BPS preons in supergravity and higher spin theories.
An overview from the hill of twistor approach

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Abstract. We review briefly the notion of BPS preons, first introduced in 11–dimensional context as hypothetical constituents of M–theory, in its generalization to arbitrary dimensions and emphasizing the relation with twistor approach. In particular, the use of a “twistor–like" definition of BPS preon (almost) allows us to remove supersymmetry arguments from the discussion of the relation of the preons with higher spin theories and also of the treatment of BPS preons as constituents. We turn to the supersymmetry in the second part of this contribution, where we complete the algebraic discussion with supersymmetric arguments based on the M–algebra (generalized Poincaré superalgebra), discuss the possible generalization of BPS preons related to the \(osp(1|n)\) (generalized AdS) superalgebra, review a twistor–like \(\kappa\)–symmetric superparticle in tensorial superspace, which provides a point–like dynamical model for BPS preon, and the rôle of BPS preons in the analysis of supergravity solutions. Finally we describe resent results on the concise superfield description of the higher spin field equations and on superfield supergravity in tensorial superspaces.

INTRODUCTION

Twistor theory [1, 2] and twistor–like methods, which are the main subjects of this Max Born symposium, are becoming now increasingly popular in the light of the work of [3, 4] on the twistor string description of the Yang–Mills scattering amplitudes [5]. This can be considered as a significant progress towards a realization of the Penrose “twistor programme” [2] aimed to describe nature in terms of twistor space rather than spacetime.

The subject of this contribution is the notion of BPS preons, introduced in [6] in an M–theoretical context, but allowing for an easy ’generalization’ to other dimensions (see [7, 8] and also [9]). In M–theory the BPS preons appeared as its (hypothetical) constituents [6]; a search 31/32 supersymmetric solutions of \(D = 11\) supergravity, which would describe BPS preons, can be witnessed [10, 11, 12]. In some other dimensions, namely in \(D = 4, 6\) and 10, the notion of BPS preons are related with higher spin theory (see [13, 14, 15, 8, 12, 16, 17, 18, 19]).

As it was noticed already in [6] the notion of BPS preons is related to the twistor approach [1] and its very simple orthosymplectic “generalization” [20] (hence the “twistorial constituents” name in the title of [6]). The discussion of this relation allows us to define the BPS preon in a simple and suggestive way, with a minimal use of supersym-
metry. This observation suggests the following structure of this contribution.

We begin in Sec. I by a brief review of the known properties of twistor approach, massless particle mechanics and their supersymmetric generalization in the form which is useful to define and to discuss the properties of BPS preons. In Sec. II we present a purely bosonic definition of the of BPS preon [6] and discuss their properties (almost) without using (more precisely, with a minimal reference to) supersymmetry. In this framework we review, in particular, the rôle of BPS preons as constituents (of M–theory for $D = 11$) and the relation of BPS preon with higher spin theories. To establish this relation we use the point–like model for BPS preon provided by the twistor–like (super)particle model in tensorial (super)space; interestingly enough, this model had been proposed in [20] before the notion of BPS preons was introduced in [6]. The same can be said (at least up to some extent, see [13] and [14, 15, 16, 18, 19]) on the relation of this model with $D = 4, 6, 10$ higher spin theories. We use here the BPS preon notion to discuss these issues as it provides a universal framework allowing to discuss the higher spin theory in $D = 4, 6, 10$ and (some issues of) M–theory in the same term. The discussion of Sec. II is completed by supersymmetry arguments in Sec. III where we start form M–algebra, discuss the rôle of BPS preons in the classification of the BPS state [6] and in the analysis of the supergravity solitons [12], review the $\kappa$–symmetry of the “preonic superparticle” model [20] and its relation with preserved supersymmetry. We finish in Sec. IV by describing recent results on the superfield description of the tower of all possible conformal massless higher spin equations in $D = 4, 6, 10$ and on supergravity in tensorial superspaces, which might be relevant both in the search for a selfconsistent supersymmetric higher spin interaction and for M–theoretical applications.

I. PRELIMINARIES. TWISTOR APPROACH, MASSLESS PARTICLE AND SUPERPARTICLE IN $D = 4$.

This section contains a review of known issues on $D = 4$ twistors and supertwistors and their relation to massless particle and superparticle mechanics [21] (see also e.g. [22, 23] and a more recent [24, 25]) in a form convenient for the discussion on BPS preons.

I.1. Cartan–Penrose representation and Penrose correspondence

Let us begin by writing two basic relation of the original Penrose twistor approach in $D = 4$ [1]. One is the Cartan–Penrose representation for a real light–like vector, e.g. the momentum of massless particle,

$$p_{\dot{A}A} := p_a \sigma^{\dot{a}}_{\dot{A}A} = \lambda_\dot{A} \bar{\lambda}_A , \quad \Leftrightarrow \quad p_a p^a = 0 \quad (1)$$

($a = 0, 1, 2, 3, A = 1, 2$ and $\dot{A} = 1, 2$ are Weyl spinor indices and $\sigma^{\dot{a}}_{\dot{A}A}$ are relativistic Pauli matrices). Another is the famous Penrose correspondence,

$$\mu^{\dot{A}} = x^{\dot{A}A} \lambda_A := \frac{1}{2} x^{\dot{a}} \sigma^{\dot{a}}_{\dot{A}A} \lambda_A . \quad (2)$$

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For a fixed \( x^a \) (real or complex), Eq. (2) is a homogeneous linear equation for the coordinates \( Y_{\alpha} = (\mu^\lambda, \lambda_A) \) of the complex space \( \mathbb{C}^4 \). Imposing the topological restriction \( Y_{\alpha} \neq (0,0,0,0) \) (that is passing from \( \mathbb{C}^4 \) to \( \mathbb{C}^4 - \{0\} \)) and using the scaling symmetry \( (\mu^\lambda, \lambda_A) \mapsto (z\mu^\lambda, z\lambda_A) \) as an identification relation, one can treat \( Y_{\alpha} = (\mu^\lambda, \lambda_A) \) as homogeneous coordinates of the projective twistor space \( \mathbb{CP}^3 \) \cite{1, 2}. Thus, as usually stated in twistor approach, Eq. (2) describes a correspondence the space of light–like lines in spacetime (which can be identified with the celestian sphere \( S^2 \)) and the set of all surfaces in the projective twistor space \( \mathbb{CP}^3 \) (“curves of genus zero and degree one” \cite{3}) which is isomorphic to \( \mathbb{CP}^1 \) (in this sense one can say that the Penrose correspondence illustrates the known identity \( S^2 = \mathbb{CP}^1 \)).

To understand that the correspondence involves light–like lines rather then points \( x^a \) of the Minkowski spacetime \( M^4 \), one notices the symmetry of Eq. (2) under

\[
\delta x^{\dot{A}A} = b \dot{\lambda}^{\dot{A}} \lambda^A ,
\]

which is usually called \( b \)–symmetry. The presence of an arbitrary parameter \( b \) as a coefficient for light–like vector \( \lambda^A \dot{\lambda}^{\dot{A}} \) implies that the orbit of the \( b \)–symmetry transformations (3) is the light–like line \( \dot{x}^{\dot{A}A}(b) = x^{\dot{A}A} + b \dot{\lambda}^{\dot{A}} \lambda^A \).

Let us notice that Eq. (2) with real \( x^a \) is the general solution of the single real equation for the twistor variables. This is usually called \emph{helicity constraint}, and reads

\[
\mathcal{S} = \bar{\mu}^\lambda \bar{\lambda}_A - \mu^A \lambda_A = 0 .
\]

If one substitute the complex vector \( x^\mu_L \) (the non–Hermitian \( x^{\dot{A}A}_L \)) for the real \( x^a \) in (2), thus studying the Penrose correspondence in the complexified Minkowski spacetime \( \mathbb{CM}^4 \) (see e.g. \cite{1}),

\[
\mu^\lambda = x^{\dot{A}A}_L \lambda_A , \quad x^{\dot{A}A}_L = x^{\dot{A}A} + iy^{\dot{A}A} ,
\]

one finds, instead of (4), \( \mathcal{S} = \bar{\mu}^\lambda \bar{\lambda}_A - \mu^A \lambda_A = 2i \dot{\lambda}^A y^{\dot{A}A} \lambda_A \), where \( y^{\dot{A}A} := 1/2i(x^{\dot{A}A}_L - (x^{\dot{A}A}_L)^*) \) is the imaginary part of \( x^{\dot{A}A}_L \). The correspondence with the complexified Minkowski spacetime \( \mathbb{CM}^4 \) can be used, in particular, to describe fields of nonzero helicity \cite{21}; to this end one sets \( \mathcal{S} = \bar{\mu}^\lambda \bar{\lambda}_A - \mu^A \lambda_A = 2is \) with some half–integer \( s \). \(^1\) Notice also that the one–parametric \( b \)–symmetry (3) in the case of complex \( x^\mu_L \) is replaced by the complex–spinor–parametric symmetry \( x^{\dot{A}A} + u^A \lambda_A \) of (5). This allows to gauge away all the imaginary part \( y^a \) of \( x^\mu_L \) except for the one enclosed in the contraction \( \lambda_A y^{\dot{A}A} \bar{\lambda}_A \equiv 1/2i \mathcal{S} \). Thus the helicity constraint with nonvanishing r.h.s., \( \mathcal{S} = 2is \) may be used, together with gauge fixing, to define \( y^a = s m(x^\mu_L) \) completely: \( y^{\dot{A}A} = sw^A \bar{\lambda}_A \) where \( w^A \lambda_A = 1 \) and \( \bar{\mu}^\lambda = (w^A)^* \).

An important observation is that, in distinction to (2), the equation (5) with an \emph{arbitrary complex} \( x^\mu_L \) does not restrict the twistor \( Y_{\alpha} = (\mu^\lambda, \lambda_A) \) at all but rather provides

\(^1\) Notice that nonzero helicities can appear as a result of the ordering ambiguity after quantization of the massless particle; the quantum consideration also indicates the quantization of helicity \( s \) in the units of \( h/2 \), see \cite{27, 28, 29} and refs. therein.

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a possibility to change a set of basic variables form the set of two spinor $(\mu^A, \lambda_A)$ to one spinor $\lambda_A$ and a complex vector $x_L^{AA} = x^{AA} + iy^{AA}$ defined modulo the (gauge) transformations $x_L^{AA} \mapsto x_L^{AA} + u^A \lambda^A$. This can be treated as a reason for the existence of the formulation of bosonic higher spin theory with an auxiliary vector variable whose AdS version allows for a nontrivial interaction [17].

I.2. Twistors and massless particle in $D = 4$ Minkowski spacetime

The one parametric $b$ symmetry (3), which is the invariance of Eq. (2), can be identified with the gauge symmetry of the massless particle action in its Ferber–Schirafuji form [21]

\[ S_0 = \int_{W^1} d\hat{x}^{AA} \hat{\lambda}_A \hat{\lambda}_A \equiv \int d\tau \partial_\tau \hat{x} - (\bar{\sigma}^A_\mu \hat{\lambda}_A)(\tau) . \]  

A simple way to obtain this "twistor–like" action is to start with the first–order form of the Brink–Schwarz formulation of the massless particle action, $S_{0, BS} = \frac{1}{2} \int_{W^1} (\hat{p}_{AA} d\hat{x}^{AA} + \frac{1}{2} d\tau e \hat{p}_{AA} \hat{p}^{AA})$, and to substitute the general solution

\[ \hat{p}_{AA} = \hat{\lambda}_A \hat{\lambda}_A \quad (\Leftrightarrow \hat{p}_{AA} \hat{p}^{AA} = 0) \]  

of the algebraic equation of motion $\delta S_{BS} = \frac{1}{2} p_a p^a = 0$ for an arbitrary $\hat{p}_{AA}$ in $S_{0, BS}$. One can also obtain the $b$–symmetry (see (5) of the action (6),

\[ \delta_b \hat{x}^{AA} = b \hat{\lambda}_A \hat{\lambda}_A , \quad \delta_b \hat{\lambda}_A = 0 \]  

by substituting the Cartan–Penrose representation (1) for the light–like $\hat{p}_a$, given by Eq. (7), in the gauge symmetry $\delta \hat{x} = b \hat{p}^a$ of the action $S_{BS}^2$.

The massless particle action (6) provides a simple way to see the relation among the two basic formulae of the twistor approach, namely among the Cartan–Penrose representation of Eq. (1) and the Penrose correspondence relation (2). First one notices that the Hamiltonian formalism for the action (6) [21] reproduces the worldline version (7) of Eq. (1) as a primary constraint (see [65]). Secondly, the above observation that the action possesses the same $b$–symmetry as the Penrose correspondence (2) (see Eqs. (8) and (3)) suggests that (6) should also reproduce the worldline version of Eq. (2),

\[ \mu^A = \hat{x}^{AA} \hat{\lambda}_A . \]  

This is indeed the case. Using the Leibnitz rule $(d\hat{x} \hat{\lambda} \hat{\lambda} = d(\hat{x} \hat{\lambda}) \hat{\lambda} - \hat{x} d\hat{\lambda})$ one can write the action (6) in the form (see [21])

\[ S_0 = \int_{W^1} (d\hat{\mu}^A \hat{\lambda}_A - \hat{\mu}^A d\hat{\lambda}_A) , \quad \hat{\mu}^A \hat{\lambda}_A - \hat{\mu}^A \hat{\lambda}_A = 0 , \]  

This symmetry also includes $\delta e = \partial_\tau b(\tau), \delta \hat{p}_a = 0$ and constitutes a "variational version" of worldline reparametrization.
where $\hat{\mu}^A$ is defined by Eq. (9); as (9) is the general solution of the helicity constraints (4), one can, alternatively, consider the twistor variables $\hat{Y}_\alpha = (\hat{\mu}^A, \hat{\lambda}_A)$ to be subject to the helicity constraint (4) [as it is written in (10)] and omit any reference on the spacetime coordinates. Taking the second point of view one finds that just the constraint (4) reduces the imaginary part of the action (10) to a total derivative. This constraint can be also incorporated into the action with a Lagrange multiplier $\Xi(\tau)$,

$$ S = \int_{W_1} (d\hat{\mu}^A \hat{\lambda}_A^\tau - \hat{\mu}^A d\hat{\lambda}_A) + \int_{W_1} d\tau \Xi(\tau) (\hat{\mu}^A \hat{\lambda}_A^\tau - \hat{\mu}^A \hat{\lambda}_A). $$

(11)

### I.3. Supersymmetry: massless superparticle and supertwistors

The supersymmetric generalization of the action (6) can be obtained $e.g.$ starting with the first order form of the Brink–Schwarz superparticle action and using there the general solution (7) of the mass shell constraints $p^2 = 0$ (see Sec.I.2). It reads [21]

$$ S = \int_{W_1} \hat{\Pi}^{AA} \hat{\lambda}_A^\tau \equiv \int d\tau \hat{\lambda}_A \hat{\sigma}^A \hat{\lambda}_A \hat{\Sigma}^\tau(\tau), $$

(12)

where $\hat{\Pi}^{AA} = d\tau \hat{\Pi}^{AA}$ is the pull–back to the particle worldline $W_1$ of the Volkov–Akulov one–form

$$ \Pi^a = dx^a - id\theta_i \sigma^a \bar{\delta}^i + i\theta_i \sigma^a d\bar{\delta}^i \quad \Leftrightarrow \quad \Pi^{AA} = dx^{\hat{\lambda}A} - id\theta_i \bar{\delta}^{\hat{\lambda}i} + i\theta_i d\bar{\delta}^{\hat{\lambda}i} $$

(13)

for the $D = 4$ N–extended superspace $\Sigma^{(4|4N)}$ with the local coordinates

$$ \Sigma^{(4|4N)} : \ z^M = (x^a, \theta^A, \bar{\theta}^{\hat{\lambda}i}); \quad a = 0, 1, 2, 3, \ A = 1, 2, \hat{A} = 1, 2, \ i = 1, \ldots, N. $$

(14)

The (N–extended) global supersymmetry transformations which leave (13) invariant are

$$ \delta x^a = -i\theta_i \sigma^a e^i + i\epsilon_i \sigma^a \bar{\delta}^i, \quad \delta \theta_i = \epsilon_i, \quad \delta \bar{\delta}^{\hat{\lambda}i} = e^{\hat{\lambda}i}. $$

(15)

As in the purely bosonic case, using the Leibnitz rule one can write the action (12) in the form (cf. (10))

$$ S = \int_{W_1} (d\hat{\mu}^A \hat{\lambda}_A^\tau - \hat{\mu}^A d\hat{\lambda}_A - 2id\hat{\eta}_i \bar{\eta}^i) = \int_{W_1} d\hat{\Sigma} \hat{\Sigma}^\tau \equiv \int_{W_1} d\hat{\Sigma} \Omega^{\Lambda\Pi} (\hat{\Sigma}^\tau)^{\ast}. $$

(16)

Here (the pull–backs of) the components of supertwistor

$$ \hat{\Sigma}_\Lambda := (Y_\alpha, \bar{\eta}_i) = (\mu^A, \lambda_A, \eta_i) \quad \left(\hat{\Pi}_\Lambda := \Omega^{\Lambda\Pi} (\hat{\Sigma}^\tau)^{\ast} \equiv (\lambda^\tau_A, \mu^A, -2i\eta^\tau)^T \right) $$

(17)

are related to the coordinates (14) (coordinate functions in (12)) by the following supersymmetric generalization of the Penrose correspondence relation (2) [21]

$$ \left\{ \begin{array}{l}
\mu^\Lambda = \hat{\Sigma}_\Lambda^\tau \lambda^\tau_A := \frac{1}{2} \hat{\Sigma}^\tau_a \sigma^A \lambda^\tau_A := (x^{\hat{\lambda}A} + i\theta_i^A \bar{\delta}^{\hat{\lambda}i}) \lambda_A, \\
\eta_i = \theta_{\hat{\lambda}i}^A \lambda_A.
\end{array} \right. $$

(18)
These expressions for the supertwistor gives the general solution of the superhelicity constraint
\[ \mathcal{S} := \mu^A \dot{\lambda}_A - \mu^A \dot{\lambda}_A - i \eta_i \bar{\eta}^i \equiv \gamma^\Lambda \bar{Y}^\Lambda \equiv Y^\Lambda \Omega^{\Lambda \Pi} (Y^\Pi)^* = 0 , \] (19)
in which (as well as in (16)) \( \Omega^{\Lambda \Pi} \)
\[ \Omega^{\Lambda \Pi} := \begin{pmatrix} \Omega^{\alpha \beta} & 0 \\ 0 & -2i \delta^i_j \end{pmatrix} = \begin{pmatrix} 0 & \delta^b_A & 0 \\ -\delta^A_B & 0 & 0 \\ 0 & 0 & -2i \delta^j_i \end{pmatrix} \] (20)
is the \( SU(2,2|N) \) invariant matrix. Such an observation allows one to write the superparticle action in an equivalent form (cf. (11))
\[ S = \int_{W^1} (d \mu^A \dot{\lambda}_A - \mu^A d \lambda_A - 2i d \eta_i \bar{\eta}^i) + \int_{W^1} d \tau \Xi (\tau) (\mu^A \dot{\lambda}_A - \mu^A \dot{\lambda}_A - 2i \eta_i \bar{\eta}^i) \equiv \int_{W^1} d \gamma^\Lambda \Omega^{\Lambda \Pi} (Y^\Pi)^* + \int_{W^1} d \tau \Xi (\tau) Y^\Lambda \Omega^{\Lambda \Pi} (Y^\Pi)^* . \] (21)

In this form the \( SU(2,2|N) \) symmetry of the superparticle action becomes manifest. The action (21), incorporating also the constraint (19) with the Lagrange multiplier \( \Xi \), involves only one constant tensor \( \Omega^{\Lambda \Pi} = -(-)^{\Lambda + \Pi} \Omega^{\Pi \Lambda} \), Eq. (20), and the invariance of such a tensor is the defining property of the \( SU(2,2|N) \) supergroup.

In relation with the evident equivalence of the action (21) (or (16)) with (12) one can ask questions about degrees of freedom. In particular, the action (12) contains 4\( N \) fermionic fields (coordinate functions) \( \theta^\alpha \) while (21) (or (16)) involves 2\( N \) fermionic \( \eta_i, \bar{\eta}^i \). This seeming mismatch indicates the presence of 2\( N \) local fermionic gauge symmetries in the action (12). These has the form
\[ \delta_\kappa \dot{x}^{AA}(\tau) = i \kappa \lambda^A \dot{\theta}^A + i \bar{\kappa} \theta^A \dot{\bar{\lambda}}^A = i \delta_\kappa \theta^A \dot{\theta}^A (\tau) - i \theta^A \delta_\kappa \dot{\theta}^A (\tau) \] (22)
\[ \delta_\kappa \theta^A = \kappa (\tau) \lambda^A , \quad \delta_\kappa \dot{\theta}^A (\tau) = \bar{\kappa} (\tau) \dot{\bar{\lambda}}^A (\tau) \] (23)

and provide an irreducible form (see [30, 31, 32, 23] and refs therein) of the seminal \( \kappa \)-symmetry [33] of the Brink–Schwarz superparticle which can be defined by \( \delta_\kappa \dot{\theta}^A \Pi_{\kappa \Lambda \bar{\Lambda}} = 0 \) and \( i_\kappa \Pi_{\Lambda \bar{A}} := \delta_\kappa \dot{x}^{\Lambda \bar{A}} - i \delta_\kappa \theta^A \dot{\bar{\lambda}}^A + i \theta^A \delta_\kappa \dot{\theta}^A = 0 \).

II. BPS PREONS WITHOUT SUPERSYMMETRY

II.1. BPS preons and generalized Cartan–Penrose representation

In this section we present the definition of the BPS preon from [6] (see also [8, 12]) in its bosonic form which makes transparent the relation with the twistor approach.

\( \diamond \) The BPS preon state can be characterized by one bosonic spinor \( \lambda_\alpha \),
\[ |BPS \ preon \rangle = |\lambda_\alpha \rangle , \quad \alpha = 1, \ldots , n , \] (24)
and is an eigenvector of the generalized momentum operator $P_{\alpha\beta} = P_{\beta\alpha}$ for the
eigenvalue $\lambda_{\alpha} \lambda_{\beta}$ determined by the above mentioned spinor $\lambda_{\alpha}$,

$$P_{\alpha\beta} |\lambda_{\alpha}\rangle = \lambda_{\alpha} \lambda_{\beta} |\lambda_{\alpha}\rangle .$$ (25)

\diamondsuit In the original M–theoretic context of [6] $\alpha = 1, \ldots, 32$ is the Majorana–Weyl spinor
index of $SO(1,10)$ ($D = 11 = 1 + 10$), but a generalization for $\alpha = 1, \ldots, n$ with other
$n = 2^k$ allowing treatment as Majorana or pseudo–Majorana spinors of $SO(t,D−t)$ with
other $D$ is straightforward.\(^3\)

In supersymmetric theory, where the generalized momentum is defined by the anti–
commutator of fermionic charges, $P_{\alpha\beta} = \{Q_{\alpha} Q_{\beta}\}$, the above definition implies (see
[6, 8]) that the BPS preon state $|\text{BPS preon}\rangle = |\lambda_{\alpha}\rangle$ preserves all but one supersymme-
tries generated by $Q_{\alpha}$ with $\alpha = 1, \ldots, n$. Hence another notation for the preonic state is
$|\text{BPS preon}\rangle = |\text{BPS} (n−1)\rangle$ reflecting the number of preserved supersymmetries; in the
M–theoretic $n = 32$ case this is $|\text{BPS preon}\rangle = |\text{BPS} 31\rangle$. This notation, however, can be
understood also without references on supersymmetry, as we will see in a moment.

\section{II.2. BPS preons as fundamental constituents}

The above definition of the BPS preon is based on the eigenvalue problem for the
generalized momentum operator $P_{\alpha\beta}$ and, hence, assumes that different components of
$P_{\alpha\beta}$ can be diagonalized simultaneously. This is the case when they are commuting,

$$[P_{\alpha\beta}, P_{\gamma\delta}] = 0 .$$ (26)

The general eigenvector $|p_{\alpha\beta}\rangle$ of the Abelian $P_{\alpha\beta}$,

$$P_{\alpha\beta} |p_{\alpha\beta}\rangle = p_{\alpha\beta} |p_{\alpha\beta}\rangle ,$$ (27)

is characterized by an eigenvalue matrix $p_{\alpha\beta}$. One can rise the question how to classify
these states. Such a classification problem looks much less academic in a supersymmet-
tric context where (see Sec. III) it is equivalent to the search for a classification of the
BPS states (M–theory BPS states for $n = 32$) [6].

The Abelian algebra of the generalized momenta, Eq. (26), possesses a manifest
$GL(n)$ symmetry. The only property of $|p_{\alpha\beta}\rangle$ states which is invariant under this
$GL(n)$ symmetry is the rank, rank($p_{\alpha\beta}$), of the eigenvalue matrix $p_{\alpha\beta}$. Let us denote the matrix
of rank $(n − k)$ by $p_{\alpha\beta}^{(k)}$, rank($p_{\alpha\beta}^{(k)}$) := $(n − k)$ ($(32 − k)$ in the M–theoretical case) and
the state with the eigenvalue matrix $p_{\alpha\beta}^{(k)}$ by $|\text{BPS} , p_{\alpha\beta}^{(k)}\rangle$ or, shortly, $|\text{BPS} k\rangle = | k\rangle$,\(^3\)

$$P_{\alpha\beta} |k\rangle = p_{\alpha\beta}^{(k)} |k\rangle , \quad \text{rank}(p_{\alpha\beta}^{(k)}) = n − k \quad ((32-k) \text{ for } n=32 \Leftarrow D=11) .$$ (28)

\(^3\) The cases of $n \neq 2^k$ allow for a treatment as multispinor index (a set of spinor indices); e.g. for
the odd values of $n$, $\lambda_{\alpha}$ one can treat $\alpha = 1, \ldots, n$ as a set of $n$ one–valued Majorana–Weyl spinors in $D = 2$. 

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The definition of BPS preon implies its identification with the state $|⟨(n−1)⟩⟩$ with a generalized momentum eigenvalue matrix $p^{|(n−1)⟩}_{αβ} = p^{(1)}_{αβ}$ of rank equal to one. Indeed, any matrix of rank one can be expressed by the direct product of two vectors,

$$p_{αβ} = λ_α λ_β \Leftrightarrow p_{αβ} = p^{|(n−1)⟩}_{αβ} := p^{(1)}_{αβ} = λ_α λ_β , \quad α, β = 1, \ldots, n . \quad (29)$$

Clearly Eq. (29) provides [20] a generalization of the Cartan–Penrose representation (1) for the light–like four vector in the Minkowski spacetime $\mathbb{M}^4$.

In the supersymmetric theory, where $P_{αβ} = \{Q_α Q_β\}$, (see Sec. III) the classification by the rank of the generalized momentum matrix $(32 − k)$ provides the classification of the BPS states by the number of preserved supersymmetry $(k)$ [6]. Here the states $k$ are the BPS states preserving $k$ of the $n$ supersymmetries generated by $Q_α$-s. The BPS preons preserves all but one supersymmetries, $|BPS preon⟩ = |⟨(n−1)⟩⟩$, which means 31 out of 32 supersymmetries in the M–theoretic ($'D = 11'$) case, $|BPS preon⟩ = |31⟩$.

Now we are ready to discuss the rôle of BPS preons as possible constituents. Notice that a symmetric $n \times n$ matrix always can be diagonalized by $GL(n, \mathbb{R})$ transformations, i.e. there exists a matrix $g_α(γ) ∈ GL(n, \mathbb{R})$ such that

$$p^{k}_{αβ} := p^{(32−k)}_{αβ} = g_α(γ) p(γ)(δ) g_β(δ) \quad (30)$$

with some diagonal matrix $p(γ)(δ) = diag(\ldots)$ holds. Moreover, this diagonal matrix can be put in the form $p(γ)(δ) = diag(1, \ldots, 1, −1, \ldots, −1, 0, \ldots, 0)$, where the number of nonvanishing elements, all +1 or −1, is equal to $\tilde{n} = (n−k) = rank(p^{(k)}_{αβ})$.

Only at this stage we really need in a reference on supersymmetry. Indeed, the usual assumptions of unitary supersymmetric quantum mechanics do not allow for negative eigenvalues of $P_{αβ} = \{Q_α, Q_β\}$. Thus, only positive eigenvalues are allowed and

$$p(γ)(δ) = diag(\underbrace{1, \ldots, 1}_{\tilde{n}=32−k}, 0, \ldots, 0) . \quad (31)$$

Substituting (31) into (30) and denoting $g^1_α = λ^1_α , \ldots, g^n_α = λ^n_α$, one finds

$$P_{αβ} |BPS, k⟩ = \sum_{r=1}^{\tilde{n}=32−k} λ^r_α λ^r_β |BPS, k⟩ \equiv (λ^1_α λ^1_β + \ldots + λ^n_α λ^n_β) |BPS, k⟩ . \quad (32)$$

Eq. (32) may be treated as a manifestation of the composite structure of any BPS state $|BPS, k⟩$ with $k < (n−1)$. To this end one solves (32) by

$$|BPS, k⟩ = |λ^1⟩ \otimes \ldots \otimes |λ^{(32−k)}⟩, \quad (33)$$

which implies that the BPS states $|BPS, k⟩$ with $k < (n−1)$ are composed from $\tilde{n} = 32 − k$ BPS preonic states $|λ^1⟩, \ldots, |λ^\tilde{n}⟩$ characterized by the spinors $λ^1_α , \ldots, λ^n_α$. Clearly for the vacuum states preserving all supersymmetries, $k = n$, Eq. (33) does not make sense; for $k = (n−1)$ it just identifies different notations for a BPS preon $|BPS, 31⟩ = |λ^1⟩$ i.e. it implies that BPS preons are fundamental.
In the light of a supersymmetric treatment this implies that [6] any BPS states preserving some (but not all) supersymmetries can be considered as a composite of BPS preons. In particular all the M–theory BPS states can be considered as composed of BPS preons, which allowed us to conjecture that the BPS preons may be considered as fundamental constituents of M–theory [6].

The supersymmetry is important in the following respect. In non–supersymmetric theory the generalized momentum matrix is not positive definite. This implies the possibility of minus signs in the diagonalized form of the generalized momentum matrix, i.e. $p(\gamma(\delta) = \text{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$ rather than (31). Then, to compose the state with such an eigenvalue of the generalized momentum one should introduce, in addition to BPS preons (25), their counterparts with negative energy, ”antipreons” $|\text{anti–BPS preon}, \lambda_\alpha\rangle \equiv |\text{anti–}\lambda_\alpha\rangle$ obeying $P_\alpha|\text{anti–}\lambda_\alpha\rangle = -\lambda_\alpha\lambda_\beta|\text{anti–}\lambda_\alpha\rangle$.

II.3. BPS preon and generalized Penrose correspondence.
Symplectic twistors and tensorial spaces

In the light of the discussion of the first sections, one may expect that some generalization of the Penrose correspondence (2) should be related with the generalization (29) of the Cartan–Penrose representation (1). As (2) includes the coordinate $x^\mu$ conjugate to the momentum $p_\mu$ entering in (1), one may expect that the desired generalization of the (2) should include a spin–tensorial coordinate $X^{\alpha\beta} = X^{\beta\alpha}$ conjugate to $p_{\alpha\beta}$ of (29). In such a way one arrives at the $n(n+1)/2$ dimensional spacetime with coordinates $X^{\alpha\beta}$,

$$\Sigma(n(n+1)/2): X^{\alpha\beta} = X^{\beta\alpha}, \quad \alpha, \beta = 1, 2, \ldots, n,$$

which is called ”tensorial space” [49, 14, 18]. To justify this name one can notice that e.g. for $n = 4$ one can decompose the symmetric spin–tensorial coordinate $X^{\alpha\beta} = X^{\beta\alpha}$ of the $\Sigma(4(4+1)/2) = \Sigma(10)$ space on the basis of $D = 4$ Dirac matrices

$$X^{\alpha\beta} = X^{\beta\alpha} = \frac{1}{2} \gamma^{\mu} \gamma^{\nu} X^{\alpha\beta}_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3; \quad \alpha, \beta = 1, 2, 3, 4, \quad (35)$$

arriving at the set of antisymmetric tensorial coordinates $\gamma^{\mu\nu} = -\gamma^{\nu\mu}$ in addition to the standard four–vector coordinates $x^\mu$ (see [49]). For $n = 16$ one can use the decomposition on the basis of $D = 10$ sigma–matrices,

$$X^{\alpha\beta} = X^{\beta\alpha} = \frac{1}{16} \gamma^{\mu} \gamma^{\nu} X^{\alpha\beta}_{\mu\nu} + \frac{1}{16 \cdot 5!} \gamma^{\mu_1 \ldots \mu_5} \gamma^{\nu_1 \ldots \nu_5} X^{\alpha\beta}_{\mu_1 \ldots \mu_5, \nu_1 \ldots \nu_5}, \quad \mu, \nu = 0, 1, \ldots, 9,$$

$$\alpha, \beta = 1, \ldots, 16, \quad \gamma^{\mu_1 \ldots \mu_5} = \gamma^{[\mu_1 \ldots \mu_5]} = (-)\left(\frac{1}{5!}\right) \gamma^{\mu_1 \ldots \mu_5} v_1 \ldots v_5 y_{v_1 \ldots v_5},\quad (36)$$

one arrives at the parametrization of $\Sigma(10(10+1)/2) = \Sigma(55)$ space by 10 usual vector and 45 antisymmetric (anti-)selfdual 5–index tensorial coordinates $y^{\mu_1 \ldots \mu_5} = y^{[\mu_1 \ldots \mu_5]} = (-)\left(\frac{1}{5!}\right) \gamma^{\mu_1 \ldots \mu_5} v_1 \ldots v_5 y_{v_1 \ldots v_5}$ (see [28]). Finally, in $n = 32$ one can use the set of $D = 11$ gamma matrices to arrive at the parametrization of $\Sigma(528)$ by the set of vectorial, $x^\mu$. 

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two–index tensorial \( y^{\mu\nu} = -y^{\nu\mu} \) and five–index tensorial \( y^{\mu_1...\mu_5} = y^{[\mu_1...\mu_5]} \) coordinates,

\[
X^{\alpha\beta} = X^{\beta\alpha} = \frac{1}{2} \delta^{\alpha\beta} + \frac{i}{64 \pi^2} y^{\mu\nu} \Gamma_{\mu\nu}^{\alpha\beta} + \frac{1}{32 \pi^2} y^{\mu_1...\mu_5} \Gamma_{\mu_1...\mu_5}^{\alpha\beta},
\]

(37)

\( \mu, \nu = 0, 1, \ldots, 10; \alpha, \beta = 1, \ldots, 32. \)

These spaces appear as the bosonic body of the ‘generalized’ [20] or ‘extended’/‘enlarged’ [36, 8] or ‘tensorial’ (see [18] and refs. therein) superspaces \( \Sigma^{(10|4)} \), \( \Sigma^{(55|16)} \) and \( \Sigma^{(528|32)} \) which we will discuss in Secs III, IV.

The generalized Penrose correspondence has the simple form [20, 6]

\[
\mu^\alpha = X^{\alpha\beta} \lambda^\beta, \quad \alpha, \beta = 1, \ldots, n
\]

(38)

involving the \( \Sigma^{(n(n+1)/2n)} \) coordinates \( X^{\alpha\beta} \) and \( 2n \) real components (\( n \) in \( \mu^\alpha \) and \( n \) in \( \lambda^\alpha \)) of symplectic twistor \( \Upsilon_0 \alpha \)

\[
\Upsilon_0 \alpha = (\mu^\alpha, \lambda^\alpha).
\]

(39)

This parametrizes the space \( \mathbb{R}^{2n} - \{0\} \) of the fundamental representation the \( Sp(2n) \) group which leaves invariant the matrix

\[
C_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & \delta_{\hat{\alpha}\hat{\beta}} \\ -\delta^\alpha_{\hat{\beta}} & 0 \end{pmatrix}, \quad \hat{\alpha}, \hat{\beta} = 1, \ldots, 2n, \quad \alpha, \beta = 1, \ldots, n.
\]

(40)

The homogeneity of Eq. (38) allows one to treat it as an equation in the projective symplectic twistor space \( \mathbb{RP}^{2n-1} \).

In distinction to (2) with real \( x^\alpha \) (and in analogy with (5) with complex \( x_I^\alpha \)), Eq. (38) with an arbitrary real \( X^{\alpha\beta} = X^{\beta\alpha} \) does not set any restrictions on the bosonic spinors (or s–vectors [14]) \( \mu \) and \( \lambda \). Indeed, defining \( \mu \) and \( \lambda \) in an arbitrary manner one always can find symmetric \( X^{\alpha\beta} \) such that (38) holds. Moreover, such \( X^{\alpha\beta} \) is not unique. The generalized Penrose correspondence (38) is invariant under the \( n(n-1)/2 \)–parametric generalization of the \( b–symmetry \) (8) (see [20, 7])

\[
\delta_b X^{\alpha\beta} = b^{IJ} u^\alpha_I u^\beta_J, \quad \delta_b \lambda^\alpha = 0, \quad \delta_b \mu^\alpha = 0.
\]

(41)

where \( u^\alpha_I \), spinors which are orthogonal to \( \lambda^\alpha \),

\[
u^\alpha_I \lambda^\alpha = 0, \quad I = 1, \ldots, (n-1).
\]

(42)

Eq. (41) provides the general solution of the condition

\[
\delta_b X^{\alpha\beta} \lambda^\alpha = 0.
\]

(43)

---

4 One may find better to call \( \lambda^\alpha \) and \( \mu^\alpha \) s–vectors [14] as the invariance of our basic equations is given by \( GL(n) \) and, non-manifestly, by \( Sp(2n) \) rather than by some thier \( SO(1, D-1) \) subgroup.
which can be used as an alternative definition of the generalized $b$–symmetry.  

Thus Eq. (38) can be considered as a correspondence between the space of $(n-1)$ dimensional hyperplanes in $\mathbf{R}^{2n-1}$ (each parameterized by spinor $\lambda$ modulo its scaling factor considered to be common with $\mu^\alpha$) and the space of $n(n-1)/2$ dimensional surfaces (given by $\hat{X}^{\alpha\beta}(b^{IJ}) = X^{\alpha\beta} + b^{IJ}u^\alpha_I u^\beta_I$) in $n(n+1)/2$ dimensional tensorial space $\Sigma^{(n(n+1)/2)[0]}$. On the language of generalized particle–like mechanics this correspondence implies that the action [20]

$$S_0 = \int_{W^1} \lambda_\alpha \lambda_\beta d\hat{X}^{\alpha\beta} \equiv \frac{1}{2} \int d\tau \lambda_\alpha(\tau) \lambda_\beta(\tau) \partial_\tau \hat{X}^{\alpha\beta}(\tau),$$

(44)

which possess a gauge $b$–symmetry given by pull–back of Eq. (43) on the worldline $W^1$, allows for the reformulation in terms of the symplectic twistor coordinate functions $\hat{Y}_{0\alpha}(\tau) = (\hat{\mu}^\alpha, \hat{\lambda}_\alpha)$. Indeed, one can use the Leibniz rule ($\hat{\lambda} \partial \hat{X} \equiv \partial (\hat{\lambda} \hat{X}) - (\partial \hat{\lambda}) \hat{X}$) to present the action (44) in the equivalent form

$$S = \int_{W^1} (d\hat{\mu}^\alpha \hat{\lambda}_\alpha - d\hat{\lambda}_\alpha \hat{\mu}^\alpha) \equiv \int_{W^1} d\hat{Y}_{0\alpha} C^{\alpha\beta} \hat{Y}_{0\beta},$$

(45)

where $\hat{\mu}^\alpha$ is defined by the pull–back of the generalized Penrose correspondence relation (38), $\hat{Y}_{0\alpha}$ by the pull–back of (39) and $C^{\alpha\beta}$ is the $Sp(2n)$ invariant of Eq. (40).

Notice that the action (44) produces the generalized Cartan–Penrose representation (29) as a primary constraint (see [65]) in the Hamiltonian formalism. This gives a reason (see the supersymmetric considerations in [8, 12] and Sec. III for more) to treat (44) as a dynamical model for a point–like BPS preon.

Hence a search for a generalized Penrose correspondence related to the generalized Cartan–Penrose representation (29) and the definition of the BPS preon (25) leads us, through Eq. (38), to the (ortho)symplectic twistors (39) and to the generalized or tensorial (super)spaces (34) which appeared also in different perspective, see [36, 37].

II.4. BPS preons and higher spin fields in $D = 4, 6, 10$

In distinction to (11), the "preonic" action (45) contains an unconstrained twistors, i.e., no counterpart of the helicity constraint (19) appears when one writes the equivalent twistor representation (45) of Eq. (44). After quantization of particle mechanics (11) one arrives at the wave function $\phi(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ subject to the quantum counterpart of the constraint (4). Just the latter constraint makes $\phi(\lambda_\alpha) = \phi(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ to describe massless

---

5 To see that the transformations (41) provide the counterpart of the 'standard' $b$–symmetry (3), one notices that, allowing for the existence of a non–degenerate antisymmetric matrix $C_{\alpha\beta} = -C_{\beta\alpha}$ [clearly, for even $n$, see footnote 3; this also reduces $GL(n)$ symmetry down to $Sp(n)$], one can use its inverse $C^{-1\beta\alpha}$ to define $\lambda^\alpha = C^{\beta\alpha}\bar{\lambda}\beta$ which clearly obeys $\lambda^\alpha\lambda_\alpha = 0$. This allows us to identify this $\lambda^\alpha$ with one of the $u^\alpha_I$ $[u^\alpha_I = (\lambda^\alpha, u^\alpha_I)$, see [12]] and to find among (41) the counterpart $\delta \phi(\lambda^\alpha, \bar{\lambda}_{\dot{\alpha}}) = b\lambda^\alpha\lambda^\beta$ of the $D = 4$ transformations (3); for $D = 3$ there is only one $u^\alpha_I$ which coincides with $\lambda^\alpha$ and thus all the $b$–symmetry is reproduced by the above formula.
particle of a certain helicity (see [21, 27, 28, 29], [13] and refs. therein), e.g. of helicity equal to zero. This explains the helicity constraint name. Now, the action (45) for the $n = 4$ case differs from that in Eq. (11) just by the absence of the helicity constraint \( (\lambda_\alpha = (\hat{\lambda}_A, \lambda_\alpha), \mu^\alpha = (\hat{\mu}_A, \mu^\alpha)) \). Hence, one may expect that the quantum state spectrum of this “preonic” particle mechanics would include a tower of massless field of all possible helicities. The analysis of [13] showed that this is indeed the case for $n = 4$ $D = 4$ and indicated an infinite tower of massless $D = 6$ and $D = 10$ ‘higher spin’ fields in the model with $n = 8$ and $n = 16$. As it is finally shown in [19] these are all the conformal massless higher spin fields in $D = 6$ and $D = 10$ dimensions, respectively.

Here we will try to create some image of the relation between preonic particle mechanics and higher spin fields with an emphasis on the rôle of twistor–like methods and notions. More technical details can be found in the original papers.

II.4.1. Higher spins from tensorial space.

Interestingly enough, the tensorial space (34) with $n = 4$, Eq. (35), was proposed in [49] as a basis for the construction of D=4 higher–spin theories. It was known that a consistent interaction of higher spin fields requires i) an infinite tower of all possible higher spin fields and ii) a spacetime with a nonzero cosmological constant (see [50, 17, 26]). The assumption of [49] was that there may exist a theory in a ten–dimensional space $\Sigma^{(10)}$ whose (alternative–to–) Kaluza–Klein reduction may lead in $D = 4$ to an infinite tower of fields with increasing spins instead of the infinite tower of Kaluza–Klein particles of increasing mass. It was argued that the symmetry group of the theory should be $Sp(8) \supset SU(2, 2)$, and $OSp(1|8)$ in supersymmetric case. The idea was that using a single representation of $OSp(1|8)$ (such that it contains each and every massless higher spin representation of the $D = 4$ superconformal group $SU(2, 2) \subset OSp(1|8)$ only once) in the ten–dimensional tensorial space one could describe an infinite tower of massless higher spin fields in $D = 4$ space-time.

In this perspective the $\Sigma^{(10(n+1)/2|n)}$ superparticle action of [20], the point–like model for BPS preon in the light of [6] and [7, 8, 12], provided (rather accidentally; in its $n = 4$ $\Sigma^{(10|4)}$ version) a dynamical realization of the proposal from [49]. This preonic superparticle action, whose purely bosonic limit is given by Eq. (44), involves the auxiliary bosonic spinor variables $\lambda_\alpha(\tau)$. These provide a twistorial dimensional reduction (for $n > 2$) and, for $n = 4, 8, 10$, also a twistorial compactification mechanism [13] which results in the discreetness of the quantum state spectrum and its identification, for $n = 4$, with the spectrum of all the massless higher spin fields in $D = 4$ [13] and, for $n = 8, 16$, with the spectrum of all conformal massless higher spin fields in $D = 6$ and $10$ [19]. The AdS generalization of the model [20] is provided by the superparticle on the $OSp(1|n)$ supergroup manifold [13, 51]. It was conjectured in [51, 14] and shown in [52, 23] that a field theory on $OSp(1|4)$ is classically equivalent to the $OSp(1|8)$–invariant free higher spin field theory in $AdS_4$.

The preonic particle model (44) possesses manifest $GL(n)$ and non–manifest $Sp(2n)$ symmetry ($OSp(1|2n)$ in supersymmetric case), thus showing the expected symmetry of higher spin theories. The latter becomes manifest symmetry after passing to an equivalent twistor form (45). The $Sp(2n)$ symmetry is also manifest in the following
equivalent form of the action (44) [14]

\[ S_0 = \int_{W^1} \left( \lambda_\alpha \lambda_\beta d\hat{X}^{\alpha\beta} + \bar{\mu}^\alpha d\lambda_\alpha \right). \] (46)

One of the simple ways to show the equivalence of (44) and (46) is to notice that, moving the derivatives, one can rewrite (46) in the equivalent form (45) of the action (44). The only difference then will be a shift in definition of \( \mu^\alpha \): Eq. (38) is replaced by

\[ \mu^\alpha = X^{\alpha\beta} \lambda_\beta + \bar{\mu}^\alpha. \]

For Hamiltonian formalism the use of the action (46) instead of (44) looks like a simple method of conversion of the second class constraints, which are present for (44), into the first class ones (see [16] for further discussion and references; the conversion was also done in [13] but in a more complicated way). The quantization with such a conversion results in the preonic wave function \( \Phi(X, \lambda) \) subject to the constraint [13]

\[ (\partial_\alpha - i \lambda_\alpha \lambda_\beta) \Phi(X, \lambda) = 0, \quad \partial_\alpha := \partial / \partial X^{\alpha\beta}. \] (47)

Eq. (47) is clearly the coordinate representation of the definition of the BPS preon (25) provided the preon is considered as a point–like object in tensorial space,

\[ \Phi(X, \lambda) = \langle X^{\alpha\beta} | \lambda_\alpha \rangle \equiv \langle X^{\alpha\beta} | \text{BPS preon } \lambda_\alpha \rangle. \] (48)

This gives one more reason to state that the generalized (super)particle model [20] with the bosonic limit (44) provides a model for a point–like BPS preon [6, 12].

The solution of Eq. (47) is given by the generalized plane wave, preonic plane wave,

\[ \Phi(X, \lambda) = \phi(\lambda) \exp\{-i\lambda_\alpha X^{\alpha\beta} \lambda_\beta\}. \] (49)

involving an arbitrary function \( \phi(\lambda) \) of the bosonic spinor \( \lambda_\alpha \). Its integration over \( \lambda \)

\[ b(X) = \int d^n \lambda \Phi(X, \lambda) = \int d^n \lambda \phi(\lambda) \exp\{-i\lambda_\alpha X^{\alpha\beta} \lambda_\beta\} \] (50)

provides the general solution of the following equations in tensorial space

\[ \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta b(X) \equiv 1/2(\partial_\alpha \partial_\beta \partial_\gamma \partial_\delta - \partial_\alpha \partial_\gamma \partial_\beta \partial_\delta)b(X) = 0. \] (51)

This was proposed in [14] as dynamical equations for massless higher spin fields. It was shown in [14] (see also [16]) that, for \( n = 4 \), decomposing the field \( b(X) = b(x^\mu, y^{\mu\nu}) \) (see (35)) in the series on \( y^{\mu\nu} = -y^{\nu\mu} \) one finds all the field strengths of the massless bosonic \( D = 4 \) higher spin fields i.e. of the higher spin fields with integer spin; their equations of motion follow from (51). The details and further discussion on relation

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\[ \text{6 } \] The choice of the class of functions where \( \phi(\lambda) \) takes its values is not unique. One should fix it to provide the convergence of integrals used on the way to a spacetime treatment which, in its turn, is also not unique. This interesting issue is beyond the scope of that contribution; see [15, 16] and also sec. IIA and footnote 5 in [13] for discussions.
between field theories in tensorial space and in the standard spacetime can be found in [15, 16] (mainly for $D = 4$) and in [19] (also for $D = 6, 10$).

The field strengths of massless fields with all the possible half–integer spins are collected in the spinorial field $f_\alpha(X)$ obeying a tensorial space counterpart of the Dirac equation

$$\partial_{\alpha}\beta f_\gamma(X) \equiv 1/2(\partial_{\alpha}\beta f_\gamma(X) - \partial_{\alpha\gamma}f_\beta(X)) = 0 .$$

The general solution of Eq. (52) is given by the integral of the preonic plane wave (49) with measure $d^n\lambda \lambda_\alpha$,

$$f_\alpha(X) = \int d^n\lambda \lambda_\alpha \Phi(X, \lambda) = \int d^n\lambda \lambda_\alpha \phi(\lambda) \exp\{-i\lambda_\alpha X^{\alpha\beta} \lambda_\beta\} .$$

Clearly, as for the even $n$ (including $n = 4$) the measure is symmetric under $\lambda \mapsto -\lambda$, the half integer fields collected in $f_\alpha(X)$ come from the odd part of $\phi(\lambda)$, $\phi_-(\lambda) = 1/2(\phi(\lambda) - \phi(-\lambda))$, while the integer fields collected in $b(X)$ come from the even part of $\phi(\lambda)$, $\phi_+(\lambda) = 1/2(\phi(\lambda) + \phi(-\lambda))$. However both integer and half–integer fields come from the same “twistorial wave function” $\phi(\lambda)$. It appears directly as a result of quantization when one starts from the action (45).

II.4.2. Twistor wave function, Cartan–Penrose representation and the Hopf fibrations. $D = 3, 4, 6$ and 10.

A simplest way to quantize the preonic particle model [20] is to use an equivalent twistor representation (45) of the preonic action (44) [13]. Indeed, the action (45) is i) written in terms of unconstrained symplectic twistor $(\mu^\alpha, \lambda_\alpha)$; ii) is the first order action, which allows to identify $(\mu^\alpha$ with the momentum conjugate to the coordinate $\lambda_\alpha$ (or vice versa); iii) the Hamiltonian of the system is identically zero, which implies that the system is free and that, after quantization, the Schrödinger equation just states an independence of the proper time parameter $\tau$ (reparametrization invariance). As a result, one sees that the wave function of the preonic particle is an arbitrary function $\phi(\lambda_\alpha)$ of the bosonic spinor $\lambda_\alpha$ (see footnote 6).

To provide the spacetime treatment of such a wavefunction one applies [13] the generalized Penrose correspondence (29) and extracts the $D$ dimensional momentum from the generalized momentum using the $n \times n$ real (or pseudo–real) representation of $D$–dimensional gamma matrices ($n = 2[D/2]$ for $D \neq 10$ and $n = 2[D/2]$ for other $D$)

$$p_{\alpha\beta} = \lambda_\alpha \lambda_\beta , \quad p_\mu \equiv \frac{1}{n} p_{\alpha\beta} \Gamma^{\alpha\beta}_\mu = \lambda \Gamma_\mu \lambda \quad \left\{ \begin{array}{l} \alpha, \beta = 1, \ldots, n \\ \mu = 0, 1, \ldots, D - 1 \end{array} \right.$$}

Thus the wave function $\phi(\lambda)$ can be treated as dependent on the spacetime momentum $p_\mu$ and the additional variables parameterizing the fibration $\mathbb{S}^D_n$ of the space $\mathcal{F} := \mathbb{R}^d$ –

---

7 In this respect one notices (see [13] and [19]) that the quantum state spectrum of the preonic particle is already supersymmetric (contains all integer and all half integer fields) while the spectrum of its supersymmetric generalization [13] is doubly degenerate. To resolve the degeneracy and to provide the physical spin–statistics correspondence one uses a projection relating the Grassmann parity and the parity with respect to $\lambda \mapsto -\lambda$; see [13], [18] and refs. therein for further discussion.
\{0\} = S^{n-1} \otimes R_+ \text{ of nonvanishing spinors } \lambda_\alpha \neq (0, \ldots, 0) \text{ over the (base) space } \omega := \{p_\mu : p_\mu = \lambda \Gamma_\mu \lambda\} \text{ of momentums determined by the Cartan–Penrose representation } p_\mu = \lambda \Gamma_\mu \lambda, \\
\phi(\lambda) = \phi(p : \mathcal{Z}_n^{D})|_{p=\lambda \Gamma_\mu \lambda} \iff \phi(\lambda) = \phi(\omega, \mathcal{Z}) , \quad \mathcal{Z} \approx \frac{\mathcal{Z}}{\rho} \quad (55)

The properties of the base space \{\omega : p_\mu = \lambda \Gamma_\mu \lambda\} \text{ depends strongly on } D \text{ and } n.

For } n = 2, 4, 8, 16 \text{ corresponding to } D = 3, 4, 6, 10 \text{ the famous identity } \Gamma^{\mu(\alpha\beta \Gamma_\mu \gamma \delta) = 0 \text{ holds. It results in a light–like momentum } p_\mu
\\D = 3, 4, 6, 10 : \quad p_\mu = \lambda \Gamma_\mu \lambda \quad \Rightarrow \quad p_\mu p_\mu = 0 , \quad (56)

Hence the space \omega spanned by momenta } p_\mu = \lambda \Gamma_\mu \lambda \text{ is } D - 1 \text{ dimensional (rather than } D-\text{dimensional), } \mathcal{Y} = R^{(D-1)} - \{0\} = S^{(D-2)} \otimes R_+. \text{ Hence the space } \mathcal{Y}_n^{D} \text{ of additional variables in (55) is the fibrations of } \mathcal{Z}^{n} \text{ of } \omega \text{ by } \mathcal{Y} = S^{(D-2)} \otimes R_+ \text{ which, in the light of identification of scales by } p_\mu = \lambda \Gamma_\mu \lambda, \text{ is the fibrations of spheres over spheres, } \mathcal{Z}_n^{D} := \mathcal{Y}_n^{D} = S^{(n-1)} \otimes R_+ = \frac{S^{(n-1)} \otimes R_+}{S^{(D-2)}}. \text{ Furthermore, as in the dimensions } D = 3, 4, 6 \text{ and } 10 \text{ the number of components of minimal real (or, in } D = 6, \text{ pseudoreal) representation can be written as } n = 2(D-2) \text{ (see [23] and refs. therein), the spaces of nonvanishing bosonic spinors are } \mathcal{Y}^{n} = \mathcal{Y}^{2(D-2)} = S^{(n-1)} \otimes R_+ = S^{(2D-5)} \otimes R_+ \text{ and the fibrations parametrized by additional variables can be presented in a more transparent form } \mathcal{Y}_D = S^{2(D-2)} = S^{(2D-5)} / S^{(D-2)}. \text{ This form makes evident that for } D = 3, 4, 6 \text{ and } 10 \text{ the spaces } \mathcal{Z} \text{ of auxilary variables in (55) are given by the Hopf fibration of spheres over spheres which are isomorphic to spheres } S^{(D-3)},
\\D = 3, 4, 6, 10 : \quad \mathcal{Z}_D = \frac{S^{(n-1)}}{S^{(D-2)}} = \frac{S^{(2D-5)}}{S^{(D-2)}} = S^{(D-3)} \quad (\approx (Z_2, S^1, S^3, S^7)) . \quad (57)

For } D = 3 \text{ Eq. (57) gives } \mathcal{Z}_3^2 = Z_2. \text{ Thus the wave function } \phi(\lambda), \text{ Eq. (55), can be treated as function of a light–like momentum (56) and a sign variable } \phi(\lambda) = \phi(p_\mu : p^2 = 0 ; \pm) \text{. In the cases of } D = 4, 6, 10 \text{ the wave function } \phi(\lambda) \text{ of Eq. (55) depends, in addition to the light–like momentum } p_\mu = \lambda \Gamma_\mu \lambda \text{, on a number of angular variables which parameterize the compact spaces } S^{(D-3)}
\\\phi(\lambda_\alpha) = \Phi(p_\mu, S^{(D-3)})|_{p_\mu p^\mu = 0} \equiv \Phi(p_\mu, \alpha_1, \ldots, \alpha_{D-3})|_{p_\mu p^\mu = 0} , \quad D = 4, 6, 10. \quad (58)

Thus for } D=4,6,10 \text{ the space of additional variable } \mathcal{Z}_D^{2(D-2)} \text{ in (55) is compact and isomorphic to the spheres } S^{(D-3)}. \text{ This phenomenon was called twistor compactification in [13]. This twistorial compactification is alternative to the Kaluza–Klein one in particular as it occurs in momentum space and hence makes the coordinates discrete. These discrete ”coordinates” can be identified with quantum numbers enumerating the possible helicity states of all massless higher spin fields in } D = 4 \text{ [13] and all conformal massless higher spin fields in } D = 6, 10 \text{ [19].}

In the quantization of preon mechanics [13] the twistor compactification occurs due to the treatment of the gamma–trace of generalized momentum } p_{\alpha\beta} \text{ as spacetime
momentum (see (54)) and due to the generalized Cartan–Penrose representation for $p_{\alpha\beta}$. It was also noticed [13] that the generalized Cartan–Penrose representation (54), $p_{\alpha\beta} = \lambda_\alpha \lambda_\beta$ provides, for $n > 2$, a mechanism of twistorial dimensional reduction which also occur in momentum space and reduces the number $n(n + 1)/2$ of degrees of freedom in $p_{\alpha\beta}$ to the smaller number $n$ of degrees of freedom in the bosonic spinor $\lambda_\alpha$.

II.4.3. Problems in M–theoretical $D = 11$ case $n = 32$

What turns out to be different in the M–theoretical $D = 11$ case is that the momentum $p_\mu = 1/32 \lambda \Gamma_\mu \lambda$ is not light–like, $p_\mu p^\mu \neq 0$. Its square (the $D = 11$ mass operator)

can be rather expressed through the values of tensorial charges, $p_\mu p^\mu = -2Z^{\mu\nu}Z_{\mu\nu} - 5! Z_{\mu_1...\mu_5}Z_{\mu_1...\mu_5}$ [48], also constructed from the bosonic spinor: $Z_{\mu\nu} = \frac{i}{32} \lambda \Gamma_{\mu\nu} \lambda$ and $Z_{\mu_1...\mu_5} = \frac{1}{32} \lambda \Gamma_{\mu_1...\mu_5} \lambda$. Thus, if one identifies $p_\mu = 1/32 \lambda \Gamma_\mu \lambda$ with the eleven–dimensional momentum, this is not restricted by a mass shell condition and parametrizes $R^{D-1} \times R_+ = S^{10} \times R_+$ (instead of $S^{D-2} \times R_+ = S^9 \times R_+$ as it would be if the momentum were light–like). Then the additional variables in (55) parametrize the fibration $S^{31}$. Such a 21 dimensional space is not (is not known to be) isomorphic to a sphere or a well–studied manifold. The indefiniteness of the $p_\mu p^\mu$ for $p_\mu = 1/32 \lambda \Gamma_\mu \lambda$ in $D = 11$ can be treated as the continuity of the mass spectrum (see [48]), which is another possibility left by conformal invariance of the particle or particle–like mechanics (the latter case includes null–(super)–$p$–branes, see [31] and refs. therein).

Notice that a similar problem appeared for eleven—dimensional supermembrane (now M2–brane) [60] and that now this is treated [61] as an indication that the supermembrane is a composite state, a system of ten–dimensional $D$–branes, in the spirit of Matrix model [62]. One might try to understand the relation between the D=11 BPS preons and spacetime fields ("higher spin" $D = 11$ or $D = 10$ fields) in such a perspective. However, for a moment we do not have much to say on this direction.

III. BPS PREONS AND SUPERSYMMETRY

We discussed the definition and some properties of BPS preons, including their relation with massless higher spin theories in $D = 4, 6, 10$ (almost) without the use of supersymmetry. Now we move to the supersymmetric aspects of the BPS preon conjecture. The pure bosonic definition of the BPS preon (25) uses the generalized momentum. As we have noticed, this appears to be related with the most general supersymmetry algebra.

III.1. Generalized momentum, M–algebra and BPS states

For any $n = 2^m$ the generalized momentum operator $P_{\alpha\beta} = P_{\beta\alpha}$ is associated with the bosonic central generator of most general supersymmetry algebra characterized by the most general form of the commutator of two fermionic supercharges $Q_\alpha$.

$$\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}, \quad P_{\alpha\beta} = P_{\beta\alpha}, \quad \alpha = 1, 2, ..., n; \quad \text{(59)}$$
the property of $P_{\alpha\beta}$ to be central is expressed by
\[
[Q_\alpha, P_{\beta\gamma}] = 0, \quad [P_{\alpha\beta}, P_{\gamma\delta}] = 0.
\]
(60)

The algebra (59) with $n = 32$ ($\alpha = 1,2, \ldots 32$) [34, 35] is usually called M–theory superalgebra or M–algebra [35]. It encodes a full information about the nonperturbative BPS states of the hypothetical underlying M–theory and also its duality symmetries [42, 35]. Indeed, for instance, treating $\alpha, \beta$ as eleven dimensional (SO(1,10)) spinor indices, one may decompose the symmetric spin–tensor generator $P_{\alpha\beta}$ on the basis provided by antisymmetric products of $D=11$ gamma–matrices
\[
P_{\alpha\beta} = P_{\mu} \Gamma_{\alpha\beta}^{\mu} + Z_{\mu\nu} \Gamma_{\alpha\beta}^{\mu\nu} + Z_{\mu_1 \ldots \mu_5} \Gamma_{\alpha\beta}^{\mu_1 \ldots \mu_5}.
\]
(61)

Here $P_{\mu}$ is treated as the $D=11$ spacetime momentum, while the additional central bosonic generators $Z_{\mu\nu} = -Z_{\nu\mu}, Z_{\mu_1 \ldots \mu_5} = Z_{[\mu_1 \ldots \mu_5]}$ can be treated [43] as topological charges of the BPS states corresponding to supersymmetric extended objects, branes, living in the eleven–dimensional world. For instance, the spacial components $Z_{\mu\nu}$ and $Z_{\mu_1 \ldots \mu_5}$ are related to the topological charges [43] of the supermembrane and the super–M5–brane [43, 44].

A simple but important observation is that, e.g. for $n = 32$, the M–algebra (59), (60) possesses the $GL(32)$ ($GL(n)$) automorphism symmetry, which is broken down to $SO(1,10)$ only upon the use of decomposition (61). This implies the “brane rotating” nature of the $GL(32)/SO(1,10)$ symmetry [64]. On the other hand it indicates that the above eleven–dimensional treatment based on the decomposition (61) is, certainly, not a unique one. Treating $\alpha, \beta$ in (59), (60) as multiindices and using the set of gamma–matrices for other $D$ in direct product with the internal space gamma–matrices one may provide the $D=10$ type IIA, $D=10$ type IIB, $D=4$ N=8 [35], as well as a more exotic $D=2+10$ treatment [63]. As a result, the information about nonperturbative BPS states of, say, $D=10$ superstring theories (including Dirichlet superbranes) can also be extracted from (59). This also explains why the M–algebra (59) encodes as well all the duality relations between different $D = 10$ and $D = 11$ superbranes.

\section{III.2. BPS preons states preserving all but one supersymmetry. A classification of M-theory BPS states}

Associating the generalized momentum matrix with the right hand side of the general supersymmetry algebra (59) one finds (see [6, 12, 8]) that the definition of the BPS preon in Eqs. (24), (25) is equivalent to defining the BPS preon as a state preserving all supersymmetries but one (hence the notation $|BPS, (n-1)\rangle$; $|BPS, 31\rangle$ in $D = 11$) [6].

---

\footnote{See [36] for a treatment of (59)–(60) as central extension of the abelian fermionic translation algebra, [38, 36] and refs. therein for further generalizations of the M–theory superalgebra and for their structure. For $n \neq 2^l$, including odd values of $n$ one can treat the algebra as (59) as $d = 2$ extended supersymmetry algebra with central charges.}
The bosonic spinor parameters $\epsilon_I^\alpha$ corresponding to the supersymmetries preserved by a BPS preon $|\lambda\rangle$, 

$$\epsilon_I^\alpha Q_\alpha |\lambda\rangle = 0, \quad I = 1, \ldots, (n-1) \quad (I = 1, \ldots, 31 \text{ in } D = 11) \quad (62)$$

are ‘orthogonal’ to the bosonic spinor $\lambda_\alpha$ that labels it,

$$\epsilon_I^\alpha \lambda_\alpha = 0, \quad I = 1, \ldots, (n-1) \quad (I = 1, \ldots, 31 \text{ in } D = 11). \quad (63)$$

Notice that these are the same bosonic spinors as in Eq. (42), which completes the definition (41) of the $b$–symmetry transformations,

$$\epsilon_I^\alpha = u_I^\alpha. \quad (64)$$

In general, the number $n-k$ $(32-k$ in $D=11)$ of supersymmetries preserved by a BPS state $|BPS, k\rangle$ coincides [6, 8, 12] with the rank of the eigenvalue matrix $p^{(k)}_{\alpha\beta}$ (28) of the generalized momentum $P_{\alpha\beta}$

$$\epsilon_I^\alpha Q_\alpha |BPS k\rangle = 0, \quad I = 1, \ldots, k \Rightarrow \begin{cases} P_{\alpha\beta} |BPS k\rangle = p^{(k)}_{\alpha\beta} |BPS k\rangle, \\ \text{rank}(p^{(k)}_{\alpha\beta}) = 32-k. \end{cases} \quad (65)$$

In this respect all the BPS states related to a general supersymmetry algebra (59), including the M-theory BPS states for $n = 32$, may be classified by the number of preserved supersymmetries [6]. This is the same as the classification by rank of the generalized momentum matrix considered in Sec. II.1. Then the discussion of section II.2. implies that a BPS state preserving $k$ supersymmetry can be treated as a composite of $n = n-k$ BPS preons

$$\epsilon_I^\alpha Q_\alpha |BPS k\rangle = 0, \quad I = 1, \ldots, k \Rightarrow |BPS, k\rangle = |\lambda^{(1)}\rangle \otimes \ldots \otimes |\lambda^{(32-k)}\rangle, (66)$$

$$\epsilon_I^\alpha \lambda_\alpha^{(1)} = 0, \ldots, \epsilon_I^\alpha \lambda_\alpha^{(n-k)} = 0.$$

### III.3. On “AdS generalizations” of the M–algebra and of the BPS preon definition

Eqs. (59)–(60) give the generalization of the super–Poincaré algebra. The corresponding generalization of the superconformal algebra is suggested to be $OSp(1|2n)$ [see [39, 34] as well as [20] and [8] for more references]. One can ask how the analogous generalization of the AdS superalgebra look like. The study of [13, 14, 52, 16] suggests that this is given by the Lie superalgebra of the $OSp(1|n)$ supergroup,

$$[P_{\alpha\beta}, P_{\gamma\delta}] = i \zeta (C_{\alpha(\gamma P_\delta)\beta} + C_{\beta(\gamma P_\delta)\alpha}), \quad (67)$$

$$[P_{\alpha\beta}, Q_\gamma] = i \zeta C_{\alpha(\gamma Q_\delta)\beta} \quad (68)$$

$$\{Q_\alpha, Q_\beta\} = P_{\alpha\beta}, \quad (69)$$
where the dimensional parameter \( \zeta \), the inverse AdS radius, is introduced to make transparent the contraction of \( OSP(1|n) \) down to the generalized superPoincaré supergroup \( \Sigma^{(n(n+1)/2|n)} \) with the algebra (59)–(60); this occurs in \( \zeta \to 0 \) limit.

Clearly, the noncommutative \( P_{\alpha \bar{\beta}} \) cannot be diagonalized and the above definition of the BPS preon should be modified for that case. The study of [52, 16] suggests the following definition of the BPS preons for that case (cf. [18])

\[
(P_{\alpha \bar{\beta}} - Y_{(\alpha} Y_{\beta)})|BPS \ \text{preon} : \lambda_{\alpha} = 0, \quad Y_{\alpha} := \lambda_{\alpha} - \zeta P_{\alpha}^{(\lambda)}.
\]

Here \( P_{\alpha}^{(\lambda)} \) is the operator of momentum conjugate to \( \lambda_{\alpha} \) and, hence, the spinorial operators \( Y_{\alpha} \) do not commute for \( \zeta \neq 0 \),

\[
[Y_{\alpha}, Y_{\beta}] = 2i\zeta C_{\alpha \beta}.
\]

Thus, in distinction to generalized super–Poincaré (\( \zeta = 0 \)) case, the definition of BPS preons for the generalized AdS or \( OSP(1|n) \) superalgebra refers to a factorization of the noncommutative spin–tensorial operator rather than to an eigenvalue problem.

Here, however, we mainly consider the case of the generalized super–Poincaré algebra with central \( P_{\alpha \bar{\beta}} \) which allows for the above simple definition (25) of the BPS preon [6].

### III.4. BPS preons and BPS states in supergravity

As discussed above the BPS states \(|BPS \ k\), \( k \neq 0 \), preserve a fraction \((k/n)\) of the supersymmetries; due to this fact they saturate the Bogomolny–Parasad–Sommerfield or BPS bound (hence the "BPS state" name) and, as a result, are stable.

In supergravity the algebraic notion of BPS state is realized as a supersymmetric solution of the supergravity equations \( \text{i.e.} \) the solitonic solution preserving a fraction \( k/n \) \((k/32 \text{ in the M–theoretic } n = 32 \text{ case})\) of the local supersymmetries characteristic of the supergravity theory. The \( k \) supersymmetries (65) preserved by the BPS state \(|BPS \ k\) are represented in this "solitonic" picture by a set of \( k \) linearly independent Killing spinors \( \bar{\varnothing}_{f}^{\alpha} (x) \) obeying the Killing spinors equation

\[
\bar{\varnothing}_{f}^{\alpha} := d \bar{\varnothing}_{f}^{\alpha} - \bar{\varnothing}_{f}^{\beta} \omega_{f}^{\alpha} \equiv D \bar{\varnothing}_{f}^{\alpha} = \bar{\varnothing}_{f}^{\beta} \omega_{f}^{\alpha} = 0, \quad f = 1, \ldots, k.
\]

In many cases, including higher dimensional supergravity \( \text{e.g.} \ D = 10, 11 \) and extended supergravity in \( D = 4 \text{ e.g.} \ N = 4, 8 \) the generalized covariant derivative \( \bar{\varnothing} \) is constructed with the use of generalized connection \( \omega_{f}^{\alpha} \) which includes, besides the Lorentz (spin) connection \( \omega_{L}^{\alpha \beta} = 1/4 \omega_{L}^{ab} \Gamma_{ab}^{\alpha \beta} \), a tensorial part \( t_{1}^{\alpha \beta} = \omega_{f}^{\alpha} - \omega_{L}^{\alpha \beta} \) constructed from the fields of supergravity multiplet or their field strengths. Among the cases where the generalized connection is reduced to the Lorentz connection \( \bar{\varnothing} = D, \quad t_{1}^{\alpha \beta} = 0 \) is the simple, \( D = 4 N = 1 \), supergravity. In \( D = 11 \) supergravity the Lorentz covariant part \( t_{1}^{\alpha \beta} \) of the generalized connection \( \omega_{f}^{\alpha} \) is constructed in terms of the field strength \( F_{abcd} \) of the three–form gauge field \( A_{3} \),

\[
\omega_{f}^{\alpha} = \frac{1}{4} \omega_{L}^{ab} \Gamma_{ab}^{\alpha \beta} + \frac{i}{18} e^{a} \left( F_{ab_{1}b_{2}b_{3}} \Gamma_{b_{1}b_{2}b_{3}} + \frac{1}{8} F_{b_{1}b_{2}b_{3}b_{4}} \Gamma_{ab_{1}b_{2}b_{3}b_{4}} \right)^{\alpha \beta}.
\]
The necessary condition for the existence of Killing spinors is given by an algebraic equation coming from the integrability condition for \((72) \mathcal{D} \mathcal{D} \in \mathbb{I} \alpha = 0\). It has the suggestive form \([10]\),

\[
\varepsilon^\beta_\alpha \mathcal{R}^\alpha_\beta = 0 \quad \Leftrightarrow \quad \mathcal{D} \mathcal{D} \in \mathbb{I} \alpha = 0 ,
\]

in terms of the generalized curvature

\[
\mathcal{R}^\alpha_\beta = d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma
\]

or curvature of generalized connection taking values in the Lie algebra of the so-called generalized holonomy group \([10, 11]\). For \(D = 11\) supergravity the generalized holonomy group has to be a subgroup of \(\text{SL}(32, \mathbb{C})\) \([11]\) (see \([56]\) for further discussion with concrete solutions). The same is true for type IIB supergravity \([59]\).

The rôle of BPS preons in the analysis of supersymmetric supergravity solutions was discussed in \([12]\). The fact that a BPS state \(|\text{BPS} k\rangle\) preserving \(k\) of 32 (in general \(k\) of \(n\)) supersymmetry can be considered as a composite of \((32 - k)\) (in general \((n - k)\)) BPS preons, Eq. (66), is reflected, in the language of supergravity solutions, by the possibility of finding \(\tilde{n} := (32 - k)\) spinors (spinor fields) \(\lambda_r^\alpha(x), r = 1, \ldots, \tilde{n}\) which are orthogonal to the Killing spinors, \(\varepsilon^\alpha_\mathbb{I}(x), \mathbb{I} = 1, \ldots, k\), Eq. (72), characterizing the solution,

\[
\tilde{n}^\alpha_\mathbb{I}(x) \lambda^\alpha_r(x) = 0 , \quad \mathbb{I} = 1, \ldots, k, \quad r = 1, \ldots, (32 - k) .
\]

Thus, BPS preonic spinors and Killing spinors provide an alternative (dual) characterization of a \(\nu = k/32\)–supersymmetric solution (\(\nu = k/n\) in general); either one can be used and, for solutions with supernumerary supersymmetries \([55]\), the description provided by BPS preons is clearly a more economic one. Moreover, the use of both BPS preonic \((\lambda^\alpha_r)\) and Killing \((\varepsilon^\alpha_\mathbb{I})\) spinors allowed us to develop \([12]\) a moving \(G\)–frame method, which may be useful in the search for new supersymmetric solutions of supergravity.

As a simplest application of the moving \(G\)–frame method let us present the general expression for the generalized curvature (75) of the \(k/32\)–supersymmetric solution of \(D = 11\) Cremmer–Julia–Scherk supergravity \([12]\). It is given by

\[
\mathcal{R}^\alpha_\beta = G^s_r \lambda^\alpha_r w^\beta_s + \nabla B^l_r \lambda^\alpha_r \in \mathbb{I} \beta ,
\]

where \(w^\beta_s\) is a set of \((32 - k)\) spinors obeying \(w^\beta_s \lambda^\alpha_r = \delta^\alpha_r\) and forming, together with the Killing spinor \(\in \mathbb{I} \alpha\), the nondegenerate matrix

\[
g^{-1}_\beta^\alpha = \begin{pmatrix} w^\alpha_s \\ \in \mathbb{I} \alpha \end{pmatrix} \quad \left( g_\alpha^\beta = \begin{pmatrix} \lambda^\alpha_s \\ w^\alpha_\mathbb{I} \end{pmatrix} \right)
\]

the moving \(G\)–frame matrix. Finally,

\[
G^s_r := (dA-A \wedge A)^s_r , \quad \nabla B^l_r := dB^l_r - A^s_r \wedge B^l_s ,
\]

where \(A^s_r\) and \(B^l_r\) are \(\tilde{n} \times \tilde{n} \equiv (32 - k) \times (32 - k)\) and \(\tilde{n} \times k \equiv (32 - k) \times k\) matrix valued one–forms which have to be fixed by the concrete \(k/32\)–supersymmetric solution. The

BPS preons. An overview from the hill of twistor approach
condition that the generalized holonomy group $H$ (as well as the generalized structure group $G$) should be inside $SL(32)$, $\mathcal{H}^\alpha_\alpha = 0$, [11] implies that $G^{rr}_r = 0 (A^{rr}_r = 0).

On one hand, Eq. (77) with $G^{rr}_r = 0$ provides an explicit expression for the results of [11, 59] that the generalized holonomies of $k/32$–supersymmetric solutions of $D = 11$ and of $D = 10$ type IIB supergravity should be $H \subset SL(32 – k, \mathbb{R}) \otimes \mathbb{R}^{(32 – k)}$. In the light of the fact that, when fermions vanish, $\Psi^\mu_\mu = 0$, all the free bosonic equations of the 11–dimensional supergravity as well as the Bianchi identities for the Riemann tensor and for the three–form gauge field strengths can be collected in the following simple equation for generalized curvature (75) [58, 12]

$$R_{\alpha \beta \gamma} \Gamma^\beta_\beta_\gamma = 0,$$

we expect that the explicit form (77) of the generalized curvature $\mathcal{R}$ to be useful in the search for new supergravity solutions.

In particular, the moving $G$–frame formalism might be useful to settle the question whether a BPS preonic solution preserving 31 out of the 32 supersymmetries exists in $D = 11$ and/or $D = 10$ type IIB supergravities. Although this problem was addressed in [10, 11, 12], neither a solution with such a property has been found nor a statement forbidding an existence of such a solution has been proved yet. However in [12] it was observed that BPS preonic configurations do solve the equations of a Chern–Simons like supergravity. This follows from the fact that the generalized curvature of a BPS preon is nilpotent, $\mathcal{R}^\alpha_\alpha \wedge \mathcal{R}^\beta_\beta = 0$. This actually follows from the statement of [11] that the generalized holonomy of (a hypothetical) $\nu = 31/32$ supersymmetric solution is a subgroup of $\mathbb{R}^{31}$. More explicitly, according to [12] the generalized curvature for the preonic ($\nu = 31/32$) solution has the form

$$\mathcal{R}^\beta_\alpha = dB^I \lambda^I_\alpha \in I^\beta,$$

which, in the light of (76), implies the nilpotency not only for the two form $\mathcal{R}^\alpha_\beta$ but also for the tensor $\mathcal{R}^\alpha_\beta$. See [6] for a further discussion on a hypothetical preonic solution in $D = 11$ Cremmer–Julia–Scherk supergravity.

III.5. Superparticle model for BPS preon

Interestingly enough, the point–like model for BPS preon [6] is provided by the action that had been proposed in [20] before the notion of BPS preons was introduced. This describes a superparticle in tensorial superspace (which was called ”generalized superspace” in [20], ”extended superspace” in [36] and enlarged superspace in [8]) with the bosonic body (34),

$$\Sigma^{(\frac{1}{2}(n+1))} \ : \ \mathcal{X}^\alpha = (X_\alpha^\beta, \theta^\alpha), \ \ X_\alpha^\beta = X_\beta^\alpha, \ \ \alpha, \beta = 1, 2, \ldots, n.$$
The action of ref. [20] is the straightforward supersymmetric generalization of the bosonic functional (44), which can be obtained by substituting the pull–back \( \hat{\Pi}^{\alpha\beta} \equiv d\tau \hat{\Pi}^{\alpha\beta}_\tau = d\hat{X}^{\alpha\beta}(\tau) - id\hat{\theta}^{(\alpha}(\hat{\theta}^{\beta)}(\tau) \) of the supersymmetric Volkov–Akulov one–form

\[
\Pi^{\alpha\beta} := dX^{\alpha\beta} - id\theta^{(\alpha} \theta^{\beta)}
\]  

(83)
of the \( \Sigma^{(n+1)/2} \) superspace for \( d\hat{X}^{\alpha\beta} \) in (44),

\[
S = \frac{1}{2} \int_{W^1} \lambda_\alpha \lambda_\beta \hat{\Pi}^{\alpha\beta} = \frac{1}{2} \int d\tau \hat{\Pi}^{\alpha\beta}_\tau (\tau) \lambda_\alpha(\tau) \lambda_\beta(\tau),
\]  

(84)
The action (84) is invariant under the global supersymmetry transformations,

\[
\delta_\epsilon \hat{X}^{\alpha\beta}(\tau) = -ie^{(\alpha}(\hat{\theta}^{\beta)}(\tau), \quad \delta_\epsilon \hat{\theta}^{\alpha}(\tau) = \epsilon^\alpha, \quad \delta_\epsilon \lambda_\alpha(\tau) = 0,
\]  

(85)
and under \((n - 1)\) local fermionic \( \kappa \)–symmetries

\[
\begin{align*}
\delta_\kappa \hat{\Pi}^{\alpha\beta} &= 0 \iff \delta_\kappa \hat{X}^{\alpha\beta} = 2i\delta_\kappa \hat{\theta}^{(\alpha}(\hat{\theta}^{\beta)}, \\
\delta_\kappa \hat{\theta}^{\alpha}(\tau) &= 0 \iff \delta_\kappa \hat{\theta}^{\alpha} = \hat{u}^{\beta}(\tau)\kappa^I(\tau), \quad I = 1, \ldots, (n - 1).
\end{align*}
\]  

(86)

(87)
In (86) \( \kappa^I(\tau) \) are the \((n - 1)\) \((31 \text{ for } D = 11)\) Grassmann parameters of the \( \kappa \)–symmetry and \( \hat{u}^{\beta}(\tau) \) are \((n - 1)\) \((31)\) auxiliary spinor fields which are orthogonal to \( \lambda_\alpha(\tau) \), Eq. (42) or Eq. (63) with (64). These can be omitted from the consideration as one can use the first equation in (87), \( \delta_\kappa \hat{\theta}^{\alpha} \lambda_\alpha(\tau) = 0 \), as the definition of the \( \kappa \)–symmetry.

Just the presence of \( 31 \)–parametric \((n \text{–parametric})\) \( \kappa \)–symmetry, allows one to treat the action (84) as a model for BPS preons: the \( \kappa \) symmetry of the worldvolume action reflects the supersymmetry preserved by the ground state of the point–like or extended object [66, 44]. Indeed the requirement of the Lorentz invariance (or more powerful \( GL(n) \) invariance) of the ground state leads to the conclusion that in this state all fermions vanish, \( \hat{\theta}^{\alpha}(\tau) = 0 \). Then, as the fermionic coordinate function transforms both under the supersymmetry and under the \( \kappa \)–symmetry, \( \delta_\kappa \hat{\theta}^{\alpha} = \epsilon^\alpha + \delta_\kappa \hat{\theta}^{\alpha} \), the invariance of the ground state of the superparticle (84) is defined by the equation

\[
\hat{\theta}^{\alpha} = 0 \Rightarrow 0 = \delta_\kappa \hat{\theta}^{\alpha} = \epsilon^\alpha + \delta_\kappa \hat{\theta}^{\alpha} = \epsilon^\alpha + \hat{u}^{\beta}(\tau)\kappa^I(\tau).
\]  

(88)
Thus the parameters of the symmetries preserving ground state solution should obey

\[
\text{susy preserved by ground state with } \hat{\theta}^{\alpha} = 0: \quad \epsilon^\alpha = -\hat{u}^{\beta}(\tau)\kappa^I.
\]  

(89)
The extended object models for BPS preons are provided by tensionless superbranes in \( \Sigma^{(528\mid32)} \) \((\Sigma^{(n+1)/2})\) superspace [53, 7].
IV. SUPERFIELDS AND SUPERGRAVITY IN TENSORIAL SUPERSPACE

IV.1. Superfield generalization of the massless higher spin equations

The superparticle models [20] with the properties of BPS preon [6] were studied in the flat tensorial superspace $\Sigma^{(n(n+1)/2n)}$ [13, 14, 16] and on the $OSp(1|2n)$ supergroup manifold [51, 52, 16]. The latter are the ”AdS–like” version of tensorial superspace (see [51, 14, 52, 16], Sec. II.4 for a brief discussion and [19] for $D=6,10$ generalization of this statement). The quantum state spectrum of the preonic superparticle in $D=4$ contains a tower of conformal massless fields of all possible ‘helicities’; which can be described all together (see [13, 19] and Sec. II.4) by the scalar bosonic and spinor (s-vector) fermionic fields obeying Eqs. (51) and (52).

This spectrum is manifestly supersymmetric. Then the question arises: is there any superfield generalization of these equations, i.e. is there a superfield equation which collects the scalar and spinor field in tensorial space and implies Eqs. (51) and (52) on these fields? The answer on this question is affirmative. As it was shown in [18], such a superfield equation does exist and has the form

$$D_{[\alpha}D_{\beta]}\Phi(X,\theta) = 0,$$

(90)

where $D_\alpha = \partial/\partial \theta^\alpha + i\theta^\beta \partial_{\beta\alpha}$ is the flat Grassmann covariant derivative in the flat tensorial superspace $\Sigma^{(n(n+1)/2n)}$, (82). $\{D_\alpha,D_\beta\} = 2i\delta_{\alpha\beta}$. Eq. (90) sets to zero all the higher components $\phi_{\alpha_1\ldots\alpha_i}(X)$, $i \geq 2$, of the scalar superfield $\Phi(X,\theta) = b(X) + f_\alpha(X) \theta^{\alpha} + \sum_{i=2}^n \phi_{\alpha_1\ldots\alpha_i}(X) \theta^{\alpha_1} \ldots \theta^{\alpha_i}$, thus reducing it to the form

$$\Phi(X^\alpha\beta,\theta^\gamma) = b(X) + f_\alpha(X) \theta^{\alpha};$$

(91)

it also imposes on the surviving components the dynamical equations (51) and (52). 10 The generalization of the “preonic equation” (47) has the form [18]

$$(D_\alpha D_\beta + \lambda_\alpha \lambda_\beta) \Phi(X,\theta,\lambda) = 0.$$  

(92)

Its antisymmetric part gives Eq. (90) while the symmetric part produces Eq. (47).

The $AdS$ generalization of Eq. (90) reads

$$\left(\nabla_{[\alpha} \nabla_{\beta]} + i\xi C_{\alpha\beta}\right)\Phi(X,\theta) = 0,$$

(93)

where $\nabla_\alpha$ are spinorial covariant derivatives on the $OSp(1|n)$ supergroup manifold obeying the superalgebra (69), (68), (67). The equation generalizing (92) for the case of $OSp(1|n)$ supergroup manifold reads can be found in [18] where it was also discussed the way of derivation of (92) from the equation for Clifford superfield wave function which appeared in the quantization [13] of the preonic superparticle [20], Eq. (84) with the conversion method.

10 One can also collect the same field content inside a spinor superfield $\Psi_\alpha$, but this should be subject to a set of two equations, $D_{[\alpha} \Psi_{\beta]}(X,\theta) = 0$ and $\partial_{\alpha\beta} \Psi_\gamma(X,\theta) = 0$ [18].
IV.2. Superfield supergravity in tensorial superspace

When one considers the standard superparticles and superbranes, the natural starting point is also an action in flat superspace. Then one finds [46, 45] that considering the natural generalization of the model in curved superspace and assuming the existence of a smooth flat superspace limit one arrives at the requirement that curved superspace has to satisfy the supergravity constraints. For higher dimensional $D > 6$ superspaces (and also for the extended, $N > 2$, superspaces in $D = 4$) these are the on-shell supergravity constraints which contain all the dynamical equations of motion among their consequences. This is not the case in $D = 4$ $N = 1$ superspace where one arrives at off-shell constraints which do not imply dynamical equations of motion (see refs. in [47] which is devoted to a complete Lagrangian description of the supergravity—superstring interaction); in a simpler $D = 3$ and $D = 2$ cases the supergravity is not dynamical.

It is natural to ask what are the generalized supergravity constraints which might appear from the consistency requirement for a preonic model in a curved tensorial superspace. Such a supergravity in a curved $\Sigma^{(528|32)}$ superspace may be interesting in an M-theoretical perspective, while the models in $\Sigma^{(n(n+1)/2|n)}$ with $n = 4, 6$ and 10 [$\Sigma^{(10|4)}, \Sigma^{(36|8)}$ and $\Sigma^{(136|16)}$] could provide a basis for interacting higher spin theories.

One might even hope that such a tensorial supergravity could itself provide an interacting higher spin theory; however, as shown in [18], this is not the case, at least when the supergravity with $SL(n)$ or $GL(n)$ holonomy groups are considered.

The natural generalization of the point like preonic action (84) for the case of curved tensorial superspace $\Sigma^{(n(n+1)/2|n)}$ reads [18]

$$S = \frac{1}{2} \int_{\mathcal{W}^1} \lambda_\alpha \lambda_\beta \hat{E}^{\alpha\beta} = \frac{1}{2} \int d\tau \hat{E}_\tau^{\alpha\beta}(\tau) \lambda_\alpha(\tau) \lambda_\beta(\tau),$$

(94)

where $\hat{E}_\tau^{\alpha\beta} := d\tau \hat{E}_\tau^{\alpha\beta} = d\tau \partial_\tau \hat{E}_\tau^{\alpha\beta}(\hat{\mathcal{X}})$ is the pull–back to the worldline $\mathcal{W}^1$ of the bosonic supervielbein form $E^{\alpha\beta} := d\mathcal{X} \cdot E^{\alpha\beta}(\mathcal{X})$ of the curved tensorial superspace $\Sigma^{(n(n+1)/2|n)}$ (82) with supervielbein

$$E_{\hat{\mathcal{X}}} = (E^{\alpha\beta}, E^\alpha) = d\mathcal{X} \cdot E_{\hat{\mathcal{X}}}^{\alpha\beta}(\mathcal{X}), \quad \alpha, \beta = 1, \ldots, n$$

(95)

including also $n$ fermionic one forms $E^\alpha$ whose pull–backs $\hat{E}^\alpha$ do not enter Eq. (94).

The supergravity in tensorial superspace should be the theory of the supervielbein superfields in (95). However, to make the formalism covariant one also introduces in superfield supergravity the connection taking values in a structure group of the superspace. For the usual superspace the structure group is the Lorentz group which in the flat superspace appears as a global symmetry of the superparticle action. With this in mind, and taking into attention that the flat superspace preonic superparticle action (84) is invariant under $GL(n)$ group, one finds natural to consider $GL(n)$ as the structure group of tensorial superspace. Hence, by analogy with the conventional spin connection of general relativity and the standard supergravity, the $GL(n)$ connection was introduced in [18],

$$\Omega^{\alpha}_\beta := d\mathcal{X} \cdot \Omega_{\hat{\mathcal{X}}}^{\alpha\beta} \equiv E_{\hat{\mathcal{X}}}^{\alpha\beta} \Omega^{\alpha\beta}.$$

(96)
The torsion 2-forms and the curvature of the $GL(n)$ connection were defined by

\[
T^{\alpha \beta} := \mathcal{D}E^{\alpha \beta} \equiv dE^{\alpha \beta} - E^\gamma \wedge \Omega_\gamma^{\beta} - E^{\beta \gamma} \wedge \Omega_\gamma^{\alpha}, \tag{97}
\]
\[
T^\alpha := \mathcal{D}E^\alpha \equiv dE^\alpha - E^\beta \wedge \Omega_\beta^{\alpha}, \tag{98}
\]
\[
\mathcal{R}_\beta^\alpha := d\Omega_\beta^{\alpha} - \Omega_\beta^{\gamma} \wedge \Omega_\gamma^{\alpha}. \tag{99}
\]

The requirement of preservation of the $\kappa$–symmetry of the superparticle (94) imposes the constraints $T^\gamma_\delta \delta^\alpha \beta \propto \delta^\gamma_\delta (\delta^\alpha \delta^\beta)$, $T^\gamma_\gamma \delta^\alpha \beta \propto \delta^\gamma_\gamma (\delta^\alpha \delta^\beta)$, on the bosonic torsion (97), $T^{\alpha \beta} := \frac{1}{2} E^\varphi \wedge E^\epsilon \Theta^\varphi_\epsilon \alpha \beta$ Then imposing the conventional constraints, which fix the freedom in redefinition of the basic superfields, and studying the Bianchi identities one finds the following complete expressions for the torsion and curvature two–forms [18]

\[
T^{\alpha \beta} = -iE^\alpha \wedge E^\beta + 2E^\gamma (\alpha \wedge E^\gamma) \delta R^{\gamma \delta}, \tag{100}
\]
\[
T^\alpha = 2E^{\alpha \beta} \wedge E^\gamma R^\beta_\gamma + E^{\alpha \beta} \wedge E^\gamma \delta U^\gamma_\delta, \tag{101}
\]
\[
\mathcal{R}_\beta^\alpha = iE^\gamma \wedge E^\alpha U^\gamma_\delta - E^\alpha \gamma \wedge E^\delta (F^\delta_\beta \gamma + D^\delta_\beta R^\gamma_\delta) - E^\alpha \gamma \wedge E^\delta (D^\gamma_\delta U^\gamma_\epsilon + D^\gamma_\epsilon R^\beta_\gamma). \tag{102}
\]

Here $R^{\gamma \delta} (\mathcal{Z}) = -R^{\gamma \delta} (\mathcal{Z})$ and $U^\alpha_\beta \gamma (\mathcal{Z}) = U^\alpha_\gamma \beta (\mathcal{Z})$ are ‘main’ superfields which are related by the equations

\[
\mathcal{D}_{[\alpha} U_{\beta] \gamma \delta} = -\mathcal{D}_\gamma R^{\alpha \beta}, \tag{103}
\]
\[
\mathcal{D}_{(\alpha} U_{\beta) \gamma \delta} = -iD_{(\gamma} F_{\delta) \alpha \beta}, \quad F_{\alpha \beta \gamma} = 2iU_{(\beta \gamma) \alpha} - iU_{\alpha \beta \gamma} - 2D_{(\beta} R_{\gamma) \alpha}, \tag{104}
\]
\[
\mathcal{D}_{\alpha} U^\gamma_\delta \sigma = -D^\gamma_\delta U_{\gamma \alpha \beta} + 2U_{\gamma \alpha} (\sigma R^\gamma_\delta) + 2U_{\gamma \beta} (\alpha R^\gamma_\delta) = 0. \tag{105}
\]

Setting the main superfields to zero, $R^{\gamma \delta} (\mathcal{Z}) = 0$, $U^\alpha_\beta \gamma (\mathcal{Z}) = 0$ and ignoring the trivial $GL(n)$ connection (setting $\Omega^\alpha_\beta = 0$) one reduces the constraints to the Maurer–Cartan equations of flat tensorial superspace $\Sigma^{(n(n+1)/2 \cdot n)}$ with the solution

\[
R^{\gamma \delta} (Z) = 0, \quad U^\alpha_\beta \gamma (Z) = 0 \quad \Rightarrow \quad E^{\alpha \beta} = \Pi^{\alpha \beta}, \quad E^\alpha = d\theta^\alpha. \tag{106}
\]

On the other hand, setting $R^{\gamma \delta} = \zeta C^{\gamma \delta}$ and $U^\alpha_\beta \gamma (Z) = 0$ one arrives in the Maurer–Cartan equations of the $OSp(1|2n)$ supergroup

\[
d\epsilon^{\alpha \beta} = -i\epsilon^{\alpha \beta} \wedge \epsilon^\gamma - \zeta \epsilon^{\alpha \gamma} \wedge \epsilon^\delta C^{\gamma \delta}, \tag{107}
\]
\[
d\epsilon^{\alpha \gamma} = -\zeta \epsilon^{\alpha \gamma} \wedge \epsilon^\delta C^{\gamma \delta}.
\]

In both cases the curvature is equal to zero which allows one to gauge away the trivial $GL(n)$ connections $\Omega^\alpha_\beta = 0$.

In the superspace subject to the constraints (100) the preonic superparticle action possesses the gauge invariance under the local fermionic $\kappa$–symmetry (cf. (86), (87))

\[
i_\kappa E^{\alpha \gamma} := \delta_{\kappa} Z^M E^{\alpha \gamma}_M = 0, \quad i_\kappa E^\alpha := \delta_{\kappa} Z^M E^\alpha_M = u^\alpha_\kappa \kappa (\tau), \tag{108}
\]

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where $u^\alpha_i$ is defined by Eq. (42), and under the $b$–symmetry transformations (cf. (41))

$$i_b E^\alpha \alpha' = \delta_\beta Z^M E^\alpha \alpha' = u^\alpha_i u^\alpha_j b^{ij} (\tau), \quad i_b E^\alpha := \delta_\beta Z^M E^\alpha = 0. \quad (109)$$

One can ask whether the scalar superfield equation (90) allows for a consistent generalization to the curved tensorial superspace. It does and the desired generalization reads

$$\mathcal{D}_\alpha \mathcal{D}_\beta \Phi = i \frac{1}{2} R^\alpha_\beta \Phi \quad (110)$$

and is consistent when the holonomy group of the tensorial superspace is restricted to be $SL(n)$ (which means that the curvature tensor is traceless $\mathcal{R}^\alpha_\alpha = 0$) [18].

The first impression might be that the tensorial supergravity defined by the constraints (100)–(102) should contain a huge number of extra nonphysical fields. However this is not the case. As it is shown in [18] the general solution of the tensorial supergravity constraints contains only two classes of superspaces: the superconformally flat superspaces and the superspaces superconformally related to the $OSp(1|n)$ supergroup manifold. The superconformally flat superspaces are described by

$$E^\alpha_\beta = e^{\frac{2w(Z)}{n}} \Pi^\alpha_\beta' L^\beta_\alpha(Z) L^{\beta'}_\alpha(Z), \quad E^\alpha = e^{\frac{w(Z)}{n}} (d\theta^\alpha - i \Pi^\alpha_\beta' D_\beta W) L^\alpha_\beta(Z),$$

$$\Omega^\alpha_\beta = \frac{1}{n} dW \delta^\alpha_\beta - L^{-1} \delta^\beta_\gamma L^{\beta'}_\alpha (d\theta^\alpha D_\beta W + \Pi^\alpha_\gamma (D_\gamma W + i \frac{1}{2} D_\gamma W D_\beta W)) L^{\alpha'}_\beta \quad (111)$$

where $L^\alpha_\beta(Z)$ is a matrix of local $SL(n)$ transformations which together with $exp\{W(Z)/n\}$ form a $GL(n)$ matrix $G^\alpha_\beta = L^\alpha_\beta exp\{W(Z)/n\}$. The extraction of the scaling factor allows to apply Eqs. (111) to supergravity with $SL(n)$ structure group.

Working with the structure group $GL(n)$ (which does not forbid reduction of the holonomy group down to its subgroup $SL(n)$), one can obtain all superspaces superconformally–related to the $OSp(1|n)$ supergroup manifold by making first the following ”generalized super–Weyl transformations” [18]

$$E^\alpha_\beta = \xi^\alpha_\beta, \quad E^\alpha = \xi^\alpha + \xi^\alpha_\beta W_\beta$$

$$\Omega^\alpha_\beta = -i \xi^\alpha W_\beta - \xi^\alpha_\gamma (\nabla_\gamma W_\beta + i W_\gamma W_\beta), \quad (112)$$

of the $OSp(1|n)$ supervielbein ($\xi^\alpha_\beta$, $\xi^\alpha$), Eq. (107), with $W_\gamma = -i \nabla_\gamma W$ and then performing a $GL(n)$ “rotation”, if needed. In (112) $\nabla_\gamma$ is the $OSp(1|n)$ covariant derivative, $d = \xi^\alpha_\beta \nabla_\alpha^\beta + \xi^\alpha \nabla_\alpha$. The flat tensorial superspaces $\Sigma^{(n(n+1)/2)|n}$ can be recovered in $\zeta = 0$ limit.

The fact that superconformally flat and $OSp(1|n)$ related superspaces provide the general (modulo topological subtleties) solution of the tensorial supergravity constraints (100)–(102) implies that the main superfields can always be expressed by

$$R^\alpha_\beta = -\frac{i}{2} C^\alpha_\beta + \mathcal{D}^\alpha_\beta W_\beta + i W_\alpha W_\beta, \quad U^\alpha_\beta = -\nabla^\beta_\gamma W_\alpha + W_\gamma (\mathcal{D}_\beta) W_\alpha \quad (113)$$

One can check that the holonomy group of the superspace reduces to $SL(n)$ (i.e. that $\mathcal{R}^\alpha_\alpha = 0$) when $W_\alpha = -i \mathcal{D}_\alpha W$. The ‘super–Weyl transformations’ (107) with $W_\alpha \neq 0$
$-i \mathcal{D}_a W$ result in the connection with $GL(n)$ holonomy. In the $OSp(1|n)$ covariant derivatives the main superfields of the superspace with $SL(n)$ holonomy group read

$$R_{\alpha\beta} = i e^{-\frac{2w}{x}} \left[ i \tilde{\mathcal{C}}_{\alpha\beta} + \nabla_{[\alpha} \nabla_{\beta]} W + \frac{i}{2} \nabla_\alpha W \nabla_\beta W \right],$$

$$U_{\beta\gamma\delta} = e^{-\frac{3w}{x}} \left[ -i \nabla_{\gamma\delta} \nabla_\beta W + \nabla_{(\gamma} W \nabla_{\delta)} \nabla_\beta W \right].$$

One can make the (seemingly important) observation that, formally, putting in (114) $R_{\alpha\beta} = -\tilde{\mathcal{C}}_{\alpha\beta} e^{(1+4/n)W/2}$ one finds an equation

$$\nabla_{[\alpha} \nabla_{\beta]} W + \frac{i}{2} \nabla_\alpha W \nabla_\beta W = -i \tilde{\mathcal{C}}_{\alpha\beta} \left( 1 - e^{-\frac{W}{2}} \right)$$

for the scalar superfield $W$. However, first one observes that, after the field redefinition $W = 2 \ln \left( \frac{a+b}{a} \right)$ (with $a > 0$) this reduces to the scalar superfield equation (93) on the $OSp(1|n)$ supergroup manifold. Moreover, this does not imply a nontrivial embedding of even the free scalar superfield (and of the higher spin theories) in tensorial supergravity. The reason can be traced to the super–Weyl invariance of both the constraints, Eqs. (100)–(102), and the scalar superfield equation in supergravity background, Eq. (110). Thus one may use (112) as a field redefinition (leaving the constraint invariant) pass form the superconformally–$OSp(1|n)$–related geometry to the rigid $OSp(1|n)$ supergroup manifold.

In other words, like the $D = 3$ $N = 1$ Poincaré and $AdS$ supergravities, the supergravity in tensorial superspace is shown to be nondynamical: the general solution of its constraints is given by superconformally flat and $OSp(1|n)$ related superspaces which may be reduced to the rigid $\Sigma^{(n(n+1)/2|n)}$ or $OSp(1|n)$ superspaces by the (super)field redefinition (Eq. (112) plus $GL(n)$ transformations or, equivalently, Eqs. (111)) [18].

This implies, in particular, that to proceed with the search for $D = 4, 6, 10$ interacting higher spin theories on the basis of curved tensorial superspaces $\Sigma^{(n(n+1)/2|n)}$ with $n = 4, 8, 16$ one has to extend the tensorial superspace rather than to restrict it, as it might be expected. On the other hand, such an extension looks natural in the light of the existing results on interacting higher spin theories [17, 50], as these imply the necessity of doubling of the auxiliary variables. Such auxiliary variables responsible for the spin degrees of freedom can be chosen to be spinors or antisymmetric tensors ($\gamma^{[\mu\nu}\}$ for $D = 4$). Hence a natural candidate for the variables to use for the extension of the tensorial superspace $X^{(\alpha\beta), \theta^\alpha} (= (x^\mu, y^{\mu\nu}, \theta^\alpha)$ for $D = 4$) in a search for consistent interacting higher spin theories are the bosonic spinors $\lambda_\alpha$ which are used to define the notion of BPS preon and are present in the action [20] for the “preonic superparticle”.

The study of supergravity and super–Yang–Mills theories in tensorial superspace enlarged by additional bosonic spinors is an interesting subject for future study. Another interesting direction is an $M$–theoretic use of the superconformally flat and $OSp(1|32)$–related superspaces in an $M$–theoretical context. This might be related with the direction described in the contribution of José A. de Azcárraga to this volume [37].
CONCLUDING REMARKS

In this contribution we made a brief review of the notion of BPS preon, both in its original $D = 11$ image as hypothetical constituents of M-theory [6] and in its natural generalization to arbitrary dimension $D$ [8, 12]. Actually the definition of BPS preon possesses a wider $GL(n)$ symmetry and, thus is rather characterized by the number $n$ of possible values of the spinor (or ’s–vector’ [14]) index $\alpha$ than by the number $D$ related to an invariance under a subgroup $SO(t, D - t) \subset GL(n)$. For $n = 4, 8, 16$ cases the BPS preon may be identified with the tower of massless higher spin fields in $D = 4, 6$ and 10. This can be established by quantization [13] of the “preonic superparticle model” [20] which, interestingly enough, had been carried out some times before the notion of BPS preon was introduced in [6]. The present treatment [8, 12, 7] of the results of [13, 20] in terms of BPS preon notion [6] is justified by a search for a universal language which might provide a bridge between (essentially) eleven–dimensional M–theory and the higher spin theories in $D = 4, 6, 10$.

We have also reviewed the “preonic superparticle” action of [20] bringing us to the tensorial superspace, as well as the rôle of BPS preons in the classification and study of supergravity solitons [12], the concise superfield description of the higher spin theories [18] and the results of the study of supergravity in tensorial superspace [18].

Actually, in the light of the algebraic classification of the M–theory BPS states proposed in [6], the possibility of treating BPS preons as constituents of M–theory is a bit more than conjecture; a conjecture concerns rather a usefulness of such a treatment. One might express doubts on such a usefulness arguing that the symmetry of the BPS preon is too high to describe the M-theory physics. However, its identification with higher spin theory in lower dimensions suggests an answer. Higher spin theories were (and are) conjectured (see [50] and e.g. [26]) to be related to the “symmetric” phase of string theory characterized by an enhanced symmetry whose spontaneous breaking should reproduce the complete string theory. In the same way one may conjecture that the $GL(32)$–invariant (actually $OSp(1|64)$–invariant) description provided by the BPS preons corresponds to a symmetric phase of M–theory, while the complete description of M–theory might require the spontaneous breaking of these $OSp(1|64)$ symmetry.

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