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Uniqueness of meromorphic functions sharing two finite sets

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Abstract: We prove uniqueness theorems of meromorphic functions, which show how two meromorphic functions are uniquely determined by their two finite shared sets. This answers a question posed by Gross. Moreover, some examples are provided to demonstrate that all the conditions are necessary.

Keywords: Meromorphic function, Shared set, Order, Uniqueness

MSC: 30D35, 30D30

1 Introduction and main results

Throughout this paper, for a meromorphic function, the word “meromorphic” means meromorphic in the whole complex plane $\mathbb{C}$. Let $\mathcal{M}(\mathbb{C})$ (resp. $\mathcal{E}(\mathbb{C})$) be the field of meromorphic (resp. holomorphic) functions in $\mathbb{C}$. The order $\lambda(f)$ and the lower order $\mu(f)$ of $f \in \mathcal{M}(\mathbb{C})$ are defined in turn as follows:

$$
\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \mu(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
$$

If $\lambda(f) < +\infty$, then we denote by $S(r, f)$ any quantity satisfying $S(r, f) = O(\log r)$, $r \to \infty$. If $\lambda(f) = +\infty$, then we denote by $S(r, f)$ any quantity satisfying $S(r, f) = O(\log(rT(r, f)))$, $r \to \infty$, $r \not\in E$, where $E$ is a set of finite linear measure not necessarily the same at every occurrence.

Denote the preimage of a subset $S \subseteq \mathbb{C} \cup \{\infty\}$ under $h \in \mathcal{M}(\mathbb{C})$ by

$$
E(S, h) = \bigcup_{a \in S} \{z \in \mathbb{C} \mid h(z) - a = 0\},
$$

where each zero of $h(z) - a$ of multiplicity $l$ appears $l$ times in $E(S, h)$. The notation $\text{E}^*(S, h)$ expresses the set containing the same points as $E(S, h)$ but without counting multiplicities. Let $f, g \in \mathcal{M}(\mathbb{C})$. If $E(S, f) = E(S, g)$, then $f$ and $g$ share the set $S$ CM (counting multiplicity). If $\text{E}^*(S, f) = \text{E}^*(S, g)$, then $f$ and $g$ share the set $S$ IM (ignoring multiplicity). For fundamental concepts and results from Nevanlinna theory and further details related to $\mathcal{M}(\mathbb{C})$, see [1, 2].

In the sequel, we mainly consider a subset $\mathcal{M}_1(\mathbb{C})$ of $\mathcal{M}(\mathbb{C})$ defined by

$$
\mathcal{M}_1(\mathbb{C}) = \{f \in \mathcal{M}(\mathbb{C}) \mid f \text{ has only finitely many poles in } \mathbb{C}\}.
$$

In 1976, Gross (see [3]) posed the following interesting question.

**Question 1.1.** Can one find two finite sets $S_i$ ($i = 1, 2$) of $\mathbb{C} \cup \{\infty\}$ such that any two elements $f$ and $g$ of a family $\mathcal{G} \subseteq \mathcal{E}(\mathbb{C})$ satisfying $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$ must be identically equal?

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More generally, Question 1.1 suggested the following question.

**Question 1.2.** For a family $\mathcal{G} \subseteq \mathcal{M}(\mathbb{C})$, determine subsets $S_1, S_2, \ldots, S_q$ of $\mathbb{C} \cup \{\infty\}$ in which the cardinality of every $S_i$ ($i = 1, 2, \ldots, q$) is as small as possible and minimise the number $q$ such that any two elements $f$ and $g$ of $\mathcal{G}$ are algebraically dependent if $E(S_i, f) = E(S_i, g)$ for every $i$ ($i = 1, 2, \ldots, q$), that is, if $f$ and $g$ share every $S_i$ ($i = 1, 2, \ldots, q$) CM (counting multiplicity).

In [4], Yi proved that there exist two finite sets $S_1$ (with 1 element) and $S_2$ (with 5 elements) of $\mathbb{C}$ such that any two elements $f$ and $g$ in $\mathcal{E}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal. In [5] and [6], Fang and Xu and independently Yi proved that there exist two finite sets $S_1$ (with 1 element) and $S_2$ (with 3 elements) of $\mathbb{C}$ such that any two elements $f$ and $g$ in $\mathcal{E}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal, which also answered Question 1.1.

For the case $\mathcal{G} = \mathcal{M}(\mathbb{C})$, choosing $S_1 = \{a_i\}$ ($i = 1, 2, \ldots, q$) for distinct elements $a_i$ of $\mathbb{C} \cup \{\infty\}$, when $q \geq 4$, Question 1.2 was completely settled by famous four-value theorem due to Nevanlinna (see e.g. [7] or [1, 2]). However, Question 1.2 is still interesting for the cases $q \leq 3$. In [8], Li and Yang proved that there exist two finite sets $S_1$ (with 15 elements) of $\mathbb{C}$ and $S_2 = \{\infty\}$ such that any two elements $f$ and $g$ in $\mathcal{M}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal. In [9] and [10], Yi and independently Li and Yang proved that there exist two finite sets $S_1$ (with 11 elements) of $\mathbb{C}$ and $S_2 = \{\infty\}$ such that any two elements $f$ and $g$ in $\mathcal{M}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal. In [11], Fang and Guo proved that there exist two finite sets $S_1$ (with 9 elements) of $\mathbb{C}$ and $S_2 = \{\infty\}$ such that any two elements $f$ and $g$ in $\mathcal{M}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal. In [12], Yi proved that there exist two finite sets $S_1$ (with 8 elements) of $\mathbb{C}$ and $S_2 = \{\infty\}$ such that any two elements $f$ and $g$ in $\mathcal{M}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal. In [13], Yi and Li recently proved that there exist two finite sets $S_1$ (with 2 element) and $S_2$ (with 5 elements) of $\mathbb{C}$ such that any two elements $f$ and $g$ in $\mathcal{M}(\mathbb{C})$ sharing $S_1$ and $S_2$ CM must be identically equal.

For the family $\mathcal{G} = \mathcal{M}_1(\mathbb{C})$, we solve Question 1.2 by proving the following theorems.

**Theorem 1.3.** Let $k$ be a positive integer and let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$, $\beta_1, \beta_2$ are $k + 2$ distinct finite complex numbers satisfying

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$ 

If two nonconstant meromorphic functions $f(z)$ and $g(z)$ in $\mathcal{M}_1(\mathbb{C})$ share $S_1$ CM, $S_2$ IM, and if the order of $f(z)$ is neither an integer nor infinite, then $f(z) \equiv g(z)$.

In order to state the next result, we need the following definition related to unique range set.

**Definition 1.4.** For a family $\mathcal{G} \subseteq \mathcal{M}(\mathbb{C})$, the subsets $S_1, S_2, \ldots, S_q$ of $\mathbb{C} \cup \{\infty\}$ such that for any two elements $f$ and $g$ of $\mathcal{G}$ the conditions $E(S_i, f) = E(S_i, g)$ for every $i$ ($i = 1, 2, \ldots, q$) imply $f(z) \equiv g(z)$ are called unique range sets (URS, in brief) of meromorphic functions in $\mathcal{M}(\mathbb{C})$.

For the case $\mathcal{G} = \mathcal{E}(\mathbb{C})$ (resp. $\mathcal{G} = \mathcal{M}(\mathbb{C})$), $q = 1$ in Definition 1.4, the best lower and upper bounds of the cardinality of the set $S_1$ known so far are 4 and 7 (resp. 5 and 11), respectively.

Choosing the family $\mathcal{G} = \mathcal{M}_1(\mathbb{C})$, $q = 2$ in Definition 1.4, from Theorems 1.3 we have the following result.

**Theorem 1.5.** Let $k$ be a positive integer and let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$, $\beta_1, \beta_2$ are $k + 2$ distinct finite complex numbers satisfying

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$ 

If the order of $f(z)$ is neither an integer nor infinite, then the sets $S_1$ and $S_2$ are the URS of meromorphic functions in $\mathcal{M}_1(\mathbb{C})$. 
Lemma 2.5 respectively.

Lemma 2.2 (see [14], p. 293) \( f(z); g(z) \) respectively.

It is easy to verify that \( f(z) \) and \( g(z) \) share every finite value infinitely often.

Remark 1.8. The assumption “nonconstant meromorphic functions \( f(z) \) and \( g(z) \) in \( M_1(\mathbb{C}) \)” in Theorems 1.3-1.5 cannot be relaxed to “nonconstant meromorphic functions \( f(z) \) and \( g(z) \) in \( M(\mathbb{C}) \)”, as shown by the following example. Fix a positive integer \( k \). Let \( f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{1/n}} \), \( g(z) = \frac{1}{\sqrt[2]{z^2}}, S_1 = \{2, 1, 3, \frac{1}{2}, \frac{1}{3} \} \), \( S_2 = \{k + 1, \frac{1}{k + 1} \} \). Then by Remark 1.6 we know \( \lambda(f) = \frac{1}{3} \) and so by Lemma 2.2 in Section 2 we see that \( g(z) \) has infinitely many poles in \( \mathbb{C} \). Moreover, \( f(z) \) and \( g(z) \) share \( S_1, S_2 \) CM. But \( f(z) \neq g(z) \).

2 Some lemmas

In this section we present some important lemmas which will be needed in the sequel.

Lemma 2.1 (see [14], p. 288). Let \( f(z) = \sum_{n=0}^{\infty} c_n z^n \in E(\mathbb{C}) \) be nonconstant and of finite order. Then

\[
\lambda(f) = \frac{1}{\liminf_{n \to \infty} \frac{\log n}{n \log n}}.
\]

Lemma 2.2 (see [14], p. 293). Let \( f(z) \in E(\mathbb{C}) \). If the order of \( f(z) \) is neither an integer nor infinite, then \( f(z) \) assumes every finite value infinitely often.

Lemma 2.3 (see [2], Theorem 1.44). Let \( h(z) \in E(\mathbb{C}) \), and let \( f(z) = e^{h(z)} \). Then (i) if \( h(z) \) is a polynomial of degree \( \deg h \), then \( \lambda(f) = \mu(f) = \deg h \); (ii) if \( h(z) \) is a transcendental entire function, then \( \lambda(f) = \mu(f) = \infty \).

Lemma 2.4 (see [15] or [2], Theorem 1.19). Let \( T_1(r) \) and \( T_2(r) \) be two nonnegative, nondecreasing real functions defined in \( r > r_0 > 0 \). If \( T_1(r) = O(T_2(r)) \) \( (r \to \infty, r \notin E) \), where \( E \) is a set with finite linear measure, then

\[
\limsup_{r \to \infty} \frac{\log^+ T_1(r)}{\log r} \leq \limsup_{r \to \infty} \frac{\log^+ T_2(r)}{\log r}
\]

and

\[
\liminf_{r \to \infty} \frac{\log^+ T_1(r)}{\log r} \leq \liminf_{r \to \infty} \frac{\log^+ T_2(r)}{\log r},
\]

which imply that the order and the lower order of \( T_1(r) \) are not greater than the order and the lower order of \( T_2(r) \) respectively.

Lemma 2.5 (see [2], Theorem 1.42). Let \( f(z) \in M(\mathbb{C}) \). If 0 and \( \infty \) are two Picard exceptional values of \( f(z) \), then \( f(z) = e^{h(z)} \), where \( h(z) \in E(\mathbb{C}) \).
Lemma 2.6 (see [2], Theorem 1.14). Let \( f(z), g(z) \in \mathcal{M}(\mathbb{C}) \). Then
\[
\lambda(f \cdot g) \leq \max\{\lambda(f), \lambda(g)\},
\]
\[
\lambda(f + g) \leq \max\{\lambda(f), \lambda(g)\}.
\]

Lemma 2.7 (see [2], Theorem 2.20). Let \( a_1, a_2, \) and \( a_3 \) be three distinct complex numbers in \( \mathbb{C} \cup \{\infty\} \). If two nonconstant meromorphic functions \( f(z) \) and \( g(z) \) in \( \mathcal{M}(\mathbb{C}) \) share \( a_1, a_2, a_3 \) CM, and if the order of \( f(z) \) and \( g(z) \) is neither an integer nor infinite, then \( f(z) \equiv g(z) \).

3 Proofs of the theorems

3.1 Proof of Theorem 1.3

First we consider the following function
\[
V(z) = \frac{H(z)(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)},
\]
where \( H(z) \) is a rational function such that \( V(z) \) has neither a pole nor a zero in \( \mathbb{C} \). It is easy to see that such an \( H(z) \) does exist since \( f(z), g(z) \in \mathcal{M}_1(\mathbb{C}) \), and a possible pole or zero of \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} \) may only come from a pole of \( f(z) \) or \( g(z) \), in view of the condition that \( f(z) \) and \( g(z) \) share \( S_1 = \{a_1, a_2, \cdots, a_k\} \) CM. Then by Lemma 2.5 there exists an entire function \( \phi(z) \in \mathcal{E}(\mathbb{C}) \) such that
\[
V(z) = \frac{H(z)(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} = e^{\phi(z)}.
\]
Noting that \( f(z) \) and \( g(z) \) have only finitely many poles, we have
\[
N(r, f) = O(\log r), \quad N(r, g) = O(\log r).
\]
Since \( f(z) \) and \( g(z) \) share \( S_2 = \{\beta_1, \beta_2\} \) IM, it follows from (2), the first and second fundamental theorems that
\[
T(r, f) \leq \frac{N \left( r, \frac{1}{f - \beta_1} \right)}{N(r, f)} + \frac{N \left( r, \frac{1}{f - \beta_2} \right)}{N(r, f)} + \frac{N \left( r, \frac{1}{g - \beta_1} \right)}{N(r, f)} + \frac{N \left( r, \frac{1}{g - \beta_2} \right)}{N(r, f)} + O(\log r) + S(r, f)
\]
\[
\leq T \left( r, \frac{1}{g - \beta_1} \right) + T \left( r, \frac{1}{g - \beta_2} \right) + O(\log r) + S(r, f)
\]
\[
\leq 2T(r, g) + O(1) + O(\log r) + S(r, f).
\]
\[
r \to \infty, r \notin E. \quad \text{Then by (3) and Lemma 2.4 we obtain}
\]
\[
\lambda(f) \leq \lambda(g).
\]
\[
\lambda(g) \leq \lambda(f).
\]
Combining (4) with (5) yields
\[
\lambda(g) = \lambda(f).
\]
From the first fundamental theorem we have
\[
T \left( r, \frac{1}{g - \alpha_i} \right) = T(r, g) + O(1)
\]
for $i = 1, 2, \cdots, k$, which implies
$$\lambda \left( \frac{1}{g - a_i} \right) = \lambda(g) \quad (7)$$
for $i = 1, 2, \cdots, k$. Moreover,
$$\lambda \left( f - a_i \right) = \lambda(f) \quad (8)$$
for $i = 1, 2, \cdots, k$. Clearly, $\lambda(H) = 0$ since $H(z)$ is a rational function. Thus it follows by (1), (6), (7), (8), and Lemma 2.6 that
$$\lambda(e^\phi) \leq \lambda(f). \quad (9)$$
In view of the assumption that $f(z)$ and $g(z)$ share $S_2 = \{\beta_1, \beta_2\}$ IM, we deduce from (1) that a zero of $(f(z) - \beta_1)(f(z) - \beta_2)$ is a zero of $H^{-1}(z)e^{\phi(z)} - 1$ or $H^{-1}(z)e^{\phi(z)} - \frac{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)}{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)}$ or $H^{-1}(z)e^{\phi(z)} - \frac{(\beta_2 - a_1)(\beta_2 - a_2) - (\beta_2 - a_k)}{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)}$. We claim that one of the following three cases holds:

(i) \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} = 1; \)

(ii) \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} \neq 1; \)

(iii) \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} \neq 1; \)

Otherwise all of the following three cases would hold:

(i') \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} \neq 1; \)

(ii') \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} \neq 1; \)

(iii') \( \frac{(f(z) - a_1)(f(z) - a_2) \cdots (f(z) - a_k)}{(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_k)} \neq 1; \)

Then, in view of the fact that $H(z)$ is rational, it follows by (i')-(iii'), (1) (2), the first and second fundamental theorems that

$$T(r, f) \leq \mathcal{N} \left( r, \frac{1}{f - \beta_1} \right) + \mathcal{N} \left( r, \frac{1}{f - \beta_2} \right) + \mathcal{N} \left( r, f \right) + S(r, f)$$

$$\leq \mathcal{N} \left( r, \frac{1}{H^{-1}e^\phi - 1} \right) + \mathcal{N} \left( r, \frac{1}{H^{-1}e^\phi - 1} \right) + \mathcal{N} \left( r, H^{-1}e^\phi - \frac{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)}{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)} \right) + O(\log r) + S(r, f)$$

$$\leq T \left( r, \frac{1}{H^{-1}e^\phi - 1} \right) + \mathcal{N} \left( r, H^{-1}e^\phi - \frac{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)}{(\beta_1 - a_1)(\beta_1 - a_2) - (\beta_1 - a_k)} \right) + O(\log r) + S(r, f)$$

$$\leq 3T \left( r, H^{-1}e^\phi \right) + O(1) + O(\log r) + S(r, f)$$

$$\leq 3T \left( r, e^\phi \right) + O(1) + O(\log r) + S(r, f).$$

$r \to \infty, r \notin E$, which together with Lemma 2.4 gives
$$\lambda(f) \leq \lambda(e^\phi). \quad (10)$$
Thus from (9) and (10) we have
$$\lambda(f) = \lambda(e^\phi).$$. 

This contradicts Lemma 2.3 since the order of $f(z)$ is neither an integer nor infinite. The claim is proved. Next we discuss the following three cases.

Case 1. Suppose that (i) occurs. Then by (i) and the assumption

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2$$

we deduce that $f(z) = \beta_1$ if and only if $g(z) = \beta_1$ since $f(z)$ and $g(z)$ share $S_2 = \{\beta_1, \beta_2\}$ CM; further, we know that $f(z) = \beta_2$ if and only if $g(z) = \beta_2$. This implies that $f(z)$ and $g(z)$ share $\beta_1, \beta_2$ CM. Again by (i) we conclude that $f(z)$ and $g(z)$ share $\beta_1, \beta_2$, and $\infty$ CM. Note that the order of $f(z)$ is neither an integer nor infinite. Thus from (6) and Lemma 2.7 we get $f(z) \equiv g(z)$.

Case 2. Suppose that (ii) occurs. Then by (ii) and the assumption

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2$$

we deduce that $f(z) = \beta_1$ if and only if $g(z) = \beta_2$ since $f(z)$ and $g(z)$ share $S_2 = \{\beta_1, \beta_2\}$ CM; further, we know that $f(z) = \beta_2$ if and only if $g(z) = \beta_1$. Since the order of $f(z)$ is neither an integer nor infinite, it follows from Lemma 2.2 that there exists $z_0 \in \mathbb{C}$ such that $f(z_0) = \beta_2$. Thus $g(z_0) = \beta_2$ and so by (ii) we obtain

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 = (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2,$$

which contradicts the assumption.

Case 3. Suppose that (iii) occurs. Then using the same manner as in Case 2, we also get a contradiction. This completes the proof of Theorem 1.3.

### 3.2 Proof of Theorem 1.5

Note that if $f$ and $g$ share the set $S$ CM (counting multiplicity) then $f$ and $g$ certainly share the set $S$ IM (ignoring multiplicity). Then $f$ and $g$ satisfy the conditions in Theorem 1.3. Therefore the conclusion of Theorem 1.5 follows from Theorem 1.3. This completes the proof of Theorem 1.5.

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