Partition asymptotics from 1D quantum entropy and energy currents

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Abstract

We give an alternative method to that of Hardy-Ramanujan-Rademacher to derive the leading exponential term in the asymptotic approximation to the partition function $p(n, a)$, defined as the number of decompositions of a positive integer $n$ into integer summands, with each summand appearing at most $a$ times in a given decomposition. The derivation involves mapping to an equivalent physical problem concerning the quantum entropy and energy currents of particles flowing in a one-dimensional (1D) channel connecting thermal reservoirs, and which obey Gentile’s intermediate statistics with statistical parameter $a$. The method is also applied to partitions associated with Haldane’s fractional exclusion statistics.
A classic result in the theory of partitions is the Hardy-Ramanujan-Rademacher formula for the unrestricted partition function \( p(n, \infty) \), wherein the latter, combinatoric quantity is represented as a power series whose terms involve elementary functions of \( n \). This series yields the following asymptotic approximation:

\[
p(n, \infty) \sim \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{2/3\sqrt{n}}}. \tag{1}
\]

A series representing \( p(n, 1) \), the number of decompositions of \( n \) into distinct summands, has also been derived (see, e.g., Sec. 24.2.2 of Ref. 4), yielding the asymptotic approximation

\[
p(n, 1) \sim \frac{1}{4 \cdot 3^{1/4} \cdot n^{3/4}} e^{\pi\sqrt{1/3 \sqrt{n}}} \tag{2}.
\]

And more recently,\(^5\) Hagis used the Hardy-Ramanujan-Rademacher method to derive a power series representation of \( p(n, a) \) for arbitrary \( a = 1, 2, \ldots \), yielding the asymptotic approximation

\[
p(n, a) \sim \frac{\sqrt{12} a^{1/4}}{(1 + a)^{3/4} (24n)^{3/4}} e^{\pi\sqrt{2a/[3(1+a)]} \sqrt{n}}, \tag{3}
\]

where \( n \gg a \). As an example, for \( a = 4 \) the number of partitions of \( n = 1000 \) to five significant figures is 2.4544 \times 10^{28}, while approximation (3) gives 2.4527 \times 10^{28}, accurate to within 0.1%.

In the present work, we give an alternative and more direct derivation of the asymptotic approximation to \( \ln p(n, a) \) which, from Eq. (3), is:

\[
\ln p(n, a) \sim \pi \sqrt{\frac{2a}{3(1 + a)}}, \sqrt{n}. \tag{4}
\]

The derivation begins by considering a 1D quantum channel which supports particles obeying Gentile’s intermediate statistics\(^6\) characterised by statistical parameter \( a \), the maximum occupation number of particles in a single particle state, with \( a = 1 \) describing fermions and \( a = \infty \) bosons. The left end of the channel is connected to a particle source and the right
end to a particle sink. The channel is dispersionless so that particle packets with different mean energies have the same velocity $c$ and hence transmission time $\tau = L/c$, where $L$ is the channel length. Imposing periodic boundary conditions on the channel length, the single-particle energies are $\epsilon_j = hf_j = h j/\tau$, $j = 1, 2, \ldots$, where $h$ is Planck's constant. The total energy $E_n$ of a given Fock state is $E_n = \sum_j \epsilon_j n_j = nh/\tau$, where $n = \sum_{j=1}^{\infty} j n_j$, and $n_j \leq a$ is the occupation number of, say, the right-propagating mode $j$.

We now suppose that the source emits a finite number of particles with fixed total energy $E_n$. The maximum possible entropy of this collection of right-propagating particles subject to the fixed energy constraint is $S(n, a) = k_B \ln p(n, a)$. Thus, the problem to determine the asymptotic approximation to $\ln p(n, a)$ is equivalent to determining the asymptotic approximation to the entropy $S(n, a)$ of the just-described physical system. (C.f. Sec. 4 of Ref. 7, where the same set-up restricted to bosons was considered in the problem to determine the optimum capacity for classical information transmission down a quantum channel.)

The crucial next step is to consider a slightly different set-up, in which the particle source and sink are replaced by two thermal reservoirs described by grand canonical ensembles, with the chemical potentials of the left and right reservoirs satisfying $\mu_L = \mu_R = 0$, the temperature of the right reservoir $T_R = 0$, and the temperature $T_L$ of the left reservoir chosen such that the thermal-averaged energy current flowing in the channel satisfies $\hat{\dot{E}}(T_L, a) = E_n/\tau$. (Note that the chemical potentials are set to zero since there is no constraint on the thermal-averaged particle number.) With this choice, the thermal-averaged, channel entropy current $\hat{\dot{S}}(T_L, a)$ coincides with $S(n, a)/\tau$ in the thermodynamic limit $E_n$ (equivalently $n$) $\to \infty$.

The advantage with using the latter, grand canonical ensemble description as opposed to the former, microcanonical ensemble description is the greater ease with which the energy and entropy currents can be calculated. The starting formula for the single channel energy current is:

$$\hat{\dot{E}}(T, a) = \sum_{j=1}^{\infty} \epsilon_j [\bar{n}_a(\epsilon_j)/L] c,$$

(5)
where we have dropped the subscript on $T_L$, and where $\bar{n}_a(\epsilon)$ is the intermediate statistics thermal-averaged occupation number of the right-moving state with energy $\epsilon$.

\begin{equation}
\bar{n}_a(\epsilon) = \frac{1}{e^{\beta E} - 1} - \frac{a + 1}{e^{\beta E(a+1)} - 1}.
\end{equation}

In the limit $L \to \infty$ (equivalently $\tau \to \infty$), we can replace the sum with an integral over $j$ and, changing integration variables $j \to \epsilon = (h/\tau) j = (hc/L) j$, we have [c.f. Eq. (13) of Ref. 8]:

\begin{equation}
\dot{\bar{E}}(T, a) = \frac{1}{\hbar} \int_0^\infty d\epsilon \epsilon \bar{n}_a(\epsilon).
\end{equation}

A formula for entropy current can be derived as follows. First note that the thermal-averaged occupation energy $\bar{\epsilon} = \epsilon \bar{n}_a(\epsilon)$ and the entropy $\bar{s}$ for a given mode with energy $\epsilon$ are related through the first law: $d\bar{s}/dT = (1/T)d\bar{\epsilon}/dT$. Integrating with respect to temperature and then summing over the right propagating channel modes, we obtain

\begin{equation}
\dot{\bar{S}}(T, a) = -\frac{k_B}{\hbar} \int_0^\infty d\epsilon \epsilon \int_{\beta} d\beta' \frac{\partial \bar{n}_a}{\partial \beta'}.
\end{equation}

The integrals are straightforwardly carried out by noting from (6) that the thermal-averaged occupation energy $\bar{\epsilon} = \epsilon \bar{n}_a(\epsilon)$ of level $\epsilon$ for statistical parameter $a$ is just the difference in the thermal-averaged occupation energies of levels $\epsilon$ and $\epsilon(a+1)$ for bosons. Thus, we require only the integrals for the bosonic case: $\dot{\bar{E}}(T, \infty) = \pi^2 (k_B T)^2/(6\hbar)$ and $\dot{\bar{S}}(T, \infty) = \pi^2 k_B^2 T/(3\hbar)$, giving

\begin{equation}
\dot{\bar{E}}(T, a) = \left(1 - \frac{1}{1 + a}\right) \frac{\pi^2 (k_B T)^2}{6\hbar}
\end{equation}

and

\begin{equation}
\dot{\bar{S}}(T, a) = \left(1 - \frac{1}{1 + a}\right) \frac{\pi^2 k_B^2 T}{3\hbar}.
\end{equation}

Comparing powers of $T$ appearing in Eqs. (8) and (9), and recalling that $\dot{\bar{E}}(T, a) = E_n/\tau$ and $\dot{\bar{S}}(T, a) \sim S(n, a)/\tau$, we learn immediately that $\ln p(n, a) \sim C(a) \sqrt{n}$, where the $n$-independent factor $C(a)$ is given by

\[\text{(9)}\]
Substituting in the expressions (9) and (10) for \( \dot{\bar{E}} \) and \( \dot{\bar{S}} \), respectively, we finally obtain \( C(a) = \sqrt{2a/[3(1+a)]} \), in agreement with Eq. (11).

We will now carry out the same steps as above for particles obeying Haldane’s fractional exclusion statistics\(^9\) to derive the asymptotic approximation to the logarithm of yet another type of partition function, \( \bar{p}(n,g) \), which also interpolates between the unrestricted and distinct partition functions [Eqs. (1) and (2), respectively]. Following the usual conventions, the statistics parameter is denoted by \( g = 1/a \) (so that \( g = 0 \) describes bosons and \( g = 1 \) fermions). Partitions associated with exclusion statistics are subject to additional constraints as compared with partitions associated with intermediate statistics (see below).

The energy and entropy currents for particles obeying exclusion statistics are\(^{10,11}\)

\[
\dot{\bar{E}}(T, g) = \left( k_B T \right)^2 \int_0^\infty dx x \bar{n}_g(x) (12)
\]

and

\[
\dot{\bar{S}}(T, g) = -\left( \frac{k_B T}{h} \right)^2 \int_0^\infty dx \{ \bar{n}_g \ln \bar{n}_g + (1 - g \bar{n}_g) \ln(1 - g \bar{n}_g) 
\]

\[
- [1 + (1 - g) \bar{n}_g] \ln[1 + (1 - g) \bar{n}_g] \}, (13)
\]

where \( x = \beta \epsilon \) and the thermal-averaged occupation number is\(^{12}\)

\[
\bar{n}_g(x) = [w(x) + g]^{-1}, (14)
\]

with the function \( w(x) \) given by the implicit equation

\[
w(x)^g[1 + w(x)]^{1-g} = e^x. (15)
\]

Again, comparing powers of \( T \) appearing in Eqs. (12) and (13), we learn immediately that \( \ln \bar{p}(n,g) \sim \dot{\bar{C}}(g) \sqrt{n} \), where the \( n \)-independent factor \( \dot{\bar{C}}(g) \) is given in terms of \( \dot{\bar{E}} \) and \( \dot{\bar{S}} \) as in Eq. (11). Substituting in the expressions for \( \dot{\bar{E}} \) and \( \dot{\bar{S}} \) and performing a change of variables from \( x \) to \( w \), Eq. (11) becomes after some algebra

\[
C(a) = \frac{\sqrt{\hbar \dot{\bar{S}}(T, a)}}{k_B \sqrt{\dot{\bar{E}}(T, a)}} (11)
\]
\[ \tilde{C}(g) = \frac{s(g)}{\sqrt{e(g)}}, \]  

(16)

where

\[ e(g) = \int_{w_g(0)}^{\infty} dw \frac{1}{w(1+w)} [(1-g) \ln(1+w) + g \ln w] \]  

(17)

and

\[ s(g) = \int_{w_g(0)}^{\infty} dw \left[ \ln(1+w)/w - \ln w/(w+1) \right]. \]  

(18)

Using the identity \( s(g) = 2e(g) \), Eq. (14) can be further simplified to

\[ \tilde{C}(g) = \sqrt{2s(g)}. \]  

(19)

Let us now describe some of the properties and consequences of result (19). Integral (18) can be rewritten in terms of dilogarithms and only for certain choices of lower integration limit do closed-form solutions exist. For example, from (13) we have \( w_{g=0}(0) = 0 \) and \( w_{g=1}(0) = 1 \) and solving the respective integrals, we obtain \( s(0) = \pi^2/3 \) and \( s(1) = \pi^2/6 \). Substituting these values into (14), we indeed obtain the arguments of the exponentials in the asymptotic approximations to the unrestricted and distinct partition functions, Eqs. (11) and (2) respectively. It is tempting to speculate that closed-form solutions to the integral \( s(g) \) exist only for \( g = 0, 1/2, 1/3, 1/4, \) and 1 in the interval \([0,1]\), since it is only for these rational values that Eq. (15) can be solved analytically for the lower integration limit \( w_g(0) \).

For \( g = 1/2 \), we have \( w_{1/2}(0) = (-1 + \sqrt{5})/2 \) and \( s(1/2) = \pi^2/5 \), so that

\[ \ln \tilde{p}(n, 1/2) \sim \pi \sqrt{2/5} \cdot \sqrt{n}. \]  

(20)

Note that \( \tilde{C}_{g=1/2} (= \pi \sqrt{2/5}) < C_{a=2} (= 2\pi/3) \), signalling the fact that \( \tilde{p}(n, g) < p(n, a = 1/g) \) for \( 0 < g < 1 \), a consequence of additional constraints on the allowed partitions associated with Haldane’s statistics. These constraints are discussed in Ref. [14]. The above, closed-form solutions for \( g = 0, g = 1/2, \) and 1 were obtained by solving the integral \( s(g) \) numerically and then noting that the result when divided by \( \pi^2 \) was rational. This method
does not work for the $g = 1/3, 1/4$ cases, however, owing to the complicated form of the lower limits $w_{1/3}(0)$ and $w_{1/4}(0)$ (they are roots of third and fourth degree polynomial equations, respectively). A more sophisticated method is required in order to determine whether or not closed-form solutions exist for these latter two cases.

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