Physically motivated ansatz for the Kerr spacetime

Joshua Baines and Matt Visser

School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand

E-mail: matt.visser@sms.vuw.ac.nz

Received 25 July 2022; revised 30 September 2022
Accepted for publication 18 October 2022
Published 3 November 2022

Abstract
Despite some 60 years of work on the subject of the Kerr rotating black hole there is as yet no widely accepted physically based and pedagogically viable ansatz suitable for deriving the Kerr solution without significant computational effort. (Typically involving computer-aided symbolic algebra.) Perhaps the closest one gets in this regard is the Newman–Janis trick; a trick which requires several physically unmotivated choices in order to work. Herein we shall try to make some progress on this issue by using a non-ortho-normal tetrad based on oblate spheroidal coordinates to absorb as much of the messy angular dependence as possible, leaving one to deal with a relatively simple angle-independent tetrad-component metric. That is, we shall write $g_{ab} = g_{AB} e^A_a e^B_b$ seeking to keep both the tetrad-component metric $g_{AB}$ and the non-ortho-normal co-tetrad $e^A_a$ relatively simple but non-trivial. We shall see that it is possible to put all the mass dependence into $g_{AB}$, while the non-ortho-normal co-tetrad $e^A_a$ can be chosen to be a mass-independent representation of flat Minkowski space in oblate spheroidal coordinates: $(g_{\text{Minkowski}})_{ab} = \eta_{AB} e^A_a e^B_b$. This procedure separates out, to the greatest extent possible, the mass dependence from the rotational dependence, and makes the Kerr solution perhaps a little less mysterious.

Keywords: Kerr spacetime, Kerr–Newman spacetime, oblate spheroidal coordinates, non-ortho-normal tetrads, Newman–Janis trick

1. Introduction

The Kerr solution was discovered in 1963 [1, 2], and quickly became a mainstay of general relativity, though it took the wider astrophysics community somewhat longer to appreciate its full significance. Understanding how the Kerr solution was first discovered is tricky [3], and

* Author to whom any correspondence should be addressed.
even to this day no really clean pedagogical first-principles derivation exists. Typically one tells the students: ‘Here is the answer, feed it into your favourite computer algebra system [Maple, Mathematica, Wolfram Alpha, whatever] and check that the Ricci tensor is zero.’ Interest in the Kerr spacetime is both intense and ongoing, with many review articles [4–12], at least two dedicated books [13, 14], and many textbook discussions [15–24].

The closest one has to a pedagogical first-principles derivation of the Kerr spacetime is via the Newman–Janis trick [25, 26], developed in 1965, which was immediately used in then deriving the electro-magnetically charged Kerr–Newman spacetime [27]. Despite many valiant efforts [28–38] it is still fair to say that no fully convincing explanation of why the Newman–Janis trick works has been forthcoming.

Herein we shall try a different approach:

• First, since we know that for a Newtonian rotating fluid body the Maclaurin spheroid is a good first approximation [41–45], and that this is an example of an oblate spheroid in flat three-space, one strongly suspects that oblate spheroidal coordinates might be useful when it comes to investigating rotating black holes. (And for that matter, other rotating bodies in general relativity.)
• Second, we know that the tetrad formalism is extremely useful [46], both in purely classical general relativity and especially when working with elementary particles with spin.
• Third, in a rather different context, the use of non-ortho-normal tetrads has recently proved to be extremely useful [47].

These observations suggest that it might be useful to write the spacetime metric in the form

$$g_{ab} = g_{AB} e^A_a e^B_b,$$  \hspace{1cm} (1.1)

where we shall seek to push all of the mass dependence into the tetrad-component metric $g_{AB}$, while pushing as much as possible of the rotational aspects of the problem into the non-ortho-normal co-tetrad $e^A_a$. Specifically we shall show that we can choose the non-ortho-normal co-tetrad to represent flat Minkowski space in oblate spheroidal coordinates

$$ (g_{\text{Minkowski}})_{ab} = \eta_{AB} e^A_a e^B_b; \quad \eta_{AB} = \text{diag}\{-1, 1, 1, 1\}. \hspace{1cm} (1.2)$$

This procedure separates out, to the greatest extent possible, the mass dependence from the rotational dependence, and makes the Kerr solution perhaps a little less mysterious.

2. Preliminaries

In any spacetime manifold one can always (at least locally) set up a flat metric $(g_{\text{flat}})_{ab}$ and from that flat metric extract a (non-unique) co-tetrad

$$ (g_{\text{flat}})_{ab} = \eta_{AB} e^A_a e^B_b; \quad \eta_{AB} = \text{diag}\{-1, 1, 1, 1\}. \hspace{1cm} (2.1)$$

Given such a co-tetrad, one can construct the associated tetrad $e^a_A$, (which is just the matrix inverse of the co-tetrad) and then for any arbitrary (non-flat) metric $g_{ab}$ one can always write:

$$g_{ab} = g_{AB} e^A_a e^B_b; \quad g_{AB} = g_{ab} e^a_A e^b_B. \hspace{1cm} (2.2)$$

1 A somewhat different ansatz, based on rather strong assumptions regarding the geodesics, has been explored in [39, 40].
We wish to derive the Kerr solution by finding a suitable flat-space tetrad, then make a natural and simple ansatz for $g_{AB}$, and check that $g_{ab}$ satisfies the vacuum Einstein equations $R_{ab} = 0$.

**Example.** Consider Schwarzschild spacetime. Order the coordinates as $\{t, r, \theta, \phi\}$. Take the co-tetrad for flat space written in spherical polar coordinates to be:

$$
e^A_a = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r & 0 \\
0 & 0 & 0 & r \sin \theta
\end{bmatrix}.
$$

(2.3)

Then, given the symmetries of the spacetime, a natural and simple ansatz for the tetrad-component metric $g_{AB}$ would be

$$
g_{AB} = \begin{bmatrix}
-f(r) & 0 & 0 & 0 \\
0 & 1/r & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

(2.4)

In this situation the Einstein equations yield

$$
f(r) = 1 - \frac{2m}{r}.
$$

(2.5)

Similarly, for the Reissner–Nordström spacetime one simply has

$$
f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}.
$$

(2.6)

We shall now seek to do something similar for the Kerr spacetime.

### 3. Oblate spheroidal coordinates

The Cartesian metric for flat spacetime can be rewritten in terms of oblate spheroidal coordinates by defining

$$
x = \sqrt{r^2 + a^2} \sin \theta \cos \phi; \\
y = \sqrt{r^2 + a^2} \sin \theta \sin \phi; \\
z = r \cos \theta.
$$

(3.1)

Then, ordering the coordinates as $\{t, r, \theta, \phi\}$, the metric is

$$
g_{ab} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} & 0 & 0 \\
0 & 0 & r^2 + a^2 \cos^2 \theta & 0 \\
0 & 0 & 0 & (r^2 + a^2) \sin^2 \theta
\end{bmatrix}.
$$

(3.2)

Setting $A \in \{0, 1, 2, 3\}$, an obvious (but naive) co-tetrad for this metric is

$$
e^A_a = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2}} & 0 & 0 \\
0 & 0 & \sqrt{r^2 + a^2 \cos^2 \theta} & 0 \\
0 & 0 & 0 & \sqrt{r^2 + a^2 \sin^2 \theta}
\end{bmatrix}.
$$

(3.3)
However, trying to use this co-tetrad would not be ideal for deriving the Kerr spacetime since the Kerr spacetime is stationary not static, meaning that the Kerr metric must have non-zero, off-diagonal components.

We could introduce non-diagonal components in our ansatz for the tetrad-component metric $g_{AB}$, however this vastly complicates the computations. In order to simplify the derivation, we will instead find a non-diagonal co-tetrad $e^A_a$ which will allow us to make $g_{AB}$ diagonal. More specifically, we will make an ansatz of the form

$$g_{AB} = \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

(3.4)

as we did for the Schwarzschild and Reissner–Nordström spacetimes.

4. An improved co-tetrad

Note that there exist an infinite number of co-tetrads for any given spacetime metric, related via local Lorentz transformations. (That is, if $e^A_a$ is a co-tetrad, then so is $L_{AB} e^B_a$, where $L_{AB}$ is a tangent-space Lorentz transformation). We wish to transform the naive tetrad given in equation (3.3) via a Lorentz transformation into a more useful form.

Since we are using an ansatz for $g_{AB}$ of the form given in equation (3.4), we wish the $e^0_t$ component of our new tetrad to be the reciprocal of the $e^1_r$ component. Furthermore, we will ask that the $g_{t\phi}$ component be the only non-zero off-diagonal component of our final spacetime metric $g_{ab}$. This then constrains the relevant local Lorentz transformation to be of the form

$$L^A_B = \begin{bmatrix} \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}} & 0 & 0 & L^0_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ L^3_0 & 0 & 0 & L^3_3 \end{bmatrix}.$$  

(4.1)

However, since $L^A_B$ is a Lorentz transformation, it must satisfy

$$L^C_A \eta_{CD} L^D_B = \eta_{AB}.$$  

(4.2)

This tightly constrains the components of $L^A_B$; in fact this requirement can be used to solve for the remaining three components. They are given by

$$L^3_3 = L^1_1 = \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}};$$

$$L^3_0 = L^0_3 = -\frac{a \sin \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}}.$$  

(4.3)

Here $a$ can be either positive or negative depending on the sense of rotation.

Hence, explicitly, we have

$$L^A_B = \begin{bmatrix} \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}} & 0 & 0 & -\frac{a \sin \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{a \sin \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}} & 0 & 0 & \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}} \end{bmatrix}.$$  

(4.4)
Note that \( \text{det}(L^A_B) = 1 \) and that this local Lorentz transformation corresponds to the velocity \( \beta = \frac{a \sin \theta}{\sqrt{r^2 + a^2}} \in (-1, 1) \).

Our new improved co-tetrad is now given by

\[
e^A_a = L^A_B \left( \epsilon_{\text{naive}} \right)_B^a
\]

\[
= \begin{bmatrix}
\sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}} & 0 & 0 & -\frac{\sqrt{r^2 + a^2 a \sin^2 \theta}}{\sqrt{r^2 + a^2 \cos^2 \theta}} \\
0 & \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2}} & 0 & 0 \\
0 & 0 & \sqrt{r^2 + a^2 \cos^2 \theta} & 0 \\
-\frac{a \sin \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}} & 0 & 0 & \frac{(r^2 + a^2) \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}
\end{bmatrix}
\]  

(4.5)

Using the ansatz for \( g_{AB} \), as given in equation (3.4), we now find

\[
g_{ab} = g_{AB} \ e^A_a \ e^B_b
\]

\[
= \begin{bmatrix}
\frac{-(f(r)(r^2 + a^2) + a^2 \sin^2 \theta)}{r^2 + a^2 \cos^2 \theta} & 0 & 0 & g_{\phi \phi} \\
0 & \frac{r^2 + a^2 \cos^2 \theta}{f(r)(r^2 + a^2)} & 0 & 0 \\
0 & 0 & r^2 + a^2 \cos^2 \theta & 0 \\
g_{\phi \phi} & 0 & 0 & g_{\phi \phi}
\end{bmatrix}
\]

(4.6)

Here

\[
g_{\phi \phi} = \frac{(r^2 + a^2) \ a \sin^2 \theta (f(r) - 1)}{r^2 + a^2 \cos^2 \theta},
\]

(4.7)

and

\[
g_{\phi \phi} = \frac{(r^2 + a^2) \ \sin^2 \theta (r^2 + a^2 - f(r)a^2 \sin^2 \theta)}{r^2 + a^2 \cos^2 \theta}.
\]

(4.8)

5. Final step: Einstein equations

We now apply the Einstein equations to the ansatz developed above. The vacuum Einstein equations then give a system of (partial) differential equations for the metric components \( g_{ab} \). Explicitly finding the set of partial differential equations (PDEs) is still best done with a computer algebra system, but one now has a well-defined and relatively simple problem to solve, and the PDEs reduce to ordinary differential equation (ODEs) for the function \( f(r) \).

If one works in a coordinate basis, then for general \( f(r) \) one finds that the five components \( R_{tt}, R_{\theta \phi}, R_{\phi \phi}, R_{t\phi}, \) and \( R_{\phi \phi} \) are all nonzero and some of them are quite messy. Setting \( R_{ab} = 0 \) leads to five compatible ODEs for \( f(r) \). The simplest of these five ODEs is

\[
R_{tt} \propto \frac{df(r)}{dr} (r^2 + a^2) + f(r)(r^2 - a^2) - r^2 + a^2 = 0.
\]

(5.1)

This is a first-order linear ODE which has the solution

\[
f(r) = 1 - \frac{2mr}{r^2 + a^2}.
\]

(5.2)

It is then easy to verify that this solution for \( f(r) \) then implies all other Ricci components are zero.

Alternatively, one could work in the tetrad basis where one has somewhat tidier results—in the tetrad basis only four Ricci components are nonzero, \( R_{tt}, R_{t\phi}, R_{\theta \phi} \) and \( R_{\phi \phi} \), and only two of them are algebraically independent. Specifically in the tetrad basis one has...
ally charged Kerr–Newman solution is to replace Lindquist ortho-normal you see the answer, the reason it works is obvious in hindsight—simply take the usual Boyer–Lindquist coordinates, which hence concludes the derivation.

This finally results in the fully explicit metric

\[
    g_{ab} = \begin{bmatrix}
        -1 & m r & 0 & 0 \\
        0 & \rho^2 & 0 & 0 \\
        0 & 0 & \rho^2 & 0 \\
        -2m a \sin^2 \theta & 0 & 0 & \Sigma \sin^2 \theta
    \end{bmatrix}.
\]

Here (as usual) we have \( \rho = \sqrt{r^2 + a^2 \cos^2 \theta} \), while \( \Delta = r^2 + a^2 - 2m r \), and in turn \( \Sigma = r^2 + a^2 + 2m a r \sin^2 \theta / \rho^2 \). Notice that equation (5.5) is just the Kerr metric written in the usual Boyer–Lindquist coordinates, which hence concludes the derivation.

6. Summary

The Kerr metric (and the Minkowski metric) can be related to the mass-independent co-tetrad

\[
    e^A_a = \begin{bmatrix}
        \sqrt{r^2 + a^2 \cos^2 \theta} & 0 & 0 & -\frac{\sqrt{r^2 + a^2 \sin^2 \theta}}{\sqrt{r^2 + a^2 \cos^2 \theta}} \\
        0 & \sqrt{r^2 + a^2 \cos^2 \theta} & 0 & 0 \\
        0 & 0 & \sqrt{r^2 + a^2 \cos^2 \theta} & 0 \\
        -\frac{a \sin \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}} & 0 & 0 & \frac{(r^2 + a^2) \sin \theta}{\sqrt{r^2 + a^2 \cos^2 \theta}}
    \end{bmatrix}.
\]

by the very simple relations

\[
    (g_{\text{Kerr}})_{ab} = g_{AB} e^A_a e^B_b; \quad (g_{\text{Minkowski}})_{ab} = \eta_{AB} e^A_a e^B_b;
\]

where the tetrad-component metric is particularly simple

\[
    g_{AB} = \begin{bmatrix}
        -f(r) & 0 & 0 & 0 \\
        0 & \frac{1}{f(r)} & 0 & 0 \\
        0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 1
    \end{bmatrix}; \quad f(r) = 1 - \frac{2m r}{r^2 + a^2}.
\]

This manifestly has the appropriate limit as \( a \to 0 \) and cleanly separates out the angular and radial behaviour. Furthermore the only change needed to accommodate the electromagnetically charged Kerr–Newman solution is to replace \( 2m r \to 2m r - Q^2 \) and so to set

\[
    f(r) = 1 - \frac{2m r - Q^2}{r^2 + a^2}.
\]

We have been relatively slow and careful in developing and presenting the analysis, trying to provide physical motivations for our choices at each step of the process. Of course, once you see the answer, the reason it works is obvious in hindsight—simply take the usual Boyer–Lindquist ortho-normal co-tetrad for Kerr, set \( m \to 0 \), and then set \( \eta_{AB} \to g_{AB} \) to compensate.
Given this, can we now generalize the ansatz to deal with other coordinate representations \[48, 49\] of the Kerr spacetime? (Or its slow-rotation Lense–Thirring \[50–55\] approximation?)

### 7. Extensions of the basic ansatz

We now develop several extensions and generalizations of the basic ansatz (6.3) presented above.

#### 7.1. Eddington–Finkelstein (Kerr–Schild) form

Take

\[
g_{AB} = \begin{bmatrix}
-1 + \Phi & 0 & 0 \\
\Phi & 1 + \Phi & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}; \quad \Phi = \frac{2mr}{r^2 + a^2}.
\]

(7.1)

Keep exactly the same non-ortho-normal co-tetrad \(e_A^a\) as above. Then the metric \(g_{ab} = g_{AB} e_A^a e_B^b\) is still Ricci flat—so it is the Kerr solution in disguise. This modified ansatz was inspired by inspecting and generalizing the Eddington–Finkelstein (Kerr–Schild) form of Schwarzschild. Defining \(\ell_A = (1, 1, 0, 0)\) we note that

\[
g_{AB} = \eta_{AB} + \Phi \ell_A \ell_B
\]

(7.2)

which is of Kerr–Schild form. Contracting with the co-tetrad and defining \(\ell_a = \ell_A e_A^a\) we have

\[
g_{ab} = (g_{\text{Minkowski}})_{ab} + \Phi \ell_a \ell_b,
\]

(7.3)

where the Minkowski space metric is written in oblate spheroidal coordinates as per (3.2) and

\[
\ell_a = \left(\sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}}, \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2}}, 0, -a \sin^2 \theta \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}}\right).
\]

(7.4)

Since \(\ell_a\) is easily checked to be a null vector, this is manifestly seen to be Kerr spacetime in Kerr–Schild form \[3, 4, 13\].

#### 7.2. Quasi-Painlevé–Gullstrand form

Take

\[
g_{AB} = \begin{bmatrix}
-1 + \Phi & \sqrt{\Phi} & 0 & 0 \\
\sqrt{\Phi} & 1 + \Phi & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad \Phi = \frac{2mr}{r^2 + a^2}.
\]

(7.5)

Keep exactly the same non-ortho-normal co-tetrad \(e_A^a\) as above. Then the metric \(g_{ab} = g_{AB} e_A^a e_B^b\) is still Ricci flat—so it is the Kerr solution in disguise. This modified ansatz was inspired by looking at and generalizing the Painlevé–Gullstrand form of Schwarzschild; though it was not derived therefrom—more on this point later. \[56, 57, 58\] The tetrad metric \(g_{AB}\) is of Painlevé–Gullstrand form, but the coordinate metric \(g_{ab}\) is not, (and, in view of the analysis by Valiente-Kroon \[59, 60\], cannot possibly be), of Painlevé–Gullstrand form.
It is convenient to introduce two vectors, $T_A = (1, 0, 0, 0)$ and $S_A = (0, 1, 0, 0)$, since then

$$g_{AB} = \eta_{AB} + \Phi \ T_A T_B + \sqrt{\Phi} (T_A S_B + S_A T_B).$$

(7.6)

We can furthermore factorize this as follows

$$g_{AB} = \eta_{CD} \left( \delta^A_C + \sqrt{\Phi} \ S^C T_A \right) \left( \delta^D_B + \sqrt{\Phi} \ S^D T_B \right),$$

(7.7)

implying the existence of a factorizable ortho-normal co-tetrad

$$(e_{ortho})^A_a = \left( \delta^A_B + \sqrt{\Phi} \ S^A T_B \right) e^B_a.$$  

(7.8)

Note that all the mass-dependence is concentrated in $\Phi$, whereas all the angular dependence is still concentrated in the usual mass-independent non-ortho-normal tetrad $e^B_a$.

Let us see what happens in the coordinate basis: Setting

$$T_a = T_A e_A^a = \left( \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}}, 0, 0, -a \sin^2 \theta \sqrt{\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}} \right),$$

(7.9)

and

$$S_a = S_A e_A^a = \left( 0, \sqrt{\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2}}, 0, 0 \right),$$

(7.10)

we see

$$g_{ab} = (g_{\text{Minkowski}})_{ab} + \Phi \ T_a T_b + \sqrt{\Phi} (T_a S_b + S_a T_b),$$

(7.11)

where the Minkowski space metric is written in oblate spheroidal coordinates as per (3.2). The vectors $T_a$ and $S_a$ are orthonormal timelike and spacelike vectors with respect to both the Minkowski metric (3.2) and the full metric (7.11). In the language of the Hamilton–Lisle ‘river model’ [61] these are easily identified as what they call the ‘twist’ and ‘flow’ vectors, and so this quasi-Painlevé–Gullstrand version of the Kerr metric is equivalent to the Doran form [62] of the Kerr metric. This is the closest we can get to putting the Kerr metric into Painlevé–Gullstrand form—partial success at the tetrad level, but failure at the coordinate level. Finally, we observe that explicit computation reveals that $g^{tt} = -1$, so this version of the metric is definitely unit lapse [48].

7.3. 1-free-function form

Let $h(r)$ be an arbitrary differentiable function and take

$$g_{AB} = \begin{bmatrix} -f(r) & -f(r) h(r) & 0 & 0 \\ -f(r) h(r) & \frac{1}{f(r)} - f(r) h(r)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$f(r) = 1 - \frac{2mr}{r^2 + a^2}.$$  

(7.12)

Keep exactly the same non-ortho-normal co-tetrad $e^A_a$ as above. Then the metric $g_{ab} = g_{AB} e^A_a e^B_b$ is still Ricci flat—so it is the Kerr solution in disguise.

This ansatz was inspired by looking at and generalizing the Boyer–Lindquist, Kerr–Schild, and quasi-Painlevé–Gullstrand forms of Kerr discussed above; not derived therefrom. With hindsight, one strongly suspects an underlying coordinate transformation is responsible for this
behaviour. Indeed after a little ‘reverse engineering’ one is lead to consider the not particularly obvious coordinate transformation
\[ t \to t + \int h(r) \, dr; \quad \phi \to \phi + \int \frac{a h(r)}{r^2 + a^2} \, dr. \] (7.13)

Writing the new coordinates as \( \bar{x}^a \) the relevant Jacobi matrix is
\[ J^a \_b = (\bar{x}^a) \_b = \frac{\partial \bar{x}^a}{\partial x^b} = \begin{bmatrix} 1 & h(r) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{a h(r)}{r^2 + a^2} & 0 & 1 \end{bmatrix}. \] (7.14)

Going to the tetrad basis an easy computation yields
\[ J^A \_B = J^a \_b \, e^A \_a \, e^b \_B = \begin{bmatrix} 1 & h(r) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \] (7.15)

But then it is easy to check that
\[ \begin{bmatrix} 1 & h(r) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & h(r) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
\[ = \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ -f(r) h(r) & \frac{1}{r^2} - f(r) h(r)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \] (7.16)

That is, the 1-free-function ansatz (7.12) can be obtained from the basic ansatz (6.3) by the very specific coordinate transformation (7.13); with the specific coordinate transformation being carefully ‘reverse engineered’ to do minimal violence to the original basic ansatz.

7.4. Lense–Thirring limit

Consider the slow rotation limit \( a \to 0 \), explicitly keeping the first two terms, while keeping \( h(r) \) arbitrary, then
\[ e^A \_a = \begin{bmatrix} 1 & 0 & 0 & -a \sin^2 \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ -\frac{a \sin \theta}{r} & 0 & 0 & r \sin \theta \end{bmatrix} + \mathcal{O}(a^3). \] (7.17)

and
\[ g_{AB} = \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ -f(r) h(r) & \frac{1}{r^2} - f(r) h(r)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \]
\[ f(r) = 1 - \frac{2m}{r} + \frac{2ma^2}{r^2} + \mathcal{O}(a^4). \] (7.18)
Thus our 1-free-function ansatz (7.12) leads to an entire class of tetrad metrics $g_{AB}$, (and implicitly, the corresponding coordinate basis metrics $g_{ab}$), that are appropriate for describing the exterior spacetime of slowly rotating objects. This Lense–Thirring slow rotation limit [50, 51], in its various incarnations [52–55] is of significant importance in observational astrophysics.

7.5. Summary

In this section we have seen how our basic ansatz (6.3), which we originally developed for physically motivating and then deriving the Kerr solution with a minimum of fuss, can be extended and modified to deal with other coordinate representations of the Kerr metric—such as the Eddington–Finkelstein (Kerr–Schild) coordinates (7.1), the quasi-Painlevé–Gullstrand (Doran) coordinates (7.5), and an entire 1-free-function class of coordinate systems (7.12) that still respect most of the fundamental symmetries of the original ansatz.

8. Discussion

We have physically motivated an ansatz for the Kerr spacetime metric, partially based on Newtonian physics, (the fact that Maclaurin’s oblate spheroids already became of interest for rotating bodies some 280 years ago), and partially based on the fact that tetrad methods are known to be useful in general relativity. Specifically, the key step is to write the coordinate metric as $g_{ab} = g_{AB} e^A_a e^B_b$, while allowing the use of non-ortho-normal tetrads.

We have seen that doing so permits one to force all of the non-trivial angular dependence into a mass-independent co-tetrad $e^A_a$ that is compatible with flat spacetime in oblate spheroidal coordinates, while forcing all of the mass-dependence (and none of the angular dependence) into the tetrad-basis metric $g_{AB}$. This clean separation between angular dependence and mass dependence greatly simplifies the computational complexity of the problem. We expect these ideas to have further applications and implications.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

J B was supported by a Victoria University of Wellington PhD Doctoral Scholarship and was also indirectly supported by the Marsden Fund, via a grant administered by the Royal Society of New Zealand. M V was directly supported by the Marsden Fund, via a grant administered by the Royal Society of New Zealand.

ORCID iDs

Joshua Baines  https://orcid.org/0000-0003-1200-7261
Matt Visser  https://orcid.org/0000-0003-1088-6485
References

[1] Kerr R 1963 Gravitational field of a spinning mass as an example of algebraically special metrics Phys. Rev. Lett. 11 237–8 Reprinted in [13]
[2] Kerr R 1965 Gravitational collapse and rotation Quasi-Stellar Sources and Gravitational Collapse: Including the Proc. 1st Texas Symp. on Relativistic Astrophysics (Austin, Texas, 16–18 December 1963) ed I Robinson, A Schild and E L Schücking (Chicago, IL: University of Chicago Press) pp 99–102 (Reprinted in [13])
[3] Kerr R 2008 Discovering the Kerr and Kerr-Schild metrics (arXiv:0706.1109 [gr-qc]) (Published in [13])
[4] Visser M 2008 The Kerr spacetime: a brief introduction (arXiv:0706.0622 [gr-qc]) (Published in [13])
[5] Teukolsky S A 2015 The Kerr metric Class. Quantum Grav. 32 124006
[6] Adamo T and Newman E T 2014 The Kerr–Newman metric: a review Scholarpedia 9 31791
[7] Heinicke C and Hehl F W 2014 Schwarzschild and Kerr solutions of Einstein’s field equation—an introduction Int. J. Mod. Phys. D 24 1530006
[8] Bambi C 2011 Testing the Kerr black hole hypothesis Mod. Phys. Lett. A 26 2453–68
[9] Johannsen T 2016 Sgr A∗ and general relativity Class. Quantum Grav. 33 113001
[10] Reynolds C S 2013 The spin of supermassive black holes Class. Quantum Grav. 30 244004
[11] Reynolds C S 2019 Observing black holes spin Nat. Astron. 3 41–47
[12] Bambi C 2018 Astrophysical black holes: a compact pedagogical review Ann. Phys. Lpz. 530 1700430
[13] Wiltshire D L, Visser M and Scott S M (eds) 2009 The Kerr Spacetime: Rotating Black Holes in General Relativity (Cambridge: Cambridge University Press)
[14] O’Neill B 1995 The Geometry of Kerr Black Holes (Wellesley, MA: Peters) Reprinted (Dover, Mineola, 2014).
[15] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Hoboken, NJ: Wiley)
[16] Misner C, Thorne K and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[17] Adler R J, Bazin M and Schiffer M 1975 Introduction to General Relativity 2nd edn (New York: McGraw–Hill) (It is important to acquire the 1975 second edition, the 1965 first edition does not contain any discussion of the Kerr spacetime)
[18] Wald R 1984 General Relativity (Chicago, IL: University of Chicago Press)
[19] D’Inverno R 1992 Introducing Einstein’s Relativity (Oxford: Oxford University Press)
[20] Hartle J 2003 Gravity: An Introduction to Einstein’s General Relativity (Reading, MA: Addison-Wesley)
[21] Carroll S 2004 An Introduction to General Relativity: Spacetime and Geometry (Reading, MA: Addison-Wesley)
[22] Hobson M P, Estathiou G P and Lasenby A N 2006 General Relativity: An Introduction for Physicists (Cambridge: Cambridge University Press)
[23] Poisson E 2010 A Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics (Cambridge: Cambridge University Press)
[24] Padmanabhan T 2010 Gravitation: Foundations and Frontiers (Cambridge: Cambridge University Press)
[25] Newman E T and Janis A I 1965 Note on the Kerr spinning particle metric J. Math. Phys. 6 915–7
[26] Newman-Janis algorithm (trick) 2022 (available at: https://en.wikipedia.org/wiki/Newman-Janis_algorithm)
[27] Newman E, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 Metric of a rotating, charged mass J. Math. Phys. 6 918–9
[28] Newman E T 1973 Complex coordinate transformations and the Schwarzschild–Kerr metrics J. Math. Phys. 14 774–6
[29] Giampietro G 1990 Introducing angular momentum into a black hole using complex variables Gravity Research Foundation Essay Contest
[30] Drake S P and Turolla R 1997 The application of the Newman–Janis algorithm in obtaining interior solutions of the Kerr metric Class. Quantum Grav. 14 1883–97
[31] Drake S P and Szekeres P 2000 Uniqueness of the Newman–Janis algorithm in generating the Kerr–Newman metric (Originally titled: “An explanation of the Newman–Janis algorithm”) Gen. Relativ. Gravit. 32 445–58
[32] Viaggiu S 2006 Interior Kerr solutions with the Newman–Janis algorithm starting with static physically reasonable spacetimes Int. J. Mod. Phys. D 15 1441–53
[33] Hansen D and Yunes N 2013 Applicability of the Newman–Janis algorithm to black hole solutions of modified gravity theories Phys. Rev. D 88 104020
[34] Ferraro R 2014 Untangling the Newman–Janis algorithm Gen. Relativ. Gravit. 46 1705
[35] Keane A J 2014 An extension of the Newman–Janis algorithm Class. Quantum Grav. 31 155003
[36] Erbin H 2016 Deciphering and generalizing Demiański–Janis–Newman algorithm Gen. Relativ. Gravit. 48 56
[37] Erbin H 2017 Janis–Newman algorithm: generating rotating and NUT charged black holes Universe 3 19
[38] Rajan D and Visser M 2017 Cartesian Kerr–Schild variation on the Newman–Janis trick Int. J. Mod. Phys. D 26 1750167
[39] Dadhich N 2013 A novel derivation of the rotating black hole metric Gen. Relativ. Gravit. 45 2383–8
[40] Dastgerdi A A, Mirza B and Dadhich N 2022 Novel way to the metric of higher dimensional rotating black holes Phys. Rev. D 105 124068
[41] Maclaurin spheroid 2022 (available at: https://en.wikipedia.org/wiki/Maclaurin_spheroid)
[42] Maclaurin C 1742 A Treatise of Fluxions: In Two Books vol 1 (Edinburg: Ruddimans) (Reprint: (Farmington Hills, MI: Gale ECCO, Print editions, 27 May 2010))
[43] Chandrasekhar S 1969 Ellipsoidal Figures of Equilibrium vol 10 (New Haven, CT: Yale University Press)
[44] Poisson E and Will C 2014 Gravity: Newtonian, Post-Newtonian, Relativistic (Cambridge: Cambridge University Press) pp 102–4
[45] Lyttleton R A 1953 The Stability Of Rotating Liquid Masses (Cambridge: Cambridge University Press)
[46] Rajan D and Visser M 2016 Global properties of physically interesting Lorentzian spacetimes Int. J. Mod. Phys. D 25 1650106
[47] Visser M 2021 Feynman’s $\epsilon$ prescription, almost real spacetimes, and acceptable complex spacetimes (arXiv:2101.14016)
[48] Baines J, Berry T, Simpson A and Visser M 2021 Unit-lapse versions of the Kerr spacetime Class. Quantum Grav. 38 055001
[49] Baines J, Berry T, Simpson A and Visser M 2021 Darboux diagonalization of the spatial 3-metric in Kerr spacetime Gen. Relativ. Gravit. 53 3
[50] Thirring H, Lense J 1918 Über den Einfluss der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie Phys. Z. 19 156–63
[51] Pfister H 2007 On the history of the so-called Lense–Thirring effect Gen. Relativ. Gravit. 39 1735–48
[52] Baines J, Berry T, Simpson A and Visser M 2021 Painlevé–Gullstrand form of the Lense–Thirring Spacetime Universe 7 105
[53] Baines J, Berry T, Simpson A and Visser M 2021 Killing tensor and Carter constant for Painlevé–Gullstrand form of Lense–Thirring spacetime Universe 7 473
[54] Baines J, Berry T, Simpson A and Visser M 2022 Geodesics for the Painlevé–Gullstrand form of Lense–Thirring spacetime Universe 8 115
[55] Baines J, Berry T, Simpson A and Visser M 2022 Constant-$r$ geodesics in the Painlevé–Gullstrand form of Lense–Thirring spacetime (title changed from: “Non-equatorial circular geodesics for the Painleve-Gullstrand form of Lense–Thirring spacetime”) Gen. Relativ. Gravit. 54 79
[56] Painlevé P 1921 La mécanique classique et la théorie de la relativité C. R. Acad. Sci., Paris 173 677–80
[57] Visser M 2005 Heuristic approach to the Schwarzschild geometry Int. J. Mod. Phys. D 14 2051–68
[58] Painlevé P 1921 La gravitation dans la mécanique de Newton et dans la mécanique d’Einstein C. R. Acad. Sci., Paris 173 873–86
[59] Valiente Kroon J A 2004 On the nonexistence of conformally flat slices in the Kerr and other stationary space-times Phys. Rev. Lett. 92 041101
[60] Valiente Kroon J A 2004 Asymptotic expansions of the Cotton–York tensor on slices of stationary space-times Class. Quantum Grav. 21 3237–50
[61] Hamilton A J S and Lisle J P 2008 The river model of black holes Am. J. Phys. 76 519–32
[62] Doran C 2000 A new form of the Kerr solution Phys. Rev. D 61 067503
[63] Gullstrand A 1922 Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie Arkiv för Matematik, Astronomi och Fysik vol 16 (Stockholm: Almqvist & Wiksell) pp 1–15