Use of a Kepler solver in $N$-body simulations

Chen Deng, Xin Wu†, Enwei Liang
School of Physical Science and Technology, Guangxi University, Nanning 530004, China
† xinwu@gxu.edu.cn

ABSTRACT

A Kepler solver is an analytical method to solve a two-body problem. It is slightly modified as a new manifold correction method. Only one change to the analytical solution lies in that the obtainment of the eccentric anomaly does not need any iteration but uses the true anomaly between the constant Laplace-Runge-Lenz (LRL) vector and a varying radial vector given by an integrator. The new method rigorously conserves seven integrals of the Kepler energy, angular momentum vector and LRL vector. This leads to the conservation of all orbital elements except the mean longitude. In the construction mechanism, the new method is unlike the existing Fukushima's linear transformation method that satisfies these properties. It can be extend to treat an $N$-body problem. The five slowly-varying orbital elements of each body relative to the central body are determined by the seven quasi-conserved quantities from their integral invariant relations, and the eccentric anomaly is calculated similarly in the above way. Substituting these values into the Kepler solver yields an adjusted solution of each body. Numerical simulations of a six-body problem of the Sun, Jupiter, Saturn, Uranus, Neptune and Pluto show that the new method can significantly improve the accuracy of all the orbital elements and positions of individual planets, as compared with the case without correction. The new method and the Fukushima’s method are almost the same in the numerical performance and require negligibly additional computational cost. They both are the best of the existing correction methods in $N$-body simulations.

Subject headings: Computational methods — Computational astronomy — Solar system astronomy

1. Introduction

In a relative coordinate system, a pure two-body problem in the solar system is a system with three degrees of freedom, which has 7 constants of motion: the Kepler energy, the relative angular momentum vector and the Laplace-Runge-Lenz (LRL) vector. Due to the conserved quantities satisfying two relations, five of them are independent. This corresponds to five invariant orbital elements. In this case, the problem is integrable and has an analytical solution. Nevertheless, $N$-body gravitational problems with $N > 2$ are always non-integrable and therefore no analytic solutions can be given. Numerical integration methods are convenient tools to solve them. Geometric integrators (Hairer et al. 1999) can preserve one or more physical/geometric properties of these systems. The properties contain structures, integrals, symmetries, reversing symmetries, phase-space volumes, etc. Symplectic integrators (Ruth 1983; Feng 1985; Wisdom & Holman 1991; Zhong et al. 2010; Mei et al. 2013a, 2013b), as a class of geometric integration methods, conserve symplectic structures and phase-space volumes of Hamiltonian systems and have no secular drift in energy errors. Because of these advantages, they are particularly suitable for studying qualitative properties on the long-term evolution of Hamiltonian systems. Symmetrical methods (Quinlan & Tremaine 1990), extended phase space methods (Pihajoki 2015; Liu et al. 2016; Luo et al. 2017; Li & Wu 2017) and energy-conserving schemes (Chorin et al. 1978; Feng 1985; Bacchini et al. 2018a, 2018b; Hu et al. 2019) should belong to the geometric integrators. Although non-geometric integrators...
such as Runge-Kutta methods do not satisfy such geometric properties, they have wider applications than the geometric integrators. In addition, they would provide more accurate numerical solutions than the same order geometric integrators (excluding those from the same non-geometric integrators reformed). Their numerical results should be reliable for a short time of numerical integration.

However, the non-geometric integrators can be reformed as the geometric ones by means of some particular treatments. As a popular treatment, the use of one or more known integrals weakening the influence of the Lyapunov’s instability of Keplerian orbits on the numerical solutions suppresses the fast growth of various numerical errors. This is the so-called stabilization methods that always confine the numerical solutions to the integral surfaces in phase space. Including the integral(s) as stabilizing term(s) in the set of differential equations is the stabilizing method of Baumgarte (1972, 1973). Its applications were given in (Ascher et al. 1995; Chin 1995; Avdyushev 2003). Another stabilization path is the manifold correction scheme of Nacozy (1971) in which the stabilizing terms are directly added to the numerical solutions. This approach makes the corrected integral accuracy become the square of the uncorrected integral accuracy by applying Lagrange multipliers to take the integrated orbits back to the original hypersurface along the least-squares shortest path. It means that the correction solutions do not very rigorously satisfy the integral(s). Following this basic idea, several authors focused on the application of and the effectiveness of the manifold correction method (Murison 1989; Chin 1995; Zhang 1996; Wu et al. 2006). The steepest descending method for approximate consistency of the Kepler energy of the two-body problem suppresses the growth of integration errors in the semimajor axis (Wu et al. 2007). The approximate conservation of the 7 integrals leads to that of the five constant elements in the two-body problem (Ma et al. 2008a). Besides these manifold correction methods that approximately satisfy the integral(s), those that rigorously satisfy the integral(s) were developed. For example, the scaling method of Fukushima (2003a) and the velocity scaling method of Ma et al. (2008b) can exactly conserve the energy (i.e. the semimajor axis) of the two-body problem. The dual scaling method for rigorous consistency of the Kepler energy and the LRL integral in the two-body problem is effective to control the growth of integration errors in the semimajor axis, eccentricity and longitude of pericenter (Fukushima 2003b). The rotation method for rigorously satisfying the angular momentum vector of the two-body problem reduces the growth of integration errors in the inclination and the longitude of the ascending node (Fukushima 2003c). The linear transformation method of Fukushima (2004) is the best of the Fukushima’s manifold correction methods because it simultaneously, rigorously satisfies the Kepler energy, angular momentum vector and the full LRL vector in the two-body problem. It can dramatically reduce the integration errors in various orbital elements at a negligibly small increase in computational cost.

Although 7 conservative integrals of motion including the total energy, the total momentum vector and the total angular momentum vector are still present in the general N-body problem, enforcing their constancy fails to work in a numerical integration (Hairer et al. 1999; Wu et al. 2007). Instead, individual Kepler energies, angular momentum vectors and LRL vectors must be corrected. However, these quantities no longer remain invariant in this case. In addition, no integrals of motion are available in dissipative and other nonconservative systems, either. The above correction methods become useless in these cases. Fortunately, these varying individual quantities obtained from their integral invariant relations (Szebehely & Bettis 1971; Huang & Innanen 1983) have much higher accuracy than those directly determined by the integrated position and velocity, and can be taken as reference values of correcting the errors. In this way, the above correction methods (e.g. Fukushima 2003a, 2003b, 2003c, 2004; Wu et al. 2007; Ma et al. 2008a, 2008b) are still valid. In particular, the velocity scaling method of Ma et al. (2008b) works well in nonconservative elliptic restricted three-body problems (Wang et al. 2016) and dissipative circular restricted three-body problems (Wang et al. 2018).

The two-body problem and each body of the
N-body problem in the relative coordinate system have a certain similarity. Noting this point, we attempt to use the analytical solvable method of the two-body problem (i.e. the Kepler solver) in N-body simulations. Based on this attempt, a new manifold correction method is developed. Each body still uses the Kepler analytical solvable form. That is, the five slowly-varying orbital elements of each body must be determined by the seven quasi-conserved quantities from their integral invariant relations, and the eccentric anomaly is not solved from the Kepler equation but is calculated by the true anomaly between the LRL vector and a radial vector. The new method and the linear transformation method of Fukushima (2004) rigorously satisfy seven integrals of the Kepler energy, angular momentum vector and LRL vector in the two-body problem. In spite of this, they both have different construction mechanisms.

The rest of this paper is organized as follows. In Section 2, a new correction method is designed to rigorously satisfy seven integrals of the Kepler energy, angular momentum vector and LRL vector in a two-body problem. Then, the new method is extended to treat N-body problems in Section 3. Finally, our main results are concluded in Section 4. The linear transformation method of Fukushima (2004) is briefly introduced in an Appendix.

2. A new manifold correction method to a Kepler problem

First, equations of motion, integrals of motion, orbital elements and analytical solutions for a two-body problem are given. Then, the Kepler solver is slightly modified as a new manifold correction method for consistency of the Kepler energy, angular momentum vector and LRL vector.

2.1. Introduction to a Kepler problem

A pure Kepler problem is a two-body problem consisting of a small body (e.g. planet) and a primary body (e.g. the Sun). In a relative coordinate system (e.g. the heliocentric coordinate system) on the small body relative to the primary body, this two-body problem is simplified as a one-body problem

\[ K = \frac{v^2}{2} - \frac{\mu}{r}, \]  

where \( r = |r| \) represents a radial separation (i.e. the length of a relative coordinate vector \( r \)), \( v \) is the magnitude of a relative velocity vector \( v \), and \( \mu = G(M + m) \) (note that \( G \) is a constant of gravity, and \( M \) and \( m \) are masses of the two bodies). The evolution of \((r,v)\) with time \( t \) satisfies the following relation

\[ \ddot{r} = -\frac{\mu}{r^3}r, \]  
equivalently,

\[ \dot{r} = v, \]

\[ \dot{v} = -\frac{\mu}{r^3}r. \]  

In terms of Equation (2) or (3), it is easy to check that \( K \) in Equation (1) is a constant of motion, called as a Kepler energy. There is also an invariant angular momentum vector

\[ L = r \times v. \]  

In fact, it contains three components. This means the existence of three integrals of motion. Let \( L \) be the magnitude of the angular momentum vector, \( L = |L| \). Besides them, three components of the LRL vector

\[ P = v \times L - \frac{\mu r}{r} \]  
do not vary with time. Take \( P \) as the magnitude of the LRL vector, \( P = |P| \). In total, this Kepler problem has 7 integrals of motion, labeled as \( K, P_x, P_y, P_z, L_x, L_y \) and \( L_z \). Since the seven quantities satisfy two relations

\[ L \cdot P = 0, \]

\[ p^2 - 2KL^2 = \mu^2, \]  
only five of them are independent.

The five independent integrals of motion correspond to five invariant elements of an elliptical orbit: the semimajor axis \( a \), the eccentricity \( e \), the inclination \( I \), the longitude of ascending node \( \Omega \) and the argument of pericentre \( \omega \). They are described as

\[ a = -\frac{\mu}{2K}, \]

\[ e = \frac{P^2}{\mu}, \]
\[ \cos I = \frac{L_z}{L}, \quad \sin I = \sqrt{1 - \cos^2 I}, \quad (10) \]
\[ \sin \Omega = \frac{L_x}{L \sin I}, \quad \cos \Omega = -\frac{L_y}{L \sin I}, \quad (11) \]
\[ \sin \omega = \frac{P_z}{e \mu \sin I}, \]
\[ \cos \omega = \frac{1}{e \mu} (P_x \cos \Omega + P_y \sin \Omega). \quad (12) \]

Note that the inclination is always in the range \(0^\circ < I < 180^\circ\), and the other angles are in the ranges \(0^\circ \leq \Omega < 360^\circ\) and \(0^\circ \leq \omega < 360^\circ\). The location of \(\Omega\) in the orbital plane system is given by the signs of \(L_x\) and \(-L_y\), and that of \(\omega\) is determined by the signs of \(P_z\) and \((P_x \cos \Omega + P_y \sin \Omega)\). However, a sixth orbital element, the mean anomaly \(M\), varies with time, that is,
\[ M = M_0 + nt, \quad (13) \]
where \(n = \sqrt{\mu/a^3}\) is a mean motion, and \(M_0\) is the initial mean anomaly. The mean anomaly and the eccentric anomaly \(E\) satisfy a Kepler equation
\[ E - e \sin E = M. \quad (14) \]

In light of Equations (13) and (14), \(M_0\) is calculated by
\[ M_0 = E_0 - e \sin E_0, \quad (15) \]
where the initial eccentric anomaly \(E_0\) is obtained from the relations \(e \cos E_0 = 1 - r_0/a\) and \(e \sin E_0 = r_0 \cdot v_0/(a^2 n)\) \((r_0\) and \(v_0\) are the initial position and velocity). The Newton-Raphson iteration method can solve the Kepler equation.

Equation (2) has an analytical solution
\[ r = \frac{a}{e \mu} (\cos E - e) P + a \sqrt{1 - e^2 \sin E} Q, \quad (16) \]
\[ v = \frac{a^2 n}{r e \mu} \sin E P + \frac{a^2 n}{r} \sqrt{1 - e^2 \cos E} Q. \quad (17) \]

Here, \(P\) is given not only by Equation (5) but in the following form
\[ P = e \mu \left( \begin{array}{c}
\cos \Omega \cos \omega - \sin \Omega \sin \omega \cos I \\
\sin \Omega \cos \omega + \cos \Omega \sin \omega \cos I \\
\sin \omega \sin I
\end{array} \right). \]

\(r\) and \(Q\) are expressed as
\[ r = a(1 - e \cos E), \quad (18) \]
\[ Q = \left( \begin{array}{c}
- \cos \Omega \sin \omega - \sin \Omega \cos \omega \cos I \\
- \sin \Omega \sin \omega + \cos \Omega \cos \omega \cos I \\
\cos \omega \sin I
\end{array} \right). \quad (19) \]

Equations (16) and (17) are a Kepler solver of the two-body problem. This Kepler solver is given completely by the seven conserved quantities, \(K, P_x, P_y, L_x, L_y\) and \(L_z\).

The above-mentioned statements are some basic concepts and theories of the two-body problem in the solar system dynamics. See the book of Murray & Dermott (1999) for more details. See also a fast and accurate universal Kepler solver without Stumpf series, which was recently given by Wisdom & Hernandez (2015).

### 2.2. Use of the Kepler solver in correction numerical solutions

When a non-geometric integration method such as an explicit Runge-Kutta integrator solves Equation (3), a numerical solution \((\mathbf{r}^*, \mathbf{v}^*)\) can be given at each step. Due to various numerical errors, the energy, angular momentum vector and LRL vector at this step cannot be equal to their initial values, i.e. \(K(\mathbf{r}^*, \mathbf{v}^*) \neq K(\mathbf{r}_0, \mathbf{v}_0), L(\mathbf{r}^*, \mathbf{v}^*) \neq L(\mathbf{r}_0, \mathbf{v}_0)\) and \(P(\mathbf{r}^*, \mathbf{v}^*) \neq P(\mathbf{r}_0, \mathbf{v}_0)\), or simply denoted as \(K^* \neq K_0, L^* \neq L_0\) and \(P^* \neq P_0\). Equivalently, numerical values of the five orbital elements \(a^*, e^*, I^*, \Omega^*\) and \(\omega^*\) are unlike their initial values \(a_0, e_0, I_0, \Omega_0\) and \(\omega_0\). If the elements \(a, e, I, \Omega\) and \(\omega\) in Equations (16) and (17) are not given by \(a^* \to a, e^* \to e, I^* \to I, \Omega^* \to \Omega\) and \(\omega^* \to \omega\), but are adjusted via \(a_0 \to a, e_0 \to e, I_0 \to I, \Omega_0 \to \Omega\) and \(\omega_0 \to \omega^\circ\) then such an adjusted numerical solution \((\mathbf{r}^*, \mathbf{v}^*)\) should be more accurate than the non-adjusted numerical solution \((\mathbf{r}^*, \mathbf{v}^*)\). If \(E\) is still solved from Equation (14), the adjusted numerical solution is completely the same as the analytical solution and the numerical integrator become useless.

To consider the use of this integrator, we provide another method how to calculate the eccentric anomaly \(E\). Its calculation needs relying on the true anomaly \(f\). Here are some related details. Obtain a unit radial vector \(\hat{\mathbf{r}}^* = \mathbf{r}^*/r^*\), given by the integrator at each step. Since the true anomaly is the angle between the two unit vectors \(\hat{\mathbf{r}}^*\) and \(\hat{\mathbf{r}}^*\),

\(\text{In this paper, the notation } A \to B \text{ means substituting } A \text{ for } B.\)
\( P/(\epsilon \mu) \), its cosine is
\[
\cos f^* = \frac{\hat{r} \cdot P_0}{\epsilon_0 \mu}.
\]
Its sine is written as
\[
\sin f^* = \frac{S \cdot \hat{r}^*}{|S|},
\]
where \( S \) is a constant vector:
\[
S = L_0 \times P_0.
\]

\( f^* \) means a correction value of \( f \), determined by Equations (20) and (21). The eccentric anomaly is expressed in terms of the true anomaly as
\[
\cos E^* = \frac{\cos f^* + \epsilon_0}{1 + \epsilon_0 \cos f^*},
\]
\[
\sin E^* = \frac{(1 - \epsilon_0 \cos E^*) \sin f^*}{\sqrt{1 - \epsilon_0^2}}.
\]

From an \((i-1)\)th step to an \(i\)th step, the numerical solution \((r^*, v^*)\) given by the adopted integrator is corrected in the following forms
\[
r^* = \frac{a_0}{\epsilon_0 \mu}(\cos E^* - \epsilon_0)P_0 + a_0 \sqrt{1 - \epsilon_0^2} \sin E^* Q_0,
\]
\[
v^* = -\frac{a_0^2 n_0}{r^* \epsilon_0 \mu} \sin E^* P_0 + a_0^2 n_0 \sqrt{1 - \epsilon_0^2} \cos E^* Q_0,
\]
where \( r^* \) in Equation (18) is written as
\[
r^* = a_0(1 - \epsilon_0 \cos E^*).
\]

It is clear that Equations (25) and (26) provide a new manifold correction scheme with the use of the Kepler solver, labeled as M1. The corrected solution in Equations (25) and (26) is almost the same as the analytical solution in Equations (16) and (17) with \( a = a_0, \epsilon = \epsilon_0, I = I_0, \Omega = \Omega_0 \) and \( \omega = \omega_0 \). Only a slight difference between them lies in different methods of computing the eccentric anomaly. For the corrected solution, a certain numerical integrator must be used to give a value of the unit radial vector \( \hat{r}^* \) in Equations (20) and (21), and an iteration method is not necessary to calculate the eccentric anomaly. However, such a computation of the radial vector for the analytical solution is unnecessary and an iteration method must be frequently employed to solve the eccentric anomaly from the Kepler equation (14). Without doubt, the corrected solutions rigorously conserve the seven integrals of motion in the whole course of numerical integrations, as the analytical solutions do but the uncorrected ones do not. In spite of this, that does not mean that the corrected solutions and the analytical ones can achieve at the same numerical accuracy. This is because the unit radial vector is given numerically and then the accuracy of the corrected solutions is decreased, compared to that of the analytical solutions. Although the unit radial vector is not corrected from one integration step, it should be frequently corrected from many integration steps. On the other hand, the new method is typically different from the linear transformation method of Fukushima (2004). The linear transformation method (M2) can also rigorously make the Kepler energy, angular momentum vector and the full LRL vector remain invariant, and is introduced briefly in an Appendix. The two correction methods have three explicit differences in their constructions. First, the new method M1 does not need any scale factor, but the Fukushima’s method M2 uses three scale factors. Second, the corrected solution seems to be explicitly dependent on the orbital elements in the method M1 but does not in the method M2. Third, the corrected solution seems to indirectly depend on the numerical solution \((r^*, v^*)\) (except the computation of \(E^*\) using the numerical unit radial vector \(\hat{r}^*\)) in M1, whereas directly comes from a linear combination of the numerical solution \((r^*, v^*)\) in M2.

Now, there is a question of whether the new method M1 is effective to dramatically suppress the fast growth of the integration errors in all orbital elements compared to the uncorrected method. There is another question of whether the new method M1 and the existing method M2 have the same numerical performance in reducing the integration errors in various orbital elements.

To answer them, we do numerical tests. Let us consider the Kepler problem with the parameter \( \mu = 1 \) and the initial orbital elements \( a_0 = 2 \) AU, \( \epsilon_0 = 0.3, I_0 = 20^\circ, \Omega_0 = 50^\circ, \omega_0 = 30^\circ \) and \( M_0 = 40^\circ \). Take a conventional fourth-order Runge-Kutta algorithm (RK4) with a fixed time
step, 1/100 of the orbital period. As shown in Figure 1, the new method M1 drastically reduces the integration errors of all the orbital elements, compared to the uncorrected algorithm RK4. In fact, all the elements (except the mean longitude) are accurate to the level of the machine double precision $\epsilon = 10^{-16}$ throughout the manifold correction. This correction numerical performance is consistent with the theoretical result. In addition, the new method M1 is almost the same as the Fukushima’s method M2 in control of the accumulation of the integration errors in all the orbital elements. Both correction methods also have the same performance in effectively suppressing the errors in the position coordinates. See Figure 2 for more information. Here, each of the three algorithms RK4, M1 and M2 yields the position errors, which are based on the analytical solutions as reference solutions.

Several facts can be concluded from the above theoretical analysis and numerical checks. The new method is very successful to rigorously conserve the seven integrals of motion in the pure Kepler problem. It can drastically suppress the rapid accumulation of the integration errors in all the orbital elements. It is almost the same as the Fukushima’s method in the two points. What about the new correction method to three- or $N$-body problems? The following discussions will deal with this question.

3. Extension to multi-body problems

In the barycentre coordinate system, an $N$-body gravitational problem in the solar system is described by the following Hamiltonian

$$H = \sum_{j=0}^{N-1} \frac{p_j^2}{2m_j} - \sum_{s=0}^{N-2} \sum_{j>s}^{N-1} \frac{Gm_s m_j}{r_{sj}}. \quad (28)$$

$j = 0$ denotes the Sun, and $j = 1, 2, \ldots$ correspond to the related planets. This system has seven isolating conservative integrals as follows: the total energy $E = H$, the total momentum vector $p = \sum_{j=0}^{N-1} p_j$ and the total angular momentum vector $L = \sum_{j=0}^{N-1} r_j \times p_j$. It was reported by Hairer et al. (1999) that a five-body integration of the Sun and four outer planets shows poor numerical performance when both the total energy and the total angular momentum are rigorously preserved through the manifold correction. It was shown in a series of publications (Fukushima 2003a, 2003b, 2003c, 2004; Wu et al. 2007; Ma et al. 2008a, 2008b) that corrections of individual Kepler energies, angular momenta and LRL integrals work well. Naturally, such an application to the new correction should be also considered.

3.1. Integral invariant relations of individual quasi-conserved quantities

In the heliocentric coordinate system, each planet relative to the Sun has a position vector $r_j$ and a velocity vector $v_j$ ($j = 1, 2, \ldots, N - 1$). The evolution of each body is governed by the equations of motion

$$\frac{dr_j}{dt} = v_j, \quad (29)$$

$$\frac{dv_j}{dt} = -\frac{\mu_j}{r_j^3}r_j + a_j, \quad (30)$$

where $\mu_j = G(m_0 + m_j)$ and $a_j$ is a perturbed acceleration in the form

$$a_j = \sum_{s=1, \neq j}^{N-1} Gm_s \left( \frac{r_s - r_j}{|r_s - r_j|^3} - \frac{r_s}{r_s^3} \right). \quad (31)$$

Equations (1), (4) and (5) are still the Kepler energy, angular momentum vector and LRL vector of the $j$th body, marked as $K_j$, $L_j$ and $P_j$. In the present case, these quantities are no longer constants of motion and slowly vary with time. Therefore, they are quasi-conserved quantities, which satisfy the following relations

$$\frac{d\Delta K_j}{dt} = \mathbf{v}_j \cdot \mathbf{a}_j, \quad (32)$$

$$\frac{d\Delta L_j}{dt} = \mathbf{r}_j \times \mathbf{a}_j, \quad (33)$$

$$\frac{d\Delta P_j}{dt} = 2(\mathbf{v}_j \cdot \mathbf{a}_j) \mathbf{r}_j - (\mathbf{r}_j \cdot \mathbf{a}_j) \mathbf{v}_j - (\mathbf{r}_j \cdot \mathbf{v}_j) \mathbf{a}_j. \quad (34)$$

In the above equations, $\Delta K_j = K_j - K_{j0}$, $\Delta L_j = L_j - L_{j0}$ and $\Delta P_j = P_j - P_{j0}$, where $K_{j0}$, $L_{j0}$ and $P_{j0}$ are the starting values of the quantities. Taking zeros as the initial values of $\Delta K_j$, $\Delta L_j$ and $\Delta P_j$ can greatly reduce the roundoff errors.
in numerical integrations. Equations (32)-(34) are called as the integral invariant relations (Szépethelyi & Bettis 1971) about the Kepler energy, angular momentum vector and LRL vector of the jth body.

Integrate Equations (29), (30), (32)-(34). As a direct result, a numerical solution \((r_j^*, v_j^*)\) is given. At the same time, a set of values of the varying integrals, \(K_j^*, L_j^*\) and \(P_j^*\), are obtained. When the numerical solution \((r_j^*, v_j^*)\) are substituted into the expressional forms of the varying integrals, the varying integrals have another set of values \(K_j^*, L_j^*\) and \(P_j^*\). Which of the two sets values from different methods are more accurate? Of course, the values \(K_j^*, L_j^*\) and \(P_j^*\) from the integral invariant relations are claimed, numerical solutions always keep each integral constant. Note that the numerical solutions \(K_j^*, L_j^*\) and \(P_j^*\) directly come from the numerical integration of Equations (29), (30), (32)-(34). Naturally, they have much higher apparent accuracy than the values \(K_j^*, L_j^*\) and \(P_j^*\) calculated by the coordinates and velocities \((r_j^*, v_j^*)\). Its truth has been confirmed via many successful examples of manifold corrections to conservative, nonconservative or dissipative systems (Wang et al. 2016; Wang et al. 2018). Implementations of the manifold correction schemes just take the values from the integral invariant relations of the varying integrals as reference values. This will provide a chance to the application of our new correction method in the present multi-body problem.

### 3.2. Implementation of the new method

Obtain five orbital elements and the mean motion of the jth planet \(a_j^*, e_j^*, I_j^*, \Omega_j^*, \omega_j^*\) and \(n_j^*\) with the integral values \(K_j^*, L_j^*\) and \(P_j^*\). Although these five elements and the mean motion of each body in the multi-body problem are unlike those of the two-body problem and vary with time, they can still be taken as reference values of manifold correction. If the eccentric anomaly \(E_j^*\) is known and when all the elements with the subscript zeros in the correction solution of the two-body problem in Equations (25) and (26) are replaced with the integral values \(K_j^*, L_j^*\) and \(P_j^*\), the new correction method M1 becomes useful. That is, the corrected numerical solution of the jth planet reads

\[
r_j^* = \frac{a_j^*}{e_j^*\mu}(\cos E_j^* - e_j^*)P_j^*
\]

\[
+ a_j^*\sqrt{1 - e_j^2} \sin E_j^*Q_j^*, \quad (35)
\]

\[
v_j^* = -\frac{a_j^*\sqrt{e_j^*}}{r_j^*e_j^*\mu} \sin E_j^*P_j^*
\]

\[
+ \frac{a_j^*e_j^*}{r_j^*} \sqrt{1 - e_j^2} \cos E_j^*Q_j^*, \quad (36)
\]

where \(r_j^*\) is of the form

\[
r_j^* = a_j^*(1 - e_j^2 \cos E_j^*). \quad (37)
\]

Notice that the angles \(\Omega_j^*\) and \(\omega_j^*\) are not necessary to be calculated, but only their sines and cosines are required. Such an operation is helpful to save computational cost and to reduce the roundoff errors. Obviously, the corrected solution in Equations (35) and (36) seems to have the same expressional forms of the Kepler analytical solution in Equations (16) and (17).

Summarized from the above demonstrations, description of the implementation of the new method is given as follows.

At an \(i\)th step, integrate Equations (29), (30), (32)-(34) and obtain \(r_j^*, v_j^*, K_j^*, L_j^*\) and \(P_j^*\).

Calculate the values of \(a_j^*, e_j^*, I_j^*, \Omega_j^*, \omega_j^*\) and \(n_j^*\) with \(K_j^*, L_j^*\) and \(P_j^*\).

Solve the eccentric anomaly \(E_j^*\) by use of a certain method.

Obtain the corrected numerical solution \((r_j^*, v_j^*)\) with Equations (35) and (36).

Take \(r_j^* \rightarrow r_j^*\) and \(v_j^* \rightarrow v_j^*\), and let a next step integration begin.

Once the eccentric anomaly \(E_j^*\) is given, the new method M1 can work. There are several possible choices of the calculation of \(E_j^*\). **Case 1**, a naturally prior choice is to apply the Kepler equation (14) to determine the value of \(E_j^*\). Why this Kepler equation was not considered in the corrected solution of the two-body problem in Equations (25) and (26) is based on the need of a numerical integrator. However, this thing does not occur in the present problem when \(E_j^*\) is solved.
from the Kepler equation. This is because obtaining
the numerical values \( r_j^*, v_j^* \), \( K_j^* \), \( L_j^* \) and \( P_j^* \)
must rely on a numerical integrator. Here, the
Kepler equation (14) becomes of the form

\[
E_{j,i}^* - e_{j,i}^* \sin E_{j,i}^* = M_{j,i-1}^* + n_{j,i-1}^* \Delta t, \tag{38}\]

where \( \Delta t \) is a time step and \( M_{j,i-1}^* \) denotes the
value of \( M_j \) at the \((i-1)\)th step. **Case 2**, because
the mean motion is not invariant, the mean motion
\( n_{j,i-1}^* \) in Equation (38) takes an average value of
the mean motions at the \((i-1)\)th, ith steps, that is,
\( n_{j,i-1}^* = (n_{j,i-1}^* + n_{j,i}^*)/2 \). In this case, the
Kepler equation reads

\[
E_{j,i}^* - e_{j,i}^* \sin E_{j,i}^* = M_{j,i-1}^* + \bar{n}_{j,i-1}^* \Delta t. \tag{39}\]

**Case 3**, calculate \( E_j^* \) with \( (r_j^*, v_j^*) \). That is,

\[
\cos E_j^* = \frac{a^* - r^*}{e^*a^*}, \quad \sin E_j^* = \frac{r_j^* \cdot v_j^*}{e^*n^*a_j^*}. \tag{40}\]

**Case 4**, calculate \( E_j^* \) with the method of calculation
of the eccentric anomaly in the correction solution
of the two-body problem. These operations \( e^* \to e_0 \), \( P_j^* \to P_0 \) and \( L_j^* \to L_0 \) are required in
Equations (20)-(24). For the sake of saving computational
cost and reducing the roundoff errors, no the eccentric anomaly \( E_j^* \) in Cases 3 and 4 but
its sine and cosine are necessarily known during
the integration.

Which of the four methods for computing
the eccentric anomaly \( E_j^* \) shows the best performance?
A key to this question awaits numerical checks. Let us take a real three-body problem
of the Sun, Jupiter and Saturn as a tested model,
and choose RK4 as a basic integrator. All data are
from the JPL’s planetary and lunar ephemerides
DE431. A step size uses 36.525 days, about 1/120
of the orbital period of Jupiter. A 12th-order Cowell
multi-step algorithm is used to provide a much
higher precision reference solution. Seen from
Figure 3, the Kepler energy accuracy of Jupiter for Case 1 is similar to that for Case 2, and is even
worse than for RK4\(^5\). In fact, the accuracy is the
poorest for Cases 1 and 2. This is due to the
mean motion varying with time. The correction effect
is the best for Case 4. In our later numerical
works, Case 4 is considered in the calculation of the eccentric anomaly.

Clearly, the Kepler solver can be used to construct
the manifold correction method for consistency of the Kepler energies, angular momentum
vectors and LRL vectors of individual planets in
the N-body problems. The effectiveness of the
new method will need more detailed numerical inves-
tigations.

### 3.3. Application to a six-body problem

Consider the application of the new method M1
to a six-body problem consisting of the Sun, four
outer planets (Jupiter, Saturn, Uranus, Neptune)
and Pluto. For comparison, the Fukushima’s
method M2 is employed.

Taking RK4 as a basic integrator, we plot the
integration errors in all orbital elements of Jupiter
in Figure 4. There are two results. When the integration
time reaches \( 10^6 \) years, the new method
improves the accuracy of each orbital element in
two or three orders of magnitude, comparable to
the uncorrected method RK4. The two correction
schemes M1 and M2 are almost the same
performance in reducing these errors. They are also
powerful to decrease the errors in the position
coordinates in Figure 6(a). When a fifth-order
Runge-Kutta integrator (RK5) is used instead of
RK4, the effectiveness of correction in Figures 5
and 6(b) is still apparent. Compared with the un-
corrected method, neither of the two correction
method has a dramatic increase in computational
cost in Table 1. Figures 7, 8 and Table 2 also show
that the new method is very effective to suppress
the errors in the orbital elements and the position
coordinates of the other planets.

### 4. SUMMARY

Based on the analytical solution of a pure two-
body problem, a new manifold correction method
is developed to rigorously conserve seven integrals,
including the Kepler energy, angular momentum
vector and LRL vector in this system. Unlike
the analytical method that solves the eccentric
anomaly from the Kepler equation with an iterative
method, the new method has no iterations and
uses the true anomaly between the constant
LRL vector and a varying radial vector to obtain
the eccentric anomaly. Here, the obtainment of

\(^5\)Without the use of manifold correction, RK4 directly solves
Equations (29) and (30) but does not need calculating the
eccentric anomaly during the integration.
the unit radial vector needs a numerical integrator. In the construction mechanism, the new method is also typically different from the Fukushima’s linear transformation method for consistency of the seven integrals. On one hand, the former does not use any scale factor, but the latter uses three scale factors; on the other hand, the former corrected solution seems to directly depend on the orbital elements rather than the numerical solution, whereas the latter one is typically a linear combination of the numerical solution. It is shown via numerical experiments of two-body problems that the new method can successfully give the level of the machine epsilon to the integration errors in all the orbital elements except the mean longitude at the epoch. In addition, the new method and the Fukushima’s method have the completely same numerical performance.

The seven integrals of the quasi-Keplerian motion of each body in an \( N \)-body problem are no longer conservative integrals and vary slowly with time, called as quasi-conserved quantities. This is an obstacle to the application of the new method. By simultaneously integrating the time evolution of the seven quasi-conserved quantities (i.e. the integral invariant relations of these quantities) and the usual equations of motion, the seven quantities have much higher accuracy than those obtained from the integrated position and velocity. These higher-precision quantities can determine the five slowly-varying orbital elements: the semi-major axis, the eccentricity, the inclination, the longitude of ascending node and the argument of pericentre. The eccentric anomaly is calculated similarly in the above way of treating the two-body problem. When these obtained values are substituted into the analytical solution of the two-body problem, an adjusted solution of each planet in the \( N \)-body problem can be available. In the expression forms, the corrected solution resembles the analytical solution of the two-body problem. This means the implementation of the new method. Numerical simulations of a six-body problem of the Sun, Jupiter, Saturn, Uranus, Neptune and Pluto show that the new method can significantly improve the accuracy of all the orbital elements and positions of each planet in two or three orders of magnitude, as compared with the case without correction. The new method and the Fukushima’s method are almost the same in the numerical performance in the present problem. They both need a negligibly small amount of additional computational cost.

The new method and the Fukushima’s linear transformation method are very perfect from the theory of celestial mechanics. In fact, they have been the best of the existing manifold correction methods in \( N \)-body simulations in the solar system (Fukushima 2003a, 2003b, 2003c, 2004; Wu et al. 2007; Ma et al. 2008a, 2008b). The new method is suitable for simulating the orbital motions of various objects, such as major and minor planets, satellites, and comets. Besides the Newtonian gravity interactions, various perturbations involving the \( J_2 \) perturbation and relativistic post-Newtonian terms (Quinn et al. 1991) are admissible. It is also applicable to systems of extrasolar planets in which the main sequence star is not the Sun. In particular, it can be applied to study the gravitational wave emission of three compact objects (Wardell 2002; Galaviz & Brügmann 2011).

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APPENDIX

THE LINEAR TRANSFORMATION METHOD OF FUKUSHIMA

The linear transformation method of Fukushima (2004) has two procedures as follows.

First, by a single-axis rotation, adjust the integrated velocity \( \mathbf{v}^* \) and position \( \mathbf{r}^* \) as the following form

\[
\mathbf{v}' = cv^* + s \times v^* + \frac{s \cdot v^*}{1 + c} s, \quad (41)
\]

\[
\mathbf{r}' = cr^* + s \times r^* + \frac{s \cdot r^*}{1 + c} s, \quad (42)
\]

where the vector \( s \) and the factor \( c \) are

\[
s = \frac{(r^* \times v^*) \times \mathbf{L}^*}{|r^* \times v^*||\mathbf{L}^*|}, \quad c = \sqrt{1 - s^2}. \quad (43)
\]
In fact, the adjusted solution \((\mathbf{r}', \mathbf{v}')\) is perpendicular to the angular momentum vector \(\mathbf{L}^*\).

Second, the above adjusted solution is used to construct a linear transformation

\[
\begin{align*}
\mathbf{r}^* &= s_r \mathbf{r}', \\
\mathbf{v}^* &= s_v (\mathbf{v}' - \alpha \mathbf{r}').
\end{align*}
\]  

(44)

(45)

The three factors \(s_r, s_v\), and \(\alpha\) are determined by the second adjusted solution \((\mathbf{r}^*, \mathbf{v}^*)\) rigorously satisfying the Kepler energy \(K^*\), the LRL integral \(P^*\) and the angular momentum vector \(\mathbf{L}^*\). They have explicit expressions

\[
\begin{align*}
s_r &= \frac{\mathbf{L}^* \cdot \mathbf{r}'}{\mathbf{F} \cdot \mathbf{r}'}, \\
\alpha &= \frac{\mathbf{F} \cdot \mathbf{v}'}{\mathbf{F} \cdot \mathbf{r}'}, \\
\mathbf{F} &= \mathbf{P}^* + \mu \left( \frac{\mathbf{r}'}{r'} \right),
\end{align*}
\]  

(46)

\[
\begin{align*}
s_v &= \sqrt{\frac{2K^* + 2\mu / (s_r \mathbf{r}')}{(v')^2 - 2\alpha (\mathbf{r}' \cdot \mathbf{v}') + \alpha^2 (r')^2}}.
\end{align*}
\]  

(47)

For a pure Kepler problem, \(K^*, \mathbf{L}^*\) and \(\mathbf{P}^*\) take their initial values \(K_0, \mathbf{L}_0\) and \(\mathbf{P}_0\), respectively. In this case, the seven integrals are always conserved with the aid of Equations (44) and (45). So are all the orbital elements except the mean longitude. These seven quantities \(K^*, \mathbf{L}^*\) and \(\mathbf{P}^*\) are given by Equations (32)-(34) for a perturbed Kepler problem or the quasi-Keplerian motion of each body in an \(N\)-body problem. They should become more accurate than those obtained from the integrated coordinates and velocities.

REFERENCES

Ascher, U. M., Chin, H., Petzold, L., & Reich, S. 1995, J. Mech. Struct. Machines, 23, 135

Avdyushev, E. A. 2003, Celest. Mech. Dyn. Astron., 87, 383

Bacchini, F., Ripperda, B., Chen, A. Y., et al. 2018a, ApJS, 237, 6. (arXiv 1801. 02378 [gr-qc])

Bacchini, F., Ripperda, B., Chen, A. Y., et al. 2018b, ApJS, 240, 4. (arXiv 1810. 00842 [astro-ph.HE])

Baumgarte, J. 1972, Comp. Math. Appl. Mech. Eng., 1, 1

Baumgarte, J. 1973, Celest. Mech., 5, 490

Chin, H. 1995, Stabilization Methods for Simulations of Constrained Multibody Dynamics, PhD Thesis, Institute of Applied Mathematics, University of British Columbia, Canada.

Chorin, A., Huges, T. J. R., Marsden, J. E., et al. 1978, Comm. Pure and Appl. Math., 31, 205

Feng, K. 1985, On difference schemes and symplectic geometry. In K. Feng, editor, Proceedings of the 1984 Beijing Symposium on Differential Geometry and Differential Equations, pages 42-58 (Science Press, Beijing China)

Fukushima, T. 2003a, AJ, 126, 1097

Fukushima, T. 2003b, AJ, 126, 2567

Fukushima, T. 2003c, AJ, 126, 3138

Fukushima, T. 2004, AJ, 127, 3638

Galaviz, P., & Brügmann, B. 2011, Phys. Rev. D, 83, 084013

Hairer E., Lubich C., & Wanner G., 1999, Geometric Numerical Integration, Springer-Verlag, Berlin

Hu, S., Wu, X., Huang, G., & Liang, E. 2019, ApJS, accepted

Huang, T.-Y., & Innanen, K. 1983, AJ, 88, 870

Li, D., & Wu, X. 2017, MNRAS, 469, 3031

Liu, L., Wu, X., Huang, G., et al. 2016, MNRAS, 459, 1968

Luo, J., Wu, X., Huang, G., et al. 2017, ApJ, 834, 64

Ma, D. Z., Wu, X., & Zhong, S. Y. 2008a, ApJ, 687, 1294

Ma, D. Z., Wu, X., & Zhu, J. F. 2008b, New Astron., 13, 216

Mei, L., Wu, X., & Liu, F. 2008b, New Astron., 13, 216

Mei, L., Ju, M., Wu, X., & Liu, S. 2013b, MNRAS, 435, 2246

Murray, C., & Dermott, S. 1999, Solar System Dynamics (Cambridge: Cambridge Univ. Press)
Table 1: CPU times (unit: second) of RK4, RK5 and their correction methods after a million years of integration of the six-body problem. The first line relates to RK4 and its correction methods: the new method M1 and the Fukushima’s linear transformation method M2. The third line is RK5 and its correction methods M1 and M2. Additional computation labor is small for M1 or M2.

| Method | RK4 | M1 | M2 |
|--------|-----|----|----|
| Time   | 8   | 11 | 11 |

| Method | RK5 | M1 | M2 |
|--------|-----|----|----|
| Time   | 12  | 16 | 16 |

Murrison, M. A. 1989, AJ, 97, 1496
Nacozy, P. E. 1971, Ap&SS, 14, 40
Pihajoki, P. 2015, Celest. Mech. Dyn. Astron., 121, 211
Quinlan, G. D., & Tremaine, S. 1990, AJ, 100, 1964
Quinn, T. R., Tremaine, S., & Duncan, M. 1991, AJ, 101, 2287
Ruth, R. D. 1983, IEEE Trans. Nucl. Sci., 30, 2669
Szebehely, V., & Bettis, D. G. 1971, Ap&SS, 13, 365
Wang, S. C., Wu, X., & Liu, F. Y. 2016, MNRAS, 463, 1352
Wang, S. C., Huang, G. Q., & Wu, X. 2018, AJ, 155, 67
Wardell, Z. E. 2002, MNRAS, 334, 149
Wisdom, J., & Holman, M. 1991, AJ, 102, 1528
Wisdom, J., & Hernandez, D. M. 2015, MNRAS, 453, 3015
Wu, X., Zhu, J. F., He, J. Z., & Zhang, H. 2006, Comput. Phys. Commun., 175, 15
Wu, X., Huang, T.-Y., Wan, X.-S., et al. 2007, AJ, 133, 2643
Zhang, F. 1996, Comput. Phys. Commun., 99, 53
Zhong, S. Y., Wu, X., Liu, S. Q., et al. 2010, Phys. Rev. D, 82, 124040

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Fig. 1.— Errors of six orbital elements for a pure Keplerian orbit. The adopted algorithms are RK4 and its correction methods: the new method M1 and the Fukushima’s linear transformation method M2. For M2, all the errors are reduced by 10 times. In fact, the two curves on the errors of M1 and M2 do basically coincide in each subfigure.
Table 2: Relative position errors of Jupiter, Saturn, Uranus, Neptune and Pluto at different times. The adopted algorithms are RK5 and the new correction method M1. The correction method M2 is not considered because it is almost the same as M1 in the numerical performance.

| time (yr) | method | Jupiter  | Saturn   | Uranus  | Neptune | Pluto   |
|-----------|--------|----------|----------|---------|---------|---------|
| 1         | RK5    | $2.8 \times 10^{-10}$ | $3.0 \times 10^{-12}$ | $8.7 \times 10^{-14}$ | $4.3 \times 10^{-14}$ | $3.8 \times 10^{-14}$ |
|           | M1     | $7.2 \times 10^{-14}$ | $1.1 \times 10^{-13}$ | $5.5 \times 10^{-14}$ | $1.0 \times 10^{-14}$ | $1.3 \times 10^{-14}$ |
| 10        | RK5    | $8.6 \times 10^{-9}$  | $4.4 \times 10^{-11}$ | $1.0 \times 10^{-11}$ | $6.0 \times 10^{-12}$ | $4.6 \times 10^{-12}$ |
|           | M1     | $2.1 \times 10^{-12}$ | $4.3 \times 10^{-12}$ | $1.4 \times 10^{-13}$ | $4.0 \times 10^{-13}$ | $6.0 \times 10^{-13}$ |
| 100       | RK5    | $1.9 \times 10^{-7}$  | $1.2 \times 10^{-9}$  | $9.5 \times 10^{-10}$ | $5.6 \times 10^{-10}$ | $1.2 \times 10^{-10}$ |
|           | M1     | $4.1 \times 10^{-11}$ | $2.2 \times 10^{-11}$ | $2.5 \times 10^{-11}$ | $3.5 \times 10^{-12}$ | $6.2 \times 10^{-11}$ |
| 1000      | RK5    | $1.9 \times 10^{-9}$  | $2.2 \times 10^{-10}$ | $4.9 \times 10^{-9}$  | $3.1 \times 10^{-8}$  | $5.5 \times 10^{-8}$  |
|           | M1     | $6.4 \times 10^{-9}$  | $1.0 \times 10^{-9}$  | $1.8 \times 10^{-10}$ | $3.3 \times 10^{-10}$ | $1.9 \times 10^{-9}$  |
| 10000     | RK5    | $1.6 \times 10^{-3}$  | $6.9 \times 10^{-5}$  | $8.8 \times 10^{-6}$  | $1.5 \times 10^{-6}$  | $4.0 \times 10^{-6}$  |
|           | M1     | $8.2 \times 10^{-7}$  | $3.2 \times 10^{-8}$  | $4.3 \times 10^{-9}$  | $2.9 \times 10^{-9}$  | $4.0 \times 10^{-8}$  |
| 100000    | RK5    | $4.3 \times 10^{-2}$  | $1.6 \times 10^{-3}$  | $3.5 \times 10^{-4}$  | $3.2 \times 10^{-4}$  | $2.3 \times 10^{-4}$  |
|           | M1     | $1.7 \times 10^{-5}$  | $3.4 \times 10^{-6}$  | $1.0 \times 10^{-7}$  | $1.5 \times 10^{-8}$  | $3.0 \times 10^{-7}$  |
| 1000000   | RK5    | $2.8 \times 10^{-4}$  | $1.1 \times 10^{-5}$  | $4.1 \times 10^{-4}$  | $7.1 \times 10^{-4}$  | $3.7 \times 10^{-4}$  |
|           | M1     | $8.7 \times 10^{-4}$  | $1.1 \times 10^{-4}$  | $9.1 \times 10^{-6}$  | $6.8 \times 10^{-6}$  | $4.3 \times 10^{-5}$  |

Fig. 2.— Same as Figure 1 but the relative position errors instead of the errors of the orbital elements.
Fig. 3.— Kepler energy errors of Jupiter in the three-body problem of the Sun, Jupiter and Saturn in four cases (labeled as C1, C2, C3 and C4) of calculation of the eccentric anomaly. The errors for Case 1 are enlarged by 10 times. Case 1 with Case 2 should have been a naturally prior choice but leads to the worst performance. Case 4 shows the best performance and is just what we want.
Fig. 4.— Errors of all orbital elements of Jupiter in the six-body problem of the Sun, Jupiter, Saturn, Uranus, Neptune and Pluto. The algorithms use RK4 and its correction methods M1 and M2.
Fig. 5.— Same as Figure 4, but the basic integrator is RK5.
Fig. 6.— Relative position errors of Jupiter in the six-body problem. The basic integrators are RK4 in panel (a) and RK5 in panel (b).

Fig. 7.— Errors of some orbital elements for individual planets in the six-body problem. (a) The semi-major axis $a$ of Pluto, (b) the eccentricity $e$ of Uranus, (c) the inclination $I$ of Neptune, and (d) the longitude of ascending node $\Omega$ of Saturn. The algorithms use RK5 and its correction method M1. The correction method M2 is not plotted because it is not typically different from M1 in the numerical performance.
Fig. 8.— Relative position errors of individual planets in the six-body problem. (a) Saturn, (b) Uranus, (c) Neptune and (d) Pluto. The algorithms use RK5 and its correction method M1.