Quantum limits on the detection sensitivity of a linear detector with feedback

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Abstract

We show that the detection sensitivity of a linear detector is lower bounded by some quantum limits. For the force sensitivity, which is relevant for atomic force microscopes, the lower bound is given by the so-called ultimate quantum limit (UQL). For the displacement sensitivity, which is relevant for detecting gravitational waves, a generalized lower bound that can overcome the usual UQL is obtained.

Keywords: quantum feedback, detection sensitivity, optomechanical system, quantum limits

(Some figures may appear in colour only in the online journal)

1. Introduction

It is known that an optomechanical system, i.e., a mechanical oscillator coupled to an optical cavity by radiation pressure, can be a sensitive detector for very weak forces or tiny displacements. An example is atomic force microscopes [1], where the mechanical oscillator is displaced by a weak force. In the case of interferometers for detecting gravitational waves [2], the cavity length is changed by the passage of gravitational waves. The corresponding detection sensitivities are called the force sensitivity and the displacement sensitivity, respectively. For a typical optomechanical detector, both are lower bounded by the so-called standard quantum limit (SQL) due to quantum noises [3-5].

However, the SQL itself is not a fundamental quantum limit. Different schemes to overcome the SQL have been proposed, such as frequency-dependent squeezing (FDS) of the input beam [6, 7], cavity detuning (CD) [8], variational measurement (VM) [9], quantum locking of the mirror [10], coherent quantum noise cancellation (CQNC) [11], etc. It was shown in [12] that the fundamental quantum limit for the force sensitivity of a linear detector is given by the so-called ultimate quantum limit (UQL) [7, 8] if the external classical force is only coupled to the position operator of the mechanical oscillator. This UQL is related to the dissipation mechanism of the oscillator, via the absolute value of the imaginary part of the inverse mechanical susceptibility. It also holds for cases with coherent quantum control and/or with simple direct quantum feedback [13, 14]. In this paper, we try to extend the result in [12] to cases with a more complicated quantum feedback loop, such as quantum locking of the mirror via a control cavity [10].

On the other hand, for the displacement sensitivity, we find that the usual UQL, given by the absolute value of the imaginary part of the mechanical susceptibility, can be overcome by some devised schemes. Moreover, a generalized quantum limit on the displacement sensitivity can be obtained by following similar arguments to those used for the force sensitivity.

The paper is organized as follows. In section 2, we prove the UQL for the force sensitivity of a linear detector in the presence of direct quantum feedback by using the general linear-response relations. This is then used to establish a general quantum limit on the displacement sensitivity. Section 3 contains two illustrative examples for the above results. Finally, a short summary is given in section 4.

2. Detection sensitivity of a linear detector with feedback

We consider a generic linear-response detector (see figure 1(a)). It is described by some unspecified Hamiltonian, $H_D$, and has both an input operator, represented by an operator $F$, and an output operator, represented by an operator $\mathcal{Z}$. We first consider the quantum limit on the force sensitivity, where the detector is used to estimate a classical force $f(t)$ applied on a mechanical oscillator, which is represented by the Hamiltonian $H_m$. The input operator $F$ is coupled with the mechanical oscillator via the interaction Hamiltonian,
This then gives an unbiased estimator $f_{\text{est}}$ of $f$, $f_{\text{est}} = f + \hat{f}$, where the added noise is defined via

$$q_j = \chi_{qq} \hat{f} = q_0 + g \chi_{qq} F_0 + \frac{Z_0}{g} (1 - g^2 \chi_{qq} \chi_{FF}).$$

Here the first term is the intrinsic mechanical noise, the second term is the backaction noise from the oscillator, and the third term is the shot noise at the output. Neglecting the intrinsic mechanical noise from now on, the force sensitivity $S_f$ can be characterized by the power spectrum $S_{ijj}$ defined by the correlation

$$\frac{1}{2} \langle O_i(\omega) O_j^\dagger(\omega') + O_j(\omega') O_i(\omega) \rangle = S_{iOj} \delta(\omega - \omega').$$

After the optimization of $S_{ijj}$ over the coupling strength $g$, the uncertainty relations for $F_0$ and $Z_0$ give [12]

$$S_f = S_{ijj} \geq |\xi_{ijj}|^2,$$

where $\xi_{ijj}$ is the imaginary part of the inverse susceptibility $\chi_{ijj} = 1/\chi_{ij} = \chi_{ij}^R + i \chi_{ij}^I$. This is the UQL for the force sensitivity.

The above result incorporated the effect of coherent quantum control [13], such as the CNQC scheme. As for the direct quantum feedback control [14], a control signal $\lambda(t)$ is fed back to the system, see figure 1(a). For a generic operator $O$, this introduces an additional term to the equation of motion,

$$\dot{O}_b(t) = i \int_{-\infty}^t dt' \hat{\lambda}(t - t') \langle \lambda(t')[P(t'), O(t)] \rangle,$$

where $\hat{\lambda}(t)$ is the feedback transfer function and $P(t)$ is the control operator. If the linear control operator $P$ is of the mechanical oscillator, $[P, q] = \text{const.}$ and $[P, F] = 0$. In the frequency domain, we have $q_{fb} = \lambda Y$, where $\lambda$ is the rescaled transfer function.

It was shown in [12] that if the measured signal is the same as the control signal fed back to the system ($Z = \lambda$), the force sensitivity does not change. On the other hand, if the measured signal is different from the control signal ($Z = \lambda$), as in the cases of quantum locking of the mirror, the equation of motion becomes

$$q = q_0 + q_f + g \chi_{qq} F + \lambda Y,$$

$$F = F_0 + g \chi_{FF} q,$$

$$Y = Y_0 + g q,$$

$$Z = Z_0 + g q.$$

We find that

$$q = \frac{q_0 + q_f + \lambda Y_0 + g \chi_{qq} F_0}{1 - g \lambda - g^2 \chi_{qq} \chi_{FF}}.$$

Then the estimator of $f$ deduced from the measured signal $Z$ is $f_{\text{est}} = f + \hat{f}$ with the added noise $\hat{f} = q_j / \chi_{qq}$, where

$$q_j = q_0 + g \chi_{qq} F_0 + \lambda Y_0 + \frac{Z_0}{g} (1 - g \lambda - g^2 \chi_{qq} \chi_{FF}).$$

We find that for $Z = Y$, the above equation reduces to equation (4).
To find a lower bound to the power spectrum of \( \hat{f} \), we first optimize \( S_{jj} \) over the transfer function \( \lambda \) and the coupling strength \( g \), respectively. This leads to

\[
S_{jj} \geq \frac{2(A_{qq} + B_{qq} l + |\lambda_{qq}|^2)}{|\lambda_{qq}|^2 (S_{YY} + S_{ZZ})},
\]

where the notations are

\[
A = S_{2Y} S_{YY} + S_{1Y} S_{ZZ} - \chi_{FF} S_{YY} S_{ZZ},
\]

\[
B = S_{2Y} S_{YY} + S_{1Y} S_{ZZ} + \chi_{FF} S_{YY} S_{ZZ},
\]

\[
C = |S_{YY} (S_{YY} + S_{ZZ}) + \chi_{FF}^2 S_{YY} S_{ZZ},
\]

\[
- (S_{YY}^l - S_{YY}^l)^2 - (S_{FF}^l - \chi_{FF}^l S_{YY} S_{ZZ}^l)
\]

\[
+ 2S_{YY} \chi_{FF}^l S_{YY}^l - \chi_{FF}^l \chi_{FF}^l S_{YY} S_{ZZ}^l.
\]

Next, we note that \([Y_0(t), Y_0(t')] = [Z_0(t), Z_0(t')] = 0\) at all times, in order for \(Y_0(t)\) and \(Z_0(t)\) to represent experimental data strings. This immediately implies that \(\chi_{YY} = \chi_{ZZ} = 0\). In addition, the causality principle imposes that the outputs \(Y_0(t)\) and \(Z_0(t)\) should not depend on the input \(F_0(t)\) for \(t < t'\), and therefore \(\chi_{YY}(\omega) = \chi_{ZZ}(\omega) = 0\). Furthermore, \(Y_0, Z_0, F_0\) should satisfy the uncertainty relations \([3, 12]\) implied by the positivity of the matrix \(M_k = S_k + i[\kappa_k - \kappa_k^*]/2\) with the indexes \(k = Y_0, Z_0, F_0\).

\[
M = \begin{bmatrix}
S_{YY} & 0 & S_{YF} + i/2 \\
0 & S_{ZZ} & S_{ZF} + i/2 \\
S_{YF} + i/2 & S_{ZF} + i/2 & S_{FF} + \chi_{FF}^l
\end{bmatrix}
\geq 0,
\]

or equivalently,

\[
S_{YY} + S_{ZZ} \geq \frac{S_{YY} + S_{ZZ} + (S_{YY}^l S_{YY} + S_{YY}^l S_{YY} S_{ZZ})}{4}
\]

\[
+ |S_{YY} S_{YY} S_{YY} + |S_{YY} S_{YY} S_{ZZ} + \chi_{FF}^l S_{YY} S_{ZZ}^l.
\]

The relations \(\chi_{YY} - \chi_{YY} = \chi_{ZZ} - \chi_{ZZ} = 1\) and \(S_k = S_k^*\) are used here.

Substituting these inequalities into equation (12), through some calculations we find

\[
C \geq A^2 + \left( B + \frac{S_{YY} + S_{ZZ}}{2} \right)^2
\]

\[
= A^2 + \left( |B| + \frac{S_{YY} + S_{ZZ}}{2} \right)^2.
\]

Then, from equation (11) we obtain the force sensitivity

\[
S_\ell = S_{jj} \geq |\lambda_{qq}|^2.
\]

which is the main result of this paper. We use the inequalities

\[
a_1 \chi_1 + \sqrt{(a_1^2 + a_2^2) \chi_1^2 + \chi_2^2} \geq |a_2 \chi_2| \geq |a| \geq a \text{ here.}
\]

Similar results can be obtained if the control operator \(P\) is from the detector. The UQL is thus established in the presence of direct quantum feedback.

As for the displacement sensitivity, the cavity length is changed by the passage of gravitational waves. Or, in other words, \(q \rightarrow q + \delta q\) after the passage of a gravitational wave, irrespective of the mechanical oscillator. For a traditional optomechanical detector, it is known that the displacement sensitivity is bounded by the SQL. Whether the displacement sensitivity is lower bounded by \(|\lambda_{qq}|^2\), namely, the UQL for the displacement sensitivity, has still not been generally established. This UQL was proved in the weak coupling limit and/or for a detector with a large power gain \([3, 5]\), and was also indicated by some numerical calculations \([8]\).

The displacement sensitivity \(S_q\) is given by the power spectrum of the added noise \(\hat{q}\) in the estimator \(\delta q_{est}\) of \(\delta q\). It is deduced from the measured signal \(Z = Z_0 + g(q + \delta q)\) and \(\delta q_{est} = Z/g = \delta q + \hat{q}\), where the position operator \(q\) is given by equation (9) with \(q_f = 0\). The added noise is found to be

\[
\hat{q} = \frac{q_j}{1 - g \lambda - g^2 \chi_{qq}\chi_{FF}},
\]

where \(q_j\) is given by equation (10). Then we obtain

\[
S_q = S_{qq} = \frac{|\lambda_{qq}|^2 S_{jj}}{1 - g \lambda - g^2 \chi_{qq}\chi_{FF}}
\]

\[
\geq \frac{|\lambda_{qq}|^2}{1 - g \lambda - g^2 \chi_{qq}\chi_{FF}}.
\]

where equation (16) is used. Therefore, the displacement sensitivity can overcome the SQL if the detector parameters are adjusted properly to make the denominator of equation (18) larger than one, as illustrated in the subsequent examples. We also note that in the weak coupling limit the denominator of the above equation is approximately equal to one, and equation (18) reduces to the UQL.

3. Cavity detuning and quantum locking of the mirror

To exemplify the results in the last section, we take the optomechanical detector as an example. The optomechanical detector consists of a high-quality Fabry–Perot cavity, with a fixed transmissive mirror in front of the cavity, and a movable, perfectly reflecting mirror \(m\) at the back (see figure 1(b)).

The cavity field described by the annihilation operator \(b = (b_1 + i b_2)/\sqrt{2}\) with resonant frequency \(\omega_b\) is fed with a driving laser \(\beta_m\). The aim is to estimate a classical force \(f(t)\) acting on the movable mirror described by the annihilation operator \(a = (q + ip)/\sqrt{2}\) with frequency \(\Omega\). In the rotating frame at the frequency \(\omega_m\) of the driving laser, the system is described by the Hamiltonian,

\[
H = H_m + H_D + H_{int}
\]

\[
= \Omega a^\dagger a + [\Delta b^\dagger b + i \sqrt{\gamma}(\beta_m b^\dagger - \beta_m^* b)]
\]

\[\quad - q [f + g_{om}(b^\dagger b - (b^\dagger b))] \],

where \(\Delta = \omega_b - \omega_m\) is the cavity detuning and \(g_{om}\) is the optomechanical coupling strength. Taking into account the thermal noises, the equations of motion are given by the quantum Langevin equations \([15]\),

\[
\dot{a} = -i \Omega a - \frac{\Gamma}{2} a + [f + g_{om}(b^\dagger b - (b^\dagger b))] / \sqrt{2}
\]

\[+ \sqrt{\Gamma} a_{in},\]

\[
\dot{b} = -i \Delta b - \frac{\gamma}{2} b + i g_{om} q b + \sqrt{\gamma} (\beta_m + b_m).
\]
where \( \Gamma(\gamma) \) and \( \alpha_{in}(b_{in}) \) are the decay rate and thermal noise operator for the oscillator (cavity), respectively. The noise correlations are given by \( \langle a_{in}(t) a_{in}^*(t') \rangle = (n_{th} + 1) \delta(t - t') \) and \( \langle b_{in}(t) b_{in}^*(t') \rangle = \delta(t - t') \), where \( n_{th} \) is the thermal occupancy of the mechanical reservoir.

Under the condition of strong laser driving, we can linearize equation (20) around the steady state, \( \langle a \rangle = 0 \) and \( \langle b \rangle = \beta = \sqrt{\gamma} \beta_{in}/(\gamma/2 + i\Delta) \), by splitting \( a \rightarrow \langle a \rangle + a \) and \( b \rightarrow \langle b \rangle + b \). Neglecting the nonlinear terms, we have

\[
\mathbf{x} = \mathbf{Ax} + \mathbf{w},
\]

where the variables \( \mathbf{x} = (q, p, b_1, b_2) \), the input \( \mathbf{w} = (\sqrt{\Gamma} q_{in}, f + \sqrt{\Gamma} p_{in}, \sqrt{\gamma} b_1^1, \sqrt{\gamma} b_2^1) \), and the matrix

\[
\mathbf{A} = \begin{pmatrix}
\frac{\Gamma}{2} & \Omega & 0 & 0 \\
-\Omega & \frac{\Gamma}{2} & g & 0 \\
0 & 0 & -\frac{\gamma}{2} & \Delta \\
g & 0 & -\Delta & -\frac{\gamma}{2}
\end{pmatrix}
\]

(22)

in terms of the effective optomechanical coupling strength \( g = \sqrt{2} g_{om} \beta \). The stability of this linearized system is guaranteed by the requirement that the real part of all the eigenvalues of \( \mathbf{A} \) must be nonpositive. The classical force is then estimated from the output current \( I_{out} \) of a photodiode that is linearly proportional to a certain optical quadrature of the output field, \( I_{out} \propto Z = b_{out}^1 \sin \phi + b_{out}^2 \cos \phi \), where the output field is obtained by the input-output relation \( b_{out} = \sqrt{\gamma} b - b_{in} \) and \( \phi \) is the adjustable readout quadrature angle via the local oscillator phase. For the stationary state, equation (21) can be solved simply in the frequency domain. The corresponding numerical results for the detection sensitivities are shown in figure 2. It can be seen that for a detuned cavity, the force sensitivity is lower bounded by the relevant UQL, \( S_f \geq \left| \chi_f \right|^2 = \omega_\Gamma/\Omega \) with \( \chi_f = \Omega(\Gamma/2 - i\omega)^{1/2} + \Omega^2 \)^{-1}, and the displacement sensitivity can overcome the corresponding UQL, namely, \( S_d < \left| \chi_d \right|^2 \).

For the displacement sensitivity, it is also found that the usual UQL is still valid for a resonant cavity. This is because the susceptibility \( \chi_{ff} \propto \Delta \) with the operator \( F = b_1 \) for the optomechanical detector, and the denominator in equation (18) becomes unity for \( \Delta = 0 \). However, there is another possibility for overcoming the usual UQL for a resonant cavity with the help of quantum feedback (\( \lambda \neq 0 \)), such as quantum locking of the mirror. In this scheme, the addition of a feedback loop is used to suppress the radiation pressure effects by freezing the motion of the mirror. The mirror motion is monitored by a feedback force \( \propto \gamma \propto c_1^1 \sin \theta + c_2^1 \cos \theta \) from another resonant cavity made up of the movable mirror \( m \) and a fixed transmissive reference mirror. The control system is governed by the Hamiltonian,

\[
H_c = \hbar \omega_c (\beta_{in}^+ c - \beta_{in}^+ c) + g_{om} q (c^1 c - (c^1 c)),
\]

where the control cavity field \( c \) with frequency \( \omega_c \) is in resonance with the driving laser \( \beta_{in}^+ \), and takes the same decay rate \( \gamma \) as the main cavity.

**Figure 2.** Plots of the force sensitivity (a) and the displacement sensitivity (b) with respect to the detection frequency for the optomechanical detector. Grey: SQL; green: UQL; blue: detuned cavity for \( \gamma = 2 \), \( g = 5 \), \( \Delta = -5 \), and \( \tan \phi = -1 \); and red: resonant cavity with feedback for \( g = g/2 = 1 \), \( \phi = 0 \), \( \gamma = 5 \), \( \tan \theta = 2 \), and \( \gamma = 2 \), \( \theta = 0 \). Here the detection sensitivities are optimized over the feedback transfer function \( \lambda(\omega) \). The common parameters are \( \Omega = \Gamma = 0.1 \) and \( n_{th} = 0 \).

The linearized equations of motion take the same form as equation (21), where

\[
\mathbf{x} = (q, p, b_1, b_2, c_1, c_2) \mathbf{y},
\]

\[
\mathbf{A} = \begin{pmatrix}
\frac{\Gamma}{2} & \Omega & 0 & 0 & 0 & 0 \\
-\Omega & \frac{\Gamma}{2} & g & 0 & -\frac{\gamma}{2} & 0 \\
0 & 0 & -\frac{\gamma}{2} & 0 & 0 & 0 \\
g & 0 & 0 & -\frac{\gamma}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\gamma}{2} & 0 \\
-g & 0 & 0 & 0 & 0 & -\frac{\gamma}{2}
\end{pmatrix}
\]

(23)

\[
\mathbf{w} = (\sqrt{\Gamma} q_{in}, f_{eff}, \sqrt{\gamma} b_1^1, \sqrt{\gamma} b_2^1, \sqrt{\gamma} c_1^1, \sqrt{\gamma} c_2^1) \mathbf{y}
\]

in terms of the effective force \( f_{eff} = f + \sqrt{\Gamma} p_{in} - \lambda(c_1^1 \sin \theta + c_2^1 \cos \theta) \) and the effective coupling strengths
g = 2g_{\text{cm}} \beta_{\text{m}} \sqrt{2/\gamma}, \quad g = 2g_{\text{cm}} \beta_{\text{m}} \sqrt{2/\gamma}, \quad g = g - \lambda \sqrt{\gamma} \sin \theta, \quad g = \lambda \sqrt{\gamma} \cos \theta. \quad \text{The detection sensitivities obtained from the output current } I_{\text{out}} \propto Z = b_{\text{in}}^\text{out} \sin \phi + b_{\text{in}}^\text{out} \cos \phi \quad \text{are also plotted in figure 2. This shows that for a resonant cavity with a feedback loop the force sensitivity is still lower bounded by the UQL, and the displacement sensitivity can overcome the usual UQL if the detector parameters are chosen properly.}

4. Conclusion

We prove the UQL for the force sensitivity of a generic linear-response detector in the presence of direct quantum feedback by using general linear-response theory. We show that the usual UQL for the displacement sensitivity can be overcome by some devised schemes. By adopting the arguments for the force sensitivity, a generalized quantum limit is obtained for the displacement sensitivity. We illustrate the utility of these quantum limits by using a detuned cavity and a resonant cavity with a feedback loop as two specific examples. We believe that our results show how to improve the performance of high-sensitivity detection schemes.

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Appendix: linear-response relations and spectral uncertainty relations

For convenience, we put the general linear-response relations and the spectral uncertainty relations in the appendix. The general linear system we described in section II is

\[ H = H_{\text{in}} + H_{\text{ap}} + H_{\text{int}} \text{ with } H_{\text{int}} = -gFq. \]

The operator in the Heisenberg picture is given by equation (1), where

\[ H_{\text{in}}^0 = -gF_0q_0. \]

Expanding the time-ordered exponential \( U(t) \), we have

\[
U(t) = 1 + \frac{1}{i} \int_{-\infty}^{t} dt_1 H_{\text{in}}(t_1) + \frac{1}{i} \int_{-\infty}^{t} dt_1 H_{\text{int}}^0(t_1)
\times \int_{-\infty}^{t} dt_2 H_{\text{int}}^0(t_2) + \cdots
\]

\[ = 1 + \frac{1}{i} \int_{-\infty}^{t} dt_1 H_{\text{in}}^0(t_1)U(t_1), \tag{25} \]

and thus

\[
O(t) = U^\dagger(t)O_0(t)U(t)
\]

\[ = O_0(t) + \frac{1}{i} \int_{-\infty}^{t} dt_1 U^\dagger(t_1)[O_0(t), H_{\text{int}}^0(t_1)]U(t_1). \tag{26} \]

For \( O \equiv q \), we note that \( [q_0(t), F_0(t')] = 0 \), since \( q_0 \) and \( F_0 \) are independent variables, and \( [q_0(t), q_0(t')] \) is a c-number for the linear operator \( q_0 \). So equation (26) gives

\[
q(t) = q_0(t) + i \int_{-\infty}^{t} dt_1 [q_0(t), x_0(t_1)]f(t_1)
+ ig \int_{-\infty}^{t} dt_1 [q_0(t), q_0(t_1)]F(t_1), \tag{27} \]

where the second term comes from the action of the external force, and the relation \( F(t_1) = U^\dagger(t_1)f_0(t_1)U(t_1) \) is used. For a stationary system, we introduce the susceptibility

\[
\chi_{XY}(t - t') = i \delta(t - t')[X(t), Y(t')]. \tag{28} \]

The Fourier transform of equation (27) immediately gives the first line of equation (2). Similarly, the equations for \( F \) and \( Z \) can be obtained.

Next we sketch the proof of the spectral uncertainty relations for arbitrary linear Hermitian operators \( O_j^\dagger \) with \( j = 1, \ldots, n \). Let us consider an operator of the form

\[
P = \sum_{j=1}^{n} \int_{-\infty}^{\infty} dt \zeta_j(t)O_j^\dagger(t), \tag{29} \]

where \( \zeta_j \) are arbitrary complex functions. The positivity of the Hermitian operator \( \langle O^\dagger O \rangle \) implies that

\[
\sum_{j,k} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \zeta_j^*(t)\zeta_k(t')\langle O_j^\dagger(t)O_k^0(t') \rangle \geq 0. \tag{30} \]

We note the identity

\[
\langle O_j^\dagger(t)O_k^0(t') \rangle = S_{jk}(t - t') + \langle O_j^\dagger(t), O_k^0(t') \rangle/2
= S_{jk}(t - t') - i [\chi_{jk}(t - t'), \chi_{jk}(t - t')]/2,
\]

where the symmetrized correlator

\[
S_{jk}(t - t') = \langle O_j^\dagger(t)O_k^0(t') + O_k^0(t')O_j^\dagger(t) \rangle/2 \tag{32} \]

is related to the power density \( S_{jk}(\omega) \) via the Fourier transform

\[
S_{jk}(t) = \int_{-\infty}^{\infty} S_{jk}(\omega)e^{-\omega t}d\omega. \tag{33} \]

In the frequency domain, equation (30) becomes

\[
\sum_{j,k} \int_{-\infty}^{\infty} d\omega \zeta_j^*(\omega)M_{jk}(\omega)\zeta_k(\omega) \geq 0
\]

with the notation

\[
M_{jk}(\omega) = S_{jk}(\omega) - i [\chi_{jk}(\omega), \chi_{jk}(\omega)]/2. \tag{34} \]

This implies that the \( n \times n \) the Hermitian matrix \( M_{jk} \) is positive. Following similar arguments for the positivity of \( \langle OO^\dagger \rangle \), we have that the Hermitian matrix \( M_{jk}^\dagger = S_{jk} + i[\chi_{jk}, \chi_{jk}]^\dagger/2 \) is also positive. The spectral uncertainty relations are implied by the positivity of \( M_{jk}^\dagger \).

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