On absolute equivalence and linearization I

Jeanne N. Clelland¹ · Yuhao Hu²

Received: 12 July 2020 / Accepted: 15 June 2023 / Published online: 5 July 2023
© The Author(s), under exclusive licence to Springer Nature B.V. 2023

Abstract
In this paper, we study the absolute equivalence between Pfaffian systems with a degree 1 independence condition and obtain structural results, particularly for systems of corank 3. We apply these results to understanding dynamic feedback linearization of control systems with 2 inputs.

Keywords Cartan prolongation · Control system · Absolute equivalence · Dynamic feedback linearization

Mathematics Subject Classification (2010) 34H05 · 37K35 · 53C10 · 58A15 · 58A17

1 Introduction
The notion of absolute equivalence between two differential systems was introduced by Élie Cartan in [2].

By the time of Cartan’s writing, Hilbert [8] asked the question: When can the general solutions of an ODE for two unknown functions be expressed in a determined way in terms of an arbitrary function and its successive derivatives? He proved that such a property is not enjoyed by the ODE

\[
\frac{dx}{dt} = \left(\frac{d^2y}{dr^2}\right)^2.
\]
This is in contrast with, for example, the ODE
\[
\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = 1,
\]
whose general (real analytic) solutions can be expressed as
\[
t = f''(\alpha) + f(\alpha),
\]
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{pmatrix} \begin{pmatrix} f'(\alpha) \\ f''(\alpha) \end{pmatrix}
\]
for an arbitrary function \( f(\alpha) \).

Cartan realized that a key to answering Hilbert’s question was to understand when one can establish a one-to-one correspondence between the solutions of two differential systems \( E_1 \) and \( E_2 \); and he called \( E_1 \) and \( E_2 \) “absolutely equivalent” if a particular kind of such correspondence exists.

Furthermore, the differential systems that enter Hilbert’s question are special cases of rank-\( n \) Pfaffian systems in \( n + 2 \) variables with a degree 1 independence condition.\(^1\) Given such a Pfaffian system \( I \), one can compute its successive derived systems \( I^{(k)} \). The integers \( r_k := \text{rank}(I^{(k)}/I^{(k+1)}) \) are important invariants of \( I \) under diffeomorphisms. Only two cases can occur: either \( r_k \leq 1 \) for all \( k \), in which case \( I \) is said to have class 0; or \( r_\ell = 2 \) for the first time for some \( \ell \) (\( \ell \geq 1 \)), in which case \( I \) is said to have class \( \text{rank}(I^{(\ell)}) \). (See [5] for details.) In the latter case, Cartan called \( I^{(\ell-1)} \) the normal system of \( I \). Furthermore, he proved the following theorem.

**Theorem 1.1** [2] Two systems \( I, \bar{I} \) of corank 2 are absolutely equivalent if and only if the following two conditions hold

(i) \( I \) and \( \bar{I} \) have the same class;
(ii) when both systems have class 0, the terminal derived systems \( I^{(\infty)}, \bar{I}^{(\infty)} \) satisfy
\[
\text{rank}(I^{(\infty)}) = \text{rank}(\bar{I}^{(\infty)});
\]
when both systems have the same positive class, the corresponding normal systems \( I^{(\ell-1)} \) and \( \bar{I}^{(\ell-1)} \) can be transformed into each other by a diffeomorphism between their underlying manifolds.

In terms of these, \( I \) satisfies Hilbert’s condition precisely when \( I \) is absolutely equivalent to the Pfaffian system generated by a single 1-form
\[
dx - u dt,
\]
which holds if and only if \( I \) has class 0 and \( I^{(\infty)} = 0 \). That the Pfaffian system corresponding to (1) does not satisfy this condition can be checked simply by computing derived systems.

Cartan’s motivation was apparently classical differential geometry, as the second half of [2] shows. This serves as an interesting contrast, as eighty years later, a significant amount of interest returned to this paper of Cartan, this time motivated by applications in control theory, represented by the works [6, 9, 10], to mention a few.

An **autonomous control system** in \( n \) states and \( m \) inputs (aka. controls) \( (n \geq m) \) is an ODE system of the form
\[
\dot{x} = f(x, u), \quad x \in \Omega \subseteq \mathbb{R}^n, \ u \in \mathbb{R}^m.
\]

\(^1\) We will call such a Pfaffian system a **system of corank 2**.
In addition, we will assume that
\[ \text{rank} \left( \frac{\partial f_i}{\partial u^\alpha} \right) = m. \]

The system is called \textit{controllable} if, given any two states \( x_1, x_2 \in \mathbb{R}^n \), there exists a solution \( x(t_1) \) and \( t_1, t_2 \in \mathbb{R} \) such that \( x(t_1) = x_1 \) and \( x(t_2) = x_2 \).

The simplest class of such systems are those that are \textit{linear}:
\[ \dot{x} = Ax + Bu, \]
where \( A, B \) are constant matrices with \( B \) attaining full rank. Controllability in this case is characterized by \textit{Kalman’s maximal rank condition}:
\[ \text{rank} \left( \begin{array}{c|c|c} B & AB & A^2 B \\ \vdots & \vdots & \vdots \\ A^{n-1} B \end{array} \right) = n. \]

It is well-known that a controllable linear system can be put into a \textit{Brunovský normal form} [1] by a linear feedback transformation of the form
\[ y = Cx, \quad v = Dx + Eu, \]
where \( C, D, E \) are constant matrices. This normal form is defined as follows:

**Definition 1.2** A control system of the following form
\[ \begin{align*}
\dot{x}_j^1 &= x_{j+1}^1 \quad (j = 0, \ldots, r_1), \\
\dot{x}_j^2 &= x_{j+1}^2 \quad (j = 0, \ldots, r_2), \\
& \quad \ldots \\
\dot{x}_j^p &= x_{j+1}^p \quad (j = 0, \ldots, r_p),
\end{align*} \]

(2)

where \( r_1, \ldots, r_p \geq 0 \), is called a \textit{Brunovský normal form}. In particular, \( x_{1j}^i \) (\( i = 1, \ldots, p; \ j = 0, \ldots, r_1 \)) are the state variables; \( x_{ri+1}^i \) (\( i = 1, \ldots, p \)) are the control variables.

The Brunovský system (2) is particularly simple, in that its general solution may be expressed in terms of \( p \) arbitrary functions \( f^1(t), \ldots, f^p(t) \) as follows:
\[ \begin{align*}
x_0^1(t) &= f^1(t), & x_1^1(t) &= (f^1)'(t), & \ldots & x_{r_1}^1(t) &= (f^1)^{(r_1)}(t), \\
& \vdots & & & \vdots & \vdots \ \vdots & \vdots \\
\end{align*} \]

\[ \begin{align*}
x_0^p(t) &= f^p(t), & x_1^p(t) &= (f^p)'(t), & \ldots & x_{r_p}^p(t) &= (f^p)^{(r_p)}(t). \end{align*} \]

More generally, it is also known when an arbitrary controllable autonomous system can be put into a Brunovský normal form by a \( t \)-independent, not necessarily linear, feedback transformation:
\[ y = \phi(x), \quad v = \psi(x, u), \]
where \( y, v \) are the new states and inputs, respectively. In fact, in [7] (see also [10, p.78, Theorem 35]), Gardner and Shadwick proved:

\textit{An autonomous control system, formulated as a Pfaffian system I with the independence condition} \( \tau = dt \), \textit{is locally feedback equivalent to a Brunovský normal form if and only if}
i. the terminal derived system $I^{(\infty)}$ vanishes;

ii. for each integer $k \geq 0$, the Pfaffian system generated by the $k$-th derived system $I^{(k)}$ and $dt$ is Frobenius.

A more general question is: Given two control systems, how to tell whether they are absolutely equivalent? The answer to this question largely depends on the number of control variables that one is dealing with. For control systems with a single control, the question reduces to the case addressed by Cartan [2]. A systematic study of cases with more than one control was carried out in the 1994 thesis [10] of Sluis. In that thesis, Sluis followed Cartan, realizing that the key to understanding absolute equivalence is to understand the so-called “Cartan prolongations”. A notable theorem that Sluis proved is Theorem 2.15 below, which allows one to understand any Cartan prolongation in terms of “prolongations by differentiation”. It is natural to wonder whether, in certain cases, the theorem can be used to classify control systems that are absolutely equivalent to a Brunovský normal form via a succession of such prolongations. This is a main motivation for the current work: In this paper we consider systems with $n$ states and 2 controls, and our main result (Theorem 3.11) gives necessary and sufficient conditions for such a system to be dynamic feedback linearizable after a particular, fixed number $K$ of differentiations. Additionally, in Theorem 3.13 we derive structure equations that provide a starting point for classifying those control systems that can be linearized via a succession of $K$ differentiations.

In future work, we hope to prove an upper bound on the maximum number of differentiations required in order to perform such a linearization. This would then provide a complete list of necessary and sufficient conditions for such a system to be dynamic feedback linearizable via a succession of prolongations by differentiation. Ideally, the structure equations of Theorem 3.13 might then be used to produce a complete classification of such systems.

This paper is organized as follows.

To start with (Sects. 2 and 3), we remind the reader of various notions of prolongation of a system (Sect. 2.1) and the extension theorem of Sluis (Sect. 2.2). Then, we introduce the notion of a relative extension (Sect. 2.4), which is a canonical construction that one can obtain from a Cartan prolongation. A relative extension can be viewed as a nested array of Pfaffian bundles. In general, it need not induce a succession of Cartan prolongations. However, when the original Cartan prolongation is regular and when the systems involved have corank 3, the relative extensions do induce a succession of Cartan prolongations (Theorem 2.35). This allows us to relate various notions of equivalence obtained from different notions of prolongation (Theorem 2.42), which in turn allows us to describe in a canonical way the necessary and sufficient conditions for a type $(n, 2)$ (that is, $n$ states and 2 controls) control-type system to be dynamic feedback linearizable via a prolongation by differentiation involving $K$ differentiations (Theorem 3.11). We then set up the structure equations (Theorem 3.13) that are associated to a linearization. In a sequel to this paper, these structure equations will serve as the basis of classifying control systems that can be linearized after a particular, fixed number of prolongations.

### 1.1 Symbols and abbreviations

- **EDS**: Exterior differential system(s).
- **CTS**: Control-type system(s).
- $(M, I)$: An EDS with manifold $M$ and differential ideal $I \subset \Omega^*(M)$.
- $\langle \eta^1, \ldots, \eta^k \rangle$: The differential ideal in $\Omega^*(M)$ generated by differential forms $\eta^1, \ldots, \eta^k \in \Omega^*(M)$ and their exterior derivatives.
\[(\eta^1, \ldots, \eta^k)_{\text{alg}}: \text{The ideal in } \Omega^*(M) \text{ algebraically generated by differential forms } \eta^1, \ldots, \eta^k \in \Omega^*(M).\]

\[(\text{pr}^{(k)}M, \text{pr}^{(k)}I): \text{The } k\text{-th (total) prolongation of an EDS } (M, I).\]

\[(M, I): \text{A Pfaffian system determined by a vector subbundle } I \subset T^*M.\]

\[(\text{pr}^{(k)}M, \text{pr}^{(k)}I): \text{The } k\text{-th (total) prolongation of a Pfaffian system } (M, I).\]

\[C(I) \text{ (resp. } C(I): \text{The Cartan system of a differential ideal } I \subset \Omega^*(M) \text{ (resp., of a Pfaffian system } I \subset T^*M). \text{ By definition, it is the Frobenius system defined on } M \text{ whose leaves are precisely the Cauchy characteristics of } (M, I) \text{ (resp. } (M, I).\]

\[C(\omega^1, \ldots, \omega^k)_{\text{alg}}: \text{The algebraic Cartan system associated to an algebraic ideal of } \Omega^*(M) \text{ generated by differential forms } \omega^1, \ldots, \omega^k. \text{ By definition, this is the Pfaffian system dual to the distribution spanned by all vector fields } X \text{ that satisfy } X \cdot \omega^i \in \langle \omega^1, \ldots, \omega^k \rangle_{\text{alg}} \text{ for all } i = 1, \ldots, k.\]

\[\theta: \text{A vector valued 1-form } (\theta^1, \ldots, \theta^k)^T.\]

\[[\theta^1, \ldots, \theta^k]: \text{The rank } k \text{ subbundle of } T^*M, \text{ or the corresponding Pfaffian system, spanned by } k \text{ linearly independent 1-forms } \theta^1, \ldots, \theta^k.\]

\[[I, \eta^1, \ldots, \eta^k]: \text{The subbundle of } T^*M \text{ spanned by the sections of a subbundle } I \subset T^*M \text{ and differential 1-forms } \eta^1, \ldots, \eta^k \in \Omega^1(M). \text{ Sometimes we write } [I, \eta^i]_{i=1}^k \text{ for brevity.}\]

\[I_k: \text{The } k\text{-th extension of a system } I \text{ relative to a Cartan prolongation.}\]

\[|S|: \text{The cardinality of a finite index set } S.\]

### 2 The structure of a Cartan prolongation

**Definition 2.1** A system is a triple \((M, I; \tau)\), where \((M, I)\) is a Pfaffian system (i.e., \(M\) is a smooth manifold, and \(I \subset T^*M\) is a vector subbundle), and \(\tau\), called the independence condition, is an exact 1-form which nowhere belongs to \(I\), such that

i. \((M, I)\) admits no Cauchy characteristics, and
ii. \((M, I)\) admits no integral surface \(\iota: S \hookrightarrow M\) that satisfies \(\iota^*\tau \neq 0\).

This definition is local. Generally, one would not require \(\tau\) to be exact (see [10, p. 35]), but an exact independence condition can always be chosen by shrinking \(M\) and allowing a negligible change in the set of integral curves of the Pfaffian system. Having an exact independence condition distinguished will be convenient when we later work with control systems, for which \(dt\), the differential of time, is the natural choice of an independence condition.

Furthermore, we will always assume that a system and its derived systems have constant ranks. This can be achieved by shrinking \(M\), if needed.

Our motivation for including condition ii in Definition 2.1 will be explained in Remark 2.9 (Sect. 2.1).

**Definition 2.2** A system \((M, I; \tau)\) is said to have type \((n, m)\) if

\[\begin{align*}
\dim(M) &= n + m + 1, \\
\text{rank}(I) &= n.
\end{align*}\]

The integer \(m + 1\) will be called the corank of \((M, I; \tau)\).
Definition 2.3  Given two systems \((M, I; \tau)\) and \((\tilde{M}, \tilde{I}; \tilde{\tau})\), we say that they are \(\tau\)-equivalent if there exists a diffeomorphism
\[
\phi : M \rightarrow \tilde{M}
\]
such that
\[
\phi^* \tilde{I} = I, \quad \phi^* \tilde{\tau} = \tau.
\]

Remark 2.4  Replacing the condition \(\phi^* \tilde{\tau} = \tau\) by
\[
\phi^* \tilde{\tau} \equiv \rho \tau \mod I
\]
for some nonzero function \(\rho\), one would obtain a more general notion of equivalence ([10, p. 38]). Because of this, in the definition above, “\(\tau\)-” is used in order to eliminate possible confusion.

Sluis proved the following result.

Theorem 2.5  [10, p.37, Theorem 14] A system \((M, I; \tau)\) admits local coordinates in which it corresponds to a (time-varying) control system
\[
\dot{x} = f(x, u, t)
\]
such that \(\tau = dt\) if and only if the Pfaffian system \(\llbracket I, \tau \rrbracket\) is Frobenius.

Motivated by this, we make the definition below.

Definition 2.6  A system \((M, I; \tau)\) of type \((n, m)\) is said to be a control-type system (CTS) if \(n \geq m\) and the Pfaffian system \(\llbracket I, \tau \rrbracket\) is Frobenius.

Remark 2.7  A. A type \((n, m)\) CTS corresponds to a control system with \(n\) states and \(m\) inputs.
B. Given two CTS \((M, I; \tau)\) and \((\tilde{M}, \tilde{I}; \tilde{\tau})\), represented in coordinates by
\[
\dot{x} = f(x, u, t)
\]
and \(\dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u}, \tilde{t})\), respectively, it is not difficult to see that they are \(\tau\)-equivalent as systems if and only if there exists a diffeomorphism of the form
\[
\phi : (x, u, t) \mapsto (\tilde{x}, \tilde{u}, \tilde{t}) = (\xi(x, t), \eta(x, u, t), t + T),
\]
where \(T\) is a constant, that transforms the second system into the first, and vice versa.

2.1 Prolongations of a system

Given a system \((M, I; \tau)\), one can define the following four types of prolongations: total prolongation, partial prolongation, prolongation by differentiation and Cartan prolongation.

Definition 2.8  Let \((M, I; \tau)\) be a system. As it will be sufficient for this paper, the following notions of prolongation are defined locally.

1. By shrinking \(M\), if needed, choose coordinates \((x^1, \ldots, x^n, u^1, \ldots, u^m, t)\) on \(M\) such that \(I, du^\alpha, \tau\) span the entire cotangent bundle \(T^*M\). Let \(pr^{(1)}M := M \times \mathbb{R}^m\), with \((\lambda^\alpha)\) being coordinates on the \(\mathbb{R}^m\)-component; let
\[
pr^{(1)}I := \llbracket I, du^\alpha - \lambda^\alpha \tau \rrbracket_{\alpha=1}^m.
\]
and let the pull-back of $\tau$ to $\text{pr}^{(1)}M$ be denoted by the same letter. The system $(\text{pr}^{(1)}M, \text{pr}^{(1)}I; \tau)$ is called the total prolongation of $(M, I; \tau)$. (Note that this system is usually just called the prolongation of $(M, I; \tau)$; we have added the adjective “total” to distinguish it from the other types of prolongations being considered.) This definition is independent of the choice of local coordinates.

2. A non-canonical way to “prolong” $(M, I; \tau)$ is by choosing $\mu^\alpha \in \Omega^1(M)$, $\alpha \in S \subseteq \{1, 2, \ldots, m\}$, linearly independent modulo $I$ and $\tau$, then letting $N := M \times \mathbb{R}^{|S|}$ and $J := \{I, \mu^\alpha - \lambda^\alpha \tau\}_{\alpha \in S}$, where $(\lambda^\alpha)$ are coordinates on the $\mathbb{R}^{|S|}$-factor. The system $(N, J; \tau)$ is called a partial prolongation of $(M, I; \tau)$.

3. Let $(x^1, \ldots, x^n, u^1, \ldots, u^m, t)$ be coordinates on $M$ such that $dt = \tau$ and that $I, du^\alpha, dt$ generate the entire $T^*M$. The system

$$(M \times \mathbb{R}^{|S|}, \{I, du^\alpha - \lambda^\alpha dt\}_{\alpha \in S}; dt),$$

where $S \subseteq \{1, 2, \ldots, m\}$ is fixed, is called a prolongation by differentiation of $(M, I; \tau)$; note that this is a particular type of partial prolongation.

4. A Cartan prolongation of $(M, I; \tau)$ is a system $(N, J; \sigma)$ together with a submersion $\pi : N \rightarrow M$ satisfying

i. $\pi^*I \subset J$;

ii. $\pi^*\tau = \sigma$;

iii. any (generic) integral curve $\gamma : (-\epsilon, \epsilon) \rightarrow M$ of $(M, I; \tau)$ has a unique lifting $\hat{\gamma} : (-\epsilon, \epsilon) \rightarrow N$ that satisfies $\hat{\gamma}^*J = 0$ and $\pi \circ \hat{\gamma} = \gamma$.

The fiber dimension of $\pi$ will be called the order of the Cartan prolongation.

**Remark 2.9** One can ask: Does a partial prolongation of a system necessarily yield a system? This is the question that motivated us to include the condition ii in Definition 2.1. Without this condition, if a Pfaffian system $(M, I)$ with an independence condition $\tau$ admits an integral surface $\iota : S \rightarrow M$ such that $\iota^*\tau$ is nonvanishing, then a partial prolongation may introduce nontrivial Cauchy characteristics, which is undesirable. As an example, consider the system

$$I = ||dz - pdx - qdt||,$$

defined on $M = \mathbb{R}^5$ with coordinates $(x, t, z, p, q)$ and with independence condition $dt$. The Pfaffian system $(M, I)$ admits integral surfaces. A partial prolongation may be achieved by adjoining to $I$ the 1-form

$$dx - \lambda dt.$$

The resulting system has nontrivial Cauchy characteristics, corresponding to those curves that annihilate the 1-forms below:

$$dz, d(\lambda p + q), dx, d\lambda, dt.$$

On the other hand, we have the following Proposition.

**Proposition 2.10** A partial prolongation of a system remains a system.

---

2 By “generic,” we mean that this condition holds for a set of integral curves that is open and dense in the space of all integral curves with respect to the $C^k$ topology for some $k$. We do not require all integral curves of $(M, I; \tau)$ to satisfy the condition, in order to make the concept useful even when there exists a negligible set of integral curves that do not admit unique liftings. For example, consider $(M, I; \tau) := (\mathbb{R}^5, \{dx - udv, dy - vdr\}; dv)$. It has a Cartan prolongation: $(N, J; \sigma) := (\mathbb{R}^5 \times \mathbb{R}, \{I, du - \lambda dv\}; dv)$, to which all integral curves of $(M, I; \tau)$ admits a unique lifting except those along which $v$ is a constant.
\textbf{Proof} Let \((M, I; \tau)\) be a system. Let \((N, J; \tau)\) be a partial prolongation of \((M, I; \tau)\), obtained by adjoining the 1-forms (as in Definition 2.8)

\[ \mu^\alpha - \lambda^\alpha \tau, \quad \alpha \in S \subseteq \{1, 2, \ldots, m\}. \]

To show that \((N, J; \tau)\) is a system, it suffices to verify that (1) it does not admit any Cauchy characteristics and (2) it does not admit any integral surfaces on which \(\tau\) is nonvanishing.

If \((N, J; \tau)\) has Cauchy characteristics \(\mathcal{C}\), then any integral curve transversal to a curve in \(\mathcal{C}\) and satisfying the independence condition \(\tau\) can be used to generate an integral surface on which \(\tau\) pulls back to be nonzero. Thus, it suffices to justify (2).

Suppose that \(\iota: S \to N\) is an integral surface of \((N, J)\) satisfying \(\iota^* \tau \neq 0\). Let \(\pi: N \to M\) be the obvious submersion. Since \(\pi^* I \subset J\) and \(\iota^* \tau \neq 0\), locally the rank of \(\pi|_S\) must be equal to 1; otherwise, \((M, I)\) would admit an integral surface where \(\tau\) pulls back to be nonzero. As a result, \(\iota^* \mu^\alpha\) are multiples of \(\tau\), and \(\iota^* d\mu^\alpha\) are equal to zero. Since

\[ 0 = \iota^* d(\mu^\alpha - \lambda^\alpha \tau) = \iota^* (d\mu^\alpha - d\lambda^\alpha \wedge \tau), \quad \alpha \in S, \]

\(d\lambda^\alpha\) must all be multiples of \(\tau\); in other words, if locally \(\tau = dt\), then \(\lambda^\alpha\) are functions of \(t\), which violates the assumption that \(S\) is a surface. \(\Box\)

\textbf{Remark 2.11} We note that each notion of prolongation in Definition 2.8 requires that the independence conditions correspond to each other via the underlying submersion. Removing this requirement leads to broader notions of prolongation that are more familiar in the literature. In this paper, we will always assume that independence conditions are matched by a prolongation.

\textbf{Remark 2.12} The notions of prolongation above satisfy the following.\(^3\)

A. Among the four notions above, Cartan prolongation is the most general.

B. An order-1 Cartan prolongation of a system \((M, I; \tau)\) need not be a partial prolongation of \((M, I; \tau)\). (See [10, p. 50].)

C. If \((M, I; \tau)\) is a CTS, then a prolongation by differentiation of \((M, I; \tau)\) is also a CTS.

D. A partial prolongation may not necessarily be realized as a prolongation by differentiation of the same system. For example, let \(\hat{M} = \mathbb{R}^7\) and \(I = \langle \theta^1, \theta^2, \theta^3 \rangle\), where \(\theta^i = dx^i - p^i dt\) \((i = 1, 2, 3)\). The system

\[ (N, J; \tau) := (\mathbb{R}^7 \times \mathbb{R}, \langle I, dp^1 + p^2 dp^3 - \lambda dt \rangle; dt) \]

is a partial prolongation of \((M, I; dt)\). However, it is not a prolongation by differentiation, since \(\langle J, dt \rangle\) is not Frobenius (in other words, \((N, J; \tau)\) is not a CTS).

E. When \(m = \dim(M) - \rank(I) - 1 = 1\), a Cartan prolongation is necessarily the result of successive total prolongations. This case is treated in [2], where Cartan did not require the matching of independence conditions.

F. A composition of Cartan prolongations is also a Cartan prolongation, as the following proposition shows.

\textbf{Proposition 2.13} If \(\pi: (N, J; \sigma) \to (M, I; \tau)\) and \(\sigma: (P, L; \rho) \to (N, J; \sigma)\) are both Cartan prolongations of systems, then the composition \(\pi \circ \sigma: (P, L; \rho) \to (M, I; \tau)\) is again a Cartan prolongation.

\(^3\) The reader may compare this with [10, p.61].
Proof By the assumption,

\[(\pi \circ \varpi)^*J = \varpi^* (\pi^* I) \subseteq \varpi^* J \subseteq L,\]

and it is clear that each generic integral curve \(\gamma\) of \((M, I; \tau)\) has a lifting to \((N, J; \sigma)\) then to \((P, L; \rho)\). To justify uniqueness, suppose that \(\gamma_1\) and \(\gamma_2\) are two liftings of \(\gamma\) to \((P, L; \rho)\). Therefore both \(\varpi \circ \gamma_i\) \((i = 1, 2)\) are integral curves of \((N, J; \sigma)\) and project via \(\pi\) to \(\gamma\). Since \(\pi\) is a Cartan prolongation, \(\varpi \circ \gamma_1 = \varpi \circ \gamma_2\); since \(\varpi\) is a Cartan prolongation, \(\gamma_1 = \gamma_2\). \(\square\)

2.2 The extension theorem of Sluis

In [2], Cartan noted that a generalization of the fact E in Remark 2.12 to the cases of \(m \geq 2\) can be quite difficult.

Sluis, in his thesis [10], considered such more general cases. In this section, we remind the reader of a notable theorem he obtained and sketch the main arguments in his proof. The theorem indicates the following. (See Theorem 2.15 for a more precise statement.)

Any Cartan prolongation of a system can be extended to a (successive) total prolongation, and such an extension itself is also a Cartan prolongation.

Consider a Cartan prolongation of order \(r\) represented by the diagram below, where the rank of \(I\) is \(n\).

\[(N^{n+m+r+1}, J; \sigma) \xrightarrow{\pi} (M^{n+m+1}, I; \tau)\]

By shrinking \(M\) if needed, choose coordinates \((x^i, u^\alpha, v^\rho, t)\) on \(M\) such that \(I, du^\alpha, dt\) span \(T^*M\), where \(dt = \tau\).

In order to understand such a Cartan prolongation, Sluis began by considering those 1-forms in \(J\) that are closest to being expressible in terms of the coordinates on \(M\). To be more specific, he asked: What is the rank of the subbundle \(\hat{J}\) of \(J\) spanned by the 1-forms that can be written as

\[f_1 du^1 + f_2 du^2 + \cdots + f_m du^m + g dt,\]

where \(f, g\) are functions on \(N\)? (Note that \(\hat{J}\) has the coordinate-free interpretation as a quotient bundle \((J \cap \pi^*(T^*M))/\pi^* I)\.)

We have the following lemma:

**Lemma 2.14** \(\text{rank}(\hat{J}) \neq 0, m + 1\).

**Proof** Let \((x^i, u^\alpha, v^\rho, t)\) \((\rho = 1, \ldots, r)\) be local coordinates on \(N\).

If \(\text{rank}(J) = 0\), then \(J\) is spanned by \(I\) and some 1-forms

\[a^k \rho dv^\rho + b^k_\alpha du^\alpha + c^k dt\]

(k = 1, \ldots, s; s \leq r),

where \(a^k, b^k_\alpha, c^k\) are functions on \(N\), and \((a^k_\rho)\) has maximum rank (i.e., rank \(s\)). Now let \(\gamma = (x(t), u(t), t)\) be any integral curve of \((M, I; \tau)\). The pullback of \(J\) to \(\pi^{-1}\gamma\) is spanned by the 1-forms

\[a^k \rho dv^\rho + (b^k_\alpha (u^\alpha)' + c^k) dt.\]
Since the row rank of \((a_k^\rho)\) is full, \(J\) induces a distribution \(J^\perp\) on \(\pi^{-1}\gamma\), whose integral curves are non-unique liftings of \(\gamma\). This violates part iii in the definition of Cartan prolongation.

If \(\text{rank}(\hat{J}) = m + 1\), then \(J\) would contain the independence condition, which violates the definition of a system.

By Lemma 2.14, the only cases that can occur are:

**I.** \(\text{rank}(\hat{J}) = m\) and **II.** \(1 \leq \text{rank}(\hat{J}) < m\).

**Case I** In this case, there exist functions \(f^\alpha : N \to \mathbb{R}\) \((\alpha = 1, \ldots, m)\) such that

\[
du^\alpha - f^\alpha dt
\]

are sections of \(J\). It is shown in [10, p. 65] that, on some dense open subset of \(N\), \(f^\alpha\) must be independent of \(x^i, u^\alpha, t\) and independent among themselves; otherwise, \((f^\alpha)\) cannot be surjective for a fixed initial point on \(M\) (by Sard’s theorem), and a generic integral curve \(\gamma\) of \((M, I; \tau)\) passing through that initial point would not admit a lifting.

Thus, by shrinking \(N\), if needed, one can define a submersion:

\[
\pi^1 : N \to M \times \mathbb{R}^m
\]

by

\[
N \ni p \mapsto (\pi(p), (f^\alpha(p))).
\]

We can identify \(M \times \mathbb{R}^m\) with the space of the total prolongation \(\text{pr}^{(1)}M\) equipped with the Pfaffian system generated by \(du^\alpha - y^\alpha dt\) \((\alpha = 1, \ldots, m)\), where \((y^\alpha)\) are coordinates on the \(\mathbb{R}^m\)-component. This makes \(\pi^1\) a Cartan prolongation. See the diagram below. 4

\[
\begin{array}{c}
(N, J) \\
\pi^1 \downarrow \\
\pi \\
\downarrow \\
(M, I) \\
\end{array}
\]

\[
\begin{array}{c}
(\text{pr}^{(1)}M, \text{pr}^{(1)}I) \\
\pi_1 \leftarrow \\
\end{array}
\]

Now note that \((\text{pr}^{(1)}M, \text{pr}^{(1)}I; dt)\) has type \((n + m, m)\). \(\pi^1\) is a Cartan prolongation of order \((r - m)\).

**Case II** In this case, \(\text{rank}(\hat{J}) = q < m\). Suppose that \(\hat{J}\) has the following basis representatives:

\[
f_1^\mu du^1 + \cdots + f_m^\mu du^m + g^\mu dt \quad (\mu = 1, \ldots, q).
\]

Let \(\mu, \nu = 1, \ldots, q\) and \(\alpha = 1, \ldots, m\). We must have \(\text{rank}(f_\alpha^\nu) = q\), since, otherwise, \(dt\) would be a section of \(J\).

By reordering \(du^\alpha\), we may assume that

\[
\det(f_\mu^\nu) \neq 0.
\]

Now consider the following prolongation of \((N, J)\) by differentiation:

\[
N_1 := N \times \mathbb{R}^{m-q},
\]

4 Here and below, we drop the independence conditions in these diagrams for clarity.
Fig. 1 Extending a Cartan prolongation in Case II

\[
\begin{array}{cccc}
(N, J) & \xleftarrow{\phi} & (N_1, J_1) \\
\pi & \downarrow & \pi^1 \\
(M, I) & \xleftarrow{\pi_1} & (pr^{(1)}M, pr^{(1)}I)
\end{array}
\]

with \((y^1, \ldots, y^{m-q})\) being coordinates on the \(\mathbb{R}^{m-q}\)-component, and

\[J_1 := [J, du^{q+1} - y^1 dt, \ldots, du^m - y^{m-q} dt].\]

Let the submersion \(N_1 \to N\) be denoted by \(\phi\).

It is easy to see that

\[\pi \circ \phi : (N_1, J_1) \to (M, I)\]

represents a Cartan prolongation that belongs to Case I. This implies the diagram in Fig. 1, where \(\pi^1\) is a Cartan prolongation of order \(r - q\).

Combining the cases I and II, for some minimal finite \(K\), we obtain the diagram in Fig. 2, where \(\pi^K\) is an isomorphism.

In Fig. 2,

1. Each \(\pi_k\) has order \(m\);
2. Each \(\phi_k\) has order \(m - q_k\), where \(1 \leq q_k \leq m\). When \(q_k = m\), \(\phi_k\) is constructed from Case I and is an isomorphism;
3. Each \(\pi^K\) has order \(r - q_1 - q_2 - \cdots - q_K\);
4. \(r = q_1 + \cdots + q_K\), and consequently \(\left\lfloor \frac{r-1}{m} \right\rfloor + 1 \leq K \leq r\).

The construction above is summarized by the following extension theorem of Sluis; it will be illustrated in Examples 2.43 and 2.44 below.

**Theorem 2.15** [10, p. 63, Theorem 24] If \(\pi : (N, J; \sigma) \to (M, I; \tau)\) is a Cartan prolongation of order \(r\), then there exists an integer \(K \leq r\) and a map \(\hat{\pi}\) given by a composition of \(K\) successive prolongations by differentiation such that the following diagram commutes in the sense of EDS, where \(\pi_{K,0} : pr^{(K)}M \to M\) is the composition of \(K\) successive total prolongations.

\[
\begin{array}{cccc}
(pr^{(K)}M, pr^{(K)}I) & \xleftarrow{\hat{\pi}} & (N, J) \\
\pi_{K,0} & \downarrow & \pi \\
(M, I) & \xleftarrow{\pi} & (N, J)
\end{array}
\]

**Corollary 2.16** If \(\pi : (N, J; \sigma) \to (M, I; \tau)\) is a Cartan prolongation, then

\[\text{rank}(J) - \text{rank}(I) = \dim N - \dim M.\]
Proof This is because, in the argument that leads to Theorem 2.15,
\[ \dim N_k - \dim N_{k-1} = \text{rank}(J_k) - \text{rank}(J_{k-1}), \]
\[ \dim \text{pr}^{(k)} M - \dim \text{pr}^{(k-1)} M = \text{rank}(\text{pr}^{(k)} I) - \text{rank}(\text{pr}^{(k-1)} I), \]
and at the $K$-th stage, $(N_K, J_K)$ and $(\text{pr}^{(K)} M, \text{pr}^{(K)} I)$ coincide. \qed

Remark 2.17 At various points in the argument above, we have applied steps such as “by shrinking to an open dense subdomain, if needed …”. This is because, as Footnote 2 indicated, we allow a negligible set of integral curves to ill-behave relating to a Cartan prolongation. Putting the example in Footnote 2 in context, one immediately notices that that Cartan prolongation belongs to Case II, and we have
\[ (N_1, J_1) = (\mathbb{R} \times \mathbb{R}, [J, dv - \mu dt]) \]
\[ = (\mathbb{R} \times \mathbb{R}, [dx - u dt, dy - v dt, dv - \lambda dv, dv - \mu dt]), \]
where $\mu$ is the coordinate on the $\mathbb{R}$-component of $N_1$. Thus, the system $(N_1, J_1)$, relative to $(M, I)$, is a Cartan prolongation in Case I. In particular,
\[ du - \lambda \mu dt, \quad dv - \mu dt \]
are sections of $J_1$. According to (4),
\[ f^1 = \lambda \mu, \quad f^2 = \mu. \]
Note that $df^1, df^2$ are only linearly independent when $\mu \neq 0$, so
\[ \pi^1 : (N_1, J_1) \xrightarrow{\cong} (\text{pr}^{(1)} M, \text{pr}^{(1)} I), \]
is understood as a diffeomorphism onto a dense open subset of $\text{pr}^{(1)} M$.

2.3 Absolute $\iff$ dynamic

A consequence of Sluis’s Extension Theorem is that absolute equivalence is an equivalence relation; this is proved in [10]. Moreover, under some mild assumptions, absolute equivalence is equivalent to the notion of dynamic equivalence, as we will demonstrate below.

Definition 2.18 Two systems $(M, I; \tau)$ and $(\bar{M}, \bar{I}; \bar{\tau})$ are said to be $\tau$-absolutely equivalent if there exists a system $(N, J; \sigma)$ and submersions $\pi : N \to M$ and $\bar{\pi} : N \to \bar{M}$ that realize $(N, J; \sigma)$ as a Cartan prolongation of both $(M, I; \tau)$ and $(\bar{M}, \bar{I}; \bar{\tau})$.

\[ \begin{array}{c}
\pi \\
(M, I; \tau)
\end{array} \xleftarrow{(N, J; \sigma)} \xrightarrow{\bar{\pi}} \begin{array}{c}
\bar{\pi} \\
(\bar{M}, \bar{I}; \bar{\tau})
\end{array} \]

Definition 2.19 Two systems $(M, I; \tau)$ and $(\bar{M}, \bar{I}; \bar{\tau})$ are said to be $\tau$-dynamically equivalent if there exist integers $p, q \geq 0$ and submersions $\Phi, \Psi$, as shown in the diagram below, such that

i. $\Phi^* \bar{\tau} = \pi^* \tau, \Psi^* \tau = \bar{\pi}^* \bar{\tau}$;
ii. $\Phi^* I \subseteq \text{pr}^{(p)} I, \Psi^* I \subseteq \text{pr}^{(q)} \bar{I}$

5 Compare with [11, Definition 3.2].
iii. for any (generic) integral curve $\gamma$ of $(M, I; \tau)$, $\Psi \circ (\Phi \circ \gamma^{(p)})^{(q)} = \gamma$, and for any (generic) integral curve $\tilde{\gamma}$ of $(\tilde{M}, \tilde{I}; \tilde{\tau})$, $\Phi \circ (\Psi \circ \tilde{\gamma}^{(p)})^{(q)} = \tilde{\gamma}$.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$M$};
  \node (B) at (2,0) {$\tilde{M}$};
  \node (C) at (0,-1) {$\mathbb{R}$};
  \node (D) at (2,-1) {$\mathbb{R}$};
  \node (E) at (1,0) {$\mathbb{N}$};
  \node (F) at (1,-1) {$\mathbb{N}$};
  \draw[->] (A) -- (B) node[midway, above] {$\Phi$};
  \draw[->] (A) -- (C) node[midway, left] {$\gamma$};
  \draw[->] (B) -- (D) node[midway, above] {$\tilde{\phi}$};
  \draw[->] (B) -- (F) node[midway, left] {$\tilde{\gamma}$};
  \draw[->] (C) -- (E) node[midway, left] {$\pi$};
  \draw[->] (D) -- (E) node[midway, left] {$\pi$};
  \draw[->] (E) -- (F) node[midway, above] {$\psi$};
\end{tikzpicture}
\end{center}

**Theorem 2.20** Two systems $(M, I; \tau)$ and $(\tilde{M}, \tilde{I}; \tilde{\tau})$ are $\tau$-absolutely equivalent if and only if they are $\tau$-dynamically equivalent.

**Proof** ($\Rightarrow$) Start with a $\tau$-absolute equivalence as described in Definition 2.18. By Theorem 2.15, there exist integers $p, q$ such that the following diagram commutes, providing a bijection between (generic) integral curves of each system involved. It follows by definition that $(M, I; \tau)$ and $(\tilde{M}, \tilde{I}; \tilde{\tau})$ are $\tau$-dynamically equivalent.

($\Leftarrow$) Conversely, assume the diagram in Definition 2.19. It suffices to show that $\Psi$ represents a Cartan prolongation.

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a generic integral curve of $I$. By the third condition in Definition 2.19, the curve

\[(\Phi \circ \gamma^{(p)})^{(q)} : (-\epsilon, \epsilon) \to \text{pr}^{(q)} \tilde{M}\]

is a lifting of $\gamma$ into $\text{pr}^{(q)} \tilde{M}$ as an integral curve of $\text{pr}^{(q)} I$.

Now suppose that there are two such liftings

$\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to \text{pr}^{(q)} \tilde{M}$.

By construction, $\tilde{\gamma}_1 := \tilde{\pi} \circ \gamma_1$ and $\tilde{\gamma}_2 := \tilde{\pi} \circ \gamma_2$ satisfy

$\Psi \circ \tilde{\gamma}_1^{(q)} = \Psi \circ \tilde{\gamma}_2^{(q)}$.

Hence,

$\tilde{\gamma}_1 = \Phi \circ (\Psi \circ \tilde{\gamma}_1^{(q)})^{(p)} = \Phi \circ (\Psi \circ \tilde{\gamma}_2^{(q)})^{(p)} = \tilde{\gamma}_2$.

Since $\tilde{\pi}$ is a total prolongation, it follows that

$\gamma_1 = \gamma_2$.

This proves that $\Psi$ is a Cartan prolongation.

We end the proof by remarking that the independence conditions are preserved by all the maps involved.

\[\square\]

---

6 This theorem may be seen as a variant of [11, Theorem 3.6].
2.4 Relative extensions

In order to understand Cartan prolongations further, we introduce the notion of a relative extension.

**Definition 2.21** Let \( \pi : (N, J, \sigma) \rightarrow (M, I, \tau) \) be a Cartan prolongation. We define the \( k \)-th extension \( I_k \) of \( I \) relative to \( \pi \) inductively:

1. \( I_0 = \pi^* I \);
2. \( I_k = C(I_{k-1}) \cap J \) (\( k \geq 1 \)).

**Example 2.22** Let \( \theta^i_j \) denote the 1-forms \( \theta^i_j = dx^i_j - x^i_{j+1} dt \).

Let \( (M, I; dt) \) be the type \((3,2)\) CTS in Brunovský normal form generated by the three 1-forms

\[
\begin{align*}
\theta^0_0 &= dx^0_1 - x^0_1 dt, \\
\theta^2_0 &= dx^2_0 - x^2_1 dt, \\
\theta^1_1 &= dx^1_1 - x^1_2 dt.
\end{align*}
\]

This system has a prolongation by differentiation to the type \((6,2)\) system \((N, J; dt)\) in Brunovský normal form generated by the six 1-forms

\[
\begin{align*}
\theta^0_0 &= dx^0_1 - x^0_1 dt, \\
\theta^2_0 &= dx^2_0 - x^2_1 dt, \\
\theta^1_1 &= dx^1_1 - x^1_2 dt, \\
\theta^2_1 &= dx^2_1 - x^2_2 dt, \\
\theta^1_2 &= dx^1_2 - x^1_3 dt, \\
\theta^3_1 &= dx^3_1 - x^3_4 dt.
\end{align*}
\]

The relative extensions of \( I \) are

\[
I_0 = \pi^* I, \quad I_1 = [I, \theta^0_1, \theta^2_1], \quad I_2 = J.
\]

Definition 2.21 is independent of the choice of coordinates. Moreover, there exists an integer \( K \geq 0 \) indicating where \( I_k \) stabilizes:

\[
\pi^* I = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_K = I_{K+1} = \cdots \subsetneq J.
\]

For simplicity, we denote \( I_\infty := I_K \), and we define the extension length of \( \pi \) to be the smallest integer \( K \) satisfying \( I_\infty = I_K \).

Each \( I_k \subseteq J \) is a subbundle on \( N \), so they may admit nontrivial Cauchy characteristics. As a result, we consider the underlying manifold of \( I_k \) as the one determined by the Cartan system \( C(I_k) \). However, since an inclusion between Pfaffian bundles does not generally imply the inclusion of their Cartan systems (for instance, \([dy - zdx] \subsetneq [dx, dy]\) but their Cartan systems satisfy \([dx, dy, dz] \supseteq [dx, dy]\)), it is not immediately clear how the \( C(I_k) \) relate to each other or whether \( I_\infty \) must be equal to \( J \) in general.

2.5 Regular and \( C \)-regular Cartan prolongations

**Definition 2.23** We call a Cartan prolongation \( \pi : (N, J; \sigma) \rightarrow (M, I; \tau) \) simple if \( I_1 = J \).
**Remark 2.24** Because a system is assumed to have no Cauchy characteristics, it is easy to see that the total prolongation, any partial prolongation or a prolongation by differentiation (Sect. 2.1) of a system are simple Cartan prolongations.

**Lemma 2.25** Any Cartan prolongation of order 1 is simple.

**Proof** In this case, if \( I_1 \neq J \), then \( I_1 = \pi^*I \), which is impossible by Lemma 2.14. \( \square \)

The following theorem, which will be useful later, can be regarded as a special case of Theorem 2.15.

**Theorem 2.26** If \( \pi : (N, J; \sigma) \rightarrow (M, I; \tau) \) is a simple Cartan prolongation, then there exists a simple Cartan prolongation \( \hat{\pi} : (pr^{(1)}M, pr^{(1)}I) \rightarrow (N, J) \) such that the diagram below commutes in the EDS sense.

\[
\begin{array}{ccc}
(pr^{(1)}M, pr^{(1)}I) & \xrightarrow{\hat{\pi}} & (N, J) \\
\downarrow{\pi_{1,0}} & & \downarrow{\pi} \\
(M, I) & & (N, J) 
\end{array}
\]

**Remark 2.27** \( \hat{\pi} \) is in fact a prolongation by differentiation.

With Theorem 2.26 in mind, we are interested in the case when a Cartan prolongation can be achieved by successively performing simple Cartan prolongations, starting from an original system. To make this point explicit, we make the definition below.

**Definition 2.28** Let \( \pi : (N, J; \sigma) \rightarrow (M, I; \tau) \) be a Cartan prolongation. It is regular if there exist vector subbundles \( I(\ell) \subset J \) \((\ell = 1, \ldots, L)\) satisfying

\[ \pi^*I = I(0) \subsetneq I(1) \subsetneq \cdots \subsetneq I(L-1) \subsetneq I(L) = J \]

such that

1. \( \sigma \) is a section of \( C(I(\ell)) \) for each \( \ell \in \{0, 1, \ldots, L\} \);
2. each \((M(\ell), I(\ell); \sigma(\ell))\) is a system, where \( M(\ell) \) is the manifold given by the quotient space of \( N \) by the leves of the distribution annihilated by \( C(I(\ell)) \), and \( \sigma(\ell) \) is a corresponding independence condition defined on \( M(\ell) \);
3. each inclusion \( I(\ell) \subsetneq I(\ell+1) \) induces an inclusion \( C(I(\ell)) \subsetneq C(I(\ell+1)) \), which in turn determines a submersion from \( M(\ell+1) \) to \( M(\ell) \) that represents a simple Cartan prolongation of \((M(\ell), I(\ell); \sigma(\ell))\).

Otherwise, \( \pi \) is called singular.

A condition that is stronger than “regular” is when the vector subbundles in Definition 2.28 can be chosen to be the canonical relative extensions \( I_k \) (Definition 2.21) and still satisfy the conditions (1)-(3). To be clear, we present the following definition.

**Definition 2.29** Let \( \pi : (N, J; \sigma) \rightarrow (M, I; \tau) \) be a Cartan prolongation. It is \( C \)-regular if the canonical relative extensions \( I_k \) \((k = 1, \ldots, K)\) (see Definition 2.21) satisfy

\[ \pi^*I = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{K-1} \subsetneq I_K = J, \]

and
(1) $\sigma$ is a section of $C(I_k)$ for each $k \in \{0, 1, \ldots, K\}$;
(2) each $(M_k, I_k; \sigma_k)$ is a system, where $M_k$ stands for the manifold determined by $C(I_k)$, and $\sigma_k$ is a corresponding independence condition defined on $M_k$;
(3) each inclusion $I_k \subset I_{k+1}$ induces a submersion from $M_{k+1}$ to $M_k$, which represents a Cartan prolongation of $(M_k, I_k; \sigma_k)$.

**Proposition 2.30** If $\pi : (N, J; \sigma) \to (M, I; \tau)$ is $C$-regular, then the extensions $I_k$ of $I$ relative to $\pi$ satisfy the condition that $I_k \subset I_{k+1}$ is a simple Cartan prolongation.

**Proof** It suffices to prove that the Cartan prolongation induced by $I_k \subset I_{k+1}$ is simple. This is immediate since $I_{k+1} = C(I_k) \cap J = C(I_k) \cap I_{k+1}$. $\square$

**Lemma 2.31** Let $\pi : (N, J; \sigma) \to (M, I; \tau)$ be a regular Cartan prolongation with an associated filtration by simple Cartan prolongations

$$\pi^* I \subset I(1) \subset \cdots \subset I(L-1) \subset I(L) = J.$$ 

Let $I_k (k = 0, 1, \ldots, K)$ be the $k$-th extension of $I$ relative to $\pi$. Then we have

$$I(1) = C(\pi^* I) \cap I(1) \subset C(\pi^* I) \cap J = I_1,$$

as desired. $\square$

**Remark 2.32** It is not yet clear to us whether the conclusion in Lemma 2.31 must hold for all $k$ in general.

**Example 2.33** There exist singular Cartan prolongations.

Consider two systems with the same independence condition $dt$:

$$I = [dx_1 - u_1 dt, \; dx_2 - u_2 dt],$$  
$$J = [I, \; du_1 + fdu_2 - gdt, \; df - hdt, \; dg - (f + h)du_2].$$

$I \subset J$ represents a Cartan prolongation. To see this, consider an integral curve $\gamma : t \mapsto (x(t), u(t), t)$ of $I$. A lifting of $\gamma$ to an integral curve of $J$ must satisfy:

$$\begin{cases}
  u'_1 + fu'_2 - g = 0, \\
  f' - h = 0, \\
  g' - (f + h)u'_2 = 0.
\end{cases}$$

From these equations, we obtain that

$$(u'_1 + fu'_2)' = (f + f')u'_2.$$ 

This implies that

$$\begin{aligned}
  f &= \frac{u''_1}{u'_2 - u''_2}, \\
  g &= u'_1 + \frac{u''_1 u'_2}{u'_2 - u''_2}, \\
  h &= \left(\frac{u''_1}{u'_2 - u''_2}\right)',
\end{aligned}$$
which is determined as long as \( u'_{2} - u''_{2} \neq 0 \) along \( \gamma \).

It is straightforward to compute that

\[
I_1 = [I, du_1 + fdu_2 - gdt].
\]

\( I \subset I_1 \) is not a Cartan prolongation, since

\[
\text{rank}(\mathcal{C}(I_1)) - \text{rank}(\mathcal{C}(I)) = 7 - 5 = 2 > 1 = \text{rank}(I_1) - \text{rank}(I),
\]

violating Corollary 2.16. (The observation that \( I \subset I_1 \) is not a Cartan prolongation can also be seen more directly from the fact that a generic integral curve of \( I \) does not have a unique lift to an integral curve of \( I_1 \).) Thus, the Cartan prolongation represented by \( I \subset J \) is not \( \mathcal{C} \)-regular.

Furthermore, if \( I \subset J \) were regular with an associated filtration \( I(\ell) \) by simple Cartan prolongations, then by Lemma 2.31, we would have \( I(1) \subset I_1 \), and therefore \( I(1) = I_1 \) (since \( \text{rank}(I_1/I) = 1 \)). However, this is impossible, since \( I \subset I_1 \) does not represent a Cartan prolongation.

**Example 2.34** There exist Cartan prolongations that are regular but not \( \mathcal{C} \)-regular.

For \( n \geq 3 \), consider the following list of 1-forms expressed in the coordinates \( (x_i, u_\alpha, v_\rho, w, t) \):

\[
\begin{align*}
\theta^i &= dx_i - u_i dt, \quad (i = 1, \ldots, n) \\
\eta^1 &= du_1 - v_1 dt, \\
\eta^2 &= du_2 - v_2 du_3 - \cdots - u_{n-1} du_n - w dt, \\
\xi^1 &= dv_1 - v_2 dt, \\
& \quad \vdots \\
\xi^{n-2} &= dv_{n-2} - v_{n-1} dt, \\
\xi^{n-1} &= dv_{n-1} - v_n dt.
\end{align*}
\]

Let

\[
I = [\theta^1, \ldots, \theta^n], \quad J = [I, \eta^1, \eta^2, \xi^1, \ldots, \xi^{n-1}].
\]

It is easy to see that \( I \subset J \) represents a Cartan prolongation. Indeed, an integral curve \( \gamma : t \mapsto (x, u, t) = (x(t), x'(t), t) \) of \( I \) has its lifting to an integral curve of \( J \) uniquely determined by the equations:

\[
\begin{align*}
v_1 &= u_1' , \\
v_2 &= u_1'' , \\
& \quad \vdots \\
v_n &= u_1^{(n)} , \\
w &= u_2' - v_2 u_3' - \cdots - u_{n-1} u_n' .
\end{align*}
\]

Now \( I_1 = [I, \eta^1, \eta^2] \), \( I_2 = J \). Since \( n \geq 3 \),

\[
\text{rank}(\mathcal{C}(I_1)) - \text{rank}(\mathcal{C}(I)) = n > 2 = \text{rank}(I_1) - \text{rank}(I),
\]

and \( I \subset I_1 \) is not a Cartan prolongation.
On the other hand, \( I \subset J \) is regular, since we can take
\[
I(0) = I, \\
I(1) = \llbracket I, \eta^1 \rrbracket, \\
I(2) = \llbracket I, \eta^1, \xi^1 \rrbracket, \\
\vdots \\
I(n) = \llbracket I, \eta^1, \xi^1, \xi^2, \ldots, \xi^{n-1} \rrbracket, \\
I(n+1) = \llbracket I, \eta^1, \xi^1, \xi^2, \ldots, \xi^{n-1}, \eta^2 \rrbracket = J,
\]
and \( I(\ell) (\ell = 0, \ldots, n + 1) \) provides a filtration of \( J \) by simple Cartan prolongations.

The requirement “\( n \geq 3 \)” in Example 2.34 is no coincidence, as the following theorem shows.

**Theorem 2.35** Let \((M, I; \tau)\) be a system of corank 3 (i.e., \( \text{rank}(C(I)/I) = 3 \)). A Cartan prolongation \( \pi : (N, J; \sigma) \rightarrow (M, I; \tau) \) is regular if and only if it is \( C \)-regular.

**Proof** \((\Leftarrow)\) is trivial; we now prove \((\Rightarrow)\) by induction on the order of \( \pi \). If \( \pi \) has order 1, then it is *simple* by Lemma 2.25, and hence it is \( C \)-regular. From now on, suppose that the theorem holds for Cartan prolongations of order less than \( r \), and suppose that \( \pi \) has order \( r \).

Let
\[
\pi^*I = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{K-1} \subseteq I_K \subseteq J
\]
be the canonical filtration of \( J \) by the relative extensions of \( I \). Moreover, by assumption there exists a filtration
\[
\pi^*I = I(0) \subseteq I(1) \subseteq \cdots \subseteq I(L-1) \subseteq I(L) = J
\]
in which \( I(\ell) \subseteq I(\ell+1) \) are all simple Cartan prolongations.

Suppose that \( \text{rank}(I_1/I) = 1 \); then by Lemma 2.31, \( I(1) = I_1 \). Thus, \( I_1 = I(1) \subset J \) represents a regular Cartan prolongation of order \( r - 1 \), which, by the inductive hypothesis, is \( C \)-regular. It immediately follows that \( \pi \) is also \( C \)-regular.

Thus, it suffices to consider the case when \( \text{rank}(I_1/I) > 1 \). Since \((M, I; \tau)\) is a system of corank 3, \( \text{rank}(I_1/I) \leq \text{rank}(C(I)) - \text{rank}(I) = 3 \). However, if \( \text{rank}(I_1/I) = 3 \), then \( I_1 \), and hence \( J \), would contain the independence condition, violating the definition of a system. Therefore, \( \text{rank}(I_1/I) = 2 \), which we assume from now on.

Thus, by Lemma 2.31, \( \text{rank}(I(1)/I) \) is either 1 or 2. If \( \text{rank}(I(1)/I) = 2 \), then \( I(1) = I_1 \).

As in the case of \( \text{rank}(I_1/I) = 1 \), one easily argues by induction that \( \pi \) is \( C \)-regular, so it remains to consider the case when \( \text{rank}(I(1)/I) = 1 \), which we now assume.

Let \( \eta \) be a nontrivial representative of \( I_1/I(1) \). In fact, suppose that, in some coordinates, \( I \) is spanned by
\[
dx_i - \sum_{j=1}^{2} A_{ij}(x, u_1, u_2, t)du_j - B_i(x, u_1, u_2, t)dt, \quad (i = 1, \ldots, n)
\]
and that \( I(1) \) is spanned by \( I \) and
\[
du_1 - f(x, u_1, u_2, w, t)du_2 - g(x, u_1, u_2, w, t)dt,
\]

\( \odot \) Springer
where \( w \) is a fiber coordinate for the projection \( M(1) \to M \). We can choose

\[
\eta = du_2 - \lambda dt,
\]

where \( \lambda \) is independent of \( x, u_1, u_2, w, t \).

Let \( \mu \) be the smallest integer such that \( \eta \) is a section of \( I(\mu) \).

Now consider the diagram below.

\[
\begin{array}{c}
\vdots \\
I_{(\mu)} \\
\subseteq \\
[ [ I_{(\mu-1)}, \eta ] ] \supset I_{(\mu-1)} \\
\cup \\
\vdots \\
[ [ I_{(1)}, \eta ] ] \supset I_{(1)} \\
\cup \\
I_{(0)} = I
\end{array}
\]

In this diagram, \( I_{(0)} \subset [ [ I_{(1)}, \eta ] ] = I_1 \) represents a total prolongation. Moreover, the 1-form \( du_2 \) must be independent of \( I_{(k)} \) and \( dt \) for \( k = 1, \ldots, \mu - 1 \); otherwise, either the minimality of \( \mu \) would be violated, or \( dt \) would be a section of \( I(\mu) \), which is impossible. Thus, each horizontal inclusion in the diagram induces a prolongation by differentiation. This implies that \( [ [ I_{(k)}, \eta ] ] (k = 1, \ldots, \mu - 1) \) are systems.

Furthermore, for each \( k \in \{1, \ldots, \mu - 2\} \),

\[
[ [ I_{(k)}, \eta ] ] \subset [ [ I_{(k+1)}, \eta ] ]
\]

represents a simple Cartan prolongation. To see this, first note that the underlying manifold of \( [ [ I_{(k)}, \eta ] ] \) is just \( M(k) \times \mathbb{R} \) with \( \xi \) as the coordinate on the \( \mathbb{R} \)-factor. Submersion at the manifold level for consecutive \( k \) follows. Because \( du_2 \) is a section of \( \mathcal{C}(I_{(k)}) \) and \( d\xi \) is not, we have

\[
\mathcal{C}( [ [ I_{(k)}, \eta ] ] ) \cap [ [ I_{(k+1)}, \eta ] ] = [ [ I_{(k+1)}, \eta ] ] \quad (k = 1, \ldots, \mu - 2).
\]

Existence and uniqueness of lifting integral curves are evident.

Now, regarding \( \text{rank}(I_{(\mu)}/I_{(\mu-1)}) \), there are two possibilities—it is either 1 or 2. (This is because \( I_{(\mu-1)} \subset I(\mu) \) represents a simple Cartan prolongation of a system of corank 3.)

1. If \( \text{rank}(I_{(\mu)}/I_{(\mu-1)}) = 1 \), then \( I_{(\mu)} = [ [ I_{(\mu-1)}, \eta ] ] \);
2. If \( \text{rank}(I_{(\mu)}/I_{(\mu-1)}) = 2 \), that is, \( I_{(\mu-1)} \subset I(\mu) \) represents a total prolongation. Since we have argued that the inclusion \( I_{(\mu-1)} \subset [ [ I_{(\mu-1)}, \eta ] ] \) is a prolongation by differentiation, Theorem 2.26 implies that \( [ [ I_{(\mu-1)}, \eta ] ] \subset I(\mu) \) represents a simple Cartan prolongation of order 1.

In summary, we have the filtration

\[
\pi^*I = I_{(0)} \subset [ [ I_{(1)}, \eta ] ] \subset \cdots \subset [ [ I_{(\mu-1)}, \eta ] ] \supseteq I_{(\mu)} \subset \cdots \subset I_{(L)} = J,
\]
where each inclusion represents a simple Cartan prolongation. In other words, there exists a new filtration \( I_{[k]} \) of \( J \) by simple Cartan prolongations that satisfies \( \text{rank}(I_{[1]}/I) = 2 \), a case already treated.

This completes the proof. \( \square \)

**Proposition 2.36** Let \((M, I; \tau)\) be a system of corank 3, and let \( \pi : (N, J; \sigma) \to (M, I; \tau) \) be a \( C \)-regular Cartan prolongation, with \( I_k \) being the canonical relative extensions of \( I \). Let \( k \) denote the smallest integer such that \( \text{rank}(I_k^{\ell+1}/I_k^{\ell}) > 1 \). Then either \( k \) does not exist, or \( k = 0 \) and \( \text{rank}(I_1/I) = 2 \).

**Proof** Suppose for the sake of contradiction that \( k > 0 \). Then \( \text{rank}(I_{\ell+1}/I_{\ell}) = 1 \) for \( \ell \in \{0, 1, \ldots, k - 1\} \). By the proof of Lemma 2.31, we have
\[
I_{(\ell)} = I_{\ell}, \quad \ell = 0, 1, \ldots, k.
\]

In other words, \( I_1, \ldots, I_k \) are successive simple Cartan prolongations starting at \( I \). By Corollary 2.16, \((M_k, I_k; \sigma_k)\) is also a system of corank 3. It follows that \( \text{rank}(I_{k+1}/I_k) \leq 2 \).

By the characterization of \( k \), we must have \( \text{rank}(I_{k+1}/I_k) = 2 \). It follows that \( I_{k+1} \) is equivalent to \( \text{pr}^{(1)}I_k \).

As we will explain below, repeated application of Theorem 2.26 yields the following diagram,

\[
\begin{array}{ccc}
I_{k+1} & \equiv & \text{pr}^{(1)}I_k \\
\downarrow & & \downarrow \\
I_{k-1} & & I_k \\
\downarrow & & \downarrow \\
I_{k-1} & \sim & \rho \\
\end{array}
\]

where all arrows are understood as submersions at the manifold level. To be clear, in the diagram, the map \( \alpha : M_k \to M_{k-1} \) is induced from the simple Cartan prolongation \( I_{k-1} \subset I_k \), and the map \( \beta : M_{k+1} \to M_k \) is induced from the inclusion \( I_k \subset I_{k+1} \), which represents a total prolongation. Moreover, let \( q_k : N \to M_k \) denote the quotient maps.

Applied to \( \alpha \), Theorem 2.26 implies the existence of \( \psi \), which is also a simple Cartan prolongation. Applied to \( \psi \), the same theorem implies the existence of \( \rho \), again a simple Cartan prolongation.

Because \( C \) commutes with pull-back, we have
\[
\text{pr}^{(1)}I_{k-1} \subset C(\phi^*I_{k-1}) = \phi^*C(I_{k-1}).
\]

Consequently,
\[
(\rho \circ q_{k+1})^*(\text{pr}^{(1)}I_{k-1}) \subset (\rho \circ q_{k+1})^*(\phi^*C(I_{k-1})) = (\alpha \circ \beta \circ q_{k+1})^*C(I_{k-1}) = q_{k-1}^*C(I_{k-1}),
\]

where the LHS is a subbundle of \( J \), and the RHS is just \( C(I_{k-1}) \), for \( I_{k-1} \subset J \) is a subbundle.

To summarize, we have
\[
(\rho \circ q_{k+1})^*(\text{pr}^{(1)}I_{k-1}) \subset C(I_{k-1}) \cap J = I_k.
\]
This implies that the rank of \( pr^{(1)}I_{k-1} \) is at most the rank of \( I_k \). But this is impossible, since \( \phi \) has order 2, while \( \sigma \) has order 1.

Therefore, if \( k \) exists, then it must be zero, and \( \text{rank}(I_1/I) = 2 \). \( \square \)

**Corollary 2.37** Let \((M, I; \tau)\) be a system of corank 3. Suppose that \( \pi : (N, J; \sigma) \to (M, I; \tau) \) is a regular Cartan prolongation with the associated relative extensions \( I_k \) of \( I \). The chain of inclusions

\[
\pi^*I = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_K = J
\]

must represent a number (possibly zero) of successive total prolongations starting from \( I \) followed by a number of successive simple Cartan prolongations of order 1 that terminate at \( J \).

**Proof** By Theorem 2.35, each inclusion in (5) is a simple Cartan prolongation. Furthermore, if \( \text{rank}(I_{k+1}/I_k) \geq 2 \) and \( \text{rank}(I_{\ell+1}/I_\ell) = 1 \) for some \( \ell < k \), then the \( C \)-regular Cartan prolongation \( I_\ell \subset J \) would violate Proposition 2.36. \( \square \)

A particular case covered by Theorem 2.35 is when \( \pi \) is obtained by successive prolongations by differentiation: \( \pi^*I \subset I(1) \subset \cdots \subset I(L) = J \). When \( \text{rank}(I(1)/I) = 2 \), \( I(1) = I_1 \) and \( I_1 \subset J \) represents a prolongation by successive differentiation (i.e., a prolongation obtained by performing a succession of prolongations by differentiation) whose order is less than that of \( I \subset J \). When \( \text{rank}(I(1)/I) = 1 \), choose \( \eta \) and determine the integer \( \mu \) as in the proof of Theorem 2.35. Since, by assumption, each prolongation \( I(\mu) \subset I(k+1) \) is obtained by differentiation, the same is true for \( \|I(\mu), \eta\| \subset \|I(k+1), \eta\|, k \in \{1, \ldots, \mu - 2\} \). Now, if \( \text{rank}(I(\mu)/I(\mu-1)) = 1 \), then \( I(\mu) = \|I(\mu-1), \eta\| \); if \( \text{rank}(I(\mu)/I(\mu-1)) = 2 \), then Theorem 2.35 and Remark 2.27 imply that \( \|I(\mu-1), \eta\| \subset I(\mu) \) is a prolongation by differentiation. As a consequence,

\[
\pi^*I = I(0) \subset \|I(1), \eta\| \subset \cdots \subset \|I(\mu-1), \eta\| \subset \|I(\mu), \eta\| \subset \cdots \subset I(L) = J
\]

is a filtration in which each inclusion represents a prolongation by differentiation, satisfying \( \text{rank}(\|I(1), \eta\|/I) = 2 \).

This argument justifies the following theorem.

**Theorem 2.38** Let \((M, I; \tau)\) be a system of corank 3. If \( \pi : (N, J; \sigma) \to (M, I; \tau) \) is a prolongation by successive differentiation, then the associated relative extensions \( I_k \) of \( I \) satisfy: each \( I_k \subset I_{k+1} \) represents a prolongation by differentiation.

**Remark 2.39** The conclusion in Theorem 2.38 holds for general corank when all prolongations by differentiation are based on a fixed set of coordinates and the \( t \)-derivatives of the control variables therein. However, when the corank of \((M, I; \tau)\) is greater than 3, the conclusion does not necessarily hold if one allows prolongation by differentiation to follow changes of coordinates. For example, consider

\[
I = [[dx_i - u_idt]]_{i=1}^3.
\]

The following is a prolongation of \( I \) by successive differentiation\(^7\):

\[
J = \|I, du_1 - \alpha dt, \quad dx_\alpha - \beta dt, \|
\]

\(^7\) In fact, this prolongation is obtained by setting \( n = 3 \) in Example 2.34.
\[ d\beta - \gamma dt, \]
\[ d(u_2 - \beta u_3) - w dt]. \]

Direct calculation yields
\[ I_1 = \left\{ \left[ I, d\alpha_1 - \alpha dt, d\alpha_2 - \beta d\alpha_3 - \hat{w} dt \right] \right\}. \]

where \( \hat{w} = \gamma u_3 + w \). It turns out that \( I_1 \) is not even a Cartan prolongation of \( I \), since, otherwise, it must be a simple Cartan prolongation, but this is impossible by Corollary 2.16.

### 2.6 Alternative descriptions of \( \tau \)-absolute equivalence

One can modify the definition of \( \tau \)-absolute equivalence by requiring the Cartan prolongations to be regular or even to be ones obtained by successive differentiations. The result is two new equivalence relations among systems. It turns out that these new equivalence relations are no more restrictive than \( \tau \)-absolute equivalence, as we will demonstrate in this section.

**Definition 2.40** Two systems \((M, I; \tau)\) and \((\bar{M}, \bar{I}; \bar{\tau})\) are said to be \( \mathcal{R} \)-related (resp. \( \mathcal{D} \)-related) if there exists a system \((N, J; \sigma)\) and submersions \( \pi : N \to M \) and \( \bar{\pi} : N \to \bar{M} \) that make \((N, J; \sigma)\) a regular Cartan prolongation (resp., a prolongation by successive differentiation\(^8\)) of both \((M, I; \tau)\) and \((\bar{M}, \bar{I}; \bar{\tau})\).

**Proposition 2.41** Being \( \mathcal{R} \)-related (resp., \( \mathcal{D} \)-related) is an equivalence relation among systems.

**Proof** Reflexivity and symmetry are trivial; it suffices to prove transitivity. Suppose that \((M, I; \tau)\) and \((\bar{M}, \bar{I}; \bar{\tau})\) are \( \mathcal{R} \)-related (resp., \( \mathcal{D} \)-related), and suppose the same for \((\bar{M}, \bar{I}; \bar{\tau})\) and \((\hat{M}, \hat{I}; \hat{\tau})\). In the diagram below, by the assumption, \( \pi, \bar{\pi}, \sigma, \hat{\sigma} \) all represent regular Cartan prolongations (resp., prolongations by successive differentiation). The rest of the diagram is constructed using the assumption that \( \bar{\pi} \) and \( \sigma \) are Cartan prolongations and Theorem 2.15 (assuming \( \bar{L} \geq L \)); in particular, \( \pi_{L,L}, \phi, \psi, \pi_{L,0} \) represent prolongations by successive differentiation.

![Diagram](http://example.com/diagram.png)

The pair of maps \( \pi \circ \phi \circ \pi_{L,L} \) and \( \hat{\sigma} \circ \psi \) thus both represent regular Cartan prolongations (resp., prolongations by successive differentiation). This completes the proof. \( \square \)

**Theorem 2.42** For systems, \( \mathcal{D} \)-related \( \iff \mathcal{R} \)-related \( \iff \tau \)-Absolutely equivalent.

**Proof** (\( \Rightarrow \)) are obvious. ‘\( \mathcal{D} \)-related \( \iff \tau \)-Absolutely equivalent’ is a consequence of Sluis’s extension theorem. In fact, suppose that \((N, J; \sigma)\) is a Cartan prolongation of both \((M, I; \tau)\) and \((\bar{M}, \bar{I}; \bar{\tau})\) with the submersions \( \pi \) and \( \bar{\pi} \), respectively. Applying the diagram in Fig. 2 to

---

\(^8\) Here we do not require each prolongation by differentiation to be constructed from a fixed set of coordinates.
Theorem 2.15. Indeed, we prolong to establish a $D$-relation between $I$ and $J$. Following the construction in Theorem 2.15, we successively obtain $I$ on which it is easy to determine a local isomorphism between $J$, and construct the 3rd total prolongation of $\alpha$ on which $\beta$, as follows:

\[
\begin{align*}
\lambda_1 &= g - f\alpha, \\
\lambda_2 &= \alpha, \\
\mu_1 &= (\alpha - \beta)f, \\
\mu_2 &= \beta, \\
\kappa_1 &= (\alpha - \beta)h + (\beta - \gamma)f, \\
\kappa_2 &= \gamma,
\end{align*}
\]

To end this section, we revisit the Cartan prolongations from Examples 2.33 and 2.34 and demonstrate explicitly that, in each case, the systems involved are $D$-related.

Example 2.43 Recall the singular Cartan prolongation in Example 2.33:

\[ I = \left\langle dx_1 - u_1 dt, dx_2 - u_2 dt \right\rangle \]
\[ \subset \left\langle I, du_1 + f du_2 - gd, df - hdt, dg - (f + h)du_2 \right\rangle = J. \]

To establish a $D$-relation between $I$ and $J$, it suffices to follow the construction leading to Theorem 2.15. Indeed, we prolong $J$ by differentiation:

\[
\begin{align*}
J_{(1)} &= \left\langle J, du_2 - \alpha dt \right\rangle, \\
J_{(2)} &= \left\langle J_{(1)}, d\alpha - \beta dt \right\rangle, \\
J_{(3)} &= \left\langle J_{(2)}, d\beta - \gamma dt \right\rangle.
\end{align*}
\]

And construct the 3rd total prolongation of $I$:

\[
\text{pr}^{(3)} I = \left\langle I, du_1 - \lambda_1 dt, du_2 - \lambda_2 dt, \\
\quad d\lambda_1 - \mu_1 dt, \quad d\lambda_2 - \mu_2 dt, \\
\quad d\mu_1 - \kappa_1 dt, \quad d\mu_2 - \kappa_2 dt \right\rangle.
\]

It is easy to determine a local isomorphism between $J_{(3)}$ and $\text{pr}^{(3)} I$, restricted to a domain on which $\alpha \neq \beta$, as follows:

\[
\begin{align*}
\lambda_1 &= g - f\alpha, \\
\lambda_2 &= \alpha, \\
\mu_1 &= (\alpha - \beta)f, \\
\mu_2 &= \beta, \\
\kappa_1 &= (\alpha - \beta)h + (\beta - \gamma)f, \\
\kappa_2 &= \gamma,
\end{align*}
\]

Example 2.44 For convenience, let $n = 3$ in Example 2.34. We have

\[ I = \left\langle dx_i - u_i dt \right\rangle_{i=1}^3 \]
\[ \subset \left\langle I, du_1 - v_1 dt, du_2 - v_2 dt, du_3 - w dt, dv_1 - v_2 dt, dv_2 - v_3 dt \right\rangle = J. \]

Following the construction in Theorem 2.15, we successively obtain

\[
\begin{align*}
J_{(1)} &= \left\langle J, du_3 - \alpha dt \right\rangle, \\
J_{(2)} &= \left\langle J_{(1)}, d\alpha - \beta dt, dw - \gamma dt \right\rangle, \\
J_{(3)} &= \left\langle J_{(2)}, d(v_3 \alpha + \gamma) - \eta dt, \quad d\beta - \xi dt \right\rangle,
\end{align*}
\]

and

\[
\text{pr}^{(3)} I = \left\langle I, du_i - \lambda_i dt, d\lambda_i - \mu_i dt, d\mu_i - \kappa_i dt \right\rangle_{i=1}^3.
\]
A $\tau$-equivalence between $J_3$ and $pr^{(3)} I$ can be established by the equations:

\[
\begin{cases}
\lambda_1 = v_1, \\
\lambda_2 = v_2\alpha + w, \\
\lambda_3 = \alpha, \\
\mu_1 = v_2, \\
\mu_2 = v_3\alpha + \gamma + v_2\beta, \\
\mu_3 = \beta, \\
\kappa_1 = v_3, \\
\kappa_2 = \eta + v_3\beta + v_2\xi, \\
\kappa_3 = \xi.
\end{cases}
\]

We point out that the prolongation by differentiation that generates $J_3$ from $J_2$ is not obtained by using the obvious coordinates in which $J_2$ is written; instead, it is obtained by first making a change of coordinates that turns $v_3\alpha + \gamma$ into a single variable.

### 3 $\tau$-Dynamic linearization

Given a control system, it is interesting to know whether we can transform it in a certain way into a (time-varying) linear system. When this is possible, such a transformation is often called a linearization of the given system.

The following notions of linearization are familiar in the literature. (See also [4].)

a. An autonomous control system $\dot{x} = f(x, u)$ is called static feedback linearizable (SFL) if there exists an invertible change of coordinates

\[
\begin{cases}
y = \phi(x), \\
v = \psi(x, u),
\end{cases}
\]

that transforms the system into a linear system

\[
\dot{y} = Ay + Bv,
\]

where $A, B$ are constant matrices.

b. A time-varying control system $\dot{x} = f(t, x, u)$ is called extended static feedback linearizable (ESFL) if there exists an ($t$-dependent) invertible change of coordinates

\[
\begin{cases}
y = \phi(t, x), \\
v = \psi(t, x, u),
\end{cases}
\]

that transforms the system into a time-varying linear system

\[
\dot{y} = A(t)y + B(t)v.
\]

In the generic case, one can find a coordinate-independent criterion that works for both notions of linearizability above, as we now explain.

**Definition 3.1** We say that a CTS $(M, I; \tau)$ is strongly linear if

i. the terminal derived system $I^{(\infty)} = 0$;
ii. each $[I^{(k)}, \tau]$ is Frobenius.

**Remark 3.2** A. A corank-$p$ Pfaffian system $I$ corresponds to a distribution $\mathcal{D}$ on $M$. In the case of a CTS, Chow’s theorem [3] implies that controllability corresponds to the bracket-generating property of $\mathcal{D}$, which is equivalent to the condition $I^{(\infty)} = 0$.

B. Strong linearity is a property of a system $(M, I; \tau)$; in particular, it is sensitive to the independence condition $\tau$. Consider, for example, $(\mathbb{R}^4, I; d\alpha)$ with

$$I = \lbrack df - gd\alpha, dg - hd\alpha \rbrack$$

and $(\mathbb{R}^4, \bar{I}; dt)$ with

$$\bar{I} = \lbrack dx - \cos \theta dt, dy + \sin \theta dt \rbrack.$$

Via the diffeomorphism given by

$$\begin{align*}
t &= f + h, \\
x &= h \cos \alpha + g \sin \alpha, \\
y &= -h \sin \alpha + g \cos \alpha, \\
\theta &= \alpha,
\end{align*}$$

$\bar{I}$ and $I$ correspond to each other. Therefore, the first derived systems

$$I^{(1)} = \lbrack df - gd\alpha \rbrack, \quad \bar{I}^{(1)} = \lbrack \cos \theta dx - \sin \theta dy - dt \rbrack$$

must also correspond under the diffeomorphism. However, $\lbrack I^{(1)}, d\alpha \rbrack$ is integrable, while $\lbrack \bar{I}^{(1)}, dt \rbrack$ is not. In other words, $(I; d\alpha)$ is strongly linear, while $(\bar{I}; dt)$ is not.

**Theorem 3.3** [7, 10] A controllable autonomous system $\dot{x} = f(x, u)$ is static feedback linearizable if and only if the corresponding CTS is strongly linear.

**Theorem 3.4** (Cf. [4, Theorem 3.11]) Let $\dot{x} = f(t, x, u)$ be a controllable system with $n$ states and $m$ inputs. Let $(M, I, dt)$ denote the corresponding CTS. The following are equivalent:

i. $(M, I, dt)$ is strongly linear;

ii. there exists a transformation (6) that turns the system into a time-varying linear system

$$\dot{y} = A(t)y + B(t)v;$$

iii. there exists a transformation (6) that turns the system into a Brunovský normal form.

**Proof** It suffices to prove $\text{ii} \Rightarrow \text{i} \Rightarrow \text{iii}$.

$(\text{ii} \Rightarrow \text{i})$ Consider a time-varying CTS $(M, I; dt)$ where $I$ is spanned by the $n$ 1-forms

$$\theta^i = dx^i - \left( A^i_j(t)x^j + B^i_\alpha(t)u^\alpha \right) dt,$$

where $i, j = 1, \ldots, n$ and $\alpha, \beta = 1, \ldots, m$. Let $n_1 := n - m$. Without loss of generality, assume that the $m$-by-$m$ minor $\det(B^{n_1+\alpha}_\beta) \neq 0$. Thus, by a change of coordinates of the form $u^\alpha \mapsto Q^\alpha_\beta(t)u^\beta$, we can arrange that $B^{n_1+\alpha}_\beta = \delta^\alpha_\beta$. Using this, one easily finds that $I^{(1)}$ is spanned by

$$\eta^\rho = \theta^\rho - B^\rho_\alpha(t)\theta^{n_1+\alpha} \quad (\rho = 1, \ldots, n_1).$$

---

9 We adopt the convention of summing over repeated indices.
By introducing $\tilde{x}^{\rho} = x^\rho - B_\rho^\nu(t)x^{\nu+1+\alpha}$, we find that each $\eta^\rho$ is of the form

$$\eta^\rho = d\tilde{x}^\rho - [A_\rho^\sigma(t)\tilde{x}^\sigma + C_\rho^\nu(t)x^{\nu+1+\alpha}]dt$$

for some functions $C_\rho^\nu(t)$. Thus, $I^{(1)}$ is also a time-varying linear system, and $[I^{(1)}, dt]$ is Frobenius. This procedure continues. Controllability implies that $I^{(\infty)} = 0$. Therefore, $I$ is strongly linear.

(i⇒iii) Let $(M, I; \tau)$ be a strongly linear CTS with $I^{(K-1)} \neq 0$ and $I^{(K)} = 0$. Suppose that $\tau = dt$, and let $s_1 = rank(I^{(K-1)})$. By assumption, $[I^{(K-1)}, dt]$ is Frobenius. Since $dt$ does not belong to $I$, we have

$$I^{(K-1)} = \|dx_i^{K-1} - x_i^{K-2}dt\|_{s_1=1}$$

for some functions $x_0^{K-1}, \ldots, x_{s_1}^{K-1}, x_1^{K-2}, \ldots, x_{s_1}^{K-2}$; these $2s_1$ functions and $t$ have linearly independent differentials, because $I^{(K)} = 0$.

Now, suppose that for some $\ell \geq 0$ we have shown (as we just did for $\ell = 1$) that

$$I^{(K-\ell-1)} = \|dx_i^{K-\ell-1} - x_i^{K-\ell-2}dt\|_{s_1=1},$$

where $dx_i^{K-\ell-1}$ and $dx_i^{K-\ell-2}$ are independent of $C(I^{(K-\ell)})$ and among themselves. Since $I^{(K-\ell-1)}$ is the derived system of $I^{(K-\ell-2)}$, we have

$$d(dx_i^{K-\ell-1} - x_i^{K-\ell-2}dt) = dt \wedge dx_i^{K-\ell-2} \equiv 0 \mod I^{(K-\ell-2)}.$$

Combined with the assumption that $[I^{(K-\ell-2)}, dt]$ is Frobenius, we see that there exist new functions $x_1^{K-\ell-2}, \ldots, x_{s_\ell}^{K-\ell-2}$ and $x_{s_\ell+1}^{K-\ell-3}, \ldots, x_{s_{\ell+1}}^{K-\ell-3}$ such that

$$I^{(K-\ell-2)} = \|I^{(K-\ell-1)}, dx_i^{K-\ell-2} - x_i^{K-\ell-3}dt\|_{s_{\ell+1}=1}$$

where $s_{\ell+1} = rank(I^{(K-\ell-2)}) - rank(I^{(K-\ell-1)}) \geq s_\ell$. Because no combination of the $dx_i^{K-\ell-2} - x_i^{K-\ell-3}dt$ occurs in the derived system $I^{(K-\ell-1)}$, the differentials $dx_i^{K-\ell-2}$ and $dx_i^{K-\ell-3}$ $(i = 1, \ldots, s_{\ell+1})$ must be independent of $C(I^{(K-\ell-1)})$ and among themselves.

Repeat this procedure until $\ell = K - 1$ in (7), at which point one recognizes that $I$ is in a Brunovský normal form. \(\Box\)

**Definition 3.5** A CTS is called $\tau$-dynamically linearizable if it is $\tau$-absolutely equivalent to a strongly linear CTS.

The following theorem and corollary, slightly modified from their original statements in [10], reduce the problem of dynamic feedback linearization to finding a particular type of Cartan prolongation.

**Theorem 3.6** [10, p.87, Theorem 41] A CTS is strongly linear if and only if its total prolongation is strongly linear.

**Proof** Let $(M, I; \tau)$ be a CTS. It is easy to verify that

$$(\text{pr}^{(1)}I)^{(1)} = I.$$ 

The conclusion follows by the definition of strong linearity. \(\Box\)

**Corollary 3.7** A CTS is $\tau$-dynamically linearizable if and only if there exists a prolongation by successive differentiation that results in a strongly linear system.
Proof For (⇐), assume that $J$ is obtained from $I$ by successive prolongation by differentiation and that $J$ is strongly linear. This implies that $I$ and $J$ are absolutely equivalent, and by definition $I$ is strongly dynamical linearizable. For (⇒), suppose that $(M, I; \tau)$ and $(\tilde{M}, \tilde{I}; \tilde{\tau})$ are strongly linear, and let $(\tilde{M}, \tilde{I}; \tilde{\tau})$ be $\tau$-absolutely equivalent. This implies that $I$ and $J$ are absolutely equivalent and by definition $I$ is $\tau$-dynamically linearizable. For $\Rightarrow$, suppose that $(\tilde{M}, \tilde{I}; \tilde{\tau})$ is strongly linear. Theorem 2.42 implies that these two systems are also $D$-related, which yields $\pi$ and $\tilde{\pi}$ (prolongations by successive differentiation) in the diagram above. Now, for $\tilde{\pi}$, we apply Theorem 2.15, which yields the prolongations $\phi$ and $\pi_{L,0}$. All the arrows in the diagram represent prolongations by differentiation. Thus, the same is true for $\pi \circ \phi$. The system $pr(L)\tilde{I}$ is strongly linear by Theorem 3.6. This completes the proof. 

Lemma 3.8 A type $(n, 2)$ CTS $(M, I; \tau)$ is $\tau$-dynamically linearizable if and only if it admits a C-regular Cartan prolongation $(N, J; \sigma)$ that satisfies

i. $(N, J; \sigma)$ is strongly linear;

ii. each relative extension $I_k$ is a CTS.

Proof For (⇐), since $J$ is a Cartan prolongation of $I$, they are absolutely equivalent, and since $J$ is assumed to be strongly linear, $I$ is $\tau$-dynamically linearizable by definition. For (⇒), Corollary 3.7 and Theorem 2.38 together imply that there exists a Cartan prolongation terminating at a strongly linear system with each associated $I_{k+1}$ being a prolongation by differentiation of $I_k$. Such a Cartan prolongation is automatically $C$-regular and satisfies i and ii, since a prolongation by differentiation of a CTS results in a CTS.

Lemma 3.9 Let $(M, I; \tau)$ be a type $(n, 2)$ CTS. Suppose that $\pi : (N, J; \sigma) \to (M, I; \tau)$ is a C-regular Cartan prolongation (with extension length $K$) that satisfies:

i. $(N, J; \sigma)$ is strongly linear;

ii. each relative extension $I_k$ is a CTS;

iii. rank$(I_1/I) = 2$;

iv. rank$(I_{k+1}/I_k) = 1$ for all $1 \leq k \leq K - 1$.

Then $I \subset J^{(1)}$ represents a C-regular Cartan prolongation satisfying conditions i and ii.

Proof The assumptions imply that one can choose a coframing

$$(\theta^1, \ldots, \theta^n, \xi^1, \eta^1, \ldots, \eta^K, \omega^1, \omega^2, \tau)$$

on $N$ such that (after dropping pull-back symbols):

$I = \[[\theta^1, \ldots, \theta^n]]$, 

$I^{(1)} = \[[\theta^2, \ldots, \theta^{n-1}]]$, 

$I_k = \[[\theta^1, \ldots, \theta^n, \xi^1, \eta^1, \ldots, \eta^k]]$, $k = 1, \ldots, K$, 

$C(I_1) = \[[I_2, \omega^1, \tau]]$.

In particular, $I_K = J$, by the definition of C-regularity.
Next, it is not difficult to see that such a coframing can be chosen to further satisfy the structure equations:

\[
\begin{align*}
    d\theta^1 &\equiv \tau \wedge \xi^1 \mod I, \\
    d\theta^\alpha &\equiv 0 \mod I, \quad (\alpha = 2, \ldots, n - 1) \\
    d\theta^n &\equiv \tau \wedge \eta^1 \mod I, \\
    d\xi^1 &\equiv \tau \wedge \omega^1 \mod I_1, \\
    d\eta^k &\equiv \tau \wedge \eta^{k+1} \mod I_k, \quad (k = 1, \ldots, K - 1) \\
    d\eta^K &\equiv \tau \wedge \omega^2 \mod I_K.
\end{align*}
\]

(8)

Note, in particular, that each $I_k$ being a CTS enforces that, in the congruences above, the right-hand-sides are multiples of $\tau$.

Note that $J^{(1)} = [I, \eta^1, \ldots, \eta^{K-1}]$.

By the assumption i, $[J^{(1)}, \tau]$ is Frobenius. It follows that there exist functions $A^k$ on $N$ such that

\[d\eta^k \equiv \tau \wedge \eta^{k+1} + A^k \tau \wedge \xi^1 \mod I, \eta^1, \ldots, \eta^k\]

for $k = 1, \ldots, K - 1$. In fact, we can arrange all $A^k (k = 1, \ldots, K - 1)$ to be zero by adding an appropriate multiple of $\theta^1$ into each $\eta^k$.

It now follows from the congruences (8) that $I \subset J^{(1)} =: \bar{J}$ represents a $C$-regular Cartan prolongation of $I$ with the relative extensions

\[\bar{I}_k = [I, \eta^1, \ldots, \eta^k].\]

It is clear that this Cartan prolongation satisfies i and ii, and its extension length is $K - 1$. \□

**Example 3.10** Consider the type $(3, 2)$ CTS $(M, I; dt)$ generated by

\[
\begin{align*}
    \theta^1 &= dx_1 - (x_2 + uv)dt, \\
    \theta^2 &= dx_2 - (u + x_1 v)dt, \\
    \theta^3 &= dx_3 - vdt.
\end{align*}
\]

Let $J = [I, \theta^4, \ldots, \theta^7]$, where

\[
\begin{align*}
    \theta^4 &= du - u_1 dt, \\
    \theta^5 &= dv - v_1 dt, \\
    \theta^6 &= dv_1 - v_2 dt, \\
    \theta^7 &= dv_2 - v_3 dt.
\end{align*}
\]

It is observed in [10, Example 48] that the inclusion $I \subset J$ induces a Cartan prolongation, and $(J; dt)$ is strongly linear. It is easy to see that

\[I_1 = [I, \theta^4, \theta^5], \quad I_2 = [I_1, \theta^6], \quad I_3 = [I_2, \theta^7] = J.\]

Lemma 3.9 applies, indicating that $I \subset [I, \theta^5, \theta^6]$ also represents a $C$-regular Cartan prolongation with $(I, \theta^5, \theta^6; dt)$ being strongly linear. A direct verification of this fact will be left to the interested reader. We note that this linearization of $(M, I; \tau)$ is different.
from the one given by [10, Example 48]; the latter is obtained by differentiating $u$ instead of $v$.

The following theorem will serve as a basis for classifying $\tau$-dynamically linearizable type $(n, 2)$ systems.

**Theorem 3.11** A type $(n, 2)$ CTS $(M, I; \tau)$ is $\tau$-dynamically linearizable if and only if either it is already strongly linear, or it admits a $C$-regular Cartan prolongation $(N, J; \sigma)$ (with extension length $K$) satisfying:

i. $(N, J; \sigma)$ is strongly linear;
ii. each relative extension $I_k$ is a CTS;
iii. $\operatorname{rank}(I_{k+1}/I_k) = 1$, for all $k = 0, \ldots, K - 1$.

**Proof** By Lemma 3.8, it suffices to justify iii in $(\Rightarrow)$. Suppose that we already have a $C$-regular Cartan prolongation of $(M, I; \tau)$ satisfying i and ii. By Corollary 2.37, as $k$ increases from 0, the simple Cartan prolongations represented by $I_k \subseteq I_{k+1}$ are a number of (if any) successive total prolongations followed by Cartan prolongations of order 1. If there is any total prolongation in this list, we can apply Lemma 3.9 to find a $C$-regular Cartan prolongation of $(M, I; \tau)$ with a lower order and still satisfying i and ii. Continue until either $J = I$ or none of $I_k \subseteq I_{k+1}$ represents a total prolongation; the former case implies that $I$ is strongly linear, by Theorem 3.6 and Lemma 3.9, and the latter case implies iii. $\square$

**Definition 3.12** We say that a type $(n, 2)$ CTS $(M, I; \tau)$ has class $K$ if it is $\tau$-dynamically linearizable with $K$ being the minimal integer such that there exists a $C$-regular Cartan prolongation (with extension length $K$) of $(M, I; \tau)$ satisfying the conditions i-iii in Theorem 3.11. If $(M, I; \tau)$ is not $\tau$-dynamically linearizable, we say that it has class $\infty$.

**Theorem 3.13** a. Suppose that $(M, I; \tau)$ is $\tau$-dynamically linearizable. If $\pi : (N, J; \sigma) \to (M, I; \tau)$ is a $C$-regular Cartan prolongation (with extension length $K$) of $(M, I; \tau)$ satisfying the conditions i-iii in Theorem 3.11, then on $N$ there exists a local coframing $$(\theta^1, \ldots, \theta^n, \eta^1, \ldots, \eta^K, \omega^1, \omega^2, \sigma)$$ satisfying

\[
I = \begin{bmatrix} \theta^1, \ldots, \theta^n \end{bmatrix},
\]

\[
I_k = \begin{bmatrix} I, \eta^1, \ldots, \eta^k \end{bmatrix}, \quad k = 1, \ldots, K,
\]

and the structure equations:

\[
\begin{aligned}
d\theta^1 &\equiv \sigma \wedge \omega^1 \mod \theta^1, \ldots, \theta^n, \\
d\theta^\alpha &\equiv 0 \mod \theta^1, \ldots, \theta^n, \quad (\alpha = 2, \ldots, n - 1) \\
d\theta^n &\equiv \sigma \wedge \eta^1 \mod \theta^1, \ldots, \theta^n, \\
d\eta^k &\equiv \sigma \wedge \eta^{k+1} \mod \theta^1, \ldots, \theta^n, \eta^1, \ldots, \eta^k, \quad (k = 1, \ldots, K - 1) \\
d\eta^K &\equiv \sigma \wedge \omega^2 \mod \theta^1, \ldots, \theta^n, \eta^1, \ldots, \eta^K.
\end{aligned}
\]

b. Conversely, if $(N, J; \sigma)$ is a strongly linear system with a coframing (9) satisfying

\[
J = \begin{bmatrix} \theta^1, \ldots, \theta^n, \eta^1, \ldots, \eta^K \end{bmatrix}
\]

and the structure equations (12), then the class of the system $I := \begin{bmatrix} \theta^1, \ldots, \theta^n \end{bmatrix}$
(with independence condition induced by $\sigma$) is at most $K$.

**Proof** a. First, by the assumption, it is clear that there exists a coframing (9) satisfying (10), (11) and

$$C(I) = \llbracket I, \eta^1, \omega^1, \sigma \rrbracket.$$

Since $I$ is a CTS with corank 3, for each $i = 1, \ldots, n$, there must exist functions $A^i, B^i$ such that

$$d\theta^i \equiv \sigma \land (A^i \eta^1 + B^i \omega^1) \mod I.$$

Moreover, the $n \times 2$ matrix $(A^i | B^i)$ must have rank 2. It follows that one can make a linear transformation of the $\theta^i$ to arrange that

$$A^n = B^1 = 1$$

and all other $A^i, B^i$ are zero.

Continuing, by the construction of $I_2$ and the expression for $d\theta^1$, we have

$$C(I_1) \subseteq \llbracket I, \eta^2, \omega^1, \sigma \rrbracket.$$

This inclusion must be an equality because $I \subset I_1$ represents a simple Cartan prolongation, which preserves corank (Corollary 2.16). Since $I_1$, by assumption, is a CTS, it follows that

$$d\eta^1 \equiv \sigma \land (C^1 \eta^2 + D^1 \omega^1) \mod I_1$$

for some functions $C^1, D^1$. We can always arrange that $D^1 = 0$ by adding a multiple of $\theta^1$ to $\eta^1$. On the other hand, $C^1$ is nonvanishing; hence, by scaling $\eta^2$, we can arrange that $C^1 = 1$.

We can continue with this type of argument and obtain the congruences

$$d\eta^k \equiv \sigma \land \eta^{k+1} \mod I_k$$

for $k = 2, \ldots, K - 1$.

Now, since $C(I_K) = T^*N$ and $I_K = J$ is a CTS, it follows that, modulo $J$, $d\eta^K$ is congruent to a linear combination of $\sigma \land \omega^i (i = 1, 2)$. We can add a multiple of $\theta^1$ to $\eta^K$ and then scale $\omega^2$ to arrange that

$$d\eta^K \equiv \sigma \land \omega^2 \mod J.$$

Thus we have obtained a desired coframing on $N$.

b. Assuming a coframing (9) on $N$ satisfying (12), and letting

$$I = \llbracket \theta^1, \ldots, \theta^n \rrbracket,$$

it is easy to see that

$$C(I) = \llbracket I, \eta^1, \omega^1, \sigma \rrbracket.$$

Furthermore, $\llbracket I, \sigma \rrbracket$ is clearly Frobenius. It follows that the system $I$ is a corank 3 CTS.

It is easy to see that $I \subset J$ represents a $C$-regular Cartan prolongation with

$$I_k = \llbracket I, \eta^1, \ldots, \eta^k \rrbracket, \quad k = 1, \ldots, K.$$

By the assumption that $(N, J; \sigma)$ is strongly linear and the structure equations, the conditions i-iii in Theorem 3.11 are satisfied.

This completes the proof. $\square$
Theorem 3.13 enables us to take the following approach towards finding type \((n, 2)\) CTS that are \(\tau\)-dynamically linearizable: For each \(n, K\), first find local coframings \((9)\) adapted to some strongly linear system \((N, J; \sigma)\) such that the structure equations \((12)\) hold; then we classify the CTS generated by \(\theta^1, \ldots, \theta^n\) with independence conditions induced by \(\sigma\).

A concrete discussion of the classification problem, particularly in the case when \(n = 3\), will be addressed in a sequel to the current paper.

Acknowledgements We would like to thank Prof. Robert L. Bryant for generously offering his time for discussion. This work also benefited from discussions with Prof. George R. Wilkens and Taylor J. Klotz. Additionally, we would like to thank the referee for their comments and questions, which have led to substantial improvements in our exposition.

References

1. Brunovský, P.: A classification of linear controllable systems. Kybernetika 6(3), 173–188 (1970)
2. Cartan, É.: Sur l’équivalence absolue de certains systemes d’équations différentielles et sur certaines familles de courbes. Bull. Soc. Math. France 42, 12–48 (1914)
3. Chow, W.-L.: Über systeme von linearen partiellen differentialgleichungen erster ordnung. Math. Ann. 117, 98–105 (1939)
4. De Doná, J., Tehseen, N., Vassiliou, P.J.: Symmetry reduction, contact geometry, and partial feedback linearization. SIAM J. Control Optim. 56(1), 201–230 (2018)
5. Gardner, R.B.: Invariants of Pfaffian systems. Trans. Am. Math. Soc. 126, 514–533 (1967)
6. Gardner, R.B., Shadwick, W.F.: Feedback equivalence for general control systems. Syst. Control Lett. 15(1), 15–23 (1990)
7. Gardner, R.B., Shadwick, W.F.: The GS algorithm for exact linearization to Brunovsky normal form. IEEE Trans. Autom. Control 37(2), 224–230 (1992)
8. Hilbert, D.: Über den begriff der klasse von differentialgleichungen. Math. Ann. 73, 95–108 (1912)
9. Shadwick, W.F.: Absolute equivalence and dynamic feedback linearization. Syst. Control Lett. 15(1), 35–39 (1990)
10. Sluis, W.M.: Absolute equivalence and its applications to control theory. Ph.D. Thesis (1994)
11. van Nieuwstadt, M., Rathinam, M., Murray, R.M.: Differential flatness and absolute equivalence of nonlinear control systems. SIAM J. Control Optim. 36(4), 1225–1239 (1998)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.