One-shot Distillation in a General Resource Theory

Madhav Krishnan Vijayan, Eric Chitambar, and Min-Hsiu Hsieh

1Centre for Quantum Software and Information, University of Technology Sydney, NSW 2007, Australia
2Department of Electrical and Computer Engineering, Coordinated Science Laboratory,
University of Illinois at Urbana-Champaign, Urbana, IL 61801

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We present a general framework of a quantum resource theory based on the assumption of a) convexity and b) that the overlap of free states with maximally resourceful state is bounded. Using this structure, we derive bounds on the one-shot distillation rate for such a resource theory, thereby reproducing known bounds in coherence and entanglement. We use the free robustness and introduce a function $G_{\min}(\rho)$ related to overlap between $\rho$ and the set of free states to express our bounds.

To deal with resource theories where the free robustness in not finite we introduce the notion of imperfect free operations which we call $\epsilon$-resource generating operations and generalize the free robustness to $\epsilon$-free robustness. We construct an $\epsilon$-resource generating map that achieves pure state transformations in the theory and derive the conditions for such a transformation to exist in terms of the $\epsilon$-free robustness.

I. INTRODUCTION

In the development of quantum information theory, the operational approach has played a crucial role. It has enabled theoreticians to describe abstract properties of quantum systems in terms of practical and well-defined information processing tasks that can be performed in a lab. From the operational approach, the idea of certain states being a resource for information processing arose and lead to the development of resource theories, most prominently the resource theory of entanglement [1]. Since the inception of entanglement theory, several other quantum resource theories (QRTs) have been identified and studied such as those of coherence [2–4], thermodynamics [5, 6], non-uniformity (purity) [7–9] and asymmetry [10, 11] to name a few. See [12] for a wider review of quantum resource theories.

While what constitutes as a resource can vary widely between different theories, the same common structure is shared among many QRTs [13]. Broadly speaking, a QRT divides states and operations in quantum mechanics into ones that we have access to freely and ones which are costly for us to use in other words, free states and resourceful states. Studying this general structure independent of specific QRTs has allowed for a better understanding of certain quantum information quantities. For example, Brandão and Gour showed that the relative entropy of resource captures the asymptotic convertibility rate between two states when one considers resource non-generating operations in a general convex QRT [14]. An operational interpretation for general resources was given in [15] by showing that for any convex QRT there exists a channel discrimination task for which a resource state will strictly outperform a free state.

In this work we define a class of convex QRTs which include those of coherence, entanglement and non-uniformity. We require a QRT to satisfy two conditions; that it is convex and the overlap between the maxmaly resourceful state and the set of free states is bounded in a specific way. For such QRTs, we obtain upper bounds for state distillation and a map for pure state concentration in the single shot regime. To measure overlap of a state $\rho$ with the set of free states we introduce a function $G_{\min}(\rho)$ which is the maximal negative logarithm of the Hilbert-Schmidt norm between $\rho$ and the set of free states and our bounds are expressed in terms of smoothed versions of $G_{\min}(\rho)$.

The robustness of resource of a given state $\rho$ measures how much mixing of the form $p_1\rho + p_2\gamma$ is required to erase the resourcefulness of $\rho$ and is an important resource monotone [16] which has found application in...
the study of general resource theories [15][17]. Allowing
the state $\gamma$ to be mixed with $\rho$ to be arbitrary leads to
the definition of the global robustness of resource while
restricting $\gamma$ to be a free state leads to the definition of
the free-robustness of resource. The global robustness is
well defined for all resource theories however the free-
robustness is not finite for all states in every resource
type. In particular for affine resource theories it can
be shown that the free-robustness will diverge for all re-
source states and even for non-affine resource theories
there can be states without finite free-robustness [18][19].

We introduce a smoothed version of the free-robustness
which we will call the $\epsilon$-free robustness which has the
property that it is well defined for all resource theories
above some threshold value $\epsilon_0$. For this purpose we
define $\epsilon$-resource states (states with less than $\epsilon$ amount
of resource) and $\epsilon$-resource generating operations (op-
erations that can only create $\epsilon$ amount of resource from
free states). We derive the condition for imperfect one-
shot pure state interconversion via $\epsilon$-resource generating
states using the $\epsilon$-free robustness.

During the completion of this work we became aware
of an independent work which also derives bounds for
the one-shot distillation rate in terms of the hypothesis
testing relative entropy [18]. We note that the hypothes-
esis testing inequality is the operator smoothed version
of $G_{min}(\rho)$ while we use the state smoothed version
$G^\epsilon_{min}(\rho)$. Similarly the achievable map the authors in
[13] use for mixed state transformation is a variation of the
one we use for pure state transformation with oper-
ator smoothing to our state smoothing and is valid for
QRTs with finite free robustness whereas ours is valid for
QRTs where the free-robustness need not be finite. The
authors define a class of QRTs where there exists pure
reference states that have constant overlap with the set
free states which is conceptually similar to the condition
of bounded overlap with maximally resourceful states we
define (Axiom[2]).

II. DEFINITIONS

A resource theory is defined by the pair $\{\mathcal{F}, \mathcal{O}\}$ where
$\mathcal{F}$ is called the set of free states and $\mathcal{O}$ are the set of
free operations. In many resource theories there exists a
maximally resourceful unit pure state $\Phi$ such as the Bell
states for entanglement and the qubit uniform superpo-
position state for coherence. The one-shot distillation rate
of the resource is optimal rate at which we can convert a
single copy of an arbitrary state into several copies of the
maximally resourceful unit state under some error thresh-
old defined as,

$$R_d(\rho, \epsilon) := \max_{m \in \mathbb{N}} \left\{ m : \max_{\Lambda \in \mathcal{O}} F^2(\Lambda(\rho), \Phi^{\otimes m}) \geq 1 - \epsilon \right\}.$$  \hspace{1cm} (1)

We will use the notation $\Phi^{\otimes m}$ and $\Phi^m$ interchangeably.
We define $G_{min}(\rho)$ as a measure of the maximum overlap
between a positive operator $\rho$ and the set of free states $\mathcal{F}$ given by,

$$G_{min}(\rho) = \min_{\gamma \in \mathcal{F}} \{- \log_2 \text{Tr}(\rho \gamma)\}. \hspace{1cm} (2)$$

The fidelity between two states is defined as,

$$F(\rho, \sigma) := \text{Tr} \left( \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right) = \| \sqrt{\rho} \sqrt{\sigma} \|_1. \hspace{1cm} (3)$$

The $\epsilon$-ball around a state $\rho$ is defined as,

$$b(\rho, \epsilon) = \{ \| I \geq \bar{\rho} \geq 0 : F(\bar{\rho}, \rho) \geq 1 - \epsilon \}. \hspace{1cm} (4)$$

The pure state ball around a state $\rho$ is defined as,

$$b_*(\rho, \epsilon) = \{ \tilde{\psi} \in b(\rho, \epsilon) \text{ s.t. } \tilde{\psi} \text{ is pure} \}. \hspace{1cm} (5)$$

The state smoothed version of $G_{min}(\rho)$ is given by,

$$G^\epsilon_{min}(\rho) = \max_{\bar{\rho} \in b_*(\rho, \epsilon)} G_{min}(\bar{\rho}). \hspace{1cm} (6)$$

The pure state smoothed version $G^\epsilon_{min, s}(\rho)$ is defined similarly with the maximization over $b_*(\rho, \epsilon)$. The mi-
entropy of a state $\rho$ is defined as,

$$S_{min}(\rho) = - \log_2(\lambda_{max}(\rho)), \hspace{1cm} (7)$$

where $\lambda_{max}(\rho)$ is the largest eigenvalue of $\rho$.
In particular we will confine our attention to QRTs that
satisfy the following axioms,
TABLE I. Value of $G_{\min}(\psi)$ in different theories

| R. Theory | Entanglement | Coherence |
|-----------|--------------|-----------|
| $G_{\min}(\psi)$ | $S_{\min}(\rho_{\psi})$ | $S_{\min}(\Delta(\psi))$ |

**Axiom 1.** The set of free states $\mathcal{F}$ is convex.

This is a natural assumption satisfied by many resource theories though there are several exceptions such as the resource theory of total correlations and the resource theory of non-Gaussianity \[12\].

**Axiom 2.** For the unit maximally resourceful state it holds that

$$G_{\min}(\Phi^{\otimes m}) = \min_{\gamma \in \mathcal{F}} \{-\log Tr(\Phi^{\otimes m}\gamma)\} \geq m.$$ \hspace{1cm} (8)

Axiom 2 is satisfied the resource theories of entanglement, coherence and non-uniformity.

**III. CONVERSE**

**Theorem 1.** For any resource theory satisfying axiom 2, the one-shot distillation of resource is bounded as,

$$R_d(\rho, \epsilon) \leq \max_{\bar{p} \in b(\rho, 2\sqrt{2}\epsilon)} G_{\min}(\bar{p}),$$ \hspace{1cm} (9)

with $\epsilon \geq 0$.

**Proof.**

Let $m$ be the highest rate achievable with error $\epsilon$. This implies that there exists a free operation $\Lambda \in \mathcal{O}$ such that $F^2(\Lambda(\rho), \Phi^m) \geq 1 - \epsilon$.

Axiom 2 can equivalently be stated as (see Appendix A),

$$\Phi^{\otimes m}\gamma\Phi^{\otimes m} \leq \frac{1}{2^m}\Phi^{\otimes m} \quad \forall \gamma \in \mathcal{F}.$$ \hspace{1cm} (10)

Multiplying both sides of equation (10) by $\Lambda(\rho)$, replacing $\gamma$ by $\Lambda(\gamma)$ and taking the trace gives,

$$Tr(\Lambda(\rho)\Phi^m\Lambda(\gamma)\Phi^m) \leq \frac{1}{2^m} Tr(\Lambda(\rho)\Phi^m),$$ \hspace{1cm} (11)

$$\leq \frac{1}{2^m} \quad \therefore \Lambda(\rho) \leq 1.$$ \hspace{1cm} (12)

Using the cyclic property of trace and denoting the dual map of $\Lambda$ as $\Lambda^*$ gives

$$m \leq -\log_2 Tr(\Phi^m\Lambda(\rho)\Phi^m\Lambda(\gamma)),$$ \hspace{1cm} (13)

$$= -\log_2 Tr(\Lambda^*(\Phi^m\Lambda(\rho)\Phi^m)\gamma),$$ \hspace{1cm} (14)

$$= -\log_2 Tr(Q\gamma),$$ \hspace{1cm} (15)

$$\leq -\log_2 Tr(\sqrt{Q}\rho\sqrt{Q}),$$ \hspace{1cm} (16)

where in the last inequality we use the fact that $\bar{p} := \sqrt{Q}\rho\sqrt{Q} \leq Q$. Since $\gamma$ is an arbitrary free state, we can say that

$$m \leq \min_{\gamma \in \mathcal{F}} \{-\log_2 Tr(\bar{p}\gamma)\} = G_{\min}(\bar{p}).$$ \hspace{1cm} (17)

We will now show that $\bar{p} \in b(\rho, 2\sqrt{2}\epsilon)$. Note that,

$$Tr(Q\rho) = Tr(\Phi^m\Lambda(\rho)\Phi^m\Lambda(\rho))),$$ \hspace{1cm} (18)

$$= \langle \Phi^m|\Lambda(\rho)\Phi^m \rangle^2,$$ \hspace{1cm} (19)

$$= (F^2(\Lambda(\rho), \Phi^m))^2 \geq 1 - 2\epsilon.$$ \hspace{1cm} (20)

where for the last inequality we use the fact that $F^2(\Lambda(\rho), \Phi^m) \geq 1 - \epsilon$. From the gentle measurement lemma \[20\] we can see that,

$$\|\rho - \bar{p}\|_1 \leq 2\sqrt{2}\epsilon.$$ \hspace{1cm} (21)

Using the bound

$$1 - F(\omega, \sigma) \leq \frac{1}{2}\|\omega - \sigma\|_1,$$ \hspace{1cm} (22)

we get,

$$F^2(\rho, \bar{p}) \geq 1 - 2\sqrt{2}\epsilon$$ \hspace{1cm} (23)

and $\bar{p} \in b(\rho, 2\sqrt{2}\epsilon)$. If $\rho$ is a pure state, we can get a quadratic improvement in this bound as (see Appendix B),

$$m \leq \max_{\psi \in b_{\psi}(\psi, 2\epsilon)} G_{\min}(\psi).$$ \hspace{1cm} (24)

Equation (24) immediately recovers known results for the one-shot concentration rate in entanglement \[21\] and coherence \[22, 23\] as shown below.

**Corollary 1.** The one-shot concentration rate for entanglement $E_c(\psi^{AB}, \epsilon)$ and the one-shot concentration rate of coherence $C_c(\psi, \epsilon)$ are given by,

$$E_c(\psi^{AB}, \epsilon) \leq \max_{\psi^{AB} \in b_{\psi}(\psi^{AB}, 2\epsilon)} S_{\min}(\rho_{\psi^{AB}}),$$ \hspace{1cm} (25)

$$C_c(\psi, \epsilon) \leq \max_{\psi \in b_{\psi}(\psi, 2\epsilon)} S_{\min}(\Delta(\psi)).$$ \hspace{1cm} (26)
where \( \rho_{\psi^{AB}} = \text{Tr}_B(\psi^{AB}) \) is the reduced density matrix of \( \overline{\psi}^{AB} \) and \( \Delta(\psi) = \sum_i |i\rangle \langle i| \psi_i \overline{\psi}_i \) is the completely dephased version of \( \psi \) in the incoherent basis.

**Proof.** See Appendix C.

### IV. DIRECT

In the one-shot setting it is established practise to allow for arbitrary error \( \epsilon \) in the final state as was done in theorem [1]. However this is not the only way we can relax the constraint of a perfect transformation. We can also allow an error \( \tilde{\epsilon} \) in the free operation used. For this purpose we define \( \tilde{\epsilon} \)-resource states as

\[
F^{\tilde{\epsilon}} = \{ \rho : C_{\tilde{\epsilon}}(\rho) \leq \tilde{\epsilon} \}
\]

where \( C_{\tilde{\epsilon}}(\rho) \) is the relative entropy of resource which is a resource monotone defined as,

\[
C_{\tilde{\epsilon}}(\rho) = \min_{\gamma \in F} S(\rho\|\gamma),
\]

and \( S(\rho\|\sigma) \) is the relative entropy. \( \tilde{\epsilon} \)-resource generating operations are defined as,

\[
O^{\tilde{\epsilon}} := \{ \Lambda : \Lambda(\gamma) \in F^{\tilde{\epsilon}}, \forall \gamma \in F \}.
\]

**Theorem 2.** For any resource theory satisfying axiom [7] and pure states \( \psi \) and \( \phi \) there exists a CPTP map \( \Lambda \) of the form

\[
\Lambda(\omega) = \text{Tr}[(I - \overline{\psi})\omega]\pi_{\phi} + \text{Tr}[\overline{\psi}\omega]\phi,
\]

where \( \overline{\psi} \) is the pure state that optimizes \( G_{\text{min,}*}(\psi) \) such that

\[
\Lambda(\psi) \in b(\phi, \epsilon)
\]

and \( \Lambda \) is an \( \tilde{\epsilon} \)-free operation iff

\[
G_{\text{min,*}}^{\tilde{\epsilon}}(\psi) \geq \log(1 + R_{\tilde{\epsilon}}^f(\phi)),
\]

where \( R_{\tilde{\epsilon}}^f(\rho) \) is the \( \tilde{\epsilon} \)-free robustness defined as,

\[
R_{\tilde{\epsilon}}^f(\rho) := \min_{\gamma \in F} \left\{ s \geq 0 : \frac{\rho + s\gamma}{1 + s} \in F^{\tilde{\epsilon}} \right\}
\]

and \( \pi_{\phi} \) is the optimal free state that achieves the minimisation for \( R_{\tilde{\epsilon}}^f(\phi) \).

**Proof.** See appendix D.

Corollary 2. For the resource theory of entanglement the perfect transformation \( \psi \to \Phi^{\otimes m} \), where \( \Phi \) is the unit maximally entangled state is achievable with a free operation \( \Lambda \in O \) with a rate

\[
E_c(\psi^{AB}, 0) = G_{\text{min}}(\psi^{AB}) = S_{\text{min}}(\rho_{\psi^{AB}}),
\]

where \( \rho_{\psi^{AB}} = \text{Tr}_B(\psi^{AB}) \).

**Proof.** From theorem 2 in the limit \( \tilde{\epsilon} \to 0 \) we know that for the transformation \( \psi \to \Phi^{\otimes m} \) there exists a free operation \( \Lambda \) iff

\[
G_{\text{min}}(\psi) \geq \log(1 + R_f(\Phi^{\otimes m})),
\]

\[
= \log(1 + R_f(\Phi^{\otimes m}))
\]

\[
= \log(1 + 2^m - 1) = m,
\]

where we have used the fact that the free robustness of entanglement is equal to the global robustness of entanglement \( R_g(\rho) \) for pure states and for the maximally entangled state of rank \( d \) it is given by \( d - 1 \) [24, 25]. Combining equations (37) and (24) in the limit \( \epsilon \to 0 \) gives the desired result.

**Remark.** For any dimension \( d \geq 2 \), the \( \tilde{\epsilon} \)-free robustness of coherence \( R_{\tilde{\epsilon}}^f(\Phi^{\otimes m}) \) is achieved by the maximally mixed state \( \mathbb{1}_d = \frac{1}{d} \sum_i |i\rangle \langle i| \), where \( m = \log_2 d \).

**Proof.** Let the optimal incoherent state achieving \( R_{\tilde{\epsilon}}^f(\Phi^{\otimes m}) \) be \( \pi_{\Phi^{\otimes m}} \). We define the set of all permutations of basis in a Hilbert space of dimension \( d \) as \( \Pi^d \). Consider the twirling operation of averaging over all possible incoherent basis permutations in \( \Pi^d \) defined as,

\[
T(\rho) = \frac{1}{d!} \sum_{\pi \in \Pi^d} \pi(\rho).
\]

Coherence measures will be invariant under twirling as it is invariant under permutations of the incoherent basis and averaging cannot increase coherence. This implies
that the set $\mathcal{I}^\varepsilon$ is closed under the twirling operation. Notice that the maximally coherent state is invariant under the twirling operation, i.e. $T(\Phi^m) = \Phi^m$. From the definition of $\varepsilon$-resource robustness we have,

$$
\rho = \frac{\Phi^m + R^\varepsilon_f(\Phi^m) \pi_{\Phi^m}}{1 + R^\varepsilon_f(\Phi^m)} \in \mathcal{I}^\varepsilon,
$$

where $\mathcal{I}^\varepsilon$ is the set of $\varepsilon$-incoherent states. Applying the twirling operation on both sides of equation (39) we have,

$$
T(\rho) = \frac{\Phi^m + R^\varepsilon_f(\Phi^m) T(\pi_{\Phi^m})}{1 + R^\varepsilon_f(\Phi^m)} \in \mathcal{I}^\varepsilon.
$$

The last inclusion follows from the fact that coherence is invariant under the twirling operation. Equation (40) implies that if mixing $R^\varepsilon_f$ amount of $\pi_{\Phi^m}$ with $\Phi^m$ gives you a state in $\mathcal{I}^\varepsilon$ then mixing $R^\varepsilon_f$ amount of $T(\pi_{\Phi^m})$ will also give you a state in $\mathcal{I}^\varepsilon$. For any incoherent state $\delta$, $T(\delta)$ will be the completely mixed state $\frac{I}{D}$. We can see this by noticing that the state $T(\delta)$ is permutation invariant by virtue of the twirling operation and the only permutation invariant incoherent state is the maximally mixed state.

V. DISCUSSION

We have given a general treatment for a class of resource theories that include that of entanglement, coherence and non-uniformity. We derive the one-shot upper and lower bounds for resource distillation of a maximally resourceful pure state and show these reduce to the known results in the resource theory of entanglement and coherence. The map we use allows for imperfections in the free map such that it can create a small amount of resource. In this way we introduce the notion of $\varepsilon$-resource states $\varepsilon$-resource generating operations and $\varepsilon$-free robustness. We show that in the resource theory of coherence the $\varepsilon$-free robustness if achieved by mixing with the maximally mixed state.

While we were able to derive the condition needed to be satisfied for pure state transformation using $\varepsilon$-resource generating operations and hence for pure state concentration $\psi \rightarrow \Phi^m$, it remains an open question from both this work and [18] what would be an analytical expression for the optimal rate $R_d(\psi, \varepsilon)$ as a function of $\psi$. In other words what is the largest integer $m_0$ such that $G_{min,\varepsilon}^\varepsilon(\psi) \geq \log(1 + R^\varepsilon_f(\Phi^{m_0})$ is satisfied. A future direction of research would be to answer this question. A starting point could to answer it for resource theories with certain symmetries or even for bounded dimension. It is also of interest to consider defining the $\varepsilon$-resource states using a different resource monotone other than the relative entropy of resource and also to explore the trade off between error in the operation used $\varepsilon$ and error in the final state $\epsilon$.

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Appendix A: Proof of equivalence of axiom 2 and equation (10)

Axiom 2 states that

\[ G_{\text{min}}(\Phi^m) \geq m. \]  

(A1)

By definition of \( G_{\text{min}} \) this is the same as,

\[ \min_{\gamma \in \mathcal{F}} - \log(\text{Tr}(\Phi^m \gamma)) \geq m, \]  

(A2)

\[ \implies - \log(\text{Tr}(\Phi^m \gamma)) \geq m \quad \forall \gamma \in \mathcal{F}, \]  

(A3)

\[ \implies \text{Tr}(\Phi^m \gamma) \leq 2^{-m}, \]  

(A4)

\[ \implies \langle \Phi^m | \gamma | \Phi^m \rangle \leq 2^{-m}, \]  

(A5)

\[ \implies \langle \Phi^m | \gamma | \Phi^m \rangle \Phi^m \leq 2^{-m} \Phi^m, \]  

(A6)

\[ \implies \Phi^m \gamma \Phi^m \leq \frac{1}{2^m} \Phi^m. \]  

(A7)

Starting from the final expression, all the steps can be reversed to obtain the initial expression hence proving the equivalence.

Appendix B: Pure state converse

The pure state converse follows the same steps as the mixed state converse in theorem 1. In equation (18), replacing the mixed state \( \rho \) with the pure state \( \psi \) we have,

\[ \text{Tr}(Q\psi) \geq 1 - 2\epsilon. \]  

(B1)

Note that \( \overline{\psi} = \sqrt{Q}\psi\sqrt{\psi} \), hence

\[ F(\psi, \overline{\psi}) = \langle \psi | \sqrt{Q} | \psi \rangle, \]  

\[ \geq \langle \psi | Q | \psi \rangle, \]  

\[ = \text{Tr}(Q\psi), \]  

\[ \geq 1 - 2\epsilon. \]  

(B2)

Hence \( \overline{\psi} \in b_\epsilon(\psi, 2\epsilon) \).

Appendix C: Converse for of one-shot concentration of Entanglement and coherence

1. Entanglement

Notice that for any separable state can be expressed as \( \gamma = \sum_i q_i \rho_i \otimes \sigma_i \). To see that entanglement theory satisfies Axiom 2 it is sufficient to show the following:

\[ \Phi^m \gamma \Phi^m \leq \frac{1}{2^m} \Phi^m, \]  

(C1)

where \( |\Phi^m\rangle = \frac{1}{\sqrt{2^m}} \sum_i |ii\rangle \) is the maximally entangled state. Using the definitions directly we have,

\[ \Phi^m \gamma \Phi^m = \langle \Phi^m | \gamma | \Phi^m \rangle \Phi^m, \]  

(C2)

where,

\[ \alpha = \max_{\gamma \in \text{SEP}} \text{Tr}(\Phi^m \gamma). \]  

(C3)
Now using the definition of $\gamma$ we can write,
\[
\max_{\gamma \in \text{SEP}} \text{Tr}(\Phi^m \gamma) = \max_{\rho, \sigma} \frac{1}{2m} \sum_k p_k \sum_{i,j} \langle j| \rho_k |i\rangle \langle j| \sigma_k |i\rangle, \quad (C4)
\]
\[
\leq \frac{1}{2m} \max_{\rho, \sigma} \sum_{i,j} \langle j| \rho |i\rangle \langle j| \sigma |i\rangle, \quad (C5)
\]
\[
= \frac{1}{2m} \max_{\rho, \sigma} \sum_{i,j} \langle j| \rho |i\rangle \langle i| \sigma^T |j\rangle, \quad (C6)
\]
\[
= \frac{1}{2m} \max_{\rho, \sigma} \text{Tr}(\rho \sigma^T), \quad (C7)
\]
\[
\leq \frac{1}{2m}, \quad (C8)
\]
where in the last inequality we have used the fact that $\rho, \sigma^T \leq I$. Thus Axiom \[2\] is satisfied.

From equation (C4) we have the ideal rate of one-shot concentration $m$ is bounded as,
\[
m \leq \max_{\overline{\psi} \in \mathcal{b}_*, \langle \psi, 2\epsilon \rangle} G_{\text{min}}(\overline{\psi}). \quad (C9)
\]

Let $\overline{\psi}$ be the state that achieves this maximisation. Then,
\[
m \leq \min_{\gamma \in \text{SEP}} \left\{ -\log_2 \text{Tr}(\Phi(\overline{\psi})\gamma) \right\}, \quad (C10)
\]
where SEP is the set of separable states. Since $\overline{\psi}$ is really a bipartite pure state $\overline{\psi}^{AB}$, we can write it as a purification of its reduced density matrix $\overline{p}^B = \text{Tr}_A(\overline{\psi}^{AB})$. Let $\overline{p}^B = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i|$, then there is some unitary $U$ such that,
\[
|\overline{\psi}^{AB}\rangle = \sum_i \sqrt{\lambda_i} U |\lambda_i\rangle^A |\lambda_i\rangle^B. \quad (C11)
\]

Now the trace in equation (C10) is maximized by a product state $\gamma = \sigma \otimes \delta$, since any convex combination can only decrease the trace. So,
\[
\max_{\gamma} \text{Tr}(\overline{\psi}^{AB} \gamma) = \max_{\sigma, \delta} \sum_{i,j} \sqrt{\lambda_i \lambda_j} \langle \lambda_j| U^{\dagger} \sigma U |\lambda_j\rangle \langle \lambda_i| \delta |\lambda_i\rangle. \quad (C12)
\]

The above sum is maximized by choosing $\sigma = U |\lambda_{\max}\rangle \langle \lambda_{\max}| U^{\dagger}$ and $\delta = |\lambda_{\max}\rangle \langle \lambda_{\max}|$ where $|\lambda_{\max}\rangle$ is understood to be the eigenvector with the largest eigenvalue. So,
\[
\max_{\gamma \in \text{SEP}} \text{Tr}(\overline{\psi}^{AB} \gamma) = \lambda_{\max}(\overline{p}^B). \quad (C13)
\]

Using this,
\[
m \leq -\log_2 \lambda_{\max}(\overline{p}^B) = S_{\text{min}}(\overline{p}^B). \quad (C14)
\]

2. Coherence

To show that the resource theory of coherence satisfies Axiom \[2\] notice that,
\[
\Phi^m \gamma \Phi^m = \text{Tr}(\Phi^m \gamma) \Phi^m, \quad (C15)
\]
\[
= \frac{1}{2m} \Phi^m, \quad (C16)
\]
where in the second line we have used the fact that for any incoherent state $\gamma$ and pure state $\psi$, $\text{Tr}(\psi \gamma) = \lambda_{\max}(\Delta(\psi))$, where $\lambda_{\max}$ gives the largest eigenvalue and $\Delta$ is the completely dephasing map in the incoherent basis. Thus Axiom \[2\] is satisfied.

Following the same line for reasoning as for entanglement, we have for some optimal state $\overline{\psi}$, the one-shot concentration of coherence with error $\epsilon$ is given by,
\[
m \leq \min_{\gamma \in \mathcal{I}} \left\{ -\log_2 \text{Tr}(\overline{\psi} \gamma) \right\}, \quad (C17)
\]
\[
= -\log_2(\lambda_{\max}(\Delta(\overline{\psi}))),
\]
\[
= S_{\text{min}}(\Delta(\overline{\psi})).
\]

Appendix D: Proof of theorem \[2\]

There are two parts to prove in theorem \[2\] namely that $\Lambda(\psi)$ is close to $\phi$ and that $\Lambda$ is an $\epsilon$-resource generating operation. To prove the former note, that
\[
F(\phi, \Lambda(\psi)) \geq F^2(\phi, \Lambda(\psi)) \quad (D1)
\]
\[
= \text{Tr}(\phi \Lambda(\psi)) \quad (D2)
\]
\[
= p_1 + (1 - p_1) \text{Tr}(\phi \pi_\phi) \quad (D3)
\]
\[
\geq p_1 \quad (D4)
\]
\[
\geq 1 - \epsilon \quad (D5)
\]
where $p_1 = \text{Tr}(\psi \overline{\psi})$. Hence $\Lambda(\psi) \in b(\phi, \epsilon)$. To show that $\Lambda$ will be an $\epsilon$-free operation we need to show that for all $\gamma \in \mathcal{F}$
\[
\Lambda(\gamma) = \text{Tr}([I - \overline{\psi}] \gamma |\pi_\phi + \text{Tr}[\overline{\psi} |\phi \in \mathcal{F}. \quad (D6)
\]
This is true if and only if

\[
\text{Tr}[(I - \overline{\psi})\gamma] \geq \frac{\mathcal{R}_f^\epsilon(\phi)}{1 + \mathcal{R}_f^\epsilon(\phi)},
\]

\[\implies 1 - \text{Tr}(\gamma\overline{\psi}) \geq \frac{\mathcal{R}_f^\epsilon(\phi)}{1 + \mathcal{R}_f^\epsilon(\phi)},\]  \hspace{1cm} (D7)

\[\implies \frac{1}{1 + \mathcal{R}_f^\epsilon(\phi)} \geq \text{Tr}(\gamma\overline{\psi}),\]  \hspace{1cm} (D8)

\[\implies - \log(1 + \mathcal{R}_f^\epsilon(\phi)) \geq \log(\text{Tr}(\gamma\overline{\psi})),\]  \hspace{1cm} (D9)

\[\implies G_{\min,s}^\epsilon(\psi) \geq \log(1 + \mathcal{R}_f^\epsilon(\phi)),\]  \hspace{1cm} (D10)

where in the last line we have used the fact that \(\gamma\) is an arbitrary free state and the definition of \(\overline{\psi}\) as the state that achieves the optimization for \(G_{\min,s}^\epsilon(\rho)\).