Abstract. We show that in a holomorphic family of compact complex connected manifolds parametrized by an irreducible complex space $S$, assuming that on a dense Zariski open set $S^*$ in $S$ the fibres satisfy the $\partial \bar{\partial} -$ lemma, the algebraic dimension of each fibre in this family is at least equal to the minimal algebraic dimension of the fibres in $S^*$. For instance, if each fibre in $S^*$ are Moishezon, then all fibres are Moishezon.

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1 Introduction

In this article we give a rather elementary proof of the following result which answers a “classical” question.

Theorem 1.0.1 Let $\pi : \mathcal{X} \to D$ be a smooth holomorphic family of compact complex connected manifolds parametrized by the unit disc $D$ in $\mathbb{C}$. Assume that for each $s \in D^*$ the fiber $X_s$ of $\pi$ at $s$ satisfies the $\partial \bar{\partial} -$ lemma. Let

$$a := \inf \{ \text{alg}(X_s), s \in D^* \}$$

where $\text{alg}(X)$ denotes the algebraic dimension of the compact complex connected manifold $X$. Then we have $\text{alg}(X_0) \geq a$.

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As a special case (when $a = n := \text{dim} X_s$) we obtain that, if for any $s \in D^*$ each $X_s$ is a Moishezon manifold, $X_0$ is also a Moishezon manifold.

The previous theorem gives easily the following corollary (see [B.15] for details).

**Corollary 1.0.2** Let $\pi : X \to S$ be a holomorphic family of compact complex connected manifolds parametrized by a reduced and irreducible complex space $\overline{S}$. Let $S^*$ be a dense Zariski open set in $S$. Assume that for each $s \in S^*$ the fiber $X_s$ of $\pi$ at $s$ satisfies the $\partial\overline{\partial}$–lemma. Let $a := \inf \{ \text{alg}(X_s), s \in S^* \}$. Then for any $s \in S$ we have $\text{alg}(X_s) \geq a$.

Note that the proof in [B.15] shows that the minimum of the algebraic dimension is obtained at the general point in $S$.

In the first section we show that the existence of a Gauduchon metric on a compact complex connected manifold $X$ implies the compactness of the connected components of the space of divisors in $X$. This new proof of this classical result is the key of our proof for the theorem above which appears as a relative version of it.

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## 2 The absolute case

Let me begin by two simple lemmas.

**Lemma 2.0.1** Let $M$ be a connected reduced complex space and let $T : M \to \mathbb{R}^+$ be a continuous function on $M$. Assume that $T$ is pluri-harmonic on the smooth part $M'$ of $M$ and that the function $T$ achieves its minimum at a point $x_0 \in M$. Then $T$ is constant on $M$.

**Proof.** This is an elementary exercise.

**Lemma 2.0.2** Let $X$ be a compact reduced complex space and let $\omega$ be a continuous real $(q, q)$–form on $X$ which is strictly positive in the Lelong sense (see section 3.1). Let $\Gamma$ be a connected component of the reduced complex space of compact $q$–cycles in $X$ and define

$$\theta : \Gamma \to \mathbb{R}^+ \quad \text{by} \quad \gamma \mapsto \theta(\gamma) := \int_\gamma \omega.$$  

Then the continuous function $\theta$ achieves its minimum on $\Gamma$.

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1 this means on the complement of a countable union of closed nowhere dense analytic subsets in $S$ (see the proposition [3.3.1] below).
Proof. Note first that $\theta$ is continuous thanks to the prop. IV 3.2.1 of [B-M 1]. Let $\alpha := \inf \{ \theta(\gamma), \gamma \in \Gamma \}$. Then the subset $A := \{ \gamma \in \Gamma \mid \theta(\gamma) \leq \alpha + 1 \}$ is a compact subset in $\Gamma$ thanks to Bishop’s theorem [Bi.64], because it is a closed subset in $\Gamma$ such that each cycle in $A$ has bounded volume (relative to $\omega$) and support in the compact space $X$ (see th. IV 2.7.23 in [B-M 1]). Then $\theta$ achieves its minimum on $A$.

Proposition 2.0.3 Let $X$ be a compact reduced complex space and let $\omega$ be a continuous real $(q, q)$−form on $X$ which is strictly positive in the Lelong sense (see section 3.1) and satisfies $\partial \bar{\partial} \omega = 0$ as a current on $X$. Then any connected component of the reduced complex space $C_q(X)$ of compact $q$−cycles in $X$ is compact.

Proof. The function $\gamma \to \theta(\gamma) := \int_\gamma \omega$ is continuous on $C_q(X)$ and $\partial \bar{\partial}$−closed in the sense of currents; as it achieves its minimum on any connected component $\Gamma$ of $C_q(X)$ thanks to the lemma 2.0.2, we conclude that it is constant on each $\Gamma$, thanks to lemma 2.0.1 and then any such $\Gamma$ is compact (see [B-M 1] ch.IV th. 2.7.20).

Remark. As any compact complex manifold $X$ of dimension $n$ admits a Gauduchon metric (see [G.77]), the proposition above gives a proof of the fact that the space of divisors of such a $X$ has always compact connected components, which is a well known classical result (see [C.82] and [Fu.82]).

3 The relative case

3.1 Relative strong positivity in the Lelong sense

Let $\pi : X \to S$ be a surjective holomorphic map between two irreducible complex spaces. Let $T_{X/D}$ be the tangent Zariski linear space of $\pi$, so the kernel of the tangent map

$$T_\pi : T_X \to \pi^*(T_D).$$

Let $1 \leq q \leq \dim X - \dim S := n$ and let $\omega$ a continuous relative $(q, q)$−differential form on $X$. Then $\omega$ induces a continuous hermitian form of the fibres on the linear space on $X$ associated to the coherent sheaf $\Omega^q_{X/S} := \pi^*(\Omega^q_S) \wedge \Omega^{n-q}_X$.

Let $p : Gr_q(X/S) \to X$ be the grassmanian of $q$−planes in $T_{X/S}$, $U$ the universal $q$−vector bundle on $Gr_q(X/S)$ and $\theta : \Lambda^q(U) \to Gr_q(X/S)$ the line bundle which is the determinant of $U$. A $\pi$−relative continuous $(q, q)$−form on $X$ defines a continuous hermitian form on $\Lambda^q(U)$.

Definition 3.1.1 We shall say that a $\pi$−relative continuous $(q, q)$−form $\omega$ on $X$ is strongly Lelong positive at $x_0$ if the hermitian form on $\Lambda^q(U)$ defined by $\omega$ is a positive hermitian form at each point of $p^{-1}(x_0)$ (so a continuous hermitian metric on this line bundle) .
In other word that means that for any \( q \)-plane \( P \) in the Zariski tangent space \( T_{X,x_0} \) which is vertical (i.e. contained in the kernel of \( T_{\pi,x_0} \)) then \( \omega_{x_0}[v_1 \wedge \cdots \wedge v_q] > 0 \) when \( v_1, \ldots, v_q \) is a basis of \( P \).

As the map \( p : Gr_q(X/S) \to X \) is proper, the condition above is open. Moreover, it is an easy exercise to prove the following properties:

1. Assume that the \( \pi \)-relative continuous \((q,q)\)-form \( \omega \) on \( X \) is strongly Lelong positive at \( x_0 \). Then for any continuous hermitian metric \( h \) on \( X \) there exists a neighbourhood \( V \) of \( x_0 \) in \( X \) and a positive constant \( C(V) \) such that on the open set \( p^{-1}(V) \) the metric induced by \( h^{\wedge q} \) and \( \omega \) on the line bundle \( \Lambda^q(U) \) satisfy
   \[
   h^{\wedge q} \leq C(V) \omega.
   \]

2. An easy consequence of the above estimate is the fact that on any closed (complex) analytic subset \( Z \) of pure dimension \( q \) in a fiber of \( \pi \) over \( V \) we have the inequality of volume forms on \( Z \)
   \[
   h^{\wedge q}|_Z \leq C(V) \omega|_Z.
   \]

3. Assume now that \( \omega \) is strongly Lelong positive on an open set \( V \) in \( X \) containing a compact set \( L \). Then there exists a positive constant \( C_L \) such that for any compact relative \( q \)-cycle \( \gamma \) contained in \( L \) we have the inequality
   \[
   \int_{\gamma} h^{\wedge q} \leq C_L \int_{\gamma} \omega.
   \]

This means that the integral over compact relative \( q \)-cycles of a continuous \( \pi \)-relative \((q,q)\)-form which is strongly Lelong positive on \( X \) controls the volume of these cycles contained in a given compact set \( L \subset X \) for any given hermitian metric on \( X \).

The following result follows easily of the description of compact subsets in the space of compact cycles which is a consequence of E. Bishop’s theorem.

**Theorem 3.1.2** Let \( \pi : X \to S \) a proper surjective map between irreducible complex spaces. Assume that there exists a smooth \( d \)-closed \( 2q \)-differential form \( \bar{\omega} \) on \( X \) such that its \((q,q)\)-part induces a relative strongly Lelong positive form on \( X \). Then any connected component of \( C_q(\pi) \), the space of relative compact \( n \)-cycles of \( \pi \), is proper over \( S \).

**Proof.** Take such a connected component \( \Gamma \). Then the continuous function given by \( \gamma \mapsto \int_{\gamma} \bar{\omega} \) is constant on \( \Gamma \). This implies that for any compact subset \( K \) in \( S \) the volume of cycles in \( \Gamma \cap \pi^{-1}(K) \) for any continuous metric is bounded. But then, as \( \pi^{-1}(K) \) is compact in \( X \) this implies that \( \Gamma \cap \pi^{-1}(K) \) is a compact subset in \( C_q(\pi) \subset C_q(X) \) (see [B-M 1] IV th.2.7.20), concluding the proof. \( \blacksquare \)
Remark. In the case of smooth family of compact connected manifolds on a complex disc (or polydisc) \( \pi : X \to D \) each of them satisfying the \( \partial \bar{\partial} \)-lemma, the existence of a smooth \((q, q)\)-relative differential form \( \omega \) such that \( \partial_{/\pi} \bar{\partial}_{/\pi} \omega = 0 \) on \( X \) which is strongly Lelong positive on the fibres of \( \pi \) allows to prove, in the same way than in the case of relative divisors, the properness over \( D \) of the connected component of the relative cycle’s space \( C_q(\pi) \):

the first step is to produce for any given \( s \in D \), using the \( \partial \bar{\partial} \)-lemma on \( X_s := \pi^{-1}(s) \) smooth forms \( \alpha \) and \( \beta \) on \( X_s \) of type \((q, q-1)\) and \((q-1, q)\) respectively such that the form \( \omega - \partial \alpha - \bar{\partial} \beta \) is \( d \)-closed on \( X_s \), but with \((q, q)\)-part strongly Lelong positive on \( X_s \). Then, using the local \( \mathcal{C}^\infty \) triviality of \( \pi \) around \( s \) to produce on \( \pi^{-1}(\Delta_s) \), where \( \Delta_s \) is a small open (poly-)disc around \( s \) in \( D \), a smooth \( d \)-closed \( 2q \)-form \( \Omega_s \) inducing \( \omega - \partial \alpha - \bar{\partial} \beta \) on \( X_s \). Then, the \((q, q)\)-part of \( \Omega_s \) is \( \omega_{|X_s} \) so it is strongly Lelong positive on \( X_s \) and then also on \( \pi^{-1}(\Delta_s) \), if \( \Delta_s \) is small enough around \( s \), because the strong Lelong positivity is an open property (and in our \( \mathcal{C}^\infty \)-trivialisation used to construct \( \Omega_s \) the complex structure varies smoothly). The theorem above gives then the properness on \( \Delta_s \) of the connected components of the space \( C_q(\pi|\Delta_s) \).

To conclude, consider now a connected component \( \Gamma \) of \( C_q(\pi) \) and remark that two \( q \)-cycles in \( \Gamma \) which are contained in \( \pi^{-1}(\Delta_s) \) have not only the same image in \( H_{2q}(X, \mathbb{C}) \) but also in \( H_{2q}(\pi^{-1}(\Delta_s), \mathbb{C}) \) because the natural map
\[
H_{2q}(\pi^{-1}(\Delta_s), \mathbb{C}) \to H_{2q}(X, \mathbb{C})
\]
is injective (in fact bijective). This implies that the integral of \( \Omega_s \) on any cycle in \( \Gamma \cap C_q(\pi^{-1}(\Delta_s)) \) is constant, so \( \Gamma \cap C_q(\pi^{-1}(\Delta_s)) \) has only finitely many connected components, each of them being proper on \( \Delta_s \). \( \blacksquare \)

Terminology. In the case of a compact complex connected manifold \( X \) of dimension \( n \) a strongly Gauduchon form in the sense of [B.15] is a smooth \( d \)-closed \((2n-2)\)-form \( \omega \) on \( X \) such that its \((n-1, n-1)\)-part is strongly positive in the sense of Lelong (in this case it is the “usual” sense). Assuming that \( X \) satisfies the \( \partial \bar{\partial} \)-lemma, any Gauduchon form\(^2\) gives rise to a strongly Gauduchon form via the method described above. And, thanks to [G.77] a Gauduchon form always exists on a compact complex connected manifold.

### 3.2 Estimation of volumes

Let now consider a holomorphic family \( \pi : X \to D \) of compact complex connected manifolds of dimension \( n \) parametrized by the unit disc in \( \mathbb{C} \). So \( X \) is a smooth complex manifold of dimension \( n+1 \), and we fix a smooth relative Gauduchon form \( \omega \) on \( X \). Then \( \omega \) is a smooth \( \pi \)-relative differential form on \( X \) on type \((n-1, n-1)\) which is positive definite in the fibres and satisfies \( \partial_{/\pi} \bar{\partial}_{/\pi} \omega \equiv 0 \) on \( X \).

\(^2\)so a \((n-1, n-1)\) smooth differential form positive definite and \( \partial \bar{\partial} \)-closed.
Using Ehresmann’s theorem we may assume that we have a $\mathcal{C}^\infty$ trivialization

$$D \times X_0 \xrightarrow{\Phi} \mathcal{X} \xrightarrow{\pi} D$$

of the map $\pi$ where $X_0 := \pi^{-1}(0)$. Then the form $\Phi^*(\omega)$ is a smooth differential form on $X_0$ of degree $2n - 2$ depending in a smooth way of the real variables $x$ and $y$ in $D$, and we then identify $\omega$ with the smooth (absolute) $(2n - 2)$--differential form of $(\Phi^{-1})^*(\Phi^*(\omega))$ on $\mathcal{X}$.

We shall define

$$\frac{\partial \omega}{\partial x} := (\Phi^{-1})^*\left(\frac{\partial (\Phi^*(\omega))}{\partial x}\right) \quad \text{and} \quad \frac{\partial \omega}{\partial y} := (\Phi^{-1})^*\left(\frac{\partial (\Phi^*(\omega))}{\partial y}\right)$$

on $\mathcal{X}$. Then $\frac{\partial \omega}{\partial x}$ and $\frac{\partial \omega}{\partial y}$ are smooth $(2n - 2)$--differential forms on $\mathcal{X}$ and satisfy

$$d\omega = \frac{\partial \omega}{\partial x} \pi^*(dx) + \frac{\partial \omega}{\partial y} \pi^*(dy) + d_{/\pi}\omega. \quad (0)$$

where $d_{/\pi}$ is the $\pi$--relative differential.

Now define the $\pi$--relative smooth differential forms $\nabla_x \omega$ and $\nabla_y \omega$ on $\mathcal{X}$ as the $(n - 1, n - 1)$--parts of the restrictions of $\frac{\partial \omega}{\partial x}$ and $\frac{\partial \omega}{\partial y}$ to each fibre of $\pi$. We may also consider $\nabla_x \omega$ and $\nabla_y \omega$ as smooth hermitian forms on the holomorphic vector bundle $\Lambda^{n-1}(T_{\mathcal{X}/D})$ on $\mathcal{X}$. Then the following lemma is given by an elementary compactness argument, because $\omega$ is a smooth positive definite hermitian form on this vector bundle.

**Lemma 3.2.1** For any compact set $K$ in $D$ there exists a constant $C_K > 0$ such on $\pi^{-1}(K)$ we have the following inequalities

$$|\nabla_x \omega| \leq C_K \omega \quad \text{and} \quad |\nabla_y \omega| \leq C_K \omega \quad (1)$$

between hermitian forms on the vector bundle $\Lambda^{n-1}(T_{\mathcal{X}/D})$ on $\pi^{-1}(K)$.

The next lemma will allow us to prove our main estimate of the integral of $\omega$ over an analytic family of relative divisors in $\mathcal{X}$.

**Lemma 3.2.2** Consider over an open disc $D' \subset D$ a closed and reduced complex hypersurface $\mathcal{H}$ in $\pi^{-1}(D')$ which is $(n - 1)$--equidimensional on $D'$. Then, for any smooth differential 1--form $\varphi := \varphi_x dx + \varphi_y dy$ on $D'$ we have at the generic point of $\mathcal{H}$ the equality

$$d\omega \wedge \pi^*(\varphi)|_{\mathcal{H}} = (\nabla_x \omega \wedge \pi^*(\varphi_y dx \wedge dy) - \nabla_y \omega \wedge \pi^*(\varphi_y dx \wedge dy)|_{\mathcal{H}} \quad (2)$$
Proof. Near a generic point in $\mathcal{H}$ we may assume that we have a local holomorphic trivialisation

$$D'' \times U \subset D'' \times V \xrightarrow{\pi} \mathcal{H} \subset \mathcal{X}$$

where $D''$ is a small open disc in $D$ and $V = U \times \Delta$ is the product of an open polydisc in $\mathbb{C}^{n-1}$ by an open disc $\Delta$ in $\mathbb{C}$. On a fibre of the restriction of $\pi$ to $\mathcal{H}$ the form $d\omega \wedge \pi^*(\varphi)$ is of pure type $(n,n)$. Remark that $d\omega \wedge \pi^*(\varphi)$ and $d\omega \wedge \pi^*(\varphi_y)$ vanish on $\mathcal{H} \cong D'' \times U$ because such forms present only the types $(n,n-1)$ and $(n-1,n)$ in the variables $t_1, \ldots, t_{n-1}$ in $U \subset \mathbb{C}^{n-1}$. Then, using (0) and the fact that the restrictions to $\mathcal{H}$ of the forms $\partial\omega \wedge \pi^*(\varphi_y dx \wedge dy)$ and $\partial\omega \wedge \pi^*(\varphi_x dx \wedge dy)$ come only from the $(n-1,n-1)-$parts of the restriction to the fibre of $\pi$ of $\partial\omega$ and $\partial\omega_y$ respectively because the restriction to $\mathcal{H}$ of $\pi^*(dx \wedge dy)$ is of type $(1,1)$, the conclusion follows, by definition of $\nabla_x \omega$ and $\nabla_y \omega$. □

The previous lemma gives the following simple estimate on $\mathcal{H}$.

**Corollary 3.2.3** We keep the notations of the previous lemma. Assuming that the 1–form $\varphi$ has a compact support in $D'$ contained in the compact set $K$ in $D$ we have on $\mathcal{H}$ the inequality

$$\left| (d\varphi \wedge \pi^*(\varphi))|_{\mathcal{H}} \right| \leq C_K \cdot (|\varphi_x| + |\varphi_y|) \cdot (\omega \wedge \pi^*(dx \wedge dy))_{|\mathcal{H}}. \tag{3}$$

between (smooth) real $2n-$differential forms on $\mathcal{H}$ where $C_K$ is the constant introduced in the lemma 3.2.1. □

The following classical lemma will be useful to conclude our estimation of the volume.

**Lemma 3.2.4** Let $U$ a bounded domain in $\mathbb{C}$. Assume that there is a point $s_0 \in U$ such that any point in $U$ can be joined to $s_0$ by a $C^1$ path in $U$ with length uniformly bounded by a number $L$. For instance, we can restrict ourself to $U := D^* \setminus A$ where $D^*$ is a punctured open disc in $\mathbb{C}$ and $A$ a closed discrete subset in $D^*$. Let $\eta : U \to \mathbb{R}^+$ be a $C^1$ function on $U$ such that

$$\left| \frac{\partial \eta}{\partial x} \right| \leq C \cdot \eta \quad \text{and} \quad \left| \frac{\partial \eta}{\partial y} \right| \leq C \cdot \eta \quad \text{in} \ U \tag{\@}$$

where $C > 0$ is a given constant. Then this implies that the function $\eta$ is uniformly bounded on $U$ by $\eta(s_0) \cdot \exp(2C.L)$. 


Proof. Firstly consider the case of an open interval \( ]a, b[ \) in \( \mathbb{R} \) and let \( L := b - a \).
If \( \eta : ]a, b[ \to \mathbb{R}^{**} \) is a \( C^1 \) function such that \( |\eta'(x)| \leq C \eta(x) \quad \forall x \in ]a, b[ \); we have

\[
-C \leq \frac{\eta'(x)}{\eta(x)} \leq C
\]

which gives, after integration on \( [x_0, x] \subset ]a, b[ \)

\[
-C |x - x_0| \leq \log \frac{\eta(x)}{\eta(x_0)} \leq C |x - x_0|
\]

and then, for any choice of \( x_0 \in ]a, b[ \) the estimate \( \eta(x) \leq \eta(x_0) \exp(C L) \) is valid.
In the case of the bounded domain \( U \subset \mathbb{C} \), choose for any \( s \in U \) a \( C^1 \) path of length strictly less than \( L \) and consider a parametrization of this path extended a little around \( s_0 \) and \( s \):

\[
\Phi : ]a, b[ \to U \quad x := \varphi(t), y := \psi(t) \quad \text{with} \quad \varphi'(t)^2 + \psi'(t)^2 \equiv 1
\]
and \( \Phi(a + \varepsilon) = s_0 \) and \( \Phi(b - \varepsilon) = s \) with \( 0 < \varepsilon \ll 1 \).

Then apply the one dimensional case to the function \( F(t) := \eta(\Phi(t)) \) which satisfies \( |F'(t)| \leq 2C F(t) \). This gives \( \eta(s) \leq \eta(s_0) \exp(2C L) \quad \forall s \in U. \)

3.3 proof of the theorem [1.0.1]

First step. For each \( s \in D^* \) we know that \( X_s \) admits a strongly Gauduchon form and then there exists a strongly Gauduchon form for the family over a small disc around \( s \) (see section 3.1). So we have a smooth \( d \)-closed \( 2(n-1) \)-form such that its \( (n-1,n-1) \) part is positive definite on each fibre. This implies that each connected component of the relative cycle space \( \mathcal{C}_{n-1}(\pi^*) \) is proper on \( D^* \), where \( \pi^* \) is the restriction of \( \pi \) to \( \pi^{-1}(D^*) \):

in a connected component \( \Gamma \) of the relative cycle space \( \mathcal{C}_{n-1}(\pi^*) \), for any disc \( \Delta \subset D^* \) all cycles in \( \pi^{-1}(\Delta) \cap \Gamma \) are homologous in \( \pi^{-1}(\Delta) \) because we have a \( C^\infty \) relative isomorphism \( X \simeq D \times X_0 \). So if we dispose of a \( d \)-closed \( 2(n-1) \)-smooth form \( \Omega \) on \( \pi^{-1}(\Delta) \) with a positive definite \( (n-1,n-1) \) part on fibres we see that the volume for \( (n-1) \)-cycles in \( \Gamma \cap \pi^{-1}(\Delta) \) relative to the “volume” defined by \( \Omega \) has to be constant, proving the properness on \( \Delta \) of \( \Gamma \cap \pi^{-1}(\Delta) \).

Second step. Applying the theorem 1.0.2 of [B.15] we obtain that the minimum of the algebraic dimension of the fibres over \( D^* \) is obtained on a dense (in fact the complement of a countable subset) subset of \( D^* \). This implies that it is enough, under the hypothesis of the theorem [1.0.1] to prove that the conclusion holds on a small open disc \( D_0 \) with center \( 0 \) in \( D \) because this “weak” version of this theorem may be apply then to a small disc centered at any point point in \( D^* \) and this gives the theorem [1.0.1].
Thanks to the second step, we may now assume that there exists a positive constant $C$ such that the inequalities of the the lemma 3.2.1 holds on $\mathcal{X}$.

Now, using the proposition 3.3.1 (recalled from [B.15]) it is enough to show that each irreducible component $\Gamma$ of the space $C_{n-1}(\pi)$ of relative $(n-1)$–cycles in $\mathcal{X}$ is proper over $D$.

**STEP 3.** Let $\Gamma$ be an irreducible component of $C_{n-1}(\pi^*)$ with reduced generic cycle and such that the projection $p : \Gamma \to D^*$ is surjective. As the map $p$ is proper, let $\tau : \pi^{-1}(D^*) \to S$ be a Stein factorization (we may assumed that $S$ is normal) of $p$ and let $\tilde{\tau} : S \to D^*$ the corresponding proper finite surjective map.

Now let $\omega$ be a smooth $\pi$–relative Gauduchon form on $\mathcal{X}$ and define on $\Gamma$ the function

$$\theta : \Gamma \to \mathbb{R}^+ \gamma \mapsto \int_{\gamma} \omega.$$ 

As the function $\theta$ is continuous, pluri-harmonic along the fibres of $p$ as a distribution and proper, it is constant in the fibres of $\tau$ which are the connected components of fibres of $p$ over $D^*$. So it defines a continuous function $\tilde{\theta} : S \to \mathbb{R}^+$. This situation corresponds to the following diagram:

![Diagram](image)

**STEP 4.** Let $\tilde{A}_0 \subset S$ be the set of points $\sigma \in S$ where the fibre $\tau^{-1}(\sigma)$ is not contained in the union of the singular set of $\Gamma$ with the critical set of $p : \Gamma \to D^*$. Then $\tilde{A}_0$ is a closed analytic subset in $S$ with no interior point. So $A_0 := \tilde{\tau}(\tilde{A}_0)$ is a closed discrete subset in $D^*$. Let $A_1 \subset D^*$ be the ramification set of $\tilde{\tau}$. It is also closed and discrete in $D^*$ and finally put $A := A_0 \cup A_1$.

We shall show now that the continuous function

$$\eta : D^* \to \mathbb{R}^{++}, s \mapsto \eta(s) := \text{Trace}_Z(\tilde{\theta})(s)$$

satisfies the following properties :

i) The partial derivatives $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$ are continuous on $D^* \setminus A$ (with $s := x + iy$).

ii) They satisfy the inequalities $|\frac{\partial \eta}{\partial x}| \leq C.\eta$ and $|\frac{\partial \eta}{\partial y}| \leq C.\eta$ on $D^* \setminus A$, where $C$ is the constant introduced above (see lemma 3.2.1 and the end of step 2).
Note that it is enough to prove the properties i) and ii) near each point in $D^* \setminus A$. Choose a point $s_0 \in D^* \setminus A$ and fix an open disc $\Delta_0 \subset D^* \setminus A$ with center $s_0$. Choose now in $p^{-1}(s_0)$ generic point $\gamma_0$ and, as $\gamma_0$ is a smooth point of $\Gamma$, such that $p$ has rank 1 at this point, we can find a smooth locally closed curve $\Sigma$ through $\gamma_0$ in $\Gamma$ such that the restriction of $p$ to $\Sigma$ induces an isomorphism of $\Sigma$ onto $\Delta_1$ where $\Delta_1 \subset \Delta_0$ is an open disc with center $s_0$.

Define, for $s \in \Delta_1$ the cycle $Y_s$ as the cycle corresponding to the point $p^{-1}_\Sigma(s)$ in $\Sigma \subset \Gamma$. This defines an analytic family of $\pi-$relative $(n-1)-$cycles. So the function $\eta_1$ which is given by $s \mapsto \int_{Y_s} \omega$ is continuous on $\Delta_1$ (see the proposition IV 2.3.1 in [B-M 1]) and coincides with $\frac{1}{\delta}.\eta$ where $\delta$ is the degree of the map $\tilde{\tau}$. We shall compute now the partial derivatives in the distribution sense of this function $\eta_1$.

Let $\mathcal{Y} \subset \Delta_1 \times \mathcal{X}$ be the graph of the analytic family $(Y_s)_{s \in \Delta_1}$ and let $\varphi$ be in $\mathcal{C}_c^\infty(\Delta_1)$. Then we have

\[
\langle d\eta_1, \varphi. dy \rangle = -\langle \eta_1, \frac{\partial \varphi}{\partial x}. dx \wedge dy \rangle = -\int_{\mathcal{Y}} \omega \wedge \pi^*(\frac{\partial \varphi}{\partial x}. dx \wedge dy)
\]

\[
= -\int_{\mathcal{Y}} \omega \wedge \pi^*(d\varphi \wedge dy) = -\int_{\mathcal{Y}} \omega \wedge d(\pi^*(\varphi.dy)) \text{ then }
\]

\[
\langle d\eta_1, \varphi. dy \rangle = \int_{\mathcal{Y}} \frac{\partial \omega}{\partial x} \wedge \pi^*(\varphi.dx \wedge dy) = \int_{\mathcal{Y}} \nabla_x \omega \wedge \pi^*(\varphi.dx \wedge dy)
\]

by Stokes formula and using the lemma $3.2.2$.

We conclude that the distribution $\frac{\partial \eta_1}{\partial x}$ is equal to the continuous function

\[
s \mapsto \int_{Y_s} \nabla_x \omega
\]

with the estimate

\[
\left| \frac{\partial \eta_1}{\partial x} \right| \leq C.\eta_1
\]

deduced from the inequality of the lemma $3.2.1$

We have the analogous result for $\frac{\partial \eta_1}{\partial y}$.

So the lemma $3.2.4$ applies and the function $\theta$ on $\Gamma$ is uniformly bounded. Then the closure of $\Gamma$ in $\mathcal{C}_{n-1}(\pi)$ is proper over $D$, using again the characterization of compact subsets in $\mathcal{C}_{n-1}(\mathcal{X})$ (see [B-M 1] IV th.2.7.20).

**Final step.** Consider now any irreducible component $\tilde{\Gamma}$ of $\mathcal{C}_{n-1}(\pi)$. Remark that to prove the properness of $\tilde{\Gamma}$ on $D$ it is enough to consider the case where the generic cycle of $\tilde{\Gamma}$ is reduced.

Then either $\tilde{\Gamma}$ is contained in some $\mathcal{C}_{n-1}(X_s)$ for some $s \in D$, and then it is a connected closed subset in $\mathcal{C}_{n-1}(X_s)$ and then it is compact, or $\tilde{\Gamma} \setminus p^{-1}(0)$ is a closed
irreducible subset in \( C_{n-1}(\pi^*) \) which is proper and surjective on \( D^* \) because we may apply the previous result. In this cases we conclude also that \( \tilde{\Gamma} \) (which is the closure of \( \Gamma \setminus p^{-1}(0) \)) is proper over \( D \).

As any irreducible component of \( C_{n-1}(\pi) \) is proper over \( D \), the conclusion follows.

For the convenience of the reader let me recall the proposition 4.0.8 of [B.15] which is an easy generalization of an old result of F. Campana (see [C.81]).

**Proposition 3.3.1** Let \( \pi : \mathcal{X} \to S \) be a proper surjective holomorphic \( n \)-equidimensional map between two irreducible complex spaces. Assume that any irreducible component of the complex space \( C_{n-1}(\pi) \) is proper over \( S \). Then there exists a countable union \( \Sigma \) of closed irreducible analytic subsets with no interior points in \( S \) and a non negative integer \( a \) such that:

1. For any \( s \in S \setminus \Sigma \) the algebraic dimension of \( X_s \) is equal to \( a \);

2. For all \( s \in S \) the algebraic dimension of \( X_s \) is at least equal to \( a \).

We conclude by noticing that there exists an analytic family of smooth complex compact surfaces of the class VII (not Kähler) parametrized by a disc such that the central fibre has algebraic dimension 0 and all other fibres have algebraic dimension equal to 1, see [F-P.09].

This shows that in our theorem 1.0.1 some “Kähler type” assumption on the general fibre \( X_s \) cannot be avoided in order that the general algebraic dimension gives a lower bound for the algebraic dimensions of all fibres.

Note that our assumption that the generic fibres satisfy the \( \partial\bar{\partial} \)-lemma (in fact for the type \((n, n-1)\)) is a rather weak such assumption.
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