Approximate nonlinear self-adjointness and approximate conservation laws

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Abstract

In this paper, approximate nonlinear self-adjointness for perturbed PDEs is introduced and its properties are studied. Consequently, approximate conservation laws which cannot be obtained by the approximate Noether theorem are constructed by means of the method. As an application, a class of perturbed nonlinear wave equations is considered to illustrate the effectiveness.

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1. Introduction

When the German mathematician Emmy Noether proved her theorem, she established a connection between symmetries and conservation laws of differential equations, provided that the equations under consideration are obtained from a variational principle, i.e. they are Euler–Lagrange equations \[1, 2\]. In order to invoke this powerful theorem, one requires a Lagrangian of the underlying differential equations which cast it as an Euler–Lagrange system \[3\]. It is well known that there is no Lagrangian for scalar evolution equations, such as the classical heat equation, the Burgers equations, etc. \[4\]. In consequence, one cannot associate conservation laws with their symmetries via Noether’s theorem. Thus, given a system without a Lagrangian formulation, one needs a corresponding algorithm to find conservation laws of the system.

In \[5, 6\], Anco and Bluman proposed a direct construction formula of local conservation laws for partial differential equations (PDEs) expressed in a standard Cauchy–Kovalevskaya form. Kara and Mahomed presented a partial Noether approach, which is efficient for Euler–Lagrange-type equations \[7\].

Recently, the general concept of nonlinear self-adjointness \[8, 9\], which includes strict self-adjointness \[10, 11\], quasi-self-adjointness \[12\] and weak self-adjointness \[13\] stated earlier, was introduced to construct conservation laws associated with symmetries of differential equations. The main idea of the method traced back to \[14, 15\] and followed in \[16\] (see exercise 5.37). The method introduced a formal Lagrangian of the system consisting of the
governing equations together with their adjoint equations and then utilized the conservation law theorem in [11] to construct local and nonlocal conservation laws of the PDEs under study.

Owing to the fast development of nonlinear self-adjointness and its subclasses, many important physical PDEs have been studied successfully [8, 17–21]. For example, the required conditions of self-adjointness and quasi-self-adjointness for a class of third-order PDEs were presented in [17]. Gandarias and Bruzón considered conservation laws of a forced KdV equation via weak self-adjointness [18]. The nonlinear self-adjointness of a generalized fifth-order KdV equation was studied in [19]. The authors in [20] showed that a (2+1)-dimensional generalized Burgers equation written as a system of two dependent variables was quasi-self-adjoint. Further examples can be found in [8, 21] and references therein.

Another vital achievement in the past several decades is the emergence of approximate symmetry, which aims to deal with the differential equations with a small parameter possessing few exact symmetries or none at all and even if exist, the small parameter also disturbs the symmetry group properties of the unperturbed equation. Consequently, two reasonably well-known approaches originating from Baikov et al [22] and Fushchich and Shtelen [23] arose, which employed standard perturbation techniques about the symmetry operator and dependent variables to obtain approximate symmetry, respectively. In [24, 25], these two methods were applied to three nonlinear PDEs which showed that the second method was superior to the first one. Systematic methods for obtaining both exact symmetries and first-order approximate symmetries for ordinary differential equations (ODEs) were available in [26].

In the meantime, the theory of approximate conservation laws associated with perturbed differential equations was introduced with regard to approximate Noether symmetries, i.e. symmetries associated with a Lagrangian of the perturbed differential equations [27, 28]. In [29], the authors studied how to construct approximate conservation laws for perturbed PDEs via approximate generalized symmetries. In [30], a basis of approximate conservation laws for perturbed PDEs was discussed. Johnpillai et al [31] showed how to construct approximate conservation laws of approximate Euler-type equations via approximate Noether type symmetry operators associated with partial Lagrangians. Quite recently, the concept of self-adjointness was extended to tackle perturbed PDEs and successfully applied to study two examples to obtain approximate conservation laws [8]. However, the study of the approximate conservation law is still a major subject for both mathematicians and physicists and should be further developed.

The purpose of the paper is to perform a further study of the properties and applications of approximate nonlinear self-adjointness for perturbed PDEs. The outline of the paper is arranged as follows. In section 2, some related basic notions and principles are reviewed and the definition of approximate nonlinear self-adjointness and its properties are given. In section 3, the method is applied to a class of perturbed nonlinear wave equations and approximate conservation laws are constructed. The last section contains a conclusion of our results.

2. Main results

We first recall some basic notions and principles associated with approximate symmetry and nonlinear self-adjointness in subsections 2.1 and 2.2, then give the main results about approximate nonlinear self-adjointness for perturbed PDEs in subsection 2.3.

Consider a system of $m$ PDEs with $r$th order

$$E_\alpha = E_\alpha^0(x, u, u_{(1)}, \ldots, u_{(r)}) + \epsilon E_\alpha^{(1)}(x, u, u_{(1)}, \ldots, u_{(r)}) = 0,$$  

(1)
where $\epsilon$ is a small parameter, $\alpha = 1, \ldots, m$, $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^m)$, $u_\sigma^i = \partial u^i / \partial x^i$, $u_{ij}^\sigma = \partial^2 u^\sigma / \partial x^i \partial x^j$, and $u_{(i)}$ denotes the collection of all $i$th-order partial derivatives of $u$ with respect to $x$, e.g., $u_{(i)} = \{ u^\sigma_i \}$ with $\sigma = 1, \ldots, m$. Note that we will use these symbols and the summation convention for repeated indices throughout the paper if no special notations are added.

The system (1) is called perturbed PDEs, while the system which does not contain the perturbed term $\epsilon E_0^\alpha(x, u, u_{(1)}, \ldots, u_{(r)})$, i.e.

$$E_0^\alpha(x, u, u_{(1)}, \ldots, u_{(r)}) = 0,$$

is called unperturbed PDEs.

The classical method for obtaining exact (Lie point) symmetries admitted by PDEs (2) is to find a one-parameter local transformation group

$$(x^i)^* = x^i + \epsilon \xi^i(x, u) + O(\epsilon^2),$$

$$(u^\sigma)^* = u^\sigma + \epsilon \eta^\sigma(x, u) + O(\epsilon^2),$$

which leaves the system (2) invariant. Lie’s method requires that the infinitesimal generator of transformation (3), i.e. $X = \xi^i(x, u) \partial / \partial x^i + \eta^\sigma(x, u) \partial / \partial u^\sigma$, satisfies Lie’s infinitesimal criterion

$$\text{pr}^k(X | E_0^\alpha) = 0, \quad \text{when } E_0^\alpha = 0,$$

where $\text{pr}^k(X)$ stands for the $k$-order prolongation of $X$ calculated by the well-known prolongation formulae [16, 32]. The infinitesimal, namely $\xi^i, \eta^\sigma$, can be found from an over-determined linear system generated by condition (4). We refer to references [32, 16] for details.

2.1. Approximate symmetry

Until now, there exist two methods to obtain an approximate symmetry of perturbed PDEs.

First, we introduce the method originating from Fushchich and Shtelen. This method employs a perturbation of dependent variables; that is, expanding the dependent variable with respect to the small parameter $\epsilon$ yields

$$u^\sigma = \sum_{k=0}^\infty \epsilon^k u_k^\sigma, \quad 0 < \epsilon \ll 1,$$

where $u_k^\sigma$ are new introduced dependent variables. After inserting expansion (5) into the system (1), approximate symmetry is defined as the exact symmetry of the system corresponding to each order in the small parameter $\epsilon$. We refer to [23] for further details.

The second approach, initiated by Baikov et al, is not a perturbation of the dependent variables but a perturbation of the symmetry generator [22].

A first-order approximate symmetry of the system (1), with the infinitesimal operator form $X = X_0 + \epsilon X_1$, is obtained by solving for $X_1$ in

$$X_1(E_0^\alpha)_{\mid_{\Delta=0}} + H = 0,$$

where the auxiliary function $H$ is obtained by

$$H = \frac{1}{\epsilon} X_0(E_0^\alpha)_{\mid_{\Delta=0}}.$$

$X_0$ is an exact symmetry of unperturbed PDEs $E_0^\alpha = 0$. The notation $\mid_{\Delta=0}$, hereinafter, means evaluation on the solution manifold of $\Delta = 0$.

We formulate the second method as follows.
Definition 1 (Approximate symmetry [22]). A first-order approximate symmetry with an infinitesimal operator $X = X_0 + \epsilon X_1$ leaves the system (1) approximate invariant if $X_0$ is an exact symmetry of unperturbed PDEs $E_u^0 = 0$ and $X_1$ is defined by (6).

The first method by Fushchich and Shtelen uses only the standard Lie algorithm and can be implemented in the computer algebra system; then, this approximate symmetry approach may readily be extended to determine infinite-dimensional and other types of approximate symmetries [33]. As for the second method, since the dependent variables are not expanded in a perturbation series, approximate solutions obtained by using a first-order approximate generator may contain higher order terms [24].

2.2. Nonlinear self-adjointness

In this subsection, we briefly recall the main idea of nonlinear self-adjointness of PDEs in order to induce approximate nonlinear self-adjointness.

Let $\mathcal{L}$ be the formal Lagrangian of the system (2) given by

$$\mathcal{L} = v^\beta E_0^\beta (x, u, u_1, \ldots, u_r);$$

then, the adjoint equations of the system (2) are defined by

$$(E_u^0)^* (x, u, v, u_1, \ldots, u_r) = \frac{\delta \mathcal{L}}{\delta u^\sigma} = 0,$$

where $v = (v^1, \ldots, v^m)$ and $u^{(i)}$ represents all $i$th-order derivatives of $v$ with respect to $x$; $\delta / \delta u^\sigma$ is the variational derivative written as

$$\frac{\delta}{\delta u^\sigma} = \frac{\partial}{\partial u^\sigma} + \sum_{s=1}^{\infty} (-1)^s D_i \ldots D_i \frac{\partial}{\partial u_{i_1} \ldots i_k},$$

where, hereinafter, $D_i$ denotes the total derivative operators with respect to $x^i$. For example, a dependent variable $w = w(y, z)$ with $y = x^1, z = x^2$, one has

$$D_y = \frac{\partial}{\partial y} + w_y \frac{\partial}{\partial w} + w_{yy} \frac{\partial}{\partial w_y} + w_{yy} \frac{\partial}{\partial w_{y}} + \ldots,$$

e tc.

In what follows, we recall the definition of nonlinear self-adjointness of differential equations.

Definition 2 (Nonlinear self-adjointness [8]). The system (2) is said to be nonlinearly self-adjoint if the adjoint system (8) is satisfied for all solutions $u$ of system (2) upon a substitution $v^\sigma = \varphi^\sigma (x, u)$, $\sigma = 1, \ldots, m$, (9)

such that $\varphi(x, u) = (\varphi^1, \ldots, \varphi^m) \neq 0$.

The substitution (9) satisfying $\varphi(x, u) \neq 0$ solves the adjoint equations (8) for all solutions of system (2), which can be regarded as an equivalent definition of nonlinear self-adjointness [8, 9]. Definition 2 is also equivalent to the following identities holding for the undetermined coefficients $\lambda^\beta_u$:

$$(E_u^0)^* (x, u, v, u_1, \ldots, u_r) = \lambda^\beta_u E_0^\beta (x, u, u_1, \ldots, u_r),$$

which is applicable in the computations. Hereinafter, $\varphi^{(i)}$, similar to $u^{(i)}$ and $u^{(i)}$, stands for all $i$th-order partial derivatives of $\varphi$ with respect to $x$. 

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Remark 1.

1. If the substitution (9) becomes $v^\sigma = u^\sigma$, then the system (2) is called strict self-adjointness.
   If $v^\sigma = \varphi^\sigma(u)$ is independent of $x$, then it is named quasi-self-adjointness. If $v^\sigma = \varphi^\sigma(x, u)$ involving all $x$ and $u$, then it is called weak self-adjointness.

2. The substitution (9) can also be extended to the case $v^\sigma = \varphi^\sigma(x, u, u^i)$, which embraces the derivatives of $u$ and is called differential substitution.

Obviously, the concept of quasi-self-adjointness and weak self-adjointness generalizes strict self-adjointness. Next, we consider an example about quasi-self-adjointness, while readers can find examples for weak self-adjointness in [13]. Consider a nonlinear PDE studied in [9]

$$u_t - u^2 u_{xx} = 0,$$

which describes the nonlinear heat conduction in solid hydrogen [10]. Let the formal Lagrangian $L = v(u_t - u^2 u_{xx})$; then by means of (8), its adjoint equation is

$$\frac{\delta L}{\delta u} = v_t + 4uu_{xx} + u^2 v_{xx} + 4uu_t v_t + 2uu_x^2 = 0,$$

which is identical to equation (11) by the substitution $v = u^{-2}$, not by $v = u$. It means that equation (11) is quasi-self-adjoint but not strictly self-adjoint.

The following theorem will be used to construct conservation laws for both unperturbed and perturbed cases [11].

**Theorem 1.** Any infinitesimal symmetry (local or nonlocal)

$$X = \xi^i(x, u, u_1, \ldots) \frac{\partial}{\partial x^i} + \eta^\sigma(x, u, u_1, \ldots) \frac{\partial}{\partial u^\sigma}$$

of the system (2) leads to a conservation law $D_i(C_i) = 0$ constructed by the formula

$$C_i = \xi^i L + W^\sigma \left[ \frac{\partial L}{\partial u_i^\sigma} - D_j \left( \frac{\partial L}{\partial u_{ij}^\sigma} \right) + D_kD_l \left( \frac{\partial L}{\partial u_{ijkl}^\sigma} \right) - \cdots \right]$$

$$+ D_j(W^\sigma) \left[ \frac{\partial L}{\partial u_{ij}^\sigma} - D_k \left( \frac{\partial L}{\partial u_{ijk}^\sigma} \right) + \cdots \right] + D_kD_l(W^\sigma) \left[ \frac{\partial L}{\partial u_{ijkl}^\sigma} - \cdots \right] + \ldots,$$

where $W^\sigma = \eta^\sigma - \xi^i u_i^\sigma$ and $L$ is the formal Lagrangian. In applying the formula, the formal Lagrangian $L$ should be written in the symmetric form with respect to all mixed derivatives $u_{ij}^\sigma, u_{ijk}^\sigma, \ldots$.

Generally speaking, the term $\xi^i L$ with $L$ in the form (7) can be omitted because it vanishes identically on the solution manifold of the studying PDEs.

In particular, a first-order approximate conserved vector $C = (C^1, \ldots, C^n)$ of the system (1) satisfies

$$D_i(C_i) = O(\epsilon^2)$$

for all solutions of $E_{\alpha} = 0$.

2.3. Approximate nonlinear self-adjointness

This subsection will concentrate on the study of approximate nonlinear self-adjointness of perturbed PDEs. The formal Lagrangian $\tilde{L}$ of the perturbed system (1) is given by

$$\tilde{L} = v^\beta \left[ E_0^\beta(x, u, u_1, \ldots, u_r) + \epsilon E_1^\beta(x, u, u_1, \ldots, u_r) \right];$$
Theorem 2. If the adjoint system (8) exists solutions in the form

then the adjoint equations of system (1) are written as

\[ E_v^\sigma(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)}) = \frac{\delta \tilde{L}}{\delta u^\sigma} = 0. \tag{13} \]

Definition 3 (Approximate nonlinear self-adjointness). The perturbed system (1) is called approximate nonlinear self-adjointness if the adjoint system (13) is approximate satisfied for all solutions \( u \) of system (1) upon a substitution

\[ v^\sigma = \psi^\sigma(x, u) + \epsilon \phi^\sigma(x, u), \quad \sigma = 1, \ldots, m, \tag{14} \]

such that not all \( \psi^\sigma \) and \( \phi^\sigma \) are identically equal to zero.

Remark 2. Denote \( \phi(x, u) = (\phi^1, \ldots, \phi^m) \) and \( \psi(x, u) = (\psi^1, \ldots, \psi^m) \). \( \phi_{(i)} \) stands for the same meaning as \( \psi_{(i)} \).

1. The required condition in definition 3 means that the adjoint equations of system (1) work out

\[ E^\sigma_v(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)})|_{v = u + \epsilon \phi(u)} = 0, \]

\[ \left( \lambda^\beta_0 + \epsilon \mu^\beta_0 \right) E^\sigma_\beta(x, u, u_{(1)}, \ldots, u_{(r)}) + \epsilon \lambda^\beta_0 E^\sigma_\beta(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \tag{15} \]

with undetermined parameters \( \lambda^\beta_0 \) and \( \mu^\beta_0 \). Equality (15) provides a computable formula to discriminate approximate nonlinear self-adjointness for perturbed PDEs.

2. Similarly to the unperturbed case, we can define approximate strict self-adjointness with \( v = u + \epsilon u \) (or \( u + \epsilon u \)), approximate quasi-self-adjointness with \( v = \psi(u) + \epsilon \phi(u) \) (or \( \psi(u) \) or \( \phi(u) \)) and approximate weak self-adjointness with \( v = \psi(x, u) + \epsilon \phi(x, u) \) (or \( \psi(x, u) \) or \( \phi(x, u) \)) containing all \( x \) and \( u \). Approximate differential substitution also holds if \( v \) contains the derivatives of \( u \).

3. If the substitution (14) does not exist for the system (1), i.e. system (1) is not approximately nonlinearly self-adjoint, then the resulting conserved vectors will be nonlocal in the sense that they involve the introduced variable \( v \) connected with the physical variable \( u \) via adjoint equations (13).

In what follows, we present some properties of approximate nonlinear self-adjointness.

Theorem 2. If the adjoint system (8) exists solutions in the form \( v^\sigma = \epsilon f^\sigma(x, u) \) with some functions \( f^\sigma(x, u) \), then the system (1) is approximately nonlinearly self-adjoint.

Proof. Observe that \( \tilde{L} = v^\beta E^\beta_0 + \epsilon v^\beta E^\beta_1 \). If the substitution for approximate nonlinear self-adjointness of system (1) is in the form \( v = \epsilon \phi(x, u) \), then by equivalent equality (15) of definition 3, \( \tilde{L} \) is simplified to \( \tilde{L} = v^\beta E^\beta_0 \) because the second part \( \epsilon v^\beta E^\beta_1 \) is second order of \( \epsilon \). Thus in this case, the adjoint equations of system (1) become \( \delta \tilde{L}/\delta u^\sigma = \delta (v^\beta E^\beta_0)/\delta u^\sigma = 0 \), which has the same form as the adjoint equation of system (2), so the solutions \( v^\sigma = \epsilon f^\sigma(x, u) \) of the adjoint system (8) also satisfy \( \delta \tilde{L}/\delta u^\sigma = 0 \). It means that \( v^\sigma = \epsilon f^\sigma(x, u) \) is just the
required substitution which makes the system (1) show approximate nonlinear self-adjointness. This proves the result.

For instance, consider a perturbed nonlinear wave equation \( F = u_t - u_{xx} + \epsilon u_t = 0 \) with formal Lagrangian \( \tilde{L} = v(u_t - u_{xx} + \epsilon u_t) \), the adjoint equation is \( F^* = \delta \tilde{L}/\delta u = v_t - v_{xx} - \epsilon v_t = 0 \). The adjoint equation of the unperturbed equation \( u_t - u_{xx} = 0 \) is \( v_t - v_{xx} = 0 \) which has a solution in the form \( v = \epsilon (c_1 x + c_2) \), where not all arbitrary constants \( c_i \) are zero. Then by theorem 2, this solution makes \( F^* = -\epsilon^2 \mu (c_1 x + c_2) \); thus, the equation \( F = 0 \) is approximately nonlinearly self-adjoint.

In particular, for linear perturbed PDEs, we have the following results.

**Corollary.** Any system of linear perturbed PDEs is approximately nonlinearly self-adjoint.

The corollary is a parallel result as unperturbed case [8] and can be shown with an almost parallel method; thus, we take an example to demonstrate it. Consider the perturbed linear wave equation \( u_t - u_{xx} + \epsilon u_t = 0 \) whose formal Lagrangian is \( \tilde{L} = v(u_t - u_{xx} + \epsilon u_t) \); then its adjoint equation is \( v_t - v_{xx} - \epsilon v_t = 0 \) which is independent of \( u \), and thus any nontrivial solution is a substitution to make \( u_t - u_{xx} + \epsilon u_t = 0 \) show approximate nonlinear self-adjointness.

For the case of one dependent variable of the system (1), namely \( u \), we have the following results.

**Theorem 3.** Equation (1) is approximately nonlinearly self-adjoint if and only if it becomes approximately strictly self-adjoint after being multiplied by an appropriate multiplier \( \mu(x, u) + \epsilon v(x, u) \).

**Proof.** Suppose equation (1) with one dependent variable written as \( E_1 = E_0^0 + \epsilon E_1^1 = 0 \) is approximately nonlinearly self-adjoint, for the substitution \( v = \phi + \epsilon \phi \), one has

\[
\frac{\delta (\epsilon \phi)}{\delta u} \bigg|_{\phi=\phi} = (\lambda_0 + \epsilon \lambda_1) E_1^1,
\]

which is equivalent to the following system after separating it with respect to \( \epsilon \):

\[
\frac{\delta (\epsilon \phi)}{\delta u} = \lambda_0 E_1^0, \quad \frac{\delta (\epsilon \phi)}{\delta u} = \lambda_1 E_1^1 + \lambda_0 E_1^1.
\]

Note that hereinafter in all equalities we neglect the terms of order \( O(\epsilon^2) \).

On the other hand, the conditions for approximate strict self-adjointness and the variational derivative associated with equations \( E_1 = 0 \) yield

\[
\frac{\delta [\mu(x, u) + \epsilon v(x, u)] E_1^1}{\delta u} = (\tilde{\lambda}_0 + \tilde{\lambda}_1) E_1^1.
\]

Inserting the dependent variable \( \omega = \omega_0 + \epsilon \omega_1 \) into (17) and splitting it with a different order of \( \epsilon \), for \( \epsilon^0 \), we have

\[
\frac{\delta (\omega_0 \mu E_1^0)}{\delta u} = \omega_0 \frac{\partial E_1^0}{\partial u} + \mu \omega_0 \frac{\partial (E_1^0)}{\partial u} - D_{ij} \mu \omega_0 \frac{\partial (E_1^0)}{\partial u_i} + D_{ij} \mu \omega_0 \frac{\partial (E_1^0)}{\partial u_j} + \cdots
\]

\[
= \omega_0 \frac{\partial E_1^0}{\partial u} + \frac{\delta (\epsilon \phi)}{\delta u}
\]

\[
= \tilde{\lambda}_0 E_1^1,
\]

where, in equation (18), we regard \( \omega_0 \) as a dependent variable in the first equality, while in the second equality, \( \phi = \omega_0 \mu(x, u) \) is taken as a new dependent variable instead of \( \omega_0 \) to obtain the second term. With the condition \( \omega = u + \epsilon u \), one has

\[
\frac{\delta (\omega_0 \mu E_1^0)}{\delta u} \bigg|_{\phi=\phi} = \mu \frac{\partial E_1^0}{\partial u} + \frac{\delta (\epsilon \phi)}{\delta u} \bigg|_{\phi=\phi} = \tilde{\lambda}_0 E_1^1.
\]
Similarly, for $\epsilon^1$, one has
\[
\frac{\delta}{\delta u} \left[ \omega_0 \mu E_1 + (\omega_0 + \mu \omega_1) E_1^0 \right] = \left. \frac{\partial \mu}{\partial u} E_1 + \frac{\partial (\mu + v)}{\partial u} E_1^0 \right|_{\omega_0 \mu = \omega_1} + \frac{\delta (\phi E_1^1 + \phi E_1^0)}{\delta u} \left|_{\omega_0 \mu = \omega_1} \right.
\]
\[
= \tilde{\lambda}_1 E_1^0 + \tilde{\lambda}_0 E_1^1,
\]
where $\phi = \omega_0 v(x, u) + \omega_1 \mu(x, u)$ is a new dependent variable.

Assume that the equation $E_1 = 0$ is approximately nonlinearly self-adjoint; then we have the multiplier $\mu + \epsilon v = \phi'/u + \epsilon (\phi - \phi)/u$. With the help of (16), (19) and (20), equation (17) becomes
\[
\frac{\delta}{\delta u} \left[ \omega \left( \mu(x, u) + \epsilon v(x, u) \right) E_1 \right] = \left. \frac{\partial \mu}{\partial u} E_1 + \frac{\partial (\mu + v)}{\partial u} E_1^0 \right|_{\omega \mu = \omega_1} + \frac{\delta (\phi E_1^1 + \phi E_1^0)}{\delta u} \left|_{\omega \mu = \omega_1} \right.
\]
\[
= \tilde{\lambda}_0 E_1^0 + \tilde{\lambda}_1 E_1^1 + \tilde{\lambda}_0 E_1^1;
\]
thus
\[
\tilde{\lambda}_0 = \lambda_0 + \frac{\partial \gamma}{\partial u} - \frac{\phi}{u}, \quad \tilde{\lambda}_1 = \lambda_1 + \frac{\partial \phi}{\partial u} - \frac{\phi}{u}.
\]
Hence, the equation $E_1 = 0$ multiplied by the multiplier $\mu + \epsilon v$ is approximately strictly self-adjoint.

Conversely, let $E_1 = 0$ with the multiplier $\mu + \epsilon v$ be approximately strictly self-adjoint; then taking $\phi = u\mu, \phi = u(\mu + v)$, equation (16) becomes
\[
\frac{\delta (\phi E_1^1)}{\delta u} = \left[ \tilde{\lambda}_0 - u \frac{\partial \mu}{\partial u} \right] E_1^0,
\]
\[
\frac{\delta (\phi E_1^1 + \phi E_1^0)}{\delta u} = \left[ \tilde{\lambda}_0 - u \frac{\partial \mu}{\partial u} \right] E_1^1 + \left[ \tilde{\lambda}_0 - u \frac{\partial (\mu + v)}{\partial u} \right] E_1^0.
\]
Alternatively,
\[
\frac{\delta (v E_1)}{\delta u} = \left[ \tilde{\lambda}_0 - u \frac{\partial \mu}{\partial u} \right] E_1^0 + \epsilon \left[ \tilde{\lambda}_0 - u \frac{\partial \mu}{\partial u} \right] E_1^1 + \epsilon \left[ \tilde{\lambda}_0 - u \frac{\partial (\mu + v)}{\partial u} \right] E_1^0.
\]
then
\[
\lambda_0 = \tilde{\lambda}_0 - u \frac{\partial \mu}{\partial u}, \quad \lambda_1 = \tilde{\lambda}_0 - u \frac{\partial (\mu + v)}{\partial u}.
\]
We conclude that the equation $E_1 = 0$ is approximately nonlinearly self-adjoint; thus completing the proof. $\square$

**Remarks 3.**

1. Theorem 3 holds for some special cases of $\lambda_i, \tilde{\lambda}_i$ ($i = 0, 1$). For example, if $\lambda_1 = \tilde{\lambda}_1 = \phi = 0$, then we find that the substitution is given by $v = \phi$ and the multiplier becomes $\mu = \phi'/u$, which has the same results as the unperturbed case [8].

2. If the substitution for approximate strict self-adjointness is adopted by $v = u$, i.e. $\omega_0 = u, \omega_1 = 0$ in the proof of theorem 3, then we have that the multiplier $\mu + \epsilon v$ takes the form $(\phi + \epsilon \phi)/u$. Similarly, if the substitution is $v = \epsilon u$, then we have the multiplier $\mu + \epsilon v = \epsilon \phi/u$. 

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3. Applications

In this section, we apply approximate nonlinear self-adjointness to construct approximate conservation laws of a class of perturbed nonlinear wave equations

\[ u_t - [F(u)u_x]_x + \epsilon u_t = 0, \quad F'(u) \neq 0, \]  

(21)

where \( F(u) \) is an arbitrary smooth function. Equation (21) describes wave phenomena in shallow water, long radio engineering lines and isentropic motion of a fluid in a pipe etc. [22, 33]. The perturbing term \( \epsilon u_t \) arises in the presence of dissipation and the function \( F(u) \) is defined by the properties of the medium and the character of the dissipation.

Equation (21) was studied by means of the stated two approximate symmetry methods and affluent approximate solutions were obtained [22, 33]. For equation (21), we take the following formal Lagrangian:

\[ \mathcal{L} = v[u_t - [F(u)u_x]_x + \epsilon u_t], \]

(22)

and work out the variational derivative of this formal Lagrangian to obtain the system of two coupled equations

\[ \frac{\delta \mathcal{L}}{\delta v} = u_t - [F(u)u_x]_x + \epsilon u_t = 0, \]

\[ \frac{\delta \mathcal{L}}{\delta u} = v_t - F(u)v_{xx} - \epsilon v_t = 0, \]

(23)

where the second equation is called the adjoint equation of equation (21).

3.1. Approximate nonlinear self-adjointness

With the help of computable formula (15), we prove the following proposition for equation (21).

**Proposition 1.** Equation (21) is approximately nonlinearly self-adjoint under the substitution \( v = (c_1t + c_2)x + c_3t + c_4 + \epsilon \left[ \left( \frac{1}{2}c_1t^2 + c_5t + c_6 \right)x + \frac{1}{2}c_7t^2 + c_7t + c_8 \right] \),

where \( c_i (i = 1, \ldots, 8) \) are arbitrary constants such that \( v \neq 0 \).

**Proof.** Assuming that \( v = \varphi(x, t, u) + \epsilon \phi(x, t, u) \) and substituting it into the adjoint equation, one obtains

\[ \varphi_t + 2\varphi_x u_t + \varphi_{xx} u_{xx} + \varphi_{uu} u_x + \epsilon \left( \varphi_t + 2\varphi_x u_t + \varphi_{xx} u_{xx} + \varphi_{uu} u_x - \varphi_t - \varphi_x u_t \right) - F(u) \left[ \varphi_{xx} + 2\varphi_x u_x + \varphi_{uu} u_x + \varphi_{xx} u_{xx} + \epsilon \left( \varphi_{xx} + 2\varphi_x u_x + \varphi_{uu} u_x + \varphi_{xx} u_{xx} \right) \right] = (\lambda_0 + \epsilon \lambda_1) \left[ u_{tt} - F'(u)u_t^2 - F(u)u_{xx} \right] + \epsilon \lambda_0 u_t, \]

(25)

where we omit the second-order terms of \( \epsilon \) in equation (25).

Comparing the coefficients for \( u_t, u_{xx}, u_x, u_t^2 \) on both sides, we obtain \( \varphi_{tt} = \varphi_t = 0 \) and \( \lambda_0 = \lambda_1 = 0 \); then the above equation (25) becomes \( \varphi_t + \epsilon (\varphi_t - \varphi_t) - F(u)(\varphi_{xx} + \epsilon \varphi_{xx}) = 0 \) and yields

\[ \varphi_{tt} = 0, \quad \phi_t = \varphi_t = 0, \quad \varphi_{xx} = 0, \quad \phi_{tt} = 0, \]

which gives the solutions \( \varphi = (c_1t + c_2)x + c_3t + c_4, \phi = (\frac{1}{2}c_1t^2 + c_5t + c_6)x + \frac{1}{2}c_7t^2 + c_7t + c_8 \).

This completes the proof.

Proposition 1 provides many choices for multipliers to make equation (21) become approximately strictly self-adjoint. For example, after assigning proper values to some parameters, we have \( v = 1 + \epsilon \). Multiplying it on the left-hand side of equation (21), by
means of theorem 3, we obtain $\mu(x, t) = 1/u$, $v(x, t) = 0$; then equation (21) multiplied by it becomes

$$
\frac{1}{u} [u_t - \mu u_x, u_v + \epsilon u_t] = 0,
$$

(26)

which is approximately strictly self-adjoint because, at this time, $L = v[u_t - \mu u_x, u_v + \epsilon u_t]/u$ and the adjoint equation of equation (26) is

$$
\frac{\delta L}{\delta u} = \frac{1}{u^2} \left[ u^2 v_t - 2uv_t v - 2uvu_t + 2uv^2 - \mu^2 v_t \right] + \frac{1}{u} [uF'(u) - 2F(u)] v u_t + \frac{1}{u} [2u_t v + 2uv_{tt} - uv_{tt}] F(u) = 0,
$$

which becomes equation (26) by the substitution $v = u$. Alternatively, one can adopt $\mu = v = 1/u$ to make equation (21) show approximate strict self-adjointness.

3.2. Approximate conservation laws

Now we turn to construct the approximate conservation laws of equation (21). The first step of the approach is to perform an approximate symmetry classification of equation (21). The exact symmetry of unperturbed equations

$$
u_t = [F(u) u_x, u_v],
$$

(27)

is well known [35]. The maximal Lie algebra is generated by a three-dimensional algebra and three special cases correspond to four- or five-dimensional Lie algebra. The results are reduced to those cases in table 1 by the equivalence transformation

$$
\tilde{e}_1 = e_1 x + e_3, \quad \tilde{e}_2 = e_3 y + e_4, \quad \tilde{e}_3 = e_5 u + e_6,
$$

where $e_i$ ($i = 1, \ldots, 6$) with $e_1 e_3 e_5 \neq 0$ being arbitrary constants.

Table 2 gives approximate symmetries obtained by the approach of Baikov et al [22].

In what follows, we will use formula (12) in theorem 1 to obtain first-order approximate conservation laws of equation (21)

$$
[D, \{C^1\} + D, \{C^2\}]_{(21)} = O(\epsilon^2),
$$

by virtue of the approximate symmetries in table 2.
Inserting the formal Lagrangian (22) into formula (12), we obtain
\[
C^1 = W(ev - v_t) + vD_v(W),
\]
\[
C^2 = W(F(u)v_t - F'(u)u_tv) - D_v(W)F(u)v.
\]

In particular, we investigate the following two cases to illustrate the effectiveness of approximate nonlinear self-adjointness in constructing approximate conservation laws, while other cases can be done by formula (28) with a similar procedure.

**Example 1.** Now, we utilize operator \( \tilde{X}_4 \) in table 2 to calculate the conserved vector. In this case, \( W = -2\epsilon t - xu_x - (t + \frac{1}{2}\epsilon t^2)u \) and \( F(u) = e^u \); then the conserved vector (28) becomes
\[
C^1 = t e^u u_x + xu_x + tu_t + xu_t v_x + \frac{1}{2} \epsilon (t^2 u_x + 4 v_t + t^2 e^u u_x - 2tu_t v - 2xu_x v - 4v),
\]
\[
C^2 = -t e^u u_x + xu_x - t e^u u_v - x e^u u_x + \epsilon \left( xu_t v + t e^u u_v - 2t e^u(tu_t u + tu_v u + 4 v) \right),
\]
where \( v \) is given by the substitution \( \epsilon = -t u_x - 2v \).

In particular, we take \( v = x \), then (29) becomes
\[
C^1_1 = xu_t + t e^u u_x + \epsilon \left( \frac{1}{2} t^2 e^u u_x - x^2 u_x - txu_u \right),
\]
\[
C^2_1 = -t e^u u_x - x e^u u_x + \epsilon \left( -\frac{1}{2} t^2 e^u u_x + x^2 u_x + tx e^u u_x - 2t e^u \right),
\]
which make
\[
[D_\mu(C_1^1) + D_\mu(C_2^1)]_{|21} = e^2 (tu_t - 2).
\]

**Example 2.** Consider \( \tilde{X}_6 \) for \( F(u) = u^\mu (\mu \neq -4, -\frac{4}{3}) \). Here, \( W = -\frac{2}{\mu} u - 2xu_x - (t + \frac{\epsilon t^2}{2(\mu + 1)}) u \); then the conserved vector (28) becomes
\[
C^1 = tu_v u_{\mu v} + tu_{v v} - tu_{x v} - 2 \frac{\mu}{2}(u_t v - 2uv_v)
\]
\[
+ \frac{\epsilon}{2\mu(\mu + 4)} (\mu^2 t^2 u_x v_{\mu v} + \mu^2 t^2 u_x v - 2\mu (\mu + 2)tu_{v v} - 8(\mu + 2)uv + 4\mu t u v_v),
\]
\[
C^2 = u^\mu u_v u_t - tu^\mu u_{v v} - tu^\mu u_{x v} + 2 \frac{\mu}{2}(u_t v - v_u u) u^\mu
\]
\[
- \frac{\epsilon t u^\mu}{2(\mu + 4)} (\mu tu_{v v} + \mu tu_{x v} + 4uv_v - 2(\mu + 2)u_v v),
\]
where \( v \) is also given by the substitution \( \epsilon = -tu_t - 2v \).

In particular, choosing \( v = x + \epsilon t \), we have
\[
C^1_2 = tu^\mu u_x = \left( 2 \frac{\mu}{\mu + 1} \right) xu_u
\]
\[
+ \frac{\epsilon}{2\mu(\mu + 4)} (\mu^2 t^2 u^\mu x - 2(\mu^2 x + 2\mu (x + 1) + 8)u_t - 4(2\mu\tau x - \mu + 4x - 4)u),
\]
\[
C^2_2 = xu^\mu u_t - tu^\mu u_x + \frac{2}{\mu} (xu_u - u) u^\mu
\]
\[
- \frac{\epsilon t u^\mu}{2\mu(\mu + 4)} (\mu^2 tu_t u - 2\mu^2 xu_x - 4(\mu + 1)u_x - 16u_x + 4\mu u).
and

\[
\left[ D_t (C_1^2) + D_x (C_2^2) \right]_{(21)}
\]

\[= - \frac{\epsilon^2}{2\mu (\mu + 4)} (2\mu^2 t u_t - \mu^2 (2x + \epsilon t) u_t + 8\mu t u_x + 4\mu u (2\epsilon + x + 1) + 16u).\]

4. Conclusion

We provide a more detailed investigation of approximate nonlinear self-adjointness and related properties, which extends the results in both the unperturbed PDEs case and the perturbed ODEs case. The results are applied to a class of perturbed nonlinear wave equations and approximate conservation laws are obtained.

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