Gravitating Q-balls in the Affleck-Dine mechanism

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We investigate how gravity affects “Q-balls” with the Affleck-Dine potential

\[ V_{AD}(\phi) := \frac{m^2}{2} \phi^2 \left[ 1 + K \ln \left( \frac{\phi}{M} \right)^2 \right]. \]

Contrary to the flat case, in which equilibrium solutions exist only if \( K < 0 \), we find three types of gravitating solutions as follows. In the case that \( K < 0 \), ordinary Q-ball solutions exist; there is an upper bound of the charge due to gravity. In the case that \( K = 0 \), equilibrium solutions called (mini-)boson stars appear due to gravity; there is an upper bound of the charge, too. In the case that \( K > 0 \), equilibrium solutions appear, too. In this case, these solutions are not asymptotically flat but surrounded by Q-matter. These solutions might be important in considering a dark matter scenario in the Affleck-Dine mechanism.

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I. INTRODUCTION

Q-balls \[1\], a kind of nontopological solitons \[2\], appear in a large family of field theories with global U(1) (or more) symmetry. In particular, it has been argued that Q-balls with the Affleck-Dine (AD) potential could play important roles in cosmology \[3\]. For example, Q-balls can be produced efficiently and could be responsible for baryon asymmetry \[4\] and dark matter \[5\]. In the AD mechanism, there are two types of potentials: gravity-mediation type and gauge-mediation type. Here, we concentrate on the gravity-mediation type,

\[ V_{AD}(\phi) := \frac{m^2}{2} \phi^2 \left[ 1 + K \ln \left( \frac{\phi}{M} \right)^2 \right] \text{ with } m^2, M > 0. \]  \hspace{1cm} (1.1)

In general, there may be nonrenormalizable terms, U(1) violation terms, and so on. Here we neglect them for simplicity. Because Q-balls are typically supposed to be microscopic objects, their self-gravity is usually ignored. Therefore, stability of Q-balls with various potentials has been intensively studied in flat spacetime \[6, 7\]. As for the AD potential \[11\], it has been known that equilibrium solutions for \( K \geq 0 \) are nonexistent while those for \( K < 0 \) are existent and stable. One may speculate that these properties are not changed by gravity if self-gravity is weak enough.

However, this speculation is not necessarily true for the following reasons. First, for the potential \( V = m^2 \phi^2/2 \), no equilibrium solution exists without gravity but equilibrium solutions, called (mini-)boson stars, exist due to self-gravity \[10\]. This is a direct evidence that there are equilibrium solutions for \( K = 0 \) with \[11\].

Second, in our previous paper \[11\], we considered gravitating Q-balls with

\[ V_4(\phi) := \frac{m^2}{2} \phi^2 - \lambda \phi^4 + \frac{\phi^6}{M^2} \text{ with } m^2, \lambda, M > 0. \]  \hspace{1cm} (1.2)

In flat spacetime Q-balls with \( V_4 \) in the thick-wall limit are unstable and there is a minimum charge \( Q_{\text{min}} \), where Q-balls with \( Q < Q_{\text{min}} \) are nonexistent. If we take self-gravity into account, on the other hand, there exist stable Q-balls with arbitrarily small charge, no matter how weak gravity is.

Therefore, it is valuable to examine the influence of gravity in AD potential \[11\]. As a result, we find that upper bound of the Q-ball charge appears due to gravity for \( K < 0 \) and there appear “Q-balls” for \( K \geq 0 \) which do not exist without gravity. Here we call all equilibrium solutions “Q-balls” collectively, though solutions supported by gravity are usually called boson stars.

This paper is organized as follows. In Sec. II, we derive equilibrium field equations. In Sec. III, we show numerical results of equilibrium Q-balls for \( K < 0 \) and discuss existence of “Q-balls” for \( K \geq 0 \) which do not exist without gravity. We explain why “Q-balls” called (mini-)boson stars exist with the influence of gravity. In the same mechanism, there appear “Q-balls” even for \( K > 0 \). In this case, “Q-balls” are surrounded by Q-matter. In Sec. IV, we devote to concluding remarks.

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II. ANALYSIS METHOD OF EQUILIBRIUM Q-BALLS

A. Equilibrium field equations

We begin with the action

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \cdot \partial_{\nu} \phi - V(\phi) \right\} , \]

where \( \phi = (\phi_1, \phi_2) \) is an SO(2)-symmetric scalar field and \( \phi := \sqrt{\phi_\prime^2 + \phi_2^2} \). We assume a spherically symmetric and static spacetime,

\[ ds^2 = -\alpha^2(r)dt^2 + A^2(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]  

For the scalar field, we assume that it has a spherically symmetric and stationary form,

\[ (\phi_1, \phi_2) = \phi(r)(\cos \omega t, \sin \omega t). \]

Then the field equations become

\[ -\frac{rA^3}{2} G_{rr}^1 := A^' + \frac{A}{2r}(A^2 - 1) \]

\[ = 4\pi GrA^3 \left( \frac{\phi_4^2}{2A^2} + \frac{\omega^2 \phi_2^2}{2\alpha^2} + V \right), \]

\[ \frac{r\alpha}{2} G_{rr}^2 := \alpha^' + \frac{\alpha}{2r}(1 - A^2) \]

\[ = 4\pi Gr\alpha A^2 \left( \frac{\phi_4^2}{2A^2} + \frac{\omega^2 \phi_2^2}{2\alpha^2} - V \right), \]

\[ \frac{A^2 \phi}{\phi_1} \Box \phi_1 := \phi^'' + \left( \frac{2}{r} + \frac{\alpha'}{\alpha} - \frac{A^'}{A} \right) \phi' + \left( \frac{\omega A}{\alpha} \right)^2 \phi \]

\[ = A^2 \frac{dV}{d\phi}, \]

where \( \prime := d/dr \). To obtain Q-ball solutions in curved spacetime, we should solve (2.4)-(2.6) with boundary conditions,

\[ A(0) = A(\infty) = \alpha(\infty) = 1, \]

\[ A'(0) = \alpha'(0) = \phi'(0) = \phi(\infty) = 0. \]

We also restrict our solutions to monotonically decreasing \( \phi(r) \). Because of the symmetry, there is a conserved charge called Q-ball charge,

\[ Q := \int d^3x \sqrt{-g} g^{\mu\nu}(\phi_1 \partial_\nu \phi_2 - \phi_2 \partial_\nu \phi_1) = \omega I, \]

where \( I := 4\pi \int \frac{A^2 \phi^2}{\alpha} dr. \)

We suppose \( V_{AD} \) Model (1.1). Rescaling the quantities as

\[ \hat{\phi} \equiv \frac{\phi}{M}, \quad \hat{V}_{AD} \equiv \frac{V_{AD}}{m^2 M^2} = \frac{\hat{\phi}^2}{2} \left( 1 + 2K \ln \hat{\phi} \right), \]

\[ \hat{\omega} \equiv \frac{\omega}{m}, \quad \hat{\kappa} = GM^2, \quad \hat{t} \equiv mt, \quad \hat{r} \equiv mr, \]

the field equations (2.4)-(2.6) are rewritten as

\[ A' + \frac{A}{2r}(A^2 - 1) = 4\pi \kappa \hat{\phi} A^3 \left( \frac{\hat{\phi}^2}{2A^2} + \frac{\hat{\omega}^2 \hat{\phi}^2}{2\alpha^2} + V_{AD} \right), \]

\[ \alpha' + \frac{\alpha}{2r}(1 - A^2) = 4\pi \kappa \hat{\phi} A^2 \left( \frac{\hat{\phi}^2}{2A^2} + \frac{\hat{\omega}^2 \hat{\phi}^2}{2\alpha^2} - V_{AD} \right), \]

\[ \hat{\phi}'' + \left( \frac{2}{\hat{r}} + \frac{\alpha'}{\alpha} - \frac{A'}{A} \right) \hat{\phi}' + \left( \frac{\hat{\omega} A}{\alpha} \right)^2 \hat{\phi} = A^2 \frac{dV_{AD}}{d\hat{\phi}}. \]

B. Equilibrium solutions in flat spacetime

FIG. 1: \(-V_\omega\) for a Q-ball in flat spacetime (\( \kappa = 0 \)). We put \( K = -0.01 \) and \( \hat{\omega}^2 = 1.04 \).

In preparation for discussing gravitating “Q-balls”, we review their equilibrium solutions in flat spacetime (\( \kappa = 0 \)). The scalar field equation (2.12) reduces to

\[ \hat{\phi}'' = \frac{2}{\hat{r}} \hat{\phi}' - \hat{\omega}^2 \hat{\phi} + \frac{dV_{AD}}{d\hat{\phi}}. \]

This is equivalent to the field equation for a single static scalar field with the potential \( V_\omega := V_{AD} - \hat{\omega}^2 \hat{\phi}^2 / 2 \). Equilibrium solutions satisfying boundary conditions exist if

\[ \max(V_\omega) > \hat{V}_{AD}(\hat{\phi} \to 0) \]

and \( \frac{d^2V_\omega}{d\hat{\phi}^2}(\hat{\phi} \to 0) > 0 \). If we introduce \( \hat{\epsilon} := 1 - \hat{\omega}^2 \), we obtain

\[ \frac{dV_\omega}{d\hat{\phi}} = \hat{\phi}(\hat{\epsilon}^2 + K + 2K \ln \hat{\phi}), \]

\[ \frac{d^2V_\omega}{d\hat{\phi}^2} = \hat{\epsilon}^2 + 3K + 2K \ln \hat{\phi}. \]
The second condition in (2.14) leads to

\[ K < 0 , \]  

or

\[ K = 0 \text{ and } \epsilon^2 > 0 . \]  

In the former case (2.17), \( V_\omega \) has a maximum at \( \tilde{\phi} = \tilde{\phi}_1 := e^{-\frac{\tilde{\epsilon}^2}{\alpha^2}} \); then the first condition in (2.14) becomes

\[ \frac{K}{2} \tilde{\phi}_1^2 < 0 , \]  

which is trivially satisfied. In the latter case (2.18), there is no maximum; then, there is no equilibrium solution.

If one regards the radius \( r \) as ‘time’ and the scalar amplitude \( \phi(r) \) as ‘the position of a particle’, one can understand Q-ball solutions in words of Newtonian mechanics, as shown in Fig. 1. Equation (2.13) describes a one-dimensional motion of a particle under the conserved force due to the potential \(-V_\omega(\phi)\) and the ‘time’-dependent friction \(-\frac{2}{r} \frac{d\phi}{dr}\). Here we put \( K = -0.01, \tilde{\omega}^2 = 1.04 \). In this case, the scalar field \( \phi \approx 0.37 \) at the initial time \( \tilde{r} = 0 \) rolls down the potential and finally reaches \( \tilde{\phi} = 0 \) at the time \( \tilde{r} \to \infty \).

### III. GRAVITATING “Q-BALLS”

The potential picture described above is also effective to argue equilibrium solutions in curved spacetime. In this case, \( \epsilon^2 \) should be redefined by

\[ \epsilon^2 := 1 - \frac{\tilde{\omega}^2}{\alpha^2} . \]  

Because ‘the potential of a particle’, \(-V_\omega\), is now ‘time’-dependent, the existence conditions of equilibrium solutions are not as simple as those in flat spacetime.

#### A. \( K < 0 \)

We discuss the existence of equilibrium solutions by analogy with Newtonian mechanics, as shown in Fig. 2 (a). We also exhibit behaviors of the metric functions in Fig. 2 (b). Because \(-V_\omega\) depends on the ‘time’ \( \tilde{r} \), it has a minimum at \( \tilde{\phi} \approx 0.08 \) when \( \tilde{r} = 100 \) while it has a minimum at \( \tilde{\phi} \approx 0 \) when \( \tilde{r} = 0 \). At the ‘initial time’ \( \tilde{r} = 0 \) the scalar field at \( \phi \approx 0.089 \) rolls down the potential and finally reaches \( \phi = 0 \) at the time \( \tilde{r} \to \infty \). We thus understand how gravity changes properties of equilibrium solutions.

As we discussed in our previous papers [11, 12], stability of Q-balls can be easily understood from the relation between \( Q \) and the Hamiltonian energy \( E \), which is defined by

\[ E = \lim_{r \to \infty} \frac{r^2 \alpha'}{2GA} = \frac{M_S}{2} , \]  

![FIG. 2](image_url)  

**FIG. 2:** (a) \(-V_\omega\) and (b) behaviors of the metric functions for a gravitating Q-balls. We put \( K = -0.01, \tilde{\omega}^2 \approx 1.045 \) and \( \kappa = 0.01 \). \( V_\omega \) changes as ‘time’ \( \tilde{r} \) goes.

![FIG. 3](image_url)  

**FIG. 3:** \( \tilde{Q} - \tilde{E} \) relation for \( K = -0.01 \).
where $M_S$ is the Schwarzschild mass. Here, stability means local stability, that is, stability against small perturbations. We also normalize $E$ and $Q$ as

$$
\tilde{E} := \frac{mE}{M^2}, \quad \tilde{Q} := \frac{m^2Q}{M^2}.
$$

(3.3)

We compare $\tilde{Q}\cdot\tilde{E}$ relation for the flat case ($\kappa = 0$) with that for the gravitating case $\kappa = 0.01$ in Fig. 3. In the case that $\kappa = 0$, $\tilde{Q}$ is almost proportional to $\tilde{E}$, and accordingly, all solutions for this parameter range are stable. In the case that $\kappa = 0.01$, however, there is a cusp structure at the point $A$, where stability changes. If there are two solutions for fixed $\tilde{Q}$, the solution with larger energy $\tilde{E}$ should be unstable. That is, the upper branch represents unstable solutions. At the same time, this cusp structure indicates that there is a maximum charge $Q_{\text{max}}$, where solutions with $\tilde{Q} > Q_{\text{max}}$ are nonexistent due to gravity. This is a common feature with $V_3$ and $V_4$ models [11–13]. If we take larger (smaller) $\kappa$, the point corresponding to $A$ has smaller (larger) $\tilde{E}$. However, qualitative features do not change.

B. $K = 0$

![Image of Fig. 4: $-V_\omega$ for a (mini-)boson star. We put $K = 0$, $\tilde{\omega}^2 \simeq 0.92$ and $\kappa = 0.01$. Because $-V_\omega$ changes as $\tilde{r}$, the scalar field with $\tilde{\phi} \simeq 0.2$ at the initial time $\tilde{r} = 0$ rolls down and finally reaches $\tilde{\phi} = 0$ at the time $\tilde{r} \to \infty$.](image)

This case corresponds to the potential for the mini-boson stars, which have been investigated in the literatures [10]. First, we explain why mini-boson stars appear if we include self-gravity. The key point is that the sign of $\epsilon^2 = d^2V/d\tilde{\phi}^2(0)$ depends on $\tilde{r}$. Figure 4 shows how the shape of $-V_\omega$ changes as $\tilde{r}$ varies. The scalar field rolls down the potential $-V_\omega$ near $\tilde{r} = 0$ while it climbs up $-V_\omega$ in the asymptotic region. As a result, the scalar field at the ‘initial position’ $\tilde{\phi}(0) \simeq 0.2$ satisfies the asymptotic boundary condition $\tilde{\phi}(\infty) = 0$. In contrast,

![Image of Fig. 5: $\tilde{Q}\cdot\tilde{E}$ relation for $K = 0$ and $\kappa = 0.01$.](image)

in the case of flat spacetime, because $-V_\omega$ is a monotonically decreasing (increasing) function of $\tilde{\phi}$ for $\kappa^2 > 0$ ($< 0$), there is no equilibrium solution.

We also show $\tilde{Q}\cdot\tilde{E}$ relation for $K = 0$ and $\kappa = 0.01$ in Fig. 5. The result is similar to that for $K = -0.01$. There is a maximum charge $Q_{\text{max}}$. The lower branch represents stable solutions while the upper branch unstable.

C. $K > 0$

![Image of Fig. 6: $-V_\omega$ for $K > 0$. We put $K = 0.1$, $\tilde{\omega}^2/\alpha^2(\tilde{r} = 0) \simeq 1.2$, and $\kappa = 0.01$. Because of $d^2V_\omega/d\tilde{\phi}^2(\tilde{\phi} \to 0) < 0$, there is no Q-ball which satisfies (2.7). However, the scalar field can stop at the maximum of $-V_\omega$, $\tilde{\phi} = \tilde{\phi}_1 \simeq 0.03$ in the large $\tilde{r}$ region, if the scalar field with initial value $\tilde{\phi}(0) \simeq 0.56$ (outside the figure) rolls down.](image)

As in the case that $K = 0$, the shape of $V_\omega$ depends on $\tilde{r}$. Figure 6 shows the potential $-V_\omega$ for $K = 0.1$, $\tilde{\omega}^2/\alpha^2 \simeq 1.2$ at $\tilde{r} = 0$ and $\kappa = 0.01$. However, because
d²V_ω/dφ²(φ → 0) < 0 regardless of ε², there is no Q-ball solution which satisfies (2.7) even if we include gravity.

However, we should notice −V_ω maximum around φ = φ₁ ≃ 0.03 in the large r region. Figure 6 indicates that if the scalar field rolls down from φ > φ₁, there is a solution which satisfies φ = φ₁ at r → ∞. We show the example of such a solution in Fig. 7 for the same parameters as in Fig. 6. We have also confirmed that this kind of solution is generic for K > 0.

Our solutions are not asymptotically flat but surrounded by Q-matter. Because the energy E and the charge Q are diverging, we cannot apply energetics or catastrophe theory to these solutions. Stability analysis is the next important issue.

IV. CONCLUSION AND DISCUSSION

We have investigated gravitating “Q-balls” in the gravity-mediated AD mechanism [11]. Contrary to the flat case, in which equilibrium solutions exist only if K < 0, we have found three types of gravitating solutions as follows. In the case that K < 0, ordinary Q-ball solutions exist; there is an upper bound of the charge due to gravity. In the case that K = 0, equilibrium solutions called (mini-)boson stars appear due to gravity; there is an upper bound of the charge, too. In the case that K > 0, equilibrium solutions appear, too. In this case, these solutions are not asymptotically flat but surrounded by Q-matter. It is worth noting that because the amplitude of the scalar field can grow even for K > 0 [14], our solution may play an important role as dark matter.

Our present work as well as previous work [11, 12] suggests that self-gravity may change properties of the solutions even if it is weak. Therefore, it may be important to extend our approach to other models such as the gauge-mediation type.

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