Structure of non-negative posets of Dynkin type $\mathbb{A}_n$

Marcin Gąsiorek
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland
mgasiorek@mat.umk.pl

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Abstract

A poset $I = (\{1, \ldots, n\}, \leq_I)$ is called non-negative if the symmetric Gram matrix $G_I := \frac{1}{2}(C_I + C_I^T) \in \mathbb{M}_n(\mathbb{R})$ is positive semi-definite, where $C_I \in \mathbb{M}_n(\mathbb{Z})$ is the (0,1)-matrix encoding the relation $\leq_I$. Every such a connected poset $I$, up to the $\mathbb{Z}$-congruence of the $G_I$ matrix, is determined by a unique simply-laced Dynkin diagram $\text{Dyn}_I \in \{A_m, D_m, E_6, E_7, E_8\}$. We show that $\text{Dyn}_I = A_m$ implies that the matrix $G_I$ is of rank $n$ or $n-1$. Moreover, we depict explicit shapes of Hasse digraphs $H(I)$ of all such posets $I$ and devise formulae for their number.

Mathematics Subject Classifications: 05C50, 06A07, 06A11, 15A63, 05C30

1 Introduction

By a finite partially ordered set (poset) $I$ of size $n$ we mean a pair $I = (V, \leq_I)$, where $V := \{1, \ldots, n\}$ and $\leq_I \subseteq V \times V$ is a reflexive, antisymmetric and transitive binary relation. Every poset $I$ is uniquely determined by its incidence matrix

$$C_I = [c_{ij}] \in \mathbb{M}_n(\mathbb{Z}), \text{ where } c_{ij} = 1 \text{ if } i \leq_I j \text{ and } c_{ij} = 0 \text{ otherwise,}$$

(1.1)
i.e., a square (0,1)-matrix that encodes the relation $\leq_I$. It is known that various mathematical classification problems can be solved by a reduction to the classification of indecomposable $K$-linear representations ($K$ is a field) of finite digraphs or matrix representations of finite posets, see [15]. Inspired by these results, here we study posets that are non-negative in the following sense. A poset $I$ is defined to be non-negative of rank $m$ if its symmetric Gram matrix $G_I := \frac{1}{2}(C_I + C_I^T) \in \mathbb{M}_n(\mathbb{R})$ is positive semi-definite of rank $m$.

Non-negative posets are classified by means of signed simple graphs as follows. One associates with a poset $I = (V, \leq_I)$ the signed graph $\Delta_I = (V, E, \text{sgn})$ with the set of edges $E = \{\{i, j\}; i <_I j \text{ or } j <_I i\}$ and the sign function $\text{sgn}(e) := 1$ for every edge (i.e., signed graph with positive edges only), see [19] and Remark 1.4. In particular, $I$ is called connected, if $\Delta_I$ is connected. We note that $\Delta_I$ is uniquely determined by its adjacency matrix $\text{Ad}_{\Delta_I} := 2(G_I - \text{id}_n)$, where $\text{id}_n \in \mathbb{M}_n(\mathbb{Z})$ is an identity matrix. Analogously as in the case of posets, a signed graph $\Delta$ is defined to be non-negative of rank $m$ if its symmetric Gram matrix $G_\Delta := \frac{1}{2}\text{Ad}_\Delta + \text{id}_n$ is positive semi-definite of rank $m$. Following
we call two signed graphs $\Delta_1$ and $\Delta_2$ weakly Gram $Z$-congruent if $G_{\Delta_1}$ and $G_{\Delta_2}$ are $Z$-congruent, i.e., $G_{\Delta_2} = B^T G_{\Delta_1} B$ for some $B \in \text{Gl}_n(Z) := \{ A \in \mathbb{M}_n(Z); \det A = \pm 1 \}$. It is easy to check that this relation preserves definiteness and rank.

We recall from [18] and [20] that every connected non-negative signed simple graph $\Delta$ of rank $m = n - r$ is weakly Gram $Z$-congruent with the canonical $r$-vertex extension of simply laced Dynkin diagram $\text{Dyn}_{\Delta} \in \{ A_m, D_m, E_6, E_7, E_8 \}$, called the Dynkin type of $\Delta$. In particular, every positive (i.e., of rank $n$) connected $\Delta$ is weakly Gram $Z$-congruent with a unique simply-laced Dynkin diagram $\text{Dyn}_{\Delta}$ of Table 1.2.

**Table 1.2: Simply-laced Dynkin diagrams**

\[
\begin{align*}
A_n &: 1 \overbrace{2 \cdots \cdots n-1}^{(n \geq 1)}; \\
D_n &: 1 \underbrace{2 \cdots \cdots n-1}_{(n \geq 1)} \quad E_6 &: 1 \overbrace{2 \cdots \cdots 5}^{6}; \\
E_7 &:\quad 1 \overbrace{2 \cdots \cdots 4}^{5} 6 7; \\
E_8 &:\quad 1 \overbrace{2 \cdots \cdots 3}^{4} 5 6 7 8.
\end{align*}
\]

Analogously, every principal (i.e., of rank $n - 1$) connected bigraph $\Delta$ is weakly Gram $Z$-congruent with $\widetilde{\text{Dyn}}_{\Delta} \in \{ \widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8 \}$ diagram of Table 1.3, which is a one point extension of a diagram of Table 1.2.

**Table 1.3: Simply-laced Euclidean diagrams**

\[
\begin{align*}
\widetilde{A}_n &: 1 \overbrace{2 \cdots \cdots n-1}^{n+1} n; \\
\widetilde{D}_n &: 1 \underbrace{2 \cdots \cdots n-1}_{(n \geq 4)} \quad \widetilde{E}_6 &: 1 \overbrace{2 \cdots \cdots 5}^{6}; \\
\widetilde{E}_7&:\quad 1 \overbrace{2 \cdots \cdots 4}^{5} 6 7 8; \\
\widetilde{E}_8 &:\quad 1 \overbrace{2 \cdots \cdots 3}^{4} 5 6 7 8 9.
\end{align*}
\]

**Remark 1.4.** We are using the following notations, see [2, 17, 18, 19].

(a) A simple graph $G = (V, E)$ is viewed as the signed graph $\Delta_G = (V, E, \text{sgn})$ with a sign function $\text{sgn}(e) := -1$ for every $e \in E$, i.e., signed graph with negative edges only.

(b) We denote positive edges by dotted lines and negative as full ones, see [2, 17].

By setting $\text{Dyn}_{I} := \text{Dyn}_{\Delta}$, one associates a Dynkin diagram with an arbitrary connected non-negative poset $I$.

In the present work, we give a complete description of connected non-negative posets $I = (V, \leq_I)$ of Dynkin type $\text{Dyn}_{I} = A_m$ in terms of their Hasse digraphs $\mathcal{H}(I)$, where $\mathcal{H}(I)$ is the transitive reduction of the acyclic digraph $\mathcal{D}(I) = (V, A_I)$, with $i \rightarrow j \in A_I$ iff $i <_I j$ (see also Definition 2.1). The main result of the manuscript is the following theorem that establishes the correspondence between combinatorial and algebraic properties of non-negative posets of Dynkin type $A_m$. 
Theorem 1.5. Assume that $I$ is a connected poset of size $n$ and $\mathcal{H}(I)$ is its Hasse digraph.

(a) $I$ is non-negative of Dynkin type $\text{Dyn}_{I} = A_{n}$ if and only if $\mathcal{H}(I)$ is a path graph.

(b) $I$ is non-negative of Dynkin type $\text{Dyn}_{I} = A_{n-1}$ if and only if $\mathcal{H}(I)$ is a cycle graph and $\mathcal{H}(I)$ has at least two sinks.

(c) If $I$ is non-negative of Dynkin type $\text{Dyn}_{I} = A_{m}$, then $m \in \{n, n-1\}$.

In particular, we confirm Conjecture 6.4 stated in [8] by showing that in the case of connected non-negative posets of Dynkin type $A_{m}$, there is a one-to-one correspondence between positive posets and connected digraphs whose underlying graph is a path. We give a similar description of principal posets: there is a one-to-one correspondence between positive posets and connected digraphs whose underlying graph is a path. We show that this characterization is complete: there are no connected non-negative posets of rank $m < n - 1$.

Moreover, using the results of Theorem 1.5, we devise a formula for the number of all, up to isomorphism, connected non-negative posets of Dynkin type $A_{m}$.

Theorem 1.6. Let $\text{Nneg}(n, A)$ be the number of all non-negative posets $I$ of size $n \geq 1$ and Dynkin type $\text{Dyn}_{I} = A_{m}$. Then

$$\text{Nneg}(n, A) = \frac{1}{2n} \sum_{d|n} (2^{\varphi(d)}) + \left[2^{n-2} + 2^{\left\lfloor \frac{n-2}{2} \right\rfloor} - \frac{n+1}{2}\right],$$

(1.7)

where $\varphi$ is Euler’s totient function.

2 Preliminaries

Throughout, we mainly use the terminology and notation introduced in [9, 10, 15, 19] (in regard to posets), [2, 16, 17, 18] (quadratic forms), and [5] (graph theory). In particular, by $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$ we denote the set of non-negative integers, the ring of integers, and the real number fields, respectively. We use a row notation for vectors $v = [v_{1}, \ldots, v_{n}]$ and write $v^{tr}$ to denote column vectors.

Two (directed) graphs $G = (V, E)$ and $G' = (V', E')$ are called isomorphic $G \simeq G'$ if there exist a bijection $f : V \rightarrow V'$ that preserves (arcs) edges. By degree $\deg_{G}(v)$ of a vertex $v \in V$ we mean the number of edges incident with $v$. We call $G$ a path graph if $|V| = 1$ and $|E| = 0$ or $G \simeq (u \rightarrow \ldots \rightarrow v)$. A graph $G$ is connected if $P(u, v) \subseteq G$ for every $u \neq v \in V$. By underlying graph $\overline{D}$ of a digraph $D$ we mean a graph obtained from $\mathcal{D}$ by forgetting the orientation of its arcs. A digraph $D$ is connected if $\overline{D}$ is connected. A connected graph $G$ is called a tree if $G$ does not contain any cycle. A digraph $D$ is called acyclic if it contains no induced subdigraph isomorphic to $\overline{D}$. We call a vertex $v$ of a digraph $D = (V, A)$ a source (minimum) if it is not a target of any arc $\alpha \in A$. Analogously, we call $v \in D$ a sink (maximum) if it is not a source of any arc.

Given two elements $x, y \in V$ of a poset $I = (V, \leq_{I})$ we write:

- $x <_{I} y$ when $x \leq_{I} y$ and $x \neq y$;
- $x \lessdot_{I} y$ when $y$ covers $x$, i.e., $x <_{I} y$ and there is no such $z \in V$, that $x <_{I} z <_{I} y$. 

Moreover, by \( N_I(x) := \{ z \in I; x <_I z \} \cup \{ z \in I; z <_I x \} \) we denote the set of elements that either cover \( x \) or are covered by \( x \) in \( I \).

**Definition 2.1.** *Hasse digraph* of a poset \( I = (V, \leq_I) \) is a simple directed graph \( \mathcal{H}(I) = (V, A) \) with the set of arcs defined as follows: \( x \to y \in A \) if \( x <_I y \).

We call a poset \( I \) *connected* if the graph \( \overline{\mathcal{H}(I)} := \overline{\mathcal{H}(I)} \) is connected (equivalently, when the signed graph \( \Delta_I \) is connected), and we note that every minimal (maximal) element in \( I \) corresponds to a source (sink) in the Hasse digraph \( \mathcal{H}(I) \). We say that \( I \) is a *one-peak poset* if \( I \) has exactly one maximal element. Every finite acyclic digraph \( D = (V, A) \) defines the poset \( I_D := (V, \leq_D) \), where \( a \leq_D b \) if \( a = b \) or there is an oriented path \( P(a, b) := a \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow b \subseteq D \). We note that \( \mathcal{H}(I_D) \neq D \) in general, see Example 2.5. By \( I(k_1, \ldots, k_r) := I \setminus \{ k_1, \ldots, k_r \} \) we denote the induced subposet of \( I \) whose set of elements equals \( \{ 1, \ldots, n \} \setminus \{ k_1, \ldots, k_r \} \).

Following [9, 17, 19], we associate with a poset \( I \) of size \( n \):

- the unit quadratic form \( q_I : \mathbb{Z}^n \to \mathbb{Z} \) defined by the formula
  \[
  q_I(x) := \sum_{i \in \{ 1, \ldots, n \}} x_i^2 + \sum_{i <_I j} x_i x_j = v \cdot G_I \cdot v^T, \tag{2.2}
  \]

- and its kernel
  \[
  \text{Ker } q_I := \{ v \in \mathbb{Z}^n; q_I(v) = 0 \} \subseteq \mathbb{Z}^n, \tag{2.3}
  \]

where \( G_I \in M_n(\mathbb{Z}) \) is the symmetric Gram matrix of \( I \). It is known that a poset \( I \) is non-negative of rank \( m \) if and only if the quadratic form \( q_I \) is positive semi-definite (i.e., \( q_I(v) \geq 0 \) for every \( v \in \mathbb{Z}^n \)) and its kernel \( \text{Ker } q_I \subseteq \mathbb{Z}^n \) is a free abelian subgroup of rank \( n - m \), see [17]. We call a non-negative poset \( I \) *positive* if \( m = n \), *principal* if \( m = n - 1 \), and *indefinite* if its symmetric Gram matrix \( G_I \) is not positive/negative semidefinite.

**Remark 2.4.** A poset \( I \) is indefinite if and only if there exist such vectors \( v, w \in \mathbb{Z}^n \) that \( q_I(v) > 0 \) and \( q_I(w) < 0 \). Since \( q_I([1, 0, \ldots, 0]) = 1 > 0 \) for every poset \( I \), to show that given \( I \) is indefinite, it suffices to show that \( q_I(w) < 0 \) for some \( w \in \mathbb{Z}^n \).

**Example 2.5.** To illustrate the definitions, we consider digraph \( D = \{ 1, 2, 3, 4 \}, \{ 2 \to 1, 2 \to 3, 2 \to 4, 1 \to 3, 4 \to 3 \} \).

We have \( I_D = \{ 1, 2, 3, 4 \}, \{ 2 \leq_D 1, 2 \leq_D 3, 2 \leq_D 4, 1 \leq_D 3, 4 \leq_D 3 \} \).

\[
\begin{align*}
D &= \begin{array}{c}
\circ & \circ & \circ \\
1 & 2 & 3
\end{array}, & 
\mathcal{H}(I_D) &= \begin{array}{c}
\circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4
\end{array}, & 
\Delta_I &= \begin{array}{c}
\circ & \circ & \circ \\
1 & 2 & 3
\end{array}, \\
G_{I_D} &= \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & 1
\end{bmatrix} = G_{\Delta_I}, & 
C_{I_D} &= \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}, & 
\text{Ad}_{I_D} &= \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix},
\end{align*}
\]

where we denote the positive edges of a signed graph by dotted lines. Moreover:

- \( 2 \) is minimal in \( I_D \) and \( 3 \) is maximal in \( I_D \), equivalently: \( 2 \) is a source in \( \mathcal{H}(I_D) \) and \( 3 \) is a sink in \( \mathcal{H}(I_D) \),

- \( q_{I_D}(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_2 + x_2 x_3 + x_3 x_4 + x_1 x_3 + x_2 x_4 = (x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3)^2 + \frac{3}{4} (x_2 + x_3 + \frac{2}{3} x_4)^2 + \frac{2}{3} (x_2 + \frac{3}{2} x_4)^2 + \frac{1}{2} x_4^2 \),

- \( \text{Ker } q_{I_D} = \{ 0 \} \subseteq \mathbb{Z}^4 \) and poset \( I_D \) is non-negative of rank 4, i.e., positive.
Since $G_{D_4} = B^r G_{\Delta_{I_P}} B$, where
\[
D_4 = \begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}, \quad G_{D_4} = \begin{bmatrix}
1 & 0 & -\frac{1}{2} & 0 \\
0 & 1 & -\frac{1}{2} & 0 \\
-\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\
0 & 0 & -\frac{1}{2} & 1
\end{bmatrix} = \frac{1}{2} A_{D_4} + \text{id}_n \text{ and } B = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 \\
-1 & -1 & 0 & -1 \\
0 & 0 & -1 & 0
\end{bmatrix},
\]
we conclude that $\text{Dyn}_{I_P} = D_4$. To finish the example, we note that elements of the adjacency matrix $A_{D_4}$ are negative because we view graph $D_4$ as a signed graph.

We recall from Section 1, that the Dynkin type of a connected non-negative poset $I$ of size $n$ and rank $m$ is such a simply-laced Dynkin diagram $\text{Dyn}_I \in \{A_m, D_m, E_6, E_7, E_8\}$, that the signed graph $\Delta_I$ is weakly Gram $\mathbb{Z}$-congruent with the canonical $r$-vertex extension of $\text{Dyn}_I$, where $r = n - m$. Equivalently, Dynkin type can be defined without referring to canonical $r$-vertex extensions, see [10, 18, 20, 21] and [2]. Although such a definition is more technical, it yields better insight into the combinatorial structure of non-negative posets. First, we need the following fact.

**Fact 2.6.** Assume that $I$ is a connected non-negative poset of size $n$ and rank $m$, $r = n - m$, and $q_1: \mathbb{Z}^n \rightarrow \mathbb{Z} (2.2)$ is the quadratic form associated with $I$.

(a) There exists such a basis $h^{k_1}, \ldots, h^{k_r}$ of the free abelian group $\text{Ker } q_1 \subseteq \mathbb{Z}^n$, that $h^{k_i} = 1$ and $h^{k_i} = 0$ for $1 \leq i, j \leq r$ and $i \neq j$, where $1 \leq k_1 < \ldots < k_r \leq n$.

(b) $I^{(a_1, \ldots, a_s)}$ is a connected and non-negative poset of size $n - s$ and rank $m$, for every $\{a_1, \ldots, a_s\} \subseteq \{k_1, \ldots, k_r\}$ and $1 \leq s \leq r$.

**Proof.** Apply [21, Lemma 2.7] and [20, Theorem 2.1] to the bigraph $\Delta_I$. \hfill \Box

The Dynkin type of a connected non-negative poset $I$ is defined to be a Dynkin diagram $\text{Dyn}_I \in \{A_m, D_m, E_6, E_7, E_8\}$ that determines $I$ uniquely, up to a weak Gram $\mathbb{Z}$-congruence.

**Definition 2.7.** Assume that $I$ is a connected non-negative poset of rank $m$ and size $n$. The Dynkin type $\text{Dyn}_I \in \{A_m, D_m, E_6, E_7, E_8\}$ is the unique simply-laced Dynkin diagram, viewed as a signed graph, such that $\Delta_I \sim_{\mathbb{Z}} \text{Dyn}_I$.

where $I' := I$ if $m = n$ and $I' := I^{(k_1, \ldots, k_r)}$ (see Fact 2.6(b)) otherwise.

We note that Dynkin type $\text{Dyn}_I$ can be calculated using the inflation algorithm [17, Algorithm 3.1] applied to the bigraph $\Delta_I$.

\[pA_r^*: \begin{array}{c}
\begin{array}{c}
\bullet \\
p_1
\end{array}
\end{array}, \quad \hat{D}_p^* \odot A_{r-p}: \begin{array}{c}
\begin{array}{c}
\bullet \\
p_1
\end{array}
\end{array}, \quad D_p^*: \begin{array}{c}
\begin{array}{c}
\bullet \\
p_1
\end{array}
\end{array}, \quad \hat{D}_p^* \odot A_{r-p}: \begin{array}{c}
\begin{array}{c}
\bullet \\
p_1
\end{array}
\end{array}
\]

Table 2.8: One-peak positive posets of Dynkin type $A_{r+1}$ and $D_{r+1}$

The aim of this manuscript is to give a full structural characterization of connected non-negative posets $I$ of Dynkin type $\text{Dyn}_I = A_m$. We note that such a result is known in the case of one-peak positive and principal posets, see [9, Theorem 5.2] and [7, Theorem 3.5]. In particular, we have the following.
Theorem 2.9. One-peak poset $I$ of size $n$ is positive if and only if its Hasse digraph $\mathcal{H}(I)$ is isomorphic to:

(a) the digraph $\tilde{\mathcal{D}}_p^* \circ \mathcal{A}_{n-1}$ (in this case the Dynkin type equals $\text{Dyn}_I = \mathcal{A}_n$);

(b) one of the digraphs $\tilde{\mathcal{D}}_p^* \circ \mathcal{A}_{n-p-1}$, $\mathcal{D}_p^* \circ \mathcal{A}_{n-p-1}$ (Dynkin type equals $\text{Dyn}_I = \mathcal{D}_n$);

(c) one of the digraphs $\mathcal{P}_1, \ldots, \mathcal{P}_{16}$ ($\text{Dyn}_I = \mathcal{E}_6$), $\mathcal{P}_{17}, \ldots, \mathcal{P}_{72}$ ($\text{Dyn}_I = \mathcal{E}_7$), $\mathcal{P}_{73}, \ldots, \mathcal{P}_{193}$ ($\text{Dyn}_I = \mathcal{E}_8$) presented in [9, Tables 6.1–6.3].

3 Hanging path in a Hasse digraph

In this section, we formalize a very useful observation that changing the orientation of arcs on the “hanging path” in the Hasse digraph $\mathcal{H}(I)$ does not change the definiteness nor rank of a poset $I$. Inspired by the ideas of [15, Prop. 16.15] and [9], we introduce the following definition.

Definition 3.1. Let $I_p \subseteq I$ be a connected subposet of a poset $I$ and $p$ be a point of $I_p$. The subposet $I_p$ is called $p$-anchored path if:

(a) for every $a \in I_p^{(p)}$ we have $N_I(a) \subseteq I_p$ and $|N_I(a)| \in \{1, 2\}$,

(b) $|N_I(p) \cap I_p| = 1$.

The following picture illustrates the definition of a $p$-anchored path.

\[ \overline{\mathcal{H}(I_p^{(p)})} \]

\[ \mathcal{H}(I) = \begin{cases} & \mathcal{H}(I_p^{(p)}) \\ & \text{p-anchored path } I_p \end{cases} \]

Definition 3.2. Let $I_p \subseteq I$ be a $p$-anchored path of a poset $I$. The $I_p$-reflection $\delta_{I_p} I$ is the poset defined by the Hasse digraph $\mathcal{H}(\delta_{I_p} I)$ obtained from $\mathcal{H}(I)$ by reversing all the arcs in the subdigraph $\mathcal{H}(I_p) \subseteq \mathcal{H}(I)$.

We call $I_p \subseteq I$ an inward (outward) $p$-anchored path if $p \in I_p$ is a unique maximal (minimal) point in $I_p$. For example, given an outward $p$-anchored path $I_p \subseteq I$ consisting of $\{p, s_1, \ldots, s_k\}$ elements, we have the following.

\[ \mathcal{H}(I) = \begin{cases} & \mathcal{H}(I_p)^{op} \\ & \text{outward p-anchored path } I_p \end{cases} \]

\[ \mathcal{H}(\delta_{I_p} I) \]

\[ \mathcal{H}(I) = \begin{cases} & \mathcal{H}(I_p)^{op} \\ & \text{inward p-anchored path } I_p \end{cases} \]

The $I_p$-reflection $I \mapsto \delta_{I_p} I$, a combinatorial operation defined at the digraph level, has an algebraic interpretation. We need one more definition from [17] to state it formally.

Definition 3.3. Two finite posets $I$ and $J$ are called strong Gram $\mathbb{Z}$-congruent $I \approx_{\mathbb{Z}} J$ if there exists such a matrix $B \in \text{Gl}_n(\mathbb{Z})$, that $C_J = B^{\text{tr}} C_I B$. 
It is straightforward to check that strong Gram $Z$-congruence of posets implies a week one and the inverse implication is not true in general [10], although it holds in the case of one-peak positive [9] and principal [7] posets. The reader is referred to [6] for a further discussion on $Z$-congruence and its applications.

**Lemma 3.4.** If $I_p \subseteq I$ is an inward or outward $p$-anchored path, then $I \approx_Z \delta_{I_p} I$. In particular, $I$ is non-negative of rank $m$ if and only if $\delta_{I_p} I$ is non-negative of rank $m$.

**Proof.** First, we note that the strong Gram $Z$-congruence of posets implies a weak Gram $Z$-congruence. Since congruent matrices have the same definiteness and rank, it suffices to show that $I \approx Z \delta_{I_p} I$.

Let $I_p \subseteq I$ be an inward $p$-anchored path of a poset $I$ and let $J := \delta_{I_p} I$. Then:

- $J_p := I_p^\text{op}$ is an outward $p$-anchored path in $J$.
- $I \setminus I_p = J \setminus J_p$.

Without loss of generality, we may assume that Hasse digraphs $\mathcal{H}(I)$ and $\mathcal{H}(J)$ have the forms

\[
\mathcal{H}(I) = \begin{array}{cccc}
I \setminus I_p & \cdots & \cdots & \cdots & I_p
\end{array}
\quad \mathcal{H}(J) = \begin{array}{cccc}
J \setminus J_p & \cdots & \cdots & \cdots & J_p
\end{array}
\]

and the incidence matrices of posets $I$ and $J$ are as follows:

$$
C_I = \begin{bmatrix}
c_{1,p} & \cdots & c_{p-1,p} & 0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix},
C_J = \begin{bmatrix}
c_{1,p} & \cdots & c_{p-1,p} & 0 & \cdots & 0 & 1 & \cdots & 1
\end{bmatrix}.
$$

It is straightforward to check that $C_J = B^\text{tr} C_I B$, where

$$
B := \begin{bmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1
0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0
0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0
\end{bmatrix} \in \text{GL}_n(Z)
$$

is an involutory matrix. It follows that posets $I$ and $J$ are strong Gram $Z$-congruent and

$$
\delta_{I_p} J = I \approx Z J = \delta_{I_p} I.
$$

Now, assume $I_p \subseteq I$ be an outward $p$-anchored path of a poset $I$. By using arguments analogous to the previous case, one easily shows that the matrix $B$ defines the strong Gram $Z$-congruence $I \approx Z \delta_{I_p} I$.

\[
\]

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Remark 3.6. The matrix $B \in \text{Gl}_n(\mathbb{Z})$, given in \eqref{eq:35}, defines the strong Gram $\mathbb{Z}$-congruence $I \approx_{\mathbb{Z}} \delta_{I_p} I$, with the assumption that $I_p \subseteq I$ is such an inward or outward $p$-anchored path, that $I_p = p \ldots n$. Assume now that $I_p$ is numbered arbitrarily, i.e., $I_p = p \ldots s_k$, where $k = |I_p| - 1$, and consider the permutation $\pi: \{s_1, \ldots, s_k\} \rightarrow \{s_1, \ldots, s_k\}$ with $\pi(s_i) := s_{k-i+1}$. One checks that for $B_{I_p} \in \text{Gl}_n(\mathbb{Z})$ composed of columns $b_1, \ldots, b_n$ with

$$b_j = \begin{cases} e_p - e_{\pi(s_i)}, & \text{if } j = s_i \in \{s_1, \ldots, s_k\}, \\ e_j, & \text{otherwise}, \end{cases}$$

where $e_i$ is the $i$-th standard basis vector in $\mathbb{Z}^n$, we have $C_{\delta I_p} = B_{I_p}^T C_I B_{I_p}$, that is, the matrix $B_{I_p}$ defines the strong Gram $\mathbb{Z}$-congruence $I \approx_{\mathbb{Z}} \delta_{I_p} I$.

Interchanging inward $p$-anchored paths with the outward ones (an operation defined at the Hasse digraph level) yields a strong Gram $\mathbb{Z}$-congruence of posets (defined at the level of incidence matrices). It is easy to generalize this fact to all, not necessarily inward/outward, $p$-anchored paths.

Corollary 3.7. Let $I_p \subseteq I$ be an arbitrary $p$-anchored path of a poset $I$.

(a) $I \approx_{\mathbb{Z}} J$, where $J$ is a poset with such an outward $p$-anchored path that $I \setminus I_p^{(p)} = J \setminus J_p^{(p)}$ and $\bar{\mathcal{H}}(J_p) = \bar{\mathcal{H}}(I_p)$.

(b) For every orientation of arcs in $\mathcal{H}(I_p) \subseteq \mathcal{H}(I)$, the resulting poset $\bar{I}$ is non-negative of rank $m$ if and only if $I$ is non-negative of rank $m$.

Proof. It is easy to see that for every orientation of edges of the path graph $\bar{\mathcal{H}}(I_p)$, there exists a series of $I_p^{(p)}$-reflections, where $I_p^{(p)} \subseteq I_p$, that carries the $p$-anchored path $I_p$ into outward $p$-anchored path $J_p$, therefore (a) follows by Lemma 3.4.

Since (b) follows from (a), the proof is finished. \hfill $\Box$

Summing up, changing the orientation of arcs in a $p$-anchored path does not change the non-negativity nor the rank.

Example 3.8. Consider the following triple of posets: $I$, $J$ and $J'$.

$$\mathcal{H}(I) = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \quad \mathcal{H}(J) = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \quad \mathcal{H}(J') = \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}$$

We have $I \approx_{\mathbb{Z}} J$ and $I \approx_{\mathbb{Z}} J'$, since $J = \delta_{\{2,5,3,1\}} \delta_{\{5,3,1\}} \delta_{\{3,1\}} I$ and $J' = \delta_{\{3,5,2,4\}} \delta_{\{5,2,4\}} \delta_{\{2,4\}} I$. Moreover, using the description given in Lemma 3.4 and Remark 3.6 we get the equality $C_J = B_{I_p}^T C_I B_1$, where

$$B_1 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$ 

Analogously, one can calculate such a matrix $B_2 \in \text{Gl}_5(\mathbb{Z})$, that $C_{J'} = B_{I_p}^T C_J B_2$.

Remark 3.9. In view of Corollary 3.7(b), we usually omit the orientation of the edges in $p$-anchored paths when presenting Hasse digraphs of finite posets.
4 Main results

The main result of this work is a complete description of connected non-negative posets $I$ of Dynkin type $\text{Dyn}_I = A_n$ in terms of their Hasse digraphs $\mathcal{H}(I)$. First, we show that Theorem 1.5 holds for “trees”, i.e., posets $I$ with graph $\overline{\mathcal{H}}(I)$ being a tree.

Lemma 4.1. If $I = (V, \leq_I)$ is such a connected poset of size $n$ that the graph $\overline{\mathcal{H}}(I)$ is a tree, then exactly one of the following conditions holds.

(a) The poset $I$ is non-negative of rank $n$ and $\overline{\mathcal{H}}(I)$ is isomorphic to a simply-laced Dynkin diagram $\text{Dyn}_I \in \{A_n, D_n, E_6, E_7, E_8\}$ of Table 1.2.

(b) The poset $I$ is non-negative of rank $n-1$ and $\overline{\mathcal{H}}(I)$ is isomorphic to a simply-laced Euclidean diagram $\tilde{\text{Dyn}}_I \in \{\tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$ of Table 1.3.

(c) The poset $I$ is indefinite, i.e., the symmetric Gram matrix $G_I$ is indefinite.

Proof. Assume that $I$ is such a connected poset that the graph $\overline{\mathcal{H}}(I)$ is a tree, where $\mathcal{H}(I) = (V, A)$ is the Hasse digraph of $I$, and let $C_I \in M_n(\mathbb{Z})$ be its incidence matrix (1.1). By [16, Proposition 2.12], the matrix $C_I^{-1} = [c_{ab}] \in M_n(\mathbb{Z})$ has coefficients

\[
c_{ab} = \begin{cases} 
1, & \text{iff } a = b, \\
-1, & \text{iff } a \rightarrow b \in A, \\
0, & \text{otherwise}, 
\end{cases}
\]

i.e., the matrix $C_I^{-1}$ uniquely encodes $\mathcal{H}(I)$. It is straightforward to check that

\[
q_{\mathcal{H}(I)}(x) := \frac{1}{2}x(C_I^{-1} + C_I^{-tr})x^{tr} = \sum_{i \in V} x_i^2 - \sum_{a \rightarrow b \in A} x_a x_b
\]  

(4.2)

is the Euler quadratic form of $\mathcal{H}(I)$ in the sense of [1, Section VII.4]. Moreover, we have

\[
C_I^{tr}G_{\mathcal{H}(I)}C_I = \frac{1}{2}C_I^{tr}(C_I^{-1} + C_I^{-tr})C_I = \frac{1}{2}(C_I^{tr} + C_I) = G_I,
\]

where $G_{\mathcal{H}(I)} := \frac{1}{2}(C_I^{-1} + C_I^{-tr})$ is the symmetric Gram matrix of the Euler quadratic form $q_{\mathcal{H}(I)}$ (4.2). Hence, the lemma follows by [9, Corollary 2.4] and [1, Proposition VII.4.5]. 

One of the key observations used in the proof of Theorem 1.5 is that the graph $\overline{\mathcal{H}}(I)$, where $I$ is a connected non-negative-poset of Dynkin type $\text{Dyn}_I = A_n$, has no vertices of degree larger than 2. In the following theorem, we give a characterization of such posets.

Theorem 4.3. Assume that $D = (V, A)$, where $V = \{1, \ldots, n\}$, is such a connected acyclic digraph that $\deg_{\overline{\mathcal{H}}}(v) \leq 2$ for every $v \in V$. Exactly one of the following conditions holds.

(a) $D \simeq A_n$ and $D$ is the Hasse digraph of the positive poset $I_D$ with $\text{Dyn}_{I_D} = A_n$.

(b) $D$ is a cycle graph. Moreover:

(b1) $D$ has exactly one sink and

(i) $\text{Dyn}_{I_D} = A_n$, $I_D$ is positive, $D \simeq 1 \cdot \cdots \cdot 2 \cdots \cdot n$, and $\mathcal{H}(I_D) \neq D$. 

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(ii) \( \text{Dyn}_{I_D} = \mathbb{D}_n \), \( I_D \) is positive, \( \mathcal{D} \simeq 1 \quad \cdots \quad \overset{n-3}{\cdots} \overset{n-2}{\cdots} \overset{n-1}{\cdots} \overset{n}{\cdots} \), and \( \mathcal{H}(I_D) = \mathcal{D} \),

(iii) \( \text{Dyn}_{I_D} = \mathbb{E}_s \), \( \mathcal{H}(I_D) = \mathcal{D} \) and

\begin{itemize}
  
  \item \( \mathcal{D} \) is positive, \( \mathcal{D} \simeq \)
  \begin{itemize}
    \item \( \overset{1}{\cdots} \overset{2}{\cdots} \overset{3}{\cdots} \overset{4}{\cdots} \overset{5}{\cdots} \overset{6}{\cdots} \overset{7}{\cdots} \overset{8}{\cdots} \overset{9}{\cdots} \overset{10}{\cdots} \)
    \item or \( \mathcal{D} \simeq \)
  \end{itemize}

with \( \text{Dyn}_{I_D} = \mathbb{E}_6 \), \( \mathbb{E}_7 \) and \( \mathbb{E}_8 \), respectively;

\item \( \mathcal{D} \) is principal, \( \mathcal{D} \simeq \)

\begin{itemize}
  \item \( \overset{1}{\cdots} \overset{2}{\cdots} \overset{3}{\cdots} \overset{4}{\cdots} \overset{5}{\cdots} \overset{6}{\cdots} \overset{7}{\cdots} \overset{8}{\cdots} \overset{9}{\cdots} \overset{10}{\cdots} \)
  \item or \( \mathcal{D} \simeq \)
\end{itemize}

with \( \text{Dyn}_{I_D} = \mathbb{E}_7 \) and \( \mathbb{E}_8 \), respectively;

(iv) otherwise, the poset \( I_D \) is indefinite.

(b2) \( \mathcal{D} \) has more than one sink, \( I_D \) is principal with \( \text{Dyn}_{I_D} = \mathbb{A}_{n-1} \) and \( \mathcal{H}(I_D) = \mathcal{D} \).

Proof. If \( \mathcal{D} \) is such a connected acyclic digraph, that \( \deg_{\bar{\mathcal{D}}}(v) \leq 2 \) for every \( v \in V \), then \( \overline{\mathcal{D}} \) is either a cycle or a path graph.

(a) If \( \overline{\mathcal{D}} \) is a path graph, then clearly \( \overline{\mathcal{D}} \) is a tree and (a) follows by Lemma 4.1(a).

(b) Assume that \( \overline{\mathcal{D}} \) is a cycle. It is easy to see that \( \mathcal{D} \) is composed of \( 2k \) oriented paths and has exactly \( k \) sources and \( k \) sinks, where \( 0 \neq k \in \mathbb{N} \). First, we assume that \( k = 1 \).

(b1) Since \( \mathcal{D} \) has exactly one sink, it is composed of two oriented paths, and we have

\[ \mathcal{D} \simeq A_{p,r} := \begin{array}{c}
  1 \\
  \vdots \\
  \vdots \\
  \vdots \\
  p \\
  \vdots \\
  p+r-1
\end{array} \begin{array}{c}
  2 \\
  \vdots \\
  \vdots \\
  \vdots \\
  p-1 \\
  \vdots \\
  p+r-1
\end{array} \begin{array}{c}
  p \\
  \vdots \\
  \vdots \\
  \vdots \\
  p+1 \\
  \vdots \\
  p+r-1
\end{array} \begin{array}{c}
  p+r=n
\end{array} \tag{4.4}
\]

where \( 1 \leq r \leq p \) and \( p + r = n \geq 3 \). We note that \( I_D \) is a one-peak poset and recall that, up to the isomorphism of Hasse digraphs, all such positive and principal posets are classified in [9, Theorem 5.2] and [10, Theorem 3.5] (see Theorem 2.9 for the positive case). This classification shows that the following cases are possible.

(i) For \( r = 1 \) (i.e., \( \mathcal{D} \simeq A_{n-1,1} \)) we have \( \mathcal{H}(I_D) = \alpha_n \mathbb{A}_{n-1} \neq \mathcal{D} \) and \( \text{Dyn}_{I_D} = \mathbb{A}_n \); (ii) for \( r = 2 \) (i.e., \( \mathcal{D} \simeq A_{n-2,2} \)) we have \( \mathcal{H}(I_D) = \hat{\mathbb{D}}_{n-2} \circ \mathbb{A}_1 = \mathcal{D} \) and \( \text{Dyn}_{I_D} = \mathbb{D}_n \), hence thesis follows by [9, Theorem 5.2].

(iii) If \( r > 2 \), then exactly 5 digraphs of the shape (4.4) define non-negative posets:

\begin{itemize}
  \item \( A_{3,3} \simeq \mathbb{P}_5 \), \( A_{4,3} \simeq \mathbb{P}_{24} \) and \( A_{5,3} \simeq \mathbb{P}_{93} \) are Hasse digraphs of a positive poset of the Dynkin type \( \mathbb{E}_6 \), \( \mathbb{E}_7 \) and \( \mathbb{E}_8 \), respectively, see [9, Tables 6.1–6.3];

  \item \( A_{4,4} \) and \( A_{6,3} \) are Hasse digraphs of a principal poset of the Dynkin type \( \mathbb{E}_7 \) and \( \mathbb{E}_8 \), respectively, see [7].
\end{itemize}

In each of the remaining cases, the poset \( I_{A_{p,r}} \) contains one of the subposets: \( I_{A_{7,3}} \) or \( I_{A_{5,4}} \).

\[ A_{7,3} = \begin{array}{c}
  1 \\
  \vdots \\
  \vdots \\
  \vdots \\
  2 \\
  \vdots \\
  10
\end{array} \begin{array}{c}
  3 \\
  \vdots \\
  \vdots \\
  \vdots \\
  4 \\
  \vdots \\
  5
\end{array} \begin{array}{c}
  5 \\
  \vdots \\
  \vdots \\
  \vdots \\
  6 \\
  \vdots \\
  7
\end{array} \begin{array}{c}
  7 \\
  \vdots \\
  \vdots \\
  \vdots \\
  8 \\
  \vdots \\
  9
\end{array} \begin{array}{c}
  9 \\
  \vdots \\
  \vdots \\
  \vdots \\
  10 \\
  \vdots \\
  1 \end{array} \]

\[ A_{5,4} = \begin{array}{c}
  1 \\
  \vdots \\
  \vdots \\
  \vdots \\
  2 \\
  \vdots \\
  10
\end{array} \begin{array}{c}
  3 \\
  \vdots \\
  \vdots \\
  \vdots \\
  4 \\
  \vdots \\
  5
\end{array} \begin{array}{c}
  5 \\
  \vdots \\
  \vdots \\
  \vdots \\
  6 \\
  \vdots \\
  7
\end{array} \begin{array}{c}
  7 \\
  \vdots \\
  \vdots \\
  \vdots \\
  8 \\
  \vdots \\
  9
\end{array} \begin{array}{c}
  9 \\
  \vdots \\
  \vdots \\
  \vdots \\
  10 \\
  \vdots \\
  1 \end{array} \]

Since

\begin{itemize}
  \item \( q_{A_{7,3}}([11, -3, -3, -3, -3, -3, -3, -7, -7, 10]) = -5 \) and

  \item \( q_{A_{5,4}}([11, -4, -4, -4, -4, -5, -5, -5, 9]) = -9 \),
\end{itemize}

these posets are indefinite and (iv) follows (see Remark 2.4).
(b2) Assume that $\mathcal{D}$ is a cycle that is composed of $s := 2k > 2$ oriented paths. Without loss of generality, we can assume that

$$\mathcal{D} \simeq \bullet \rightarrow \cdots \rightarrow \bullet$$

where every $r_1, \ldots, r_s \in \{1, \ldots, n\}$ is either a source or a sink and

- $1 = r_1 < r_2 < \cdots < r_t < \cdots < r_s,$
- $\{1, \ldots, r_2\}, \{r_2, \ldots, r_3\}, \ldots, \{r_{s-1}, \ldots, r_s\}, \{r_s, \ldots, n, 1\} \simeq \bullet \rightarrow \cdots \rightarrow \bullet.$

Since the quadratic form $q_{I_D} : \mathbb{Z}^n \rightarrow \mathbb{Z}$ (2.2) is given by the formula:

$$q_{I_D}(x) = \sum_{i=1} \sum_{1 \leq i < s} x_i x_j + \sum_{r_s \leq i < j \leq r_{s+1}} x_i x_j + x_1 (x_{r_s} + \cdots + x_n)$$

$$= \sum_{i \notin \{r_1, \ldots, r_s\}} \frac{1}{2} x_i^2 + \frac{1}{2} \sum_{1 \leq i < s} x_i^2 + \frac{1}{2} \sum_{r_s \leq i < j \leq r_{s+1}} x_i^2 + \frac{1}{2} (x_{r_s} + \cdots + x_n + x_1)^2,$$

the poset $I_D$ is non-negative. Consider the vector $h_{I_D} = [h_1, \ldots, h_n] \in \mathbb{Z}^n$, where $h_i = 0$ if $i \notin \{r_1, \ldots, r_s\}$, $h_{r_i} = 1$ if $i$ is odd and $-1$ otherwise. That is, $h_{I_D} \in \mathbb{Z}^n$ has $s = 2k$ non-zero coordinates (equal 1 and $-1$ alternately). It is straightforward to check that

$$q_{I_D}(h_{I_D}) = \frac{1}{2}(h_{r_1} + h_{r_2})^2 + \cdots + \frac{1}{2}(h_{r_{s-1}} + h_{r_s})^2 + \frac{1}{2}(h_{r_s} + 1)^2 = 0,$$

i.e., $I_D$ is not positive and the first coordinate of the vector $h_{I_D} \in \text{Ker} q_{I_D}$ equals $h_1 = 1$. Since $\mathcal{D}^{(1)} \simeq \mathbb{A}_{n-1}$, by (a) the poset $I_{D^{(1)}} \subset I_D$ is positive and $\text{Dyn}_{I_{D^{(1)}}} = \mathbb{A}_{n-1}$. Hence, we conclude that $I_D$ is principal of Dynkin type $\text{Dyn}_{I_D} = \mathbb{A}_{n-1}$, see Definition 2.7.

One of the methods to prove that a particular poset $I$ is indefinite is to show that it contains a subposet $J \subseteq I$ that is indefinite (we use this argument in the proof of Theorem 4.3(iii)). The following lemma presents a list of indefinite posets used further in the paper.

**Lemma 4.5.** If $I$ is such a finite poset, that its Hasse digraph $\mathcal{H}(I)$ is isomorphic with any of the digraphs $\mathcal{F}_1, \ldots, \mathcal{F}_7$ given in Table 4.6, then $I$ is indefinite.

$$\begin{align*}
\mathcal{F}_1: & \quad \begin{array}{c}
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\begin{array{
• \( \mathcal{F}_1 \) has three sinks,
• \( \mathcal{F}_2 \) has two sinks and
• \( \mathcal{F}_3, \ldots, \mathcal{F}_7 \) have exactly one sink.

It is straightforward to check that \( q_{\vec{F}}(v^i) < 0 \), where \( I_i := I_{\vec{F}_i} \) is a poset defined by a digraph \( \vec{F}_i \in \{ \mathcal{F}_1, \ldots, \mathcal{F}_7 \} \) and \( v^i \in \mathbb{Z}^n \) is an integer vector whose coordinates are given as the labels of vertices of diagram \( \mathcal{F}_i \) in Table 4.6. Therefore, by Remark 2.4, we conclude that poset \( I_i \) is indefinite. Since isomorphic posets are weakly Gram \( \mathbb{Z} \)-congruent (the congruence is defined by a permutation matrix), it follows that every poset that has its Hasse digraph isomorphic with \( \mathcal{F}_i \) is indefinite. \( \square \)

We need one more technical result to prove Theorem 1.5. In Lemma 4.7 we describe connected posets \( I \) with the following property: for some \( p \in \{1, \ldots, n \} \) the graph \( H(I(p)) \) is isomorphic to a path graph.

**Lemma 4.7.** Assume that \( I \) is such a connected poset, that for some \( p \in \{1, \ldots, n \} \) the graph \( H(I(p)) \) is isomorphic to a path graph. Exactly one of the following conditions holds.

(a) \( I \) is positive and:

(a1) \( \text{Dyn}_I = \mathcal{A}_n \) with \( H(I) \simeq \mathcal{A}_n = \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & n-1 & n \end{array} \);

(a2) \( \text{Dyn}_I = \mathcal{D}_n \) and the Hasse digraph \( H(I) \simeq \mathcal{D}_I \) has a shape \( \mathcal{D}_I \in \{ \mathcal{D}^{[1]}_n, \mathcal{D}^{[2]}_{n,s}, \mathcal{D}^{[3]}_{n,s} \} \),

\[
\begin{align*}
\mathcal{D}^{[1]}_n &= \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & n \end{array}, \\
\mathcal{D}^{[2]}_{n,s} &= \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & n \\ s, s+1 & s+2 & \cdots & n-1 \end{array}, \\
\mathcal{D}^{[3]}_{n,s} &= \begin{array}{ccccccc} 1 & 2 & 3 & \cdots & n \\ s, s+1 & s+2 & \cdots & n-1 \end{array},
\end{align*}
\]

where \( n \geq 4 \) for \( \mathcal{D}_I = \mathcal{D}^{[1]}_n \); \( n \geq 4 \), \( s \geq 1 \) for \( \mathcal{D}^{[2]}_{n,s} \), and \( n \geq 5 \), \( s \geq 2 \) for \( \mathcal{D}^{[3]}_{n,s} \).

(a3) \( \text{Dyn}_I = \mathcal{E}_n \), where \( n \in \{6, 7, 8\} \), and the Hasse digraph \( H(I) \), up to isomorphism, is one of 498 digraphs \( \{86 \text{ up to orientation of hanging paths, see Corollary 3.7(b)}\} \), i.e., there are 38, 145, 315 \( \{11, 30, 45\} \) digraphs with \( \text{Dyn}_I = \mathcal{E}_6, \mathcal{E}_7 \) and \( \mathcal{E}_8 \), respectively. In particular, Hasse digraphs \( H(I) \) of all posets \( I \) with \( \text{Dyn}_I = \mathcal{E}_n \) are depicted below.

![Hasse digraphs](image)

We note that, up to isomorphism, the first Hasse digraph describes exactly 20 posets and the second exactly 3 posets.

(b) \( I \) is principal and:

(b1) \( \text{Dyn}_I = \mathcal{A}_{n-1} \), \( H(I) \) is a cycle graph and \( H(I) \) has at least two sinks.

(b2) \( \text{Dyn}_I = \mathcal{E}_{n-1} \), where \( n \in \{8, 9\} \), and the Hasse digraph \( H(I) \), up to isomorphism, is one of 850 digraphs \( \{98 \text{ up to orientation of hanging paths}\} \), i.e., there are exactly 185, 665 \( \{36, 62\} \) digraphs with \( \text{Dyn}_I = \mathcal{E}_7 \) and \( \mathcal{E}_8 \), respectively.

(c) \( I \) is indefinite.
Proof. Assume that \( I \) is a connected poset of size \( n \). By Theorem 2.9 and Corollary 3.7(b), we know that posets \( I \) with \( \overline{H}(I) \) isomorphic to \( \mathcal{A}_n, \mathcal{D}^{[1]}_{n,s}, \mathcal{D}^{[2]}_{n,s}, \mathcal{D}^{[3]}_{n,s} \) are positive, with \( \text{Dyn}_I = \mathcal{A}_n \) if \( \overline{H}(I) \cong \mathcal{A}_n \) and \( \text{Dyn}_I = \mathcal{D}_n \) otherwise. Furthermore, Theorem 4.3(b2) asserts that posets \( I \) with \( \overline{H}(I) \) isomorphic to a cycle graph and \( \mathcal{H}(I) \) having at least two sinks are principal with \( \text{Dyn}_I = \mathcal{A}_{n-1} \).

The proof is divided into two parts. First, we prove the thesis by analyzing all posets having at most 11 elements. Then, using induction, we prove it for posets \( I \) of size \( |I| > 11 \).

Part 1° It is easy to see that all connected posets \( I \) of size \( n \leq 3 \) satisfy the assumptions: in this case \( \mathcal{H}(I) \in \{ \mathcal{1} \cdot \cdots \cdot \mathcal{2}, \mathcal{1} \cdot \cdots \cdot \mathcal{2} \leftarrow \mathcal{3}, \mathcal{1} \cdot \cdots \cdot \mathcal{2} \rightarrow \mathcal{3}, \mathcal{1} \cdot \cdots \cdot \mathcal{2} \rightarrow \mathcal{3} \} \). Since \( \overline{H}(I^{(p)}) \) is isomorphic to a path graph, \( \text{Dyn}_I = \mathcal{A}_n \) and the thesis follows. Now, using Computer Algebra System (e.g. SageMath, Maple), we compute all (up to isomorphism) posets \( I \) of size at most 11 using a suitably modified version of [9, Algorithm 7.1] (see also [4] for a different approach). There are exactly 49 519 383 [46 485 488 connected] posets \( I \) of size \( 4 \leq |I| \leq 11 \). Moreover, 58 723 [58 198 connected] posets \( I \) have the graph \( \overline{H}(I^{(p)}) \) isomorphic to a path graph, for some \( p \in I \).

In particular, there are 46 749 427 [43 944 974 connected] non-isomorphic posets of size 11. In 39 335 [39 079] cases, for some \( p \in \{1, \ldots, 11\} \), the graph \( \overline{H}(I^{(p)}) \) is isomorphic to a path graph and, up to isomorphism, there are exactly:

- **256 disconnected positive posets** \( I \) with \( \mathcal{H}(I) \cong \mathcal{A}_n \).
- **2 575** connected positive posets \( I \), of which 528, 768, 1024 and 255 have its Hasse digraph \( \mathcal{H}(I) \) isomorphic to \( \mathcal{A}_n, \mathcal{D}^{[1]}_{n,s}, \mathcal{D}^{[2]}_{n,s}, \mathcal{D}^{[3]}_{n,s} \), respectively;
- **88** connected principal posets \( I \), where \( \overline{H}(I) \) is a cycle graph and \( \mathcal{H}(I) \) has at least two sinks;
- **36 416** connected indefinite posets.

A more precise analysis of connected posets \( I \) is given in the following table.

| \( n \) | \#I | \( \mathcal{A}_n \) | \( \mathcal{D}^{[1]}_{n,s} \) | \( \mathcal{D}^{[2]}_{n,s} \) | \( \mathcal{D}^{[3]}_{n,s} \) | \( \mathcal{E}_n \) | \( \mathcal{A}_{n-1} \) | \( \mathcal{E}_{n-1} \) | \#I |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 10 | 4 | 4 | 1 | 1 | 1 | 1 |
| 5 | 34 | 10 | 12 | 4 | 3 | 1 | 4 |
| 6 | 129 | 16 | 24 | 12 | 7 | 38 | 5 | 27 |
| 7 | 413 | 36 | 48 | 32 | 15 | 145 | 6 | 131 |
| 8 | 1 369 | 64 | 96 | 80 | 31 | 315 | 17 | 185 | 581 |
| 9 | 4 184 | 136 | 192 | 192 | 63 | 256 | 655 | 2 911 |
| 10 | 12 980 | 256 | 384 | 448 | 127 | 56 | 11 709 |
| 11 | 39 079 | 528 | 768 | 1 024 | 255 | 88 | 36 416 | 39 079 |

This computer-assisted analysis completes the proof for posets \( I \) of size \( |I| \leq 11 \).

Part 2° We proceed by induction. Assume that \( I \) is such a finite connected poset of size \( |I| = n > 11 \) that for some \( p \in \{1, \ldots, n\} \) the graph \( \overline{H}(I^{(p)}) \) is isomorphic to a path graph, and the thesis holds for posets of size \( n - 1 \). To prove the inductive step we
show that $\mathcal{H}(I)$ has one of the forms described in (a1), (a2) and (b1), or is indefinite (c).
Without loss of generality, we may assume that $p = 1$ and $\deg_{\mathcal{H}(I^{(1)})}(n) = 1$, i.e.,
$$\mathcal{H}(I^{(1)}) \simeq P(2,n) = 2 \rightarrow 3 \rightarrow \ldots \rightarrow n-1 \rightarrow n \simeq A_{n-1}.$$ Consider the poset $J := I^{(n)}$.

(A) If $J$ is not connected, then the element $n$ is an articulation point in the graph $\overline{\mathcal{H}}(J)$. Since degree of $n$ in $\overline{\mathcal{H}}(I^{(1)})$ equals one, we conclude that $J$ has two connected components: $\{2, \ldots, n-1\}$ and $\{1\}$. Moreover, the graph $\overline{\mathcal{H}}(I)$ have the shape
$$\overline{\mathcal{H}}(I) \simeq \frac{2 \rightarrow 3 \rightarrow \ldots \rightarrow n-1 \rightarrow n \rightarrow 1}{\simeq A_{n}},$$
where the elements of the poset $J$ are highlighted. Hence the thesis follows.

(B) Assume that $J$ is connected. Since $J^{(1)} = I^{(1,n)}$ is such a poset of size $n-1$, that the graph $\overline{\mathcal{H}}(J^{(1)})$ is isomorphic to a path graph, by the inductive hypothesis one of the following conditions holds:

(i) $\mathcal{H}(J) \simeq D_{J}$, where $D_{J} \in \{A_{n}, D_{n}^{[1]}, D_{n,s}^{[2]}, D_{n,s}^{[3]}\}$, as described in (a1) and (a2);

(ii) $\overline{\mathcal{H}}(J)$ is a cycle graph and $\mathcal{H}(J)$ has at least two sinks, as described in (b1);

(iii) $J$ is indefinite, as described in (c).

We analyze these cases one by one. Since the digraphs $\mathcal{H}(I)$ and $\mathcal{H}(J)$ are connected, $\overline{\mathcal{H}}(I^{(1)}) \simeq A_{n-1}$ and $\deg_{\overline{\mathcal{H}}(I^{(1)})}(n) = 1$, we conclude that the degree of the vertex $n$ in the graph $\overline{\mathcal{H}}(I)$ equals two, if elements $1$ and $n$ are in relation, and one otherwise.

(i) Assume that $\mathcal{H}(J) \simeq D_{J} \in \{A_{n}, D_{n}^{[1]}, D_{n,s}^{[2]}, D_{n,s}^{[3]}\}$. We have the following:

(1) if $D_{J} = A_{n-1}$, then either $\mathcal{H}(I) \simeq D_{I} \in \{A_{n}, D_{n}^{[1]}, D_{n,s}^{[3]}\}$, where $2 \leq s' \leq n - 2$, or $\mathcal{H}(I)$ has at least two sinks and $\overline{\mathcal{H}}(I)$ is a cycle graph;

(2) if $D_{J} = D_{n}^{[1]}$, then either $\mathcal{H}(I) \simeq D_{I} \in \{D_{n}^{[1]}, D_{n,1}^{[2]}, D_{n,s}^{[3]}, D_{n,2}^{[3]}\}$, or $I$ is indefinite, as it contains (as a subposet) some $F \in \{F_{1}, \ldots, F_{6}\}$;

(3) if $D_{J} = D_{n,s}^{[2]}$, then either $\mathcal{H}(I) \simeq D_{n,s}^{[2]}$, where $1 \leq s' \leq n - 3$, or $I$ is indefinite, as it contains (as a subposet) some $F \in \{F_{3}, \ldots, F_{6}\}$;

(4) if $D_{J} = D_{n-1,s}^{[3]}$, then either $\mathcal{H}(I) \simeq D_{n-1,s}^{[3]}$, where $2 \leq s' < n - 1$, or $I$ is indefinite, as it contains (as a subposet) some $F \in \{F_{2}, \ldots, F_{7}\}$.

We describe the case (3) in detail. One has to consider all such finite posets $I$, that $\overline{\mathcal{H}}(I^{(1)}) \simeq P(2,n)$, $\mathcal{H}(I^{(n)}) \simeq D_{n-1,s}^{[2]}$ and $\deg_{\overline{\mathcal{H}}(I^{(1)})}(n) = 1$. These are the only possibilities:

(3a) $\mathcal{H}(I) \simeq D_{n-1,s}^{[2]}$ and $\mathcal{H}(I)$ has one of the following shapes:

(3b) $I$ is indefinite, since $F_{3} \subseteq I$:
(3°c) $I$ is indefinite, since $F_4 \subseteq I$: 

(3°d) $I$ is indefinite as $F_5 \subseteq I$ or $F_6 \subseteq I$, respectively:

The cases (1°), (2°) and (4°) follow by similar arguments. Details are left to the reader.

(ii) Assume that $J$ is principal, i.e., $\overline{H}(J)$ is a cycle graph and $H(J)$ has at least two sinks. There are two possibilities: degree of vertex $n$ in the graph $\overline{H}(I)$ equals either one or two. In the first case the poset $I$ is indefinite, as it contains a subposet $F_1$, $F_2$, $F_3$ or $F_4$. In the second case, $\overline{H}(I)$ is a cycle graph and $H(I)$ contains the identical number of sinks as $H(J)$. Therefore, by Theorem 4.3(b2), $I$ is principal and statement (b) follows.

(iii) Since poset $J \subseteq I$ is indefinite, the poset $I$ is also indefinite.

Proof of Theorem 1.5

Now we have all the necessary tools to prove the main result of this work.

Proof of Theorem 1.5. Let $I = (V, \leq_I)$ be a finite connected non-negative poset of size $n$, rank $m$ and Dynkin type $\text{Dyn}_I = A_m$.

(a) Our aim is to show that $m = n$ if and only if 

$$\overline{H}(I) \simeq P(1, n) = 1 \overbrace{\cdots \cdots n}.$$ 

Since the implication “$\Leftarrow$” is a consequence of Lemma 4.1(a), it is sufficient to prove “$\Rightarrow$”. First, we show that for every vertex $v \in V$ we have $\deg_{\overline{H}(I)}(v) \leq 2$. We proceed by contradiction. Assume that there exists a vertex $v$ of degree at least 3. If that is the case, there exists such a subposet $J \subseteq I$ that its Hasse digraph $H(J)$ has the form

$$H(J): \bullet \overbrace{\cdots \cdots \bullet}.$$ 

By Lemma 4.1(a), $\text{Dyn}_J = D_4$. Since $J \subseteq I$, [2, Proposition 2.25] yields $\text{Dyn}_I \in \{D_n, E_n\}$, contrary to our assumptions. We conclude that $\deg_{\overline{H}(I)}(v) \leq 2$ for every vertex $v \in V$ and, in view of Theorem 4.3, statement (a) follows.

(b) We need to show that $m = n - 1$ if and only if $\overline{H}(I)$ is a cycle graph and $H(I)$ has at least two sinks. To prove “$\Rightarrow$” assume that $I$ is a connected principal poset of the Dynkin type $\text{Dyn}_I = A_m$. By definition, there exists such a $k \in I$, that the poset $J := I(k)$ is positive of the Dynkin type $\text{Dyn}_J = A_m$. By (a) we know that $\overline{H}(J) \simeq P(1, n)$ thus, by Lemma 4.7(b), $\overline{H}(I)$ is a cycle graph, $H(I)$ has at least two sinks, and “$\Rightarrow$” follows.

“$\Leftarrow$” By Theorem 4.3(b2), every poset $I$ with $\overline{H}(I)$ being a cycle graph and $H(I)$ having at least two sinks is principal of the Dynkin type $\text{Dyn}_I = A_{m-1}$.

(c) Our aim is to show that the assumption $\text{Dyn}_I = A_m$ yields $m \in \{n, n - 1\}$, i.e., $I$ is either positive or principal. The proof is divided into two parts. First, we show that $m \neq n - 2$. Then, using this result, we show that $m > n - 2$. 


Part 1° \( m \neq n - 2 \) Assume, by contradiction, that \( I \) is a connected non-negative poset of rank \( n - 2 \) and Dynkin type \( \text{Dyn}_n = A_{n-2} \). Since there are no such posets of size \( n \leq 16 \) (see [10, Corollary 4.4(b)]), without loss of generality, we can assume that \( n > 16 \).

By Fact 2.6(a), there exists such a basis \( h_1, h_2 \) of the free abelian group \( \text{Ker} q_I \subseteq \mathbb{Z}^n \), that \( h_1 = h_2 = 1 \) and \( h_1 = h_2 = 0 \) where \( 1 < k_1 < k_2 \leq n \). Consider the posets \( J_1 := I^{(k_1)} \) and \( J_2 := I^{(k_2)} \). Since \( J_1^{(k_1)} = J_2^{(k_2)} = I^{(k_1, k_2)} \), the posets \( J_1 \) and \( J_2 \) are connected principal of Dynkin type \( \mathbb{A}_m \), see Fact 2.6(b) and Definition 2.7. It follows that \( \overline{\mathcal{H}}(J_1) \) and \( \overline{\mathcal{H}}(J_2) \) are cycle graphs and the Hasse digraphs \( \mathcal{H}(J_1) \) and \( \mathcal{H}(J_2) \) have at least two sinks, see (b). That is, one of the following conditions holds for the poset \( I \):

(i) \( \overline{\mathcal{H}}(I) \) is a cycle graph and \( \mathcal{H}(I) \) has at least two sinks;

(ii) \( \mathcal{H}(I) \) has at least two sinks and is of the shape \( \mathcal{H}(I) \): \[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

(iii) \( I \) contains an indefinite subposet \( F_1, F_2, F_3 \) or \( F_4 \).

In the case (i), the poset \( I \) is principal, i.e., \( m = n - 1 \), which contradicts the assumption. Now, we show that the same goes for (ii). Without loss of generality, we may assume that the Hasse digraph \( \mathcal{H}(I) \) has the following form

\[
\mathcal{H}(I) : \begin{array}{c}
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\bullet \\
\end{array}
\]

where:

- every \( r_1, \ldots, r_s \in \{1, \ldots, n\} \setminus \{j + 1, j + 2\} \) is either a source or a sink,
- \( 1 = r_1 < r_2 < \cdots < r_t < \cdots < r_s \) where \( s > 2 \) is an even number,
- subdigraphs \( \{1, \ldots, j, j + 1, j + 3, \ldots, r_2\}, \{1, \ldots, j, j + 2, j + 3, \ldots, r_2\}, \{r_s, \ldots, n, 1\} \) and \( \{r_{t-1}, \ldots, r_t\} \), where \( 3 \leq t \leq s \), have exactly one sink.

Since the quadratic form \( q_I : \mathbb{Z}^n \rightarrow \mathbb{Z} \) (2.2) is given by the formula:

\[
q_I(x) = \sum_{1 \leq i \leq n} x_i^2 + \sum_{1 \leq i < r_2} \sum_{1 < j \leq r_2} x_{i+1} x_{j+1} + \sum_{1 \leq i < r_2} x_i \sum_{2 \leq s \leq n} x_{i+1} x_{i+2} + \sum_{r_s \leq i < k \leq n} x_i x_k + x_1 (x_{r_s} + \cdots + x_n)
\]

\[
= \sum_{i \neq (r_1, \ldots, r_s, j+1, j+2)} \frac{1}{2} x_i^2 + \frac{1}{2} (x_{j+1} - x_j)^2 + \frac{1}{2} \sum_{1 \leq s < r_1} \sum_{r_s \leq i \leq r_1} x_i^2 + \frac{1}{2} (x_{r_s} + \cdots + x_n + x_1)^2,
\]

then \( q_I(v) \geq 0 \) for every \( v \in \mathbb{Z}^n \) and \( I \) is non-negative. Consider the non-zero vector \( h = [h_1, \ldots, h_n] \in \mathbb{Z}^n \), where \( h_i = 1 \) if \( i \) is odd, \( -1 \) if \( i \) is even, and \( h_k = 0 \) for \( k \neq r_i \). It is straightforward to check that

\[
q_I(h) = \frac{1}{2} (h_{r_1} + h_{r_2})^2 + \cdots + \frac{1}{2} (h_{r_{t-1}} + h_{r_t})^2 + \frac{1}{2} (h_{r_s} + h_1)^2 = 0.
\]
i.e., the poset $I$ is not positive and $h \in \ker q_I$. Since the vector $h$ has the first coordinate equal $h_1 = 1$ and $f_1^{(1)} \simeq \mathcal{D}_{n-r,s}$ is a positive poset of Dynkin type $\mathbb{D}_{n-1}$ (see Lemma 4.7(a2) and Definition 2.7), it follows that $I$ is principal of Dynkin type $\mathbb{D}_{n-1}$. This contradicts the assumption that $\text{Dyn}_I = \mathbb{A}_{n-2}$.

To finish this part of the proof, we note that every poset $I$ that is not described in (i) or (ii) contains (as a subposet) one of the posets $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ or $\mathcal{F}_4$ presented in Table 4.6, hence is indefinite. This follows by the standard case-by-case inspection, as described in the proof of Lemma 4.7. Details are left to the reader.

**Part 2° $[m > n - 2]$** Let $I$ be a connected non-negative poset of rank $m$ and Dynkin type $\text{Dyn}_I = \mathbb{A}_m$. We show that the assumption $m \in \{1, \ldots, n - 3\}$ yields a contradiction. It follows from Fact 2.6(a) that there exists such a basis $h^{k_1}, \ldots, h^{k_r}$ of the free abelian group $\ker q_I \subseteq \mathbb{Z}^n$, that $h^{k_i}_{k_i} = 1$ and $h^{k_i}_{k_j} = 0$, for $1 \leq i, j \leq r$ and $i \neq j$, where $r = n - m$ and $1 \leq k_1 < \ldots < k_r \leq n$. Moreover, by Fact 2.6(b), the poset $J := I^{(k_1, \ldots, k_r)}$ is connected non-negative of size $n' = m + 2$ and rank $m = n' - 2$. Since $J^{(k_1, k_2)} = I^{(k_1, \ldots, k_r)} \simeq_{\mathbb{Z}} \mathbb{A}_m$, it follows that $\text{Dyn}_J = \mathbb{A}_{n'-2}$ which yields a contradiction with Part 1°. □

## 5 Enumeration of $\mathbb{A}_n$ Dynkin type non-negative posets

We finish the manuscript by giving explicit formulae (5.2) and (5.6) for the number of all possible orientations of the path and cycle graphs, up to isomorphism of unlabeled digraphs. We apply these results to devise the formula (1.7) for the number of non-negative Dynkin type $\mathbb{A}_m$ posets of size $n$.

**Fact 5.1.** There are exactly

$$ONum(P_n) = \begin{cases} 2^{\frac{n^2 - n}{2}} + 2^{n-2}, & \text{if } n \geq 1 \text{ is odd}, \\ 2^{n-2}, & \text{if } n \geq 2 \text{ is even}, \end{cases} \quad (5.2)$$

digraphs $D$ with $\overrightarrow{D} \simeq P_n := 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$, up to the isomorphism.

**Proof.** Here we follow arguments given in the proof of [8, Proposition 6.7]. To calculate the number of non-isomorphic orientations among all $2^{n-1}$ possible orientations of edges of $P_n$, we consider two cases.

(i) First, assume that $|I| = n \geq 2$ is an even number. In this case, every digraph $I$ has exactly two representatives among $2^{n-1}$ edge orientations: one drawn “from the left” and the other “from the right” (i.e., symmetric along the path). Therefore, the number $ONum(P_n)$ of all such non-isomorphic digraphs equals $2^{n-2}$.

(ii) Now we assume that $|I| = n \geq 1$ is an odd number. If $n = 1$, then, up to isomorphism, there exists exactly $1 = 2^{-1} + 2^{-1}$ digraph. Otherwise, $n \geq 3$ and among all $2^{n-1}$ edge orientations:

- digraphs that are “symmetric” along the path have exactly one representation, and
- the rest of digraphs have exactly two representations, analogously as in (i).

It is straightforward to check that there are $2^{\frac{n-1}{2}}$ “symmetric” path digraphs, therefore in this case we obtain

$$ONum(P_n) = \frac{2^{n-1} - 2^{\frac{n-1}{2}}}{2} + 2^{\frac{n-1}{2}} = 2^{\frac{n-3}{2}} + 2^{n-2}. \quad \square$$
Remark 5.3. The formula (5.2) describes the number of various combinatorial objects. For example the number of linear oriented trees with $n$ arrows or unique symmetrical triangle quilt patterns along the diagonal of an $n \times n$ square, see [12, OEIS sequence A051437].

By Theorem 1.5(a), the Hasse digraph $H(I)$ of every positive connected poset $I$ of Dynkin type $\text{Dyn}_I = A_n$ is an oriented path graph, as suggested in [8, Conjecture 6.4]. Hence, Fact 5.1 gives an exact formula for the number of all, up to the poset isomorphism, such connected posets $I$ (see also [8, Proposition 6.7]).

Corollary 5.4. Given $n \geq 1$, the total number of all finite non-isomorphic connected positive posets $I$ of Dynkin type $A_n$ and size $n$ equals $N_{\text{neg}}(n, A_n) := \text{ONum}(P_n)$.

Similarly, the description given in Theorem 1.5(b) makes it possible to count all connected principal posets $I$ of Dynkin type $A_{n-1}$. First, we need to know the exact number of all, up to isomorphism, orientations of the cycle graph $C_n$.

Fact 5.5. Let $C_n := 1 \rightarrow \cdots \rightarrow n$ be the cycle graph on $n \geq 3$ vertices. The number $\text{ONum}(C_n)$ of digraphs $D$ with $\overline{D} = C_n$, up to the isomorphism, equals

$$\text{ONum}(C_n) = \begin{cases} \frac{1}{2n} \sum_{d|n} (2\pi \varphi(d)), & \text{if } n \geq 3 \text{ is odd,} \\ \frac{1}{2n} \sum_{d|n} (2\pi \varphi(d)) + 2\pi^{-2}, & \text{if } n \geq 4 \text{ is even,} \end{cases} \quad (5.6)$$

where $\varphi$ is the Euler’s totient function.

Proof. Assume that $D$ is such a digraph that $\overline{D} = C_n$. Without loss of generality, we may assume that $D$ is depicted in the circle layout (on the plane), and its arrows are labeled with two colors:

- **black**: if the arrow is clockwise oriented, and
- **white**: if the arrow is counterclockwise oriented.

That is, every $D$ can be viewed as a binary combinatorial necklace $N_2(n)$. For example, for $n = 5$ there exist 32 orientations of edges of the cycle $C_5$ that yield exactly 8 different binary necklaces of length 5 shown in Table 5.7.

Table 5.7: Binary combinatorial necklaces of length 5

Moreover, up to digraph isomorphism, every $D$ has in this case exactly two representations among the necklaces: “clockwise” and “anticlockwise” ones, as shown in the Table 5.7 (isomorphic digraphs are gathered in boxes).
On the other hand, if \(|D| = n \geq 4\) is an even number, certain digraphs have exactly one representation among necklaces. As an illustration, let us consider the \(n = 6\) case.

Table 5.8: Binary combinatorial necklaces of length 6

Every such a “rotationally symmetric” digraph is uniquely determined by a directed path graph of length \(\frac{n}{2} + 1\) and, by Fact 5.1, there are exactly \(2 \frac{n}{2} - 1\) such digraphs.

Now we show that every isomorphism \(f: \{1, \ldots, n\} \to \{1, \ldots, n\}\) of digraphs \(D_1\) and \(D_2\) with \(\overline{D}_1 = \overline{D}_2 = C_n\) has a form of a “clockwise” or “anticlockwise” rotation. Fix a vertex \(v_1 \in D_1\) and consider the sequence \(v_1, v_2, \ldots, v_n\) of vertices, where \(v_{i+1}\) is a “clockwise” neighbour of \(v_i\) (i.e., we have either \(v_i \to v_{i+1}\) black arrow or \(v_i \leftarrow v_{i+1}\) white arrow in digraph \(D_1\)). One of two possibilities holds: \(f(v_2)\) is either a “clockwise” neighbor of \(f(v_1)\) or an “anticlockwise” one. If the first possibility holds, then \(f(v_{i+1})\) is a “clockwise” neighbour of \(f(v_i)\) for every \(2 \leq i < n\), hence isomorphism \(f\) encode a “clockwise” rotation. On the other hand, the assumption that \(f(v_2)\) is an “anticlockwise” neighbour of \(f(v_1)\) implies that \(f(v_{i+1})\) is an “anticlockwise” neighbour of \(f(v_i)\), i.e., \(f\) encodes an “anticlockwise” rotation.

Summing up, if \(|D| = n \geq 3\) is an odd number, the digraph \(D\) has exactly two representatives among binary necklaces \(N_2(n)\): one “clockwise” and the other “anticlockwise”. Since \(|N_2(n)| = \frac{1}{n} \sum_{d \mid n} (2 \frac{n}{2} \varphi(d))\), see [14], the first part of the equality (5.6) follows. Assume now that \(|D| = n \geq 4\) is an even number. In the formula \(|N_2(n)|/2\) we count only half (i.e., \(2 \frac{n}{2} - 2\)) of “rotationally symmetric” digraphs. Hence, \(ONum(C_n) = \frac{1}{2n} \sum_{d \mid n} (2 \frac{n}{2} \varphi(d)) + 2 \frac{n}{2} - 2\) in this case.

In the proof of Fact 5.5 we show that the number of cyclic graphs with oriented edges, up to the symmetry of the dihedral group, coincides with the number of such digraphs, up to isomorphism of unlabeled digraphs.

Remark 5.9. The formula (5.6) is described in [13, OEIS sequence A053656] and, among others, counts the number of minimal fibrations of a bidirectional \(n\)-cycle over the 2-bouquet (up to precompositions with automorphisms of the \(n\)-cycle), see [3].

Corollary 5.10. Let \(n \geq 3\) be an integer. Then, up to isomorphism, there exists exactly:

(a) \(ONum(C_n) - 1\) directed acyclic graphs \(D\) whose underlying graph is \(\overline{D} = C_n\),

(b) \(N_{\text{neg}}(n, \mathbb{A}_{n-1}) = ONum(C_n) - \left\lceil \frac{n+1}{2} \right\rceil\) principal posets \(I\) of Dynkin type \(\mathbb{A}_{n-1}\),

where \(ONum(C_n)\) is given by the formula (5.6).

Proof. Since, up to isomorphism, there exists exactly one cyclic orientation of the \(C_n\) graph, (a) follows directly from Fact 5.5.
To prove (b), we note that by Theorem 1.5(b) it is sufficient to count all oriented cycles that have at least two sinks. Since, among all possible orientations of a cycle, there are:

- \( \left\lfloor \frac{n}{2} \right\rfloor \) cycles with exactly one sink,
- 1 oriented cycle

and \( \left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n+1}{2} \right\rceil \), the statement (b) follows from Fact 5.5.

Proof of Theorem 1.6

Now, we can devise an exact formula for the total number of non-negative posets of size \( n \) and Dynkin type \( A_m \).

Proof of Theorem 1.6. We note that \( \text{Neg}(n, A_m) = \text{Neg}(n, A_n) + \text{Neg}(n, A_{n-1}) \) by Theorem 1.5, hence by Corollary 5.4 and Corollary 5.10(b), for \( n \geq 3 \) we have

\[
\text{Neg}(n, A_m) = \begin{cases} 
\frac{1}{2n} \sum_{d | n} \left( 2 \frac{n}{d} \varphi(d) \right) + 2^{n-2} + 2^{\frac{n}{2}} - \left\lceil \frac{n+1}{2} \right\rceil , & \text{if } n \geq 3 \text{ is odd,} \\
\frac{1}{2n} \sum_{d | n} \left( 2 \frac{n}{d} \varphi(d) \right) + 2^{n-2} + 2^{\frac{n}{2}} - \left\lceil \frac{n+1}{2} \right\rceil , & \text{if } n \geq 4 \text{ is even.}
\end{cases}
\]

Since \( \frac{n-3}{2} = \left\lfloor \frac{n}{2} - 2 \right\rfloor \) for odd values of \( n \) and \( \text{Neg}(1, A_m) = \text{Neg}(2, A_m) = 1 \), it follows that

\[
\text{Neg}(n, A_m) = \frac{1}{2n} \sum_{d | n} \left( 2 \frac{n}{d} \varphi(d) \right) + \left[ 2^{n-2} + 2^{\frac{n}{2}} - \frac{n+1}{2} \right]
\]

for any \( n \geq 1 \).

Summing up, we get the following asymptotic description of connected non-negative posets \( I \) of Dynkin type \( A_m \).

Corollary 5.11. Let \( \text{Neg}(n, A_{n-1}) \) be the number of principal posets \( I \) of Dynkin type \( A_{n-1} \) and \( \text{Neg}(n, A_m) \) be the number of all non-negative posets \( I \) of size \( n \) and Dynkin type \( A_m \). Then

(a) \( \lim_{n \to \infty} \frac{\text{Neg}(n+1, A_m)}{\text{Neg}(n, A_m)} = 2 \) and \( \text{Neg}(n, A_m) \approx 2^{n-2} \),

(b) the number of connected non-negative posets of Dynkin type \( A_m \) grows exponentially,

(c) \( \lim_{n \to \infty} \frac{\text{Neg}(n, A_{n-1})}{\text{Neg}(n, A_m)} = 0 \), hence almost all such posets are positive.

Proof. Apply Corollary 5.10(b) and Theorem 1.6.
Figure 5.12: Logarithmic scale plot of the number of connected non-negative posets $I$ of Dynkin type $\text{Dyn}_I = A_m$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| positive | 1 | 1 | 3 | 4 | 10 | 16 | 36 | 64 | 136 | 256 | 528 | 1024 | 2080 | 4096 | 8256 | 16384 | 32768 | 65536 | 131072 |
| principal | 0 | 0 | 0 | 1 | 1 | 5 | 6 | 17 | 25 | 56 | 88 | 185 | 309 | 615 | 1088 | 2113 | 3847 | 7419 | 13788 |
| all | 1 | 1 | 3 | 5 | 11 | 21 | 42 | 81 | 161 | 312 | 616 | 1209 | 2389 | 4711 | 9344 | 18497 | 36743 | 72955 | 145116 | 288633 |

Table 5.13: Number of connected non-negative posets $I$ of Dynkin type $\text{Dyn}_I = A_m$ and size $1 \leq |I| \leq 20$
6 Future work

In the present work, we give a complete description of connected non-negative posets $I$ of Dynkin type $\text{Dyn}_I = A_m$ and, in particular, we show that $m \in \{n, n-1\}$. Computer experiments suggest that there is an upper bound for the rank of $E_n$ type posets as well.

**Conjecture 6.1.** If $I$ is a Dynkin type $E_m$ non-negative connected poset, then $m \geq n - 3$.

The conjecture yields $|I| \leq 11$, and consequently, we get the following.

**Conjecture 6.2.** If $I$ is a non-negative connected poset of size $n > 11$ and rank $m < n-1$, then $\text{Dyn}_I = D_m$.

In other words, checking the Dynkin type of a connected non-negative poset $I$ that has at least $n \geq 12$ elements is straightforward (compare with [11] and [21]).

**Proposition 6.3.** If Conjecture 6.1 holds, the Dynkin type of a connected non-negative poset $I$ of size $n \geq 12$ and rank $m$, encoded in the form of the adjacency list of the Hasse digraph $\mathcal{H}(I)$, can be calculated in $O(n)$. Moreover, assuming that this adjacency list is sorted by degrees of vertices, $\text{Dyn}_I$ can be calculated in $O(1)$.

**Proof.** First, we note that the assumptions yield $\text{Dyn}_I \in \{A_m, D_m\}$. Moreover, $\text{Dyn}_I = A_m$ if and only if one of the following conditions hold:

(i) $\text{deg}_{\mathcal{H}(I)}(v) = 2$ for all $v \in \mathcal{H}(I)$, or

(ii) $\text{deg}_{\mathcal{H}(I)}(v) = 2$ for all but two $v_i \in \mathcal{H}(I)$ with $\text{deg}_{\mathcal{H}(I)}(v_i) = 1$,

see Theorem 1.5. In the pessimistic case, to verify these conditions, one has to examine the degrees of all $n$ vertices, thus we have $O(n)$ complexity. In the case of the adjacency list sorted by degrees of vertices, this can be simplified to checking degrees of at most two vertices, which yields $O(1)$ complexity.

Nevertheless, this description does not give any insights into the structure of $D_m$ type non-negative connected posets.

**Open problem 6.4.** Give a structural description of Hasse digraphs of $D_m$ type non-negative connected posets.

References

[1] I. Assem, A. Skowroński, and D. Simson. *Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory*, volume 65 of London Math. Soc. Student Texts. Cambridge University Press, Cambridge, 2006.

[2] M. Barot, J. A. Jiménez González, and J.-A. de la Peña. *Quadratic Forms: Combinatorics and Numerical Results*, volume 25 of Algebra and Applications. Springer International Publishing, Cham, 2019.

[3] P. Boldi and S. Vigna. Fibrations of graphs. *Discrete Math.*, 243(1-3):21–66, 2002.

[4] G. Brinkmann and B. D. McKay. Posets on up to 16 points. *Order*, 19(2):147–179, 2002.
[5] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2017.

[6] M. Gąsiorek. Congruence of rational matrices defined by an integer matrix. *Appl. Math. Comput.*, 440:127639, 2023.

[7] M. Gąsiorek. A Coxeter type classification of one-peak principal posets. *Linear Algebra Appl.*, 582:197–217, 2019.

[8] M. Gąsiorek. On algorithmic Coxeter spectral analysis of positive posets. *Appl. Math. Comput.*, 386:125507, 2020.

[9] M. Gąsiorek and D. Simson. One-peak posets with positive quadratic Tits form, their mesh translation quivers of roots, and programming in Maple and Python. *Linear Algebra Appl.*, 436(7):2240–2272, 2012.

[10] M. Gąsiorek and K. Zając. On algorithmic study of non-negative posets of corank at most two and their Coxeter-Dynkin types. *Fundam. Inform.*, 139(4):347–367, 2015.

[11] B. Makuracki and A. Mróz. Quadratic algorithm to compute the Dynkin type of a positive definite quasi-Cartan matrix. *Math. Comp.*, 90(327):389–412, 2020.

[12] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2021. Available from: [https://oeis.org/A051437](https://oeis.org/A051437).

[13] OEIS Foundation Inc. The on-line encyclopedia of integer sequences, 2021. Available from: [https://oeis.org/A053656](https://oeis.org/A053656).

[14] J. Riordan. The combinatorial significance of a theorem of Pólya. *J. Soc. Ind. Appl. Math.*, 5(4):225–237, 1957.

[15] D. Simson. *Linear representations of partially ordered sets and vector space categories*, volume 4 of *Algebra, logic and applications*. Gordon and Breach Science Publishers, Montreux, 1992.

[16] D. Simson. Incidence coalgebras of intervally finite posets, their integral quadratic forms and comodule categories. *Colloq. Math.*, 115(2):259–295, 2009.

[17] D. Simson. A Coxeter-Gram classification of positive simply laced edge-bipartite graphs. *SIAM J. Discrete Math.*, 27(2):827–854, 2013.

[18] D. Simson. Symbolic algorithms computing Gram congruences in the Coxeter spectral classification of edge-bipartite graphs, I. A Gram classification. *Fundam. Inform.*, 145(1):19–48, 2016.

[19] D. Simson and K. Zając. A framework for Coxeter spectral classification of finite posets and their mesh geometries of roots. *Int. J. Math. Math. Sci.*, Article ID 743734, 22 pages, 2013.

[20] K. Zając. On the structure of loop-free non-negative edge-bipartite graphs. *Linear Algebra Appl.*, 579:262–283, 2019.

[21] K. Zając. On polynomial time inflation algorithm for loop-free non-negative edge-bipartite graphs. *Discrete Appl. Math.*, 283:28–43, 2020.