EXISTENCE, REGULARITY AND APPROXIMATION OF GLOBAL ATTRACTORS FOR WEAKLY DISSIPATIVE P-LAPLACE EQUATIONS

YANGRONG LI* AND JINYAN YIN

School of Mathematics and Statistics
Southwest University
Chongqing 400715, China

Abstract. A global attractor in $L^2$ is shown for weakly dissipative $p$-Laplace equations on the entire Euclid space, where the weak dissipativeness means that the order of the source is lesser than $p - 1$. Half-time decomposition and induction techniques are utilized to present the tail estimate outside a ball. It is also proved that the equations in both strongly and weakly dissipative cases possess an $(L^2, L^r)$-attractor for $r$ belonging to a special interval, which contains the critical exponent $p$. The obtained attractor is proved to be approximated by the corresponding attractor inside a ball in the sense of upper strictly and lower semicontinuity.

1. Introduction. This paper is concerned with the dynamical behavior of the following $p$-Laplace equation

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|u_{x_i}|^{p-2}u_{x_i}) + \lambda u + f(x, u) = g(x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n,$$

where $p > 2$, $\lambda > 0$. The source $f$ is a continuously differential function satisfying:

$$f(x, s)s \geq \alpha_1|s|^q + \psi_1(x), \quad |f(x, s)| \leq \alpha_2|s|^{q-1} + \psi_2(x), \quad \frac{\partial f}{\partial s}(x, s) \geq -\alpha_3,$$

where $q > 2$, $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\psi_2 \in L^2(\mathbb{R}^n)$. The force $g$ and $\psi_1$ satisfy

$$\psi_1 \in L^1 \cap L^{N+1}, \quad g \in L^2 \cap L^{N+1} \quad \text{with} \quad N = N(p, q) = \left\{ \frac{p-2}{q-2} \right\},$$

where $\{r\}$ denotes the minimal integer no lesser than $r$. We say the equation is strongly dissipative if $p \leq q$ (equivalently $N = 1$) and weakly dissipative if $p > q$ (equivalently $N \geq 2$).

Recently, Krause et al [11, 12] and Li et al [14, 15] have investigated the long-time behavior for the stochastic $p$-Laplace equation on the whole Euclid space. Khanmamedov et al [8, 9, 10] had earlier obtained a global attractor for the deterministic equation (i.e. (1)) on the whole space still. However, all these papers have assumed the equation was strongly dissipative. In this case, the tail estimate

2010 Mathematics Subject Classification. Primary: 37L30; Secondary: 35B40, 35B41.

Key words and phrases. Global attractors, p-Laplace equations, weak dissipativeness, lower semicontinuity of attractors, regularity of attractors, unbounded domains.

This work is supported by National Natural Science Foundation of China grant 11571283.

* Corresponding author: Yangrong Li.
outside a ball can be achieved directly from the absorption of $p$-norm by using an interpolation. By the way, the reaction-diffusion equation (i.e. Eq. (1) with $p = 2$) is always strongly dissipative, its dynamical behavior in stochastic or deterministic case has been investigated by many papers, see e.g. [1, 13, 16, 22, 25, 26, 29, 34]. The strong dissipativeness was also assumed in [19, 20, 21, 27, 32] for $p$-Laplace equations defined on a bounded domain with a possibly different principal part.

The weakly dissipative case was considered by Gess et al [4, 5, 6] and Wang [23] for stochastic $p$-Laplace equations but restricted on a bounded domain. In this case, it does not require the tail estimate. So the techniques are similar in both strongly and weakly dissipative cases.

In a word, it remains open whether there is an attractor for the weakly dissipative $p$-Laplace equation on the entire Euclid space even though the equation is non-random.

In this paper, we first give an affirmative answer to the above question. The main difficulty arises from estimating the tail of a solution outside a ball. This estimate requires an auxiliary absorption under the $p$-norm. In general, the absorption may be achieved easily under the $q$-norm. If the equation is strongly dissipative ($q \geq p$), this absorption reaches automatically the level of $p$-norm by an interpolation. This is the reason why the literatures mentioned above assumed the strong dissipativeness.

If the equation is weakly dissipative ($q < p$), the interpolation lose its effectiveness. To overcome this difficulty, we define an increasing sequence $q_m = m(q - 2) + 2$ and prove the absorption under $q_N$-norm by an induction argument, where $N$ is given in (3) and it is just the minimal integer such that $q_N \geq p$. So the absorption can reach the level of $p$-norm. We then use this absorption together with an half-time decomposition to derive that the $p$-Laplace system is asymptotically small outside a large ball and thus obtain a global attractor in the weakly dissipative case.

Our second purpose is to consider the regularity problem after proving the existence of a global attractor in $L^2$. The main difficulty to consider regularity of attractors is the continuity of systems in the terminate space. Several continuity concepts were introduced in the literatures to overcome the difficulty. For instance, norm-to-weak continuity and quasi-continuity were introduced in [34] and in [17] respectively to deal with the reflexive phase space (such as $L^r$ for $1 < r < +\infty$ or Sobolev spaces), while quasi-weak-star-continuity (i.e. a generalization of quasi-continuity) was introduced recently in [7] to deal with non-reflexive phase space (such as $L^1$ or $L^\infty$). The authors in a recent paper [15] gave an abstract existence result of random bi-spatial attractors without any continuity assumption in the terminate space (it depends on continuity of systems in the initial space only). The present paper will discuss the regularity of global attractors for Eq.(1) by using the abstract result given in [15] and developed by [18]. We will show by an induction method that the obtained attractor is in fact an $(L^2, L^r)$-attractor for any $r \in [2, q_N]$ with $q_N = N(q - 2) + 2$ in both strongly and weakly dissipative cases. Since $q_N$ is no lesser than both $p$ and $q$, our result contains two critical exponents ($r = p$ and $r = q$), which generalizes the main results in [8, 9, 28] even in the strongly dissipative case.

Our third purpose is to consider whether the obtained attractor (denoted by $A_\infty$) can be approximated by the corresponding attractor (denoted by $A_R$) for the equation defined on a ball with the radius $R$. We will use the method given in [15] to prove that the attractor $A_\infty$ is constructed as the closure of union of attractors $A_R$ over all $R \in [R_{f,g}, +\infty)$, where the constant $R_{f,g}$ is related to both functions...
Let \(\mathcal{A}_R\) be the family of attractors. This means that the family \(\mathcal{A}_R\) of attractors is strictly upper semi-continuous and lower semi-continuous as the radius \(R\) tends to infinity. This continuity of attractors holds true under the \(r\)-norm with any \(r \in [2, q_N]\). The result generalizes an earlier result in [2] for the lattice dynamical system with \(q \geq p = 2\) and generalizes the corresponding result in [15] to the weakly dissipative case. More importantly, the result presents an actual example of lower semi-continuity, even though it is extremely hard in practical examples to prove lower semi-continuity or equivalently equi-attraction of attractors as pointed out in [3, P.65].

2. Preliminaries and semigroups of solutions. We denote the norms in \(L^2\) and \(L^r\) by \(\|\cdot\|\) and \(\|\cdot\|_r\), respectively, where \(1 \leq r \neq 2\). We also set \(\|u\|_W^p := \sum_{i=1}^n \|u_{x_i}\|_{p_i}^p\) and thus \(\|\cdot\|_p + \|\cdot\|_W\) defines a norm in the Sobolev space \(W^{1,p}([R^n])\). A semilinear \(p\)-Laplace operator is defined by \(A : W^{1,p} \to W^{-1,p'}\),

\[
Au = -\sum_{i=1}^n \frac{\partial}{\partial x_i} (|u_{x_i}|^{p-2} u_{x_i}), \quad (Au, v) = \sum_{i=1}^n \int_{R^n} |u_{x_i}|^{p-2} u_{x_i} v_{x_i} \text{ for } u, v \in W^{1,p}.
\]

In particular, \((Au, u) = \|u\|_W^p\). We know that \(A\) is semi-continuous and monotone (see [24]).

By using the subsequent estimates in Lemma 2.1 and Lemma 3.1, one can easily establish the existence result of solutions for problem (1), precisely, for every \(u_0 \in L^2(R^n)\) and \(T > 0\), the problem (1) has a unique solution

\[
u := u(t, u_0) \in L^\infty(0, T; L^2) \cap L^p(0, T; W^{1,p}) \cap L^{q_N}(0, T; L^{q_N})\]

such that \(u(0, u_0) = u_0\). Continuity of solutions in \(L^2\) is also easily established from the third hypothesis in (2). In fact, the well-posed property of (1) in the weakly dissipative case is easily obtained by a similar argument as given in [11, 12] for the strongly dissipative case. The unique solution defines a continuous semigroup \(S(\cdot)\) on \(L^2\) by \(S(t)u_0 = u(t, u_0)\) for each \(t \geq 0\) and \(u_0 \in L^2\).

The following lemma shows that the semigroup has absorption on any sufficiently large interval.

**Lemma 2.1.** Let (2) hold true and \(\psi_1 \in L^1, g \in L^2\). Then there is a positive constant \(C_0\) such that, for every bounded set \(B\) in \(L^2\), there is a large time \(T_0 := T_0(B) \geq 2\) such that

\[
\sup_{s \in [t/2, t]} \|u(s, u_0)\| + \sup_{u_0 \in B} \int_{t-1}^t \|u(r, u_0)\|_W^q dr \leq C_0, \quad \text{for all } t \geq T_0.
\]

**Proof.** It is elementary from condition (2) to establish the following energy inequality:

\[
\frac{d}{dt} \|u\|^2 + \|u\|_W^p + \lambda \|u\|^2 + \alpha_1 \|u\|_q^2 \leq C,
\]

where \(C := C([\psi_1, |g|])\) is a generic constant. Applying the classical Gronwall lemma to (7) (omitting the second terms), we have, for all \(t > 0\),

\[
\|u(t)\|^2 + \alpha_1 \int_0^t e^{\lambda(s-t)} \|u(s)\|_q^2 ds \leq \|u_0\|^2 e^{-\lambda t} + C(1 - e^{-\lambda t})/\lambda.
\]
Let $M = \| B \| := \sup_{v \in B} \| v \|$. We choose a $T$ large enough such that $M^2 e^{-\lambda T} \leq 1$, then by (8),
\[
\sup_{u_0 \in B} (\| u(t, u_0) \|^2 + \int_{t-1}^{t} \| u(s, u_0) \|^2 ds) \leq C, \quad \text{for all } t \geq T.
\] (9)
Therefore, we obtain (6) by taking $T_0 = 2T$. \hfill \Box

The following Gronwall-type lemma from [30, 33] will be used frequently later.

**Lemma 2.2.** Let $y, h$ be nonnegative, locally integrable on $[a, +\infty)$ and $y'$ locally integrable such that
\[
y'(\tau) + by(\tau) \leq h(\tau) \quad \text{for } \tau \geq a,
\] (10)
where $b \in \mathbb{R}$. Then for every $t \geq s \geq a$ we have
\[
y(t) \leq e^{-bt} \left( \frac{1}{t-s} \int_{s}^{t} y(\tau) e^{b\tau} d\tau + \int_{s}^{t} h(\tau) e^{b\tau} d\tau \right).
\] (11)
In particular, if $t \geq a + 1$ we have
\[
y(t) \leq C(b)(\int_{t-1}^{t} y(\tau) d\tau + \int_{t-1}^{t} h(\tau) d\tau),
\] (12)
where $C(b) = 1$ if $b \geq 0$ and $C(b) = e^{-b}$ if $b < 0$.

**Proof.** It is easy to prove (11) by integrating (10) twice (see [30]). Since, for all $\tau \in [t-1, t]$, $e^{-bt} e^{b\tau} \leq 1$ if $b \geq 0$ and $e^{-bt} e^{b\tau} \leq e^{-b}$ if $b < 0$, we obtain (12) from (11). \hfill \Box

We also recall the following embedding relationship among integrable spaces, which generalizes slightly the corresponding formulation in [11, 12].

**Lemma 2.3.** (i) For any $r > 1$, $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ with $\| u \|_r \leq \| u \|_1 \| u \|_\infty^{-1}$.

(ii) If $0 < r_1 < r_2 < r_3 < +\infty$ then $L^{r_1}(\mathbb{R}^n) \cap L^{r_3}(\mathbb{R}^n) \subset L^{r_2}(\mathbb{R}^n)$ with the following inequality:
\[
\| u \|_{r_2} \leq \varepsilon \| u \|_{r_1} + C(\varepsilon) \| u \|_{r_3}.
\] (13)
The number $\varepsilon$ may be small enough.

3. **Absorption by induction.** This section presents the absorption of the semigroup in $W^{1,p}$ and especially in $L^r$ for any $r \in (2, q_N)$, where $N = \{(p-2)/(q-2)\}$ and we define successively $q_m = m(q - 2) + 2$, $m \in \mathbb{N}$.

**Lemma 3.1.** Suppose (2) and (3) hold true. Then for each $m \in \{1, \ldots, N\}$ there is a positive constant $C_m$ such that, for every bounded $B \subset L^2$ and for all $t \geq T_m := 2^m T_0$ (where $T_0$ is given in Lemma 2.1),
\[
\sup_{s \in [t/2, t]} \sup_{u_0 \in B} \| u(s, u_0) \|_{q_m}^{q_m} + \sup_{u_0 \in B} \int_{t-1}^{t} \| u(s, u_0) \|_{q_{m+1}}^{q_{m+1}} ds \leq C_m.
\] (14)
In particular, the solution $u(t, u_0) \in L^{q_N}$ for all $t > 0$ and $u_0 \in L^2$. 

Proof. We first prove (14) for \( m = 1 \). Note that \( q_1 = q \). We multiply Eq.(1) by \( |u|^{q-2}u \) and then integrate it on \( \mathbb{R}^n \) to find
\[
\frac{1}{q} \frac{d}{dt} \|u\|_q^q + (A u, |u|^{q-2}u) + \lambda \|u\|_q^q + (f(x, u), |u|^{q-2}u) = (g, |u|^{q-2}u). \tag{15}
\]
By (4), we have
\[
(A u, |u|^{q-2}u) = (q-1) \sum_{i=1}^n \int_{\mathbb{R}^n} |u|^{q-2}|u_x|^p dx \geq 0. \tag{16}
\]
By the first hypothesis in (2) and the Young inequality, we have
\[
(f(x, u), |u|^{q-2}u) \geq \alpha_1 \|u\|_{q_2}^{q_2} + (\psi_1, |u|^{q-2}) \geq \frac{\alpha_1}{2} \|u\|_{q_2}^{q_2} - C \|\psi_1\|_{\mu_1},
\]
where \( \mu_1 = 1/(1 - \frac{q-2}{q_2}) = 2 - \frac{2}{q} \) and thus \( 1 < \mu_1 < 2 \). So, by Lemma 2.3 (ii), we have
\[
(f(x, u), |u|^{q-2}u) \geq \frac{\alpha_1}{2} \|u\|_{q_2}^{q_2} - C(\|\psi_1\|_1 + \|\psi_1\|^2). \tag{17}
\]
By the Young inequality again,
\[
(g, |u|^{q-2}u) \leq \frac{\alpha_1}{4} \|u\|_{q_2}^{q_2} + C \|g\|^2. \tag{18}
\]
We see from (15)-(18) that
\[
\frac{d}{dt} \|u\|_q^q + \frac{q \alpha_1}{4} \|u\|_{q_2}^{q_2} \leq C. \tag{19}
\]
Applying now Lemma 2.2 (the Gronwall-type lemma) to (19), we see from (6) that, for all \( t \geq T_0 \) and \( u_0 \in B \),
\[
\|u(t, u_0)\|_q^q \leq \int_{t-1}^t \|u(\tau)\|_q^q d\tau + \int_{t-1}^t C d\tau \leq C. \tag{20}
\]
By taking \( T_1 = 2T_0 \), we see that, for all \( t \geq T_1 \) and for some positive constant \( C_1 \),
\[
\sup_{s \in [t/2, t]} \sup_{u_0 \in B} \|u(s, u_0)\|_q^q \leq C_1. \tag{21}
\]
On the other hand, integrating (19) over \([t-1, t]\) yields for all \( t \geq T_1 \) (then \( t-1 \geq T_0 \)),
\[
\int_{t-1}^t \|u(s, u_0)\|_{q_2}^{q_2} \leq C \|u(t-1)\|_q^q + C \leq C. \tag{22}
\]
We have proved (14) for \( m = 1 \).

We then assume (14) is true for \( m = k \), i.e. \( k+1 < N \). Since \( k+1 \leq N+1 \), by (3) and Lemma 2.3 (ii), we have
\[
\psi_1 \in L^1 \cap L^{k+1}, \quad g \in L^2 \cap L^{k+1}. \tag{23}
\]
Multiplying Eq.(1) with \( |u|^{q_2-2}u \) to obtain a similar formulation as (15):
\[
\frac{1}{q_k} \frac{d}{dt} \|u\|_{q_k}^{q_k} + (A u, |u|^{q_k-2}u) + \lambda \|u\|_{q_k}^{q_k} + (f(x, u), |u|^{q_k-2}u) = (g, |u|^{q_k-2}u). \tag{24}
\]
By the first inequality in (2), we have
\[
(f(x, u), |u|^{q_k-2}u) \geq \alpha_1 \|u\|_{q_{k+1}}^{q_{k+1}} + (\psi_1, |u|^{q_k-2}) \geq \frac{\alpha_1}{2} \|u\|_{q_{k+1}}^{q_{k+1}} - C \|\psi_1\|_{\mu_k}. 
\]
where \( \mu_k = 1/(1 - \frac{q_k-2}{q_k+1}) = k + 1 - \frac{2k}{q} \) and thus \( 1 \leq \mu_k \leq k + 1 \). By Lemma 2.3,

\[
(f(x, u), |u|^{q_k-2}u) \geq \frac{\alpha_1}{2} \|u\|^{q_{k+1}}_{q_{k+1}} - C(\|\psi_1\|_1 + \|\bar{\psi}_1\|_{k+1}^{k+1}).
\]

By the Young inequality again,

\[
(g, |u|^{q_k-2}u) \leq \frac{\alpha_1}{4} \|u\|^{q_{k+1}}_{q_{k+1}} + C\|g\|_{\nu_k},
\]

where \( \nu_k = 1/(1 - \frac{q_k-1}{q_k+1}) = k + 1 - \frac{k-1}{q} \) and thus \( 2 \leq \nu_k \leq k + 1 \). So,

\[
(g, |u|^{q_k-2}u) \leq \frac{\alpha_1}{4} \|u\|^{q_{k+1}}_{q_{k+1}} + C(\|g\|^2 + \|g\|_{k+1}^{k+1}).
\]

The same calculation as (16) yields \((Au, |u|^{q_k-2}u) \geq 0\). Therefore, by (23)-(26), we have

\[
\frac{d}{dt}\|u\|_{q_k}^q + \frac{\alpha_1 q_k}{4} \|u\|_{q_{k+1}}^{q_{k+1}} \leq C,
\]

Applying Lemma 2.2 to (27), we see from the inductive hypothesis (i.e. the second term in (14) with \( m = k - 1 \)) that for all \( t \geq T_{k-1} \),

\[
\|u(t)\|_{q_k}^q \leq \int_{t-1}^t \|u(\tau)\|_{q_k}^q d\tau + C \leq C.
\]

If taking \( T_k = 2T_{k-1} \), then we have \( \|u(s)\|_{q_k}^q \leq C \) for all \( s \in [t/2, t] \) with \( t \geq T_k \), which proves the first term in (14) for \( m = k \). On the other hand, we integrate (27) over \([t-1, t]\) with \( t \geq T_k \) to find

\[
\int_{t-1}^t \|u(\tau)\|_{q_{k+1}}^{q_{k+1}} d\tau \leq \frac{4}{\alpha_1 q_k} \|u(t-1)\|_{q_k}^{q_k} + C \leq C.
\]

We finish the induction proof.

The following corollary establishes the absorption in \( L^p \) whether the dissipative-ness is strong or weak (i.e. \( q \geq p \) or \( q < p \)).

**Corollary 1.** There is a constant \( C_p \) such that, for each bounded set \( B \) in \( L^2 \), there is a \( T_p := T_p(B) \) such that

\[
\sup_{s \in [t/2, t]} \sup_{u_0 \in B} \|u(s, u_0)\|_p \leq C_p \text{ for all } t \geq T_p.
\]

**Proof.** Since \( q_N = N(q - 2) + 2 \geq \frac{p-2}{q-2}(q - 2) + 2 = p \), by Lemma 2.3(ii), we have

\[
\|u\|_p^p \leq C\|u\|^2 + \|u\|_{q_N}^{q_N}.
\]

Therefore, the required conclusion follows from Lemma 3.1 and Lemma 2.1 immediately.

To close this section, we prove the absorption in \( W^{1,p} \).

**Lemma 3.2.** Assume both (2) and (3) holds true. Then there is a positive constant \( C_W \) such that, for each bounded set \( B \) in \( L^2 \), there is a time \( T_W := T_W(B) \) such that

\[
\sup_{s \in [t/2, t]} \sup_{u_0 \in B} \|u(s, u_0)\|_W \leq C_W \text{ for all } t \geq T_W.
\]
Proof. By integrating (7) on \([t-1, t]\) with \(t \geq T_0\), we see from (6) that for all \(t \geq T_0\) and \(u_0 \in B\),

\[
\int_{t-1}^{t} \|u(s, u_0)\|_W^p \, ds \leq \|u(t-1)\|^2 + C \leq C.
\]  \hfill (33)

We then multiply Eq.(1) by \(u_t\) to obtain

\[
\frac{1}{p} \frac{d}{dt} \|u\|_W^p + \|u_t\|^2 + \lambda(u, u_t) + (f(x, u), u_t) = (g, u_t).
\]  \hfill (34)

By the second inequality in condition (2), we have

\[
|(f(x, u), u_t)| \leq \frac{1}{4} \|u_t\|^2 + \|f(x, u)\|^2 \leq \frac{1}{4} \|u_t\|^2 + C(\|u\|_{q_2}^2 + \|\psi_2\|^2),
\]  \hfill (35)

where \(q_2 = 2q - 2\) is just the second number as defined in Lemma 3.1. Thus, by the Young inequality, we see from (34) and (35) that

\[
\frac{d}{dt} \|u\|_W^p \leq C(\|u\|_{q_2}^2 + \|u\|^2 + 1).
\]  \hfill (36)

Applying Lemma 2.2 to (36), we find

\[
\|u(t)\|_W^p \leq \int_{t-1}^{t} \|u(\tau)\|_W^p \, d\tau + C\int_{t-1}^{t} (\|u(\tau)\|_{q_2}^2 + \|u(\tau)\|^2 + 1) \, d\tau.
\]  \hfill (37)

Therefore, the needful conclusion follows from (6),(33) and (14) with \(m = 1\), where we use the fact that (14) holds true for \(m = 1\) at least since \(N \geq 1\). \hfill \(\square\)

4. Asymptotic compactness and global attractors in \(L^2\). This section proves the asymptotic compactness of the semigroup \(S(t)\) in \(L^2(\mathbb{R}^n)\). We need to estimate the integral of solutions outside a ball. The following lemma shows that the semigroup \(S(t)\) is asymptotically small outside a ball whether the dissipativeness is strong or weak. A half-time decomposition technique is used in the proof.

**Lemma 4.1.** Let (2) and (3) hold true and \(\varepsilon\) be small. Then, for every bounded subset \(B \subset L^2(\mathbb{R}^n)\), there are \(T := T(\varepsilon, B)\) and \(K := K(\varepsilon, B)\) such that

\[
\sup_{u_0 \in B} \int_{|x| \geq k} |u(t, u_0)|^2 \, dx < \varepsilon, \quad \text{for all } t \geq T, \ k \geq K.
\]  \hfill (38)

**Proof.** We take a continuously differential function \(\xi : [0, +\infty) \to [0, 1]\) such that \(\xi \equiv 0\) on \([0, 1]\) and \(\xi \equiv 1\) on \([2, +\infty)\). Then we know \(\xi'\) is bounded on \(\mathbb{R}^+\) and \(\xi' \equiv 0\) on \([0, 1] \cup [2, +\infty)\). For each \(k > 0\), define a function by \(\xi_k(x) = \xi(|x|^2/k^2), \ x \in \mathbb{R}^n\).

We now take the inner product of Eq.(1) with \(\xi_k u(t, u_0)\) (where \(u_0 \in B\)) to obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \xi_k u^2 \, dx + \lambda \int_{\mathbb{R}^n} \xi_k u^2 \, dx + (Au, \xi_k u) + (f(x, u), \xi_k u) = (g, \xi_k u).
\]  \hfill (39)

By (4), the semilinear Laplacian term can be calculated as follows

\[
(Au, \xi_k u) = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \xi_i |u_{x_i}|^p \, dx + \sum_{i=1}^{n} \int_{\mathbb{R}^n} |u_{x_i}|^{p-2} u_{x_i} u_{x_{i'}} (\frac{|x_i|^2}{k^2}) \frac{2x_{i'}}{k^2} \, dx
\]

\[
\geq - \frac{C}{k} \sum_{i=1}^{n} \int_{|x| \leq \sqrt{2k}} |u_{x_i}|^{p-1} |u| \, dx \geq - \frac{C}{k}(\|u\|_W^p + \|u\|_{p}^p).
\]  \hfill (40)
By the condition (2), the nonlinear term can be estimated as follows:

\[(f(x,u),\xi_k u) \geq \alpha_1 \int_{\mathbb{R}^n} \xi_k |u|^q dx - \int_{\mathbb{R}^n} \xi_k |\psi_1| dx \geq -\int_{\mathbb{R}^n} \xi_k |\psi_1| dx.\]  

(41)

\[|(g,\xi_k u)| \leq \frac{\lambda}{2} \int_{\mathbb{R}^n} \xi_k u^2 dx + \frac{2}{\lambda} \int_{\mathbb{R}^n} \xi_k g^2 dx.\]  

(42)

All above estimates imply the following differential inequality:

\[\frac{d}{dt} \int_{\mathbb{R}^n} \xi_k u^2 dx + \lambda \int_{\mathbb{R}^n} \xi_k u^2 dx \leq C \frac{\|u\|_W^p + \|u\|_p^p}{K} + C \int_{\mathbb{R}^n} \xi_k (|\psi_1| + g^2) dx.\]  

(43)

Therefore, we obtain the following inequality by applying the classical Gronwall lemma to (43) on the interval \([\frac{t}{2}, t]\) (rather than on \([0, t]\), which is different from the strongly dissipative case, see, e.g. [8, 9, 11, 12, 14, 15]).

\[\int_{\mathbb{R}^n} \xi_k u^2(t) dx \leq e^{-\frac{\lambda}{2}} \int_{\mathbb{R}^n} \xi_k u^2(\frac{t}{2}) dx + C \int_{t/2}^{t} e^{-\lambda(t-s)} \|u(s)\|_W^p + \|u(s)\|_p^p ds + C \int_{\mathbb{R}^n} \xi_k (|\psi_1| + g^2) dx.\]  

(44)

We take \(T\) sufficiently large, say, \(T = \max\{T_0, T_p, T_W\}\), where \(T_0, T_p, T_W\) are defined as in Lemma 2.1, Corollary 3.2 and Lemma 3.3 respectively. Then we see from (6) that, for all \(t \geq T\),

\[e^{-\frac{\lambda}{2}} \int_{\mathbb{R}^n} \xi_k u^2(\frac{t}{2}) dx \leq e^{-\frac{\lambda}{2}} \|u(\frac{t}{2})\|_W^p \leq C_0^2 e^{-\frac{\lambda}{2} t} \to 0 \text{ as } t \to +\infty.\]  

(45)

By (30) and (32), for each \(t \geq T\), the second term in the righthand side of (44) is less than

\[\frac{C}{k} \int_{t/2}^{t} e^{-\lambda(t-s)} (C_W^p + C_p^p) ds \leq \frac{C}{k} \int_{0}^{+\infty} e^{-\lambda s} ds \to 0 \text{ as } k \to +\infty.\]  

(46)

Finally, since \(\psi_1 \in L^1\) and \(g \in L^2\), we have

\[\int_{\mathbb{R}^n} \xi_k (|\psi_1| + g^2) dx = \int_{|x| \geq k} (|\psi_1| + g^2) dx \to 0 \text{ as } k \to +\infty.\]  

(47)

Therefore, we see from (44)-(47) that there exist \(T := T(\varepsilon, B) \geq \max\{T_0, T_p, T_W\}\) and \(K := K(\varepsilon) > 0\) such that, for all \(k \geq K\), \(t \geq T\) and \(u_0 \in B\),

\[\int_{|x| \geq \sqrt{2}k} u^2(t, u_0) dx \leq \int_{\mathbb{R}^n} \xi_k u^2(t, u_0) dx < \varepsilon,\]  

(48)

which finishes the proof of Lemma 4.1.

We obtain the following existence result of global attractors by using Lemma 2.1, Lemma 3.3, Lemma 4.1 and the classical existence theorem of global attractors (see, e.g. [3, 24]). Because the following result will be included in Theorem 5.7, so we omit the proof.

**Theorem 4.2.** Suppose (2) and (3) hold true. No matter how dissipative the equation is (that is, no matter \(q \geq p\) or \(q < p\)), there is a unique global attractor \(A_\infty\) in \(L^2(\mathbb{R}^n)\) for the semigroup generated by the p-Laplace equation.

**Remark.** To guarantee existence of a global attractor in \(L^2(\mathbb{R}^n)\), we require only the force \(g \in L^2\) in the strongly dissipative case, but we require at least \(g \in L^3\) in the weakly dissipative case.
5. Bi-spatial attractors for the $p$-Laplace equation. This section proves the existence of an $(L^2, L^*)$-attractor for the semigroup $S(t)$ induced by Eq.(1) for every $r \in [2, q_N]$. For this result with $r > 2$, we need the higher regularity of $\psi_1$, i.e. $\psi_1 \in L^\infty$, which is assumed in the literature investigating regularity of attractors.

5.1. Asymptotic estimates in $L^r$. To obtain a bi-spatial attractor with the terminate space $L^r$ for each $r \in [2, q_N]$, it is necessary to present some asymptotic estimates for the unbounded part of the modulus of solutions in $L^r$. We start at the following auxiliary lemma, which presents the asymptotic estimate in $L^2$.

Lemma 5.1. Let $B$ be a bounded set in $L^2(\mathbb{R}^n)$. We have

$$\lim_{t,M \to +\infty} \sup_{u_0 \in B} m(|u(t, u_0)| \geq M) = 0,$$

(49)

$$\lim_{t,M \to +\infty} \sup_{u_0 \in B} \int_{|u| \geq M} |u(t, u_0)|^2 dx = 0,$$

(50)

where the subscript $|u| \geq M$ denotes the set $\{x \in \mathbb{R}^n; |u(x)| \geq M\}$ and $m(\cdot)$ denotes the Lebesgue measure.

Proof. By Lemma 2.1, we know that, for all $t \geq T_0$ and $u_0 \in B$,

$$m(|u(t, u_0)| \geq M) \leq \frac{1}{M^2} \int_{|u| \geq M} |u(t, u_0)|^2 dx \leq \frac{1}{M^2} \|u(t, u_0)\|^2 \leq \frac{C_0}{M^2}.$$

(51)

For each $\varepsilon > 0$, if we choose $M$ large enough such that $M \geq \sqrt{C_0/\varepsilon}$, then $m(\mathbb{R}^n(|u(t, u_0)| \geq M) < \varepsilon$ for all $t \geq T_0$ and $u_0 \in B$, which proves (49).

To proves (50), we define a decreasing family of subsets of $L^2$ (when $T$ increases) as follows:

$$K(T) := \bigcup_{t \geq T} \bigcup_{u_0 \in B} \{u(t, u_0)\} \text{ for each } T > 0.$$

(52)

For each $\varepsilon > 0$, by Lemma 4.1, there are $T > 0$ and $R > 0$ such that

$$\int_{|x| \geq R} w^2(x) dx < \frac{\varepsilon^2}{16}$$

for all $w \in K(T) \subset L^2(\mathbb{R}^n))$.

(53)

By Lemma 3.3 and the compact embedding $W^{1,p}(Q_R) \hookrightarrow L^2(Q_R)$, we know the restricted set $K_R(T)$ of $K(T)$ is pre-compact in $L^2(Q_R)$ for some larger $T$, where $Q_R = \{x \in \mathbb{R}^n; |x| < R\}$ and the restricted set is defined by

$$K_R(T) = \{w \in L^2(Q_R); \exists w_1 \in K(T) \text{ s.t. } w(x) = w_1(x) \text{ for } x \in Q_R\}.$$

(54)

Therefore, $K_R(T)$ has a finite $\varepsilon/4$-net in $L^2(Q_R)$, which together with (53) implies that $K(T)$ has a finite $\varepsilon$-net in $L^2(\mathbb{R}^n)$ with the centers $w_i \in L^2(\mathbb{R}^n)$ ($i = 1, 2, ..., k$).

We then choose a $\delta > 0$ such that, for any set $e \subset \mathbb{R}^n$ with $m(e) < \delta$,

$$\sup_{1 \leq i \leq k, e} \int_{e} w_i^2(x) dx < \varepsilon^2.$$

(55)

By (49), we can choose an $M$ such that $m(|w| \geq M) < \delta$ for all $w \in K(T)$ with a larger $T$. Given now $w \in K(T)$, there is a center $w_i \in K(T)$ such that $\|w - w_i\| \leq \varepsilon$.

Therefore, by (55), we have

$$\int_{|w| \geq M} w^2(x) dx \leq 2\|w - w_i\|^2 + 2\int_{|w| \geq M} w_i^2(x) dx \leq 4\varepsilon^2,$$

which proves (50).
The following lemma shows that the nonlinear function \( f(x, s) \) (given by (2)) keeps in plus or minus sign if \(|s|\) is large enough.

**Lemma 5.2.** Let \( f := f(x, s) \) be a function satisfying (2) with \( \psi_1 \in L^\infty \) additionally. Then there is a positive constant \( M_f \) such that, for a.e. \( x \in \mathbb{R}^n \),

\[
f(x, s) \geq \frac{\alpha_1}{2} |s|^{q-1} \quad \text{if } s \geq M_f \quad \text{and} \quad f(x, s) \leq -\frac{\alpha_1}{2} |s|^{q-1} \quad \text{if } s \leq -M_f. \tag{56}
\]

**Proof.** Since \( \psi_1 \in L^\infty \), we set \( M_f = \max \{ 1, \sqrt{2\|\psi_1\|_\infty \alpha_1^{-1}} \} \) to find

\[
\frac{\alpha_1}{2} |s|^q - |\psi_1(x)| \geq 0 \quad \text{for a.e. } x \in \mathbb{R}^n \quad \text{and for all } |s| \geq M_f.
\]

Therefore, by the first formulation in (2), we have for a.e. \( x \in \mathbb{R}^n \) and for all \( |s| \geq M_f \),

\[
f(x, s) \geq \frac{\alpha_1}{2} |s|^q + \frac{\alpha_1}{2} |s|^q - |\psi_1(x)| \geq \frac{\alpha_1}{2} |s|^q,
\]

which implies (56) immediately. \( \Box \)

The following lemma is a key to prove the asymptotic compactness in \( L^r \) for \( r \in [2, q_N] \).

**Lemma 5.3.** Suppose (2) holds and \( \psi_1 \in L^1 \cap L^\infty \), \( g \in L^2 \cap L^{N+1} \). Let \( B \subset L^2(\mathbb{R}^n) \) be bounded, then we have

\[
\lim_{t, M \to +\infty} \sup_{u_0 \in B} \int_{|s| \geq M} |u(t, u_0)(x)|^r \, dx = 0 \quad \text{for each } r \in [2, q_N]. \tag{57}
\]

**Proof.** The case \( r = 2 \) has been proved by Lemma 5.1 and thus, by Lemma 2.3, it suffices to prove (57) with \( r = q_N \). To this end, we will prove successively about \( m \in \{ 1, 2, \ldots, N \} \) that, for each \( \varepsilon > 0 \), there exist both increasing sequences \( \{ T_m \} \) and \( \{ M_m \} \) such that, for an intrinsical constant \( C \),

\[
\sup_{t \geq T_m} \sup_{u_0 \in B} \int_{u \geq M_m} |u(t, u_0)|^{q_m} \, dx < C \varepsilon, \tag{58}
\]

\[
\sup_{t \geq T_m} \sup_{u_0 \in B} \int_{t-1}^{t} \int_{u \geq M_m} (u-M_m)^{q_m-1} |u(s, u_0)|^{q-1} \, dxds < C \varepsilon, \tag{59}
\]

where \( q_m = m(q-2) + 2, \ m \in \{ 1, 2, \ldots, N \} \) and \( v_+ = \max \{ v, 0 \} \). We only consider here the positive part since the negative part is similar.

We start at proving the case of \( m = 1 \). Since \( g \in L^2 \cap L^{N+1} \), by Lemma 2.3, for \( \varepsilon > 0 \), we can take \( \delta > 0 \) such that, for any set \( e \subset \mathbb{R}^n \) with \( m(e) < \delta \),

\[
\int_e (g^2 + |g|^3 + \cdots + |g|^{N+1}) \, dx < \varepsilon. \tag{60}
\]

By Lemma 5.1, there are both positive constants \( M' \) and \( T \) such that, for all \( t \geq T \),

\[
\sup_{u_0 \in B} m(|u(t, u_0)| \geq M') < \delta, \tag{61}
\]

\[
\sup_{u_0 \in B} \int_{|s| \geq M'} |u(t, u_0)|^2 \, dx < \varepsilon. \tag{62}
\]
Let $M = \max\{M', M_f\}$, where $M_f$ is the constant given in Lemma 5.2. We multiply Eq. (1) by $(u - M)_+$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \| (u - M)_+ \|^2 + \| (u - M)_+ \|_{W}^2 + \lambda(u, (u - M)_+) \\
+ (f(x, u), (u - M)_+) = (g, (u - M)_+).
\] (63)

By (61), $m(u(t)) \geq M \leq m(u(t)) \leq M' < \delta$ for $t \geq T$. Hence, we see from (60) that
\[
(g, (u - M)_+) \leq \lambda \|(u - M)_+\|^2 + C \int_{(u \geq M)} g^2 dx \leq \lambda(u, (u - M)_+) + C \varepsilon. \quad (64)
\]

By (56), we have
\[
(f(x, u), (u - M)_+) \geq \frac{\alpha_1}{2} \int_{u \geq M} u^{q-1} (u - M)_+ dx,
\] (65)

which together with (63) and (64) imply that, for $t \geq T$,
\[
\frac{1}{2} \frac{d}{dt} \| (u - M)_+ \|^2 + \alpha_1 \int_{u \geq M} u^{q-1} (u - M)_+ dx \leq C \varepsilon. \quad (66)
\]

Let $t \geq T + 1$ be fixed. We integrate (66) over $[t - 1, t]$ and use (62) to find
\[
\int_{t-1}^{t} \int_{u \geq M} |u|^{q-1} (u - M)_+ dx ds \leq \frac{2}{\alpha_1} \|(u(t) - 1) - M)_+\|^2 + C \varepsilon \leq C \varepsilon. \quad (67)
\]

We then take the inner product of Eq. (1) with $(u - M)_+^{q-1}$ to obtain
\[
\frac{1}{q} \frac{d}{dt} \| (u - M)_+ \|_{q}^q + (Au, (u - M)_+^{q-1}) + \lambda(u, (u - M)_+^{q-1}) \\
+ (f(x, u), (u - M)_+^{q-1}) = (g, (u - M)_+^{q-1}). \quad (68)
\]

It is easy to see $(u, (u - M)_+^{q-1}) \geq 0$ and
\[
(Au, (u - M)_+^{q-1}) = (q - 1) \sum_{i=1}^{n} \int_{(u \geq M)} |u_x|^{p} (u - M)^{q-2} dx \geq 0. \quad (69)
\]

By (56) again, we have
\[
(f(x, u), (u - M)_+^{q-1}) \geq \frac{\alpha_1}{2} \int_{u \geq M} u^{q-1} (u - M)_+^{q-1} dx. \quad (70)
\]

By the Young inequality, we see from (60)-(61) that, for $t \geq T$,
\[
|g, (u - M)_+^{q-1}| \leq \frac{\alpha_1}{4} \|(u - M)_+\|_{2q-2}^{2q-2} + C \int_{u \geq M} g^2 dx \\
\leq \frac{\alpha_1}{4} \int_{u \geq M} u^{q-1} (u - M)_+^{q-1} dx + C \varepsilon. \quad (71)
\]

Therefore, we see from (68)-(71) that
\[
\frac{d}{dt} \| (u - M)_+ \|_{q} + \frac{q \alpha_1}{4} \int_{u \geq M} u^{q-1} (u - M)_+^{q-1} dx \leq C \varepsilon. \quad (72)
\]
Applying Lemma 2.2 to (72) and using the result in (67), we find, for \( t \geq T + 1 \),
\[
\|(u(t) - M)_+\|^q \leq \int_{t-1}^t \|(u(s) - M)_+\|^q ds + C \varepsilon \\
\leq \int_{t-1}^t \int_{u \geq M} u^{q-1}(u - M)_+ dx ds + C \varepsilon < C \varepsilon.
\] (73)
If \( u \geq 2M \) then \( u \leq 2(u - M) \) and so for \( t \geq T + 1 \),
\[
\int_{u \geq 2M} u^q(t) dx \leq C \int_{u \geq M} (u - M)^q dx = C \|(u - M)_+\|^q \leq C \varepsilon,
\] (74)
which proves (58) for \( m = 1 \) and \( M_1 = 2M \). Integrating (72) over \([t-1, t]\), we find, for \( t \geq T + 2 \),
\[
\int_{t-1}^t \int_{u \geq M} u^{q-1}(u - M)^{q_+ - 1} dx ds \leq C \|(u(t-1) - M)_+\|^q + C \varepsilon \leq C \varepsilon,
\] (75)
which proves (59) for \( m = 1 \) and \( T_1 = T + 2 \).

We now assume that both (58) and (59) hold for \( m = k - 1 < N \) and choose afresh a constant \( M \geq M_{k-1} \geq M_1 \). We then take the inner product of (1) with \((u - M)^{q_k - 1}_+\) to get
\[
\frac{1}{q_k} \frac{d}{dt} \|(u - M)_+\|^{q_k} + (f(x, u), (u - M)^{q_k - 1}_+) \leq (g, (u - M)^{q_k - 1}_+).
\] (76)
where we have thrown away both nonnegative terms \((Au, (u - M)^{q_k - 1}_+)\) and \(\lambda (u, (u - 2M)^{q_k - 1}_+)\). By (56) again,
\[
(f(x, u), (u - M)^{q_k - 1}_+) \geq \frac{\alpha_1}{2} \int_{u \geq M} u^{q-1}(u - M)^{q_k - 1} dx.
\] (77)
Let \( \nu_k = 1/(1 - \frac{q_k - 1}{q_k + 1}) = k + 1 - \frac{k - 1}{q - 1} \) as given in Lemma 3.1 and thus \( 2 \leq \nu_k \leq k + 1 \). So, by Lemma 2.3 and (60)-(61), we have
\[
|\langle g, (u - M)^{q_k - 1}_+ \rangle| \leq \frac{\alpha_1}{4} \|(u - M)_+\|^{q_k + 1} + \int_{u \geq M} |g|^{q_k + 1} dx ds \\
\leq \frac{\alpha_1}{4} \int_{u \geq M} u^{q-1}(u - M)^{q_k - 1}_+ dx + \int_{u \geq M} (g^2 + |g|^{k+1}) dx \\
\leq \frac{\alpha_1}{4} \int_{u \geq M} u^{q-1}(u - M)^{q_k - 1}_+ dx + C \varepsilon,
\] (78)
where we use \( k + 1 \leq N + 1 \) in the last step. From (76)-(78), we know \( t \geq T \geq T_k \),
\[
\frac{d}{dt} \|(u - M)_+\|^{q_k} + \frac{q_k \alpha_1}{4} \int_{u \geq M} u^{q-1}(u - M)^{q_k - 1}_+ dx \leq C \varepsilon.
\] (79)
Applying Lemma 2.2 to (79) and using the inductive hypothesis (59) for \( m = k - 1 \), we find, for \( t \geq T + 1 \),
\[
\|(u(t) - M)_+\|^{q_k} \leq \int_{t-1}^t \|(u(s) - M)_+\|^{q_k} ds + C \varepsilon \\
\leq \int_{t-1}^t \int_{u \geq M} u^{q-1}(u - M)^{q_k - 1}_+ dx ds + C \varepsilon < C \varepsilon.
\] (80)
By the same argument as given in (74)-(75), we see both (58) and (59) hold true for \( m = k \), which finishes the induction proof. \( \square \)
5.2. Abstract results on bi-spatial attractors. This subsection recalls some concepts and results about bi-spatial attractors from [15]. Let $S(\cdot)$ be a semigroup on a Banach space $X$. Assume the semigroup takes its values (for strictly positive times) in another Banach space $Y$ in the following sense:

$$S(t)u_0 \subset Y \quad \text{for all} \quad t > 0 \quad \text{and} \quad u_0 \in X.$$  \hfill (81)

We also assume that the pair $(X,Y)$ of Banach spaces has the following limit-identical property:

$$u_n \in X \cap Y; \|u_n - u_0\|_X \to 0, \|u_n - v_0\|_Y \to 0 \quad \Rightarrow \quad u_0 = v_0 \in X \cap Y.$$  \hfill (82)

**Definition 5.4.** Let $S(\cdot)$ be a semigroup on $X$ taking its values in $Y$ in the sense of (81). A set $A \subset X \cap Y$ is called to be an $(X,Y)$-attractor for $S(\cdot)$ if

(i) $A$ is compact in $X \cap Y$,

(ii) $A$ is invariant under $S(\cdot)$, i.e. $S(t)A = A$ for all $t \geq 0$,

(iii) $A$ attracts every bounded subset in $X$, precisely, for each bounded set $B \subset X$, we have

$$\lim_{t \to \infty} \text{dist}_Y (S(t)B, A) := \lim_{t \to \infty} \sup_{y \in S(t)B} d_Y(y, A) = 0.$$  \hfill (83)

**Definition 5.5.** We say that a semigroup $S(\cdot)$ has an absorbing set $B$ if for every bounded set $B \subset X$ there is a $T := T(B)$ such that $S(t)B \subset B$ for all $t \geq T$. We say a semigroup $S(\cdot)$ is asymptotically compact in $X$ (resp. in $Y$) if $\{S(t_n)u_n\}$ is pre-compact in $X$ (resp. in $Y$) whenever $t_n \to \infty$ and $\{u_n\}$ is bounded in $X$.

The following existence result of bi-spatial attractors is the weaker form of [15, Theorem 3.1] restricted on the deterministic case (the original theorem discussed random dynamical systems).

**Proposition 1.** Let $(X,Y)$ be a limit-identical pair of Banach spaces. Let $S(\cdot)$ be a continuous semigroup on $X$ but take its values in $Y$ in the sense of (81). Assume further

(i) $S(\cdot)$ has an absorbing set $B$, which is bounded in $X$;

(ii) $S(\cdot)$ is asymptotically compact in $X$;

(iii) $S(\cdot)$ is asymptotically compact in $Y$.

Then $S(\cdot)$ has a unique $(X,Y)$-attractor $A$, it is just the unique global $(X,X)$-attractor (as the same set).

5.3. Bi-spatial attractors for p-Laplacian equations. By using the above abstract result, we obtain the following existence theorem of bi-spatial attractors for Eq.(1). Recall $q_N = N(q - 2) + 2 = \left\{\frac{p-2}{q-2}\right\}(q - 2) + 2$.

**Theorem 5.6.** Suppose (2) holds and $\psi_1 \in L^1 \cap L^\infty$, $g \in L^2 \cap L^{N+1}$. Then for each $r \in [2, q_N]$ there is a unique $(L^2(\mathbb{R}^n), L'(\mathbb{R}^n))$-attractor $A_r$ for the semigroup $S(\cdot)$ generated by Eq.(1) no matter how dissipative the equation is. Furthermore, all attractors $A_r, r \in [2, q_N]$ are the same set.

**Proof.** We first explain that $(L^2(\mathbb{R}^n), L'(\mathbb{R}^n))$ is a limit-identical pair, although $L^2(\mathbb{R}^n)$ and $L'(\mathbb{R}^n)$ may not be contained in any direction. Both $L^2(\mathbb{R}^n)$ and $L'(\mathbb{R}^n)$ are continuously embedded into the distribution space $D'(\mathbb{R}^n)$ and thus the limit-identical property follows from uniqueness of limits (see [31, Lemma 2.7]).

We then see from Lemma 3.1 that $S(\cdot)$ takes its values in $L'$ for all positive times. We see also from Lemma 2.1 that the ball $B(0, C_0)$ is a bounded absorbing set in
where \( M \) decompose the domain \( \mathbb{R}^n \)
see from Lemma 5.3 that, for each \( \varepsilon > 0 \), there are \( I_1 \in \mathbb{N} \) and \( M > 0 \) such that
\[
\int_{|u_i| \geq M} |u_i(x)|^r \, dx < \varepsilon^r \quad \text{for all } i \geq I_1,
\]
(84)
On the other hand, by the foregoing proof, the sequence \( \{u_i\} \) has a convergent subsequence (denoted by itself) in \( L^2 \). Hence the subsequence \( \{u_i\} \) is a Cauchy sequence in \( L^2 \), which implies that there is an integer \( I_2 \geq I_1 \) such that, for all \( i_1, i_2 \geq I_2 \),
\[
\|u_{i_1} - u_{i_2}\|^2 \leq M^{2-r} \varepsilon^r,
\]
(85)
where \( M \) is the constant given in (84). According to the pair \((i_1, i_2)\) of integers, we decompose the domain \( \mathbb{R}^n \) into four parts \( \mathbb{R}^n = \bigcup_{j=1}^{4} D_j \) as follows:
\[
D_1 = \mathbb{R}^n(\{u_{i_1} \mid \leq M\}) \cap \mathbb{R}^n(\{u_{i_2} \leq M\}),
D_2 = \mathbb{R}^n(\{u_{i_1} \leq M\}) \cap \mathbb{R}^n(\{u_{i_2} \geq M\}),
D_3 = \mathbb{R}^n(\{u_{i_1} \geq M\}) \cap \mathbb{R}^n(\{u_{i_2} \leq M\}),
D_4 = \mathbb{R}^n(\{u_{i_1} \geq M\}) \cap \mathbb{R}^n(\{u_{i_2} \geq M\}).
\]
(86)
By (85) we have
\[
\int_{D_1} |u_{i_1} - u_{i_2}|^r \, dx \leq \int_{u_{i_1} - u_{i_2} \leq 2M} |u_{i_1} - u_{i_2}|^r \, dx \leq (2M)^{r-2} \|u_{i_1} - u_{i_2}\|^2 \leq 2^r \varepsilon^r.
\]
(87)
Note that \( |u_{i_1}(x)| \leq |u_{i_2}(x)| \) if \( x \in D_2 \) and \( |u_{i_1}(x)| \geq |u_{i_2}(x)| \) if \( x \in D_3 \). We see from (84) that
\[
\int_{D_2} |u_{i_1} - u_{i_2}|^r \, dx \leq 2^r \int_{u_{i_2} \geq M} |u_{i_2}|^r \, dx \leq 2^r \varepsilon^r,
\]
(88)
\[
\int_{D_3} |u_{i_1} - u_{i_2}|^r \, dx \leq 2^r \int_{u_{i_1} \geq M} |u_{i_1}|^r \, dx \leq 2^r \varepsilon^r,
\]
(89)
\[
\int_{D_4} |u_{i_1} - u_{i_2}|^r \, dx \leq 2^{-r-1} \int_{u_{i_1} \geq M} |u_{i_1}|^r \, dx + \int_{u_{i_2} \geq M} |u_{i_2}|^r \, dx \leq 2^r \varepsilon^r.
\]
(90)
All above estimates imply that \( \|u_{i_1} - u_{i_2}\|_r \leq 4 \varepsilon \) if \( i_1, i_2 \geq I_2 \). This means that \( \{u_i\} \) has a Cauchy subsequence in \( L^r(\mathbb{R}^n) \) and thus the original sequence \( \{u_i\} \) is pre-compact in \( L^r \) as required. So far, we have verified all conditions in Proposition 5.6. Therefore the semigroup \( S(\cdot) \) has a unique \( (L^2(\mathbb{R}^n), L^r(\mathbb{R}^n)) \)-attractor \( \mathcal{A}_r \) and all these attractors \( \mathcal{A}_r \) are the same set for different \( r \in [2, q_N] \).

6. **Approximation of attractors.** This section considers the question whether the obtained attractor in Theorem 5.7 is approximated by the corresponding attractor for Eq.(1) inside a ball. It is necessary to give an additional assumption: the function \( f(\cdot,0) - g(\cdot) \) has a compact support in \( \mathbb{R}^n \), more precisely, there is a positive number \( R_{f,g} \) such that
\[
f(x,0) = g(x) \quad \text{for a.e. } |x| \geq R_{f,g}
\]
(91)
In the sequel, one may regard a set $K \subset L^2(Q_R)$ as a subset of $L^2(\mathbb{R}^n)$ by null-expanding, where $Q_R = \{x \in \mathbb{R}^n; |x| < R\}$ and the null-expansion $\tilde{K}$ of $K$ is defined by

$$\tilde{K} := \{u \in L^2(\mathbb{R}^n) | \exists v \in K \text{ s.t. } u(x) = v(x) \text{ if } |x| < R, \ u(x) = 0 \text{ if } |x| \geq R\}. \quad (92)$$

It is obvious that $\tilde{K} \subset L^r(\mathbb{R}^n)$ if $K \subset L^r(Q_R)$ for $r \geq 2$.

**Theorem 6.1.** Suppose (2), (91) hold and $\psi_1 \in L^1 \cap L^\infty$, $g \in L^2 \cap L^{N+1}$. Let $A_\infty$ be the obtained attractor in Theorem 5.7 for Eq. (1) defined on $\mathbb{R}^n$ and let $A_R$ be the global attractor for Eq. (1) inside a ball $Q_R$ with the Dirichlet boundary condition. Then $A_R$ is strictly upper semi-continuous to $A_\infty$ under the Hausdorff semi-distance of any $r$-norm, $r \in [2, q_N]$:

$$\text{dist}_r(A_R, A_\infty) = 0 \text{ for each } R \geq R_{f,g}, \quad (93)$$

where $R_{f,g}$ is given in (91). The family $A_R$ is also lower semi-continuous to $A_\infty$ under any $r$-norm, $r \in [2, q_N]$, that is,

$$\lim_{R \to +\infty} \text{dist}_r(A_\infty, A_R) = 0. \quad (94)$$

Furthermore, we have the following construction of $A_\infty$:

$$A_\infty = \overline{\bigcup_{R_{f,g} \leq R < +\infty} A_R}, \quad (95)$$

where the closure is taken in the topology of $L^r(\mathbb{R}^n)$ and each $A_R$ is regarded as its null-expansion to the entire Euclid space.

**Proof.** Let $S_\infty(t)$ be the semigroup generated by Eq. (1) in $L^2(\mathbb{R}^n)$ and let $S_R(t)$ be the semigroup generated by Eq. (1) in $L^2(Q_R)$ ($0 < R < \infty$) with the Dirichlet boundary condition:

$$u(t, x) = 0, \quad \text{if } t \geq 0 \text{ and } |x| = R. \quad (96)$$

We first explain that, if $R \geq R_{f,g}$, $L^2(Q_R)$ is a positively invariant subspace (in the null-expanding sense) under the operator $S_\infty(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for each $t \geq 0$. Indeed, suppose $u_{0,R} \in L^2(Q_R)$ and let $u_R(t, \cdot, u_{0,R})$ denote the solution of Eq. (1) defined on $Q_R$ which satisfies (96).Then it is easy from the assumption (91) to verify that the null-expansion $\tilde{u}_R(t, \cdot, u_{0,R})$ of $u_R(t, \cdot, u_{0,R})$ is a solution of Eq. (1) defined on $\mathbb{R}^n$ such that $\tilde{u}_R(0, u_{0,R}) = \tilde{u}_{0,R}$, the null-expansion of $u_{0,R}$. Then the uniqueness of solutions implies that $S_\infty(t)\tilde{u}_{0,R} = \tilde{u}_R(t, u_{0,R})$ for each $t \geq 0$, which proves $S_\infty(t)L^2(Q_R) \subset L^2(Q_R)$ as required.

We then prove the upper semi-continuity (93). Let $\tilde{A}_R$ be the null-expansion of $A_R$ as defined by (92). It is easy to see that both sets have the same (maximal) norms $\|\tilde{A}_R\|_{L^2(\mathbb{R}^n)} = \|A_R\|_{L^2(Q_R)}$, which implies that $\tilde{A}_R$ is a bounded set in $L^2(\mathbb{R}^n)$ since the attractor $A_R$ is bounded in $L^2(Q_R)$. We will prove that $\tilde{A}_R$ is (strictly) invariant under the operator $S_\infty(t)$ for each $t \geq 0$ and $R \geq R_{f,g}$. Indeed, if $u_R \in \tilde{A}_R$, then there is a $u_{0,R} \in A_R$ such that $\tilde{u}_0(x) = u_{0,R}(x)$ for all $x \in Q_R$. Then, by the invariance of $A_R$ under $S_R(t)$, we have $\tilde{w} := S_R(t)\tilde{u}_{0,R} \in A_R$. Let $\tilde{w}$ be the null-expansion of $w$. By the foregoing proof, $L^2(Q_R)$ is an invariant subspace under $S_\infty(t)$, then we have $S_\infty(t)\tilde{w} = \tilde{u}_R(t, u_{0,R}) = \tilde{w} \in \tilde{A}_R$ and so $S_\infty(t)\tilde{A}_R \subset \tilde{A}_R$. Conversely, by the invariance of $A_R$ again, for each $u_{0,R} \in A_R$, there is a $v \in A_R$ such that $S_R(t)v = u_{0,R}$. By null-expanding, we have $S_\infty(t)v = \tilde{w} = u_{0,R}$, which means...
that \(u_{0,R} \in S_\infty(t)\tilde{A}_R\) and thus \(\tilde{A}_R \subseteq S_\infty(t)\tilde{A}_R\) for all \(t \geq 0\). We have proved that \(\tilde{A}_R\) is indeed an invariant set under \(S_\infty(\cdot)\) for each \(R \geq R_{f,g}\). Therefore, we see from the attraction of \(A_\infty\) that, for all \(R \geq R_{f,g}\),

\[
\text{dist}(\tilde{A}_R, A_\infty) = \text{dist}(S_\infty(t)\tilde{A}_R, A_\infty) \to 0 \text{ as } t \to \infty, \tag{97}
\]

which implies that the strictly upper semi-continuity holds in \(L^2(Q_R)\):

\[
\text{dist}(\tilde{A}_R, A_\infty) = 0 \quad \text{and thus } \tilde{A}_R \subseteq A_\infty \text{ for each } R \geq R_{f,g}. \tag{98}
\]

In addition, we see from (98) that the strictly upper semi-continuity holds for all \(r\)-norms with \(r \in [2, q_N]\) (since by Theorem 5.7 \(A_R \subseteq L^r(Q_R)\)), that is,

\[
\text{dist}_r(\tilde{A}_R, A_\infty) = 0 \quad \text{for each } R \geq R_{f,g} \text{ and } r \in [2, q_N]. \tag{99}
\]

To prove the lower semi-continuity, we need to prove \(A_R\) is the same set as the restriction \(A_{\infty,R}\) of \(A_\infty\) on \(L^2(Q_R)\) for each \(R \geq R_{f,g}\), where the restricted set is defined by

\[
A_{\infty,R} := \{w \in L^2(Q_R) \mid \exists v \in A_\infty \text{ s.t. } w(x) = v(x) \text{ if } |x| < R\}. \tag{100}
\]

By the same method as above, one can prove that each \(A_{\infty,R}\) \((R_{f,g} \leq R < \infty)\) is an invariant set under \(S_R(\cdot)\) since \(A_\infty\) is invariant under \(S_\infty(\cdot)\). It is obviously that \(\|A_{\infty,R}\| \leq \|A_\infty\|\) and so \(A_{\infty,R}\) is a bounded set in \(L^2(Q_R)\). Hence we see from the attraction of \(A_R\) that

\[
\text{dist}(A_{\infty,R}, A_R) = \text{dist}(S_R(t)A_{\infty,R}, A_R) \to 0 \text{ as } t \to \infty, \tag{101}
\]

which implies that \(A_{\infty,R} \subseteq A_R\). On the other hand, we have proved in (98) that the null-expansion \(\tilde{A}_R \subseteq A_\infty\). Hence, by considering their restrictions in both sides of the included relationship, we have \(A_R \subseteq A_{\infty,R}\) and thus \(A_R = A_{\infty,R}\) for all \(R \geq R_{f,g}\). By this fact, we can calculate the Hausdorff semi-distance under the topology of \(L^2(\mathbb{R}^n)\) between \(A_\infty\) and \(\tilde{A}_R\) \((R \geq R_{f,g})\) as follows:

\[
\text{dist}(A_\infty, \tilde{A}_R) = \sup_{u \in A_\infty} \inf_{v \in \tilde{A}_R} \|u - v\| \\
\leq \sup_{u \in A_\infty} \inf_{v \in \tilde{A}_R} \|\tilde{u}_R - v\| + \sup_{u \in A_\infty} \|u - \tilde{u}_R\| \\
= \sup_{u \in A_{\infty,R}} \inf_{v \in \tilde{A}_R} \|u - v\| + \sup_{u \in A_\infty} \|u\|_{L^2(Q_R^c)} = \sup_{u \in A_\infty} \|u\|_{L^2(Q_R^c)}, \tag{102}
\]

where \(\tilde{u}_R(x) = u(x)\) if \(|x| < R\) and \(\tilde{u}_R(x) = 0\) if \(|x| \geq R\) and we use the fact \(A_R = A_{\infty,R}\) in the last step. Since \(A_\infty\) is bounded and invariant in \(L^2(\mathbb{R}^n)\), it follows from Lemma 4.1 that for each \(\varepsilon > 0\) there are \(R_0 \geq R_{f,g}\) and \(T > 0\) such that, for all \(R > R_0\),

\[
\sup_{u \in A_\infty} \|u\|_{L^2(Q_R^c)} = \sup_{u \in A_\infty} \|S_\infty(T)u\|_{L^2(Q_R^c)} < \varepsilon. \tag{103}
\]

Both (102) and (103) imply the lower semi-continuity in \(L^2(\mathbb{R}^n)\):

\[
\lim_{R \to \infty} \text{dist}(A_\infty, A_R) = 0, \tag{104}
\]

where \(A_R\) is regarded as its null-expansion. By [3, Lemma 3.2 (ii)], we see from (104) that for each \(u \in A_\infty\) there is a \(u_R \in A_R\) for each \(R \in [R_{f,g}, +\infty)\) such that \(u_R \to u\) in \(L^2(\mathbb{R}^n)\) as \(R \to \infty\). Hence \(A_\infty\) is included into the closure of the union
of all \(A_R\) over all \(R \in [R_{f,g}, \infty)\), which together with (98) implies (95) for \(r = 2\), i.e.

\[
A_\infty = \bigcup_{R_{f,g} \leq R < \infty} A_R. \tag{105}
\]

Finally, we prove that both (94) and (95) hold true under \(r\)-norms for each \(r \in (2, q_N]\). To this end, let \(\varepsilon > 0\) be arbitrarily small. Since \(A_\infty\) is bounded in \(L^2(\mathbb{R}^n)\) and invariant under \(S_\infty(t)\), we see from Lemma 5.3 that there is an \(M\) such that, for some \(T\) large enough,

\[
\sup_{u \in A_\infty} \int_{|u| \geq M} |u(x)|^r dx = \sup_{u \in A_\infty} \int_{|S_\infty(T)u| \geq M} |(S_\infty(T)u)(x)|^r dx < \varepsilon^r. \tag{106}
\]

For such a number \(M\), by (103), one can choose a \(R' \geq R_{f,g}\) such that for all \(R > R'\),

\[
\sup_{u \in A_\infty} \int_{Q^r_R} |u(x)|^2 dx < \varepsilon^r. \tag{107}
\]

Then we obtain from (106) and (107) that for all \(R > R'\) and \(u \in A_\infty\),

\[
\int_{Q^r_R} |u(x)|^r dx \leq \int_{Q^r_{R'}(|u| \leq M)} |u(x)|^r dx + \int_{Q^r_{R'}(|u| \geq M)} |u(x)|^r dx
\]

\[
\leq M^{r-2} \int_{Q^r_{R'}} |u(x)|^2 dx + \varepsilon^r \leq 2\varepsilon^r \leq (2\varepsilon)^r. \tag{108}
\]

Considering the Hausdorff semi-distance of \(r\)-norm, by a similar estimate as (102), we have for all \(R > R'\)

\[
\text{dist}_r(A_\infty, A_R) \leq \sup_{u \in A_\infty} \|u\|_{L^r(Q^r_R)} \leq 2\varepsilon. \tag{109}
\]

Therefore \(A_R\) is lower semi-continuous under \(r\)-norm for any \(r \in (2, q_N]\), which proves (94). By the lower semi-continuity under \(r\)-norm, we see from [3, Lemma 3.2 (ii)] that \(A_\infty \subset \text{cl}_r(\bigcup_{R_{f,g} \leq R < \infty} A_R)\), which together with (98) implies that (95) holds true for any \(r \in [2, q_N]\) and finishes the proof.

\[\square\]

**Remark.** It remains open whether there is a random attractor even in \(L^2(\mathbb{R}^n)\) for stochastic \(p\)-Laplacian equations in the weakly dissipative case \((q < p)\). In the stochastic case, one may not derive the uniform absorption as given in (30) over a large interval \([t/2, t]\) since the sample is varied in a cocycle. The half-time decomposition technique may lose its efficacy in the stochastic case. But our method may be effective for deterministic non-autonomous systems provided the non-autonomous force satisfies some additional conditions.

**REFERENCES**

[1] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differ. Equ.*, 246 (2009), 845–869.

[2] P. W. Bates, K. Lu and B. Wang, Attractors for lattice dynamical systems, *International J. Bifur. Chaos*, 11 (2001), 143–153.

[3] A. N. Carvalho, J. A. Langa and J. C. Robinson, *Attractor for Infinite-dimensional Non-autonomous Dynamical Systems*, Appl. Math. Sciences, Vol. 182, Springer, 2013.

[4] B. Gess, W. Liu and M. Rockner, Random attractors for a class of stochastic partial differential equations driven by general additive noise, *J. Differ. Equ.,* 251 (2011), 1225–1253.

[5] B. Gess, Random attractors for degenerate stochastic partial differential equations, *J. Dyn. Differ. Equ.,* 25 (2013), 121–157.
[6] B. Gess, Random attractors for singular stochastic evolution equations, *J. Differ. Equ.*, **255** (2013), 524–559.

[7] B. Gess, Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise, *Annals Probability*, **42** (2014), 818–864.

[8] A. K. Khanmamedov, Existence of a global attractor for the parabolic equation with nonlinear Laplacian principal part in an unbounded domain, *J. Math. Anal. Appl.*, **316** (2006), 601–615.

[9] A. K. Khanmamedov, Global attractors for one dimensional p-Laplacian equation, *Nonlinear Anal. TMA*, **71** (2009), 155–171.

[10] P. G. Geredeli and A. Khanmamedov, Long-time dynamics of the parabolic p-Laplacian equation, *Commun Pure Appl Anal*, **12** (2013), 735–754.

[11] A. Krause, M. Lewis and B. Wang, Dynamics of the non-autonomous stochastic p-Laplace equation driven by multiplicative noise, *Appl. Math. Comput.*, **246** (2014), 365–376.

[12] A. Krause and B. Wang, Pullback attractors of non-autonomous stochastic degenerate parabolic equations on unbounded domains, *J. Math. Anal. Appl.*, **417** (2014), 1018–1038.

[13] J. Li, Y. R. Li and B. Wang, Random attractors of reaction-diffusion equations with multiplicative noise in $L^p$, *Appl. Math. Comput.*, **215** (2010), 3399–3407.

[14] J. Li, Y. R. Li and H. Y. Cui, Existence and upper semi-continuity of random attractors for stochastic p-Laplacian equations on unbounded domains, *Electronic J. Differ. Equ.*, **B**, (2014), 1–27.

[15] Y. R. Li, A. H. Gu and J. Li, Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations, *J. Differ. Equ.*, **258** (2015), 504–534.

[16] Y. R. Li, H. Y. Cui and J. Li, Upper semi-continuity and regularity of random attractors on p-times integrable spaces and applications, *Nonlinear Anal. TMA*, **109** (2014), 33–44.

[17] Y. R. Li and B. L. Guo, Random attractors for quasi-continuous random dynamical systems and applications to stochastic reaction-diffusion equations, *J. Differ. Equ.*, **245** (2008), 1775–1800.

[18] Y. R. Li and J. Y. Yin, A modified proof of pullback attractors in a Sobolev space for stochastic Fitzhugh-Nagumo equations, *Discrete Contin. Dyn. Syst. B*, **21** (2016), 1203–1223.

[19] T. F. Ma and M. L. Pelicer, Attractors for weakly damped beam equations with p-Laplacian, *Discrete Contin. Dyn. Sys., SI*, (2013), 525–534.

[20] J. Simsen, A note on p-Laplacian parabolic problems in R-n, *Nonlinear Anal. TMA*, **75** (2012), 6620–6624.

[21] J. Simsen, M. J. D. Nascimento and M. S. Simsen, Existence and upper semi-continuity of pullback attractors for non-autonomous p-Laplacian parabolic problems, *J. Math. Anal. Appl.*, **413** (2014), 685–699.

[22] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differ. Equ.*, **253** (2012), 1544–1583.

[23] B. Wang and B. Guo, Asymptotic behavior of non-autonomous stochastic equations with nonlinear Laplacian principal part, *Electronic J. Differ. Equ.*, **191** (2013), 1–25.

[24] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Second ed., Springer-Verlag, New York, 1997.

[25] G. L. Wang, B. L.Guo and Y. R. Li, The asymptotic behavior of the stochastic Ginzburg-Landou equation with additive noise, *Appl. Math. Comput.*, **198** (2008), 849–857.

[26] Z. Wang and S. Zhou, Random attractors for stochastic reaction-diffusion equations with multiplicative noise on unbounded domains, *J. Math. Anal. Appl.*, **384** (2011), 160–172.

[27] M. Yang, C. Sun and C. Zhong, Global attractors for p-Laplacian equation, *J. Math. Anal. Appl.*, **327** (2007), 1130–1142.

[28] X. Yan and C. Zhong, $L^p$-uniform attractor for nonautonomous reaction-diffusion equations in unbounded domains, *J. Math. Phys.*, **49** (2008), 102705, 17pp.

[29] J. Y. Yin, Y. R. Li and H. J. Zhao, Random attractors for stochastic semi-linear degenerate parabolic equations with additive noise in $L^q$, *Appl. Math. Comput.*, **225** (2013), 526–540.

[30] W. Q. Zhao, Regularity of random attractors for a degenerate parabolic equations driven by additive noise, *Appl. Math. Comput.*, **239** (2014), 358–374.

[31] W. Zhao and Y. R. Li, $(L^2, L^p)$-random attractors for stochastic reaction-diffusion on unbounded domains, *Nonlinear Anal. TMA*, **75** (2012), 485–502.

[32] W. Q. Zhao and Y. R. Li, Existence of random attractors for a p-Laplacian-type equation with additive noise, *Abstr. Appl. Anal.*, **10** (2011), Article ID 616451, 21pp.

[33] W. Q. Zhao and Y. R. Li, Random attractors for stochastic semi-linear degenerate parabolic equations with additive noises, *Dyn. Partial Differ. Equ.*, **11** (2014), 269–298.
[34] C. Zhong, M. Yang and C. Sun, The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations, *J. Differ. Equ.*, **223** (2006), 367–399.

Received June 2015; revised August 2016.

*E-mail address: liyr@swu.edu.cn*

*E-mail address: yjy111@email.swu.edu.cn*