On Kummer-like surfaces attached to singularity and modular forms

Atsuhira Nagano¹ | Hironori Shiga²

¹Faculty of Mathematics and Physics, Kanazawa University, Ishikawa, Japan
²Graduate School of Science, Chiba University, Chiba, Japan

Correspondence
Atsuhira Nagano, Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University, Kakuma, Kanazawa, Ishikawa 920-1192, Japan.
Email: atsuhira.nagano@gmail.com

Funding information
Japan Society for the Promotion of Science, Grant/Award Numbers: Grant-in-Aid for Scientific Research (18K13383), Grant-in-Aid for Scientific Research (19K03396); Ministry of Education, Culture, Sports, Science and Technology, Grant/Award Number: LEADER

Abstract
We study a family of lattice polarized $K3$ surfaces which is an extension of the family of Kummer surfaces derived from principally polarized Abelian surfaces. Our family has two special properties. First, it is coming from a resolution of a simple $K3$ singularity. Second, it has a natural parameterization by Hermitian modular forms of four complex variables. In this paper, we show two results: (1) we determine the transcendental lattice and the Néron–Severi lattice of a generic member of our family. (2) We give a detailed description of the double covering structure associated with our $K3$ surfaces.

KEYWORDS
Hermitian modular forms, $K3$ singularities, Kummer surfaces

MSC (2020)
14J28, 32S25, 11F11

1 | INTRODUCTION

A Kummer surface $\text{Kum}(\mathfrak{A})$ for a principally polarized Abelian surface $\mathfrak{A}$ is an algebraic $K3$ surface with an explicit defining equation parameterized by Siegel modular forms of degree 2 (see [4] Section 5.1). Let $\mathcal{G}_{\text{Kum}}$ be the family of such Kummer surfaces. The Picard number of a generic member of $\mathcal{G}_{\text{Kum}}$ is 17. Since $\mathcal{G}_{\text{Kum}}$ is important in various areas in mathematics (e.g., see [1] or [20]), it is meaningful to extend $\mathcal{G}_{\text{Kum}}$. In this paper, we study a family $\mathcal{G}_1$, which is a natural extension of $\mathcal{G}_{\text{Kum}}$ as shown in the diagram (1.3).

Now, we give a brief summary of our family $\mathcal{G}_1$. The family $\mathcal{G}_1$ has a particular property: it is coming from a resolution of a simple $K3$ singularity (see Section 2). It has the following explicit expression. Letting $T$ be a Zariski open set of the weighted projective space $\mathbb{P}(4,6,10,12,18)$, this family is given by $\pi_{\mathcal{G}_1} : \mathcal{G}_1 = \{ K(t) | [t] = (t_4 : t_6 : t_{10} : t_{12} : t_{18}) \in T \} \to T$, where $K(t)$ is an elliptic surface given by the Weierstrass equation

$$K(t) : Z^2 = Y^3 + a_K(U)Y + b_K(U) \quad \text{with} \quad a_K(U) = \left( t_4 + \frac{t_{10}}{U^2} \right), \quad b_K(U) = \left( t_6 + U^2 + \frac{t_{12}}{U^2} + \frac{t_{18}}{U^4} \right), \quad (1.1)$$

which is birationally equivalent to a $K3$ surface of Picard number 16 (for detail, see Equation (2.3) and Section 4.2). Indeed, the subfamily $\pi_{\mathcal{G}_1}^{-1} \{ t_{18} = 0 \} \cap T$ coincides with the family $\mathcal{G}_{\text{Kum}}$. So, we regard $\mathcal{G}_1$ as an extension of $\mathcal{G}_{\text{Kum}}$ and we call a member of $\mathcal{G}_1$ a Kummer-like surface.
In order to study $G_1$, we use another family $\pi_{\mathcal{F}_1} : \mathcal{F}_1 = \{ S(t) \mid [t] \in T \} \to T$ of elliptic surfaces, where $S(t)$ is given by the Weierstrass equation

$$S(t) : Z^2 = Y^3 + a_5(X)Y + b_5(X) = \left( t_4 + \frac{t_{10}}{X} \right) b_5(X) = \left( t_6 + X + \frac{t_{12}}{X} + \frac{t_{18}}{X^2} \right).$$ (1.2)

Here, $S(t)$ gives a $K3$ surface (for detail, see Equation (3.1) and Section 4.2). For a fixed point $[t] \in T$, $K(t)$ is a rational double covering of $S(t)$ (for this terminology, see Remark 4.5). The surface $S(t)$ has a simple structure as a lattice polarized $K3$ surface and the period mapping for $F_1$ is studied in [14]. Especially, the period mapping characterizes the tuple $t$ of the parameters. Namely, $t_4, t_6, t_{10}, t_{12}, t_{18}$ are regarded as Hermitian modular forms for $SU(2, 2)$ (see Theorem 3.2).

Our families $G_1$ and $F_1$ are extensions of already known families of $K3$ surfaces as in the following diagram:

\[
\begin{array}{ccc}
G_1 & \longrightarrow & F_1 \\
U & \U & U \\
G_{\text{Kum}} & \longrightarrow & F_{\text{CD}}
\end{array}
\] (1.3)

Here, $F_{\text{CD}}$ is the family studied in [4] (see Section 6.1). Also, the horizontal arrows are corresponding to the rational double coverings for each member. We remark that the double covering induces a parameterization of $G_1$ ($G_{\text{Kum}}$, resp.) by Hermitian (Siegel, resp.) modular forms derived from the period mapping of $F_1$ ($F_{\text{CD}}$, resp.).

The first result of this paper is to show the precise lattice structure of our family $G_1$. We determine the transcendental lattice and the Néron–Severi lattice for $G_1$ (Theorem 5.1). In fact, it is a non-trivial problem to determine them. The situation is explained by the following facts:

• The lattice structure of $F_1$ is already known (see Section 3). However, as mentioned at the beginning of Section 5, the lattice structure of $G_1$ is not directly calculated from that of $F_1$.
• The Néron–Severi lattices for families coming from simple $K3$ singularities are studied by Belcastro [2]. However, unfortunately, her result for $G_1$ is not correct (see Remark 5.2).

In Section 5, we will give a geometric construction of 2-cycles on a generic member of $G_1$, in order to determine the precise structure of the transcendental lattice. Also, by applying arithmetic properties of even lattices, we will calculate the Néron–Severi lattice.

The second result of this paper is to obtain a detailed description of the double covering between $G_1$ and $F_1$. The double covering between the subfamilies $G_{\text{Kum}}$ and $F_{\text{CD}}$ induces an interesting phenomenon called the Kummer sandwich (for detail, see Section 5). This phenomenon is studied by many researchers (e.g., see [3, 10, 24] or [11]). We will prove that we cannot find a complete sandwich phenomenon for our families $G_1$ and $F_1$ (Theorem 6.3). In the meanwhile, the restrictions of our expressions (1.1) and (1.2) to $t_{18} = 0$ are useful to consider the Kummer sandwich between $G_{\text{Kum}}$ and $F_{\text{CD}}$ (see Section 6.1). Furthermore, we will see a handy description of a sandwich between the family $G_{\text{MSY}}$ of [12] and the family $F_{\text{CMS}}$ of [5] (see Section 6.3).

In Table 1, we show the properties of our pair $(G_1, F_1)$ compared with those of other pairs which are already known. Here, $U$ means the transcendental lattice of a generic member. Also, $U$ is the unimodular hyperbolic lattice of rank 2. Moreover, for a lattice $L$ with the intersection matrix $(c_{j,k})_{j,k}$ and $n \in \mathbb{Z}$, we denote by $L(n)$ the lattice given by the intersection matrix $n(c_{j,k})_{j,k}$.

Our family $G_1$ must be one of the most reasonable extension of $G_{\text{Kum}}$. It has several good properties. The moduli space of $G_1$ is a natural extension of that of the family of principally polarized Abelian surfaces. Also, the Hermitian modular forms

| $(G, F)$ | $(G_{\text{Kum}}, F_{\text{CD}})$ | $(G_{\text{MSY}}, F_{\text{CMS}})$ |
|---------|-------------------------------|-------------------------------|
| $\text{Tr for } G$ | $U(2)^{\oplus 2} \oplus A_1(-2)$ | $U(2)^{\oplus 2} \oplus A_1(-2)$ |
| $\text{Tr for } F$ | $U^{\oplus 2} \oplus A_2(-1)$ | $U^{\oplus 2} \oplus A_1(-1)$ |
| Weights of parameters | 4, 6, 10, 12, 18 | 4, 6, 12 | 4, 6, 8, 10, 12 |
| Sandwich | Only one side | Both sides | Both sides |
be given by a hypersurface singularity. Let \( (X', x) \) be a normal isolated singularity in an analytic space \( X' \) of three-dimension. Let \( \rho : (\tilde{X}, E) \to (X', x) \) be a good resolution, where \( E \) is the exceptional set. For any positive integer \( m \), the plurigenera \( \delta_m(\tilde{X}, x) \) is defined as \( \dim \mathbb{C}(\Gamma(X' - \{ x \}, O(mK^\circ))/L^m(\tilde{X} - \{ x \})) \). Here, \( K^\circ \) is the canonical bundle on \( X' - \{ x \} \) and \( L^m(\tilde{X} - \{ x \}) \) is the set of holomorphic \( m \)-ple 3-forms on \( \tilde{X} - \{ x \} \) which are \( L^2 \)-integrable at \( x \). A singularity \( (X', x) \) is said to be purely elliptic if it holds \( \delta_m(\tilde{X}, x) = 1 \) for any \( m \).

Suppose that \( (X', x) \) is quasi-Gorenstein. Letting \( E = \bigcup E_j \) be the irreducible decomposition, we have an expression in the form \( K_{\tilde{X}} = \pi^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j \), with \( m_i, m_j \in \mathbb{Z}_{>0} \). If \( (X', x) \) is purely elliptic, we can prove that \( m_j = 1 \) holds for any \( j \in J \). Now, setting \( E_j = \sum_{j \in J} E_j \), there is an unique \( j \in \{0, 1, 2\} \) such that \( H^2(E_j, \mathcal{O}_j) \cong H^0,1(E_j) \cong 0 \). Such a singularity \( (X', x) \) is said to be of \((0, j)\)-type. Now, we have the following result.

**Proposition 2.1** ([9] Section III). For a three-dimensional isolated singularity \((X', x)\), the following two conditions are equivalent.

(i) \((X', x)\) is Gorenstein, purely elliptic and of \((0,2)\)-type,  
(ii) \((X', x)\) is quasi-Gorenstein and the exceptional divisor \(E\) for any minimal resolution \(\rho : (\tilde{X}, E) \to (X', x)\) is a normal K3 surface.

If \((X', x)\) satisfies the conditions of the above proposition, it is called a simple K3 singularity.

In the following, we assume that a K3 simple singularity \((X', x)\) is given by a hypersurface singularity. Let \( X' \) be a hypersurface \( X' = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 | F(x_1, x_2, x_3, x_4) = 0\} \), where \( F \) is a non-degenerate polynomial given by

\[
F(x_1, x_2, x_3, x_4) = \sum_{(p_1, p_2, p_3, p_4)} \lambda_{p_1, p_2, p_3} p_1^{p_1} p_2^{p_2} p_3^{p_3} p_4^{p_4} = \sum_p \lambda_p \xi^p. \tag{2.1}
\]

Also, let \( x \) be the origin of the \((x_1, x_2, x_3, x_4)\)-space. Here, the points \( p = (p_1, p_2, p_3, p_4) \in \mathbb{Z}^4_{\geq 0} \) with \( \lambda_p \neq 0 \) are integral points of a three-dimensional Newton polytope \( \Delta \). Such a polytope \( \Delta \) is determined by a weight vector \( a = (a_1, a_2, a_3, a_4) \in \mathbb{Q}^4_{>0} \) as follows. Let \( \Delta' \) be the convex hull of

\[
\left\{ \left( \frac{1}{a_1}, 0, 0, 0 \right), \left( 0, \frac{1}{a_2}, 0, 0 \right), \left( 0, 0, \frac{1}{a_3}, 0 \right), \left( 0, 0, 0, \frac{1}{a_4} \right) \right\}.
\]

Then, \( \Delta \) is the convex hull of all integral points of \( \Delta' \). We can prove that \((X', x)\) is a simple K3 singularity if and only if the corresponding three-dimensional polytope \( \Delta \) contains the point \((1, 1, 1, 1)\) in the relative interior (see [26]). We note that such a weight vector \( a \) satisfies \( \sum_{i=1}^{\dim \Delta} a_i p_i = 1 \) for every integral point \( p \in \Delta \). Especially, it holds \( \sum_{i=1}^{\dim \Delta} a_i = 1 \). Yonemura [27] classified such weight vectors into 95 types.

We can obtain minimal resolutions of the above simple K3 singularities as follows. Let the weight vector \( a \) be given by \((w_1, w_2, w_3, w_4)\) such that \( \gcd(w_i, w_j, w_k) = 1 \) for distinct \( i, j, k \in \{1, 2, 3, 4\} \). Then, from the 4 unit vectors in \( \mathbb{Z}^4_{\geq 0} \) and the integral vector \((w_1, w_2, w_3, w_4)\), we can obtain a four-dimensional fan \( \mathcal{Y} \subset \mathbb{R}^4_{\geq 0} \). So, we have a four-dimensional toric variety \( V_{\mathcal{Y}} \). There is a canonical morphism \( \hat{\rho} : V_{\mathcal{Y}} \to \mathbb{C}^4 \) such that \( V_{\mathcal{Y}} - \hat{\rho}^{-1}(0) \) is isomorphic to \( \mathbb{C}^4 - \{0\} \) and \( \hat{\rho}^{-1}(0) \) is a three-dimensional weighted projective space \( \mathbb{P}(w_1, w_2, w_3, w_4) \). Letting \( \tilde{X} \) be the proper transform of \( X' \) by \( \hat{\rho} \), set \( \rho = \hat{\rho}|_{\tilde{X}} \). Then, we have the following result.
Proposition 2.2 ([27] Section 3, see also [9] Section IV). The morphism \( \rho : (\tilde{X}, E) \to (X, x) \) gives a minimal resolution of \((X, x)\). Here, the exceptional divisor is given by a two-dimensional set \( \rho^{-1}(0) = \tilde{\rho}^{-1}(0) \cap \tilde{X} \), which is a hypersurface in \( \mathbb{P}(w_1, w_2, w_3, w_4) \) corresponding to Equation (2.1).

Let \( \mathfrak{R}_\lambda \) be the hypersurface in \( \mathbb{P}(w_1, w_2, w_3, w_4) \) in the above proposition. So, we obtain a family \( \{ \mathfrak{R}_\lambda \}_{\lambda} \) of K3 hypersurfaces in \( \mathbb{P}(w_1, w_2, w_3, w_4) \) with complex parameters \( \lambda = \lambda_{p_1, p_2, p_3, p_4} \). In fact, each weight vector of the 95 vectors of [27] corresponds to one of the “famous 95” families of K3-weighted projective hypersurfaces with Gorenstein singularities due to Reid (e.g., see [21], Section 4). Moreover, Belcastro [2] studied the lattice structure of a generic member of each family of 95 families from the viewpoint of mirror symmetry of K3 surfaces.

2.2 | Family of K3 surfaces corresponding to a weight vector

In this paper, we focus on the weight vector

\[
a = \left( \frac{11}{27}, \frac{1}{3}, \frac{5}{27}, \frac{2}{27} \right).
\]

This is the No. 88 in Yonemura’s list of the weight vectors (see [27]). The purpose of this paper is to show that the family coming from the singularity of No. 88 gives a natural extension of the family of Kummer surfaces.

In this case, the polynomial \( F(\zeta) \) of Equation (2.1) is given by a linear combination of 11 monomials

\[
\zeta_1^2, \zeta_2^3, \zeta_3^5, \zeta_4^7, \zeta_1^2 \zeta_2, \zeta_1^2 \zeta_3, \zeta_2 \zeta_3, \zeta_2 \zeta_4, \zeta_2^2 \zeta_3, \zeta_2^2 \zeta_4, \zeta_2^2 \zeta_5,
\]

over \( \mathbb{C} \). We regard \( F(\zeta) \) as a quasi-homogeneous polynomial of weight 27, where the weight of \( \zeta_1, (\zeta_2, \zeta_3, \zeta_4, \text{resp.}) \) is 11 (9, 5, 2, resp.). A K3 surface \( \mathfrak{R}_\lambda \) is corresponding to the hypersurface \( \{ F(\zeta) = 0 \} \) in \( \mathbb{P}(11, 9, 5, 2) \). So, by putting \( \zeta_1 = x_0, \zeta_2 = y_0, \zeta_3 = z_0 \) and \( \zeta_4 = 4 \), \( \mathfrak{R}_\lambda \) is given by a hypersurface in \( \mathbb{C}^3 \), which is defined by a linear combination of monomials

\[
x_0^2z_0, x_0, y_0^3, y_0, z_0^5, x_0, x_0y_0z_0, x_0^2z_0, x_0z_0^2, x_0, y_0z_0^2, z_0.
\]

(2.2)

A generic member of this family of complex surfaces is a K3 surface. By an appropriate birational transformation, the defining equation of a generic member can be transformed to an equation

\[
\hat{K}(t) : x_1^2 = y_1^3 + (t_4z^4 + t_{10}z^2)y_1 + (z^8 + t_6z^6 + t_{12}z^4 + t_{18}z^2).
\]

(2.3)

We note that Equation (2.3) defines an elliptic surface with five complex parameters \( t_4, t_6, t_{10}, t_{12}, \text{and } t_{18} \).

Remark 2.3. Let \( R(x_0, y_0, z_0) = 0 \) be a linear combination of the monomials in (2.2). Then, one can find appropriate constants \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, \text{and } c_8 \) and polynomial \( \psi(y_j, z) \in C[y_j', z] \) such that \( R(x_0, y_0, z_0) = 0 \) is transformed to

\[
x_1^2 = y_1^3 + (t'_2z)y'_2 + (t'_4z^4 + t'_{10}z^2)y'_1 + (z^8 + t'_6z^6 + t'_2z^4 + t'_{18}z^2)
\]

by the birational transformation \( (x_0, y_0, z_0) \mapsto (x_1, y'_1, z) \) with \( x_0 = \frac{x_1}{z^2} + \frac{\psi(y'_1, z)}{z}, y_0 = c_0 \frac{y'_1}{z}, z_0 = c_2 z \). Here, \( t'_j \) (\( j \in \{2, 4, 6, 10, 12, 18\} \)) are complex numbers. The form (2.4) can be birationally transformed to Equation (2.3).

3 | PARTNER SURFACES AND HERMITIAN MODULAR FORMS

In order to investigate the K3 surfaces \( \hat{K}(t) \), let us consider the partner surfaces \( \hat{S}(t) \) of Equation (3.1). We can consider the moduli of the family of \( \hat{S}(t) \), since they have clear structures of marked lattice polarized K3 surfaces as in [14], Section 2. In this section, we survey those results. For detailed proofs, see [14, 15].
Let \( t = (t_4, t_6, t_{10}, t_{12}, t_{18}) \in \mathbb{C}^5 - \{0\} \). Let us consider the elliptic surfaces
\[
\hat{S}(t) : \quad z_2^2 = y_2^3 + (t_4 x_2^4 + t_{10} x_2^5) y_2 + (x_2^7 + t_6 x_2^6 + t_{12} x_2^5 + t_{18} x_2^4).
\] (3.1)

Suppose the weight of \( x_2 \) (\( y_2, z_2, t_j \), resp.) is \( 6 (14, 21, j \), resp.), Equation (3.1) is quasi-homogeneous of weight 42.

So, we have the family of the complex surfaces \( \hat{S}(t) \) over \( \mathbb{P}(4, 6, 10, 12, 18) = \text{Proj}(\mathbb{C}[t_4, t_6, t_{10}, t_{12}, t_{18}]) \). The point of \( \mathbb{P}(4, 6, 10, 12, 18) \) corresponding to \( t = (t_4, t_6, t_{10}, t_{12}, t_{18}) \in \mathbb{C}^5 - \{0\} \) is denoted by \([t] = (t_4 : t_6 : t_{10} : t_{12} : t_{18})\). Set
\[
T = \mathbb{P}(4, 6, 10, 12, 18) - \{ [t] \in \mathbb{P}(4, 6, 10, 12, 18) \mid t_{10} = t_{12} = t_{18} = 0 \} .
\] (3.2)

If \([t] \in T\), then the complex surface (3.1) is a \( K_3 \) surface.

The \( K_3 \) lattice \( L_{K_3} \), which is isomorphic to the 2-homology group of a \( K_3 \) surface, is isometric to \( II_{3,19} = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \). Also, the lattice
\[
A = U \oplus U \oplus A_2(-1)
\] (3.3)
of signature (2,4) is necessary for our study.

Proposition 3.1.

1. ([14], Corollary 1.1) For a generic point \([t] \in T\), the transcendental lattice (the Néron–Severi lattice, resp.) of the \( K_3 \) surface \( \hat{S}(t) \) of Equation (3.1) is given by the intersection matrix \( A \) of Equation (3.3) (\( U \oplus E_8(-1) \oplus E_8(-1) \), resp.).

2. ([14], Proposition 2.3) For a generic point \([t] \in \{t_{18} = 0\} \cap T\), the transcendental lattice (the Néron–Severi lattice, resp.) of \( \hat{S}(t) \) is given by the intersection matrix \( U \oplus U \oplus A_1(-1) \) (\( U \oplus E_8(-1) \oplus E_7(-1) \), resp.). Such a surface coincides with the \( K_3 \) surfaces studied in [4].

We remark that \( A \) is the simplest lattice satisfying the Kneser conditions, which are arithmetic conditions for quadratic forms. Due to the Kneser conditions of \( A \), we can prove that \( \Gamma = \hat{O}(A) \) is generated by the reflections of \((-2)\)-vectors and \( \text{Char}(\Gamma) = \text{Hom}(\Gamma, \mathbb{C}^\times) \) is equal to \( \{\text{id}, \text{det}\} \).

We can define the multivalued period mapping \( \Phi : T \to D \) defined by
\[
[t] \mapsto \left( \int_{\Delta_{1, i}} \omega^S_{[t]} : \cdots : \int_{\Delta_{6, i}} \omega^S_{[t]} \right),
\] (3.5)
where \( \omega^S_{[t]} \) is the unique holomorphic 2-form up to a constant factor and \( \Delta_{1, i}, \ldots, \Delta_{6, i} \) are appropriate 2-cycles on \( \hat{S}(t) \).

We note that the cycles \( \Delta_{1, i}, \ldots, \Delta_{6, i} \) are constructed geometrically and explicitly in [15], Section 2. We can prove that this mapping induces the biholomorphic isomorphism
\[
\hat{\Phi} : T \simeq D / \Gamma
\] (3.6)
by a detailed argument of the period mappings for lattice polarized \( K_3 \) surfaces. Especially, the isomorphism (3.6) means that \( T \) of (3.2) gives the moduli space of marked pseudo-ample \((U \oplus E_8(-1) \oplus E_8(-1))\)-polarized \( K_3 \) surfaces (for detail, see [14], Section 2). Thus, we have a clear expression of the moduli space of our partner surface (3.1).

The period mapping for the family of \( \hat{S}(t) \) is closely related to modular forms. Let \( D^\times \) be the \( \mathbb{C}^\times\)-bundle of \( D \). If a holomorphic function \( f : D^\times \to \mathbb{C} \) given by \( Z \mapsto f(Z) \) satisfies the conditions
(i) \( f(\lambda Z) = \lambda^{-k} f(Z) \) (for all \( \lambda \in \mathbb{C}^* \)),
(ii) \( f(\gamma Z) = \chi(\gamma) f(Z) \) (for all \( \gamma \in \Gamma \)),

where \( k \in \mathbb{Z} \) and \( \chi \in \text{Char}(\Gamma) \), then \( f \) is called a modular form of weight \( k \) and character \( \chi \) for the group \( \Gamma \).

**Theorem 3.2** ([14], Theorem 5.1).

1. The ring \( \mathcal{A}(\Gamma, \text{id}) \) of modular forms of character \( \text{id} \) is isomorphic to the ring \( \mathbb{C}[t_4, t_6, t_{10}, t_{12}, t_{18}] \). Namely, via the inverse of the period mapping \( \Phi \) of the isomorphism (3.6), the correspondence \( Z \mapsto t_k(Z) \) gives a modular form of weight \( k \) and character \( \text{id} \).

2. There is a modular form \( s_{54} \) of weight 54 and character \( \det \). Here, \( s_{54} \) is defined by \( s_9 s_{45} \), where \( s_9 \) and \( s_{45} \) are holomorphic functions on \( \mathbb{H} \) such that \( s_9^2 = t_{18}, s_{45}^2 = (a \text{ polynomial in } t_4, t_6, t_{10}, t_{12}, t_{18} \text{ of weight } 90 \text{ in [14] Section 1}) \).

These relations determine the structure of the ring \( \mathcal{A}(\Gamma) \) of modular forms with characters.

There is a biholomorphic mapping between \( \mathbb{H} \) and the four-dimensional complex bounded symmetric domain \( \mathbb{H}_I \) of type \( I \). Via this biholomorphic mapping, the modular forms in Theorem 3.2 are identified with the Hermitian modular forms for the imaginary quadratic field of the smallest discriminant. Moreover, we have an explicit expression of the Hermitian modular forms in Theorem 3.2 by theta functions introduced by Dern-Krieg [6] (see [15]).

Thus, the family of K3 surfaces \( \hat{S}(t) \) is very closely related to Hermitian modular forms and theta functions. Such a relation is a natural and non-trivial counterpart of a classical relation among the Weierstrass form of elliptic curves, elliptic modular forms and Jacobi theta functions.

### 4 | LATTICE THEORY FOR DOUBLE COVERINGS FOR K3 SURFACES

#### 4.1 | Lattice theoretic properties

Let \( L \) be a non-degenerate even lattice. Let \( (s_+, s_-) \) be the signature of \( L \). We have a natural embedding \( L \hookrightarrow L^\vee = \text{Hom}(L, \mathbb{Z}) \). We set \( s(L) = L^\vee/L \). The length \( l(s(L)) \) of \( s(L) \) is the minimum number of generators of \( s(L) \).

Let \( q \) be the quadratic form which defines the lattice \( L \). Then, it induces the discriminant form \( q_L : s(L) \to \mathbb{Q}/2\mathbb{Z} \). We have \( q_{L_1} \oplus q_{L_2} \cong q_{L_1} \oplus q_{L_2} \).

**Proposition 4.1** ([18], Corollary 1.13.3, [13], Theorem 2.2). Let \( L \) be an even lattice with the conditions \( s_+ > 0, s_- > 0 \) and \( l(s(L)) \leq \text{rank}(L) - 2 \). Then, \( L \) is the unique lattice with invariants \( (s_+, s_-, q_L) \) up to isometry.

**Proposition 4.2** ([18], Proposition 1.6.1, [13], Lemma 2.4). For a unimodular lattice \( L \) and a primitive embedding \( M \hookrightarrow L \), it holds \( q_{M^\perp} \cong -q_M \).

Conversely, if \( M_1 \) and \( M_2 \) are non-degenerate even lattices satisfying \( q_{M_1} \cong -q_{M_2} \), then there are a unimodular lattice \( L \) and a primitive embedding \( M_1 \hookrightarrow L \) such that \( M_1^\perp \cong M_2 \).

Next, let us summarize results of Nikulin involutions. For detail, see [10, 17, 19]. Let \( \mathfrak{X} \) be an algebraic K3 surface and \( \omega \) be the unique holomorphic 2-form on \( \mathfrak{X} \) up to a constant factor. An involution \( \iota \in \text{Aut}(\mathfrak{X}) \) is called a Nikulin involution (or symplectic involution), if it holds \( \iota^* \omega = \omega \). If \( \iota \) is a Nikulin involution on \( \mathfrak{X} \), then the minimal resolution \( \mathfrak{Y} = \mathfrak{X}/(\iota) \) is also an algebraic K3 surface. Namely, we have a rational quotient mapping \( \mathfrak{X} \to \mathfrak{Y} \). Conversely, any given rational double covering \( \mathfrak{X} \to \mathfrak{Y} \) of K3 surfaces is derived from a Nikulin involution.

Let \( \iota \) be a Nikulin involution on \( \mathfrak{X} \) and set \( \mathfrak{Z} = H_2(\mathfrak{X}, \mathbb{Z})^{\perp} \). Then, we can see that the transcendental lattice \( \text{Tr}(\mathfrak{X}) \) is a primitive sublattice of \( \mathfrak{Z} \). The orthogonal complement \( (\mathfrak{Z}^\perp)^\perp \) of \( \mathfrak{Z}^\perp \) in \( L_{K3} \) has remarkable properties. The lattice \( (\mathfrak{Z}^\perp)^\perp \) is an even negative definite lattice and has no \((-2)\)-vectors (see [17], Lemma 4.2). Also, this is a lattice of rank 8 and determinant \( 2^8 \). Moreover, we have \( ((\mathfrak{Z}^\perp)^\perp)^\vee/(\mathfrak{Z}^\perp)^\perp \cong \mathbb{Z}/2\mathbb{Z}^8 \) (see [17], Proposition 10.1). According to these properties, we can see
that \((\mathfrak{X}')^\perp\) is isometric to \(E_8(-2)\) and

\[
\mathfrak{X}' \cong U \oplus U \oplus U \oplus E_8(-2).
\] (4.1)

Thus, the lattice \(E_8(-2)\) is important to study Nikulin involutions. We will use the following result.

**Lemma 4.3** ([19] Section 2, see also [10] Proposition 2.1). Let \(\mathfrak{X}\) be an algebraic K3 surface.

1. If \(\mathfrak{X}\) admits a Nikulin involution \(\iota\), there is a primitive embedding

\[
\text{Tr}(\mathfrak{X}) \hookrightarrow U \oplus U \oplus U \oplus E_8(-2).
\] (4.2)

Also, one has the structure of the transcendental lattice of \(\mathfrak{Y} = \mathfrak{X} / \langle \iota \rangle\):

\[
\text{Tr}(\mathfrak{Y}) \cong \left( (\text{Tr}(\mathfrak{X}) \otimes \mathbb{Q}) \cap \left( U \oplus U \oplus U \oplus \frac{1}{2} E_8(-2) \right) \right) (2)
\] (4.3)

2. Conversely, if the transcendental lattice of \(\mathfrak{X}\) admits a primitive embedding (4.2), there is a Nikulin involution \(\iota\).

### 4.2 Double covering for our families

In this subsection, let us give a relation between the K3 surface \(\hat{K}(t)\) of Equation (2.3), which is coming from the simple K3 singularity of No. 88, and the partner surface \(\hat{S}(t)\) of Equation (3.1), which is closely related to Hermitian modular forms and the simplest lattice \(A\) of Equation (3.3) with the Kneser conditions.

**Theorem 4.4.** Suppose \([t]\) be a point \(T\) of (3.2).

1. The K3 surface \(\hat{K}(t)\) of Equation (2.3) is birationally equivalent to the surface

\[
K(t) : Z^2 = Y^3 + \left( t_4 + \frac{t_{10}}{U^2} \right) Y + \left( t_6 + U^2 + \frac{t_{12}}{U^2} + \frac{t_{18}}{U^4} \right).
\] (4.4)

2. The K3 surface \(\hat{S}(t)\) of Equation (3.1) is birationally equivalent to the surface

\[
S(t) : Z^2 = Y^3 + \left( t_4 + \frac{t_{10}}{X} \right) Y + \left( t_6 + X + \frac{t_{12}}{X} + \frac{t_{18}}{X^2} \right).
\] (4.5)

3. One has an explicit double covering \(K(t) \rightarrow S(t)\) given by \((U, Y, Z) \mapsto (X, Y, Z) = (U^2, Y, Z)\), which is coming from a Nikulin involution \(\iota\) on \(K(t)\).

**Proof.**

1. By putting

\[
x_1 = U^3 Z, \quad y_1 = U^2 Y, \quad z = U,
\]

Equation (2.3) is transformed to Equation (4.4).

2. By putting

\[
x_2 = X, \quad y_2 = X^2 Y, \quad z_2 = X^3 Z,
\]
Equation (3.1) is transformed to Equation (4.5).

(3) Since the unique holomorphic 2-form on $K(t)$ is given by $\frac{dY \wedge dU}{UZ}$ up to a constant factor, an explicit involution given by $(U, Y, Z) \mapsto (-U, Y, Z)$ is a Nikulin involution on $K(t)$.

The family $\pi_{C_1} : C_1 \to (\pi_{F_1} : F_1 \to T, \text{resp.})$ in Section 1 is equal to the family of $K(t)$ ($S(t), \text{resp.}$).

Remark 4.5. The reason why $S(t)$ is named the partner surface is the following. The subfamily $\pi_{F_1}^{-1} \{ t_{18} = 0 \} \cap T$ is equal to the family $F_{CD}$ of $K3$ surfaces studied in [4] (see Proposition 3.1 (2) and Section 6.1). Since every member of $F_{CD}$ has a van Geemen–Sarti involution (see [25], Section 4), it is called a van Geemen–Sarti partner of the Kummer surface (e.g., see [5]). However, as we will see in Section 6.3, a generic member of our family $F_1$ admits no van Geemen–Sarti involutions. Hence, we simply call $F_1$ the family of partner surfaces of $C_1$.

Remark 4.6. The reason why we use expressions (4.4) and (4.5) with denominators is the following. First, the double covering $S(t) \to K(t)$ is described in a very simple form by using Equations (4.4) and (4.5). Also, our expressions are natural extensions of the results of [24], in which he explicitly studies a subfamily $C_{Shio}$ of $C_{Kum}$ consisting of the Kummer surfaces attached to direct products of two elliptic curves and a subfamily $F_{Shio}$ of $F_{CD}$ corresponding to $C_{Shio}$ (for detail, see the beginning of Section 5).

Remark 4.7. The authors found the very simple expressions (4.4) and (4.5) of $K3$ surfaces during our research in order to describe our period mappings via solutions of a system of GKZ hypergeometric differential equations. This investigation will be published elsewhere.

Proposition 4.8. For a generic point $[t] = (t_4 : t_6 : t_{10} : t_{12} : t_{18}) \in T$, the Picard number of the $K3$ surface $K(t)$ of (4.4) is equal to 16.

Proof. By Proposition 3.1 and Lemma 4.3 (especially (4.3)), the rank of $\text{Tr}(K(t))$ is equal to that of $\text{Tr}(S(t))$. From Proposition 3.1, we have the assertion.

5 GEOMETRIC CONSTRUCTION OF TRANSCENDENTAL LATTICE

In this section, we will determine the lattice structure of the surface $K(t)$ of Equation (4.4).

According to Proposition 4.8, the rank of the transcendental lattice of $K(t)$ is generically equal to 6. If we had a double covering $S(t) \to K(t)$, we could determine the transcendental lattice of $K(t)$ by applying Lemma 4.3 to Proposition 3.1. However, in practice, $S(t)$ does not have any Nikulin involutions, as we will see in Section 6.3. Therefore, in this section, we will construct 2-cycles on $K(t)$ geometrically, in order to determine the lattice structure of $K(t)$ directly. Eventually, we will prove the following theorem.

Theorem 5.1.

(1) For a generic point $[t] = (t_4 : t_6 : t_{10} : t_{12} : t_{18}) \in T$, the transcendental lattice of the $K3$ surface $K(t)$ of Equation (4.4) is given by the intersection matrix $A(2) = U(2) \oplus U(2) \oplus A_2(-2)$. This is a primitive sublattice of $U \oplus U \oplus U \oplus E_8(-2)$.

(2) For a generic point $[t] \in T$, the Néron–Severi lattice of $K(t)$ is isometric to the lattice $U(2) \oplus E_8(-1) \oplus L_6$, where

$$L_6 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 2 & 0 \\ 1 & 0 & -4 & 0 & 2 \\ 0 & 2 & 0 & -4 & 2 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \quad (5.1)$$
Remark 5.2. Belcastro [2] determines the structures of the lattices for the K3 surfaces which are coming from resolutions of simple K3 singularities. Especially, in the list of [2], it is stated that the transcendental lattice corresponding to the K3 surfaces for the singularity of No. 88 is isomorphic to A of Equation (3.3). If that is true, the family of $K(t)$ should coincide with that of $S(t)$.

We tried to follow such a story, but it was not successful. On the contrary, we found that the transcendental lattice of the family of $K(t)$ should be $A(2)$ and this family naturally contains the family of the algebraic Kummer surfaces.

Thus, at least for the case of No. 88, the statement of [2] is not correct. Our Theorem 5.1 gives a correction of this defect.

In order to calculate the transcendental lattice of $K(t)$, we consider the elliptic K3 surface $\hat{K}(t)$ of Equation (2.3), which is birationally equivalent to $K(t)$. There is an involution

$$J : (x_1, y_1, z) \mapsto (x_1, y_1, -z). \quad (5.2)$$

Also, on $\hat{K}(t)$, there is the unique holomorphic 2-form

$$\omega_1 = \omega = \frac{dy_1 \wedge dz}{x_1} \quad (5.3)$$

up to a constant factor. Then, we have $J^*(\omega) = -\omega$. Namely, $J$ is not a Nikulin involution on the K3 surface (2.3). The quotient surface

$$\Sigma(t) : x_1^2 = y_1^3 + (t_4w^2 + t_{10}w)y_1 + (w^4 + t_6w^3 + t_{12}w^2 + t_{18}w) \quad (5.4)$$

by the involution $J$ is a rational surface.

Remark 5.3. We remark that the period mapping for our family $\mathcal{P}_1$ is connected to the complex reflection group of No. 33 in the list of [23]. For detail, see [15].

On the other hand, Sekiguchi [22] studied the complex reflection groups of No. 33 from the viewpoint of Frobenius potentials and obtains a family of rational surfaces given by the Weierstrass equation

$$z^2 = f_{E_6(1)} = y^3 + (t_2x^2 + t_5x)y + (x^4 + t_3x^3 + t_6x^2 + t_9x) + s_4y^2.$$ 

If $s_4 = 0$, we have our rational surface $\Sigma(t)$ of Equation (5.4). Although our method and viewpoint are very different from those of Sekiguchi, our surface $\Sigma(t)$ is very close to his surface.

By putting $(t_4, t_6, t_{10}, t_{12}, t_{18}) = (0, 0, 3, -1, -1)$, we take a generic member

$$\hat{K}_0 : x_1^2 = y_1^3 + 3z^2y_1 + z^8 - z^4 - z^2 \quad (5.5)$$

of the family of the surfaces of Equation (2.3). Namely, we regard $\hat{K}_0$ as a reference surface of the family. By virtue of a consideration of local period mapping, as in the argument in [14], Section 1.3, we can see that the lattice structure for the reference surface $\hat{K}_0$ is valid for generic members of the family of $K(t)$ of Equation (4.4). We obtain the corresponding quotient surface $\Sigma_0$ of $\hat{K}_0$ by $J$:

$$\Sigma_0 : x_1^2 = y_1^3 + 3wy_1 + w^4 - w^2 - w. \quad (5.6)$$

Properties of $\Sigma_0$ will be useful to construct 2-cycles on $\hat{K}_0$.

### 5.1 Rational surface $\Sigma_0$

Let us observe the elliptic surface $(\Sigma_0, \pi, P^1(\mathbb{C}))$ over $w$-sphere $P^1(\mathbb{C})$. Let $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ and $\tau_6$ be the solutions of $1 + 6w + w^2 - 2w^3 - 2w^4 + w^5 = 0$ such that approximate values are $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) \approx \ldots$
We have singular fibers of $\Sigma_0$ of type $I_1$ over these points. Moreover, there is a singular fiber of type $II$ ($IV$, resp.) over $w = \tau_0 = 0$ ($w = \tau_\infty = \infty$, resp.). Set $\tau_b = -0.5$ be a base point on the $w$-space. We have a regular fiber

$$
\pi^{-1}(\tau_b) : x_1^2 = y_1^3 - 1.5y_1 + 0.3125.
$$

It has four ramification points on the $y_1$-plane: $(v_1, v_2, v_3, v_m) \approx (-1.31799, 0.214955, 1.10304, \infty)$. On $\pi^{-1}(\tau_b)$, set an oriented closed arc $\gamma_1$ ($\gamma_2$, resp.) such that its projection goes around $v_1$ and $v_2$ ($v_2$ and $v_3$, resp.) in the positive direction. We define their branch and orientation so that the intersection number $(\gamma_1 \cdot \gamma_2)$ is equal to 1 (see Figure 1).

We make an oriented closed circuits $\delta_i$ ($i = 0, 1, \ldots, 6, \infty$) on the $w$-plane which goes around $\tau_i$ in the positive direction with the starting point $\tau_b$ (see Figure 2).

Let $M_i$ be the matrix which represents the monodromy transformation of the homology basis $(\gamma_1, \gamma_2)$ by the continuation of the fiber $\pi^{-1}(\tau_b)$ along $\delta_i$. We call it the circuit matrix. They are given in Table 2. Note that they are assumed to be left actions.
5.2 | Construction of cycles on $\hat{K}_0$

We have the elliptic $K3$ surface $(\hat{K}_0, \pi_\mathcal{Z}, \mathbb{P}^1(\mathbb{C}))$ of Equation (5.5), where the base space $\mathbb{P}^1(\mathbb{C})$ is the $z$-sphere and $\pi_\mathcal{Z}$ is the natural projection. There is the double covering $\hat{K}_0 \to \Sigma_0$ given by the involution (5.2). Recalling the singular fibers over $\Sigma_0$, we find singular fibers of the elliptic $K3$ surface $\hat{K}_0$ over the points

$$
\zeta_j = \sqrt{\tau_j}, \quad \zeta'_j = -\sqrt{\tau_j} \quad (j \in \{1, \ldots, 6\}), \quad \zeta_0 = 0, \quad \zeta_{\infty} = \infty.
$$

**Remark 5.4.** We have a singular fiber of type $IV$ ($IV^*$, resp.) over $\zeta_0$ ($\zeta_{\infty}$, resp.). So, besides the general fiber and the global section, we have two (six, resp.) components of $\pi_\mathcal{Z}^{-1}(0)$ ($\pi_\mathcal{Z}^{-1}(\infty)$, resp.). Hence, we obtain a sublattice of $\text{NS}(\hat{K}_0)$ of rank 10 which is generated by these divisors. We denote it by $L_B$.

For $i \in \{1, \ldots, 0, \infty\}$, let $\ell_i$ be a line segment connecting a fixed point $B$ in the $w$-sphere and $\tau_i$ respectively. We lift up the cut lines $\ell_1, \ldots, \ell_6, \ell_0, \ell_\infty$, to $m_1, \ldots, m_6, m'_1, \ldots, m'_6, m_0, m_\infty$. Those are indicated in Figure 3. We take an oriented arc $\alpha$ from $\zeta'_b = \sqrt{\tau_b}$ to $\zeta'_b = -\sqrt{\tau_b}$ in the simply connected region $\mathbb{P}' = \mathbb{P}^1(\mathbb{C}) - (\bigcup m_i \cup \bigcup m'_i \cup B B')$, where $B (B',$ resp.) is the initial points of the cut lines $m_1, m_2, m_4, m'_3, m'_5, m'_6$ and $m_0 (m_3, m_5, m_6, m'_1, m'_2, m'_4$ and $m'_\infty$, resp.) and $BB'$ is an arc connecting them indicated in Figure 3.

We make the liftings $\delta_{2i} (i \in \{1, \ldots, 6\})$ of $\delta_i$ indicated in Figure 4. In any case, we take $\zeta'_b$ as their starting point. By a similar manner, we make circuits $\delta'_{2i}$ starting from $\zeta'_b$ as indicated in Figure 4. Also, we take a closed circle $\delta_{\infty} (\delta_{\infty},$ resp.) around $z = 0 (z = \infty,$ resp.).
Then, \( \pi_x^{-1}(\xi_b) \) and \( \pi_x^{-1}(\xi'_b) \) are the same elliptic curve which is identified with \( \pi^{-1}(\tau_b) \) on \( \Sigma_0 \). So, we can define \( \gamma_1 \) and \( \gamma_2 \) on them by this identification. On the other hand, the involution \( \iota \) of (5.2) induces an isomorphism \( \pi_x^{-1}(\xi_b) \cong \pi_x^{-1}(\xi'_b) \) on \( \hat{K}_0 \). So, we obtain the 1-cycles \( \gamma'_1 = j(\gamma_1) \) and \( \gamma'_2 = j(\gamma_2) \) on \( \pi_x^{-1}(\xi'_b) \).

We have the local monodromy of the system \( \{\gamma_1, \gamma_2\} \) along every circuit \( \delta_{zi} \). We denote the matrix which represents it by \( M_{zi} \). By observing the covering structure \( \hat{K}_0 \to \Sigma_0 \) and Figure 2, we have \( M_{zi} = M_i \) (\( i = 1, \ldots, 6 \)) and \( M_{z0} = M^2_0 \).

Since the local monodromy induced from \( \delta'_{zi} \) on the system \( \{\gamma'_1, \gamma'_2\} \) is just the copy of those in Table 3, we have the same matrix for \( i \in \{1, \ldots, 6\} \). We have the transformation between two systems \( \{\gamma_1, \gamma_2\} \) and \( \{\gamma'_1, \gamma'_2\} \) on \( \pi_x^{-1}(\xi'_b) \) induced from the arc \( \alpha \):

\[
\begin{pmatrix} \gamma'_1 \\ \gamma'_2 \end{pmatrix} = M_\alpha \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad \text{where } M_\alpha = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.7}
\]

Then, the local monodromy \( M'_{zi} \) induced from \( \delta'_{zi} \) on the system \( \{\gamma'_1, \gamma'_2\} \) is given by \( M'_{zi} = M^{-1}_\alpha M_{zi} M_\alpha \). Hence, we obtain Table 4 for them.

**Remark 5.5.** We can determine \( M_{\alpha} \) by the facts that \( M'_{z0} = M_{z0} \) and that the fiber \( \pi_x^{-1}(0) \) is a singular fiber of type IV. By the relation \( M_{z0} M'_{z0} M'_{z1} M'_{z2} M'_{z3} M'_{z4} M'_{z5} M'_{z6} M_{z6} M_{z5} M_{z4} M_{z3} M_{z0} = \text{id} \), the matrix \( M_{z0} \) is determined as in Table 3.

Let \( \rho \) be an oriented arc in the \( z \)-sphere \( \mathbb{P}^1(\mathbb{C}) \) from \( \xi \) to \( \eta \), where \( \xi \) and \( \eta \) are two points in \( \mathbb{P}^1(\mathbb{C}) \). Take \( \gamma \) be a 1-cycle on the fiber \( \pi_x^{-1}(\xi) \). We let \( U(\rho, \gamma) \) denote the 2-chain obtained by the analytic continuation of \( \gamma \) along \( \rho \). Here, we define the orientation of \( U(\rho, \gamma) \) as the ordered pair of the orientation of \( \rho \) and that of \( \gamma \) as in [12], Chapter 2. We note that if \( \gamma \) is a vanishing cycle at \( \xi \) and \( \eta \), then \( U(\rho, \gamma) \) becomes to be a 2-cycle on \( \hat{K}_0 \).

Let us construct a 2-cycle \( \Gamma_1 \). We take the points \( Q_1, Q_2, Q_3, \) and \( Q_4 \) on the \( z \)-plane indicated in Figure 5. Then, we make an “8-shaped” closed arc \( \rho_1 \) connecting them in this order returning to the initial point \( Q_1 \) (see Figure 5). We take a 1-cycle \( \gamma_1 \) on the fiber \( \pi_x^{-1}(Q_1) \). Then, we obtain a 2-cycle \( \Gamma_1 = U(\rho_1, \gamma_1) \). According to Table 2, we can see that the continuation returns back to the original \( \gamma_1 \). Namely, the cycle \( \gamma_1 \) changes to \( \gamma_1 - \gamma_2 \) after crossing the cut line \( m_1 \) and it returns back to \( \gamma_1 \) after crossing \( m_2 \). So, \( \Gamma_1 \) is a 2-cycle on \( \hat{K}_0 \).

By the same way, according to the indication in Figure 6, we can obtain 2-cycles \( \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \).

Next, we construct a 2-cycle \( \Gamma_5 \). We take the points \( P_1 \) and \( P_2 \) illustrated in Figure 7. Take a small circle \( c_1 \) (\( c_2 \), resp.), which starts from \( P_1 \) and goes around \( \xi_3 \) (\( \xi_5 \), resp.) in the positive (negative, resp.) direction. Let \( \beta_5 \) be an oriented arc from \( P_1 \) to \( P_2 \) crossing the cut line \( m_0 \). So, we have the 2-chains \( U(c_1, \gamma_2), U(\beta_5, \gamma_1) \) and \( U(c_2, \gamma_1) \) on \( \hat{K}_0 \). Here, we use the circuit matrices in Table 3. For example, \( \gamma_2 \) on \( P_1 \) is transformed to \( \gamma_1 + \gamma_2 \) after crossing the cut line \( m_3 \). Thus, we obtain a 2-cycle

\[
\Gamma_5 = U(c_1, \gamma_2) \cup U(\beta_5, \gamma_1) \cup U(c_2, \gamma_1),
\]

that is illustrated by Figure 7.
Also, let us construct a 2-cycle \( \Gamma_6 \). Take the points \( P_2, P_3, P_4, \) and \( P_5 \) indicated in Figure 8. Let \( c_2 \) be as above. Let \( c_3 \) (\( c_5 \), resp.) be a small circle which starts from \( P_3 \) (\( P_5 \), resp.) and goes around \( \zeta_6 \) (\( \zeta'_6 \), resp.) in the negative (positive, resp.) direction. For \((l, m) \in \{(4, 2), (4, 3), (5, 4)\}, \) let \( \beta_{lm} \) be an arc from \( P_l \) to \( P_m \). We have the 2-chains \( U(c_2, \gamma_1 - \gamma_2), U(c_3, \gamma_1 + \gamma_2), U(c_5, \gamma'_2), U(\beta_{42}, -\gamma_2), U(\beta_{43}, \gamma_1) \) and \( U(\beta_{54}, \gamma'_1) \). Here, according to the relation \((5.7)\), we have \( \gamma'_2 = \gamma_1 \) and \( \gamma'_1 = \gamma_1 - \gamma_2 \). As in the case of \( \Gamma_5 \), we obtain a 2-cycle

\[
\Gamma_6 = U(c_2, \gamma_1 - \gamma_2) \cup U(c_3, \gamma_1 + \gamma_2) \cup U(c_5, \gamma'_2) \cup U(\beta_{42}, -\gamma_2) \cup U(\beta_{43}, \gamma_1) \cup U(\beta_{54}, \gamma'_1),
\]

which is illustrated in Figure 8.
5.3 Intersection numbers

We put $\Gamma'_i = \varphi(\Gamma_i)$ ($i \in \{1, 2, 3, 4, 5, 6\}$). Set $L_{GG'} = \{\Gamma_1, \ldots, \Gamma_6, \Gamma'_1, \ldots, \Gamma'_6\}$. (5.8)

**Proposition 5.6.** The rank of $L_{GG'}$ is equal to 12. It is orthogonal to the system $L_B$ in Remark 5.4. Hence, it holds $\langle L_{GG'}, L_B \rangle \otimes \mathbb{Q} = H_2(\hat{K}_0, \mathbb{Q})$.

**Proof.** We have $\text{rank}(H_2(\hat{K}_0, \mathbb{Q})) = 22$ and $\text{dim}_\mathbb{Q}(L_B \otimes \mathbb{Q}) = 10$. By the construction, any member of $L_{GG'}$ is orthogonal to $L_B$. So, it is enough to check that $\Gamma_1, \ldots, \Gamma_6, \Gamma'_1, \ldots, \Gamma'_6$ are independent. Since we have Proposition 5.7, it follows that the intersection matrix to be nonsingular. Hence, we have the assertion.

**Proposition 5.7.** The intersection matrix $M_{GG'}$ of the system (5.8) is given by $M_{GG'} = \left( \begin{array}{cc} C_G & C_{GG'} \\ C_{GG'} & C_G \end{array} \right)$, where

$C_G = ((\Gamma_i \cdot \Gamma_j))_{1 \leq i, j \leq 6} = \begin{pmatrix} -2 & -1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & -1 \\ 1 & 1 & -2 & -1 & -1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & -1 \\ 0 & -1 & 0 & 0 & -1 & -2 \end{pmatrix}

$C_{GG'} = ((\Gamma'_i \cdot \Gamma'_j))_{1 \leq i, j \leq 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$.

Especially, $M_{GG'}$ is nonsingular.

**Proof.** Let us calculate the intersection number $(\Gamma_1 \cdot \Gamma_2)$. We have two geometric intersections $R_1$ and $R_2$ on the $z$-plane as described in Figure 9. We observe both local intersections $(\Gamma_1 \cdot \Gamma_2)_{R_1}$ and $(\Gamma_1 \cdot \Gamma_2)_{R_2}$. At the point $R_1$, $\Gamma_1 \cap \pi^{-1}(R_1)$ is the 1-cycle $\gamma_1 - \gamma_2$ and $\Gamma_2 \cap \pi^{-1}(R_1)$ is the same 1-cycle $\gamma_1 - \gamma_2$. So, it holds $(\Gamma_1 \cdot \Gamma_2)_{R_1} = 0$. At the point $R_2$, $\Gamma_1 \cap \pi^{-1}(R_2)$ is the 1-cycle $\gamma_1$ and $\Gamma_2 \cap \pi^{-1}(R_2)$ is the 1-cycle $\gamma_1 - \gamma_2$. Hence, we have $(\Gamma_1 \cdot \Gamma_2)_{R_2} = (-1) \cdot (\gamma_1 \cdot (\gamma_1 - \gamma_2)) \cdot (\varphi_1 \cdot \varphi_2)_{R_2} = -1$.

So $(\Gamma_1 \cdot \Gamma_2) = -1$ holds. We can similarly calculate other intersection numbers.
By (5.3), we have
\[
\int_{\Gamma'_i} \omega = - \int_{\Gamma_i} \omega \quad (i = 1, \ldots, 6).
\]
It means that every \( \Gamma_i + \Gamma'_i \) (\( i = 1, \ldots, 6 \)) is an algebraic cycle.

**Proposition 5.8.** Every \( \Gamma_i - \Gamma'_i \) (\( i = 1, \ldots, 6 \)) is an element of the orthogonal complement of the Néron–Severi lattice \( \text{NS}(\hat{K}_0) \).

**Proof.** By Proposition 4.8, the rank of \( \text{NS}(\hat{K}_0) \) is 16. By the construction, it is apparent that \( \Gamma_i - \Gamma'_i \) is orthogonal to the lattice \( L_B \). Also, we obtain \( \left( (\Gamma_i - \Gamma'_i) \cdot (\Gamma_j + \Gamma'_j) \right) = 0 \) from Proposition 5.7. Hence, the assertion is proved. \( \square \)

We make oriented arcs \( h_0, h_1, \ldots, h_5 \) on the \( z \)-plane given by Figure 10. Each arc starts from \( \zeta_i \) with the terminal \( \zeta_\infty \). Over these arcs, we make 2-cycles:

\[
\begin{align*}
C_1 &= U(h_1, \gamma_2), \quad C_2 = U(h_2, -\gamma_2), \quad C_3 = U(h_3, -\gamma_1), \\
C_4 &= U(h_4, \gamma_1), \quad C_5 = U(h_5, -\gamma_2), \quad C_6 = U(h_0, -\gamma_2).
\end{align*}
\]

(5.9)

For \( C_i \) (\( i = 1, \ldots, 6 \)), each 1-cycle on the fiber on \( h_i \) is a vanishing cycle at the starting point \( \zeta_i \) (\( i = 0, 1, \ldots, 5 \)). So, every \( C_i \) determines a 2-cycle on \( \hat{K}_0 \). By a direct calculation as in the proof of Proposition 5.7, we obtain the following result.
Proposition 5.9. The intersection matrix $M_{CG} = ((C_i \cdot (\Gamma_j - \Gamma'_j)))_{i,j \in \{1, \ldots, 6\}}$ is given by

$$M_{CG} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

It holds $\det(M_{CG}) = 1$.

By this proposition, it is guaranteed that the system $\{\Gamma_1 - \Gamma'_1, \ldots, \Gamma_6 - \Gamma'_6\}$ becomes to be a $\mathbb{Z}$-basis of $H_2(\bar{K}_0, \mathbb{Z})/\text{NS}(\bar{K}_0)$. Set $G_j = \Gamma_j - \Gamma'_j (j \in \{1, \ldots, 6\})$. Also, setting $t(C'_1, \ldots, C'_6) = M_{CG}^{-1} t(C_1, \ldots, C_6)$, it holds

$$(C'_i \cdot G_j) = \delta_{ij} \quad (i, j \in \{1, \ldots, 6\}).$$

We set $M_G = (G_i \cdot G_j)_{1 \leq i, j \leq 6}$. By a direct calculation, we have the following proposition.

Proposition 5.10.

$$M_G = 2 \begin{pmatrix} -2 & -1 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 1 & 0 & -1 \\ 1 & 1 & -2 & -1 & -1 & 0 \\ 0 & 1 & -1 & -2 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -2 \end{pmatrix}.$$

Theorem 5.11. The system $\{G_1, \ldots, G_6\}$ gives a basis of the transcendental lattice $\text{Tr}(\bar{K}_0)$ of the reference surface with the intersection matrix $A(2) = U(2) \oplus U(2) \oplus A_2(-2)$.

Proof. Set $M_f = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$. This is a unimodular matrix. By a direct calculation, it holds $M_f M_G M_f^t = \begin{pmatrix} U(2) \oplus U(2) \oplus \begin{pmatrix} -4 & 2 \\ 2 & -4 \end{pmatrix} \end{pmatrix}$. □

Theorem 5.1 (1) immediately follows from Theorem 5.11 and Lemma 4.3.

5.4 Néron–Severi lattice for $G_1$

In this subsection, we prove Theorem 5.1 (2).

Lemma 5.12.

(1) There is a primitive embedding $i_1 : U(2) \leftrightarrow U \oplus U$.
(2) There is a primitive embedding $i_2 : U(2) \oplus A_2(-2) \leftrightarrow U \oplus E_8(-1)$. 
Proof.

(1) If $V$ be an even unimodular lattice of rank 2, then there exists a primitive embedding $V \hookrightarrow U \oplus U$. The assertion follows from this fact.

(2) Let $\{e, f\}$ be a basis of $U$ similar as above. Let $\{p_1, \ldots, p_8\}$ be a basis of $E_8(-1)$ with the Dynkin diagram in Figure 11. Set $w_1 = e + f + p_1, w_2 = e + f + p_3, w_3 = p_6 + p_8$ and $w_4 = e - f + p_7$. Then, $\{w_1, w_2, w_3, w_4\}$ gives a basis of $U(2) \oplus A_2(-2)$ which induces a primitive embedding $U(2) \oplus A_2(-2) \hookrightarrow U \oplus E_8(-1)$.

Lemma 5.13.

(1) The intersection matrix of the orthogonal complement of $i_1(U(2))$ is given by $U(2)$.

(2) The intersection matrix of the orthogonal complement of $i_2(U(2) \oplus A_2(-2))$ is given by $L_6$ of Equation (5.1).

Proof. (1) is obvious. We give a proof of (2). Under the notation of the proof of Lemma 5.12, we have a basis $\{b_1, \ldots, b_6\}$ of the orthogonal complement of $\langle w_1, \ldots, w_4 \rangle$, where $b_1 = e + f + p_1 + p_3, b_2 = -e - p_1 - p_2 - 3p_3 - 2p_4 - 2p_5 - p_6, b_3 = -f + p_2 + 3p_3 + p_6, b_4 = -p_1 - 2p_2 - p_3, b_5 = -p_3 - 2p_5 - 2p_6 - p_7$ and $b_6 = p_6 - p_8$. The intersection matrix of this basis is $L_6$ of Equation (5.1).

For an even lattice $L$, let $\mathcal{A}_L$ and $q_L$ be as in Section 4.1.

Lemma 5.14.

(1) The group $\mathcal{A}_{A_2(-1)}$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. The discriminant form $q_{A_2(-1)}$ is a quadratic form corresponding to the intersection matrix $\begin{pmatrix} -1/3 \\ 0 \\ 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}$.

(2) The group $\mathcal{A}_{U(2)}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. The discriminant form $q_{U(2)}$ is a quadratic form corresponding to the intersection matrix $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

Proof.

(1) Let $\{n_1, n_2\}$ be a basis of $A_2(-2)$ such that $(n_1 \cdot n_1) = (n_2 \cdot n_2) = -4$ and $(n_1 \cdot n_2) = 2$. Then, $\mathcal{A}_{A_2(-2)} = \langle n_1 + 2n_2 \rangle \mathbb{Z}$. The form of $q_{A_2(-2)}$ follows immediately.

(2) Let $\{E, F\}$ be a basis of $U(2)$ such that $(E \cdot E) = (F \cdot F) = 0$ and $(E \cdot F) = 2$. Then, $\mathcal{A}_{U(2)} = \langle E \rangle \mathbb{Z} \cdot \langle F \rangle \mathbb{Z}$. The form of $q_{U(2)}$ follows immediately.

Now, we give a proof of Theorem 5.1 (2).

Proof of Theorem 5.1 (2). Letting $\tilde{M}$ be the orthogonal complement of $U(2) \oplus U(2) \oplus A_2(-2)$ in $L_{K3}$, according to Proposition 4.2 and Lemma 5.14, we have

$$l(\mathcal{A}_{\tilde{M}}) = 5 \leq 14 = \text{rank}(\tilde{M}) - 2.$$ 

From Proposition 4.1, $\tilde{M}$ is uniquely determined by the invariants $(1, 15, -(q_{U(2)} \oplus q_{U(2)} \oplus q_{A_2(-2)}))$. 

\[ \]
Since we have the primitive embeddings $i_1$ and $i_2$ in Lemma 5.12, together with Lemma 5.13, $\tilde{M}$ is regarded as a primitive sublattice of $L' = U \oplus U \oplus U \oplus E_8(-1)$ with the orthogonal complement $U(2) \oplus L_6$. Also, the orthogonal complement of $L'$ in $L_{K3}$ is $E_8(-1)$. By virtue of Proposition 4.2, the lattice $U(2) \oplus E_8(-1) \oplus L_6$ has the invariants $(1, 15, -(q_{U(2)} \oplus q_{U(2)} \oplus q_{A_2(-2)}))$. So, $\tilde{M}$ is isometric to this lattice.

### 5.5 Multivalued period mapping for $G_t$

As in [14] Section 1.3, we can define a marking $H_2(\hat{K}(t), \mathbb{Z}) \rightarrow L_{K3}$ for $[t] \in T$ using a basis of $\text{NS}(\hat{K}(t))$ and the analytic continuation in the parameter space $T$ of (3.2). Here, the 2-cycles on the reference surface $\hat{K}_0$ give the initial marking. Via such a marking, we can define the period mapping for the family of $\hat{K}(t)$, also. We can obtain an expression of the period mapping for $\hat{K}(t)$ under the notation of this section as follows. Set $(H_1, \ldots, H_6) = (G_1, \ldots, G_6)^t M_f$. We can expand the system $\{H_1, \ldots, H_6\}$ to $\{H_1, \ldots, H_{22}\}$, which gives a basis of $H_2(\hat{K}_0, \mathbb{Z})$. Let $\{D_1, \ldots, D_{22}\}$ be the dual basis of $\{H_1, \ldots, H_{22}\}$ with respect to the unimodular lattice $L_{K3}$. By using the above-mentioned marking, we can naturally obtain 2-cycles $D_{1,t}, \ldots, D_{6,t} \in H_2(\hat{K}(t), \mathbb{Z})$. Now, let us recall the period mapping (3.5) for the family of the partner surface $S(t)$. Since it holds $\text{Tr}(\hat{K}(t)) = \text{Tr}(S(t))(2)$, the Riemann–Hodge relations for $\hat{K}(t)$ are equal to those for $S(t)$. Especially, the period domain for the family of $\hat{K}(t)$ coincides with $\mathfrak{R}$ which appears in Section 3. Hence, we obtain the following theorem.

**Theorem 5.15.** The multivalued period mapping for the family of $\hat{K}(t)$, which is birationally equivalent to $K(t)$ of Equation (4.4), coincides with Equation (3.5):

$$ T \ni [t] \mapsto \left( \int_{D_{1,t}} \omega_t : \cdots : \int_{D_{6,t}} \omega_t \right) = \left( \int_{\Delta_{1,t}} \omega_{S,t}^S : \cdots : \int_{\Delta_{6,t}} \omega_{S,t}^S \right) \in D, $$

where $\omega_t$ is the holomorphic 2-form of (5.3) on $\hat{K}(t)$.

By virtue of the argument at the end of Section 3, we have the following corollary.

**Corollary 5.16.** One has an explicit expression of the inverse correspondence of the period mapping for the family of $K(t)$ by the Dern–Krieg theta functions. The theta expression is the same as that of [15], Theorem 4.1.

Thus, we have an analytic correspondence between the period integrals on $K(t)$ and the parameters $t_j$ via the theta functions.

We note that the family of our explicit surfaces $K(t)$ does not coincide with the set of equivalence classes of marked lattice polarized $K3$ surfaces for the lattice $A(2)^4(\subset L_{K3})$ in the sense of [7]. The moduli space of such polarized $K3$ surfaces is given by a covering of the parameter space $T$ of (3.2). Recall that $T$ just gives the moduli space of marked $(U \oplus E_8(-1) \oplus E_6(-1))$-polarized $K3$ surfaces via the isomorphism (3.6). Nevertheless, the expression of the multivalued period mapping in Theorem 5.15 and the theta expression mentioned in Corollary 5.16 are valid.

### 6 SANDWICHPHENOMENON

Shioda [24] studied the following explicit defining equations of the Kummer surface $\text{Kum}(E_1 \times E_2)$ for the product of two elliptic curves and a $K3$ surface $S_{\text{Shio}}$:

$$ \text{Kum}(E_1 \times E_2) : \ Z^2 = Y^3 - 3\alpha Y + \left(U^2 + \frac{1}{U^2} - 2\beta \right), $$

$$ S_{\text{Shio}} : \ Z^2 = Y^3 - 3\alpha Y + \left(X + \frac{1}{X} - 2\beta \right), $$

where these equations define elliptic surfaces and $\alpha$ and $\beta$ are complex parameters. Let $G_{\text{Shio}} (F_{\text{Shio}}, \text{resp.})$ be the family of all $\text{Kum}(E_1 \times E_2)$ ($S_{\text{Shio}}, \text{resp.}$). He shows that there are double coverings $\varphi_{\text{Shio}}$ and $\psi_{\text{Shio}}$ such that

$$ \text{Kum}(E_1 \times E_2) \xrightarrow{\varphi_{\text{Shio}}} S_{\text{Shio}} \xrightarrow{\psi_{\text{Shio}}} \text{Kum}(E_1 \times E_2). $$
Explicitly, \( \varphi_{\text{Shio}} \) and \( \psi_{\text{Shio}} \) are derived from the involutions

\[
(U, Y, Z) \mapsto (-U, Y, Z), \quad (X, Y, Z) \mapsto \left( \frac{1}{X}, Y, -Z \right),
\]

respectively. He calls this phenomenon the Kummer sandwich. We note that studies for the Kummer sandwich are active and there are several recent works (e.g., see \([3]\) or \([11]\)).

The surface \( K(t) \) of Equation (4.4) (\( S(t) \) of Equation (4.5), resp.) can be regarded as a natural extension of \( \text{Kum}(E_1 \times E_2) \) (\( S_{\text{Shio}}, \) resp.). In this section, we will study natural and explicit counterparts of the Kummer sandwich phenomenon.

### 6.1 Explicit expression of Kummer sandwich of general type

Following the work \([24]\), Ma \([10]\) proved that an arbitrary Kummer surface \( \text{Kum}(\mathfrak{A}) \), where \( \mathfrak{A} \) is a principally polarized Abelian surface, admits the Kummer sandwich. Namely, there are double coverings \( \varphi_{\text{Kum}} \) and \( \psi_{\text{Kum}} \) such that

\[
\text{Kum}(\mathfrak{A}) \xrightarrow{\varphi_{\text{Kum}}} S_{\text{CD}} \xrightarrow{\psi_{\text{Kum}}} \text{Kum}(\mathfrak{A})
\]

(see \([10]\), Theorem 2.5). Here, \( S_{\text{CD}} \) is a lattice polarized K3 surface with the transcendental lattice \( U \oplus U \oplus A_1(-1) \). The family of such K3 surfaces coincides with the family studied in \([4]\) from the viewpoint of the Siegel modular forms. This is the reason why we use the notation \( S_{\text{CD}} \). The family \( \varphi_{\text{Kum}} (F_{\text{CD}}, \text{resp.}) \) in Section 1 is the family of \( \text{Kum}(\mathfrak{A}) \) (\( S_{\text{CD}}, \text{resp.} \)). We note that the proof of \([10]\) is based on a lattice theoretic argument, but he does not give explicit forms of the defining equations and the double coverings like Equations (6.1), (6.2), and (6.3).

As an application of Theorem 4.4, let us give a simple expression of \( \varphi_{\text{Kum}} \) and \( \psi_{\text{Kum}} \) in (6.4). This explicit result gives a natural extension of (6.3) of \([24]\).

**Theorem 6.1.** The Kummer surface \( \text{Kum}(\mathfrak{A}) \) for a principally polarized Abelian surface \( \mathfrak{A} \) is given by the Weierstrass equation

\[
Z^2 = Y^3 + \left( t_4 + \frac{t_{10}}{U^2} \right) Y + \left( t_6 + U^2 + \frac{t_{12}}{U^2} \right).
\]

Also, the K3 surface \( S_{\text{CD}} \) is given by the Weierstrass equation

\[
Z^2 = Y^3 + \left( t_4 + \frac{t_{10}}{X} \right) Y + \left( t_6 + X + \frac{t_{12}}{X} \right).
\]

The mappings \( \varphi_{\text{Kum}} \) and \( \psi_{\text{Kum}} \), which give the Kummer sandwich (6.4), are explicitly given by the Nikulin involutions

\[
t_{\varphi, \text{Kum}} : (U, Y, Z) \mapsto (-U, Y, Z), \quad t_{\psi, \text{Kum}} : (X, Y, Z) \mapsto \left( \frac{t_{10} Y + t_{12}}{X}, Y, -Z \right),
\]

respectively.

**Proof.** In \([14]\), the K3 surface (3.1) with the transcendental lattice \( U \oplus U \oplus A_2(-1) \) is introduced as an extension of the K3 surface \( S_{\text{CD}} \) with the transcendental lattice \( U \oplus U \oplus A_1(-1) \) (see also Proposition 3.1 (2)). Namely, if \( t_{18} = 0 \), then Equation (3.1) degenerates to \( S_{\text{CD}} \). So, together with Theorem 4.4, the surface defined by Equation (6.6) is birationally equivalent to \( S_{\text{CD}} \).

The involution \( t_{\psi, \text{Kum}} \) defines a Nikulin involution on Equation (6.6). Since \( S_{\text{CD}} \) admits a Shioda–Inose structure in the sense of \([13]\), the minimal resolution of the quotient \( S_{\text{CD}} / (t_{\psi, \text{Kum}}) \) coincides with \( \text{Kum}(\mathfrak{A}) \). This image is defined by the equation

\[
U_0 V_0^2 - U_0^2 + 4 t_{12} + t_6 V_0^2 + 4 t_{10} Y + t_4 V_0^2 Y + V_0^2 Y^3 = 0,
\]
where \( U_0 = X + \frac{t_{10}Y+ t_{12}}{X} \) and \( V_0 = \frac{1}{2} (X - \frac{t_{10}Y + t_{12}}{X}) \). By the birational transformation \( (U_0, V_0) = (2U(U + Z), 2U) \), Equation (6.7) is transformed to Equation (6.5) with the transcendental lattice \( U(2) \oplus U(2) \oplus A_1(-2) \) by virtue of [13], Theorem 5.7. Therefore, Equation (6.5) defines a general algebraic Kummer surface.

The involution \( \iota_{\phi, \text{Kum}} \) is just a special case of the involution of Theorem 4.4.

\[ \text{Remark 6.2.} \] The involution \( \iota_{\phi, \text{Kum}} \) is equal to the van Geemen–Sarti involution (see [25], Section 4) for the family of \( S_{\text{CD}} \) studied in [4].

Theorem 6.1 also gives a natural visualization of the Kummer sandwich phenomenon and Siegel modular forms of degree 2. Recall that the ring of Siegel modular forms on the three-dimensional Siegel upper half plane \( G_2 \) of the trivial character is generated by the modular forms of weight 4, 6, 10, 12, and 35. In fact, each of the parameters \( t_j \) (\( j \in \{4, 6, 10, 12\} \)) in Theorem 6.1 gives a member of a system of generators of the ring of Siegel modular forms via the period mapping for the family of \( S_{\text{CD}} \). Also, the generator of weight 35 and modular forms of the non-trivial character can be calculated by considering the degeneration of \( S_{\text{CD}} \) (see [4], see also [14], Proposition 2.3).

6.2 Nonexistence of sandwich between \( G_1 \) and \( F_1 \)

The family of \( K(t) \) (\( S(t) \), resp.) is a natural extension of that of \( \text{Kum}(\mathfrak{M}) \) (\( S_{\text{CD}} \), resp.). The argument in the last subsection guarantees that there is the sandwich when \( t_{18} = 0 \). However, if \( t_{18} \neq 0 \), we have only one side

\[ K(t) \to S(t) \]

of the sandwich, because Theorem 4.4 and the following theorem hold.

**Theorem 6.3.** The parameter \( t_{18} \) gives an obstruction to the existence of a double covering \( S(t) \to K(t) \) for the K3 surfaces (4.4) and (4.5). Namely, if \( t_{18} \neq 0 \), there is no Nikulin involution on \( S(t) \).

**Proof.** In order to apply the results of Section 4.1, we assume that there is a primitive embedding

\[ i : U \oplus U \oplus A_2(-1) \hookrightarrow U \oplus U \oplus U \oplus E_8(-2). \quad (6.8) \]

Now, this direct summand \( E_8(-2) \) should be a primitive sublattice of \( E_8(-1) \oplus E_8(-1) \subset L_{K3} \):

\[ E_8(-2) \hookrightarrow E_8(-1) \oplus E_8(-1). \quad (6.9) \]

So, we have a primitive embedding

\[ i' : U \oplus U \oplus A_2(-1) \hookrightarrow L_{K3} = II_{3,19} \]

as the composition of the mappings (6.8) and (6.9). The orthogonal complement of \( U \oplus U \) in \( II_{3,19} \) is isometric to \( U \oplus E_8(-1) \oplus E_8(-1) \), because it is an even unimodular lattice of signature (1,17). Hence, we can regard the direct summand \( U \oplus U \) of \( L_{K3} \) as the image of \( i'|_{U \oplus U} \). Thus, we can suppose that the embedding \( i \) satisfies

\[ i(A_2(-1)) \subseteq U \oplus E_8(-2). \quad (6.10) \]

Now, let \( \{e, f\} \) be a basis of \( U \) such that \( (e \cdot e) = (f \cdot f) = 0 \) and \( (e \cdot f) = 1 \). Also, let \( \{\nu_1, \nu_2\} \) be a basis of \( A_2(-1) \) satisfying \( (\nu_1 \cdot \nu_1) = (\nu_2 \cdot \nu_2) = -2 \) and \( (\nu_1 \cdot \nu_2) = 1 \). If the condition (6.10) holds, we have an expression

\[ i(\nu_j) = l_je + m_jf + \mu_j \]

for \( j = 1, 2 \), where \( l_j, m_j \in \mathbb{Z} \) and \( \mu_j \in E_8(-2) \). Then, we have

\[ -2 = (i(\nu_1) \cdot i(\nu_1)) = 2l_1m_1 + (\mu_1 \cdot \mu_1). \]
Since self-intersection numbers of $E_8(-2)$ are in $4\mathbb{Z}$, it follows that both $l_1$ and $m_1$ are odd numbers. Similarly, both of $l_2$ and $m_2$ are odd numbers. On the other hand, we obtain

$$1 = (i(v_1) \cdot i(v_2)) = l_1 m_2 + l_2 m_1 + (\mu_1 \cdot \mu_2).$$

Here, $(\mu_1 \cdot \mu_2) \in 2\mathbb{Z}$, because $\mu_j \in E_8(-2)$. Hence, $l_1 m_2 + l_2 m_1$ is an odd number. This is a contradiction. Therefore, there is no primitive embedding (6.8). So, by Lemma 4.3, there is no Nikulin involution on the K3 surface (3.1).

\[\square\]

### 6.3 Sandwich phenomenon for the families of [5, 12]

As a by-product of our explicit expression of Nikulin involutions, we can obtain a handy description of a sandwich phenomenon for another families of K3 surfaces.

Clingher, Malmendier and Shaska [5] obtained another natural extension of the story for the K3 surfaces $S_{CD}$. They consider the family of lattice polarized K3 surfaces $S_{CMS}$ with the transcendental lattice $U \oplus U \oplus A_1(-1) \oplus A_1(-1)$. Also, they study the relation between $S_{CMS}$ and $K_{MSY}$, where $K_{MSY}$ is the K3 surfaces with the transcendental lattice $U(2) \oplus U(2) \oplus A_1(-1) \oplus A_1(-1)$. The family of $K_{MSY}$ is precisely studied by Matsumoto, Sasaki and Yoshida [12] on the basis of the viewpoint of the double covering of $\mathbb{P}^2(\mathbb{C})$ branched along six lines and hypergeometric differential equations. In [5], they show that there is a van Geemen–Sarti involution on $S_{CMS}$ and the corresponding double covering $S_{CMS} \rightarrow K_{MSY}$. They call the K3 surface $S_{CMS}$ the van Geemen–Sarti partner of $K_{MSY}$.

In this section, we use the Weierstrass equation of the K3 surfaces $S_{CMS}$ given in [16]:

$$z^2 = y^3 + (u_{5,3}x^5 + u_{4,4}x^4 + u_{3,5}x^3)y + (u_{7,5}x^7 + u_{6,6}x^6 + u_{5,7}x^5),$$

(6.11)

Set

$$t_4 = u_{4,4}, \quad t_6 = u_{6,6}, \quad t_8 = u_{5,3}u_{3,5}, \quad t_{10} = u_{5,3}u_{5,7} + u_{3,5}u_{7,5}, \quad t_{12} = u_{5,7}u_{7,5}.$$  

Via the inverse of the period mapping for the family of Equation (6.11), $t_4, t_6, t_8, t_{10},$ and $t_{12}$ give modular forms on the four-dimensional symmetric domain $D$ of type $IV$ for the orthogonal group $O(2, 4; \mathbb{Z})$. Also, there are three modular forms for $O(2, 4; \mathbb{Z})$ of the three non-trivial characters (see [16], Theorem 1.1). Now, remark that the K3 surface $S_{CMS}$ given by Equation (6.11) degenerates to the K3 surface $S_{CD}$ if $u_{5,3} = 0$ and $u_{7,5} = 1$. Thus, the family of $S_{CMS}$, which is characterized by the connection with the configuration of six lines in $\mathbb{P}^2(\mathbb{C})$ and modular forms for $O(2, 4; \mathbb{Z})$, gives another natural extension of the family of $S_{CD}$.

By the birational transformation $(x, y, z) = \left(\frac{Y}{u_{5,3}x+u_{7,5}}, \frac{XY^2}{(u_{5,3}x+u_{7,5})^2}, \frac{Y^3Z}{(u_{5,3}x+u_{7,5})^3}\right)$, Equation (6.11) is transformed to

$$S_{CMS} : Z^2 = Y^3 + u_{4,4}Y + u_{6,6} + \left(\frac{X + \frac{u_{5,3}u_{3,5}Y^2 + (u_{5,3}u_{5,7} + u_{3,5}u_{7,5})Y + u_{5,7}u_{7,5}}{X}}{u_{5,3}u_{3,5}Y^2 + (u_{5,3}u_{5,7} + u_{3,5}u_{7,5})Y + u_{5,7}u_{7,5}}\right).$$

(6.12)

There is a Nikulin involution

$$\iota_{\psi, CMS} : (X, Y, Z) \mapsto \left(\frac{u_{5,3}u_{3,5}Y^2 + (u_{5,3}u_{5,7} + u_{3,5}u_{7,5})Y + u_{5,7}u_{7,5}}{X}, Y, -Z\right)$$

(6.13)

on the K3 surface (6.12). Note that this is equal to the Nikulin involution which appears in [5]. As in the proof of Theorem 6.1, the Nikulin involution $\iota_{\psi, CMS}$ induces a double covering $\psi_{CMS} : S_{CMS} \rightarrow K_{MSY}$, where $K_{MSY}$ is a K3 surface given by

$$K_{MSY} : Z^2 = Y^3 + u_{4,4}Y + u_{6,6} + \left(U^2 + \frac{u_{5,3}u_{3,5}Y^2 + (u_{5,3}u_{5,7} + u_{3,5}u_{7,5})Y + u_{5,7}u_{7,5}}{U^2}\right).$$

(6.14)

The transcendental lattice of $K_{MSY}$ is given by $U(2) \oplus U(2) \oplus A_1(-1) \oplus A_1(-1)$. If $u_{5,3} = 0$ and $u_{7,5} = 1$, $K_{MSY}$ degenerates to Kum(\(\mathbb{W}\)) of Equation (6.5). So, we can regard the family of $K_{MSY}$ as another extension of $G_{Kum}$. There is a Nikulin
involution

\[ t_{\varphi_{MSY}} : (U, Y, Z) \mapsto (-U, Y, Z) \quad (6.15) \]

which induces a double covering \( \varphi_{MSY} : K_{MSY} \rightarrow S_{CMS} \). Summarizing this argument, we have the following theorem.

**Theorem 6.4.** One has the sandwich for \( K_{MSY} \) of Equation (6.12) and \( S_{CMS} \) of Equation (6.14):

\[ K_{MSY} \xrightarrow{\varphi_{MSY}} S_{CMS} \xrightarrow{\psi_{CMS}} K_{MSY}. \]

Here, \( \psi_{CMS} (\varphi_{MSY}, \text{resp.}) \) is the double covering given by the involution (6.13) ((6.15), resp.).

**ACKNOWLEDGMENTS**

The first author is supported by JSPS Grant-in-Aid for Scientific Research (18K13383) and MEXT LEADER. The second author is supported by JSPS Grant-in-Aid for Scientific Research (19K03396). The authors are thankful to Professor Jiro Sekiguchi for valuable discussions about the surface \( \Sigma(t) \) in Section 5 from the viewpoint of complex reflection groups, and the anonymous referee for suggestions for improvement.

**REFERENCES**

[1] W. P. Barth, et al., *Compact complex surfaces*, Springer, New York, 2004.
[2] S. M. Belcastro, *Picard lattices of families of K3 surfaces*, Commun. Algebr. 30 (2002), 61–82.
[3] N. Braeger, A. Malmendier, and Y. Sung, *Kummer sandwiches and Greene–Plesser constructions*, J. Geom. Phys. 154 (2020), 103718.
[4] A. Clinger and C. Doran, *Mirror symmetry for lattice polarized K3 surfaces*, Adv. Math. 231 (2012), 172–212.
[5] A. Clingher, A. Malmendier, and T. Shaska, *Six line configurations and string dualities*, Commun. Math. Phys. 271 (2019), no. 1, 159–196.
[6] T. Dern and A. Krieg, *Graded rings of Hermitian modular forms of degree 2*, Manuscr. Math. 110 (2003), 251–272.
[7] I. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. 81 (1996), no. 3, 2599–2630.
[8] C. Ingalls, A. Logan, and O. Patashnick, *Explicit coverings of families of elliptic surfaces by squares of curves*, Math. Z. 302 (2022), 103718.
[9] S. Ishii and K. Watanabe, *On simple elliptic K3 singularities* (in Japanese), Proceedings of Conference on Algebraic Geometry at Tokyo Metropolitan University, Tokyo, Japan, pp. 20–31.
[10] S. Ma, *On K3 surfaces which dominate Kummer surfaces*, Proc. Amer. Math. Soc. 141 (2013), 131–137.
[11] A. Malmendier and T. Shaska, *The Satake sextic in F-theory*, J. Geom. Phys. 120 (2017), 290–305.
[12] K. Matsumoto, T. Sasaki, and M. Yoshida, *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3, 6)*, Int. J. Math. 3 (1992), 1–164.
[13] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. 75 (1984), 105–121.
[14] A. Nagano, *Inverse period mappings of K3 surfaces and a construction of modular forms for a lattice with the Kneser conditions*, J. Algebr. 565 (2021), 33–63.
[15] A. Nagano and H. Shiga, *Geometric interpretation of Hermitian modular forms via burkhardt invariants*, Transform. Groups (2022), https://doi.org/10.1007/s00031-021-09681-w.
[16] A. Nagano and K. Ueda, *The ring of modular forms of O(2, 4; Z) with characters*, Hokkaido Math. J. 51 (2022), no. 2, 275–286.
[17] V. V. Nikulin, *Finite groups of automorphisms of Kählerian K3 surfaces*, Trudy Moskov Mat. Obshch. 38 (1979), 75–137.
[18] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Izv. Akad. Nauk SSSR. 43 (1979), 111–177.
[19] V. V. Nikulin, *On rational maps between K3 surfaces*, Constantin Carathéodory: an international tribute II, World Sci. Publ., pp. 964–995.
[20] I. I. Piatetski-Shapiro and I. R. Shafarevich, *A Torelli theorem for algebraic surfaces of type K3*, Math. USSR Izv. 5 (1971), 547–588.
[21] M. Reid, *Canonical 3-folds*, Journées Géométrie algébraique d’Angers, Angers, France, pp. 273–310.
[22] J. Sekiguchi, *The construction problem of algebraic potentials and reflection groups*, 2021, preprint.
[23] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math. 6 (1954), 274–304.
[24] T. Shioda, *Kummer sandwich theorem of certain elliptic K3 surfaces*, Proc. Japan Acad. Ser. A Math. Sci. 82 (2006), 137–140.
[25] B. van Geemen and A. Sarti, *Nikulin involutions on K3 surfaces*, Math Z. 255 (2007), 731–753.
[26] K. Watanabe, *On plurigenera of normal isolated singularities II*, Complex Analytic Singularities, Adv. Studies Pure Math., 8, Elsevier Sci. Ltd., pp. 671–685.
[27] T. Yonemura, *Hypersurface simple K3 singularities*, Tohoku Math. J. 42 (1990), 351–380.

**How to cite this article:** A. Nagano and H. Shiga, *On Kummer-like surfaces attached to singularity and modular forms*, Math. Nachr. 296 (2023), 2513–2534. https://doi.org/10.1002/mana.202100552