Two patterns of $\mathcal{PT}$–symmetry breakdown in a non-numerical six-state simulation

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Abstract

Three-parametric family of non-Hermitian but $\mathcal{PT}$–symmetric six-by-six matrix Hamiltonians $H^{(6)}(x, y, z)$ is considered. The $\mathcal{PT}$–symmetry remains spontaneously unbroken (i.e., the spectrum of the bound-state energies remains real so that the unitary-evolution stability of the quantum system in question is shown guaranteed) in a non-empty domain $\mathcal{D}^{(\text{physical})}$ of parameters $x, y, z$. The construction of the exceptional-point (EP) boundary $\partial \mathcal{D}^{(\text{physical})}$ of the physical domain is preformed using an innovative non-numerical implicit-function-construction strategy. The topology of the resulting EP boundary of the spontaneous $\mathcal{PT}$–symmetry breakdown (i.e., of the physical “horizon of stability”) is shown similar to its much more elementary $N = 4$ predecessor. Again, it is shown to consist of two components, viz., of the region of the quantum phase transitions of the first kind (during which at least some of the energies become complex) and of the quantum phase transitions of the second kind (during which some of the level pairs only cross but remain real).
keywords

quantum theory; non-Hermitian observables; discrete models of bound states; typology of instabilities;
1 Introduction

In 1998, Bender with Boettcher \[1\] conjectured that the reality of the bound state energy spectra (i.e., the unitarity of the evolution) might be attributed to the unbroken $\mathcal{PT}$—symmetry (i.e., parity times time-reversal symmetry) of the underlying phenomenological Hamiltonian $H$. Mathematical formulation as well as implementations of the newly developed theory were, twelve years later, reviewed and summarized by Mostafazadeh \[2\]. At present it is widely accepted that the manifest non-Hermiticity of the $\mathcal{PT}$—symmetric Hamiltonians with real spectra is fully compatible with the Stone’s theorem \[3\] and with the unitarity of the evolution of the quantum system in question \[4\].

The price to pay for the resolution of the apparent paradox lies in the necessity of an \textit{ad hoc} amendment of the Hilbert space. Simply stated (see, e.g., \[5\]), one has to distinguish between the naively preselected initial, unphysical, “friendly but false” Hilbert space $\mathcal{H}^{(F)}$ (in which our $\mathcal{PT}$–symmetric Hamiltonian with real bound-state spectrum appears manifestly non-Hermitian, $H \neq H^\dagger$) and its “standard physical” amendment $\mathcal{H}^{(S)}$ (here, the inner product is amended in such a way that the \textit{same} operator becomes self-adjoint, $H = H^\dagger$).

The innovative picture of quantum dynamics led to a perceivable extension of the class of tractable quantum Hamiltonians. For example, in the traditional unitary quantum theory of textbooks the linear differential Hamiltonians

$$H = -\triangle + V(\vec{x})$$

must be kept self-adjoint in $\mathcal{H}^{(S)} = \mathcal{H}^{(F)} = L^2(\mathbb{R}^d)$. In the new context the constraint was relaxed. The progress was rendered possible by the separation of $\mathcal{H}^{(F)} = L^2(\mathbb{R}^d) \neq \mathcal{H}^{(S)}$. This resulted in the representation of unitary systems in two different Hilbert spaces, viz., in physical $\mathcal{H}^{(S)}$ and, simultaneously, in auxiliary unphysical $\mathcal{H}^{(F)}$. A number of innovative model-building activities followed \[6\].

Successfully, the mathematical meaning of $\mathcal{PT}$—symmetry $H \mathcal{PT} = \mathcal{PT} H$ was identified with the older concepts of pseudo-Hermiticity $H^\dagger \mathcal{P} = \mathcal{P} H$ \[2\] \textit{alias} Krein-space self-adjointness \[7\] of the Hamiltonian. Still, for the generic non-Hermitian Hamiltonians the physical essence of quantum dynamics in $\mathcal{H}^{(S)}$ appeared counterintuitive and deeply non-local \[8\]. It has been revealed that for many non-Hermitian local potentials $V(\vec{x})$ (some of which played the role of benchmark toy models) the amended physical Hilbert space $\mathcal{H}^{(S)}$ need not exist, in mathematical sense, at all \[9\].

One of the ways out of the crisis has been found in a return to the more restricted class of the so called quasi-Hermitian Hamiltonians $H$. In nuclear physics, for example,
these operators were obligatorily assumed bounded in $H^{(F)}$ \[10\]. Often, they were even represented by the mere finite, $N$—dimensional matrices $H^{(N)}$. In what follows, we shall also proceed along this line.

In the historical perspective \[11\] the inspiration of the latter strategy can be traced back to the Kato’s rigorous mathematical monograph \[12\]. Many illustrative Hamiltonians were chosen there in the form of matrices with minimal dimension $N = 2$. Also in Ref. \[13\] devoted to the study of several manifestly non-Hermitian differential operators \(\mathcal{P}\mathcal{T}\), several anomalous spectral features caused by $\mathcal{P}\mathcal{T}$—symmetry were sucessfully mimicked by certain most elementary benchmark matrices $H^{(N)}$ with $N = 2$.

Due to $\mathcal{P}\mathcal{T}$—symmetry, the bound-state-energy spectrum of $H^{(N)}$ can be either “physical” (i.e., real, compatible with unitarity) or “unphysical” (i.e., containing one or several non-real, complex conjugate pairs). This leads to one of the most interesting mathematical questions and challenges in the $\mathcal{P}\mathcal{T}$—symmetric quantum mechanics: Once we assume that a given $\mathcal{P}\mathcal{T}$—symmetric Hamiltonian depends on a $J$–plet of couplings or dynamics-specifying real parameters $g_1 = a, g_2 = b, \ldots, g_J = z$, we must be able to separate the Euclidean space $\mathbb{R}^J$ of these parameters into an open domain $\mathcal{D}^{(N)}_{(\text{physical})}$ (in which our Hamiltonian $H^{(N)} = H^{(N)}(a, b, \ldots, z)$ is diagonalizable and in which its spectrum is real) and its unphysical complement (in which the necessary physical Hilbert space $\mathcal{H}^{(S)}$ does not exist).

The determination of the domain $\mathcal{D}^{(N)}_{(\text{physical})}$ and the localization and description of its boundary $\partial \mathcal{D}^{(N)}_{(\text{physical})}$ are two rather difficult mathematical tasks in general. The (technically much easier) determination of at least one of the non-empty subdomains $\mathcal{D}^{(N)}_{(0)} \subset \mathcal{D}^{(N)}_{(\text{physical})}$ is in fact one of the most important necessary conditions of the very applicability of the formalism. This is one of the reasons of the above-mentioned popularity of the benchmarks with minimal $N = 2$. For $N = 2$, indeed, the construction of $\mathcal{D}^{(2)}_{(\text{physical})}$ proves always feasible by non-numerical means \[14\].

The difficulty of the analysis does not grow too much at $N = 3$ but the three-dimensional (and, in general, all odd-dimensional) matrix models are not too instructive because their “added” energy level remains always real \[15\]. In this context, several existing studies \[16, 17\] of the next, $N = 4$ benchmark matrix spectra seem to represent a feasibility limit and a transition point between the numerical and non-numerical descriptions of the “quantum phase transition” boundaries $\partial \mathcal{D}^{(N)}_{(\text{physical})}$.

In our present paper we intend to develop, in some detail, the idea as mentioned in Refs. \[18, 19\] where it enabled us to push the non-numerical description of domain $\mathcal{D}^{(N)}_{(\text{physical})}$ beyond the $N = 4$ boundary. In the language of mathematics the essence of the idea lies in
an *ad hoc* lowering of the number $J$ of the variable parameters in matrices $H^{(N)}(a, b, \ldots, z)$. In the context of physics, a very strong motivation of such a project may be seen in the fact that the boundary $\partial D_{(physical)}^{(N)}$ should be, in general, composed of the hierarchy of the lower-dimensional subdomains of the parameters at which the quantum system in question encounters a genuine phase transition.

A complementary, more specific reason for the study of the $N \geq 4$ models has been found in Ref. [17]. In a model with $N = 4$, the above-mentioned quantum phase transitions appeared there to be of *two different kinds*. We expect that such an observation should and could be also reconfirmed at $N = 6$. In what follows we will extend, therefore, some of the qualitative non-numerical $N = 4$ results of Ref. [17] to the next, phenomenologically richer model with $N = 6$. The model will be introduced in Section 2 and its properties will be described and discussed in subsequent Sections 3 and 4.

## 2 Six-state $\mathcal{PT}$–symmetric model

### 2.1 Hamiltonian

We intend to work with the real matrices $H^{(N)}(a, b, \ldots, z)$ for which the time-reversal $\mathcal{T}$ is realized by the matrix transposition and for which the role of the parity-reversal is played by the antidiagonal $N$ by $N$ square-of-unit matrix

$$
\mathcal{P} = \mathcal{P}^{(N)} = \begin{bmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & \ldots & 0 & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix}
$$

with the first nontrivial choice of dimension $N = 6$. In another methodical constraint we recall Eq. [11] and assume that our toy model Hamiltonian $H^{(N)}$ will be split into an arbitrary (i.e., non-Hermitian but $\mathcal{PT}$–symmetric) interaction matrix $V^{(N)} = V_{ij}^{(N)}$ and a kinetic-energy-simulating term $\Delta^{(N)}$ given in the form of the standard discrete Laplacean $\Delta_{ij}^{(N)} = \delta_{i,j+1} + \delta_{i,j-1}$, i.e., as matrix

$$
\Delta^{(N)} = \begin{bmatrix}
0 & -1 & 0 & \ldots & 0 \\
-1 & 0 & -1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & -1 & 0 & -1 \\
0 & \ldots & 0 & -1 & 0
\end{bmatrix}
$$

where
Although the general form of matrix \( V^{(N)} = V^{(N)}_{ij} \) could contain, in principle, up to \( J_{\text{max}} = N^2 \) (i.e., in the present paper, \( J_{\text{max}} = 36 \)) free real parameters, we shall follow the guidance by paper [17] and keep the matrix form of the real interaction term strictly tridiagonal and antisymmetric. Together with the requirement of \( \mathcal{PT} \)–symmetry this makes our ultimate \( N = 6 \) toy model Hamiltonian strictly \( J \)–parametric with \( J = N/2 = 3 \),

\[
H^{(6)}(x, y, z) = \begin{pmatrix}
0 & -1 + z & 0 & 0 & 0 & 0 \\
-1 - z & 0 & -1 + y & 0 & 0 & 0 \\
0 & -1 - y & 0 & -1 + x & 0 & 0 \\
0 & 0 & -1 - x & 0 & -1 + y & 0 \\
0 & 0 & 0 & -1 - y & 0 & -1 + z \\
0 & 0 & 0 & 0 & -1 - z & 0
\end{pmatrix}.
\]  

(4)

2.2 Bound-state energies

The general discrete Schrödinger equations for bound states

\[
H^{(N)} \ket{\psi^{(N)}_n} = E^{(N)}_n \ket{\psi^{(N)}_n}, \quad n = 0, 1, \ldots, N - 1, \quad N < \infty
\]

may be solved, in practice, by various computer-assisted numerical methods. Under the present choice of Hamiltonian [6] with \( N = 6 \), fortunately, the related spectrum-determining secular equation

\[
\det \left[ \begin{bmatrix} \mathcal{H}^{(N)}(x, y, z) - E^{(N)}_n I^{(N)} \end{bmatrix} \right] = 0
\]

admits a perceivable simplification mediated by the change of variables

\[
z \to C = 1 - z^2, \quad y \to B = 1 - y^2, \quad x \to A = 1 - x^2.
\]

(7)

After some algebra one converts the secular equation into polynomial relation

\[
E^6 + [-2C - 2B - A] E^4 + [2BC + 2AC + C^2 + B^2] E^2 - AC^2 = 0.
\]

(8)

The equation is solvable in terms of Cardano formulae. This indicates the non-numerical origin of the pictures which sampled the parametric-dependence of the energies \( E^{(6)}_n (C, B, A) \) in Ref. [17]. Unfortunately, the shape of domain \( D_{\text{physical}} \) as well as the specification of its boundaries \( \partial D_{\text{physical}} \) were omitted there as tractable by the purely numerical constructive means.

The general numerical algorithm of the latter construction was described (but not numerically tested) in Ref. [20]. Thus, the conclusions of the two studies [17, 20] were
discouraging: the construction of the boundary $\partial D_{(\text{physical})}$ remains a purely numerical task at $N = 6$. Moreover, even a sufficiently transparent presentation of the results of the numerical construction seem to require an active use of some interactive graphical software.

In what follows we intend to demonstrate that the sceptical conclusions of Ref. [20] resulted from the consideration of too broad a class of matrices $H^{(6)}$. In this context, the main result of our present study will lie in the discovery that due to certain specific features of our choice of Hamiltonians (8), the construction of the physical unitary evolution domain $D_{(\text{physical})}$ may be made much more straightforward. We shall show that up to the necessary determination of certain auxiliary constants, this construction also remains strictly non-numerical at $N = 6$.

3 Domain $D_{(\text{physical})}^{(6)}(A, B, C)$

From the matrix form of our present $N = 6$ Hamiltonian (6) one can immediately deduce that in a way generalizing the results of Ref. [17] the boundary $\partial D_{(\text{physical})}$ will contain, $i.e.$, the three planes $A = 0$, $B = 0$ and $C = 0$ representing the Kato’s [12] exceptional points (EPs) at which the matrix ceases to be diagonalizable.

Such an overall reality-domain-structure conjecture will be given a more complete, constructive and explicit form in what follows. It is also time for us to point out now that our present approach will be based on the replacement of the tedious and practically useless explicit Cardano formulae for $E_n(C, B, A)$ by their perceivably simpler and analytically tractable implicit-function alternatives.

In this setting, one of the other and also one of the most important properties of the set $D_{(\text{physical})}$ will be guessed (i.e., conjectured and, later, proved) via an analogy with the $N = 4$ results of Ref. [17]. This property is that the admissible innermost coupling $A$ must be always positive.

3.1 An elimination of the outermost coupling $C = 1 - z^2$

In a preparatory step let us replace the (positive) parameter $A$ by a pair of its real square roots $\alpha = \pm \sqrt{A}$ (by our above-mentioned conjecture, these roots remain always real). Via the subsequent application of the computer-assisted factorization techniques (followed by their not too difficult backward check) we managed to reduce our secular Eq. (8) to the pair of formulae

$$C = C^{(\pm)}(E, B, A) = E^2 - \frac{B}{E - \alpha^{(\pm)}(A)} E$$

(9)
for the outermost coupling (one for each sign of $\alpha$). Such an $N = 6$ analogue of its $N = 4$ predecessor exhibits a parabolic asymptotic growth $C^{(\pm)}(E, B, A) = E^2 + \mathcal{O}(1)$ at $|E| \gg 1$. One of the generic zeros of $C = C(E)$ occurs at $E = E_{\text{zero}}^{(0)} = 0$. The resulting estimated shape is further modified by a single first-order pole at (positive or negative) $E_{\text{singular}} = \alpha^{(\pm)}(A)$. This means that at the large $C$s and for each sign of $\alpha \lesssim 0$ the curve will be intersected by the horizontal line $C = \text{const}$ at the three different real values of the bound state energy $E_n$. Thus, after one (numerically) determines the appropriate EP minimum $E_{\text{EP}}^{(B, A)}$ of the function $C = C(E)$, all of the points $C \in (E_{\text{EP}}^{(B, A)}, \infty)$ (and only these points) will belong to $\mathcal{D}_{\text{(physical)}}$.

Without loss of generality (i.e., due to left-right symmetry of the spectrum of energies) let us now consider just the branch of Eq. (9) with positive $\alpha > 0$ (i.e., with the pole to the right from the origin). We then can distinguish between the two alternative scenarios in which the remaining two zeros of the curve $C(E)$, viz., values

\[ E_{\text{zero}}^{(\pm)} = \frac{1}{2} \left( \alpha \pm \sqrt{\alpha^2 + 4B} \right) \]  

are separated by the origin (for $B > 0$) or not (for $B < 0$; remember that the two respective limits with $B = 0$ become unphysical).

In the former case with $B > 0$ and to the right from $E = \alpha$ the real intersection of the curve $C(E)$ with the horizontal line $C = \text{const.}$ will exist at any $C \in (-\infty, \infty)$, defining the largest (and always real) energy root. The other two bound state energies will be smaller than $\alpha$, originating from the intersection of the fixed horizontal line $C > C_{\text{(EP)}}(B, A)$ with the left, U-shaped part of the curve (9). This curve will attain its unique minimum (equal to $C_{\text{(EP)}}(B, A)$) at the negative value of energy $E_{\text{min}}$. This constant may be evaluated using its implicit-function definition

\[ \alpha B = -2E_{\text{min}}(E_{\text{min}} - \alpha)^2, \quad B > 0, \quad E_{\text{min}} < 0. \]  

The other, more interesting scenario occurs at the negative values of $B < 0$. To the right from $E = \alpha$, the curve $C(E)$ will then become U-shaped, with a real minimum $E_{\text{right}} = C_{\text{(EP)}}(B, A) > \alpha^2$. This minimum is attained at the largest root $E_{\text{right}} > \alpha$ of Eq. (11) and it moves slowly upwards with the decrease of $B < 0$.

At the small negative $B$s and to the left from the pole at $E = \alpha$, the curve $C(E)$ decreases, with the growth of $E$ from its zero at $E = 0$ to its local minimum $C_{\text{(min)}}$ at $E_{\text{min}}$. It subsequently grows to its local maximum $C_{\text{(max)}}$ at $E_{\text{max}}$ and, finally, it decreases to minus infinity at $E = \alpha$. The values $E_{\text{min}}$ and $E_{\text{max}}$ are the two remaining roots of the cubic Eq. (11). In and only in the interval of (negative) $B \in (B_{\text{(EP)}}, 0)$ these roots
remain real. In such an interval, the domain $\mathcal{D}_{\text{(physical)}}$ also contains an additional, anomalous, non-empty interval of the acceptable couplings $C \in (C_{(\text{min})}(B, A), C_{(\text{max})}(B, A))$, separated from the above-mentioned upper interval by a non-empty gap of unphysical $C \in (C_{(\text{max})}(B, A)), C_{(\text{EP})}(B, A))$.

One can conclude that the existence of the gap of a non-zero width in $\mathcal{D}_{\text{(physical)}}$ should be perceived as a phenomenologically highly interesting consequence of the simultaneous variability of the three parameters in our six-site discrete $\mathcal{PT}$–symmetric quantum lattice.

### 3.2 The innermost coupling $A = 1 - x^2$

Our constructive considerations of the preceding paragraph were based on the assumption that the domain $\mathcal{D}_{\text{(physical)}}$ does not contain a part with the negative values of the innermost parameter $A$. At $N = 6$ the validity of this property appears as surprising and serendipitous as at $N = 4$. Also its proof is again easy to verify only after a nontrivial preliminary computer-assisted factorization of certain components of the secular polynomial.

The net result of these algebraic manipulations may be expressed by the following elementary formula

$$\pm \alpha^{(\pm)}(A) = \alpha(E, C, B) = \left(1 - \frac{B}{E^2 - C}\right) E.$$  \hspace{1cm} (12)

Its form looks very similar to its $C$–related predecessor so that also the analysis of the shape of this function is feasible and routine.

The analysis is more straightforward at the positive $C = \gamma^2$ when it leads to the simplification

$$\alpha(E, C, B) = E - \frac{B}{2} \left(\frac{1}{E + \gamma} + \frac{1}{E - \gamma}\right).$$  \hspace{1cm} (13)

At the positive $B$s this function of $E$ is composed of the three separate branches, each of which grows from $-\infty$ to $+\infty$. As long as the sextuplet of the bound state energies $E_n$ is specified by the intersections of this curve with the pair of horizontal lines $\alpha = \pm \sqrt{A} = \text{const.}$, the conclusion is that all of these energies are real iff $A \geq 0$.

After we move to the negative $B$s, the branches of $\alpha(E)$ will all flip and decrease near the singularities (i.e., at $E \approx \pm \gamma$). One can only guarantee the entirely robust existence of a single maximum at $\alpha = \alpha_{(\text{max})} < 0$ for $E = E_{\text{max}} < -\gamma$ and of a single minimum $\alpha_{(\text{min})} > 0$ at $E_{\text{min}} > \gamma > 0$ (this minimum may be found sampled in the right upper corner of Fig. 1). Thus, the robust reality of the sextuplet of the bound state energies can be safely guaranteed in the interval of the sufficiently large $A \in (\alpha_{(\text{max})}^2, \infty)$.

As long as the negative $B$s remain small, $B \in (B_{(\text{EP})}, 0)$, the curve $\alpha(E)$ may also exhibit a negative minimum $\alpha_{(-)} < 0$ at some $E_{-} \in (-\gamma, 0)$, followed by a positive maximum.
Figure 1: The graph of function $\alpha(E)$ of Eq. (13) at $B = B_0 = 1/10$ and $C = C_0 = 1$ (i.e., $\gamma = 1$). The two thin horizontal lines sample the choice of $A = A_0 = 0.09$ (i.e., of $\alpha_0^{(\pm)} = \pm \sqrt{A_0} = \pm 0.3$). The intersections (= small circles) determine the six real bound-state energies $E = E_n(A_0, B_0, C_0)$ with $n = 0, 1, \ldots, 5$.

$\alpha_+ = -\alpha_- > 0$ at some $E_+ = -E_-$ (this situation is sampled in Fig. 1). One may conclude that domain $D_{(\text{physical})}$ will contain another component of $A \in (0, \alpha_{2(\pm)}^2)$. This will be an anomalous subdomain separated from the upper bulk part by the non-empty gap of unphysical values of $A \in (\alpha_{(+)2}, \alpha_{(\text{max})2})$.

In the last step of our analysis we have to move to the range of negative Cs. In this case the correction to the linear term is a bounded function of $E$ so that the function $\alpha(E)$ itself (having no real singularities) can only develop a negative local minimum $\alpha_{\text{left}} < 0$ at a negative energy (plus, symmetrically, a positive local maximum $-\alpha_{\text{left}} > 0$ at an opposite positive energy). This can only happen at some sufficiently large $B > B_{(\text{EP})} > 0$. In such a case one notices the emergence of a new piece of domain $D_{(\text{physical})}$ with $A \in (0, \alpha_{\text{left}}^2)$.

3.3 The intermediate coupling $B = 1 - y^2$

For the completion of our description of the geometric shape of $D_{(\text{physical})}$ it is necessary to derive and recall also the last pair of the eligible explicit definitions

$$B^{(\pm)}(E, C, A) = (E - \alpha^{(\pm)}(A))(E^2 - C)/E.$$  \hspace{1cm} (14)

Such a function with the single singularity in the origin (with the most elementary growing or decreasing shape $B(E) = \alpha C/E - C + O(E)$) and with the parabolic asymptotic behavior $B(E) = E^2 + O(E)$ always contains a U-shaped part (to the right from the origin for the positive product $\alpha C$ and vice versa). The other part decreases or, respectively, increases from infinity to infinity so that at least two energy levels are always real. At the positive Cs the minimum of the U-shaped part is zero so that in $D_{(\text{physical})}$ we have all $B \in (0, \infty)$. 
At the negative $C$'s the minimum $B_{(EP)}$ of the U-shaped part of the curve $B(E)$ is positive so that the spectral reality constraint implies that $B \in (B_{(EP)}, \infty)$.

4 Discussion

4.1 Stable quantum systems in non-Hermitian representations

Quantum systems exhibiting $\mathcal{PT}$ symmetry were made popular, by Bender and Boettcher \cite{1}, via a one-parametric family of non-Hermitian quantum Hamiltonians $H = H(\lambda) \neq H^\dagger(\lambda)$ such that $H(\lambda)\mathcal{PT} = \mathcal{PT}H(\lambda)$. The spectrum was shown real and discrete if and only if $0 < \lambda < \infty$, i.e., if and only if the value of the parameter belonged to an “admissible” open set $\mathcal{D}_{(\text{physical})}$. The authors conjectured that whenever $\lambda \in \mathcal{D}_{(\text{physical})}$, these Hamiltonians may be given the conventional unitary-evolution-generator interpretation.

The expectations were confirmed. Several reviews \cite{2, 4, 6} may be consulted for a detailed account of the well-developed quantum theory covering $\mathcal{PT}$ symmetric quantum systems. In the Bender’s and Boettcher’s toy model, in particular, the physical domain of parameters $\mathcal{D}^{(BB)}_{(\text{physical})} = (0, \infty)$ is a semi-infinite interval. As a consequence the decrease of $\lambda$ and its passage through the boundary $\lambda^{(BB)} = 0$ results in the loss of the reality of the spectrum and, simultaneously, in the breakdown of the $\mathcal{PT}$ symmetry of the system. This breakdown is spontaneous, i.e., the wave functions suddenly lose the symmetry while the Hamiltonian itself remains formally $\mathcal{PT}$ symmetric.

The knowledge of the boundaries of the physical domain of parameters becomes important when one turns attention to the study of mechanisms of the loss of quantum stability. An exactly solvable illustrative example may be provided by the non-self-adjoint but $\mathcal{PT}$-symmetric harmonic-oscillator Hamiltonian

$$H^{(HO)}(\lambda) = -\frac{d^2}{dx^2} + \frac{\lambda^2 - 1/4}{(x - i\varepsilon)^2} + (x - i\varepsilon)^2, \quad x \in (-\infty, \infty)$$

(15)

in which the shift $\varepsilon \neq 0$ is arbitrary \cite{21}. In this model the parameter $\lambda = 0$ is still the point of spontaneous breakdown of $\mathcal{PT}$-symmetry. Below this value the energy spectrum ceases to be real. Nevertheless, the physical domain of parameters cannot contain any positive integers (see the proof in \cite{21}) so that it becomes “punctured” and topologically nontrivial,

$$\mathcal{D}^{(HO)}_{(\text{physical})} = \{ \lambda \in (0, \infty), \lambda \notin \{1, 2, \ldots\} = \mathbb{Z}^+ \}.$$  

(16)

The excluded integer parameters have a number of interesting properties. Although the energies merge at these EP singularities, they do not complexify in their vicinity. The
operator $H^{(HO)}(\lambda_{EP})$ itself ceases to be diagonalizable and it acquires a Jordan-block canonical form at these values. Naturally \cite{2}, such an operator does not admit any acceptable probabilistic physical interpretation.

4.2 Exactly solvable models: phenomenological appeal

The exact solvability of quantum systems is a vague concept. It covers the most elementary harmonic oscillators as well as certain truly complicated many-body systems characterized by sophisticated symmetries. In between the two extremes one finds a number of systems called quantum lattices or $N$–site quantum chains, the dynamics of which is controlled by a finite-dimensional Schrödinger equation \cite{5}. Indeed, the choice of a finite matrix dimension $N < \infty$ makes these models solvable, numerically, with arbitrary precision.

Recently, the practical studies of the quantum models of this type found a challenging new motivation in the context of the growth of interest in the non-Hermitian quantum Hamiltonians with real spectra \cite{11}. Some of the puzzling obstacles posed by rigorous mathematics were successfully circumvented by the restriction of attention either to the $\mathcal{PT}$–symmetric Hamiltonians \cite{6} or to the class of bounded operators \cite{10} or, in an extreme case, to the finite-dimensional matrices as sampled in Eq. (5). On phenomenological side people managed to connect some of these branches of the theory with experiments. In the laboratory they confirmed various predicted properties of $N$–point-lattice structures characterized by the $\mathcal{PT}$–symmetric balance between gain (sources) and loss (sinks) \cite{22}. One of the most interesting as well as challenging features of all of the classical as well as quantum non-Hermitian lattices may be seen in the possibility of having the latter symmetry, at certain values of parameters, spontaneously broken \cite{23}. In the most common scenario this breakdown means that some eigenenergies complexify and survive as complex conjugate pairs (cf., e.g., the implementation of the idea of the possible complexification in the context of non-Hermitian quantum thermodynamics \cite{24}).

4.3 Quantum physics near the real exceptional points

For couple of years the problem of the passage of $\lambda$ through the level-crossing points $\lambda^{(k)}_{EP} \in \mathbb{Z}^+$ of model \cite{15} remained unclarified. The changes of physics at the level-crossing EP boundaries $\partial D$ were known to be model-dependent and technically difficult to describe \cite{25, 26}. Many researches came to the conclusion that by non-numerical means, the explicit constructive explanation of the underlying physics may not even be feasible at all. One of the key sources of the scepticism lied in the complicated mathematics.
Dieudonné [27], for example, discouraged the applications of the non-Hermitian operators in physics rather persuasively.

The resolution of the contradictions was provided by the quasi-Hermitian quantum theory [2 5 10 28]. In the framework of this theory it has been clarified that the energies may remain real even if the Hamiltonian itself appears non-selfadjoint. It became widely known that this operator can be made self-adjoint via an \textit{ad hoc} change of the physical inner product in Hilbert space. Thus, whenever our parameters stay inside the physical domain $D$, many non-Hermitian Hamiltonians $H(\lambda)$ may be assigned the status of an acceptable quantum observable.

One of the most important keys to the applicability of the theory lies in the correct localization of the physical domains $D^{(HO)}$. Naturally, the theory ceases to be applicable in the limit $\lambda \to \lambda_{(EP)}$. In the language of physics, the \textit{observable aspects} of the quantum system in question may change whenever its real variable parameter $\lambda$ \textit{crosses} the singular dynamical boundary at $\lambda_{(EP)}$.

In the related literature the attention is almost exclusively payed to the scenarios in which the loss of the observability involves the Hamiltonian. In the language of Ref. [17] one can speak about the quantum phase transition of the first kind. Typically, a new degree of freedom emerges and must necessarily be included in an amended Hamiltonian. In model (15) the latter change only takes place at the leftmost point $\lambda_{(EP)} = \lambda_{(EP)}^{(0)} = 0$ of the boundary $\partial D^{(HO)}$ at which all of the energies cease to be real. In contrast, the passage of the parameter through any other EP value $\lambda_{(EP)} = \lambda_{(EP)}^{(k)} = k, k = 1, 2, \ldots$ does not lead to any complexification. The observability of the energy survives and we may hold the Hamiltonian unaltered.

### 4.4 Matrix models: mathematical appeal

The advantage of solvability of the extremely elementary $N = 2$ matrix models is already playing the role of an inspiring methodical guide for years [2 12 29]. Unfortunately, one has to pay for the advantage. The facilitated mathematical tractability of the low-dimensional models may be more than counterbalanced by their perceivably smaller phenomenological appeal. One also has to mention their limited capability of leading to a deeper insight or of enhancing the predictive power of quantum theory.

Using the toy models it has been pointed out [30 31] that the quantum lattices can support not only the well known spontaneous breakdown of $\mathcal{PT}$-symmetry but also another, alternative version of phase transition, \textit{not accompanied} by the complexification of the energies. In review [11], for example, it was emphasized that both of the complexifying
and non-complexifying quantum phase transitions have the same mathematical origin and that both of them may be attributed to the non-Hermiticity of at least one of the observables. This opened a new viable model-building perspective in which one of the Kato’s complex points of degeneracy loses its characteristic imaginary part and becomes real.

In place of the realistic but difficult differential-operator Hamiltonians sampled by Eq. (15) we recommended, in Ref. [17], the use of certain elementary $N = 4$ predecessor of our present $N = 6$ model. Re-written in the form

$$H^{(4)}(\lambda) = \begin{pmatrix}
0 & -1 + \sqrt{1 - \lambda} & 0 & 0 \\
-1 - \sqrt{1 - \lambda} & 0 & -1 + \sqrt{1 - A} & 0 \\
0 & -1 - \sqrt{1 - A} & 0 & -1 + \sqrt{1 - \lambda} \\
0 & 0 & -1 - \sqrt{1 - \lambda} & 0
\end{pmatrix}, \quad (17)$$

the latter Hamiltonian was found observable for $\lambda$ in

$$\mathcal{D}^{(4)}_{(\text{physical})} = (-A/4, 0) \cup (0, \infty)$$

i.e., inside an elementary but still “punctured” analogue of the harmonic-oscillator physical domain (16). After such a simplification of mathematics an important progress was achieved in physics because we were able to prove that the new system’s passage through the unavoided-level-crossing point $\lambda^{(4)}_{(EP)} = 0$ does change the system of observables (other than Hamiltonian). One can certainly speak about the quantum phase transition of the second kind.

In Ref. [17] we did not manage to make our argumentation sufficiently model-independent. For the purely technical reasons the extension of our matrix model from $N = 4$ to $N = 6$ was temporarily found too difficult. In this sense we just filled the gap in our present paper. We showed that the main phenomenological observations about the coexistence of the quantum phase transitions of the first and second kind in a single quantum system may be expected to be generic.

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