On the Bose–Einstein distribution and Bose condensation

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ABSTRACT. For a system of identical Bose particles sitting on integer energy levels, we give sharp estimates for the convergence of the sequence of occupation numbers to the Bose–Einstein distribution and for the Bose condensation effect.

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1. Introduction

The Bose–Einstein distribution has been studied by physicists (here we only mention the well-established textbooks by Landau and Lifshits [1] and Kvasnikov [2]) as well as mathematicians (e.g., see Vershik’s papers [3, 4], where further references can be found). It is well known that the vector of occupation numbers tends, in a certain sense, to the Bose–Einstein distribution as the total energy of the system and the number of particles tend to infinity. It is, however, of interest to establish sharp estimates for this convergence, including the case of Bose condensation. This has been done in our papers [5–9]. The present exposition is mainly based on these papers but contains a number of improvements and has the important advantage of being largely self-contained (at least as far as Bose condensation itself and distributions with variable number of particles are concerned). Moreover, in contrast to other mathematical treatments of the subject, it does not use any but very elementary mathematical tools. (For example, we avoid

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resorting to number-theoretic results like the Meinardus theorem \cite{10} or methods of analytic number theory \cite{11}.)

Before proceeding to the results themselves, let us recall what Bose condensation is by using an elementary combinatorial model, namely, that of balls distributed over boxes.

Suppose that there is a sequence of boxes $U_j$, $j = 0, 1, 2, \ldots$, and each box $U_j$ is divided into $q_j$ compartments. We take $N$ identical balls and put them into the boxes at random observing the only condition that

\begin{equation}
\sum_{j=0}^{\infty} jN_j \leq M,
\end{equation}

where $N_j$ is the number of balls in the box $U_j$ and $M$ is a positive integer specified in advance. As an outcome, we obtain a sequence of nonnegative integers $N_j$, $j = 0, 1, 2, \ldots$, such that

\begin{equation}
\sum_{j=0}^{\infty} N_j = N
\end{equation}

and condition (1) is satisfied. It is easily seen that, given $M$ and $N$, there are finitely many such sequences. Suppose that all allocations of balls to compartments are equiprobable. Since the number of ways to distribute $N_j$ indistinguishable balls over $q_j$ compartments is equal to

\[
\binom{N_j + q_j - 1}{N_j} = \frac{(q_j + N_j - 1)!}{N_j!(q_j - 1)!},
\]

it follows that each sequence $\{N_j\}$ can be realized in

\begin{equation}
W(\{N_j\}) = \prod_{j=0}^{\infty} \binom{N_j + q_j - 1}{N_j}
\end{equation}

ways, and the probability of this sequence is equal to $W(\{N_j\})$ divided by the sum of expressions similar to (3) over all sequences of nonnegative integers satisfying the constraints (1) and (2). This makes the set of all such sequences a probability space; the corresponding probability measure will be denoted by $P_{M,N}$. The positive integers $q_j$ are called the multiplicities. We will assume that

\begin{equation}
q_j = Qj^{d-1} + o(j^{d-1}), \quad j \to \infty,
\end{equation}

where $d > 1$ is a given parameter (which we refer to as dimension) and $Q \geq 1$ is a positive constant.

What happens as $M, N \to \infty$? It turns out that the so-called condensation phenomenon can occur: if $N$ tends to infinity too rapidly, namely, if it exceeds some threshold $\overline{N} = \overline{N}(M)$, then a majority of the excessive $N - \overline{N}$ balls end up landing in the box $U_0$; more precisely, with probability asymptotically equal to 1, the number of balls in $U_0$
is close to \( N - \overline{N} \) (and accordingly, the total number of balls in all the other boxes is close to \( \overline{N} \), no matter how large \( N \) itself is). Let us state the corresponding assertion (which is a special case of Corollary 4 and the subsequent argument in Sec. 2.2.2).

**Theorem 1.** Define \( \overline{N} = \overline{N}(M) \) by the formula

\[
\overline{N} = \sum_{j=1}^{\infty} \frac{q_j}{e^{b j} - 1},
\]

where \( b \) is the unique positive root of the equation

\[
\sum_{j=1}^{\infty} \frac{j q_j}{e^{b j} - 1} = M,
\]

and set

\[
\Delta = \begin{cases} 
\overline{N} \ln \overline{N} \chi(\overline{N}) & \text{if } d > 2, \\
\overline{N}^{1/d} \ln \overline{N} \chi(\overline{N}) & \text{if } 1 < d \leq 2,
\end{cases}
\]

where \( \chi(x), x \geq 0, \) is an arbitrary positive function arbitrarily slowly tending to infinity as \( x \to \infty \). There exist constants \( C_m, \) independent of \( M \) and \( N, \) such that if \( N > \overline{N}, \) then

\[
P_{M,N}(\left| N_0 - (N - \overline{N}) \right| > \Delta) \leq C_m \overline{N}^{-m}, \quad m = 1, 2, \ldots.
\]

Nothing of this sort happens if \( N \leq \overline{N}. \) In this case, the limit distribution as \( M, N \to \infty \) is the Bose–Einstein distribution with parameters \( \beta, \mu > 0 \) (see formulas (22) and (23) below), and no condensation on the zero level occurs.

It is not hard to write out an asymptotic formula for \( \overline{N}. \) To this end, one substitutes (4) into (5) and (6) and applies the Euler–Maclaurin formula to the resulting series so as to transform them into integrals. The result (see formula (19) in Sec. 2.1.2) is that

\[
\overline{N} = C(d) M^{\frac{d}{d+1}} Q^{\frac{1}{d+1}} (1 + o(1)),
\]

where \( C(d) \) is a constant\(^1\) depending only on the dimension \( d. \)

Suppose that, for given \( M \) and \( N > \overline{N}, \) we wish to avoid Bose condensation. How can we do that?

One way would be to increase \( \overline{N} \) so as to ensure that \( \overline{N} \geq N. \) To this end, let us partly “Boltzmannize” the system, i.e., make the balls partly distinguishable. More precisely, suppose that the model is basically the same, but we are additionally allowed to paint each of the \( N \) balls at random into one of \( K \) distinct colors. Now that we can

\[^1\text{The explicit expression is } C(d) = \frac{\Gamma(d) \zeta(d)}{(\Gamma(d + 1) \zeta(d + 1))^{\frac{1}{d+1}}}, \]

where \( \Gamma(x) \) is the gamma function and \( \zeta(x) \) is the Euler zeta function.
distinguish between balls of different colors (but balls of a same color are still indistinguishable), the Bose condensation threshold $\overline{N}$ should change.

Let us compute how exactly it changes. To make the computation, instead of painting the balls, we mentally divide each of the $q_j$ compartments in the $j$th box into $K$ sub-compartments and put the uncolored balls there (with the understanding that putting a ball into the $k$th sub-compartment is equivalent to painting the ball into the $k$th color). Now there are $Kq_j$ compartments in the $j$th box, and we see that, all in all, the introduction of $K$ colors has the only effect that all multiplicities $q_j$ are multiplied by $K$.

Let us apply Theorem 1 (with $q_j$ replaced by the new multiplicities $\tilde{q}_j = Kq_j$). Formula (19) gives an asymptotic expression for the new threshold, which we denote by $\tilde{N}$. All we have to do is to replace $Q$ by $KQ$ in formula (7); then we obtain

$$
\tilde{N} = C(d)M^{\frac{d}{d+1}}(KQ)^{\frac{1}{d+1}}(1 + o(1)) = K^{\frac{1}{d+1}}\overline{N}(1 + o(1)).
$$

Thus, the introduction of $K$ distinct colors has raised $\overline{N}$ by the factor $K^{\frac{1}{d+1}}$.

2. Main Results

In this section, we state our main results. All proofs are given in Sec. 3. The simplest physical model to imagine behind our mathematical constructions is that of a system of identical Bose particles sitting on integer energy levels.

2.1. System with a variable number of particles. We start our analysis by considering the “photonic” case, where the number of particles in the system is not fixed and only a constraint on the overall system energy is given.

2.1.1. Definition of the system. Let $M \geq 0$ be an integer. We denote the set of all sequences $\{N_j\} \equiv \{N_j\}_{j=1}^{\infty}$ of nonnegative integers satisfying the condition

$$
\sum_{j=1}^{\infty} jN_j \leq M
$$

by $\Omega_M$. Note that all such sequences are finitely supported and $\Omega_M$ is finite.

Next, let positive real numbers $q_j > 0$, $j = 1, 2, \ldots$, be given. We introduce the probability space $X_M = (\Omega_M, \mathcal{F}_M, \mathcal{P}_M)$, where $\mathcal{F}_M = 2^{\Omega_M}$ is the powerset of $\Omega_M$ (as is customary with discrete probability spaces).
and the probability $P_M$ is defined as follows. We assign the weight

$$w(\{N_j\}) = \prod_{j=1}^{\infty} \left( \frac{N_j + q_j - 1}{N_j} \right),$$

where $\binom{z}{n}$ is the generalized binomial coefficient,\(^3\) to each element $\{N_j\} \in \Omega_M$ and the weight

$$w(A) = \sum_{\{N_j\} \in A} w(\{N_j\})$$

to each subset $A \subset \Omega_M$ and set\(^4\)

$$P_M(A) = \frac{w(A)}{w(\Omega_M)}, \quad A \subset \Omega_M.$$  

The numbers $q_j$ will be referred to as multiplicities.\(^5\) We assume that they have the asymptotics\(^6\)

$$q_j = Qj^{d-1}(1 + o(1)) \quad \text{as } j \to \infty$$

for some real constants $d > 1$ and $Q > 0$. In particular, there exists constants $B_1, B_2 > 0$ such that\(^7\)

$$B_1j^{d-1} \leq q_j \leq B_2j^{d-1}, \quad j = 1, 2, \ldots .$$

2.1.2. Limit distribution. It turns out that if $M$ is large, then a randomly chosen sequence $\{N_j\} \in \Omega_M$ is with high probability close, in the sense described in Theorem 2 below, to the nonrandom sequence $\{\overline{N}_j\}$ defined as follows. For $M > 0$, the equation

$$\sum_{j=1}^{\infty} \frac{j^{q_j}}{e^{q_j} - 1} = M$$

\(^3\)Recall that $\binom{z}{n} = \prod_{j=0}^{n-1} \frac{z-j}{n-j}$, the empty product (for $n = 0$) being by definition equal to 1. Since $\{N_j\}$ is finitely supported, it follows that only finitely many factors in (9) are different from 1.

\(^4\)In what follows, we also feel free to write the condition determining the set $A$ instead of the argument $A$ itself in expressions like $P_M(A)$.

\(^5\)If the $q_j$ are integers, then $\binom{N_j + q_j - 1}{N_j}$ is exactly the number of ways in which $N_j$ indistinguishable particles can be placed on an energy level of multiplicity $q_j$, and $w(\{N_j\})$ is the number of distinct system states corresponding to the sequence $\{N_j\}$ of occupation numbers.

\(^6\)The asymptotics (12) with $Q = \Gamma(d^{-1})$ holds, for example, for the multiplicities $q_j = (j^{d-1})$ of energy levels of the $d$-dimensional quantum-mechanical harmonic oscillator.

\(^7\)Most of the results stated below, except for some explicit formulas like (19), remain valid if we drop (12) and only require that the estimates (13) be true.
has a unique solution \( b > 0 \) (which tends to zero as \( M \to \infty \)), and we set

\[
N_j = \frac{q_j}{e^{bj} - 1}.
\]

In the theorem below, we also need the sum

\[
N = \sum_{j=1}^{\infty} N_j \equiv \sum_{j=1}^{\infty} \frac{q_j}{e^{bj} - 1},
\]

which tends to infinity together with \( M \). Take a positive function \( \chi(x) \), \( x \geq 0 \), tending (arbitrarily slowly) to infinity as \( x \to \infty \) and set

\[
\Delta = \begin{cases} 
(N \ln N)^{1/2} \chi(N) & \text{if } d > 2, \\
N^{d/d} \ln N \chi(N) & \text{if } 1 < d \leq 2.
\end{cases}
\]

Now we are in a position to state our assertions.

**Theorem 2.** Let the numbers \( q_j \) satisfy (12). Then there exist constants \( C_s > 0 \), \( s = 1, 2, \ldots \), such that the estimates

\[
P_M \left( \left| \sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) \right| > \Delta \right) \leq C_s N^{-s}, \quad s = 1, 2, \ldots,
\]

where \( \overline{N}_j, \overline{N}, \) and \( \Delta \) are defined in (15), (16), and (17), hold for an arbitrary \( M > 0 \) and an arbitrary sequence \( \{f_j\}, j = 1, 2, \ldots \), of complex numbers satisfying the condition \( \sup_j |f_j| \leq 1 \).

By taking \( f_j = 0 \) for \( j < l \) and \( f_j = 1 \) for \( j > l \) in (18), we obtain the following assertion.

**Corollary 1.** Under the assumptions of the theorem, for arbitrary integer \( l \geq 1 \) one has

\[
P_M \left( \left| \sum_{j=l}^{\infty} N_j - \sum_{j=l}^{\infty} \overline{N}_j \right| > \Delta \right) \leq C_s N^{-s}, \quad s = 1, 2, \ldots.
\]

Essentially, Theorem 2 and Corollary 1 say that for, large \( M \), a random element \( \{N_j\} \) in \( \mathcal{X}_M \) is well approximated by the Bose–Einstein distribution (15) with parameter \( b \). The following closed-form asymptotic expressions relating \( M, b, \overline{N}, \) and the cumulative distribution

---

8Indeed, it follows from (13) that the series on the left-hand side in (14) converges for all \( b > 0 \); moreover, the sum of this series is easily seen to be a function monotone decreasing from \( \infty \) to 0 as \( b \) goes from 0 to \( \infty \).

9One should bear in mind that \( b, \overline{N}_j, \overline{N}, \) and \( \Delta \) are functions of \( M \), even though we do not always write out the argument \( M \) explicitly.
\[ \sum_{j\geq 1} \mathcal{N}_j \text{ hold as } M \to \infty \text{ by Proposition 4, (i) in Sec. 4: } \\
M = b^{-d-1}Q \Gamma(d+1)\zeta(d+1)(1+o(1)), \\
\mathcal{N} = b^{-d}Q \Gamma(d)\zeta(d)(1+o(1)) \text{ and hence } \\
\mathcal{N} = M \frac{\Gamma(d)\zeta(d)(1+o(1))}{\Gamma(d+1)\zeta(d+1) + o(1)} \frac{\Gamma(d+1)}{\Gamma(d+1)\zeta(d+1)}, \\
\sum_{j=1}^{\infty} \mathcal{N}_j = Qb^{-d} \int_{b}^{\infty} \frac{x^{d-1} dx}{e^x - 1} + o(b^{-d}). \]

The last formula in (19) reveals the role of the nondimensionalized Bose–Einstein distribution function

\[ \phi(x) = \frac{x^{d-1}}{e^x - 1}. \]

It is no surprise that, being appropriately normalized, the random elements \( \{N_j\} \in \Omega_M \), in a sense, tend as \( M \to \infty \) to the function (20). More precisely, let us define a sequence of independent random functions \( \phi_M(x), M = 1, 2, \ldots, \) on the positive real line \( \mathbb{R}_+ \) by setting

\[ \phi_M(x) = Q^{-1}b^{-d-1}N_j, \quad x \in [b(j-1), b), \quad j = 1, 2, \ldots, \]

where \( M \) and \( b \) are related by (14) and \( \{N_j\} \in \Omega_M \) is a random element of the probability space \( \mathcal{X}_M \). Then the following assertion holds.

**Corollary 2.** The sequence \( \{\phi_M\} \) almost surely \( \ast \)-weakly converges in the space \( (C^1(\mathbb{R}_+))^* \) of continuous linear functionals on the space \( C^1(\mathbb{R}_+) \) to the function (20). Namely, for each differentiable function \( f(x), x \in \mathbb{R}_+ \), bounded together with its first derivative uniformly on \( \mathbb{R}_+ \), one has

\[ \langle \phi_M, f \rangle \xrightarrow{a.s.} \langle \phi, f \rangle, \]

where the angle brackets denote the pairing

\[ \langle u, v \rangle = \int_{0}^{\infty} u(x)v(x) \, dx. \]

The convergence to the limit distribution (20) can also be stated in a somewhat different manner, as was done by Vershik [3].

**Corollary 3 (cf. [3, Theorem 4.4]).** For any \( \varepsilon > 0 \) and any closed interval \( [x_1, x_2] \), \( 0 < x_1 < x_2 < \infty \), there exists an \( M_0 \) such that

\[ \mathbb{P}_M \left( \sup_{x \in [x_1, x_2]} |Q^{-1}b^{-d} \sum_{j > x/b} N_j - \int_{x}^{\infty} \phi(x) \, dx| > \varepsilon \right) < \varepsilon \]

for \( M > M_0 \).
Remark. Note that the precise information on the convergence rate contained in Theorem 2 and Corollary 1 has been lost in Corollaries 2 and 3.

2.2. System with a fixed number of particles. Now let us consider systems in which two constraints, one on the total energy and one on the number of particles, are given.

2.2.1. Definition of the system. Let \( M, N \geq 0 \) be integers. By \( \Omega_{M,N} \) we denote the set of all sequences \( \{N_j\} \equiv \{N_j\}_{j=0}^{\infty} \) of nonnegative integers satisfying the conditions\(^10\)

\[
\sum_{j=0}^{\infty} N_j = N, \quad \sum_{j=0}^{\infty} jN_j \leq M.
\]

Again, such sequences are finitely supported, and \( \Omega_{M,N} \) is finite. We take the same multiplicities \( q_j > 0, \ j = 1,2,\ldots \), as in Sec. 2.1 and supplement them with some number \( q_0 \geq 1 \). Next, we introduce the probability space \( \mathcal{X}_{M,N} = (\Omega_{M,N}, \mathcal{F}_{M,N}, \mathcal{P}_{M,N}) \), where \( \mathcal{F}_{M,N} = 2^{\Omega_{M,N}} \) and the probability \( \mathcal{P}_{M,N} \) is defined as follows:

\[ (21) \quad \mathcal{P}_{M,N}(A) = \frac{W(A)}{W(\Omega_{M,N})}, \quad A \subset \Omega_{M,N}, \]

where, for every \( A \subset \Omega_{M,N} \), the weight \( W(A) \) is given by\(^11\)

\[
W(A) = \sum_{\{N_j\} \in A} W(\{N_j\}), \quad W(\{N_j\}) = \prod_{j=0}^{\infty} \left( N_j + q_j - 1 \right). \]

2.2.2. Limit distribution and Bose condensation. Let us study the behavior of random elements \( \{N_j\} \in \Omega_{M,N} \) as \( M \to \infty \) and \( N \to \infty \). This problem involves two large parameters \( M \) and \( N \) rather than one, and it is natural to expect that the answer is more complicated than in the case of one large parameter \( M \), considered in Sec. 2.1. It turns out that the asymptotic behavior of our sequences \( \{N_j\} \) strongly depend on how the rates at which \( M \) and \( N \) tend to infinity are related. Namely, let \( \bar{N} = \bar{N}(M) \) be defined by (14) and (16). Recall that, by (19),

\[
\bar{N}(M) = M^\frac{d}{d+1} Q^\frac{1}{d+1} \frac{\Gamma(d) \zeta(d)}{(\Gamma(d+1) \zeta(d+1))^{\frac{d}{d+1}}} (1 + o(1)).
\]

\(^{10}\)It is merely a matter of convenience that we have decided to start the indexing from \( j = 0 \), that is, have chosen zero for the ground energy level. Should we wish to start from \( j = 1 \), it suffices to change the notation as follows: \( \bar{N} = N, \ \bar{M} = M + N, \ \bar{N}_j = N_{j-1}, \ j = 1,2,\ldots \). In terms of the variables with tildes, the sums start from \( j = 1 \).

\(^{11}\)We denote the weights in this section by the capital letter \( W \) so as to avoid confusion with the weight \( w \) introduced in Sec. 2.1 for sequences starting from \( j = 1 \).
There are two possible types of asymptotic behavior of random elements \( \{N_j\} \in \Omega_{M,N} \) as \( M, N \to \infty \) depending on whether \( N \) is smaller or greater than \( N(M) \).

(i) If \( N \leq N(M) \), then the limit distribution is the Bose–Einstein distribution

\[
\overline{N}_j = \frac{q_j}{e^{\beta j + \mu} - 1}, \quad j = 0, 1, 2, \ldots ,
\]

where the parameters \( \beta, \mu > 0 \) are determined from the system of equations

\[
\sum_{j=0}^{\infty} \frac{q_j}{e^{\beta j + \mu} - 1} = N, \quad \sum_{j=0}^{\infty} \frac{j q_j}{e^{\beta j + \mu} - 1} = M.
\]

(ii) If \( N > N(M) \), then Bose condensation occurs: the occupation numbers \( N_j \) with \( j \geq 1 \) in the limit distribution no longer depend on \( N \) and coincide with the numbers (15), while the excessive particles, however many, occupy the zero level,

\[
\overline{N}_j = \frac{q_j}{e^{\beta j} - 1}, \quad j \geq 1, \quad \overline{N}_0 = N - \sum_{j=1}^{\infty} \overline{N}_j = N - \overline{N}.
\]

Here we do not consider case (i) in detail and refer the reader to Theorem 5 in [9], where the probabilities of deviations from the limit distribution (22), (23) are estimated.

In case (ii), the following theorem holds.

**Theorem 3.** Let the numbers \( q_j \) satisfy (12), and let \( q_0 \geq 1 \). Then there exist constants \( C_s > 0, s = 1, 2, \ldots \), such that the estimates

\[
P_{M,N} \left( \left| \sum_{j=0}^{\infty} f_j (N_j - \overline{N}_j) \right| > \Delta \right) \leq C_s \overline{N}^{-s}, \quad s = 1, 2, \ldots ,
\]

where the \( \overline{N}_j \) are defined in (24) and \( \Delta \) is the same as in Theorem 2, hold for arbitrary \( M > 0 \) and \( N > \overline{N}(M) \) and an arbitrary sequence \( \{f_j\}, j = 0, 1, 2, \ldots \), of complex numbers satisfying the condition \( \sup_j |f_j| \leq 1 \).

In the same way as in Sec. 2.1.2, we obtain the following corollary.

**Corollary 4.** Under the assumptions of the theorem, for arbitrary integer \( l \geq 0 \) one has

\[
P_M \left( \left| \sum_{j=l}^{\infty} N_j - \sum_{j=l}^{\infty} \overline{N}_j \right| > \Delta \right) \leq C_s \overline{N}^{-s}, \quad s = 1, 2, \ldots .
\]

In particular, this proves Theorem 1. Indeed, to derive the estimate in that theorem from Corollary 4, it suffices to set \( l = 1 \) and notice
that

$$|N_0 - (N - \overline{N})| = \left| \sum_{j=1}^{\infty} N_j - \sum_{j=1}^{\infty} \overline{N}_j \right|.$$ 

**Remark.** One could also state the counterparts of Corollaries 2 and 3; we do not dwell upon this.

3. Proofs

In this section, the letter $C$ is used to denote various *positive constants* independent of $N$, $b$, $M$, etc. These constants are not assumed to be the same in all formulas! If we need to keep track of several constants simultaneously, we equip $C$ with subscripts. We also widely use the following standard notation: we write $f \asymp g$ if the ratio $f/g$ is bounded above and below by positive constants; in other words, $f$ and $g$ never have opposite signs, $f = O(g)$, and $g = O(f)$.

3.1. Proof of Theorem 2. Instead of (18), it suffices to prove that (the modulus sign is removed)

$$P_M \left( \sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) > \Delta \right) \leq C_s N^{-s}, \quad s = 1, 2, \ldots,$$

where the $f_j$ are assumed to be real. Then we obtain (18) for real $f_j$ (with the constants $C_s$ multiplied by 2) by combining (26) with the similar inequality where each $f_j$ has the same modulus and the opposite sign, and finally reach the case of complex $f_j$ using Pythagoras’ theorem (with further increase in the constants). So we concentrate on the proof of (26).

By (11), the probability on the left-hand side in (26) is given by

$$P_M \left( \sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) > \Delta \right) = \frac{w(\Omega_M(\Delta))}{w(\Omega_M)},$$

where $\Omega_M(\Delta) \subset \Omega_M$ is the set of all sequences $\{N_j\}$ for which

$$\sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) > \Delta.$$

To prove (26), we will obtain a lower bound for $w(\Omega_M)$ and an upper bound for $w(\Omega_M(\Delta))$.

3.1.1. A lower bound for $w(\Omega_M)$. It is not so easy to estimate $w(\Omega_M)$ directly. Instead, we will estimate the weight $w(\Omega_M^0)$, where $\Omega_M^0 \subset \Omega_M$ is the subset formed by the sequences for which $\sum_{j=1}^{\infty} j N_j = M$ (i.e., equality takes place in (8)). This weight obviously does not exceed $w(\Omega_M)$. First, let us write out an exact formula for $w(\Omega_M^0)$. Let

$$F(z) = \sum_{M=0}^{\infty} w(\Omega_M^0) z^M$$
be the generating function of the numbers $w(\Omega^0_M)$.

**Lemma 1.** The series (28) converges in the disk $\{|z| < 1\}$, and the sum is given by the formula

$$F(z) = \prod_{j=1}^{\infty} \frac{1}{(1 - z^j)^{q_j}}.$$  

This assertion is well known for integer $q_j$ (e.g., see [3] and [10, Chap. 1]), and the proof for noninteger $q_j$ is also easy.  

Now we can express $w(\Omega^0_M)$ by the Cauchy formula

$$w(\Omega^0_M) = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z) \, dz}{z^{M+1}} = \frac{1}{2\pi i} \int \left[ \prod_{j=1}^{\infty} \frac{1}{(1 - z^j)^{q_j}} \right] \frac{dz}{z^{M+1}},$$

where $0 < r < 1$. The change of variables $z = e^{-\xi}$ yields

$$w(\Omega^0_M) = \frac{1}{2\pi i} \int_{\gamma_0} e^{\Phi(\xi)} \, d\xi, \quad \Phi(\xi) = M\xi + \sum_{j=1}^{\infty} q_j \ln \frac{1}{|1 - e^{-j\xi}|}.$$  

Here the main branch of the logarithm is taken, and the integration contour $\gamma_0$ is the circle $\{\Re \xi = -\ln r\}$ on the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$ with coordinate $\xi \mod 2\pi i$. To estimate the integral, we use a saddle-point argument (cf. [13, Chap. 4]). Let us deform $\gamma_0$ into a contour $\gamma^*$ on which

$$\min_{\gamma} \max_{\xi \in \gamma} \Re \Phi(\xi)$$

is attained, where the minimum is taken over all contours $\gamma$ lying in the right half-cylinder $\{\Re \xi > 0\}$ and homotopic to $\gamma_0$. We have

$$\Re \Phi(\xi) = M \Re \xi + \sum_{j=1}^{\infty} q_j \ln \frac{1}{|1 - e^{-j\xi}|}.$$  

To find $\gamma^*$, the following lemma will be of help.

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12Here is the proof. From (28), using (9), (10), and the fact that the sets $\Omega^0_M$ are disjoint, we obtain

$$F(z) = \sum_{M=0}^{\infty} \sum_{\{N_j\} \in \Omega^0_M} z^M \prod_{j=1}^{\infty} \left( \frac{N_j + q_j - 1}{N_j} \right) = \sum_{\{N_j\}} \prod_{j=1}^{\infty} \left[ \left( \frac{N_j + q_j - 1}{N_j} \right) \frac{1}{z^{N_j}} \right],$$

where the sum is taken over all finitely supported sequences $\{N_j\}$ of nonnegative integers. Since the $q_j$ grow polynomially by condition (13), it follows by routine estimates that the sum and the product can be interchanged, and we obtain

$$F(z) = \prod_{j=1}^{\infty} \left( \sum_{N_j=0}^{\infty} \left( \frac{N_j + q_j - 1}{N_j} \right) \frac{1}{z^{N_j}} \right) = \prod_{j=1}^{\infty} \frac{1}{(1 - z^j)^{q_j}},$$

by the binomial series formula.
Lemma 2. (i) For fixed $\text{Re} \xi > 0$, the maximum of $\text{Re} \Phi(\xi)$ is attained on the real axis.

(ii) The minimum of $\text{Re} \Phi(\xi)$ on the positive real axis is attained at the point $\xi = b$, where $b$ is the root of Eq. (14).

Proof. (i) In fact, a stronger assertion is true: if $\text{Re} \xi > 0$, then

$$\text{Re} \Phi(\text{Re} \xi) - \text{Re} \Phi(\xi) \geq \frac{1}{5} \sum_{j=1}^{\infty} q_j e^{-j \text{Re} \xi} (1 - \cos(j \text{Im} \xi)).$$

All terms on the right-hand side in (29) are nonnegative, and the first term ($j = 1$) is strictly positive unless $\text{Im} \xi \equiv 0 \mod 2\pi$. This proves (i).

(ii) For real $\xi$, we have $\text{Re} \Phi(\xi) = \Phi(\xi)$. Next,

$$\Phi'(\xi) = M - \sum_{j=1}^{\infty} \frac{j q_j}{e^{j \xi} - 1}, \quad \Phi''(\xi) = \sum_{j=1}^{\infty} \frac{j^2 q_j e^{j \xi}}{(e^{j \xi} - 1)^2}.$$  

The function $\Phi'(\xi)$ has the unique positive zero $\xi = b$ (cf. (14)), and this zero is the unique point of strict minimum of $\text{Re} \Phi(\xi)$ on the positive real axis, because the second derivative is strictly positive for real $\xi$. This gives (ii) and completes the proof of Lemma 2. $\square$

It follows from Lemma 2 that for $\gamma^*$ we can take the circle $\{\text{Re} \xi = b\}$ on the cylinder $\mathbb{C}/2\pi i \mathbb{Z}$. Hence we write

$$w(\Omega^0_M) = \frac{1}{2\pi i} \int_{\gamma^*} e^{\Phi(\xi)} d\xi = \frac{\Phi(b)}{2\pi} \int_{-\pi}^{\pi} e^{S(\varphi)} d\varphi,$$

Indeed, let $\gamma$ be any contour homotopic to $\gamma_0$. Then $\gamma$ necessarily contains a point $\eta$ of the real axis, and

$$\max_{\xi \in \gamma^*} \text{Re} \Phi(\xi) \leq \Phi(\eta) \leq \max_{\xi \in \gamma} \text{Re} \Phi(\xi),$$

which shows that $\gamma^*$ is a saddle-point contour.
where
\[ S(\varphi) = \Phi(b + i\varphi) - \Phi(b) = iM\varphi + \sum_{j=1}^{\infty} q_j \ln \frac{1 - e^{-bj}}{1 - e^{-bj - ij\varphi}}. \]

Let us estimate the integral in (31). By construction, we have \( S(0) = 0 \), and
\[ \text{Re} \, S(\varphi) \leq -\frac{1}{5} \sum_{j=1}^{\infty} q_j e^{-bj} (1 - \cos(j\varphi)) \]
by (29). Next,
\[ S'(0) = i\Phi'(b) = 0, \quad S''(0) = -\Phi''(b) = -\sum_{j=1}^{\infty} \frac{j^2 q_j e^{bj}}{(e^{bj} - 1)^2} \]
(cf. (30)), and finally,
\[ S'''(\varphi) = i \sum_{j=1}^{\infty} \frac{j^3 q_j (e^{2bj + ij\varphi} + e^{bj + ij\varphi})}{(e^{bj + ij\varphi} - 1)^3}. \]

**Lemma 3.** There exist positive constants \( c_1, c_2, c_3 \) such that
\[ -c_1 b^{-d-2} \leq S''(0) \leq -c_2 b^{-d-2}, \quad \sup_{\varphi} |S'''(\varphi)| \leq c_3 b^{-d-3} \]

**Proof.** We use inequalities (13). Then it follows from (33) and (34) that
\[ B_1 \sum_{j=1}^{\infty} \frac{j^{d+1} e^{bj}}{(e^{bj} - 1)^2} \leq -S''(0) \leq B_2 \sum_{j=1}^{\infty} \frac{j^{d+1} e^{bj}}{(e^{bj} - 1)^2} \]
and (since \( |e^{bj + ij\varphi} - 1| \geq e^{bj} - 1 \))
\[ |S'''(\varphi)| \leq B_2 \sum_{j=1}^{\infty} \frac{j^{d+2} (e^{2bj} + e^{bj})}{(e^{bj} - 1)^3}. \]
Now it remains to use Proposition 4, (ii) in Sec. 4. This proves the lemma. \( \square \)

We split the integration interval \([-\pi, \pi]\) in (31) into three zones,
\[ D_1 = \{|\varphi| < \delta_1 b^{1+d/3}\}, \quad D_2 = \{\delta_1 b^{1+d/3} \leq |\varphi| \leq \delta_2 b\}, \quad D_3 = \{\delta_2 b \leq |\varphi| \leq \pi\}. \]
Here \( \delta_1 \) and \( \delta_2 \) are sufficiently small positive constants, independent of \( b \), to be chosen later. Let us represent \( S(\varphi) \) by Taylor’s formula with remainder of order 3,
\[ S(\varphi) = \frac{1}{2} S''(0) \varphi^2 + R_3(\varphi), \quad \text{where} \quad |R_3(\varphi)| \leq \frac{c_3}{6} b^{-d-3} |\varphi|^3 \]
by the second inequality in (35). In particular,

(36) \[ |R_3(\varphi)| \leq \frac{c_3\delta_3}{6} \quad \text{in } D_1. \]

Take \( \delta_3 \) so small that the right-hand side in (36) is smaller than \( \pi/4 \). Then

\[ c_4 e^{-c_5 b^{-d-2} \varphi^2} \leq \text{Re } e^{S(\varphi)} \leq c_6 e^{-c_7 b^{-d-2} \varphi^2} \quad \text{in } D_1 \]

for some positive constants \( c_4, \ldots, c_7 \). We have

\[
\int_{D_1} e^{-cb^{-d-2} \varphi^2} d\varphi = \frac{b^{d/2+1}}{b^{-1/6}} \int_{-\delta_1 b^{-1/6}}^{\delta_1 b^{-1/6}} e^{-cy^2} dy \asymp b^{d/2+1},
\]

and hence

\[
\int_{D_1} e^{S(\varphi)} d\varphi \asymp b^{d/2+1}.
\]

Now consider the zone \( D_2 \). There we have

\[ \text{Re } S(\varphi) \leq -(c_2 b^{-d-2} - c_3 b^{-d-3} |\varphi|) \varphi^2 \]

\[ \leq -(c_2 - \delta_2 c_3) b^{-d-2} \varphi^2 \leq -(c_2 - \delta_2 c_3) b^{-d/3} \delta_2^2, \]

and if we take \( \delta_2 \) small enough that \( c_8 = c_2 - \delta_2 c_3 > 0 \), then the integral over \( D_2 \) decays exponentially (at the rate of \( e^{-c_8 b^{-d/3}} \)) as \( b \to 0 \).

Finally, consider the zone \( D_3 \). Here we use inequality (32). Since all terms in the series on the right-hand side in (32) are nonnegative, we can drop some terms and write

\[ \text{Re } S(\varphi) \leq -\frac{1}{5} \sum_{x_1 < b j < x_2} q_j e^{-bj} (1 - \cos(j \varphi)) \]

with some positive \( x_1 \) and \( x_2 \) to be chosen later. By (13), \( q_j e^{-bj} \geq c_9 b^{-d+1} \) for \( x_1 < b j < x_2 \), where the constant \( c_9 \) depends on \( x_1 \) and \( x_2 \), and so

\[ \text{Re } S(\varphi) \leq -\frac{c_9 b^{-d+1}}{5} \sum_{x_1 < b j < x_2} (1 - \cos(j \varphi)). \]

Using [12, 1.341.3], we obtain

\[
\sum_{x_1 < b j < x_2} (1 - \cos(j \varphi)) = \frac{\cos((j_1 + j_2) \varphi/2)}{\sin(\varphi/2)} \sin((j_2 - j_1 + 1) \varphi/2) \sin(\varphi/2) \geq b^{-1}(x_2 - x_1) - 1 - |\sin(\varphi/2)|^{-1},
\]

where \( j_1 \) and \( j_2 \) are the first and the last integer, respectively, in the interval \((x_1/b, x_2/b)\). Since \( \varphi \in D_3 \), we have \(|\sin(\varphi/2)| \geq \sin(\delta_2 b/2) \geq \delta_2 b/2 \) for \( \delta_2 b/2 \)
\[ \delta_2 b / \pi, \text{ and so the sum (37) is greater than } c_{10} b^{-1} \text{ provided that we take a sufficiently large } x_2 - x_1. \] 
Thus, we see that
\[ \Re S(\varphi) \leq c_{11} b^{-d} \quad \text{in } D_3, \]
so that the integral over \( D_3 \) also decays exponentially as \( b \to 0 \). Now we summarize the preceding and see that we have proved the following assertion:

**Proposition 1.** There exists a constant \( C > 0 \) such that
\[ w(\Omega_M) \geq C b^{d/2 + 1} e^{S(M)}, \]
where
\[ S(M) = bM + \sum_{j=1}^{\infty} q_j \ln \frac{1}{1 - e^{-bj}} \]
and \( b \) is related to \( M \) by formula (14).

This is the desired lower bound for \( w(\Omega_M) \).

3.1.2. An upper bound for \( w(\Omega_M(\Delta)) \). It follows from (8) and (27) that if \( \{N_j\} \in \Omega_M(\Delta) \) and \( c \geq 0 \), then
\[ b \left( M - \sum_{j=1}^{\infty} jN_j \right) + c \sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) - c\Delta \geq 0. \]
Accordingly, the exponential of the left-hand side of (39) is greater than 1 for any \( \{N_j\} \in \Omega_M(\Delta) \); of course, it is positive for any (finitely supported) \( \{N_j\} \). Hence
\[ w(\Omega_M(\Delta)) \leq \sum_{\{N_j\}} \left[ e^{b \left( M - \sum_{j=1}^{\infty} jN_j \right) + c \sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) - c\Delta} w(\{N_j\}) \right] \]
\[ = e^{bM - c\Delta - c \sum_{j=1}^{\infty} f_j \overline{N}_j} \sum_{\{N_j\}} \prod_{j=1}^{\infty} \left[ \frac{(N_j + q_j - 1)}{N_j} e^{(-bj + cf_j)N_j} \right], \]
where the sum extends over all finitely supported sequences \( \{N_j\} \) of nonnegative integers. Note that if \( c \leq b/2 \), then
\[ e^{-bj + cf_j} < (e^{-b/2})^j \]
(recall that \( |f_j| < 1 \)), and hence the series on the right-hand side in (40) is dominated by the series (28) with \( z = e^{-b/2} < 1 \), which is convergent by Lemma 1. Arguing as in the proof of Lemma 1, we interchange the
sum and the product and use the binomial formula to obtain
\[
\sum_{j=1}^{\infty} f_j N_j \prod_{j=1}^{\infty} \left( \sum_{N_j=0}^{\infty} \binom{N_j + q_j - 1}{N_j} e^{(-b_j + c)N_j} \right)
\]
\[
= e^{bM - c\Delta - c\sum_j^\infty f_j N_j} \prod_{j=1}^{\infty} \frac{1}{(1 - e^{-b_j + c})q_j}
\]
\[
= \exp\left\{ bM - c\Delta + \sum_{j=1}^{\infty} q_j \left[ \ln \frac{1}{1 - e^{-b_j + c}N_j} - \frac{c N_j}{e^{b_j} - 1} \right] \right\}.
\]

By Taylor’s formula with remainder,
\[
\ln \frac{1}{1 - e^{-b_j + c}N_j} = \ln \frac{1}{1 - e^{-b_j}} + \frac{c f_j}{e^{b_j} - 1} + \frac{(c f_j)^2}{2} \frac{e^{b_j - \theta_j c f_j}}{(e^{b_j - \theta_j c f_j} - 1)^2},
\]
where \( \theta_j \in [0, 1] \). Since \( |f_j| \leq 1 \) and \( c \in [0, b/2] \), we obtain, transposing the term \( c f_j (e^{b_j} - 1)^{-1} \),
\[
\ln \frac{1}{1 - e^{-b_j + c}f_j} - \frac{c f_j}{e^{b_j} - 1} \leq \ln \frac{1}{1 - e^{-b_j}} + \frac{c^2}{2} \frac{e^{b(j+1/2)}}{(e^{b(j-1/2)} - 1)^2}
\]
\[
\leq \ln \frac{1}{1 - e^{-b_j}} + \frac{c^2}{2} \frac{e^{b e^{b/2}}}{(e^{b/2} - 1)^2},
\]
and thus we arrive at the following assertion.

**Proposition 2.** One has the estimate
\[
\sum_{j=1}^{\infty} f_j(N_j - N_j) > \Delta \leq C b^{-1-1/d} e^{-c \Delta + c^2 K}, \quad c \in [0, b/2],
\]
where \( S(M) \) is given by (38) and
\[
K = \frac{e^b}{2} \sum_{j=1}^{\infty} \frac{q_j e^{b/2}}{(e^{b/2} - 1)^2}.
\]

**3.1.3. Completion of the proof.** By combining Propositions 1 and 2, we obtain
\[
P_M\left(\sum_{j=1}^{\infty} f_j(N_j - N_j) > \Delta \right) \leq C b^{-1-1/d} e^{-c \Delta + c^2 K}, \quad c \in [0, b/2].
\]

Now we will prove that there exists a \( c \in [0, b/2] \) such that
\[
e^{-c \Delta + c^2 K} \leq C_k \chi^{-s} N, \quad s = 1, 2, \ldots.
\]
Then the assertion of the theorem readily follows, since \( b \simeq N^{-1/d} \).

We consider three cases.

1. **\( d > 2 \).** In this case, \( K < C b^{-d} \) by Proposition 4, (ii) in Sec. 4, and so \( K \leq C \chi(N) \). Set
\[
c = \frac{1}{2C} N^{-1/2} \sqrt{\ln N \chi(N)}.
\]
Since $\overline{N} \asymp b^{-d}$, it follows that $c < b/2$ for sufficiently large $M$. Next,

$$c\Delta - c^2 K \geq \frac{1}{4C} \ln \overline{N} \chi(\overline{N}),$$

and (41) holds.

2. $d = 2$. In this case, $K \leq C b^{-2} \ln b$ (Proposition 4, (ii)) and hence $K \leq C \overline{N} \ln \overline{N}$. Set $c = b/2$. Then

$$c \Delta - c^2 K \geq \frac{b}{2} \sqrt{\overline{N}} \ln \overline{N} \chi(\overline{N}) - C \frac{b^2}{4} \overline{N} \ln \overline{N}.$$

The first term is of the order of $\ln \overline{N} \chi(\overline{N})$, and the subtrahend is of the smaller order of $\ln \overline{N}$, so that (41) again holds.

3. $1 < d < 2$. In this case, $K \leq C b^{-2}$ (Proposition 4, (ii)), and hence $K \leq C \overline{N}^{2/d}$. We again take $c = b/2$. Then

$$c \Delta - c^2 K \geq \frac{b}{2} \overline{N}^{1/d} \ln \overline{N} \chi(\overline{N}) - C \frac{b^2}{4} \overline{N}^{1/d}.$$

The first term is of the order of $\ln \overline{N} \chi(\overline{N})$, and the subtrahend is $O(1)$, so that (41) again holds.

The proof of Theorem 2 is complete. \(\square\)

3.2. Proof of Corollary 2. The proof of this corollary is based on the following lemma.

**Lemma 4.** One has

$$\langle \phi - \phi_M, f \rangle = b^{-d} Q^{-1} \sum_{j=1}^{\infty} (N_j - \overline{N}_j) f(bj) + R_1(M) + R_2(M),$$

where $R_1(M)$ is a random variable on $\Omega_M$ such that

$$|R_1(M)| \leq Ct^{d+1} \left( \overline{N} - \sum_{j=1}^{\infty} N_j \right)$$

and $R_2(M)$ is a (nonrandom) function such that

$$R_2(M) \to 0 \quad \text{as} \quad M \to \infty.$$

**Proof.** Since the function $f(x)$ is bounded together with the first derivative and $\phi(x)$ is given by (20) with $d > 1$, it follows from the Euler–Maclaurin formula (Proposition 3) that

$$\langle \phi, f \rangle \equiv \int_0^{\infty} \phi(x) f(x) \, dx = b \sum_{j=1}^{\infty} \phi(bj) f(bj) + r(M),$$

where $r(M) \to 0$ as $M \to \infty$. Next, by (12),

$$b \phi(bj) \equiv \frac{b^{(bj)_{d-1}}}{e^{bj} - 1} = b^d Q^{-1} \overline{N} j (1 + \theta_j),$$

and (42) holds.
where \( \theta_j \to 0 \) as \( j \to \infty \). We claim that
\[
b^d Q^{-1} \sum_{j=1}^{\infty} N_j \theta_j f(b_j) \to 0 \quad \text{as } M \to \infty.
\]
Indeed, since \( f \) is uniformly bounded, we have
\[
\left| b^d Q^{-1} \sum_{j=1}^{\infty} N_j \theta_j f(b_j) \right| \leq C b^d \left( \sum_{j=1}^{k} N_j |\theta_j| + \sum_{j=k+1}^{\infty} N_j |\theta_j| \right),
\]
where \( k \) is arbitrary. Take \( k \) so large that \(|\theta_j| < \varepsilon \) for \( j > k \). Then the second sum does not exceed \( \varepsilon N \). For fixed \( k \), the first sum does not exceed \( C_1 b^{-1} \) (where the constant depends on \( k \)) in view of formula (15) for \( N_j \). Thus, the right-hand side does not exceed \( CC_1 b^{-1} + C b^d \varepsilon \).

Recall that \( N \approx b^{-d} \); hence the second term can be made as small as desired by an appropriate choice of \( \varepsilon \), and then we take \( M \) large enough (i.e., \( b \) small enough) to ensure that the first term is also small. We conclude that
\[
\langle \phi, f \rangle = b^d Q^{-1} \sum_{j=1}^{\infty} N_j f(b_j) + r_1(M),
\]
where \( r_1(M) \to 0 \) as \( M \to \infty \).

Now let us study \( \langle \phi_M, f \rangle \). We represent \( f \) in the form \( f(x) = f_1(x) + f_2(x) \), where \( f_1(x) \) is the step function taking the value \( f(b_j) \) on the interval \([b(j-1), b_j)\) and \( f_2(x) = f(x) - f_1(x) \) satisfies the estimate \(|f_2(x)| \leq C b\), since the derivative \( f'(x) \) is uniformly bounded. Then, in view of the definition of \( \phi_M(x) \), we have
\[
\langle \phi_M, f \rangle = \langle \phi_M, f_1 \rangle + \langle \phi_M, f_2 \rangle = b^d Q^{-1} \sum_{j=1}^{\infty} N_j f(b_j) + R(M),
\]
where
\[
|R(M)| \leq C b \int_0^{\infty} \phi_M(x) \, dx = C b^{d+1} Q^{-1} \sum_{j=1}^{\infty} N_j = C b^{d+1} Q^{-1} \sum_{j=1}^{\infty} N_j
\]
\[
+ C b^{d+1} Q^{-1} \left( N - \sum_{j=1}^{\infty} N_j \right) \equiv r_2(M) + r_3(M).
\]
Now we can set \( R_1(M) = r_2(M) \) and \( R_2(M) = r_1(M) + r_2(M) \). The proof of the lemma is complete. \( \square \)

We assume without loss of generality that \(|f(x)| \) is bounded by 1.

It follows from Theorem 2 and Lemma 4 that for each \( \varepsilon > 0 \) there exists an \( M_0 = M_0(\varepsilon) \) such that
\[
P_M(\langle \phi - \phi_M, f \rangle > \varepsilon) \leq C M^{-2} \quad \text{for } M \geq M_0.
\]
Indeed, the third term on the right-hand side in (42) is necessarily less than \( \varepsilon/3 \) in modulus for sufficiently large \( M \). Let us study the first term. It is necessarily less than \( \varepsilon/3 \) in modulus provided that

\[
\left| \sum_{j=1}^{\infty} f(b_j)(N_j - N_j) \right| \leq C_0 \mathcal{N}_0,
\]

where \( C_0 \) is some constant (depending on \( Q \) and on the constants in the relation \( \mathcal{N}_0 \asymp b^{-d} \)). If \( M \) is large enough that \( C_0 \mathcal{N}_0 > \Delta \) (which can always be achieved, because \( \Delta \) grows slower than \( \mathcal{N}_0 \) as \( M \to \infty \)), we can apply Theorem 2 with appropriate \( s \) and conclude that

\[
P_M \left( \left| b^{-d} \sum_{j=1}^{\infty} (N_j - N_j) f(b_j) \right| > \varepsilon/3 \right) \leq C M^{-2}.
\]

The same argument applies to the second term on the right-hand side in (42), and we arrive at (45) (with some new \( C \) and \( M_0(\varepsilon) \)).

Consider the product \( \prod_{M=1}^{\infty} \mathcal{X}_M \) of the probability spaces \( \mathcal{X}_M \). The probability measure on this product will be denoted by \( P \). Consider the event

\[
A_{m \varepsilon} = \{ \langle \phi - \phi_M, f \rangle \leq \varepsilon \text{ for all } M \geq m \}.
\]

It follows from (45) that

\[
P(A_{m \varepsilon}) \geq 1 - C \sum_{M=m}^{\infty} M^{-2}
\]

for \( m \geq M_0(\varepsilon) \). The series on the right-hand side converges, and so

\[
P(A_{m \varepsilon}) \to 1 \quad \text{as } m \to \infty.
\]

Next, for the probability of the desired convergence \( \langle \phi_M, f \rangle \to \langle \phi, f \rangle \) we have the expression

\[
P(\langle \phi - \phi_M, f \rangle \to 0) = P(\forall \varepsilon > 0 \exists m: A_{m \varepsilon}) = P\left( \bigcap_{\varepsilon > 0} \bigcup_{m > 0} A_{m \varepsilon} \right).
\]

Note that the sets \( A_{m \varepsilon} \) are nested,

\[
A_{m \varepsilon} \subset A_{m' \varepsilon} \quad \text{for } m \leq m',
\]

and that the sets

\[
B_{\varepsilon} = \bigcup_{m > 0} A_{m \varepsilon}
\]

are nested as well,

\[
B_{\varepsilon} \subset B_{\varepsilon'} \quad \text{for } \varepsilon \leq \varepsilon'.
\]

It follows that

\[
P(B_{\varepsilon}) = \lim_{m \to \infty} P(A_{m \varepsilon}) = 1, \quad P(\langle \phi - \phi_M, f \rangle \to 0) = \lim_{\varepsilon \to 0} P(B_{\varepsilon}) = 1.
\]

The proof of Corollary 2 is complete. \( \square \)
3.3. Proof of Corollary 3. Using Proposition 4 and arguing as in the proof of (44), we obtain
\[ \int_{x}^{\infty} \phi(x) \, dx = b^d Q^{-1} \sum_{j > x/b} N_j + r(M), \]
where \( r(M) \to 0 \) as \( M \to \infty \). Hence it suffices to prove that
\[ P_M \left( \sup_{x \in [x_1, x_2]} \left| \sum_{j > x/b} (N_j - \overline{N}_j) \right| > \varepsilon C \overline{N} \right) \leq \varepsilon, \]
where \( C \) is a constant depending on \( Q \) and the constants in the relation \( \overline{N} \asymp b^{-d} \). For sufficiently large \( M \), we have \( \varepsilon C \overline{N} > \Delta \), and Theorem 2 can be used to estimate the probability for each fixed \( x \). This probability is less that \( C_s \overline{N}^{-s} \) for all \( s = 0, 1, 2, \ldots \). The supremum is actually taken over the finite set of integer points on the interval \([x_1/b, x_2/b]\), and the number of these points is of the order of \( 1/b \), i.e., of the order of \( \overline{N}^{1/d} \). Thus, passing from individual points to the supremum over \( O(\overline{N}^{1/d}) \) points, we make the estimate of the probability slightly worse (by the factor equal to the number of these points); i.e., the probability estimate becomes \( \tilde{C}_s \overline{N}^{-s+1/d} \) with some new constants \( \tilde{C}_s \). However, this does not matter, and we take \( s = 1 \), which provides the desired estimate:
\[ P_M \left( \sup_{x \in [x_1, x_2]} \left| \sum_{j > x/b} (N_j - \overline{N}_j) \right| > \varepsilon C \overline{N} \right) \leq \tilde{C}_1 \overline{N}^{-1+1/d} \leq \varepsilon \]
for sufficiently large \( M \) (and hence \( N \)).

The proof of Corollary 3 is complete.

3.4. Proof of Theorem 3. Just as in the proof of Theorem 2, it suffices to show that
\[ (46) \quad P_{M,N} \left( \sum_{j=0}^{\infty} f_j(N_j - \overline{N}_j) > \Delta \right) \leq C_s \overline{N}^{-s}, \quad s = 1, 2, \ldots, \]
i.e., drop the modulus sign. Next, we will assume without loss of generality that \( f_0 = 0 \). Indeed,
\[ \sum_{j=0}^{\infty} N_j = \sum_{j=0}^{\infty} \overline{N}_j = N, \]
and hence
\[ \sum_{j=0}^{\infty} f_j(N_j - \overline{N}_j) = \sum_{j=1}^{\infty} (f_j - f_1)(N_j - \overline{N}_j). \]
The new numbers $f'_j = f_j - f_1$ are bounded in absolute value by 2 rather than 1, but we can pass to the numbers $\tilde{f}_j = f'_j / 2$ and simultaneously divide the function $\chi(x)$ occurring in definition of $\Delta$ by 2.

By definition (21), the probability on the left-hand side in (46) has the form

$$P_{M,N}\left(\sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) > \Delta\right) = \frac{W(\Omega_{M,N}(\Delta))}{W(\Omega_{M,N})},$$

where $\Omega_{M,N}(\Delta) \subset \Omega_{M,N}$ is the set of all sequences $\{N_j\}_{j=0}^{\infty} \in \Omega_{M,N}$ such that

$$\sum_{j=1}^{\infty} f_j (N_j - \overline{N}_j) > \Delta.$$

Consider the mapping

$$\tau : \Omega_{M,N} \longrightarrow \Omega_M$$

that takes each sequence $\{N_j\} = \{N_j\}_{j=0}^{\infty} \in \Omega_{M,N}$ to the sequence $\{N_j\}' = \{N_j\}_{j=1}^{\infty} \in \Omega_M$ obtained by throwing away the first element $N_0$. This mapping is one-to-one onto its range, because $N_0 = N - \sum_{j=1}^{\infty} N_j$ is uniquely determined by $\{N_j\}'$. Since $f_0 = 0$, it follows that

$$\tau(\Omega_{M,N}(\Delta)) \subset \Omega_M(\Delta).$$

(Inequality (48) does not involve $N_0$.)

For each $\{N_j\} \in \Omega_{M,N}$, we have

$$W(\{N_j\}) = \binom{N_0 + q_0 - 1}{N_0} w(\{N_j\}') .$$

Since, by assumption, $q_0 > 1$, we see that the binomial $\binom{N_0 + q_0 - 1}{N_0}$ is a monotone increasing function of $N_0$; by Stirling’s formula,

$$\binom{N_0 + q_0 - 1}{N_0} \approx N_0^{q_0 - 1}.$$

Now we can write

$$W(\Omega_{M,N}(\Delta)) = \sum_{\{N_j\} \in \Omega_{M,N}(\Delta)} \binom{N_0 + q_0 - 1}{N_0} w(\{N_j\}') \leq \left\{ \max_{\Omega_{M,N}(\Delta)} \binom{N_0 + q_0 - 1}{N_0} \right\} \sum_{\{N_j\} \in \Omega_{M,N}(\Delta)} w(\{N_j\}') \leq C N^{q_0 - 1} w(\Omega_M(\Delta)).$$
On the other hand,
\[ W(\Omega_{M,N}) = \sum_{\{N_j\} \in \Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right) w(\{N_j\}') \]
\[ \geq \left\{ \min_{\Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right) \right\} \sum_{\{N_j\} \in \Omega_{M,N}} w(\{N_j\}') = \left\{ \min_{\Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right) \right\} w(\tau(\Omega_{M,N})). \]

We have \( N > \overline{N} \). Let us consider two cases, \( N > \overline{N} + \Delta \) and \( \overline{N} \leq N < \overline{N} + \Delta \).

1. Let \( N > \overline{N} + \Delta \). The set \( \Omega_M \setminus \tau(\Omega_{M,N}) \) consists of all sequences \( \{N_j\} \in \Omega_M \) for which
\[ \sum_{j=1}^{\infty} (N_j - \overline{N}_j) = \sum_{j=1}^{\infty} N_j - \overline{N} > \Delta. \]
Hence
\[ \frac{w(\Omega_M) - w(\tau(\Omega_{M,N}))}{w(\Omega_M)} \equiv P_M(w(\Omega_M) - w(\tau(\Omega_{M,N}))) \leq C_s \overline{N}^{-s}, \]
where \( s = 1, 2, \ldots \), by Corollary 1 (with \( l = 1 \)). It follows that, for sufficiently large \( M \) and \( N \geq \overline{N}(M) + \Delta \),
\[ W(\Omega_{M,N}) \geq \frac{1}{2} \left\{ \min_{\Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right) \right\} w(\Omega_M) \]
and
\[ P_{M,N} \left( \sum_{j=1}^{\infty} f_j(N_j - \overline{N}_j) > \Delta \right) \]
\[ \leq \frac{2CN_0^q - 1}{\min_{\Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right)} P_M \left( \sum_{j=1}^{\infty} f_j(N_j - \overline{N}_j) > \Delta \right) \]
\[ \leq C_s \frac{N_0^q - 1}{\min_{\Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right)} \overline{N}^{-s}. \]

If \( N < M \), then the minimum is attained at \( N_0 = 0 \) and is equal to 1. If \( N \geq M \), then the minimum is attained at \( N_0 = N - M \), and we have, by Stirling’s formula,
\[ \frac{N_0^q - 1}{\min_{\Omega_{M,N}} \left( \binom{N_0 + q_0 - 1}{N_0} \right)} \leq \left( \frac{N}{N - M} \right)^{q_0 - 1} \leq C M^{q_0 - 1} \leq \overline{C} N^{(q_0 - 1)(d+1)/d}. \]
In any case, we obtain
\[ P_{M,N} \left( \sum_{j=1}^{\infty} f_j(N_j - \overline{N}_j) > \Delta \right) \leq C_s \overline{N}^{-s}. \]
(with some new constants $C_s$). This proves the theorem for this case.

2. Let $\overline{N} < N \leq \overline{N} + \Delta$. Here we need a finer argument.

We carry out the same computations for the upper bound of the numerator, but to estimate the denominator, we write

$$W(\Omega_{M,N}) \geq \left( \frac{N - \overline{N} + q_0 - 1}{N - \overline{N}} \right) \sum_{\{N_j\} \in \Omega'_{M,\overline{N}}} w(\{N_j\})$$

$$= \left( \frac{N - \overline{N} + q_0 - 1}{N - \overline{N}} \right) w(\Omega'_{M,\overline{N}}),$$

where $\Omega'_{M,\overline{N}} \subset \Omega_M$ is the subset of sequences satisfying the conditions

$$\sum_{j=1}^{\infty} N_j = \overline{N}, \quad \sum_{j=1}^{\infty} jN_j = M.$$

It follows from Lemma 6 and formula (40) in [9] that, for some $m_0$,

$$w(\Omega'_{M,\overline{N}}) \geq C e^{S(M)} \overline{N}^{-m_0},$$

where $S(M)$ is defined in (38). By combining this with the estimate for $w(\Omega_M(\Delta))$ obtained in the proof of Theorem 2 (Proposition 2), we arrive at the desired estimate.

The proof of Theorem 3 is complete.

4. Appendix

We shall use the following two versions of the Euler–Maclaurin formula.

**Proposition 3.** Let $f(x)$ be a continuously differentiable function on $(0, \infty)$ absolutely integrable together with $f'(x)$ at infinity. Then the series $\sum_{j=1}^{\infty} f(j)$ converges absolutely, and its sum can be computed by the formula

$$\sum_{j=1}^{\infty} f(j) = \int_{1}^{\infty} f(x) \, dx + R_1,$$

where the remainder $R_1$ satisfies the estimate

$$|R_1| \leq \int_{1}^{\infty} |f'(x)| \, dx.$$

If, in addition, $f'(x)$ is absolutely integrable at zero, then

$$\sum_{j=1}^{\infty} f(j) = \int_{0}^{\infty} f(x) \, dx + R_2.$$
where the remainder $R_2$ satisfies the estimate

$$|R_2| \leq \int_0^\infty |f'(x)| \, dx.$$ 

**Proof.** The obtain the desired error estimates, one can use the Stieltjes integral representations

$$
\sum_{j=1}^\infty f(j) = \int_0^\infty f(x) \, d[x] = -\int_1^\infty f(x) \, d[-x],
$$

where $[x]$ is the integer part of $x$, and then integrate by parts to estimate the remainders. \(\square\)

Let us use Proposition 3 to carry out some computations needed in the main text.

**Proposition 4.** The following asymptotic formulas hold as $b \to +0$.

(i) Let $s > 0$. Then

$$
\sum_{j=1}^\infty \frac{j^s}{e^{bj} - 1} = b^{-s-1} \Gamma(s + 1) \zeta(s + 1)(1 + o(1))
$$

and, more generally,

$$
\sum_{j=t}^\infty \frac{j^s}{e^{bj} - 1} = b^{-s-1} \int_t^\infty \frac{y^s \, dy}{e^y - 1} + o(b^{-s-1}).
$$

(ii) Let $d > 1$. Then

$$
\sum_{j=1}^\infty \frac{j^{d+1} e^{bj}}{(e^{bj} - 1)^2} = b^{-d-2} \int_0^\infty \frac{y^{d+1} e^y \, dy}{(e^y - 1)^2} (1 + o(1)),
$$

$$
\sum_{j=1}^\infty \frac{j^{d+2} (e^{2bj} + e^{bj})}{(e^{bj} - 1)^3} = b^{-d-3} \int_0^\infty \frac{y^{d+2} (e^{2y} + e^y) \, dy}{(e^y - 1)^3} (1 + o(1)),
$$

$$
\sum_{j=1}^\infty \frac{j^{d-1} e^{bj/2}}{(e^{bj/2} - 1)^2} \leq \begin{cases} 
Cb^{-d} & \text{if } d > 2, \\
Cb^{-2} |\ln b| & \text{if } d = 2, \\
Cb^{-2} & \text{if } 1 < d < 2.
\end{cases}
$$

**Proof.** (i) If $s > 1$, then, by (51),

$$
\sum_{j=1}^\infty \frac{j^s}{e^{bj} - 1} = \int_0^\infty \frac{x^s \, dx}{e^{bx} - 1} + R_2 = b^{-s-1} \int_0^\infty \frac{y^s \, dy}{e^y - 1} + R_2
$$

$$
= b^{-s-1} \Gamma(s + 1) \zeta(s + 1) + R_2
$$
(we have used [12, 3.411.1]), where

$$|R_2| \leq b^{-s} \int_0^\infty \frac{sy^{s-1}(e^y - 1) + y^s e^y}{(e^y - 1)^2} dy,$$

and we have the desired estimate, because the integral converges. If $0 < s \leq 1$, then this computation does not work, but we can use (50) and write

$$\sum_{j=1}^{\infty} \frac{j^s}{e^{bj} - 1} = \int_1^{\infty} \frac{x^s}{e^{bx} - 1} dx + R_1 = b^{-s-1} \int_0^\infty \frac{y^s}{e^y - 1} dy + R_1,$$

where

$$|R_1| \leq b^{-s} \int_0^b \frac{sy^{s-1}(e^y - 1) + y^s e^y}{(e^y - 1)^2} dy,$$

and so $|R_1| \leq Cb^{-1}$ for $s \in (0, 1)$ and $|R_1| \leq Cb^{-1} |\ln b|$ for $s = 1$. Now note that

$$\int_0^b \frac{y^s}{e^y - 1} \leq Cb^s,$$

and we obtain the desired estimate.

The estimate for the sum starting form $j = l$ can be obtained in a similar way; one only need to shift the summation index by $l - 1$.

(ii) The proof goes by the same scheme as for (i). We omit the details.\qed

** Remark.** The integrals in (ii) can also be expressed via special functions, but we do not need these expressions.

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