Graphs of Transportation Polytopes

Jesús A. De Loera ∗ Edward D. Kim † Shmuel Onn ‡ Francisco Santos §

Abstract

This paper discusses properties of the graphs of 2-way and 3-way transportation polytopes, in particular, their possible numbers of vertices and their diameters. Our main results include a quadratic bound on the diameter of axial 3-way transportation polytopes and a catalogue of non-degenerate transportation polytopes of small sizes. The catalogue disproves five conjectures about these polyhedra stated in the monograph by Yemelichev et al. (1984). It also allowed us to discover some new results. For example, we prove that the number of vertices of an $m \times n$ transportation polytope is a multiple of the greatest common divisor of $m$ and $n$.

1 Introduction

This paper takes a new look at the graphs of transportation polytopes. Transportation polytopes are well-known objects in operations research and mathematical programming (see e.g., [1, 2, 3, 10, 24, 26, 27, 32, 33] and references therein). Statisticians have also interest in them of their own (see e.g., [8, 9, 16, 17, 21, 25] and references therein).

During the 1970’s and 1980’s the study of the classical 2-way transportation polytopes, i.e., those polytopes of $m \times n$ tables satisfying row-sum and column-sum conditions, was very active. The state of the art of that research was carefully summarized in the comprehensive book by Yemelichev, Kovalev, and Kratsov [33] and Vlach’s survey [32]. On the other hand, 3-way transportation polytopes, whose feasible points are $l \times m \times n$ arrays of non-negative numbers satisfying certain sum conditions, are less understood. We define the polyhedra whose points are $l \times m \times n$ tables satisfying certain sum conditions. They come in two main varieties:

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1. First, consider the **axial** 3-way transportation polytope: Let \( x = (x_1, \ldots, x_l) \), \( y = (y_1, \ldots, y_m) \), and \( z = (z_1, \ldots, z_n) \) be three rational vectors of lengths \( l \), \( m \) and \( n \), respectively, with non-negative entries. Let \( T_{x,y,z} \) be the polytope defined by 1-marginals; that is, the following \( l + m + n \) equations in the \( l \times m \times n \) real variables \( a_{i,j,k} \) \((i = 1, \ldots, l; j = 1, \ldots, m; k = 1, \ldots, n)\):

\[
0 \leq a_{i,j,k}, \forall i, j, k \quad \sum_{j,k} a_{i,j,k} = x_i, \forall i \quad \sum_{i,k} a_{i,j,k} = y_j, \forall j \quad \sum_{i,j} a_{i,j,k} = z_k, \forall k.
\]

Observe that a necessary and sufficient condition for \( T_{x,y,z} \) to be non-empty is that

\[
\sum_i x_i = \sum_j y_j = \sum_k z_k,
\]

and that, consequently, \( T_{x,y,z} \) is defined by only \( l + m + n - 2 \) independent equations.

2. Similarly, **planar** 3-way transportation polytopes can be defined by specifying three matrices \( U \in M_{m,n}(\mathbb{Q}) \), \( V \in M_{l,n}(\mathbb{Q}) \), \( W \in W_{l,m}(\mathbb{Q}) \) for the line-sums resulting from fixing two of the indices of entries and adding over the remaining index. That is, we have the following \( lm + ln + mn \) equations (2-marginals) in the same \( lmn \) variables \( a_{i,j,k} \):

\[
0 \leq a_{i,j,k}, \forall i, j, k \quad \sum_i a_{i,j,k} = U_{j,k}, \forall j \quad \sum_j a_{i,j,k} = V_{i,k}, \forall i \quad \sum_k a_{i,j,k} = W_{i,j}, \forall i, j.
\]

One can see that in fact only \( lm + ln + mn - l - m - n + 1 \) of the defining equations are linearly independent for feasible systems.

Observe that the axial 3-way transportation polytopes generalize the classical transportation polytope of size \( m \times n \), which coincides with \( T_{x,y,z} \) for \( l = 1 \) and \( x = \sum_j y_j = \sum_k z_k \). A less trivial rewriting of the classical \( 2 \times n \) transportation polytope as a \( 2 \times 2 \times n \) 3-way planar transportation polytope is given in Theorem 1.4 below.

Recall that the 1-**skeleton** or **graph** of a convex polytope \( P \) is the set of all 0-dimensional and 1-dimensional faces (vertices and edges) of \( P \), with their natural incidence relation. The main focus of this paper is to investigate the number of vertices and the diameters of the graphs of classical (that is, 2-way) and 3-way transportation polytopes.

Some of the statements below require our transportation polytopes to be non-degenerate. By this we mean that the polytope is simple (i.e., every vertex is adjacent to dimension many other vertices) and it is of maximal possible dimension (that is, dimension \( lmn - l - m - n + 2 \) for \( l \times m \times n \) axial transportation polytopes, and dimension \( (l - 1)(m - 1)(n - 1) \) for \( l \times m \times n \) planar transportation polytopes). Graphs of non-degenerate transportation polytopes are of particular interest because they
have the largest possible number of vertices and largest possible diameter among the graphs of all transportation polytopes of given type and parameters. Indeed, if a transportation polytope $P$ is degenerate, by carefully perturbing the marginals that define it we can get a non-degenerate one $P'$. The perturbed marginals are obtained by taking a feasible point $X$ in $P$, perturbing the entries in the table and using the recomputed sums as the new marginals for $P'$. The graph of $P$ can be obtained from that of $P'$ by contracting certain edges, which cannot increase either the diameter nor the number of vertices.

Our main result is a bound on the diameter of axial 3-way transportation polytopes:

**Theorem 1.1** The graph of the $l \times m \times n$ axial transportation polytope $T_{x,y,z}$ has diameter at most $2(l + m + n - 3)^2$.

A similar result for the graph of a classical transportation polytope was given by Brightwell et al. (see [7]), who proved an upper bound of $8(m + n - 2)$ for the diameter. More recently, Hurkens (see [20]) has obtained a bound of $4(m + n - 1)$, a factor of four away from the predicted value of the Hirsch conjecture.

Using computational tools, we also give a complete catalogue of non-degenerate 2-way and 3-way transportation polytopes (both axial and planar) of small sizes. This allowed us to explore properties of transportation polytopes (e.g., their diameters and how close they were to the Hirsch conjecture bound). The summary of the catalogue is:

**Theorem 1.2**  
- The only possible numbers of vertices of non-degenerate $2 \times 3$, $2 \times 4$, $2 \times 5$, $3 \times 3$, and $3 \times 4$ classical transportation polytopes are those given in Table 1.
  
- The only possible numbers of vertices of non-degenerate $2 \times 2 \times 2$, $2 \times 2 \times 3$, $2 \times 2 \times 4$, $2 \times 2 \times 5$, and $2 \times 3 \times 3$ planar transportation polytopes are those given in Table 2.
  Every non-degenerate $2 \times 3 \times 4$ planar transportation polytope has between 7 and 480 vertices.

- The only possible numbers of vertices of non-degenerate $2 \times 2 \times 2$ and $2 \times 2 \times 3$ axial transportation polytopes are those given in Table 3.
  Every non-degenerate $2 \times 2 \times 4$ axial transportation polytope has between 32 and 504 vertices.
  Every non-degenerate $2 \times 3 \times 3$ axial transportation polytopes has between 81 and 1056 vertices.
  The number of vertices of non-degenerate $3 \times 3 \times 3$ axial transportation polytopes is at least 729.

The catalogue was obtained via the exhaustive and systematic computer enumeration of all combinatorial types of non-degenerate transportation polytopes. The theoretical foundations of it are the notions of parametric linear programming, chamber complex, Gale diagrams and secondary polytopes.
| Size      | Dimension | Possible numbers of vertices                        |
|-----------|-----------|----------------------------------------------------|
| $2 \times 3$ | 2         | 3 4 5 6                                           |
| $2 \times 4$ | 3         | 4 6 8 10 12                                      |
| $2 \times 5$ | 4         | 5 8 11 12 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 |
| $3 \times 3$ | 4         | 9 12 15 18                                      |
| $3 \times 4$ | 6         | 16 21 24 26 27 29 31 32 34 36 37 39 40 41 42 44 45 46 48 49 50 52 53 54 56 57 58 60 61 62 63 64 66 67 68 70 71 72 74 75 76 78 80 84 90 96 |

Table 1: Numbers of vertices possible in non-degenerate classical transportation polytopes

| Size      | Dimension | Possible numbers of vertices                        |
|-----------|-----------|----------------------------------------------------|
| $2 \times 2 \times 2$ | 1         | 2                                                  |
| $2 \times 2 \times 3$ | 2         | 3 4 5 6                                           |
| $2 \times 2 \times 4$ | 3         | 4 6 8 10 12                                      |
| $2 \times 2 \times 5$ | 4         | 5 8 11 12 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 |
| $2 \times 3 \times 3$ | 4         | 5 8 9 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 |

Table 2: Numbers of vertices possible in non-degenerate planar transportation polytopes

| Size      | Dimension | Possible numbers of vertices                        |
|-----------|-----------|----------------------------------------------------|
| $2 \times 2 \times 2$ | 4         | 8 11 14                                           |
| $2 \times 2 \times 3$ | 7         | 18 24 30 32 36 38 40 42 44 46 48 50 52 54 56 58 60 62 64 66 68 70 72 74 76 78 80 84 86 96 108 |

Table 3: Numbers of vertices possible in non-degenerate axial transportation polytopes
We present them in Section 2. The full catalogue of transportation polytopes (including other families, such as $3 \times 5$, $4 \times 4$, $4 \times 5$, etc.) is available in a searchable web database at: http://www.math.ucdavis.edu/~ekim/transportation_polytope_database/

Based on the data we collected, we discovered and proved (in Section 3) the following results:

**Theorem 1.3** The number of vertices of a non-degenerate $m \times n$ classical transportation polytope is divisible by $\text{GCD}(m,n)$.

**Theorem 1.4** The $2 \times 2 \times n$ planar transportation polytopes are in $1-1$ correspondence with the $2 \times n$ classical transportation polytopes, with corresponding pairs being linearly isomorphic.

Note that Theorem 1.4 is best possible in the sense that for $m,n \geq 3$ there are many more types of planar $2 \times m \times n$ transportation polytopes than types of $m \times n$ transportation polytopes - see the rows of the $3 \times 3$ and $2 \times 3 \times 3$ polytopes in Tables 1 and 2. Finally, we state two open conjectures at the end of Section 2.

We close the introduction explaining the relevance of our results to current research. Bounding the diameter of graphs of polytopes has received a lot of attention because of its connection to the performance of the simplex method for linear programming and, most especially, to try to understand the Hirsch conjecture (see [23, 22] and references therein).

We recall that the Hirsch conjecture asserts that the diameter of the graph of any polytope of dimension $d$ and with $f$ facets is bounded above by $f - d$. Not only is this conjecture open, but even the weaker statement asserting a polynomial upper bound (in $f$ and $d$) for the diameters of graphs of polytopes is unknown (although a quasi-polynomial bound appeared in [22]).

As we observed, [7] provided the first linear bound for the diameter of the graphs of 2-way transportation polytopes. Theorem 1.1 provides a quadratic one for axial 3-way transportation polytopes and, moreover, a sublinear one if we assume that the three parameters $l$, $m$ and $n$ are approximately the same. (Observe that the number of facets of a 3-way transportation polytope is bounded above by the product $l mn$ of its size parameters).

Bounding the diameter of 3-way transportation polytopes is particularly interesting because of the following results recently proved by two of the authors in [13]:

1. Any rational convex polytope can be rewritten as a face $F$ of an axial 3-way transportation polytope. The sizes $l, m, n$, 1-marginals $x, y, z$, and the entries $a_{i,j,k}$ that are prescribed to be zero in the face $F$ can be computed in polynomial time on the size of the input.

2. More dramatically, any convex rational polytope is isomorphically representable as a planar 3-way transportation polytope.
That is to say, a version of Theorem 1.1 for the 3-way planar case, or a version for the axial case that allows one to prescribe some variables to be zero, would provide a polynomial upper bound on the diameter of the graph of every convex rational polytope.

Another consequence of these results is that the method of Section 2 for enumerating all combinatorial types of planar 3-way transportation polytopes, yields, in particular, an enumeration of all types of rational convex polytopes.

Let us finally mention that our systematic listing of non-degenerate transportation polytopes provides the solution to at least four open problems and conjectures about transportation polytopes stated in the monograph [33]:

1. Klee and Witzgall in [24] prove that the largest possible number of vertices in classical transportation polytopes of size $m \times n$ is achieved by the generalized Birkhoff polytope (the transportation polytope with parameters $x_i = n \forall i$, $y_j = m \forall j$). Problem 32 in page 400 of [33] conjectured that the same holds in general.

But in Example 2.7 we provide an explicit counterexample of this for planar 3-way transportation polytopes. (In this case, the generalized Birkhoff polytope is the planar 3-way transportation polytope whose 2-marginals are given by the $m \times n$ matrix $U(j,k) = l$, the $l \times n$ matrix $V(i,k) = m$, and the $l \times m$ matrix $W(i,j) = n$.)

2. Question 36 on page 396 of [33] asked: Is it true that every integer of the form $(l - 1)(m - 1)(n - 1) + t$ where $1 \leq t \leq ml + nl + mn - l - m - n$, and only these integers, can equal the number of facets of a non-degenerate planar 3-way transportation polytope of order $l \times m \times n$, where $l, m, n \geq 2$?

For the case $l = m = 2$ and $n = 3$, the conjecture asks if every integer from 3 to 11, and only these integers, equal the number of facets of non-degenerate $2 \times 2 \times 3$ planar transportation polytopes. Since in this case the number of facets equals the number of vertices (because the polytopes are two dimensional) Table 2 answers the question negatively: only facet-counts from 3 to 6 occur, while 7 through 11 are in fact missing.

3. Similarly, Conjecture 33 on page 400 of [33] asked: Is it true that every integer from 1 to $ml + nl + mn - l - m - n + 1$, and only these numbers, are realized as the diameter of a planar 3-way transportation polytope of order $l \times m \times n$?

The same case $l = m = 2$, and $n = 3$ shows that this is false. The transportation polytopes obtained are polygons with up to six sides, hence of diameter at most three, instead of 10.
4. Open problem 37 in page 396 of [33] asks whether the numbers of vertices of \( l \times m \times n \) non-degenerate planar transportation polytopes satisfy:

\[(l - 1)(m - 1)(n - 1) + 1 < f_0 < 2(l - 1)(m - 1)(n - 1).\]

We show the answer is no even in the case \( 2 \times 2 \times 4 \).

In addition to the four solved problems above, Theorems 1.2 and 1.3 are initial steps on the solution of Problem 25 in page 399 of [33]. It asks to find the complete distribution of possible number of vertices for transportation polytopes.

2 Classifying Transportation Polytopes

Theorem 1.2 was obtained through an exhaustive enumeration whose foundation is the theory of secondary polytopes and parametric linear programming. In some cases when the full enumeration was impossible we at least get lower and upper bounds for the number of vertices that these polytopes can have. In this section we discuss the necessary background to understand the construction of the complete catalogue.

2.1 Enumeration via Regular triangulations and Secondary polytopes

We begin by recalling some basic facts about convex polytopes presented, as all transportation polytopes are, in the form \( P_c = \{ x : Bx = c, x \geq 0 \} \). For the case of transportation polytopes, the vector \( c \) is the vector given by the demand/supply quantities. Fix a matrix \( B \) of full row rank. Most of our results are obtained by studying what happens to the combinatorics of \( P_c \) as the vector \( c \) changes while we fix the matrix \( B \). This study, for general matrices, is known as parametric linear programming (see Chapter 1 of [15]).

A subset of \( \mathbb{R}^n \) that is closed under addition and under multiplication by positive scalars is a cone. For any set \( L \) of vectors in \( \mathbb{R}^n \), the cone generated by \( L \), denoted \( \text{cone}(L) \), is the set of all vectors that can be expressed as non-negative linear combinations of the members of \( L \). Abusing notation, for a matrix \( B \), by \( \text{cone}(B) \) we mean the cone generated the set of column vectors of \( B \).

A maximal linearly independent subset \( b \) of \( B \) is a basis of \( B \). Geometrically, each basis of the matrix \( B \) spans a simple cone inside \( \text{cone}(B) \). Every basis \( b \) of \( B \) defines a basic solution of the system as the unique solution of the \( m \) linearly independent equations \( bx_b = c \) and \( x_j = 0 \) for \( j \) not in \( b \). A basic solution is feasible if in addition, \( x \geq 0 \). Geometrically, a basic feasible solution corresponds to a simple cone that contains \( c \). In fact, one can see that \( P_c \) is non-empty if and only if \( c \in \text{cone}(B) \).
A fundamental fact in linear programming is that, for a given right-hand-side vector \( c \), all vertices of the polyhedron \( P_c \) are basic feasible solutions (see [30, 33]). Moreover, if the polyhedron \( P_c \) is assumed to be non-degenerate then the basic feasible solution must be strictly positive on the entries corresponding to the basis \( b \). Geometrically, a basis \( b \subseteq B \) produces a vertex of a non-degenerate \( P_c \) if and only if \( c \) lies in the interior of the cone generated by \( b \). In conclusion, the vertices of a non-degenerate polyhedron \( P_c \), from this family of parametric polytopes, are in bijection with the bases that contain the right-hand-side vector \( c \) within their interior.

We now look at what happens when we let \( c \) vary. If \( P_c \) is non-degenerate and the change in \( c \) is small, the facets of \( P_c \) move but the combinatorial type of \( P_c \) does not change. Only when a basic solution changes from being feasible to not feasible, or vice versa, the combinatorics of \( P_c \) (that is, the face lattice and, in particular, the graph of \( P_c \)) can change.

Put differently: Denote by \( \Sigma_B \) the set of all cones generated by bases of \( B \). Let \( \partial \Sigma_B \) denote the union of the boundaries of all elements of \( \Sigma_B \). The connected components of \( \text{cone}(B) \setminus \partial \Sigma_B \) are open convex cones called the chambers of \( B \). Equivalently, we call the chamber associated to a given right-hand side vector \( c \) the intersection of the interiors of simple cones that contain \( c \) in their interior. We remark that every feasible and sufficiently generic \( c \) is in a chamber (as opposed to lying on \( \partial \Sigma_B \)). Two vectors \( c_1 \) and \( c_2 \) in the same chamber determine non-degenerate polytopes \( P_{c_1} \) and \( P_{c_2} \) that are equivalent up to combinatorial type. The collection of all the chambers is the chamber complex or chamber system associated with \( B \). Putting all this together we conclude

**Proposition 2.1** To represent all the possible combinatorial types of non-degenerate polytopes of the form \( P_c = \{ x : Bx = c, x \geq 0 \} \), for a fixed \( B \) and varying vector \( c \), it is enough to choose one \( c \) from each chamber of the chamber complex of \( B \).

**Example 2.2** Consider the matrix \( B_{2,3} \), the constraint matrix of all \( 2 \times 3 \) transportation polytopes. That is:

\[
B_{2,3} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

This means, the system \( \{ B_{2,3} y = c, \ y \geq 0 \} \) defines the \( 2 \times 3 \) transportation polytopes with marginals \( c \).
The columns of the matrix $B_{2,3}$ span a four-dimensional cone in $\mathbb{R}^5$. It will be relevant later that if we slice this cone by an affine hyperplane (such as $\sum y_i = 1$) we obtain the three-dimensional triangular prism shown in Figure 1, but embedded in $\mathbb{R}^4$.

The chamber complex can be obtained by slicing the prism with the six planes containing a vertex of the prism and the edge "opposite" to it. The resulting chamber complex is hard to visualize or draw, even in this small case, but we will see below how to recover the structure of the chamber complex for this example using Gale transforms. In particular, as we will see, this decomposes the triangular prism into 18 chambers.

It is very easy to "sample" inside the chamber complex and find chambers of different numbers of bases, i.e., transportation polytopes with different number of vertices. One can simply throw random positive values to the cell entries of an $l \times m \times n$ table and then compute the 1-marginals or 2-marginals associated to it. But with this method it is not obvious how to guarantee that one has obtained all the possible chambers. For this we use the approach based on Gale transforms and regular triangulations, that we now explain.

A vector configuration $A$ of $r$ vectors in $\mathbb{R}^{(r-d)}$ is called a Gale transform of another vector configuration $B$ of $r$ vectors in $\mathbb{R}^d$ if the row space of the matrix with columns given by $A$ is the orthogonal complement in $\mathbb{R}^r$ of the row space of the matrix with columns given by $B$. Gale transforms are essential tools in the study of convex polytopes because the combinatorial properties of $B$ and $A$ are intimately related (see Chapter 6 in [34] for details). Proofs of the following statements can be found in [5, 19]. See also Chapters 4 and 5 in [15].

Lemma 2.3  
• The chambers of $B$ are in bijection with the regular triangulations of the Gale transform $\hat{B}$ of $B$: From a chamber in $B$, one can recover a regular triangulation of $\hat{B}$ via complementation, namely for a basis $\sigma$ of vectors in $B$ the elements of $\hat{B}$ not belonging to $\sigma$ form a basis for $\hat{B}$. The collection of those bases gives a triangulation of $\hat{B}$ (see below for an example).

• There exists a polyhedron, the secondary polyhedron, whose vertices are in bijection with the regular triangulations of the Gale transform $\hat{B}$.

• The face lattice of the chamber complex of the vector configuration $B$ is anti-isomorphic to the face lattice of the secondary polyhedron of the Gale transform $\hat{B}$ of $B$. The latter is, in turn, isomorphic to the refinement poset of all regular subdivisions of $\hat{B}$.

• If $B$ defines a pointed polyhedral cone (for example, if all its entries are non-negative as it is the case for transportation polytopes), then its Gale transform $\hat{B}$ is a totally cyclic vector configuration. That is, the cone spanned by $\hat{B}$ is the whole of $\mathbb{R}^n$. 

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Example 2.4 (Example 2.2 continued) We take again the matrix $B_{2,3}$ (the case of $2 \times 3$ transportation polytopes). A Gale transform consists of the columns of the matrix

$$\hat{B}_{2,3} = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$ 

In Figure 1 we represent the Gale diagram $\hat{B}_{2,3}$ and its 18 regular triangulations, each one providing a combinatorial type of non-degenerate $2 \times 3$ transportation polytope, although repeated combinatorial types occur. The chamber adjacency, which corresponds to bistellar flips, is indicated by dotted edges.

![Figure 1: The 18 regular triangulations of $\hat{B}_{2,3}$, which correspond to the 18 combinatorial types of non-degenerate $2 \times 3$ transportation polytopes.](image-url)

Thus, generating all the combinatorial types of non-degenerate transportation polytopes is the same as listing the distinct regular triangulations of the Gale transform of the defining matrix $B$. Note
that in our case $B$ depends only on the sizes $l, m, n$ and the type, axial or planar, of transportation polytopes we look at. That is, we have one $B$ for each row of Tables 1, 2, and 3.

Now, it is well-known that the regular triangulations of a vector configuration can all be generated by applying bistellar flips to a seed regular triangulation (see [5, 15, 34]). Bistellar flips are combinatorial operations that transform one triangulation into another and regularity of triangulations can be determined by checking feasibility of a certain linear program (in our case, the very one that defines $P_c$). An example of the linear programming feasibility problem is given in Chapter 5 of [15].

**Example 2.5 (Example 2.2 continued)** Consider the only triangulation of $\hat{B}_{2,3}$ with six cones (the first one of the middle row in Figure 7). The necessary and sufficient conditions in the non-negative vector $(c_1, c_2, \ldots, c_6)$ in order to produce this triangulation are that each $c_i$ be smaller than the sum of the two adjacent to it. That is,

\[
\begin{align*}
    c_1 &< c_5 + c_6, \\
    c_2 &< c_4 + c_6, \\
    c_3 &< c_4 + c_5, \\
    c_4 &< c_2 + c_3, \\
    c_5 &< c_1 + c_3, \\
    c_6 &< c_1 + c_2.
\end{align*}
\]

Thus, these conditions on the marginals characterize the $2 \times 3$ transportation polytopes that are hexagons.

To implement this method we have written a C++ program that is available from the web page of the second author. This program calls TOPCOM (see [29]), a package for triangulations that computes, among other things, the list of all regular triangulations of a configuration. Our program also calls polymake (see [18]) for the Gale transform, and computes one vector $c$ per chamber. The output is a list of transportation polytopes, one per chamber, given in the polymake file format.

### 2.2 Lower and upper bounds via integer programming

Even for seemingly small cases, such as $3 \times 3 \times 3$ transportation polytopes, listing all chambers (and thus all combinatorial types of transportation polytopes) is practically impossible. In these cases we have followed a different approach to at least obtain upper and lower bounds for the number of vertices of transportation polytopes. By the discussion above, this is the same as finding bounds for the number of simplices in triangulations of the Gale transform $\hat{B}$. Here we follow the method proposed in [11], based on the universal polytope. This universal polytope, introduced by Billera, Filliman and Sturmfels in [4], has all triangulations (regular or not) of a given vector configuration $A$ in $\mathbb{R}^n$ as vertices, and projects to the secondary polytope. The universal polytope has much higher dimension than the secondary polytope, in fact its ambience dimension is the number of possible bases
of the configuration \( A \), thus no more than \( \binom{4}{4} \). It has the advantage that the number of simplices in different triangulations is given by the values of a linear functional \( \psi \).

More precisely, we think of the chambers of \( \text{cone}(B) \) as the vertices of the following high-dimensional 0/1-polytope: Assume \( B \) is a vector configuration with \( n \) vectors inside \( \mathbb{R}^d \). Let \( N \) be the number of \( d \)-dimensional simple cones in \( B \). We define \( U_B \) as the convex hull in \( \mathbb{R}^N \) of the set of incidence 0/1 vectors of all chambers of \( B \). For a chamber \( T \) the incidence vector \( v_T \) has coordinates \((v_T)_\sigma = 1\) if the basis \( \sigma \in T \) and \((v_T)_\sigma = 0\) if \( \sigma \) is not a basis of \( T \). The polytope \( U_B \) is the universal polytope defined in general by Billera, Filliman and Sturmfels in [31] (although there it is defined in terms of the triangulations of the Gale transform of \( B \)).

In [31], it was shown that the vertices of the universal polytope of \( B \) are exactly the integral points inside a polyhedron that has a simple inequality description in terms of the oriented matroid of \( B \) (see [11] [34] for information on oriented matroids). The concrete integer programming problems in question were solved using C-plex Linear Solver™. The program to generate the linear constraints is a small C++ available from the web page of the first author (see [31]).

Example 2.6 (Example 2.2 continued) Continuing with the running example, if \( B \) is \( B_{2,3} \) from above, then \( U_B \) is defined in \( \mathbb{R}^{15} \), where each coordinate is indexed by a 2-subset \( \sigma \) of \{1, \ldots, 6\}. Let \( S \) denote the set of all bases. Then \( N = |S| = 15 \). Thus \( U_B \) is the convex hull in \( \mathbb{R}^N \) of the incidence vectors \( v_T \) corresponding to the 18 chambers of \( B \). By Lemma 2.3 this is equivalent to the convex hull of the incidence vectors \( v_T \) of the 18 triangulations of \( \hat{B} \). For example, the triangulation \( T = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\} \) in Figure 1 gives the incidence vector \( v_T = e_{\{1,2\}} + e_{\{1,3\}} + e_{\{2,4\}} + e_{\{3,4\}} \) (where \( e_\sigma \) is the basis unit vector in the direction \( \sigma \)) as one of the vectors of the convex hull.

The convex hull of these 18 incidence vectors is a 6-dimensional 0/1 polytope \( U_B \) in \( \mathbb{R}^{15} \). That the dimension is (at most) six follows from the following considerations:

- Since the pairs \{1, 4\}, \{2, 5\} and \{3, 6\} are not full-dimensional and thus never appear as a simplex in any triangulation \( T \) of \( \hat{B} \), \( U_B \) is contained in the subspace \( x_{\{1,4\}} = x_{\{2,5\}} = x_{\{3,6\}} = 0 \).

- Since the vector 1 has 2 and 6 on one side and 5 and 3 on the other, in every triangulation the sum \( x_{\{1,2\}} + x_{\{1,6\}} \) equals the sum \( x_{\{1,3\}} + x_{\{1,5\}} \) (and it equals zero or one depending on whether the triangulation uses the vector 1 or not). This implies the first of the following equalities, the rest being the analogue statement for the other five vectors.

\[
\begin{align*}
x_{\{1,2\}} + x_{\{1,6\}} - x_{\{1,3\}} - x_{\{1,5\}} &= 0 \\
x_{\{2,3\}} + x_{\{2,4\}} - x_{\{1,2\}} - x_{\{2,6\}} &= 0 \\
x_{\{1,3\}} + x_{\{3,5\}} - x_{\{2,3\}} - x_{\{3,4\}} &= 0 \\
\end{align*}
\]
\begin{align*}
x_{\{3,4\}} + x_{\{4,5\}} - x_{\{2,4\}} - x_{\{4,6\}} &= 0 \\
x_{\{1,5\}} + x_{\{5,6\}} - x_{\{3,5\}} - x_{\{4,5\}} &= 0 \\
x_{\{2,6\}} + x_{\{4,6\}} - x_{\{1,6\}} - x_{\{5,6\}} &= 0
\end{align*}

Observe that one of these equations is redundant, since the sum of the left-hand sides is already zero.

- Since every triangulation needs to cover the angle between, for example, vectors $1$ and $6$, and this angle is covered only by the cones $16$, $12$ and $56$ (see Figure 1), we have that

\[ x_{\{1,6\}} + x_{\{1,2\}} + x_{\{5,6\}} = 1. \]

The results in [11] say that $U_B$ is the convex hull of the non-negative integer points in $\mathbb{R}^{15}$ satisfying this list of equations.

We now denote by $\psi \in (\mathbb{R}^N)^*$ the cost vector defining the linear function

\[ \psi(x) = (1,1,\ldots,1) \cdot x = \sum_{\sigma \in S} x_\sigma \]

Then the values of $\psi(x)$ on $U_B \cap \{0,1\}^N$ are the only possible values for the number $f_0$ of vertices of non-degenerate polytopes of the form $P_c = \{x | Bx = c, x \geq 0\}$. In particular, the solutions to the linear programming relaxations: “minimize (respectively maximize) $\psi(x)$ subject to $x \in U_B$” give lower (respectively upper) bounds to the possible values for the number $f_0$ of vertices of non-degenerate polytopes of the form $P_c = \{x | Bx = c, x \geq 0\}$. In the running example, $3 \leq \psi(x) \leq 6$ whenever $x \in U_B \cap \{0,1\}^N$. From Table 1 we observe that the number $f_0$ of vertices of a non-degenerate $2 \times 3$ transportation polytope equals $3, 4, 5$ or $6$.

**Example 2.7** Here is an application of our method. Table 4 is an explicit vector of $2$-marginals for a $3 \times 3 \times 3$ transportation polytope which has more vertices (270 vertices) than the generalized Birkhoff polytope, with only 66 vertices.

| 164424 | 324745 | 127239 | 163445 | 49395 | 403568 | 184032 | 123585 | 269245 |
| 262784 | 601074 | 9369116 | 1151824 | 767866 | 8313284 | 886393 | 6722333 | 935582 |
| 149654 | 7618489 | 1736281 | 1609500 | 6331023 | 1563901 | 1854344 | 302366 | 9075926 |

**Table 4: Counterexample to [33, open problem 37]**

Based on the data collected from the enumeration process, we also conjecture to be true:
Conjecture 2.8 The graph of every non-degenerate \( m \times n \) transportation polytope has a Hamiltonian cycle \((mn > 4)\).

Conjecture 2.9 If \( P \) is a non-degenerate \( l \times m \times n \) axial transportation polytope \((l, m, n \geq 3)\), then the diameter of its graph \( G(P) \) is equal to \( f - d \), where \( d = lmn - l - m - n + 2 \) is the dimension and \( f \) is the number of facets of \( P \).

For \( m \times n \) classical transportation polytopes \((m, n \leq 5)\), there are non-degenerate polytopes where the diameter of the graph \( G(P) \) is strictly less than \( f - d \).

3 Proofs of Theorems 1.3 and 1.4

We start with Theorem 1.4: The \( 2 \times 2 \times n \) planar transportation polytopes are linearly isomorphic to the \( 2 \times n \) classical transportation polytopes.

Lemma 3.1 The planar \( 2 \times m \times n \) transportation polytopes are exactly the \( m \times n \) transportation polytopes with bounded entries.

Proof. Every planar \( 2 \times m \times n \) transportation polytope

\[
P = \left\{ (a_{i,j,k}) \in \mathbb{R}^{2 \times m \times n}_+ : \sum_k a_{i,j,k} = x_{i,j}, \sum_j a_{i,j,k} = y_{i,k}, a_{1,j,k} + a_{2,j,k} = z_{j,k} \right\}
\]

is linearly isomorphic to an \( m \times n \) transportation polytope with bounded entries,

\[
Q = \left\{ (a_{1,j,k}) \in \mathbb{R}^{m \times n}_+ : \sum_k a_{1,j,k} = x_{1,j}, \sum_j a_{1,j,k} = y_{1,k}, a_{1,j,k} \leq z_{j,k} \right\},
\]

via the projection \( \mathbb{R}^{2 \times m \times n} \to \mathbb{R}^{m \times n} \) taking \((a_{i,j,k}) \mapsto (a_{1,j,k})\), which maps \( P \) bijectively onto \( Q \). Conversely, every \( m \times n \) transportation polytope \( Q \) with bounded entries is linearly isomorphic to a planar \( 2 \times m \times n \) transportation polytope \( P \) by defining \( x_{2,j} := (\sum_k z_{j,k}) - x_{1,j}, \ j = 1, \ldots, m \) and \( y_{2,k} := (\sum_j z_{j,k}) - y_{1,k}, \ k = 1, \ldots, n \).

Proof of Theorem 1.4 Consider any planar \( 2 \times 2 \times n \) transportation polytope

\[
P = \left\{ (a_{i,j,k}) \in \mathbb{R}^{2 \times 2 \times n}_+ : \sum_k a_{i,j,k} = x_{i,j}, a_{1,i,k} + a_{2,i,k} = y_{i,k}, a_{1,j,k} + a_{2,j,k} = z_{j,k} \right\}.
\]
The equations of the last two types imply that for each \( k \) we can express all the \( a_{i,j,k} \) in terms of \( a_{1,1,k} \) as follows:

\[
\begin{align*}
    a_{1,2,k} &= y_{1,k} - a_{1,1,k}, \\
    a_{2,1,k} &= z_{1,k} - a_{1,1,k}, \\
    a_{2,2,k} &= a_{1,1,k} + z_{2,k} - y_{1,k} = a_{1,1,k} + y_{2,k} - z_{1,k}.
\end{align*}
\]

In particular, \( P \) is linearly isomorphic to its projection

\[
Q = \left\{ (a_{1,1,k}) \in \mathbb{R}^{2\times 2\times n}_+: \alpha_k \leq a_{1,1,k} \leq \beta_k, \sum_k a_{1,1,k} = x_{1,1} \right\},
\]

where \( \alpha_k = \max\{0, z_{1,k} - y_{2,k}\} = \max\{0, y_{1,k} - z_{2,k}\} \) and \( \beta_k = \min\{y_{1,k}, z_{1,k}\} \). Now, by applying a translation to \( Q \), there is no loss of generality in assuming that \( \alpha_k = 0 \) for all \( k \). Then \( Q \) is a 1-way transportation polytope with bounded entries, isomorphic (by Lemma 3.1) to a \( 2 \times n \) transportation polytope.

Conversely, any \( 2 \times n \) transportation polytope

\[
Q = \left\{ (a_{j,k}) \in \mathbb{R}^{2\times n}_+: \sum_k a_{j,k} = x_j, \ a_{1,k} + a_{2,k} = y_k \right\},
\]

is linearly isomorphic to the following planar \( 2 \times 2 \times n \) transportation polytope:

\[
P = \left\{ (a_{i,j,k}) \in \mathbb{R}^{2\times 2\times n}_+: \sum_k a_{1,j,k} = x_j, \ \sum_i a_{i,1,k} = \sum_j a_{i,2,k} = \sum_j a_{1,j,k} = \sum_j a_{2,j,k} = y_k \right\}.
\]

The equations relating the solutions of \( Q \) to those of \( P \) are \( a_{j,k} = a_{1,j,k} = a_{2,3-j,k} \). \( \Box \)

One final comment. The above result is best possible since the list of \( 2 \times 3 \times 3 \) planar transportation polytopes presented in Table 2 is not the same as the list of \( 3 \times 3 \) classical transportation problems presented in Table 1.

We now move to Theorem 1.3. The number of vertices of a non-degenerate \( m \times n \) classical transportation polytope is divisible by \( \text{GCD}(m,n) \). The first observation, already hinted in Example 2.2, is that the vector configuration \( B_{m,n} \) associated to these transportation polytopes is (a cone over) the set of vertices of the product \( \Delta_{m,n} \) of two simplices of dimensions \( m-1 \) and \( n-1 \). So, we are interested in the cardinalities of chambers in the product of two simplices. Here and in what follows we call the cardinality of a chamber \( c \) of \( B_{m,n} \) the number of bases of \( B_{m,n} \) that contain the chamber \( c \). We denote it by \( |c| \). The proof of Theorem 1.3 consists of the following two steps, which are established respectively in the two lemmas below:
• There is a “seed” chamber in \( \Delta_{m,n} \) whose cardinality is indeed a multiple of \( \text{GCD}(m,n) \).

• The difference in the cardinalities of any two adjacent chambers of \( \Delta_m \times \Delta_n \) is a multiple of \( \text{GCD}(m,n) \).

Since the chamber complex is a connected polyhedral complex (where two adjacent chambers are divided by a hyperplane supported on the vector configuration) the two lemmas settle the proof.

Let us define the lexicographic chamber of \( \Delta_{m,n} \) recursively as the (unique) chamber incident to the lexicographic chamber of \( \Delta_{m,n} - 1 \). The recursion starts with \( \Delta_{m,1} \), which is an \((m-1)\)-simplex and contains a unique chamber. Observe that the definition of the lexicographic chamber is not symmetric in \( m \) and \( n \). For example, the lexicographic chamber of the triangular prism \( \Delta_{3,2} \) is the one incident to a basis of the prism, and has cardinality 3. The lexicographic chamber of \( \Delta_{2,3} \) is incident to one of the edges parallel to the axis of the prism, and has cardinality four.

**Lemma 3.2** The lexicographic chamber of \( \Delta_{m,n} \) is contained in exactly \( m^{n-1} \) simplices.

**Proof.** The cardinality of the lexicographic chamber of \( \Delta_{m,n} \) equals the cardinality of the lexicographic chamber of \( \Delta_{m,n-1} \) times the number of vertices of \( \Delta_{m,n} \) not lying in its facet \( \Delta_{m,n-1} \). The latter equals \( m \). \( \square \)

When moving from a chamber \( c_- \) to an adjacent one \( c_+ \) we “cross” a certain hyperplane \( \mathcal{H} \) spanned by all except one of the elements of any basis containing \( c_+ \) but not containing \( c_- \). Let us denote by \( C_+ \) and \( C_- \) the subsets of \( B_{m,n} \) lying in the sides of \( \mathcal{H} \) containing \( c_+ \) and \( c_- \) respectively (Remember that, in our case, \( B_{m,n} \) equals the set of vertices of \( \Delta_{m,n} \)). Observe also that the common boundary

![Figure 2: A cross-section of the chamber complex of some cone with two adjacent chambers](image)

\( c_0 \subset \mathcal{H} \) of \( c_+ \) and \( c_- \) is a chamber in the vector configuration \( B_{m,n} \cap \mathcal{H} \).

**Lemma 3.3**

1. \( |c_+| - |c_-| = |c_0||(C_+ - C_-)| \).
2. If $B_{m,n}$ is the set of vertices of $\Delta_{m,n}$, then $|C_+| - |C_-|$ is a multiple of $\text{GCD}(m,n)$.

Proof. A basis $b_+$ contains $c_+$ but not $c_-$ if and only if $b_+$ is of the form $b_0 \cup \{v_+\}$, where $b_0$ is a basis of $B \cap \mathcal{H}$ containing $c_0$ and $v_+$ is an element of $C_+$. This and the analogous property for $c_-$ proves the first part.

For the second part, we restate a few facts in the terminology of oriented matroids. This makes the proof easier to write (for details we recommend [6]):

- In oriented matroid terminology a pair $(C_+, C_-)$ consisting of the subconfigurations on one and the other side of a hyperplane $\mathcal{H}$ spanned by a subset of $B_{m,n}$ is called a cocircuit of $B_{m,n}$. That is, part 2 is a statement about the cocircuits in the oriented matroid $\mathcal{M}_{m,n}$ associated to the vertices of the product of two simplices.

- The oriented matroid $\mathcal{M}_{m,n}$ coincides with the one associated to the complete directed bipartite graph $K_{m,n}$. (i.e., the complete bipartite graph with all of its edges oriented from one part to the other). Thus, part 2 is a statement about the cocircuits in the oriented matroid of the directed $K_{m,n}$.

- The cocircuits of a directed graph $G = (V,E)$ are all read off from cuts in the graph. By this we mean that the vertex set $V$ is decomposed into two parts $(V_+, V_-)$. The cocircuit $(C_+, C_-)$ associated to the cut $(V_+, V_-)$ has $C_+$ consisting of all the edges directed from $V_+$ to $V_-$ and $C_-$ consisting of all the edges directed from $V_- to V_+$.

Using the dictionary between the directed graph $K_{m,n}$ and the product of simplices we can finish the proof. Let $(V_+, V_-)$ be a cut in the complete directed bipartite graph $K_{m,n}$. Since our graph is bipartite, we have $V_+$ and $V_-$ naturally decomposed as $V_+^{(m)} \cup V_+^{(n)}$ and $V_-^{(m)} \cup V_-^{(n)}$, respectively. The sizes of $C_+$ and $C_-$ are then:

$$|C_+| = |V_+^{(m)}| \cdot |V_-^{(n)}| \quad \text{and} \quad |C_-| = |V_-^{(m)}| \cdot |V_+^{(n)}|.$$  

Now, using that $|V_+^{(m)}| + |V_-^{(m)}| = m$ and $|V_+^{(n)}| + |V_-^{(n)}| = n$ we get:

$$|C_+| - |C_-| = |V_+^{(m)}| \cdot (n - |V_-^{(n)}|) - |V_-^{(n)}| \cdot (m - |V_+^{(m)}|) = |V_+^{(m)}| \cdot n - |V_-^{(n)}| \cdot m,$$

which is clearly a multiple of $\text{GCD}(m,n)$. \(\square\)
4 The diameter of 3-way axial transportation polytopes

Here we consider a 3-way axial transportation polytope $T_{x,y,z}$ defined by certain 1-marginal vectors $x$, $y$ and $z$. Recall that for bounding its diameter there is no loss of generality in assuming $T_{x,y,z}$ non-degenerate, that is, that $x$, $y$ and $z$ are sufficiently generic. In the non-degenerate case, at every vertex $V$ of our polytope exactly $lmn - l - m - n + 2$ variables are zero, and exactly $l + m + n - 2$ are non-zero. The set of triplets $(i,j,k)$ indexing non-zero variables will be called the support of the vertex $V$.

We say that a vertex $V$ of $T_{x,y,z}$ is well-ordered if the triplets $(i,j,k)$ that form its support are totally ordered with respect to the following coordinate-wise partial order:

$$(i,j,k) \leq (i',j',k') \quad \text{if} \quad i \leq i', \, j \leq j', \, \text{and} \, k \leq k'.$$ 

(1)

Observe that a set of $l + m + n - 2$ triplets satisfying this must contain exactly one triplet $(i,j,k)$ with $i + j + k = p$ for each $p = 3, \ldots, l + m + n$. Actually, supports of well-ordered vertices are the monotone staircases from $(1,1,1)$ to $(l,m,n)$ in the $l \times m \times n$ grid.

**Lemma 4.1** If $x$, $y$ and $z$ are generic, then $T_{x,y,z}$ has a unique well-ordered vertex $\hat{V}$.

**Proof.** Existence is guaranteed by the “northwest corner rule algorithm”, which fills the entries of the table in the prescribed order (see survey [28] or exercise 17 in Chapter six of [33]). More explicitly: let $\hat{V}_{l,m,n} = \min \{x_l, y_m, z_n\}$. Genericity implies that the three values $x_l$, $y_m$ and $z_n$ are different. Without loss of generality we assume that the minimum is $z_n$. Then, our choice of $\hat{V}_{l,m,n}$ makes $\hat{V}_{i,j,n} = 0$ for every other pair $(i,j)$. The rest of our vertex $\hat{V}$ is a vertex of the $l \times m \times (n-1)$ axial transportation polytope with margins $x' = (x_1, \ldots, x_{l-1}, x_l - z_n), \, y' = (y_1, \ldots, y_{m-1}, y_m - z_n)$, and $z' = (z_1, \ldots, z_{n-1})$.

Uniqueness follows from the same argument, simply noticing that the support of a well-ordered vertex always contains the entry $(l,m,n)$, and no other entry from one of the three planes $(l,*,*)$, $(*,m,*)$ and $(*,*,n)$. This, recursively, implies that the vertex can be obtained by the northwest corner rule.

**Remark 4.2** Another proof of Lemma 4.1 can be done using the formalism of chambers developed in the previous sections: it is obvious (and is proved in [11]) that if $c$ denotes a chamber of $B$, and $T$ is a triangulation of $\text{cone}(B)$, then there is a unique maximal-dimension simplex in $T$ that contains $c$. Thus, Lemma 4.1 follows from the fact that monotone staircases in the $l \times m \times n$ grid form a triangulation of the vector configuration $B_{l,m,n}$ of axial $l \times m \times n$ transportation polytopes. The latter is well-known, once we observe that $B_{l,m,n}$ is the vertex set of a product of three simplices. The triangulation in question is called the “staircase triangulation” of it (see Chapter 6 of [13]).
Example 4.3 To illustrate Lemma 4.1 consider the non-degenerate $3 \times 3 \times 3$ axial transportation polytope $T_{x,y,z}$ with:

$$
\sum_{j,k} a_{1,j,k} = 112 \quad \sum_{j,k} a_{2,j,k} = 18 \quad \sum_{j,k} a_{3,j,k} = 30 \\
\sum_{i,k} a_{i,1,k} = 40 \quad \sum_{i,k} a_{i,2,k} = 6 \quad \sum_{i,k} a_{i,3,k} = 114 \\
\sum_{i,j} a_{i,j,1} = 82 \quad \sum_{i,j} a_{i,j,2} = 44 \quad \sum_{i,j} a_{i,j,3} = 34
$$

The unique well-ordered vertex $\hat{V}$ of $T_{x,y,z}$ has the non-zero coordinates $a_{(1,1,1)} = 40$, $a_{(1,2,1)} = 6$, $a_{(1,3,1)} = 36$, $a_{(1,3,2)} = 30$, $a_{(2,3,2)} = 14$, $a_{(2,3,3)} = 4$, and $a_{(3,3,3)} = 30$. Note that the non-zero entries of $\hat{V}$ are totally ordered (they are presented above in increasing order) with respect to (1). Figure 3 depicts the associated monotone staircase.

Figure 3: A well-ordered vertex and its staircase.

Our bound on the diameter of $T_{x,y,z}$ is based on an explicit path that goes from any initial vertex $V$ of $T_{x,y,z}$ to the unique well-ordered vertex $\hat{V}$. To build this path we rely on the following stratified version of the concept of well-ordered vertex. We say that a vertex $V$ of $T_{x,y,z}$ is well-ordered starting at level $p$, where $p$ is an integer between 3 and $l + m + n$ if:

1. For each $q = p, \ldots, l + m + n$, the support of $V$ contains exactly one triplet $(i, j, k)$ with $i + j + k = q$.

2. Those triplets are well-ordered. (The partial order given in (1) is a total order on these triplets.)
3. All other triplets in the support have entries which are index-wise smaller than or equal to those of the unique triplet \((i_0, j_0, k_0)\) with \(i_0 + j_0 + k_0 = p\).

For example, the only vertex “well-ordered starting at level 3” is the well-ordered vertex \(\hat{V}\). Slightly less trivially, it is also the unique vertex “well-ordered starting at level 4”. On the other extreme, all vertices that contain \((l, m, n)\) as a support triplet are well-ordered starting at level \(l + m + n\). Observe that from any vertex of \(T_{x,y,z}\) we can move, by a single pivot edge in the sense of the simplex method, to another vertex containing any prescribed entry \((i, j, k)\) to be non-zero. In particular, we can move to a vertex that has \((l, m, n)\) in its support. So, we can assume from the beginning that \((l, m, n)\) is in the support of our initial vertex \(V\), and will add one to the count of edges traversed to arrive to \(\hat{V}\).

**Lemma 4.4** If \(V\) is a vertex of \(T_{x,y,z}\) that is well-ordered starting at level \(p \in \{5, \ldots, l + m + n\}\), then there is a path of at most \(2(p - 4)\) edges of \(T_{x,y,z}\) that leads from \(V\) to a vertex that is well-ordered starting at level \(p - 1\).

**Proof.** Let \((i_0, j_0, k_0)\) be the unique triplet in the support of \(V\) with \(i_0 + j_0 + k_0 = p\). We first observe that there is no loss of generality in assuming that \(p = l + m + n\) (that is, \((i_0, j_0, k_0) = (l, m, n)\)). This is because the vertices of \(T_{x,y,z}\) that are well-ordered starting at level \(p\) and agree with \(V\) in all the triplets with sum of indices greater than or equal to \(p\) are the vertices of a non-degenerate \(i_0 \times j_0 \times k_0\) axial transportation polytope, obtained as in the proof of Lemma 4.1.

So, from now on we assume that \(V\) is well-ordered starting at level \(p = l + m + n\). Let \(S_1\) be the set of support triplets in \(V\), other than \((l, m, n)\), that have first index equal to \(l\). Similarly, let \(S_2\) and \(S_3\) be the sets of support triplets that have, respectively, second and third indices equal to \(m\) and \(n\).

Our goal is to modify \(V\) until one of \(S_1\), \(S_2\) or \(S_3\) becomes empty, but always keeping the triplet \((l, m, n)\) in the support. Once this is done, a single pivot step can be used to obtain a vertex that is well-ordered starting at level \(p - 1\) as follows: Without loss of generality assume that \(S_1\) is empty (the cases when \(S_2\) or \(S_3\) are empty are treated identically). In particular, neither \((l, m - 1, n)\) nor \((l, m, n - 1)\) are in the support. If \((l - 1, m, n)\) is in the support then our vertex is already well-ordered starting at level \(p - 1\). If not, we do the pivot step that inserts \((l - 1, m, n)\). This pivot step cannot remove \((l, m, n)\) or insert \((l, m - 1, n)\) or \((l, m, n - 1)\) in the support. (The \((l, m, n)\) coordinate is not removed from the support since the entry remains constant in this pivot. The \((l, m - 1, n)\) and \((l, m, n - 1)\) coordinates remain zero because only non-zero entries of \(V\) and the entry \((l - 1, m, n)\) change in the pivot). This pivot produces a vertex well-ordered starting at level \(p - 1\). Figure 4 gives a picture for this case.

Given a vertex \(V\) well-ordered at level \(p\), we specify a sequence of pivots in the graph of \(T_{x,y,z}\) to a vertex \(V'\) such that one of \(S_1\), \(S_2\) or \(S_3\) is empty for \(V'\). Lemma 4.5 below shows how to get such a
Figure 4: A well-ordered vertex starting at level $p - 1$.

$V'$ in a number of steps bounded above by

$$2|S_1 \cup S_2 \cup S_3| - 3 \leq 2(p - 3) - 3 = 2p - 9.$$  

In one more step, that is, at most $2p - 8$, we get to a vertex that is well-ordered starting at level $p - 1$. This completes the proof of our lemma.  

For Lemma 4.5 let us introduce the following notation:

$$R_i := S_i \setminus (S_2 \cup S_3), \quad R_2 := S_2 \setminus (S_1 \cup S_3), \quad R_3 := S_3 \setminus (S_1 \cup S_2),$$  

$$R_{12} := S_1 \cap S_2, \quad R_{13} := S_1 \cap S_3, \quad R_{23} := S_2 \cap S_3.$$  

That is, $R_i$ consists of the elements of $S_1 \cup S_2 \cup S_3$ that belong only to $S_i$, and $R_{ij}$ of those that belong to $S_i$ and $S_j$. Observe that, by definition, no element belongs to the three $S_i$'s, so that $S_1 \cup S_2 \cup S_3$ is the disjoint union of these six subsets.

**Lemma 4.5** With the above notation and the conditions of the proof of the previous lemma, suppose that no $S_i$ is empty. Then:

1. If both $R_i$ and $R_{jk}$ are non-empty, with $\{i,j,k\} = \{1,2,3\}$, then there is a single pivot step that decreases $|S_1 \cup S_2 \cup S_3|$.

2. If the three $R_{ij}$'s are non-empty, then there is a sequence of two pivot steps that decreases $|S_1 \cup S_2 \cup S_3|$.

3. If the three $R_i$'s are non-empty, then there is a sequence of two pivot steps that decreases $|S_1 \cup S_2 \cup S_3|$.

4. If none of the above happens, then $S_1 \cup S_2 \cup S_3$ is contained in one of the $S_i$'s, say $S_1$. Then, there is a sequence of $|S_1| - 1$ pivot steps that makes $S_1 \cup S_2 \cup S_3$ empty.
All in all, there is a sequence of at most $2|S_1 \cup S_2 \cup S_3| - 3$ pivot steps that makes some $S_i$ empty.

**Proof.** Let us first show how the conclusion is obtained. We argue by induction on $|S_1 \cup S_2 \cup S_3|$, the base case being $|S_1 \cup S_2 \cup S_3| = 2$, which is the minimum to have no $S_i$ empty. The base case implies we are in the situation of either part (1) or part (4), and a single pivot step makes an $S_i$ empty.

If $|S_1 \cup S_2 \cup S_3| > 2$ and one of the conditions (1), (2) or (3) holds, then we do the step or the two pivot steps mentioned there and apply induction. If none of these three conditions hold then it is easy to see that (4) must hold. (Remember that we are assuming that no $S_i$ is empty, and $S_i = R_i \cup R_{ij} \cup R_{ik}$). Part (4) guarantees we have a sequence of $|S_1| - 1 \leq 2|S_1 \cup S_2 \cup S_3| - 3$ pivot steps that makes an $S_i$ empty.

So, let us prove each of the four items in the lemma. Let $\alpha$ denote the entry $(l, m, n)$.

1. Suppose without loss of generality that $R_{12}$ and $R_3$ are not empty. Let $\beta \in R_{12}$ and $\gamma \in R_3$. The reader may find it useful to follow our proof using Figure 5 which depicts the situation. Observe that $\alpha = (l, m, n)$ is the index-wise maximum of $\beta$ and $\gamma$. Let $\delta$ be the index-wise minimum of them. First observe that $\delta$ is not in the support of vertex $V$. Otherwise, we could add $\pm \frac{1}{2} \min\{a_\alpha, a_\beta, a_\gamma, a_\delta\} (e_\alpha + e_\delta - e_\beta - e_\gamma)$ to $V$ and stay in $T_{x,y,z}$. Hence, $V$ would be a convex combination of two other points from $T_{x,y,z}$ (and thus not a vertex), parallel to the direction of $e_\alpha + e_\delta - e_\beta - e_\gamma$ (here $e_{i,j,k}$ denotes the basis unit vector in the direction of the variable $a_{i,j,k}$).

Next, consider $V' = V + \min\{a_\beta, a_\gamma\} (e_\alpha + e_\delta - e_\beta - e_\gamma)$. Observe that $V'$ has different support than $V$ since either $\beta$ or $\gamma$ has been removed (not both, because $a_\beta \neq a_\gamma$ by non-degeneracy). Also, since $V'$ cannot have support strictly contained in that of $V$, $\delta$ must have been added. That is, the supports of the vertices $V$ and $V'$ differ in the deletion and insertion of a single element, which means they are adjacent in the graph of the polytope $T_{x,y,z}$. As desired, when going from $V$ to $V'$ the cardinality of $S_1 \cup S_2 \cup S_3$ is decreased by one.
Example 4.6 (Example 4.3 continued) Consider the vertex $V$ with non-zero coordinates $a(1,2) = 28$, $a(2,1) = 12$, $a(2,2,3) = 6$, $a(1,3,2) = 2$, $a(1,3,1) = 82$, $a(3,3,2) = 2$, and $a(3,3,3) = 28$.

In this example, $\alpha = (3,3,3)$, $\beta = (3,3,2)$, $\gamma = (2,2,3)$, and $\delta = (2,2,2)$. After clearing $a_\beta$, we arrive at the vertex $V'$ with non-zero coordinates $a'_(1,1) = 28$, $a'_(2,1) = 12$, $a'(2,2,3) = 4$, $a'(1,3,2) = 2$, $a'(1,3,1) = 82$, $a'(2,2,2) = 2$, and $a'(3,3,3) = 28$.

2. Suppose now that none of the $R_{ij}$'s is empty, and let $\beta \in R_{13}$, $\gamma \in R_{23}$, and $\delta' \in R_{12}$. We apply the same pivot as in case one, which makes $\delta$, the coordinate-wise minimum of $\beta$ and $\gamma$, enter the support. This pivot does not decrease $|S_1 \cup S_2 \cup S_3|$, but it leads to a situation where we have $\delta \in R_3$ and $\delta' \in R_{12}$. Hence, we can apply part one and decrease $|S_1 \cup S_2 \cup S_3|$ with a second step.

4. Let us prove now part (4) and leave (3), which is more complicated, for the end. Observe that if $S_1 \cup S_2 \cup S_3 = S_1$ but $S_2$ and $S_3$ are not empty, then necessarily $R_{12}$ and $R_{13}$ are both non-empty. While this holds, we can do the same pivot steps as before with a $\beta \in R_{12}$ and a $\gamma \in R_{13}$. Each step decreases by one the cardinality of $R_{12} \cup R_{13}$, increasing the cardinality of $R_1$. The process finishes when $R_{12}$ (hence $S_2$) or $R_{13}$ (hence $S_3$) becomes empty, which happens, in the worst case, in $|S_1| - 1$ steps.

3. Finally, consider the case where the three $R_i$'s are non-empty. Let $\beta = (l,j_1,k_1) \in R_1$, $\gamma = (i_2,m,k_2) \in R_2$, and $\delta = (i_3,j_3,n) \in R_3$, and, as before, $\alpha = (l,m,n)$. Figure 6 depicts the situation to help the reader with following our proof. Let $\epsilon_1$ and $\epsilon_2$ be two triplets of indices with the property that $\{\alpha, \epsilon_1, \epsilon_2\}$ and $\{\beta, \gamma, \delta\}$ use exactly the same three first indices, the same three second indices, and the same three third indices. For example, let us make the choice

![Figure 6](image-url)
\( \epsilon_1 = (i_2, j_1, k_1) \) and \( \epsilon_2 = (i_3, j_3, k_2) \) as in the left part of Figure 6. By non-degeneracy, the smallest value among \( a_\beta, a_\gamma \) and \( a_\delta \) at \( V \) is unique. We assume without loss of generality that the smallest among them is \( a_\beta \). Let \( W \) be the point of \( T_{x,y,z} \) obtained by changing the following six coordinates:

\[
\begin{align*}
    a'_\alpha &= a_\alpha + a_\beta, \\
    a'_\beta &= a_\beta - a_\beta = 0, \\
    a'_\gamma &= a_\gamma - a_\beta, \\
    a'_\delta &= a_\delta - a_\beta, \\
    a'_{\epsilon_1} &= a_{\epsilon_1} + a_\beta, \\
    a'_{\epsilon_2} &= a_{\epsilon_2} + a_\beta.
\end{align*}
\]

It may occur that \( \epsilon_1 = \epsilon_2 \), as shown in the right side of Figure 6. Then we do the same pivot except we increase the corresponding entry \( a_{\epsilon_1} = a_{\epsilon_2} \) twice as much.

Observe that one of \( \epsilon_1 \) or \( \epsilon_2 \) may already be in the support of \( V \), but not both: Otherwise \( W \) would have support strictly contained in that of \( V \), which is impossible because \( V \) is a vertex and has minimal support. If one of \( \epsilon_1 \) or \( \epsilon_2 \) were already in the support of \( V \), or if \( \epsilon_1 = \epsilon_2 \), then \( W \) is a vertex and we take \( V' = W \). As in the first case, \( V' \) is obtained from \( V \) by traversing a single edge and has one less support element in \( S_1 \cup S_2 \cup S_3 \) than \( V \): None of \( \epsilon_1 \) or \( \epsilon_2 \) can have a common entry with \( \alpha \), since none of \( \beta, \gamma \) and \( \delta \) has two common entries with \( \alpha \).

However, if \( \epsilon_1 \) and \( \epsilon_2 \) are different and none of them was in \( V \), then \( W \) has one-too-many elements in its support to be a vertex, which means it is in the relative interior of an edge \( E \) and \( L = VW \) is not an edge. See Figure 7. Moreover, both \( E \) and \( VW \) lie in a two-dimensional face \( F \). This is so because every support containing the support of a vertex defines a face of dimension equal the excess of elements it has. In our case, \( F \) is the face with support \( \text{support}(V) \cup \text{support}(W) = \text{support}(V) \cup \{\epsilon_1, \epsilon_2\} \).

![Figure 7: The octagon containing segment VW arising when entries \( \epsilon_1 \) and \( \epsilon_2 \) are different and none of them was in the support of \( V \)](image)

We now look more closely at the structure of \( F \). Each edge \( H \) of \( F \) is the intersection of \( F \) with a facet of our transportation polytope. That is, there is a unique variable \( \eta \) that is constantly
zero along $H$ but not zero as we move on $F$ in other directions. For example, since $\epsilon_1$ and $\epsilon_2$ are zero at $V$ but not constant on $F$ (they increase along $L$), $V$ is the common end of the edges defined by $\epsilon_1$ and $\epsilon_2$.

Our goal is to show that there is a vertex $V'$ of $F$ at distance at most two from $V$ and incident to the edge defined by one of the variables $\beta$, $\gamma$ and $\delta$. At such a vertex $V'$ we will have decreased $|S_1 \cup S_2 \cup S_3|$ by one, as claimed. The key remark is that there are at most two edges of $F$ not produced by one of the variables $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon_1$, and $\epsilon_2$: every variable $\eta$ other than those six is constant along $L$, so it either produces an edge parallel to $L$ or no edge at all. In particular, $F$ is at most an octagon, as in Figure 7. Now:

- If $F$ has five or less edges, then every vertex of $F$ is at distance one or two from $V$. Take as $V'$ either end of the end-point $W$ of $L$. This works because at $W$ one of $\beta$, $\gamma$ or $\delta$ is zero, by construction.
- If $F$ has six or more edges, then the two vertices $V'$ and $V''$ of $F$ at distance two from $V$ are at distance at least two from each other. So, together they are incident to four different edges, none of which is the edge of $\epsilon_1$ or $\epsilon_2$. (Remember that the edges of $\epsilon_1$ and $\epsilon_2$ are incident to $V$). At least one of these four edges is defined by $\beta$, $\gamma$, or $\delta$, because there are (at most) three other possible edges: the one of $\alpha$ and two parallel to $L$.

Example 4.7 To make ideas completely clear, using the same polytope $T_{x,y,z}$ as in Example 4.3, we consider its vertex $V$ with non-zero coordinates $v_{(1,3)} = 25$, $v_{(3,1,1)} = 15$, $v_{(3,2,1)} = 6$, $v_{(1,3,1)} = 61$, $v_{(1,3,2)} = 26$, $v_{(2,3,2)} = 18$ and $v_{(3,3,3)} = 9$. Here, $\alpha = (3,3,3)$, $\beta = (3,2,1)$, $\gamma = (2,3,2)$, $\delta = (1,1,3)$, $\epsilon_1 = (2,2,1)$, and $\epsilon_2 = (1,1,2)$. The triplet $\beta$ is not in the support of $W$, and $W$ has non-zero coordinates $w_{(1,1,3)} = 19$, $w_{(1,1,2)} = 6$, $w_{(3,1,1)} = 15$, $w_{(2,2,1)} = 6$, $w_{(1,3,1)} = 61$, $w_{(1,3,2)} = 26$, $w_{(2,3,2)} = 12$, and $w_{(3,3,3)} = 15$.

The vertices of $T_{x,y,z}$ with support contained in support($V$) $\cup \{\epsilon_1, \epsilon_2\}$ form the 4-gon $F = VBCD$ where $B$ is the vertex with non-zero coordinates

$$b_{(1,3)} = 12, b_{(1,1,2)} = 26, b_{(3,1,1)} = 2, b_{(3,2,1)} = 6, b_{(3,3,3)} = 22, b_{(2,3,2)} = 18, b_{(1,3,1)} = 74,$$

$C$ is the vertex with non-zero coordinates

$$c_{(1,1,3)} = 22, c_{(3,1,1)} = 18, c_{(2,2,1)} = 6, c_{(3,3,3)} = 12, c_{(1,3,2)} = 32, c_{(2,3,2)} = 12, c_{(1,3,1)} = 58$$

25
and $D$ is the vertex with non-zero coordinates

\[ d_{(1,1,3)} = 6, d_{(1,1,2)} = 32, d_{(3,1,1)} = 2, d_{(2,2,1)} = 6, d_{(3,3,3)} = 28, d_{(2,3,2)} = 12, d_{(1,3,1)} = 74. \]

Note $W$ is in the edge $E = CD$. We let $V' = D$, the endpoint of $E$ closer to $V$. Thus, we use one edge to go from $V$ to $V'$.

**Proof of Theorem 1.1** Starting with any vertex “well-ordered starting at level $p = l + m + n$” (which can be reached in a single step) we use Lemma 4.4 to decrease one unit by one the level at which our vertex starts to be well-ordered until we reach the unique well-ordered vertex $\hat{V}$. Thus, the number of steps needed to go from an arbitrary vertex $V$ to $\hat{V}$ is at most

\[ 1 + \sum_{q=5}^{p} 2(q - 4) = 1 + 2 \sum_{q=1}^{p-4} q = 1 + 2 \left( \frac{p - 3}{2} \right)^2 \leq (p - 3)^2. \]

To go from one arbitrary vertex to another, twice as many steps suffice. ☐

**Remark:** The whole proof can be generalized to arbitrary axial $d$-way tables, instead of $d = 3$, without much effort. Everything in Lemma 4.1 goes through without change, as well as the definition of “well-ordered starting at level $p$”. In the other arguments, the first change is that we have $d$ sets $S_1, \ldots, S_d$ instead of just three. In particular, in the proof of Lemma 4.5 the worst case will be that of $d - 1$ different $\epsilon$’s, which gives a face of dimension $d - 1$. Hence, the bound given in the statement of Lemma 4.4 can be substituted to the maximum diameter of a simple polytope of dimension $d - 1$ with at most $p$ facets. This still yields a polynomial bound for any fixed value of $d$. We leave the details for the interested reader.

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