Twisted Spectral Triples
without First-Order Condition

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Abstract

We extend twisted inner fluctuations to twisted spectral triples that do not meet the twisted first order condition, following what has been done in [6] for the non twisted case. We find a similar non-linear term in the fluctuation, and work out the twisted version of the semi-group of inner perturbations.

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1 Introduction

Twisted spectral triples have been introduced by Connes and Moscovici in [11] to incorporate type III algebras in the paradigm of spectral triples. Instead of requiring that the commutator $[D,a]$ is bounded for any $a \in A$, one asks for an algebra automorphism $\sigma$ that makes the twisted commutator

$$[D,a]_\sigma := Da - \sigma(a)D$$

bounded for any $a \in A$. Other properties of spectral triples have been later generalised to the twisted case, in particular the real structure [9]. This leads to a twisted version of the first order condition [25], [26]

$$[[D,a]_\sigma,b]_\sigma = 0 \quad \forall a, b \in A$$

where $\sigma^\circ$ is the automorphism induced by $\sigma$ on the opposite algebra $A^\circ$ (for a twisted version of the the regularity condition, see [28]).

In [6], the authors have shown how the removal of the first order condition for a usual - i.e. non twisted - spectral triple yields a non-linear term in the inner fluctuation of the Dirac operator. This term turns out to be important for the application of noncommutative geometry to high-energy physics, since it paves the way to models “Beyond the Standard Model” of fundamental interactions [5].

In this paper, we show that a similar phenomenon occurs for twisted spectral triples. The twisted inner fluctuations of [26] generalise in case the twisted first-order condition (2.2) does not hold, yielding a non-linear term. Twisted gauge transformations as well still make sense, and are well encoded by a semi-group structure, similar as the one worked out in [6] in the non-twisted case.

The paper is organised as follows. In §2 we recall some basics on real twisted spectral triples (§2.1), including generalised 1-forms and connections (§2.2). We discuss in particular in §2.3 the conditions under which the twisting automorphism $\sigma$ lifts to $A$-modules, showing that the assumption made in [25] is no the only possibility. The definition of hermitian connections in the twisted context is discussed in §2.4. Fluctuations without the first order conditions are generalised to the twisted case in section 3. First, we work out in details how to export a twisted spectral triple between Morita equivalent algebras using a right module (§3.1), then taking into account the real structure and assuming the condition of order 1 (§3.2). All this is done following what has been done in [6] for the non-twisted case, and provides an extension of the results of [26] beyond self-Morita equivalence. The twisted first-order condition is removed in §3.3. One obtains a non-linear component in the twisted fluctuation in proposition 3.7. Section 4 deals with gauge transformation. It begins with a brief recalling of gauge transformations for twisted spectral triples in §4.1 that are extended to the non-linear term in §4.2. The equivalence between gauge transformations and the twisted conjugate action of unitaries on the Dirac operator is shown in Prop 4.2. The loss of selfadjointness of the Dirac operator under gauge transformation is discussed in §4.3; one finds the same limitations as when the condition of order 1 holds. In section 5 we work out the structure of semi-group associated to twisted inner fluctuations. The normalisation condition defining the semi-group is discussed in §5.1 and the semi-group is explicitly built in §5.2. All is summarised in Prop 5.8, which also shows how to describe gauge transformations by actions of unitary elements of the semi-group. Finally, in §6 we adapt to the twisted case the concluding $U(1) \times U(2)$ example of [6], showing that we obtain a similar field contains. The appendices contain technical results on fluctuations implemented by a left module.

In all the paper, we assume that the algebras are unital.
2 Twisted Spectral Triples

We begin with a brief summary of the results of [25, 26] on real twisted spectral triples (§2.1) and twisted 1-forms (§2.2). We discuss the lift of the twisting automorphism to modules in §2.3 and hermitian connections in the twisted context in §2.4.

2.1 Real twisted spectral triples

For simplicity we work with complex algebras, but the definitions below make sense for real algebras as well (this is important for applications to physics, since the algebra describing the Standard Model of fundamental interactions is a real one).

Definition 2.1. [11] A twisted spectral triple \((A, H, D, \sigma)\) consists in a unital, involutive, complex algebra \(A\) acting faithfully on a separable Hilbert space \(H\) via an involutive representation \(\pi\), together with a self-adjoint densely defined operator with compact resolvent \(D\) (called Dirac operator) and an automorphism \(\sigma \in \text{Aut}(A)\) satisfying

\[
\sigma(a^*) = \sigma^{-1}(a)^* \quad \forall a \in A,
\]

and such that for any \(a \in A\) the twisted commutator

\[
[D, \pi(a)]_{\sigma} := D\pi(a) - \pi(\sigma(a))D
\]

is bounded.

The automorphism \(\sigma\) is not asked to be involutive, but rather to satisfy the regularity property (2.1) following from considerations on local index theory [11].

As in the non-twisted case, the spectral triple is graded when there is a self-adjoint operator \(\Gamma\) on \(H\) that squares to the identity, anticommutes with \(D\) and commutes with \(A\). The real structure [9] as well makes sense without change: this is an antilinear isometry \(J\) on \(H\) such that:

\[
J^2 = \varepsilon 1 \quad JD = \varepsilon' DJ \quad J\Gamma = \varepsilon'' \Gamma J
\]

where \(\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}\) defines the \(KO\)-dimension of the triple. It implements a representation

\[
\pi^0(a^0) := J\pi(a)^*J^{-1}
\]

of the opposite algebra \(A^0\), where the map \(\circ : A \to A^0\) identifies any \(a \in A\) as an element \(a^0\) of \(A^0\).

To \(\sigma\) is associated the automorphism of \(A^0\)

\[
\sigma^0(a^0) := (\sigma^{-1}(a))^0, \quad \text{with inverse} \quad \sigma^{-1}(a^0) = \sigma(a)^0.
\]

This automorphism satisfies a regularity condition similar as (2.1)

\[
\sigma^0((a^0)^*) = \sigma^0((a^*)^0) = (\sigma^{-1}(a^*))^0 = (\sigma(a)^*)^0 = (\sigma(a)^*)^* = \left(\sigma^{-1}(a^0)\right)^*,
\]

where we used the commutation of \(\circ\) with the involution: \(J = J^*\) hence \((a^0)^* = (a^*)^0\). Moreover, by (2.4) and (2.1) one has

\[
\pi^0(\sigma^0(a^0)) = J\pi(\sigma(a^*))J^{-1},
\]

which guarantees the boundedness, for any \(a^0 \in A^0\), of the twisted commutator

\[
[D, \pi^0(a^0)]_{\sigma^0} := D\pi^0(a^0) - \pi^0(\sigma^0(a^0))D = \varepsilon' J[D, \pi(a^*)]_{\sigma} J^{-1}.
\]

To define an (ordinary) real spectral triple, \(J\) is asked to satisfy two conditions of order zero and one. The former passes to the twist without modification, the latter is modified as follows.

3
Definition 2.2. \cite{25} A twisted spectral triple \((A, \mathcal{H}, D), \sigma\) is real when it comes with a real structure \(J\) which satisfies the conditions of

\begin{align}
\text{order-zero:} & \quad [\pi(a), \pi^0(b^0)] = 0, \\
\text{order-one:} & \quad [[D, \pi(a)], \pi^0(b^0)]_{\sigma^0} = 0 \quad \forall a \in A, b^0 \in A^0. \tag{2.9} \tag{2.10}
\end{align}

The order zero condition guarantees that the right action of \(A\) on \(\mathcal{H}\) defined by

\[
\psi \pi(a) := \pi^0(a^0)\psi = J\pi(a^*)J^{-1}\psi \quad \forall a \in A, \psi \in \mathcal{H},
\tag{2.11}
\]

commutes with the left action induced by the representation \(\pi\).

Let \(\hat{\pi}(a) := J\pi(a)J^{-1} = \pi^0(\pi^0(a^0)^*\) denote the conjugation by the real structure. Dropping the representation \(\pi\), this equation, eq. \(\text{(2.11)}\), the zero and first order conditions are equivalent to

\[
\psi a = a^0\psi, \quad \hat{a} = (a^*)^0 = (a^0)^*, \quad [a, \hat{b}] = 0, \quad [[D, a], \sigma^0, b]_{\sigma^0} = 0 \quad \forall a, b \in A, \psi \in \mathcal{H}. \tag{2.12}
\]

2.2 Twisted one-forms and connections

The twisted commutators \(\text{(2.2)}\) and \(\text{(2.3)}\) are derivations on the algebra \(A\),

\[
\delta(\cdot) := [D, \cdot]_\sigma, \quad \delta^0 := [D, (\cdot)^0]_{\sigma^0} \tag{2.13}
\]

which take value in the \(A\)-bimodule of twisted one-forms \(\Omega\) and its opposite

\[
\Omega := \Omega_0^1(A, \sigma) \bigg\{ \sum_j a_j [D, b_j]_\sigma, a_j, b_j \in A \bigg\}, \quad \Omega^0 := \Omega_0^1(A^0, \sigma^0) = \bigg\{ \sum_j a^0_j [D, b^0_j]_{\sigma^0}, a^0_j, b^0_j \in A^0 \bigg\},
\]

where the bimodule structure is, for any \(\omega \in \Omega, \omega^0 \in \Omega^0\) and \(a, b \in A\),

\[
a \cdot \omega \cdot b := \sigma(a) \omega b \quad a \cdot \omega^0 \cdot b := \sigma^0(b^0) \omega^0 a^0. \tag{2.14}
\]

Remark 2.3. One twists the left action of \(A\) in order to have the Leibniz rules

\[
\delta(ab) = \delta(a)b + \sigma(a)\delta(b) = \delta(a) \cdot b + a \cdot \delta(b), \tag{2.15}
\]

\[
\delta^0(ab) = \sigma^0(b^0)\delta^0(a) + \delta^0(b)a^0 = \delta^0(a) \cdot b + a \cdot \delta^0(b) \quad \forall a, b \in A. \tag{2.16}
\]

Unlike usual commutators, these derivations are not anti-hermitian but rather satisfy

\[
\delta(a^*) = Da^* - \sigma(a^*)D = -(D\sigma^{-1}(a) - aD)^* = -[D, \sigma^{-1}(a)]^*_\sigma = -\delta(\sigma^{-1}(a))^*, \tag{2.17}
\]

\[
\delta^0(a^*) = Da^{*0} - \sigma^0(a^{*0})D = (a^0 D - D\sigma^0(a^{*0})^*^* = -[D, a^{*0}]^*_{\sigma^0} = -\delta^0(\sigma(a))^*, \tag{2.18}
\]

where in \(\text{(2.17)}\) we use the regularity \(\text{(2.1)}\) and in \(\text{(2.18)}\) the commutativity \(\circ, \ast\), then \(\sigma^0(a^{*0}) = (\sigma(a^0))^*\) extracted from \(\text{(2.6)}\). These rules, as well as the bimodule laws, do not require the zero nor the first order conditions but rely only on the properties of the twisted commutators: \([D, ab]_\sigma = [D, a]_\sigma b + \sigma(a) [D, b]_\sigma\) and \([D, (ab)^0]_{\sigma^0} = \sigma^0(b^0) [D, a^{*0}]_{\sigma^0} + [D, b^0]_{\sigma^0} a^0\).

The derivations \(\delta, \delta^0\) serve to define connections required to export spectral triples between Morita equivalent algebras. Recall that a \(\Omega\)-valued, resp. \(\Omega^0\)-valued, connection on a right \(A\)-module \(\mathcal{E}\), resp. a left \(A\)-module \(\mathcal{F}\), are \(\mathbb{C}\)-linear maps

\[
\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_A \Omega, \quad \nabla^0 : \mathcal{F} \rightarrow \Omega^0 \otimes_A \mathcal{F} \tag{2.19}
\]

which fulfill the Leibniz rules, for any \(\xi \in \mathcal{E}, \zeta \in \mathcal{F}\) and \(a \in A\),

\[
\nabla(\xi a) = (\nabla\xi)a + \xi \otimes \delta(a), \quad \nabla^0(a\zeta) = a(\nabla^0\zeta) + \delta^0(a) \otimes \zeta, \tag{2.20}
\]

which define connections required to export spectral triples between Morita equivalent algebras. Recall that a \(\Omega\)-valued, resp. \(\Omega^0\)-valued, connection on a right \(A\)-module \(\mathcal{E}\), resp. a left \(A\)-module \(\mathcal{F}\), are \(\mathbb{C}\)-linear maps
where one defines
\[(\nabla \xi) a = \xi(0) \otimes (\xi(1) \cdot a), \quad a(\nabla^o \zeta) := (a, \zeta_{(-1)}) \otimes \zeta(0)\] (2.21)

using the Snyder notations
\[\nabla \xi = \xi(0) \otimes \xi(1), \quad \nabla^o \zeta = \zeta_{(-1)} \otimes \zeta(0)\] (2.22)

with
\[\xi(0) \in \mathcal{E}, \; \xi(1) \in \Omega \quad \text{and} \quad \zeta(0) \in \mathcal{F}, \; \zeta_{(-1)} \in \Omega^o.\] (2.23)

### 2.3 Lift of automorphism

Inner fluctuations consist in exporting a noncommutative geometry from an algebra \(A\) to a Morita equivalent algebra \(B\). In case of twisted geometries, this requires as a preamble to lift the twisting automorphism \(\sigma\) first to the module \(E\) implementing Morita equivalence, then to \(B\).

To this aim, recall that two algebras \(A, B\) are (strongly) Morita equivalent when there exists a full Hilbert \(B\)-\(A\)-module \(E\) (that is \(B\)-left, \(A\)-right), such that the algebra \(\text{End}_A(E)\) of \(A\)-linear, adjointable, endomorphisms of \(E\) is isomorphic to \(B\). If both \(A\) and \(B\) are unital then \(E\) is finitely projective as right \(A\)-module, i.e. there is an idempotent \(e = e^2 = e^* \in M_n(A)\) for some \(n \in \mathbb{N}\) such that
\[E \simeq eA^n.\] (2.24)

Any element of \(E\), viewed as a vector \(\xi = e\xi \in A^n\), has components \(\xi_i = e^i_j \xi_j \in A\) \((i = 1, ..., n)\) with \(e^i_j \in A\) the \(i\)th-line, \(j\)th-column component of the idempotent \(e\) and we use Einstein summation on indices in alternate up/down position. Identifying a vector with its components, we write
\[\xi = (\xi_i) = (e^i_j \xi_j).\] (2.25)

The module \(E\) is hermitian for the \(A\)-valued inner product
\[(\xi', \xi) := \sum_i \xi'^*_i \xi_i.\] (2.26)

An automorphism \(\sigma\) of \(A\) lifts to \(E\) by defining a \(A\)-module morphism \(\Sigma : E \to E\) such that
\[\Sigma(\xi a) := \Sigma(\xi)\sigma(a) \quad \forall \xi \in E, a \in A.\] (2.27)

Explicitly,
\[\Sigma \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} := e \begin{pmatrix} \sigma(\xi_1) \\ \vdots \\ \sigma(\xi_n) \end{pmatrix} = e\sigma(\xi) \quad \forall \xi = e\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}\] (2.28)

where \(\sigma(\xi)\) is a shorthand notation for the vector in \(A^n\) with components \(\sigma(\xi_i)\).

To lift \(\sigma\) to \(B\), recall the later is the subalgebra of the \(n\)-square matrices with entries in \(A\), invariant under the conjugate action of \(e\), namely
\[B \simeq \text{End}_A(E) \simeq eM_n(A)e = \{b \in M_n(A) \mid \text{such that } ebe = b\}.\] (2.29)
It is equipped with an involution: the matrix transpose composed with the $\mathcal A$-involution of every entry. The algebra $\mathcal B$ acts on $\mathcal E$ by left multiplication: with $b_i^j \in \mathcal A$ the components of $b$, one has
\[
  b\xi = e\begin{pmatrix} b_1^1\xi_1 \\ \vdots \\ b_n^\xi_n \end{pmatrix} \quad \forall b \in \mathcal B, \xi \in \mathcal E. \tag{2.30}
\]

The automorphism $\sigma$ extends to $M_n(\mathcal A)$ acting on each entry: for $m \in M_n(\mathcal A)$ with entries $m_i^j \in \mathcal A$, we define $\sigma(m)$ as the matrix with entries $\sigma(m_i^j)$. However this does not define an automorphism of $\mathcal B$, for $b = e\sigma(b)e$ does not guarantee that $\sigma(b)$ equals $e\sigma(b)e$. To lift $\sigma$ to an automorphism of $\mathcal B$, one should first ensure that its lift $\Sigma$ to $\mathcal A$ commutes with the inverse. By this, one intends that the lift to $\mathcal A$ of $\sigma^{-1}$,
\[
  \Sigma^{-1}\xi := e\sigma^{-1}(\xi)
\]
coincides with the inverse of the lift $\Sigma$ (2.28).

**Remark 2.5.** Conditions (2.32) are true if $e = e\sigma(e)$ is invariant under the twist, as assumed in [26]. However, this may not be the only solution.

Assuming that the lift $\Sigma$ to $\mathcal A$ of the twisting automorphism $\sigma$ is invertible in the sense of lemma 2.4 then one is now able to define its lift $\sigma'$ to $\mathcal B$ as
\[
  \sigma'(b) := e\sigma(b)e \quad \forall b = e\sigma(b)e \in \mathcal B. \tag{2.35}
\]

**Proposition 2.6.** $\sigma'$ is an automorphism of $\mathcal B$, with inverse $\sigma'^{-1}(b) = e\sigma^{-1}(b)e$. If $\sigma$ is regular in the sense of [2.1] then $\sigma'$ is regular as well:
\[
  \sigma'(b^*) = \sigma'^{-1}(b^*) \quad \forall b \in \mathcal B. \tag{2.36}
\]

**Proof.** For $a, b \in \mathcal B$ one has one has $eb = b$ and $ae = a$ thus
\[
  \sigma'(a)\sigma'(b) = e\sigma(a)e\sigma(b)e = e\sigma(a)\sigma(e)\sigma(e)\sigma(b)e,
\]
where, to get the second line, we used $\sigma(e)\sigma(e) = \sigma(e)^2$ obtained applying $\sigma$ on (2.32), then using $\sigma(e) = \sigma(e)^2 = \sigma(e)^2$. This shows that $\sigma'$ is an automorphism of $\mathcal B$. That $\sigma'^{-1}$ is its inverse comes from
\[
  \sigma'(\sigma'^{-1}(b)) = e\sigma(e\sigma^{-1}(b)e)e = e\sigma(e)b\sigma(e)e = e\sigma(e)ebe\sigma(e)e = ebe = b
\]
and a similar result for $\sigma'^{-1}(\sigma'(b))$.

For $\sigma$ regular, the matrix $\sigma(b^*)$ has components $\sigma(b^*)_{ij} = \sigma(b_{ij}^*) = (\sigma^{-1}(b_{ij}))^*$, which is the component $ij$ of $(\sigma^{-1}(b))^*$. Hence
\[
  \sigma'(b^*) = e\sigma(b^*)e = e(\sigma^{-1}(b))^*e = (e\sigma^{-1}(b)e)^* = (\sigma'^{-1}(b))^*. \quad \square
\]
2.4 Twisted hermitian connection

The connections $\nabla$ relevant for inner fluctuations are the hermitian ones, that is those compatible with the inner product of $\mathcal{E}$ in that [7 Chap.6, Def.10]

$$(\xi', \nabla \xi) - (\nabla \xi', \xi) = [D, (\xi', \xi)]$$  \hspace{1cm} (2.39)

where

$$(\nabla \xi, \xi') = \xi_{(1)}(\xi_{(0)}), \xi' \quad \quad (\xi, \nabla \xi') = (\xi, \xi_{(0)}) \xi_{(1)}'. \hspace{1cm} (2.40)$$

As explained in [8], the minus sign in (2.39) is because $[D, a^*] = -[D, a]^*$, and guarantees that any such connection is the sum of the Grassmann connection with a selfadjoint element of $\Omega^1_D(\mathcal{A})$.

For a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D), \sigma$, the derivations $\delta$ is not anti-hermitian but rather satisfy (2.17). In addition, one needs to modify (2.40) to take into account the bimodule structure, this means

$$(a \cdot \omega)^* = (\sigma(a) \omega)^* = \omega^* \sigma(a)^* = \omega^* \cdot \sigma(a)^*, \hspace{1cm} (2.42)$$

$$(\omega \cdot a)^* = (\omega a)^* = a^* \omega^* = \sigma^{-1}(a^*) \cdot \omega^* = \sigma(a)^* \cdot \omega^*. \hspace{1cm} (2.43)$$

Notice these laws are compatible since $\sigma (\sigma(a)^*)^* = (\sigma^{-1}(\sigma(a))^*)^* = (a^*)^* = a$.

We look for a definition of a $\Omega$-hermitian connection which guarantees the same properties as in the non-twisted case, namely that any such connection is the sum of the Grassmann connection

$$\nabla_0 \xi := \begin{pmatrix} e_1^X \\ \vdots \\ e_n^X \end{pmatrix} \otimes \delta(\xi_j) \simeq e \cdot \begin{pmatrix} \delta(\xi_1) \\ \vdots \\ \delta(\xi_n) \end{pmatrix} = e \cdot \delta(\xi) \quad \forall \xi = e \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathcal{E}, \hspace{1cm} (2.44)$$

with a selfadjoint element of $M_n(\Omega)$. Note that the second equality in (2.44) is the identification of $\mathcal{E} \otimes \Omega$ with $e \cdot \Omega^n$ (beware the matrix $e$ acts by the module law 2.14 $e$-actually is the usual matrix multiplication by $\sigma(e)$), while the last one is a shorthand notation with $\delta(\xi)$ the vector of $\Omega^n$ with components $\delta(\xi_i) \in \Omega$.

**Definition 2.7.** An $\Omega$-connection $\nabla$ on an hermitian right $\mathcal{A}$-module $\mathcal{E}$, with lift $\Sigma$ invertible in the sense of lemma 2.4 is hermitian if

$$(\xi', \nabla \xi) - (\nabla(\Sigma^{-1} \xi'), \xi) = \delta((\xi', \xi)) \quad \forall \xi, \xi' \in \mathcal{E}. \hspace{1cm} (2.45)$$

As long as the idempotent $e$ is twist-invariant or twist-commutes componentwise with $D$,

$$\sigma(e) = e \quad \text{or} \quad \delta(e_i^X) = 0 \quad \forall i, j = 1, \ldots, n, \hspace{1cm} (2.46)$$

then definition 2.7 is precisely the one guaranteeing similar properties as in the non-twisted case.

**Lemma 2.8.** Assuming (2.46), the Grassmann connection $\nabla_0$ is hermitian. Furthermore, any hermitian connection is of the form $\nabla = \nabla_0 + M$ where $M$ is a selfadjoint element of $M_n(\Omega)$. 


Proof. By (2.31), \(\Sigma^{-1}\xi'\) has components \(S'_j = e^j_k \sigma^{-1}(\xi'_k)\) such that \(e^j_i S'_j = S'_i\). Therefore (2.44) yields
\[
\nabla_0(\Sigma^{-1}\xi') = (e^j_i) \otimes \delta(S'_k)
\]
(2.47)
where \((e^j_i)\) denotes the element \(\xi^j_k \in \mathcal{E}\) with components \((\xi^j_k)_i = e^j_i\). If \(e = \sigma(e)\), then \(\delta(S'_j) = \delta(\sigma^{-1}(e^j_k \xi'_k)) = \delta(\sigma^{-1}(\xi'_j))\). Otherwise \(e\) twist commutes with \(D\) so that \(\delta(S'_j) = e^j_k \cdot \delta(\sigma^{-1}(\xi'_k))\). In any case,
\[
\nabla_0(\Sigma^{-1}\xi') = (e^j_i) \otimes \delta(\sigma^{-1}(\xi'_j)) = e \cdot \delta(\sigma^{-1}(\xi')).
\]
(2.48)
The product (2.40) yields
\[
(\xi', \nabla_0 \xi) = (\xi', (e^j_i)) \cdot \delta(\xi_j) = \left(\sum_i \xi^i_j \overline{e^j_i}\right) \cdot \delta(\xi_j) = \left(\sum_i (e^j_i \xi'_i)^*\right) \cdot \delta(\xi_j) = \sum_j \xi^i_j \overline{\delta(\xi_j)},
\]
(2.49)
using (2.17) for the last equality. Since \(\delta((\xi', \xi)) = \delta(\sum_i \xi^i_j \xi'_i) = \sum_i \xi^i_j \overline{\delta(\xi_i)} + \delta(\xi'_i) \cdot \xi_i\), one has that \(\nabla_0\) satisfies (2.45), hence is hermitian.

From the Leibniz rule (2.20), the difference \(\nabla := \nabla - \nabla_0\) of the two connections is \(\mathcal{A}\)-linear,
\[
\nabla(\xi a) = (\nabla_0 \xi)a - (\nabla \xi)a = \nabla(\xi)a,
\]
(2.50)
meaning that \(\nabla\) is an \(\mathcal{A}\)-linear endomorphism from \(\mathcal{E}\) to \(\mathcal{E} \otimes \Omega\), that is an element of \(M_n(\mathcal{A}) \otimes_A \Omega \simeq M_n(\Omega)\) invariant by the conjugate action of \(e\). More precisely, there exists a matrix \(M \in M_n(\Omega)\) with entries \(m^j_i \in \Omega\), such that \(e \cdot M \cdot e = M\) and
\[
\nabla_\xi = (e^j_i) \otimes (m^j_k \cdot \xi_k) \simeq M \cdot \xi,
\]
(2.51)
\[
\nabla(\Sigma^{-1}\xi') = (e^j_i) \otimes (m^j_k \cdot \sigma^{-1}(\xi'_k)) \simeq M \cdot \Sigma^{-1}(\xi')
\]
Being both \(\nabla_0\) and \(\nabla\) hermitian, one has
\[
0 = (\xi' \cdot \nabla \xi) - \left(\nabla(\Sigma^{-1}\xi')\right),
\]
(2.52)
\[
= \sum_j \xi^i_j \overline{\delta(\xi_j)} + \left(\sum_j \xi^i_j \overline{m^j_k \cdot \xi_k} - \left(\sum_j \xi^i_j \overline{m^j_k \cdot \sigma^{-1}(\xi'_k)} \right) \cdot \xi_j = \sum_j \xi^i_j \overline{m^j_k \cdot \xi_k - \xi^i_j \overline{m^j_k \cdot \xi'_k}} \right)
\]
(2.53)
where, for the last equality, we use (2.43) as \((m^j_k \cdot \sigma^{-1}(\xi'_k))^* = \xi^i_j \overline{m^j_k \cdot \xi'_k}\), then exchange the indices \(j\) and \(k\). Being (2.53) true for any \(\xi, \xi'\), one obtains \(m^j_k = (m^j_k)^*\), meaning the matrix \(M\) is selfadjoint.

All the results of (2.3) and (2.4) makes sense with minimal modifications for left \(\mathcal{A}\)-module and \(\Omega^*\)-connections, as shown in the appendix A.1. This is important later, in order to export a real twisted geometry to a Morita equivalent algebra.

**Remark 2.9.** Here, we have absorbed the twist in the bimodule laws (2.14), and modified accordingly the definition of hermitian connections. An alternative - which should be equivalent - consists in letting the module law untouched and twist the connection, as done in (2.9).
3 Twisted fluctuation with a non-linear term

Inner fluctuations follow from self-Morita equivalence and have been adapted to the twisted case in [26]. We extend these results to a wider class of Morita equivalence, namely the one implemented by a bimodule which satisfies (2.33), (2.46), following what has been done in [6] for the non-twisted case. We begin with twisted spectral triples in §3.1, then take into account the real structure in §3.2. In §3.3 we go back to self-Morita equivalence and show how the removal of the first-order condition yields an extra non-linear term in the fluctuation, similar as the one worked out in [6].

3.1 Morita equivalent twisted geometries

We first recall inner fluctuations of a usual (i.e. non twisted) geometry \((A, \mathcal{H}, D)\). Take \(B = \text{End}_A(\mathcal{E})\) for a hermitian right \(A\)-module \(\mathcal{E}\) with inner product (2.26). Then

\[
\mathcal{H}_R := \mathcal{E} \otimes_A \mathcal{H}
\]

is a (pre)-Hilbert space for the product (denoting \(\langle , \rangle_{\mathcal{H}}\) the inner product on \(\mathcal{H}\)):

\[
\langle \xi' \otimes \psi', \xi \otimes \psi \rangle := \langle \psi', (\xi', \xi)\psi \rangle_{\mathcal{H}}.
\]

(3.1)

Its completion, still denoted \(\mathcal{H}_R\), carries both a representation of the algebra \(B\)

\[
\pi_R(b)(\xi \otimes \psi) := b\xi \otimes \psi \quad \forall b \in B, \xi \in \mathcal{E}, \psi \in \mathcal{H},
\]

(3.3)

and an action of the operator \(D\) as

\[
(\mathbb{1} \otimes \nabla D)(\xi \otimes \psi) := \xi \otimes D\psi + (\nabla\xi)\psi,
\]

(3.4)

with \(\mathbb{1}\) the identity endomorphism on \(\mathcal{E}\), \(\nabla\) a \(\Omega^1_B(A)\)-valued connection on \(\mathcal{E}\), and

\[
(\nabla\xi)\psi := \xi_{(0)} \otimes \xi_{(1)}\psi
\]

(3.5)

(using (2.22)) where the action of \(\xi_{(1)} \in \Omega^1_B(A)\) on \(\mathcal{H}\) follows from the representation of \(a_j\) and \([D, b_j]\) as bounded operators). Then \((B, \mathcal{H}_R, 1 \otimes \nabla D)\) is a spectral triple ([10, §10.8], see also [6]).

The construction is similar for a twisted spectral triple \((A, \mathcal{H}, D), \sigma\), provided \(\sigma\) satisfies the compatibility conditions (2.46) with the idempotent and its lift \(\Sigma\) to \(\mathcal{E}\) defined (2.35) is invertible. Make \(B\) act on \(\mathcal{H}_R\) as in (3.3), but instead of (3.4) consider an \(\Omega\)-connection \(\nabla\) and define

\[
D_R := (\Sigma \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \nabla D).
\]

(3.6)

This is a well densely-defined operator on \(\mathcal{H}_R\) [26, Prop.3.5].

**Proposition 3.1.** Assume the lift \(\Sigma\) is invertible in the sense of lemma [2.4] and that (2.46) holds. Then \(\left[D_R, \pi_R(b)\right]_\sigma\), with \(\sigma'\) the lift (2.35) of \(\sigma\) to \(B\) is bounded for any \(b \in B\).

**Proof.** With \(\xi^p = e\xi^p\) and \(\psi_p\) generic elements of \(\mathcal{E}\) and \(\mathcal{H}\), a generic element of \(\mathcal{H}_R\) is

\[
\Psi = \xi^p \otimes \psi_p = (e_j^1)\xi^p \otimes \psi_p = (e_j^1) \otimes \psi_j \simeq \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = e\Psi
\]

(3.7)

where \(e_j^1\) are the components of \(\xi^p\) and \(\psi_j := \xi_j^p \psi_p \in \mathcal{H}\). The former last equality is the identification \(\mathcal{E} \otimes_A \mathcal{H} \sim e\mathcal{H}^n\). Denoting \(\tilde{\nabla}\) the difference of \(\nabla\) with the Grassmann connection (2.44), then for \(\Psi\) with components \(\psi_j\) in \(\text{Dom}(D)\), eq. (3.4) yields

\[
(\mathbb{1} \otimes \nabla D)\Psi = (e_j^1) \otimes D\psi_j + \tilde{\nabla}_0(e_j^1)\psi_j + \tilde{\nabla}(e_j^1)\psi_j,
\]

(3.8)
By (2.44), $\nabla_0(e_i^j) = (e_i^k) \otimes \delta(e_k^j)$ is zero in case $e$ twist-commutes with $D$. Otherwise by (3.5)

$$\langle \nabla_0(e_i^j) \rangle_{\sigma_j} = (e_i^k) \otimes \delta(e_k^j) \psi_j \simeq e\delta(e)\Psi$$

with $\delta(e) \in M_n(\Omega)$ with components $\delta(e)^j_i := \delta(e_i^j)$. Being $e$ twist-invariant, (2.15) gives

$$\delta(e^j_i) = \delta \left( e_i^k e_k^j \right) = \delta(e_i^k) \delta(e_k^j) + \delta(e_i^k) \delta(e_k^j) = e_i^k \delta(e_k^j) + \delta(e_i^k) \delta(e_k^j),$$

that is $\delta(e) = e \delta(e) + \delta(e)e$. Multiplying on the right by $e$ shows that $e \delta(e)e = 0$, that is (3.9) is zero. So in any case, the second term in the r.h.s. of (3.8) vanishes.

The third term is, by (2.50),

$$\frac{\nabla(e_i^j)) \psi_j}{\psi_j} = (e_i^k) \otimes (m_k^j \cdot e_j^i) \psi_j \simeq eM\Psi$$

Applying $\Sigma \otimes I$ to (3.8) (with $D\Psi \in eH^n$ the action of $D$ on each components of $\Psi$) one gets

$$D_R\Psi = e(\sigma(e_i^j)) \otimes (D\psi_j + e(\sigma(e_i^j)) \otimes m_k^j \psi_j) \simeq e\sigma(e)(D + M)\Psi.$$  

(3.11)

If $e$ is twist-invariant, this becomes

$$D_R\Psi = (e_i^k) \otimes \left( D\psi_k + m_k^j \psi_j \right) \simeq e(D + M)\Psi.$$  

(3.12)

The same is true if $e$ twist-commutes with $D$ since then $e\sigma(e)D\Psi = eD(c)e = eD\Psi$ whereas $\sigma(e)M = e.M = M$ by definition of $M$.

Consider now $b = ebe$ in $B$ with components $b_i^j \in A$. From (3.3) and (3.7) one has

$$\pi_R(b)\Psi = b\pi^p \otimes \psi_p = (e_i^j) \otimes b_j^k \pi^p \psi_p = (e_i^j) \otimes b_j^k \pi^p \psi_p \simeq b\Psi$$

where we use $((b\pi^p)_i^j) = (e_i^j) \otimes b_j^k \pi^p \simeq (e_i^j) \otimes b_j^k \pi^p$. Thus (3.12) gives

$$D_R \pi_R(b)\Psi = e(D + M)b\Psi, \quad \pi_R(\sigma'(b))D_R\Psi = e\sigma(b)\sigma(e)(D + M)\Psi = e\sigma(b)(D + M)\Psi,$$

where for the second equation we used (2.35) together with (3.11), then (2.32) as $\sigma(b)\sigma(e) = \sigma(e)\sigma(b)\sigma(e)\sigma(e) = \sigma(e)\sigma(b) = \sigma(b)$. Hence

$$[D_R, \pi_R(b)]_{\sigma} \Psi \simeq e[D + M, b]_{\sigma} \Psi.$$  

(3.14)

Since $[D, b]_{\sigma}$ is a matrix with components $[D, b]_{\sigma}^j_i$, bounded by hypothesis, while $[M, b]_{\sigma}$ is bounded being both $M$ and $b$ bounded, then (3.14) is bounded.

Proposition 3.1 is not sufficient to build a twisted spectral triple, for there is no guarantee that $D_R$ is a selfadjoint. This happens however if one restricts to hermitian connections.

**Proposition 3.2.** In the conditions of Prop. 3.1 and with $\nabla$ hermitian in the sense of Def. 2.7, then $(B, H_R, D_R, \sigma')$ with $B$ acting on $H_R$ as in (3.3) is a twisted spectral triple.

**Proof.** In view of props 2.6 and 3.1 the only point to check is that $D_R$ is selfadjoint with compact resolvent. The operator (3.12) coincides with the operator (7) in [6], which is shown to be part of a spectral triple, hence selfadjoint with compact resolvent. For completeness we develop the proof here. For $\Psi' = (e_i^j) \otimes \psi_j'$ with $\psi_j' \in \text{Dom } D$, (3.2) yields

$$\langle \Psi', D_R\Psi \rangle = \langle \psi_j', (e_i^j)(D\psi_k + m_k^j \psi_j) \rangle_{\sigma_j} = \sum_k \langle e_i^j \psi_j', D\psi_k + m_k^j \psi_j \rangle_{\sigma_j} = \sum_k \langle \psi_k', D\psi_k + m_k^j \psi_j \rangle_{\sigma_j},$$

(3.15)

$$\langle (D_R\Psi', \Psi) = \sum_j \langle D\psi_j' + m_j^j \psi_j', e_j^i \psi_j \rangle_{\sigma_j} = \sum_j \langle D\psi_j' + m_j^j \psi_j', \psi_j \rangle_{\sigma_j},$$

(3.16)
where we compute the $A$-valued inner product (2.26) (remembering $e_i^* e_i = e_k^* e_k = e_i^* e_k = e^* e_i$).

$$\left( e_i^s, e_i^t \right) = \sum_i e_i^* e_i^s e_i^* e_i^t = e_i^* e_i^t = e^* e_i.$$

(3.17)

$D$ and $m_k^l$ being selfadjoint, (3.15) equals (3.16), meaning $D_R$ is symmetric. Furthermore, $\text{Ran}(D + m_k^l) = \mathcal{H}$ [30] Theo. 8.3, so $\text{Ran}(D_R) = \mathcal{H}_R$, hence $D_R$ is selfadjoint.

Regarding the compact resolvent, let us denote $D_R = D_0$ in case $\nabla$ is the Grassmann connection. For $\lambda$ in the resolvent set of $D$, $D_0 - \lambda I$ is invertible with inverse $\left( (D - \lambda I)^{-1} \psi_1 \\
\vdots \\
(D - \lambda I)^{-1} \psi_n \right) = e \left( (D - \lambda I)^{-1} \psi_1 \\
\vdots \\
(D - \lambda I)^{-1} \psi_n \right).$

(3.18)

$e$ being the identity on $\mathcal{H}_R$ and a finite direct sum of compact operators being compact, $(D_0 - \lambda I)^{-1}$ is compact. The passage to an arbitrary hermitian connection $\nabla$ is similar as for [31, Thm. 6.15], having in mind Prop. 2.8 which guarantees that the difference $\nabla - \nabla_0$ is similar as for the non-twisted case.

3.2 Morita equivalence for real twisted geometries

Provided the initial twisted spectral triple is real, then the previous construction holds as well if the Morita equivalence between $A$ and $B$ is implemented by a left $A$-module (details are in A.2). In particular, instead of the right-Morita equivalent triple of proposition 3.2, one may built a left-Morita equivalent triple using the $A$-$B$-module $\bar{E}$ conjugate to $E$ defined in (A.40). The latter is hermitian for the $A$-valued pairing

$$\{ \xi', \xi \} := (\xi', \xi) \forall \xi, \xi' \in E,$$

(3.19)

and the Hilbert space $\mathcal{H}_L := \mathcal{H} \otimes_A \bar{E}$ carries the representation (A.20) of $B$

$$\pi_L(b)(\Psi \otimes \bar{\xi}) := \Psi \otimes \bar{\xi} b \forall b \in B, \xi \in \bar{E}, \Psi \in \mathcal{H}_R.$$

(3.20)

Consider an hermitian $\Omega^o$-connection on $\bar{E}$, for instance the conjugate $\bar{\nabla}$ of an hermitian connection $\nabla$ on $E$ defined in lemma A.4. Assuming the idempotent $e$ satisfies the conditions of proposition 3.2, one makes the operator $D$ act on $\mathcal{H}_L$ as $D_L$ in (A.23). Then proposition A.3 shows that $(B, \mathcal{H}_L, D_L), \Sigma^{o-1}$ is a twisted spectral triple.

However, the real structure $J$ of the initial triple has no reason to be a real structure, neither for this left-Morita equivalent nor for the right-Morita one of Prop. 3.2. Actually this is not even true for self-Morita equivalence [Lem.3.7][26]). So, to export a real twisted spectral triple one needs to combines the two previous constructions, following what has been done for usual spectral triples in [8] (later explained in greater details in [10, 3]).

Explicitly, given a real, graded, twisted spectral triple

$$(A, \mathcal{H}, D), \sigma, \Gamma, J,$$

(3.21)

one considers the Hilbert space

$$\mathcal{H}' := \mathcal{H}_R \otimes_A \bar{E}$$

(3.22)

for $\mathcal{H}_R$ in (3.1). The tensor product makes sense with respect to the right action of $A$ on $\mathcal{H}_R$

$$(\xi \otimes \psi)a = \xi \otimes \psi a$$

(3.23)
well defined by the order zero condition of (3.21) (see (3.26) below). On \( \mathcal{H}' \), one makes the Dirac operator \( D \) act as the operator

\[
D' := (\mathbb{1} \otimes \Sigma^{-1}) \circ \left( D_R \otimes \nabla_R 1 \right)
\]

(3.24)

with \( D_R \) given in (3.6), \( \nabla \) an \( \Omega \)-valued hermitian connection on \( \mathcal{E} \), \( \Sigma \) the lift (A.5) of \( \sigma \) to \( \mathcal{E} \), and \( \nabla_R \) an hermitian connection on \( \mathcal{E} \) with value in the bimodule

\[
\Omega^0_R := \Omega^0_{D_R}(A^0, \sigma^0) = \left\{ \sum_j \hat{\pi}^0(a_j^0) \left[ D_R, \hat{\pi}^0(b_j^0) \right]_{\sigma^0}, a_j^0, b_j^0 \in A^0 \right\}
\]

(3.25)

generated by the derivation \( \delta^0_R(a) := [D_R, \hat{\pi}^0(a^0)] \), where the action of \( A^0 \) on \( \mathcal{H}_R \) is

\[
\hat{\pi}^0(a)(\xi \otimes \psi) := \xi \otimes a^0 \psi \quad \forall a \in A, \xi \otimes \psi \in \mathcal{H}.
\]

(3.26)

This action coincides with the right action (3.23) and is well defined, for

\[
\hat{\pi}^0(a^0)(\xi' \otimes \psi) = \xi \otimes a^0 \xi' \psi = \hat{\pi}^0(a^0)(\xi \otimes a' \psi) \quad \forall a' \in A.
\]

(3.27)

by the order zero condition for (3.21).

In order for \( D' \) to make sense, one has to make sure that \( \Omega^0_R \) acts on \( \mathcal{H}_R \) as bounded operators. To this aim, let us denote \( \tilde{R} \) the bimodule morphism \( \Omega^0 \to \Omega^0_R \)

\[
R(\omega^0) := \sum_j \hat{\pi}(a_j^0) \delta^0_R(b_j) \in \Omega^0_R \quad \forall \omega^0 = \sum_j a_j^0 \delta^0(b_j) \in \Omega^0 \n\]

(3.28)

(\text{one shows this is a morphism by considerations as in remark (2.3).})

**Lemma 3.3.** For any \( a \in A \), \( \delta^0_R(a) \) is a bounded operator on \( \mathcal{H}_R \) and acts as \( \Sigma \otimes \delta^0(a) \).

Any element of \( \Omega^0_R \) is of the form \( R(\omega^0) \) for some \( \omega^0 \in \Omega^0 \), and acts on \( \mathcal{H}_R \) as

\[
R(\omega^0)\Psi = \Sigma \xi \otimes \omega^0 \psi \quad \forall \Psi = \xi \otimes \psi \in \mathcal{H}_R.
\]

(3.29)

**Proof.** From (3.4) one has

\[
[\mathbb{1} \otimes \nabla D, \hat{\pi}^0(a^0)]_{\sigma^0}(\xi \otimes \psi) = (\mathbb{1} \otimes \nabla D)(\xi \otimes a^0 \psi) - \hat{\pi}^0(\sigma^0(a^0)) (\mathbb{1} \otimes \nabla D)(\xi \otimes \psi),
\]

(3.30)

\[
= \xi \otimes D\omega \psi + \nabla(\xi) a^0 \psi - \xi \otimes \sigma^0(a^0)D \psi - \hat{\pi}(\sigma^0(a^0)) \nabla(\xi) \psi = \xi \otimes [D, a^0]_{\sigma^0} \psi,
\]

where we noticed that

\[
\nabla(\xi) a^0 \psi - \hat{\pi}(\sigma^0(a^0)) \nabla(\xi) \psi = \xi(0) \otimes \xi(1) a^0 \psi - \xi(0) \otimes \sigma^0(a^0) \xi(1) \psi = \xi(0) \otimes \xi(1), [\xi(1), a^0]_{\sigma^0} \psi \quad (3.31)
\]

vanishes by the first order condition satisfied by (3.21). Thus (3.6) yields

\[
[D_R, \hat{\pi}^0(a^0)]_{\sigma^0}(\xi \otimes \psi) = \Sigma(\xi) \otimes [D, a^0]_{\sigma^0} \psi,
\]

(3.32)

which is bounded, being \( [D, a^0]_{\sigma^0} \) bounded by definition of the triple (3.21). Notice that this action is well defined thanks to the twisted-first order condition, rewritten as \( [[D, a^0]_{\sigma^0}, a']_{\sigma} = 0 \) (see [25, Def.2.1]).

\( \square \)

In the language of [6], \( \omega^0 \) and \( R(\omega^0) \) are representations of the same universal 1-form: on \( \mathcal{H} \) using the twisted commutator with \( D \), on \( \mathcal{H}_R \) using the one with \( D_R \).

Given an \( \Omega \)-hermitian connection \( \nabla \) on \( \mathcal{E} \), we denote

\[
\nabla_R = (R \otimes \mathbb{1}) \circ \nabla
\]

(3.33)

the \( \Omega^0_R \)-connection on \( \tilde{\mathcal{E}} \) defined, for any \( \tilde{\eta} \in \tilde{\mathcal{E}} \) with \( \nabla \tilde{\eta} = \eta(0) \otimes \eta(1) \), as

\[
\nabla_R \tilde{\eta} = R(\tilde{\eta}(-1)) \otimes \tilde{\eta}(0)
\]

(3.34)

where \( \tilde{\eta}(-1) = e^J_1 \eta(1) J^{-1} \in \Omega^0 \) is defined in (A.45). This is an hermitian connection (one checks (A.11) using \( R \) is a bimodule morphism). This permits to conclude the construction of twisted fluctuations of real twisted spectral triples.
Proposition 3.4. Consider a real, graded twisted spectral triple \( [3.21] \) and a finite projective right \( A \)-module \( \mathcal{E} = eA^\nu \) such that the lift \( \Sigma \) of \( \sigma \) is invertible in the sense of lemma \( [2.4] \) and \( [2.46] \) holds. Let \( \mathcal{B} = \text{End}_A(\mathcal{E}) \) act on \( \mathcal{H}' \) as

\[
\pi' := \pi_R \otimes \mathbb{I}
\]  

(3.35)

with \( \pi_R \) defined in \( [3.3] \) and \( \sigma' \) the lift \( [2.36] \) of \( \sigma \) to \( \mathcal{B} \). Given an \( \Omega \)-connection \( \nabla \) on \( \mathcal{E} \), define \( D' \) as in \( [3.24] \) with \( \nabla_R = \nabla_R \) given in \( [3.34] \). Then

\[
(\mathcal{B}, \mathcal{H}', D'), \ \sigma'
\]  

(3.36)

is a real, graded, twisted spectral triple with grading and real structure

\[
\Gamma'(\xi \otimes \psi \otimes \bar{\eta}) := \xi \otimes \Gamma \psi \otimes \bar{\eta},
\]  

(3.37)

\[
J'(\xi \otimes \psi \otimes \bar{\eta}) := \eta \otimes J\psi \otimes \bar{\xi}, \quad \forall \xi \otimes \psi \in \mathcal{H}_R, \bar{\eta} \in \bar{\mathcal{E}}
\]  

(3.38)

and the same \( K \)-dimension as \( [3.21] \).

Proof. For \( \Psi' = \Psi \otimes \bar{\eta} \in \mathcal{H}' \) with \( \Psi \in \mathcal{H}_R \) and \( \bar{\eta} \in \bar{\mathcal{E}} \), one gets from \( [A.21], [A.22], [A.44] \)

\[
D'\Psi' = D_R\Psi \otimes \Sigma^{-1}\eta + (\mathbb{I} \otimes \Sigma^{-1}) \circ \nabla_R(\bar{\eta})\Psi = D_R\Psi \otimes \Sigma\eta + R(\bar{\eta}_{(-1)})\Psi \otimes \Sigma\eta_{(0)},
\]  

(3.39)

so that

\[
[D', \pi'(b)]_{\sigma'} \Psi' = [D_R, \pi_R(b)]_{\sigma'} \Psi \otimes \Sigma\eta + [R(\bar{\eta}_{(-1)}), \pi_R(b)]_{\sigma'} \Psi \otimes \Sigma\eta_{(0)}.
\]  

(3.40)

The first term is bounded by Prop\( [3.1] \), the second because \( \pi_R(b) \) is bounded, as well as \( R(\bar{\eta}_{(-1)}) \) by lemma \( [3.3] \). That \( D' \) is selfadjoint with compact resolvent is shown as in the proof of Prop\( [3.2] \). The operator \( \Gamma' \) is well defined (\( \Gamma'(a \otimes \psi \otimes \bar{\eta}) = \Gamma'(a \otimes \psi \otimes \bar{\eta}) \) for \( \Gamma \) commutes with \( A \)). In addition, \( \Gamma'^2 = \mathbb{I} \) and \( [\Gamma', \pi'(a)] = 0 \). Since \( \Gamma \) anticommutes with both \( \Omega \) and \( \Omega^o \), then \( \mathbb{I} \otimes \Gamma \) anticommutes with \( D_R \) and \( \bar{\eta}_{(-1)} \), thus \( \Gamma' \) anticommutes with \( D \). In other terms \( (\mathcal{B}, \mathcal{H}', D'), \sigma' \) is a graded twisted spectral triple.

To show that it is real, first note that \( J' \) is well defined on \( \mathcal{H}' \), for \( [A.40] \) yields

\[
J'(a \otimes \psi \otimes \bar{\eta}) = \eta \otimes J\psi \otimes \bar{\xi} = \eta \otimes (J\psi)a^* \otimes \bar{\eta} = \eta \otimes J\psi \otimes \bar{\xi} = J'(a \otimes \psi \otimes \bar{\eta}),
\]  

\[
J'(a \otimes \psi \otimes \bar{\eta}) = J'(a \otimes \psi \otimes \bar{\eta}) = \eta \otimes a^* \otimes \bar{\psi} \otimes \xi = \eta \otimes J\psi \otimes \bar{\xi}.
\]  

It induces a representation of the opposite algebra \( B^o \), following \( [2.4] \),

\[
\pi^o(c^o)\Psi' = J'\pi'(c)J'\Psi' = \epsilon J'\pi'(c^o)(a \otimes J\psi \otimes \bar{\xi}) = \epsilon J'(c^o \otimes J\psi \otimes \bar{\xi}) = \xi \otimes \psi \otimes \bar{\eta}c
\]  

(3.41)

for any \( c \in \mathcal{B} \) and \( \Psi' = \xi \otimes \psi \otimes \bar{\eta} \in \mathcal{H}' \), where we have used \( J'^{-1} = \epsilon J' \) as well as

\[
\bar{\eta}c = \begin{pmatrix} (c^o)^1 \bar{\eta}_1 \\ \vdots \\ (c^o)^m \bar{\eta}_m \end{pmatrix} = (\bar{\eta}_j^*((c^o)^1) = \cdots, \bar{\eta}_j^*((c^o)^m))^* = (\eta_j^c)(c_j^c) = \eta_j^c.
\]  

(3.42)

For the order zero condition, it is convenient to identify \( \mathcal{H}' \) with \( \epsilon M_n(\mathcal{H})e \) (\( n \)-square matrices on \( \mathcal{H} \), invariant by conjugation with \( e \)). Indeed, a generic element of \( \mathcal{H}' \) is

\[
\Psi' = \xi^p \otimes \psi^q \otimes \bar{\eta}q = (c^k) \otimes \psi^q \otimes (c^l) \approx \begin{pmatrix} \psi^1 \cdots \psi^q \\ \vdots \\ \psi^m \cdots \psi^m \end{pmatrix} = e\Psi'e
\]  

(3.43)
where $\xi^i_j = e^k_j \xi^k_i$ and $\bar{\eta}_j = \bar{\eta}^l_k \xi^l_j$ are the components of generic elements $\xi^i_j \in \mathcal{E}$, $\bar{\eta}_j \in \bar{\mathcal{E}}$, $\psi^d$ is a generic element of $\mathcal{H}$ and we denote $\psi^d_k = e^p_k \psi^d_p \bar{\eta}_q \in \mathcal{H}$ (unambiguously defined by the order zero condition of (3.21)). The action (3.38) of $J'$ then amounts to acting with $J$ on each components of the transpose of $\Psi'$: from (A.43) one has

$$J' \Psi' = \sum_{k,l} (e^l_j) \otimes J \psi^l_k \otimes (e^k_j) = (e^l_j) \otimes J^T(\psi^l_k) \otimes (e^k_j) \simeq e(J^T \Psi')e$$

(3.44)

where $J$ is the $n$-diagonal matrix with $J$ on the diagonal. Meanwhile, the action of $\pi'(b)$, $\pi^0(c)$ are the left and right matrix multiplications

$$\pi'(b) \Psi' = (e^k_j) \otimes b^l_k \psi^l_j \otimes (e^k_j) \simeq b \Psi'$$

(3.45)

$$\pi^0(c) \Psi' = (e^l_j) \otimes \psi^m_l \otimes (e^k_j) c = (e^k_i) \otimes \psi^m_k \sigma^l_i \otimes (e^k_j) \simeq \Psi' c$$

(3.46)

where the first equation comes from (3.13), while for the second we use $ec = ce$ as $(e^l_j) c = (e^m_l) c_m = (e^m_k) c_m = (c^{m}_l) e_m$ then exchange $l$ with $m$. The order zero condition of 3.21 guarantees that the $i,j$ component $e^k_i(b^l_k \psi^m_j)\psi^j_l$ of $\pi'(b)\Psi'$ equals the one $e^k_i(b^l_k \psi^m_j)\psi^j_l$ of $\pi^0(c)\pi'(b)\Psi'$ for any $b, c \in B$. This means that 3.36 satisfies the order zero condition $[\pi'(b), \pi^0(c)] = 0$.

Regarding the condition of order 1, given a generic $\Psi \in \mathcal{H}$ by (3.44), the first term on the r.h.s. of (3.40) is - denoting $X^i_{\pi} : = \left[ D, b^l_k \right]_l + \left[ m, b^l_k \right]_m$ and using (3.12) and (3.13) -

$$X \Psi' = [D, \pi_R(\Psi)]_l \Psi \otimes \sigma^{-1}(e^k_j) = (e^k_j) \otimes X^i_{\pi} \psi^l_j \otimes (\sigma^{-1}(e^k_j))c.$$

(3.47)

Together with (3.46) this gives

$$X \pi^0(c) \Psi' = (e^k_j) \otimes \left( X^i_{\pi} \psi^m_j \right) \otimes (\sigma^{-1}(e^k_j))c,$$

(3.48)

$$\pi^0(c) \pi' X \Psi' = (e^k_j) \otimes \left( X^i_{\pi} \psi^m_j \sigma^{-1}(e^k_j) \right) \otimes (\sigma^{-1}(e^k_j))c$$

(3.49)

where to get (3.49) we multiply $X \Psi'$ on the left by $\pi^0(c') \pi^0(c) = \pi^0(c', c)$, using

$$(\sigma^{-1}(e^k_j))c \sigma^{-1}(c) = (\sigma^{-1}(e^k_j))c \sigma^{-1}(c) = (\sigma^{-1}(e^k_j) \sigma(c^d_j) e^k_j) = (\sigma^{-1}(e^k_j) \sigma(c^d_j) e^m_k c_m \psi^l_j)$$

(3.50)

$$= (\sigma^{-1}(e^k_j) c_m \psi^l_j) = (\sigma^{-1}(e^m_l) c_m \psi^l_j) = (\sigma^{-1}(e^m_l) c_m \psi^l_j)$$

(3.51)

following from the definition (2.35) of $\sigma'$ together with $e = ec = ce$ and (2.32), then exchanging $l$ with $m$. The twisted-first order condition from the initial triple guarantees that $X^i_{\pi} \psi^m_j \sigma^{-1}(c_m) = X^i_{\pi} \psi^m_j \sigma^{-1}(c_m)$ and $X^i_{\pi} \psi^m_j \sigma^{-1}(c_m)$ for the component $[D, b^l_k]_l$ of $X^i_{\pi}$ this is precisely the order one condition, for the component $[m, b^l_k]_l$ this is because both $m$ and $b$ twist commute with $c'$ by the order zero and the first order conditions. Hence the first term in the r.h.s. of (3.40) twist-commutes with $\pi^0(c)$.

The twisted first order condition for (3.36) then follows noticing that the second term in the r.h.s. of (3.40) actually vanishes. Indeed, for $\Psi = (e^k_j) \otimes \psi_j$ in $\mathcal{H}_R$, one has

$$\bar{\eta}_{(-1)} \Psi = (e^k_j) \otimes \sigma(e^k_j) \bar{\eta}_{(-1)} \psi_j$$

(3.52)

where for $\bar{\eta}_{(-1)} = \sum \alpha^i_i \delta^i_R(b_i) \in \Omega_R$, one denotes $\bar{\eta}_{(-1)} = \sum \alpha^i_i \delta^i(b_i) \in \Omega$: From (3.13) one obtains

$$\pi_R(b) \bar{\eta}_{(-1)} \Psi = (e^k_j) \otimes b^l_k \sigma(e^k_j) \bar{\eta}_{(-1)} \psi_j,$$

(3.53)

$$\bar{\eta}_{(-1)} \pi_R(\Psi) = (e^k_j) \otimes \sigma(e^k_j) \bar{\eta}_{(-1)} \sigma(b^k_j) \psi_k$$

(3.54)

Finally, for the real structure, one has $J^2 = \mathbb{I} \otimes J^2 \otimes \mathbb{I} = e \mathbb{I}$, as well as

$$J' \Gamma' \Psi' = \eta \otimes J \Gamma \psi \otimes \bar{\xi} = e'' \eta \otimes \Gamma \psi \otimes \bar{\xi} = e' \Gamma' \Psi'.$$

(3.55)
There remains only to check that \( J'D' = \epsilon'D'J' \). The construction of \cite{[A.2]} applies because \( \Omega^\circ_R \) satisfies the same module laws as \( \Omega^0 \) and \( \delta^0(e) = 0 \) is equivalent to \( \delta^\circ_R(e) = 0 \) by lemma \cite{[3.3]}. Developing in \cite{[3.39]} the actions on \( \mathcal{H}_R \) of \( D_R \) (by \cite{[3.34],[3.29]}) yields

\[
D'\Psi' = \sum \xi \otimes D\psi \otimes \overline{\sum \eta} + \sum \xi(0) \otimes \xi(1)\psi \otimes \overline{\sum \eta} + \sum \xi \otimes J\overline{\eta(-1)}\psi \otimes \overline{\sum \eta(0)},
\]

so that by \cite{[3.44]} one obtains

\[
J'D'\Psi' = \sum \eta \otimes JD\psi \otimes \overline{\sum \xi} + \sum \eta \otimes J\xi(1)\psi \otimes \overline{\sum \xi(0)} + \sum \eta(0) \otimes J\overline{\eta(-1)}\psi \otimes \overline{\sum \xi},
\]

\[
D'J'\Psi' = \sum \eta \otimes DJ\psi \otimes \overline{\sum \xi} + \sum \eta(0) \otimes \eta(1)J\psi \otimes \overline{\sum \xi} + \sum \eta \otimes \overline{\xi}(-1)J\psi \otimes \overline{\sum \xi(0)},
\]

From \cite{[A.45]} follows \( \tilde{\eta}(-1) = \epsilon' J\eta(1)J^{-1} = \epsilon' J\eta(1)J \) (from \( J^{-1} = \epsilon J \), with \( \epsilon^2 = 1 \)), so that

\[
J\tilde{\eta}(-1)\psi = \epsilon' \tilde{\eta}(1)J\psi.
\]

Similarly, \( \tilde{\xi}(-1)J\psi = J\xi(1)\psi \). Together with \( DJ = \epsilon' JD \), this give \( J'\Psi' = \epsilon'D'J'\Psi' \). Hence \( J' \) is a real structure for \cite{[3.36]}, for the same \( KO \)-dimension as the initial triple \cite{[3.21]}.

This proposition is both a generalization of \cite{[26]} which dealt with twist but only for self Morita equivalence, and of \cite{[6]} which dealt with general Morita equivalence but with no twist.

**Remark 3.5.** Whether conditions \cite{[2.33]} and \cite{[2.46]} are necessary restrictions on the module implementing Morita equivalence in order to export a twisted spectral triple should be investigated elsewhere. Notice, however, that any idempotent whose non-zero components are the identity of \( \mathcal{A} \) satisfy all these conditions. This is in particular true for self-Morita equivalence, in which case \( \epsilon \) is the unit element of \( \mathcal{A} \).

The construction above is symmetric from the left/right module points of view. Namely one may view the total Hilbert space as

\[
\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}_L
\]

and, given an hermitian \( \Omega \)-connection on \( \mathcal{E} \), define the Dirac operator

\[
D'' = (\Sigma \otimes I) \circ (I \otimes \nabla_L) D_L,
\]

where

\[
\nabla_L = (I \otimes L) \circ \nabla
\]

is a connection on \( \mathcal{E} \) valued in the bimodule \( \Omega_L \) generated by the derivation \( \delta_L(a) := [D_L, \tilde{\pi}(a)] \),

\[
\tilde{\pi}(a)(\psi \otimes \overline{\eta}) = a\psi \otimes \overline{\eta}
\]

is the representation of \( \mathcal{A} \) on \( \mathcal{H}_L = \mathcal{H} \otimes \mathcal{E} \), and \( L \) is the map sending any \( \omega = \sum_j a_j\delta(b_j) \in \Omega \) to

\[
L(\omega) := \sum_j \tilde{\pi}(a_j)\delta_L(b_j) \in \Omega_L.
\]

One checks that \( \delta_L(a) \) acts on \( \mathcal{H}_L \) as \( \delta(a) \otimes \Sigma^{\circ-1} \), so that \( L(\omega) \) acts on \( \mathcal{H}_L \) as

\[
L(\omega)\varphi = \omega\psi \otimes \Sigma^{\circ-1}\overline{\eta} \quad \forall \varphi = \psi \otimes \overline{\eta} \in \mathcal{H}_L.
\]

\[
\nabla_L \xi = \xi(0) \otimes L(\xi(1)).
\]

This yields a twisted version of the first statement of \cite{[6], Prop. 1].
Proposition 3.6. Let $\nabla$ be an hermitian connection on $\mathcal{E}$ with $\nabla_L$ the associated $\Omega_L$-connection (3.62), and $D_L$ defined by (A.23) with $\nabla^o = \nabla$ the conjugate to $\nabla$ of lemma (A.3). Then $D'' = D'$.

Proof. For $\Psi' = \xi \otimes \varphi \in \mathcal{H}'$ with $\varphi = \psi \otimes \bar{\eta} \in \mathcal{H}_L$, one has

$$D'' \Psi' = \Sigma \xi \otimes D_L \varphi + (\Sigma \otimes 1) \circ (\nabla_L \xi) \varphi,$$

(3.67)

$$= \Sigma \xi \otimes D \psi \otimes \Sigma^{\omega-1} \bar{\eta} + \Sigma \xi \otimes \bar{\eta}(-1) \psi \otimes \Sigma^{\omega-1} \bar{\eta}(0) + \Sigma \xi(0) \otimes \epsilon(1) \psi \otimes \Sigma^{\omega-1} \bar{\eta},$$

(3.68)

$$= \Sigma \xi \otimes D \psi \otimes \Sigma \eta + \Sigma \xi \otimes \bar{\eta}(-1) \psi \otimes \Sigma \eta(0) + \Sigma \xi(0) \otimes \epsilon(1) \psi \otimes \Sigma \eta.$$  

(3.69)

This coincides with the formula (3.56) of $D'$.

3.3 Twisted fluctuations without first order condition

Let us apply the preceding construction to self-Morita equivalence, that is

$$\mathcal{B} = \mathcal{A} \quad \text{with} \quad \mathcal{E} = \mathcal{A} = \bar{\mathcal{E}}.$$  

(3.70)

Any hermitian $\Omega$-connection $\nabla$ on $\mathcal{E}$ and its conjugate $\nabla$ on $\bar{\mathcal{E}}$ are such that

$$\nabla(ea) = e \otimes \delta(a) + e \otimes \omega a,$$

(3.71)

$$\nabla(e^a) = \delta(a^\omega) \otimes \bar{\eps} + \epsilon J \omega J^{-1} a^\omega \otimes \epsilon \quad \text{with} \quad \omega = \omega^o \in \Omega$$

(3.72)

where $e$ is the unit of $\mathcal{A}$ and (3.72) follows from (3.71), using $\epsilon J \delta(a^\omega) J^{-1} = \delta(a)$ (see (2.8)). Modulo the identification $\mathcal{H}_R \simeq \mathcal{H} \simeq \mathcal{H}_L$, one gets (e.g. [26, Cor. 3.6.3.11])

$$D_R = (\Sigma \otimes 1) \circ (1 \otimes \nabla) = D + \omega,$$

(3.73)

$$D_L = (1 \otimes \Sigma^{-1}) \circ (D \otimes \bar{\nabla}) = D + \epsilon J \omega J^{-1},$$

(3.74)

in agreement with the formula (3.12) for $D_R$ and (A.29) for $D_L$. The left and right Morita equivalent triples of Prop. 3.2, 3.13 thus differ from the initial triple by the substitution of $D$ with $D + \omega$ and $D + \epsilon J \omega J^{-1}$. The latter are called twisted fluctuations of $D$ by $\mathcal{A}$ and $\mathcal{A}^o$.

For a real spectral triple (3.21), the Hilbert space $\mathcal{H}'$ (3.36) coincides with the initial one $\mathcal{H}$ and $J'$, $\Gamma'$ in (3.37), (3.38) with $J$, $\Gamma$. The Dirac operator (3.24) is (see e.g. [26, Prop. 3.13])

$$D' = D + \omega(1) + \hat{\omega}(1)$$

(3.75)

where, to match the notations of (3), one uses (2.12) and denote

$$\omega(1) := \omega = \sum_i a_i [D, b_i]_{\sigma}, \quad \hat{\omega}(1) = \sum_i \hat{a}_i [D, \hat{b}_i]_{\sigma^o} = \epsilon J \omega(1) J^{-1}.$$  

(3.76)

Exporting the real twisted spectral triple $(\mathcal{A}, \mathcal{H}, D), \sigma$ via self Morita equivalence thus amounts to substituting $D$ with $D'$. This is a twisted inner fluctuation, first introduced by imitation of the ordinary case in [25], then rigorously derived by Morita equivalence in [26].

What happens if one no longer assumes the twisted first-order condition? This has been investigated in [3] for the non-twisted case and leads to Pati-Salam extensions of the Standard Model [5]. The process is similar in the twisted case, as described below. Some of the physical consequences are investigated in [20].
Thus, instead of (3.29), the action of (3.31) has no reason to vanish any more. In particular, for \( \xi = \epsilon \), one has
\[
\delta R^\epsilon(a)(\xi \otimes \psi) = \xi \otimes \delta^\epsilon(a) \psi + \xi(0) \otimes [\xi(1),a^\epsilon]_{\sigma^\epsilon} \psi \quad \forall \xi \otimes \psi \in H_R, 
\]
for (3.31) has no reason to vanish any more. In particular, for \( \xi = e \), one has
\[
\delta R^e(a)(e \otimes \psi) = e \otimes \delta^e(a) \psi + e \otimes [\omega,a^\epsilon]_{\sigma^\epsilon} \psi. 
\]
Thus, instead of (3.29), the action of \( R(\omega^\epsilon) \) on \( H_R \), for \( \omega^\epsilon = \sum_j a_j^\epsilon \delta^\epsilon(b_j) \), is
\[
R(\omega^\epsilon) \Psi = e \otimes \omega^\epsilon \psi + e \otimes \sum_j a_j^\epsilon[\omega,b_j^\epsilon]_{\sigma^\epsilon} \psi 
= e \otimes \left( \omega^\epsilon + \sum_j a_j^\epsilon[\omega,b_j^\epsilon]_{\sigma^\epsilon} \right) \psi \quad \forall \Psi = e \otimes \psi \in H_R. 
\]
In particular, for \( \nabla_R^\epsilon = \nabla_R \) as in (3.33), one has
\[
\omega^\epsilon = \tilde{e}_{(-1)} = \epsilon' J e_{(1)} J^{-1} = \epsilon' J \omega J^{-1} = \tilde{\omega}_{(1)}. 
\]
In other terms, \( \omega^\epsilon = \sum_j \tilde{a}_j[\tilde{D},b_j^\epsilon]_{\sigma^\epsilon} \), meaning that the parenthesis of (3.82) is \( \tilde{\omega}_{(1)} + \tilde{\omega}_{(2)} \).
For \( \Psi' = e \otimes \psi \otimes \tilde{e} \) a generic element of \( \mathcal{H}' \), the Dirac operator \( D' \) in (3.56) reads
\[
D' \Psi' = e \otimes D \psi \otimes \tilde{e} + e \otimes \omega_{(1)} \psi \otimes \tilde{e} + e \otimes (\tilde{\omega}_{(1)} + \tilde{\omega}_{(2)}) \psi \otimes \tilde{e} 
= e \otimes D \psi \otimes e + e \otimes \omega_{(1)} \psi \otimes e + e \otimes (\omega_{(1)} + \omega_{(2)}) \psi \otimes e 
\]
Identifying \( e \otimes \psi \otimes e \simeq \psi \) amounts to identifying the operator \( D' \) with \( D_\omega \).

If the twisted-first condition holds, then \( \omega_{(2)} \) vanishes and one finds back (3.75). Therefore, as in the non-twisted case, the term \( \omega_{(2)} \) breaks the linearity of the map \( \omega \mapsto D + \omega_{(1)} + \omega_{(1)} \) between twisted 1-forms and fluctuations. To 7truecm, one has the concluding

**Proposition 3.8.** The triple \((A,H,D_\omega)\) together with the automorphism \( \sigma \) and the operators \( \Gamma, J \) has all the properties of a real twisted spectral triple, but the twisted first order condition.

*Proof.* The proof of Prop 3.4 does not refer to the first order condition, except to show that the fluctuated triple satisfies the first order condition. So by applying this proposition to self Morita equivalence, one obtains that \((A,H,D_\omega),\sigma \) together with \( \Gamma \) is a graded twisted spectral triple. As in the proof of proposition 3.4, one checks that \( J^2 = \epsilon, J\Gamma = \epsilon' \Gamma J \). The only point is to check that \( JD_\omega = \epsilon' D_\omega J \). Actually Prop. 3.4 guarantees that \( (D + \omega_{(1)} + \omega_{(1)}) J = \epsilon'(D + \omega_{(1)} + \omega_{(1)}) J \), so one just needs to show that
\[
\omega_{(2)} J = \epsilon' \omega_{(2)} J. 
\]
Omitting the summation symbol, one has
\[ J \omega(2) J^{-1} = J \hat{a}_i [\omega(1), \hat{b}_i]_{\sigma^0} J^{-1} = a_i J \left( \omega(1) \hat{b}_i - \sigma^0(\hat{b}_i) \omega(1) \right) J^{-1} = \varepsilon' a_i (\omega(1) \hat{b}_i - \sigma(\hat{b}_i) \omega(1)) = \varepsilon' a_i [\hat{w}(1), \hat{b}_i]_{\sigma}, \]
where we used \( J^{-1} = \varepsilon J \), then \( \omega(1) = \varepsilon' J \hat{w}(1) J^{-1} \) together with \( J^2 = J^2 = \varepsilon \mathbb{I} \), as well as
\[ \sigma^0(\hat{b}_i) = \sigma^0 \left( (b_i^*)^0 \right) = (\sigma(b_i))^0 = J \sigma(b_i) J^{-1}. \] (3.86)

The result follows noticing that,
\[ \omega(2) = \sum_{i,j} \hat{a}_i [a_j [D, b_j]_{\sigma}, \hat{b}_i]_{\sigma^0} = \sum_{i,j} a_j \hat{a}_i [D, b_j]_{\sigma}, \hat{b}_i]_{\sigma^0} = \sum_{i,j} a_j \hat{a}_i [D, \hat{b}_i]_{\sigma^0}, b_j]_{\sigma} = \sum_{j} a_j [\hat{w}(1), \hat{b}_j]_{\sigma}, \]
where the second equality follows from the order-zero condition, and the third from
\[ [[D, b]_{\sigma}, \hat{a}]_{\sigma^0} = [[D, \hat{a}]_{\sigma^0}, b]_{\sigma} \] (3.77)
that is checked by direct computation.

**Remark 3.9.** In the non-twisted case, a self-adjoint element of \( \Omega \) is the image of a self-adjoint universal 1-form (i.e. a self-adjoint element of the differential algebra \( \Omega(A) \) [24]). In the twisted case this is no longer the case, for the representation
\[
\pi : \Omega(A) \rightarrow \mathcal{B}(\mathcal{H})
\]
\[
\pi(a_0 \delta a_1 \ldots \delta a_n) \mapsto \pi(a_0)[D, \pi(a_1)]_{\sigma} \ldots [D, \pi(a_n)]_{\sigma}
\] (3.88)
is not in general an \( \ast \)-homomorphism.

## 4 Gauge transformation

We investigate in §4.2 how a twisted spectral triple that does not meet the twisted first-order condition behaves under a gauge transformation, still following the strategy of [6]. The loss of selfadjointness is discussed in §4.3. We begin in §4.1 by recalling the definition of gauge transformations for twisted spectral triples [26], which is a straightforward adaptation of the non-twisted case [8].

### 4.1 Twisted gauge transformation

A gauge transformation on a module \( \mathcal{E} \) equipped with a connection \( \nabla \) is a change of connection, obtained by acting on \( \nabla \) with a unitary endomorphism \( u \) of \( \mathcal{E} \) (see e.g. [24] for more details).
\[
\nabla \rightarrow \nabla^u := u \nabla u^*. \] (4.1)

For self-Morita equivalence, \( \mathcal{E} = A \) and the set of unitary endomorphisms of \( \mathcal{E} \) is the group
\[
\mathcal{U}(A) = \{ u \in A, \pi(u) \pi(u^*) = \pi(u^*) \pi(u) = \mathbb{I} \} \] (4.2)
of unitary elements of \( A \).

Given the real twisted spectral triple \((A, \mathcal{H}, D'), \sigma\) of Prop. 3.7 obtained by inner fluctuations, with \( D' \) given in (3.75) and \( \Gamma, J \) the grading and real structure of the initial triple, then a gauge transformation amounts to substituting \( \omega(1) \) and \( \hat{w}(1) \) in the twist-fluctuated operator \( D' \) with (details are given for instance in [26 §A.2])
\[
\omega(1) \rightarrow \omega'(1) := \sigma(u) [D, u^*]_{\sigma} + \sigma(u) \omega(1) u^*; \] (4.3)
\[
\hat{w}(1) \rightarrow \hat{w}'(1) := \sigma(\hat{u}) [D, \hat{u}^*]_{\sigma^0} + \sigma(\hat{u}) \hat{w}(1) \hat{u}^* = \varepsilon' J \omega'(1) J^{-1}. \] (4.4)
This substitution turns out to be equivalent to the conjugate action (twisted on the left) on $D'$ of

$$\text{Ad}(u)\psi := u\psi u^* = u\hat{\psi}$$ \quad \forall \psi \in \mathcal{H}. \quad (4.5)$$

Namely one has [26, Prop. 4.5]

$$\text{Ad}(\sigma(u)) D_\omega \text{Ad}(u)^* = D + \omega_1^u + \hat{\omega}_1^u. \quad (4.6)$$

The twisted first-order condition is required for this result. If it does not hold, then there is no reason for (4.6) to be true. Actually this is not a surprise nor a problem since, as discussed in the previous section, this is not the operator $D'_\omega$ that is relevant, but the operator $D_\omega$ in (3.77). By relaxing the twisted first-order condition, we show below that the twisted action of $\text{Ad}(u)$ on $D_\omega$ induces a transformation of the non-linear term $\omega(2)$ to

$$\sigma^o(\hat{u})\omega(2)\hat{u}^* + \sigma^o(\hat{u})[\sigma(u)[D,u^*]_\sigma,\hat{u}^*]_{\sigma^o}. \quad (4.7)$$

This is a twisted version of the non-linear gauge transformation of [6] (formula for $A(2)$ before lemma 3), and gives a non-linear correction to the first-order, linear fluctuation of $\omega(1) + \hat{\omega}(1)$.

### 4.2 Non-linear gauge transformation

The inner twisted fluctuation of proposition [3.4] is a map

$$\omega \to D_\omega \quad (4.8)$$

that associates to any $\omega = \sum_{j=1}^{n} a_j [D,b_j]_{\sigma}$ the operator $D_\omega$ defined by (3.77), with $\omega(1)$, $\hat{\omega}(1)$ and $\omega(2)$ the functions of the components $a_j, b_j$ of $\omega$ given in (3.76) and (3.78). A gauge transformation thus amounts to substituting $D_\omega$ with $D_{\omega^u}$ for $\omega^u$ in (4.3). We show in proposition [4.2] below that this is equivalent to the twisted adjoint action of $\text{Ad}(u)$ on $D_\omega$.

To prove that, we need the preliminary

**Lemma 4.1.** Let $(\mathcal{A}, \mathcal{H}, D), \sigma, J$ be a real, twisted, spectral triple that does not necessarily fulfils the first-order condition. Then, for any $u \in \mathcal{U}(\mathcal{A}),$

$$\text{Ad}(\sigma(u)) D \text{Ad}(u)^* \text{Ad}(u)^* = D + \sigma(u)[D,u^*]_{\sigma} + \sigma^o(\hat{u})[D,u^*]_{\sigma^o}$$

$$+ \sigma^o(\hat{u})[\sigma(u)[D,u^*]_{\sigma},\hat{u}^*]_{\sigma^o}. \quad (4.9)$$

$$+ \sigma^o(\hat{u})[\sigma(u)[D,u^*]_{\sigma},\hat{u}^*]_{\sigma^o}. \quad (4.10)$$

**Proof.** Remembering [2.72.12] the right action [2.11] of $\sigma(u)^*$ is the left multiplication by

$$\sigma(\hat{u}) = J\sigma(u)J^{-1} = \sigma^o((u^*)^o) = \sigma^o(\hat{u}) \quad (4.11)$$

so that the adjoint action [4.5] of $\sigma(u)$ writes

$$\text{Ad}(\sigma(u)) = \sigma(u)\sigma^o(\hat{u}) = \sigma^o(\hat{u})\sigma(u) \quad (4.12)$$

(the second equality comes from the order-zero condition). As well

$$\text{Ad}(u)^* = (u\hat{u})^* = \hat{u}^*u^* = u^*\hat{u}^*, \quad (4.13)$$

where we use the commutation of the involution with the conjugation by $J,$

$$\hat{u}^* = u^*. \quad (4.14)$$
Therefore
\[ \text{Ad}(\sigma(u))\text{Ad}(u)^* = \sigma^0(\hat{u})\sigma(u) D u^* \overline{u}^* = \sigma^0(\hat{u})\sigma(u) (\sigma(u^*) D + [D, u^*]_\sigma) \overline{u}^*, \]
\[ = \sigma^0(\hat{u}) \left( D u^* + \sigma(u)[D, u^*]_\sigma \overline{u}^* \right) = \sigma^0(\hat{u}) \left( \sigma(u^*) D + [D, u^*]_\sigma \right), \]
\[ + \sigma^0(\hat{u}) \left( \sigma(u^*) \sigma(u)[D, u^*]_\sigma + \sigma(u)[D, u^*]_\sigma, \overline{u}^* \right), \]
\[ = D + \sigma^0(\hat{u})[D, u^*]_\sigma + \sigma(u)[D, u^*]_\sigma + \sigma^0(\hat{u}) \sigma(u)[D, u^*]_\sigma, \overline{u}^* \sigma. \]

\[ \square \]

We now come to the main result of this section, which shows that even if the condition of order one is not met, a twisted gauge transformation is equivalent to the adjoint action of \( \text{Ad}(u) \) (twisted on the right) on the Dirac operator.

**Proposition 4.2.** Let \((A, \mathcal{H}, D), \sigma, J\) be a real twisted spectral triple that does not necessarily satisfy the twisted first-order condition, and \( \omega = \sum_{j=1}^n a_j[D, b_j]_\sigma \) a twisted 1-form. Then for any unitary \( u \) in \( \mathcal{U}(A) \) one has
\[ \text{Ad}(\sigma(u))D_\omega \text{Ad}(u)^* = D_\omega u \]

for
\[ \omega^u = \sigma(u)\omega u^* + \sigma(u)[D, u^*]_\sigma, \]  

(4.15)

**Proof.** We adapt the method of [6] to compute
\[ D_\omega u = \omega_{(1)}^u + \hat{\omega}_{(1)}^u + \omega_{(2)}^u \]  

(4.16)
as the image of \( \omega^u \) under the map (4.8). The terms \( \omega_{(1)}^u \) and \( \hat{\omega}_{(1)}^u \) are given by (4.3), (4.4).

To compute \( \omega_{(2)}^u \), it convenient to rewrite \( \omega_{(1)}^u \) as
\[ \omega_{(1)}^u = \sigma(u) \left( \sum_{j=1}^n a_j[D, b_j]_\sigma \right) u^* + \sigma(u)[D, u^*]_\sigma, \]
\[ = \sigma(u) \sum_{j=1}^n a_j ([D, b_j u^*]_\sigma - \sigma(b_j)[D, u^*]_\sigma) + \sigma(u)[D, u^*]_\sigma = \]
\[ = \sigma(u) \left( \mathbb{I} - \sum_{j=1}^n a_j \sigma(b_j) \right) [D, u^*]_\sigma + \sum_{j=1}^n \sigma(u)a_j[D, b_j u^*]_\sigma = \sum_{j=0}^n a'_j[D, b'_j]_\sigma \]

where we defined
\[ a'_0 = \sigma(u) \left( \mathbb{I} - \sum_{j=1}^n a_j \sigma(b_j) \right) \]
\[ b'_0 = u^*, \]  

(4.17)
\[ a'_j = \sigma(u)a_j \]
\[ b'_j = b_j u^* \quad \forall j \geq 1. \]  

(4.18)

Notice that the same relation holds for any operator \( T \in \mathcal{L}(\mathcal{H}) \), namely
\[ \sum_{j=0}^n a'_j[T, b'_j]_\sigma = \sigma(u) \left( \sum_{j=1}^n a_j[T, b_j]_\sigma \right) u^* + \sigma(u)[T, u^*]_\sigma. \]  

(4.19)
Thus $\omega^u_{(2)}$ is given by \ref{om2} with $a_j', b_j'$ instead of $a_j, b_j$:

$$\omega^u_{(2)} = \sum_{j=0}^n a_j' [\hat{\omega}^u_{(1)}, b_j']_\sigma = \sigma(u) \left( \sum_{j=1}^n a_j [\hat{\omega}^u_{(1)}, b_j]_\sigma \right) u^* + \sigma(u)[\hat{\omega}^u_{(1)}, u^*]_\sigma, \tag{4.20}$$

$$= \sum_{j=1}^n \sigma(u)a_j [\sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, b_j]_\sigma u^* + \sum_{j=1}^n \sigma(u)a_j [\sigma^\circ(\hat{u}) [D, \hat{u}^*]_\sigma, b_j]_\sigma u^* + \sigma(u)[\sigma^\circ(\hat{u}) [D, \hat{u}^*]_\sigma, u^*]_\sigma \tag{4.21}$$

$$+ \sigma(u)[\sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, u^*]_\sigma + \sigma(u)[\sigma^\circ(\hat{u}) [D, \hat{u}^*]_\sigma, u^*]_\sigma \tag{4.22}$$

where \ref{4.20} comes from \ref{4.17}, \ref{4.18}, while \ref{4.21} is obtained substituting $\hat{\omega}^u_{(1)}$ with its explicit form \ref{om1}, using also \ref{4.14}. Let us compute these four terms separately, dropping the summation index.

- The first one is

$$\sigma(u)a [\sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, b]_\sigma u^* = \sigma(u)a [\sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, u^*] \tag{4.23}$$

$$= \sigma(u)[\sigma^\circ(\hat{u})a[\hat{u}^*, u^*]_\sigma = \sigma(u)a[\hat{u}^*, u^*]_\sigma = \text{Ad}(\sigma(u))\omega^u_{(2)} \text{Ad}(u^*) \tag{4.24}$$

where the first equalities are obtained by explicit computation, using that $b$ commutes with $\hat{u}^*$ and $\sigma^\circ(\hat{u})$ with $\sigma(b)$ (then with $a$) by the order zero condition. The last one follows from the definition \ref{om2} of $\omega^u_{(2)}$ and the adjoint actions \ref{4.12}-\ref{4.13}.

- The second term is

$$\sigma(u)a[\sigma^\circ(\hat{u})[D, \hat{u}^*]_\sigma, b]_\sigma u^* = \sigma(u)a[\sigma^\circ(\hat{u})[D, \hat{u}^*]_\sigma, u^*] \tag{4.25}$$

$$= \sigma(u)[\sigma^\circ(\hat{u})a[D, b]_\sigma u^* - \sigma(u)a[D, b]_\sigma u^*] \tag{4.26}$$

$$= \text{Ad}(\sigma(u))\omega^u_{(1)} \text{Ad}(u^*) - \sigma(u)\omega^u_{(1)} u^* \tag{4.27}$$

- The third and fourth terms give

$$\sigma(u)[\sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, u^*]_\sigma = \sigma(u)[\sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, u^*] \tag{4.28}$$

$$= \text{Ad}(\sigma(u)) \text{Ad}(u^*) - \sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^*, \tag{4.29}$$

$$\sigma(u)[\sigma^\circ(\hat{u})[D, \hat{u}^*]_\sigma, u^*]_\sigma = \sigma(u)[\sigma^\circ(\hat{u})[D, \hat{u}^*]_\sigma, u^*]_\sigma \tag{4.30}$$

$$= \sigma^\circ(\hat{u}) \sigma(u)[D, u^*]_\sigma, \hat{u}^* u^*]_\sigma \tag{4.31}$$

where \ref{4.30} is proven using $[a, b, c]_\sigma = a[b, c]_\sigma + [a, c]_\sigma b$, and \ref{4.31} using \ref{4.7}.

Collecting \ref{4.24}, \ref{4.27}, \ref{4.29} and \ref{4.31}, one obtains

$$\omega^u_{(2)} = \text{Ad}(\sigma(u)) \left( \omega^u_{(1)} + \omega^u_{(2)} \right) \text{Ad}(u^*) - \sigma(u)\omega^u_{(1)} u^* - \sigma^\circ(\hat{u}) \hat{\omega}_{(1)} \hat{u}^* + \sigma^\circ(\hat{u}) [\sigma(u)[D, u^*]_\sigma, \hat{u}^*]_\sigma \tag{4.22}$$

Adding $\omega^u_{(1)}$ and $\hat{\omega}^u_{(1)}$ in \ref{4.3} and \ref{om1}, one obtains

$$D_{\omega^u} = D + \omega^u_{(1)} + \omega^u_{(2)} = D + \text{Ad}(\sigma(u)) \left( \omega^u_{(1)} + \omega^u_{(2)} \right) \text{Ad}(u^*) + \sigma(u)[D, u^*]_\sigma + \sigma^\circ(\hat{u}) [\sigma(u)[D, D^*]_\sigma, u^*]_\sigma.$$}

The result then follows by lemma \ref{lemma1}.

\begin{remark}
A gauge transformation for the twisted covariant Dirac operator $D_\omega$ is implemented by the twisted conjugate action of $\text{Ad}(u)$. This is the same law \ref{4.6} as when the twisted first-order condition holds. As a consequence, the gauge invariance of the fermionic action defined in \cite{15,16} still holds, even if the twisted first-order condition is violated.
\end{remark}
4.3 Self-Adjointness

A twisted gauge transformation does not preserve selfadjointness: starting with a selfadjoint operator $D_\omega$, one has that $\text{Ad}(\sigma(u))D_\omega \text{Ad}(u)^* = $ selfadjoint if and only if \[ [D_\omega, u\hat u]_\sigma = 0 \text{ with } u := \sigma(u)^* u \text{ and } \sigma(u\hat u) := \sigma(u)\sigma(\hat u). \tag{4.32} \]
This relation is trivially satisfied if the unitary $u$ is twist-invariant, that is $\sigma(u) = u$. But this is not the only solution, as shown below.

**Proposition 4.4.** Let $(A, \mathcal{H}, D), \sigma, J$ be a twisted real spectral triple not necessarily fulfilling the twisted first-order condition, and $D_\omega$ a selfadjoint twisted inner fluctuation \[5.11\]. Then the gauge-transformed Dirac operator $D_\omega u^\sigma$ is self-adjoint if and only if \[
\gamma(u) + \epsilon' J\gamma(u)J^{-1} + [[D_\omega, u]_{\sigma^*}, \hat u]_{\sigma^*} = 0 \tag{4.33} 
\]
where \[
\gamma(u) := \sigma^0(\hat u)[D_\omega, u]_{\sigma}. \tag{4.34} 
\]

**Proof.** By \[4.32\] and \[4.11\], the gauge-transformed Dirac operator is self-adjoint iff \[
[D_\omega, u\hat u]_{\sigma} = [D_\omega, u]_{\sigma} \hat u + \sigma(u)[D_\omega, \hat u]_{\sigma^*} = 0. \tag{4.35} 
\]
The result follows from \[
[D_\omega, u\hat u]_{\sigma} = [[D_\omega, u]_{\sigma}, \hat u]_{\sigma^*} + \sigma^0(\hat u)[D_\omega, u]_{\sigma}, \tag{4.36} 
\]
so that \[4.11\] yields \[
\sigma^0(\hat u)[D_\omega, u]_{\sigma} = J u^* \sigma(u)^{-1} u^* \sigma(u) J^{-1}, \tag{4.37} \]
noticing that \[
\sigma^0(\hat u) = \sigma^0((u^*)^o) = \sigma^{-1}(u^*)^o = \sigma^{-1}(u^*\sigma(u))^o = (\sigma(u)^* u)^o = J u^* \sigma(u) J^{-1}, \tag{4.38} \]
so that \[4.11\] yields \[
\sigma^0(\hat u)[D_\omega, u]_{\sigma} = J u^* \sigma(u)^{-1} [D, u]_{\sigma} = u^* \sigma^0(\hat u)[D, u]_{\sigma} \tag{4.39} \]
By the condition of order one, the right term in the equation above vanishes, as well as the last term in the r.h.s. of \[4.33\]. One is left with \[
\gamma(u) + \epsilon' J\gamma(u)J^{-1} = 0 \text{ with } \gamma(u) = u^*[[D, u]_{\sigma}, \hat u]_{\sigma^*} \tag{4.40} \]
which is precisely Prop. 5.2 of \[26\].

5 Twisted semi-group of inner perturbations

In this section we adapt to the twisted case the semi-group of inner perturbations of \[6\]. The normalisation condition is twisted in \[5.11\] and the structure of semi-group for twisted 1-forms is worked out in \[5.2\]. Its interpretation in terms of twisted inner fluctuation is the object of \[5.3\].

There are two notable differences with the non-twisted case: the semi-group structure depends on the twisting automorphism, and we do not restrict its definition to selfadjoint elements, for reasons explained below.

In all this section, we consider a real twisted spectral triple \[(A, \mathcal{H}, D), \sigma, J \tag{5.1} \]
that does not necessarily satisfy the twisted first-order condition. The unit of $A$ is $\epsilon$. 

5.1 Twisted normalised condition

Let $A^e := A \otimes C_A$ denote the enveloping algebra of $A$ [24], with product
\[
(a_1 \otimes b_1^\circ) \cdot (a_2 \otimes b_2^\circ) := a_1 a_2 \otimes b_1^\circ b_2^\circ.
\] (5.2)

The normalisation condition imposed in [6] easily generalises to the twisted case.

**Definition 5.1.** A combination $\sum_j a_j \otimes b_j^\circ \in A^e$ is twisted normalised iff
\[
\sum_j a_j \sigma(b_j) = e.
\]

In [6], the semi-group of inner fluctuations is defined as the set of self-adjoint normalised elements of $A^e$. The definition we propose in the twisted case is similar, except that we do not restrict to selfadjoint elements. Indeed, as explained in §4.3, the twisted gauge transformations do not preserve selfadjointness of 1-forms, so there is no reason to consider only selfadjoint normalised elements of $A^e$.

**Proposition 5.2.** The set of twisted normalised elements of $A^e$,

\[
\text{Pert}(A, \sigma) := \left\{ \sum_j a_j \otimes b_j^\circ \in A^e \text{ such that } \sum_j a_j \sigma(b_j) = e \right\}
\]

(5.3)
is a semi-group for the product of the enveloping algebra.

**Proof.** Let $\sum_j a_j \otimes b_j^\circ$ and $\sum_i a_i' \otimes b_i'^\circ$ be normalised elements of $A^e$. Their product $\sum_{j,i} a_j a_i' \otimes b_j^\circ b_i'^\circ = \sum_{j,i} a_j a_i' \otimes (b_j b_i'^\circ)$ is normalised since
\[
\sum_{j,i} a_j a_i' \sigma(b_j b_i'^\circ) = \sum_j a_j \left( \sum_i a_i' \sigma(b_i'^\circ) \right) \sigma(b_j) = \sum_j a_j \sigma(b_j) = e.
\]

Hence Pert$(A, \sigma)$ is stable by the product of the enveloping algebra. \qed

**Remark 5.3.** Since we assume the algebra are unital, Pert$(A, \sigma)$ is actually a monoid, with unit $e \otimes e$.

We show below that the action of this semi-group on the Dirac operator $D$ coincides with the twisted fluctuations, and the multiplication by unitaries gives back the gauge transformation, as in the non-twisted case. This justifies to call Pert$(A, \sigma)$ the semi-group of twisted inner fluctuations of the twisted spectral triple (5.1). To show this, we begin with working out the relation between the semi-group and twisted 1-forms.

5.2 From the semi-group to twisted one-forms

One defines a map from the semi-group to the twisted one-forms,
\[
\eta : \text{Pert}(A, \sigma) \longrightarrow \Omega_D(A, \sigma), \quad \eta \left( \sum_j a_j \otimes b_j^\circ \right) := \sum_j a_j [D, b_j]_{\sigma},
\] (5.4)

which has similar properties as in the non-twisted case (the following lemma extends to the twisted case lemma 4 of [6]).
Lemma 5.4. i) The map \( \eta \) is surjective. ii) The adjoint is given by

\[
\left( \eta \left( \sum_j a_j \otimes b_j^* \right) \right)^* = \eta \left( \sum_j b_j^* \otimes (a_j)^* \right). \tag{5.5}
\]

iii) The gauge transformed \[4.3\] of \( \omega = \eta \left( \sum_j a_j \otimes b_j^* \right) \) is

\[
\omega^u = \eta \left( \sum_j \sigma(u)a_j \otimes (b_j u^*)^\circ \right) \quad \forall u \in \mathcal{U}(A). \tag{5.6}
\]

Proof. This is a straightforward adaptation of the proof of [6, Lemma. 4].

i) Any twisted 1-form is a finite sum \( \sum_{j=1}^n a_j \delta(b_j) \) with \( a_j, b_j \) arbitrary elements of \( A \).

The point is to write it as a sum such that \( \sum_j a_j \sigma(b_j) \) is twisted normalised. This is obtained adding to the sum

\[
a_0 := e - \sum_{j=1}^n a_j \sigma(b_j), \quad b_0 = e.
\]

Indeed, since \( \delta(e) = 0 \), one has

\[
\sum_{j=1}^n a_j \delta(b_j) = \sum_{j=1}^n a_j \delta(b_j) + \left( e - \sum_{j=1}^n a_j \sigma(b_j) \right) \delta(e) = \sum_{j=0}^n a_j \delta(b_j)
\]

where, by construction, \( \sum_{j=0}^n a_j \sigma(b_j) \) is twisted normalised.

ii) The Leibniz rule \[2.15\] for \( \delta(\sigma^{-1}(a_j)b_j) = \delta(\sigma^{-1}(a_j\sigma(b_j))) = \delta(\sigma^{-1}(e)) = 0 \) (we omitted the symbol of summation) reads

\[
\sum_j a_j \delta(b_j) = - \sum_j \delta(\sigma^{-1}(a_j))b_j. \tag{5.7}
\]

Therefore, for \( \sum_j a_j \otimes b_j^* \) in \( \text{Pert}(A, \sigma) \), one has (using \[2.17\])

\[
\left( \eta \left( \sum_j a_j \otimes b_j^* \right) \right)^* = \left( \sum_j a_j \delta(b_j) \right)^* = - \left( \sum_j \delta(\sigma^{-1}(a_j))b_j \right)^* = \sum_j b_j^* \sigma(a_j^*) = \eta \left( \sum_j b_j^* \otimes (a_j^*)^\circ \right). \tag{5.8}
\]

The result follows noticing that \( \sum_j b_j^* \otimes (a_j^*)^\circ \) is normalised, for

\[
\sum_j b_j^* \sigma(a_j^*) = \sum_j \left( \sigma^{-1}(a_j)b_j \right)^* = \sum_j \sigma^{-1}(a_j \sigma(b_j))^* = \sigma^{-1} \left( \sum_j a_j \sigma(b_j) \right)^* = e.
\]

iii) We first check that \( \sum_j \sigma(u)a_j \otimes (b_j u^*)^\circ \) is twisted normalised:

\[
\sum_j \sigma(u)a_j \sigma(b_j u^*) = \sigma(u) \left( \sum_j a_j \sigma(b_j) \right) \sigma(u^*) = \sigma(u) \sigma(u^*) = \sigma(u u^*) = e.
\]
Then, by the Leibniz rule and the normalisation condition one obtains
\[
\eta \left( \sum_j \sigma(u)a_j \otimes (b_j u^*) \right) = \sum_j \sigma(u)a_j \delta(b_j u^*) = \sum_j \sigma(u)a_j \delta(b_j) u^* + \sum_j \sigma(u)a_j \sigma(b_j) \delta(u^*),
\]
which is precisely the gauge transform (4.3) of \( \omega \).

The group \( \mathcal{U}(A) \) of unitaries of \( A \) maps to \( \text{Pert}(A, \sigma) \) via the semi-group homomorphism
\[
u = p(u) := \sigma(u) \otimes (u^*). \tag{5.10}\]
The gauge transformed (5.6) corresponds to the product by \( p(u) \) in the semi-group:
\[
\omega^u = \eta(p(u) \omega). \tag{5.11}
\]

A similar construction holds for the opposite algebra. The subset
\[
\text{Pert}(A^o, \sigma^o) := \left\{ \sum_j a_j^o \otimes b_j \in A^o \otimes_C A \text{ such that } \sum_j a_j^o \sigma^o(b_j^o) = e \right\} \tag{5.12}\]
of the enveloping algebra of \( A^o \) forms a semi-group, for the product
\[
\sum_{ij} a_j^o a_i^o \otimes b_j b_i = \sum_{ij} (a_i^o a_j^o) \otimes b_j b_i \tag{5.13}
\]
of two of its elements \( \sum_j a_j^o \otimes b_j \) and \( \sum_i a_i^o \otimes b_i^o \) is in \( \text{Pert}(A^o, \sigma^o) \), since
\[
(a_i^o a_j^o) \sigma^o((b_j b_i^o)^o) = a_j^o (a_i^o \sigma^o(b_i^o)) \sigma^o(b_j^o) = a_j^o a_i^o \sigma^o(b_j^o) = 1. \tag{5.14}
\]
Moreover, the surjective map
\[
\eta^o : \text{Pert}(A^o, \sigma^o) \longrightarrow \Omega^1_D(A^o, \sigma^o), \tag{5.15}
\]
defined as
\[
\eta^o \left( \sum_j a_j^o \otimes b_j \right) := \sum_j a_j^o [D, b_j^o]_{\sigma^o} \tag{5.16}
\]
satisfies similar properties as the map \( \eta \) in lemma 5.4 (see the proof in appendix B). In particular, the unitary group \( \mathcal{U}(A) \) maps to this semi-group via
\[
u = p^o(u) := \sigma^o(\hat{u}) \otimes \hat{u}^*; \tag{5.17}
\]
and the image (4.4) of the opposite 1-form \( \hat{\omega} \in \text{Pert}(A^o, \sigma^o) \) under a gauge transformation is
\[
\hat{\omega}^u = \eta^o(p^o(u) \hat{\omega}). \tag{5.18}
\]
5.3 Twisted fluctuations by action of the semi-group

The action of the semi-group of perturbations on the Dirac operator of a twisted spectral triple (5.1) yields the twisted fluctuation (hence justifying the name of the semi-group), similarly to what happens in the non-twisted case. This is shown below by adapting propositions 5 and 6 of [6].

One defines the action of \( \text{Pert}(A, \sigma) \) and \( \text{Pert}(A^0, \sigma^0) \) on \( \mathcal{L}(\mathcal{H}) \) as

\[
(A, T) := \sum_j a_j T b_j \quad \forall A = \sum_j a_j \otimes b_j \in \text{Pert}(A, \sigma),
\]

\[
(A^0, T) := \sum_j a_j^0 T b_j^0 \quad \forall A^0 = \sum_j a_j^0 \otimes b_j \in \text{Pert}(A^0, \sigma^0), \quad T \in \mathcal{L}(\mathcal{H})
\]

(omitting the representation symbol on the r.h.s.).

**Lemma 5.5.** These actions are transitive, namely

\[
(A', (A, T)) = (A'A, T), \quad (A^0, (A^0, T)) = (A'^0 A^0, T)
\]

for any \( A, A' \in \text{Pert}(A, \sigma) \) and \( A^0, A'^0 \in \text{Pert}(A^0, \sigma^0) \).

**Proof.** For \( A = \sum_j a_j \otimes b_j, A' = \sum_i a_i^0 \otimes b_i^0 \) in \( \text{Pert}(A, \sigma) \), one has

\[
(A', (A, T)) = (A', \sum_j a_j T b_j) = \sum_{i,j} a_i' a_j T b_i b_j' = \left( \sum_{i,j} a_i' a_j \otimes (b_j b_i')^0, T \right),
\]

\[
= \left( \sum_i a_i' \otimes b_i^0 \right) \left( \sum_j a_j \otimes b_j^0, T \right) = (A'A, T).
\]

The transitivity of the action of \( \text{Pert}(A^0, \sigma^0) \) is shown in a similar way. \( \square \)

For \( T \) the Dirac operator \( D \), one has

\[
(A, D) = \sum_j a_j D b_j = \sum_j a_j \sigma(b_j) D + \sum_j a_j [D, b_j]_\sigma = D + \sum_j a_j [D, b_j]_\sigma = D + \eta(A),
\]

\[
(A^0, D) = \sum_i a_i^0 D b_i^0 = \sum_i a_i^0 \sigma^0(b_i^0) D + \sum_i a_i^0 [D, b_i^0]_{\sigma^0} = D + \sum_i a_i^0 [D, b_i]_{\sigma^0} = D + \eta^0(A^0).
\]

In other terms, the action of \( A \in \text{Pert}(A, \sigma) \) on \( D \) yields the twisted fluctuation (3.73) of \( D \) by \( A \), with \( \omega = \eta(A) \); while the action of \( A^0 \in \text{Pert}(A^0, \sigma^0) \) on \( D \) yields the twisted fluctuation (3.74) of \( D \) by \( A^0 \), with \( \epsilon J \omega J^{-1} = \eta^0(A^0) \). The transitivity of these actions means that twisted fluctuations of twisted fluctuations are twisted fluctuations.

All these results extend to the twisted fluctuations (3.77) of real spectral triples. To take into account the real structure, one introduces

\[
\text{Pert}(A \otimes_C A^0, \sigma) := \left\{ \sum_k A_k \otimes B_k \text{ such that } A_k \in \text{Pert}(A, \sigma), B_k \in \text{Pert}(A^0, \sigma^0) \right\}
\]

which is a semi group for the natural product \( (A \otimes B)(A' \otimes B') = AA' \otimes BB' \).
Lemma 5.6. For $A = \sum_j a_j \otimes b_j^\circ \in \text{Pert}(A, \sigma)$, denote $\hat{A} := \sum_j \hat{a}_j \otimes \hat{b}_j^\circ$. Then the map
\begin{equation}
\mu : \ \text{Pert}(A, \sigma) \longrightarrow \text{Pert}(A \otimes_C A^0, \sigma) \tag{5.24}
\end{equation}
\begin{equation}
A \mapsto A \otimes \hat{A}, \tag{5.25}
\end{equation}
is a semi-group homomorphism where,

Proof. One first checks that $A \otimes \hat{A}$ is in $\text{Pert}(A \otimes_C A^0, \sigma)$, which is equivalent to show that $\hat{A}$ belongs to $\text{Pert}(A^0, \sigma^0)$. That $\hat{A}$ belongs to $A^0 \otimes A$ comes from $\hat{a}_j = (a_j^*)^0 \in A^0$ and $\hat{b}_j^\circ = b_j^\circ \in A$. The normalisation \textbf{5.12} follows from $\sigma^0((b_j^\circ)^0) = (\sigma^{-1}(b_j^\circ))^0 = (\sigma(b_j)^\circ)^0$:
\begin{equation}
\sum_j (a_j^*)^0 \sigma^0((b_j^\circ)^0) = \sum_j (a_j^*)^0 (\sigma(b_j)^\circ)^0 = \sum_j (\sigma(b_j) a_j^*)^0 = \left( \left( \sum_j a_j \sigma(b_j) \right)^\circ \right)^0 = e^\circ = e. \tag{5.27}
\end{equation}

To show that $\mu$ preserves the product of semi-group, notice that with the notations of lemma \textbf{5.5} and omitting the summation index, one has
\begin{equation}
\hat{A} \hat{A}' = (\hat{a} \otimes \hat{b}^\circ)(\hat{a}' \otimes \hat{b}'^\circ) = \hat{a}\hat{a}' \otimes \hat{b}\hat{b}'^\circ = \hat{a}\hat{a}' \otimes \hat{b}^\circ \hat{b}'^\circ = \hat{A} \hat{A}' \quad \forall A, A' \in \text{Pert}(A, \sigma), \end{equation}
where we used $\hat{a}\hat{a}' = \hat{a}\hat{a}'$ and $\hat{b}\hat{b}'^\circ = (\hat{b}^\circ)^0(\hat{b}')^0 = (\hat{b}^\circ b)^0 = (\hat{b}b)^0 = \hat{b} \hat{b}^\circ = \hat{b}^\circ \hat{b}'^\circ$. Hence
\begin{equation}
\mu(AA') = AA' \otimes \hat{A} \hat{A}' = (A \otimes \hat{A})(A' \otimes \hat{A}') = \mu(A)\mu(A'). \tag{5.28}
\end{equation}

\textbf{Remark 5.7.} The semi-group defined in \textbf{3} is $\text{Pert}(A \otimes_C \hat{A})$ where $\hat{A}$ denotes the image of $A$ under the conjugation by $J$. This notation is somehow more coherent with the map $\mu$ defined above. However, here we prefer to define $\text{Pert}(A \otimes_C A^0)$ for it is more coherent with the mapping to opposite twisted 1-forms. One should be careful that the "natural map" between $\text{Pert}(A, \sigma)$ and $\text{Pert}(A \otimes_C A^0)$
\begin{equation}
A \longrightarrow A \otimes A^0 \tag{5.26}
\end{equation}
is not a semi-group homomorphism for $A \in \text{Pert}(A, \sigma)$ does not imply $A^0 \in \text{Pert}(A^0, \sigma^0)$. This is because the normalisation condition defining $\text{Pert}(A^0, \sigma^0)$ is not equivalent to the one defining $\text{Pert}(A^0, \sigma^0)$ (see \textbf{B.2}).

The action of $\text{Pert}(A \otimes_C \hat{A}, \sigma)$ on $L(H)$ is defined by combining the actions \textbf{5.19} and \textbf{5.20}
\begin{equation}
\left( \sum_k A_k \otimes B_k^\circ, T \right) := \sum_k (A_k, (B_k^\circ, T)) \quad \forall \sum_k A_k \otimes B_k^\circ \in \text{Pert}(A \otimes_C \hat{A}, \sigma). \tag{5.27}
\end{equation}
By the order zero condition, this action is equal to
\begin{equation}
\left( \sum_k A_k \otimes B_k^\circ, T \right) = \sum_k (B_k^\circ, (A_k, T)). \tag{5.28}
\end{equation}

\textbf{Proposition 5.8.} The action \textbf{5.27} of $\mu(\text{Pert}(A, \sigma))$ on the Dirac operator $D$ of a real twisted spectral triple \textbf{5.1} yields the twisted fluctuation \textbf{3.77}:
\begin{equation}
(\mu(A), D) = D_\omega \quad \text{for} \quad \omega = \eta(A). \tag{5.29}
\end{equation}
Moreover, the product \textbf{5.2} in $\text{Pert}(A, \sigma)$ encodes the transitivity of the fluctuations (a twisted fluctuation of a twisted fluctuation is a twisted fluctuation).
Proof. Let $A = \sum_j a_j \otimes b_j^\gamma$ in Pert$(A, \sigma)$. Then
\[
\mu(A) = A \otimes \hat{A} = \sum_{j,i} a_j \otimes b_j^\gamma \otimes \hat{a}_i \otimes \hat{b}_i^\gamma
\]  
(5.30)
so that (using the twisted normalisation conditions for $A$ and $\hat{A}$)
\[
(\mu(A), D) = \sum_{j,i} a_j \hat{a}_i \sigma(b_j) b_j = \sum_{j,i} a_j \hat{a}_i \big( \sigma^\circ(b_j) D + [D, b_j]_\sigma \big) b_j,
\]
\[
= \sum_{j} a_j D b_j + \sum_{j,i} a_j \hat{a}_i [D, \hat{b}_i]_\sigma b_j,
\]
\[
= D + \sum_{j} a_j [D, b_j]_\sigma + \sum_{j,i} a_j \big( \sigma(b_j) \hat{a}_i [D, \hat{b}_i]_\sigma + [\hat{a}_i, [D, \hat{b}_i]]_\sigma^\circ, b_j \big),
\]
\[
= D + \sum_{j} a_j [D, b_j]_\sigma + \sum_{i,j} \hat{a}_i [D, \hat{b}_i]_\sigma + \sum_{j,i} a_j [\hat{a}_i [D, \hat{b}_i]]_\sigma^\circ b_j,
\]
\[
= D + \omega(1) + \hat{\omega}(1) + \omega(2) = D_\omega,
\]
where the last equation follows from the formula of $\omega(2)$ below (3.86).

For the second statement, we need to show that the action (5.27) is transitive. For
$M = A \otimes B^0, M' = A' \otimes B'^0$ in Pert$(A \otimes C, \hat{A}, \sigma)$, one has
\[
(M, (M', T)) = (M, (A', (B'^0, T))) = (B^0, (A, (A', (B'^0, T)))) = (B^0, (B^0, (AA', T))) = (AA', (B^0 B'^0, T)),
\]  
(5.31)
\[
= (AA' \otimes B^0 B'^0, T) = ((A \otimes B^0)(A' \otimes B'^0), T) = (MM', T).
\]  
(5.34)
Together with lemma (5.6) this yields
\[
(\mu(A'), (\mu(A), D)) = (\mu(A')\mu(A), D) = (\mu(A'A), D).
\]
\[
\square
\]
This proposition (5.8) is a straightforward generalization to the twisted case of proposition 5 of [6]. This shows that in the twisted case as well, the transition from ordinary to real spectral triples is encoded by the homomorphism $\mu$.

The group of unitary $\mathcal{U}(A)$ maps to Pert$(A \otimes C, \hat{A}, \sigma)$ composing the inclusion (5.10) of $\mathcal{U}(A)$ in Pert$(A, \sigma)$) with the homomorphism $\mu$, that is
\[
\mu(p(u)) = \mu(\sigma(u) \otimes u^\sigma) = \sigma(u) \otimes u^\sigma \otimes \sigma^\circ(\hat{u}) \otimes u,
\]
where we used $\hat{\sigma}(u) = (\sigma(u)^*)^\circ = (\sigma^{-1}(u^*))^\circ = \sigma^\circ(\sigma(u) u^\sigma) = \sigma^\circ(\hat{u})$ and $u^\circ = \hat{u} = u$. Its action on action $\mathcal{L}(\mathcal{H})$ is - remembering (4.12) and (4.13) -
\[
(\mu(p(u)), T) = \sigma(u) \sigma^\circ(\hat{u}) T u^\sigma u^* = \text{Ad}(\sigma(u)) T \text{Ad}(u)^* \forall T \in \mathcal{L}(\mathcal{H}).
\]  
(5.35)
Thus the gauge transformation of Prop[4.2] is given by the action of $\mu(p(u))$ on the twisted fluctuate Dirac operator $D_\omega$. The latter being obtained by the action of $\mu(A)$ for $\omega = \eta(A)$, one has
\[
D \xrightarrow{\mu(A)} D_\omega \xrightarrow{\mu(p(u))} D_{\omega^u}.
\]
6 Example: the twisted \( U(1) \times U(2) \) model

To illustrate our results, we work out in this final part the twisting of the \( U(1) \times U(2) \) model presented in \([6]\), applying the minimal twist introduced in \([25]\). Just to mention it, the minimal twist of the canonical triple associated to a closed spin\(^c\) Riemannian manifold satisfies the twisted first-order condition, so it is a trivial example in the present context.

The model starts with the general classification of irreducible finite geometries of \( KO \)-dimension 6 done in \([3]\). In the simplest interesting case, the algebra and Hilbert space are

\[
\mathcal{A} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), \quad \mathcal{H} = (\mathbb{C}^2 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^2).
\] (6.1)

The elements of \( \mathcal{H} \) are labelled by two multi-indices \( A = \alpha I, A' = \alpha'I' \) where \( \alpha = 1, 2 \) and \( I = 1, 2 \) label the first summand, while \( \alpha' = 1, 2, I' = 1, 2 \) label the second one. Any \( \psi \in \mathcal{H} \) thus writes

\[
\psi = \begin{pmatrix} \psi_A \\ \psi_{A'} \end{pmatrix}
\] (6.2)

where \( \psi_{A'} \) is the conjugate spinor to \( \psi_A \). The Dirac operator is

\[
D = \begin{pmatrix} D^B_A & D^{B'}_{A'} \\ D^{B'}_A & D^B_{A'} \end{pmatrix} \quad \text{with} \quad \begin{cases} D^B_A = D^B_{\alpha I} = \begin{pmatrix} 0 & k_x \\ k_x & 0 \end{pmatrix} \delta_I^\alpha, \\
D^{B'}_{A'} = D^{B'}_{\alpha'I'} = \begin{pmatrix} k_y & 0 \\ 0 & 0 \end{pmatrix} \delta_{I'}^{\alpha'}, \end{cases}
\] (6.3)

where \( k_x, k_y \) are two complex constants. The grading and real structure are

\[
\Gamma = \begin{pmatrix} \Gamma_A^B & 0 \\ 0 & -\Gamma_{A'}^{B'} \end{pmatrix} \quad \text{where} \quad \Gamma_A^B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta_I^\alpha, \text{ and similarly for } \Gamma_{A'}^{B'}; \quad (6.4)
\]

\[
J = \begin{pmatrix} 0 & J^A_{B'} \\ J^B_A & 0 \end{pmatrix} \quad \text{where} \quad J^B_A = \delta_B^\alpha \delta_I^\alpha, \text{ and similarly for } J^B_{A'}. \quad (6.5)
\]

They satisfy \( J^2 = I, JD = DJ \) and \( J\Gamma = -\Gamma J \). The first \( M_2(\mathbb{C}) \) in \( \mathcal{A} \) acts on the indices \( \alpha \). Its commuting with \( \Gamma \) makes it break into two copies of \( \mathbb{C} \), thus defining the even part of \( \mathcal{A} \).

\[
\mathcal{A}_{ev} = \mathbb{C} R \oplus \mathbb{C} L \oplus M_2(\mathbb{C}), \quad (6.6)
\]

whose element \( a = (\lambda_R, \lambda_L, m) \) with \( \lambda_R \in \mathbb{C}^R, \lambda_L \in \mathbb{C}^L \) and \( m \in M_2(\mathbb{C}) \) act on \( \mathcal{H} \) as

\[
\pi_0(a) = \begin{pmatrix} a^B_A & 0 \\ 0 & a^{B'}_{A'} \end{pmatrix} \quad \text{where} \quad a^B_A = \begin{pmatrix} \lambda_R & 0 \\ 0 & \lambda_L \end{pmatrix} \delta_I^\alpha \text{ and } a^{B'}_{A'} = \delta_B^\alpha m_{I'}. \quad (6.7)
\]

The spectral triple \( (\mathcal{A}_{ev}, \mathcal{H}, D) \) does not satisfy the first order condition \([6, \text{Prop.7}]\).

The twisting by grading, formalised in \([25]\), consists in letting two copies of \( \mathcal{A}_{ev} \) act independently on the eigenspaces of \( \Gamma \). The latter are the image of \( \mathcal{H} \) under the projections

\[
p_+ = \frac{1}{2}(I + \Gamma) = \begin{pmatrix} \delta_1^I & 0 \\ 0 & \delta_2^I \end{pmatrix} \quad \text{and} \quad p_- = \frac{1}{2}(I - \Gamma) = \begin{pmatrix} 0 & \delta_1^I \\ \delta_2^I & 0 \end{pmatrix}. \quad (6.8)
\]

For \( a^r = (\lambda_R^r, \lambda_L^r, m^r) \), \( a^l = (\lambda_R^l, \lambda_L^l, m^l) \) in \( \mathcal{A}_{ev} \), one thus defines the representation \( \pi \) of \( \mathcal{A}_{ev} \oplus \mathcal{A}_{ev} \),

\[
\pi((a^r, a^l)) = p_+ \pi_0(a^r) + p_- \pi_0(a^l) = \begin{pmatrix} Z^B_A & 0 \\ 0 & M^B_{A'} \end{pmatrix}. \quad (6.9)
\]
where
\[ Z_A^B = \left( \begin{array}{cc} \lambda_R^r & 0 \\ 0 & \lambda_L^l \end{array} \right) \delta^I_{\alpha} \text{ and } M_{A'}^B = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) (m^r)^I_{\alpha'} + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \delta^I_{\alpha} (m^l)^I_{\alpha'} = \left( \begin{array}{cc} (m^r)^I_{\alpha'} & 0 \\ 0 & (m^l)^I_{\alpha'} \end{array} \right) \delta^I_{\alpha'} . \]

The twisting automorphism is the flip \( \sigma((a^r,a^l)) = (a^l,a^r) \), so that
\[ \pi(\sigma((a^r,a^l))) = \left( \begin{array}{cc} W_A^B & 0 \\ 0 & N_A^B \end{array} \right) \]

where
\[ W_A^B = \left( \begin{array}{cc} \lambda_L^r & 0 \\ 0 & \lambda_L^l \end{array} \right) \delta^I_{\alpha} \text{ and } N_A^B = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) (m^l)^I_{\alpha'} + \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) (m^r)^I_{\alpha'} = \left( \begin{array}{cc} (m^l)^I_{\alpha'} & 0 \\ 0 & (m^r)^I_{\alpha'} \end{array} \right) \delta^I_{\alpha'} . \]

The resulting triple \((A_{ev} \oplus A_{ev}, H, D)\), \(\sigma,J,\Gamma\) is a real, graded, twisted spectral triple with the same \(KO\)-dimension of \((A,H,D)\), \(J,\Gamma\). However it does not satisfy the twisted first-order condition, as can be checked computing the inner fluctuation of \(D\).

**Proposition 6.1.** An inner twisted fluctuation of \((A_{ev} \oplus A_{ev}, H, D)\), \(\sigma\) is parametrised by \(6\) complex parameters \(\phi, \phi', \sigma_1, \sigma_2, \sigma_1', \sigma_2'\) that enter the components of \(D_{\omega} (3.77)\) as
\[
\begin{align*}
(D_{\omega})_{\alpha I}^J &= \left( \delta^I_{\alpha} \delta^J_{\epsilon} k_x (1 + \phi) + \delta^I_{\alpha} \delta^J_{\epsilon} k_x (1 + \phi') \right) \delta^J_{\epsilon}, \\
(D_{\omega})_{\alpha' I'}^J &= \left( \delta^J_{\epsilon} \delta_{\alpha' \epsilon} k_x (1 + \phi) + \delta^J_{\epsilon} \delta_{\alpha' \epsilon} k_x (1 + \phi') \right) \delta^J_{\epsilon'}, \\
(D_{\omega})_{\alpha I}^{J'} &= k_y \delta^J_{\epsilon} \delta^J_{\epsilon} (\sigma_1 + \delta^J_{\epsilon}) (\sigma_1' + \delta^J_{\epsilon'}), \\
(D_{\omega})_{\alpha' I'}^{J'} &= k_y \delta^J_{\epsilon} \delta^J_{\epsilon} (\sigma_1 + \delta^J_{\epsilon}) (\sigma_1' + \delta^J_{\epsilon'}).
\end{align*}
\]

*Imposing the twisted 1-form \(\omega(1)\) to be self-adjoint implies \(\phi = \overline{\phi}\) and \(\sigma = \overline{\sigma} \) so that the number of free parameters reduces to \(3\).*

**Proof.** We first compute the twisted one form \(\omega(1) = \sum_j \pi(a_j)[D,\pi(b_j)]_\sigma\). From (6.3),(6.8),(6.9), one has for \(a = ((\lambda^r_R, \lambda^l_L, m^r), (\lambda^r_R, \lambda^l_L, m^l))\) and \(b = ((\lambda^r_R, \lambda^l_L, m^r), (\lambda^l_R, \lambda^r_L, m^l))\)
\[
[D,\pi(b)]_\sigma = \left( D_A^B Z_A^B - W_A^B D_A^B \right) \left( D_A^B Z_A^B - N_A^B D_A^B \right) \left( D_A^B M_A^B - W_A^B D_A^B \right) ;
\]

where one computes
\[
\begin{align*}
D_A^B Z_A^B - W_A^B D_A^B &= \left( k_x (\lambda^r_R - \lambda^l_L) \right) \delta^J_{\alpha}, \\
D_A^B Z_A^B - N_A^B D_A^B &= \left( k_y (\lambda^r_R \delta^J_{\alpha} - (m^r)^I_{\alpha'}) \right) \delta^J_{\alpha'}, \\
D_A^B M_A^B - W_A^B D_A^B &= \left( \bar{k}_y ((m^l)^I_{\alpha'} - \lambda^l_R \delta^J_{\alpha}) \right) \delta^J_{\alpha}.
\end{align*}
\]

Hence \(\omega(1)\) has diagonal components \((\omega(1))_{\alpha I}^{J'} = 0\), \((\omega(1))_{\alpha I}^{J'} = Z_A^B (D_A^B Z_A^B - W_A^B D_A^B)\) (we omit the summation index \(j\), that is
\[
\begin{align*}
(\omega(1))_{\alpha I}^{J'} &= 0, \\
(\omega(1))_{\alpha I}^{J'} &= k_x \delta^I_{\alpha} \delta^J_{\alpha} \phi + \bar{k}_x \delta^J_{\alpha} \delta^I_{\alpha} \phi'.
\end{align*}
\]

where one defines the complex parameters
\[
\begin{align*}
\phi := \sum_j \lambda^r_R (|\lambda^l_L - \lambda^r_R|) \\
\phi' := \sum_j \lambda^l_L (|\lambda^r_R - \lambda^l_L|).
\end{align*}
\]
The off-diagonal components of $\omega(1)$ are $Z_B^B(D_A^B M_A^B - W_A^B D_A^B)$, $M_B^B(D_A^B Z_A^B - N_A^B D_A^B)$, that is

$$(\omega(1))_{\alpha I}^{\beta J} = \tilde{b}_y \delta_1^\alpha \delta_1^\beta \delta_1^J \sigma^J, \quad (\omega(1))_{\alpha' I'}^{\beta' J'} = \delta_2^\alpha \delta_2^\beta \delta_2^J \sigma^J$$

where

$$(\sigma^J := \sum_j \chi^R_j (m^J)^1_j - \chi^I_j (m^J)^1_j \sigma^J, \quad (\alpha') := \sum_j (m^J)^1_j \chi^R_j - (m^J)^1_j \chi^I_j).$$

The next term is $\hat{\omega}(1) = J\omega(1)J^{-1} = J\omega(1)J$. By (6.5) one easily obtains

$$(\hat{\omega}(1))_{\alpha I}^{\beta J} = 0, \quad (\hat{\omega}(1))_{\alpha' I' J'} = (\omega(1))_{\alpha I}^{\beta J} = \tilde{b}_y \delta_1^\alpha \delta_2^\beta \delta_1^J \delta_2^J \tilde{\phi}, \quad (\hat{\omega}(1))_{\alpha' I' J'} = (\omega(1))_{\alpha' I' J'} = \delta_2^\alpha \delta_2^\beta \delta_2^J \delta_2^J \sigma^J.$$ (6.22)

The quadratic term $\omega(2) = \sum_i a_i [\hat{\omega}(1), b_i]_\sigma$ is computed as $\hat{\omega}(1)$, substituting the components of $D$ with those of $\omega(1)$. The latter have the same indices structure as the components of $D$ so the computation is similar and one obtains

$$(\omega(2))_{\alpha I}^{\beta J} = (\omega(2))_{\alpha' I' J'} = 0, \quad (\omega(2))_{\alpha' I' J'} = \tilde{b}_y \delta_1^\alpha \delta_2^\beta \delta_1^J \delta_2^J \sigma^J.$$ (6.23)

The result follows summing up (6.24)-(6.25), (6.22)-(6.23), (6.18)-(6.20) with (6.3). □

In case $\omega(1)$ is selfadjoint (i.e. $\phi' = \tilde{\phi}$ and $\sigma^I = \tilde{\sigma}^I$), the components (6.10)-(6.13) of $D_\omega$ are the same as in the non-twisted case. The only difference is in the relation (6.19), (6.21) between the complex parameters $\phi, \sigma^I$ and the algebra elements. One finds back the formula of the non-twisted case [6] identifying the $l$ and $r$ indices.

7 Outlook

There is a way to twist the spectral triple of the Standard Model in order to generate an extra scalar field, as investigated in [20]. In the $U(1) \times U(2)$ model above there is no such extra field, for the part $D^B_A$ of the Dirac operator that commutes with the algebra also twist-commutes with it. Instead, the part of the Dirac operator of the Standard Model that commutes with the algebra (namely the one containing the Majorana mass of the neutrino) no longer twist-commutes with the algebra. A systematic study of the minimal twisting of almost commutative geometries, in relation with the generation of extra scalar fields and the twisted first-order condition is on its way [21].

In the non-twisted case, the first order condition is retrieved dynamically by minimising the spectral action. A similar thing occurs with the partial twist of the Standard Model performed in [13]. Whether this happens in other examples of minimal twist (as for the $U(1) \times U(2)$ model above should be investigated in a systematic way; but this requires first to stabilise a definition for the spectral action in a twisted context. Some ideas have been proposed in [13] but deserve more study, especially in the light of a possible signature transition towards the lorentzian [15][27].

Finally let us mention another alternative to [6] proposed in [12] based on spectral triples in which the real structure is twisted [2] (see recent developments in [13] and [14]). A link with the twisted spectral triples used in this paper is worked out in [1] (see also [22] for another approach on untwisting a twisted spectral triple).
Appendices

A  Morita equivalence by left module

As announced at the beginning of §3.2, we show that a twisted spectral triple \((A, \mathcal{H}, D), \sigma\) with real structure \(J\) can be exported to a Morita equivalent algebra \(B\) - as a twisted spectral triple, but not as a real one - when the module implementing the equivalence is \(A\)-left finite projective,

\[ \mathcal{F} \simeq A^n e, \quad (A.1) \]

whose generic element is a raw vector \(\zeta = (\zeta^1, \ldots, \zeta^n)\) with components \[\text{(1)}\]

\[ \zeta^i = \zeta^i e_j^* \in A. \quad (A.2) \]

It is hermitian for the product

\[ \{\zeta^i, \zeta^j\} = \sum_i \zeta^i \zeta^i, \quad (A.3) \]

which satisfies the left version of (2.26)

\[ \{a \zeta^i, \zeta^j\} = a \{\zeta^i, \zeta^j\}, \quad \{\zeta^i, a \zeta^j\} = \{\zeta^i, \zeta^j\} a^* \quad a \in A. \quad (A.4) \]

A.1  Twisted hermitian connection for left modules

We adapt the results of sections 2.3 and 2.4 to left-modules. The lift \(\Sigma^0\) of \(\sigma\) to \(\mathcal{F}\),

\[ \Sigma^0((\zeta^1, \ldots, \zeta^n)) := (\sigma(\zeta^1), \ldots, \sigma(\zeta^n)) e \quad (A.5) \]

is invertible if and only if conditions (2.32) holds (the proof is as for lemma 2.4). It lifts to \(B = \text{End}_A(\mathcal{F}) \simeq \mathcal{M}(A)\) as defined in (2.35),

\[ \Sigma^0(b) = e\sigma(b)e, \quad (A.6) \]

and satisfies the regularity condition (2.36) (proof as in Prop. 2.6).

To define a hermitian \(\Omega^1_j(A^0, \sigma^0)\)-value connection \(\nabla^0\) on \(\mathcal{F}\), recall that the derivation \(\delta^0\) is not anti-hermitian but satisfies (2.18), and that the involution of \(\Omega^1_j(A^0, \sigma^0)\) which follows from its action on \(\mathcal{H}\) - \((a^0 \delta^0(b^0))^* = \delta^0(b^0)^*(a^0)^*\) - is such that

\[ (\omega^0 a)^* = (\sigma^0(a^0)) a^0, \quad (A.7) \]

\[ (a \cdot \omega^0)^* = (\omega^0 a)^* = (a^0)^* a^0, \quad (A.8) \]

where we use the module law (3.10), then from (2.5) and (2.1)

\[ \sigma^0(a^0)^* = (\sigma^{-1}(a)^0)^* = (\sigma^{-1}(a)^0)^0 = (\sigma(a^0))^0, \quad (A.9) \]

\[ \sigma^0(a^0) = (\sigma^{-1}(a^0)) = (\sigma(a^0))^0. \quad (A.10) \]

These laws are compatible, for \(\sigma(\sigma(a^0)^*) = \sigma^{-1}(\sigma(a^0)^*) = (a^0)^* = a.\)

This motivates to adapt Definition 2.7 posing that a \(\Omega^1_j(A^0, \sigma^0)\)-connection \(\nabla^0\) on \(\mathcal{F}\) is hermitian if it satisfies

\[ -\{\zeta^i, \nabla^0(\Sigma^0 \zeta)\} + \{\nabla^0 \zeta^i, \zeta\} = \delta^0(\{\zeta^i, \zeta\}) \quad \forall \zeta, \zeta' \in \mathcal{F}, \quad (A.11) \]

where one defines (notice the difference with (2.41)

\[ \{\nabla^0 \zeta^i, \zeta\} = \zeta^i(-1) \cdot \{\zeta^i(0), \zeta\}, \quad \{\zeta^i, \nabla^0 \zeta\} = \{\zeta^i, \zeta(0)\} \cdot \zeta^i(-1). \quad (A.12) \]

\(^1\)We use a similar convention as in relativity, changing the component index from low to up when passing from a right to a left module.
Indeed, (A.11) is precisely the compatibility condition with the inner product (A.3) which is satisfied by the $\Omega^1_D(A^0, \sigma^o)$-value Grassmann connection on $\mathcal{F}$

$$\nabla_0^\Sigma \zeta := \delta^o(\zeta') \otimes (e_j, \ldots, e_j) \simeq \delta^o(\zeta'), \ldots, \delta^o(\zeta^n) \cdot e \quad \forall \zeta = \zeta \in \mathcal{F}.$$  

(A.13)

**Lemma A.1.** Assuming the idempotent $e$ satisfies (2.46), then the Grassmann connection (A.13) is hermitian. Furthermore, any hermitian connection $\nabla^\Sigma$ on $\mathcal{F}$ is the sum of $\nabla^\Sigma_0$ with a selfadjoint element $N^o$ of $M_n(\Omega^1_D(A^0, \sigma^o))$.

**Proof.** The proof is similar as in lemma (2.8) $\Sigma^o \zeta'$ has components $S^{ij} = \sigma(\zeta') e_j^i$, so by (A.13) $\nabla^\Sigma_0(\Sigma^o \zeta') = \delta^o(S^j) \cdot e$. If $e = \sigma(e)$, then $\delta^o(S^j) = \delta^o(\sigma(\zeta')e_j)$. Otherwise $e$ twist-commuting with $D$ implies $\delta^o(e) = e' \delta(e) = 0$ and $\delta^o(S^j) = \delta^o(\sigma(\zeta')e_j) \cdot e$. In any case

$$\nabla^\Sigma_0(\Sigma^o \zeta') = \delta^o(\sigma(\zeta')) \cdot e = \delta^o(\sigma(\zeta')) \otimes e.$$  

(A.14)

By (2.18) one gets

$$\{\zeta', \nabla^\Sigma_0(\Sigma^o \zeta')\} = \{\zeta', (e_j)\} \cdot \delta^o(\sigma(\zeta')) \simeq \{\zeta', (e_j)\} \cdot \delta^o(\zeta,e),$$

$$\{\nabla^\Sigma_0 \zeta', \zeta\} = \delta^o(\zeta') \cdot \{(e_j), \zeta\} = \delta^o(\zeta') \cdot \zeta',$$

where we compute $\{(\zeta', (e_j))\} = \sum \zeta' e_j \cdot \zeta' = \zeta'$ and $\{(e_j), \zeta\} = \zeta'$. Hence the l. h. s. of (A.11) is $\zeta' \cdot \delta^o(\zeta) + \delta^o(\zeta') \cdot \zeta' = \delta^o(\zeta')$, meaning $\nabla^\Sigma_0$ is hermitian.

By Leibniz rule (2.20) the difference $\tilde{\nabla}^\Sigma := \nabla^\Sigma - \nabla^\Sigma_0$ is $A$-linear - $\nabla^\Sigma(\alpha \zeta) = \alpha \tilde{\nabla}^\Sigma(\zeta)$ - meaning that

$$\tilde{\nabla}^\Sigma \zeta = (\zeta' \cdot n_k^j) \otimes (e_j) \simeq \zeta \cdot N^o \quad \forall \zeta \in \mathcal{F}$$  

(A.15)

with $N^o$ a matrix in $M_n(\Omega^o)$ with components $n_k^j \in \Omega^o$ such that $N^o = e \cdot N^o \cdot e$. Thus $\tilde{\nabla}^\Sigma(\zeta') = (\zeta' \cdot n_k^j) \otimes (e_j)$ and $\tilde{\nabla}^\Sigma(\Sigma^o \zeta) = (\sigma(\zeta') \cdot n_k^j) \otimes (e_j)$. So the hermicity implies

$$0 = \{\zeta', \tilde{\nabla}^\Sigma(\Sigma^o \zeta)\} - \{\tilde{\nabla}^\Sigma \zeta', \zeta\},$$

$$= \sum \zeta' \cdot (\sigma(\zeta') \cdot n_k^j)^* - (\zeta' \cdot n_k^j)^* \cdot \zeta' = \sum \zeta' \cdot (n_k^j)^* \cdot (n_{j_1}^{k_1} - n_{j_2}^{k_2}) \cdot \zeta'$$

(A.16)

where, for the last equality, we use (A.8) as $(\sigma(\zeta') \cdot n_k^j)^* = n_k^j \cdot \sigma(\zeta') \cdot n_k^j$ then exchange $k$ with $j$. This should be true for any $\zeta$, hence the matrix $N^o$ is selfadjoint. $$\square$$

**A.2 Twisted fluctuation by left module**

This is an adaptation of §3.1 to left $A$-module. Given a twisted spectral triple $(A, \mathcal{H}, D, \sigma$ with real structure $J$ and the left $A$-module (A.1), then

$$\mathcal{H}_L := \mathcal{H} \otimes_A \mathcal{F}$$  

(A.18)

is a (pre)-Hilbert space for the inner product

$$\langle \psi' \otimes \zeta', \psi \otimes \zeta \rangle := \langle \psi' \langle \zeta', \zeta \rangle, \psi \rangle_{\mathcal{H}}$$  

(A.19)

where the right action of $A$ on $\mathcal{H}$ is given in (2.11). This carries a representation of $\mathcal{B}$,

$$\pi_L(b)(\psi \otimes \zeta) := \psi \otimes \zeta b \quad \forall \psi \in \mathcal{H}, \quad \zeta \in \mathcal{F}, \quad b \in \mathcal{B},$$  

(A.20)

an the action of $D$

$$(D \otimes \nabla^\Sigma)(\psi \otimes \zeta) := D\psi \otimes \zeta + \nabla^\Sigma(\zeta)\psi$$  

(A.21)
where $\nabla^\circ$ is an $\Omega^\circ$-connection and (remembering (2.22))
\[
(\nabla^\circ \zeta) \psi = \zeta(-1) \psi \otimes \zeta(0).
\tag{A.22}
\]
where the action of $\zeta(-1)$ on $H$ comes from the representation of $\Omega^\circ$ as bounded operator on $H$. Denoting $\Sigma^\circ$ the lift (A.5) of $\sigma$ on $F$, then the operator
\[
D_L := (\mathbb{I} \otimes \Sigma^{-1}) \circ (D \otimes \nabla^\circ \mathbb{I})
\tag{A.23}
\]
is well defined on $H_L$ [26, Prop.3.9].

**Proposition A.2.** Assume the lift $\Sigma^\circ$ is invertible (that is $\Sigma^\circ(2.32)$ holds) and the idempotent satisfies $\Sigma^\circ(2.46)$. Then the twisted commutator $[D_L, b]$ is bounded for any $b \in B$ acting on $H_L$ according to $\Sigma^\circ(2.20)$, with $\sigma'$ the lift of $\sigma$ to $B$ defined in Prop. $\Sigma^\circ(2.6)$.

**Proof.** This is a straightforward adaptation of the proof of Prop. $\Sigma^\circ(3.1)$. A generic element of $H_L = H \otimes_A F \cong H^n e$ is
\[
\Phi := \psi^p \otimes \zeta_p = \psi^p \otimes (\zeta_p^i e^i_j) = \psi^j \otimes (e^i_j) \simeq (\psi^1, \ldots, \psi^n)e
\tag{A.24}
\]
where $\psi^p$ is generic element of $H$ and $\psi^j := \psi^p e^p_j \in H$. Denoting $\nabla^\circ = \nabla^\circ - \nabla_0^\circ$, one gets
\[
(D \otimes \nabla_0^\circ \mathbb{I}) \Phi = D \psi^j \otimes (e^i_j) + \nabla_0^\circ(e^i_j) \psi^j + \nabla_0^\circ(e^i_j) \psi^j
\tag{A.25}
\]
for any $\Phi$ with components $\psi^j$ in the domain of $D$. The second term vanishes. This is obvious in case $e$ twist commutes with $D$, for by $\Sigma^\circ(3.13)$ one as $\nabla_0^\circ((e^i_j)) = \delta^j(e^i_j) \otimes (e^k_j)$. Otherwise, $\Sigma^\circ(2.22)$ and $\Sigma^\circ(2.2)$ yield (remembering $\Sigma^\circ(2.11)$)
\[
\nabla_0^\circ((e^i_j)) \psi^j = \delta^j(e^i_j \psi^j) \otimes (e^k_j) = \delta^j(e^i_j \psi^j) \otimes \delta^k(e^i_j \psi^j) \otimes (e^k_j e^k_m) = \delta^k(e^i_j \psi^j) \otimes (e^k_m)
\tag{A.26}
\]
by $e^i_j$ and summing on $j$ yields $0 = \sigma(e^i_m \psi^j) \psi^j(e^i_j) = \sigma(e^i_m \psi^j) \psi^j(e^i_j)$ for all $m, l$. So in any case the second term in the r.h.s of (A.25) vanishes. The third term reads, by $\Sigma^\circ(3.15)$, $\Sigma^\circ(3.4)$ and denoting $N$ the matrix with components $n^l_i \in \Omega$,
\[
(\nabla_0^\circ((e^i_j))) \psi^j = (e^i_j \cdot n^l_i) \psi^j \otimes (e^l_j) = n^l_i \psi^j \otimes (e^l_j) = \psi^j n^l_i \otimes (e^l_j) \simeq \Phi N e
\tag{A.27}
\]
Applying $\mathbb{I} \otimes \Sigma^{-1}$ on (A.25) yields
\[
D_L \Phi = D \psi^j \otimes (\sigma^{-1}(e^i_j)) e + \psi^j n^l_i \otimes (\sigma^{-1}(e^l_j)) e \simeq (D \Phi + \Phi N) e
\tag{A.28}
\]
where $D \Phi$ denotes the operator $D$ acting on each components $\psi^j$ of $\Phi$. If $e$ is twist-invariant, then $\sigma^{-1}(e) = e$ and the above reads
\[
D_L \Phi = (D \psi^j + \psi^j n^l_i) \otimes (e^l_j) \simeq (D \Phi + \Phi N) e.
\tag{A.29}
\]
The same is true of $e$ twist-commutes with $D$, since
\[
(D \psi^j) \sigma^{-1}(e^i_j) = \sigma^{-1}(e^i_j) \psi^j = D(e^i_j \psi^j) = D(e^i_j e^i_j) = D(e^i_j)
\tag{A.30}
\]
\[
(n^l_i \psi^j) \sigma^{-1}(e^i_j) = \sigma^{-1}(e^i_j) n^l_i \psi^j = \sigma(e^i_j) \sigma^{-1}(e^i_j) n^l_i \psi^j = n^l_i \psi^j = n^l_i \psi^j
\tag{A.31}
\]
One twists with the inverse of $\Sigma^\circ$ so that the incompatibility of the operator $D \otimes 1$ with the tensor product is captured by an $\Omega^\circ_1(A^\circ, \sigma^\circ)$ connection [26] remark 3.12].
where we use $N^0 \cdot e = N^0$ as well as
\[
\sigma^{-1}(e_j^0) \circ D = J \sigma^{-1}(e_j^0) J^{-1} = D J e_j^1 J^{-1} = D(e_j^1) \circ .
\] (A.32)

Consider now $b = e b$ in $B$ with components $b_i^1 \in A$. Repeating the computation of
(A.24) with (A.20) yields
\[
\pi_L(b) \Phi = \psi^p \otimes \zeta_p b = \psi^p \otimes (\zeta_p b_k^b e_j^0) = \psi^k b_k^b \otimes (e_j^0) \simeq \Phi b.
\] (A.33)

Denoting $D_0$ the operator $D_L$ when $\nabla^0 = \nabla^0_0$ is the Grassmann connection (i.e. $N = 0$), one has
\[
D_0 \pi_L(b) \Phi = D \left( \psi^i b_i^1, \ldots, \psi^i b_i^0 \right) e = \left( D b_1^0 \psi^1, \ldots, D b_n^0 \psi^1 \right) e,
\]
\[
\pi_L(\sigma^{-1}(b)) D_0 \psi = (D \psi^1, \ldots, D \psi^n) \sigma^{-1}(e) e (e \sigma^{-1}(b) e) = (D \psi^1, \ldots, D \psi^n) \sigma^{-1}(b) e,
\]
\[
= \left( (D \psi^1) \sigma^{-1}(b_1^1), \ldots, (D \psi^1) \sigma^{-1}(b_n^1) \right) e = \left( \sigma^0 (b_1^1 \circ D \psi^1), \ldots, \sigma^0 (b_n^1 \circ D \psi^1) \right) e,
\]

where to get the second equation we use (2.32) together with $b = e b$ to write
\[
\sigma^{-1}(e) e \sigma^{-1}(b) = \sigma^{-1}(e \sigma^{-1}(b) e) = \sigma^{-1}(e \sigma^{-1}(e) e) = \sigma^{-1}(e) e = \sigma^{-1}(b).
\] (A.34)

Therefore
\[
[D_0, \pi_L(b)] = \psi = \begin{bmatrix} [D, b_1^0] \sigma^0 & \cdots & [D, b_n^0] \sigma^0 \\ \vdots & \ddots & \vdots \\ [D, b_1^0] \sigma^0 & \cdots & [D, b_n^0] \sigma^0 \end{bmatrix} \begin{bmatrix} \psi^1, \ldots, \psi^n \end{bmatrix} \sigma^{-1}(e).
\]

This shows that $[D_0, \pi_L(b)]$ is bounded. The same holds true for a generic connection, as in the proof of [3.1] $\square$

**Proposition A.3.** In the conditions of Prop. A.2 and with $\nabla^0$ hermitian, then $(B, H_L, D_L)$, $\sigma^{-1}$ is a twisted spectral triple.

**Proof.** The proof is similar as in Prop [3.2]. The only point is to check that $D_L$ is self-adjoint which is obtained as in the right module case, noticing that (A.28) reduces to
\[
D_0 \psi = (D \psi_1, \ldots, D \psi^n) e
\] (A.35)

(either obviously in case $e$ is twist-invariant, or because
\[
(D \psi^j) \sigma^{-1}(e_j^1) = (\sigma^{-1}(e_j^1) \circ D \psi^j = \sum_j J \sigma(e_j^1) J^{-1} D \psi^j = \sum_j e_j^1 J \sigma \sigma^{-1}(e_j^1) J D^{-1} D \psi^j,
\]
\[
= \sum_j (e_j^1 D \psi^j J^{-1} D \psi^j = D e_j^1 \psi^j = D (\psi^j e_j^1) = D \psi^j
\]

in case $e$ twist-commutes with $D$). Therefore
\[
\langle \psi', D_0 \psi \rangle = \langle \psi' \otimes (e_1^1), D \psi^k \otimes (e_k^1) \rangle = \langle \psi' \{ (e_1^1), (e_k^1) \}, D \psi^k \rangle_H,
\]
\[
= \sum_k \langle \psi' \otimes (e_k^1), D \psi^k \rangle_H = \sum_k \langle \psi' \otimes (e_k^1), D \psi^k \rangle_H
\] (A.38)

and
\[
\langle D_0 \psi', \psi \rangle = \langle D \psi'^j \otimes (e_j^1), D \psi^k \otimes (e_k^1) \rangle = \sum_k \langle (D \psi'^j) e_j^1, D \psi^k \rangle_H = \sum_k \langle D \psi'^j, D \psi^k \rangle_H
\]

where we use $\langle \psi' \otimes (a^0 \psi'), \psi \rangle_H = \langle \psi', (a^0) \psi \rangle_H = \langle \psi', (a^0) \psi \rangle_H = \langle \psi', (a^0) \psi \rangle_H = \langle \psi', (a^0) \psi \rangle_H$. We are back to eqs. (3.15), (3.16), and the rest of the proof is as in the right module case. $\square$
A.3 Hermitian connection on the conjugate module

The conjugate of the right \( \mathcal{A} \)-module \( \mathcal{E} = e\mathcal{A}^n \) of column vectors with entries in \( \mathcal{A} \) - invariant by left multiplication by \( e \) - is the left \( \mathcal{A} \)-module \( \bar{\mathcal{E}} = \mathcal{A}^n e \) of raw vectors invariant by right multiplication by \( e \). Explicitly, given

\[
\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathcal{E}, \quad \text{then} \quad \bar{\xi} = (\xi_1^*, \ldots, \xi_n^*). \tag{A.39}
\]

In particular the left \( \mathcal{A} \) product is such that

\[
a \bar{\xi} = \bar{\xi} a^* \quad \forall a \in \mathcal{A}, \xi \in \mathcal{E}. \tag{A.40}
\]

The module \( \bar{\mathcal{E}} \) is hermitian for the product (A.3), and one checks by (3.56) that (3.19) holds:

\[
\{ \bar{\xi}', \bar{\xi} \} = \sum_i \bar{\xi}'_i \bar{\xi}^*_i = \sum_i (\xi'_i)^* \xi_i = (\xi', \xi). \tag{A.41}
\]

The selfadjointness of \( e \) makes that, for any \( j = 1, \ldots, n \), one has

\[
\begin{pmatrix} e_j^1 \\ \vdots \\ e_j^n \end{pmatrix} = (e_j^1)^*, \ldots, (e_j^n)^* \tag{A.42}
\]

Identifying module elements with their components, that is \( \xi = (\xi_i) \) and \( \bar{\xi} = (\bar{\xi}^i) \), the equation above writes

\[
(e_j^i) = (e_j^i)^*, \quad \forall j = 1, \ldots, n. \tag{A.43}
\]

The lift \( \Sigma \) to \( \mathcal{E} \) and \( \Sigma^0 \) to \( \bar{\mathcal{E}} \) of an automorphism \( \sigma \) of \( \mathcal{A} \), as defined (2.28) and (A.5), are inverse of one another in that

\[
\Sigma \xi = e \begin{pmatrix} \sigma(\xi_1) \\ \vdots \\ \sigma(\xi_n) \end{pmatrix} = (\sigma(\xi_1)^*, \ldots, \sigma(\xi_n)^*) e = (\sigma^{-1}(\xi_1^*), \ldots, \sigma^{-1}(\xi_n^*)) e = \Sigma^0 \bar{\xi}. \tag{A.44}
\]

then any connection \( \nabla \) on \( \mathcal{E} \) induces a connection \( \bar{\nabla} \) on \( \bar{\mathcal{E}} \) defined as follows.

**Lemma A.4.** Given a \( \Omega^1_D(\mathcal{A}, \sigma) \)-connection \( \nabla \) on \( \mathcal{E} \) as in (2.22), then

\[
\bar{\nabla}(\bar{\xi}) = \bar{\nabla}(\xi a^*), \tag{A.45}
\]

is an \( \Omega^1_D(\mathcal{A}^0, \sigma^0) \)-connection on \( \bar{\mathcal{E}} \) defined by the derivation \( \delta^0 \).

**Proof.** By the twisted first order condition, one has that \( J\xi(1) J^{-1} \) belongs to \( \Omega^1_D(\mathcal{A}^0, \sigma^0) \). The only point is to check the Leibniz rule (2.20) By (A.40).

\[
\bar{\nabla}(a\bar{\xi}) = \bar{\nabla}(\xi a^*), \tag{A.46}
\]

while for the Leibniz rule (2.20) for \( \nabla \)

\[
\nabla(\xi a^*) = \xi(0) \otimes (\xi(1) \cdot a^*) + \xi \otimes \delta(a^*). \tag{A.47}
\]

Hence (A.45) yields

\[
\bar{\nabla}(a\bar{\xi}) = \bar{\nabla}(\xi(1) \cdot a^*) J^{-1} \otimes \bar{\xi}(0) + \bar{\nabla}(\delta(a^*) J^{-1} \otimes \bar{\xi}, \tag{A.48}
\]

\[
= \epsilon' a \cdot J\xi(1) J^{-1} \otimes \bar{\xi}(0) + \delta^0(a) \otimes \bar{\xi}. \tag{A.49}
\]

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where we used
\[ J((\xi_1 \cdot a^*)J^{-1} = J(\xi_1) a^*)J^{-1} = J(\xi_1)J^{-1} a^* = a \cdot J_1 J^{-1} \] (A.50)
which comes from (2.14) and (3.10), as well as
\[ J_\delta(a^*)J^{-1} = J[D, a^*]J^{-1} = \epsilon'[D, J a^*]J^{-1}_\sigma = \epsilon'[D, a^*]J^{-1}_\sigma = \epsilon' \delta^0(a) \] (A.51)
that follows from (2.3) and (2.7). Eq. (A.49), rewritten as
\[ \nabla(a\xi) = a \nabla(\xi) + \delta^0(a) \otimes \xi \] (A.52)
is the Leibniz rule for an \( \Omega^1_D(A^0, \sigma^\omega) \)-connection generated by \( \delta^0 \).

**Lemma A.5.** Let \( \nabla \) be an hermitian connection on \( \mathcal{E} \). Then the connection \( \nabla \) defined in lemma (A.4) is hermitian on \( \mathcal{E} \).

**Proof.** Eq. (A.12) together with the definition of \( \nabla \) in (A.45) yields
\[
\{ \nabla \nabla, \xi \} = \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\} = \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\}, \\
= \epsilon' \sigma^0 \left( \left( \nabla J^{-1} \right)^\sigma \right) J(\xi(1)J^{-1} = \epsilon' \sigma \left( \left( \nabla J^{-1} \right)^\sigma \right) J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\} = \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\}, \\
= \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\} = \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\}, \\
= \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\} = \epsilon' J(\xi(1)J^{-1} \cdot \left\{ \nabla J^{-1} \right\}
\]
where we use (A.41), then (3.10), (2.5), the properties of the map \( \circ \) and finally (2.41). Thus, being \( \nabla \) hermitian by hypothesis, it follows from (2.45) that
\[
- \{ \nabla J^{-1}, \nabla J^{-1}, \xi \} = \epsilon' J(\xi) \left( - \left( \nabla J^{-1}, \xi \right) + \left( \xi, \nabla J^{-1} \right) \right), \\
= \epsilon' J(\xi) \left( - \left( \nabla J^{-1}, \xi \right) + \left( \xi, \nabla J^{-1} \right) \right) = \epsilon' J(\xi) \left( - \left( \nabla J^{-1}, \xi \right) + \left( \xi, \nabla J^{-1} \right) \right)
\] (A.53)
where the former last equation follows from (2.8) written as \( \delta^0(a) = \epsilon' J(\xi) J^{-1} \). Hence \( \nabla \) is hermitian in the sense of (A.11).

In particular, the conjugate of the Grassmann connection \( \nabla_0 \) on \( \mathcal{E} \) is the Grassmann connection \( \nabla_0^\ast \) on \( \mathcal{F} \).

**Lemma A.6.** One has \( \nabla_0 = \nabla_0^\ast \).

**Proof.** From (2.44) and (A.13) one has
\[
\nabla_0(\xi) = \epsilon' J(\xi) J^{-1} \cdot \left( \begin{array}{c} e_{i1} \\ \vdots \\ e_{in} \end{array} \right) = \epsilon^0(\xi^* \otimes (e_{j1}, \ldots, e_{jn})) = \nabla_0^\ast(\xi).
\]
B Semi-group for opposite twisted one-forms

The map $\eta^o$ defined in (5.16) has similar properties as the lap $\eta$ (5.5).

**Lemma B.1.** i) The map $\eta^o$ is surjective; ii) The adjoint is given by

$$
\left( \eta^o \left( \sum_j a_j^o \otimes b_j \right) \right)^* = \eta^o \left( \sum_j b_j^{o*} \otimes a_j^o \right).
$$

iii) The gauge-transformed (4.4) of $\check{\omega} = \eta^o \left( \sum_j a_j^o \otimes b_j \right) \in \Omega^1_D(A^o, \sigma^o)$ is

$$
\check{\omega}^u = \eta^o \left( \sum_j \sigma^o(\upsilon^o) a_j^o \otimes ub_j \right) \quad \forall u \in \mathcal{U}(A).
$$

**Proof.** i) Surjectivity is proven as in lemma (5.4).

ii) The normalisation condition in (5.12) is equivalent to $\sum_j b_j^o \sigma(a_j) = e$, for

$$
(b_j^o \sigma(a_j))^o = \sigma(a_j)^o b_j^o = \sigma^{-1}(a^o) b_j^o = \sigma^{-1}(a^o \sigma(b_j)).
$$

(B.2)

The Leibniz rule (2.16) for $\delta^o(b_j \sigma(a_j)) = \delta^o(e) = 0$ (omitting the symbol of summation) yields $\sigma^o(\sigma(a_j)^o) \delta^o(b_j) + \delta^o(\sigma(a_j)) b_j^o = 0$, that is

$$
a_j^o \delta^o(b_j) = -\delta^o(\sigma(a_j)) b_j^o.
$$

(B.3)

Therefore, for $\sum_j a_j^o \otimes b_j$ in Pert$(A^o, \sigma^o)$ and using (2.18), one has

$$
\left( \eta^o \left( \sum_j a_j^o \otimes b_j \right) \right)^* = \left( \sum_j a_j^o \delta^o(b_j) \right)^* = - \left( \sum_j \delta^o(\sigma(a_j)) b_j^o \right)^* = \sum_j b_j^{o*} \delta^o(a_j^o) = \eta^o \left( \sum_j b_j^{o*} \otimes a_j^o \right).
$$

(B.4)

(B.5)

The result follows noticing that $\sum_j b_j^{o*} \otimes a_j^o$ is normalised, for (2.6) yields

$$
\sum_j b_j^{o*} \sigma^o(a_j^o) = \sum_j b_j^{o*} \sigma^{-1}(a_j^o)^* = \sum_j \left( \sigma^{-1}(a_j^o)^o b_j^o \right)^* = \sum_j \sigma^{-1}(a_j^o) \sigma^o(b_j^o)^* = \sigma^{-1}(e)^* = e.
$$

iii) Let us first check that for $a_j^o \otimes b_j$ in Pert$(A^o, \sigma^o)$ (omitting the symbol of summation), then the argument of $\eta^o$ in (B.4) is normalised:

$$
\sigma^o(\upsilon^o) a_j^o \sigma^o((\upsilon^o) \upsilon^o) = \sigma^o(\upsilon^o) a_j^o \sigma^o(b_j^o) \sigma^o(\upsilon^o) = \sigma^o(\upsilon^o) \sigma^o(\upsilon^o) = e.
$$

(B.6)

From the Leibniz rule (2.16), one gets

$$
\eta^o \left( \sum_j \sigma^o(\upsilon^o) a_j^o \otimes ub_j \right) = \sum_j \sigma^o(\upsilon^o) a_j^o \delta^o(ub_j),
$$

(B.7)

$$
= \sum_j \sigma^o(\upsilon^o) a_j^o \left( \sigma^o(b_j^o) \delta^o(u) + \delta^o(b_j) \upsilon^o \right),
$$

(B.8)

$$
= \sigma^o(\upsilon^o) \delta^o(u) + \sigma^o(\upsilon^o) \left( \sum_j a_j^o \delta^o(b_j^o) \right) \upsilon^o,
$$

(B.9)

$$
= \sigma^o(\upsilon) \delta^o(u) + \sigma^o(\upsilon) \check{\omega} \upsilon^*,
$$

(B.10)

where in the last line we use $\upsilon^o = \check{u}$, $\upsilon^o = \check{u}^*$. This coincides with the formula (4.4) of $\check{\omega}^u$, noticing that $\delta^o(u) = [D, \upsilon^o]_{\sigma^o}$. 

$\square$
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