A new analytical model of ultimate water cut for light oil reservoirs with bottom-water

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Abstract

Ultimate water cut (WCult) defines well’s maximum water production for uncontained oil pay with bottom-water. The WCult is important to determine if the reservoir development is economical. Since presently-used WCult formula derives from simplifying assumption ignoring the effect of non-radial inflow, the formula needs to be redefined. A new analytical formula of WCult is developed by considering the inflow of oil and water into separate completions at the top of oil-zone and aquifer respectively. Then the formula is verified using the design of 46 simulated experiments representing wide variety of reservoir-bottomwater systems. It was found out that for light-oil reservoirs, the presently-used theoretical formula may significantly diverge from the proposed formula which closely matches the simulated data and is more physics driven. Hence the proposed formula should be preferred. However, for the viscous oil reservoirs, the presently used formula conforms to the proposed formula, which is also proved mathematically.

Keywords: Ultimate water-cut, light oil, bottom-water reservoir, water coning, partial penetration

Introduction

Ultimate water-cut is a maximum stabilized water cut in an oil-pay affected by water coning. The scenario is physically modeled by setting a balanced-oil-rate (BOR) boundary of the well’s drainage area by replacing the produced oil at the the drainage boundary. After the water break-through time, there is an initial rapid increase of water-cut representing the water cone development stage, followed by the stabilization period until the WC value becomes constant, WCult.

Kuo and Desbrisay introduced the concept and formula of ultimate water-cut:

\[ WCult = \frac{M_{h_w}}{M_{h_o} + h_o} \]  

(1)

Shirman and Wojtanowicz showed that WCult in DWS wells is always lower than that in conventional wells. Their experimental results revealed that it is possible to completely reduce WCult to zero at high drainage rates. Other authors showed the dependence of ultimate water-cut on production rate. For production rates slightly higher than critical rates (maximum possible production rate without water breakthrough), water-cut would stabilize at value lower than that in Eq. (1). After conducting laboratory experiments, Shirman and Wojtanowicz found out that the water-cut stabilization value may not predict the Kuo and Desbrisay model at low production rate. They modified Eq. (1) by including the effect of production-rate as, 

\[ WCult = \left(1 - \frac{q_o}{Q_o} \right) \frac{M_{h_w}}{M_{h_o} + h_o} \]  

(2)

Both Eqs. (1) and (2) assume the radial flow in the oil-zone and aquifer having a BOR boundary depicted in Figure 1, and there by ignores any nonradial distorted inflows (in oil-zone and aquifer) to a partially penetrating well. Prasun and Wojtanowicz attempted to include the effect of partial-penetration in the closed-boundary reservoirs. However, they found that the new modified WCult formula reduces back to the original formula (Eq. (1)); thus disapproving any effect of partial-penetration on ultimate water-cut in these reservoirs. Apparently, they verified the effect of partial penetration by comparing the formula with the results from the wide variety of NFRs. However, they failed to understand that the generalized consideration of all attributes of reservoirs while verification, may conceal the partial-penetration effects for certain types of reservoirs. So, this study derives a new model of ultimate water-cut for the BOR systems considering the non-radial inflow to a partial-penetrating well, and then verifies it with particular types of reservoirs classified as light oil and viscous oil reservoirs. A good match for the particular reservoir, would justify the relevance of the partial penetration effects for this reservoir.
Modified analytical formula of ultimate water-cut

In derivation of a new ultimate water-cut model for a partially penetrating well in BOR system, we consider the following assumptions:

There is a piston-like displacement of oil by coned water flowing into the well. So, the rising water cone development covers larger area of oil completion before final stabilization. Eventually, the ratio of well completion producing oil and water becomes equal to the ratio of oil and water zone thickness, when ultimate water-cut is reached.\(^3\)

In a piston-like displacement, there is almost no mixing between the flow regions of oil and water. Assumption 1 follows that the partially penetrating oil completion region (producing only oil) is at the top of oil-zone, whereas, for simplicity, we assume the partially penetrating water completion region (producing only water) is displaced from the oil-zone to the top of aquifer as shown in Figure 2. This assumption ignores the additional skin due to the water inflow from aquifer to the completion in oil-zone.

![Figure 2](image)

**Figure 2** Equivalence of oil and water inflow schematic between combined and separate systems.

Darcy-law flow-rate equations of oil \( (q_o) \) and water \( (q_w) \) well-inflow (into their respective completions) during ultimate water-cut stage, at surface conditions, can be given by (Appendix A),

\[
q_o = \frac{2\pi k_o k_e h_o (p_e - p_w)}{\mu_o B_o (\ln \frac{r_e}{r_o} + s_o)}
\]

\[
q_w = \frac{2\pi k_w k_e h_w (p_e - p_w)}{\mu_w B_w (\ln \frac{r_e}{r_w} + s_w)}
\]

where, \( r_e \) is the radial size of reservoir, ft; \( S_o \) is the skin factor due to oil-inflow defined by Eq. (A-4); \( S_w \) is the skin factor due to water-inflow defined by Eq. (A-7); \( r_o \) is the well radius, ft. Now, after incorporating the above formulas into the ultimate water-cut equation (as shown in Appendix A), a new model of ultimate water-cut is developed, given by,

\[
WC_{ult} = (1 - \frac{q_o}{Q}) \left( \frac{Mh_o}{\ln \frac{r_e}{r_o} + s_o} + \frac{h_o}{\ln \frac{r_e}{r_o} + s_o} \right)
\]

\[
\frac{\ln \frac{r_e}{r_o} + s_w}{\ln \frac{r_e}{r_o} + s_o}
\]

**Validation of the proposed models using experiments**

For simulation experiments, a 2-D radial-cylindrical model is built with IMEX simulation model depicted in Figure 3 using the base case reservoir properties, PVT and simulation grid data presented in Appendix C. In the model, transition zone is neglected and the produced oil and water is injected back to the oil drainage boundary and aquifer respectively at the constant pressure boundary (representing BOR boundary). The production well is completed in 50% of the total oil-zone thickness.

![Figure 3](image)

**Figure 3** Radial model of oil with bottom water.
anisotropy ratio, as shown in Table 1. For 5 parameters chosen in this study, the design stipulates 46 number of runs (reservoir systems).

Critical-rate values, \( q_{cr} \), for different reservoir systems used in Eq. (5) are estimated using Eq. A-12.

### Table 1: Three-level values of different reservoir/aquifer system parameters

| Levels       | Mobility (M) | Aquifer thickness (h<sub>a</sub>) | Horizontal permeability (k<sub>h</sub>) | Penetration ratio \( \left( \frac{h_{wp}}{h_{a}} \right) \) | Anisotropy ratio \( \left( \frac{k_{a}}{k_{h}} \right) \) |
|--------------|--------------|-----------------------------------|----------------------------------------|------------------------------------------------|------------------------------------------------|
| Low (-1)     | 1            | 20                                | 50                                      | 0.2                                             | 0.01                                             |
| Intermediate (0) | 3            | 75                                | 100                                     | 0.5                                             | 0.1                                             |
| High (+1)    | 10           | 500                               | 500                                     | 0.8                                             | 1                                               |

### Table 2: Simulated and predicted data (WCult, oil-rate and water-rate) for an experimental matrix: \( h = 25 \text{ ft} ; Q = 20 \text{M}bbl/\text{d} \)

| Reservoir-aquifer system ID | Mobility (M) | Aquifer-thickness (h<sub>a</sub>) | Horizontal permeability (k<sub>h</sub>) | Penetration ratio \( \left( \frac{h_{wp}}{h_{a}} \right) \) | Anisotropy ratio \( \left( \frac{k_{a}}{k_{h}} \right) \) | Simulated WCult | WCult (from Eq. 2) | Abs. Discrepancy (WCult) | Penetration directions (\( \eta \), \( \mu \)) | Simulated oil-rate | Simulated water-rate | Predicted Oil-rate (from Eq. 7) | Predicted water-rate (from Eq. 4) |
|---------------------------|--------------|-----------------------------------|----------------------------------------|------------------------------------------------|------------------------------------------------|-----------------|----------------------|---------------------------------|---------------------------------|-----------------|------------------|--------------------------|--------------------------|
| 1                         | 10           | 75                                | 100                                    | 0.5                                            | 1.0                                            | 0.948           | 0.967                | 0.958                           | 0.018                           | 609             | 64               | 1936                     | 70                       | 1945             |
| 2                         | 10           | 75                                | 100                                    | 0.5                                            | 0.8                                           | 0.965           | 0.795                | 0.713                           | 0.096                           | 668             | 563              | 1493                     | 408                      | 1465             |
| 3                         | 10           | 75                                | 100                                    | 0.5                                            | 0.8                                           | 0.961           | 0.888                | 0.891                           | 0.003                           | 1178            | 196              | 1004                     | 182                      | 1805             |
| 4                         | 10           | 75                                | 500                                    | 0.5                                            | 0.8                                           | 0.969           | 0.946                | 0.958                           | 0.007                           | 152             | 82               | 918                      | 67                       | 940              |
| 5                         | 1           | 75                                | 50                                     | 0.5                                            | 0.8                                           | 0.970           | 0.750                | 0.780                           | 0.057                           | 1417            | 563              | 1493                     | 521                      | 1475             |
| 6                         | 1           | 75                                | 50                                     | 0.8                                            | 0.8                                           | 0.965           | 0.710                | 0.884                           | 0.018                           | 792             | 70               | 1804                     | 193                      | 1760             |
| 7                         | 1           | 75                                | 100                                    | 0.5                                            | 0.8                                           | 0.963           | 0.899                | 0.879                           | 0.053                           | 702             | 74               | 1066                     | 205                      | 1805             |
| 8                         | 1           | 75                                | 100                                    | 0.5                                            | 0.8                                           | 0.965           | 0.842                | 0.850                           | 0.017                           | 629             | 1070             | 958                      | 970                      | 960              |
| 9                         | 1           | 75                                | 100                                    | 0.5                                            | 0.8                                           | 0.974           | 0.895                | 0.869                           | 0.004                           | 625             | 52               | 1940                     | 19                      | 1910             |
| 10                        | 1           | 75                                | 100                                    | 0.5                                            | 1.0                                           | 0.946           | 0.962                | 0.946                           | 0.038                           | 480             | 10               | 1703                     | 98                       | 1493             |
| 11                        | 1           | 75                                | 100                                    | 0.5                                            | 0.8                                           | 0.963           | 0.899                | 0.879                           | 0.023                           | 710             | 194              | 1004                     | 206                      | 1805             |
| 12                        | 1           | 75                                | 100                                    | 0.6                                            | 1.0                                           | 0.969           | 0.889                | 0.880                           | 0.022                           | 410             | 182              | 1004                     | 205                      | 1805             |
| 13                        | 1           | 75                                | 50                                     | 0.5                                            | 1.0                                           | 0.946           | 0.899                | 0.873                           | 0.030                           | 1137            | 168              | 956                      | 218                      | 1810             |
| 14                        | 1           | 75                                | 50                                     | 0.5                                            | 1.0                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 15                        | 1           | 75                                | 50                                     | 0.5                                            | 1.0                                           | 0.946           | 0.896                | 0.960                           | 0.014                           | 714             | 163              | 956                      | 48                       | 1800             |
| 16                        | 1           | 75                                | 50                                     | 0.8                                            | 1.0                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 17                        | 1           | 75                                | 50                                     | 0.8                                            | 1.0                                           | 0.946           | 0.960                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 18                        | 1           | 75                                | 50                                     | 0.5                                            | 0.8                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 19                        | 1           | 75                                | 50                                     | 0.5                                            | 0.8                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 20                        | 1           | 75                                | 50                                     | 0.5                                            | 0.8                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 21                        | 1           | 75                                | 50                                     | 0.8                                            | 0.8                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 22                        | 1           | 75                                | 100                                    | 0.8                                            | 0.8                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |
| 23                        | 1           | 75                                | 100                                    | 0.8                                            | 0.8                                           | 0.946           | 0.896                | 0.960                           | 0.007                           | 1490            | 74               | 1529                     | 45                       | 1910             |

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Using the pressure drawdown simulation data for different runs, oil and water production rates were calculated using Eqs. (3) and (4) as shown in Table 2, which were then subsequently compared with their simulated data (from Table 2) shown in Figures 4 and 5. Near unit-slope correlation plot and high R² value close to 1, approve the validity of underlying assumptions of these proposed models (Eqs. (3) and (4)) to a larger extent. The slight discrepancy is due to the assumptions of 1) piston-like displacement process and 2) displaced water completion as shown in Figure 2 that neglects the additional skin due to water inflow from aquifer to the oil-zone. Further, the comparison plot between the predicted values of WCult from Eqs. (2) and (5) and the simulated values (from Table 2) is shown in Figure 6.

![Figure 4](image-url) Simulated vs. predicted oil production rate (Eq. 3).

![Figure 5](image-url) Simulated vs. predicted water production rate (Eq. 4).

![Figure 6](image-url) Simulated vs predicted ultimate water-cut with Eq. (2) and Eq. (5).

It is clear from the unit-slope correlation plot (Figure 6) that both the formulas give practically the same result. This infers that though the formula 2 ignores the inevitable non-radial flow to a partially penetration well, it still manages to conform to a more realistic physics-based formula 5 and hence predict the simulated WCult value.
Figure 7a shows the average absolute discrepancy (error), in percentage between the presently-used formula 2 and the proposed formula 5 using the data from Table 2. Also, Figure 7b shows the discrepancy between the formulas Eq. (2) and Eq. (5) for light oil reservoirs (M<3). From these two figures, it can be inferred that for the light oil reservoirs (when the mobility ratio is <3), the theoretical formula 2 may significantly deviate from the better (physically accurate) formula 5 for some cases (Figure 7a) with discrepancy as high as 8% (Figure 7b), which may not be reflected in Figure 6 due to considerable wide variety of sample size. In this study, any discrepancy exceeding the limit of 5% would be considered significant. This implies that for the light oil reservoir, the simplified assumptions of formula 2 may no longer allow it to better predict the actual WCult values, for which the formula 5 can serve better. This can be also be justified by the mathematical proof in Appendix B. So, in practice, formula 5 should be preferred for general use.

On the other hand, for moderate to high mobility ratio reservoirs (M≥3), Figure 7a shows that the average discrepancy between the formulas is less than 5%, which is insignificant. This implies that in those conditions, formula (5) can be reduced to formula (2), which is also shown mathematically in Appendix B. So, Eq. (2), being simpler than Eq. (5), suffices to predict WCult for viscous oil reservoirs (M≥3).

Conclusions

Results of the study are summarized in the following conclusions:

1. A new analytical formula for WCult has been proposed including the physical effect ignored in the presently-used formula: partial penetration of oil zone, and aquifer. The formula utilizes the new models of oil and water production-rates during the ultimate water-cut stage. The derivation of models considers the piston-like displacement process and the inflow of oil and water into separate completions at the top of oil-zone and aquifer respectively.

2. The proposed formulas are systematically verified for wide variety of reservoir systems using design of simulated experiments (IMEX). High R² value for the plot between the simulated and the predicted oil and water production-rates approves the validity of the proposed model’s underlying assumptions to a large extent. However, sight discrepancy can be attributed to the above assumptions.

3. In general, both the formulas (proposed and presently-used) of WCult predicts almost the same results which matches the simulated WCult values. However, for the light oil reservoirs (mobility ratio<3), simulations showed that the theoretical presently used-formula may significantly deviate from the (physically accurate) proposed formula. This is also confirmed by mathematical proof, so in practice, proposed formula should be preferred for the possible avoidance of errors.

4. On the other hand, for viscous oil reservoirs (Mobility ratio≥3), comparison of the simulations with the predicted values showed that the presently-used formula suffices to predict the WCult values. This fact that the proposed formula reduces to presently-used formula for the above reservoirs, can be justified mathematically.

Nomenclature

- \( \mu_o \) = viscosity of oil, cp
- \( \mu_w \) = viscosity of water, cp
- \( \Delta \rho \) = density difference between water and oil, lb/ft³
- \( B_o \) = oil formation volume factor, bbl/stb
- \( B_w \) = water formation volume factor, bbl/stb
- BOR = balanced-oil-rate
- \( h_o \) = oil-zone thickness, ft
- \( h_p \) = perforated length, ft
- \( h_{opa} \) = length of well-completion occupied by oil during WCult stage, ft
- \( h_{upa} \) = length of well-completion occupied by water during WCult stage, ft
- \( h_q \) = aquifer thickness, ft
- \( k_h \) = horizontal permeability, md
- \( k_e \) = effective permeability of oil, md
- \( k_{re} \) = relative permeability of oil
Appendix A: Derivation of new analytical WCult formula

Assuming piston-like displacement process, the rise of water cone before final stabilization covers larger area of oil completion. Eventually, the ratio of well completion producing oil and water becomes equal to the ratio of oil and water zone thickness, when ultimate water-cut is reached. So, the length of well-completion occupied by oil during WCult stage:

\[ h_{opw} = h_o \frac{h_o + h_w}{h_o + h_w} \]  \hspace{1cm} (A-1)

And, the length of well-completion occupied by water during WCult stage:

\[ h_{wpw} = h_w \frac{h_o + h_w}{h_o + h_w} \]  \hspace{1cm} (A-2)

This follows that the well completion system during water cone stabilization stage can be assumed to be the combination of the oil completion (producing only oil) at the top of oil-zone and the displaced water completion (producing only water) at the top of aquifer (Figure 2). So, oil inflow rate due to partial penetration in oil-zone (producing only oil) is given by:

\[ q_o = \frac{2\pi k_w h_o (p_o - p_w)}{\mu_o (\ln \frac{r_o}{r_w} + s_w)} \]  \hspace{1cm} (A-3)

Since, \( k_w = k_R k_o \), we get:

\[ q_o = \frac{2\pi k_R k_o h_o (p_o - p_w)}{\mu_o \ln \frac{r_o}{r_w} + s_w} \]  \hspace{1cm} (A-3)

Where, \( s_w \) is the skin factor due to oil-inflow and is given by,

\[ S_w = \left(1 - \frac{1}{h_{opD}}\right) - 1 \ln \frac{\pi}{2h_{opD}} + \frac{1}{h_{opD}} \ln \left[1 + \frac{h_{opD}}{2 + h_{opD}} \left(\frac{A - 1}{B - 1}\right)^{\frac{1}{2}}\right] \]  \hspace{1cm} (A-4)

\[ h_{opD} = \frac{h_{opw}}{h_o \frac{h_o + h_w}{h_o + h_w}} \]  \hspace{1cm} (From Eq. (A-1)) \hspace{1cm} (A-5)

Now, again water inflow rate due to partial penetration in an aquifer (producing only water) is given by,

\[ q_w = \frac{2\pi k_w h_w (p_o - p_w)}{\mu_w (\ln \frac{r_w}{r_w} + s_w)} \]  \hspace{1cm} (A-6)

So, the skin factor, \( s_w \) due to water-inflow can be represented by:

\[ S_w = \left(1 - \frac{1}{h_{wD}}\right) - 1 \ln \frac{\pi}{2h_{wD}} + \frac{1}{h_{wD}} \ln \left[1 + \frac{h_{wD}}{2 + h_{wD}} \left(\frac{Aw - 1}{Bw - 1}\right)^{\frac{1}{2}}\right] \]  \hspace{1cm} (A-7)

\[ h_{wD} = \frac{h_{wpw}}{h_w \frac{h_o + h_w}{h_o + h_w}} \]  \hspace{1cm} (From Eq. (A-2)) \hspace{1cm} (A-8)

\[ r_{wD} = \left(\frac{r_w}{h_w}\right) \left(\frac{k}{k_w}\right)^{\frac{1}{2}} ; \hspace{0.5cm} Aw = 4/h_{wD} ; \hspace{0.5cm} Bw = 4/3h_{wD} \]  \hspace{1cm} (A-9)

From Eqs. (A-5) and (A-8), we get:

\[ h_{wD} = h_{wpD} = h_{wiD} \]  \hspace{1cm} (A-9)

Ultimate Water-cut, during water-cut stabilization stage is given by:

\[ WCult = \left(1 - \frac{q_o}{Q}\right)q_w + \left(1 - \frac{q_w}{Q}\right) \frac{1}{1 + \frac{q_o}{q_w}} \]  \hspace{1cm} (A-10)

Substituting \( q_o \) and \( q_w \) from Eqs. (A-3) and (A-6) in (A-10), we get:

\[ WCult = \left(1 - \frac{q_o}{Q}\right) \frac{q_w}{h_o} + \left(1 - \frac{q_w}{Q}\right) \frac{1}{1 + \frac{q_o}{q_w}} \]  \hspace{1cm} (A-11)

\[ \text{Since, } k_w = k_R k_o, \text{ we get:} \]

\[ q_o = \frac{2\pi k_R k_o h_o (p_o - p_w)}{\mu_o (\ln \frac{r_o}{r_w} + s_w)} \]  \hspace{1cm} (A-3)

Where, \( s_w \) is the skin factor due to oil-inflow and is given by,

\[ S_w = \left(1 - \frac{1}{h_{opD}}\right) - 1 \ln \frac{\pi}{2h_{opD}} + \frac{1}{h_{opD}} \ln \left[1 + \frac{h_{opD}}{2 + h_{opD}} \left(\frac{A - 1}{B - 1}\right)^{\frac{1}{2}}\right] \]  \hspace{1cm} (A-4)

\[ h_{opD} = \frac{h_{opw}}{h_o \frac{h_o + h_w}{h_o + h_w}} \]  \hspace{1cm} (From Eq. (A-1)) \hspace{1cm} (A-5)

Now, again water inflow rate due to partial penetration in an aquifer (producing only water) is given by,

\[ q_w = \frac{2\pi k_w h_w (p_o - p_w)}{\mu_w (\ln \frac{r_w}{r_w} + s_w)} \]  \hspace{1cm} (A-6)

So, the skin factor, \( s_w \) due to water-inflow can be represented by:

\[ S_w = \left(1 - \frac{1}{h_{wD}}\right) - 1 \ln \frac{\pi}{2h_{wD}} + \frac{1}{h_{wD}} \ln \left[1 + \frac{h_{wD}}{2 + h_{wD}} \left(\frac{Aw - 1}{Bw - 1}\right)^{\frac{1}{2}}\right] \]  \hspace{1cm} (A-7)

\[ h_{wD} = \frac{h_{wpw}}{h_w \frac{h_o + h_w}{h_o + h_w}} \]  \hspace{1cm} (From Eq. (A-2)) \hspace{1cm} (A-8)

\[ r_{wD} = \left(\frac{r_w}{h_w}\right) \left(\frac{k}{k_w}\right)^{\frac{1}{2}} ; \hspace{0.5cm} Aw = 4/h_{wD} ; \hspace{0.5cm} Bw = 4/3h_{wD} \]  \hspace{1cm} (A-9)

Ultimate Water-cut, during water-cut stabilization stage is given by:

\[ WCult = \left(1 - \frac{q_o}{Q}\right)q_w + \left(1 - \frac{q_w}{Q}\right) \frac{1}{1 + \frac{q_o}{q_w}} \]  \hspace{1cm} (A-10)

Substituting \( q_o \) and \( q_w \) from Eqs. (A-3) and (A-6) in (A-10), we get:

\[ WCult = \left(1 - \frac{q_o}{Q}\right) \frac{q_w}{h_o} + \left(1 - \frac{q_w}{Q}\right) \frac{1}{1 + \frac{q_o}{q_w}} \]  \hspace{1cm} (A-11)
Critical rate, $q_{cr}$ in above Eq. (A-11) can be substituted by the following formula:

$$q_{cr} = 0.0783 \times 10^{-4} \left[ \frac{\Delta \rho k_o \left( h_{cr}^2 - h_{op}^2 \right)}{\mu_o B_o} \right] \left[ 0.7311 + \frac{1.943}{r_w k_o / h_o \sqrt{k_h}} \right]$$  \hspace{1cm} (A-12)

Where, all the parameters are in field units.

**Appendix B: Mathematical convergence of new formula to presently-used formula**

Using Eqs. (A-4), (A-7) and (A-9), Eq. 5 can be rewritten as:

$$\ln \left( \frac{h_{cr}}{h_o} \right) = \ln \left( \frac{r_w}{r_{cr}} + \frac{s_o}{h_o} \right) = \frac{M_{h_o}}{\ln \left( \frac{r_w}{r_{cr}} + s_o \right)} \left( \frac{Q_{cr}}{Q} \right) = \frac{M_{h_o}}{\ln \left( \frac{r_w}{r_{cr}} + s_o \right)} \left( 1 - \frac{q_{cr}}{Q} \right)$$  \hspace{1cm} (B-1)

Substituting $T = h_{cr} / h_o$, and $C = \ln \left( \frac{r_w}{r_{cr}} + 1 \right) \ln \frac{\pi h_o}{2 r_w \sqrt{k_c / k_h}} + \ln \frac{h_{opD}}{2 + h_{opD}} \left( \frac{A - 1}{B - 1} \right)^{1/2}$ in Eq. (B-1), we get:

$$\frac{M_{h_o}}{\ln \left( \frac{r_w}{r_{cr}} + s_o \right)} \left( \frac{Q_{cr}}{Q} \right) = \frac{M_{h_o}}{\ln \left( \frac{r_w}{r_{cr}} + s_o \right)} \left( 1 - \frac{q_{cr}}{Q} \right)$$  \hspace{1cm} (B-2)

Figure B-1 clearly shows the maximum value of $\ln T / T$ is 0.37. Subsequently, the approximate maximum possible value of $\ln T / T$ is 0.15 for the practical field operating range values of $h_{opD}$ (between 0.1 and 1) and for practical value of $T > 0.8$. Minimum possible value of $\frac{1}{r}$ tends to 0 for infinite thick aquifers.

Now, assuming 5% maximum possible error is permissible in predicted WCut value given by Eq. (B-2); for viscous reservoirs (when mobility ratio $\geq 3$), any value of $\frac{\ln T}{T}$ would lie withing this error margin of Eq. (B-2) and hence, the part $\frac{1}{h_{opD}}$ can be ignored. So, Eq. (B-2) or Eq. (5) can be rewritten as:

$$\ln \left( \frac{T}{T_{min}} \right) = \frac{M_{h_o}}{\ln \left( \frac{r_w}{r_{cr}} + s_o \right)} \left( 1 - \frac{q_{cr}}{Q} \right) \frac{M_{h_o}}{h_o}$$  \hspace{1cm} (B-3)
Above derivation mathematically proves that Eq. (5) reduces to Eq. (2) in case of viscous oil reservoirs. However, for mobility ratio<3, Eq. (5) may or may not reduce to Eq. (2) depending upon the ratio of aquifer to oil-zone thickness.

**Appendix C: Complete Reservoir Simulation Input Data**

**Table C-1** Reservoir and Well Input data

| Parameter                        | Unit     | Value |
|----------------------------------|----------|-------|
| Datum depth                      | ft       | 5000  |
| Thickness of oil zone            | ft       | 25    |
| Depth of WOC                     | ft       | 5025  |
| Thickness of water zone          | ft       | 75, varied |
| Reservoir pressure at datum depth| psi      | 6000  |
| Position of top completion from formation top | ft | 0 |
| Perforated length                | ft       | 12, varied |
| Horizontal permeability          | md       | 100, varied |
| Anisotropy ratio                 | md       | 0.1, varied |
| Porosity                         | fraction | 0.3   |
| Well radius                      | ft       | 0.25  |
| Outer radius of oil-zone         | ft       | 1000  |
| Outer radius of water zone       | ft       | 1000  |
| Total liquid Production rate     | bpd      | 2000  |

**Table C-2** Fluid Properties Input Data

| Property                          | Unit     | Value |
|-----------------------------------|----------|-------|
| Reference pressure                | psi      | 6000  |
| Formation oil volume factor       | rb/stb   | 1.2   |
| Relative oil permeability at connate water saturation | fraction | 1 |
| Water compressibility             | 1/psi    | 3.3202e-6 |
| Oil compressibility               | 1/psi    | 1.50E-05 |
| water viscosity                   | cp       | 0.5   |
| Oil viscosity                     | cp       | 1.5, varied |
| oil density                       | lb/cuft  | 43.65 |
| Water density                     | lb/cuft  | 60.55 |
| Bubble point                      | psi      | 100   |

**Table C-3** Simulation Grid Data

| Region      | Direction | Grid Number |
|-------------|-----------|-------------|
| R           | O         | 20          |
| Oil zone    | Φ         | 1           |
| Z           |           | 25          |
| R           |           | 29          |
| Water zone  | Φ         | 1           |
| Z           |           | 15          |

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