Bures measure of entanglement of an arbitrary state of two qubits

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In [Phys. Rev. Lett. 80, 2245 (1998)] an explicit expression for entanglement of formation for any two qubit state was given. Based on this result we present an expression for the Bures measure of entanglement for two qubit states. This measure was proposed in [Phys. Rev. A 57, 1619 (1998)], where the authors showed that it satisfies all properties every entanglement measure must have.

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A bipartite mixed state $\rho$ on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is in general called entangled if it can not be written in the form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B,$$

with nonnegative probabilities $p_i$, $\sum_i p_i = 1$, and $\rho_i^{A,B}$ being states on $\mathcal{H}_A, \mathcal{H}_B$. Otherwise the state is called separable.

Entanglement of pure bipartite states $|\psi\rangle$ is usually quantified by the entanglement entropy

$$E(|\psi\rangle) = -\text{Tr}\left[\rho^A \log_2 \rho^A\right],$$

$$\rho^A = \text{Tr}_B |\psi\rangle\langle \psi|.$$

For mixed states many different measures were proposed, two of them are entanglement of formation $E_F$ and Bures measures of entanglement $E_B$.

$E_F$ is defined as the minimal entanglement needed on average to create the state:

$$E_F(\rho) = \min_{\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|} \sum_i p_i E(|\psi_i\rangle).$$

For two qubit states a simple expression is known, it can be found in [2].

Bures measure of entanglement was proposed in [3] and is defined as the Bures distance from the set of separable states:

$$E_B(\rho) = \min_{\sigma \in S} D_B(\rho, \sigma).$$

Here $S$ is the set of separable mixed states. With fidelity $F(\rho, \sigma) = (\text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}})^2$ Bures distance is defined as

$$D_B(\rho, \sigma) = 2 - 2F(\rho, \sigma).$$

It is trivial to see that $E_B(\rho)$ is zero on separable states only. Further $E_B$ is invariant under local unitary operations and nonincreasing under LOCC operations, proofs can be found in [3].

In this paper we will give an expression for $E_B$ for an arbitrary state of two qubits. This result is heavily based on results in [2]. There the concurrence $C(\rho)$ for a two qubit state $\rho$ was defined as

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where $\lambda_i$ are squareroots of the eigenvalues of $\rho (\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y)$ in decreasing order. Complex conjugation is taken in the standard basis, and $\sigma_y = \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right)$.

**Theorem.** For an arbitrary two qubit state $\rho$ holds

$$E_B(\rho) = 2 - 2\sqrt{\frac{1 + \sqrt{1 - C(\rho)^2}}{2}}.$$

**Proof.** Let $\sigma$ be a separable state that maximizes the fidelity among all separable states. We will show that $F(\rho, \sigma) = \frac{1 + \sqrt{1 - C(\rho)^2}}{2}$ which will end the proof.

According to [4, Theorm 9.4] holds:

$$F(\rho, \sigma) = \max_{\phi} |\langle \psi | \phi \rangle|^2,$$

where $|\psi\rangle$ is a purification of $\rho$ and maximization is done over all purifications of $\sigma$ denoted by $|\phi\rangle$. In the following $|\phi\rangle$ will denote a particular purification of $\sigma$ that reaches the maximum, that is $F(\rho, \sigma) = |\langle \psi | \phi \rangle|^2$.

If $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$ and $\sigma = \sum_i q_i |\phi_i^{(1)}\rangle|\phi_i^{(1)}\rangle \otimes |\phi_i^{(2)}\rangle|\phi_i^{(2)}\rangle$, then

$$|\psi\rangle = \sqrt{p_i} |\psi_i^{(0)}\rangle |\psi_i\rangle,$$

$$|\phi\rangle = \sum_i \sqrt{q_i} |\phi_i\rangle.$$

Here $p_i \geq 0$, $q_i \geq 0$, $\sum_i p_i = \sum_i q_i = 1$, $\langle \psi_i^{(0)} | \psi_j^{(0)} \rangle = \delta_{ij}$, and $T_{R_0} |\psi\rangle = \rho$, $T_{R_0} |\phi\rangle = \sigma$, $|\phi_i\rangle = |\phi_i^{(0)}\rangle |\phi_i^{(1)}\rangle |\phi_i^{(2)}\rangle$.

The states $|\phi_i^{(1)}\rangle$ and $|\phi_i^{(2)}\rangle$ can always be chosen such that $\langle \psi | \phi_i\rangle \geq 0$, thus using

$$|\langle \psi | \phi \rangle|^2 = \sum_i \sqrt{q_i} |\langle \psi | \phi_i \rangle|^2.$$
Using lagrange multipliers it can easily be shown that \[\text{is maximal if } \sqrt{h_i} = \frac{\langle \psi | \phi_i \rangle}{\sqrt{\sum_i \langle \psi | \phi_i \rangle^2}}. \]
Using this we get

\[\langle \psi | \phi_i \rangle^2 = \sum_i \langle \psi | \phi_i \rangle^2. \tag{9}\]

Note that there always is a unitary matrix \(u\) such that

\[|\psi_i(0)\rangle = \sum_j u_{ij}|\phi_j(0)\rangle. \tag{10}\]

Using (10) in (6) we get

\[|\psi\rangle = \sum_i \sum_j \sqrt{u_{ij}}|\phi_j(0)\rangle|\psi_i\rangle \]
\[= \sum_j \sqrt{P_{ij}}|\phi_j(0)\rangle|\psi_j\rangle, \tag{11}\]

where \(\rho = \sum_j P_{ij}|\psi_j\rangle\langle \psi_j|\). With this result we see that it is sufficient to restrict ourselves to the case where \(|\psi_i(0)\rangle = |\phi_i(0)\rangle\). Then

\[\langle \psi | \phi_i \rangle^2 = p_i \langle \psi_i | \phi_i(1) \rangle \langle \phi_i(2) | \psi_i \rangle^2. \tag{12}\]

Let now

\[|\psi_i\rangle = \sqrt{\lambda_i}|u_i(1)\rangle|u_i(2)\rangle + \sqrt{1 - \lambda_i}|v_i(1)\rangle|v_i(2)\rangle \tag{13}\]

be the Schmidt decomposition of \(|\psi_i\rangle\), \(\lambda_i \geq \frac{1}{2}\). It is very easy to show that \(\langle \psi_i | \phi_i(1) \rangle \langle \phi_i(2) | \psi_i \rangle^2\) is maximal for \(|\phi_i(1) \rangle = |u_i(1)\rangle\), \(|\phi_i(2) \rangle = |u_i(2)\rangle\) with the maximal value \(\lambda_i\).

Noting this we get

\[\langle \psi | \phi_i \rangle^2 = p_i \lambda_i. \tag{14}\]

Using (14) in (6) we get

\[\langle \psi | \phi \rangle^2 = \sum_i p_i \lambda_i. \tag{15}\]

From [2] we know that every two qubit state \(\rho\) has a decomposition into four pure states all having equal entanglement and thus equal Schmidt coefficients, denoted here by \(\mu\) and \(1 - \mu\). Further this decomposition minimizes average entanglement, which means that with \(h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)\) holds:

\[h(\mu) \leq \sum_i p_i h(\lambda_i) \tag{16}\]

for any decomposition of the given two qubit state into pure states with Schmidt coefficients \(\lambda_i\) and \(1 - \lambda_i\) and probabilities \(p_i\). Using [16] we will now show that [15] is maximal if all \(\lambda_i\) are equal to \(\mu\).

Suppose the opposite, that is there is a decomposition into states having Schmidt coefficients \(\lambda_i\) and probabilities \(p_i\) such that \(\sum_i p_i \lambda_i > \mu\). Then holds:

\[\sum_i p_i h(\lambda_i) \leq h \left( \sum_i p_i \lambda_i \right) < h(\mu). \tag{17}\]

The first inequality is true because \(h\) is concave, the second inequality is true because \(h(x)\) decreases for \(x \geq \frac{1}{2}\). This is in contradiction to [16].

The consequence is that [15] is maximal if all \(\lambda_i\) are equal to \(\mu\). In [2] it was shown that in this case

\[\mu = \lambda_i = \frac{1 + \sqrt{1 - C(\rho)^2}}{2}. \tag{18}\]

With [15] we have showed that \(F(\rho, \sigma) = \langle \psi | \phi \rangle^2 = 1 + \sqrt{1 - C(\rho)^2}\). This completes the proof. \(\square\)

In this paper a formula for Bures measure of entanglement for two qubit states was presented and proved. It is interesting to note that the proof is based on the fact that a two qubit state can always be decomposed into pure states, all having equal entanglement. This is in general true only for two qubits, this technique will not work for higher dimensions [1].

For the most measures of entanglement presented so far no explicit expressions are known even for two qubit states [1]. It is very probable that the concurrence also plays an important role in other entanglement measures, further work is needed in this direction.

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