YET ANOTHER INDUCTION SCHEME
FOR NON-UNIFORMLY EXPANDING TRANSFORMATIONS

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Abstract. We introduce a new induction scheme for non-uniformly expanding maps $f$ of compact Riemannian manifolds, relying upon ideas of [30] and [10]. We use this induction approach to prove that the existence of a Gibbs-Markov-Young structure is a necessary condition for $f$ to preserve an absolutely continuous probability with all its Lyapunov exponents positive.

1. Introduction

A Borel probability $\mu$ is $f$-invariant if $\mu(f^{-1}(B)) = \mu(B)$ for every Borel subset $B \subset M$. Invariant probabilities provide quantitative information about the spatial distribution of orbits of $f$ and are crucial in the study of statistical properties of the dynamics. See [13] and [19, Chapter 4].

Absolutely continuous invariant measures (acim) are of interest because they are generated by sets of positive Lebesgue measure, that is, they are experimentally observable.

This paper is concerned with induction, a useful notion aimed at the study of ergodic properties of dynamical systems. Namely, suppose that every point in a non-empty subset $B \subset M$ returns infinitely many often to $B$. This is the case, for example, of a set $B$ with $\mu(B) > 0$, by Poincaré’s recurrence theorem. See [13]. Then it is natural to consider the induced dynamical system $f^R : B \to B$ defined as a first return map $f^R(x) = f^{R(x)}(x)$, where $R(x) = \min\{n > 0 : f^n(x) \in B\}$ is the first return times from $x \in B$ to $B$.

First return maps were originally introduced by Poincaré to study nearby orbits of limit cycles. It was later, in the work of Kakutani [17], that they revealed their usefulness in the study of ergodic properties of the dynamics. Transformations induced by more general return times were also considered in the work of J. Neveu and Schweiger between 1969 and 1981. See references in [26].

To the best of our knowledge, Jakobson’s theorem [15] was the first to show the close connection between induced maps and the existence of acim. Jakobson’s theorem says that given a one-parameter family of real quadratic maps $f_a(x) = 1 - ax^2$, there exists a Cantor set $\Lambda$ with positive Lebesgue measure such that for every $a \in \Lambda$, $f = f_a$ has an ergodic acim with positive Lyapunov exponent. See [15, Theorem A, Theorem B].

The idea of the proof is to construct $\Lambda$ in such way that for every $a \in \Lambda$, there exists a closed interval $J$ and an induced expanding Markov transformation $f^R : \bigcup J_i \to J$ with bounded distortion (see subsection 1.1) and integrable return times $R$. By a folklore theorem $f^R$ has a
(unique) ergodic \( f^R \)-invariant acim. Integrability of return times permits to “coinduce” a \( f \)-invariant acim \( \mu \), by an standard procedure (see below). We refer to Yoccoz's College de France manuscript [33] for an up-to-date and quite clear exposition of this celebrated result.

Jakobson’s paper inspired a number of results concerning the application of induction method to prove the existence of acim for different classes of one-dimensional dynamical systems. See for instance [3] Chapter V, Section 3, [12], [10], [14] and [32].

The next breakthrough in the induction approach was Lai Sang Young’s “horseshoes with infinitely many branches and variable return times”. See [34], [35]. Generally speaking, Young’s horseshoes are Cantor sets endowed with an induced return map with good hyperbolic and distortion properties with respect to a reference measure \( m \). The return structures are made of infinitely many ‘vertical strips’ mapped onto ‘horizontal strips’ similarly to Smale’s horseshoes. Projecting along stable invariant manifolds one gets a uniformly expanding map with good distortion properties \( f^R : \bigcup_i \Delta_i \to \Delta \) defined over a decomposition of a Cantor set \( \Delta \) with positive reference measure. These Markov structures are used to prove the existence of equilibrium measures, showing the connection between the speed of convergence to the equilibrium and its rates of mixing with the rate of decaying of return times tails.

The following is a partial sample of the growing body of literature on the applications of also called Markov towers or Gibbs-Markov-Young structures (GMY) in the study of statistical properties of the dynamics. See [1], [2], [4], [5], [6], [9], [10], [11], [21], [22], [23], [24], [25], [26], [28], [29].

In [34] Young describes informally diffeomorphisms exhibiting GMY structures as those being “hyperbolic in large part of the phase space without being uniformly hyperbolic”, however, up to the best of our knowledge, a precise characterization of the systems displaying Markov structures is still lacking.

Recently, it was conjectured that a \( C^{1+\alpha} \) diffeomorphism of compact Riemannian manifold has a GMY structure if and only if it is non-uniformly hyperbolic, meaning a system with an invariant measure with non zero Lyapunov exponents. The conjecture was proved in [1] for a large class of non-uniformly expanding systems. In particular, let \( f \) be a \( C^{1+\alpha} \) map of compact Riemannian manifold of arbitrary dimension with critical points. If the criticalities satisfy non-degeneracy conditions similar to non-flatness in one-dimensional maps, then it is proved that \( f \) preserves an ergodic acim \( \mu \) with all its Lyapunov exponents positive if and only if it there exists a GMY structure having Lebesgue as a reference measure. If, in addition, \( f \) has singularities, then a similar result was proved whenever \( \mu \) satisfies a regularity condition. See [1, Theorem 2, Theorem 3] and subsection 1.1.

The paper [1] proposes a simplified and conceptually clearer method to induce an expanding Markov map with good distortion properties, in comparison with the technicalities involved in most of the results on the subject. The proposed algorithm is essentially the most naive strategy to construct an induced expanding transformation in the non-uniformly expanding setting. The idea is to fix an open ball \( B \) and iterate it until the return of some subset of \( B' \subset B \) covers \( B \), expanding with bounded distortion. The set \( B' \) thus selected becomes an element of a partition and the argument is repeated for the points that were outside \( B' \). However these returns have, in general, overlapping domains. This requires an additional effort to construct a countable family of expanding branches \( f^{R_i} : B_i \to B \) with pairwise disjoint domains covering a full measure subset of \( B \), assuring the integrability of the return times. [1, Section 4.1].

We introduce in this paper an alternative induction scheme, motivated by results of [36], giving an independent proof of [1, Theorem 2, Theorem 3]. Roughly speaking our induced maps are projections of the first return map of the natural extension \( F : Z \to Z \) of \( f \) to a suitable
subset $\Sigma \subset \mathbb{Z}$. Schematically,
\begin{align}
(Z,F) \xrightarrow{\text{first return}} (\Sigma,F^R) \\
\pi \downarrow \pi_{\Sigma} \\
(M,f) \xrightarrow{\text{induction}} (B,f^R)
\end{align}

See subsection 1.2.

Crucially, $\Sigma$ is chosen to be a ‘rectangle with regular returns’ (RRR). These geometrical structures were introduced in [10] to study of ergodic properties of certain equilibrium states of non-uniformly expanding rational maps of the Riemann sphere proving, among other things, the existence of Markov structures with a non-exceptional conformal measure as reference.

Although [10] is mostly based on geometrical and analytical properties of conformal maps, the proof of the existence of a RRR essentially relies upon smooth ergodic theory methods. This suggests the possibility to extend its construction for a more general class of non conformal mappings. This is precisely what we do for non-uniformly expanding maps with non-degenerate criticalities and/or singularities.

We should mention too [5] as an immediate antecedent to [30]. There the question was raised as to whether the integrability of the return times of an induced Markov map is a necessary condition for the existence of an acim $\mu$. It is proved that this is indeed the case, for interval piecewise monotonic maps preserving a conservative acim with positive entropy, whenever the induced Markov map satisfies some natural condition.

The induction method proposed in [5] can be summarized in the following diagram:
\begin{align}
(\hat{I},\hat{f}) \xrightarrow{\text{first return}} (\hat{J},\hat{f}^R) \\
\pi \downarrow \pi \\
(I,f) \xrightarrow{\text{induction}} (J,f^R)
\end{align}

Here $\hat{f} : \hat{I} \to \hat{I}$ is a Markov extension of $f : I \to I$, a construction due to Keller and Hofbauer. $\hat{f} : \hat{I} \to \hat{I}$ is a Markov transformation of a non-compact metric space $\hat{I}$ endowed with a naturally-defined countable partition defined using the monotonicity intervals of $f$. Moreover $\hat{I}$ comes endowed with a continuous and surjective map $\pi : \hat{I} \to I$ commuting with $f$ and $\hat{f}$, that is, $f \circ \pi = \pi \circ \hat{f}$. It is proved that if $f^R : J \to J$ is natural or naturally extendible then there exists an interval $\hat{J} \subset \hat{I}$ making this diagram commutative. An induced Markov map is natural or naturally extendible if the return times are as small as possible and the branch-domains are as large as possible. See [5, Lemma 2]. If $f$ preserves a conservative acim then the return times of a natural or naturally extendible induced Markov map is Lebesgue integrable. See [5, Theorem 1].

This induction approach has received much attention recently in the work of Pesin, Senti and Zhang on the thermodynamics of non-uniform hyperbolic systems. See [23], [24], [25], [26].

The induction method presented in this paper has some advantages over [5] approach, at least for smooth non-uniformly expanding maps. To begin with the phase space of the natural extension is compact, in contrast with Keller-Hofbauer’s construction. Moreover, every measure $f$-invariant $\mu$ admits a (unique!) lift $\bar{\mu}$ to $\hat{Z}$ meanwhile non every $f$-invariant Borel probability admits an elevation to a $\bar{f}$-invariant measure in the Markov extension. Moreover, our induced maps may or may not be natural or naturally extendible, a technical condition required to carry on the induction process under [5] premises.

Finally, let us underline that the present approach can be used to prove the existence of Markov structures for maps preserving a singular (wrt Lebesgue) $f$-invariant Borel probability $\mu$ with positive entropy and all its Lyapunov exponents positive. The question here is the choice of a good reference measure. We will discuss this problem in a forthcoming paper [7].
1.1. Statement of main result.

**Definition 1.1.** We will say that a probability of Borel $f$-invariant $\mu$ is expanding if its Lyapunov exponents

$$\chi^+(x, v) = \lim_{n \to +\infty} \frac{1}{n} \log \|Df^n(x)v\|$$

exists and they are positive for every $v \in T_x M - \{0\}$ and for $\mu$-a.e. $x \in M$.

**Definition 1.2.** Let $B \subset M$ an open simply connected subset. We say that $f^R : \bigcup B_i \to B$ is an induced expanding Markov map with bounded distortion and return times $R$ if $\{B_i \subset B\}$ is a collection of pairwise disjoint open subsets covering $B$ up-to zero volume, such that $R|B_i = R_i$ and $f^R : B_i \to B$ are $C^2$ expanding diffeomorphisms with bounded distortion, for every $i > 0$. That is, there exists constants $\lambda > 1$ and $C > 0$ such that

$$\|Df^{R_i}(x)v\| \geq \lambda \|v\|, \quad \forall v \in T_x M, \forall x \in B_i;$$

and

$$\frac{Jf^{R_i}(x)}{Jf^{R_i}(y)} \leq \exp(Cd(f^{R_i}(x), f^{R_i}(y))), \quad \forall x, y \in B_i, \forall i > 0,$$

where $Jf = |\det(Df)|$ is the Jacobian of $f$ with respect to the Riemannian volume. We say that $f^R : \bigcup B_i \to B$ has an integrable return times if

$$\int RdVol = \sum R_i Vol(B_i) < +\infty.$$

We shall refer to an induced expanding Markov map $f^R : \bigcup B_i \to B$ with bounded distortion and integrable return times as a \textit{GMY structure with Vol as reference measure}.

Let $f^R : \bigcup B_i \to B$ be a GMY structure. By a folklore theorem there exists a unique ergodic, expanding $f^R$-invariant acim $\mu^*$ with a density bounded away from zero and $\infty$. In particular, $R$ is integrable wrt $\mu^*$, i.e. $\int Rd\mu^* < +\infty$. We say that a $f$-invariant Borel probability $\mu$ is coinduced or generated by $f^R : \bigcup B_i \to B$ if

$$\mu = \sum_i \sum_{j=0}^{R_i-1} \mu^* \circ f^j \bigg/ \sum_i R_i \mu^*(B_i).$$

It that case $\mu$ is expanding and ergodic, as it is easy to check. See [33, Section 2.9] or [8, Lemma 3.1].

The following integrability condition is widely used in smooth ergodic theory and plays a crucial role in our approach.

**Definition 1.3.** We will say that a Borel $f$-invariant probability is regular if

$$\int \max \{\log \|Df(x)\|, \log \|Df(x)^{-1}\|\} d\mu(x) < +\infty.$$

Next are the non-degeneracy conditions for critical and/or singular points.

**Non-degenerate critical points** We will say that

$$\mathcal{C} = \{x \in M : \ker Df(x) \neq 0\},$$

the critical point set, is not degenerate if they exist $\beta, A, k > 0$ and an open neighborhood $V$ of $\mathcal{C}$ such that:

$$0 < \sup_{x \in V} \left| \frac{\|Df(x)^{-1}\|^{-1}}{d(x, \mathcal{C})^\beta} \right| < +\infty$$
\[ |\log |\det(Df(x))| - \log |\det(Df(y))| | \leq A \frac{d(x,y)}{d(x,\mathcal{O})^k}, \ \forall \ x, y \in M. \]

**Non-degenerate singular points** If \( f \) has singularities
\[ \mathcal{S} = \{ x : \|Df(x)\| = +\infty \text{ or is not defined} \} \]
we will say that these are non-degenerate if \( (9) \) holds and, instead \( (8) \),
\[ 0 < \sup_{x \in W} \left| \frac{\|Df(x)\|}{d(x,\mathcal{S})^{-1}} \right| < +\infty, \]
where \( W \) is an open neighborhood of \( \mathcal{S} \).

The next is our main result.

**Theorem 1.1.** Let \( f : M \to M \) be a \( C^2 \) transformation with non-degenerate criticalities and/or singularities that preserves an ergodic, expanding, regular acim \( \mu \). Then there exists a GMY structure \( f^R : \bigcup_i B_i \to B \) generating \( \mu \).

Critical and singular cases were discussed separately in [1]. In particular, the critical case [2] does not need \( \mu \) to be regular, in contrast with theorem 1.1. Actually, our induction approach depends heavily on the existence and expanding properties of local unstable manifolds. This requires some regularity conditions on \( \mu \) aimed at taking advantage of standard smooth ergodic theory tools, as Pesin’s \( \epsilon \)-reduction theorem, Lyapunov charts and so on. This allows for a unified treatment of the critical and singular cases. On the other hand, we do not require \( \log \|Df(x)\| \) to be locally Lipschitz.

### 1.2. Strategy of the proof: the induction scheme.

Let us start by recalling the natural extension of a non-invertible map. The phase space is the inverse limit of \( f : M \to M \):
\[ Z = \{(z_n)_{n \geq 0} : f(z_{n+1}) = z_n \}, \]
i.e., points \( z \in Z \) are simply pre-orbits of \( f \). \( Z \) is a compact metric space with distance \( d(z, z') = \sum_{n=0}^{+\infty} d_M(z_n, z'_n)/2^n \), where \( d_M \) is the natural metric of \( M \) as the Riemann space. \( F : Z \to Z \) defined as
\[ F(z) = (f(z_0), z_0, \cdots) \]
is an invertible transformation, with inverse \( F^{-1}(z) = (z_1, z_2, \cdots) \) which extends to \( f \), namely \( f \circ \pi = \pi \circ F \), where \( \pi : Z \to M \) is the projection on the first coordinate. If \( f \) is continuous, then \( F \) is a homeomorphism of \( Z \) itself.

Given a probability of Borel \( \mu \) in \( M \) there is a unique \( \bar{\mu} \) in \( Z \) such that \( \bar{\mu}(\pi_n^{-1}(B)) = \mu(B) \), for all Borel subset \( B \subset M \) and for every \( n \geq 0 \), where \( \pi_n : Z \to M \) is the projection on the \( n \)-th coordinate: \( \pi_n(z_0, z_1, z_2, \cdots) = z_n \). \( \bar{\mu} \) is \( F \)-invariant (resp. ergodic) if \( \mu \) is \( f \)-invariant (resp. ergodic).

**Definition 1.4.** \( F : (Z, B, \bar{\mu}) \to (Z, B, \bar{\mu}) \) is the natural extension of \( (f, \mu) \).

\( \pi : Z \to M \) is a Cantor fiber bundle, that is, every fiber \( F(z) = \pi^{-1}(\pi(z)) \) is a Cantor set. If \( f \) is uniformly expanding these fibers vary continuously and coherently so that the fiber bundle is locally trivial: for every \( x \in M \) there exists an open neighborhood \( U_x \) such that \( \pi^{-1}(U_x) \) is homeomorphic to the cartesian product \( U \times \mathcal{F} \), where \( \mathcal{F} \) is a typical ’vertical’ fiber of \( Z \). This gives to \( Z \) a structure of continuous lamination. Every local plaque \( U_z \subset Z \) passing by \( z \) is a local unstable manifold, that is, the backward orbit under \( F \) of every \( w \in U_z \) is exponentially asymptotic to the backward orbit of \( z \).
When \( f \) is non-uniformly expanding we get instead a measurable lamination of local unstable manifolds, that is, \( \pi^{-1}(U_z) \) is no longer homeomorphic but measurable isomorphic to \( U \times K \), that is, there exist an invertible measurable map \( \varphi : \pi^{-1}(U_z) \to U \times F \) with measurable inverse. Nevertheless, \( Z \) is covered by ‘flow-boxes’ \( \Sigma \) which are homeomorphic to cartesian products \( B \times K \) of an open set \( B \subset M \) times a Cantor subset \( K \subset F \). Furthermore, \( \pi|\Sigma_z : \Sigma_z \to B \) is a diffeomorphism for every \( z \in \Sigma \), being \( \Sigma_z \subset \Sigma \) the plaque of ‘height’ \( z \) in \( \Sigma \). We will call rectangles to these ‘flow-boxes’ and say that a map is \( \varphi \) is a diffeomorphism for every \( z \in Z \).

As we mention before, our induced Markov maps are suitable projections of the first return map \( F^j : \Sigma \to \Sigma \) to a rectangle with regular returns \( \Sigma \). Additionally to a uniform control on the rate of contraction along inverse branches and good control on the forward and backward distortion of horizontal plaques of \( \Sigma \), \( \varphi \) has the crucial property that, for every \( n, m \geq 0 \) and for every \( z, z' \in \Sigma \), the preimages \( F^{-n}\Sigma_z, F^{-m}\Sigma_{z'} \) and its projections \( \pi F^{-n}\Sigma_z, \pi F^{-m}\Sigma_{z'} \) are disjoint or nested, similarly to the ‘nice sets’ appearing in the study of rational maps of the Riemann sphere, from which Yoccoz’s puzzles are prominent and well known examples. See \([10], [28] \) and \([29]\).

We will prove in subsection 2.3 the existence of \( \varphi \) with positive measure for non-uniformly expanding transformations of compact Riemannian manifolds with non-degenerate criticalities and/or singularities.

The ‘disjoint or nested’ property of \( \varphi \) gives an elegant solution to the overlapping of expanding branches which appear naturally in the induction process, allowing for a great simplification of the construction of a GMY structure.

Our crucial observation is that the first return map to a \( \varphi \) is a piecewise hyperbolic map. That is, there exists a decomposition into countably many rectangles \( S_j \) which covers \( \Sigma \) up-to a \( \mu \)-zero set and laminated diffeomorphisms \( F^j : S_j \to U_j \) which maps hyperbolically \( S_j \) onto ‘horizontal’ strips \( U_j \) crossing \( \Sigma \) such that \( F^j|S_j = F^j|\Sigma \). Here hyperbolic means, as usual, that it expands (resp. contracts) uniformly the ‘horizontal’ (resp. ‘vertical’) plaques (resp. Cantor fibers) preserving the laminated structure of \( \Sigma \). This is our main technical lemma 3.1

This structure is replicated by iterations of the first return map. So we take a suitable iterate such that the corresponding hyperbolic branches \( F^j : S_j \to U_j \) projects onto extendable expanding branches \( f^j : B_j \to B \) with good distortion properties. One get thus an induced Markov map \( f^R : \bigcup B_j \to B \) with integrable return times \( R \) choosing the \( S_j \)’s which intersects a suitable unstable plaque \( \Sigma_z \) and then projecting the corresponding branches onto \( B \).

The ‘disjoint or nested’ property of \( \varphi \) has several advantages. In first place, it gives an elegant solution to the overlapping of expanding branches domains appearing in this type of constructions. On the other hand, it provides a partial ordering to the family of expanding branches, proving therefore that there are maximal induced maps. These are precisely the induced Markov maps appearing in \([10]\). However, in contrast with Doob’s approach, our induced maps are not necessarily maximal. Moreover, even though the maximal induced Markov maps associated to non-uniformly expanding rational maps of the Riemann sphere are natural and naturally extendible, this might not be the case in our setting, due to the non conformal character of \( f \). See section 4.

Finally, as we shall see in section 4, the first return map \( F^R : \Sigma \to \varphi \) is a “horseshoe with infinitely many branches” but, even though ‘vertical rectangles’ are mapped hyperbolically onto ‘horizontal rectangles’ they do not return at the same time. This correct a gap in the proof and statement of \([30], \text{Lemma } 4.1\].

1.3. Overview. In section 2 we collect some technical preliminary results which will be used in the proof of theorem 1.1. In subsection 2.1 we prove some technical lemmas on slowly varying \( \epsilon \)-tempered functions. In subsection 2.2 we use non-degeneracy conditions and regularity of \( \mu \).
to show that preorbits approaches the critical/singular set with subexponential speed. This is important in the local unstable manifolds existence theorem. **Subsection 2.3** is dedicated to state and outline the proof of the local unstable manifold theorem for the non-uniformly expanding maps in consideration. **Subsection 2.4** establishes a tempered estimate for distortion of forward and backward iterates of local unstable manifolds, using the non-degeneracy conditions. In **subsection 2.5** we prove the existence of rectangles with regular returns. In **subsection 2.6** we construct an increasing generating measurable partition \( \xi \) of \( Z \) called Pesin partition and recall the densities of conditional measures of \( \tilde{\mu} \) wrt \( \xi \). Then we use Rochlin’s decomposition theorem to prove that the push-forward \( \pi^*\tilde{\mu}_\Sigma \) of the restriction of \( \tilde{\mu} \) to \( \Sigma \) is equivalent to the Riemannian volume. This is a crucial step in the proof of theorem 1.1. Then we prove that return times of maximal induced maps is integrable. Finally, in **section 4** we discuss the structure of ‘horseshoe with infinitely many branches’ of the first return map to \( \Sigma \).

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2. Preliminary technical results

**Definition 2.1.** We say that a Borel function \( \phi > 0 \) is \( \epsilon \)-tempered for \( F \) if

\[
\left( 11 \right) \quad e^{-\epsilon} \leq \frac{\phi(F(z))}{\phi(z)} \leq e^\epsilon
\]

**Lemma 2.1.** Let \( \phi : Z \to (0, +\infty) \) be a Borel function such that

\[
\liminf_{n \to \pm\infty} \frac{\log \phi(F^n(z))}{n} = 0 \quad \text{(resp. } \limsup \text{)} \quad \tilde{\mu} - \text{c.t.p. } z \in Z_0.
\]

Then, for every \( \epsilon > 0 \) small, there are \( \epsilon \)-tempered Borel functions \( \phi_\epsilon > 0 \) (resp. \( \tilde{\phi}_\epsilon > 0 \)) such that

\[
\left( 12 \right) \quad \phi_\epsilon \leq \phi \quad \text{(resp. } \tilde{\phi}_\epsilon \leq \tilde{\phi} \text{)} \quad \tilde{\mu} - \text{c.t.p. } z \in Z_0.
\]

**Proof.** For every \( \epsilon > 0 \) small and \( \tilde{\mu} \)-a.e. \( Z \in Z \) there exists \( N = N(\epsilon, z) \) such that

\[
e^{-\epsilon |n|} \leq \inf_{|k| \geq n} \phi(F^n(z)) \leq e^{\epsilon |n|} \quad \forall \ n \geq N.
\]
Therefore, we define
\[
\phi(\epsilon)(z) = \inf_{n \in \mathbb{Z}} \phi(F^n(z)) e^{[n] \epsilon}
\]
(24) and (25). This is a well defined and a positive Borel function. Then,
\[
\phi(\epsilon)(F(z)) = \inf_{n \in \mathbb{Z}} \phi(F^{n+1}(z)) e^{[n] \epsilon} = \inf_{m \in \mathbb{Z}} \phi(F^m(z)) e^{[m-1] \epsilon}.
\]
as \( |m| - 1 \leq |m - 1| \leq |m| + 1, \forall \ m \in \mathbb{Z} \), clearly we have
\[
e^{-\epsilon} \phi(\epsilon)(z) \leq \phi(\epsilon)(F(z)) \leq e^{\epsilon} \phi(\epsilon)(z)
\]
Similarly, if
\[
\limsup_{n \to \pm \infty} \frac{\log \phi(F^n(z))}{n} = 0, \ \bar{\mu} - c.t.p. \ z \in Z_0
\]
we have that for almost every \( z \in Z_0 \) there is \( N = N(\epsilon, z) \) such that
\[
e^{-\epsilon[n]} \leq \sup_{|k| \geq n} \phi(F^k(z)) \leq e^{\epsilon[n]} \ \forall \ n \geq N.
\]
Then we define
\[
\bar{\phi}(\epsilon)(z) = \sup_{n \in \mathbb{Z}} \phi(F^n(z)) e^{-[n] \epsilon}
\]
and we prove in a similar manner that \( \bar{\phi} \) is an \( \epsilon \)-tempered positive Borel function. \( \square \)

As an immediate corollary we get that, if
\[
\lim_{n \to \pm \infty} \frac{\log \phi(F^n(z))}{n} = 0, \ \bar{\mu} - c.t.p. \ z \in Z_0,
\]
then there are \( \epsilon \)-tempered positive Borel functions \( \underline{\phi} \) and \( \bar{\phi} \) such that
\[
\underline{\phi} \leq \phi \leq \bar{\phi}.
\]

2.2. Approach to the critical set. From now on, we denote \( C \) the critical and/or singular point set satisfying hypothesis (8) and (10).

The next lemma proves that, generally, the pre-orbits approach with subexponential speed to the critical/singular set \( C \).

Lemma 2.2. Let
\[
\delta_C(z) := d(\pi(z), C).
\]
then,
\[
\lim_{n \to \pm \infty} \frac{\log \delta_C(F^n(z))}{|n|} = 0, \ \bar{\mu} - c.t.p. \ z \in Z_0.
\]
(13)

Proof. Follow from Birkhoff’s theorem since \( \delta_C = \delta_C(z) \) is \( \bar{\mu} \)-integrable by the regularity of \( \mu \) and hypothesis (8) / (10). \( \square \)
2.3. Local unstable manifold scheme.

Theorem 2.1. Let $f$ be a $C^2$ transformation with non degenerate critical/singular set $C$ leaving invariant a regular ergodic expanding Borel probability $\mu$. Then for every $\epsilon > 0$ there exists $\epsilon$-tempered Borel functions $\alpha, \beta, \gamma : Z \to (0, +\infty)$ such that $\mu$-a.e.:

1. the local unstable set
\[
W^u_\alpha(z) = \{ w \in Z : d(z_n, w_n) \leq \beta(z) e^{-n\chi/2} d(z_0, w_0) \quad \forall n \geq 0 \}
\]

is a regular submanifold embedded in $Z$;

2. $\pi|W^u_\alpha(z) : W^u_\alpha(z) \to B(\pi(z), \alpha(z)) \subset M$ is a diffeomorphism;

3. $B(\pi(z), \alpha(z)) \subset B(\pi(z), \delta_C(z)/2)$; in particular $B(\pi(z), \alpha(z))$ does not intersect the critical/singular set $C$;

4. the family of local unstable manifolds is invariant, in that, $W^u_\alpha(f(z)) \subset f(W^u_\alpha(z))$ and $f^n : \pi(F^{-n}W^u_\alpha(z)) \to W^u_\alpha(z)$ is an expanding diffeomorphism, modulated by $\gamma(z)$, i.e.
\[
\inf_{w \in \pi^{-1}[W^u_\alpha(z)]} \| (f^n)^\prime(\pi(w))v \| \geq \gamma(z) e^{n\chi/2} \| v \|
\]

for every $v \in T_{\pi(w)} M$ and for every $n \geq 0$.

This result is a well known result. See for example [10 Proposition 14], [20 Theorem 8], [31 Theorem 6.1], [Ruelle - Shub, 1980], [37 Theorem 1]. For the sake of completeness we outline a proof. Details will appear elsewhere.

Outline of a proof. We shall prove that there are Borel functions $\alpha$ and $\beta$ with subexponential growth along the orbits of $F$ and $C^2$ diffeomorphisms
\[
\Phi_z : B(\pi(z), \alpha(z)) \to W^u_\alpha(z).
\]

Here $\Phi_z(x) = (f_z^{-n}(x))_{n \geq 0}$ where $f_z^{-n} : B(\pi(z), \alpha(z)) \to M$ is the (unique) local branch of $f^{-n}$ such that $f_z^{-n}(z_0) = z_n$, for every $n \geq 0$.

For this we first choose Lyapunov’s charts $\psi_z : B(0, \rho(z)) \to M$ for $\mu$-a.e. $z \in Z$ such that, if $\tilde{f}_z = \psi^{-1}_{f(z)} \circ f \circ \psi_z$ is the local representative of $f$ then
\[
e^{-\chi/2} \| u - v \| \leq \| \tilde{f}_z(u) - \tilde{f}_z(v) \| \leq e^{3/2\chi^+} \| u - v \|,
\]

for every $u, v \in B(0, \rho(z))$, where $\chi > 0$ (resp. $\chi^+$) are the least (resp. largest) Lyapunov exponent of $\mu$ and $\| \cdot \|$ is the norm defined by the standard inner product of $\mathbb{R}^m$.

Then $\tilde{f}_z^{-n} = \psi^{-1}_{f^{-n}(z)} \circ f_z^{-n} \circ \psi_z$, the local representative of $f_z^{-n}$, is a contraction on a suitable domain, that is,
\[
e^{-3/2n\chi^+} \leq \| \tilde{f}_z^{-n}(u) - \tilde{f}_z^{-n}(v) \| \leq e^{-n\chi/2} \| u - v \|,
\]

for every $u, v \in B(0, \rho_z(z))$, where $\rho_z \leq \rho$ is a positive $\epsilon$-tempered Borel function provided by lemma 2.1.

The distortion between the standard metric of $\mathbb{R}^m$ and the Riemannian metric in $M$ due to the charts $\psi_z$ can be estimated as follows:
\[
C d(\psi_z(x), \psi_z(y)) \leq \| x - y \| \leq D(z) d(\psi_z(x), \psi_z(y)),
\]

for every $x, y \in B(0, \rho(z))$, where $C > 0$ is a suitable constant and $D(z)$ a positive Borel function with subexponential growth along the orbits of $F$.

We use (18) and lemma 2.1 to define
\[
\alpha(z) := \min \left\{ \frac{\rho_z/2(z)}{D_z/2(z)}, 2\chi/2(z) \right\}
\]
where \( \rho_{l/2} \leq \rho \) and \( \delta_{l/2} \geq D \) are \( \epsilon/2 \)-tempered functions and \( \delta_{l} \leq \delta_{2l/2} \) is an \( \epsilon \)-tempered function. Then, \( B(\pi(z), \alpha(z)) \subset \psi_{l}(B(0, \rho(z))) \) and condition (3) in the theorem 2.1 holds true.

To get estimative (14) we use (17) and (18) and define

\[
\beta(z) := \frac{\bar{D}_{l}(z)}{C}
\]

where \( \bar{D}_{l} \geq D \) is an \( \epsilon \)-tempered positive Borel function given by lemma 2.1.

The construction of Lyapunov’s charts is standard. See [3] and [18], for instance. The first step is to define an invertible linear cocycle \( \mathcal{L} \) which covers the natural extension \( F : Z_{0} \to Z_{0} \). \( \mathcal{L} \) is the natural lift of the linear cocycle defined by the derivative \( DF \). \( Z_{0} \) is a totally \( F \)-invariant set of total measure, formed by the preorbits \( z \in Z \), which do not intersect the critical/singular set \( \mathcal{C} \).

By Pesin’s \( \epsilon \)-reduction theorem there exists a tempered linear change of coordinates, \( C_{\epsilon}(z) : \mathbb{R}^{m} \to \mathbb{R}^{m} \) which transforms \( \mathcal{L} \) into a uniformly expanding linear cycle \( A_{\epsilon}(z) \) in \( \mathbb{R}^{m} \) such that:

\[ e^{\chi+\epsilon}v \leq ||A_{\epsilon}(z)v|| \leq e^{\chi+\epsilon^2}, \quad \mu - \text{a.e. } z \in Z_{0}. \]

To do this we define a Lyapunov metric \( | \cdot |_{\epsilon} \) by averaging the iterates of the linear cocycle \( \mathcal{L} \) weighted with a suitable kernel. Then we get a linear isometry \( C_{\epsilon} : \mathbb{R}^{m} \to T_{\pi(z)}M \) between \( (\mathbb{R}^{m}, || \cdot ||) \) and \( (T_{\pi(z)}M, | \cdot |_{\epsilon}) \). See [15] Theorem S.2.10.

Lyapunov’s charts are the composition \( \psi_{\epsilon} := \exp_{\pi(z)} \circ C_{\epsilon}(z) \) of the linear coordinate change \( C_{\epsilon}(z) \) and a geodesic chart in \( M \) defined at a (uniform) open ball in the tangent space \( T_{\pi(z)}M \). Given \( \epsilon > 0 \) we choose \( \rho(z) > 0 \) in such way that the representative of \( f \) in this local coordinate system is a \( C^{2} \) nonlinear perturbation of the expanding linear map \( A_{\epsilon}(z) : \mathbb{R}^{m} \to \mathbb{R}^{m} \) with small \( C^{1} \) norm. This chart satisfies the distortion estimates [18]. See [15] Theorem S.3.1. Then, for a well chosen and sufficiently small \( \epsilon > 0 \) the local representative \( f_{\epsilon} \) (resp. \( f_{\epsilon}^{-\epsilon} \)) satisfies [15] (resp. [17]).

Remark 2.1. By [17], [18] and (20) it holds

\[
\frac{1}{\beta(z)} e^{\chi x/2}d(x,y) \leq d(f^{n}(x), f^{n}(y)) \leq \beta(z) e^{2\chi x}d(x,y),
\]

\( \forall \ x, y \in \pi(F^{-n}(W^{s}_{\alpha}(z))) \), for \( \mu \)-a.e. \( z \in Z \). Indeed, let \( \epsilon > 0 \) sufficiently small such that \( 3/2\chi + \epsilon < 2\chi + \epsilon \) and \( \bar{D}_{l} \geq D \) be an \( \epsilon \)-tempered function provided by lemma 2.1. Then, for every \( x, y \in \pi(F^{-n}(W^{s}_{\alpha}(z))) \)

\[
d(f^{n}(x), f^{n}(y)) \leq \frac{\bar{D}_{l}(F^{-n}(z))}{C} e^{3\chi n/2}d(x,y)
\]

\[
\leq \frac{\bar{D}_{l}(F^{-n}(z))}{C} e^{\chi n/2}d(x,y)
\]

\[
\leq \frac{\bar{D}_{l}(z)}{C} e^{3\chi n/2}d(x,y)
\]

\[
\leq \frac{\bar{D}_{l}(z)}{C} e^{2\chi n}d(x,y).
\]

Likewise the lower bound.

2.4. Tempered Distortion. Let us denote \( JF(z) = Jf(\pi(z)) \) the Jacobian of \( F|W^{s}_{\alpha}(z) \), putting in \( W^{s}_{\alpha}(z) \) the volume element obtained as a push-forward of the Riemanian volume element in \( M \) by the parameterization \( \Phi_{z} : B(\pi(z), \alpha(z)) \to W^{s}_{\alpha}(z) \).
Lemma 2.3. There exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ there is an $\epsilon$-tempered Borel function $\gamma_\epsilon : Z \to (0, +\infty)$, such that

$$\limsup_{i \to +\infty} \frac{J F(F^{-i}w)}{J F(F^{-i}w')} \leq \exp\{\gamma_\epsilon(z)d(w, w')\},$$

for every $w, w' \in W^u_\alpha(z)$ and $n \geq 0$,

$$\frac{|J F^m(w)|}{|J F^m(w')|} \leq \exp\{\gamma_\epsilon(z)d(f^n(\pi(w)), f^n(\pi(w')))\}$$

for every $w, w' \in F^{-n}W^u_\alpha(z)$.

See [30, Lemma 3.3]. In particular, for every $0 < k < n$, $w, w' \in F^{-n}W^u_\alpha(z)$:

$$\frac{|J F^k(w)|}{|J F^k(w')|} \leq \exp\{\gamma_\epsilon(z)\beta(z)\lambda^{n-k}d(f^n(\pi(w)), f^n(\pi(w')))\}$$

with $\lambda = e^{-\chi/3} \in (0, 1)$.

Proof. By the hypothesis (9) and the estimate (14) in the theorem (2.1), for every $w, w' \in F^{-n}W^u_\alpha(z)$:

$$\log\frac{|J F^{m}(w)|}{|J F^{m}(w')|} \leq \sum_{j=0}^{n-1} |\log J F(F^j(w)) - \log J F(F^j(w'))|$$

$$\leq \sum_{j=0}^{n-1} A \frac{d_M(\pi(F^j(w)), \pi(F^j(w')))}{d_M(\pi(F^j(w)), C)^k}$$

$$\leq \sum_{j=0}^{n-1} 2^k A \frac{\beta(z)e^{-\epsilon/2}d_M(f^n(\pi(w)), f^n(\pi(w')))}{d_M(\pi(F^{-n+j}(z)), C)^k},$$

because

$$d_M(\pi(F^j(w)), C) \geq d_M(\pi(F^{-n+j}(z)), C) - \alpha(F^{-n+j}(z)) \geq 1/2d_M(\pi(F^{-n+j}(z)), C)$$

since $\alpha(z) < 1/2\delta_\epsilon(z)$ and $F^j(w) \in W^u_\alpha(F^{-n+j}(z))$. By lemma (2.2) and lemma (12) for every $\epsilon > 0$ there exist $\epsilon/2k$-tempered functions $\delta^*(z) = \delta_{1/2k}(z) \geq \delta_\epsilon(z)$, for $\mu$-a.e. Therefore,

$$\frac{1}{d_M(\pi(F^{-n+j}(z)), C)} \leq \frac{1}{d_M(\pi(F^{-n+j}(z)), C)} \frac{\delta^*(z)}{\delta^*(z) \delta^*(F^{-n+j}(F^n(z)))} \leq \frac{e^{\epsilon/2k}}{\delta^*(z)}.$$
Let $\tilde{\beta} \geq \beta$ be an $\epsilon/2$-tempered Borel function given by lemma 21. Then, choosing $\epsilon_0 > 0$ such that $\chi/3 < \chi/2 - \epsilon/2k$ whenever $0 < \epsilon < \epsilon_0$, we have

$$\log \left| \frac{JF^n(w)}{JF^n(w')} \right| \leq \frac{2^k A \tilde{\beta}(z)}{(\delta^*(z))^k} d_M(f^n(\pi(w)), f^n(\pi(w'))) \sum_{j=0}^{n-1} e^{-(n-j)(\chi/2 - \epsilon/2k)}$$

$$\leq \frac{2^k A \tilde{\beta}(z)}{(\delta^*(z))^k} d_M(f^n(\pi(w)), f^n(\pi(w'))) \sum_{j=0}^{n-1} e^{-(n-j)(\chi/2 - \epsilon/2k)}$$

$$\leq \frac{2^k A \tilde{\beta}(z)}{(\delta^*(z))^k} d_M(f^n(\pi(w)), f^n(\pi(w'))) \sum_{n=0}^{+\infty} e^{-n\chi/3}$$

$$\leq \frac{2^k A}{1 - e^{-\chi/3}} \frac{\tilde{\beta}(z)}{(\delta^*(z))^k} d_M(f^n(\pi(w)), f^n(\pi(w')))$$

Now, $\beta$ is $\epsilon/2$-tempered and $\delta^*$ is $\epsilon/2k$-tempered then

$$\gamma_d(z) := \frac{2^k A}{1 - e^{-\chi/3}} \frac{\tilde{\beta}(z)}{(\delta^*(z))^k},$$

is $\epsilon$-tempered, proving (23). Similarly,

$$\log \prod_{i=0}^{+\infty} \frac{JF(F^{-i}w)}{JF(F^{-i}w')} \leq \sum_{i=0}^{+\infty} \frac{2^k A \beta(z) e^{-ix/2} d_M(\pi(w), \pi(w'))}{d_M(\pi(F^{-i}z), C)^k}$$

$$\leq \frac{2^k A \beta(z)}{(\delta^*(z))^k} d_M(\pi(w), \pi(w')) \sum_{i=0}^{+\infty} e^{-i(\chi/2 - \epsilon/2k)}$$

$$\leq \frac{2^k A \beta(z)}{(\delta^*(z))^k} d_M(\pi(w), \pi(w')) \sum_{i=0}^{+\infty} e^{-i\chi/3}$$

$$= \frac{2^k A}{1 - e^{-\chi/3}} \frac{\beta(z)}{(\delta^*(z))^k} d_M(\pi(w), \pi(w'))$$

so proving (22). \hfill \Box

2.5. Rectangles with regular returns. By the non-uniform expanding character of the dynamics to $(f, \mu)\ Z$ produces a measurable lamination by local unstable manifolds, that is, we can cover $Z$, up to a zero measure set, by 'flow boxes' which are homeomorphic to the cartesian product of a Cantor set and an open subset of $M$. We call rectangles to these 'flow boxes'. Let us state this more precisely. For this we recall that $B \subset M$ is an open disc if it is diffeomorphic to an open ball $B(x, r) \subset M$.

Definition 2.2. Let $B \subset M$ an open disc and $K \subset F(z) := \pi^{-1}(\pi(z))$ a non trivial Cantor subset. We say that $\Sigma \subset Z$ is a rectangle with base $B$ and height $K$ if there exists a homeomorphism $\Phi : \Sigma \to B \times K$ such that

\[ \forall w \in K : \quad \pi_w := \pi|_{\Sigma_w} : \Sigma_w \to B, \text{ is a diffeomorphism onto } B. \]

where $\Sigma_w = \Phi^{-1}(B \times \{w\})$ is the 'plaque' of 'height' $w \in K$.

$\Sigma$ is a regular rectangle if the cross sections $K_w := \pi^{-1}(\pi(w)) \cap \Sigma$ are homeomorphic to each other by holonomy, that is,

\[ \phi_{w,w'} : K_w \to K_{w'} \quad \text{is a homeomorphism } \forall \ w, w' \in \Sigma, \]

where $\phi_{w,w'}(w'') = \Sigma_{w''} \cap K_{w'}$, and $\Sigma_{w''}$ is the plaque containing $w'' \in K_w$. 

Remark 2.2. In particular, if \( \Sigma \) it is regular, \( \Phi : \Sigma \longrightarrow B \times K \), defined as \( \Phi(w) = (\pi(w), \phi_{w,z}(w)) \) is a homeomorphism.

**Important warning:** All the rectangles appearing in our construction are regular, so we shall omit to mention this in our statements, unless notice in contrary.

Remark 2.3. It follows from the local unstable manifold theorem that \( Z \) has a measurable laminated structure: for every \( z \in M \) there exists an open ball \( B_z \subset M \) such that there exists an invertible measurable map with measurable inverse, \( \phi : \pi^{-1}(B_z) \rightarrow B_z \times \mathcal{F} \). Moreover, for every \( n > 0 \) there exists a compact set \( Z_n \subset Z \) with \( \mu(Z_n) \geq 1 - 2^{-n} \) such that \( Z_n \) is covered by rectangles \( \Sigma_i \) which are domains of local charts \( \Phi_i : \Sigma_i \rightarrow B_i \times K_i \), where \( \Phi_i \) are homeomorphisms. For this we just take \( Z_n \) a large compact set such that \( \alpha \) and \( z \mapsto W^u_\alpha(z) \) restricted to \( Z_n \) are continuous.

**Definition 2.3.** We say that a rectangle \( \Sigma \subset Z \) of base \( B \) and fiber \( K \subset \mathcal{F}(z) \) has the property of \textit{regular returns} if for every \( z, z' \in \Sigma \) and for all \( n, m > 0 \):

1. \( F^{-n}\Sigma_z \) and \( F^{-m}\Sigma_{z'} \) (resp. \( \pi F^{-n}\Sigma_z \) and \( \pi F^{-m}\Sigma_{z'} \)) are disjoint or nested;
2. \( \pi|F^{-n}\Sigma_z : F^{-n}\Sigma_z \rightarrow \pi F^{-n}\Sigma_z \) is a diffeomorphism;
3. for every \( z \in \Sigma \) and every \( n > 0 \), \( f^n : \pi F^{-n}\Sigma_z \rightarrow B \) is an expanding diffeomorphism with bounded volume distortion, that is, there exists \( C > 0 \) such that for every \( z \in \Sigma \):

\[
\|Df^n(\pi(w))v\| \geq C^{-1}e^{\alpha n/2}\|v\|,
\]

for every \( v \in T_{\pi(w)}M \) and \( w \in F^{-n}\Sigma_z \);

\[
\frac{Jf^n(\pi(w))}{Jf^n(\pi(w'))} \leq \exp(Cd(f^n(\pi(w)), f^n(\pi(w'))), \forall w, w' \in F^{-n}\Sigma_z,
\]

for every \( w, w' \in F^{-n}\Sigma_z \);

\[
\prod_{i=0}^{\infty} \frac{Jf(\pi(F^{-i}w))}{Jf(\pi(F^{-i}w'))} \leq \exp(Cd(\pi(w), \pi(w'))),
\]

for every \( w, w' \in F^{-n}\Sigma_z \).

See [10] Definition 23.

**Lemma 2.4.** Let \((f, \mu)\) be a dynamical system satisfying the hypotheses of the theorem [13]. Then there exists a rectangle with regular returns \( \Sigma \subset Z \) with positive measure.

Essentially this was proved in [10] Theorem 24, so we simply outline main steps of the proof and refer to that paper for details.

**Proof.** The idea is to prove that there exists an open disc \( B \) and a rectangle of positive measure \( \Sigma \) over \( B \) such that

\[
\forall z \in \Sigma : \pi(F^{-n}W(z)) \cap \partial B = \emptyset, \forall n \geq 0,
\]

for it follows easily from the above condition that for every \( z, z' \in \Sigma \) and for every \( n, m \geq 0 \), \( F^{-n}B_z \) (resp. \( \pi(F^{-n}B_z) \)) and \( F^{-m}B_{z'} \) (resp. \( \pi(F^{-m}B_{z'}) \)) are disjoint or nested, where \( B_z = \pi^{-1}(B) \) is a diffeomorphic copy of \( B \) in the unstable plaque \( W^u(z) \).

We start choosing a compact set \( \Sigma_0 \) with \( \mu(\Sigma_0) \geq 1 - \delta \) such that they are continuous the local unstable manifolds \( W^u_\alpha \) and the functions \( \alpha, \beta, \gamma \) and \( \gamma_0 \) in theorem 2.1 and lemma 2.3 vary continuously with \( z \in \Sigma_0 \). This is possible by Egorov-Lusin’s theorem.

In particular, there exists \( C > 0 \) such that for every \( z \in \Sigma_0 \):

\[
\|Df^n(\pi(w))v\| \geq C^{-1}e^{\alpha n/2}\|v\|
\]
for every $v \in T_{\pi(w)}M$ and $w \in F^{-n}W^u_\alpha(z)$;

$$
\left( Jf^n(\pi(w)) \right) \leq \exp(Cd(f^n(\pi(w)), f^n(\pi(w'))),
$$

for every $w, w' \in F^{-n}W^u_\alpha(z)$ by

$$
\prod_{i=0}^{\infty} \left( \frac{Jf(\pi(F^{-i}w))}{Jf(\pi(F^{-i}w'))} \right) \leq \exp(Cd(\pi(w), \pi(w'))),
$$

for every $w, w' \in F^{-n}W^u_\alpha(z)$. For this it suffices to take $C = \max\{\gamma(z)^{-1}, \gamma_d(z)\}$.

Let us denote $W(z) = W^u_\alpha(z)$ and let $\alpha_0 > 0$ be the minimum of $\alpha|\Sigma_0$. We choose $x \in M$ such that

$$
\Sigma_1 = \pi^{-1}(B(x, \alpha_0/2)) \cap \bigcup_{z \in \Sigma_0} W(z)
$$

has positive measure. Then $\Sigma_1$ is a rectangle.

Now we prove that there exists an open ball $B \subset B(x, \alpha_0/2)$ and rectangle $\Sigma_2 \subset \Sigma_1$ over $B$, with positive measure and $N_0 > 0$ such that (27) holds for every $n \geq N_0$.

First Claim: there exists $\alpha_0/8 < r < \alpha_0/4$, an integer $N_0 > 0$ and a subset $\Sigma_{0,N_0} \subset \Sigma_0$ of large measure such that,

$$
\forall \ z \in \Sigma_{0,N_0} : \text{diam}(\pi(F^{-n}W(z))) < d(\pi(F^{-n}(z)), \partial B(x, r))/2, \forall \ n \geq N_0.
$$

By (10) Lemma 21] $d(\pi(F^{-n}(z)), \partial B(x, r))$ decays subexponentially for $\mu$-a.e. for Lebesgue almost every $r \in (\alpha_0/8, \alpha_0/4)$. Fix one such $r$. Then for any small $\epsilon > 0$ there exists a Borel function $\delta$ with subexponential growth such that

$$
d(\pi(F^{-n}(z)), \partial B(x, r)) \geq \delta(z)e^{-\epsilon n}.
$$

By the local unstable manifold theorem the diam $d(\pi(F^{-n}W(z)))$ decays exponentially. Moreover, by (21) and definition (20) of $\beta$,

$$
\text{diam}(\pi(F^{-n}W(z))) \leq \beta(z)\alpha(z)e^{-n\chi/2}, \forall n > 0.
$$

To find $\Sigma_{0,N_0}$ we let $0 < \eta < \chi$ be sufficiently small, define $\epsilon = \chi/2 - \eta/2$, $\alpha_{\max} = \max\alpha|\Sigma_0$, $\beta_{\max} = \max\beta|\Sigma_0$ and observe that

$$
\text{diam}(\pi(F^{-n}W(z))) < d(\pi(F^{-n}(z)), \partial B(x, r))/2 \quad \text{whenever} \quad \beta_{\max}\alpha_{\max} < \frac{\delta(z)e^{\epsilon n/2}}{2}.
$$

We define a non decreasing sequence of Borel sets

$$
X(n) = \left\{ z \in Z : \beta_{\max}\alpha_{\max} < \frac{\delta(z)e^{\epsilon n/2}}{2} \right\}
$$

covering $Z$ up-to a $\mu$-zero set. Let $\Sigma_{0,n} := \Sigma_0 \cap X(n)$ and choose $N_0 > 0$ sufficiently large such that $\mu(\Sigma_{0,N_0}) \geq (1 - \delta)\mu(\Sigma_0)$. This proves the first claim. Therefore,

$$
\Sigma_2 = \pi^{-1}(B(x, r)) \cap \bigcup_{z \in \Sigma_{0,N_0}} W(z)
$$

is a rectangle $\Sigma_2 \subset \Sigma_1$ with positive measure such that

$$
\forall \ z \in \Sigma_2 : \pi(F^{-n}W(z)) \cap \partial B(x, r) = \emptyset, \forall n \geq N_0.
$$

Now, notice that (31) is equivalent to the following: if $n \geq 0$ and there exists $k > 0$ such that

$$
\pi(F^{-n-k}W(z)) \cap \partial \pi(F^{-n}B^u_\alpha) \neq \emptyset.
$$
then $k \leq N_0$. Therefore, for each $z \in \Sigma_2$ there exists a non decreasing sequence of positive integers $\{n_j(z)\}$ such that, for every $j > 0$ there exists $0 < k_j \leq N_0$ with

$$\pi(F^{-n_j(z)-k_j}W(z)) \cap \partial\pi(F^{-n_j(z)}B'_z) \neq \emptyset.$$ 

The sequence $\{n_j(z)\}$ is either finite or infinite.

**Second Claim:** \[
\{n_j(z)\} \text{ is finite } \mu\text{-a.e. } z \in \Sigma_2.
\]

Indeed, let us suppose that $C \subset \Sigma_2$ is a Borel subset with positive measure such that $\{n_j(z)\}$ is infinite for every $z \in C$. By [10] Lemma 22 there exists a continuous positive function $\theta : [0, +\infty) \to [0, +\infty)$ with $\theta(0) = 0$ such that for every set $V$ such that $V \cap f^{-k}(V) \neq \emptyset$ for some $0 < k \leq N$ then $\text{dist}(V, \text{Fix}(f^{N_1})) < \theta(\text{diam}(V))$. As $\text{diam}(\pi(F^{-n}W(z))) \to 0$ exponentially then

$$\text{dist}(\pi(F^{-n}W(z)), \text{Fix}(f^{N_1})) \to 0,$$

uniformly in $z \in C$. Hence, for every $\epsilon > 0$

$$\mu(B(\text{Fix}(f^{N_1}), \epsilon)) = \mu(\pi^{-1}(B(\text{Fix}(f^{N_1}), \epsilon))) \geq \mu(F^{-N}C) = \mu(C) > 0.$$

Therefore $\mu$ must be concentrated on a repelling periodic orbit contradicting that $\mu$ is absolutely continuous. This proves the Second Claim. Let $Y_N = \{z \in \Sigma_2 : \max_j n_j(z) \leq N\}$. Then there exists $N_1 > 0$ such that $\mu(Y_{N_1}) \geq (1 - \delta)\mu(\Sigma_2)$. Hence

$$\Sigma_3 = \pi^{-1}(B') \cap \bigcup_{z \in Y_{N_1}} W(z)$$

is a subrectangle of $\Sigma_2$ with $\bar{\mu}(\Sigma_3) \geq (1 - \delta)\bar{\mu}(\Sigma_2)$ such that

$$\forall z \in \Sigma_3 : \pi(F^{-n-k}W(z)) \cap \partial\pi(F^{-n}B'_z) = \emptyset, \forall n \geq N_1, k > 0.$$

Notice that $F^{-N_1}(\Sigma_3)$ is a rectangle satisfying (24), (25) and (20). Let $B$ be a connected component of $f^{-N_1}(B')$ such that

$$\Sigma = \pi^{-1}(B') \cap F^{-N_1}(\Sigma_3)$$

has positive measure. $\Sigma$ is a rectangle satisfying (24), (25) and (20) and (27) and has therefore the regular returns property. \qed

### 2.6. Pesin partitions and density of conditional measures.

As $\bar{\mu}(\Sigma) > 0$, by the ergodicity of $\mu$, there exists for $\mu$-a.e. $z \in Z$ a first time of entry $e(z) > 0$ of the positive orbit of $z$ into $\Sigma$:

$$e(z) = \min\{n > 0 : F^n(z) \in \Sigma, F^k(z) \notin \Sigma, 0 \leq k < n\}$$

We define a partition $\xi$ of $Z$ as

$$\xi(z) = F^{-e(z)}\Sigma_{F^e(z)}, \text{ if } z \notin \Sigma \text{ and } \xi(z) = \Sigma_2, \text{ if } z \in \Sigma.$$

**Lemma 2.5.** (1) $\xi$ is a measurable partition of $Z$ subordinate to the unstable lamination of $Z$, this is $\xi(z) \subset W^u_n(z)$, $\mu$-a.e.;

(2) $\xi$ is increasing: $F^{-1}\xi \geq \xi$: $F^{-1}(P) \cap Q = \emptyset$ implies $F^{-1}(P) \subset Q$, for every $P, Q \in \xi$;

(3) $\xi$ generate the $\sigma$-algebra of Borelians of $Z$: $\bigvee_{n=0}^{+\infty} F^{-n}\xi = \epsilon$, where $e(x) = \{x\}$ is the largest partition of $Z$ into points;

(4) $\pi|\xi(z): \xi(z) \to M$ is a diffeomorphism onto its image;

(5) $F^e(z): \xi(z) \to \Sigma_{F^e(z)}$ is projected onto a diffeomorphism $f^e(z): \pi\xi(z) \to B$ satisfying (24), (25) and (27).

See [20 Proposition 3.2]. Compare also [10] Section 4 and Section 6.

**Definition 2.4.** $\xi$ is a Pesin partition for $(f, \mu)$. 
Let $\xi(z)$ the atom of the Pesin partition $\xi$ which contains $z \in Z$. By Rochlin's theorem, there is a system of conditional measures $\bar{\mu}_\xi(z)$ such that
\[
\bar{\mu}(A) = \int_{Z/\xi} \bar{\mu}_\xi(z) (A \cap \xi(z)) d\bar{\mu}_\xi(\xi(z)),
\]
where $\bar{\mu}$ is the quotient measure on $X/\xi$, that is,
\[
\int_Z \phi(w) d\bar{\mu}(w) = \int_{Z/\xi} \left( \int_{\xi(z)} \phi(w) d\bar{\mu}_\xi(z)(w) \right) d\bar{\mu}_\xi(\xi(z)),
\]
for every Borel measurable real function $\phi$.

Proof. Let $\phi : B(\pi(z), \alpha(z)) \to W^u(z)$ be the parametrization of the local unstable manifolds $W^u(z)$ containing $\xi(z)$.

Since $\mu$ is absolutely continuous with respect to the Riemannian volume, then it satisfies the Pesin-Rochlin formula $h(\mu) = \sum_{\chi(z) > 0} \lambda_i(x) \dim E_i(x) d\mu(x)$. This allows to compute explicitly the density of the conditional measures $\bar{\mu}_\xi(z)$ with respect to the volume $\Vol \bar{\mu}_\xi(z)$.

\textbf{Lemma 2.6.}

(34)
\[
\bar{\mu}_\xi(z)(A) = \frac{\Delta(z,w) \Vol \bar{\mu}_\xi(z)(w)}{\int_{\xi(z)} \Delta(z,w) \Vol \bar{\mu}_\xi(z)(w)},
\]
for every Borel subset $A \subset \xi(z)$, where
\[
\Delta(z,w) = \prod_{i=0}^{+\infty} \frac{\int \phi(F^{-i}w)}{\int \phi(F^{-i}z)}.
\]

See \cite{20} Proposition 3.6] and \cite{10} Proposition 32].

Let $\pi^* \bar{\mu}_\Sigma(A) = \bar{\mu}_\Sigma(\pi^{-1}(A))$ be the push-forward of $\bar{\mu}_\Sigma$, the restriction $\bar{\mu}\Sigma$ normalized to a probability. The following shall be used later.

\textbf{Lemma 2.7.} Let $\Sigma$ be a rectangle with regular returns. Then, there are constants $C_0, C_1 > 0$ such that,

(35)
\[
C_0 \Vol(A) \leq \pi^* \bar{\mu}_\Sigma(A) \leq C_1 \Vol(A) \quad \text{for every Borel subset } A \subset B.
\]

Proof. Let $\{\bar{\mu}_\xi(z)\}$ be the Rochlin decomposition of $\bar{\mu}\Sigma$ with respect to $\xi$:
\[
\bar{\mu}(\pi^{-1}(A) \cap \Sigma) = \int_{Z/\xi} \bar{\mu}_\xi(z)(\pi^{-1}(A)) d\bar{\mu}_\xi(\xi(z)).
\]

Then, from \textbf{(34)} in lemma \textbf{2.6} and \textbf{2.9} in the definition of a rectangle with regular returns we have
\[
e^{-2c\diam(B)\Vol(A)} \Vol(B) \leq \bar{\mu}_\xi(z)(\pi^{-1}(A)) \leq e^{2c\diam(B)\Vol(A)} \Vol(B),
\]
for every Borel subset $A \subset B$. Therefore, using the Rochlin decomposition,
\[
C_0 \Vol(A) \leq \bar{\mu}_\Sigma(\pi^{-1}(A)) \leq C_1 \Vol(A),
\]
for every Borel subset $A \subset B$, where
\[
C_0 := \frac{e^{-2c\diam(B)\bar{\mu}(\Sigma/\xi)}}{\Vol(B)\bar{\mu}(\Sigma)} \quad \text{and} \quad C_1 := \frac{e^{2c\diam(B)\bar{\mu}(\Sigma/\xi)}}{\Vol(B)\bar{\mu}(\Sigma)}.
\]

\hfill $\Box$
3. Proof of theorem \[1.1\]

Let \( \Sigma \subset Z \) be a rectangle with regular returns and \( \bar{\mu}(\Sigma) > 0 \) and \( \tau : \Sigma \to \Sigma \) the first return map, that is, \( \tau(z) = F^{\overline{R}(z)}(z) \), where \( \overline{R}(z) = \min\{n > 0 : F^n(z) \in \Sigma\} \) is the first return times to \( \Sigma \).

**Definition 3.1.** Let \( \Sigma \) a rectangle. We say that a subset \( U \subset \Sigma \) is a \( u \)-rectangle if is a rectangle and \( U \cap \Sigma_w = \Sigma_w \), for every \( w \in U \).

Briefly an \( u \)-rectangle \( U \) is a ‘horizontal strip’ crossing \( \Sigma \) from ‘left to right’.

**Lemma 3.1.** (Main technical lemma) There exists a decomposition of \( \Sigma \) into subrectangles \( S_j \subset \Sigma \) and \( u \)-rectangles \( U_j \subset \Sigma \) and non negative integers \( T_j \in \mathbb{Z}^+ \) such that

1. \( \bar{\mu}(\Sigma - \bigcup_j S_j) = \bar{\mu}(\Sigma - \bigcup_j U_j) = 0 \);
2. \( F^{T_j} : S_j \to U_j \) is a laminated diffeomorphism such that, \( \tau|S_j = F^{\overline{T}_j} \), and

\[
\|D_f^{\overline{T}_j}(\pi(w))\| \geq C^{-1}e^{\overline{T}_j|x/3}\|v\|, \quad \forall v \in T_{\pi(w)}M, \forall w \in S_j(z)
\]

for every \( z \in S_j \) and

\[
\frac{J_f^{\overline{T}_j}(\pi(w))}{J_f^{\overline{T}_j}(\pi(w'))} \leq \exp(Cd(f^{\overline{T}_j}(\pi(w)), f^{\overline{T}_j}(\pi(w')))), \quad \forall w, w' \in S_j(z), z \in S_j,
\]

where \( C > 0 \) is the constant provided by lemma \[2.4\] and \( S_j(z) := S_j \cap \Sigma_z \) is the leave of level \( z \) on the rectangle \( R_j \);
3. for every \( m > 0 \), \#\{\( j > 0 : \overline{T}_j = m \} < +\infty \).

The next well known result will be used in the proof of theorem \[1.1\].

**Kac’s lemma** Let \( T : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu) \) be a measurable, measure preserving transformation of a probability space, \( A \in \mathcal{A} \) a subset with \( \mu(A) > 0 \) and \( R_A(x) = \inf\{n > 0 : T^n(x) \in A\} \) the first return times to \( A \). Then,

\[
\int_A R_A d\mu = 1
\]

See \[27\] Theorem 2.4.6.

**Proof of theorem \[1.1\]**

Let \( \tau : \Sigma \to \Sigma \) the first return map to a rectangle with regular returns. Then, \( \tau^n(z) = F^{\overline{R}_n(z)}(z) \), where

\[
\overline{R}_n(z) = \sum_{k=0}^{n-1} \overline{R}_0(\tau^k(z)), \quad \forall n \geq 0.
\]

**Claim:** for every \( n > 0 \) there exist decompositions \( \mathcal{R}^n = \{S^n_i\} \) (resp. \( U^n = \{U^n_j\} \)) of \( \Sigma \) into subrectangles (resp. \( u \)-rectangles) and non-negative integers \( \overline{T}_j^n \) such that \( \overline{R}_n|S^n_i = \overline{T}_j^n \) and

\[
F^{\overline{T}_j^n} : S^n_i \to U^n_j
\]

is a laminated diffeomorphism satisfying estimates \[30\] and \[37\].

Indeed, by the main technical lemma, there exist collections of subrectangles \( \mathcal{R} \) and \( u \)-rectangles \( U \) of first level and times of return \( \overline{R}_j \) such that \( \tau|S_j = F^{\overline{R}_j} : S_j \to U_j \) is a laminated diffeomorphism satisfying \[30\] and \[37\]. This is the first step of an induction argument.

Now, suppose that we have done the construction up to \( n \)-th step. For every \( j > 0 \), we label \( S_{nij} := U_j \cap S^n_i \) the subrectangles obtained by intersecting \( U_j \) with the subrectangles of \( n \)-th
level in $\mathcal{R}^n$. The collection $\{S_{nij}\}$ covers $U_j$ up-to a $\mu$-zero measure set. Then, we decompose $S^n_j$ into subrectangles $S^n_{ij+1} \subset S^n_j$ of $n + 1$-th generation such that $\tau(S^n_{ij+1}) = S_{nij}$. Let $U^n_{ij+1} := \tau^n(S_{nij})$ be the $n$-th first return of $S_{nij}$. Then $U^n_{ij+1}$ is an $n$-rectangle contained in $U^n_i$ and $\tau^{n+1}(S^n_{ij+1}) = U^n_{ij+1}$ and

$$\mathcal{T}_{n+1}|S^n_{ij+1} = \mathcal{T}_j + \mathcal{T}^n_i := \mathcal{T}^{n+1}_{ij}.$$ Estimations (36) and (37) follows immediately from (24) and (25) in definition 2.3.

Let $N > 0$ be the smallest positive integer such that

$$\lambda := C^{-1} e^{\frac{j}{N^3/3}} > 1$$

and notice that $\mathcal{T}_N(z) \geq N$ for $\mu$-a.e. $z \in \Sigma$. For every $j > 0$, $F^{\mathcal{T}^N_j} : S^N_j \rightarrow U^N_j$ projects onto a diffeomorphism covering $B$, that is,

$$S^N_j \xrightarrow{\pi} U^N_j \xrightarrow{\pi} B$$

such that,

$$\|D\mathcal{T}^N_j(x)v\| \geq \lambda \|v\|, \quad \forall \ v \in T_x M, \ \forall x \in B^N_j$$

and

$$\mathcal{J}\mathcal{T}^N_j(x) \leq \exp(Cd(f\mathcal{T}^N_j(x), f\mathcal{T}^N(x))), \quad \forall x, y \ in B^N_j.$$ In other words, $f\mathcal{T}^N_j : B^N_j \rightarrow B$ is an expanding branch with bounded distortion.

Notice that these branches are extendible: choose $z \in S_j$ and $w = F^{\mathcal{T}^N_j}(z) \in U_j$. The plaque $U_j(w) = U_j \cap \Sigma_w$ is contained in some local stable manifold $W^s_{\alpha}(w')$ and therefore, the plaque of level $z$ in $S_j$, $S_j(z) = S_j \cap \Sigma_z$ is contained in $F^{-\tilde{\mathcal{T}^N_j}}W^s_{\alpha}(w')$. By (28) and (29), estimations (36) and (37) also holds for the extension $f\mathcal{T}^N_j : B_j \rightarrow \hat{B}$, where

$$\hat{B}_j = \pi(F^{-\tilde{\mathcal{T}^N_j}}W^s_{\alpha}(w')) \quad \text{and} \quad \hat{B} = \pi(W^s_{\alpha}(w')).$$

From now on we drop the superscript $N$ on the return times $\mathcal{T}^N_j$, subrectangles $S^N_j$ and $n$-rectangles $U^N_j$ and denote $\mathcal{R} = \{S_j\}$ the family of rectangles of $N$-th generation. Notice that, by Kac’s lemma

$$\int_{\Sigma} \mathcal{T}d\tilde{\mu} = N,$$

since $\mathcal{T} = \mathcal{T}_N$ is the return times of $\tau^N$ and $\tilde{\mu}$ is $\tau$-invariant. Then, by Rochlin’s decomposition theorem,

$$\int_{\Sigma} \mathcal{T}(w)d\tilde{\mu}(w) = \int_{\Sigma/\xi} \mathcal{T}(w)d\tilde{\mu}(z(w)) < +\infty,$$

consequently,

$$\int_{\Sigma} \mathcal{T}(w)d\tilde{\mu}(z(w)) < +\infty,$$

for $\tilde{\mu} - \text{a.e.} \ z \in \Sigma$. Now we choose a family of rectangles

$$\mathcal{R}(z) = \{S_i \in \mathcal{R} : S_i \cap \xi(z) \neq \emptyset\}. $$
Moreover, for $\tilde{\mu}$-a.e. $z \in \Sigma$, 

\begin{equation}
\tilde{\mu}_{\xi}(z) \left( \xi(z) - \bigcup_{j \in \mathcal{R}(z)} S_j \right) = 0.
\end{equation}

This follows from Rochlin’s theorem since 

$$
\tilde{\mu} \left( \Sigma - \bigcup_j S_j \right) = \int_{\Sigma(\xi)} \tilde{\mu}_{\xi}(z) \left( \Sigma - \bigcup_j S_j \right) d\bar{\mu}_{\xi}(\xi(z)) = 0.
$$

Let $z$ be a point in $\Sigma$ satisfying (40) and (41) and let $f^{R_i} : B_i \to B$ be the family of branches so chosen. Then $f^R : \bigcup_i B_i \to B$ is an induced expanding Markov map with bounded distortion and integrable return map. Indeed, as we argue in lemma 2.7 there exists a constant $K_0 > 0$ such that $Vol_{\xi}(A) \leq K_0 \tilde{\mu}_{\xi}(A)$, for every Borel subset $A \subset \xi(z)$. Therefore, 

$$
Vol \left( B - \bigcup_i B_i \right) = Vol_{\xi}(z) \left( \xi(z) - \bigcup_{S_j \in \mathcal{R}(z)} S_j \right) \leq K_0 \tilde{\mu}_{\xi}(z) \left( \xi(z) - \bigcup_{S_j \in \mathcal{R}(z)} S_j \right) = 0.
$$

Let $R$ be the return times of the induced map $f^R : \bigcup_i B_i \to B$. Then $R(\pi(w)) = \overline{R}(w)$ for every $w \in \xi(z)$ and thus, by (40) and lemma 2.7 

$$
\int_B R(x) dVol(x) = \int_{\xi(z)} \overline{R}(w) dVol_{\xi}(w) \leq K_0 \int_{\xi(z)} \overline{R}(w) d\tilde{\mu}_{\xi}(w) < +\infty.
$$

The next arguments are due to Doob. See [10, Proposition 34]. The domains $\{\pi(S_j) : S_j \in \mathcal{R}\}$ associated to $\tau^N : \Sigma \subset$ are partially ordered by inclusion, since $\pi(S_j)$ are disjoint or nested. Then there exists a maximal family $\{B_i\}$ of pairwise disjoint open domains and associated extendable expanding branches with bounded distortion $f^{R_i} : B_i \to B$. We call such $f^R : \bigcup_i B_i \to B$ a maximal induced map. Next proposition proves that a maximal induced map is a generating GMY structure for $\mu$.

**Proposition 3.1.** Let $f^R : \bigcup_i B_i \to B$ be a maximal induced map. Then, $Vol \left( B - \bigcup_i B_i \right) = 0$ and $R$ is Lebesgue integrable.

**Proof.** By lemma 2.7 Vol $|B|$ is equivalent to $\pi^* \tilde{\mu}_{\Sigma}$. Then, 

$$
Vol \left( B - \bigcup_i B_i \right) = \tilde{\mu}_{\Sigma} \left( \Sigma - \bigcup_i \pi^{-1}(B_i) \right) = 0,
$$

since, by the disjoint or nested property of the rectangle $\Sigma$, 

$$
\pi^{-1}(B_i) = \bigcup_{S_j \subset \pi^{-1}(B_i)} S_j.
$$
Riemann sphere. It is proved in [10, Proposition 34] that
\[ R \]
\[ \forall x, y \in B_j, \text{ for every expanding branch } f^{B_j} : B_j \rightarrow B, \text{ where } R_j = R_N | S_j \text{ and } B_j = \pi(S_j). \]

Therefore,
\[ 1 \leq \beta_0 e^{-\pi/x^2} \text{ diam } (B) \leq \text{ diam } (B_j) \leq \beta_0 e^{-\pi/x^2}. \]

Thus, \( B_j \subset B_i \) whenever
\[ 1 \leq \beta_0 e^{-\pi/x^2} \leq \beta_0 e^{-\pi/x^2}. \]

Hence, if \( f^{R_i} : B_i \rightarrow B \) belongs to a maximal family, then
\[ R_i \leq \inf_{B_i \subset B_i} \frac{4 \log(\beta_0) + 4 \chi^+ R_j}{\chi}. \]

Equivalently, let \( R : B \rightarrow \mathbb{Z}^+ \) such that \( R | B_i = R_i \), then
\[ R(x) \leq \inf_{z \in \Sigma_{\pi(z)} = x} \frac{4 \log(\beta_0) + 4 \chi^+ R_N(z)}{\chi}, \quad \forall x \in B_i, \]

since \( \pi^{-1}(B_i) = \bigcup_{B_i \subset B_i} S_j \), up to a \( \mu \)-zero measure set, by the 'disjoint or nested' property. Thus, by lemma 2.7,
\[ \sum_i R_i \text{Vol}(B_i) = \int_B R(x) d\text{Vol}(x) \]
\[ \leq C_0^{-1} \int_B R(x) d\pi^* \mu_{\Sigma}(x) \]
\[ = C_0^{-1} \int_{\Sigma} R(\pi(z)) d\mu_{\Sigma}(z) \]
\[ \leq C_0^{-1} \int_{\Sigma} \frac{4 \log(\beta_0) + 4 \chi^+ R_N(z)}{\chi} d\mu_{\Sigma}(z) \]
\[ = C_0^{-1} \frac{4 \log(\beta_0) + 4 \chi^+ N}{\chi \mu(\Sigma)} \]
\[ < +\infty, \]

by (39). \hfill \Box

Remark 3.1. Let \( f^{B_i} : \bigcup_{B_i \subset B_i} \rightarrow B \) be a maximal induced Markov map for a rational map of the Riemann sphere. It is proved in [10] Proposition 34] that
\[ R(x) = \inf_{z \in \Sigma_{\pi(z)} = x} R_N(z) \]

This is a consequence of the conformal character of \( f \). In particular, \( R_i = \inf_{B_i \subset B_i} R_j \) and therefore, those maximal induced Markov maps are natural or naturally extendible. See [5, Lemma 2]. Nevertheless, we notice that our maximal induced Markov maps are not necessarily natural neither naturally extendible. So far if \( f^{R_i} : B_i \rightarrow B \) is a branch of a maximal induced Markov map then there might be \( 0 < k < R_i \) and subsets \( A \subset B_i \) such that \( f^k : A \rightarrow B \) is a diffeomorphism, in other words, [12] do not exclude the possibility of earlier returns. This seems to be a consequence of the non conformal character of \( f \) and the non-uniformity of expansion.
follows from (24) and (25) in the definition of rectangles with regular returns, since
$S_i$ is precisely the horizontal plaque $u_i$ is an
$B$ so there are at most finitely many open sets
to get a family of hyperbolic branches
$F$ that
which, essentially, amounts to say that
Therefore, for every
$i > 3.2$
proof of main technical lemma 3.1.

Notice that this may be the case even if $f$ is asymptotically conformal, that is, if all its Lyapunov exponents are equal.

Let us denote $\Sigma_m = \{ z \in \Sigma : \overline{F}(z) = m \}$. Then, we claim that there exists a finite collection of sub-rectangles $S_{i}^{m} \subset \Sigma$ such that
$\Sigma_m = \bigcup_{i} S_{i}^{m} \mod \mu$.

and $u$-rectangles $U_{i}^{m}$ such that $F_{i}^{m} : S_{i}^{m} \rightarrow U_{i}^{m}$ is a laminated diffeomorphism satisfying (30) and (37) in the horizontal plaques $S_{i}^{m}(z)$.

The proof of the lemma follows since $\Sigma = \bigcup_{m \geq 0} \bigcup_{i} S_{i}^{m}$, $\mu$-modulo zero. We get a family of laminated diffeomorphisms satisfying (30) and (37) by relabeling the family $\{ F_{i}^{m} : S_{i}^{m} \rightarrow U_{i}^{m} ; m > 0, i > 0 \}$.

By the property of regular returns, if $\overline{R}(z) = m$ then $\overline{R}(w) = m$ for every $w \in \overline{F}^{-m} \Sigma F_{m}(z)$.

In particular,

$\Sigma_m = \bigcup_{z \in \Sigma_m} F^{-m} \Sigma F_{m}(z)$.

However \{ $\pi F^{-m} \Sigma F_{m}(z) : z \in \Sigma_m$ \} is an at most countable family of open disjoint subsets. Indeed, if $\pi F^{-m} \Sigma F_{m}(z) \subset \pi F^{-m} \Sigma F_{m}(z')$ then $\pi F^{-m} \Sigma F_{m}(z) = \pi F^{-m} \Sigma F_{m}(z')$ since $f^{m}[\pi F^{-m} \Sigma F_{m}(z)]$ and $f^{m}[\pi F^{-m} \Sigma F_{m}(z')]$ are diffeomorphisms onto $B$. Otherwise, there would be a contradiction since $f^{m}[\pi F^{-m} \Sigma F_{m}(z)]$ is one-to-one.

Therefore, $\Sigma_m$ is a countable disjoint union of rectangles $S_{i}^{m} := \pi^{-1}(B_{i}^{m}) \cap \Sigma_m$, that is,

$\Sigma_m = \bigcup_{i} S_{i}^{m} \mod \mu$.

By the bounded distortion property there is $C(m) > 0$ such that $\text{Vol } (B_{i}^{m}) \geq C(m)$ for every $i$, then there are at most finite $i$ for each $m > 0$. Actually, there exists a constant $C > 0$ such that

$e^{-C} \leq \text{Vol } (B_{i}^{m}) J f^{m}(x) \leq e^{C}$

Therefore, for every $i > 0$,

$\text{Vol } (B_{i}^{m}) \geq C(n) := e^{-C} \sup_{x \in M} J f(x)^{-m} > 0,$

so there are at most finitely many open sets $B_{i}^{m}$'s.

We define $U_{i}^{m} := F^{m}(S_{i}^{m})$, where $S_{i}^{m}$ has $\mu(S_{i}^{m}) > 0$. As $S_{i}^{m} = \bigcup_{z \in S_{i}^{m}} F^{-m} \Sigma F_{m}(z)$ then $U_{i}^{m}$ is an $u$-rectangle and $F_{i}^{m} : S_{i}^{m} \rightarrow U_{i}^{m}$ a laminated diffeomorphism. Estimations (30) and (37) follows from (24) and (25) in the definition of rectangles with regular returns, since $F^{-m} \Sigma F_{m}(z)$ is precisely the horizontal plaque $S_{i}^{m}(z)$ of height $z$ in $S_{i}^{m}$ and $\Sigma F_{m}(z)$ the plaques of $U_{i}^{m}$ of level $F^{m}(z)$. To conclude, we simply relabel the family

$\{ F^{m} : S_{i}^{m} \rightarrow U_{i}^{m} : \mu(S_{i}^{m}) > 0, m > 0, i > 0 \}$

to get a family of hyperbolic branches $\{ F \overline{F}_{j} : j \rightarrow U_{j} \}$ as claim lemma 3.1 to exists.

Remark 3.2. Main technical lemma corrects a gap in the statement and proof of [20] Lemma 4.3 which, essentially, amounts to say that

$\pi^{-1}(B_{i}^{m}) \cap \Sigma_m = \pi^{-1}(B_{i}^{m}) \cap \Sigma$

that is, every point $z$ in the 'vertical' rectangle $U_{i}^{m} = \pi^{-1}(B_{i}^{m}) \cap \Sigma$ returns for the first time to $\Sigma$ after $m$ iterates. However, we can not exclude the possibility that some points $z \in U_{i}^{m}$ return later to $\Sigma$. Indeed the image of a vertical fiber $K_{i} = U_{i}^{m} \cap F(z)$ might not match the vertical fiber $K_{F^{m}(z)} = \Sigma \cap F(F^{m}(z))$, even if $F^{m}(z) \in \Sigma$, since $F^{m}(w)$ could fall into a gap of $K_{F^{m}(z)}$. 


All what we know for certain is that the set $S^m_1 := \pi^{-1}(B^m_i) \cap S_m$ is a rectangle with positive $\mu$-measure, contained in the vertical strip $U^m_i$. However, as we will show in next section, $\tau$ still has the structure of a horseshoe with infinitely many branches and variable return times which projects onto an induced Markov map, as stated in [30, Lemma 4.3], even though points in the branches of the horseshoe return for the first time at different moments. We explain this in next section.

4. First return map is a horseshoe with infinitely many branches

Definition 4.1. We say that $S \subset \Sigma$ is a $s$-rectangle if it is a rectangle over an open set $\pi(S) \subset B$ and $K \cap S = K_z$, where $K_z := \Sigma \cap F(z)$ is the vertical fiber of $\Sigma$ passing by $z$, $F(z) = \pi^{-1}(\pi(z))$.

In other words, $S \subset \Sigma$ is a 'vertical strip' crossing $\Sigma$ from 'top to bottom'. We denote $S(z)$ the plaque of 'height' $z \in \Sigma$ of a $s$-rectangle $S_s : S(z) := S \cap \Sigma_z$. Likewise for $u$-rectangles.

A hyperbolic branch is a laminated map $\phi : S \to U$ which expands uniformly horizontal leaves of $S$ and contracts uniformly the vertical fibers. More precisely,

Definition 4.2. Let $S$ (resp. $U$) be an $s$-rectangle (resp. $u$-rectangle) of $\Sigma$. A laminated $C^2$ map $\phi : S \to U$ is a multivalued hyperbolic branch with bounded distortion if there exists constants $C > 0$ and $\lambda > 1$ such that:

1. for every $z \in S$ there is an open subset $R(z) \subset S(z)$ such that $\phi(R(z)) = U(F^n(z))$ and

$$\| D\phi(w)v \| \geq C^{-1}\lambda \| v \| , \quad \forall \ v \in T_w S(z), \ \forall \ w \in R(z);$$

(43)

2. let $J \phi(z)$ the Jacobian of $\phi|S(z)$ with respect to the volume element on $S_z$ given by the parametrization $\Phi_z : B \to \Sigma_z$. Then,

$$\frac{J \phi(w)}{J \phi(w')} = \exp(C d(\phi(w), \phi(w'))), \quad \forall \ w, w' \in R(z), \ \forall \ z \in S;$$

(44)

$\phi$ contracts exponentially the fibers $K_z \subset S$.

Remark 4.1. $\phi : S \to U$ is 'multivalued' in the following sense: every horizontal plaque $S(z)$ is decomposed into at most countably many open disjoint $i$-rectangles $R_i \subset S(z)$ such that for each one of them is mapped onto a horizontal plaque $\Sigma_i$.

The following result clarifies the structure of the first return transformation to $\Sigma$.

Proposition 4.1. (Horseshoe lemma) Let $\tau : \Sigma \to \Sigma$ be the first return map to a rectangle with regular returns $\Sigma$. Then, there exists a decomposition of $\Sigma$ into countably many disjoint $s$-rectangles $S_i$ and $u$-rectangles $U_i$ and multivalued hyperbolic branches $\phi_i : S_i \to U_i$ such that

$$\bar{\mu} \left( \Sigma - \bigcup_i S_i \right) = \bar{\mu} \left( \Sigma - \bigcup_i U_i \right) = 0$$

and $\tau|S_i = \phi_i$, that is, $\tau$ has the structure of a 'horseshoe with infinitely many branches'.

Remark 4.2. Notice that the hyperbolic branches $\phi_i : S_i \to U_i$ do not expand strictly in the horizontal direction, due to the presence of a constant $C^{-1}$ in (43). However, as we argued in the proof of theorem [13] we can choose a suitable iterate $\tau^N$ endowed with a decomposition into countably many multivalued hyperbolic branches mapping 'vertical' rectangles $S_i$ onto 'horizontal' rectangles $U_i$ in $\Sigma$ with good distortion which expands uniformly the horizontal plaques.
Proof. By lemma 3.4, \( \tau \) is the union of a countable family of laminated diffeomorphisms \( F^\tau_j : S_j \to U_j \), satisfying (33) and (44) with \( \lambda = e^{-\chi/3} \). Let \( f^R : \bigcup B_i \to B \) be a maximal induced Markov map and define
\[
S_i = \pi^{-1} (B_i) \cap \Sigma.
\]
By the main technical lemma each \( S_i \) has a decomposition \( S_i = \bigcup \{ S_j : S_j \subset S_i \} \), \( \bar{\mu} \)-modulo zero. Let \( U_i = \bigcup \{ U_j : \pi(S_j) \subset B_i \} \) and define
\[
\phi_i : S_i \to U_i \quad \text{as} \quad \phi_i | S_j = F^\tau_j : S_j \to U_j.
\]
Then, \( S_i \) (resp. \( U_i \)) is a \( s \)-rectangle (resp. \( u \)-rectangle) and clearly \( \tau = \bigcup \phi_i \). By (36) and (37), the maps \( \phi_i : S_i \to U_i \) satisfy (33) and (44), up-to parametrizations of the horizontal plaques.

To prove that each \( \phi_i \) contracts exponentially the vertical fibers in \( \Sigma \) we notice that, by a simple calculation, for every \( n > 0 \),
\[
d(F^n(w),F^n(w')) \leq 1/2^nd(w,w'),
\]
for every \( w, w' \in \mathcal{F}(z) \).

Now, let \( w, w' \in K_z \) be two points on the same vertical fiber \( K_z = \mathcal{F}(z) \cap \Sigma \). Then,
\[
d(\tau^n(w),\tau^n(w')) = d \left( F^{\tau_n}(w), F^{\tau_n}(w') \right) 
\leq 2^{-n}d(w,F^{\tau_n}(w')-F^{\tau_n}(w)) 
\leq 2^{-n}diam(Z)
\]
since \( \tau_n \geq n \). That is, \( d(\tau^n(w),\tau^n(w')) \to 0 \) exponentially, for every pair of points \( w, w' \in K_z \) at the same vertical fiber of \( \Sigma \). As \( \tau|S_i = \phi_i \), this shows that \( \phi_i \) contracts vertical fibers exponentially. \( \square \)

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