Strong gravitational lensing in 5D Myers-Perry black hole spacetime

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We study the motion of massless test particles in a five dimensional (5D) Myers-Perry black hole spacetime with two spin parameters. The behaviour of the effective potential in view of different values of black hole parameters is discussed in the equatorial plane. The frequency shift of photons is calculated which is found to depend on the spin parameter of black hole and the observed redshift is discussed accordingly. The deflection angle and the strong deflection limit coefficients are also calculated and their behaviour with the spin parameters is analysed in detail. It is observed that the behaviour of both deflection angle and strong field coefficient differs from Kerr black hole spacetime in four dimensions (4D) in General Relativity (GR) mainly due to the presence of two spin parameters in higher dimension.

Keywords: 5D Myers-Perry BH, Null geodesics, Cone of avoidance, Frequency shift

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1. Introduction

The Black holes (BHs) in Einstein’s General Relativity (GR) are one of the most strange and mysterious objects in the universe.\textsuperscript{1,2} The most general spherically symmetric, vacuum solution of the Einstein field equations in GR is the well known Schwarzschild BH spacetime\textsuperscript{1,5,6} in four dimensions. The study of Schwarzschild BH solution and its applications to the solar system gives the accurate tests of the predictions made by GR. Further, a static solution to the Einstein-Maxwell field equations, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body is the Reissner-Nordström spacetime\textsuperscript{1,5,6}. The rotating generalization of the Schwarzschild BH spacetime is Kerr BH spacetime in GR while the spacetime geometry in the region surrounding by a charged rotating BH is represented by the Kerr-Newman BH spacetime which is a solution of Einstein-Maxwell equations in GR.\textsuperscript{7,8}

In recent years, various interesting BH solutions in more than four spacetime dimensions, especially in five dimensions,\textsuperscript{9} have been the subject of intensive research, motivated by various ideas in brane-world cosmology, string theory and gauge/gravity duality.\textsuperscript{10} The uniqueness theorem does not hold in higher dimensions due to the fact that there are more degrees of freedom. However, the discovery of black-ring solutions in five dimensions asserts that the non-trivial topologies are allowed in higher dimensions.\textsuperscript{11} In particular, the Myers-Perry black hole (MPBH) spacetime\textsuperscript{12} is higher dimensional generalization of the four-dimensional Kerr BH spacetime in GR. The study of geodesic structure of massless particles in a given BH spacetime is one of the important ways to understand the gravitational field around a BH spacetime. The geodesic motion around various BH spacetimes in a variety of contexts (for timelike as well as null geodesics), both in GR and in alternating theories of gravity, are studied widely time and again.\textsuperscript{13-40} The geodesic motion of massive and massless test particles in the background of equally spinning Myers Perry anti-de Sitter black hole spacetimes is also recently studied in more detail.\textsuperscript{41-43} The GR which has revolutionized our understanding of the universe as a whole is now more than one hundred years old and the recent advancement in understanding the gravitational collapse and nature of BH solutions in diversified scenario is remarkable.\textsuperscript{44-47}

The deflection of light ray in a gravitational field is one of the crucial predictions of GR and the gravitational lensing is an important phenomena resulting due to the bending of light in gravitational field of a massive object while passing close to that object. The strong gravitational lensing is caused by a compact object viz. BHs with a photon sphere and when the photons pass close to the photon sphere, the deflection angles become so large that an observer would detect two infinite sets of faint relativistic images on each side of the BH, which are produced by photons that make complete loops around the BH before reaching to the observer. These relativistic images may therefore provide us not only some important signatures about BHs in the universe, but might also helpful in verification of the alternative
theories of gravity in their strong field regime. The gravitational lensing in weak field approximation studies the properties of galaxies and stars, but when BHs is treated as a lens, it is no longer valid and the strong field limit is needed which is referred as strong deflection limit. Thus, it acts as a powerful indicator of the physical nature of the central celestial objects and then has been used to study in various theories of gravity. The studies of the strong field limit lensing due to different BHs have received considerable attentions in recent years. The development of lensing theory in the strong-field regime started with the study of gravitational lensing due to a Schwarzschild BH spacetime and it is also shown that a supermassive BH like at the center of our Galaxy may be a suitable lens candidate.

The main objective of this paper is to study the strong lensing in a 5D MPBH spacetime. We have used the units that fix the speed of light and the gravitational constant via $8\pi G = c^4 = 1$. In Section II, the first integral of the geodesic equations and the effective potential in MPBH spacetime are discussed. We have discussed the optical properties like frequency shift and the cone of avoidance from the null geodesics in Section III. The gravitational lensing aspects in the strong field limit are then discussed in detail in Section IV. Finally, the results obtained are summarised in Section V.

2. Equations of Motion in 5D MPBH Spacetime

To study the geodesics and strong lensing in 5D rotating MPBH spacetime background, we begin with the following metric of MPBH spacetime in the Boyer-Lindquist coordinates:

\[
\begin{align*}
\text{ds}^2 &= \frac{\rho^2}{4\Delta} dx^2 + \rho^2 d\theta^2 - dt^2 + (x + a^2) \sin^2 \theta d\phi^2 + (x + b^2) \cos^2 \theta d\psi^2 \\
&\quad + \frac{2m}{\rho^2} \left[ dt + a \sin^2 \theta d\phi + b \cos^2 \theta d\psi \right]^2,
\end{align*}
\]

with $\rho^2$ and $\Delta$ are defined as

\[
\begin{align*}
\rho^2 &= x + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \\
\Delta &= (x + a^2)(x + b^2) - r_0^2 x.
\end{align*}
\]

The metric is singular when $\Delta = g_{rr} = 0$ and $\rho^2 = 0$. Here $a$ and $b$ are two spin parameters, and $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq \pi/2$ are the angles those are bounded as $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq \pi/2$. Following, we use the coordinate $x = r^2$ instead of the radius $r$ in order to simplify the calculations. It is worth noticing here that the metric reduces to 5D Tangherlini solution for $a = b = 0$. For the horizon structure of MPBH spacetime with $\Delta = 0$, we obtain

\[
x_{\pm} = \frac{1}{2} \left[ 2m - (a^2 + b^2) \pm \sqrt{(2m - (a^2 + b^2))^2 - 4a^2 b^2} \right].
\]

Here, $x_+$ and $x_-$ denotes the outer horizon and the inner horizon respectively. The metric describes non-extremal BH for $x_+ > x_-$ and when $x_+ = x_-$, one can
obtain an extremal BH spacetime. The horizon exists when \(a^2 + b^2 < 2m\) and 
\[ [2m - (a^2 + b^2)]^2 \geq 4a^2b^2. \]
Now in the equatorial plane, i.e., \(\theta = \pi/2\), \(L_\psi = 0\) and 
\(K = L_\phi^2 - E^2b^2\), the metric (1) reduces to
\[
ds^2 = -A(x)dt^2 + B(x)dx^2 + C(x)d\phi^2 - D(x)dtd\phi,
\]
where the metric coefficients are described as below,
\[
A(x) = (1 - \frac{2m}{\rho^2}),
\]
\[
B(x) = \frac{\rho^2}{4\Delta},
\]
\[
C(x) = x + a^2 + \frac{2ma^2}{\rho^2},
\]
\[
D(x) = -\frac{4ma}{\rho^2}.
\]
The first integral of geodesic equations may then be expressed in terms of the above 
mentioned metric coefficients, in the following form,
\[
\dot{t} = \frac{4C(x)E - 2D(x)L}{4A(x)C(x) + D(x)^2},
\]
\[
\dot{x} = \pm 2\sqrt{\frac{C(x)E^2 - D(x)EL - A(x)L^2}{B(4A(x)C(x) + D(x)^2)}},
\]
\[
\dot{\phi} = \frac{2D(x)E + 4A(x)L}{4A(x)C(x) + D(x)^2}.
\]
For null geodesics, \(\dot{x}\) from Eq. (10), can be reconstructed as
\[
\dot{x}^2 + V_{eff} = 0,
\]
where,
\[
V_{eff} = -4\left[\frac{C(x) - D(x)L - A(x)L^2}{B(x)(4A(x)C(x) + D(x)^2)}\right],
\]
\[
= -4\left[\frac{2ma^2 + x^2 + xb^2 + a^2x + a^2b^2E^2}{x + b^2} + \frac{4maEL}{x + b^2} - \frac{(x + b^2 - 2m)\Delta^2}{x + b^2}\right].
\]
The general behavior of effective potential as a function of \(x\) for different values of rotation parameter is presented in Fig. (1). In particular, Fig (1a) here represents 
the variation of potential with the spin parameter \(b\) for fixed value of \(a(= 0.1)\) while 
Fig. (1b) represents the variation of the potential with spin parameter \(a\) for fixed 
value of \(b(= 0.1)\). The effective potential shows a maxima which corresponds to an 
unstable circular orbit. It is also observed that with the increase in the value of 
parameter \(b\), the maximum of the effective potential is shifting towards the left (see 
Fig. (1a)), i.e., the circular orbits also shift towards the central object accordingly 
whereas with the increase in the value of spin parameters \(a\) at fixed \(b\), the peak is 
shifting towards the right (see Fig. (1b)), which signifies the shifting of circular orbit
Fig. 1. Variation of effective potential with radius (a) at different values of spin parameter $b$ for a fixed value of spin parameter $a$ (= 0.1) (b) at different values of spin parameter $a$ for a fixed value of spin parameter $b$ (= 0.1).

away from the central object.

3. Observables for Photons

In order to discuss the optical properties in 5D MPBH spacetime, the frequency shift and cone of avoidance are discussed below:

3.1. Frequency shift

A more physically meaningful quantity associated with the circular orbit is the angular frequency. The angular frequency of the circular geodesics relative to a distant observer for unequal spin is defined by

$$\Omega = \frac{d\phi}{dt}. \quad (14)$$

Thus, using Eq. (9) and Eq. (11), the angular frequency given by Eq. (14) can be calculated as below,

$$\Omega = \frac{(x + b^2 - 2m)d - 2ma}{(x^2 + (a^2 + b^2 - 2m)x + a^2b^2 + 2ma)d + 2mx + 2ma^2}, \quad (15)$$

where $d = L_\phi/E$. The frequency shift may however be expressed as

$$g = \frac{k_\mu u^\mu_o}{k_\mu u^\mu_e}, \quad (16)$$

where $k_\mu$ are the covariant components of the photon four-momentum and $u^\mu_o$ ($u^\mu_e$, respectively) are the contravariant components of the four-velocity of the observer.
Fig. 2. Variation of frequency shift: (a) with spin parameter \( b \) for different values of spin parameter \( a \), (b) Variation of frequency shift with spin parameter \( a \) for different values of spin parameter \( b \).

In case of static distant observer, the four-velocity reads \( u_0 = (1, 0, 0, 0, 0) \) and in the case of emitter the four-velocity reads as \( u_e = (u^t_e, 0, 0, u^\phi_e, 0) \). The frequency shift then acquires the following form,

\[
g = \frac{1}{u^t_e(1 - d\Omega)}. \tag{17}
\]

The temporal component of the emitter four-velocity can be obtained from the norm of the four-velocity:

\[
u^t_e = \left[1 - \frac{2m}{\rho^2} - \left(x + a^2 + \frac{2ma^2}{\rho^2}\right)\Omega^2\right]^{-1/2}, \tag{18}
\]

such that the expression for frequency shift now reads as,

\[
g = \left[1 - \frac{2m}{\rho^2} - \left(x + a^2 + \frac{2ma^2}{\rho^2}\right)\Omega^2\right]^{1/2}. \tag{19}
\]

Here, by considering the value of \( a \) in between 0 and 1, one automatically has a range for \( b \) (from the expressions \( a^2 + b^2 < 2m \) and \( [2m - (a^2 + b^2)]^2 \geq 4a^2b^2 \)) with \( m = 1 \) as, \(-0.4 \leq b \leq 0.4 \) or \(-2.412 \leq b \geq 2.412 \). The behaviour of the frequency shift for different values of different spin parameters is illustrated in Figs. (2) - (4). In particular in Fig. (2), we have presented the frequency shift for different values of spin parameter \( a \) and \( b \) while keeping the other parameter constant respectively. The Fig. (2a), shows the variation of frequency shift with spin parameter \( b \) at different values of spin parameter \( a \), whereas Fig. (2b), shows the variation of frequency shift with \( a \) at different values of \( b \). As the frequency shift increases, the observed frequency of the photon decreases which in turn gives an equivalent increase in the corresponding wavelength of the photon. Hence, for the positive frequency shifts, the photons get redshifted. One may note that the stronger is the gravitational field
Fig. 3. Variation of frequency shift with $x$: (a) for different values of spin parameter $b$, with $a = 0.1$ and (b) for different values of spin parameter $a$, with $b = 0.4$.

Fig. 4. Variation of frequency shift with $x$ for different values of $b$, with $a = 0.4$.

of a source, larger will be the energy loss of an incoming photon and also larger will be the observed redshift.

From Fig. 2, it can also be observed that for a particular value of parameter $a$, the redshift for photons around a MPBH spacetime increases with an increase in the value of spin parameter $b$ while with the increase in value of spin parameter $a$ it decreases with an increase in the value of parameter $a$. It therefore signifies the strength of gravitational field depends strongly on both the parameters at a fixed value of $x$. In Fig. 3 and Fig. 4, the variation of frequency shift with $x$ for different values of spin parameters $a$ and $b$ is shown respectively. It is also observed that for a particular value of spin parameter $b$, there is a decrease in the redshift for
the photons with an increase in the value of $x$, while for fixed for any fixed value of $a$, the redshift first increases and then decreases afterwards with $x$.

### 3.2. Null geodesics in the observers frame

If a point source at a given distance $r$ of the centre emits light isotropically into all directions, a part of the light will be captured by the BH, while another part can escape. It is clear that the critical orbits are at the limit of infall or escape. These orbits define a cone with half opening angle and $\psi$ denotes the half angle of the cone then, the expression of angle $\psi$ in the observer’s frame comes out as,

$$ \tan \psi = \frac{k(x)}{k(\phi)}. $$  \hspace{1em} (20)

Here, $k_x$ and $k_\phi$ are the components of the null vector field $k = k^\mu \partial_\mu$ for null geodesic in this spacetime. In general case, the null vector field $k$ has four components ($k^t, k^x, k^\theta, k^\phi$). Using the equations for constant of motion i.e. $E = -g_{\mu\nu}k^\mu$ and $L = g_{\phi\mu}k^\mu$, we obtain the relations,

$$ k^t = \frac{G_t(E, L)}{G}, \quad k^\phi = \frac{G_\phi(E, L)}{G}. $$  \hspace{1em} (21)

where the functions $G_t$ and $G_\phi$ are defined as below,

$$ G_t(E, L) = C(x)E - \frac{D(x)L}{2}, $$  \hspace{1em} (22)

$$ G_\phi(E, L) = \frac{D(x)E}{2} + A(x)L. $$  \hspace{1em} (23)

Using the constraint $k \cdot k = 0$ (i.e. $(k^t)^2 + (k^x)^2 + (k^\theta)^2 + (k^\phi)^2 = 0$) for null geodesics, we can then obtain an expression for $k^x$ in equatorial plane,

$$ (k^x)^2 = \frac{W}{B(x)G}, $$  \hspace{1em} (24)

where $W = C(x)E^2 - D(x)EL - A(x)L^2$.

Further, $k_{(\alpha)} = g_{\mu\nu}k^\mu u^{(\alpha)}_{\nu}$ leads to the components of $k$ in observers frame as follows,

$$ k_{(x)} = \frac{G_t}{\sqrt{-GC(x)}} \sinh q - \left( \sqrt{\frac{W}{G}} \cos \chi - \frac{L}{\sqrt{-C(x)}} \right) \cosh q, $$  \hspace{1em} (25)

$$ k_{(\theta)} = 0, \quad k_{(\phi)} = \sqrt{\frac{W}{G}} \sin \chi + \frac{L}{\sqrt{-C(x)}} \cos \chi. $$  \hspace{1em} (26)

Using Eqs. (25) and (26) in Eq. (20), one can then obtain

$$ \tan \psi = \frac{G_t \sinh q - \left( \sqrt{\frac{W}{G}} \cos \chi - \frac{L}{\sqrt{-C(x)}} \right) \cosh q}{\left( \sqrt{\frac{W}{G}} \sin \chi + \frac{L}{\sqrt{-C(x)}} \cos \chi \right).} $$  \hspace{1em} (27)
Here, we have considered the null geodesics in observer’s frame moving with constant acceleration. However, an arbitrary observer in the spacetime is determined by a velocity timelike vector field $u$ (i.e. $u^2 = 1$) as,

$$u = \alpha \partial_t + \beta \partial_x + \gamma \partial_\phi.$$  

(28)

The vector field can be parametrized by a pair of functions $(q(x), \chi(x))$ (see) as given below,

$$u = \sqrt{-C(x)} \frac{\cosh q}{G} \partial_t + \frac{1}{\sqrt{-B(x)}} \sinh q \cos \chi \partial_x + \frac{1}{\sqrt{-C(x)}} \left( \sinh q \sin \chi - \frac{D(x)}{2\sqrt{G}} \cosh q \right) \partial_\phi,$$  

(29)

where, $G = A(x)C(x) + \frac{D(x)^2}{4}$. The other basis vectors $u_i$ are,

$$u_x = \sqrt{-C(x)} \frac{\sinh q}{G} \partial_t + \frac{1}{\sqrt{-B(x)}} \cosh q \cos \chi \partial_x + \frac{1}{\sqrt{-C(x)}} \left( \cosh q \sin \chi - \frac{D(x)}{2\sqrt{G}} \sin q \right) \partial_\phi,$$

$$u_\phi = -\sin \chi \sqrt{-B(x)} \partial_x + \cos \chi \sqrt{-C(x)} \partial_\phi.$$  

(30)

The (equatorial) trajectories of the observer are given by following equations,

$$\frac{dt}{d\lambda} = \sqrt{-C(x)} \frac{\cosh q}{G},$$  

(31)

$$\frac{dx}{d\lambda} = \frac{1}{\sqrt{-B(x)}} \sinh q \cos \chi,$$  

(32)

$$\frac{d\theta}{d\lambda} = 0,$$  

(33)

$$\frac{d\phi}{d\lambda} = \frac{1}{\sqrt{-C(x)}} \left( \sinh q \sin \chi - \frac{D(x)}{2\sqrt{G}} \cosh q \right),$$  

(34)

where $\lambda$ is an affine parameter. From Eqs. (31)-(34), the static observer (for $\chi = \pi/2$ and $\tanh q = \frac{D(x)}{2\sqrt{G}}$) is determined as follows,

$$\cosh q = \frac{D(x)}{\sqrt{4G - D(x)^2}},$$  

(35)

$$\sinh q = \frac{2\sqrt{G}}{\sqrt{4G - D(x)^2}}.$$  

(36)

The radial trajectories ($\theta = \text{constant}$) are given by the conditions,

$$\tanh q \sin \chi = \frac{D(x)}{2\sqrt{G}}, \quad \chi \neq 0.$$  

(37)
For the radial trajectories of the observer in equatorial plane i.e., $\chi = \pi/2$, $\tanh q = \frac{D(x)}{\sqrt{2G}}$. However for $\chi = \pi/4$, the relation $\sinh q = \frac{D(x)}{\sqrt{2G}} \cosh q$ holds and therefore the observer’s velocity vector $u$ takes the following form,

$$u = \sqrt{-C(x)} \cosh q \partial_t + \frac{D(x)}{2\sqrt{-B(x)G}} \cosh q \partial_x,$$  

(38)

and from the condition $u^2 = +1$, one can easily obtain $\cosh q = \sqrt{\frac{G}{A(x)C(x) - \frac{D^2(x)}{4}}}$. Thus the expression $[27]$ now reads as,

$$\tan \psi = \frac{1}{G} D(x) - \left( \sqrt{\frac{WC(x)}{G}} - L \right) G $$ 

(39)

which clearly indicates that the angle $\psi$ depends on the two parameters of the null geodesics i.e. $E$ and $L$.

4. Strong field lensing by 5D MPBH Spacetime

In this section, we investigate the strong field lensing by a 5D MPBH spacetime given by Eq. (1) for the case where both the observer and the source lie in the equatorial plane i.e., $\theta = \pi/2$. Where we follow the Bozza’s method to obtain the deflection angle and other parameters of strong gravitational lensing.$[51, 52]$ The impact parameter is related to the minimum distance reached by the photon. In general, a light ray coming from infinity approaches the BH, reaches the minimum distance and then leaves again towards infinity. Using the geodesic equations, we find an implicit relation between angular momentum and the closest approach distance. Here for simplicity, we are considering $E = 1$ and at the minimum distance (say $x_0$) of photon trajectory (where $V_{\text{eff}} = 0$), we have$[51]$  

$$L = \frac{-D(x_0) + \sqrt{4A(x_0)C(x_0) + D(x_0)^2}}{2A(x_0)} = \frac{2ma - \rho \sqrt{x_0\rho^2 + a^2\rho^2 - 2mx_0}}{2m - \rho^2}.$$  

(40)

Now following$[52]$ the condition for the radius of photon sphere is,  

$$A(x_p)C'(x_p) - A'(x_p)C(x_p) + L(x_p) \left( A'(x_p)D(x_p) - A(x_p)D'(x_p) \right) = 0.$$  

(41)

Thus, the equation for the radius of photon sphere takes the following form,

$$x_p^4 + (-8m + 4b^2)x_p^3 + (6b^4 - 20mb^2 + 16m^2 - 8ma^2)x_p^2 + (-16mb^4 + 4b^6 + 16b^2m^2 - 16b^2ma^2)x_p + b^8 - 8b^4ma^2 - 4b^6m + 4b^4m^2 = 0.$$  

(42)

In the limit $a = 0$ and $b = 0$, the radius of the photon sphere comes out as $x_p = 4m$, i.e., the radius of photon sphere for the Tangherlini spacetime.$[53]$ The radius of the
Fig. 5. Variation of radius of photon sphere as a function of spin parameter $b$ for different values of spin parameter $a$ with $m = 1$.

The photon sphere is plotted with respect to the spin parameter $b$ in Fig. 5 for different values of spin parameter $a$. One can easily notice from the plots that on increasing the value of spin parameter $a$ the radius of photon sphere decreases.

Depending on the direction of the rotation of BH there are two types of photon spheres, one for the photons winding in the same direction of rotation as the BH known as direct photons and the other one for photons winding in the opposite direction known as retrograde photons. One may also notice that both direct and retrograde photons have same impact, i.e., in both the cases the photons are not easily captured by increasing the value of spin parameter $a$.

The deflection angle for photons coming from infinity can be written as,

$$\alpha(x_0) = \phi(x_0) - \pi,$$

where $\phi(x_0)$ is the total azimuthal angle, which evaluates to $\pi$ for a straight line and becomes larger with the bending of light ray in gravitational field. As the distance of closest approach $x_0$ decreases, the deflection angle increases accordingly. When $x_0$ reaches a minimum value, i.e., the radius of the photon sphere, the deflection angle becomes very large, and a photon will be captured by the BH. Now, using Eq. (43), azimuthal angle is given by,

$$\phi(x_0) = 2 \int_{x_0}^{\infty} \frac{d\phi}{dx} dx,$$

where $P = C(x)A(x_0) - A(x)C(x_0) + L [A(x)D(x_0) - A(x_0)D(x)]$.

We can find the behaviour of the deflection angle very close to the photon sphere following Bozza. The divergent integral is first split into two parts, one of which contains the divergence and the other is regular. Both these parts are expanded.
around the radius of photon sphere and approximated with the leading term. We first define two new variables $y$ and $z$ as,

$$y = A(x),$$

$$z = \frac{y - y_0}{1 - y_0},$$

where $y_0 = A(x_0)$. The total azimuthal angle in terms of two new variables can be expressed as,

$$\phi(r_0) = \int_0^1 R(z, x_0)f(z, x_0) dz,$$
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with

\[ R(z, x_0) = \frac{2(1 - y_0) \sqrt{B(x)A(x_0)}}{A'(x)} \left[ \frac{D(x) + 2LA(x)}{\sqrt{4A(x)C^2(x) + C(x)D^2(x)}} \right], \]  

(48)

\[ f(z, x_0) = \frac{\sqrt{C(x)}}{\sqrt{[C(x)A(x_0) - A(x)c(x_0)] + L(A(x_0)D(x_0) - A(x_0)D(x) \right]}}. \]  

(49)

Here, the function \( R(z, x_0) \) is regular for all the values of \( z \) and \( x_0 \), whereas \( f(z, x_0) \) diverges at \( z = 0 \). So, the integral \([47]\) can be separated into two parts

\[ \phi(x_0) = \phi_R(x_0) + \phi_D(x_0), \]  

(50)

with

\[ \phi_D(x_0) = \int_0^1 R(0, x_0)f_0(z, x_0)dz, \]  

(51)

and the regular part

\[ \phi_R(x_0) = \int_0^1 g(z, x_0)dz, \]  

(52)

with \( g(z, x_0) = R(z, x_0) f(z, x_0) - R(0, x_0) f_0(z, x_0) \). In order to find the divergence of the integrand in Eq. \([52]\), we expand the argument of the square root of \( f(z, x_0) \) to second order in \( z \):

\[ f(z, x_0) \sim f_0(z, x_0) = \frac{1}{\sqrt{\alpha z + \beta z^2 + O(z^3)}}, \]  

(53)

where

\[ \alpha = \left(1 - A(x_0)\right)^2 \left( A(x_0)C'(x_0) - A'(x_0)C(x_0) + L[A'(x_0)D(x_0) - A(x_0)D'(x_0)] \right), \]  

(54)

\[ \beta = \frac{(1 - A(x_0))^2}{2C(x_0)^2A''(x_0)} \left( 2C(x_0)C'(x_0)A''(x_0) + (C(x_0)C''(x_0) - 2C'^2(x_0))A(x_0)A'(x_0) \right. \]

\[ - C(x_0)C'(x_0)A(x_0)A''(x_0) + L[A(x_0)C(x_0)(A''(x_0)D'(x_0) - A'(x_0)D''(x_0))] \]

\[ + 2A'(x_0)C'(x_0)(A(x_0)D'(x_0) - A'(x_0)D(x_0)) \right). \]  

(55)

Here, prime denotes the derivative with respect to \( x \). At \( x_0 = x_{ps} \), \( \alpha \) vanishes. The outermost solution of \( \alpha = 0 \) defines the photon sphere. In the strong deflection limit, the expression for the deflection angle \([51]\) around the radius of the photon sphere reads as follows,

\[ \alpha(u) = -\bar{a} \log \left( \frac{u}{u(x_p)} - 1 \right) + \bar{b} + O(u - u(x_p)). \]  

(56)
The strong field coefficients $u(x_p)$, $\bar{a}$ and $\bar{b}$ are then given by

$$u(x_p) = L|_{x_0 = x_p},$$

$$\bar{a} = \frac{R(0, x_p)}{2\sqrt{\beta_p}} = \sqrt{\frac{2A(x_p)B(x_p)}{A(x_p)C(x_p)'' - A(x_p)''C(x_p) + u(x_p)Q}},$$

$$\bar{b} = -\pi + b_R + \bar{a} \log \left(\frac{4\beta_p C(x_p)}{u(x_p)[A(x_p)](D(x_p) + 2u(x_p)A(x_p))}\right),$$

where, $Q = A(x_p)''D(x_p) - A(x_p)D(x_p)''$ and $b_R$ goes to zero for the case of MPBH spacetime. It can be due to the presence of two spin parameters. Here, we have plotted the deflection angle and the parameters $\bar{a}$ and $\bar{b}$, with respect to both the spin parameters $a$ and $b$. The behaviour of deflection angle is qualitatively similar for direct as well as retrograde photons. In Fig. (6), the deflection angle with respect to spin parameter $b$ for different values of spin parameter $a$ is graphically presented and one can observe that the deflection angle first increases followed by a sharp decrease. The decreasing bending angle depicts the decrease in the strength of the gravitational field of MPBH spacetime. It is observed that in comparison to Kerr BH where qualitatively the variation of deflection angle is different for direct and retrograde photons, the variation in MPBH is identical for both direct and retrograde case. Fig. (7), shows the variation of $\bar{a}$, $\bar{b}$ with spin parameter $b$ for different values of spin parameter $a$. It can also be seen from the Fig. (7) that apart from the Kerr case here initially $\bar{a}$ decreases upto $b = 0$ and then increases, whereas $\bar{b}$ increases upto $b = 0$ and then increases. One can also observe that both the coefficients are converging first and then diverging. Thus, it can be concluded that the behaviour of deflection angle and the strong field coefficients changes in the case of MPBH spacetime where the case of two spin parameters $a$ and $b$ is being considered.

Deimer et. al. also studied massive as well as massless test particles in the general 5D MPBH spacetime. Where the defection angle is calculated for general case by using Mino time. However, in the present work, we have calculated the deflection angle and other strong lensing parameters by using Bozza’s method and the variation of deflection angle with spin parameter is investigated. We believe that this study might be useful for the investigation of relativistic images in context of mentioned BH spacetime in higher dimension.

5. Summary and Conclusions

In this article, we have investigated the frequency shift, cone of avoidance and strong gravitational lensing in the background of a MPBH spacetime. Some of the important results obtained are summarised below.

(i) The effective potential has a maximum which corresponds to an unstable circular orbits. It is observed that with the increase in the value of spin parameters $a$ and $b$ circular orbit shifts towards and away from the central object respectively.
(ii) The frequency shift depends on the spin parameters $a$ and $b$. The redshift becomes stronger with the increase in the value of spin parameter $a$ whereas blueshift becomes stronger with the increase in the value of spin parameter $b$. For positive frequency shift the spin parameter $a$ strengthens the gravitational field of a MPBH spacetime.

(iii) The behaviour of radius of photon sphere indicates that the photons are not easily captured with increasing the value of spin parameter $a$ in case of both type of photon i.e., the direct and retrograde photon.

(iv) The deflection angle first increases and then decreases with the increase in the value of spin parameter $b$ for fixed values of parameter $b$. There is a significant effect of spin on deflection angle. A decrease in the bending angle shows the decrease in the gravitational strength of the MPBH spacetime. However, the deflection angle and strong field coefficients both changes in a similar way qualitatively with an increase in the spin parameters $a$ and $b$.

(v) The behaviour of deflection angle and strong field coefficients in 5D MPBH spacetime differs from the Kerr BH spacetime in 4D in GR due to the presence of two spin parameters and the observable quantities are more complex because the spin ($a$ and $b$) breaks the spherical symmetry of the system.

We believe that the results obtained herewith would be useful in the study of gravitational lensing around the rotating BHs in higher dimensions in future.

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