Perturbative $c$-theorem in $d$-dimensions

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Abstract

We study perturbative behavior of free energies on a $d$-dimensional sphere $S^d$ for theories with marginal interactions. The free energies are interpreted as the “dilaton effective action” with the dilaton having a nontrivial background vacuum expectation value. We compute the dependence of the free energies on the radius of the sphere by using dimensional regularization. It is shown that the first (second) derivative of the free energies in odd (even) dimensions with respect to the radius of the sphere are proportional to the square of the beta functions of coupling constants. The result is consistent with the $c$, $F$ and $a$-theorems in two, three, four and six dimensions. The result is also used to rule out a large class of scale invariant theories which are not conformally invariant.
1 Introduction

In two dimensional quantum field theory, two elegant theorems are known. Zamolodchikov showed [1] that there exists a function $c(r)$ of a length scale $r$ which monotonically decreases as $r$ is increased, and becomes constant only on conformal fixed points. Roughly speaking, this result indicates that a number of “degrees of freedom” monotonically decreases along renormalization group (RG) flows. This is the famous Zamolodchikov’s $c$-theorem. Then, Polchinski proved [2] that all scale invariant theories (with discrete spectrum of scaling dimensions) are also conformally invariant. The result of Ref. [1] played a crucial role in the proof of Ref. [2].

There have also been significant developments in the study of monotonically decreasing quantities in higher dimensions. In even dimensional CFT, the trace of the energy-momentum tensor has anomaly when the CFT is coupled to external gravitational background. It is given as [3]

$$T_{\mu}^{\mu} = (-1)^{\frac{d}{2}+1}a E_d + \cdots, \quad (1)$$

where $E_d \propto \epsilon_{\mu_1 \mu_2 \cdots \mu_{d-1} \mu_d} \epsilon^{\nu_1 \nu_2 \cdots \nu_{d-1} \nu_d} R_{\mu_1 \mu_2 \nu_1 \nu_2} \cdots R_{\mu_{d-1} \mu_d \nu_{d-1} \nu_d}$ is the Euler density, and the dots indicate terms which vanish in conformally flat background. In two dimensional CFTs, the coefficient of the Euler density $E_d$ in the trace anomaly (written as $a$ in Eq. (1)) coincides with the Zamolodchikov’s $c$ function. In general even dimensional field theories, it was conjectured [4] that $a$ decreases along RG flows in theories which interpolate UV and IR CFTs, that is, $a_{\text{IR}} < a_{\text{UV}}$.

The quantity $a$ may be extracted in the following way. Let us put a CFT on a $d$-dimensional sphere $S^d$ with radius $r$ and consider the partition function on the sphere,

$$Z = \int [D\varphi] e^{-S}, \quad (2)$$

where $\varphi$ denotes dynamical fields of the theory, $S$ is the action on the sphere, and $\int [D\varphi]$ is the path integral. Using the fact that the change of the radius $r$ as $r \rightarrow e^{\sigma}r$ for a constant $\sigma$ is equivalent to the Weyl rescaling of the metric as $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$, the free energy $F = -\log Z$ satisfies

$$\frac{dF}{d\log r} = -\left( \int d^d \sqrt{g} T_{\mu}^{\mu} \right) \propto (-1)^{\frac{d}{2}} a, \quad (3)$$
where we have used the fact that the metric of the sphere is conformally flat in projective coordinates, \( ds^2 = (2r^2/(x^2 + r^2))^2 dx^2 \), and hence the terms denoted by the dots in Eq. (11) do not contribute. Therefore, the above conjecture may be interpreted as the conjecture that the function \((-1)^{d/2} dF/d\log r\) decreases as \( r \) is increased.

In odd dimensions, there is a similar conjecture about the free energy \( F \) \([5, 6]\). In this case, it is \((-1)^{d+1}/2 F\) that is conjectured to decrease. Therefore, in both even and odd dimensions, the free energy on the sphere, \( F = -\log Z \), plays an important role in the study of monotonically decreasing quantities.

Recently, a proof that \( a \) satisfies \( a\text{IR} < a\text{UV} \) was given in four dimensions \([7]\) and further discussed in Refs. \([8, 9]\). Also, a monotonically decreasing quantity was constructed in three dimensions \([10]\) which coincides with \( F \) in CFT \([11]\). Completely different methods were used in the proofs in two \([1]\), three \([10]\) and four \([7]\) dimensions. There is still no proof in general space-time dimensions, although holography suggests the existence of a monotonically decreasing function in arbitrary dimensions \([12, 13, 14]\). See Refs. \([15, 16, 17]\) for recent works in six and higher dimensions.

Progress has also been made regarding the equivalence of scale and conformal invariance in four dimensions. A proof of the equivalence was given in perturbation theory \([9, 18]\) (see also Refs. \([19, 2, 20, 21, 22]\) \([19] \)). Ref. \([9]\) also gave a non-perturbative argument in favor of the equivalence. In that proof, the existence of a monotonically decreasing quantity \( a \) (or more precisely the dilaton forward scattering amplitude) is essential. This is similar to the proof of the equivalence in two dimensions \([11, 2]\).

However, much less is known in other dimensions. In particular, there are many perturbative field theories in three dimensions, and there is a possibility that some of them could be scale invariant without conformal invariance by the same mechanism discussed in Refs. \([23, 24, 25, 26]\).

As discussed above, one of the ways to generalize the two dimensional theorems to arbitrary dimensions may be to use the free energy on the sphere. The flow of the free

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1. Although the existence of explicit counterexamples are discussed \([23, 24, 25, 26]\), they are argued to be conformally invariant \([27, 28, 18]\) based on the results of Refs. \([29, 30]\).

2. There exist free field theory counterexamples in \( d > 4 \) \([31, 32]\). But there is no local current operator for scaling symmetry and only the charge is well-defined in those theories. There still remains a possibility that every scale invariant theory with a scaling current is conformally invariant.
Table 1: Schematic forms of the Lagrangians of perturbative theories with marginal interactions. The fields $\phi$ are bosons, $\psi$ are fermions, $A$ are gauge bosons, $F$ are gauge field strengths, and possible indices specifying these fields are suppressed. $G_b(\phi)$, $G_f(\phi)$ and $H(\phi)$ are arbitrary functions of scalar fields $\phi$. All the coupling constants are dimensionless in these Lagrangians.

| Dimensions | Lagrangians |
|------------|-------------|
| $d = 2$    | $G_b(\phi)(\partial_\mu \phi)^2 + G_f(\phi)\psi \bar{\psi} + H(\phi)\psi^4$ |
| $d = 3$    | $(A \partial A + \frac{3}{2} A^3) + (D_\mu \phi)^2 + \psi \bar{\psi} + \phi^6 + \phi^2 \psi^2$ |
| $d = 4$    | $(F_{\mu\nu})^2 + (D_\mu \phi)^2 + \psi \bar{\psi} + \phi^4 + \phi^2 \psi^2$ |
| $d = 6$    | $(\partial_\mu \phi)^2 + \phi^4$ |

energy was studied when a CFT is deformed by adding slightly marginal operators $\mathcal{O}$ to the Lagrangian $[4, 6]$. The operators were assumed to have scaling dimensions $d - y$ with $y \ll 1$.

In this paper, we study the free energy for general weakly interacting field theories with marginal interactions. A list of such theories is given in Table 1. We show that $(-1)^{\frac{d-2}{2}} F$ (in odd dimensions) or $(-1)^{\frac{d}{2}} dF/d\log r$ (in even dimensions) decreases monotonically in these theories. Furthermore, following Ref. 9, we argue that scale invariance is equivalent to conformal invariance in these theories.

The rest of the paper is organized as follows. In section 2 we give a relation between the free energy on the sphere and the "dilaton effective action" which was used in the proof of the $a$-theorem in four dimensions $[7, 8, 9]$. It enables us to compute perturbative flows of the free energy by using the method of Refs. 8, 9. We obtain the dilaton effective action in dimensional regularization. In section 3, we compute the flow of the free energy using the dilaton effective action. We check our result in two, three, four and six dimensions. Using the result, we argue the equivalence of scale and conformal invariance. Section 4 is devoted to conclusions.

## 2 Dilaton effective action

We define the free energy of a theory on a $d$-dimensional sphere as a dilaton effective action in the following way. We first consider the partition function as a functional of a background metric $\hat{g}_{\mu\nu}$. (The hat is used on the metric following the notation of
It is given as

\[ Z = \int [D\varphi] \exp \left( -S[\varphi, \hat{g}_{\mu\nu}] - S_{\text{c.t.}}[\hat{g}_{\mu\nu}] \right) = Z_0 \exp \left( -S_{\text{eff},0}[\hat{g}_{\mu\nu}] - S_{\text{c.t.}}[\hat{g}_{\mu\nu}] \right), \]  

where \( \varphi \) denotes dynamical fields of the theory, and \( S[\varphi, \hat{g}_{\mu\nu}] \) is the action of the fields \( \varphi \) coupled to the metric \( \hat{g}_{\mu\nu} \). The factor \( Z_0 \) is the contribution to the partition function which does not depend on the background metric, and \( S_{\text{eff},0} \) is the (bare) effective action of the metric obtained as a result of the path integral. The counterterm \( S_{\text{c.t.}} \) is taken so that the functional

\[ S_{\text{eff}}[\hat{g}_{\mu\nu}] = S_{\text{eff},0}[\hat{g}_{\mu\nu}] + S_{\text{c.t.}}[\hat{g}_{\mu\nu}], \]  

becomes finite. We will impose further condition on the counterterms \( S_{\text{c.t.}} \) later.

We introduce a dilaton field \( \tau \) and a new metric \( g_{\mu\nu} \), as \( \hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu} \). Then the dilaton effective action is defined as

\[ S[\tau, g_{\mu\nu}] = S_{\text{eff}}[\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}]. \]

This definition of the dilaton effective action is emphasized in Ref. [9]. When \( g_{\mu\nu} = \eta_{\mu\nu} \), it gives the dilaton effective action in flat space, and this definition makes clear the invariance of the dilation effective action under conformal transformations. This is because conformal transformations are just the subgroup of the diffeomorphism of the original metric \( \hat{g}_{\mu\nu} \) which preserves the form \( ds^2 = e^{-2\tau} dx^2 \).

The metric of the sphere with radius \( r \) can be written using the projective coordinates as \( ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = [2r^2/(x^2 + r^2)]^2 dx^2 \). However, we may also interpret this as a flat metric \( g_{\mu\nu} = \eta_{\mu\nu} \) with a nontrivial background for the dilaton, \( e^{-\tau} = 2r^2/(x^2 + r^2) \). Then, the free energy of the theory on the sphere, \( F = -\log Z \), is given as

\[ F(r) = -\log Z_0 + S \left[ e^{-\tau} = \frac{2r^2}{x^2 + r^2}, \eta_{\mu\nu} \right]. \]

By this interpretation, we can use the results of Refs. [8, 9] for the dilaton effective action to compute the free energy on the sphere.

It is clear that the dependence of \( F(r) \) on the radius of the sphere \( r \) should be contained in the second term of Eq. (7). In this paper we attempt to calculate only the derivatives.
of $F(r)$ with respect to $r$. Then we may neglect the term $\log Z_0$, and focus on the dilaton effective action.

The above definition still has an ambiguity regarding the choice of the counterterms in $S_{\text{c.t.}}$. Although the divergent part of $S_{\text{c.t.}}$ is determined uniquely so that it makes the metric effective action $S_{\text{eff}}$ finite, the finite part of $S_{\text{c.t.}}$ is not fixed. We impose the following requirement on the finite part. In this paper we only consider massless theories which do not contain dimensionful parameters (see Table. 1). Furthermore, we always use dimensional regularization as a regularization method. Then, by using mass-independent renormalization scheme (such as minimal subtraction), counterterms which contain dimensionful coefficients are not necessary (see e.g. Ref. [33]). That is, we can set all the counterterms to zero aside from counterterms with dimensionless coefficients which are schematically given as

$$S_{\text{c.t.}}[\hat{g}_{\mu\nu}] \sim \int d^d x \sqrt{\hat{g}} (R_{\mu\rho\sigma}(\hat{g}))^{\frac{d}{2}},$$

(8)

where $R_{\mu\rho\sigma}$ is the Riemann tensor and indices are contracted in arbitrary ways. Therefore, we only introduce counterterms of the form (8). This is our criterion for choosing the counterterms.

In the case of odd dimensions, terms like Eq. (8) do not exist and hence we need no counterterms at all. Therefore $F(r)$ is uniquely determined by our criterion. In even dimensions, finite counterterms of the form (8) are allowed\(^\text{3}\) and hence the ambiguity in defining $F(r)$ remains. However, one can see that the contributions coming from these finite counterterms disappear if we take the derivative of the free energy, $dF/dr$\(^4\). As discussed in the introduction, the important quantity in even dimensions is $dF/dr$ rather than $F$ itself, and hence the remaining ambiguity in choosing the counterterms does not matter.

The above requirement on the counterterms is a little technical. More physical requirement may be that the free energy $F$ (in odd dimensions) or its derivative $dF/dr$ (in

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\(^3\)The finite counterterms are in fact necessary in order for the effective action $S_{\text{eff}}(\hat{g}_{\mu\nu})$ to be RG invariant. Even if they are set to zero at some RG scale, they are generated along RG flows.

\(^4\)Strictly speaking, these contributions are not precisely zero in dimensional regularization. They are suppressed by $\epsilon$, where the space-time dimensions is given by $d = (\text{integer}) - 2\epsilon$. Then, the contributions of the finite part of the counterterms become zero when we take $\epsilon \to 0$, but the contributions from divergent part of the counterterms are important.
even dimensions) becomes constant on UV/IR fixed points. This physical requirement will be satisfied by the above choice of the counterterms.

Now let us study the dilaton effective action $S[\tau]$ in flat (Euclidean) space-time. The most important part of $S[\tau]$ in perturbation theory has been given in Refs. [8, 9]. We use dimensional regularization where we work in $d = d_0 - 2\epsilon$ dimensions with $d_0$ an integer.

We expand the action as

$$S[\varphi, \eta_{\mu\nu}] = S[\varphi, \eta_{\mu\nu}] + \int d^d x \tau T^\mu_{\mu} + O(\tau^2),$$

where $T^\mu_{\mu}$ is the energy-momentum tensor. The linear term in $\tau$ is proportional to the trace anomaly. We assume that interaction terms are present in the Lagrangian as $\lambda_0^{i} O_i$, where $\lambda_0^{i}$ are bare couplings and $O_i$ are bare operators. For example, in a four-dimensional scalar $\phi^4$ theory, we may define $O = \frac{1}{4!} \phi^4$. Then, if the energy-momentum tensor is improved appropriately, the trace anomaly may be given as

$$T^\mu_{\mu} = -\sum_i B^i [O_i],$$

where $[O_i]$ are the renormalized operators corresponding to the bare operator $O_i$, and $B^i$ are the beta functions of the renormalized coupling constants $\lambda^i$. In cases where there are many flavors of matter fields, there are ambiguities in the definition of usual beta functions $\beta^i$, while the beta functions $B^i$ appearing in Eq. (10) is unambiguous [29, 30]. Following the notation of Refs. [29, 30], we denote this unambiguous beta functions as $B^i$ rather than $\beta^i$.

The improvement of the energy-momentum tensor is related to the term $R\phi^2$ in the Lagrangian, where $R$ is the Ricci scalar and $\phi$ are scalar fields of the theory. For a moment, let us assume that this term is chosen so that Eq. (10) holds. We will revisit this point at the end of this section.

The higher order terms of $\tau$ in Eq. (9) are accompanied by additional powers of $\epsilon$ or the coupling constants [9]. The reason is the following. In theories with only dimensionless parameters which are listed in Table. [1] the dilaton appears in the combination $\epsilon \tau$ in the bare Lagrangian after performing appropriate Weyl rescaling of matter fields. For example, in the case of a four dimensional $\phi^4$ theory, the bare Lagrangian of the theory
is given as

\[ \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{d-2}{8(d-1)} R(\hat{g}) \hat{\phi}^2 + \frac{\lambda_0}{4!} \hat{\phi}^4 \right) = \left( \eta^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi + e^{-2\tau} \frac{\lambda_0}{4!} \phi^4 \right) \]  

(11)

where \( \hat{\phi} \) is the bare field, \( \hat{g}_{\mu \nu} = e^{-2\tau} \eta_{\mu \nu} \) and \( \phi = e^{-(d-2)\tau/2} \hat{\phi} \). Loop calculations give divergences which may cancel the factor \( \epsilon \) in \( \epsilon \tau \). However, whenever \( \epsilon \) is cancelled, there is always an additional loop suppression factor \( L \) (e.g., \( L = \lambda/16\pi^2 \) in the \( \phi^4 \) theory).

Therefore, \( \tau \) appears only in the combination \( \epsilon \tau \) or \( L \tau \).

Then, neglecting the higher order terms, the leading order term in the dilaton effective action is given by

\[ S_{\text{eff},0}[\hat{g}_{\mu \nu} = e^{-2\tau} \eta_{\mu \nu}] = -\frac{1}{2} \int d^d x d^d y \tau(x) \tau(y) \sum_{i,j} \mathcal{B}^i \mathcal{B}^j \langle [\mathcal{O}_i(x)][\mathcal{O}_i(y)] \rangle + \cdots \]  

(12)

where dots denote higher order terms in \( \epsilon \) or loop factors.

At the leading order of perturbation theory, correlation functions of \( [\mathcal{O}_i] \) are given as

\[ \langle [\mathcal{O}_i(x)][\mathcal{O}_i(y)] \rangle = \mu^2 \frac{(d-2-d_i)c_i \delta_{ij}}{|x-y|^{2d_i}} \]  

(13)

where \( c_i \) are dimensionless constants (e.g., \( c = \frac{1}{4\pi} (\Gamma(d/2 - 1)/4\pi^{d/2})^4 \) for \( \mathcal{O} = \frac{1}{4!} \phi^4 \)), \( \mu \) is the unit of mass of dimensional regularization (or in other words the RG scale), and \( d_i \) is the classical scaling dimension of \( \mathcal{O}_i \). Although we are considering only marginal interactions, \( d_i \) differs from \( d \) by order \( \epsilon \) (e.g., \( d_i = 2(d-2) = 4 - 4\epsilon \) for \( \mathcal{O} = \frac{1}{4!} \phi^4 \)). The operators \( [\mathcal{O}_i] \) are assumed to be normalized so that \( \langle [\mathcal{O}_i(x)][\mathcal{O}_i(y)] \rangle \) is proportional to \( \delta_{ij} \) at the leading order. The constants \( c_i \) are ensured to be positive by reflection positivity. Then the dilaton effective action becomes

\[ S_{\text{eff},0}[\hat{g}_{\mu \nu} = e^{2\tau} \eta_{\mu \nu}] = -\frac{1}{2} \int d^d x d^d y \tau(x) \tau(y) \sum_i \frac{\mu^{2(d-d_i)} c_i \mathcal{B}^2_i}{|x-y|^{2d_i}} + \cdots \]  

\[ = -\int \frac{d^d k}{(2\pi)^d} \frac{|\tilde{\tau}(k)|^2}{2\Gamma(d_i)} \sum_i \pi^{\frac{d-2d_i}{2}} \frac{\Gamma(d/2 - d_i)}{2\Gamma(d_i)} c_i \mathcal{B}^2_i \mu^{2(d-d_i)} k^{2d_i - d} + \cdots \]  

(14)

\[ \text{Here we pretend as if the term } R \phi^2 \text{ is chosen as the conformal coupling of a free scalar, } \frac{(d-2)}{8(d-1)} R \phi^2. \]  

This is not correct at higher orders of perturbation theory \([34]\), but the corrections occur at sufficiently higher orders so that the following discussion is not violated.
where we have Fourier-transformed the dilaton as \( \tilde{\tau}(k) = \int d^dxe^{-ikx}\tau(x) \).

In odd dimensions, the above dilaton effective action is finite in the limit \( \epsilon \to 0 \). This is consistent with the fact that we need no counterterms in odd dimensions as discussed above. In even dimensions, there is a divergence coming from the factor \( \Gamma(d/2 - d_i) \) and we have to renormalize it. The counterterm should be local and is given as

\[
S_{c.t.}[\hat{g}_{\mu\nu}] = e^{-2\tau}\eta_{\mu\nu} \propto \int d^dxd^2\sqrt{\hat{g}}R(\hat{g}).
\]

where \( a_0 \) is a constant which is finite in the limit \( \epsilon \to 0 \). This counterterm makes the dilaton effective action finite.

Although it is not immediately evident whether the counterterm (15) can be obtained from counterterms for the metric \( S_{c.t.}[\hat{g}_{\mu\nu}] \) by replacing the metric as \( \hat{g}_{\mu\nu} \to e^{-2\tau}\eta_{\mu\nu} \), it is known to be possible [15, 16]. It may be instructive to see it explicitly in the simplest case where the space-time dimension is \( d = 2 - 2\epsilon \). There is only one candidate for the counterterm which is given by

\[
S_{c.t.}[\hat{g}_{\mu\nu}] \propto \int d^d x \sqrt{\hat{g}}R(\hat{g}).
\]

Then, the dilaton counterterm is obtained as

\[
S_{c.t.}[\hat{g}_{\mu\nu}] = e^{-2\tau}g_{\mu\nu} \propto \int d^d x \sqrt{g}e^{2\tau} \left[R(g) - 2\epsilon(1 - 2\epsilon)(\nabla\tau)^2\right]
\]

\[
 = \int d^d x \sqrt{g} \left[R(g) + 2\epsilon \left(\tau R(g) - (\nabla\tau)^2\right) + O(\epsilon^2)\right].
\]

Thus, by taking \( g_{\mu\nu} \to \eta_{\mu\nu} \), the counterterm of the form \( \frac{1}{\epsilon}\tau\partial^2\tau \) (a single pole term in \( \epsilon \)) is obtained from \( \frac{1}{\epsilon}\sqrt{g}R(\hat{g}) \) (a double pole term in \( \epsilon \)).

One should also notice that the finite term in the dilaton counterterm \( S_{c.t.}[\hat{g}_{\mu\nu}] = e^{-2\tau}\eta_{\mu\nu} \) actually comes from the divergent term in the metric counterterm \( S_{c.t.}[\hat{g}_{\mu\nu}] \). In fact, this is how the Wess-Zumino action for the dilaton [35, 7] arises in dimensional regularization. The integral of the Euler density \( E_{d_0} \) is a topological quantity in \( d_0 \)-dimensions. In the case of \( d = d_0 - 2\epsilon \) dimensional space-time, this topological property
is broken by $\epsilon$, and the change of the metric $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$ gives

$$\int d^d x \sqrt{\hat{g}} E_{d_0}(\hat{g}) = \int d^d x \sqrt{g} E_{d_0}(g) + 2\epsilon S_{\text{WZ}} + O(\epsilon^2),$$

(18)

where

$$S_{\text{WZ}} = \int d^d x \sqrt{g} \tau E_{d_0}(g) + \cdots$$

(19)

is the Wess-Zumino action for the dilaton. See Eq. (17) for the case of $d_0 = 2$. In CFTs, we need a counterterm of the form $\int d^d x (a/\epsilon) E_{d_0}$ to make the energy-momentum tensor finite \[36\]. This counterterm leads to the trace anomaly $T_\mu^\mu \sim aE_{d_0}$. One can see that the presence of this counterterm gives the finite Wess-Zumino action $aS_{\text{WZ}}$ for the dilaton by using Eq. (18).

Let us return to the computation of the dilaton effective action. We have neglected higher order terms in $\epsilon\tau$ and loop factors. We continue to neglect the higher order corrections of the loop factors. However, for our purposes it is important to recover the higher order terms of $\epsilon\tau$. Actually, there are divergences when we compute the free energy by substituting $e^{-\tau} = 2r^2/(x^2 + r^2)$. It turns out that terms containing extra powers of $\tau = \log((x^2 + r^2)/(2r^2))$, $\tau^k$ ($k = 0, 1, 2, \cdots$), give additional divergences $\frac{1}{\epsilon}$. Therefore it is necessary to retain higher order terms in $\epsilon\tau$.

To recover the dependence on $\epsilon\tau$, we use the conformal invariance of the dilaton effective action. As discussed above, the dilaton effective action should be conformally invariant since the conformal transformations are just the subgroup of the diffeomorphism of the original effective action for the metric. We can make Eqs. (14) and (15) conformally invariant by replacing them as

$$\int d^d x \tau(x) \frac{1}{|x-y|^{2d_i}} \tau(y) \rightarrow \int d^d x d^d y \left( \frac{e^{-(d-d_i)\tau(x)}}{d-d_i} \right) \frac{1}{|x-y|^{2d_i}} \left( \frac{e^{-(d-d_i)\tau(y)}}{d-d_i} \right),$$

(20)

and

$$\int d^d x \tau(x) (-\partial^2)^{d_0/2} \tau(y) \rightarrow \int d^d x \left( \frac{e^{-(d-d_0)/2} \tau(x)}}{d-d_0/2} \right) (-\partial^2)^{d_0/2} \left( \frac{e^{-(d-d_0)/2} \tau(y)}}{d-d_0/2} \right),$$

(21)

By expanding in $\tau$, one can check that the linear terms in $\tau$ are absent due to the properties of dimensional regularization. The quadratic terms in $\tau$ just reproduce the original ones.
Higher order terms in $\tau$ are accompanied by appropriate powers of $d - d_i \propto \epsilon$ as expected. The modified action in the right-hand-side of Eq. (20) is conformally invariant since the field $e^{-(d-d_i)\tau}$ has the scaling dimension $(d - d_i)$. By performing Fourier transformation to momentum space, the right-hand-side of Eq. (21) is just the same as that of Eq. (20) by analytically continuing $d_i \to (d + d_0)/2$ (up to a field-independent factor). Therefore it is also conformally invariant. See Refs. [15, 16] for the construction of this counterterm from $S_{c.t.}[\hat{g}_{\mu\nu}]$. By using the replacements (20) and (21) in Eqs. (14) and (15) respectively, we obtain the desired dilaton effective action with nonzero $\epsilon$.

Before closing this section, let us discuss the term $R\phi^2$ in the matter action. In general, the existence of this term gives an additional ambiguity in the definition of the free energy because we can introduce new parameters $\xi$ as $\xi R\phi^2$. This is related to the ambiguity of the energy-momentum tensor $T_{\mu\nu}$, since the addition of the term $\xi R\phi^2$ changes $T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$ by improvement term of the form $(\partial^\mu \partial^\nu - \eta^\mu^\nu \partial^2)\xi \phi^2$.

We assume the existence of the renormalization scheme for the energy-momentum tensor $T_{\mu\nu}$ such that

1. $T_{\mu\nu}$ is RG invariant, i.e., $\mu \frac{\partial}{\partial \mu} T_{\mu\nu} = 0$. In other words, there is no operator mixing with $(\partial^\mu \partial^\nu - \eta^\mu^\nu \partial^2)\phi^2$.

2. The trace anomaly (with nontrivial metric) is given by

$$T_\mu^\mu = -B_i [\mathcal{O}_i]' + \text{purely metric terms},$$

where $[\mathcal{O}_i]'$ are operators which coincide with $[\mathcal{O}_i]$ in the flat space limit. In particular, $T_\mu^\mu$ becomes metric dependent c-number when $B_i = 0$. (Actually, in $d \leq 4$ dimensions it is enough that Eq. (22) is satisfied in flat space, since the flat space trace anomaly combined with Wess-Zumino consistency condition for Weyl transformations leads to Eq. (22) in nontrivial metric background [30].) We always couple the metric to the energy-momentum tensor satisfying the above assumptions.

The existence of the energy-momentum tensor satisfying the above assumptions can be explicitly proved for some theories. For example, a proof was given in Ref. [34] for

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6 In the case of the six dimensional $\phi^3$ theories, there can also exist linear terms of $\phi$ given by $\eta R \nabla^2 \phi$ and $\zeta R^2 \phi$. Just for simplicity, we consider the case $d \leq 4$ in the following discussion.
the case of $\phi^4$ theory in four dimensions. A large class of four dimensional renormalizable supersymmetric theories also satisfies the assumptions. In this case there is a
Ferrara-Zumino supercurrent multiplet \[37\], $J_{\alpha\dot{\alpha}}^{FZ}$ (if the theory does not contain FI-terms \[38\]. See also Refs. \[39, 40\] for recent comprehensive discussions on supercurrents.) The Ferrara-Zumino supercurrent multiplet can mix with other operators only as
$J_{\alpha\dot{\alpha}}^{FZ} \rightarrow J_{\alpha\dot{\alpha}}^{FZ} + [D_\alpha, \bar{D}_{\dot{\alpha}}](\Phi + \Phi^\dagger)$, where $\Phi$ is a chiral superfield, i.e, $\bar{D}_{\dot{\alpha}}\Phi = 0$, and $\Phi$ has mass dimension two. If there is no candidate for $\Phi$ which is invariant under any global or local symmetries, the multiplet $J_{\alpha\dot{\alpha}}^{FZ}$ cannot mix with any other operators. Then the energy-momentum tensor contained in $J_{\alpha\dot{\alpha}}^{FZ}$ satisfies the first assumption. It also satisfies
the second assumption \[41, 42, 43\]. Thus a large class of supersymmetric theories has
the energy-momentum tensor satisfying the above assumptions. The situation is similar
in three dimensions as long as the Ferrara-Zumino multiplet exists.

If we introduce additional parameters $\xi$, it is possible to construct an RG invariant
energy-momentum tensor. Let $\mathcal{O}_a^{\phi^2}$ denote the set of operators $\phi^2$, where $a$ is the label
specifying the operators. If the energy-momentum tensor satisfies Eq. (22), the operator
mixing is in general given by (see e.g., Ref. \[30\])

$$
\frac{\partial}{\partial \mu} \begin{pmatrix}
T_{\mu\nu}^{[\mathcal{O}_i]} \\
[\mathcal{O}_a^{\phi^2}]
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -B^2 \delta_i^b (\eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu)/(d - 1) \\
0 & -\partial_i B^j & \delta_i^b \partial^2 \\
0 & 0 & \gamma_{\alpha a}^b
\end{pmatrix}
\begin{pmatrix}
T_{\mu\nu}^{[\mathcal{O}_j]} \\
[\mathcal{O}_j^{\phi^2}]
\end{pmatrix},
$$

(23)

where $\delta_i^b$ and $\gamma_{\alpha a}^b$ are some anomalous dimension matrices, and $\partial_i B^j$ is the derivative of $B^j$ with respect to the coupling $\lambda^i$. We introduce new parameters $\xi_a^i$ which are defined to satisfy the RG equation

$$
\mu \frac{\partial}{\partial \mu} \xi_a^i + \gamma_{\alpha a}^b \xi_a^b + \partial_i B^j \xi_{j a}^i + \delta_i^a = 0.
$$

(24)

Then, we define the new operators as

$$
[\mathcal{O}_i^{(\text{new})}] = [\mathcal{O}_i] + \xi_a^i \partial^2 [\mathcal{O}_a^{\phi^2}],
$$

(25)

$$
T_{\mu\nu}^{(\text{new})} = T_{\mu\nu} - \frac{1}{d - 1} B^a \xi_{a}^i (\eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu)[\mathcal{O}_a^{\phi^2}].
$$

(26)

This new energy-momentum tensor is invariant under RG and has a trace $-B^i [\mathcal{O}_i^{(\text{new})}]$. If we can find $\xi_a^i$ as a function of the coupling constants $\lambda^i$ with a well-defined perturbative

\[7\]In higher orders of perturbation theory, there is a notorious problem called the anomaly puzzle \[44\]. See Refs. \[45, 46, 47, 48, 49, 50\] and references therein for discussions on this problem.
expansion, the difference between $T_{\mu\nu}$ and $T_{\mu\nu}^{(\text{new})}$ is just a change of renormalization prescription and $T_{\mu\nu}^{(\text{new})}$ is the desired energy-momentum tensor. This was indeed shown to be possible in $\phi^4$ theory [34]. Even if $\xi^a_i$ is not a function of $\lambda^i$, our computation of the free energy is valid as long as we can find a solution to Eq. (24) such that $\xi^a_i$ remain small along RG flows. One can check that $\delta^a_i$ vanish at the one loop level, and hence nonzero $\xi^a_i$ are generated only at higher orders of perturbation theory.

3 Perturbative free energy and $c$-theorems

3.1 Free energy in $d$-dimensions

Let us summarize the result for the leading term of the dilaton effective action obtained in the previous section. The unrenormalized effective action is given by

$$S_{\text{eff},0}[\hat{g}_{\mu\nu} = e^{-2\tau} \eta_{\mu\nu}] = -\frac{1}{2} \sum_i c_i B_i^2 I_{d_i},$$

(27)

and the counterterm (for $d_0=$even) is given by

$$S_{\text{c.t.}}[\hat{g}_{\mu\nu} = e^{-2\tau} \eta_{\mu\nu}] = \left( a_0 + \sum_i \pi^{d/2} 2^{d-2d_i} \frac{d}{2} \Gamma(d/2 - d_i) c_i B_i^2 \right) J,$$

(28)

where $I_{d_i}$ and $J$ are defined as

$$I_{d_i} = \mu^{2(d-d_i)} \int d^d x d^d y \left( e^{-d(d-d_i)\tau(x)} \frac{1}{|x-y|^{2d_i}} \right) \left( e^{-d(d-d_i)\tau(y)} \right),$$

(29)

$$J = \mu^{(d-d_0)} \int d^d x \left( e^{-(d-d_0)\tau(x)} \right) \left( -\partial^2 \right)^{d_0/2} \left( e^{-(d-d_0)\tau(y)} \right),$$

(30)

In this section we evaluate the explicit values of $I_{d_i}$ and $J$ when we substitute the dilation vacuum expectation value $e^{-\tau} = 2r^2/(x^2 + r^2)$.

The computation of $I_{d_i}$ is the same as in Refs. [4, 6]. First we rewrite $I_{d_i}$ as

$$I_{d_i} = \frac{\mu^{2(d-d_i)}}{(d-d_i)^2} \int d^d x \sqrt{\hat{g}(x)} d^d y \sqrt{\hat{g}(y)} \frac{1}{s(x,y)^{2d_i}},$$

(31)

$$s(x,y) = \frac{2r^2}{(x^2 + r^2)^{\frac{d}{2}} (y^2 + r^2)^{\frac{d}{2}}} |x-y|.$$

(32)
where $\hat{g}_{\mu\nu} = e^{-2\tau}\eta_{\mu\nu}$ is the metric of the sphere, and $s(x, y)$ is the “chordal distance” between points $x$ and $y$ if the sphere is embedded in a flat Euclidean space. Then, by using the rotational invariance of the sphere, we may set $y = \infty$ to obtain

$$I_{d_i} = \frac{\mu^2(d-d_i)\text{Vol}(S^d)}{(d-d_i)^2} \int d^d x \sqrt{\hat{g}(x)} \frac{1}{s(x, \infty)^{2d_i}}$$

$$= \frac{\mu^2(d-d_i)}{(d-d_i)^2} \frac{2\pi^{d/2} r^d}{\Gamma(d/2 + 1)} \int d^d x \frac{(2r^2)^{d-2d_i}}{x^2 + r^2}$$

$$= (2\mu r)^{2(d-d_i)} \frac{\pi^d \Gamma(d/2) \Gamma(d/2 - d_i)}{(d-d_i) \Gamma(d) \Gamma(d - d_i + 1)},$$  

(33)

where $\text{Vol}(S^d)$ is the volume of the sphere with radius $r$, and in the process of the computation we have used some identities of the gamma function such as $\Gamma(d/2) \Gamma(d/2 + 1) = \pi^{d/2} 2^{1-d} \Gamma(d)$.

It is also easy to compute $J$. By Fourier transforming to momentum space and looking at the expressions for $I_{d_i}$ and $J$ in momentum space, we find that $I_{d_i}$ and $J$ are related as

$$J = \lim_{d_i \to d_0} \frac{\Gamma(d_i)}{\pi^{d/2} 2^{d-2d_i} \Gamma(d/2 - d_i)} I_{d_i}$$

$$= (2\mu r)^{d-d_0} \frac{2d_0 \pi^{d/2} \Gamma(d/2) \Gamma(d/2 + d_0)}{(d-d_0) \Gamma(d) \Gamma(d - d_0 + 1)}.  

(34)

By combining the above results, we finally get the free energy $F = -\log Z$ for odd dimensions or its derivative $dF/d \log r$ for even dimensions as follows.

**Odd dimensions**

$$F_{d_0=\text{odd}} = -\log Z_0 - \frac{1}{2} \sum_i c_i B^2_i (2\mu r)^{2(d-d_i)} \frac{\pi^d \Gamma(d/2) \Gamma(d/2 - d_i)}{(d-d_i) \Gamma(d) \Gamma(d - d_i + 1)} + \cdots$$

$$= (\text{const.}) + (-1)^{d_0-1} \frac{2\pi^d d_0}{d_0!} \log(\mu r) \sum_i c_i B^2_i + \cdots.  

(35)

**Even dimensions**

$$\frac{dF_{d_0=\text{even}}}{d \log r} = (2\mu r)^{d-d_0} \frac{2d_0 \pi^{d/2} \Gamma(d/2) \Gamma(d/2 + d_0)}{\Gamma(d) \Gamma(d - d_0 + 1)} \left(2a_0 + \sum_i \frac{\pi^{d/2} 2^{d-2d_i} \Gamma(d/2 - d_i)}{\Gamma(d_i)} c_i B^2_i \right)$$

$$- \sum_i c_i B^2_i (2\mu r)^{2(d-d_i)} \frac{\pi^d \Gamma(d/2) \Gamma(d/2 - d_i)}{\Gamma(d) \Gamma(d - d_i + 1)} + \cdots$$

$$= (\text{const.}) + (-1)^{d_0} \frac{4\pi^d d_0}{d_0!} \log(\mu r) \sum_i c_i B^2_i + \cdots.  

(36)
The dots denote higher order corrections in $\epsilon$ or loop factors.

The constant terms in the above equations depend on $\log Z_0$ (in odd dimensions) or $a_0$ (in even dimensions) which we have not computed. However, from the above result we can obtain the flows of $F$ or $dF/d\log r$ as

\[
(-1)^{d_0+1} \frac{dF_{d_0=\text{odd}}}{d\log r} = -\frac{2\pi^{d_0+1}}{d_0!} \sum_i c_i B_i^2 + \cdots
\]

\[
(-1)^{d_0} \frac{d^2 F_{d_0=\text{even}}}{d(\log r)^2} = -\frac{4\pi^{d_0}}{d_0!} \sum_i c_i B_i^2 + \cdots.
\]

As is usual in perturbation theory, the higher order terms contain powers of the logarithm $\log(\mu r)$ and we may set the renormalization scale as $\mu \to r^{-1}$ to avoid large logarithmic corrections in perturbation theory. Then the coupling constants $\lambda(\mu)$ become functions of $r$ as $\lambda(\mu) \to \lambda(r^{-1})$.

Eqs. (37) and (38) are our main result. Similar results were obtained in Refs. [4, 6] when a CFT is deformed by slightly marginal operators. Eqs. (37) and (38) extend those results to theories with marginal interactions. In particular, these equations show that $(-1)^{d_0+1} F_{d_0=\text{odd}}$ and $(-1)^{d_0} dF_{d_0=\text{even}}/d\log r$ decreases monotonically as we increase the radius of the sphere $r$ because the coefficients $c_i$ are positive.

### 3.2 c-theorems in various dimensions

**Two dimensions** In two dimensions, the trace anomaly in CFT is given as

\[
T_\mu^\mu = \frac{c}{24\pi} R,
\]

where $c$ is the central charge. The derivative of the free energy with respect to the radius of the sphere, $r$, is given by the one-point function of $T_\mu^\mu$ on the sphere $S^2$, and hence

\[
\frac{dF_{d_0=2}}{d\log r} = -\left\langle \int d^2 x \sqrt{g} T_\mu^\mu \right\rangle_{S^2} = -\frac{c}{3}.
\]

This relation is valid for CFT.

The Zamolodchikov’s c-function [1] is defined as

\[
c(|x|) = (2\pi)^2 \left[ 2z^4 \left< T_{zz}(x) T_{zz}(0) \right> - 4z^2 x^2 \left< T_{zz}(x) T_{zz}(0) \right> - 6x^4 \left< T_{z\bar{z}}(x) T_{z\bar{z}}(0) \right> \right],
\]

\[\text{Eq. (41)}\]

\[\text{We do not introduce an additional factor of } 2\pi \text{ in the definition of the energy-momentum tensor which often appears in the literature of two dimensional CFT.}\]
where $z = x^1 + ix^2$. The function $c(|x|)$ coincides with the central charge $c$ in Eq. (39) at conformal fixed points. The flow of this function is given by

$$\frac{\delta c(|x|)}{\delta \log |x|} = -6\pi^2 x^4 \langle T^\mu_\mu(x)T^\nu_\nu(0) \rangle = -6\pi^2 \sum_{i,j} B_i B_j x^4 \langle O_i(x)O_j(0) \rangle.$$  (42)

At the leading order, the operator correlation functions are given in Eq. (13). Therefore, by comparing Eqs. (38) and (42), we find

$$d^2 F_{d_0=2}(r) = \frac{2}{d \log r \sqrt{g}} \langle \int d^2 x \sqrt{\hat{g}} T^\mu_\mu \rangle_{S^2} = 4a.$$  (43)

This is consistent with Eq. (40). Although the two functions $d^2 F_{d_0=2}/d \log r$ and $-c(r)/3$ need not precisely be the same in non-CFT, the above result shows that they indeed coincide at the order of perturbation theory we are considering. In particular, our formula (38) correctly reproduces the difference of UV and IR central charges $c_{UV} - c_{IR}$ if the UV and IR theories are conformal.

**Four dimensions** The case of four dimensions is similar to that of two dimensions. The trace anomaly in CFT is given as

$$T^\mu_\mu = \frac{1}{16\pi^2} (-aE_4 + cW_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}),$$  (44)

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor and $E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Euler density. Putting the theory on the sphere $S^4$, we obtain

$$\frac{dF_{d_0=4}}{d \log r} = -\left\langle \int d^2 x \sqrt{\hat{g}} T^\mu_\mu \right\rangle_{S^2} = 4a.$$

The change of $a$ as we vary some length scale $r$ is given by

$$\frac{da(r)}{d \log r} = -\frac{\pi^4}{24} \sum_i c_i B_i^2.$$  (46)

Therefore, by comparing Eqs. (38) and (46) we get

$$\frac{d^2 F_{d_0=4}(r)}{d(\log r)^2} = 4 \frac{da(r)}{d \log r}.$$  (47)

This relation is consistent with Eq. (45).

---

9 More precisely, $a$ is defined by the dilaton forward scattering amplitude, and $r$ should be the inverse of the center-of-mass energy in that scattering process. See Ref. [9] for details.
Three dimensions In three dimensional $\mathcal{N} = 2$ supersymmetric theories there are exact results for the partition functions $[51, 52, 53]$. We check our result for a simple case by comparing it to the exact result.

Let us consider an $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theory with gauge group $U(N_c)$. We introduce chiral superfields $(Q, \tilde{Q})$ which are in a representation of the gauge group, $R \oplus \tilde{R}$, where we take $R$ as $N_f$ copies of some irreducible representation $r$, i.e., $R = N_f \times r$. The Chern-Simons level is denoted as $k$, and we take $k \gg N_c, N_f$ so that perturbation theory is applicable. We take the superpotential as

$$W = \frac{\lambda}{2}(\tilde{Q}T_aQ)(\tilde{Q}T_aQ)$$

where $T_a$ are generators of the gauge group $U(N_c)$ normalized as $\text{tr}_{\text{fund}}(T_aT_b) = \delta_{ab}$ for a fundamental representation. Without loss of generality we take $\lambda$ to be real and positive. See Ref. [55] for details of this theory.

The theory has $\mathcal{N} = 2$ supersymmetry for a general value of the yukawa coupling constant $\lambda$, and the supersymmetry is enhanced to $\mathcal{N} = 3$ when $\lambda = 4\pi/k$. The RG equation for $\lambda$ is given by [55]

$$B_\lambda = \frac{d\lambda}{d\log \mu} = \frac{b_0}{16\pi^2} \lambda \left( \lambda^2 - \left( \frac{4\pi}{k} \right)^2 \right)$$

where $b_0 = \frac{2}{\text{dim} R}(\text{tr}_R(T_aT_b)\text{tr}_R(T_aT_b) + \text{tr}_R(T_aT_bT_aT_b))$ and $\text{dim} R$ is the dimension of the representation $R$. Therefore, this model connects two different superconformal fixed points; $\lambda = 0$ in the UV and $\lambda = 4\pi/k$ in the IR.

Let us apply our formula (37) to this model. First, we define the operator $O_\lambda$ as

$$O_\lambda = \frac{1}{2}(\tilde{Q}T_aQ)(\tilde{Q}T_aQ)|_{\theta^2} + \text{h.c.},$$

where $|\theta^2|$ means that we take the $\theta^2$ component of a chiral field. The Lagrangian of this theory contains the interaction term $\lambda O_\lambda$. By computing the correlation function $\langle O_\lambda(x)O_\lambda(0) \rangle$ at the leading order, we find that the constant $c_\lambda$ defined as $\langle O_\lambda(x)O_\lambda(0) \rangle = c_\lambda/x^6$ is given by

$$c_\lambda = \frac{6b_0 \text{dim} R}{(4\pi)^4};$$

---

$^{10}$See also Refs. [6, 54] for calculations of $F$ in non-supersymmetric theories.

$^{11}$Note that our normalization of $T_a$ is different from Ref. [55], and we have also corrected some errors in the beta function of Ref. [55].
where we have neglected $O(\epsilon)$ corrections. Then, the difference of the UV and IR free energies, $F_{UV} = F_{d_0=3}(r \to 0)$ and $F_{IR} = F_{d_0=3}(r \to \infty)$, is given as

$$F_{IR} - F_{UV} = -\frac{\pi^4}{3} \int_0^\infty \frac{dr}{r} c_\lambda B^2_\lambda$$

$$= \frac{\pi^4}{3} \int_0^{4\pi/k} d\lambda c_\lambda B_\lambda$$

$$= -\frac{\pi^2 b_0^2 \dim R}{2^5 k^4}. \quad (52)$$

Now let us restrict our attention to the case that the gauge group is U(1) (i.e., $N_c = 1$) and there are $N_f$ pairs of chiral fields $(Q, \tilde{Q})$ with charge $\pm 1$. In this case, the gauge group generator is $T = 1_{N_f \times N_f}$, and we have $b_0 = 2(N_f + 1)$ and $\dim R = N_f$. Therefore, we obtain

$$F_{IR} - F_{UV} = -\frac{\pi^2 N_f (N_f + 1)^2}{8k^4}. \quad (53)$$

On the other hand, the exact partition function for the model is explicitly obtained in Ref. [52]. Denoting the superconformal R-charge of $(Q, \tilde{Q})$ as $\Delta = \frac{1}{2} - a$, the real part of the free energy is given as

$$\text{Re} F(a) = \log(2^{N_f} \sqrt{k}) - \frac{\pi^2 N_f a^2}{2} + \frac{\pi^2 N_f (N_f + 1)}{16k^2} (1 + 8a) + O(k^{-5}), \quad (54)$$

where we have assumed $a = O(k^{-2})$, which will be justified below. In the UV CFT ($\lambda = 0$), the value of $a$ is determined by the solution of $d\text{Re} F(a)/da = 0$ [52, 56] and is given by $a = \frac{N_f + 1}{2k^2}$. In the IR CFT ($\lambda = 4\pi/k$), we should have $a = 0$ for the superpotential to be invariant under the $R$-symmetry. Then, we obtain

$$\text{Re} F_{IR} - \text{Re} F_{UV} = \text{Re} F(a = 0) - \text{Re} F(a = \frac{N_f + 1}{2k^2})$$

$$= -\frac{\pi^2 N_f (N_f + 1)^2}{8k^4}. \quad (55)$$

This result completely agrees with Eq. (53).

---

\footnote{The imaginary part is just an artifact of imaginary supergravity background; see Refs. [57, 56, 58] for details.}
Six dimensions  In six dimensions, scalar field theories with $\phi^3$ interactions can be treated perturbatively. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \sum_a \partial_\mu \phi_a \partial_\mu \phi_a + \frac{1}{6} \sum_{a,b,c} \lambda_{a,b,c} \phi_a \phi_b \phi_c. \quad (56)$$

Our result shows that $-dF_{d=6}/d \log r$ decreases monotonically as we increase $r$. Notice that the conformal coupling $\frac{d-2}{8(d-1)} R \phi^2$ makes the vacuum perturbatively stable when the theory is put on the sphere, and hence we need not worry about infrared divergences.

This model is asymptotically free at the one-loop level \[59\], but unfortunately, no Banks-Zaks type IR fixed point is known. However, it is at least encouraging for the six dimensional $a$-theorem \[15\] (see also Ref.\[16\]) that $dF_{d=6}/d \log r$ decreases monotonically as a function of $r$.

### 3.3 Scale versus conformal invariance

Now we discuss the equivalence between scale and conformal invariance in the class of theories studied in this paper (see Table.\[1\]). More generally, we study the possible IR (or UV) asymptotics of perturbative quantum field theories. Our discussion follows the one in Ref.\[9\] which studied the same problem in four dimensions.

First let us briefly review the mechanism by which a theory could have scale invariance without conformal invariance \[23, 24, 25, 26\]. If the trace of the energy-momentum tensor is given by a total derivative, $T^\mu_\mu = \partial_\mu j^\mu_V$, where $j^\mu_V$ is some vector field called the virial current, the theory is scale invariant because we can define a conserved current of scaling transformation as $T^\mu_\mu x_\nu - j^\mu_V$. In theories studied in this paper, this requirement is given as

$$T^\mu_\mu = - \sum_i \mathcal{B}^i [O_i] = \partial_\mu j^\mu_V. \quad (57)$$

If $\partial_\mu j^\mu_V \neq 0$, the beta functions $\mathcal{B}^i$ are nonzero and the theory is not conformally invariant. If we see the coupling constants of the theory as spurions, there is a symmetry associated with the current $j_V$ under which the coupling constants transform nontrivially. When Eq. \(57\) holds, the RG flow is generated by that symmetry acting on the coupling constants. Our purpose is to show that such RG flows are impossible and $T^\mu_\mu$ actually vanishes when the theory has scale invariance.
The free energy $F$ in general depends on the parameters in the finite part of the counterterms $S_{\text{c.t.}}(g_{\mu\nu})$. These new parameters are absent in the original flat space theories. However, as we discussed in section 2, there is a way to define the free energy in which $F$ in odd dimensions and $dF/d\log r$ in even dimensions do not contain any such new parameters. In that definition, they only depend on the coupling constants $\lambda_i(\mu)$, the renormalization scale $\mu$, and the radius of the sphere $r$ as

$$C = C(\lambda(\mu), \log(r/\mu)) = C(\lambda(r^{-1}))$$

(58)

where we define $C$ as

$$C = \begin{cases} (-1)^{d_0+1} \frac{d_0!}{2\pi^{d_0+1}} F_{d_0=\text{odd}} & (d_0 = \text{odd}) \\ (-1)^{d_0} \frac{d_0!}{4\pi^{d_0}} \frac{dF_{d_0=\text{even}}}{d\log r} & (d_0 = \text{even}) \end{cases}$$

(59)

In the last equality in Eq. (58) we have used RG invariance and set $\mu = r^{-1}$. Therefore, the function $C$ defined as in Eq. (59) is only a function of $\lambda_i(r^{-1})$.

From Eqs. (37) and (38), we see that $C$ satisfies

$$\frac{dC}{d\log r} = -\sum_i c_i B_i^2$$

(60)

with $c_i$ all positive. Suppose that the theory is weakly coupled in the IR limit $r \to \infty$. (The discussion is completely parallel in the UV.) Then, we can trust perturbation theory in the IR, and $C(\lambda(r^{-1}))$ remains finite in the IR since all the couplings $\lambda_i(r^{-1})$ are small. Then, from Eq. (60), it is necessary that $\int_{\infty}^\infty d\log r \sum_i c_i B_i^2$ is finite, and hence $c_i B_i^2$ should vanish faster than $1/\log r$ in the limit $r \to \infty$ for $C$ to be finite in the IR. Since

$$\left\langle T_{\mu}^{\mu}(x)T_{\nu}^{\nu}(0) \right\rangle = \sum_i c_i B_i^2/x^{2d},$$

the trace of the energy-momentum tensor should vanish in the IR limit and hence the theory is conformal. We conclude that the IR limit of the class of theories studied in this paper is either conformal or strongly coupled so that perturbation breaks down.

Although we have neglected higher order corrections, they do not change the conclusion. As long as the energy-momentum tensor satisfies Eq. (22), couplings between the dilaton and dynamical fields in Eq. (9) are always proportional to the beta functions $B_i$. 
Then, higher order corrections to Eq. (60) are of the form \( \sum_{i,j} \delta c_{ij} B_i B_j \), where \( \delta c_{ij} \) are suppressed by loop factors compared with \( c_i \). These corrections are always smaller than the leading contribution and the above discussion is valid even if we include them. See Ref. [9] for detailed discussions. Furthermore, the ambiguity of the energy-momentum tensor related to improvement discussed in section 2 does not invalidate the above argument if there exists a solution to Eq. (24) in which \( \xi^a \) remains small in the IR. In particular, in a scale invariant theory in which the energy-momentum tensor and the virial current \( j^\mu \) in Eq. (57) are eigenstates of dilatation with eigenvalues \( d \) and \( d - 1 \) respectively, there is no operator mixing and the above argument of the equivalence between scale and conformal invariance is valid.

4 Conclusions

In this paper we have studied the free energy \( F = -\log Z \) on a \( d \)-dimensional sphere with radius \( r \) for theories which have marginal interactions. Such theories are listed in Table. [1]. If we couple the metric of the sphere to the energy-momentum tensor \( T_{\mu\nu} \) satisfying \( T^\mu_{\mu} = -\sum_i \mathcal{B}_i \mathcal{O}_i \), the free energy \( F \) satisfies

\[
(-1)^{\frac{d_0+1}{2}} \frac{dF}{d \log r} \bigg|_{d_0=\text{odd}} = -\frac{2\pi^{d_0+1}}{d_0!} \sum_i c_i B_i^2 + \cdots
\]

\[ (61) \]

\[
(-1)^{\frac{d_0}{2}} \frac{d^2F}{d(\log r)^2} \bigg|_{d_0=\text{even}} = -\frac{4\pi^{d_0}}{d_0!} \sum_i c_i B_i^2 + \cdots,
\]

\[ (62) \]

where \( d_0 \) is the space-time dimension, \( \mathcal{B}_i \) are beta functions of coupling constants, \( c_i \) are positive constants defined as \( \langle \mathcal{O}_i(x) | \mathcal{O}_j(0) \rangle = c_i \delta_{ij} / x^{2d} + \cdots \), and dots indicate subleading terms which are always smaller than the leading term. In particular, \( (-1)^{\frac{d_0+1}{2}} F \) (in odd dimensions) or \( (-1)^{\frac{d_0}{2}} dF/d \log r \) (in even dimensions) decreases monotonically in perturbation theory. This result extends the perturbative \( c \) (\( a \)) theorem in two (four) dimensions to other dimensions. Using this result, we have extended the work of Ref. [9, 18] to other dimensions and argued that scale invariance is equivalent to conformal invariance in perturbation theory.
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