MONTE CARLO ALGORITHM FOR THE EXTREMA OF TEMPERED STABLE PROCESSES

JORGE I. GONZÁLEZ CÁZARES & ALEKSANDAR MIJATOVIĆ

Abstract. We develop a novel Monte Carlo algorithm for the vector consisting of the supremum, the time at which the supremum is attained and the position of an exponentially tempered Lévy process. The algorithm, based on the increments of the process without tempering, converges geometrically fast (as a function of the computational cost) for discontinuous and locally Lipschitz functions of the vector. We prove that the corresponding multilevel Monte Carlo estimator has optimal computational complexity (i.e. of order $\epsilon^{-2}$ if the mean squared error is at most $\epsilon^2$) and provide its central limit theorem (CLT). Using the CLT we construct confidence intervals for barrier option prices and various risk measures based on drawdown under the tempered stable (CGMY) model calibrated/estimated on real-world data. We provide non-asymptotic and asymptotic comparisons of our algorithm with existing approximations, leading to rule-of-thumb guidelines for users to the best method for a given set of parameters, and illustrate its performance with numerical examples.

1. Introduction

1.1. Setting and motivation. The class of tempered stable processes is very popular in the financial modelling of asset prices of risky assets, see e.g. [CT15]. A tempered stable process $X = (X_t)_{t \geq 0}$ naturally address the shortcomings of diffusion models by allowing the large (often heavy-tailed and asymmetric) sudden movements of the asset price observed in the markets, while preserving the exponential moments required in exponential Lévy models $S_0 e^{X}$ of asset prices [CGMY02, Sch03, Kou14, CT15]. Of particular interest in this context are the expected drawdown (the current decline from a historical peak) and its duration (the elapsed time since the historical peak), see e.g. [Sor03, Vec06, CZH11, BPP17, LLZ17], as well as barrier option prices [ACU02, Sch06, KL09, GX17] and the estimation of ruin probabilities in insurance [Mor03, KKM04, LZZ15]. In these application areas, the key quantities are the historic maximum $X_T$ at time $T$, the time $\tau_T(X)$ at which this maximum was attained during the time interval $[0, T]$ and the value $X_T$ of the process $X$ at time $T$.

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In this paper we focus on the Monte Carlo estimation of $\mathbb{E}[g(X_T, \overline{X}_T, \tau_T(X))]$, where the payoff $g$ is discontinuous or locally Lipschitz, covering the aforementioned applications. We construct a novel algorithm TSB-Alg, applicable to a tempered Lévy process, if the increments of the process without tempering can be simulated, which clearly holds for the tempered stable processes $X$. We show that the bias of TSB-Alg decays geometrically fast in its computational cost for functions $g$ that are either locally Lipschitz, or discontinuous (see Subsection 3 for details). We prove that the corresponding multilevel Monte Carlo estimator has optimal computational complexity (i.e. of order $\epsilon^{-2}$ if the mean squared error is at most $\epsilon^2$) and establish its central limit theorem (CLT). Using the CLT we construct confidence intervals for barrier option prices and various risk measures based on drawdown under the tempered stable (CGMY) model. TSB-Alg combines the stick-breaking algorithm in [GCMUB21] with the exponential change-of-measure for Lévy processes, also applied in [PT06] for the Monte Carlo pricing of European options.

1.2. Comparison with the literature. The stick-breaking (SB) representation (see (2.2) below) was used in [GCMUB21] to obtain an approximation $\chi_n$ of $\chi := (X_T, \overline{X}_T, \tau_T(X))$ that converges geometrically fast in the computational cost when the increments of $X$ can be simulated exactly. In [GCM20], the same representation was used in conjunction with a small-jump Gaussian approximation for more general processes. In the present work we address a situation in between the two aforementioned papers using TSB-Alg below. TSB-Alg preserves the geometric convergence in the computational cost of SB-Alg but is applicable to general tempered stable processes, unlike SB-Alg.

Under the assumption that the increments of the Lévy process $X$ can be simulated, the algorithm SB-Alg in [GCMUB21] outperforms other algorithms for which the computational complexity analysis has been done. For instance, the computational complexity analyses for the procedures presented in [FLT14, KK16], applicable to tempered stable processes of finite variation, have not been done and we cannot compare directly against them. If the increments cannot be sampled, one generally uses the Gaussian approximation of small jumps, in which case the algorithm SBG-Alg in [GCM20] outperforms its competitors. In this sense, it suffices to compare TSB-Alg below with those two. Table 1 shows the asymptotic computational complexities of the corresponding Monte Carlo (MC) and multilevel Monte Carlo (MLMC) estimators based on TSB-Alg, SB-Alg and SBG-Alg.

In those cases, simulation algorithms provide a natural alternative [PT06, Che11, GX17, GCMUB21, GCM20]. Exact simulation of the drawdown is currently out of reach. Among the approximate simulation algorithms, the recently developed stick-breaking (SB) approximation [GCMUB21] is the fastest in terms of its computational complexity, as it samples from the law of $(X_T, \overline{X}_T, \tau_T(X))$ with a geometrically decaying bias. However, like most approximate simulation algorithms for a
| Estimator     | Monte Carlo Case | Infinite variation | Multilevel Monte Carlo Case | Finite variation | Infinite variation |
|--------------|-----------------|--------------------|----------------------------|-----------------|-------------------|
| TSB-Alg      | $O(\epsilon^{-2} \log(1/\epsilon))$ | $O(\epsilon^{-2} \log(1/\epsilon))$ | $O(\epsilon^{-2})$ | $O(\epsilon^{-2})$ |
| SB Algorithm | $O(\epsilon^{-2} \log(1/\epsilon))$ | NA                 | $O(\epsilon^{-2})$ | NA              |
| SBG-Alg      | $O(\epsilon^{-2-\beta_*})$ | $O(\epsilon^{-2-\beta_*})$ | $O(\epsilon^{-2})$ | $O(\epsilon^{-2\beta_*})$ |

Table 1. Computational complexities of MC and MLMC estimators. For SBG-Alg, the results only apply assuming that the payoff is locally Lipschitz (see [GCM20, Subsec. 4.2.1]), otherwise, the corresponding complexities are larger. The index $\beta_*$, defined in (5.2) below, is slightly larger than the Blumenthal-Getoor index $\beta$.

The statistic of the entire trajectory, it is only valid for Lévy process whose increments can be sampled, an assumption not satisfied by tempered stable models of infinite variation, making a direct application of the algorithm in [GCMUB21] infeasible. The stick-breaking Gaussian (SBG) approximation [GCM20] is applicable to tempered stable processes; however, the algorithm has only polynomial decay in its computational complexity.

1.3. **Organisation.** The remainder of the paper is structured as follows. In Section 2 we recall the SB representation and construct TSB-Alg. In Section 3 we describe the geometric decay of the bias and the strong error in $L^p$ and explain what the computational complexities of the MC and MLMC estimators are. In Subsection 3.3 we provide an in-depth comparison of TSB-Alg with the SB and SBG algorithms, identifying where each algorithm outperforms the others. In Section 4 we consider the case of tempered stable processes and illustrate with numerical examples the previously described results.

## 2. The tempered stick-breaking algorithm

Let $T > 0$ be a time horizon and $\lambda = (\lambda_+, \lambda_-) \in \mathbb{R}_+^2$, a vector with non-negative coordinates, different from the origin $0 = (0, 0)$. A generating (or Lévy) triplet $(\sigma^2, \nu_\lambda, b_\lambda)$ (see [Sat13, Def. 8.2] for definition) of a tempered Lévy process $X = (X_t)_{t \in [0,T]}$ is given by

$$
\nu_\lambda(dx) := e^{-\lambda \text{sgn}(x)|x|} \nu(dx) \quad \text{and} \quad b_\lambda := b + \int_{(-1,1)} x(e^{-\lambda \text{sgn}(x)|x|} - 1) \nu(dx),
$$

where $\sigma^2 \in \mathbb{R}_+$, $b \in \mathbb{R}$ and $\nu$ is a Lévy measure on $\mathbb{R} \setminus \{0\}$, i.e. $\nu$ satisfies $\int_{(-1,1)} x^2 \nu(dx) < \infty$ and $\nu(\mathbb{R} \setminus (-1,1)) < \infty$ (all Lévy triplets in this paper are given with respect to the cutoff function $x \mapsto 1_{(-1,1)}(x)$ and the sign function in (2.1) is defined as $\text{sgn}(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$). The triplet $(\sigma^2, \nu_\lambda, b_\lambda)$ determines uniquely the law of $X$, see e.g. [Sat13, Thms 7.10 & 8.1].
Our aim is to sample from the law of the statistic \((X_T, \bar{X}_T, \tau_T)\) consisting of the position \(X_T\) of the process \(X\) at \(T\), the supremum \(\bar{X}_T := \sup\{X_s : s \in [0, T]\}\) of \(X\) on the time interval \([0, T]\) and the time \(\tau_T := \inf\{s \in [0, T] : \bar{X}_s = \bar{X}_T\}\) at which the supremum was attained in \([0, T]\).

By [GCMUB21, Thm 1] there exists a coupling \((X, Y, \ell)\) under a probability measure \(\mathbb{P}_\lambda\), such that \(\ell = (\ell_n)_{n \in \mathbb{N}}\) is a stick-breaking process on \([0, T]\) based on the uniform law \(U(0, 1)\) (i.e. \(L_0 := T, L_n := L_{n-1}U_n\) and \(\ell_n := L_{n-1} - L_n\) for \(n \in \mathbb{N}\) where \(U_n\) are iid \(U(0, 1)\)), independent of the Lévy process \(Y\) with law equal to that of \(X\), and the SB representation holds \(\mathbb{P}_\lambda\)-a.s.:

\[
(2.2) \quad \chi := (X_T, \bar{X}_T, \tau_T) = \sum_{n=1}^{\infty} \left( \xi_n, \max\{\xi_n, 0\}, \ell_n \cdot 1\{\xi_n > 0\} \right), \quad \text{where} \quad \xi_n := Y_{L_{n-1}} - Y_{L_n}, \ n \in \mathbb{N}.
\]

We stress that \(\ell\) is not independent of \(X\). In fact \((\ell, Y)\) can be expressed naturally through the geometry of the path of \(X\) (see [PUB12, Thm 1] and the coupling in [GCMUB21]), but further details of the coupling are not important for our purposes. The key step in the construction of our algorithm is given by the following theorem. Its proof in Section 5 below is based on the coupling described above and the change-of-measure theorem for Lévy processes [Sat13, Thms 33.1 & 33.2].

**Theorem 1.** Denote by \(\sigma B, Y^{(+)}, Y^{(-)}\) the independent Lévy processes with generating triplets \((\sigma^2, 0, 0), (0, \nu \chi|_{(0, \infty)}), (0, \nu \chi|_{(-\infty, 0)}), 0)\), respectively, satisfying \(Y_t = \sigma B_t + Y_t^{(+)} + Y_t^{(-)} + b \lambda t\) for all \(t \in [0, T]\), \(\mathbb{P}_\lambda\)-a.s. Let \(\mathbb{E}_\lambda\) (resp. \(\mathbb{E}_0\)) be the expectation under \(\mathbb{P}_\lambda\) (resp. \(\mathbb{P}_0\)) and define

\[
(2.3) \quad \mathbb{Y}_\lambda := \exp(-\lambda_+ Y_T^{(+) +} + \lambda_- Y_T^{(-)} - \mu_\lambda T), \quad \text{where}
\]

\[
(2.4) \quad \mu_\lambda := \int_{\mathbb{R}} (e^{-\lambda \text{sgn}(x)|x|} - 1 + \lambda \text{sgn}(x)|x| \cdot 1_{(-1,1)}(x)) \nu(dx),
\]

Then for any \(\sigma(\ell, \xi)\)-measurable random variable \(\zeta\) with \(\mathbb{E}_\lambda |\zeta| < \infty\) we have \(\mathbb{E}_\lambda [\zeta] = \mathbb{E}_0[\zeta Y_\lambda]\).

By the equality in (2.2), the measurable function \(\zeta\) of the sequences \(\ell\) and \(\xi\) in Theorem 1 may be either equal to \(g(\chi)\) (for any integrable function \(g\) of the statistic \(\chi\)) or its approximation \(g(\chi_n)\), where \(\chi_n\) is as introduced in [GCMUB21]:

\[
(2.5) \quad \chi_n := (Y_{L_n}, \max\{Y_{L_n}, 0\}, L_n \cdot 1\{Y_{L_n} > 0\}) + \sum_{k=1}^{n} \left( \xi_k, \max\{\xi_k, 0\}, \ell_k \cdot 1\{\xi_k > 0\} \right).
\]

Theorem 1 enables us to sample \(\chi_n\) under the probability measure \(\mathbb{P}_0\), which for any tempered stable process \(X\) makes the increments of \(Y\) stable and thus easy to simulate. Under \(\mathbb{P}_0\), the law of \(Y_t\) equals that of \(Y_t^{(+)} + Y_t^{(-)} + \sigma B_t + bt\), where \(\sigma B_t + bt\) is normal \(N(bt, \sigma^2 t)\) with mean \(bt\) and variance \(\sigma^2 t\) and the Lévy processes \(Y^{(+)\text{ and }} Y^{(-)}\) have triplets \((0, \nu|_{(0, \infty)}), 0)\) and \((0, \nu|_{(-\infty, 0)}), 0)\), respectively. Denote their distribution functions by \(F^{(\pm)}(t, x) := \mathbb{P}_0(Y_t^{(\pm)} \leq x), x \in \mathbb{R}, t > 0\).

**Algorithm (TSB-Alg).** Unbiased simulation of \(g(\chi_n)\) under \(\mathbb{P}_\lambda\)
Require: Tempering parameter \( \lambda \in \mathbb{R}^2 \setminus \{0\} \), generating triplet \((\sigma^2, \nu, b)\), time horizon \( T > 0 \), test function \( g \), approximation level \( n \in \mathbb{N} \)

1: Set \( L_0 = T \) and compute \( \mu_\lambda \) in (2.4)

2: for \( k = 1, \ldots, n \) do

3: Sample \( U_k \sim U(0,1) \) and put \( L_k = U_k L_{k-1} \) and \( \ell_k = L_{k-1} - L_k \)

4: Sample \( \xi_k^{(\pm)} \sim F^{(\pm)}(\ell_k, \cdot) \), \( \eta_k \sim N(\ell_k b, \sigma^2 \ell_k) \) and put \( \xi_k = \xi_k^{(+) + \xi_k^{(-)} + \eta_k} \)

5: end for

6: Sample \( \zeta_n^{(-)} \sim F^{(-)}(L_n, \cdot) \), \( \vartheta_n \sim N(L_n b, \sigma^2 L_n) \) and put \( \zeta_n = \zeta_n^{(+)} + \zeta_n^{(-)} + \vartheta_n \)

7: Set \( \chi_n = \sum_{k=1}^n (\xi_k, \max\{\xi_k, 0\}, \ell_k 1\{\xi_k > 0\}) + (\zeta_n, \max\{\zeta_n, 0\}, L_n 1\{\xi_n > 0\}) \) \( g(Y_T^{(\pm)}) = \zeta_n^{(+)} + \sum_{k=1}^n \xi_k^{(\pm)} \)

8: return \( g(\chi_n) \exp(-\lambda_+ Y_T^{(\pm)} + \lambda_- Y_T^{(\pm)} - \mu_\lambda T) \)

In [GCMUB21], it was shown that the approximation \( \chi_n \) converges geometrically fast in computational effort (or equivalently as \( n \to \infty \)) to \( \chi \) if the increments of \( Y \) can be sampled under \( \mathbb{P}_\lambda \) (see [GCMUB21] for more details and a discussion of the benefits of the “correction term” \( (Y_{L_n}, \max\{Y_{L_n}, 0\}, L_n \cdot 1\{Y_{L_n} > 0\}) \) in (2.5)). Theorem 1 allows us to weaken this requirement in the context of tempered Lévy processes, by requiring that we be able to sample the increments of \( Y \) under \( \mathbb{P}_0 \). The main application of TSB-Alg is to general tempered stable processes as the simulation of their increments is currently out of reach for many cases of interest (see Section 3.3 below for comparison with existing methods when it is not), making the main algorithm in [GCMUB21] not applicable. Moreover, Theorem 1 allows us to retain the geometric convergence of \( \chi_n \) established in [GCMUB21], see Section 3 below for details.

3. MC and MLMC estimators based on TSB-Alg

3.1. Bias of TSB-Alg. An application of Theorem 1 implies that the bias of TSB-Alg equals

\[
\mathbb{E}_\lambda[g(\chi)] - \mathbb{E}_0[g(\chi_n) Y_\lambda] = \mathbb{E}_0[\Delta_n^g], \quad \text{where} \quad \Delta_n^g := (g(\chi) - g(\chi_n)) Y_\lambda.
\]

The natural coupling \((\chi, \chi_n, Y_T^{(\pm)}, Y_T^{(-)})\) in (3.1) is defined by \( Y_T^{(\pm)} := \sum_{k=1}^\infty \xi_k^{(\pm)} \), \( \xi_k := \xi_k^{(+)} + \xi_k^{(-)} + \eta_k \)

for all \( k \in \mathbb{N} \), \( \chi \) in (2.2) and \( \chi_n \) in (2.5) with \( Y_{L_n} := \sum_{k=n+1}^\infty \xi_k \), where, conditional on the stick-breaking process \( \ell = (\ell_k)_{k \in \mathbb{N}} \), the random variables \( \{\xi_k^{(\pm)}, \eta_k : k \in \mathbb{N}\} \) are independent and distributed as \( \xi_k^{(\pm)} \sim F^{(\pm)}(\ell_k, \cdot) \) and \( \eta_k \sim N(\ell_k b, \sigma^2 \ell_k) \) for \( k \in \mathbb{N} \).

The following result presents the decay of the strong error \( \Delta_n^g \) for Lipschitz, locally Lipschitz and two classes of barrier-type discontinuous payoffs that arise frequently in applications.

Proposition 2. Let \( \lambda = (\lambda_+, \lambda_-), \nu \) and \( \sigma^2 \) be as in Section 2 and fix \( p \geq 1 \). Then, for the classes of payoffs \( g(\chi) = g(X_T, \overline{X}_T, \tau_T) \) below, the strong error of TSB-Alg decays as follows (as \( n \to \infty \)).
(Lipschitz): Suppose $|g(x,y,t) - g(x',y',t')| \leq K(|y - y'| + |t - t'|)$ for some $K$ and all $(x,y,t), (x',y',t') \in \mathbb{R} \times \mathbb{R}_+^2 \times [0,T]^2$. Then $\mathbb{E}_0[|\Delta_n^g|^p] = O(\eta_p^{-n})$, where $\eta_p \in [3/2,2]$ is in (5.4) below.

(locally Lipschitz): Let $|g(x,y,t) - g(x',y',t')| \leq K(|y - y'| + |t - t'|)e^{\max\{y,y'\}}$ for some constant $K > 0$ and all $(x,y,y',t,t') \in \mathbb{R} \times \mathbb{R}_+^2 \times [0,T]^2$. If $\lambda_+ \geq q > 1$ and $\int_{[1,\infty)} e^{p(q-\lambda_+)x} \nu(dx) < \infty$, then $\mathbb{E}_0[|\Delta_n^g|^p] = O(\eta_{pq}^{-n/q'})$, where $q' := (1 - 1/q)^{-1} > 1$ and $\eta_{pq} \in [3/2,2]$ is given in (5.4).

(barrier-type 1): Suppose $g(x) = h(X_T) \cdot 1\{X_T \leq M\}$ for a measurable and bounded function $h : \mathbb{R} \to \mathbb{R}$ and some $M > 0$. If $\mathbb{P}_0(M < X_T \leq M + x) \leq Kx$ for some $K > 0$ and all $x \geq 0$, then for $\alpha_x \in (1,2]$ in (5.3) and any $\gamma \in (0,1)$ we have $\mathbb{E}_0[|\Delta_n^g|^p] = O(2^{-n\gamma/(\gamma + \alpha_x)})$. Moreover, we may take $\gamma = 1$ if any of the following hold: $\sigma^2 > 0$ or $\int_{(-1,1)} |x|^\gamma \nu(dx) < \infty$ or Assumption (S) below.

(barrier-type 2): Suppose $g(x) = h(X_T, X_T') \cdot 1\{\tau_T \leq s\}$, where $s \in (0,T)$, $h$ is measurable and bounded with $|h(x,y) - h(x,y')| \leq K|y - y'|$ for some $K > 0$ and all $(x,y,y') \in \mathbb{R} \times \mathbb{R}_+^2$. If $\sigma^2 > 0$ or $\nu(\mathbb{R} \setminus \{0\}) = \infty$, then $\mathbb{E}_0[|\Delta_n^g|^p] = O(e^{-n/\varepsilon})$.

In Proposition 2 and throughout the paper, the notation $f(n) = O(g(n))$ as $n \to \infty$ for functions $f, g : \mathbb{N} \to (0,\infty)$ means $\limsup_{n \to \infty} f(n)/g(n) < \infty$. Put differently, $f(n) = O(g(n))$ is equivalent to $f(n)$ being bounded above by $C_0g(n)$ for some constant $C_0 > 0$ and all $n \in \mathbb{N}$.

Remark 1. (i) The proof of Proposition 2, given in Section 5 below, is based on Theorem 1 and analogous bounds in [GCMUB21] (for Lipschitz, locally Lipschitz and barrier-type 1 payoffs) and [GCM20] (for barrier-type 2 payoffs). In particular, in the proof of Proposition 2 below, we need not assume $\lambda_+ > 0$ to apply [GCMUB21, Prop. 1], which works under our standing assumption $\lambda \neq 0$.

(ii) For barrier option payoffs under a tempered stable process $X$ (i.e. barrier-type 1 class in Proposition 2), we may take $\gamma = 1$ since $X$ satisfies either $\int_{(-1,1)} |x|^\gamma \nu(dx) < \infty$ or Assumption (S).

(iii) The restriction $p \geq 1$ is not essential. It covers the cases $p \in \{1,2\}$ required for the MC and MLMC complexity analyses and ensures that the corresponding rate $\eta_r$ in (5.4) lies in $[3/2,2]$. In fact, for any payoff $g$ in Proposition 2 we have $\mathbb{E}_0[|\Delta_n^g|^p] = O(\eta_g^{-k})$ with $\eta_g > 1$ bounded away from one: $\eta_g \geq 1.25$ (resp. $3/2$, $(3/2)^{1-1/\lambda_+}$) for barrier-type 1 & 2 (resp. Lipschitz, locally Lipschitz) payoffs.

3.2. Computational complexity and the CLT for the MC and MLMC estimators. Consider the MC estimator

$$\hat{\theta}_{MC}^{g,n} := \frac{1}{N} \sum_{i=1}^{N} \theta_i^{g,n},$$

(3.2)
where \( \{g_{i}^k\}_{i \in \mathbb{N}} \) is iid output of TSB-Alg with \( \theta_{1}^{g,n} \overset{d}{=} g(\chi_{n})Y_{\lambda} \) (under \( P_{0} \)) and \( n, N \in \mathbb{N} \). The corresponding MLMC estimator is given by

\[
\hat{\theta}_{\text{ML}}^{g,n} := \sum_{k=1}^{n} \frac{1}{N_{k}} \sum_{i=1}^{N_{k}} D_{k,i}^{g,n},
\]

where \( \{D_{k,i}^{g}\}_{k,i \in \mathbb{N}} \) is an array of independent variables satisfying \( D_{k,i}^{g} \overset{d}{=} (g(\chi_{k}) - g(\chi_{k-1}))Y_{\lambda} \) and \( D_{1,i}^{g} = g(\chi_{1})Y_{\lambda} \) (under \( P_{0} \)), for \( i \in \mathbb{N}, k \geq 2 \) and \( n, N_{1}, \ldots, N_{n} \in \mathbb{N} \). Note that TSB-Alg can be easily adapted to sample the variable \( D_{k,i}^{g} \) by drawing the ingredients for \( (\chi_{k}, Y_{\lambda}) \) and computing \( (\chi_{k-1}, \chi_{k}, Y_{\lambda}) \) deterministically from the output, see [GCMUB21, Subsec. 2.4] for further details.

The computational complexity analysis of the MC and MLMC estimators is given in the next result (the usual notation \([x] := \inf\{k \in \mathbb{N} : k \geq x\}, x \in \mathbb{R}_{+}\), is used for the ceiling function).

**Proposition 3.** Let the payoff \( g \) from Proposition 2 also satisfy \( \mathbb{E}_{0}[g(\chi)^2Y_{\lambda}^2] < \infty \). For any \( \epsilon > 0 \), let \( n(\epsilon) := \inf\{k \in \mathbb{N} : |\mathbb{E}_{0}[g(\chi_{k}) Y_{\lambda}] - \mathbb{E}_{0}[g(\chi)]| \leq \epsilon/\sqrt{2} \} \). Denote by \( c \) the computational cost of line 6 in TSB-Alg and by \( \mathbb{V}_{0}[] \) the variance under the probability measure \( P_{0} \).

**MC** Suppose \( n = n(\epsilon) \) and \( N = \lfloor 2\epsilon^{-2}\mathbb{V}_{0}[g(\chi_{k}) Y_{\lambda}] \rfloor \), then the MC estimator \( \hat{\theta}_{\text{MC}}^{g,n} \) is \( \epsilon \)-accurate, that is \( \mathbb{E}_{0}[(\hat{\theta}_{\text{MC}}^{g,n} - \mathbb{E}_{0}[g(\chi)])^2] \leq \epsilon^2 \), at the cost \( C_{\text{MC}}(\epsilon) := c(n+1)N = \mathcal{O}(\epsilon^{-2}\log(1/\epsilon)) \) as \( \epsilon \to 0 \).

**MLMC** Suppose \( n = n(\epsilon) \) and set

\[
N_{k} := \left\lfloor 2\epsilon^{-2}\sqrt{\mathbb{V}_{0}[D_{k,1}^{g}]/k \left( \sum_{j=1}^{n} \sqrt{\mathbb{V}_{0}[D_{j,1}^{g}] / k} \right)} \right\rfloor, \quad k \in \{1, \ldots, n\}.
\]

Then the multilevel Monte Carlo estimator \( \hat{\theta}_{\text{ML}}^{g,n} \) is \( \epsilon \)-accurate and the corresponding cost equals

\[
C_{\text{ML}}(\epsilon) := 2c\epsilon^{-2} \left( \sum_{k=1}^{n} \sqrt{k \mathbb{V}_{0}[D_{k,1}^{g}]} \right)^2 = \mathcal{O}(\epsilon^{-2}) \quad \text{as} \quad \epsilon \to 0.
\]

Proposition 2 implies \( n(\epsilon) = \mathcal{O}(\log(1/\epsilon)) \) as \( \epsilon \to 0 \) and, together with Remark 1(i) above, that the variance of \( \theta_{1}^{g,k} \) is bounded in \( k \in \mathbb{N} \):

\[
\mathbb{V}_{0}[\theta_{1}^{g,k}] \leq \mathbb{E}_{0}[g(\chi_{k})^2Y_{\lambda}^2] \leq 2\mathbb{E}_{0}[g(\chi)^2Y_{\lambda}^2] + 2\mathbb{E}_{0}[(\Delta_{k}^{g})^2] \to 2\mathbb{E}_{0}[g(\chi)^2Y_{\lambda}^2] < \infty \quad \text{as} \quad k \to \infty.
\]

Note that barrier-type payoffs \( g \) considered in Proposition 2 satisfy the second moment assumption, while in the Lipschitz or locally Lipschitz cases it is sufficient to assume additionally that \( \lambda_{+} \) is either positive or strictly greater than one, respectively. Moreover, \( \mathbb{V}_{0}[D_{k,1}^{g}] \leq 2\mathbb{E}_{0}[(\Delta_{k}^{g})^2 + (\Delta_{k-1}^{g})^2] = \mathcal{O}(n_{g}^{-k}) \) for \( n_{g} > 1 \) bounded away from one (see Remark 1(iii) above). These facts and the standard complexity analysis for MLMC (see e.g. [GCM20, App. A] and the references therein) imply that the estimators \( \hat{\theta}_{\text{MC}}^{g,n} \) and \( \hat{\theta}_{\text{ML}}^{g,n} \) are \( \epsilon \)-accurate with the stated computational costs, proving Proposition 3.

We stress that payoffs \( g \) in Proposition 3 include discontinuous payoffs in the supremum \( X_{T} \) (barrier-type 1) or the time \( \tau_{T} \) this supremum is attained (barrier-type 2), with the corresponding
computational complexities of the MC and MLMC estimators given by $O(\epsilon^{-2} \log(1/\epsilon))$ and $O(\epsilon^{-2})$, respectively. This theoretical prediction matches the numerical performance of TSB-Alg for barrier options and the modified ulcer index, see Section 4.2 below.

In order to obtain confidence intervals (CIs) for the MC and MLMC estimators $\hat{\theta}_{\text{MC}}^{g, n}$ and $\hat{\theta}_{\text{ML}}^{g, n}$, a central limit theorem can be very helpful. In fact, the CLT is necessary to construct a CI if the constants in the bounds on the bias in Proposition 2 are not explicitly known (e.g. for barrier-type 1 payoffs, the constant depends on the unknown value of the density of the supremum $\overline{X}_T$ at the barrier), see the discussion in [GCMUB21, Sec. 2.3]. Moreover, even if the bias can be controlled explicitly, the concentration inequalities typically lead to wider CIs than those based on the CLT, see [BAK15, HK19]. The following result establishes the CLT for the MC and MLMC estimators valid for payoffs considered in Proposition 2.

**Theorem 4 (CLT for TSB-Alg).** Let the payoff $g$ be as in Proposition 2 with $p = 2$ and $\mathbb{E}_0 [g(\chi)^2 \gamma^2] < \infty$. Let $\eta_g \in (1, 2]$ be the rate satisfying $\mathbb{E}_0 [|| \Delta \chi ||] = O(\eta_g^{-n})$, given in Proposition 2 and Remark 1(iii) above. Fix $c_0 > 1/ \log \eta_g$, let $n = n(\epsilon) := \lceil c_0 \log(1/\epsilon) \rceil$ and suppose $N$ and $N_1, \ldots, N_n$ are as in Proposition 3. Then the MC and MLMC estimators $\hat{\theta}_{\text{MC}}^{g, n}$ and $\hat{\theta}_{\text{ML}}^{g, n}$ respectively satisfy the following CLTs ($Z$ is a standard normal random variable):

\[
\sqrt{2\epsilon^{-1}} (\hat{\theta}_{\text{MC}}^{g, n}(\epsilon) - \mathbb{E}_\lambda [g(\chi)]) \xrightarrow{d} Z, \quad \text{and} \quad \sqrt{2\epsilon^{-1}} (\hat{\theta}_{\text{ML}}^{g, n}(\epsilon) - \mathbb{E}_\lambda [g(\chi)]) \xrightarrow{d} Z, \quad \text{as } \epsilon \to 0.
\]

Theorem 4 works well in practice: in Figure 4.7 of Section 4.2 below we construct CIs (as a function of decreasing accuracy $\epsilon$) for an MLMC estimator of a barrier option price under a tempered stable model. The rate $c_0$ can be taken arbitrarily close to $1/ \log \eta_g$, where $\eta_g$ is the corresponding geometric rate of decay of the error for the payoff $g$ in Proposition 2 ($\eta_g$ is bounded away from one by Remark 1(iii) above), ensuring that the bias of the estimators vanishes in the limit.

By Lemma 7 below, the definition of the sample sizes $N$ and $N_1, \ldots, N_n$ in Proposition 3 implies that the variances of the estimators $\hat{\theta}_{\text{MC}}^{g, n}(\epsilon)$ and $\hat{\theta}_{\text{ML}}^{g, n}(\epsilon)$ (under $\mathbb{P}_\theta$) satisfy

\[
\frac{\mathbb{V}_0[\hat{\theta}_{\text{MC}}^{g, n}(\epsilon)]}{\epsilon^2/2} = \frac{\mathbb{V}_0[\hat{\theta}_{1}^{g, n}(\epsilon)]}{\epsilon^2 N/2} \to 1 \quad \& \quad \frac{\mathbb{V}_0[\hat{\theta}_{\text{ML}}^{g, n}(\epsilon)]}{\epsilon^2/2} = \sum_{k=1}^{n(\epsilon)} \frac{\mathbb{V}_0[D_{k,1}^g]}{\epsilon^2 N_k/2} \to 1 \quad \text{as } \epsilon \to 0.
\]

Hence, the CLT in (3.6) can also be expressed as

\[
(\hat{\theta}_{\text{MC}}^{g, n}(\epsilon) - \mathbb{E}_\lambda [g(\chi)])/\mathbb{V}_0[\hat{\theta}_{\text{MC}}^{g, n}(\epsilon)]^{1/2} \xrightarrow{d} Z \quad \text{and} \quad (\hat{\theta}_{\text{ML}}^{g, n}(\epsilon) - \mathbb{E}_\lambda [g(\chi)])/\mathbb{V}_0[\hat{\theta}_{\text{ML}}^{g, n}(\epsilon)]^{1/2} \xrightarrow{d} Z, \quad \text{as } \epsilon \to 0.
\]

Since the variances $\mathbb{V}_0[\hat{\theta}_{\text{MC}}^{g, n}(\epsilon)]$ and $\mathbb{V}_0[\hat{\theta}_{\text{ML}}^{g, n}(\epsilon)]$ can be estimated from the sample, this is in fact how the CLT is often applied in practice. The proof of Theorem 4 is based on the CLT for triangular arrays and amounts to verifying Lindeberg’s condition for the estimators $\hat{\theta}_{\text{MC}}^{g, n}$ and $\hat{\theta}_{\text{ML}}^{g, n}$; see Section 5 below.
3.3. Comparisons. This subsection performs non-asymptotic and asymptotic performance comparisons of estimators based on TSB-Alg. The main aim is to develop rule-of-thumb principles guiding the user to the most effective estimator. In Subsection 3.3.1, for a given value of accuracy \( \epsilon \), we compare the computational complexity of the MC and MLMC estimators based on TSB-Alg. The MLMC estimator based on TSB-Alg is compared with the ones based on SB-Alg \([GCMUB21]\) with rejection sampling (available only when the jumps of \( X \) are of finite variation) and SBG-Alg \([GCM20]\) in Subsections 3.3.2 and 3.3.3, respectively. In both cases we analyse the behaviour of the computational complexity in two regimes: (I) \( \epsilon \) tending to 0 and fixed time horizon \( T \); (II) fixed \( \epsilon \) and time horizon \( T \) tending to 0 or \( \infty \).

Regime (II) is useful when there is a limited benefit to arbitrary accuracy in \( \epsilon \) but the constants may vary greatly with the time horizon \( T \). For example, in option pricing, estimators with accuracy smaller than a basis point are of limited interest. For simplicity, in the remainder of this subsection the payoff \( g \) is assumed to be Lipschitz. However, analogous comparisons can be made for other payoffs under appropriate assumptions.

3.3.1. Comparison between MC and MLMC estimators. Propositions 2 and 3 imply that MLMC estimator outperforms the MC estimator as \( \epsilon \to 0 \). Moreover, since \( \mathbb{V}_0[g(\chi_n)\Upsilon_\lambda] \to \mathbb{V}_0[g(\chi)\Upsilon_\lambda] \) and \( c^2c_{\text{ML}}(\epsilon) \to 2c(\sum_{k=1}^{\infty}(k\mathbb{V}_0[D_{k,1}^g])^{1/2})^2 < \infty \) as \( \epsilon \to 0 \), the MLMC estimator is preferable to the MC estimator for \( \epsilon > 0 \) satisfying

\[
n(\epsilon) > \left( \sum_{k=1}^{\infty} \sqrt{k\mathbb{V}_0[D_{k,1}^g]} \right)^2 / \mathbb{V}_0[g(\chi)\Upsilon_\lambda].
\]

Since the payoff \( g \) is Lipschitz, Proposition 2 implies that this condition is close to

\[
\log(1/\epsilon) > \log(\eta_1) \left( \sum_{k=1}^{\infty} \sqrt{k(2^{-k} - \eta_1^{-2k})} \right)^2,
\]

where we recall that \( \eta_1 \in [3/2, 2] \) is defined in (5.4) below. In particular, the latter condition is equivalent to \( \epsilon < 0.0915 \) if \( \eta_1 = 3/2 \), or \( \epsilon < 5.06 \times 10^{-5} \) if \( \eta_1 = 2 \).

3.3.2. Comparison with SB-Alg. In the special case when the jumps of \( X \) have finite variation (equivalently, \( \int_{(-1,1)} |x|\nu(dx) < \infty \)), the increments \( X_t \) can be simulated under \( \mathbb{P}_\lambda \) using rejection sampling (see \([KM11, Gra19]\)), making SB-Alg \([GCMUB21]\) applicable to sample \( \chi_n \) (see (2.5) for definition) under \( \mathbb{P}_\lambda \). The rejection sampling is performed for each of the increments of the subordinators

\[
\tilde{Y}_t^{(\pm)} := \pm Y_t^{(\pm)} + d_\pm t, \quad \text{where} \quad d_+ := \int_{(0,1)} x \nu(dx), \quad \text{and} \quad d_- := \int_{(-1,0)} |x| \nu(dx),
\]

and the processes \( Y^{(\pm)} \) are as in Theorem 1. The algorithm proposes samples under \( \mathbb{P}_0 \) and rejects independently with probability \( \exp(-\lambda \pm \tilde{Y}_t^{(\pm)}) \). Let \( \lambda_+ := (\lambda_+, 0) \) and \( \lambda_- := (0, \lambda_-) \), then the
expected number of proposals required for each sample equals \( \exp(\gamma_{\lambda_+} t) = 1/E_0[\exp(-\lambda_+ \tilde{Y}_t^{(+)}]) \), where we define
\[
\gamma_{\lambda_+} := \lambda_+ d_+ - \mu_{\lambda_+} = \int_{\mathbb{R}_+} (1 - e^{-\lambda_+ |x|}) \nu(dx) \in [0, \infty).
\]
(Note that \( \mu_{\lambda_+} = p_{\lambda_+} = p(\gamma_{\lambda_+} + \gamma_{\lambda_-}) - (p_{\lambda_+} + p_{\lambda_-}) \), see (2.4).)

We need the following elementary lemma to analyse the computational complexity of SB-Alg with rejection sampling.

**Lemma 5.** (a) Let \( \ell \) be a stick-breaking process on \([0, 1]\), then for any \( n \in \mathbb{N} \) we have
\[
0 \leq n + \int_0^1 \frac{1}{x} (e^{c^x} - 1) dx - \sum_{k=1}^{n} E[e^{c_{\ell_k}}] \leq 2^{-n} \int_0^1 \frac{1}{x} (e^{c^x} - 1) dx.
\]
(b) We have \( c^{-1} e^{-c} \int_0^1 x^{-1} (e^{c^x} - 1) dx \to 1 \) as either \( c \to 0 \) or \( c \to \infty \).

Assume that the simulation of the increments \( Y_t^{(\pm)} \) under \( \mathbb{P}_0 \) have constant cost independent of the time horizon \( t \) (we also assume without loss of generality that the simulation of uniform random variables and the evaluation of operators such as sums, products and the exponential functions have constant cost). Since the SB-Alg requires the rejection sampling to be carried out over random stick-breaking lengths, the expected cost of the SB-Alg with rejection sampling is, by (3.8) in Lemma 5, asymptotically proportional to
\[
\sum_{k=1}^{n} E[e^{\gamma_{\lambda_+} \ell_k} + e^{\gamma_{\lambda_-} \ell_k}] = 2n + (1 + O(2^{-n})) \int_0^1 \frac{1}{x} (e^{\gamma_{\lambda_+} T x} + e^{\gamma_{\lambda_-} T x} - 2) dx, \quad n \to \infty.
\]
In fact, by Lemma 5(a), the absolute value of the term \( O(2^{-n}) \) is bounded by \( 2^{-n} \). Moreover, by Lemma 5(b), the integral in (3.9) may be replaced with an explicit expression
\[
\Gamma_{\lambda}(T) := \frac{\gamma_{\lambda_+} T}{1 + (\gamma_{\lambda_+} T)^2} e^{\gamma_{\lambda_+} T} + \frac{\gamma_{\lambda_-} T}{1 + (\gamma_{\lambda_-} T)^2} e^{\gamma_{\lambda_-} T},
\]
as \( T \) tends to either 0 or \( \infty \).

Table 2 shows how SB-Alg with rejection sampling compares to TSB-Alg above. The bias has been omitted as they are equal for both algorithms.

*Regime (I).* By Table 2, we can deduce that the MC and MLMC estimators of both algorithms have the complexities \( O(\epsilon^{-2} \log(1/\epsilon)) \) and \( O(\epsilon^{-2}) \) as \( \epsilon \to 0 \), respectively, for all the payoffs considered in Proposition 2.

*Regime (II).* Assume \( \epsilon \) is fixed. The biases of both algorithms agree and equal \( O(\eta_1 n \sqrt{T + T}) \), implying \( n = \log((\sqrt{T + T})/\epsilon)/\log(\eta_1) + O(1) \) (with \( \eta_1 \) defined in (5.4)). The level variances of SB-Alg and TSB-Alg are \( O(2^{-n} (T + T^2)) \) and \( O(2^{-n} (T + T^2)) e^{(\mu_\lambda - 2\mu_\lambda T)} \), with costs \( O(n + \Gamma_{\lambda}(T)) \) and \( O(n) \), respectively. Thus, by (3.10), the ratios of the level variance and cost converge to 1 as \( T \to 0 \), so the
Table 2. Level variance and expected cost of SB-Alg and TSB-Alg, up to multiplicative constants that do not depend on the time horizon $T$. The bounds on the level variances follow from [GCMUB21, Thm 2] for SB-Alg and Proposition 2 for TSB-Alg.

3.3.3. Comparison with SBG-Alg. Given any cutoff level $\kappa \in (0,1]$, the algorithm SBG-Alg approximates the Lévy process $X$ (under $P_\lambda$ with the generating triplet $(\sigma^2, \nu_\lambda, b_\lambda)$) with a jump-diffusion $X^{(\kappa)}$ and samples exactly the vector $(X_T^{(\kappa)}, \overline{X}_T^{(\kappa)}, \tau_T(X^{(\kappa)}))$, see [GCM20] for details.

**Regime (I).** The complexities of the MC and MLMC estimators based on the SBG-Alg are by [GCM20, Thm 22] equal to $O(\epsilon^{-2-\beta_s})$ and $O(\epsilon^{-\max\{2,2\beta_s\}}(1 + 1_{\{\beta_s=1\}} \log(\epsilon)^2))$, respectively, where $\beta_s$, defined in (5.2) below, is slightly larger than the Blumenthal-Getoor index of $X$. Since, by Subsection 3.2 above, the complexities of the MC and MLMC estimators based on TSB-Alg are $O(\epsilon^{-2} \log(1/\epsilon))$ and $O(\epsilon^{-2})$, respectively, the complexity of TSB-Alg is always dominated by that of SBG-Alg as $\epsilon \to 0$.

**Regime (II).** Suppose $\epsilon > 0$ is fixed. Then, as in Subsection 3.3.2 above, the computational complexity of the MLMC estimator based on TSB-Alg is $O(\epsilon^{-2}(T + T^2)e^{(\mu_\lambda^2 - 2\mu_\lambda)T})$, where the multiplicative constant does not depend on the time horizon $T$. By [GCM20, Thms 3 & 22] and [GCM20,
where \( C \) performance of TSB-Alg deteriorates if the expectation variance reduction via exponential martingales. below, we apply this criterion to tempered stable process, see Figure 4.8.

method, a variance reduction technique, based on exponential computational cost.

\[ \text{Suppose we are interested in estimating } \mathbb{E}_0[\zeta] = \mathbb{E}_0[\zeta \mathcal{Y}_\lambda], \text{ where } \zeta \text{ is either } g(\chi_n) \text{ (MC) or } g(\chi_n) - g(\chi_{n-1}) \text{ (MLMC). We propose drawing samples of } \tilde{\zeta} \text{ under } \mathbb{P}_0, \text{ instead of } \zeta \mathcal{Y}_\lambda, \text{ where} \]

\[ \tilde{\zeta} := \zeta \mathcal{Y}_\lambda - w_0(\mathcal{Y}_\lambda - 1) - w_+(\mathcal{Y}_{\lambda+} - 1) - w_-(\mathcal{Y}_{\lambda-} - 1), \]

and \( w_0, w_+, w_- \in \mathbb{R} \) are constants to be determined (recall \( \lambda_+ = (\lambda_+,0) \) and \( \lambda_- = (0,\lambda_-) \)). Clearly \( \mathbb{E}_0[\tilde{\zeta}] = \mathbb{E}_0[\zeta \mathcal{Y}_\lambda] \) since the variables \( \mathcal{Y}_\lambda, \mathcal{Y}_{\lambda+} \) and \( \mathcal{Y}_{\lambda-} \) have unit mean under \( \mathbb{P}_0 \). We choose \( w_0, w_+ \) and \( w_- \) to minimise the variance of \( \tilde{\zeta} \), by minimising a quadratic form [Gla04, Sec. 4.1.2]. The solution, in terms of the covariance matrix (under \( \mathbb{P}_0 \)) of the vector \( (\zeta \mathcal{Y}_\lambda, \mathcal{Y}_{\lambda+}, \mathcal{Y}_{\lambda-}) \), is:

\[
\left( \begin{array}{c}
w_0 \\
w_+ \\
w_-
\end{array} \right) = \left( \begin{array}{ccc}
2\mathbb{V}_0[\mathcal{Y}_\lambda] & \text{cov}_0(\mathcal{Y}_\lambda, \mathcal{Y}_{\lambda+}) & \text{cov}_0(\mathcal{Y}_\lambda, \mathcal{Y}_{\lambda-}) \\
\text{cov}_0(\mathcal{Y}_\lambda, \mathcal{Y}_{\lambda+}) & 2\mathbb{V}_0[\mathcal{Y}_{\lambda+}] & \text{cov}_0(\mathcal{Y}_{\lambda+}, \mathcal{Y}_{\lambda-}) \\
\text{cov}_0(\mathcal{Y}_\lambda, \mathcal{Y}_{\lambda-}) & \text{cov}_0(\mathcal{Y}_{\lambda+}, \mathcal{Y}_{\lambda-}) & 2\mathbb{V}_0[\mathcal{Y}_{\lambda-}]
\end{array} \right)^{-1} \cdot \left( \begin{array}{c}
\text{cov}_0(\zeta \mathcal{Y}_\lambda, \mathcal{Y}_\lambda) \\
\text{cov}_0(\zeta \mathcal{Y}_\lambda, \mathcal{Y}_{\lambda+}) \\
\text{cov}_0(\zeta \mathcal{Y}_\lambda, \mathcal{Y}_{\lambda-})
\end{array} \right).
\]

In practice, these covariances are estimated from the same samples that were drawn to estimate \( \mathbb{E}_0[\zeta \mathcal{Y}_\lambda] \). The additional cost is (nearly) negligible as all the variables in the exponential martingales are by-products of TSB-Alg. It is difficult to establish theoretical guarantees for the improvement
of $\zeta$ over $\zeta \Upsilon_\lambda$. However, since most of the variance of the estimator based on $\zeta \Upsilon_\lambda$ comes from $\Upsilon_\lambda$, $\zeta$ typically performs well in applications, see e.g. the CIs in Figures 4.3 and 4.4 for a tempered stable process.

4. Extrema of tempered stable processes

4.1. Description of the model. In the present section we apply our results to the tempered stable process $X$. More precisely, given $\lambda \in \mathbb{R}_+^d \setminus \{0\}$, the tempered stable Lévy measure $\nu_\lambda$ specifies the Lévy measure $\nu$ via (2.1):

$$\nu(dx) = \frac{c_+}{x^{a_+ + 1}} \cdot 1_{(0,\infty)}(x) + \frac{c_-}{|x|^{a_- + 1}} \cdot 1_{(-\infty,0)}(x),$$

where $c_\pm \geq 0$ and $\alpha_\pm \in (0,2)$. The drift $b$ is given by the tempered stable drift $b_\lambda \in \mathbb{R}$ via (2.1) and the constant $\mu_\lambda$ is given in (2.4). Both $b$ and $\mu_\lambda$ can be computed using (4.2) and (4.3) below. TSB-Alg samples from the distribution functions $F^{(\pm)}(t,x) = \mathbb{P}_0(Y^{(\pm)}_t \leq x)$, where $Y^{(\pm)}$ are the spectrally one-sided stable processes defined in Theorem 1, using the Chambers-Mallows-Stuck algorithm [CMS76]. We included a version of this algorithm in the appendix, given explicitly in terms of the drift $b$ and the parameters in the Lévy measure $\nu$, see Algorithm 1 below.

Next, we provide explicit formulae for $b$ and $\mu_\lambda$ in terms of special functions. We begin by expressing $b$ in terms of $b_\lambda$ (see (2.1) above):

$$b = b_\lambda - c_+ B_{a_+,\lambda_+} - c_- B_{a_-,\lambda_-}, \quad \text{where} \quad B_{a,r} := \int_0^1 (e^{-rx} - 1)x^{-a}dx.$$  

We have $B_{a,0} = 0$ for any $a \geq 0$ and, for $r > 0$,

$$B_{a,r} = \sum_{n=1}^{\infty} \frac{(-r)^n}{n!(n-a-1)} = \begin{cases} (e^{-r} - 1 + r^{a-1} \gamma(2-a,r))/(1-a), & a \in (0,2) \setminus \{1\}, \\ -\gamma - \log r - E_1(r), & a = 1, \end{cases}$$

where $\gamma(a,r) = \int_0^r e^{-x}x^{a-1}dx = \sum_{n=0}^{\infty}(-1)^n r^{n+a}/(n!(n+a))$, $a > 0$, is the lower incomplete gamma function, $E_1(r) = \int_r^{\infty} e^{-x}x^{-1}dx$, $r > 0$, is the exponential integral and $\gamma = 0.57721566\ldots$ is the Euler-Mascheroni constant. Similarly, to compute $\mu_\lambda$ from (2.4), note that

$$\mu_\lambda = c_+ C_{a_+,\lambda_+} + c_- C_{a_-,\lambda_-}, \quad \text{where} \quad C_{a,r} := \int_0^{\infty} (e^{-rx} - 1 + r x \cdot 1_{(0,1)}(x))x^{-a-1}dx.$$  

Clearly, $C_{a,0} = 0$ for any $a \geq 0$ and, for $r > 0$,

$$C_{a,r} = -\frac{1}{a} \left( e^{-r} + r(1 + B_{a,r}) \right) + r^a \Gamma(-a,r),$$

where $\Gamma(a,r) = \int_r^{\infty} e^{-x}x^{a-1}dx$ is the upper incomplete gamma function. When $a < 1$, this simplifies to $C_{a,r} = r^a \Gamma(-a) + r/(1-a)$, where $\Gamma(\cdot)$ is the gamma function.
As discussed in Subsection 3.3 (see also Table 2 above), the performance of TSB-Alg deteriorates for large values of the constant $\mu_2 - 2\mu_\lambda$. As a consequence of the formulae above, it is easy to see that this constant is proportional to $\lambda^{\alpha_+}/(2 - \alpha_+) + \lambda^{\alpha_-}/(2 - \alpha_-)$ as either $\lambda_+ \rightarrow \infty$ or $\alpha_\pm \rightarrow 2$, see Figure 4.1.

It is natural to expect the performance of TSB-Alg to deteriorate as $\lambda_\pm \rightarrow \infty$ or $\alpha_\pm \rightarrow 2$ because the variance of the Radon-Nikodym derivative $\Upsilon_\lambda$ tends to $\infty$, making the variance of all the estimators very large.

4.2. Monte Carlo and multilevel Monte Carlo estimators for tempered stable processes.

Let $X$ denote the tempered stable process with generating triplet $(\sigma^2, \nu_\lambda, b_\lambda)$ given in Subsection 4.1. Then $S_t = S_0 e^{X_t}$ models the risky asset for some initial value $S_0 > 0$ and all $t \geq 0$. Observe that $\underline{S}_t = S_0 e^{\overline{X}_t}$ and that the time at which $S$ attains its supremum on $[0, t]$ is also $\tau_t$. Since the tails $\nu(\mathbb{R} \setminus (-x, x))$ of $\nu$ decay polynomially as $x \rightarrow \infty$ (see (4.1)), $S_t$ and $\underline{S}_t$ satisfy the following moment conditions under $P_\lambda$: $\mathbb{E}_\lambda[\underline{S}_t^r] < \infty$ (resp. $\mathbb{E}_\lambda[\overline{S}_t^r] < \infty$) if and only if $r \in [-\lambda_-, \lambda_+]$ (resp. $r \leq \lambda_+$). In the present subsection we apply TSB-Alg to estimate $\mathbb{E}_\lambda[g(\chi)]$ using the MC and MLMC estimators in (3.2) and (3.3), respectively, for payoffs $g$ that arise in applications and calibrated/estimated values of tempered stable model parameters.

4.2.1. MC estimators. Consider the estimator in (3.2) for the following payoff functions $g$: (I) the payoff of the up-and-out barrier call option $g(\chi) = \max\{S_T - K, 0\} 1\{\underline{S}_T \leq M\}$ and (II) the function $g(\chi) = (S_T/\underline{S}_T - 1)^2$ associated to the ulcer index (UI) given by $100\sqrt{\mathbb{E}_\lambda[g(\chi)]}$ (see [Dow]). We plot the estimated value along with a band, representing the empirically constructed CIs of level 95% via Theorem 4, see Figures 4.2 and 4.3. We set the initial value $S_0 = 100$, assume $b_\lambda = 0$ and consider multiple time horizons $T$: we take $T \in \{30, 90\}$ in (I) and $T \in \{14, 28\}$ in (II). The values of the other parameters, given in Table 3 below, are either calibrated or estimated: in (I)
we use calibrated parameters with respect to the risk-neutral measure obtained in [AL13, Table 3] for the USD/JPY foreign exchange (FX) rate and in (II) we use the maximum likelihood estimates in [CGMY02, Table 2] for various time series of historic asset returns. In both, (I) and (II), it is assumed that $\alpha_\pm = \alpha$; (II) additionally assumes that $c_+ = c_-$ (i.e., $X$ is a CGMY process). The stocks MCD, BIX and SOX in (II) were chosen with $\alpha > 1$ to stress test the performance of TSB-Alg when $\mu_{2\lambda} - 2\mu_\lambda$ is big, forcing the variance of the estimator to be large, see the discussion at the end of Subsection 4.1 above and Figure 4.1.

| Stock     | $\sigma$ | $\alpha$ | $c_+$ | $c_-$ | $\lambda_+$ | $\lambda_-$ | $\mu_{2\lambda} - 2\mu_\lambda$ |
|-----------|----------|----------|-------|-------|-------------|-------------|-------------------------------|
| USD/JPY (v1) | 0.0007   | 0.66     | 0.1305| 0.0615| 6.5022      | 3.0888      | 0.9658                        |
| USD/JPY (v2) | 0.0001   | 1.5      | 0.0069| 0.0063| 1.932       | 0.4087      | 0.0395                        |
| MCD       | 0        | 1.50683  | 0.08  | 0.08  | 25.4        | 25.4        | 41.47                         |
| BIX       | 0        | 1.2341   | 0.32  | 0.32  | 37.42       | 47.76       | 96.6                          |
| SOX       | 0        | 1.3814   | 0.44  | 0.44  | 34.73       | 34.76       | 196.81                        |

Table 3. Calibrated/estimated parameters of the tempered stable model. The first two sets of parameters were calibrated in [AL13, Table 3] based on FX option data. The bottom three sets of parameters were estimated using the MLE in [CGMY02, Table 2], based on a time series of equity returns.

The visible increase in the variance of the estimations for the SOX, see Figure 4.3, as the maturity increases from 14 to 28 days is a consequence of the unusually large value of $\mu_{2\lambda} - 2\mu_\lambda$, see Table 3 above. If $\mu_{2\lambda} - 2\mu_\lambda$ is not large, we may take long time horizons and relatively few samples to obtain...
an accurate Monte Carlo estimate. In fact this appears to be the case for calibrated risk-neutral parameters (we were unable to find calibrated parameter values in the literature with a large value of $\mu_2 \lambda - 2 \mu \lambda$). However, if $\mu_2 \lambda - 2 \mu \lambda$ is large, then for any moderately large time horizon an accurate Monte Carlo estimate would require a large number of samples. In such cases, the control variates method from Subsection 3.4 becomes very useful.

To illustrate the control variates method from Subsection 3.4, we apply it in the setting of Figure 4.3, where we observed the widest CIs. Figure 4.4 displays the resulting estimators and CIs, showing that this method is beneficial in the case where the variance of the Radon-Nikodym derivative $\Upsilon_\lambda$ is large, i.e., when $(\mu_2 \lambda - 2 \mu \lambda) T$ is large. The confidence intervals for the SOX asset at $n = 12$ shrank by a factor of 4.23, in other words, the variance became 5.58% of its original value.

4.2.2. Multilevel Monte Carlo estimators. We will consider the MLMC estimator in (3.3) with parameters $n, N_1, \ldots, N_n \in \mathbb{N}$ given by [GCM20, Eq. (A.1)–(A.2)]. In this example we chose (I)
the payoff \( g(\chi) = \max\{S_T - 95, 0\} \mathbb{1}\{S_T \leq 102\} \) from Subsection 4.2.1 above and (II) the payoff 
\( g(\chi) = (S_T/\overline{S}_T - 1)^2 \mathbb{1}\{\tau_T < T/2\} \), associated to the modified ulcer index (MUI) 
100\sqrt{\mathbb{E}_\lambda[g(\chi)]}, a risk measure which weighs trends more heavily than short-time fluctuations (see \cite{GCM20} and the references therein).

The payoff in (I) is that of a barrier up-and-out option, so it is natural to use the risk-neutral 
parameters for the USD/JPY FX rate (see (v2) in Table 3) over the time horizon \( T = 90/365 \). For 
(II) we take the parameter values of the MCD stock in Table 3 with \( T = 28/365 \). In both cases 
we set \( S_0 = 100 \). Figure 4.5 (resp. Figure 4.6) shows the decay of the bias and level variance, the 
corresponding value of the constants \( n, N_1, \ldots, N_n \) and the growth of the complexity for the first 
(resp. second) payoff.

![Bias decay](image1)

![Variance decay](image2)

**Figure 4.5.** The top pictures show the bias and level variance decay as a function of \( k \) 
for the payoff \( g(\chi) = \max\{S_T - K, 0\} \mathbb{1}\{S_T \leq M\} \) with \( S_0 = 100, T = 90/365, K = 95, 
M = 102 \) and the parameter values for the USD/JPY FX rate (v2) in Table 3. The theoretical 
predictions (dashed) are based on Proposition 2 for barrier-type 1 payoffs. The bottom 
pictures show the corresponding value of the complexities and parameters \( n, N_1, \ldots, N_n \) 
associated to the precision levels \( \epsilon \in \{2^{-9}, 2^{-10}, \ldots, 2^{-19}\} \).

In Figure 4.7 we plot the estimator \( \hat{\theta}_{\text{MC}}^{g_{n}} \) in Theorem 4 for (I), parametrised by \( \epsilon \to 0 \). To further 
illustrate the CLT in Theorem 4, the figure also shows the CIs of confidence level 95% constructed 
using the CLT.
Figure 4.6. The top pictures show the bias and level variance decay as a function of $k$ for the payoff $g(\chi) = (S_T/S_T - 1)^2 \mathbb{1}_{\{\tau^T < T/2\}}$ with $S_0 = 100$, $T = 28/365$ and the parameter values for MCD in Table 3. The theoretical predictions (dashed) are based on Proposition 2 for barrier-type 2 payoffs. The bottom pictures show the corresponding value of the complexities and parameters $n, N_1, \ldots, N_n$ associated to the precision levels $\epsilon \in \{2^{-9}, 2^{-10}, \ldots, 2^{-13}\}$.

Figure 4.7. The solid line indicates the estimated value of the expectation $\mathbb{E}_\lambda g(\chi_n)$ for the payoff $g(\chi) = \max\{S_T - K, 0\}\mathbb{1}_{\{S_T \leq M\}}$ with $S_0 = 100$, $T = 90/365$, $K = 95$, $M = 102$ and the parameter values for the USD/JPY FX rate (v2) in Table 3. We use the MLMC estimator $\hat{\theta}_g^{n,M}$ in (3.3) and the confidence intervals (dotted lines) are constructed using Theorem 4 with confidence level 95%.

4.3. Comparing TSB-Alg with existing algorithms for tempered stable processes. In this subsection we take the analysis from Subsection 3.3 and apply it to the tempered stable case.
4.3.1. Comparison with SB-Alg. Recall from Subsection 3.3.2 that SB-Alg is only applicable when \( \alpha_\pm < 1 \) and, under Regime (II), SB-Alg is preferable over TSB-Alg for all sufficiently large \( T \) if and only if \( \max\{\gamma^{(+)}_\lambda, \gamma^{(-)}_\lambda\} \leq \mu_2\lambda - 2\mu_\lambda \), where \( \gamma^{(\pm)}_\lambda = -c_+\lambda^{\alpha_\pm}_\lambda - c_-\lambda^{\alpha_-}_\lambda \geq 0 \) is defined in (3.7). By the formulae in Subsection 4.1, it is easily seen that \( \max\{\gamma^{(+)}_\lambda, \gamma^{(-)}_\lambda\} \leq \mu_2\lambda - 2\mu_\lambda \) is equivalent to

\[
\min\{c_+\lambda^{\alpha_+}_\lambda, c_-\lambda^{\alpha_-}_\lambda\} \geq c_+\lambda^{\alpha_+}_\lambda (2 - 2^{\alpha_+})\Gamma(-\alpha_+) + c_-\lambda^{\alpha_-}_\lambda (2 - 2^{\alpha_-})\Gamma(-\alpha_-).
\]

Assuming that \( \alpha_\pm = \alpha \), the inequality simplifies to \( \alpha \leq \phi(\varrho) \), where we define \( \phi(x) := \log\left(1 + \frac{x}{1+x}\right) \) and \( \varrho := \min\{c_+\lambda^{\alpha}_\lambda, c_-\lambda^{\alpha}_\lambda\}/\max\{c_+\lambda^{\alpha}_\lambda, c_-\lambda^{\alpha}_\lambda\} \). In particular, a symmetric Lévy measure yields \( \varrho = 1 \) and \( \phi(1) = \log_2(3/2) = 0.58496\ldots \), and a one-sided Lévy measure gives \( \varrho = 0 \) and \( \phi(0) = 0 \).

\[\text{Figure 4.8. The picture shows the map } \varrho \mapsto \phi(\varrho), \varrho \in [0, 1]. \text{ Assuming } \alpha_\pm = \alpha \text{ and defining } \varrho := \min\{c_+\lambda^{\alpha}_\lambda, c_-\lambda^{\alpha}_\lambda\}/\max\{c_+\lambda^{\alpha}_\lambda, c_-\lambda^{\alpha}_\lambda\}, \text{ TSB-Alg is preferable to SB-Alg when } (\varrho, \alpha) \text{ lies in the shaded region.}\]

4.3.2. Comparison with SBG-Alg. Recall from Subsection 3.3.3 that TSB-Alg is preferable to SBG-Alg when \( \max\{\alpha_+, \alpha_-\} \geq 1 \) (as it is equivalent to \( \beta_* \geq 1 \)). On the other hand, if \( \alpha_\pm < 1 \) (equivalently, \( \beta_* < 1 \)), TSB-Alg outperforms SB-Alg if and only if \( (1 + T)e^{(\mu_2\lambda - 2\mu_\lambda)T} \leq C_1 + C_2T \).

For large enough \( T \), the SB-Alg will outperform TSB-Alg; however, it is generally hard to determine when this happens. We illustrate the region of parameters \((T, \alpha)\) where TSB-Alg is preferable in Figure 4.9, assuming \( \alpha_\pm = \alpha \in (0, 1) \) and all other parameters are as in the USD/JPY (v1) currency pair in Table 3.

5. Proofs

Proof of Theorem 1. The exponential change-of-measure theorem for Lévy processes (see [Sat13, Thms 33.1 & 33.2]) implies that for any measurable function \( F \) with \( \mathbb{E}_\lambda|F((Y_t)_{t \in [0,T]})| < \infty \), we have the identity \( \mathbb{E}_\lambda[F((Y_t)_{t \in [0,T]})] = \mathbb{E}_0[F((Y_t)_{t \in [0,T]}) \mathbb{Y}_\lambda] \), where \( \mathbb{Y}_\lambda \) is defined in (2.3) in the statement of Theorem 1. Since the stick-breaking process \( \ell \) is independent of \( Y \) under both \( \mathbb{P}_\lambda \)
and $\mathbb{P}_0$, this property extends to measurable functions of $(\ell, (Y_t)_{t \in [0,T]})$ and thus to the measurable functions of $(\ell, \xi)$, as claimed.

We now introduce the geometric rate $\eta_p$ used in Proposition 2 above. Let $\beta$ be the Blumenthal-Getoor index \cite{BG61}, defined as

\begin{equation}
\beta := \inf\{p > 0 : I_0^p < \infty\}, \quad \text{where} \quad I_0^p := \int_{(-1,1)} |x|^p \nu(dx), \quad \text{for any} \ p \geq 0,
\end{equation}

and note that $\beta \in [0,2]$ since $I_0^2 < \infty$. Moreover, $I_0^1 < \infty$ if and only if the jumps of $X$ have finite variation. In the case of (tempered) stable processes, $\beta$ is the greatest of the two activity indices of the Lévy measure. Note that $I_0^p < \infty$ for any $p > \beta$ but $I_0^\beta$ can be either finite or infinite. If $I_0^\beta = \infty$ we must have $\beta < 2$ and can thus pick $\delta \in (0, 2 - \beta)$, satisfying $\beta + \delta < 1$ whenever $\beta < 1$, and define

\begin{equation}
\beta^* := \beta + \delta \cdot \mathbf{1}_{\{I_0^\beta = \infty\}} \in [\beta, 2].
\end{equation}

The index $\beta^*$ is either equal to $\beta$ or arbitrarily close to it. In either case we have $I_0^{\beta^*} < \infty$. Define $\alpha \in [\beta, 2]$ and $\alpha^* \in [\beta^*, 2]$ by

\begin{equation}
\alpha := 2 \cdot \mathbf{1}_{\sigma \neq 0} + \mathbf{1}_{\sigma = 0} \begin{cases} 
1, & I_0^1 < \infty \text{ and } b_0 \neq 0 \\
\beta, & \text{otherwise,}
\end{cases}
\end{equation}

and

\begin{equation}
\alpha^* := \alpha + (\beta^* - \beta) \cdot \mathbf{1}_{\alpha = \beta}.
\end{equation}

Finally, we may define

\begin{equation}
\eta_p := 1 + \mathbf{1}_{p > \alpha} + \frac{p}{\alpha^*} \cdot \mathbf{1}_{p \leq \alpha} \in (1, 2], \quad \text{for any} \quad p > 0,
\end{equation}

and note that $\eta_p \geq 3/2$ for $p \geq 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{shaded_region}
\caption{The shaded region is the set of points $(T, \alpha)$ where TSB-Alg is preferable to SBG-Alg assuming $\alpha_\pm = \alpha \in (0, 1)$ and all other parameters are as in the USD/JPY (v1) currency pair in Table 3.}
\end{figure}
In order to prove Proposition 2 for barrier-type 1 payoffs, we need to ensure that \( \bar{X}_T \) has a sufficiently regular distribution function under \( \mathbb{P}_{p\lambda} \). The following assumption will help us establish that in certain cases of interest.

**Assumption (S).** Under \( \mathbb{P}_0 \), the Lévy process \( X = (X_t)_{t \in [0,T]} \) is in the domain of attraction of an \( \alpha \)-stable process as \( t \to 0 \) with \( \alpha > 1 \).

When \( X \) is tempered stable, Assumption (S) holds trivially if \( \max\{\alpha_+, \alpha_-\} > 1 \) or \( \sigma \neq 0 \). The index \( \alpha \) in Assumption (S) necessarily agrees with the one in (5.3), see [BI20, Subsec. 2.1]. For further sufficient and necessary conditions for Assumption (S), we refer the reader to [Iva18, BI20]. In particular, Assumption (S) remains satisfied if the Lévy measure \( \nu \) is modified away from 0 or the law of \( X \) is changed under an equivalent change of measure, see [BI20, Subsec. 2.3.4].

**Lemma 6.** For any Borel set \( A \subset \mathbb{R} \times [0,T] \) and \( p > 1 \) we have

\[
(5.5) \quad \mathbb{P}_\lambda(\chi \in A) \leq e^{(\mu_\lambda - p\mu_\lambda)/p}\mathbb{P}_0(\chi \in A)^{1-1/p},
\]

where the constants \( \mu_\lambda \) and \( \mu_\lambda \) are defined in (2.4). Moreover, if \( I_0^1 < \infty \), then we also have

\[
(5.6) \quad \mathbb{P}_\lambda(\chi \in A) \leq e^{\gamma_\lambda^+ + \gamma_\lambda^-} \mathbb{P}_0(\chi \in A),
\]

where the constants \( \gamma_\lambda^\pm \) are defined in (3.7).

The proofs of Lemma 6 and Proposition 2 rely on the identity \( \Upsilon^\rho_\lambda = \Upsilon_{p\lambda} e^{\mu_\lambda - p\mu_\lambda} \), valid for any \( \lambda \in \mathbb{R}_+^2 \) and \( p \geq 1 \).

**Proof of Lemma 6.** Fix the Borel set \( A \). Applying Theorem 1 and Hölder’s inequality with \( p > 1 \), we obtain

\[
\mathbb{P}_\lambda(\chi \in A) = \mathbb{E}_0[\Upsilon^\rho_\lambda 1_{\{\chi \in A\}}] \leq \mathbb{E}_0[\Upsilon^\rho_\lambda^{1/p}\mathbb{P}_0(1_{\{\chi \in A\}})^{1/q}] = e^{(\mu_\lambda - p\mu_\lambda)/p}\mathbb{P}_0(\chi \in A)^{1/q},
\]

where \( 1/q = 1 - 1/p \), implying (5.5). If \( I_0^1 < \infty \), then \( \mu_\lambda - p\mu_\lambda = p(\gamma_\lambda^+ + \gamma_\lambda^-) - (\gamma_{p\lambda}^+ + \gamma_{p\lambda}^-) \leq p(\gamma_\lambda^+ + \gamma_\lambda^-) \). Thus, taking \( p \to \infty \) (and hence \( q \to 1 \)) in (5.5) yields (5.6). \( \square \)

**Proof of Proposition 2.** Theorem 1 implies that all the expectations in the statement of Proposition 2 can be replaced with the expectation \( \mathbb{E}_{p\lambda}[|g(\chi) - g(\chi_n)|] \). Since \( \lambda \neq 0 \), implying that \( |\mathbb{E}[\max\{X_t, 0\}], \mathbb{E}[\max\{-X_t, 0\}]| < \infty \), [GCMUB21, Prop. 1] yields the result for Lipschitz payoffs. By the assumption in Proposition 2 for the locally Lipschitz case, the Lévy measure \( \nu_{p\lambda} \) in (2.1) satisfies the assumption in [GCMUB21, Prop. 2], implying the result for locally Lipschitz payoffs. The result for barrier-type 2 payoffs follows from a direct application of [GCM20, Lem. 14 & 15] and [GCMUB21, Thm 2].
The result for barrier-type 1 payoffs follows from [GCMUB21, Prop. 3 & Rem. 6] if we show the existence of a constant $K'$ satisfying $P_{pA}(M < \bar{X}_T \leq M + x) \leq K'x^\gamma$ for all $x > 0$. If $\gamma \in (0, 1)$, such $K'$ exists by (5.5) in Lemma 6 above with $p = (1 - \gamma)^{-1} > 1$ and $A = \mathbb{R} \times (M, M + x] \times [0, T]$. If $\gamma = 1$ and $I^1_k < \infty$, the existence of $K'$ follows from the assumption in Proposition 2 and (5.6) in Lemma 6. If $\gamma = 1$ and Assumption (S) holds, then [BI20, Thm. 5.1] implies the existence of $K'$.

\begin{lemma}
Let the payoff $g$ be as in Proposition 2 with $p = 2$ and $\mathbb{E}_0[g(\chi)^2\gamma_\lambda^2] < \infty$. Let $n = n(\epsilon)$, $N$ and $N_1, \ldots, N_n$ be as in Proposition 3, then the following limits hold as $\epsilon \to 0$:

\begin{equation}
\epsilon^2 N_k \to 2\sqrt{\mathbb{V}_0[D^g_{k,1}]/k} \sum_{j=1}^{\infty} j\mathbb{V}_0[D^g_{j,1}], \quad (0, \infty), \quad k \in \mathbb{N},
\end{equation}

\begin{equation}
\frac{\mathbb{V}_0[\theta_{\text{MC},n}(\epsilon)]}{\epsilon^2/2} = \frac{\mathbb{V}_0[\theta_{\text{ML},n}(\epsilon)]}{\epsilon^2 N/2} \to 1, \quad \text{and} \quad \frac{\mathbb{V}_0[\hat{\theta}_{\text{ML},n}(\epsilon)]}{\epsilon^2/2} = \sum_{k=1}^{n} \frac{\mathbb{V}_0[D^g_{k,1}]}{\epsilon^2 N_k/2} \to 1.
\end{equation}

\end{lemma}

\textbf{Proof.} Since $x + 1 \geq [x] \geq x$, we have $B_k(\epsilon) \geq \epsilon^2 N_k \geq B_n, k(\epsilon)$, where

\begin{equation}
B_k(\epsilon) := \epsilon^2 + 2\sqrt{\mathbb{V}_0[D^g_{k,1}]/k} \sum_{j=1}^{\infty} j\mathbb{V}_0[D^g_{j,1}], \quad \text{and} \quad B_{n, k}(\epsilon) := \epsilon^2 + 2\sqrt{\mathbb{V}_0[D^g_{k,1}]/k} \sum_{j=1}^{n} j\mathbb{V}_0[D^g_{j,1}],
\end{equation}

implying the limit in (5.7) (note that since $\mathbb{V}_0[D^g_{k,1}] \leq 2\mathbb{E}_0[(\Delta^g_k)^2 + (\Delta^g_{k-1})^2] = O(\eta^{-k})$ for some $\eta_g > 1$, the limiting value in (5.7) is finite). The first limit in (5.8) follows similarly: $\epsilon^2/2 + \mathbb{V}_0[g(\chi)\gamma_\lambda] \geq \epsilon^2 N/2 \geq \mathbb{V}_0[g(\chi)\gamma_\lambda]$, where $\mathbb{V}_0[\theta_{\text{ML},n}] = \mathbb{V}_0[g(\chi)\gamma_\lambda] \to \mathbb{V}_0[g(\chi)\gamma_\lambda] > 0$ as $\epsilon \to 0$ by the convergence in $L^2$ of Proposition 2. By the same inequalities, we obtain

\begin{equation}
\sum_{k=1}^{n} \frac{\mathbb{V}_0[D^g_{k,1}]}{B_k(\epsilon)/2} \leq \sum_{k=1}^{n} \frac{\mathbb{V}_0[D^g_{k,1}]}{\epsilon^2 N_k/2} \leq \sum_{k=1}^{n} \frac{\mathbb{V}_0[D^g_{k,1}]}{B_{n, k}(\epsilon)/2} = 1.
\end{equation}

The left-hand side converges to 1 by the monotone convergence theorem with respect to the counting measure, implying the second limit in (5.8) and completing the proof. \hfill \Box

\textbf{Proof of Theorem 4.} We first establish the CLT for the MLMC estimator $\hat{\theta}_{\text{ML},n}(\epsilon)$, where $n = n(\epsilon)$ is as stated in the theorem and the numbers of samples $N_1, \ldots, N_n$ are given in (3.4). By (3.1) and (3.3) we have

\begin{equation}
\sqrt{2\epsilon^{-1}} \left( \hat{\theta}_{\text{ML},n}(\epsilon) - \mathbb{E}_0[g(\chi)] \right) = \sqrt{2\epsilon^{-1}} \mathbb{E}_0[\Delta^g_{n}(\epsilon)] + \sum_{k=1}^{n} \sum_{i=1}^{N_k} \xi_{k,i} \quad \text{where} \quad \xi_{k,i} := \sqrt{\frac{2}{\epsilon N_k}} (D^g_{k,i} - \mathbb{E}_0[D^g_{k,i}]).
\end{equation}

By assumption we have $c_0 > 1/\log \eta_g$. Thus, the limit $\sqrt{2\epsilon^{-1}} \mathbb{E}_0[\Delta^g_{n}(\epsilon)] = O(\epsilon^{-1+c_0 \log(\eta_g)}) \to 0$ as $\epsilon \to 0$ follows from Proposition 2. Hence the CLT in (3.6) for the estimator $\hat{\theta}_{\text{ML},n}(\epsilon)$ follows if we prove

\begin{equation}
\sum_{k=1}^{n} \sum_{i=1}^{N_k} \xi_{k,i} \xrightarrow{d} Z \quad \text{as} \quad \epsilon \to 0,
\end{equation}

where
where \(Z\) is a normal random variable with mean zero and unit variance. Thus, by [Kal02, Thm. 5.12], it suffices to note that \(\zeta_{k,i}\) have zero mean \(\mathbb{E}_0[\zeta_{k,i}] = 0\),

\[
\sum_{k=1}^{n(\epsilon)} \sum_{i=1}^{N_k} \mathbb{E}_0[\zeta_{k,i}^2] = \sum_{k=1}^{n(\epsilon)} \frac{2}{\epsilon^2 N_k} \mathbb{V}_0[D^g_{k,1}] \to 1 \quad \text{as} \: \epsilon \to 0,
\]

which holds by (5.8), and establish the Lindeberg condition: for any \(r > 0\) the following limit holds

\[
\sum_{k=1}^{n(\epsilon)} \sum_{i=1}^{N_k} \mathbb{E}_0[\zeta_{k,i}^2 \mathbb{1}_{\{|\zeta_{k,i}|>r\}}] \to 0 \quad \text{as} \: \epsilon \to 0.
\]

To prove the Lindeberg condition first note that

\[
\sum_{i=1}^{N_k} \mathbb{E}_0[\zeta_{k,i}^2 \mathbb{1}_{\{|\zeta_{k,i}|>r\}}] = N_k \mathbb{E}_0[\zeta_{k,1}^2 \mathbb{1}_{\{|\zeta_{k,1}|>r\}}] \leq N_k \mathbb{E}_0[\zeta_{k,1}^2] \quad \text{for any} \: k \in \mathbb{N}.
\]

By (5.7), the bound \(N_k \mathbb{E}_0[\zeta_{k,1}^2] = 2 \mathbb{V}_0[D^g_{k,1}]/(\epsilon^2 N_k)\) converges for all \(k \in \mathbb{N}\) as \(\epsilon \to 0\) to some \(c_k \geq 0\) and \(\sum_{k=1}^{n(\epsilon)} N_k \mathbb{E}_0[\zeta_{k,1}^2] \to 1 = \sum_{k=1}^{\infty} c_k\). Lemma 7 also implies that \(\epsilon N_k \to \infty\) and \(\epsilon^2 N_k\) converges to a positive finite constant as \(\epsilon \to 0\). Since \(\mathbb{V}_0[D^g_{k,1}] < \infty\) for all \(k \in \mathbb{N}\), the dominated convergence theorem implies

\[
N_k \mathbb{E}_0[\zeta_{k,1}^2 \mathbb{1}_{\{|\zeta_{k,1}|>r\}}] = \frac{2 \mathbb{E}_0[(D^g_{k,1} - \mathbb{E}[D^g_{k,1}])^2 \mathbb{1}_{\{|D^g_{k,1}|>\epsilon r \epsilon N_k/2\}}]}{\epsilon^2 N_k} \to 0, \quad \text{as} \: \epsilon \to 0.
\]

Thus, the inequality in (5.9) and the dominated convergence theorem [Kal02, Thm 1.21] with respect to the counting measure yield the Lindeberg condition, establishing the CLT for \(\hat{\theta}_k^g, n(\epsilon)\).

Let us now establish the CLT for the MC estimator \(\hat{\theta}_k^g, n(\epsilon)\), with the number of samples \(N\) given in Proposition 3. As before, by Proposition 2 and the definition of \(n(\epsilon)\) in the theorem, the bias satisfies

\[
\sqrt{2} \epsilon^{-1} \mathbb{E}_0[|\Delta^g_{n(\epsilon)}|] = \mathcal{O}(\epsilon^{-1} \log(n(\epsilon))) \to 0 \quad \text{as} \: \epsilon \to 0.
\]

Thus, by [Kal02, Thm. 5.12], it suffices to show that

\[
2 \mathbb{V}_0[\mathbb{g}(\chi_n) \mathbb{Y}_\lambda]/(\epsilon^2 N) \to 1 \quad \text{as} \: \epsilon \to 0 \quad \text{and the Lindeberg condition holds: for any} \: r > 0,
\]

\[
C(\epsilon) := \sum_{i=1}^{N} \mathbb{E}[|\zeta_{i,n(\epsilon)}|^2 \mathbb{1}_{\{|\zeta_{i,n(\epsilon)}|>r\}}] \to 0, \quad \text{as} \: \epsilon \to 0, \quad \text{where} \quad \zeta_{i,n(\epsilon)} := \frac{\sqrt{2}}{\epsilon N}(\theta_i^g,n - \mathbb{E}_0[\theta_i^g,n]).
\]

The limit \(2 \mathbb{V}_0[\mathbb{g}(\chi_n) \mathbb{Y}_\lambda]/(\epsilon^2 N) \to 1 \quad \text{as} \: \epsilon \to 0\) follows from Lemma 7. To establish the Lindeberg condition, let \(\bar{\theta}_\epsilon := \theta_1^g,n(\epsilon) - \mathbb{E}_0[\theta_1^g,n(\epsilon)]\) and note that

\[
C(\epsilon) = N \mathbb{E}[|\zeta_{1,n(\epsilon)}|^2 \mathbb{1}_{\{|\zeta_{1,n(\epsilon)}|>r\}}] = 2 \mathbb{E}[|\bar{\theta}_\epsilon|^2 \mathbb{1}_{\{|\bar{\theta}_\epsilon|>r \epsilon N/2\}}]/(\epsilon^2 N).
\]

Since \(\theta_1^g,n \xrightarrow{L^2} \mathbb{g}(\chi) \mathbb{Y}_\lambda\) as \(n \to \infty\) by Proposition 2, we get \(\bar{\theta}_\epsilon \xrightarrow{L^2} \mathbb{g}(\chi) \mathbb{Y}_\lambda - \mathbb{E}_0[\mathbb{g}(\chi) \mathbb{Y}_\lambda]\) as \(\epsilon \to 0\). By Lemma 7, \(\epsilon^2 N \to 2 \mathbb{V}_0[\mathbb{g}(\chi) \mathbb{Y}_\lambda] > 0\) and the indicator function in the integrand vanishes in probability since \(\epsilon N \to \infty\). Thus, the dominated convergence theorem [Kal02, Thm 1.21] yields \(C(\epsilon) \to 0\), completing the proof. \(\square\)
Proof of Lemma 5. (a) Note that the density of \( \ell_n \) is given by \( x \mapsto \log(1/x)^{n-1}/(n-1)! \), \( x \in (0,1) \). Note that \( x^{-1} = e^{\log(1/x)} = \sum_{k=0}^{\infty} \log(1/x)^k/k! \), hence

\[
n + \int_0^1 \frac{1}{x}(e^{cx} - 1)dx - \sum_{k=1}^{n} \mathbb{E}[e^{\ell_k}] = \sum_{k=n+1}^{\infty} \mathbb{E}[e^{\ell_k} - 1] = \sum_{k=n+1}^{\infty} \sum_{j=1}^{\infty} \frac{c^j}{j!} \mathbb{E}[\ell_k^j] = \sum_{j=1}^{\infty} \frac{c^j}{j!} \sum_{k=n+1}^{\infty} (1+j)^{-k} = \sum_{j=1}^{\infty} \frac{c^j}{j!} (1+j)^{-n}.
\]

Since \( \int_0^1 x^{-1}(e^{cx} - 1)dx = \sum_{j=1}^{\infty} c^j/(j!)^2 \) and \( (1+j)^{-n} \leq 2^{-n} \) for all \( j \in \mathbb{N} \), the inequality in (3.8) holds, implying (a).

(b) Note that \( \int_0^1 x^{-1}(e^{cx} - 1)dx = \int_0^c x^{-1}(e^x - 1)dx \) and apply l'Hôpital’s rule. □

Appendix A. Simulation of stable laws

In this section we adapt the Chambers-Mellows-Stuck simulation of the increments of a Lévy process \( Z \) with generating triplet \((0, \nu, b)\), where

\[
\frac{\nu(dx)}{dx} = \frac{c_+}{x^{\alpha+1}} \mathbb{1}_{(0,\infty)}(x) + \frac{c_-}{x^{\alpha+1}} \cdot \mathbb{1}_{(-\infty,0)}(x),
\]

for arbitrary \((c_+, c_-) \in \mathbb{R}^2 \setminus \{0\} \) and \( \alpha \in (0,2) \). First, we introduce the constant \( \nu \) given by (see (14.20)–(14.21) in [Sat13, Lem. 14.11]):

\[
\nu = \int_1^\infty x^{-2} \sin(x)dx + \int_0^1 x^{-2}(\sin(x) - x)dx = 1 - \gamma.
\]

Then the characteristic function of \( Z_t \) is given by (see [Sat13, Thm 14.15])

\[
\mathbb{E}[e^{iuZ_t}] = \exp(i\Psi(u)), \quad u \in \mathbb{R},
\]

\[
\Psi(u) = i\mu u - \begin{cases}
\zeta |u|^\alpha (1 - i\theta \tan(\frac{\pi \alpha}{2}) \text{sgn}(u)), & \alpha \in (0, 2) \setminus \{1\}, \\
\zeta |u|(1 + i\theta \frac{2\pi}{\alpha} \text{sgn}(u) \log |u|), & \alpha = 1,
\end{cases}
\]

where \( \theta = (c_+ - c_-)/(c_+ + c_-) \) and the constants \( \mu \) and \( \zeta \) are given by

\[
(\mu, \zeta) = \begin{cases}
(b + \frac{c_- - c_+}{1-\alpha}, -(c_+ + c_-)\Gamma(-\alpha) \cos \left(\frac{\pi \alpha}{2}\right)), & \alpha \in (0, 2) \setminus \{1\}, \\
(b + \nu(c_+ - c_-), \frac{\pi}{2}(c_+ + c_-)), & \alpha = 1.
\end{cases}
\]

Finally, we define Zolotarev’s function

\[
A_{\alpha, r}(u) = \left(1 + \theta^2 \tan^2 \left(\frac{\pi \alpha}{2}\right)\right)^{\frac{1}{2\alpha}} \frac{\sin(a(r + u)) \cos(ar + (a - 1)u)^{1/\alpha - 1}}{\cos(u)^{1/\alpha}}, \quad u \in (-\frac{\pi}{2}, \frac{\pi}{2}).
\]
Algorithm 2. (Chambers-Mallows-Stuck) Simulation of $Z_t$ with triplet $(0, \nu, b)$

**Require:** Parameters $(c_\pm, \alpha, b)$ and time horizon $t > 0$

1. Compute $\theta = (c_+ - c_-)/(c_+ + c_-)$ and $(\mu, \varsigma)$ in (A.1)
2. Sample $U \sim U(-\pi/2, \pi/2)$ and $E \sim \text{Exp}(1)$
3. if $\alpha \neq 1$ then
   4. Compute $\delta = \arctan \left( \theta \tan \left( \frac{\pi \alpha}{2} \right) \right)/\alpha$ and return $(\varsigma t)^{1/\alpha} A_{0, \delta}(U) E^{1-1/\alpha} + \mu t$
4. else
   5. return $\frac{2}{\pi} t \left( \left( \frac{\pi}{2} + \theta U \right) \tan(U) - \theta \log \left( \frac{\pi E \cos(U)}{\sin(\pi + 2\theta U)} \right) \right) + \mu t$
7. end if

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Department of Statistics, University of Warwick, & The Alan Turing Institute, UK
Email address: jorge.gonzalez-cazares@warwick.ac.uk

Department of Statistics, University of Warwick, & The Alan Turing Institute, UK
Email address: a.mijatovic@warwick.ac.uk