On the functional determinant of a special operator with a zero mode in cosmology

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Abstract. The functional determinant of a special second order quantum-mechanical operator is calculated with its zero mode gauged out by the method of the Faddeev-Popov gauge fixing procedure. This operator subject to periodic boundary conditions arises in applications of the early Universe theory and, in particular, determines the one-loop statistical sum in quantum cosmology generated by a conformal field theory (CFT). The calculation is done for a special case of a periodic zero mode of this operator having two roots (nodes) within the period range, which corresponds to the class of cosmological instantons in the CFT driven cosmology with one oscillation of the cosmological scale factor of its Euclidean Friedmann-Robertson-Walker metric.

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The theory of long-wavelength perturbations in early Universe cosmology, including the formation of observable CMB spectra [1, 2], essentially relies on the differential equation with the operator of the form

$$F = -\frac{1}{g} \frac{d}{d\tau} \frac{g^2}{d\tau} \frac{1}{g} = -\frac{d^2}{d\tau^2} + \ddot{g} \frac{\dot{g}}{g}, \quad (1.1)$$

where $g = g(\tau)$ is a rather generic function of the cosmic time depending on the behavior of the cosmological scale factor $a$ and its time derivative, $g \propto \dot{a}$. In particular, for superhorizon cosmological perturbations of small momenta $k^2 \ll \dot{g}/g$ their evolution operator only slightly differs from (1.1) by adding $k^2$ to its potential term, whereas in the minisuperspace sector of cosmology, corresponding to spatially constant variables, the operator has exactly the above form.

Up to an overall sign, this operator is the same both in the Lorentzian and Euclidean signature spacetimes with the time variables related by the Wick rotation $\tau = it$. In the Euclidean case it plays a very important role in the calculation of the statistical sum for the microcanonical ensemble in cosmology — a concept of initial conditions, which is very promising from the viewpoint of the cosmological constant, inflation and dark energy problems [3–7]. However, in this statistical theory context, when $\tau$ plays the role of the Euclidean time, the properties of this operator essentially differ from the Lorentzian dynamics. In the latter case the function $g$ is a monotonic function of time because of the monotonically growing cosmological scale factor, whereas in the Euclidean case $g(\tau)$ is periodic just as the scale factor $a(\tau)$ itself and, moreover, has zeroes at turning points of the Euclidean evolution with $\dot{a} = 0$, because $g(\tau) \propto \dot{a}(\tau)$.

This does not lead to a singular behavior of $F$ because $\ddot{g}$ also vanishes at the zeroes of $g$ [3, 7], and the potential term of (1.1) remains analytic (both $g$ and $\ddot{g}$ simultaneously have
a first-order zero as it follows from the dynamical equation for \( a(\tau) \) in [3]). Nevertheless, the calculation of various quantities associated with this operator becomes cumbersome due to the roots of \( g(\tau) \). Among such quantities is the functional determinant of \( F \) which determines the one-loop contribution to the statistical sum of the CFT driven cosmology of [7]. Its peculiarity is that the operator has an obvious zero mode which is the function \( g(\tau) \) itself,

\[
F g(\tau) = 0,
\]

(1.2)

and the functional determinant of \( F \) should, of course, be understood as calculated on the subspace of its nonzero modes. Thus, the focus of this paper is the calculation of such a restricted functional determinant of (1.1), denoted below by \( \text{Det}_* F \). This determinant is calculated on the space of functions periodic on a compactified range of the Euclidean time \( \tau \) (forming a circle) with its zero mode \( g \) removed or gauged out.

There exist several different methods for restricted functional determinants. When the whole spectrum of the operator is known this is just the product of all non-zero eigenvalues. With the knowledge of only the zero mode, one can use the regularization technique of [8–10] to extract the regulated zero-mode eigenvalue from the determinant and subsequently take the regularization off. Here we use another approach to the definition of \( \text{Det}_* F \) dictated by the gauge-fixing procedure for the path integral in cosmology [7].

The zero mode of (1.1) arises in quantum cosmology as a generator of the global gauge transformation of the gravitational variable \( \varphi \) (a canonically normalized perturbation of the cosmological scale factor in the Friedmann-Robertson-Walker metric [7]), \( \Delta^\varepsilon \varphi(\tau) = \varepsilon g(\tau) \), which is the residual symmetry of the action

\[
S[\varphi] = \frac{1}{2} \oint d\tau \varphi(\tau) F\varphi(\tau),
\]

(1.3)

remaining after gauge-fixing the local diffeomorphism invariance. Here the integration runs over the full period of time compactified to a circle according to the definition of the cosmological statistical sum [4], and the variable \( \varphi(\tau) \) as well as the zero mode \( g(\tau) \) are periodic on this circle. Therefore, this symmetry is also subject to the Faddeev-Popov gauge fixing procedure consisting of imposing the gauge on the integrated (1-dimensional) field and inserting in the path integral the the relevant Faddeev-Popov factor.

This gauge condition \( \chi(\varphi) \) and the corresponding Faddeev-Popov ghost factor \( Q \) can be chosen in the form

\[
\chi = \oint d\tau g(\tau) \varphi(\tau),
\]

(1.4)

\[
\Delta^\varepsilon \chi = Q\varepsilon, \quad Q = \oint d\tau g^2(\tau),
\]

(1.5)

and the restricted functional determinant of \( F \) can be determined as the following Faddeev-Popov Gaussian functional integral with the delta-function type gauge

\[
(\text{Det}_* F)^{-1/2} = \text{const} \times \int D\varphi \delta\left( \oint d\tau g\varphi \right) Q \exp \left\{ -\frac{1}{2} \oint d\tau \varphi F\varphi \right\}.
\]

(1.6)

This definition is in fact independent of the choice of gauge by the usual gauge independence mechanism for the Faddeev-Popov integral. In particular, enforcing the gauge \( \chi = 0 \) means that the field \( \varphi \) is functionally orthogonal to the zero mode \( g(\tau) \) in a trivial \( L^2 \) metric on
a circle, and the above definition is independent of the choice of this metric — a possible function weighting the integrand of (1.4).

In this paper we undertake the calculation of this path integral for the restricted functional determinant of (1.1) as an explicit functional of $g(\tau)$. The result will be obtained in quadratures for a particular case when the function $g(\tau)$ has one oscillation in the full range of the Euclidean time forming a circle. Therefore, the function $g(\tau)$ has two zeroes at the points labeled by $\tau_\pm$, $g(\tau_\pm) = 0$, which will mark the boundaries of the half period of the total time range, $T = 2(\tau_+ - \tau_-)$. For brevity of the formalism we shift the first root of $g(\tau)$ to zero, $\tau_- = 0$, and let the coordinate $\tau$ run in the total range $-\tau_+ \leq \tau \leq \tau_+$ with the points $\pm \tau_+$ identified. Then $g(\tau)$ is an odd function of $\tau$ which is periodic with all its derivatives and has two first degree zeros at antipodal points $\tau = \tau_- \equiv 0$ and $\tau = \tau_+$ of this circle

$$g(\tau) = -g(-\tau),$$

$$g(\tau_\pm) = 0, \quad \dot{g}(\tau_\pm) \equiv \dot{g}_\pm \neq 0. \quad \tag{1.8}$$

As mentioned above, in spite of singularity of $1/g$ at $\tau_\pm$ the operator (1.1) is everywhere regular (analytic) on the circle, because $\dot{g}(\tau_\pm) = 0$.

The final result of this paper — the restricted functional determinant of the operator (1.1) with the zero mode $g$ gauged out — is determined by a special solution of the homogeneous equation $F \Psi = 0$. This is a two-point function $\Psi(\tau, \tau_s)$

$$\Psi(\tau, \tau_s) \equiv g(\tau) \int_{\tau_s}^{\tau} dy \frac{\dot{g}(y)}{g^2(y)}, \quad \tau_- \equiv 0 < \tau < \tau_+, \quad \tau_- < \tau_s < \tau_+, \quad \tag{1.9}$$

with some fixed point $\tau_s$ in the half period range of $\tau$. The determinant of (1.1) in terms of this solution equals

$$\operatorname{Det}_* F = \text{const} \times \left| \Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_- \right|, \quad \tag{1.10}$$

$$\Psi_\pm \equiv \Psi(\tau_\pm, \tau_s), \quad \dot{\Psi}_\pm \equiv \dot{\Psi}(\tau_\pm, \tau_s). \quad \tag{1.11}$$

The main property of the function $\Psi(\tau, \tau_s)$ is that it is smoothly defined in the half-period range of $\tau$ and $\tau_s$, where the integral (1.9) is convergent because the roots of $g(\tau)$ do not occur in the integration range. It cannot be smoothly continued beyond the half-period $\tau_- \leq \tau \leq \tau_+$, though its limits are well defined for $\tau \to \tau_\pm \equiv 0$,

$$\Psi(\tau_\pm, \tau_s) = -\frac{1}{g(\tau_\pm)} \equiv -\frac{1}{\dot{g}_\pm}, \quad \tag{1.12}$$

because the factor $g(\tau)$ tending to zero compensates for the divergence of the integral at $\tau \to \tau_\pm$. Moreover, because of $\dot{g}(\tau_\pm) = 0$ the function $\Psi(\tau, \tau_s)$ is differentiable at $\tau \to \tau_\pm$, and all the quantities which enter the algorithm (1.10) are well defined. These properties of $\Psi(\tau, \tau_s)$ guarantee the independence of the obtained result from an arbitrary choice of the point $\tau_s$, which can be easily verified by using a simple relation $d\Psi_\pm / d\tau_s = -\dot{\Psi}_\pm / g^2(\tau_s)$.

## 2 Variational expression for the determinant

Representing the delta function of the gauge condition in (1.6) via the integral over the Lagrangian multiplier $\pi$ we get the Gaussian path integral over the periodic function $\varphi(\tau)$ and the numerical variable $\pi$ which are collectively denoted by $\Phi = (\varphi(\tau), \pi)$,

$$(\operatorname{Det}_* F)^{-1/2} = \text{const} \times Q \int D\varphi D\pi \exp \left( -S_{\text{eff}}[\varphi(\tau), \pi] \right) = \text{const} \times Q \left( \operatorname{Det} F \right)^{-1/2}. \quad \tag{2.1}$$
Here $S_{\text{eff}}[\varphi(\tau), \pi]$ is the effective action of these variables and $F$ is the matrix valued Hessian of this action with respect to $\Phi$,

$$S_{\text{eff}}[\varphi(\tau), \pi] = \int d\tau \left( \frac{1}{2} \varphi F \varphi - i \pi g \varphi \right),$$

$$F = \frac{\delta^2 S_{\text{eff}}}{\delta \Phi \delta \Phi'} = \begin{bmatrix} F \delta(\tau, \tau') & -ig(\tau) \\ -ig(\tau') & 0 \end{bmatrix}$$

(note the position of time entries associated with the variables $\Phi = (\varphi(\tau), \pi)$ and $\Phi' = (\varphi(\tau'), \pi)$).

A dependence of this determinant on $g$ can be found from its functional variation with respect to $g(\tau)$. From (2.1) we have

$$\delta \ln \left( \text{Det}_* F \right) = -2 \delta \ln Q + \text{Tr} \left( \delta F G \right),$$

where $G$ is the Green’s function of $F$, $FG = I$. The block structure of this matrix Green’s function has the form

$$G = \begin{bmatrix} G(\tau, \tau') & ig(\tau) \\ ig(\tau') & 0 \end{bmatrix},$$

where the Green’s function $G(\tau, \tau')$ in the diagonal block satisfies the system of equations

$$FG(\tau, \tau') = \delta(\tau, \tau') - \frac{g(\tau)g(\tau')}{Q},$$

$$\oint d\tau g(\tau) G(\tau, \tau') = 0,$$

which uniquely fix it. The second equation imposes the needed gauge, whereas the right hand side of the first equation implies that $G(\tau, \tau')$ is the inverse of the operator $F$ on the subspace orthogonal to its zero mode. In what follows it will be useful to express $G(\tau, \tau')$ in terms of another auxiliary Green’s function $\tilde{G}(\tau, \tau')$. It arises as follows. From (2.6)–(2.7) it follows that the Green’s function $G(\tau, \tau')$ gives the solution

$$\tilde{\varphi}(\tau) = \oint d\tau' G(\tau, \tau') J(\tau')$$

to the following problem

$$F \tilde{\varphi}(\tau) = \tilde{J}(\tau),$$

$$\oint d\tau g \tilde{\varphi} = 0,$$

$$\tilde{J}(\tau) \equiv J(\tau) - \frac{g(\tau)}{Q} \oint d\tau_1 g(\tau_1) J(\tau_1),$$

with the modified source $\tilde{J}(\tau)$ which is functionally orthogonal to the zero mode $g(\tau)$ — the property that guarantees the existence of the solution of eq. (2.9) whose left hand side is also...
functionally orthogonal to \( g \) in view of integration by parts in \( \oint d\tau g(F,\tilde{\varphi}) = 0 \). If we denote by \( \tilde{G}(\tau, \tau') \) an auxiliary Green’s function which solves this last problem in the form

\[
\tilde{\varphi}(\tau) = \oint d\tau \tilde{G}(\tau, \tau') \tilde{J}(\tau'),
\]

then the comparison with (2.8) shows the relation between the two Green’s functions via the projection of \( \tilde{G}(\tau, \tau') \) with respect to the second argument onto the subspace orthogonal to the zero mode

\[
G(\tau, \tau') = \tilde{G}(\tau, \tau') - \oint d\tau_1 \tilde{G}(\tau, \tau_1) \frac{g(\tau_1) g(\tau')}{Q}.
\]

(2.13)

The introduction of \( \tilde{G}(\tau, \tau') \) is justified by a number of simplifications which one has when finding the solution \( \tilde{\varphi}(\tau) \) in terms of \( \tilde{J}(\tau) \) rather than in terms of the original source. This remains true even despite the fact that this auxiliary Green’s function is not unique because of the freedom in the transformation \( \tilde{G}(\tau, \tau') \rightarrow \tilde{G}(\tau, \tau') + \epsilon(\tau) g(\tau') \) preserving (2.12). This freedom results in the ambiguity of the equation for \( \tilde{G}(\tau, \tau') \),

\[
F \tilde{G}(\tau, \tau') = \delta(\tau, \tau') + \omega(\tau) g(\tau'),
\]

with some function \( \omega(\tau) \) which is transformed as \( \omega(\tau) \rightarrow \omega(\tau) + F \epsilon(\tau) \). The function \( \omega(\tau) \) can be rather general and is only subject to the condition \( \oint d\tau g \omega = -1 \) which follows from integrating the above equation with \( g(\tau) \) — the zero mode of \( F \).\(^1\) This ambiguity, of course, goes away in the Green’s function (2.13) due to the projection operation.

Now we can express the variation (2.4) in terms of the above Green’s functions. Using (2.3) and (2.5) in (2.4) we have

\[
\delta \ln \left( \text{Det}_s F \right) = \oint d\tau \delta F G(\tau, \tau') \bigg|_{\tau' = \tau} - \delta \ln Q.
\]

(2.15)

Also substituting (2.13) in the first term and integrating by parts the action of the operator \( \delta F \) we have

\[
\oint d\tau \delta F \bigg|_{\tau' = \tau} = \oint d\tau \delta F \tilde{G} \bigg|_{\tau' = \tau} - \frac{1}{Q} \oint d\tau dy \left( \delta F g(\tau) \right) \tilde{G}(\tau, y) g(y),
\]

(2.16)

whence on account of the obvious variational relation \( \delta F g = -F \delta g \) and again integrating by parts we have

\[
\oint d\tau \delta F \bigg|_{\tau' = \tau} = \oint d\tau \delta F \tilde{G} \bigg|_{\tau' = \tau} + \frac{1}{2} \delta \ln Q + \oint d\tau \delta g(\tau) \omega(\tau),
\]

(2.17)

where we have used (2.14). Therefore finally

\[
\delta \ln \left( \text{Det}_s F \right) = \oint d\tau \delta F \tilde{G}(\tau, \tau') \bigg|_{\tau' = \tau} - \frac{1}{2} \delta \ln Q + \oint d\tau \delta g(\tau) \omega(\tau).
\]

(2.18)

One can check that this expression is also invariant with respect to the \( \epsilon \)-transformation of \( \tilde{G}(\tau, \tau') \) and \( \omega(\tau) \) of the above type. Below we explicitly find \( \tilde{G}(\tau, \tau') \) along with its \( \omega(\tau) \) and calculate the above variation.

\(^1\)This does not necessarily imply that \( \omega(\tau) = -g(\tau)/\oint dy g^2(y) \), as one would expect from the functional orthogonality of \( g(\tau) \) and \( \tilde{G}(\tau, \tau') \). This will be confirmed by the calculation of a specific \( \omega(\tau) \) below.
3 The periodic boundary conditions problem and its Green’s function

We will look for the periodic solution of the problem (2.9)–(2.10) as a linear combination of the
partial solution of the inhomogeneous equation (2.9) reads

\[
\Phi(\tau) = -g(\tau) \int_0^\tau \frac{dy}{g^2(y)} \int_0^y dr' gJ(r') = -\int_0^\tau dr' \Psi(\tau, r') g(r') \tilde{J}(r')
\]

\[
= -\int_0^\tau dr' \left[ \theta(\tau - r') \theta(r') - \theta(\tau' - \tau) \theta(-\tau') \right] \Psi(\tau, r') g(r') \tilde{J}(r').
\]

(3.1)

It is defined for all \(-\tau_+ \leq \tau \leq \tau_+\) and satisfies at \(\tau = \tau_-\) the initial conditions

\[
\Phi(0) = 0, \quad \dot{\Phi}(0) = 0.
\]

However it is not periodic, because \(\Phi(-\tau_+) = \Phi(\tau_+), \text{ but } \Phi(-\tau_+) \neq \tilde{\Phi}(\tau_+)\). Indeed, in view of (1.12) and (2.11)

\[
\Phi(\tau_+) - \Phi(-\tau_+) = \frac{1}{g_+} \left( \int_{\tau_+}^{\tau_+} - \int_{\tau_+}^{-\tau_+} \right) d\tau gJ = \frac{1}{g_+} \int d\tau gJ(\tau) = 0,
\]

(3.3)

\[
\dot{\Phi}(\tau_+) - \ddot{\Phi}(-\tau_+) = -\int_0^{\tau_+} dy \dot{\Psi}(\tau_+, y) g(y) \tilde{J}(y) + \int_0^{-\tau_+} dy \ddot{\Psi}(-\tau_+, y) g(y) \tilde{J}(y)
\]

\[
= -\int_0^{\tau_+} dy \left[ \ddot{\Psi}(\tau_+, y) \theta(y) - \ddot{\Psi}(\tau_+, y) \theta(-y) \right] g(y) \tilde{J}(y) \neq 0.
\]

(3.4)

Here we used the fact that in view of the asymmetry of \(g(\tau), g(-\tau) = -g(\tau), \Psi(-\tau, y) = \Psi(\tau, -y)\) and \(\dot{\Psi}(-\tau, y) = -\dot{\Psi}(\tau, -y)\).

Thus, we shall look for the solution in question as a linear combination of \(\Phi(\tau)\) and two basis functions of \(F\) — the periodic function \(g(\tau)\) and the non-periodic \(\Psi(\tau, \tau')\) with some \(\tau' = \tau_+ \geq 0\). For negative \(\tau\) the role of this second basis function will be played by \(\Psi(-\tau, \tau_+ = 0)\) in view of the odd nature of \(g(\tau)\) in (1.1). Thus the solution has the following piecewise smooth form

\[
\tilde{\varphi}(\tau) = \Phi(\tau) + C_+ \Psi(\tau, \tau_+) \theta(\tau) + C_- \Psi(-\tau, \tau_+) \theta(-\tau)
\]

\[
+ D_+ g(\tau) \theta(\tau) + D_- g(\tau) \theta(-\tau),
\]

(3.5)

where all four coefficients should be determined from the periodic boundary conditions at \(\tau = \pm \tau_+\),

\[
\tilde{\varphi}(\tau_+) = \tilde{\varphi}(-\tau_+),
\]

(3.6)

\[
\dot{\tilde{\varphi}}(\tau_+) = \dot{\tilde{\varphi}}(-\tau_+),
\]

(3.7)

the gauge condition (2.10) and also the conditions of smoothness at \(\tau = \tau_- \equiv 0\)

\[
\tilde{\varphi}(0) = \dot{\tilde{\varphi}}(0) = 0.
\]

(3.8)

(3.9)

In view of \(g(0) = 0\) together with (3.2) and the fact that \(\Psi(\tau, \tau_+ \in (3.5) is continued to negative values of \(\tau\) as an even function, the condition (3.8) implies that \(C_+ = C_-\). The
second of smoothness conditions reads \( C_+ \dot{\Psi}_+ + D_+ \dot{g}_- = -C_- \dot{\Psi}_- + D_- \dot{g}_- \), \( \dot{\Psi}_- \equiv \dot{\Psi}(0, \tau_*) \), and results in

\[
2C_+ \dot{\Psi}_- = -(D_+ - D_-) \dot{g}_-. \tag{3.10}
\]

The first of periodicity conditions (3.6) is identically satisfied because \( g(\pm \tau_+) = 0 \), \( C_- = C_+ \) and (3.3), while the second condition (3.7) reads \( \dot{\Phi}(\tau_+) + C_+ \dot{\Psi}_+ + D_+ \dot{g}_+ = \dot{\Phi}(-\tau_-) - C_- \dot{\Psi}_+ + D_- \dot{g}(-\tau_+) \), where \( \dot{\Psi}_+ \equiv \dot{\Psi}(\tau_+, \tau_*) \), or

\[
\dot{\Phi}(\tau_+) - \dot{\Phi}(-\tau_-) + 2C_+ \dot{\Psi}_+ + (D_+ - D_-) \dot{g}_+ = 0. \tag{3.11}
\]

Therefore in view of (1.12)

\[
C_\pm = -\frac{1}{2} \frac{\Psi_+}{\Psi_+ - \Psi_-} \left[ \dot{\Phi}(\tau_+) - \dot{\Phi}(-\tau_-) \right], \tag{3.12}
\]

\[
D_+ - D_- = -\frac{\Psi_-}{\Psi_+ - \Psi_-} \left[ \dot{\Phi}(\tau_+) - \dot{\Phi}(-\tau_-) \right]. \tag{3.13}
\]

In view of the odd nature of \( g(\tau) \), \( g(-\tau) = -g(\tau) \), the \( C_\pm \) coefficients do not contribute to the last equation on the coefficients of (3.5) — the gauge condition (2.10). This gauge condition leads to

\[
\frac{D_+ + D_-}{2} = -\frac{1}{Q} \oint d\tau g \Phi. \tag{3.14}
\]

Thus, all coefficients are uniquely determined by the source \( J \). Since \( \Phi, C_\pm \) and \( D_\pm \) are all linear in this source the Green’s function \( \tilde{G}(\tau, \tau') \) can be read off (3.5) as

\[
\tilde{G}(\tau, \tau') = \frac{\delta \tilde{\varphi}(\tau)}{\delta J(\tau')}, \tag{3.15}
\]

where of course the functional derivative should not be understood literally because the modified source is functionally restricted by the condition of orthogonality to the zero mode (2.11). Rather, this expression should be regarded as a coefficient of \( \tilde{J}(\tau') \) in eq. (2.12).

Now we have to determine the function \( \omega \) in the equation (2.14) for this Green’s function. From (3.5) it follows that \( \tilde{G}(\tau, \tau') \) is a priori smooth and satisfies the homogeneous equation \( F \tilde{G}(\tau, \tau') = 0 \) for all \( \tau \neq \tau', 0, \tau_+ \). At \( \tau = \tau' \) it continues but has a jump in the first order derivative which contributes the delta \( \delta(\tau, \tau') \) to the right hand side of (2.14). The continuity of \( \tilde{\varphi}(\tau) \) and its derivative at \( \tau = \tau_- \) and \( \tau = \pm \tau_+ \) could have implied complete smoothness of \( \tilde{G}(\tau, \tau') \) at these points if the expression (2.12) for \( \tilde{\varphi}(\tau) \) would hold for an arbitrary source \( \tilde{J} \). However this source is not arbitrary — it is functionally orthogonal to \( g(\tau), \int d\tau g \tilde{J} = 0 \), and the continuity of \( \tilde{\varphi}(\tau) \) at \( \pm \tau_+ \), (3.6), holds only in virtue of this orthogonality (cf. eq. (3.3)). Therefore the Green’s function \( \tilde{G}(\tau, \tau') \) can have a discontinuity at this point proportional to \( g(\tau') \), which disappears in \( \tilde{\varphi} \) after being integrated with \( \tilde{J}(\tau') \). This discontinuity is contributed by the \( \Phi \)-part of \( \tilde{\varphi} \) or by the jump of the kernel in the last line of the equation (3.1)

\[
\tilde{G}(\tau_+, \tau') - \tilde{G}(-\tau_+, \tau') = -\left[ \theta(\tau - \tau') \theta(\tau') - \theta(\tau - \tau) \theta(-\tau') \right] \Psi(\tau, \tau') g(\tau') \bigg|_{\tau=\tau_+}^{\tau=-\tau_+} = \frac{1}{\dot{g}_+} g(\tau'). \tag{3.16}
\]
On the contrary, the first order derivative of $\tilde{G}(\tau, \tau')$ at $\pm \tau_+$ is continuous, because the continuity of $\dot{\varphi}$ at this points is enforced irrespective of the choice of the source.

The discontinuity (3.16) contributes the second term on the right hand side of the equation (2.14) for $\tilde{G}(\tau, \tau')$

$$F \dot{G} = \delta(\tau, \tau') + \frac{1}{g_+} \dot{\delta}(\tau, \tau_+) g(\tau'),$$

(3.17)

when treated as a generalized function in the vicinity of the point $\tau = \pm \tau_+$. Indeed, since the interval $-\tau_+ < \tau < \tau_+$ forms a circle with the points $\pm \tau_+$ identified, a small negative vicinity of the point $\tau_+$ becomes adjacent to a small positive vicinity of $-\tau_+$. Shifting for brevity this point to zero, $\tau_+ \to \tau = 0$, we then have such a situation. The full Greens function has a form $G = G_1 \theta(\tau) + G_2 \theta(-\tau)$ with two different functions $G_1(\tau)$ and $G_2(\tau)$ having different limits at zero, $G_1(0) \neq G_2(0)$, but the same derivatives, $\dot{G}_1(0) = \dot{G}_2(0)$, and satisfying one and the same equation $FG G_1,2 = \delta(\tau, \tau')$. Then

$$FG = (FG_1)(\theta(\tau) + (FG_2)(\theta(-\tau) - 2[\dot{G}_1(0) - \dot{G}_2(0)] \delta(\tau)$$

$$- [G_1(0) - G_2(0)] \delta(\tau) = \delta(\tau, \tau') - [G_1(0) - G_2(0)] \delta(\tau),$$

(3.18)

which gives (3.17). Therefore, $\omega(\tau)$ in eq. (2.14) equals

$$\omega(\tau) = \frac{1}{g_+} \delta(\tau, \tau_+),$$

(3.19)

and of course satisfies the consistency condition $\oint d\tau \omega(\tau) g(\tau) = -1$.

4 The variation of the determinant

In view of (3.15) and (3.5) and the fact that $\delta F = \delta(\dot{g}/g)$ one can write

$$\oint d\tau \delta \dot{F} \tilde{G}(\tau, \tau') |_{\tau' = \tau} = \oint d\tau \delta \left( \frac{\dot{g}}{g} \right) \delta \Phi(\tau)$$

$$+ \oint d\tau \delta \left( \frac{\dot{g}}{g} \right) \left( \Psi(\tau, \tau_+) \frac{\delta C_+}{\delta J(\tau)} \theta(\tau) + \Psi(-\tau, \tau_+) \frac{\delta C_-}{\delta J(\tau)} \theta(-\tau) \right)$$

$$+ \oint d\tau \delta \left( \frac{\dot{g}}{g} \right) g(\tau) \left( \frac{\delta D_+}{\delta J(\tau)} \theta(\tau) + \frac{\delta D_-}{\delta J(\tau)} \theta(-\tau) \right).$$

(4.1)

From (3.1) it follows that $\delta \Phi(\tau)/\delta J(\tau) \propto \Psi(\tau, \tau) = 0$, and the first term here vanishes.

Since $C_- = C_+$ the second term in (4.1) is just the value of $C_+ = C_+\left[J(\tau)\right]$ under a special choice of the source $J$,

$$\oint d\tau \delta \left( \frac{\dot{g}}{g} \right) \left( \Psi(\tau, \tau_+) \frac{\delta C_+}{\delta J(\tau)} \theta(\tau) + \Psi(-\tau, \tau_+) \frac{\delta C_-}{\delta J(\tau)} \theta(-\tau) \right)$$

$$= C_+ \left| J(\tau) = \delta(\dot{g}/g) (\Psi(\tau, \tau_+) \theta(\tau) + \Psi(-\tau, \tau_+) \theta(-\tau)) \right. \right. \right.$$

(4.2)

This expression is calculated in appendix A and reads

$$C_+ \left| J(\tau) = \delta(\dot{g}/g) (\Psi(\tau, \tau_+) \theta(\tau) + \Psi(-\tau, \tau_+) \theta(-\tau)) \right.$$

$$= - \frac{1}{\Psi_+ \Psi_- - \Psi_- \Psi_+} \left( \Psi_+ \delta \Psi_- - \delta \Psi_- \Psi_+ - \Psi_+ \delta \Psi_- + \frac{\Psi_+ \dot{\Psi}_+}{\Psi_-} \delta \Psi_- \right).$$

(4.3)
D-terms of (4.1) can also be reorganized as
\[
\oint d\tau \delta \left( \frac{\dot{g}}{g} \right) g(\tau) \left( \frac{\delta D_+}{\delta \tilde{J}(\tau)} \theta(\tau) + \frac{\delta D_-}{\delta \tilde{J}(\tau)} \theta(-\tau) \right) = \frac{D_+ + D_-}{2} \left| \tilde{J} = \delta(\dot{g}/g) g(\tau) \right| + \frac{D_+ - D_-}{2} \left| \tilde{J} = \delta(\dot{g}/g) g \varepsilon(\tau) \right|, \tag{4.4}
\]
where \(\varepsilon(\tau) = \theta(\tau) - \theta(-\tau)\). The calculations of appendix C give
\[
\frac{D_+ + D_-}{2} \left| \tilde{J} = \delta(\dot{g}/g) g(\tau) \right| = \frac{1}{2} \delta \ln Q + \frac{\delta \Psi_-}{\Psi_-}, \tag{4.5}
\]
\[
\frac{D_+ - D_-}{2} \left| \tilde{J} = \delta(\dot{g}/g) g \varepsilon(\tau) \right| = \frac{\Psi_- \dot{\Psi}_-}{\Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_-} \left( \frac{\delta \Psi_+ - \delta \Psi_-}{\Psi_+ - \Psi_-} \right). \tag{4.6}
\]
Collecting (4.2)–(4.3), (4.5) and (4.6) in (4.1) we have
\[
\oint d\tau \delta F \tilde{G}(\tau, \tau') \bigg|_{\tau' = \tau} = \delta \ln \left| \frac{\Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_-}{\Psi_+} \right| + \frac{1}{2} \delta \ln Q. \tag{4.7}
\]
After substituting this expression in (2.18) with \(\omega(\tau)\) given by (3.19) one can see that the contributions of the Faddeev-Popov factor \(Q\) completely cancel out. Therefore, as expected gauging out the zero mode of \(F\) is gauge independent, and we get
\[
\delta \ln \left( \text{Det}_s F \right) = \delta \ln \left| \frac{\Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_-}{\Psi_+} \right| + \frac{1}{g_+} \oint d\tau \delta g(\tau) \delta(\tau, \tau_+) \]
\[
= \delta \ln \left| \frac{\Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_-}{\Psi_+} \right| - \delta \dot{\Psi}_+ \frac{\dot{g}_+}{g_+} = \delta \ln \left| \Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_- \right|, \tag{4.8}
\]
which finally proves (1.10).

5 The case of the second zero mode

In the special case when the basic quantity of eq. (1.10) degenerates to zero,
\[
\Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_- = 0, \tag{5.1}
\]
the coefficients \(C_{\pm}\) and \(D_{\pm}\) in the solution (3.5), defined by (3.12)–(3.14), become singular and also the restricted functional determinant vanishes
\[
\text{Det}_s F = 0. \tag{5.2}
\]
All this points out to the fact that the operator \(F\) has an additional zero mode different from \(g(\tau)\). This mode is not gauged out in the path integral, and the corresponding functional determinant vanishes due to its zero eigenvalue.

This zero mode can be constructed as follows. Take the basis function \(\Psi(\tau, \tau_+)\) of the operator, defined by eq. (1.9) on the half period \(\tau_+ \geq \tau \geq 0\). It satisfies initial conditions \((\Psi_-, \dot{\Psi}_-)\) at \(\tau_+ = 0\). Now, smoothly continue this function to the negative half period of \(\tau\)
by evolving these initial conditions by the equation of motion backward in time. To be more precise, this function denoted below as $\Psi(\tau)$ will be a solution of the Cauchy problem

$$F\Psi(\tau) = 0, \quad -\tau_+ < \tau < \tau_+,$$

$$\Psi(0) = \Psi_-, \quad \dot{\Psi}(0) = \dot{\Psi}_-, \quad (5.3)$$

with the Cauchy data surface in the middle of the time range at $\tau_- = 0$. For a positive $\tau$ it coincides with $\Psi(\tau, \tau_*)$, whereas for $\tau < 0$ this is a linear combination of $\Psi(|\tau|, \tau_*)$ and $g(\tau)$ with the coefficients following from the initial data at $\tau_-,

$$\Psi(\tau) = \Psi(|\tau|, \tau_*) - 2\Psi_- \theta(-\tau) g(\tau).$$

Generically this function is not periodic on a circle $-\tau_+ \leq \tau \leq \tau_+$ with the points $\pm \tau_+$ identified, because $\Psi(+\tau_+) = \Psi(-\tau_+)$ in view of $g(\pm \tau_+) = 0$, but

$$\dot{\Psi}(-\tau_+) = \dot{\Psi}(\tau_+) - \frac{2}{\Psi_+} (\Psi_+ \dot{\Psi}_+ - \Psi_- \dot{\Psi}_-) \quad (5.6)$$

However, for a special case of (5.1) the derivatives also match at $\tau = \pm \tau_+$, and (5.5) becomes completely smooth and periodic and forms the second zero mode of $F$. This explains the vanishing value of the restricted determinant and, in fact, confirms the validity of its algorithm (1.10).

A simple example of the operator (1.1) with two zero modes is given by the case of its constant potential term $\ddot{g}/g = -w^2 = \text{const}$, corresponding to a harmonic function

$$g(\tau) = d \sin(w\tau), \quad -\pi/w \leq \tau \leq \pi/w.$$ 

(5.7)

In this case the function $\Psi(\tau, \tau_*)$ is exactly calculable,

$$\Psi(\tau, \tau_*) = \frac{\sin(w(\tau - \tau_*))}{d \sin(w\tau_*)},$$

(5.8)

and it exactly satisfies the relation (5.1). This function by itself is periodic on a circle and comprises the second linear independent basis function of the equation $\ddot{\Psi} + w^2\Psi = 0$, complementary to (5.7). Therefore, it can be taken as the second zero mode which would differ from (5.5) only by a linear recombination with $g(\tau)$.

The situation with the second zero mode of $F$ serves as a check of consistency of the obtained algorithm (1.10). Otherwise, its role should not be overestimated. In particular, this mode does not have to be gauged out in the path integral for the statistical sum in cosmology [7]. Unlike $g(\tau)$ this second zero mode cannot be associated with the generator of the invariance transformation for the whole cosmological action. It leaves invariant the action of the $\varphi$-variable (1.3), but the full cosmological action contains also the part with the global (modular) degree of freedom – the proper time period of the Euclidean FRW metric — which is not invariant under the transformations associated with the second zero mode [7]. Consequently, the one-loop preexponential factor of the statistical sum remains well-defined also in the limit of (5.1), because the determinant (2.1) enters this prefactor in a way more complicated than a simple overall factor [7]. This is the result of the additional integration over the global modular variable of the FRW metric. In applications of the above technique to the microcanonical ensemble in cosmology the second zero mode of $F$ arises for a special subset of the cosmological instantons. As shown in [11], it leads in view of (5.1)–(5.2) to a certain suppression of loop corrections in the cosmological statistical sum, but does not completely nullify them.
6 Conclusions

Though very involved, the above calculation of the functional determinant of the operator (1.1) with its zero mode gauged out leads to a rather concise answer (1.10) in terms of the special basis function of this operator (1.9) and the zero mode itself. The determinant $\text{Det}_\pi \mathcal{F}$ as well as the solution of the periodic boundary conditions problem with this operator enter the construction of the statistical sum in quantum cosmology [7]. In particular, the form of (1.10) is crucial for the partial cancelation of contributions to its one-loop preexponential factor. This cancelation is associated with the absence of the local degrees of freedom in the Friedmann-Robertson-Walker sector of cosmology. The remaining part is due to the global degree of freedom of the FRW metric, and it is also determined by the contribution of (1.10).

Important limitation of the obtained result is, however, that it includes only the case of two roots of the zero mode $g(\tau)$ in the full period of the Euclidean time, whereas the path integral in cosmology driven by a conformal field theory suggests cosmological instantons with an arbitrary number of oscillations of the cosmological scale factor, corresponding to numerous roots of the oscillating zero mode $g$. For the number of these oscillations tending to infinity this CFT driven cosmology approaches a new quantum gravity scale — the maximum possible value of the cosmological constant [3, 4] — where the physics and, in particular, the effects of the quantum prefactor become very interesting and important. Thus the extension of the above technique for $\text{Det}_\pi \mathcal{F}$ to an arbitrary number of roots of the zero mode of $\mathcal{F}$ becomes important, as this extension might be relevant to the cosmological constant problem.

The invariant definition of the restricted functional determinant with the zero mode gauged out by the Faddeev-Popov method is likely to be equivalent to the regularization technique of [8–10] for similar operators with a zero eigenvalue. When combined with a monodromy method of [12] this technique might allow us to make this extension manageable, which we hope to attain in a foreseeable future.

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A The contribution of the C-terms

The contribution of the C-terms (4.2) can be obtained by using (3.4) and (3.12)

$$C_+ \bigg| \tilde{f}(\tau) = \delta \left( \frac{\delta g}{g} \right) \left( \Psi(\tau, \tau_*) \Psi(\tau, -\tau) - \Psi(-\tau, \tau_*) \Psi(-\tau, -\tau) \right)$$

$$= \frac{\Psi_+}{\Psi_+ \Psi_- - \Psi_- \Psi_-} \int_{\tau_-}^{\tau_+} d\tau \left( \delta g \left( \frac{\delta g}{g} \right) \Psi(\tau, \tau_*) - \frac{\delta g}{g} \Psi(\tau, \tau_*) \delta g \right) \Psi(\tau, \tau). \quad (A.1)$$

In view of the equation $\mathcal{F} \Psi(\tau, \tau_*) = 0$ we have $(\delta g/g) \Psi(\tau, \tau_*) = \tilde{\Psi}(\tau, \tau_*)$, whence

$$\delta g \frac{\delta g}{g} \Psi(\tau, \tau_*) - \frac{\delta g}{g} \Psi(\tau, \tau_*) \delta g = \delta g \Psi(\tau, \tau_*) - \delta g \tilde{\Psi}(\tau, \tau_*)$$

$$= \frac{d}{d\tau} \left( \delta g \Psi(\tau, \tau_*) - \delta g \tilde{\Psi}(\tau, \tau_*) \right). \quad (A.2)$$
Therefore

\[
\int d\tau \delta \left( \frac{\dot{g}}{g} \right) \left( \Psi(\tau, \tau_s) \frac{\delta C_+}{\delta J(\tau)} \theta(\tau) + \Psi(-\tau, \tau_s) \frac{\delta C_-}{\delta J(\tau)} \theta(-\tau) \right) = \frac{\Psi_+}{\Psi_+ - \Psi_-} \int_{\tau_-}^{\tau_+} d\tau \frac{d}{d\tau} \left( \delta \dot{g} \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) \right) \dot{\Psi}(\tau_+, \tau). \tag{A.3}
\]

Straightforward integration by parts is impossible here, because the relevant surface terms at \( \tau = \tau_\pm \) are divergent. Indeed, the function \( g(\tau) \) and its variation \( \delta g(\tau) \) in view of (1.8) have expansions at \( \tau = \tau_- \)

\[
g(\tau) = \delta_0 + O(\tau^3), \quad \delta g(\tau) = \delta \delta_0 + O(\tau^3), \tag{A.4}
\]

and \( \Psi(\tau, \tau_s) = \Psi_+ + \hat{\Psi}_+ + O(\tau^2) \), so that \( \delta \dot{g} \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) = \delta \dot{g}_- \Psi_- + O(\tau^2) \), whereas

\[
\dot{\Psi}(\tau_+, \tau) = \dot{g}_+ \int_\tau^{\tau_+} \frac{dy}{\dot{g}_+^2 y^2 + \ldots} + \ldots = \frac{\dot{g}_+}{\dot{g}_+^2 \tau} + \ldots, \quad \tau \to \tau_- = 0. \tag{A.5}
\]

Similarly

\[
\delta \dot{g} \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) = \delta \dot{g}_+ \Psi_+ + O(\Delta^2), \quad \dot{\Psi}(\tau_+, \tau) = \frac{1}{\dot{g}_+}, \quad \Delta \equiv \tau - \tau_+ \to 0. \tag{A.6}
\]

Consequently, the integration by parts in eq. (A.3) is possible only after making the subtractions of constant terms inside the parenthesis in the right hand side of (A.3) — \( \delta \dot{g}_- \Psi_- \) at \( \tau = \tau_- \) and \( \delta \dot{g}_+ \Psi_+ \) at \( \tau_+ \). Thus we identically rewrite the right hand side of (A.3) as

\[
\int_{\tau_-}^{\tau_+} d\tau \frac{d}{d\tau} \left( \delta \dot{g} \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) \right) \dot{\Psi}(\tau_+, \tau)
\]

\[
= \int_{\tau_-}^{\tau_+} d\tau \frac{d}{d\tau} \left( \delta \dot{g} \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) - \delta \dot{g}_- \Psi_- \right) \dot{\Psi}(\tau_+, \tau)
\]

\[
+ \int_{\tau_-}^{\tau_+} d\tau \frac{d}{d\tau} \left( \delta g \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) - \delta \dot{g}_+ \Psi_+ \right) \dot{\Psi}(\tau_+, \tau) \tag{A.7}
\]

and integrate by parts taking into account that \( (d/d\tau) \dot{\Psi}(\tau_+, \tau) = -\dot{g}_+ / g^2(\tau) \). This gives

\[
\int_{\tau_-}^{\tau_+} d\tau \frac{d}{d\tau} \left( \delta \dot{g} \Psi(\tau, \tau_s) - \delta g \dot{\Psi}(\tau, \tau_s) \right) \dot{\Psi}(\tau_+, \tau)
\]

\[
= \left( \delta \dot{g}(\tau_s) \Psi(\tau_s, \tau_s) - \delta g(\tau_s) \dot{\Psi}(\tau_s, \tau_s) - \delta \dot{g}_- \Psi_- \right) \dot{\Psi}(\tau_+, \tau_s)
\]

\[
- \left( \delta \dot{g}(\tau_s) \Psi(\tau_s, \tau_s) - \delta g(\tau_s) \dot{\Psi}(\tau_s, \tau_s) - \delta \dot{g}_+ \Psi_+ \right) \dot{\Psi}(\tau_+, \tau_s)
\]

\[
+ \dot{g}_+ \left( I_+ + I_- \right), \tag{A.8}
\]

where

\[
I_- = \int_{\tau_-}^{\tau_+} \frac{d\tau}{g^2(\tau)} \left( \delta \dot{g}(\tau) \Psi(\tau, \tau_s) - \delta g(\tau) \dot{\Psi}(\tau, \tau_s) - \delta \dot{g}_- \Psi_- \right), \tag{A.9}
\]

\[
I_+ = \int_{\tau_-}^{\tau_+} \frac{d\tau}{g^2(\tau)} \left( \delta \dot{g}(\tau) \Psi(\tau, \tau_s) - \delta g(\tau) \dot{\Psi}(\tau, \tau_s) - \delta \dot{g}_+ \Psi_+ \right). \tag{A.10}
\]
Note that the surface terms arising from integration are contributed only by the intermediate point \( \tau_\ast \), and they equal
\[
\left( \delta \dot{g}(\tau_\ast) \Psi(\tau_\ast, \tau_\ast) - \delta g(\tau_\ast) \dot{\Psi}(\tau_\ast, \tau_\ast) - \delta \dot{g}_- \dot{\Psi}_- \right) \dot{\Psi}(\tau_+, \tau_\ast) \\
- \left( \delta \dot{g}(\tau_\ast) \Psi(\tau_\ast, \tau_\ast) - \delta g(\tau_\ast) \dot{\Psi}(\tau_\ast, \tau_\ast) - \delta \dot{g}_+ \dot{\Psi}_+ \right) \dot{\Psi}(\tau_+, \tau_\ast) \\
= \left( \frac{\delta \dot{g}_-}{\dot{\Psi}} - \frac{\delta \dot{g}_+}{\dot{\Psi}_+} \right) \dot{\Psi}_+ = \dot{\Psi}_+ \delta \left( \ln \frac{\Psi_+}{\Psi_-} \right).
\] (A.11)

The calculation of the integral terms with \( I_\pm \) is much trickier and is presented in appendix B. It is based on a systematic conversion of their integrands to the form of total derivatives in order to generate easily calculable surface terms at \( \tau = \tau_\pm \). The result is, however, rather simple,
\[
I_- = \Psi_- \delta \dot{\Psi}_- - \delta \Psi_- \dot{\Psi}_-, \quad (A.12)
I_+ = -\Psi_+ \delta \dot{\Psi}_+ + \delta \Psi_+ \dot{\Psi}_+. \quad (A.13)
\]

Collecting (A.11), (A.12), (A.13) and (A.8) we get the total contribution of C-terms (4.3).

B The contribution of integral terms with \( I_\pm \)

The strategy of calculating the integrals (A.9)–(A.10) consists in a systematic conversion of their integrands to the form of total derivatives which yield easily calculable surface terms at \( \tau = \tau_\pm \). For this purpose we use the corollaries of equation (1.9) for \( \Psi = \Psi(\tau, \tau') \),
\[
\dot{\Psi} = \frac{\dot{g}}{g} \Psi + \frac{1}{g}, \quad (B.1)
1 = \frac{d}{d\tau} \left( \frac{\Psi}{g} \right). \quad (B.2)
\]
In view of the last equation the last term in the integrand of \( I_- \) is the total derivative
\[
- \frac{d}{d\tau} \left( \frac{\Psi}{g} \Psi_- \frac{\delta \dot{g}_-}{\dot{\Psi}} \right) = \frac{d}{d\tau} \left( \frac{\Psi}{g} \delta \dot{g}_- \right), \quad (B.3)
\]
whereas the first two terms can be transformed by using (B.1) as
\[
\frac{1}{g^2} \left( \delta \dot{g} \Psi - \delta g \dot{\Psi} \right) = \frac{d}{d\tau} \left( \frac{\delta g \Psi}{g^2} \right) + \frac{2 \delta \dot{g} \dot{g}}{g^3} \Psi - \frac{2 \delta \dot{g}}{g^2} \dot{\Psi} = \frac{d}{d\tau} \left( \frac{\delta g \Psi}{g^2} \right) - \frac{2 \delta g}{g^3}, \quad (B.4)
\]
whence
\[
I_- = \int_{\tau_-}^{\tau_+} d\tau \left\{ \frac{d}{d\tau} \left[ \frac{\Psi}{g} \left( \frac{\delta g}{g} + \frac{\delta \dot{g}_-}{\dot{\Psi}_-} \right) \right] - \frac{2 \delta g}{g^3} \right\}. \quad (B.5)
\]

The integrand is not yet a total derivative. Moreover, the total derivative term cannot be reduced to the surface term, because the latter diverges at \( \tau = \tau_- \) as \(-2\delta \dot{g}_-/g^3 \tau, \tau \to 0\) (remember that \( \Psi_-(\tau) = -1/\dot{g}, g(\tau) \sim \dot{g}_- \tau \) and \( \delta g(\tau) \sim \delta \dot{g}_- \tau \)). This divergence is in fact canceled in the integrand by the last term \(-2\delta g\dot{g}/g^3 \sim -2\delta \dot{g}_-/g^2 \tau^2 \). Thus, these two terms
cannot be treated separately, because of this subtraction mechanism. Instead of splitting them we will use another quantity in which \( \delta g/g^3 \) also enters the integrand. This quantity is a variation of the expression

\[
\Psi \dot{\Psi} \bigg|_{\tau = \tau_*} = \int_{\tau_-}^{\tau_*} d\tau \left( \Psi \dot{\Psi} \right) = \int_{\tau_-}^{\tau_*} d\tau \left( \Psi^2 + \dot{\Psi} \ddot{\Psi} \right) = \int_{\tau_-}^{\tau_*} d\tau \left( \dot{\Psi}^2 + \frac{\dot{g}}{g} \dot{\Psi}^2 + \frac{\ddot{g}}{g} \dot{\Psi}^2 + \frac{\dot{g} \dddot{g}}{g} \dot{\Psi}^2 \right) = \int_{\tau_-}^{\tau_*} d\tau \left( \dot{\Psi}^2 + \frac{\dot{g}}{g} \dot{\Psi}^2 \right),
\]

in which we used the equation for \( \Psi, \dot{\Psi} = (\dot{g}/g)\Psi \). Using (B.1) to express \( \dot{\Psi} \) in terms of \( \Psi \) one easily shows that

\[
\Psi \dot{\Psi} \bigg|_{\tau = \tau_*} = \int_{\tau_-}^{\tau_*} d\tau \left\{ \left( \frac{d}{d\tau} \left( \frac{\dot{g}}{g} \dot{\Psi}^2 \right) \right) + \left( \frac{1}{g^2} \right) \right\}.
\]

The variation of this expression has the same structure as (B.5)

\[
\delta \left( \Psi \dot{\Psi} \bigg|_{\tau = \tau_*} \right) = \int_{\tau_-}^{\tau_*} d\tau \left\{ \left( \frac{d}{d\tau} \left( \frac{\dot{g}}{g} \dot{\Psi}^2 \right) \right) - \frac{2\dot{g}g^2}{g^3} \right\}.
\]

Therefore, the subtraction of this expression from (B.5) allows one to exclude the integral of \(-2\dot{g}g^3\) and to obtain

\[
I_- - \delta \left( \Psi \dot{\Psi} \bigg|_{\tau = \tau_*} \right) = \int_{\tau_-}^{\tau_*} d\tau \left[ \frac{\Psi}{g} \left( \frac{\delta \dot{g}}{g} + \frac{\delta \dot{g}_-}{g} \right) \right] - \frac{\delta \left( \frac{\dot{g}}{g} \dot{\Psi}^2 \right) \bigg|_{\tau = \tau_*} \right]
= \left\{ \delta \left( \frac{\dot{g}}{g} \dot{\Psi}^2 \right) - \frac{\Psi}{g} \left( \frac{\delta \dot{g}}{g} + \frac{\delta \dot{g}_-}{g} \right) \right\} \bigg|_{\tau = \tau_*} ,
\]

where the surface term at \( \tau_* \) vanishes in view of \( \Psi_* \equiv \Psi(\tau_+, \tau_*) = 0 \) and the surface term at \( \tau_- \) is now finite, because of the cancelation of divergences between the two terms in the last line of the above equation. To find the remnant of this cancelation we write

\[
\left[ \delta \left( \frac{\dot{g}}{g} \dot{\Psi}^2 \right) - \frac{\Psi}{g} \left( \frac{\delta \dot{g}}{g} + \frac{\delta \dot{g}_-}{g} \right) \right] \bigg|_{\tau = \tau_*} = \frac{1}{\dot{g}_-} \frac{d}{d\tau} \left[ \delta g \Psi^2 - \frac{\delta g \Psi}{g} \dot{g} \dot{\Psi} - \frac{\delta g \dot{\Psi}}{g} \Psi \dot{g} - \frac{\delta g \ddot{\Psi}}{g} \right] \bigg|_{\tau = \tau_*} = 2\Psi_- \delta \dot{\Psi}_-, \quad \text{(B.10)}
\]

where we took into account that in view of (A.4) \( \delta \dot{g}_- = \dot{g}_- = 0, \left( \frac{d}{d\tau} \delta (\dot{g}/g) \right)|_{\tau = 0} = 0 \) and \( \delta \dot{\Psi}_- = -\delta (1/\dot{g}_-) \). Therefore, because \( \Psi(\tau_+, \tau_*) = \delta \Psi(\tau_+, \tau_*) = 0 \), we finally have the expression (A.12) for \( I_- \) used above.

A similar calculation for the second integral term gives the equations analogous to (B.9) and (B.10)

\[
I_+ - \delta \left( \Psi \dot{\Psi} \bigg|_{\tau = \tau_*} \right) = - \left[ \delta \left( \frac{\dot{g}}{g} \dot{\Psi}^2 \right) - \frac{\Psi}{g} \left( \frac{\delta \dot{g}}{g} + \frac{\delta \dot{g}_+}{g} \right) \right] \bigg|_{\tau = \tau_*} = -2\Psi_+ \delta \dot{\Psi}_+, \quad \text{(B.11)}
\]

except the opposite sign caused by the upper integration limit (rather than by the lower one). This leads to (A.13).
\section{The contribution of the D-terms}

By using (3.14) one has

\[
Q \frac{D_+ + D_-}{2} \frac{\partial J}{\partial g} = \int dy \, g(y) \int_{\tau_-}^{\tau_+} d\tau \, \frac{d}{d\tau} \left( g \frac{\delta g}{g} - \dot{g} \delta g \right)
\]

where when integrating by parts no surface terms arise, because \( \Psi(y, y) = 0 \) and \( (g \delta g - \dot{g} \delta g) = O(\tau^2) \) at \( \tau \to 0 \), and we used the fact that \( (d/d\tau)\Psi(y, \tau) = -g(y)/g^2(\tau) \).

Here we perform the variation of \( J \), so that with \( D_+ \equiv 0 \) and \( D_- = \delta \equiv \Delta \) at \( \Delta \)

\[
\frac{D_+ + D_-}{2} \frac{\partial J}{\partial g} = \frac{1}{2} \delta \ln Q - \frac{\delta g}{g}(\tau_-). \tag{C.2}
\]

Here we perform the variation of \( g(\tau) \) in the class of functions satisfying \( \delta g(\tau) = \delta \dot{g} - O(\tau^2) \), so that with \( g(\tau) = \dot{g} \tau + O(\tau^2) \) we have \( (\delta g/g)(\tau_-) = \delta \dot{g}/\dot{g}_- = -\delta \Psi_\Delta/\Psi_- \), which finally leads to (4.5).

The second term in eq. (4.4) in view of (3.13) and (3.4) reads

\[
\frac{D_+ - D_-}{2} \frac{\partial J}{\partial g} = \frac{1}{2} \frac{\Psi_\Delta \Psi_- - \Psi_-}{\Psi_\Delta \Psi_- - \Psi_\Psi_-} \left( \int_{\tau_-}^{\tau_+} dy \, \dot{\Psi}(\tau_+, y) g^2 \frac{\delta g}{g} (y) \right) + \int_{\tau_-}^{-\tau_+} dy \, \dot{\Psi}(\tau_+, y) g^2 \frac{\delta g}{g} (y).
\tag{C.3}
\]

In view of the symmetries of the functions in the integrand, \( \dot{\Psi}(\tau_+, y) = -\dot{\Psi}(\tau_+, -y) \) and \( \delta (\dot{g}/g)(-y) = \delta (\dot{g}/g)(y) \) the two integrals above coincide and

\[
\frac{D_+ - D_-}{2} \frac{\partial J}{\partial g} = \frac{\Psi_\Delta \Psi_- - \Psi_-}{\Psi_\Delta \Psi_- - \Psi_\Psi_-} \int_{\tau_-}^{\tau_+} dy \, \dot{\Psi}(\tau_+, y) \frac{d}{dy} (g \frac{\delta g}{g} - \dot{g} \delta g). \tag{C.4}
\]

Here integration by parts does not give surface terms because \( g \delta \dot{g} - \dot{g} \delta g = O(\Delta^2) \), \( \dot{\Psi}(\tau_+, y) = O(1/\Delta) \) at \( \Delta \equiv y - \tau \to 0 \). Therefore

\[
\frac{D_+ - D_-}{2} \frac{\partial J}{\partial g} = -\frac{\Psi_\Delta \Psi_- - \Psi_-}{\Psi_\Delta \Psi_- - \Psi_\Psi_-} \int_{\tau_-}^{\tau_+} dy \, \frac{d}{dy} \dot{\Psi}(\tau_+, y) (g \delta \dot{g} - \dot{g} \delta g)
\]

\[
= \frac{\Psi_-}{\Psi_\Delta \Psi_- - \Psi_\Psi_-} \left( \frac{\delta \Psi_+}{\Psi_+} - \frac{\delta \Psi_-}{\Psi_-} \right), \tag{C.5}
\]

where we took into account that \( (d/dy)\dot{\Psi}(\tau_+, y) = -\dot{g}_+/g^2(y) \) and \( \delta g/g(\tau_+) = \delta \dot{g}_+/\dot{g}_+ \) and \( \delta g/g(0) = \delta \dot{g}_-/\dot{g}_- \). This gives (4.6).
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