Self-energy of Heavy Quark

M. A. Ivanov* and T. Mizutani

Department of Physics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

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Abstract

We demonstrate that to calculate the self-energy of a heavy quark in the heavy quark limit (or the heavy fermion limit in what is called the Baryon Chiral Perturbation Theory), the use of standard dimensional regularization provides only the quantum limit: opposite to the heavy quark (or classical) limit that one wishes to obtain. We thus devised a standard ultraviolet/infrared regularization procedure in calculating the one- and two-loop contributions to the heavy quark self-energy in this limit. Then the one-loop result is shown to provide the standard classical Coulomb self-energy of a static colour source that is linearly proportional to the ultraviolet cutoff $\Lambda$. All the two-loop contributions are found to be proportional to $\Lambda \ln(\Lambda/\lambda)$ where $\lambda$ is the infrared cutoff. Often only the contribution from the bubble (light quarks, gluon and ghost) insertion to the gluon propagator has been considered as the $O(\alpha_s)$ correction to the Coulomb energy to this order. Our result shows that other contributions are of the same magnitude, thus have to be taken into account.

*Permanent address: Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna (Moscow region), Russia
1 Introduction

Recently, many discussions have been devoted to the definition of the heavy quark mass. This quantity is the basic parameter in the heavy quark physics. Generally (not only for the heavy quarks in the heavy quark limit in which we are interested, in the following) one of the most popular choices within the perturbative scheme is the pole mass. In reference [1] it has been shown that the pole mass is gauge independent and infrared finite at the two-loop level. It was checked at the three-loop level in [2] where the relation between the mass of the modified minimal subtraction scheme and the scheme-independent pole mass was also obtained. Later authors of work [3] argue that precise definition of the pole mass in the heavy quark limit may not be given once nonperturbative effects are included (a more analytical approach leading to the identical conclusion has been given independently in [4]). To demonstrate that, they adopted the standard effective running constant (as a typical non-perturbative quantity) obtained from the summation of leading logarithms to calculate the Coulomb energy of a static (or infinitely heavy) colour source. This has produced an auxiliary singularity for small momentum (the so-called, infrared renormalon) which, they claimed, leads to the uncertainty of order $\Lambda_{\text{QCD}}/m_Q$ in the definition of the pole mass. An important observation which served as one of the bases to this demonstrative calculation is that the widely used dimensional regularization may not be used to analyze self-energy diagrams in the heavy quark limit, in particular, for separating the infrared and ultraviolet domains. The best example, as we shall also present in the following section, is that this regularization method leaves out the linear divergence that defines the self-energy of a static quark in the lowest order: the Coulomb energy. Also it should be emphasized here that the Heavy Quark Limit does make sense only after introducing a scale parameter to the theory with which the quark mass may be compared, and there is something awkward about it in the dimensional regularization method.

It may be useful to point out here that a quite analogous situation to the heavy quark limit can be found in the problem of the classical limit $\hbar \to 0$ for the self-energy of an electron in QED [5]. It was found that in the limit $\hbar \to 0$, or more precisely for the electron Compton wave length $\hbar/mc \ll r_0$, where parameter $r_0$ regularizes small distances, the energy of the Coulomb field by a change of radius $r_0$ is reproduced in the second order of
perturbative theory whereas the higher order contributions were shown to vanish in this limit.

Here we first study the quantum and classical limits for the lowest order contributions to the quark self-energy and quark-gluon vertex, and show that the standard dimensional regularization method only reproduces the quantum limit. Since one may clearly identify the classical limit to be equal to the currently fashionable heavy quark limit, we could conclude that the dimensional regularization method is not suited for studying this case. Then in the following section we calculate the two-loop contributions to the quark self-energy in the heavy quark limit, exploiting the conventional ultraviolet and infrared regularization method. We have found that each contribution to this order is all proportional to $\Lambda \ln (\Lambda/\lambda)$ where $\Lambda$ and $\lambda$ are the ultraviolet and infrared cutoffs, respectively and that, unlike in QED, these contributions do not mutually cancel out. We point out that the gluonic bubble diagrams (Fig.4d) (implicitly) taken often as the major contribution in some processes is of the same magnitude as those from vertex and quark self-energy diagrams (Figs. 4a-4c) for the self-energy in the heavy quark limit. In the last section some comments will be given regarding the alleged infrared renormalon and the Baryon (or heavy fermion) Chiral Perturbation Theory.

2 One-loop diagram

The self-energy of quark with mass $m$ in the second order of perturbative theory is defined by the diagram in Fig.1,

$$\delta m_2 = \Sigma_2(m) = C_F g_s^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(-k^2)} \gamma^\mu \left( \frac{1}{m-k-p} \gamma^\mu \right) \Big|_{p=m}$$ (1)

where $C_F = (N_c^2 - 1)/2N_c$, for a gauge group $SU(N_c)$. First, we note that this expression coincides with the electromagnetic self-energy except for the trivial factor $C_F g_s^2$. While the integral has no infrared divergence, it develops the linear ultraviolet divergence. We are interested in this quantity in the heavy quark limit: $m \to \infty$. As was mentioned in [3], the dimensional regularization cannot serve this purpose. To make this point more clear, we calculate the quark self-energy by way of introducing an ultraviolet regularization,
\[
\frac{1}{(-k^2)} \rightarrow F(-k^2/\Lambda^2) \quad (2)
\]

Here, the function \( F(-k^2/\Lambda^2) \) is assumed to decrease rapidly in the Euclidean region \( k^2_E = -k^2 \rightarrow +\infty \) providing the ultraviolet convergence of the Feynman integrals. Now, the heavy quark limit (or classical limit) means the case where \( m \gg \Lambda \) in comparison with the quantum limit: \( m \ll \Lambda \). The both limits may be realized using the Mellin representation for the form factor \( F(-k^2/\Lambda^2) \) under calculation of the integral in eq. (1).

We can write in the Euclidean region

\[
F(k^2_E) = \sum_{n=0}^{\infty} c_n(-)^n (k^2_E)^n = \frac{1}{2i} \int_{-\delta+i\infty}^{-\delta-i\infty} \frac{d\zeta}{\sin \pi \zeta} c(\zeta)(k^2_E)^\zeta, \quad (3)
\]

where \( c(n) \equiv c_n \) for \( n \) being positive integers, and \( 0 < \delta < 0.5 \).

Finally, we have

\[
\delta m^2 = C_F g_s^2 \frac{1}{2i} \int_{-\delta+i\infty}^{0} \frac{d\zeta}{\sin \pi \zeta} c(\zeta) \frac{1}{\Lambda^{2\zeta}} \int \frac{d^4k}{(2\pi)^4 i} \frac{4m - 2(k+p)}{(-k^2)^{1-\zeta}(m^2 - (k+p)^2)}
\]

\[
= C_F g_s^2 m \frac{1}{2i} \int_{-\delta+i\infty}^{0} \frac{d\zeta}{\sin \pi \zeta} c(\zeta) \frac{1}{\Lambda^{2\zeta}} \frac{\Gamma(2 - \zeta)}{\Gamma(1 - \zeta)} \int d\alpha (1 - \alpha)^{-\zeta} \int \frac{d^4k}{(2\pi)^4 i} \frac{4 - 2(1 - \alpha)}{(\alpha^2 m^2 - k^2)^{-\zeta}}
\]

\[
= 3\alpha_s \sqrt{2\pi} C_F m \frac{1}{2i} \int_{-\delta+i\infty}^{-0.5} \frac{d\zeta}{\sin \pi \zeta} c(\zeta) \frac{\Gamma(1 + 2\zeta)}{(-\zeta)} \frac{(m^2)^\zeta}{\Lambda^{2\zeta}} \frac{\Gamma(1 - \zeta)(1 + \zeta)}{\Gamma(3 + \zeta)}. \quad (4)
\]

To get the leading term in \( m/\Lambda \) in the quantum limit, we move the integration counter to the right and take into account of the first double pole at \( \zeta = 0 \). Assuming that \( F(0) = 1 \), we have

\[
\delta m_{2}^{qu} = \frac{3\alpha_s}{4\pi} C_F m \left\{ \ln \frac{\Lambda^2}{m^2} + O(1) \right\}. \quad (5)
\]

To get the leading term in \( \Lambda/m \) in the classical limit, we move the integration counter to the left and take into account the first pole at \( \zeta = -0.5 \). From the standard relation between the Mellin and its inverse transformations
\[ c(-0.5) = 2 \int_0^\infty dt F(t^2), \]

we have

\[ \delta m^2_{cl} = \frac{\alpha_s}{2\pi} C_F \Lambda c(-0.5) = \frac{\alpha_s}{\pi} C_F \Lambda \int_0^\infty dt F(t^2). \] (6)

If we compare the above result with the calculation of the expression in Eq.(1) devising the dimensional regularization,

\[ \delta m_2 = \frac{3\alpha_s}{4\pi} C_F m \left\{ N_\epsilon + \frac{5}{3} - \ln \frac{m^2}{\mu^2} \right\} \] (7)

where \( N_\epsilon = 1/\epsilon + \ln 4\pi - \gamma \) (\( \gamma \): the Euler-Mascheroni constant) and \( \mu \) is the subtraction point, one can see that it only reproduces the quantum limit Eq.(5) for the quark self-energy by identifying \( \mu \) as the ultraviolet cutoff. Thus, the naive dimensional regularization does not allow to see the heavy quark limit.

Ordinarily, studies of heavy quark limit may be carried out by using the expansion of quark propagator in the inverse quark mass:

\[
S(\not{p} + \not{k})|_{p=m} = i \frac{M_0 + k/2m}{[1 + i/2m + f(k^2/m^2)][k + 2m \bar{f}(k^2/m^2)]}
\]

\[
= \frac{iM_0}{k_4 + i\epsilon} + O\left(\frac{1}{m}\right) \] (8)

Here, \( k_4 = -ik_0, M_0 = (1/2)(I + \gamma^0), \) and \( f(x) = (1/2)(\sqrt{1+x} - 1). \) It is readily seen that the Dirac spinor \( u(\vec{p}) \) for quark may be chosen as

\[
u(\vec{p}) = \begin{pmatrix} I \\ 0 \end{pmatrix}
\]

so that \( \bar{u}(\vec{p}) u(\vec{p}) = \bar{u}(\vec{p}) \gamma^0 u(\vec{p}) = 1, \) and \( \bar{u}(\vec{p}) \gamma u(\vec{p}) = 0. \)

In the second order: Eq.(1), upon introducing the ultraviolet regulator to the otherwise linearly divergent expression, the leading contribution reads

\[ \delta m^2_{cl} = \Sigma_2(m) = 2\pi \alpha_s C_F \Lambda \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\vec{k}^2} \] (9)
which obviously coincides with the Eq.(6). One can see that this expression is the classical Coulomb energy of a static color source. To give more physical meaning to Eq.(9), one may introduce the charge density defined as

$$\rho(\vec{r}^2) = \int \frac{d^3 \vec{k}}{4\pi^3} \frac{e^{i\vec{k}\cdot\vec{r}}}{\sqrt{F(k^2)}} \tag{10}$$

supposing that $F(k^2) > 0$, and the electron (quark) "radius" $r_0 = 1/\Lambda$.

Then, we have

$$\delta m_{2}^{cl} = \frac{g_{s}^{2}}{2r_0} C_{F} \int d\vec{r}_1 d\vec{r}_2 \rho(\vec{r}_1^2) \left[ \frac{1}{4\pi} \frac{1}{|\vec{r}_1 - \vec{r}_2|} \right] \rho(\vec{r}_2^2) \tag{11}$$
in accordance with the classical result.

As was mentioned above, there is no infrared divergence in the second order, but it plays the crucial role in higher orders in perturbative expansion.

To understand its role, we compute the quark-gluon vertex induced by both quark-gluon and self-gluon interactions: Figs.2-3. For simplicity we omit all the common coefficients and color factors.

(a) Quark-gluon interaction (diagram in Fig.2):

By introducing the gluon mass $\lambda$ serving to regularize the infrared singularity we get

$$\Lambda_{1}^{\nu}(p, p) = \frac{1}{2\pi i} \frac{1}{\lambda^2 - k^2} \gamma^{\mu}(m + \vec{k} + \vec{p}) \gamma^{\nu}(m + \vec{k} + \vec{p}) \gamma^{\mu} \bigg|_{\gamma = m} \tag{12}$$

Using the form factor as in Eq. (2) one obtains

$$\Lambda_{1}^{\nu}(p, p) = \begin{cases} 
\gamma^{\nu} \frac{1}{(4\pi)^2} \left\{ 2 \ln \frac{\Lambda^2}{\lambda^2} - \ln \frac{\Lambda^2}{m^2} + O(1) \right\} & \text{if } \Lambda \gg m \\
\gamma^{\nu} \frac{1}{(4\pi)^2} \left\{ 2 \ln \frac{\Lambda^2}{\lambda^2} + O(1) \right\} & \text{if } \Lambda \ll m
\end{cases}$$

One can see that the infrared singularity defines the behavior of the vertex in the heavy quark limit. Again result obtained from the dimensional regularization for ultraviolet divergence corresponds only to the quantum case:

$$\Lambda_{1}^{\nu}(p, p) = \gamma^{\nu} \frac{1}{(4\pi)^2} \left\{ 2 \ln \frac{\Lambda^2}{\lambda^2} - N_{c} + \ln \frac{m^2}{\mu^2} + O(1) \right\}. \tag{13}$$
Note that if we do not introduce the infrared cutoff within the dimensional regularization scheme, we find the auxiliary \(1/\epsilon\) pole in the expression for the vertex in Eq.(12) leading to

\[
\Lambda^\nu_1(p, p) = \gamma^\nu \frac{1}{(4\pi)^2} \left\{ -2N^\epsilon_{IR} - N^\epsilon_{UV} + 3\ln \frac{m^2}{\mu^2} + O(1) \right\}.
\]

(b) Self-gluon interaction (diagram in Fig.3):

This contribution may be written as

\[
\Lambda^\nu_2(p, p) = \int \frac{d^4k}{(2\pi)^4 i} \frac{1}{(\lambda^2 - k^2)^2} \left\{ -2k^2\gamma^\nu - 4k^\nu k \frac{\gamma^\nu - \gamma^\nu}{\gamma^\nu - \gamma^\nu}\right\} |_{\gamma=m} \tag{14}
\]

This integral turns out to be free from infrared singularity. Using the ultraviolet form factor gives

\[
\Lambda^\nu_2(p, p) = \begin{cases} 
\gamma^\nu \frac{1}{(4\pi)^2} \left\{ 3\ln \frac{\Lambda^2}{m^2} + O(1) \right\} & \text{if } \Lambda \gg m \\
O(\frac{\Lambda}{m}) & \text{if } \Lambda \ll m
\end{cases}
\]

It is readily seen that this diagram gives no contribution in the heavy quark limit. Again result obtained from the dimensional regularization for ultraviolet divergence corresponds only to the quantum case:

\[
\Lambda^\nu_2(p, p) = \gamma^\nu \frac{1}{(4\pi)^2} \left\{ 3N_\epsilon - 3\ln \frac{m^2}{\mu^2} + O(1) \right\}.
\]  \tag{15}

3 Two-loop Diagrams in the Heavy Quark Limit

The contributions to the quark self-energy from the two-loops (or fourth order in coupling constant) are defined by the diagrams in Fig.4. As a consequence of our discussion in the previous section, we shall not adopt the dimensional regularization, but merely use the ultraviolet form factor Eq.(2) together with the infrared regularization of the gluon propagator in the following.
(a) Diagram in Fig. 4a.

With $\Lambda$ and $\lambda$ being the ultraviolet and infrared cutoff masses, as in the previous section, one finds

$$\delta m_4^{(a)} = \Sigma_4^{(a)}(m) = C_4^{(a)} g_s^4 \Lambda I_4^{(a)}(\kappa^2)$$  \hspace{1cm} (16)

where $\kappa^2 \equiv \lambda^2/\Lambda^2$; $C_4^{(a)} = 1$ (for QED), and $t^a t^b t^c = -C_F/2N_c$ (for QCD). The integral $I_4^{(a)}(\kappa^2)$ is equal to

$$I_4^{(a)}(\kappa^2) = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{F(k_1^2)}{k_1^2} \frac{F(k_2^2)}{k_2^2} \frac{1}{k_1^2 + \kappa^2} \frac{1}{k_2^2 + \kappa^2} \frac{1}{(k_{14} + i\epsilon)(k_{14} + k_{24} + i\epsilon)(k_{24} + i\epsilon)}$$ \hspace{1cm} (17)

Here, to be definite, the integration is over the dimensionless Euclidean variables denoted as $(E)$ under the integration sign. The calculation of this integral is given in Appendix.

We have

$$\delta m_4^{(a)} = 2\pi \alpha_s C_F \Lambda \int d^3k F(\vec{k}^2) \left[ -\frac{\alpha_s}{4\pi N_c} \ln(1/\kappa^2) \right].$$ \hspace{1cm} (18)

(b) Diagram in Fig. 4b.

Here one obtains

$$\delta m_4^{(b)} = \Sigma_4^{(b)}(m) = C_4^{(b)} g_s^4 \Lambda I_4^{(b)}(\kappa^2)$$ \hspace{1cm} (19)

where $C_4^{(b)} = -if^{abc} t^a t^b t^c = (N_c/2)C_F$ (of course there is no QED counterpart to this diagram). The integral $I_4^{(b)}(\kappa^2)$ is given as

$$I_4^{(b)}(\kappa^2) = \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{F(k_1^2)}{k_1^2} \frac{F(k_2^2)}{k_2^2} \frac{F((k_1 - k_2)^2)}{k_1^2 + \kappa^2} \frac{1}{k_2^2 + \kappa^2} \frac{1}{(k_{14} + k_{24} + i\epsilon)(k_{24} + i\epsilon)}$$ \hspace{1cm} (20)

with the numerator $N_b$ equal to

$$N_b = (k_1 - 2 k_2) M_0 \gamma^\nu M_0 \gamma^\nu + \gamma^\nu M_0(k_1 + k_2) M_0 \gamma^\nu + \gamma^\nu M_0 \gamma^\nu M_0(k_2 - 2 k_1).$$

7
Since the integration is over Euclidean variables here, $k = \gamma^0 i k_4 - \vec{\gamma} \vec{k}$. With this it is easy to check that $\bar{u}(\vec{p}) N_b u(\vec{p}) = 0$, which means that the contribution from the diagram in the Fig.4b behaves as $\Lambda/m$ for large $m$, thus vanishes in the heavy quark limit.

(c) Diagram in Fig. 4c.

This contribution may be written as

$$
\delta m_4^{(c)}(m) = C_4^{(c)} g_4^4 \Lambda I_4^{(c)}(\kappa^2)
$$

where $C_4^{(c)} = 1$ (for QED), and $t^a t^a t^b t^b = C_F^2$ (for QCD). The integral $I_4^{(c)}(\kappa^2)$ is equal to

$$
I_4^{(c)}(\kappa^2) = i \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{F(k_1^2)}{k_1^4} \frac{F(k_2^2)}{k_2^4} \left\{ \frac{1}{(k_1^4 + i\epsilon)(k_1^4 + k_2^4 + i\epsilon)} \right\} = -I_4^{(a)}(\kappa^2)
$$

Here, we have taken into account the mass renormalization in the second order (the dark disk in Fig. 4c). From the above result, it is easy to see that in QED the contributions coming from Fig. 4a and Fig. 4c cancel. This is a crucial point in solving the problem of classical limit $\hbar \to 0$ in QED \cite{5} since the contribution to the self-energy of a massive fermion from the photon vacuum polarization is proportional to

$$
\Pi_m^{\mu\nu}(k) \propto (g^{\mu\nu} k^2 - k^\mu k^\nu) \ln \left[ 1 + \frac{k^2}{m^2} \right]
$$

and vanishes in the infinite mass limit: $m \to \infty$.

For QCD one finds

$$
\delta m_4^{(c)} = 2\pi\alpha_s C_F \Lambda \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\vec{k}^2} \left[ -\frac{\alpha_s}{4\pi} 2C_F \ln(1/\kappa^2) \right].
$$

(d) Diagram in Fig. 4d.

The contribution from the diagrams in Fig. 4d, involving the vacuum polarization by massless quarks, gluons, and ghost is written as

$$
\delta m_4^{(d)} = \Sigma_4^{(d)}(m) = 2\pi\alpha_s C_F \Lambda \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\vec{k}^2} \Pi^{\text{ren}}(-\vec{k}^2)
$$
In the above expression, the renormalized vacuum polarization reads (see Appendix for a somewhat detailed calculation)

\[
\Pi_{\text{ren}}(\vec{k}^2) = -\frac{\alpha_s}{4\pi} \int_0^1 d\alpha \ln \left[ 1 + \alpha (1 - \alpha) \frac{\vec{k}^2}{\kappa^2} \right] \left\{ 2N_c \left[ 1 + 4\alpha (1 - \alpha) \right] - 4n_f \alpha (1 - \alpha) \right\} 
\]

\[
= -\frac{\alpha_s}{4\pi} \tilde{b} \ln \frac{\vec{k}^2}{\kappa^2} + O(1) 
\]

upon expanding in small parameter \(\kappa^2\). Here the appearance of the coefficient \(\tilde{b} = (10N_c - 2n_f)/3\) is due to the use of the Feynman gauge.

Summing up all the contributions from the one- and two-loops, one finds

\[
\delta m = 2\pi \alpha_s C_F \Lambda \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\vec{k}^2} \left\{ 1 - \frac{\alpha_s}{4\pi} \left[ N_c \ln(1/\kappa^2) + \frac{\tilde{b}}{N_c} \ln(\vec{k}^2/\kappa^2) \right] \right\} 
\]

\[
\to 2\pi \alpha_s C_F \Lambda \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\vec{k}^2} \left\{ 1 - \frac{\alpha_s}{4\pi} \ln(\Lambda^2/\lambda^2) N_c \left[ \frac{2}{3} + \frac{b}{N_c} \right] \right\}. 
\]

Here, we express the final result in term of the first coefficient in the Gell-Mann-Low function \(b = (11N_c - 2n_f)/3\). By adopting the number (flavor) of light quarks to be three: \(n_f = 3\), we can see for \(N_c = 3\) that the \(O(\alpha_s)\) corrections to the leading static Coulomb self-energy coming from the bubble insertions to the gluon propagator is of the same order of magnitude as those from other diagrams: Figs.4a-c, so the latter contribution should be included.

In ref. [3] the self-energy of the heavy quark in the heavy quark limit was calculated by introducing the running constant \(\alpha_s(\vec{k}^2)\) in the Coulomb energy expression: Eq.(9), and by introducing the explicit integration cutoff in place of the ultraviolet form factor \(F(-k^2/\Lambda^2)\) (Fig.5),

\[
\delta m = \frac{8\pi}{3} \int_{|\vec{k}|<\mu_0} \frac{d^3 \vec{k}}{(2\pi)^3} \frac{\alpha_s(\vec{k}^2)}{\vec{k}^2} 
\]

\[
\text{with} \quad \alpha_s(\vec{k}^2) = \frac{\alpha_s(\mu_0^2)}{1 - (\alpha_s(\mu_0^2)b/4\pi) \ln(\mu_0^2/\vec{k}^2)}. 
\]
According to Ref. [3], the zero of the denominator in the above expression defines the position of the infrared renormalon, leading to the uncertainty in the pole mass. Here, we are interested in the $O(\alpha_s^2(\mu_0^2))$ contribution from Eq. (27),

$$\delta m_4 = \left(\frac{\alpha_s}{\pi}\right)^2 \frac{b}{3} \mu_0 \int_0^1 dt \ln \frac{1}{t^2} = \left(\frac{\alpha_s}{\pi}\right)^2 \frac{2b}{3} \mu_0.$$  

(29)

Here one might regard $\mu_0$ as corresponding to our $\Lambda$ (in this respect one could argue that the value of the integration cutoff in Eq. (27) should be different from the scale $\mu_0$ introduced in Eq. (28). This would introduce an additional contribution of a logarithmic type to Eq. (29), which would however not prohibit the appearance of the alleged renormalon). This result may be compared with the part of the expression in Eq. (26) which is proportional to $b/N_c$. Since we have not calculated the higher loop contributions, we cannot draw any definite conclusion. However, in view of the difference in our result: Eq. (26) and that of ref. [3], it might be possible that due to the contributions other than the bubble insertions in the gluon propagator the characteristics (position, etc.) of the alleged infrared renormalon in the heavy quark self-energy would be different from the original one, or even the renormalon might not be there.

4 Discussion

To summarize our present note, first we have demonstrated that the popular dimensional regularization method yeilds only the quantum limit, thus unsuited for the calculation of the self-energy of the heavy quark in the heavy quark (or the classical) limit which is just the opposite of the quantum limit. Thus we employed next the conventional method for ultraviolet and infrared regularizations to calculate the one- and two-loop contributions to the self-energy of the heavy quark in the heavy quark limit. The one-loop result is found as the standard Coulomb-like self-energy of a static colour source whose radius is characterized by the ultraviolet regularization mass $\Lambda$ of the theory. Then all the two-loop contributions have turned out to be proportional to $\Lambda \ln(\Lambda/\lambda)$ with the infrared cutoff $\lambda$, while the corresponding contributions in QED internally cancel out to vanish. Of those two-loop contributions, only the bubble insertion to the gluon propagator in the lowest order (Coulomb) contribution (Fig. 4d) has been discussed often. We have found
that other contributions are of the same size and have to be taken into account.

From our study we have reached a couple of observations (possibly, speculations). The first is that since the contributions depicted in Figs.4a-c are not very different in magnitude from the bubble insertions to the gluon propagator at the two-loop level, the infrared renormalon in the heavy quark self energy resulting from the simple bubble chain summation, due to the adoption of the running coupling constant, might be modified regarding its position, etc. or might even be absent once a due consideration of the remaining higher-loop effects are incorporated.

The second observation is concerned with the dimensional regularization. Our assertion regarding the inappropriateness of in this method in the context of the heavy quark limit should be valid also in what is called the heavy baryon chiral perturbation theory (HBCHPT), (see for example a review [6] and references therein) which combines the ordinary Chiral Perturbation theory (CHPT) for the octet pseudo scalar mesons coupled with baryonic (nucleons, deltas, etc.) degrees of freedom. Here very often the Lagrangian (or the S-matrix) for the system is expanded in the inverse power of the baryon mass(es), exactly like in the heavy quark expansion realized in the heavy quark limit. The the meson loop integrals arising from the meson-baryon interactions are conventionally carried out using the \textit{dimensional regularization}. Now according to our study in this note, this method is only compatible with the opposite quantum limit ! Thus it is our feeling that the existing results in HBCHPT should be examined again carefully from the present context.

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Appendix

(a) The calculation of the integral \( I_4^{(a)}(\kappa^2) \).

Using the obvious equality in Eq. (17)

\[
\frac{1}{k_{14} + k_{24} + i\epsilon k_{24} + i\epsilon} = \left( \frac{1}{k_{24} + i\epsilon} - \frac{1}{k_{14} + k_{24} + i\epsilon} \right) \frac{1}{k_{14}}
\]

and the well-known formula

\[
\frac{1}{k_{4} + i\epsilon} = \frac{P}{k_{4}} - i\pi\delta(k_{4})
\]

one finds

\[
I_4^{(a)}(\kappa^2) = \frac{1}{4\pi} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\vec{k}^2} J(\kappa^2)
\]

where

\[
J(\kappa^2) = 2 \int_0^\infty \frac{f(0) - f(t^2)}{t^2} = -4 \int_0^\infty dt f'(t^2).
\]

We define the function \( f(t^2) \) as

\[
f(t^2) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{F(t^2 + \vec{k}^2)}{t^2 + \vec{k}^2 + \kappa^2}.
\]

Using the Mellin representation: Eq.(3) for the form factor \( F(k^2) \) and integrating over \( \vec{k} \), one can get

\[
J(\kappa^2) = \frac{\sqrt{\pi}}{2\pi^2} \frac{1}{2i} \int_{-\delta+i\infty}^{-\delta-i\infty} \frac{d\zeta \sin \pi\zeta}{c(\zeta)} \frac{\Gamma(1/2 - \zeta)}{\Gamma(-\zeta)} \int_0^1 d\alpha (1 - \alpha)^{-\zeta-1} \int_0^\infty dt [t^2 + \alpha \kappa^2]^{-\zeta-1/2}
\]

\[
= -\frac{1}{4\pi} \frac{1}{2i} \int_{-\delta+i\infty}^{-\delta-i\infty} \frac{d\zeta \sin \pi\zeta}{c(\zeta)(\kappa^2)^\zeta} \frac{\Gamma(1 + \zeta)\Gamma(1 - \zeta)}{\zeta} \to \frac{1}{4\pi} \ln \frac{1}{\kappa^2},
\]

for small \( \kappa \). Finally, we have

\[
I_4^{(a)}(\kappa^2) = \frac{1}{(4\pi)^2} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{F(\vec{k}^2)}{\kappa^2} \ln \frac{1}{\kappa^2} + O(1).
\]
(a) The calculation of $\Pi_{\text{ren}}^{(q^2)}$.

As an example of the calculation of diagrams in Fig.4, we here present the details of the calculation of the vacuum polarization $\Pi_{\text{ren}}^{(q^2)}$ by massless gluons, ghosts, and quarks (see, insertions to the gluon propagator in Fig.4d). We proceed in adopting the Feynman gauge. We shall use the dimensional regularization in the intermediate calculations, but the final result will not depend on the choice of the regularization.

We define the S-matrix element describing the vacuum polarization as

$$S = \frac{i}{2} \int dx_1 dx_2 A_{\nu_1}^c(x_1) \Pi_{\nu_1 \nu_2}^{c_1 c_2}(x_1 - x_2) A_{\nu_2}^{c_2}(x_2)$$

with its Fourier transform defined as

$$\Pi_{\nu_1 \nu_2}^{c_1 c_2}(q) = \int dx e^{-i q x} \Pi_{\nu_1 \nu_2}^{c_1 c_2}(x).$$

For simplicity we do not introduce new notation for this value.

The contribution coming from the gluons and ghosts may be written in the form

$$\left[\Pi_{\nu_1 \nu_2}^{c_1 c_2}(q)\right]_{\text{gluon+ghost}} = g_s^2 f^{abc} f^{abc} \int_0^1 d\alpha \frac{1}{\mu^2} \int \frac{d^D k}{(2\pi)^D i} \frac{N_{\text{gluon}} + N_{\text{ghost}}}{\left[-k^2 - \alpha(1 - \alpha) q^2\right]^2}$$

where

$$N_{\text{gluon}} = g^{\nu_1 \nu_2} (D - 1) \left\{ \frac{6}{D} k^2 + [1 - 4\alpha (1 - \alpha)]q^2 \right\}$$

$$+ [q^2 g^{\nu_1 \nu_2} - q^{\nu_1} q^{\nu_2}] \left[ 6 - D + (4D - 6)\alpha (1 - \alpha) \right],$$

$$N_{\text{ghost}} = g^{\nu_1 \nu_2} \left\{ -\frac{2}{D} k^2 + 2\alpha (1 - \alpha)q^2 \right\} - 2\alpha (1 - \alpha) [q^2 g^{\nu_1 \nu_2} - q^{\nu_1} q^{\nu_2}].$$

First, we show that the apparent non-gauge invariant term proportional to $g^{\nu_1 \nu_2}$ does vanish. For this objective we use the identity:

$$\int d^D k \frac{ak^2 + bq^2}{[-k^2 - \alpha \beta q^2]^2} = -\frac{1}{2\alpha \beta} \int d^D k \frac{D\alpha \beta a + (2 - D)b}{[-k^2 - \alpha \beta q^2]}$$
which may be proved by using the following expression,

\[
\frac{k^2}{[-k^2 - \alpha \beta q^2]^2} = k^2 \frac{d}{dk^2} \left[ \frac{1}{-[k^2 - \alpha \beta q^2]} \right] = \frac{1}{2} k^\mu \frac{d}{dk^\mu} \left[ \frac{1}{-[k^2 - \alpha \beta q^2]} \right],
\]

and integrating it by parts. Then the coefficient of the apparently non-gauge invariant contribution becomes

\[
\int_0^1 \frac{d\alpha}{\alpha(1-\alpha)} \int \frac{d^Dk}{[-k^2 - \alpha(1-\alpha)q^2]^2} \left[ -1 + 4 \frac{(D-1)}{(D-2)} \alpha(1-\alpha) \right]
= \int d^Dk \frac{1}{[-k^2 - q^2]^2} \left[ -1 + 4 \frac{(D-1)}{(D-2)} \alpha(1-\alpha) \right]
= \int d^Dk \frac{1}{[-k^2 - q^2]^2} \left\{ -\frac{\Gamma^2(D/2 - 1)}{\Gamma(D-2)} + 4 \frac{(D-1)}{(D-2)} \frac{\Gamma^2(D/2)}{\Gamma(D)} \right\} \equiv 0.
\]

Finally one finds the gauge invariant form for the vacuum polarization including both gluons and ghosts

\[
\left[ \Pi_{\mu_1 \nu_1 \mu_2 \nu_2}^\text{gluon+ghost} (q) \right] = f^{abc} f^{abc_2} [q^2 g_{\mu_1 \nu_1} - q_{\nu_1} q_{\mu_2}] \Pi_{\text{gluon+ghost}} (q^2)
\]

with \( \Pi_{\text{gluon+ghost}} \) being equal to

\[
\Pi_{\text{gluon+ghost}} (q^2) = \int_0^1 d\alpha \mu^2 \int \frac{d^Dk}{(2\pi)^D i} \left[ \frac{D - 6 - 4(D-2) \alpha(1-\alpha)}{[-k^2 - \alpha(1-\alpha)q^2]^2} \right]
\]

Note that this expression has both infrared and ultraviolet logarithmic divergences in four dimension. To regularize the infrared divergence we introduce the gluon (ghost) mass \( \lambda \) to the denominator. Then the renormalized vacuum polarization is written as

\[
\Pi_{\text{gluon+ghost}}^\text{ren} (q^2) = \Pi_{\text{gluon+ghost}} (q^2) - \Pi_{\text{gluon+ghost}} (0)
= -\frac{\alpha_s}{4\pi} \int_0^1 d\alpha \ln[1 - \alpha(1-\alpha) \frac{q^2}{\lambda^2}] \left\{ 2N_c[1 + 4\alpha(1-\alpha)] \right\}
= -\frac{\alpha_s}{4\pi} q^2 \int_{4\lambda^2}^\infty \frac{dm^2}{m^2} \sqrt{1 - 4\lambda^2/m^2} N_c \left[ 10 + 8 \frac{\lambda^2}{m^2} \right].
\]
The calculation of the vacuum polarization by quarks turns out to go along the similar line with the result identical to the above one up to a constant factor. The combined result is the one in Eq.(25). Note that in this equation the momentum is measured in units of $\Lambda$.

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Fig. 4. Quark self-energy diagrams, to two loops. The mass counter term from the lowest order contribution is indicated as a blob in Diagram c. In Diagram d inserted in the gluon propagator are the gluon, ghost and light quark loops, respectively.

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Fig. 5