SUPERSYMMETRIC BALANCE OF FORCES
AND CONDENSATION OF BPS STATES

Renata Kallosh and Andrei Linde

Physics Department, Stanford University, Stanford CA 94305

ABSTRACT

Until now all known static multi black hole solutions described BPS states with charges of the same sign. Such solutions could not be related to flat directions in the space of BPS states: The total number of such states could not spontaneously increase because of the charge conservation.

We show that there exist static BPS configurations which remain in equilibrium even if they consist of states with opposite electric (or magnetic) charges from vector multiplets. This is possible because of the exact cancellation between the Coulomb and scalar forces. In particular, in the theories with N=4 or N=2 supersymmetry there exist stable massless multi center configurations with vanishing total charge. Since such configurations have vanishing energy and charge independently of their number, they can be associated with flat directions in the space of all possible BPS states. For N=2 case this provides a realization of the idea that BPS condensates could relate to each other different vacua of the string theory.

\(^1\)E-mail: kallosh@physics.stanford.edu
\(^2\)E-mail: linde@physics.stanford.edu
1 Introduction

Recently it was found that there exist supersymmetric BPS states with vanishing ADM mass. They include the solutions with one half of the N=4 \cite{1, 2, 3, 4} or N=2 \cite{5} supersymmetry unbroken. As usual for supersymmetric configurations, the multi-center solutions are also available \cite{2}. Those solutions have various extraordinary features; in particular they do not represent an extreme limit of any known non-extreme black holes. When considered in the four-dimensional canonical frame, they are singular and the singularity has a universal repulsive character. This is why it is not appropriate to call them "black" holes, and we called them "white" holes, or repulsons \cite{6}. In what follows we will try to avoid attributing any color to the massless supersymmetric configurations under study to minimize the abuse of terminology \cite{3}, but we will keep calling them holes, or massless BPS states.

The existence of such solutions may look surprising \cite{6}. However one should take into account several considerations:

i) The existence of such states in exact quantum theory is expected according to conjectures of dualities and enhanced symmetries \cite{7, 8, 9}.

ii) BPS states of extended supersymmetry, if they exist as solutions of classical theory, may remain exact in quantum theory, at least for N=4 case according to supersymmetric non-renormalization theorems.

iii) If they would not be found at least one of the standard lores i) or ii) would be incorrect.

With all this in view one may take the following attitude. Supersymmetric black&white holes may have naked singularities and a better description of physical states of supersymmetric gravity is desirable. However, so far they served well by explicitly realizing various mass formulas for BPS states and mass and charge relations. The present investigation will show again that the new type of mass-charge relations which are believed to remain exact in quantum theory can be discovered when new soliton configurations are found.

The purpose of this note is to exhibit some dramatic differences between new, massless supersymmetric configurations and the ones which are known as extreme massive multi black hole solutions. The difference is due to the fact that massless solutions exist in case that we have more than one supersymmetric multiplet involved in the solution, let us say $1+n$ multiplets. This leads to the existence of $1+n$ independent supersymmetric balance of forces conditions. The analysis of these conditions shows that massless BPS states may consist of the hole-anti-hole configurations with the vanishing net charge which are in equilibrium. We will indeed find the multi-center BPS configurations with the vanishing total mass and charge. The mass of every hole vanishes. However, the sign of the charges with respect to vector fields other than graviphoton
may alternate from one hole to another. One may argue, therefore that the massless holes may form a condensate. The argument for this is simple: if there is a state with $s$ holes which form a massless neutral set and the other state with $s+t$ holes which are also massless and neutral, it will not cost any energy to produce them, and no charge conservation will be violated when the system changes from an $s$-hole state to $(s+t)$-hole state. In particular, the total number of massless neutral clusters of $s$ holes may take any value. The effective theory will have a non-trivial ground state with the massless mode corresponding to the condensation of the neutral massless BPS configurations.

We would like to explain here under which conditions the massless neutral solutions can be found. Four-dimensional theories with one gravitational multiplet do not have non-trivial massless BPS configurations. Indeed, with one multiplet of $N=4$ supersymmetry any solution is characterized by the mass, which due to supersymmetry is related to the graviphoton charge and to the dilaton charge. When this parameter vanishes the solution becomes trivial. With one gravitational multiplet of $N=2$ supersymmetry we have the relation between the mass and the graviphoton charge. When the mass goes to zero, the charge goes to zero.

In $N=8$ case only the gravitational multiplet exists. It is characterized by one parameter and when the mass goes to zero, only trivial solutions can be expected. Unless more than one half of $N=8$ supersymmetry is broken, one cannot expect any interesting massless solutions.

The situation changes for $N=4$ theory since one can consider the interaction of $N=4$ supergravity with any number of $N=4$ vector multiplets. In particular, one can consider the theory with 22 $N=4$ vector multiplets. Together with 6 graviphotons in the gravitational multiplet they form the original 28 vectors which previously were all in one gravitational multiplet of $N=8$ supergravity. However, when the symmetry between 6 and 22 is broken by placing them into different multiplets, we may expect that there exists a configuration with 6 graviphoton charges vanishing but with 22 non-vanishing vector multiplet charges. This is indeed the case [1, 2, 3, 4].

If even more supersymmetries are broken and we are looking for configurations with one half of $N=2$ supersymmetry unbroken, we may consider one gravitational multiplet of $N=2$ supergravity interacting with some number of vector multiplets of $N=2$ supersymmetry. During this step of breaking supersymmetry, the dilaton which was in one multiplet with graviton is now in a different multiplet, in a vector one. The dilaton charge, as well as the charges of the vector fields in the vector multiplets, are not related to the graviphoton charge anymore. Therefore one may have expected that when supersymmetric solutions in this theory will be found, that they will remain non-trivial when the mass of the configuration will tend to zero, since the breaking of supersymmetry has made the charges of some vectors and scalars not related to that of the graviphoton anymore. This is indeed the case: there is a rich variety of massless solutions in $N=2$ theory [5].
2 Supersymmetric balance of forces for massive extreme black holes: gravitational multiplet solutions

Consider an example of supersymmetric balance of forces between two $a=1$ stringy black holes of Gibbons-Maeda-Garfinkle-Horowitz-Strominger. These solutions are supersymmetric when embedded into pure N=4 supergravity without vector multiplets. Any extreme solution in this class with one half of N=4 supersymmetry unbroken saturate the following supersymmetric bound [10] (we consider the electric solutions first):

$$M^2 + (\Sigma_{dil})^2 - (Q_{gr})^2 = 0 .$$  \hspace{1cm} (1)

The mass $M$ is related to the graviphoton charge $Q_{gr}$ and to the dilaton charge $\Sigma_{dil}$ as follows:

$$M = \frac{|Q_{gr}|}{\sqrt{2}} = -\Sigma_{dil} .$$  \hspace{1cm} (2)

If there are two supersymmetric black holes, each of them has to saturate the same type of bounds:

$$m_1^2 + \sigma_1^2 - q_1^2 = 0 ,$$

$$m_2^2 + \sigma_2^2 - q_2^2 = 0 .$$  \hspace{1cm} (3)

For each multiplet we have the same set of relations between charges in every hole.

$$m_1 = -\sigma_1 = \frac{|q_1|}{\sqrt{2}} ,$$

$$m_2 = -\sigma_2 = \frac{|q_2|}{\sqrt{2}} .$$  \hspace{1cm} (4)

We can study the forces between two such black holes and find whether the holes are in equilibrium. Consider Newtonian, Coulomb and dilatonic forces between two distant objects of masses and charges $(m_1, q_1, \sigma_1)$ and $(m_2, q_2, \sigma_2)$:

$$F_{12} = -\frac{m_1 m_2}{r_{12}^2} + \frac{q_1 q_2}{r_{12}^2} - \frac{\sigma_1 \sigma_2}{r_{12}^2} .$$  \hspace{1cm} (5)

The dilatonic force is attractive for charges of the same sign and repulsive for charges of opposite sign. For our configurations the dilaton charge does not change the sign when the electric charge does. Therefore if the electric charges of the two black holes are of the same sign we get

$$F_{12} = -\frac{m_1 m_2}{r_{12}^2} + \frac{|q_1||q_2|}{r_{12}^2} - \frac{\sigma_1 \sigma_2}{r_{12}^2} .$$  \hspace{1cm} (6)
According to eq. (4),
\[ F_{12} = -\frac{m_1 m_2}{r_{12}^2} (1 - 2 + 1) = 0. \] (7)
Thus we see that \( F_{12} \) vanishes and we conclude that two holes with the same sign of the electric charge are in equilibrium with each other, since the attractive force between them due to gravity and dilaton force is cancelled by Coulomb repulsion of two equal sign charges.

Now consider the situation when the electric charges have opposite sign. All three forces are attractive this time:
\[ F_{12} = -\frac{m_1 m_2}{r_{12}^2} (1 + 2 + 1) = \frac{4m_1 m_2}{r_{12}^2} \neq 0. \] (8)
This configuration is unstable since there is no balance of forces. Instead of performing this balance of forces analysis one could simply look into explicit form of the two-hole solution \[10].
There is no static solution available for two holes with positive masses and with opposite electric charges.

For a magnetic BPS state with the mass \( M \), the magnetic graviphoton charge \( P_{gr} \) and the dilaton charge \( \Sigma_{dil} \) the relation between charges is similar to (2):
\[ M = \left| P_{gr} \right| \sqrt{2} = \Sigma_{dil}. \] (9)
The conclusions of the balance of force analysis would be the same: one can get two holes in equilibrium under the condition that they have magnetic charges of the same sign.

Thus one could have concluded that two supersymmetric black holes may be in equilibrium only when their charges have the same sign. Therefore the sum of all masses of BPS states in equilibrium and the absolute value of their graviphoton charges is always positive:
\[ \sum_{a=1}^{n} m_a > 0, \quad \left| \sum_{a=1}^{n} (q_{gr})_a \right| > 0. \] (10)

This is where the massless configurations brought a new surprise.

### 3 Supersymmetric balance of forces: gravitational and vector multiplet solutions

As explained above we have explicit solutions with vanishing ADM mass only in case of few multiplets, one gravitational and \( n \) vector multiplets. In case of N=4 supersymmetry our solutions saturate \( 1 + n \) independent bounds, one for each multiplet.
\[ M^2 + \left( \Sigma_{dil} \right)^2 - (Q_{gr})^2 = 0, \] (11)
\[ (\Sigma_{vec}^I)^2 - (Q_{vec}^I)^2 = 0, \quad I = 1, \ldots, n. \] (12)
The first bound relates three types of charges: the mass, the dilaton charge and the electric charge of the graviphoton in the gravitational multiplet. Every other bound relates the modulus charge $\Sigma_{\text{vec}}$ with the vector charge $Q_{\text{vec}}$ inside each vector multiplet.

The nature of the second type of supersymmetric bounds turned out to be very different from what may have been expected. The important property of the supersymmetric solutions is the following: the sign of the charge of the modulus field $\Sigma^I_{\text{vec}}$ is correlated with the sign of the charge of the electric field of the same vector multiplet.

$$\Sigma^I_{\text{vec}} = Q^I_{\text{vec}} .$$  \hspace{1cm} (13)

When the sign of the vector charge changes, the sign of the scalar charge also changes. Therefore it is possible to consider the situation when there are two massive black holes, saturating all bounds. The picture described above for one gravitational multiplet remains intact: the graviphoton charge must always be of the same sign for two holes. However, the vector multiplet charges do not have to be constrained that way. Each black hole has to saturate its own vector multiplet bound.

$$(\sigma^I_{\text{vec}})_1^2 - (q^I_{\text{vec}})_1^2 = 0 , \quad (\sigma^I_{\text{vec}})_2^2 - (q^I_{\text{vec}})_2^2 = 0 .$$  \hspace{1cm} (14)

Given that the vector charge of every hole equals its scalar charge,

$$(\sigma^I_{\text{vec}})_1 = (q^I_{\text{vec}})_1 , \quad (\sigma^I_{\text{vec}})_2 = (q^I_{\text{vec}})_2 ,$$  \hspace{1cm} (15)

all bounds are saturated and there is no restriction on the relative sign of vector charges of two holes. There is always the balance of forces: whether we have the Coulomb attraction for opposite sign electric charges or Coulomb repulsion for same sign electric charges, we get the same picture from modulus field force. Since the sign of the scalar charge is correlated with the sign of the vector charge the corresponding part of the force between two holes vanishes

$$F^i_{12} = \frac{(q^I_{\text{vec}})_1(q^I_{\text{vec}})_2}{r^2_{12}} - \frac{(\sigma^I_{\text{vec}})_1(\sigma^I_{\text{vec}})_2}{r^2_{12}} = 0 .$$  \hspace{1cm} (16)

Thus we have found that for the massive extreme black holes with graviphoton charges as well as with vector multiplet charges there are two possible configurations in equilibrium: The sign of graviphoton charge for two holes is always the same, but the sign of the vector multiplet charge may or may not alternate between two holes. One may have either

$$(q_{\text{gr}})_1(q_{\text{gr}})_2 > 0 , \quad (q^I_{\text{vec}})_1(q^I_{\text{vec}})_2 > 0 ,$$  \hspace{1cm} (17)

or

$$(q_{\text{gr}})_1(q_{\text{gr}})_2 > 0 , \quad (q^I_{\text{vec}})_1(q^I_{\text{vec}})_2 < 0 .$$  \hspace{1cm} (18)

In N=2 case we have found extreme black hole solutions which saturate the gravitational bound as well as the vector multiplet bound:

$$M^2 - (Q_{\text{gr}})^2 = 0 ,$$  \hspace{1cm} (19)

$$\Sigma^I_{\text{vec}} - (Q^I_{\text{vec}})^2 = 0 , \quad I = 1, \ldots, n.$$  \hspace{1cm} (20)
The difference from N=4 case is in the structure of the gravitational bound which does not include the dilaton charge anymore. It is saturated when $M = |Q_{\text{gr}}|$. The dilaton (in heterotic theory) is now placed in one of the vector multiplets and as such, behave as any other scalar of the vector multiplets when the vector charge changes the sign. Alternatively, from type II string theory point of view, the dilaton is in one of the hypermultiplets. Apart from this there is no difference in the picture of balance of forces, as discussed above: the graviphoton charge splits into the same sign charges whereas the vector multiplet charge may split into same sign or opposite sign charges.

The crucial feature of the configuration with the vanishing mass, dilaton charge and graviphoton charge is the fact that the gravitational supersymmetric bound is saturated trivially for each hole. There is no gravitational, dilaton and electric graviton force to be compensated, all of them vanish in order $\frac{1}{r^2}$. Thus the conclusion from the balance of forces study is: the configurations with the vanishing total mass and all charges may exist.

\begin{equation}
M = \sum_{a=1}^{n} m_a = 0, \quad Q_{\text{gr}} = \sum_{a=1}^{n} (q_{\text{gr}})_a = 0, \quad Q^I_{\text{vec}} = \sum_{a=1}^{n} (q^I_{\text{vec}})_a = 0.
\end{equation}

The massless solutions with the vanishing vector multiplet charge $Q^I_{\text{vec}} = 0$ in all gauge groups were not known to exist before. The mechanism of the black hole condensation proposed by Greene, Morrison and Strominger [13] (GMS mechanism) is based on the assumption that such configurations exist. Our analysis of the balance of forces proves that they may exist. In what follows we will describe them.

## 4 Hole-anti-hole solution in N=4 theory

We would like to consider first the two-hole and 2s-hole supersymmetric solution which may have an interpretation of the particle-antiparticle state. The class of massive black holes with non-vanishing graviphoton charge will not allow us to find such, as explained above. However, if we use directly the massless solutions, those with two opposite charges can be found. Starting with the Maharana-Schwarz-Sen action for toroidally compactified heterotic string theory we can get a two-center supersymmetric massless solution by slightly generalizing the construction presented in [11].

The generic pure magnetic $(6,22)$ symmetric solution is described completely in terms of magnetic potentials $\vec{\chi}$:

\begin{equation}
\vec{\chi}(x) = \begin{pmatrix} \vec{\chi}_{\text{vec}}(x) \\ \vec{\chi}_{\text{gr}}(x) \end{pmatrix}, \quad \partial_i \partial_i \vec{\chi}(x) = 0.
\end{equation}

The 28-dimensional harmonic $O(6,22)$-vector $\vec{\chi}$ consists of the 22-dimensional vector $\vec{\chi}_{\text{vec}}$, or $\chi^I$, $I = 1, \ldots, 22$, describing the vector multiplets and of the 6-dimensional vector $\vec{\chi}_{\text{gr}}(x)$, or...
\( x^\alpha, \alpha = 1, \ldots 6 \), describing the gravitational multiplet. The metric, the dilaton, the moduli fields \( M \) and the magnetic field \( \vec{H}_i = \frac{1}{2} \epsilon_{ijk} \vec{F}_{jk} \) are

\[
ds^2_{\text{can}} = -e^{2U} dt^2 + e^{-2U} \, dx^2,
\]

\[
e^{-4U} = 2 \left( (\chi^{\alpha})^2 - (\chi^I)^2 \right) = e^{4\phi},
\]

\[
\mathcal{M} = 1_{28} + 4e^{4U} \left( \chi^I \chi^J \left( \xi \chi^\alpha \chi^{\beta} \right) \right), \quad \vec{H}_i = \partial_i \vec{\chi},
\]

where \( \xi \equiv (\chi^I)^2 / (\chi^m)^2 \).

Consider the simplest case of asymptotically flat geometry and vanishing at infinity scalar fields. The massless solution with \( a = 2s \) holes and vanishing net charge for each of the 28 gauge groups is given by

\[
\chi^I = \sum_{a=1}^{a=2s} \frac{q^I_a}{|\vec{x} - \vec{x}_a|}, \quad \chi^\alpha = \frac{1}{\sqrt{2}} n^\alpha, \quad (n^\alpha)^2 = 1,
\]

\[
Q^I \equiv \sum_a q^I_a = 0, \quad m_a = 0, \quad M = \sum m_a = 0.
\]

The graviphoton charge vanishes for each hole since each of them is massless. However, the vector multiplet charge of each individual hole does not vanish, its sign alternates and only the total sum over all holes vanishes. For example, the simplest monopole-anti-monopole solution has two holes of the opposite charge,

\[
\chi^I = \frac{q^I_1}{|\vec{x} - \vec{x}_1|} - \frac{q^I_2}{|\vec{x} - \vec{x}_2|}, \quad \chi^\alpha = \frac{1}{\sqrt{2}} n^\alpha, \quad (n^\alpha)^2 = 1,
\]

\[
Q^I \equiv q^I_1 + q^I_2 = 0, \quad m_1 = m_2 = 0, \quad M = m_1 + m_2 = 0.
\]

To confirm our balance of force condition analysis we must check that the moduli field charges indeed compensate the Coulomb forces between the monopole-anti-monopole pair. For this purpose we will write down the total solution, corresponding to the potential of the pair above. We have for the metric, the dilaton and the magnetic fields:

\[
e^{-4U} = 1 - 2 \left( \frac{q^I_1}{|\vec{x} - \vec{x}_1|} - \frac{q^I_2}{|\vec{x} - \vec{x}_2|} \right)^2 = e^{4\phi}, \quad H_i^a = \frac{(x - x_2)^i q^I}{|\vec{x} - \vec{x}_2|^3} - \frac{(x - x_1)^i q^I}{|\vec{x} - \vec{x}_1|^3},
\]

and there are no graviphoton magnetic fields, \( H_a^i = 0 \). The moduli fields are given in eq. (23). We are particularly interested here in the moduli field charges. Therefore we will write down explicitly only the terms required for defining the scalar field charges. Those are

\[
\mathcal{M} = 1_{28} + 2\sqrt{2} \left( \begin{array}{cc}
0 & n^\beta \left( \frac{q^I}{|\vec{x} - \vec{x}_1|} - \frac{q^I}{|\vec{x} - \vec{x}_2|} \right) \\
n^\alpha \left( \frac{q^I}{|\vec{x} - \vec{x}_1|} - \frac{q^I}{|\vec{x} - \vec{x}_2|} \right) & 0 \end{array} \right) + \ldots
\]

\( \text{This solution with same sign charges was found in [11], however it remained unnoticed that alternating signs are also possible.} \)
Here ... stays for terms which will not contribute to scalar charges of each monopole. To find the charges of the $\mathcal{M}^{I\beta}$ and $\mathcal{M}^{\alpha J}$ we will take the following two limits. First, we choose $\vec{x}_1 = 0$, i.e. we place the first hole in the center of coordinates. The distance between two holes is given by the vector $\vec{l}$. The scalar $\mathcal{M}^{\alpha J}$ becomes

$$\mathcal{M}^{\alpha J} = 2\sqrt{2} n^{\alpha} \left( \frac{q^J}{|\vec{x}|} - \frac{q^J}{|\vec{x} - \vec{l}|} \right).$$

(30)

The scalar charge of the first hole is defined as follows. We remove the second hole far away, i.e. we consider $l \to \infty$. In this limit

$$\frac{1}{2\sqrt{2}} (\mathcal{M}^{\alpha J})_1 \to \frac{n^{\alpha} q^J}{|\vec{x}|} \equiv \left( \Sigma^{\alpha J} \right)_1.$$  

(31)

This defines the scalar charge of the first hole. In our case this leads to

$$(\Sigma^{\alpha J})_1 = n^{\alpha} q^J = n^{\alpha} (q^J)_1.$$  

(32)

To find the scalar charge of the second hole we place it in the beginning of coordinates, remove the first hole far away and consider the $(\Sigma^{\alpha J})_2/|\vec{x}|$ term. We get

$$(\Sigma^{\alpha J})_2 = - n^{\alpha} q^J = n^{\alpha} (q^J)_2.$$  

(33)

Thus we have confirmed the balance of force analysis: the scalar charge of the vector multiplet is sensitive to the sign of the magnetic charge of the same multiplet. Therefore the supersymmetric positivity bound permits massless holes with opposite charges to be in equilibrium.

The reader familiar with Sen’s spherically symmetric extreme massive supersymmetric black holes [12] may easily verify that the dilaton charge of the electric (magnetic) solution is not sensitive to the sign of the electric (magnetic) graviphoton charge $Q_R$, whereas the charge of the modulus $\mathcal{M}$ does change when the sign of the charge of the vector in the vector multiplet $Q_L$ changes. This may give an additional explanation of why the BPS solutions with half of unbroken supersymmetry include massless neutral configurations with arbitrary number of centers.

We should emphasize, that the solutions discussed above are obtained by solving exact non-linear equations. The balance of force analysis, which describes asymptotic behavior of forces between the two holes, is necessary only to interpret our exact solutions and to explain why they are consistent despite describing configurations with opposite charges.

Black hole multiplets with unbroken $N=4$ supersymmetry form the vector multiplets of $N=4$ supersymmetry. It is likely that the dynamics of such multiplets will show the condensation of the massless states. Indeed, it does not cost any energy to create as many pairs of massless oppositely charged BPS states as one wishes, and it does not violate charge conservation since each of these pairs is electrically and magnetically neutral. We will consider below the case of $N=2$ theory which is simpler from the point of view of the effective theory and which also has the massless neutral BPS states.


\section{Condensation of Massless Holes in N=2 theory}

The GMS mechanism of black hole condensation in N=2 theory is the following \cite{13}. One starts with the system of 16 massless black holes of unbroken N=2 supersymmetry. The low energy theory should contain 15 $U(1)$ gauge groups. The total charge of the system of 16 holes in each of the 15 gauge groups has to vanish,

$$Q^I \equiv \sum_{a=1}^{16} (q^I)_a = 0 \ .$$ \hspace{1cm} (34)

A specific example in \cite{13} was to have the first hole with the charge $+1$ in the first group, the second one with the charge $+1$ in the second group, the third one with the charge $+1$ in the third group, etc. The 16-th hole, however, had to be negatively charged in all 15 groups. The problem which was not clearly resolved there was whether massless black holes can actually exist and whether they can be in equilibrium despite the Coulomb attraction between objects with opposite charges.

The Calabi-Yau manifold in question was described in a way that it is difficult to find what kind of a Kahler manifold corresponds to it in the low energy four-dimensional theory. Therefore, we will not consider the study below as the one describing a particular Calabi-Yau manifold. However, we will give an example of N=2 black holes which have the massless limit and have the set of 16 holes with the charges satisfying the constraint (34) in equilibrium. The simplest case is to use the example of $\frac{SU(1,15)}{SU(15)}$ N=2 black holes \cite{5}.

The prepotential and the Kahler potential are

$$F(X^0, X^I) = (X^0)^2 - (X^I)^2 \ , \quad e^{-K(Z, \bar{Z})} = 1 - |Z|^2 \ .$$ \hspace{1cm} (35)

We choose 15 harmonic scalars $Z^I = \frac{X^I}{X^0}$ to vanish at infinity and to describe a 16-hole massless configuration:

$$Z^I = \sum_{a=1}^{16} \frac{q^I_a}{|\vec{x} - \vec{x}_a|} \ , \quad \sum_{a=1}^{16} q^I_a = 0 \ ,$$ \hspace{1cm} (36)

$$ds^2 = (1 - |Z|^2)^{-1} dt^2 - (1 - |Z|^2) d\vec{x}^2 \ .$$ \hspace{1cm} (37)

The magnetic charges of 15 vector fields are given by $q^I_a$. In a more detailed form and assigning only $\pm 1$ charges we have

$$Z^1 = \frac{1}{|\vec{x} - \vec{x}_1|} - \frac{1}{|\vec{x} - \vec{x}_{16}|} \ ,$$ \hspace{1cm} (38)

$$Z^2 = \frac{1}{|\vec{x} - \vec{x}_2|} - \frac{1}{|\vec{x} - \vec{x}_{16}|} \ ,$$ \hspace{1cm} (39)

$$\ldots$$ \hspace{1cm} (40)

$$Z^{15} = \frac{1}{|\vec{x} - \vec{x}_{15}|} - \frac{1}{|\vec{x} - \vec{x}_{16}|} \ .$$ \hspace{1cm} (41)
The total ADM mass, the total magnetic charge in each of the 15 gauge groups, as well as the total scalar charge in every gauge group vanish. Still we have a rather non-trivial configuration. If we take only one hole, for example the first one, remove all other 15 far away, we will find that it has a non-vanishing positive magnetic as well as the scalar charge. The same with the second and other 13 holes. If we finally consider the 16-th one, we would find that when the other 15 holes are far away, this one happens to be negatively charged (both in magnetic and scalar charges) in all 15 gauge groups. The system of these 16 holes is in equilibrium.

It is natural therefore to move to an alternative picture: associate with each hole a massless hypermultiplet as suggested in [13]. Each hypermultiplet contains two charged complex scalars $h^{a\alpha}$ where $\alpha = 1, 2$ is the global $SU(2)_R$ index of N=2 representation. This gives a total of 32 complex scalar fields. The potential describing the interactions between these holes in agreement with N=2 supersymmetry is given by

$$V \sim \sum_{I,J=1}^{15} M_{IJ} D^{\alpha\beta I} D_{\alpha\beta J},$$

(42)

where $M_{IJ}$ is a positive definite matrix and

$$D^{\alpha\beta I} = \sum_{a=1}^{16} q_I a (h^{*\alpha a} h^{a\beta} + h^{*\alpha a} h^{a\alpha}).$$

(43)

A remarkable feature of this potential is the existence of flat directions along which the bilinear combinations of scalar fields $D^{\alpha\beta I}$ vanish,

$$D^{\alpha\beta I} = 0.$$  

(44)

This gives 45 real constraints on 32 complex fields. In addition there are 15 gauge transformations which rotate the fields, leaving 4 real vacuum parameters. Up to a gauge transformation the general solution of (44) is defined by a complex two-vector $v$.

$$h^{a\alpha} = v^\alpha$$ for all $a$.  

(45)

Moving along the flat direction, the holes condense and their moduli space is parametrized by a single hypermultiplet. The point $v = 0$ was associated in [13] with the conifold point in the space of quintics (at which all 16 cycles vanish). Moving away from this point along the flat direction corresponds to giving a vev’s to the charged hypermultiplets which break all 15 U(1)’s. Thus a second branch of the moduli space corresponding to a charged black hole condensate was discovered in [13]. This branch has $101 - 15 = 86$ massless vector multiplets, and $2 + 1 = 3$ massless hypermultiplets.

We would like to stress here that the total picture depends crucially on the fact that $D^{\alpha\beta I} = 0$ when the condensate ansatz is introduced into the expression for the flat direction condition. The reason it works is that

$$D^{\alpha\beta I} |_{h^{a\alpha} = v^\alpha} = (v^{*\alpha} v^\beta + v^{*\beta} v^\alpha) \sum_{a=1}^{16} q_I a = (v^{*\alpha} v^\beta + v^{*\beta} v^\alpha) Q^I = 0.$$  

(46)
Thus it is clear that if the total charge $Q_I$ of all holes in every gauge group would not vanish, we would be unable to have a condensate with $v^\alpha \neq 0$. But now that we know that such 16 holes may exist in equilibrium the total picture of condensation of massless holes with vanishing total charge looks much more plausible.

### 6 Massless Mode as a Goldstone Boson

Being supported by the existence of exact massless neutral multi-center solutions, we may study the general case of spontaneous symmetry breaking which brings one theory with $h_{21}$ massless vector multiplets and $h_{11}$ massless hypermultiplets into the phase with $h_{21}'$ massless vector multiplets and $h_{11}'$ massless hypermultiplets.

Let us first try to understand if there is any magic here about the numbers 15 and 16 in the example above. If we would start with $n$ gauge groups instead of 15 and $n+1$ holes (hypermultiplets) we would easily construct an $n+1$-hole solution using the example in [3] with $SU(1,n)/SU(n)$. This would be the generalization of eqs. (72), (38). Thus we can consider the case of $n$ gauge groups and $n+1$-hole solution with the total mass and charges in every gauge group vanishing. However, what will happen with the counting above which worked well for 15,16 case? It actually works in the general case as well. We start with $4 \times (n+1)$ real scalars (for $n+1$ hypermultiplets). We have to impose $3 \times n$ conditions for a direction to be flat. In addition we can use $n$ gauge transformations. Thus we are left with 4 real vacuum parameters $v^\alpha$ as before, since previously we had

$$[4 \times (15 + 1)] - [3 \times 15] - [15] = 4 \ , \quad \text{(47)}$$

and now we have

$$[4 \times (n+1)] - [3 \times n] - [n] = 4 \ . \quad \text{(48)}$$

This would describe a condensate of a system of the original $n+1$ holes. The moduli space of the condensed holes is parametrized as before by a single hypermultiplet. The condensate breaks $n$ gauge groups this time.

Thus, suppose we start with $n$ gauge groups and choose the $n+1$-hole solution with the total charge in all groups vanishing. The distribution of the charges has the same structure as before: the first hole has charge 1 in the first group, etc, the $n$-th one has the charge 1 in the $n$-th group. The last one has the negative charge in all groups. One can represent it as a charge matrix with $n$ columns (for $n$ groups) and $n+1$ rows (for $n+1$ holes)

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-1 & -1 & \cdots & -1
\end{pmatrix}
\quad \text{(49)}
$$
The counting above shows that again we get one massless mode and \( n \) massive ones. The relevant question to ask is: does this situation allow the interpretation of the massless mode as a Goldstone boson? For this purpose we may check the symmetries of the potential in the form in which we first perform the shift of the fields. The form of the hyper-hole condensate \( \langle h^a \rangle = v \) suggests the natural combinations of fields:

\[
\phi^{\alpha I} \equiv \sum_{a=1}^{n} q^I_a h^{a\alpha}, \quad G^\alpha \equiv h^{(n+1)\alpha},
\]

such that the ground state is defined by

\[
\langle \phi^{\alpha I} \rangle = 0, \quad \langle G^\alpha \rangle = v^\alpha.
\]

The potential in these variables is given by eq. (42) with

\[
D^{\alpha \beta I} = \phi^{* \alpha I} (\phi^{\beta I} + 2G^\beta I) + (\phi^{* \beta I} + 2G^{* \beta}) \phi^{\alpha I}.
\]

In terms of fields with vanishing vacuum expectation values

\[
G^\alpha = \tilde{G}^\alpha + v^\alpha, \quad \langle \tilde{G}^\alpha \rangle = 0,
\]

we have

\[
D^{\alpha \beta I} = \phi^{* \alpha I} (\phi^{\beta I} + \tilde{G}^\beta + v^\beta) + (\phi^{* \beta I} + \tilde{G}^{* \beta} + v^{* \beta}) \phi^{\alpha I}.
\]

The relevant Goldstone-type continuous global symmetry of this theory is a \( U(2) \) symmetry given by

\[
\Delta \phi^{\alpha I} = \lambda^{\alpha \beta} \phi^{\beta I}, \quad \Delta G^\alpha = \lambda^{\alpha \beta} G^\beta, \quad \lambda^I = \lambda.
\]

This symmetry has exactly four parameters in agreement with the fact that one massless hypermultiplet has four massless scalars. This symmetry is broken spontaneously. To prove that there is one massless Goldstone hypermultiplet (four scalars) in such theory one can use the standard procedure. The values of the variations of the fields at the ground state are

\[
\langle \Delta \phi^{\alpha I} \rangle = \lambda^{\alpha \beta} \langle \phi^{\beta I} \rangle = 0, \quad \Delta \langle G^\alpha \rangle = \lambda^{\alpha \beta} \langle G^\beta \rangle = \lambda^{\alpha \beta} v^\alpha.
\]

From the symmetry of the potential we get

\[
\frac{\partial V}{\partial \phi^I} \Delta \phi^I + \frac{\partial V}{\partial G} \Delta G + \frac{\partial V}{\partial \phi^I \phi^I} \Delta \phi^I + \frac{\partial V}{\partial G^* G^*} \Delta G^* = 0.
\]

The second derivatives of this equation over \( G^* \) and \( \phi^I \) taken at the ground state give the consistency conditions for the mass matrix:

\[
\langle \frac{\partial^2 V}{\partial G^\alpha \partial G^\gamma} \rangle \lambda^{\alpha \beta} v^\beta + \langle \frac{\partial^2 V}{\partial G^{* \alpha} \partial G^{* \gamma}} \rangle (\lambda^{\alpha \beta} v^\beta)^* = 0,
\]

\[
\langle \frac{\partial^2 V}{\partial G^* \partial \phi^{* I}} \rangle \lambda^{\alpha \beta} v + \langle \frac{\partial V}{\partial G^{* \alpha} \partial \phi^{* I \gamma}} \rangle (\lambda^{\alpha \beta} v)^* = 0.
\]

\footnote{One can choose different forms of the hyper-hole condensate, e.g. \( \langle h^1 \rangle = v, \quad \langle h^2 \rangle = -v, \) etc. For all of this choices one can find the natural combinations of fields in terms of which the theory near each of these ground states is described as the one with \( \langle h^\alpha \rangle = v \).}
This tells us that there is one Goldstone massless hypermultiplet $G^\alpha$ with four massless scalars whose mass matrix is decoupled from the other $n$ massive hypermultiplets $\phi^\alpha_I$. This is in agreement with the fact that there is one four-dimensional spontaneously broken continuous symmetry $U(2)$ (53). The spontaneously generated mass term for $n$ hypermultiplets is

$$\sim [\phi^{*\alpha I} v_\beta + v^{*\beta} \phi^{\alpha I}] M_{IJ} [\phi^{* J} v_\beta + v^{* \beta} \phi^{J}] .$$  

If we would start with the neutral set of holes but with the different distribution of charges over the holes we would have to follow a slightly more complicated procedure of diagonalizing the massive and massless modes. However we will get again $n$ massive and one massless mode.

Thus we have found how to decrease any number $h_{21} - h'_{21} = n$ of massless vector multiplets and increase the number of massless hypermultiplets by one ($h'_{11} - h_{11} = 1$). The rule was to create one neutral cluster of $n + 1$ massless charged holes with the configuration of charges described in (49). If one would like to have more than one, $k = h'_{11} - h_{11}$, massless hypermultiplet as a result of the hole condensation, this is also possible. One has to start with $k$ clusters of neutral system of holes of the type described above. Each cluster has some number of columns $n_k$ and rows $(n_k + 1)$.

$$\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
-1 & \cdots & -1 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
-1 & -1 & \cdots & -1
\end{pmatrix}$$

The matrix of charges is a direct product of the matrices of the type given for one neutral set. The total number of columns in this matrix equals the number of vector multiplets $n = n_1 + n_2 + \ldots + n_k$ which become massive in the process of spontaneous breaking of symmetry. The number of submatrices in (60) equals $k$, the number of massless hypermultiplets. If the hypermultiplets in each cluster interact only with the vector multiplets and hypermultiplets of its own group one can write down the potential with $[U(2)]^k$ global symmetry, which is spontaneously broken and as the result, $k$ massless hypermultiplets describe the condensed state of the original neutral $k$ clusters. All of such multi-hole configurations are available.
7 After Condensation

After the condensate of one cluster of massless holes has been formed we end up by a set of holes which are all massive except one. It is interesting to go back from the effective action which was treating the holes as hypermultiplets to the original action of supergravity interacting with $n$ vector multiplets and identify the configuration which describes the state of a system after the condensate has been formed. For simplicity consider the case of two sets of 4 gauge groups and again, the simplest for our purpose Kahler manifold describing $SU(1,4+4) \overline{SU(4+4)}$ N=2 black holes \[5\]. The first four massless vector multiplets will be excited in the initial configuration and the second four in the final\[]\]. This will allow us to have an example of 5 holes with the required properties. The original configuration of 5 massless holes is given by

$$m_1 = m_2 = m_3 = m_4 = m_5 = 0, \quad Z_\infty^1 = Z_\infty^2 = Z_\infty^3 = Z_\infty^4 = 0.$$ \hspace{1cm} (61)

with the charge matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & -1 & -1 & -1
\end{pmatrix} \hspace{1cm} (62)$$

The charge of the graviphoton in all 5 holes equals the mass and vanishes. The mass formula is

$$m_a = \frac{\sum_{i=1}^{I=4} Z_{\infty}^I q_a^I}{(1 - |Z_\infty|^2)}.$$ \hspace{1cm} (63)

The massless solution is

$$Z_I^a = \sum_{a=1}^{a=5} \frac{q_a^I}{|\vec{x} - \vec{x}_a|}, \quad \sum_{a=1}^{a=5} q_a^I = 0, \quad I = 1, 2, 3, 4.$$ \hspace{1cm} (64)

$$ds^2 = (1 - |Z|^2)^{-1} dt^2 - (1 - |Z|^2) d\vec{x}^2.$$ \hspace{1cm} (65)

The magnetic charges of 4 vector fields are given by $q_a^I$. The scalars are

$$Z^1 = \frac{1}{|\vec{x} - \vec{x}_1|} - \frac{1}{|\vec{x} - \vec{x}_5|},$$ \hspace{1cm} (66)

$$Z^2 = \frac{1}{|\vec{x} - \vec{x}_2|} - \frac{1}{|\vec{x} - \vec{x}_5|},$$ \hspace{1cm} (67)

$$Z^3 = \frac{1}{|\vec{x} - \vec{x}_3|} - \frac{1}{|\vec{x} - \vec{x}_5|},$$ \hspace{1cm} (68)

$$Z^4 = \frac{1}{|\vec{x} - \vec{x}_4|} - \frac{1}{|\vec{x} - \vec{x}_5|}.$$ \hspace{1cm} (69)

\[5\] We need two sets of vector multiplets since the fields in the first group will become massive when the condensate is formed.
After the condensation which is described in the dual picture of interacting hypermultiplets the final system corresponds to the 5-hole solution with 4 massive and one massless hole. One possible example of such configuration is:

\[ m_1 = m_2 = m_3 = m_4 = \frac{C}{\sqrt{1-4C^2}} \quad m_5 = 0 \quad Z_5^5 = -Z_5^6 = Z_5^7 = -Z_5^8 = C \], (70)

with the charge matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & -1 & -1 & -1
\end{pmatrix}
\] (71)

The new 5 holes are sitting in different places:

\[
Z^I = Z_\infty^I + \sum_{a=1}^{a=5} \frac{q^I_a}{|x - \vec{x}_a|} \quad Q^5 = \sum_{a=1}^{a=5} q^5_a = Q^7 = \sum_{a=1}^{a=5} q^7_a = 0
\], (72)

\[
Q^6 = \sum_{a=1}^{a=5} q^6_a = -2 \quad Q^8 = \sum_{a=1}^{a=5} q^8_a = -2
\] (73)

The metric is

\[
ds^2 = \left( \frac{1 - |Z|^2}{1 - |Z_\infty|^2} \right)^{-1} dt^2 - \left( \frac{1 - |Z|^2}{1 - |Z_\infty|^2} \right) d\vec{x}^2 , \quad I = 5, 6, 7, 8
\] (74)

\[
Z^5 = +C + \frac{1}{|x - \vec{x}_1|} - \frac{1}{|x - \vec{x}_5|} ,
\]

\[
Z^6 = -C - \frac{1}{|x - \vec{x}_2|} - \frac{1}{|x - \vec{x}_5|} ,
\]

\[
Z^7 = +C + \frac{1}{|x - \vec{x}_3|} - \frac{1}{|x - \vec{x}_5|} ,
\]

\[
Z^8 = -C - \frac{1}{|x - \vec{x}_4|} - \frac{1}{|x - \vec{x}_5|} ,
\] (78)

There are many other solutions with 4 massive and 1 massless holes but with the different total charge in various gauge groups. All of these configurations have the limit when the scalars in the vector multiplets at infinity vanish \( C \to 0 \) and all five holes become massless. However, these solutions still carry a non-vanishing total charge in some of the gauge groups. In the example

\[\text{6The positions of holes in the original multi-center solutions } x_a \text{ is arbitrary. We choose a different set of centers } \vec{x}_a \text{ for the "after phase transition solution" to stress the fact that the positions do not have to coincide with the original ones.}\]
above, the charge in the 6-th and 8-th direction is not vanishing. If we would try to study this system as we did with the previous one we would find that

\[
< D^{\alpha \beta 6} > = \sum_{a=1}^{5} q_a^6 < (h^{* \alpha a} h^{a \beta} + h^{* \beta a} h^{a \alpha}) > = -2 (v^{* \alpha} v^{\beta} + v^{* \beta} v^{\alpha}) \neq 0. \tag{79}
\]

Thus \( v \neq 0 \) is not a ground state of these theory. Thus the condensation consistent with N=2 supersymmetric potential of hypermultiplets only occurs when we start with an \( n \)-hole solution with the total mass and all charges vanishing. For this configuration one can write down the potential for the hypermultiplets with the global \( U(2) \) symmetry with flat directions along which the holes condense. This symmetry is broken explicitly either when the mass terms for the hypermultiplets is added to the action or spontaneously, as considered in this paper. After the spontaneous generation of the condensate (vacuum expectation value of the scalar part of the hypermultiplets in the effective theory) the new state may be described as an \( n \)-hole solution of the same theory but with the different values of masses and charges. The values of the masses are defined by the condensate.

8 Discussion

The main result of this work is the realization of the fact that supersymmetric multi-hole solutions with the vanishing mass and total charges in all gauge groups do exist. They are massless and neutral, still it is not a flat space: the solution has an arbitrary number of centers with a particular distribution of charges. Such configurations plays the central role in the picture of the black hole condensation. This picture reflects nicely various properties of unbroken supersymmetry, in particular, the subtleties of the supersymmetric balance of forces in different multiplets.

We may qualify the condensation of holes as a process which drives the transition from one multi-hole configuration to another. After the formation of the condensate in the neutral system of \((n + 1)\) types of massless holes, we obtain a new vacuum state with \( n \) types of massive holes and one massless hole. The original gravitational system has various solutions of this kind which are also supersymmetric but whose total mass and charges are not vanishing anymore. The initial as well as the final configurations are both exact multi–hole solutions of the same effective Lagrangian. However these two sets have a dramatic difference in the masses and charges. In particular one can show that the distribution of charges in the final state differs from the original one by specific sign-flip of charges in some holes and by different asymptotic behavior of the scalars in the vector multiplets. Therefore both the original as well as the final set of holes are in equilibrium according to the balance of forces analysis in supersymmetric systems. The picture of transition between two different sets of supersymmetric holes presented above relies on the effective theory and may suggest the possible link between various supersymmetric solitons which are known to exist in supergravities and in string theories. The ultimate importance of
the spontaneous generation of the central charges in supersymmetric theories reveals itself in this picture: the massive supersymmetric configurations are created in the process of spontaneous breaking of symmetries of the theory. The mass parameters of the theory are not fixed at the fundamental level, but appear in the theory with the various choices of the ground states. We have found that the picture of condensation of N=2 supersymmetric BPS states is related to the spontaneous breaking of a global $U(2)$ symmetry. The presence of central charges in $N$-extended supersymmetry is known to break the global $U(N)$ symmetry. One can expect that the dynamics of the massive BPS states which form short massive multiplets in general may be understood via spontaneous breaking of the $U(N)$ symmetry which is present in massless theories with global extended $N$ supersymmetries.

We would like to add, from a somewhat different perspective, that the existence of flat directions often has important cosmological implications such as inflation, Polonyi field problem, etc. It would be very interesting to study this question in the context of the theory of BPS condensation. As a first step one may try to obtain solutions describing massless multi center BPS states in de Sitter space, like it was done in [14] for extreme Reissner-Nordström black holes. Since massless black holes do not modify metric of space-time far away from them, it may happen that they will not affect de Sitter expansion, and, vice versa, flat directions which we discussed above, will remain flat in de Sitter space. If this is the case, then one may expect that during inflation quantum fluctuations will move the BPS field $h^a$ along all possible flat directions in different causally disconnected parts of the universe. This would divide the universe after inflation into exponentially large domains corresponding to different stringy vacua. One should note that it is not so easy to obtain inflation in string theory. But here again the existence of flat directions of a new type may occur to be very useful. The best way of unifying string theory and inflation would be to consider inflation along flat directions. If some of the BPS flat directions become not exactly flat either due to expansion of the universe or because of the supersymmetry breaking, one may try to investigate the possibility that inflation occurs during the rolling of the black hole condensate towards the minimum of its effective potential. This is admittedly a very speculative possibility, but it sounds so interesting that it would be hard not to mention it here.

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