ON A PARTIALLY OVERDETERMINED PROBLEM IN A CONE

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ABSTRACT. We prove a rigidity result for Serrin’s overdetermined problem in a cone that is contained in a half-space in arbitrary dimensions. In the special case where the cone is an epigraph, this result was shown previously in low dimensions with a different approach.

INTRODUCTION AND PROOF OF THE MAIN RESULT

In Corollary 9 of their paper [7], Farina and Valdinoci considered the following partially overdetermined problem in the cone

\[ \Omega = \{ x = (x', x_n) \in \mathbb{R}^n, n \geq 2 : x_n > \alpha |x'| \} \]

with \( \alpha \geq 0 \):

\[ \begin{aligned}
\Delta u + f(u) &= 0, \quad u > 0 \text{ in } \Omega; \\
u &= 0, \quad \partial_\nu u = c \text{ on } \partial \Omega \setminus \{0\},
\end{aligned} \]

where \( \nu \) denotes the exterior unitary normal vector on \( \partial \Omega \setminus \{0\} \), \( c \in \mathbb{R} \) is any fixed constant, and \( f \in C^1(\mathbb{R}) \). It is shown therein that \( \alpha = 0 \), provided that \( n \leq 3 \). It is worth mentioning that the above problem was initially studied in [6], motivated by a question of Vazquez. Their result represents a further extension of the famous Serrin’s problem [12] in unbounded epigraphs, in the spirit of [3]. We point out that the characterization “partially overdetermined” comes from the fact that the overdetermined boundary conditions are not prescribed in the entire \( \partial \Omega \).

In this short note, using a completely different approach, we prove a generalization of this result in any dimension. In fact, as will be apparent, our arguments go through with a bit weaker regularity assumptions on \( f \). Moreover, \( W^{1,\infty}_{\text{loc}}(\Omega) \) suffices for our purposes (see also [3] below). Our approach is greatly motivated from the study of the regularity properties of free boundaries in one-phase and obstacle-type problems, and hinges on the fact that \( \lambda \Omega \equiv \Omega \) for any cone \( \Omega \) with vertex at the origin and \( \lambda > 0 \).

**Theorem 1.** Let \( \Omega \) be an open cone in \( \mathbb{R}^n \), \( n \geq 2 \), that is contained in the half-space \( \{ x_n > 0 \} \) and such that \( \partial \Omega \setminus \{0\} \) has \( C^{1,\beta} \) regularity for some \( \beta > 0 \). Moreover, let \( c, f \) be as above and let \( u \) satisfy (1) and (2). Then, the cone \( \Omega \) coincides with the half-space \( \{ x_n > 0 \} \).

**Proof.** We first consider the case \( c \neq 0 \) i.e. \( c < 0 \).
Motivated from the study of one-phase free boundary problems [1], for \( r > 0 \) small, we consider the following blow-up of \( u \):

\[
u_r(y) = \frac{u(ry)}{r}, \quad y \in \Omega.
\]

We readily find that

\[
\begin{aligned}
\Delta u_r + rf(ru_r) &= 0, \quad u_r > 0 \text{ in } \Omega; \\
u_r &= 0, \quad \partial_n u_r = c \text{ on } \partial \Omega \setminus \{0\}.
\end{aligned}
\]

By virtue of (2), which implies that

\[(3) \quad u(x) \leq C|x| \text{ for some } C > 0 \text{ near the origin (see also [8, Thm. 4.1]),} \]

and standard elliptic estimates [9, Ch. 9-10] (interior \( W^{2,p} \) estimates and boundary \( C^{1,\beta} \) Schauder estimates, keeping in mind that the cone becomes flatter and flatter at infinity), along a sequence \( r_j \to 0 \), \( u_{r_j} \) converges in \( C^1_{\text{loc}}(\Omega \setminus \{0\}) \) to some blow-up limit \( u_0 \in C^2(\Omega) \cap C(\bar{\Omega}) \cap C^1(\bar{\Omega} \setminus \{0\}) \) which solves

\[
\begin{aligned}
\Delta u_0 &= 0, \quad u_0 > 0 \text{ in } \Omega; \\
u_0 &= 0, \quad \partial_n u_0 = c \text{ on } \partial \Omega \setminus \{0\}.
\end{aligned}
\]

We point out that we got a nontrivial limit \( u_0 \) because of the assumption that \( c < 0 \).

By a result of [2], all positive harmonic functions in \( \Omega \) that vanish on \( \partial \Omega \) must be homogeneous, i.e.,

\[
u_0(y) = |y|^{\gamma} \Phi_0 \left( \frac{y}{|y|} \right),
\]

for some \( \gamma > 0 \) and \( \Phi_0 \in C(\mathbb{S}^{n-1} \cap \Omega) \cap C^2(\mathbb{S}^{n-1} \cap \Omega) \) which vanishes on \( \mathbb{S}^{n-1} \cap \partial \Omega \). Observe that, since \( \nabla u_0 \) is a homogeneous function of degree \( \gamma - 1 \), in order for the overdetermined boundary conditions in (4) to be satisfied, we must have

\[
\gamma = 1.
\]

So, \( \Phi_0 \) is a positive eigenfunction with Dirichlet boundary conditions to the Laplace-Beltrami operator \( -\Delta_{\mathbb{S}^{n-1}} \) on \( \mathbb{S}^{n-1} \cap \Omega \), corresponding to the eigenvalue \( n - 1 \) (see for instance the proof of [3 Lem. 2.1]). Hence, recalling that \( n - 1 \) is the principal Dirichlet eigenvalue of \( -\Delta_{\mathbb{S}^{n-1}} \) on the upper half-sphere \( \mathbb{S}^n_+ \), we deduce that

\[
|\mathbb{S}^{n-1} \cap \Omega| = |\mathbb{S}^{n-1}_+| = \frac{1}{2} |\mathbb{S}^{n-1}|.
\]

The above relation, however, is only possible if \( \Omega \) coincides with the half-space \( \{x_n > 0\} \) as desired.

It remains to consider the case

\[
c = 0.
\]

Firstly, by Hopf’s boundary point lemma (at some point on \( \partial \Omega \setminus \{0\} \) which belongs to the boundary of a ball contained in \( \Omega \)), we deduce that

\[(5) \quad f(0) < 0.\]
It follows readily from (1) and (2) that \( u \) (extended trivially outside of \( \Omega \)) satisfies, in the weak sense, the following problem:

\[
\Delta u = -H(u)f(u) \quad \text{in } \mathbb{R}^n,
\]

where \( H \) stands for the usual Heaviside function. We just point out that near the origin one uses that

\[
\lim_{\rho \to 0} \int_{B_{\rho}(0)} \nabla u \nabla \varphi \, dx = 0,
\]

for any \( \varphi \in C_0^\infty(\mathbb{R}^n) \), which holds since \( u \in W^{1,\infty} \) (see also [5, Thm. 1.3] for a related argument).

Using (5) and the assumed regularity on \( f, u \), it follows from [10, Ch. 2] that

\[
u \in C^{1,1}(B_\delta(0)) \quad \text{for some small } \delta > 0,
\]

(see also [11] for a more general approach).

This time, as in the study of free boundary problems of obstacle type [10], for \( r > 0 \) small, we consider the following blow-up of \( u \):

\[
u_r(y) = \frac{u(ry)}{r^2}, \quad y \in \Omega.
\]

We readily find that

\[
\begin{cases}
\Delta u_r + f(r^2u_r) = 0, & u_r > 0 \text{ in } \Omega; \\
u_r = 0, & \nabla u_r = 0 \text{ on } \partial \Omega \setminus \{0\}.
\end{cases}
\]

By virtue of (7) and standard elliptic estimates, along a sequence \( r_j \to 0 \), \( u_{r_j} \) converges in \( C^1_{loc}(\mathbb{R}^n) \) to some blow-up limit \( u_0 \in C^2(\Omega) \cap C^{1,1}(\mathbb{R}^n) \) (globally) which satisfies

\[
\begin{cases}
\Delta u_0 = -f(0), & u_0 > 0 \text{ in } \Omega; \\
u_0 = 0, \ n & \nabla u_0 = 0 \text{ on } \partial \Omega \setminus \{0\}.
\end{cases}
\]

We point out that we got a nontrivial limit \( u_0 \) because of (5).

Let \( e \) be an arbitrary direction in \( \mathbb{R}^n \) and let

\[
v = \partial_e u_0.
\]

Then, the function \( v \) is harmonic in the cone \( \Omega \), vanishes on its boundary, and is globally Lipschitz continuous. To conclude, let us suppose, to the contrary, that the cone \( \Omega \) does not coincide with the half-space \( \{x_n > 0\} \). Then, since \( v \) has at most linear growth, we get from Lemma 2.1 in [3] and the comments after it that

\[
v \equiv 0.
\]

On the other hand, recalling that the direction \( e \) is arbitrary, this contradicts the fact that \( u_0 \) is nontrivial.

The proof of the theorem is complete. \( \square \)

**Remark 1.** The initial value problem

\[
\ddot{u} = -f(u); \quad u(0) = 0, \ \dot{u}(0) = c,
\]

has a unique local solution \( U \) (not necessarily positive) with maximal interval of existence \([0, T), T \leq +\infty\). By Hopf’s boundary lemma, applied to the difference \( u(x) - U(x_n) \), we infer that \( u \) coincides with \( U \) for \( 0 \leq x_n < T \). In particular, the partial derivatives \( \partial_x u, i < n \)
are identically zero in this strip, and thus in the entire half-space \( \{ x_n > 0 \} \) (by the unique continuation principle applied in the linear equation that each one satisfies). Consequently, \( u \) depends only on \( x_n \).

**Remark 2.** It is well known that, in any dimension \( n \geq 3 \), there exists an \( \alpha_n > 0 \) such that the cone described by \( |x_n| < \alpha_n |x'| \) supports a one-homogeneous solution to (1) with \( f \equiv 0 \), see [4] [4].

**Remark 3.** Our assumption that the boundary of the cone is smooth except from its vertex is not restrictive by means of a dimension reduction argument as in [13].

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