Hoyle-analogue state in $^{13}\text{C}$
studied with Antisymmetrized Molecular Dynamics

Y. Chiba

Department of Physics, Hokkaido University, 060-0810 Sapporo, Japan

M. Kimura

Department of Physics, Hokkaido University, 060-0810 Sapporo, Japan and
Nuclear Reaction Data Centre, Faculty of Science,
Hokkaido University, Sapporo 060-0810, Japan

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Abstract

The cluster states in $^{13}\text{C}$ are investigated by antisymmetrized molecular dynamics. By investigating the spectroscopic factors, the cluster configurations of the excited states are discussed. It is found that the $1/2^+_2$ state is dominantly composed of the $^{12}\text{C}(0^+_2)\otimes s_{1/2}$ configuration and can be regarded as a Hoyle analogue state. On the other hand, the $p$-wave states ($3/2^-$ and $1/2^-$) do not have such structure, because of the coupling with other configurations. The isoscalar monopole and dipole transition strengths from the ground to the excited states are also studied. It is shown that the excited $1/2^-$ states have strong isoscalar monopole transition strengths consistent with the observation. On the other hand, the excited $1/2^+$ states unexpectedly have weak isoscalar dipole transitions except for the $1/2^+_1$ state. It is discussed that the suppression of the dipole transition is attributed to the property of the dipole operator.
I. INTRODUCTION

In these decades, the Hoyle state of $^{12}\text{C}$ \cite{1-6} attracts much interests as a possible bosonic condensate. A natural extension of the discussion is the search for the analogue states in heavier $4n$ nuclei such as $^{16}\text{O}$ and $^{20}\text{Ne}$. Recently, a possible candidate in $^{16}\text{O}$ \cite{7-10} is being intensively discussed, and a theoretical study \cite{11} predicted the existence of the $\alpha$ particle condensate up to approximately 10 $\alpha$ system $^{40}\text{Ca}$.

Another direction of the research is the study of $N \neq Z$ nuclei in which nucleon particles or holes can be injected into the $\alpha$ particle condensate as an impurity. In the case of $^{11}\text{B}$ which has a proton hole coupled to $^{12}\text{C}$, the theoretical studies based on antisymmetrized molecular dynamics (AMD) \cite{12, 13} pointed out that the $3/2^{-}$ state located just below the $^{7}\text{Li} + \alpha$ threshold has pronounced $2\alpha + t$ clustering with large radius. Hence, the state was suggested as a candidate of the Hoyle analogue state. More recently, T. Yamada et al. performed the orthogonality condition model (OCM) calculation \cite{14} and predicted the $1/2^{+}$ state as another candidate of Hoyle analogue state in which all of $2\alpha$ and triton particles occupy the s-wave state.

Several discussions have also been made for $^{13}\text{C}$ which has an extra neutron. T. Yamada et al. \cite{15} discussed the possible reduction of spin-orbit splitting in the Hoyle analogue states. Namely, they suggested that the spin-orbit splitting between the $p_{1/2}$ and $p_{3/2}$ coupled to the Hoyle state ($1/2^{-}$ and $3/2^{-}$ states) will be reduced, because the splitting is dependent on the first derivative of the density distribution and the Hoyle state has dilute density profile. In addition to this, T. Yamada et al. performed the OCM calculation \cite{16} and predicted the Hoyle-analogue $1/2^{+}$ state in which all of three $\alpha$ particles and a neutron occupy the same s-wave state, that is quite similar to the discussion made for $^{11}\text{B}$. Thus, the $3/2^{-}$, $1/2^{-}$ and $1/2^{+}$ states in $^{13}\text{C}$ are of particular interest and importance for the understanding of the Hoyle analogue state in $N \neq Z$ nuclei.

Up to now, the existence of the Hoyle-analogue $3/2^{-}$, $1/2^{-}$ and $1/2^{+}$ states in $^{13}\text{C}$ is still ambiguous, because the information is not enough to identify them. Therefore, in this work, we conduct the AMD calculation to supply further theoretical information. We investigate the spectroscopic factors ($S$-factors) in $^{12}\text{C} + n$ and $^{9}\text{Be} + \alpha$ channels to identify the Hoyle analogue states. Furthermore, we focus on the isoscalar dipole (IS1) transition strength as well as the isoscalar monopole (IS0) transition strength, which are known to be enhanced
for the cluster states $^{18-23}$. We expect that they are useful to identify the Hoyle analogue states in $^{13}$C.

This paper is organized as follows: First, we explain theoretical framework of AMD and how to calculate $S$-factors of the $^{12}$C + $n$ and $^{9}$Be + $\alpha$ channels. Second, we present our numerical calculation results and compare it to experimental data. We analyze nuclear structure of $3/2^-$, $1/2^-$ and $1/2^+$ states in detail using $S$-factors and identify the Hoyle-analogue states. We also discuss on the IS0 and IS1 transition strengths to supply theoretical information for forthcoming experiments. Finally, we summarize our paper.

II. AMD FRAMEWORK

A. Hamiltonian and model wave function

The Hamiltonian employed in this work is

$$\hat{H} = \sum_{i=1}^{A} \hat{t}_i - \hat{t}_{\text{c.m.}} + \sum_{i<j} \hat{v}_{NN} + \sum_{i<j} \hat{v}_{\text{Coul}},$$

(1)

where $\hat{t}_i$ is the $i$-th nucleon kinetic energy and $\hat{v}_{NN}$ and $\hat{v}_{\text{Coul}}$ are the Gogny D1S nucleon-nucleon interaction [24] and Coulomb interaction, respectively. The center-of-mass kinetic energy $\hat{t}_{\text{c.m.}}$ is subtracted from the Hamiltonian.

The intrinsic AMD wave function is a Slater determinant of nucleon Gaussian wave packets [25–27],

$$\Phi_{AMD} = A \{ \varphi_1 \varphi_2 \ldots \varphi_A \},$$

(2)

$$\varphi_i = \phi_i \otimes \chi_i \otimes \xi_i,$$

(3)

$$\phi_i = \left( \frac{\pi^3}{8|\nu|} \right)^{-\frac{1}{4}} \exp \left[ - \sum_{\sigma=xyz} \nu_\sigma \left( r_{i\sigma} - \frac{Z_{i\sigma}}{\sqrt{\nu_\sigma}} \right) \right],$$

(4)

$$\chi_i = \alpha_i |\uparrow\rangle + \beta_i |\downarrow\rangle,$$

$$\xi_i = |p\rangle \text{ or } |n\rangle.$$  

(5)

It is noted that the center-of-mass wave function $\Phi_{c.m.}$ is analytically separable from the intrinsic wave function,

$$\Phi_{AMD} = \Phi_{\text{int}} \Phi_{c.m.},$$

(6)

$$\Phi_{c.m.} = \left( \frac{\pi^3}{8A^3|\nu|} \right)^{-\frac{1}{4}} \exp \left[ -A \sum_{\sigma=xyz} \nu_\sigma R_\sigma^2 \right].$$

(7)
Here, $\Phi_{\text{int}}$ is the internal wave function, and we assume that the relation $\sum_i Z_i = 0$ holds. Therefore, the AMD framework is completely free from spurious motion. This is an important advantage when we calculate the IS1 transition strengths. The parameters of AMD wave function $\nu$, $Z_i$, $\alpha_i$ and $\beta_i$ are determined so as to minimize the energy after parity-projection,

$$\Phi^\pi = \frac{1 + \pi \hat{P}_x}{2} \Phi_{\text{int}}, \; \pi = \pm,$$

$$E^\pi = \frac{\langle \Phi^\pi | \hat{H} | \Phi^\pi \rangle}{\langle \Phi^\pi | \Phi^\pi \rangle}.$$  \hfill (8) \hfill (9)

To describe the various states of $^{13}$C, we impose the constraint on the expectation values of harmonic oscillator quanta $N$, $\lambda$ and $\mu$ which are defined by using the harmonic oscillator quanta in Cartesian coordinate $N_x, N_y$ and $N_z$,

$$N = N_x + N_y + N_z, \; \lambda = N_z - N_y, \; \mu = N_y - N_x.$$ \hfill (10)

Here, we assume the relation $N_x \leq N_y \leq N_z$. Roughly speaking, the excitation of system is expressed by $N$, and $\lambda$ and $\mu$ indicate the asymmetries around the longest and shortest deformed axis. The detail of this constraint is described in Ref. [28].

Compared with the constraint on the quadrupole deformation parameters ($\beta\gamma$-constraint), which is often used in mean-field and AMD calculations, the constraint on $N$, $\lambda$ and $\mu$ is appropriate for the description of the highly excited states. The $\beta\gamma$-constraint is useful to describe the low-lying quadrupole collectivities but it often fails to describe highly excited states. On the other hand, the constraint on $N$, $\lambda$ and $\mu$ is capable to describe the highly excited states with many-particle many-hole configurations. In this study, the possible combinations of values for $N$, $\lambda$ and $\mu$ up to $N = 18$ ($9\hbar \omega$ excitation) are adopted as the constraint. We denote thus-obtained basis wave function as $\Phi^\pi(N\lambda\mu)$.

After energy variation, the basis wave functions are projected to angular-momentum eigenstates and superposed to obtain excitation spectra and eigen wave functions (generator coordinate method (GCM)).

$$\Phi^J_M(N_i \lambda_i \mu_i) = \mathcal{N}_K^{-\frac{1}{2}} P^J_M \Phi^\pi(N_i \lambda_i \mu_i),$$

$$\mathcal{N}_K = \langle \Phi^J_M(N \lambda \mu) | \Phi^J_M(N \lambda \mu) \rangle,$$  \hfill (11) \hfill (12)

$$\Psi^J_n = \sum_{K_i} g_{K_i n}^J \Phi^J_M(N_i \lambda_i \mu_i),$$ \hfill (13)
where $\hat{P}_{MK}^J$ is the angular momentum projection operator. The coefficients $g_{Kn}^{J\pi}$ is determined by diagonalizing the Hamiltonian,

\[
\sum_{\ell' \ell} H_{\ell K' \ell' K}^J g_{\ell' K'n}^{J\pi} = E_n^{J\pi} \sum_{\ell' \ell} N_{\ell K' \ell' K}^J g_{\ell' K'n}^{J\pi},
\]

\[
H_{\ell K' \ell' K}^J = \langle \Phi_{MK}^J(N_i, \lambda_i, \mu_i) | \hat{H} | \Phi_{MK'}^J(N_{\ell'}, \lambda_{\ell'}, \mu_{\ell'}) \rangle,
\]

\[
N_{\ell K' \ell' K}^J = \langle \Phi_{MK}^J(N_i, \lambda_i, \mu_i) | \Phi_{MK'}^J(N_{\ell'}, \lambda_{\ell'}, \mu_{\ell'}) \rangle.
\]

### B. Reduced width amplitudes and spectroscopic factors

To search for Hoyle-analogue states, we calculate the reduced width amplitudes and $S$-factors in the $^{12}$C+$n$ and $^9$Be+$\alpha$ channels. The reduced width amplitude in the $^{12}$C+$n$ channel is defined as

\[
y_{jC=\pi=jl}(a) = \sqrt{13} \left\langle \frac{\delta(r-a)}{r^2} \left[ \Phi_C^{jC=\pi=C} \left[ Y_l(\hat{r}) \chi_{1/2} \right] \right]_J \right| \Psi_n^{J\pi} \rangle
\]

where $\Phi_C^{jC=\pi=C}$ is the wave function of $^{12}$C, and $\chi_{1/2}$ is the spin-isospin wave function of neutron. $j_C$ and $\pi_C$ are angular momentum and parity of $^{12}$C, and $j$ and $l$ are total and orbital angular momenta of neutron. In a same manner, the reduced width amplitude in the $^9$Be+$\alpha$ channel is defined as

\[
y_{jBe=\pi=jl}(a) = \sqrt{\frac{13!}{9!4!}} \left\langle \frac{\delta(r-a)}{r^2} \Phi_\alpha \left[ \Phi_{Be}^{jBe=\pi=Be} Y_l(\hat{r}) \right]_J \right| \Psi_n^{J\pi} \rangle
\]

where $\Phi_{Be}^{jBe=\pi=Be}$ is the wave function of $^9$Be with angular momentum $j_{Be}$ and parity $\pi_{Be}$. $\Phi_\alpha$ is the wave function of the ground state of $\alpha$ cluster. The $S$-factors of the $^{12}$C+$n$ and $^9$Be+$\alpha$ channels are defined as the integral of the reduced width amplitudes,

\[
S_{jC=\pi=jl}^{J\pi n} = \int_0^\infty da \left| ay_{jC=\pi=jl}(a) \right|^2,
\]

\[
S_{jBe=\pi=Be}^{J\pi n} = \int_0^\infty da \left| ay_{jBe=\pi=Be}(a) \right|^2.
\]

To evaluate the reduced width amplitudes, we use Laplace expansion method proposed in Ref. [29], which can treat the deformed and the excited cluster wave functions without any approximations. In this study, the $\alpha$ cluster is described by the $(0s)^4$ configuration with oscillator parameter $\nu = m\omega/(2\hbar) = 0.25 \text{ fm}^{-1}$ while the wave functions of $^9$Be and $^{12}$C are obtained by AMD+GCM calculations.
III. RESULTS AND DISCUSSIONS

A. Excitation spectra

We performed the energy variation under constraint on the harmonic oscillator quanta \( N, \lambda \) and \( \mu \). Using the basis wave functions generated by the energy variation, we performed the GCM calculation and obtained excitation spectra of \(^{13}\)C. The observed data \(^{30}\) and calculated excitation spectra up to \( E_x = 20 \text{ MeV} \) with \( J^\pi \leq 5/2^\pm \) are shown in Fig. 1.

The calculated yrast states reasonably agree with the observed spectra. On the other hand, the energies of the non-yrast states above 10 MeV are overestimated. For example, the \( 3/2^- \) state observed at 9.9 MeV is located at 13.2 MeV in the present calculation. As shown later, many of the excited states located above 10 MeV have cluster structure. Thus, we can say that the present calculation overestimates the energies of the cluster states. This is mainly due to the limitation of our model space. The restriction up to \( N = 18 \) configuration may not be sufficient to describe the relative motion of clusters.

B. Structure of 1/2^- states

In this section, we discuss the structure of the 1/2^- states, which are the candidates of the Hoyle-analogue state having \(^{12}\)C(0^+_2) \( \otimes p_{1/2} \) configuration. The calculated root-mean-square (rms) radii, \( S \)-factors in the \(^{12}\)C + \( n \) and \(^9\)Be + \( \alpha \) channels and the IS0 transition matrix...
FIG. 2. (Color online) The excitation energies and the properties of the $1/2^-$ states obtained in the present work. In the upper panels, the calculated excitation spectra, matter rms radii $r_{\text{rms}}$, IS0 transition matrix from the ground state $M(IS0)$ are shown from left to right. The calculated $S$-factors in the $^{12}\text{C} + n$ and $^{9}\text{Be} + \alpha$ channels are presented in the lower panels. In the last panel, $l$ denotes the orbital angular momentum between $\alpha$ and $^9\text{B}$ clusters.

The ground state (1/2$^-$ state) has the compact shell structure with the radius of 2.52 fm. This state has large overlap (0.94) with the basis wave function obtained by the energy variation with the constraint $(N, \lambda, \mu) = (9, 0, 3)$ whose intrinsic density distribution is shown in Fig. 3 (a). The shell-model like character of the ground state can be confirmed by the large $S$-factors in the $^{12}\text{C}(0^+_1) \otimes p_{1/2}$ and $^{12}\text{C}(2^+_2) \otimes p_{3/2}$ channels, which are 0.81 and 0.94 respec-
FIG. 3. (Color online) Intrinsic matter and valence neutron density distributions of the basis wave functions obtained with the variation under the constraint on the H.O. quanta on $z = 0$ plane. Contour plot indicates the matter density distribution while the color plot indicates the valence neutron density distribution.

It is noted that $^{12}\text{C}(0^+_1) \otimes p_{1/2}$ and $^{12}\text{C}(2^+_1) \otimes p_{3/2}$ channels identically corresponds to $(0s)^4(0p_{3/2})^8(0p_{1/2})^1$ configuration if the $^{12}\text{C}(0^+_1)$ and $^{12}\text{C}(2^+_1)$ have the $(0s)^4(0p_{3/2})^8$ and $(0s)^4(0p_{3/2})^7(0p_{1/2})^1$ configurations, respectively.

While the ground state has the compact shell structure, the excited $1/2^-$ states have the larger rms radii than 2.70 fm. The enhancement of the rms radii in the excited $1/2^-$ states implies their developed cluster structure. The $1/2^-_2$ state at $E_x = 13.8$ MeV largely overlaps with wave function having $^9\text{Be} + \alpha$ cluster configuration shown in Fig. 3(b), which amounts to 0.46. Hence, this state has large $S$-factors in the $^9\text{Be}(3/2^-) \otimes l = 2$ and $^9\text{Be}(5/2^-) \otimes l = 2$ channels that are 0.11 and 0.09, respectively. The RWAs in the $^9\text{Be}(3/2^-) \otimes l = 2$ and $^9\text{Be}(5/2^-) \otimes l = 2$ channels have two nodes ($N = 6$) as shown in Fig. 4(b), while the those in the ground state have one node ($N = 4$). This means that the $1/2^-_2$ state is regarded as the nodal excitation of the inter-cluster motion between $^9\text{Be}$ and $\alpha$ clusters. Therefore, the $1/2^-_2$ state is not a Hoyle-analogue state but an excited $^9\text{Be} + \alpha$ cluster state, although it also has non-negligible $S$-factors in the $^{12}\text{C} + n$ channels.

The $1/2^-_3$, $1/2^-_4$ and $1/2^-_5$ states have large overlap with the basis wave functions displayed in Fig. 3(c), (d) and (e), respectively. These states have non-negligible $S$-factors in the $^{12}\text{C}(0^+_2) \otimes p_{1/2}$ channel but the magnitudes are less than 0.20. This means that the $^{12}\text{C}(0^+_2) \otimes$
\( p_{1/2} \) configuration (Hoyle-analogue configuration) does not manifest as a single excited state, but it is fragmented into these \( 1/2^- \) states. Thus, we conclude that there is no Hoyle-analogue \( 1/2^- \) state. Interestingly, these states also have the \( S \)-factors in the \( ^{12}\text{C}(2^+_{2}) \otimes p_{1/2} \) channel, which corresponds to the rotational excited state of the Hoyle state. Difference between the \( 1/2^-_3 \), \( 1/2^-_4 \) and \( 1/2^-_5 \) states is seen in different magnitudes of the coupling to the \( ^9\text{Be} + \alpha \) channels. Similar result was also obtained by T. Yamada et al. [16]. They argued that Hoyle-analogue state does not appear in the \( 1/2^- \) states because of the enhanced

FIG. 4. (Color online) The calculated RWAs of the \( 1/2^- \) states in the \( ^{12}\text{C} + n \) and \( ^9\text{Be} + \alpha \) channels. The RWAs which yields the larger \( S \) than 0.04 are displayed.
\( ^9\text{Be} + \alpha \) correlation induced by the attractive odd-parity \( \alpha - n \) interaction. We also confirm this as the non-negligible \( S \)-factors in the \( ^{12}\text{C}(0^+_2) \otimes p_{1/2}, ^{12}\text{C}(2^+_2) \otimes p_{3/2} \) and \( ^9\text{Be} + \alpha \) channels. In addition, our result shows the shrinkage of the rms radii compared to the Hoyle state (2.94 fm), which is also consistent to the interpretation suggested by T.Yamada et al. \[16\].

Although there is no Hoyle-analogue state, it is interesting to note that all of four \( 1/2^- \) states have large monopole transition matrix comparable with the Hoyle state, which, in total, exhaust 24% of the energy weighted sum rule which is consistent with the experiment \[31,32\]. One may wonder why number of the excited state with large monopole matrix is increased in \( ^{13}\text{C} \) than in \( ^{12}\text{C} \) despite of the fragmentation of the \( ^{12}\text{C}(0^+_2) \otimes 0p_{1/2} \) configuration into many states. The reason of the increase and the origin of the monopole strength of each state are explained as follows.

The origin of the monopole strength of the \( 1/2^-_2 \) state is the excitation of the relative motion between \( ^9\text{Be} \) and \( \alpha \) clusters. As already mentioned, the \( 1/2^-_2 \) state has a \( ^9\text{Be} + \alpha \) cluster structure in which the inter-cluster motion is excited by \( 2\hbar \omega \) from the ground state. Therefore, it naturally has the enhanced monopole strength. Different from the \( 1/2^-_2 \) state, the monopole strengths of the other \( 1/2^- \) states originates in the excitation of the \( ^{12}\text{C} \) core. In particular, we found that the monopole excitation of \( ^{12}\text{C}(2^+_1) \rightarrow ^{12}\text{C}(2^+_1) \) plays an important role as well as the excitation of \( ^{12}\text{C}(0^+_1) \rightarrow ^{12}\text{C}(0^+_1) \). To elucidate this, we here show a simple estimation of the monopole transition matrix. First, let us assume that the ground state of \( ^{13}\text{C} \) (the \( 1/2^-_1 \) state) has a \( (0s_{1/2})^4(0p_{3/2})^8(0p_{1/2})^1 \) configuration, and \( ^{12}\text{C}(0^+_1) \) and \( ^{12}\text{C}(2^+_1) \) respectively have a \( (0s_{1/2})^4(0p_{3/2})^8 \) and \( (0s_{1/2})^4(0p_{3/2})^7(0p_{1/2})^1 \) configurations. Then, \( ^{13}\text{C}(1/2^-_1) \) can be written as,

\[
\ket{^{13}\text{C}(1/2^-_1)} = n_0 \ket{\mathcal{A} \{ ^{12}\text{C}(0^+_1) \otimes 0p_{1/2} \}},
\]

\[
= n_2 \ket{\mathcal{A} \{ ^{12}\text{C}(2^+_1) \otimes 0p_{3/2} \}},
\]

where \( n_0 \) and \( n_2 \) denote the normalization factors defined as,

\[
n_0 = \langle \mathcal{A} \{ ^{12}\text{C}(0^+_1) \otimes 0p_{1/2} \} | \mathcal{A} \{ ^{12}\text{C}(0^+_1) \otimes 0p_{1/2} \} \rangle^{-1/2},
\]

\[
n_2 = \langle \mathcal{A} \{ ^{12}\text{C}(2^+_1) \otimes 0p_{3/2} \} | \mathcal{A} \{ ^{12}\text{C}(2^+_1) \otimes 0p_{3/2} \} \rangle^{-1/2}.
\]

Second, the excited \( 1/2^- \) states (\( 1/2^-_3, 1/2^-_4 \) and \( 1/2^-_5 \)) may be written as,

\[
\ket{^{13}\text{C}(1/2^-_n)} = a_n \ket{\mathcal{A} \{ ^{12}\text{C}(0^+_1) \otimes 0p_{1/2} \}}
+ b_n \ket{\mathcal{A} \{ ^{12}\text{C}(2^+_1) \otimes 0p_{3/2} \}} + \text{(other configurations)}
\]
since they are dominated by the $^{12}\text{C}(0_2^+) \otimes 0p_{1/2}$ and $^{12}\text{C}(2_2^+) \otimes 0p_{3/2}$ configurations. Here, $n_0'$ and $n_2'$ are the normalization factors defined in a similar manner, and we assumed that the neutron orbits are unchanged from the ground state. We also assume that $n_0' \{^{12}\text{C}(0_1^+) \otimes 0p_{1/2}\}$ and $n_2' \{^{12}\text{C}(2_1^+) \otimes 0p_{3/2}\}$ are orthogonal, and their amplitudes are represented by $a$ and $b$.

Finally, following the discussion made by T. Yamada et al. \[20\], we rewrite the monopole operator as

$$
\mathcal{M}^{I=0}(13\text{C}) = \mathcal{M}^{I=0}(12\text{C}) + \frac{12}{13} r^2, \tag{28}
$$

where $\mathcal{M}^{I=0}(12\text{C})$ acts on the $^{12}$C core, while $r$ denotes the coordinate between $^{12}$C core and valence neutron. With these expressions, we can derive an estimation for the monopole transition matrix,

$$
M(IS0) = \langle 13\text{C}(1/2^-_ex) \mid \mathcal{M}^{I=0}(13\text{C}) \mid 13\text{C}(1/2^-_1) \rangle \\
= a^* \frac{n_0'}{n_0} \langle 12\text{C}(0_1^+) \mid \mathcal{M}^{I=0}(12\text{C}) \mid 12\text{C}(0_1^+) \rangle \\
+ b^* \frac{n_2'}{n_2} \langle 12\text{C}(2_1^+) \mid \mathcal{M}^{I=0}(12\text{C}) \mid 12\text{C}(2_1^+) \rangle \\
+ (\text{other channels}). \tag{29}
$$

The derivation of the Eq. (29) is almost same with that explained in Ref. \[20\]. Thus, the monopole strengths of the excited $1/2^-$ states can be related to the monopole transitions of $^{12}$C. Here, it is noted that $n_0'/n_0$ and $n_2'/n_2$ are almost equal to 1, and $\langle 12\text{C}(2_2^+) \mid \mathcal{M}^{I=0}(12\text{C}) \mid 12\text{C}(2_1^+) \rangle$ is as large as or even larger than $\langle 12\text{C}(0_2^+) \mid \mathcal{M}^{I=0}(12\text{C}) \mid 12\text{C}(0_1^+) \rangle$. Therefore, if $a$ and $b$ are not small and have the same phase, the transition matrix can be large. From this simple estimation, it is also clear that the $^{12}\text{C}(2_2^+) \otimes 0p_{3/2}$ channel increases the number of the $1/2^-$ states having large monopole transition strengths.

### C. Structure of the $3/2^-$ states

The $3/2^-$ states are also candidates of Hoyle-analogue state with $P$-wave valence neutron. The properties of the $3/2^-$ states below 20 MeV are summarized in Fig. 5. In our calculation, except for the $3/2^-_1$ and $3/2^-_4$ states, the $3/2^-$ states have larger matter rms radius than 2.75 fm.
The $3/2^-$ state is obviously dominated by the $^{12}\text{C}(2^+_1) \otimes p_{1/2}$ channel ($S = 0.87$), and its configuration is concluded as $(0p_{3/2})^{-1}(0p_{1/2})^{2}$ because $^{12}\text{C}(2^+_1)$ is dominated by the $(0p_{3/2})^{-1}(0p_{1/2})^{1}$ configuration. The properties of the other $3/2^-$ states are not clear, since their $S$-factors are small in all calculated channels ($S \leq 0.14$). In particular, there are no state having sizable $S$-factor in the $^{12}\text{C}(0^+_2) \otimes p_{3/2}$ channel except for the $3/2^-$ state. Therefore, we conclude that there is no Hoyle-analogue $3/2^-$ state below 20 MeV.

As explained above, the Hoyle-analogue state with $P$-wave neutron does not appear. This is due to the strong attractive interaction between $\alpha$ cluster and $P$-wave neutron, which induces the coupling with many different channels. As a result, the $^{12}\text{C}(0^+_2) \otimes p_{1/2}$ and $^{12}\text{C}(0^+_2) \otimes p_{3/2}$ configurations are fragmented into many states.

FIG. 5. (Color online) The calculated excitation spectra, the rms matter radii $r_{rms}$ and the $S$-factors of $3/2^-$ states below 20 MeV. The $S$-factors smaller than 0.05 are not displayed.
D. Structure of the $1/2^+$ states

In the above discussion, we showed that there is no Hoyle-analogue state in the $1/2^-$ and $3/2^-$ states, where the strong attractive interaction between $\alpha$-clusters and neutron induces the coupling with many different channels. On other hand, because the interaction between $\alpha$ and $S$-wave neutron is weaker than that for $P$-wave neutron, we expect that the $1/2^+$ state is promising candidate of the Hoyle-analogue state.

The calculated rms radii and $S$-factors are shown in Fig. 6. The rms radius of the $1/2^+_1$ state is only 2.62 fm and suggests that $1/2^+_1$ state has a compact shell structure. On the other hand, the radii of the $1/2^+_2$ and $1/2^+_3$ states are larger than 2.75 fm, which indicates their developed cluster structure. This point can be clearly confirmed by analysis of the $S$-factor. The $1/2^+_1$ state is a particle-hole excited state, because its $S$-factor in the $^{12}\text{C}(0^+_1) \otimes s_{1/2}$
channel is 0.84 and the other channel contributions are relatively small. The RWA of the 
$1/2^+_1$ state in the $^{12}\text{C}(0^+_1) \otimes s_{1/2}$ has one node (Fig. 7 (a)), and hence, its particle-hole 
configuration is $(0p_{1/2})^{-1}(1s_{1/2})^1$.

The $1/2^+_2$ state is located 3.0 MeV above $^{12}\text{C}(0^+_2) + n$ threshold energy and has the largest 
$S$-factor of 0.64 in the $^{12}\text{C}(0^+_2) \otimes s_{1/2}$ channel among the calculated $1/2^+$ states. The RWA 
in the $^{12}\text{C}(0^+_2) \otimes s_{1/2}$ channel has one node (Fig. 7 (b)), which indicates that the $1/2^+_2$ state is 
the Hoyle-analogue $1/2^+$ state with $^{12}\text{C}(0^+_2) \otimes 1s_{1/2}$ configuration. It has the largest overlap 
with the basis wave function shown in the Fig. 3 (g) but its magnitude is 0.48. This state 
also has non-negligible overlaps with the various basis wave functions having $3\alpha + n$ cluster 
structure, which suggests the dilute gas-like nature of the $1/2^+_2$ state. However, the rms 
radius of the $1/2^+_2$ state (2.76 fm) is shrank compared to that of the Hoyle state (2.94 fm) in 
our calculation. This shrinkage indicates that the Hoyle-analogue nature is weakened by the 
interaction between $^{12}\text{C}$ and valence neutron. This point can be seen in the coupling with 
the other $^{12}\text{C} + n$ channels. For example, the $1/2^+_2$ state has the non-negligible $S$-factors of 
0.11 and 0.18 in the $^{12}\text{C}(1^-_1) \otimes p_{3/2}$ and $^{12}\text{C}(2^+_2) \otimes d_{5/2}$ channels, respectively.

The $1/2^+_3$ state has quite different nature from the $1/2^+_1$ and $1/2^+_2$ states. This state has 
almost zero $S$-factors in the $^{12}\text{C} + n$ channels, but the $S$-factors in the $^9\text{Be} + \alpha$ channels 
amounts to 0.32 in total. Therefore, we conclude that the $1/2^+_3$ state has the $^9\text{Be} + \alpha$ cluster 
structure. Interestingly, its density distribution (Fig. 3 (h)) shows the similar structure to the 
$1/2^+_2$ state (Fig. 3 (b)). Furthermore, the $S$-factors indicate that they are dominated by 
the $^9\text{Be}(3/2^-_1) + \alpha$ and $^9\text{Be}(5/2^-_1) + \alpha$ channels. Therefore, we consider that the $1/2^+_2$ and 
$1/2^+_3$ state constitute the parity-doublet having the $^9\text{Be} + \alpha$ cluster structure. A similar bent-
armed $3\alpha + n$ cluster structure in negative parity states was also discussed by N. Furutachi et al. [17] in relation with the inversion-doublet of $^9\text{Be} + \alpha$ cluster band suggested by M. Millin 
and W. von Oertzen [34]. The predicted $3/2^-_2$ state has the similar bent-armed $3\alpha + n$ 
cluster structure to the $1/2^+_2$ and $1/2^+_3$ states in the present calculation.

The calculated IS1 transition matrix $M(IS1)$ (Fig. 6) indicate that the $1/2^+_1$ state is 
strongly populated by IS1 transition from the ground state ($M(IS1) = 0.95$ W.u.) while 
the $1/2^+_2$ and $1/2^+_3$ states are not. In particular, despite of its Hoyle-analog structure, the 
IS1 transition strength of the $1/2^+_3$ state is unexpectedly small. This may be explained as 
follows. Following the discussion in Refs. [22, 35], we decompose the system into $^{12}\text{C}$ core
FIG. 7. (Color online) The calculated RWAs of the $1/2^+$ states in the $^{12}\text{C}+n$ and $^9\text{Be}+\alpha$ channels. The RWAs which yields the larger $S$ than 0.04 are displayed.

and the valence neutron, and rewrite the IS1 operator as

$$
\mathcal{M}_{\mu}^{IS1} = \frac{132}{169} r^2 Y_{1\mu}(r) - \frac{5}{39} \sum_{i\in^{12}\text{C}} \xi_i^2 Y_{1\mu}(r)
+ \frac{4\sqrt{2\pi}}{39} \left[ \sum_{i\in^{12}\text{C}} Y_2(\xi_i) \otimes Y_1(r) \right]_{1\mu}
+ \sum_{i\in^{12}\text{C}} \xi_i^2 Y_{1\mu}(\xi_i)
$$

(30)

where $\xi_i$ denote the internal coordinates of the $^{12}\text{C}$ core, while $r$ denotes the valence neutron coordinate. The first term of the Eq. (30) is dependent only on $r$ and induces the IS1
transition of the valence neutron. Therefore, the $1/2^+_1$ state which has the $1p1h$ configuration is mainly excited by this term. This is the reason why $1/2^+_1$ state has strong IS1 transition matrix comparable with Weisskopf estimate.

On the other hand, the second and third terms induce the monopole and quadrupole transitions of $^{12}$C core. Therefore, if these terms act on the ground state wave function (Eq. (30)), they bring about the core excitations $^{12}$C$(0^+_1) \rightarrow ^{12}$C$(0^+_2)$ and $^{12}$C$(2^+_1) \rightarrow ^{12}$C$(0^+_2)$ combined with the valence neutron excitations to yield the Hoyle-analogue $1/2^+_2$ state. Since the transition matrix for the core excitations are large, we expect that the IS1 excitation from the ground state to the $1/2^+_2$ state is enhanced. However, the coefficients for these two terms are rather small ($5/39$ and $4\sqrt{2\pi}/39$). As a result, the $1/2^+_2$ state has relatively small transition strength despite of its dilute gas-like nature.

Thus, the Hoyle-analogue $1/2^+_2$ state has unexpectedly small $M(IS1)$. However, in the preliminary reported IS1 transition strength distribution of $^{13}$C, there is a small peak around 13 MeV [31–33], which is close to the our prediction and may correspond to the Hoyle-analogue $1/2^+_2$ state.

Finally, we discuss the decay width of the $1/2^+$ states. Owing to the cluster structure of the $1/2^+_2$ and $1/2^+_3$ states, they have unique decay patterns. The calculated the reduced width $\gamma^2$ of the $1/2^+_2$ and $1/2^+_3$ states in the $^{12}$C + n and $^9$Be + $\alpha$ channels are shown in Fig. 8. The $1/2^+_2$ state has largest reduced width of 0.96 MeV in the $^{12}$C$(0^+_2) \otimes s_{1/2}$ channel. The reduced widths in the other channels are negligibly small. On the other hand, The reduced decay widths in the $^9$Be$(3/2^-_1) \otimes l = 1$ and $^9$Be$(5/2^-_1) \otimes l = 3$ channels are largest in the $1/2^+_3$ state. Because of the larger $Q$-value, the $1/2^+_3$ state dominantly decays via the $^9$Be$(3/2^-_1) \otimes l = 1$ channel. Therefore, the strong decays via the $^{12}$C$(0^+_2) \otimes s_{1/2}$ and $^9$Be$(3/2^-_1) \otimes l = 1$ channels are signature of the $1/2^+_2$ and $1/2^+_3$ states, respectively.

IV. SUMMARY

We studied the Hoyle-analogue states in $^{13}$C based on AMD. The basis wave functions are obtained by the energy variation with constraint on the expectation values of harmonic oscillator quanta. Using these basis wave functions, GCM calculation was performed to obtain the excitation energies and the eigen wave functions.

The analysis of the $S$-factors in the $^{12}$C + n and $^9$Be + $\alpha$ channels revealed the char-
FIG. 8. (Color online) The calculated reduced widths $\gamma^2$ of the $1/2^+_2$ and $1/2^+_3$ states for $^{12}$C + $n$ and $^9$Be + $\alpha$ channels. The matching radius $a = 4.5$ fm is applied.

Actors of the ground and excited states of $^{13}$C. The ground state ($1/2^-_1$ state) has the $(0s)^4(0p_{3/2})^8(0p_{1/2})^1$ configuration, and the $3/2^-_1$ and $1/2^+_1$ states have the $1p1h$ configurations. In contrast to these shell model like states, the non-yrast states have developed cluster structure. The $1/2^-_2$ and $1/2^+_3$ states constitute the inversion-doublet of the bent-armed $^9$Be + $\alpha$ cluster structure. The $1/2^-_3$, $1/2^-_4$ and $1/2^-_5$ states are $3\alpha + n$ cluster states in which the $^{12}$C$(0^+_2) \otimes 0p_{1/2}$ and $^{12}$C$(2^+_2) \otimes 0p_{3/2}$ configurations are mixed. However, they cannot be regarded as the Hoyle-analogue state because the $S$-factors in the $^{12}$C$(0^+_2) \otimes p_{1/2}$ channel are small. Similarly, there is no Hoyle-analogue state in $3/2^-$ states, because of
the fragmentation of $^{12}\text{C}(0^+ \rightarrow 2s_{1/2} \otimes 0p_{3/2}$ configuration into many states. The absence of the Hoyle-analogue states in $P$-wave states is attributed to the strong $\alpha - n P$-wave interaction. On the hand, the $1/2_2^+$ state located at 15.4 MeV is Hoyle-analogue state dominated by the $^{12}\text{C}(0^+ \rightarrow 2s_{1/2} \otimes 1s_{1/2}$ configuration with $S = 0.64$.

The characters of the $1/2^-$ and $1/2^+$ states are reflected to the IS0 and IS1 transitions. The IS0 transitions to the excited $1/2^-$ states are comparable to the Hoyle state in $^{12}\text{C}$. The origins of the enhanced IS0 transitions are clustering nature of the excited $1/2^-$ states. In particular, the enhanced $M(IS0)$ of the $1/2_3^-, 1/2_4^-$ and $1/2_5^-$ originate in the coupling of the $^{12}\text{C}(0^+ \rightarrow 2s_{1/2} \otimes 2p_{3/2}$ configurations. Contrary, the IS1 transition to the Hoyle-analogue $1/2_2^+$ state is suppressed due to the property of the IS1 transition operator.

The decay widths of the $1/2^+$ states show very unique patterns. The Hoyle-analogue $1/2_2^+$ state dominantly decays via $^{12}\text{C}(0^+ \rightarrow 2s_{1/2}$ channel but the $1/2_3^+$ state decays via $^9\text{Be}(3/2^+ \rightarrow 1s_{1/2}$ channel. These unique decay patterns are key observable to identify the Hoyle-analogue $1/2^+$ state.

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