A CHARACTERIZATION OF RIGHT 4-NAKAYAMA ARTIN ALGEBRAS

ALIREZA NASR-ISFAHANI AND MOHSEN SHEKARI

Abstract. We characterize right 4-Nakayama artin algebras which appear naturally in the study of representation-finite artin algebras. For a right 4-Nakayama artin algebra \( \Lambda \), we classify all finitely generated indecomposable right \( \Lambda \)-modules and then we compute all almost split sequences over \( \Lambda \). We also give a characterization of right 4-Nakayama finite dimensional \( K \)-algebras in terms of their quivers with relations.

1. Introduction

Let \( R \) be a commutative artinian ring. An artin algebra is an \( R \)-algebra \( \Lambda \) that is a finitely generated \( R \)-module. Let \( \Lambda \) be an artin algebra. A right \( \Lambda \)-module \( M \) is called uniserial (1-factor serial) if the lattice of its submodules is totally ordered under inclusion. An artin algebra \( \Lambda \) is called Nakayama algebra if the indecomposable projective right \( \Lambda \)-modules as well as the indecomposable projective left \( \Lambda \)-modules are uniserial. This then implies that all the finitely generated indecomposable right \( \Lambda \)-modules are uniserial. Nakayama algebras are representation-finite algebras whose module categories completely understood (see [3], [7] and [11]). A non-uniserial right \( \Lambda \)-module \( M \) of length \( l \) is called \( n \)-factor serial \((l \geq n > 1)\), if \( \rad^{l-n}(M) \) is uniserial and \( \rad^{l-n+1}(M) \) is not uniserial ([6, Definition 2.1]). An artin algebra \( \Lambda \) is called right \( n \)-Nakayama if every indecomposable right \( \Lambda \)-module is \( i \)-factor serial for some \( 1 \leq i \leq n \) and there exists at least one indecomposable \( n \)-factor serial right \( \Lambda \)-module ([6] Definition 2.2]. The authors in [6] proved that an artin algebra \( \Lambda \) is representation-finite if and only if \( \Lambda \) is right \( n \)-Nakayama for some positive integer \( n \) ([6] Theorem 2.18]. Indecomposable modules and almost split sequences over right 2-Nakayama and right 3-Nakayama artin algebras are studied in [6] and [3]. In this paper we will study the module categories of right 4-Nakayama algebras. We first show that an artin algebra \( \Lambda \) which is neither Nakayama nor right 2-Nakayama nor right 3-Nakayama is right 4-Nakayama if and only if every indecomposable right \( \Lambda \)-module of length greater than 5 is uniserial and every indecomposable right \( \Lambda \)-module of length 5 is local. Then we classify all indecomposable modules and almost split sequences over a right 4-Nakayama artin algebra. We also show that finite dimensional right 4-Nakayama algebras are special biserial and characterize quivers and relations of finite dimensional right 4-Nakayama algebras. The paper is organized as follows. In Section 2 we study 4-factor serial right \( \Lambda \)-modules and then we give a characterization of right 4-Nakayama artin algebras.

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In Section 3 we classify all indecomposable modules over right $4$-Nakayama artin algebras, up to isomorphisms.

In Section 4 we compute all almost split sequences over right $4$-Nakayama artin algebras.

Finally in the last section we describe the structure of quivers and their relations of finite dimensional right $4$-Nakayama algebras.

1.1. notation. Throughout this paper all modules are finitely generated right $\Lambda$-modules and all fields are algebraically closed fields. For a $\Lambda$-module $M$, we denote by $soc(M)$, $top(M)$, $rad(M)$, $l(M)$ and $ll(M)$ its socle, top, radical, length and Loewy length of $M$, respectively. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\alpha : i \to j$ be an arrow in $Q$. One introduces a formal inverse $\alpha^{-1}$ with $s(\alpha^{-1}) = j$ and $t(\alpha^{-1}) = i$. An edge in $Q$ is an arrow or the inverse of an arrow. To each vertex $i$ in $Q$, one associates a trivial path, also called trivial walk, $\varepsilon_i$ with $s(\varepsilon_i) = t(\varepsilon_i) = i$. A non-trivial walk $w$ in $Q$ is a sequence $w = c_1c_2\cdots c_n$, where the $c_i$ is an edge such that $t(c_i) = s(c_{i+1})$ for all $i$, whose inverse $w^{-1}$ is defined to be the sequence $w^{-1} = c_n^{-1}c_{n-1}^{-1}\cdots c_1^{-1}$. A walk $w$ is called reduced if $c_{i+1} \neq c_i^{-1}$ for each $i$. For $i \in Q_0$, we denote by $i^+$ and $i^-$ the set of arrows starting in $i$ and the set of arrows ending in $i$, respectively.

2. right $4$-Nakayama artin algebras

In this section we first describe right $4$-factor serial modules and then we give a characterization of right $4$-Nakayama artin algebras.

Definition 2.1. [6] Definitions 2.1 and 2.2] Let $\Lambda$ be an artin algebra and $M$ be a right $\Lambda$-module of length $l$.

(1) $M$ is called 1-factor serial (uniserial) if $M$ has a unique composition series.

(2) Let $l \geq n > 1$. $M$ is called $n$-factor serial if $\frac{M}{rad^{i-n}(M)}$ is uniserial and $\frac{M}{rad^{i-n+1}(M)}$ is not uniserial.

(3) $\Lambda$ is called right $n$-Nakayama if every indecomposable right $\Lambda$-module is $i$-factor serial for some $1 \leq i \leq n$ and there exists at least one indecomposable $n$-factor serial right $\Lambda$-module.

Lemma 2.2. Let $\Lambda$ be an artin algebra and $M$ be a right $\Lambda$-module of length $r$ and Loewy length $t$. The following conditions are equivalent.

(a) $M$ is a $4$-factor serial right $\Lambda$-module.

(b) One of the following conditions hold.

(i) For every $0 \leq i \leq r - 5$, $rad^i(M)$ is local, $rad^{r-4}(M)$ is not local and length of $M$ is either $t + 3$ or $t + 2$ or $t + 1$. Moreover in this case if $r = t + 3$, then $rad^{r-4}(M) = soc(M) = S_1 \oplus S_2 \oplus S_3 \oplus S_4$, where $S_i$ is a simple submodule of $M$ for each $1 \leq i \leq 4$.

(ii) $M$ is not local, $r = 4$ and the Loewy length of $M$ is either 2 or 3. Moreover in this case if $soc(M)$ is simple, then $ll(M) = 3$, otherwise $ll(M) = 2$.

Proof. (a) $\implies$ (b). Assume that $M$ is a local right $\Lambda$-module. Then by [6] Theorem 2.6], for every $0 \leq i \leq r - 5$, $rad^i(M)$ is local and $rad^{r-4}(M)$ is not local. On the other hand by [6] Lemma 2.21], $t + 1 \leq r \leq t + 3$. If $r = t + 3$, then by [6] Remark 2.7],
soc(M) ⊆ rad^{r-4}(M) and t = r - 3. Therefore soc(M) = rad^{r-4}(M) = S_1 ⊕ S_2 ⊕ S_3 ⊕ S_4. This finishes the proof of (i). Assume that M is not local. So by [6 Corollary 2.8], r = 4 and the result follows.

(b) ⇒ (a). If M is not local and r = 4, then by [6 Corollary 2.8], M is a 4-factor serial right Λ-module. If M satisfies the condition (i), then $\frac{M}{rad^{r-4}(M)}$ is uniserial and $\frac{M}{rad^{r-3}(M)}$ is not uniserial. Therefore M is a 4-factor serial right Λ-module.

An Artin algebra Λ is of right n-th local type if for every indecomposable right Λ-module M, $top^i(M) = \frac{M}{rad^i(M)}$ is indecomposable [2].

**Lemma 2.3.** Let Λ be a right 4-Nakayama artin algebra. Then Λ is of 3-ed local type.

**Proof.** Let M be an indecomposable right Λ-module. If M is local, then $\frac{M}{rad^i(M)}$ is indecomposable. If M is not local, then by [6 Lemma 5.2], M is either 3-factor serial or 4-factor serial. By [5 Lemma 2.1] and Lemma 2.2, $ll(M) \leq 3$, so $\frac{M}{rad^i(M)} \cong M$ is indecomposable and the result follows.

Let Λ be an artin algebra. The valued quiver of Λ is a quiver with n vertices, where n is the number of the isomorphism classes of simple right Λ-modules and with at most one arrow from a vertex i to the vertex j and with an ordered pair of positive integers associated with each arrow. This is done by writing an arrow from i to j if $Ext^1_\Lambda(S_i, S_j) \neq 0$, where $S_i$ and $S_j$ are simple Λ-modules corresponding to the vertices i and j and assigning to this arrow the pair of integers ($dim_{End_\Lambda(S_i)}Ext^1_\Lambda(S_i, S_j)$, $dim_{End_\Lambda(S_j)}Ext^1_\Lambda(S_i, S_j)$) [3]. Note that by [3 Proposition III.1.15], $Ext^1_\Lambda(S_i, S_j) \neq 0$ if and only if $S_j$ is a direct summand of $\frac{rad(P_i)}{rad^2(P_i)}$, where $P_i$ is a projective cover of $S_i$.

Let Λ be an artin algebra and M be a right Λ-module. M has a waist if there is a nontrivial proper submodule N of M such that every submodule of M contains N or is contained in N. In this case, N is called a waist in M.

**Proposition 2.4.** Let Λ be a right 4-Nakayama artin algebra and M be an indecomposable local right Λ-module. Then the following statements hold.

(a) If M is 4-factor serial, then $l(M) = 5$.

(b) If M is 3-factor serial, then $l(M) = 4$.

(c) If M is 2-factor serial, then $l(M) = 3$.

**Proof.** (a). By [5 Lemma 2.3], M is a projective right Λ-module and by Lemma 2.3, Λ is of 3-ed local type. If $rad^2(M) \neq 0$, then by [11 Theorem 2.5], $rad^3(M)$ is uniserial and waist of M. Also by [6 Theorem 2.6], $rad(M)$ is 4-factor serial. Assume that $rad(M)$ is local. Then by [6 Theorem 2.6], $rad^2(M)$ is 4-factor serial. If $rad^2(M)$ is local, then by [6 Theorem 2.6], $rad^2(M)$ is 4-factor serial which gives a contradiction. Therefore $rad^2(M)$ is non-local and by [6 Corollary 2.8] $l(rad^2(M)) = 4$. This implies that $l(M) = 6$. Let $top(M) = S_1$, $top(rad^2(M)) = S_2$ and $S_3, S_4$ be direct summands of $top(rad^2(M))$. Since M is projective, by [3 Proposition III.1.15], there exists one arrow from the vertex 1 to the vertex 2 in the valued quiver of Λ. Since $rad(M)$ is local, $rad(M)$ is either projective or quotient of an indecomposable projective right Λ-module $P_2$. Assume that $rad(M)$ is projective. Since $S_3$ and $S_4$ are direct summands of $top(rad^2(M))$, by [3] Proposition...
III.1.15], there are one arrow from the vertex 2 to the vertex 3 and one arrow from the vertex 2 to the vertex 4 in the valued quiver of \( \Lambda \). Then the valued quiver of \( \Lambda \) has a subquiver of the form

\[
\begin{array}{c}
4 \\
\uparrow \\
1 \rightarrow 2 \\
\downarrow \\
3
\end{array}
\]

which implies that there exists a non-local indecomposable right \( \Lambda \)-module \( N \) of length 5. Therefore by [6 Corollary 2.8], \( N \) is a 5-factor serial right \( \Lambda \)-module which is a contradiction. Now assume that \( \text{rad}(M) \) is not projective. Then \( \text{rad}(M) \cong \frac{P_2}{K} \), where \( P_2 \) is the indecomposable projective right \( \Lambda \)-module and \( K \) is a submodule of \( P_2 \) and so \( \text{rad}^2(M) \cong \frac{\text{rad}(P_2)}{K} \). Then \( S_3 \) and \( S_4 \) are direct summands of \( \text{rad}(P_2) \) and by [3 Proposition III.1.15], there are one arrow from the vertex 2 to the vertex 3 and one arrow from the vertex 2 to the vertex 4 in the valued quiver of \( \Lambda \). Then the valued quiver of \( \Lambda \) has a subquiver of the form

\[
\begin{array}{c}
4 \\
\uparrow \\
1 \rightarrow 2 \\
\downarrow \\
3
\end{array}
\]

which implies that there exists a non-local indecomposable right \( \Lambda \)-module \( N \) of length 5. Then by [6 Corollary 2.8], \( N \) is a 5-factor serial right \( \Lambda \)-module which is a contradiction. Therefore \( \text{rad}(M) \) is non-local and by [6 Corollary 2.8], \( \text{rad}(M) \) is of the length 4. It implies that \( l(M) = 5 \).

In the parts (b) and (c), \( M \) is local. Then \( M \) is either projective or quotient of an indecomposable projective. Then by the similar argument as in the proof of the part (a), we can prove parts (b) and (c).

\[\square\]

Now we give a characterization of right 4-Nakayama artin algebras.

**Theorem 2.5.** Let \( \Lambda \) be an artin algebra which is neither Nakayama, nor right 2-Nakayama, nor right 3-Nakayama. Then \( \Lambda \) is right 4-Nakayama if and only if every indecomposable right \( \Lambda \)-module \( M \) of length greater than 5 is uniserial and every indecomposable right \( \Lambda \)-module \( M \) of length 5 is local.

**Proof.** Let \( \Lambda \) be a right 4-Nakayama algebra. It follows from Proposition[24] that, every indecomposable right \( \Lambda \)-module \( M \) of length greater than 5 is uniserial. If there exists an indecomposable non-local right \( \Lambda \)-module \( M \) of length 5, then by [6 Corollary 2.8], \( M \) is 5-factor serial which gives a contradiction. Conversely, assume that any indecomposable right \( \Lambda \)-module of length greater than 5 is uniserial and every indecomposable right \( \Lambda \)-module of length 5 is local, so by [6 Corollary 2.8] and [6 Lemma 2.21], every indecomposable right \( \Lambda \)-module is \( t \)-factor serial for some \( t \leq 4 \). Since \( \Lambda \) is neither Nakayama, nor right 2-Nakayama, nor right 3-Nakayama, there exists an indecomposable \( t \)-factor serial right \( \Lambda \)-module \( M \) for some \( t \geq 4 \). Therefore \( \Lambda \) is right 4-Nakayama. \[\square\]
3. INDECOMPOSABLE MODULES OVER RIGHT 4-NAKAYAMA ARTIN ALGEBRAS

In this section we give a classification of finitely generated indecomposable modules over right 4-Nakayama artin algebras.

**Proposition 3.1.** Let Λ be a right 4-Nakayama artin algebra and M be an indecomposable right Λ-module. Then \( l(\text{soc}(M)) \leq 2 \).

**Proof.** If \( M \) is uniserial, then \( l(\text{soc}(M)) = 1 \) and if \( M \) is 2-factor serial, then by [3, Lemma 5.1], \( l(\text{soc}(M)) = 2 \). Assume that \( M \) is 3-factor serial. If \( M \) is non-local, then \( l(M) = 3 \) and by [3, Lemma 2.1], \( l(\text{soc}(M)) = 1 \) and if \( M \) is local, then by Proposition 2.4, \( l(M) = 4 \) and \( l(\text{soc}(M)) \leq 3 \). Assume that on the contrary that \( \text{rad}(M) = \text{soc}(M) = S_1 \oplus S_2 \oplus S_3 \) and \( \text{top}(M) = S_4 \), where \( S_i \) is a simple \( \Lambda \)-module for each \( 1 \leq i \leq 4 \). Then by the similar argument as in the proof of the Proposition 2.4, the valued quiver of \( \Lambda \) has a subquiver of the form

```
1
 ↙
4 ─ 2
 ↙
 3
```

Then there exists a non-local indecomposable right \( \Lambda \)-module of length 5 that by [3, Corollary 2.8] is 5-factor serial which gives a contradiction. Now assume that \( M \) is 4-factor serial. If \( M \) is non-local, then by Lemma 2.2, \( l(M) = 4 \) and since \( l(\text{top}(M)) \geq 2 \), \( l(\text{soc}(M)) \leq 2 \). If \( M \) is local, then by [5, Lemma 2.2], \( M \) is projective and by Proposition 2.4, \( l(M) = 5 \). Assume on the contrary that \( l(\text{soc}(M)) = 4 \). Then \( \text{rad}(M) = \text{soc}(M) = S_1 \oplus S_2 \oplus S_3 \oplus S_4 \), where \( S_i \) is a simple \( \Lambda \)-module for each \( 1 \leq i \leq 4 \). Therefore we have almost split sequences

\[
0 \rightarrow S_1 \rightarrow M \rightarrow \tau^{-1}(S_1) \rightarrow 0 \\
0 \rightarrow S_2 \rightarrow M \rightarrow \tau^{-1}(S_2) \rightarrow 0 \\
0 \rightarrow S_3 \rightarrow M \rightarrow \tau^{-1}(S_3) \rightarrow 0 \\
0 \rightarrow S_4 \rightarrow M \rightarrow \tau^{-1}(S_4) \rightarrow 0
\]

Then \( \tau^{-1}(M) \) is an indecomposable right \( \Lambda \)-module of length 11 and by Proposition 2.4, \( \tau^{-1}(M) \) is uniserial. Also for each \( 1 \leq i \leq 4 \), irreducible morphisms \( \tau^{-1}(S_i) \rightarrow \tau^{-1}(M) \) are monomorphisms. For each \( 1 \leq i \leq 4 \), \( \tau^{-1}(S_i) \cong \frac{M}{S_i} \) and by [6, Theorem 2.13], there exists \( 1 \leq i \leq 4 \) such that \( \tau^{-1}(S_i) \cong \frac{M}{S_i} \) is not uniserial which gives a contradiction to the Corollary 2.17 of [6]. Now assume that \( l(\text{soc}(M)) = 3 \) and \( \text{soc}(M) = S_1 \oplus S_2 \oplus S_3 \) where \( S_i \) is a simple \( \Lambda \)-module for each \( 1 \leq i \leq 3 \). If \( \text{rad}(M) = N \oplus S_2 \oplus S_3 \), where \( N \) is uniserial of length 2 and \( \text{top}(N) = S_4 \), \( \text{soc}(N) = S_1 \) and \( \text{top}(M) = S_5 \), then by the similar
argument as in the proof of the Proposition 2.4, the valued quiver of Λ has a subquiver of the form

\[
\begin{array}{c}
3 \\
\downarrow \\
4 \leftarrow 5 \\
\downarrow \\
2
\end{array}
\]

Therefore there exists a non-local indecomposable right Λ-module \( L \) of length 5. By [6, Corollary 2.8], \( L \) is 5-factor serial which gives a contradiction. If \( \text{rad}(M) = N \oplus S_3 \) where \( N \) is a 2-factor serial right Λ-module of length 3, \( \text{top}(N) = S_4 \) and \( \text{rad}(N) = \text{soc}(N) = S_1 \oplus S_2 \). The similar argument as in the proof of Proposition 2.4 shows that the valued quiver of Λ has a subquiver of the form

\[
\begin{array}{c}
1 \\
\downarrow \\
5 \rightarrow 4 \\
\downarrow \\
2
\end{array}
\]

which gives a contradiction. Therefore \( l(\text{soc}(M)) \leq 2 \) and the result follows. □

**Proposition 3.2.** Let \( Λ \) be a right 4-Nakayama artin algebra and \( M \) be an indecomposable 4-factor serial right Λ-module. Then the following statements hold.

(a) If \( M \) is local and \( \text{rad}(M) \) is indecomposable, then \( \text{soc}(M) \) is simple.

(b) If \( M \) is non-local, then \( l(\text{top}(M)) = 2 \).

**Proof.** (a) By [5, Lemma 2.3], \( M \) is projective and by the Proposition 2.4 \( l(M) = 5 \). By the Proposition 3.1 \( l(\text{soc}(M)) \leq 2 \) and by [6, Theorem 2.6, Corollary 2.8], \( \text{rad}(M) \) is non-local 4-factor serial of length 4. Assume that \( \text{soc}(M) = S_1 \oplus S_2 \), \( \text{top}(\text{rad}(M)) = S_3 \oplus S_4 \) and \( \text{top}(M) = S_5 \), where \( S_i \) is a simple Λ-module for each \( 1 \leq i \leq 5 \). Since \( M \) is projective, by [3, Proposition III.1.15] there are one arrow from the vertex 5 to the vertex 3 and one arrow from the vertex 5 to the vertex 4 in the valued quiver of Λ. Since \( \text{top}(\text{rad}(M)) = S_3 \oplus S_4 \), \( \frac{P_3 \oplus P_4}{L} \cong \text{rad}(M) \) where \( P_i \) is indecomposable projective that \( \text{top}(P_i) = S_i \) and \( L \) is a submodule of \( P_3 \oplus P_4 \). Then \( S_1 \oplus S_2 = \text{soc}(M) = \frac{\text{rad}(M)}{\text{rad}^2(M)} \subseteq \frac{\text{rad}(P_3 \oplus P_4)}{\text{rad}^2(P_3 \oplus P_4)} \). Since \( \text{rad}(M) \) is indecomposable, \( S_1 \oplus S_2 \) must be a direct summand of either \( \text{top}(\text{rad}(P_3)) \) or \( \text{top}(\text{rad}(P_4)) \). We can assume that \( S_1 \oplus S_2 \) is a direct summand of \( \text{top}(\text{rad}(P_3)) \). By [3, Proposition III.1.15], there are one arrow from the vertex 3 to the vertex 2 and one arrow from the vertex 3 to the vertex 1 in the valued quiver of Λ. Then the valued quiver of Λ has a subquiver of the form

\[
\begin{array}{c}
1 \\
\downarrow \\
5 \rightarrow 3 \\
\downarrow \\
2
\end{array}
\]

which implies that there is a non-local indecomposable right Λ-module \( N \) of length 5. By [6, Corollary 2.8], \( N \) is 5-factor serial which gives a contradiction. Therefore \( \text{soc}(M) \) is simple.
(b) By Lemma 2.2, \( l(M) = 4 \). Assume on the contrary that \( \text{top}(M) = S_1 \oplus S_2 \oplus S_3 \) and \( \text{soc}(M) = S_4 \), where \( S_i \) is a simple \( \Lambda \)-module for each \( 1 \leq i \leq 4 \). \( M \) has a projective cover of the form \( f = (f_1, f_2, f_3) : P_1 \oplus P_2 \oplus P_3 \to M \). Then \( \frac{P_1 \oplus P_2 \oplus P_3}{L} \cong M \), where \( L = \text{Ker}(f) \) and \( S_4 = \text{rad}(M) \cong \frac{\text{rad}(P_1) \oplus \text{rad}(P_2) \oplus \text{rad}(P_3)}{L} \). We claim that for each \( 1 \leq i \leq 3 \), \( S_4 \) is a direct summand of \( \text{rad}(P_i) \). Assume that \( S_4 \) is not a direct summand of \( \text{rad}(P_i) \). Since \( S_4 = \text{rad}(M) \), \( \text{rad}(P_i) \subset L \). We consider the \( \Lambda \)-homomorphism \( f_1 : P_1 \to M \). Then \( \frac{P_1}{\text{rad}(P_i)} \cong \text{Im}f_1 \leq M \). Since \( \frac{P_1}{\text{rad}(P_i)} \cong \text{Im}f_1 \) is simple, \( \text{Im}f_1 \) is direct summand of \( \text{soc}(M) = S_4 \) and \( \text{Im}f_1 = S_4 \) which gives a contradiction. The similar argument shows that \( S_4 \) is a direct summand of \( \text{rad}(P_2) \) and \( \text{rad}(P_3) \). So by [3, Proposition III.1.15], the valued quiver of \( \Lambda \) has a subquiver of the form

\[
\begin{align*}
1 & \quad \downarrow \\
2 & \quad \rightarrow \\
3 & \quad \uparrow \\
\end{align*}
\]

which gives a contradiction. \( \square \)

The following theorem gives a classification of submodules of indecomposable modules over right 4-Nakayama artin algebras.

**Theorem 3.3.** Let \( \Lambda \) be a right 4-Nakayama artin algebra and \( M \) be an indecomposable right \( \Lambda \)-module. Then the following statements hold.

(a) If \( M \) is a 4-factor serial right \( \Lambda \)-module, then one of the following situations hold:

(i) \( M \) is a local and colocal. Submodules of \( M \) are \( \text{rad}(M) \) which is indecomposable non-local 4-factor serial, two indecomposable submodules \( M_1 \) and \( M_2 \) of length 3 that \( M_1 \) is uniserial and \( M_2 \) is non-local 3-factor serial, two indecomposable uniserial submodules \( N_1 \) and \( N_2 \) of length 2 which \( N_1 \) is a submodule of \( M_1 \) and \( M_2 \), and \( N_2 \) is a submodule of \( M_2 \) and \( \text{soc}(M) = S \) which is simple.

(ii) \( M \) is local and non-colocal. Submodules of \( M \) are \( \text{rad}(M) = N \oplus S_1 \) that \( N \) is uniserial of length 3 and \( S_1 \) is a simple right \( \Lambda \)-module, uniserial submodule \( N_1 \) of length 2 which is a submodule of \( N \) and \( \text{soc}(M) = S_1 \oplus S_2 \) that \( \text{soc}(N) = S_2 \).

(iii) \( M \) is local and non-colocal. Submodules of \( M \) are \( \text{rad}(M) = N_1 \oplus N_2 \) that for each \( 1 \leq i \leq 2 \), \( N_i \) is uniserial of length 2 and \( \text{soc}(M) = S_1 \oplus S_2 \) where for each \( 1 \leq i \leq 2 \), \( \text{soc}(N_i) = S_i \).

(iv) \( M \) is non-local of length 4 that \( \text{soc}(M) \) is simple. Submodules of \( M \) are two submodules \( M_1 \) and \( M_2 \) of length 3 that \( M_1 \) is uniserial and \( M_2 \) is non-local 3-factor serial, two indecomposable uniserial submodules \( N_1 \) and \( N_2 \) of length 2 that \( N_1 \) is a submodule of \( M_1 \) and \( M_2 \) and \( N_2 \) is a submodule of \( M_2 \) and \( \text{soc}(M) = S \) which is simple.

(v) \( M \) is non-local of length 4 that \( \text{soc}(M) \) is non-simple. Submodules of \( M \) are indecomposable modules \( N_1 \) of length 3 which is 2-factor serial, uniserial module \( N_2 \) of length 2 and \( \text{soc}(M) = S_1 \oplus S_2 \) where \( S_i \) is a simple submodule of \( M \) for each \( 1 \leq i \leq 2 \) and \( \text{soc}(N_2) = S_2 \).

(b) If \( M \) is a 3-factor serial right \( \Lambda \)-module, then one of the following situations hold.
(i) $M$ is local and colocal. Submodules of $M$ are $\text{rad}(M)$ which is non-local 3-factor serial of length 3, two uniserial submodules $N_1$ and $N_2$ of length 2 and $\text{soc}(M) = S$ which is simple.

(ii) $M$ is local and $\text{soc}(M)$ is not simple. Submodules of $M$ are $\text{rad}(M) = N \oplus S_1$ that $N$ is uniserial of length 2, $S_1$ is simple and $\text{soc}(M) = S_1 \oplus S_2$ that $\text{soc}(N) = S_2$ which is simple.

(iii) $M$ is non-local of length 3. Submodules of $M$ are two uniserial submodules $N_1$ and $N_2$ of length 2 and $\text{soc}(M) = S$ which is simple.

(c) If $M$ is a 2-factor serial local module, then submodules of $M$ are $\text{rad}(M) = \text{soc}(M) = S_1 \oplus S_2$ that $S_1$ and $S_2$ are simple right $\Lambda$-modules.

Proof. We prove only parts (a)(i) and (a)(v), the proof of the other parts is similar.

(a)(i). Since $M$ is a local 4-factor serial right $\Lambda$-module, by the Proposition 2.4, $l(M) = 5$ and it implies that $l(\text{rad}(M)) = 4$. By [6] Theorem 2.6, $\text{rad}(M)$ is 4-factor serial and so by [6] Corollary 2.8, $\text{rad}(M)$ is non-local indecomposable. By the Proposition 3.2, $l(\text{top}(\text{rad}(M))) = 2$. Assume that $\text{top}(\text{rad}(M)) = S_1 \oplus S_2$, where $S_1$ and $S_2$ are simple $\Lambda$-modules. We claim that $\text{rad}(M)$ has two maximal submodules. Since $\text{rad}(M)$ is non-local, $\text{rad}(M)$ has at least two maximal submodules $M_1$ and $M_2$ that $\frac{\text{rad}(M_1)}{M_1} \cong S_2$ and $\frac{\text{rad}(M_2)}{M_2} \cong S_1$. It implies that $S_1$ is a direct summand of $\text{top}(M_1)$, $S_2$ is a direct summand of $\text{top}(M_2)$ and we have an exact sequence

$$0 \longrightarrow \text{rad}^2(M) \longrightarrow \text{rad}(M) \longrightarrow \frac{\text{rad}(M)}{\text{rad}(M)} \longrightarrow 0$$

Hence $l(\text{rad}^2(M)) = 2$. $l(\text{rad}(M)) = 4$ and the length of any maximal submodule of $\text{rad}(M)$ is 3, then $\text{rad}(M)$ has two maximal submodules $M_1$ and $M_2$. Since for every $1 \leq i \leq 2$, $\text{soc}(M) = \text{soc}(M_i)$ which is simple, $M_i$ is indecomposable and by [6] Theorem 5.2, $M_1$ and $M_2$ are not 2-factor serial. If $M_i$ is local, then $M_i$ is uniserial and if $M_i$ is non-local, then $M_i$ is 3-factor serial for $1 \leq i \leq 2$. We show that $M_1$ is uniserial and $M_2$ is 3-factor serial. Assume that $S_3 = \text{top}(\text{rad}(M))$ and $S_4 = \text{soc}(M)$, where $S_3$ and $S_4$ are simple. If $M_1$ and $M_2$ are uniserial, then $\text{top}(M_1) = S_3$, $\text{top}(M_2) = S_3$ and $P_1, P_2$ are projective covers of $M_1, M_2$, respectively. Since $\text{rad}^2(M) \subset M_i$, $S_3$ is a direct summand of $\text{top}(\text{rad}(P_i))$ for each $1 \leq i \leq 2$. By [3] Proposition III.1.15], there is one arrow from the vertex 1 to the vertex 3 and one arrow from the vertex 2 to the vertex 3 in the valued quiver of $\Lambda$. Also $P_3$ is the projective cover of $\text{rad}^2(M)$ and hence $S_4$ is a direct summand of $\text{top}(\text{rad}(P_3))$. By [3] Proposition III.1.15], there is one arrow from the vertex 3 to the vertex 4 in the valued quiver of $\Lambda$. Then the valued quiver of $\Lambda$ has a subquiver of the form

$$\begin{array}{ccc}
1 & \rightarrow & 3 \\
\uparrow & & \uparrow \\
2 & \rightarrow & 4
\end{array}$$

Therefore, there exists a non-local indecomposable right $\Lambda$-module $L$ of length 5. By [6] Corollary 2.8, $L$ is 5-factor serial which gives a contradiction. Now assume that $M_1$ and $M_2$ are 3-factor serial. Since $\frac{\text{rad}(M_1)}{M_1} \cong S_2$ and $\frac{\text{rad}(M_2)}{M_2} \cong S_1$, $S_2$ is not a direct summand
of \( \text{top}(M_1) \) and \( S_1 \) is not a direct summand of \( \text{top}(M_2) \). Since \( \text{rad}(M_i) \subset M_i \) for every \( 1 \leq i \leq 2 \), \( \text{top}(M_1) = S_1 \oplus S_3 \) and \( \text{top}(M_2) = S_2 \oplus S_3 \) and so \( P_1 \oplus P_3, P_2 \oplus P_3 \) are projective covers of \( M_1 \) and \( M_2 \), respectively. Then by [3, Proposition III.1.15], the valued quiver of \( \Lambda \) has a subquiver of the form

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}
\]

which gives a contradiction. So one of the maximal submodules \( M_1 \) and \( M_2 \) is uniserial and the other one is 3-factor serial.

(a)(v). Assume that \( M \) is non-local of length 4 that \( \text{soc}(M) \) is not simple. By the Proposition [3, 2] \( l(\text{top}(M)) = 2 \) and it implies that \( \text{rad}(M) = \text{soc}(M) = S_3 \oplus S_4 \) and \( \text{top}(M) = S_1 \oplus S_2 \), where \( S_i \) is simple for each \( i \). \( M \) has two maximal submodules \( M_1 \) and \( M_2 \) of length 3 that \( \text{rad}(M) = \text{soc}(M) \subset M_i \) for each \( 1 \leq i \leq 2 \), \( M_i \cong S_2, M_2 \cong S_1 \) and by [6, Lemma 5.2] if \( M_i \) is indecomposable for each \( 1 \leq i \leq 2 \), then \( M_i \) is 2-factor serial. We show that only one of the \( M_1 \) and \( M_2 \) is indecomposable. Assume on the contrary that, \( M_1 \) and \( M_1 \) are not indecomposable. \( P_1 \oplus P_2 \) is a projective cover of \( M \) and so \( P_1 \oplus P_2 \cong M \), where \( L \) is a submodule of \( P_1 \oplus P_2 \) and \( \frac{\text{rad}(P_1 \oplus P_2)}{L} \cong \text{rad}(M) \cong \text{soc}(M) = S_3 \oplus S_4 \). Therefore each \( S_i, 3 \leq i \leq 4 \) is a direct summand of one of the \( \text{top}(\text{rad}(P_i)) \) and \( \text{top}(\text{rad}(P_2)) \). Assume that \( S_3 \) is direct summand of \( \text{top}(\text{rad}(P_1)) \) and \( S_4 \) is direct summand of \( \text{top}(\text{rad}(P_2)) \). Since \( M_1 \) and \( M_2 \) are not indecomposable, \( S_4 \) is not a direct summand of \( \text{top}(\text{rad}(P_1)) \) and \( S_3 \) is not a direct summand of \( \text{top}(\text{rad}(P_2)) \). Then \( M \) is a direct summand of two uniserial modules of length 2 which gives a contradiction. Now assume that \( M_1 \) and \( M_2 \) are both indecomposable 2-factor serial right \( \Lambda \)-modules. Since for every \( 1 \leq i \leq 2 \), \( S_1 \oplus S_2 = \text{rad}(M) \subset M_i \) and \( P_1, P_2 \) are projective covers of \( M_1, M_2 \), respectively. Then \( S_3 \) and \( S_4 \) are direct summands of \( \text{top}(\text{rad}(P_i)) \) for each \( 1 \leq i \leq 2 \) and by [3, Proposition III.1.15], the valued quiver of \( \Lambda \) has a subquiver of the form

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}
\]

Therefore \( \Lambda \) is not representation-finite which gives a contradiction. Then we can assume that \( M_1 \) is indecomposable and 2-factor serial and \( M_2 = N \oplus S \) that \( N \) is uniserial of length 2 and \( S \) is a simple right \( \Lambda \)-module. \( \square \)

Now by using the Theorem [3, 3] we give a characterization of indecomposable non-projective modules over right 4-Nakayama artin algebras.

**Theorem 3.4.** Let \( \Lambda \) be a right 4-Nakayama artin algebra and \( M \) be an indecomposable right \( \Lambda \)-module. Then \( M \) is non-projective if and only if one of the following situations hold.
(A) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is uniserial and $M \cong \frac{P}{\text{rad}(P)}$ for some $1 \leq t < l(P)$.

(B) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is a 2-factor serial right $\Lambda$-module that $\text{rad}(P) = \text{soc}(P) = S_1 \oplus S_2$ and $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{S_i}$ for some $1 \leq i \leq 2$.

(C) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is a 3-factor serial colocal right $\Lambda$-module such that submodules of $P$ are $\text{rad}(P)$ which is 3-factor serial of length 3 and non-local, two uniserial submodules $N_1$ and $N_2$ of length 2 and $\text{soc}(P) = S$ which is simple. $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{N_i}$ for some $1 \leq i \leq 2$ or $\frac{P}{S_i}$.

(D) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is a 3-factor serial non-colocal right $\Lambda$-module such that submodules of $P$ are $\text{rad}(P) = N \oplus S_1$ that $N$ is uniserial of length 2, $S_1$ is simple and $\text{soc}(P) = S_1 \oplus S_2$ that $\text{soc}(N) = S_2$. $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{N}$ or $\frac{P}{\text{soc}(P)}$ or $\frac{P}{S_i}$ for some $1 \leq i \leq 2$.

(E) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is colocal 4-factor serial right $\Lambda$-module such that submodules of $P$ are $\text{rad}(P)$ which is 4-factor serial of length 4 and non-local, two submodules $M_1$ and $M_2$ of length 3 that $M_1$ is uniserial and $M_2$ is non-local 3-factor serial, two uniserial submodules $N_1$ and $N_2$ of length 2 that $N_1$ is a submodule of $M_1$ and $M_2$ and $N_2$ is a submodule of $M_2$ and $\text{soc}(P) = S$ which is simple. $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{M_i}$ for some $1 \leq i \leq 2$ or $\frac{P}{N_i}$ for some $1 \leq i \leq 2$ or $\frac{P}{S_i}$.

(F) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is a non-colocal 4-factor serial right $\Lambda$-module such that submodules of $P$ are $\text{rad}(P) = N \oplus S_1$ that $N$ is uniserial of length 3 and $S_1$ is a simple right $\Lambda$-module, an indecomposable uniserial submodule $N_1$ of length 2 and $\text{soc}(P) = S_1 \oplus S_2$ that $\text{soc}(N) = S_2$. $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{N}$ or $\frac{P}{N_1}$ or $\frac{P}{S_i}$ or $\frac{P}{\text{soc}(P)}$ or $\frac{P}{S_i}$ for some $1 \leq i \leq 2$.

(G) $M$ is a factor module of an indecomposable projective right $\Lambda$-module $P$ which is a non-colocal 4-factor serial right $\Lambda$-module such that submodules of $P$ are $\text{rad}(P) = N_1 \oplus N_2$ that $N_i$ is uniserial of length 2 for each $1 \leq i \leq 2$ and $\text{soc}(P) = S_1 \oplus S_2$ that $\text{soc}(N_i) = S_i$ for each $1 \leq i \leq 2$. $M$ is isomorphic to either $\frac{P}{\text{rad}(P)}$ or $\frac{P}{N_i}$ for some $1 \leq i \leq 2$ or $\frac{P}{N_1 \oplus N_2}$ or $\frac{P}{N_2 \oplus N_1}$ or $\frac{P}{\text{soc}(P)}$ or $\frac{P}{S_i}$ for some $1 \leq i \leq 2$.

(H) $M$ is a non-local 3-factor serial right $\Lambda$-module whose submodules are two uniserial modules $M_1$ and $M_2$ and $\text{soc}(M) = S$ which is simple. Indecomposable quotients of $M$ are $\frac{M}{M_i}$ for $1 \leq i \leq 2$.

(I) $M$ is a non-local 4-factor serial right $\Lambda$-module with simple socle whose submodules are two maximal submodules $M_1$ and $M_2$ of length 3 that $M_1$ is uniserial and $M_2$ is non-local 3-factor serial, two uniserial submodules $N_1$ and $N_2$ of length 2 which $N_1$ is a uniserial submodule of $M_1$ and $M_2$ and $N_2$ is a submodule of $M_2$ and $\text{soc}(M) = S$, which is simple. Indecomposable quotients of $M$ are $\frac{M}{M_i}$ for $1 \leq i \leq 2$ and $\frac{M}{N_2}$. 
(J) $M$ is a non-local 4-factor serial right $\Lambda$-module that $\text{soc}(M)$ is not simple. Submodules of $M$ are, an indecomposable module $N_1$ of length 3 which is 2-factor serial, a uniserial module $N_2$ of length 2 and $\text{soc}(M) = S_1 \oplus S_2$ that $\text{soc}(N_2) = S_2$. Indecomposable quotients of $M$ are $M_{i}^{N}$ for $1 \leq i \leq 2$, $\frac{M_{i}}{S_{1}}$ and $\frac{M_{i}}{S_{2}}$.

4. ALMOST SPLIT SEQUENCES OF RIGHT 4-NAKAYAMA ARTIN ALGEBRAS

Now we compute almost split sequences of right 4-Nakayama artin algebras.

**Theorem 4.1.** Let $\Lambda$ be a right 4-Nakayama artin algebra and $M$ be an indecomposable non-projective right $\Lambda$-module. Then one of the following situations hold.

(A) Assume that $M \cong \frac{P}{\text{rad}^d(P)}$, where $P$ is uniserial projective, $1 \leq i < l(P)$ and $M$ is not isomorphic to the quotient of an indecomposable non-local 3-factor serial module. Also assume that $M$ is not isomorphic to the either $\frac{M_{1}}{M_{i}}$ or $\frac{M_{2}}{M_{i}}$, where $L$ is an indecomposable colocal non-local 4-factor serial, $M_1$ is a uniserial submodule of $L$ and $N_2$ is a uniserial submodule of $L$ which is not a submodule of $M_1$. Then the following sequence is an almost split sequence.

\[
0 \rightarrow \frac{\text{rad}(P)}{\text{rad}^{d+1}(P)} \rightarrow \frac{\text{rad}(P)}{\text{rad}^{d+1}(P)} \oplus \frac{P}{\text{rad}^{d+1}(P)} \rightarrow \frac{P}{\text{rad}^{d+1}(P)} \rightarrow 0
\]

(B) Assume that $M$ is a factor of an indecomposable projective 2-factor serial right $\Lambda$-module $P$. Submodules of $P$ are $\text{rad}(P) = \text{soc}(P) = S_1 \oplus S_2$ that $S_i$ is a simple submodule of $P$ for each $1 \leq i \leq 2$. Also assume that $P$ is not a submodule of an indecomposable non-local 4-factor serial right $\Lambda$-module $L$ that $\text{soc}(L)$ is not simple.

(i) If $M \cong \frac{P}{S_i}$ for some $1 \leq i \leq 2$, then the sequence

\[
0 \rightarrow S_i \rightarrow P \rightarrow \frac{P}{S_i} \rightarrow 0
\]

is an almost split sequence.

(ii) If $M \cong \frac{P}{\text{rad}(P)}$, then the sequence

\[
0 \rightarrow P \rightarrow \frac{P}{S_1} \oplus \frac{P}{S_2} \rightarrow \frac{P}{\text{rad}(P)} \rightarrow 0
\]

is an almost split sequence.

(C) Assume that $M$ is a factor of an indecomposable projective 3-factor serial colocal right $\Lambda$-module $P$. Submodules of $P$ are $\text{rad}(P)$ which is indecomposable non-local 3-factor serial of length 3, two uniserial modules $M_1$ and $M_2$ of length 2 and $S = \text{soc}(P)$ which is simple.

(i) If $M \cong \frac{P}{\text{rad}(P)}$, then the sequence

\[
0 \rightarrow \frac{P}{S} \rightarrow \frac{P}{M_1} \oplus \frac{P}{M_2} \rightarrow \frac{P}{\text{rad}(P)} \rightarrow 0
\]

is an almost split sequence.

(ii) If $M \cong \frac{P}{M_i}$, for some $1 \leq i \leq 2$, then the sequence
is an almost split sequence.

(iii) If \( M \cong \frac{P}{S} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{M}{S} & \rightarrow & \frac{P}{S} & \rightarrow \frac{P}{M} & \rightarrow 0 \\
\end{array}
\]

is an almost split sequence.

(D) Assume that \( M \) is a factor of an indecomposable projective 3-factor serial non-colocal right \( \Lambda \)-module \( P \). Submodules of \( P \) are \( \text{rad}(P) = N \oplus S_1 \) that \( N \) is uniserial of length 2, \( S_1 \) is simple and \( \text{soc}(P) = S_1 \oplus S_2 \) that \( \text{soc}(N) = S_2 \).

(i) If \( M \cong \frac{P}{\text{rad}(P)} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{P}{S_2} & \rightarrow & \text{soc}(P) & \rightarrow \frac{P}{\text{rad}(P)} & \rightarrow 0 \\
\end{array}
\]

is an almost split sequence.

(ii) If \( M \cong \frac{P}{N} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{N}{S_2} & \rightarrow & \frac{P}{S_2} & \rightarrow \frac{P}{N} & \rightarrow 0 \\
\end{array}
\]

is an almost split sequence.

(iii) If \( M \cong \frac{P}{\text{soc}(P)} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & P & \rightarrow & \text{soc}(P) & \rightarrow \frac{P}{\text{rad}(P)} & \rightarrow 0 \\
\end{array}
\]

is an almost split sequence.

(iv) If \( M \cong \frac{P}{S_1} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & S_1 & \rightarrow & P & \rightarrow \frac{P}{S_1} & \rightarrow 0 \\
\end{array}
\]

is an almost split sequence.

(v) If \( M \cong \frac{P}{S_2} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & N & \rightarrow & \frac{N}{S_2} & \rightarrow \frac{P}{S_2} & \rightarrow 0 \\
\end{array}
\]

is an almost split sequence.

(E) Assume that \( M \) is a factor of an indecomposable projective colocal 4-factor serial right \( \Lambda \)-module \( P \). Submodules of \( P \) are \( \text{rad}(P) \) that is non-local 4-factor serial of length 4, two submodules \( M_1 \) and \( M_2 \) of length 3 that \( M_1 \) is uniserial and \( M_2 \) is non-local 3-factor serial, two uniserial submodules \( N_1 \) and \( N_2 \) of length 2 that \( N_1 \) is a submodule of \( M_1 \) and \( M_2, N_2 \) is a submodule of \( M_2 \) and \( \text{soc}(P) = S \) which is simple.

(i) If \( M \cong \frac{P}{\text{rad}(P)} \), then the sequence

\[
\begin{array}{cccccc}
0 & \rightarrow & \frac{M_1}{S} & \rightarrow & \frac{P}{S} & \rightarrow \frac{P}{M_1} & \rightarrow 0 \\
\end{array}
\]
A CHARACTERIZATION OF RIGHT 4-NAKAYAMA ARTIN ALGEBRAS

is an almost split sequence.

(ii) If $M \cong \frac{P}{M_2}$, then the sequence

$$
0 \rightarrow \frac{P}{S} \rightarrow \frac{P}{N_1} \oplus \frac{P}{N_2} \rightarrow \frac{P}{N_1} \oplus \frac{P}{N_2} \rightarrow \frac{P}{M_2} \rightarrow 0
$$

is an almost split sequence.

(iii) If $M \cong \frac{P}{M_1}$, then the sequence

$$
0 \rightarrow \frac{P}{S} \rightarrow \frac{M_1}{N_1} \rightarrow \frac{P}{N_1} \rightarrow \frac{P}{M_1} \rightarrow 0
$$

is an almost split sequence.

(iv) If $M \cong \frac{P}{N_1}$, then the sequence

$$
0 \rightarrow \frac{M_1}{S} \rightarrow \frac{M_1}{N_1} \oplus \frac{P}{S} \rightarrow \frac{P}{N_1} \rightarrow 0
$$

is an almost split sequence.

(v) If $M \cong \frac{P}{N_2}$, then the sequence

$$
0 \rightarrow \frac{N}{S} \rightarrow \frac{P}{S} \rightarrow \frac{P}{N_2} \rightarrow 0
$$

is an almost split sequence.

(vi) If $M \cong \frac{P}{S}$, then the sequence

$$
0 \rightarrow \text{rad}(P) \rightarrow \frac{P}{S} \rightarrow 0
$$

is an almost split sequence.

(F) Assume that $M$ is a factor of an indecomposable projective non-colocal 4-factor serial right $\Lambda$-module $P$. Submodules of $P$ are $\text{rad}(P) = N \oplus S_1$ that $N$ is a uniserial module of length 3 and $S_1$ is a simple right $\Lambda$-module, an indecomposable uniserial submodule $N_1$ of length 2 and $\text{soc}(P) = S_1 \oplus S_2$ that $S_2 = \text{soc}(N)$.

(i) If $M \cong \frac{P}{\text{rad}(P)}$, then the sequence

$$
0 \rightarrow \frac{P}{N_1} \rightarrow \frac{P}{N_1 \oplus S_1} \rightarrow \frac{P}{N_1 \oplus S_1} \rightarrow \frac{P}{\text{rad}(P)} \rightarrow 0
$$

is an almost split sequence.

(ii) If $M \cong \frac{P}{N}$, then the sequence

$$
0 \rightarrow \frac{N}{N_1} \rightarrow \frac{P}{N_1} \rightarrow \frac{P}{N} \rightarrow 0
$$

is an almost split sequence.

(iii) If $M \cong \frac{P}{N_1 \oplus S_1}$, then the sequence
0 \rightarrow \frac{P}{S_2} \rightarrow \frac{P}{soc(P)} \oplus \frac{P}{N_1} \rightarrow \frac{P}{N_1 \oplus S_1} \rightarrow 0

is an almost split sequence. 

(iv) If \( M \cong \frac{P}{soc(P)} \), then the sequence

0 \rightarrow P \rightarrow \mathbb{P} \rightarrow \frac{P}{soc(P)} \rightarrow 0

is an almost split sequence. 

(v) If \( M \cong \frac{P}{N_2} \), then the sequence

0 \rightarrow N \rightarrow \mathbb{P} \rightarrow \frac{P}{soc(P)} \rightarrow 0

is an almost split sequence. 

(vi) If \( M \cong \frac{P}{S_1} \), then the sequence

0 \rightarrow S_1 \rightarrow P \rightarrow \frac{P}{S_1} \rightarrow 0

is an almost split sequence. 

(vii) If \( M \cong \frac{P}{S_2} \), then the sequence

0 \rightarrow N \rightarrow \mathbb{P} \rightarrow \frac{P}{soc(P)} \rightarrow 0

is an almost split sequence.

(G) Assume that \( M \) is a factor of an indecomposable projective non-colocal 4-factor serial right \( \Lambda \)-module \( P \). Submodules of \( P \) are \( rad(P) = N_1 \oplus N_2 \) that for each \( 1 \leq i \leq 2 \), \( N_i \) is uniserial of length 2 and \( soc(P) = S_1 \oplus S_2 \) that for each \( 1 \leq i \leq 2 \), \( soc(N_i) = S_1 \).

(i) If \( M \cong \frac{P}{rad(P)} \), then the sequence

0 \rightarrow \frac{P}{soc(P)} \rightarrow \frac{P}{N_1 \oplus S_2} \oplus \frac{P}{N_2 \oplus S_1} \rightarrow \frac{P}{rad(P)} \rightarrow 0

is an almost split sequence. 

(ii) If \( M \cong \frac{P}{N_i \oplus S_j} \) for some \( 1 \leq i, j \leq 2 \) and \( i \neq j \), then the sequence

0 \rightarrow \frac{P}{S_2} \rightarrow \frac{P}{soc(P)} \oplus \frac{P}{N_1} \rightarrow \frac{P}{N_i \oplus S_j} \rightarrow 0

is an almost split sequence. 

(iii) If \( M \cong \frac{P}{N_i} \) for some \( 1 \leq i \leq 2 \), then the sequence

0 \rightarrow \frac{N}{S_1} \rightarrow \frac{P}{S_1} \rightarrow \frac{P}{N_i} \rightarrow 0

is an almost split sequence. 

(iv) If \( M \cong \frac{P}{soc(P)} \), then the sequence
0 \rightarrow P \xrightarrow{\pi_{70}} P_{S_1} \oplus P_{S_2} \xrightarrow{[-\pi_{72}, \pi_{73}]} P_{soc(P)} \rightarrow 0

is an almost split sequence.

(v) If $M \cong \frac{P}{S_i}$ for some $1 \leq i \leq 2$, then the sequence

$$0 \rightarrow N_i \xrightarrow{\pi_{74}} N_i \oplus P \xrightarrow{[-\pi_{75}, \pi_{76}]} \frac{P}{S_i} \rightarrow 0$$

is an almost split sequence.

(H) Assume that $M$ is an indecomposable non-local 3-factor serial right $\Lambda$-module which is not isomorphic to the quotient of an indecomposable non-local non-colocal 4-factor serial right $\Lambda$-module $L$. $M$ is of length 3 and submodules of $M$ are two uniserial maximal submodules $M_1$ and $M_2$ of length 2 and $\text{rad}(M) = \text{soc}(M) = S$ which is simple.

(i) The exact sequence

$$0 \rightarrow S \xrightarrow{i_{29}} M_1 \oplus M_2 \xrightarrow{[i_{31}, i_{32}]} M \rightarrow 0$$

is an almost split sequence.

(ii) For an indecomposable right $\Lambda$-module $\frac{M}{M_j}$, $1 \leq j \leq 2$, the exact sequence

$$0 \rightarrow M_j \xrightarrow{i_{28}} M \xrightarrow{\pi_{76}} \frac{M}{M_j} \rightarrow 0$$

is an almost split sequence.

(I) Assume that $M$ is an indecomposable non-local 4-factor serial right $\Lambda$-module that $\text{soc}(M)$ is simple. Submodules of $M$ are two indecomposable submodules $M_1$ and $M_2$ of length 3 that $M_1$ is uniserial and $M_2$ is non-local, two uniserial submodules $N_1$ and $N_2$ of length 2 that $N_1$ is a submodule of $M_1$ and $M_2$, $N_2$ is a submodule of $M_2$ and $\text{soc}(M) = S$ which is simple.

(i) The exact sequence

$$0 \rightarrow N_1 \xrightarrow{i_{33}} M_1 \oplus M_2 \xrightarrow{[-i_{35}, i_{36}]} M \rightarrow 0$$

is an almost split sequence.

(ii) For an indecomposable module $\frac{M}{M_1}$, the exact sequence

$$0 \rightarrow M_1 \xrightarrow{i_{37}} M \xrightarrow{\pi_{77}} \frac{M}{M_1} \rightarrow 0$$

is an almost split sequence.

(iii) For an indecomposable module $\frac{M}{N_2}$, the exact sequence

$$0 \rightarrow M_2 \xrightarrow{\pi_{78}} \text{soc}(\frac{M}{N_2}) \oplus M \xrightarrow{[-i_{38}, \pi_{79}]} \frac{M}{N_2} \rightarrow 0$$

is an almost split sequence.
(J) Assume that $M$ is an indecomposable non-local 4-factor serial right $\Lambda$-module of length 4 that $soc(M)$ is not simple. Submodules of $M$ are $M_1$ and $M_2$ that $M_1$ is 2-factor serial of length 3, $M_2$ is uniserial of length 2 and $soc(M) = S_1 \oplus S_2$ that $soc(M_2) = S_2$.

(i) The sequence

$$0 \rightarrow S_2 \xrightarrow{i_{40}} M_1 \oplus M_2 \xrightarrow{[i_{41}, \pi_{81}]} M \rightarrow 0$$

is an almost split sequence.

(ii) For an indecomposable module $M_{S_1}$, the sequence

$$0 \rightarrow M_1 \xrightarrow{i_{42}} M_{S_1} \oplus M \xrightarrow{[i_{43}, \pi_{83}]} \frac{M}{S_1} \rightarrow 0$$

is an almost split sequence.

(iii) For an indecomposable module $M_{M_2}$, the sequence

$$0 \rightarrow M_2 \xrightarrow{i_{44}} M \xrightarrow{\pi_{84}} \frac{M}{M_2} \rightarrow 0$$

is an almost split sequence.

(iv) For an indecomposable module $M_{M_2 \oplus S_1}$, the sequence

$$0 \rightarrow M \xrightarrow{\pi_{85}} \frac{M}{S_1} \oplus \frac{M}{M_2} \xrightarrow{[\pi_{87}, \pi_{88}]} \frac{M}{M_2 \oplus S_1} \rightarrow 0$$

is an almost split sequence.

(v) For an indecomposable module $M_{M_1}$, the sequence

$$0 \rightarrow \frac{M_1}{S_1} \xrightarrow{i_{45}} \frac{M}{S_1} \xrightarrow{\pi_{89}} \frac{M}{M_1} \rightarrow 0$$

is an almost split sequence.

Where $i_j$ is an inclusion for each $1 \leq j \leq 45$ and $\pi_j$ is a canonical epimorphism for each $1 \leq j \leq 89$.

Proof. We only prove the parts $(H)(i), (H)(ii), (I)(i), (I)(ii), (I)(iii), (J)(i), (J)(ii), (J)(iii), (J)(iv)$ and $(J)(v)$. In the other parts, since $M$ is a quotient of an indecomposable projective right $\Lambda$-module without any condition, the proof is easy. Put $g_1 = [-i_{31}, i_{32}], g_2 = [-i_{35}, i_{36}], g_3 = [-i_{39}, \pi_{79}], g_4 = [-i_{41}, \pi_{81}], g_5 = [-i_{43}, \pi_{83}]$ and $g_6 = [-\pi_{87}, \pi_{88}]$. It is easy to see that all given sequences are exact, non-split and have indecomposable end terms. Then it is enough to show that homomorphisms $g_1, \pi_{76}, g_2, \pi_{77}, g_3, g_4, g_5, \pi_{84}, g_6$ and $\pi_{89}$ are right almost split morphisms.

$(H)(i)$. Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow M$ be a non-isomorphism. If $\nu$ is an epimorphism, then by [6, Lemma 2.11], $V$ is a 4-factor serial right $\Lambda$-module and by the Theorem 3.3 $V$ is non-local non-colocal 4-factor serial. Then $M$ isomorphic to the quotient of an indecomposable non-local non-colocal 4-factor serial right $\Lambda$-module which is a contradiction. Thus $Im(\nu)$ is a proper submodule of $M$. Since $M_1$ and $M_2$
are maximal submodules of $M$, $Im(\nu)$ is a submodule of $M_1 \oplus M_2$. Then there exists a homomorphism $h : V \rightarrow M_1 \oplus M_2$ such that $\nu = g_1 h$.

(H)(ii). Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{M}{M_j}$ be a non-isomorphism. Since $\frac{M}{M_j}$ is a simple right $\Lambda$-module, $\nu$ is an epimorphism. If $j = 1$, then $V$ is isomorphic to either $M$ or $M_2$ or a quotient of an indecomposable projective right $\Lambda$-module $P$. If $V$ is isomorphic to either $M$ or $M_2$, then there exists a homomorphism $h : V \rightarrow M$ such that $\pi_{\tau_6} h = \nu$. Now assume that $V$ is isomorphic to the quotient of $P$. We show that in this case $P$ is uniserial. Let $\frac{M}{M_1} \cong S_2$, $top(M) = S_1 \oplus S_2$ and $rad(M) = soc(M) = S_3$, then $P_1 \oplus P_2$ is a projective cover of $M$ and $P = P_2$. By \cite[Proposition III.1.15]{3}, the valued quiver of $\Lambda$ has a subquiver of the form

![Diagram](https://via.placeholder.com/150)

If $P_2$ is not uniserial, then by Theorem \cite[3, top(rad(P_2))] = S_3 \oplus S_4 and by \cite[Proposition III.1.15]{3}, the valued quiver of $\Lambda$ has a subquiver of the form

![Diagram](https://via.placeholder.com/150)

Thus $M$ is isomorphic to the quotient of an indecomposable non-local non-colocal 4-factor serial right $\Lambda$-module $L$, which gives a contradiction. Then $P$ is uniserial. Also $M_2$ is a submodule of $M$ and $top(\frac{P}{rad(P)})$ is direct summand of $top(M)$ for some $2 \leq i < l(P)$.

Therefore there exists a homomorphism $h : V \rightarrow M$ such that $\pi_{\tau_6} h = \nu$.

(I)(i). The proof is similar to the proof of (H)(i).

(I)(ii). Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{M}{M_1}$ be a non-isomorphism. Since $\frac{M}{M_1}$ is simple, $V$ is isomorphic to either $M$ or $M_2$ or $N_2$ or a quotient of an indecomposable projective right $\Lambda$-module $P$. The similar argument as in the proof of (H)(ii) shows that $P$ is uniserial. Then there exists a homomorphism $h : V \rightarrow M$ such that $\pi_{\tau_7} h = \nu$.

(I)(iii). Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{M}{N_2}$ be non-isomorphism. If $\nu$ is an epimorphism, then $V$ is isomorphic to either $M$ or $M_1$ or a quotient of an indecomposable projective right $\Lambda$-module $P$. The similar argument as in the proof of (H)(ii) shows that $P$ is uniserial. Since $top(\frac{P}{rad(P)}) \cong top(\frac{M}{N_2})$, there exists a homomorphism $h : V \rightarrow soc(\frac{M}{N_2}) \oplus M$ such that $g_3 h = \nu$. If $\nu$ is not an epimorphism, then $Im \nu = soc(\frac{M}{N_2})$ and so there exists a homomorphism $h : V \rightarrow soc(\frac{M}{N_2}) \oplus M$ such that $g_3 h = \nu$.

(J)(i). The proof is similar to the proof of (H)(i).

(J)(ii). Let $V$ be an indecomposable right $\Lambda$-module and $\nu : V \rightarrow \frac{M}{S_1}$ be a non-isomorphism. If $\nu$ is an epimorphism, then $V \cong M$ and there exists a homomorphism $h : V \rightarrow \frac{M}{S_1} \oplus M$ such that $g_5 h = \nu$. Now assume that $\nu$ is not an epimorphism. Maximal submodules
of \( \frac{M}{S_1} \) are \( M_2 \) and \( \frac{M_1}{S_1} \), then there exists a homomorphism \( h : V \rightarrow \frac{M_0}{S_1} \oplus M \) such that \( gh = \nu \).

(J)(iii). Let \( V \) be an indecomposable right \( \Lambda \)-module and \( \nu : V \rightarrow \frac{M}{M_2} \) be a non-isomorphism. If \( \nu \) is an epimorphism, then \( V \) is isomorphic to either \( M \) or \( M_1 \) or a quotient of an indecomposable projective right \( \Lambda \)-module \( P \). The similar argument as in the proof of (H)(ii) shows that \( M_1 \) is a quotient of \( P \). Then there exists a homomorphism \( h : V \rightarrow M \) such that \( \pi_{S_4} h = \nu \). Now assume that \( \nu \) is not an epimorphism. Since \( \text{Im}(\nu) = \text{soc}(\frac{M}{M_2}) \subset \text{soc}(M) \), \( \text{Im}(\nu) \) is a submodule of \( M \) and so there exists a homomorphism \( h : V \rightarrow M \) such that \( \pi_{S_4} h = \nu \).

(J)(iv). Let \( V \) be an indecomposable right \( \Lambda \)-module and \( \nu : V \rightarrow \frac{M}{M_2 \oplus S_1} \) be a non-isomorphism. Since \( \frac{M}{M_2 \oplus S_1} \) is simple, \( V \) is isomorphic to either \( M \) or \( M_1 \) or \( \frac{M}{S_1} \) or \( \frac{M}{M_2} \) or a quotient of an indecomposable projective right \( \Lambda \)-module \( P \). The similar argument as in the proof of (H)(ii) shows that \( M_1 \) is a quotient of \( P \). Then there exists a homomorphism \( h : V \rightarrow \frac{M}{S_1} \oplus \frac{M}{M_2} \) such that \( gh = \nu \).

(J)(v). The proof is similar to the proof of (J)(iii). \( \square \)

5. Quivers and Relations of Right 4-Nakayama Finite Dimensional \( K \)-algebras

Let \( \Lambda \) be a basic connected finite dimensional \( K \)-algebra. It is known that there exist a quiver \( Q \) and an admissible ideal \( I \) of the path algebra \( KQ \) such that \( \Lambda \cong \frac{KQ}{I} \). In this section, we give a necessary and sufficient conditions for the quiver \( Q \) and the admissible ideal \( I \) that \( \frac{KQ}{I} \) be a right 4-Nakayama algebra.

A finite dimensional \( K \)-algebra \( \Lambda = \frac{KQ}{I} \) is called special biserial algebra provided \((Q, I)\) satisfying the following conditions:

1. For any vertex \( a \in Q_0 \), \( |a^+| \leq 2 \) and \( |a^-| \leq 2 \).
2. For any arrow \( a \in Q_1 \), there is at most one arrow \( \beta \) and at most one arrow \( \gamma \) such that \( a\beta \) and \( \gamma a \) are not in \( I \).

Let \( \Lambda = \frac{KQ}{I} \) be a special biserial finite dimensional \( K \)-algebra. A walk \( w = c_1c_2 \cdots c_n \) is called string of length \( n \) if \( c_i \neq c_{i+1}^{-1} \) for each \( i \) and no subwalk nor its inverse is in \( I \). In addition, we have strings of length zero, for any \( a \in Q_0 \) we have two strings of length zero, denoted by \( 1_{(a,1)} \) and \( 1_{(a,-1)} \). We have \( s(1_{(a,1)}) = t(1_{(a,1)}) = s(1_{(a,-1)}) = t(1_{(a,-1)}) = a \) and \( 1_{(a,1)}^{-1} = 1_{(a,-1)} \). A string \( w = c_1c_2 \cdots c_n \) with \( s(w) = t(w) \) such that each power \( w^m \) is a string, but \( w \) itself is not a proper power of any strings is called band. We denote by \( \mathcal{S}(\Lambda) \) and \( \mathcal{B}(\Lambda) \) the set of all strings of \( \Lambda \) and the set of all bands of \( \Lambda \), respectively. Let \( \rho \) be the equivalence relation on \( \mathcal{S}(\Lambda) \) which identifies every string \( w \) with its inverse \( w^{-1} \) and \( \sigma \) be the equivalence relation on \( \mathcal{B}(\Lambda) \) which identifies every band \( w = c_1c_2 \cdots c_n \) with the cyclically permuted bands \( w(i) = c_1c_{i+1} \cdots c_nc_1 \cdots c_{i-1} \) and their inverses \( w(i)^{-1} \), for each \( i \). Butler and Ringe in [4] for each string \( w \) defined a unique string module \( M(w) \) and for each band \( v \) defined a family of band modules \( M(v, m, \varphi) \) with \( m \geq 1 \) and \( \varphi \in Aut(K^m) \). Let \( \tilde{\mathcal{S}}(\Lambda) \) be the complete set of representatives of strings relative to \( \rho \) and \( \tilde{\mathcal{B}}(\Lambda) \) be the complete set of representatives of bands relative to \( \sigma \). Butler and Ringe in [4] proved that, the modules \( M(w), w \in \tilde{\mathcal{S}}(\Lambda) \) and the modules \( M(v, m, \varphi) \) with
v \in \tilde{B}(\Lambda)$, $m \geq 1$ and $\varphi \in Aut(K^m)$ provide complete list of pairwise non-isomorphic indecomposable $\Lambda$-modules. Indecomposable right $\Lambda$-modules are either string modules or band modules or non-uniserial projective-injective modules (see [4] and [10]). If $\Lambda$ is a representation-finite special biserial algebra, then any indecomposable right $\Lambda$-module is either string or non-uniserial projective-injective.

**Proposition 5.1.** Any basic connected finite dimensional right 4-Nakayama $K$-algebra is representation-finite special biserial.

**Proof.** The proof is similar to the proof of [5, Proposition 3.2]. □

**Theorem 5.2.** Let $\Lambda = \frac{KQ}{I}$ be a basic, connected and finite dimensional $K$-algebra. Then $\Lambda$ is a right 4-Nakayama algebra if and only if $\Lambda$ is a representation-finite special biserial algebra that $(Q, I)$ satisfying the following conditions:

(i) If there exist a walk $W$ and two different arrows $w_1$ and $w_2$ with the same target such that $w_1^{-1}w_2$ is a subwalk of $W$, then $\text{length}(W) \leq 3$.

(ii) If there exist a walk $W$ and two different arrows $w_1$ and $w_2$ with the same source such that $w_1^{-1}w_2$ is a subwalk of $W$, then $\text{length}(W) \leq 4$.

(iii) If there exist two paths $p$ and $q$ with the same target and the same source such that $p - q \in I$. Then $\text{length}(p) + \text{length}(q) \leq 5$.

(iv) At least one of the following conditions hold:

(a) There exist a walk $W$ of length 3 and two different arrows $w_1$ and $w_2$ with the same target such that $w_1^{-1}w_2$ is a subwalk of $W$.

(b) There exist a walk $W$ of length 4 and two different arrows $w_1$ and $w_2$ with the same source such that $w_1^{-1}w_2$ is a subwalk of $W$.

(c) There exist two paths $p$ and $q$ with the same target and the same source such that $p - q \in I$ and $\text{length}(p) + \text{length}(q) = 5$.

**Proof.** Let $\Lambda$ be a right 4-Nakayama algebra. By Proposition 5.1 $\Lambda$ is a special biserial algebra of finite type. Assume on the contrary that the condition (i) does not hold. Then there exists a walk $W$ of length greater than or equal 4 which has a subwalk $W'$ of length 3 that $W'$ has a subwalk of the form $w_1^{-1}w_2$. Since $\Lambda$ is a special biserial algebra of finite type, the walk $W'$ is one of the following forms.

- First case: The walk $W'$ is of the form

```
  w_1  1  w_2  w_3  4
   ▼      ▼      ▼
  2   ▼  3
  ▼
  2  3
```

In this case $W$ has a subwalk of one of the following forms:

(i) $a$

```
  a  w_1  1  w_2  w_3  4
   ▼      ▼      ▼
  2   ▼  3
  ▼
  2  3
```

In this case the vertex $a$ can be either 1 or 2 or 5.
(ii)

\[ \begin{array}{c}
  a \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_1 \hspace{1cm} \downarrow \\
  w_2 \hspace{1cm} 1 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 4 \\
\end{array} \]

In this case the vertex \( a \) can be either 2 or 3 or 4 or 5.

(iii)

\[ \begin{array}{c}
  1 \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_1 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_2 \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_4 \hspace{1cm} \downarrow \\
  a \\
\end{array} \]

In this case the vertex \( a \) can be either 1 or 3 or 4 or 5.

(iv)

\[ \begin{array}{c}
  1 \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_1 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_2 \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_4 \hspace{1cm} \downarrow \\
  a \\
\end{array} \]

In this case the vertex \( a \) can be either 2 or 4 or 5.

- Second case: The walk \( W' \) is of the form

\[ \begin{array}{c}
  1 \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_1 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_2 \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_4 \hspace{1cm} \downarrow \\
  3 \\
\end{array} \]

In this case \( W \) has a subwalk of one of the following forms:

(i)

\[ \begin{array}{c}
  a \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_1 \hspace{1cm} \downarrow \\
  w_2 \hspace{1cm} 1 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 3 \\
\end{array} \]

In this case the vertex \( a \) can be either 1 or 2 or 4.

(ii)

\[ \begin{array}{c}
  a \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_1 \hspace{1cm} \downarrow \\
  w_2 \hspace{1cm} 1 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 3 \\
\end{array} \]

In this case the vertex \( a \) can be either 2 or 4.

(iii)

\[ \begin{array}{c}
  1 \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_1 \hspace{1cm} 2 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} w_2 \hspace{1cm} \downarrow \\
  w_3 \hspace{1cm} 3 \hspace{1cm} \downarrow \\
  \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
  w_4 \hspace{1cm} 3 \\
\end{array} \]

In this case the vertex \( a \) can be either 1 or 4.

- Third case: The walk \( W' \) is of the form
In this case $W$ has a subwalk of one of the following forms:

(i) For $i = 1, 2$. If $i = 1$, then $a = 2$ and if $i = 2$, then $a = 3$.

(ii) In this case the vertex $a$ can be either 2 or 4.

(iii) In this case the vertex $a$ can be either 2 or 3 or 4.

(iv) In this case the vertex $a$ can be either 2 or 4.

- Forth case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:

(i) In this case the vertex $a$ can be either 2 or 3 or 4 or 5.
In this case the vertex $a$ can be either 1 or 4 or 5.

In this case the vertex $a$ can be either 1 or 2 or 5.

In this case the vertex $a$ can be either 2 or 3 or 5.

- Fifth case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:

In this case the vertex $a$ can be either 1 or 4.

for $i = 2, 3$. If $i = 2$, then $a = 1$ and if $i = 3$, then $a = 3$.

In this case the vertex $a$ can be either 1 or 2 or 4.
(iv)

\[ a \xleftarrow{w_4} 2 \xrightarrow{w_1} 1 \xrightarrow{w_2} 3 \xrightarrow{w_3} (i) \]

In this case the vertex \( a \) can be either 2 or 4.

- Sixth case: The walk \( W' \) is of the form

\[ a \xleftarrow{w_1} 2 \xrightarrow{w_2} 3 \xrightarrow{w_3} 2 \]

In this case \( W \) has a subwalk of one of the following forms:

(i)

\[ w_1 \xrightarrow{w_2} w_3 \xrightarrow{w_4} a \]

In this case the vertex \( a \) can be either 3 or 4.

(ii)

\[ w_1 \xrightarrow{w_2} w_3 \xrightarrow{w_4} a \]

In this case the vertex \( a \) can be either 1 or 4.

- Seventh case: The walk \( W' \) is of the form

\[ a \xleftarrow{w_4} 2 \xrightarrow{w_1} 1 \xrightarrow{w_2} 3 \xrightarrow{w_3} (i) \]

In this case \( W \) has a subwalk of one of the following forms:

(i)

\[ 1 \xleftarrow{w_1} w_1 \xrightarrow{w_2} w_3 \]

\[ 1 \xleftarrow{w_1} w_1 \xrightarrow{w_2} 2 \]

\[ 1 \xleftarrow{w_1} w_1 \xrightarrow{w_2} 3 \]

(ii)

\[ a \xleftarrow{w_4} 2 \xrightarrow{w_1} 1 \xrightarrow{w_2} 3 \xrightarrow{w_3} (i) \]

In this case the vertex \( a \) can be either 2 or 3 or 4.
1 In this case the vertex $a$ can be either 3 or 4.

(iv)

In this case the vertex $a$ can be either 2 or 3 or 4.

- Eighth case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:

(i)

(ii)

(iii)

- Ninth case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:

(i)
In this case the vertex $a$ can be either 2 or 3 or 4.

(ii)

```
1  ↗
  ↘
1  ↗
  ↘
```

In this case the vertex $a$ can be either 3 or 4.

(iii)

```
1  ↗
  ↘
1  ↗
  ↘
```

(iv)

```
1  ↗
  ↘
1  ↗
  ↘
```

In this case the vertex $a$ can be either 2 or 4.

- Tenth case: The walk $W'$ is of the form

```
1  ↗
  ↘
1  ↗
  ↘
```

In this case $W$ has a subwalk of one of the following forms:

(i)

```
1  ↗
  ↘
1  ↗
  ↘
```

(ii)

```
1  ↗
  ↘
1  ↗
  ↘
```

If $i = 2$, then $a = 1$ and if $i = 3$, then $a = 2$.

(iii)

```
1  ↗
  ↘
1  ↗
  ↘
```

If $i = 2$, then $a = 2$ and if $i = 1$, then $a = 1$. 
• Eleventh case: The walk $W'$ is of the form

\[
\begin{align*}
3 & \quad w_4 \quad w_1 \\
1 & \quad 2
\end{align*}
\]

In this case $W$ has a subwalk of one of the following forms:

(i)

\[
\begin{align*}
1 & \quad w_1 \\
1 & \quad 2
\end{align*}
\]

(ii)

\[
\begin{align*}
a & \quad w_4 \\
1 & \quad 2
\end{align*}
\]

In this case the vertex $a$ can be either 2 or 3.

(iii)

\[
\begin{align*}
1 & \quad w_1 \\
1 & \quad 2
\end{align*}
\]

In this case the vertex $a$ can be either 2 or 3.

(iv)

\[
\begin{align*}
1 & \quad w_1 \\
1 & \quad 2
\end{align*}
\]

In this case the vertex $a$ can be either 2 or 3.

• Twelfth case: The walk $W'$ is of the form

\[
\begin{align*}
3 & \quad w_3 \quad w_1 \\
1 & \quad 2
\end{align*}
\]

In this case $W$ has a subwalk of one of the following forms:
(i) $\begin{array}{c}
3 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
1 & 1 & 2 & a \\
\end{array}$

In this case the vertex $a$ can be either 2 or 4.

(ii) $\begin{array}{c}
3 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
1 & 1 & 2 & a \\
\end{array}$

In this case the vertex $a$ can be either 2 or 3 or 4.

(iii) $\begin{array}{c}
w_4 & 3 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
a & 1 & 2 \\
\end{array}$

In this case the vertex $a$ can be either 3 or 4.

(iv) $\begin{array}{c}
w_4 & 3 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
a & 1 & 2 \\
\end{array}$

In this case the vertex $a$ can be either 2 or 3 or 4.

• Thirteenth case: The walk $W'$ is of the form $\begin{array}{c}
2 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
1 & 1 & 2 \\
\end{array}$

In this case $W$ has a subwalk of one of the following forms:

(i) $\begin{array}{c}
2 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
1 & 1 & 2 & 3 \\
\end{array}$

(ii) $\begin{array}{c}
2 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
1 & 1 & 2 & 3 \\
\end{array}$

(iii) $\begin{array}{c}
w_4 & 2 \\
\downarrow \swarrow \downarrow \swarrow \downarrow \\
3 & 1 & 2 \\
\end{array}$
\( (iv) \)

In all the above cases, there exists an indecomposable 5-factor serial right \( \Lambda \)-module which gives a contradiction.

Assume on the contrary that the condition \((ii)\) does not hold. Then there exists a walk \( W \) of length greater than or equal to 5 which has a subwalk \( W' \) of length 4 that \( W' \) has a subwalk of the form \( w_1^{-1}w_2^{+1} \). By the condition \((i)\), \( W \) has not a subwalk \( W'' \) of length greater than or equal to 4 that \( W'' \) has a subwalk of the form \( w_1^{+1}w_2^{-1} \). Since \( \Lambda \) is an algebra of finite type, the walk \( W' \) is one of the following forms.

- First case: The walk \( W' \) is of the form

```
  2  \( w_1 \) \( w_2 \)  3  \( w_3 \) \( w_4 \)  5  \\
  1  \( w_5 \)  4
```

In this case \( W \) has a subwalk of one of the following forms:

\( (i) \)

```
  2  \( w_1 \) \( w_2 \)  3  \( w_3 \) \( w_4 \)  5  \( w_5 \)  a  \\
  1  \( \text{a} \)  4
```

In this case the vertex \( a \) can be either 6 or 1.

\( (ii) \)

```
  \( w_5 \)  2  \( w_1 \) \( w_2 \)  3  \( w_3 \) \( w_4 \)  5  \\
  a  1  \( \text{a} \)  4
```

In this case the vertex \( a \) can be either 6 or 1.

- Second case: The walk \( W' \) is of the form

```
  2  \( w_1 \) \( w_2 \)  3  \( w_3 \) \( w_4 \)  1  \\
  1  \( \text{a} \)  4
```

In this case \( W \) has a subwalk of one of the following forms:

\( (i) \)

```
  2  \( w_1 \) \( w_2 \)  3  \( w_3 \) \( w_4 \)  1  \( w_i \)  a  \\
  1  \( \text{a} \)  4
```

for \( i = 1, 2 \). If \( i = 1 \), then \( a = 2 \) and if \( i = 2 \), then \( a = 3 \).
• Third case: The walk $W'$ is of the form

In this case $W$ has a subwalk of the following form:

In this case the vertex $a$ can be either 6 or 1.

• Fourth case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:

(i)

(ii)

If $i = 1$, then $a = 2$ and if $i = 2$, then $a = 3$.

• Fifth case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:
• Sixth case: The walk $W'$ is of the form

\[
\begin{array}{c}
\text{2} \\
\uparrow w_1 \downarrow \end{array}
\begin{array}{c}
\text{3} \\
\uparrow w_2 \downarrow \end{array}
\begin{array}{c}
\text{2} \\
\uparrow w_3 \downarrow \end{array}
\begin{array}{c}
\text{4} \\
\uparrow w_4 \downarrow \end{array}
\begin{array}{c}
\text{1} \\
\uparrow w_1 \downarrow \end{array}
\begin{array}{c}
\text{1} \\
\uparrow w_2 \downarrow \end{array}
\begin{array}{c}
\text{1} \\
\uparrow w_3 \downarrow \end{array}
\begin{array}{c}
\text{1} \\
\uparrow w_4 \downarrow \end{array}
\end{array}
\]

In this case $W$ has a subwalk of one of the following forms:

• Seventh case: The walk $W''$ is of the form

\[
\begin{array}{c}
\text{4} \\
\downarrow w_4 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_1 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_2 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_3 \uparrow \end{array}
\begin{array}{c}
\text{3} \\
\downarrow w_2 \uparrow \end{array}
\begin{array}{c}
\text{3} \\
\downarrow w_3 \uparrow \end{array}
\begin{array}{c}
\text{3} \\
\downarrow w_4 \uparrow \end{array}
\end{array}
\]

In this case $W$ has a subwalk of one of the following forms:

- If $i = 1$, then $a = 2$ and if $i = 2$, then $a = 3$. 

\[
\begin{array}{c}
\text{4} \\
\downarrow w_4 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_1 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_2 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_3 \uparrow \end{array}
\begin{array}{c}
\text{a} \\
\downarrow w_i \uparrow \end{array}
\end{array}
\]

- If $i = 1$, then $a = 2$ and if $i = 2$, then $a = 3$. 

\[
\begin{array}{c}
\text{5} \\
\downarrow w_5 \uparrow \end{array}
\begin{array}{c}
\text{4} \\
\downarrow w_4 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_1 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_2 \uparrow \end{array}
\begin{array}{c}
\text{1} \\
\downarrow w_3 \uparrow \end{array}
\end{array}
\]
Eighth case: The walk $W'$ is of the form

```
1  w_1 w_2  2  w_3 w_4  4
\downarrow       \downarrow       \downarrow
1               3               5
```

In this case $W$ has a subwalk of one of the following forms:

(i) 

```
1  w_1 w_2  2  w_3 w_4  4  w_5
\downarrow       \downarrow       \downarrow
1               3               5
```

(ii) 

```
1  \quad \quad \quad 2  \quad \quad \quad 4
\downarrow       \downarrow
1               3
```

If $i = 1$, then $a = 1$ and if $i = 2$, then $a = 2$.

Ninth case: The walk $W'$ is of the form

```
1  w_1 w_2  2  w_3
\downarrow       \downarrow
1               3
```

In this case $W$ has a subwalk of one of the following forms:

(i) 

```
1  w_1 w_2  2  w_3 w_4  4
\downarrow       \downarrow       \downarrow
1               3               5
```

(ii) 

```
1  \quad \quad \quad 2  \quad \quad \quad \quad \quad 3
\downarrow       \downarrow       \downarrow
1               3               3
```

Tenth case: The walk $W'$ is of the form

```
1  w_1 w_2  2
\downarrow       \downarrow
1               1
```

In this case $W$ has a subwalk of one of the following forms:
• Eleventh case: The walk $W'$ is of the form

In this case $W$ has a subwalk of one of the following forms:

• Twelfth case: The walk $W''$ is of the form

In this case $W$ has a subwalk of one of the following forms:
In all the above cases there exists an indecomposable 5-factor serial right \( \Lambda \)-module which gives a contradiction.

Now assume that the condition \((iii)\) does not hold. Then there exist paths \( p = p_1 \ldots p_t \) and \( q = q_1 \ldots q_s \) with the same target and the same source that \( p_i \) and \( q_j \) are arrows, \( s \geq 2 \), \( t \geq 2 \), \( p - q \in I \) and \( s + t > 5 \).

We have two cases. In the first case, we have \( t \geq 2 \) and \( s \geq 4 \). Then there exists a string \( w = q_{s-1}^{-1}q_{s-2}^{-1}q_{s-3}^{-1}p_t^{-1} \) of \( KQ \), that \( M(w) \) is a 5-factor serial right \( \Lambda \)-module which gives a contradiction. In the second case, we have \( t \geq 3 \) and \( s \geq 3 \). Then there exists a string \( w = p_t^{-1}p_{t-1}^{-1}q_{s-1}^{-1}q_{s-2}^{-1} \) of \( KQ \), that \( M(w) \) is a 5-factor serial right \( \Lambda \)-module which gives a contradiction. Now assume that the condition \((iv)\) does not hold. By \([6\), Theorem 5.13\] and \([5\) Theorem 3.3\] in this case \( \Lambda \) is a \( t \)-Nakayama algebra for some \( t \leq 3 \), which gives a contradiction.

Conversely, assume that \( \Lambda \) is a special biserial algebra of finite type that \((Q, I)\) satisfies the conditions \((i) - (iv)\). By \([4\) and \([10\), every indecomposable right \( \Lambda \)-module is either a string \( M(w) \), \( w \in \hat{S}(\Lambda) \) or a band module \( M(v, m, \varphi) \), \( v \in B(\Lambda) \), \( m \geq 1 \) and \( \varphi \in Aut(K^m) \) or non-uniserial projective-injective. Since \( \Lambda \) is representation-finite, \( B(\Lambda) = \emptyset \). For any \( w \in \hat{S}(\Lambda) \), \( \varphi \) is either \( w_{1+1} \ldots w_n \) or \( w_{1-1}w_2w_4 \) or \( w_{1+1}w_2^{-1}w_3w_4 \) or \( w_{1+1}w_2+1w_3+1w_4+1 \) or \( w_{1+1}w_2+1w_3+1w_4+1 \) or \( w_{1+1}w_2+1w_3+1 \). If \( w = w_{1+1} \) and \( M(w) \) is uniserial. If \( w = w_{1+1}w_2^{-1} \), then \( M(w) \) is 2-factor serial. If either \( w = w_{1+1}w_2^{-1} \) or \( w = w_{1-1}w_2+1w_3+1 \), then \( M(w) \) is 3-factor serial. If either \( w_{1-1}w_2^{-1}w_3+1w_4+1 \) or \( w_{1+1}w_2+1w_3+1 \) or \( w_{1-1}w_2+1w_3+1 \), then \( M(w) \) is a 4-factor serial right \( \Lambda \)-module. By the condition \((iv)\), there exists at least one string module \( M(w) \), where \( w \) is either \( w_{1-1}w_2-1w_3+1w_4+1 \) or \( w_{1-1}w_2-1w_3+1w_4+1 \) or \( w_1+1w_2+1w_3+1 \). Therefore \( \Lambda \) is a right 4-Nakayama and the result follows.

\[\square\]

Remark 5.3.

(1) Let \( \Lambda = KQ/I \) be a right 4-Nakayama finite dimensional \( K \)-algebra that the condition \((iv)(c)\) of the theorem \([5,2\] holds. Then there exists a non-uniserial projective-injective right \( \Lambda \)-module \( M \), that \( M \) is either 3-factor serial or 4-factor serial.

(2) Let \( \Lambda = KQ \) be a basic, connected and finite dimensional \( K \)-algebra. By \([3\) Proposition 4.2\] \( \Lambda \) is right 4-Nakayama self-injective if and only if \( Q \) is the following quiver with \( s \geq 1 \).
and \( m, n \geq 2 \),

and \( I \) is the ideal generated by the following relations:

(i) \( \alpha_{1}^{[i]} \cdots \alpha_{m}^{[i]} = \beta_{1}^{[i]} \cdots \beta_{n}^{[i]} \) for all \( i \in \{0, \ldots, s-1\} \);

(ii) \( \beta_{1}^{[i]} \alpha_{1}^{[i+1]} = 0, \alpha_{m}^{[i+1]} \beta_{n}^{[i+1]} = 0 \) for all \( i \in \{0, \ldots, s-2\} \), \( \beta_{n}^{[s-1]} \alpha_{1}^{[0]} = 0 \) and \( \alpha_{m}^{[s-1]} \beta_{1}^{[0]} = 0 \);

(iii) (a) Paths of the form \( \alpha_{1}^{[j]} \cdots \alpha_{h}^{[f]} \) of length \( m+1 \) are equal to 0.
(b) Paths of the form \( \beta_{1}^{[j]} \cdots \beta_{h}^{[f]} \) of length \( n+1 \) are equal to 0.

that \( m + n = 5 \).

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Department of Mathematics, University of Isfahan, P.O. Box: 81746-73441, Isfahan, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

*E-mail address*: nasr-a@sci.ui.ac.ir / nasr@ipm.ir

Department of Mathematics, University of Isfahan, P.O. Box: 81746-73441, Isfahan, Iran

*E-mail address*: mshekari@sci.ui.ac.ir