Personalized Federated Learning: A Meta-Learning Approach

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Abstract

The goal of federated learning is to design algorithms in which several agents communicate with a central node, in a privacy-protecting manner, to minimize the average of their loss functions. In this approach, each node not only shares the required computational budget but also has access to a larger data set, which improves the quality of the resulting model. However, this method only develops a common output for all the agents, and therefore, does not adapt the model to each user data. This is an important missing feature especially given the heterogeneity of the underlying data distribution for various agents. In this paper, we study a personalized variant of the federated learning in which our goal is to find a shared initial model in a distributed manner that can be slightly updated by either a current or a new user by performing one or a few steps of gradient descent with respect to its own loss function. This approach keeps all the benefits of the federated learning architecture while leading to a more personalized model for each user. We show this problem can be studied within the Model-Agnostic Meta-Learning (MAML) framework. Inspired by this connection, we propose a personalized variant of the well-known Federated Averaging algorithm and evaluate its performance in terms of gradient norm for non-convex loss functions. Further, we characterize how this performance is affected by the closeness of underlying distributions of user data, measured in terms of distribution distances such as Total Variation and 1-Wasserstein metric.

1 Introduction

In Federated Learning (FL), we consider a network of \( n \) users that are all connected to a central node (i.e., a star connectivity graph) where each user has access only to its local data (Konečný et al., 2016). In this setting, the goal of the users is to come up with a model that is trained over all the data points in the network without exchanging their local data with other users or the central node, i.e., the server, due to privacy issues or communication limitations.

More formally, the classic FL setting studies a star-shaped network with \( n \) users and one server, and they all coordinate to solve the following optimization problem:

\[
\min_{w \in \mathbb{R}^d} f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w),
\]

where \( f_i : \mathbb{R}^d \rightarrow \mathbb{R} \) denotes the loss function corresponding to user \( i \). In particular, consider a supervised learning application, where \( f_i \) represents expected loss over the data distribution of user \( i \), i.e.,

\[
f_i(w) := \mathbb{E}_{(x,y) \sim p_i} [l_i(w; x, y)],
\]

where \( l_i(w; x, y) \) measures the error of model \( w \) in predicting the true label \( y \in \mathcal{Y}_i \) given the input \( x \in \mathcal{X}_i \), and \( p_i \) is the distribution over \( \mathcal{X}_i \times \mathcal{Y}_i \). We would like to emphasize that in this paper we study the case that the probability distribution \( p_i \) of users in the network are not identical and we face a heterogeneous data probability distribution.

To illustrate this formulation, as an example, consider the problem of training a Natural Language Processing (NLP) model over the devices of a set of users. In this problem, \( p_i \) represents the

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empirical distribution of words and expressions used by user \(i\), and hence, \(f_i(w)\) can be expressed as

\[
f_i(w) = \sum_{(x,y) \in \mathcal{S}_i} p_i(x,y) l_i(w; x, y),
\]

where \(\mathcal{S}_i\) is the data set corresponding to user \(i\) and \(p_i(x,y)\) is the probability that user \(i\) assigns to a specific word which is proportional to the frequency of using this word by user \(i\).

In most algorithms designed for FL, the problem in (1) is solved in multiple rounds, where at each round the center sends the current model to a fraction of the users and those users update the model with respect to their own loss functions, usually by performing a few steps of a gradient-based method. Then, these users return their updated models to the center, and the center combines the received models to update the global model (for example by averaging, as in FedAvg Algorithm (McMahan et al., 2017a)) and sends the updated model to a (possibly different) fraction of the users for the next round. This way, the computational power of all the users in the network are used to train the global model. In addition, the shared model is trained over a larger data set which could lead to a better model. Indeed, this approach leads to a model that solves the problem in (1) and the resulted solution \(w^*\) performs well over all users on average.

Closeness of data distributions of users is crucial for the success of the federated learning framework. However, it is not necessarily the case that the data samples of all users are drawn from a common underlying distribution. This heterogeneity leads to an issue with formulation (1) in that the resulting model is only good on average and it does not take into account the heterogeneity of data distribution of users. In other words, the solution of problem (1) is not personalized for each user. To better highlight this point, recall the NLP example above, where although the distribution over the words and expressions varies from one person to another, the solution to problem (1) only provides a shared answer for all users, and therefore, it is not fully capable of achieving a user-adapted model.

Hence, in the setting that the underlying distribution of data points of the users are not identical, solving the average problem defined in (1) could lead to poor local performance for each user. In this paper, we overcome this issue by considering a new problem formulation. We further introduce an efficient method for solving the proposed formulation and characterize its convergence properties. A detailed list of our contributions follows:

1. We consider a modified formulation of the federated learning problem which incorporates personalization (Section 2). Building on the Model-Agnostic Meta-Learning (MAML) problem formulation introduced by Finn et al. (2017), the goal of our formulation is to find an initial point shared between all users which performs well after each user updates it with respect to its own loss function, potentially by performing a few steps of a gradient-based method. This way, while the initial model is derived in a distributed manner over the whole network (same as the classic FL setting), the final model implemented by each user differs from other ones based on his or her own data.

2. We also propose a Personalized variant of the FedAvg algorithm, called Per-FedAvg, designed for solving the proposed personalized FL problem (Section 3). In particular, we elaborate on its connections with the original FedAvg algorithm (McMahan et al., 2017a), and also, discuss a number of considerations that one need to take into account for implementing Per-FedAvg.

3. We study the convergence properties of the proposed Per-FedAvg algorithm for solving non-convex loss functions in terms of the objective function gradient norm (Section 4). In particular, we characterize the role of data heterogeneity and closeness of data distribution of different users, measured by distribution distances, such as Total Variation (TV) or 1-Wasserstein, on convergence of Per-FedAvg method.

1.1 Related Work

As mentioned earlier, McMahan et al. (2017a) proposed the FedAvg algorithm, where the global model is updated by averaging local SGD updates. Later, Guha et al. (2019) proposed one-shot Federated Learning (FL) in which the master node learns the model after a single round of communication. Also, several approaches have been used to address the communication limitations in FL. This includes quantization and compression ideas (Reisizadeh et al., 2019; Dai et al., 2019) as well as performing multiple local updates before communicating with the master (Stich, 2018; Lin
Several works have studied the problem of preserving privacy in federated learning (Duchi et al., 2014; McMahan et al., 2017b; Agarwal et al., 2018; Zhu et al., 2019). More related to our paper, there are several works that study statistical heterogeneity of users’ data points in FL (Zhao et al., 2018; Sahu et al., 2018; Karimireddy et al., 2019; Haddadpour and Mahdavi, 2019; Khaled et al., 2019; Li et al., 2019), but they do not attempt to find a personalized solution for each user. In addition, Smith et al. (2017) used multi-task learning framework and proposed a new method, MOCHA, to address these statistical and systems challenges (including data heterogeneity as well as communication efficiency).

The idea of personalization in FL and its connections with meta-learning has recently gained attention in a number of papers. Khodak et al. (2019) proposed ARUBA, a meta-learning algorithm inspired by online convex optimization, and showed how applying it to FedAvg method improves its performance empirically. Jiang et al. (2019) proposed a personalized FedAvg algorithm in which the classic FedAvg is first deployed, and then they switch to Reptile, a meta-learning algorithm proposed in (Nichol et al., 2018), and finally run local updates to achieve personalization. Note that this approach is different from our proposed framework, as in this paper we do not perform the classic FedAvg and instead we look for a good initial point which performs well after it is fine-tuned for each user. Moreover, Chen et al. (2018) focused on recommendation systems and proposed a meta-federated learning framework in which a parameterized meta-algorithm is used to train parameterized recommendation models and both meta-algorithm and local models’ parameters need to be optimized. For the special case that the meta-algorithm parameter is its initialization, this framework reduces to our formulation. The authors evaluated the success of this framework empirically over various data sets and by taking different meta-algorithms. However, in our work, we specifically focus on the case that the meta-algorithm parameter is the initial point, and characterize its convergence theoretically, and highlight the role of different parameters including heterogeneity of data distributions. We further provide empirical results for our proposed method. For a detailed survey on the connections of FL and multi-task and meta-learning check Section 3.3 of (Kairouz et al., 2019).

2 Personalized Federated Learning via Model-Agnostic Meta-Learning (MAML)

As we stated in Section 1, our goal in this section is to show how the fundamental idea behind the Model-Agnostic Meta-Learning (MAML) framework in (Finn et al., 2017) can be exploited to design a personalized variant of the FL problem. To do so, let us first briefly recap the MAML formulation. Given a set of tasks drawn from an underlying distribution, in MAML, in contrast to the traditional supervised learning setting, the goal is not finding a model which performs well on all the tasks in expectation. Instead, in MAML, we assume we have a limited computational budget to update our model after a new task arrives, and in this new setting, we look for an initialization which performs well after it is updated with respect to this new task, possibly by one or a few steps of gradient descent. In particular, if we assume each user takes the initial point and updates it using one step of gradient descent with respect to its own loss function, then problem (1) changes to

\[
\min_{w \in \mathbb{R}^d} F(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w - \alpha \nabla f_i(w))
\]

where \(\alpha \geq 0\) is the learning rate (stepsize). The strength of this formulation is that, not only it allows us to maintain the advantages of FL (limited communication), but also it captures the difference between users as either existing or new users can take the solution of this new problem as an initial point and slightly update it with respect to their own data. Going back to the NLP example (3), this means that each users could take this resulting initialization and update it by going over her/his own data and performing just one or few steps of gradient descent to obtain a model that works well for her/his own dataset.

As we mentioned earlier, for the considered heterogeneous model of data distribution, solving problem (1) is not the ideal choice as it returns a single model that even after a few steps of local gradient may not quickly adjust to each users local data, but by solving (4) we find an initial model (Meta-model) which is trained in a way that after one step of local gradient leads to a good model for each individual user. Indeed, this formulation can also be extended to the case that each...
user runs a few steps of gradient update, but to simplify our notation we only focus on the single gradient update case.

The centralized version of this formulation was first proposed by Finn et al. (2017) and followed by a number of papers studying its empirical characteristics (Antoniou et al., 2019; Li et al., 2017; Grant et al., 2018; Nichol et al., 2018; Zintgraf et al., 2019; Behl et al., 2019) as well as its convergence properties (Fallah et al., 2019). In this work, we focus on exploiting the MAML formulation to introduce a personalized solution for the federated learning setting. The analysis of the proposed algorithm for the FL setting is more challenging than the centralized case as we discuss in Section 4.

3 Personalized FedAvg

In this section, we introduce our proposed Personalized FedAvg method for solving problem (4). This algorithm is inspired by the FedAvg algorithm originally proposed for the classic federated learning problem (1), but it has been modified in a way that the resulting method finds the optimal solution of (4) instead of (1). To better highlight this connection, let us recap the main steps of the FedAvg algorithm. In FedAvg, at each round, server chooses a fraction of users with size \( r \) (with \( 1 \geq r > 0 \)) and sends its current model to these users. Each selected user \( i \) updates this model according to its own loss function \( f_i \) and by running \( \tau \geq 1 \) steps of stochastic gradient descent. Then, the users return their updated models to the server. Finally, the server updates the global model by computing the average of the models received from these selected users, and then the next round follows.

The proposed personalized FedAvg method follows the same principle and it aims to implement a similar algorithm for minimizing the function \( F \) defined in (4). Before formally stating the update of personalized FedAvg let us mention that the global objective function \( F \) in (4) can be written as the average of meta-functions \( F_1, \ldots, F_n \) where the meta-function \( F_i \) associated with user \( i \) is defined as

\[
F_i(w) := f_i(w - \alpha\nabla f_i(w)).
\]

In other words, in this case, each local function is defined as the value of the local loss function after running one step of gradient descent.

To follow a similar scheme as FedAvg for solving problem (4), the first step is to compute the gradient of local functions, which in this case, is an estimate of the local learning rate (stepsize) and by running \( \tau \geq 1 \) steps of stochastic gradient descent, we therefore, take a batch of data \( D_i \) with respect to distribution \( p_i \) to obtain an unbiased estimate \( \hat{\nabla} f_i(w, D_i) \) given by

\[
\hat{\nabla} f_i(w, D_i) := \frac{1}{|D_i|} \sum_{(x,y) \in D_i} \nabla l_i(w; x, y).
\]

Similarly, we could replace the Hessian \( \nabla^2 f_i(w) \) in (6) by its unbiased estimate \( \hat{\nabla}^2 f_i(w, D_i) \) over the batch \( D_i \).

At round \( k \) of Personalized FedAvg algorithm, similar to FedAvg, first the server sends the current global model \( w_k \) to a fraction of users \( A_k \) chosen uniformly at random with size \( r n \). Each user \( i \) in \( A_k \) performs \( \tau \) steps of stochastic gradient descent locally and with respect to \( F_i \). In particular, these local updates generate a local sequence \( \{w_{k+1,t}^i\}_{t=0}^\tau \) where \( w_{k+1,0}^i = w_k \) and, for \( \tau \geq t \geq 1 \),

\[
w_{k+1,t}^i = w_{k+1,t-1}^i - \beta \hat{\nabla} F_i(w_{k+1,t-1}^i)
\]

where \( \beta \) is the local learning rate (stepsize) and \( \hat{\nabla} F_i(w_{k+1,t-1}^i) \) is an estimate of \( \nabla F_i(w_{k+1,t-1}^i) \) in (6). Note that the stochastic gradient \( \hat{\nabla} F_i(w_{k+1,t-1}^i) \) for all local iterates is computed using independent batches \( D_i^1, D_i^2, \ldots, D_i^{\tau} \) as follows

\[
\hat{\nabla} F_i(w_{k+1,t-1}^i) := \left( I - \alpha \hat{\nabla}^2 f_i(w_{k+1,t-1}^i, D_i^t) \right) \hat{\nabla} F_i(w_{k+1,t-1}^i - \alpha \hat{\nabla} f_i(w_{k+1,t-1}^i, D_i^1), D_i^t).
\]
Indeed, it can be verified the outcome of these two steps is equivalent to the update in (8). To this end, we formally state the assumptions required for proving our main results.

4 Theoretical Results

In this section, we study the convergence properties of our proposed Personalized FedAvg (Per-FedAvg) method. We focus on nonconvex settings, and characterize the overall communication rounds between server and users for achieving first-order stationarity. To do so, we first formally define the notion of an $\epsilon$-approximate first-order stationary point.

4.1 Definitions and Assumptions

**Definition 4.1.** A random vector $w_\epsilon \in \mathbb{R}^d$ is called an $\epsilon$-approximate First-Order Stationary Point (FOSP) for problem (4) if it satisfies

$$\mathbb{E}[\|\nabla F(w_\epsilon)\|] \leq \epsilon.$$
Assumption 1. Function $F$ is bounded below, i.e., $\min_{w \in \mathbb{R}^d} F(w) > -\infty$.

Assumption 2. For every $1 \leq i \leq n$, $f_i$ is twice continuously differentiable and $L_i$-smooth, and also, its gradient is bounded by a nonnegative constant $B_i$, i.e.,

$$
\|\nabla f_i(w)\| \leq B_i, \quad \forall w \in \mathbb{R}^d, \quad (11a)
$$

$$
\|\nabla f_i(w) - \nabla f_i(u)\| \leq L_i \|w - u\|, \quad \forall w, u \in \mathbb{R}^d. \quad (11b)
$$

It is worth noting that (11b) also implies that $f_i$ satisfies the following conditions for all $w, u \in \mathbb{R}^d$:

$$
-L_i I_d \preceq \nabla^2 f_i(w) \preceq L_i I_d, \quad (12a)
$$

$$
\|f_i(w) - f_i(u) - \nabla f_i(u)^\top (w - u)\| \leq \frac{L_i}{2} \|w - u\|^2. \quad (12b)
$$

As we discussed in Section 3, the second-order derivative of all functions appears in the update rule of Per-FedAvg Algorithm. Hence, in the next Assumption, we impose a regularity condition on the Hessian of each $f_i$ which is also a customary assumption in the analysis of second-order methods.

Assumption 3. For every $1 \leq i \leq n$, the Hessian of function $f_i$ is $\rho_i$-Lipschitz continuous, i.e.,

$$
\|\nabla^2 f_i(w) - \nabla^2 f_i(u)\| \leq \rho_i \|w - u\|, \quad \forall w, u \in \mathbb{R}^d. \quad (13)
$$

To simplify the analysis, in the rest of the paper, we define $B := \max_i B_i$, $L := \max_i L_i$, and $\rho := \max_i \rho_i$ which can be, respectively, considered as a bound on the norm of gradient of $f_i$, smoothness parameter of $f_i$, and Lipschitz continuity parameter of Hessian $\nabla^2 f_i$, for all $1 \leq i \leq n$.

Now, we state the next assumption which provides upper bounds on the variances of gradient and Hessian estimation.

Assumption 4. For any $i$ and any $w \in \mathbb{R}^d$, the stochastic gradient $\nabla l_i(x, y; w)$ and Hessian $\nabla^2 l_i(x, y; w)$, computed with respect to a single data point $(x, y) \in X_i \times Y_i$, has bounded variance, i.e.,

$$
\mathbb{E}_{(x, y) \sim p_i} \left[ \|\nabla l_i(x, y; w) - \nabla f_i(w)\|^2 \right] \leq \sigma_G^2, \quad (14)
$$

$$
\mathbb{E}_{(x, y) \sim p_i} \left[ \|\nabla^2 l_i(x, y; w) - \nabla^2 f_i(w)\|^2 \right] \leq \sigma_H^2, \quad (15)
$$

where $\sigma_G$ and $\sigma_H$ are non-negative constants.

Finally, we state our last assumption which characterizes the similarity between the tasks of users.

Assumption 5. For any $w \in \mathbb{R}^d$, the variance of gradient $\nabla f_i(w)$ and Hessian $\nabla^2 f_i(w)$ are bounded, i.e., for some non-negative $\gamma_G$ and $\gamma_H$, we have

$$
\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(w) - \nabla f(w)\|^2 \leq \gamma_G^2, \quad (16a)
$$

$$
\frac{1}{n} \sum_{i=1}^{n} \|\nabla^2 f_i(w) - \nabla^2 f(w)\|^2 \leq \gamma_H^2, \quad (16b)
$$

for any $w \in \mathbb{R}^d$.

Note that Assumption 2 implies that this assumption holds automatically for $\gamma_G = 2B$ and $\gamma_H = 2L$. However, we state this assumption separately to highlight the role of similarity of functions corresponding to different users in convergence analysis of Per-FedAvg. In particular, in the following subsection, we highlight the connections between this assumption and the similarity of distributions $p_i$ for the case of supervised learning (2) under two different distribution distances.
4.2 On the Connections of Task Similarity and Distribution Distances

Recall the definition of \( f_i \) for the supervised learning problem stated in (2). As mentioned above, Assumption 5 captures the similarity of loss functions of different users, and one fundamental question here is whether this has any connection with the closeness of distributions \( p_i \). We study this connection by considering two different distances: Total Variation (TV) distance and 1-Wasserstein distance. Throughout this subsection, we assume all users have the same loss function.

Assumption 5 captures the similarity of loss functions of different users, and one fundamental question here is whether this has any connection with the closeness of distributions \( p_i \). We study this connection by considering two different distances: Total Variation (TV) distance and 1-Wasserstein distance. Throughout this subsection, we assume all users have the same loss function \( l(\cdot,\cdot) \) over the same set of inputs and labels, i.e., \( f_i(w) := \mathbb{E}_{z \sim P_i}[l(z; w)] \) where \( z := (x, y) \in \mathcal{Z} := \mathcal{X} \times \mathcal{Y} \). Also, let \( p = \frac{1}{n} \sum p_i \) denote the average of all users’ distributions.

- **Total Variation (TV) Distance**: For distributions \( q_1 \) and \( q_2 \) over countable set \( \mathcal{Z} \), their TV distance is given by

  \[
  \|q_1 - q_2\|_{TV} = \frac{1}{2} \sum_{z \in \mathcal{Z}} |q_1(z) - q_2(z)|. \tag{17}
  \]

  If we further assume a stronger version of Assumption 2 holds where for any \( z \in \mathcal{Z} \) and \( w \in \mathbb{R}^d \), we have

  \[
  \|\nabla_w l(z; w)\| \leq B, \quad \|\nabla_w^2 l(z; w)\| \leq L, \tag{18}
  \]

  then, Assumption 5 holds with (check Appendix A for the proof)

  \[
  \gamma_G = 2B \sqrt{\frac{1}{n} \sum_{i=1}^{n} \|p_i - p\|^2_{TV}}, \tag{19a}
  \]

  \[
  \gamma_H = 2L \sqrt{\frac{1}{n} \sum_{i=1}^{n} \|p_i - p\|^2_{TV}}. \tag{19b}
  \]

  This simple derivation shows that \( \gamma_G \) and \( \gamma_H \) exactly capture the difference between the probability distributions of the users in a heterogeneous setting.

- **1-Wasserstein Distance**: The 1-Wasserstein distance between two probability distributions measures \( q_1 \) and \( q_2 \) over a metric space \( \mathcal{Z} \) defined as \(^1\)

  \[
  W_1(q_1, q_2) := \left( \inf_{Q \in \mathcal{Q}(q_1, q_2)} \int_{\mathcal{Z} \times \mathcal{Z}} d(z_1, z_2) \; dq(z_1, z_2) \right) \tag{20}
  \]

  where \( d(\cdot, \cdot) \) is a distance function over metric space \( \mathcal{Z} \) and \( Q(q_1, q_2) \) denotes the set of all measures on \( \mathcal{Z} \times \mathcal{Z} \) with marginals \( q_1 \) and \( q_2 \) on the first and second coordinate, respectively. Here, we assume all \( p_i \) have bounded support (note that this assumption holds in many cases as either \( \mathcal{Z} \) itself is bounded or because we normalize the data). Also, we assume that for any \( w \), the gradient \( \nabla_w l(z; w) \) and the Hessian \( \nabla_w^2 l(z; w) \) are both Lipschitz with respect to parameter \( z \) and distance \( d(\cdot, \cdot) \), i.e,

  \[
  \|\nabla_w l(z_1; w) - \nabla_w l(z_2; w)\| \leq L_Z d(z_1, z_2), \tag{21a}
  \]

  \[
  \|\nabla_w^2 l(z_1; w) - \nabla_w^2 l(z_2; w)\| \leq \rho_Z d(z_1, z_2). \tag{21b}
  \]

  Then, Assumption 5 holds with (check Appendix A for the proof)

  \[
  \gamma_G = L_Z \sqrt{\frac{1}{n} \sum_{i=1}^{n} W_1(p_i, p)^2}, \tag{22a}
  \]

  \[
  \gamma_H = \rho_Z \sqrt{\frac{1}{n} \sum_{i=1}^{n} W_1(p_i, p)^2}. \tag{22b}
  \]

  It is worth noting that this derivation does not use other Assumptions such as Assumption 2 and holds in general when (21a) and (21b) are satisfied.

\(^1\)The integral can be replaced by sum if \( \mathcal{Z} \) is countable.
4.3 Convergence Analysis of Per-FedAvg Algorithm

In this subsection, we derive the overall complexity of Per-FedAvg for achieving an ε-first-order stationary point. To do so, we first prove the following intermediate result which shows that under Assumptions 2 and 3, the local meta-functions \( F_i(w) \) defined in (5) and their average function \( F(w) = (1/n) \sum_{i=1}^{n} F_i(w) \) are smooth.

**Lemma 4.2.** Recall the definition of \( F_i(w) \) (5) with \( \alpha \in [0,1/L] \). Suppose that the conditions in Assumptions 2 and 3 are satisfied. Then, \( F_i \) is smooth with parameter \( L_F := 4L + \alpha \gamma B \). As a consequence, their average \( F(w) = (1/n) \sum_{i=1}^{n} F_i(w) \) is also smooth with parameter \( L_F \).

*Proof.* Check Appendix B.

The conditions in Assumption 4 provide upper bounds on the variances of gradient and Hessian estimation for functions \( f_i \). To analyze the convergence of Per-FedAvg, however, we need an upper bound on the variance of gradient estimation of the functions \( F_i \). We derive such an upper bound in the following lemma.

**Lemma 4.3.** Recall (9) that we estimate \( \nabla F_i(w) \) by

\[
\nabla F_i(w) = \left( I - \alpha \nabla^2 f_i(w, D') \right) \nabla f_i \left( w - \alpha \nabla f_i(w, D) \right) \nabla f_i \left( w - \alpha \nabla f_i(w, D') \right)
\]

where \( D, D' \), and \( D'' \) are independent batches with size \( D, D' \), and \( D'' \), respectively. Suppose that the conditions in Assumptions 2-4 are satisfied. Then, for any \( \alpha \in [0,1/L] \) and \( w \in \mathbb{R}^d \), we have

\[
\mathbb{E} \left[ \| \nabla F_i(w) - \nabla F_i(w) \|^2 \right] \leq \sigma_{F_i}^2\gamma_i, \quad \text{where} \quad \sigma_{F_i}^2 \text{ is given by}
\]

\[
\sigma_{F_i}^2 := 3 \left[ B^2 + \sigma_{G_i}^2 \left[ \frac{1}{D'} + \frac{(\alpha L)^2}{D} \right] \right] \left[ 4 + \sigma_{H_i}^2 \frac{\alpha^2}{D''} \right] - 12\beta^2
\]

*Proof.* Check Appendix C.

To measure the tightness of this result, we consider two special cases. First, if the exact gradients and Hessians are available, i.e., \( \sigma_{G} = \sigma_{H} = 0 \), then \( \sigma_{F} = 0 \) as well which is expected as we can compute exact \( \nabla F_i \). Second, for the classic federated learning problem, i.e., \( \alpha = 0 \) and \( F_i = f_i \), we have \( \sigma_{F} = \mathcal{O}(1)\sigma_{G}^2/\gamma \) which is tight up to constants.

Next, we use the similarity conditions for the functions \( f_i \) in Assumption 5 to study the similarity between gradients of the functions \( F_i \).

**Lemma 4.4.** Recall the definition of \( F_i(w) \) in (5) and assume that \( \alpha \in [0,1/L] \). Suppose that the conditions in Assumptions 2, 3, and 5 are satisfied. Then, for any \( w \in \mathbb{R}^d \), we have

\[
\frac{1}{\tau} \sum_{i=1}^{n} \| \nabla F_i(w) - \nabla F(w) \|^2 \leq \gamma_F^2
\]

with \( \gamma_F^2 := 3B^2\alpha^2\gamma_H^2 + 192\gamma_{G_i}^2 \).

*Proof.* Check Appendix D.

It is worth going over the two special cases that we discussed for Lemma 4.3 to see how tight Lemma 4.4 is. First, if \( \nabla f_i \) are all equal, i.e., \( \gamma_{G_i} = \gamma_H = 0 \), then \( \gamma_{F} = 0 \) as well. This is indeed expected as all \( \nabla F_i \) are equal to each other in this case. Second, for the classic federated learning problem, i.e., \( \alpha = 0 \) and \( F_i = f_i \), we have \( \gamma_{F} = \mathcal{O}(1)\gamma_{G_i} \) which is optimal up to a constant factor given the conditions in Assumption 5.

Now, we are ready to state the main result of our paper on the convergence of our proposed Per-FedAvg method.

**Theorem 4.5.** Consider the objective function \( F \) defined in (4) for the case that \( \alpha \in [0,1/L] \). Suppose that the conditions in Assumptions 1-4 are satisfied, and recall the definitions of \( L_F, \sigma_F, \) and \( \gamma_F \) from Lemmas 4.3-4.4. Consider running Algorithm 1 for \( K \) rounds with \( \tau \) local updates in each round and with \( \beta \leq 1/(10\tau L_F) \). Then, the following first-order stationary condition holds

\[
\frac{1}{\tau K} \sum_{k=0}^{K-1} \sum_{t=0}^{\tau-1} E \left[ \| \nabla F(\bar{w}_{k+1,t}) \|^2 \right] \leq \frac{4(F(w_0) - F^*)}{\beta \tau K} + 60 \left( \sigma_F^2 + \gamma_F^2 \right)
\]
where $\bar{w}_{k+1,t}$ is the average of iterates of users in $A_k$ at time $t$, i.e.,
$$
\bar{w}_{k+1,t} = \frac{1}{\tau} \sum_{i \in A_k} w_{k+1,t}^i,
$$
and in particular, $\bar{w}_{k+1,0} = w_k$ and $\bar{w}_{k+1,\tau} = w_{k+1}$.

Proof. Check Appendix F.

The result in Theorem 4.5 shows that if each user runs $\tau$ local updates at each iteration, after $K$ rounds of communication between users and server the average squared gradient norm in expectation converges at a sublinear rate of $O(1/K\tau)$ to a neighborhood of 0 with radius $O(\sigma_F^2 + \gamma_F^2)$. This result shows to find an $O(\epsilon + \sigma_F + \gamma_F)$-FOSP, we need to ensure that the parameters $K$ and $\tau$ satisfy the condition $K\tau = O(1/\epsilon^2)$.

Note that $\sigma_F$ is not a constant, and as expressed in Lemma 4.3, we can make it arbitrary small by choosing batch sizes $D, D'$, or $D''$ large enough. Also, and as we discussed after Lemma 4.4, $\sigma_F$ would be zero if we assume we have access to the exact the gradient and Hessians. Similarly, Lemma 4.4 implies that having small values for $\gamma_G$ and $\gamma_H$ would imply that $\gamma_F$ is also small.

As we discussed in Section 4.2, this observation is related to the closeness of data distribution of agents with respect to distribution measures such as Total Variation or 1-Wasserstein metric. In particular, consider the special case when $f_i$ admits the finite sum representation (3) and the data distributions are homogeneous, i.e., all users data distributions are drawn from an underlying distribution $p_u$. Then, having more samples for each user, i.e., larger $S_i$ in (3), will lead to smaller $\gamma_G$ and $\gamma_H$ as the empirical distribution of each user becomes closer to $p_u$ (see (Reisizadeh et al., 2019)).

Remark 4.6. The result of Theorem 4.5 provides an upper bound on the average of $E[\|\nabla F(\bar{w}_{k+1,t})\|^2]$ for all $k \in \{0, 1, ..., K\}$ and $t \in \{0, 1, ..., \tau - 1\}$. However, one concern here is that due to the structure of Algorithm 1, for any $k$, we only have access to $\bar{w}_{k+1,t}$ for $t = 0$.

To address this issue, at any iteration $k$, the center can choose $t_k \in \{0, 1, ..., \tau - 1\}$ uniformly at random, and ask all the users in $A_k$ to send $w_{k+1,t_k}^i$ back to the server, possibly in addition to $w_{k+1,\tau}^i$. If follow such a scheme then we can ensure that
$$
\frac{1}{\tau K} \sum_{k=0}^{K-1} E[\|\nabla F(\bar{w}_{k+1,t_k})\|^2] \leq \frac{4(F(w_0) - F^*)}{\beta \tau K} + 60(\sigma_F^2 + \gamma_F^2).
$$

5 Numerical Experiments

In this section, we design a numerical setting to highlight the role of personalization when the data distributions are heterogeneous. In particular, we consider the problem of classifying handwritten digits from the MNIST dataset (LeCun, 1998) and distribute the training data between $n$ users as follows:

- Half of the users have $a$ images of each of the digits 0-4.
- The rest, each have $a/2$ images from one of 0-4 digits and $2a$ images from one of 5-9 digits.

This way, we create an example where the distribution of images over all the users are different from each other. Similarly, we divide the test data over the nodes with the same distribution as the one for the training data.

We consider three algorithms in this setting: First, the classic FedAvg method, where the users find a shared model which all implement without any update during the test timet. Second, we take the output of the FedAvg method, and update it with one step of gradient descent with respect to the test data, and then evaluate its performance. Third, we consider our proposed algorithm, Per-FedAvg, and update its output, again with one step of gradient descent, during the test time. Similar to MAML, implementation of Per-FedAvg requires access to second-order information which is computationally costly. To address this issue, we replace the gradient estimate at each iteration with its first-order approximation which ignores the Hessian term, i.e., $\nabla F_i(w_{k+1,t-1})$ in (9) is approximated by
$$
\nabla f_i \left( w_{k+1,t-1}^i - \alpha \nabla f_i(w_{k+1,t-1}^i, D_i^t), D_i^t \right).
$$  (23)
This is the same idea deployed in First-Order MAML (FO-MAML) in (Finn et al., 2017), and it has been shown that it almost achieves the same level of performance as MAML when the the learning rate $\alpha$ is small (Fallah et al., 2019). Also, in Appendix G, we discuss how our analysis can be extended to first-order approximations of Per-FedAvg, such as the one implemented for this experiment.

For this experiment, we use a neural network classifier with two hidden layers with sizes 80 and 60, respectively, and we use Exponential Linear Unit (ELU) activation function. We run all three algorithms for $K = 1000$ rounds. At each round, we assume a fraction of agents with size $r_n$ are chosen to run $\tau$ local updates. The batch sizes $D = D' = 50$ and the learning rates are chosen as $\alpha = 0.01$ and $\beta = 0.001$. Further, we consider the case that there are $n = 10$ users in the network. We would like to mention that part of the code is adopted from (Langelaar, 2019).

The results for different values of number of local updates $\tau$ and ratio of active users $r$ are illustrated in Figure 1. As expected, in all considered cases, the model trained by running the update of FedAvg to solve the classic FL problem in (1) performs worse than the same model after running one step of local gradient in the test phase. Hence, if extra computation is available at the test time, the model of FedAvg after one step of gradient descent leads to a more personalized solution.

More importantly, the Per-FedAvg method, which is originally designed to find a point which performs well once it is updated using one step of local gradient descent has the best performance among the three considered approaches. In other words, its model has a better test accuracy compared to the model that is obtained by running one step of local gradient over the solution of FedAvg. These experiments show that by solving the MAML variant of the FL problem we obtain a solution that performs better in heterogeneous settings.
6 Conclusion

In this paper, we studied the Federated Learning (FL) problem in a heterogeneous case that the probability distribution of the users in the network are not identical and could be different. To solve this problem, we studied a personalized variant of the classic FL formulation in which our goal is to find a proper initialization model for the users in the network that can be quickly adapted to the local data of each user after the training phase. In particular, we introduced a Model-Agnostic Meta-Learning (MAML) variant of FL in which instead of minimizing the average loss over the data of all users, we find the best initial model that after one step of local gradient leads to a good model for each individual user. As expected, this approach leads to a more personalized model for each user. We then introduced a personalized variant of the FedAvg algorithm, called Per-FedAvg, to solve the proposed personalized FL problem. We also characterized the overall complexity of the Per-FedAvg method for nonconvex settings. Specifically, for the case that each user runs $\tau$ local updates at each iteration, we showed that after $K$ rounds of communication between users and server Per-FedAvg converges to a neighborhood of a first-order stationary point at a rate of $O(1/K\tau)$, where the radius of this neighborhood depends on the closeness of data distribution of different users. Finally, we provided a numerical experiment to illustrate the performance of Per-FedAvg and its comparison with FedAvg method.

Appendix

A Proofs of results in Subsection 4.2

A.1 TV Distance

Note that

$$\| \nabla f_i(w) - \nabla f(w) \| = \left\| \sum_{z \in Z} \nabla w_l(z; w) (p_i(z) - p(z)) \right\|$$

$$\leq \sum_{z \in Z} \| \nabla w_l(z; w) \| \| p_i(z) - p(z) \|$$

$$\leq B \sum_{z \in Z} | p_i(z) - p(z) | = 2B \| p_i - p \|_{TV}$$ (24)

where for the inequality we used the assumption that $\| \nabla w_l(z; w) \| \leq B$ for any $w$ and $z$. Plugging (24) in

$$\frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(w) - \nabla f(w) \|^2$$ (25)

gives us the desired result. The other result on Hessians can be proved similarly.

A.2 1-Wasserstein Distance

We claim that for any $i$ and $w \in \mathbb{R}^d$, we have

$$\| \nabla f_i(w) - \nabla f(w) \| \leq L_Z W_1(p_i, p)$$ (26)

which will immediately gives us one of the two results. To show this, first, note that

$$\| \nabla f_i(w) - \nabla f(w) \| = \sup_{v \in \mathbb{R}^d : \|v\| \leq 1} v^\top (\nabla f_i(w) - \nabla f(w))$$

$$= \sup_{v \in \mathbb{R}^d : \|v\| \leq 1} E_{z \sim p_i} [v^\top \nabla l(z; w)] - E_{z \sim p} [v^\top \nabla l(z; w)]$$

Thus, we need to show for any $v \in \mathbb{R}^d$ with $\|v\| \leq 1$, we have

$$E_{z \sim p_i} [v^\top \nabla l(z; w)] - E_{z \sim p} [v^\top \nabla l(z; w)] \leq L_Z W_1(p_i, p).$$ (27)
Next, note that since $p_i$ and $p$ both have bounded support, by Kantorovich–Rubinstein Duality Theorem Villani (2008), we have

$$W_1(p_i, p) = \sup \{\mathbb{E}_{z \sim p_i} [g(z)] - \mathbb{E}_{z \sim p} [g(z)] \mid \text{continuous } g : \mathbb{R} \to \mathbb{R}, \text{Lip}(g) \leq 1\}. \quad (28)$$

Using this result, to show (27), it suffices to show $g(z) = v^T \nabla l(z; w)$ is $L_Z$-Lipschitz. Note that Cauchy-Schwarz inequality implies

$$\|v^T \nabla l(z_1; w) - v^T \nabla l(z_2; w)\| \leq \|v\|\|\nabla l(z_1; w) - \nabla l(z_2; w)\| \leq L_Z d(z_1, z_2) \quad (29)$$

where the last inequality is obtained using $\|v\| \leq 1$ along with (21).

Finally, note that we can similarly show the result for $\gamma_H$ by considering the fact that

$$\|\nabla^2 f_i(w) - \nabla^2 f(w)\| = \sup_{v \in \mathbb{R}^d : \|v\| \leq 1} v^T (\nabla f_i(w) - \nabla f(w)) v$$

$$= \sup_{v \in \mathbb{R}^d : \|v\| \leq 1} \mathbb{E}_{z \sim p_i} [v^T \nabla l(z; w)v] - \mathbb{E}_{z \sim p} [v^T \nabla l(z; w)v]$$

and taking the function $g(z) = v^T \nabla l(z; w)v$ and using Kantorovich–Rubinstein Duality Theorem again.

**B Proof of Lemma 4.2**

Recall that

$$\nabla F_i(w) = (I - \alpha \nabla^2 f_i(w)) \nabla f_i(w - \alpha \nabla f_i(w)). \quad (30)$$

Given this, note that

$$\begin{align*}
\|\nabla F_i(w_1) - \nabla F_i(w_2)\| &= \|(I - \alpha \nabla^2 f_i(w_1)) \nabla f_i(w_1 - \alpha \nabla f_i(w_1)) - (I - \alpha \nabla^2 f_i(w_2)) \nabla f_i(w_2 - \alpha \nabla f_i(w_2))\|
\end{align*}$$

$$= \|(I - \alpha \nabla^2 f_i(w_1)) (\nabla f_i(w_1 - \alpha \nabla f_i(w_1)) - \nabla f_i(w_2 - \alpha \nabla f_i(w_2))) + (I - \alpha \nabla^2 f_i(w_1)) - (I - \alpha \nabla^2 f_i(w_2))\| \quad (31)
$$

$$\leq \|(I - \alpha \nabla^2 f_i(w_1))\| \|(\nabla f_i(w_1 - \alpha \nabla f_i(w_1)) - \nabla f_i(w_2 - \alpha \nabla f_i(w_2)))\| + \alpha \|(\nabla^2 f_i(w_1) - \nabla^2 f_i(w_2))\| \|(\nabla f_i(w_2 - \alpha \nabla f_i(w_2))\| \quad (32)
$$

where (31) is obtained by adding and subtracting $(I - \alpha \nabla^2 f_i(w_1)) \nabla f_i(w_2 - \alpha \nabla f_i(w_2))$ and the last inequality follows from the triangle inequality and the definition of matrix norm. Now, we bound two terms of (32) separately.

First, note that by (12a), $\|(I - \alpha \nabla^2 f_i(w_1))\| \leq 1 + \alpha L$. Using this along with smoothness of $f_i$, we have

$$\begin{align*}
\|(I - \alpha \nabla^2 f_i(w_1))\| \|(\nabla f_i(w_1 - \alpha \nabla f_i(w_1)) - \nabla f_i(w_2 - \alpha \nabla f_i(w_2)))\|
\leq (1 + \alpha L)\|w_1 - \alpha \nabla f_i(w_1) - w_2 + \alpha \nabla f_i(w_2)\|
\leq (1 + \alpha L)\|\|w_1 - w_2\| + \alpha \|\nabla f_i(w_1) - \nabla f_i(w_2)\|\|
\leq (1 + \alpha L)(1 + \alpha L)\|w_1 - w_2\| \leq 4L\|w_1 - w_2\| \quad (33)
\end{align*}$$

where we used smoothness of $f_i$ along with $\alpha \leq 1/L$ for the last line.

For the second term, using (11a) in Assumption 2 along with Assumption 3 implies

$$\alpha \|\nabla^2 f_i(w_1 - \nabla^2 f_i(w_2))\| \|\nabla f_i(w_2 - \alpha \nabla f_i(w_2))\| \leq \alpha \rho B\|w_1 - w_2\|. \quad (34)$$

Putting (33) and (34) together, we obtain the desired result.

**C Proof of Lemma 4.3**

Recall that the expression for the stochastic gradient $\hat{\nabla} F_i(w)$ is given by

$$\hat{\nabla} F_i(w) = \left(I - \alpha \nabla^2 f_i(w, D')\right) \nabla f_i\left(w - \alpha \nabla f_i(w, D), D'\right) \quad (35)$$

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which can be written as
\[
\nabla F_i(w) = (I - \alpha \nabla^2 f_i(w) + e_1)(\nabla f_i(w - \alpha \nabla f_i(w)) + e_2). 
\]

(36)

Note that in the above expression \(e_1\) and \(e_2\) are given by
\[
e_1 = \alpha \left( \nabla^2 f_i(w) - \tilde{\nabla}^2 f_i(w, \mathcal{D}') \right),
\]
and
\[
e_2 = \nabla f_i(w - \alpha \tilde{\nabla} f_i(w, \mathcal{D}'), \mathcal{D}') - \nabla f_i(w - \alpha \nabla f_i(w)).
\]

It can be easily shown that
\[
\mathbb{E} \|e_1\|^2 \leq \alpha^2 \frac{\sigma_H^2}{\mathcal{D}''}.
\]

(37)

Next, we proceed to bound the second moment of \(e_2\). To do so, first note that \(e_2\) can also be written as
\[
e_2 = \left( \nabla f_i \left( w - \alpha \tilde{\nabla} f_i(w, \mathcal{D}') \right) - \nabla f_i \left( w - \alpha \nabla f_i(w, \mathcal{D}) \right) \right) + \left( \nabla f_i \left( w - \alpha \tilde{\nabla} f_i(w, \mathcal{D}) \right) - \nabla f_i \left( w - \alpha \nabla f_i(w) \right) \right).
\]

(38)

Note that, conditioning on \(\mathcal{D}\), the first term is zero mean and the second term is deterministic. Therefore,
\[
\mathbb{E} \|e_2\|^2 = \mathbb{E} \left[ \mathbb{E} \|e_2\|^2 | \mathcal{D} \right] \leq \sigma_G^2 \left( \frac{1}{\mathcal{D}'} + \frac{(\alpha L)^2}{D} \right).
\]

(39)

where (39) is obtained using smoothness of \(f_i\) along with the fact that
\[
\mathbb{E} \left[ \left\| \nabla f_i \left( w - \alpha \tilde{\nabla} f_i(w, \mathcal{D}), \mathcal{D}' \right) - \nabla f_i \left( w - \alpha \nabla f_i(w, \mathcal{D}) \right) \right\|^2 \right] \leq \sigma_G^2.
\]

The last inequality is also obtained using
\[
\mathbb{E} \left[ \left\| \nabla f_i(w, \mathcal{D}) - \nabla f_i(w) \right\|^2 \right] \leq \sigma_G^2.
\]

Next, note that, by comparing (36) and (6), along with the matrix norm definition, we have
\[
\left\| \nabla F_i(w) - \nabla F_i(w) \right\| \leq \|I - \alpha \nabla^2 f_i(w)\| e_2 + \|e_1\| \| \nabla f_i(w - \alpha \nabla f_i(w)) \| + \|e_1\| \|e_2\|.
\]

(41)

As a result, by the Cauchy-Schwarz inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) for \(a, b, c \geq 0\), we have
\[
\left\| \nabla F_i(w) - \nabla F_i(w) \right\|^2 \leq 3 \|I - \alpha \nabla^2 f_i(w)\| \|e_2\|^2 + 3 \|e_1\|^2 \| \nabla f_i(w - \alpha \nabla f_i(w)) \|^2 + 3 \|e_1\|^2 \|e_2\|^2.
\]

(42)

By taking expectation, and using the fact that \(\|I - \alpha \nabla^2 f_i(w)\| \leq 1 + \alpha L \leq 2\) and
\[
\| \nabla f_i(w - \alpha \nabla f_i(w)) \| \leq B,
\]
we have
\[
\mathbb{E} \left[ \left\| \nabla F_i(w) - \nabla F_i(w) \right\|^2 \right] \leq 3B^2 \mathbb{E} \left[ \|e_1\|^2 \right] + 12 \mathbb{E} \left[ \|e_2\|^2 \right] + 3 \mathbb{E} \left[ \|e_1\|^2 \right] \mathbb{E} \left[ \|e_2\|^2 \right]
\]

(43)

where we also used the fact that \(e_1\) and \(e_2\) are independent as \(\mathcal{D}''\) is independent from \(\mathcal{D}\) and \(\mathcal{D}'\).

Plugging (37) and (40) in (43), we obtain
\[
\mathbb{E} \left[ \left\| \nabla F_i(w) - \nabla F_i(w) \right\|^2 \right] \leq 3B^2 \alpha^2 \frac{\sigma_H^2}{\mathcal{D}''} + 12 \sigma_G^2 \left( \frac{1}{\mathcal{D}'} + \frac{(\alpha L)^2}{D} \right) + 3 \alpha^2 \sigma_G^2 \left( \frac{1}{\mathcal{D}'\mathcal{D}''} + \frac{(\alpha L)^2}{D\mathcal{D}''} \right),
\]

which gives us the desired result.
D Proof of Lemma 4.4

Recall that
\[ \nabla F_i(w) = (I - \alpha \nabla^2 f_i(w)) \nabla f_i(w - \alpha \nabla f_i(w)). \]  
which can be expressed as
\[ \nabla F_i(w) = (I - \alpha \nabla^2 f(w) + E_i) (\nabla f(w - \alpha \nabla f(w)) + r_i) \]  
where
\[ E_i = \alpha (\nabla^2 f(w) - \nabla^2 f_i(w)) , \]
\[ r_i = \nabla f_i(w - \alpha \nabla f_i(w)) - \nabla f(w - \alpha \nabla f(w)). \]

First, note that, by Assumption 5, we have
\[ \sum_n \| \nabla f_i(w - \alpha \nabla f_i(w)) - \nabla f_i(w - \alpha \nabla f(w)) \| \leq \alpha L \sum_n \| \nabla f_i(w) - \nabla f(w) \|. \]

Second, note that
\[ \| r_i \| \leq \| \nabla f_i(w - \alpha \nabla f_i(w)) - \nabla f_i(w - \alpha \nabla f(w)) \| + \| \nabla f_i(w - \alpha \nabla f(w)) - \nabla f(w - \alpha \nabla f(w)) \| \]
\[ \leq \alpha L \| \nabla f_i(w) - \nabla f(w) \| + \| \nabla f_i(w - \alpha \nabla f(w)) - \nabla f(w - \alpha \nabla f(w)) \| \]

where the last inequality is obtained using (11b) in Assumption 2. Now, by using \((a + b)^2 \leq 2(a^2 + b^2)\), we have
\[ \frac{1}{n} \sum_i \| r_i \|^2 \leq \frac{2}{n} \sum_i \left( (\alpha L)^2 \| \nabla f_i(w) - \nabla f(w) \|^2 + \| \nabla f_i(w - \alpha \nabla f(w)) - \nabla f(w - \alpha \nabla f(w)) \|^2 \right) \]
\[ \leq 2 \left( 1 + (\alpha L)^2 \right) (\gamma_0^2 + \gamma_i^2) \]
\[ \leq 8 \gamma_0^2. \]

where the second inequality follows from Assumption 5 and the last inequality is obtained using \(\alpha L \leq 1\). Next, recall that the goal is to bound the variance of \(\nabla F_i(w)\) when \(i\) is drawn from a uniform distribution. We know that by subtracting a constant from a random variable, its variance does not change. Thus, variance of \(\nabla F_i(w)\) is equal to variance of \(\nabla F_i(w) - (I - \alpha \nabla^2 f(w)) \nabla f(w - \alpha \nabla f(w))\). Also, the variance of the latter is bounded by its second moment, and hence,
\[ \frac{1}{n} \sum_i \| \nabla F_i(w) - \nabla F(w) \|^2 \leq \frac{1}{n} \sum_i \| E_i \nabla f(w - \alpha \nabla f(w)) + (I - \alpha \nabla^2 f(w)) r_i + E_i r_i \|^2 \]
\[ \leq \frac{1}{n} \sum_i \left( \| E_i \nabla f(w - \alpha \nabla f(w)) \| + \| (I - \alpha \nabla^2 f(w)) r_i \| + \| E_i r_i \| \right)^2 \]

Therefore, using \(\| \nabla f(w - \alpha \nabla f(w)) \| \leq B\) along with \(\| I - \alpha \nabla^2 f(w) \| \leq 1 + \alpha L\) and Cauchy-Schwarz inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) for \(a, b, c \geq 0\), we obtain
\[ \frac{1}{n} \sum_i \| \nabla F_i(w) - \nabla F(w) \|^2 \leq 3 \left( B^2 \frac{1}{n} \sum_i \| E_i \|^2 + \left( 1 + \alpha L \right)^2 \frac{1}{n} \sum_i \| r_i \|^2 + \frac{1}{n} \sum_i \| E_i r_i \|^2 \right) \]
\[ \leq 3 \left( B^2 \frac{1}{n} \sum_i \| E_i \|^2 + 4 \frac{1}{n} \sum_i \| r_i \|^2 + \frac{1}{n} \sum_i \| E_i r_i \|^2 \right) \]

where the last inequality is obtained using \(\alpha L \leq 1\) along with \(\| E_i r_i \| \leq \| E_i \| \| r_i \|\) which comes from the definition of matrix norm. Finally, to complete the proof, notice that we have
\[ \frac{1}{n} \sum_i \| E_i \|^2 \| r_i \|^2 \leq \max_i \| E_i \|^2 \left( \frac{1}{n} \sum_i \| r_i \|^2 \right) \]
\[ \leq \max_i \| E_i \|^2 (8 \gamma_0^2) \]
\[ \leq 32 (\alpha L)^2 \gamma_0^2 \leq 32 \gamma_0^2 \]
where (55) follows from (51) and the last line is obtained using $\alpha L \leq 1$ along with the fact that $\|\nabla^2 f_i(w)\| \leq L$, and thus,
\[
\frac{\|E_i\|}{\alpha} = \|\nabla^2 f(w) - \nabla^2 f_i(w)\| \leq 2L. \tag{57}
\]
Plugging (55) in (53) along with (48) and (51), we obtain the desired result.

**E An Intermediate Result**

**Proposition E.1.** Recall from Section 3 that at any round $k \geq 1$, and for any agent $i \in \{1, \ldots, n\}$, we can define a sequence of local updates $\{w^i_{k,t}\}_{t=1}^T$ where $w^i_{k,0} = w_{k-1}$ and, for $t \geq 1$,
\[
w^i_{k,t} = w^i_{k,t-1} - \beta \nabla F_i(w^i_{k,t-1}). \tag{58}
\]
We further define the average of these local updates at round $k$ and time $t$ as $w_{k,t} = 1/n \sum_{i=1}^n w^i_{k,t}$. Suppose that the conditions in Assumptions 2-4 are satisfied. Then, for any $\alpha \in [0, 1/L]$ and any $t \geq 1$, we have
\[
\begin{align*}
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|w^i_{k,t} - w_{k,t}\| \right] &\leq (1 + 2\beta L^2) \left(\sigma F + \gamma F\right)/L F, \tag{59a}
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|w^i_{k,t} - w_{k,t}\|^2 \right] &\leq \left(1 + \phi + 16(1 + \frac{1}{\phi})\beta^2 L^2 F\right)^t \frac{2\sigma_F^2 + \gamma_F^2}{4L^2 F} \tag{59b}
\end{align*}
\]
where $\phi > 0$ is an arbitrary positive constant and $L_F$, $\sigma_F$, and $\gamma_F$ are given in Lemmas 4.2, 4.3, and 4.4, respectively.

Before stating the proof, note that an immediate consequence of this result is the following corollary:

**Corollary E.2.** Under the same assumptions as Proposition E.1, and for any $\beta \leq 1/(10\tau L F)$, we have
\[
\begin{align*}
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|w^i_{k,t} - w_{k,t}\| \right] &\leq 2(\sigma_F + \gamma F)/L F, \tag{60a}
\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|w^i_{k,t} - w_{k,t}\|^2 \right] &\leq \frac{2\sigma_F^2 + \gamma_F^2}{L^2 F} \tag{60b}
\end{align*}
\]
for any $1 \leq t \leq \tau$.

**Proof.** Let
\[
S_t := \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\|w^i_{k,t} - w_{k,t}\| \right] \tag{61}
\]
where $S_0 = 0$ since $w^i_{k,0} = w_{k-1}$ for any $i$. Note that
\[
S_{t+1} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\|w^i_{k,t+1} - w_{k,t+1}\| \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\|w^i_{k,t} - \beta \nabla F_i(w^i_{k,t}) - \frac{1}{n} \sum_{j=1}^n \left( w^i_{k,t} - \beta \nabla F_j(w^j_{k,t}) \right) \| \right] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\|w^i_{k,t} - \frac{1}{n} \sum_{j=1}^n w^j_{k,t}\| \right] + \beta \frac{1}{n} \sum_{i=1}^n \mathbb{E}\left[\|\nabla F_i(w^i_{k,t}) - \frac{1}{n} \sum_{j=1}^n \nabla F_j(w^j_{k,t})\| \right]. \tag{62}
\]
As a result, and by using (64), we have
\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \| \right] \]
\[ \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \| \right] + \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \nabla F_i(w_{k,t}) \| \right] \]
\[ + \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} \| \nabla F_j(w_{k,t}) - \nabla F_j(w_{k,t}) \| \right] \]
\[ \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \| \right] + 2 \beta \sigma_F \]
where the last inequality is obtained using Lemma 4.3. By substituting this in (62), we obtain
\[ S_{t+1} \leq S_t + 2 \beta \sigma_F + \beta \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \| \right] . \] (63)
If we define \( \eta_i := \nabla F_i(w_{k,t}) - \nabla F_i(w_{k,t}) \), using (63), we obtain
\[ S_{t+1} \leq S_t + 2 \beta \sigma_F + \beta \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \| \right] + \beta \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \eta_i - \frac{1}{n} \sum_{j=1}^{n} \eta_j \| \right] . \] (64)
Note that, by Lemma 4.2,
\[ \| \eta_i \| \leq L_F \| w_{k,t}^i - w_{k,t} \| , \] (65)
and thus,
\[ \frac{1}{n} \sum_{i=1}^{n} \| \eta_i \| \leq L_F S_t . \] (66)
As a result, and by using (64), we have
\[ S_{t+1} \leq (1 + 2 \beta L_F) S_t + 2 \beta \sigma_F + \beta \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \nabla F_i(w_{k,t}) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \| \right] , \]
\[ \leq (1 + 2 \beta L_F) S_t + 2 \beta (\sigma_F + \gamma_F) \] (67)
where the last inequality is obtained using Lemma 4.4. Using (67) inductively, we obtain
\[ S_{t+1} \leq \left( \sum_{j=0}^{t} (1 + 2 \beta L_F)^j \right) 2 \beta (\sigma_F + \gamma_F) = \left( 1 + 2 \beta L_F \right)^{t+1} - 1 \]
\[ \leq \frac{2 \beta (\sigma_F + \gamma_F)}{L_F} \]
(68)
which completes the proof of (59a). To prove (59b), let
\[ \Sigma_t := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| w_{k,t}^i - w_{k,t} \|^2 \right] . \] (69)
Similarly $\Sigma_0 = 0$. Note that

$$\Sigma_{t+1} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| w_{k,t+1}^i - w_{k,t+1} \|^2 \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| w_{k,t}^i - \beta \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \left( w_{k,t}^j - \beta \nabla F_j(w_{k,t}^j) \right) \right\|^2 \right]$$

$$\leq \frac{1 + \phi}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| w_{k,t}^i - \frac{1}{n} \sum_{j=1}^{n} w_{k,t}^j \right\|^2 \right] + \beta^2 \frac{1 + 1/\phi}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}^j) \right\|^2 \right]$$

$$\leq (1 + \phi) \Sigma_t + \beta^2 \frac{1 + 1/\phi}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}^j) \right\|^2 \right]$$

(70)

where (70) is obtained using $\|a + b\|^2 \leq (1 + \phi)\|a\|^2 + (1 + 1/\phi)\|b\|^2$ for with $\phi > 0$ an arbitrary positive real number. To bound the second term in (71), note that

$$\mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}^j) \right\|^2 \right] \leq 2 \mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}^j) \right\|^2 \right]$$

$$+ 2 \mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \nabla F_i(w_{k,t}^j) \right\|^2 \right] + \frac{1}{n} \sum_{j=1}^{n} \left( \nabla F_j(w_{k,t}^i) - \nabla F_j(w_{k,t}^j) \right) \right\|^2 \right].$$

(72)

Now, we bound the second term in (72). Using Cauchy-Schwarz inequality

$$\left\| \sum_{i=1}^{n+1} a_i b_i \right\| \leq \left( \sum_{i=1}^{n+1} \| a_i \|^2 \right)^{1/2} \left( \sum_{i=1}^{n+1} \| b_i \|^2 \right)^{1/2}$$

(73)

with $a_i = \nabla F_i(w_{k,t}^i) - \nabla F_i(w_{k,t}^j)$, $b_1 = 1$ and $a_i = 1/\sqrt{n} (\nabla F_i(w_{k,t}^i) - \nabla F_i(w_{k,t}^j))$, $b_i = 1/\sqrt{n}$, for $l = 2, ..., n + 1$, implies

$$\mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \nabla F_i(w_{k,t}^j) \right\|^2 \right]$$

$$\leq 2 \mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \nabla F_i(w_{k,t}^j) \right\|^2 \right] + \frac{1}{n} \sum_{j=1}^{n} \left( \nabla F_j(w_{k,t}^i) - \nabla F_j(w_{k,t}^j) \right) \right\|^2 \right]$$

$$\leq 4\sigma_F^2$$

(74)

where the last inequality is obtained using Lemma 4.3. Plugging (74) in (72) and using (71), we obtain

$$\Sigma_{t+1} \leq (1 + \phi) \Sigma_t + 8\frac{1}{\beta^2} + 2\frac{1}{\phi^2} + 2\frac{1}{\phi} \sum_{i=1}^{n} \mathbb{E} \left[ \left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}^j) \right\|^2 \right].$$

(75)

Now, it remains to bound the last term in (75). Recall $\eta_i = \nabla F_i(w_{k,t}^i) - \nabla F_i(w_{k,t})$. First, note that, using $\|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$, we have

$$\left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}^j) \right\|^2 \leq \left\| \nabla F_i(w_{k,t}^i) - \frac{1}{n} \sum_{j=1}^{n} \nabla F_j(w_{k,t}) \right\|^2 + 2\|\eta_i - \frac{1}{n} \sum_{j=1}^{n} \eta_j\|^2.$$

(76)
Substituting this bound in (75) and using Lemma 4.4 yields

$$\Sigma_{t+1} \leq (1 + \phi) \Sigma_t + 4(1 + \frac{1}{\phi}) \beta^2 (2\sigma_F^2 + \gamma^2) + 4(1 + \frac{1}{\phi}) \beta^2 \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \eta_i - \frac{1}{n} \sum_{j=1}^{n} \eta_j \|^2 \right].$$  \hspace{1cm} (77)

Note that, using Cauchy-Schwarz inequality (73) with \( a_1 = \eta_i, b_1 = 1 \) and \( a_t = 1/\sqrt{m} \eta_{t-1}, b_t = 1/\sqrt{n} \) for \( l = 2, ..., n+1 \), implies

$$\| \eta_i - \frac{1}{n} \sum_{j=1}^{n} \eta_j \|^2 \leq 2 \left( \| \eta_i \|^2 + \frac{1}{n} \sum_{j=1}^{n} \| \eta_j \|^2 \right) \leq 2L_F^2 \left( \| w_{k,t} - w_{k,t} \|^2 + \frac{1}{n} \sum_{j=1}^{n} \| w_{k,t} - w_{k,t} \|^2 \right)$$ \hspace{1cm} (78)

where the last inequality is obtained using Lemma 4.2 which states

$$\| \eta_i \| \leq L_F \| w_{k,t} - w_{k,t} \|. \hspace{1cm} (79)$$

Plugging (78) in (77) implies

$$\Sigma_{t+1} \leq \left( 1 + \phi + 16(1 + \frac{1}{\phi}) \beta^2 L_F^2 \right) \Sigma_t + 4(1 + \frac{1}{\phi}) \beta^2 (2\sigma_F^2 + \gamma^2).$$ \hspace{1cm} (80)

As a result, using induction similar to (68), we obtain

$$\Sigma_{t+1} \leq \left( 1 + \phi + 16(1 + \frac{1}{\phi}) \beta^2 L_F^2 \right)^{t+1} \frac{2\sigma_F^2 + \gamma^2}{4L_F^2}$$ \hspace{1cm} (81)

which gives us the desired result (59b).

Finally, to show (60), first note that for any \( n \), we know

$$\left( 1 + \frac{1}{n} \right)^n \leq e.$$ \hspace{1cm} (82)

Using this, along with the assumption \( \beta \leq 1/(10L_F \tau) \) and the fact that \( e^{0.2} \leq 2 \), we immediately obtain (60a). To show the other one (60b), we use (59b) with \( \phi = 1/(2\tau) \):

$$\phi + 16(1 + \frac{1}{\phi}) \beta^2 L_F^2 = \frac{1}{2\tau} + 16(1 + 2\tau) \beta^2 L_F^2 \leq \frac{1}{2\tau} + 16(1 + 2\tau) \frac{1}{100\tau^2} \leq \frac{1}{\tau}.$$ \hspace{1cm} (83)

where the first inequality follows from the assumption \( \beta \leq 1/(10L_F \tau) \) and the last inequality is obtained using the trivial bound \( 1 + 2\tau \leq 3\tau \). Finally, using (83) along with (82) completes the proof.

\[ \square \]

F Proof of Theorem 4.5

Although we only ask a fraction of agents to compute their local updates in Algorithm 1, here, and just for the sake of analysis, we assume all agents perform local updates. This is just for our analysis and we will not use all agents’ updates in computing \( w_{k+1} \). Also, from Proposition E.1, recall that \( w_{k,t} = 1/n \sum_{i=1}^{n} w_{k,t} \).

Let \( F_{k+1} \) denote the \( \sigma \)-field generated by \( \{ w_{k+1,t} \}_{i=1}^{n} \). Note that, by Lemma 4.2, we know \( F \) is smooth with gradient Lipschitz parameter \( L_F \), and thus, by (12b), we have

$$F(\bar{w}_{k+1,t+1}) \leq F(\bar{w}_{k+1,t}) + \nabla F(\bar{w}_{k+1,t})^\top (\bar{w}_{k+1,t+1} - \bar{w}_{k+1,t}) + \frac{L_F}{2} \| \bar{w}_{k+1,t+1} - \bar{w}_{k+1,t} \|^2$$ \hspace{1cm} (84)

$$\leq F(\bar{w}_{k+1,t}) - \beta \nabla F(\bar{w}_{k+1,t})^\top \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) + \frac{L_F}{2} \beta^2 \frac{1}{rn} \sum_{i \in A_k} \| \nabla F_i(\bar{w}_{k+1,t}) \|^2$$
where the last inequality is obtained using the fact that
\[ \bar{w}_{k+1,t+1} = \frac{1}{rn} \sum_{i \in A_k} w_{k+1,t+1} = \frac{1}{rn} \sum_{i \in A_k} \left( w_{k+1,t} - \beta \nabla F_i (w_{k+1,t}) \right) = \bar{w}_{k+1,t} - \beta \frac{1}{rn} \sum_{i \in A_k} \nabla F_i (w_{k+1,t}). \]

Taking expectation from both sides of (84) yields
\[
\mathbb{E} [F(\bar{w}_{k+1,t+1})] \leq \mathbb{E} [F(\bar{w}_{k+1,t})] - \beta \mathbb{E} \left[ \nabla F(\bar{w}_{k+1,t})^\top \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i (w_{k+1,t}) \right) \right] + \frac{L_F}{2} \beta^2 \mathbb{E} \left[ \| \frac{1}{rn} \sum_{i \in A_k} \nabla F_i (w_{k+1,t}) \|^2 \right].
\]

Next, note that
\[
\frac{1}{rn} \sum_{i \in A_k} \nabla F_i (w_{k+1,t}) = X + Y + Z + \frac{1}{rn} \sum_{i \in A_k} \nabla F_i (\bar{w}_{k+1,t})
\]

where
\[
X = \frac{1}{rn} \sum_{i \in A_k} \left( \nabla F_i (w_{k+1,t}) - \nabla F_i (w_{k+1,t}) \right), \quad Y = \frac{1}{rn} \sum_{i \in A_k} \left( \nabla F_i (w_{k+1,t}) - \nabla F_i (w_{k+1,t}) \right), \quad Z = \frac{1}{rn} \sum_{i \in A_k} \left( \nabla F_i (w_{k+1,t}) - \nabla F_i (\bar{w}_{k+1,t}) \right).
\]

We next bound the second moment of X, Y, and Z, condition on \( F_{k+1} \). First, recall the Cauchy-Schwarz inequality
\[
\left\| \sum_{i=1}^{rn} a_i b_i \right\|^2 \leq \left( \sum_{i=1}^{rn} \| a_i \|^2 \right) \left( \sum_{i=1}^{rn} \| b_i \|^2 \right).
\]

- Using this inequality with \( a_i = (\nabla F_i (w_{k+1,t}) - \nabla F_i (w_{k+1,t})) / \sqrt{rn} \) and \( b_i = 1 / \sqrt{rn} \), we obtain
\[
\| X \|^2 \leq \frac{1}{rn} \sum_{i \in A_k} \left\| \nabla F_i (w_{k+1,t}) - \nabla F_i (w_{k+1,t}) \right\|^2,
\]

and hence, by using Lemma 4.3 along with the tower rule, we have
\[
\mathbb{E} [\| X \|^2] = \mathbb{E} [\mathbb{E} [\| X \|^2 | F_{k+1}]] \leq \sigma_F^2.
\]

- Regarding Y, note that by using Cauchy-Schwarz inequality (similar to what we did above) along with smoothness of F, we obtain
\[
\| Y \|^2 \leq \frac{1}{rn} \sum_{i \in A_k} \left\| \nabla F_i (w_{k+1,t}) - \nabla F_i (w_{k+1,t}) \right\|^2 \leq \frac{L_F^2}{rn} \sum_{i \in A_k} \| w_{k+1,t} - w_{k+1,t} \|^2.
\]

Again, taking expectation and using the fact that \( A_k \) is chosen uniformly at random, implies
\[
\mathbb{E} [\| Y \|^2] = \mathbb{E} [\mathbb{E} [\| Y \|^2 | F_{k+1}]] \leq L_F^2 \mathbb{E} \left[ \frac{1}{rn} \sum_{i \in A_k} \| w_{k+1,t} - w_{k+1,t} \|^2 \right] = L_F^2 \mathbb{E} \left[ \frac{1}{rn} \sum_{i=1}^{n} \| w_{k,t} - w_{k,t} \|^2 \right] \leq 2\sigma_F^2 + \gamma_F^2.
\]

- Regarding Z, first recall that if we have n numbers \( a_1, ..., a_n \) with mean \( \mu = 1/n \sum_{i=1}^{n} a_i \) and variance \( \sigma^2 = 1/n \sum_{i=1}^{n} |a_i - \mu|^2 \), and we take a subset of them \( \{a_i\}_{i \in A} \) with size \( |A| = rn \) by sampling without replacement, then we have
\[
\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A} a_i - \mu \right)^2 \right] = \frac{\sigma^2}{rn} \left( 1 - \frac{rn - 1}{n - 1} \right) \leq \frac{\sigma^2}{rn}.
\]
Using this, we have
\[
E \left[ \| \bar{w}_{k+1,t} - w_{k+1,t} \|^2 \mid \mathcal{F}_{k+1} \right] \leq \frac{1}{n} \sum_{i=1}^{n} \| w_{i,k+1,t}^t - w_{k+1,t} \|^2 \frac{1}{\sqrt{n}},
\]  
(96)
and hence, by taking expectation from both sides and using the tower rule along with (60b) in Corollary E.2, we obtain
\[
E \left[ \| \bar{w}_{k+1,t} - w_{k+1,t} \|^2 \right] \leq \frac{2\sigma_F^2 + \gamma_F^2}{rnL_F^2}.
\]  
(97)
Next, note that by using Cauchy-Schwarz inequality (90), with \( a_i = (\nabla F_i(w_{k+1,t}) - \nabla F_i(\bar{w}_{k+1,t})) / \sqrt{rn} \) and \( b_i = 1 / \sqrt{rn} \), we have
\[
\| Z \|^2 \leq \frac{1}{rn} \sum_{i \in A_k} \| \nabla F_i(w_{k+1,t}) - \nabla F_i(\bar{w}_{k+1,t}) \|^2 \leq \frac{L_F^2}{rn} \sum_{i \in A_k} \| w_{k+1,t} - \bar{w}_{k+1,t} \|^2 = L_F^2 \| \bar{w}_{k+1,t} - w_{k+1,t} \|^2
\]  
(98)
where the last inequality is obtained using smoothness of \( F_i \) (Lemma 4.2). Now, taking expectation from both sides and using (97) yields
\[
E[\| Z \|^2] \leq \frac{2\sigma_F^2 + \gamma_F^2}{rn}.
\]  
(99)
Now, getting back to (85), we first lower bound the term
\[
E \left[ \nabla F(\bar{w}_{k+1,t})^T \left( \frac{1}{rn} \sum_{i \in A_k} \tilde{\nabla} F_i(w_{k+1,t}^i) \right) \right].
\]  
To do so, note that, by (86), we have
\[
E \left[ \nabla F(\bar{w}_{k+1,t})^T \left( \frac{1}{rn} \sum_{i \in A_k} \tilde{\nabla} F_i(w_{k+1,t}^i) \right) \right] = E \left[ \nabla F(\bar{w}_{k+1,t})^T \left( X + Y + Z + \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) \right] \geq E \left[ \nabla F(\bar{w}_{k+1,t})^T \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) \right] - \frac{1}{2} E[\| \nabla F(\bar{w}_{k+1,t}) \|^2] - \frac{1}{2} E[\| X + Y + Z \|^2]
\]  
(100)
where the last inequality is obtained using the fact that
\[
E \left[ \nabla F(\bar{w}_{k+1,t})^T (X + Y + Z) \right] \leq \frac{1}{2} \left( E[\| \nabla F(\bar{w}_{k+1,t}) \|^2] + E[\| X + Y + Z \|^2] \right).
\]
Now, we bound terms in (100) separately. First, note that by tower rule we have
\[
E \left[ \nabla F(\bar{w}_{k+1,t})^T \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) \right] = E \left[ \nabla F(\bar{w}_{k+1,t})^T \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) \mid \mathcal{F}_{k+1} \right] = E \left[ \nabla F(\bar{w}_{k+1,t})^T \mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) \mid \mathcal{F}_{k+1} \right] \right] = E \left[ \| \nabla F(\bar{w}_{k+1,t}) \|^2 \right]
\]  
(101)
where the last equality is obtained using the fact that $A_k$ is chosen uniformly at random, and thus,
\[
\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right) \mathcal{F}_{k+1}^t \right] = \frac{1}{n} \sum_{i=1}^n \nabla F_i(\bar{w}_{k+1,t}).
\]
Second, note that by Cauchy-Schwarz inequality,
\[
\mathbb{E}[\|X + Y + Z\|^2] \leq 3 \left( \mathbb{E}[\|X\|^2] + \mathbb{E}[\|Y\|^2] + \mathbb{E}[\|Z\|^2] \right)
\leq 3 \left( (3 + 2/rn)\sigma_F^2 + (1 + 2/rn)\gamma_F^2 \right) \leq 15\sigma_F^2 + 9\gamma_F^2 \tag{102}
\]
where second inequality is obtained using (92), (94), and (99). Plugging (101) and (102) in (100) implies
\[
\mathbb{E} \left[ \nabla F(\bar{w}_{k+1,t})^\top \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(w_{k+1,t}) \right) \right] \geq \frac{1}{2} \mathbb{E}[\|\nabla F(\bar{w}_{k+1,t})\|^2] - \frac{15}{2}(\sigma_F^2 + \gamma_F^2). \tag{103}
\]
Next, we characterize an upper bound for the other term in (85):
\[
\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(w_{k+1,t}) \right)^2 \right]
\]
Note that, by (86) we have
\[
\left\| \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(w_{k+1,t}) \right\|^2 \leq 2\|X + Y + Z\|^2 + 2\left\| \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right\|^2, \tag{104}
\]
and thus, by (102), we have
\[
\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(w_{k+1,t}) \right)^2 \right] \leq 2\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right)^2 \right] + 30\sigma_F^2 + 18\gamma_F^2. \tag{105}
\]
Note that, $\mathbb{E} \left[ 1/r \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \mid \mathcal{F}_{k+1}^t \right] = \nabla F(\bar{w}_{k+1,t})$, since $A_k$ is chosen uniformly at random. Also, by Lemma 4.4, we have
\[
\frac{1}{n} \mathbb{E} \left[ \|\nabla F_i(\bar{w}_{k+1,t}) - \nabla F(\bar{w}_{k+1,t})\|^2 \mid \mathcal{F}_{k+1}^t \right] \leq \gamma_F^2,
\]
and thus, by (95), we have
\[
\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(\bar{w}_{k+1,t}) \right)^2 \right] \leq \mathbb{E} \left[ \|\nabla F(\bar{w}_{k+1,t})\|^2 \right] + \frac{\gamma_F^2}{rn}. \tag{106}
\]
Plugging (106) in (105), we obtain
\[
\mathbb{E} \left[ \left( \frac{1}{rn} \sum_{i \in A_k} \nabla F_i(w_{k+1,t}) \right)^2 \right] \leq 2\mathbb{E} \left[ \|\nabla F(\bar{w}_{k+1,t})\|^2 \right] + 30(\sigma_F^2 + \gamma_F^2). \tag{107}
\]
Substituting (107) and (103) in (85) implies
\[
\mathbb{E}[F(\bar{w}_{k+1,t+1})] \leq \mathbb{E}[F(\bar{w}_{k+1,t})] - \beta(1/2 - \beta L_F)\mathbb{E} \left[ \|\nabla F(\bar{w}_{k+1,t})\|^2 \right] + 15(\frac{1}{2} + \beta L_F)\beta(\sigma_F^2 + \gamma_F^2)
\leq \mathbb{E}[F(\bar{w}_{k+1,t})] - \frac{\beta}{4} \mathbb{E} \left[ \|\nabla F(\bar{w}_{k+1,t})\|^2 \right] + 15\beta(\sigma_F^2 + \gamma_F^2) \tag{108}
\]
where the last inequality is obtained using $\beta \leq 1/(10rL_F)$. Summing up (108) for all $t = 0, \ldots, \tau - 1$, we obtain
\[
\mathbb{E}[F(w_{k+1})] \leq \mathbb{E}[F(w_k)] - \frac{\beta\tau}{4} \left( \frac{1}{\tau} \sum_{t=0}^{\tau-1} \mathbb{E} \left[ \|\nabla F(\bar{w}_{k+1,t})\|^2 \right] \right) + 15\beta\tau(\sigma_F^2 + \gamma_F^2) \tag{109}
\]
where we used the fact that $\tilde{w}_{k+1,\tau} = w_{k+1}$. Finally, summing up (109) for $k = 0, ..., K-1$ implies

$$\mathbb{E}[F(w_K)] \leq F(w_0) - \frac{\beta \tau K}{4} \left( \frac{1}{\tau K} \sum_{k=0}^{K-1} \sum_{t=0}^{\tau-1} E \left[ \| \nabla F(w_{k+1,t}) \|^2 \right] \right) + 15\beta \tau K (\sigma_F^2 + \gamma_F^2).$$  \hspace{1cm} (110)

As a result, we have

$$\frac{1}{\tau K} \sum_{k=0}^{K-1} \sum_{t=0}^{\tau-1} E \left[ \| \nabla F(w_{k+1,t}) \|^2 \right] \leq \frac{4}{\beta \tau K} (F(w_0) - \mathbb{E}[F(w_K)] + 15\beta \tau K (\sigma_F^2 + \gamma_F^2))$$

$$\leq \frac{4(F(w_0) - F^*)}{\beta \tau K} + 60(\sigma_F^2 + \gamma_F^2)$$ \hspace{1cm} (111)

which gives us the desired result.

### G On First-Order Approximations of Per-FedAvg

As we stated previously, the Per-FedAvg method, same as MAML, requires computing Hessian-vector product which is computationally costly in some applications. As a result, one may consider using the first-order approximation of the update rule for the Per-FedAvg algorithm. The main goal of this section is to show how our analysis can be extended to the case that we either drop the second-order term or approximate the Hessian-vector product using first-order techniques.

To do so, we show that it suffices to only extend the result in Lemma 4.3 for the first-order approximation settings and find $\bar{\sigma}_F$ such that $E[\|\nabla F_i(w) - \nabla F_i(\bar{w})\|^2] \leq \bar{\sigma}_F^2$. One can easily check that the rest of analysis does not change, and the final result (Theorem 4.5) holds if we just replace $\sigma_F$ by $\bar{\sigma}_F$.

We next focus on two different approaches, developed for MAML formulation, for approximating the Hessian-vector product, and show how we can characterize $\bar{\sigma}_F$ for both cases:

- **Ignoring the second-order term:** Finn et al. (2017) suggested to simply ignore the second-order term in the update of MAML to reduce the computation cost of MAML, i.e., to replace $\nabla F_i(w)$ with

$$\tilde{\nabla} f_i \left( w - \alpha \tilde{\nabla} f_i(w, \mathcal{D}, \mathcal{D}') \right).$$ \hspace{1cm} (112)

This approach is known as First-Order MAML (FO-MAML), and it has been shown that it performs relatively well in many cases (Finn et al., 2017). In particular, Fallah et al. (2019) characterized the convergence properties of FO-MAML for the centralized MAML problem. Next, we characterize the variance of this gradient approximation.

**Lemma G.1.** Assume that we estimate $\nabla F_i(w)$ by (112) where $\mathcal{D}$ and $\mathcal{D}'$ are independent batches with size $D$ and $D'$, respectively. Suppose that the conditions in Assumptions 2-4 are satisfied. Then, for any $\alpha \in [0, 1/L]$ and $w \in \mathbb{R}^d$, we have $E \left[ \| \tilde{\nabla} F_i(w) - \nabla F_i(w) \|^2 \right] \leq \bar{\sigma}_F^2$, where $\bar{\sigma}_F^2$ is given by

$$\bar{\sigma}_F^2 := 2\sigma_G^2 \left( \frac{1}{D'} + \frac{(\alpha L)^2}{D} \right) + 2(\alpha LB)^2.$$

**Proof.** In fact, in this case, $\tilde{\nabla} F_i(w)$ is approximating

$$G_i(w) := \nabla f_i \left( w - \alpha \nabla f_i(w) \right).$$ \hspace{1cm} (113)

To characterize $\bar{\sigma}_F$ note that

$$E \left[ \| \tilde{\nabla} F_i(w) - \nabla F_i(w) \|^2 \right] \leq 2E \left[ \| \tilde{\nabla} F_i(w) - G_i(w) \|^2 \right] + 2E \left[ \| G_i(w) - \nabla F_i(w) \|^2 \right].$$ \hspace{1cm} (114)

We bound these two terms separately. Note that we have already bounded the first term in Appendix C (see (40)), and we have

$$E \left[ \| \tilde{\nabla} F_i(w) - G_i(w) \|^2 \right] \leq \sigma_G^2 \left( \frac{1}{D'} + \frac{(\alpha L)^2}{D} \right).$$ \hspace{1cm} (115)
To bound the second term in (114), note that
\[
\|G_i(w) - \nabla F_i(w)\| = \alpha \|\nabla^2 f_i(w)\nabla f_i(w - \alpha \nabla f_i(w))\| \\
\leq \alpha \|\nabla^2 f_i(w)\| \cdot \|\nabla f_i(w - \alpha \nabla f_i(w))\| \leq \alpha LB
\]
where the first inequality follows from the matrix norm definition and the last inequality is obtained using Assumption 2. Plugging (115) and (116) into (114), we obtain the desired result. \(\square\)

Note that while the first term in \(\tilde{\sigma}_F\) can be made arbitrary small by choosing \(D\) and \(D'\) large enough, this is not the case for the second term. However, the second term is also negligible if \(\alpha\) is small enough. Yet this bound suggests that this approximation introduces a non-vanishing error term which is directly carried to the final result (Theorem 4.5).

- **Estimating Hessian-vector product using gradient differences:** In the context of MAML problem, it has been shown that the update of FO-MAML leads to an additive error that does not vanish as time progresses. To resolve this matter, Fallah et al. (2019) introduced another variant of MAML, called HF-MAML, which approximates the Hessian-vector product by gradient differences. More formally, the idea behind their method is that for any function \(g\), the product of the Hessian \(\nabla^2 g(w)\) by any vector \(v\) can be approximated by
\[
\frac{\nabla g(w + \delta v) - \nabla g(w - \delta v)}{2\delta}
\]
with an error of at most \(\rho \delta \|v\|^2\), where \(\rho\) is the parameter for Lipschitz continuity of the Hessian of \(g\). Building on this idea, in Per-FedAvg update rule, we can replace \(\nabla F_i(w)\) by
\[
\tilde{\nabla} f_i \left( w - \alpha \tilde{\nabla} f_i(w, D), D' \right) - \alpha d_i(w)
\]
where
\[
d_i(w) := \frac{\tilde{\nabla} f_i \left( w + \delta \tilde{\nabla} f_i(w, D'), D'' \right) - \tilde{\nabla} f_i \left( w - \delta \tilde{\nabla} f_i(w, D'), D'' \right)}{2\delta}
\]
(119)

For this approximation, we have the following result:

**Lemma G.2.** Assume that we estimate \(\nabla F_i(w)\) by (118) where \(D, D',\) and \(D''\) are independent batches with size \(D, D',\) and \(D''\), respectively. Suppose that the conditions in Assumptions 2-4 are satisfied. Then, for any \(\alpha \in [0, 1/L]\) and \(w \in \mathbb{R}^d\), we have
\[
\mathbb{E} \left[ \|\tilde{\nabla} F_i(w) - \nabla F_i(w)\|^2 \right] \leq \tilde{\sigma}_F^2, \text{ where} \quad \tilde{\sigma}_F^2 \text{ is given by}
\]
\[
\tilde{\sigma}_F^2 := 6\sigma_G^2 \left( \frac{2(\alpha L)^2}{D} + \frac{2}{D'} + \frac{\alpha^2}{2\delta^2 D''} \right) + 2(\alpha \rho \delta)^2 B^4.
\]

**Proof.** Note that, this time \(\tilde{\nabla} F_i(w)\) is approximating
\[
G'_i(w) := \nabla f_i \left( w - \alpha \nabla f_i(w) \right) - \alpha d_i(w)
\]
where
\[
d_i(w) := \frac{\nabla f_i \left( w + \delta \nabla f_i \left( w - \alpha \nabla f_i(w) \right) \right) - \nabla f_i \left( w - \delta \nabla f_i \left( w - \alpha \nabla f_i(w) \right) \right)}{2\delta}
\]
(121)
is the term approximating \(\nabla^2 f_i(w)\nabla f_i \left( w - \alpha \nabla f_i(w) \right)\). To characterize \(\tilde{\sigma}_F\), again and similar to (114), we have
\[
\mathbb{E} \left[ \|\tilde{\nabla} F_i(w) - \nabla F_i(w)\|^2 \right] \leq 2\mathbb{E} \left[ \|\tilde{\nabla} F_i(w) - G'_i(w)\|^2 \right] + 2\mathbb{E} \left[ \|G'_i(w) - \nabla F_i(w)\|^2 \right].
\]
(122)

We again bound both terms separately. To simplify the notation, let us define
\[
g_i(w) := \nabla f_i \left( w - \alpha \nabla f_i(w) \right), \quad \tilde{g}_i(w) := \tilde{\nabla} f_i \left( w - \alpha \tilde{\nabla} f_i(w, D), D' \right).
\]
(123)
First, note that, using \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) for \(a, b, c \geq 0\), we have
\[
\left\| \nabla F_i(w) - G'_i(w) \right\|^2 \leq 3\|\tilde{g}_i(w) - g_i(w)\|^2 + \frac{3\sigma^2}{4D^2} \left\| \nabla f_i (w + \delta \tilde{g}_i(w), \mathcal{D}^\prime) - \nabla f_i (w + \delta g_i(w)) \right\|^2 \\
+ \frac{3\sigma^2}{4D^2} \left\| \nabla f_i (w - \delta \tilde{g}_i(w), \mathcal{D}^\prime) - \nabla f_i (w - \delta g_i(w)) \right\|^2.
\] (124)
Taking expectation from both sides, along with using (115), we have
\[
\mathbb{E} \left[ \left\| \nabla F_i(w) - G'_i(w) \right\|^2 \right] \\
\leq 3\sigma^2 \left( \frac{1}{D^2} + \frac{(\alpha L)^2}{D^2} \right) + \frac{3\sigma^2}{4D^2} \left( \mathbb{E} \left[ \left\| \nabla f_i (w + \delta \tilde{g}_i(w), \mathcal{D}^\prime) - \nabla f_i (w + \delta g_i(w)) \right\|^2 \right] \\
+ \mathbb{E} \left[ \left\| \nabla f_i (w - \delta \tilde{g}_i(w)) - \nabla f_i (w - \delta g_i(w)) \right\|^2 \right] \right) \\
\leq 3\sigma^2 \left( \frac{\alpha^2}{2D^2} + \frac{1}{D^2} + \frac{(\alpha L)^2}{D^2} \right) + \frac{3\sigma^2}{4D^2} \left( \mathbb{E} \left[ \left\| \nabla f_i (w + \delta \tilde{g}_i(w)) - \nabla f_i (w + \delta g_i(w)) \right\|^2 \right] \\
+ \mathbb{E} \left[ \left\| \nabla f_i (w - \delta \tilde{g}_i(w)) - \nabla f_i (w - \delta g_i(w)) \right\|^2 \right] \right) \\
\] (125)
where (125) is obtained using the fact that \(\mathcal{D}^\prime\) is independent from \(\mathcal{D}\) and \(\mathcal{D}^\prime\) which implies
\[
\mathbb{E} \left[ \left\| \nabla f_i (w + \delta \tilde{g}_i(w), \mathcal{D}^\prime) - \nabla f_i (w + \delta g_i(w)) \right\|^2 \right] \leq \frac{\sigma^2}{D^2} \\
+ \mathbb{E} \left[ \left\| \nabla f_i (w + \delta \tilde{g}_i(w)) - \nabla f_i (w + \delta g_i(w)) \right\|^2 \right].
\]
Next, note that Assumption 2 yields
\[
\left\| \nabla f_i (w + \delta \tilde{g}_i(w)) - \nabla f_i (w + \delta g_i(w)) \right\| \leq \delta \|\tilde{g}_i(w) - g_i(w)\|.
\]
Plugging this bound into (125) and using (114) implies
\[
\mathbb{E} \left[ \left\| \nabla F_i(w) - G'_i(w) \right\|^2 \right] \leq 3\sigma^2 \left( \frac{\alpha^2}{2D^2} + \frac{1}{D^2} + \frac{(\alpha L)^2}{D^2} \right) \\
\leq 3\sigma^2 \left( \frac{2(\alpha L)^2}{D^2} + \frac{2}{D^2} + \frac{\alpha^2}{2D^2} \right) \\
\] (126)
where the last inequality is obtained using \(\alpha L \leq 1\).
Bounding the second term in (122) is more straightforward as we have
\[
\left\| G'_i(w) - \nabla F_i(w) \right\| = \alpha \left\| d_i(w) - \nabla^2 f_i(w) \nabla f_i (w - \alpha \nabla f_i(w)) \right\| \leq \alpha \rho \delta \|g_i(w)\|^2 \leq \alpha \rho \delta B^2.
\] (127)
Plugging (126) and (127) into (122) gives us the desired result. \(\square\)
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