Topology Learning in Radial Dynamical Systems With Unreliable Data

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Abstract—Many complex engineering systems admit bidirectional and linear couplings between their agents. Blind and passive methods to identify such influence pathways/couplings from data are central to many applications. However, dynamically related data streams originating at different sources are prone to corruption caused by asynchronous time stamps of different streams, packet drops, and noise. Such imperfect information may be present in the entire observation period and, hence, is not detected by change detection algorithms that require an initial clean observation period. In this article, we provide a novel approach to detect the location of corrupt agents as well as present an algorithm to learn the structure of radial dynamical systems despite corrupted data streams. In particular, we show that our approach provably learns the true radial structure if the unknown corrupted nodes are at least three hops away from each other. Our theoretical results are further validated in a test dynamical network.

Index Terms—Fault detection, network topology, time series analysis, uncertain systems.

I. INTRODUCTION

NETWORKS provide an effective representation framework to analyze the interactions in complex systems, such as neuroscience [1], financial networks [2], and power grids [3]. Learning the network representation can provide insights into the analysis of system behavior, detect faults, and optimize flows. Initial research on learning the network representation involved considering the outputs of the system as random variables [4]. However, such an approach requires independence among time-lagged observations and is not applicable in the presence of dynamics in the system. In many applications, such as metabolic pathways, financial networks, and power grids, it is impractical or impermissible to externally influence the system by applying control inputs. Here, passive means that structure inference is to be adopted.

Radial networks constitute an important class of networks that are prevalent across many applications, such as power distribution grids [3], gas transmission networks [5], water networks [6], and information broadcasting [7]. Such systems have an operational configuration that has a tree network structure. Furthermore, there are several physical flow-driven systems, such as the power grid; fluid flow [8], [9]; and heat transfer networks [10], where all dynamic couplings between the agents/nodes are bidirectional. Bidirected networks are also prevalent in engineered systems, such as networks of oscillators [11] and consensus networks [12]. We consider such radial dynamical systems. Here, identifying the true Boolean structure (presence or absence of undirected edges) of the network correctly identifies all the influence pathways in the network.

Network identification for dynamical systems from time-series measurements is extensively studied in the control community [13], [14], [15], [16], [17], [18]. In the context of radial networks, Cavraro and Kekatos [19] use inverter probing techniques for exact inference of the power distribution topology in an active manner. Materassi and Salapaka [20] use multivariate Wiener filters to infer the network structure in a passive manner in the context of linear time-invariant (LTI) systems. Phase-based results of the estimated Wiener filters are shown to enable the removal of spurious edges in nonradial bidirectional LTI systems [21]. For a network of interacting agents with nonlinear dynamic dependencies, strictly causal interactions, and temporally uncorrelated process noise, Quinn et al. [22] use the directed information metric to determine the directed structure of the network. In this article, we admit noncausal relations in the generative model and allow temporal correlations in the process noise. The problem of learning polytree structures is studied in [23] and [24]. All the above work assumes that the measurements are uniformly sampled and are available without any nonidealities such as packet drops or delays. Often, the data streams in large systems are not immune to the effects of noise [25]; asynchronous sensor clocks [26], [27]; and packet drops [28], [29]. Subramanian et al. [30] focused on directed networks with LTI interactions, and provided the characterization of the extent of spurious links that can appear due to data corruption in the moral graph. However, little is known if these spurious edges can be eliminated to infer the exact network structure even in the presence of data corruptions, thus establishing consistency guarantees. In [31], focusing on bidirectional networks, the...
location of corrupt nodes is shown to be detected by combining tools from information theory and graph theory. However, a method to eliminate spurious edges was not presented.

Our contribution: In this article, the objective is to determine the exact network representation of radial bidirectional LTI systems, using passive means from corrupt data streams. We show that for a radial bidirected network of LTI systems where corrupt nodes are located deep in the network, at least three hops away from the leaf nodes, the spurious edges owing to data corruption can be eliminated, and the exact network structure can be inferred. We present novel topological characterizations and phase-based properties to determine the location of corrupt nodes. Finally, we propose an algorithm called “hide and learn” to determine the exact topology generating the time-series observations. To this, we follow a similar topology learning algorithm presented in our prior conference paper [32] that considered hidden nodes. However, in [32], there was a tighter assumption on the distance between hidden nodes restricting them to be at least four hops away from each other, and the measurements were assumed to be perfect. Moreover, rigorous proofs were not presented. Here, we consider time series with imperfect information and relax the assumption on the location of corrupt nodes and provide rigorous proofs to our results.

The rest of this article is organized as follows. Section II describes the generative model and the graphical representation and reviews earlier work on network identification using power spectra. Section III describes the data corruption models and its effect on structure inference. In Section IV-A, we present the main result to determine the location of corrupt nodes. The exact topology learning algorithm is presented in Section IV-B. Simulation results are provided in Section V. Finally, Section VI concludes this article.

I. Notation

$Y$ denotes a vector with $y_i$ being the $i$th element of $Y$. Unordered pair $(i, j)$ denotes an undirected edge between nodes $i$ and $j$ in an undirected graph, while an ordered pair $(i, j)$ denotes a directed edge from $i$ to $j$ in a directed graph. We also use $i \rightarrow j$ to denote a directed edge from $i$ to $j$. If $M(z)$ is a transfer function matrix, then $M(z)^T = M(z^{-1})^T$ is the conjugate transpose. $M(i, j)$ denotes the matrix entry at the $i$th row and the $j$th column. $E[ \cdot ]$ denotes the expectation operator. $R_{XY}(k) := E[X[n+k]Y[n]]$ is the cross-correlation function of jointly wide-sense stationary (WSS) processes $X$ and $Y$. If $Y = X$, then $R_{XX}(k)$ is called the autocorrelation. $\Phi_{XY}(z) := Z(R_{XY}(k))$ represents the cross-power spectral density, while $\Phi_{XX}(z) := Z(R_{XX}(k))$ denotes the power spectral density (PSD). $Z(\cdot)$ is the $Z$-transform operator. $b_i$ represents the $i$th element of the canonical basis of $\mathbb{R}^n$.

II. PRELIMINARIES

In this section, the generative model and the generative graph that represents the networked system are presented.

A. Generative Model

Consider $N$ agents that interact over a network. Consider the following discrete-time linear dynamics for each agent $i \in \{1, \ldots, N\}$:

\[
\sum_{m=0}^{n} a_{m,i} x_i[t-m] + a_{0,i} x_i[t] = \sum_{j=1,j \neq i}^{N} b_{ij} x_j[t] + w_i[t].
\]

(1)

The process $w_i[t]$ is considered to be zero-mean WSS process innate to agent $i$ and, thus, $w_i$ is independent of $w_j$ if $i \neq j$. Thus, the PSD of $w = (w_1, w_2, \ldots, w_N)^T, \Phi_w(z)$ is a diagonal matrix. Above, $a_{m,i}, b_{ij} \in \mathbb{R}$ for all $m \in \{0, \ldots, n\}$ such that at least one $a_{m,i}$ is nonzero. We assume that the signals are bounded in a mean-square sense: $E[\| x_i[t] \|^2] < \infty$ and $E[\| w_i[t] \|^2] < \infty$. Taking the $Z$-transform of (1), we obtain

\[
\sum_{m=0}^{n} a_{m,i} z^{-m} x_i(z) = \sum_{j=1,j \neq i}^{N} b_{ij} x_j(z) + w_i(z)
\]

(2)

where $x_i(z)$ and $w_i(z)$ are the $Z$-transform of $x_i[t]$ and $w_i[t]$, respectively. Rewriting (2), we obtain

\[
x_i(z) = \sum_{j=1,j \neq i}^{N} G_{ij}(z) x_j(z) + e_i(z), \quad \text{for } i = 1, \ldots, N.
\]

(3)

Here, $S_i(z) = \sum_{m=0}^{n} a_{m,i} z^{-m}, G_{ij}(z) = \frac{b_{ij}}{S_i(z)},$ and $e_i(z) = w_i(z) \frac{1}{S_i(z)}.$

Compactly, (3) is equivalent to

\[
x = G(z)x + e
\]

(4)

where $x = (x_1(z), x_2(z), \ldots, x_N(z))^T$ and $e = (e_1(z), e_2(z), \ldots, e_N(z))^T$ and $G(i, j) = G_{ij}(z).$ We call the pair $(G(z), e)$ the generative model. We consider bidirected generative models such that $G_{ij}(z) \neq 0$ if and only if $G_{ji}(z) \neq 0.$ Such models are prevalent in linearized models of engineering systems operating around an equilibrium point. For example, consider swing dynamics for power systems and heat transfer dynamical systems.

B. Graphical Representation

The structural description of (3) induces a generative graph, $G = (V, A)$, formed by identifying the set of vertices, $V = \{1, 2, \ldots, n\}$, with random processes $x_i$ and the set of directed links, $\overrightarrow{A}$ given by $\overrightarrow{A} = \{(i \rightarrow j) | G_{ji} \neq 0\}$. Given the generative graph $G$, its generative topology is the undirected graph $G_T = (V, A)$, formed by replacing all the directed edges in $\overrightarrow{A}$ with undirected edges such that multiple undirected edges between any two nodes are replaced by a single undirected edge between those same nodes. Fig. 1 represents a bidirected system.

For a bidirected system, $A = \{(i, j) | i \rightarrow j \in \overrightarrow{A} \} \cup \{(i, j) | j \rightarrow i \in \overrightarrow{A} \}$
Consider a generative model \((G(z), e)\) consisting of \(N\) nodes with generative graph \(G\). Let \(x = (x_1, \ldots, x_N)^T\) denote the time-series measurements. Let \(\Phi_{xx}\) be the PSD matrix of the vector process \(x\). Then, the \((j, i)\) entry of \(\Phi_{xx}^{-1}\) is nonzero implies that \(i\) is a kin of \(j\).

The basis of the above result comes from the structure of the matrix \(G(z)\). Recall that \(G_{ij}(z) = 0\) if and only if \((i, j)\) holds in \(G_T\). From (4) we can express \(\Phi_{xx}\) as follows:

\[
\Phi_{xx} = (I - G(z))^{-1}\Phi_e(I - G(z))^{-T}.
\]

The inverse PSD, \(\Phi_{xx}^{-1}\), is given by

\[
\Phi_{xx}^{-1} = (I - G(z))^T\Phi_e^{-1}(I - G(z)).
\]

For a radial system with generative model (1), we can express \(\Phi_{xx}^{-1}\) more explicitly, where the nonzero entries only correspond to diagonal entries, neighbors, and two-hop neighbors in \(G_T\). Here

\[
\Phi_{xx}^{-1}(i, i) = \Phi_{w_i}^{-1}|S_i|^2 + \sum_{j \in \text{hop}_n(i)} \Phi_{w_j}^{-1}B_j^2 [S_j]^{-2}.
\]

\[
\Phi_{xx}^{-1}(i, j) = -\frac{b_{ij}}{\Phi_{w_i}(z)} \sum_{m=0}^\infty a_{m,i}z^m - \frac{b_{ij}}{\Phi_{w_j}(z)} \sum_{m=0}^\infty a_{m,j}z^{-m}
\]

and if \(i\) and \(j\) are two-hop neighbors with unique path \(i \rightarrow k \rightarrow j\) in \(G_T\), then

\[
\Phi_{xx}^{-1}(i, j)(z) = \frac{b_{ki}b_{kj}}{\Phi_{w_i}(z)}.
\]

Remark 1: We consider that the measurements’ sampling resolution is high and sampling-related nonzero entries in the inverse PSD are negligible.

III. UNCERTAINTY DESCRIPTION

The general class of measurement perturbation/corruption models is presented here. Consider the \(i\)th node in a network, and let its associated unperturbed time series be \(x_i\).

A. General Perturbation Models

The corrupt data stream \(u_i\) associated with \(i\) is considered to follow the stochastic linear system described below:

\[
z_i[t+1] = A_i[t]z_i[t] + B_i[t]x_i[t] + r_i[t] \tag{10a}
\]

\[
u_i[t] = C_i[t]z_i[t] + D_i[t]x_i[t] + v_i[t] \tag{10b}
\]

where \(z_i\) denotes hidden states in the stochastic linear system that describes the corruption. Only \(u_i\) is measured. Here, the matrices \(M_i[t] = \begin{bmatrix} A_i[t] & B_i[t] \\ C_i[t] & D_i[t] \end{bmatrix}\) are independent and identically distributed (IID) and independent of \(x_i[t]\). The terms \(r_i[t]\) and \(v_i[t]\) are zero-mean IID noise terms, which are independent of...
For distinct perturbed nodes, \( i \neq j \), we assume that \( M_i[t] \), \( r_i[t] \), and \( v_i[t] \) are independent of \( M_j[t] \), \( r_j[t] \), and \( v_j[t] \).

Define the means of the state-space matrices by

\[
\begin{align*}
A_i &= \mathbb{E}[A_i[t]], & B_i &= \mathbb{E}[B_i[t]], & C_i &= \mathbb{E}[C_i[t]], & D_i &= \mathbb{E}[D_i[t]].
\end{align*}
\]

Let \( h_i(k) \) be the impulse response of the system defined by \( A_i, B_i, C_i, \) and \( D_i \)

\[
h_i(k) = \frac{A_i}{C_i} B_i + D_i \cdot (k). \tag{12}
\]

Note that \( \bar{u}_i[t] = \mathbb{E}[u_i[t]|x_i] = (h_i * x_i)[t] \). The following result from [30] states how randomized perturbations in (10) corrupt the power spectra of measured signals.

**Theorem 2:** Assume that \( M_i[t] \) has bounded second moments. Define \( \Delta u_i[t] := u_i[t] - \bar{u}_i[t] \). Then, the signals \( u_i \) will be WSS with cross-spectra and power spectra

\[
\Phi_{u_i,u_j}(z) = H_i(z) \Phi_{x_i,x_j}(z) H_i(z^{-1}) + \theta_i(z) \tag{13a}
\]

\[
\Phi_{u_i,x_i}(z) = H_i(z) \Phi_{x_i,x_i}(z) \tag{13b}
\]

where \( H_i(z) = \mathbb{E}(h_i) \) and \( \theta_i(z) = \mathbb{E}(|R_{\Delta u_i, \Delta u_i}|) \).

**B. Uncertainty Examples**

1) **Temporal Uncertainty:** Randomized delays are modeled as

\[
u_i[t] = x_i[t - \delta[t]] \tag{14}
\]

where \( \delta[t] \) is a random variable. For example, if \( \delta[t] \in \{1, 2, 3\} \), then the randomized delay model is given by

\[
\begin{bmatrix}
A_i[t] & B_i[t] \\
C_i[t] & D_i[t]
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \( b_1, b_2, \) and \( b_3 \) are the standard basis vectors of \( \mathbb{R}^3 \).

2) **Noisy Filtering:** White measurement noise can be represented in the form of (10) by setting \( C_i[t] = 0 \), \( D_i[t] = 1 \):

\[
u_i[t] = x_i[t] + r_i[t]. \tag{15}
\]

Colored measurement noise is represented by setting \( B_i[t] = 0 \) and \( D_i[t] = 1 \), and the matrices \( A_i[t] \) and \( C_i[t] \) as constants.

3) **Packet Drops:** Here, the data stream suffers from randomly dropping measurement packets. The corrupted data stream \( u_i \) is obtained from \( x_i \) as follows:

\[
u_i[t] = \begin{cases} 
x_i[t], & \text{with probability } p_i \\
u_i[t - 1], & \text{with probability } (1 - p_i).
\end{cases} \tag{16}
\]

**C. Network Identification in the Presence of Corruption**

Structure learning using sparsity in the inverse PSD of corrupted data streams leads to inference of spurious links that correspond to edges in the perturbed graph, defined next.

**Definition 8 (Perturbed graph):** Let \( G_M = (V, A_M) \) be a moral graph. Suppose that \( Y \subset V \) is the set of corrupt nodes satisfying (13). Then, the perturbed graph of \( G_M \) with respect to set \( Y \) is the graph, \( G_U = (V, A_U) \) such that \((i, j) \in A_U \) if \((i, j) \in A_M \) or there is a path from \( i \) to \( j \) in \( G_M \), or both, such that all intermediate nodes are in \( Y \).

We have the following result from [30].

**Theorem 3:** Consider a generative model \((G(z), e)\) with the corresponding moral graph, \( G_M = (V, A_M) \). Let \( Y \) be the set of perturbed nodes where each perturbation satisfies (13). Then, \( \Phi_{-1}^{-1}(p, q) \neq 0 \) implies that \( p \) and \( q \) are neighbors in the perturbed graph \( G_U \).

Consider a bidirectional generative graph, as shown in Fig. 2(a). Its true moral graph is depicted in Fig. 2(b). Suppose that node 4 is corrupted. Applying Theorem 3, the inferred perturbed graph is shown in Fig. 2(c) containing spurious links.

**Assumption 1:** Assume that for all \( z \) on the unit circle and all nodes \( i \), we have \( \Phi_{w_i}(z) > 0 \) and \( |S_i(z)| > 0 \). Furthermore, if \( i \) is corrupted, then \( |H_i(z)| > 0 \) and \( \theta_i(z) > 0 \) for all \( z \) on the unit circle.

**Assumption 2:**

P1) If \( i \) and \( j \) are one-hop neighbors, then \( b_{ij} \neq 0 \) and \( b_{ji} \neq 0 \).

P2) If \( i \) and \( j \) are one-hop neighbors, then \( \angle \Phi_{xx}(i, j)(z) \) is not constant on the unit circle.

P3) If \( i \) and \( j \) are one-hop neighbors and \( i \) is corrupted, then \( \angle \Phi_{xx}(i, j)(z) \) is not constant on the unit circle.

P4) If \( i \neq j \), then \( k \) is a path in \( G_T \), then \( \angle(\Phi_{xx}(i, j)\Phi_{xx}(j, k)) \) is not constant on the unit circle.

**Remark 2:** Assumptions 1 and 2 and (7) imply that \( \Phi_{xx}^{-1} \) is positive everywhere. With (8), the assumptions imply that if \( i \) and \( j \) are one-hop neighbors, then \( \Phi_{xx}^{-1}(i, j) \neq 0 \). Furthermore if \( i \) and \( j \) are two-hop neighbors, the assumptions combined with (9) imply that \( \Phi_{xx}^{-1}(i, j) \) is real and nonzero everywhere.

**Remark 3:** If \( \Phi_{xx}^{-1}(i, j)(z) \) were constant over the unit circle, then \( \angle \Phi_{xx}^{-1}(i, j)(z) \) is \( \angle \Phi_{xx}^{-1}(i, j)(1) \) when \( |z| = 1 \). In particular, \( \Phi_{xx}^{-1}(i, j)(1) \in \mathbb{R} \), so we must have \( \text{Im}(\Phi_{xx}^{-1}(i, j)(z)) = 0 \), where \( \text{Im}(\cdot) \) gives the imaginary part. Therefore, each \( |z| = 1 \) with \( z \notin \{-1, 1\} \) defines an algebraic constraint on the parameters, which is only satisfied by a set of measure zero in the space of parameters corresponding to (1) with graph \( G \). Physically, an example of such a pathological parameter set includes the case of static systems. Similarly, if \( \angle \Phi_{xx}^{-1}(i, j)(z) \) is constant on the unit circle, then \( \text{Im}(\Phi_{xx}^{-1}(i, j)(z)) = 0 \) on the unit circle. As above, the corresponding constraints are only satisfied by a set of measure zero of the parameters.

**IV. EXACT TOPOLOGY LEARNING**

The first step toward exact topology learning is to determine the location of all the corrupt nodes. This is presented in the
following subsection. We consider the following assumption on the location of corrupt nodes.

**Assumption 3:**

C1) Corrupt nodes are at least three hops away from all leaf nodes in the generative topology.

C2) Corrupt nodes are at least three hops away from each other in the generative topology.

C3) There is at least one corrupt node.

**Remark 4:** Condition C1) implies that the corrupt nodes are located deep in the network such that their effects are felt by the agents that have perfect measurements. Moreover, C1) implicitly implies that every node has a two-hop neighbor in the generative topology. We consider arbitrarily large networks satisfying this, and with at least one corrupt node. By C1) and C3), the smallest network permitted is a seven-node chain, as shown in Fig. 2(a), with node 4 being corrupted.

### A. Corruption Detection

A method to locate the corrupt nodes in the inferred perturbed graph for radial dynamical systems is presented in this section. To this, we first characterize the topological properties of the neighborhood set of leaf and corrupt nodes in the inferred perturbed graph.

#### 1) Neighborhood Characterization:

The following proposition states that if \( i \) is a leaf or a corrupt node, its neighborhood set in the inferred perturbed graph, \( G_U \), is completely determined by the set of one-hop and two-hop neighbors of \( i \) in the generative topology, \( G_T \).

**Proposition 1:** Consider a radial system with generative topology \( G_T = (V, A) \) with the moral graph \( G_M = (V, A_M) \). Let \( hop_1(i) \) and \( hop_2(i) \) denote the set of one-hop and two-hop neighbors of \( i \) in \( G_T \). Let \( Y \subseteq V \) be the set of perturbed nodes where each perturbation satisfies (13) and Assumption 3. Suppose that \( G_U = (V, A_U) \) is the corresponding perturbed graph. Let \( B \) be the set of nodes detected using Lemma 1 whose neighborhood \( N_U(i) \) forms a clique with \( \{i\} \) in \( G_U \). Take a node \( i \in B \). Then, \( i \) has at least two neighbors \( p \) and \( q \) in \( G_U \) with \( \Phi_{uu}^{-1}(i, p)(z) \) nonconstant over \( \{z \in \mathbb{C} ||z| = 1\} \) if and only if \( i \) is a corrupt node in \( G_T \).

The above result detects the set of corrupt nodes, \( y \), from the candidate set, \( B \), and hence, the remaining nodes \( B \setminus y \) are the leaf nodes. Furthermore, Theorem 4 delineates that only leaf nodes have one unique entry in \( \Phi_{uu}^{-1} \) with a nonconstant phase response. This corresponds to the true edge associated with the leaf node. Let \( E_L \) be this set of edges. Thus, in addition to corrupt nodes, we also detect leaf nodes and remove spurious edges associated with leaf nodes applying Theorem 4.

### B. Hide and Learn Algorithm

The steps to recover the exact topology of the radial linear dynamical system using imperfect information are presented in this section. To accomplish this, we follow the hide and learn strategy. This is described in Algorithm 1. This is accomplished in three stages: First, hide the measurements of the corrupt nodes. We infer the graphical structure of the network by observing sparsity pattern in inverse PSD using only the nodes that have perfect information by marginalizing out the corrupt node measurements. That is, the corrupt nodes will be treated as latent nodes. This graph will contain spurious edges. This constitutes lines 1–7 in Algorithm 1. Second, identify the true edges in the graph obtained from the previous step. This constitutes lines 8–15 in Algorithm 1. Finally, place the corrupt nodes back at the correct location in the structure resulting from previous step, as described in lines 16–29 in Algorithm 1. We will now elaborate on each stage and provide technical results that yield the hide and learn algorithm.

#### 1) Learning With Latent Corrupt Nodes:

Let \( y \) be the time-series measurements of corrupt nodes, \( Y \), detected after Theorem 4. Let \( o \) denote the set of measurements without \( y \). That is, \( o = u \setminus y \). We compute the inverse PSD of \( o \). Now, using sparsity pattern in inverse PSD of \( o \) as adjacency matrix constructs an undirected graph, \( T_m = (V_o, A_o) \). The following result shows that \( T_m \) has spurious edges connecting up to four-hop neighbors in \( G_T \). The proof is given in Appendix C.

**Lemma 2:** Consider a radial dynamical system with generative topology \( G_T \). Then, \( \Phi_{oo}^{-1}(i, j) \neq 0 \) if and only if one of the following conditions holds.
Algorithm 1: Exact Topology Learning: Hide and Learn.

Input: Time series measurements, $u$, set of perturbed nodes, $Y$, set of leaf nodes, $L$, and set of true edges, $E_L$ associated with leaf nodes from Theorem 4.

Output: Set of true edges, $A$, in generative topology $G_T$.

Init: Set of observed edges, $A_o \leftarrow \emptyset$.

1: Isolate noncorrupt measurements, $o = u \setminus Y$. Observed nodes $V_o = V \setminus Y$.
2: Using measurements $o$, compute inverse PSD, $\Phi_{oo}^{-1}$.
3: for all $i \in V_o$, $i \neq j$ do
4: \quad if $\Phi_{oo}^{-1}(i, j) \neq 0$ then
5: \quad \quad $A_o \leftarrow A_o \cup \{(i, j)\}$
6: \quad end if
7: end for
8: Nonleaf nodes, $V_{nl} = V \setminus L$. True edge set, $E_T \leftarrow E_L$.
9: for all $p, q \in V_{nl} \text{ such that } (p, q) \in A_o$ do
10: \quad if There exist $K \neq \emptyset$ and $S \neq \emptyset$ such that $\text{sep}(K, S)\{p, q\}$ holds then
11: \quad \quad $E_T \leftarrow E_T \cup \{(p, q)\}$, $A \leftarrow A \cup E_T$
12: \quad end if
13: end for
14: $d \leftarrow$ number of disconnected components in the graph, $\Theta = (V_o, E_T)$ (i.e., $\Theta = \bigcup_{i=1}^{d} \Theta_i$).
15: for all $i \in \{1, 2, \ldots, d\}$ and $j \in \{i + 1, \ldots, d\}$ do
16: \quad if There exists nodes $q \in \Theta_i$ and $r \in \Theta_j$ such that $\text{sep}(q, r, s)\{p, q\}$ holds for some other observed nodes $p, s$ then
17: \quad \quad for all $l \in Y$ do
18: \quad \quad \quad if $(p, q, l, r, s)$ forms a clique in $G_Z$ and if $\Phi_{uu}^{-1}(p, s)(\omega)$ is constant for all $\omega \in (-\pi, \pi]$ then
19: \quad \quad \quad \quad $A \leftarrow A \cup \{q - l, l - r\}$
20: \quad \quad \quad end if
21: \quad \quad end for
22: \quad end if
23: end for

1) $i$ and $j$ are at most two hops apart.
2) $i$ and $j$ are three or four hops apart and the path from $i$ to $j$ has a hidden node, $k$, which is at most two hops away from $i$ and at most two hops away from $j$.

2) True Edge Set Discovery Between Observed Nodes:
The graph $T_m$ inferred from Lemma 2 contains spurious edges. The objective here is to eliminate the spurious edges and, thus, identify the true edges. To this, the following notion of separation in undirected graphs is introduced.

Definition 10 (Separation): Given an undirected graph $G = (V, E)$, the set of nodes $Z \subset V$ is said to separate the path between nodes $i$ and $j$, if there exist no path between $i$ and $j$ in $G$ after removing the set of nodes $Z$. We denote this by $\text{sep}(i, j|Z)$, which is read as $i$ and $j$ are separated by $Z$.

The following result from [32] provides a topological method based on the separation property to identify the observed nonleaf nodes and identify the true edges between them. The proof is given in Appendix D.

Lemma 3: Suppose that $T_m$ is the graph inferred using measurements $o$ in Lemma 2. Then, there exist observed nodes $c$ and $d$ distinct from observed nodes $a$ and $b$ such that $\text{sep}(c, d)\{a, b\}$ holds in $T_m$ if and only if $(a, b)$ is a true edge in $G_T$ and $a$ and $b$ are nonleaf nodes.

Combining Lemma 3 with the result of Theorem 4 that detected the only true edge associated with all the leaf nodes, we have thus identified all true edges associated with the observed nodes. Denote this graph as $\Theta$.

3) Placement of Corrupt Nodes: The graph $\Theta$ will have multiple radial disconnected components denoted as $\Theta_j$, with the disconnections being at the location of the latent corrupt nodes, $Y$. Based on our assumptions, it can be shown that each disconnected component has at least two observed nodes. Thus, for all node $p \in \Theta_j$, there is another node $q \in \Theta_j$ such that $(p, q) \in G_T$. Since $G_T$ is a connected graph, the final step is to connect the disconnected components by placing the corrupt nodes at the disconnected locations. We make use of the prior knowledge gained by inferring the perturbed graph $G_U$, and we map every corrupt node $i \in Y$ to its corresponding neighborhood $N_a(i)$ in $G_U$. To do this, we will use Lemma 4 that precisely characterizes the location to place the corrupt nodes. The proof is given in Appendix E.

Lemma 4: Let $\Theta$ be the disconnected network inferred after removing spurious edges between the observed nodes based on Lemma 3. Consider two disconnected components $\Theta_i$ and $\Theta_j$ in $\Theta$ with observed nodes $q \in \Theta_i$ and $r \in \Theta_j$. Consider all $p \in \Theta_j$ and all $s \in \Theta_j$ such that $(p, q)$ and $(r, s)$ are edges in $\Theta_j$ and $\Theta_j$, respectively. Consider a corrupt node $l \in Y$. Suppose that $(p, q, l, r, s)$ forms a clique in the perturbed graph, $G_U$. Then, $p - q - l - r - s$ holds in $G_T$ if and only if $Z \Phi_{uu}^{-1}(p, s)$ is constant.

Theorem 5 is our main result, which states that Algorithm 1 precisely learns the exact topology of a radial system with unreliable data. The proof is given in Appendix E.

Theorem 5: Suppose that $Y$ is the set of perturbed nodes, $L$ is the set of leaf nodes, and $E_L$ is the set of true edges associated with leaf nodes detected from Theorem 4. Then, Algorithm 1 results in learning the true generative topology $G_T = (V, A)$ for the corresponding radial system.

V. Simulation Result

In this section, we demonstrate the topological learning algorithm via a numerical example. Let the true generative graph, $G$, be as shown in Fig. 2(a) with the following dynamics:

\[
x_1[t] = 0.5x_2[t-1] + e_1[t]
x_2[t] = 0.36x_1[t-1] + 0.6x_3[t-1] + e_2[t]
x_3[t] = 0.95x_2[t-1] - 1.7x_4[t-1] + e_3[t]
x_4[t] = 0.51x_3[t-1] + 0.55x_5[t-1] + e_4[t]
x_5[t] = 1.5x_4[t-1] + 0.6x_6[t-1] + e_5[t]
x_6[t] = 0.7x_5[t-1] + 0.5x_7[t-1] + e_6[t]
x_7[t] = 0.65x_6[t-1] + e_7[t]
\]  

(17)
where $e_i$ are white noise sequences. The corruption model for node 4 is

$$u_4[t] = \begin{cases} x_4[t-2], & \text{with probability 0.7} \\ x_4[t], & \text{with probability 0.3} \end{cases}.$$  

From a trajectory length of $10^7$, the estimates for PSD was obtained using MATLAB “cpsd” command. The plot for magnitude of the inverse PSD estimates is shown in Fig. 3. The inferred perturbed graph is shown in Fig. 2(c). Fig. 4 shows the phase response of the inverse PSD estimated. Using Theorem 4, node 4 is isolated as the corrupt node. Next, follow the hide and learn Algorithm 1. The magnitude of $\Phi_{uo}^{-1}$ is shown in Fig. 5 and infer the undirected graph shown in Fig. 6(a) following Lemma 2.

**APPENDIX A**

**PROOF OF LEMMA 1**

$(\Rightarrow)$ We will show that if $i$ is neither a leaf node nor a corrupt node, then $N_u(i) \cup \{i\}$ does not form a clique in $G_U$. All nodes in hop$_1(i)$ and all nodes in hop$_2(i)$ will be neighbors of $i$ in moral graph and, hence, are neighbors of $i$ in $G_U$. We will show that there exists a pair of nodes $a, b \in N_u(i)$ such that $(a, b)$ does not hold true in $G_U$, and thus, $N_u(i) \cup \{i\}$ cannot form a clique in $G_U$.

By Assumption 3, we have that there is at least one one-hop neighbor and one two-hop neighbor for every node in $G_T$. Consider $a \in$ hop$_1(i)$ and $b \in$ hop$_2(i)$. Then, either $a$ and $b$ are neighbors or not in $G_T$. Consider $a$ and $b$ are not neighbors. Then, a path $a - i - p - b$ exists in $G_T$ for some node $p \in V$. Either $a$ is a leaf node or not in $G_T$. Suppose that $a$ is a leaf node.
node. As $G_T$ is a tree, the path $a - i - p - b$ between $a$ and $b$ is unique. Thus, all the possible paths between $a$ and $b$ in $G_M$ go through at least one of $i, p$. As $a$ is a leaf node, by condition C1, $i$ and $p$ are not corrupt. Thus, $(a, b) \notin A_U$.

Now say that $a$ is not a leaf node. Then, there is some distinct node $c$ that is a neighbor of $a$ in $G_T$, and there exists a path $c - a - i - p - b$ in $G_T$. Thus, $c \in \text{hop}_c(i)$ and a neighbor of $i$ in the perturbed graph, $G_U$. Therefore, $c$ and $b$ are neighbors in $G_U$. We will show that $(c, b) \notin G_U$. Since $G_T$ is a tree, the path connecting $c$ and $b$, $c - a - i - p - b$ is unique in $G_T$. Thus, all possible paths between $c$ and $b$ in $G_M$ go through at least one of $a, p$, and $i$. By condition C2), both $a$ and $p$ cannot be corrupt. Thus, all possible paths between $c$ and $b$ in $G_M$ go through at least one nonperturbed node. Thus, $(c, b) \notin A_U$.

Suppose that $a$ and $b$ are neighbors in $G_T$. Then, the path $i - a - b$ exists in $G_T$. As $i$ is not a leaf node, then there exists some node $q$ such that $q - i - a - b$ exists in $G_T$. Here, we consider $q \in \text{hop}_q(i)$ and $b \in \text{hop}_b(i)$. Using similar arguments as before, we can show that $(q, b) \notin A_U$.

(⇒) Suppose that $i$ is a leaf node in $G_T$ or a corrupt node. We will show that $N_u(i) \cup \{i\}$ forms a clique in the perturbed kin graph, $G_U$. Using Proposition 1, $N_u(i) \cup \{i\} = \text{hop}_i(i) \cup \text{hop}_p(i) \cup \{i\}$.

$i$ is a leaf node: There is only one nonleaf node $n_l$, which is a neighbor of $i$ in $G_T$. Any pair of two-hop neighbors of $i$ in $G_T$, $k_1, k_2$, has a common parent of $n_l$ in the generative graph. Thus, $k_1 - n_l - k_2$ holds in $G_T$ and so the path also exist in $G_M$ and $G_U$. Thus, $N_u(i) \cup \{i\}$ forms a clique in $G_U$.

$i$ is a corrupt node: For any $k_1, k_2 \in N_u(i)$, there is a path $k_1 - i - k_2$ in the moral graph $G_M$. As $i$ is a corrupt node, $(k_1, k_2)$ holds in $G_M$. Thus, $N_u(i) \cup \{i\}$ forms a clique in $G_U$.

APPENDIX B
PROOF OF THEOREM 4

Structure of inverse power spectra due to corruption

The expressions for $\Phi^{-1}_{uu}(z)$ are explicitly required for the results. First, we will describe the general structure of $\Phi^{-1}_{uu}(z)$. For compact notation, we will often drop the $z$ arguments.

Consider a network of $N$ nodes and $Y = \{v_1, \ldots, v_N\}$. For $p = 1, \ldots, N$, if $p$ is not a perturbed node, set $h_p \equiv 1$ and $\theta_p \equiv 0$. With this notation, (13) implies that the entries of $\Phi_{uu}$ are given by

$$(\Phi_{uu})_{pq} = \begin{cases} H_p (\Phi_{xx})_{pq} H_p^+, & \text{if } p \neq q \\ H_p (\Phi_{xx})_{pp} H_p^+ + \theta_p, & \text{if } p = q. \end{cases}$$

We can express $\Phi_{uu}$ in matrix form as

$$\Phi_{uu} = H \Phi_{xx} H^* + E \Theta E^\top,$$

where $H$ is the diagonal matrix with $H_p$ on the diagonal, $\Theta$ is the matrix with $\theta_p$ on the diagonal for $p \in Y$, and $E = \begin{bmatrix} b_{v_1} & \cdots & b_{v_N} \end{bmatrix}$, where $b_v$ are the standard unit basis vectors.

Setting $\Psi = H \Phi_{xx} H^*$, the Woodbury matrix identity gives

$$\Phi^{-1}_{uu} = \Psi^{-1} - \Psi^{-1} E (\Theta^{-1} + E^\top \Psi^{-1} E)^{-1} E^\top \Psi^{-1}.$$

Now, we show that the matrix $\Theta^{-1} + E^\top \Psi^{-1} E$ is diagonal. $\Theta^{-1}$ is diagonal since $\Theta$ is diagonal. The entries of $E^\top \Psi^{-1} E$ correspond to $\Psi^{-1}(i, j) = \Phi^{-1}_{xx}(i, j)/(H_i H_j)$, where $i$ and $j$ are both corrupted. Now, by Assumption 3, if $i \neq j$, then they are at least three hops apart and so $\Phi^{-1}_{xx}(i, j) = 0$.

Thus, we have that the entries of $\Phi^{-1}_{uu}$ are given by

$$\Phi^{-1}_{uu}(i, j) = \Phi^{-1}_{xx}(i, j)/(H_i H_j) - \sum_{k \in Y} \Phi^{-1}_{xx}(i, k)(\Phi^{-1}_{xx}(k, j)/(H_i H_j))^2 \Phi^{-1}_{xx}(k, k).$$

Expression for $\Phi^{-1}_{uu}(i, j)$ for $i \in B$

Now, we will show that for each candidate node $i \in B$ and $j \in V$

$$\Phi^{-1}_{uu}(i, j) = \begin{cases} \Phi^{-1}_{xx}(i, j)/(H_i H_j), & \text{if } i \text{ is a leaf} \\ \Phi^{-1}_{xx}(i, j)/(H_i H_j) \left(1 - \frac{\Phi^{-1}_{xx}(i, i)}{H_i^2 + \Phi^{-1}_{xx}(i, i)} \right), & \text{if } i \text{ is corrupt}. \end{cases}$$

First say that $i$ is a leaf. Then, by Assumption 3, $i$ is not corrupted, and every $k \in V$ is at least three hops away from $i$. This implies that $H_i = 1$ and $\Phi^{-1}_{xx}(i, k) = 0$, so the form follows from (19).

Now say that $i$ is corrupt. Then, by Assumption 3, $i$ is at least three hops away from every other corrupt node, so that the only term in the sum from (19) does not vanish corresponds to $k = i$. The form in (20) follows by factoring out $\Phi^{-1}_{xx}(i, i)/(H_i H_j)$.

Proof of Theorem 4

Recall that a node $i \in B$ can either be perturbed or a leaf. (⇒)
We will show that if $i \in B$ is a leaf node in $G_T$, then there is at most one node $j \in N_u(i)$ such that $\angle(\Phi^{-1}_{uu}(i, j), i)$ is nonconstant. By Proposition 1, $N_u(i) = \text{hop}_i(i) \cup \text{hop}_p(i)$. Moreover, as $i$ is a leaf node, any node $j \in N_u(i)$ is not a corrupt node.

Since $i$ is a leaf node, there is only node in hop$_i(i).$ Call this node $r$. We will show that for all $j \neq r \in N_u(i)$ (this means $j \in \text{hop}_p(i)$), $\angle(\Phi^{-1}_{uu}(i, j), i)$ is a constant, while $\angle(\Phi^{-1}_{uu}(i, r), i)$ is nonconstant.

Take any $j \in \text{hop}_p(i)$. Node $j$ cannot be corrupt, and so $H_j = 1$. Let $q \in V$ be the common neighbor of $i$ and $j$ in $G_T$. Combining (9) and (20), we have

$$\Phi^{-1}_{uu}(i, j) = \Phi^{-1}_{xx}(i, j) + b_{ij}b_{jj}(\Phi^{-1}_{uu}(z))$$

which is a real scalar with constant sign. Thus, $\angle(\Phi^{-1}_{uu}(i, j), z) = 0$ or $\pi$.

Now, consider the node $r$. Since $r$ cannot be corrupt, we have $H_r = 1$. Then, $\Phi^{-1}_{uu}(i, r) = \Phi^{-1}_{xx}(i, r)$, which has nonconstant phase over the unit circle by Assumption 2 since $i$ and $r$ are one-hop neighbors.

(⇒) We will show that if $i \in B$ is a corrupt node, then there are at least two neighbors of $i$ in $G_T$ such that the corresponding entries in the inverse PSD have nonconstant transfer functions. By Assumption 2, every corrupt node has at least two one-hop neighbors in $G_T$. By the construction of perturbed graph, $G_U$, these nodes will also be neighbors of $i$ in $G_U$. Call these nodes as $p$ and $r$. We will now show that $\Phi^{-1}_{uu}(i, p)$ and $\Phi^{-1}_{uu}(i, r)$ are nonconstant transfer functions. Consider $\Phi^{-1}_{uu}(i, p)$. Then, since

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cannot be corrupted, (20) implies that
\[
\Phi_{xu}^{-1}(i, p) = \frac{\Phi_{xx}^{-1}(i, i) - \Phi_{xx}^{-1}(i, i)}{H_i}.
\]
Assumption 2 implies that \((1 - \frac{\Phi_{xx}^{-1}(i, i)}{\theta_i H_i^2 + \Phi_{xx}^{-1}(i, i)}) > 0\). It follows that:
\[
\angle \Phi_{xu}^{-1}(i, p) = \angle \left( \frac{\Phi_{xx}^{-1}(i, p)}{H_i} \right).
\]
The phase of the latter function is nonconstant on the unit circle by Assumption 2.
As \(p\) and \(r\) are both one-hop neighbors of \(i\) in \(G_T\), similar arguments hold true for node \(r\).

**Remark 5:** We remark here that to prove \((\Leftarrow)\), it suffices to show that the leaf node, \(i\), has at most one neighbor, \(p\), such that \(\angle \Phi_{xu}^{-1}(i, p)\) is not a constant for all \(\omega\). However, we showed that there is exactly one such neighbor.

**APPENDIX C**

**PROOF OF LEMMA 3**

Recall the generative model, \((G(z), e)\), from (4) in Section II-A. Let \(x(z) \in \mathbb{C}^N\) denote the vector of \(z\) transform of \(N\) nodal measurements, with \(x(z) = [x_o(z)^T, x_h(z)^T]^T\), where \(x_o(z) \in \mathbb{C}^m\) and \(x_h(z) \in \mathbb{C}^{N-m}\) are the \(z\) transform of the nodal measurements corresponding to \(m\) observed and \(n-m\) unobserved (corrupt) nodes, respectively. The network dynamics is represented in a compact form as
\[
\begin{bmatrix}
    x_o(z) \\
    x_h(z)
\end{bmatrix} = \begin{bmatrix}
    G_o(z) & G_o(h(z)) \\
    G_h(o(z)) & G_h(h(z))
\end{bmatrix} \begin{bmatrix}
    x_o(z) \\
    x_h(z)
\end{bmatrix} + \begin{bmatrix}
    e_o(z) \\
    e_h(z)
\end{bmatrix}
\]
(22)
where \(e_o(z)\) and \(e_h(z)\) denote the process noise at the observed and hidden nodes, respectively. We assume that the unobserved (corrupt) nodes are not neighbors in \(G_T\), that is, \(G_h(h(z)) = 0\). Recall that \(o\) denotes the set of observed nodes, and let \(h = y\) denote the set of unobserved nodes. Let \(u = o \cup h\). Let \(J\) denote the inverse PSD matrix. Using (6) and (22), we have
\[
J = \begin{bmatrix}
    J_{oo}(z) & J_{oh}(z) \\
    J_{ho}(z) & J_{hh}(z)
\end{bmatrix} = \Phi_{xx}^{-1} = \begin{bmatrix}
    \Phi_{oo}(z) & \Phi_{oh}(z) \\
    \Phi_{ho}(z) & \Phi_{hh}(z)
\end{bmatrix}^{-1}
= (I - G(z)) \Phi_{xx}^{-1}(I - G(z)).
\]
Using the matrix inversion lemma, it follows that:
\[
\Phi_{oo}^{-1} = J_{oo} - J_{oh} J_{hh}^{-1} J_{ho}.
\]
(23)
Note that if \(i\) and \(j\) are hidden nodes, then they must be at least three hops apart by Assumption 3. It follows that \(J_{hh}(i, j) = \Phi_{xx}^{-1}(i, j) = 0\). Therefore, we have that \(J_{hh}\) is diagonal.
Since \(J_{hh}\) is diagonal, if \(i\) and \(j\) are observed, then (23) can be expressed as
\[
\Phi_{oo}^{-1}(i, j) = \Phi_{xx}^{-1}(i, j) - \sum_{k \in h} \frac{\Phi_{xx}^{-1}(i, k) \Phi_{xx}^{-1}(k, j)}{\Phi_{xx}(k, k)}.
\]
(24)
This can only be nonzero if \(i\) and \(j\) are separated by at most four hops. We will describe the precise cases in which it will be nonzero.
Suppose \(i - j\) holds in \(G_T\) or there is an observed node \(\ell\) such that \(i - \ell - j\) holds in \(G_T\). Recall that \(G - T\) is a tree. Then, the uniqueness of paths implies that
\[
\Phi_{oo}^{-1}(i, j) = \Phi_{xx}^{-1}(i, j)
\]
which is nonzero by Remark 2.
If \(i - k - j\) holds in \(G_T\) with \(k\) hidden, we have
\[
\Phi_{oo}^{-1}(i, j) = \Phi_{xx}^{-1}(i, j) - \frac{\Phi_{xx}^{-1}(i, k) \Phi_{xx}^{-1}(k, j)}{\Phi_{xx}(k, k)}
\]
(26)
where \(\Phi_{xx}^{-1}(i, k)\) and \(\Phi_{xx}(k, k)\) are real and nonzero by Remark 2, and \(\angle (\Phi_{xx}^{-1}(i, k) \Phi_{xx}^{-1}(k, j))\) is not constant by Assumption 2. It follows that \(\Phi_{oo}^{-1}(i, j) \neq 0\) in this case.
Now, say that \(i\) and \(j\) are three or four hops apart. Assumption 3 implies that the path from \(i\) to \(j\) can have at most one hidden node. If the path had no hidden node, then (24) will be zero. The only paths for which (24) could possibly be nonzero have one hidden node, \(k\), which is at most two hops away from both \(i\) and \(j\). These paths take the forms \(i - a - k - j\), \(i - k - a - j\), and \(i - a - k - b - j\), where \(a\) and \(b\) are observed nodes. In each of these cases, we have
\[
\Phi_{oo}^{-1}(i, j) = -\frac{\Phi_{xx}^{-1}(i, k) \Phi_{xx}^{-1}(k, j)}{\Phi_{xx}(k, k)}.
\]
(27)
Remark 2 implies that \(\Phi_{xx}^{-1}(k, k)\) is strictly positive. Consider a path of the form \(i - a - k - j\), where \(a\) is observed. Then, \(\Phi_{xx}^{-1}(i, k)\) real and nonzero everywhere and \(\Phi_{xx}^{-1}(k, j)\) \(\neq 0\) by Remark 2. Therefore, \(\Phi_{oo}^{-1}(i, j) \neq 0\) in this case. The case of a path \(i - k - a - j\) is similar. For a path of the form \(i - a - k - b - j\), all of the corresponding functions are real and nonzero by Remark 2. Thus, \(\Phi_{oo}^{-1}(i, j) \neq 0\) in this case.

**APPENDIX D**

**PROOF OF LEMMA 4**

\((\Leftarrow)\) Suppose that \((a, b)\) is not a link in \(G_T\). We will show that \(\text{sep}(c, d\{a, b\})\) does not hold in \(T_m\). Let \(p := c - \pi_1 - \pi_2 - \pi_3 - \cdots - \pi_m - d\) be the unique path between \(c\) and \(d\) in \(G_T\). Set \(c = \pi_0\) and \(d = \pi_{m+1}\). Here, \(\{\pi_1, \pi_2, \ldots, \pi_m\}\) can have both observed nodes and unobserved corrupt nodes.
First, say that \(a\) and \(b\) do not belong to \(\{\pi_1, \ldots, \pi_m\}\). Then, Lemma 2 implies that the path formed by deleting hidden nodes from \(p\) is a path from \(c\) to \(d\) in \(T_m\). Since this path does not contain \(a\) or \(b\), \(\text{sep}(c, d\{a, b\})\) does not hold.
Now, we will show that if \(a, b \in \{\pi_1, \ldots, \pi_m\}\), then there must be an edge in \(T_m\) from \(\{c, \pi_1, \ldots, \pi_m\} \setminus \{b\}\) to \(\{\pi_k, \pi_l, \ldots, \pi_m, d\} \setminus \{b\}\). Such an edge “skips” over \(a\) and possibly \(b\).
At least one of \(\pi_k - 1\) and \(\pi_k + 1\) must be observed by Assumption 3. Say that both \(\pi_k - 1\) and \(\pi_k + 1\) are observed. Since \(a\) and \(b\) are not neighbors, we have \(\pi_k - 1 \neq b\) and \(\pi_k + 1 \neq b\). Furthermore, \(\pi_k - 1 \neq \pi_k + 1\) is an edge of \(T_m\) by Lemma 2.
If \(\pi_k - 1\) is not observed, then \(\pi_k - 1 \neq c\) and \(\pi_k - 2\) and \(\pi_k + 1\) are both observed. Thus, \(\pi_k - 2 - \pi_k + 1\) is an edge in \(T_m\) by Lemma 2.
Now, $\pi_{k+1} \neq b$ since $a$ and $b$ are not neighbors, but we could have $\pi_{k-2} = b$. However, if $\pi_{k-2} = b$, then $\pi_{k-3} \neq b$ must be observed, and Lemma 2 implies that $\pi_{k-3} - \pi_{k+1}$ is also an edge of $T_m$. The case with $\pi_{k+1}$ not observed is similar.

Similarly, if $b = \pi_\ell \in \{\pi_1, \ldots, \pi_m\}$, then there must be an edge in $T_m$ from $\{c, \pi_1, \ldots, \pi_{\ell-1}\} \setminus \{a\}$ to $\{\pi_{\ell+1}, \ldots, \pi_m, d\} \setminus \{a\}$.

Thus, the path formed deleting hidden nodes from $p$ and using the edges that skip over $a$ and $b$ constructed above is a path from $c$ to $d$ in $T_m$, which does not contain $a$ or $b$. Thus, $\text{sep}(c, d; \{a, b\})$ does not hold.

($\Rightarrow$) Suppose that $(a, b)$ is a true link (that is a link in $G_T$), and $a$ and $b$ are not leaf nodes. We will show that there are observed nodes on either side of $a$ and $b$ in $T_m$ such that they are separated when $\{a, b\}$ are removed. As $a$ and $b$ are nonleaf nodes, there exist nodes $p$ and $q$ on either side of $a$ and $b$, respectively, in $G_T$.

There are three possibilities: 1) $p$ and $q$ are both observed nodes; 2) only one of them is observed; or 3) both are unobserved.

**Case 1:** Both $p$ and $q$ are observed. $p - a - b - q$ is the unique path between $p$ and $q$ in $G_T$. Lemma 2 implies that $(p, q)$ is not a link of $T_m$, so any path from $p$ to $q$ must go through at least one of $a$ or $b$. Therefore, $\text{sep}(p, q; \{a, b\})$ holds.

**Case 2:** One of $p$ and $q$ is unobserved. Without loss of generality, let $p$ be the unobserved node. Since $p$ is at least three hops from a leaf node, there exists an observed nonleaf node $g$ such that $g - p - a - b - q$ is the unique path in $G_T$ between $g$ and $q$. It follows from Lemma 2 that links $(g, a), (g, b), (a, b), (a, q),$ and $(b, q)$ are all present in $T_m$, and $(g, q)$ is not. It follows that at least one of $a$ or $b$ is present in each path between $(g, q)$ in $T_m$. Removing nodes $a$ and $b$ thus separate $g$ and $q$ in $T_m$.

**Case 3:** Both $p$ and $q$ are unobserved. By Assumption 3, there are observed and uncorrupt nodes $\pi_p$ and $\pi_q$ on either side of $p$ and $q$, respectively. Lemma 2 implies that $(\pi_p, a), (\pi_p, b), (a, \pi_q),$ and $(b, \pi_q)$ are links in $T_m$, but $(\pi_p, \pi_q)$ is not. Thus, all paths between $\pi_p$ and $\pi_q$ in $T_m$ will involve at least $a$ or $b$, and these paths will be separated if the set $\{a, b\}$ is removed.

**Appendix E: Proof of Theorem 5**

Following the discussion in Section IV-B, the only result that needs to be proved is Lemma 4, which we will now prove. Since $(p, q, l, r, s)$ forms a clique in $G_U$ and $G_T$ is a tree, it follows that $l$ is located at the point of disconnection between $(p, q)$ and $(r, s)$. What needs to be shown is the correct alignment among the paths $p - q - l - r - s, p - q - l - s - r, q - p - l - s - r,$ and $p - q - l - r - s$ in $G_T$. For this, we will analyze the phase of inverse PSD entry corresponding to pairs from $(p, q) \times \{r, s\}$ described in Proposition 3. Before that, we will need the following proposition that gives an expression for inverse PSD entries corresponding to pairs from $(p, q) \times \{r, s\}$.

**Proposition 2:** Suppose that $p - q - l - r - s$ holds in $G_T$, where $l$ is a corrupt node. Then, for any $a \in (p, q)$ and $b \in \{r, s\}$

$$\Phi_{uu}^{-1}(a, b) = \Phi_{xx}^{-1}(a, b) - \frac{\Phi_{xx}^{-1}(a, l)\Phi_{xx}^{-1}(l, b)}{\theta_l^{-1}|H_l|^2 + \Phi_{xx}^{-1}(l, l)}$$

(28)

**Proof:** We will apply (19). Note that $a$ and $b$ cannot be corrupted by Assumption 3. Thus, $H_p = H_b = 1$.

We claim that for any corrupt node $k$ with $k \neq l$, either $\Phi_{xx}^{-1}(a, k) = 0$ or $\Phi_{xx}^{-1}(b, k) = 0$. For the sake of contradiction, assume that $\Phi_{xx}^{-1}(a, k) \neq 0$ and $\Phi_{xx}^{-1}(b, k) \neq 0$. Then, by Assumption 3, $a$ and $b$ must be at most two hops apart, and $k$ and $b$ must be at least two hops apart. This implies that there must be a path from $a$ to $b$, which goes through $k \neq l$, which contradicts the assumption of a radial network. Thus, the claim holds, and therefore, the only term in the sum from (19) that does not vanish corresponds to $k = l$, and the result holds.

Our next result, Proposition 3, analyzes the phase values of entries in $\Phi_{uu}^{-1}$.

**Proposition 3:** If $p - q - l - r - s$ holds in $G_T$, then $\angle \Phi_{uu}^{-1}(p, s)$ is a constant, while $\angle \Phi_{uu}^{-1}(p, r), \angle \Phi_{uu}^{-1}(q, s)$, and $\angle \Phi_{uu}^{-1}(q, r)$ are nonconstant.

**Proof:** First, consider the case that $a = p$ and $b = s$, so that $p$ and $s$ are four hops apart, while $p$ and $l$ are two-hop neighbors and $s$ and $l$ are two-hop neighbors. Then

$$\Phi_{uu}^{-1}(p, s) = -\frac{\Phi_{xx}^{-1}(p, l)\Phi_{xx}^{-1}(l, s)}{\theta_l^{-1}|H_l|^2 + \Phi_{xx}^{-1}(l, l)}$$

where all of the functions are real and nonzero by Remark 2. Thus, the phase is constant.

Now, consider the case that $a = q$ and $l$ are one-hop neighbors and $b = s$ and $l$ are two-hop neighbors, and so $q$ and $s$ are three hops apart. Thus

$$\Phi_{uu}^{-1}(q, s) = -\frac{\Phi_{xx}^{-1}(q, l)\Phi_{xx}^{-1}(l, s)}{\theta_l^{-1}|H_l|^2 + \Phi_{xx}^{-1}(l, l)}.$$  

Then, by Assumption 2 and Remark 2, $\Phi_{xx}^{-1}(q, l)$ has nonconstant phase, while all of the other functions from (28) are real and nonzero everywhere. Thus, $\angle \Phi_{uu}^{-1}(q, s)$ is not constant. The case of $a = p$ and $b = r$ is similar.

Finally, consider the case that $a = q$ and $b = r$. In this case, $q$ and $l$ are one-hop neighbors, $r$ and $l$ are one-hop neighbors, and $q$ and $r$ are two-hop neighbors. Thus

$$\Phi_{uu}^{-1}(q, r) = \Phi_{xx}^{-1}(q, r) - \frac{\Phi_{xx}^{-1}(q, l)\Phi_{xx}^{-1}(l, r)}{\theta_l^{-1}|H_l|^2 + \Phi_{xx}^{-1}(l, l)}.$$  

By Assumption 2, $\angle (\Phi_{xx}^{-1}(q, l)\Phi_{xx}^{-1}(l, r))$ is nonconstant, while all of the other functions are real and nonzero by Remark 2. It follows that $\angle \Phi_{uu}^{-1}(q, r)$ is not constant. It follows from the above proposition that only if the corresponding phase properties hold in $p - q - l - r - s$, then it is the only correct alignment as any other alignment will have non-two-hop neighbors as four hops away and, hence, will violate the constant phase argument. This verifies Lemma 4. □

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