Two-component quantum Hall effects in topological flat bands

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We study quantum Hall states for two-component particles (hardcore bosons and fermions) loading in topological lattice models. By tuning the interplay of interspecies and intraspecies interactions, we demonstrate that two-component fractional quantum Hall states emerge at certain fractional filling factors \( \nu \) for fermions \( (\nu = 2/3 \text{ for bosons}) \) in the lowest Chern band, classified by features from ground states including the unique Chern number matrix \((\nu \text{ for fermions})\) and the fractional charge and spin pumpings, and two parallel propagating edge modes. Moreover, we also apply our strategy to two-component fermions at integer filling factor \( \nu = 2 \), where a possible topological Neel antiferromagnetic phase is under intense debate very recently. For the typical \( \pi \)-flux checkerboard lattice, by tuning the onsite Hubbard repulsion, we establish a first-order phase transition directly from a two-component fermionic \( \nu = 2 \) quantum Hall state at weak interaction to a topologically trivial antiferromagnetic insulator at strong interaction, and therefore exclude the possibility of an intermediate topological phase for our system.

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I. INTRODUCTION

Fractionalized topological ordered phases in topological flat bands have attracted intense attention in the past few years [1]. They emerge as the ground state of interacting many-body systems at the fractional fillings of topological bands in the absence of a magnetic field, in analogy to the fractional quantum Hall (FQH) states in two-dimensional Landau levels [2–5]. Topological flat bands with higher Chern number \( C \) have been revealed to host a series of Abelian single-component FQH states [14] in a single Chern band [15–17]. Much stronger evidence comes from a hidden symmetry of the particle entanglement spectrum, as discussed in Ref. [9]. However, a closely related problem is that no direct evidence for the integer valued symmetric \( K \) matrix has been revealed, which is believed to classify the topological order at different fillings for multicomponent systems [18–21].

Here we numerically address the possibility of multicomponent quantum Hall states in topological flat band models filled with interacting two-component (or bilayer) particles, where “two-component” serves as a generic label for spin or pseudospin (bilayer, etc.) quantum number. Different from the single-component case, there are more tunable parameters in two-component systems, like interspecies interaction whose magnitude can be tuned through Feshbach resonance in cold atom setting [22]. When the spin degrees of freedom are included, one would expect many more exotic phases to occur, such as quantum Hall ferromagnetism [23–25], and a rich class of fractional quantum Hall states including Halperin (331) states for two-component fermions [26–31] and spin-singlet incompressible states including Halperin (221) states for two-component bosons [32–36] previously studied for the lowest Landau level systems. Thus it is interesting to study the effect of interspecies interactions on two-component quantum Hall states in topological lattice models. Experimentally, the Haldane honeycomb insulator has been achieved from two-component fermionic \(^{40}\)K atoms with a tunable Hubbard repulsion in a periodically modulated honeycomb optical lattice [37]. For two-component bosonic \(^{87}\)Rb in different hyperfine spin channels, the two-dimensional Hofstadter-Harper Hamiltonian is also engineered in the optical lattice with time-reversal symmetry [38]. These advances would open up new possibilities of studying the multicomponent quantum Hall effect for bosons and fermions in topological lattice models.

The aim of this paper is to provide compelling numerical evidence of \( K \)-matrix classifications for both fractional Halperin and integer quantum Hall states in several microscopic topological lattice models through exact diagonalization (ED) and density-matrix renormalization group (DMRG) methods. By tuning the interspecies and intraspecies interactions, we show that for a given fractional filling factor, the many-body ground states in the decoupled limit evolve to a set of degenerate states separated from the higher-energy spectrum by a finite gap in the strongly interacting regime, whose topological nature is described by the \( K \) matrix [20]. In addition, the topological properties of these states are also characterized by (i) fractional quantized topological invariants related to Hall conductance, and (ii) degenerate ground-state manifold under the adiabatic insertion of flux quanta. For integer quantum Hall states, we mainly focus on their phase transition nature driven by onsite interspecies interactions, and demonstrate their first-order characteristics from discontinuous behaviors of related physical quantities at the transition point.

This paper is organized as follows. In Sec. II, we give a description of the Hamiltonian of two-component quantum particles in two types of topological lattice models, such as \( \pi \)-flux checkerboard and Haldane-honeycomb lattices. In Sec. III, we study the ground states of these two-component particles in the strong interaction regime, present numerical results of the \( K \) matrix by exact diagonalization at fillings \( \nu = 2/3 \) for spinful hardcore bosons and \( \nu = 1/2 \) for spinful fermions in Sec. III B, and discuss the properties of these ground states under the insertion of flux quanta. In Sec. III C,
we calculate the adiabatic charge and spin pumping from DMRG, and demonstrate the quantized drag Hall conductance. In Sec. III D, we discuss the momentum-resolved entanglement spectrum of these two-component FQH states. In Sec. IV, we discuss the role of interspecies onsite interaction on the $C_{q}=2$ integer quantum Hall state for two-component fermions at $v = 2$. Finally, in Sec. V, we summarize our results and discuss the prospect of investigating nontrivial topological states in multicomponent quantum gases.

II. THE MODEL HAMILTONIAN

Our starting point is the following noninteracting Hamiltonian of two-component particles (hardcore bosons and fermions) in topological lattice models, such as the $\pi$-flux checkerboard (CB) lattice,

$$H_{CB} = -t \sum_{\sigma \langle r,r' \rangle} [c_{r,\sigma}^\dagger c_{r',\sigma} \exp(i\phi_{rr'}) + H.c.]$$

$$\pm t' \sum_{\sigma \langle r,r' \rangle} \sum_{\sigma} c_{r,\sigma}^\dagger c_{r',\sigma} - t'' \sum_{\sigma \langle r,r' \rangle} \sum_{\sigma} c_{r,\sigma}^\dagger c_{r,\sigma} + H.c.,$$

and Haldane-honeycomb (HC) lattice,

$$H_{HC} = -t \sum_{\sigma \langle r,r' \rangle} [c_{r,\sigma}^\dagger c_{r',\sigma} \exp(i\phi_{rr'}) + H.c.]$$

$$-t \sum_{\langle r,r' \rangle} c_{r,\sigma}^\dagger c_{r',\sigma} - t'' \sum_{\sigma \langle r,r' \rangle} \sum_{\sigma} c_{r,\sigma}^\dagger c_{r,\sigma} + H.c.,$$

where $c_{r,\sigma}$ is the particle creation operator of spin $\sigma = \uparrow, \downarrow$ at site $r$, $\langle \ldots \rangle, \langle \langle \ldots \rangle \rangle$ denote the nearest-neighbor, the next-nearest-neighbor, and the next-next-nearest-neighbor pairs of sites, respectively. We take the parameters $t' = 0.3t$, $t'' = 0.5t$ for the checkerboard lattice, as in Ref. [39], while $t = 0.6t, t'' = 0.58t$ for the honeycomb lattice, as in Refs. [40,41]. These parameters enhance the flatness of the topological band and make quantum Hall states more robust.

Consider the onsite interspecies and nearest neighboring intraspecies interactions,

$$V_{int} = U \sum_{r} n_{\uparrow r} n_{\downarrow r} + V \sum_{\sigma \langle r,r' \rangle} n_{\sigma r} n_{\sigma r'},$$

where $n_{\sigma r}$ is the particle number operator of spin $\sigma$ at site $r$. The model Hamiltonian becomes $H = H_{CB} + V_{int}$ ($H = H_{HC} + V_{int}$). Here, $U$ is the strength of the interspecies interaction while $V$ is the strength of intraspecies correlations in topological flat bands, playing the analogous role of Haldane pseudopotentials for the two-component FQHE system in Landau levels [30]. For strong Hubbard repulsion, two-component fermions in the Haldane-honeycomb model may exhibit various chiral magnetic orderings or topological Mott insulator phase at half-filling [42,43].

In the ED study, we explore the many-body ground state of $H$ in a finite system of $N_x \times N_y$ unit cells (the total number of sites is $N_s = 2 \times N_x \times N_y$). The total filling of the lowest Chern band is $\nu = 2N_s/N$, where $N = N_1 + N_2$ is the total particle number with global conservation $U(1)$ symmetry. With the translational symmetry, the energy states are labeled by the total momentum $K = (K_x, K_y)$ in units of $(2\pi/N_x, 2\pi/N_y)$ in the Brillouin zone. For larger systems, we exploit DMRG on cylinder geometry, and keep the number of states 1200–2400 to obtain accurate results.

III. HALPERIN (221) AND (331) STATES

In this section, we present the numerical evidences of two-component FQH states at filling factors $v = 2/3$ and $v = 1/2$ for bosons and fermions, respectively. Importantly, the Chern number matrix (the inverse of the $K$ matrix) uniquely identifies these two-component FQH states. The further information from ground-state degeneracies and charge (spin) pumpings is complementary to and consistent with the Chern number matrix. The Chern number matrix also provides an accurate prediction for the transport measurement for experimental systems. Thus it is very important to numerically extract this topological information for two-component particles in topological flat band models.

A. Ground-state degeneracy

First, we demonstrate the ground-state degeneracy on torus geometry, which serves as a primary signature of an incompressible FQH state. We consider finite-size systems up to maximum particle number $N = 8$. In Figs. 1(a) and 1(b), we show the energy spectrum of several typical systems in the strong interacting regime. The key feature is that there exists a well-defined and degenerate ground-state manifold separated from higher-energy levels by a robust gap. For two-component hardcore bosons at $v = 2/3$, the ground states show threefold quasidegeneracies; for two-component fermions at $v = 1/2$, we find that the ground-state manifold hosts eightfold quasidegeneracies. We also calculate the density and spin structure factors for the ground states, and exclude any possible charge or spin density wave orders as the competing ground states, due to the absence of the Bragg peaks in the results. (We check them up to $2 \times 4 \times 4$ sites using DMRG with periodic boundary conditions.)

Next, we consider the effects of interspecies repulsion on these topological phases. We first use ED to study the evolution of the energy spectrum with $U$ for the periodic system. As
shown in Figs. 2(a) and 2(b), with the decrease of \( U \), the degeneracy is lifted and finally disappears at \( U = 0 \), where the ground state becomes a possibly metallic phase with a vanishingly small excitation energy gap. In usual FQH states, the occupation of each single-particle orbital is constant and equal to the filling factor [44]. By diagonalizing the \( N_x \times N_z \) matrix \( \rho_{\sigma} = (c_\sigma^i, c_{\bar{\sigma}}^i) \), we obtain reduced single-particle eigenstates \( \rho_{\sigma} |\phi_\alpha\rangle = \rho_\sigma^\alpha |\phi_\alpha\rangle \) where \( |\phi_\alpha\rangle (\alpha = 1, \ldots, N_z) \) are the effective orbitals as eigenvectors for \( \rho_\sigma \) and \( \rho_\sigma^\alpha \); \( \rho_{\sigma}^\alpha \) are interpreted as occupations. We find that the occupations are close to the uniform filling \( \rho_{\sigma}^\alpha \approx \nu/2 \) for \( \alpha \leq N_z/2 \) with standard deviations of the order 0.01, while \( \rho_{\sigma} \ll 1 \) for \( \alpha > N_z/2 \) in the strongly interacting regime, indicating an incompressible liquid with particles uniformly occupying the \( N_z/2 \) orbitals of the lowest Chern band. However, for \( U \ll t \), these eigenvalues form a nonuniform distribution with standard deviations of the order 0.2, which are possibly compressible liquid states. Thus, we remark that the emergence of two-component FQH states is induced by suitable interspecies and intraspecies repulsions.

B. Chern number matrix and K matrix

To uncover the topological nature of the ground-state manifold we extract the Chern number matrix. Here, we utilize the scheme proposed by one of the current authors in Refs. [45,46]. With twisted boundary conditions \( \psi(\mathbf{r}_\sigma + N_z) = \psi(\mathbf{r}_\sigma) \exp(i \theta_\sigma^\alpha) \) where \( \theta_\sigma^\alpha \) is the twisted angle for spin-\( \sigma \) particles in the direction \( \alpha \), we plot the low energy spectra under the variation of \( \theta_\sigma^\alpha \). As shown in Figs. 3(a) and 3(b), these ground states evolve into each other without mixing with the higher levels. For two-species hardcore bosons at \( \nu = 2/3 \), the energy recovers itself after the insertion of three flux quanta for \( \theta_\sigma^\alpha = 0 \), indicating its 1/4 fractional quantization of quasiparticles. Meanwhile, the many-body Chern number of the ground-state wave function \( \psi_i \) is defined as [45,46]

\[
C_{\sigma,\sigma'}^{i} = \frac{1}{2\pi} \int d\theta^{\sigma} d\theta^{\sigma'} \text{Im} \left( \frac{\partial \psi^{i} \cdot \partial \psi^{i'}}{\partial \theta_{\sigma} \partial \theta_{\sigma'}} - \frac{\partial \psi^{i} \cdot \partial \psi^{i'}}{\partial \theta_{\sigma'} \partial \theta_{\sigma}} \right).
\]

(4)

For the three ground states with \( K = (0,0) \), the above equation implies a symmetric \( C \) matrix in the spanned Hilbert space, namely,

\[
C_{\sigma,\sigma'}^{i} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.
\]

(5)

where the off-diagonal part \( C_{\sigma,\sigma'}^{i} \) is related to the drag Hall conductance. Thus we can obtain the \( K \) matrix which is the inverse of the \( C \) matrix, namely \( K = C^{-1} = (\frac{1}{3} - \frac{1}{3}) \). Therefore, we establish that threefold ground states for two-species hardcore bosons at \( \nu = 1/3 \) are indeed Halperin (221) states in the lattice version, and the threefold degeneracy coincides with the determinant \( \det K \), as predicted in Ref. [47].

Similarly, for the eight ground states with \( K = (0,i), (i = 0,1,2,3) \) of two-species fermions at \( N = 6, N_z = 2 \times 3 \times 4 \), by numerically calculating the Berry curvatures (using \( m \times m \) mesh squares in the boundary phase space with \( m \geq 9 \)) we obtain \( C_{\sigma,\sigma'}^{i} = 3 \), and \( C_{\sigma,\sigma'}^{i} = -1 \). The above results imply a symmetric \( C \) matrix, namely,

\[
C_{\sigma,\sigma'}^{i} = \frac{1}{8} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.
\]

(6)

Thus we can obtain the \( K \) matrix which is the inverse of the \( C \) matrix, namely \( K = C^{-1} = (\frac{1}{3} - \frac{1}{3}) \). Therefore, we establish that eightfold ground states for the two-species fermion at \( \nu = 1/2 \) are indeed Halperin (331) states in the lattice version, and the eightfold degeneracy coincides with the determinant \( \det K \), as predicted in Ref. [47].
With \( \theta_1^U = \theta_1^V = \theta^U \) and \( \theta_2^U = \theta_2^V = \theta^V \), the many-body charge Chern number of the ground-state wave function, related to charge Hall conductance, reads

\[
C_q = \int \frac{d\theta^x d\theta^y}{2\pi} \text{Im} \left( \frac{\partial \psi}{\partial \theta^x} \frac{\partial \psi}{\partial \theta^y} - \frac{\partial \psi}{\partial \theta^y} \frac{\partial \psi}{\partial \theta^x} \right)
\]

\[
= q \cdot C \cdot q^T = \sum_{\sigma,\sigma'} C_{\sigma,\sigma'} = \nu,
\]

where \( q = (1,1) \) is the charge eigenvector of \( K \) matrix. Similarly, with \( \theta_1^U = -\theta_2^U = \theta^U \) and \( \theta_1^V = -\theta_2^V = \theta^V \), we can also define the many-body spin Chern number of the ground-state wave function, related to spin Hall conductance, as

\[
C_s = \int \frac{d\theta^x d\theta^y}{2\pi} \text{Im} \left( \frac{\partial \psi}{\partial \theta^x} \frac{\partial \psi}{\partial \theta^y} - \frac{\partial \psi}{\partial \theta^y} \frac{\partial \psi}{\partial \theta^x} \right)
\]

\[
= s \cdot C \cdot s^T = C_{\uparrow,\uparrow} + C_{\downarrow,\downarrow} - C_{\uparrow,\downarrow} - C_{\downarrow,\uparrow},
\]

where \( s = (1,-1) \) is the spin eigenvector of the \( K \) matrix. From Eqs. (7) and (8), we conclude that to identify the nature of these degenerate states, one can calculate the adiabatic charge and spin pumping by performing different flux insertion simulations, which identify the total fractionally quantized Hall and drag Hall conductances in experiments, as will be shown below in Sec. III C.

C. Fractional charge and spin pumpings

To uncover the topological nature of two-component FQH states, we further calculate the charge pumping under the insertion of flux quanta on cylinder systems based on the newly developed adiabatic DMRG [48] in connection to the quantized Hall conductance. It is expected that a quantized charge will be pumped from the right side to the left side by inserting a U(1) charge flux \( \theta = 0 \rightarrow 2\pi \). The net transfer of the total charge from the right side to the left side is encoded by \( Q(\theta) = N^U_1 + N^V_1 = \text{tr} [\hat{\rho}_1(\theta) \hat{Q}] \) \((\hat{\rho}_1, \text{the reduced density matrix of the left part})[49]\). In order to quantify the drag Hall conductance, we also define the spin transfer \( \Delta S \) by \( S(\theta) = N^U_1 - N^V_1 = \text{tr} [\hat{\rho}_1(\theta) \hat{S}] \) in analogy to the charge transfer.

As shown in Figs. 4(a) and 4(b), for bosons at \( \nu = 2/3 \), a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.33 \) is pumped by threading one flux quanta with \( \theta_1 = \theta_2 = 0 \) in one species of two-component gases, and a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.66 \) would be pumped by threading two flux quanta with \( \theta_1 = \theta_2 = \theta \) in both of the two-component gases. For the fermion at \( \nu = 1/2 \), a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.25 \) is pumped by threading one flux quanta with \( \theta_1 = \theta_2 = 0 \) in one species of two-component gases, and a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.50 \) would be pumped by threading one flux quanta with \( \theta_1 = \theta_2 = \theta \) in both of the two-component gases.

The dynamical pumping process reveals the fractional statistics of the pumped quasiparticle. Based on these observations, we claim that the number of physically distinct stable quasiparticles is equal to the rank of the \( K \) matrix, and establish the pumping relationship that (i) by threading the flux \( \theta_1 = \theta_2 = 0 \) from \( \theta = 0 \) to \( \theta = 2\pi \),

\[
\Delta Q = C_{\uparrow,\uparrow} + C_{\downarrow,\downarrow},
\]

\[
\Delta S = C_{\uparrow,\downarrow} - C_{\downarrow,\uparrow},
\]

and (ii) by threading the flux \( \theta_1 = \theta_2 = \theta \) from \( \theta = 0 \) to \( \theta = 2\pi \),

\[
\Delta Q = \sum_{\sigma,\sigma'} C_{\sigma,\sigma'},
\]

\[
\Delta S = 0.
\]

D. Chiral edge spectrum

Another “fingerprint” of chiral topological order is the characteristic edge state usually described by the specific Luttinger liquid theory, which can be revealed through the low-lying entanglement spectrum (ES) [50]. Here we examine the ES of these FQH states based on the DMRG method. Due to its difficulty of DMRG convergence for two-component gases, we stack the two-component gases. For the fermion, the edge excitations would also have a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.33 \) pumped by threading one flux quanta with \( \theta_1 = \theta_2 = 0 \) in one species of two-component gases, and a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.66 \) would be pumped by threading two flux quanta with \( \theta_1 = \theta_2 = \theta \) in both of the two-component gases. For the fermion at \( \nu = 1/2 \), a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.25 \) is pumped by threading one flux quanta with \( \theta_1 = \theta_2 = 0 \) in one species of two-component gases, and a fractional charge \( \Delta Q = Q(2\pi) - Q(0) \approx 0.50 \) would be pumped by threading one flux quanta with \( \theta_1 = \theta_2 = \theta \) in both of the two-component gases. The dynamical pumping process reveals the fractional statistics of the pumped quasiparticle. Based on these observations, we claim that the number of physically distinct stable quasiparticles is equal to the rank of the \( K \) matrix, and establish the pumping relationship that (i) by threading the flux \( \theta_1 = \theta_2 = 0 \) from \( \theta = 0 \) to \( \theta = 2\pi \),

\[
\Delta Q = C_{\uparrow,\uparrow} + C_{\downarrow,\downarrow},
\]

\[
\Delta S = C_{\uparrow,\downarrow} - C_{\downarrow,\uparrow},
\]

and (ii) by threading the flux \( \theta_1 = \theta_2 = \theta \) from \( \theta = 0 \) to \( \theta = 2\pi \),

\[
\Delta Q = \sum_{\sigma,\sigma'} C_{\sigma,\sigma'},
\]

\[
\Delta S = 0.
\]

For hardcore bosons, as shown in Figs. 5(a)–5(c), the two branches of low-lying bulk ES appear with the level counting 1,2,5,\ldots,\, implying the gapless edge modes. Since \( K \) has two positive eigenvalues, the edge excitations would have two forward-moving branches in the same direction in the charge sector \( \Delta N = 0 \) as shown in Fig. 5(b), consistent with theoretical analysis of the spin-singlet quantum Hall state in Refs. [51–53].

Similarly for fermions, the edge excitations would also have two forward-moving branches in the same direction. As shown in Figs. 6(a) and 6(b), for odd charge sector \( \Delta N = -1 \), two symmetric branches of ES appear with the same level counting 1,2,5,\ldots,\, differing by a phase \( \pi \) in the momentum. However, for even charge sector \( \Delta N = 0 \), two asymmetric branches of ES appear with different level countings 1,1,4,\ldots,\, and 125134-4
0, 1, 3, ..., differing by a phase $\pi$ in the momentum. Thus, without the $\pi$-phase shift, the total counting of two branches of ES would be (a) 2, 4, 10, ... for odd charge sector $\Delta N = -1$ and (b) 1, 2, 7, ... for even charge sector $\Delta N = 0$, which are consistent with the analysis of the counting of the root configurations of the bilayer Halperin (331) edge excitations in the lowest Landau level at half filling [54].

**IV. INTEGER QUANTUM HALL STATE**

In this section, we consider the ground states at integer filling $v = 2N/N_s = 2$ for two-component fermions ($N = N_1 + N_2$), namely, one fermion per site. In noninteracting cases $U = 0, V = 0$, the system is a topological Chern insulator with quantized Hall conductivity two. In the large repulsive case $U \gg t$, it would be a trivial antiferromagnetic Mott insulator as expected. Several mean field studies of the Haldane-honeycomb model indicate a topologically nontrivial Néel antiferromagnetic insulating phase in the intermediate interaction regime [55–57], supported by quantum cluster methods [43]. However, recent dynamical cluster approximations have revealed a first-order transition from the $C_q = 2$ Chern insulator into a trivial Mott insulator, with no evidence of a topological antiferromagnetic insulator as an intermediate phase [58,59]. In the following discussion, we take the typical example of the $\pi$-flux checkerboard lattice, and investigate the nature of the interaction-driven transition using both ED and DMRG calculations.

For small system sizes, we present an ED diagnosis of quantum phase transition of Fermi-Hubbard model from weak interactions to strong interactions in the $\pi$-flux checkerboard lattice. Our current system size limit for ED calculation is $N_s = 16, N_1 = N_2 = 8$ at $v = 2$. For simplicity, we take $t' = 0.3t, t'' = 0$. The lattice geometry is indicated in Fig. 7(a), with two different lattice sizes $N_s = 16, 12$. In Fig. 7(b), we plot its low energy as onsite repulsion $U$ increases. For weak interactions, there always exists a stable unique ground state with a large gap separated from higher levels. When one flux quanta is inserted, it would evolve back to itself. With twisted boundary conditions as described in Sec. III B, we obtain the $K$ matrix as a unit matrix,

$$K = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (13)$$

When $U$ increases further, this ground state undergoes an energy level crossing with other levels around $U_c \simeq 9t$. In order to clarify the nature of the phase transition, we calculate its Chern numbers $C_{\uparrow,\uparrow}, C_{\downarrow,\downarrow}$ and antiferromagnetic spin
The first-order transition is characterized by the discontinuous
structure factor,

$$S(Q) = \frac{1}{N \nu} \sum_{\nu \nu'} e^{i Q \cdot (\nu - \nu')} <\bar{n}_n^\nu \bar{n}_{n'}^\nu'^*>,$$

(14)

where $Q = (\pi, \pi), S_L = (n_{11} - n_{1\bar{1}})/2$. As indicated in Fig. 7(c),
both $C_{t,\uparrow} = C_{\bar{1},\uparrow}$ and $S(Q)$ experience a sudden dramatic
jump as the interaction $U$ increases across the critical threshold
$U_\nu$, demonstrating the discontinuous first-order transition. In
the Mott regime, the antiferromagnetic order dominates and we
obtain the charge Hall conductance $C_q = C_{t,\uparrow} + C_{\bar{1},\uparrow} = 0$
$\partial S_L/\partial U$ as expected. Meanwhile, the spin gap
$\Delta_1 \propto 0$ in weak interacting regime
$U_\nu$, $\Delta_1$ is almost a constant for a given system, indicating
that the same topological properties as integer quantum Hall state.
$\Delta_1$ starts to drop at $U \approx U_c$, with a discontinuous first-order
derivative $\partial S_L/\partial U$ due to the change of topology [63]. By
comparing the complete spectra of reduced density matrices
between two ground states, the first-order transition can be
extracted from the discontinuous jump of majorization [62].
Recently, a diagnostic of phase transition driven by disorder
from quantum Hall states to an insulator via entanglement
entropy is also examined [64].

The above pictures from both ED and DMRG calculations
indicate a first-order phase transition directly from a $C_q = 2$
integer quantum Hall state to a Mott insulator driven by onsite
repulsion. We did not observe any evidence of topological
Neel antiferromagnetic insulating phase in the intermediate
interaction regime, which is consistent with recent studies
on the Haldane-honeycomb lattice from different methods
[58,59]. Further, with the inclusion of next-nearest-neighbor
hopping $t'' = -0.2t$, we numerically arrive at the
same phase transition nature as above. For the future work,
it would be interesting to consider the effects of the band
flatness modulated by next-nearest-neighbor hopping $t'$
and next-next-nearest-neighbor hopping $t''$ on the quantum
phase diagram and phase transition.

\section*{V. SUMMARY AND DISCUSSIONS}

In summary, we show that two-component hardcore bosons
and fermions in topological lattice models could host Halperin
FQH states at a partial filling of the lowest Chern band,
with fractional topological properties characterized by the $K$
matrix, including the degeneracy, fractional quantized Hall
conductance, and chiral edge modes. The role of onsite
terspecies repulsion on integer quantum Hall state on the
$\pi$-flux checkerboard topological lattice is examined, and
shown to lead to a first-order phase transition from a $C_q = 2$
integer quantum Hall state to a trivial Mott insulator for
two-component fermions at integer filling factor $v = 2$. Another
interesting issue for two-component hardcore bosons at $v = 2$
filling, but not discussed here, is related to the possibility for
a bosonic integer quantum Hall state [34,65–71], which is left
for future study.

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