Homogenization of elliptic equations with large random potential

Guillaume Bal ∗ Ningyao Zhang †

November 26, 2013

Abstract

We consider an elliptic equation with purely imaginary, highly heterogeneous, and large random potential with a sufficiently rapidly decaying correlation function. We show that its solution is well approximated by the solution to a homogeneous equation with a real-valued homogenized potential as the correlation length of the random medium $\varepsilon \to 0$ and estimate the size of the random fluctuations in the setting $d \geq 3$.

1 Introduction

We study the asymptotic behavior of the solution to the equations parameterized by $\varepsilon$

$$
\left(-\Delta + 1 - iV_\varepsilon\right)u_\varepsilon(x) = f(x)
$$

for $x \in \mathbb{R}^d$ as $\varepsilon \to 0$ in dimension $d \geq 3$ with $V_\varepsilon = \varepsilon^{-1}V(\varepsilon x)$. Here, $i = \sqrt{-1}$. As a possible application for (1.1), we may rewrite it as the system

$$
\begin{pmatrix}
-\Delta + 1 & 0 \\
0 & -\Delta + 1
\end{pmatrix}
\begin{pmatrix}
u_{1,\varepsilon} \\
u_{2,\varepsilon}
\end{pmatrix}
+
\begin{pmatrix}
0 & V_\varepsilon \\
-V_\varepsilon & 0
\end{pmatrix}
\begin{pmatrix}
u_{1,\varepsilon} \\
u_{2,\varepsilon}
\end{pmatrix}
= 
\begin{pmatrix}
f_r \\
f_i
\end{pmatrix},
$$

where we have defined $f = f_r + if_i$ and where $V_\varepsilon$ may model the (linear) interaction between two populations represented by the densities $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$. In the absence of interactions, the two populations follow independent diffusions. Assuming that the interaction is modeled by a large, highly oscillatory, random, and mean zero field $V_\varepsilon$, we wish to understand the limit as the correlation length $\varepsilon \to 0$ of such interactions.

It turns out that the limiting behavior of $u_\varepsilon$ depends on the correlation properties of $V$. When the latter decay slowly (of the form $|x|^{-\gamma}$ as $|x| \to \infty$ with $\gamma < 2$), we expect $u_\varepsilon$ to converge to the solution of a stochastic partial differential equation; see [2, 18] for such results in a time-dependent setting. In dimension $d = 1$, we also expect the solution $u_\varepsilon$ to

---

∗Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027, (gb2030@columbia.edu).

†Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027, (nz2164@columbia.edu).
remain stochastic in the limit $\varepsilon \to 0$ [16]. We consider here the setting where the correlation function decays sufficiently rapidly so that $u_\varepsilon$ is expected to converge to a deterministic, homogenized, solution. The main objective of this paper is to present such a convergence result in the setting $d \geq 3$ and to provide an optimal rate of convergence when the potential $V$ is assumed to be sufficiently mixing. A similar result, not considered here, is expected to hold in the critical dimension $d = 2$ with the strength of the random potential $\varepsilon^{-1}$ in (1.1) replaced by $\varepsilon^{-1} |\ln \varepsilon|^{-\frac{1}{2}}$ [3].

The above problems are written on $\mathbb{R}^d$ to simplify the presentation. Our convergence result would also hold for a problem posed on a bounded open domain $X$ with, say, Dirichlet conditions on $\partial X$. The operator $-\Delta + 1$ could also be replaced by any operator of the form $-\nabla \cdot a \nabla + b$ with $a$ (as a symmetric tensor) and $b$ sufficiently smooth and bounded above and below by positive constants.

The homogenization of partial differential equations in periodic or random media has a long history; see for instance [1, 6, 13]. The homogenization of elliptic equations with random diffusion coefficients was treated in [14, 15]. Rates of convergence to homogenization in similar settings are proposed in [7, 8, 9, 17]. The homogenization of elliptic and parabolic equations with large random potential has also been studied recently in different contexts. Convergence to stochastic limits is considered in [2, 16, 18]. Convergence to homogenized solutions is treated in [3, 19] by diagrammatic techniques, in [10, 11] using probabilistic representations, and in [12] using a multi-scale method; see also the review [4].

We now present our main hypotheses on the potential $V$ and our main results.

The potential $V(x, \omega)$ is defined, following [15], on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathfrak{U}(\omega)$ a bounded measurable function on $\Omega$. We assume the existence of a translation group $\tau : \Omega \to \Omega$ for all $x \in \mathbb{R}^d$ leaving $\mathbb{P}$ invariant and being ergodic in the sense that for all $A \in \mathcal{F}$ such that $\tau_x A \subset A$ for all $x \in \mathbb{R}^d$, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. Let $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$. For $f \in \mathcal{H}$ and $x \in \mathbb{R}^d$, we define the unitary operator $T_x$ on $\mathcal{H}$ as $T_x f(\omega) = f(\tau_x \omega)$. The stationary, bounded, potential $V$ is then defined as $V(x, \omega) = T_x \mathfrak{U}(\omega) = \mathfrak{U}(\tau_x \omega)$. The group (in $x$) of unitary operators $T_x$ admits a spectral resolution

\[ T_x = \int_{\mathbb{R}^d} e^{i \xi \cdot x} U(d\xi) \]

for $U(d\lambda)$ the associated projection valued measure and the $s$ powers of the (positive) Laplacian $L$ are given by

\[ L^s = \int_{\mathbb{R}^d} |\xi|^s U(d\xi). \]

Note that for $f(x, \omega) = T_x f(\omega)$, we have $(-\Delta)^s f(x, \omega) = T_x L^s f(\omega) = L^s T_x f(x, \omega)$, $dx \times \mathbb{P}$--a.s., where $\Delta$ is the usual (negative) Laplacian in $\mathbb{R}^d$.

The correlation function of $\mathfrak{U}$ (and $V$) is defined as

\[ R(x) = \mathbb{E}\{\mathfrak{U} T_x \mathfrak{U}\} = \mathbb{E}\{V(0, \cdot)V(x, \cdot)\}. \quad (1.3) \]

The power spectrum $\hat{R}(\xi)$ is the (rescaled) Fourier transform of $R$ defined by

\[ (2\pi)^d \hat{R}(\xi) = \int_{\mathbb{R}^d} e^{-i \xi \cdot x} R(x) dx. \quad (1.4) \]
The main assumption we make on the correlation function is that
\[ \rho := \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi < \infty. \] (1.5)

This may be recast by Plancherel as \( \rho = \int_{\mathbb{R}^d} \Phi(x) R(x) dx \) for \( \Phi(x) = \Delta^{-1} \delta \) the fundamental solution to the Laplace equation in dimension \( d \geq 3 \). Such an assumption is satisfied when \( R(x) \) decays like \( \kappa |x|^{-\gamma} \) as \( |x| \to \infty \), or equivalently when \( \hat{R}(\xi) \) behaves as \( \kappa' |\xi|^{-d} \) as \( \xi \to 0 \), with \( \gamma > 2 \).

The bound (1.5) is the main hypothesis we impose on \( \mathcal{V} \), beyond stationarity and ergodicity. When the latter fails, for instance when \( \gamma < 2 \) in the above example, then we do not expect \( u_\varepsilon \) to converge to a homogenized solution [2, 16, 18]. For technical reasons, we also need in the convergence result to make some regularity assumptions on \( \mathcal{V} \) and assume that \( \mathcal{V}_s := \mathbb{L}^s \mathcal{V} \) satisfies the same hypothesis as \( \mathcal{V} \) for some \( s > \frac{d-2}{4} \). By construction, the power spectrum of \( \mathcal{V}_s \) is given by \( |\xi|^{2s} \hat{R}(\xi) \) so we also impose that \( |\xi|^{2s-2} \hat{R}(\xi) \) is integrable.

With these hypotheses, we can state the following result.

\textbf{Theorem 1.1.} Let us assume that \( V \) is a stationary, bounded, random field such that
\[ \int_{\mathbb{R}^d} \frac{1+|\xi|^{2s}}{|\xi|^2} \hat{R}(\xi) < \infty \quad \text{for some} \quad s > \frac{d-2}{4}. \] (1.6)

Then \( u_\varepsilon \) the unique solution to (1.1) with \( f \in H^{-1}(\mathbb{R}^d) \) converges weakly in \( H^1(\mathbb{R}^d; \mathcal{H}) \) and strongly in \( L^2_{\text{loc}}(\mathbb{R}^d; \mathcal{H}) \) to the unique solution of the deterministic equation
\[ -\Delta u + (1+\rho)u = f, \quad \mathbb{R}^d \] (1.7)
with \( \rho \) defined in (1.5).

When the decay rate of the correlation function \( R \) is sufficiently large and \( \mathcal{V} \) satisfies additional technical assumptions, then we obtain an optimal rate of convergence of \( u_\varepsilon \) to \( u \) in \( H^1(\mathbb{R}^d; \mathcal{H}) \). More precisely, we now assume that \( V \) is bounded \( \mathbb{P} \)-a.s. (although this specific bound does not appear in subsequent estimates), that \( R(x) \in L^1(\mathbb{R}^d) \) and that for all \( (x_1, x_2, x_3, x_4) \in (\mathbb{R}^d)^4 \),
\[ |\mathbb{E}\{\prod_{j=1}^4 V(x_j)\} - R(x_1-x_2)R(x_3-x_4)| \leq \eta(|x_1-x_3|)\eta(|x_2-x_4|) + \eta(|x_1-x_4|)\eta(|x_2-x_3|), \] (1.8)
for some integrable function \( \eta \) from \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \).

A large class of mixing potentials with sufficiently rapidly decaying maximal correlation function was shown to satisfy (1.8) in [12]; see also [5] for similar bounds for specific distributions. Our main convergence result is then the following theorem.

\textbf{Theorem 1.2.} We assume that \( V \) is bounded, that the correlation function \( R(x) \in L^1(\mathbb{R}^d) \) and that there is an integrable function \( \eta \) such that (1.8) holds. In dimension \( d \geq 3 \), the solution \( u_\varepsilon(x) \) to (1.1) with \( f \in L^2(\mathbb{R}^d) \) converges to the solution \( u \) of (1.7). Moreover, we have the estimate
\[ ||u_\varepsilon - u||_{L^2(\mathbb{R}^d; \mathcal{H})} \leq C\left\{ \begin{array}{ll} \varepsilon^{\frac{3}{2}} & d = 3 \\ \varepsilon \sqrt{\ln \varepsilon} & d = 4 \\ \varepsilon & d > 4. \end{array} \right. \] (1.9)
In fact, for a vector field $\Xi(x)$ in $L^2(\mathbb{R}^d)$ formally defined as $-\nabla \Delta^{-1}V$ (see Lemma 2.1 for a more precise statement), then we obtain that $\|\nabla u_\varepsilon - \nabla u - u\Xi(\varepsilon)\|_{L^2(\mathbb{R}^d;H)}$ satisfies the same bound as $\|u_\varepsilon - u\|_{L^2(\mathbb{R}^d;H)}$.

The rest of the paper is organized as follows. The proof of theorem 1.1 is presented in section 2. The proof of theorem 1.2 is given in section 3 with technical calculations involving fourth moments postponed to section 4.

2 Energy and perturbed test function methods

Let us consider the problem (1.1) with $f \in H^{-1}(\mathbb{R}^d) := H^{-1}(\mathbb{R}^d;\mathbb{C})$. We assume that $V$ is bounded on $\mathbb{R}^d$ $P$–a.s. to simplify the presentation. Multiplying the equation by $u_\varepsilon^*$ with $u_\varepsilon \in H^1(\mathbb{R}^d) := H^1(\mathbb{R}^d;\mathbb{C})$ solution of the above equation and integrating by parts gives us the a priori estimate

$$\int_{\mathbb{R}^d} \left( |\nabla u_\varepsilon|^2 + |u_\varepsilon|^2 - iV_\varepsilon |u_\varepsilon|^2 \right) dx = \int_{\mathbb{R}^d} f u_\varepsilon^* dx. \quad (2.1)$$

Upon taking the real part, we obtain by Cauchy-Schwarz that

$$\|u_\varepsilon\|_{H^1(\mathbb{R}^d)} \leq \|f\|_{H^{-1}(\mathbb{R}^d)} \quad P - a.s. \quad (2.2)$$

By the Lax-Milgram theory, we thus obtain that (1.1) admits a unique solution in $H^1(\mathbb{R}^d)$ $P$–a.s. for any source $f \in H^{-1}(\mathbb{R}^d)$. Note that when the source $f \in H^{-1}(\mathbb{R}^d;H)$ (defined as the dual to $H^1(\mathbb{R}^d;H)$), then the solution $u_\varepsilon$ is bounded in $H^1(\mathbb{R}^d;H)$ by the preceding estimate.

From the previous estimate, we deduce that $u_\varepsilon$ converges weakly in $H^1(\mathbb{R}^d)$ $P$–a.s. to a limit $u \in H^1(\mathbb{R}^d)$ (after possible extraction of a subsequence, though the limit $u$ will be proved to be unique and hence the whole sequence converges). Moreover, for $\theta$ a smooth function with compact support, we have by the Rellich-Kondrachov embedding that $\theta u_\varepsilon$ converges strongly in $L^p(\mathbb{R}^d)$ to its limit $\theta u$ for all $1 \leq p < \frac{2d}{d-2}$. Our aim is now to pass to the limit in a variant of (2.1) and obtain the limiting equation for $u$.

Let $\theta_\varepsilon \in H^1(\mathbb{R}^d;H)$ be a (complex-valued) test function. We thus find that

$$\mathbb{E} \int_{\mathbb{R}^d} (\nabla u_\varepsilon \cdot \nabla \theta_\varepsilon^* + u_\varepsilon \theta_\varepsilon^* - iV_\varepsilon u_\varepsilon \theta_\varepsilon^*) dx = \mathbb{E} \int_{\mathbb{R}^d} f \theta_\varepsilon^* dx. \quad (2.3)$$

In order to pass to the limit in the above expression, we need to replace the highly oscillatory $V_\varepsilon$ by a better-behaving function, and as it turns out, we need to choose $\theta_\varepsilon$ as an $\varepsilon$–dependent function to help cancel out large contributions.

Our first task is to replace $V_\varepsilon$ by an object of the form $\Delta \psi_\varepsilon$ so that after integrations by parts, the resulting $\nabla \psi_\varepsilon$ is bounded in an appropriate manner as $\varepsilon \to 0$. We introduce the \textit{corrector} (following standard terminology in homogenization theory) $\psi_\varepsilon$ solution of

$$(-\Delta + 1)\psi_\varepsilon + V_\varepsilon = 0. \quad (2.4)$$
By an application of the Lax-Milgram lemma, the real-valued function \( \psi_\varepsilon \) is uniquely defined in \( H^1(\mathbb{R}^d; \mathcal{H}) \). Moreover, in the variables \( y = \frac{x}{\varepsilon} \), let us define

\[
(-\Delta + \varepsilon^2)\psi_\varepsilon(y) + V(y) = 0.
\]  

(2.5)

Therefore, \( \psi_\varepsilon \) is morally an approximation of \( \Delta^{-1}V \), which is not defined and thus regularized with the small absorption coefficient \( \varepsilon^2 \).

We verify that \( \psi_\varepsilon(x) = \varepsilon \psi_\varepsilon(\frac{x}{\varepsilon}) \) so that \( \nabla \psi_\varepsilon(x) = \nabla \psi_\varepsilon(\frac{x}{\varepsilon}) \), which as we now see is a well defined object in \( L^2_{\text{loc}}(\mathbb{R}^d; \mathcal{H}) \) uniformly in \( \varepsilon \).

**Lemma 2.1.** Let \( \psi_\varepsilon \) be the unique solution of (2.5). We assume that \( V \) is such that (1.5) holds. Then \( \varepsilon \psi_\varepsilon \) converges to 0 in \( L^2_{\text{loc}}(\mathbb{R}^d; \mathcal{H}) \) as \( \varepsilon \to 0 \). Moreover, \( \nabla \psi_\varepsilon(y, \omega) \) converges in \( L^2_{\text{loc}}(\mathbb{R}^d; \mathcal{H}) \) to a stationary process \( \Xi(y, \omega) = \mathcal{X}(\tau_y \omega) \) with \( \mathcal{X} \in \mathcal{H}^d \).

More precisely, we have the estimates for any open domain \( D \in \mathbb{R}^d \) and \( |D| = \int_D dx \),

\[
\|\psi_\varepsilon\|_{L^2(D; \mathcal{H})} \leq C\sqrt{|D|}, \quad \|\psi_\varepsilon\|_{L^2(D; \mathcal{H})} \leq C\varepsilon\sqrt{|D|}, \quad \|\nabla \psi_\varepsilon\|_{L^2(D; \mathcal{H})} \leq C\sqrt{|D|}.
\]  

(2.6)

**Proof.** The equation (2.5) may be equivalently cast as

\[
(L + \varepsilon^2)\rho_\varepsilon + \mathfrak{W} = 0.
\]  

(2.7)

With \( D \) the vector valued infinitesimal generators of \( T_x \) so that \( D \cdot D = -\mathbb{L} \) and with \( \mathcal{H}^1 \) the Hilbert space of functions \( \mathfrak{f} \) in \( \mathcal{H} \) such that \( D\mathfrak{f} \in (\mathcal{H})^d \), we obtain from the Lax-Milgram theory that the above equation admits a unique solution \( \rho_\varepsilon \in \mathcal{H}^1 \) [15]. Moreover, it is given by

\[
\rho_\varepsilon = -(L + \varepsilon^2)^{-1}\mathfrak{W} = \int_{\mathbb{R}^d} \frac{-1}{|\xi|^2 + \varepsilon^2} U(d\xi)\mathfrak{W}.
\]

This shows that

\[
\mathbb{E}|\varepsilon \rho_\varepsilon|^2 = \varepsilon^2 \mathbb{E}|(L + \varepsilon^2)^{-1}\mathfrak{W}|^2 = \varepsilon^2 \mathbb{E}\{\mathfrak{W}(L + \varepsilon^2)^{-2}\mathfrak{W}\} = \int_{\mathbb{R}^d} \frac{\varepsilon^2 R(\xi)}{|(\xi)^2 + \varepsilon^2|^2} d\xi \leq \rho,
\]

the latter bound coming from separating the contributions \( |\xi| < \varepsilon \) and \( |\xi| > \varepsilon \). The integrand, which converges to 0 point-wise, is dominated by \( R(\xi)|\xi|^{-2} \). This implies by the dominated Lebesgue convergence theorem that \( \mathbb{E}|\varepsilon \rho_\varepsilon|^2 \to 0 \) as \( \varepsilon \to 0 \). Similarly,

\[
\mathbb{E}|D\rho_\varepsilon|^2 = \mathbb{E}|D(L + \varepsilon^2)^{-1}\mathfrak{W}|^2 = \int_{\mathbb{R}^d} \frac{|\xi|^2}{|(\xi)^2 + \varepsilon^2|^2} R(\xi) d\xi \leq \rho.
\]

By dominated convergence, we thus again observe that \( D\rho_\varepsilon \) converges to \( \mathcal{X} = \int_{\mathbb{R}^d} \frac{d\xi}{|\xi|^2} U(d\xi)\mathfrak{W} \) in \( \mathcal{H} \) with \( \mathbb{E}|\mathcal{X}|^2 = \rho \). It now remains to define \( \psi_\varepsilon(y, \omega) = T_y \rho_\varepsilon(\omega) \) and \( \Xi(y, \omega) = T_y \mathcal{X}(\omega) \) to deduce (2.6).

The above regularity properties of \( \psi_\varepsilon \) are not quite sufficient for our convergence proof. We assume more regularity on \( V \) and obtain a stronger result on \( \psi_\varepsilon \) as follows.
Corollary 2.2. Let us assume that the stationary potential $V$ is such that $V_s := (-\Delta)^s V$ satisfies the hypotheses of Lemma 2.1. Then $\nabla \psi_\varepsilon \in L^2(\Omega; H^s_{loc}(\mathbb{R}^d))$ for $\psi_\varepsilon$ the solution of (2.4).

By Sobolev embedding, then $\nabla \psi^\varepsilon(y)$ is bounded in $L^2(\Omega; L^2_{loc}(\mathbb{R}^d))$ with the norm on a bounded domain $D$

$$\left(\int_\Omega \left( \int_D |u(x,\omega)|^{2q} dx \right)^{\frac{1}{q}} d\mathbb{P}(\omega) \right)^{\frac{1}{2}} \leq C \left( \int_\Omega ||u(\cdot,\omega)||_{H^s(D)}^{2q} d\mathbb{P}(\omega) \right)^{\frac{1}{2}}$$

for $q = \frac{d}{d-2s}$ (and bounded in $L^2(\Omega; L^\infty(\mathbb{R}^d))$ when $s > \frac{d}{2}$) and converges strongly to its limit $\Xi(y)$ in the $L^2(\Omega; L^2_{loc}(\mathbb{R}^d))$ sense. This implies that

$$||\nabla \psi_\varepsilon(x)\varepsilon||^2 - |\Xi(x)\varepsilon|^2||_{L^2(\Omega; L^2(D))} \xrightarrow{\varepsilon \to 0} 0$$

(2.8) for any bounded domain $D$.

By an application of the Birkhoff ergodic theorem, we deduce that

$$|\Xi|^2(x) \xrightarrow{\varepsilon \to 0} \rho \quad (weak) L^q_{loc}(\mathbb{R}^d), \quad \mathbb{P} - a.s.$$

(2.9)

Proof. We observe that

$$(-\Delta + \varepsilon^2)(-\Delta)^s \psi^\varepsilon + (-\Delta)^s V = 0.$$

As a consequence, we obtain that $\psi^\varepsilon \in L^2(\Omega; H^s(\Omega))$. The regularity results follow by Sobolev embedding. Then (2.8) follows from the result in $L^1$ and the dominated Lebesgue convergence theorem. \hfill \Box

At this stage, (2.3) may be replaced by

$$\mathbb{E} \int_{\mathbb{R}^d} (\nabla u_\varepsilon \cdot \nabla \theta_\varepsilon + u_\varepsilon \theta_\varepsilon + i\psi_\varepsilon u_\varepsilon \theta_\varepsilon^* + i\nabla \psi_\varepsilon \cdot \nabla (u_\varepsilon \theta_\varepsilon^*))dx = \mathbb{E} \int_{\mathbb{R}^d} f \theta_\varepsilon^* dx.$$  

(2.10)

It remains to exhibit the limit of $\nabla \psi_\varepsilon \cdot \nabla u_\varepsilon$, which is non-trivial. In order to do so, we introduce the following perturbed test function

$$\theta_\varepsilon(x,\omega) = \theta(x)e^{i\psi_\varepsilon(x,\omega)}, \quad \theta \in L^2(\Omega; C^\infty(\mathbb{R}^d)).$$

(2.11)

The motivation for the above choice may be explained by formal multi-scale expansions as done in [6]. Formally assuming that $u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, y)$, we find that $\varepsilon u_1(x, y) = -u_0(x)e^{i\psi_\varepsilon(y)} = -u_0(x)i\psi_\varepsilon(x)$. Moreover, $1 - i\psi_\varepsilon$ is the Taylor expansion of $e^{-i\psi_\varepsilon(x,\omega)}$. Now, the latter quantity is uniformly bounded whereas the former may not be. A similar choice of correctors was considered for a time dependent problem in [12].

We then obtain that

$$\nabla \theta_\varepsilon = e^{i\psi_\varepsilon} \nabla \theta + i\theta_\varepsilon \nabla \psi_\varepsilon, \quad \nabla \theta_\varepsilon^* = e^{-i\psi_\varepsilon} \nabla \theta^* - i\theta_\varepsilon^* \nabla \psi_\varepsilon$$

6
We observe that
\[
\nabla u_{\varepsilon} \cdot \nabla \theta^* + i \nabla \psi_{\varepsilon} \cdot \nabla (u_{\varepsilon} \theta^*) \\
= \ e^{-i\psi_{\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla \theta^* + i \nabla \psi_{\varepsilon} \cdot \nabla (u_{\varepsilon} \theta^*) \\
= \ e^{-i\psi_{\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla \theta^* + ie^{-i\psi_{\varepsilon}} u_{\varepsilon} \nabla \psi_{\varepsilon} \cdot \nabla \theta^* + \theta_{\varepsilon}^* u_{\varepsilon} |\nabla \psi_{\varepsilon}|^2.
\]

We may now recast (2.10) as
\[
\mathbb{E} \int_{\mathbb{R}^d} (e^{-i\psi_{\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla \theta^* + u_{\varepsilon} \theta^*(1 + i\psi_{\varepsilon}) + ie^{-i\psi_{\varepsilon}} u_{\varepsilon} \nabla \psi_{\varepsilon} \cdot \nabla \theta^* + \theta_{\varepsilon}^* u_{\varepsilon} |\nabla \psi_{\varepsilon}|^2 - f \theta_{\varepsilon}) dx = 0. \tag{2.12}
\]

It remains to pass to the limit in each of the terms above. Since \(|e^{-i\psi_{\varepsilon}} - 1| \leq |C\psi_{\varepsilon}|\), we deduce from lemma 2.1 that \(\theta \psi_{\varepsilon}\) converges to 0 in \(L^2(\mathbb{R}^d; \mathcal{H})\) and hence that \(\theta_{\varepsilon}^* = e^{-i\psi_{\varepsilon}} \theta^*\) and \(e^{-i\psi_{\varepsilon}} \nabla \theta^*\) converge to \(\theta^*\) and \(\nabla \theta^*\), respectively, in the same sense. Similarly, \(\psi_{\varepsilon} \theta_{\varepsilon}^*\) converges to 0 in the same sense. This shows that
\[
\mathbb{E} \int_{\mathbb{R}^d} (e^{-i\psi_{\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla \theta^* + u_{\varepsilon} \theta_{\varepsilon}^*(1 + i\psi_{\varepsilon}) - f \theta_{\varepsilon}) dx \xrightarrow{\varepsilon \to 0} \mathbb{E} \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \theta^* + u \theta - f \theta) dx.
\]

Let us consider the term \(T_1 = ie^{-i\psi_{\varepsilon}} u_{\varepsilon} \nabla \psi_{\varepsilon} \cdot \nabla \theta^*.\) On the support of \(\theta\), \(e^{-i\psi_{\varepsilon}} \nabla \psi_{\varepsilon}\) is bounded in the \(L^2(D; \mathcal{H})\) sense. Since \((u_{\varepsilon} - u) \nabla \theta^*\) converges to 0 in that sense, the limit of the integral of \(T_1\) is the same as that of \(T_2 = ie^{-i\psi_{\varepsilon}} u \nabla \psi_{\varepsilon} \cdot \nabla \theta^*\). For the same reason, we may now replace \(e^{-i\psi_{\varepsilon}}\) by its limit 1 so the limit of the integral of \(T_1\) is the same as that of \(T_3 = iu \nabla \psi_{\varepsilon} \cdot \nabla \theta^*,\) and by integrations by parts the same as that of \(T_4 = -i\psi_{\varepsilon} \nabla \cdot (u \nabla \theta^*).\) Since \(\psi_{\varepsilon}\) goes to 0 in \(L^2_{\text{loc}}\) and \(\theta\) is smooth and compactly supported, we obtain that
\[
\mathbb{E} \int_{\mathbb{R}^d} (ie^{-i\psi_{\varepsilon}} u_{\varepsilon} \nabla \psi_{\varepsilon} \cdot \nabla \theta^*) dx \xrightarrow{\varepsilon \to 0} 0.
\]

Finally, we consider the convergence of the integral of \(\theta_{\varepsilon}^* u_{\varepsilon} |\nabla \psi_{\varepsilon}|^2\). We have that \(\theta_{\varepsilon}^* u_{\varepsilon}\) converges strongly to \(\theta u\) in \(L^p\) for \(1 < p < \frac{2d}{d-2}\). We thus need some regularity on \(|\nabla \psi_{\varepsilon}|^2(x) = |\nabla \psi|^2(\tilde{x})\). From corollary 2.2, we deduce that \(\nabla \psi(\tilde{x})\) is bounded in \(L^q_{\text{loc}}\) for \(2 \leq q \leq \frac{4d}{d+4}\) (or \(L^\infty\) when \(s > \frac{d}{4}\)) and hence that \(|\nabla \psi|^2(\tilde{x})\) is bounded in \(L^q_{\text{loc}}\). Choosing \(q = p' > \frac{2d}{d+4}\), which holds when \(s > \frac{d-2}{4}\), we obtain from (2.8) that the integral of \(\theta_{\varepsilon}^* u_{\varepsilon} |\nabla \psi_{\varepsilon}|^2\) has the same limit as the integral of \(\theta u|\Xi|^2(\tilde{x})\).

Since \(\theta u \in L^2(\Omega; L^p(\mathbb{R}^d))\), we obtain from the Birkhoff ergodic theorem in (2.9) that
\[
\mathbb{E} \int_{\mathbb{R}^d} u \theta |\Xi|^2(\tilde{x}) dx \xrightarrow{\varepsilon \to 0} \mathbb{E} \int_{\mathbb{R}^d} u \theta dx \tag{2.13}
\]

This shows that for all \(\theta \in L^2(\Omega; C^\infty_c(\mathbb{R}^d))\), we have that
\[
\mathbb{E} \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \theta + (1 + \rho)u \theta - f \theta) dx = 0.
\]

This is the weak formulation in \(H^1(\mathbb{R}^d; \mathcal{H})\) (valid for all \(\theta \in L^2(\Omega; C^\infty_c(\mathbb{R}^d))\) and by density for all \(\theta \in H^1(\mathbb{R}^d; \mathcal{H})\)) of the (unique and deterministic) solution to the equation (1.7). This proves theorem 1.1.
3 Decorrelation properties and rate of convergence

We now prove theorem 1.2. Our main assumption on the coefficients is a control of the fourth-order moments of the potential \( V(x) \) as well as some regularity on the unique solution \( u_0 \) of the limiting equation. More precisely, we assume that \( f \in L^2(\mathbb{R}^d) \) and denote by \( u_0 \) the solution in \( H^2(\mathbb{R}^d) \) of (1.7).

Let \( G \) be the Green’s function defined as the fundamental solution of \( (-\Delta + 1)G(x) = \delta(x) \). It is given by the explicit expression \( G(x) = c_n e^{-|x|^2/2n} \) for a normalizing constant \( c_n > 0 \). Then we find that for \( \nu > 0 \) and \( C > 0 \) that

\[
G(x)|x| + |\nabla G|(x) \leq C^\exp(-\nu|x|)
\]

(3.1)

Define \( \chi_\varepsilon = G * \left( \frac{x}{\varepsilon^2} V(\frac{x}{\varepsilon}) \right) = \frac{1}{\varepsilon} \psi_\varepsilon \) and \( u_1(x) = -\chi_\varepsilon(x)u_0(x) \). Some algebra shows that

\[
(\Delta - 1 + iV_\varepsilon)(u_0 + \varepsilon u_1 - \varepsilon \chi_\varepsilon) = (\rho - iV(\frac{x}{\varepsilon})\chi_\varepsilon(x))u_0 - \varepsilon(\chi_\varepsilon \Delta u_0 + 2\nabla \chi_\varepsilon \cdot \nabla u_0).
\]

(3.2)

In other words, \( u_0 + \varepsilon u_1 = u_0(1 - \varepsilon \chi_\varepsilon) \) is the leading expansion of \( u_\varepsilon \). In the preceding section, we proved that \( \varepsilon \chi_\varepsilon \) converged to 0 in the \( L^2(D; \mathcal{H}) \) sense for \( D \) a bounded domain. We also observe that \( \nabla (u_\varepsilon u_{\varepsilon 1}) \) is well approximated by \( \nabla u_0 - u_0 \Xi(\varepsilon) \).

When the potential \( V \) decorrelates sufficiently rapidly, then we can obtain optimal rates of convergence of \( u_\varepsilon \) to \( u_0 \) in \( L^2(\mathbb{R}^d) \) and error estimates between \( u_\varepsilon \) and \( u_0 + \varepsilon u_1 \) in \( H^1(\mathbb{R}^d) \).

Let us assume that the correlation function \( R(x) \) is integrable. Then the size of \( \varepsilon u_1 \) may be estimated as

\[
E \int |\varepsilon u_1(x)|^2 dx = \frac{1}{\varepsilon^2} \int G(x - y_1)G(x - y_2)R(\frac{y_1 - y_2}{\varepsilon})|u_0(x)|^2 dy_1 dy_2 dx
\]

\[
= \frac{1}{\varepsilon^2} \int G(y_1)G(y_2)R(\frac{y_1 - y_2}{\varepsilon})|u_0(x)|^2 dy_1 dy_2 dx
\]

\[
\leq \frac{1}{\varepsilon^2} \int \frac{\exp(-\nu|y_1|)}{|y_1|^d} \frac{\exp(-\nu|y_2|)}{|y_2|^d} R(\frac{y_1 - y_2}{\varepsilon})|u_0(x)|^2 dy_1 dy_2 dx
\]

\[
= \frac{1}{\varepsilon^2} \int \frac{\exp(-\nu|y_1|)}{|y_1|^d} \frac{\exp(-\nu|y_2|)}{|y_2|^d} R(\frac{y_2}{\varepsilon})|u_0(x)|^2 dy_1 dx
\]

\[
\leq \left\{ \begin{array}{ll}
\frac{1}{\varepsilon^2} \left( \varepsilon^d \right) = O(\varepsilon) & d = 3 \\
\frac{1}{\varepsilon^d} \left( \log \varepsilon \right) = O(\varepsilon^2) & d = 4 \\
\frac{1}{\varepsilon^d} \left( \log \varepsilon \right) = O(\varepsilon^2) & d > 4.
\end{array} \right.
\]

(3.3)

The latter estimates easily follow from the integrability of the correlation function in dimension \( d = 3 \) and \( d = 4 \). For \( d > 4 \), we decompose the integral into two parts as

\[
\frac{1}{\varepsilon^2} \int e^{-\nu|y_2|}(|y_2|^{-(d-4)} + 1)R(\frac{y_2}{\varepsilon})dy_2 = \frac{1}{\varepsilon^2} \int_{|y_2| \geq 1} e^{-\nu|y_2|}(|y_2|^{-(d-4)} + 1)R(\frac{y_2}{\varepsilon})dy_2
\]

\[+ \frac{1}{\varepsilon^2} \int_{|y_2| < 1} e^{-\nu|y_2|}(|y_2|^{-(d-4)} + 1)R(\frac{y_2}{\varepsilon})dy_2.\]

(3.4)
We recast this as \((i) + (ii)\) and \((i)\) and \((ii)\) are estimated respectively as

\[
(i) \leq \frac{2}{\varepsilon^2} \int_{|y_2| \geq 1} \exp(-\nu |y_2|) R(\frac{y_2}{\varepsilon}) dy_2 \leq 2\varepsilon^{d-2} \exp(-\nu) \|R\|_1, \quad (3.5)
\]

\[
(ii) \leq \frac{1}{\varepsilon^2} \int_{|y_2| \leq 1} (|y_2|^{-d-4} + 1) R(\frac{y_2}{\varepsilon}) dy_2 \leq \frac{2}{\varepsilon^2} \int_{|y_2| \leq 1} |y_2|^{-d-4} R(\frac{y_2}{\varepsilon}) dy_2 \leq 2\varepsilon^2 \int_{|y_2| \leq \frac{1}{\varepsilon}} |y_2|^{-d-4} R(y_2) dy_2 \leq 2\varepsilon^2 \int_{|y_2| \leq \frac{1}{\varepsilon}} |y_2|^{-d-4} R(y_2) dy_2 \leq 2\varepsilon^2 \|R\|_1. \quad (3.6)
\]

By replacing the Green’s function with its gradient in \((3.3)\) we find that

\[
\mathbb{E} \int |\varepsilon \nabla u_1(x)|^2 dx \sim O(1). \quad (3.7)
\]

This shows that \(\varepsilon u_1\) is negligible in the \(L^2\) sense but not in the \(H^1\) sense. We now estimate the error \(v_\varepsilon := u_0 + \varepsilon u_1 - u_\varepsilon\) using \((3.2)\). Multiplying \((3.2)\) by \(-v_\varepsilon^*\) and integrating by parts, we know from the analysis in the preceding section that

\[
\|v_\varepsilon\|_{H^1(\mathbb{R}^d; H)}^2 \leq \mathbb{E} \int (\rho - iV(\frac{x}{\varepsilon}) \chi_\varepsilon(x)) u_0 v_\varepsilon^* dx \leq \mathbb{E} \int (\chi_\varepsilon \Delta u_0 + 2\nabla \chi_\varepsilon \cdot \nabla u_0) v_\varepsilon^* dx. \quad (3.8)
\]

Let us consider the second-term on the above right-hand side. The term \(\chi_\varepsilon \Delta u_0\) can be estimated in the same way as \(u_1\) and by using the Cauchy-Schwarz inequality, we have

\[
\mathbb{E} \int \chi_\varepsilon \Delta u_0 v_\varepsilon^* dx \leq C \mathbb{E} \|v_\varepsilon\|_{L^2(\mathbb{R}^d; H)} \times \begin{cases} \sqrt{\varepsilon}, & d = 3 \\ \varepsilon \log \varepsilon, & d = 4 \\ \varepsilon, & d > 4. \end{cases} \quad (3.9)
\]

The integral \(\mathbb{E} \int \nabla \chi_\varepsilon \cdot \nabla u_0 v_\varepsilon^* dx\) is estimated using integrations by parts as

\[
\mathbb{E} \int \nabla \chi_\varepsilon \cdot \nabla u_0 v_\varepsilon^* dx = \mathbb{E} \int (\nabla \cdot (\chi_\varepsilon \nabla u_0) - \chi_\varepsilon \Delta u_0) v_\varepsilon^* dx \leq \mathbb{E} \int \nabla v_\varepsilon \cdot \nabla u_0 \chi_\varepsilon dx + \mathbb{E} \int \chi_\varepsilon \Delta u_0 v_\varepsilon^* dx \leq C \mathbb{E} \|v_\varepsilon\|_{L^2(\mathbb{R}^d; H)} \times \begin{cases} \sqrt{\varepsilon}, & d = 3 \\ \varepsilon \sqrt{|\ln \varepsilon|}, & d = 4 \\ \varepsilon, & d > 4. \end{cases} \quad (3.10)
\]

The first term on the right-hand side in \((3.8)\) is bounded by

\[
\|v_\varepsilon\|_{H^1(\mathbb{R}^d; H)} \int (\rho - iV(\frac{x}{\varepsilon}) \chi_\varepsilon(x)) u_0 \|_{H^{-1}(\mathbb{R}^d; H)}.
\]

Recalling that \(G\) is the fundamental solution of \(-\Delta + 1\), we obtain that

\[
\|\rho - iV(\frac{x}{\varepsilon}) \chi_\varepsilon(x)\|_{H^{-1}(\mathbb{R}^d; H)} = \|G \ast ((\rho - iV(\frac{x}{\varepsilon}) \chi_\varepsilon(x)) u_0)\|_{H^1(\mathbb{R}^d; H)},
\]

since \(-\Delta + 1\) is an isomorphism from \(H^1(\mathbb{R}^d)\) to \(H^{-1}(\mathbb{R}^d)\).
Define $f_\varepsilon(x) = G * ((\rho - iV(\frac{x}{\varepsilon})) \chi_\varepsilon(x))u_0$. We show in the next section that $\|f_\varepsilon\|_{H^1(\mathbb{R}^d; H)}$ is bounded by a constant times $\sqrt{\varepsilon}$ in $d = 3$, $\varepsilon|\ln \varepsilon|^{\frac{1}{2}}$ in $d = 4$ and $\varepsilon$ in $d > 4$. Note that $\rho = \lim_{\varepsilon \to 0} \mathbb{E}\{iV(\frac{x}{\varepsilon}) \chi_\varepsilon(x)\}$ so that $f_\varepsilon$ is asymptotically mean-zero.

Collecting the previous bounds, we obtain that

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{H^1(\mathbb{R}^d; H)} + \|u_\varepsilon - u_0\|_{L^2(\mathbb{R}^d; H)} \leq C \begin{cases} \sqrt{\varepsilon} & d = 3 \\ \varepsilon \sqrt{|\ln \varepsilon|} & d = 4 \\ \varepsilon & d > 4. \end{cases}$$

(3.12)

This concludes the proof of theorem 1.2.

4 Estimation of fourth order moments

In this section we discuss the estimation of $\mathbb{E}\int |\nabla f_\varepsilon|^2 dx$ and $\mathbb{E}\int |f_\varepsilon|^2 dx$ when the potential $V$ satisfies (1.8). Following [12], we first recall that the latter estimate holds for a large class of sufficiently mixing coefficients.

**Definition 4.1.** For any $r > 0$, $\gamma(r)$ is the smallest value such that the bound

$$\mathbb{E}(\phi_1(V)\phi_2(V)) \leq \gamma(r) \sqrt{\mathbb{E}\phi_1^2(V)\mathbb{E}\phi_2^2(V)},$$

(4.1)

holds for any two compact sets $K_1, K_2$ such that

$$d(K_1, K_2) = \inf_{x_1 \in K_1, x_2 \in K_2} (|x_1 - x_2|) \geq r;$$

(4.2)

for any two random variables $\phi_i(V)$ such that $\phi_i(V)$ is $\mathcal{F}_{K_i}$-measurable and $\mathbb{E}\phi_i(V) = 0$.

It is shown in [12] that (1.8) holds for a function $\eta: \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\eta(r) = \sqrt{K\gamma(r/3)}; \quad \text{with} \quad K = 4(\|V(x)\|_2\|V^3(x)\|_2 + \|V^2(x)\|_2^2).$$

(4.3)

Note that when $V(\cdot)$ is a Gaussian random field, inequality (1.8) becomes an equality with $\eta$ replaced by $R$. We assume that $\eta \in L^1(\mathbb{R}^d)$, and hence that $\sqrt{\gamma} \in L^1(\mathbb{R}^d)$ for the following estimation to hold.
We have the following decomposition for $\|\nabla f_\varepsilon\|^2_{L^2(\mathbb{R}^d;\mathcal{H})}$

\[
\mathbb{E} \int |\nabla f_\varepsilon|^2 \, dx \\
= \mathbb{E} \rho \int \nabla G(x, y) u_0(y) dy - \frac{1}{\varepsilon^2} \int \int \nabla G(x, y) V \left( \frac{y}{\varepsilon} \right) G(y, z) V \left( \frac{z}{\varepsilon} \right) u_0(y) dy \, dz \bigg|_{x}^2 dx
\]

\[
= \frac{1}{\varepsilon^4} \mathbb{E} \int \nabla G(x, y_1) G(y_1, z_1) \nabla G(x, y_2) G(y_2, z_2) \left( \frac{y_1}{\varepsilon} \right) V \left( \frac{z_1}{\varepsilon} \right) V \left( \frac{y_2}{\varepsilon} \right) V \left( \frac{z_2}{\varepsilon} \right) u_0(y_1) u_0(y_2) \, dy
\]

\[
- \frac{1}{\varepsilon^4} \int \nabla G(x, y_1) G(y_1, z_1) \nabla G(x, y_2) G(y_2, z_2) R \left( \frac{y_1 - z_1}{\varepsilon} \right) R \left( \frac{y_2 - z_2}{\varepsilon} \right) u_0(y_1) u_0(y_2) \, dy
\]

\[
\leq \frac{1}{\varepsilon^4} \int \nabla G(x, y_1) G(y_1, z_1) \nabla G(x, y_2) G(y_2, z_2) \eta \left( \frac{y_1 - y_2}{\varepsilon} \right) \eta \left( \frac{z_1 - z_2}{\varepsilon} \right) u_0(y_1) u_0(y_2) \, dy
\]

\[
+ \frac{1}{\varepsilon^4} \int \nabla G(x, y_1) G(y_1, z_1) \nabla G(x, y_2) G(y_2, z_2) \eta \left( \frac{y_1 - y_2}{\varepsilon} \right) \eta \left( \frac{y_2 - z_1}{\varepsilon} \right) u_0(y_1) u_0(y_2) \, dy
\]

\[
:= (I) + (II)
\]

(4.4)

with $dy = dy_1 dy_2 dz_1 dz_2 dx$.

**Estimation of (I).** Changing variables $y_i$ and $z_i$ to $x - y_i$ and $x - y_i - z_i$ for $i = 1, 2$ gives

\[
\begin{aligned}
|(I)| & \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1) G(z_1) \nabla G(y_2) G(z_2) \eta \left( \frac{y_1 - y_2}{\varepsilon} \right) \eta \left( \frac{y_1 - y_2}{\varepsilon} - \frac{z_1 - z_2}{\varepsilon} \right) \\
& \quad \times \left| u_0(x - y_1) \right| \left| u_0(x - y_2) \right| dy_1 dy_2 dz_1 dz_2 dx
\end{aligned}
\]

(4.5)

Using $u_0$ to integrate in $x$, we then have

\[
\begin{aligned}
|(I)| & \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1) G(z_1) \nabla G(y_2) G(z_2) \eta \left( \frac{y_1 - y_2}{\varepsilon} \right) \eta \left( \frac{y_1 - y_2}{\varepsilon} - \frac{z_1 - z_2}{\varepsilon} \right) dy_1 dy_2 dz_1 dz_2.
\end{aligned}
\]

(4.6)

Changing variables $y_2$ and $z_2$ to $y_1 - y_2$ and $z_1 - z_2$, and using (3.1) yields

\[
\begin{aligned}
|(I)| & \leq C \frac{1}{\varepsilon^4} \int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-1}} \frac{\exp(-\nu|z_1|)}{|z_1|^{d-1}} \frac{\exp(-\nu|y_1 - y_2|)}{|y_1 - y_2|^{d-1}} \frac{\exp(-\nu|z_1 - z_2|)}{|z_1 - z_2|^{d-1}} \\
& \quad \times \eta \left( \frac{y_2}{\varepsilon} \right) \eta \left( \frac{z_2}{\varepsilon} \right) dy_1 dy_2 dz_1 dz_2.
\end{aligned}
\]

(4.7)

Now we may apply Lemma A.1 to integrate in $y_1$ and $z_1$:

\[
\int \frac{\exp(-\nu|y_1|)}{|y_1|^{d-1}} \frac{\exp(-\nu|y_1 - y_2|)}{|y_1 - y_2|^{d-1}} dy_1 \leq C \exp(-\nu|y_2|)(1 + |y_2|^{-(d-2)}),
\]

(4.8)

\[
\int \frac{\exp(-\nu|z_1|)}{|z_1|^{d-2}} \frac{\exp(-\nu|z_1 - z_2|)}{|z_1 - z_2|^{d-2}} dz_1 \leq \begin{cases}
C \exp(-\nu z_2), & d = 3 \\
C \exp(-\nu z_2)(1 + \log|z_2|), & d = 4 \\
C \exp(-\nu z_2)(1 + |z_2|^{-(d-4)}), & d > 4.
\end{cases}
\]

(4.9)

This estimate of (I) can be recast as

\[
\begin{aligned}
|(I)| & \leq C \frac{1}{\varepsilon^4} \int \int dy_2 dz_2 \exp(-\nu|y_2|)(1 + |y_2|^{-(d-2)}) \exp(-\nu|z_2|) \eta \left( \frac{y_2}{\varepsilon} \right) \eta \left( \frac{z_2}{\varepsilon} \right) \\
& \quad \times \begin{cases}
1, & d = 3 \\
\log(|z_2|), & d = 4 \\
(1 + |z_2|^{-(d-4)}), & d > 4.
\end{cases}
\end{aligned}
\]

(4.10)
It remains to integrate in $y_2$ and $z_2$ to obtain

$$|(I)| \sim \begin{cases} O(\varepsilon), & d = 3 \\ O(\varepsilon^2 \log \varepsilon), & d = 4 \\ O(\varepsilon^2), & d > 4. \end{cases} \tag{4.11}$$

**Estimation of (II).** After changing variables $y_i$ and $z_i$ to $x - y_i$ and $x - y_i - z_i$ for $i = 1, 2$, and integrating in $x$ using $u_0$, we have

$$|(II)| \leq C \frac{1}{\varepsilon^4} \int \nabla G(y_1) G(z_1) \nabla G(y_2) G(z_2) \eta \left( \frac{y_1 + y_2 + z_2}{\varepsilon} \right) |\eta| \left( \frac{y_2 + y_1 + z_1}{\varepsilon} \right) d\mathcal{Y}$$

with $d\mathcal{Y} = dy_1 dy_2 dz_1 dz_2$. Changing variable $y_2$ to $y_1 - y_2$ and using (3.1) gives

$$|(II)| \leq C \frac{1}{\varepsilon^4} \int \frac{\exp(-\nu |y_1|) \exp(-\nu |z_1|) \exp(-\nu |y_1 - y_2|) \exp(-\nu |z_2|)}{|y_1|^{d-1} |z_1|^{d-2} |y_1 - y_2|^{d-1} |z_2|^{d-2}} |\eta| \left( \frac{z_2 - y_2}{\varepsilon} \right) |\eta| \left( \frac{z_1 + y_2}{\varepsilon} \right) dy_1 dy_2 dz_1 dz_2. \tag{4.12}$$

We now integrate in $y_1$ and $z_1$:

$$\int \frac{\exp(-\nu |y_1|) \exp(-\nu |y_1 - y_2|)}{|y_1|^{d-1} |y_1 - y_2|^{d-1}} dy_1 \leq C \exp(-\nu |y_2|)(1 + |y_2|^{-(d-2)}),$$

$$\int \frac{\exp(-\nu |z_1|)}{|z_1|^{d-2}} |\eta| \left( \frac{z_1 + y_2}{\varepsilon} \right) dz_1 \leq C \varepsilon^2. \tag{4.14}$$

The estimate is then recast as

$$|(II)| \leq C \frac{1}{\varepsilon^2} \int d\mathcal{Y} \eta \left( \frac{z_2}{\varepsilon} \right) \times \begin{cases} 1, & d = 3 \\ \log(|z_2|), & d = 4 \\ (1 + |z_2|^{-(d-4)}), & d > 4. \end{cases} \tag{4.16}$$

Changing variable $z_2$ to $y_2 - z_2$, and integrating in $y_2$ using Lemma A.1 yields

$$|(II)| \leq C \frac{1}{\varepsilon^2} \int d\mathcal{Y} \eta \left( \frac{z_2}{\varepsilon} \right) \times \begin{cases} O(\varepsilon), & d = 3 \\ O(\varepsilon^2 \log \varepsilon), & d = 4 \\ O(\varepsilon^2), & d > 4. \end{cases} \tag{4.17}$$

Collecting (4.11) and (4.17), we find that

$$\mathbb{E} \int |\nabla f_\varepsilon|^2 dx \sim \begin{cases} O(\varepsilon), & d = 3 \\ O(\varepsilon^2 \log \varepsilon), & d = 4 \\ O(\varepsilon^2), & d > 4. \end{cases} \tag{4.18}$$

The estimate of $\mathbb{E} \int |f_\varepsilon|^2 dx$ can be obtained by replacing $\nabla G$ by $G$ in (4.4) and estimating every term in the same way. The result is

$$\mathbb{E} \int |f_\varepsilon|^2 dx \sim \begin{cases} O(\varepsilon^2), & d = 3 \\ O(\varepsilon^4 \log \varepsilon^2), & d = 4 \\ O(\varepsilon^4), & d > 4. \end{cases} \tag{4.19}$$

This concludes the proof of theorem 1.2.
Acknowledgments

The authors would like to thank Yu Gu or multiple discussions on the homogenization of equations with random potentials. This work was partially funded by AFOSR Grant NSSEFF- FA9550-10-1-0194 and NSF Grant DMS-1108608.

A Appendix

The following lemma is proved in [5].

Lemma A.1. Let us fix two distinct points \( x, y \in \mathbb{R}^d \). Let \( \alpha \), \( \beta \) be positive numbers in \((0,d)\), and \( \lambda \) another positive number. We have the following convolution results.

\[
\int_{\mathbb{R}^d} e^{-\lambda|z-x|} e^{-\lambda|z-y|} \frac{1}{|z-x|^\alpha |z-y|^\beta} dz \leq \begin{cases} 
C \exp(-\lambda|x-y|)(|x-y|^{d-(\alpha+\beta)}+1), & \text{if } \alpha + \beta > d; \\
C \exp(-\lambda|x-y|)(|\log |x-y||+1), & \text{if } \alpha + \beta = d; \\
C \exp(-\lambda|x-y|), & \text{if } \alpha + \beta < d.
\end{cases}
\]

(A.1)

The above constants depend only on the \( \text{diam}(X) \), \( \alpha \), \( \beta \), \( \lambda \), and dimension \( d \) but not on \( |x-y| \).

References

[1] G. Allaire, Shape optimization by the homogenization method, vol. 146 of Applied Mathematical Sciences, Springer-Verlag, New York, 2002.
[2] G. Bal, Convergence to SPDEs in Stratonovich form, Comm. Math. Phys., 212(2) (2009), pp. 457–477.
[3] ———, Homogenization with large spatial random potential, Multiscale Model. Simul., 8(4) (2010), pp. 1484–1510.
[4] G. Bal and Y. Gu, Limiting models for equations with large random potential: a review, Submitted, (2013).
[5] G. Bal and W. Jing, Corrector theory for elliptic equations in random media with singular Green’s function. Application to random boundaries, Comm. Math. Sci., 9(2) (2011), pp. 383–411.
[6] A. Bensoussan, J.-L. Lions, and G. C. Papanicolaou, Homogenization in deterministic and stochastic problems, in Stochastic problems in dynamics (Sympos., Univ. Southampton, Southampton, 1976), Pitman, London, 1977, pp. 106–115.
[7] L. A. Caffarelli and P. E. Souganidis, Rates of convergence for the homogenization of fully nonlinear uniformly elliptic PDE in random media, Inventiones Math., 180 (2010), pp. 301–360.
[8] J. G. Conlon and A. Naddaf, On homogenization of elliptic equations with random coefficients, Electron. J. Probab., 5 (2000), pp. 9–58.
[9] A. Gloria and F. Otto, *An optimal variance estimate in stochastic homogenization of discrete elliptic equations*, Ann. of Probab., 39 (2011), pp. 779–856.

[10] Y. Gu and G. Bal, *An invariance principle for Brownian motion in random scenery*, Submitted.

[11] ———, *Weak convergence approach to a parabolic equation with large random potential*, Submitted, arXiv preprint arXiv:1304.5005.

[12] M. Hairer, E. Pardoux, and A. Piatnitski, *Random homogenization of a highly oscillatory singular potential*, To appear in Stoch. Partial Diff. Equ., arXiv preprint arXiv:1303.1955, (2013).

[13] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, New York, 1994.

[14] S. M. Kozlov, *The averaging of random operators*, Math. Sb. (N.S.), 109 (1979), pp. 188–202.

[15] G. C. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, in Random fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai, 27, North Holland, Amsterdam, New York, 1981, pp. 835–873.

[16] E. Pardoux and A. Piatnitski, *Homogenization of a singular random one dimensional PDE*, GAKUTO Internat. Ser. Math. Sci. Appl., 24 (2006), pp. 291–303.

[17] V. V. Yurinskii, *Averaging of symmetric diffusion in a random medium*, Siberian Math. J., 4 (1986), pp. 603–613. English translation of: Sibirs. Mat. Zh. 27 (1986), no. 4, 167–180 (Russian).

[18] N. Zhang and G. Bal, *Convergence to SPDE of the Schrödinger equation with large, random potential*, To appear in Comm. Math. Sci., (2013).

[19] ———, *Homogenization of a Schrödinger equation with large, random, potential*, To appear in Stochastics and Dynamics, (2013).