Momentum of charge-magnetic coil systems

Francis Redfern

Texarkana College, Texarkana, Texas 75599

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Abstract. Solenoids and toroidal solenoids exposed to an electric field have been thought to contain hidden momentum. Here I show that there is no hidden momentum in these systems.

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1 Introduction

In 1891 J.J. Thompson pointed out an apparent paradox where electromagnetic systems at rest could contain non-zero electromagnetic momentum. This result languished for many years until, in 1967, Shockley and James examined a charge-magnet system and claimed it had to contain a form of mechanical momentum they termed “hidden momentum” in order for momentum conservation not to be violated. A year later Coleman and Van Vleck published a detailed analysis on the question of a point charge in the vicinity of a magnet and concluded that the conjecture of Shockley and James was correct.

Since these papers were published, charge-magnet systems have been thought to contain this new form of linear mechanical momentum. (Also, there is supposed to be hidden angular momentum in many of these systems.) I have published work on several charge-magnet systems that shows they contain no such momentum. What has been ignored are the electromagnetic forces that arise when an electric field is applied to a magnet or a magnet is formed in an electric field. When these systems originate, mechanical momentum due to these forces, as well as electromagnetic momentum, is imparted to them such that you can’t consider them to be at rest in the reference frame in which they originated. Ignoring the appearance of mechanical momentum has resulted in the invention of hidden momentum, which I argue is an unphysical concept. When forces of origination are taken into account, it is clear that the momentum that is stored in the electromagnetic field is equal and opposite to the mechanical momentum the system gains as a result of those forces. If the system (or parts thereof) is kept stationary, the so-called hidden momentum actually resides in the environment responsible for keeping the system at rest.

In the following sections I will show that systems consisting of a charge in the vicinity of a solenoid and toroidal solenoid contain no hidden momentum. I examine several cases. First I look at the situation where a charge is in place near the coil (in the geometrical center of the coil in the case of the toroid) and magnetic flux is built up in the coil. Then I look at cases where a charge is moved from infinity to the vicinity of coils already containing flux. In none of these cases is hidden momentum necessary for momentum conservation.

2 Activating a solenoid near a charged particle

Consider a charge \( q \) a distance \( R \) from a long air solenoid. Both charge and coil are at rest and current is flowing in the solenoid. Assuming no electrical shielding, there is an electric field inside the solenoid due to the presence of the charge, thus there should be an electromagnetic momentum given (in the Lorentz formulation) by the formula

\[
P_{\text{em}} = \epsilon_0 \int_V (E \times B) dV = \frac{q}{4\pi} \int_V \frac{\hat{r} \times \hat{k}}{r^2} \cdot B \hat{k} dV,
\]

where \( \hat{r} \) is the position vector from the charge to a point on the axis of the solenoid. Have the solenoid be centered on the \( z \) axis with its magnetic field in the positive \( z \) direction and the charged particle along the \( x \) axis at \( x = -R \). In this configuration \( \hat{r} \times \hat{k} = -\cos \alpha \hat{j} \), where the angle \( \alpha \) is measured from the \( x \) axis to \( r \), \( a \) is the radius of the solenoid (\( a \ll R \)), and \( dV = \pi a^2 dz \). With these expressions, the electromagnetic momentum becomes

\[
P_{\text{em}} = -\frac{q\pi\Phi_B}{4\pi R} \int_{-\pi/2}^{\pi/2} \cos \alpha d\alpha = -\frac{aq\Phi_B}{2\pi R} \hat{j},
\]

where \( \Phi_B = \pi a^2 B \) is the magnetic flux inside the solenoid.

Since the system is at rest, there appears to be no mechanical momentum present to balance the electromagnetic momentum. To conserve momentum the conventional view is that there is hidden momentum in the solenoid, equal and opposite to the electromagnetic momentum. However, consider the following scenario.
Have the charge positioned as before, but let there be no current in the solenoid. Let a current build up in the solenoid such that the flux in the coil goes from zero to $\Phi_B$. Faraday’s law says there will be an induced electric field at the position of the charge given by

$$E = -\frac{\partial A}{\partial t},$$

(3)

where $A$ is the vector potential at that point. The vector potential for a solenoid can be found as follows.

$$\Phi_B = \oint B \cdot dS = \oint \nabla \times A \cdot dS = \oint A \cdot d\ell = 2\pi r A.$$  

(4)

In the above equation the first integral is a surface integral over the cross-sectional area of the solenoid, the second uses the fact that $B = \nabla \times A$, and the third, a line integral, follows from Stokes’ theorem. The final result follows from symmetry: The right hand rule shows that if the magnetic field at the position of the charge given by $\nabla \times A \cdot dS = 2\pi r A$, so that

$$A = \Phi_B / 2\pi R \hat{j},$$

(5)

where $\Phi = -\hat{j}$ at the position $(-R, 0, 0)$ of the charge.

The electric field acting on the charge a distance $R$ from the coil is

$$E = \frac{q\Phi_B}{2\pi R} \hat{j},$$

(6)

with $\Phi_B = \partial \Phi_B / \partial t$. With an electric field acting on the charge as the magnetic flux increases, there must be an equal and opposite force if the charge is to be held stationary. This force is

$$F = \frac{q\Phi_B}{2\pi R} \hat{j}.$$  

(7)

As a result the external agent necessary to supply this force will gain an amount of mechanical momentum given by

$$P_{mm} = \int \frac{q\Phi_B}{2\pi R} \hat{j} \cdot \hat{v} = \frac{q\Phi_B}{2\pi R} \hat{j}.$$  

(8)

Note that this is equal and opposite to the electromagnetic momentum contained in the solenoid. Hence the total momentum of the system — charge, solenoid, and external agent — is conserved without the need for a hidden form.

### 3 Activating a toroidal solenoid near a charged particle

Consider a current loop of radius $a$ lying in the $x$-$y$ plane and centered at the origin with the $z$ axis marking the axis of symmetry of the loop. According to Jackson, the vector potential a distance $r$ from the center of the loop in the $x$-$y$ plane ($\theta = \pi/2$) and where $a << r$ is approximately, in SI units,

$$A = \frac{\mu_o I a^2}{4\pi^2} \Phi.$$  

(9)

The vector potential at the location $(R, 0, 0)$ is given by the above equation with $\Phi = \hat{j}$. Imagine this loop is an element of a toroidal solenoid with $N$ turns whose center is a distance $R$ from the loop. The coil will be oriented such that its symmetry axis will be in the $z$ direction and with a current such that the magnetic field is in the $\Phi$ direction. The vector potential due to all the current loops in the coil will now be directed in the positive $z$ direction. Since each loop contributes the same quantity to the vector potential along the coil’s symmetry axis, the vector potential along the axis will be

$$A_z = \frac{\mu_o NI a^2}{4\pi R^2}.$$  

(10)

Along this axis $r = R/\cos \theta$ such that $A_z$ becomes

$$A_z = \frac{\mu_o NI a^2}{4R^2} \cos^2 \theta.$$  

(11)

At the position of the charge, $\theta = 0$ so that

$$A_z = \frac{\mu_o NI a^2}{4R^2} = \Phi_B / 2R,$$  

(12)

since $B = \mu_o NI / 2\pi R$ inside the coil and the cross-sectional area of the coil is $\pi a^2$, again taking the radius of the coil $a << R$. Following the same procedure as for the solenoid, it is easily seen that the electromagnetic linear momentum in the coil is

$$P_{em} = \frac{q\Phi_B}{2R} k,$$  

(13)

whereas the mechanical momentum obtained by an external agent required to keep the charge at rest while the flux in the coil increases is

$$P_{mm} = -\frac{q\Phi_B}{2R} k.$$  

(14)

Once again, the total momentum — the mechanical momentum of the external agent plus the electromagnetic momentum — is zero such that no hidden momentum is necessary to achieve momentum conservation.

### 4 Bringing a charged particle into the vicinity of a solenoid

Placing a charged particle near a coil and turning on the electricity is not the only way to put together a charge-magnetic coil system and derive the momentum changes involved. You can also move the charge from a sufficiently great distance, where there is no interaction between the charge and coil, to the vicinity of the coil.

I will now show that there is no electric or magnetic field to act on the charged particle as it moves in a region external to an infinite solenoid or a toroidal solenoid. The potential four-vector is given in general by

$$A' = (A_x', A_y', A_z', V'/c),$$  

(15)

where $A'$ is the three-vector potential and $V'$ is the scalar potential. Since there is no net charge on the solenoid, the scalar potential is zero.
It can be shown that, in a certain reference frame where the scalar potential is zero (or constant), the vector potential is not a function of time (no electric field), and the curl of the vector potential is zero (no magnetic field), there can be no electric or magnetic field in any other inertial reference frame. This follows from the fact, taking \( \mathbf{v} \) arbitrarily to be in the positive \( x \) direction, that \( \mathbf{A} \) is a function of \( \gamma(x-vt) \).

Let \( \mathbf{A}' \) be a position-dependent but time-independent vector potential in a certain (primed) frame of reference; that is, a function of \((x', y', z')\) but not \( t' \); and let \( V' = 0 \). Transform the primed frame to the lab (unprimed) frame by having it move with speed \( v \) in the positive \( x \) direction. In the lab frame the four-vector potential will be \((A_x, A_y, A_z, V/c) = (\gamma A_{x'}, A_{y'}, A_{z'}, \gamma (v/c) A_{x'})\), where \( x' = \gamma(x-vt) \) and \( V \) is the scalar potential. The electric field in the lab frame will be given by

\[
\mathbf{E} = -\frac{\partial A_x}{\partial t} \hat{i} - \frac{\partial A_y}{\partial t} \hat{j} - \frac{\partial A_z}{\partial t} \hat{k} - \nabla V. \tag{16}
\]

Note that
\[
\begin{align*}
\frac{\partial A_x}{\partial t} &= \gamma \frac{\partial A_{x'}}{\partial t'}, \\
\frac{\partial A_y}{\partial t} &= \gamma \frac{\partial A_{y'}}{\partial t'}, \\
\frac{\partial A_z}{\partial t} &= \gamma \frac{\partial A_{z'}}{\partial t'}.
\end{align*}
\]

Also
\[
\begin{align*}
\frac{\partial V}{\partial x} &= \gamma v \frac{\partial A_{x'}}{\partial x'}, \\
\frac{\partial V}{\partial y} &= \gamma v \frac{\partial A_{y'}}{\partial y'}, \\
\frac{\partial V}{\partial z} &= \gamma v \frac{\partial A_{z'}}{\partial z'},
\end{align*}
\]

where \( y \) and \( z \) have been replaced by \( y' \) and \( z' \) for clarity in recognizing that substituting the quantities in the above two sets of equations into the electric field of Eq. (16) and using the fact that \( \nabla' \times \mathbf{A}' = 0 \) results in the electric field in the lab frame (and in any other inertial frame, since the lab frame is arbitrary) being zero. The magnetic field is also zero since \( \mathbf{B} = \mathbf{v} \times \mathbf{E}/c^2 \). A moving charge will therefore experience no electromagnetic force in either the vicinity of a long solenoid or a toroidal solenoid (with no magnetic field leakage).

Next, look at the effect the moving charge has on the solenoid. In Fig. 1 showed that a charge moving toward a current loop in plane of the loop will impart an impulse to the loop given by

\[
\Delta \mathbf{P} = \frac{\mu q I A}{4 \pi r^2} \mathbf{j} = -\frac{1}{2} \mu \varepsilon_0 \mathbf{E} \times \mathbf{m}, \tag{19}
\]

where \( \mathbf{m} \) is the magnetic moment of the current loop. The loops of a solenoid will be off-center from the direction of motion of the charge, so a new calculation taking this into account is in order. The electric field of moving charge, \( q \), at point \( r \) in the lab frame is given by the following [8].

\[
E = \frac{1}{4 \pi \varepsilon_0} \frac{q r}{\gamma^2 r^3 (1 - \beta^2 \sin^2 \alpha)^{3/2}}, \tag{20}
\]

where \( \beta = v/c \) and \( \alpha \) is the angle between the direction of motion of the charge and \( \mathbf{r} \). The motion of the charge will produce a magnetic field given by

\[
B = \frac{1}{c^2 v} \mathbf{E} = \frac{1}{4 \pi \varepsilon_0} \frac{q v \times r}{\gamma^2 c^2 r^3 (1 - \beta^2 \sin^2 \alpha)^{3/2}}. \tag{21}
\]

where \( v \) is the velocity of the charge. If the charge is moving in the positive \( x \) direction, \( v \times r = v(-\mathbf{j} + y \mathbf{k}) \).

The solenoid is stationary in the lab frame with its axis parallel to the \( z \) direction and its magnetic field in the positive \( z \) direction. Consider a current loop of the coil with its center located at the origin of a cylindrical coordinate system. The current density of the loop will be

\[
\mathbf{J} = J \hat{\phi} = 16 \delta(\rho - a) \delta(z)(-\sin \phi \hat{i} + \cos \phi \hat{j}). \tag{22}
\]

Here, the Dirac delta function is used, \( \rho \) is the cylindrical radial coordinate, \( a \) is the radius of the loop, and \( \phi \) is the azimuthal coordinate. The center of the current loop is located at the position \( r = (R, 0, h) \) in a Cartesian coordinate system centered on the charge. An element of current on the loop is located at \((R + a \cos \phi, a \sin \phi, h)\). In the slow motion situation, \( \gamma \) can be taken as one and \( \beta \approx 0 \). (This will be the case for the rest of this article.) The force per unit volume acting on the loop is then approximately given by

\[
f = \mathbf{J} \times \mathbf{B} = \frac{1}{4 \pi \varepsilon_0} \frac{q v I \delta(\rho - a) \delta(z)}{c^2 r^3} ((a \sin \phi \hat{i} + a \cos \phi \hat{j} + h \sin \phi \hat{k}). \tag{23}
\]

It is tempting to take \( r \approx \sqrt{r^2 + h^2} \) when \( a << r \), but this makes the magnetic field larger at locations on the loop farther away from the charge. The proper way to do the calculation is to maintain the accuracy of \( r^3 \). However, you can expand \( r^3 \) and drop terms the order of \( a^2/r^2 \).

It turns out this is not necessary here, since either side of the loop is the same distance from \( q \). This will not be the case when the toroidal solenoid is considered later on.

The magnetic field forms circular loops centered on the line of action of the velocity of the charge. If you use the thumb of your right hand to point in the direction of the charge’s motion, then your curled fingers will represent the magnetic field lines. Note that there will be a torque acting on a current loop on the positive \( z \) axis with the torque vector in the same direction as the velocity of the charge. For a current loop on the negative \( z \) axis the torque is in the opposite direction such that no net torque is applied to an infinite solenoid or to a charge approaching the halfway point of a finite solenoid, although the coil will experience stress as the charge moves. An off-center approach of the charge toward a finite solenoid will produce a net torque on the coil.
Realizing that you don’t need to engage in the extra effort required to expand \( r^{-3} \), the force acting on the loop is

\[
F = \int_V J \cdot \rho \cdot dV = \frac{1}{4\pi \epsilon_0} \frac{qvI \pi a^2}{c^2 r^3} \hat{z}.
\]  

(24)

To get the impulse applied to the current loop, you must move the charge in from infinity in a direction perpendicular to the solenoid axis and directly at it. Take \( R \) as the final distance from the charge to the solenoid. Replace \( h \) with \( z \), the location of a loop from the intersection of the line of action of \( v = \hat{v} \) and the axis of the solenoid as this will later become a variable. The speed can be given in terms of the time rate of change of the angle \( \alpha \), the angle between \( v \) and \( r \), as \( v = z/(\cos^2 \alpha)(d\alpha/dt) \). Also note that \( r = z/\sin \alpha \). The force on the current loop is then

\[
F = \frac{1}{4\pi \epsilon_0} \frac{qI \pi a^2}{c^2 z^2} \sin \alpha \frac{d\alpha}{dt} \hat{j}.
\]  

(25)

The impulse delivered to the current loop will follow from the integral of \( F \hat{t} \) over the distance traveled by the charge. Let \( \alpha_R \) be the value of \( \alpha \) when the charge reaches the distance \( R \) from the axis of the coil. The mechanical momentum received by the loop is then

\[
P_{\text{loop}} = \frac{1}{4\pi \epsilon_0} \frac{qI \pi a^2}{c^2 z^2} \int_0^{\alpha_R} \sin \alpha d\alpha = \frac{1}{4\pi \epsilon_0} \frac{qI \pi a^2}{c^2 z^2} (1 - \cos \alpha_R).
\]  

(26)

Next, you turn this equation into another differential equation for the momentum received by the entire solenoid. To do this you can replace \( \alpha_R \) with \( \alpha \), now a variable, and \( I \) by \( dI = K d\theta \) where \( K = NI/L \) is the current per unit length of a solenoid of length \( L \) with \( N \) turns. Also, to perform the integral involved, you make the substitutions \( z = R \tan \alpha \) and \( dz = R d\alpha/(\cos^2 \alpha) \). The loop now contributes an amount of mechanical momentum \( dP_{\text{mm}} \) to the coil of

\[
dP_{\text{mm}} = \frac{1}{4\pi \epsilon_0} \frac{qK \pi a^2}{c^2 R \tan^2 \alpha \cos^2 \alpha} \left(1 - \cos \alpha \right) d\alpha.
\]  

(27)

To integrate this note that \( \cos^2 \tan^2 \alpha = \sin^2 \alpha \) and that \( (1 - \cos \alpha) \sin \alpha = d\alpha/(1 + \cos \alpha) = d\theta/\cos^2 \theta = d(\tan \theta) \), where \( \theta = \alpha/2 \). The integral is taken from \( \theta = -\pi/4 \) to \( \theta = \pi/4 \). The result is

\[
P_{\text{mm}} = \frac{q \Phi_B}{2\pi R^2} \hat{j},
\]  

(28)

where \( \Phi_B = \mu_0 K \pi a^2 = \mu_0 NI \pi a^2/L = \pi a^2 B \), and where \( c^{-2} = \epsilon_0 \mu_0 \) has been used. Note that this mechanical momentum applied to the solenoid is equal and opposite to the electromagnetic momentum in the coil given by Eq. (2). Again, no hidden momentum is necessary to preserve momentum conservation as the total momentum is zero.

### 5 Bringing a charged particle into the vicinity of a toroidal solenoid

For this calculation the charge is moved from infinity to a point in the plane of and outside a current loop, and then \( N \) loops are arranged to form the toroid. You saw in the previous section that the electric field acting on the charge has to be zero, so the only mechanical momentum is imparted to the loop. The magnetic field and current density are given by Eqs. (21) and (22) as before, but now the loop is at the position \( r = (s, R, 0) \) with respect to the charge, where \( s \) is originally very far away and goes to zero where the charge finally comes to rest. \( R \) will turn out to be the radius of the toroid.

In this instance you cannot take \( r \) to be the distance from the charge to the center of the current loop when calculating the force on the loop. Instead, \( r = (s + acos\phi, R + asin\phi, 0) \) must be used in the denominator, but you can expand the denominator since it is assumed that \( a << R \). Following standard expansion procedure, you can write \( r^{-3} \) as

\[
\frac{1}{r^3} = \frac{1}{r_o^3} \left[ 1 - \frac{3(\cos \phi + R \sin \phi)}{r^2_o} \right],
\]  

(29)

where \( r_o = \sqrt{s^2 + R^2} \). The magnetic field due to the motion of the charge is then

\[
B = \frac{1}{4\pi \epsilon_0} \frac{qv(R + \sin \phi)k}{c^2 r_o^2} \times \left[ 1 - \frac{3(\cos \phi + R \sin \phi)}{r^2_o} \right].
\]  

(30)

The force per unit volume on the loop is given by

\[
f = J \times B = \frac{1}{4\pi \epsilon_0} \frac{qv(\delta(\rho - a)\delta(z)(R + \sin \phi)}{c^2 r_o^3} \left[ 1 - \frac{3(\cos \phi + R \sin \phi)}{r^2_o} \right] (\cos \phi \hat{i} + \sin \phi \hat{j}).
\]  

(31)

Performing the volume integral of the above equation yields the force on the current loop. Only those terms that aren’t zero when the integral over \( \phi \) is taken are shown in the equation below.

\[
F = \left. \frac{1}{4\pi \epsilon_0} \frac{qvNI \pi a}{c^2 r^3} \right|^{r_o} \int_0^{\pi} \left[ \sin^2 \phi \hat{j} - \frac{3a}{r_o^2} R \cos^2 \phi \hat{i} - \frac{3a}{r_o^2} R^2 \sin^2 \phi \hat{j} \right] d\phi.
\]  

(32)
When \( N \) current loops are put together to form the toroid, the contributions to the force acting on the coil in the \( y \) direction cancel, leaving only the contribution in the \( x \) direction. Now, I permute the coordinate axes to bring the axis of symmetry of the coil to coincide with the \( z \) axis. The magnetic field is then in the \( \hat{\phi} \) direction. The result of the integration is

\[
F = -\frac{1}{4\pi\epsilon_0} \frac{3qviN\pi a^2 R s}{c^2 r_o^2} \hat{k}.
\]  
(33)

The speed of the charge in terms of \( s \) is \( v = -\frac{ds}{dt} \) such that the impulse delivered to the coil is given by

\[
P_{mm} = -\frac{1}{4\pi\epsilon_0} \frac{qNI\pi a^2 R \hat{k}}{c^2} \int_{-\infty}^{0} -(R^2 + s^2)^{-5/2} ds.  
\]  
(34)

This integration yields

\[
P_{mm} = -\frac{\mu_o}{4\pi} \frac{qNI\pi a^2}{R^2} \hat{k}. 
\]  
(35)

In terms of the magnetic flux in the coil, this mechanical momentum is

\[
P_{mm} = -\frac{q\Phi_B}{2R} \hat{k}. 
\]  
(36)

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