Abstract

Generalized Ward-Takahashi identity (gWTI) in the pion sector for broken isotopic symmetry is derived and used for the model-independent calculation of the longitudinal form factor $f_-$ of the $\pi e_3$ vector vertex. The on-shell $f_-$ is found to be proportional to the mass difference of pions and the difference between vector isospin $T = 1$ and scalar isospin $T = 2$ pion radii. A numerical estimate of the form factor gives a value two times higher than the earlier estimate in the quark model. Off-shell form factors are known to be ambiguous because of the gauge dependence and the freedom in parameterization of the fields. The near-mass-shell $f_-$ appears to be an exception, allowing the experimental verification of the gWTI consequences. We calculate the near-mass-shell $f_-$ using the gWTI and dispersion techniques. The results are discussed in the context of the conservation of vector current (CVC) condition.
1 Introduction

The $\pi^+ \to \pi^0 e^+ \nu_e$ decay ($\pi_{e3}$) is one of the main semileptonic electroweak processes. Vector nature of the transition, simple kinematics, and the precision measurement of the partial width make this decay particularly attractive for testing the standard model.

The decay amplitude is proportional to the $V_{ud}$ element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Radiative corrections, as well as the pion structure effects, have been calculated in the $\pi_{e3}$ decay with high accuracy\(^1\)\(^-\)\(^5\), sufficient for verification of the unitarity of the CKM matrix. Experimental data, however, are not yet precise enough for this purpose.\(^6\)

Measurement of the $\pi_{e3}$ decay is also motivated by the possibility of testing the conservation of vector current (CVC) in the meson sector. CVC hypothesis\(^9\)\(^-\)\(^10\) suggests that the isovector piece of the electromagnetic current and the charged components of the weak vector current belong to the same isospin triplet. In the limit of exact isotopic symmetry, conservation of the electromagnetic current gives the conservation of the weak vector current.

Off the mass shell, the CVC is equivalent to the Ward-Takahashi identity (WTI) for isospin $SU(2)$ group. WTI is, however, of greater generality and leads to useful relationships between off-shell form factors, including those which vanish when some of the external legs are on-shell. Violation of isotopic symmetry, associated with a small mass difference of up and down quarks and electromagnetic and weak interactions, results in the non-conservation of the charge-changing components of the weak vector current. For broken isotopic symmetry, the CVC is replaced by the partial CVC, while the WTI is replaced by the generalized WTI (gWTI). The partial CVC and gWTI are especially sensitive to the pattern of isotopic symmetry breaking.

Off-shell form factors enter the description of nucleon knockout reactions\(^11\)\(^-\)\(^12\) as well as bremsstrahlung reactions.\(^13\) As a result of shifting from the mass shell, electromagnetic currents of bound nucleons are different from the free currents, which also leads to observable effects.\(^14\)

Parameterization of the degrees of freedom associated with the pion field can be done in various ways, which produces off-shell ambiguity in the amplitudes. On-shell form factors are related to the asymptotic states and uniquely defined. This statement is known as the equivalence theorem (ET).\(^15\)\(^-\)\(^17\) Off-shell form factors, while contribute to the physical amplitudes, depend on the parameterization and cannot be measured experimentally.\(^18\)\(^-\)\(^19\)

We report a notable exception to this rule. The longitudinal part of the $\pi_{e3}$ vertex is shown to be uniquely defined near mass shell, fundamentally accessible for measurement, and can thus be investigated for matching with the gWTI.

In quark models, pions are usually poorly described because of their Goldstone boson nature. As an application of the gWTI, we derive a model-independent expression and give numerical estimate for the longitudinal form factor $f_-$. The plan of this paper is as follows. Sect. 2 starts from a discussion of constraints imposed by WTI of the symmetry group $U(1)$ on the electromagnetic pion form factors. In Sect. 3, a generalization of the WTI for broken isotopic symmetry is derived and, in Sect. 4, used to find relationships between weak vector form factors. The near-mass-shell form factor $f_-$ is then expressed in terms of the pion mass difference and the pion radii in the isospin $T = 1$ and 2 channels.

In Sect. 5, we focus on extracting the isospin $T = 2$ pion radius from the experimental data on $\pi\pi$ scattering phase shifts and evaluate numerically $f_-$. Finally we conclude in Sect. 6 with a summary and a discussion.

2 U(1) vector vertex

On-shell conserved vector current of a charged scalar particle is determined by one form factor. Off the mass shell, there are two form factors. In the most general case, the current can be written as follows

$$\Gamma_\mu (p', p) = (p' + p)_\mu F_1 + q_\mu (p'^2 - p^2) F_2,$$

(2.1)
where \( q = p' - p \) is the momentum transfer. Form factors \( \mathcal{F}_i \) are symmetric functions of \( p'^2 \) and \( p^2 \) and arbitrary functions of \( q^2 \) and the physical mass \( m \) of the charged pion. The factor \( p'^2 - p^2 \) in the second term is added to provide the negative \( C \)-parity of the current. The form factor decomposition (2.1) occurs in the scalar quantum electrodynamics (QED) (see, e.g. \( ^{20} \)) and in the chiral perturbation theory (ChPT) (see, e.g. \( ^{21} \)). WTI of the symmetry group \( U(1) \) establishes a relationship between \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) identical with Eq. (2.5).

Isotopic symmetry implies that the mass difference of non-strange quarks is neglected and the electromagnetic and weak interactions are switched off beyond tree level. In this limit, the weak interaction is described by an isovector current \( \Gamma_\mu(p', p) = T^\alpha \Gamma_\mu(p', p) \), where \( T^\alpha \) are isospin generators. As a result, the exact CVC occurs: the dressed vertex forms an isospin triplet, while the weak vector current is conserved.

WTI associated with the symmetry group \( U(1) \) imposes a constraint on the vertex (see, e.g. \( ^{22} \))

\[
\Delta^{-1}(p') - \Delta^{-1}(p) = q_\mu \Gamma_\mu(p', p), \tag{2.2}
\]

where \( \Delta(p) = p^2 - m^2 - \Sigma(p^2, m) \) is the renormalized pion propagator. The self-energy operator satisfies

\[
\Sigma(m^2, m) = 0, \tag{2.3}
\]

\[
\frac{\partial}{\partial p^2} \Sigma(p^2, m) \bigg|_{p^2 = m^2} = 0. \tag{2.4}
\]

Equation (2.2) being combined with Eq. (2.1) gives

\[
q^2 \mathcal{F}_2(p'^2, p^2, q^2) = \frac{\Delta^{-1}(p') - \Delta^{-1}(p)}{p'^2 - p^2} - \mathcal{F}_1(p'^2, p^2, q^2). \tag{2.5}
\]

In the limit \( p'^2 = p^2 = m^2 \), we obtain

\[
\mathcal{F}_2(m^2, m^2, q^2) = \frac{1 - \mathcal{F}_1(m^2, m^2, q^2)}{q^2}. \tag{2.6}
\]

In a vicinity of \( q^2 = 0 \) the form factor \( \mathcal{F}_1 \) can be expanded to give

\[
\mathcal{F}_2(m^2, m^2, 0) = -\frac{1}{6} \langle r^2 \rangle_v, \tag{2.7}
\]

where \( \langle r^2 \rangle_v \) is the vector charge radius.

The pion form factor \( \mathcal{F}_1(p'^2, p^2, q^2) \), describing strong interaction effects, is an analytic function of the complex variable \( p^2 \) at least in the circle \( |p^2 - m^2| < 8m^2 \). The nearest to the mass shell singularity at \( p^2 = 9m^2 \) is caused by the three-pion threshold. Taylor series in powers of \( p^2 - m^2 \) converges in that circle. The same is true for the variable \( p'^2 \). Taylor series in powers of \( q^2 \) converges for \( |q^2| < 4m^2 \), the radius of convergence is determined by the two-pion threshold.

The equivalence of the Coulomb and Lorentz gauges in QED was rigorously proved in Refs. \( ^{23} \) \( ^{24} \) The amplitudes are gauge-invariant on-shell, while off-shell dependence on the gauge persists. \( \mathcal{F}_1 \) and, by virtue of Eq. (2.6), \( \mathcal{F}_2 \) are thus uniquely defined, when both legs of the charged pion are on the mass shell. To the first order in the displacement from the mass shell, the longitudinal component of the vertex also is gauge invariant as evident from Eq. (2.1). Apart from these cases, \( \Gamma_\mu(p', p) \) depends on the gauge and cannot be measured. From a mathematical point of view, one can speak of the uniqueness of the longitudinal component of \( \Gamma_\mu(p', p) \) in an infinitesimal neighborhood of the mass shell, or, equivalently, of the uniqueness of the longitudinal component of \( \Gamma_\mu(p', p) \) and its first derivatives with respect to \( p'_\mu \) and \( p_\nu \) on the mass shell.

\( \mathcal{F}_2 \) with two on-shell legs \( p'^2 = p^2 = m^2 \) does not contribute to the current. Off the mass shell, \( \mathcal{F}_2 \) does contribute and its contribution is uniquely determined by the WTI. Isotopic rotation of \( \mathcal{F}_2 \) is not enough to get a full weak-interaction vertex. We show that violation of isotopic symmetry generates an isospin \( T = 2 \) contribution not connected with the isotopic rotation.
3 Generalized Ward-Takahashi identity for broken isotopic symmetry

In order to find the longitudinal form factor of the weak vector current, we derive a generalization of the WTI, associated with the replacement of the exact $U(1)$ symmetry by the broken $SU(2)$ symmetry.

Let us consider the variation of the pion propagator

$$i\delta \Delta(x', x) = \langle 0 | T\phi^a(x')\phi^b(x) | 0 \rangle$$

under the $SU(2)$ transformation

$$\phi \to \phi' = e^{-i\chi}\phi.$$ (3.1)

In terms of a Cartesian basis, the pion field $\phi^a(x)$ is Hermitian, $\chi$ is an infinitesimal imaginary anti-symmetric matrix that can be expanded in the group generators with real coefficients $\chi = \sum_a \chi^a T^a$.

In the conventional normalization, $\text{Tr}(T^a T^b) = 2\delta^{ab}$. The variation of the pion propagator is given by

$$i\delta \Delta(x', x) = \chi(x')\Delta(x', x) - \Delta(x', x)\chi(x).$$ (3.2)

In matrix notation, $\Delta^{-1}i\delta \Delta \Delta^{-1} = -i\delta \Delta^{-1} = [\Delta^{-1}, \chi]$.

Independent calculation of $\delta \Delta$ involves the effective Lagrangian, which we consider to be of the form

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}((D_\mu \phi \dagger), D_\mu \phi, \phi \dagger, \phi),$$ (3.3)

where $iD_\mu = i\partial_\mu - eA_\mu - B_\mu$, $A_\mu$ is the electromagnetic field, $e = T^3$, $B_\mu = B^a T^a$ is the weak vector field associated with $Z^0$- and $W^\pm$-bosons. In a Cartesian basis, $D_\mu$ is real and $\phi \dagger = \phi$. The isotopic invariance is broken in $\mathcal{L}_{\text{eff}}$ by the mass term $\phi \dagger m^2 \phi$ with $[m^2, T^a] \neq 0$ and by the electromagnetic and weak interactions. In other respects, $\mathcal{L}_{\text{eff}}$ is an arbitrary function, which reflects the ambiguity in the parameterization of the field.

The combination of $eA_\mu + B_\mu$ ensures implementation of the CVC condition to the bare interaction vertex: the electromagnetic current coincides with the third component of the weak vector current by the construction. If isotopic symmetry in $\mathcal{L}_{\text{eff}}$ were not broken, the CVC condition could also hold in the dressed vertex, while the weak vector current could also be conserved. In case of the broken symmetry, the dressed vertex acquires an admixture of tensor components, while the charge-changing weak vector current is no longer conserved. The WTI can be viewed as a generalization of the CVC for the off-shell vertex. It admits generalizations for broken symmetries, so when the isotopic symmetry is broken the CVC becomes the partial CVC, while WTI of the exact isotopic symmetry turns to WTI of the broken isotopic symmetry.

Transformation (3.1) is equivalent to the compensating transformation in $\mathcal{L}_{\text{eff}}$:

$$\phi \to \phi' = e^{i\chi}\phi.$$ (3.4)
which generates variations of the pion mass term and the vector fields in $\mathcal{L}_{\text{eff}}$:

\[
m^2 \rightarrow m'^2 = m^2 + [m^2, i\chi],
\]

\[
e A_\mu \rightarrow e A'_\mu = e A_\mu + [e, i\chi] A_\mu,
\]

\[
B_\mu \rightarrow B'_\mu = B_\mu + \partial_\mu \chi + [B_\mu, i\chi].
\]

As a result, $\mathcal{L}_{\text{eff}}$ acquires correction

\[
\delta \mathcal{L}_{\text{eff}} = -\text{Tr} (\mathcal{J}_\mu (\partial^\mu \chi + [e, i\chi] A_\mu + [B_\mu, i\chi])) - \text{Tr} (\mathcal{J}[m^2, i\chi]),
\]

where

\[
\mathcal{J}^\alpha_{\mu\beta} = -\frac{\partial \mathcal{L}_{\text{eff}}}{\partial B_{\alpha \beta}}, \quad \mathcal{J}^\alpha = -\frac{\partial \mathcal{L}_{\text{eff}}}{\partial (m^2_{\alpha})}
\]

are vector and scalar currents of the pion.

The response of the pion propagator to the compensating transformation is

\[
i \delta \Delta(x', x) = \langle 0 | T \varphi(x') \tilde{\varphi}(x) i \int d^4 y \delta \mathcal{L}_{\text{eff}}(y) | 0 \rangle
\]

\[
= \int d^4 y d^4 z' d^4 z i \Delta(x', z') [-i \Gamma^a_{\mu}(z', x, y) \partial^\mu \chi^a(y) + \Theta^a(z', z, y) \chi^a(y) + \Omega^a(z', z, y)] i \Delta(z, x),
\]

with vertex functions defined by

\[
\Gamma^a_\mu(z', z, y) = -\int d^4 x' d^4 x \Delta^{-1}(z', x') \langle 0 | T \varphi(x') \tilde{\varphi}(x) \text{Tr} (\mathcal{J}_\mu(y) T^a) | 0 \rangle \Delta^{-1}(x, z),
\]

\[
\Theta^a(z', z, y) = -\int d^4 x' d^4 x \Delta^{-1}(z', x') \langle 0 | T \varphi(x') \tilde{\varphi}(x) \text{Tr} (\mathcal{J}(y)[m^2, T^a]) | 0 \rangle \Delta^{-1}(x, z),
\]

\[
\Omega^a(z', z, y) = -\int d^4 x' d^4 x \Delta^{-1}(z', x') \langle 0 | T \varphi(x') \tilde{\varphi}(x) \text{Tr} (\mathcal{J}_\mu(y)[e A^\mu(y) + B^\mu(y), T^a]) | 0 \rangle \Delta^{-1}(x, z).
\]

We pass over to the momentum space

\[
(2\pi)^4 \delta^4(p' - p - q) \Gamma^a_\mu(p', p) = \int d^4 z d^4 z' d^4 y e^{i p' z' - i p z - i q y} \Gamma^a_\mu(z', z, y),
\]

\[
(2\pi)^4 \delta^4(p' - p - q) \Theta^a(p', p) = \int d^4 z d^4 z' d^4 y e^{i p' z' - i p z - i q y} \Theta^a(z', z, y),
\]

\[
(2\pi)^4 \delta^4(p' - p - q) \Omega^a(p', p) = \int d^4 z d^4 z' d^4 y e^{i p' z' - i p z - i q y} \Omega^a(z', z, y).
\]

Equations 3.2 and 3.10 being combined give

\[
\Delta^{-1}(p') T^a - T^a \Delta^{-1}(p) = q_\mu \Gamma^a_{\mu}(p', p) - \Theta^a(p', p) - \Omega^a(p', p).
\]

This equation generalizes the WTI for the broken $SU(2)$ symmetry. It is shown graphically in Fig. 1.

## 4 SU(2) vector vertex

In a Cartesian basis $\Delta(x, y) = \tilde{\Delta}(y, z)$, which means $\Delta(p) = \tilde{\Delta}(-p)$. Given that $\Delta^{-1}(p) = p^2 - m^2 - \Sigma(p^2)$, the pion propagator is symmetric under transposition of isospin indices: $\Delta(p) = \tilde{\Delta}(p)$. 
The isospin content of a $3 \times 3$ matrix, $\mathcal{M}$, can be found with the use of projection operators

\[
\begin{align*}
(\mathcal{M})^{T=0} & = \frac{1}{2} \text{Tr}[\mathcal{M}], \\
(\mathcal{M})^{T=1}_{a} & = \frac{1}{2} \text{Tr}[\mathcal{M}T_{a}], \\
(\mathcal{M})^{T=2}_{ab} & = \frac{1}{2} \text{Tr}[\mathcal{M}(T_{a}T_{b} + T_{b}T_{a} - \frac{2}{3} \delta_{ab}T(T+1))].
\end{align*}
\] (4.1)

Among the symmetric matrices, there are only isospin 0 or 2 matrices proportional to the unit matrix or quadratic forms of $T^{a}$. The admissible pion mass operator is therefore $m^{2} + \Sigma(m^{2}) = C_{0} + C_{1}(T^{3})^{2}$, where $C_{i}$ are numbers. The charged pions are degenerate in mass, so the $C$-invariance holds.

The vertex functions obey

\[
\begin{align*}
\Gamma_{\mu}^{a}(p', p) & = \tilde{\Gamma}_{\mu}^{a}(-p, -p'), \\
\Theta^{a}(p', p) & = \tilde{\Theta}^{a}(-p, -p'), \\
\Omega^{a}(p', p) & = \tilde{\Omega}^{a}(-p, -p').
\end{align*}
\] (4.2)

Functions $\Theta^{a}(p', p)$ and $\Omega^{a}(p', p)$ depend on the variables $p'^{2}$, $p^{2}$ and $q^{2}$. Symmetry (antisymmetry) in isospin indices implies that vertex function is symmetric (antisymmetric) under the permutation of $p'^{2}$ and $p^{2}$. Symmetric functions in isospin describe states with isospin $T = 0$ and 2, the antisymmetric ones describe isospin $T = 1$ states. Similar properties characterize form factors of $\Gamma_{\mu}^{a}(p', p)$.

The vertex functions can be expanded in scalar functions $F^{a}_{\pm}$ symmetric in $p'^{2}$ and $p^{2}$. The lower index $\pm$ indicates the symmetry with respect to permutation of isospin indices: $F^{a}_{\pm} = \pm F^{a}_{\pm}$ (i.e. 1, 2, 3). The expansion takes the form

\[
\begin{align*}
\Gamma_{\mu}^{a}(p', p) & = (p' + p)_{\mu} (F^{a}_{1-} + (p'^{2} - p^{2})F^{a}_{1+}) + q_{\mu}((p'^{2} - p^{2})F^{a}_{2-} + F^{a}_{2+}), \\
\Theta^{a}(p', p) + \Omega^{a}(p', p) & = F^{a}_{3+} + (p'^{2} - p^{2})F^{a}_{3-}.
\end{align*}
\] (4.5)

If there were no isospin symmetry breaking, we could have $F^{a}_{1-} = T^{a}F_{1-}$, since there are no other $SU(2)$ invariant tensors, and $F^{a}_{1+} = 0$, implying $\Gamma_{\mu}^{a}(m^{2}, m^{2}, q^{2}) \propto T^{a}$ and $\Theta^{a}(m^{2}, m^{2}, q^{2}) + \Omega^{a}(m^{2}, m^{2}, q^{2}) = 0$.

The gWTI, Eq. (3.17), can be split into isospin symmetric and antisymmetric parts:

\[
\begin{align*}
-\frac{1}{2}(\Sigma(p') + \Sigma(p), T^{a}) & = (p'^{2} - p^{2})^{2}F^{a}_{1+} + q^{2}F^{a}_{2+} - F^{a}_{3+}, \\
T^{a} - \frac{1}{2}(\Sigma(p') - \Sigma(p), T^{a}) & = F^{a}_{1-} + q^{2}F^{a}_{2-} - F^{a}_{3-}.
\end{align*}
\] (4.7)

where [·, ·] is commutator and {·, ·} is anticommutator.

Consider some of the consequences of Eq. (4.7). For $p' = p$, $q = 0$, we obtain

\[
[m^{2} + \Sigma(p^{2}), T^{a}] = F^{a}_{3+}(p^{2}, p^{2}, 0).
\] (4.9)

In a cyclic basis, the $SU(2)$ generators equal $T_{\pm 1} = \mp(T^{3} \pm iT^{2})/\sqrt{2}$ and $T_{0} = T^{3}$. The left side of Eq. (3.9), when sandwiched between the initial and final states in the pion $\beta$ decay, gives the pion mass difference:

\[
\langle \pi^{0}|m^{2} + \Sigma(p^{2}), T_{-1}|\pi^{+}\rangle = m_{T}^{2} - m_{I}^{2} = -\Delta m^{2},
\]

where $m_{\pi^{0}} = m_{f}$ and $m_{\pi^{+}} = m_{i}$. In a more general case $p'^{2} = p^{2}$ and $q \neq 0$, one has

\[
[m^{2} + \Sigma(p^{2}), T^{a}] = -q^{2}F^{a}_{2+}(p^{2}, p^{2}, q^{2}) + F^{a}_{3+}(p^{2}, p^{2}, q^{2}).
\] (4.10)

Trace of commutator of finite dimensional matrices vanishes, so the left side of Eqs. (4.9) and (4.10) describes a state of total isospin 2. The isospin zero component of Eq. (4.10) satisfies the equation
of the electromagnetic form factor \(F(q^2)\) is determined by the Jost function \(D_j^T(q^2)\) that can be constructed in terms of phase shift in the corresponding channel (see, e.g., [22]). We work with an \(S\)-wave because the last three vertices in \(\delta L_{\text{eff}}\) induced by the transformation \(\chi^a\) do not contain derivatives of \(\chi^a\), and so the \(S\)-wave \(T = 2\) channel of the pion-pion scattering is relevant. We thus write

\[
\left( F_{3+}^2(m_f^2, m_i^2, q^2) \right)_{T=2} = \frac{F_{3+}^2(m_f^2, m_i^2, 0)}{D_{T=2}^T(q^2)},
\]

where \(D_{T=2}^T(0) = 1\). The value of \(F_{3+}^2(m_f^2, m_i^2, 0)\) is proportional to the mass splitting in the pion multiplet. We restrict ourselves to the first order in \(\Delta m_\pi^2\). Accordingly, the physical pion masses in the arguments of the vertex functions can be replaced by a mean value e.g., \(\mu = (m_f + m_i)/2\), provided these functions like \(F_{2+}^a\) and \(F_{3+}^a\) already contain the smallness. Using Eqs. (4.9 - 4.11), we obtain

\[
\left( F_{3+}^a(\mu^2, \mu^2, q^2) \right)_{T=2} = \frac{m_\pi^2 + \Sigma(\mu^2), T^a}{D_{T=2}^T(q^2)},
\]

\[
\left( F_{2+}^a(\mu^2, \mu^2, q^2) \right)_{T=2} = \frac{m_\pi^2 + \Sigma(\mu^2), T^a}{F_{2+}(\mu^2, \mu^2, q^2)},
\]

where

\[
F_{2+}(\mu^2, \mu^2, q^2) = \frac{1}{D_{T=2}^T(q^2)} - 1.
\]

Let us consider consequences of Eq. (4.10). Equation (4.10) shows that with the required accuracy \(F_{2-}^a\) can be evaluated in the limit of exact isotopic symmetry with \([m_\pi^2 + \Sigma(\mu^2), T^a] = F_{1+}^a = F_{3+}^a = 0\) and \(F_{1-}^a = T^a F_{1-}\). On the mass shell, moreover, \(\Sigma(p) = 0\), and so

\[
F_{2-}(\mu^2, \mu^2, q^2) = \frac{1}{q^2} \left( 1 - \frac{F_{2+}(\mu^2, \mu^2, q^2)}{q^2} \right).
\]

In the leading order of the expansion in \(\Delta m_\pi^2\), form factor \(F_{2-}^a\) is determined by the isotopic rotation of the electromagnetic form factor \(F_{2+}^a\).

Form factors \(F_{2\pm}\) defined by Eqs. (4.12) and (4.13) are not singular at \(q^2 = 0\). For small \(q^2\) they are fixed by the pion radii. The on-shell weak vector current is usually parameterized in the form

\[
\langle \pi^0(p')|d\gamma_\mu(1 - \gamma_5)u|\pi^0(p)\rangle = \sqrt{2}((p' + p)_\mu f_+ + q_\mu f_-),
\]

where \(q_\mu = (p' - p)_\mu\).

The exact CVC condition implies

\[
f_- = 0.
\]

The partial CVC, as follows from the comparison of Eqs. (4.10) and (4.14), gives

\[
f_-(m_{\pi^0}^2 - m_{\pi^+}^2) = \left( \frac{\langle r^2 \rangle_{T=1}}{r_{T=2}^2} \right)
\]

where \(r_{T=1}^2 = \left( \frac{m_{\pi^0}^2 - m_{\pi^+}^2}{6} \right)
\]

The isovector part is restored form the electromagnetic vertex, while the isotensor part is independent of it. It is remarkable that in the dressed vertex \(W^\pm\)-boson, being a member of the weak isospin triplet, is coupled to both the strong isospin triplet and the strong isospin quintet.

Equation (4.10) can be derived in a simpler and more direct way, in analogy to the standard analysis of the \(K\) decays[22]
Firstly, we introduce form factors
\[
\begin{align*}
  f_+(q^2) &= F_+(m_f^2, m_1^2, q^2), \quad (4.17) \\
  f_-(q^2) &= F_-(m_f^2, m_1^2, q^2) + q^2 (F_2-(m_f^2, m_2^2, q^2) + F_2+(m_2^2, m_1^2, q^2)), \quad (4.18)
\end{align*}
\]
with \( f_+ = f_+(0) = f_-(0) = 1 \). In the t-channel, the space-like component of the current is proportional to \( f_+(q^2) \), while the time-like component is proportional to \( f_-(q^2) \). Applying the t-channel unitarity condition for the form factors, we find that \( f_+ \) is determined by the \( J = T = 1 \) scattering amplitude, whereas \( f_- \) is determined by the \( J = 0, T = 2 \) scattering amplitude, where \( J \) is the angular momentum of the pions. \( f_+ \) and \( f_- \) can be identified thereby as vector and scalar form factors, respectively, which is in accord with the gWTI.

Secondly, assume an expansion in \( q^2 \) for both \( f_+(q^2) \) and \( f_-(q^2) \). This assumption conveys the analyticity of the form factors. Equation (4.18) gives
\[
1 + \frac{1}{6} (r^2)_{s=2} q^2 = 1 + \frac{1}{6} (r^2)_{v=1} q^2 + \frac{q^2}{m_f^2 - m_i^2} f_- + O(q^4),
\]
so that we recover Eq. (4.16) and, in addition, observe that it is exact on-shell. Connection with the CVC condition in this simplified approach remains hidden.

The nature of the form factor \( f_+ \) can also be clarified with the help of Eq. (4.14). Taking the scalar product of both sides of the equation with the vector \( q \mu \) yields
\[
\langle \pi^0(p') | i \bar{\psi} \mu \left( \bar{d} \gamma^5 (1 - \gamma_5) u \right) | \pi^+ (p) \rangle = -\Delta m_\pi^2 \sqrt{2} f_+(q^2).
\]
Equation (4.19) is the formal statement of the non-conservation of the lowering component of the weak vector current.

Equation (4.18) can therefore be derived using either the gWTI or the decomposition of Eqs. (4.17) and (4.18). Since recourse to the gWTI is optional, Eq. (4.18) is less interesting for testing the gWTI. We thus analyze the form factors off the mass shell.

The WTI of exact symmetry implies, to the first orders in the displacement \( p'^2 - p^2 \) and for low momentum transfers,
\[
f_- = -\frac{p'^2 - p^2}{6} (r^2)_{v=1}. \quad (4.20)
\]

The WTI of broken symmetry implies, to the first orders in \( \Delta m_\pi^2 \), in the displacement \( p'^2 - p^2 \), and for low momentum transfers,
\[
f_- = -\frac{p'^2 - p^2}{6} (r^2)_{v=1} + \frac{m_\pi^2 - m_\pi^2}{6} (r^2)_{s=2}. \quad (4.21)
\]
The \( \pi^+ \rightarrow \pi^0 e^+ \nu_e \gamma \) decay rate depends on the near-mass-shell behavior of the form factor \( f_- \). Any such reaction may serve for testing the gWTI.

For large momentum transfers, the pion radii in Eqs. (4.16), (4.20) and (4.21) should be replaced by form factors of Eqs. (4.12) and (4.13). Expressions (4.15), (4.16), (4.20) and (4.21) correspond to the various versions of the CVC condition.

On-shell form factor \( F^\pm_0 \) is independent of the gauge and the parameterization. By virtue of Eqs. (4.12) and (4.13), on-shell form factors \( F^\pm \) are uniquely defined too. The longitudinal component of the vertex (4.3) contains the factor \( p'^2 - p^2 \) in \( F^2_\pm \) while \( F^2_\pm \) has the smallness \( O(\Delta m_\pi^2) \). We thus conclude that the longitudinal component of \( \Gamma^\pm_\mu(p', p) \) is uniquely defined in the neighborhood of the mass shell to the first orders in the displacement \( p'^2 - p^2 \) and the pion mass splitting. In the neighborhood of the mass shell, the form factor \( f_- \) thus escapes a general rule, according to which amplitudes are ambiguous off-shell. The near-mass-shell representation of \( f_- \) in terms of the physical masses and radii of the pion demonstrates explicitly independence of the results on the gauge and the parameterization of the pion field.

The transverse component of the vertex does not show up the anomalous behavior, so we consider \( f_+ \) to be ambiguous off-shell.
5 Numerical estimate

In the scattering of $T = 2$ pions the inelastic channels do not show up until the invariant mass of 1.2 GeV \[27,28,29\]. In the approximation neglecting inelastic effects, the scattering amplitude is determined by the phase shift through the Jost function

$$D_J^T(s) = \exp\left(-\frac{s}{\pi} \int_4 \frac{\delta^T(s')ds'}{s'(s'-s)}\right). \tag{5.1}$$

Unitarity relation for the form factor implies (cf. \ref{1.11})

$$(F_{2\pm}(s))^T = P(s)/D_J^T(s), \tag{5.2}$$

where $P(s)$ is a finite-degree polynomial. The degree of $P(s)$ can be fixed by quark counting rules, which in our case have the form $F_{2\pm}(s) \sim 1/s$ for $s \to -\infty$. Asymptotes of the Jost function are determined by the asymptotic phase at infinity. The experimental scattering phase is known at energies up to $\sqrt{s_{\text{max}}} = 2.1$ GeV. At this energy $\delta(s_{\text{max}}) \approx 0$. If one takes $\delta(s) = 0$ for $\sqrt{s} > 2.1$ GeV, the quark counting rules are not satisfied, even for zero-degree polynomial $P(s) = 1$. Minimal modification is to introduce a hypothetical resonance with mass $m_X > \sqrt{s_{\text{max}}}$. Such a resonance should have a large width because it is not observed experimentally. Note that in the Veneziano model there are no resonances in the $T = 2$ channel. The contribution of large $s$ is suppressed in the dispersion integral by the inverse of the second power of $s$, and one can assume that this contribution is not dominant. The contribution of the region of large $s$ can be evaluated by comparing the contributions of intervals $\sqrt{s} = 2\mu - 1.5$ GeV and $1.5 - 2.1$ GeV, where the experimental data are available. The polynomial $P(s)$ of degree $n$ requires $n + 1$ resonances. If there are primitives, the number of resonances gets higher. The number of resonances and primitives correlates with the number of Castillejo-Dalitz-Dyson poles \[29\].

Expanding the Jost function in the vicinity of $s = 0$, we obtain

$$\langle r^2 \rangle_s^T = \frac{6}{\pi} \int_4^{s_{\text{max}}} \frac{\delta^T(s')ds'}{s'^2} = \frac{6}{\pi} \int_4^{s_{\text{max}}} \frac{\delta^T(s')ds'}{s'^2}. \tag{5.3}$$

For numerical estimates the upper limit in the integral is replaced by $s_{\text{max}}$; the scattering phase shift between the experimental points is interpolated linearly. The results are shown in Table 1 for the data sets of Refs. \[27,28,29\].

The negative ”mean square radius” occurs because of the negative $t$-channel scattering phase, which is a signature of repulsion. In the general case, the mean square radius is defined by the first expansion coefficient of the corresponding form factor in powers of $q^2$, the sign of the expansion coefficient is not fixed a priori.

Equation \ref{5.3} has a variety of applications. In the zero-width approximation with $\delta^{T=1}(s) = \pi \delta(s - m^2_p)$, where $m_p$ is the $p$-meson mass, one arrives at the well-known expression for the pion charge radius $\langle r^2 \rangle_v^{T=1} = 6/m^2_p$. In the channel $T = 1$ with an attraction $\langle r^2 \rangle_v^{T=1}$ gets positive and close to the experimental value, thereby supporting the vector meson dominance model. The pion scalar radius is determined using the pion-pion scattering data in the $T = 0$ channel to give $\langle r^2 \rangle_s^{T=0} = 0.61 \pm 0.04 \text{ fm}^2$. \[29\]

The positive value of $\langle r^2 \rangle_s^{T=0}$ points to the dominance of attraction, which is consistent with the existence of the $\sigma$-meson.

The data in Table 1 show that the contribution of the domain $1.5 - 2.1$ GeV is small, indicating that the high-energy contribution is also small. The error associated with the region $s > s_{\text{max}}$ can be estimated by assuming existence of a hypothetical resonance $X$ with mass $m_X > \sqrt{s_{\text{max}}}$. In the zero-width approximation the error is $6/m^2_X$. For $m_X = 3$ GeV, we obtain 0.03 fm$^2$, which is comparable with the experimental error in the calculation using Eq. \ref{5.3}.

Experimental value of the $\pi^+$ charge radius equals $\langle r^2 \rangle_v^{T=1} = (0.672 \pm 0.008 \text{ fm})^2$. \[31\] Using the value $\langle r^2 \rangle_s^{T=2} = -0.10 \pm 0.03 \text{ fm}^2$, we obtain with the help of Eq. \ref{4.11}

$$f_- = (2.97 \pm 0.17) \times 10^{-3}, \tag{5.4}$$
Table 1: Lorentz scalar isospin $T = 2$ mean square radius of the pion determined with the use of Eq. (5.3). The phase-shift analyzes of Refs. [27,28,29] are used; (a) and (b) denote non-equivalent datasets of scattering phases. $\sqrt{s_{\text{max}}}$ is the maximum energy up to which phase analysis is performed.

| $(\langle r^2 \rangle)^{\frac{T}{2}}$ [fm$^2$] | $\sqrt{s_{\text{max}}}$ [GeV] | Ref. |
|---------------------------------------------|----------------|-----|
| $-0.09 \pm 0.05$                           | 1.4            | 27  |
| $-0.10 \pm 0.01$                           | 2.1            | 28 (a) |
| $-0.10 \pm 0.03$                           | 2.1            | 28 (b) |
| $-0.11 \pm 0.02$                           | 1.5            | 29 (a) |
| $-0.13 \pm 0.01$                           | 1.5            | 29 (b) |

which is a factor of two greater than the light-front quark model prediction\cite{55}

Contribution of the longitudinal form factor to the $\pi^+ \rightarrow \pi^0 e^+ \nu_e$ decay rate can be estimated to give $\Delta B/B = -0.94 \times 10^{-3} f_-^e$; the additional small factor in $f_-^e$ arises for kinematic reasons. The experimental error in $B$ is 0.6\%.\cite{51} We thus reaffirm earlier conclusions that $f_-^e$ is currently beyond the capabilities of the experimental study. The possibility of measuring the longitudinal weak vector current in the neutron $\beta$-decay\cite{32}, muon capture\cite{33} and in the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ and $\tau^- \rightarrow K^- K^0 \nu_\tau$ decays is, perhaps, more promising.

A favorable experimental situation exists in the $K_{e3}$ decays. The large difference in the masses of $\pi$- and $K$-mesons hampers the near-mass-shell expansion, so the $K_{e3}$ decays seem to be not suitable for the measuring $f_-$ off-shell and testing the gWTI. At the same time the dispersion techniques allowing to determine $f_-$ on-shell have found in these decays a successful application\cite{34,35}.

The dominant part of $\Delta m^2_{\pi}$ is electromagnetic in origin\cite{36,37,38}. Small mass difference between up and down quarks leads to the $\pi^0 - \eta$ mixing, which also contributes to $\Delta m^2_{\pi}$. Equation (4.16) describes therefore the electromagnetic contribution of order $O(\alpha)$ to the longitudinal weak vector current and the small $\pi^0 - \eta$ mixing effect.

6 Conclusion

In this paper, the longitudinal component of weak vector current in the $\pi_{e3}$ decay was calculated. The longitudinal component arises due to violation of isotopic symmetry by pion masses and electromagnetic and weak interactions. A generalization of the WTI was derived in the pion sector with account of the isotopic symmetry breaking. It was shown that isovector $T = 1$ part of the current can be reconstructed by isotopic rotation of the off-shell pion electromagnetic form factors, while isotensor $T = 2$ part has no analogues, but is uniquely determined by the gWTI combined with the elastic unitarity, analyticity, and pion-pion scattering data. Due to low momentum transfers, the on-shell form factor is proportional to the pion mass difference and the difference of vector isovector and scalar isotensor pion radii. The relationship of the near-mass-shell form factor with the pion radii, Eq. (4.21), is the consequence of the CVC condition going beyond the approximation of exact isotopic symmetry.

The various versions of the CVC condition are distinguished according as to whether the isotopic symmetry is exact or broken and whether the outer legs in the vertex are on- or off-shell. The corresponding predictions for the longitudinal form factor $f_-$ are given in Eqs. (4.15), (4.16), (4.20), and (4.21). In the case of the exact isotopic symmetry, the bare and dressed weak vertices are pure isospin triplets; the CVC and the WTI hold on- and off-shell, respectively. In the case of the broken isotopic symmetry, the dressed weak vertex is no longer the pure isospin triplet; the partial CVC and the gWTI hold on- and off-shell.

Favorable conditions for testing the gWTI by measuring the longitudinal weak vector current appear to exist in the $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau \gamma$ and $\tau^- \rightarrow K^- K^0 \nu_\tau \gamma$ decays with a soft photon emission, on account
of the involvement of the near-mass-shell form factors, the relatively large momentum transfer in the weak vertex, and the small mass differences within the pion and kaon multiplets.

A non-trivial consequence of the partial CVC occurs off-shell, where the vertex in general is not gauge invariant and depends on the parameterization of the pion field. The only exception is the longitudinal component of the vertex in the neighborhood of the mass shell. To the first orders in the displacement and the pion mass splitting, the longitudinal component is independent of both the gauge and the parameterization, so the near-mass-shell form factor $f_-$ appears to be a unique object whose properties are unambiguously determined by the partial CVC (gWTI) and at the same time fundamentally allow for experimental verification.

Acknowledgments

This work was supported by RFBR grant No. 13-02-01442 and grant No. 3172.2012.2 for Leading Scientific Schools.
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