Polyhedron phase space using 2-groups:
\(\kappa\)-Poincaré as a Poisson 2-group

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Abstract

We construct a phase space for a three dimensional cellular complex with decorations on edges and faces using crossed modules (strict 2-groups) equipped with a (non-trivial) Poisson structure. We do not use the most general crossed module, but only the ones where the target map \(t\)-map) is trivial. As a particular case, we recover that deformations of the Poincaré group can be exported to deformations of the Poincaré 2-group. The \(\kappa\)-deformation case provides a natural candidate for describing discrete geometries with curved edge decorations (but still flat face decorations). Our construction generalizes the classical phase space defined in [5] for a 3d triangulation in terms of an un-deformed Poincaré 2-group.

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1 Introduction

Topological quantum field theories (TQFT) [6] have gathered a lot of interest for many different reasons. From a mathematical point of view they can be used to construct topological invariants [76, 25]. From a physical perspective, they can be used to construct quantum gravity models. 3d gravity is itself a topological theory [75]. 4d gravity can be seen as a constrained topological theory [63] so one can constrain the relevant TQFT to build a quantum gravity model [24, 15] — this is the spinfoam approach [68]. From the condensed matter perspective, topological order have been an active and fruitful research direction [73, 74], with application in quantum information theory [42].

TQFT’s in 3d are naturally described in terms of monoidal (spherical) categories [16], which appear when dealing with representations of quantum groups or Hopf algebras [56]. The TQFT is encoded in a state sum, a triangulation/cellular complex decorated by representations of the (quantum) group such as the Turaev-Viro model [72]. The state sum can be seen as a transition amplitude between initial and final states which are given in terms of triangulation/cellular decomposition of 2d surfaces which bound the complex. These states have sometimes been called spin networks in the gravity context. They were first introduced by Penrose [62] who intended to reconstruct space(-time) through colliding spinning tops with conserved angular momentum. The spin network states naturally encode quantum states for the 2d triangulation/cellular complex. In particular an intertwiner encodes the quantum state of a (curved) polygon [32]. We can construct a classical picture behind these quantum states, ie a phase space for the 2d triangulation [19, 18]. This phase space can be deduced from the discretization of a BF action, possibly with a non-zero cosmological constant (which specifies then the quantum group deformation)[31].

We emphasize that the phase spaces we consider are more general than the typical cotangent bundle phase space. Indeed, the classical version of a quantum group is a Lie group equipped with a non-trivial Poisson bracket. Quantum group symmetries are classically given by a group of symmetries equipped with non-trivial Poisson bracket. Poisson Lie groups encode symmetries of phase spaces which have curvature in momentum space. The relevant phase space is then called a Heisenberg double [1, 70, 70].

Spin networks also naturally appear when quantizing 4d BF theory. In fact, by discretizing 4d BF theory
(with no cosmological constant) we recover the same type of phase space we have obtained when dealing with a 2d triangulation. The reason is simple a posteriori and is encoded in Minkowski’s theorem [3]. This theorem states that given a (non-planar) polygon one can construct a polyhedron where the edges of the polygon are interpreted as the area normals of the faces. Hence if we describe a polyhedron in terms of its faces and have no information regarding the edges, it is essentially the same as a polygon. Bluntly, given a spin network, we actually cannot know whether we are dealing with a 2d or 3d space due to this equivalence between polygon/polyhedron.

Describing the quantum states of 3d geometry (for a 4d spacetime/manifold) in terms of spin networks can be viewed as problematic for several reasons. From a topological perspective, one expects that the proper way to capture topological features are through 2-categories [7, 54, 55, 53], where both the edges and the faces of the triangulation are decorated. These 2-categories should be encoding the representations of a quantum 2-group or Hopf 2-algebra. While a definition of such quantum strict 2-group has been proposed in [59], there are not so many explicit examples, especially involving Lie groups, to the best of our knowledge. Furthermore, classifications of representations are lacking for most Lie 2-groups, to the notable exception of the Poincaré 2-group [10]. There is also no generalization of the Peter-Weyl theorem – which is instrumental in defining quantum states in terms of spin networks – for general 2-groups. Quantum 2-groups should have a classical version, given by Poisson 2-groups. Again, while they have been formally defined [22], to the best of our knowledge there are not many explicit examples. A state sum model using 2-complexes has been built by Yetter [77, 78] using finite groups. Yetter’s state sum model was later showed to be a discretization of the transition amplitude characterized by an action with 2-group symmetries called the BFCG action when dealing with Lie groups [37, 60]. The initial and final states associated with this transition amplitude are now given in terms of a categorification of the notion of spin networks, which we would coin 2-spin networks. One expects that the classical picture behind these states to be given in terms of a phase space defined in terms of 2-groups. While it is tempting to say that it could be some type of 2-Heisenberg double, such object has not been defined to the best of our knowledge.

If spin networks cannot sufficiently capture topological features of 3d geometries, they may also be lacking as the main tool of a quantum gravity model. In fact we can argue further that only having access to the face information and no decorations on edges is problematic. After all, edge decorations should be related to some discretization of the metric degrees of freedom, which we would like to have. Again, putting decorations on both edges and faces naturally points to the concept of 2-group, which in turn naturally points to using a BFCG type action with constraints to recover gravity and build the gravity model [61]. With this mind, some phase space structure associated to a 3d triangulation was introduced in [5] to relate the discretization of a Poincaré 2-group BFCG discretization to a state sum model based on 2-representations of the Poincaré 2-group. This phase space is the classical version of 2-spin networks defined for the Poincaré 2-group and describes “flat” triangulation, in the sense that the variables decorating the edges are element in an abelian group.

The key question we want to address here is whether we can generalize this construction to curved polyhedra, namely with edge decorations with value in a non-abelian group. This would imply in particular to have non-trivial Poisson brackets for the decorations associated to the faces dual to the edges. Hence, upon quantization this would amount to dealing with a quantum 2-group.

We do not solve this problem in full generality, but we provide some new examples based on a specific class of 2-groups, namely strict 2-groups with trivial t-map, which we will call skeletal 2-groups following [10]. skeletal 2-groups always have abelian group decorations on the faces. The Poincaré 2-group or the (co-)tangent 2-group are natural examples of skeletal 2-groups [12]. Because the t-map is trivial, a skeletal 2-group behaves very much as a group, the main difference being in the geometrical interpretation of the different products of the (sub)groups. This is the technical insights that allows us to circumvent the general

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1They would possibly be related to the notion of G-networks introduced in [5], when dealing with the Poincaré 2-group.

2In the following, we will use edges and faces or triangles to denote resp. 1d and 2d objects in the triangulation. Links and wedges (sometimes dual faces) will denote resp. 1d and 2d objects in the dual 2-complex.

3The Minkowski theorem was generalized to the curved case [40], where decorations of the faces are specified by a non-abelian group. However, there is an hidden dependence on some edge variables with value in a non-abelian group. It would be interesting to see how this is consistent with a 2-group picture. We thank A. Riello for the clarification.

4We will essentially work with crossed modules as strict 2-groups.
definition of a “2-Heisenberg double” and deal with usual Heisenberg doubles, built from skeletal 2-groups. We can then use symplectic reduction to fuse different phase spaces to construct the general phase space of a 3d triangulation with both decoration on edges and faces. Note that symplectic reduction is a natural tool to glue regions containing gauge degrees of freedom, see for example [65, 66].

An initially unexpected outcome of the construction is that one natural deformation of the Poincaré 2-group is actually specified by the \(\kappa\)-Poincaré deformation \([52, 57]\). Such deformation is a well known deformation of the Poincaré group, which generates a non-commutative Minkowski space-time and is used to study quantum gravity phenomenology\([4]\). Here it naturally arises when switching curvature on the edge decorations. Once again, due to the similarity between groups and skeletal 2-groups, it is only a matter of interpretation to see the (classical) \(\kappa\)-Poincaré group as a Poincaré 2-group equipped with a non-trivial Poisson bracket. The Heisenberg double we will consider as our main example and will use as our basic building block to build the triangulation phase space, will be based on the \(\kappa\)-Poincaré (2-)group and its dual (called the \(\kappa\)-Poincaré algebra).

In Section 2, we provide all the mathematical ingredients we will need to define the phase space for a 3d triangulation. We recall the definitions of Poisson Lie groups, Heisenberg double and the concept of symplectic reduction when the momentum map is with value in a (non-)abelian group. We explain how this symplectic reduction allows to fuse/glue phase spaces. We move then to recall the notion of strict 2-groups and Poisson strict 2-groups. We provide some examples when the strict 2-group is simple, ie the \(t\)-map is trivial. We show then how the strict skeletal 2-groups can be embedded in a Heisenberg double and focus on the construction of the \(\kappa\)-Poincaré 2-group and its dual. This provides the main example of a non-trivial Poisson 2-group which can be used to generate non-trivial decorations on the triangulation edges.

In Section 3, we recall how the notion of a Heisenberg double together with symplectic reduction can be used to build the phase space of a 2d triangulation. We then argue using Minkowski’s theorem how this construction can be extended to the 3d case.

Section 4 is in some sense the main section of the paper. It details the construction of the phase space for a 3d triangulation in terms of Heisenberg doubles defined in terms of skeletal 2-groups. We provide the general gluings we need to do to construct the phase space of an arbitrary triangulation. We show in particular how the phase space constructed in \([5]\) is a special example of the construction.

In the concluding section, we discuss the many new directions that this work opens up.

2 Mathematical tool box

In this section, we review the notions of Poisson groups and 2-groups, as well as Heisenberg/Drinfeld doubles. We highlight that for a certain class of 2-groups, namely the ones for which the \(t\)-map is trivial, they can be seen as groups and hence embedded in the Heisenberg/Drinfeld doubles framework. We show that the Poincaré 2-group can be equipped with a non-trivial Poisson bracket, which is the same as the ones deforming the standard Poincaré group into the \(\kappa\)-Poincaré group. Furthermore we discuss how Majid’s semi-dualization \([56, 58]\) can be used to define different 2-groups. We recall the symplectic reduction framework, in the case of group valued momentum maps, which will be the main tool we will use to build phase spaces for a triangulation.

2.1 Heisenberg double, symmetries and symplectic reduction

Poisson Lie group.

**Definition 1.** \([20, 56]\) Let \(G\) be a Lie group with Poisson brackets \(\{\cdot,\cdot\}_G\). If the group multiplication \(\mu: G \times G \to G\) is a Poisson map then \(G\) is said to be a Poisson Lie group. Explicitly, for \(g,h \in G\) and \(f_1,f_2 \in C^\infty(G)\), the compatibility is

\[
\{f_1,f_2\}_G(\mu(h,g)) = \{f_1 \circ R_g, f_2 \circ R_g\}_G(h) + \{f_1 \circ L_h, f_2 \circ L_h\}_G(g),
\]

(1)
where $L_h$ is the left translation by $h$ and $R_g$ is the right translation by $g$.

We can equivalently describe the Poisson brackets by the Poisson bivector $\pi$, given by $\{f_1, f_2\}_G(y) = \pi_g(df_1 \wedge df_2)$. By evaluating (1) at $g = h = e$ (where $e$ is the identity of $G$) we find that the Poisson bivector vanishes at the identity, $\pi_e = 0$. It follows that the Poisson structure of a Poisson Lie group is not symplectic.

**Definition 2.** [20, 56] The associated Lie algebra $\mathfrak{g}$ of a Poisson Lie group $G$ is a Lie bi-algebra; it is the Lie algebra of $G$, with its Lie bracket and an additional structure, the co-cycle map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ derived from the Poisson bivector of $G$. The co-cycle map satisfies the co-Jacobi identity: cyclic permutations of $(\delta \otimes \text{id}) \circ \delta$ sum to zero. It also satisfies the co-cycle condition,

$$\delta[x, y] = ad_x \delta y - ad_y \delta x, \quad \forall x, y \in \mathfrak{g}. \quad (2)$$

**Classical double.** Consider the pair of Lie bi-algebras $\mathfrak{g}$ and $\mathfrak{g}^*$, such that there exists a bilinear map $\{\cdot, \cdot\} : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ as

$$\{e^i, e^*_j\} = \delta^j_i, \quad (3)$$

where $e^i \in \mathfrak{g}$ and $e^*_j \in \mathfrak{g}^*$ are the generators of their respective Lie bi-algebras. Let the Lie brackets and co-cycles of the Lie bi-algebra $\mathfrak{g}$ be

$$[e^i, e^j] = d^k_{ij} e^k, \quad \delta^i_0(e^j) = \frac{1}{2} d^i_{jk} e^j \wedge e^k. \quad (4)$$

Let the Lie brackets and co-cycles of $\mathfrak{g}^*$ be defined by the conditions

$$\{e^i, [e^*_k, e^*_j]\} = \langle \delta^k_0(e^i), e^*_j \otimes e^*_k \rangle, \quad \langle [e^j, e^i], e^*_k \rangle = \langle e^i \otimes e^j, \delta^*_0(e^*_k) \rangle. \quad (5)$$

The Lie bi-algebra $\mathfrak{g}^*$ is thus the Lie bi-algebra with Lie brackets and co-cycles respectively dual to the co-cycles and Lie brackets of $\mathfrak{g}$:

$$[e^*_i, e^*_j] = d^k_{ij} e^*_k, \quad \delta^*_0(e^*_i) = \frac{1}{2} e^j k d^j_{ik} e^*_k \wedge e^*_k. \quad (6)$$

The co-adjoint actions between $\mathfrak{g}$ and $\mathfrak{g}^*$ are

$$ad^*_e(e^i) = e^*_i \otimes e^j := \{e^*_i, [e^j, e^k]\} e^*_k = d^j_{ki} e^*_k, \quad ad^*_e(e^j) = e^*_i \triangleright e^j := -\langle [e^*_i, e^*_k], e^j \rangle e^k = d^j_{ik} e^k. \quad (7)$$

The adjoint action of any Lie algebra on itself is given by its Lie brackets. The bilinear map (3) encodes the fact that the adjoint action of a Lie algebra on itself, is dual to the co-adjoint action of that algebra on its dual:

$$\{e^i, [e^*_k, e^*_j]\} = \langle \delta^k_0(e^i), ad^*_e(e^*_j) \rangle, \quad \langle [e^j, e^i], e^*_k \rangle = \langle ad^*_e(e^*_i), e^*_k \rangle. \quad (8)$$

**Definition 3** (Classical double.). [20, 56] The classical double of $\mathfrak{g}$ is the double cross sum Lie bi-algebra $\mathfrak{d} = \mathfrak{g}^{\text{cop}} \rtimes \mathfrak{g}^*$, equipped with with the element $r = e^i \otimes e^j \in \mathfrak{g}^{\text{cop}} \otimes \mathfrak{g}^*$, called classical r-matrix. Here $\mathfrak{g}^{\text{cop}}$ is the Lie bi-algebra with the same Lie brackets of $\mathfrak{g}$ and opposite co-cycle: $\delta^\text{\text{cop}}(e^i) = -\delta_0(e^i)$. As a Lie algebra, $\mathfrak{d}$ is defined by the brackets

$$[e^i, e^j] = d^j_{ij} e^k, \quad [e^*_i, e^*_j] = d^k_{ij} e^*_k, \quad [e^*_i, e^j] = e^*_i \triangleright e^j + e^*_j \otimes e^j = d^j_{ki} e^k + d^k_{ij} e^*_k, \quad (9)$$

where the cross brackets are induced by the co-adjoint actions (7). As a Lie co-algebra, $\mathfrak{d}$ is defined by the co-cycles

$$\delta_0(e^i) = -\delta_0(e^i), \quad \delta_0(e^*_i) = \delta^*_0(e^*_i). \quad (10)$$

The classical r-matrix can be split into its symmetric and antisymmetric parts

$$r_- = \frac{1}{2} (r - r^t) = \frac{1}{2} \sum_i e^*_i \wedge e^i, \quad r_+ = \frac{1}{2} (r + r^t). \quad (11)$$

The co-cycles (10) can equivalently be expressed through the co-boundary condition,

$$\delta_0(x) = [x \otimes 1 + 1 \otimes x, r_-], \quad \forall x \in \mathfrak{d}, \quad (12)$$

making $\mathfrak{d}$ a co-boundary Lie bi-algebra.
2.1.1 Heisenberg and Drinfeld doubles

In the following we consider the cases where the cocycle of \( \mathfrak{d} \) is given by a \( r \)-matrix.

**Heisenberg double.** Upon exponentiation, the components, \( g \) and \( g^* \) of the classical double \( \mathfrak{d} \) generate respectively the group \( G \) and \( G^* \). Similarly, we define the (simply connected) Lie group generated by \( \mathfrak{d} \) to be \( D \). The group \( D \) equipped with the Poisson bi-vector

\[
\pi^+(d) = -[d \otimes d, r_-]^+, \quad d \in D,
\]

is called the Heisenberg double of \( D \) [20, 56, 70, 1] and is denoted \( D^+ \) (or simply \( D \) if there is no confusion with the Drinfeld double — to be defined shortly). \([\cdot, \cdot]^+\) stands for the anticommutator. The Heisenberg double can be factorized\(^5\) as \( D^+ \cong G \bowtie G^* \cong G^* \bowtie G \). With \( h, \tilde{h} \in G \) and \( \ell, \tilde{\ell} \in G^* \), the so-called ribbon relation follows from these two splittings:

\[
d = \ell h = \tilde{h} \tilde{\ell}.
\]

The ribbon relation can be written diagramatically as in Fig. 1. The ribbon decomposition allows us to define the mutual actions between \( G \) and \( G^* \):

\[
\tilde{h} = \ell \triangleright h, \quad \tilde{\ell} = \ell \triangleleft h, \quad \ell = \tilde{h} \triangleright \tilde{\ell}, \quad h = \tilde{h} \triangleleft \tilde{\ell}.
\]

From the Poisson bivector \( \pi^+ \), we can derive the Poisson brackets of the Heisenberg double; these Poisson brackets are symplectic, thus even if \( G \) and \( G^* \) are Poisson Lie groups, the Heisenberg double is not. Being symplectic, \( D^+ \) is instead considered as a phase space with symplectic form [1, 70]

\[
\Omega = \frac{1}{2} (\langle \Delta \ell \wedge \Delta \tilde{h} \rangle + \langle \Delta \tilde{\ell} \wedge \Delta h \rangle)
\]

where \( \Delta u = \delta uu^{-1} \) and \( \Delta u = u^{-1}\delta u \) are respectively the right and left Maurer Cartan forms.

**Drinfeld double.** The Lie group \( D \) equipped with the Poisson bivector

\[
\pi_-(d) = [d \otimes d, r_-], \quad d \in D,
\]

is a Poisson Lie group called Drinfeld double [20, 56, 70, 1] and denoted \( D_- \). Here, \([\cdot, \cdot]\) are the usual commutators. The co-cycles of the classical double \( \mathfrak{d} \) are the infinitesimal limit of the Poisson bivector \( \pi_- \) associated to the Poisson brackets of \( D_- \).

The Lie group \( D \) naturally acts on itself by the left or right multiplication. The action of the Drinfeld double \( D_- \) on the Heisenberg double \( D^+ \) is a Poisson map, i.e. the symplectic Poisson bracket is covariant under the action of \( D_- \) [70]. We can focus in particular at the action of the subgroups \( G \) and \( G^* \) on \( D^+ \).

\(^5\)This splitting will not be global in general. For our purpose it is sufficient to consider elements \( d \in D \) close to the identity so that this splitting is true. We refer to [1] for the general case.
G transformations on the left. Let $\alpha \in \mathfrak{g}$. The infinitesimal (left) action $\delta^L_\alpha$ of $G \ni h' \sim 1 + \alpha$ on $\mathcal{D}_+ \ni d$, induced by the left group multiplication is

$$h'd \sim (1 + \alpha)d \quad \text{with} \quad d = \ell h = \hat{h} \hat{\ell}.$$

We deduce then the transformations for the sub-components [31]

$$\delta^L_\alpha g = ag \quad \Rightarrow \quad \begin{cases} \delta^L_\alpha \hat{h} = \alpha \hat{h}, & \delta^L_\alpha \hat{\ell} = 0, \\ \delta^L_\alpha \ell = \alpha \ell - \ell (\alpha \circ \ell), & \delta^L_\alpha h = (\alpha \circ \ell)h. \end{cases}$$

$G^*$ transformations on the left. A similar calculation can be performed for the infinitesimal (left) transformations $\delta^L_\phi$ of $G^* \ni \ell' \sim 1 + \phi$, with $\phi \in \mathfrak{g}^*$

$$\ell'd \sim (1 + \phi)d \quad \text{with} \quad d = \ell h = \hat{h} \hat{\ell},$$

from which, the transformations for the sub-components are [31]

$$\delta^L_\phi d = \phi d \quad \Rightarrow \quad \begin{cases} \delta^L_\phi \ell = \phi \ell, & \delta^L_\phi \hat{h} = 0, \\ \delta^L_\phi \hat{\ell} = \phi \hat{\ell} - \hat{\ell} (\hat{h}^{-1} \circ \phi) = \phi \hat{h} - \hat{h} (\phi \circ \hat{h}), & \delta^L_\phi \hat{h} = (\phi \circ \hat{h}) \hat{\ell}. \end{cases}$$

2.1.2 Momentum maps and symplectic reduction

In this section we review the concepts of momentum maps and symplectic reduction applied to our specific case of interest. We refer to [2] for the general set up. The symplectic reduction is the essential tool to build up the phase spaces for the triangulation we will consider. To take an analogy, if Heisenberg doubles can be seen as bricks then the symplectic reduction can be interpreted as the cement.

Momentum maps. For what concerns us, momentum maps are functions on phase space which generate the infinitesimal symmetry transformations of interest using the symplectic Poisson bracket. We will be exclusively focused on the symmetry transformations given by the left or right translations on $\mathcal{D}$, the Heisenberg double, given by the action of $G$, $G^*$ and possibly some of their subgroups. These actions are free and proper.

In the simple example of a phase space $T^* \mathbb{R} \sim \mathbb{R} \times \mathbb{R}$ with coordinates $(x, p)$, the momentum $p$ generates the infinitesimal translations in $x$, while $x$ generates the infinitesimal translations in $p$. Hence these coordinates can be seen as momentum maps. More explicitly, if each type of infinitesimal translation is such that

$$\delta_x \epsilon = \epsilon, \delta_p \epsilon = 0, \quad \delta_x \eta = 0, \delta_p \eta = \eta,$$

where $\delta_x, \delta_p$ is understood as the vector field generating the transformation, and $\Omega = \delta x \wedge \delta p$, then

$$\delta_x \epsilon \Omega = \epsilon \delta p = \langle \epsilon, \delta p \rangle, \quad \delta_x \eta \Omega = -\eta \delta x = -\langle \eta, \delta x \rangle,$$

where we introduce the canonical pairing between $\mathbb{R}$ and $\mathbb{R}^* \sim \mathbb{R}$ and took $\epsilon \in \mathbb{R}$ and $\eta \in \mathbb{R}^*$. Hence when the manifold of interest is given by a phase space, we see that some momentum maps are just given by the projection either on configuration or momentum coordinates. This concept will extend to the more general Heisenberg double we will consider.

We denote by $\sigma$ the left Poisson action of the Poisson Lie group $G$ on a phase space $\mathcal{P}$, and $G^*$ the Poisson Lie group dual to $G$. We denote by $X^L$ the left invariant 1-form on $G^*$ which takes value $\alpha$ at the identity of $e$. $\delta_X$ is the vector field generating the infinitesimal version of $\sigma$.

Definition 4. The momentum map of the action $\sigma$ is the $C^\infty$ map $\mathcal{J} : \mathcal{P} \to G^*$ if

$$\delta_X \circ \Omega = \mathcal{J}^* (X^L).$$

We say that the momentum map is equivariant if the momentum map commutes with the symmetry action.
Proposition 1. We consider the group element $d = \ell h = \hat{h} \ell \in D$. The $G^*$ (resp. $G$) group element $\ell$ (resp. $\hat{h}$) generates the infinitesimal left $G$ transformation (19) (resp. the infinitesimal left $G^*$ transformation (21))

$$\delta^L_\alpha : \Omega = (\alpha, \Delta \ell), \quad \delta^L_\phi : \Omega = -\langle \phi, \Delta \hat{h} \rangle, \quad \alpha \in \mathfrak{g}, \, \phi \in \mathfrak{g}^*.$$ \hfill (25)

$\ell := J^L_G (d)$ and $\hat{h} := J^L_G (d)$ are therefore momentum maps. In a similar manner, $h$ and $\bar{\ell}$ are respectively the momentum map for the right transformations of $G^*$ and $G$.

Proof of this proposition is given in [31]. Hence the momentum maps we will consider are given by the projections

$$\mathcal{J}^L_G : D \sim G^* \bowtie G \to G, \quad \mathcal{J}^L_{G^*} : D \sim G \bowtie G^* \to G^*,$$

$$\mathcal{J}^R_G : D \sim G \bowtie G^* \to G, \quad \mathcal{J}^R_{G^*} : D \sim G^* \bowtie G \to G^*.$$ \hfill (26)

These momentum maps are clearly equivariant. For example, consider $J^L_{G^*}$, (we recall that $G$ acts on $D$ by left or right multiplication),

$$\mathcal{J}^L_G (h' \ell \ell h) = \mathcal{J}^L_G (h' \ell h) = \mathcal{J}^L_{G^*} ((h' \ell)(h' \ell h)) = h' \ell = \ell = \ell \mathcal{J}^L_G (d), \quad d \in D, \, h', \, h \in G, \, \ell \in G^*.$$ \hfill (28)

Some properties of momentum maps. We can list other relevant properties of momentum maps, all discussed in [2]. The following statements, made for the action of $G$, also apply to the action of $G^*$.

- If we can decompose the group $G$ (resp. $G^*$) into subgroups $G \sim G_1 \bowtie G_2$ (resp. $G^* \sim G_1^* \bowtie G_2^*$), then the projection of $D$ into $G_i$ also defines a momentum map and generates the infinitesimal right $G_i^*$ transformations.
- We have seen that $\ell \in G^*$ generates the infinitesimal right $G$ translations through the symplectic form $\Omega$. Then $\ell^{-1}$ is a momentum map infinitesimal left $G$ translations through the symplectic form $-\Omega$.
- If we consider the phase space $D^{(n)}$ made of $n$ copies of $D$, $D^{(n)} = D \times \ldots \times D$ with symplectic form $\Omega^{(n)} = \sum_1^n \Omega$, then the global right infinitesimal $G$ translation is generated by the momentum map $\ell_1 \ldots \ell_n$.

Symplectic reduction. Symplectic reduction is the process of taking a symplectic space with some symmetries and reducing the symmetries (ie by considering functions on the symplectic space which are invariant under the symmetries) such that the resulting space is still symplectic. An invariant function can be characterized as a function that commutes with momentum map associated to the symmetries. This is just a different language to discuss Dirac’s approach to constrained systems [27]. Indeed the constraint corresponds to the requirement that the momentum map is constrained to take a specific value, for example the unit in $G^*$.

Theorem 1. [50, 2]. Assume $\sigma : G \times D \to D$ is a Poisson action of a Poisson Lie group $G$ on the Poisson space $D$, with its associated equivariant momentum map $\mathcal{J}_G$. We consider an element $\ell_0 \in G^*$ and $G_{\ell_0}$ its isotropy subgroup under the action of $G$ on $G^*$. If $G$ acts freely and properly on $\mathcal{J}^{-1}(\ell_0)$, then the symplectic form $\Omega$ of $D$ is pulled back to the quotient space $\mathcal{J}^{-1}(\ell_0)/G_{\ell_0}$, which is denoted $D//G_{\ell_0}$.

The general proof of this theorem can be found for example in [50].

The actions we are considering, namely the left or right translations, are free and proper. The associated momentum maps are equivariant. We can therefore use the symplectic reduction. We are going to illustrate the most important cases we will use later to build the phase space for a triangulation.

\(^6\)Since $G$ acts on $G^*$, $G_{\ell_0}$ is the set of elements which leave $\ell_0$ invariant.

\(^7\)If $\ell_0$ is the unit, then we write $D//G$. 

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2.1.3 Relevant examples of symplectic reduction

The symplectic reduction will be used extensively to glue/fuse phase spaces together and still recover a phase space. There will be three main cases to consider. We emphasize that momentum and configuration variables are treated on an equal footing. Hence while we formulate the constraints in terms of “momenta” in $G^*$, it would similarly apply for “configuration” variables in $G$.

**Gluing rectangular ribbons: identifying momenta pairwise.** Let us consider a direct product of symplectic phase spaces $D \times D = D^{(2)}$, with respective symplectic form and ribbon constraint

\[
\Omega_a = \frac{1}{2} \left( (\Delta \ell_a \wedge \Delta \tilde{h}_a) + (\Delta \tilde{\ell}_a \wedge \Delta h_a) \right), \quad \ell_a h_a = \tilde{h}_a \tilde{\ell}_a, \quad a = 1, 2. \tag{29}
\]

The momentum map generating the global (right) $G$ transformation on $D^{(2)}$ is $\tilde{\ell}_1 \tilde{\ell}_2$ [2]. We consider the associated constraint, or identification,

\[
C = \tilde{\ell}_1 \tilde{\ell}_2 = 1 \iff \tilde{\ell}_1 = \tilde{\ell}_2^{-1}. \tag{30}
\]

We can now construct the symplectic form for the new phase space when we have performed the symplectic reduction.

**Proposition 2.** The reduced phase space $D^{(2)}//G$ is isomorphic to $D$.

We are going to proceed to prove this proposition in two ways, a long and a short one. We still include the long way as it is still instructive to see what happens.

**Proof.** When reducing it is convenient to introduce the new variables

\[
h_{12} = h_1 h_2^{-1}, \quad \tilde{h}_{12} = \tilde{h}_1 \tilde{h}_2^{-1}. \tag{31}
\]

From the ribbon constraint and the definition of $h_{12}$, we have the identities

\[
\Delta \ell = \Delta (\tilde{h} \ell h^{-1}) = \Delta \tilde{h} + \tilde{h} \Delta \tilde{h} h^{-1} - (\tilde{h} \ell) \Delta (\tilde{h} \ell) h^{-1}, \tag{32}
\]

\[
\Delta h_{12} = -\Delta \tilde{h} + h_{12}^{-1} \Delta h_1 h_{12}, \quad \Delta \tilde{h}_{12} = \Delta \tilde{h}_1 - \tilde{h}_{12} \Delta h_2 \tilde{h}_{12}^{-1}. \tag{33}
\]

Let us evaluate $\Omega_{12}$ on the constraint surface,

\[
\Omega_{12} = \Omega_1 + \Omega_2 = \frac{1}{2} \left( (\Delta \ell_1 \wedge \Delta \tilde{h}_1) + (\Delta \tilde{\ell}_1 \wedge \Delta h_1) + (\Delta \ell_2 \wedge \Delta \tilde{h}_2) + (\Delta \tilde{\ell}_2 \wedge \Delta h_2) \right)
\]

\[
= \frac{1}{2} \left( (\Delta \ell_1 \wedge \Delta \tilde{h}_{12}) + (\Delta \tilde{\ell}_1 \wedge \Delta h_{12}) \right) + \frac{1}{2} \left( (\Delta \tilde{\ell}_1 + \tilde{h}_1 \Delta \tilde{\ell}_1 \tilde{h}_1^{-1}) \wedge \Delta h_2 + (\Delta \tilde{\ell}_1 + \tilde{h}_2 \Delta \tilde{\ell}_1 \tilde{h}_2^{-1}) \wedge \Delta h_1) \right)
\]

\[
= \frac{1}{2} \left( (\Delta \ell_1 \wedge \Delta \tilde{h}_{12}) + (\Delta \tilde{\ell}_1 \wedge \Delta h_{12}) \right)
\]

\[
\approx \frac{1}{2} \left( (\Delta \ell_1 \wedge \Delta \tilde{h}_{12}) + (\Delta \tilde{\ell}_1 \wedge \Delta h_{12}) \right) \approx \frac{1}{2} \left( (\Delta \ell_1 \wedge \Delta \tilde{h}_{12}) + (\Delta \tilde{\ell}_1 \wedge \Delta h_{12}) \right). \tag{34}
\]

where we evaluated the constraint, $\tilde{\ell}_1 \tilde{\ell}_2 = 1$ and used that the bilinear form between terms in $g$ are zero. In the constrained mechanics language, the constraints (31) can be viewed as the second class constraint. We
note that the constraint $\tilde{h}_{12} = \tilde{h}_1 \tilde{h}_2^{-1}$ is not an independent constraint so that we do have indeed the right number of constraint.

$$
\begin{align*}
\begin{array}{c}
\tilde{h}_1 = \ell_1 \triangleright h_1 \\
\tilde{h}_2^{-1} = \ell_2^{-1} \triangleright \tilde{h}_2^{-1}
\end{array}
\end{align*}
\Rightarrow \tilde{h}_{12} = \tilde{h}_1 \tilde{h}_2^{-1} = (\ell_1 \triangleright h_1)(\ell_2^{-1} \triangleright \tilde{h}_2^{-1}) \approx (\ell_1 \triangleright h_1)(\ell_1 \triangleright \tilde{h}_2^{-1})
\end{align*}
$$

$$
= (\ell_1 \triangleright h_1)(\ell_1 \triangleright (h_1 \triangleright \tilde{h}_2^{-1})) = \ell_1 \triangleright (h_1 \triangleright \tilde{h}_2^{-1}) \approx \ell_1 \triangleright h_{12}. \quad (35)
$$

It is interesting to note that the constraints $\tilde{\ell}_1 \tilde{\ell}_2 = 1$ or $h_1 \tilde{h}_2^{-1} = h_{12}$ can be treated symmetrically. Indeed, we could have considered instead the constraint $h_1 \tilde{h}_2^{-1} = h_{12}$ and dealt with $\tilde{\ell}_1 \tilde{\ell}_2 = 1$ as the second class constraint. This would have led to the same reduced phase space. Indeed the symplectic reduction also works when the momentum map is not constrained to be the unit (see Theorem 1).

$$
\mathcal{D}^{(2)}//G \cong \mathcal{D}^{(2)}//G^*_h_{12} \cong \mathcal{D}. \quad (36)
$$

There is a quicker way to derive the reduced symplectic structure. Indeed, we know that for each phase space we have a ribbon constraint.

$$
\ell_1 h_1 = \tilde{h}_1 \ell_1, \quad \ell_2 h_2 = \tilde{h}_2 \ell_2. \quad (37)
$$

If we impose the constraint $\tilde{\ell}_1 \tilde{\ell}_2 = 1$, then we can combine these ribbon constraints into a single one.

$$
\begin{align*}
\begin{array}{c}
\ell_1 h_1 = \tilde{h}_1 \ell_1 \\
\ell_2 h_2 = \tilde{h}_2 \ell_2
\end{array}
\end{align*}
\Rightarrow
\begin{align*}
\ell_1 h_1 \rightarrow \tilde{h}_1 \rightarrow \ell_1 h_1 = \tilde{\ell}_1 \\
\ell_2 h_2 \rightarrow \tilde{h}_2 \rightarrow \ell_2 h_2 = \tilde{\ell}_2
\end{align*}
\Rightarrow
\ell_1 h_1 \rightarrow \tilde{\ell}_1 h_2^{-1} = (\tilde{h}_1 \tilde{h}_2^{-1}) \ell_2^{-1} \Leftrightarrow \ell_1 h_1 = \tilde{h}_1 \ell_2^{-1}. \quad (38)
$$

Since we have a ribbon constraint, we can deduce then the associated symplectic form if this ribbon constraint defines a Heisenberg double. This is exactly the symplectic form we derived in (34). Hence combining ribbon constraints using a given constraint allows to identify the reduced Heisenberg double/phase space.

The reduction has therefore in this specific case a simple geometrical interpretation. We consider two ribbons and the constraint $\tilde{\ell}_1 \tilde{\ell}_2 = 1$ allows to glue them into a new ribbon, with a long ribbon edge given by the other constraint $h_1 \tilde{h}_2^{-1} = h_{12}$. This is illustrated in Fig. 2.

We note that this gluing is the classical counterpart of the fusion product [2]. Such gluing is associative, and if we were considering the other order for the constraint, we would still get an isomorphic phase space [2].

In the following, we will essentially combine the ribbon constraints to determine the reduced phase space since then the symplectic structure follows when we have a ribbon constraint.

**Gluing octagonal ribbons: identifying some sub-components of momentum pairwise.** Instead of identifying the full momentum variables, we can just identify a subset of them. Consider the case where

---

**Figure 2:** Each phase space is represented by a ribbon diagram. Two ribbons are glued along an edge to create a single ribbon. The geometric gluing is the same as demanding the constraint $\tilde{\ell}_1 \tilde{\ell}_2 = 1$. 

- $h_1 \rightarrow \tilde{h}_1 \rightarrow h_1$ 
- $\ell_1 \rightarrow \tilde{\ell}_1 \rightarrow \ell_1$ 
- $h_2 \rightarrow \tilde{h}_2 \rightarrow h_2$ 
- $\ell_2 \rightarrow \tilde{\ell}_2 \rightarrow \ell_2$
Figure 3: The new ribbon diagram representing the division of $G$ and $G^*$ into subgroups.

$G \cong G_1 \bowtie G_2$ and dually $G^* \cong G_1^* \bowtie G_2^*$. The momentum map given by the projection of $\mathcal{D}$ into $G_1$ generates the infinitesimal right $G_1^*$ transformations. Let us introduce some notations:

$$h = u\lambda, \quad \tilde{h} = \tilde{u}\tilde{\lambda}, \quad \ell = \beta y, \quad \tilde{\ell} = \tilde{\beta}\tilde{y}, \quad u, \tilde{u} \in G_1, \lambda, \tilde{\lambda} \in G_2, \beta, \tilde{\beta} \in G_1^*, y, \tilde{y} \in G_2^*.$$  \hfill (39)

In terms of geometric interpretation, we can deform the ribbon into an octagon, which keeps track of all the different group elements, see Fig. 3.

Let us consider again $\mathcal{D}^{(2)} = \mathcal{D} \times \mathcal{D}$. The global left infinitesimal transformation of $G_1^*$ on $\mathcal{D}^{(2)}$ is given by the momentum map $u_1 u_2$.

**Proposition 3.** The symplectic reduction $\mathcal{D}^{(2)}/G_1^*$ is isomorphic to the Heisenberg double $(G_1 \bowtie (G_2 \bowtie G_2)) \bowtie (G_1^* \bowtie (G_2^* \bowtie G_2^*))$.

Since we are identifying only a subspace of momentum space, the reduced phase space is larger than $\mathcal{D}$. The symplectic reduction is nicely interpreted again as the gluing of (octagonal) ribbons, as illustrated in Fig. 4. The new ribbon after fusion has twelve sides.

**Proof.** To prove the proposition, we take this time the short way only, that is we only focus on the ribbon constraints. Fig. 4 provides a diagrammatic proof of the new ribbon constraint.

As we alluded earlier, once one has identified the ribbon constraint, the symplectic form follows. \hfill $\square$

**Closure constraints.** The last example does not involve explicit parameterization of the reduced phase space. Given $n$ copies of $\mathcal{D}$, we can look at the global $G$ transformation induced by the momentum map $\ell_1...\ell_n$ on $\mathcal{D}^{(n)} = \mathcal{D} \times ... \times \mathcal{D}$. The symplectic reduction $\mathcal{D}^{(n)}/G$ induced by the constraint $\ell_1...\ell_n = 1$ is then diagrammatically represented in Fig. 5.
2.2 2-groups

2.2.1 Definition and examples

In this section we review the definition and some examples of strict 2-groups which will be relevant to decorate our polyhedra.

Definition 5. [8, 12] A strict 2-group, also called a crossed module, is a pair of groups, $G$ and $H$ with a group action $\triangleright$ of $G$ on $H$ such that the map $h \mapsto g \triangleright h$ is a homomorphism for any $g \in G$. In addition a 2-group also includes a group homomorphism $t : H \to G$. The map $t$ is compatible with the action in the following sense:

$$t(g \triangleright h) = gt(h)g^{-1}, \quad t(h) \triangleright h' = hh'h^{-1}, \quad g \in G, \quad h, h' \in H$$

(40)

Since the above definition is the only type of 2-group we consider, we will often omit the word “strict”. A
crossed module is equipped with a pair of multiplication rules. We can represent an element of the 2-group as a surface with boundary. The two multiplications allow us to compose surfaces in a consistent way.

A diagrammatic representation of an element \((h, g) \in H \times G\) is given by a surface decorated by an element \(h \in H\). On the boundary of the surface, we specify a point \(s\), called the source, and a target point. The boundary is divided into two paths originating from the source. One of the two paths is labelled by \(g_1 \in G\) and is seen as the source of the surface decoration. The other path label is defined by the map \(t: g_2 = t(h)g_1\). The setup is shown in Fig. 6.

![Figure 6: Graphical representation of \((h, g_1)\) in the 2-group given by \((G, H, t, \triangleright)\), with \(t(h) = g_2g_1^{-1}\).](image)

Now we consider the first of the two products on the crossed module, the horizontal multiplication. Let \((h_1, g_1), (h_2, g_2) \in H \times G\), then horizontal multiplication is given by

\[
(h_1, g_1) \bullet (h_2, g_2) = (h_1(g_1 \triangleright h_2), g_1g_2).
\]  

This multiplication has an unit given by \((1_H, 1_G)\), where \(1_H\) and \(1_G\) are the identities in \(H\) and \(G\) respectively. For each \((h, g) \in H \times G\), there is a left and a right horizontal inverse. The left inverse is \((h, g)^{-1}\L = ((g^{-1} \triangleright h)^{-1}, g^{-1})\) and the right inverse is \((h, g)^{-1}\R = (g^{-1} \triangleright h^{-1}, g^{-1})\). The geometrical picture for this product is shown in figure 7

![Figure 7: Horizontal product \((h_1, g_1) \bullet (h_2, g_2) = (\tilde{h}, g_1g_2)\), where \(\tilde{h} = h_1(g_1 \triangleright h_2)\).](image)

The second multiplication is the vertical multiplication. For \((h_1, g_1), (h_2, g_2) \in H \times G\) satisfying \(t(h_1) = g_2g_1^{-1}\), we define

\[
(h_2, g_2) \diamond (h_1, g_1) = (h_2h_1, g_1).
\]  

The diagrammatic depiction of the vertical multiplication is the composition of two surfaces sharing an edge. This is shown in figure 8.
Figure 8: The diagrammatic depiction of the vertical composition \((h_2, g_2) \circ (h_1, g_1) = (h_2 h_1, g_1)\). The condition \(t(h_1) = g_2 g_1^{-1}\) is expressed graphically by requiring that the two surfaces being composed share an edge.

When vertically composing surfaces, we must ensure that the source points of the surfaces match. It is therefore convenient to detail how one goes about changing the source point, a process called whiskering. Let the original source point be at \(s\) on the boundary, and say that we would like to move the source to a new point \(s'\) also on the boundary. If the path along the boundary from \(s\) to \(s'\) is decorated by \(g'\), then we have the relation \((h, g_1)_s = (h', g_1 g'^{-1})_{s'}\), where the subscripts indicate the base point. The new surface variable, \(h'\) satisfies \(t(h') = t(h)\), as we can see from figure 9.

Figure 9: Diagrammatic representation of changing the source point of a surface from \(s\) to \(s'\), where the two sources are separated by \(g'\).

In a similar vein, we can change the target point, the point on the boundary where the two paths \(g_1\) and \(g_2\) meet. Let the original target point be denoted by \(\tau\) and the new target point by \(\tau'\), and say the path connecting \(\tau\) to \(\tau'\) is given by \(g'\). Then the new crossed module element is given by \(\tau(h, g_1) \rightarrow_{\tau'} (g' \triangleright h, g' g_1)\). This is shown in figure 10. Since the map \(t\) is calculated by taking the path around the loop starting at the target point, it is no surprise that the new surface variable is given by the action of \(g'\).

Figure 10: Changing the target point by a path \(g'\). We have that \(h' = g' \triangleright h\).

In fact, the new target or source point need not be on the boundary of the surface. We can choose a new target or source which lies off the boundary altogether without changing the calculation. We call this kind
of source/target changing whiskering and is shown in Fig. 11.


g_1 \rightarrow \tau' \quad g' \rightarrow \tau

Figure 11: Whiskering on the target of the surface. Once again we have that $h' = g' \triangleright h$.

Let us focus on a special class of strict 2-groups, which will be the main objects we will work with.

**Definition 6.** If a crossed module has $t(h) = 1$ for all $h \in H$, we say that it is a skeletal crossed module.

This restriction on the $t$-map has several implications.

First the compatibility relations (40) imply that the group $H$ has to be abelian. Second this constraint amounts to have a flat holonomy along the closed loop.

$$g_2 = t(h) g_1 \leftrightarrow g_2 g_1^{-1} = 1.$$  \hspace{1cm} (43)

Finally, in this case, we can relate the vertical multiplication to the horizontal one. Indeed, since $g_1 = g_2$ for each $g \in G$, the vertical composition $(h_1, g) \circ (h_2, g) = (h_1 h_2, g)$ simply corresponds to the multiplication of the group $H$. We can also write

$$(h_2, g) \circ (h_1, g) = (h_2, 1) \bullet (h_1, g).$$  \hspace{1cm} (44)

In this sense, the horizontal product which can be seen as the product of the group $G \times H$ can be used to specify the vertical product. Roughly said, skeletal crossed modules are not groups but are not so far away from being a group. We will use this extensively to associate a phase space structure to a pair of 2-groups.

**Examples** Let us consider some examples of skeletal crossed modules. \[12\]

**Trivial 2-groups.** Consider the crossed module where $H = 1$ is the trivial group and $G$ any group. In this case, both the action and map $t$ are trivial. This 2-group is essentially a group. This is the usual framework for lattice gauge theory with $G$ decorating paths and no information on the faces (or trivial group elements on faces).

Alternatively, we may consider $G$ to be the trivial group. The compatibility conditions mean that $H$ must then be abelian. Once again, this 2-group is simply a group. Geometrically, in analogy with the lattice picture before, we have decorations on plaquettes, but not on paths.

**Semi-direct product.** Let $G$ be a semi direct product group $G = H \rtimes G$ with $H$ abelian, so that for any $k \in G$ there is the unique factorization $k = hg$ with $h \in H$ and $g \in G$. Such a group can be identified as a skeletal crossed module with the action given by conjugation. The identification is $hg \mapsto (h, g)$. Under this representation, multiplication in $G$ corresponds to horizontal multiplication:

$$(hg)(h'g') = h(gh'g^{-1})gg' \mapsto (h(gh'g^{-1}), gg') = (h, g) \bullet (h', g'),$$  \hspace{1cm} (45)

with the action being conjugation. It is clear that the left and right horizontal inverses are the same and is consistent with the mapping: $(hg)^{-1} = (g^{-1}h^{-1}g)^{-1} \mapsto (g^{-1}h^{-1}g, g^{-1})$.

The vertical multiplication can be seen as simply the multiplication in the subgroup $H$.

**Poincaré and Euclidean 2-group.** The Poincaré (or Euclidean) group can be seen as examples of semi-direct product 2-groups, with $H$ the group of translations and $G$ the Lorentz (or rotation) group.
**Co-tangent 2-group.** Let $G$ be a Lie group and $H = g^*$ be its dual Lie algebra, which we see as an abelian group. Then, $G$ acts on $g^*$ via the co-adjoint action. Since the $t$-map is trivial, $g^* \rtimes G$ is a skeletal crossed module.

**Lie 2-algebras** For our concerns, the pair of groups which make up a 2-group are most of the time Lie groups. As such, it is useful to define the infinitesimal version of crossed modules coming from Lie groups.

**Definition 7.** [11, 12] A strict Lie 2-algebra is a pair of Lie algebras with a Lie algebra homomorphism $dt : h \to g$ and an action $\triangleright : g \times h \to h$ satisfying

$$[x_1, x_2] \triangleright y = x_1 \triangleright (x_2 \triangleright y) - x_2 \triangleright (x_1 \triangleright y) \quad x \triangleright [y_1, y_2] = [x \triangleright y_1, y_2] + [y_1, x \triangleright y_2]$$

(46)

for all $x_i \in g$ and $y_i \in h$. The homomorphism and the action must be compatible in the sense that

$$dt(x \triangleright y) = [x, dt(y)] \quad dt(y_1) \triangleright y_2 = [y_1, y_2].$$

(47)

A skeletal strict Lie 2-algebra is an infinitesimal crossed module $g \rtimes h$ with a trivial $dt$-map, so that $h$ is abelian.

Given a Lie 2-group, one can construct a corresponding Lie 2-algebra [11]. The Lie algebras $g$ and $h$ are taken to be Lie algebras of the groups $G$ and $H$ respectively. The action and the Lie algebra homomorphism are the infinitesimal versions of the action and homomorphism of the 2-group.

**2.2.2 Poisson Lie 2-groups and 2-bi-algebras**

Just as a Poisson Lie group was a Lie group equipped with a Poisson bracket respecting group multiplication, we can define the same for crossed modules.

**Definition 8.** [22] A Poisson Lie 2-group is a Lie 2-group $G$ equipped with a Poisson bracket that is compatible with both product laws.

Explicitly, for any functions $f_1$ and $f_2$ on $G$ and $x, y \in G$,

$$\{f_1, f_2\}(x \bullet y) = \{f_1 \circ L^*_x, f_2 \circ L^*_y\}(y) + \{f_1 \circ R^*_y, f_2 \circ R^*_x\}(x)$$

(48)

$$\{f_1, f_2\}(x \circ y) = \{f_1 \circ L^*_x, f_2 \circ L^*_y\}(y) + \{f_1 \circ R^*_y, f_2 \circ R^*_x\}(x)$$

(49)

where $L^*_x (R^*_y)$ is the left (right) horizontal multiplication by $x$ ($y$), similarly for $L^*$.

**Examples.**

- If $G$ is any 2-group, equipping it with the trivial Poisson bracket $\{\cdot, \cdot\} = 0$ makes $G$ a Poisson Lie 2-group.

- If $G$ is a trivial 2-group in the sense that either $G$ or $H$ is the trivial group, then if the other group is a Poisson Lie group, $G$ is a Poisson Lie 2-group.

- skeletal crossed modules provide some easy examples of such Poisson 2-groups. Indeed, consider $H$ a Poisson Lie group and $G = H \rtimes G$ a Poisson Lie group such that the induced Poisson bracket on $H$ makes it a Poisson Lie group. Then $G$ is can be viewed as a Poisson 2-group. Indeed, $H \rtimes G$ being a Poisson group means that the horizontal product is compatible with the Poisson structure. If $H$ is a Poisson Lie group, then the vertical product is also compatible with the Poisson structure.

- The co-tangent 2-group as a non-trivial Poisson 2-group. The semi-direct product $G \rtimes g^*$ can be seen as a the Drinfeld double associated to the Heisenberg double $G \rtimes g^*$ which is nothing else than $T^*G$. Explicitly, $p^i$ are the coordinate functions of $g^*$ and $g^{ij}_j$ are the matrix elements of $G$ then the following Poisson structure (the Drinfeld double Poisson brackets) gives a Poisson 2-group

$$\{g^{ij}_j, g^{hk}_i\} = \{g^{ij}_j, p^k\} = 0, \quad \{p^i, p^j\} = f^{ij}_{\ k}p^k,$$

(50)

where $f^{ij}_{\ k}$ is the structure constant of $g$. 

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The infinitesimal version of a Poisson Lie group is a Lie bi-algebra. Similarly, the infinitesimal version of a Poisson 2-group is a Lie 2-bi-algebra. We defined a bi-algebra as Lie algebra equipped with a a cocyle. We can equivalently define it as a pair of Lie algebra, \((\mathfrak{g}, \mathfrak{g}^*)\) in duality, where the Lie bracket of \(\mathfrak{g}^*\) is defined by the cocycle of \(\mathfrak{g}\). This is this type of definition used in [23].

**Definition 9.** [23] A Lie 2-bi-algebra (aka a Lie bi-algebra crossed module) is a pair of infinitesimal crossed modules in duality, \((\mathfrak{g} \ltimes \mathfrak{h}, dt)\), \((\mathfrak{h}^* \ltimes \mathfrak{g}^*, -dt^*)\), such that \((\mathfrak{g} \ltimes \mathfrak{h}, \mathfrak{h}^* \ltimes \mathfrak{g}^*)\) form a Lie bi-algebra.

In particular the pairs \((\mathfrak{g}, \mathfrak{g}^*)\) and \((\mathfrak{h}, \mathfrak{h}^*)\) are Lie bi-algebras [23]. The following theorem will be quite useful when constructing explicit Lie 2-bi-algebras in Sec. 2.3.1.

**Theorem 2.** [23] Assume that \((\mathfrak{g} \ltimes \mathfrak{h}, dt)\) and \((\mathfrak{h}^* \ltimes \mathfrak{g}^*, -dt^*)\) are infinitesimal crossed modules, then they form a Lie 2-bi-algebra if and only if \((\mathfrak{g}, \mathfrak{h}^*)\) form a matched pair of Lie algebras \(\mathfrak{g} \bowtie \mathfrak{h}^*\), where the action of \(\mathfrak{g}\) on \(\mathfrak{h}^*\) is the dual of the action of \(\mathfrak{g}\) on \(\mathfrak{h}\), and the action of \(\mathfrak{h}^*\) on \(\mathfrak{g}\) is dual to the action of \(\mathfrak{h}\) on \(\mathfrak{g}^*\).

### 2.3 Heisenberg double and skeletal crossed modules

In this section, we discuss how we can construct Heisenberg doubles out of a pair of skeletal crossed modules. The key idea is that if a skeletal crossed module still behaves as a group then it can be embedded into an Heisenberg double. The simplest example is when the 2-group is trivial, namely just a group. For example, \(T^* G \cong \mathfrak{g}^* \ltimes \mathfrak{g}\) can be seen as built from a pair of trivial 2-groups, \(G\) and \(\mathfrak{g}^*\). This is the typical structure that appears when discretizing a BF theory [39]. skeletal 2-groups provide an another example.

In particular, we will recover the Poisson version of the \(\kappa\)-Poincaré group and its dual, the \(\kappa\)-Poincaré algebra. Said differently, we will recover the Poincaré group equipped with a non-trivial Poisson structure which can then be interpreted as a non trivial Poisson 2-group.

#### 2.3.1 Semi-dualized Lie bi-algebras and skeletal Lie 2-bialgebras

Consider the Lie bialgebras \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) with generators \(e^i\) and \(f_i\) respectively, with dual spaces \(\mathfrak{g}_1^*\) and \(\mathfrak{g}_2^*\) generated by \(e_i^*\) and \(f^*_i\), equipped with a bi-linear map, defined by

\[
\langle e^i, e_j^* \rangle = \langle f^i, f_j^* \rangle = \delta^i_j,
\]

and all the other pairings between generators vanish.

We demand that \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) have trivial co-cycles\(^9\), so that the \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\) bialgebra structures are respectively

\[
[e^i, e^j] = c^{ij}{}_k e^k, \quad \delta_{\mathfrak{g}_1}(e^i) = 0, \tag{51}
\]

\[
[f^i, f^j] = d^{ij}{}_k f^k, \quad \delta_{\mathfrak{g}_2}(f^i) = 0. \tag{52}
\]

**Semi-dualized Lie bi-algebras.** First we introduce the concept of matched pair of Lie algebras, we will focus on the cocycle sector after that.

**Definition 10** (Matched pair). [56] Consider a pair of Lie algebras \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\), they form a matched pair, noted \(\mathfrak{g}_1 \bowtie \mathfrak{g}_2\), if there exist actions

\[
\triangleleft : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_2 \quad \quad \triangleright : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \to \mathfrak{g}_1 \quad \quad f^i \triangleleft e^j = \alpha^{ij}{}_k f^k \quad \quad f^i \triangleright e^j = \beta^{ij}{}_k e^k. \tag{54}
\]

\(^8\)Interestingly, \(T^* G\) can also be seen as a 2-group. It would be interesting to determine whether general notion of 2-Heisenberg double, eg a 2-group built out of a pair of 2-groups equipped with a symplectic structure, can be defined in some way.

\(^9\)This condition might be released, we leave it for further investigations.
that obey to the conditions
\[
[f^i, f^j] \triangleright e^k = f^i \triangleright (f^j \triangleright e^k) - f^j \triangleright (f^i \triangleright e^k), \quad f^i \triangleright [e^j, e^k] = (f^i \triangleright e^j) \triangleright e^k - (f^i \triangleright e^k) \triangleright e^j, \quad (55)
\]
\[
f^i \triangleright [e^j, e^k] = [f^i \triangleright e^j, e^k] + [e^j, f^i \triangleright e^k] + (f^i \triangleright e^j) \triangleright e^k - (f^i \triangleright e^k) \triangleright e^j, \quad (56)
\]
\[
[f^i, f^j] \triangleright e^k = [f^i, f^j] \triangleright e^k + [f^i, f^j] \triangleright e^k - f^i \triangleright (f^j \triangleright e^k) - f^j \triangleright (f^i \triangleright e^k). \quad (57)
\]

Given a matched pair, we can consider the dual picture.

**Definition 1.1 (Dual of a matched pair). [56]** Consider a matched pair \((g_1, g_2)\) equipped with trivial co-cycles. The dual of such matched pair is noted \(\triangleright\triangleright g_2\) and is a pair of Lie bi-algebras \(g_1^*\) and \(g_2^*\), with Lie brackets and co-cycles inferred by (5) and (51),
\[
[e^*_i, e^*_j] = 0, \quad \delta g_1(e^*_i) = 1/2 \alpha^{ki}e^*_j \wedge e^*_k, \quad (58)
\]
\[
[f^*_i, f^*_j] = 0, \quad \delta g_2(f^*_i) = 1/2 \beta^{ki}f^*_j \wedge f^*_k, \quad (59)
\]
equipped with the pair of co-actions
\[
\alpha : \ g_2 \to g_1 \otimes g_1^*, \quad \beta : \ g_1^* \to g_2 \otimes g_2^* \quad (60)
\]
derived, by dualization of the actions (54), as
\[
\langle f^i \triangleright e^j, f^*_k \rangle = \langle f^i \triangleright e^j, \alpha(f^*_k) \rangle, \quad \langle f^i \triangleright e^j, e^*_k \rangle = \langle f^i \triangleright e^j, \beta(e^*_k) \rangle, \quad (61)
\]

We emphasize that if actions are denoted by triangles of the type \(\blacktriangleleft\) or \(\blacktriangleright\), co-actions are denoted by black triangles of the type \(\blacktriangleleft\) or \(\blacktriangleright\).

From a matched pair \((g_1, g_2)\), we just built its “full” dual \((g_1^*, g_2^*)\). Instead we can only dualize a given component of the matched pair, say \(g_2\) for example. We get in this case a bicross sum.

**Definition 1.2 (Bicross sum). [56]** Consider the matched pair \((g_1, g_2)\), there exist two bicross sum Lie bi-algebras dual to each other\(^{10}\): the Lie bi-algebra \(g_2^* \triangleright g_1\), equipped with an action of \(g_1\) on \(g_2^*\) and a co-action \(\beta\) of \(g_2^*\) on \(g_1\),
\[
\triangleright^* : \ g_1 \otimes g_2^* \to g_2^* \quad \beta : \ g_1 \to g_1 \otimes g_2^* \quad (62)
\]
and the Lie bi-algebra \(g_2 \triangleright g_1^*\), equipped with an action of \(g_2\) on \(g_1^*\) and a co-action of \(g_1^*\) on \(g_2\),
\[
\blacktriangleleft^* : \ g_1^* \otimes g_2 \to g_1^* \quad \beta : \ g_2 \to g_2 \otimes g_1^* \quad (63)
\]
The actions of \(g_1\) on \(g_2^*\) and of \(g_2\) on \(g_1^*\) are derived by dualization of the actions in the matched pair (54):
\[
\langle e_i \blacktriangleright^* f^*_k, f^j \rangle = \langle f^*_k, f^j \triangleright e^i \rangle, \quad \langle e_i \blacktriangleleft^* f^*_j, e^k \rangle = \langle e_i, f^j \blacktriangleright e^k \rangle, \quad (64)
\]
The co-actions of \(g_2^*\) on \(g_1\) and of \(g_1^*\) on \(g_2\) are derived by dualization of the actions \(\blacktriangleleft^*, \blacktriangleright^*\) in (64):
\[
\langle \beta(e^*_i), e^*_j \otimes f^k \rangle = \langle e^*_i, e^*_j \triangleright f^k \rangle, \quad \langle \alpha(f^*_i), e^j \otimes f^*_k \rangle = \langle f^*_i, e^j \blacktriangleright f^*_k \rangle, \quad (65)
\]
The cross Lie brackets of \(g_2^* \triangleright g_1\) and \(g_2 \triangleright g_1^*\) are induced by the actions
\[
[e^*_i, f^*_j] = e^*_i \blacktriangleright^* f^*_j = \alpha^{ki}f^*_j, \quad \quad [e^*_i, f^j] = e^*_i \blacktriangleright f^j = \alpha^{ki}e^*_j, \quad (66)
\]
and the co-cycle of \(g_1\) and \(g_2\) are corrected by the co-actions
\[
\delta g_1 \circ \alpha(e^*_i) = \delta g_2 (f^*_i) = \delta g_2 (f^*_i) + (1 - \tau) \circ \alpha(f^*_i) = \alpha^{ij}e^*_j \wedge f^*_k, \quad (67)
\]
\(^{10}\)The co-action is encoded by the black triangle.
We can now construct two types of matched pairs. The first one consists in putting together the original (matched pair) $g_1 \bowtie g_2$ and its dual. Doing this we recover the Lie algebra $\mathfrak{d}$ which was used to generate the Heisenberg double associated to $\mathcal{D} \cong T^* (G_1 \bowtie G_2)$. As such this is a pretty standard construction.

**Proposition 5** (Classical double behind $T^* (G_1 \bowtie G_2)$). Given a matched pair $(g_1, g_2)$ endowed with trivial co-cycles, the bi-algebra

$$\mathfrak{d} = (g_1 \bowtie g_2)^{\text{cop}} \bowtie (g_1^* \bowtie g_2^*) = (g_1 \bowtie g_2) \times (g_1^* \bowtie g_2^*)$$

with the Lie brackets and co-cycles of $g_1 \bowtie g_2$ and $g_1^* \bowtie g_2^*$ being

$$[e^i, e^j] = c^{ij}_k e^k, \quad [f^i, f^j] = d^{ij}_k f^k,$$

$$[e^i, f^j] = \beta^{ij}_k e^k, \quad [f^i, e^j] = \alpha^{ij}_k f^k,$$

$$\delta_{g_1} (e^i) = 0, \quad \delta_{g_2} (f^i) = 0,$$

$$\delta_{g_1^*} (e^*_i) = \frac{1}{2} c^{ij}_k e^*_j \wedge e^*_k, \quad \delta_{g_2^*} (f^*_i) = \frac{1}{2} d^{ij}_k f^*_j \wedge f^*_k.$$

is a classical double. The action of $g_1 \bowtie g_2$ on $g_1^* \bowtie g_2^*$ is the co-adjoint action $ad^*: ((g_1 \bowtie g_2)^{\text{cop}} \bowtie (g_1 \bowtie g_2)) \rightarrow (g_1^* \bowtie g_2^*)$ defined in (7), and induces the cross Lie brackets

$$[f^*_i, f^*_j] = d^{ij}_k f^*_k + \alpha^{ij}_k f^*_k, \quad [e^*_i, f^*_j] = \beta^{ij}_k e^*_k.$$

The co-cycle of the Lie bi-algebra $\mathfrak{d}$ can be expressed in terms of the r-matrix

$$r = e^*_i \otimes e^i + f^*_i \otimes f^i.$$

**Proof.** The matched pair $(g_1, g_2)$ described in Def. 10, defined on the vector space $g_1 \oplus g_2$ and equipped with trivial co-cycles, as a Lie bi-algebra is equivalent to a classical double (introduced in Def. 3) with cross Lie brackets induced by the actions of the matched pair (54).

Similarly, the dual of the matched pair is a Lie bi-algebra with trivial Lie brackets and co-cycles given by the co-cycles of $g_1^*$ and $g_2^*$, corrected by the co-actions (60):

$$\delta_{g_1^* \bowtie g_2^*} (e^*_i) = \delta_{g_1^*} (e^*_i) + (1 - \tau) \circ \beta (e^*_i), \quad \delta_{g_1^* \bowtie g_2^*} (f^*_i) = \delta_{g_2^*} (f^*_i) + (1 - \tau) \circ \alpha (f^*_i).$$

The Lie bi-algebra $\mathfrak{d}$ is thus constructed as the cross sum of $g_1 \bowtie g_2$ and $g_1^* \bowtie g_2^*$, with action of the first on the second given by the co-adjoint actions defined in (7). One can check that the co-adjoint action of $g_1 \bowtie g_2$ on $g_1^* \bowtie g_2^*$ satisfies the conditions (55)-(57) for $\mathfrak{d}$ to be a matched pair; to carry out this computation it is enough to use the Jacobi identity for $g_1$ and $g_2$, plus the axioms (55)-(57) for the matched pair $(g_1, g_2)$.

The co-cycles (72) can be derived through the co-boundary equation (12) using the classical $r$-matrix (74). □

The second match pair we can construct can be viewed as a semi-dualization of the previous option: we interchange the $g_2$ and $g_2^*$ sectors. We then construct a classical double out of a pair of bicross algebras. This is the structure we will use to deal with non-trivial Poisson 2-groups. This can generate curvature in momentum space. As we will see below, it is the key structure relevant to study some deformation of special relativity [52, 4].

**Proposition 5** (Semi-dualized case). Given a matched pair $(g_1, g_2)$ endowed with trivial co-cycles, the bi-algebra

$$\mathfrak{b} = (g_2 \bowtie g_1)^{\text{cop}} \bowtie (g_1^* \bowtie g_2)$$

with the Lie brackets and co-cycles of $(g_2 \bowtie g_1)^{\text{cop}}$ and $g_1^* \bowtie g_2$ being

$$[f^i, f^j] = d^{ij}_k f^k, \quad [e^i, e^j] = 0, \quad [e^i, f^j] = 0,$$

$$[e^i, f^j] = \beta^{ij}_k e^k, \quad [f^i, e^j] = \alpha^{ij}_k f^k,$$

$$\delta_{g_2^* \bowtie g_1} (e^*_i) = \frac{1}{2} c^{ij}_k e^*_j \wedge e^*_k, \quad \delta_{g_2^* \bowtie g_1} (f^*_i) = \frac{1}{2} d^{ij}_k f^*_j \wedge f^*_k.$$
is a classical double.

Since the Lie bi-algebras $g_2 \lhd g_1^*$ and $g_2^* \rhd_{\mathrm{co-adj}} g_1$ are dual each other, their mutual actions are encoded in (54) and in the respective co-adjoint actions, that induce the cross Lie brackets

$$[f_i^*, f_j^*] = d^{jk} f_i^* + \alpha^{jk} e_k^*,$$
$$[e_i^*, e_j^*] = e^{ji} f_k^* + \beta^{ji} e_k^*,$$
$$[e_i^*, f_j^*] = -\alpha^{ij} f_k^* - \beta^{ij} e_k^*, \quad [e_i^*, f_j^*] = 0$$

(79)

The co-cycle of the Lie bi-algebra $b$ can be expressed in terms of the $r$-matrix

$$r = e_i^* \otimes e_i^* + f_i^* \otimes f_i^*$$.  

(80)

**Proof.** Given the matched pair $(g_1, g_2)$ equipped with trivial co-cycles, we first build the bicross sum Lie bi-algebras $g_2 \lhd g_1^*$ and $g_2^* \rhd_{\mathrm{co-adj}} g_1$ as explained in Def. 12.

Since these two Lie bi-algebras are dual each other, we can use them to construct a classical double as in Def. 3, or equivalently a matched pair as in Def. 10, with mutual actions given by the actions of the matched pair $(g_1, g_2)$, eq. (54), plus the co-adjoint actions of $g_1$ on $g_2^*$ and of $g_2$ on $g_1^*$. One can check that the mutual action between $g_2 \lhd g_1^*$ and $g_2^* \rhd_{\mathrm{co-adj}} g_1$ satisfy the properties (55)-(57); to carry out this computation it is enough to use the Jacobi identity for $g_1$ and $g_2$, plus the axioms (55)-(57) for the matched pair $(g_1, g_2)$.

The co-cycles (78) can be derived through the co-boundary equation (12) using the classical $r$-matrix (80).

We emphasize for the following that the Lie algebras $g_1^*$ and $g_2^*$ are always abelian Lie algebras. We note that $d$ and $b$ are identical as Lie algebras, but we have been identifying two different types of sub Lie algebras. Indeed, the list of Lie brackets is always the same for $d$ or $b$ and we have sliced the total algebra in two different ways. $d$ and $b$ are however different as Lie bi-algebras.

Let us show now how the semi-dualized case can be interpreted in terms of the 2-group framework.

**Lie 2-bialgebra from semi-dualization.** According to Thm. 2, $g \ltimes h$ and $h^* \ltimes g^*$ are both Lie 2-bi-algebras if $g$ and $h^*$ form a matched pair. If we set

$$g \equiv g_1, \quad h^* \equiv g_2, \quad g^* \equiv g_1^*, \quad h \equiv g_2^*,$$

(81)

then the theorem states that $g_1 \ltimes g_1^*$ and $g_2 \ltimes g_1^*$ are both Lie 2-bi-algebras if $g_1$ and $g_2$ form a matched pair, $g_1 \ltimes g_2$, which is exactly the choice we made. More exactly, we also need to have that the action of $g_1$ on $h^* \equiv g_2$ is the dual of the action of $g_1$ on $h \equiv g_2^*$ and the action of $h^* \equiv g_2$ on $g_1$ is dual to the action of $h^* \equiv g_2$ on $g_1^* \equiv g_1^*$. This is exactly what the semi-dualization implements.

Having this in mind, we can reconsider the semi-dualized bi-algebra of Prop. 5. The components, $(g_2 \ltimes g_1)$ and $(g_2^* \ltimes g_1^*)$ can be viewed individually as infinitesimal crossed modules, or 2-Lie bi-algebras, with trivial $t$-map. Hence we can interpreted $b$ as a classical double, defined in terms of simple infinitesimal crossed modules.

We can exponentiate the Lie (2-)bi-algebras to recover groups or skeletal crossed modules. Upon exponentiation, we define

$$g_1 \ltimes g_2^* \to G = G_1 \ltimes G_2, \quad g_2 \ltimes g_1^* \to G^* = G_2 \ltimes G_1^*.$$ 

(82)

Hence $b$ becomes the group\(^\dagger\) $B = (G_1 \ltimes G_2^*) \ltimes (G_2 \ltimes G_1^*)$, which can be equipped with a symplectic Poisson bracket or a Poisson bracket which makes $B$ a Poisson Lie group (Drinfeld double). Since both $G_2$ and $G_1^*$ are abelian groups, both $(G_1 \ltimes G_2)$ and $(G_2 \ltimes G_1^*)$ can be viewed as skeletal Poisson crossed modules [22].

As we already alluded, the Poincaré case is a key example of bicross bi-algebra. We are going to illustrate the construction in this case, and as such recover the Poincaré 2-group equipped with a non-trivial Poisson bracket.

\(^\dagger\)Once again, we will neglect the possible subtleties that might arise if such splitting cannot be done globally.
2.3.2 Example: Poincaré case

We use the metric $\eta_{\mu\nu}$ as the Minkowski metric. We take the Lie bi-algebra $\mathfrak{g}_2$ to be the $n$ dimensional Lie algebra $\mathfrak{an}$ with generators $P^\mu$ having the same dimension as the scale $\kappa^{-1}$, satisfying\(^{(12)}\)

$$[P^\mu, P^\nu] = \kappa^{-1}(\eta^{\rho\sigma} P^\rho - \eta^{\rho\sigma} P^\mu), \quad \delta_{\mathfrak{an}}(P^\mu) = 0, \quad \delta_{\mathfrak{an}}(P^\nu) = 0, \quad (83)$$

and the Lorentz Lie bi-algebra $\mathfrak{g}_1$ to be $\mathfrak{so}$ of dimension $\frac{n(n-1)}{2}$, generated by $J^{\mu\nu}$, obeying

$$[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\nu\sigma} J^{\rho\mu} - \eta^{\mu\sigma} J^{\rho\nu} - \eta^{\nu\rho} J^{\mu\sigma}, \quad \delta_{\mathfrak{so}}(J^{\mu\nu}) = 0, \quad (84)$$

The Lie bi-algebras $(\mathfrak{an}, \mathfrak{so})$, equipped with the classical structures:

$$\langle P^\mu, P^\nu \rangle = \eta_{\mu\nu}, \quad \langle J^{\mu\nu}, J^{\rho\sigma} \rangle = \eta_{\mu\nu}\rho\sigma - \eta_{\mu\rho}\mu\sigma. \quad (86)$$

Using (5), we derive the classical double of $\mathfrak{an}$ and $\mathfrak{so}$ Lie bi-algebra structures:

$$[P^\mu, P^\nu] = 0, \quad \delta_{\mathfrak{an}}(P^\mu) = \kappa^{-1} P^\mu \wedge P^\mu, \quad (87)$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = 0, \quad \delta_{\mathfrak{so}}(J^{\mu\nu}) = J^{\mu\nu} \wedge J^{\mu\nu}. \quad (88)$$

Finally, by dualization of the actions (85), we derive the co-actions

$$\alpha : \mathfrak{so} \rightarrow \mathfrak{so} \otimes \mathfrak{an}^*, \quad \beta : \mathfrak{an}^* \rightarrow \mathfrak{so} \otimes \mathfrak{an}^*$$

$$\alpha(J^{\mu\nu}) = \kappa^{-1}(J_{0\mu} \otimes P^\nu - J_{0\nu} \otimes P^\mu), \quad \beta(P^\mu) = -J^{\mu\nu} \otimes P^\nu \wedge P^\nu. \quad (89)$$

**Classical double of $\mathfrak{an} \bowtie \mathfrak{so}$.** Following Prop. 4, from the matched pair $(\mathfrak{an}, \mathfrak{so})$ with trivial co-cycles, we can construct the cotangent bundle of $\mathfrak{an} \bowtie \mathfrak{so}$ as a classical double

$$\mathfrak{d} = (\mathfrak{an} \bowtie \mathfrak{so}) \times (\mathfrak{an}^* \bowtie \mathfrak{so}^*), \quad (90)$$

equipped with the classical $r$-matrix

$$r = P^\mu \otimes P^\mu + \frac{1}{2} J^{\mu\nu} \otimes J^{\mu\nu}. \quad (91)$$

**Semi-dualized case.** According to Prop. 5, given the matched pair $(\mathfrak{an}, \mathfrak{so})$ with trivial co-cycles, we can construct the classical double

$$\mathfrak{b} = (\mathfrak{so} \bowtie \mathfrak{an}^*)^{\text{cop}} \bowtie (\mathfrak{so} \bowtie \mathfrak{an}), \quad (92)$$

equipped with the classical $r$-matrix

$$r = P^\mu \otimes P^\mu + \frac{1}{2} J^{\mu\nu} \otimes J^{\mu\nu}. \quad (93)$$

In particular, the Lie bi-algebras $\mathfrak{so} \bowtie \mathfrak{an}^*$ and $\mathfrak{so}^* \bowtie \mathfrak{an}$ are given by

$$[P^\mu, P^\nu] = 0, \quad [J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\nu\sigma} J^{\rho\mu} - \eta^{\mu\sigma} J^{\rho\nu} - \eta^{\nu\rho} J^{\mu\sigma}, \quad [P^\mu, P^\nu] = \kappa^{-1}(\eta^{\rho\sigma} P^\rho - \eta^{\rho\sigma} P^\mu), \quad (84)$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = 0, \quad [P^\mu, J^{\rho\sigma}] = \kappa^{-1}(\eta^{\rho\sigma} J^{\mu\nu} - \eta^{\mu\nu} J^{\rho\sigma}), \quad \delta_{\mathfrak{so} \bowtie \mathfrak{an}^*}(P^\mu) = P^\mu \wedge J^{\mu\nu}. \quad (94)$$

$$\delta_{\mathfrak{so} \bowtie \mathfrak{an}^*}(J^{\mu\nu}) = \kappa^{-1}(J^{\rho\nu} J^{\mu\sigma} - J^{\rho\sigma} J^{\mu\nu}), \quad \delta_{\mathfrak{so} \bowtie \mathfrak{an}^*}(J^{\mu\nu}) = \frac{1}{2} J^{\mu\nu} \otimes J^{\mu\nu}. \quad (95)$$

\(^{(12)}\)We are using the time-like deformation but other deformations, such as space or light-like are also possible [13, 51]. The Euclidean signature can also be considered in a similar manner.
The cross Lie brackets, induced by the actions of the matched pair and by the co-adjoint actions, are
\[
\begin{align*}
[P^\rho, J^\mu] &= \kappa^{-1} (\eta^\rho{}^0 J^\mu{}^0 - \eta^\rho{}^0 J^\mu{}^0) + (\eta^\rho{}^\nu P^\mu - \eta^\mu{}^\rho P^\nu), \\
[J^\mu\nu, P^\rho] &= 0, \\
[J^\rho\sigma, J^\mu\nu] &= (\eta^\nu{}^\rho J^\mu\sigma + \eta^\sigma{}^\rho J^\mu\nu - \eta^\sigma{}^\nu J^\mu\rho - \eta^\mu{}^\rho J^\nu\sigma) \\
&+ \kappa^{-1} (\eta^\nu{}^0 (\eta^\mu{}^\sigma P^\rho - \eta^\mu{}^\rho P^\sigma) - \eta^\mu{}^0 (\eta^\nu{}^\rho P^\sigma - \eta^\nu{}^\sigma P^\rho)), \\
[P^\mu, P^\nu] &= -\kappa^{-1} (\eta^\mu{}^0 P^\nu + \eta^\nu{}^0 P^\mu) + J^\mu\nu.
\end{align*}
\]

As a Lie algebra, \( so \circ an^* \cong so \circ \mathbb{R}^n \) is the Poincaré/Euclidean Lie algebra. So upon exponentiation it would give the Poincaré/Euclidean Lie group. Since its co-cycle is non trivial, we would get the Poincaré group equipped with a non-trivial Poisson bracket. This is in fact the classical version of the so-called \( \kappa \)-Poincaré group [79].

Since it is only the geometrical interpretation that distinguishes the Poincaré group from the Poincaré 2-group, this construction allows to consider the Poincaré 2-group equipped with a non-trivial Poisson structure, giving rise upon quantization to a quantum strict 2-groups [59].

Similarly, the Lie bi-algebra \( so^* \circ an \) upon exponentiation gives rise to a group \( AN \circ SO^* \cong AN \circ \mathbb{R}^{(n^2-n)/2} \) equipped with a non-trivial Poisson structure. Upon quantization this gives rise to the \( \kappa \)-Poincaré algebra [79, 46]. Once again, this is a semi-direct product of groups, with \( SO^* \) being an abelian group. Hence this can also be interpreted as a skeletal crossed module, equipped with a non-trivial Poisson bracket.

Upon exponentiation, \( b \) becomes the Lie group noted \( B \), which is isomorphic to \( D \) as a Lie group, since as Lie algebras, \( b \) and \( d \) are the same. The Poisson Lie group associated with \( b \) is the Drinfeld double of the Poincaré group; the quantization of such Drinfeld double gives rise to the quantum double built as the double cross product of the \( \kappa \)-Poincaré group and the \( \kappa \)-Poincaré algebra. The exponentiation of \( b \), equipped with a symplectic Poisson structure, is the Heisenberg double of the Poincaré group \( ISO \), with a non-trivial Poisson brackets, with a non-abelian dual group \( ISO^\ast \cong AN \circ SO^* \).

Finally, as we emphasized many times now, the difference between the Poincaré group and Poincaré 2-group is mainly in the geometrical interpretation of the products. The construction we have presented can therefore be re-interpreted in terms of skeleton crossed modules and we have a Drinfeld double or Heisenberg double defined in terms of a pair of skeletal crossed module, with non-trivial Poisson structures. This will be the main object to construct the phase space for a three dimensional triangulation with decorations both on faces and edges.

3 Polygon and polyhedron phase space

In this section we describe the phase space of a two dimensional discrete geometry. For simplicity, we focus on a triangulation, but our considerations extend directly to an arbitrary polygonal decomposition. We will focus on the flat case to set up the framework and then move to the case of a triangulation with homogeneously curved triangles.

We will then recall how a polygon can be used to specify a polyhedron, thanks to the Minkowski theorem. This allows to obtain directly a phase space for a polyhedron.

3.1 Polygon phase space

First, we review the seminal work by Kapovich and Millson, restricted to the case of a triangle [41]. In the following we will deal the Euclidian picture. Discussion of a Lorentzian picture can be found in [47].

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Consider the $\mathbb{R}^3$ vector space in the $\mathfrak{su}(2)$ basis: $\ell \equiv \vec{\ell} \cdot \vec{\sigma}$, where $\sigma_I$, $I = 1, 2, 3$ are the Pauli matrices. The Killing form on $\mathfrak{su}(2)$ allows to encode the scalar product:

$$\langle \ell, y \rangle = \vec{\ell} \cdot \vec{y} \quad \text{with} \quad \langle \sigma_I, \sigma_J \rangle = \delta_{IJ}.$$  

(96)

Consider a set of three vectors $\ell_i$ in the Euclidian plane $\mathbb{R}^3$, which close, that is

$$C = \ell_1 + \ell_2 + \ell_3 = 0.$$  

(97)

This constraint defines a triangle. We interpret the vectors as edges of the triangle, with edge length $l_i \equiv |\ell_i|$. The vectors $\ell_i$ are equivalent to defining geodesics in $\mathbb{R}^3$. We note that the triangle can be embedded in $\mathbb{R}^3$ up to global rotations and translations in $\mathbb{R}^3$.

We now introduce a phase space structure for such triangle. In the Hamiltonian formalism, a constraint is usually associated to a symmetry invariance [27]. We would like to pick the Poisson structure such that the geometric constraint $C$ becomes a momentum map generating the symmetry freedom of the triangle embedding in $\mathbb{R}^3$. In particular, we want that this constraint implements the rotational symmetry.

As argued by Kapovich and Millson [41], the following Poisson structure

$$\{\ell^I_i, \ell^J_j\} = \delta_{ij} \epsilon^{K} \ell^K_i, \quad \{l_i, \ell^I_j\} = 0,$$  

(98)

is precisely such that $C$ generates the infinitesimal rotations in space. Hence $C$ can be viewed as the relevant momentum map. Note that since we have three edges, each with three components, the dimensions of this space is odd, hence it cannot have a symplectic Poisson bracket. Instead, Kapovich and Millson [41] restricted the vectors to their direction only, so that their norm, the edge length, is fixed. As such the vector $\ell_i$ is an element of the sphere $S^2$ and it is well known that the Poisson bracket (98) is symplectic on such sphere. Hence the final phase space $P^{KM}$ for a triangle with fixed edge lengths is given by the symplectic reduction

$$P^{KM} = (S^2 \times S^2 \times S^2) / \text{SO}(3).$$  

(99)

From the gravity perspective, it would be natural to introduce the length as a degree of freedom. However by doing so we do not get a phase space in general, since the reduced space will have dimension $3N - 6$, with $N$ being the total number of edges.

To avoid this issue, the simplest way is to add extra variables. The interest of doing so is twofold.

- We can introduce more geometric information which would allow us to extend the phase space to a full 2d cellular decomposition for example.
- We can have a realization of the rotational and translational symmetries in terms of our phase space variables.

We can consider two different extensions which are actually related. The first consists of adding some holonomy information, and we extend $\mathbb{R}^3$ for each edge to $T^* \text{SO}(3) \cong \text{SO}(3) \ltimes \mathfrak{so}^*(3) \cong \text{SO}(3) \ltimes \mathbb{R}^3$. The chosen holonomy provides a way to transport vectors on a path dual to the edge. In this sense the holonomy contains some non-local information, which allow us to discuss about adjacent triangles/polygons sharing the edge dual to the holonomy.

The second extension consists of extending $\mathbb{R}^3$ to $\mathbb{R}^4 \sim \mathbb{C}^2$, where the extra degree of liberty is a phase. The new phase space is therefore parametrized in terms of a pair of complex numbers $(z^-, z^+) \in \mathbb{C}^2$, usually called spinor variables (since they will transform as spinor under the infinitesimal rotations). This is a more local picture, working for each triangle/polygon and it is very convenient to describe the algebra of observables for a given polygon [38, 34, 33]. This is the root of the spinor approach to loop quantum gravity [48, 49].

Let us focus on the first option.
**Flat triangle phase space.** Label the edges of the triangle by 1, 2, 3 and associate to each of them a vector \( \ell_i \in \mathbb{R}^3 \). Denote the center of the triangle by \( c \) and consider holonomies \( h_{ci} \in \text{SO}(3) \) joining the center to a point in the edge \( i \). Such path is called (half) link.

We take for phase space \( T^*\text{SO}(3) \cong \text{SO}(3) \times \mathbb{R}^3 \), with symplectic Poisson bracket generated by the coordinate functions \( \ell^I \) of \( \mathfrak{so}^*(3) \cong \mathbb{R}^3 \) and \( h \in \text{SO}(3) \):

\[
\{\ell^I, \ell^J\} = \epsilon^{IJ} \ell^K, \quad \{h^I, h^J\} = 0, \quad \{\ell^I, h^K\} = (h^I\ell^J)\ell^K.
\] (100)

We note that in the \( \ell \) sector (“momentum sector”) we have used the same Poisson bracket as used by Kapovich and Millson.

To construct the phase space for the full triangle, we consider three of these phase spaces and impose the constraint that the edges close.

\[
\ell_1 + \ell_2 + \ell_3 = 0.
\] (101)

Physically, the triangle phase space is nothing other than three spinning tops constrained to have a total angular momentum to be zero. The fact that geometry can be recovered from spinning tops with angular momentum conservation laws, was key to inception of the so-called spin networks by Penrose [62] to describe angular momentum to be zero. The fact that geometry can be recovered from spinning tops with angular momentum conservation laws, was key to inception of the so-called spin networks by Penrose [62] to describe a quantum space.

To generalize the construction to the curved case in particular, it is useful to remember that \( T^*\text{SO}(3) = \mathcal{D}_0 \) can be seen as a Heisenberg double. The group element \( \ell_1 + \ell_2 + \ell_3 \in \mathbb{R}^3 \) can be interpreted as the momentum map generating the global (left) rigid rotations denoted \( \text{SO}^t(3) \). Hence we can define a new phase space\(^{13}\) for the triangle \( t \) as

\[
\mathcal{P}_t^0 = (T^*\text{SO}(3))^\times 3 //\text{SO}^t(3) = \mathcal{D}_0^3 //\text{SO}^t(3).
\] (102)

**Curved triangle phase space.** We move on to the curved case, focusing on the triangle with negative curvature. We refer to [71] for the case of positive curvature. The triangle with negative curvature has been treated in [19, 21]. Let us review the construction which is relevant to us [21].

The triangle edges are given in terms of geodesics in the hyperboloid. Since the hyperboloid can be seen as the quotient \( \text{SL}(2, \mathbb{C})//\text{SU}(2) \cong \text{AN}(2) \). Such geodesics can be characterized by elements/holonomies in \( \text{AN}(2) \) which we will again denote by \( \ell \).

Our strategy to construct a phase space is based on using Heisenberg doubles. Hence we deform \( \mathbb{R}^3 \) into the non-abelian group \( \text{AN}(2) \). The fact that the triangle is closed is then recovered in the closure constraint

\[
\ell_1 \ell_2 \ell_3 = 1.
\] (103)

The new phase space associated to each edge is given by a deformation of the standard cotangent bundle into a Heisenberg double with curvature both in momentum and configuration space.

\[
\mathcal{D}_0 = T^*\text{SO}(3) \cong \text{SO}(3) \times \mathbb{R}^3 \rightarrow \mathcal{D} = \text{SL}(2, \mathbb{C}) \cong \text{SU}(2) \cong \text{AN}_2.
\] (104)

By definition, \( \ell_1 \ell_2 \ell_3 \) is still the momentum map generating the global (left) rotations \( \text{SU}^t(2) \). Hence we can again define the symplectic reduction to define the phase space for a curved triangle.

\[
\mathcal{P} = \text{SL}(2, \mathbb{C})^\times 3 //\text{SU}(2) \cong \mathcal{D}^3 //\text{SU}^t(2).
\] (105)

We can then extend the construction even more. We can consider for every pair of link/edge a Heisenberg double \( \mathcal{D} = G \bowtie G^* = G^* \bowtie G \), endowed with the ribbon factorization (14), with actions given by (15) and a symplectic form (16). By convention, \( G^* \) decorates the link while \( G^* \) decorate the edge and provide the information to reconstruct the geometry of the polygon. If \( G^* \) is abelian, we will say that the polygon is flat.

The phase space of the polygon with \( n \) edges is given by the symplectic reduction

\[
\mathcal{P} = \mathcal{D}^{(n)} //G^t,
\] (106)

\(^{13}\)We will note in the following \( \mathcal{D}^{(n)} \equiv \mathcal{D} \times ... \times \mathcal{D} = \mathcal{D}^n \).
where the constraint \( G^* \ni \ell_1 \ldots \ell_n = 1 \) is the momentum map generating the global left \( G \) transformations, denoted \( G^* \).

We note that once we generalize the construction to have possibly non-abelian groups both in configuration and momentum spaces, the construction becomes symmetric. In particular, we can implement a global translational symmetry instead of a global rotational symmetry by symplectic reduction over \( G^* \) which could be either of \( \mathbb{R}^3 \) or \( \mathsf{AN}_2 \) for example. The momentum map implementing the symplectic reduction is the flatness constraint \( g_1 \ldots g_n = 1, \ g_i \in \mathsf{SU}(2) \) [26].

Reference frames for 2d discrete geometry decorated by Heisenberg doubles. To glue triangles or polygons together to construct the phase space for a 2d cellular complex, it is useful to define the notion of a reference frame, to make sure we identify quantities associated to different triangles consistently.

Let us consider an edge \( e = (v_1, v_2) \) connecting vertices \( v_1 \) and \( v_2 \) and its dual, called a link, \( l = (v_1, v_2)^* = (c_1, c_2) \), which connects nodes \( c_1 \) and \( c_2 \).

As mentioned earlier, we decorate the links with \( G \) elements and the edges with \( G^* \) elements so that a Heisenberg double \( D \) decorates the (edge, link) pair.

**Representations.** Both the edge and the link carry an orientation with one vertex being the source or “initial vertex” of the edge and the other being the target or “final vertex”, and similarly for the nodes of a link. We can use representations of \( D \) to induce representations \((\varphi, \rho)\) of \( G \) and \( G^* \) respectively.

- On the initial node (source) of the link, \( c_1 \), we consider a vector space \( V_{c_1} \) which carries a representation \( \rho^{c_1} \) of the group \( G^* \) coming from the right decomposition \( D = G^* \bowtie G \).

\[
\ell \to \rho^{c_1}(\ell) \equiv \ell^{c_1}
\]

- On the final node (target) of the link, \( c_2 \), we consider a vector space \( V_{c_2} \) which carries a representation \( \rho^{c_2} \) of the group \( G^* \) coming from the left decomposition \( D = G \bowtie G^* \).

\[
\tilde{\ell} \to \rho^{c_2}(\tilde{\ell}) \equiv \tilde{\ell}^{c_2}
\]

- On the initial vertex (source) of the edge, \( v_1 \), we consider a vector space \( W_{v_1} \) which carries a representation \( \varphi^{v_1} \) of the group \( G \) coming from the right decomposition \( D = G \bowtie G^* \).

\[
\tilde{h} \to \varphi^{v_1}(\tilde{h}) \equiv \tilde{h}^{v_1}
\]

- On the final vertex (target) of the edge, \( v_2 \), we consider a vector space \( W_{v_2} \) which carries a representation \( \varphi^{v_2} \) of the group \( G \) coming from the left decomposition \( D = G^* \bowtie G \).

\[
h \to \varphi^{v_2}(h) \equiv h^{v_2}
\]

*Action and parallel transport.* The left or right mutual actions between \( G \) and \( G^* \) can be interpreted as parallel transport. For example, the relation

\[
\ell = \tilde{h} \triangleright \ell = \tilde{h}^{-1} \ell h^{-1} \iff \rho^{c_1}(\ell) = \varphi^{v_1}(h) \triangleright \rho^{c_2}(\tilde{\ell}) \quad (107)
\]

encodes how the edge variable is transported from the representation at \( c_2 \) to the representation at \( c_1 \), through the representation of \( h \) at \( v_1 \).

The left action of \( G \) on \( G^* \) (resp. \( G^* \) on \( G \)) transports the representation of the edge (resp. link) from the final node \( c_2 \) (resp. final vertex \( v_2 \)) to the initial node \( c_1 \) (resp. initial vertex \( v_1 \)) of the link (resp. edge); the right action of \( G \) on \( G^* \) (resp. \( G^* \) on \( G \)) transports the representation of the edge (resp. link) from \( c_1 \) to \( c_2 \) (resp. from \( v_1 \) to \( v_2 \)). We summarize the role of left and right actions in Table 1. Moreover, note that
The identification condition is therefore that the corresponding edge decorations equal in some common representation. This is done by identifying the nodes at the vertex $v_2$ to $v_1$; the right action of a link (resp. edge) on the edge (resp. link) transports the representation from $c_1$ to $c_2$ (resp. from $v_1$ to $v_2$).

| Left action $\triangleright$ | $c_2 \rightarrow c_1$ | $v_2 \rightarrow v_1$ |
|----------------------------|----------------------|----------------------|
| Right action $\triangleleft$ | $c_1 \rightarrow c_2$ | $v_1 \rightarrow v_2$ |

Table 1: Geometric interpretation of the left and right mutual actions between $G$ and $G^*$. Geometrically, the left action of a link (resp. edge) on the edge (resp. link) transports the representation of the edge (resp. link) from the node $c_2$ to $c_1$ (resp. from the vertex $v_2$ to $v_1$); the right action of a link (resp. edge) on the edge (resp. link) transports the representation from $c_1$ to $c_2$ (resp. from $v_1$ to $v_2$).

The left action of $\hat{h}$ (resp. $\ell$) is the inverse of the right action of $h$ (resp. $\hat{\ell}$):

$$
\ell = \hat{h} \triangleright \ell = \hat{h} \triangleright (\ell \triangleleft h), \quad h = \hat{h} \triangleleft \ell = (\ell \triangleright h) \triangleleft \hat{\ell},
$$

(108)

If we change the orientation of the link, $c_2 \rightarrow c_1$, which implies a change of orientation for the edge $v_2 \rightarrow v_1$, we use the inverse map for the full Heisenberg double.

$$
d = \ell h = \hat{h} \hat{\ell} \quad \rightarrow \quad d^{-1} = \hat{\ell}^{-1} \hat{h}^{-1} = h^{-1} \ell^{-1}.
$$

(109)

This agrees with the interpretation of left and right actions given in Table 1. For example, the relation

$$
\hat{\ell}^{-1} = h^{-1} \triangleright \ell^{-1} \iff \rho^e(\hat{\ell}^{-1}) = \phi^e(h^{-1}) \triangleright \rho^{e'}(\ell^{-1})
$$

(110)

encodes how the the inverse edge variable is transported from the representation at $c_1$ (final node of $h^{-1}$) to the representation at $c_2$ (initial node of $h^{-1}$), through the representation of the inverse link at $v_2$.

Specifying these different representations is also at the root of the spinor approach [48, 49]. Indeed in this case, one deals explicitly with the spinor representations and can reconstruct the full geometric data from the spinor data.

This identification of representation is important to understand how to glue/fuse terms. Indeed we need to identify the gluing constraint for terms living in the same representation.

**Phase space for a 2 dimensional triangulation.** We intend now to construct the phase space for a triangulation. The main idea is to glue/fuse triangles along common edges and build from there the total phase space. Since an edge is shared by a pair of triangles, the gluing is done pairwise.

Consider two triangles dual to the nodes $c_1$ and $c_2$. The triangle $c'_1$ has an edge denoted $e$ and the triangle $c'_2$ has an edge denoted $e'$. We denote by $c_e$ (resp. $c_{e'}$) the intersection of edge $e$ (resp. $e'$) and its dual half link, so that the half link dual to the edge $e$ (resp. $e'$) is denoted $(c_1c_e)$ (resp. $(c_2c_{e'})$). We denote by $D_{c_1,e} \cong G \bowtie G^*$ (resp. $D_{c_2,e'} \cong G \bowtie G^*$) the Heisenberg double associated to the edge $e$ and half link $(c_1c_e)$ (resp. to the edge $e'$ and half link $(c_2c_{e'})$).

We therefore have the pair of ribbon constraints

$$
\tilde{D}_{c_1,e} \ni \ell_1 h^{v_2} = \tilde{h}^{v_1} \tilde{\ell}_e, \quad \tilde{D}_{c_2,e'} \ni \ell_2 h^{v_2} = \tilde{h}^{v_1} \tilde{\ell}_e' .
$$

(111)

To glue $c'_1$ and $c'_2$ we must identify $e = (v_1v_2)$ and $e' = (v'_1v'_2)$. To make the identification, we must set the corresponding edge decorations equal in some common representation. This is done by identifying the nodes $c_e$ and $c_{e'}$. This means that the orientation of the edges being identified are opposite,

$$
e = -e' \rightarrow v_1 = v'_2, \quad v_2 = v'_1.
$$

(112)

As such, at the node $c_{e'}$, we have a representation $\rho^{c_{e'}}(\tilde{\ell}_1^{-1} \tilde{v}_{1v_2})$ and at $c_e$ we have the representation $\rho^{c_{e}}(\tilde{h}_{v_1v_2})$. The identification condition is therefore that

$$
\tilde{\ell}_e = \tilde{\ell}_e' \tilde{v}_{v_1v_2}, \quad \tilde{h}_e = \tilde{\ell}_e'^{-1}.
$$

(113)

(114)
Figure 12: The phase space of a triangle is given by a set of three ribbons. The phase space of two triangles is obtained by gluing ribbons associated to the shared edge.

We recover the constraint that we used to glue ribbons in Proposition 2. This is the momentum generating the left global $G$ transformations and we can perform the symplectic reduction.

\[(D_{c_1,e} \times D_{c_2,e'})/G \cong D_{c_1 c_2, e} = D_{l,e}, \quad l = (c_1 c_2).\]  

In particular, putting the two ribbon constraints together with the gluing constraint, we identify the reduced variables,

\[
\begin{align*}
\ell^{v_1} h^{v_2} &= \ell^{c_{v_1}} h^{c_{v_2}}, \\
(h^{c_{v_2}})^{-1} (\ell^{c_{v_1}})^{-1} h^{v_1} &= (\ell^{c_{v_1}})^{-1} (\ell^{c_{v_2}})^{-1} h^{v_2} \quad \Rightarrow \quad \ell^{c_{v_1}} h^{v_2} = \ell^{c_{v_2}} h^{v_1}, \\
\ell_{v_1}^{v_2} &= \ell_{v_2}^{v_1}, \\
h_{v_1}^{v_2} &= (\ell_{v_1}^{v_2})^{-1}, \\
\tilde{h}_{v_1}^{v_2} &= (\tilde{h}_{v_2}^{v_1})^{-1}.
\end{align*}
\]

The phase space obtained after gluing is simply the Heisenberg double with $G$ decorations on the (full) link $l = (c_1 c_2)$ and $G^*$ decorations on the edge $e = (v_1 v_2)$. See Fig. 12. Note that a similar construction was done at the quantum level using fusion product in [26]. Note also that, unsurprisingly, we have recovered the variables obtained after discretization of 3d gravity with a non-zero cosmological constant [31]. These variables are also called triangle operators in Kitaev’s model [42].

By repeating this gluing we can recover the full phase space $P^{\text{tot}}$ for the 2d triangulation $\mathcal{T}$ with dual $\mathcal{T}^*$.

\[
P^{\text{tot}} = \left( \bigotimes_{t \in \mathcal{T}^*} D_{t,e} \right) / \left( \bigotimes_{t \in \mathcal{T}} G^t \right),
\]

where $D_{t,e}$ are the Heisenberg doubles for each full link of the dual complex of the triangulation and $G^t$ is the global $G$ transformation induced by the closure constraint of each triangle $t$ of the triangulation.

Dynamics can then be implemented by imposing some constraint on the closed $G$ holonomies. From the usual gravity picture, this would be typically the flatness constraint. If we swap what we call $G$ and $G^*$, dynamics would be implemented by implementing that closed $G^*$ holonomies should be flat, this would be implementing the flatness constraint.

### 3.2 Polyhedron phase space using groups

Minkowski proved in 1897 that given a set of vectors $\ell_i \in \mathbb{R}^3$ which close to form a (non-planar) polygon,

\[
\sum_i \ell_i = 0
\]

then we can uniquely (up to translations and rotations) reconstruct a polyhedron, provided we interpret these vectors $\ell_i$ as the normals of the (flat) faces of the polyhedron, and their magnitude to be the area of
the relevant face [3]. Given such set of vectors, we can use the Lasserre reconstruction program — revisited in [17], [69] — to reconstruct the polyhedron.

For our concern, it means that the polygon data is sufficient to reconstruct a polyhedron. Hence, all the phase space structures we discussed in the previous section can be also used to describe a polyhedron or a 3d triangulation (or a 3d cellular decomposition).

This equivalence between polyhedron and polygon reappears at the the quantum level in the ambiguity of interpreting an intertwiner. If one does not specify the dimension at first, we can interpret it either a quantum state of a polygon or of a polyhedron. This extends for the full spin network, it can encode the quantum state of a 2d triangulation or of a 3d triangulation.

Let us sketch how the construction of the phase space works for a 3d triangulation $T$ and its dual $T^\ast$.

We focus first on a link and its dual, given by the triangle. We consider the Heisenberg double $D \cong G \bowtie G^\ast$, with both $G$ and $G^\ast$ three dimensional Lie groups. $G$ elements still decorate the links in $T^\ast$, which are dual to the triangles/faces in the triangulation; hence, $G^\ast$ elements decorate the faces. There are no decorations on the edges of the triangle meaning that $G^\ast$ must be abelian in order to have a consistent way of composing face decorations, due to the Eckmann-Hilton argument, see Fig 13. The Heisenberg double must therefore be of the type $D \cong T^\ast G$ (if we deal with a Lie group $G$). In the 3d case, the magnitude of the vector in $\ell \in G^\ast$ encodes the area instead of a length in 2d, and the direction of the vector itself, instead of being the edge direction, will indicate the (outgoing) direction normal to the face.

Given four links rooted at the common root $c$, we interpret the constraint (119) as the closure of the tetrahedron. From the phase space perspective, the closure of the polyhedron encoded in the constraint (119) is still a constraint on the momentum map generating $G$ left global transformations, which we denote $G^\tau$. The phase space of a tetrahedron $\tau$ can then be recovered from the symplectic reduction

$$\mathcal{P} = (T^\ast G)^{\times 4} / G^\tau.$$ (120)

Physically, the tetrahedron phase space is nothing else than four spinning tops constrained to have a total angular momentum to be zero. The difference between the phase space of a quadrilateral (ie a polygon) is simply in the way we geometrically interpret the angular momentum $\ell$. In the polygon case, it is associated to the edge, while in the tetrahedron case, it is associated to the normal of the face.
To get the full phase space of a 3d triangulation in this context, we glue tetrahedra by identifying the face information. The mathematical description is exactly the same as for gluing triangles. This is again the ambiguity between 2d and 3d showing.

One might argue that this ambiguity between 2d and 3d shows that such description of the phase space of a 3d triangulation is not the best one. We are likely missing some information for the 3d description. Furthermore, for a gravity point of view it is a bit of a draw-back to not have access to some edge decoration, which should encode some information about the metric. Finally, to actually define the notion of curved polyhedron, we need to have some data decorating its edges, just like we defined a curved triangle by decorating its edges with a non-abelian group.\footnote{\[40, 65\] generalized the Minkowski theorem with non-abelian group decorations apparently solely on the faces. However, it is likely that some edge dependence was actually hidden \[67\].}

All these arguments point to the need to decorate the edges of the triangulation.

In order to decorate both the faces and the edges, we require compatibility rules such as those given by the 2-group structure in Sec. 2.2. 2-groups have been argued to be the natural generalization of the notion of (quantum) groups to discuss topological invariant of a 4d manifold (hence in terms of states describing 3d triangulations). Just as quantum groups are the natural object to deal with (homogeneously) curved 3d manifold, one might expect that quantum 2-groups will be the right structure to deal with (homogeneously) curved 3d manifold. Finally, it was argued in \[61, 5\] that a maybe more adequate construction of spin foam model for 4d gravity could be done using 2-group structure.

As a summary, it seems natural to construct the notion of phase space for a 3d triangulation using 2-groups. At this time, we will still be focusing on abelian decoration on the faces, that is we focus on simple 2-groups. As a consequence, we will not put curvature on the triangles yet. Nevertheless, we expect that our set up could be generalized to allow curved triangles.

4 Polyhedron phase space from skeletal 2-groups

4.1 General construction

In this section we explain how to build the phase space of a 3 dimensional (curved) cellular decomposition as the Heisenberg double $\mathcal{B} = \mathcal{G} \bowtie \mathcal{G}^* \cong \mathcal{G}^* \bowtie \mathcal{G}$, given by the double cross product of two skeletal crossed modules, $\mathcal{G}$ and $\mathcal{G}^*$, as explained in Sec. 2.2.

According to the 2-group picture introduced in Sec. 2.2, we can interpret each crossed module as decorating 1d and 2d objects, as part of the cellular decomposition $\mathcal{T}$ and its dual $\mathcal{T}^*$. The skeletal crossed module $\mathcal{G}^*$ decorates $\mathcal{T}$ and $\mathcal{G}$ decorates $\mathcal{T}^*$.

Just like for the 2d case, it is convenient to fix the reference frame conventions in order to glue/fuse structures in a consistent manner.

4.1.1 Reference frames for 3d geometries decorated by skeletal crossed modules.

We consider an (oriented) edge $e = (v_1 v_2)$, a face $t_e$ which contains the edge $v_1$ and $v_2$ in its boundary. Dually we deal with a node $n_1$, and the node $n_2$ which is a point somewhere on the interior of the face $t_e$. The wedge $w_l$ is a dual face which contains $l = (c_1 c_2)$, called a (half) link, in its boundary. This set up is shown in Fig. 14.
In 2-group theory, changing the decompositions of $G$ face (resp. wedge) rooted either in the source or in the target of the edge (resp. link). The different $G$ decompositions of model [42].

These 2-representations can be seen as the generalization of the triangular operators appearing in the Kitaev model [42].

The skeletal crossed modules $G$ and $G^*$, in turn, can be split in the left or right decompositions. Different decompositions of $G$ (resp. $G^*$) are decoration of the same pair edge/face (resp. link/wedge), with the face (resp. wedge) rooted either in the source or in the target of the edge (resp. link). The different decompositions of $G$ and $G^*$ are distinguished by use of an overline ($\overline{\cdot}$). In 2-group theory, changing the decomposition of a crossed module can be viewed as whiskering, see Fig. 9.

$$B = G^* \bowtie G = G \bowtie G^* \quad \rightarrow \quad \ell h = \tilde{h} \tilde{\ell}$$

Since there is no back action of $G_1^*$ on $G_2$ (resp. of $G_2^*$ on $G_1$), we note that $u = \tilde{u}$, $\tilde{u} = \tilde{u}$, $\lambda = \tilde{\lambda}$ and $\tilde{\lambda} = \tilde{\lambda}$.

---

**Figure 14:** The building block we will glue to reconstruct a full 3d triangulation. The (half) link and wedge will be decorated by $G$ elements, while the edge and face will be decorated by elements in $G^*$.
Table 2: List of the group elements used to decorate edges, links, faces and wedges. Decorations of links (resp. edges) are labelled by a single index, related to vertex (resp. node) where the link (resp. edge) is represented. Decorations of wedges (resp. faces) are labelled by an index that specifies the vertex (resp. node) where it is represented, and a subscript that specifies the node (resp. vertex) where it is rooted.

| Source of $l$: $c_1$ | Target of $l$: $c_1$ |
|----------------------|----------------------|
| Source of $e$: $v_1$ | $\ell^c = \beta_{\ell_1}^c \chi^{c_1}$, $\hat{h}^{c_1} = \tilde{g}_{v_1}^c \tilde{u}^{c_1}$ | $\hat{\ell}^{c_1} = \beta_{\ell_1}^{c_1} \hat{\lambda}^{c_1}$, $\tilde{h}^{c_1} = \tilde{u}^{c_1} \tilde{g}_{v_1}^{c_1}$ |
| Target of $e$: $v_2$ | $\ell^c = \chi^c \beta_{\ell_2}^c$, $h^{v_2} = \tilde{g}_{v_1}^{v_2} u^{v_2}$ | $\hat{\ell}^{c_1} = \chi^{c_1} \beta_{\ell_2}^{c_1}$, $\tilde{h}^{v_2} = u^{v_2} \tilde{g}_{v_1}^{v_2}$ |

• Take the skeletal crossed module $G$ that decorates a pair link/wedge.
  
  – The right decomposition $G = G'_2 \rtimes G_1$ implies that $G'_2$ elements (decoration of wedges) are rooted at the source of the link (node $c_1$).
  – The left decomposition $G = G_1 \rtimes G'_2$ implies that $G_2$ elements (decoration of wedges) are rooted at the target of the link (node $c_t$).

• Take the skeletal crossed module $G^*$ that decorates a pair edge/face.
  
  – The right decomposition $G = G^*_1 \rtimes G_2$ implies that $G^*_1$ elements (decoration of faces) are rooted at the source of the edge (vertex $v_1$).
  – The left decomposition $G = G_2 \rtimes G^*_1$ implies that $G^*_2$ elements (decoration of faces) are rooted at the target of the edge (vertex $v_2$).

Therefore, the choice of decoration of an edge (resp. link) is specified by a single node (resp. vertex) as in the 2d case; while decorations of faces and wedges are specified by a pair of node and vertex. In Table 2 we list the decorations for the pair edge/face, as well as the pair link/wedge, according to the choice of representation and root.

**Action and parallel transport.** In analogy with the geometric interpretation given in the 2 dimensional case, here the left action of $G$ on $G^*$ (resp. $G^*$ on $G$) transports the decoration of the edge and face (resp. link and wedge) from the target $c_t$ to the source $c_1$ of the link (resp. from the target $v_2$ to the source $v_1$ of the edge), and similarly the right action of $G$ on $G^*$ (resp. $G^*$ on $G$) transports the decoration of edge and face (resp. link and wedge) from $c_1$ to $c_t$ (resp. from $v_1$ to $v_2$).

Since we deal with parallel transport between vertices and nodes (sources and targets of edges and links), the action responsible for the parallel transport between nodes is that of the group $G_1 \subset G$ on $G^*$ (rather than the action of $G$ on $G^*$) and the action responsible for the parallel transport between vertices is that of the group $G_2 \subset G^*$ on $G$ (rather than the action of $G^*$ on $G$).

Moreover, the left action of $G_1$ on $G'_2$ (resp. $G_2$ on $G^*_1$) transports the root of the wedge (resp. face) from the target of the link $c_t$ to its source $c_1$ (resp. from the target of the edge $v_2$ to its source $v_1$) and the right action transports the root of the wedge (resp. face) from the source $c_1$ to the target $c_t$ (resp. from the source $v_1$ to the target $v_2$).

We summarize the geometric role of left and right actions in Table 3. Note that the geometric interpretation given in Table 3 agrees with the convention in Table 2.

**Gluing.** According to the notation in Table 2, decorations of links and edges carry a single index while decorations of faces and wedges carry two indices.

To fuse phase spaces (Heisenberg doubles), we impose a geometric constraint, identifying some objects in the two phase spaces. In order to properly impose these conditions, the decorations of such objects need to be represented and rooted in the same reference frame, therefore we also demand that all the indices of such decorations match.
Table 3: Geometrically, the left action of $G_1$ (resp. $G_2$) on the other groups, transports the decoration of the objects from the node $c_2$ to $c_1$ (resp. from the vertex $v_2$ to $v_1$); the right action of $G_1$ (resp. $G_2$) on the other groups, transports the decoration from $c_1$ to $c_2$ (resp. from $v_1$ to $v_2$).

When $B$ is decorating the pair ((edge, face); ((half-)link, wedge)), we will call it the atomic phase space. We will glue/fuse such atomic phase spaces to construct the phase space associated to the full cellular complex.

The atomic phase space is encoded in the octagonal ribbon constraint, as in Fig. 3. We note that there are actually many possible constraints, depending on the choice of representation of each group.

As in the 2 dimensional case the symplectic structure was given in terms of a symplectic form, here the symplectic structure for the atomic phase space is induced by the form

$$\Omega = \frac{1}{2} \left( \langle \Delta t \wedge \Delta \hat{h} \rangle + \langle \Delta \hat{t} \wedge \Delta \hat{h} \rangle \right) = \frac{1}{2} \left( \langle \Delta (\beta \lambda) \wedge \Delta (\hat{y} \hat{u}) \rangle + \langle \Delta (\lambda \hat{\beta}) \wedge \Delta (uy) \rangle \right)$$

$$= \frac{1}{2} \left( \langle (\delta \beta + \Delta \lambda + [\beta, \Delta \lambda]) \wedge (\delta \hat{y} + \Delta \hat{u} + [\hat{y}, \Delta \hat{u}]) \rangle + \langle (\Delta \lambda + \delta \hat{\beta} - [\hat{\beta}, \Delta \hat{\lambda}]) \wedge (\Delta u + \delta y - [y, \Delta u]) \rangle \right)$$

$$= \frac{1}{2} \left( \langle (\delta \beta - [\beta, \Delta \lambda]) \wedge \Delta \hat{u} \rangle + \langle \Delta \lambda \wedge (\delta \hat{y} - [\hat{y}, \Delta \hat{u}]) \rangle + \langle (\delta \hat{\beta} - [\hat{\beta}, \Delta \hat{\lambda}]) \wedge \Delta u \rangle + \langle \Delta \hat{\lambda} \wedge (\delta y - [y, \Delta u]) \rangle \right).$$

Let us now define the ways to glue/fuse the different geometric objects.

### 4.1.2 Gluing rules

In the previous subsection, we characterized how the Heisenberg double of a skeletal 2-group, what we called the atomic phase space, decorates the single building block of our phase space, composed by edge, face, link and wedge, see Fig. 14.

To build the full phase space for a polyhedron or a 3d cellular complex, we will glue these atomic phase spaces to build the phase space for the structure of interest. The gluing can be decomposed according to four types of gluing/fusion. We recall from the previous discussion that the gluing comes from the identification of the variables dual to the ones we want to glue. The gluing operations, achieved through symplectic reduction, are associative and commutative.

- **Link gluing.** Links are decorated by $G_1$ elements, so this gluing is performed by identifying elements in $G_1^*$. We perform a symplectic reduction with respect to the $G_1$ symmetry.
- **Face gluing.** Faces are decorated by $G_1^*$ elements, so this gluing is performed by identifying elements in $G_1$. Note that this amounts to perform a vertical product from the 2-group picture. We perform a symplectic reduction with respect to the $G_1^*$ symmetry.
• **Wedge gluing.** Wedges are decorated by $G_2^*$ elements, so this gluing is performed by identifying elements in $G_2^*$. Note that this amounts to perform a vertical product from the 2-group picture. We perform a symplectic reduction with respect to the $G_2^*$ symmetry.

• **Edge gluing.** Edges are decorated by $G_2$ elements, so this gluing is performed by identifying elements in $G_2$. We perform a symplectic reduction with respect to the $G_2$ symmetry.

Upon performing the symplectic reduction, we can construct the new ribbon constraints to determine the symplectic structure in terms of the fused variables. We emphasize however that when we keep repeating the different gluings, the ribbon equations will be mostly useful for expressing all the variables in the right frame and root. This will be useful for example to define the closure constraints where we must ensure that the different variables are expressed in the right frame (and properly rooted).

Let us illustrate explicitly these gluings. We consider the atomic phase space $B_{c_1,e}$ associated to the edge $e = (v_1v_2)$, face $t_e$ and (half) link $l = (c_1c_t)$, wedge $w_l$ (resp. $B_{c_2,e'}$ associated to the edge $e' = (v'_1v'_2)$, face $t'_e$ and half link $l' = (c_2c'_t)$, wedge $w_{l'}$), as in Fig. 14.

For the sake of clarity, it is useful to introduce the following notation. Consider the cross Lie bracket (79). Following the Def. 3 of a classical double, the Lie bracket between elements of $g_1$ and $g_1^*$ (resp. $g_2$ and $g_2^*$) are constructed as the mutual co-adjoint actions between $G_2^* \trianglerightarrow g_1$ and $g_2^* \trianglerightarrow G_1^*$; however, these brackets are not stable in $g_1$ and $g_1^*$ (resp. $g_2$ and $g_2^*$). At the group level, the exponentiation of such Lie brackets leads to the group conjugations

$$u\beta u^{-1} = \beta' \leftrightarrow \begin{cases} \beta' = (u\beta u^{-1})|_{G_1^*}, \\ y' = (u\beta u^{-1})|_{G_2^*}, \end{cases}$$

$$\lambda y^{-1} = y'' \leftrightarrow \begin{cases} \beta'' = (\lambda y^{-1})|_{G_1^*}, \\ y'' = (\lambda y^{-1})|_{G_2^*}. \end{cases}$$

Therefore, in order to express the conjugations $u\beta u^{-1}$ and $\lambda y^{-1}$, we will use their projections into the groups $G_1^*$ and $G_2^*$.

![Figure 15: We fuse two links by identifying the dual faces $t_e$ and $t_{e'}$. We require that the points $c_t$ and $c'_{t'}$ in the respective face to match. As we discussed before, face decorations are rooted at a point. Matching the face decorations imposes that they have the same root. Picking $v_2$ and $v'_2$ as respectively root of $t_e$ and $t_{e'}$, we therefore must have $v_2 = v'_2$.](image)

**Link gluing.** The gluing of two links, as illustrated in Fig. 15, is done by identifying the face data dual to each link.

The geometric identification consists in setting\(^{15}\)

$$t_e = -t_{e'}.$$  \hspace{1cm} (124)

\(^{15}\)Faces are identified with opposite outgoing directions.
We enforce this geometric constraint as a condition on \( G_1^* \) elements. Let us represent the decorations of the faces \( t_e \) and \( t_{e'} \) at the nodes \( c_t \) and \( c_{t'} \). We root them respectively at the vertices \( v_2 \) and \( v_2' \).

\[
B_{t,e} : \quad G^* = G_2 \times G_1^* \ni \tilde{e}t_e = \tilde{e}_1 = \tilde{\lambda}_1 \tilde{\beta}_1 \equiv \tilde{\lambda}^{e_t} \tilde{\beta}^{e_t}, \quad G = G_1 \times G_2^* \ni h^{v_2} \equiv h_1 = u_1 \ y_1 \equiv u^{v_2} y^{v_2},
\]

\[
B_{t',e'} : \quad G^* = G_2 \times G_1^* \ni \tilde{e}t_{e'} = \tilde{e}_2 = \tilde{\lambda}_2 \tilde{\beta}_2 \equiv \tilde{\lambda}^{e_{t'}} \tilde{\beta}^{e_{t'}}, \quad G = G_1 \times G_2^* \ni h^{v_2} \equiv h_2 = u_2 \ y_2 \equiv u^{v_2} y^{v_2}.
\]  

(125)

In order to express the face decorations in the same reference frame, their indices have to match, hence we further demand that

\[
c_t = c_{t'}, \quad v_2 = v'_2.
\]  

(126)

The condition \( c_t = c_{t'} \) provides the gluing the half links. The proper geometric condition for the face decorations is encoded in the identification

\[
\tilde{\beta}^{e_t}_{v_2} = (\tilde{\beta}^{e_{t'}})_{v_2}^{-1} \Leftrightarrow \tilde{\beta}_1 = \tilde{\beta}_2^{-1}.
\]  

(127)

Using the above constraint and the Heisenberg ribbon equation, we obtain the extended ribbon equation

\[
i = 1, 2, \quad \ell_i h_i = \tilde{h}_i \tilde{e}_i = \tilde{h}_i (\lambda_i \tilde{\beta}_i) \Leftrightarrow \tilde{\beta}_1 = \tilde{h}_1^{-1} \lambda_1^{-1} c_i h_i \quad \tilde{\beta}_2 = \tilde{h}_2^{-1} \lambda_2^{-1} c_i h_i = \tilde{h}_2^{-1} \lambda_2^{-1} \lambda_1 h_1 = \tilde{h}_2^{-1} \lambda_2^{-1} \lambda_1 \tilde{h}_2
\]

\[
\Leftrightarrow \tilde{\lambda}_1^{-1} (\tilde{u}_1 \tilde{y}_1)^{-1} (\lambda_1 \tilde{\beta}_1) (\tilde{y}_1 u_1) = (\tilde{y}_2 u_2)^{-1} (\lambda_2 \tilde{\beta}_2) (\tilde{y}_2 u_2)^{-1} (\tilde{u}_2 \tilde{y}_2) \lambda_2
\]

\[
(\lambda_1 \tilde{\beta}_1) (\tilde{y}_1 u_1) \tilde{\lambda}_2^{-1} (\tilde{u}_2 \tilde{y}_2)^{-1} = (\tilde{u}_1 \tilde{y}_1) \tilde{\lambda}_1 (\tilde{y}_2 u_2)^{-1} (\lambda_2 \tilde{\beta}_2)^{-1} \lambda_1 \tilde{\beta}_1 (u_1 \tilde{\lambda}_2^{-1}) \tilde{y}' \tilde{\beta}' ((u_1 \tilde{\lambda}_2^{-1}) \tilde{y}_2^{-1} (u_1 \tilde{\lambda}_2^{-1}) \tilde{u}_2^{-1} = \tilde{u}_1 \tilde{y}_1 (\tilde{\lambda}_1 \tilde{\beta}_1 (u_1 \tilde{\lambda}_2^{-1}) \tilde{y}_2^{-1} \tilde{u}_2^{-1}) \tilde{\lambda}_1 \tilde{\beta}_1 (u_1 \tilde{\lambda}_2^{-1}) \tilde{y}_2^{-1} \tilde{u}_2^{-1}
\]

\[
\lambda_1 \tilde{\beta}_1 (u_1 \tilde{\lambda}_2^{-1}) y' \tilde{\beta}' ((u_1 \tilde{\lambda}_2^{-1}) \tilde{y}_2^{-1} (u_1 \tilde{\lambda}_2^{-1}) \tilde{u}_2^{-1} = \tilde{u}_1 \tilde{y}_1 (\tilde{\lambda}_1 \tilde{\beta}_1 (u_1 \tilde{\lambda}_2^{-1}) \tilde{y}_2^{-1} \tilde{u}_2^{-1}) (\tilde{y}_1 u_1 \beta'' \tilde{\lambda}_1 u_1 \tilde{\lambda}_2^{-1} \tilde{\beta}_2^{-1} \lambda_2^{-1},
\]

(128)

\[
(\lambda_1 (u_1 \tilde{\lambda}_2^{-1})) (\tilde{\beta}_1 \tilde{u}_2^{-1}) ((\tilde{y}_1 u_1 \tilde{\lambda}_2^{-1}) \tilde{y}' ((u_1 \tilde{\lambda}_2^{-1}) \tilde{u}_2^{-1}))) \quad (\tilde{u}_1^{-1} \tilde{y}_1 u_1 \beta'' \tilde{\lambda}_1 u_1 \tilde{\lambda}_2^{-1} \tilde{\beta}_2^{-1} \lambda_2^{-1}),
\]

with \( (u_1 \tilde{\lambda}_2^{-1})^{-1} \tilde{y}_1 (u_1 \tilde{\lambda}_2^{-1}) = y' \tilde{\beta}' \) and \( (\tilde{\lambda}_1 u_1 \tilde{\lambda}_2^{-1}) \tilde{y}_2 (\tilde{\lambda}_1 u_1 \tilde{\lambda}_2^{-1})^{-1} = y'' \tilde{\beta}'' \). The link fusion is therefore obtained through the symplectic reduction

\[
(B_{c_1,e} \times B_{c_2,e'}) / G_1
\]  

(129)

with momentum map \( \tilde{\beta}^{e_t}_{v_2} \tilde{\beta}^{e_{t'}}_{v_2} = \tilde{\beta}^{e_t}_{v_2} + \tilde{\beta}^{e_{t'}}_{v_2} \in G_1^* \) and fused link variable \( \tilde{u}_i^{v_1} = \tilde{u}_i^{v_1} (\tilde{\lambda}^{e_t} (u^{v_2})^{-1}) \in G_1 \).

Figure 16: We fuse two wedges by identifying the edges \( e \) and \( e' \), so that they have opposite orientation. To fuse the wedges, we require that the two wedges are rooted at the same point, so we identify the nodes \( c_1 \) and \( c_2 \).

**Wedge gluing Part 1.** The gluing of two wedges, as illustrated in Fig. 16, is done by identifying the edge data, dual to each wedge.
The geometric identification consists in setting\textsuperscript{16}
\[ e = -e' \iff v_1 = v'_2, \ v_2 = v'_1. \] (130)

We enforce the geometric constraint as a condition on \( G_2 \) elements. Let us represent the decorations of the edges \( e \) and \( e' \) at the nodes \( c_1 \) and \( c_2 \).

\[
\begin{align*}
B_{1,e} : \ G^* = G_2 \times G_1^* \ni \ell_1 &\equiv \ell_1 = \lambda_1 \tilde{\lambda}_1 \equiv \lambda^c_1 \tilde{\lambda}^c_1, \quad G = G_2^* \times G_1 \ni \tilde{h} \equiv \tilde{h}_1 = \tilde{y}_1 u_1 \equiv \tilde{y}^v_1 \tilde{u}^v_2, \\
B_{1,e'} : \ G^* = G_1^* \times G_2 \ni \ell_2 &\equiv \ell_2 = \beta_2 \lambda_2 \equiv \beta^c_2 \lambda^c_2, \quad G = G_2^* \times G_1 \ni \tilde{h} \equiv \tilde{h}_2 = \tilde{y}_2 u_2 \equiv \tilde{y}^{v'}_2 \tilde{u}^{v'}_1.
\end{align*}
\] (131)

In order to express the edge decorations in the same reference frame, their indices have to match, hence we further demand that
\[ c_1 = c_2. \] (132)

The proper geometric condition is encoded in the identification
\[ \lambda^c_1 = (\lambda^c_2)^{-1} \iff \lambda_1 = \lambda_2^{-1}. \] (133)

Using the above constraint and the Heisenberg ribbon equation, we obtain the extended ribbon equation
\[
\begin{cases}
\tilde{h}_1 \ell_1 = \ell_1 h_1 = (\lambda_1 \tilde{\lambda}_1) h_1 &\iff \lambda_1 = \tilde{h}_1 \ell_1 h_1^{-1} \tilde{\lambda}_1^{-1} = h_2^{-1} \tilde{\lambda}_1^{-1} \tilde{h}_1^{-1} \tilde{\lambda}_1 = h_2^{-1} \tilde{h}_2^{-1} \tilde{\lambda}_2^{-1} \\
\tilde{h}_2 \ell_2 = \ell_2 h_2 = (\beta_2 \lambda_2) h_2 &\iff \lambda_2 = \tilde{h}_2 \ell_2 h_2^{-1} \tilde{\lambda}_2^{-1} = h_2^{-1} \tilde{h}_2^{-1} \tilde{\lambda}_2^{-1} = h_2^{-1} \tilde{h}_2^{-1} \tilde{\lambda}_2^{-1}
\end{cases}
\] (134)

with \( \tilde{u}_2^{-1} \tilde{\beta}_1 \tilde{u}_2 = (y' \beta')^{-1} \) and \( \tilde{u}_1^{-1} \tilde{\beta}_2 \tilde{u}_1 = y'' \beta''. \) The wedge gluing is therefore obtained through the symplectic reduction
\[
(B_{c_1,e} \times B_{c_2,e'})//G_2^* \] (135)
with momentum map \( \lambda^c_1 \lambda^c_2 \in G_2 \) and fused wedge variable \( (\tilde{y}^{v''}_2)^2|_{c_1} = -\tilde{y}^{v''}_1 \tilde{u}^{v''}_2 + \tilde{y}^{v''}_1 \tilde{u}^{v''}_2 - (\tilde{u}^{v''}_2)^2 |_{G_2^*} \in G_2^* \).

**Wedge gluing Part 2.** The gluing of wedges is done by identifying the edge data dual to each wedge. It is mostly relevant for building the holonomy around an edge.

The geometric identification consists in setting\textsuperscript{17}
\[ e = e' \iff v_1 = v'_2, \ v_2 = v'_1. \] (136)

We enforce the geometric constraint as a condition on \( G_2 \) elements. Let us represent the decorations of the edges \( e \) and \( e' \) at the nodes \( c_1 \) and \( c_2 \).

\[
\begin{align*}
B_{1,e} : \ G^* = G_2 \times G_1^* \ni \ell_1 &\equiv \ell_1 = \tilde{\beta}_1 \lambda_1 \equiv \tilde{\beta}^c_1 \lambda^c_1, \quad G = G_2^* \times G_1 \ni \tilde{h} \equiv \tilde{h}_1 = \tilde{y}_1 \tilde{y}_1 \equiv \tilde{y}^v_1 \tilde{y}^v_1, \\
B_{1,e'} : \ G^* = G_1^* \times G_2 \ni \ell_2 &\equiv \ell_2 = \beta_2 \lambda_2 \equiv \beta^c_2 \lambda^c_2, \quad G = G_2^* \times G_1 \ni \tilde{h} \equiv \tilde{h}_2 = \tilde{y}_2 \tilde{u}_2 \equiv \tilde{y}^{v'}_2 \tilde{u}^{v'}_1.
\end{align*}
\] (137)

So that the variables we wish to identify are in the same representation, we set \( c_2 = c_1 \). The proper geometric condition is encoded in the identification
\[ \tilde{\lambda}^c_1 = \lambda^c_2 \iff \tilde{\lambda}_1 = \lambda_2. \] (138)

\textsuperscript{16}In this example the edges are identified with opposite orientations.
\textsuperscript{17}In this example the edges are identified with the same orientation.
Using the above constraint and the Heisenberg ribbon equation, we obtain the extended ribbon equation

\[
\begin{align*}
\ell_1 h_1 = \tilde{h}_1 \tilde{\ell}_1 = \tilde{h}_1 (\tilde{\beta}_1 \lambda_1) \quad &\iff \lambda_1 = \tilde{\beta}_1^{-1} h_1^{-1} \ell_1 h_1 \quad \lambda_1 = \lambda_2 \quad \tilde{\beta}_1^{-1} h_1^{-1} \ell_1 h_1 = \tilde{h}_2 \tilde{\ell}_2 h_2^{-1} \tilde{\beta}_2 \\
\tilde{h}_2 \tilde{\ell}_2 = \ell_2 h_2 = (\lambda_2 \tilde{\beta}_2) h_2 \quad &\iff \lambda_2 = \tilde{h}_2 \tilde{\ell}_2 h_2^{-1} \tilde{\beta}_2^{-1}
\end{align*}
\]

\[
\Rightarrow \quad \tilde{\beta}_1^{-1} (\tilde{u}_1 \tilde{y}_1)^{-1} (\lambda_1 \tilde{\beta}_1) (\tilde{y}_1 u_1) = (\tilde{u}_2 \tilde{y}_2) (\tilde{\beta}_2 \lambda_2) (\tilde{y}_2 u_2)^{-1} \tilde{\beta}_2 \\
(\lambda_1 \tilde{\beta}_1) (\tilde{y}_1 u_1) (\tilde{y}_2 u_2) = (\tilde{u}_1 \tilde{y}_1) (\tilde{\beta}_1 \lambda_1) (\tilde{u}_2 \tilde{y}_2) (\tilde{\beta}_2 \lambda_2) \\
\lambda_1 \tilde{\beta}_1 \tilde{y}_1 \tilde{y}' u_1 \tilde{y}_2 u_2 = \tilde{u}_1 \tilde{y}_1 \tilde{u}_2 \tilde{y}' \tilde{y}_2 \tilde{\beta}_2 \lambda_2 \\
\lambda_1 (\tilde{\beta}_1 \tilde{y}') (\tilde{y}_1 y' (u_1 \tilde{y}_2)) (u_1 u_2) = (\tilde{u}_1 \tilde{u}_2) (\tilde{y}_1 \tilde{y}_2) (\tilde{\beta}_2 \lambda_2) \lambda_2,
\end{align*}
\]

with \( u_1 \tilde{\beta}_2 u_1^{-1} = \beta' y' \) and \( \tilde{u}_2^{-1} \tilde{\beta}_1 \tilde{u}_2 = \tilde{y}' \tilde{\beta}' \). The wedge gluing is therefore obtained through the symplectic reduction

\[
(\mathcal{B}_{c_1, e} \times \mathcal{B}_{c_2, e'}) // G_2^* \tag{140}
\]

with momentum map \( \tilde{\lambda}_1 \tilde{\lambda}_2^{-1} \in G_2 \) and fused wedge variable \( (\tilde{y}_w w')^{v_1}_{c_1} = \tilde{y}_w^{v_1} + u^{v_2} \tilde{y}_w^{v_2} + (u^{v_2} \beta^{v_2 c_2} (u^{v_2})^{-1})G_2^* \in G_2^* \).

![Figure 17: We fuse two faces by identifying the links \( l \) and \( l' \). To merge the faces, we also require them to have the same root. We choose to root \( t_e \) and \( t_{e'} \) respectively at \( v_2 \) and \( v_1' \). We then identify \( v_2 = v_1' \).](image)

**Face gluing.** The gluing of two faces, as illustrated in Fig. 17, is done by identifying the link data dual to each face.

The geometric identification consists in setting

\[
l = l' \iff c_1 = c_2, \ c_4 = c_{e'}. \tag{141}
\]

We enforce the geometric constraint as a condition on \( G_1 \) elements. Let us represent the decorations of the links \( l \) and \( l' \) at the vertices \( v_2 \) and \( v_1' \).

\[
\mathcal{B}_{l, c} : \quad G^* = G_2 \ltimes G_1^* \ni \tilde{c}^{v_1} \equiv \tilde{\ell}_1 = \tilde{\lambda}_1 \tilde{\beta}_1 \equiv \tilde{\lambda}_1 \tilde{c}_1 \tilde{v}_c^{v_1}, \quad \mathcal{G} = G_1 \ltimes G_2^* \ni h^{v_2} \equiv h_1 y_1 \equiv u^{v_2} y_c^{v_2}
\]

\[
\mathcal{B}_{l', c'} : \quad \tilde{G}^* = G_1^* \ltimes G_2 \ni \tilde{c}^{v_1} \equiv \tilde{\ell}_2 = \tilde{\beta}_2 \lambda_2 \equiv \tilde{\beta}_2 \tilde{c}_1 \tilde{v}_c^{v_1}, \quad \tilde{G} = G_1 \ltimes G_2 \ni \tilde{h}^{v_2} \equiv \tilde{h}_2 y_2 \equiv \tilde{u}^{v_2} y_c^{v_2}. \quad \tag{142}
\]

In order to express the link decorations in the same reference frame, their indices have to match, hence we further demand that

\[
v_2 = v_1'.
\]

The proper geometric condition is encoded in the identification

\[
u^{v_2} = \tilde{u}^{v_1'} \iff u_1 = \tilde{u}_2. \tag{144}
\]

36
Using the above constraint and the Heisenberg ribbon equation, we obtain the extended ribbon equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\ell_1 h_1 = \ell_1 (u_1 y_1) \\
\ell_2 h_2 = \ell_2 (u_2 y_2)
\end{array} \right. \iff \begin{array}{l}
u_1 = \ell_1^{-1} h_1 \tilde{y}_1^{-1} \\
u_2 = \ell_2 h_2 \tilde{y}_2^{-1}
\end{array}
\]

with \( \tilde{y}_1^{-1}(\lambda_1 \tilde{\beta}_1)^{-1}(\bar{u}_1 \bar{y}_1) = (\lambda_2 \tilde{\beta}_2)(\bar{y}_2)\).}

\[
(145)
\]

The condition \( \bar{u}_1 \bar{y}_1 \bar{\lambda}_1 = \bar{y}_1 \bar{\beta}' \) and \( \lambda_2^{-1} \bar{y}_1 \lambda_2 = \beta' y' \). The face fusion is therefore obtained through the symplectic reduction

\[
(B_{c_1,e} \times B_{c_2,e'})//G_1^* \quad (146)
\]

with momentum map \( u^{v_2}(\bar{u}^{v_1})^{-1} \in G_1 \) and fused face variable \( (\bar{\beta}_{u''})_{v_1} = \bar{\beta}_{v_1} + \lambda^{v_1} \bar{\lambda}_{v_1} + (\bar{\lambda}_{v_1} \bar{y}_{v_1} (\bar{\lambda}_{v_1})^{-1}) |_{G_1^*} \in G_1^* \).

Figure 18: We fuse two edges by identifying the wedges \( w_1 \) and \( w_e \). Matching the wedge decorations imposes that they have the same root. Picking \( c_1 \) and \( c_2 \) as respectively root of \( w_1 \) and \( w_e \), we therefore must have \( c_2 = c_1 \). To fuse the edges, we must have that the target of the first edge, \( e \) is the source of the second edge \( e' \). Hence we must have \( v_2 = v'_{2} \)

**Edge gluing.** The gluing of two edges, as illustrated in Fig. 18, is done by identifying the wedge data, dual to each edge.

The geometric identification consists in setting

\[
(147)
\]

We enforce this geometric constraint as a condition on \( G_2^* \) element. Let us represent the decorations of the wedges \( w_1 \) and \( w_{e'} \) at the vertices \( v_2 \) and \( v'_{2} \). We root them respectively at the nodes \( c_1 \) and \( c_2 \).

\[
B_{1,e} : \ G^* = G_2 \times G_1^* \ni \ell = \lambda_1 \tilde{\beta}_1 = \lambda_2 \tilde{\beta}_2 \leftarrow h^v = h_1 = u_1 y_1 = u_{v_2} y_{v_2}, \quad G = G_1 \times G_2 \ni h^v \leftarrow h_2 = \bar{y}_2 u_2 = \bar{y}_{v_2}^\prime u_{v_2}^\prime.
\]

\[
B_{e,e'} : \ G^* = G_2 \times G_1^* \ni \ell = \lambda_2 \tilde{\beta}_2 = \lambda_2 \tilde{\beta}_2 \leftarrow h^v = h_2 = \bar{y}_2 u_2 = \bar{y}_{v_2}^\prime u_{v_2}^\prime.
\]

In order to express the wedge decorations in the same reference frame, their indices have to match, hence we further demand that

\[
(149)
\]

The condition \( v_2 = v'_{2} \) provides the gluing of the edges. The proper geometric condition is encoded in the identification

\[
(150)
\]

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Using the above constraint and the Heisenberg ribbon equations, we obtain the extended ribbon equation

\[
\begin{align*}
\tilde{h}_1 \tilde{\ell}_1 &= \ell_1 h_1 = \ell_1 (u_1 y_1) \quad \iff \quad y_1 = u_1^{-1} \ell_1^{-1} \tilde{h}_1 \tilde{\ell}_1 \quad \iff \quad y_1 \equiv y_2 \\
\tilde{h}_2 \tilde{\ell}_2 &= \ell_2 h_2 = \ell_2 (\tilde{y}_2 u_2) \quad \iff \quad \tilde{y}_2 = \ell_2^{-1} \tilde{h}_2 \tilde{\ell}_2 u_2^{-1}
\end{align*}
\]

\[
\begin{align*}
\forall & \quad u_1^{-1} (\lambda_1, \tilde{\beta}_1)^{-1} (u_1 \tilde{y}_1) (\tilde{\beta}_1^{-1} \lambda_1) = (\beta_2 \lambda_2)^{-1} (\tilde{y}_2 u_2) (\tilde{\beta}_2^{-1} \lambda_2) \\
& \quad (u_1 \tilde{y}_1) (\tilde{\beta}_1 \lambda_1) u_2 (\beta_2 \tilde{\beta}_2)^{-1} = (\lambda_1 \tilde{\beta}_1) u_1 (\beta_2 \lambda_2)^{-1} (\tilde{y}_2 u_2) \\
& \quad \tilde{u}_1 \tilde{y}_1 \tilde{\beta}_1 (\lambda_1 \triangleright u_2) (\tilde{\lambda}_2^{-1} \beta_2^{-1}) = \lambda_1 \tilde{\beta}_1 (u_1 \triangleright \lambda_2^{-1}) (\beta_2^{-1} \tilde{y}_2 u_2) \tilde{y}_2 \\
& \quad \tilde{u}_1 \tilde{y}_1 (\tilde{\lambda}_1 \triangleright u_2) \tilde{\beta}_1^{-1} (\tilde{\lambda}_1^{-1} \beta_2) = \lambda_1 (u_1 \triangleright \lambda_2^{-1}) (\tilde{\beta}_1^{-1} \tilde{y}_2 u_2) \tilde{y}_2 \\
& \quad \quad \quad (\tilde{\lambda}_1 \triangleright u_2) \tilde{\beta}_1^{-1} (\tilde{\lambda}_1^{-1} \beta_2) = \lambda_1 (u_1 \triangleright \lambda_2^{-1}) (\tilde{\beta}_1^{-1} \tilde{y}_2 u_2) \tilde{y}_2
\end{align*}
\]

(151)

with \((u_1 \triangleleft \lambda_2^{-1}) \beta_2 (u_1 \triangleleft \lambda_2^{-1})^{-1} = \beta' y'\) and \((\tilde{\lambda}_1 \triangleright u_2) \tilde{\beta}_1 (\tilde{\lambda}_1 \triangleright u_2)^{-1} = \tilde{\beta}' \tilde{y}'\). The wedge fusion is therefore obtained through the symplectic reduction

\[
(B_{c_1, e} \times B_{c_2, e'}) / G_2
\]

(152)

with momentum map \(y_{c_2}^{e_2} (\tilde{y}_{c_2}')^{-1} = y_{c_2}^{e_2} - \tilde{y}_{c_2}' \in G_2^*\) and fused edge variable \(\lambda_{c_2} \triangleleft e_2 = \lambda_{c_2} (u^{e_2} \triangleright (\lambda_{c_2}')^{-1}) \in G_2\).

### 4.1.3 Triangulation phase space

By consecutively applying the different gluings, which are associative and commutative [2], we can construct the phase space for a given triangulation/cellular complex. To have a simpler final formula, it is useful to introduce the following phase space

\[
B_{t,e} \cong (B_{c_1, e} \times B_{c_2, e'}) / \mathcal{G}, \quad \text{with} \quad c_{t'} = c_t, \quad e_t = -t_{e'}, \quad \text{and} \quad e = -e',
\]

(153)

which decorates a full link, instead of a half link. This phase space is obtained by gluing both the wedge and the link data at once, hence by identifying both edge and face data. The resulting phase space is then isomorphic to the atomic phase space (see Proposition 2). Instead of being associated to half link, it is associated to a full link and its dual wedge, as well as edge and face.

To build the 3d triangulation phase space from a set of atomic phase spaces, we need

- three edges closing into a triangle \(t\). We consider three atomic phase spaces and fuse them along a common link, recovering a phase space with one single fused face, with one dual link and three edges together with their dual wedges. This fusion is implemented by a symplectic reduction along two copies of \(G_1^*\). Finally we impose a constraint to close the edges. This is implemented by constraining the product of the three \(G_2\) elements and hence inducing a symplectic reduction with respect to \(G_2^*\), which we will denote \(G_2^{*t}\). Hence the phase space of the triangle \(\mathcal{P}_t\) is obtained by the symplectic reduction along the group \(H^t \equiv \left( (G_1^* \times G_1^*) \times G_2^{*t} \right) \)

\[
\mathcal{P}_t = B_{t, e}^{(3)} / H^t, \quad H^t \equiv \left( (G_1^* \times G_1^*) \times G_2^{*t} \right).
\]

(154)

We provide the explicit construction in Sec. 4.2.

- four triangles closing into a tetrahedron \(\tau\). Once we have the triangle phase space, we glue four of them by identifying the edges pairwise to form the tetrahedron. This is implementing six wedge gluings, by constraining momentum maps in \(G_2\). Finally we impose the closure constraint to close the faces. This is implemented by constraining the product/sum of four \(G_1^*\) elements, which is implementing a symplectic reduction along the group \(G_1^*\).
Hence the phase space of the tetrahedron $P^\tau$ is obtained by the symplectic reduction along the group $H^\tau = ((G_2^*)^{\times 0}) \times G_1^\tau$,

$$P^\tau = P^\tau_1//H^\tau, \quad H^\tau = ((G_2^*)^{\times 0}) \times G_1^\tau.$$

We provide the explicit construction in Sec. 4.2.

Since we have both decoration on the triangulation and its dual, we also have to impose geometric constraints that arise on the dual, as well as some possible compatibility between the two. We have

- (1-)flatness constraint. When gluing the wedges together, we need to remember that we have a trivial $t$-map, which implies that the holonomy that decorates a closed link around the wedge $w$ has to be flat. First, we consider $n$ atomic phase spaces and fuse them along a common edge, to recover the phase space with one single fused wedge, its dual edge, and $n$ links around it. This fusion is implemented by a symplectic reduction along $n$ copies of $G_2^*$. Finally we impose a condition to constraint the closed holonomy. This is implemented by constraining the product of $n$ elements in $G_1$ and hence inducing a symplectic reduction with respect to $G_1^\tau$, which we will note $G_1^\tau_w$. Therefore, the phase space of a closed holonomy around an edge $P^\partial_w$ is obtained by the symplectic reduction along the group $H^\partial_w = ((G_2^*)^{\times n-1} \times G_1^\tau_w)$

$$P^\partial_w = P^\partial_w_{(n)}//H^\partial_w, \quad H^\partial_w = ((G_2^*)^{\times n-1} \times G_1^\tau_w).$$

- Edge simplicity. The edge simplicity constraint discussed in [30, 29, 5] is weaker than the flatness constraint. It mixes information between the triangulation and its dual. It encodes that the transport of the edge data around a closed link is trivial. If we already assumed the (1-)flatness constraint, then this is automatically implemented. In fact, not demanding the flatness condition first, but instead the edge simplicity condition would amount to allow for the possibility of a non trivial flatness constrain along a closed link, which would be equivalent to have a non-trivial $t$-map. This would then be outside the realm of skeletal crossed modules which we focused on exclusively.

The following definition constitutes the main result of the paper.

**Definition 13.** We define the phase space $P$ of a triangulation $T$, with dual $T^*$ to be

$$P = \left( \bigotimes_{l \in T^*} B_{l,e} \right) // \left( \bigotimes_{\partial w \in T^*} H^\partial_w \times H^t \times H^\tau \right).$$

We emphasize that the triangulation is not yet fully geometrically consistent. Indeed, we could expect that the triangle decoration in $G_1^\tau$ to be related to the normal which should be constructed from the edge information (in $G_2$). This would amount to impose the simplicity constraints [63], which can be a subtle issue [39].

We could demand that the decoration of the faces/wedges of the dual polyhedron close. This would be equivalent to demand 2-flatness in the 2-gauge theory picture. Typically this is the condition that encodes dynamics. This is similar to the 2d case, where the 1-flatness condition, around the closed links, encodes dynamics.

Upon quantization, we expect this phase space to describe possibly quantum curved 3d discrete geometries. The states, spin 2-networks, will be given in terms of the 2-representations of the crossed module $\mathcal{G}$. These provide a natural generalization of the notion of spin networks introduced by Penrose [62].

### 4.2 Examples

Given the general construction of the phase space of a triangulation, in this section we show some concrete examples. We construct the phase space for a triangle and a tetrahedron. These explicit cases allow us to identify the shape of the momentum map constraints inducing the closure and generating the symplectic reduction. We also recover the phase space leading to the $G$-networks when focusing on the Poincaré 2-group equipped with a trivial Poisson structure. We recall that while we exclusively discuss triangulations, our construction holds for any cellular decomposition and its dual with minor adjustments.
4.2.1 Holonomy around an edge.

For the (1-)flatness constraint, it is useful to derive the phase space of a closed loop of links (holonomies) around an edge. For the sake of simplicity, we first consider the shortest non trivial loop, made of two links. To determine this phase space we consider then two atomic phase spaces, Heisenberg doubles $B_{l_1,e_1}$, $B_{l_2,e_2}$, associated to the full links $l_1 = (c_1 c_1')$, $l_2 = (c_2 c_2')$ and to the edges $e_1 = (v_1 v_1')$, $e_2 = (v_2 v_2')$. According to the Wedge Gluing Part 2 of Sec. 4.1.2, we impose the identification $e = e'$, representing the edges at the common nodes $c_1' = c_2$. The proper geometric identification is encoded in the gluing condition

$$\tilde{\lambda}_1 = \lambda_2.$$

(158)

In order to enforce the closure of the fused holonomy, we identify $c_2' = c_1$, by imposing the momentum map

$$u_1 u_2 = 1.$$  

(159)

The symplectic reduction for a closed holonomy made of two links around an edge is thus

$$\mathcal{P}_{\partial w} = (B_{l_1,e_1} \times B_{l_2,e_2})/(G_2^* \times G_1^*),$$

(160)

with momentum maps (162) and (163), and fused wedge variable (decoration of a closed dual face)

$$G_2^* \ni \bar{y} = \bar{y}_1 + u_1 \triangleright \bar{y}_2 + (u_1 \bar{\beta}_2 u_1^{-1})|G_2^*.$$  

(161)

The generalization to a loop of $n$ links is straightforward: impose $n-1$ momentum maps to identify the $n$ edges

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n,$$

(162)

and then impose the closure of the holonomy

$$u_1 u_2 \cdots u_n = 1.$$  

(163)

The symplectic reduction for the phase space of a closed holonomy around an edge is

$$\mathcal{P}_{\partial w} = B_{l,e}^n/(G_2^*)^{n-1} \times G_1^*,$$

(164)

with momentum maps (162) and (163), and fused wedge variable rooted at the node $c_1 = c_n'$

$$\bar{y} = \bar{y}_1 + u_1 \triangleright \bar{y}_2 + \cdots + (u_1 \cdots u_{n-1}) \triangleright \bar{y}_n + (u_1 \bar{\beta}_2 u_1^{-1})|G_2^* + ((u_1 \cdots u_{n_1}) \bar{\beta}_n (u_1 \cdots u_{n_1})^{-1})|G_2^*.  

(165)

Figure 19: Identifying three links and fusing three faces to create the phase space associated to a triangle.
To determine the phase space of a triangle, we take three Heisenberg doubles $B_{i,e_i}$, with $i = 1, 2, 3$: three atomic phase spaces are associated to the full links\(^\text{18}\) $l_i = (c_i, c'_i)$ and edges $e_i = (v_i, v'_i)$ as well as their respective dual structures.

The three faces of the atomic pieces are fused together into a single surface — the triangle face. We will fuse the face decoration in pairs since the fusion process is associative. We will then require that the three edge decorations close.

First, we glue two phase spaces by identifying the links $l_1 = (c_1, c'_1) = (c_2, c'_2) = l_2$ in the representation sitting at the common vertices $v'_1 = v_2$ (we refer to the paragraph 4.1.2 for a more detailed explanation). The geometric identification induces the momentum constraint

$$u_1 = \tilde{u}_2. \quad (166)$$

Then we add the third phase space, by imposing the condition $l_1 = l_2 = l_3$ on the links represented at the common vertices $v'_2 = v_3$. This time the identification induces the momentum constraint

$$u_2 = \tilde{u}_3. \quad (167)$$

Putting everything together, we get the full ribbon constraint

$$\tilde{u}_1(\tilde{y}_1 y' y'')(\tilde{\beta}_1((\lambda_1 \lambda_2 \lambda_3)((\tilde{\beta}_2 \tilde{\beta}_3)^{-1} \lambda_1 \lambda_2 \lambda_3)) = (\lambda_1 \lambda_2 \lambda_3)((\tilde{\beta}_2 \tilde{\beta}_3)^{-1} \lambda_1 \lambda_2 \lambda_3)) \lambda_1 \lambda_2 \lambda_3 \tilde{y}_3, \quad (168)$$

with $\tilde{\lambda}_1 \tilde{y}_2 \tilde{\lambda}_2^{-1} = y' \tilde{y}_2$, $\lambda_2^{-1} \tilde{y}_1 \lambda_2 = y' \tilde{y}_2$, $\tilde{\lambda}_1 \tilde{y}_2 \tilde{\lambda}_2^{-1} = y'' \tilde{y}_2$ and $\lambda_3^{-1} \tilde{y}_3 \lambda_3 = y'' \tilde{y}_2$.

Last, in order to close the triangle we need to identify the vertices $v'_3 = v_1$. This identification allows us to define the closure constraint (we recall that the $t$-map is trivial)

$$G_2 \triangleright \lambda_1 \lambda_2 \lambda_3 = 1. \quad (169)$$

The symplectic reduction for a triangle is thus

$$\mathcal{H}_t = (B_{1,e_1} \times B_{2,e_2} \times B_{3,e_3}) \parallel (G_1^1 \times G_1^2 \times G_2^2), \quad (170)$$

with momentum maps $(u_1 \tilde{u}_2^{-1}, u_2 \tilde{u}_3^{-1}, \lambda_1 \lambda_2 \lambda_3)$ and triangle fused face variable

$$\tilde{\beta}_t = \tilde{\beta}_1 \triangleleft \lambda_2 \lambda_3 + \tilde{\beta}_2 \triangleleft \lambda_3 + \tilde{\beta}_3 + (\lambda_3^{-1} \tilde{y}_3 \lambda_3) \big|_{G_1^1} + ((\lambda_2 \lambda_3)^{-1} \tilde{y}_1 \lambda_2 \lambda_3)) \big|_{G_1^2}. \quad (171)$$

Figure 20: The phase space of a tetrahedron is constructed by gluing four triangles. To properly glue triangles, we must root the face variables at a common point. Three triangles are glued by identifying vertices $v'_1$. The fourth triangle must have its root transported along an edge $(e_1)$ before being identified. The links are connected at a common point $c \equiv c_1 = c_2 = c_3 = c_4$.

---

\(^{18}\)The result would be similar if only using half links.
4.2.3 Tetrahedron.

In order to derive the phase space of a tetrahedron, we first consider twelve copies of the Heisenberg double, $B_{i,a,A}$, for $a = 1, 2, 3, 4$ and $i = 1, 2, 3$, associated to full links $l_{i,a} = (c_{a}, c_{a,i}')$ and edges $e_{a,i} = (v_{a,i}, v_{a,i}')$. Later on, the nodes $c_{a,1}, c_{a,2}, c_{a,3}$ will be identified as the source of the link dual to triangle $a$ and noted as $c_{a}$; then, the sources of the four links $c_{1}, c_{2}, c_{3}, c_{4}$ will be further identified as the center of the tetrahedron and noted as $c$.

To construct a tetrahedron, we have to impose first the momentum maps necessary to construct the four triangle phase spaces:

$$u_{a;1} = \tilde{u}_{a;2}, \quad u_{a;2} = \tilde{u}_{a;3} \quad \text{and} \quad \lambda_{a;1} \lambda_{a;2} \lambda_{a;3} = 1, \quad \text{for} \ a = 1, 2, 3, 4. \quad (172)$$

Then, we have to fuse the triangle phase spaces by identifying the three edges of each triangle with one of the edges of the other three triangles: use the Wedge Gluing Part 1 of Sec. 4.1.2 to glue the triangles. According to Fig. 20, we impose the six momentum maps

$$\lambda_{1;1} = \lambda_{3;3}^{-1}, \quad \lambda_{1;2} = \lambda_{4;3}^{-1}, \quad \lambda_{1;3} = \lambda_{2;1}^{-1}, \quad \lambda_{2;2} = \lambda_{4;2}^{-1}, \quad \lambda_{2;3} = \lambda_{3;1}^{-1}, \quad \lambda_{3;2} = \lambda_{4;1}^{-1}. \quad (173)$$

Since the symplectic reduction prescription is commutative, the order in which we implement the above eighteen momentum maps does not matter. Computing the associated extended ribbon equation provides us the expression of the fused decoration. For instance, we can compute the extended ribbon equation for the four triangles (as in (168)) to derive the expression of the four fused face decorations

$$\begin{align*}
\bar{\beta}_{t_{1}} &= \beta_{1;1} + \beta_{1;2} \triangleright \lambda_{1;3} + \beta_{1;3} + (\lambda_{1;3}' \bar{y}_{1;2} \lambda_{1;3})|_{G_{1}}, \\
\bar{\beta}_{t_{2}} &= \beta_{2;1} + \beta_{2;2} \triangleright \lambda_{2;3} + \beta_{2;3} + (\lambda_{2;3}' \bar{y}_{2;2} \lambda_{2;3})|_{G_{1}}, \\
\bar{\beta}_{t_{3}} &= \beta_{3;1} + \beta_{3;2} \triangleright \lambda_{3;3} + \beta_{3;3} + (\lambda_{3;3}' \bar{y}_{3;2} \lambda_{3;3})|_{G_{1}}, \\
\bar{\beta}_{t_{4}} &= \beta_{4;1} + \beta_{4;2} \triangleright \lambda_{4;3} + \beta_{4;3} + (\lambda_{4;3}' \bar{y}_{4;2} \lambda_{4;3})|_{G_{1}}.
\end{align*} \quad (174)$$

From the ribbon equation we also know that the above face decorations are represented at the source of the four links, common node $c \equiv c_{1} = c_{2} = c_{3} = c_{4}$ and rooted respectively at the vertices $v_{1;1} = v_{1;3}', v_{2;1} = v_{2;3}', v_{3;1} = v_{3;3}', v_{4;1} = v_{4;3}'$ (red vertex in Fig. (20)).

The extended ribbon equation is also useful to change the reference frame: we act by some edge or link decoration on the ribbon equation. For instance, starting by the twelve atomic phase spaces and fusing them pairwise by the momentum maps (173), we compute the six extended ribbon equations associated to the six edges of a tetrahedron; in principle, they are given in different reference frames. Then, acting with some edge decorations on such ribbon equations, is then possible to derive the expression of the six fused wedge decorations dual to the six edges, all rooted at the node $c$, and represented at the common vertex $v \equiv v_{1;1} = v_{2;1} = v_{3;1}$ (red vertex in Fig. (20)).

$$\begin{align*}
\bar{y}_{12} &= -\bar{y}_{2;1} + (\lambda_{1;3} \bar{y}_{1;3} \lambda_{1;3}^{-1})|_{G_{2}}, \quad \bar{y}_{23} = -\bar{y}_{3;1} + (\lambda_{2;3} \bar{y}_{2;3} \lambda_{2;3}^{-1})|_{G_{2}}, \\
\bar{y}_{13} &= \bar{y}_{1;1} - (\lambda_{3;3} \bar{y}_{3;3} \lambda_{3;3}^{-1})|_{G_{2}}, \quad \bar{y}_{24} = \bar{y}_{2;2} - (\lambda_{4;2} \bar{y}_{4;2} \lambda_{4;2}^{-1})|_{G_{2}}, \\
\bar{y}_{14} &= \bar{y}_{1;2} - (\lambda_{4;3} \bar{y}_{4;3} \lambda_{4;3}^{-1})|_{G_{2}}, \quad \bar{y}_{34} = \bar{y}_{3;2} - (\lambda_{4;1} \bar{y}_{4;1} \lambda_{4;1}^{-1})|_{G_{2}}.
\end{align*} \quad (175)$$

where we denoted $\bar{y}_{ab}$ the fused wedge decoration shared by triangles $a$ and $b$.

The construction of the tetrahedron phase space is completed with the last momentum map, known as 1-Gauss constraint, that enforces the closure of the four faces. According to Fig. 20, three of the four triangles share the same reference frame, while the fourth is at a different vertex (vertex $v_{4;3}$ instead of $v$). To impose the 1-Gauss constraint, we thus act from the left on the ribbon equation of the fourth triangle by the edge decoration that connect the vertices $v_{4;1}$ and $v$, $\lambda_{1;1}$. Once we have the expression of the four face decorations all at the same reference frame, we can correctly impose the last momentum map

$$\bar{\beta}_{t_{1}} + \bar{\beta}_{t_{2}} + \bar{\beta}_{t_{3}} + \lambda_{1;1} \triangleright \bar{\beta}_{t_{4}} + (\lambda_{1;1} \bar{y}_{4;1} \lambda_{1;1}^{-1})|_{G_{1}} = 0, \quad (176)$$

42
where we used the expressions in (174) and
\[ \tilde{y}_{t4} = \tilde{y}_{4:3} + (\lambda_{4:3}^{-1} \tilde{y}_{4:2} \lambda_{4:3})|_G^2 + ((\lambda_{4:2} \lambda_{4:3})^{-1} \tilde{y}_{4:1} (\lambda_{4:2} \lambda_{4:3}))|_G^1. \] (177)

The symplectic reduction for a tetrahedron phase space is thus
\[ \mathcal{P}_t = B_{1,t}^{\times 12} // ((G^*_1 \times G^*_1) \times G^*_2)^{\times 4} \times ((G^*_2)^{\times 6} \times G^*_1), \] (178)

with momentum maps (172), (173) and (176). We can alternatively express the constraint (176) as a combination of four face decorations plus the contribution of the six fused wedge decorations (175). This expression will be particularly useful in the next subsection. As we did for the fused wedges, we denote \( \lambda_{ab} \) the edge shared by triangles \( a \) and \( b \); in particular we have
\[ \lambda_{12} \equiv \lambda_{2,1}, \quad \lambda_{13} \equiv \lambda_{1,1}, \quad \lambda_{14} \equiv \lambda_{1,2}, \quad \lambda_{23} \equiv \lambda_{3,1}, \quad \lambda_{24} \equiv \lambda_{2,2}, \quad \lambda_{34} \equiv \lambda_{3,2}. \] (179)

The constraint (176) then becomes
\[ b_1 + b_2 + b_3 + \lambda_{13} \triangleright b_4 + (\lambda_{12} \tilde{y}_{12})|_G^1 + (\lambda_{13} \tilde{y}_{13} \lambda_{13}^{-1})|_G^1 + \lambda_{13} \triangleright (\lambda_{14} \tilde{y}_{14} \lambda_{14}^{-1})|_G^1 \]
\[ + (\lambda_{23} \tilde{y}_{23} \lambda_{23}^{-1})|_G^1 + \lambda_{12} \triangleright (\lambda_{24} \tilde{y}_{24} \lambda_{24}^{-1})|_G^1 + \lambda_{23} \triangleright (\lambda_{34} \tilde{y}_{34} \lambda_{34}^{-1})|_G^1 = 0, \] (180)

with the face decorations
\[ b_1 = \beta_{1:1} + \lambda_{13} \triangleright \beta_{1:2} + \tilde{\beta}_{1:3} + \lambda_{12} \triangleright (\lambda_{12}^{-1} \tilde{y}_{1:2} \lambda_{12})|_G^1 + (\lambda_{12} \lambda_{13} \lambda_{13}^{-1})|_G^1 \]
\[ b_2 = -\beta_{2:1} + \lambda_{12} \triangleright \beta_{2:2} + \tilde{\beta}_{2:3} + \lambda_{23}^{-1} \triangleright (\lambda_{23}^{-1} \tilde{y}_{2:3} \lambda_{23})|_G^1 + (\lambda_{12} \lambda_{24} \lambda_{24}^{-1})|_G^1 \]
\[ b_3 = -\beta_{3:1} + \lambda_{23} \triangleright \beta_{3:2} - \tilde{\beta}_{3:3} - \lambda_{34} \triangleright (\lambda_{34}^{-1} \tilde{y}_{3:3} \lambda_{34})|_G^1 + (\lambda_{23} \lambda_{34} \lambda_{34}^{-1})|_G^1 \]
\[ b_4 = -\beta_{4:1} - \lambda_{34}^{-1} \triangleright \beta_{4:2} - \tilde{\beta}_{4:3} - \lambda_{34} \triangleright (\lambda_{34}^{-1} \tilde{y}_{4:3} \lambda_{34})|_G^1 + (\lambda_{23} \lambda_{34} \lambda_{34}^{-1})|_G^1 \]
\[ + (\lambda_{12} \tilde{y}_{4:2} \lambda_{12}^{-1})|_G^2 + \lambda_{23} \triangleright (\lambda_{24} \tilde{y}_{24} \lambda_{24}^{-1})|_G^1 + (\lambda_{23} \tilde{y}_{4:1} \lambda_{23}^{-1})|_G^1 \] (181)

Note that, we derived the expression (180), such that each fused wedge decoration \( \tilde{y}_{ab} \) is conjugated by the respective dual edge decoration \( \lambda_{ab} \).

### 4.2.4 4-simplex boundary.

As we managed to construct the tetrahedron phase space, the fundamental building block of a 3d space, we can use it to build any triangulation phase space. Let us give here an example of a closed piece of triangulation: we build the phase space of the boundary of a 4-simplex which was the main example considered in [5].

The boundary of a 4-simplex is given in terms of five tetrahedra, so we consider five tetrahedron phase spaces and fuse them by identifying each of the faces of each tetrahedron with a face of one of the other four tetrahedra. The starting point to derive the 4-simplex boundary phase space are then sixty copies of the Heisenberg double \( B_{(4,t),c(a,1)} \), for \( A = 1, 2, 3, 4, 5, a = 1, 2, 3, 4, i = 1, 2, 3 \). For simplicity and to not be redundant, it is simpler to start by five copies of tetrahedron phase space \( \mathcal{P}_{\tau_A} \), as given in (178).

We represent the 4-simplex boundary construction in Fig. 21. We denote by \( c_A \) the center of the tetrahedron \( \tau_A \). The five tetrahedra share five vertices: \( v \) (in black) is shared by tetrahedra 1,2,3,5, \( v' \) (in white) is shared by tetrahedra 1,3,4,5, \( v'' \) (in gray) is shared by tetrahedra 1,2,4,5, \( v''' \) (in yellow) is shared by tetrahedra 2,3,4,5 and \( v'''' \) (in green) is shared by tetrahedra 1,2,3,4. Let \( t_{A,a} \) be the triangle \( a \) of tetrahedron \( A \) and \( \tilde{\beta}_{t_{A,a}} \) of (174) (or \( b_{A,a} \) of (181)) is the respective decoration represented at the center of tetrahedron \( A \). \( y_{A,ab} \) is instead the variable that decorates the fused wedge shared by triangles \( a \) and \( b \) of tetrahedron \( A \), rooted at the center of it.

In order to fuse the tetrahedron phase spaces, we use the Link Gluing of Sec. 4.1.2; according to the Fig. 21, we denote \( t_{AB} \) the face shared by tetrahedron \( A \) and \( B \):
\[ t_{12} \equiv t_{1:1} = -t_{2:4}, \quad t_{13} \equiv t_{1:2} = -t_{3:2}, \quad t_{14} \equiv t_{1:3} = -t_{4:2}, \quad t_{15} \equiv t_{1:4} = -t_{5:1}, \quad t_{23} \equiv t_{2:1} = -t_{3:4}, \]
\[ t_{24} \equiv t_{2:2} = -t_{4:3}, \quad t_{25} \equiv t_{2:3} = -t_{5:2}, \quad t_{34} \equiv t_{3:1} = -t_{4:4}, \quad t_{35} \equiv t_{3:2} = -t_{5:3}, \quad t_{45} \equiv t_{4:1} = -t_{5:4}. \] (182)
Figure 21: 4-simplex boundary construction: five tetrahedra share five vertices. Each of the four faces of each tetrahedron is identified with one of the faces of the other four tetrahedra. We use the same color and a double dotted line for the identified faces.
We enforce these geometric conditions on the variables $\bar{\beta}_{t_{4,4}}$ through the ten momentum maps

$$\bar{\beta}_{12} = \bar{\beta}_{1,1} = -(u_{12} \bar{\beta}_{4,2} u_{12}^{-1})|G_i^1$$,  
$$\bar{\beta}_{31} = \bar{\beta}_{3,1} = -(u_{13} \bar{\beta}_{1,3} u_{13})|G_i^3$$,  
$$\bar{\beta}_{14} = \bar{\beta}_{1,4} = -(u_{14} \bar{\beta}_{4,4} u_{14}^{-1})|G_i^4$$,  
$$\bar{\beta}_{51} = \bar{\beta}_{5,1} = -(u_{15} \bar{\beta}_{1,5} u_{15})|G_i^5$$,  
$$\bar{\beta}_{23} = \bar{\beta}_{2,3} = -(u_{23} \bar{\beta}_{3,4} u_{23}^{-1})|G_i^6$$,  
$$\bar{\beta}_{42} = \bar{\beta}_{4,2} = -(u_{24} \bar{\beta}_{4,2} u_{24})|G_i^7$$,  
$$\bar{\beta}_{25} = \bar{\beta}_{2,5} = -(u_{25} \bar{\beta}_{5,4} u_{25}^{-1})|G_i^8$$,  
$$\bar{\beta}_{34} = \bar{\beta}_{3,4} = -(u_{34} \bar{\beta}_{4,4} u_{34}^{-1})|G_i^9$$,  
$$\bar{\beta}_{53} = \bar{\beta}_{5,3} = -(u_{35} \bar{\beta}_{4,2} u_{35})|G_i^{10}$$,  
$$\bar{\beta}_{45} = \bar{\beta}_{4,5} = -(u_{45} \bar{\beta}_{5,5} u_{45}^{-1})|G_i^{11}$$,

where we denoted $\bar{\beta}_{AB}$ the decoration of the fused face shared by tetrahedra $A$ and $B$, rooted at the center of tetrahedron $A$. Using a similar notation, for later purpose, it is useful to introduce the variable $b_{AB}$:

$$b_{12} = b_{1,1} = -(u_{12} b_{2,4} u_{12}^{-1})|G_i^1$$,  
$$b_{31} = b_{3,1} = -(u_{13} b_{1,3} u_{13})|G_i^3$$,  
$$b_{14} = b_{1,4} = -(u_{14} b_{4,4} u_{14}^{-1})|G_i^4$$,  
$$b_{51} = b_{5,1} = -(u_{15} b_{1,5} u_{15})|G_i^5$$,  
$$b_{23} = b_{2,3} = -(u_{23} b_{3,4} u_{23}^{-1})|G_i^6$$,  
$$b_{42} = b_{4,2} = -(u_{24} b_{4,2} u_{24})|G_i^7$$,  
$$b_{25} = b_{2,5} = -(u_{25} b_{5,4} u_{25}^{-1})|G_i^8$$,  
$$b_{34} = b_{3,4} = -(u_{34} b_{4,4} u_{34}^{-1})|G_i^9$$,  
$$b_{53} = b_{5,3} = -(u_{35} b_{3,2} u_{35})|G_i^{10}$$,  
$$b_{45} = b_{4,5} = -(u_{45} b_{5,5} u_{45}^{-1})|G_i^{11}$$.

According to the Link Gluing in Sec. 4.1.2, for each $o$ these momentum maps we have a fused link decoration

$$u_{12} = u_{1,1} u_{12}^{-1}, \quad u_{13} = u_{1,2} u_{13}^{-1}, \quad u_{14} = u_{1,3} u_{14}^{-1}, \quad u_{15} = u_{1,4} u_{15}^{-1}, \quad u_{23} = u_{2,1} u_{23}^{-1}, \quad u_{24} = u_{2,2} u_{24}^{-1}, \quad u_{25} = u_{2,3} u_{25}^{-1}, \quad u_{34} = u_{3,1} u_{34}^{-1}, \quad u_{35} = u_{3,2} u_{35}^{-1}, \quad u_{45} = u_{4,1} u_{45}^{-1},$$

where $u_{AB}$ decorates the fused link with source at the center of tetrahedron $A$ and target at the center of tetrahedron $B$. Last, to complete the gluing of the five tetrahedra, we identify the boundary of the faces, too. According to Fig. 21, denote $\lambda_{AB}$ the edge shared by faces $a$ and $b$ of tetrahedron $A$, represented at the center of it. Let $e_{ABC}$ be the edge shared by tetrahedra $A, B, C$ and $\lambda_{ABC}^D$ be the respective edge decoration represented at the node $D$ defined through the following momentum maps

$$\lambda_{123} = \lambda_{1,12} = u_{12} \lambda_{2,14} = u_{13} \lambda_{3,34}, \quad \lambda_{124} = \lambda_{1,14} = u_{14} \lambda_{2,24} = u_{15} \lambda_{3,43},$$

$$\lambda_{125} = \lambda_{1,15} = u_{15} \lambda_{2,51} = u_{16} \lambda_{3,12}, \quad \lambda_{134} = \lambda_{1,34} = u_{14} \lambda_{2,43} = u_{15} \lambda_{3,21},$$

$$\lambda_{145} = \lambda_{1,45} = u_{15} \lambda_{2,54} = u_{16} \lambda_{3,13}, \quad \lambda_{234} = \lambda_{2,34} = u_{23} \lambda_{3,12} = u_{24} \lambda_{3,21},$$

$$\lambda_{245} = \lambda_{2,45} = u_{24} \lambda_{3,13} = u_{25} \lambda_{3,24}, \quad \lambda_{345} = \lambda_{3,45} = u_{24} \lambda_{3,13} = u_{35} \lambda_{3,24},$$

From the Wedge Gluing Part 1 of Sec. 4.1.2, we have a fused wedge decoration for each momentum map

$$\bar{y}_{123} = \bar{y}_{1,12} + u_{12} \bar{y}_{2,14} + u_{13} \bar{y}_{3,34} + (u_{12} \bar{\beta}_{23} u_{12}^{-1})|G_i^1 + (u_{13} \bar{\beta}_{31} u_{13}^{-1})|G_i^2,$$

$$\bar{y}_{124} = \bar{y}_{1,14} + u_{14} \bar{y}_{2,24} + u_{15} \bar{y}_{4,23} + (u_{14} \bar{\beta}_{42} u_{14}^{-1})|G_i^2,$$

$$\bar{y}_{125} = \bar{y}_{1,15} + u_{15} \bar{y}_{2,51} + (u_{12} \bar{\beta}_{25} u_{12}^{-1})|G_i^2 + (u_{15} \bar{\beta}_{51} u_{15}^{-1})|G_i^3,$$

$$\bar{y}_{134} = \bar{y}_{1,23} + u_{13} \bar{y}_{3,14} + u_{14} \bar{y}_{4,24} + (u_{13} \bar{\beta}_{34} u_{13}^{-1})|G_i^2 + (u_{14} \bar{\beta}_{43} u_{14}^{-1})|G_i^3,$$

$$\bar{y}_{145} = \bar{y}_{1,34} + u_{14} \bar{y}_{4,12} + u_{15} \bar{y}_{5,14} + (u_{14} \bar{\beta}_{45} u_{14}^{-1})|G_i^2 + (u_{15} \bar{\beta}_{51} u_{15}^{-1})|G_i^3,$$

$$\bar{y}_{234} = \bar{y}_{2,34} + u_{23} \bar{y}_{3,24} + u_{24} \bar{y}_{5,23} + (u_{23} \bar{\beta}_{34} u_{23}^{-1})|G_i^3 + (u_{24} \bar{\beta}_{43} u_{24}^{-1})|G_i^4,$$

$$\bar{y}_{245} = \bar{y}_{2,45} + u_{24} \bar{y}_{4,13} + u_{25} \bar{y}_{5,24} + (u_{24} \bar{\beta}_{45} u_{24}^{-1})|G_i^3 + (u_{25} \bar{\beta}_{54} u_{25}^{-1})|G_i^5,$$

$$\bar{y}_{345} = \bar{y}_{3,45} + u_{34} \bar{y}_{4,12} + u_{35} \bar{y}_{5,24} + (u_{34} \bar{\beta}_{45} u_{34}^{-1})|G_i^3 + (u_{35} \bar{\beta}_{54} u_{35}^{-1})|G_i^5,$$

where the variable $\bar{y}_{ABC}^D$ decorates the fused wedge dual to the edge $e_{ABC}$ (closed surface in the dual complex shared by tetrahedra $A, B, C$), it is rooted at the center of tetrahedron $D$.  

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The symplectic reduction for a 4-simplex boundary phase space is thus
\[
\mathcal{P}_\tau = \mathcal{B}_\tau^{\times 60} \big/ \big( \big( (G_1^* \times G_2^*) \times (G_2^*)^4 \times ((G_2^*)^6 \times G_1^* \times G_2^* \times G_1^* \times G_2^*) \big) \big)^{\times 5} \times ((G_1^*)^{10} \times (G_2^*)^{10})
\]
with momentum maps (183) and (186). Using (180), the 1-Gauss constraints of the five tetrahedra, each of them represented at the center of the respective tetrahedron, are
\[
\tau_1: \quad b_{12} - (u_{13} b_{51} u_{15}^*)|_{G_1^*} + \lambda_{135}^1 \triangleright b_{14} - (u_{15} b_{51} u_{15}^*)|_{G_1^*}
+ (\lambda_{123}^1 \tilde{y}_{1:12} (\lambda_{123}^1)^{-1})|_{G_1^*} + \lambda_{125^3} \triangleright (\lambda_{124} \tilde{y}_{1:13} (\lambda_{124}^1)^{-1})|_{G_1^*}
+ \lambda_{135^3} \triangleright (\lambda_{134} \tilde{y}_{1:23} (\lambda_{134}^1)^{-1})|_{G_1^*} + \lambda_{135^5} \triangleright (\lambda_{145} \tilde{y}_{1:34} (\lambda_{145}^1)^{-1})|_{G_1^*} = 0,
\]
\[
\tau_2: \quad b_{23} - \lambda_{125^3} \triangleright (u_{24} b_{42} u_{24}^*)|_{G_1^*} + b_{25} - (u_{12} b_{12} u_{12}^*)|_{G_1^*}
+ \lambda_{235^3} \triangleright (\lambda_{234} \tilde{y}_{2:12} (\lambda_{234}^1)^{-1})|_{G_1^*} + \lambda_{235} \triangleright (\lambda_{235} \tilde{y}_{2:13} (\lambda_{235}^1)^{-1})|_{G_1^*}
+ \lambda_{235^3} \triangleright (\lambda_{245} \tilde{y}_{2:23} (\lambda_{245}^1)^{-1})|_{G_1^*} + \lambda_{235^5} \triangleright (\lambda_{245} \tilde{y}_{2:34} (\lambda_{245}^1)^{-1})|_{G_1^*} = 0,
\]
\[
\tau_3: \quad \lambda_{135^3} \triangleright b_{34} - (u_{35} b_{53} u_{35}^*)|_{G_1^*} + b_{31} - (u_{23} b_{23} u_{23}^*)|_{G_1^*}
+ \lambda_{335^3} \triangleright (\lambda_{345} \tilde{y}_{3:12} (\lambda_{345}^1)^{-1})|_{G_1^*} + \lambda_{335} \triangleright (\lambda_{345} \tilde{y}_{3:13} (\lambda_{345}^1)^{-1})|_{G_1^*}
+ \lambda_{335^3} \triangleright (\lambda_{345} \tilde{y}_{3:23} (\lambda_{345}^1)^{-1})|_{G_1^*} + \lambda_{335^5} \triangleright (\lambda_{345} \tilde{y}_{3:34} (\lambda_{345}^1)^{-1})|_{G_1^*} = 0,
\]
\[
\tau_4: \quad \lambda_{135^3} \triangleright (b_{45} - (u_{14} b_{14} u_{14}^*)|_{G_1^*} + \lambda_{145^3} \triangleright (u_{23} b_{34} u_{34}^*)|_{G_1^*}
+ \lambda_{135} \triangleright (\lambda_{145} \tilde{y}_{4:12} (\lambda_{145}^1)^{-1})|_{G_1^*} + \lambda_{125^3} \triangleright (\lambda_{145} \tilde{y}_{4:13} (\lambda_{145}^1)^{-1})|_{G_1^*}
+ \lambda_{135^3} \triangleright (\lambda_{145} \tilde{y}_{4:23} (\lambda_{145}^1)^{-1})|_{G_1^*} + \lambda_{135^5} \triangleright (\lambda_{145} \tilde{y}_{4:34} (\lambda_{145}^1)^{-1})|_{G_1^*} = 0,
\]
\[
\tau_5: \quad b_{51} - (u_{25} b_{25} u_{25}^*)|_{G_1^*} + b_{53} - \lambda_{135^3} \triangleright (u_{45} b_{45} u_{45}^*)|_{G_1^*}
+ (\lambda_{35} \tilde{y}_{5:12} (\lambda_{35}^1)^{-1})|_{G_1^*} + (\lambda_{35} \tilde{y}_{5:13} (\lambda_{35}^1)^{-1})|_{G_1^*}
+ (\lambda_{35} \tilde{y}_{5:23} (\lambda_{35}^1)^{-1})|_{G_1^*} + \lambda_{135^3} \triangleright (\lambda_{35} \tilde{y}_{5:34} (\lambda_{35}^1)^{-1})|_{G_1^*} + \lambda_{135^5} \triangleright (\lambda_{35} \tilde{y}_{5:34} (\lambda_{35}^1)^{-1})|_{G_1^*} = 0.
\]

### 4.2.5 Recovering the un-deformed case.

In this section, we consider the explicit example of the Poincaré 2-group with a trivial Poisson structure. This is to recover the phase space that was introduced in [5] and recovered again in [39]. The most difficult constraints to recover are the 1-Gauss constraints since they are non-trivial.

Let \( G = \text{ISO}(4) \cong \text{SO}(4) \ltimes \mathbb{R}^4 \cong \mathbb{R}^4 \ltimes \text{SO}(4) \) and \( G^* = \text{ISO}(4)^* \cong \text{SO}^*(4) \ltimes \mathbb{R}^4 \cong \mathbb{R}^4 \ltimes \text{SO}^*(4) \). Let \( \mathcal{T} \) be a triangulation with edges and faces decorated by elements of \( \mathbb{R}^4 \) and \( \text{SO}^*(4) \) respectively. The dual 2-complex \( \mathcal{T}^* \) has links and wedges decorated by holonomies of \( \text{SO}(4) \) and \( \mathbb{R}^4 \). Specifically, we have

\[
G_1 = \text{SO}(4) \ni u, \quad G_2^* = \mathbb{R}^4 \ni y, \quad y, \quad G_2 = \mathbb{R}^4 \ni \lambda, \quad G_1^* = \text{SO}^*(4) \ni \beta, \tilde{\beta}.
\]

Note that there is no action of \( \mathbb{R}^4 \) on \( \text{SO}(4) \) nor on \( \text{SO}^*(4) \) and the conjugations (123) take a simpler shape

\[
u \beta u^{-1} = \beta y \gamma' \Leftrightarrow \left\{ \begin{array}{l}
\beta' = (u \beta u^{-1})|_{\text{SO}^*(4)} \\
y' = (u \beta u^{-1})|_{\mathbb{R}^4} = 1
\end{array} \right.
\]

\[\lambda y \lambda^{-1} = y'' \beta'' \Leftrightarrow \left\{ \begin{array}{l}
\beta' = (\lambda y \lambda^{-1})|_{\text{SO}^*(4)} \equiv [\lambda, y] \\
y' = (\lambda y \lambda^{-1})|_{\mathbb{R}^4} = y,
\end{array} \right.
\]

where for the second conjugation we used a convenient representation\(^{19}\) of \( \mathbb{R}^4 \). First, to construct the triangle

\(^{19}\)If \( X_\mu \) is a Lie algebra generator of \( \mathbb{R}^4 \), we use a representation such that \( X_\mu X_\nu = 0 \). This implements that \( e^X = 1 + X \). If \( Y_\nu \) are the generators of another \( \mathbb{R}^4 \), then \( e^Y e^{-Y} = 1 + X + [Y, X] \).
phase space we impose the momentum maps \( u_1 = \tilde{u}_2 \) and \( u_2 = \tilde{u}_3 \), plus the closure constraint that here takes the form \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \). The fused face decoration (171) is thus

\[
\beta_t = \beta_1 + \beta_2 + \beta_3 + \beta_4 - [\lambda_2, \tilde{y}_1] - [\lambda_3, \tilde{y}_1] - [\lambda_3, \tilde{y}_2].
\]  

(196)

Similarly, to construct the tetrahedron phase space we first consider four triangle phase spaces and fuse them imposing the six momentum maps (173),

\[
\lambda_{1,1} = -\lambda_{3,3}, \quad \lambda_{1,2} = -\lambda_{4,3}, \quad \lambda_{1,3} = -\lambda_{2,1}, \quad \lambda_{2,2} = -\lambda_{4,2}, \quad \lambda_{2,3} = -\lambda_{3,1}, \quad \lambda_{3,2} = -\lambda_{4,1}.
\]  

(197)

From these momentum maps we then derive the six fused wedge decorations (175) represented at the center of the tetrahedron:

\[
\tilde{y}_{12} = -\tilde{y}_{21} + \tilde{y}_{13}, \quad \tilde{y}_{13} = \tilde{y}_{11} - \tilde{y}_{33}, \quad \tilde{y}_{14} = \tilde{y}_{12} - \tilde{y}_{43},
\]

\[
\tilde{y}_{23} = -\tilde{y}_{31} + \tilde{y}_{23}, \quad \tilde{y}_{24} = \tilde{y}_{23} - \tilde{y}_{42}, \quad \tilde{y}_{34} = \tilde{y}_{32} - \tilde{y}_{41}.
\]  

(198)

The last momentum map that we impose for the construction of the tetrahedron is the 1-Gauss constraint (180), that now reduces to

\[
b_1 + b_2 + b_3 + b_4 + [\lambda_{12}, \tilde{y}_{12}] + [\lambda_{13}, \tilde{y}_{13}] + [\lambda_{14}, \tilde{y}_{14}] + [\lambda_{23}, \tilde{y}_{23}] + [\lambda_{24}, \tilde{y}_{24}] + [\lambda_{34}, \tilde{y}_{34}] = 0,
\]  

(199)

where

\[
b_1 = \beta_{1;1} + \beta_{1;2} + \beta_{1;3} - [\lambda_{21}, \tilde{y}_{11}] + [\lambda_{12}, \tilde{y}_{12}],
\]

\[
b_2 = -\beta_{2;1} + \beta_{2;2} + \beta_{2;3} - [\lambda_{32}, \tilde{y}_{23}] + [\lambda_{12}, \tilde{y}_{22}],
\]

\[
b_3 = -\beta_{3;1} + \beta_{3;2} - \beta_{3;3} - [\lambda_{43}, \tilde{y}_{33}] + [\lambda_{23}, \tilde{y}_{32}],
\]

\[
b_4 = -\beta_{4;1} - \beta_{4;2} - \beta_{4;3} - [\lambda_{43}, \tilde{y}_{43}] - [\lambda_{12}, \tilde{y}_{42}] - [\lambda_{23}, \tilde{y}_{41}].
\]  

(200)

Once we derived the tetrahedron phase space, we can use it to build the 4-simplex boundary phase space.

In order to do it, we first consider five tetrahedra phase spaces and fuse them through the momentum maps (183) and (186), so that we derive the fused link decorations (185) and the fused wedge decorations (187). Then we consider the following change of variables for the face decorations

\[
b'_{12} = b_{12} - u_{12} [\lambda_{12}^{2}, \tilde{y}_{21};4] u_{12} - u_{12} [\lambda_{12}^{2}, \tilde{y}_{21};4] u_{12} - u_{12} [\lambda_{12}^{2}, \tilde{y}_{21};4] u_{12} - u_{12} [\lambda_{12}^{2}, \tilde{y}_{21};4] u_{12},
\]

\[
b'_{13} = b_{13} + [\lambda_{13}, \tilde{y}_{13}] + [\lambda_{13}, \tilde{y}_{13}] - u_{12} [\lambda_{13}^{2}, \tilde{y}_{12};3] u_{12},
\]

\[
b'_{14} = b_{14} - u_{14} [\lambda_{14}, \tilde{y}_{41};2] u_{14} - u_{14} [\lambda_{14}, \tilde{y}_{41};2] u_{14} - u_{14} [\lambda_{14}, \tilde{y}_{41};2] u_{14},
\]

\[
b'_{15} = b_{15} + [\lambda_{15}, \tilde{y}_{15};1] + [\lambda_{15}, \tilde{y}_{15};1],
\]

\[
b'_{23} = b_{23} + [\lambda_{23}, \tilde{y}_{32};4] u_{23} - u_{23} [\lambda_{23}, \tilde{y}_{32};4] u_{23} - u_{23} [\lambda_{23}, \tilde{y}_{32};4] u_{23} - u_{23} [\lambda_{23}, \tilde{y}_{32};4] u_{23},
\]

\[
b'_{24} = b_{24} + [\lambda_{24}, \tilde{y}_{42};3] + [\lambda_{24}, \tilde{y}_{42};3],
\]

\[
b'_{25} = b_{25} + [\lambda_{25}, \tilde{y}_{52};3] u_{25} - u_{25} [\lambda_{25}, \tilde{y}_{52};3] u_{25} - u_{25} [\lambda_{25}, \tilde{y}_{52};3] u_{25} - u_{25} [\lambda_{25}, \tilde{y}_{52};3] u_{25},
\]

\[
b'_{34} = b_{34} - u_{34} [\lambda_{34};4, \tilde{y}_{44};1] u_{34},
\]

\[
b'_{35} = b_{35} + [\lambda_{35}, \tilde{y}_{53};1] + [\lambda_{35}, \tilde{y}_{53};1].
\]  

(201)

We use these new variables in the five 1-Gauss constraints (189)-(193), in order to express the closure constraints in terms of the closed faces (wedges) variables (187).

\[
\tau_1: \quad b'_{14} = -[u_{13} b'_{31} u_{13}^{-1}] + [\lambda_{123}, \tilde{y}_{123}] + [\lambda_{124}, \tilde{y}_{124}] + [\lambda_{125}, \tilde{y}_{125}] + [\lambda_{134}, \tilde{y}_{134}] + [\lambda_{145}, \tilde{y}_{145}],
\]

\[
\tau_2: \quad b'_{24} = -[u_{24} b'_{42} u_{24}^{-1}] + [u_{13} b'_{12} u_{12}^{-1}] + [\lambda_{235}, \tilde{y}_{235}] + [\lambda_{245}, \tilde{y}_{245}] + [\lambda_{254}, \tilde{y}_{254}],
\]

\[
\tau_3: \quad b'_{34} = -[u_{35} b'_{53} u_{35}^{-1}] + [u_{24} b'_{23} u_{23}^{-1}] + [\lambda_{345}, \tilde{y}_{345}] + [\lambda_{345}, \tilde{y}_{345}],
\]

\[
\tau_4: \quad b'_{45} = -[u_{14} b'_{14} u_{14}^{-1}] + b'_{42} - [u_{23} b'_{23} u_{23}],
\]

\[
\tau_5: \quad b'_{51} = -[u_{25} b'_{25} u_{25}] + b'_{53} - [u_{45} b'_{45} u_{45}],
\]  

(202)
Our notation & Notation in [39] \\
| Face (triangle) | $b_{ij}'$ | $b_{ij}^\ast$ | \\
| Link | $u_{ij}$ | $h_{ij}$ | \\
| Edge $e$ | $\lambda_i^\ast$ | $\ell_i^\ast$ | \\
| Wedge/dual face | $Y_i^\ast$ | $V_i^\ast$ |

Table 4: Dictionary between the variables used here and the ones in [39].

Comparing this result with that of [39], we derive the dictionary between the variables here and in [39] in Table 4.

Hence our prescription allows to recover exactly phase space of [5, 39] in the case of the Euclidean Poisson Lie 2-group with a trivial Poisson structure.

4.2.6 Other choices of groups $G_i$.

There are a number of other interesting cases that we can highlight.

- **Curved case.** We have described the flat case and we can input curvature in the triangulation, ie dealing with non-abelian groups on the edges, in different manners.
  
  - *Use the $\kappa$-Poincaré deformation.* This would amount to consider the case we described in more details in Sec. 2.3. We would take
    
    $$G_1 = \text{SO}, \quad G_2 = \text{AN}, \quad G_1^\ast = \text{SO}^\ast, \quad G_2^\ast = \text{AN}^\ast.$$ 
    
    This type of deformation would work for both sign of the cosmological constant in the Lorentzian case, and for a negative cosmological constant in the Euclidian case.
    
  - *Use a self dual case.* We could start the classical double based on $T^\ast\text{Spin}(4) \cong T^\ast(\text{SU}(2) \times \text{SU}(2)) \cong (\text{SU}(2) \times \text{SU}(2)) \ltimes \mathbb{R}^6$ and semi-dualize it, to recover $(\text{SU}(2) \ltimes \mathbb{R}^3) \ltimes (\text{SU}(2) \ltimes \mathbb{R}^3)$, which can be seen as the double of the Drinfeld double of $\text{SU}(2)$ (as a Poisson Lie group).

- **Dual case.** Instead of using $\mathcal{G}$ to decorate $\mathcal{T}$, we use $\mathcal{G}^\ast$ and $\mathcal{G}$ decorates $\mathcal{T}^\ast$. This would amount to dualize the cases we considered and can be seen as the analogue of the BF vacuum picture [28].

5 Conclusion

Building a phase space associated to a triangulation is relevant both for topological studies and quantum gravity models. Upon quantization, the phase space leads to the relevant notion of quantum state of discrete geometry. Spin networks are graphs decorated by (quantum) group representations and encode the naturally quantum states of (curved) discrete geometries in a 2+1 dimensional space. While spin networks can be generalized to the 3+1 dimensional case, they are likely not the best tool to discuss discrete geometries in this case. We expect instead spin 2-networks based on representations of (quantum) 2-groups to be relevant. This means that we should have some phase space based on 2-groups associated with a triangulation. This is the construction we proposed in Def. 13.

We did not deal with the most general type of 2-groups. Instead we dealt with skeletal strict 2-groups, ie strict 2-groups such that the $t$-map is trivial, such as the Poincaré 2-group. We focused on this class of 2-groups as they are similar to groups in many aspects. Because of this similarity, we could recycle many structures such as Heisenberg double and symplectic reduction to achieve our construction.
The skeletal 2-groups we dealt with will lead upon quantization to quantum 2-groups in general. Indeed, they are in general equipped with a non-trivial Poisson structure. As an example we showed how the \( \kappa \)-Poincaré deformation arises naturally, when dealing with the Poincaré 2-group. But in fact, any deformation (ie non-trivial Poisson structure) of the Poincaré group could be used here and exported to the Poincaré 2-group setting.

We gave the explicit construction for the main building blocks of a 3d triangulation, namely the triangle and the tetrahedron. We also recovered the phase space introduced in [5] based on the un-deformed Poincaré 2-group and discussed how they can be generalized to different cases. We highlighted the \( \kappa \)-Poincaré case which can be viewed as defining polyhedra with curved edges, but still flat face decorations.

The construction of the triangulation phase space and the construction of examples of Poisson 2-groups relevant for describing the 3d triangulation phase space constitute the main results of the paper.

They open many new directions and beg for many questions to be answered.

**Generalization, away from skeletal 2-groups.** Our construction is based on the fact that skeletal Poisson 2-groups share many similarities with Poisson Lie groups. The first question to explore is how can we extend our construction to a non-trivial \( t \)-map? This would allow to have a non-abelian decorations on the faces. In this case it is unlikely that we can use the group structures and some type of semi-dualization as we did. In a sense, we expect non skeletal 2-groups to be more 2-groups than groups.

In the construction we proposed, the edge variables in \( G_2 \) are always Poisson commutative since dually the wedge decoration in \( G_2^* \) is abelian. Switching on curvature on the wedge side — making \( G_2^* \) non abelian and dealing with a non-trivial \( t \)-map in the crossed module \( G = G_1 \ltimes G_2 \) — would generate some Poisson non-commutativity for the edge variables. With the proper choice of group \( G_2^* \), such as SU(2), this could imply a discrete spectrum for the length operator. This interesting aspect is currently under investigation.

Another interesting generalization is to consider the case of weak 2-groups [8]. Finally, to have a full description of the 3d triangulation, it seems that a 3-group structure might exhaust all the possible triangulation decorations, namely, 1d, 2d, and 3d. We expect that the volume decoration would be dually equivalent to considering particle like defects. A different argument, also pointing to the use of 3-groups for matter is [64].

**Derivation from a classical action.** The phase space defined in terms of the un-deformed Poincaré 2-group, proposed in [5], can be derived from the BFCG action [37] defined as a 2-gauge theory for the Poincaré 2-group, or from the BF action seen as a gauge theory for the Poincaré group [39]. Indeed, such BFCG and BF actions can be related by a change of polarization in the translation sector only [61]. This is the continuum version of the semi-dualization relating the Lie bi-algebras \( \mathfrak{d} \) and \( \mathfrak{b} \). It would be interesting to see whether we could derive the other cases of skeletal 2-groups. In particular, we expect that the \( \kappa \)-Poincaré 2-group should arise when dealing with a SO(4, 1) BF theory, where we would do a change of polarization in the AN sector. This would allow to define a BFCG type of action of a slightly more complicated shape than the usual one, due to the presence of the mutual actions.

Another natural example to consider is SO(3, 1) BF theory which can be constrained to recover 4d gravity (with no cosmological constant). We conjecture that such theory can be discretized in terms of the 3d \( \kappa \)-Poincaré 2-group. This would provide potentially an interesting new route to define a spinfoam model for 4d gravity, when implementing the simplicity constraints.

**Simplicity constraints.** The polyhedron we defined is not really geometrical since the face decorations are independent of the edge decorations. We would like to have — on faces — some data encoding the normal and the area of the face, as done in the Minkowski theorem, while maintaining the edge information. Imposing such geometricity constraint is equivalent to the simplicity constraint which are essential to project the topological sector to the gravity sector. This type of constraint usually breaks the 2-gauge invariance. It would be interesting to see whether we can still define a 2-group where the face decoration, the normal,
is defined in terms of the edges. This would potentially provide an interesting structure to discuss discrete gravity.

**Revisiting twisted geometries.** In the usual loop quantum gravity picture, one uses $T^*\text{SU}(2)$ to decorate the link and triangle data. This phase space can be decomposed in a different manner to have discrete geometry variables, such as the dihedral angle, the normal and the area of the triangle [35, 36]. This discrete geometry description goes under the name of **twisted geometries**. We have increased the phase space dimension to include variables describing the edges (and the dual faces) which also allows to discuss curved geometries. It would be interesting to explore how this can generate a more elaborate version of the twisted geometries, possibly also incorporating curvature.

**Quantization: spin 2-networks, 2-representations of deformed Poincaré 2-group, state sum.** Studying the (2-)representation theory of a 2-group is a difficult task in general. The case of skeletal 2-groups was done in [10]. It would be interesting to see how such structure is deformed when dealing with the deformed Poincaré 2-groups. Strict quantum 2-groups have been defined in [59], but their representation theory is still an open question to the best of our knowledge. Using similar arguments as in [5], one could discuss the spectrum of the length operator and assess whether it becomes discrete or not according to the choice of Poisson 2-group.

Starting from $G$-networks, it was shown how we could recover the KBF amplitude/state sum model [43, 44, 45, 14], defined in terms of the Poincaré 2-group representations. Interestingly, there was no need to use some type of Plancherel/Peter-Weyl formula to recover it. It would be interesting to see whether we can apply this type calculations to determine the deformed version of the KBF model.

**Semi-dualization, relation between spin networks and spin 2-networks?** As we alluded several times now, skeletal 2-groups such as the Poincaré 2-group behave in some respect as groups. Since the Lie algebra case $\mathfrak{g}$ and the skeletal Lie 2-algebra case, $\mathfrak{b}$, can be related by a semi-dualization map, we conjecture that the BF amplitude based on spin networks defined for $G_1 \bowtie G_2$ could be related to a BFCG type amplitude based on spin 2-networks defined for the skeletal 2-group $G_1 \bowtie G_2^*$. We leave this for later studies.

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