The power index of a graph

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Abstract

The \textit{power index} $\Theta(\Gamma)$ of a graph $\Gamma$ is the least order of a group $G$ such that $\Gamma$ can embed into the power graph of $G$. Furthermore, this group $G$ is $\Gamma$-\textit{optimal} if $G$ has order $\Theta(\Gamma)$. We say that $\Gamma$ is \textit{power-critical} if its order equals to $\Theta(\Gamma)$. This paper focuses on the power indices of complete graphs, complete bipartite graphs and 1-factors. We classify all power-critical graphs $\Gamma'$ in these three families, and give a necessary and sufficient condition for $\Gamma'$-optimal groups.

\textit{Keywords:} Power graph, embedding, power index, power-critical graph.
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1 Introduction

Each graph $\Gamma$ considered in this paper is a finite, simple and undirected graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Let $\Gamma_1$ and $\Gamma_2$ be graphs. We call that $\Gamma_1$ is a \textit{spanning subgraph} of $\Gamma_2$ if $V(\Gamma_1) = V(\Gamma_2)$ and $E(\Gamma_1) \subseteq E(\Gamma_2)$. An \textit{embedding} from $\Gamma_1$ to $\Gamma_2$ is an injection $f : V(\Gamma_1) \rightarrow V(\Gamma_2)$ such that $\{x, y\} \in E(\Gamma_1)$ implies $\{f(x), f(y)\} \in E(\Gamma_2)$. A graph $\Gamma_1$ can \textit{embed} into a graph $\Gamma_2$ if there is an embedding from $\Gamma_1$ to $\Gamma_2$.

Let $G$ be a group. The \textit{power graph} $\Gamma_G$ of $G$ has the vertex set $G$ and two distinct elements are adjacent if one is a power of the other. In 2000, Kelarev and Quinn [12] introduced the concept of a (directed) power graph. With this motivation, Chakrabarty, Ghosh and Sen [7] introduced the undirected power graph of a group.

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For convenience throughout we use the term “power graph” to refer to an undirected power graph defined as above. Recently, many interesting results on power graphs have been obtained, see [1–6, 9, 10, 13–17]. A detailed list of results and open questions on power graphs can be found in [2].

Note that $\Gamma_G$ is complete if and only if $G$ is isomorphic to the cyclic group of prime power order (see [7, Theorem 2.12]). This means that each graph can embed into $\Gamma_G$ for some finite group $G$. The motivation for this research is to decide the minimum order of a group $G$ such that a given graph can embed into $\Gamma_G$.

**Definition 1.1.** The power index of a graph $\Gamma$, denoted by $\Theta(\Gamma)$, is the least order of a group $G$ such that $\Gamma$ can embed into $\Gamma_G$. Furthermore, this group $G$ is $\Gamma$-optimal if $G$ has order $\Theta(\Gamma)$.

Let $K_n$ and $\mathbb{Z}_n$ be the complete graph and cyclic group of order $n$, respectively. Since the power graph of any group of order 6 is not complete, we have $\Theta(K_6) = \Theta(K_7) = 7$, and $\mathbb{Z}_7$ is $K_6$-optimal and $K_7$-optimal. For any graph $\Gamma$, one has

$$|V(\Gamma)| \leq \Theta(\Gamma).$$ (1)

Another motivation of this paper is to study which graphs satisfy the equality in (1).

**Definition 1.2.** A graph $\Gamma$ is power-critical if its order equals to $\Theta(\Gamma)$.

All power graphs are power-critical, but not vice versa. Actually, given a graph $\Gamma$, determining whether $\Gamma$ is power-critical is equivalent to determining whether $\Gamma$ is isomorphic to a spanning subgraph of the power graph of some finite group. Many researchers [6, 9, 19] investigated two groups which have isomorphic power graphs. Actually, this problem is to find two finite groups $G$ and $H$ such that $G$ is $\Gamma_H$-optimal and $H$ is $\Gamma_G$-optimal.

A cycle and a path of length $n$ are denoted by $C_n$ and $P_n$, respectively. It follows from [7, Theorem 4.13] that $\Gamma_{\mathbb{Z}_n}$ has a Hamiltonian cycle, i.e., a cycle visiting each vertex of the graph. Thus, all paths and cycles are power-critical. Furthermore, a group $G$ of order $n$ is $P_n$-optimal (resp. $C_n$-optimal) if and only if $\Gamma_G$ has a Hamilton path (resp. Hamilton cycle).

Let $K_{s,t}$ be the complete bipartite graph and $nK_2$ be the 1-factor, i.e., the graph union of $n$ copies of $K_2$. This paper focuses on the power indices of $K_n$, $K_{s,t}$ and $nK_2$. In Section 2, we compute $\Theta(K_n)$ and show that a $K_n$-optimal group is cyclic. In Section 3, we give a necessary and sufficient condition for power-critical $K_{s,t}$, and under this condition, all $K_{s,t}$-optimal groups are classified. In Section 4, we show that $nK_2$ is power-critical, and give a necessary and sufficient condition for $nK_2$-optimal groups.
2 Complete graphs

Denote by $\rho_n$ the smallest prime power at least $n$. Since $\rho_n$ is a prime power, any graph $\Gamma$ of order $n$ can embed into $\Gamma_{\mathbb{Z}_{\rho_n}}$. Hence, we have

$$n = \Theta(\overline{K}_n) \leq \Theta(\Gamma) \leq \Theta(K_n) \leq \rho_n,$$

where $\overline{K}_n$ is the null graph of order $n$. Moreover, any group of order $n$ is $\overline{K}_n$-optimal.

Observation 2.1. Let $n$ be a positive integer.

(i) The complete graph $K_n$ is power-critical if and only if $n$ is a prime power.

(ii) If $n$ is a prime power, then any graph of order $n$ is power-critical.

In the following, we compute $\Theta(K_n)$ and find all groups which are $K_n$-optimal. Let $n = p_1^{r_1}p_2^{r_2} \cdots p_m^{r_m}$, where $p_1 < p_2 < \cdots < p_m$ are primes and $r_i \geq 1$ for $1 \leq i \leq m$. Write

$$\chi_n = \varphi(n) + \varphi\left(\frac{n}{p_1}\right) + \cdots + \varphi\left(\frac{n}{p_1^{r_1}}\right) + \varphi\left(\frac{n}{p_1^{r_1}p_2}\right) + \varphi\left(\frac{n}{p_1^{r_1}p_2^{r_2}}\right) + \varphi\left(\frac{n}{p_1^{r_1}p_2^{r_2}p_3}\right) + \cdots + \varphi\left(\frac{n}{p_1^{r_1}p_2^{r_2} \cdots p_m^{r_m-1}}\right) + \varphi(1),$$

where $\varphi$ is the Euler totient function. Recently, this number $\chi_n$ is studied by Curtina and Pourgholi [8].

Lemma 2.2. [8] (i) $\chi_n = \varphi(n) + \chi_{\frac{n}{p}}$, where $p$ is the least prime factor of $n$.

(ii) $\chi_n \leq n$, with equality if and only if $n$ is a prime power.

(iii) $\chi_n = n - 1$ if and only if $n$ is twice an odd prime.

Given a graph $\Gamma$, a subset of $V(\Gamma)$ is a clique if its induced subgraph is complete. The clique number of $\Gamma$, denoted by $\omega(\Gamma)$, is the maximum size of a clique in $\Gamma$.

Lemma 2.3. [18] (i) $\omega(\Gamma_{\mathbb{Z}_n}) = \chi_n$.

(ii) For a finite group $G$, each clique in $\Gamma_G$ is a clique in the power graph of a cyclic subgroup of $G$.

Theorem 2.4. For a positive integer $n$, we have

$$\Theta(K_n) = \min\{k : \chi_k \geq n\}.$$}

Moreover, a group $G$ is $K_n$-optimal if and only if $G$ is a cyclic group of order $\Theta(K_n)$. 3
Proof. Write
\[ t = \min\{k : \chi_k \geq n\}. \]
Then \( \omega(\Gamma_{\mathbb{Z}_t}) = \chi_t \) by Lemma 2.3 (i). Note that \( \chi_t \geq n \). Thus, there exists an embedding from \( K_n \) to \( \Gamma_{\mathbb{Z}_t} \). This implies that \( \Theta(K_n) \leq t \).

Suppose that \( G \) is a \( K_n \)-optimal group. Then \( \omega(\Gamma_G) \geq n \) and \( |G| = \Theta(K_n) \). By Lemma 2.3, there exists an element \( x \in G \) such that
\[ \chi_{|x|} = \omega(\Gamma_{\langle x \rangle}) = \omega(\Gamma_G) \geq n. \]
It follows that
\[ |G| \geq |x| \geq t \geq \Theta(K_n) = |G|, \]
which implies that \( \Theta(K_n) = t \) and \( G \) is cyclic.

Now suppose that \( G \) is a cyclic group of order \( t \). Then \( \omega(\Gamma_G) = \chi_t \geq n \), and so \( G \) is \( K_n \)-optimal.

We accomplish our proof.\( \square \)

By Observation 2.1, we have \( n + 1 \leq \Theta(K_n) \leq \rho_n \) for a positive integer \( n \) which is not a prime power. We determine all \( n \) satisfying \( \Theta(K_n) = n + 1 \).

**Corollary 2.5.** Let \( n \) be a positive integer which is not a prime power. Then \( \Theta(K_n) = n + 1 \) if and only if \( n + 1 \) is a prime power or twice an odd prime.

**Proof.** Assume that \( \Theta(K_n) = n + 1 \). Then \( n \leq \chi_{n+1} \leq n + 1 \). If \( \chi_{n+1} = n + 1 \), then \( n + 1 \) is a prime power Lemma 2.2 (ii). If \( \chi_{n+1} = n \), then \( n + 1 \) is twice an odd prime by Lemma 2.2 (iii).

Conversely, we have \( \chi_{n+1} = n + 1 \) or \( n \). Note that \( \chi_n < n \). It follows that \( \Theta(K_n) = n + 1 \), as desired. \( \square \)

It follows from Theorem 2.4 and Corollary 2.5 that \( \Theta(K_{14}) = 16 = \rho_{14} \). In view of \( \chi_{36} = 27 \), we get \( \Theta(K_{34}) = 37 = \rho_{34} \). In view of \( \chi_{93} = 91 \), one has \( \Theta(K_{91}) = 93 < \rho_{91} \). It is interesting to determine all \( n \) such that \( \Theta(K_n) = \rho_n \).

### 3 Complete bipartite graphs

We begin this section by observing that the star \( K_{1,t} \) can embed into the power graph of each group of order at least \( 1 + t \).

**Observation 3.1.** The star \( K_{1,t} \) is power-critical. Particularly, any group of order \( 1 + t \) is \( K_{1,t} \)-optimal.
In the remaining of this section, we always assume that the complete bipartite graph $K_{s,t}$ have the vertex set partition $\{U, W\}$, and $2 \leq s \leq t$. For a subset $S \subseteq V(K_{s,t})$ and an embedding

$$f : V(K_{s,t}) \rightarrow V(\Gamma_G),$$

write $f(S) = \{f(v) : v \in S\}$.

**Lemma 3.2.** Let $u$ and $v$ be vertices of $K_{s,t}$. With reference to (2), if $|f(u)| = |f(v)| = p$ for some prime $p$, then $\langle f(u) \rangle = \langle f(v) \rangle$. Moreover, if $|G| = s + t$ and $p$ is a prime divisor of $s + t$, then $G$ has a unique subgroup of order $p$.

**Proof.** If $u$ and $v$ are adjacent, then one of $f(u)$ and $f(v)$ is a power of the other, and so $\langle f(u) \rangle = \langle f(v) \rangle$. Now suppose that $u$ and $v$ are nonadjacent. Without loss of generality, let $\{u, v\} \subseteq U$. Note that $2 \leq s \leq t$. Pick $w \in W$ such that $f(w)$ is not the identity of $G$. Then both $f(u)$ and $f(v)$ are adjacent to $f(w)$ in $\Gamma_G$. Note that $|f(u)| = |f(v)| = p$. It is easy to check that $\langle f(u) \rangle = \langle f(v) \rangle$.

If $|G| = s + t$, then $\{f(U), f(W)\}$ is a partition of $G$. Hence, the desired result follows.

**Lemma 3.3.** As refer to (2), if $G$ is not a $p$-group and $|G| = s + t$, then $G$ is cyclic. In particular, one of $f(U)$ and $f(W)$ is a subset of $A$, where $A$ is the set of generators and the identity of $G$.

**Proof.** Let $x$ be the set of all elements of prime order of $G$. Note that $X$ contains two elements with distinct orders. Then $X \subseteq f(U)$ or $f(W)$. Without loss of generality, let $X \subseteq f(U)$. Take any non-identity element $x$ in $f(W)$. Then each element of $X$ belongs to $\langle x \rangle$.

We now claim that every element of $f(U)$ belongs to $\langle x \rangle$. In fact, suppose that there exists $y$ in $f(U)$ such that $y \notin \langle x \rangle$. Then $x \in \langle y \rangle$. Hence, there exist prime $p$ and positive integer $m$ such that $|y|$ is divisible by $p^m$ and $|x|$ is not divisible by $p^m$. Let $z$ be an element of $\langle y \rangle$ of order $p^m$. Since $|x|$ is not a prime power, $x$ and $z$ are not adjacent in $\Gamma_G$, which means that $z \in f(W)$. Moreover, it is clear that there exists an element $z'$ of order $q$ in $G$, where $q$ is a prime divisor of $|G|$ and $q \neq p$. Since $z' \in X \subseteq f(U)$, $z'$ is adjacent to $z$ in $\Gamma_G$, and so $|z'|$ divides $|z|$ or $|z|$ divides $|z'|$, which contradicts the orders of $z$ and $z'$. Thus, our claim is valid.

Take any non-identity element $x'$ in $f(W) \setminus \{x\}$, and let $|x'| = p_1^{r_1}p_2^{r_2} \cdots p_l^{r_l}$, where $p_1, \ldots, p_l$ are pairwise distinct primes and $r_i \geq 1$ for each $1 \leq i \leq l$. Let $u_i$ be an element of $\langle x' \rangle$ of order $p_i^{r_i}$. Since there exists an element of $X$ of prime order which is different from $p_i$, one has $u_i \in f(U)$. By the claim above, we have $u_i \in \langle x \rangle$ for each $1 \leq i \leq l$. Since $\langle u_1 \rangle \langle u_2 \rangle \cdots \langle u_l \rangle = \langle x' \rangle$, one gets $x' \in \langle x \rangle$. It follows that any element of $G$ belongs to $\langle x \rangle$, and so $G = \langle x \rangle$, as wanted.
Lemma 3.4. Let \( 2 \leq s \leq t \). If \( \varphi(s + t) \geq s - 1 \), then \( Z_{s+t} \) is \( K_{s,t} \)-optimal.

Proof. Without loss of generality, assume that \( |U| = s \) and \( |W| = t \). Let \( A \) be the set of generators and the identity of \( Z_{s+t} \). Then \( |A| = \varphi(s + t) + 1 \). Hence, there exists a bijection \( h \) from \( V(K_{s,t}) \) to \( Z_{s+t} \) such that \( h(U) \subseteq A \). It is easy to verify that \( h \) is an embedding from \( K_{s,t} \) to \( \Gamma Z_{s+t} \). Hence, the required result follows.

Theorem 3.5. Let \( 2 \leq s \leq t \). Then \( K_{s,t} \) is power-critical if and only if \( \varphi(s + t) \geq s - 1 \).

Proof. Suppose that \( K_{s,t} \) is power-critical. With reference to (2), assume that \( |G| = s + t \). If \( G \) is a \( p \)-group, then \( s + t \) is a prime power, which implies that

\[
\varphi(s + t) + 1 \geq \frac{s + t}{2} \geq s.
\]

If \( G \) is not a \( p \)-group, it follows from Lemma 3.3 that \( G \) is cyclic and

\[
s = \min\{|f(U)|, |f(W)|\} \leq \varphi(s + t) + 1.
\]

For the converse, the required result follows from Lemma 3.4.

By Theorem 3.5 and some properties of the Euler's totient function, we get

Corollary 3.6. (i) For any positive integer \( s \), there exists a positive integer \( t_s \) such that \( K_{s,t_s} \) is power-critical for each integer \( t \geq t_s \).

(ii) Let \( s \geq 2 \). Then \( K_{s,s} \) is power-critical if and only if \( s \) is an odd prime or a power of 2.

For any \( 2 \leq s \leq 6 \) and \( t \geq 7 \), \( K_{s,t} \) is power-critical. However, \( \Theta(K_{6,6}) = 13 \) and \( \Theta(K_{9,9}) = 19 \).

For \( n \geq 2 \), the generalized quaternion group \( Q_{4n} \) is defined by

\[
Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle.
\]

Corollary 3.7. Let \( 3 \leq k \) and \( 2 \leq s \leq t \). Suppose that \( K_{s,t} \) is power-critical.

(i) A group \( G \) is \( K_{2,2^{k-2}} \)-optimal if and only if \( G \) is isomorphic to \( Z_{2^k} \) or \( Q_{2^k} \).

(ii) If \( (s,t) \neq (2,2^k - 2) \), then a group \( G \) is \( K_{s,t} \)-optimal if and only if \( G \) is isomorphic to \( Z_{s+t} \).

Proof. (i) It is clear that \( Z_{2^k} \) is \( K_{2,2^{k-2}} \)-optimal. Since \( Q_{2^k} \) has a unique involution which is a power of any other nonidentity elements of \( Q_{2^k} \), it is easy to check that \( Q_{2^k} \) is \( K_{2,2^{k-2}} \)-optimal. Now suppose that \( G \) is \( K_{2,2^{k-2}} \)-optimal. Then \( |G| = 2^k \). It follows from Lemma 3.2 that \( G \) has a unique subgroup of order 2, and so \( G \) is isomorphic to \( Z_{2^k} \) or \( Q_{2^k} \) by [11, Theorem 5.4.10 (ii)].
(ii) It follows from Lemma 3.4 that $\mathbb{Z}_{s+t}$ is $K_{s,t}$-optimal. In the following, suppose that $G$ is $K_{s,t}$-optimal. Then $|G| = s + t$. If $G$ is not a $p$-group, then the required result holds by Lemma 3.3. Now let $G$ be a $p$-group. Then $G$ is cyclic or generalized quaternion by Lemma 3.2 and [11, Theorem 5.4.10 (ii)]. Suppose that $G$ is isomorphic to $Q_{2^k}$ for some integer $k \geq 3$. Since $(s, t) \neq (2, 2^k - 2)$ and $s + t = 2^k$, we have $3 \leq s \leq 2^{k-1}$. Hence, the power graph $\Gamma_G$ has 3 distinct vertices with degree at least $2^{k-1}$, which contradicts the structure of $\Gamma_{Q_{2^k}}$ as shown in Figure 1.

![Figure 1: The power graph $\Gamma_{Q_{2^k}}$, where $e$ is the identity of $Q_{2^k}$ and $\Delta$ is a complete graph of order $2^{k-2}$.](image)

Therefore, the desired result follows.

The following result is immediate by Corollary 3.7.

**Corollary 3.8.** Let $G$ be a group of order $n$. Then there exists a nonidentity element with degree at least $n - 1$ in $\Gamma_G$ if and only if $G$ is a cyclic group or a generalized quaternion $2$-group.

### 4 1-factors

Let $K_1 + nK_2$ be the graph obtained from $nK_2$ by adding a vertex and joining this vertex to each vertex of $nK_2$. In this section, we always use $e$ to denote the identity of a group $G$.

**Proposition 4.1.** The graph $K_1 + nK_2$ is power-critical, and any group $G$ of order $2n + 1$ is $(K_1 + nK_2)$-optimal.
Proof. We only need to show that any group $G$ of order $2n+1$ is $(K_1 + nK_2)$-optimal. For each $x \in G \setminus \{e\}$, we get $x \neq x^{-1}$. Write

$$V(K_1 + nK_2) = \{u, v_1, \ldots, v_n, w_1, \ldots, w_n\},$$

$$E(K_1 + nK_2) = \bigcup_{i=1}^{n} \{\{u, v_i\}, \{u, w_i\}, \{v_i, w_i\}\}.$$

Then there exists a bijection $f$ from $V(K_1 + nK_2)$ to $G$ such that $f(u) = e$ and $f(v_i) = f(w_i)^{-1}$ for each $i \in \{1, \ldots, n\}$. It is easy to verify that $f$ is an embedding from $K_1 + nK_2$ to $\Gamma_G$. Hence, the desired result follows.

A matching in a graph is a set of pairwise nonadjacent edges. A perfect matching of a graph $\Gamma$ is a matching if its size equals to $\frac{|V(\Gamma)|}{2}$, and a near-perfect matching of $\Gamma$ is a matching if its size equals to $\frac{|V(\Gamma)|-1}{2}$. The following result is immediate by Proposition 4.1.

Corollary 4.2. If $G$ is a group of odd order, then $\Gamma_G$ has a near-perfect matching.

A path $P$ in a power graph $\Gamma_G$ is inverse-closed if $V(P) = \{x^{-1} : x \in V(P)\}$. Actually, for any distinct elements $u$ and $v$ in $G$, there exists an inverse-closed path between $u$ and $v$ in $\Gamma_G$. Denote by $L(P)$ the set of the endpoints of a path $P$.

Algorithm 1 Find $x_1, \ldots, x_m$

1: Input $u_1, \ldots, u_r$
2: Put $x_1 = u_1$, $i = 1$ and $m = 1$
3: for $x_m \neq u_r$ and $x_m \neq u_r^{-1}$ do
4: Let $l_i$ be the index with $u_i = u_i^{-1}$.
5: $i \leftarrow \max\{i, l_i\} + 1$ and $m \leftarrow m + 1$
6: $x_m = u_i$
7: end for
8: if $x_m = u_r$ then
9: $x_m \leftarrow u_r^{-1}$
10: end if
11: Output $x_1, \ldots, x_m$

Lemma 4.3. Let $P$ be an inverse-closed path in a power graph $\Gamma_G$ such that $|x| \geq 3$ for each $x \in V(P)$. Then there exists an inverse-closed path

$$P' = (x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_m, x_m^{-1})$$

such that $V(P') \subseteq V(P)$ and $L(P') = L(P)$.  

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Proof. Write $P = (u_1, \ldots, u_r)$. By Algorithm 1, we obtain some vertices $x_1, \ldots, x_m$ from $V(P)$. From Step 4, each vertex in $\{u_i, u_i\}$ is adjacent to $u_{i+1}$ and $u_{i+1}$ in $\Gamma_G$. It follows from Steps 5 and 6 that $\{x_j^{-1}, x_{j+1}\} \in E(\Gamma_G)$ for $1 \leq j \leq m-1$. Note that $\{x_1, \ldots, x_m\} \cap \{x_1^{-1}, \ldots, x_m^{-1}\} = \emptyset$. Then $(x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_m, x_m^{-1})$ is the desired path. □

For a group $G$ of even order, we always use $U$ to denote the set of all involutions, and write $\overline{U} = U \cup \{e\}$. Note that a group of even order has odd number of involutions.

**Theorem 4.4.** Let $G$ be a group of order $2n$ with $2k - 1$ involutions. Then the followings are equivalent.

(i) The group $G$ is $nK_2$-optimal.

(ii) The power graph $\Gamma_G$ has a perfect matching.

(iii) There exist $k$ vertex-disjoint and inverse-closed paths $P_1, \ldots, P_k$ in $\Gamma_G$ such that $$\bigcup_{i=1}^k L(P_i) = \overline{U}.$$ 

Proof. It is clear that (i) and (ii) are equivalent. In the following, we shall show that (ii) and (iii) are equivalent.

Suppose (ii) holds. Let $\mathcal{M}$ be a perfect matching of $\Gamma_G$. Using Algorithm 2, we obtain $k$ vertex-disjoint paths $P_1, \ldots, P_k$ such that $\bigcup_{i=1}^k L(P_i) = \overline{U}$. Hence (iii) holds.

Suppose (iii) holds. For $i \in \{1, \ldots, k\}$, write $$P_i = (u_{i1}, x_{i1}, x_{i2}, \ldots, x_{it_i}, u_{i2}).$$ Without loss of generality, assume that $u_{k1} = e$. For $1 \leq i \leq k - 1$, we have $t_i \geq 2$. By Lemma 4.3, there exists $$P'_i = (u_{i1}, y_{i1}, y_{i1}^{-1}, \ldots, y_{im_i}, y_{im_i}^{-1}, u_{i2})$$ such that $V(P'_i) \subseteq V(P_i)$. Let $P'_k = (u_{k1}, u_{k2})$. Now we may assume that $$G \setminus \bigcup_{i=1}^k V(P'_i) = \{v_1, v_2, \ldots, v_r\} \cup \{v_1^{-1}, v_2^{-1}, \ldots, v_r^{-1}\},$$ where $|v_i| \geq 3$ for $1 \leq i \leq r$. Then

$$\{u_{(k-1)1}, y_{(k-1)1}, y_{(k-1)1}^{-1}, y_{(k-1)2}, y_{(k-1)2}^{-1}, \ldots, y_{(k-1)m_{k-1}}, u_{(k-1)2}, u_{k1}, v_1, v_1^{-1}, \ldots, v_r, v_r^{-1}\}$$
Algorithm 2 Find $k$ vertex-disjoint paths $P_1, \ldots, P_k$ such that $\bigcup_{i=1}^{k} L(P_i) = \overline{U}$

1: Input $\overline{U}$ and $\mathcal{M}$
2: Set $A := \overline{U}$  // $|A| = 2k$
3: for $i = 1, \ldots, k$ do
4: Choose $u \in A$
5: Put $P_i := (u, x)$ for $\{u, x\} \in \mathcal{M}$
6: while $x \notin A$ do
7: $P_i \leftarrow (P_i, x^{-1}, y)$ for $\{x^{-1}, y\} \in \mathcal{M}$  // Note that $x \notin \overline{U} \iff x \neq x^{-1}$
8: $x \leftarrow y$
9: end while
10: $A \leftarrow A \setminus \{u, x\}$
11: end for  // $A = \emptyset$
12: Output $P_1, \ldots, P_k$

is a perfect matching of $\Gamma_G$.

Note that $\mathbb{Z}_{2n}$ has a unique involution, which is adjacent to the identity in $\Gamma_{\mathbb{Z}_{2n}}$. The following result holds from Theorem 4.4.

**Theorem 4.5.** The 1-factor is power-critical and $\mathbb{Z}_{2n}$ is $nK_2$-optimal.

Actually, any group of order $2n$ having a unique involution is $nK_2$-optimal. Hence, $Q_{4n}$ is $2nK_2$-optimal. Let $D_{2n}$ be the dihedral group of order $2n$. In $\Gamma_{D_{2n}}$, any path between two distinct involutions contains the identity, so $D_{2n}$ is not $nK_2$-optimal.

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