Impact of turbulence on the stratified flow around small particles

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We study the turbulent flow of the density-stratified fluid around a small translating (either passively or self-propelled) particle. It was found recently [A. M. Ardekani and R. Stocker, Phys. Rev. Lett. \textbf{105}, 084502 (2010)] that without turbulence, the familiar Stokes flow is dramatically altered by the stratification. Stratification-induced inhomogeneity “turns on” the buoyancy introducing a new “cutoff” or screening length scale for the flow, yielding closed streamlines and a faster (exponential-like) decay of velocity. This result, however, did not account for the potential role of the background turbulence, intrinsically present in many aquatic environments. Turbulence mixes the density opposing the effect. Here we derive and solve the advection-diffusion equation that describes the interplay of turbulent mixing, diffusion of the stratifying agent and buoyancy. We derive an exact expression for fluctuations due to weak background turbulence and show that stronger turbulence can completely change the flow around the particle, canceling the effect of stratification and restoring the unstratified Stokes flow.

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Translation of small particles through the viscous fluid has been studied intensively for centuries already \cite{1}. This is both due to the huge range of practical applications that includes swimming of small organisms, sedimentation and pollution, and due to the theoretical challenge posed by the problem. While the case of one particle (considered spherical for simplicity) translating either passively or self-propelled (e.g. “spherical squirmers”\cite{2}) is described by a closed form solution of the Stokes equations of viscous flow, the case of two hydrodynamically interacting particles does not possess a closed-form analytical solution \cite{3}. The flow with a larger number of particles, such as sedimenting or sheared suspension, is a many-body problem with long-range interactions and is one of the most challenging problems at the frontier of the modern research.

Unexpectedly, it was found recently that even the problem of one particle that translates in the quiescent fluid can produce a non-trivial flow pattern when fluid’s density stratification (present invariably in many aquatic environments) is taken into account \cite{3}. The flow induced by the particle in the stratified fluid is nothing similar to the familiar unstratified Stokes flow: instead of the streamlines that are open to infinity, one gets closed streamlines and toroidal eddies.

The obtained flow showed that in spite of wide separation of scales of stratification (kilometers) and swimmers (0.1–1 mm) in typical marine environment, the stratification cannot be neglected. The reason is that the spatial inhomogeneity of the stratifying agent (temperature or salinity) opens a new pathway of interaction - the buoyancy. This force has no impact on the flow if the distribution of the stratifying agent is spatially uniform, but it affects the flow when inhomogeneity is present. A combination of buoyancy, diffusivity, viscosity and the stratification agent’s gradient (measuring non-uniformity) creates a new length-scale \(L\) of the order of \(\sim 1\) mm. Beyond this “screening” \(L\) the perturbation flow induced by the moving particle decays much faster than the unstratified Stokes flow, that holds at scales much smaller than \(L\). We show here that the decay is exponential-like which is in sharp contrast to the slow algebraic velocity decay (i.e. inverse with the distance for passively translating particle) in the unstratified fluid.

The scale \(L\) is the typical size of the toroidal eddies mentioned above. It was suggested that this previously unnoticed feature of the flow around small particles in the ocean may affect propulsion of small organisms and sinking of marine snow particles, diminish the effectiveness of mechanosensing in the ocean \cite{4}, stifle nutrient uptake of small motile organisms \cite{4} or potentially hinder the drift-induced biogenic mixing \cite{5}.

The above effect should be very sensitive to the presence of a background turbulent flow that would mix the stratifying agent, decreasing the role of buoyancy and restoring the unstratified Stokes flow. Since turbulence is present in natural environments invariably, it is obligatory for applications to examine its impact on the flow.

It is to be noted immediately that the value of the smallest scale of turbulence (the Kolmogorov’s scale \(\ell_\eta\), see \cite{6}) is irrelevant for mixing as long as that scale is much larger than \(L\), which we assume below. Turbulence at scales smaller than \(\ell_\eta\) is a large-scale chaotic flow that mixes at the scale-independent rate \(\lambda\) (in contrast to the mixing in the inertial range) \cite{7,8}. That rate is given by the characteristic value \(\lambda\) of the gradient of turbulent velocity field. Note that \(\lambda^2\) is the energy dissipation per unit mass \(\epsilon\) divided by the kinematic viscosity \(\nu\).

It is anticipated that the importance of turbulence can be evaluated by the “time criterion”. If the mixing timescale \(\lambda^{-1}\) is smaller than the characteristic time-scale of the setting of the stratified flow \(L^2/\kappa\), where \(\kappa\) is the
diffusivity of the stratifying agent, then turbulence homogenizes the agent’s distributions canceling the effect of buoyancy, so the usual unstratified Stokes flow holds. In contrast, if $L^2/\kappa \ll \lambda^{-1}$ then turbulent mixing is slow and one can neglect turbulence in describing the flow. Thus, the importance of turbulence can be measured by the dimensionless parameter $\beta \equiv L^2 \lambda/\kappa$.

This paper describes consistent, quantitative analysis of the flow around small either passively moving (e.g. sinking) or self-propelled particles (e.g. small motile organisms) in the presence of both the density stratification and turbulence. We derive the governing advection-diffusion equation by a consistent reduction of the full system of hydrodynamic equations. The main modification compared to the theory describing the passive scalar field mixed by turbulence \cite{12} is the non-trivial wave-number dependence of the diffusion coefficient. Solving the equation we describe the flow field and distribution of the stratifying agent (i.e. temperature or salt concentration) fields around the moving particle. The solution depends on one dimensionless parameter $\beta$. The effect of turbulence is negligible at $\beta \ll 1$, but at $\beta \gg 1$ the streamlines corresponding to the Stokes’ flow without stratification are recovered, confirming the above time criterion.

It should be emphasized that the results are obtained without modeling the statistics of turbulence and they can be more directly applied to natural environments. Our estimates show that, for instance, in typical marine environments $\beta \gtrsim 1$, indicating that the effect of turbulence is of order one, so it can not be entirely neglected in the analysis of the flow.

We use the Boussinesq approximation \cite{12} to describe the interaction of the flow $\mathbf{v}$ with the stratified agent $\theta$, \begin{align}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla (p/\rho) + \Theta \mathbf{g} + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0,
\partial_t \theta + \mathbf{v} \cdot \nabla \theta &= \kappa \nabla^2 \theta; \quad \mathbf{v}(|\mathbf{x} - \mathbf{Y} | = a, t) = \mathbf{V}
\end{align}
where $p/\rho$ is the pressure divided by the density, $\mathbf{g} = -g\hat{z}$ is the gravitational acceleration, $\mathbf{Y}$ and $a$ are the particle’s coordinate, velocity and radius respectively \cite{11}. We decompose the flow into the background turbulent flow $\mathbf{u}$, $\mathbf{P}$, $\theta_0$ and the perturbation induced by the boundary condition describing the particle, $\mathbf{v}(x) = \mathbf{u}(x) + \mathbf{w}(x) = \mathbf{P}(x) + \Theta (x - \mathbf{Y}(t))$ and $\partial_t \theta = \theta_0(x) + \Theta |x - \mathbf{Y}(t)|$, where $\mathbf{w}$, $\mathbf{P}$, $\Theta$ decay at large $r \equiv |\mathbf{x} - \mathbf{Y}(t)|$. The perturbations obey, cf. \cite{11},
\begin{align}
\partial_t \mathbf{w}' + \mathbf{w}' \cdot \nabla \mathbf{w}' + \mathbf{u}(\mathbf{r} + \mathbf{Y}[t]) - \mathbf{V}(t) \cdot \nabla \mathbf{w}' + \sigma \mathbf{w}' &= -\nabla \mathbf{P} + \Theta \mathbf{g} + \nu \nabla^2 \mathbf{w}', \quad \partial_t \Theta + \mathbf{w}' \cdot \nabla \Theta + \mathbf{w}' \cdot \nabla \theta_0
+ \mathbf{u}(\mathbf{r} + \mathbf{Y}[t]) - \mathbf{V}(t) \cdot \nabla \Theta = \kappa \nabla^2 \Theta, \quad \nabla \cdot \mathbf{w}' = 0, \quad \mathbf{w}'(|\mathbf{r}| = a) = -(\mathbf{u} - \mathbf{V}).
\end{align}
Since we are interested in the impact of turbulence on the stratified flow with the characteristic scale $L$ then the treatment below is done at scales $r \lesssim L$. We make several assumptions. The experimentally relevant combination corresponds to $L \ll \ell_0$, so the latter inequality is assumed below (note that $a \ll L$ is assumed too).

Thus, the importance of turbulence can be measured by the relative motion of the particle with respect to the flow. The effective description of the boundary condition describing the particle, \begin{equation}
\nabla \mathbf{P} + \Theta \mathbf{g} + \nu \nabla^2 \mathbf{w}' + f \delta (\mathbf{r} \cdot \hat{z}), \quad \nabla \cdot \mathbf{w}' = 0, \quad \partial_t \Theta + (\sigma \mathbf{r} \cdot \nabla) \Theta - \gamma \omega_\alpha^2 = \kappa \nabla^2 \Theta,
\end{equation}
comparable with the diffusive term at a characteristic time scale of the stratified flow around small particles. This term is crucial for the description of the impact of turbulence on the flow, see below. We concentrate here on the role of the vertical component of \( \mathbf{u} - \mathbf{V} \), see \( \mathcal{L} \). To demonstrate the relevance of turbulence in the simplest context, this component is considered time-independent in the derivation of the solution below. Though the average value of \( \mathbf{u} - \mathbf{V} \) is time-independent for stationary turbulence that occurs in most applications, the fluctuations do depend on time. The impact of this time-dependence is considered in the Conclusion.

We list the assumptions that underlie the system \( \mathcal{L} \).

These are: the flow happens well below the Kolmogorov scale, \( L \ll \ell_\eta \), the Peclet number \( UL/\kappa \) is small, the Prandtl number is large, \( Pr^{1/2} \gg 1 \), and turbulence does not break the approximate constancy of gradients of the stratified agent over the scale \( L \). These assumptions are not too restrictive and should describe situations in nature, see the discussion below. Note that there is no assumption on the Reynolds number of the background turbulence itself, that can be arbitrarily large.

Turbulent transport is described by \( \partial_t \Theta + (\mathbf{r} \cdot \nabla) \Theta \), terms that occurs universally in the description of the advection of the passive scalar fields by turbulence at large Prandtl numbers, see \( \mathcal{L} \) and references therein. The appearance of these terms in the equations is inevitable, so that the system \( \mathcal{L} \) can be said to be the minimal model for the description of the impact of turbulence on the stratified flow around small particles. This term is comparable with the diffusive term at a characteristic scale \( \ell_d = \sqrt{\kappa/\lambda} \) and it is dominating at larger scales.

This is the term which effect we study in present work.

The fundamental Stokeslet solution described by Eq. \( \mathcal{L} \) corresponds to a flow induced by a small particle moving under the action of an external force (e.g. passive sinking under the action of gravity). The flow field around self-propelled objects could be, however, quite different from that of a passively towed particle. A steadily self-propelled swimmer generates no net momentum flux, since the thrust is counter-balanced by the drag force. The two forces of equal magnitude, acting in opposite directions and separated by some distance, constitute a force dipole of strength \( \hat{s} \), so that the flow induced by a self-propelled swimmer can be approximately described by

\[
\nabla \hat{P}_s = \Theta_s g + \nu \nabla^2 \mathbf{w}' + \hat{s}_{ij} \nabla \delta(r),
\]

where \( \hat{s}_{ij} \) is a diagonal matrix \( \hat{s}_{ij} = \text{diag}[\hat{s}/3, \hat{s}/3, 2\hat{s}/3] \), and the subscript “s” will denote the fields of the force-doublet flow. Passing to \( \hat{P}_s = P_s + \hat{s} \delta(r)/3 \) we find that the flow around the swimmer is determined from

\[
\nabla \hat{P}_s = \Theta_s g + \nu \nabla^2 \mathbf{w}' + \hat{s} \delta(r),
\]

\[
\partial_t \Theta_s + (\sigma \mathbf{r} \cdot \nabla) \Theta_s - \gamma w_z' = \kappa \nabla^2 \Theta_s,
\]

\( \mathcal{L} \)
The relation between the Stokeslet and the force-doublet flows is trivial without turbulence when the term \( \partial_t \Theta + (\sigma \mathbf{r} \cdot \nabla) \Theta \) is missing in the equations. The equations corresponding to the flow induced by the force-doublet can be simply obtained by differentiating the equations for Stokeslet with respect to \( z \), so \( \mathbf{w}' = (\hat{s}/f) \partial_z \mathbf{w}, \Theta_s = (\hat{s}/f) \partial_z \Theta \) and \( \hat{P}_s = (\hat{s}/f) \partial_z P \). When turbulence is considered, the coefficients in Eqs. \( \mathcal{L} \) depend on the coordinate, so differentiation over \( z \) does not produce Eqs. \( \mathcal{L} \). Thus in the presence of turbulence, the relation between the Stokeslet and the force-doublet flows is non-trivial. Below we consider both the Stokeslet solution and the force-doublet solution.

We briefly consider Stokeslet in the case \( L \ll \ell_d \) where to leading order the advection term can be dropped at \( r \ll L \). This is the case without turbulence considered in \( \mathcal{L} \). Taking the Laplacian of the first of Eqs. \( \mathcal{L} \) we find

\[
\nabla \nabla^2 \mathbf{w}' = \mathbf{g}/\kappa + \nu \nabla^2 \mathbf{w}' + \nabla^2 \mathbf{f} \delta(r).
\]

\( \mathcal{L} \)
The squared Laplacian term and the buoyancy term are of the same order at the length scale \( L \approx \kappa/g \). At smaller scales viscosity dominates and the unstratified Stokes flow holds. At \( r \sim L \) a non-trivial change in the flow pattern around the particle occurs, see \( \mathcal{L} \) and below.

We return to the full system \( \mathcal{L} \). Taking the Fourier transform and using the incompressibility \( \sigma = 0 \) yields

\[
\mathbf{k} \cdot \mathbf{w}' = 0.
\]

\( \mathcal{L} \)

Introducing \( \hat{k} \equiv k/k \) and the projection \( \Pi_{ij}(k) \) leads to

\[
\nu \mathbf{k}^2 \mathbf{w}' = \Theta \Pi(k) g, \quad \Pi_{ij}(k) = \delta_{ij} - \hat{k}_i \hat{k}_j.
\]

\( \mathcal{L} \)

We note that obtaining Eq. \( \mathcal{L} \) involves division by \( k^2 \) making it necessary to consider the point \( k = 0 \) separately. Inserting \( \mathbf{w}' \) in Eq. \( \mathcal{L} \), we obtain the following closed advection-diffusion equation for \( \Theta \)

\[
\partial_t \Theta - (\sigma \mathbf{r} \cdot \nabla) \Theta = -\alpha(k) \Theta + \phi(k), \quad \alpha(k) \equiv \kappa k^2 d(k)
\]

\( \mathcal{L} \)

\[
d(k) \equiv 1 + \frac{k^2}{L^2 k^6}, \quad \phi(k) \equiv \frac{f \gamma k^4}{\nu k^4}.
\]
where $k_2^2 = k^2 - k_1^2$. The stratification produces wave-number dependent diffusion coefficient $d(k)$ and the source of the fluctuations $\phi$. Although both quantities diverge at $k = 0$, the time evolution of $\Theta(k)$ at $k \neq 0$ decouples from $k = 0$, which is clear from relations below, so one can consider the solution at $k \neq 0$ and then continue it. Note that Eq. (11) has the solution $\Theta(k) = \text{const} \times \delta(k)$ at zero stratification with $\gamma = 0$ that describes uniform distribution of $\Theta$ in the real space.

Following the same steps for the flow induced by the force-doublet one finds the same form of Eqs. (11) with the forcing term $\phi(k)$ replaced with $\phi_s(k) = i\gamma sk_2k_1^2/\nu k^4$. The corresponding expressions for pressure and velocity are (we write directly the pressure $P_s$, rather than $P_s$),

$$P_s = \frac{igk\Theta_s + k_s\delta_{ij}k_j}{k^2}, \quad \nu k^2\mathbf{w}_s' = \Pi(k) [\Theta_s \mathbf{g} + i\kappa \mathbf{k}].$$

Returning to the Stokeslet solution, one has

$$\partial_t \Theta' - (\sigma^t \mathbf{k} \cdot \nabla) \Theta' = -\alpha(k)\Theta' - \kappa k^2 f/g.$$  \(\text{(13)}\)

To find the solution we pass to the moving frame $\hat{\Theta}(k, t) = \Theta'(k(t), t)$ where $k(t) = W^{-1,t}(t)k$ with

$$\hat{W} = \sigma W, \quad W^{-1,t} = -\sigma^t W^{-1,t}, \quad W_{ij}(t=0) = \delta_{ij}. \quad \text{(14)}$$

The matrix $\sigma$, and thus also $W$, have to be considered random for turbulence and described statistically. The properties of the statistics of $W$ that are relevant here do not depend on the details of the statistics of $\sigma$ due to universality [9], yet for clarity we assume that the statistics of $\sigma$ is close to the Lagrangian statistics of $\nabla_j u_i$ (the statistics in the frame of fluid particle). This holds provided $U$ is much less than the characteristic velocity $\nu_\gamma \sim \lambda_\rho v$ of the viscous scale eddies of turbulence [9]. It seems that this assumption is not restrictive and it is obeyed in typical natural situations. Since $\sigma$ is statistically the same as the velocity gradient of $\mathbf{u}$ in the fluid particle’s frame, then $W$ is statistically the same as the Jacobian matrix of the turbulent flow backward in time [9]. That is, if we consider the Lagrangian trajectories $q(t, r)$ defined by $\partial_t q(t, r) = u[t, q(t, r)]$ and $q(t = 0, r) = r$, then $W_{ij}(t, r) = \partial_j q_i(t, r)$ at $t < 0$ describes the evolution of small volumes in the turbulent flow backward in time and obeys Eq. (14). In particular, since the Lyapunov exponents of the backward in time flow are $(-\lambda_3, -\lambda_2, -\lambda_1)$ where $(\lambda_1, \lambda_2, \lambda_3)$ are the Lyapunov exponents of the forward in time flow, then $k(t)$, which is governed by $W^{-1,t}$ rather than $W(t)$, obeys

$$\lim_{t \to -\infty} (1/|t|) \ln[k(t)/k(0)] = \lambda_1, \quad \text{(15)}$$

see details in [9]. Thus the growth of $k(t)$ with $|t|$ is similar to the exponential growth of the separation between two infinitesimally close fluid particles in turbulence (governed by the principal Lyapunov exponent $\lambda_1$). The limit in Eq. (15) holds for almost every realization of $\sigma(t)$ and does not involve the randomness of turbulence that disappears after taking the infinite time limit. To describe the fluctuations of $k(t)$ when $t$ is finite, one introduces the polar representation $k(t) = k \exp[\rho(t)] \hat{n}(t)$, where $|\hat{n}| = 1$. Using $\hat{k} = -\alpha' \hat{k}$ one finds [9]

$$\hat{n} = -\sigma' \hat{n} + \hat{n}_\zeta, \quad \partial_t \hat{n} = \sigma' \hat{n}, \quad \zeta = \gamma \hat{n} \sigma \hat{n}.$$ \(\text{(16)}\)

It follows that $\ln[k(t)/k] = \int_0^t \zeta(t')dt'$ where $\zeta$ is a finite-correlated noise which correlation time $\tau_\sigma$ is of order of the correlation time of $\sigma$, so that $\tau_\sigma \sim \lambda^{-1}$. Thus Eq. (15) resembles the law of large numbers. To find the moments of $k(t)$ one introduces

$$\lim_{t \to -\infty} (1/|t|) \ln[k(t)] = \varphi(t).$$ \(\text{(17)}\)

The function $\varphi(t)$ is convex and obeys $\varphi(0) = \varphi(-3) = 0$, so it is negative at $-3 < n < 0$ and positive otherwise. This holds independently of the statistics of turbulence (see [9] for details). In the moving frame Eq. (13) becomes

$$\partial_t \Theta' = -\alpha[k(t)]\hat{\Theta} - \kappa k^2 f/g.$$ \(\text{(18)}\)

We consider $\Theta$ at $t = 0$, taking the initial condition at $t = -T$ and studying the limit $T \to \infty$, i.e. we focus on the steady state solution. Using $\hat{\Theta}(t = 0) = \Theta'(t = 0)$,

$$\Theta' = -\frac{k^3 f/g}{\int_0^\infty dt \exp \left[ -\int_t^0 \alpha[k(t')] dt' \right] k^2(t').$$ \(\text{(19)}\)

The above together with Eqs. (19)-(20) give implicit solution to the system (10) in the Fourier space. The corresponding solution for the force-dipole swimmer is

$$\Theta_s = \int_{-\infty}^0 dt \exp \left[ -\int_t^0 \alpha[k(t')] dt' \right] \phi_s[k(t)].$$ \(\text{(20)}\)

together with Eqs. (12).

Since for turbulence $k(t)$ is a random vector, then $\Theta, \Theta_s$ and $\mathbf{w}, \mathbf{w}_s$ are random too and should be studied statistically. The computation of the statistics however cannot be done due to the complex dependence on $\sigma$. Thus we consider the limiting cases, which will allow us to understand the behavior of the solution in detail. We introduce integration variable $s_k(t) = \int_0^0 \alpha[k(t')] dt'$ that depends on $k$ via the final condition $k(0) = k$. The inverse transformation is denoted by $t_k(s)$. We have

$$\Theta'(k) = -\frac{f}{g} \int_0^\infty \frac{\exp[-s]}{1 + k_s^2(k(s))} \frac{ds}{L_k \kappa^2(t_k(s))},$$ \(\text{(21)}\)

Using in $s_k(t) = \int_0^0 \alpha[k(t')] dt'$ the definition of $\alpha(k)$ and passing to integration variable $l = \lambda t$, we find

$$s = \frac{1}{\beta} \int_{\lambda t / L(s)} q^2(l) \left[ 1 + \frac{q^2(l)}{q^2(0)} \right] dl, \quad q(l) = W^{-1,t} \left( \frac{l}{\lambda} \right) q,$$
where $\beta = L^2/\ell^2 = \sqrt{\nu \lambda^2/\kappa \eta q}$ and $q(l)$ is dimensionless. We introduce the dimensionless time variable $\tau_q(s) = \lambda q/L(s/\beta)$ that obeys

$$\tau_q(s) = \int_0^s q^2(l) \left[ 1 + \frac{q^2(l)}{q_0^2(l)} \right] dl.$$  

It follows that

$$\Theta' \left( \frac{q}{L} \right) = -\frac{f}{g} \int_0^\infty F(\beta s) e^{-s} ds, \quad F(s) = \frac{q^2(s)}{q_0^2(s) + q^2(s)},$$

where $q(s) = W^{-1,t} \left[ \tau_q(s)/\lambda \right] q$.

This form of the solution is particularly well-suited for the study of the impact of turbulence because $F(s)$ is determined by turbulence only. If intermittency (dependence of the statistics of $\sigma/\lambda$ on the Reynolds number, $Re$) can be neglected, then $q(s)$ and $F(s)$ vary at scales of order one. We consider this case first, since intermittency is negligible up to rather high Reynolds numbers of $10^5 - 10^6$ due to the smallness of the corresponding anomalous exponents $[3]$. The solution in this case depends on one dimensionless parameter, $\beta$ and to study the role of turbulence we study how the solution changes when $\beta$ is increased from 0 (the stratified flow without turbulence) to $\infty$. The function $F(\beta s)$ in Eq. (22) varies at the scale $1/\beta$. When $\beta \ll 1$ the integral in Eq. (22) converges over the scale of order one that is much less than $1/\beta$. Correspondingly the asymptotic series that describes the solution in the limit of small $\beta$ is obtained by Taylor-expansion of $F(\beta s)$ in the integral. One finds

$$\Theta' \left( \frac{q}{L} \right) = -(f/g) \sum_{n=0}^\infty \beta^n F^{(n)}(0), \quad \Theta = \Theta' + f/g. \quad (23)$$

To find the solution to order $\beta$ one can use $-F'(0) = F^2(0) / F(0) = F^2(0) [q_0^2 + q^2]/q_0^2$, writing the result in terms of the solution without turbulence $\Theta_0(q/L) = (f/g) q_0^2 [q_0^2 + q^2]^{-1}$, one finds

$$\Theta(q/L) = \Theta_0(q/L) \left[ 1 + \beta \Theta(q/L) \right], \quad \delta \Theta(q/L) = 2\beta \lambda^{-1} q^2 q_{\perp}^{-2} [q_0^2 + q^2]^{-2} f \cdot \sigma q,$$

where $f \equiv [(q_0^2 - 3q_{\perp}^2) q_x, (q_0^2 - 3q_{\perp}^2) q_y, -3q_{\perp}^2 q_z]$. Thus in this order the relative correction $\delta \Theta$ to the solution without turbulence is a linear function of the current value of the gradient of the turbulent velocity $\nabla_j u_i$ at the location of the particle.

The correction is a random field that depends on the random value of the matrix of velocity gradients $\sigma$ at the location of the particle. One has $\langle \sigma_{ij} \rangle = \langle \nabla_j u_i \rangle = 0$ and $\langle \sigma_{ij} \sigma_{mn} \rangle = \langle \nabla_j u_i \nabla_m u_n \rangle$, where the last average can be taken in the Eulerian frame due to incompressibility (we use here that the statistics of $\sigma$ is assumed to be close to the one of the matrix of velocity gradients in the frame of the fluid particle). One finds

$$30 \nu \langle \sigma_{ij} \sigma_{mn} \rangle = \epsilon [4\delta_{im} \delta_{jn} - \delta_{im} \delta_{jm} - \delta_{ij} \delta_{mn}], \quad (25)$$

where isotropy (typically valid for small-scale turbulent fluctuations that determine $\nabla u$ [2]) and spatial uniformity are assumed. The form of $\langle \nabla_j u_i \nabla_m u_n \rangle$ is fixed uniquely by the demands of isotropy, incompressibility and spatial uniformity that imply $\langle \nabla_m u_i \nabla_n u_m \rangle = 0$. The relation is exact due to stationarity condition $\nu \langle \nabla_j u_i \nabla_j u_i \rangle = \epsilon$. Thus we obtain the exact result for fluctuations of $\Theta$ (clearly $\langle \delta \Theta \rangle = 0$) around $\Theta_0$ that are caused by the turbulent fluctuations of the background velocity field in the limit of small $\beta$,

$$\langle \delta \Theta^2 \rangle^{1/2} = \beta q^2 q_{\perp}^{-2} \left[ q_0^2 + q^2 \right]^{-2} \sqrt{4q^2 f^2 - 2(q \cdot f)^2}/15.$$

This formula continued asymptotically to $\beta \sim 1$ demonstrates that the impact of turbulence is of order one at $\beta \sim 1$.

Thus in the case of weak turbulent background the correction to the flow around the particle is determined uniquely by the local instantaneous value of the gradient of the turbulent velocity field. This robust result involves only quite safe, general, assumptions on turbulence. We pass to study this correction in details.

In the absence of turbulence the flow is axially symmetric, so that one can introduce the stream function $\psi_0(r, z) = \int_0^r r' w_z(r', z) dr'$ that obeys in cylindrical coordinates

$$w'_z = \frac{1}{r} \frac{\partial \psi_0}{\partial r}, \quad w_r = -\frac{1}{r} \frac{\partial \psi_0}{\partial z}.$$ \quad (26)

The scalar stream function gives concise description of the flow. In the presence of turbulence, however, the symmetry holds only statistically, so one cannot obtain the flow from a stream function. However, we can still introduce the scalar stream function that will concisely describe the distortion of the streamlines by turbulence via the angle-averaging,

$$\psi = \int_0^r \int_0^{2\pi} r' w_z(r', z, \phi) \frac{dr' d\phi}{2\pi}.$$ \quad (27)

This is a random function which (random) isolines give counterpart to the streamlines without turbulence. When the isolines of the typical realization of $\psi$ differ strongly from those of $\psi_0$, it is guaranteed that the impact of turbulence on the flow is considerable and cannot be entirely neglected as done in [1]. We have

$$\psi = \int_0^r \int_0^{2\pi} r' \frac{dr' d\phi}{2\pi} \int \frac{dk}{(2\pi)^3} \exp [ikz + ik_{\perp} r' \cos \phi] w'_z(k) = \frac{2}{\nu} \int \frac{k^2 dk}{(2\pi)^3 k^4} \Theta'(k) \int \exp [ikz] \int_0^r r' J_0(k_{\perp} r') dr', \quad (28)$$
where we used \( w' = -gk^2 \Theta'(k)/\nu k^4 \) and \( J_0(x) \) is the Bessel function of zeroth order. Noting that \( \int_0^{k_r} \int_0^{k_r} J_0(k \cdot r') d^2 r' = \int_0^{k_r} J_0(r) d x = k \cdot r J_1(k \cdot r) \), passing to the dimensionless integration variable \( q = L k \) and scaling all lengths (i.e. \( r, z \)) with \( L \),

\[
\psi(Lr) = -\frac{g L r}{\nu} \int_0^{q \frac{d q}{(2\pi)^3} q^q \psi \left( \frac{q}{L} \right) \exp [i q z] J_1(q \cdot r) \]

\[
= \psi_0(Lr) - \frac{g L r}{\nu} \int_0^{q \frac{d q}{(2\pi)^3} q^q \Theta_0 \left( \frac{q}{L} \right) \delta \Theta \left( \frac{q}{L} \right) \exp [i q z] \]

\[
\times J_1(q \cdot r) \equiv \psi_0(Lr) - \frac{2\beta f L r}{\nu} \delta \psi(Lr) . \tag{28}
\]

Using the obtained expression of \( \Theta_0(k) \) we find for the stream function \( \psi_0 \) of the flow without turbulence obeys

\[
\psi_0(Lr) = \frac{f L r}{\nu} \int_0^{\infty} \frac{q^2 d q}{(2\pi)^3} \int_{-\infty}^{\infty} [q^2 + q^6] \exp [i q z] J_1(q \cdot r) d q .
\]

The dimensionless correction due to turbulence, \( \delta \psi \), obeys

\[
\delta \psi(Lr) = \int_0^{\infty} \frac{q^2 d q}{(2\pi)^3} \int_{-\infty}^{\infty} [q^2 + q^6] \exp [i q z] J_1(q \cdot r) \]

\[
\times \int \frac{d \phi}{2 \pi} f \cdot \sigma^q q . \tag{29}
\]

To further simplify the expression for \( \psi_0 \) we introduce polar coordinates in \((q_z, q_r)\) plane by \( q_r = \lambda \sin \theta, q_z = q \cos \theta \), so that

\[
\psi_0(Lr) = \frac{f L r}{\nu} \int_0^{\pi/2} \frac{\sin^2 \theta d \theta}{2\pi} \int_0^{\infty} q^3 \cos \left[ q z \cos \theta \right] \sin^2 \theta + q^4 d q \]

\[
\times J_1(q \cdot r \sin \theta) \tag{30}
\]

When \( q \) is large the integrand is proportional to \( q^{-3/2} \) times an oscillating function of \( q \), so the convergence is slow. We rewrite the integral so the convergence is fast and convenient for the numerical evaluation. The denominator has simple poles in the upper half-plane at \( q_1 = \sqrt{\sin \theta} (1 + i)/\sqrt{2} \) and \( q_2 = -q_1^* \). We write

\[
\frac{1}{\sin^2 \theta + q^4} = \left( q - q_1 \right) \left( q + q_1 \right) \left( q - q_1^* \right) \left( q + q_1^* \right) . \tag{31}
\]

Closing the contour in the upper half-plane is not straightforward because the integrand has growing exponents when continued onto the complex plane. We first write

\[
\psi_0(Lr) = \frac{f L r}{\nu} \int_0^{\pi/2} \frac{\sin^2 \theta d \theta}{2\pi} \int_0^{\infty} q^3 J_1(q \cdot r \sin \theta) \]

\[
\times \left( \exp [i q z \cos \theta] + \exp [-i q z \cos \theta] \right) , \tag{32}
\]

where we used that \( J_1(q \cdot r \sin \theta) \) is odd function of \( q \) to continue the integral over \( q \) to \((-\infty, \infty)\). Using the integral representation of \( J_1(z) \) (Re stands for real part)

\[
\psi_0(Lr) = f L r \Re \left[ J_0(y_+) + J_0(y_-) \right] \tag{33}
\]

where \( y_\pm = r \sin \theta \sin \phi \pm z \cos \theta \) and

\[
I_0(y) = \int_0^{\pi/2} \frac{\sin^2 \theta d \theta}{2(2\pi)^2} \int_0^{\pi/2} \frac{d \phi}{\pi} \exp [-i \phi] \]

\[
\int_{-\infty}^{\infty} q^3 d q \cos \left[ y \sqrt{\sin \theta} \right] . \tag{34}
\]

The integral is purely imaginary due to the parity properties of the integrand, so \( I_0(y) \) is odd function of \( y \). We consider \( y > 0 \) when we can close the contour in the upper half-plane,

\[
I_1 = \int_0^{\pi/2} \frac{\sin^2 \theta d \theta}{2(2\pi)^2} \int_0^{\pi/2} \frac{d \phi}{\pi} \exp [-i \phi] \]

\[
\int_{-\infty}^{\infty} q^3 d q \cos \left[ y \sqrt{\sin \theta} \right] . \tag{35}
\]

In the case \( y < 0 \) we close the contour in the lower half-plane which gives

\[
I_1 = -\frac{q^3}{2q_1(q_1 - q_1^{-1})(q_1 + q_1^{-1})} - c.c. \tag{36}
\]

Therefore we can write

\[
I_1 = \text{sign}(y) \exp \left[ -|y| \sqrt{\sin \theta} \right] \cos \left[ y \sqrt{\sin \theta} \right] . \tag{37}
\]

Thus we finally obtain

\[
\psi_0(Lr) = \frac{f L r}{\nu} \int_0^{\pi/2} \frac{\sin^2 \theta d \theta}{2(2\pi)^2} \int_0^{\pi/2} d \phi \exp [-h_+ \sin \phi \cos(h_+)s_+] \]

\[
+ \frac{f L r}{\nu} \int_0^{\pi/2} \frac{\sin^2 \theta d \theta}{2(2\pi)^2} \int_0^{\pi/2} d \phi \exp [-h_- \sin \phi \cos(h_-)s_-] . \tag{38}
\]

where \( h_\pm = |r \sin \theta \sin \phi \pm z \cos \theta| \sqrt{\sin \theta} \) and \( s_\pm = \text{sign}(r \sin \theta \sin \phi \pm z \cos \theta) \). This form of \( \psi_0 \) suits well the numerical evaluation.

We now pass to consider \( \delta \psi \). Using the formula for \( f \) we obtain (note \( d \phi d q_z d y \propto \int d \phi \cos \phi \sin \phi = 0 \))

\[
\int \frac{d \phi}{2\pi} f \cdot \sigma^q q = \int \frac{d \phi}{2\pi} \left[ \sigma_{1q}^q \left( q^2 - 3q_1^2 \right) + \sigma_{2q}^q \left( q^2 - 3q_1^2 \right) \right] / \lambda = -\tilde{\sigma} q_1^3 \left( q^2 - 3q_1^2 \right) / 2 - 3\tilde{\sigma} q_1^3 q_z^2 \]

\[
= \tilde{\sigma} \left( q_1^4 - 7q_1^2 q_z^2 / 2 \right) , \tag{39}
\]
where we introduced the random dimensionless factor \( \bar{\sigma} \equiv \sigma_{33}/\lambda \) and used that \( \sigma_{11} + \sigma_{22} = -\sigma_{33} \) by incompressibility. Thus, we find

\[
\psi = \psi_0 - \frac{2\beta f Lr \bar{\sigma}}{\nu} \delta\psi', \quad \delta\psi' \equiv \int_0^\infty \int_0^\infty \frac{dq_\perp dq_z}{2(2\pi)^2} \frac{q_\perp^4 q_z^4 [2q_\perp^2 - 7q_z^2]}{[q_\perp^4 + q_z^4]^3} \exp [iq_z z] J_1(q_r r).
\]

We observe that in the angle-averaged stream-function the whole dependence on the local value of the turbulent velocity gradient reduces to one random factor \( \bar{\sigma} \) which is of order unity if intermittency is negligible.

The resulting plot of the flow \( \psi_0 + \psi \) (both scaled with \( fL/\nu \)) induced by a passively translating particle (Stokeslet) are shown in the Fig. 1a and b, respectively. The streamline pattern in Fig. 1 is the same as reported before (see Fig. 1 in [3], the axial velocity at \( r = 0 \) vanishes at \( z \approx 3.8 \)). The streamline pattern corresponding to weakly turbulent case in Fig. 1 for \( \beta\bar{\sigma} = -0.5 \) shows unequivocally that the flow differs quite a lot from that where the effect of turbulence is entirely neglected, in particular, at distances \( r \approx L \).

The stream-function provides a global criterion of the importance of the background turbulent flow. Further measure of the strength of turbulence’s impact on the flow is provided by the axial velocity averaged over the angles

\[
\bar{w}_z' (Lr) = \int_0^{2\pi} w_z' (Lr, Lz, \phi) \frac{d\phi}{2\pi} = -\frac{g}{\nu L} \int \frac{q_\perp^3 dq}{(2\pi)^3 q^4} \Theta' \left( \frac{q_r}{L} \right) \exp [iq_z z] J_0 (q_r r) = \frac{f}{\nu L} \int \frac{q_\perp^2 q_z^2 dq}{(2\pi)^3 [q_\perp^4 + q_z^4]} \exp [iq_z z] J_0 (q_r r)
\]

where we used \( q_z' = -gk_\perp^2 \Theta'(k)/\nu k^4 \). The flow without turbulence is described by the first term that is given by

\[
\bar{w}_z = \frac{f}{\nu L} \int_0^\infty dq_z \int_0^\infty dq_\perp \frac{q_\perp q_z^2}{2\pi^2 [q_\perp^4 + q_z^4]} \cos [q_z z] J_0 (q_r r),
\]

where we used that the integrand is even function of \( q_z \). We consider the asymptotic forms that \( \bar{w}_z \) takes at large and small distances, cf. [3]. Passing to polar coordinates in \((q_\perp, q_z)\) plane we find

\[
\bar{w}_0 = \int_0^{\pi/2} \frac{\sin^3 \theta d\theta}{(2\pi)^2} \int_0^\infty dq_\perp \frac{q_\perp^4 \cos [q_z \cos \theta] J_0 (q_r \sin \theta)}{[\sin^2 \theta + q_z^4]} = \left. \int_0^{\pi/2} \frac{\sin^3 \theta d\theta}{(2\pi)^2} \int_0^\infty dq \cos [q_z \cos \theta] J_0 (q_r \sin \theta) \right|_{\theta = \pi/2} - \left. \int_0^{\pi/2} \frac{\sin^3 \theta d\theta}{(2\pi)^2} \int_0^\infty dq \cos [q_z \cos \theta] J_0 (q_r \sin \theta) dq \right|_{\theta = \pi/2, \sin^2 \theta + q_z^4}.
\]

One finds

\[
\int_0^{\pi/2} \frac{\sin^3 \theta d\theta}{(2\pi)^2} \int_{-\infty}^\infty dq \cos [q_z \cos \theta] J_0 (q_r \sin \theta) = \int_0^{\pi/2} d\alpha \left( \frac{2 + r^2 / z^2}{8\pi |z| (1 + r^2 / z^2)^{3/2}} \right),
\]

which is nothing but the Stokeslet flow that holds without stratification. Thus we obtained the representation of the velocity as the sum of the flow without stratification and

\[
\begin{align*}
\int_0^\infty dq_\perp dq_z & \equiv \int_0^\infty dq_\perp dq_z, \\
\int_0^{\pi/2} d\theta \int_{-\infty}^\infty dq (q_\perp^4 + q_z^4) & \equiv \int_0^{\pi/2} d\theta \int_{-\infty}^\infty dq(q_\perp^4 + q_z^4), \\
\int_0^\infty dq_\perp dq_z & \equiv \int_0^\infty dq_\perp dq_z.
\end{align*}
\]
the correction due to the stratification,
\[
\tilde{w}_0 = \frac{2 + r^2/z^2}{8\pi z(1 + r^2/z^2)^{3/2}} - \int_0^{\pi/2} \frac{\sin^5 \theta d\theta}{(2\pi)^2} \times \int_{-\infty}^{\infty} \cos \left[ qz \cos \theta \right] J_0(qr \sin \theta) dq \left[ \sin^2 \theta + q^4 \right].
\]  

(37)

This representation is useful for studying the role of stratification. At \( z \ll 1, r \ll 1 \), one has
\[
\tilde{w}_0(rL, zL) \approx \frac{2 + r^2/z^2}{8\pi z(1 + r^2/z^2)^{3/2}} - \int_0^{\pi/2} \frac{\sin^5 \theta d\theta}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dq}{\sin^2 \theta + q^4} = B(9/4, 9/4)/\pi \approx 0.0531.
\]

Thus introducing a small uniform correction to the flow. It should be noted that though the correction is small, it can have finite effect on the motion of the particles due to the persistent drift that it induces. The study of such drift is left for future work.

On the other hand, the stratification’s contribution is dominant at scales larger than \( L \), screening the Stokeslet flow, so that the resulting flow is fully determined by the stratification. To demonstrate this we consider the axial velocity along the axis of symmetry \( r = 0 \),
\[
\tilde{w}_0(0, zL) = \frac{1}{4\pi |z|} \int_0^{\pi/2} \frac{\sin^5 \theta d\theta}{(2\pi)^2} \int_{-\infty}^{\infty} \exp \left[ iq |z| \cos \theta \right] dq \left[ \sin^2 \theta + q^4 \right].
\]

We note that
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[ iq |z| \cos \theta \right] dq = \exp \left[ iq_1 |z| \cos \theta \right] = \frac{4q_1 (q_1^2 - q_1^2)}{2i |z| \sin^3 \theta} - \text{c.c.}
\]
\[
\times \exp \left[ -|z| \cos \theta \sqrt{\frac{\sin \theta}{2} \cos \left( \frac{\sin \theta}{2} - \frac{\pi}{4} \right) \right].
\]

(38)

Thus, introducing \( \varphi = \pi/2 - \theta \),
\[
\tilde{w}_0(r = 0) = \frac{1}{4\pi |z|} \int_0^{\pi/2} \cos \left[ |z| \sin \varphi \sqrt{\cos \varphi} - \frac{\pi}{4} \right] \exp \left[ -|z| \sin \varphi \sqrt{\cos \varphi} \right] d\varphi.
\]

(39)

When \( |z| \) is large the integral is determined by the minima of \( \sin \varphi \sqrt{\cos \varphi} / 2 \) which are equal to zero and attained at \( \varphi = 0 \) and \( \varphi = \pi/2 \). The contribution of the saddle-point \( \varphi_0 \) where \( \sin \varphi \sqrt{\cos \varphi} \) has zero derivative is to be considered too, though it includes the exponentially small factor, see below. The contribution of the leading order term that comes from the neighborhood of \( \varphi = 0 \) is
\[
\int_0^{\pi/2} \cos \left[ |z| \varphi \sqrt{2 - \pi/4} \right] \exp \left[ -|z| \varphi \sqrt{2} \right] d\varphi
\]
\[
= Re \frac{\sqrt{2} \exp[\pi/4]}{4\pi |z| (1 + i)} = \frac{1}{4\pi |z|},
\]

(40)

which is the exponential expansion of the contribution of the leading order term coming from the neighborhood of \( \varphi = \pi/2 \) is
\[
\int_0^{\pi/2} \cos \left[ |z| \varphi \sqrt{2 - \pi/4} \right] \exp \left[ -|z| \varphi \sqrt{2} - \pi/4 \right] d\varphi
\]
\[
= Re \frac{\sqrt{2} \exp[\pi/4]}{4\pi |z| (1 + i)} = \frac{1}{4\pi |z|}.
\]

(41)

We conclude that the velocity field decays faster than the Stokeslet solution at large distances from the particle. To find the leading order term at large \( |z| \) we split the domain of integration over \( \varphi \) into \( \varphi < \varphi_0 \) and \( \varphi > \varphi_0 \), where \( \varphi_0 \equiv \arctan(\sqrt{2}) \).

\[
\tilde{w}_0(r = 0) = \frac{1}{4\pi |z|} \int_0^{\varphi_0} \cos \left[ |z| \sin \varphi \sqrt{\cos \varphi} / 2 \right] d\varphi
\]
\[
= Re \frac{\sqrt{2} \exp[\pi/4]}{4\pi |z| (1 + i)} = \frac{1}{4\pi |z|}.
\]

(42)

Introducing \( y_0 \equiv y(\varphi_0) = \sqrt{2/(3\sqrt{3})} \) (we use \( \cos[\arctan(\sqrt{2})] = 1/\sqrt{3} \) one finds
\[
I' = \int_0^{y_0} h(y) \exp \left[ \frac{-|z| y}{\sqrt{2}} + \frac{i\pi}{4} \right] dy / 2\pi,
\]

where we defined
\[
h(y) = \frac{x_1^4(y)}{3x_1^4(y) - 1}, \quad y = \sqrt{x_1(y) - x_1^3(y)}
\]

(43)

with \( x_1 = \cos \varphi(y) \), so that \( x_1(y) \) is the branch of the solution of the cubic equation \( x^3 - x = -y^2 \) that obeys \( x(0) = 1 \). One has
\[
\frac{dx}{dy} = \frac{2y}{1 - 3x^2}, \quad \frac{dh}{dy} = \frac{4x^3(3x^2 - 2)}{(1 - 3x^2)^2} y.
\]

(44)
The large $|z|$ expansion is obtained by introducing the Taylor expansion of $h(y)$ into $I'$,

$$I' \approx \sum_{n=0}^{\infty} \frac{h^{(2n)}(0)}{(2n)!} \Re \int_{0}^{\infty} y^{2n+1} \exp \left[ -\frac{(1+i)|z|y}{\sqrt{2}} + \frac{i\pi}{4} \right] dy \frac{1}{2\pi},$$

$$= \sum_{n=0}^{\infty} h^{(2n)}(0) \Re \frac{1}{2^{n+1}|z|^{2n+1}} \frac{(-1)^{k}h^{(4k)}(0)}{2|z|^{4k+1}}, \quad (45)$$

where we used that $h^{(2n)}(0) = 0$ by $h(y) = h(-y)$. The $k = 0$ term reproduces $(4\pi |z|)^{-1}$ found previously. The $k = 1$ term turns out to be vanishing because a direct computation reveals that $h^{(4)}(0) = 0$. The next order term is proportional to $|z|^{-9}$, which is the same order as in Eq. (41), necessitating the consideration of the contribution of the neighborhood of $\varphi = \pi/2$ into $\tilde{w}_0$. This is described by

$$I'' \equiv \Re \int_{0}^{\pi/2-z} \frac{\sin \frac{1}{2} \varphi}{4\pi} \exp \left[ -\frac{(1+i)|z|\tilde{y}}{\sqrt{2}} + \frac{i\pi}{4} \right] d\tilde{\varphi},$$

where $\tilde{y} = \cos \tilde{\varphi} \sqrt{\sin \varphi}$. Passing to the integration variable $\tilde{y}$, using

$$\frac{d\tilde{\varphi}}{dy} = \frac{2\sqrt{\sin \varphi}}{\cos^2 \tilde{\varphi} - 2\sin^2 \tilde{\varphi}},$$

we obtain

$$I'' = -\Re \int_{0}^{\pi/2-z} \tilde{h}(\tilde{y}) \exp \left[ -\frac{(1+i)|z|\tilde{y}}{\sqrt{2}} + \frac{i\pi}{4} \right] d\tilde{y},$$

where we defined

$$\tilde{h}(\tilde{y}) = \frac{x_3(\tilde{y})}{3x_2(\tilde{y}) - 1}, \quad y = \sqrt{x_2(y) - x_3^2(y)},$$

with $x_3(\tilde{y}) = \sin \tilde{\varphi}(\tilde{y})$ given by the branch of the solution of the cubic equation $\tilde{x}^3 - \tilde{x} = -\tilde{y}^2$ that obeys $x(0) = 0$. Combining $I'$ and $I''$ the following representation of $\tilde{w}_0$ is obtained

$$\tilde{w}_0 = \frac{1}{4\pi|z|} - \int_{0}^{\pi/2-z} \frac{x_3(y)}{3x_1(y) - 1} - \frac{x_2(y)}{3x_2(y) - 1} \exp \left[ -\frac{|z|y}{\sqrt{2}} \right] \cos \left( \frac{|z|y}{\sqrt{2}} - \frac{i\pi}{4} \right) dy \frac{1}{2\pi}.$$

Introducing

$$l(y) = \frac{x_1(y)}{3x_1(y) - 1} - \frac{x_2(y)}{3x_2(y) - 1}, \quad (48)$$

we find the asymptotic series for $\tilde{w}_0$ at large $|z|$,

$$\tilde{w}_0 = -\sum_{k=2}^{\infty} \frac{(-1)^{k}l^{(4k)}(0)}{2|z|^{4k+1}}, \quad (49)$$

Thus at very large $|z|$ the axial velocity $\tilde{w}_0(r = 0)$ is expected to decay as $\sim |z|^{-9}$, as $l^{(8)}(0) \neq 0$. The absolute value of the axial velocity at $r = 0$ determined by numerical integration of Eq. (47) is shown in Fig. 2b. It agrees with the earlier results in [3] and shows that at length scales below $L$ the flow is just that due to unstratified Stokeslet solution $\sim 1/|z|$ (dashed red line in Fig. 2a). At scales $\gtrsim L$ the Stokeslet flow is screened by the buoyant flux due to vertical stratification resulting in a series of eddies with velocity decaying much faster than $\sim 1/|z|$. The numerical results suggest that at large, but finite $|z|$, the saddle-point contribution dominates the integral in (47) so that the velocity decays exponentially fast $\exp(-y_0|z|/\sqrt{2})$ (dashed blue line in Fig. 2b), and not $\sim |z|^{-9}$ as was suggested above. It seems that the involved numbers are such that the power-law will be seen only at very large $|z|$ when the solution is vanishingly small. For practical purposes, therefore, one can say that the velocity decays exponentially at scales larger than $L$.

The turbulence correction $\delta \tilde{w} \equiv \delta w_{z} \nu L/f$ to the velocity is

$$\delta \tilde{w} = -\frac{q}{f} \int_{0}^{\infty} \frac{q^2 dq}{(2\pi)^3} \Theta_0 \left( \frac{q}{L} \right) \delta q \cdot f \cdot \sigma' q \cdot \exp[iqz] J_0(q_1 \pm),$$

$$= -2\beta \int_{0}^{\infty} \frac{q^3 dq}{(2\pi)^3} \delta q \cdot \sigma' q \cdot \exp[iqz] J_0(q_1 \pm),$$

$$= -2\beta \int_{-\infty}^{\infty} \frac{q^3 dq}{(2\pi)^3} \int_{-\infty}^{\infty} dq_1 \cdot \sigma' q \cdot \exp[iqz] J_0(q_1 \pm) \int \frac{d\phi}{2\pi} f \cdot \sigma' q.$$
where we used
\[
\frac{dq}{ds} (s = 0) = \frac{\sigma^4 q^4}{\lambda q^4_\perp + q^6_\perp}
\]  
(53)

that follows by differentiation of the definition of \( q(s) \). We obtain
\[
\Theta_s(q/L) = \Theta_s^0(q/L) [1 + \delta \Theta_s(q/L)],
\]
(54)
\[
\delta \Theta_s(q/L) = \frac{\beta \sigma_{zq} q^4}{q_\perp \lambda} q^4_\perp + q^6_\perp + \delta \Theta(q/L).
\]
(55)

This result can be written in the following form
\[
\Theta_s(k) = \frac{i k_{z}s \Theta(k)}{f} \left[ 1 + \frac{\beta \sigma_{zq} k_{z} q^4}{k_{\perp} \lambda} k_{\perp}^4 L^4 + k_{\perp}^6 L^6 \right] + O(\beta^2),
\]

that shows the leading order deviation due to turbulence from the relation \( \Theta_s = i k_{z} \delta \Theta / f \) between the Stokeslet and force-doublet solutions that holds without turbulence. Similarly to the calculation for \( \Theta \) one can write down the exact relation on \( (\delta \Theta_s)^{1/2} \) which shows that turbulence’s impact on the flow is order one when \( \beta \sim 1 \).

We now consider the limit of strong turbulence \( \beta = (\nu / k \gamma g)^{1/2} \gg 1 \) when intermittency is negligible or \( Re \) is fixed. One observes that this limit is equivalent to the one of small stratification \( \gamma \rightarrow 0 \) in agreement with consideration that strong mixing opposes stratification canceling its effects completely at \( \beta \rightarrow \infty \). We consider
\[
\frac{g \Theta(q/L)}{f} = \frac{1}{\beta} \int_0^\infty \frac{q_{\perp}^2(s) ds}{q_{\perp}^4(s) + q_\perp^6(s)} + o(1/\beta).
\]
(56)

Returning to the original integration variable, \( t = \tau_k(s) / \lambda \),
\[
\int_0^\infty \frac{q_{\perp}^2(s) ds}{q_{\perp}^4(s) + q_\perp^6(s)} = \frac{\lambda}{k_{\perp}^2} \int_0^\infty e^{-2 \rho(t)} \left[ n_2^2(t) + n_4^2(t) \right] dt.
\]

We can use \( \langle e^{-2 \rho(t)} \rangle \approx \exp[\varphi(-2)t] \) following from Eq. (17) at \( \lambda t \gg 1 \). Since \( \hat{n}(t) \) and \( \rho(t) \) can be considered as independent, while \( \hat{n}(t) \) is distributed isotropically, we find \( \langle e^{-2 \rho(t)} \rangle n_2^2(t) = \exp(-2t) / 3 \). Thus,
\[
\langle \Theta(q/L) \rangle \sim 2 \lambda f /[3 \beta g q^2 \varphi(-2)].
\]
(57)

The integral at \( q = 0 \) is made convergent by \( \exp[-s/\beta] \) term, that has to be kept, while the integral at \( q \sim 1 \) is obtained by setting \( \exp[-s/\beta] \approx 1 \). It can be seen that the transition between the two asymptotic regions
is where the two answers are of the same order, that is \( \beta q^2 \approx 1 \). The result in terms of \( \Theta' \) is particularly simple,

\[
\Theta'(k) \approx -\frac{f}{g}, \quad k^2 \gg \frac{1}{L^2 \beta}; \quad |\Theta'(k)| \ll \frac{f}{g}, \quad k^2 \ll \frac{1}{L^2 \beta}.
\]

For finding the inverse Fourier transform, one can put \( \Theta' \approx -f/g \) at large \( \beta \) uniformly. Substituting \( \Theta' = -f/g \) in Eq. (10) recovers the unstratified Stokes flow \( v \kappa^2 \omega = f \Pi(k) \hat{z} \).

Thus the solution depends on one dimensionless parameter \( \beta \) so that the unstratified Stokes flow with streamlines open everywhere holds at \( \beta \gg 1 \) and the stratified flow with closed streamlines holds at \( \beta \ll 1 \). It follows that at \( \beta \gtrsim 1 \) turbulence cannot be disregarded. In particular, there is a critical value \( \beta_c \approx 1 \) such that the streamlines (statistically) open at \( \beta = \beta_c \) (there are closed streamlines at \( \beta < \beta_c \), but not at \( \beta > \beta_c \)). The account of intermittency (the dependence on the Reynolds number) is expected to produce stronger fluctuations of \( \sigma \) and \( \Theta \). It should lower the value of \( \beta \) where the account of turbulence is necessary so that in the limit of very high \( \text{Re} \) turbulence can be important already at \( \beta < 1 \).

Similar result holds for the force-doublet flow. The leading order term in the limit \( \beta \gg 1 \) is obtained by discarding the exponential term (corresponding to \( \exp[-s/\beta] \) before the transformation of variables in the integral) in the integral Eq. (20).

\[
\Theta_s = \int_{-\infty}^{0} dt \phi_s [k(t)] = \frac{i \gamma s}{\nu} \int_{-\infty}^{0} dt \frac{k_s(t)k_2(t)}{k^2(t)}. \tag{58}
\]

This integral converges over the time-scale \( \lambda^{-1} \) so that \( \Theta_s \propto 1/\beta \). Discarding \( \Theta \) in the expression for the velocity in Eq. (12) one finds that the force-doublet flow tends to the flow without the stratification in the limit \( \beta \gg 1 \).

We conclude that turbulence is important at \( \beta \gtrsim 1 \). It can also be important at \( \beta \ll 1 \). In this case turbulence is negligible in \( \mathbf{w}' \), but the total flow \( \mathbf{v} = \mathbf{u} + \mathbf{w}' \) can differ from \( \mathbf{u}' \) significantly due to \( \mathbf{u} \). The flow \( \mathbf{u} \) produces the characteristic difference \( \lambda L \) of velocities of particles separated by \( L \). The corresponding relative velocity induced by \( \mathbf{w}' \) is estimated by \( U a/L \). The ratio of the two differences \( \frac{U a}{U a} = (\nu/U a)(L^2/L^2_{a}) \) is the product of the large parameter \( \nu/U a = 1/\text{Re}_{\text{inj}} \) and the small parameter \( L^2/L^2_{a} \). If it is large turbulence is important. Thus turbulence is important at \( (\nu/U a)(L^2/L^2_{a}) \gtrsim 1 \) or \( |\beta| \gtrsim 1 \).

Note that whether turbulence is relevant for the flow around small particles does not depend on the parameters of the particles themselves: neither the particle’s size, mass or velocity enter \( \beta \). The size is irrelevant as long as \( a \ll L \) is obeyed, since then the flow at the scale \( L \) is sensitive only to the integral characteristics of the particle - the force \( f \), while the mass is irrelevant because the particle influences the flow only via the no-slip boundary condition. The perturbed flow around the particle is caused by its motion relative to the flow, so that the resulting perturbation is influenced both by turbulence and stratification. Both impacts are proportional to the magnitude of the relative motion but their relative importance, which we studied in this work, is independent of that magnitude, i.e. of the particle’s velocity.

We now consider the possible role of the time-dependence in \( f \) caused by the temporal fluctuations of the turbulent velocity field. It is clear that these fluctuations can only make the effect of turbulence stronger by introducing one more term in the force besides the time-independent average term (remind that the amplitude of the force is irrelevant due to linearity). It is implied however in linearity in force that the temporal variations of the latter have no relevance on the relative importance of turbulence and stratification.

Let us now estimate the typical values of \( \beta \) in various aquatic environments. Using the extreme value of the density gradient \( \gamma \rho_0 = 1 \text{ kg m}^{-2} \) that may occur locally in fjords \([13]\), lakes and reservoirs \([16]\) with \( \mu = 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1} \) yields \( L \approx 0.6 \text{ mm} \) for salt-stratified water \((\kappa \approx 1.3 \times 10^{-9} \text{ m}^2 \text{ s}^{-1})\) and \( L \approx 2 \text{ mm} \) for temperature-stratified water \((\kappa \approx 1.4 \times 10^{-7} \text{ m}^2 \text{ s}^{-1})\). Further considering weakly turbulent conditions with the dissipation rate per unit mass \( \epsilon \approx 10^{-10} \text{ m}^2 \text{ s}^{-3} \) \((\text{e.g. Kunze et al.}[14]) \) measured \( \epsilon \approx 10^{-9} \text{ m}^2 \text{ s}^{-3} \) in a coastal inlet gives \( \lambda = \sqrt{\epsilon/\nu} \approx 0.01 \text{ s}^{-1} \). Thus, the corresponding values of \( \beta = \lambda L^2/\kappa \approx 0.3 \) and \( \approx 2.8 \) for temperature- and salt-stratified water, respectively. Furthermore, in the marine environment the buoyancy frequency \( N = \sqrt{g/\gamma} \) corresponding to the marginal oscillations which the stable stratification supports \([17]\) is typically in the range between \( 10^{-4} \) and \( 10^{-2} \text{ s}^{-1} \), yielding density gradients \( \gamma \rho_0 \) that ranges between \( 10^{-6} \) and \( 10^{-2} \text{ kg m}^{-4} \), several orders of magnitude lower than that considered in \( \beta \). In some extreme cases, however \((\text{e.g. during seasonal thermocline}[17]) \) \( N \) may exceed \( 0.05 \text{ s}^{-1} \) so \( \gamma \rho_0 \) may reach \( \approx 0.3 \text{ kg m}^{-4} \). Using this extreme value of density stratification and \( \epsilon \approx 10^{-10} \text{ m}^2 \text{ s}^{-3} \) we arrive at \( \beta \approx 0.5 \) and \( \approx 5.5 \) for temperature- and salt-stratified water, respectively. However, for the less extreme conditions of marine turbulence and/or stratification typically \( \beta > 1 \). For example, for \( \epsilon \approx 10^{-9} \text{ m}^2 \text{ s}^{-3} \) and \( \gamma \rho_0 \approx 0.01 \text{ kg m}^{-4} \) we find \( \beta \approx 8 \) and \( \approx 90 \) for temperature and salt stratification, respectively.

We derived and solved the advection-diffusion equation that describes the turbulent flow around small translating particles in the stratified fluid in the limit of large Prandtl numbers. We showed that when intermittency is negligible \((\text{which allows very high \( \text{Re} \)) \) the solution is determined by a single dimensionless parameter \( \beta \), whereas turbulence is important when \( \beta \geq 1 \). Intermittency, important at higher \( \text{Re} \), only strengthens the impact of turbulence. We therefore conclude that the account of turbulence is necessary in natural environments.

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[18] We concentrate here for simplicity, as in [3], on the impact of the vertical component of the motion, i.e. on the effect of the \(z\)-component of \(\mathbf{u} - \mathbf{V}\). The transversal motion can be included by introducing transversal component of the force and using the method of superposition. This will not be considered here since it is irrelevant for our conclusions, while making the analysis unnecessarily more cumbersome.