Remarks on 1-D Euler Equations with Time-Decayed Damping

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Abstract

We study the 1-d isentropic Euler equations with time-decayed damping
\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x p(\rho) &= -\frac{\mu}{1+t} \rho u, \\
\rho|_{t=0} &= 1 + \varepsilon \rho_0(x), u|_{t=0} = \varepsilon u_0(x).
\end{aligned}
\]

This work is inspired by a recent work of F. Hou, I. Witt and H.C. Yin [3]. In [3], they proved a global existence and blow-up result of 3-d irrotational Euler flow with time-dependent damping. In the 1-d case, we will prove a different result when the damping decays of order $-1$ with respect to the time $t$. More precisely, when $\mu > 2$, we prove the global existence of the 1-d Euler system. While when $0 \leq \mu \leq 2$, we will prove the blow up of $C^1$ solutions.

Keywords: Euler equations, global existence, blow up, time-decayed damping.

Mathematical Subject Classification 2010: 35L70, 35L65, 76N15.

1 Introduction

This paper deals with the isentropic Euler equations with time-decayed damping in 1 dimension:
\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x p(\rho) &= -\frac{\mu}{1+t} \rho u, \\
\rho|_{t=0} &= 1 + \varepsilon \rho_0(x), u|_{t=0} = \varepsilon u_0(x),
\end{aligned}
\]

(1.1)

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where \( \rho_0(x), u_0(x) \in C_0^\infty(\mathbb{R}) \), supported in \( |x| \leq R \) and \( \varepsilon > 0 \) is sufficient small. Here \( \rho(x), u(x) \) and \( p(x) \) represent the density, fluid velocity and pressure respectively and \( \mu \) is a positive number to describe the scale of the damping. We assume the fluid is a polytropic gas which means we assume \( p(\rho) = \frac{1}{\gamma} \rho^\gamma, \gamma > 1 \). We denote \( c^2 = p'(\rho) \).

As is well known, when the damping is vanishing, the smooth solution of compressible Euler flow will blow up in finite time. For the extensive literature on the blow-up results and the blow-up mechanism, Readers can see [1], [2], [14, 15], and [21, 22] and references therein for more details.

While 1-d Euler equations with linear damping read as

\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x P(\rho) &= -\kappa \rho u.
\end{aligned}
\]  

(1.2)

Here \( \kappa \) is a constant. There are many results to prove the global existence and convergence rates of solutions to system (1.2) with small data. Readers can turn to [4], [11] and [12] for more information.

In multi space dimensions, there are also some results to Euler equations with linear damping. Wang and Yang proved the stability of the planar diffusion wave for the 2D Euler system with linear damping in [19] and studied the pointwise estimates of solutions for the 3-dimension Euler equations in [20]. Similar convergence rates in 3 dimensions are obtained by Tan and Wang by using a different method in [17]. Also in [6], Jiu and Zheng proved the global existence of the 3D Euler systems with linear damping in Besov spaces. While Sideris, Thomases and Wang in [13] showed that the smooth solutions of the linear-damped Euler equations do not decay in exponentially in time and may blow up if the initial data is sufficient large.

It is nature to ask whether the global solution exists when the damping is decayed and what is the critical decayed rate to separate the global existence and the finite-time blow up of solutions with small data. Recently F. Hou, I. Witt and H.C. Yin in [3] proved a global existence(the damping is time-decayed smaller than or equal to order -1) and blow-up(the damping is time-decayed larger than order -1) result for the 3d irrotational Euler flow. Our article discusses the 1-dimension Euler equations with damping of time-decayed order -1. Compared with their work, we will get a different result in 1-dimensional case. The global existence and blow up depend on the scale of \( \mu \) and \( \mu = 2 \) is the critical value.

Throughout this paper we denote a generic constant by \( C \). It may be different line by line. \( H^m(\mathbb{R}) \) denotes the usual Sobolev space with its norm

\[ \|f\|_{H^m} \triangleq \sum_{k=0}^m \|\partial_x^k f\|_{L_p}. \]

For convenience, we use \( \| \cdot \| \) to denote \( \| \cdot \|_{L_2} \) and \( \| \cdot \|_m \) for \( \| \cdot \|_{H^m} \).

We state our main results as following.

**Theorem 1.1 (Global existence for \( \mu > 2 \))** Suppose \( (\rho_0, u_0) \in H^m(\mathbb{R}), m \geq 3 \) and \( \mu > 2 \). Then there exists a unique global classical solution \( (\rho(x, t), u(x, t)) \) of (1.1) satisfies

\[
\|(1 + t)\partial_t \rho(t)\|_{m-1}^2 + \|(1 + t)\partial_x \rho(t)\|_{m-1}^2 + \|(1 + t)\partial_x u(t)\|_{m-1}^2
\]
+\| (\rho(t) - 1) \|^2 + \| u(t) \|^2 \\
+ \int_0^t (1 + \tau) (\| \partial_t \rho(\tau) \|_{m-1}^2 + \| \partial_x \rho(\tau) \|_{m-1} + \| \partial_x u(\tau) \|_{m-1}) d\tau \\
\leq C \varepsilon^2 (\| \rho_0 \|_m^2 + \| u_0 \|_m^2).

**Remark 1.1** By Sobolev inequality, we have the pointwise estimates

\[ \sum_{\alpha \leq m-1} \sup \left\{ (1 + t)^{\frac{1 + \min\{\alpha, 1\}}{2}} (\| \partial^\alpha_x v \| + \| \partial^\alpha_x u \|) \right\} \leq C \varepsilon. \]

**Remark 1.2** Inspired by the work of F. Hou, I. Witt and H.C. Yin, we believe that by introducing a “potential function” \( \phi(x, t) = \int_0^\infty u(y, t) dy \), we can prove the global existence when the damping decays like \( \frac{\mu}{(1 + t)^{\lambda}} \) \( (0 < \lambda < 1, \mu > 0) \). We will present it in our later work.

The idea of proving Theorem 1.1 comes from [20]. In the first part of their paper, they proved the global existence of multi-dimensional Euler equations with constant linear damping by an energy estimate method. We revise their method with a time-weighted energy estimate to make it suitable for the proving of Theorem 1.1.

Next we discuss the blow up of system (1.1) when \( 0 \leq \mu \leq 2 \). Define \( T_\varepsilon \) be the lifespan, the largest existing time, of \( C^1 \) solutions to system (1.1). Define two functions

\[ q^0(r) = \int_r^\infty (x - r)^2 \rho_0(x) dx, \]
\[ q^1(r) = 2 \int_r^\infty (x - r)(\rho_0 u_0)(x) dx. \]

**Theorem 1.2** Suppose for some \( 0 < R_0 < R \),

\[ q^0(r) > 0, \quad q^1(r) \geq 0, \tag{1.3} \]

for \( R_0 < r < R \). Then the lifespan \( T_\varepsilon \) of \( C^1 \) solutions of (1.1) is finite.

**Remark 1.3** the proof of Theorem 1.2 can find its original version in T. Sideris [15] for 3-d compressible Euler equations. It can not be used directly for the 1-d damped Euler equations, but we can revise the test function there to similarly prove the finite-time blow up.

**Remark 1.4** Actually our method can deal with the case when the damping decays like \( \frac{\mu}{(1 + t)^{\lambda}} \) when \( \lambda > 1, \mu \geq 0 \).

We arrange our paper as following. In Section 2, we prove Theorem 1.1 for the global existence with relatively “large” damping. In Section 3, we prove Theorem 1.3 for the blow up of solutions with relatively “small” damping.
2 PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 by a time-weighted energy method. Remember \( c = \sqrt{P'(\rho)} = \rho^{\frac{\gamma-1}{2}} \). First we change (1.1) into

\[
\begin{aligned}
\frac{2}{\gamma-1} \partial_t c + c \partial_x u + \frac{2}{\gamma-1} u \partial_x c &= 0, \\
\partial_t u + u \partial_x u + \frac{2}{\gamma-1} c \partial_x c + \frac{\mu}{1+t} u &= 0,
\end{aligned}
\]

(2.1)

where \( c_0(x) \in C^\infty_0(R) \), supported in \( |x| \leq R \).

Let \( v = \frac{2}{\gamma-1} (c - 1) \), then \( v, u \) satisfy

\[
\begin{aligned}
\partial_t v + \partial_x u &= -u \partial_x v - \frac{\gamma-1}{2} v \partial_x u, \\
\partial_t u + \partial_x v + \frac{\mu}{1+t} u &= -u \partial_x u - \frac{\gamma-1}{2} v \partial_x v, \\
v|_{t=0} &= \varepsilon v_0(x), u|_{t=0} = \varepsilon u_0(x),
\end{aligned}
\]

(2.2)

where \( v_0(x) = \frac{2}{\gamma-1} c_0(x) \).

From (2.2), we have

\[
\partial_t v - \partial_{xx} v + \frac{\mu}{1+t} \partial_t v = Q(v, u),
\]

(2.3)

where

\[
Q(v, u) = \frac{\mu}{1+t} (-u \partial_x v - \frac{\gamma-1}{2} v \partial_x u) \\
- \partial_t (u \partial_x v) - \frac{\gamma-1}{2} \partial_t (v \partial_x u) + \partial_x (u \partial_x u) + \frac{\gamma-1}{2} \partial_x (v \partial_x v).
\]

In the following, we will estimate \((v, u)\) under the a priori assumption

\[
E_m(T) = \sup_{0 < t < T} \left\{ \| (1+t) \partial_x v \|_{m-1}^2 + \| (1+t) \partial_x u \|_{m-1}^2 + \| v \|^2 + \| u \|^2 \right\}^{\frac{1}{2}} \\
\leq M \varepsilon.
\]

(2.4)

Where \( M \), independent of \( \varepsilon \), will be determined later. By choosing \( M \) suitably, we will prove

\[
E_m(T) \leq \frac{1}{2} M \varepsilon.
\]

(2.5)

By Sobolev inequality, we know that

\[
\sum_{|\alpha| \leq 1} \sup_{|x| \leq R} \left\{ (1+t) \frac{2}{m+1} (|\partial_x^\alpha v| + |\partial_x^\alpha u|) \right\}
\]

Since (2.2) implies
\[ |(1 + t)\partial_t v| + |(1 + t)\partial_t u| \leq C \{ |u| + (1 + t)|\partial_x v| + (1 + t)|\partial_x u| \}, \]
we also have
\[ \sum_{\alpha + \beta \leq 1} \sup \left\{ (1 + t) \frac{\alpha + \beta + 1}{2} (|\partial_x^\alpha \partial_t^\beta v| + |\partial_x^\alpha \partial_t^\beta u|) \right\} \leq CM\varepsilon. \quad (2.6) \]

In the following, we will first obtain some elementary estimates for the 1-order derivatives of the solution. Then the higher derivatives will be handled in the similar way.

### 2.1 Estimate 1

For some constant \( \eta \), to be determined later, multiplying (2.3) by \( \eta (1 + t)^2 \partial_t v \) yields
\[
\frac{\eta}{2} \partial_t [(1 + t)^2 (\partial_t v)^2] + \eta (\mu - 1)(1 + t)(\partial_t v)^2 - \eta (1 + t)^2 \partial_t v \partial_{xx} v
= \eta (1 + t)^2 \partial_t v Q(v, u). \quad (2.7)
\]

Integrating it over \( R \times [0, t] \) and using integration by parts give
\[
\frac{\eta}{2} \int_R (1 + t)^2 (\partial_t v)^2 dx + \eta (\mu - 1) \int_0^t \int_R (1 + \tau) (\partial_\tau v)^2 dxd\tau
+ \frac{\eta}{2} \int_R (1 + t)^2 (\partial_x v)^2 dx - \eta \int_0^t \int_R (1 + \tau) (\partial_x v)^2 dxd\tau
= \frac{\eta}{2} \int_R (\partial_t v)^2|_{t=0} dx + \frac{\eta}{2} \int_R (\partial_x v)^2|_{t=0} dx + \int_0^t \int_R \eta (1 + \tau)^2 \partial_\tau v Q(v, u) dxd\tau
\leq C\varepsilon^2 (\|v_0\|_1^2 + \|u_0\|_1^2) + \int_0^t \int_R \eta (1 + \tau)^2 \partial_\tau v Q(v, u) dxd\tau. \quad (2.8)
\]

Also multiplying (2.3) by \( (1 + t)v \), we get
\[
\partial_t [(1 + t)v \partial_t v] + \frac{\mu - 1}{2} \partial_t v^2 - (1 + t)v \partial_{xx} v - (1 + t)(\partial_t v)^2
= (1 + t)v Q(v, u). \quad (2.9)
\]

Then integrating (2.9) over \( R \times [0, t] \) and using integration by parts give
\[
\int_R (1 + t)v \partial_t v dx + \frac{\mu - 1}{2} \int_R v^2 dx
+ \int_0^t \int_R (1 + \tau)(\partial_\tau v)^2 dxd\tau - \int_0^t \int_R (1 + \tau)(\partial_\tau v)^2 dxd\tau
= \int_R (v \partial_t v)|_{t=0} dx + \frac{\mu - 1}{2} \int_R v^2|_{t=0} dx + \int_0^t \int_R (1 + t)v Q(v, u) dxd\tau.
\]
\[
\leq C \varepsilon^2 (\|v_0\|_1^2 + \|u_0\|_1^2) + \int_0^t \int_R (1 + \tau) v Q(v, u) dx d\tau. \tag{2.10}
\]

Adding (2.8) and (2.10), we have
\[
\int_R (1 + t)^2 (\partial_v^2 v + \frac{\mu - 1}{2} + (\partial_x v)^2) dx + \int_0^t \int_R (1 + \tau)^2 (\partial_v^2 v + \gamma - \frac{1}{2}) d\tau dx + \int_0^t \int_R (1 + \tau)^2 (\partial_x v)^2 dx d\tau
\]
\[
\leq C \varepsilon^2 (\|v_0\|_1^2 + \|u_0\|_1^2) + \int_0^t \int_R [(1 + \tau)^2 \partial_v + (1 + \tau) v] Q(v, u) dx d\tau. \tag{2.11}
\]

If \( \mu = 2 + 4\delta \) for some \( \delta > 0 \), using Cauchy-Schwartz inequality, we have
\[
\int_R (1 + t) v \partial_v dx \geq -\frac{1 + 4\delta}{2(1 + \delta)} \int_R v^2 dx - \frac{1 + \delta}{2(1 + 4\delta)} \int_R (1 + t)^2 (\partial_v^2 v) dx. \tag{2.12}
\]

From (2.11) and (2.12) by choosing \( \eta = \frac{1 + 2\delta}{1 + 4\delta} \), we have
\[
\frac{\delta}{2(1 + 4\delta)} \int_R (1 + t)^2 (\partial_v^2 v) dx + \frac{1}{2(1 + \delta)} \int_R v^2 dx + \frac{1 + 2\delta}{2(1 + 4\delta)} \int_R (1 + t)^2 (\partial_x v)^2 dx + \frac{1 + \delta}{2(1 + 4\delta)} \int_R (1 + t)^2 (\partial_x v)^2 dx
\]
\[
\leq C \varepsilon^2 (0) + \int_0^t \int_R [(1 + \tau)^2 \partial_v + (1 + \tau) v] Q(v, u) dx d\tau. \tag{2.13}
\]

So we have
\[
\int_R (1 + t)^2 (\partial_v^2 v) dx + \int_R (1 + t)^2 (\partial_x v)^2 dx + \int_R v^2 dx
\]
\[
+ \int_0^t \int_R (1 + \tau)^2 (\partial_v^2 v) dx d\tau + \int_0^t \int_R (1 + \tau)^2 (\partial_x v)^2 dx d\tau
\]
\[
\leq C \varepsilon^2 (0) + CI. \tag{2.14}
\]

where \( C \) depends on \( \mu \) and

\[
I = \int_0^t \int_R [(1 + \tau)^2 \partial_v + (1 + \tau) v] Q(v, u) dx d\tau
\]
\[
= \int_0^t \int_R [(1 + \tau)^2 \partial_v + (1 + \tau) v] \frac{\mu}{1 + \tau} (-u \partial_x v - \gamma - \frac{1}{2} v \partial_x u) dx d\tau
\]
\[
- \int_0^t \int_R [(1 + \tau)^2 \partial_v + (1 + \tau) v] [\partial_y (u \partial_x v) + \gamma - \frac{1}{2} \partial_y (v \partial_x u)] dx d\tau
\]
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\[ + \int_0^t \int_R [(1 + \tau)^2 \partial_\tau v + (1 + \tau) v] [\partial_x (u \partial_x u) + \frac{\gamma - 1}{2} \partial_x (v \partial_x v)] dx d\tau \]

\[ = I_1 + I_2 + I_3. \]

Now we estimate \( I_1, I_2 \) and \( I_3 \).

\[ I_1 = \mu \int_0^t \int_R [(1 + \tau) \partial_\tau v (-u \partial_x v - \frac{\gamma - 1}{2} v \partial_x u)]dx d\tau + \mu \int_0^t \int_R v (-u \partial_x v - \frac{\gamma - 1}{2} v \partial_x u) dx d\tau. \]

From (2.6), using Cauchy-Schwartz inequality and integration by parts, we have

\[ I_1 \leq CM \varepsilon \int_0^t \int_R \left[(1 + \tau) \left| \partial_\tau v \right|^2 + \left| \partial_x v \right|^2 + \left| \partial_x u \right|^2 + \frac{u^2}{1 + \tau} \right] dx d\tau + C \int_0^t \int_R uv \partial_x v dx d\tau \]

\[ \leq C M \varepsilon \int_0^t \int_R \left[(1 + \tau) \left| \partial_\tau v \right|^2 + \left| \partial_x v \right|^2 + \left| \partial_x u \right|^2 + \frac{u^2}{1 + \tau} \right] dx d\tau. \quad (2.15) \]

Now we focus on the estimate of \( I_2 \), then \( I_3 \) will be essentially the same with \( I_2 \). Dealing \( I_2 \) the same with \( I_1 \), we have

\[ I_2 \leq C M \varepsilon \int_0^t \int_R \left[(1 + \tau) \left| \partial_\tau u \right|^2 + \left| \partial_x v \right|^2 + \left| \partial_\tau v \right|^2 \right] dx d\tau \]

\[ - \int_0^t \int_R (1 + \tau)^2 u \partial_\tau v \partial_x^2 v dx d\tau - \int_0^t \int_R (1 + \tau) v u \partial_x^2 v dx d\tau - \frac{\gamma - 1}{2} \int_0^t \int_R (1 + \tau)^2 \partial_x^2 u dx d\tau \]

\[ = C M \varepsilon \int_0^t \int_R \left[(1 + \tau) \left| \partial_\tau u \right|^2 + \left| \partial_x v \right|^2 + \left| \partial_\tau v \right|^2 \right] dx d\tau \]

\[ + \frac{1}{2} \int_0^t \int_R (1 + \tau)^2 \partial_x u (\partial_x v)^2 dx d\tau + \int_0^t \int_R (1 + \tau) \partial_x (vu) \partial_\tau v dx d\tau \]

\[ - \frac{\gamma - 1}{2} \int_0^t \int_R (1 + \tau)^2 v \partial_\tau v \partial_x^2 u dx d\tau + \frac{\gamma - 1}{2} \int_0^t \int_R (1 + \tau) \partial_x v^2 \partial_\tau u dx d\tau. \]

Using (2.6) again, we have

\[ I_2 \leq C M \varepsilon \int_0^t \int_R \left[(1 + \tau) \left| \partial_\tau u \right|^2 + \left| \partial_x v \right|^2 + \left| \partial_\tau v \right|^2 \right] dx d\tau \]

\[ - \frac{\gamma - 1}{2} \int_0^t \int_R (1 + \tau)^2 v \partial_\tau v \partial_x^2 u dx d\tau. \quad (2.16) \]
From (2.2)

$$\partial_x u = \frac{-\partial_t v - u \partial_x v}{1 + \frac{\gamma - 1}{2} v}.$$ 

Then

$$\partial^2_{xt} u = -\frac{\partial^2_t v + \partial_t u \partial_x v + u \partial^2_x v}{1 + \frac{\gamma - 1}{2} v} + \frac{\gamma - 1}{2} \partial_t v (\partial_x v + u \partial_x v) \frac{(1 + \frac{\gamma - 1}{2} v)^2}{(1 + \frac{\gamma - 1}{2} v)^2}.$$ 

From (2.6), we have

$$\frac{1}{1 + \frac{\gamma - 1}{2} v} \leq \frac{1}{1 - CM\varepsilon}.$$ 

Inserting (2.17) and (2.18) into (2.16), we have

$$\int_0^t \int_R (1 + \tau)^2 v \partial_\tau v \partial^2_{xx} u dxd\tau$$

$$= \int_0^t \int_R (1 + \tau)^2 v \partial_\tau \left[ -\frac{\partial^2_t v + \partial_t u \partial_x v + u \partial^2_x v}{1 + \frac{\gamma - 1}{2} v} + \frac{\gamma - 1}{2} \partial_t v (\partial_x v + u \partial_x v) \frac{(1 + \frac{\gamma - 1}{2} v)^2}{(1 + \frac{\gamma - 1}{2} v)^2} \right] dxd\tau$$

$$\leq - \int_0^t \int_R (1 + \tau)^2 \frac{v}{1 + \frac{\gamma - 1}{2} v} \partial_\tau v (\partial^2_x v + u \partial^2_x v) dxd\tau$$

$$+ \frac{CM^2 \varepsilon^2}{(1 - CM\varepsilon)^2} \int_0^t \int_R (1 + \tau) (|\partial_\tau v|^2 + |\partial_x v|^2) dxd\tau.$$ 

Combining (2.16) and (2.19), we have

$$I_2 \leq C \left[ M\varepsilon + \frac{M^2 \varepsilon^2}{(1 - CM\varepsilon)^2} \right] \int_0^t \int_R (1 + \tau) (|\partial_\tau v|^2 + |\partial_x v|^2 + |\partial_r u|^2) dxd\tau$$

$$- \int_0^t \int_R (1 + \tau)^2 \frac{v}{1 + \frac{\gamma - 1}{2} v} \partial_\tau v \partial^2_x v + (1 + \tau)^2 \frac{vu}{1 + \frac{\gamma - 1}{2} v} \partial_\tau v \partial^2_x v dxd\tau$$

$$= C \left[ M\varepsilon + \frac{M^2 \varepsilon^2}{(1 - CM\varepsilon)^2} \right] \int_0^t \int_R (1 + \tau) (|\partial_\tau v|^2 + |\partial_x v|^2 + |\partial_r u|^2) dxd\tau$$

$$+ I_{2,1} + I_{2,2}. \quad (2.20)$$

Using integration by parts with respect to $\tau$, we have

$$I_{2,1} = -\frac{1}{2} \int_0^t \int_R (1 + \tau)^2 \frac{v}{1 + \frac{\gamma - 1}{2} v} \partial_\tau (\partial_\tau v)^2 dxd\tau$$

$$= -\frac{1}{2} \int_R (1 + \tau)^2 \frac{v}{1 + \frac{\gamma - 1}{2} v} (\partial_\tau v)^2 dx + \frac{1}{2} \int_R \frac{\varepsilon v_0}{1 + \frac{\gamma - 1}{2} \varepsilon v_0} (\partial_\tau v (\cdot, 0))^2 dx$$

$$+ \frac{1}{2} \int_0^t \int \partial_\tau \left[ (1 + \tau)^2 \frac{v}{1 + \frac{\gamma - 1}{2} v} \right] (\partial_\tau v)^2 dxd\tau$$

$$\leq \frac{C\varepsilon}{1 - C\varepsilon} E_1^2(0) + \frac{CM\varepsilon}{1 - CM\varepsilon} \int_R (1 + t)^2 (\partial_t v)^2 dx.$$
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\[ + \frac{CM\varepsilon}{(1-CM\varepsilon)^2} \int_0^t \int_R (1 + \tau)(\partial_\tau v)^2 dx d\tau. \]

(2.21)

And using integration by parts, we have

\[ I_{2,2} = -\frac{1}{2} \int_0^t \int_R (1 + \tau)^2 \frac{vu}{1 + \frac{\gamma - 1}{2}} \partial_x (\partial_\tau v)^2 dx d\tau \]
\[ = \frac{1}{2} \int_0^t \int_R (1 + \tau)^2 \partial_x \left[ \frac{vu}{1 + \frac{\gamma - 1}{2}} \right] (\partial_\tau v)^2 dx d\tau \]
\[ \leq \frac{CM^2\varepsilon^2}{(1-CM\varepsilon)^2} \int_0^t \int_R (1 + \tau)^2 (\partial_\tau v)^2 dx d\tau. \]

(2.22)

Inserting (2.21) and (2.22) into (2.20), we have

\[ I_2 \leq \frac{CM\varepsilon}{(1-CM\varepsilon)^2} \int_0^t \int_R (1 + \tau)(|\partial_\tau v|^2 + |\partial_x v|^2 + |\partial_x u|^2) dx d\tau \]
\[ + \frac{C\varepsilon}{1-C\varepsilon} E_1^2(0) + \frac{CM\varepsilon}{1-CM\varepsilon} \int_R (1 + t)^2 (\partial_x v)^2 dx. \]

(2.23)

We can deal \( I_3 \) almost the same with \( I_2 \). Then we can get

\[ I_3 \leq \frac{CM\varepsilon}{(1-CM\varepsilon)^2} \int_0^t \int_R (1 + \tau)(|\partial_\tau v|^2 + |\partial_x v|^2 + |\partial_x u|^2) dx d\tau \]
\[ + \frac{C\varepsilon}{1-C\varepsilon} E_1^2(0) + \frac{CM\varepsilon}{1-CM\varepsilon} \int_R (1 + t)^2 (\partial_x v)^2 dx. \]

(2.24)

Remember that in (2.2), we have

\[ |\partial_\tau u| \leq C \left( |\partial_x v| + |\partial_x u| + \frac{|u|}{1+t} \right). \]

Inserting the estimates of \( I_1 \) (2.15), \( I_2 \) (2.23) and \( I_3 \) (2.24) into (2.14), we have

\[ \int_R (1 + t)^2 (\partial_\tau v)^2 dx + \int_R (1 + t)^2 (\partial_x v)^2 dx + \int_R v^2 dx \]
\[ + \int_0^t \int_R (1 + \tau)(\partial_\tau v)^2 dx d\tau + \int_0^t \int_R (1 + \tau)(\partial_x v)^2 dx d\tau \]
\[ \leq C \frac{1 + \varepsilon}{1-C\varepsilon} E_1^2(0) + \frac{CM\varepsilon}{1-CM\varepsilon} \int_R (1 + t)^2 \left[ (\partial_\tau v)^2 + (\partial_x v)^2 \right] dx \]
\[ + \frac{CM\varepsilon}{(1-CM\varepsilon)^2} \int_0^t \int_R \left\{ (1 + \tau) \left[ (\partial_\tau v)^2 + (\partial_x v)^2 + (\partial_x u)^2 \right] + \frac{u^2}{1 + \tau} \right\} dx d\tau. \]

(2.25)
\section*{2.2 Estimate 2}

Multiplying (2.2) \textsubscript{2} by \( u \) and integrating on \( R \times [0, t] \) yield

\[
\int_0^t \int_R \left( \frac{1}{2} \partial_\tau u^2 + u \partial_\tau v + \frac{\mu}{1 + \tau} u^2 \right) dx d\tau = \int_0^t \int_R (-u \partial_\tau u - \frac{\gamma - 1}{2} v \partial_\tau v) u dx d\tau.
\]

Then using integration by parts, we have

\[
\frac{1}{2} \int_R u^2 dx - \frac{\varepsilon^2}{2} \int_R u_0^2 dx - \int_0^t \int_R v \partial_\tau u dx d\tau + \int_0^t \int_R \frac{\mu}{1 + \tau} u^2 dx d\tau
\]

\[
= \int_0^t \int_R (-u \partial_\tau u - \frac{\gamma - 1}{2} v \partial_\tau v) u dx d\tau.
\]

Using (2.2) and (2.6), we have

\[
\frac{1}{2} \int_R u^2 dx + \int_0^t \int_R \left\{ \frac{1}{1 + \tau} u^2 + (1 + \tau) \left[ (\partial_\tau v)^2 + (\partial_x u)^2 \right] \right\} dx d\tau.
\]

Then we can have

\[
\int_R (u^2 + v^2) dx + \int_0^t \int_R \frac{\mu}{1 + \tau} u^2 dx d\tau
\]

\[
\leq CE_1^2(0) + CM \varepsilon \int_0^t \int_R \left\{ \frac{1}{1 + \tau} u^2 + (1 + \tau) \left[ (\partial_\tau v)^2 + (\partial_x u)^2 \right] \right\} dx d\tau. \quad (2.26)
\]

\section*{2.3 Estimate 3}

By differentiating (2.2) \textsubscript{2} with respect to \( x \) and integrating its product with \((1 + \tau)^2 \partial_x u\) on \( R \times [0, t] \), we have

\[
\int_0^t \int_R (1 + \tau)^2 \partial_x u \left( \partial_\tau^2 u + \partial_x^2 v + \frac{\mu}{1 + \tau} \partial_x u \right) dx d\tau
\]

\[
= \int_0^t \int_R (1 + \tau)^2 \partial_x u \partial_x (-u \partial_\tau u - \frac{\gamma - 1}{2} v \partial_\tau v) dx d\tau.
\]

Then

\[
\frac{1}{2} \int_R (1 + t)^2 (\partial_x u)^2 dx + (\mu - 1) \int_0^t \int_R (1 + \tau) (\partial_x u)^2 dx d\tau
\]

\[
+ \int_0^t \int_R (1 + \tau)^2 \partial_x (\partial_\tau v + u \partial_x v + \frac{\gamma - 1}{2} v \partial_\tau v) \partial_x u dx d\tau
\]

\[
\leq CE_1^2(0) + \int_0^t \int_R (1 + \tau)^2 \partial_x u \partial_x (-u \partial_\tau u - \frac{\gamma - 1}{2} v \partial_\tau v) dx d\tau.
\]
So
\[
\frac{1}{2} \int_R (1+t)^2 \left[ (\partial_x u)^2 + (\partial_x v)^2 \right] \, dx \\
+ (\mu - 1) \int_0^t \int_R (1 + \tau) (\partial_x u)^2 \, dx \, d\tau - \int_0^t \int_R (1 + \tau) (\partial_x v)^2 \, dx \, d\tau \\
\leq - \int_0^t \int_R (1 + \tau)^2 \partial_x (u \partial_x v + \gamma \frac{1}{2} v \partial_x u) \partial_x v \, dx \, d\tau \\
+ C E_1^2(0) + \int_0^t \int_R (1 + \tau)^2 \partial_x (u \partial_x v - \gamma \frac{1}{2} v \partial_x u) \partial_x v \, dx \, d\tau.
\]

We can deal with the right terms almost the same with \( I_2 \) and \( I_3 \). Then we can get the estimate
\[
\int_R (1+t)^2 \left[ (\partial_x u)^2 + (\partial_x v)^2 \right] \, dx \\
+ \int_0^t \int_R (1 + \tau) (\partial_x u)^2 \, dx \, d\tau - \int_0^t \int_R (1 + \tau) (\partial_x v)^2 \, dx \, d\tau \\
\leq C \frac{1 + \varepsilon}{1 - C \varepsilon} E_1^2(0) + \frac{CM \varepsilon}{1 - CM \varepsilon} \int_R (1+t)^2 \left[ (\partial_t v)^2 + (\partial_x v)^2 \right] \, dx \\
+ \frac{CM \varepsilon}{(1 - CM \varepsilon)^2} \int_0^t \int_R \left\{ (1 + \tau) \left[ (\partial_t v)^2 + (\partial_x v)^2 + (\partial_x u)^2 \right] + \frac{u^2}{1 + \tau} \right\} \, dx \, d\tau.
\]
(2.27)

Let (2.25) + (2.26) + \( \frac{1}{2} \times (2.27) \), then we get
\[
\int_R \left\{ (1+t)^2 \left[ (\partial_t v)^2 + (\partial_x v)^2 + (\partial_x u)^2 \right] + v^2 + u^2 \right\} \, dx \\
+ \int_0^t \int_R \left\{ (1 + \tau) \left[ (\partial_t v)^2 + (\partial_x v)^2 + (\partial_x u)^2 \right] + \frac{u^2}{1 + \tau} \right\} \, dx \, d\tau \\
\leq C \frac{1 + \varepsilon}{1 - C \varepsilon} E_1^2(0) + \frac{CM \varepsilon}{1 - CM \varepsilon} \int_R (1+t)^2 \left[ (\partial_t v)^2 + (\partial_x v)^2 \right] \, dx \\
+ \frac{CM \varepsilon}{(1 - CM \varepsilon)^2} \int_0^t \int_R \left\{ (1 + \tau) \left[ (\partial_t v)^2 + (\partial_x v)^2 + (\partial_x u)^2 \right] + \frac{u^2}{1 + \tau} \right\} \, dx \, d\tau.
\]
(2.28)

2.4 Estimates for Higher Derivatives

The estimates as (2.25) and (2.27) can also be obtained for higher derivatives. In fact, by multiplying (2.3) by \( \partial_x^2 [\eta(1 + \tau)^2 \partial_x v + (1 + \tau) v] \) and integrating it on \( R \times [0, t] \), we can get
\[
\int_R (1+t)^2 (\partial_x^2 v)^2 \, dx + \int_R (1+t)^2 (\partial_x^2 u)^2 \, dx + \int_R (\partial_x v)^2 \, dx
\]
By differentiating (2.2) two times with respect to \( x \) and integrating its product with \((1 + \tau)^2 \partial_x^2 u\) on \( R \times [0, t] \), we have

\[
\int_R (1 + t)^2 \left[ (\partial_x^2 u)^2 + (\partial_x^2 v)^2 \right] dx + \int_0^t \int_R (1 + \tau)(\partial_x^2 u)^2 dxd\tau
\leq C \frac{1 + \varepsilon}{1 - C_\varepsilon} E_2^2(0) + \frac{CM\varepsilon}{1 - CM\varepsilon} \int_R (1 + t)^2 \left[ (\partial_x v)^2 + (\partial_x^2 v)^2 \right] dx
+ \frac{CM\varepsilon}{(1 - CM\varepsilon)^2} \int_0^t \int_R \left\{ (1 + \tau) \left[ (\partial_x^2 v)^2 + (\partial_x^2 v)^2 + (\partial_x^2 u)^2 \right] \right\} dxd\tau.
\]

(2.29)

Combining (2.28), (2.29) and (2.30) gives

\[
\|(1 + t)\partial_t v\|_{m-1}^2 + \|(1 + t)\partial_x v\|_{m-1}^2 + \|(1 + t)\partial_x u\|_{m-1}^2 + \|v\|^2 + \|u\|^2
+ \int_0^t \left[ (1 + \tau)(\|\partial_t v\|_{m-1}^2 + \|\partial_x v\|_{m-1}^2 + \|\partial_x u\|_{m-1}^2) + \frac{\|u\|^2}{1 + \tau} \right] d\tau
\leq C \frac{1 + \varepsilon}{1 - C_\varepsilon} E_2^2(0) + \frac{CM\varepsilon}{1 - CM\varepsilon} \left[ \|(1 + t)\partial_t v\|_{m-1}^2 + \|(1 + t)\partial_x v\|_{m-1}^2 \right]
+ \frac{CM\varepsilon}{(1 - CM\varepsilon)^2} \int_0^t \left[ (1 + \tau)(\|\partial_t v\|_{m-1}^2 + \|\partial_x v\|_{m-1}^2 + \|\partial_x u\|_{m-1}^2) + \frac{\|u\|^2}{1 + \tau} \right] d\tau.
\]

(2.31)

Actually, we can prove for \( m \)

\[
\|(1 + t)\partial_t v\|_{m-1}^2 + \|(1 + t)\partial_x v\|_{m-1}^2 + \|(1 + t)\partial_x u\|_{m-1}^2 + \|v\|^2 + \|u\|^2
+ \int_0^t \left[ (1 + \tau)(\|\partial_t v\|_{m-1}^2 + \|\partial_x v\|_{m-1}^2 + \|\partial_x u\|_{m-1}^2) + \frac{\|u\|^2}{1 + \tau} \right] d\tau
\leq C \frac{1 + \varepsilon}{1 - C_\varepsilon} E_2^2(0) + \frac{CM\varepsilon}{1 - CM\varepsilon} \left[ \|(1 + t)\partial_t v\|_{m-1}^2 + \|(1 + t)\partial_x v\|_{m-1}^2 \right]
+ \frac{CM\varepsilon}{(1 - CM\varepsilon)^2} \int_0^t \left[ (1 + \tau)(\|\partial_t v\|_{m-1}^2 + \|\partial_x v\|_{m-1}^2 + \|\partial_x u\|_{m-1}^2) + \frac{\|u\|^2}{1 + \tau} \right] d\tau.
\]

(2.32)

When \( \varepsilon \) is small, for some \( C_0 \), we get

\[
\|(1 + t)\partial_t v\|_{m-1}^2 + \|(1 + t)\partial_x v\|_{m-1}^2 + \|(1 + t)\partial_x u\|_{m-1}^2 + \|v\|^2 + \|u\|^2
\]
\[
+ \int_0^t \left[ (1 + \tau)(\|\partial_\tau v\|_{m-1}^2 + \|\partial_\tau v\|_1^2 + \|\partial_x u\|_{m-1}^2) + \frac{\|u\|^2}{1 + \tau} \right] d\tau \\
\leq \frac{(1 - CM\varepsilon)^2}{(1 - CM\varepsilon)^2 - CM\varepsilon} C_0 \varepsilon^2.
\]

Let \( M^2 = 5C_0 \). By using the smallness of \( \varepsilon \), we can have

\[
E_m^2(t) \leq \frac{1}{4} M^2 \varepsilon^2.
\] (2.33)

The local existence of symmetrizable hyperbolic equations have been proved by using the fixed point theorem. In order to get the global existence of the system, we only need a priori estimate. Based on our above estimate (2.33) and the continuation argument, we finish the prove of Theorem 1.1. □

3 PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2 when \( 0 \leq \mu \leq 2 \). We first deal with the case \( \gamma = 2 \) and later indicate the modification for general case.

Proof. Let \((\rho, u)\) be a \( C^1 \) solution. By the finite propagation property, we have \( \rho - 1 \) supported in \( B(t) = \{x \mid |x| \leq t + R\} \). We define

\[
P(r, t) = \int_{x > r} (x - r)^2 (\rho(x, t) - 1) dx.
\] (3.1)

Using (1.1) and integration by parts, we have

\[
\partial_t P(r, t) = \int_{x > r} (x - r)^2 \rho_t dx
= -\int_{x > r} (x - r)^2 (\rho u)_x dx
= \int_{x > r} 2(x - r)(\rho u) dx.
\]

Then \( P(r, t) \) is \( C^2 \) in \( t \). Differentiating it again, using (1.1) and integration by parts, we have

\[
\partial^2_t P(r, t) = \int_{x > r} 2(x - r)(\rho u)_t dx
= \int_{x > r} 2(x - r) \left( -\partial_x (\rho u^2) - \partial_x p - \frac{\mu}{1 + t} \rho u \right)_t dx
= \int_{x > r} 2\rho u^2 dx + \int_{x > r} 2p dx + \frac{\mu}{1 + t} \int_{x > r} (x - r)^2 (\rho u)_x dx
\]

\[
= \int_{x > r} 2\rho u^2 dx + \int_{x > r} 2p dx - \frac{\mu}{1 + t} \int_{x > r} (x - r)^2 (\rho - 1) dx.
\]

Hence we have
\[
\partial_t^2 P(r, t) + \frac{\mu}{1 + t} \partial_t P(r, t) = \int_{x > r} 2\rho u^2 \, dx + \int_{x > r} 2p \, dx \geq 0.
\]

Due to our initial data assumption (1.3), by integrating the above differential inequality, we have
\[
\partial_t P(r, t) \geq 0 \quad \text{and} \quad P(r, t) > 0.
\]

Now we come to estimate a lower bound for \( P(r, t) \). Rewriting \( \partial_t^2 P(r, t) \) as following.

\[
\partial_t^2 P(r, t) = \int_{x > r} 2(x - r)(\rho u_t) \, dx
\]
\[
= \int_{x > r} 2(x - r) \left( -\partial_x (\rho u^2) - \partial_x p - \frac{\mu}{1 + t} \rho u \right) \, dx
\]
\[
= \int_{x > r} 2\rho u^2 \, dx + \int_{x > r} 2(p - \overline{p}) \, dx + \frac{\mu}{1 + t} \int_{x > r} (x - r)^2 (\rho u)_x \, dx
\]
\[
\geq \int_{x > r} 2(p - \overline{p}) \, dx - \frac{\mu}{1 + t} \int_{x > r} (x - r)^2 (\rho - 1) \, dx,
\]

where \( \overline{p} = p(1) \). Then we have

\[
\partial_t^2 P(r, t) - \partial_t^2 P(r, t) + \frac{\mu}{1 + t} \partial_t P(r, t) \geq \int_{x > r} 2(p - \overline{p}) \, dx - \int_{x > r} 2(\rho - 1) \, dx
\]
\[
= \frac{2}{\gamma} \int_{x > r} [(\rho^\gamma - 1) - \gamma(\rho - 1)] \, dx \\
\triangleq G(r, t).
\]

When \( \gamma = 2 \),
\[
G(r, t) = \int_{x > r} (\rho - 1)^2 \, dx \geq 0.
\]

When \( 0 \leq \mu \leq 2 \), due to the nonnegativity of \( \partial_t P \), we have
\[
\partial_t^2 P(r, t) - \partial_t^2 P(r, t) + \frac{2}{1 + t} \partial_t P(r, t) \geq G(r, t) \quad (3.3)
\]

Define \( W(r, t) = (1 + t)P(r, t) \). From the above inequality, one get
\[
\partial_t^2 W(r, t) - \partial_t^2 W(r, t) \geq (1 + t)G(r, t).
\]

We see
\[
W(r, 0) = \varepsilon q^0(r), \quad (\partial_t W)(r, 0) = \varepsilon (q^0(r) + q^1(r)).
\]

\( ^3 \text{If the damping decays like } \frac{\mu}{(1 + t)^{\lambda}} \text{ where } \lambda > 1, \mu \geq 0 \text{, we can choose a } t_1 \text{ such that when } t \geq t_1, \frac{\mu}{(1 + t)^{\lambda}} \leq \frac{2}{1 + t} \text{. We can still get (3.3).} \)
EULER EQUATIONS WITH TIME-DECAYED DAMPING

Inversion of 1-d d’Alembertian operator gives (for \( r > R_0 + t \))

\[
W(r, t) = W^0(r, t) + \frac{1}{2} \int_0^t \int_{r-(t-\tau)}^{r+t-\tau} \Box W(y, \tau) dy d\tau
\]

\[
\geq W^0(r, t) + \frac{1}{2} \int_0^t \int_{r-(t-\tau)}^{r+t-\tau} (1 + \tau) G(y, \tau) dy d\tau,
\]

where \( W^0(r, t) = \frac{\varepsilon}{2} \left\{ q^0(r + t) + q^0(r - t) + \int_{r-t}^{r+t} (q^0(y) + q^1(y)) dy \right\}. \) (3.6)

Now define

\[
F(t) = \int_0^t (t - \tau) \int_{R_0+\tau}^{R+t} r^{-1} W(r, t) dr d\tau.
\]

We see that

\[
F''(t) = \int_{R_0+t}^{R+t} r^{-1} W(r, t) dr
\]

\[
\geq \int_{R_0+t}^{R+t} r^{-1} W^0(r, t) dr
\]

\[
+ \frac{1}{2} \int_{R_0+t}^{R+t} \int_{r-(t-\tau)}^{r+t-\tau} (1 + \tau) G(y, \tau) dy d\tau dr
\]

\[
= J_1 + J_2.
\] (3.7)

From our assumption (1.3), we have

\[
J_1 \geq \frac{\varepsilon}{2} \int_{R_0+t}^{R+t} r^{-1} q^0(r - t) dr
\]

\[
\geq \varepsilon (R + t)^{-1} \int_{R_0+t}^{R+t} q^0(r - t) dr
\]

\[
\geq \varepsilon (R + t)^{-1} B_0
\]

\[
> 0,
\] (3.8)

where \( B_0 = \frac{1}{2} \int_{R_0}^{R} q^0(r) dr. \)

Exchanging the order of integration in \( J_2 \) and remembering that \( G(y, \tau) \) is supported in \( \{ y \mid |y| \leq \tau + R \} \), we have

\[
J_2 \geq \frac{1}{2} \int_0^t \int_{r-R_0}^{r+R} (1 + \tau) G(y, \tau) \int_{\max[t+R_0, y-(t-\tau)]}^{y+t-\tau} r^{-1} dr dy d\tau.
\]

If we set \( t \geq t_1 = \frac{1}{2}(R - R_0) \), by direct computation, we have

\[
\int_{\max[t+R_0, y-(t-\tau)]}^{y+t-\tau} r^{-1} dr \geq C(t + R)^{-2} (t - \tau)(y - \tau - R_0)^2.
\] (3.9)
Since $G(y, \tau) \geq 0$, we have

$$J_2 \geq C(t + R)^{-2} \int_0^t \int_{\tau+R_0}^{\tau+R} (t - \tau)(y-\tau-R_0)^2(1+\tau)G(y, \tau)dyd\tau, \quad (3.10)$$

when $t > t_1$. We know that $G(y, \tau)$ is supported in $\{y \mid |y| \leq \tau + R\}$ and

$$G(y, \tau) = \partial_y^2 \int_{x>y} (x-y)^2(\rho - 1)^2dx.$$  

Using integration by parts in $(3.10)$, we have

$$J_2 \geq C(t + R)^{-2} \int_0^t (t-\tau) \int_{\tau+R_0}^{\tau+R} (1+\tau)(x-y)^2(\rho - 1)^2dxdyd\tau = C(t + R)^{-2}J_3. \quad (3.11)$$

Recall that

$$F(t) = \int_0^t (t-\tau) \int_{\tau+R_0}^{\tau+R} y^{-1}(1+\tau) \int_{x>y} (x-y)^2(\rho - 1)dxdyd\tau.$$  

Using Cauchy-Schwartz’s inequality, we have

$$F^2(t) \leq J_3 \int_0^t (t-\tau) \int_{\tau+R_0}^{\tau+R} y^{-2} \int_{\tau+R_0}^{\tau+R} (x-y)^2dxdyd\tau$$

$$= J_3J_4. \quad (3.12)$$

We compute $J_4$ as follows

$$J_4 = \frac{1}{3} \int_0^t (t-\tau)(1+\tau) \int_{\tau+R_0}^{\tau+R} y^{-2}(\tau + R - y)^3dyd\tau$$

$$\leq \frac{(R - R_0)^3}{3} \int_0^t (t-\tau)(1+\tau) \int_{\tau+R_0}^{\tau+R} y^{-2}dyd\tau$$

$$\leq C \int_0^t (t-\tau)(1+\tau) \frac{1}{(\tau + R_0)^2}d\tau$$

$$\leq C(t + R) \ln(t + R). \quad (3.13)$$

Combining $(3.11)$, $(3.12)$, $(3.13)$ and $(3.7)$, we get

$$F''(t) \geq C[(t + R)\ln(t + R)]^{-1}F^2(t), \quad t \geq t_1. \quad (3.14)$$

From $(3.7)$, $(3.8)$ and the fact $J_2 \geq 0$, $F'(0) = F(0) = 0$, we have

$$F''(t) \geq \varepsilon B_0(t + R)^{-1}, \quad t \geq 0, \quad (3.15)$$

$$F'(t) \geq \varepsilon B_0 \ln \left( \frac{t + R}{R} \right), \quad t \geq 0, \quad (3.16)$$
\[ F(t) \geq C \varepsilon B_0 (t + R) \ln \left( \frac{t + R}{R} \right), \quad t \geq t_2, \quad (3.17) \]

where \( t_2 = \max \{ t_1, R(e^2 - 1) \} \).

Actually from (3.14), (3.15), (3.16) and (3.17), we can deduce the blow up as in [16]. However for completion of our paper, we sketch the proof in the following. For simplicity, we set \( R = 1 \).

Inserting (3.17) into (3.14), one obtain the improvement for \( F''(t) \)
\[ F''(t) \geq C \varepsilon^2 B_0^2 (t + 1)^{-1} \ln(t + 1), \quad t \geq t_2. \quad (3.18) \]

Integrating (3.18) twice, we have
\[ F(t) \geq C \varepsilon^2 B_0^2 (t + 1) \ln(t + 1)^2, \quad t \geq t_2. \quad (3.19) \]

Inserting (3.19) into (3.14), we have
\[ F''(t) \geq C \varepsilon^2 B_0^2 (t + 1)^{-2} \ln(t + 1) F(t), \quad t \geq t_2. \quad (3.20) \]

Multiplying both sides of (3.20) by \( F'(t) (\geq 0) \), we have
\[ [(F'(t))^2]' \geq C \varepsilon^2 B_0^2 (t + 1)^{-2} \ln(t + 1) [(F(t))^2]' . \quad (3.21) \]

Integrating (3.21) from some \( t_3 \geq t_2 \) to \( t \), we have
\[ (F'(t))^2 \geq (F'(t_3))^2 + C \varepsilon^2 B_0^2 \int_{t_3}^{t} (\tau + 1)^{-2} \ln(\tau + 1) [F^2(\tau)]' d\tau. \]

First choosing \( t_3 \) such that
\[ C \varepsilon^2 B_0^2 \ln(t_3 + 1) = 1. \quad (3.22) \]

Then we have
\[ (F'(t))^2 \geq (F'(t_3))^2 + \frac{1}{\ln(t_3 + 1)} \int_{t_3}^{t} (\tau + 1)^{-2} \ln(\tau + 1) [F^2(\tau)]' d\tau. \]

Using integration by parts, we have
\[ (F'(t))^2 \geq (F'(t_3))^2 + \frac{(t + 1)^{-2} \ln(t + 1)}{\ln(t_3 + 1)} F^2(t) \]
\[ - (t_3 + 1)^{-2} F^2(t_3) \quad - \frac{1}{\ln(t_3 + 1)} \int_{t_3}^{t} F^2(\tau) \frac{1 - 2 \ln(t + 1)}{(t + 1)^3} d\tau \]
\[ \geq \frac{(t + 1)^{-2} \ln(t + 1)}{\ln(t_3 + 1)} F^2(t) \]
\[ + (F'(t_3))^2 \quad - (t_3 + 1)^{-2} F^2(t_3). \quad (3.23) \]
Here we have use the fact when $t \geq t_3$, we have $1 - 2 \ln(t + 1) \leq 0$. Noting that $F'(t)$ is increasing and $F(0) = 0$, we have
\[ F(t_3) \leq t_3 F'(t_3). \] (3.24)
Substituting (3.24) into (3.23), we have
\[ F'(t) \geq C(t + 1)^{-1} (\ln(t + 1))^{\frac{3}{2}} F(t) \quad t \geq t_3. \]
Integrating this from $t_3$ to $t$, one obtain
\[ \ln \frac{F(t)}{F(t_3)} = \frac{2}{3} C (\ln(t + 1))^{3/2} - \frac{2}{3} C (\ln(t_3 + 1))^{3/2}. \]
Choosing $t_4 = 2t_3^2$ and noting (3.19), we have
\[ F(t) \geq C \varepsilon^2 B_0^2 (t + 1)^8, \quad t \geq t_4. \]
Inserting this into (3.14), we get
\[ F''(t) \geq C \varepsilon B_0 F(t)^{3/2}, \quad t \geq t_4. \]
Integrating, as before, the above differential inequality, we get
\[ (F'(t))^2 \geq C \varepsilon B_0 \left( (F(t))^{5/2} - (F(t_4))^{5/2} \right), \quad t \geq t_4. \]
On the other hand, due to the nonnegative of $F'(t)$ and $F''(t)$, we have
\[ F(t) \geq F(t_4) + F'(t_4)(t - t_4) \geq F'(t_4)(t - t_4) \geq F(t_4) \frac{t - t_4}{t_4}. \]
Then choosing $t_5 = 3t_4$, we get
\[ F'(t) \geq C \sqrt{\varepsilon B_0} (F(t))^{5/4}, \quad t \geq t_5. \]
If the lifespan $T_\varepsilon \geq 2t_5$, integrating the above inequality from $t_5$ to $T_\varepsilon$ gives
\[ (F(t_5))^{-1/4} - (F(T_\varepsilon))^{-1/4} \geq C \sqrt{\varepsilon B_0} T_\varepsilon. \] (3.25)
Noting that we have chosen $t_5 = 6t_3^2$ and the inequality (3.19) and (3.22), Then we have
\[ F(t_5) \geq C \varepsilon^2 B_0^2 e^{C \varepsilon B_0}, \]
which combined with (3.25) can deduce that
\[ T_\varepsilon \leq C e^{C \varepsilon B_0}. \]
However if $T_{\epsilon} \leq 2t_5$, then by our choice of $t_5$ and (3.22)
\[ T_{\epsilon} \leq C t_5^2 \leq C e^{\frac{C}{\epsilon^2 t_5^6}}. \]

For the general case, we need to adjust the function $G(r, t)$ in (3.3). Using Taylor’s theorem, we have
\[ (\rho^\gamma - 1) - \gamma(\rho - 1) = \gamma(\gamma - 1) \int_1^{\rho} \tau^{\gamma-2}(\rho - \tau)d\tau. \]

It is easy to see that
\[ \int_1^{\rho} \tau^{\gamma-2}(\rho - \tau)d\tau \geq C(\gamma) \varphi_{\gamma}(\rho), \]
where $C(\gamma)$ is a constant and $\varphi_{\gamma}$ is given by
\[
\varphi_{\gamma}(\rho) = \begin{cases} 
(1 - \rho)^\gamma, & 0 < \rho < \frac{1}{2}, \\
(\rho - 1)^2, & \frac{1}{2} \leq \rho \leq 2, \\
(\rho - 1)^\gamma, & \rho > 2.
\end{cases}
\]

Then
\[ G(r, t) \geq C(\gamma) \int_{x > r} \varphi_{\gamma}(\rho)dx. \]

Young inequalities will be used in (3.13). We still can get similar inequalities as (3.14), (3.15), (3.16) and (3.17) to prove the finite-time blow up, although the upper bound for the lifespan will be a little different. We omit the details.

This finishes the proof of Theorem 1.2.

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