Cutting and slicing weak solids

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Cutting soft solids with a sharp knife is quicker and smoother if the blade is sliding rapidly parallel to its edge in addition to the normal squeezing motion. We explain this common observation with a consistent theory suited for soft gels and departing from the standard theories of elastic fracture mechanics developed for a century. The gel is assumed to locally fails when submitted to stresses exceeding a threshold $\sigma_1$. The changes in its structure generate a liquid layer coating the blade and transmitting the stress through viscous forces. The driving parameters are the ratio $U/W$ of the normal to the tangential velocity of the blade, and the characteristic length $\eta W/\sigma_1$, with $\eta$ the viscosity of the liquid. The existence of a maximal value of $U/W$ for a steady regime explains the crucial role of the tangential velocity for slicing biological and other soft materials.

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Cutting soft materials has been done forever and is nowadays encountered in a large range of processes, from precision surgery \textsuperscript{[1]}, histology \textsuperscript{[2]}, to food industry \textsuperscript{[3],[4]}, and also in everyday life where it is commonly observed that cutting cheese or meat is made much easier by sliding rapidly the knife \textsuperscript{[5]}. Pulling a sharp knife on an elastic material yields a very large stress at the tip just because a finite force (per unit length along the edge) is exerted at the contact line. A supplementary sliding of the knife along this contact line produces an additional shear stress. The resulting tensile stress is far more efficient than the compressive stress produced by the single normal displacement of the knife. Hence, cutting soft materials is facilitated by the sliding, requiring a much lower normal force compared to a simple compression \textsuperscript{[6]}. Beyond this base explanation, the origin of a critical tensile stress needed for cutting a soft gel, as well as the effect of the tangential velocity of the blade, have to be elucidated. The explanation to this intriguing question necessarily comes from the specific properties shared by the class of material considered here, \textit{i.e.} soft gels. It must be consistent with the large deformations these materials can usually withstand before being cut. In addition, due to the low values of the stresses involved during the failure of these weak solids, macroscopic length-scales emerges, for instance for the energy dissipation zone \textsuperscript{[7],[8]}. Hence, a consistent description of the dissipative zone is required to properly describe the mechanics of the cutting. Indeed, the effect of the velocity of the blade is likely to be closely related to the rate of energy dissipation. For these reasons, the physics of slicing soft gels with a sharp wedge has little chance to be well captured by the overspread theory of the linear elastic fracture mechanics, or even its extension to the large deformations \textsuperscript{[9],[10]}. Here, we give a coherent explanation to the role of the tangential velocity of the blade during the slicing of a soft gel, based on a self consistent theory. The basis of the idea is as follows. We think to a model material which can withstand stress until a given maximum value, called $\sigma_1$ and defined more accurately later. For stresses lower than this maximum, the material under consideration can return reversibly to its unperturbed state when unstressed. For stresses larger than $\sigma_1$, the material melts, like a solid above the melting temperature. This is a reasonable assumption for gels: gels are made of a network filling a liquid (usually water). Once the links between a big enough proportion of links are broken, the network disappears and the gel becomes a liquid suspension of small particles, without the cohesiveness of a solid. Obviously this does not take into account a likely transition from solid to liquid through a kind of intermediate state, which certainly takes place but we assume that the thickness of the transition region is small enough to be neglected, at least in a first approximation. Moreover to simplify the matter we shall consider steady situations only. The material can melt reversibly or irreversibly. If it does it reversibly, it heals once the stress decreases below $\sigma_1$, recovering its elastic properties \textsuperscript{[11],[12]}. If not, it remains permanently melted whatever the forthcoming changes of the stress \textsuperscript{[12]}. It is also possible that the gel cannot stand stress beyond a critical value without expelling some of its liquid part by permeation \textsuperscript{[13]}. Most likely the two phenomena (melting and permeation) occur simultaneously in a stressed gel. The different possible mechanisms inducing the formation of the liquid layer does not make a difference with the cutting mechanism considered here, and so we consider these possibilities as a single one.

The physical explanation of why it is easier to cut soft materials by sliding rapidly the knife parallel to itself follows from our assumption that the material to be cut cannot stand stresses beyond a critical value. Such a stress is transmitted from the knife to the gel by a thin viscous layer of liquid made either by the broken gel or/and by expelled liquid (Fig.\textsuperscript{[1]}). This stress depends on viscosity, knife velocity, and on the geometry of the layer. It
includes a component coming from the forward cutting motion of the knife toward it, but also a component due to the sliding motion, usually at a far bigger velocity than in the cutting direction.

FIG. 1: (a) Sketch of a blade (in red) moving at velocity $U$ in the cutting (normal) direction $x$, and velocity $W$ in the slicing direction $z$ (parallel to the initial surface of the gel). (b) View in plane $xz$. In the frame moving with the blade, gel velocity is $-U$ along $x$, and $-W$ along $z$. The system is assumed to be invariant along $z$.

We consider a thin blade consisting in a sharp wedge with an infinitely small angle so that its geometry can be approached, in a first approximation, as a half plane of zero thickness. This half plane (the blade) is assumed to move parallel to its own plane both in the $z$ direction (the sliding motion) and inward a gel in the $x$ direction perpendicular to the edge (Fig. 1). Lastly $y$ is the direction perpendicular to the plane of the knife. Let $U$ be the velocity of the knife in the direction of cutting and $W$ in the direction of slicing. In Cartesian coordinates this is the velocity of the frame of reference of the knife. In Cartesian coordinates this is the velocity $(-U, 0, -W)$. This makes also the boundary conditions for the velocity field in the liquid on $\Gamma$: $u|_\Gamma = -U$, $v|_\Gamma = 0$, and $w|_\Gamma = -W$ (Fig. 2). Given $\Gamma$ this yields the right number of conditions for the problem.

There remains to get the condition that determines $\Gamma$. This is the condition that the stress there is exactly at the given critical value $\sigma_1$. Since the gel is mainly composed of solvent, the same as the liquid phase, the interfacial tension between the two phases is negligible in a first approximation, and the stress has to take the same value on the gel-side and on the liquid-side of the boundary. Because the stress on the liquid side is given by the standard formulae for viscous fluids, this stress is proportional to local values of the velocity derivatives times the shear viscosity of the fluid. The Cauchy stress tensor in the liquid phase writes:

$$\sigma = -\eta \begin{pmatrix} u_x & u_y + v_x & w_x \\ u_y + v_x & v_y & w_y \\ w_x & w_y & 0 \end{pmatrix}$$

where $\eta$ is the shear viscosity of the liquid. There remains to give the expression of the critical stress, $\Sigma(\ell)$, that is to be made equal to the critical value $\sigma_1$. For an isotropic solid, this stress has to be independent on the choice of coordinates. This is achieved by taking as measure of the stress the von Mises stress defined by $\Sigma^2 = \frac{3}{2} tr(\sigma^2)$ [14] [15]. In the geometry in consideration this yields:

$$\Sigma^2 = 3\eta^2 (w_x^2 + w_y^2 + v_x^2 + u_y^2 + u_x^2 - 2v_xu_y) .$$

The equation of the curve $\Gamma$ is found by imposing

$$|\Sigma| = \sigma_1$$

on $\Gamma$, $\sigma_1$ being a given positive quantity with the physical dimension of a stress. This amounts to assuming that failure occurs as the elastic energy density of distortion reaches a critical value [12].

Taking $\ell = \sqrt{3\eta W}/\sigma_1$ as unit length and $W$ as unit speed, the set of equations Eqs. [14] and [17] can be reformulated with a unique parameter, the ratio $r = U/W$.
of the sliding velocity to the cutting velocity. Taking $\sigma_1$ of the order of magnitude of the shear modulus of a soft hydrogel (a reasonable assumption for polymeric gels) e.g. $\sigma_1 = 100$ Pa, and $\eta = 10$ mPas (the liquid layer being a mixture of water and molecules resulting from gel breakage), we find, with $W = 1 \text{ m/s}$ for the tangential velocity, $\ell = 0.1 \text{ mm}$, a small but measurable macroscopic length.

In the following, the thickness $h$ of the liquid layer along the blade, the normal distance $a$ between the edge of the blade and the bottom of the channel, and the maximum curvature $1/r_0$ of the gel surface, are computed as function of $r$ and $\ell$.

Let us first consider the special situation in which the blade is pushed normally toward the gel without sliding, $W = 0$. Due to mirror symmetry with respect to plane $y = 0$, $u, y = 0$ and $v = 0$ along axis $y = 0$ for $x > 0$ (see Fig. 2) hence, Eq. 7 simplifies in $\Sigma = 3\eta^2 u^2 y$ or equivalently (from Eq. 1)$\Sigma = 3\eta^2 u^2 y$ ($x > 0$ and $y = 0$). The tangent of $\Gamma$ at $(x, y) = (a, 0)$ being parallel to $y$ by symmetry, one deduces from $v_{\Gamma} = 0$ that $v, y = 0$ at $(x, y) = (a, 0)$ and then, that the stress $\Sigma$ at $(x, y) = (a, 0)$ has to be equal to zero. Since $\sigma_1 \neq 0$, the condition $\Sigma = \sigma_1$ on $\Gamma$ cannot be fulfilled. This means that no stationary state exists for $W = 0$, and highlights the overriding role played by the sliding of the blade.

In what follows, the tangential velocity is non zero. Far behind the edge of the blade ($x \to -\infty$), derivatives along $x$-direction are zero and $u(y)$ is parabolic. According to the boundary conditions at $y = 0$ and $y = h$ and the incompressibility condition (the average value of $u$ along $y$ is equal to $-U$), $u = -U (4y/h - 3y^2/h^2)$. This is a simple combination of Couette and Poiseuille flows, the pressure gradient along the $x$-direction being constant. In addition, $v \sim 0$ and $w(y)$ is linear (Couette flow), $w = -Wy/h$. One concludes that, far from the edge of the blade ($x \to -\infty), |\Sigma| \sim (\eta/h)^3 (4U^2 + W^2)$ and from Eq. 7

$$h = \ell \sqrt{1 + 4r^2}.$$ (8)

Eqs. 1 and 7 are solved using the finite-element method, implemented in the open-source finite-element library FEniCS [17]. We adopt a set of units such that $\ell = 1$ and $W = 1$. Assuming reflectional symmetry, we consider domains $\Omega$ defined by

$$(x, y) \in \Omega \Leftrightarrow 0 \leq y \leq h \text{ and } x_{\text{min}} \leq x \leq \Gamma(y)$$ (9)

$\Gamma(y)$ is a decreasing function for $y \in [0, h]$. $x_{\text{min}}$ is the lower $x$-coordinates considered in the simulation ($x_{\text{min}} < \Gamma(h)$, see Fig. 2). The boundary conditions are detailed in Fig. 2. We take for $\Gamma(y)$ a series expansion in the form

$$\Gamma(y) = \sum_{k=0}^{n-1} a_{2k} (y/h)^{2k} + a_n \sum_{k=n}^{m} (y/h)^{2k}. \quad (10)$$

Coefficients $a_{2k}$ are fitted in order to fulfill Eq. 7. The fits are done by carrying out a systematic exploration of the coefficients $a_k$ and by minimizing the variance $s^2 = \frac{1}{0.95(s)} \int_0^{0.95(s)} (\Sigma(y) - \Sigma_0)^2 dy$. The last sum in Eq. 10 has no significant effect on the best $\Gamma(y)$, except for $y \sim h$: neither a change in $n$ (provided that $n > 8$) nor a change in $m$ (provided that $m > 30$) has an effect on the values found for $a_0$ or $a_2$.

Fig. 3 gives an example, for $r = 0.2$, of the reduced
stress $\Sigma/\sigma_1$ calculated after the fitting procedure. As required, $\Sigma/\sigma_1$ is almost constant and equal to one. Fig. 3a shows that the variance $s^2$ calculated for the best fit starts to increase rapidly beyond a threshold value of $r$, $r^* \approx 0.27$. This indicates that solution of Eq. 2 exists only for $r < r^*$, i.e. for $U < 0.27W$, suggesting that steady states exist only if the sliding velocity is large enough compared to the normal velocity, $W > U/0.27$.

The reduced normal distance $a/\ell = a_0$ and the reduced radius of curvature $r_0/\ell = h^2/(2a_0^2)$ are plotted as a function of $r$ (for $r < r^*$) in Fig. 3b. The rate of increase of $a/\ell$ as a function of $r$ is lower than the rate for $r_0/\ell$ and $h/\ell$: the profile of the channel is more blunt as $r$ increases. Velocity components and pressure computed for $r = 0.2$ are shown in Fig. 4. Note that in the limit $r = 0$, $\Sigma$ is the unique non-zero scalar invariant of $\sigma$ and taking the von Mises criteria among the others criteria does not limit the generality.

To sumarize, the cut made by pressing and sliding a sharp wedge on a soft materials has been described by considering the viscous liquid layer surrounding the wedge. This layer results from the transformation of the soft solid to a liquid when the applied stress exceeds a prescribed value. The stress exerted on the wedge is transmitted to the soft solid thought the liquid layer, hence the prevailing role of the wedge velocities in the cutting process.

No steady state in the melting of the solid can be induced by a pure normal indentation of the sharp wedge ($W = 0$). Indeed, a steady regime requires a large enough ratio of the sliding velocity to the normal velocity. The minimal value of the normal velocity (the cutting speed) is proportional to the tangential velocity, $U_{\text{min}} \approx 0.27W$. Hence, the maximum cutting speed is directly related to the sliding velocity: quicker dicing requires higher tangential velocity.

For given imposed normal and tangential velocities in a steady regime, the condition that the critical stress of the gel has to be reached fixes the shape of the transition zone. The thickness of the fluid layer far behind the edge of the blade, the thickness in the direction normal to the edge, and also the radius of curvature of the transition zone, have been computed.

Whether the deformations of the solid phase are large or small does not matter in the theory, the important property being that the gel remains elastic below a critical stress is reached. In that sense, the theory fundamentally departs from standard theories of fracture mechanics that are based on the calculation of elastic deformations together with an estimation of the energy release rate taking place in the plastic zone.

Extending the theory introduced here to unsteady states would be useful to capture the nucleation stages, to unfold the cases steady states do not exist (e.g. normal dicing), and to explain how a tangential vibration can improve dicing, has evidenced in the puncture of soft gels [12, 13] or practiced in surgery.

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**FIG. 4:** Reduced components of the velocity $u$ (a), $v$ (b) and $w$ (c), and pressure $p$ (d), computed from Eqs. 1-2 and 4 using the finite element method, calculated for $r = 0.2$.

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