Resolution of Orbifold Singularities in String Theory

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ABSTRACT

In this paper the relationship between the classical description of the resolution of quotient singularities and the string picture is discussed in the context of $N=(2,2)$ superconformal field theories. A method for the analysis of quotients locally of the form $\mathbb{C}^d/G$ where $G$ is abelian is presented. Methods derived from mirror symmetry are used to study the moduli space of the blowing-up process. The case $\mathbb{C}^2/\mathbb{Z}_n$ is analyzed explicitly.

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1 Introduction

One of the earliest manifolds considered for a superstring target space was a blown-up orbifold \[1\]. In this case the orbifold (the “Z-orbifold”) was constructed by modding out a torus of 6 real dimensions by a group isomorphic to \(\mathbb{Z}_3\). This group action has 27 fixed points leading to 27 isolated quotient singularities in the resulting space. Algebraic geometry then tells us that these singularities can be smoothed away (in particular, “blown-up”) to give us a Calabi-Yau manifold. In this context the orbifold itself is little more than a step in the construction of a smooth Calabi-Yau manifold.

Shortly afterwards it was realized \[2\] that string theory is well-behaved (at least in some sense) on the orbifold itself before the smoothing process. It was also noticed that calculations of such quantities as the field-theory version of the Euler characteristic when performed on the orbifold agreed with the same geometric calculation on the target space after blowing up. It is as if the string theory “knows” about the blow-up even before it has been performed. Thus the Euler characteristic formula of \[2\] can be taken as one of the earliest indications of the power of “stringy” geometry. By “stringy” or “quantum” geometry we mean geometry as indicated via field theory analysis of a string theory rather than direct analysis of the geometry of the string’s target space.

In many respects toroidal orbifolds such as the Z-orbifold provide attractive models for string target spaces. The torus itself is a little too trivial whereas a general Calabi-Yau manifold can render many calculations very difficult. The toroidal orbifold may be thought of as a good compromise since the field theory view of the quotient singularities is rather straight-forward. Having said this, proponents of Calabi-Yau manifolds would say that the blown-up version of the orbifold is more general — the orbifold itself is just a limiting case of the blown-up orbifold. Thus people who study orbifolds are artificially singling out a special class of string target spaces. We will argue below that this point of view might be a little unfair.

Before discussing orbifolds any further we should first state clearly what class of spaces we are studying. Orbifolds were introduced many years ago \[3\]. As originally defined an orbifold may be taken to be a space (an algebraic variety) whose only singularities are locally of the form of quotient singularities. In the context of this paper we will only consider “Calabi-Yau orbifolds”, that is an orbifold which is a complex algebraic variety of complex dimension \(d\) with holonomy group \(G \subseteq SU(d)\). The class of Calabi-Yau orbifolds clearly covers the case of a complex \(d\)-torus divided by a discrete group \(G \subseteq SU(d)\) and indeed any Calabi-Yau \(d\)-fold divided by \(G\).

As an example of an orbifold not of the form \(M/G\) for \(M\) a smooth Calabi-Yau manifold

\[1\] The original name was “V-manifold”. The term “orbifold” appears to be due to W.P. Thurston.
consider again the case of the $\mathbb{Z}$-orbifold. Clearly the $\mathbb{Z}$-orbifold is of the form $M/G$ but if we blow-up just one of the 27 quotient singularities we destroy this property. The resultant space still has 26 quotient singularities locally of the form $\mathbb{C}^3/\mathbb{Z}_3$ however and is thus an orbifold. String theory is still well-defined on this space indicating that it is the local rather global form of the singularities that is important. We therefore believe that the mathematicians definition of an orbifold should also be the appropriate definition in string theory.

In order to focus on purely local aspects of quotient singularities we will restrict ourselves to isolated singularities in this paper. This is done mainly for reasons of clarity and it should be straight-forward to generalize most of what is said in this paper to non-isolated quotient singularities. We also assume that any smooth manifold appearing in this paper should be Kähler.

Let us now clarify the term “blowing-up”. A simple example of a blow-up, which will be the main focus of this paper, acts to alter slightly a singular space so that it becomes smooth. The singularity is thus resolved. A blow-up is a particular kind of resolution of singularity and so we require more explanation of the term. It is actually easier to describe the reverse process of “blowing-down”. That is we take a smooth manifold and change it so that it becomes singular.

Given a Calabi-Yau manifold $X$ with a metric there are two types of continuous changes we consider. Firstly, deformation of complex structure of $X$ (i.e., what geometers would normally call simply a deformation) and secondly deformation of the Kähler form on $X$. It turns out that the blowing-down process corresponds to a deformation of the Kähler form. The moduli space of Kähler forms on $X$ has a cone structure as is easy to see — if $g_{ij}$ is a valid Hermitian metric on $X$ then so is $\lambda g_{ij}$ for $\lambda$ a positive real number. As we approach the wall of the Kähler cone of $X$ from the interior, some algebraically embedded subspace of $X$ or $X$ itself will shrink down acquiring zero size on the wall. If it is a subspace that shrinks down rather than the whole of $X$, we say that this subspace has been blown-down when we reach the wall. Thus the orbifolds in classical geometry which may be blown-up into smooth Calabi-Yau manifolds lie at the edge of the Kähler cone as depicted in figure ??.

At first sight the analysis of the string theory on an orbifold appears to have a similar structure to that suggested by the description of classical geometry above. One can build the $\mathcal{N}=(2,2)$ superconformal field theory corresponding to the orbifold and find the dimension of the moduli space in which it lives by counting the truly marginal operators preserving the $\mathcal{N}=(2,2)$ world-sheet supersymmetry in the theory. When one does this one finds the correct number for the dimension of the moduli space of the maximally resolved orbifold. By maximally resolved we mean that the most blow-ups have been done to resolve the singularities consistent with the $K = 0$ (vanishing first Chern class) condition. Thus it appears that the some of the marginal operators in the orbifold conformal field theory correspond to
blow-up modes. Indeed, as we will show later in this paper they are in natural one-to-one correspondence.

This is not the full story however. When one analyzes the structure of the moduli space around an orbifold point, the conformal field theory picture and the classical geometric picture do not appear to agree \[^4\]. In particular it is not clear how a cone-like structure appears in the moduli space spanned by the marginal operators when one looks at a neighbourhood of the orbifold point. This problem was effectively resolved in \[^5, 6, 7\] where it was shown how cone structures did appear naturally in the moduli space of \(N=(2,2)\) superconformal field theories as one moves sufficiently far away from the orbifold point. This allows one to recover the classical picture of a blow-up in the conformal field theory language once one has moved sufficiently far in moduli space. The problem is that the cone structure found in \[^5, 6, 7\] is in complete disagreement with the picture of the orbifold point in figure \[^??\]!

The situation discovered in \[^5, 6, 7\] may be stated roughly as follows. Given a certain class of theories (corresponding to Calabi-Yau manifolds which may be realized as hypersurfaces in toric varieties) the moduli space of suitably defined Kähler forms (to be precise the “algebraic measures” of \[^8\]) is isomorphic to \(\mathbb{R}^{h_1,1}\). One can now subdivide this space into a fan-like structure consisting of cones with their apex at the origin of \(\mathbb{R}^{h_1,1}\). This decomposition divides the moduli space into so-called “phases”. One of these cones will be congruent to the Kähler cone of the Calabi-Yau manifold. The other cones appear to be best thought of as associated to other geometries. One of the other geometries is an orbifold. In such a picture, the orbifold point itself lies in the deep interior of the orbifold cone.

In figure \[^??\] we show how this phase picture modifies the classical picture of figure \[^??\]. Only part of the fan is shown for clarity. In order to reach the orbifold point one must leave the Calabi-Yau Kähler cone.

The above state of affairs was clarified in \[^8\] by thinking about correlation functions arising from string theory on a Calabi-Yau manifold, \(X\). The string theory is given in terms of a non-linear \(\sigma\)-model with target space \(X\). Some correlation functions in this theory are prone to instanton corrections which correspond to rational curves. The precise form of the instanton correction is as a power series in \(q_j\) where \(q_j = \exp(-A_j)\) and \(A_j\) is the area of a curve representing the \(j\)th generator of \(H_2(X,\mathbb{Z})\). If all the rational curves are sufficiently large then this power series will converge (assuming a non-zero radius of convergence). In this case there is no problem analyzing the \(\sigma\)-model. By varying the Kähler form on \(X\), we can shrink some of the rational curves on \(X\). As we do this we may eventually move outside the region of convergence of instanton power series. Now the analysis of the \(\sigma\)-model is unclear. It might happen however that the field theory can be interpreted more simply in terms of a different target space. This is exactly what happens in the case of an orbifold. When we are in the orbifold phase, correlation functions may be determined in terms of the
orbifold theory perturbed in some way (actually by twist fields, see for example [3]). The orbifold cone marks roughly, in some way, the region of convergence of this new perturbation theory.

Even though the phase picture naturally divides up the moduli space into regions with different geometrical interpretations one should not take the word “phase” too seriously — a generic path from one phase region into another will not encounter a theory that is singular in any way. Thus one can choose one geometrical interpretation and “analytically continue” it into other regions. In terms of the correlation functions above this analytic continuation is quite literal. Thus, for example, we might describe the whole moduli space in terms of the Calabi-Yau region. This is what we will do later on this paper to determine the precise conformal field theory picture of blowing-up. This will allow us to make contact with figure ?? by redefining the Kähler form. It should be emphasized however that this is an artificially biased approach in favour of smooth target spaces. One could equally choose the orbifold region to analytically continue over the whole space. It is in this sense that the Calabi-Yau proponent has no claims being “more general” than the orbifold proponent. It should also be noted that there are examples where at least one of the proponents would fail, i.e., there are Calabi-Yau manifolds which cannot be blown down to orbifolds and orbifolds which cannot be blown up into smooth Calabi-Yau manifolds. It is therefore important to always consider both methods of approach.

In section 2 we will review the classical theory of blow-ups of singularities formed by quotients by abelian groups in a language that will be useful later in the paper. In section 3 we will look at the same problem from the conformal field theory point of view and derive a direct map correspondence between the two approaches. In section 4 we will analytically continue the Calabi-Yau region into the orbifold region to measure the size of the blown-up parts of the orbifold. Finally in section 5 we will study some details of the structure of the moduli space of blowing-up codimension 2 singularities.

2 Classical Geometry

Let us first outline the concept of a (Kähler) blow-up. In much of what follows we will consider the case where $X$ is not compact. In particular $X$ will correspond to $\mathbb{C}^d/G$ for some group $G$. Since we are only interested in the local behaviour of string theory around a quotient singularity this will suffice. Given a Kähler manifold $X$, one can vary the sizes of algebraically embedded submanifolds of $X$ by varying the Kähler form on $X$. In particular one might have a divisor (i.e., an algebraically embedded codimension one subspace) which can be shrunk down to an arbitrarily small size while keeping other divisors (not in the same

\footnote{Dimensions of the target space will always be counting complex degrees of freedom in this paper.}
homology class) at finite size. Such a divisor is *exceptional*. If one shrinks an exceptional divisor down to zero size then the limit of $X$ in this process may be singular. We say then that $X$ has been blown-down along this exceptional divisor. As we shall see, if one blows down along the right combination of divisors then the resultant singularity is a quotient singularity. Thus we have described the reverse of the process of blowing-up a quotient singularity to obtain a smooth manifold $X$. The singular point set in an orbifold when blown-up appears to be replaced by an exceptional divisor (which may be the sum of many irreducible divisors) in $X$.

We will now review how any singularity locally in the form of $\mathbb{C}^d/G$ where $G$ is a discrete *abelian* group can be blown-up. We will focus on the situation of an isolated singularity (i.e., the origin is the only fixed point of any non-trivial element of $G$) but the following method also can be used in the non-isolated case. We also want to consider the case in which we do not affect the canonical class so that the resultant space, if smooth, will be a Calabi-Yau manifold. A simple holonomy argument shows that in this case, $G \subset SU(d)$. To do the blowing-up we will use the language of toric geometry. It is beyond the scope of this article to explain toric geometry — for that the reader is referred to [10] or [6]. See also [11] for an early account of this in the physics literature. To fully understand the following arguments one should consult these references but it is hoped that the following be self-contained if one requires only a schematic understanding.

A toric variety $X_\Delta$ is a $d$-dimensional complex space specified by a fan $\Delta$. In this paper, the fan $\Delta$ is a collection of *simplex*-based cones (together with their faces) in $\mathbb{R}^d$. One also puts a lattice structure in the same space, i.e., $\mathbb{Z}^d \subset \mathbb{R}^d$. This lattice is denoted $\mathbb{N}$. $X_\Delta$ is compact if and only if $\Delta$ spans the whole of $\mathbb{R}^d$. There is a correspondence between $p$-codimensional algebraic subspaces of $X_\Delta$ and $p$-(real)-dimensional subcones of $\Delta$. Thus, divisors in $X_\Delta$ correspond to 1 dimensional cones, i.e., rays in $\Delta$.

In figure ?? we show an example of a fan $\tilde{\Delta}$ describing a toric variety $X_{\tilde{\Delta}}$. (We use $\tilde{\Delta}$ rather than $\Delta$ for this particular example to fit in with notation used below.) In this case $\tilde{\Delta}$ lives in $\mathbb{R}^3$ and consists of three 3-dimensional cones. The configuration of these cones is shown by slicing $\tilde{\Delta}$ by a hyperplane $\Pi$ as shown in the figure. Since $\tilde{\Delta}$ does not fill $\mathbb{R}^3$, $X_{\tilde{\Delta}}$ is not compact. $X_{\tilde{\Delta}}$ contains a divisor, $E$, which corresponds to the ray indicated in the figure. In this case the other 3 rays correspond to divisors not homologically distinct from $E$.

We now discuss the toric picture of blowing up. In the context of this paper, a blow-up consists of adding a divisor with one or more irreducible component into $X_\Delta$. Thus we must add rays to the fan $\Delta$. This process is easily visualized in terms of the intersection of $\Delta$ with $\Pi$. To add an irreducible divisor to $X$, draw a point in the interior of $\Delta \cap \Pi$ and then draw lines from it to form a triangulation of the $(d-1)$-dimensional simplex (or simplices) in which the point lies. Thus the simplex containing the new point is subdivided into a
set of simplices each having this new point as a vertex. This simple case is known as a star subdivision of the fan. A sequence of many star-subdivisions gives a blow-up whose exceptional divisor consists of many irreducible divisors. We can illustrate an example of the process of a single star-subdivision by the toric variety pictured in figure ??.

In this case we consider $X_\Delta$ to be the blown-up toric variety from a variety $X_\Delta$. The exceptional divisor is $E$. The process is shown in figure ?? where we draw only the intersection of the fan with $\Pi$ rather than the whole fan itself.

Now let us consider the questions of when $X_\Delta$ is singular and when $X_\Delta$ has trivial canonical class. For both of these questions we need to look at the lattice structure $N$ and the way in which it relates to $\Delta$. Consider following each ray in $\Delta$ out from the origin of the fan, which is itself a lattice point, until we reach another lattice point. Mark this point in $\mathbb{R}^d$. The condition for trivial canonical class is simple to state — all these points must lie in a hyperplane. We will imagine $\Pi$ to be this plane from now on. Now consider each simplex in $\Delta \cap \Pi$. Each $(d-1)$-dimensional simplex forms a $d$-dimensional simplex by including the origin of the fan as another vertex. A volume (or length or area etc.) of this $d$-simplex may be calculated where the volume is defined in terms of units defined by the lattice $N$. $X_\Delta$ is smooth if and only if each such simplex has volume $1/d!$. Thus we see from figure ?? that if $X_\Delta$ is smooth then $X_\Delta$ is singular since the tetrahedron subtended by the triangle in $\Delta \cap \Pi$ has volume $\frac{1}{2} \neq \frac{1}{3!}$.

Now all we need to complete our discussion of resolving quotient singularities is the form in which $\Delta$ represents a quotient singularity. It turns out that a singularity of the type $\mathbb{C}^d/G$ for abelian $G$ is represented by a fan where $\Delta \cap \Pi$ consists of just one $(d-1)$-dimensional simplex. Suppose we consider the space $\mathbb{C}^d$ with coordinates $(z_1, z_2, \ldots, z_d)$ and $X_\Delta \cong \mathbb{C}^d/G$ with coordinates $(x_1, x_2, \ldots, x_d)$ away from the origin. Let $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_d$ be the vertices of the simplex $\Delta \cap \Pi$ which have coordinates

\begin{align*}
\vec{\alpha}_1 &= (a_{11}, a_{12}, \ldots, a_{1d}) \\
\vec{\alpha}_2 &= (a_{21}, a_{22}, \ldots, a_{2d}) \\
&\vdots \\
\vec{\alpha}_d &= (a_{d1}, a_{d2}, \ldots, a_{dd}).
\end{align*}

The coordinates above are defined with respect to the lattice $N$, i.e., the points with integer coordinates are the points in $N$. This implies that the $a_{ij}$'s are integers given our definition of $\Pi$ above.

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3 This hyperplane must also be distance “one” in suitable units from the origin to ensure that $K = 0$ rather than only $nK = 0$ for some $n > 1$. See, for example, the discussion of $\mathbb{Q}$-Cartier divisors in [10]. This situation will not arise for the class of examples we consider here.

4 Note that the conventions in this paper differ by a factor of $d!$ from those of [6].

5 To be more precise, $(x_1, x_2, \ldots, x_d)$ are the coordinates on the algebraic torus.
We now state that the transformation between $X_\Delta$ and $\mathbb{C}^d$ is given by
\begin{align*}
  z_1^{a_{11}} z_2^{a_{12}} \ldots z_d^{a_{1d}} &= x_1 \\
  z_1^{a_{21}} z_2^{a_{22}} \ldots z_d^{a_{2d}} &= x_2 \\
  & \vdots \\
  z_1^{a_{d1}} z_2^{a_{d2}} \ldots z_d^{a_{dd}} &= x_d.
\end{align*}
(2)

The above relationships are sufficient to determine $G$ for a given set of $a_{ij}$’s. Suppose an element $g \in G$ acts on $\mathbb{C}^d$ as
\begin{equation}
  g : (z_1, z_2, \ldots, z_d) \mapsto (e^{2\pi i g_1} z_1, e^{2\pi i g_2} z_2, \ldots, e^{2\pi i g_d} z_d),
\end{equation}
(3)

where $g_i$ are rational numbers, $0 \leq g_i < 1$. The relations (2) imply that
\begin{equation}
  \sum_i g_i a_{ij} \in \mathbb{Z} \quad \forall j.
\end{equation}
(4)

We can now blow-up $X_\Delta$ by subdividing the simplex with corners $\bar{\alpha}_i$ with points that lie in the intersection of this simplex with $\mathbb{N}$. Such a blow-up will not affect the canonical class. Such a point $\bar{\beta}_k$ is given by
\begin{equation}
  \bar{\beta}_k = \sum_i h_i^{(k)} \bar{\alpha}_i,
\end{equation}
(5)

\begin{equation}
  \sum_i h_i^{(k)} = 1,
\end{equation}

\begin{equation}
  0 \leq h_i^{(k)} < 1,
\end{equation}

and that $\bar{\beta}_k$ has integer coordinates. From (4) we see that each point $\bar{\beta}_k$ is given by $h^{(k)} = g \in G$ such that
\begin{equation}
  \sum_i g_i = 1.
\end{equation}
(6)

As a simple example, suppose $d = 3$ and $\Delta$ is given by
\begin{align*}
  \bar{\alpha}_1 &= (3, -1, -1) \\
  \bar{\alpha}_2 &= (0, 1, 0) \\
  \bar{\alpha}_3 &= (0, 0, 1).
\end{align*}
(7)

This implies that $G$ is isomorphic to $\mathbb{Z}_3$ and is generated by
\begin{equation}
  \zeta : (z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, \omega z_3), \quad \omega = e^{2\pi i/3},
\end{equation}
(8)
i.e., $\zeta_i = \frac{1}{3}$ for $i = 1, 2, 3$. In fact figure ?? shows the blow-up of this singularity. The point added, $\vec{\beta}_1$ has coordinates $(1, 0, 0)$ and is given from (3) by $h^{(1)} = \zeta$.

As a more complicated example consider the case where $d = 3$ and $G \cong \mathbb{Z}_{11}$ and is generated by

$$(\zeta_1, \zeta_2, \zeta_3) = \left(\frac{1}{11}, \frac{3}{11}, \frac{7}{11}\right).$$

In this case

$$\vec{\alpha}_1 = (11, -3, -7)$$
$$\vec{\alpha}_2 = (0, 1, 0)$$
$$\vec{\alpha}_3 = (0, 0, 1).$$

The points, $\vec{\beta}_k$, added to $\Delta \cap \Pi$ to form the blow-up together with the corresponding element of $G$ are

$$\vec{\beta}_1 = (1, 0, 0) \quad \text{given by } \zeta$$
$$\vec{\beta}_2 = (2, 0, -1) \quad " \quad \zeta^2$$
$$\vec{\beta}_3 = (4, -1, -2) \quad " \quad \zeta^4$$
$$\vec{\beta}_4 = (5, -1, -3) \quad " \quad \zeta^5$$
$$\vec{\beta}_5 = (8, -2, -5) \quad " \quad \zeta^8.$$

A blow-up of this $\mathbb{Z}_{11}$ quotient singularity is shown in figure ?? . Note that there is more than one way to subdivide this triangle to add in the points $\vec{\beta}_k$. This corresponds to the fact that there is more than one way of blowing up this singularity. No matter how we do the subdivision we will always divide the big triangle $\Delta \cap \Pi$ into 11 triangles subtending volumes of $\frac{1}{6}$. This means that all the blow-ups completely resolve the $\mathbb{Z}_{11}$ singularity to a smooth space.

Note that in the case of the $\mathbb{Z}_{11}$, the complete resolution required an exceptional divisor with 5 irreducible components which correspond to $\vec{\beta}_1, \ldots, \vec{\beta}_5$. Each of these components is independent with regards to homology. The precise geometry of these 5 divisors depends on the triangulation of $\Delta \cap \Pi$ and figure ?? shows just one possibility.

It is a matter of combinatorics to show that this process of blowing up will always fully resolve a singularity of the form $\mathbb{C}^3/G$ where $G \subset SU(3)$ is abelian. It is interesting to note however that this is not the case for more than 3 dimensions. Consider the case of $\mathbb{C}^4/\mathbb{Z}^2$ given by the generator

$$\zeta : (z_1, z_2, z_3, z_4) \mapsto (-z_1, -z_2, -z_3, -z_4).$$

In this case

$$\vec{\alpha}_1 = (2, -1, -1, -1)$$
$$\vec{\alpha}_2 = (0, 1, 0, 0)$$
$$\vec{\alpha}_3 = (0, 0, 1, 0)$$
$$\vec{\alpha}_4 = (0, 0, 0, 1).$$
The tetrahedron $\Delta \cap \Pi$ subtends a 4-simplex with the origin with volume $\frac{1}{12}$ implying, as expected, that $X_\Delta$ is singular, but this time there are no points in $\mathbb{N}$ inside this tetrahedron on which one might subdivide. Thus there is no toric resolution of this singularity preserving vanishing canonical class. In fact D. Morrison [14] has shown that the results of [13] may be used to show that there is no resolution of this singularity of any kind which preserves $K = 0$.

Let us now discuss the way in which the Kähler form controls the blow-up. The Kähler form, $J$, may be expanded as

$$J = \sum_{\xi=1}^{h_1,1} J_\xi e_\xi,$$

where $J_\xi \in \mathbb{R}$ and the $e_\xi$’s form basis of $H^2(X,\mathbb{Z})$. Imposing the condition that all curves, surfaces, etc., have positive volume will put restrictions on the allowed range of values of $J_\xi$ to give the Kähler cone. In the simple case of a curve having homology class $\sum c_\xi e_\xi$, the area will be $\sum J_\xi c_\xi$.

We may also calculate the size of any divisor, $D_l$, as follows. Suppose that $D_l$ is a generator of $H_{2(d-1)}(X,\mathbb{Z})$ and is dual to a cycle with homology class $e_l$. We then have (with suitable normalizations of volume)

$$\text{Vol}(D_l) = \int_{D_l} J^{\wedge (d-1)} = \sum_{n_1,n_2,...} J_{n_1} J_{n_2} \cdots \int_{D_l} e_{n_1} \wedge e_{n_2} \wedge \cdots$$

$$= \sum_{n_1,n_2,...} (D_l \cap D_{n_1} \cap D_{n_2} \cap \ldots) J_{n_1} J_{n_2} \ldots,$$

where $(D_1 \cap D_2 \cap \ldots \cap D_d)$ represents the intersection form on $X$. Thus we see that the volume of a divisor is given by an expression more complicated than that for curves and it depends on the intersection numbers.

Suppose we take a smooth manifold, $X$, and blow-down an exceptional divisor to form an isolated quotient singularity. Let $D_{r_1}, D_{r_2}, \ldots$ represent the irreducible components of the exceptional divisor which we assume are some of the generators of $H_{2(d-1)}(X,\mathbb{Z})$ and are thus dual to $e_{r_1}, e_{r_2}, \ldots \in H^2(X,\mathbb{Z})$. The part of $H_{2(d-1)}(X,\mathbb{Z})$ not spanned by $D_{r_1}, D_{r_2}, \ldots$ may be thought of as the homology which descends from the covering space of our quotient singularity and this has no intersection with $D_{r_1}, D_{r_2}, \ldots$. Thus we see that for $J_{r_1} = J_{r_2} = \ldots = 0$ we have $\text{Vol}(D_{r_1}) = \text{Vol}(D_{r_2}) = \ldots = 0$. Thus the orbifold is described by points corresponding to $J_{r_1} = J_{r_2} = \ldots = 0$ on the edge of the Kähler cone. It is important to realize however that the parameters $J_{r_1}, J_{r_2}, \ldots$ do not independently control the sizes of $D_{r_1}, D_{r_2}, \ldots$. 9
As an example consider a $\mathbb{C}^3/\mathbb{Z}_5$ singularity. There are two components of the exceptional divisor, $A$ and $B$, such that

\begin{align*}
(A \cap A \cap A) &= 9 \\
(B \cap B \cap B) &= 8 \\
(A \cap A \cap B) &= -3 \\
(A \cap B \cap B) &= 1.
\end{align*}

(16)

Let us put $J = J_0 + ae_a + be_b$, where $J_0$ is the part of the Kähler form given by cohomology elements away from the quotient singularity. We then have

\begin{align*}
\text{Vol}(A) &= (3a - b)^2 \\
\text{Vol}(B) &= (a - 2b)(-3a - 4b).
\end{align*}

(17)

We also have an algebraic curve homologous to $A \cap B$ which has area $b - 3a$ and another curve within $B$ with area $a - 2b$. Thus we have constraints on the Kähler cone $a - 2b > 0$, and $b - 3a > 0$. These are sufficient to give positive volumes to $A$ and $B$.

Finally in this section let us introduce the notion of a “dual curve”. We have seen that given a component $D_l$ of the exceptional divisor, the associated Kähler form gives the volume of the divisor itself in terms of an expression like (17). If there were a curve in $X$ whose homology class was dual to that of $D_l$, the description of its area in terms of the Kähler form expansion would be more direct. Thus when thinking of performing a blow-up by switching on components of the Kähler form it is probably best to think in terms of the dual curves growing rather than the exceptional divisor itself. This will be implicit in much of what follows. Note that $D_l$ might be any element of $H_{2(d-1)}(X, \mathbb{Z})$ and so in general the dual must be thought as a rational combination of algebraic curves. In the $\mathbb{C}^3/\mathbb{Z}_5$ example above, the Kähler cone imposes the condition $a, b < 0$. Thus there are no actual curves dual to $A$ or $B$ since they would have negative area.

3 Conformal Field Theory

In this section we will review and study the quotient singularities from the point of view of the conformal field theory of string propagating on the singular target space. The reader is referred to [14] for example for a more complete discussion. We should also point out that we will not consider the case of nontrivial “discrete torsion” in the sense of [15] in this paper.

The field theory on $M/G$ is studied in terms of the theory on the smooth covering space $M$. In addition to the $G$-invariant states of $M$, the $M/G$ theory also has “twisted” states. These may be thought of as open strings in $M$ whose ends are identified by the element $g \in G$. Such a state is said to be in the $g$-twisted “sector”. As we shall see, it is the twisted states which correspond to blow-up modes.
Let us analyze the chiral ring of the conformal field theory which is expected to correspond to the cohomology ring of the target space. It is not too hard to see that the $G$-invariant elements of the cohomology of $M$ will be precisely the untwisted modes on $M/G$ and so the twist fields are somehow “extra” cohomology on $M/G$. Since the counting of chiral fields should not change under a deformation of a theory, the count for a blown-up orbifold should be the same as the orbifold itself. Thus, the chiral fields which are twist fields must count elements of cohomology which are generated during the blowing-up process.

In conformal field theory language, the deformations are done by truly marginal operators which, in the context of $N=(2,2)$ theories, takes a particularly simple form. That is, an action $S_0$ may be deformed into $S$ as

$$S = S_0 + \sum_k a_k \int \Phi_k d^2\theta^+ d^2z + \sum_r \tilde{a}_r \int \tilde{\Phi}_r d\bar{\theta}^+ d\theta^- d^2z + \text{h.c.} \quad (18)$$

The $\Phi_k$ and $\tilde{\Phi}_r$ are (anti)chiral superfields with $U(1)$ charges (left,right) equal to $(1,1)$ and $(-1,1)$ respectively. As is well-known, the $\sigma$-model interpretation of these superfields are as elements of $H^{1,1}$ and $H^1(T)$ and thus as deformations of Kähler form and complex structure respectively. In order to focus on blow-ups, which are to be regarded as deformations of the Kähler form, we will concentrate on the fields $\Phi_k$.

Let us consider the spectrum of chiral twist-fields in an orbifold $\mathbb{C}^d/G$ with only the origin as an isolated fixed point. Consider an element $g \in G$ which acts on $\mathbb{C}^d$ with coordinates $(z_1, z_2, \ldots, z_d)$ as in (3) subject again to the condition that $0 \leq g_i < 1$. One can then show (see, for example, [17] or [18]) that the $g$-twisted state associated with the origin has charge $(Q, \bar{Q})$ where

$$Q = \bar{Q} = \sum g_i. \quad (19)$$

Thus, in order to obtain twisted marginal operators associated to blow-up modes we require precisely equation (11) from the previous section. That is, the classification of twisted marginal operators associated to blow-ups has exactly matched the determination of irreducible components of the exceptional divisor.

We have thus arrived at what might be called the “(twist-field)-divisor” map. Since the classifications are isomorphic one can naturally identify a natural one-to-one correspondence. E.g., for the $\mathbb{Z}_{11}$ example one may consider figure ?? and equation (11) where the last column in (11) now represents the group element by which the twist-field is twisted.

If we consider the $\mathbb{C}^4/\mathbb{Z}_2$ singularity in the last section which had no $K = 0$ resolution then we see that in the language of conformal field theory, there are no twisted marginal

---

6The choice of signs of these charges is a matter of convention.
operators associated with the fixed point. Thus geometry and conformal field theory again agree.

Actually what we have done thus far is little more than paraphrase the work of \cite{6, 19} where the “monomial-divisor mirror map” was introduced. The rôle of monomials in a Landau-Ginzburg theory has been replaced by twist fields in the mirror of that theory. In fact, it can be seen from the example studied in \cite{3} that the equivalence of monomials in a Landau-Ginzburg theory and twist-fields in the mirror arises naturally and can be traced back to the work of \cite{20}. All we have done in this paper is to free the formalism of \cite{6, 19} from references to the global space as a whole so we need concentrate only on the quotient singularities locally.

We can now continue our study of the conformal field theory of blow-ups by continuing to copy results from the monomial-divisor mirror map to the (twist-field)-divisor map. In particular we can study quantitatively the relationship between the coefficients $a_k$ in (18) and exactly the size and shape of the exceptional divisor.

The question of determining the size and shape of the exceptional divisor immediately forces one to face questions to do with quantum verses classical geometry. One knows that if the coefficients $a_k$ are small then the exceptional divisor will be small and we are thus dealing with lengths possibly near the Planck scale. This is the point at which we must address the issues raised in the introduction concerning the phase picture of the moduli space. We are going to try to interpret the orbifold phase in terms of sizes of exceptional divisors and thus are biasing ourselves in favour of the smooth Calabi-Yau phase. In order to do this we must begin in the Calabi-Yau phase, i.e., when the singularity is blown-up so that all radii are large and we can make contact with classical notions of length.

Within the Calabi-Yau phase there is a natural definition of Kähler form. This is given by the “$\sigma$-model measure” defined in \cite{8}. It cannot be emphasized too strongly that this is not the same Kähler form that was used in the introduction to draw figure ???. The Kähler form used in the introduction was abstractly defined and treats each phase equally. See \cite{8} for a more complete discussion of these two Kähler forms. The Kähler form derived from the $\sigma$-model arises as follows. Suppose, $d = 3$ and, for simplicity, that $H^{2,0}(X) \cong 0$. Let us consider the 3-point function between three chiral fields $\Phi_l$, $\Phi_m$ and $\Phi_n$. Assuming we have a purely geometrical interpretation, $X$, of the theory, we can associate these fields to elements of $H^{1,1}(X)$ and thus divisors $D_l$, $D_m$ and $D_n$. It can then be shown that \cite{21, 22, 23}

$$
\langle \Phi_l \Phi_m \Phi_n \rangle = (D_l \cap D_m \cap D_n) + \sum_{\Gamma} \frac{q^{\Gamma}}{1 - q^{\Gamma}} (D_l \cap \Gamma)(D_m \cap \Gamma)(D_n \cap \Gamma),
$$

(20)

where $\Gamma$ is a holomorphically embedded $\mathbb{P}^1$ in $X$ and $q^{\Gamma}$ is a monomial in the variables $q_\xi$. 12
We define the parameters $q_\xi$ by

$$q_\xi = \exp\{2\pi i (B_\xi + i J_\xi)\},$$

(21)

where the Kähler form, $J$, on $X$ has been expanded as in (14). The antisymmetric tensor, “$B$-field”, on $X$ is also an element of $H^2(X; \mathbb{R})$ on $X$ and is similarly expanded to give $B_\xi$. We have implicitly used the same normalization $4\pi^2 \alpha' = 1$ as in [8].

In the neighbourhood of a large radius limit, the expansion (20) is sufficiently single-valued to allow a determination of $J$ from (14). This then allows us to work out the size of any divisor or, to be more precise, the area of the dual curve as in section 2. This is the sense in which the conformal field theory data, i.e., the three point functions $\langle \Phi_l \Phi_m \Phi_n \rangle$ can be used to “measure” sizes within $X$.

4 Exploring the Moduli Space

We will now discuss the relationship between the coupling constants $a_k$ in (18) and the parameters $J_l$. This will tell us the precise way in which the twisted marginal operators perform the blow-up of the orbifold. In general there is a very complicated relationship between the $a_k$’s and $J_l$’s but in this paper we will study the situation when only one component is added to the exceptional divisor at a time. This is the picture illustrated in figure ???. In terms of toric geometry this resolution consists of a sequence of star-subdivisions of the fan corresponding to the orbifold. One should note that not all resolutions can be generated by a sequence of star-subdivisions and that we have thus lost some generality in the following discussion. The reader might wish to verify that the resolution in figure ?? can be obtained by a sequence of star-subdivisions, e.g., $\vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_1, \vec{\beta}_4, \vec{\beta}_5$.

Consider the situation in which

$$S = S_1 + \sum_{u} a_u \int \Phi_u d^2 \theta + d^2 z,$$

(22)

and suppose each field $\Phi_u$ corresponds to a ray in a fan $\Delta$. Let the label $l$ denote which point we are going to add to give our star subdivision and let the field $\Phi_l$ be the corresponding twisted marginal operator. In this case there will be a plane $\Pi$ as discussed in the previous section intersecting this ray and its neighbours (but not necessarily all the rays in the fan). Associate the position vectors $\vec{\alpha}_u$ as the points of intersection between $\Pi$ and each ray. There will then be some minimal relationship

$$\sum_{u \in U_l} N_u \vec{\alpha}_u - N_l \vec{\alpha}_l = \vec{0},$$

(23)
for the set of rays $U_l$ which give the vertices of the minimum dimension simplex in $\Pi$ which is being star-subdivided. The integers $N_u, N_l$ are positive and have highest common factor 1. We then define

$$z_l = (-a_l)^{-N_l} \prod_{u \in U_l} a_u^{N_u}. \tag{24}$$

The hypothesis of the monomial-divisor mirror map [19] then tells us [8] that

$$(B + iJ)_l = \frac{1}{2\pi i} \log z_l + O(z_l), \tag{25}$$

for $|z_l| \ll 1$.

The above expansion is expected to converge for $|z_l| \ll 1$, that is, $J_l \gg 0$. This is the sense in which we have had to go to the large-radius limit to make contact with classical geometry. Note that, as one would expect, this limit is achieved by adding in a large amount of twist field to the action, i.e., having a large coefficient $a_l$.

The situation explored in [19] was for the mirror of a Landau-Ginzburg theory. In this situation the form (22) made sense — each $\Phi_u$ corresponds to a monomial in the superpotential of the mirror. In a general orbifold theory, it is not clear how to write an action in the form (22). Since we expect the analysis of a blow-up to not depend on global properties of a manifold however we claim that some version of the above analysis can be performed for any abelian quotient singularity. That is, we may obtain some “normalized” variable $z_l$ which controls the size of a blow-up at the large radius limit in terms of the coefficient of the marginal operator $a_l$. One should also note that it is by use of the mirror map that we are effectively putting a connection on the bundle of fields over the moduli space. When a marginal operator is used to perturb a theory a finite distance in moduli space, one should specify the way in which the marginal operator itself is changing along this path. In the context of Landau-Ginzburg theories, one can write down a superpotential in terms of parameters — the coefficients within the superpotential. By varying these parameters one moves a finite distance in moduli space. If one identifies fields with the monomials in the superpotential one is also implicitly defining the way in which these fields transform along the path. Thus in this paper we are asserting that the twist-fields transform the same way in which the monomials do in the Landau-Ginzburg-type theory.

Once we have the relationship (25) we may use the local geometry of the moduli space to determine the form of the $O(z_l)$ term. Again we will use mirror symmetry to motivate our argument. It was conjectured in [24] and demonstrated in [25] that given a pair of mirror spaces, $X$ and $Y$, with the Kähler form on $X$ being given by the above notation then

$$(B + iJ)_l = \frac{\int_{\gamma_l} \Omega}{\int_{\gamma_0} \Omega}, \tag{26}$$
where $\Omega$ is a $(3,0)$-form on $Y$ and $\gamma_0, \gamma_l$ are elements of $H_3(Y,\mathbb{Z})$. In the case that $Y$ is represented by a Landau-Ginzburg theory with superpotential (22) we can consider a period $\int \gamma_\Omega$ to be a function of the variables $a_u$. In this case, the periods satisfy a set of coupled linear Fuchsian partial differential equations known as the Picard-Fuchs equations. In [26] it was shown that this set of differential equations were of the hypergeometric type studied by Gel’fand et al in [27, 28]. However, in [28] these differential equations were written down purely in terms of the toric data of $X$. This means that in determining the periods as a function of $a_u$, no direct reference need be made to the geometry of $Y$ — only that of $X$. This is exactly what we want in the context of this paper since we are not directly interested in the structure of $Y$.

The Picard-Fuchs equations will have many solutions but the singularity structure of the equations dictates that the solutions are classified by their behavior near $z_l = 0$. In fact, the equation (25) is sufficient to determine precisely which two periods are required for the ratio in (26).

We are now in a position to track the entire process of a blow-up of a singularity. The general solution of the Picard-Fuchs equation is rather awkward to handle and so we will study only the decoupled situations when we need only consider ordinary differential equations (although it should be noted that a full multiparameter system can been studied [29, 30, 31, 32]). It was shown in [8] that if all the components of $J$ were held at $\infty$ except for one, $J_l$, the the Picard-Fuchs equations decoupled leaving an ordinary differential equation of which two of the solutions would provide the ratio for (26). It is also the case that if a point is not included in the triangulation of $\Delta \cap \Pi$ then that point plays no rôle in the formulation of the Picard-Fuchs equations. Thus, if we consider the process of a star-subdivision on a point, we may consider this process decoupled from the rest of the system both before the star-subdivision is included (since then the point is not in the triangulation) and when the large $J$ limit of this process has been taken.

In summary then we consider the following process. Consider an orbifold which is at its large radius limit, i.e., all components of the Kähler form not associated with blow-ups are infinite. Then blow-up the singularities by a sequence of processes as follows: take a single star-subdivision and slowly take that component of the Kähler form (i.e., the area of the dual curve) to infinity. In this way only one component of the Kähler form will be finite at any given time and we will only ever have to deal with ordinary differential equations.

We now state the ordinary differential equations required (see [8] for more details). Given
a star-subdivision on $\tilde{\alpha}_l$ given by (23) then

\[
\left\{ \prod_{u \in U_l} N_u^N_u \left( z_l \left( \frac{d}{dz_l} - \frac{N_u - 1}{N_u} \right) \left( \frac{d}{dz_l} - \frac{N_u - 2}{N_u} \right) \ldots \left( \frac{d}{dz_l} - \frac{1}{N_u} \right) \right) \right. \\
\left. - z_l N_l^N_l \left( z_l \left( \frac{d}{dz_l} + \frac{N_l - 1}{N_l} \right) \left( \frac{d}{dz_l} + \frac{N_l - 2}{N_l} \right) \ldots \left( \frac{d}{dz_l} + \frac{1}{N_l} \right) \right) \right\} f(z_l) = 0,
\]

where $f(z_l)$ is the required period. It is trivial to show that $f(z_l) = 1$ is a solution and we take this to be the period in the denominator of (26).

We can write down another solution if we impose that one of the $N_u$'s is equal to 1. Let us denote the set $U_l$ with this element removed by $\bar{U}_l$. With a little effort, using the ideas in [28] and that $\sum_{u \in U_l} N_u = N_l$, one can show that

\[
h_l(z) = \frac{N_l}{2\pi i} \int_{-\infty}^{+\infty} \prod_{u \in \bar{U}_l} \left( \frac{\Gamma(N_u s)\Gamma(-s)}{\Gamma(N_u s + 1)} \right) (-z)^s ds - \pi i
\]

\[
= \log z + \sum_{n=1}^{\infty} \frac{N_l(N_l n + 1)!}{n!} \prod_{u \in \bar{U}_l} (N_u n)! z^n
\]

is a solution. The integration path is taken along the imaginary line deformed around the origin to the left and the series is obtained by completing this path to the right if we impose the condition $\arg(-z) < \pi$. Hence from (25) we deduce that the precise Kähler form is given by

\[
(B + iJ)_l = \frac{1}{2\pi i} h_l(z_l).
\]

The region of convergence of the series in (28) is given by $|\hat{z}| < 1$ or $\hat{z} = 1$ where

\[
\hat{z} = \prod_{u \in \bar{U}_l} \frac{N_l^N_l}{N_u^N_u} z.
\]

We can analytically continue $h_l(z)$ into the region $|\hat{z}| > 1$ in the usual way by closing the integration path in (28) to the left. We then obtain

\[
h_l(z) = -\pi i - \frac{\Gamma(1 - \frac{N_l}{N_u})}{\prod_{u \in \bar{U}_l} \Gamma(1 - \frac{N_u}{N_s})} \psi_l + O(\psi_l^2),
\]
where $\psi^{-N_l} = z_l$ and $0 < \arg(\psi_l) < \frac{2\pi}{N_l}$. Note that if we normalize $a_u = 1$ for $u \in U_l$, then we have $\psi_l = a_l$. This means we can think of $\psi_l$ as the twist-field coupling constant.

Without the blow-up, $a_l = 0$ and thus $\psi_l = 0$. This means that before blowing up we have

$$B_l = -\frac{1}{2}, \quad J_l = 0.$$  \hspace{1cm} (32)

This value of $J_l$ agrees with classical geometry. That is, the boundary of the Kähler cone at which $J_l = 0$ is precisely where the classical orbifold lives. This may be thought of in terms of the commutativity of the following diagram:

$$
\begin{array}{ccc}
M & \stackrel{\sigma\text{-model}}{\longrightarrow} & S_M \\
\downarrow/G & & \downarrow/G \\
M/G & \stackrel{\text{"sigma-model"}}{\longrightarrow} & S_M/G
\end{array}
$$

In this diagram, $M$ is a smooth Calabi-Yau manifold, $G$ is a symmetry by which we can mod out and $S_M$ represents the conformal field theory associated to $M$. The quotients taking us from the top row to the bottom row are effected geometrically on the left and in terms of field theory (i.e., twist-fields) on the right. The arrow on the top line represents a $\sigma$-model interpretation which is straight-forward for large $M$ whereas the arrow in the bottom line represents the “analytic continuation” of a $\sigma$-model to small radii (of exceptional divisor).

It is important to notice that we have only established the commutativity of (33) when the orbifold is at large radius limit away from the quotient singularities. That is, all the components of the Kähler form associated to untwisted elements of $H^2(X)$ are infinite. It is an interesting question to see if commutativity still holds away from this limit. That is, given a finite sized orbifold, do we get $J_l = 0$ for the Kähler form analytically continued from the large blow-up limit. This is addressed in [33].

It is not yet clearly understood what the geometrical meaning of the value $B_l = -\frac{1}{2}$ at the orbifold point is.

Let us discuss more carefully the way in which we have performed the analytic continuation in obtaining (31). The Fuchsian differential equation (27) has singularities at $\hat{z}_l = 0, 1, \infty$ and thus we expect non-trivial monodromy of the solutions around these points. We may think of $z_l$ (together with $z_l = \infty$) as parametrizing a $\mathbb{P}^1$ in the compactified moduli space of conformal field theories. We show this $\mathbb{P}^1$ in figure ??.

If we wish to ascribe a single value of $(B + iJ)_l$ to each point on the $\mathbb{P}^1$, we need to define branch cuts. Around $z_l = 0$ we have infinite monodromy familiar from the symmetry $B_l \cong B_l + 1$ of the quantum field theory. This tells us that there should be one branch line coming out from this point. We effectively perform this cut by imposing $-1 < B_l < 0$ which may be thought of as the $\arg(-z) < \pi$ condition implicit in (28). The other end of this
branch cut is $\hat{z}_l = 1$ — the point at the edge of the convergence of the series. The analytic continuation then implies a branch cut from $\hat{z}_l = 1 \rightarrow z_l = \infty$. This cut was imposed by the condition $0 < \arg(\psi_l) < \frac{2\pi}{N_l}$. These branch cuts are shown in figure ??.

One would naturally expect the smallest values of $J_l$ to be close to the orbifold point $\psi_l = 0$. From (31) however we see that for small values of $\psi_l$, our branch cut imposes the condition $J_l \geq 0$. This indicates strongly that $J_l \geq 0$ over the entire $\mathbb{P}^1$. It also indicates that all the $J_l$’s may be non-negative over the whole moduli space. This may be checked in simple examples but there is no general proof of this fact yet. Thus it would appear that no measurements of negative distance (area, etc.) are possible in this scenario.

The monodromy around $z_l = \infty$ is finite of order $N_l$. We may therefore remove the branch cut locally by taking a finite cover branched around this point in the moduli space. Thus the orbifold point may be thought of as a $\mathbb{Z}_{N_l}$-quotient singularity itself within the moduli space. Note that the structure of the singularity in the moduli space is generally different from the the singularity in the target space although there are similarities. There will be a $\mathbb{Z}_{N_l}$-quotient singularity in moduli space for each irreducible component of the exceptional divisor of the target space quotient singularity.

These orbifold points in the moduli space may also be seen to arise in terms of “quantum symmetries” [34]. It is known that for string theory with a target space in the form of an orbifold $M/G$ where $G$ is a cyclic group, then $\tilde{G}$ is part of the symmetry group of the theory and orbifolding by this group will retrieve the string with target space $M$. The symmetry group $G$ acts nontrivially on the twist fields and thus on the tangent bundle of the moduli space at the orbifold point giving a quotient singularity in the moduli space. Note however that this argument relies on global symmetries. If we return to the example of the $Z$-orbifold, we have a $\mathbb{Z}_3$ quantum symmetry. This is destroyed if any twist-field marginal operator is switched on to resolve one of the 27 quotient singularities. Thus if we consider switching on a second twist-field to blow-up another fixed point we cannot use the quantum symmetry argument to find the structure of the moduli space in the vicinity of this region. Our analysis of the differential equations above however tells us that there is indeed a $\mathbb{Z}_3$ quotient singularity in the moduli space for each of the 27 blow-ups.

5 Quotient Singularities in Two Dimensions

In this section we will illustrate the results of the last few sections by considering the simplest set of quotient singularities — namely those of the form $\mathbb{C}^2/\mathbb{Z}_n$ given by the action

$$\zeta : (z_1, z_2) \mapsto (e^{2\pi i/n} z_1, e^{-2\pi i/n} z_2).$$

(34)

Our analysis does not determine the full codimension of this singularity. One should remember that codimensional one quotient singularities may be removed by coordinate redefinition.
Figure 1: The fan for the resolution of $\mathbb{C}^2/\mathbb{Z}_6$.

This corresponds to a matrix

$$A = \begin{pmatrix} n & 1 \\ 0 & 1 \end{pmatrix}. \quad (35)$$

The lattice points in $\Delta \cap \Pi$ are thus given by

$$\vec{\beta}_m = (m, 1), \quad m = 1, \ldots, n - 1, \quad (36)$$

corresponding to marginal operators in the $\zeta^m$-twisted sector. The subdivision of the fan $\Delta$ to smooth this quotient is unique and is shown (for $n = 6$) in figure 1.

The geometric interpretation of the blow-up in figure 1 is of a chain of $\mathbb{P}^1$'s each one touching its neighbour at one point. This corresponds to the well-known Hirzebruch-Jung string for the resolution of this singularity (see for example [35]). Another way of viewing this is as follows. Begin with a $\mathbb{C}^2/\mathbb{Z}_n$ quotient and star subdivide on $\vec{\beta}_1$. One can show that the resulting blow-up has an exceptional divisor of $\mathbb{P}^1$ (corresponding to $\vec{\beta}_1$) and a singularity locally of the form $\mathbb{C}^2/\mathbb{Z}_{n-1}$ lying on this $\mathbb{P}^1$. We can now blow-up this singularity with $\vec{\beta}_2$ to give another $\mathbb{P}^1$, touching the first one, this time with a $\mathbb{C}^2/\mathbb{Z}_{n-2}$ quotient singularity. This process is repeated giving a chain of $n - 1 \mathbb{P}^1$'s for which the space is smooth.

In order to find the twist-field version of the resolution of $\mathbb{C}^2/\mathbb{Z}_n$ we thus need only study the star-subdivision on $\vec{\beta}_1$ since the complete blow-up may be viewed in terms of a sequence.
of such events for decreasing $n$. The relationship (23) then becomes

$$ (n-1)\vec{\alpha}_1 + \vec{\alpha}_2 = n\vec{\beta}_1. $$ (37)

Assuming our twist-field is normalized correctly, which would amount to set $a_1 = a_2 = 1$ in the Landau-Ginzburg theory of the mirror, we may call the coupling constant in front of the marginal operator $\psi$, in which case

$$ z = \psi^{-n}, $$ (38)

and

$$ (B + iJ)_1 = -\frac{n}{4\pi^2} \int_{-i\infty}^{+i\infty} \frac{\Gamma(ns)\Gamma(-s)}{\Gamma(ns - s + 1)(-z)^s} ds - \frac{1}{2}, $$ (39)

which may be expanded as

$$ (B + iJ)_1 = \frac{1}{2\pi i} \left( \log z + \sum_{p=1}^{\infty} \frac{n(np + 1)!}{p!(n-1)p!} z^p \right), $$ (40)

for $|z| < (n-1)^{n-1}/n^n$ or as

$$ (B + iJ)_1 = -\frac{1}{2} + \frac{1}{2\pi} e^{(\frac{1}{2} + \frac{1}{n})\pi i} \psi + O(\psi^2), $$ (41)

for $|\psi| < n/(n-1)^{n-1}/n$ and $0 < \arg(\psi) < 2\pi/n$.

It was shown in [8] that for $n = 2$ the Kähler form could be given in closed form as

$$ (B + iJ)_1 = \frac{i}{\pi} \cosh^{-1} \left( \frac{1}{2\sqrt{z}} \right) = -\frac{1}{\pi} \cos^{-1} \frac{\psi}{2}. $$ (42)

Finally, it is interesting consider the value of $J_1$ at the “phase transition” point between the orbifold and the Calabi-Yau manifold. That is, the point at which string theory becomes singular and we are on the boundary of the region of convergence of (40) and (41). This occurs for $z = (n-1)^{n-1}/n^n$. In table 1 we give $J_1$ for the first few values of $n$. The value of $B_1$ at this point is 0 for any $n$. It is interesting to note that for $n = 2$, the Calabi-Yau phase extends all the way down to zero distance and so the orbifold theory at $\psi = 0$ and the phase transition point differ only in the value of $B_1$. This is no longer the case for $n > 2$ and the size of the exceptional divisor for the singular theory gets progressively larger as $n$ increases.

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| $n$ | $J_i$ at transition |
|-----|----------------------|
| 2   | 0                    |
| 3   | 0.11                 |
| 4   | 0.18                 |
| 5   | 0.22                 |

Table 1: Values of the Kähler form for $z = (n - 1)^{n-1}/n^n$.

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