Improving regularity of solutions of a difference equation

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Dedicated to Professor János Aczél on the occasion of his 90th birthday

Abstract. Using some results on convex and almost convex functions defined on a locally compact Abelian group, we prove a theorem showing a “measurability implies continuity” effect for non-negative solutions of the difference equation \( \varphi(x) = \sum_{i=1}^{k} p_i \varphi(x + a_i) \), where \( p_1, \ldots, p_k \in (0, \infty) \) and non-zero elements \( a_1, \ldots, a_k \) of the group are given.

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Introduction

Given an Abelian group \( G \), non-zero elements \( a_1, \ldots, a_k \in G \) and positive numbers \( p_1, \ldots, p_k \) we are interested in non-negative solutions \( \varphi: G \to \mathbb{R} \) of the difference equation

\[
\varphi(x) = \sum_{i=1}^{k} p_i \varphi(x + a_i).
\]

(E)

In the case when \( G = \mathbb{R} \) all non-negative Lebesgue measurable solutions of (E) were determined by Laczkovich [12]. Later, another proof was given by the present author (see [5, Th. 3.1]), and then by Grinč [2] when \( G = \mathbb{R}^n \).

The main step in the reasoning presented there (cf. [5, Prop. 3.3]) is an improvement of regularity of non-negative solutions of (E) provided the subgroup generated by \( a_1, \ldots, a_k \) is dense in \( G \). Such a “measurability implies continuity” effect is well-known in the theory of functional equations in several variables (cf. for instance, the book [4] by A. Járai; also [1] by J. Aczél and [11] by M. Kuczma) but for equations in a single variable it is rather unexpected.
In the present paper we show how to improve regularity of non-negative solutions of (E) in the case when $G$ is a locally compact Abelian group. Some arguments presented here take the pattern of those used in [5] in the case $G = \mathbb{R}$.

In the whole paper *measurability* of a function defined on $G$ means $\mathcal{M}_\lambda$-measurability, where $\mathcal{M}_\lambda$ stands for the completion of the $\sigma$-algebra $\mathcal{B}(G)$ of Borel subsets of $G$ with respect to the Haar measure $\lambda$. Equivalently this is *measurability in the sense of Carathéodory*, i.e.

$$\mathcal{M}_\lambda = \{ A \subset G : \lambda^* (Z \cap A) + \lambda^* (Z \setminus A) \leq \lambda^* (Z) \text{ for every } Z \subset G \},$$

where $\lambda^* : 2^G \to [0, \infty]$ is the outer measure generated by $\lambda$:

$$\lambda^* (A) = \inf \{ \lambda(B) : A \subset B \in \mathcal{B}(G) \}$$

for every $A \subset G$. Measurability of a function defined on $G^2$ is meant with respect to the $\lambda^2$-completion $\mathcal{M}_{\lambda^2}$ of the $\sigma$-algebra $\mathcal{B}(G^2)$, where $\lambda^2$ is the product measure built with two copies of $\lambda$.

The main result of the paper reads as follows.

**Theorem.** Let $G$ be an Abelian 2-divisible group, $\sigma$-compact and locally compact, with Haar measure $\lambda$. Assume that the subgroup generated by $a_1, \ldots, a_k$ is dense in $G$.

If $\varphi : G \to \mathbb{R}$ is a non-negative measurable solution of equation (E), then either $\varphi = 0 \lambda$-a.e., or there is a positive continuous geometrically convex solution $\psi : G \to \mathbb{R}$ of (E) such that $\varphi = \psi \lambda$-a.e.

Geometric convexity of $\psi : G \to \mathbb{R}$ means here that

$$\psi (x)^2 \leq \psi (x + h) \psi (x - h)$$

for all $x, h \in G$.

The proof of the Theorem is split into some lemmas presented in Sect. 2. Moreover, the following remarks will be recalled in Sect. 1 while proving some auxiliary facts.

**Remark 0.1.** It is well-known (cf. [3, (15.8) and (11.34)]) that the Haar measure on any Abelian locally compact group $G$ is regular, i.e.

$$\lambda (B) = \inf \{ \lambda(U) : B \subset U \subset G \text{ and } U \text{ is open} \}$$

for every $B \in \mathcal{B}(G)$ and

$$\lambda (B) = \sup \{ \lambda(C) : C \subset B \text{ and } C \text{ is compact} \}$$

for every set $B \subset G$ which is open or of finite measure. Moreover, $\lambda$ takes finite values on compacts. So, if $K \subset G$ is compact, then for all $B \in \mathcal{B}(K)$ and $\varepsilon \in (0, \infty)$ there exist a set $U$, open in $K$, and a compact set $C$ such that

$$C \subset B \subset U \quad \text{and} \quad \lambda (U \setminus C) < \varepsilon.$$

The next two remarks concern folk-theorems.
Remark 0.2. Repeating the proof of [14, Th. 8.2] step by step we come to the following version of the classical Lusin’s theorem.

Let $X$ and $Y$ be topological spaces, the second one with a countable base, and let $\mu$ be a measure defined on a $\sigma$-algebra $\mathcal{M}$ of subsets of $X$ containing all Borel sets. Assume that for all $B \in \mathcal{M}$ and $\varepsilon \in (0, \infty)$ there exist an open set $U \subset X$ and a closed set $F \subset X$ such that

$$F \subset B \subset U \quad \text{and} \quad \mu (U \setminus F) < \varepsilon.$$ 

If $f : X \to Y$ is an $\mathcal{M}$-measurable function, then for every $\varepsilon \in (0, \infty)$ there exists a closed set $F \subset X$ such that

$$\mu (X \setminus F) < \varepsilon \quad \text{and the function } f|_F \text{ is continuous.}$$

Remark 0.3. The standard argument, proving that in a metric setting any continuous function defined on a compact set is uniformly continuous, allows to obtain the following group version of this fact.

Any continuous function $f$, mapping a compact subset $C$ of an Abelian topological group $G$ into an Abelian topological group $H$, is uniformly continuous: for every neighbourhood $W \subset H$ of 0 there exists a neighbourhood $V \subset G$ of 0 such that

$$\bigwedge_{x_1, x_2 \in C} (x_1 - x_2 \in V \Rightarrow f(x_1) - f(x_2) \in W).$$

1. Auxiliary results

We start with two general facts, not immediately connected with the problem of solutions of equation (E). The first one is a simple purely topological observation.

Lemma 1.1. Let $G$ be an Abelian $\sigma$-compact and locally compact group. Then there exists a sequence $(K_n)_{n \in \mathbb{N}}$ of compacts and a neighbourhood $U$ of 0 such that $\text{cl}U$ is compact,

$$K_n + U \subset K_{n+1}, \quad n \in \mathbb{N}, \quad (1.1)$$

and

$$G = \bigcup_{n=1}^\infty K_n. \quad (1.2)$$

Proof. The group $G$, being $\sigma$-compact, is the union of a sequence $(C_n)_{n \in \mathbb{N}}$ of compacts. Take any neighbourhood $U$ of 0 such that $\text{cl}U$ is compact. For every $n \in \mathbb{N}$ put

$$K_n = \bigcup_{i=1}^n C_i + [n] \text{cl}U,$$
where \([n]A\) stands for the sum \(A + \cdots + A\) of \(n\) copies of \(A\). Clearly the sets 
\(K_n, n \in \mathbb{N}\), are compact. Moreover, for every \(n \in \mathbb{N}\) we have
\[
C_n \subset K_n \subset K_n + U = \bigcup_{i=1}^{n} C_i + [n]clU + U \subset K_{n+1}
\]
and the desired properties (1.1) and (1.2) follow.

The next result is an extension of [12, Lemma 2] to a group setting (see also [4, Theorems 19.3 and 19.5]).

**Lemma 1.2.** Let \(G\) be an Abelian \(\sigma\)-compact and locally compact group with Haar measure \(\lambda\) and let \(\varphi: G \to \mathbb{R}\) be a measurable function. Then, for every \(y_0 \in G\) and for every sequence \((y_n)_{n \in \mathbb{N}}\) of elements of \(G\) converging to \(y_0\), there exists a strictly increasing sequence \((m_n)_{n \in \mathbb{N}}\) of positive integers such that
\[
\lim_{n \to \infty} \varphi (x + y_{m_n}) = \varphi (x + y_0) \quad \text{for } \lambda\text{-a.a. } x \in G.
\]

**Proof.** Since \(\lambda\) is translation invariant, we may additionally assume that \(y_0 = 0\). Define functions \(\varphi_n: G \to \mathbb{R}, n \in \mathbb{N}\), by
\[
\varphi_n (x) = \varphi (x + y_n).
\]
By virtue of Lemma 1.1 we find a sequence \((K_i)_{i \in \mathbb{N}}\) of compacts in \(G\) and a neighbourhood \(U \subset G\) of \(0\) satisfying (1.1) and (1.2). We prove that for every \(i \in \mathbb{N}\) the sequence \((\varphi_n|_{K_i})_{n \in \mathbb{N}}\) converges in measure to the function \(\varphi|_{K_i}\).

Fix any \(i \in \mathbb{N}\) and a positive number \(\varepsilon\). Following Remarks 0.1 and 0.2 we find a closed subset \(F\) of \(K_{i+1}\) such that
\[
\lambda (K_{i+1} \setminus F) < \frac{\varepsilon}{2} \quad \text{and the function } \varphi|_F \text{ is continuous.}
\]
Since the set \(F\) is compact, \(\varphi|_F\) is actually uniformly continuous (cf. Remark 0.3). Thus we can find a neighbourhood \(V \subset U\) of \(0\) such that
\[
|\varphi (x_1) - \varphi (x_2)| < \varepsilon \quad \text{for all } x_1, x_2 \in F \text{ satisfying } x_1 - x_2 \in V.
\]
Define a sequence \((A_n)_{n \in \mathbb{N}}\) of subsets of \(G\) by
\[
A_n = \{ x \in K_i : |\varphi_n (x) - \varphi (x)| \geq \varepsilon \}.
\]
Since \((y_n)_{n \in \mathbb{N}}\) converges to 0, there exists a positive integer \(n_0\) such that \(y_n \in V\) for every \(n \geq n_0\). Take any integer \(n \geq n_0\) and point \(x \in A_n\). Suppose that \(x \in F \cap (F - y_n)\). Then \(x, x + y_n \in F\) and \((x + y_n) - x = y_n \in V\), and thus
\[
|\varphi_n (x) - \varphi (x)| = |\varphi (x + y_n) - \varphi (x)| < \varepsilon,
\]
which is impossible. This shows that
\[
A_n \subset (K_i \setminus F) \cup [(K_i + V) \setminus F] - y_n \subset (K_i \setminus F) \cup [(K_{i+1} \setminus F) - y_n].
\]
Consequently, we have
\[
\lambda (A_n) \leq \lambda (K_i \setminus F) + \lambda ((K_{i+1} \setminus F) - y_n)
\]
\[
= \lambda (K_i \setminus F) + \lambda (K_{i+1} \setminus F) \leq 2 \lambda (K_{i+1} \setminus F) < \varepsilon.
\]
This proves that the sequence \((\varphi_n|_{K_i})_{n \in \mathbb{N}}\) converges in measure to \(\varphi|_{K_i}\).

Consequently, every subsequence of \((\varphi_n|_{K_i})_{n \in \mathbb{N}}\) has a subsequence converging to \(\varphi|_{K_i}\). Using induction and a standard diagonal method we complete the proof. \(\Box\)

Now we remark that the group addition and substraction are transformations preserving measurability.

**Lemma 2.1.** Let \(G\) be an Abelian locally compact group with Haar measure \(\lambda\). Then

\[
\phi_+^{-1}(A) \in \mathcal{M}_{\lambda^2} \quad \text{and} \quad \phi_-^{-1}(A) \in \mathcal{M}_{\lambda^2}, \quad A \in \mathcal{M}_\lambda,
\]

where \(\phi_+: G^2 \to G\) and \(\phi_-: G^2 \to G\) are given by \(\phi_+(x, y) = x + y\) and \(\phi_-(x, y) = x - y\), respectively.

**Proof.** Take any \(A \in \mathcal{M}_\lambda\). Then \(A = B \cup M\), where \(B \in \mathcal{B}(G)\) and \(M \subset N \in \mathcal{B}(G)\) with \(\lambda(N) = 0\). Clearly,

\[
\phi_+^{-1}(A) = \phi_+^{-1}(B \cup M) = \phi_+^{-1}(B) \cup \phi_+^{-1}(M)
\]

and \(\phi_+^{-1}(M) \subset \phi_+^{-1}(N)\). Since \(\phi_+\) is continuous, we have \(\phi_+^{-1}(B), \phi_+^{-1}(N) \in \mathcal{B}(G^2)\). Moreover, for any \(x \in G\) the \(x\)-section \((\phi_+^{-1}(N))_x\) of \(\phi_+^{-1}(N)\) is

\[
(\phi_+^{-1}(N))_x = \{y \in G: \phi_+(x, y) \in N\} = \{y \in G: x + y \in N\} = N - x,
\]

and thus, by Fubini’s Theorem,

\[
\lambda_2(\phi_+^{-1}(N)) = \int_G \lambda((\phi_+^{-1}(N))_x) d\lambda(x)
= \int_G \lambda(N - x) d\lambda(x) = \int_G \lambda(N) d\lambda(x) = 0.
\]

Consequently, \(\phi_+^{-1}(A) \in \mathcal{M}_{\lambda^2}\). Similarly one can prove that \(\phi_-^{-1}(A) \in \mathcal{M}_{\lambda^2}\). \(\square\)

**2. Proof of the Theorem**

The first of the lemmas, dealing with solutions of \((E)\), is purely algebraic: no topology in the group \(G\) is assumed. However, the non-negativity of a solution turns out to be crucial for the assertion.

**Lemma 2.1.** Let \(G\) be an Abelian group. If \(\varphi: G \to \mathbb{R}\) is a non-negative solution of equation \((E)\), then

\[
\varphi(x)^2 \leq \varphi(x + h) \varphi(x - h)
\]

(2.1)

for every \(x \in G\) and all \(h\)’s running through the subgroup of \(G\) generated by \(a_1, \ldots, a_k\).
Proof. Take any \( x \in G \) and define \( c: \mathbb{Z}^k \to \mathbb{R} \) by
\[
c(n) = \varphi(x + n_1a_1 + \cdots + n_ka_k)
\]
[here \( n = (n_1, \ldots, n_k) \)]. One can check that, by (E), \( c \) is a non-negative solution of the recurrent equation
\[
c(n) = \sum_{i=1}^{k} p_i c(n + e_i),
\]
where \((e_1, \ldots, e_k)\) stands for the canonical zero-one basis of the space \( \mathbb{R}^k \). It follows from [5, Th. 1.1] that \( c \) is geometrically convex, that is
\[
c(m)^2 \leq c(m + n)c(m - n), \quad m, n \in \mathbb{Z}^k.
\]
Putting here \( m = (0, \ldots, 0) \) we see that
\[
\varphi(x)^2 \leq \varphi(x + n_1a_1 + \cdots + n_ka_k) \varphi(x - n_1a_1 - \cdots - n_ka_k), \quad n \in \mathbb{Z}^k,
\]
which was to be proved. \( \square \)

The next result shows that, under suitable assumptions on the group \( G \) and the function \( \varphi \), if inequality (2.1) holds on a set which is large in a certain topological sense, then it is satisfied on a set of full measure.

**Lemma 2.2.** Let \( G \) be an Abelian \( \sigma \)-compact and locally compact group with Haar measure \( \lambda \). Let \( \varphi: G \to \mathbb{R} \) be a measurable function. If inequality (2.1) holds for every \( x \in G \) and \( h \)'s running through a dense subset of \( G \), then (2.1) is satisfied for all \( \lambda^2 \)-a.a. \((x, h) \in G^2\).

**Proof.** According to Lemma 1.3 the set
\[
T = \{(x, h) \in G^2 : \varphi(x)^2 > \varphi(x + h)\varphi(x - h)\}
\]
is measurable. Fix an \( h \in G \) and a sequence \((h_n)_{n \in \mathbb{N}}\) of elements of \( G \) converging to \( h \) and satisfying the condition
\[
\varphi(x)^2 \leq \varphi(x + h_n) \varphi(x - h_n), \quad x \in G, n \in \mathbb{N}. \quad (2.3)
\]
On account of Lemma 1.2 there exists a strictly increasing sequence \((m_n)_{n \in \mathbb{N}}\) of positive integers such that
\[
\lim_{n \to \infty} \varphi(x + h_{m_n}) = \varphi(x + h), \quad x \in G \setminus E(h),
\]
and
\[
\lim_{n \to \infty} \varphi(x - h_{m_n}) = \varphi(x - h), \quad x \in G \setminus E(h).
\]
Thus, by (2.3), we have
\[
\varphi(x)^2 \leq \varphi(x + h)\varphi(x - h), \quad x \in G \setminus E(h).
\]
This means that
\[
\{ x \in G : \varphi(x)^2 > \varphi(x + h)\varphi(x - h) \} \subset E(h), \quad h \in \mathbb{R},
\]
which was to be proved.
and, consequently, all $h$-sections of the set $T$ are null sets. By Fubini’s Theorem
we infer that also $T$ is a null set. □

The final lemma below completes the proof of the Theorem.

**Lemma 2.3.** Let $G$ be an Abelian 2-divisible group, locally compact, with Haar
measure $\lambda$. Let $\varphi: G \to \mathbb{R}$ be a non-negative measurable function satisfying
inequality (2.1) for $\lambda^2$-a.a. $(x, h) \in G^2$. Then either $\varphi = 0$ $\lambda$-a.e., or there
is a positive continuous geometrically convex function $\psi: G \to \mathbb{R}$ such that
$\varphi = \psi \lambda$-a.e.

**Proof.** The set $Z = \{ x \in G: \varphi(x) = 0 \}$ is measurable. If $\lambda(G \setminus Z) = 0$, then
$\varphi(x) = 0$ for $\lambda$-a.a. $x \in G$. Now assume that $\lambda(G \setminus Z) > 0$. Since the set $T$
defined by (2.2) is of measure $\lambda^2$ zero, Fubini’s Theorem allows to find a null
set $N \subset G$ such that

$$
\lambda\left( \left\{ h \in G: (x, h) \in T \right\} \right) = 0, \quad x \in G \setminus N.
$$

As $\lambda(G \setminus (Z \cup N)) > 0$ we can take an $x_0 \in G \setminus (Z \cup N)$. Then $\varphi(x_0) > 0$
and $\lambda\left( \left\{ h \in G: (x_0, h) \in T \right\} \right) = 0$ which means that

$$
0 < \varphi(x_0)^2 \leq \varphi(x_0 + h) \varphi(x_0 - h) \quad \text{for} \quad \lambda-\text{a.a.} \ h \in G.
$$

Thus $\varphi(x_0 + h) > 0$ for $\lambda$-a.a. $h \in G$, whence $\varphi$ is positive a.e., that is $Z$
is a null set.

Define the function $f: G \to \mathbb{R}$ by

$$
f(x) = \begin{cases} 
\log \varphi(x), & \text{if } x \in G \setminus Z, \\
0, & \text{if } x \in Z.
\end{cases}
$$

Since $\lambda^2(T) = 0$, where $T_0 = T \cup (Z \times G)$, we have

$$
0 < \varphi(x)^2 \leq \varphi(x + h) \varphi(x - h), \quad (x, h) \in G^2 \setminus T_0,
$$

whence

$$
2f(x) \leq f(x + h) + f(x - h) \quad \text{for} \quad \lambda^2$-a.a. $(x, h) \in G^2.
$$

In other words, the function $f$ is almost convex, and thus, by [9, Th. 1] (see
also [7]), there exists a convex function $g: G \to \mathbb{R}$:

$$
2g(x) \leq g(x + h) + g(x - h), \quad (x, h) \in G^2,
$$

such that $g = f \lambda$-a.e. In particular, $g$ is measurable. Making use of the
extended version of the Blumberg–Sierpiński theorem [8, Th. 4.1] we infer
that $g$ is continuous. Now it is enough to observe that the function $\psi = \exp \circ g$
is positive, continuous, geometrically convex, and $\psi = \varphi \lambda$-a.e. □
3. Concluding remarks and an open problem

The main tools used here have topological counterparts. The topological version of Lusin’s Theorem can be easily proved in topological spaces (cf. [14, Th. 8.1]). An analog of Lemma 1.2 for Baire measurable functions defined on an arbitrary linear topological space was given by M. Grinč (see [2, Lemma 2]); however, the argument used by him works for topological groups, too. Theorem 1 from [9], stating that every almost convex function is λ-a.e. equal to a convex function and used in the proof of Lemma 2.3, is a generalization of the Kuczma theorem [11, Th. 17.8.2] (cf. also [10]). Its topological version for functions defined on groups was proved in [6] (see also [7]). Finally, also the Blumberg–Sierpiński theorem has a topological version in a group setting which can be found in [8] (see also [7]). Making use of these results one can obtain a suitable counterpart of the Theorem where solutions of Eq. (E) are assumed to be Baire measurable. In the case $G = \mathbb{R}^n$ such a result was proved by Grinč in [2].

Using some versions of the Kuczma theorem one can prove also results improving the regularity of non-negative solutions of the following extension of Eq. (E), called the integrated Cauchy functional equation:

$$\varphi(x) = \int_G \varphi(x + y)d\mu(y);$$  \hfill (I)

here $\mu$ is a regular Borel measure on the group $G$. Its locally $\lambda$-integrable solutions were determined in [13] by Ka-Sing Lau and Wei-Bin Zeng.

Improvement of regularity of non-negative solutions of (E) is a crucial step in determining them in the cases $G = \mathbb{R}$ (see [5, Prop. 3.3]) and $G = \mathbb{R}^n$ (see [2, Theorem]). It seems that the Theorem could play an analogous role while looking for the form of solutions defined on groups. Also determining all non-negative measurable solutions of (I) might run in a similar way.

**Open problem.** The assumption of 2-divisibility of the group $G$ in the Theorem is caused by the same condition imposed on $G$ in [9, Th.1]. The authors of [9] still do not know if this assumption is essential there. However, the following question is natural: *is the 2-divisibility of $G$ essential for the validity of the Theorem?*

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