A landscape in boundary string field theory: New class of solutions with massive state condensation

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We solve the equation of motion of boundary string field theory allowing generic boundary operators quadratic in $X$, and explore string theory non-perturbative vacua with massive state condensation. Using numerical analysis, a large number of new solutions are found. Their energies turn out to distribute densely in the range between the D-brane tension and the energy of the tachyon vacuum. We discuss an interpretation of these solutions as perturbative closed string states. From the cosmological point of view, the distribution of the energies can be regarded as the so-called landscape of string theory, as we have a vast number of non-perturbative string theory solutions including one with small vacuum energy.

Subject Index B26, B28

1. Introduction

As a non-perturbative formulation of open bosonic string theory, boundary string field theory (BSFT) \cite{1,2} was proposed, as was cubic string field theory (CSFT) \cite{3}. In general, solutions of string field theories are quite important as they provide non-perturbative vacua of string theory, to look at the true capability of string theory.

Recently, multiple D-brane solutions which have greater energies than the trivial vacuum were proposed \cite{4,5} in CSFT. This would have a significance equivalent to the proof of Sen's conjecture \cite{6–10}, since D-brane creation is thought of as a necessary ingredient for a complete non-perturbative formulation of string theory. To climb up the SFT potential hill instead of rolling down the hill to get to the tachyon vacuum, it is indispensable to treat the string massive modes.

After the construction of the analytic solution for tachyon condensation \cite{11}, various analytic solutions in CSFT have been found \cite{12–14}. In recent times, analytic forms of lump solutions \cite{15,16} and multiple D-brane solutions were proposed. In BSFT, as well, an analytic solution for tachyon condensation and lump solutions have been found \cite{17–19}.

To solve the equation of motion of CSFT, we encounter an infinite-dimensional equation, which is hard to solve. In fact, there are some subtleties of proposed solutions \cite{5,24–28}. On the other hand, there is a consistent truncation scheme which reduces BSFT to a standard field theory with a finite number of fields. The BSFT action was also constructed for boundary interactions quadratic in the worldsheet field $X$, corresponding to a subset of massive modes of open strings \cite{20}.

The purpose of this paper is to solve the equation of motion of the BSFT action for quadratic boundary operators. In contrast to CSFT, only the tachyon field plays a significant role in the BSFT
exact solution for tachyon condensation and the lump solutions, so that the analysis is rather simple. For this reason, it is natural to expect that one may obtain a new class of solutions by involving some more boundary operators, aiming at new string vacua and the construction of a multiple-D-brane solution.

We adopt the BSFT action for quadratic boundary interactions with arbitrary number of derivatives on the worldsheet given in [20], and solve the equation of motion numerically to find homogeneous static solutions. The condensation of the massive fields is taken care of to all their orders. So the solutions are non-perturbative ones at the classical level of SFT, in the same sense as for the non-perturbative tachyon vacuum solutions of BSFT. We discover a large number of new solutions of BSFT. Interestingly, those energies turn out to be smaller than the D-brane energy. Our analysis strongly suggests the existence of an infinite number of solutions.

We also find an approximately uniform distribution of the energies of the solutions, which suggests a relation to closed string excitations at the tachyon vacuum. Furthermore, from a cosmological point of view, the distribution of infinitely many solutions is reminiscent of the so-called string landscape. It is intriguing that any solution with small energy may be possible in BSFT, to reconcile the cosmological constant problem. We also find solutions with a part of Lorentz symmetry broken, which would serve as a realization of the old idea of spontaneous Lorentz symmetry breaking (and CPT breaking) through SFT [21–23].

This paper is organized as follows. In the next section we review the derivation of the BSFT action and derive the potential for the tachyon field and massive fields associated with generic quadratic boundary interactions on the worldsheet. From the potential, we obtain the equations of motion and solve them numerically in Sect. 3. We show plots of numerical results as well, to show the energy distribution of the solutions. In Sect. 4, we study properties of the solutions and present a possible interpretation of the solutions as relevant to closed string states. Finally, Sect. 5 is devoted to discussions.

2. Review: the BSFT action

We give a short review of boundary string field theory (BSFT) based on [1,2].\(^1\) In addition, we summarize the derivation of the BSFT action for quadratic boundary interactions following [20].

2.1. Generic formulation of BSFT

The dynamical variables of BSFT are boundary coupling constants \(\lambda_i\) associated with the boundary operators \(O^i\) of ghost number 1. The BSFT action is given by the solution to the equation

\[
\frac{\partial S}{\partial \lambda_i} = -\frac{1}{2} T_{25} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \langle O^i(\theta) \{Q_B, O(\theta')\}\rangle_{\lambda}.
\]

(1)

Here, \(T_{25}\) is the tension of the D25-brane, \(O = \sum_i \lambda_i O^i\) and \(Q_B\) is the BRST charge. \(\langle \cdots \rangle_{\lambda}\) denotes the correlation function in two-dimensional field theory on a unit disk, described by a bulk worldsheet action \(S_{\text{bulk}}\) with boundary interaction terms:

\[
S_{\text{bulk}} + \sum_i \lambda_i \int_0^{2\pi} d\theta' V^i(\theta).
\]

\(^1\) See [29,30] for relevant formulas and derivations. For the supersymmetric formulation, see, for example, [19,31–33]. Recent proposals of the formulation include [34].
Here, $\theta$ is the angle parametrizing the boundary of disk. $V^i$ is the vertex operator associated with the open string state $|V^i\rangle$:

$$|V^i\rangle = b_{-1} |O^i\rangle,$$  \hspace{1cm} (3)

where $|O^i\rangle$ is also associated with the boundary operator $O^i$. One can show that the action $S$ satisfies the Batalin–Vilkovisky (BV) master equation and so has a gauge symmetry [1].

Reference [2] describes how to write down the action directly for general $V$ constructed only from matter fields. For general choice of $V$, 

$$\{Q, O(\theta)\} = \sum_{n=1}^{\infty} c \partial^n c F_n(\theta),$$  \hspace{1cm} (4)

where $F_n$ are matter operators. Hence, the ghost correlation functions appearing in (1) are of the form $\langle c(\theta) c(\theta') c(\theta'') \rangle = 2 \left( \sin(\theta - \theta') + \sin(\theta' - \theta'') + \sin(\theta'' - \theta) \right)$, 

(5)

(1) can be written in the form 

$$\frac{\partial S}{\partial \lambda_i} = -T_{25} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \left( V^i(\theta) \left( A(\theta') + \cos(\theta - \theta') B(\theta') + \sin(\theta - \theta') C(\theta') \right) \right)_\lambda,$$  \hspace{1cm} (6)

where $A, B, C$ are linear combinations of $F_n$. The operator $A$ has an expansion in terms of a basis $\{V_i(\theta)\}$ of matter operators,

$$A(\theta) = \sum_i \alpha_i V^i(\theta).$$  \hspace{1cm} (7)

Then the action is given by 

$$S = -T_{25} \left( \sum_i \alpha_i \frac{\partial}{\partial \lambda_i} + g \right) Z,$$  \hspace{1cm} (8)

where $Z$ is the partition function of the worldsheet theory (2) and $g$ is a constant.

2.2. BSFT action with generic quadratic boundary interactions

Next, following Li and Witten [20], we derive the BSFT action for the most general quadratic boundary operators. Note that the quadratic part gives a free CFT on the worldsheet, so the truncation of string theory to the one with generic quadratic boundary interactions is a consistent truncation [18]. Solutions of BSFT with generic quadratic boundary interactions amount to solutions of the full theory.2

The generic quadratic operators are:3

$$O = c(\theta) V(\theta), \hspace{0.5cm} V(\theta) = \frac{a}{2\pi} + \frac{1}{4\pi \alpha'} : X^\mu(\theta) \int_0^{2\pi} d\theta' \, u^{\mu\nu}(\theta - \theta') X^\nu(\theta') :.$$  \hspace{1cm} (9)

Here, $: :$ stands for the normal ordering as 

$$: X^\mu(\theta) X^\nu(\theta') : = X^\mu(\theta) X^\nu(\theta') + 2\alpha' \eta_{\mu\nu} \ln |1 - e^{i(\theta - \theta')}|. \hspace{1cm} (10)$$

\[^2\] This was the important observation for the proof of Sen's conjecture [6] by BSFT [17–19].

\[^3\] It is not necessary to normal order $V_{\mu\nu}^k$ since it has no singularity for generic $u^{\mu\nu}(\theta)$ regular at $\theta = 0$. In fact, in [20], $V_{\mu\nu}^k$ was not normal ordered. The normal ordered form, however, is useful to see how the action reduces to the one associated with $V = \frac{a}{2\pi} + \frac{1}{4\pi \alpha'} \sum_{\mu=0}^{25} u^{\mu\nu} \eta_{\mu\nu} : X^2(\theta) :$, so we adopt it in this paper.
We use the standard closed string action

\[ S_{\text{bulk}} = \int d^2 \sigma \sqrt{h} \left( \frac{1}{4 \pi \alpha'} h^{ab} \eta^{\mu\nu} \partial_a X_\mu \partial_b X_\nu + \frac{1}{2 \pi} b_{ab} \nabla^a c^b \right). \]  

(11)

Here, \( \{a, u^{\mu\nu}(\theta - \theta')\} \) is a set of boundary couplings and \( h_{ab} \) and \( \eta_{\mu\nu} = \text{diag}(\ldots, +, +, \ldots) \) are the metrics on the worldsheet and the target space respectively. Without loss of generality, we can assume \( u^{\mu\nu}(\theta) = u^{\nu\mu}(\theta) \).

Following the above terminology, \( \{\lambda_i\} = \{a, u_{k}^{\mu\nu}\} \) with \( u_{k}^{\mu\nu} = u_{-k}^{\nu\mu} = \int_0^{2\pi} d\theta \, u^{\mu\nu}(\theta)e^{-ik\theta} \), and

\[ V^a(\theta) = \partial_0 V = \frac{1}{2\pi}, \quad V^k_{\mu\nu} = \partial_{u_{k}^{\mu\nu}} V = \frac{1}{8\pi^2 \alpha'} : X_\mu(\theta) \int_0^{2\pi} d\theta' e^{ik(\theta - \theta')}:. \]  

(12)

It is notable that the \( V^k_{\mu\nu} \) can be formally expressed as a linear combination of quadratic boundary operators \( X_\mu \delta^r X_\nu \), which are the vertex operators corresponding to a constant field strength and to a set of massive modes of open string for \( r > 0 \). Since the worldsheet action is quadratic in \( X \), we can solve this theory. The boundary condition is deformed by the boundary interaction as

\[ 0 = \partial_r X^\mu(\theta) + \int_0^{2\pi} d\theta' u^{\mu\nu}(\theta - \theta') X_\nu(\theta'), \]  

(13)

where \( r \) is the radial coordinate of the unit disk. The Green's function satisfying this boundary condition is

\[ G_{\mu\nu}(z, w) = \langle X_\mu(z, \bar{z}) X_\nu(w, \bar{w}) \rangle_{\lambda}, \]

\[ = \frac{-\alpha'}{2} \eta_{\mu\nu} \left( \ln |z - w|^2 + \ln |1 - \bar{z}\bar{w}|^2 \right) - \alpha' A_{0,\mu\nu} - \alpha' \sum_{k=1}^{\infty} \left( A_{k,\mu\nu}(\bar{z}\bar{w})^k + A_{-k,\mu\nu}(\bar{z}\bar{w})^k \right), \]  

(14)

where

\[ A_{0,\mu\nu} = -(u_0^{-1})_{\mu\nu}, \quad A_{k,\mu\nu} = \frac{1}{|k|} \eta_{\mu\nu} - \left( \frac{1}{|k|\eta + u_k} \right)_{\mu\nu} \text{ for } k \neq 0. \]  

(15)

Here, \( (1/(|k|\eta + u_k))_{\mu\nu} \) is the inverse matrix of \( (|k|\eta + u_k)_{\mu\nu} \):

\[ \left( \frac{1}{|k|\eta + u_k} \right)_{\mu\nu} (|k|\eta_{\rho\nu} + u_{k}^{\rho\nu}) = \delta_{\mu}^{\nu}. \]

Notice that the correlation function in (14) is evaluated with the bulk action (11) with the boundary terms given by (9). The partition function of the worldsheet theory is determined from the differential equation

\[ \frac{\partial}{\partial u_{k}^{\mu\nu}} \ln Z = -\frac{1}{8\pi^2 \alpha'} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' e^{ik(\theta - \theta')} : X_\mu(\theta) X_\nu(\theta') :_{\lambda} = \frac{1}{2} A_{k,\nu\mu}. \]  

(16)

In the last equation, we have used (14) and \( A_{k,\mu\nu} = A_{-k,\nu\mu} \). By integrating this differential equation, we obtain the partition function

\[ Z = N \det(u_0)^{-1/2} e^{-a \sum_{k=1}^{\infty} e^{-k^{-1}tr(\eta_{\mu\nu}u_k)}} \det(1 + k^{-1}\eta \cdot u_k)^{-1}. \]  

(17)

Here, \( (\eta \cdot u_k)_{\mu}^{\nu} = \eta_{\mu\rho} u_{k}^{\rho\nu} \) and \( N \) is the normalization constant determined by demanding that the partition function reduces to \( V_{26} \), the volume of the target space, for \( a = u_k = 0 \). The factor \( \det(u_0)^{-1/2} \)

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4 Until just before (27), we treat \( u_{k}^{\mu\nu} \) for all \( k \in Z \) as independent valuables when we take derivatives.
in the partition function originates from the integration over the zero modes
\[
\int d^{2\theta} x \ e^{-\frac{1}{2}\partial_0^\mu x_\mu x_\nu} = (2\pi \alpha')^{13} \det(u_0)^{-1/2}.
\] (18)

Thus, in the \( u_0 \to 0 \) limit, the factor \( \det(u_0)^{-1/2} \) should be replaced by \((2\pi \alpha')^{-13}V_{26}\). This fact determines \( \mathcal{N} = (2\pi \alpha')^{13} \). It is interesting to see how (17) reduces to the partition function with the boundary interaction
\[
V_2 = \frac{a}{2\pi} + \frac{1}{4\pi \alpha'} \sum_{\mu=0}^{25} u^{\mu} \eta^{\mu\mu} : X^2_\mu(\theta) :,
\] (19)

which was given in [2]. The vertex operator \( V_2 \) is given by (9) by choosing \( u^{\mu\nu}(\theta) = u^{\mu} \eta^{\mu\nu} \delta(\theta) \) or choosing \( u_k^{\mu\nu} = u^{\mu} \eta^{\mu\nu} \) for all \( k \). The so-called Weierstrass' product formula,
\[
\Gamma(x) = \frac{1}{x} e^{-\gamma x} \prod_{k=1}^{\infty} \left( e^{k^{-1}(1 + k^{-1}x)^{-1}} \right),
\] (20)

where \( \gamma \) is Euler's constant, leads to
\[
Z|_{u^{\mu\nu}(\theta) = u^{\mu} \eta^{\mu\nu} \delta(\theta)} = \mathcal{N} e^{-a} \prod_{\mu=0}^{25} \left( \sqrt{\mu} e^{\gamma \mu} \Gamma(\mu) \right).
\] (21)

This is nothing but the partition function for (19) given in [2].

The remaining task in deriving the BSFT action is to find \( \{\alpha_i\} = \{\alpha_a, \alpha_k^{\mu\nu}\} \) and \( g \). By applying \( \{Q_B, X_\mu(\theta)\} = c \partial_{\theta} X_\mu(\theta) \), one gets
\[
\{Q_B, O(\theta)\} = c \partial_{\theta} c V(\theta) - \frac{1}{4\pi \alpha'} c X_\mu(\theta) \int_0^{2\pi} d\theta' u^{\mu\nu}(\theta - \theta') c \partial_{\theta'} X_\nu(\theta') :.
\] (22)

Substituting this into (1) and using the ghost correlation function (5), we obtain
\[
A(\theta) = -V(\theta) - \frac{1}{4\pi \alpha'} : X_\mu(\theta) \int_0^{2\pi} d\theta' \sin(\theta - \theta') u^{\mu\nu}(\theta - \theta') \partial_{\theta'} X_\nu(\theta') :.
\] (23)

This implies
\[
\alpha_a = -a, \quad \alpha_k^{\mu\nu} = \frac{1}{2} k \left( u_k^{\mu\nu} - u_{k-1}^{\mu\nu} \right) - u_k^{\mu\nu}.
\] (24)

g can be determined as follows. According to (6),
\[
\frac{\partial S}{\partial a} = -\frac{T_{25}}{2\pi} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \langle A(\theta') \rangle = T_{25} \sum_i \alpha_i \frac{\partial}{\partial \lambda_i} Z,
\] (25)

where we have used (7) and the fact that \( \theta \)-integrals of \( \cos(\theta - \theta') \) and \( \sin(\theta - \theta') \) vanish. On the other hand, using \( \partial_a Z = -Z \) and (24), the derivative of (8) with respect to \( a \) is
\[
\frac{\partial S}{\partial a} = T_{25} \left( \sum_i \alpha_i \frac{\partial}{\partial \lambda_i} + (g - 1) \right) Z.
\] (26)

Consequently, we obtain \( g = 1 \) and
\[
S = -T_{25} \left( a + 1 + \sum_{k=0}^{\infty} \left( \frac{1}{2} k \left( u_{k+1}^{\mu\nu} - u_{k-1}^{\mu\nu} \right) - u_k^{\mu\nu} \right) \partial u_k^{\mu\nu} \right) Z.
\] (27)
Since we are interested in homogeneous static solutions, we remove \(- (u_0^{\mu\nu}/2) \partial Z / \partial u_1^{\mu\nu}\), which leads to the kinetic term of the tachyon field \(T\), from (27). Hence the potential term is

\[
U = T_{25} \int d^{26}x \ e^{-T} \left( T + 1 + \sum_{k=1}^{\infty} \beta_k^{\mu\nu} \partial u_k^{\mu\nu} \right) \prod_{k=1}^{\infty} e^{k^{-1}u(\eta u_k)} \det(1 + k^{-1} \eta \cdot u_k)^{-1},
\]

where

\[
\beta_k^{\mu\nu} = \frac{1}{2} u_2^{\mu\nu} - u_1^{\mu\nu}, \quad \beta_{k \geq 2}^{\mu\nu} = u_1^{\mu\nu} = \frac{1}{2} k \left( u_{k+1}^{\mu\nu} - u_{k-1}^{\mu\nu} \right) - u_k^{\mu\nu}.
\]

Setting \(T = a + u_0^{\mu\nu} x_\mu x_\nu / 2 \alpha'\), one can reproduce (27) (apart from the kinetic term). In particular, \(- u_0^{\mu\nu} \partial Z / \partial u_0^{\mu\nu}\) in (27) is obtained from \(T\) in the parenthesis in the potential (28).

We further restrict our attention to the the case where \(u_k^{\mu\nu} = u_k^{\mu} \eta^{\mu\nu}\). Since the non-diagonal parts of \(u_k^{\mu\nu}\) always accompany the other non-diagonal elements of \(u_k^{\mu\nu}\), this restriction is consistent in the sense that \(\partial V / \partial u_k^{\mu\nu} = 0\) for \(\mu \neq \nu\). The potential for the tachyon field \(T\) and the diagonal elements of \(u_k\) is

\[
U = T_{25} V_{26} e^{-T} \left( T + 1 - \sum_{\mu=0}^{25} \sum_{k=1}^{\infty} \beta_k^{\mu} \left( \frac{1}{k + u_k^{\mu}} - \frac{1}{k} \right) \right) \prod_{k=1}^{\infty} \prod_{\mu=0}^{25} e^{k^{-1}u_k^{\mu}} (1 + k^{-1} u_k^{\mu})^{-1},
\]

where

\[
\beta_1^{\mu} = \frac{1}{2} u_2^{\mu} - u_1^{\mu}, \quad \beta_{k \geq 2}^{\mu} = \frac{1}{2} k \left( u_{k+1}^{\mu} - u_{k-1}^{\mu} \right) - u_k^{\mu}.
\]

Here we focused on the homogeneous static fields and performed the integration \(\int d^{26}x\).

### 3. The solutions of the equations of motion

In this section, we solve the equations of motion derived from the non-perturbative potential (30) of the BSFT. Since we have infinitely many degrees of freedom, we adopt an approximation and solve them numerically. We find a large number of solutions, and those solutions have peculiar properties. First, we present the equations of motion, and then show how to solve them numerically with an estimate of the validity of the approximation. Finally, we study the peculiar properties of the energy distribution of the solutions.

#### 3.1. The equation of motion

To find solutions of the equations of motion with non-vanishing \(u_k\), we first solve \(\partial U / \partial T = 0\). This gives two solutions, the first one is

\[
T = \sum_{\mu=0}^{25} \sum_{k=1}^{\infty} \beta_k^{\mu} \left( \frac{1}{k + u_k^{\mu}} - \frac{1}{k} \right),
\]

and the second one is \(T = \infty\) with \(u_k^{\mu}\) arbitrary.\(^5\) Obviously, the second one is the tachyon vacuum solution \([18]\) since the potential energy (30) vanishes, and here we see explicitly the consistency with the truncated solution of \([18]\).\(^6\)

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\(^{\text{5}}\) It is worth noting that both solutions lead to \(S = T_{25} Z\). This equality was also found in BSFT with the boundary interaction \([19]\)\(^2\). Since the difference between \(S\) and \(T_{25} Z\) was given by \(dZ / d \ln \mu\), this equality implies that the solutions correspond to the conformal fixed points of worldsheet theory.

\(^{\text{6}}\) It is interesting that the second solution \(T = \infty\) lets all the other equations of motion for \(u_k^{\mu}\) be trivially satisfied, for any value of \(\{u_k^{\mu}\}\). This is a generalization of the fact that at the tachyon vacuum any constant field strength of the massless gauge field is a degenerate solution. This is probably related to the fact that there is no open string excitation at the tachyon vacuum.
In the following, we consider only the first solution, (32). Substituting this solution back into (30) gives

\[ U = T_{25} V_{26} e^{-\sum_{\mu=0}^{25} f(u_\mu)}, \]  

where

\[ f(u) = \sum_{k=1}^{\infty} \left( \beta_k \left( \frac{1}{k+u_k} - \frac{1}{k} \right) - k^{-1}u_k + \log(1 + k^{-1}u_k) \right) \]

\[ = \frac{1}{2}u_1 + \sum_{k=1}^{\infty} \left( \beta_k \left( \frac{1}{k+u_k} \right) + \log(1 + k^{-1}u_k) \right), \]  

(34)

and

\[ \beta_1 = \frac{1}{2}u_2 - u_1, \quad \beta_{k \geq 2} = \frac{1}{2}k(u_{k+1} - u_{k-1}) - u_k. \]  

(35)

Here, \( u \) in \( f(u) \) is the shorthand notation of \( \{u_k\} \). In the second line of (34), we have used \( \sum_{k=1}^{\infty} (u_{k+1} - u_{k-1}) = -u_1 \). Clearly, solutions of the equations of motion derived from the potential \( U \) are the stationary points of \( \sum_{\mu=0}^{25} f(u_\mu) \). Now the potential energy at a stationary point is in the form of

\[ U_u = T_{25} V_{26} e^{-\sum_{\mu=0}^{25} f(u^\mu_u)}, \]  

(36)

where \( \{u^\mu_u\} \) is a complete set of solutions of \( \partial f/\partial u = 0 \) labeled by \( n \).

Our next task is to solve \( \partial f/\partial u = 0 \). Since this is an infinite-dimensional equation, it is difficult to solve it analytically. However, we first note that we can find a solution

\[ u^\mu_k = 0 \]  

for all \( k \geq 1 \) and \( \mu \). This implies \( T = 0 \) by using (32), so, the solution is nothing but the trivial vacuum of the original D25-brane. It is important that the trivial D25-brane solution and the tachyon vacuum solution are allowed in our generalized scheme, as a check of the consistent truncation of the BSFT.

To find nontrivial solutions with massive state condensation, we solve this \( \partial f/\partial u = 0 \) numerically by truncating the fields as

\[ u_k = 0 \quad \text{for} \ k > k_c. \]  

(38)

The fact that the variation of \( f(u) \) with respect to \( u_k \) consists of only \( u_k \) and \( u_{k \pm 1} \) implies \( \partial f/\partial u_{k > k_c + 1} = 0 \). Hence, the nontrivial equations we have to solve are

\[ \frac{\partial f}{\partial u_k} = 0, \quad \text{for} \ k \leq k_c + 1. \]  

(39)

In general, there is no solution since the number of equations is \( k_c + 1 \) and is bigger than the number of degrees of freedom \( k_c \). We first neglect the stationary condition with respect to \( u_{k_c+1} \) and find the numerical solution of

\[ \text{EOM}(k_c) : \frac{\partial f}{\partial u_k} = 0, \quad \text{for} \ k \leq k_c. \]  

(40)

Let \( v^{k_c,s} \) be the solution of EOM\((k_c)\), where \( s \) is the natural number labeling the solutions. We sort solutions in ascending order in their values of \( f \), i.e. \( f(v^{k_c,s_1}) < f(v^{k_c,s_2}) \) for any choice of \( s_1 < s_2 \) in the set \( \{s\} \). If \( \partial f/\partial u_{k_c+1} \) is sufficiently small at \( u = v^{k_c,s} \), we can regard \( v^{k_c,s} \) as the approximate solution of the whole system of equations (39).
3.2. Numerical solutions

We now present numerical solutions. We shall also explain how we can make the solution accurate, in spite of the introduction of the effective cut-off $k_c$. We find a large number of solutions having different energies.

Since the energy is written in terms of the function $f$ as in (36), first we study the numerical solutions in terms of the values of $f$, which can help to make the analysis easier.

We solve EOM($k_c$) numerically for $k_c = 2, 3, \ldots, 6$. The values of $f(u)$ and $\left| \partial f/\partial u_{k_c+1} \right|$ for the real solutions are plotted in Fig. 1. The number of real solutions are $3, 5, 11, 21, 44$ for $k_c = 2, 3, 4, 5, 6$ respectively. So the number of solutions diverges rapidly as we increase $k_c$.

At this stage, $\partial f/\partial u_{k_c+1}$ is not so small and is about the same magnitude as $f(u)$ itself. So, we cannot tell that these numerical solutions solve the full equation of motion. We shall improve the situation below.

For $k_c \geq 7$, the computations become much more complicated. Hence we take an alternative approach. We begin with the $s$th solution of EOM($k_c$), $v^{k_c,s}$. Using Newton’s method with the initial value

$$ u_k = v^{k_c,s}_k \quad \text{for } k \leq k_c, u_{k_c+1} = 0, \quad (41) $$

we obtain the solution of EOM($k_c+1$) which we call $w^{k_c,s,k_c+1}$. In the same way, we can find the solution of EOM($k_c+2$) by applying Newton’s method with the initial value

$$ u_k = w^{k_c,s,k_c+1}_k \quad \text{for } k \leq k_c+1, u_{k_c+2} = 0. \quad (42) $$

By iterating this procedure, we can obtain the solution of EOM($L$) for any $L \geq k_c$.

The above procedure is shown to work well as an approximation, and provides a convergent solution solving the full equation of motion. Let $w^{k_c,s,L}$ be the solution of EOM($L$) found by the iteration procedure which begins with $v^{k_c,s}$, the solution of EOM($k_c$). We found that as a function of $L$, $f(w^{k_c,s,L})$ converges to a nonzero finite value as $L \to \infty$, while $\partial f/\partial u_{L+1}$ tends to vanish as $O(1/L^2)$. In Fig. 2, we show an example (among many solutions) of the plots of $f$ and $|\partial f/\partial u_{L+1}|$

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7 From this data, approximately, the number of the solutions $N(k_c)$ increases by a factor of 2 when we increase $k_c$ by 1, so one can approximate $N(k_c) \propto 2^{k_c}$. However, the data can be fit well also with $N(k_c) \propto k_c^2$. So we cannot conclude whether the growth of the number of solutions is exponential or power-law-like. See the discussions at Fig. 7.

8 It is, however, important to note that we get only a subset of all solutions of EOM($L$), since EOM($L$) is expected to have many more solutions than EOM($k_c$).

9 It is not clear from first principles that vanishing of $\partial f/\partial u_{L+1}$ as $O(1/L^2)$ is enough, since the number of fields diverges as $O(L)$. The nontriviality of the equation of motion due to an infinite number of fields also appears in cubic string field theory. We leave further study of this problem to future work.
Fig. 2. The plots of $f(u^{6.5,L})$ (the left graph) and $|\frac{\partial f}{\partial u_{L+1}}(u^{6.5,L})|^{-1}$ (the right graph) as functions of $1/L$ and $1/L^2$ respectively.

Fig. 3. $f_\infty(k_c, s)$ extrapolated from the data for $30 \leq L \leq 100$. The horizontal lines are drawn in accordance with the values of $f_\infty(4, s)$.

for $k_c = 6, s = 5$ as a function of $1/L$ and $1/L^2$ respectively. In this example, $f \sim 0.0071$ and $|\frac{\partial f}{\partial u_{L+1}}| \sim 0.000017$ at $L = 100$. Therefore we claim that the equation of motion is approximately satisfied. By fitting $f(u^{6.5,L})$ for large $L$, say $30 \leq L \leq 100$, to a quadratic function of $1/L$, it turns out to approach to a non-trivial value $0.006887$.

For any $k_c$ and $s$ within $2 \leq k_c \leq 6$, we extrapolate the values of $f$ at $L = \infty$, named $f_\infty(k_c, s)$, from the data for $30 \leq L \leq 100$. Here, the label $s$ of the solution is chosen in the same manner as before: $f_\infty(k_c, s_1) < f_\infty(k_c, s_2)$ for any $s_1 < s_2$. The result is shown in Fig. 3.

The horizontal lines reveal that solutions of EOM(4) include those of EOM(2) and EOM(3) and are included in those of EOM(5) and EOM(6). For this reason, we expect that $\{w^{k_c,s,\infty}\}$ with fixed $k_c$ forms a subset of solutions of EOM($k_c = \infty$).

3.3. Distribution of the energies of the solutions
Based on the above results, we study the energies of the solutions. We shall concentrate on Lorentz-invariant solutions for simplicity in this subsection. Recall that the potential energy is written in terms of $f(u)$ as (33). The Lorentz-invariant solutions are given by choosing $u^\mu = u$ for all $\mu$. Now the energy in units of $T_{25}V_{26}$ is given by $e^{26f(u)}$ and so the extrapolated value at $L = \infty$ is $e^{26f_\infty(k_c,s)}$. The plots of the energies are given in Fig. 4.

We notice the following interesting features of the distribution of the energies of the solutions:

(1) Almost uniform distribution of the energy spectrum
Although we have found a lot of solutions, the energy of those solutions do not overlap with each other, while tending to be uniformly distributed. This can be seen in Fig. 4 as linear profiles of each dotted sector.
(2) Energies below the D-brane tension
All the solutions we found have energies which take the values between the D-brane tension (normalized as 1.0 in Fig. 4) and the tachyon vacuum (no D-brane).

(3) Solution with a very small energy density
The lowest value of the energies among a fixed set decreases as $k_c$ increases. It suggests that bringing up $k_c$ further would lower the lowest value of the possible range of the energy distribution.

(4) The presence of the “desert”
As seen from the plots in Fig. 3, there exists a “desert” in $0.25 < f_{\infty} < 0.4$ where there appears no solution up to $k_c = 6$. The desert, however, gets narrow as $k_c$ increases. For this reason, we expect that this desert is an artifact of finite $k_c$.

These features of the energies are interesting, besides the surprising fact that we have obtained a large number of solutions in string field theory. The large set of the solutions would serve as a “string landscape”.

This requires a possible interpretation in terms of string theory. In the next section, we provide a possible interpretation of the solutions, with a detailed analysis of the intervals of the energies and the degeneracies of the solutions.

4. Properties of the solutions

4.1. Uniformity of the energy intervals
We first analyze the distribution of the function $f$ for the solutions. To investigate the distribution of $f_{\infty}(k_c, s)$, we evaluate the intervals of pairs of $f_{\infty}(k_c, s)$:

$$g_{\infty}(k_c, s_1, s_2) = \frac{1}{s_2 - s_1} \left( f_{\infty}(k_c, s_2) - f_{\infty}(k_c, s_1) \right),$$

for any pair $s_1 < s_2$. Henceforth, we don’t take into account the pairs separated by the desert. The result is shown in Fig. 5. It turns out that the intervals get smaller as $k_c$ increases. This is the outcome of the increase of the number of solutions. It is important to note that the intervals become almost identical to each other. This implies that the distribution of $f_{\infty}$ becomes uniform.

To see the intervals of the energies we define, in the same manner,

$$E(k_c, s_1, s_2) = \frac{1}{s_2 - s_1} \left( e^{-26f_{\infty}(k_c, s_1)} - e^{-26f_{\infty}(k_c, s_2)} \right),$$

where we don’t take into account the pairs separated by the desert. Again, the distribution of the energies is seemingly uniform, except for the desert; see Fig. 6.
Fig. 5. The left graph shows the intervals of any pairs of $f_{\infty}(k_c, s)$ next to each other, namely, $g_{\infty}(k_c, s, s + 1)$, while $g_{\infty}(k_c, s_1, s_2)$ for general $s_1 < s_2$ are shown in the right graph.

Fig. 6. The energy intervals in units of $T_{25}V_{26}$. The left graph shows $E(k_c, s, s + 1)$, the energy intervals of any pairs next to each other, while the intervals of any pairs are shown in the right graph.

So, as expected from a brief look at the energy distribution of Fig. 4, it is indeed the case that the intervals between energy levels for different solutions are approximately identical to each other, and the quantized distribution of the energy levels are uniform.

The fact that the distribution of the energies is uniform appears to be inconsistent with the uniformness of the distribution of $f_{\infty}$. For the solutions we found, however, the values of $f_{\infty}$ are small, so that $e^{-26f_{\infty}} \sim 1 - 26f_{\infty}$, and so there is not much numerical difference between the intervals of $f_{\infty}$ and those of the energies. At this stage, we can’t precisely conclude whether $f$ or some function of $f$ like $e^{-26f}$ has a uniform distribution. To obtain a more precise prediction, we need to extend our computations to higher $k_c$ where we might get solutions with a sufficiently large value of $f$ to investigate which intervals are uniform.

The averaged interval is smaller for larger values of $k_c$. Figure 7 shows that the averaged interval can be fit well as a function $2^{-k_c}$. This is consistent with the fact that the number of solutions grows as $2^{k_c}$ as stated earlier, assuming that finally in the $k_c \to \infty$ limit the energy levels are uniformly distributed between the D-brane tension and the tachyon vacuum.

4.2. Degeneracy among the solutions

Previously, we have considered only a solution with Lorentz invariance; we took all $u^\mu$ to be equal to each other. In general, however, by choosing different solutions of $f(u)$ for different components of $u^\mu$ we obtain solutions breaking Lorentz invariance. Apparently, the energy spectrum degenerates since the equation of motion is still satisfied when we permute the components of $u^\mu$. This is an exact symmetry among solutions.

Suppose we decompose 26 $u^\mu$ into two subsets. The $l$ components in the first subset take $u^\mu = 0$, while the other $26 - l$ components take a single solution $u^\mu = u_s^\mu$. Obviously, we can find $26!/(26 - l)!l!$ solutions with degenerate energy from the way we arrange the subsets. So, allowing a breaking
of the Lorentz symmetry provides a huge degeneracy in the energies of the solutions. Note that we can decompose 26 components further into a large number of subsets.

There is an additional form of an approximate degeneracy which comes with our numerical finding mentioned earlier, the approximate uniformity of the distribution of the solutions in the $f_\infty$ space. Let us formally express solutions of EOM($k_c = \infty$) as $\{f_\infty(\infty, s)\}$. The uniformity of the distribution implies

$$f_\infty(\infty, s) = sf_1, \quad \text{for } s = 0, 1, \ldots ,$$

where $f_1 = f_\infty(\infty, 1)$. It turns out that there are two types of solution which have the same energy $e^{-2f_1}$:

1: $f(u^\mu) = f(u^\nu) = f_1$, \quad the other components = 0,

2: $f(u^\mu) = 2f_1$, \quad the other components = 0. \hfill (46)

Therefore, the energy spectrum degenerates. This degeneracy adds up to the exact degeneracy explained above. So, in total, we would have a huge degeneracy in energy among our solutions.

### 4.3. An interpretation: Closed string states?

In this subsection, we discuss that the properties studied in the previous subsections may serve as indirect evidence for our interpretation of our solutions as closed string states, but the true connection to the closed string states in the tachyon vacuum is yet to be unraveled.

Closed strings, which should live in the tachyon vacuum as physical excitations, remain a mystery in string field theory. There should exist closed string excitations, in particular at the tachyon vacuum where the open string degrees of freedom should go away, along with the disappearance of the original D-branes à la Sen’s conjecture. Now, a first look at the energy plot of Fig. 4 would suggest a set of closed string states.

First, let us see the uniform distribution of the energy of the solutions. One should note that as the D-brane tension is inversely proportional to the string coupling constant $g_s$, the perturbative spectrum should appear infinitely dense if we normalize the D-brane tension to be unity and take the perturbative string limit $g_s \to 0$. So, the increase of the number of energy levels for larger $k_c$ is consistent with the interpretation that these are some perturbative excitations of string theory.

The uniformity first reminds us of energy levels of harmonic oscillators. Harmonic oscillators appear in string theory as excitations of closed strings. This fact also suggests that our solutions are closed string excitations. However, our energy is the bulk energy of the BSFT solutions, while in closed string excited states what is uniform is the closed string hamiltonian as a single-body problem.
Since we do not know why closed strings should be homogeneously distributed in space, we lack a direct connection between closed strings and our BSFT solutions.

The standard closed string states have a huge degeneracy among states sharing the same energy. As we saw above, we have found a similar degeneracy in our spectrum of the energies of the solutions. The properties in (46) are reminiscent of the closed string spectrum. In fact, we can reproduce a part of the closed string degeneracies by our solutions, and the representations under Lorentz transformation of a part of the closed string spectrum by the following simple rule of replacement: \[ f(u^\mu) = n f_1 \rightarrow A^\mu_{-n} = \alpha^-_{-n} \alpha^\mu_{-n}. \]

For example, two solutions given in (46) correspond to \( A^\mu_{-1} A^\nu_{-1} |0\rangle \) and \( A^\mu_{-2} |0\rangle \), respectively.

There are, however, two important discrepancies: First, among our solutions there seems to be no solution which amounts to the closed string state \( A^\mu_{-1} A^\nu_{-1} |0\rangle \). Second, more notably, the degeneracy increases as the energy decreases, in contrast to the closed string spectrum, since \( f(u) \) contributes to the energy in the form of \( e^{-f(u)} \). The latter problem would be complicatedly related to the infinitely large density of energy levels in the \( k_c \rightarrow \infty \) limit, which may require more exploration of the solution space at larger \( k_c \).

In sum, although the uniformity and the degeneracy are quite suggestive, they are not sufficient to claim that our solutions are closed string states. Further study is required for a conclusive interpretation.

5. Discussion

In this paper we have solved the equations of motion derived from the BSFT action associated with general quadratic boundary operators. By means of numerical analysis, we have found a large number of solutions whose energies are uniformly distributed between the energy of the tachyon vacuum and the D-brane tension. As the quadratic boundary interactions give a free worldsheet theory, the solutions we obtained are solutions of the full string theory.

As discussed in Sect. 4, our solutions are possibly related to non-trivial closed string excitations at the tachyon vacuum (alternatively called the closed string vacuum). In fact, it was discussed in [35] that the non-local open string background implements shifts in the closed string background. By extending our analysis to larger \( k_c \), we may make progress on the interpretation of the solutions.

Our solutions in BSFT do not have any counterparts among known solutions in CSFT. Although there are some suggestions on possible relations between the two SFTs [36], it is difficult to see how our solutions may be mapped to CSFT. It would be interesting if one could construct CSFT solutions sharing properties with our solutions. In addition, to gain insight on what our solutions mean, it is important to determine physical excitations around the found solutions. Whether the closed string excitations actually can be identified would be the key point. Furthermore, an analogue of rolling tachyon solutions in BSFT [37–39], and possible deformations of solutions in BSFT [40], would provide more insight. One of our Lorentz-violating solutions resembles tachyon matter solutions [41].

While we worked with non-local boundary operators in this paper, the BSFT action associated with the “local” boundary operators can be obtained by setting \( u^{\mu \nu}(\theta) = \sum_{r=0}^{r_s} t_r \delta^{(r)}(\theta) \). Due to the contact divergence we need to introduce the short-distance cut-off \( \epsilon \) and renormalize the coupling \( a \) as \( a' = a + \Delta a(t, \epsilon) \).\(^{10}\) With a proposed form of the counter term \( \Delta a \) in [20], the BSFT action

\(^{10}\) A different form of the renormalization using so-called \( \zeta \)-function renormalization is discussed in [42].
$S_s(t_0, \ldots, t_s)$ turns out to manifestly depend on $s$: $S_s(t_0, \ldots, t_s \to 0) \neq S_{s-1}(t_0, \ldots, t_{s-1})$. Since the counter term $\Delta a$ just shifts $a$, after we integrate out $a$, the action is independent of the choice of $\Delta a$. We found that this action again depends on $s$ in the above sense. The physical interpretation of this fact is not clear and this is why we didn’t work with the local boundary interactions.

In the introduction, we mentioned that massive mode condensation would be related to multiple-D-brane solutions in SFT. Although in Sect. 4 we described how our solutions may be interpreted as closed string states, they may still allow another interpretation as multiple D-branes. Although we have not obtained a solution with energy larger than the original D25-brane, this does not mean that our solutions are not multiple-D-brane solutions. The reason is that since we are working in bosonic string theory, multiple-D-branes are unstable and would form a bound state whose energy may be much smaller than just the multiple of the D-brane tension. This kind of question can be answered only in superstring field theory, as superstrings should have stable multiple-D-branes with energy protected by the BPS property, and further study is necessary. It is, however, nontrivial to extend our analysis to superstrings. In fact, finding the action is even more involved since the associated boundary operators are no longer quadratic in the matter operator $X^\mu$, contrary to the bosonic string (see [43] for an explicit treatment of massive states in super BSFT).

At a glance, our main result of Fig. 4 resembles a band structure of electrons in materials. The fermion band structure is related to matrix models, and indeed some matrix models represent tachyons and unstable D-branes (see for example [44,45]). Suppose our solutions with the massive field condensation are bound states of multiple unstable D-branes discussed above; then it is natural that matrix models appear as a low energy description of the multiple D-branes. The excitations of the matrix models may look like a band structure. In general, the number of excitations of the matrix model increases as the rank of the matrix $N$ increases. In this sense, $k_c$ plays a similar role to $N$. It is interesting to seek the counterpart of the “desert” with given $k_c$ in the matrix models with finite $N$.

From the cosmological point of view, the distribution of infinitely many solutions is suggestive of the so-called landscape, which was found in superstring theory. Our solutions could be called a landscape of BSFT.\footnote{It is important to note again that to find the excitations around our solutions is important. There will be tachyonic excitations.} Among its peculiar properties, it is intriguing that the lowest value of the energy decreases when we increase $k_c$. If we take a large enough $k_c$, one may have a BSFT solution with a very small cosmological constant. In Sect. 4, we also commented that we have solutions breaking Lorentz invariance. This fact suggests that generic non-perturbative vacua may spontaneously break Lorentz symmetry. It reminds us of the original motivation for exploring non-perturbative vacua in string field theories in late 1980s: spontaneous Lorentz and CPT violation [21–23]. It would be interesting to further explore our Lorentz-symmetry-violating (metastable) vacua and their relevance to the baryon asymmetry of the universe.

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References

[1] E. Witten, Phys. Rev. D 46, 5467 (1992).
[2] E. Witten, Phys. Rev. D 47, 3405 (1993).
[3] E. Witten, Nucl. Phys. B 268, 253 (1986).
[4] M. Murata and M. Schnabl, Prog. Theor. Phys. Suppl. 188, 50 (2011).
[5] M. Murata and M. Schnabl, J. High Energy Phys. 1207, 063 (2012).
[6] A. Sen, Int. J. Mod. Phys. A 20, 5513 (2005).
[7] A. Sen, J. High Energy Phys. 9808, 012 (1998).
[8] A. Sen, Int. J. Mod. Phys. A 14, 4061 (1999).
[9] A. Sen, hep-th/9904207.
[10] A. Sen, J. High Energy Phys. 9912, 027 (1999).
[11] M. Schnabl, Adv. Theor. Math. Phys. 10, 433 (2006).
[12] Y. Okawa, J. High Energy Phys. 0604, 055 (2006).
[13] M. Kiermaier, Y. Okawa, L. Rastelli, and B. Zwiebach, J. High Energy Phys. 0801, 028 (2008).
[14] M. Schnabl, Phys. Lett. B 654, 194 (2007).
[15] I. Ellwood, J. High Energy Phys. 0905, 037 (2009).
[16] L. Bonora, C. Maccaferri, and D.D. Tolla, J. High Energy Phys. 1111, 107 (2011).
[17] A. A. Gerasimov and S.L. Shatashvili, J. High Energy Phys. 0010, 034 (2000).
[18] D. Kutasov, M. Marino, and G. W. Moore, J. High Energy Phys. 0010, 045 (2000).
[19] D. Kutasov, M. Marino, and G.W. Moore, hep-th/0010108.
[20] K. Li and E. Witten, Phys. Rev. D 48, 853 (1993).
[21] V. A. Kostelecky and S. Samuel, Phys. Rev. D 39, 683 (1989).
[22] V. A. Kostelecky and R. Potting, Nucl. Phys. B 359, 545 (1991).
[23] V. A. Kostelecky and R. Potting, Phys. Lett. B 381, 89 (1996).
[24] T. Erler and C. Maccaferri, J. High Energy Phys. 1111, 092 (2011).
[25] D. Takahashi, J. High Energy Phys. 1111, 054 (2011).
[26] T. Masuda, T. Noumi, and D. Takahashi, J. High Energy Phys. 1210, 113 (2012).
[27] H. Hata and T. Kojita, J. High Energy Phys. 1201, 088 (2012).
[28] H. Hata and T. Kojita, arXiv:1209.4406 [hep-th].
[29] S. L. Shatashvili, Alg. Anal. 6, 215 (1994).
[30] S. L. Shatashvili, Phys. Lett. B 311, 83 (1993).
[31] M. Marino, J. High Energy Phys. 0106, 059 (2001).
[32] V. Niarchos and N. Prezas, Nucl. Phys. B 619, 51 (2001).
[33] D. Ghoshal, hep-th/0106231.
[34] S. Teraguchi, J. High Energy Phys. 0702, 017 (2007).
[35] M. Baumgartl, I. Sachs, and S.L. Shatashvili, J. High Energy Phys. 0505, 040 (2005).
[36] E. Coletti, I. Sigalov, and W. Taylor, J. High Energy Phys. 0508, 104 (2005).
[37] S. Sugimoto and S. Terashima, J. High Energy Phys. 0207, 025 (2002).
[38] J. A. Minahan, J. High Energy Phys. 0207, 030 (2002).
[39] G. Gibbons, K. Hashimoto, and P. Yi, J. High Energy Phys. 0209, 061 (2002).
[40] K. Hashimoto and S. Hirano, Phys. Rev. D 65, 026006 (2002).
[41] A. Sen, J. High Energy Phys. 0207, 065 (2002).
[42] O. Andreev, Nucl. Phys. B 598, 151 (2001).
[43] K. Hashimoto and S. Terashima, J. High Energy Phys. 0410, 040 (2004).
[44] J. McGreevy and H.L. Verlinde, J. High Energy Phys. 0312, 054 (2003).
[45] T. Takayanagi and N. Toumbas, J. High Energy Phys. 0307, 064 (2003).