Triangle inequalities in coherence measures and entanglement concurrence

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We provide detailed proofs of triangle inequalities in coherence measures and entanglement concurrence. If a rank-2 state \( \rho \) can be expressed as a convex combination of two pure states, i.e., \( \rho = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2| \), a triangle inequality can be established as \( |E(|\Psi_1\rangle) - E(|\Psi_2\rangle)| \leq E(\rho) \leq E(|\Psi_1\rangle) + E(|\Psi_2\rangle) \), where \( |\Psi_1\rangle = \sqrt{p_1}|\psi_1\rangle \) and \( |\Psi_2\rangle = \sqrt{p_2}|\psi_2\rangle \), \( E \) can be considered either coherence measures or entanglement concurrence. This inequality displays mathematical beauty for its similarity to the triangle inequality in plane geometry. An illustrative example is given after the proof.

I. INTRODUCTION

Coherence and entanglement are two crucial quantum mechanical properties which are widely used in quantum information processing and quantum computation [1]. While quantum coherence is defined for single systems, quantum entanglement is adopted to describe bipartite or multipartite systems. In earlier research, coherence is usually a main concern of quantum optics. But a new resource theory. We review some measures of coherence and entanglement at first.

A widely used measure of coherence is the distance-based measure [2]. The starting point for the definition of coherence is the identification of the set \( I \) of incoherent states. The incoherent states are diagonal in the reference basis \( \{|i\rangle\}_{i=1}^{d} \) (which are chosen according to the practical physical problem), and take the form

\[
\delta = \sum_{i=1}^{d} \delta_{i} |i\rangle \langle i|,
\]

for a \( d \)-dimensional Hilbert space. A measure of coherence of a state \( \rho \) can be defined by the minimal distance between \( \rho \) and the set \( I \) of incoherent states, i.e.,

\[
C_D(\rho) = \min_{\delta \in I} D(\rho, \delta),
\]

where \( D(\rho, \delta) \) denotes certain distance measures of quantum states. If \( \rho \) is a incoherent state, i.e., \( \rho \in I \), \( C_D(\rho) \) must be zero with \( \rho = \delta \).

Concurrence applies to the measure of entanglement [3, 4]. The definition of concurrence is based on the convex-roof construction. It is suitable for use in both pure states and mixed states [5–7]. Unfortunately, analytical solutions of concurrence can only be obtained in 2-qubit states (2 \( \otimes \) 2 dimensions) [3, 4] and some high-dimensional bipartite states with high symmetries, such as isotropic states and Werner states [10, 11]. For general high-dimensional mixed states, it is not fully explored with only a little knowledge [12, 13].

For a general high-dimensional bipartite pure state \( \rho_{AB} = |\psi\rangle \langle \psi| \), which is expanded in a finite \( d_1 \otimes d_2 \)-dimensional Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \), the concurrence is defined as \( C(|\psi\rangle) = \sqrt{2(1 - \text{Tr} \rho_{AB}^2)} \), with \( \rho_{AB} = \text{Tr}_B \rho_{AB} \) being the reduced density matrix [14]. Moreover, a pure state can be generally expressed as \( |\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \psi_{ij} |i\rangle |j\rangle \) (\( \psi_{ij} \in \mathbb{C} \)) in the computational bases \( |i\rangle \) and \( |j\rangle \) of \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, where \( i = 1, \ldots, d_1 \) and \( j = 1, \ldots, d_2 \). Then the squared concurrence of a pure state can be written as [15]:

\[
C^2(|\psi\rangle) = \sum_{m=1}^{D_1} \sum_{n=1}^{D_2} |C_{mn}|^2 = 4 \sum_{i<j}^{d_1} \sum_{k<l}^{d_2} |\phi_{ik} \phi_{jl} - \phi_{il} \phi_{jk}|^2, \tag{3}
\]

where \( D_1 = d_1 (d_1 - 1)/2 \), \( D_2 = d_2 (d_2 - 1)/2 \), \( C_{mn} = \langle \psi | \tilde{w}_{mn} \rangle \), \( \tilde{w}_{mn} = (L_m \otimes L_n) |\psi\rangle \), and \( \ast \) denotes complex conjugate. Here \( L_m, m = 1, \ldots, d_1 (d_1 - 1)/2 \) and \( L_n, n = 1, \ldots, d_2 (d_2 - 1)/2 \) are the generators of group \( SO(d_1) \) and \( SO(d_2) \): \( \tau_{ij} = |i\rangle \langle j| - |j\rangle \langle i| \), \( L_n = |k\rangle \langle l| - |l\rangle \langle k| \).

The concurrence of a pure state can be easily calculated as zero if the state is separable. For a mixed state \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \), \( \sum p_i = 1 \), the concurrence is defined by the convex-roof [10] as follows

\[
C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum p_i C(|\psi_i\rangle). \tag{4}
\]

The minimization is taken over all possible decompositions of \( \rho \) into pure states. For a 2-qubit mixed state \( \rho \), an analytic solution of concurrence can be calculated:

\[
C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{5}
\]

where \( \{\lambda_i\} \) are the eigenvalues, in decreasing order, of the Hermitian matrix \( R \equiv \sqrt{\rho} \rho^{\dagger} \sqrt{\rho} \). \( \rho \) is the spin-flipped state \( \tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho (\sigma_y \otimes \sigma_y) \). It is worth noticing that \( \{\lambda_i\} \) are also the singular values of a complex symmetric matrix \( \tau \), where \( \tau_{ij} = \langle u_i | u_j \rangle \). The states \( |u_i\rangle \) are the eigenstates of \( \rho \) [9].

For a high-dimensional mixed state, the minimum decomposition is very cumbersome to detect. We usually provide a bound for concurrence to analyze mixed states. Until now, several bounds of concurrence have been introduced [20–35].

In our research, we find that if a rank-2 mixed state \( \rho \) can be expressed as a convex combination of two pure states, \( \rho = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2| \), a triangle inequality can be established as \( |E(|\Psi_1\rangle) - E(|\Psi_2\rangle)| \leq E(\rho) \leq E(|\Psi_1\rangle) + E(|\Psi_2\rangle) \), where \( |\Psi_1\rangle = \sqrt{p_1}|\psi_1\rangle \) and \( |\Psi_2\rangle = \sqrt{p_2}|\psi_2\rangle \), \( E \) can be considered either coherence measures or entanglement concurrence. This inequality displays mathematical beauty for its similarity to the triangle inequality in plane geometry. An illustrative example is given after the proof.

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i.e., \( \rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| \) (\( |\psi_1\rangle \) and \( |\psi_2\rangle \) are linearly independent), the triangle inequality can be established as

\[
|E(|\Psi_1\rangle) - E(|\Psi_2\rangle)| \leq E(\rho) \leq E(|\Psi_1\rangle) + E(|\Psi_2\rangle),
\]

where \( |\Psi_1\rangle = \sqrt{p_1}|\psi_1\rangle \) and \( |\Psi_2\rangle = \sqrt{p_2}|\psi_2\rangle \). \( E(\rho) \) can be considered either coherence measures or entanglement concurrence. It is similar to the triangle inequality in geometry. In Sec. II and III, the triangle inequality is proven in coherence measures and entanglement concurrence, respectively.

**II. TRAINGLE INEQUALITIES IN COHERENCE MEASURES**

Several measures of coherence are proposed based on the distance-based measures, such as the relative entropy, the \( l_1 \) norm and the trace norm \( \|\cdot\| \). Here we only focus on the form of the \( l_1 \) norm

\[
C_l(\rho) = \min_{\delta \in \mathcal{C}} \|\rho - \delta\|_{l_1} = \sum_{i,j} |\langle i|\rho|j\rangle|,
\]

which is equal to sum of the absolute values of all off-diagonal elements of \( \rho \).

For mixed states, the convex-roof \( l_1 \) norm is adopted as a different measure of coherence \( \tilde{C}_l(\rho) \). The convex-roof construction is used to define concurrence and some other measures of entanglement previously. Taking into account the resource theory for coherence, we can make use of the convex-roof construction to measure coherence similar to its application in entanglement. The convex-roof \( l_1 \) norm of a mixed state \( \rho \) takes the form

\[
\tilde{C}_l(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_l(|\psi_i\rangle),
\]

where the minimization is taken over all pure state decompositions of \( \rho = \sum p_i|\psi_i\rangle\langle\psi_i| \), \( \sum p_i = 1 \). \( C_l(|\psi_i\rangle) \) is the \( l_1 \) norm of the state \( |\psi_i\rangle \). It is very similar to the entanglement concurrence.

We begin with a general triangle inequality in the \( l_1 \) norm measure of coherence.

**Theorem 1.** If a state \( \rho \) can be expressed as a convex combination of two states \( \rho = p_1\rho_1 + p_2\rho_2 \), the \( l_1 \) norm of \( \rho \), i.e., \( C_l(\rho) \), satisfies the triangle inequality

\[
|C_l(p_1\rho_1) - C_l(p_2\rho_2)| \leq C_l(\rho) \leq C_l(p_1\rho_1) + C_l(p_2\rho_2).
\]

**Proof.** The \( l_1 \) norm of \( \rho \) can be expressed as

\[
C_l(\rho) = C_l(p_1\rho_1 + p_2\rho_2) = \sum_{i,j} |p_1\langle i|\rho_1|j\rangle + p_2\langle i|\rho_2|j\rangle|.
\]

Considering absolute value inequality, the right hand side of Eq. \([10]\) should conform to the inequality

\[
\left| \sum_{i,j} |p_1\langle i|\rho_1|j\rangle| - \sum_{i,j} |p_2\langle i|\rho_2|j\rangle| \right|
\leq \sum_{i,j} |p_1\langle i|\rho_1|j\rangle + p_2\langle i|\rho_2|j\rangle| 
\leq \sum_{i,j} |p_1\langle i|\rho_1|j\rangle| + \sum_{i,j} |p_2\langle i|\rho_2|j\rangle|.
\]

According to the definition of the \( l_1 \) norm of coherence, \( C_l(p_1\rho_1) = \sum_{j \neq i} p_1|\langle i|\rho_1|j\rangle| \) and \( C_l(p_2\rho_2) = \sum_{j \neq i} p_2|\langle i|\rho_2|j\rangle| \). Then the triangle inequality can be established

\[
|C_l(p_1\rho_1) - C_l(p_2\rho_2)| \leq C_l(\rho) \leq C_l(p_1\rho_1) + C_l(p_2\rho_2).
\]

Note that for arbitrary \( \rho \), \( \rho_1 \) and \( \rho_2 \) can be alternatively pure states or mixed states.

If \( \rho \) is a rank-2 mixed state and the decomposition parts of \( \rho \) are two pure states \( |\psi_1\rangle \) and \( |\psi_2\rangle \), then the convex-roof \( l_1 \) norm of \( \rho \) should also satisfy the triangle inequality.

**Theorem 2.** For a rank-2 mixed state \( \rho \), if it can be decomposed into two pure states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) with linear independence: \( \rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| \), let \( |\Psi_1\rangle = \sqrt{p_1}|\psi_1\rangle \) and \( |\Psi_2\rangle = \sqrt{p_2}|\psi_2\rangle \), the convex-roof \( l_1 \) norm of \( \rho \), i.e., \( \tilde{C}_l(\rho) \) satisfies the triangle inequality:

\[
|\tilde{C}_l(\rho)| \leq |\tilde{C}_l(|\Psi_1\rangle) - \tilde{C}_l(|\Psi_2\rangle)| \leq \tilde{C}_l(|\Psi_1\rangle) + \tilde{C}_l(|\Psi_2\rangle).
\]

where \( \tilde{C}_l(|\Psi_1\rangle) \) and \( \tilde{C}_l(|\Psi_2\rangle) \) are the convex-roof \( l_1 \) norm of \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \) respectively.

**Proof.** Note that the convex-roof \( l_1 \) norm equals \( l_1 \) norm of coherence for pure states. So Eq. \([13]\) can be read as

\[
|\tilde{C}_l(\rho)| \leq |\tilde{C}_l(|\Psi_1\rangle) - \tilde{C}_l(|\Psi_2\rangle)| \leq \tilde{C}_l(|\Psi_1\rangle) + \tilde{C}_l(|\Psi_2\rangle).
\]

The right hand side of Eq. \([14]\) can be proven by the definition of the convex-roof \( l_1 \) norm. \( \tilde{C}_l(\rho) \) is a sum of the minimal decomposition of \( \rho \), \( \tilde{C}_l(|\Psi_1\rangle) + \tilde{C}_l(|\Psi_2\rangle) \) can be regarded as a sum of a general decomposition of \( \rho \). So the right hand side is established.

The convex-roof \( l_1 \) norm is not less than the \( l_1 \) norm of coherence for mixed state, i.e., \( C_l(\rho) \leq \tilde{C}_l(\rho) \) for any state \( \rho \). Here we give a simple proof of this corollary.

Assume that \( \rho = \sum_k q_k|\phi_k\rangle\langle\phi_k| \) is the sum of minimal decomposition, the convex-roof \( l_1 \) norm follows

\[
\tilde{C}_l(\rho) = \sum_k q_k C_l(|\phi_k\rangle) = \sum_k q_k \sum_{i,j} |\langle i|\phi_k\rangle\langle\phi_k|j\rangle| 
\geq \sum_{i,j} \sum_k |\langle i|\phi_k\rangle\langle\phi_k|j\rangle| 
= \sum_{i,j} |\langle i|\rho|j\rangle| 
= C_l(\rho)
\]

Considering theorem 1, it is correct that \( C_l(|\Psi_1\rangle) - C_l(|\Psi_2\rangle) \leq C_l(\rho) \leq \tilde{C}_l(\rho) \). The proof is over.

**Remark.** The convex-roof \( l_1 \) norm was named coherence concurrence for its similarity to entanglement concurrence. It is interesting that both coherence concurrence and entanglement concurrence satisfy the triangle inequality with a rank-2 state. There is a potential question that whether other measures of coherence, such as the relative entropy and the trace norm, are suitable to the triangle inequality. The triangle inequality in the \( l_1 \) norm is simple to prove for the easy computation of the \( l_1 \) norm, but other measures are not so simple to be analyzed.
III. TRIANGLE INEQUALITY IN ENTANGLEMENT CONCURRENCE

As we mentioned in the introduction section that the matrix $\tau$ is a complex symmetric matrix, and $\{\lambda_i\}$ can be alternatively considered as the singular values of $\tau$. At the beginning, we propose a lemma for complex symmetric matrices which is helpful to prove the triangle inequality.

A. Two-qubit states

**Lemma 1.** For a 2 by 2 complex symmetric matrix with nonzero diagonal elements $x_1, x_2$ and singular values $\sigma_1, \sigma_2$ (set $\sigma_1 \geq \sigma_2$), the inequality can be established as

$$|x_1| - |x_2| \leq \sigma_1 - \sigma_2. \quad (16)$$

Proof. A 2 by 2 complex symmetric matrix $\tau$ can be expressed as $\tau = U\Sigma U^T$ by Singular Value Decomposition (SVD), $UU^T = I$ and $\Sigma = \text{diag}\{\sigma_1, \sigma_2\}$. The 2 by 2 unitary matrix $U$ can read

$$U = e^{i\tau} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad (17)$$

where $|a|^2 + |b|^2 = 1, r^* = r$. Then the matrix $\tau$ reads

$$\tau = e^{i2\tau} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} a & -b^* \\ b^* & a^* \end{pmatrix}. \quad (18)$$

Based on Eq. (18), the diagonal elements $x_1$ and $x_2$ can be expressed as $x_1 = e^{i\pi}(\sigma_1 a^2 + \sigma_2 b^2)$ and $x_2 = e^{i\pi}(\sigma_1 b^2 + \sigma_2 a^2)$. For $|a|^2 + |b|^2 = 1$, set $a = e^{i\theta_1} \cos \alpha$ and $b = e^{i\theta_2} \sin \alpha$. Then we have

$$\begin{align*}
|x_1| - |x_2| &= \sqrt{(\sigma_1 e^{i2\theta_1} \cos^2 \alpha + \sigma_2 e^{i2\theta_2} \sin^2 \alpha)(\sigma_1 e^{-i2\theta_1} \cos^2 \alpha + \sigma_2 e^{-i2\theta_2} \sin^2 \alpha)} \\
&\quad - \sqrt{\sigma_1 e^{-i2\theta_1} \sin^2 \alpha + \sigma_2 e^{i2\theta_1} \cos^2 \alpha}(\sigma_1 e^{i2\theta_2} \sin^2 \alpha + \sigma_2 e^{-i2\theta_2} \cos^2 \alpha) \\
&\quad \leq \sqrt{(\sigma_1 \cos^2 \alpha - \sigma_2 \sin^2 \alpha)^2 + (\sigma_1 \sin^2 \alpha - \sigma_2 \cos^2 \alpha)^2}. \quad (19)
\end{align*}$$

For the right hand side of Eq. (19), we discuss the results with four situations:

i. If $\cos^2 \alpha \geq \sin^2 \alpha$ and $\sigma_1 \sin^2 \alpha \geq \sigma_2 \cos^2 \alpha$:

$$R.H.S = (\sigma_1 + \sigma_2)(\cos^2 \alpha - \sin^2 \alpha) \leq \sigma_1 - \sigma_2. \quad (20)$$

ii. If $\cos^2 \alpha \geq \sin^2 \alpha$ and $\sigma_1 \sin^2 \alpha \leq \sigma_2 \cos^2 \alpha$:

$$R.H.S = \sigma_1 - \sigma_2. \quad (21)$$

iii. If $\sin^2 \alpha \geq \cos^2 \alpha$ and $\sigma_1 \cos^2 \alpha \geq \sigma_2 \sin^2 \alpha$:

$$R.H.S = (\sigma_1 + \sigma_2)(\sin^2 \alpha - \cos^2 \alpha) \leq \sigma_1 - \sigma_2. \quad (22)$$

iv. If $\sin^2 \alpha \geq \cos^2 \alpha$ and $\sigma_1 \cos^2 \alpha \leq \sigma_2 \sin^2 \alpha$:

$$R.H.S = \sigma_1 - \sigma_2. \quad (23)$$

In conclusion, for a 2 by 2 complex symmetric matrix, the inequality $|x_1| - |x_2| \leq \sigma_1 - \sigma_2$ is always correct.

**Remark.** Lemma 1 is the crux of this paper. It can be derived by Thompson theorem [147]. But it is so important that we give a detailed proof by ourself. This lemma can be directly applied in a rank-2 mixed state $\varrho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$. For the matrix $\tau$, the diagonal elements ($d_1$ and $d_2$) are the concurrence of $\sqrt{p_1}|\psi_1\rangle$ and $\sqrt{p_2}|\psi_2\rangle$. The difference of the two singular values ($\sigma_1 - \sigma_2$) of $\tau$ is the concurrence of the mixed state $\varrho$.

**Theorem 3.** For a 2-qubit mixed state $\varrho$, if it can be decomposed into two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ with linear independence: $\varrho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$, let $|\Psi_1\rangle = \sqrt{p_1}|\psi_1\rangle$ and $|\Psi_2\rangle = \sqrt{p_2}|\psi_2\rangle$, the bound of concurrence $C(\varrho)$ satisfies the triangle inequality:

$$C(|\Psi_1\rangle) - C(|\Psi_2\rangle) \leq C(\varrho) \leq C(|\Psi_1\rangle) + C(|\Psi_2\rangle). \quad (24)$$

where $C(|\Psi_1\rangle)$ and $C(|\Psi_2\rangle)$ are the concurrences of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ respectively.

**Proof.** The proof of the right hand side follows Theorem 2. We give a detailed proof of the left side of the formula below. For a mixed state $\varrho = \rho_1|\psi_1\rangle\langle\psi_1| + \rho_2|\psi_2\rangle\langle\psi_2|$, the complex symmetric matrix $\tau$ is established as

$$\tau = \begin{pmatrix} \rho_1 & \rho_1 |\Psi_1\rangle \langle\Psi_2| \\ \rho_2 & |\Psi_2\rangle \langle\Psi_2| \end{pmatrix}, \quad (25)$$

where $C(|\Psi_1\rangle) = |\langle\Psi_1|\Psi_1\rangle|$, $C(|\Psi_2\rangle) = |\langle\Psi_2|\Psi_2\rangle|$. By the lemma 1, the inequality is established:

$$C(|\Psi_1\rangle) - C(|\Psi_2\rangle) \leq |\langle\Psi_1|\tilde{\Psi}_1\rangle| - |\langle\Psi_2|\tilde{\Psi}_2\rangle| \leq \lambda_1 - \lambda_2 = C(\varrho), \quad (26)$$

where $\lambda_1$ and $\lambda_2$ are the singular values of $\tau$, $\lambda_1 \geq \lambda_2$. Theorem 3 is finished.

**Remark.** Consider a rank-$S$ state $\varrho = \sum_{i=1}^S p_i|\psi_i\rangle\langle\psi_i|$, where the number $K$ is called the cardinality of the ensemble. Necessarily, $K$ cannot be smaller than the rank $S$. In Theorem 3, the rank of the density matrix $\varrho$ in Eq. (24) equals 2 and the rank of matrix $R \equiv \sqrt{\rho_1^T \rho_2}$ equals 2. Eq. (24) is rewritten to suit the rank-$2$ state $C(\varrho) = \lambda_1 - \lambda_2, \lambda_1 \geq \lambda_2 \geq 0$. 


Concurrence must exist in the mixed state. We extend the theorem 4.

\[ C_{mn} = \sqrt{\lambda_1^{mn} - \lambda_2^{mn}} \]

where \( \lambda_1^{mn} \) and \( \lambda_2^{mn} \) are the singular values of the matrix \( \tau_{mn} \):

\[ \tau_{mn} = \begin{pmatrix} \langle \Psi_1 | L_m \otimes L_n | \Psi_1^* \rangle & \langle \Psi_2 | L_m \otimes L_n | \Psi_1^* \rangle \\ \langle \Psi_2 | L_m \otimes L_n | \Psi_2^* \rangle & \langle \Psi_2 | L_m \otimes L_n | \Psi_2^* \rangle \end{pmatrix}. \]

Eq. (27) provides a lower bound of squared concurrence of \( g \) [25].

By the lemma 1, the diagonal elements and the singular values of the complex symmetric matrix \( \tau_{mn} \) satisfy

\[ \left| \langle \Psi_1 | L_m \otimes L_n | \Psi_1^* \rangle \right| - \left| \langle \Psi_2 | L_m \otimes L_n | \Psi_2^* \rangle \right| \leq \lambda_1^{mn} - \lambda_2^{mn}. \] (29)

The sum of all squared \( m, n \) items of Eq. (29) is calculated:

\[ \sum_{m,n} \left| \langle \Psi_1 | L_m \otimes L_n | \Psi_1^* \rangle \right| - \left| \langle \Psi_2 | L_m \otimes L_n | \Psi_2^* \rangle \right|^2 \leq C^2(g). \] (30)

For simplicity, we set \( \langle \Psi_1 | L_m \otimes L_n | \Psi_1^* \rangle = T_{11}^{mn} \), \( \langle \Psi_2 | L_m \otimes L_n | \Psi_2^* \rangle = T_{22}^{mn} \), and \( \sum_{m,n} T_{11}^{mn} \) instead of \( \sum_{m,n} T_{11}^{mn} \). Then the left hand side of Eq. (31) reads:

\[ L.H.S = \sum_{m,n} \left( |T_{11}^{mn}|^2 + |T_{22}^{mn}|^2 - 2|T_{11}^{mn}||T_{22}^{mn}| \right) \]

\[ \geq \sum_{m,n} |T_{11}^{mn}|^2 + \sum_{m,n} |T_{22}^{mn}|^2 \]

\[ - 2 \sum_{m,n} |T_{11}^{mn}|^2 - \sum_{m,n} |T_{22}^{mn}|^2. \] (31)

where the inequality \( \sum a_i b_i \leq \sum a_i^2 + b_i^2 \) is applied.

It is worth noticing that \( |\Psi_1 \rangle \) and \( |\Psi_2 \rangle \) are pure states so that \( \sum_{m,n} |T_{11}^{mn}|^2 = C^2(|\Psi_1 \rangle) \) and \( \sum_{m,n} |T_{22}^{mn}|^2 = C^2(|\Psi_2 \rangle) \). The right hand side of Eq. (31) reads:

\[ R.H.S = C^2(|\Psi_1 \rangle) + C^2(|\Psi_2 \rangle) - 2C(|\Psi_1 \rangle)C(|\Psi_2 \rangle) \]

\[ = C(|\Psi_1 \rangle) - C(|\Psi_2 \rangle). \] (32)

So the inequality \( |C(|\Psi_1 \rangle) - C(|\Psi_2 \rangle)| \leq C(g) \) is tenable.

Remark. The triangle inequality in concurrence reveals the relation between pure states and mixed states. Bipartite mixed states are inclined to entanglement. If a bipartite mixed state can decomposed into two pure states and the concurrences of the two pure states are different, this mixed state must have entanglement.

B. Bipartite high-dimensional states

Theorem 4. For a \( d_1 \otimes d_2 \)-dimensional mixed state \( \rho \), if it can be decomposed into two pure states \( |\psi_1 \rangle \) and \( |\psi_2 \rangle \) which are linearly independent, the triangle inequality Eq. (24) still holds.

Proof. For a \( d_1 \otimes d_2 \)-dimensional mixed state \( \rho \), the concurrence \( C(\rho) \) satisfies

\[ d_1(d_1-1)/2 \sum_{n=1}^{d_2(d_2-1)/2} C_{mn}^2 \leq C^2(\rho). \] (27)

The right hand side of Eq. (31) reads

\[ R.H.S = C^2(|\Psi_1 \rangle) + C^2(|\Psi_2 \rangle) - 2C(|\Psi_1 \rangle)C(|\Psi_2 \rangle) \]

\[ = |C(|\Psi_1 \rangle) - C(|\Psi_2 \rangle)|. \] (32)

C. Example

Consider a 2-qubit mixed state \( \rho = P|\psi_1 \rangle \langle \psi_1 | + (1 - P)|\psi_2 \rangle \langle \psi_2 | \), where

\[ |\psi_1 \rangle = \sqrt{\frac{2}{8}}(|00 \rangle + |11 \rangle) + \sqrt{\frac{2}{8}}i(|01 \rangle + |10 \rangle), \]

\[ |\psi_2 \rangle = \sqrt{\frac{3}{8}}(|00 \rangle + |11 \rangle) + \sqrt{\frac{1}{8}}(|01 \rangle + |10 \rangle). \] (33)

\( |\psi_1 \rangle \) and \( |\psi_2 \rangle \) are pure states and each qubit has two orthonormal bases \( |0 \rangle \) and \( |1 \rangle \). Let \( |\Psi_1 \rangle = \sqrt{P}|\psi_1 \rangle, |\Psi_2 \rangle = \sqrt{1-P}|\psi_2 \rangle \). \( C(\rho), C(|\Psi_1 \rangle) \) and \( C(|\Psi_2 \rangle) \) are computed with \( P \) going from 0 to 1. \( C(\rho) \) and \( C(|\Psi_1 \rangle) \) plots on the vertical y-axis against \( P \) on the horizontal x-axis in Fig. 1. \( C(|\Psi_2 \rangle) \) are plotted in Fig. 2.

In Fig. 1 the blue line indicates \( C(\rho) \). Red points in the diagram represent \( C(|\Psi_1 \rangle) + C(|\Psi_2 \rangle) \) with each point indicating one kind of pure state decomposition for \( \rho \). All red points are distributed above the blue line with a same \( P \), implying \( C(\rho) \leq C(|\Psi_1 \rangle) + C(|\Psi_2 \rangle) \). (35)

In Fig. 2 red points represent \( |C(|\Psi_1 \rangle) - C(|\Psi_2 \rangle)| \) with each point indicating one kind of pure state decomposition for \( \rho \). All red points are distributed below the blue line with the same \( P \). The inequality \( |C(|\Psi_1 \rangle) - C(|\Psi_2 \rangle)| \leq C(g) \) is correct in this system.
FIG. 2: The blue line indicates $C(\rho)$. Red points represent $|C(|\Psi_1\rangle) - C(|\Psi_2\rangle)|$ with each point indicating one kind of pure state decomposition for $\rho$. All points are distributed below the blue line with a same $P$, implying $|C(|\Psi_1\rangle) - C(|\Psi_2\rangle)| \leq C(\rho)$.

IV. DISCUSSIONS AND CONCLUSION

In this paper, we have reviewed the definition of some measures in coherence and entanglement. We provide a general triangle inequality in coherence measures and entanglement concurrence. Then we give an example of entanglement concurrence in a 2-qubit system. This inequality is formally perfect for its similarity to the triangle inequality in geometry. Both coherence and entanglement are quantum characteristics and they can be measured for the same state. For a bipartite or multipartite state, coherence is defined as an integral property while entanglement describes the relation among its subsystems. The relation between coherence and entanglement attracts much attention to study. Finding common laws between coherence and entanglement may be an available way to discover more intrinsic connections between them.

Concurrence is a widely used entanglement measure built on the convex-roof construction for mixed states. An attractive question is that whether other coherence or entanglement measures built by the convex-roof construction are suitable for this kind of triangle inequality, such as the entanglement of formation (EOF) [16, 17], the geometric measure of entanglement (GME) [18, 19], the convex-roof extended negativity (CREN) [20, 21], the G-concurrence [22] and so on. The right hand side of the inequality Eq. (24) must be correct for the convex-roof construction. But it is an open question that whether the left hand side is workable in other measures.

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