Uniform non-convex optimisation via Extremum Seeking

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Abstract—The paper deals with a well-known extremum seeking scheme by proving uniformity properties with respect to the amplitudes of the dither signal and of the cost function. Those properties are then used to show that the scheme guarantees the global minimiser to be semi-global practically stable despite the presence of local minima. Under the assumption of a globally Lipschitz cost function, it is shown that the scheme, improved through a high-pass filter, makes the global minimiser practically stable with a global domain of attraction.

Keywords—Extremum Seeking, Fourier Series, Non-Convex Optimisation.

I. INTRODUCTION

The early research on the Extremum Seeking (ES) dates back to the 1920s [1] and since then this strategy has been extensively exploited to solve several optimisation problems in electronics [2], mechatronics [3], mechanics [4], aerodynamics [5], thermohydraulics [6], and thermoacoustic [7]. Some of the most popular ES schemes are those proposed in [8], [9], which represent the subject of the proposed analysis, although a remarkable variety of schemes were proposed, such as the adoption of integral action in [10], the use of a cost function’s parameter estimator [11], [12], the introduction of an observer [13], the extension to fractional derivatives in [3], the use of a predictor to compensate output delays in [14], the implementation of a Newton-based algorithm avoiding the Hessian matrix inversion [15], [16], and the concurrent use of a simplex-method to find the global minimiser [17].

All the methods, in a way or another, share the common philosophy of perturbing the system subject to optimisation using the so-called dither signal, which is periodic in most of the proposed solutions, to unveil in which direction the associated cost function decreases (in the case of a minimisation problem). The analysis tool that is typically adopted to prove practical convergence to the minimiser exploits the averaging theory [18]. It is shown that the average system asymptotically converges to the minimiser and that the trajectories of the original system remain closed to the average ones if a design parameter, namely γ, is kept sufficiently small [19], [20], [9], [21], [22].

A further fundamental parameter of most of the ES schemes is represented by the dither amplitude, named δ from now on. Roughly, it represents the local range in which the cost function is evaluated to estimate the local cost variation and so the decreasing direction. Differently from γ, δ is subject to a design compromise [23]. From one hand, the smaller δ the more the estimation of the local cost variation approximates the local cost gradient [20], [24]. Moreover, assuming the current optimisation variable sufficiently close to the actual minimiser, the smaller δ the better is the cost optimisation. On the other hand, as far as local minima are conceived, small δ could trap the scheme at a local minimiser due to a local inversion of the cost gradient. So, to overcome this issue, the dither amplitude should be sufficiently large to make the estimation of the cost variation robust in respect of local cost fluctuations [25]. For this reason, some authors suggested an adaptation policy for the dither amplitude which shrinks δ over time so that at the beginning a large dither lets the ES overcome local minima while asymptotically a small dither guarantees good optimisation performances [26], [27].

For sake of completeness, it is worth mentioning the results in [28], [29] in which the value of the cost function is directly used to perturb the phase of the dither signal rather than to estimate the local gradient. Lie-derivative arguments, instead of averaging techniques, are then adopted in the analysis.

Within the previous research context, this paper proposes three contributions.

First, We propose to study the average systems presented in [8] via Fourier series arguments. This alternative approach allows handling a class of non-convex cost functions with a unique global minimiser. We show that the average system trajectories converge to a neighbourhood of the minimiser. The size of this neighbourhood is proportional to δ, which is not required to be small. As for the second contribution, it is shown that the addition of a high-pass filter makes γ independent (uniform) of the value of the cost function.

We prove that semi-global and practical convergence to the minimiser is achieved with the parameter γ only depending on the Lipschitz constant of the cost function in the domain of interest and not on its value. This allows for tuning of the algorithm that preserves good convergence speed even in presence of a large domain of attractions. Third and final, we show that if the cost function is globally Lipschitz and under certain regularity conditions on the average system, the investigated ES scheme makes the global minimiser practically stable with a global domain of attraction.

The rest of this paper is organised as follows. Section II provides the formulation of the problem and reviews the basic ES scheme proposed by [8]. In this section, we show that the ES scheme applicability can be extended to non strictly convex cost functions via averaging analyses, which are not based on Taylor’s arguments. Section III describes the ES scheme improved with the high-pass filter and states
II. PROBLEM FORMULATION

The ES problem consists of the optimisation of an unknown cost function \( h : \mathbb{R} \to \mathbb{R} \) satisfying the following two assumptions.

**Assumption 1** The function \( h \) is smooth and there exists a \( x^* \in \mathbb{R} \) such that
\[
h(x) - h(x^*) > 0 \quad \forall x \in \mathbb{R} : x \neq x^*.
\]

**Assumption 2** There exist a locally Lipschitz and strictly quasi-convex function \( m : \mathbb{R} \to \mathbb{R} \), a class-\( K_{\infty} \) function \( \alpha(\cdot) \), and a \( A \geq 0 \) such that
1. \( m(x) - A \leq h(x) \leq m(x) + A \) for all \( x \in \mathbb{R} \)
2. for all \( x_1, x_2 \in \mathbb{R} : (x_1 - x^*)(x_2 - x^*) \geq 0 \)
\[
|m(x_2) - m(x_1)| \geq \alpha(|x_2 - x_1|)
\]

**Remark 1** As for Assumption 2, we observe that the function \( h(\cdot) \) might have local minima whose “depth” is bounded by the number \( A \). When \( A = 0 \) (which implies \( m(\cdot) = h(\cdot) \)) Assumption 2 asks for a (not necessarily strict) monotone behaviour of \( h \), where the latter could have isolated saddle points. In other words, \( m(\cdot) \) belongs to the class of the so-called strictly quasi-convex functions [30]. Assumption 2 is weaker than the common assumption \( (\partial h(x)/\partial x) x > 0 \) for any \( x \neq x^* \) typically present in literature, (see, among the others, [8], Assumptions 3 and 4) ruling out the existence of local minima or even saddle points.

The problem of semi-global extremum seeking can be formulated in the following way. For any \( \epsilon > 0 \) and \( r_0 > 0 \), design a system of the form
\[
\dot{x} = \varphi_{\epsilon, r_0}(x, h(x), t) \quad x(0) = x_0,
\]
so that for all \( x_0 \) satisfying \( |x_0 - x^*| \leq r_0 \) the resulting trajectories \( x(t) \) are bounded and satisfy \( \lim_{t \to \infty} \sup |x(t) - x^*| \leq \epsilon \).

If the convergence property is sought \( \forall x_0 \in \mathbb{R} \) with \( \varphi \) independent of \( \epsilon \) and \( x_0 \), then the problem is referred to as global extremum seeking.

Among the different ES schemes proposed in the literature to solve the previous problem, a common one is given by (see [8])
\[
\dot{x} = -\gamma y_s(x, t) u(t) \quad x(0) = x_0 \quad (1)
\]
in which \( y_s(x, t) := h(x + \delta u(t)) \), \( u(t) := \sin(2\pi t) \) is the dither signal and \( \gamma, \delta > 0 \) are tunable parameters. The block diagram of this algorithm is represented in Figure 3. In the next part, we briefly comment on the main properties, as available in the literature, of this algorithm and strengthen them.

Since the right-hand side of (1) is 1-periodic, the average system linked to (1) is given by ([31], §10.4)
\[
\dot{x}_a = -\gamma \int_0^1 y_s(x_a, t) u(t) dt.
\]

In the remaining part of the section, we present a different route for the analysis of (2) based on a Fourier series expansion rather than on the Taylor one used in the available literature. This analysis allows one to claim stronger results on (1).

The smoothness of \( h(\cdot) \) in Assumption 1 is essentially asked to guarantee the existence of the Fourier series of the function \( y_s \)\(^2\) (plus other regularity properties used in the proof).

\(^1\)The general case of a dither takes the form \( \tau \mapsto \sin(\omega \tau) \) with \( \omega > 0 \), \( \tau \in \mathbb{R} \) (as considered in [8]) can be always obtained by rescaling the time as \( t = \tau 2\pi/\omega \).

\(^2\)Milder regularity properties guaranteeing the existence of the series could be assumed.
of the forthcoming Lemma 1 and Proposition 1). Then, since
\[ y_\delta(x_a, t) \] and its time derivatives are continuous and periodic,
y_\delta(x_a, t) can be expressed in terms of its Fourier series as
\[ y_\delta(x_a, t) = \frac{a_0,\delta(x_a)}{2} + \sum_{k=1}^{\infty} a_{k,\delta}(x_a) \cos(k2\pi t) + b_{k,\delta}(x_a) \sin(k2\pi t) \] (3)

where
\[ a_{k,\delta}(x_a) := 2\int_0^1 y_\delta(x_a, t) \cos(k2\pi t) \, dt \]
\[ b_{k,\delta}(x_a) := 2\int_0^1 y_\delta(x_a, t) \sin(k2\pi t) \, dt . \] (4)

Embedding (3) in (2) it is immediately seen that the average system linked to (1) reads as
\[ \dot{x}_a = \frac{-\gamma}{2} b_{1,\delta}(x_a) . \] (5)

For this system the following result holds. In the result, we refer to the class-\( K_\infty \) function \( \delta^*(\cdot) \) defined as
\[ \delta^*(s) := 2\int_0^{1/2} \alpha(s \sin(2\pi t)) \, dt . \] (6)

**Lemma 1** Let \( h(\cdot) \) be such that Assumptions 1-2 are satisfied. Then:

a) for all positive \( \delta \) and \( A \) such that \( \delta^*(\delta) \geq A + \bar{b} \) for some \( b > 0 \), there exists a compact set \( A_\delta \subseteq [x^* - \delta, x^* + \delta] \) that is globally asymptotically and locally exponentially stable for (5).

b) There exists a \( \delta^* > 0 \) such that, for all positive \( \delta \) and \( A \) such that \( \delta \leq \delta^* \) and \( \delta^*(\delta) \geq A + \bar{b} \) for some \( b > 0 \), there exists an equilibrium point \( x^*_{\delta} \in \text{int} A_\delta \) that is locally exponentially stable for system (5). If, in addition, the function \( h(s) := h(x^* + s) - h(x^*) \) is even, then \( x^*_{\delta} = x^* \).

The Lemma is proved in Appendix I.

Item a) of the previous lemma states that the trajectories of the average system reach a compact set \( A_\delta \) that is contained in a \( \delta \) neighbourhood of \( x^* \) for all \( \delta \) that are sufficiently large with respect to the “depth” \( A \) of the local minima in a global way. This, in particular, implies that there exists a class \( K\mathcal{L} \) function \( \beta(\cdot, \cdot) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \) such that
\[ |x_a(t)|_{A_\delta} \leq \beta(|x_a(0)|_{A_\delta}, t) \]
where \( | \cdot |_{A_\delta} \) denotes the distance to the set \( A_\delta \). In case \( A = 0 \), namely only saddle points can be present, then \( \delta \) is only required to be positive. Otherwise, \( \delta \) must be taken sufficiently large to not get stuck in local minima.

Item b) claims that the set \( A_\delta \) collapses to an equilibrium point if \( \delta \) is also taken sufficiently small, besides being, as before, sufficiently large according to \( A \), which could require Assumption 2 to hold with a sufficiently small \( A \). Moreover, the last point of item b) shows that \( x^* \) represents the equilibrium point only for cost functions that are locally symmetric around the optimum.

Standard averaging results can be then used to show that the same property is preserved also for the trajectories of the original system (1) for sufficiently small \( \gamma \) but in a semi-global and practical way. This is detailed in the next Proposition 1 where we refer to the class \( K_\infty \) function \( \chi(s) \) defined as
\[ \chi(s) := \beta^{-1}(s, 0) . \]

In the following analysis we denote by \( L_r > 0 \) and \( M_r > 0 \), respectively the local Lipschitz constant and the upper bound of the function \( h(\cdot) \) on a closed interval of length \( r \). In particular, regularity of \( h \) implies that

- for all \( r > 0 \) there exists \( L_r > 0 \) such that for all \( x_1, x_2 \in [x^* - r, x^* + r] \)
  \[ |h(x_1) - h(x_2)| \leq L_r|x_1 - x_2| . \] (7)
- for all \( r > 0 \), there exists \( M_r > 0 \) such that for all \( x \in [x^* - r, x^* + r] \)
  \[ |h(x)| \leq M_r . \] (8)

**Proposition 1** Let \( h(\cdot) \) be such that Assumptions 1-2 hold and let \( r, \delta, d \) be arbitrary positive numbers such that \( r - \delta - 2\bar{b} > 0 \) and \( \delta^*(\delta) \geq A + \bar{b} \) for some \( \bar{b} > 0 \). Let \( r_0 := \chi(r - \delta - 2\bar{b}) \).

There exist \( L(r_0, d) \) and \( \gamma^*(M_r, L_r, \delta, d) > 0 \) such that for any \( \gamma \in (0, \gamma^*) \), any \( x_0 \in \mathbb{R} : |x_0 - x^*| \leq r_0 \), the trajectories of (1) are bounded and
\[ |x(t)|_{A_\delta} \leq d \quad \forall t \geq \frac{\bar{t}}{\gamma} . \]

Appendix II details the proof of this Proposition.

An immediate consequence of the previous result is the next corollary showing that under Assumptions 1-2 system (1) solves the problem of semi-global extremum seeking formulated before provided that Assumption 2 is fulfilled with a sufficiently small \( A \).

**Corollary 1** Let \( h(\cdot) \) be such that Assumptions 1-2 are fulfilled and let \( r_0 \) and \( \varepsilon \) be arbitrary positive numbers. Then, there exist \( L(r_0, \varepsilon) > 0 \), \( \delta^*(\varepsilon) > 0 \) and \( \gamma^*(r_0, \delta^*, \varepsilon) > 0 \) such that for any \( \delta \in (0, \delta^*) \) and \( \delta^*(\delta) \geq A + \bar{b} \) for some \( \bar{b} > 0 \), any \( \gamma \in (0, \gamma^*) \) and any \( x_0 \in \mathbb{R} : |x_0 - x^*| \leq r_0 \), the trajectories of (1) are bounded and
\[ |x(t) - x^*| \leq \varepsilon \quad \forall t \geq \frac{\bar{t}}{\gamma} . \]

By going through the proof of Proposition 1, it is immediately seen that \( \gamma^* \) is inversely proportional to \( M_r \). As a consequence, the higher the cost function is within the set where \( x(t) \) ranges, the lower the value of \( \gamma \) and, in turn, the slower the convergence rate of \( x \) to the neighbourhood of the optimum. Section III presents an improvement of (1) overtaking this limitation and, in turn, paving the way for a global result.
A. Comments on Taylor expansion-based averaging analyses

The analysis of (2) is typically approached, see [8], by using a Taylor expansion of \( h(\cdot) \) to obtain a system of the form

\[
\dot{x}_a = -\gamma c_1 \delta \frac{\partial h}{\partial x} \bigg|_{x_a} \left( -\gamma \sum_{k=2}^{\infty} c_k \delta^{2k-1} \frac{\partial^{2k-1} h}{\partial x^{2k-1}} \right)_{x_a}
\]

(9)

where \( c_k \) are suitably defined positive coefficients. A key role in the study of this system is played by the first-order approximation

\[
\dot{x}_a = -\gamma c_1 \delta \frac{\partial h}{\partial x} \bigg|_{x_a} .
\]

(10)

In fact, if our Assumption 2 is strengthened by asking that, for any \( x \neq x^* \), it holds \( (\partial h(x)/\partial x)(x - x^*) > 0 \) (respectively \( > \alpha |x - x^*| \)) with \( \alpha \) a class-\( K \) function, see [8], Assumptions 3 and 4, then the Lyapunov arguments of [8], eq. (45) demonstrate \( x^* \) to be a stable (respectively globally asymptotically stable) equilibrium point for (10). These stability properties are transferred to (2) for sufficiently small \( \delta \). Then, averaging techniques [31] can be used to prove that, for sufficiently small \( \gamma \), the trajectories of (1) and (2) remain arbitrarily close. Namely, the semi-global extremum seeking problem is solved.

The fact that the asymptotic properties of the average system (2) are just ensured by the first-order term (10), and thus by the gradient of \( h \), implies that isolated local minima, or even saddle points, of \( h \), cannot be handled by that proof technique. This justifies why [[8], Assumption 3], which is stronger than our Assumption 2, is needed. Furthermore, we observe that the previous analysis requires that the dither amplitude is kept sufficiently small for the higher-order terms of the average dynamics to be negligible.

III. THE HIGH-PASS FILTER (HPF)-ES ALGORITHM

In [8] the authors proposed the next modification of (1)

\[
\begin{align}
\dot{x} &= -\gamma (y_{\delta}(x,t) - \bar{y}) \ u(t) \quad x(0) = x_0 \quad (11a) \\
\dot{\bar{y}} &= \gamma (y_{\delta}(x,t) - \bar{y}) \quad \bar{y}(0) = \bar{y}_0 \quad (11b)
\end{align}
\]

where \( u(t) \) is the dither signal defined before, and \( (x, \bar{y}) \in \mathbb{R} \times \mathbb{R} \). A block representation of (11) is depicted in Figure 3.

The intuition behind the previous scheme is to interpret \( y_{\delta}(x,t) - \bar{y} \) as the output of a high pass filter of \( y_{\delta} \). Moreover, the difference \( y_{\delta}(x,t) - \bar{y} \) represents a proxy of the local mean variation of \( h(x) \), directly proportional to the local mean Lipschitz constant. In the following, we show how this feature guarantees that the upper bound for the value of \( \gamma \) is not dependent on \( M_r \) (eq. (8)) but rather only on \( L_r \) (eq. (7)).

As for \( \gamma \), similarly to (1), it must be small to let \( x \) and \( \bar{y} \) be sufficiently slow to preserve the correlation between the oscillations of \( y_{\delta}(x,t) - \bar{y} \) and those of \( u(t) \). The average system of (11) is defined as

\[
\begin{align}
\dot{x}_a &= -\gamma \int_0^1 (y_{\delta}(x_a,\tau) - \bar{y}_a) \ u(\tau) \ d\tau \\
\dot{\bar{y}}_a &= -\gamma \bar{y}_a + \gamma \int_0^1 y_{\delta}(x_a,\tau) \ d\tau.
\end{align}
\]

(12a) \quad (12b)

By expanding \( y_{\delta}(x_a,t) \) in terms of the Fourier series as in the previous section, it turns out that

\[
\begin{align}
\int_0^1 y_{\delta}(x_a,\tau) \ d\tau &= \frac{a_{0,\delta}(x_a)}{2} \\
\int_0^1 (y_{\delta}(x_a,\tau) - \bar{y}_a) \ u(\tau) \ d\tau &= \frac{b_{1,\delta}(x_a)}{2}.
\end{align}
\]

(13a) \quad (13b)

where in the latter we exploited \( \int_0^1 \bar{y}_a u(\tau) d\tau = 0 \) and (4). Hence, the average system reads as

\[
\begin{align}
\dot{x}_a &= -\gamma \frac{b_{1,\delta}(x_a)}{2} \\
\dot{\bar{y}}_a &= -\gamma \bar{y}_a + \gamma \frac{a_{0,\delta}(x_a)}{2}
\end{align}
\]

(14a) \quad (14b)

which is a cascade where the first subsystem coincides with (5) and the second subsystem is linear and asymptotically stable. From this, the next result follows from Lemma 1.

**Lemma 2** Let \( h(\cdot) \) be such that Assumptions 1-2 are satisfied. Then:

a) for any positive \( \delta > 0 \) and \( A > 0 \) such that \( \delta^* A > A + b \) for some \( b > 0 \), there exist a compact set \( A_\delta \subseteq [x^* - \delta, x^* + \delta] \) and a continuous function \( \tau : \mathbb{R} \to \mathbb{R} \) such that the set

\[
\text{graph } \tau|_{A_\delta} = \{(x_a, y_a) \in A_\delta \times \mathbb{R} : y_a = \tau(x_a)\}
\]

is globally asymptotically and locally exponentially stable for (14).

b) There exists \( \delta^* > 0 \) such that, for any \( \delta \in (0, \delta^*) \) and for any \( A > 0 \) such that \( \delta^* A > A + b \) for some \( b > 0 \), there exists an equilibrium point \( (x_{a,\delta}, \bar{y}_{a,\delta}) \in \text{graph } \tau|_{A_\delta} \) that is globally asymptotically and locally exponentially stable for system (5). If, in addition, the function \( h(s) := h(x^* + s) - h(x^*) \) is even, then \( x_{a,\delta} = x^* \).

A sketch of the proof of this Lemma, with special regard to the definition of the function \( \tau(\cdot) \), is presented in Appendix III.

From this, the following result mimics the one of Proposition 1 with the remarkable difference that the upper bound \( \gamma^* \) of \( \gamma \) is uniform with respect to \( M_r \).
Theorem 1 Let $h(\cdot)$ be such that Assumptions 1-2 are fulfilled and let $r, \delta, d$ be arbitrary positive numbers such that $r - d - 2\delta > 0$ and $\delta^*(\delta) \geq A + b$ for some $b > 0$. Let $r_0 := \chi(r - d - 2\delta)$. There exist $\epsilon(\delta, r, d)$ and $\gamma^*(L_r, \delta, d) > 0$ such that for any $\gamma \in (0, \gamma^*)$ and any $(x_0, \bar{y}_0)$ fulfilling $|x_0 - x^*| \leq r_0$ and $|\bar{y}_0 - a_0, \delta(x_0)/2| \leq \gamma^*$, then the trajectories of (11) are bounded and

$$ ||(x(t), \bar{y}(t))||_{\text{graph } \gamma A} \leq d \quad \forall t \geq \frac{\bar{t}}{\gamma}. $$

This Theorem is proved in Appendix ???. The fact that $\gamma^*$ does not depend anymore on $M_r$ but only on $L_r$ suggests that in presence of a globally Lipschitz cost function the HPF extremum seeking scheme in (11) is global. This intuition is confirmed in the following theorem.

Theorem 2 Let $h(\cdot)$ be such that Assumptions 1-2 are fulfilled. Moreover, assume $h(\cdot)$ to be globally Lipschitz. Then, for any $\epsilon > 0$ there exist $\gamma^*(\epsilon) > 0$ and $\overline{\delta}(\epsilon)$ such that for any $\gamma \in (0, \gamma^*)$, for any $\delta$ and $A > 0$ fulfilling $\delta \leq \overline{\delta}(\epsilon)$ and $\delta^*(\delta) \geq A + \overline{\delta}$ for some $b > 0$, and for any $x_0 \in \mathbb{R}$ and $\bar{y}_0 \in \mathbb{R}$ : $|\bar{y}_0 - a_0, \delta(x_0)/2| \leq \gamma$, then the trajectories of (11) are bounded and

$$ \limsup_{t \to \infty} |x(t) - x^*| \leq \epsilon. $$

This theorem is proved in Appendix ???.

IV. NUMERICAL RESULTS

This section presents numerical results obtained adopting the following cost function $h(\cdot) : \mathbb{R} \to \mathbb{R}$

$$ h(x) = h_0 + A \sin(10x) $$

$$ + \begin{cases} 
(x - \pi)^2 - 1 & x < \pi \\
\cos(x - \pi) - 2 & x \in [\pi, 2\pi) \\
(x - 2\pi)^2 - 3 & x \geq 2\pi 
\end{cases} $$

where $h_0 \in \mathbb{R}$. This function, which verifies Assumptions 1 and 2, is depicted in Figure 4a for $A = 0$ and $h_0 = 10$, and in Figure 5a for $A = 1/4$ and $h_0 = 10$, on $[-\pi, 3\pi]$.

The tests are grouped into two categories, the first of which highlights the performance of (1) whereas the second deals with the behaviour of (11).

As for the performance of (1), the results are presented in agreement with an incremental complexity policy. First, for the case of a local strongly convex cost, we show that the Fourier-based averaging (5) predicts the asymptotic equilibrium of (1) more accurately than the first-order Taylor expansion-based averaging (10). Then, assuming a quasi-strongly convex cost, we test that the trajectory of (10) does not track the trajectory of (1) as good as done by (5). Moreover, we introduce a non-convex cost function and we show that increasing $\delta$ lets (1) to pass over local minima.

To simulate the case of a strictly convex function, $x_0 = -\pi$ is picked sufficiently close to the minimiser of the cost function in Figure 4a. Then, Figures 4b-4c show the convergence performance of the basic ES together with the average computed by (10) (yellow) and by (5) (red). These simulations confirm that, in agreement with Lemma 1 and Proposition 1, the optimisation variable $x$ converges to $A \delta \subset [x^* - \delta, x^* + \delta]$. In more detail, the asymmetry of $h(\cdot)$ in the neighbourhood of $x^*$ implies that the equilibrium point of (2) does not correspond to $x^*$, see yellow lines in Figures 4c and 4b.

To simulate a quasi-strictly convex function, $x_0 = -\pi$ is picked sufficiently large to make (1) passing through the saddle point of the cost function depicted in Figure 4a. In this scenario, Figure 4d shows that the trajectory of (10) (yellow) gets stacked in presence of local extrema whereas the trajectories of the proposed Fourier-based (5) (red) remain close to those of (1), as foreseen by Proposition 1.

The behaviour of (1) in the case of local minima is shown in Figure 5, for the cost function depicted in Figure 5a. A selection of a too-small $\delta$ could trap the ES (1) (blue) in the neighbourhood of local minima, see Figure 5b. Then, following Lemma 1, for sufficiently large $\delta$, the scheme (1) (blue) can pass through local minima, see Figures 5c and 5d. It is worth noting that, also in this case, the averaging
Based on the first-order Taylor expansion (10) (yellow) is less accurate than (5) (red) in describing the trajectories of (1) (blue). A comparison of Figures 5c and 5d confirms the correctness of the bounds provided in Lemma 1.

Before investigating the performance of (11), the following test is performed to show that the classic ES suffers of large values of $|h(x)|$. Indeed, as depicted in Figure 6 (blue lines in subplots (a) and (c)), while keeping $\gamma$ fixed, larger $M_r$ lead to more oscillatory behaviours. To mathematically support this result we observe that

$$h(x + \delta u) = h(x) + R(x, \delta u)$$

where $R(\cdot, \cdot)$ represents the remainder of the Taylor expansion around $x$ of $h(\cdot)$. Exploit the definition of the Lipschitz constant of $h(\cdot)$ and $|u(t)|_{\infty} \leq 1$ to bound the remainder from above as

$$|h(x + \delta u) - h(x)| = |R(x, \delta u)| \leq L_r \delta.$$  \hspace{1cm} (17)

Substitute (16) into (1)

$$\dot{x} = -\gamma h(x + \delta u) u = -\gamma (h(x) + R(x, \delta u)) u$$

and investigate the following support system

$$\dot{x}_1 = -\gamma h(x_1) u \quad x(0) = x_{10}$$

conceivable as approximation of (18) for $|h(x)| \gg L_r \delta$. Let $H(x) := \int h(x)^{-1} dx$ and solve (19) by parts as

$$x_1(t) = H^{-1}(H(x_{10}) - \gamma(\cos(t) - 1))$$

which is a pure oscillation whose amplitude is proportional to $|H(x_{10})|$, evident in subplot (c) of Figure 6.

The performance of the ES improved with the high-pass filter (11) is tested increasing the cost function. Figure 6 shows that the convergence rate of (11) is uniform with the amplitude of the cost. To conclude, the stability properties of the dynamics of the high-pass filter are tested in the simulations of Figure 7 for the same test conditions of Figure 6. These tests show that the average (14b) (red) accurately tracks the exact dynamics (11b) (blue).

Through the proposed simulations we confirm three results: a) the classic ES can deal with cost functions with local minima where, on the opposite, the average of the classic ES obtained through the Taylor expansion gets stacked (Figure 4d and Figure 5); b) the classic ES and its average via Taylor expansion do not converge to the same equilibrium point (Figure 4c); c) in case of large cost functions the classic ES behaves as an oscillator whose amplitude is proportional to the value of the cost function (Figure 6c). Vice versa, the adoption of the high-pass filter makes the basic ES convergence rate uniform with respect to the amplitude of the cost function (Figures 6b and 6d).
Fig. 7: The behaviour of the HPF-ES does not change for increasing $M_r$. In these simulations the value of $h_0$ is increased from $10$ to $10^3$ demonstrating that, keeping fixed $\gamma$, the average (14b) (red) accurately tracks the exact dynamics (11b) (blue). These simulations are performed with $\gamma = \delta = 0.1$.

A. Solar Panel Optimisation

As demonstrated in [32], the ES algorithm can be efficiently applied to optimise the performance of solar panels. In this section, we compare the performance of (1) and (11) when applied to the photovoltaic array model presented in [32], and briefly recalled hereafter.

Let $\zeta \in \mathbb{R}^2$ be the state of the photovoltaic array modelling the dynamics of the output voltage $x \in \mathbb{R}$, forced by the input voltage $z \in \mathbb{R}$. Then, the plant is modelled as a linear time-invariant system

$$\dot{\zeta} = A\zeta + Bz$$
$$x = C\zeta$$

(21)

with $A$ Hurwitz. Since $A$ is Hurwitz, there exists a linear map $L : \mathbb{R} \rightarrow \mathbb{R}$ such that the set $\Omega := \{x = Lz, z \in \mathbb{R}\}$ is globally exponentially stable. On the other hand, let $w \in \mathbb{R}$ be the solar panel output current, then there exists a nonlinear smooth map $q : \mathbb{R} \rightarrow \mathbb{R}$ such that $w = q(x)$. The cost function (to be maximised) is represented by the output $h := xw$ which, constrained on $\Omega$, reads as $y|\Omega = h(z) := Lz q(Lz)$. Figure 8 (top-left) graphically represents $h(z)$. Use (1) and (21), and let $(x_b(t), y_b(t))$ be provided by the solutions of

$$\dot{z}_b = -\gamma y_b u(t) \quad z(0) = z_0$$
$$\dot{\zeta}_b = A\zeta_b + Bz_b \quad \zeta_b(0) = \zeta_0$$
$$x_b = C\zeta_b$$
$$y_b = x_b q(x_b) + \nu(t)$$

where $\nu(t) \in \mathbb{R}$ represents the watt-meter noise. Moreover, use (11) and (21), and let $(x_h(t), y_h(t))$ be provided by the solutions of

$$\dot{z}_h = -\gamma (y_h - \bar{y}) u(t) \quad z(0) = z_0$$
$$\dot{\zeta}_h = A\zeta_h + Bz_h \quad \zeta_h(0) = \zeta_0$$
$$x_h = C\zeta_h$$
$$y_h = x_h q(x_h) + \nu(t)$$

Then, the performance indices of these two schemes are compared in Figure 8 in which the following parameters have been adopted: $\omega = 115.2 \text{ rad/s}$, $\gamma = 0.5 \text{ J}^{-1}$, $\delta = 0.05 \text{ V}$, and $||\nu(t)||_\infty = 1 \text{ W}$. As previously described, the convergence speed of the basic ES (1) is deeply influenced by the magnitude of the cost function whereas (11) is not. Indeed, as can be seen in the bottom-right plot of Figure 8, the average speed of the ES with the high-pass filter is nearly constant and motivated by the quasi-linearity of the map $h(x)$ for $x \in [0, 20] \text{ V}$. At the opposite, and accordingly to the study of the toy-example of Figure 6, as soon as the cost increases the zero-mean oscillations of (1) dominate the slope-related contents thus stopping the seeking machinery. The final result is that (11) performs much better than (1).

V. Conclusions

This paper deals with two well-known extremum seeking schemes to show that they work under less restrictive assumptions. Relying on averaging and Fourier-series arguments, it is demonstrated that these schemes can deal with non-convex cost functions making the global minimiser (assumed to be unique) semi-global practically stable. Moreover, it is shown that the presence of a high-pass filter elaborating the cost function makes the tuning of the parameters independent of the cost magnitude. This feature is then exploited to demonstrate that the global minimiser has a global domain of attraction if the cost function is globally Lipschitz and certain regularity conditions are verified by the
average system.

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By using point 2) of Assumption 2, with \( x_2 = x + \delta \sin(2\pi t) \) and \( x_1 = x \) in the first term, and with \( x_2 = x \) and \( x_1 = x - \delta \sin(2\pi t) \) in the second one, it holds that

\[
\int_0^1 [m(x + \delta \sin(2\pi t)) - m(x)] \sin(2\pi t) \, dt \geq \Delta^*(\delta),
\]

with \( \Delta^*(\delta) \) is defined in (6), for all \( x \) such that \( x \geq x^* + \delta \). Similarly, using the second part of Assumption 2 with the same choices of \( x_1 \) and \( x_2 \), we observe that

\[
\int_0^1 [m(x + \delta \sin(2\pi t)) - m(x)] \sin(2\pi t) \, dt \leq -\Delta^*(\delta)
\]

for all \( x \) such that \( x \leq x^* - \delta \). Overall, we thus obtain that

\[
\frac{b_1,\delta(x)}{2} \geq -A + \Delta^*(\delta) \quad \forall x \geq x^* + \delta
\]

\[
\frac{b_1,\delta(x)}{2} \leq A - \Delta^*(\delta) \quad \forall x \leq x^* - \delta,
\]

namely, using the constraint on \( A \) and \( \delta \) specified in the claim,

\[
\frac{b_1,\delta(x)}{2} \geq \beta \quad \forall x \geq x^* + \delta
\]

\[
\frac{b_1,\delta(x)}{2} \leq -\beta \quad \forall x \leq x^* - \delta.
\]

(22)

Standard arguments can be then used to claim that the trajectories of (5) are ultimately bounded and enter in finite time (dependent on \( \gamma \)) and it is locally asymptotically stable for the trajectories of (5) and with \( x^* \in \Delta \), such that \( \partial h(x)/\partial s \) implies that there exists an interval \( \Delta \), independent of \( \delta \) and with \( x^* \in \Delta \), such that \( \partial^2 h(x)/\partial s^2 \geq -\varepsilon/4 \) for all \( s \in \Delta \) from this, it follows that

\[
\frac{\partial b_1,\delta(x)}{\partial x} \geq \varepsilon + O(\delta^2)
\]

for all \( x \in \Delta \). The result claimed in item b then follows by taking \( \delta^* \) sufficiently small so that \( [x^* - \delta^*, x^* + \delta^*] \subseteq \Delta \) and \( \partial^2 h(x)/\partial s^2 \geq -\varepsilon/4 \). To prove the last point of item b) we evaluate

\[
\frac{b_1,\delta(x^*)}{2} = 2 \int_0^1 h(x^* + \delta \sin(2\pi t)) \sin(2\pi t) \, dt
\]

\[
= 2 \int_1^{1/2} h(x^* + \delta \sin(2\pi t)) \sin(2\pi t) \, dt
\]

\[
- 2 \int_1^{1/2} h(x^* - \delta \sin(2\pi t)) \sin(2\pi t) \, dt
\]

in which we have exploited \( \sin(2\pi t) = -\sin(2\pi t + \pi) \) for all \( t \in \mathbb{R} \). Then, assuming \( \delta \leq \delta^* \), and \( h(x^* - s) = h(x^* + s) \) for all \( s \in [0, \delta] \) it follows that

\[
\frac{b_1,\delta(x^*)}{2} = 2 \int_0^{1/2} h(x^* + \delta \sin(2\pi t)) \sin(2\pi t) \, dt
\]

\[
- 2 \int_0^{1/2} h(x^* + \delta \sin(2\pi t)) \sin(2\pi t) \, dt = 0.
\]

**APPENDIX II**

**PROOF OF PROPOSITION 1**

The proof of Proposition 1 follows by standard averaging results as C1) and C2) of Lemma 3 (or, for instance, Theorem 2.6.1 and Theorem 4.1.1 in [18]) and it is thus just sketched with a particular eye in showing the dependence of certain key quantities from \( M_r \).

In the first part of the proof we show that trajectories of (1) and (5) originating from \( x(0) = x_a(0) \) are arbitrarily closed for an arbitrary large finite timespan if \( \gamma \) is taken sufficiently small. By using the definition of \( \gamma_0 \), the fact that \( A \subseteq [x^* - \delta, x^* + \delta] \) and \( |x_a(t)|_{A^4} \leq \beta(|x_a(0)|_{A^4}, 0) \) for all \( t \geq 0 \), it turns out that

\[
|x_a(t) - x^*| \leq \beta(|x_a(0)|_{A^4}, 0) + \delta \leq \beta(|x_a(0)|_{A^4}, 0) + \delta \leq \beta(r, 0) = r - d - \delta
\]

for all \( t \geq 0 \). This implies that \( |x_a(t) + \delta u(t) - x^*| \leq r - d < r \) for all \( t \geq 0 \). Furthermore, for all \( x(t) \) such that \( |x(t) - x_a(t)| \leq d \) we have \( |x(t) + \delta u(t) - x^*| \leq r \). These facts will used later in conjunction with Assumption 2.
Let
\[ \epsilon(x_a, t) := \int_0^t h(x_a + \delta(u(\tau)))u(\tau) - \frac{b_{1, \delta}(x_a)}{2} d\tau \]  
and note that \( \epsilon(x_a, \cdot) \) is 1-periodic with \( \epsilon(x_a, n) = 0 \) for all \( n \in \mathbb{N} \). Pick \( N \in \mathbb{N} \) as the largest integer lower than \( t \) and decompose the domain \([0, t]\) as the union of \([n - 1, n)\) and \([N, t]\), with \( n = 1, \ldots, N \). Then
\[ \epsilon(x_a, t) = \int_N^t h(x_a + \delta(u(\tau)))u(\tau) - \frac{b_{1, \delta}(x_a)}{2} d\tau. \]  

Now, add and subtract \( h(x_a)u(\tau) \) into the integral (24), and note that
\[ \left| (h(x_a + \delta(u(\tau)))u(\tau) - \frac{b_{1, \delta}(x_a)}{2} \right| \leq \left| (h(x_a + \delta(u(\tau)))u(\tau) - h(x_a)u(\tau)) + \frac{b_{1, \delta}(x_a)}{2} \right| + |h(x_a)||u(\tau)|. \]

By Assumption 1, the first term in the previous relation can be bounded as
\[ \left| (h(x_a + \delta(u(\tau))) - h(x_a))u(\tau) \right| \leq L_r\delta. \]  
Similarly, by definition (4) and Assumption 1, the second term can be bounded as
\[ \frac{|b_{1, \delta}(x_a)|}{2} = \left| \int_0^1 h(x_a + \delta(u(t)))u(t) dt \right| = \int_0^1 |h(x_a + \delta(u(t))) - h(x_a))u(t) dt| \leq \int_0^1 |h(x_a + \delta(u(t)) - h(x_a))| dt \leq L_r\delta. \]  
As for the third term, we have \( |h(x_a)||u(\cdot)| \leq M_r \). Overall, \( |\epsilon(x_a, t)| \) can be bounded as
\[ |\epsilon(x_a, t)| \leq \epsilon(L_r, M_r, \delta) := 2L_r\delta + M_r. \]  

Define \( z(t) := x_a(t) - \gamma \epsilon(x_a(t), t) \), and note that, using (28),
\[ |x(t) - x_a(t)| = |x(t) - z(t) - x_a(t) + z(t)| \leq |x(t) - z(t)| + \gamma \epsilon(L_r, M_r, \delta). \]  

We consider now the relation
\[ x(t) - z(t) = x(0) - z(0) + \int_0^t \dot{x}(\tau) - \dot{z}(\tau) d\tau \]
and we observe that
\[ \dot{x}(t) - \dot{z}(t) = -\gamma \left( h(x(t) + \delta(u(t)))u(t) - \frac{b_{1, \delta}(x_a(t))}{2} \right) - \gamma \left( \frac{\partial \epsilon(x, t)}{\partial x} \right)_{x=x_a(t)} \frac{b_{1, \delta}(x_a(t))}{2} \]
\[ + \gamma \left( \frac{\partial \epsilon(x, t)}{\partial u} \right)_{x=x_a(t)}. \]

By adding and subtracting the term
\[ \gamma \left( h(z(t) + \delta(u(t))) + h(x_a(t) + \delta(u(t))) \right) u(t) \]
to (30) and exploiting (23), we obtain
\[ \dot{x}(t) - \dot{z}(t) = -\gamma(h(x(t) + \delta(u(t))) - h(x_a(t) + \delta(u(t))))u(t) \]
\[ - \gamma(h(z(t) + \delta(u(t))) - h(x_a(t) + \delta(u(t))))u(t) \]
\[ - \gamma \left( \frac{\partial \epsilon(x, t)}{\partial x} \right)_{x=x_a(t)} \frac{b_{1, \delta}(x_a(t))}{2}. \]

By using Assumption 1 we observe that for all \( x(t) \) satisfying \( |x(t) - x_a(t)| \leq d \) and for all \( z(t) \) satisfying \( |z(t) + \delta(u(t)) - x^*| \leq r \), we have
\[ |h(x(t) + \delta(u(t))) - h(x_a(t) + \delta(u(t)))| \leq L_r|x(t) - z(t)| \]
\[ |h(z(t) + \delta(u(t))) - h(x_a(t) + \delta(u(t)))| \leq \gamma L_r|\epsilon(x_a(t), t)|. \]

Relation (33) is fulfilled if \( |z(t) - x_a(t)| \leq d \), which, by recalling the definition of \( z \) and the bound (28), is true if \( \gamma \leq \gamma_1^* \)
\[ \gamma_1^* := \frac{\bar{c}}{\epsilon(L_r, M_r, \delta)}. \]

in which \( \bar{c} \) is any positive number with \( \bar{c} < d \). Now use (4) into (24) and use \( |u(\cdot)| \leq 1 \) to obtain
\[ \left| \frac{\partial \epsilon(x, t)}{\partial x} \right|_{x=x_a(t)} \leq \int_N^t \left| \frac{\partial h(x)}{\partial x} \right|_{x=x_a(t)+\delta u(\tau)} u(\tau) dt \]
\[ + \frac{1}{2} \int_N^t \left| \partial b_{1, \delta}(x) \right|_{x=x_a(t)} u(\tau) dt \]
\[ + \int_N^t \int_0^1 \left| \frac{\partial h(x)}{\partial x} \right|_{x=x_a(t)+\delta u(\tau)} ds d\tau \]
\[ \leq 2L_r. \]

Overall, using (27), (28), (34) and (36), it turns out that
\[ |x(t) - z(t)| \leq |x(0) - z(0)| + \int_0^t |\dot{x}(\tau) - \dot{z}(\tau)| d\tau \]
\[ \leq |x(0) - z(0)| + \gamma L_r \int_0^t |x(\tau) - z(\tau)| d\tau \]
\[ + \gamma^2 \bar{k}(L_r, M_r, \delta)d. \]

where \( \bar{k}(L_r, M_r, \delta) := L_r \epsilon(L_r, M_r, \delta) + 2L_r^2 \delta \), namely, by the Gronwall Lemma [18], §1.3] to obtain
\[ |x(t) - z(t)| \leq |x(0) - z(0)| e^{\gamma L_r t} \]
\[ + (e^{\gamma L_r t} - 1) \frac{\bar{k}(L_r, M_r, \delta)}{L_r}. \]

Now let \( \bar{t} > 0 \) and \( \bar{c} \in (0, d) \) be arbitrary numbers, and let \( \gamma^* := \min\{\gamma_1^*, \gamma_2^*, \gamma_3^*\} \) in which
\[ \gamma_1^* := \frac{\bar{c}}{3e^{L_r \bar{t}}} \]
\[ \gamma_2^* := \frac{\bar{c} L_r}{3k(L_r, M_r, \delta)(e^{L_r \bar{t}} - 1)} \]
and $\gamma_4^* := \gamma_1^*/3$. Using (29) and (37), simple computations show that for any $\gamma \in (0, \gamma^*)$
\[ |x(t) - x_\bar{a}(t)| \leq \bar{c} \quad \forall t \in [0, t/\gamma]. \]

We use now the local exponential stability of the attractor $\mathcal{A}_\bar{a}$ to extend the previous bound for any $t \geq t/\gamma$. The assumption in question implies the existence of $\tilde{c}_0 > 0$, $\kappa > 0$, $\lambda > 0$ such that the trajectories of (5) starting at $x_\bar{a}(0) |_{\mathcal{A}_\bar{a}} \leq \tilde{c}_0$ are bounded by
\[ |x_\bar{a}(t)|_{\mathcal{A}_\bar{a}} \leq \kappa |x_\bar{a}(0)|_{\mathcal{A}_\bar{a}} e^{-\gamma \lambda t}. \]

Without loss of generality let $\tilde{c}_0 < d/2\kappa$ and fix $\bar{c} \in (0, \tilde{c}_0)$. Furthermore, let $\bar{t}_1, \bar{t}_2 > 0$ be such that
\[ \beta(t, \bar{t}_1) < \tilde{c}_0 - \bar{c} \quad \bar{t}_2 > \frac{1}{\lambda} \ln \left( \frac{\kappa \tilde{c}_0}{\tilde{c}_0 - \bar{c}} \right), \]

fix $\bar{t} = \max\{\bar{t}_1, \bar{t}_2\}$ and fix once for all $\gamma \in (0, \min\{\gamma^*, 1\})$, with $\gamma^* = (\bar{t}, \bar{c}, L_r, M_r, \delta)$ defined before. Now divide the time axis into sub-intervals of the form
\[ I_n := \left[ \frac{n}{\gamma}, (n + 1)\frac{\bar{t}}{\gamma} \right) \quad n \in \mathbb{N} \]

and, with $x_\bar{a}(t, x_{a0})$ the trajectory of the average system at time $t$ with initial condition $x_{a0}$ at time $t = 0$, let
\[ x_u(t) := x_\bar{a}(t - n\bar{t}/\gamma, x(n\bar{t}/\gamma)) \quad \forall t \in I_n. \]

Because of the first part of the proof and by the definition of $\bar{c}$, it turns out that $|x(\bar{t}/\gamma)|_{\mathcal{A}_\bar{a}} < \tilde{c}_0$ and thus $|x_\bar{a}(t)|_{\mathcal{A}_\bar{a}} < \kappa \bar{c}_0 e^{-\gamma \lambda (t-\bar{t})/\gamma}$ for all $t \in I_1$. The same arguments used in the first part of the proof show that
\[ |x(t) - x_1(t)| \leq \bar{c} \quad \forall t \in I_1. \]

Moreover,
\[ |x(t)|_{\mathcal{A}_\bar{a}} \leq |x_1(t)|_{\mathcal{A}_\bar{a}} + |x(t) - x_1(t)| \leq \kappa \bar{c}_0 e^{-\gamma \lambda (t-\bar{t})/\gamma} + \bar{c} \leq \kappa \bar{c}_0 + \bar{c} \leq d \quad \forall t \in I_1 \]
and
\[ |x(2\bar{t}/\gamma)|_{\mathcal{A}_\bar{a}} \leq \kappa \bar{c}_0 e^{-\gamma \lambda t/\gamma} + \bar{c} \leq \kappa \bar{c}_0 e^{-\gamma \lambda t/\gamma} + \bar{c} \leq \bar{c}_0. \]

The last steps can be then iterated for all $n = 2, 3, \ldots$ and this proves the Proposition.

**APPENDIX III**

**PROOF OF LEMMA 2**

This proof follows from standard cascade arguments in view of the result in Lemma 1 and from the fact that $\gamma$ is a positive parameter. We only present some detail behind the construction of the (non-unique) function $\tau(\cdot)$. By (22) in Lemma 1 we have that the set $\tilde{\Delta} := [x_\star - 2\bar{\delta}, x_\star + 2\bar{\delta}]$ is forward invariant and reached in a finite time $t^*(\gamma, x_{a0})$ for the trajectories $x_\bar{a}(t, x_{a0})$ of (14a) with initial condition $x_{a0}$. Let $\tilde{\Delta}$ a superset of $\Delta$ and let $b_{1, \delta} : \mathbb{R} \to \mathbb{R}$ a smooth function defined as
\[ b_{1, \delta}(x) = \begin{cases} b_{1, \delta}(x) & x \in \tilde{\Delta} \\ 0 & x \not\in \tilde{\Delta} \end{cases}. \]

Denote with $\hat{x}_a(t, x_{a0})$ the flow of $\hat{x}_a = -\gamma b_{1, \delta}(\hat{x}_a)/2$ at time $t$ with initial condition $x_{a0}$ and note that for all $x_{a0} \in \tilde{\Delta}$ we have $\hat{x}_a(t, x_{a0}) = x_\bar{a}(t, x_{a0})$. Define the map $\tau : \mathbb{R} \to \mathbb{R}$ with
\[ \tau(x) = \gamma \int_0^t e^{\gamma t} a_{0, \delta}(\hat{x}_a(\tau, x)) d\tau. \]

Let $(x_a(t, x_{a0}), \bar{y}_a(t, \bar{y}_{a0}, x_{a0}))$ be the solution of (14) with initial conditions $(x_{a0}, \bar{y}_{a0})$ and note that for any $x_{a0} \in \mathcal{A}_{\bar{a}}$ the following holds
\[ \bar{y}_a(t, \tau(x_a(t), x_{a0})) = \tau(x_{a0}) e^{-\gamma t} \]
\[ + \frac{\gamma}{2} \int_0^t e^{-\gamma(t-\tau)} a_{0, \delta}(x_a(\tau, x_{a0})) d\tau \]
\[ = \frac{\gamma}{2} \int_0^t e^{-\gamma(t-\tau)} a_{0, \delta}(\hat{x}_a(\tau, x_{a0})) d\tau \]
\[ = \frac{\gamma}{2} \int_0^t e^{\gamma \tau} a_{0, \delta}(\hat{x}_a(\tau + t, x_{a0})) d\tau \]
\[ = \tau(\hat{x}_a(t, x_{a0})) \]
\[ = \tau(x_a(t, x_{a0})). \]

Since $\mathcal{A}_{\bar{a}}$ is forward invariant for (14a), this implies that $\text{graph}(\tau |_{\mathcal{A}_{\bar{a}}})$ is forward invariant for (14). Point a) then follows by using, for instance, the arguments in [36]. As for point b), adopt the same arguments to prove point b) of Lemma 1, use the cascade connection, and the forward invariance of $\text{graph}(\tau |_{\mathcal{A}_{\bar{a}}})$.

**APPENDIX IV**

**PROOF OF THEOREM 1**

First, the trajectories of (11) are demonstrated to remain close to those of (14) for a finite timespan. Second, exploiting the local exponential stability of graph $\tau |_{\mathcal{A}_{\bar{a}}}$, the trajectories of (11) are demonstrated to remain bounded and close to this set for an infinite time horizon.

Define $\eta = (x, \bar{y})$ and $\eta_a = (x_a, \bar{y}_a)$, and let
\[ F(\eta) := \begin{pmatrix} -\varepsilon_{\delta}(x, \bar{y}, t) u(t) \\ \varepsilon_{\delta}(x, \bar{y}, t) \end{pmatrix} \]
and
\[ F_a(\eta_a) := \begin{pmatrix} -b_{1, \delta}(x_{a0}) \\ -\bar{y}_{a0} + \frac{a_{0, \delta}(x_{a0})}{2} \end{pmatrix}. \]

Moreover, let
\[ \epsilon(\eta_a, t) := \int_0^t F(\eta_a, \tau) - F_a(\eta_a) d\tau \]
and note that $\epsilon(\eta_a, t)$ is 1-periodic and $\epsilon(\eta_a, t) = 0$ for each $n \in \mathbb{N}$ and for any $\eta_a \in \mathbb{R}^2$. Let $N \in \mathbb{N}$ be the largest
Let \( a \) be an integer such that \( N \leq t \), then
\[
\epsilon(\eta_a, t) = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} F(\eta_a, \tau) - F_a(\eta_a) \, d\tau \\
+ \int_{t_{N}}^{t} F(\eta_a, \tau) - F_a(\eta_a) \, d\tau \\
= \int_{t_{N}}^{t} F(\eta_a, \tau) - F_a(\eta_a) \, d\tau.
\]

Let \( \epsilon_1(\eta_a, t) \) and \( \epsilon_2(\eta_a, t) \) be the first and second entries of \( \epsilon(\eta_a, t) \). We now find a bound for these two quantities.

As a starting point, let \( \epsilon_{y_a} := \frac{a_0, \delta(x_a)}{2} - \tilde{y}_a \) and compute
\[
\dot{\epsilon}_{y_a} = -\gamma \epsilon_{y_a} - \frac{\gamma}{4} \frac{\partial a_0, \delta(x_a)}{\partial x_a} b_{1, \delta}(x_a) \epsilon_{y_a}(0) = 0 \tag{38}
\]
in which, without loss of generality, we set \( \tilde{y}_a 0 = \frac{a_0, \delta(x_a(0))}{2} \). Now, assume \( |x_a|_{A_3} < r \) and use the definition of the Lipschitz constant of \( h(\cdot) \) to bound
\[
\left| \frac{\partial a_0, \delta(x)}{\partial x} \right| \leq 2 \int_0^1 \left| \frac{\partial h(x)}{\partial x} \right|_{x=x_a+\delta u(t)} \, dt \leq 2L_r.
\]

Solve (38) and use (27) to obtain
\[
|\epsilon_{y_a}(t)| \leq \delta L_r^2 \quad \forall t \geq 0. \tag{39}
\]

As for
\[
\epsilon_1(\eta_a, t) = \int_{t_{N}}^{t} \epsilon_{\delta}(x_a, \tilde{y}_a, \tau) u(\tau) - b_{1, \delta}(x_a) / 2 \, d\tau \tag{40}
\]
we add and subtract \( h(x_a) u(t) \) and \( a_0, \delta(x_a) u(t) / 2 \) inside the integral and use the triangle inequality to bound
\[
|\epsilon_1(\eta_a, t)| \leq \\
\int_{t_{N}}^{t} |\epsilon_{\delta}(x_a, \tilde{y}_a, \tau) u(\tau) - b_{1, \delta}(x_a) / 2| \, d\tau \\
+ \int_{t_{N}}^{t} |h(x_a + \delta u(t)) - h(x_a)| u(t) \, d\tau \\
+ \int_{t_{N}}^{t} |a_0, \delta(x_a) / 2 - \tilde{y}_a| \, d\tau. \tag{41}
\]

Then, in order, for the terms appearing in (41) we use the definition of the Lipschitz constant of \( h(\cdot) \), (27), the Mean Value Theorem, and (39) to obtain
\[
|\epsilon_1(\eta_a, t)| \leq \delta L_r(3 + L_r). \tag{42}
\]

The benefit introduced by the presence of \( \tilde{y} \) is evident when (42) is compared to (28). Indeed, in (42) only the Lipschitz constant of \( h(\cdot) \) appears whereas (28) is bounded by the supremum of \( |h(\cdot)| \) which is radially unbounded. As for
\[
\epsilon_2(\eta_a, t) = \int_{t_{N}}^{t} h(x_a + \delta u(t)) - a_0, \delta(x_a) / 2 \, d\tau \tag{43}
\]
thanks to the Mean Value Theorem we note that there exists \( \bar{x}_1 \in [x_a - \delta, x_a + \delta] \) such that \( h(\bar{x}_1) = \frac{a_0, \delta(x_a)}{2} \). Then
\[
|\epsilon_2(\eta_a, t)| \leq \int_{t_{N}}^{t} |h(x_a + \delta u(t)) - h(\bar{x}_1)| \, d\tau \leq L_r 2\delta. \tag{44}
\]

Define \( z(t) = \eta_a(t) - \gamma \epsilon(\eta_a(t), t) \), exploit the triangle inequality, and use (42) and (44) to write
\[
\|\eta(t) - \eta_a(t)\| = \|\eta(t) - \eta_a(t) + z(t)\| \\
\leq \|\eta(t) - z(t)\| + \|z(t) - \eta_a(t)\| \\
\leq \|\eta(t) - z(t)\| + \gamma \|\epsilon(\eta_a(t), t)\| \\
\leq \|\eta(t) - z(t)\| + \gamma \bar{k}_3(L_r, \delta). \tag{45}
\]
with \( \bar{k}_3(L_r, \delta) := \frac{\delta L_r}{\sqrt{4 + (3 + L_r)^2}} \). Denote with \( z_1 \) and \( z_2 \) the first and second entry of \( z \). In the next steps we find a bound for \( x - z_1 \) and \( \bar{y} - z_2 \) which represent the first and second entry of \( \eta - z \). Let us compute
\[
x(t) - z_1(t) = x(0) - z_1(0) + \int_0^t \dot{x}(\tau) - \dot{x}_a(\tau) \, d\tau \\
- \gamma \int_0^t \frac{\partial \epsilon_1(\eta, t)}{\partial \eta} \dot{\eta}_a(\tau) \, d\tau \tag{46}
\]
- \gamma \int_0^t \frac{\partial \epsilon_1(\eta, t)}{\partial \tau} \epsilon_1(\eta_a, \tau) \, d\tau.

Then, we use (11a), (14a), and (40) to rewrite (46) as
\[
x(t) - z_1(t) = x(0) - z_1(0) \\
+ \gamma \int_0^t \epsilon_{\delta}(x(\tau), \bar{y}(\tau), \tau) u(\tau) - b_{1, \delta}(x(\tau)) / 2 \, d\tau \\
- \gamma \int_0^t \frac{\partial \epsilon_1(\eta, t)}{\partial \eta} \dot{\eta}(\tau) d\tau \tag{47}
\]
- \gamma \int_0^t \frac{\partial \epsilon_1(\eta, \tau)}{\partial \eta} \dot{\eta}(\tau) d\tau.

Add and subtract the term
\[
\epsilon_{\delta}(z_1(\tau), \bar{y}(\tau), \tau) u(\tau)
\]
inside the first integral of the right member of (47) and rearrange this latter as follows
\[
x(t) - z_1(t) = x(0) - z_1(0) \\
+ \gamma \int_0^t \epsilon_{\delta}(x(\tau), \bar{y}(\tau), \tau) u(\tau) - \epsilon_{\delta}(z_1(\tau), \bar{y}(\tau), \tau) u(\tau) \, d\tau \\
+ \gamma \int_0^t \epsilon_{\delta}(z_1(\tau), \bar{y}(\tau), \tau) u(\tau) - \epsilon_{\delta}(x(\tau), \bar{y}(\tau), \tau) u(\tau) \, d\tau \\
- \gamma \int_0^t \frac{\partial \epsilon_1(\eta, \tau)}{\partial \eta} \dot{\eta}_a(\tau) d\tau. \tag{48}
\]
In analogy with (33)-(35), let \( \tilde{c} \in (0, d) \) and define
\[
\gamma_0^* = \frac{1}{2L_r} \left( \sqrt{9 + 4\tilde{c}/\delta} - 3 \right). \tag{49}
\]
Then, in agreement with the definition of \( z_1, |z_1(t) - x_a(t)| \leq d \) for all \( \gamma \in (0, \gamma_0^*) \).

As consequence,
\[
|z_1(t) + \delta u(t) - x^*| \leq r \tag{50}
\]
and so for all x(t) satisfying |x(t) − x_a(t)| ≤ d we can exploit the Lipschitz constant of h(·) to derive

\[ |\varepsilon(x, y, t) - \varepsilon_\delta(x, y, t)| \leq L_r|x - z_1|. \] (51)

On the other hand, the use of the triangle inequality, the definition of z_2, and (44) lead to

\[ |\bar{y} - \bar{y}_a| = |\bar{y} - \bar{y}_a + z_2| \leq |\bar{y} - z_2| + |z_2 - \bar{y}_a| \leq |\bar{y} - z_2| + \gamma \varepsilon_\delta(x_a, t)| \leq |\bar{y} - z_2| + \gamma L_r 2\delta \]

which, with the use of the definitions of \( \varepsilon_\delta \) and z_1, jointly with (42), implies

\[ |\varepsilon(x, y, t) - \varepsilon_\delta(x_a, y_a, t)| \leq L_r \gamma |\varepsilon_\delta(x_a, t)| + |\bar{y}(t) - \bar{y}_a(t)| \leq \gamma \delta L_r (2 + L_r (3 + L_r) + |\bar{y} - z_2|). \] (52)

Moreover, through the triangle inequality we get

\[ |\partial_\varepsilon_1(\eta, t)| \leq \gamma |a_{0, \delta}(x_a)/2 - \bar{y}_a| + \gamma \varepsilon_\delta(x_a, t)/2 \]

Now, since \( \partial_\varepsilon_\delta(x, y, t)/\partial x = \partial_\varepsilon_\delta(x)/\partial x \) and (5) coincides with (14a), we can exploit (36) to bound

\[ |\partial_\varepsilon_\delta(x, y, t) - b_{1, \delta}(x)/2| \leq 2L_r. \] (54)

Then, substitute (54) into (53) and use (27) and (39) to bound

\[ |\partial_\varepsilon_1(\eta, t)| \leq \gamma \delta L_r 3 \]

and use (42), (51), (52) and (55) to bound (48) as

\[ |x(t) - z_1(t)| \leq |x(0) - z_1(0)| + t \gamma^2 \bar{k}_3(\bar{L}_r, \delta) + \gamma \int_0^t L_r|x(x)| - z_1(\tau)| + |\bar{y}(\tau) - z_2(\tau)|d\tau \]

where \( \bar{k}_3(\bar{L}_r, \delta) := \delta L_r (2 + L_r (3 + L_r) + 3L_r) \).

We now use the same conceptual steps (46)-(56) to find a bound for |\( \bar{y}(\tau) - z_2(\tau) |.

Compute

\[ \bar{y}(\tau) - z_2(\tau) = \bar{y}(0) - z_2(0) + \int_0^t \dot{\bar{y}}(\tau) - \dot{z}_2(\tau) \]

\[ = \bar{y}(0) - z_2(0) + \gamma \int_0^t h(\tau) + \delta u(\tau) - \frac{a_{0, \delta}(x_a(\tau))}{2}d\tau \]

\[ - \gamma \int_0^t \bar{y}(\tau) - \bar{y}_a(\tau) d\tau \]

\[ + \gamma \int_0^t \frac{\partial_\varepsilon_2(\eta, \tau)}{\eta = \eta_\alpha(\tau)} \dot{\eta}_\alpha(\tau) d\tau \]

\[ + \gamma \int_0^t \frac{\partial_\varepsilon_2(\eta, \tau)}{\eta = \eta_\alpha(\tau)} \dot{\eta}_\alpha(\tau) d\tau \]

Add and subtract to the first and the second integral \( h(x_a(\tau) + \delta u(\tau)) \) and \( z_2(\tau) \) respectively, use (43) and rearrange (57) as follows

\[ \bar{y}(t) - z_2(t) = \bar{y}(0) - z_2(0) \]

\[ + \gamma \int_0^t h(\tau) + \delta u(\tau) - h(x_a(\tau) + \delta u(\tau))d\tau \]

\[ - \gamma \int_0^t \bar{y}(\tau) - z_2(\tau) d\tau + \gamma^2 \int_0^t \varepsilon_\delta(y_a(\tau), \tau) d\tau \]

\[ + \gamma \int_0^t \frac{\partial_\varepsilon_2(\eta, \tau)}{\eta = \eta_\alpha(\tau)} \dot{\eta}_\alpha(\tau) d\tau. \]

Use the definition of the Lipschitz constant of h, the definition of \( z_1 \), the triangle inequality, (39), and (42) to bound

\[ |h(x + \delta u(\tau)) - h(x_a + \delta u(\tau))| \leq L_r|x - x_a| \]

\[ \leq L_r|x - z_1| + \gamma|\varepsilon_\delta(x_a, \tau)| \leq L_r|x - z_1| + \gamma \delta L_r (3 + L_r). \] (59)

On the other hand, we use (4) into (43), the Lipschitz constant of h(·), and (27) to write

\[ |\partial_\varepsilon_2(\eta, \tau)| \leq \gamma \left| \frac{\partial_\varepsilon_2(\eta, \tau)}{\eta = \eta_\alpha(\tau)} \right| \left| \frac{|b_{1, \delta}(x_a)|}{2} \right| \leq \gamma 2L_r^2 \delta^2. \] (60)

Use (44), (59), (60), to bound (58) as

\[ |\bar{y}(t) - z_2(t)| \leq |\bar{y}(0) - z_2(0)| + t \gamma^2 \bar{k}_2(L_r, \delta) \]

\[ + \gamma \int_0^t L_r|x - z_1| + |\bar{y}(\tau) - z_2(\tau)|d\tau \]

\[ + \gamma \int_0^t \frac{\partial_\varepsilon_2(\eta, \tau)}{\eta = \eta_\alpha(\tau)} \dot{\eta}_\alpha(\tau) d\tau, \]

where \( \bar{k}_2(L_r, \delta) := \delta L_r (2 + L_r (2\delta + 3 + L_r)) \).

Now define \( f(t) := |x(t) - z_1(t)| + |\bar{y}(t) - z_2(t)| \) and \( A = \max \{ 1, L_r \} \), sum (56) and (61) to obtain

\[ f(t) \leq f(0) + t \gamma^2 \bar{k}_2(L_r, \delta) + \gamma A \int_0^t f(\tau) d\tau, \]

and use the specific Gronwall Lemma [[18], §1.3] to determine

\[ f(t) \leq \frac{\bar{k}_3(L_r, \delta) + \bar{k}_2(L_r, \delta)}{A} \left( e^{\gamma A t} - 1 \right) + f(0) e^{\gamma A t} \]

where, since \( \varepsilon(\eta_\alpha, 0, 0) = 0 \), we have

\[ f(0) = |x(0) - z_1(0)| + |\bar{y}(0) - z_2(0)| \]

\[ = |x(0) - x_a(0) + \gamma \varepsilon_\delta(x_a, 0)| + |\bar{y}(0) - \bar{y}_a(0) + \gamma \varepsilon_\delta(y_a(0), 0)| \]

\[ = |x(0) - x_a(0)| + |\bar{y}(0) - \bar{y}_a(0)|. \]

Use (45) and \( \| \cdot \| \leq \| \cdot \|_1 \) to bound

\[ \|\eta(t) - \eta_\alpha(t)\|_1 \leq \|\eta(t) - z(t)\|_1 + \gamma \bar{k}_3(L_r, \delta) \]

\[ = f(t) + \gamma \bar{k}_3(L_r, \delta). \]
Finally, let $\bar{t} > 0$ and define

$$
\gamma_1^* = \frac{\bar{c}}{3} k_1(L_r, \delta, \bar{c}/3) + \frac{\bar{c}}{3} k_2(L_r, \delta, \bar{c}/3) e^{At} - 1
$$

$$
\gamma_2^* = \frac{\bar{c}}{3} k_1(L_r, \delta) + \frac{\bar{c}}{3} k_2(L_r, \delta) e^{At} - 1
$$

$$
\gamma_3^* = \frac{\bar{c}}{3} k_2(L_r, \delta) e^{At} - 1
$$

$$
\gamma_*^* = \min\{\gamma_0^*, \gamma_1^*, \gamma_2^*, \gamma_3^*\}
$$

then for any $t \in [0, \bar{t}/\gamma]$, for any $\|\eta(0) - \eta_a(0)\|_1 \leq \gamma^*$, and for any $\gamma \in (0, \gamma^*)$

$$
\|\eta(t) - \eta_a(t)\|_2 \leq f(t) + \gamma \bar{k}_3(L_r, \delta)
$$

It is worth noting that $A$, $\bar{k}_1$, $\bar{k}_2$, and $\bar{k}_3$ are not dependent on the cost function magnitude $M_r$. To stress this point let

$$
\gamma^*(t, \bar{c}, L_r, \delta, \gamma) := \min\{\gamma_0^*, \gamma_1^*, \gamma_2^*, \gamma_3^*\}.
$$

Finally, we use the same arguments adopted at the end of the proof of Proposition 1 to exploit the local exponential stability of the attractor graph $\tau|_{A^*}$ to extend the previous bound for any $t \geq \bar{t}/\gamma$.

Appendix V

Results on Averaging

Let $f(\cdot, \cdot) : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$, let $\gamma > 0$ and consider the system

$$
\dot{x} = \gamma f(x, t) \quad x(0) = x_0.
$$

Let $T > 0$ and assume $f$ to be $T$-periodic in $t$, then the associated average system is defined as

$$
\dot{x}_a = \gamma f_a(x_a) \quad x_a(0) = x_0
$$

with

$$
f_a(x) := \frac{1}{T} \int_0^T f(x, t) \, dt.
$$

The results provided in Lemma 3 are grounded on the following assumptions:

A1) $f(x, t)$ is Lipschitz continuous and bounded on any compact set $D \subset \mathbb{R}^n$, uniformly in $t$, i.e. there exist (finite) $L_D, M_D > 0$ such that

$$
\|f(x_1, t) - f(x_2, t)\| \leq L_D \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in D \text{ and for all } t \geq 0
$$

$$
\|f(x, t)\| \leq M_D \text{ for all } x \in D \text{ and for all } t \geq 0
$$

A2) There exists $A \subset \mathbb{R}^n$, and $\beta(\cdot, \cdot) \in \mathcal{K}L$ such that

$$
\|x_a(t)\|_A \leq \beta(\|x_a(0)\|_A, \gamma t) \quad \forall x_a(0) \in \mathbb{R}^n.
$$

A3) Let A1) be verified for $D = \mathbb{R}^n$. Let A2) and assume that for any $\rho, d > 0$ with $\rho > d$ there exists (finite) $\tilde{t} > 0$ such that

$$
\beta(\rho, \tilde{t}) < \rho - d \quad \forall \tilde{t} > \tilde{t}^*.
$$

Lemma 3 Let (62), then the following claims are true:

C1) Closeness. Let A1) be verified and pick $d > 0$ and $D \subset \mathbb{R}^n$. Define $D$ as the smallest set such that if $\|x\|_D \leq d$ then $x \in D$. Then for any $t > 0$ there exists $\gamma^*(\tilde{t}, \overline{L_D}, L_D) > 0$ such that for any $\gamma \in (0, \gamma^*)$

$$
\|x(t) - x_a(t)\| \leq d
$$

for all $t \in [0, \tilde{t}/\gamma]$ such that $x(t), x_a(t) \in D$.

C2) Semi-Global Practical Stability. Let A1) and A2) be verified. Then, for any $d, \rho > 0$ there exists $\tilde{t} > 0$ and $\gamma^*(\tilde{t}, M_D, L_D)$ such that for any $x_0 \in \mathbb{R}^n$

$$
\|x_0\|_A \leq \rho \text{ and for any } \gamma \in (0, \gamma^*) \text{ the trajectories } t \mapsto x(t) \text{ are bounded and}
$$

$$
\|x(t)\|_A \leq d \quad \forall t \geq \tilde{t}/\gamma
$$

C3) Global Practical Stability. Let A3) be verified. Then, there exists $\gamma^* > 0$ such that for any $\gamma \in (0, \gamma^*)$, and for any $x_0 \in \mathbb{R}^n$ the trajectories $t \mapsto x(t)$ are bounded and

$$
\limsup_{t \to \infty} \|x(t)\|_A \leq d.
$$

Proof: Let A1) be verified. As a starting point, we note that $\|f_a(x)\| \leq M_D$ for all $x \in D$ thanks to the $t$-uniform boundedness of $f(x, t)$ on $D$ and thanks to the definition of $f_a(x)$. Let

$$
\epsilon(x, t) := \int_0^t f(x, \tau) - f_a(x) \, d\tau,
$$

and define $N \in \mathbb{N}$ as the largest integer such that $NT \leq t$. Then, thanks to the $T$-periodicity of $f(x, t)$ we have

$$
\epsilon(x, t) = \int_0^t f(x, \tau) - f_a(x) \, d\tau = \int_{NT}^{t} f(x, \tau) - f_a(x) \, d\tau,
$$

from which, using the triangle inequality,

$$
\epsilon(x, t) \leq \int_{NT}^{t} \|f(x, \tau)\| + \|f_a(x)\| \, d\tau \leq 2M_D T.
$$

Exploit $z(t) := x_a(t) + \gamma \epsilon(x_a(t), t)$ to bound

$$
\|x(t) - x_a(t)\| \leq \|x(t) - z(t)\| + \|z(t) - x_a(t)\|
$$

$$
\leq \|x(t) - z(t)\| + \gamma \epsilon(x_a(t), t)
$$

$$
\leq \|x(t) - z(t)\| + \gamma 2M_D T.
$$

Finally, let $\bar{t} > 0$ and define
On the other hand

\[
x(t) - z(t) = x(0) - z(0) + \int_0^t \dot{x}(\tau) - \dot{z}(\tau) \, d\tau
\]

\[
= x(0) - z(0) + \gamma \int_0^t f(x(\tau), \tau) \, d\tau
\]

\[
- \int_0^t \dot{x}_a(\tau) \, d\tau
\]

\[
- \gamma \int_0^t \frac{\partial \epsilon(s, \tau)}{\partial s} \bigg|_{s=x_a(\tau)} \dot{x}_a(\tau) \, d\tau
\]

\[
- \gamma \int_0^t \frac{\partial \epsilon(x_a(\tau), s)}{\partial s} \bigg|_{s=\tau} d\tau
\]

\[
= x(0) - z(0) + \gamma \int_0^t f(x(\tau), \tau) - f(x_a, \tau) \, d\tau
\]

\[
- \gamma^2 \int_0^t \frac{\partial \epsilon(s, \tau)}{\partial s} \bigg|_{s=x_a(\tau)} f_a(x_a(\tau)) \, d\tau
\]

Add and subtract within the first integral \( f(z(\tau), \tau) \), let \( \bar{\epsilon} \in (0, d) \), and \( \gamma^*_1 := \bar{\epsilon}/(2M_D) \). Define \( L_D \) as the Lipschitz constant of \( f(x, t) \) for \( x \in D \). Then, for any \( \gamma \in (0, \gamma^*_1) \) and for any \( x_a(t) \in D \) we have \( z(t) \in D \) and

\[
\|f(z(\tau), \tau) - f(x_a, \tau)\| \leq L_D \|z(\tau) - x_a(\tau)\| \\
\leq 2L_D M_D T
\]

\[
\|f(x(\tau), \tau) - f(z(\tau), \tau)\| \leq L_D \|x(\tau) - z(\tau)\|
\]

Moreover, note that

\[
\left\| \frac{\partial \epsilon(s, \tau)}{\partial s} \right\| \leq \int_0^t \left\| \frac{\partial f(s, \tau)}{\partial s} \right\| + \left\| \frac{\partial f_a(s)}{\partial s} \right\| \, d\tau \leq 2L_D T,
\]

then

\[
\|x(t) - z(t)\| \leq \|x(0) - z(0)\|
\]

\[
+ \gamma \int_0^t \|f(x(\tau), \tau) - f(z(\tau), \tau)\| \, d\tau
\]

\[
+ \gamma \int_0^t \|f(z(\tau), \tau) - f(x_a(\tau), \tau)\| \, d\tau
\]

\[
+ \gamma^2 \int_0^t \left\| \frac{\partial \epsilon(s, \tau)}{\partial s} \bigg|_{s=x_a(\tau)} \right\| \|f_a(x_a(\tau))\| \, d\tau
\]

\[
\leq \|x(0) - z(0)\|
\]

\[
+ \gamma L_D \int_0^t \|x(\tau) - z(\tau)\| \, d\tau
\]

\[
+ \gamma^2 3M_D M_D T t
\]

Applying the Gronwall Lemma to compute

\[
\|x(t) - z(t)\| \leq \|x(0) - z(0)\| + \gamma^2 3M_D M_D T t
\]

\[
+ \gamma L_D \int_0^t \|x(0) - z(0)\| + \gamma^2 3M_D M_D T t \, e^{\gamma L_D (t-\tau)} \, d\tau
\]

\[
= \|x(0) - z(0)\| e^{\gamma L_D t} + \gamma 3M_D T \left( e^{\gamma L_D t} - 1 \right)
\]

from which

\[
\|x(t) - x_a(t)\| \leq \|x(0) - z(0)\| e^{\gamma L_D t}
\]

\[
+ \gamma 3M_D T \left( e^{\gamma L_D t} - 1 \right)
\]

\[
+ \gamma 2M_D T.
\]

Then, let \( \bar{t} > 0 \),

\[
\gamma^* := \frac{d}{6M_D T e^{\gamma L_D t}}
\]

\[
\gamma^* := \frac{d}{6M_D T (e^{\gamma L_D t} - 1)}
\]

\[
\gamma^* := \frac{d}{6M_D T},
\]

and define \( \gamma^*(d, \bar{t}, M_D, L_D) = \min\{1, \gamma^*_1, \gamma^*_2, \gamma^*_3, \gamma^*_4\} \). Without loss of generality pick \( x_a(0) = x_0 \). Then for any \( \gamma \in (0, \gamma^*) \)

\[
\|x(t) - x_a(t)\| \leq \gamma 2M_D T e^{\gamma L_D t}
\]

\[
+ \gamma 3M_D T \left( e^{\gamma L_D t} - 1 \right)
\]

\[
+ \gamma 2M_D T
\]

\[
\leq \gamma^*_2 2M_D T e^{\gamma L_D t}
\]

\[
+ \gamma^*_2 3M_D T \left( e^{\gamma L_D t} - 1 \right)
\]

\[
+ \gamma^*_2 2M_D T
\]

\[
\leq \frac{d}{3} \left( \frac{e^{\gamma L_D t}}{e^{\gamma L_D t} - 1} + 1 \right) \leq d
\]

for all \( t \in [0, \bar{t}/\gamma] \) as long as \( x(t), x_a(t) \in D \). This proves C1).

To prove C2) let

\[
D := \{ x \in \mathbb{R}^n : \|x\|_A \leq \beta(\rho, 0) + d \}.
\]

Let \( \rho \) be the largest real such that

\[
\beta(\rho, 0) < d/2.
\]

Define \( \tilde{d} \in (0, \rho) \) and find \( \bar{\bar{t}} > 0 \) such that

\[
\beta(\rho, \tau) + \tilde{d} \leq \rho \quad \forall \tau \geq \bar{\bar{t}}\beta.
\]

Let \( \bar{\bar{t}} > \bar{\bar{t}}^* \), then for any \( \gamma \in (0, \gamma^*(\tilde{d}, \bar{t}, M_D, L_D)) \) and any \( x_0 \in \mathbb{R}^n \) such that \( \|x_0\|_A \leq \rho \) we have

\[
\|x(t)\|_A \leq \|x(t) - x_a(t)\| + \|x_a(t)\|_A
\]

\[
\leq \bar{\bar{d}} + \beta(\|x_0\|_A, \gamma t) \quad \forall t \in [0, \bar{\bar{t}}/\gamma]
\]

and

\[
\|x(\bar{\bar{t}}/\gamma)\|_A \leq \rho.
\]

Now divide the time axis into sub-intervals of the form

\[
I_n := \left[ \frac{n \bar{\bar{t}}}{\gamma}, (n + 1) \frac{\bar{\bar{t}}}{\gamma} \right] \quad n \in \mathbb{N}
\]

and, with \( x_a(t, x_0) \) the trajectory of the average system at time \( t \) with initial condition \( x_0 \) at time \( t = 0 \), let

\[
x_n(t) := x_a(t - n\bar{\bar{t}}/\gamma, x(n\bar{\bar{t}}/\gamma)) \quad \forall t \in I_n.
\]

The same arguments used in the first part of the proof show that

\[
\|x(t) - x_n(t)\| \leq \bar{\bar{d}} \quad \forall t \in I_n, \forall n \in \mathbb{N}
\]
Moreover,
\[ \|x(t)\|_A \leq \|x_n(t)\|_A + \|x(t) - x_n(t)\| \]
\[ \leq \beta(\|x(n\bar{t}/\gamma)\|_A, \gamma t - n\bar{t}) + \tilde{d} \]
\[ \leq \beta(\|x(n\bar{t}/\gamma)\|_A, 0) + \tilde{d} \quad \forall t \in I_n, \forall n \in \mathbb{N} \]
and
\[ \|x((n+1)\bar{t}/\gamma)\|_A \leq \beta(\|x(n\bar{t}/\gamma)\|_A, \bar{t}) + \tilde{d} \quad \forall n \in \mathbb{N}. \]

Finally, since for \( n = 1 \)
\[ \|x(t)\|_A \leq \beta(\|x(\bar{t}/\gamma)\|_A, 0) + \tilde{d} \]
\[ \leq \bar{d} \quad \forall t \in I_1 \]
and
\[ \|x(2\bar{t}/\gamma)\|_A \leq \beta(\|x(\bar{t}/\gamma)\|_A, \bar{t}) + \tilde{d} \]
\[ \leq \beta(\rho, \bar{t}) + \tilde{d} \]
\[ \leq \rho \]
the arguments can be iterated for any \( n > 1 \). This proves C2).

To prove C3), define \( \rho, \tilde{d} \) and \( \bar{t}^* \) by following the same steps used before to prove C2). Using A3) define \( \bar{t}^* > 0 \) such that
\[ \beta(r, \bar{t}) < r - \rho \quad \forall r > \rho, \forall \bar{t} > \bar{t}^*. \]
Define
\[ \bar{t} > \max\{\bar{t}^*, \bar{t}^*_2\} \]
and pick \( \gamma \in (0, \gamma^*(\tilde{d}, \bar{t}, M_{\mathbb{R}^n}, L_{\mathbb{R}^n})) \). As done to prove C2) divide the time axis into sub-intervals \( I_n \) and on each interval define \( x_n(t) \). As consequence, for any \( n \in \mathbb{N} \) and any \( x_0 \in \mathbb{R}^n \), we have that \( x_{a_0} = x_0 \) implies
\[ \|x(t) - x_n(t)\| \leq \tilde{d} \quad \forall t \in I_n. \]
Then, as far as \( \|x((n\bar{t}/\gamma))\|_A > \rho \)
\[ \|x((n+1)\bar{t}/\gamma)\|_A \leq \|x_n((n+1)\bar{t}/\gamma)\|_A \]
\[ + \|x((n+1)\bar{t}/\gamma) - x_n((n+1)\bar{t}/\gamma)\| \]
\[ \leq \|x_n(\bar{t}/\gamma, x(n\bar{t}/\gamma))\|_A + \tilde{d} \]
\[ \leq \beta(\|x(n\bar{t}/\gamma)\|_A, \bar{t}/\gamma) + \tilde{d} \]
\[ \leq \beta(\|x(n\bar{t}/\gamma)\|_A, \bar{t}/\gamma) + \tilde{d} \]
\[ < \|x(n\bar{t}/\gamma)\|_A \]
Then we can conclude that as far as \( \|x(n\bar{t}/\gamma)\|_A > \rho \) the sequence
\[ \{\|x(n\bar{t}/\gamma)\|_A\}_{n \in \mathbb{N}} \]
is strictly decreasing and
\[ \limsup_{n \to \infty} \|x(n\bar{t}/\gamma)\|_A \leq \rho. \]
The proof of C3) can be then obtained by following the same final steps used to prove C2).