ABSOLUTE CONTINUITY AND CONVERGENCE IN VARIATION FOR DISTRIBUTIONS OF A FUNCTIONALS OF POISSON POINT MEASURE

ALEXEY M. KULIK

Abstract. General sufficient conditions are given for absolute continuity and convergence in variation of distributions of a functionals on a probability space, generated by a Poisson point measure. The phase space of the Poisson point measure is supposed to be of the form $\mathbb{R}^+ \times U$, and its intensity measure to be equal $dt \Pi(du)$. We introduce the family of time stretching transformations of the configurations of the point measure. The sufficient conditions for absolute continuity and convergence in variation are given in the terms of the time stretching transformations and the relative differential operators. These conditions are applied to solutions of SDE’s driven by Poisson point measures, including an SDE’s with non-constant jump rate.

1. Introduction

In this paper, we give a general and transparent sufficient conditions for absolute continuity and convergence in variation of a distributions of a functionals on the probability space, generated by a Poisson point measure. The phase space of the Poisson point measure is supposed to be of the form $\mathbb{R}^+ \times U$, and its intensity measure to be equal $dt \Pi(du)$, with $(U, \mathcal{U})$ being Borel measurable space and $\Pi$ being a $\sigma$-finite measure on $\mathcal{U}$. The Poisson point measures of such a type arise naturally when the Levy processes or their various modifications are considered; typically, $U = \mathbb{R}^m \setminus \{0\}$. For the Poisson point measures of such a type, we introduce the family of time stretching transformations. The sufficient conditions for absolute continuity and convergence in variation are given in the terms of the time stretching transformations and the relative differential operators. We illustrate the sufficient conditions obtained in this paper by applying them to solutions of SDE’s driven by Poisson point measures, including SDE’s with non-constant jump rate.

Our approach strongly relies on an appropriate modification of Yu.Davydov’s stratification method. This method is based on disintegration of the probability space and finite-dimensional change-of-variables formula. It is known that the stratification method, unlike the Malliavin calculus, does not allow one to prove the distribution density to be bounded, smooth, etc. The main advantage of the stratification method is that it can be applied under a very mild differentiability conditions on the functionals under investigation, while, as we will see below, the differential properties of the functionals of the Poisson point measure typically are rather poor. In addition, this method appears to be powerful enough to provide not only absolute continuity for an individual distribution, but also convergence in variation for a sequence of distributions. The latter finds a very natural and useful applications in ergodic theory for SDE’s with jump noise. See [29], where the time stretching transformations and associated stratifications are used to provide the local Doeblin condition for the solution to an SDE with jumps, considered as a Markov process, and then to establish ergodic and mixing rates for this process.

This paper unifies and generalizes the previous papers [21, 22, 24] by the same author. It contains partially the unpublished preprint [25]. The paper is also closely related to the papers [11] and [29]. Statements 1 and 3 of Theorem 4.1 below contain Theorem A [29] as a partial case. Statement 1 of Theorem 4.2 below is

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a generalization of Theorem 3.3.2 [11]. However, Theorem 3.3.2 [11] has a serious "gap" in its proof, that seemingly can not be fixed up in the framework of [11], based on the Dirichlet form technique (see discussion in subsection 4.3 below). The discussion of the relation between Theorem 4.2 and the recent papers [2], [15], devoted to investigation of the laws of solutions to SDE’s with non-constant jump rate, is given in subsection 4.2 below.

Let us give a brief overview of the other references related to our investigation. The integration-by-parts structure for the pure Poisson process was introduced independently in [3] and [12]. One can say that this method, introduced by J.Picard in [31] (see also [16], [17]), that, in the case of a one-dimensional group of transformations of the time axis \( \mathbb{R} \), denoted \( \{H\} \), these suppositions, if to compare with those made in the Introduction, do not restrict generality, since one can reduce Borel measurable space \((\mathcal{U}, \mathcal{U})\) to \( ((0, 1), \mathcal{B}(0, 1)) \) with a locally finite measure \( \Pi' \) by an appropriate Borel isomorphism.

By \( \nu \), we denote the Poisson point measure on \( \mathbb{R}^+ \times \mathcal{U} \) with its intensity measure equal \( dt\Pi(du) \). By \( \mathcal{O} \equiv \mathcal{O}(\mathbb{R}^+ \times \mathcal{U}) \), we denote the space of configurations over \( \mathbb{R}^+ \times \mathcal{U} \), i.e. a family of locally finite subsets of \( \mathbb{R}^+ \times \mathcal{U} \). The space \( \mathcal{O} \) is equipped with the vague topology, i.e. the weakest topology such that every function \( \mathcal{O} \ni \varpi \mapsto \sum_{(t, u) \in \varpi} f(t, u) \) with \( f : \mathbb{R}^+ \times \mathcal{U} \rightarrow \mathbb{R} \) being a continuous function with bounded support, is continuous. We denote \( \mathcal{B}(\mathcal{O}) \) the Borel \( \sigma \)-algebra on \( \mathcal{O} \) and write \( P_\nu \) for the distribution of the random element \( \mathcal{O}, \mathcal{B}(\mathcal{O}) \) generated by \( \nu \). For more details, see e.g. [18]. In the sequel, we suppose the basic probability space to have the form \((\Omega, \mathcal{F}, P) = (\mathcal{O}, \mathcal{B}(\mathcal{O}), P_\nu)\) and put \( \nu(\omega) = \omega \).

Denote \( H = L_2(\mathbb{R}^+), H_0 = L_\infty(\mathbb{R}^+ \cap L_2(\mathbb{R}^+) \cap \mathcal{L}_2(\mathbb{R}^+), Jh(\cdot) = \int_0^1 h(s) \, ds, h \in H \). For a fixed \( h \in H_0 \), we define the family \( \{T_{th}^t, t \in \mathbb{R}\} \) of transformations of the axis \( \mathbb{R}^+ \) by putting \( T_{th}^t x, x \in \mathbb{R}^+ \) equal to the value at the point \( s + t \) of the solution of the Cauchy problem

\[
(2.1) \quad z'_{x,h}(s) = \int_0^s h(z_{x,h}(s)), \quad s \in \mathbb{R}, \quad z_{x,h}(0) = x.
\]

Since \( 2.1 \) is the Cauchy problem for the time-homogeneous ODE, one has that \( T_{h}^{s+t} = T_{h}^s \circ T_{h}^t \), and in particular \( T_{h}^{-t} \) is the inverse transformation to \( T_{h}^t \). By multiplying \( h \) by some \( a > 0 \), we multiply, in fact, the symbol \( Jh(\cdot) \) of the equation by \( a \). Now, making the time change \( \tilde{s} = \frac{x}{a} \), we see that \( T_{h}^t = T_{ah}^\frac{a}{a}, a > 0 \), which together with the previous considerations gives that \( T_{h}^t = T_{h}^{ah}, h \in H_0, t \in \mathbb{R} \).

Denote \( T_{h} \equiv T_{h}^1 \), we have just demonstrated that \( T_{sh} \circ T_{th} = T_{(s+t)h} \). This means that \( T_{h} \equiv \{T_{th}, t \in \mathbb{R}\} \) is a one-dimensional group of transformations of the time axis \( \mathbb{R}^+ \). It follows from the construction that

\[
(2.2) \quad \frac{d}{dt}\big|_{t=0} T_{th} x = Jh(x), \quad x \in \mathbb{R}^+.
\]

Remark 2.1. We call \( T_{h} \) the time stretching transformation because, for \( h \in C(\mathbb{R}^+) \cap H_0 \), it can be constructed in a more illustrative way: take the sequence of partitions \( \{S^n\} \) of \( \mathbb{R}^+ \) with \( |S_n| \rightarrow 0, n \rightarrow +\infty \). For every
n, we make the following transformation of the axis: while preserving an initial order of the segments, every segment of the partition should be stretched by \( e^{h(\theta)} \) times, where \( \theta \) is some inner point of the segment (if \( h(\theta) < 0 \) then the segment is in fact contracted). After passing to the limit (the formal proof is omitted here in order to shorten the exposition) we obtain the transformation \( T_h \). Thus one can say that \( T_h \) performs the stretching of every infinitesimal segment \( dx \) by \( e^{h(x)} \) times.

Denote \( U_{fin} = \{ \Gamma \in \mathcal{B}(U), \Gamma \text{ is bounded} \} \) and define, for \( h \in H_0, \Gamma \in U_{fin} \), a transformation \( T^\Gamma_h \) of the random measure \( \nu \) by

\[
[T^\Gamma_h \nu](\{0, t] \times \Delta) = \nu([0, t] \times (\Delta \cap \Gamma)) + \nu([0, t] \times (\Delta \setminus \Gamma)), \quad t \in \mathbb{R}^+, \Delta \in U_{fin}.
\]

This transformation is generated by the following transformation of the space of the configuration: \( (\tau, x) \in \omega \) with \( x \notin \Gamma \) remains unchanged; for every point \( (\tau, x) \in \omega \) with \( x \in \Gamma \), its “moment of the jump” \( \tau \) is transformed to \( T_{-h} \tau \); neither any point of the configuration is eliminated nor any new point is added to the configuration. In the sequel we denote, by the same symbol \( T^\Gamma_h \), the bijective transformation of the space of configurations described above.

The image \( T^\Gamma_h \nu \) is again a random Poisson point measure, and its intensity measure can be expressed through \( \Pi \) and \( r_h(x) \equiv \frac{dF}{dx}(T_h x) \) explicitly. An easy calculation gives that

\[
(3.3) \quad r_h(x) = \int_0^1 h(T_{sh} x) \, ds, \quad x \in \mathbb{R}^+.
\]

Thus the following statement is a corollary of the classical absolute continuity result for Lévy processes, see [33], Chapter 9. We put

\[
p^\Gamma_h = \exp \left\{ \int_{\mathbb{R}^+} r_h(t) \nu(dt, \Gamma) - \lim_{t \to +\infty} [T_h t - t] \Pi(\Gamma) \right\}.
\]

**Lemma 2.1.** The transformation \( T^\Gamma_h \) is admissible for the distribution of \( \nu \) with the density \( p^\Gamma_h \), i.e., for every \( \{t_1, \ldots, t_n\} \subset \mathbb{R}^+, \{\Delta_1, \ldots, \Delta_n\} \subset U_{fin} \) and Borel function \( \phi : \mathbb{R}^n \to \mathbb{R}, \)

\[
E \phi([T^\Gamma_h \nu](\{0, t_1] \times \Delta_1), \ldots, [T^\Gamma_h \nu](\{0, t_n] \times \Delta_n)) = E p^\Gamma_h \phi(\nu([0, t_1] \times \Delta_1), \ldots, \nu([0, t_n] \times \Delta_n)).
\]

The lemma implies that every transformation \( T^\Gamma_h \) generates the corresponding transformation of the random variables. In the sequel, we denote the latter transformation by the same symbol \( T^\Gamma_h \).

**Definition 2.1.** Let \( h \in H_0, \Gamma \in U_{fin} \) be fixed.

1. The functional \( f \in L_0(\Omega, \mathcal{F}, P) \) is said to be **almost surely (a.s.) differentiable** in the direction \((h, \Gamma)\) and to have almost sure (a.s.) derivative \( D^h_h f \), if

\[
(2.4) \quad \frac{T^\Gamma_h f - f}{\varepsilon} \to D^\Gamma_h f, \quad \varepsilon \to 0
\]

almost surely.

2. Let \( p \in [1, +\infty) \). The functional \( f \in L_p(\Omega, \mathcal{F}, P) \) is said to be **\( L_p \)-differentiable** in the direction \((h, \Gamma)\) and to have \( L_p \) derivative \( D^\Gamma_p f \), if convergence (2.4) holds in \( L_p \) sense.

Let us give an example demonstrating one specific property of the family \( \{T_h, h \in H_0\} \).

**Example 2.1.** Let \( f = \tau^\Gamma_n \equiv \inf\{t : \nu(t, \Gamma) = n\} \), and let \( h, g \in C(\mathbb{R}^+) \cap H_0 \) be such that \( h(t) \int_0^t g(s) \, ds \neq g(t) \int_0^t g(s) \, ds, t > 0 \). Then

\[
D^\Gamma_h D^\Gamma_g f = h(\tau^\Gamma_n) \int_0^{\tau^\Gamma_n} g(s) \, ds \neq g(\tau^\Gamma_n) \int_0^{\tau^\Gamma_n} h(s) \, ds = D^\Gamma_g D^\Gamma_h f
\]
almost surely (this follows from the relation \([4.3]\) given below). In particular, the family of transformations \(\{T^h, h \in H_0\}\) is not commutative and therefore cannot be considered as an infinite-dimensional additive group of transformations.

The non-commutative structure of the family \(\{T_h, h \in H_0\}\) does not allow one to apply the stratification method for study of the absolute continuity of the laws of differentiable functionals straightforwardly. In order to overcome this difficulty, we introduce an additional construction based on the notion of a differential grid.

**Definition 2.2.** A family \(\mathcal{G} = \{(a_i, b_i) \subset \mathbb{R}^+, h_i \in H_0, \Gamma_i \in U_{fin}, i \leq m\}\) is called a differential grid (or simply a grid) if
1. for every \(i \neq j\), \(\{a_i, b_i\} \times \Gamma_i \cap \{a_j, b_j\} \times \Gamma_j = \emptyset\);
2. for every \(i \in \mathbb{N}\), \(Jh_i > 0\) inside \((a_i, b_i)\) and \(Jh_i = 0\) outside \((a_i, b_i)\).

The number \(m \in \mathbb{N}\) is called a dimension of the grid \(\mathcal{G}\).

Denote \(T_i^j = T_{th_i}^{T_i^j}\). It follows from the construction of the transformations \(T^h\) that, for a given \(i \in \mathbb{N}, t \in \mathbb{R}\),

\[
T_i^j \tau_n^r = T_{th_i}^{T_i^j} \tau_n^r, \quad \tau_n^r \notin [a_i, b_i], \quad \tau_n^r \in [a_i, b_i)
\]

for every \(n\).

In other words: a grid \(\mathcal{G}\) generates a partition of some part of the phase space \(\mathbb{R}^+ \times U\) of the random measure \(\nu\) into the non-intersecting cells \(\{S_i = [a_i, b_i] \times \Gamma_i\}\). The transformation \(T_i^j\) does not change points of configuration outside the cell \(S_i\) and keeps the points from this cell in it. In addition, for every \(i \leq m, t, t_i \in \mathbb{R}\), the transformations \(T_i^j T_i^j\) commute because so do the time axis transformations \(T_{th_i} T_{th_i}\). Therefore, for every \(i, i_1, t, t_1 \in \mathbb{R}\), the transformations \(T_i^j T_i^j\) commute. This implies the following proposition.

**Proposition 2.1.** For a given grid \(\mathcal{G}\) and \(z = (z_i)_{i=1,...,m} \in \mathbb{R}^m\), define the transformation

\[
T^G_z = T_{z_1}^1 \circ T_{z_2}^2 \circ \ldots \circ T_{z_m}^m.
\]

Then \(\mathcal{T}^G = \{T^G_z, z \in \mathbb{R}^m\}\) is the group of admissible transformations of \(\Omega\) which is additive in the sense that \(T^G_{z_1 + z_2} = T^G_{z_1} \circ T^G_{z_2}\), \(z_1, z_2 \in \mathbb{R}^m\).

It can be said that, by fixing the grid \(\mathcal{G}\), we choose from the whole variety of admissible transformations \(\{T^h, h \in H_0, \Gamma \in U_{fin}\}\) the additive subfamily that is more convenient to deal with.

**Definition 2.3.** 1. The functional \(f \in L_0(\Omega, \mathcal{F}, P)\) is a.s. stochastically differentiable w.r.t. differential grid \(\mathcal{G}\) if \(f\) is a.s. differentiable in every direction \((h_i, \Gamma_i), i = 1, \ldots, m\). The random vector \(D^G f = (D^G_{h_1} f, \ldots, D^G_{h_m} f)\) is called the a.s. stochastic derivative of \(f\).

2. Let \(p \in [1, +\infty)\). The functional \(f \in L_p(\Omega, \mathcal{F}, P)\) is stochastically differentiable in \(L_p\) sense w.r.t. differential grid \(\mathcal{G}\) if \(f\) is \(L_p\)-differentiable in every direction \((h_i, \Gamma_i), i = 1, \ldots, m\). The random vector \(D^G f = (D^G_{h_1} f, \ldots, D^G_{h_m} f)\) is called the \(L_p\) stochastic derivative of \(f\).

We denote \(D^G_{h_i} f = D^G_{h_i, i}, i = 1, \ldots, m\).

2.2. **Sufficient conditions for absolute continuity and convergence in variation.** The proofs for the following theorems are given in Section 3 below.

**Theorem 2.1.** Consider an \(\mathbb{R}^m\)-valued random vector \(f = (f_1, \ldots, f_m)\) and a grid \(\mathcal{G}\) of dimension \(m\). Let every component of the vector \(f\) to be differentiable w.r.t. \(\mathcal{G}\) either in a.s. or in \(L_p\) sense for some \(p \geq 1\).

Denote \(\Sigma^G = (D^G f)^T\) and put \(N(f, \mathcal{G}) = \{\omega: the\ matrix\ \Sigma^G(\omega)\ is\ non-degenerate\}\). Then

\[
P\big|_{\mathcal{N}(f, \mathcal{G})} f^{-1} \ll \lambda^m.
\]
Theorem 2.2. Consider a sequence of $\mathbb{R}^m$-valued random vectors $\{f^n, n \geq 1\}$ such that, for a given grid $\mathcal{G}$ of dimension $m$, every component $f^n_j$, $j = 1, \ldots, m$ of the vector $f^n$ is $L_m$ differentiable w.r.t. $\mathcal{G}$. Suppose that

$$f^n_j \to f_j, \quad D^\mathcal{G}_i f^n_j \to D^\mathcal{G}_i f_j \text{ in } L_m, \quad n \to +\infty, \quad i, j = 1, \ldots, m.$$  

Then, for every $A \subset \mathcal{N}(f, \mathcal{G})$,

$$P\big|_A \circ f^{-1}_n \to P\big|_A \circ f^{-1}, \quad n \to +\infty$$

in variation.

We remark that the type of differentiability of the components of $f$ is unimportant in the condition for absolute continuity, given in Theorem 2.1. On the contrary, this type is crucial in the condition for convergence in variation. For instance, the immediate analogue of Theorem 2.2, with the $L_m$ derivatives replaced by the a.s. ones, fails to be true. One can construct the counterexample to such a statement using Example 1.2 [24]. In order to formulate the correct version of Theorem 2.2 in the terms of a.s. derivatives, we need an auxiliary notion.

Definition 2.4. The sequence of the measurable functions $\{f_n : \Omega \to \mathbb{R}, n \geq 1\}$ is said to have a uniformly dominated increments w.r.t. the grid $\mathcal{G}$ on the set $\Omega' \in \mathcal{F}$, if there exist a random variable $\varrho$ and a family of jointly measurable functions $\{g_i : \Omega \times \mathbb{R} \to \mathbb{R}\}$ such that

(i) for every $i$ and almost every $\omega$, the function $g_i(\omega, \cdot)$ is an increasing one;

(ii) $\varrho > 0$ almost surely and, for every $n \geq 1, \omega \in \Omega$,

$$|T^\mathcal{G}_i f_n(\omega) - T^\mathcal{G}_i f_n(\omega)| \leq \sum_{i=1}^m \left| g_i(\omega, t_i \lor s_i) - g_i(\omega, t_i \land s_i) \right|, \quad \|t\|, \|s\| < \varrho(\omega), \quad T^\mathcal{G}_i \omega \in \Omega', \quad T^\mathcal{G}_i \omega \in \Omega'. \quad (2.5)$$

A sequence of $\mathbb{R}^m$-valued random vectors $\{f^n, n \geq 1\}$ is said to have a uniformly dominated increments w.r.t. the grid $\mathcal{G}$ on the set $\Omega' \in \mathcal{F}$ if every sequence $\{f^n_j, n \geq 1\}, j = 1, \ldots, m$ has a uniformly dominated increments w.r.t. the grid $\mathcal{G}$ on this set.

Theorem 2.3. Consider a sequence of $\mathbb{R}^m$-valued random vectors $\{f^n, n \geq 1\}$ such that, for a given grid $\mathcal{G}$ of dimension $m$, every component $f^n_j$, $j = 1, \ldots, m$ of the vector $f^n$ is a.s. differentiable w.r.t. $\mathcal{G}$. Suppose that

$$f^n_j \to f_j, \quad D^\mathcal{G}_i f^n_j \to D^\mathcal{G}_i f_j \text{ in probability, } \quad n \to +\infty, \quad i, j = 1, \ldots, m.$$  

Suppose additionally that $\{f^n\}$ has a uniformly dominated increments w.r.t. $\mathcal{G}$ on the set $\Omega'$.

Then, for every $A \subset \mathcal{N}(f, \mathcal{G}) \cap \Omega'$,

$$P\big|_A \circ f^{-1}_n \to P\big|_A \circ f^{-1}, \quad n \to +\infty$$

in variation.

3. The stratification method and proofs of Theorems 2.1 – 2.3

In this section, we prove the general statements, formulated in section 2.2. Our main tool is a certain version of Yu.Davydov’s stratification method; for the basic constructions of this method, references and further discussion, we refer the reader to the Chapter 2 of the monograph [10]. Some steps in our considerations have analogues in the available literature. For instance, the trick with using Theorem 3.1.16 [14] in order to replace a function differentiable in some weak sense by a $C^1$ one, was used in [6] in the context of stratifications generated by linear shifts and in [7], Chapter II.5 in the context of Dirichlet forms on vector spaces.
3.1. Stratifications, generated by differential grids. Let $\mathcal{G}$ be a differential grid of dimension $m$. For every $\omega \in \Omega$, consider the set $v = v(\omega) = \{T^\mathcal{G}_\omega z, z \in \mathbb{R}^m\}$. This set is called the orbit of the group $\mathcal{T}^\mathcal{G}$, corresponding to $\omega$. The set of all such orbits is denoted $\mathcal{T}$, and $\Omega$ is represented as the disjunctive union

$$\Omega = \bigcup_{v \in \mathcal{T}} v.$$  

The decomposition (3.1) is called the stratification of $\Omega$ to the orbits of the group $\mathcal{T}^\mathcal{G}$.

Every orbit $v$ has a simple structure. Denote $\mathcal{D}^\mathcal{G} \Delta \{\tau \in \mathcal{D} : p(\tau) \in \Gamma\}$,

$$I(\omega) = \left\{ i = 1, \ldots , m : (a_i, b_i) \cap \mathcal{D}^\mathcal{G}_i(\omega) \neq \emptyset \right\}, \quad \omega \in \Omega.$$  

By condition (ii) of Definition 2.2 for every $i = 1, \ldots , m$, the mapping

$$\mathbb{R} \ni t \mapsto T^\mathcal{G}_{th} x \in \mathbb{R}^+$$

is the identical one if $x \notin (a_i, b_i)$ and is strictly monotonous if $x \in (a_i, b_i)$. This implies the following equivalence: for every $z^1, z^2 \in \mathbb{R}^m, \omega \in \Omega$,

$$T^\mathcal{G}_{z^1} \omega = T^\mathcal{G}_{z^2} \omega \iff z^1 = z^2, \quad i \in I(\omega).$$

Therefore, the orbit $v(\omega)$ is the bijective image of $\mathbb{R}^{\#(\omega)}$ (here and below, $\#$ is used for the number of elements of the set).

Denote $\Omega^\mathcal{G} = \left\{ \omega : I(\omega) = \{1, \ldots , m\} \right\}$, one can see that $\Omega^\mathcal{G}$ is measurable. For our further purposes, it would be enough to restrict the initial probability $P$ to $\Omega^\mathcal{G}$ and to describe the stratification of $\Omega^\mathcal{G}$, only. Such a restriction simplifies the exposition, since, for every point $\omega \in \Omega^\mathcal{G}$, the corresponding orbit is a bijective image of $\mathbb{R}^m$.

**Lemma 3.1.** There exists a complete separable metric space $\mathcal{Y}$ and a bijection $\vartheta : \Omega \to \mathcal{Y} \times \mathbb{R}^m$ such that $\vartheta$ is $\mathcal{F} \setminus \mathcal{B}(\mathcal{Y}) \otimes \mathcal{B}(\mathbb{R}^m)$ measurable, $\vartheta^{-1}$ is $\mathcal{B}(\mathcal{Y}) \otimes \mathcal{B}(\mathbb{R}^m) - \mathcal{F}$ measurable and

$$\vartheta \left( T^\mathcal{G}_{z} \omega \right) = \left( \pi_1(\vartheta(\omega)), \pi_2(\vartheta(\omega)) + z \right), \quad \omega \in \Omega^\mathcal{G}, \quad z \in \mathbb{R}^m,$$

where $\pi_1, \pi_2$ denote the projections on the first and the second coordinates in $\mathcal{Y} \times \mathbb{R}^m$ respectively.

**Proof.** First of all, we mention that $\Omega = \emptyset$ can be considered as a Polish space via the following construction. For two configurations $\omega', \omega'' \in \emptyset$, we put

$$d_\emptyset(\omega', \omega'') = \sum_{m=1}^{\infty} 2^{-m}[1 \wedge d_H(\omega' \cap K_m, \omega'' \cap K_m)],$$

where $d_H$ is the Hausdorff metrics on the set of closed subsets of $\mathbb{R}^+ \times \mathbb{U}$, and $\{K_m\}$ is a sequence of compacts such that $\bigcup_m K_m = \mathbb{R}^+ \times \mathbb{U}$. Then $(\emptyset, d_\emptyset)$ is a Polish space, and one can deduce from 28, Propositions 1.4.1 and 1.4.4, that the Borel structure on $\emptyset$ generated by $d_\emptyset$ coincides with the one generated by the vague topology.

For $\omega \in \Omega^\mathcal{G}$, define $\tau_i(\omega) = \min \left[ (a_i, b_i) \cap \mathcal{D}^\mathcal{G}_i(\omega) \right], i = 1, \ldots , m$. We have already mentioned that, for every $i = 1, \ldots , m$ and $x \in (a_i, b_i)$, the transformation $\mathbb{R} \ni t \mapsto T^\mathcal{G}_{th} x \in (a_i, b_i)$ is strictly monotonous. In addition, it is bijective and continuous together with its inverse. Therefore, for every $i = 1, \ldots , m$ there exists unique $z_i(\omega) \in \mathbb{R}$ such that $T^\mathcal{G}_{-z_i(\omega)} \tau_i(\omega) = \frac{1}{2}(a_i + b_i)$. Denote $z(\omega) = (z_1(\omega), \ldots , z_m(\omega)) \in \mathbb{R}^m$. Denote by $\mathcal{Y}$ the family of all configurations satisfying the following additional condition: for every set $(a_i, b_i) \times \Gamma_i, i = 1, \ldots , m$, the configuration is not empty in this set, and the smallest time coordinate of the point in this set is equal to $\frac{1}{2}(a_i + b_i)$. This family is a complete separable metric space w.r.t. the local Hausdorff metrics described above.
Now put
\[ \vartheta(\omega) = \left( T^G_{\xi(\omega)} \omega, z(\omega) \right) \in \mathcal{Y} \times \mathbb{R}^m, \quad \omega \in \Omega^G. \]
The map \( \vartheta \) is a bijection between \( \Omega^G \) and \( \mathcal{Y} \times \mathbb{R}^m \). One can easily see that both \( \vartheta \) and \( \vartheta^{-1} \) are measurable (moreover, continuous). At last, if \( \tilde{\omega} = T^G_1 \omega \) then, by the group property of the family \( \mathcal{T}^G \),
\[ z(\tilde{\omega}) = z(\omega) + \tilde{z}. \]

This proves (3.2). The lemma is proved.

In a sequel, we denote the points of \( \mathcal{Y} \) by \( \upsilon \) in order to emphasize that \( \mathcal{Y} \), in fact, is the set of the orbits. We also omit \( \vartheta \) in the notation and identify \( \omega \in \Omega^G \) with its image \( (\upsilon, z) \in \mathcal{Y} \times \mathbb{R}^m \).

For \( A \subset \Omega^G, \upsilon \in \mathcal{Y} \) denote \( A_\upsilon = \{ z \in \mathbb{R}^m : (\upsilon, z) \in A \} \). Similarly, for \( \upsilon \in \mathcal{Y} \) and the function \( f : \Omega^G \to \mathbb{R} \), define the function \( f_\upsilon : \mathbb{R}^m \ni l \mapsto f((\upsilon, z)) \in \mathbb{R}^m \). It follows from (3.2) that
\[ [T^G_\upsilon f](\cdot) = f_\upsilon(\cdot + z), \quad \upsilon \in \mathcal{Y}, \; z \in \mathbb{R}^m. \]

Denote, by \( P^G \), both \( P|_{\Omega^G} \) and its image under \( \vartheta \). Denote, by \( P^\mathcal{Y} \), the projection of \( P^G \) on the first coordinate in \( \mathcal{Y} \times \mathbb{R}^m \) (i.e., the image of \( P^G \) under the projection \( \pi_1 \)). The following statement is a version of the well known theorem on existence of the family of conditional distribution (e.g. [30], Chapter 5).

**Proposition 3.1.** There exists a family \( \{ P_\upsilon, \upsilon \in \mathcal{Y} \} \) of finite measures on \( \mathcal{B}(\mathbb{R}^m) \) such that

1. for every \( B \in \mathcal{B}(\mathbb{R}^m) \), the function \( \upsilon \mapsto P_\upsilon(B) \) is Borel measurable;

2. for every \( A \in \mathcal{T}, A \subset \Omega^G \),
\[ P(A) = \int \mathcal{Y} P_\upsilon(A_\upsilon) P^\mathcal{Y}(d\upsilon). \]

**Lemma 3.2.** 1. For \( P^\mathcal{Y} \) – almost all \( \upsilon \in \mathcal{Y} \), the measure \( P_\upsilon \) possesses a continuous strictly positive density w.r.t. \( \lambda^m \).

2. If \( f \) is a.s. stochastically differentiable w.r.t. the grid \( \mathcal{G} \), then, for \( P^\mathcal{Y} \) – almost all \( \upsilon \in \mathcal{Y} \), the function \( f_\upsilon \ \lambda^m \)-almost everywhere possesses partial derivatives \( \frac{\partial}{\partial z_i} f_\upsilon, \ldots, \frac{\partial}{\partial z_m} f_\upsilon \). In addition,
\[ \frac{\partial}{\partial z_i} f_\upsilon = [D^{\mathcal{G}}_{h_i} f]_\upsilon, \quad i = 1, \ldots, m \]
almost surely.

3. If \( f \) is stochastically differentiable in \( L_p \) sense w.r.t. the grid \( \mathcal{G} \), then, for \( P^\mathcal{Y} \) – almost all \( \upsilon \in \mathcal{Y} \), the function \( f_\upsilon \) belongs to the local Sobolev space \( W^{1,p}_{\mathcal{G}_{\{0\}}}(\mathbb{R}^m, \lambda^m) \). In addition, the relation (3.4) holds true almost surely (in this case, \( \frac{\partial}{\partial z_i} \) is the Sobolev partial derivative and \( D^{\mathcal{G}}_{h_i} \) is the \( L_p \) stochastic derivative).

**Proof.** Let \( C^1_0(\mathbb{R}^m) \) denote the set of continuously differentiable functions \( \mathbb{R}^m \to \mathbb{R} \) with a compact supports. Denote, by \( \mathcal{C}_0 \), the set of measurable functions \( g \) on \( \Omega \) such that \( g_\upsilon \in C^1_0(\mathbb{R}^m) \), \( \upsilon \in \mathcal{Y} \) and
\[ \sup_{\upsilon \in \mathcal{Y}} \sup_{z \in \mathbb{R}^m} \left[ |g_\upsilon(z)| + \| \nabla g_\upsilon(z) \|_{\mathbb{R}^m} \right] < +\infty. \]

By (3.3) and the dominated convergence theorem, every \( g \in \mathcal{C}_0 \) is \( L_p \) differentiable w.r.t. the grid \( \mathcal{G} \) for every \( p \geq 1 \), and (3.4) holds true at every point.

Denote \( \rho_i = \int_0^\infty h_i(t) \tilde{\upsilon}(t, \Gamma_i) \), \( i = 1, \ldots, m \). One can verify that
\[ \frac{1 - p^\mathcal{G}_{h_i}}{\varepsilon} \to \rho_i, \quad \varepsilon \to 0, \quad i = 1, \ldots, m \]
in \( L_p \) sense for every \( p \geq 1 \). Therefore, for every \( g \in \mathcal{C}_0 \),
\[ E D^{\mathcal{G}}_{h_i} g = \lim_{\varepsilon \to 0} E \frac{T^\mathcal{G}_{h_i}}{\varepsilon} g \frac{g - g}{\varepsilon} = \lim_{\varepsilon \to 0} E g \frac{p^\mathcal{G}_{h_i}}{\varepsilon} - 1 = -Eg_{\rho_i}, \quad i = 1, \ldots, m. \]
Consider a countable dense subset $\Phi$ of $C_0^1(\mathbb{R}^m)$. For every $\phi \in \Phi, c \in \mathbb{B}(y)$, consider the function

$\phi_C: (v, z) \mapsto I_C(v)\phi(z)$. This function belongs to $c_0$ and integration-by-parts formula (3.3) for this function has the form

$$\int_C \int_{\mathbb{R}^m} \frac{\partial}{\partial z_i} \phi(z) P_v(dz) P^y(dv) = - \int_C \int_{\mathbb{R}^m} [\rho_1]_v(z) \phi(z) P_v(dz) P^y(dv), \quad i = 1, \ldots, m.$$  

Since $C \in \mathbb{B}(y)$ is arbitrary, we conclude that, for a given $\phi \in \Phi$,

$$\int_{\mathbb{R}^m} \frac{\partial}{\partial z_i} \phi(z) P_v(dz) P^y(dv) = - \int_{\mathbb{R}^m} [\rho_1]_v(z) \phi(z) P_v(dz) P^y(dv), \quad i = 1, \ldots, m$$

for $P^y$-almost all $v$. Denote, by $\mathcal{Y}_\phi$, the set of $v \in \mathcal{Y}$ such that (3.6) holds.

Every $\rho_i$ is an integral of a bounded function over a compensated Poisson point measure of the finite intensity. Therefore, $E \exp |\rho_1| < +\infty, i = 1, \ldots, m$. Then there exists a set $\mathcal{Y}_* \in \mathbb{B}(\mathcal{Y})$ such that $P^y(\mathcal{Y} \setminus \mathcal{Y}_*) = 0$ such that

$$\int_{\mathbb{R}^m} \exp |\rho_1| P_v(dz) < +\infty, \quad i = 1, \ldots, m, \quad v \in \mathcal{Y}_*.$$  

Thus, for $v \in \mathcal{Y}_* \cap \mathcal{Y}_\phi$, the relation (3.6) holds true for every $\phi \in C_0^1(\mathbb{R}^m)$ and every function $[\rho_i]_v$ possesses an exponential moment. In other words: for every $v \in \mathcal{Y}_*$, the measure $P_v$ is differentiable w.r.t. the basic directions in $\mathbb{R}^m$ and its logarithmic derivative possesses an exponential moment. Then Proposition 3.1.5 provides that $P_v$ possesses a continuous strictly positive density. This completes the proof of the statement 1. This statement provides that, for $P^y$-almost all $v \in \mathcal{Y}$, $P_v$-a.s. convergence is equivalent to $\lambda^m$-a.s. convergence and $L_p(\mathbb{R}^m, P_v)$-convergence implies $L_{p, loc}(\mathbb{R}^m, \lambda^m)$-convergence. Now the statements 2.3 follow from 3.3 and Fubini theorem. The lemma is proved.

3.2. Proof of Theorem 2.1. For every $\omega \notin \Omega^3$, there exists $i \in \{1, \ldots, m\}$ such that $T^{G_{\delta}}_{T^3}, \omega, t \in \mathbb{R}$. This implies that, for such an $\omega$, at least one column in the matrix $\Sigma_{T^3}$ contains zeroes only. Thus, $N(f, G) \subset \Omega^3$ and

$$P(\{f \in B\} \cap N(f, G)) = \int_{\mathcal{Y}} P_v(f_v \in B, \nabla f_v \text{ is non-degenerate}) P^y(dv), \quad B \in \mathbb{B}(\mathbb{R}^m),$$

here $\nabla f_v$ denotes the matrix that contains the partial derivatives of $f_v$, either a.s. or Sobolev ones. Here we have used that, by Lemma 3.2 $\nabla f_v = [\Sigma_{T^3}]_v$, almost surely on $\mathbb{R}^m$ for $P^y$-almost all $v$.

By 3.3 and Fubini theorem, it is enough to prove that, for almost all $v$, the image under the mapping $f_v$ of the Lebesgue measure restricted to $N(f_v) = \{x : \nabla f_v(x), \text{is non-degenerate}\}$ is absolutely continuous. The crucial step in the proof of the latter fact is provided by the following statement.

**Lemma 3.3.** Let $F : \mathbb{R}^m \to \mathbb{R}^n$ to have the approximative partial derivative w.r.t. every basic direction at $\lambda^m$-almost all points of $x \in \mathbb{R}^m$ and $G$ be the corresponding approximative gradient. Then for every $\varepsilon > 0$ there exists $F_\varepsilon \in C^1(\mathbb{R}^m, \mathbb{R}^n)$ such that

$$\lambda^m(\{x : F(x) \neq F_\varepsilon(x)\} \cup \{x : G(x) \neq \nabla F_\varepsilon(x)\}) < \varepsilon.$$  

Lemma 3.3 is a corollary of the following two statements, given in [13].

**Proposition 3.2.** I. ([13], Theorem 3.1.4). Let the function $F : \mathbb{R}^m \to \mathbb{R}^n$ to possess the approximative partial derivative w.r.t. every basic direction at all the points of a set $A \subset \mathbb{R}^m$. Then, for $\lambda^m$-almost all points $a \in A$, the function $F$ possesses the approximative differential.

II. ([13], Theorem 3.1.16). Let $A \subset \mathbb{R}^m, f : A \to \mathbb{R}^n$ and

$$\limsup_{x \to a} \frac{\|F(x) - F(a)\|_{\mathbb{R}^n}}{\|x - a\|_{\mathbb{R}^m}} < +\infty$$

(3.8)
for \( \lambda^m \)-almost all \( a \in A \), then, for every \( \varepsilon > 0 \), there exists \( F_\varepsilon \in C^1(\mathbb{R}^m, \mathbb{R}^n) \) such that
\[
\lambda^m(\{x : F(x) \neq F_\varepsilon(x)\}) < \varepsilon.
\]

We do not discuss here the notions of the approximative upper limit (\( \text{ap lim sup} \)), approximative partial derivative and approximative differential, referring to \cite{13}. Chapter 3. We just mention that, if \( F \) either belong to \( W^1_{p,\text{loc}}(\mathbb{R}^m) \) or possesses partial derivatives w.r.t. basic directions at \( \lambda^m \)-almost all points, then \( F \) possesses approximative partial derivatives w.r.t. basic directions at \( \lambda^m \)-almost all points. Moreover, if the function \( F \) possesses the approximative differential at the point \( a \), then \( \text{3.3} \) holds true at this point.

Thus, for every \( \varepsilon > 0 \) and almost every \( v \), there exists \( f^v_\varepsilon \in C^1(\mathbb{R}^m, \mathbb{R}^n) \) such that the Lebesgue measure of the set
\[
C(v, \varepsilon) := \{z \in \mathbb{R}^m : f^v(z) \neq f^v_\varepsilon(z)\}
\]
is less that \( \varepsilon \). By the Lebesgue theorem, almost every point of \( \mathbb{R}^m \setminus C(v, \varepsilon) \) is a Lebesgue point (i.e. a density point), and therefore \( \nabla f^v = \nabla f^v_\varepsilon \) a.e. on \( \mathbb{R}^m \setminus C(v, \varepsilon) \).

Now, the image measure of \( \lambda^m|_{N(v_\varepsilon)} \) under \( f_\varepsilon \) can be represented as the sum of the two measures
\[
\lambda_m|_{\mathbb{R}^m \setminus C(v, \varepsilon) \cap N(v_\varepsilon)} \circ [f^v_\varepsilon]^{-1} \quad \text{and} \quad \lambda_m|_{C(v, \varepsilon) \cap N(v_\varepsilon)} \circ [f^v_\varepsilon]^{-1}.
\]
The first one is absolutely continuous by the standard change-of-variables formula for \( C^1 \)-transformations. The second one has its total mass being less than \( \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, this proves the required absolute continuity. The theorem is proved.

3.3. **Proofs of Theorems** \cite{22,23,24} Theorem 2.1 \cite{11} provides the criterium for convergence in variation of induced measures on a finite-dimensional space. In our considerations, we use two following sufficient conditions, based on this criterium.

**Proposition 3.3.** 1. (\cite{11}, Corollary 2.7). Let \( F, F_n \in W^1_{p,\text{loc}}(\mathbb{R}^m, \mathbb{R}^m) \) with \( p \geq m \), and \( F_n \to F, n \to \infty \) w.r.t. Sobolev norm \( \| \cdot \|_{W^1_p(\mathbb{R}^m, \mathbb{R}^m)} \) on every ball. Then
\[
\lambda^m|A | F^{-1} \circ F_n^{-1} \rightharpoonup \lambda^m|A | F^{-1}, n \to +\infty \quad \text{for every measurable } A \subset \{\det \nabla F \neq 0\}.
\]

II. (\cite{24}, Theorem 3.1). Let \( F_n, F : \mathbb{R}^m \to \mathbb{R}^n \) possess approximative partial derivatives at \( \lambda^m \)-almost every point and \( F_n \to F, \nabla F_n \to \nabla F \) in a sense of convergence in measure \( \lambda^m \). Let, in addition, the sequence \( \{F_n\} \) be uniformly approximatively Lipschitz. This, by definition, means that, for every \( \delta > 0, R < +\infty \), there exist a compact set \( K_{\delta, R} \) and a constant \( L_{\delta, R} < +\infty \) such that \( \lambda^m(\mathbb{B}(0, R) \setminus K_{\delta}) < \delta \) and every function \( F_n|_{K_{\delta}} \) is a Lipschitz function with the Lipschitz constant \( L_{\delta, R} \). Then (\text{3.3}) holds true.

Under conditions of Theorem 2.2 the statements 1 and 3 of Lemma 3.2 provide that, for \( P^y \)-almost all \( v \), \( f^n \to f_v \) w.r.t. Sobolev norm \( \| \cdot \|_{W^1_p(\mathbb{R}^m, \mathbb{R}^m)} \) on every ball. Then the statement I of Proposition 3.3 and the statement 1 of Lemma 3.2 provide that, for every \( B \in \mathcal{B}(\mathbb{R}^m) \),
\[
P_{v|N(f_\varepsilon) \cap B} \circ [f^n_\varepsilon]^{-1} \rightharpoonup P_v|_{N(f_\varepsilon) \cap B} \circ [f^n_\varepsilon]^{-1} \quad \text{for } P^y \text{-almost all } v \in \mathcal{Y}.
\]

By applying the decomposition formula (3.7) and Fubini theorem, we complete the proof of Theorem 2.2.

Via the same arguments, the statement II of Proposition 3.3 would provide the proof of Theorem 2.3, but we have to verify additionally that, for \( P^y \)-almost all \( v \), the sequence \( \{f^n_\varepsilon \mathbf{1}_\mathcal{Y}\} \) is uniformly approximatively Lipschitz.

Recall that \( N(f, \mathcal{Y}) \subset \Omega^\mathcal{Y} \) and thus we can exclude \( \omega \notin \Omega^\mathcal{Y} \) from the consideration. By the analogy with Definition 2.2, we say that the sequence of measurable functions \( \{F_n : \mathbb{R}^m \to \mathbb{R}, n \geq 1\} \) has a **uniformly dominated increments** on the set \( O \in \mathcal{B}(\mathbb{R}) \) if there exist a measurable function \( \varphi \) and a family of jointly measurable functions \( \{G_i : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}\} \) such that
(i) for every \( i \) and \( \lambda^m \)-almost every \( z \), the function \( G_i(z, \cdot) \) is an increasing one;
(ii) \( \varrho > 0 \ \lambda^m \)-almost surely and, for every \( n \geq 1, z \in \mathbb{R}^m \),

\[
|F_n(z + t) - F_n(z + s)| \leq \sum_{i=1}^{m} \left| G_i(z_i, t_i \vee s_i) - G_i(z_i, t_i \wedge s_i) \right|, \quad \|t\|, \|s\| < \varrho(z), z + t \in O, z + s \in O.
\]

It follows from (2.25), (3.3) and Fubini theorem that, for every \( i = 1, \ldots, m \) and \( P^\prime \)-almost all \( v \in \mathcal{Y} \), the sequence \( \{f^n_i|v\} \) has a uniformly dominated increments on \( [\Omega]_v \). Since \( \lambda^m \)-almost every point of \( [\Omega]_v \) is a density point for \( [\Omega]_v \), the approximative partial derivatives of the functions \( f^n_i|v \), \( n \geq 1, i = 1, \ldots, m \) coincide with those of the functions \( f^n_i|v \), \( n \geq 1, i = 1, \ldots, m \). Thus Theorem 2.3 follows from the Fubini theorem, the statement II of Proposition 3.3 and the following lemma.

**Lemma 3.4.** Suppose the sequence \( \{F_n : \mathbb{R}^m \rightarrow \mathbb{R} \} \) to converge \( \lambda^m \)-a.s. and to have a uniformly dominated increments on \( O \in \mathcal{B}(\mathbb{R}^m) \). Then the sequence \( \{F_n I_O \} \) is uniformly approximatively Lipschitz.

**Proof.** Let \( \delta, R > 0 \) be fixed. Denote \( A_{\epsilon, R} = \{z : \|z\| \leq R, \varrho(z) > 2m\epsilon \} \) and take \( \varepsilon > 0 \) such that \( \lambda^m(B^m_\varepsilon(0, R) \setminus A_{\epsilon, R}) < \frac{\delta}{6} \). Consider the family of a rectangles of the type \( \prod_{i=1}^{m}(n_i\varepsilon, (n_i + 1)\varepsilon), n_1, \ldots, n_m \in \mathbb{Z} \).

This family performs a partition of \( \mathbb{R}^m \) up to a set of zero Lebesgue measure. In this family, consider the sets that provide non-empty intersections with \( A_{\epsilon, R} \) and denote these sets by \( B^j_1, \ldots, B^j_t \), here \( J < +\infty \) is the total number of the sets.

Let \( j \in \{1, \ldots, J\} \) be fixed. Then there exists \( z^j_i \in B^j_i \) such that \( \varrho(z^j_i) > 2m\epsilon \). Since the diameter of \( B^j_i \) does not exceed \( 2m\epsilon \), this provides that every point \( x \in B^j_i \) can be written to the form \( x = z^j_i + t \) with \( |t| < \varrho(z^j_i) \). Denote, for \( i = 1, \ldots, m \), \( G^j_i(r) = G_i(z^j_i - r, -z^j_i \vee r), r \in \mathbb{R} \).

Then, from (3.10), we have that

\[
|F_n(x) - F_n(y)| \leq \sum_{i=1}^{m} \left| G^j_i(x_i \vee y_j) - G^j_i(x_i \wedge y_j) \right|, \quad x, y \in B^j_i \cap O.
\]

The set \( B^j_1 \) is a product of intervals \( (c^1_i, d^1_i), i = 1, \ldots, m \). Every function \( G^j_i \) is monotonous and thus differentiable at \( \lambda^i \)-almost all points of the interval \( (c^1_i, d^1_i) \). Thus Lemma 3.3 provides that, for every \( \gamma > 0 \), there exists a function \( G^j_{i, \gamma} \in C^1(\mathbb{R}) \) and a compact set \( K^j_{i, \gamma} \subset (c^1_i, d^1_i) \) such that \( G^j_i \) is continuous at every point of \( K^j_{i, \gamma} \), \( G^j_i = G^j_{i, \gamma} \) on \( K^j_{i, \gamma} \) and \( \lambda^i((c^1_i, d^1_i) \setminus K^j_{i, \gamma}) \leq \gamma \). Denote \( K^j_{i} = \prod_{i=1}^{J} K^j_{i, \gamma} \). One can choose \( \gamma \) small enough for

\[
\sum_{j=1}^{J} \lambda^m(B^j \setminus K^j_i) < \frac{\delta}{6}.
\]

The function \( G^j_{i, \gamma} \) is locally Lipschitz, and therefore the restriction of \( G^j_i \) to \( K^j_{i, \gamma} \) is Lipschitz with some constant \( L^j_{i, \gamma} \). Then, by (3.11), the restriction of \( F_n I_O \) to \( K^j_i \cap O \) is Lipschitz with the constant \( L^j_i = \sum_{\gamma} L^j_{i, \gamma} \). The restriction of \( F_n I_O \) to \( B(0, R) \setminus O \) is also Lipschitz with the constant 0.

By Ulam theorem, there exist compact sets \( \tilde{K} \subset \bigcup_{j}(K^j_i \cap O) \) and \( \tilde{K} \subset B(0, R) \setminus O \) such that

\[
\lambda^m \left( \bigcup_{j}(K^j_i \cap O) \setminus \tilde{K} \right) + \lambda^m \left( B(0, R) \setminus (O \cup \tilde{K}) \right) < \frac{\delta}{6}.
\]

At last, by Egorov theorem, there exist \( C > 0 \) and a compact set \( K_* \) with \( \lambda^m(B^m_\varepsilon(0, R) \setminus K_*) < \frac{\delta}{6} \) such that \( |F_n(x)| \leq C, x \in K_*, C > 0 \). By the construction, there exists \( \theta > 0 \) such that \( \|x - y\| \geq \theta \) as soon as \( x \in K^j_i, y \in K^j_i \) with \( j_1 \neq j_2 \) or \( x \in \tilde{K}, y \in \tilde{K} \). Therefore, for every \( n \), the restriction of \( F_n I_O \) to \( K^j_{i, \gamma} \) is Lipschitz with the constant \( L^j_{i, \gamma} \). By the construction, \( A_{\epsilon, R} \subset \bigcup_{j=1}^{J} B^j_1 \) and thus

\[
\lambda^m(B(0, R) \setminus K^j_{i, \gamma}) < \lambda^m(B(0, R) \setminus A_{\epsilon, R}) + \frac{\delta}{6} + \frac{\delta}{6} < \frac{\delta}{3}.
\]
This completes the proofs of Lemma 3.4 and Theorem 2.3.

4. Absolute continuity and convergence in variation of distributions to SDE’s with jumps

In this section, applications of Theorems 2.1–2.3 to solutions of SDE’s with jumps are given. We consider separately two classes of SDE’s. The first one contains SDE’s with additive noise of the type (4.1). The second one contains SDE’s with non-additive noise, including SDE’s with non-constant jump rate, of the type (4.16). The latter class does not cover the former one because the conditions imposed on the measure $\nu$ and the coefficients of (4.16) imply that the solution to (4.16) possesses trajectories with bounded variation, while the Lévy process $Z$ in (4.1) may be arbitrary.

Let us introduce notational conventions. Any time the functional $f$ of $\nu$ is expressed explicitly through the coefficients $a, b, c$ and the measure $\nu$, $f^n$ denotes the functional of the same form with the coefficients $a^n, b^n, c^n$ and the same point measure. We introduce conditions $H_1, H_2, \ldots$ for a one functional $f$ in the terms of the coefficients involved into expression for this functional $(a, b, c$ etc.). Then we write $H^*_1, H^*_2, \ldots$ for the uniform analogues of these conditions, imposed on the sequence $\{f^n\}$. The constants in these conditions, as well as the auxiliary functions $\alpha, \beta, \ldots$, are the same with those in conditions $H_1, H_2, \ldots$. The partial derivative w.r.t. time variable is denoted by $\partial_t$. The gradient w.r.t. phase variable $x \in \mathbb{R}^m$ is denoted by $\nabla$.

The unit sphere in $\mathbb{R}^m$ is denoted by $S^m$.

4.1. SDE’s with additive noise. Let $U = U_1 \cup U_2$ with $\Pi(U_1) < +\infty$. Denote

$$Z(t) = \int_0^t \int_{U_1} c(u)\nu(ds, du) + \int_0^t \int_{U_2} c(u)\nu(ds, du), \quad t \in \mathbb{R}^+,$$

where $c : U \to \mathbb{R}^m, \|c\|_{U_2} \in L_2(\Pi)$. Consider SDE driven by the Lévy process $Z$:

$$X(x, t) = x + \int_0^t a(X(x, s))\, ds + Z(t), \quad x \in \mathbb{R}^m, t \in \mathbb{R}^+.$$  \hspace{1cm} (4.1)

Under condition

$$H_1, a \in C^1(\mathbb{R}^m, \mathbb{R}^m), \|a(x)\| \leq C(1 + \|x\|),$$

equation (4.1) possesses unique strong solution. Put $f = X(x, t), f^n = X^n(x^n, t^n), \Delta(x, u) = a(x+c(u)) - a(x),$

$$\mathcal{N}(f) = \{ g(f) = \mathbb{R}^m \}, \quad \mathcal{S}(f) = \text{span} \{ \mathcal{E}_s^t \Delta(X(\tau^-), p(\tau)), \tau \in \mathcal{D} \cap [0, t] \},$$

where $\mathcal{E}_s^t, 0 \leq s \leq t$ denotes the stochastic exponent, i.e. the $m \times m$-matrix valued process defined by the equation

$$\mathcal{E}_s^t = \mathcal{I}_{\mathbb{R}^m} + \int_s^t \nabla a(X(x, r))\mathcal{E}_r^s \, dr, \quad t \geq s.$$ 

**Theorem 4.1.** 1. Under condition $H_1$, $P_{[X(x, t)]} \circ f^{-1} \ll \lambda^m$.

2. Let $x^n \to x, t^n \to t, c^n(\cdot) \to c(\cdot)$ $\Pi$-almost everywhere and $a^n \to a, \nabla a^n \to \nabla a$ uniformly on every compact set. Suppose also that $H^*_1$ holds true and

$$\|c^n(u)\|_{U_2} \leq \alpha(u), \quad u \in U_2 \quad \text{with} \quad \alpha \in L_2(\Pi).$$

Then, for every $A \subset \mathcal{N}(f)$,

$$P_A \circ [X^n(x^n, t^n)]^{-1} \to P_A \circ [X(x, t)]^{-1}, \quad n \to +\infty$$
in variation.

3. Let there exist $\varepsilon > 0$ such that

$$\Pi\left( u : (\Delta(y, u), l)_{\mathbb{R}^m} \neq 0 \right) = +\infty, \quad l \in \mathbb{S}^m, y \in \bar{B}(y, \varepsilon) \overset{df}{=} \{ y : \|y - x\| \leq \varepsilon \}. $$  \hspace{1cm} (4.2)
Then $P(N(f)) = 1$.

**Corollary 4.1.** Let conditions (4.2) hold true for every $x \in \mathbb{R}^m$. Then the transition probability for the process $X$, considered as a Markov process, possesses a density: $P(X(x,t) \in dy) = p_{x,t}(y) dy$. Moreover, the mapping $\mathbb{R}^m \times (0, +\infty) \ni (x,t) \mapsto p_{x,t}(\cdot) \in L_1(\mathbb{R}^m, \lambda^m)$ is continuous and, consequently, $X$ is a strongly Feller process.

**Remark 4.1.** Examples are available (see [23], Example 1.4 and Proposition 1.2), such that $p_{x,t} \not\in L_{p,\text{loc}}(\mathbb{R}^m)$ for every $p > 1, x \in \mathbb{R}^m, t > 0$. This means that, in some sense, the continuity property exposed in the Corollary 4.1 is the best possible one when no additional restrictions on the measure $\Pi$ are imposed.

The proof of Theorem 4.1 contains several steps. First, we prove differentiability of $f$ and the property of $\{f^n\}$ to have a uniformly dominated increments. Let a grid $\mathcal{G}$ of dimension $m$ be fixed.

**Proposition 4.1.** Under conditions of Theorem 4.1, every component of the vector $X(x,t)$ is a.s. differentiable w.r.t. $\mathcal{G}$ for every $x \in \mathbb{R}^m, t \in \mathbb{R}^+$. For every $i = 1, \ldots, m$, the process $Y_i(x, \cdot) = (D^y_i X_j(x,t))_{j=1}^m$ satisfies the equation

$$
Y_i(x,t) = \int_0^t \int_{\Gamma_i} \Delta(X(x,s-), u) J_i(s) \nu(ds, du) + \int_0^t [\nabla a](X(s,x)) Y_i(x,s) ds, \quad t \geq 0.
$$

**Remark 4.2.** In the case $m = 1$, the analogous result was proved in [29]. We cannot use here the result from [29] straightforwardly, since the proof there contains some specifically one-dimensional features such as an exponential formula for the derivative of the flow corresponding to ODE (Lemma 1 [29]).

**Proof.** We fix $i \in \{1, \ldots, m\}$ and omit the subscript $i$ in the notation. Denote $\nu^\tau(t, A) = \nu(t, A \setminus \Gamma)$,

$$
Z^\tau(t) = \int_0^t \int_{U_i \setminus \Gamma} c(u) \nu(ds, du) + \int_0^t \int_{U_j \setminus \Gamma} c(u) \nu(ds, du) - \int_0^t \int_{U_i \cap \Gamma} c(u) \Pi(du) ds.
$$

For a given $t > 0, \tau \in (0, t), p \in U, x \in \mathbb{R}^m$, consider the process $X^\tau$ on $[0, t]$ defined by

$$
X^\tau_r = \begin{cases} 
x + \int_0^r a(X^\tau_s) ds + Z^\tau(r), & r < \tau \\
x + \int_0^r a(X^\tau_s) ds + c(p) + Z^\tau(r), & r \geq \tau
\end{cases}
$$

Denote $\Omega_k = \{\mathcal{D} \cap \{0, t\} = \emptyset, \#(\mathcal{D} \cap (0, t)) = k\}, k \geq 0$. Since $\Gamma \in U_{fix}, \Omega = \bigcup_k \Omega_k$ almost surely. Thus, it is enough to prove a.s. differentiability on every $\Omega_k$, separately. The case $k = 0$ is trivial, let us consider the case $k = 1$.

By the construction of the transformations $T_{th}^\tau$,

$$
[T_{th}^\tau \tau^\tau_j](\omega) = T_{-th}(\tau^\tau_j(\omega)), \quad \omega \in \Omega.
$$

The point process $\{p(r), r \in \mathcal{D}^\tau\}$ is independent of $\nu^\tau$, and the distribution of the variable $\tau^\tau = \min \mathcal{D}^\tau$ is absolutely continuous. Thus a.s. differentiability of $X(x,t)$ on $\Omega_1$ follows immediately from (4.4) and the following lemma.

**Lemma 4.1.** With probability 1, for $\lambda^1$-almost all $\tau \in (0, t)$,

$$
\left. \frac{d}{d \varepsilon} \right|_{\varepsilon=0} X^{\tau+\varepsilon} = -\Delta(X^{\tau-}, p) \mathcal{E}_t
$$

with $\mathcal{E}_t$ defined by the equation

$$
\mathcal{E}_r = I_{\mathbb{R}^m} + \int_\tau^r \nabla a(X^\tau(s)) \mathcal{E}_s ds, \quad r \geq \tau.
$$
The distributions of the variables \( \tau, q \) of \( X \). Proposition 4.2. The proposition is proved. This implies the statement of the lemma. The lemma is proved.

The estimates analogous to ones made before show that, up to the \( \varepsilon \) terms, the process \( Z^\Gamma \) has càdlàg trajectories, and therefore almost surely the set of discontinuities for its trajectories is at most countable. In addition, every given point \( s \in \mathbb{R}^+ \) is a continuity point for the trajectory of \( Z^\Gamma \) almost surely. Therefore, there exists a set \( \mathbb{T} = \mathbb{T}(\omega) \subset \mathbb{R}^+ \) of the full Lebesgue measure such that

\[
\delta(t, \gamma) \equiv \sup_{|s-t| \leq \gamma} \left[ \| Z^\Gamma(s) - Z^\Gamma(t) \| \right] \to 0, \quad \gamma \to 0, \quad t \in \mathbb{T}
\]

and \( \tau \in \mathbb{T} \) a.s. Then \( \| X^\tau_\tau - X^\tau_\tau \| + \| X^\tau+\varepsilon_\tau - X^\tau+\varepsilon_\tau - c(p) \| \leq C \{ |\varepsilon| + \delta(\tau, |\varepsilon|) \} \) for \( s \in (\tau + \varepsilon, \varepsilon) \). Here and below, \( C \) denotes any constant such that it can be expressed explicitly, but its exact form is not needed in a further exposition. Thus, for \( \tau \in \mathbb{T} \),

\[
\| \Phi(\tau, \varepsilon) + \varepsilon [a(X^\tau_\tau + c(p)) - a(X^\tau_\tau)] \| \leq C |\varepsilon| \{ |\varepsilon| + \delta(\tau, |\varepsilon|) \}.
\]

The solution to (4.3) with the starting point \( \tau \) is differentiable w.r.t. initial value with the derivative being equal \( \mathcal{E} \). This statement is quite standard and we omit the proof. This together with (4.6) implies the needed statement.

The case \( \varepsilon > 0 \) is analogous, let us discuss it briefly. Again, take \( \tau \in \mathbb{T} \) and represent \( X^\tau_\tau \) as the solution to (4.3) with the initial value \( X^\tau_\tau + p \). \( X^\tau+\varepsilon \) is also the solution to (4.3) but with the other starting point \( \tau + \varepsilon \). The estimates analogous to ones made before show that, up to the \( o(|\varepsilon|) \) terms,

\[
X^\tau+\varepsilon_\tau - X^\tau+\varepsilon_\tau = \varepsilon \left\{ -a(X^\tau_\tau + p) + a(X^\tau_\tau) \right\},
\]

which implies the statement of the lemma. The lemma is proved.

Now let \( k > 1 \) be fixed. Consider the countable family \( \Omega_k \) of partitions \( Q = \{ 0 = q_0 < q_1 \cdots < q_k = t \} \) with \( q_1, \ldots, q_{k-1} \in \mathcal{Q} \) and denote

\[
\Omega_Q = \{ \mathcal{D} \cap \{ q_i, i = 0, k \} = \varnothing, \mathcal{D} \cap \{ q_{i-1}, q_i \} = 1, i = 1, \ldots, k \}, \quad Q \in \Omega_k.
\]

We have \( \Omega_k = \bigcup_{Q \in \Omega_k} \Omega_Q \). Therefore, it is enough to verify a.s. differentiability of \( X(x, t) \) on \( \Omega_Q \) for a given \( Q \). The distributions of the variables \( \tau_j^\Gamma, j = 1, \ldots, k \) are absolutely continuous. Then one can write the statements analogous to the one of Lemma 4.1 on the intervals \( [0, q_1], [q_1, q_2], \ldots, [q_{k-1}, t] \) and obtain a.s. differentiability of \( X(x, t) \) from (4.4) and the theorem on differentiability of the solution to (4.1) w.r.t. initial value. The proposition is proved.

**Proposition 4.2.** Let conditions of Theorem 4.1 hold true. Let, in addition, the sequence \( \{ a^n \} \) be uniformly bounded and \( J h_n(t_n) = 0, n \geq 1, i = 1, \ldots, m \). Then the sequence \( \{ X^n(x^n, t^n) \} \) has a uniformly dominated increments w.r.t. \( \mathbb{S} \) on \( \Omega' = \Omega \).

**Proof.** Again, we omit \( i \) in notation. In the framework of Lemma 4.1 one has the estimate

\[
\| X^{\tau+\varepsilon}_\tau \| \leq C |\varepsilon|, \quad \tau, \tau + \varepsilon \in (0, t),
\]
valid point-wise. Indeed, both $X_i^{τ+ε}$ and $X_i^{τ}$ are the solutions to (4.3) with the same initial point $(τ$ for $ε < 0$ and $τ + ε$ for $ε > 0$) and different initial values. The difference between the initial values are estimated by

$$
\left\| \int_τ^{τ+ε} [a(X_i^{τ+ε}) - a(X_i^{τ})] ds \right\| \leq -2||a||_\infty for ε < 0 and \\
\left\| \int_τ^{τ+ε} [a(X_i^{τ+ε}) - a(X_i^{τ})] ds \right\| \leq 2||a||_\infty for ε > 0.
$$

Thus inequality (4.7) follows from the Gronwall lemma. Using the described above technique, involving partitions $Q \in Q_k$, and applying the Gronwall lemma once again, we obtain that, almost surely on the set $Ω_k$,

$$||T_hX(x, t) - X(x, t)|| \leq kC sup_s |Jh(s)||ε|.
$$

(4.8)

Here we have used that $Jh(t) = 0$ and thus $T_hx \in (0, t)$ as soon as $x \in (0, t)$.

The same estimate holds true for every $n$. Thus every sequence $\{X^n_i(x^n, t^n)\}, j = 1, \ldots, m$, satisfies (4.5) with $ϕ \equiv +∞$ and $g_i(t) = t C sup_s |Jh_i(s)| f_i ν([0, T] × Γ_i), t \in \mathbb{R}, i = 1, \ldots, m, T = sup_n t_n$. The proposition is proved.

Now we apply Theorems 2.1 and 2.2 in order to prove statements 1 and 2 of Theorem 4.1.

Proof of statement 1 of Theorem 4.1. Consider the family $\{U_N, N \geq 1\}$ of bound measurable subsets of $\mathbb{U}$ such that $\mathbb{U}_N \uparrow \mathbb{U}, N \rightarrow +∞$. Denote, by $S_N$, a linear span of the set of vectors $\{E^1, \Delta(X(τ_), p(τ)), τ \in \mathbb{D}^U \cap (0, t)\}$ and put $Ω_N = \{ω : S_N(ω) = \mathbb{R}^m\}$. It is clear that $N(f) = \bigcup_{N \geq 1} Ω_N$. Thus, in order to prove statement 1 of Theorem 4.1 it is enough to prove that $P[Ω_N \cap [X(x, t)]^{-1} ≤ λ^m$ for a given $N$.

Let $N$ be fixed. Denote by $L^M_t$ the set of all vectors $l = (l_1, \ldots, l_m) \in \mathbb{N}^m$ with $l_1 < l_2 \cdots < l_m$ and $Ml_m < t$. Consider the family of differential grids $\{G^M, t \in L^M_t, M \geq 1\}$ of the form $Γ^M_i = \mathbb{U}_N$,

$$a_i^{M,l} = \frac{l_i - 1}{M}, b_i^{M,l} = \frac{l_i}{M}, h_i^{M,l}(s) = h \left( \frac{s - a_i^{M,l}}{b_i^{M,l} - a_i^{M,l}} \right), \quad s \in \mathbb{R}^+, i = 1, \ldots, m, l \in L^M_t, M \geq 1,$$

where $h \in H_0$ is some fixed function such that $Jh > 0$ inside $(0, 1)$ and $Jh = 0$ outside $(0, 1)$.

Our aim is to show that almost surely

$$Ω^N \subset \bigcup_{l,M} \{ω : Σ^{X(x, t), G^M} \text{ is non-degenerate} \},
$$

(4.9)

see Theorem 2.1 for the notation $Σ^f, g$. Theorem 2.1 together with (4.5) immediately imply the needed statement.

Denote $A^N_M = \{ω : Δ(X(τ_), p(τ)), τ \in \mathbb{D}^U \cap (0, 1], i = 1, \ldots, [Mt + 1]\}$. Since $Ω_N \in \mathbb{U}_{fin}$, one has that almost surely $Ω^N \subset \bigcup_{M} Ω^N \cap A^N_M$. On the other hand, $Ω^N \cap A^N_M = \bigcup_{l,M} A^N_M$ with

$$A^N_M = \bigcup_{l,M} \{ω : \text{span } \{E^1, \Delta(X(τ_), p(τ)), τ \in \mathbb{D}^U \cap (0, 1], i = 1, \ldots, m\} = \mathbb{R}^m \}.
$$

Thus, in order to prove (4.9), it is sufficient to show that, for every $M, l$, the matrix $Σ^{X(x, t), G^M}$ is non-degenerate on the set $Ω^N \cap A^N_M$. By (4.3),

$$D_i^{G^M} X(x, t) = Jh(τ_i^{M,l})E_i^{M,l}Δ(X(x, τ_{i-1}^{M,l})), \quad i = 1, \ldots, m,
$$

on the set $A^N_M$, where $τ_{i-1}^{M,l}$ denotes the (unique) point from $\mathbb{D}^U \cap (\frac{l_i - 1}{M}, \frac{l_i}{M})$. By the construction, $Jh(τ_{i-1}^{M,l}) > 0, i = 1, \ldots, m$ and the family

$$E_i^{M,l}Δ(X(x, τ_{i-1}^{M,l})), \quad i = 1, \ldots, m,
$$

has the maximal rank on the set $Ω^N \cap A^N_M$. Thus the matrix $Σ^{X(x, t), G^M}$ is non-degenerate on this set. This completes the proof of statement 1.
Proof of statement 2 of Theorem 4.4. Consider first the case with \{a^n\} uniformly bounded. The standard limit theorem for SDE’s provide that \(X^n(x^n,t^n) \to X(x,t)\) in probability and, for every grid \(S\), \(Y^n_i(x^n,t^n) \to Y_i(x,t)\) in probability. Then Theorem 2.3 and Proposition 4.2 imply immediately that, for every \(B \in \mathcal{F}\),

\[
P|_{B \cap \mathcal{A}_{M,l}^{N,N,t}} \circ [f^n]^{-1} \varpi P|_{B \cap \mathcal{A}_{M,l}^{N,N,t}} \circ f^{-1}, \quad M, N \geq 1, l \in \mathbb{L}^M_i.
\]

(4.10)

We have already proved that \(N(f) = \bigcup_{M,N,l} A^{N,N,t}_{M,l}\), and thus (4.10) provides the required statement. The additional limitation on \(\{a^n\}\) to be uniformly bounded can be removed via the following standard localisation procedure. Take, for \(R > 0\), the function \(a_R\) and the uniformly bounded sequence \(\{a^n_R\}\) such that \(a^n_R \to a_R, \nabla a^n_R \to \nabla a_R\) uniformly over every bounded set and \(a^n_R(x) = a^n(x), a_R(x) = a(x), \|x\| \leq R\). Then, on the set \(\{\sup_n \sup_s \leq t_n \|X^n(x^n,s)\| \leq R\}\), solutions to (4.1) with the coefficients \(a^n\) coincide with the solutions to (4.1) with the coefficients \(a^n_R\), respectively. Thus, for every \(A \subset N(f)\),

\[
P|_{A \cap \{\sup_n \sup_s \leq t_n \|X^n(x^n,s)\| \leq R\}} \circ [f^n]^{-1} \varpi P|_{A \cap \{\sup_n \sup_s \leq t_n \|X^n(x^n,s)\| \leq R\}} \circ f^{-1}.
\]

One can see (the proof is standard and omitted) that \(P(\sup_n \sup_s \leq t_n \|X^n(x^n,s)\| \leq R) \to 1, R \to +\infty\). This completes the proof of statement 2.

Proof of statement 3 of Theorem 4.4. We have that, under conditions of Theorem 4.1

\[
\gamma_N \equiv \inf_{y \in \mathbb{B}(x,v), v \in \mathbb{S}^m} \Pi \left( u \in \mathbb{U}_N : (\Delta(y,u,v) \mathbb{R}^m) \neq 0 \right) \to +\infty, \quad N \to +\infty.
\]

(4.11)

This statement follows immediately from the Dini theorem applied to the monotone sequence of lower semi-continuous functions

\[
\phi_N : \mathbb{B}(x,v) \times \mathbb{S}^m \ni (y,v) \mapsto \Pi \left( u \in \mathbb{U}_N : (\Delta(y,u,v) \mathbb{R}^m) \neq 0 \right).
\]

With probability 1, the matrix \(\mathcal{E}_0^t\) is invertible for every \(r\) and the function \(r \mapsto \mathcal{E}_0^t\) is continuous (e.g. [52], Chapter 5, §10). In addition, \(\mathcal{E}_0^t = \mathcal{E}_0^t[\mathcal{E}_0^t]^{-1}, r \in [0, t]\). Therefore,

\[
\mathbb{S}(f) = \mathbb{R}^m \iff \text{span} \left\{ [\mathcal{E}_0^t]^{-1} \cdot \Delta(X(\tau), p(\tau)), \tau \in \mathbb{D} \cap (0, t) \right\} = \mathbb{R}^m
\]

almost surely. Denote by \(\mathbb{S}\) the set of all proper subspaces of \(\mathbb{R}^m\). This set can be parameterized in such a way that it becomes a Polish space, and, for every family of random vectors \(\xi_1, \ldots, \xi_k\), the map \(\omega \mapsto \text{span} (\xi_1(\omega), \ldots, \xi_k(\omega))\) defines the random element in \(\mathbb{S}\).

For every \(N \geq 1\), consider the set \(\mathbb{D}^U_N = \{\tau_1^N, \tau_2^N, \ldots\}\). Denote

\[
\mathbb{S}_N^0 = \text{span} \left\{ [\mathcal{E}_0^t]^{-1} \cdot \Delta(X(\tau), p(\tau)), \tau \in \mathbb{D}^U_N \cap (0, t) \right\}, \quad \mathbb{S}_\delta = \text{span} (\bigcup_N \mathbb{S}_N^0).
\]

For a given \(S^* \in \mathbb{S}\), \(\delta > 0\), consider the event

\[
D_N^\delta = \{S_N^0 \not\subset S^*\} = \{\exists k : \tau_k^N \leq \delta, [\mathcal{E}_0^t]^{-1} \Delta(X(\tau_k^N), p(\tau_k^N)) \notin S^*\}
\]

One has that \(\Omega \setminus D_N^\delta \subset \mathbb{B}_N \cup \mathbb{C}_N^\delta\), where \(\mathbb{B}_N = \{\exists s \in [0, \delta] : X(s) \notin \mathbb{B}(x, \varepsilon)\}, \mathbb{C}_N^\delta = \bigcap_k \{\tau_k^N > \delta\} \cup \{X(\tau_k^N) \notin \mathbb{B}(x, \varepsilon), [\mathcal{E}_0^t]^{-1} \Delta(X(\tau_k^N), p(\tau_k^N)) \in S^*, \tau_k^N \leq \delta\}\}.

The distribution of the value \(p(\tau_k^N)\) is equal to \(\lambda_N^{-1} \Pi |_{\mathbb{U}_N}\), where \(\lambda_N = \Pi |_{\mathbb{U}_N}\). Moreover, this value is independent with the \(\sigma\)-algebra \(\mathcal{F}_{\tau_k^N}\), and, in particular, with \(X(\tau_k^N), \mathcal{E}_0^t\). This provides the estimate

\[
P \left[ \{\tau_k^N > \delta\} \cup \{X(\tau_k^N) \in \mathbb{B}(x, \varepsilon), [\mathcal{E}_0^t]^{-1} \Delta(X(\tau_k^N), p(\tau_k^N)) \in S^*, \tau_k^N \leq \delta\} \right| \mathcal{F}_{\tau_k^N}\right] 
\]

\[
\leq \mathbf{1}_{\{\tau_k^N > \delta\}} + (1 - \frac{\gamma_N}{\lambda_N}) \mathbf{1}_{\{\tau_k^N \leq \delta\}}
\]
with $\gamma_N$ defined in (4.11). It follows from (4.1) that

$$P(C^N_\delta) \leq E \left( 1 - \frac{\gamma_N}{N} \right)^{\nu([0,\delta] \times U_n)} = \exp\{-\delta \gamma_N\} \to 0, \quad N \to +\infty.$$ 

Then $D^N_\delta \subset \{S_\delta \not\subset S^*\}$, and almost surely

$$(4.13) \quad \{S_\delta \subset S^*\} \subset B_\delta.$$ 

Now we take $\delta < \frac{1}{m}$ and iterate (4.13) on the time intervals $[0,\delta], [\delta, 2\delta], \ldots, [(m-1)\delta, m\delta]$ with $S^*_1 = \{0\}, S^*_2 = S_\delta, \ldots, S^*_m = S_{(m-1)\delta}$ (we can do this due to the Markov property of $X$). We obtain that

$$\{\dim S_t < m\} \subset \bigcup_{k=1}^m \{\dim S_{(k-1)\delta} = \dim S_{k\delta} < m\} \subset B_{m\delta}.$$ 

Since $P(B_{m\delta}) \to 0, \delta \to 0+$, this provides that $P\{\dim S_t < m\} = 0$ and completes the proof of Theorem 4.1.

Condition (4.2) involves both the Lévy measure of the noise and the coefficient $a$. In some cases, it would be convenient to have a more explicit sufficient conditions for (4.12), with the restrictions on $a$ and $\Pi$ separated one from another.

The first condition is given in the case $m = 1$. Denote $N(a, z) = \{y \in \mathbb{R} : a(y) = z\}$.

**Proposition 4.3.** Suppose that $\Pi(u : c(u) \neq 0) = +\infty$ and there exists some $\delta > 0$ such that

$$\forall z \in \mathbb{R} \quad \# [N(a, z) \cap (x-\delta, x+\delta)] < +\infty.$$ 

Then (4.2) holds true, and therefore $P(N(f)) = 1$.

**Remark 4.3.** In [29], the law of the solution to one-dimensional SDE (4.1) was proved to be absolutely continuous under condition that $a(\cdot)$ is strictly monotonous at some neighborhood of $x$. One can see that this condition is somewhat more restrictive than the one of Proposition 4.3.

**Proof of the Proposition.** Take $\varepsilon = \frac{\delta}{2}$. Then, for every $y \in \bar{B}(x, \varepsilon)$,

$$\{u : \Delta(y, u) = 0\} = \{u : a(y + c(u)) = a(y)\} \subset \{u : |c(u)| > C \delta\} \cup \{u : x + c(u) \in N(a, a(y)) \cap (x-\delta, x+\delta)\} = \Delta_1 \cup \Delta_2.$$ 

Here we have used that $a$ is Lipschitz. We have that, for every $d > 0$, the set $\{u : |c(u)| > d\}$ has finite measure $\Pi$, and therefore $\Pi(\Delta_1) < +\infty$. The set $N(a, a(y)) \cap (x-\delta, x+\delta) \backslash \{x\}$ is finite and, therefore, separated from $x$. Thus $\Pi(\Delta_2 \backslash \{u : c(u) = 0\}) < +\infty$. Since $\Pi(u : c(u) \neq 0) = +\infty$, this means that $\Pi(\Delta(x, u) \neq 0) = +\infty$. The proposition is proved.

The second sufficient condition is formulated in the multidimensional case. Define a *proper smooth surface* $S \subset \mathbb{R}^m$ as any set of the type $S = \{x : \phi(x) \in L\}$, where $L$ is a proper linear subspace of $\mathbb{R}^m$ and $\phi \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ is such that $\det \nabla \phi(0) \neq 0$ and $\phi^{-1}(\{0\}) = \{0\}$.

**Proposition 4.4.** Suppose that one of the following group of conditions holds true:

a. $a \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, det $\nabla a(x_*) \neq 0$ and

$$(4.14) \quad \Pi(u : c(u) \in \mathbb{R}^m \backslash S) = +\infty \quad \text{for every proper smooth surface } S;$$

b. $a(x) = Ax, A \in \mathbb{L}(\mathbb{R}^m, \mathbb{R}^m)$ is non-degenerate and

$$(4.15) \quad \Pi(u : c(u) \in \mathbb{R}^m \backslash L) = +\infty \quad \text{for every proper linear subspace } L \subset \mathbb{R}^m.$$ 

Then (4.2) holds true, and therefore $P(N(f)) = 1$. 
4.2. Solutions to SDE’s with non-additive noise and non-constant jump rate.

Example 4.1. Let \( \mathcal{U} = \mathbb{R}^2 \setminus \{0\}, c(u) = u, m = 2, \Pi = \sum_{k \geq 1} \delta_{z_k} \), where \( z_k = \left( \frac{1}{k!}, \frac{1}{(k!)^2} \right), k \geq 1 \). Every point \( z_k \) belongs to the parabola \( \{ z = (x, y) : y = x^2 \} \). Since every line intersects this parabola at most at two points, condition (4.15) and assumption (b) given before hold true. On the other hand, for any \( t > 0 \), it is easy to calculate the Fourier transform of the first coordinate \( Z_1(t) \) of \( Z(t) = (Z_1(t), Z_2(t)) \) and show that

\[
\lim_{N \to +\infty} E \exp \{ i2\pi N!|Z_1(t)\} = 1.
\]

This means that the law of \( Z(t) \) is singular.

Although condition (4.15) is not strong enough to provide the Levy process \( Z \) itself to possess an absolutely continuous distribution, Proposition 4.4 shows that this condition appears to be a proper one for the solution to an Orstein-Uhlenbeck type SDE driven by this process to possess a density as soon as the drift coefficient is non-degenerate. At this time, we cannot answer the question whether (4.15) is strong enough to handle the non-linear case, i.e. whether statement a of Proposition 4.3 is valid with (4.14) replaced by (4.15).

4.2. Solutions to SDE’s with non-additive noise and non-constant jump rate. Suppose \( \mathcal{U} \) to have the form \( \mathcal{U} = \mathcal{V} \times \mathbb{R}^+ \) and the measure \( \Pi \) to have the form \( \Pi = \pi \times \lambda^1 \). Denote \( \nu(dt, dv) \equiv \nu(dt, dv, dp), u \equiv (v, p) \) and consider SDE of the type

\[
X(x, t) = x + \int_0^t a(X(s, s)) \, ds + \int_0^t \int_{\mathcal{V}} b(X(s, s^-), v) \, c(X(s, s^-), v) \nu(ds, dv, dp), \quad x \in \mathbb{R}^m, t \in \mathbb{R}^+.
\]

The following conditions are imposed.

\( H_2. \) \( a \in C^1_b(\mathbb{R}^m, \mathbb{R}^m), b(\cdot, v) \in C_b(\mathbb{R}^m, \mathbb{R}^+), c(\cdot, v) \in C^1_b(\mathbb{R}^m, \mathbb{R}^m), \) \( v \in \mathcal{V}. \)
$H_3$. There exist $\beta, \gamma: \mathbb{V} \to \mathbb{R}^+$ such that $\beta \gamma \in L_1(\mathbb{V}, \pi)$ and

$$b(x, v) \leq \beta(v), \quad \|c(x, v)\| \leq \gamma(v), \quad x \in \mathbb{R}^m, v \in \mathbb{V},$$

$$|b(x, v) - b(y, v)| \leq \|x - y\| \beta(v), \quad \|c(x, v) - c(y, v)\| \leq \|x - y\| \gamma(v), \quad x, y \in \mathbb{R}^m, v \in \mathbb{V}.$$  

$H_4$. There exists a representation $b(x, v) = b_0(v) + b_1(x, v)$ such that $b_0 \gamma \in L_1(\mathbb{V}, \pi)$ and, for some $\beta_1, \gamma_1: \mathbb{V} \to \mathbb{R}^+$, $\beta_1 \gamma_1 \in L_2(\mathbb{V}, \pi)$ and

$$|b_1(x, v)| \leq \beta_1(v), \quad \|c(x, v)\| \leq \gamma_1(v), \quad x, y \in \mathbb{R}^m, v \in \mathbb{V}.$$  

$H_5$. $a^n \to a, \nabla a^n \to \nabla a$ and, for $\pi$-almost all $v \in \mathbb{V}$, $b^n(\cdot, v) \to b(\cdot, v), c^n(\cdot, v) \to c(\cdot, v), \nabla c^n(\cdot, v) \to \nabla c(\cdot, v)$ uniformly on every compact set.

Under conditions $H_2, H_3$, equation (4.16) possesses unique strong solutions being a strong Markov processes with càdlàg trajectories. Moreover, under conditions $H_2^*, H_3^*, H_5$, $X^n(x^t, t^n) \to X(x, t)$ in probability for any sequences $x_n \to x$ and $t_n \to t$ (recall that $X^n$ denotes the solution to (4.16) with the coefficients $a, b, c$ replaced by $a^n, b^n, c^n$). We omit the proofs of these statements, referring to [14], Section 2 for the proof of a similar statement.

Put $f = X(x, t)$. Denote by $p_1(\cdot), p_2(\cdot)$ the projections of the point process $p(\cdot)$ on the first and second coordinates in $\mathbb{U} = \mathbb{V} \times \mathbb{R}^+$, respectively. For $y \in \mathbb{R}^m$, denote $\mathbb{V}_y = \{v \in \mathbb{V} : I_{\mathbb{R}^m} + \nabla c(y, v) \text{ is invertible}\}$, $\Pi_y(dy) = b(y, v)\pi(dy)$ and put

$$\Delta(y, v) = a(y + c(y, v)) - a(y) - \nabla c(y, v)a(y, v), \quad \tilde{\Delta}(y, v) = [I_{\mathbb{R}^m} + \nabla c(y, v)]^{-1}\Delta(y, v), \quad v \in \mathbb{V}_y,$$

$$\mathcal{N}(f) = \{S(f) = \mathbb{R}^m\}, \quad S(f) = \text{span} \left\{ \mathcal{E}_t^x \Delta(X(\tau^-), p_1(\tau)), \tau \in \mathbb{D} \cap [0, t] : p_2(\tau) \in [0, b(X(\tau^-), p_1(\tau))] \right\},$$

where the stochastic exponent $\mathcal{E}_s^x, 0 \leq s \leq t$ is defined by the equation

$$\mathcal{E}_t^x = I_{\mathbb{R}^m} + \int_s^t (\mathcal{E}_r^x \nabla a(X(r, r)) + \int_r^t \mathcal{E}_s^x \nabla c(X(s, r^-), \nu) = \int_s^t \nu ds, dv, dp, \quad t \geq s. \quad (4.17)$$

**Theorem 4.2.**

1. Under conditions $H_2, H_3$,

$$P \|X(x, t)\| \leq f^{-1} \ll \lambda^m.$$  

2. Under conditions $H_2^* - H_4^*, H_5$, for any sequences $x_n \to x, t^n \to t$,

$$P \bigg|_{A} \|X^n(x^t, t^n)\|^{-1} \to P \bigg|_{A} \|X(x, t)\|^{-1}$$

in variation for every $A \subset \mathcal{N}(f)$.

3. Suppose that, for every $y \in \mathbb{R}^m, l \in S^m$,

$$\Pi_y \left( v \in \mathbb{V}_y : (\tilde{\Delta}(y, v), l)_{\mathbb{R}^m} \neq 0 \right) = +\infty. \quad (4.18)$$

Then $P(\mathcal{N}(f)) = 1$.

**Remark 4.4.** The statements 2 of Theorems 4.1.1.2 can be used efficiently in order to provide the local Doeblin condition to hold true for the Markov processes $X$, see [26]. In such a set up, the sufficient conditions for $P(\mathcal{N}(f)) > 0$ are required rather than the conditions for $P(\mathcal{N}(f)) = 1$. Here we formulate one condition of such a type:

$$\Pi_y \left( v \in \mathbb{V}_x : (\tilde{\Delta}(x, u), l)_{\mathbb{R}^m} \neq 0, \|c(x, v)\| < \varepsilon \right) > 0, \quad l \in S^m, \varepsilon > 0. \quad (4.19)$$

We do not give the proof here, referring to the similar proof of Proposition 4.3 [26]. See also Proposition 4.8 [26] for a refinement of condition (4.19) in the one-dimensional case.
Remark 4.5. Theorem 4.2 still holds true with the uniform bounds on \(a, b, c, \nabla c\) replaced by the linear growth conditions

\[
\|a(x)\| \leq L(1 + \|x\|), \quad b(x, v) \leq (1 + \|x\|)\beta(v), \quad \|c(x, v)\| \leq (1 + \|x\|)\gamma(v),
\]

\[
b(x, v) \leq (1 + \|x\|)\beta_1(v), \quad \|c(x, v)\| \leq (1 + \|x\|)\gamma_1(v).
\]

One can prove this via the localization procedure analogous to the one used in the proof of statement 2 of Theorem 4.1

Conditions \(H_3, H_4\) cover a large variety of SDE’s, let us emphasize some particular classes of equations.

A. Let \(b(x, v) = b_0(v)\), then (4.10) is an SDE with constant jump rate. The Lévy measure of the noise is given by \(V(dv) = b(v)\pi(dv)\) and condition \(H_4\) holds true with \(\beta_1 = 0, \gamma_1 = \gamma\). Condition \(H_3\) now has the form

\[
\|c(x, v)\| \leq \gamma(v), \quad \|c(x, v) - c(y, v)\| \leq \|x - y\|\gamma(v), \quad x, y \in \mathbb{R}^m, v \in \Omega, \quad \gamma \in L_1(\Omega, d\Pi).
\]

For such an SDE’s, Theorem 4.2 is already proved in [22] (statements 1 and 3) and [24] (statement 2). Also, in this case statement 1 is closely related to Theorem 3.3.2 [11], but the later theorem has the ”gap” in its proof, discussed in subsection 4.3 below.

B. Let \(\sup_x |b_1(x, v)| \in L_1(\Omega, \pi)\), i.e. the jump rate varies moderately, in a sense. In this case, \(H_3\) yields \(H_4\) with \(\beta_1(v) = \sup_x |b_1(x, v)|\) and \(\gamma_1 = \gamma\). A class of equations satisfying, among others, the condition analogous to the one indicated above is studied in [2] (the so called case without blow up).

C. Let \(b(\cdot, v) = b(\cdot) \in C^1_b(\mathbb{R}^m)\). Such class of (one-dimensional) equations is studied in [15]. In this case, one can put \(\beta = \beta_1 = \text{const}, \gamma = \gamma_1\) and claim \(\gamma \in L_1(\Omega, \pi) \cap L_2(\Omega, \pi)\).

We remark that in [2] and [15], for the cases B and C respectively, existence of a smooth distribution density for the solution to (4.10) is proved (see also references therein for some previous results on absolute continuity of the law of the solution). This is an essentially stronger result than statement 1 of Theorem 4.2 but the conditions, imposed in [2] and [15], are much more restrictive. This is substantial, because the solution to (4.10) may possess a distribution density, but this density may fail to be smooth (see Remark 4.1 and [22], Section 5). We turn the reader’s attention to the fact that the convergence in variation holds true under the same weak assumptions that provide absolute continuity (statement 2 of Theorem 4.2). This allows one to study ergodic properties of the solution to (4.10), considered as a Markov process, under these weak assumptions (20).

The main difficulty in the proof of Theorem 4.2 is to get the differentiability properties, analogous to those given by Propositions 4.14.2. We expose this step in details and then sketch the rest of the proof. In order to get an analogues of Propositions 4.14.2 we have to establish the properties of the solution to (4.10) considered as a function of \(x\). Consider SDE of the type (4.10) with the starting time moment \(s\):

\[
X(x, s, t) = x + \int_s^t a(X(x, s, r)) \, ds + \int_s^t \int_0^{b(X(x, s, r), v)} c(X(x, s, r), v) \nu(dr, dv, dp), \quad x \in \mathbb{R}^m, t \geq s \geq 0.
\]

We write \(E\) for the solution to equation of the type (4.17) with \(X(x, r)\) replaced by \(X(x, s, r)\).

Lemma 4.2. Under conditions \(H_2 - H_4\), the following properties hold true.

1. For every \(T \in \mathbb{R}^+\), there exists \(C \in \mathbb{R}^+\) such that

\[
E\|X(x, s, t) - X(y, s, t)\| \leq C\|x - y\|, \quad x, y \in \mathbb{R}^m, \quad 0 \leq s \leq t \leq T.
\]

2. There exists an increasing process \(\eta(\cdot)\) such that

\[
\|X(x, s, t_1) - X(x, s, t_2)\| \leq \eta(t_2) - \eta(t_1), \quad s \leq t_1 \leq t_2.
\]
3. For every $x \in \mathbb{R}^m, s \leq t$, the function $X(\cdot, \cdot, \cdot)$ is differentiable w.r.t. every variable at the point $(x, s, t)$ with probability 1 and
\begin{equation}
\frac{\partial X}{\partial t}(x, s, t) = a(X(x, s, t)), \quad \frac{\partial X}{\partial x}(x, s, t) = \mathcal{E}^t_s, \quad \frac{\partial X}{\partial s}(x, s, t) = -\mathcal{E}^t_s a(x).
\end{equation}

We remark that the function $X(\cdot, \cdot, \cdot)$ may fail to possess a continuous trajectories. The situation here is like the one for the Poisson process $N$: the trajectories $\mathbb{R}^+ \ni t \mapsto N(t)$ are a.s. discontinuous, but $N'(t) = 0$ a.s. for every fixed $t \in \mathbb{R}^+$.

Proof. Denote $A = \sup_x \|a(x)\| + \sup_x \|\nabla a\|$. We have
\begin{align*}
\|X(x, s, t) - X(y, s, t)\| & \leq \|x - y\| + A \int_s^t \|X(x, s, r) - X(y, s, r)\| \, dr + \\
& + \int_s^t \int_V \int_0^{b(X(x, s, r), v)} \|X(x, s, r) - X(y, s, r)\| \gamma(v) \nu(dr, dv, dp) + \int_s^t \int_V \int_0^{b(X(y, s, r), v)} \gamma(v) \nu(dr, dv, dp),
\end{align*}

here we have used the notation $\int_b^a = \int_a^b$, $a < b$. Put $D(x, y, t) = \sup_{0 \leq s \leq t} E\|X(x, s, r) - X(y, s, r)\|$ and take the expectation in the previous inequality. Then we have
\begin{align*}
D(x, y, t) & \leq \|x - y\| + L \int_0^t D(x, y, s) \, ds + 2 \int_0^t D(x, y, s) \int_V \beta(v) \gamma(v) \nu(dv) \, ds.
\end{align*}

Now the statement 1 follows from the Gronwall lemma.

The statement 2 obviously holds true with $\eta(t) = At + \int_0^t \int_V \beta(v) \gamma(v) \nu(dr, dv, dp)$. The second summand $\eta_2(\cdot)$ in the expression for $\eta(\cdot)$ is a Lévy process with almost all its trajectories being a singular functions with locally bounded variation. Then Lebesgue theorem combined with Fubini theorem provides that, for $\lambda^1$-almost all $t \in \mathbb{R}^+$, $\eta_2(t) = 0$ almost surely. Since $\eta_2$ is time homogeneous, this yields that $\eta_2(t) = 0$ almost surely for every $t \in \mathbb{R}^+$. This provides the first relation in (4.21). In order to prove the second and the third relations in (4.21), we need an auxiliary construction. In order to shorten the notation, we suppose $s = 0$ and omit $s$ in the notation.

Denote $\theta = \sqrt{\beta \gamma_1} \in L_1(\mathcal{V}, \pi)$. For a given $\varepsilon > 0$, consider the random set
\[D^\varepsilon_{x, t} = \{(r, v, p) : r \in [0, t], |p - b(X(x, r), v)| \leq \varepsilon \theta(v)\} \subset \mathbb{R}^+ \times \mathcal{V} \times \mathbb{R}^+
\]
and put
\[\Omega^\varepsilon_{x, t} = \{\omega : \forall r \in \mathcal{D}, (\tau, p_1(\tau), p_2(\tau)) \notin D^\varepsilon_{x, t}\}.
\]

Consider the sequence $\mathcal{V}_n \uparrow \mathcal{V}$ with $\pi(\mathcal{V}_n) < +\infty$ and denote $\mathcal{D}^n = \{\tau \in \mathcal{F} \cap \mathcal{V}_n, p_2(\tau) \in [0, n]\}$,
\[\Omega^\varepsilon_{x, t} = \{\omega : \forall r \in \mathcal{D}^n, (\tau, p_1(\tau), p_2(\tau)) \notin D^\varepsilon_{x, t}\}.
\]

With probability 1, the set $\mathcal{D}^n$ can be represented as $\mathcal{D}^n = \{\tau^n_j, j \geq 1\}, \tau^n_1 < \tau^n_2 < \ldots$ and
\[\Omega^\varepsilon_{x, t} = \bigcap_{j=1}^{\infty} B_j, \quad B_j \overset{df}{=} \{\tau^n_j > t \cup \{\tau^n_j \leq t, |p_2(\tau^n_j) - b(X(x, \tau^n_j) - t)| > \varepsilon \theta(p_1(\tau^n_j))\}, \quad j \geq 1.
\]

Denote, by $\{\mathcal{F}_t\}$, the flow of $\sigma$-algebras generated by $X(\cdot, \cdot, \cdot)$. For every $j$, the variable $X(x, \tau^n_j - t)$ is $\mathcal{F}_{\tau^n_j - t}$ measurable. On the other hand, the variables $p_1(\tau^n_j), p_2(\tau^n_j)$ are jointly independent on $\mathcal{F}_{\tau^n_j - \mathcal{V} \bigcap \sigma(\tau^n_j)}$ and have their distributions equal $\frac{\beta(v) \gamma_1}{\pi(\mathcal{V}_n)}$ and $\frac{\lambda_1(1) \gamma_1(0, n)}{n}$, respectively. For every given $z \in \mathbb{R}^+$ and every $v \in \mathcal{V}_n$, the set $\{p \in [0, n] : |p - z| \leq \varepsilon \theta(v)\}$ is the interval of the length $\leq 2\varepsilon$. Then the probability for $(p_1(\tau^n_j), p_2(\tau^n_j))$ to satisfy the relation $|p_2(\tau^n_j) - z| \leq \varepsilon \theta(p_1(\tau^n_j))$ does not exceed $\frac{2\varepsilon}{n \pi(\mathcal{V}_n)} \int_{\mathcal{V}_n} \theta(v) \gamma(v) \, dv$, and thus
\begin{equation}
P(B_j | \mathcal{F}_{\tau^n_j - t}) \geq \mathbf{1}_{\tau^n_j > t} = \mathbf{1}_{\tau^n_j \leq t} \left[1 - \frac{2\varepsilon}{n \pi(\mathcal{V}_n)} \int_{\mathcal{V}_n} \theta(v) \, dv\right], \quad j \geq 1.
\end{equation}
Since $B_{j-1}$ is $\mathcal{F}_{j-1}$ measurable for $j > 1$, \((1.22)\) imply the estimate

$$P(\Omega_{t, \varepsilon}^{\varepsilon}) \geq E\left[1 - \frac{2\varepsilon}{n^2} \int_{V_n} \theta(v) dv \right]^{\nu([0, t] \times V_n \times [0, n])} = \exp\left[-2\varepsilon \int_{V_n} \theta(v) dv\right].$$

After passing to the limit as $n \to +\infty$, we get

\[(4.23) \quad P(\Omega_{t, \varepsilon}^{\varepsilon}) \geq \exp\left[-2\varepsilon \int_{V} \theta(v) dv\right].\]

Therefore, for every $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^+$, almost every $\omega$ belongs to some $\Omega_{x, t}^{\varepsilon}$ with $\varepsilon > 0$.

Consider the linear SDE $E(r) = 1 + \int_{0}^{r} E(s-) d\eta(s)$ and write $L = L(t, \omega) = E(t, \omega) < +\infty$ a.s. Write

$$\zeta(\kappa, t) = \int_{0}^{t} \int_{B(\varepsilon)} \gamma_1(v) \beta(v) \frac{1}{\theta(v)} \nu(ds, dv, dp), \quad \kappa, t \in \mathbb{R}^+.$$

We have

$$E\zeta(\kappa, t) \leq 2t \int_{B(\varepsilon)} \beta(v) \gamma_1(v) \frac{1}{\theta(v)} \pi(dv) = 2t \int_{B(\varepsilon)} \theta(v) \pi(dv) \to 0, \quad \kappa \to 0.$$

Since $\zeta(\cdot, t)$ is monotonous, this provides that $\zeta(\kappa, t) \to 0, \kappa \to 0^+$ almost surely.

Let $x \in \mathbb{R}^m$, $t \in \mathbb{R}^+, \varepsilon > 0$ and $\omega \in \Omega_{x, t}^{\varepsilon}$ be fixed. Suppose that $L(t, \omega) < +\infty$ and $\zeta(\kappa, t, \omega) \to 0, \kappa \to 0^+$. For a given $y \in \mathbb{R}^m$, we consider $X(y, \cdot, \omega)$ and put $\zeta = \inf\{r : \|X(x, r) - X(y, r)\| > 2L \|x - y\|\}$. We remark that the random variable $\zeta$ is not a stopping time since $L$ is defined through the whole configuration of $\nu$. All the integrals over $\nu$ throughout the rest of the proof should be understood, for the fixed $\omega$, in the point-wise sense. We have

$$X(y, r) - X(x, r) = (y - x) + \int_{0}^{r} \left(a(X(y, s)) - a(X(y, s))\right) ds +$$

$$+ \int_{0}^{r} \int_{0}^{V} b(X(x, s-), v) c(X(y, s-), v) - c(X(x, s-), v) \nu(ds, dv, dp) +$$

$$+ \int_{0}^{r} \int_{0}^{V} b(X(y, s-), v) c(X(y, s-), v) \nu(ds, dv, dp).$$

The last integral in \((4.24)\), for every $\delta > 0$, is dominated by $I_1(\delta, r) + I_2(\delta, r)$,

$$I_1(\delta, r) = \int_{0}^{r} \int_{\theta(v) \leq L \delta(v)} \int_{(b(v) - \beta_1(v)) \nu(dv, ds, dp)},$$

$$I_2(\delta, r) = \int_{0}^{r} \int_{\theta(v) > L \delta(v)} \int_{b(X(x, s-), v) \gamma_1(v) \nu(ds, dv, dp).}$$

Take $\delta = 2\|y - x\|$. Then, for $s \leq \zeta$, we have $\|X(x, s) - X(y, s)\| \leq L \delta$. Thus, as soon as $\varepsilon \theta(v) > L \delta \beta(v)$,

$$|b(X(x, s-), v) - b(X(y, s-), v)| \leq L \delta \beta(v) < \varepsilon \theta(v).$$

This means that $I_2(\delta, r) = 0, r \leq \zeta$ on the set $\Omega_{x, \varepsilon}^{\varepsilon}$.

For $\varepsilon \theta(v) \leq L \delta \beta(v)$, we have $\delta \geq \frac{L \delta \theta(v)}{\varepsilon \delta} \geq 1$, and therefore, for $r < t$, $I_1(\delta, r)$ can be estimated by $\frac{L \delta}{\varepsilon \delta} \zeta(\frac{L \delta}{\varepsilon \delta}, t)$.

Since $\zeta(\kappa, t) \to 0, \kappa \to 0^+$, there exists $\delta_0 > 0$ such that $\frac{L \delta}{\varepsilon \delta} \zeta(\frac{L \delta}{\varepsilon \delta}, t) < \frac{1}{2}, \delta < \delta_0$.

The first two integrals in \((4.24)\) are dominated by

$$\int_{0}^{r} \|X(x, s) - X(y, s)\| d\eta(s).$$
Therefore, if \(2\|y - x\| < \delta_0\), then, up to the time moment \(\zeta\), the values of the process \(\|X(x, \cdot) - X(y, \cdot)\|\) are dominated by the solution to the equation
\[
Z(r) = \frac{5}{3}\|x - y\| + \int_0^r Z(s-)ds(s).
\]
But \(Z(r) = \frac{5}{3}\|x - y\|E(r) < 2L\|y - x\|, r \leq t\). This means that \(\zeta > t\), and the estimates given before show that
\[
\Delta(x, y, t) = \sup_{r \leq t} \left[I_1(\|y - x, r\|) + I_2(\|y - x\|, r)\right] = o(\|y - x\|), \quad y \to x.
\]

Now we fix \(X(x, \cdot)\) and consider \((4.2)\) as a family of the equations on \(X(y, \cdot)\) with the parameter \(y\) involved both into the initial value and the small interfering term (i.e., the last summand in \((4.2)\)) dominated by \(\Delta(x, y, t)\.

Since the coefficients of these equations are smooth, \((4.25)\) and the standard considerations provide that the solution is differentiable w.r.t. \(y\) at the point \(y = x\) and the derivative is equal \(\mathcal{E}'_t\).

At last, the discussed above differentiability properties of the process \(\eta\) provide that, for every \(x \in \mathbb{R}^n, s > 0, \frac{1}{\Delta}\left(X(x, s, s + \Delta s) - x\right) \to a(x), \quad \frac{1}{\Delta s}\left(X(x, s - \Delta s, s) - x\right) \to a(x), \quad \Delta s \to 0+
\]
with probability 1. This, together with the already proved relation \(\frac{\partial X}{\partial x}(x, s, t) = \mathcal{E}'_s\), provides that \(\frac{\partial X}{\partial x}(x, s, t) = -\mathcal{E}'_s(a(x))\) with probability 1. The lemma is proved.

Inequality \(\zeta > t\) can be rewritten to the following form:
\[
\|X(x, r) - X(y, r)\| \leq 2L\|x - y\|, \quad \|x - y\| \leq \frac{1}{2} \delta_0, \quad r \in [0, t].
\]

We remark that the same arguments with those used in the proof of \(\zeta > t\) can be applied, for the same \(\omega, L, \delta_0\), to SDE of the type \((4.20)\) with the initial value \(x' = X(x, 0, s)\). These estimates provide that, for every \(s \in [0, t], \sup_{r \in [s, t]} \|X(x, 0, t) - X(y, s, t)\| \leq 2L\|X(x, 0, s) - y\|\) as soon as \(\|X(x, 0, s) - y\| \leq \frac{1}{2} \delta_0\).

In order study the properties of the sequence of the solutions to SDE’s of the type \((4.10)\), we need the following uniform version of \((4.20)\).

**Lemma 4.3.** Let conditions \(H_2^* - H_5^*\) hold true and \(x^n \to x, t^n \to t\) be arbitrary sequences. Then there exist an a.s. positive random variable \(\sigma\), a random variable \(\zeta\) and a subsequence \(\{n(k)\}, k \geq 1\) \(\subset \mathbb{N}\) such that, for every \(k \geq 1, s \in [0, t^n(k)], \)
\[
\sup_{r \in [s, t^n(k)]} \|X^n(x^n(k), 0, r) - X^n(y, s, r)\| \leq \zeta \|X^n(x^n(k), 0, s) - y\|, \quad \|X^n(x^n(k), 0, s) - y\| \leq \sigma.
\]

**Proof.** In order to shorten exposition, we consider the case \(t^n \equiv t\); one can easily see that this restriction is not essential in the considerations given below. We omit the starting moment in the notation and write \(X(x, t)\) for \(X(x, 0, t)\). First, let us show briefly that
\[
\sup_{r \leq t} \|X^n(x^n, r) - X(x, r)\| \to 0 \quad \text{in probability.}
\]

Estimates analogous to those given at the beginning of the proof of Lemma \((4.2)\) provide that \(\sup_{r \leq t} E\|X^n(x^n, r) - X(x, r)\| \to 0\). We write
\[
X^n(x^n, r) = x^n + \int_0^r \tilde{a}^n(X^n(x^n, s))ds + M^n(r), \quad M^n(r) \overset{df}{=} \int_0^r \int_0^r b^n(X^n(x^n, s), v)c^n(X^n(x^n, s), v)d\nu(ds, dv, dp),
\]
\[
\tilde{a}^n \overset{df}{=} a^n + \int_0^r \int_0^\gamma c^n(v, d\pi)dv = a^n + \int_0^\gamma b^n(v)c^n(v, d\pi).
\]
Let \( M, \tilde{a} \) be defined by the same formulae with \( a^n, b^n, c^n, X^n(x^n, \cdot) \) replaced by \( a, b, c, X(x, \cdot) \). Then \( \tilde{a}^n \) are uniformly Lipschitz and converge to \( \tilde{a} \) uniformly on every compact. Thus \( \sup_{r \leq t} E\|M^n(r) - M(r)\| \to 0 \). The Doob martingale inequality provides that \( E\sup_{r \leq t} \|M^n(r) - M(r)\| \to 0 \) in probability. This, via Gronwall lemma, provides (4.28).

Next, for a given sequences \( \{n(k), k \geq 1\} \subset \mathbb{N}, \{q(k), k \geq 1\} \subset (0, +\infty) \) and \( \varepsilon > 0 \) we consider the random sets

\[
D_{x,t}^{\varepsilon, \ell} = \bigcup_k \{(r, v, p) : r \in [0, t], |p - b(X^{n(k)}(x^{n(k)}, r^{-}), v)| \leq \varepsilon \theta(v)\},
\]

\[
D_{x,t}^0 = \bigcup_k \{(r, v, p) : r \in [0, t], |p - b(X^{n(k)}(x^{n(k)}, r^{-}), v)| \leq \varepsilon \theta(v), \theta(v) > q(k)\beta(v)\}
\]

\[
D_{x,t}^{k, \varepsilon} = \{(r, v, p) : r \in [0, t], |p - b(X^{n(k)}(x^{n(k)}, r^{-}), v)| \leq \varepsilon \theta(v), \theta(v) \leq q(k)\beta(v)\}, \quad k \geq 1
\]

and put \( \Omega_{x,t}^{\varepsilon, \ell} = \{\forall \tau \in \mathcal{D}, (\tau, p_1(\tau), p_2(\tau)) \not\in D_{x,t}^{\varepsilon, \ell}\}, \Omega_{x,t}^{\varepsilon} = \{\forall \tau \in \mathcal{D}, (\tau, p_1(\tau), p_2(\tau)) \not\in D_{x,t}^{\varepsilon}\}, \Omega_{x,t}^{k, \varepsilon} = \{\forall \tau \in \mathcal{D}, (\tau, p_1(\tau), p_2(\tau)) \not\in D_{x,t}^{k, \varepsilon}\} \). Our aim is to construct \( \{n(k)\} \) in such a way that

\[
P(\Omega_{x,t}^{\varepsilon, \ell}) \to 1, \quad \varepsilon \to 0 + .
\]

Once this construction is complete, the considerations analogous to those given in the proof of Lemma 4.2 can be made uniformly over \( k \) and provide (4.27). We have

\[
\Omega_{x,t}^{\varepsilon, \ell} = \bigcap_k \Omega_{x,t}^{k, \varepsilon}.
\]

Since \( \{\theta(v) \leq q(\beta(v))\} \downarrow \{\theta = 0\}, q \downarrow 0 \) and \( \theta \in L_1(\mathbb{V}, \pi) \), one can choose a monotonically decreasing sequence \( \{q(k)\} \) in such a way that \( \int (\theta(v) \leq q(k)\beta(v)) \theta(v)\pi(\beta(v)) \leq 2^{-k}, k \geq 1 \). Analogously to (4.23), we have

\[
1 - P(\Omega_{x,t}^{k, \varepsilon}) \leq 1 - \exp\left[-2\varepsilon \int (\theta(v) \leq q(k)\beta(v)) \theta(v)\pi(\beta(v)) dv\right] \leq 2\varepsilon \cdot 2^{-k}, \quad k \geq 1.
\]

Therefore, \( P\left(\bigcap_k \Omega_{x,t}^{k, \varepsilon}\right) \geq 1 - \sum_k (2\varepsilon \cdot 2^{-k}) = 1 - 2\varepsilon \). Next, let \( \varepsilon > 0 \) be fixed. By the condition \( H_3^\varepsilon \),

\[
\Omega_{x,t}^{2^\varepsilon} \setminus \Omega_{x,t}^{\varepsilon, \ell} \subset \bigcup_k \left\{\sup_{r \leq t} \|X^{n(k)}(x^{n(k), r}) - X(x, r)\| \geq \varepsilon q(k)\right\}.
\]

It follows from (4.28) that the sequence \( \{n(k)\} \) can be constructed in such a way that

\[
P(\Omega_{x,t}^{\varepsilon, \ell}) \geq P(\Omega_{x,t}^{2^\varepsilon}) - \sum_{k=1}^{\infty} P(\sup_{r \leq t} \|X^{n(k)}(x^{n(k), r}) - X(x, r)\| \geq \varepsilon q(k)) \geq 1 - \exp\left[-4\varepsilon \int \theta(v) dv\right] - \varepsilon,
\]

and therefore

\[
P(\Omega_{x,t}^{\varepsilon, \ell}) \geq 1 - \exp\left[-4\varepsilon \int \theta(v) dv\right] - 3\varepsilon.
\]

We remark that the sequence \( \{n(k)\} \) has been built for a given \( \varepsilon > 0 \), and for the other values of \( \varepsilon \) (4.30) may fail. Now we proceed in the following way. We take \( \varepsilon_j = 2^{-j}, j \in \mathbb{N} \) and construct consequently the sequences \( \{n_1(k)\}, \{n_2(k)\}, \ldots \) in such a way that every \( \{n_j+1(k)\} \) is a subsequence of \( \{n_j(k)\} \) and, for every \( j \), (4.30) holds true with \( \{n(k)\} = \{n_j(k)\} \) and \( \varepsilon = \varepsilon_j \). Then we put \( \{n(k) = n_k(k), k \geq 1\} \). For this sequence, (4.30) holds true for \( \varepsilon = \varepsilon_j, j \in \mathbb{N} \) by the construction. This implies (4.29). The lemma is proved.

Consider a grid \( \mathfrak{G} \) with \( \Gamma_i = \Theta_i \times I_i, i = 1, \ldots, m \), where \( \Theta_1, \ldots, \Theta_m \subset \mathbb{V} \) are a bounded measurable sets and \( I_1, \ldots, I_m \subset \mathbb{R}^+ \) are a finite segments.
Proposition 4.5. 1. Under conditions $H_2 - H_4$, every component of the vector $X(x,t)$ is a.s. differentiable w.r.t. $\mathcal{G}$ for every $x \in \mathbb{R}^m$, $t \in \mathbb{R}^+$ and

$$
(4.31) \quad (D^i_j X_j(x,t))^m_{i,j} = \int_0^t \int_{\Gamma \cap (\mathbb{V} \times [0,b(X(s,x,-)),v))} E_s^i \Delta (X(s,x_-,v) J h_i(s) \nu(ds,du), \quad i = 1, \ldots, m.
$$

2. Let conditions $H^*_2 - H^*_4$, $H_5$ hold true and sequences $x^n \to x$, $t^n \to t$ and $\{u'(k), k \geq 1\} \subset \mathbb{N}$ be given. Then there exist subsequences $\{n(k), k \geq 1\} \subset \{u'(k), k \geq 1\}$ and sets $\Omega'_j \in \mathcal{F}, j \geq 1$ such that $P(\bigcup \Omega'_j) = 1$ and the family $\{X^{n(k)}(x^{n(k)}, v^{n(k)})\}, k \geq 1\}$ has uniformly dominated increments w.r.t. $\mathcal{G}$ on every $\Omega'_j$.

Proof. Let $i \in \{1, \ldots, m\}$ be fixed, denote $\nu' = \nu(\cdot, (\mathbb{R}^+ \times \Gamma))$. One can replace $\nu$ by $\nu_i$ in SDE (4.20) and apply Lemma 4.2 for this new equation. Then, by the statements 1,2 of this lemma and standard theorem on measurable modification, there exists a function $\Psi : (s,t,x,\omega) \mapsto \Psi_{x,t}^i(x,\omega)$ such that

1) $\Psi^i : \mathcal{B}([0,\infty)) \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \sigma(\nu') - \mathcal{B}(\mathbb{R}^m)$ measurable;

2) for every $s \leq t$, the function $\Psi_{s,t}^i$ is $\mathcal{B}(\mathbb{R}^m) \otimes \sigma(\nu')|_{s,t} \otimes \mathcal{B}(\mathbb{R}^m)$ measurable;

3) the process $X_i(x,\cdot) = \Psi_{x,t}^i(x)$ is the solution to (4.20) with $\nu$ replaced by $\nu'$.

Define the function $\Phi : \mathbb{R}^+ \times \mathcal{V} \times \mathcal{R}^+ \times \mathcal{R}^m$ by

$$
\Phi : (s,v,p,x) \mapsto \Phi_{s,v,p}^i(x) \overset{df}{=} x + c(x,v) I_{p \leq b(x,v)}.
$$

Then solution of (4.10) can be represented at the form

$$
X(x,t) = \left[\Psi_{0,t}^i \circ \Phi_{t_1 \cdots t} \cdots \circ \Phi_{t_1 t_2} \circ \Phi_{t_1} \circ \Psi_{0} \right](x),
$$

where $t^i_1 = t$ and $k$ is such that $t^i_1 \leq t, t^i_{k+1} > t$. Let us deduce statement 1 from Lemma 4.2. In order to shorten notation, we suppose $k = 1$, the general case can be treated using the technique involving partitions $Q \in \Omega$, introduced in the proof of Proposition 4.1. The variables $t^i_1, p_1(t^i_1), p_2(t^i_1)$ are jointly independent with $\nu'$. In addition, $p_2(t^i_1)$ possesses a distribution density, thus $p_2(t^i_1) \neq b(\Psi_{0,t}^i(x), p_1(t^i_1)v)$ a.s. Therefore, since $\frac{\partial}{\partial x}$ $T_s x_j = -J h_i(t^i_1)$ and $T^i_1$ do not change $\nu', p_1(t^i_1)$ and $p_2(t^i_1)$, it is enough to prove that, for every $s,v,p$,

$$
(4.32) \quad \frac{\partial}{\partial s} \left[\Psi_{0,s}^i \circ \Phi_{s,v,p} \circ \Psi_{s,t}^i \right](x) = -E_s^i \Delta (\Psi_{0,s}^i(x), v) I_{p \leq b(\Psi_{0,s}^i(x),v)} \quad \text{a.s. on the set } \{p \neq b(\Psi_{0,s}^i(x), v)\}.
$$

Statement 3 of Lemma 4.2 and the chain rule yield that

$$
(4.33) \quad \frac{\partial}{\partial s} \left[\Psi_{0,s}^i \circ \Phi_{s,v,p} \right](x) = a(\Psi_{0,s}^i(x)) + \nabla c(\Psi_{0,s}^i, v)(\Psi_{0,s}^i(x)) I_{p \leq b(\Psi_{0,s}^i(x),v)} \quad \text{a.s. on the set } \{p \neq b(\Psi_{0,s}^i(x), v)\}.
$$

Denote, by $\chi$, the distribution of $\left[\Psi_{0,s}^i \circ \Phi_{s,v,p} \right](x)$. It follows from the statement 3 of Lemma 4.2 and Fubini theorem that, on some set $\Omega^*_i \in \sigma(\nu')|_{s,t} \times \mathcal{V} \times \mathcal{R}^+$ with $P(\Omega^*_i) = 1$,

$$
\frac{\partial}{\partial y} \Psi_{s,t}^i(y) = \chi^i_s, \quad \frac{\partial}{\partial s} \Psi_{s,t}^i(y) = -E_s^i a(y) \quad \text{for } \chi\text{-almost all } y.
$$

Since $\Psi_{0,s}^i \circ \Phi_{s,v,p}(x)$ and $\nu'\|_{s,t} \times \mathcal{V} \times \mathcal{R}^+$ are independent, this together with (4.33), Fubini theorem and the chain rule provides (4.32). This proves statement 1.

In order to shorten notation, we prove statement 2 in the case $m = 1$. In this case, $\mathcal{G} = \{a_1, b_1, h_1, \Gamma_1\}$. In addition, we consider the particular case $\omega \in \Omega = \{\#(D^{r^n} \cap [0,t^n]) = 1, n \geq 1\}$. On can see that the considerations given below can be extended to $\omega \in \Omega$ and arbitrary $m$ straightforwardly. Consider the functions $\Psi^{1,n}$ satisfying the properties 1,2) listed above (with $i = 1$), such that the processes $X^{1,n}(x,\cdot) = \Psi_{x,t}^{1,n}(x)$ are the solutions to (4.20) with $\nu$ replaced by $\nu^1$ and $a, b, c$ replaced by $a^n, b^n, c^n$. Put $\Phi_{s,v,p}^n(x) \overset{df}{=} x + c^n(x,v) I_{p \leq b^n(x,v)}$. 


Denote by $\tau$ the (unique) point from $\mathcal{D}^{11} \cap [0, t^n]$. Then

$$X^n(x^n, t^n) = \left[ \Psi_{0, \tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n).$$

By taking a subsequence, we can suppose that $\sup_{r \leq \sup_n t^n} \| X^n(x^n, r) - X(x, r) \| \to 0$ almost surely, see (4.28). With probability 1, $\mathcal{D}^{11} \cap \{ t \} = \emptyset$ and thus a.s. there exists $n = n(\omega) \in \mathbb{N}$ and $\varepsilon = \varepsilon(\omega) > 0$ such that $t_n > T_{l_1, \tau, t_n} < T_{l_1, \tau, t_2}, |l| < \varepsilon$, where $\tau_2$ denotes the second point from $\mathcal{D}^{11}$. Denote $\tilde{\tau}_1 = T_{l_1, \tau, t_1}, T_l = T_{l_1, \tau, l} \in \mathbb{R}$. Since $T_l$ does not change $p(\tau)$ and $\Psi^{1, n}$, we have

$$T_l X^n(x^n, t^n) = \left[ \Psi_{0, \tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n), \quad |l| < \varepsilon.$$

For every $s < r$ and $y \in \mathbb{R}^m$, we have $| \Psi_{s, \tilde{\tau}_1}^{1, n}(y) | \leq \eta(r) - \eta(s)$ (the process $\eta(\cdot)$ is given in Lemma 4.2). The function $l \mapsto \eta(\tilde{\tau}_1)$ is a.s. continuous at the point 0, see the proof of Lemma 4.2 In addition, $p_2(\tau) \neq b^n \{ X(x, \tau^-), p_1(\tau) \} \neq b^n \{ X^n(x^n, \tau^-), p_1(\tau) \}, n \geq 1$ a.s. Thus the variable $\varepsilon(\omega)$ can be chosen in such a way that $\varepsilon > 0$ a.s. and

$$p_2(\tau) < b^n \{ X^n(x^n, \tilde{\tau}_1, p_1(\tau) \} \Leftrightarrow p_2(\tau) < b^n \{ X^n(x^n, \tau^-), p_1(\tau) \}, \quad |l| < \varepsilon,$$

Since every mapping $y \mapsto y + c^n(y, p_1(\tau)), n \geq 1$ is Lipschitz with the constant $\gamma(p_1(\tau))$, the previous considerations provide that, for every $l_1 < l_2$ with $|l_{1,2}| < \varepsilon$, a random variable $L = 1 + \gamma(p_1(\tau)) < +\infty$ a.s. Therefore, for every $j \in \mathbb{N}$, there exists an a.s. positive random variable $\vartheta_j$ such that, for every $l_1, l_2$ with $|l_{1,2}| < \vartheta_j(\omega)$,

$$\left| \left[ \Psi_{0, \tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n) - \left[ \Psi_{0, \tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n) \right| \leq L | \eta(\tilde{\tau}_1) - \eta(\tilde{\tau}_1) |$$

with a random variable $L = 1 + \gamma(p_1(\tau)) < +\infty$ a.s. Therefore, for every $j \in \mathbb{N}$, there exists an a.s. positive random variable $\vartheta_j$ such that, for every $l_1, l_2$ with $|l_{1,2}| < \vartheta_j(\omega)$,

$$\left| \left[ \Psi_{0, \tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n) - \left[ \Psi_{0, \tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n) \right| \leq j^{-1}.$$

The construction of the sequence $\{ n(k) \}$ at the end of the proof of Lemma 4.3 can be modified easily in order to provide an additional requirement $\{ n(k) \} \subset \{ n'(k) \}$. We suppose further that $n'(k) = n(k) = k$ (this supposition is made for notational convenience, only). Let $\zeta, \sigma$ be the random variables given by Lemma 4.3 Denote $\Omega'_j = \{ \zeta \leq j, \sigma \geq j^{-1} \}$, then $P(\bigcup_j \Omega'_j) = 1$. For every given $l \in \mathbb{R}$ and $r > 0$, we have that, on the set

$$\{ \zeta < r \}$$

Thus we can apply Lemma 4.3 and write that, for $T_{l, \omega} \in \Omega'_j$ and $r > \zeta_1,$

$$\left| \left[ \Psi_{\tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n) - \left[ \Psi_{\tilde{\tau}_1}^{1, n} \circ \Phi_{\tilde{\tau}_1, \tilde{\tau}_2}^{n, p_1(\tau), p_2(\tau)} \circ \Psi_{\tau, \tilde{\tau}_1, \tilde{\tau}_2}^{1, n} \right] (x^n) - y \right| \leq j^{-1}.$$
Consider the set $\Omega$, the function $R$ not use Theorem 2.2 in order to prove statements 1, 2 of Theorem 4.1 under this additional supposition and then Example 4.2. 

\[
\text{Example 4.2. Suppose that } U = \{u_1, u_2\} \text{ and } \Pi(\{u_1, u_2\}) \in (0, +\infty) \text{ and consider SDE}
\]

\[
X(x, t) = x + \int_0^t c(X(x, s-), u)\nu(ds, du)
\]

with $c(x, u_1) = 1, c(x, u_2) = \arctg x$. One can see that this SDE can be considered as an equation of the type \((4.16)\) and $H_2 - H_4$ hold true. Consider the grid $\mathcal{G} = \{[0, t], h, \Gamma\}$ with $\Gamma = \{u_1\}$, let us show that $X(x, t)$ is not $L_p$-differentiable w.r.t. $\mathcal{G}$ for any $p \geq 1$. It follows from Lemma 5.2 that, for any $f$ being $L_1$-differentiable, the function $\mathbb{R} \ni l \mapsto T^f_{\theta} \in \mathbb{R}$ belongs to $W_{1,\text{loc}}^1(\mathbb{R}, \lambda^1)$ and therefore is continuous for almost all $\omega \in \Omega$. Consider the set $\Omega_{1,1} = \{\nu([0, t] \times \{u_1\}) = 1, i = 1, 2\}$. This set has positive probability. In addition, on this set,

\[
T^\Gamma_{\theta} X(x, t) = \phi(T_{\theta} \tau_1, \tau_2), \quad \phi(s_1, s_2) \equiv \begin{cases} 
 x + 1 + \arctg (x + 1), & s_1 < s_2 \\
 x + \arctg x + 1, & s_1 > s_2
\end{cases}
\]

where $\tau_1 (\tau_2)$ denotes the time moment of unique jump with the value of the jump equal $u_1 (u_2)$. Thus the function $\mathbb{R} \ni l \mapsto T^\Gamma_{\theta} X(x, t)$ is discontinuous with positive probability and therefore $X(x, t)$ fails to be $L_1$ differentiable w.r.t. $\mathcal{G}$. 

4.3 A discussion on the differential properties of the solution to SDE with jumps. The given above proofs of Theorems 4.1, 4.2 are based on Theorem 2.2. The question whether the Theorem 2.2 in the proofs can be replaced by Theorem 2.3 is quite natural, since the latter theorem has a simpler formulation and does not require any additional technical assumptions like the one given in Definition 2.4. For an SDE with additive noise, the answer is positive: under additional conditions sup $s \leq t E\|Z(s)\|^m < +\infty$, Proposition 4.1 and (4.8) provide $L_m$ differentiability of every component of $X(x, t)$ via the dominated convergence theorem. One can use Theorem 2.2 in order to prove statements 1, 2 of Theorem 4.1 under this additional supposition and then remove this supposition via a localization procedure.

The following example shows that, for an SDE with non-additive noise, even when the jump rate is constant, the situation is essentially different.
We remark that Example 4.2 gives a counterexample to Theorem 2.1.3 [11], also. Let, in the notation of [11], $M_1 = \{u_1\}, M_2 = \{u_2\}, M = M_1 \cup M_2$, then one has $X(x, t) \in d$ but $X(x, t) \notin \tilde{d}$. The latter statement follows from the definition of the tensor product of two Dirichlet forms ([11], section 2.1.1), Proposition 1.2.2 [11] and the fact that, for every $s_2 \in (0, 1)$, the function

$$[0, 1] \ni s \mapsto \phi(s, s_2)$$

is discontinuous and therefore does not belong to $H(\Delta_1)$. This ”gap” is crucial and, in general, $(D^n, \mathcal{E}^n) \not\subset (\mathcal{D}^n, \mathcal{E}^n)$ ([11], subsection 2.2). Thus Theorem 3.2.2 [11] does not imply Theorem 3.3.1 [11], the latter being crucial for the proof of Theorem 3.3.2 [11].

The situation exposed in Example 4.2 is quite typical, and the solution to SDE with non-additive noise, in general, neither is $L_1$-differentiable in the sense of Definition 2.1 above, nor belong to the domain $\mathcal{E}$ of the Dirichlet form $\mathcal{D}$ in the sense of [11]. Thus the notion of a.s. derivative appears to be an efficient tool that allows one to consider the functionals with quite poor differential properties and extends the domain of possible applications significantly.

References

[1] Alexandrova D.E., Bogachev V.I., and Pilipenko A.Yu. (1999) On convergence in variation of the induced measures. Mat. Sbornik, 190, no. 9, 3-20. (Russian, English transl in Sbornik: Mathematics 190 (1999), 1229-1245).
[2] Bally V. (2007) Integration by parts formula for locally smooth laws and applications to equations with jumps I. Preprint, Mittag-Leffler Institute, Report No. 14, 2007/2008, fall.
[3] Bichteler K., Gravereaux J.-B., and Jacod J. (1987) Malliavin calculus for processes with jumps. New York, Gordon and Breach.
[4] Bismut J.M. (1983) Calcul des variations stochastiques et processus de sauts. Zeit. fur Wahr, 63, 147-235.
[5] Bogachev V.I. (1997) Differentiable measures and the Malliavin calculus. Jour. of Math. Sci., 87, no. 5, 3577-3731.
[6] Bogachev V.I., and Smolyanov O.G. (1990) Analytical properties of infinite-dimensional distributions. Uspekhi mat. nauk, 45, no. 3, 3-83 (Russian, Engl. transl. in Russian Math. Surveys 45 (1990), no. 3, 1-104).
[7] Bouleau N., Hirsch F. (1991) Dirichlet Forms and Analysis on Wiener Space, De Gruyter Studies in Math.
[8] Carlen E., Pardoux E. (1990) Differential calculus and integration-by-parts on Poisson space. Stoch. Algebra and Analysis in Classical and Quantum Dynamics (Marseille, 1988), Math. Appl., 59, 63-67.
[9] Davydov Yu.A., and Lifshits M.A. (1984) Stratification method in some probability problems. In Prob. Theory, Math. Statist., Theor. Cybernetics, 22, 61-137. (Russian).
[10] Davydov Yu.A., Lifshits M.A., and Smorodina N.V.(1995) Local Properties of Distributions of Stochastic Functionals, Moscow, Nauka (Russian, English transl. in Translations of Mathematical Monographs, 173. American Mathematical Society, Providence, RI, 1998).
[11] Denis L. (2000) A criterion of density for solutions of Poisson-driven SDE’s. Probab. Theory Relat. Fields, 118, 406-426.
[12] Elliott R.J., Tsoi A.H. (1993) Integration-by-parts for Poisson processes. Jour. of Multivar. Analysis, 44, 179-190.
[13] Federer G. (1987) Geometric Measure Theory. Moscow, Nauka (Russian, translated from Federer G. (1969) Geometric Measure Theory. New York, Springer).
[14] Fournier N. (2002) Jumping SDEs: absolute continuity using monotonicity. Stochastic Processes Appl., 98, no. 2, 317-330.
[15] Fournier N. (2008) Smoothness of the law of some one-dimensional jumping S.D.E.s with non-constant rate of jump. Electr. Jour. Probab., 13, 135-156.
[16] Ishikawa Y. (2001) Density estimate in small time for jump processes with singular Lévy measures. Tohoku Math. Jour., 53, 183-202.
[17] Ishikawa Y., and Kunita H. (2006) Existence of density for canonical differential equations with jumps. Stochastic Processes Appl., 116, no. 12, 1743-1769.
[18] Kerstan J., Mattes K., and Mecke J. (1978) Infinite divisible point processes, Akademie-Verlag, Berlin.
[19] Komatsu T., and Takeuchi A. (2001) On the smoothness of PDF of solutions to SDE of jump type. Int. Jour. Differ. Equ. Appl., 2, no. 2, 141–197.
[20] T. Komatsu, A. Takeuchi. (2001) Simplified probabilistic approach to the Hörmander theorem. Osaka Jour. Math., 38, 681-691.
[21] Kulik A.M. (1999) Admissible transformations and Malliavin calculus for compound Poisson process. Theory of stochastic processes, 5(21), no. 3-4, 120-126.
[22] Kulik A.M. (2005) Malliavin Calculus for Lévy Processes With Arbitrary Lévy Measures. Probability Theor. Math. Statist., 72, 67-83.
[23] Kulik A.M. (2005) On a regularity of distribution for solution of SDE of a jump type with arbitrary Lévy measure of the noise, Ukrainian Math. Jour., 57, no. 9, 1261-1283.
[24] Kulik A.M. (2005) On a convergence in variation for distributions of solutions of SDE’s with jumps. Random Operators and Stoch. Equations, 13, no. 3, 297-312.
[25] Kulik A.M. (2006) Stochastic calculus of variations for general Lévy processes and its applications to jump-type SDEs with non degenerated drift. Preprint, arxiv.org PR/0606427.
[26] Kulik A.M. (2007) Exponential ergodicity of the solutions to SDE’s with a jump noise. Preprint, [arXiv:math/0701747](http://arxiv.org/abs/math/0701747) to appear in Stochastic Processes Appl.
[27] Léandre R. (1998) Regularites de processus de sauts degeneres (II). Ann. Inst. Henri Poincare Prob. Stat., 24, 209-236.
[28] Matheron G. (1978) Random sets and integral geometry. Moscow, Mir (Russian, translated from Matheron G. (1975) Random sets and integral geometry. New York, Wiley).
[29] Nourdin I. and Simon T. (2006) On the absolute continuity of Lévy processes with drift. Annals of Probability, 34, no. 3, 1035-1051.
[30] Parthasarathy K. (1978) Introduction to Probability and Measure. New York, Springer-Verlag.
[31] Picard J. (1996) On the existence of smooth densities for jump processes. Probab. Theory Rel. Fields, 105, 481-511.
[32] Protter P.E. (2004) Stochastic Integration and Differential Equations. Applications of Mathematics, Stochastic Modelling and Applied Probability, 21. Berlin, Springer.
[33] Skorokhod A.V. (1967) Random Processes with Independent Increments. Moscow, Nauka (Russian, English transl. A.V.Skorokhod (2001) Random Processes with Independent Increments. Kluwer).
[34] Yamazato M. (1994) Absolute continuity of transition probabilities of multidimensional processes with independent increments. Probab. Theor. Appl., 38, no. 2, 422-429.

Institute of Mathematics, Ukrainian National Academy of Sciences, 3, Tereshchenkivska Str., Kyiv 01601, Ukraine
E-mail address: kulik@imath.kiev.ua