AN INDEX THEORY FOR ZERO ENERGY SOLUTIONS OF THE
PLANAR ANISOTROPIC KEPLER PROBLEM

XI JUN HU
Department of Mathematics, Shandong University, P.R. China

GUOWEI YU
Dipartimento di Matematica “Giuseppe Peano”, Università degli Studi di Torino,
Italy

Abstract. In the variational study of singular Lagrange systems, the zero
energy solutions play an important role. In this paper we find a simple way
of computing the Morse indices of these solutions for the planar anisotropic
Kepler problem. In particular an interesting connection between the Morse in-
dices and the oscillating behaviors of these solutions discovered by the physicist
M. Gutzwiller is established.

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tions, index theory

1. Introduction

Lagrangian systems with singular potentials have been studied by many authors
due to their connection with celestial mechanics and relevant problems in physics,
see [5], [6], [2], [3], [33] and the references within. In this paper, we study the
2-dimension singular Lagrangian system
\[ \ddot{x}(t) = \nabla U(x(t)), \quad x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2, \]
with \( U \) being a positive, \((-\alpha)\)-homogeneous potential for some \( 0 < \alpha < 2 \), i.e.
\[ U(x) = \frac{\Omega(|x|)}{|x|^{\alpha}}, \quad \text{where} \quad \Omega \in C^2(S^1, (0, +\infty)) \text{ and } S^1 := [-\pi, \pi]/\{\pm \pi\}. \]

This can be seen as a generalization of the planar anisotropic Kepler problem,
introduced by physicist Gutzwiller ([20], [21]) and further studied by Devaney ([17],
[18]), where
\[ U(x) = \frac{1}{\sqrt{\mu x_1^2 + x_2^2}}, \quad \text{for some } \mu > 1. \]

E-mail addresses: xjhu@sdu.edu.cn, guowei.yu@unito.it.
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Such a potential describes the motion of an electron in a semiconductor by an impurity of the donor type and reveals the connection between chaotic behaviors in classic and quantum mechanics. The general case we are considering also applies to the Kepler problem and the isosceles three body problem, see Section 5.

Solutions of (11) are critical points of the action functional

\[ \mathcal{F}(x; t_1, t_2) := \int_{t_1}^{t_2} L(x(t), \dot{x}(t))dt, \quad x \in W^{1,2}([t_0, t_1], \mathbb{R}^2 \setminus \{0\}) \]

where the Lagrangian

\[ L(x, \dot{x}) = K(\dot{x}) + U(x) = \frac{1}{2} |\dot{x}|^2 + U(x). \]

The corresponding Hamiltonian \( H(\dot{x}(t), x(t)) = K(\dot{x}(t)) - U(x(t)) \) represents the total energy and is a constant along a solution. Under polar coordinates \( x = (r, \theta) = (r \cos \theta, r \sin \theta) \), \( (r, \theta) \in [0, +\infty) \times \mathbb{S}^1 \), we find these zero energy solutions interesting due to the following reasons: first, in the variational study of the singular Lagrange systems, they be used to build up complex trajectories, see [28] and [29]; second, in [9], [10] and [16], the existence/absence of parabolic solutions is shown to be connected with the absence/existence of collision in the action minimizers of the Bolza problem (fixed-end); third, in the variational study of the singular Lagrange systems, they...
are usually what one gets after the blow-up argument\cite{33, 19} and play a key role in proving the absence of collision in the corresponding critical points.

The main novelty of our paper is to study these solutions from an index theory point of view. To be precise, given a zero energy solution $x(t), \ t \in (T^-, T^+)$, we define its Morse index as

$$m^-(x) = \lim_{n \to \infty} m^-(x; t_n^-, t_n^+).$$

where $T^- < t_n^- < t_n^+ < T^+$ satisfies $\lim_{n \to +\infty} t_n^\pm = T^\pm$. For any $t_1 < t_2$, $m^-(x; t_1, t_2)$ is the dimension of the largest subspace of $W^{1,2}_0([t_1, t_2], \mathbb{R}^2 \setminus \{0\})$, where the second derivative $d^2F(x; t_1, t_2) < 0$. By the monotone property in \cite{15},

$$m^-(x; t_1, t_2) \leq m^-(x; t_1^+, t_2^+), \ \text{if} \ \ t_1^+ \leq t_1, t_2 \leq t_2^+.$$

Hence $m^-(x)$ is well defined and independent of the choice of $t_n^\pm$.

The computation of Morse index is not an easy job, especially along the directions that are not orthogonal to the solution. Our result gives a simple way of computing the Morse index of a zero energy solution, and quite interestingly it is connected with the oscillating behavior of the solution discovered numerically by Gutzwiller and proven analytically by Devaney:

Let $\alpha = 1$ and $\Phi(x) = (\mu \cos^2 \theta + \sin^2 \theta)^{-\frac{1}{2}}$ with $\mu > 1$, then $\{-\pi/2, 0, \pi/2, \pi\}$ are the critical points of $\Phi$. If $x(t) = (r \cos \theta, r \sin \theta)(t)$ is a collision solution of (1) with $x(0) = 0$ (not necessarily with zero energy), as $t \to 0$, $\theta(t)$ converges to one of the critical point. When the the critical point belongs to $\{\pm \pi/2\}$, then when $\mu > 9/8$, the corresponding trajectory in $\mathbb{R}^2$ oscillates along the vertical axis $\{x_1 \equiv 0\}$, as it approaches to the origin; meanwhile when the critical point belongs to $\{0, \pi\}$, then such oscillating behavior does not exist along the horizontal axis $\{x_2 \equiv 0\}$. See Figure 4 for corresponding numerically simulations, where the corresponding graphs of the function $\theta(\tau)$ are given ($\tau$ is a new time parameter that will be given later). This may also be seen from the phase portrait given in Figure 4.

Inspired by the above phenomena, we call $i(x)$ the oscillation index of $x(t)$:

$$i(x) := \begin{cases} \#\{t \in (T^-, T^+) | \dot{\theta}(t) = 0\}, & \text{if} \ x(t) \ \text{is non-homothetic,} \\ 0, & \text{if} \ x(t) \ \text{is homothetic.}\end{cases}$$

Remark 1.1. If $x(t)$ is homothetic, $\dot{\theta}(t) \equiv 0, \ \forall t$, so there is no oscillation at all. Meanwhile if $x(t)$ is non-homothetic, by Remark 2.1, $\{t \in (T^-, T^+) | \dot{\theta}(t) = 0\}$ is isolated in $(T^-, T^+)$. 

\begin{center}
\begin{tabular}{cc}
(A) $\theta(t) \to \pi/2$ & (B) $\theta(t) \to 0$
\end{tabular}
\end{center}

\textbf{Figure 1}
Theorem 1.2. Let $x(t) = (r\cos\theta, r\sin\theta)(t)$, $t \in (T^-, T^+)$, be a non-homothetic zero energy solution of $\mathcal{U}$ with $\lim_{t \to T^+} \theta(t) = \theta_0^+$. Then $\theta_0^\pm$ are critical points of $\mathcal{U}$. Moreover when both of them are non-degenerate, i.e. $\mathcal{U}_{\theta\theta}(\theta_0^\pm) \neq 0$, then

(a) if at least one of $\Delta(\theta_0^\pm)$ is negative, then $m^-(x) = i(x) = +\infty$, where

\[ \Delta(\theta) := \frac{(2 - \alpha)^2}{2}\mathcal{U}(\theta) + 4\mathcal{U}_{\theta\theta}(\theta), \quad \forall \theta \in \mathbb{S}^1, \]

(b) if both $\Delta(\theta_0^\pm)$ are positive, then $m^-(x) - i(x) = 0$ or 1, and in particular, $m^-(x) - i(x) = 0$, when $\theta_0^-$ is a local minimizer of $\mathcal{U}$.

Because of degeneracy, Theorem 1.2 does not hold for homothetic solutions. Instead we have the following result.

Theorem 1.3. Let $\bar{x}(t) = \bar{r}(t)(\cos\theta_0, \sin\theta_0)$ be a homothetic zero energy solution of $\mathcal{U}$, where $\theta_0$ is a critical point of $\mathcal{U}$, then

(a) if $\Delta(\theta_0) < 0$, $m^-(\bar{x}) = +\infty$,

(b) if $\Delta(\theta_0) \geq 0$, $m^-(\bar{x}) = 0$.

Remark 1.2. (1) Each critical point $\theta_0$ of $\mathcal{U}$ corresponds to two equilibria on the collision manifold and the sign of $\Delta(\theta_0)$ is related to the spectra of the linearized vector field at those equilibria: when $\Delta(\theta_0) < 0$, $(\psi_0, \theta_0)$ is a stable (or unstable) focus with the nearby orbits asymptotically spiral into (or away from) $(\psi_0, \theta_0)$ (see Section 2).

(2) When $\Delta(\theta_0) < 0$, the Morse index of a collision solution of the $N$-body problem was first investigated in [8], where results similar to property (a) in both Theorem 1.2 and 1.3 were obtained.

(3) Recently in [7] the Morse indices of both collision and complete parabolic solutions of the $N$-body problem are studied in more details. In particular the case with $\Delta(\theta_0) > 0$ (called [BS]-condition) is also considered there.

Although we require the corresponding critical points of $\mathcal{U}$ to be non-degenerate in Theorem 1.2, our approach may still work even when they are not. This is important as the $N$-body problem is highly degenerate due to symmetries. As an example, the Kepler-type problem with $\mathcal{U}(\theta)$ being a constant, will be considered in Section 3.2.

Theorem 1.2 has the following corollary (for a proof see Section 4). A related result has been obtained recently in [30] for the planar three body problem.

Corollary 1.1. Following the notations from Theorem 1.2, if $x(t)$ be a parabolic solution with $\theta(t)$ converges to two non-degenerate global minimizers of $\mathcal{U}$, then $m^-(x) = i(x) = 0$.

The existence of parabolic solutions connecting two non-degenerate global minimizers of $\mathcal{U}$ have been studied for the anisotropic Kepler problem with two degrees of freedom in [9] and arbitrary finite degrees of freedom in [10], where they are found as collision-free minimizers in the entire domain of time (under additional topological constraints in [9]), so naturally their Morse index must be zero. Corollary 1.1 can be seen as a complementation of their results, as it says any parabolic solution connecting two global minimizers of $\mathcal{U}$ must have zero Morse index.

We believe our result could be useful in deepening the variational study of the singular Lagrange systems including the classic $N$-body problem. In recent years, many new periodic and quasi-periodic solutions have been found as collision-free...
minimizers in the $N$-body problem under symmetric and/or topological constraints (see [14], [19], [13], [35]). However no result is available through minimax methods due to the problem of collision. Results from [33], [11] and [34] show that the Morse indices of zero energy solutions could be used to rule collisions in minimax approaches.

Our paper is organized as follows: Section 2 contains a brief introduction of the McGehee coordinates; Section 3 gives the asymptotic analysis of the linear system along non-homothetic zero energy solutions, as they approach to the collision or infinity; Section 4, studies the relations between various indices and contains proofs of our main results; Section 5 contains some applications of our results in celestial mechanics; Section 6 gives a brief introduction of the Maslov index.

2. McGehee coordinates and dynamics on the collision manifold

This section is an introduction to McGehee coordinates [26]. The results are not new and essentially due to Devaney ([17] and [18]). Their proofs either can be found in the above references or follow from direct computations, so will be omitted.

The Hamiltonian corresponds to $L(r,\theta,\dot{r},\dot{\theta})$ given in (5) is

$$H(p_1,p_2,r,\theta) = \frac{1}{2}p_1^2 + \frac{1}{2}r^{-2}p_2^2 - r^{-\alpha}\mathcal{U}(\theta),$$

where $p_1 = \dot{r}, p_2 = r^2\dot{\theta}$.

Let $z = (p_1,p_2,r,\theta)^T$, the corresponding Hamiltonian system of (1) is

$$\dot{z} = J\nabla H(z),$$

where $J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$, and

$$\nabla H(z) = (p_1, r^{-2}p_2, \alpha r^{-1-\alpha}\mathcal{U}(\theta) - r^{-3}p_2^2, -r^{-\alpha}\mathcal{U}_\theta(\theta))^T.$$

Under the McGehee coordinates $(v,u,r,\theta)$

$$v = r^{\alpha/2}p_1 = r^{\alpha/2}\dot{r}, \quad u = r^{-1+\frac{\alpha}{2}}p_2 = r^{-1+\frac{\alpha}{2}}\dot{\theta},$$

and the new time parameter $\tau$ given by $dt = r^{1+\frac{\alpha}{2}}d\tau$, equation (11) becomes

$$\begin{aligned}
\dot{v}' &= \alpha u^2 + u^2 - \alpha\mathcal{U}(\theta), \\
\dot{u}' &= (\frac{\alpha}{2} - 1)uv + \mathcal{U}_\theta(\theta), \\
\dot{r}' &= rv, \\
\dot{\theta}' &= u,
\end{aligned}$$

where $'$ means $\frac{d}{d\tau}$ throughout the paper.

The vector field now is well-defined on the singular set $\mathfrak{M} := \{(v,u,r,\theta) : r = 0\}$.

Moreover it is an invariant sub-manifold of (13), which will be called the collision manifold. In McGehee coordinates, the energy identity reads

$$\frac{1}{2}(u^2 + v^2) - \mathcal{U}(\theta) = r^\alpha H.$$

As a result whenever $r = 0$ or $H = 0$,

$$u^2 + v^2 = 2\mathcal{U}(\theta).$$

Plug this into the first equation of (13), we get

$$v' = (1 - \frac{\alpha}{2})u^2,$$

so $v$ is a Lyapunov function of (13), i.e. it is non-decreasing along any orbit.
By (15), $\mathcal{M}$ is a 2-dim torus homeomorphic to $S^1 \times S^1$. We introduce a global coordinates $(\psi, \theta)$ with $\theta$ as above and $\psi$ as
\begin{equation}
\cos \psi = \frac{u}{\sqrt{2\Im(\theta)}}, \quad \sin \psi = \frac{\nu}{\sqrt{2\Im(\theta)}}.
\end{equation}
Then on $\mathcal{M}$, the vector field (13) has the following expression:
\begin{equation}
\begin{cases}
\psi' = (1 - \frac{\nu}{\sqrt{2\Im(\theta)}}) \cos \psi - \frac{\Im(\theta)}{\sqrt{2\Im(\theta)}} 
\sin \psi, \\
\theta' = \frac{\Im(\theta)}{\sqrt{2\Im(\theta)}} \cos \psi.
\end{cases}
\end{equation}

**Lemma 2.1.** (a). $(\psi_0, \theta_0) \in \mathcal{M}$ is an equilibrium of (18), if and only if $\psi_0 \in \{\pm \pi/2\}$ and $\theta_0$ is a critical point of $\Im$;
(b). If $(\psi, \theta)(\tau), \tau \in \mathbb{R}$, is a non-equilibrium solution of (18), then $\{\tau \in \mathbb{R} : \theta'(\tau) = 0\}$ is an isolated set in $\mathbb{R}$.

Consider the linearization of (18) at an equilibrium $(\psi_0, \theta_0) \in \mathcal{M}$:
\begin{equation}
M(\psi_0, \theta_0) = \begin{pmatrix}
\left( \frac{\alpha}{2} - 1 \right) \sqrt{2\Im(\theta_0)} \sin \psi_0 & -\frac{\Im(\theta_0) \sin \psi_0}{\sqrt{2\Im(\theta_0)}} \\
-\frac{\Im(\theta_0) \sin \psi_0}{\sqrt{2\Im(\theta_0)}} & 0
\end{pmatrix}.
\end{equation}

**Notation 2.1.** We set $\lambda_\pm(\psi_0, \theta_0)$ as the two eigenvalues of $M(\psi_0, \theta_0)$, and $e_\pm(\psi_0, \theta_0)$ the corresponding eigenvectors. If $\lambda_\pm(\psi_0, \theta_0)$ are real numbers, we always assume $\lambda_- (\psi_0, \theta_0) \leq \lambda_+ (\psi_0, \theta_0)$. When there is no confusion, we may omit $(\psi_0, \theta_0)$ in these notations.

For $\Delta(\theta_0)$ given in [9], whenever it is negative, $\sqrt{\Delta(\theta_0)}$ should be understood as the imaginary number $i \sqrt{|\Delta(\theta_0)|}$.

**Lemma 2.2.** Following the notations given as above, we have
\begin{align*}
\lambda_\pm &= -\frac{(2 - \alpha)}{4} \sqrt{2\Im(\theta_0)} \sin \psi_0 \pm \frac{1}{2} \sqrt{\Delta(\theta_0)}; \\
e_\pm &= \begin{pmatrix}
\frac{2 - \alpha}{4} \sin \psi_0 + \frac{\Delta(\theta_0)}{2 \sqrt{2\Im(\theta_0)}}, \\
\frac{\Delta(\theta_0)}{2 \sqrt{2\Im(\theta_0)}}, 1
\end{pmatrix}.
\end{align*}
Furthermore,
(a). when $\Im(\theta_0) > 0$, $\Delta(\theta_0) > 0$ and $\lambda_- < 0 < \lambda_+$;
(b). when $\Im(\theta_0) = 0$, $\Delta(\theta_0) > 0$ and
\begin{align*}
\lambda_- &< \lambda_+ = 0, \text{ if } \psi_0 = \pi/2; 0 = \lambda_- < \lambda_+, \text{ if } \psi_0 = -\pi/2;
\end{align*}
(c). when $0 > \Im(\theta_0) > -\frac{(2 - \alpha)^2}{8} \Im(\theta_0)$, $\Delta(\theta_0) > 0$ and
\begin{align*}
\lambda_- &< \lambda_+ < 0, \text{ if } \psi_0 = \pi/2; 0 < \lambda_- < \lambda_+, \text{ if } \psi_0 = -\pi/2;
\end{align*}
(d). when $\Im(\theta_0) = -\frac{(2 - \alpha)^2}{8} \Im(\theta_0)$, $\Delta(\theta_0) = 0$ and
\begin{align*}
\lambda_- &< \lambda_+ = 0, \text{ if } \psi_0 = \pi/2; \lambda_- = \lambda_+ > 0, \text{ if } \psi_0 = -\pi/2;
\end{align*}
(e). when $\Im(\theta_0) < -\frac{(2 - \alpha)^2}{8} \Im(\theta_0)$, $\Delta(\theta_0) < 0$ and
\begin{align*}
\Re(\lambda_\pm) &< 0, \text{ if } \psi_0 = \pi/2; \Re(\lambda_\pm) > 0, \text{ if } \psi_0 = -\pi/2.
\end{align*}
The following result is well-known, for a proof see [36].

**Lemma 2.3.** When $\theta_0$ is a non-degenerate critical point of $\Im$. Then
(a) If $\lambda_- < 0 < \lambda_+$, then $(\psi_0, \theta_0)$ is a saddle, with a 1-dim stable manifold and a 1-dim unstable manifold, which are tangent of linear subspace $\langle e_- \rangle$ and $\langle e_+ \rangle$ at $(\psi_0, \theta_0)$ respectively. See Figure 2.

(b) If $\lambda_- < \lambda_+ < 0$, then $(\psi_0, \theta_0)$ is a stable node. It is asymptotically stable with all the orbits asymptotically converge to $(\psi_0, \theta_0)$, when $t$ goes to positive infinity, along the linear subspace $\langle e_+ \rangle$, except two orbits which asymptotically converge to $(\psi_0, \theta_0)$ along the linear subspace $\langle e_- \rangle$. See Figure 3a.

(c) If $0 < \lambda_- < \lambda_+$, then $(\psi_0, \theta_0)$ is a unstable node. It is asymptotically unstable with all the orbits asymptotically converge to $(\psi_0, \theta_0)$, when $t$ goes to negative infinity, along the linear subspace $\langle e_+ \rangle$, except two orbits which asymptotically converge to $(\psi_0, \theta_0)$ along the linear subspace $\langle e_- \rangle$. See Figure 3b.

(d) If $\lambda_\pm \in \mathbb{C} \setminus \mathbb{R}$, with $\Re(\lambda_-) < 0$, then $(\psi_0, \theta_0)$ is a stable focus. It is asymptotically stable with all the orbits spiral into $(\psi_0, \theta_0)$. See Figure 3c.

(e) If $\lambda_\pm \in \mathbb{C} \setminus \mathbb{R}$, with $\Re(\lambda_+) > 0$, then $(\psi_0, \theta_0)$ is an unstable focus. It is asymptotically unstable with all the orbits spiral away from $(\psi_0, \theta_0)$. See Figure 3d.

Since $v$ is a Lyapunov function of the vector field on the collision manifold, besides the equilibria, there are no closed or recurrent orbits. As a result

**Corollary 2.1.** If the critical point of $U$ are isolated, any orbit in $\mathcal{M}$ is either an equilibrium or a heteroclinic orbit connecting two different equilibria.

Lemma 2.1, 2.2 and 2.3 give us a complete picture of the phase portraits of the vector field on $\mathcal{M}$ (see Figure 4 for numerical pictures when the potential is defined as in (3)). Let $(\psi, \theta)(\tau)$ be a heteroclinic orbit and $(\psi_0^\pm, \theta_0^\pm)$ two equilibria in $\mathcal{M}$ satisfying

$$\lim_{\tau \to \pm \infty} (\psi, \theta)(\tau) = (\psi_0^\pm, \theta_0^\pm),$$

![Figure 2. $(\psi_0, \theta_0) = (\pi/2, 0)$](image)
then correspondingly

\[ \lim_{\tau \to \pm \infty} (v, u, \theta)(\tau) = (\sqrt{2\Omega(\theta_0^\pm)} \sin \psi_0^\pm, \sqrt{2\Omega(\theta_0^\pm)} \cos \psi_0^\pm, \theta_0^\pm). \]

Since \( v \) is a Lyapunov function,

\[ \sqrt{2\Omega(\theta_0^-)} \sin \psi_0^- < \sqrt{2\Omega(\theta_0^+)} \sin \psi_0^+. \]

As a result, there are three different types of heteroclinic orbits in \( \mathcal{M} \):

- **type-I.** \( \psi_0^- = -\pi/2, \psi_0^+ = \pi/2; \)
- **type-II.** \( \psi_0^- = \psi_0^+ = \pi/2 \) and \( \Omega(\theta_0^-) < \Omega(\theta_0^+) \);
- **type-III.** \( \psi_0^- = \psi_0^+ = -\pi/2 \) and \( \Omega(\theta_0^-) > \Omega(\theta_0^+) \).

**Figure 3**

\[ (A) \ (\psi_0, \theta_0) = (\pi/2, \pi/2), \Delta > 0 \]

\[ (B) \ (\psi_0, \theta_0) = (-\pi/2, \pi/2), \Delta > 0 \]

\[ (C) \ (\psi_0, \theta_0) = (\pi/2, \pi/2), \Delta < 0 \]

\[ (D) \ (\psi_0, \theta_0) = (-\pi/2, \pi/2), \Delta < 0 \]
Lemma 2.4. Given a heteroclinic orbit \((\psi, \theta)(\tau)\) in \(\mathcal{M}\), if \((r, t)(\tau)\) satisfies
\[
\begin{aligned}
r' &= r\sqrt{2U(\theta)}\sin \psi, \\
t' &= r^{1+\alpha/2},
\end{aligned}
\]
then
(a). when \((\psi, \theta)(\tau)\) is type-I, \(\lim_{\tau \to \pm \infty} r(\tau) = \pm \infty\), \(\lim_{\tau \to \pm \infty} t(\tau) = \pm \infty\);
(b). when \((\psi, \theta)(\tau)\) is type-II, \(\lim_{\tau \to +\infty} r(\tau) = +\infty\), \(\lim_{\tau \to +\infty} t(\tau) = +\infty\), and \(\lim_{\tau \to -\infty} r(\tau) = 0\), \(\lim_{\tau \to -\infty} t(\tau) = T_0^- > -\infty\);
(c). when \((\psi, \theta)(\tau)\) is type-III, \(\lim_{\tau \to +\infty} r(\tau) = 0\), \(\lim_{\tau \to +\infty} t(\tau) = T_0^+ < +\infty\) and \(\lim_{\tau \to -\infty} r(\tau) = +\infty\), \(\lim_{\tau \to -\infty} t(\tau) = -\infty\).

Let \(x(t), t \in (T^-, T^+)\), be a zero energy solution of (11) and \(z(\tau), \tau \in \mathbb{R}\), the corresponding orbit of (11), we define \(\pi(z)(\tau) := (\psi, \theta)(\tau)\) as the projection of \(z(\tau)\) in the collision manifold.

Proposition 2.1. If the critical points of \(\Omega\) are isolated in \(S^1\), then
(a). \(\pi(z)(\tau)\) is an equilibrium in \(\mathcal{M}\), if and only if \(x(t)\) is homothetic;
(b). \(\pi(z)(\tau)\) is a type-I heteroclinic orbit, if and only if \(x(t)\) is a non-homothetic parabolic solution;
(c). \(\pi(z)(\tau)\) is a type-II heteroclinic orbit, if and only if \(x(t)\) is a non-homothetic collision-parabolic solution;
(d). \(\pi(z)(\tau)\) is a type-III heteroclinic orbit, if and only if \(x(t)\) is a non-homothetic parabolic-collision solution.

Remark 2.1. The above proposition implies Theorem 1.1 and for a non-homothetic zero energy solution \(x(t)\), Lemma 2.1 and Proposition 2.1 imply \(\{t \in (T^-, T^+): \dot{\theta}(t) = 0\}\) is isolated in \((T^-, T^+)\), as \(dt = r^{1+\frac{\alpha}{2}} d\tau\) and \(r(\tau) > 0\), for any \(\tau\).

Remark 2.2. For systems with arbitrary finite degrees of freedom, one can define the McGehee coordinates similarly with the corresponding \(v\) being a Lyapunov
function. Moreover the connection between zero energy solutions and orbits on the collision manifold still exist, so we expect results from this section will still hold.

3. ASYMPOTIC ANALYSIS OF THE LINEAR HAMILTONIAN SYSTEM

Throughout this section let $x(t), t \in (T^-, T^+)$, be a non-homothetic zero energy solution of (11) and $z(t) = (p_1, p_2, r, \theta)^T(t)$ the corresponding zero energy orbit of (11). Consider the linearized equation of (11) along $z(t)$

$$\dot{\xi}(t) = J\nabla^2 H(z(t))\xi(t).$$

Under the time parameter $\tau$ (notice that $t \to \pm \infty$, as $t \to T^\pm$),

$$\xi'(\tau) = J B(\tau) \xi(\tau) := r^{1+\frac{1}{2}}(\tau)J\nabla^2 H(z(\tau))\xi(\tau),$$

where

$$B(\tau) = \begin{pmatrix} r^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & r^{-\frac{1}{2}} & -2r^{\frac{1}{2}} p_2 & 0 \\ 0 & 0 & -\frac{2p_2}{r} & \frac{3p_2}{r} - \frac{\alpha(\alpha + 1)U(\theta)}{\alpha r^{-\frac{1}{2}}U_\theta(\theta)} \\ 0 & 0 & \frac{2p_2}{r} & -\frac{\alpha(\alpha + 1)U(\theta)}{\alpha r^{-\frac{1}{2}}U_\theta(\theta)} \end{pmatrix} (\tau).$$

Our main goal is to understand the asymptotic behavior of the above linear Hamiltonian system, as $\tau$ goes to $\pm \infty$. To separate the variable $r$, we define the following symplectic matrix

$$R(\tau) = \text{diag}(r^{\frac{1}{2}}, r^{\frac{1}{2}} - \frac{1}{2}, r^{\frac{1}{2}} - \frac{3}{2}, r^{\frac{1}{2}} - \frac{5}{2})(\tau).$$

Then for $\xi(\tau)$ satisfying (24), $\eta(\tau) = R(\tau)\xi(\tau)$ is a solution of

$$\eta'(\tau) = J \hat{B}(\tau)\eta(\tau),$$

where $\hat{B}(\tau) = -J R'(\tau) R^{-1}(\tau) + R^{-1}(\tau) B(\tau) R^{-1}(\tau)$.

Under McGehee coordinates (12),

$$\hat{B}(\tau) = \begin{pmatrix} 1 & 0 & -\frac{2+\alpha}{4} & 0 \\ 0 & 1 & -\frac{2-\alpha}{4} & 0 \\ -\frac{2+\alpha}{4} & -\frac{2-\alpha}{4} & 3\alpha^2 - \alpha(\alpha + 1)U(\theta) & \frac{2+\alpha}{4} \\ 0 & 0 & \alpha U_\theta(\theta) & -\alpha U_\theta(\theta) \end{pmatrix} (\tau).$$

Recall that the projection of $z(\tau)$ on the collision manifold, $(\psi, \theta)(\tau) = \pi(z)(\tau)$ is a heteroclinic orbit between two equilibria. Let $T^* = \pm \infty$, then

$$\lim_{\tau \to T^*} (\psi, \theta)(\tau) = (\psi_0^*, \theta_0^*),$$

where $\psi_0^* \in \{\pm \pi/2\}$ and $U_\theta(\theta_0^*) = 0$.

By (21),

$$\lim_{\tau \to T^*} (v, u, \theta)(\tau) = (\sin \psi_0^* \sqrt{2\lambda(\theta_0^*)}, 0, \theta_0^*).$$

This implies $\hat{B}_* := \lim_{\tau \to T^*} \hat{B}(\tau)$ exists. Moreover $\hat{B}_* = \hat{B}_*^{(1)} \circ \hat{B}_*^{(2)}$ with

$$\hat{B}_*^{(1)} := \begin{pmatrix} 1 & 0 & -\frac{2+\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* \\ 0 & 1 & -\frac{2-\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* \\ -\frac{2+\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* & -\frac{2-\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* \\ 0 & 0 & -\alpha(\alpha + 1)U(\theta_0^*) \end{pmatrix},$$

$$\hat{B}_*^{(2)} := \begin{pmatrix} 1 & 0 & -\frac{2+\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* \\ 0 & 1 & -\frac{2-\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* \\ -\frac{2+\alpha}{4} \sqrt{2\lambda(\theta_0^*)} \sin \psi_0^* & -\alpha(\alpha + 1)U(\theta_0^*) \\ 0 & 0 & -\alpha U_\theta(\theta_0^*) \end{pmatrix}. $$
The symplectic sum $\oplus$ is defined as in \cite{25}: for any two $2m_k \times 2m_k$ square block matrices, $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$, $k = 1, 2$, $M_1 \oplus M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$.

For $i = 1$ or 2, let $\hat{\lambda}_i^\pm(\psi_0^*, \theta_0^*)$ be the two eigenvalues of $J\hat{B}_i^\dagger(\psi_0^*, \theta_0^*) \leq \hat{\lambda}_i^\pm(\psi_0^*, \theta_0^*)$, when both of them are real, and $\hat{e}_i^\pm(\psi_0^*, \theta_0^*)$ the corresponding eigenvectors. When there is no confusion, we may omit $(\psi_0^*, \theta_0^*)$ in these notations. Direct computations give us the following lemma.

**Lemma 3.1.** $J\hat{B}_1^\dagger$ is a hyperbolic matrix with

$$\hat{\lambda}_1^\pm = \pm \frac{2 + 3\alpha}{4} \sqrt{2\mu(\theta_0^*)}, \quad \hat{e}_1^\pm = \left( \frac{2 + 3\alpha}{4} \sin \psi_0^* \pm \frac{2 + 3\alpha}{4} \sqrt{2\mu(\theta_0^*)}, 1 \right)^T.$$

$J\hat{B}_2^\dagger$ is a hyperbolic matrix, when $\Delta(\theta_0^*) > 0$, with

$$\hat{\lambda}_2^\pm = \pm \frac{1}{2} \sqrt{\Delta(\theta_0^*)}, \quad \hat{e}_2^\pm = \left( -\frac{2 - \alpha}{4} \sqrt{2\mu(\theta_0^*)} \sin \psi_0^* \pm \frac{1}{2} \sqrt{\Delta(\theta_0^*)}, 1 \right)^T.$$

Since $U$ is $(-\alpha)$-homogeneous, when $x(t) = (r(t), \theta(t))$ is solution of (11), so is $x_h(t) = (r_h(t), \theta_h(t)) := (h^{-\frac{2\alpha}{\alpha+2}} r(ht), \theta(ht))$, for any $h > 0$. This means

$$z_h(t) = J\nabla H_h(z_h(t)),$$

where $z_h(t) = (p_{1,h}, p_{2,h}, r_h, \theta_h)^T(t)$ with

$$p_{1,h}(t) = \dot{r}_h(t) = h^{-\frac{2\alpha}{\alpha+2}} \dot{r}(ht); \quad p_{2,h}(t) = r_h^2(t) \dot{\theta}_h(t) = h^{-\frac{2\alpha}{\alpha+2}} r^2(ht) \dot{\theta}(ht),$$

and

$$H_h(z_h(t)) = \frac{1}{2} \left( p_{1,h}^2(t) + r_h^{-2}(t) p_{2,h}^2(t) \right) - r_h^{-\alpha}(t) \mu(\theta_h(t)).$$

Let $h = 1$ and differentiate $(32)$ with respect to $t$, we get a solution of (23):

$$\zeta_1(t) := \dot{z}_1(t) = (\dot{r}, 2r \dot{\theta} + r^2 \dot{\theta}, \dot{r}, \dot{\theta})^T(t).$$

Meanwhile by differentiating $(32)$ with respect to $h$, we get

$$\frac{d\dot{z}_h}{dh}|_{h=1}(t) = J\nabla^2 H(z_1(t)) \left( \frac{dz_h}{dh}|_{h=1}(t) \right).$$

Hence $\zeta_3(t) := \frac{d\dot{z}_h}{dh}|_{h=1}(t)$ is another solution of (24). Define

$$\zeta_2(t) := \zeta_3(t) - t\zeta_1(t) = \left( \frac{\alpha}{2 + \alpha} \dot{r}, \frac{\alpha - 2}{\alpha + 2} r^2 \dot{\theta} - \frac{\alpha}{2 + \alpha} r, 0 \right)^T(t).$$

Under the time parameter $\tau$, using $R(\tau)$ given in (27), we find the following two solutions of the linear system (28):

$$\eta_1(\tau) = R(\tau)\zeta_1(\tau) = \frac{2\alpha}{\alpha+2} (\tau)(u^2 - \alpha \mu(\theta), \mu(\theta), v, u)^T(\tau),$$

$$\eta_2(\tau) = R(\tau)\zeta_2(\tau) = \frac{2\alpha}{\alpha+2} (\tau) \left( \frac{\alpha v}{2 + \alpha}, \frac{\alpha - 2}{\alpha + 2} u, -\frac{2}{2 + \alpha} 0 \right)^T(\tau).$$

**Definition 3.1.** For each $\tau \in \mathbb{R}$, we define $V(\tau) := \text{span}\{\eta_1(\tau), \eta_2(\tau)\}$ as the linear space generated by $\eta_1(\tau)$ and $\eta_2(\tau)$ defined as above.
Notice that \( \eta_1(\tau) \) and \( \eta_2(\tau) \) are linear independent if and only if \( x(t) \) is a non-homothetic solution.

Let \( (\mathbb{R}^4, \omega) \) with \( \omega(x, y) = (Jx, y) \) being the standard symplectic form on \( \mathbb{R}^4 \). A subspace \( V \subset \mathbb{R}^2 \) in Lagrangian, if \( \dim(V) = 2 \) and \( \omega|_V = 0 \). We denote by \( \text{Lag}(\mathbb{R}^4) \) the Lagrangian Grassmannian, i.e. the set of all Lagrangian subspaces of \( (\mathbb{R}^4, \omega) \). For any \( V \in \text{Lag}(\mathbb{R}^4) \), let \( P_V \) be the orthogonal projection of \( \mathbb{R}^4 \) to \( V \), then
\[
\text{dist}(W, W^*) := \| P_W - P_{W^*} \|, \quad \text{for any } W, W^* \in \text{Lag}(\mathbb{R}^4),
\]
gives a complete metric on \( \text{Lag}(\mathbb{R}^4) \). Here \( \| \cdot \| \) represents the metric on the space of bounded linear operators from \( \mathbb{R}^4 \) to itself.

**Lemma 3.2.** If \( x(t) \) is a non-homothetic zero energy solution, \( V(\tau) \in C^0(\mathbb{R}, \text{Lag}(\mathbb{R}^4)) \).

**Proof.** By a direct computation,
\[
\omega(\eta_1, \eta_2) = \frac{2\alpha}{2 + \alpha} r^{-\alpha} \left( \frac{1}{2} (y^2 + v^2) - \Omega(\theta) \right).
\]
Then the result follows from [14] and \( x(t) \) with 0 energy. \( \square \)

We will study the limit of \( V(\tau) \), as \( \tau \) goes to \( T^* \). For it to exist, \( J\hat{B}_s \) needs to be hyperbolic, and the precise limit depends on how the corresponding heteroclinic orbit \( (\psi, \theta)(\tau) \) approaches to the equilibrium \( (\psi_0^*, \theta_0^*) \) on the collision manifold.

When \( \Delta(\theta_0^*) > 0 \), by Lemma 2.2 and 2.3 \( (\psi, \theta)(t) \) converges to \( (\psi_0^*, \theta_0^*) \) either along the subspace \( \langle e_\pm \rangle \) or \( \langle e_\pm^* \rangle \), where \( e_\pm^* = e_\pm(\psi_0^*, \theta_0^*) \), see Notation 2.1

**Proposition 3.1.** Assume \( \Delta(\theta_0^*) > 0 \) and \( \Delta_\theta(\theta_0^*) \neq 0 \), when \( (\psi, \theta)(\tau) \to (\psi_0^*, \theta_0^*) \) along \( \langle e_\pm \rangle \), as \( \tau \to T^* \),
\[
\lim_{\tau \to T^*} V(\tau) = \text{span}\{ \hat{e}_1(\psi_0^*), \hat{e}_2(\psi_0^*) \}, \quad \text{where} \quad \hat{e}_1(\psi_0^*) = \begin{cases} e_1 & \text{if } \psi_0^* = -\pi/2 \\ e_1^* & \text{if } \psi_0^* = \pi/2 \end{cases}.
\]

We first give a proof of the above proposition using the following lemma.

**Lemma 3.3.** Assume \( \Delta(\theta_0^*) > 0 \) and \( \Delta_\theta(\theta_0^*) \neq 0 \), if \( (\psi, \theta)(\tau) \to (\psi_0^*, \theta_0^*) \) along \( \langle e_\pm^* \rangle \), as \( \tau \to T^* \), then
\[
\lim_{\tau \to T^*} \frac{\Lambda_\theta(\theta)}{u} = -\lambda(\psi_0^*, \theta_0^*) = \frac{2 - \alpha}{4} \sqrt{2\Omega(\theta_0^*)} \sin \psi_0^* + \frac{1}{2} \Delta(\theta_0^*).
\]

**Proof.** We only give details for \( \psi_0^* = \pi/2 \) and \( (\psi, \theta)(\tau) \) converges to \( (\psi_0^*, \theta_0^*) \) along \( \langle e_\pm^* \rangle \), while the others are similarly. Let \( e_i \in \mathbb{R}^4 \), \( i = 1, 2, 3, 4 \), be an orthogonal basis of \( \mathbb{R}^4 \) with the \( i \)-th component equal to 1 and the others all being zero.

Let \( V(\tau) = \text{span}\{ \eta_1(\tau), \eta_2(\tau) \} \) be defined as in Definition 3.1 then
\[
\eta_1 \wedge \eta_2 = \frac{u}{(2 + \alpha)^{\alpha}} \left( (2 - \alpha)(\alpha \Omega(\theta) - u^2) - \alpha v \frac{\Lambda_\theta(\theta)}{u} \right) e_1 \wedge e_2
\]
\[
- (2 - \alpha)u e_2 \wedge e_3 - \alpha v e_1 \wedge e_4 + (2 - \alpha)v - \frac{\Lambda_\theta(\theta)}{u} e_2 \wedge e_3
\]
\[
+ (2 - \alpha)u e_2 \wedge e_4 + 2e_3 \wedge e_4 \right\}.
\]
By [31] and Lemma 3.3 a direct computation shows

\[
\lim_{\tau \to T^*} V(\tau) = \text{span}\{ \hat{e}_1(\psi_0^*), \hat{e}_2(\psi_0^*) \}.
\]
\[ (37) \lim_{\tau \to T^*} \frac{(2 + \alpha)\rho^2}{u} = \left( \frac{\alpha(2 - \alpha)}{2} \mu(\theta_0^*) + \frac{\alpha}{2} \sqrt{2\mu(\theta_0^*)\Delta(\theta_0^*)} \right) e_1 \wedge e_2 \]

\[ - \alpha \sqrt{2\mu(\theta_0^*)} e_1 \wedge e_4 + \left( \frac{2 - \alpha}{2} \sqrt{2\mu(\theta_0^*)} + \sqrt{\Delta(\theta_0^*)} \right) e_2 \wedge e_3 + 2e_3 \wedge e_4. \]

Meanwhile for \( \dot{e}_1^1, \dot{e}_2^2 \) given in Lemma 3.1, a straightforward computation shows 2\( \dot{e}_1^1 \wedge \dot{e}_2^2 \) is the same as what we got in (37). This finishes our proof. \( \square \)

**Proof of Lemma 3.3** We will only give the details for \( T^* = +\infty \). As both \( \mu_0(\theta) \) and \( u \) goes to 0, when \( \tau \to +\infty \), by L'Hospital's rule,

\[ (38) \lim_{\tau \to +\infty} \frac{\mu_0(\theta)}{u} = \lim_{\tau \to +\infty} \frac{\mu_{00}(\theta)\theta'}{(\sqrt{2\mu(\theta)} \cos \psi)'} = -\frac{\mu_{00}(\theta_0^*)}{\sqrt{2\mu(\theta_0^*)} \sin \psi_0^*}, \]

If \( (\psi, \theta)(\tau) \) converges to \( (\psi_0^*, \theta_0^*) \) along \( \langle e_+^* \rangle \), as \( \tau \to +\infty \), by Lemma 2.2,

\[ \lim_{\tau \to +\infty} \frac{\psi_0^*}{\theta'} = \frac{2 - \alpha}{4} + \frac{\sin \psi_0^*}{2} \sqrt{\frac{\Delta(\theta_0^*)}{2\mu(\theta_0^*)}}. \]

Plug this into (38), we get

\[ \lim_{\tau \to +\infty} \frac{\mu_0(\theta)}{u} = \frac{2 - \alpha}{4} \sqrt{2\mu(\theta_0^*)} \sin \psi_0^* - \frac{1}{2} \sqrt{\Delta(\theta_0^*)} = -\lambda_+(\psi_0^*, \theta_0^*). \]

The second equality follows from Lemma 2.2.

Similarly if \( (\psi, \theta)(\tau) \) converges to \( (\psi_0^*, \theta_0^*) \) along \( \langle e_-^* \rangle \), as \( \tau \to +\infty \), then

\[ \lim_{\tau \to +\infty} \frac{\psi_0^*}{\theta'} = \frac{2 - \alpha}{4} - \frac{\sin \psi_0^*}{2} \sqrt{\frac{\Delta(\theta_0^*)}{2\mu(\theta_0^*)}}, \]

and

\[ \lim_{\tau \to +\infty} \frac{\mu_0(\theta)}{u} = \frac{2 - \alpha}{4} \sqrt{2\mu(\theta_0^*)} \sin \psi_0^* + \frac{1}{2} \sqrt{\Delta(\theta_0^*)} = -\lambda_-(\psi_0^*, \theta_0^*). \]

\( \square \)

4. Connect the Morse and oscillation indices by Maslov indices

In this section, except the last proof, which deals with the homothetic solution, we always assume \( x(t), t \in (T^-, T^+) \), is a non-homothetic zero energy solution of (1) with \( z(t) \) being the corresponding zero energy orbit of (11) and \( \pi(z)(\tau) \) the heteroclinic orbit on \( \mathfrak{m} \) satisfying \( \lim_{\tau \to \pm\infty} \pi(z)(\tau) = (\psi_0^+, \theta_0^+) \).

We need the *Maslov index* to connect the Morse and oscillation indices. For details of the Maslov index, see [12] or Section 5. Let \( \gamma(t, t_1) \) be the fundamental solution of the linear Hamiltonian equation (24):

\[ (39) \gamma(t, t_1) = J\nabla^2 H(z(t))\gamma(t, t_1), \quad \gamma(t_1, t_1) = I_4. \]

For any \( t_1 < t_2 \), we define the Maslov index of \( x(t), t \in [t_1, t_2] \) as

\[ \mu(V_d, \gamma(t, t_1)V_d; [t_1, t_2]), \quad \text{where } V_d := \mathbb{R}^2 \oplus 0. \]

By the *Morse Index Theorem* (see [24])

\[ (41) m^-(x; t_1, t_2) + 2 = \mu(V_d, \gamma(t, t_1)V_d; [t_1, t_2]). \]
Under the time parameter $\tau$, the corresponding $\gamma(\tau, \tau_1) := \gamma(t(\tau), t_1)$, where $t_1 = t(\tau_1)$, is the fundamental solution of (25), and $\hat{\gamma}(\tau, \tau_1) = R(\tau)\gamma(\tau, \tau_1)R^{-1}(\tau_1)$ ($R(\tau)$ is the matrix defined in (27)) is the fundamental solution of equation (25):

$$\hat{\gamma}(\tau, \tau_1) = J\hat{B}(\tau)\hat{\gamma}(\tau, \tau_1), \quad \hat{\gamma}(\tau_1, \tau_1) = I_4.$$  

**Lemma 4.1.** When $t_i = t(\tau_i)$, $i = 1, 2$,  
$$\mu(V_d, \hat{\gamma}(\tau, \tau_1) V_d; [\tau_1, \tau_2]) = \mu(V_d, \gamma(t, t_1) V_d; [t_1, t_2]).$$

**Proof.** First as the Maslov index is invariant under the change of time parameter,

$$\mu(V_d, \hat{\gamma}(t, \tau_1) V_d; [t_1, t_2]) = \mu(V_d, \gamma(\tau, \tau_1) V_d; [\tau_1, \tau_2]),$$

Meanwhile

$$\mu(V_d, \hat{\gamma}(\tau, \tau_1) V_d; [\tau_1, \tau_2]) = \mu(V_d, R(\tau)\gamma(\tau, \tau_1)R^{-1}(\tau_1) V_d; [\tau_1, \tau_2])$$

$$= \mu(R^{-1}(\tau) V_d, \gamma(\tau, \tau_1) R^{-1}(\tau_1) V_d; [\tau_1, \tau_2]) = \mu(V_d, \gamma(\tau, \tau_1) V_d; [\tau_1, \tau_2]).$$

The last equality follows from the fact that $R^{-1}(\tau)V_d = V_d$, for any $\tau$, as $R(\tau)$ is a diagonal matrix. \hfill \square

By the above lemma,

$$m^-(x; t_1, t_2) + 2 = \mu(V_d, \hat{\gamma}(\tau, \tau_1) V_d; [\tau_1, \tau_2]).$$

Then for any sequences $\tau_n^- < \tau_n^+$ satisfying $\lim_{n\to+\infty} \tau_n^\pm = \pm\infty$,

$$m^-(x) + 2 = \lim_{n\to+\infty} \mu(V_d, \hat{\gamma}(\tau, \tau_1) V_d; [\tau_n^-, \tau_n^+]).$$

To compute the above limit, we need another Maslov index. For any $\tau \in \mathbb{R}$, define the stable/unstable subspace $V^+(\tau)/V^-(\tau)$ of the linear system (42) as

$$V^\pm(\tau) := \{v \in \mathbb{R}^4 | \lim_{\sigma \to \pm\infty} \hat{\gamma}(\sigma, \tau)v = 0\}.$$ 

Notice that $V^\pm(\tau) = \hat{\gamma}(\tau, \sigma)V^\pm(\sigma)$, for any two $\sigma, \tau \in \mathbb{R}$.

**Definition 4.1.** We define the **Maslov index** $\mu(x)$ of $x$ as

$$\mu(x) := \mu(V_d, V^-(\tau); \mathbb{R}) = \lim_{T \to +\infty} \mu(V_d, V^-(\tau); [-T, T]).$$

The index $\mu(x)$ defined above was introduced in the study of heteroclinic orbits (see [22, 23] or the Appendix for more details).

At this moment it is not clear whether $\mu(x)$ is well defined. We will show this shortly. Following the notations from the previous section, we set $T^* = \pm\infty$. Recall that $J\hat{B}_\pm = \lim_{\tau \to T^*} J\hat{B}(\tau)$ is a hyperbolic matrix, when $\Delta(\theta_\ast) > 0$. Let $V^+(J\hat{B}_\pm)$ and $V^-(J\hat{B}_\pm)$ be the $J\hat{B}_\pm$ invariant subspaces of $\mathbb{R}^4$ corresponding to eigenvalues with positive and negative real part respectively. By Lemma 3.1

$$V^\pm(J\hat{B}_\pm) = \text{span}\{e^1_\pm, e^2_\pm\}.$$ 

In the following, we may need to specify the value of $T^*$, in those cases we set $J\hat{B}_{\pm} := \lim_{\tau \to \pm\infty} J\hat{B}(\tau)$. The next lemma follows from [1, Theorem 2.1].

**Lemma 4.2.** When $J\hat{B}_\pm$ are hyperbolic matrices.

(a) $V^\pm(\tau)$ is the only linear subspace of $\mathbb{R}^4$ satisfying $\hat{\gamma}(\sigma, \tau)V^\pm(\tau) \to V^\mp(J\hat{B}_{\pm})$, as $\sigma \to \pm\infty$.
(b). If a linear subspace \( W \) of \( \mathbb{R}^4 \) is topologically complement of \( V^+(\tau) \) (or \( V^-(\tau) \)), then for any \( v \in W \setminus \{0\} \), \( \gamma(\sigma, \tau)v \to +\infty \) exponentially fast, as \( \sigma \to +\infty \) (or \( \tau \to -\infty \)), and \( \hat{\gamma}(\sigma, \tau)W \to V^+(J\hat{B}_+) \) (or \( V^-(J\hat{B}_-) \)) at the same time.

In the following, for \( i = 1, 2 \), let \( \eta_i(\tau), \tau \in \mathbb{R} \), be two solutions of the linear system \( (28) \) given in \( (31) \) and \( (35) \), and \( V(\tau) = \text{span}\{\eta_1(\tau), \eta_2(\tau)\} \) is defined in Definition \( (4.1) \). Since \( x(t) \) is non-homothetic, By Lemma \( (4.2) \), \( V(\tau) \in C^0(\mathbb{R}, \text{Lag}(\mathbb{R}^4)) \).

**Lemma 4.3.** Assume \( J\hat{B}_\pm \) are hyperbolic and \( W(\tau) \in C^0(\mathbb{R}, \text{Lag}(\mathbb{R}^4)) \) is invariant under the flow of \( \hat{\gamma}(\sigma, \tau) \), if \( \eta_i(\tau) \in W(\tau), \forall \tau \in \mathbb{R} \), then

\[
W(T^+) := \lim_{\tau \to T^+} W(\tau) = \text{span}\{\hat{e}_1(\psi_0^+), \hat{e}_2(\psi_0^+)\} \text{ or span}\{\hat{e}_1(\psi_0^-), \hat{e}_2(\psi_0^-)\},
\]

where
\[
\hat{e}_j(\psi_0^+) = \hat{e}_1^+, \quad \text{if } \psi_0^+ = -\pi/2 \quad \text{and} \quad \hat{e}_j(\psi_0^-) = \hat{e}_2^-, \quad \text{if } \psi_0^- = \pi/2.
\]

**Proof.** We will only give the details for \( T^+ = +\infty \) and \( \psi_0^+ = \pi/2 \). The others are similar. Recall that \( \eta_1 = \hat{r}^{-\frac{\pi}{\hat{\omega}^4 \hat{V}(\theta)}}(u^2 - \alpha \hat{\xi}(\theta), \hat{\omega}(\theta), v, u)^T \). Then

\[
\lim_{\tau \to +\infty} \frac{\eta(\tau)}{|\eta(\tau)|} = \left( -\frac{\pi}{\sqrt{2\hat{2}\hat{\xi}(\theta_0^+) - 0, 1, 0} \tau}, \frac{\hat{e}_1^+}{\sqrt{\alpha^2 \hat{\xi}(\theta_0^+)^2/2 + 1}} \right).
\]

As \( \psi_0^+ = \pi/2, (\pi(\tau)) \) is either a type-I or type-II heteroclinic orbit. By Lemma \( (4.2) \), \( \lim_{\tau \to +\infty} \hat{r} = +\infty \), which implies \( \lim_{\tau \to +\infty} \eta_1(\tau) = 0 \).

Since \( W(\tau) \) is a Lagrangian subspace, we can always find another path \( \eta(\tau) \in W(\tau), \tau \in \mathbb{R} \), invariant under the flow of \( \hat{\gamma}(\sigma, \tau) \), independent of \( \eta_1(\tau) \) and satisfying \( \eta(\tau) \in V^w(\eta_1(\tau)) \), i.e., the \( \omega \) orthogonal space of \( \eta_1(\tau) \) in \( \mathbb{R}^4 \).

If \( \lim_{\tau \to +\infty} \eta(\tau) = 0 \), since the dimension of \( W(\tau) \) is two, \( \lim_{\tau \to +\infty} \gamma(\sigma, \tau) v = 0 \), for any \( v \in W(\tau) \). By Lemma \( (4.2) \), \( W(\tau) = V^+(\tau) \) and \( W(\tau) \to V^-(J\hat{B}_+) = \text{span}\{\hat{e}_1^+, \hat{e}_2^+\} \), when \( \tau \to +\infty \).

If \( \lim_{\tau \to +\infty} \eta(\tau) \neq 0 \), then we can find a two dimensional linear subspace of \( \mathbb{R}^4 \), which is a topological complement of \( V^+(\tau) \) and contains \( \eta(\tau) \). By Lemma \( (4.2) \) for \( \tau \) large enough, \( \eta(\tau) \in \mathcal{N}_\varepsilon(V^+(J\hat{B}_+)) \), i.e., the \( \varepsilon \) neighborhood of \( V^+(J\hat{B}_+) \), for some \( \varepsilon > 0 \) small enough. As a result, \( \eta(\tau) \in \mathcal{N}_\varepsilon(V^+(J\hat{B}_+)) \cap V^w(\eta_1) \). Since \( \varepsilon \) can be arbitrarily small, \( \lim_{\tau \to +\infty} \eta(\tau)/|\eta(\tau)| \in V^+(J\hat{B}_+) \cap V^w(\hat{e}_1^+) \).

Recall that \( V^+(J\hat{B}_+^+) = \text{span}\{\hat{e}_1^+, \hat{e}_2^+\} \), by a direct computation,

\[
\omega(\hat{e}_1^-, \hat{e}_2^-) = 0, \quad \omega(\hat{e}_2^-, \hat{e}_1^+) = -\frac{2 + 3\alpha}{\sqrt{2\hat{2}\hat{\xi}(\theta_0^+) - 0}} \neq 0.
\]

Hence \( \lim_{\tau \to +\infty} \eta(\tau)/|\eta(\tau)| = \hat{e}_1^+/\hat{e}_2^+ \), and together with \( (18) \), it shows \( W(\tau) \to \text{span}\{\hat{e}_1^+, \hat{e}_2^+\} \) as \( \tau \to +\infty \). This finishes our proof.

Under the assumption of Lemma \( (4.3) \), \( W(T^+) \cap V_d \) (\( \cap \) means transversal intersection). Notice that \( \eta_1(\tau) \in V^+(\tau) \) (or \( V^-(\tau) \)) implies \( V^+(T^+) \cap V_d \) (or \( V^-(T^+) \cap V_d \)), where \( V^\pm(T^+) := \lim_{\tau \to T^+} V^\pm(\tau) \). Then Lemma \( (2.4) \) and \( (4.3) \) tell us.

**Corollary 4.1.** (a). If \( \pi(z)(\tau) \) is type-I, then \( V^\pm(T^+) \cap V_d \).
(b). If \( \pi(z)(\tau) \) is type-II, then \( V^+(T^+) \cap V_d \).
(c). If \( \pi(z)(\tau) \) is type-III, then \( V^-(T^+) \cap V_d \).

By the above corollary, when \( \pi(z)(\tau) \) is type-I or III, \( \mu(V_d, V^-(\tau); [-T, T]) \) is a constant for \( T > 0 \) large enough, so \( \mu(x) \) given in Definition \( (4.1) \) is well defined.
**Theorem 4.1.** If $\pi(\tau)(\tau)$ is type-I or III and $\Delta(\theta^+_{0}) > 0$, then $m^{-}(x) = \mu(x)$.

**Proof.** When $\varepsilon > 0$ is small enough,

$$
\mu(V_{d}, e^{\varepsilon J} V_{d}; s \in [0, 1]) = 2.
$$

Then for any $T > 0$ large enough, by (70),

$$
\mu(V_{d}, \gamma(T, -T)e^{\varepsilon J} V_{d}; s \in [0, 1]) = 0.
$$

since the path is transversal. Fix an $\varepsilon > 0$ small enough, for any $T > 0$ large enough, from the homotopy property of Maslov index, we have

$$
\mu(V_{d}, \gamma(T, -T)e^{\varepsilon J} V_{d}; [-T, T]) = \mu(V_{d}, \gamma(T, -T)V_{d}; [-T, T]) = 2.
$$

Together with (45), it shows

$$
m^{-}(x; -T, T) = \mu(V_{d}, \gamma(T, -T)e^{\varepsilon J} V_{d}; [-T, T]).
$$

Now we will try to estimate $\mu(V_{d}, \gamma(T, -T)e^{\varepsilon J} V_{d}; [-T, T]) - \mu(V_{d}, V^{-}(\tau); [-T, T])$.

For this, let $\Lambda_{s}$, $s \in [0, 1]$ be a path of Lagrangian subspaces of $\mathbb{R}^{d}$ with $\Lambda_{0} = V^{-}(-T)$ and $\Lambda(1) = e^{\varepsilon J} V_{d}$, then by the homotopy invariant property of Maslov index,

$$
\mu(V_{d}, \Lambda_{s}; [0, 1]) + \mu(V_{d}, \gamma(T, -T)e^{\varepsilon J} V_{d}; [-T, T]) = \mu(V_{d}, V^{-}(\tau); [-T, T]) + \mu(V_{d}, \gamma(T, -T)\Lambda_{s}; [0, 1]).
$$

Then we have,

$$
\mu(V_{d}, \gamma(T, -T)e^{\varepsilon J} V_{d}; [-T, T]) - \mu(V_{d}, V^{-}(\tau); [-T, T])
$$

$$
= \mu(V_{d}, \gamma(T, -T)\Lambda_{s}; [0, 1]) - \mu(V_{d}, \Lambda_{s}; [0, 1]) = s(\gamma(T, -T)^{-1} V_{d}, V_{d}; V^{-}(-T), e^{\varepsilon J} V_{d}),
$$

where $s(\ldots, \ldots)$ is the Hörmander index (see (72) in Appendix). As $V_{d} \oplus V^{-}(T)$,

$$
\lim_{T \to +\infty} \gamma(T, -T)^{-1} V_{d} = V^{-}(J \hat{B}_{-}).
$$

Since the above hold for any $T$ large enough, we get

$$
\lim_{T \to +\infty} s(\gamma(T, -T)^{-1} V_{d}, V_{d}; V^{-}(-T), e^{\varepsilon J} V_{d}) = s(V^{-}(J \hat{B}_{-}), V_{d}; V^{-}(-\infty), e^{\varepsilon J} V_{d}).
$$

Recall that

$$
V^{-}(J \hat{B}_{-}) = \hat{e}_{1}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \wedge \hat{e}_{2}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}), \quad V^{-}(\infty) = \hat{e}_{1}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \wedge \hat{e}_{2}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}).
$$

Let $V_{d} = V_{d}^{1} \oplus V_{d}^{2}$, where $V_{d}^{1} = \mathbb{R} \oplus 0$ and $V_{d}^{2} = \mathbb{R} \oplus 0$, then

$$
s(V^{-}(J \hat{B}_{-}), V_{d}; V^{-}(-\infty), e^{\varepsilon J} V_{d}) = s(\langle \hat{e}_{1}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, V_{d}^{1}; \langle \hat{e}_{1}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, e^{\varepsilon J} V_{d}^{1})
$$

$$
+ s(\langle \hat{e}_{2}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, V_{d}^{2}; \langle \hat{e}_{2}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, e^{\varepsilon J} V_{d}^{2}).
$$

A simple computation shows

$$
s(\langle \hat{e}_{1}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, V_{d}^{1}; \langle \hat{e}_{1}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, e^{\varepsilon J} V_{d}^{1}) = 0;
$$

$$
s(\langle \hat{e}_{2}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, V_{d}^{2}; \langle \hat{e}_{2}^{\perp}(\psi^{-}_{0}, \theta^{-}_{0}) \rangle, e^{\varepsilon J} V_{d}^{2}) = 0.
$$

This means $m^{-}(x) = \mu(x).$
While the above theorem connects \( m^-(x) \) with \( \mu(x) \), the next one will does the same for \( i(x) \) and \( \mu(V_d, V(\tau); \mathbb{R}) \), where

\[
\mu(V_d, V(\tau); \mathbb{R}) = \lim_{T \to +\infty} \mu(V_d, V(\tau); [-T, T]).
\]

The limit exists, when \( \Delta(\theta^\pm_0) > 0 \), as it implies \( V(\pm \infty) \cap V_d \), see Proposition \[3.1\].

**Theorem 4.2.** When \( \Delta(\theta^\pm_0) > 0 \), \( i(x) = \mu(V_d, V(\tau); \mathbb{R}) \).

**Proof.** From (70), we have

\[
(53) \quad \mu(V_d, V(\tau); R) = \sum_{\tau \in \mathbb{R}} \text{dim}(V_d \cap V(\tau)),
\]

Recall that \( V(\tau) = \text{span}\{\eta_1(\tau), \eta_2(\tau)\} \), where \( \eta_1(\tau), \eta_2(\tau) \) are defined in (34) and (55). Obviously \( \eta_2(\tau) \notin V_d, \forall \tau \in \mathbb{R} \), which implies \( \text{dim}(V(\tau) \cap V_d) \leq 1 \).

We claim \( \text{dim}(V(\tau) \cap V_d) = 1 \), if and only if \( u(\tau) = 0 \). Assume there is a non-zero \( \eta(\tau) = \beta_1 \eta_1(\tau) + \beta_2 \eta_2(\tau) \), which is also contained in \( V_d \). Then it must satisfies the following two equations

\[
(54) \quad \beta_1 v - \frac{2\beta_2}{2 + \alpha} = 0; \quad \beta_1 u = 0.
\]

However the above equations has a solution if and only if \( u = 0 \).

Meanwhile by the fourth equation in (13),

\[
u(\tau) = \theta'(\tau) = \hat{\theta}(t(\tau)) r^{-(1+\frac{\alpha}{2})}(t(\tau)).
\]

Since \( r(\tau) > 0 \), for any \( \tau \in \mathbb{R} \), we have \( \frac{dt}{\tau} = r^{1+\frac{\alpha}{2}} > 0 \) and

\[
(55) \quad i(x) = \#\{t \in (T^-, T^+) | \hat{\theta}(t) = 0\} = \#\{\tau \in \mathbb{R} | u(\tau) = 0\}.
\]

This finishes our proof.

With the above result, we just need to estimate the difference between \( \mu(x) \) and \( \mu(V_d, V(\tau); \mathbb{R}) \), which is exactly the purpose of our next lemma.

**Lemma 4.4.** Assume \( \pi(z)(\tau) \) is type-I or III and \( \Delta(\theta^\pm_0) > 0 \), when \( \tau \to -\infty \),

(a) if \( \pi(z)(\tau) \to (\psi_0^-, \theta_0^-) \) along \( \ell_+(\psi_0^-, \theta_0^-) \), then \( \mu(x) - \mu(V_d, V(\tau); \mathbb{R}) = 0 \);

(b) if \( \pi(z)(\tau) \to (\psi_0^-, \theta_0^-) \) along \( \ell_-(\psi_0^-, \theta_0^-) \), then \( \mu(x) - \mu(V_d, V(\tau); \mathbb{R}) = 0 \) or 1.

**Proof.** Notice that when \( \pi(z)(\tau) \) is type-I or III, \( \psi_0^- = -\pi/2 \). If \( \pi(z)(\tau) \to (\psi_0^-, \theta_0^-) \) along \( \ell_+(\psi_0^-, \theta_0^-) \), as \( \tau \to -\infty \), by Proposition \[3.1\]

\[
\lim_{\tau \to -\infty} V(\tau) = \text{span}\{\hat{e}_1(\psi_0^-, \theta_0^-), \hat{e}_2(\psi_0^-, \theta_0^-)\} = V^+(J\hat{B}_-).
\]

Then Lemma \[1.2\] implies \( V(\tau) = V^-(\tau), \forall \tau \in \mathbb{R} \), so \( \mu(x) = \mu(V_d, V(\tau); \mathbb{R}) \). This proves property (a).

Now assume \( \pi(z)(\tau) \to (\psi_0^-, \theta_0^-) \) along \( \ell_-(\psi_0^-, \theta_0^-) \), as \( \tau \to -\infty \). Fix an arbitrary \( T > 0 \) large enough in the following. It will be enough for us to prove

\[
(56) \quad \mu(V_d, V^u(\tau); [-T, T]) - \mu(V_d, V(\tau); [-T, T]) = 0 \text{ or } 1.
\]

By the proof of Theorem \[4.1\] for a given \( \varepsilon > 0 \) small enough,

\[
\mu(V_d, V^u(\tau); [-T, T]) = \mu(V_d, \hat{\gamma}(\tau, -T)e^{x^j}V_d; [-T, T]).
\]
Hence instead of (60), we will show the following

\[ \mu(V_d, \gamma(\tau, -T)e^{\varepsilon J}V_d; [-T, T]) - \mu(V_d, V(\tau); [-T, T]) = 0 \text{ or } 1. \]

Let \( \Lambda_s, s \in [0, 1] \) be a path of Lagrangian subspaces of \( \mathbb{R}^4 \) with \( A_0 = V(-T) \) and \( A_1 = e^{\varepsilon J}V_d. \) Similar to (51), we have

\[
\mu(V_d, \gamma(\tau, -T)e^{\varepsilon J}V_d; [-T, T]) - \mu(V_d, V(\tau); [-T, T])
= s(\gamma(T, -T)^{-1}V_d, V_d; V(-T), e^{\varepsilon J}V_d).
\]

Notice that \( \lim_{T \to +\infty} \gamma(T, -T)^{-1}V_d = V^-(J\hat{B}_-) = e^1(\psi_0^-, \theta_0^-) \wedge e^2(\psi_0^-, \theta_0^-), \) and under the condition of property (b), \( \lim_{T \to +\infty} V(-T) = V(-\infty) = e^1(\psi_0^-, \theta_0^-) \wedge e^2(\psi_0^-, \theta_0^-). \) Therefore

\[
s(\gamma(T, -T)^{-1}V_d, V(-T), e^{\varepsilon J}V_d) = s(\gamma(T, -T)^{-1}V_d, e^{\varepsilon J}V_d; V(-T), e^{\varepsilon J}V_d).
\]

When \( \psi_0^- = -\pi/2, \) by Lemma 3.4,

\[
e^1(\psi_0^-, \theta_0^-) = \left( -(\alpha + 1)\sqrt{2\lambda(\theta_0^-)}, 1 \right)^T, \quad e^1(\psi_0^-, \theta_0^-) = \left( \frac{\alpha}{2} \sqrt{2\lambda(\theta_0^-)} \right),
\]

\[
e^2(\psi_0^-, \theta_0^-) = \left( 2 - \frac{\alpha}{4} \sqrt{2\lambda(\theta_0^-)} - \frac{1}{2} \sqrt{\Delta(\theta_0^-)} \right)^T.
\]

For simplicity, set \( b = \sqrt{2\lambda(\theta_0^-)}, c = \frac{2-\alpha}{2} \sqrt{2\lambda(\theta_0^-)} - \frac{1}{2} \sqrt{\Delta(\theta_0^-)}. \)

Write the Lagrangian subspaces as graphs of linear maps: \( 0 \oplus \mathbb{R}^2 \to V_d: \)

\[
V^-(J\hat{B}_-) = \text{Gr}(A_0), e^{\varepsilon J}V_d = \text{Gr}(A_1); \quad V(-\infty) = \text{Gr}(B_0), e^{\varepsilon J}V_d = \text{Gr}(B_1),
\]

where

\[
A_0 = \left( \begin{array}{ccc} -(\alpha + 1)b & 0 & 0 \\ 0 & \cot(\varepsilon/2) & 0 \\ 0 & 0 & \cot(\varepsilon/2) \end{array} \right), \quad A_1 = \left( \begin{array}{ccc} \cot(\varepsilon/2) & 0 & 0 \\ 0 & \cot(\varepsilon/2) & 0 \\ 0 & 0 & \cot(\varepsilon) \end{array} \right);
\]

\[
B_0 = \left( \begin{array}{ccc} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{array} \right), \quad B_1 = \left( \begin{array}{ccc} \cot(\varepsilon) & 0 & 0 \\ 0 & \cot(\varepsilon) & 0 \\ 0 & 0 & \cot(\varepsilon) \end{array} \right).
\]

Let \( A_{0,T}, B_{0,T} \) be the matrices, such that \( \gamma(T, -T)^{-1}V_d = \text{Gr}(A_{0,T}) \) and \( V(-T) = \text{Gr}(B_{0,T}). \) Then for \( T \) large enough, \( A_{0,T}, B_{0,T} \) are in the \( \varepsilon/2 \)-neighborhood of \( A_0, B_0 \) correspondingly. By the property of Hörmander index (see (74)),

\[
s(\gamma(T, -T)^{-1}V_d, e^{\varepsilon J}V_d; V(-T), e^{\varepsilon J}V_d)) = \frac{1}{2} \text{sign}(B_{0,T} - A_1) + \frac{1}{2} \text{sign}(B_1 - A_0,T) - \frac{1}{2} \text{sign}(B_1 - A_1) - \frac{1}{2} \text{sign}(B_{0,T} - A_{0,T}).
\]

Notice that \( B_{0,T} - A_1, B_1 - A_1 \) are negative definite, and \( B_1 - A_{0,T} \) is positive definite. Hence

\[
-\frac{1}{2} \text{sign}(B_{0,T} - A_1) = \frac{1}{2} \text{sign}(B_1 - A_0,T) = -\frac{1}{2} \text{sign}(B_1 - A_1) = 1.
\]

Since \( B_{0,T} - A_{0,T} \) is in the \( \varepsilon \)-neighborhood of \( B_0 - A_0, \) which has a positive eigenvalue \( (1 + \frac{3}{2} \alpha), \) we have

\[
\frac{1}{2} \text{sign}(B_{0,T} - A_{0,T}) = 0, \quad \text{or} \quad 1.
\]

This completes our proof. \( \square \)
Corollary 4.2. Assume $\Delta(\theta^+_0) > 0$. When $\pi(z)(\tau)$ is a type-I or III heteroclinic orbit,

(a) If $\pi(z)(\tau) \to (\psi^-_0, \theta^-_0)$ along $(e_+(\psi^-_0, \theta^-_0))$, as $\tau \to -\infty$, then $m^-(x) = i(x)$;

(b) If $\pi(z)(\tau) \to (\psi^-_0, \theta^-_0)$ along $(e_-(\psi^-_0, \theta^-_0))$, as $\tau \to -\infty$, then $m^-(x) - i(x) = 0$ or 1.

When $\pi(z)(\tau)$ is a type-II heteroclinic orbit,

(c) If $\pi(z)(\tau) \to (\psi^+_0, \theta^+_0)$ along $(e_-(\psi^+_0, \theta^+_0))$, as $\tau \to +\infty$, then $m^-(x) = i(x)$;

(d) If $\pi(z)(\tau) \to (\psi^+_0, \theta^+_0)$ along $(e_+(\psi^+_0, \theta^+_0))$, as $\tau \to +\infty$, then $m^-(x) - i(x) = 0$ or 1.

Proof. Property (a) and (b) follows directly from Theorem 4.1 and Lemma 4.3.

For property (c) and (d), as the corresponding $x(t)$ is a collision-parabolic solution, $x(t) = x(-t)$ will be a collision-parabolic solution. By their definitions, it is not hard to see $m^-(\hat{x}) = m^-(x)$ and $i(\hat{x}) = i(x)$.

Let $\hat{z}$ be the zero energy orbit of (11) corresponding to $\hat{x}$, and $(\hat{v}, \hat{u}, \hat{r}, \hat{\theta})(\tau)$ the corresponding orbit in McGehee coordinates, then by the computation given at the beginning of Section 2, we have

$$(\hat{v}, \hat{u}, \hat{r}, \hat{\theta})(\tau) = (-\psi, -u, \tau, \theta)(\tau).$$

As a result, on the collision manifold $\mathcal{M}$ with coordinates defined in (17), we have

$$(\hat{\psi}, \hat{\theta})(\tau) = (\psi + \pi, \theta)(\tau).$$

Then

$$(\hat{\psi}^-_0, \hat{\theta}^-_0) := \lim_{\tau \to -\infty} (\hat{\psi}, \hat{\theta})(\tau) = \lim_{\tau \to -\infty} (\psi + \pi, \theta)(\tau) = (\psi^+_0 + \pi, \theta^+_0).$$

By (15),

$$M(\hat{\psi}^-_0, \hat{\theta}^-_0) = M(\psi^+_0 + \pi, \theta^+_0) = -M(\psi^-_0, \theta^-_0).$$

As a result,

$$e_+(\hat{\psi}^-_0, \hat{\theta}^-_0) = e_-(\psi^+_0, \theta^+_0), \quad e_-(\hat{\psi}^-_0, \hat{\theta}^-_0) = e_+(\psi^+_0, \theta^+_0).$$

Then the rest follows from property (a) and (b), which we have already proven.

In the above we always assume $\Delta(\theta^+_0) > 0$, to deal the non-hyperbolic case, i.e., $\Delta(\theta^+_0) < 0$, we have the next proposition

Proposition 4.1. If at least one of $\Delta(\theta^+_0)$ is negative, then $m^-(x) = i(x) = +\infty$.

Proof. We only give the details for the case $\Delta(\theta^+_0) < 0$, while the proof for the other case is exactly the same.

For $\varepsilon > 0$ small enough, we can find a $\tau_0 > 0$, such that $\|\hat{B}(\tau) - \hat{B}_+\| < \varepsilon$, for any $\tau \in [\tau_0, +\infty)$. By (15), if $\lim_{\tau \to +\infty} \mu(V_{d}, \hat{\gamma}(\tau, \tau_0))V_d; [\tau_0, \tau_1]) = +\infty$, then $m^-(x) = +\infty$. Since $\hat{B}(\tau) > \hat{B}_+ - \varepsilon I_4$, from the monotonic property of Maslov index,

$$\mu(V_d, e^{(\tau-\tau_0)(\hat{B}_+ - \varepsilon I_4)} V_d; [\tau_0, \tau_1]) \geq \mu(V_d, e^{(\tau-\tau_0)(\hat{B}_+ - \varepsilon I_4)} V_d; [\tau_0, \tau_1]), \forall \tau_1 > \tau_0.$$ 

By the symplectic additivity property,

$$\mu(V_d, e^{(\tau-\tau_0)(\hat{B}_+ - \varepsilon I_4)} V_d) = \mu(V_d, e^{(\tau-\tau_0)(\hat{B}_+ - \varepsilon I_4)} V_d) + \mu(V_d, e^{(\tau-\tau_0)(\hat{B}_+ - \varepsilon I_4)} V_d).$$
Since in this case the crossing form is always positive, \( \mu(V_d, \exp((\tau - \tau_0)(\tilde{B}^{(i)}_+ - \varepsilon I_2))V_d) \) is the summation of \( \dim(\exp((\tau - \tau_0)(\tilde{B}^{(i)}_+ - \varepsilon I_2)V_d \cap V_d) \) for \( \tau \in [\tau_0, \tau_1] \), for \( i = 1, 2 \). As a result, 
\[
\mu(V_d, e^{(\tau - \tau_0)(\tilde{B}+ - \varepsilon I_4)}V_d) \geq \sum_{\tau \in [\tau_0, \tau_1]} \dim e^{(\tau - \tau_0)(\tilde{B}^{(2)}_+ - \varepsilon I_2)}V_d \cap V_d.
\]

Notice that 
\[
\tilde{B}^{(2)}_+ - \varepsilon I_2 = \left( 1 - \varepsilon \begin{pmatrix} 2 - \alpha & 2\sqrt{2\mu(\theta_0^+)} \\ 2\sqrt{2\mu(\theta_0^+)} & -\mu_{\theta^0}(\theta_0^+) \end{pmatrix} \right).
\]

For \( \varepsilon \) small enough, \( \tilde{B}^{(2)}_+ - \varepsilon I_2 > 0 \), a direct computation shows the summation of crossing time is unbounded as \( \tau_1 \to +\infty \). Hence \( m^-(x) = +\infty \).

Let’s assume \( x(t) \) is either a parabolic or a parabolic-collision solution (a collision-parabolic solution becomes parabolic-collision after reversing time). Since \( x(t) \) is not homothetic, the corresponding orbit \( \pi(z)(\tau) \) on the collision manifold \( M \) is a type-I or II heteroclinic orbit.

Now if one of \( \Delta(\theta_0^+) \) is negative, then \( m^-(x) = i(x) = +\infty \), by Proposition 1.1. This proves property (a). If both \( \Delta(\theta_0^+) \) are positive, then \( m^-(x) \) follows from Lemma 2.2. In particular, when \( \theta_0^+ \) is a non-degenerate local minimizer, \( \mu_{\theta^0}(\theta_0^+) > 0 \). Then by Lemma 2.2 \( \lambda_-(\psi^-_0, \theta_0^+) < 0 < \lambda_+(\psi^-_0, \theta_0^+) \), and by Lemma 2.3 the unstable manifold of \( \psi^-_0, \theta_0^+ \) is a stable focus. Hence \( \pi(z)(\tau) \) approaches to \( \psi^-_0, \theta_0^+ \) in the collision manifold is tangent to \( \hat{e}_+(\psi^-_0, \theta_0^+) \). Then by property (a) in Corollary 1.2, \( m^-(x) = i(x) \).}

Prove Theorem 1.3. By Theorem 1.2 it is enough to show \( i(x) = 0 \). Meanwhile by 5.5, and 1.7, this is equivalent to \( \psi(\tau) \neq \pm \pi/2 \), for any \( \tau \in \mathbb{R} \).

Assume \( \psi(\tau_0) = \pm \pi/2 \). Then \( \psi(\tau_0) = \sqrt{\mu(\theta(\tau_0))} \). Recall that \( \mu(\theta(\tau_0)) \) is a non-decreasing function of \( \tau \), so \( \psi(\tau_0) = \sqrt{\mu(\theta(\tau_0))} \leq \psi(\infty) = \sqrt{\mu(\theta_0^+)} \). This means \( \theta(\tau_0) \) must be a global minimizer of \( \mu \). Then by Lemma 2.1 \( \psi(\tau_0), \theta(\tau_0) \) is a equilibrium in the collision manifold, which is absurd.

Our next proof follows ideas from [8] and [17].

Prove Theorem 1.3. Without loss of generality let’s assume \( \bar{x}(t) \) is a collision-parabolic solution defined on \( \mathbb{R}^+ = (0, +\infty) \). With the energy being zero, we have 
\[
\bar{r}(t) = (\kappa t)^{(2 + \alpha)/2}, \quad \text{where} \quad \kappa = \frac{2 + \alpha}{2}\sqrt{2\mu(\theta_0^+)}. 
\]

Recall that in polar coordinates, the action functional is 
\[
\mathcal{F}(r, \theta) = \int \frac{1}{2} \bar{r}^2 + \frac{1}{2} \bar{\theta}^2 + \bar{r}^{\alpha-2} \mu(\theta) \, dt.
\]
By results from [10], for any \([t_0, t_1] \subset (0, +\infty), \bar{x}(t)\) is a minimizer of \(\mathcal{F}\) in 
\[
\{(r, \theta) \in W^{1,2}([t_0, t_1], \mathbb{R}^+ \times S^1): \ r(t_0) = \bar{r}(t_0), r(t_1) = \bar{r}(t_1), \theta(t) = \theta_0\}.
\]
Therefore we only need to consider variations of \(\mathcal{F}\) along \(\rho \in C_0^\infty(\mathbb{R}^+, S^1)\) (smooth functions with compact supports). The second derivative of \(\mathcal{F}\) along such a \(\rho\) is 
\[
d^2\mathcal{F}(r, \theta)[\rho, \phi] = \int r^2 \rho^2 + r^{-\alpha} \mathcal{L}(\theta) \phi^2 \, dt = \int r^{\frac{2-\alpha}{\rho}} ((\phi')^2 + \mathcal{L}(\theta) \phi^2) \, dr,
\]
(57) 
If \(\xi(t) = \bar{r}(t) r^{-\alpha} \phi(t)\), then \(\xi' = \bar{r}^{\frac{2-\alpha}{\rho}} \phi' + \frac{2-\alpha}{2} r^{-\frac{2-\alpha}{\rho}} r' \phi\), and 
\[
\bar{r}^{-\frac{2-\alpha}{\rho}} (\phi')^2 = (\xi')^2 + \frac{(2-\alpha)^2}{16} \left(\frac{\rho'}{r}\right)^2 \xi^2 - \frac{2-\alpha}{2} \left(\frac{\rho'}{r}\right) \xi \xi' - \frac{(\xi')^2}{8} \mathcal{L}(\theta) \xi^2 - \frac{2-\alpha}{2} \sqrt{2\mathcal{L}(\theta)} \xi \xi',
\]
(58) 
where the second equality following from 
\[
\frac{\rho'}{r} = \bar{r}^{-\frac{1}{\rho}} \frac{dt}{d\tau} = \bar{r}^{-\frac{2}{\rho}} = \frac{2}{2+\alpha} = \sqrt{2\mathcal{L}(\theta)}.
\]
Plug (58) into (57), we get 
\[
d^2\mathcal{F}(\bar{r}, \theta_0)[\rho, \phi] = \int (\xi')^2 + \frac{1}{4} \mathcal{L}(\theta) \xi^2 - \frac{2-\alpha}{2} \sqrt{2\mathcal{L}(\theta)} \xi \xi' \, d\tau.
\]
As \(\xi\) has a compact support in \(\mathbb{R}^+\), using integration by parts, 
\[
d^2\mathcal{F}(\bar{r}, \theta_0)[\rho, \phi] = \int (\xi')^2 + \frac{1}{4} \mathcal{L}(\theta_0) \xi^2 - \frac{2-\alpha}{2} \sqrt{2\mathcal{L}(\theta_0)} \xi \xi' \, d\tau.
\]
When \(\mathcal{L}(\theta_0) \geq 0\), \(d^2\mathcal{F}(\bar{r}, \theta_0)[\rho, \phi] \geq 0\), for any \(\phi\), so \(m^-(x) = 0\). When \(\mathcal{L}(\theta_0) < 0\), there is a countable set of linear independent functions \(\{\phi_n \in C_0^\infty(\mathbb{R}^+, S^1): n \in \mathbb{Z}^+\}\) satisfying \(d^2\mathcal{F}(\bar{r}, \theta_0)[\phi_n, \phi_n] < 0\) (see [3] Theorem 4.3), which implies \(m^-(x) = +\infty\). \(\square\)

5. Application in Celestial Mechanics

In this section, we give some applications of our results to celestial mechanics.

5.1. The planar isosceles three body problem. Consider the problem of three point masses, \(m_i, i = 1, 2, 3\), in a plane moving under the Newtonian gravitational force of each other. Let \(q = (q_1, q_2, q_3)\), where \(q_i\) represents the position of \(m_i\), and \(p = M \dot{q}\), where \(M = \text{diag}(m_1, m_1, m_2, m_2, m_3, m_3)\), then 
\[
\dot{p} = \nabla q \bar{U}(q): \ \dot{q} = M^{-1} p,
\]
(59) 
where \(\bar{U}(q) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|q_i - q_j|}\), is the (negative) potential. This is equivalent to the Euler-Lagrangian equation \(\frac{d}{dt} L(q, \dot{q}) = L(q, \dot{q})\) with 
\[
L(q, \dot{q}) = K(q) + \bar{U}(q) = \frac{1}{2} \|\dot{q}\|^2_M + \bar{U}(q),
\]
(60) 
where \(|w|_M := (\sum_{i=1}^3 m_i |w_i|^2)\), for any \(w = (w_1, w_2, w_3) \in \mathbb{R}^{2 \times 3}\). The above problem has six degrees of freedom. It can be reduced to four after fixing the center of mass at the origin, \(\sum_{i=1}^3 m_i q_i = 0\). Moreover when two of the masses are equal \((m_1 = m_2)\), it has an invariant sub-system with two degrees of freedom, where the three masses form an isosceles triangle all the time: 
\[
\{q = (q_1, q_2, q_3) | q_2 = \mathfrak{R}(q_1), q_3 = \mathfrak{R}(q_3)\}.
\]
(61)
Here $\mathcal{R}$ represents the reflection in $\mathbb{R}^2$ with respect to the vertical axis.

For simplicity, we assume $m_1 = m_2 = m$ and $m_3 = 1$. Let

$$r = |q|_M \quad \text{and} \quad s_i = q_i/r, \quad \forall i = 1, 2, 3.$$

Then $s = (s_1, s_2, s_3)$ satisfies $|s|_M = 1$. Set $s_1 = (\xi, \eta)$, by (61),

$$s_2 = (-\xi, \eta), \quad s_3 = (0, -2m\eta),$$

which means

$$|s|_M^2 = 2m\xi^2 + 2m(2m + 1)\eta^2 = 1.$$

This allows us to introduce an angular variable $\theta \in S^1$ by

$$\xi = \frac{\cos \theta}{\sqrt{2m}}, \quad \eta = \frac{\sin \theta}{\sqrt{2m(2m + 1)}}.$$

Under the new variables $(r, \theta)$, the Lagrangian of the isosceles three problem has the following expression which fits the framework of this paper:

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\Omega(\theta)}{r}, \quad \text{where} \quad \Omega(\theta) = \frac{m\dot{\theta}}{\sqrt{2|\cos \theta|}} + \frac{2\sqrt{2m^3}}{(1 + 2m\sin^2 \theta)^{3/2}}.$$

However besides the singularity at the origin, $r = 0$, corresponding a triple collision. There are additional singularities at $\theta = \pm \pi/2$ due to binary collisions between $m_1$ and $m_2$. Although a double collision can be regularized (see [32, Section 7] or [27]), it is not so clear how to define the corresponding Morse index in this case, so when applying our results, we have to restrict ourselves to a domain of the zero energy solution, where there is no binary collision.

It is easy to see $\Omega(\theta)$ has four different non-degenerate global minima:

$$-\pi + \theta^* < -\theta^* < \theta < \pi - \theta^*, \quad \text{for some} \quad \theta^* \in (0, \pi/2),$$

which are the Lagrangian configurations, where the three masses form an equilateral triangle. The second derivatives of $\Omega(\theta)$ at these critical points all are positive, so the condition required in Lemma 3.1 always holds at these points.

Meanwhile there are two non-degenerate critical points at $\theta = 0$ or $\pi$, which are local maxima of $\Omega$. They are the Euler configurations with $m_3$ at the origin. By a direct computation,

$$\Omega(0) = \Omega(\pi) = \frac{m\dot{\theta}}{\sqrt{2}} + 2\sqrt{2m^3}, \quad \Omega_{\theta\theta}(0) = \Omega_{\theta\theta}(\pi) = -\frac{7}{\sqrt{2}}m^3.$$

Recall that for $\alpha = 1$, $\Delta(\theta_0) = \frac{1}{2}\Omega(\theta_0) + 4\Omega_{\theta\theta}(\theta_0)$. Then $\Delta(0) = \Delta(\pi)$ are positive, when $m < 4/55$, and negative, when $m > 4/55$. As shown by Moeckel [27], if a zero energy solution (non-homothetic) approaches to the origin or the infinity along the horizontal axis (or equivalently the configuration formed by the three masses converges to a Euler configuration), then for a generic $m > 4/55$, during the process, the three masses oscillate frequently along the horizontal axis. This corresponds to the change of the sign of $\dot{\theta}(t)$, which by our results gives an estimate of the Morse index of the solution.
5.2. The Kepler-type problem. In our results, we require the critical points of $\Omega$ to be non-degenerate. In general our approach may still work even when this condition does not hold. What we need is the knowledge of the asymptotic behavior of $V(\tau)$ defined in Lemma 3.2 as $\tau$ goes to infinity. This is important as in celestial mechanics these critical points correspond to central configurations, which are degenerate due to symmetries. As an example, we will consider the Kepler-type problem, where each $\theta$ is a degenerate critical point of $\Omega$:

$$\Omega(\theta) \equiv m, \ \forall \theta \in S^1, \text{ for some constant } m > 0.$$ 

Now the vector field $\mathbf{K}$ on the collision manifold $\mathfrak{M}$ becomes

$$\left\{ \begin{array}{l} \psi' = (1 - \frac{\alpha}{2}) \sqrt{2m} \cos \psi, \\ \theta' = \sqrt{2m} \cos \psi, \end{array} \right.$$ 

and very $(\psi_0, \theta_0)$ with $\psi_0 \in \{ \pm \frac{\pi}{2} \}, \theta_0 \in S^1$, is an equilibrium. Let $M(\psi_0, \theta_0)$ be defined as in (19). Following Notation 2.1 by Lemma 2.2

$$\left\{ \begin{array}{l} -(2 - \alpha) = \lambda_-(\frac{\pi}{2}, \theta_0) < \lambda_+(\frac{\pi}{2}, \theta_0) = 0; \\ e_-(\frac{\pi}{2}, \theta_0) = (\frac{\pi}{2m}, 1)^T, \ e_+(\frac{\pi}{2}, \theta_0) = (0, 1)^T, \end{array} \right.$$ 

$$\left\{ \begin{array}{l} 0 = \lambda_-(\mp \frac{\pi}{2}, \theta_0) < \lambda_+(\mp \frac{\pi}{2}, \theta_0) = (2 - \alpha); \\ e_-(\mp \frac{\pi}{2}, \theta_0) = (0, 1)^T, \ e_+(\mp \frac{\pi}{2}, \theta_0) = (\frac{\pi}{2m}, 1)^T \end{array} \right.$$ 

Let $x(t)$ be a parabolic solution of the Kepler-type problem, then its projection to the collision manifold is a heteroclinic orbit going from $(-\frac{\pi}{2}, \theta^+_0)$ to $(\frac{\pi}{2}, \theta^-_0)$. Since $\theta^\pm_0$ are degenerate, Lemma 2.3 does not apply. However by (22),

$$\frac{\psi'}{\theta'}(\tau) = \frac{2 - \alpha}{2}, \ \forall \tau \in \mathbb{R}.$$ 

Hence the heteroclinic orbit converges to $(\frac{\pi}{2}, \theta^+_0)$ along the subspace $\langle e_-(\frac{\pi}{2}, \theta^+_0) \rangle$, as $\tau \to +\infty$, and converges to $(-\frac{\pi}{2}, \theta^-_0)$ along the subspace $\langle e_+(\frac{\pi}{2}, \theta^-_0) \rangle$, as $\tau \to -\infty$, which means it is a type-I heteroclinic orbit.

Let $V(\tau) = \text{span}\{\eta_1(\tau), \eta_2(\tau)\}$ be the path of Lagrangian subspaces given in Definition 3.1. With (63), the same computation used in the proof of Lemma 3.3 shows $\lim_{\tau \to \pm \infty} \mathfrak{l}_{\text{un}}(\eta)_{\eta} = 0$. Recall that $\lambda_+(\frac{\pi}{2}, \theta^+_0) = \lambda_-(\frac{\pi}{2}, \theta^-_0) = 0$, so results of Lemma 3.3 still hold. Then by Proposition 3.4

$$\lim_{\tau \to \pm \infty} V(\tau) = \text{span}\{\hat{e}_1^\pm(\pm \frac{\pi}{2}, \theta^+_0), \hat{e}_2^\pm(\pm \frac{\pi}{2}, \theta^+_0)\}.$$ 

Notice that for the Kepler-type potential,

$$\hat{e}_1^\pm(\pi/2, \theta^+_0) = (-\alpha \sqrt{m/2}, 1)^T, \ e_2^\pm(\pi/2, \theta^+_0) = ((2 - \alpha) \sqrt{m/2}, 1)^T; \ e_1^\pm(-\pi/2, \theta^-_0) = (-\sqrt{m/2}, 1)^T, \ e_2^\pm(-\pi/2, \theta^-_0) = ((2 - \alpha) \sqrt{m/2}, 1)^T.$$

This means the corresponding results in Section 4 will still hold. In particular, by Corollary 4.2 $i(x) = \mu(x) = m^{-}(x)$.

Since the angular momentum is a first integral of the Kepler-type problem, for a parabolic solution (so non-homothetic), $\dot{\theta}(t)$ is always positive or negative. Together with the above result they imply

**Corollary 5.1.** For a Kepler-type problem, the Morse index of a parabolic solution is always zero.
6. APPENDIX: A BRIEF INTRODUCTION TO THE MASLOV INDEX FOR HETEROCLINIC ORBITS

We start with a brief review of the Maslov index theory from [12, 31]. Let \((\mathbb{R}^{2n}, \omega)\) be the standard symplectic space, and \(\text{Lag}(2n)\) the Lagrangian Grassmannian, i.e., the set of Lagrangian subspaces of \((\mathbb{R}^{2n}, \omega)\). Given two continuous paths \(L_1(t), L_2(t), t \in [a, b]\), in \(\text{Lag}(2n)\), the Maslov index \(\mu(L_1(t), L_2(t))\) is an integer invariant. There are several different ways to define such an invariant. Here we use the one given in [12]. Following are some properties of the Maslov index (for the details see [12]).

**Property I. (Reparametrization invariance)** Let \(\varphi : [c, d] \to [a, b]\) be a continuous and piecewise smooth function satisfying \(\varphi(c) = a, \varphi(d) = b\), then

\[
\mu(L_1(t), L_2(t)) = \mu(L_1(\varphi(t)), L_2(\varphi(t))).
\]

**Property II. (Homotopy invariant with end points)** If two continuous families of Lagrangian paths \(L_1(s, t), L_2(s, t), 0 \leq s \leq 1, a \leq t \leq b\) satisfies

\[
\dim(L_1(s, a) \cap L_2(s, b)) = C_1, \dim(L_1(s, b) \cap L_2(s, b)) = C_2, \quad \text{for any } 0 \leq s \leq 1,
\]

where \(C_1, C_2\) are two constant integers, then

\[
\mu(L_1(0, t), L_2(0, t)) = \mu(L_1(1, t), L_2(1, t)).
\]

**Property III. (Path additivity)** If \(a < c < b\), then

\[
\mu(L_1(t), L_2(t)) = \mu(L_1(t), L_2(t); [a, c]) + \mu(L_1(t), L_2(t); [c, b]).
\]

**Property IV. (Symplectic invariance)** Let \(\gamma(t), t \in [a, b]\) be a continuous path of symplectic matrices in \(\text{Sp}(2n)\), then

\[
\mu(L_1(t), L_2(t)) = \mu(\gamma(t)L_1(t), \gamma(t)L_2(t)).
\]

**Property V. (Symplectic additivity)** Let \(W_i, i = 1, 2\), be two symplectic spaces, if \(L_i \subset C([a, b], \text{Lag}(W_i))\) and \(\hat{L}_i \subset C([a, b], \text{Lag}(W_2))\), \(i = 1, 2\), then

\[
\mu(L_1(t) \oplus \hat{L}_1(t), L_2(t) \oplus \hat{L}_2(t)) = \mu(L_1(t), L_2(t)) + \mu(\hat{L}_1(t), \hat{L}_2(t)).
\]

**Property VI. (Symmetry)** If \(L_i \subset C([a, b], \text{Lag}(2n))\), \(i = 1, 2\), then

\[
\mu(L_1(t), L_2(t)) = \dim(L_1(a) \cap L_2(a) - \dim(L_1(b) \cap L_2(b) - \mu(L_2(t), L_1(t)).
\]

When the Hamiltonian system is given by the Legendre transformation of a Sturm-Liouville system, then

\[
\mu(V_d, \Lambda(t)) = \dim(\Lambda(a) \cap V_d) + \sum_{a < t < b} \dim(\Lambda(t) \cap V_d).
\]

For the detail see [31, 22].

Given a Lagrangian path \(t \mapsto \Lambda(t)\), the difference of the Maslov indices of it with respect to two Lagrangian subspaces \(V_0, V_1 \in \text{Lag}(2n)\), is given in terms of the Hörmander index (see [31 Theorem 3.5])

\[
s(V_0, V_1; \Lambda(0), \Lambda(1)) = \mu(V_0, \Lambda(t)) - \mu(V_1, \Lambda(t)).
\]

Obviously for \(\varepsilon > 0\) small enough,

\[
s(V_0, V_1; \Lambda(0), \Lambda(1)) = s(V_0, V_1; e^{-\varepsilon J}\Lambda(0), e^{-\varepsilon J}\Lambda(1)),
\]
The Hörmander index is independent of the choice of the path connecting $\Lambda(0)$ and $\Lambda(1)$. Under the non-degenerate condition, i.e., $V_0, V_1$ are transversal to $\Lambda(0), \Lambda(1)$ correspondingly, it has the following two basic properties

$$s(V_0, V_1; \Lambda(0), \Lambda(1)) = -s(V_1, V_0; \Lambda(0), \Lambda(1)),$$

$$s(\Lambda(0), \Lambda(1); V_0, V_1) = -s(V_0, V_1; \Lambda(0), \Lambda(1)).$$

If $V_i = \text{Gr}(A_i), \Lambda(i) = \text{Gr}(B_i)$ for symmetry matrices $A_i$ and $B_i$, $i = 0, 1$, then

$$s(V_0, V_1; \Lambda(0), \Lambda(1)) = \frac{1}{2}\text{sign}(B_0 - A_1) + \frac{1}{2}\text{sign}(B_1 - A_0)$$

$$- \frac{1}{2}\text{sign}(B_1 - A_1) - \frac{1}{2}\text{sign}(B_0 - A_0),$$

where for a symmetric matrix $A$, sign$(A)$ is the signature of the symmetric form $\langle A, \cdot \rangle$. A direct corollary shows that

$$|s(V_0, V_1; \Lambda(0), \Lambda(1))| \leq 2n.$$

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