Protecting unknown two-qubit entangled states by nesting Uhrig’s dynamical decoupling sequences

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Future quantum technologies rely heavily on good protection of quantum entanglement against environment-induced decoherence. A recent study showed that an extension of Uhrig’s dynamical decoupling (UDD) sequence can (in theory) lock an arbitrary but known two-qubit entangled state to the Nth order using a sequence of N control pulses [Mukhtar et al., Phys. Rev. A 81, 012331 (2010)]. By nesting three layers of explicitly constructed UDD sequences, here we first consider the protection of unknown two-qubit states as superposition of two known basis states, without making assumptions of the system-environment coupling. It is found that the obtained decoherence suppression can be highly sensitive to the ordering of the three UDD layers and can be remarkably effective with the correct ordering. The detailed theoretical results are useful for general understanding of the nature of controlled quantum dynamics under nested UDD. As an extension of our three-layer UDD, it is finally pointed out that a completely unknown two-qubit state can be protected by nesting four layers of UDD sequences. This work indicates that when UDD is applicable (e.g., when environment has a sharp frequency cut-off and when control pulses can be taken as instantaneous pulses), dynamical decoupling using nested UDD sequences is a powerful approach for entanglement protection.

I. INTRODUCTION

Virtually all quantum systems are coupled to an environment and hence suffer from decoherence. Even more troublesome, the implication of decoherence for entangled states is far more severe than for a single quantum system. For example, even though a superposition state of one qubit cannot be completely decohered within any finite time, the entanglement between two such qubits may be totally destroyed by decoherence within a very short time [1–3]. Evidently then, developing useful schemes to protect quantum entanglement from environment-induced decoherence is crucial for entanglement-based quantum technologies.

One promising approach towards decoherence suppression is dynamical decoupling (DD) [4], which advocates the application of a sequence of instantaneous control pulses to effectively average out the system-environment coupling. Important extensions of the original DD approach have also been developed, e.g., “concatenated dynamical decoupling” pulses [5], soft but optimized pulses under an energy cost constraint [6] or a minimal leakage requirement [7], and Uhrig’s dynamical decoupling (UDD) [8–10] sequence that can achieve a very high efficiency of decoherence suppression, i.e., decoherence suppression to the Nth order with a sequence of N instantaneous control pulses. Although UDD requires a sharp frequency cut-off in the bath spectrum [11–13] and cannot operate well if we set limitations on the control pulse width [14], UDD has attracted substantial interests soon after its discovery. In addition to its high-order suppression of decoherence (at least in theory), UDD for single-qubit decoherence suppression could be powerful because it works for most general system-bath coupling [10], for a bath that has unknown spectral density (but with a sharp cut-off), and for time-dependent system-bath Hamiltonians as well [15]. Experimental studies of UDD under two specific situations have been reported [16–18]. For a recent concise review on UDD-related theoretical studies, see Ref. [19].

So far the majority of DD studies have focused on single-qubit systems [20]. Hence it is urgent to investigate if quantum entanglement, e.g., two-qubit entangled states, can be well protected by DD. Because preserving two-qubit entanglement is more subtle than preserving single-qubit coherence, we wish to find a general control scheme to achieve good entanglement preservation without assuming any particular form of system-environment coupling. Indeed, given so many different ways of coupling a composite quantum system to an environment, in general a specific assumption about system-environment coupling may over-simplify the issue of entanglement protection. As such, a universal and efficient DD scheme for entanglement protection should be of sufficient interest. Certainly, if under certain environments some crucial information about system-environment coupling becomes available, then a general DD scheme may be further reduced, a situation exploited in the first experimental study of entanglement protection by DD in a solid-state environment [21].
In our early study [22], it was shown that an extended UDD can also lock a known two-qubit entangled state to the Nth order with N control pulses. The present study is concerned with the protection of unknown entangled states, a situation that is more relevant for quantum information processing. To that end we extend the recent work by West et al. [23], where a scheme based on two layers of UDD sequences is proposed to suppress both the population relaxation and transverse dephasing of one single qubit. In particular, we show that by nesting three layers of UDD sequences, it is possible to lock a two-qubit entangled state as an unknown superposition of two basis states, to the Nth order, using about $N^3$ control pulses in total. This entanglement protection scheme is independent of how the two-qubit system is coupled to its environment. The control operators used in our nested UDD are also explicitly constructed based on two arbitrary, but known, basis states. We shall also show that there are different scenarios in constructing the control operators. Interestingly, it is found that the ordering of the nested UDD layers can be a crucial factor for achieving high-order entanglement protection. This intriguing ordering dependence is absent in two-layer UDD for single-qubit decoherence control, thus offering more insights into quantum decoherence control via nested UDD.

As a further extension of our three-layer UDD, we show that it is possible to construct a four-layer UDD scheme such that a totally unknown two-qubit state can be protected using about $N^4$ pulses of single-qubit control operators. Experimentally, this might be even more challenging than realizing $N^3$ pulses of two-qubit operators. However, such a theoretical possibility is hoped to motivate future studies.

This paper is organized as follows. In Sec. II, after introducing the most general system-environment coupling for two-qubit systems, we briefly review our previous extension of UDD from one-qubit to two-qubit systems. Emphasis is placed on the key requirements to achieve such an extension. In Sec. III, we consider two schemes for nesting UDD in three layers in order to protect unknown two-qubit entangled states. Supporting numerical results are also presented. Section IV discusses a four-layer UDD scheme, followed by Sec. V that concludes this paper.

II. PROTECTING A KNOWN TWO-QUBIT STATE BY UDD

A. General total Hamiltonian of a two-qubit system interacting with a bath

In terms of system-environment coupling, two-qubit systems are far more complex than one-qubit systems. A general total Hamiltonian describing a two-qubit system interacting with a bath can be written as

$$H = \sum_{\gamma,\delta=x,y,z} c_{\gamma}\sigma^{\gamma\delta}_{x} c_{x,1} + \sigma^{1}_{y}\sigma_{y,1} + \sigma_{z,1}^{1} + \sigma_{z,2}^{2} c_{x,2} + \sigma^{2}_{y}\sigma_{y,2} + \sigma^{2}_{z,2} + \sigma^{2}_{x,2} \sigma_{xx} + \sigma^{2}_{y,1} \sigma_{xy} + \sigma^{2}_{x,1} \sigma_{xz} + \sigma^{1}_{y,1} \sigma_{yx} + \sigma^{1}_{y,2} \sigma_{yy} + \sigma^{1}_{y,2} \sigma_{yz} + \sigma^{1}_{z,1} \sigma_{zx} + \sigma^{1}_{z,2} \sigma_{zy} + \sigma^{1}_{y,1} \sigma_{zy} + \sigma^{1}_{z,2} \sigma_{zz}. \quad (1)$$

For convenience each term in the above total Hamiltonian is assumed to be time independent (this assumption can be lifted). Here $c_{0}$ represents the self-Hamiltonian of the bath, $\sigma^{1}_{x}, \sigma^{1}_{y},$ and $\sigma^{1}_{z}$ are the standard Pauli matrices for the first ($j = 1$) or the second ($j = 2$) qubit, $c_{\gamma,j}$ and $c_{\gamma,\delta}$ ($\gamma, \delta = x, y, z$) represent arbitrary smooth bath operators. From Eq. (1), it is seen that in general, the bath may interact with each individual qubit, or modulate the mutual interaction between the two qubits. The latter situation naturally arises if, for example, the bath can induce phonon excitations in a solid and hence perturb the relative distance between the two qubits embedded in the solid. The above total Hamiltonian is in the most general form, because it can be regarded as a linear expansion over all possible 16 linearly independent basis operators operating on a four-dimensional Hilbert space, with the expansion coefficients containing arbitrary bath operators. Note however, the frequency spectrum of the bath is assumed to have a hard cutoff so that the general theory of UDD is applicable.

The basis operators used in the above-mentioned expansion can be taken as $\{R_{i}\}_{i=1,2,\ldots,16} = \{\sigma_{k} \otimes \sigma_{l}\}$, with $\sigma_{k}, \sigma_{l} \in \{I, \sigma_{x}, \sigma_{y}, \sigma_{z}\}$ (I the unity operator for the two-qubit Hilbert space) and the orthogonality condition $\text{Tr}(R_{i} R_{k}) = 4\delta_{ik}$. This choice of basis operators is rather arbitrary. Purely for the sake of discussions below, we find it convenient to define two new sets of basis operators. Let $|0\rangle, |1\rangle, |2\rangle,$ and $|3\rangle$ be the four orthogonal basis states of the two-qubit Hilbert space, we define the...
following two new sets of basis operators,

\begin{align}
Y_1 &= \hat{Y}_1 = I, \\
Y_2 &= \hat{Y}_2 = |0\rangle\langle 0| + |1\rangle\langle 1|, \\
Y_3 &= \hat{Y}_3 = |2\rangle\langle 2| - |3\rangle\langle 3|, \\
Y_4 &= \hat{Y}_4 = |2\rangle\langle 3|, \\
Y_5 &= \hat{Y}_5 = |3\rangle\langle 2|, \\
Y_6 &= \hat{Y}_6 = |0\rangle\langle 0| - |1\rangle\langle 1|, \\
Y_7 &= |1\rangle\langle 2|, \\
Y_8 &= |1\rangle\langle 1|, \\
Y_9 &= |0\rangle\langle 3| - |1\rangle\langle 3|, \\
Y_{10} &= |3\rangle\langle 1|, \\
Y_{11} &= |0\rangle\langle 2| + |1\rangle\langle 2|, \\
Y_{12} &= |2\rangle\langle 0| + |2\rangle\langle 1|, \\
Y_{13} &= |0\rangle\langle 3| + |1\rangle\langle 3|, \\
Y_{14} &= |3\rangle\langle 0| - |3\rangle\langle 1|, \\
Y_{15} &= \hat{Y}_{15} = |0\rangle\langle 1| + |1\rangle\langle 0|, \\
Y_{16} &= \hat{Y}_{16} = -i(|1\rangle\langle 0| - |0\rangle\langle 1|). \\
\end{align}

Then our general total Hamiltonian \( H \) may be re-expressed as

\[ H = \sum_{i=1}^{16} W_i Y_i = \sum_{k=1}^{16} \tilde{W}_k \tilde{Y}_k, \]

where \( W_i \) or \( \tilde{W}_k \) are the associated new expansion coefficients containing bath operators. The motivation of using \( Y_i \) \((i = 1 - 16)\) as the basis operators is purely for convenience. For example, most of them have only one nonzero matrix element and some others are chosen to form a standard SU(2) subalgebra. Similarly, the basis operators \( \tilde{Y}_i \) \((i = 7 - 14)\) are chosen to simplify our later calculations involving states \((|0\rangle + |1\rangle)/\sqrt{2}\) [see Eq. (13)]. Note also that the new basis operators defined in Eq. (2), which still form the generating algebra of \( H \), may not take a Hermitian form. This is not an issue because their linear superpositions still generate all possible Hermitian operators for a two-qubit system.

### B. Locking a known two-qubit state by an extended UDD scheme

Given a known but arbitrary two-qubit state, here assumed to be \(|0\rangle\rangle \) without loss of generality, we first construct a control operator

\[ X_0 = 2|0\rangle\langle 0| - I, \]

with \( X_0^2 = I \). As recently pointed out in Ref. [19], such a control operator was also considered in Ref. [24] before UDD was discovered. We can now split \( H \) into two parts,

\[ H = H_0 + H_1, \]

\[ H_0 = \sum_{i=1}^{10} W_i Y_i, \]

\[ H_1 = \sum_{i=11}^{16} W_i Y_i, \]

with the commuting relation \([X_0, H_0] = 0\), and the anti-commuting relation \([X_0, H_1] = 0\). We proposed in Ref. [23] the following control Hamiltonian describing a sequence of extended UDD \( \pi \)-pulses over a duration of \( T \), i.e.,

\[ H_c = \sum_{j=1}^{N} \pi \delta(t - T_j) X_0, \]

with the UDD timing \( T_j \) given by

\[ T_j = T \sin^2 \left( \frac{j\pi}{2N + 2} \right), \]

\( j = 1, 2, \ldots, N \).

For odd \( N \), an additional control pulse is applied in the end. Then the unitary evolution operator for the whole system of two qubits in a bath for the period \( t = 0 \) to \( t = T \) is given by \((h = 1\) throughout\)

\[ U_N(T) = X_0^N \exp[-i(H_0 + H_1)(T - T_N)(-iX_0)] \times \exp[-i(H_0 + H_1)(T_{N-1} - T_N)(-iX_0)] \times \cdots \times \exp[-i(H_0 + H_1)T], \]

Exploiting \([H_0, X_0] = 0\) and \([H_1, X_0] = 0\), one can directly use the UDD universality proof developed by Yang and Liu [16, 19], yielding

\[ U_N(T) = U_N^{\text{even}} + O(T^{N+1}), \]

where

\[ U_N^{\text{even}} = \exp(-iH_0T) \sum_{k=0}^{+\infty} (-i)^{2k} \Delta_{2k}, \]

with \( \Delta_{2k} \) only containing even powers of \( H_1^k(t) \), \( k \) defined by \( H_1^k(t) = \exp(iH_0t)H_1^k(t)\exp(-iH_0t) \).

Because \([H_1, X_0] = 0\), one has \([H_1^k(t), X_0] = 0\). As such, any even power of \( H_1^k(t) \) will commute with \( X_0 \), e.g., \([H_1^k(t_1)H_1^k(t_2), X_0] = [H_1^k(t_1)H_1^k(t_2), X_0] = [H_1^k(t_1), X_0] + H_1^k(t_2), X_0] = 0 \). This important observation indicates that \( \Delta_{2k} \) can be expanded as a linear superposition of all possible basis operators that commute with \( X_0 \). That is,

\[ \Delta_{2k} = \sum_{i=1}^{10} A_i Y_i, \]

with the commuting relation \([X_0, H_0] = 0\), and the anti-commuting relation \([X_0, H_1] = 0\). We proposed in Ref. [23] the following control Hamiltonian describing a sequence of extended UDD \( \pi \)-pulses over a duration of \( T \), i.e.,

\[ H_c = \sum_{j=1}^{N} \pi \delta(t - T_j) X_0, \]

where \( T_j = T \sin^2 \left( \frac{j\pi}{2N + 2} \right), \] \( j = 1, 2, \ldots, N \). For odd \( N \), an additional control pulse is applied in the end. Then the unitary evolution operator for the whole system of two qubits in a bath for the period \( t = 0 \) to \( t = T \) is given by \((h = 1\) throughout\)

\[ U_N(T) = X_0^N \exp[-i(H_0 + H_1)(T - T_N)(-iX_0)] \times \exp[-i(H_0 + H_1)(T_{N-1} - T_N)(-iX_0)] \times \cdots \times \exp[-i(H_0 + H_1)T], \]

Exploiting \([H_0, X_0] = 0\) and \([H_1, X_0] = 0\), one can directly use the UDD universality proof developed by Yang and Liu [16, 19], yielding

\[ U_N(T) = U_N^{\text{even}} + O(T^{N+1}), \]

where

\[ U_N^{\text{even}} = \exp(-iH_0T) \sum_{k=0}^{+\infty} (-i)^{2k} \Delta_{2k}, \]

with \( \Delta_{2k} \) only containing even powers of \( H_1^k(t) \), \( k \) defined by \( H_1^k(t) = \exp(iH_0t)H_1^k(t)\exp(-iH_0t) \).

Because \([H_1, X_0] = 0\), one has \([H_1^k(t), X_0] = 0\). As such, any even power of \( H_1^k(t) \) will commute with \( X_0 \), e.g., \([H_1^k(t_1)H_1^k(t_2), X_0] = [H_1^k(t_1)H_1^k(t_2), X_0] = [H_1^k(t_1), X_0] + H_1^k(t_2), X_0] = 0 \). This important observation indicates that \( \Delta_{2k} \) can be expanded as a linear superposition of all possible basis operators that commute with \( X_0 \). That is,

\[ \Delta_{2k} = \sum_{i=1}^{10} A_i Y_i, \]
where $A_i$ are the expansion coefficients containing bath operators. Clearly then, to the $N$th order, $U_N(T)$ can be expressed as a combination of $Y_1$, $Y_2$, \ldots, $Y_{10}$ only. Using the closure of this set of operators, i.e.,

$$\left(\sum_{i=1}^{10} A_i Y_i \right) \left(\sum_{k=1}^{10} B_k Y_k \right) = \sum_{i=1}^{10} C_i Y_i,$$  \hspace{1cm} (12)

we further obtain

$$U_N(T) = \exp(-iH_{\text{eff}}^{UDD-1} T) + O(T^{N+1}),$$  \hspace{1cm} (13)

where

$$H_{\text{eff}}^{UDD-1} = \sum_{i=1}^{10} D_{1,i} Y_i,$$  \hspace{1cm} (14)

with $D_{1,i}$ being the expansion coefficients.

The outcome of applying a UDD sequence of $X_0$ is now evident by comparing the original total Hamiltonian $H$ in the absence of control with the effective Hamiltonian $H_{\text{eff}}^{UDD-1}$ realized by UDD. In essence, the UDD sequence based on $X_0$ efficiently removes the operators $Y_1$, $Y_2$, \ldots, $Y_{10}$ from the initial generating algebra of $H$, thus suppressing all possible coupling between the pre-chosen state $|0\rangle$ and all other states. Therefore, given a known entangled state $|0\rangle$, we can protect this state to the $N$th order with $N$ or $(N + 1)$ instantaneous control pulses, a result analogous to single-qubit UDD.

For later use, we list below three key requirements in achieving a UDD-reduced effective Hamiltonian $H_{\text{eff}}^{UDD-1}$ to the $N$th order, from a general Hamiltonian describing two qubits plus a bath:

(i) Construction of a control operator (e.g., $X_0$) whose square equals the unity operator. This control operator will be used to form a UDD sequence of $N$ instantaneous pulses [e.g., Eq. (13)]

(ii) Separation of the bare system-bath Hamiltonian into two terms, say $H_0$ and $H_1$, with $H_0$ commuting with the control operator and $H_1$ anti-commuting with the control operator.

(iii) Algebra closure of the operators forming $H_0$, which becomes the generating algebra of a UDD-reduced effective Hamiltonian.

In our following considerations we will make a number of references to these three requirements.

III. PROTECTING UNKNOWN TWO-QUBIT ENTANGLED STATES BY NESTED UDD

A. First scheme for nesting three UDD layers

The $X_0$ control operator in Sec. II is based on the knowledge that the state to be preserved is $|0\rangle$. It is even more useful if we can develop a scheme to protect unknown two-qubit entangled states. To that end, let us assume that an unknown two-qubit state $|\psi(0)\rangle$ is a superposition of two orthogonal basis states, e.g., $|0\rangle$ and $|1\rangle$. Though our considerations below are general, to be specific we assume $|0\rangle \equiv |\uparrow\uparrow\rangle$, and $|1\rangle \equiv |\downarrow\downarrow\rangle$, where $\uparrow$ and $\downarrow$ represent spin-up and spin-down states of each qubit. The unknown two-qubit state to be protected can be written as

$$|\psi(0)\rangle = \alpha |0\rangle + \beta |1\rangle = \alpha |\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\rangle,$$  \hspace{1cm} (15)

where the two unknown coefficients $\alpha$ and $\beta$ satisfy $|\alpha|^2 + |\beta|^2 = 1$ at time zero. Can we efficiently protect such type of unknown entangled states by further extending UDD? Note that this problem is different from a single-qubit case because the population can leak out from the initial two-dimensional subspace.

We use $\rho(t)$ to represent the total density matrix of the system and the bath at time $t$, evolving from a direct product state of $|\psi(0)\rangle$ and some initial state of the bath. The protection of the state $|\psi(0)\rangle$ requires to freeze multiple coherence properties, i.e., diagonal populations $\text{Tr}[\rho(t)\uparrow\uparrow\langle\uparrow\uparrow\uparrow\rangle] \approx |\alpha|^2$ and $\text{Tr}[\rho(t)\downarrow\downarrow\langle\downarrow\downarrow\downarrow\rangle] \approx |\beta|^2$, as well as the off-diagonal phase property $\text{Tr}[\rho(t)\uparrow\uparrow\langle\downarrow\downarrow\downarrow\rangle] \approx \alpha \beta^*$. This motivates us to extend the UDD nesting scheme in Ref. 22, where both single-qubit population relaxation and single-qubit transverse dephasing are suppressed in a near-optimal fashion. Certainly, our problem here is more demanding: in single-qubit systems with a two-dimensional Hilbert space, the locking of one projection probability onto one basis state automatically freezes the projection probability onto a second basis state, whereas here the locking of the diagonal probabilities at $|\alpha|^2$ and $|\beta|^2$ should be respectively achieved by control pulses. Given that two layers of UDD sequences are needed for complete single-qubit decoherence control in Ref. 23, it is a natural guess that we will at least need three layers of nested UDD sequences.

We can now directly make use of our results in the previous section. In the first step, we lock the diagonal property $\text{Tr}[\rho(t)\uparrow\uparrow\langle\uparrow\uparrow\uparrow\rangle]$. This can be achieved by considering an innermost layer of UDD sequence of $X_0$, such that all possible coupling between $|0\rangle$ and all other orthogonal states can be efficiently removed. Doing so, we reduce $H$ to $H_{\text{eff}}^{UDD-1}$ to the $N$th order, as elaborated in Sec. II. The population on state $|0\rangle$ is hence locked.

In the second step, we treat a decoherence control problem for the effective Hamiltonian $H_{\text{eff}}^{UDD-1}$. We assume that $H_{\text{eff}}^{UDD-1}$ resulting from the innermost layer of UDD is a sufficiently smooth function of time, such that a second layer of UDD can be applied to $H_{\text{eff}}^{UDD-1}$. Note however, though this is a natural assumption and Ref. 18 reasoned about the smoothness of analogous UDD-reduced effective Hamiltonians, some counter examples might exist 19. With this smoothness assumption exercised with caution, we now introduce a second UDD layer to lock the second diagonal property $\text{Tr}[\rho(t)\downarrow\downarrow\langle\downarrow\downarrow\downarrow\rangle]$. \textit{...}
As is clear from Sec. II, in order to remove all possible couplings between $|1\rangle$ and all other states, one may apply the control operator $X_{1} = 2|1\rangle\langle 1| - I$ with $X_{1}^{2} = I$. To examine if this is feasible, we decompose $H^{\text{UDD-1}}_{\text{eff}}$ into two terms, i.e.,

$$H^{\text{UDD-1}}_{\text{eff}} = \sum_{i=1}^{10} D_{1,i}Y_{i}$$

$$= H^{\text{UDD-1}}_{\text{eff,0}} + H^{\text{UDD-1}}_{\text{eff,1}}, \quad (16)$$

with

$$H^{\text{UDD-1}}_{\text{eff,0}} = \sum_{i=1}^{6} D_{1,i}Y_{i};$$

$$H^{\text{UDD-1}}_{\text{eff,1}} = \sum_{i=7}^{10} D_{1,i}Y_{i}.$$  (17)

Because $[H^{\text{UDD-1}}_{\text{eff,0}}, X_{1}] = 0$ and $\{H^{\text{UDD-1}}_{\text{eff,1}}, X_{1}\}_{+} = 0$, it is seen that $H^{\text{UDD-1}}_{\text{eff}}$ and $X_{1}$ guarantee requirement (ii) outlined in Sec. II. Finally, it is straightforward to see that the operators $Y_{i}, i = 1 - 6$ form a closed algebra [requirement (iii) above]. We hence expect that if a second layer of UDD sequence of $X_{1}$ is applied, to the $N$th order the dynamics of $H^{\text{eff}}_{\text{UDD-1}}$ under the second control layer becomes that of a simpler effective Hamiltonian (denoted $H^{\text{UDD-2}}_{\text{eff}}$) generated by a further reduced algebra. That is,

$$H^{\text{UDD-2}}_{\text{eff}} = \sum_{i=1}^{6} D_{2,i}Y_{i}, \quad (18)$$

where $D_{2,i}$ represent the expansion coefficients due to two UDD layers.

Examining the self-closed set of operators that form $H^{\text{UDD-2}}_{\text{eff}}$, one sees that only the component $D_{2,6}Y_{6} = D_{2,6}[|0\rangle\langle 0| - |1\rangle\langle 1|]$ can affect the initial superposition state $\alpha|0\rangle + \beta|1\rangle$. Further, this $Y_{6}$ component does not change the populations on states $|0\rangle$ and $|1\rangle$, so it represents a pure dephasing mechanism. To efficiently suppress this pure dephasing, we now consider a third, outermost UDD layer. Assuming again that $H^{\text{UDD-2}}_{\text{eff}}$ is sufficiently smooth for UDD to apply, we introduce the following “phase” control operator

$$X_{\phi} = [|0\rangle + |1\rangle][|0\rangle + |1\rangle] - I,$$  (19)

with $X_{\phi}^{2} = I$. Separating $H^{\text{UDD-2}}_{\text{eff}}$ into two terms, we obtain

$$H^{\text{UDD-2}}_{\text{eff}} = H^{\text{UDD-2}}_{\text{eff,0}} + H^{\text{UDD-2}}_{\text{eff,1}}, \quad (20)$$

with

$$H^{\text{UDD-2}}_{\text{eff,0}} = \sum_{i=1}^{5} D_{2,i}Y_{i};$$

$$H^{\text{UDD-2}}_{\text{eff,1}} = D_{2,6}Y_{6}.$$  (21)

Interestingly, $[H^{\text{UDD-2}}_{\text{eff,0}}, X_{\phi}] = 0$ and $\{H^{\text{UDD-2}}_{\text{eff,1}}, X_{\phi}\}_{+} = 0$. Further, the operators $Y_{i}, i = 1 - 5$ form a closed algebra. All the three key requirements for UDD are again met for this outermost layer. The final reduced effective Hamiltonian after three layers of UDD is hence formed by five operators, i.e.,

$$H^{\text{UDD-3}}_{\text{eff}} = \sum_{i=1}^{5} D_{3,i}Y_{i}, \quad (22)$$

where $D_{3,i}$ represent the expansion coefficients due to three UDD layers. Referring to the subspace spanned by $|0\rangle$ and $|1\rangle$, $H^{\text{UDD-3}}_{\text{eff}}$ only contains an identity operator for that subspace. Hence any unknown initial superposition state $\alpha|0\rangle + \beta|1\rangle$ is well preserved to the $N$th order.

In terms of step-by-step reduction of the generating algebra associated with the effective Hamiltonians at each level, the following chart summarizes how our nesting scheme reduces a general total Hamiltonian to a much simplified form:

- For completeness, we also explicitly present here the timing of the control pulses within each UDD layer. In particular, the control operator $X_{0}$ in the innermost layer is applied at

$$T_{j,k,l} = T_{j,k} + (T_{j,k+1} - T_{j,k}) \sin^{2} \left( \frac{l\pi}{2N+2} \right), \quad (23)$$

where $T_{j,k}$ is the UDD timing for $X_{1}$ in the middle layer. $T_{j,k}$ is given by

$$T_{j,k} = T_{j} + (T_{j+1} - T_{j}) \sin^{2} \left( \frac{k\pi}{2N+2} \right), \quad (24)$$

where $T_{j}$ represents the timing for $X_{6}$ in the outermost layer and it is already given by Eq. [10]. Similar to single-qubit cases, for each layer, if $N$ is odd then an additional control operator is applied at the end of each sequence. Overall, $N^{3}$ [or $(N + 1)^{3}$] control pulses are applied to achieve decoherence suppression to the $N$th order. Because states $|0\rangle$ and $|1\rangle$ play a similar role here, our analysis above equally applies if $X_{1}$ and $X_{0}$ are exchanged.
TABLE I: Explicit construction of control operators for nesting three layers of UDD sequences in order to protect unknown two-qubit entangled states $\alpha |0\rangle + \beta |1\rangle$, with $|0\rangle = |\uparrow\uparrow\rangle$ and $|1\rangle = |\downarrow\downarrow\rangle$.

| Constructed Operator | Explicit Form |
|----------------------|--------------|
| $X_0$                | $\frac{(\sigma_1^x \sigma_2^z + \sigma_1^z \sigma_2^x - I)}{2}$ |
| $X_1$                | $\frac{(\sigma_1^x \sigma_2^z - \sigma_1^z \sigma_2^x - I)}{2}$ |
| $X_\phi$             | $\frac{(\sigma_1^x \sigma_2^z - \sigma_1^z \sigma_2^x + \sigma_1^z \sigma_2^z - I)}{2}$ |
| $X_{0,1}$            | $\frac{\sigma_1^z}{2}$ |

To realize such a UDD nesting scheme with this high-order decoherence suppression, the involved two-qubit control operators are nonlocal control operators in general. Taking $|0\rangle = |\uparrow\uparrow\rangle$ and $|1\rangle = |\downarrow\downarrow\rangle$ as an example, Table I lists the three control operators (another operator to be explained later) in terms of the familiar Pauli matrices. The identity operator in the expressions for $X_0$, $X_1$ and $X_\phi$ in Table I is not important. Experimentally, realizing such two-qubit control operators is analogous to realizing quantum computation in a two-qubit system. The true challenge might lie in realizing a sufficient speed of such two-qubit operations.

B. Wrong ordering of three UDD layers

At this point an interesting question arises. That is, does the ordering of the three nested UDD sequences matter or not? To answer this question let us first exchange the ordering of the two sequences of $X_0$ and $X_\phi$, such that $X_\phi$ is in the innermost layer and $X_0$ is in the outermost layer. We denote this ordering as $X_0 - X_1 - X_\phi$. In the following we shall stick to this convention for ordering, i.e., an operator appearing at the rightmost (leftmost) will be placed in the innermost (outermost) layer. Because the $X_\phi$ layer is now operating directly on the bare Hamiltonian $H$, we re-partition $H$ in Eq. (6) into the following two terms, i.e.,

$$H = \left( \sum_{k=1}^{5} \tilde{W}_k \tilde{Y}_k + \sum_{k=7}^{10} \tilde{W}_k \tilde{Y}_k + \tilde{W}_{15} \tilde{Y}_{15} \right) + \left( \tilde{W}_6 \tilde{Y}_6 + \tilde{W}_{16} \tilde{Y}_{16} + \sum_{k=11}^{14} \tilde{W}_k \tilde{Y}_k \right),$$

(25)

where operators $\tilde{Y}_k$ are defined in Eq. (2). The first term in the above equation commutes with $X_\phi$, whereas the second term anti-commutes with $X_\phi$. All the operators contained in the first term form a closed algebra. The innermost UDD sequence of $X_\phi$ hence yields an effective Hamiltonian

$$\tilde{H}_{\text{eff}}^{\text{UDD-1}} = \sum_{i=1}^{5} \tilde{D}_{i,1} \tilde{Y}_i + \sum_{i=7}^{10} \tilde{D}_{i,1} \tilde{Y}_i + \tilde{D}_{1,15} \tilde{Y}_{15}$$

(26)

where $\tilde{D}_{1,i}$ are the expansion coefficients. To consider the second UDD layer, we rewrite $\tilde{H}_{\text{eff}}^{\text{UDD-1}}$ in terms of $Y_k$, i.e.,

$$\tilde{H}_{\text{eff}}^{\text{UDD-1}} = \left( \sum_{i=1}^{5} D'_{i,1} Y_i + \sum_{i=11}^{14} D'_{i,1} Y_i \right) + \left( \sum_{i=7}^{10} D'_{i,1} Y_i + D'_{1,15} Y_{15} \right),$$

(27)

where $D'_{1,i}$ is connected with $\tilde{D}_{1,i}$ via a simple relation between $Y_i$ and $\tilde{Y}_i$ [see Eq. (2)].

The middle layer of UDD sequence would be based on the control operator $X_1$. Among those basis operators that form $\tilde{H}_{\text{eff}}^{\text{UDD-1}}$, $X_1$ commutes with those in the first line of Eq. (27) and anti-commutes with those in the second line of Eq. (27). This indicates that the final evolution operator associated with this UDD layer can be cast into a form similar to Eq. (10). We next investigate if the key requirement (iii) of UDD can be satisfied. Interestingly and somewhat unexpectedly, this is not the case: operators $Y_i$ with $i = 1, 2, \ldots, 5$ together with $Y_i$ with $i = 11 - 14$ cannot form a closed algebra. For example, $2Y_{11}Y_{12} - Y_2$ will yield $Y_0$, which is already outside this collection of basis operators. As a result, the application of this second control layer will not yield a further reduced effective Hamiltonian. Such a nesting scheme then breaks down due to its wrong ordering! Indeed, in the correctly ordered case, the dephasing operator $Y_0$ will be suppressed by $X_\phi$, but here it resurfaces (from even powers of those operators that commute with $X_1$) after the $X_\phi$ layer is already applied.

Note however, due to this wrong ordering, the undesired operators (such as $Y_6$ that cannot be suppressed by the outermost $X_0$ layer) reemerge from multiplications of a set of basis operators. We hence intuitively expect its impact on decoherence control to be at least a second-order effect. We will come back to this when discussing our numerical results.

One may wonder what happens to other ordering? Because states $|0\rangle$ and $|1\rangle$ play the same role here, there is only one non-equivalent ordering left, i.e., $X_1 - X_\phi - X_0$ (or equivalently, $X_0 - X_\phi - X_1$). The effective Hamiltonian after the innermost layer of UDD sequence is hence still given by $\tilde{H}_{\text{eff}}^{\text{UDD-1}} = \sum_{i=1}^{10} D_{i,1} Y_i$. Since the next layer of control is $X_\phi$, one checks if $H_{\text{eff}}^{\text{UDD-1}}$ can be decomposed into two terms in accord with the properties of $X_\phi$ [to fulfill requirement (ii) outlined above]. Interestingly, though this procedure can be easily done in the entire two-qubit operator space, it cannot be done here within the reduced generating algebra of $H_{\text{eff}}^{\text{UDD-1}}$. That is, among the set of basis operators that form $H_{\text{eff}}^{\text{UDD-1}}$, some operators or their arbitrary combinations are neither commuting nor anti-commuting with $X_\phi$. For example,

$$[Y_7, X_\phi] \neq 0; \quad \{Y_7, X_\phi\}_+ \neq 0.$$
So even without checking if there is a self-closed set of operators that commute with \(X_\phi\), it is already seen that the nesting scheme breaks down directly in this ordering. Thus, out of three non-equivalent ordering of the three control operators considered here, only the ordering advocated in the previous subsection may achieve high-order protection of two-qubit states.

From a more general perspective, the dependence of the controlled dynamics on the ordering of the three UDD layers can be explained as follows. Consider a particular time interval \(\Delta T\) for a time-independent Hamiltonian \(H\), during which each of the two control operators \(A\) and \(B\) \((A^2 = 1\) and \(B^2 = 1\)) are applied twice with a certain ordering. If \(B\) is nested inside, the associated unitary evolution is given by

\[
U_{A \rightarrow B} = AB e^{-iH\Delta T} BA = e^{-i[(AB)H(BA)]\Delta T}; \quad (29)
\]

whereas for the other ordering, the unitary evolution is given by

\[
U_{B \rightarrow A} = B Ae^{-iH\Delta T} AB = e^{-i[(BA)H(AB)]\Delta T}. \quad (30)
\]

As seen from the right hand side of Eqs. (29) and (30), the two ordering leads to two effective Hamiltonians \((AB)H(BA)\) and \((BA)H(AB)\). In general these two effective Hamiltonians are different, thus giving rise to the ordering dependence. Note however, if for a concerned Hilbert subspace \(AB = BA\) or \(AB = -BA\), i.e., if the two control operators commute or anti-commute, then these two effective Hamiltonians are identical and hence the ordering dependence no longer exists. This further explains why we cannot exchange the ordering between \(X_\phi\) and \(X_0\) or the ordering between \(X_\phi\) and \(X_1\), but can exchange the ordering between \(X_0\) and \(X_1\).

C. Alternative Nesting Scheme

For the two-dimensional subspace spanned by states \(|0\rangle\) and \(|1\rangle\), any two orthogonal states \(|0'\rangle\) and \(|1'\rangle\) can be adopted to construct analogous control operators \(X_0'\), \(X_{1'}\), and \(X_{\phi'}\) for three UDD layers. Therefore, when it comes to an actual implementation, there are infinite possibilities to realize three nested UDD layers in order to protect an unknown entangled state in that subspace.

Even for fixed basis states \(|0\rangle\) and \(|1\rangle\), the nesting scheme with the right ordering analyzed above is not the only solution. Consider the following control operator

\[
X_{0,1} = 2(|0\rangle\langle 0| + |1\rangle\langle 1|) - I = X_0 + X_1 + I \quad (31)
\]

with \(X_{0,1}^2 = I\). The explicit form of \(X_{0,1}\) is also given in Table I. Following the analysis in Sec. II, it is straightforward to confirm that a UDD sequence of \(X_{0,1}\) can efficiently freeze the total population in the two-dimensional subspace spanned by \(|0\rangle\) and \(|1\rangle\). We can now construct an alternative nesting scheme using \(X_{0,1}\), \(X_\phi\) and \(X_1\) (or \(X_0\)).

The ordering of \(X_\phi - X_1 - X_{0,1}\) is studied first. It is found that each of the three UDD layers satisfies the three requirement outlined in Sec. II and hence yields a simple effective Hamiltonian to the \(N\)th order. Qualitatively, it is also obvious why this scheme is expected to work. The innermost layer effectively lock the population in a two-dimensional Hilbert subspace. Then, in essence, the next two layers are similar to those in single-qubit two-layer UDD, insofar as a second layer locks one population and the outermost layer freezes the relative phase between the two projection amplitudes. Extending this analogy, it is expected that the ordering of the middle layer and the outermost layer can be exchanged (indeed, the associated control operators anti-commute in the two-dimensional subspace). This can be more formally analyzed by working out the detailed algebra layer-by-layer. In particular, for the ordering of \(X_\phi - X_1 - X_{0,1}\), the generating algebra for the effective Hamiltonians after each control layer reduces in the following fashion:

\[
Y_{i}, i = 1, 2, \cdots, 16
\]

\[
\downarrow X_{0,1}, \text{ UDD-1}
\]

\[
Y_{i}, i = 1, 2, \cdots, 6; Y_{15}, Y_{16}
\]

\[
\downarrow X_{1}, \text{ UDD-2}
\]

\[
Y_{i}, i = 1, 2, \cdots, 6
\]

\[
\downarrow X_{\phi}, \text{ UDD-3}
\]

\[
Y_{i}, i = 1, 2, \cdots, 5
\]

For the ordering of \(X_1 - X_\phi - X_{0,1}\), we have

\[
Y_{i}, i = 1, 2, \cdots, 16
\]

\[
\downarrow X_{0,1}, \text{ UDD-1}
\]

\[
Y_{i}, i = 1, 2, \cdots, 6; Y_{15}, Y_{16}
\]

\[
\downarrow X_{\phi}, \text{ UDD-2}
\]

\[
Y_{i}, i = 1, 2, \cdots, 5; Y_{15}
\]

\[
\downarrow X_{1}, \text{ UDD-3}
\]

\[
Y_{i}, i = 1, 2, \cdots, 5
\]

As seen above, both nesting strategies are successful and in the end the same generating algebra for the final effective Hamiltonian is reached.

Can we place a sequence of \(X_{0,1}\) in the middle layer instead? For the ordering of \(X_\phi - X_{0,1} - X_1\), since the population on state \(|1\rangle\) is already locked by the innermost layer, the role of the second layer is equivalent to further locking the population on state \(|0\rangle\), thus playing a similar role as \(X_0\). The outermost layer then freezes the
relative phase between the two projection amplitude analogous to the case of $X_\phi - X_0 - X_1$. In this sense, tl ordering of $X_\phi - X_{0,1} - X_1$ does not provide any new. Indeed, based on the discussion at the end of tl last subsection, because $[X_{0,1}, X_1] = 0$, their ordering expected to be exchangeable.

Consider then the other ordering $X_1 - X_{0,1} - X_\phi$ (which is also equivalent to $X_0 - X_{0,1} - X_\phi$). In this case, tl effective Hamiltonian reduced by the innermost layer is expected to be exchangeable.

where the first term commutes with $X_{0,1}$ and the second term anti-commutes with $X_{0,1}$. As an interesting outcome, now the set of operators that commute with the second-layer control operator also form a closed algebra, thus paving the way for the third UDD layer. Indeed, the outermost $X_0$ layer further reduces the algebra by removing the $Y_{15}$ component and hence yields an effective Hamiltonian seen before. Summarizing, the explicit algebra reduction route for $X_1 - X_{0,1} - X_\phi$ becomes

\[ Y_i, \ i = 1, 2, \cdots, 16 \]
\[ \downarrow X_{0,1}, \text{UDD-1} \]
\[ Y_i, \ i = 1, \cdots, 5; \ Y_i, \ i = 7, 8, \cdots, 15 \]
\[ \downarrow X_0, \text{UDD-2} \]
\[ Y_i, \ i = 1, 2, \cdots, 5; \ Y_{15} \]
\[ \downarrow X_1, \text{UDD-3} \]
\[ Y_i, \ i = 1, 2, \cdots, 5 \]

We have also examined what happens if the $X_{0,1}$ layer is placed in the outermost layer. For reasons analogous to our first nesting scheme using $X_0$, $X_1$ and $X_\phi$, such type of ordering cannot simplify the effective Hamiltonians layer-by-layer. This concludes this subsection.

D. Numerical study

Similar to our previous work [22], we use a five-spin system to carry out simple numerical experiments. Two of the five spins are identified as our two-qubit system, and the other three spins are regarded as the bath. To avoid assumptions about how the system is coupled to the bath, we work with the following general total Hamiltonian:

\[ H = \sum_{m=1}^{5} \sum_{\gamma=(x,y,z)} b_{\gamma,m} \sigma_\gamma^m + \sum_{m=1}^{5} \sum_{\gamma=(x,y,z)} \sum_{n>m} \sum_{\beta=(x,y,z)} c_{\gamma\delta}^{mn} \sigma_\gamma^m \sigma_\delta^n \] (33)

in dimensionless units, where all the coefficients $b_{\gamma,m}$ and $c_{\gamma\delta}^{mn}$ are randomly sampled from the range $[-0.5, 0.5]$. We average our results over ten random realizations of this five-spin system-bath Hamiltonian. In addition, to demonstrate that our approach does not depend on the actual form of an initial superposition state $\alpha | \uparrow\uparrow \rangle + \beta | \downarrow\downarrow \rangle$, we further average our results over ten initial states with randomly sampled coefficients $\alpha$ and $\beta$ under the constraint $|\alpha|^2 + |\beta|^2 = 1$.

Figure 1 depicts the averaged trace distance (denoted $D$) between the system’s reduced density matrix at time $t = 0.1$ and its initial state, for $N = 1 - 10$. Different ordering of three UDD sequences $X_0$, $X_1$, and $X_\phi$ are plotted together for comparison. Consider first the two cases (bottom two curves) with the correct ordering, i.e., $X_\phi - X_1 - X_0$ and $X_0 - X_\phi - X_1$. For these two cases a remarkably high fidelity is achieved in locking the initial
unknown superposition state. For \( N = 10 \) (totally \( N^3 \) UDD pulses), \( D \) already reaches the \( 10^{-10} \) level. The almost linear scaling of \( \log(D) \) vs \( N \) is consistent with the expectation that the extent of the decoherence suppression for a working nesting scheme is to the \( N \)th order.

Turning to the top two flat curves associated with the wrong ordering \( X_1 - X_\phi - X_0 \) and \( X_0 - X_1 - X_\phi \). Their \( D \) values do not decrease with \( N \) and stay at about \( 10^{-1} \). Therefore, for these two cases the three-layer nested UDD does not work at all due to the wrong ordering. This directly confirms our early insights into the issue. In particular, from the algebra considerations we observe that for the incorrect ordering here, the second layer of UDD directly breaks down because the effective Hamiltonian reduced from the innermost layer does not meet requirement (ii).

Still referring to Fig. 1, let us now discuss the middle curves associated with another type of wrong ordering, i.e., \( X_1 - X_0 - X_\phi \) and \( X_0 - X_1 - X_\phi \). It is seen that their \( D \) values first decrease and then tend to saturate as \( N \) increases. The smallest \( D \) values for \( N = 10 \) is about \( 10^{-4} \), which is about six orders of magnitude larger than in previous correctly ordered cases. However, this performance is at the same time better than the top two flat curves. As such, the wrong ordering here represents a weak deviation from an ideal nesting. This is consistent with our early intuition that for the current ordering, the UDD nesting scheme breaks down due to a high-order effect, i.e., the non-closure of a set of operators that commute with the control operator in the middle layer.

As a comparison with a conventional dynamical decoupling approach based on control pulses equally spaced in time, Fig. 2 presents the parallel results if, within each layer, the control operator is applied periodically. Clearly, in this case, irrespective of the ordering of the control operators, the performance of decoherence control for \( N = 10 \) in all cases is about nine orders of magnitude worse than the best two cases in Fig. 1. It is also observed that the \( D \) values are very weakly dependent on \( N \). Results here remind us that in addition to the ordering of the three layers, the timing of the control operators is essential.

Finally, results for an alternative UDD nesting scheme based on \( X_{0,1} \), \( X_1 \), and \( X_\phi \) are shown in Fig. 3. The top two curves are for two incorrect ordering \( X_{0,1} - X_1 - X_\phi \) and \( X_0 - X_{0,1} - X_1 \), with their \( D \) values saturating at about \( 10^{-4} \) as \( N \) increases. It is noted that the \( X_{0,1} - X_1 - X_\phi \) curve is similar to the \( X_0 - X_1 - X_\phi \) case in Fig. 1. This is understandable because the underlying mechanism for unsuccessful nesting is the same. However, the \( X_{0,1} - X_\phi - X_1 \) case here has better performance than the \( X_0 - X_\phi - X_1 \) case in Fig. 1. This is interesting because in both cases, the effective Hamiltonian reduced from the innermost layer does not satisfy requirement (ii) for the second layer. Clearly then, for the outermost layer, a sequence of \( X_{0,1} \) turns out to be superior to a sequence of \( X_0 \). This is somewhat expected because \( X_{0,1} \) in the outermost layer can still freeze the total population in the two-dimensional subspace whereas \( X_0 \) cannot.

All other four curves in Fig. 3 display a roughly linear scaling of \( \log(D) \) vs \( N \), indicating the success of three-layer nested UDD. Indeed, the layer ordering associated with these curves is all predicted to be correct in our theoretical analysis above. Interestingly, for fixed \( N \), the performance for the ordering of \( X_\phi - X_1 - X_{0,1} \) or \( X_\phi - X_{0,1} - X_1 \) can differ from that for \( X_1 - X_\phi - X_{0,1} \) or \( X_1 - X_{0,1} - X_\phi \) by about two orders of magnitude. Comparing with the best performance here with that in Fig. 1, a difference about two orders of magnitude is also observed. These numerical details indicate that even with the same scaling with \( N \), the actual performance of a correctly ordered three-layer UDD may depend on the specific algebra reduction route. More insights into this intriguing finding might help to further understand the nature of decoherence dynamics under nested multi-layer UDD.

**IV. FROM THREE-LAYER UDD TO FOUR-LAYER UDD**

We are optimistic that for \( N \sim 10 \) a total of \( N^3 \) UDD control pulses as proposed in this work might be achievable in some systems in the near future. If this is the...
case, then as the next step one wonders if there exist even better schemes, at least in theory. In particular, as a result of three correctly ordered UDD layers, the generating algebra for the final effective Hamiltonian $H_{\text{eff}}^{\text{UDD-3}}$ contains only five operators. Can we construct better control operators to reduce the algebra more rapidly? Can we even consider one more UDD layer to remove all possible system-environment coupling?

As a brief summary of our latest progress along these two questions, we first note that, the two nesting schemes proposed in Sec. III treat the subspace spanned by $|0\rangle$ and $|1\rangle$ differently than the subspace spanned by $|2\rangle$ and $|3\rangle$. Indeed, we have assumed that the initial state is a superposition of states $|0\rangle$ and $|1\rangle$. If the initial state is totally unknown, then it is best to construct control operators that are symmetric with respect to the two subspaces. Upon completion of our studies of the two nesting schemes discussed in Sec. III, we find that the following three symmetry-adapted control operators can form another nesting scheme for three-layer UDD, i.e.,

$$
Z_1 \equiv |0\rangle\langle 0| + |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3| = X_{0,1},
$$

$$
Z_2 \equiv |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2| - |3\rangle\langle 3|
$$

$$
Z_3 \equiv |0\rangle\langle 1| + |1\rangle\langle 0| + |2\rangle\langle 3| + |3\rangle\langle 2|.  \quad (34)
$$

That is, for each UDD layer, the three key requirements of UDD outlined in Sec. II are satisfied. Dramatically, the form of the three control operators $Z_1$, $Z_2$, and $Z_3$ is explicitly symmetric with respect to an exchange between the ($|0\rangle$, $|1\rangle$) subspace and the ($|2\rangle$, $|3\rangle$) subspace.

So what happens to the first subspace also applies to the second subspace. Their physical meaning is also clear: $Z_1$ locks the total population within each subspace spanned by $|0\rangle$ and $|1\rangle$ or by $|2\rangle$ and $|3\rangle$. $Z_2$ locks the individual populations on each state, and $Z_3$ finally suppresses the pure dephasing within each of the two subspaces of dimensional two. Therefore, this symmetry-adapted three-layer UDD scheme should have more efficiency in reducing the algebra layer-by-layer. For the ordering of $Z_3 - Z_2 - Z_1$, the associated algebra reduction route is found to be:

$$
\downarrow Z_1, \text{UDD-1} \\
Y_i, i = 1, 2, \cdots, 16 \\
\downarrow Z_2, \text{UDD-2} \\
Y_1, Y_2, Y_3, Y_6 \\
\downarrow Z_3, \text{UDD-3} \\
Y_1, Y_2 \\
\downarrow UDD-3
$$

The final effective Hamiltonian $H_{\text{eff}}^{\text{UDD-3}}$ after such three UDD layers is hence a combination of only two operators: $Y_2$ and the unity operator $Y_1$ (or equivalently, $|2\rangle\langle 2| + |3\rangle\langle 3|$ and $Y_1$). Note that the dephasing between the ($|0\rangle$, $|1\rangle$) subspace and the ($|2\rangle$, $|3\rangle$) subspace is still not suppressed. Perhaps even more remarkable, for these three symmetry-adapted control operators, they either commute or anti-commute, and consequently different orderings of $Z_1$, $Z_2$ and $Z_3$ can produce the same final generating algebra.

One can further rewrite $H_{\text{eff}}^{\text{UDD-3}}$ in a more enlightening and symmetric form, i.e.,

$$
H_{\text{eff}}^{\text{UDD-3}} = D_{3,1}^Z I + D_{3,2}^Z |0\rangle\langle 0| + |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3|,  \quad (35)
$$

This finally brings us to our last theoretical question: can we further reduce $H_{\text{eff}}^{\text{UDD-3}}$ by adding one more UDD layer? Our answer is yes in theory. This is somewhat obvious if one introduces the fourth control operator

$$
Z_4 \equiv |0\rangle\langle 2| + |2\rangle\langle 0| + |1\rangle\langle 3| + |3\rangle\langle 1|  \quad (36)
$$

with $Z_4^2 = 1$. Clearly, because $Z_4$ anti-commutes with the second operator in Eq. (35) and commutes with the identity operator, the fourth UDD layer based on $Z_4$ will finally yield an effective Hamiltonian as a certain bath operator multiplied by a unity-operator in the four-dimensional Hilbert space for a two-qubit system! So it is theoretically possible to protect a totally unknown two-qubit state against most general system-environment coupling, using about $N^4$ control pulses in total. Further, since this four-layer scheme is intended to lock any two-qubit state, one may now arbitrarily choose the four orthogonal basis states in order to simplify the four control
operators (this is not allowed in three-layer UDD because a two-dimensional subspace is chosen beforehand). In particular, if we consider a new set of basis states different than above, e.g., \(|0\rangle = |\uparrow\downarrow\rangle, |1\rangle = |\uparrow\rangle, |2\rangle = |\downarrow\rangle,\) and \(|3\rangle = |\downarrow\downarrow\rangle,\) then one obtains \(Z_1 = \sigma_z^1, Z_2 = \sigma_z^2, Z_3 = \sigma_z^3,\) and \(Z_4 = \sigma_z^4,\) which are only local control operators in this new representation. Such a four-layer solution is also numerically checked.

V. CONCLUSION

With both theoretical analysis and numerical study, we have shown how nested three-layer UDD can protect unknown two-qubit entangled states as a superposition of two known basis states, to the \(N\)th order with about \(N^3\) control pulses, without assuming a specific form of system-environment coupling. This is of much interest to current theoretical investigations of entanglement protection. As a remarkable side result, it is found that the ordering of the three UDD layers can be a crucial factor. Numerical results support our theoretical considerations.

Given the theoretical feasibility of extending UDD beyond single-qubit systems, decoherence control via nested UDD should be of experimental interest as well. Though a rigorous mathematical foundation for nested UDD is still under development \([19, 25]\), the success of three-layer UDD demonstrated here in two-qubit systems further strengthens the view that nested UDD can be a good strategy for decoherence suppression. Our approach can also be extended to protect an unknown superposition of two known basis states in an arbitrary multi-level system.

As a final extension, in Sec. IV we also discussed how a totally unknown two-qubit entangle state can be protected by applying four layers of UDD sequences that involve local operations only. Applying \(N^4\) pulses can be highly demanding in experiments, but the existence of such a theoretical solution should offer a useful reference point for future studies of entanglement protection. On the other hand, it becomes interesting to compare our main topic of this work, i.e., three-layer UDD, with the ultimate four-layer solution. The four-layer solution can be realized by local control operators but the required number of pulses may present experimental difficulties. By treating a particular class of two-qubit states, three-layer UDD can achieve entanglement protection of unknown states using much less control pulses, with the price that rapid nonlocal operations are required.

VI. ACKNOWLEDGMENTS

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