Resonances in random reactance networks with fluctuating entries

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Abstract. It is well known that disordered LC networks are an appropriate model for describing giant fluctuations of electric fields in a random metal-dielectric composite. The fluctuations reflect an inherent multifractal internal structure of the local fields which arise due to dipole-dipole interactions between the clusters with “dielectric resonances”. The resonances are poles of the conductance of disordered dissipationless LC networks. Though this topic is widely presented in the literature, described methods allow to study only the cases with fixed L and C values randomly distributed on a lattice. In this paper we generalize this approach and study random LC networks with fluctuating values of L and C entries. We demonstrate anticrossing of resonances due to their dipole-dipole interaction and splitting of the typical resonances for a discrete set of fluctuating L or C (or both) entries.

1. Introduction

Optical properties of metal-dielectric composites consisting of metallic nanoparticles incorporated into dielectric matrix are attracting considerable attention last years both from theoretical and experimental point of view [1, 2, 3, 4, 5, 6].

A simplest theoretical approach to such systems is to replace a continuous medium to a large discrete random network made of capacitors C and inductances L, the latter being in series with a weak resistor R. This network description is a discrete version of continuous Maxwell equations for scalar potential of electric field. The capacitors here model dielectric bridges between metallic nanoparticles, whereas isolated metallic granules are characterized by almost purely inductive response. This approximation is valid for frequencies ω of electromagnetic radiation satisfying to condition ωτ ≪ ω < ωp. Here ωp is the plasma frequency and ωτ is the plasmon relaxation rate [3].

Such large random reactance networks (that is networks made of random mixture of capacitances C and inductances L) have a lot of resonance clusters with any size and shape and different resonance frequencies. The clusters have a finite real conductance at resonance frequencies and thus can disperse an electric power. These resonances show up as narrow peaks with Lorentzian line shapes, e.g. in the weak-dissipation regime of the RL − C model [4]. The dielectric resonances provide a natural explanation for the anomalous fluctuations of the local electric field [1], which are responsible for giant surface-enhanced Raman scattering observed in semicontinuous metal films [7].

In the papers [4, 5] considering as an example a 2d square lattice it was shown that the problem of resonances of a random LC network (with random positions of capacitors C and
inductances \( L \) can be reduced to some generalized eigenvalue problem. However this reduction was done only for the very particular case when all values of the inductances \( L \) (and capacitors \( C \)) are equal to each other. Then in the system there is a typical frequency \( \omega_0 = 1/\sqrt{LC} \). In the present paper we have extended these results to the general case when values of inductances \( L \) and capacitors \( C \) are also random and can fluctuate from bond to bond. First we consider the case of two different interacting resonances and show the typical anticrossing effect when two frequencies approach each other. Then it is shown that for discrete set of different \( L \) or \( C \) (or both) values the known resonances observed for equal values of \( L \) and \( C \) are split and spectrum becomes more reach than for non fluctuating cases investigated before. Finally, we have also studied the spatial structure of these resonances for 2d and 3d random \( LC \) lattices with fluctuating entries and have shown that they also experience giant fluctuations of local electric fields.

2. Kirchhoff equations

![Figure 1. A piece of random LC network. Shaded are regions with charge accumulated on sites 6 and 7 respectively.](image)

To derive Kirchhoff equations in the general case, let us numerate sites of our \( LC \) network using integer index \( i \) taking values from 1 to \( N \), where \( N \) is the total number of cites (see Fig. 1). Let \( \varphi_i \) being the potential of site \( i \). Then let us attribute to each site \( i \) a charge \( q_i \). It is equal to the sum of charges of capacitor plates adjoined site \( i \) (see shaded regions shown on Fig. 1 around sites 6 and 7). If no capacitor adjoins site \( i \), then the corresponding charge \( q_i = 0 \). The charges \( q_i \) and the potentials \( \varphi_i \) are satisfying to the system of equations

\[
q_i = \sum_{j \neq i} C_{ij} (\varphi_i - \varphi_j),
\]

(1)

where \( C_{ij} \) is a capacity of the capacitor connecting sites \( i \) and \( j \). The matrix \( C \) is by definition symmetric \( C_{ij} = C_{ji} \). If there is no capacitor between sites \( i \) and \( j \) then \( C_{ij} = 0 \). If we define the diagonal matrix element \( C_{ii} \) as a minus sum of the non-diagonal elements

\[
C_{ii} \equiv -\sum_{j \neq i} C_{ij},
\]

(2)

then as follows from Eq. (1) we get

\[
q_i = -\sum_j C_{ij} \varphi_j.
\]

(3)
The time derivative \( \dot{q}_i \) by definition is equal to the sum of incoming electric currents to the site \( i \) from all neighboring sites \( j \)

\[
\dot{q}_i = \sum_{j \neq i} I_{ji}.
\]

Here \( I_{ji} \) is the electric current from site \( j \) to site \( i \). The matrix \( I_{ji} \) by definition is antisymmetric \( I_{ij} = -I_{ji} \). In its turn the time derivative of electric current \( \dot{I}_{ij} \) is related to voltage \( \varphi_i - \varphi_j \) in the corresponding inductance coil

\[
L_{ij}\dot{I}_{ij} = \varphi_i - \varphi_j.
\]

(5)

Here \( L_{ij} \) is an inductance of the coil between sites \( i \) and \( j \). The matrix \( L \) is a symmetric matrix, \( L_{ij} = L_{ji} \). If there is no coil between sites \( i \) and \( j \), then the corresponding value of \( L_{ij} = \infty \).

Now using equations (4) and (5) we can relate the second time derivative of charge \( \ddot{q}_i \) with voltages \( \varphi_j - \varphi_i \)

\[
\ddot{q}_i = \sum_{j \neq i} \frac{1}{L_{ij}}(\varphi_j - \varphi_i).
\]

(6)

To proceed further let us introduce matrix \( K \) of inverse inductances (between sites \( i \) and \( j \))

\[
K_{ij} = \frac{1}{L_{ij}}.
\]

(7)

The matrix \( K \) is symmetric, i.e. \( K_{ij} = K_{ji} \). If sites \( i \) and \( j \) are not connected by inductance coil, then by definition \( K_{ij} = 0 \). Similar with the case of matrix \( C \) let us define the diagonal matrix elements of the matrix of inverse inductances \( K \) as minus sum of the non-diagonal elements (compare with Eq. (2))

\[
K_{ii} \equiv -\sum_{j \neq i} K_{ij}.
\]

(8)

Then from (6) we get

\[
\ddot{q}_i = \sum_j K_{ij}\varphi_j.
\]

(9)

Comparing Eq. (3) and Eq. (9), we obtain finally a linear system of ordinary differential equations for site potentials

\[
\sum_j C_{ij}\ddot{\varphi}_j = -\sum_j K_{ij}\varphi_j.
\]

(10)

Looking for solution of these equations in the exponential form \( \varphi_i \propto \exp(i\omega t) \) we get

\[
\omega^2 \sum_j C_{ij}\varphi_j = \sum_j K_{ij}\varphi_j.
\]

(11)

Non-zero solution of this linear system of equations exist when

\[
\det(K - \omega^2 C) = 0.
\]

(12)

As a result we have reduced calculation of eigenfrequencies of our random impedance network to a generalized eigenvalue problem.

Let us show finally that Eqs. (11) are equivalent to the Kirchhoff equations investigated in [3, 4, 5] and similar papers. For this purpose we can present them in the equivalent form

\[
\sum_j i\omega C_{ij}\varphi_j = -\sum_j \frac{1}{i\omega}K_{ij}\varphi_j.
\]

(13)
Making use of properties (2) and (8) we transform them as follows

\[ \sum_j i\omega C_{ij} (\varphi_i - \varphi_j) = -\sum_j \frac{1}{i\omega} K_{ij} (\varphi_i - \varphi_j) \]  

(14)

or

\[ \sum_j \sigma_{ij} (\varphi_i - \varphi_j) = 0, \quad \text{where} \quad \sigma_{ij} = i\omega C_{ij} + \frac{1}{i\omega L_{ij}}. \]  

(15)

In the papers [3, 4, 5] it was investigated a particular case when all different from zero values \( C_{ij} = C \) and all different from zero values \( 1/L_{ij} = 1/L \). Thus we have generalized these results to the general case of different values of capacitances and inductances.

3. Anticrossing effect between two resonances

![Figure 2](attachment:image.png)

**Figure 2.** Anticrossing of frequencies for two interacting inductances \( L_1 \) and \( L_2 \) in a 2d lattice (50 x 50) of capacitors with \( C = 1 \) separated by 3 lattice constants.

Let us now apply the developed method to some particular cases. First we consider two inductance coils with inductances \( L_1 \) and \( L_2 \) (so that \( L_1L_2 = 1 \)) embedded into 2d lattice of capacitors with equal capacities \( C \). To avoid interaction with boundaries we will use doubly periodic boundary conditions. As it follows from paper [5] in the case of one inductance coil \( L \) in an infinite 2d lattice of capacitors with equal capacities \( C \) we have one resonance frequency \( \omega_0 = 1/\sqrt{LC} \). In the following instead of frequency \( \omega \) it is convenient to use the dimensionless value of \( \lambda \) [4], so that \( \lambda = 0.5 \) for \( \omega = \omega_0 \)

\[ \lambda = \left(\frac{\omega/\omega_0}{1 + (\omega/\omega_0)^2}\right)^2. \]  

(16)

As it was shown in [4, 8] the electric field from one inductance coil embedded into an infinite lattice of equal capacitors is a field of electric dipole with dipole moment parallel to the coil direction. In the case of two identical coils (oriented parallel to each other) we have two identical dipole oscillators which interact with each other. The interaction strength in 2d case is \( I \propto 1/R^2 \), where \( R \) is a distance between the dipoles. As a result the resonance frequency \( \omega_0 \) splits on two values separated by interaction strength \( I \). In the case of two different coils with inductances \( L_1 \) and \( L_2 \) we have two different harmonic oscillators interacting with each other. As a result changing bare frequencies of these oscillators we observe a typical anticrossing effect seen clearly on Fig. 2. The two asymptotes on the figure show two bare frequencies (or \( \lambda \)'s) for non interacting oscillators \( \lambda_1 = 1/(1 + L_1) = L_2/(1 + L_2) \) and \( \lambda_2 = 1 - \lambda_1 = 1/(1 + L_2) \).
4. Splitting of the resonances

Figure 3. Splitting of the resonances in a random 2d network with fluctuating entries. (a) Random lattice with equal inductances $L = 1$, and concentration $p = 0.1$ embedded in a 2d square lattice with equal capacitances [5]; (b) The same lattice with two sorts of coils with inductances $L_1 = 1.13$ and $L_2 = 0.89$, and with concentrations of each type $p = 0.05$; (c) The same lattice with three sorts of coils with inductances $L_1 = 1.25$, $L_2 = 1$ and $L_3 = 0.8$, and with concentration of each type $p = 0.033$. In all three cases all capacitances $C$ in the lattice are equal to each other ($C = 1$).

As it is well known random reactance networks have a complicated structure of "dielectric resonances" depending on the concentration of $L$ and $C$ entries [3]. In the case when all $L$ and $C$ values are equal the resonance structure for a 2d case was analyzed in [5]. But if $L$ and $C$ values can fluctuate from bond to bond the typical resonances should split and their structure becomes more reach. To demonstrate the splitting let us consider resonances in a 2d lattice of capacitors with a small concentration of inductance coils. On Fig. 3(a) is shown the case investigated in [5] when inductance coils with equal inductances are embedded in a 2d lattice with equal capacitances. On Fig. 3(b) is shown the same lattice but with fluctuating values of inductances $L_1$ and $L_2$ (so that $L_1L_2 = 1$). We see how the central peak for $\lambda = 0.5$ splits into two peaks. The Fig. 3(c) shows splitting of the central peak into three peaks for the case when we use three different values of coil inductances $L_1$, $L_2$ and $L_3$ embedded randomly into the capacitors matrix. The density of states (DOS) shown on this figure was obtained on a lattice $(10 \times 10)$ with double periodic boundary conditions after averaging over $10^5$ realizations.

5. Fluctuations of local fields

The last question we are going to touch in the paper is about the geometrical structure of resonances. Due to long-range dipole-dipole interaction the resonances are not localized [9] and have multifractal structure of eigenmodes [5, 10]. In the case with equal values of $L$ and $C$ entries the geometry of local fields was qualitatively described in [1] and investigated quantitatively in [5]. We have investigated the local structure of resonances in the case when values of $L$ and $C$ can fluctuate. For example in a 2d square lattice with equal capacitances $C = 1$ and concentration of inductance coils $p = 0.1$ for $L_1 = 0.8$, $p = 0.1$ for $L_2 = 1$ and $p = 0.1$ for $L_3 = 1.25$ the structure of resonances is shown on Fig. 4. Similar with the case of non fluctuating entries, we can see strong fluctuations of the local electric fields in the system demonstrating a multifractal structure of the eigenstates.

6. Conclusion

We generalized the problem of random $LC$ networks to the case when values of capacitances $C$ and inductances $L$ can fluctuate in a lattice from bond to bond. We demonstrated the
Figure 4. The geometry of local electric fields $|E|^2$ for the random lattice with 5000 bonds and total inductance concentration $p = 0.3$ of three different types (see text) for $\lambda = 0.48$. 

anticrossing of two resonances and showed how typical resonances investigated before split in several peaks corresponding to different values of inductance $L$ (or capacitance $C$) entries. The fluctuations of $L$ and $C$ values do not change qualitatively the multifractal local structure of resonances.

We are very grateful to Prof. V. I. Kozub for many fruitful discussions. One of the authors (YMB) thanks Dynasty Foundation for financial support.

References
[1] Brouers F, Sarychev A K, Blacher S and Lothaire O 1997 J. Phys. A 241 146
[2] Sarychev A K and Shalaev V M 1999 Physica A 266 115
[3] Fyodorov V V 2001 Physica E 9 609
[4] Clerc J P, Giraud G, Luck J M and Robin Th 1996 J. Phys. A 29 4781
[5] Jonckheere Th and Luck J M 1998 J. Phys. A 31 3687
[6] Clerc J P, Giraud G, Laugier J M and Luck J M 1990 Adv. Phys. 39 191
[7] Brouers F, Blacher S, Lagarkov A N, Sarychev A K, Gadenne P and Shalaev V M 1997 Phys. Rev. B 55 13234
[8] Schafer S, Raymond L and Albinet G, 2005 Eur. Phys. J. B 43 81
[9] Levitov L S, 1989 Europhys. Lett. 9 83
[10] Parshin D A and Schober H R 1998 Phys. Rev. B 57 10232