \hat{c} \hat{d} \) describes coordinate transformations along the fibre, and it is given by

\[
u_a \hat{c} \hat{d} = -u^b_{\, ,a} \omega_b \hat{c} \hat{d} + (D_a \lambda) \hat{c} \hat{d}. (A2)
\]

Moreover, the spinorial form of the holomorphic vector field \( u \) (see (3.6)) is obtained from the familiar relation \( u^a e_a A A' = u^{A A'} \). On integrating \( \tilde{J}(u) \) on \( \Sigma_c \) and setting such an integral to zero, the holomorphic constraint equations (4.1)-(4.2) are obtained.

References

[1] Penrose R and Rindler W 1986 Spinors and Space-Time II: Spinor and Twistor Methods in Space-Time Geometry (Cambridge: Cambridge University Press)

[2] Ward R S and Wells R O 1990 Twistor Geometry and Field Theory (Cambridge: Cambridge University Press)

[3] Ashtekar A 1991 Lectures on Non-Perturbative Canonical Gravity (Singapore: World Scientific)

[4] Esposito G 1994 Quantum Gravity, Quantum Cosmology and Lorentzian Geometries (Lecture Notes in Physics m12) (Berlin: Springer)

[5] Esposito G 1995 Complex General Relativity (Fundamental Theories of Physics 69) (Dordrecht: Kluwer)

[6] Esposito G, Gionti G and Stornaiolo C Space-Time Covariant Form of Ashtekar’s Constraints (DSF preprint 95/7)

[7] Kaiser G 1981 J. Math. Phys. 22 705