Partial-wave decomposition of the Faddeev amplitude of a Poincaré-invariant three-fermion bound state

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March 30, 2022

1 Motivation

The composite nature of the nucleon can be considered as the main source of motivation to study of the relativistic three-fermion bound state problem. On the other hand, this compositeness is also the source of our still incomplete understanding of the nucleon structure. Although protons and neutrons (together with electrons) form the building blocks of matter most of their properties are poorly understood. E.g. the experimental results on the spin structure of the proton have been so surprising to model builders that they named the problem of explaining them the proton spin crisis, see [1, 2] and references therein.

The aim of this lecture is much more modest than trying to explain the proton’s substructure in terms of quarks and gluons. Taking the simplest relativistic and also realistic approach to model nucleons, namely, a Poincaré covariant Faddeev approach to describe the binding of three valence quarks, we will demonstrate that even with relativistic valence quarks only the nucleon has quite a rich structure embodied in its wave function. This will be exemplified in the nucleon’s rest frame by a decomposition into partial waves w.r.t. the motion of one of the valence quark relative to the complementary pair of quarks. As will be seen this analysis (without referring to a specific dynamical model) also answers the question whether the nucleon is spherically symmetric: It is not – due to the highly relativistic motion of quarks within the nucleon.

∗Talk given by R.A. at the 43rd International Winter School on Theoretical Physics, Schladming, Styria, Austria, February 26 - March 4, 2005
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2 Spin of elementary particles

To the best of our knowledge quarks are pointlike Dirac fermions with spin $\frac{1}{2}$. It will prove helpful for the following to recall a few facts about their relativistic description based on the solution of Dirac’s equation. First, Dirac wavefunctions are four-component spinors. The physical reason for this is the simultaneous description of particles and antiparticles. These four-component spinors are reduced to two components by a projection onto positive energy states yielding a formalism akin to Pauli’s equation. Second, in the rest frame (in standard representation) the lower two components vanish. Third, the upper (“nonrelativistic”) and the lower (“relativistic”) components carry different angular momentum, e.g. the lower components represent a $p$-wave if the upper component corresponds to an $s$-wave. Fourth, interactions between the fermions can be incorporated unambiguously, the prime example is the causal coupling to the electromagnetic field.

Coupling the three spin $\frac{1}{2}$ quarks to a composite spin $\frac{1}{2}$ nucleon such that Poincaré covariance is maintained we will see that [3, 4, 5]:

- due to the compositeness, we need more components than four in total or two for the positive energy states.
- the lower components will not vanish in the rest frame thus giving rise to the unavoidable presence of at least a (relativistic) $p$-wave contribution.
- the difference of one angular momentum unit between upper and lower components remains, however, there will be also a $d$-wave contribution.
- the coupling to the electromagnetic field can be chosen such to maintain causality, however, at the expense of a fairly complicated structure of the nucleon–photon vertex containing one– and two-loop contributions.

As a remark we want to mention that three quarks may couple to form a total spin $\frac{3}{2}$, and, of course, they should do so to form the $\Delta$ baryon. However, already for elementary spin–$\frac{3}{2}$ objects, the Rarita-Schwinger fields, there are a number of problems: First, the corresponding spinors have sixteen components but only eight of them are physical. Second, a “non-relativistic” limit does not exist. Third, (if the field is not part of a supergravity multiplet) interactions are not well-defined, e.g. the coupling to the electromagnetic field is not causal. From this point of view it may appear surprising that within the Poincaré covariant Faddeev approach $\Delta$ baryons can be described [3, 5] – which can be understood, however, from the finite extension of these composite objects.

3 Relativistic angular momentum: Pauli-Lubanski vector

In a nonrelativistic setting angular momentum is defined w.r.t. a fixed origin. This makes plain why the concept of angular momentum has to be generalized properly in a relativistic setting. Formally one sees the effect of relativity from the fact that the the Casimir operator of the (non-relativistic) rotation group, $J^2$, does not commute with boosts.
Describing the angular momentum with the help of a vector operator orthogonal to the particle momentum will cure the underlying problem. Thus we will start our considerations from the Pauli-Lubanski (axial-)vector:

$$W_{\mu} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} P^\sigma$$

where $J^{\mu\nu}$ is the Noether charge of rotations and boosts. We note that

$$C_2 = W_{\mu}W^\mu = m^2 j(j + 1)$$

is a (second) Casimir invariant of the Poincaré group, and that in the rest frame it reduces to a quantity proportional to the usual spin:

$$W_{\mu} = (0, \vec{W}) \quad W_i = -\frac{1}{2} \epsilon_{ijk} J^{jk} P^0 = -m \Sigma_i .$$

### 4 Three-fermion states:

**Partial wave decomposition in rest frame**

For the problem at hand we can reduce the complexity by noting that in a baryon every quark pair is in a colour antitriplet state, and that the corresponding interaction between the quarks is attractive. (This can be surmised by calculating the group-theoretical factors for a one–gluon exchange diagram; additionally, the attraction has been corroborated by lattice calculations.) Considering only states with vanishing orbital angular momentum these quark pairs form scalar (spin 0) and axialvector (spin 1) “diquarks”. Furthermore, the Pauli principle requires flavour antisymmetry for scalar and flavour symmetry for axialvector “diquarks”.

To obtain a Poincaré covariant Faddeev equation we consider Dyson’s equation for the 6-point function and neglect genuinely irreducible three-quark interactions in its kernel. This leads to the equation depicted diagrammatically in Fig. 1.

By assuming that the two-body $t$-matrix can be approximated well by a finite number of separable quark–quark correlations\(^1\) into the Poincaré covariant Faddeev equation is mapped to a set of coupled Bethe–Salpeter equations. Its structure is diagrammatically represented in Fig. 2. The corresponding interaction is quark exchange. This reinstates the Pauli principle at the level of all three valence quarks. As the colour quantum number is antisymmetric, and thus all other quantum numbers are symmetric, the correlations induced by the Pauli principle amount to an attractive interaction (as confirmed by the group–theoretical factor in Fig. 2).

\(^1\)A rigorous separability expansion generates actually an infinite number of terms out of which we consider the (presumably) dominant ones.
Figure 2: Pictorial representation of the coupled quark-diquark Bethe–Salpeter equations.

Figure 3: A pictorial representation of the decomposition of the upper components of the nucleon’s bispinors.

At this point we note that through our quite drastically simplifying assumptions we have reduced the number of wave function components for the composite nucleon from $4^3 = 64$ to still $4 \cdot (1_{\text{sc},dq} \cdot 3_{\text{ax},dq}) = 16$ components. A projection onto positive energy reduces the number of components to eight which can be grouped into a set of four bispinors consisting of an upper and a lower component similar to Dirac spinors.

To assess their physical properties we will first present a partial wave decomposition in the rest frame. Note that due to the fact that neither orbital angular momentum nor spin are good quantum numbers independent of the frame the following discussion is specific to this frame. First, we decompose the Pauli-Lubanski vector into orbital and spin part

$$W_i = L_i + S_i = \frac{1}{2} \epsilon_{ijk} (L^{jk} + S^{jk})$$

(4)

$$L^{jk} = \left( p' \frac{\partial}{\partial p^k} - p^k \frac{\partial}{\partial p'} \right),$$

(5)

$$S^{jk} = \frac{1}{2} \sigma^{jk} \otimes 1 \otimes 1 + \text{permutations}.$$ (6)

Here, $p$ is the relative momentum between the quark and the “diquark” composite. Second, we apply these operators onto the Bethe–Salpeter wave functions

$$L^i L^j \Phi_{\alpha \beta \gamma} = l(l+1) \Phi_{\alpha \beta \gamma},$$

(7)

$$S^i S^j \Phi_{\alpha \beta \gamma} = s(s+1) \Phi_{\alpha \beta \gamma},$$

(8)
Table 1: The eight components of the nucleon’s Faddeev amplitude derived with the simplifications described in the text, given as trispinors $\Phi_{\alpha\beta\gamma} = U^i_{\alpha}(\gamma^i C)_{\beta\gamma}$. Scalar diquark correlations correspond to $i = 5$, axialvector ones to $i \equiv \mu = 1 \ldots 4$.

| nucleon wave function components in the rest frame | eigenvalue $l(l+1)$ of $L^2$ | eigenvalue $s(s+1)$ of $S^2$ |
|-----------------------------------------------|-------------------|-------------------|
| $S_1 u(\gamma_5 C) = (\chi_0)(\gamma_5 C)$ | 0 s $\frac{3}{4}$ | |
| $S_2 u(\gamma_5 C) = (\frac{i}{\sqrt{2}}(\sigma_\mu)\chi)(\gamma^\mu C)$ | 2 p $\frac{3}{4}$ | |
| $A_1^\mu u(\gamma^\mu C) = \hat{P}_0 (\frac{i}{\sqrt{2}}(\sigma_\mu)\chi)(\gamma^i C)$ | 2 p $\frac{3}{4}$ | |
| $A_2^\mu u(\gamma^\mu C) = \hat{P}_0 (0)(\gamma^i C)$ | 0 s $\frac{3}{4}$ | |
| $B_1^\mu u(\gamma^\mu C) = (i\sigma^\iota\chi)(\gamma^i C)$ | 0 s $\frac{3}{4}$ | |
| $B_2^\mu u(\gamma^\mu C) = (\frac{i}{\sqrt{2}}(\sigma_\mu)\chi)(\gamma^i C)$ | 2 p $\frac{3}{4}$ | |
| $C_1^\mu u(\gamma^\mu C) = (i(\hat{P}^i(\sigma_\mu) - \frac{4}{3}\sigma^i)\chi)(\gamma^i C)$ | 6 d $\frac{15}{4}$ | |
| $C_2^\mu u(\gamma^\mu C) = (\frac{i}{\sqrt{2}}(\sigma_\mu)\chi)(\gamma^i C)$ | 2 p $\frac{15}{4}$ | |

To obtain for the upper components three $s$-waves and one $d$-wave, the lower components being four $p$-waves. A pictorial representation of the decomposition of the upper components is given in Fig. 3.

5 Nucleons: Poincaré-covariant amplitudes

In the Bethe–Salpeter wave functions there appear two independent momenta, the total (nucleon) momentum $P$ and the quark-diquark relative momentum $p$. Even in the rest frame the lower components are non-vanishing. Similar as in Dirac’s case the lower components carry Pauli spin matrices but now these are contracted with the relative momentum, see Table 1. Thus we may summarize shortly: A minimum of eight components is needed to describe the nucleon as Poincaré-covariant bound state of three quarks. The lower components of this bispinorial quantities do NOT vanish, even not in the rest frame. From this we conclude that if no mysterious cancelations occur the nucleon is a non-spherical,
The construction of the electromagnetic coupling of the nucleon within this approach can be found in the literature, see e.g. Refs. [4, 6]. Hereby the photon does not only couple to the nucleons' constituents but also to the exchange quark and the quark-diquark vertex functions. The corresponding terms can be either derived from the electromagnetic Ward identity [4] or one employs the gauging-of-equations method [6] which guarantees the validity of the Ward identity from the beginning.

Calculations of the nucleons' electromagnetic form factors employing these amplitudes and the consistent coupling of the photon are in agreement with experimental data for larger momentum transfers [7, 8, 9]. A failure at lower $Q^2$ had to be expected as mesonic contributions are not present in this description. However, as mesons are composite objects their contribution dies out fast for $Q^2$ above one to two GeV$^2$. A prediction of Ref. [9] is a zero of the proton's electric form factor at $Q^2 \approx 8$ GeV$^2$. In this calculation this zero can be traced back to the above described spinorial structure. Thus its experimental verification or falsification may constitute a test of the considerations given above.

Acknowledgements

Support by a grant from the Ministry of Science, Research and the Arts of Baden-Württemberg (Az: 24-7532.23-19-18/1 and 24-7532.23-19-18/2) is acknowledged.

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Rotational symmetry is of course guaranteed by the fact that all possible orientations are energetically degenerate, cf. the physics of deformed nuclei.