Power functions of the Shewhart control chart

M B C Khoo
School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Minden, Penang, Malaysia
Email: mkbc@usm.my

Abstract. The Shewhart control chart is used to check for a lack of control (a shift in the process). However, it is insensitive to small shifts. To increase the sensitivity of the Shewhart control chart, some actions can be taken. These include reducing the width of the control limits, increasing the subgroup size to reduce the variance of the sample mean and making use of detection rules to increase the sensitivity of the chart. All of these actions will influence the power functions of the Shewhart control chart. A probability table showing the probabilities of detecting sustained shifts in the process mean, calculated using the formulae have been recommended. However, the calculations of the probabilities using the formulae are complicated and time consuming. In this paper, a Monte Carlo simulation using the Statistical Analysis System (SAS) is conducted to compute the necessary probabilities. These probabilities are close to that obtained by using the formulae. The Monte Carlo simulation method is recommended as a better way to calculate the probabilities, as it provides savings, in terms of time and cost. Besides, the Monte Carlo simulation method also provides a higher flexibility in calculating the probabilities, for different combinations of the detection rules. The results obtained will assist practitioners in designing and implementing the Shewhart chart effectively.

1. Introduction
The control charting method for controlling the quality of a process in manufacturing was introduced by Dr. Walter A. Shewhart in the 1920s. According to [1], Shewhart defined the product attributes, types of product variations and suggested methods on how to collect, plot and analyze data. In fact, control charts used by quality practitioners from all around the world are based on the process control methods provided by Shewhart.

The Shewhart control chart plays an important role in Statistical Process Control (SPC). It is able to detect the occurrence of assignable causes quickly in a process control so that investigations of the process and corrective actions can be made promptly before many defective products are produced [2]. The main objective of applying the Shewhart chart is not to detect defective products but to prevent failure so that production of low quality products will not happen. This move can help in cost savings, as well as helping to produce high quality products.

The Shewhart chart is powerful in the detection of large shifts in the process mean. Nonetheless, it exhibits a lack of sensitivity to detect small shifts. Many researches have been made by quality experts to improve the chart’s sensitivity. Decreasing the variance of the sample mean by increasing the subgroup size, employing a narrower width of the control limits and applying sensitizing rules, such as those by [3], [4], [5], and [6], to the Shewhart chart are some of the popular approaches made to
enhance the performance of the chart. All these approaches can influence the power functions of the Shewhart chart. A probability table showing the probabilities of detecting sustained shifts in the process mean, computed by means of formulae was presented by [7] to serve as guidelines to quality practitioners in constructing the Shewhart chart. In this paper, we present a Monte Carlo simulation method to obtain similar results. This research is motivated by the fact that the Shewhart chart is the most well known control charting tool among practitioners and that an in-depth understanding of the power function of this chart will enable a more efficient use of the chart in process monitoring.

This paper is organized as follows: The Shewhart control chart is presented in Section 2. In Section 3, the power functions of the Shewhart chart are discussed. The probability table of the power functions of the Shewhart $\bar{X}$ chart is explained in Section 4. In Section 5, performance comparisons are made between the probabilities obtained via formulae and that obtained by means of Monte Carlo simulation. Finally, conclusions are drawn in Section 6.

2. The Shewhart control chart

The Shewhart control chart is a graphical plot that helps in the study on how a process changes over time. The Shewhart chart consists of three lines that are the lower control limit (LCL), upper control limit (UCL) and center line (CL). The construction of the Shewhart chart is based on statistical principles. Typically, the UCL is plotted $+3\sigma$ above the CL whereas the LCL is plotted $-3\sigma$ below the CL. Here, CL represents the mean of the sample data. We intend to detect a process that is out-of-control when there are sample points plotting beyond the control limits, but when the process is in-control, we wish that the probability of detecting a false alarm is as small as possible [8].

Assume that a quality characteristic follows a normal distribution with mean $\mu$ and standard deviation $\sigma$, where both $\mu$ and $\sigma$ are known. If $X_1, X_2, ..., X_n$ is a sample of size $n$, then the average of this sample is [9]

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$  \hspace{1cm} (1)

and we know that $\bar{X}$ is normally distributed with mean $\mu$ and standard deviation $\sigma_\bar{X} = \frac{\sigma}{\sqrt{n}}$. Besides, the probability is $1 - \alpha$ that any sample mean will fall between

$$\mu + Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$ \hspace{1cm} (2a)$$

and

$$\mu - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$ \hspace{1cm} (2b)$$

Hence, if $\mu$ and $\sigma$ are known, equations (2a) and (2b) can be used as upper and lower control limits of the Shewhart $\bar{X}$ chart. As mentioned earlier, it is customary to replace $Z_{\alpha/2}$ by 3 so that the $\pm 3\sigma$ limits are used. The process mean is out-of-control if the sample mean falls outside these limits. In practice, $\mu$ and $\sigma$ are usually unknown. Consequently, these parameters are estimated from a set of in-control Phase-I data. These estimates should usually be based on at least 20 to 25 samples.

3. Power functions of the Shewhart control chart

In statistical quality control, an inference on the mean of a population can be summarized as the following hypotheses testing:

$$H_0 : \mu = \mu_0,$$

$$H_1 : \mu \neq \mu_0.$$  \hspace{1cm} (3)

The definition of power is given by [9]
Power = 1 − β
= P \{ \text{reject } H_0 \mid H_0 \text{ is false} \},

where β is the probability of a Type-II error. In industries, the main goal of using a control chart is to minimize the number of defective products produced prior to the detection of a shift in the process mean [10]. Hence, it is vital to identify and understand the methods that help to increase the power of a control chart so that a quicker detection of process shifts can be attained.

It is found that the subgroup size, n is used in the calculation of the control limits and sample standard deviation and for this reason, n is one of the factors that has a significant impact on the power of a control chart. As n increases, the power of a control chart increases. The application of detection rules can help to enhance the sensitivity of a control chart in detecting process mean shifts. Some of the recent works on detection rules, also called runs rules were made by [11], [12], [13], [14] and [15].

[9] and [16] provided an overall Type-I error probability when more detection rules are used. The overall Type-I error probability is

\[ \alpha = 1 - \prod_{i=1}^{r} (1 - \alpha_i), \]

where
\[
\alpha \text{ = Type-I error probability,} \\
r \text{ = number of detection rules used,} \\
\alpha_i \text{ = Type-I error probability of the } i^{th} \text{ rule, for } i = 1, 2, \ldots, r.
\]

Note that equation (5) assumes that the r detection rules are independent of one another.

[7] had estimated the probability of the occurrence of a Type-I error in the first k subgroups by using the following four detection rules given in table 1. The four detection rules considered by [7] are as follows:

Rule 1: Process is out-of-control when one or more points fall above the +3σ control limit.

Rule 2: Process is out-of-control when 2 of 3 consecutive points fall in the same zone and these points are between the +2σ and +3σ control limits.

Rule 3: Process is out-of-control when 4 of 5 consecutive points fall in the same zone and these points are beyond the +1σ control limits.

Rule 4: Process is out-of-control when 8 consecutive points fall in the same zone, either above or below the CL.

Using more detection rules will increase the sensitivity of a control chart but it will also inflate the overall Type-I error probability.

Table 1. The probabilities of a Type-I error for several combinations of detection rules 1 to 4 [7].

| Detection rules | Number of subgroups |
|-----------------|---------------------|
|                 | k = 1   | k = 2   | k = 3   | k = 4   | k = 5   | k = 6   | k = 7   | k = 8   | k = 9   | k = 10  |
| 1               | 0.003   | 0.005   | 0.008   | 0.011   | 0.013   | 0.016   | 0.019   | 0.021   | 0.024   | 0.027   |
| 1 and 2         | 0.003   | 0.006   | 0.011   | 0.015   | 0.020   | 0.024   | 0.028   | 0.034   | 0.039   | 0.043   |
| 1, 2 and 3      | 0.003   | 0.006   | 0.011   | 0.016   | 0.025   | 0.032   | 0.040   | 0.050   | 0.060   | 0.060   |
| 1, 2, 3 and 4   | 0.003   | 0.006   | 0.011   | 0.016   | 0.025   | 0.032   | 0.040   | 0.060   | 0.070   | 0.080   |

4. Probability table for the power functions of a Shewhart X̄ chart

In this section, the method by means of formulae to compute the probabilities of several combinations of detection rules applied on the Shewhart X̄ chart will be discussed. The probability table given by [7] will also be discussed.

4.1. Formulae and probability computation for the power functions

The symbols that will be used in the following discussion are defined as follows:
\( k \) = number of subgroups.

\( PDS(k) \) = probability of detecting an out-of-control signal within \( k \) subgroups.

\( p_k \) = probability of detecting an out-of-control signal at the \( k^{th} \) subgroup.

The four detection rules employed by [7] as stated in Section 3 are still considered in the following discussion:

4.1.1. The use of detection rule 1. Assume that \( "a" \) is the probability that an \( \bar{X} \) sample falls above the \(+3\sigma\) limit.

A general formula for detecting an out-of-control signal within \( k \) subgroups is given by [7]

\[
PDS(k) = \sum_{i=1}^{k} p_i = a + a(1-a) + a(1-a)^2 + a(1-a)^3 + \ldots + a(1-a)^{k-1}
\]

\[
= \sum_{i=1}^{k} a(1-a)^{i-1} = 1 - (1-a)^k .
\]

Note that the case discussed here is only valid for the one sided control chart (upper sided).

4.1.2. The use of detection rules 1 and 2. Assume that
a = the probability that an \( \bar{X} \) sample falls above the \(+3\sigma\) limit,
b = the probability that an \( \bar{X} \) sample falls between the \(+2\sigma\) and the \(+3\sigma\) limits,
c = the probability that an \( \bar{X} \) sample falls between the CL and the \(+2\sigma\) limit.
When detection rule 2 is applied, at least 2 subgroups are required in the analysis. The probability obtained will be similar to that in equation (6), except for cases where \( k = 2 \) or more.

For case \( k = 2 \), \( p_2 \) and \( PDS(2) \) can be computed using [7]
\[
p_2 = ca + ba + b^2
\]
(7)
and
\[
PDS(2) = p_1 + p_2 = a + ca + ba + b^2 = a + b^2 + (b+c)a
\]
(8)

For case \( k = 3 \), \( p_3 \) and \( PDS(3) \) are given by [7]
\[
p_3 = 2abc + 2b^2c + ac^2
\]
(9)
and
\[
PDS(3) = p_1 + p_2 + p_3 = a + b^2 + (b+c)a + 2abc + 2b^2c + ac^2 = a + b^2 + ac + ab(1 + 2c) + 2b^2c + ac^2
\]
(10)

From a similar manner, \( p_k \), for \( k = 4, 5, \ldots, 10 \) can be obtained using the following equations [7]:
\[
p_4 = 3abc^2 + 2b^2c^2 + ac^3
\]
(11)
\[
p_5 = ab^3c^2 + b^3c^2 + 4abc^3 + 2b^2c^3 + ac^4
\]
(12)
\[
p_6 = 3ab^2c^3 + 3b^3c^3 + 5abc^4 + 2b^2c^4 + ac^5
\]
(13)
\[
p_7 = 6ab^2c^4 + 5b^3c^4 + 6abc^5 + 2b^2c^5 + ac^6
\]
(14)
\[
p_8 = ab^3c^4 + b^4c^4 + 10abc^5 + 5b^3c^5 + 7abc^6 + 4b^2c^6 + ac^7
\]
(15)
\[
p_9 = 4ab^3c^5 + 4b^4c^5 + 15abc^6 + 9b^3c^6 + 8abc^7 + 2b^2c^7 + ac^8
\]
(16)
\[
p_{10} = 11ab^3c^6 + 10b^4c^6 + 21ab^3c^7 + 10b^2c^7 + 8abc^8 + 2b^2c^8 + ac^9
\]
(17)

4.1.3. The use of detection rules 1, 2 and 3. Assume that
\( a = \) the probability that an \( \bar{X} \) sample above the +3\( \sigma \) limit,
\( b = \) the probability that an \( \bar{X} \) sample falls between the +2\( \sigma \) and the +3\( \sigma \) limits,
\( c = \) the probability that an \( \bar{X} \) sample falls between the +\( \sigma \) and the +2\( \sigma \) limits,
\( d = \) the probability that an \( \bar{X} \) sample falls between the CL and the +\( \sigma \) limit.

![Diagram](image.png)

Figure 3. Probabilities of an \( \bar{X} \) sample on the Shewhart chart, based on detection rules 1, 2 and 3.

When detection rule 3 is used, at least 4 subgroups are required in the analysis. Thus, the probabilities obtained are similar to that in the equations given in Section 4.1.2, except for cases where \( k = 4 \) or more. When \( k \geq 4 \), \( p_k \) can be computed using the equations as follows [7]:
\[ p_4 = 3abc^2 + 6abcd + 3abbd + 3ac^2d + 3acd^2 + ac^3 + ad^3 + 3b^2cd + 2b^2d^2 + c^4 + 4bc^3. \]  
\[ p_5 = 2ab^2cd + 2b^2cd + 12abc^2d + 14b^2c^2d + 16bc^2d + 4ac^2d + 4ac^2d + ab^2d^2 + b^2d^2 + 12abcd^2 + 6b^2cd^2 + 6acd^2 + 4bc^3 + 2b^2d^3 + 4acd^2 + ad^4 \]  
\[ p_6 = 9ab^2cd^2 + 9b^2cd^2 + 30abc^2d^2 + 20b^2c^2d^2 + 10ac^2d^2 + 16bc^2d^2 + 4ac^2d^2 + 3ab^2d^3 + 3b^2d^3 + 20abcd^3 + 8b^2cd^3 + 10ac^2d^3 + 5abd^4 + 2b^2d^4 + 5acd^4 + ad^5 \]  
\[ p_7 = 19ab^2c^2d^3 + 18bc^2d^3 + 24abc^2d^3 + 25bc^2d^3 + 15bc^2d^3 + 6a^2c^2d^3 + 3c^2d^3 + 24ab^2cd^3 + 20b^2cd^3 + 60abc^2d^3 + 28b^2c^2d^3 + 20ac^2d^3 + 16bc^2d^3 + 4ac^2d^3 + 6ab^2d^4 + 5b^2d^4 + 30abcd^4 + 10b^2cd^4 + 15ac^2d^4 + 6abd^5 + 2b^2d^5 + 6acd^5 + ad^6. \]  
The probabilities \( p_4, p_5, \text{and } p_6 \) are computed using Monte Carlo simulation.

4.1.4. The use of detection rules 1, 2, 3 and 4. When detection rule 4 is employed, at least 8 subgroups are needed in the analysis. Thus, the probabilities obtained are similar to that in the equations given in Section 4.1.3, except for cases where \( k = 8 \) or more. Due to the complexity of the equations, [7] confirmed that the computation needs more than 12,000 combinations to detect a shift and thus, instead of using formulae, the Monte Carlo simulation method will be used to calculate the probabilities for \( k = 8, 9 \text{ and } 10 \).

4.2. Probability tables of the power functions

A probability table to show the probability of detecting a sustained shift in the process mean, calculated using the formulae has been recommended by [7]. The characteristics of the probability table include:

- Size of the shift: 11 sizes of shifts are considered in the probability table. They are \( 0.42\sigma, 0.67\sigma, 0.95\sigma, 1.25\sigma, 1.52\sigma, 1.72\sigma, 1.96\sigma, 2.16\sigma, 2.48\sigma, 2.75\sigma \) and \( 3.00\sigma \).
- Number of subgroups, \( k \): The number of subgroups displayed in the probability table ranges from \( k = 1 \) to \( k = 10 \).
- Subgroup size, \( n \): The subgroup sizes shown by [7] in the probability table are \( n = 1 \) to \( n = 10 \), \( n = 12 \), \( n = 15 \) and \( n = 20 \).
- Detection rules: Several combinations of the detection rules have been used to obtain the necessary probabilities.

5. Performance comparison

From the framework of the probability table and the formulae (see also Sections 4.1.1, 4.1.2, 4.1.3 and 4.1.4) given in [7], computer programs are written in the Statistical Analysis System (SAS) software to compute the probabilities of detecting a sustained shift in the mean within certain number of subgroups following the shift. Tables 2 to 9 show the results obtained.

From tables 2 to 9, we notice that

- as the shift size increases, the probability of detecting a process mean shift increases, regardless of the values of \( n \) and \( k \), and the combination of the detection rules being used.
- the probability of detecting a process mean shift within \( k \) subgroups increases as \( k \) increases, for all \( n \), shift sizes and combinations of the detection rules.
- the control chart will become more sensitive if we use more detection rules, for any \( k \), \( n \) and sizes of shifts.
Furthermore, if we compare table 2 with table 6, we see that the probabilities in table 6 are generally greater than the corresponding probabilities in table 2. Similar trends are observed if we compare the probabilities in table 3 with table 7, table 4 with table 8 and table 5 with table 9. This means that the power of a control chart increases with an increase in the subgroup size, irrespective of the value of $k$, size of the shift and detection rules being used.

**Table 2.** Probabilities of detecting a sustained shift in the process mean within $k$ subgroups, following a shift, for $n = 2$, using detection rule 1 only.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0.010   | 0.017   | 0.024   | 0.033   | 0.041   | 0.048   | 0.056   | 0.064   | 0.070   | 0.078   |
| 0.67σ      | 0.022   | 0.041   | 0.058   | 0.078   | 0.096   | 0.116   | 0.132   | 0.150   | 0.165   | 0.186   |
| 1.52σ      | 0.197   | 0.351   | 0.477   | 0.580   | 0.665   | 0.732   | 0.789   | 0.827   | 0.863   | 0.893   |
| 1.72σ      | 0.289   | 0.481   | 0.629   | 0.735   | 0.812   | 0.867   | 0.902   | 0.928   | 0.948   | 0.963   |
| 3.00σ      | 0.893   | 0.988   | 0.999   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   |

**Table 3.** Probabilities of detecting a sustained shift in the process mean within $k$ subgroups, following a shift, for $n = 2$, using detection rules 1 and 2.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0.010   | 0.021   | 0.037   | 0.054   | 0.071   | 0.087   | 0.104   | 0.118   | 0.133   | 0.147   |
| 0.67σ      | 0.022   | 0.054   | 0.097   | 0.142   | 0.184   | 0.219   | 0.251   | 0.289   | 0.320   | 0.354   |
| 1.52σ      | 0.197   | 0.481   | 0.696   | 0.811   | 0.877   | 0.923   | 0.956   | 0.973   | 0.982   | 0.988   |
| 1.72σ      | 0.289   | 0.626   | 0.831   | 0.910   | 0.954   | 0.978   | 0.989   | 0.994   | 0.998   | 0.999   |
| 3.00σ      | 0.893   | 0.998   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   |

**Table 4.** Probabilities of detecting a sustained shift in the process mean within $k$ subgroups, following a shift, for $n = 2$, using detection rules 1, 2 and 3.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0.010   | 0.021   | 0.037   | 0.064   | 0.112   | 0.143   | 0.171   | 0.196   | 0.227   | 0.260   |
| 0.67σ      | 0.022   | 0.054   | 0.097   | 0.172   | 0.285   | 0.348   | 0.403   | 0.462   | 0.507   | 0.558   |
| 1.52σ      | 0.197   | 0.481   | 0.696   | 0.882   | 0.960   | 0.980   | 0.991   | 0.997   | 0.998   | 0.999   |
| 1.72σ      | 0.289   | 0.626   | 0.831   | 0.951   | 0.991   | 0.997   | 0.999   | 1.000   | 1.000   | 1.000   |
| 3.00σ      | 0.893   | 0.998   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   |

**Table 5.** Probabilities of detecting a sustained shift in the process mean within $k$ subgroups, following a shift, for $n = 2$, using detection rules 1, 2, 3 and 4.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0.010   | 0.021   | 0.037   | 0.064   | 0.112   | 0.143   | 0.171   | 0.233   | 0.275   | 0.320   |
| 0.67σ      | 0.022   | 0.054   | 0.097   | 0.172   | 0.285   | 0.348   | 0.403   | 0.541   | 0.593   | 0.643   |
| 1.52σ      | 0.197   | 0.481   | 0.696   | 0.882   | 0.960   | 0.980   | 0.991   | 0.999   | 1.000   | 1.000   |
| 1.72σ      | 0.289   | 0.626   | 0.831   | 0.951   | 0.991   | 0.997   | 0.999   | 1.000   | 1.000   | 1.000   |
| 3.00σ      | 0.893   | 0.998   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   | 1.000   |
Table 6. Probabilities of detecting a sustained shift in the process mean within \( k \) subgroups, following a shift, for \( n=10 \), using detection rule 1 only.

| Shift size | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) | \( k = 6 \) | \( k = 7 \) | \( k = 8 \) | \( k = 9 \) | \( k = 10 \) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| \( 0.42\sigma \) | 0.047 | 0.090 | 0.132 | 0.173 | 0.210 | 0.253 | 0.287 | 0.318 | 0.350 | 0.383 |
| \( 0.67\sigma \) | 0.189 | 0.340 | 0.466 | 0.568 | 0.652 | 0.715 | 0.769 | 0.810 | 0.843 | 0.872 |
| \( 1.52\sigma \) | 0.967 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 1.72\sigma \) | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 3.00\sigma \) | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 7. Probabilities of detecting a sustained shift in the process mean within \( k \) subgroups, following a shift, for \( n=10 \), using detection rules 1 and 2.

| Shift size | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) | \( k = 6 \) | \( k = 7 \) | \( k = 8 \) | \( k = 9 \) | \( k = 10 \) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| \( 0.42\sigma \) | 0.047 | 0.132 | 0.234 | 0.317 | 0.388 | 0.452 | 0.519 | 0.572 | 0.619 | 0.656 |
| \( 0.67\sigma \) | 0.189 | 0.466 | 0.682 | 0.797 | 0.876 | 0.921 | 0.950 | 0.966 | 0.979 | 0.988 |
| \( 1.52\sigma \) | 0.967 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 1.72\sigma \) | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 3.00\sigma \) | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 8. Probabilities of detecting a sustained shift in the process mean within \( k \) subgroups, following a shift, for \( n=10 \), using detection rules 1, 2 and 3.

| Shift size | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) | \( k = 6 \) | \( k = 7 \) | \( k = 8 \) | \( k = 9 \) | \( k = 10 \) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| \( 0.42\sigma \) | 0.047 | 0.132 | 0.234 | 0.389 | 0.566 | 0.645 | 0.715 | 0.771 | 0.817 | 0.855 |
| \( 0.67\sigma \) | 0.189 | 0.466 | 0.682 | 0.868 | 0.957 | 0.979 | 0.990 | 0.995 | 0.998 | 0.999 |
| \( 1.52\sigma \) | 0.967 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 1.72\sigma \) | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 3.00\sigma \) | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 9. Probabilities of detecting a sustained shift in the process average within \( k \) subgroups, following a shift, for \( n=10 \), using detection rules 1, 2, 3 and 4.

| Shift size | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) | \( k = 6 \) | \( k = 7 \) | \( k = 8 \) | \( k = 9 \) | \( k = 10 \) |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| \( 0.42\sigma \) | 0.047 | 0.132 | 0.234 | 0.389 | 0.566 | 0.645 | 0.715 | 0.846 | 0.883 | 0.914 |
| \( 0.67\sigma \) | 0.189 | 0.466 | 0.682 | 0.868 | 0.957 | 0.979 | 0.990 | 0.999 | 0.999 | 0.999 |
| \( 1.52\sigma \) | 0.967 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 1.72\sigma \) | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( 3.00\sigma \) | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

To compare between Wheeler's formulae method [7] and the Monte Carlo simulation method, the percentage of difference is computed. Comparisons in terms of percentages are made for each probability table using the following formula:

\[
\text{Percentage of difference} = \left| \frac{s - t}{t} \right| \times 100\% , \tag{22}
\]

where
- \( s \) = probability obtained from Monte Carlo simulation,
- \( t \) = probability calculated using formula.

The results obtained are shown in tables 10 to 17. From tables 10 to 17, it is obvious that the probabilities obtained by Monte Carlo simulation are closed to that obtained by means of formulae.
The boldfaced entries in tables 10 to 17 denote cases, where differences in the probabilities between the two methods exist. Note that the differences in the probabilities between the two methods are small.

**Table 10.** Percentages of difference in probabilities, for subgroup size, $n = 2$ using detection rule 1 only.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 20%     | 5%      | 2%      | 3%      | 1%      | 3%      | 3%      | 1%      | 0%      | 0%      |
| 0.67σ      | 9%      | 1%      | 2%      | 1%      | 0%      | 1%      | 0%      | 0%      | 1%      | 2%      |
| 1.52σ      | 1%      | 1%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 1%      | 0%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 3.00σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

**Table 11.** Percentages of difference in probabilities, for subgroup size, $n = 2$ using detection rules 1 and 2.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 20%     | 2%      | 5%      | 1%      | 1%      | 0%      | 2%      | 4%      | 4%      | 3%      |
| 0.67σ      | 9%      | 3%      | 5%      | 1%      | 1%      | 0%      | 1%      | 3%      | 3%      | 2%      |
| 1.52σ      | 1%      | 1%      | 1%      | 0%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 1%      | 1%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 3.00σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

**Table 12.** Percentages of difference in probabilities, for subgroup size, $n = 2$ using detection rules 1, 2 and 3.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 20%     | 2%      | 5%      | 2%      | 3%      | 1%      | 1%      | 2%      | 1%      | 0%      |
| 0.67σ      | 9%      | 3%      | 5%      | 2%      | 0%      | 1%      | 2%      | 2%      | 3%      | 2%      |
| 1.52σ      | 1%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 1%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 3.00σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

**Table 13.** Percentages of difference in probabilities, for subgroup size, $n = 2$ using detection rules 1, 2, 3 and 4.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 20%     | 2%      | 5%      | 2%      | 3%      | 1%      | 1%      | 2%      | 5%      | 0%      |
| 0.67σ      | 9%      | 3%      | 5%      | 2%      | 0%      | 1%      | 2%      | 2%      | 1%      | 3%      |
| 1.52σ      | 1%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 1%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.96σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 3.00σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
Table 14. Percentages of difference in probabilities, for subgroup size, $n = 10$ using detection rule 1 only.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0%      | 3%      | 2%      | 2%      | 2%      | 0%      | 0%      | 1%      | 1%      | 0%      |
| 0.67σ      | 0%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 1%      | 1%      |
| 1.52σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.96σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

Table 15. Percentages of difference in probabilities, for subgroup size, $n = 10$ using detection rules 1 and 2.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0%      | 1%      | 1%      | 1%      | 1%      | 1%      | 1%      | 1%      | 1%      | 1%      |
| 0.67σ      | 0%      | 1%      | 1%      | 0%      | 1%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.52σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.96σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

Table 16. Percentages of difference in probabilities, for subgroup size, $n = 10$ using detection rules 1, 2 and 3.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0%      | 1%      | 1%      | 0%      | 0%      | 1%      | 1%      | 1%      | 0%      | 0%      |
| 0.67σ      | 0%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 1%      | 0%      | 0%      |
| 1.52σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.96σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

Table 17. Percentages of difference in probabilities, for subgroup size, $n = 10$ using detection rules 1, 2, 3 and 4.

| Shift size | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ | $k = 6$ | $k = 7$ | $k = 8$ | $k = 9$ | $k = 10$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.42σ      | 0%      | 1%      | 1%      | 0%      | 0%      | 1%      | 1%      | 1%      | 0%      | 0%      |
| 0.67σ      | 0%      | 1%      | 1%      | 0%      | 0%      | 0%      | 0%      | 1%      | 0%      | 0%      |
| 1.52σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.72σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |
| 1.96σ      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      | 0%      |

6. Conclusions

The Shewhart chart is the simplest and most popular process monitoring tool in SPC. In view of this, tables 2 – 17 are provided to help practitioners in a quick and effective design and implementation of the Shewhart chart, for a more efficient process monitoring system. The approach by means of formulae is extremely complicated, where it becomes impossible when detection rule 4 is employed (see Section 4.1.4). The results of this study given in tables 2 – 17 show that the Monte Carlo simulation method is an easier and quicker way to calculate the probabilities of detecting a sustained shift in the process mean within $k$ subgroups, for the various detection rules, as compared to the calculation of these probabilities using the formulae proposed by [7]. On the contrary, the Monte Carlo simulation method is not only simple, but it is also accurate and less time consuming. Furthermore, a
higher flexibility in obtaining the probabilities for different combinations of the detection rules can be achieved by means of the Monte Carlo simulation method.

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