SU(\(N\))-symmetric quasi-probability distribution functions

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Abstract
We present a set of \(N\)-dimensional functions, based on generalized SU(\(N\))-symmetric coherent states, that represent finite-dimensional Wigner, Q- and P-functions. We then show the fundamental properties of these functions and discuss their usefulness for analyzing \(N\)-dimensional pure and mixed quantum states.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Recent developments in quantum technology have provided us with the capability of manipulating and measuring a quantum system larger than two qubits using a number of different physical systems. The quantum states of such high-dimensional systems have been experimentally measured and characterized [1]. The standard method used to evaluate quantum states in experiments is state tomography. With this, we can reconstruct the density matrix of the system. Since the density matrix contains all the information of the quantum state we have, we are, in principle, able to calculate any characteristic of the system. The only problem is that the larger the system gets, there are exponentially more elements that we have to measure and compute to characterize the system, and the analysis of the quantum nature of a state quickly becomes intractable.

Fortunately, there has been some recent progress in state characterization and visualization. A tomographic method for large systems with certain symmetries [2] and a state visualization method for discrete systems that extends the Wigner function [3, 4] have been developed. By contrast, for qunats (continuous variables), a long history exists for establishing a tool set that efficiently represents and analyzes quantum states in an infinite-dimensional Hilbert space. In particular, the standard method in quantum optics is based on mapping operator functions to corresponding \(c\)-number functions. For example, the Wigner [5–9], Q- [10] and P-functions [11, 12], which have been widely used in both theoretical and experimental
analysis, are all c-number functions using coherent states. The coherent state is defined as
\(|\alpha\rangle = D(\alpha, \alpha^*)|0\rangle = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle|13\rangle. Here, \(D(\alpha, \alpha^*)\) is a displacement operator in the phase space where \(|0\rangle\) is the vacuum. It can also be taken as the kernel that generates the
Wigner function; hence \([14]\)

\[ W(\alpha, \alpha^*) = \text{Tr}[\rho \cdot 2D(\alpha, \alpha^*)(-1)^{\hat{a}^\dagger\hat{a}}D^{-1}(\alpha, \alpha^*)]. \]  \tag{1.1}

Similarly, the Q-function and P-function can be obtained through (1.1) by replacing \(D(\alpha, \alpha^*)\) with the appropriate operator, respectively.

In this paper, we expand the use of these functions to general SU\((N)\) systems by using SU\((N)\)-symmetric coherent states. To begin, SU\((2)\) coherent states for \(d\)-level systems can be generalized as the trajectory of the SU\((2)\) group action \(U^{M/2}_2\) on the lowest weight state \(|\psi_0\rangle\) \([16, 17]\); hence

\[ |\theta, \phi\rangle \equiv U^{M/2}_2(\theta, \phi)|\psi_0\rangle = \sum_{m=-\frac{M}{2}}^{\frac{M}{2}} \left( \frac{M}{2 + m} \right) \left[ \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right) e^{i\left( \frac{\theta}{2} - m \right)\phi} \right] |\frac{M}{2} + m\rangle. \]  \tag{1.2}

Here an Euler angle decomposition was used for \(U^{M/2}_2(\theta, \phi)\) and \(d = M + 1\), which is usually denoted as \(d = 2j + 1\), where \(j\) is a quantum number. We have introduced the integer parameter \(M\) only for convenience in the generalization later in this paper.

With the SU\((2)\) coherent state given in (1.2), the Wigner, Q- and P-functions have been defined \([18, 19]\). They are successfully used to analyze atoms in a trap \([19]\) and spin-squeezed states of two-component Bose–Einstein condensates \([20, 21]\). However, the main obstacle in using this state representation is the difficulty in adopting its composite structure into the analysis of the quantum system. For example, in quantum information processing, it is important to maintain the properties dependent on the composite structure of the system; quantities such as entanglement only have meaning with it. To accommodate a more detailed system structure, we first need to generalize the state representation method to SU\((N)\) systems.

Our starting point is to generalize (1.2) to SU\((N)\). This generalization can be done as \([22]\)

\[ |(\theta, \phi)^N_M\rangle = U^M_N(\theta, \phi)|\psi_0\rangle. \]  \tag{1.3}

Here, \(\theta \equiv \theta_1, \theta_2, \ldots, \theta_{N-1}\), \(\phi \equiv \phi_1, \phi_2, \ldots, \phi_{N-1}\) and \(|\psi_0\rangle\) is the lowest weight state in terms of the group operation \(U^M_N\). The quantum number \(M\) defines the dimension \(d\) of the representation; hence, the system size is given as

\[ d = b_{N,M} = \left( \begin{array}{c} N + M - 1 \\ M \end{array} \right). \]  \tag{1.4}

We will revisit this and explicitly define the coherent states with an Euler angle parametrization \([23]\) in section 2.

Using SU\((3)\) coherent states, a Winger function has recently been constructed \([24]\). More general SU\((N)\)-symmetric distribution functions have been shown to exist \([25, 26]\). Furthermore, the Q-function can be generalized rather straightforwardly with these coherent states; however, no general quasi-probability functions are constructed using the coherent states defined in (1.3). In this paper, we construct Wigner and P-functions for SU\((N)\) in both the \(M = 1\) and 2 cases.

This paper is organized as follows. We first review the construction and properties of our generalized coherent states and then show how they help build an SU\((N)\)-symmetric Wigner, Q- and P-functions through the Stratonovich–Weyl correspondence \([15]\). We then discuss some general properties of the functions and conclude with an example.
2. SU(N)-symmetric coherent states

We start by explicitly defining our SU(N) coherent states for d-dimensional systems, where d is given in (1.4). First we denote the SU(N) generators [22] by the set
\[ \{ \Lambda_{N,M}(k) \}, \quad \text{where} \quad k = 1, 2, \ldots, N^2 - 1. \] (2.1)

This set is made up of off-diagonal generators\[ \Lambda_{N,M}^{(1)}(a, b) \quad \text{and} \quad \Lambda_{N,M}^{(2)}(a, b), \quad \text{for} \ a, b = 1, 2, 3, \ldots, N; \ a < b \] (2.2)
and diagonal generators\[ \Lambda_{N,M}^{(3)}(c) \quad \text{for} \ 1 \leq c \leq N - 1. \] (2.3)

A detailed procedure to construct these matrices and their properties is given in the appendix. When \( d = N, i.e. M = 1, \) the representation is fundamental and the generators above reduce to the generalized Gell–Mann matrices [\( \lambda_k \)] for SU(N) [27, 28]. In particular, when \( d = 3, i.e. M = 1 \) and \( N = 3, \) this procedure generates the standard Gell–Mann matrices [\( \lambda_k \)] for SU(3):

\[ \Lambda_{3,1}^{(1)}(1, 2) \equiv \Lambda_{3,1}(1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{3,1}^{(2)}(1, 2) \equiv \Lambda_{3,1}(2) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
\[ \Lambda_{3,1}^{(1)}(1, 3) \equiv \Lambda_{3,1}(4) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{3,1}^{(2)}(1, 3) \equiv \Lambda_{3,1}(5) = \begin{pmatrix} 0 & 0 & -i \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]
\[ \Lambda_{3,1}^{(1)}(2, 3) \equiv \Lambda_{3,1}(6) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_{3,1}^{(2)}(2, 3) \equiv \Lambda_{3,1}(7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \]
and
\[ \Lambda_{3,1}^{(3)}(3) \equiv \Lambda_{3,1}(3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{3,1}^{(3)}(8) \equiv \Lambda_{3,1}(8) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \] (2.5)

Given the generators \( \{ \Lambda_{N,M}(k) \} \), we can employ the parametrization given in [23] for SU(N). Using this parametrization, an SU(N) operator for \( M \) can be decomposed as
\[ U_{N}^{M}(\theta, \phi) = \left( \prod_{N \geq z \geq 2} \prod_{2 \leq y \leq z} A_M(y, j(z)) \right) \times B_M, \] (2.6)
where
\[ A_M(y, j(z)) = e^{i A_{N,M}^{(3)}(3) \theta_{y-1+j(z)}} e^{i A_{N,M}^{(2)}(1,y) \theta_{y-1+j(z)}}, \] (2.7)
and
\[ B_M = \prod_{1 \leq c \leq N-1} e^{i A_{N,M}^{(3)}(c+1) \phi_{(N-1)/2+c}} \] (2.8)

Here, \( j(z) = 0 \) for \( z = N \) and \( j(z) = \sum_{i=1}^{N-z} (N-i) \) for \( z \neq N \). The \( A_M(y, j(z)) \) terms are from the off-diagonal generators and the \( B_M \) term is from the diagonal generators. For example, for SU(3), (2.6) gives us [29]
\[ U = U_{3}^{3}(\theta, \phi) \]
\[ = e^{i A_{3,1}^{(1)}(3) \theta_1} e^{i A_{3,1}^{(2)}(1,2) \theta_1} e^{i A_{3,1}^{(2)}(3) \theta_1} e^{i A_{3,1}^{(2)}(1,3) \theta_1} e^{i A_{3,1}^{(2)}(3) \theta_1} e^{i A_{3,1}^{(2)}(1,2) \theta_1} e^{i A_{3,1}^{(3)}(3) \theta_1} e^{i A_{3,1}^{(3)}(8) \theta_1}. \] (2.9)
Using (2.6), the generalized coherent state for $SU(N)$ in the fundamental representation $|\theta, \phi\rangle_N$ can explicitly be written as

$$
|\theta, \phi\rangle_N = \varphi \begin{bmatrix}
\varphi^{i_1 \theta_1 + i_2 \theta_2 + \cdots + i_N \theta_N} 
\varphi^{i_1 \theta_1 + i_2 \theta_2 + \cdots + i_N \theta_N} 
\vdots 
\varphi^{i_1 \theta_1 + i_2 \theta_2 + \cdots + i_N \theta_N}
\end{bmatrix} \begin{bmatrix}
\cos[\theta_1] \cos[\theta_2] \cdots \cos[\theta_{N-1}] 
\sin[\theta_N] \cos[\theta_1] \cdots \cos[\theta_{N-1}] 
\vdots 
\sin[\theta_{N-1}] \cos[\theta_1] \cdots \cos[\theta_{N-2}] \sin[\theta_{N-1}]
\end{bmatrix},
$$

(2.10)

where $\varphi = e^{-i \sqrt{\sum_{N} \sum_{-N}^{2} \phi \sin[\theta_1 + \cdots + \theta_N]}}$ is an overall global phase. More generally, since we are looking at the lowest weight state as the reference state, $SU(N)$ coherent states for $d$-dimensional systems can be easily shown to be equal to the $d$th column of $U^M_N(\theta, \phi)$,

$$
|\theta, \phi\rangle_N^M = U^M_N(\theta, \phi)|\psi_0\rangle = [U^M_N(\theta, \phi)]_{d}.
$$

(2.11)

This is easy to see when $U^M_N(\theta, \phi)$ and $|\psi_0\rangle$ are represented in a matrix form. The column vector of the lowest weight state has zero for all components apart from the bottom row element which is 1. The only elements of the matrix $U^M_N(\theta, \phi)$ that are therefore relevant are those in the $d$th column. If, on the other hand, we had chosen the highest weight state as the reference state, the resulting coherent state would be the first column of $U^M_N(\theta, \phi)$, i.e. $[U^M_N(\theta, \phi)]_1$.

Lastly, the coherent state (2.11) is equivalent to (1.2), regardless of the value of $M$, for $N = 2$ [16–18], as well as more general coherent states for larger values of $N$ and $M$ [22, 30], if one makes the appropriate coordinate transformations on $\theta$ and $\phi$.

Lastly, by using the invariant volume element for the complex projective space in $N - 1$ dimensions ($dV_C^{p-1}$) [31],

$$
dV_C^{p-1} = \left( \prod_{2 \leq j \leq N} K(y) \right) d\theta_{N-1} d\theta_{N-2} \cdots d\theta_1.
$$

(2.12)

derived from (2.6) we have the following resolution of unity for $|\theta, \phi\rangle_N^M$:

$$
\frac{(N + M - 1)!}{2\pi^{N-1}(M!)} \int_{\theta_1, \phi_1} \cdots \int_{\theta_{N-1}, \phi_{N-1}} |(\theta, \phi\rangle_N^M) \langle(\theta, \phi\rangle_N^M| dV_C^{p-1} = 1_d.
$$

(2.13)

We denote the identity matrix of size $d \times d$ by $1_d$ and use the following integration ranges [31]:

$$
0 \leq \theta_i \leq \frac{\pi}{2} \quad \text{and} \quad 0 \leq \phi_i \leq 2\pi, \quad \text{for} \quad 1 \leq i \leq N - 1,
$$

(2.14)

to evaluate the integral.

As a final set of observations, we note the following properties of our coherent states:

$$
\frac{(N + M - 1)!}{2\pi^{N-1}(M!)} \int_{\theta_1, \phi_1} \cdots \int_{\theta_{N-1}, \phi_{N-1}} \langle(\theta, \phi\rangle_N^M| \Lambda_{N,M}(k) \langle(\theta, \phi\rangle_N^M| dV_C^{p-1} = 0
$$

(2.15)

for all $k$, and, as a special case when $M = 1$, 

$$
\frac{N!}{2\pi^{N-1}} \int_{\theta_1, \phi_1} \cdots \int_{\theta_{N-1}, \phi_{N-1}} \langle(\theta, \phi\rangle_N^1| \Lambda_{N,1}(\alpha) \langle(\theta, \phi\rangle_N^1|
$$

$$
\times \langle(\theta, \phi\rangle_N^1| \Lambda_{N,1}(\beta) \langle(\theta, \phi\rangle_N^1| dV_C^{p-1} = \frac{2}{N + 1} \delta_{ab},
$$

(2.16)

where $\delta_{ab}$ is Kronecker's delta.
3. SU(N)-symmetric distribution functions

Having defined our generalized coherent states, we can immediately generalize the Q-function to SU(N) systems [20]. The Q-function of a density matrix $\rho$ may be written as

$$Q(\theta, \phi) = \langle (\theta, \phi)_N^M | \rho | (\theta, \phi)_N^M \rangle.$$  

(3.1)

Here, $\text{Tr} [\rho] = 1$ and, using (2.12),

$$\frac{(N + M - 1)!}{2\pi^{N-1}(M!)} \int_{\theta_1, \phi_1} \ldots \int_{\theta_{N-1}, \phi_{N-1}} Q(\theta, \phi) \, d\nu_{\text{CIR}-1} = 1.$$  

(3.2)

Noting that

$$\langle (\theta, \phi)_N^M | \rho | (\theta, \phi)_N^M \rangle = \text{Tr} \left[ \rho \cdot | (\theta, \phi)_N^M \rangle \langle (\theta, \phi)_N^M | \right],$$  

(3.3)

we see that we have the relation

$$Q(\theta, \phi) = \text{Tr} \left[ \rho \cdot | (\theta, \phi)_N^M \rangle \langle (\theta, \phi)_N^M | \right].$$  

(3.4)

Equation (3.4) is a special case of the more general Stratonovich–Weyl correspondence [15] that describes mappings between any Hilbert space operator $X$ and a characteristic function $f^s(\beta)$ on the classical phase space $X$ ($\beta = \beta_1, \beta_2, \ldots, \beta_N$ and $\beta \in X$) by

$$f^s(\beta) = \text{Tr} [X \cdot F^s(\beta)].$$  

(3.5)

This mapping is very useful in that it allows us to represent a density matrix as a distribution function in the phase space. The quasi-distribution functions are information complete to their original density matrix. This means that one can reconstruct the density matrix from its quasi-distribution function, $f^s(\beta) \mapsto X$, and the generating kernels $F^s(\beta)$ of $f^s(\beta)$, which we will build from a set of Hermitian generators (2.1), satisfy

$$F^s(\beta) = F^s(\beta)^\dagger \quad \text{and} \quad \int_{\beta_1} \ldots \int_{\beta_N} F^s(\beta) \, d\nu(\beta) = \mathbb{I}_N.$$  

(3.6)

Here, $\mathbb{I}_N$ is the $N$-dimensional identity matrix and $s$ determines the types of distribution functions being described [32] as

$$f^{-1}(\beta) \mapsto \text{P-function},$$  

$$f^{+1}(\beta) \mapsto \text{Q-function}$$  

and

$$f^0(\beta) \mapsto \text{Wigner function}.$$  

(3.7)

Because of the way the $f^s(\beta)$ are defined, they exhibit all the properties of the distribution functions they represent; however, from the application point of view, the value $s$ should be chosen dependent on the properties to be investigated. For instance, the Q-function is easy to calculate, but often obscures the quantum nature in the states. In these cases, the Wigner function tends to be preferred to represent the signature of quantum properties by interference fringes. For finite-dimensional systems, (3.5) is a natural way to analyze the Wigner, Q- and P-functions, and we will build our generalized functions based on the various cases of (3.5). In detail,

$$f^s_{N,M,\rho}(\theta, \phi) = \text{Tr} \left[ \rho \cdot F^s_{N,M}(\theta, \phi) \right],$$  

(3.8)

$F^s_{N,M}(\theta, \phi)$ is an operator of an $M$ representation of SU(N) and $(\theta, \phi)$ denotes the parameters from the coherent states given in (1.3).

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1 See also reference [43] in Briët C and Mann A 1999 Phase-space formation of quantum mechanics and quantum-state reconstruction for physical systems with Lie-group symmetries Phys. Rev. A 59 971.
Following (3.2) and (3.6), and using (2.12) as our integral measure, we will demand that $F_{N,M}^r(\theta, \phi)$ and $F_{N,M}^\rho(\theta, \phi)$ satisfy
\[
\frac{(N + M - 1)!}{2\pi^{N-1}(M!)} \int_{\theta, \phi} \cdots \int_{\theta_{N-1}, \phi_{N-1}} f_{N,M,r}^s(\theta, \phi) \, dV_{C_{N-1}} = 1 \tag{3.9}
\]
and
\[
\frac{(N + M - 1)!}{2\pi^{N-1}(M!)} \int_{\theta, \phi} \cdots \int_{\theta_{N-1}, \phi_{N-1}} F_{N,M,\rho}^s(\theta, \phi) \, dV_{C_{N-1}} = 1. \tag{3.10}
\]
For example, if we define $F_{N,M}^r(\theta, \phi) = |(\theta, \phi)_{M,N}^r|/(\theta, \phi)_{M,N}^r$ and $F_{N,M,\rho}^r(\theta, \phi) = Q(\theta, \phi)$ then we can see that (3.9) and (3.10) are satisfied by (2.13) and (3.2). Lastly, to recover the density matrix, the relation
\[
\rho = \frac{(M + N - 1)!}{2\pi^{N-1}(M!)} \int_{\theta, \phi} \cdots \int_{\theta_{N-1}, \phi_{N-1}} f_{N,M,r}^{(s,0)}(\theta, \phi) F_{N,M}^{(s,0)}(\theta, \phi) \, dV_{C_{N-1}} \tag{3.11}
\]
has to be satisfied.

As we have seen in the beginning, the generalized Q-function is rather easy to construct; however, the Wigner and P-functions are somewhat more complicated. Thus, we will first look into the $M = 1$ case for arbitrary $SU(N)$ systems.

### 3.1. Fundamental representation

As before, we start with the Q-function, which, in the fundamental representation, can be written as
\[
f_{N,1,r}^0(\theta, \phi) = \text{Tr}\left[\rho \cdot F_{N,1}^r(\theta, \phi)\right], \tag{3.12}
\]
where
\[
F_{N,1}^r(\theta, \phi) = |(\theta, \phi)_{N,1}^r|/(\theta, \phi)_{N,1}^r = \frac{1}{N} \mathbb{1}_N + \frac{1}{2} \sum_{k=1}^{N-1} |(\theta, \phi)_{N,1}^r \Lambda_{N,1}(k)|/(\theta, \phi)_{N,1}^r \Lambda_{N,1}(k). \tag{3.13}
\]

Now, in order for (3.12) to be useful, we need to be able to evaluate (3.11). We can only do this if we know $F_{N,1}^{(s,0)}(\theta, \phi)$, which is the generating kernel for the P-function. Demanding that we satisfy (3.9) and (3.10), we see that $F_{N,1}^{(s,0)}(\theta, \phi)$ is
\[
F_{N,1}^{(s,0)}(\theta, \phi) = \frac{1}{N} \mathbb{1}_N + \frac{N + 1}{2} \sum_{k=1}^{N^2-1} |(\theta, \phi)_{N,1}^s \Lambda_{N,1}(k)|/(\theta, \phi)_{N,1}^s \Lambda_{N,1}(k). \tag{3.14}
\]

With (3.14) done, the remaining function to generate is the Wigner function. Requiring that we again satisfy (3.9) and (3.10), we obtain for the Wigner function’s generating kernel
\[
F_{N,1}^{(s,1)}(\theta, \phi) = \frac{1}{N} \mathbb{1}_N + \frac{\sqrt{N + 1}}{2} \sum_{k=1}^{N^2-1} |(\theta, \phi)_{N,1}^s \Lambda_{N,1}(k)|/(\theta, \phi)_{N,1}^s \Lambda_{N,1}(k). \tag{3.15}
\]

Combining these results yields
\[
F_{N,1}^r(\theta, \phi) = \frac{1}{N} \mathbb{1}_N + \frac{\Omega(s)}{2} \sum_{k=1}^{N^2-1} |(\theta, \phi)_{N,1}^s \Lambda_{N,1}(k)|/(\theta, \phi)_{N,1}^s \Lambda_{N,1}(k),
\]
where
\[
\Omega(s) = \begin{cases} 
\sqrt{N + 1} & s = 0, \\
1 & s = +1, \\
N + 1 & s = -1. 
\end{cases} \tag{3.16}
\]
Substitution of (3.16) into (3.8), with $M = 1$, yields results that agree with existing Wigner, Q- and P-functions for $N = 2$ and 3 [24, 33]. Lastly, (3.16) with (3.8) satisfies (3.9), (3.10) and (3.11) for all three $s$ values.

Next, in the fundamental representation, a density matrix $\rho$ can be represented by [34]

$$\rho = \frac{1}{N} \mathbb{1}_N + \sqrt{\frac{N - 1}{2N}} \sum_{k=1}^{N^2-1} n_k \Lambda_{N,1}(k).$$  \tag{3.17}

For $N = 2$, the set of all pure states are characterized by $\mathbf{n} \cdot \mathbf{n} = |n_k|^2$, while for $N > 2$, the set of all pure states are characterized by $\mathbf{n} \cdot \mathbf{n}$ and $\mathbf{n} \times \mathbf{n} = \mathbf{n}$, with the star product being defined in (A.2). By substituting (3.17) into (3.8) and using (3.16), as well as exploiting (A.3), we obtain

$$f_{N,1,s}^{\rho}(\theta, \phi) = \frac{1}{N} \mathbb{1}_N + \sqrt{\frac{N - 1}{2N}} \Omega(s) \sum_{k=1}^{N^2-1} \langle (\theta, \phi)_{N,1}^1 | \Lambda_{N,1}(k) \rangle \langle \theta, \phi \rangle_{N,1}^1 n_k. \tag{3.18}$$

To recover the density matrix from this function, we substitute (3.16) and (3.18) into (3.11), as well as exploit (2.15) and (2.16), to obtain (for the $s = 0$ case)

$$\rho = \frac{N!}{2^N N!} \int_{\theta_1, \phi_1} \cdots \int_{\theta_{N-1}, \phi_{N-1}} f_{N,1,0}^{\rho}(\theta, \phi) F_{N,1}^{\rho}(\theta, \phi) dV_{CP^{N-1}} = \frac{1}{N} \mathbb{1}_N + \sqrt{\frac{N - 1}{2N}} \sum_{j=1}^{N^2-1} n_k \Lambda_{N,1}(k). \tag{3.19}$$

For the pure state case, the density matrix $\rho_{PS} = |\psi\rangle\langle \psi|$ has the following $\mathbf{n}$ decomposition [35]:

$$n_k = \frac{N}{2(N - 1)} \langle \psi | \Lambda_{N,1}(k) | \psi \rangle. \tag{3.20}$$

Substitution of (3.20) into (3.18) and (3.19) yields

$$f_{N,1,ps}^{\rho}(\theta, \phi) = \frac{1}{N} \mathbb{1}_N + \frac{\Omega(s)}{2} \sum_{k=1}^{N^2-1} \langle (\theta, \phi)_{N,1}^1 | \Lambda_{N,1}(k) \rangle \langle (\theta, \phi)_{N,1}^1 | \psi \rangle | \Lambda_{N,1}(k) \rangle \langle \psi \rangle, \tag{3.21}$$

and

$$\rho_{PS} = \frac{1}{N} \mathbb{1}_N + \frac{1}{2} \sum_{k=1}^{N^2-1} \langle \psi | \Lambda_{N,1}(k) | \psi \rangle \Lambda_{N,1}(k). \tag{3.22}$$

which is equivalent to (3.13). Similar calculations can be performed for the $s = \pm 1$ cases.

### 3.2. Higher dimensional representations

This problem cannot be simultaneously resolved for all three $s$ values by simply replacing $|(\theta, \phi)^1_N \rangle$ with more general spin coherent state representations from (1.3), or like those defined in [22, 30], the full form of the kernels must be calculated. To accomplish this, we start by generalizing (3.16) to generate an $M = 2$ representation of $F_{N,M}^{\rho}(\theta, \phi)$,

$$F_{N,2}^{\rho}(\theta, \phi) = \frac{1}{d} \mathbb{1}_d + \sum_{c=1}^{2} \omega_{N,2}^{(c)}(c) v_{N,2}(\theta, \phi, c), \tag{3.23}$$
where, using (1.4) and section 2,

\[ \nu_{N,2}(\theta, \phi, c) = \sum_{k=1}^{(b_{N+2,-(c-1)})^2-1} \langle (\theta, \phi) | A_{b_{N+2,-1},c} \rangle \langle k | A_{b_{N+2,-1},c} \rangle (k) \]  \tag{3.24}

and

\[ \omega_{N,2}^c(c) = \begin{cases} 
\frac{1}{\sqrt{2}} b_{N+2,2} & c = 1, \\
\frac{1}{2b_{N+2,2}} \left( \frac{b_{N+2,1}}{2} - 2\omega_{N,2}^0(1) \right) & c = 0, \\
\frac{1}{2} & c = 1, \\
0 & c = 2, \\
(-1)^{c+1} & c = -1.
\end{cases} \tag{3.25}

It can be shown that (3.23) with (3.8) satisfies (3.9), (3.10) and (3.11) for all three values.

As an example, we note that for an \( M = 2 \) \( SU(2) \) system, when \( s = 0 \), our new \( F_{N,2}^c(\theta, \phi) \) gives us

\[ F_{2,2}^0(\theta, \phi) = \frac{1}{d} \left[ \sum_{c=1}^{2} \omega_{2,2}^0(c) \nu_{2,2}(\theta, \phi, c) \right], \]

\[ = \frac{1}{3} + \omega_{2,2}^0(1) \sum_{k=1}^{8} \langle (\theta, \phi) | A_{3,1} \rangle \langle k | A_{3,1} \rangle A_{3,1}(k) \]

\[ + \omega_{2,2}^0(2) \sum_{k=1}^{3} \langle (\theta, \phi) | A_{2,2} \rangle \langle k | A_{2,2} \rangle A_{2,2}(k), \]

\[ = \frac{1}{3} + \frac{\sqrt{10}}{2} \sum_{k=1}^{8} (\theta, \phi) | A_{3,1} \rangle \langle k | A_{3,1} \rangle \lambda_k \]

\[ + \frac{\sqrt{2} - \sqrt{10}}{8} \sum_{k=1}^{3} (\theta, \phi) | A_{2,2} \rangle \langle k | A_{2,2} \rangle J_k. \tag{3.26}
\]

Here, we have again used (1.4) and section 2 to recognize that \( A_{2,2}(K) \equiv J_k \), the spin-1 representation of the \( SU(2) \) Pauli spin matrices, and that \( A_{3,1}(K) \equiv \lambda_k \), the standard Gell–Mann matrices for \( SU(3) \). Lastly, evaluating (1.3) for \( N, M = 2 \) yields

\[ |(\theta, \phi)_{2,2}^c \rangle = e^{-2i\Phi} \begin{bmatrix} e^{2i\Phi} \sin[\theta_1]^2 \\
\sin[2\theta_1] \\
e^{-2i\Phi} \cos[\theta_1]^2 \end{bmatrix}. \tag{3.27}
\]

It is easy to verify that (3.26) yields an equivalent operator for the Wigner function as that given in [33, 36, 37] and, using (3.8), gives equivalent Wigner functions as that given in [19]. Furthermore, a similar calculation for the \( s = \pm 1 \) cases yields equivalent operators as that given in [33] for the Q- and P-functions. Therefore, despite the apparent differences between expressions, (3.23) is the appropriate generalization of (3.16) to \( M = 2 \).
4. Discussion and conclusion

In this section, we mainly discuss two issues: the graphical representation of states of $N$-level systems using our distribution functions and the correspondence relation between the various distribution functions. First, we show some examples of the graphical representation for an $SU(4)$ system by considering a Werner state [38] of two qubits:

$$
\rho_{\text{Werner}} = \frac{1}{4} \begin{pmatrix}
1 - \gamma & 0 & 0 & 0 \\
0 & 1 + \gamma & -2\gamma & 0 \\
0 & 2\gamma & 1 + \gamma & 0 \\
0 & 0 & 0 & 1 - \gamma
\end{pmatrix}.
$$

(4.1)

Here, the parameter $\gamma$ defines the purity of the state. In particular, $\gamma = 1$ corresponds to a pure state, whereas $\gamma = 0$ gives the completely mixed state. Evaluating $\rho_{\text{Werner}}$ via (3.17) gives us

$$
n_1 = -\frac{\gamma}{\sqrt{6}}, \quad n_6 = -\gamma\sqrt{\frac{2}{3}}, \quad n_8 = -\frac{\gamma}{3\sqrt{2}}, \quad \text{and} \quad n_{15} = \frac{\gamma}{3},
$$

(4.2)

which can be substituted into (3.18) to yield

$$
f_{s,1,\text{Werner}}^4(\theta, \phi) = \frac{1}{2^4} [6 - \gamma(2 + 4\cos[2\theta_1] - (1 - 6\cos[2\theta_1]\cos[\theta_2])^2 - 3\cos[2\theta_2] - 12\cos[\phi_1 - \phi_2]\sin[\theta_1]\sin[2\theta_2])\sin[\theta_1]^2\Omega(s)].
$$

(4.3)

Here, $\Omega(s)$ is as defined in (3.16) with $N = 4$, i.e. $\Omega(0) = \sqrt{5}$, $\Omega(+1) = 1$ and $\Omega(-1) = 5$.

The function given in (4.3) is expressed in a five-parameter space; thus, it is not easy to represent its entire property in one figure. So we take cross sections of the function. To do this efficiently, we look at the element of the phases $\phi_1$ and $\phi_2$ in (4.3), that is, $12\cos[\phi_1 - \phi_2]$. This element is effectively one parameter ($\phi_1 - \phi_2$), so we can set it to have two extreme cases: $\phi_1, \phi_2 = 0$ and $\phi_1 = \pi, \phi_2 = 0$. Doing this gives

$$
f_{s,1,\text{Werner}}^4(\theta, \phi) = \frac{1}{2^4} [6 - \gamma(2 + 4\cos[2\theta_1] - (1 - 6\cos[2\theta_1]\cos[\theta_2])^2 - 3\cos[2\theta_2] - 12\sin[\theta_1]\sin[2\theta_2])\sin[\theta_1]^2\Omega(s)].
$$

(4.4)

Figures 1 and 2 show the parameter regions where these quasi-distribution functions exhibit negative values for both cases $\phi_1, \phi_2 = 0$ and $\phi_1 = \pi, \phi_2 = 0$, respectively. As we expect, the Q-function does not yield negative values in the entire parameter regime; however, when the states are pure enough, the P-function and the Wigner function can be negative. When the purity of the Werner state, indicated by $\gamma$, becomes small enough, i.e. when the state is more mixed, the parameter regime for the negative value disappears. In fact, the P-function can be negative when $p \geq 1/4$, while the Wigner function shows a negative region when $p \geq 1/2$. Such negativity in these quasi-distribution functions is often considered evidence of the quantum nature in the states; however, our results indicate that such a simple explanation does not apply to $SU(N)$ systems.

There are of course other ways to reduce the dimensionality of the parameter space. The optimality of a representation is dependent on the properties we are after. Taking a cross section is the easiest way to generate a graphical representation to discern rough properties of the state. Next, we briefly discuss the correspondence between different distribution functions. As we mentioned before, the concrete expression of the various distribution functions depends on the parametrization of the $SU(N)$ group operators. In this paper, we employed the parametrization given in [23], which is an extension of the ones used for $SU(2)$ and $SU(3)$. This parametrization gives us a way to write the Wigner, Q- and P-functions that, in the $M = 1$ and $M = 2$ cases, makes their correspondence easy to see. Furthermore, it shows us how more general $M$ representations should be related.
To begin, for the $M = 1$ case, we start by making the following redefinition of (3.23):

$$F_{N,1}^\prime(\theta, \phi) = F_{N,1}(\theta, \phi) - \frac{1}{N} 1_N. \quad (4.5)$$

such that

$$\tilde{F}_{N,1}^\prime(\theta, \phi) = \frac{\Omega(s')}{\Omega(s)} F_{N,1}(\theta, \phi) \quad (4.6)$$

is true. This can only be done because our $\Omega(s)$ terms in (3.16) are all positive definite. Using (4.5) and (4.6), we therefore obtain

$$F_{N,1}^\prime(\theta, \phi) = \frac{\Omega(s')}{\Omega(s)} \tilde{F}_{N,1}(\theta, \phi) + \frac{1}{N} 1_N,$n

$$= \frac{\Omega(s')}{\Omega(s)} \left( F_{N,1}(\theta, \phi) - \frac{1}{N} 1_N \right) + \frac{1}{N} 1_N,$n

$$= \frac{\Omega(s')}{\Omega(s)} F_{N,1}(\theta, \phi) + \left( 1 - \frac{\Omega(s')}{\Omega(s)} \right) \frac{1}{N} 1_N. \quad (4.7)$$

From this, we can see how we can convert between the various $F_{N,1}(\theta, \phi)$ operators and, through (3.8), the various $f_{N,1}(\theta, \phi)$.

A similar procedure can be done for the $M = 2$ case. In detail, following (4.5) we redefine (3.23) to give us

$$\tilde{F}_{N,2}^\prime(\theta, \phi) = F_{N,2}^\prime(\theta, \phi) - \frac{1}{d} 1_d. \quad (4.8)$$
Figure 2. Graphical representation of the Q-function (top), P-function (middle) and Wigner (bottom) function from (4.4) for $\phi_1 = \pi$, $\phi_2 = 0$ and various values of $\gamma$. The three plots at the far left show the case of the completely mixed states with $\gamma = 0$ and the far right plots show the pure states with $\gamma = 1$. The value of $\gamma$ for each plot is at the right of the graphic. The colored areas represent those values of $\theta_1$, $\theta_2$ and $\theta_3$ where the corresponding function is negative.

By construction, $\tilde{F}_{N,2}^\rho(\theta, \phi)$ is non-singular and of rank $d$. It is therefore invertible, allowing us to have

$$\tilde{F}_{N,2}^\rho(\theta, \phi) = \tilde{F}_{N,2}^\rho(\theta, \phi) \cdot \tilde{F}_{N,2}^\rho(\theta, \phi)^{-1} \cdot \tilde{F}_{N,2}^\rho(\theta, \phi).$$

This allows us to state (via (4.8)) that

$$\tilde{F}_{N,2}^\rho(\theta, \phi) + \frac{1}{d^d} = \tilde{F}_{N,2}^\rho(\theta, \phi) \cdot \tilde{F}_{N,2}^\rho(\theta, \phi)^{-1} \cdot \tilde{F}_{N,2}^\rho(\theta, \phi) + \frac{1}{d^d},$$

$$F_{N,2}^\rho(\theta, \phi) = \tilde{F}_{N,2}^\rho(\theta, \phi) \cdot \tilde{F}_{N,2}^\rho(\theta, \phi)^{-1} \left( F_{N,2}^\rho(\theta, \phi) - \frac{1}{d^d} \right) + \frac{1}{d^d},$$

$$= \Upsilon(s', s, 2) \cdot F_{N,2}^\rho(\theta, \phi) + (1 - \Upsilon(s', s, 2)) \cdot \frac{1}{d^d},$$

where we have made the following definition: $\Upsilon(s', s, 2) = \tilde{F}_{N,2}^\rho(\theta, \phi) \cdot \tilde{F}_{N,2}^\rho(\theta, \phi)^{-1}$. Conversion between the various $F_{N,2}^\rho(\theta, \phi)$ via (3.8) is now straightforward.

In general, we can see that the transformation sequence between the various functions in the general $M$ case is equivalent to (4.10) if we make the following definition: $\Upsilon(s', s, M) = \tilde{F}_{N,M}^\rho(\theta, \phi) \cdot \tilde{F}_{N,M}^\rho(\theta, \phi)^{-1}$. For example, when $M = 1$, $\Upsilon(s', s, 1)$ reduces to $\Omega(s')/\Omega(s)$ as expected.

To conclude, in this paper, we have given an explicit set of $SU(N)$-symmetric functions that represent finite-dimensional versions of the Wigner, Q- and P-functions by using generalized coherent states. In the case of the general $M$ $SU(2)$ and $M = 1$ $SU(3)$ representations, these functions are equivalent to previously derived finite-dimensional Wigner, Q- and P-functions with an appropriate parameter change [24, 33, 36, 37, 39, 40]. The quasi-probability...
distribution functions in this paper have been generalized to a higher quantum number $M = 2$. Such quasi-probability distribution functions may also have some benefits to characterize qubit-qutrit systems. These hybrid systems are becoming extensively investigated in the context of quantum information devices. For more complex systems, there are possibilities to further generalize the formula to an arbitrary $M$. However, the analysis has showed that the process is not as straightforward as the $SU(2)$ case [19] and further work will be necessary to complete the generalization.

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Appendix

The $\{\Lambda_{N,M}(k)\}$ matrices given in (2.1) are a subset of the corresponding Lie algebra of $SU(N)$, a set of Hermitian, traceless matrices of size $d \times d$ that are defined in the following way [22].

(i) Define a general basis $|m_1, m_2, \ldots, m_N\rangle$, where $M = \sum_{k=1}^{N} m_k$ and $M \in \mathbb{Z}^+$. 
(ii) Define the following three operators:

\[ J^a_\mu |m_1, \ldots, m_a, m_b, \ldots, m_N\rangle = \sqrt{(m_a + 1)m_b} |m_1, \ldots, m_a + 1, m_b - 1, \ldots, m_N\rangle \text{ for } 1 \leq a < b \leq N, \]
\[ J^b_\mu |m_1, \ldots, m_b, m_a, \ldots, m_N\rangle = \sqrt{m_a(m_b + 1)} |m_1, \ldots, m_a - 1, m_b + 1, \ldots, m_N\rangle \text{ for } 1 \leq b < a \leq N \text{ and} \]
\[ J^c_\mu |m_1, \ldots, m_c, \ldots, m_N\rangle = \sqrt{\frac{2}{c(c+1)}} \sum_{k=1}^{N} m_k |m_1, \ldots, m_c, \ldots, m_N\rangle \text{ for } 1 \leq c \leq N - 1. \]

(iii) Using the basis given in (i) and the operators given in (ii), define the following matrices:

\[ \Lambda_{N,M}^{[1]}(a, b) \equiv J^b_\mu + J^a_\mu, \]
\[ \Lambda_{N,M}^{[2]}(a, b) \equiv -i(J^b_\mu - J^a_\mu), \]
\[ \Lambda_{N,M}^{[3]}((c + 1)^2 - 1) \equiv J^c_\mu, \] \hspace{1cm} (A.1)

for $a, b = 1, 2, 3, \ldots, N$; $a < b$ and $c = 1, 2, \ldots, N - 1$.

(iv) Combine the three matrices given in (A.1) to yield the set $\{\Lambda_{N,M}(k)\}$, where $k = 1, 2, \ldots, N^2 - 1$.

In general, our lambda matrices can be used to define the $M = 1$ star product,

\[ (x \star y)_k = \sqrt{\frac{N(N-1)}{2(N-2)^2}} \text{ Tr}[(\Lambda_{N,1}(i), \Lambda_{N,1}(j)) \cdot \Lambda_{N,1}(k)] x_i y_j, \] \hspace{1cm} (A.2)

as well as be shown to satisfy

\[ \text{Tr}[\Lambda_{N,M}(i) \cdot \Lambda_{N,M}(j)] = \frac{2M}{N+1} b_{N+1,M} \delta_{ij}, \]
\[ [\Lambda_{N,M}(i), \Lambda_{N,M}(j)] = c \times f_{ijk} \Lambda_{N,M}(k), \]
\[ f_{ijk} = \frac{1}{2c} \times \text{Tr}[(\Lambda_{N,M}(i), \Lambda_{N,M}(j)) \cdot \Lambda_{N,M}(k)]. \] \hspace{1cm} (A.3)
thus forming a basis for the corresponding vector space and a representation of the spin generators of SU(N). For example, when $M = 1$ and $c = 2i$, (A.1) and (A.3) reproduce the form, and properties, of the generalized Gell–Mann matrices [27, 28].

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