Model-Free Algorithm and Regret Analysis for MDPs with Peak Constraints

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Abstract

In the optimization of dynamic systems, the variables typically have constraints. Such problems can be modeled as a constrained Markov Decision Process (MDP). This paper considers a model-free approach to the problem, where the transition probabilities are not known. In the presence of peak constraints, the agent has to choose the policy to maximize the long-term average reward as well as satisfy the constraints at each time. We propose a novel algorithm that converts the constrained problem to an unconstrained problem using a modification of the reward function, and a Q-learning based approach is used on the unconstrained problem. The proposed algorithm is shown to achieve $O(\sqrt{HSA\ell T})$ bound for both the obtained reward and constraint violations with probability at-least $1 - 2p$, where $T$ is the time-horizon, $A$ is the number of actions, $S$ is the number of states, $H$ is the number of steps in each episode, and $\ell = \log(\frac{2SA}{p})$. We note that these are the first results on regret analysis for constrained MDP, where the transition problems are not known apriori. We demonstrate the proposed algorithm on an energy harvesting problem where it outperforms state-of-the-art and performs close to the theoretical upper bound of the studied optimization problem.

Keywords: Reinforcement Learning, Constrained Markov Decision Process, Q-learning Algorithm

1. Introduction

Optimization of dynamic systems typically have constraints, e.g., battery capacity for robots. As an example, if a robot is powered by battery, which is also being charged with an external power supply, the amount of energy used at each time is limited by the battery capacity. The dynamical systems are typically modeled as a Markov Decision Process (MDP), while the transition probabilities may not be known apriori (or may be dynamic). In the absence of knowledge of transition probabilities, the MDP is modeled as a Reinforcement Learning
(RL) problem which aims to maximize the long-time reward by making actions given the state of the process to be controlled. RL algorithm can be divided into model-based and model-free, where the model-based approaches estimate the transition probabilities, while model-free approaches do not. In this paper, we consider a model-free approach to RL in the presence of peak constraints.

One of the model-free approaches for reinforcement learning is based on the Q-learning algorithm (Watkins and Dayan, 1992). Analysis for such algorithms has been studied in (Strehl et al., 2006; Azar et al., 2017b; Jin et al., 2018), where sub-linear regret is derived. However, such analysis in the presence of constraints is open, which is considered in this paper.

This paper considers peak constraints, which is an important constraint in many dynamical systems. For instance, algorithms with peak constraints have been studied for communications (Shamai and Bar-David, 1995), flow-shop scheduling (Fang et al., 2013), thermostatically-controlled systems (Karmakar et al., 2013), economics (Bailey, 1972), robotics (Li et al., 1997), etc. The peak constraints have been considered for Markov Decision Processes (Altman, 1999). However, these require complete knowledge of the transition probabilities. In the absence of such knowledge, algorithms have been proposed (Geibel and Wysotzki, 2005; Geibel, 2006). However, to the best of our knowledge, none of the algorithms so far has provably sub-linear regret for objective and constraint violations.

**Contributions:** In this paper, we propose a novel model-free algorithm, based on the modification of the Q-learning algorithm that accounts for the constraints. We assume that the transition probability is unknown, the reward function is observed, and the constraint function can be queried but does not need to be known in closed form. Using the modification of the reward function, we convert the constrained problem to the unconstrained problem. The proposed algorithm is analyzed and found to have sub-linear regret for both the objective and constraint violations. More precisely, our algorithm achieves $O\left(\sqrt{H^6SAT}\ell\right)$ regret bound for both the obtained reward and the constraint violation for the time-horizon $T$. Further, the proposed algorithm is evaluated on an energy-harvesting transmitter studied in (Wang et al., 2014). It’s found that the proposed algorithm outperforms baseline strategies and performs close to the genie-aided upper bound for the problem.

### 2. Related Work

**Online Convex Optimization (OCO):** OCO problem is an extension of the constrained convex optimization. In this problem, we wish to optimize $\sum_{t=1}^{T} f_t(x)$ for given functions $f_t$, $t \in \{1, \cdots, T\}$ such that $x \in K$. In online convex optimization, we select $x_t$ at time $t$, such that the regret in objective is minimized, which is defined as

$$\text{Regret}(T) = \sum_{t=1}^{T} f_t(x_t) - \min_{x \in K} \sum_{t=1}^{T} f_t(x).$$

Further, $x_t$ may not satisfy constraints, and thus there may be a constraint violation.

By changing the problem into an online convex-concave optimization problem, (Malhavi et al. 2011) proposed an algorithm which achieves the $O(\sqrt{T})$ bound for the regret and $O(T^{3/4})$ bound on the violation of constraints. Further, they proposed another algorithm based on the mirror-prox method (Nemirovski, 2004) that achieves $O(T^{2/3})$ bound on both
regret and constraints when the domain can be described by a finite number of linear constraints. The authors of (Jenatton et al., 2016) proposed an algorithm which achieves $O(T^{\max(\beta,1-\beta)})$ objective regret and $O(T^{1-\beta/2})$ constraint violations for $\beta \in (0,1)$. Further, the authors of (Yu and Neely, 2016) proposed an algorithm with $O(\sqrt{T})$ regret bound for objective with finite constraint violations. However, the problem in MDP is different from that in OCO, since $f_t$ also depends on previous actions. Further, the functions and constraints are not known explicitly in reinforcement learning (RL). Thus, the problem of MDP does not follow from that of OCO.

**Constrained Markov Decision Process (CMDP):** When the system model (the transition probability distribution, the reward function, and the constraint function) is known, the problem is generally considered as CMDP. CMDP in the form of discounted and average reward has been deeply studied in (Altman, 1999). It is well known that CMDP problem is convex and can be converted into an equivalent unconstrained MDP problem by using the method of Lagrange multipliers. Thus, when the model is known, CMDP can be solved using linear programming (LP) or dynamic programming (DP). In addition to the LP method, (Geibel, 2006) proposed three different algorithms, WeiMDP, AugMDP, and RecMDP, to solve CMDP in different settings.

The key difference between these works and reinforcement learning (RL) approach is that the transition probabilities for the next state given the previous state and action are assumed to be known in CMDP approaches, while are not known apriori in RL approaches. They may be learnt in model-based RL, while not learnt at all in model-free RL approaches. In this paper, we consider the reinforcement learning based approaches.

**Regret Bounds for Reinforcement Learning:** Regret Analysis for the Reinforcement Learning has been considered for both the model-based approaches (Jaksch et al., 2010; Agrawal and Jia, 2017; Azar et al. 2017b; Kakade et al., 2018) and the model-free approaches (Kearns and Singh 2002; Strehl et al., 2006; Jin et al. 2018). Our paper extends the episodic reinforcement learning setup with the addition of peak constraints.

**Model-free Reinforcement Learning Algorithm for CMDP with Peak Constraints:** Q-learning based methods with peak constraints have been studied (Bouton et al., 2019; Wang et al., 2014), where the Q function in each epoch is projected to the constraint set. These algorithms involve knowledge of constraint functions explicitly (since projection to the constraint set is needed) to make decisions at each time. In contrast, we do not require knowledge of constraint function. Recently, based on the primal-dual method, (Paternain et al., 2019) proposed an algorithm with policy descent to prove $1-\delta$ safe algorithm, which gives $P(\cap_{t\geq 0}\{s_t \in \mathcal{S}_0\}|\pi_0) \geq 1-\delta$, where $\mathcal{S}_0$ is the safe region. Besides, (Gattami, 2019) related CMDP with peak constraints to the unconstrained zero-sum game where the objective is the Lagrangian of the optimization problem and applied max-min Q-learning to CMDP to prove convergence. However, none of the works in this direction have shown a sub-linear regret for objectives and constraints, which is the focus of our paper. To the best of our knowledge, this paper provides the first regret analysis for model-free reinforcement learning with peak constraints.
3. Problem Formulation and Assumption

We consider an episodic setting of the Constrained Markov Decision Process with finite state and action space, defined by CMDP$(S, A, H, P, f_i)$, where $S$ is the state space with $|S| = S$, $A$ is the set of actions with $|A| = A > 1$, $H$ is the number of steps in each episode, and $P$ is the transition matrix so that $P_h(\cdot|s, a)$ gives the probability distribution over next state based on the state and action pair $(s, a)$ at the step $h$. Further, $r: S \times A \rightarrow \mathbb{R}$ is the deterministic reward function and $f_i: S \times A \rightarrow \mathbb{R}, i = 1, ..., I$ are the peak constraint functions. In the RL setting, both the reward function and constraint functions are unknown to the agent but can be measured when a state action pair $(s, a)$ is observed. In this paper, we make the following two assumptions.

Assumption 1 The absolute values of the reward function $r$ and constraint functions $f_i, i = 1, \cdots, I$ are strictly bounded by a constant known to the agent. Without loss of generality, we let this constant be $1$.

Assumption 2 The values of the reward function $r$ is non-negative, i.e., $0 \leq r(s, a) \leq 1, \forall (s, a)$.

These assumptions on reward function are typical in reinforcement learning Jin et al. (2018); Yang et al. (2019); Azar et al. (2017a), and the bound of reward function can be normalized. Further, the reward can be shifted up by adding a constant to make the reward function non-negative.

We define the policy as a function that maps a state $s \in S$ to a probability distribution of the actions with a probability assigned to each action $a \in A$. In episodic setting, the whole policy $\pi$ is a collection of $H$ policy functions $\pi_h$ at each step, that is $\pi_h(s) = a$ with probability $Pr(a|s)$.

Constrained RL problem is concerned with finding the optimal policy to achieve the highest total reward subject to a set of constraints, which can be formally stated as

$$\max_{\pi} \mathbb{E}\left[ \sum_{h=1}^{H} r(s_h, \pi_h(s_h)) \right]$$

s.t. $f_i(s_h, \pi_h(s_h)) \geq 0 \text{ w.p.1 } \forall h \in [H], \forall i \in [I]$ (2)

where the expectation is taken with respect to the randomness introduced by the policy $\pi$ and the transition mapping $P$. At the beginning of each episode of Constrained MDP, an initial state is chosen arbitrarily, then an action $a_h$ is taken by the agent using the policy $\pi_h(\cdot|s_h)$, and the MDP transits to another state $s_{h+1}$ with the probability $P_h(\cdot|s_h, a_h)$. Let $\tilde{r}(s, a) = -H$ if any of the constraint is violated and $\tilde{r}(s, a) = r(s, a)$ if all constraints are satisfied. Thus, $\tilde{r}(s, a)$ modifies the reward so that we punish the reward to the minimal value $-H$ when the constraints are not satisfied. The (2) can also be re-written as

$$\max_{\pi} \mathbb{E}\left[ \sum_{h=1}^{H} \tilde{r}(s_h, \pi_h(s_h)) \right]$$

s.t. $f_i(s_h, \pi_h(s_h)) \geq 0 \text{ w.p.1 } \forall h \in [H], \forall i \in [I]$ (3)

This is because when the constraints are satisfied, $\tilde{r}(s, a) = r(s, a)$. We use the state value function $V_h^\pi : S \rightarrow \mathbb{R}$ to denote the value function at step $h$ under policy $\pi$, where $V_h^\pi(s)$ is
given as
\[ V^\pi_h(s) := \mathbb{E} \left[ \sum_{h'=h}^H \tilde{r}(s_{h'}, \pi_{h'}(s_{h'})) \big| s_h = s \right] \]  

Denote the set Π as the constraint set in which the policy satisfies the constraints in the Eq. (3). We denote an optimal policy as \( \pi^* \), which gives the optimal value \( V^*_h(s) = \sup_{\pi \in \Pi} V^\pi_h(s) \) for all \( s \in S \) and \( h \in [H] \). We note that the proposed Peak Constrained MDP problem is a special case of Constrained MDP problem mentioned in (Altman, 1999). The difference is that the constraint functions need to be satisfied in each step \( h \) in our formulation, while it is only needed to be satisfied in the average in (Altman, 1999). It is well known that the optimal policy for the Constrained MDP with average constraint functions could be stochastic. However, it is shown that the optimal policy for the Peak Constraint MDP is deterministic (Gattami, 2019), which makes the policy \( \pi \) to be a 0,1 policy in each step. Assuming that the agent plays the game for \( K \) episodes \( k = 1, 2, ...K \), we define the regret and constraint violations as,

\[
\text{Regret}(K) = \sum_{k=1}^K [V^*_1(s^k_1) - V^\pi_k(s^k_1)] \\
\text{Violation}(K) = \sum_{k=1}^K \sum_{h=1}^H H \mathbb{E} |f_i^-(s^k_h, \pi^k_h(s^k_h))|
\]  

where the notation \( f_i^- (\cdot) \) is defined as \( f_i^- (\cdot) := \min \{0, f_i(\cdot)\} \) and the expectation is taken over the transition probability.

We note that the definition of \( V^\pi_h(s) \) enforces that Regret\( (K) \) is non-negative when a feasible solution exists, which is assumed in this paper. This is because when the constraints are not satisfied, reward \(-H\) will punish the reward collected in the episode to be negative. Thus, the reward using any policy \( \pi \) is only counted when the constraints are satisfied. Thus, the policy \( \pi \) can be adapted to a policy where the action given by \( \pi \) is chosen when the constraints are satisfied and an arbitrary feasible action is chosen when the constraints are not satisfied. This modified policy is feasible and achieves no worse average future reward than \( \pi \). Thus, the optimal policy will achieve no worse reward than any strategy \( \pi \). We also note that if the \( V^\pi_h(s) \) in (4) was defined using \( r(s, a) \) instead of \( \tilde{r}(s, a) \), the same result would not hold since the policy \( \pi \) could achieve larger rewards by violating constraints, thus motivating the definitions of \( V^\pi_h(s) \) and Regret\( (K) \).

4. Proposed Algorithm

For any state action pair \( (s, a) \), we define a modified reward function as,

\[
R(s, a) = \tilde{r}(s, a) - \frac{1}{2I} \sum_{i=1}^I |f_i^-(s, a)|.
\]  

This modified reward function gives the original reward function \( r(s, a) \) when the constraints are satisfied while provides a negative reward when the constraints are not satisfied. Based
on the modified reward function, we define a counterpart of the value function $W^\pi_h(s)$ as

$$W^\pi_h(s) := \mathbb{E}\left[ \sum_{h'=h}^{H} R(s_{h'}, \pi_h(s_{h'})) | s_h = s \right]$$ (7)

Similarly, with the notation $[\mathbb{P}_h V_{h+1}](s,a) := \mathbb{E}_{s' \sim \mathbb{P}_h(\cdot | s,a)} V_{h+1}(s')$, we define a counterpart of the state-action function $Q_\pi(h,s,a)$ as

$$Q_\pi(h,s,a) := R(s,a) + \mathbb{E}\left[ \sum_{h'=h+1}^{H} R(s_{h'}, \pi_{h'}(s_{h'})) \bigg| s_h = s, a_h = a \right] = (R + \mathbb{P}_h W^\pi_{h+1})(s,a) \quad \text{(8)}$$

With these notations, we are able to define a modified unconstrained MDP problem as

$$\max_\pi \mathbb{E}\left[ \sum_{h=1}^{H} R(s_h, \pi_h(s_h)) \right]$$ (9)

Recalling the assumption that the original reward function $r(\cdot)$ is bounded by 1, we now show the absolute value of the modified reward function $R$ is also bounded.

**Lemma 1** The absolute value of the modified reward function $R(s,a)$ is bounded by $-H - 1 \leq R(s,a) \leq 1$ for all $(s,a) \in S \times A$

**Proof** If the constraints are satisfied, $R(s,a) = r(s,a)$ and thus $0 \leq R(s,a) \leq 1$.

If the constraints are not satisfied, we have from (6) that

$$R(s,a) = -H - \frac{1}{2T} \sum_{i=1}^{T} |f_i^{-}(s,a)|$$ (10)

Since $0 \leq |f_i^{-}(s,a)| \leq 1$, we have $-H - \frac{1}{2} \leq R(s,a) \leq 0$.

The two cases together provides the result as in the statement of the Lemma.

We use the modified reward function to provide a Q-learning based algorithm as described in Algorithm 1. The basic steps of Q-learning follow from that in (Jin et al., 2018), while are adapted to incorporate constraints. In line 1, the agent initializes the Q-table and $N_h(s,a)$, which is the notation for the times that the state-action pair is taken at step $h$. In line 3, the agent is given an initial state at the beginning of each episode. Then, in line 5, the agent takes an action to maximize the current state-value function $Q_h(s_h,a_h)$ and observes the next state. $N_h(s,a)$ is updated in line 6. Q-table and the W-table are then updated according to the line 8 and line 9, where $b_t$ is the upper confidence bound $b_t = 4\sqrt{H^3\ell/t}$, $\ell = \log(\frac{28AT}{p})$, $p$ is the probability parameter which will bound the probability of regret in the results, and $\alpha_t$ is the learning rate defined as

$$\alpha_t := \frac{H + 1}{H + t}$$ (11)

Given a Markov Decision Problem with constraints, this paper shows that Algorithm 1 converges to the optimal policy and it can be seen that the policy from the Algorithm 1 is a deterministic policy. The regret bound and constraint violations of the proposed algorithm will be analyzed in the next section.
Algorithm 1 Constrained Q-Learning Algorithm

1: Initialize $Q_h(s, a) \leftarrow H$ and $N_h(s, a) \leftarrow 0$ for all $(s, a, h) \in S \times A \times [H]$  
2: for episode $k = 1,...K$ do  
3: Observe $s_1$  
4: for step $h = 1,...H$ do  
5: Take action $a_h \leftarrow \arg \max_{a'} Q_h(s_h, a')$ and observe $s_{h+1}$  
6: $t = N_h(s_h, a_h) \leftarrow N_h(s_h, a_h) + 1$  
7: $b_t \leftarrow 4(H + 1)^3 \ell/t$  
8: $Q_h(s_h, a_h) \leftarrow (1 - \alpha_t)Q_h(s_h, a_h) + \alpha_t[R(s_h, a_h) + W_{h+1}(s_{h+1}) + b_t]$  
9: $W_h(s_h) \leftarrow \min\{H, \max_{a' \in A} Q_h(s_h, a')\}$  
10: end for  
11: end for

5. Regret Bound Analysis
First, we will provide the connections between the original constrained problem and modified problem. Then, we will derive guarantees for the modified unconstrained problem. The guarantees for the unconstrained problem and its connection with the original constrained problem will then be used to provide the overall regret result in this paper.

5.1 Connecting Modified Unconstrained Problem and Original CMDP
The two following results, Lemma 2 and Lemma 3, describe the relationship between the optimal policy and that given by the proposed algorithm.

Lemma 2 If the problem is feasible, the optimal value function $V^*$ for the original CMDP problem is less than or equal to the new optimal value function $W^*$. More formally,

$$V_1^*(s_1^k) \leq W_1^*(s_1^k) \quad (12)$$

Proof Firstly, consider the optimal policy $\pi^*$ in the modified unconstrained RL problem, we have

$$V_1^*(s_1^k) = \mathbb{E}\left[ \sum_{h=1}^{H} r(s_h^k, \pi_{s_h}^k) \right] = \mathbb{E}\left[ \sum_{h=1}^{H} R(s_h^k, \pi_{s_h}^k) \right] \leq W_1^*(s_1^k) \quad (13)$$

The equality holds because with feasible optimal policy in original CMDP, $r(s, a) = R(s, a)$. Moreover the final inequality holds because the optimal policy for original problem and modified problem can be different and any other policy will achieve less reward than the optimal policy for $W_1$. Thus, this gives the result as in the statement of the Lemma.

Lemma 3 The value function given by policy $\pi^k$ in the proposed algorithm for the modified problem, $W_1^{\pi_k}$, can be expressed by the value function for the original CMDP problem, $V_1^{\pi_k}$, minus a term describing the violation of constraints. Formally,

$$W_1^{\pi_k}(s_1^k) = V_1^{\pi_k}(s_1^k) - \frac{1}{2I} \mathbb{E}\left[ \sum_{i=1}^{I} \sum_{h=1}^{H} |f_{s_h}^{-}(s, a)| \right] \quad (14)$$
Proof According to the definition of function $W$, we expand it as follows.

\[
W_{1}^{\pi_{k_{1}}} (s_{k_{1}}) = \mathbb{E} \left[ \sum_{h=1}^{H} R(s_{h}, \pi_{h}(s_{h})) \right] = \mathbb{E} \left[ \sum_{h=1}^{H} \bar{r}(s_{h}, \pi_{h}(s_{h})) \right] - \frac{1}{2T} \mathbb{E} \left[ \sum_{i=1}^{I} \sum_{h=1}^{H} |f_{i}^{-} (s_{h}, a_{h})| \right]
\]

which is the result as in the statement of the Lemma.

5.2 Guarantees for the Modified Unconstrained Problem

In this subsection, we will provide guarantees for the modified unconstrained problem. For ease of analysis, we define the related quantities $\alpha_{0}^{t}$ and $\alpha_{i}^{t}$.

\[
\alpha_{0}^{t} = \prod_{j=1}^{t} (1 - \alpha_{j}), \quad \alpha_{i}^{t} = \alpha_{i} \prod_{j=i+1}^{t} (1 - \alpha_{j})
\]

Further properties for $\alpha_{0}^{t}$ and $\alpha_{i}^{t}$ are given in Appendix A, which will be used in the proofs.

Let $[\mathbb{P}_{h} W_{h+1}] (s, a)$ be defined as

\[
[\mathbb{P}_{h} W_{h+1}] (s, a) := \mathbb{E}_{s' \sim \mathbb{P}_{h}(\cdot | s, a)} W_{h+1} (s'),
\]

and define its empirical counterpart of episode $k$ as

\[
[\hat{\mathbb{P}}_{h} W_{h+1}] (s, a) := W_{h+1} (s_{h+1}).
\]

Assume the state-action pair $(s_{h}^{k}, a_{h}^{k})$ is visited at the step $h$ in episode $k$, and let $Q_{h}^{k}, W_{h}^{k}, N_{h}^{k}$ be the values of $Q_{h}, W_{h}, N_{h}$ functions at the beginning of episode $k$, respectively. Define $Q_{h}^{*}$ and $W_{h}^{*}$ as the Q-function and W-function with optimal policy at step $h$, respectively. With properties in Lemma 8 and update rule in the algorithm, we have following result.

Lemma 4 For any $(s, a, h) \in S \times A \times [H]$ and episode $k \in [K]$, let $t = N_{h}^{k}(s, a)$ and suppose that $(s, a)$ was previously taken at step $h$ of episodes $k_{1}, ..., k_{t} < k$. Then:

\[
(Q_{h}^{k} - Q_{h}^{*})(s, a) = \alpha_{0}^{t} (H - Q_{h}^{*}(s, a)) + \sum_{i=1}^{t} \alpha_{i}^{t} \left[ (W_{h+1}^{k_{i}} - W_{h+1}^{*}) (s_{h+1}^{k_{i}}) + [(\hat{\mathbb{P}}_{h}^{k_{i}} - \mathbb{P}_{h}) W_{h+1}^{*} (s, a)] + b_{i} \right]
\]

Proof The detailed proof of this Lemma is provided in the Appendix B.

Lemma 4 gives a recursive form of the $Q$ function. Based on this form, the following lemma gives a bound on $Q^{k} - Q^{*}$, which is the key step for achieving sub-linear regret for the modified unconstrained problem.
Lemma 5 For any \( p \in (0,1) \), let \( \ell = \log(2SAT/p) \), \( b_t = 4(H + 1)\sqrt{H^3\ell/t} \), and \( \beta_t = 12(H + 1)\sqrt{H^3\ell/t} \). With probability at least \( 1 - p \), the following holds simultaneously for all \((s, a, h, k) \in S\times A \times [H] \times [K] \):

\[
0 \leq (Q^k_h - Q^*_h)(s, a) \leq \alpha^0_t(H + 1)^2 + \sum_{i=1}^{t} \alpha^i_t(W^k_{h+1} - W^*_h(s_{h+1})) + \beta_t
\]

(18)

Proof The detailed proof of this Lemma is provided in the Appendix C.

Equipped with Lemmas 8, 4, and 5, we get the first main result that the regret for the modified problem is sub-linear, which is formally written as follows.

Theorem 6 For any \( p \in (0,1) \), let \( \ell = \log(2SAT/p) \), \( b_t = 4(H + 1)\sqrt{H^3\ell/t} \), and \( \beta_t = 12(H + 1)\sqrt{H^3\ell/t} \), the bound on the total regret using Algorithm 1 is

\[
\sum_{k=1}^{K} [W^*_1(s^k_1) - W^*_1(s^k_1)] \leq O(\sqrt{H^{6}SAT\ell})
\]

with probability at least \( 1 - 2p \).

Proof The detailed proof is provided in Appendix D.

5.3 Overall Regret Bound

Having considered the results for the modified unconstrained problem, and its relation to the original CMDP, the next theorem provides the regret bounds for the original problem.

Theorem 7 The regret bound in the CMDP and the bound on the violation of constraints are sub-linear. In particular, for any \( p \in (0,1) \), the regret bound on reward and constrain violation are both \( O(\sqrt{H^{6}SAT\ell}) \) with probability at least \( 1 - 2p \).

Proof Combining the results in Lemma 2, Lemma 3, and Lemma 5, for large enough \( T \), we have

\[
\sum_{k=1}^{K} [V^*_1(s^k) - V^*_1(s^k)] + \frac{1}{2T} \mathbb{E} \left[ \sum_{i=1}^{I} \sum_{k=1}^{K} \sum_{h=1}^{H} [f^k_i(s, a)] \right] \leq \sum_{k=1}^{K} [W^*_1(s^k) - W^*_1(s^k)] \leq O(\sqrt{H^{6}SAT\ell})
\]

(19)

We note that the second term (the violation of peak constraints) on the left hand side of (19) is non-negative. Thus, we obtain

\[
\sum_{k=1}^{K} [V^*_1(s^k) - V^*_1(s^k)] \leq O(\sqrt{H^{6}SAT\ell})
\]

(20)

This gives the result of the sub-linear property for the value function in original CMDP. Furthermore, due to the definition of \( V^*_h(s) \), we have

\[
\sum_{k=1}^{K} [V^*_1(s^k) - V^*_1(s^k)] \geq 0
\]

(21)
This holds because if the policy $\pi_k$ violates any constrain at any step $h$, then $\tilde{r}_h(s_h^k, a_h^k) = -H$ and then $V^*_1 \leq -H + (H - 1) < 0 \leq V^*_1$. Using Eq. (19) with this lower bound, we have the following inequality

$$\frac{1}{2I} \mathbb{E} \left[ \sum_{i=1}^{I} \sum_{k=1}^{K} \sum_{h=1}^{H} |f_i^- (s, a)| \right] \leq O(\sqrt{H^6 SATl}) \quad (22)$$

This provides a sub-linear bound for the peak constraint violations, which is given as

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} |f_i^- (s_h^k, \pi_h^k(s_h^k))| \leq O(\sqrt{H^6 SATl}) \quad (23)$$

6. Simulations

Figure 1: This figure describes the model of energy harvesting communication system used in the evaluations.

In this section, we evaluate the proposed algorithm on a communication channel, where the transmitter is powered by renewable energy. Such a model has been studied widely in communication systems (Tutuncuoglu et al., 2015; Blasco et al., 2013; Yang and Ulukus, 2012; Wang et al., 2015; Wang et al., 2014). In this model, we assume that the time is divided into time-slots. As shown in Fig. 1, in each time-slot, the transmitter can send data over an Additive Gaussian White Noise (AWGN) channel, where the signal transmitted by the transmitter gets added by a noise given by complex normal with zero mean and unit variance $\mathcal{CN}(0, 1)$ at each time instance within the time-slot. We assume that the transmitter can use a power of $P_h$ in time-slot $h$, where the transmission is constrained by a maximum power of $P_{\text{max}}$.

We assume that the transmitter is powered by a renewable energy source, where energy $E_h$ arrives during time-slot $h - 1$ and can be used for time-slot $h$. Further, the transmitter is attached to a battery, which has a capacity of $B_{\text{max}}$. The transmitter can use the energy from the existing battery capacity at the start of time-slot $h$, $B_h$, or the new energy arrival $E_h$. The energy from $E_h$ that is not utilized is stored in the battery. Thus, the battery state evolves as

$$B_{h+1} = \min\{B_{\text{max}}, B_h + E_h - P_h\}. \quad (24)$$
We wish to optimize an upper bound on the reliable transmission rate (Wang et al., 2015), given as
\[ C = \sum_h \log(1 + P_h). \] (25)

We note that the battery and the transmission constraints can be modeled as peak constraints. Thus, the overall optimization problem is given as
\[
\max_{P_{h, h=1,\ldots,H}} \mathbb{E} \left[ \sum_{h=1}^{H} \log(1 + P_h) \right] \\
\text{s.t.} \\
0 \leq B_h \leq B_{\text{max}} \\
0 \leq P_h \leq P_{\text{max}} \\
B_{h+1} = \min\{B_{\text{max}}, B_h + E_h - P_h\}
\] (26)

We note that the expectation in the above is over the energy arrivals \( E_h \), which makes the choice of \( P_h \) stochastic. If the energy arrivals \( E_h \) are known non-causally (known at \( h = 1 \) for the entire future), the problem is convex and can be solved efficiently using the dynamic water-filling algorithm proposed in (Wang et al., 2015). However, in realistic systems, \( E_h \) is only known at time-slot \( h \). When the energy is known causally, dynamic programming based solutions have been proposed (Blasco et al., 2013 Wang et al., 2014).

We will now model the problem as an MDP. The state at time-slot \( h \) is given as \( S_h = (B_h, E_h) \), which are the current battery level and the energy arrival. The energy \( E_h \) is known causally, and the distribution is unknown. Based on the state, the action is the transmission power \( P_h \). Based on the state and the action, the battery state evolves as Eq. (24), and the \( E_h \) may evolve based on some Markov process in general. Based on the state and action, the reward is given by the objective in (26), where the peak constraints are also given in (26).

We let the distribution of \( E_h \) as truncated Gaussian of mean \( \mu \) and standard deviation \( \sigma \), where the truncation levels are 0 and \( E_{\text{max}} \), and we let it be independent across episodes. The problem is discretized to integers in order to apply the proposed algorithm. According to the selection of the parameters in (Wang et al., 2014), we set the horizon \( H = 20 \) time-slots, battery capacity \( B_{\text{max}} = 20 \), power constraint \( P_{\text{max}} = 15 \), maximal harvest energy \( E_{\text{max}} = 20 \), mean and standard deviation \( \mu = 10, \sigma = 5 \), respectively. For our algorithm, we let \( \gamma = 0.25 \).

In order to compare the proposed algorithm, we consider three other baseline algorithms, the greedy policy, the balanced policy, and the optimal non-causal algorithm. The greedy policy tries to consume the harvested energy as much as possible in each slot, as calculated by \( P_h = \min(P_{\text{max}}, B_h + E_h) \). We also consider a balanced policy that consumes the fixed amount of energy in each slot if available, where the fixed value is calculated by \( \sum_{h=1}^{H} E_h / H \), while that is limited by the available energy at each time. However, the balanced algorithm uses the future energy arrivals and is not a causal strategy. Further, the optimal strategy when all future energy arrivals are non-causally known is also used to show the performance of the proposed algorithm. We note that the proposed algorithm only assumes that the constraint function in state \( s \) and action \( a \) can be queried, but the function is not explicitly known, thus, the algorithms that project to the constraint function are not considered as they require complete knowledge of the function.
In Fig. 2, we plot the sum of the transmission rate in each episode, and the number of constraint violations in each episode (the number of constraint violations in each episode is between 0 and \( H \)). The plotted results are averaged over 1000 runs. We see that the reward converges around 10,000 episodes and the constraint violations go to zero. Thus, the policy converges, and the final policy satisfies the peak constraints.

![Figure 2](image1)

**Figure 2**: This figure shows the convergence of the reward function and the number of constraint violations for the proposed algorithm.

Based on the convergence results, we choose \( K = 12,000 \). In Fig. 3, we set the mean value of the harvested energy as 8, 9, 10, 11, and 12, and show the performances of different algorithms. We see that the balanced policy achieves higher performance than greedy policy because the energy can be allocated more reasonably while requiring non-causal information of energy arrival. The performance of the non-causal convex solver achieves the highest reward since it is an upper bound on the performance. However, we see that the proposed

![Figure 3](image2)

**Figure 3**: This figure compares the average per-episode transmission rate for the different algorithms. We note that the proposed algorithm outperforms the baseline algorithms, and is nearly same as genie-aided optimal approach.
algorithm achieves nearly the same performance as the upper bound, which shows that our algorithm is able to achieve the optimal solution. Furthermore, the proposed algorithm doesn’t need any prior knowledge of the harvested energy and the constraint functions, which is a great advantage over the convex solver.

7. Conclusion

In this paper, we formulate a constrained MDP problem with a set of peak constraints. By using a modified reward function, we convert the problem into a modified bounded and unconstrained RL problem. An algorithm for this unconstrained problem is proposed. Using guarantees for the unconstrained problem, and its relation to the original problem, the regret bound for both the objective and the constraint violation is provided. This paper proves a bound of $O(T^{1/2})$ on both the objective regret and the constraint violation. We note that this is the first result of the regret analysis of CMDP with constraints when the state evolution and the constraint functions are unknown. The results are applied to the energy harvesting communication link, and the proposed algorithm is shown to be close to the non-causal optimal solution.

The authors of (Gattami, 2019) related the problem of MDP with peak constraints to a zero-sum game. Thus, the proposed results could give insights to the regret bounds for zero-sum games, which is left as future work.

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Appendix A. A Result that will be used in the Proofs

Lemma 8 The following properties hold for $\alpha_t^0$ and $\alpha_t^i$

(a) $\alpha_t^0 = 0$ for $t \geq 1$, $\alpha_t^0 = 1$ for $t = 0$
(b) $\sum_{i=1}^{t} \alpha_t^i = 1$ for $t \geq 1$, $\sum_{i=1}^{t} \alpha_t^i = 0$ for $t = 0$
(c) $\frac{1}{\sqrt{t}} \leq \sum_{i=1}^{t} \frac{\alpha_t^i}{\sqrt{i}} \leq \frac{2}{\sqrt{t}}$
(d) $\max_{i \in [t]} \alpha_t^i \leq \frac{2H}{t}$ and $\sum_{i=1}^{t} (\alpha_t^i)^2 \leq \frac{2H}{t}$ for every $t \geq 1$
(e) $\sum_{i=t}^{\infty} \alpha_t^i = 1 + \frac{1}{H}$ for every $i \geq 1$

Proof These properties have been derived in (Jin et al., 2018) (See (4.2) in (Jin et al., 2018) for (a)-(b), Lemma 4.1 in (Jin et al., 2018) for proof of (c)-(e)), and hence the proof is omitted.

Appendix B. Proof of Lemma 4

Proof By the update rule in line 8 in the algorithm, we see that the value of $Q_{k+1}^h(s, a)$ will be updated if and only if $(s, a) = (s^k_h, a^k_h)$ and the value is updated to

\[ Q_{k+1}^h(s, a) = (1 - \alpha_t)Q_{k}^h(s, a) + \alpha_t[R(s, a) + W_{h+1}^k(s_{h+1}^k) + b_t] \]  

By recursively using the update rule and the notation we defined in Eq. (16), we have

\[ Q_{k}^h(s, a) = (1 - \alpha_t)Q_{k}^h(s, a) + \alpha_t[R(s, a) + W_{h+1}^k(s_{h+1}^k) + b_t] \]
\[ = (1 - \alpha_t)\left[(1 - \alpha_{t-1})Q_{k}^{h-1}(s, a) + \alpha_{t-1}[R(s, a) + W_{h+1}^{k-1}(s_{h+1}^{k-1}) + b_t]\right] \]
\[ + \alpha_t[R(s, a) + W_{h+1}^{k-1}(s_{h+1}^{k-1}) + b_t] \]
\[ = \prod_{i=t-1}^{t} [1 - \alpha_i]Q_{h}^{k_{i-1}}(s, a) + \sum_{i=t-1}^{t} \alpha_t \prod_{j=i+1}^{t} (1 - \alpha_j) \cdot [R(s, a) + W_{h+1}^{k_i}(s_{h+1}^{k_i}) + b_t] \]
\[ = ... \]
\[ = \alpha_t^0Q_{h}^{k_1}(s, a) + \sum_{i=1}^{t} \alpha_t^i[R(s, a) + W_{h+1}^{k_i}(s_{h+1}^{k_i}) + b_t] \]

where the term $Q_{h}^{k_1} = H$ because there is no update when we bump into $(s, a)$ for the first time and it should be the initial value. Thus,

\[ Q_{h}^{k}(s, a) = \alpha_t^0H + \sum_{i=1}^{t} \alpha_t^i[R(s, a) + W_{h+1}^{k_i}(s_{h+1}^{k_i}) + b_t] \]  

(28)
Then, considering the Bellman equation with optimal policy, for $t \geq 1$, we have

$$Q^*_h(s, a) = (R + \mathbb{P}_h W^*_{h+1})(s, a)$$

$$= \sum_{i=1}^{t} \alpha_i^{(a)} [(R + \mathbb{P}_h W^*_{h+1})(s, a)]$$

$$= \sum_{i=1}^{t} \alpha_i^{(b)} [R(s, a) + (\mathbb{P}_h - \hat{P}_k^i) W^*_{h+1}(s, a) + W^*_{h+1}(s_{k_i}^{i+1})],$$

where step (a) holds due to the lemma 8(a) and step (b) holds by the definition of $\hat{P}_k^i$. For $t = 0$, we have $Q^*_h(s, a) = \alpha_0^0 Q^*_h(s, a)$. Further,

$$Q^*_h(s, a) = \alpha_0^0 Q^*_h(s, a) + \sum_{i=1}^{t} \alpha_i^{(a)} [R(s, a) + (\mathbb{P}_h - \hat{P}_k^i) W^*_{h+1}(s, a) + W^*_{h+1}(s_{k_i}^{i+1})]$$

(31)

Combing Eq. (29) and Eq. (31), we obtain the result as in the statement of Lemma 4.

**Appendix C. Proof of Lemma 5**

**Proof** For each fixed $(s, a, h) \in S \times A \times [H]$ and a fixed $k \in [K]$, let $t = N^k_h(s, a)$, and suppose that $(s, a)$ was previously taken at step $h$ of episodes $k_1, ..., k_t < k$. Let $\mathcal{F}_i$ be the sigma field generated by all the random variables until episode $k_i$, step $h$. Then,

$$(\alpha_i^{(a)} [(\hat{P}_k^i - \mathbb{P}_h) W^*_{h+1}](s, a))_{i=1}^{t}$$

is clearly a martingale difference sequence w.r.t. the filtration $\mathcal{F}_i$. According to the result in Lemma 1, the $i^{th}$ term in the martingale difference sequence is bounded by $c_i = 2H\alpha_i^{(a)}$. Let $E$ be defined as

$$E = 2H(H + 1) \sqrt{\sum_{i=1}^{t} (\alpha_i^{(a)})^2 \cdot \ell}.$$

Using Azuma’s inequality, we have.

$$\mathbb{P} \left[ \sum_{i=1}^{t} \alpha_i^{(a)} [(\hat{P}_k^i - \mathbb{P}_h) W^*_{h+1}](s, a) \leq E \right] \geq 1 - 2 \exp \left( \frac{-E^2}{2 \sum_{i=1}^{t} c_i^{(a)^2} } \right)$$

$$= 1 - 2 \exp \left( \frac{-4H^2(H + 1)^2 \sum_{i=1}^{t} (\alpha_i^{(a)})^2 \cdot \ell}{4H^2(H + 1)^2 \sum_{i=1}^{t} (\alpha_i^{(a)})^2} \right)$$

$$= 1 - \frac{p}{SAT}.$$

(32)

By union bound, we know with probability at least $1 - p$, the following holds for all $(s, a, h, k) \in S \times A \times [H] \times [K]$

$$\left| \sum_{i=1}^{t} \alpha_i^{(a)} [(\hat{P}_k^i - \mathbb{P}_h) W^*_{h+1}](s, a) \right| \leq E < 4(H + 1) \sqrt{\frac{H^3 \ell}{t}}$$

(33)
where the last step comes from the result in Lemma 8(d). Finally, we have

\[
(Q_h^k - Q_h^*(s, a))^2 = \alpha_t^0(H - Q_h^*(s, a)) + \sum_{i=1}^{t} \alpha_i^t \cdot [(W_h^{k_i} - W_h^*(s_h^{k_i}))(s_h^{k_i} + 1) + ([\bar{P}_h^{k_i} - P_h]W^{*}_{h+1})(s, a) + b_i]
\]

(a) \leq \alpha_t^0(H + 1)^2 + \sum_{i=1}^{t} \alpha_i^t(W_h^{k_i} - W_h^*(s_h^{k_i}))(s_h^{k_i} + 1) + 4(H + 1)\sqrt{\frac{H^3\ell}{t}} + \sum_{i=1}^{t} \alpha_i^t b_i

(b) \leq \alpha_t^0(H + 1)^2 + \sum_{i=1}^{t} \alpha_i^t(W_h^{k_i} - W_h^*(s_h^{k_i}))(s_h^{k_i} + 1) + 12(H + 1)\sqrt{\frac{H^3\ell}{t}}

= \alpha_t^0(H + 1)^2 + \sum_{i=1}^{t} \alpha_i^t(W_h^{k_i} - W_h^*(s_h^{k_i}))(s_h^{k_i} + 1) + \beta_t

(34)

where step (a) follows from Eq. (33) and \(Q_h^*(s, a) \geq -H(H + 1)\) for all \((s, a) \in S \times A\) due to the bound of \(R\) in Lemma 1. Step (b) follows from Lemma 8(c). This proves the second inequality in the statement of Lemma 5. Similarly, we have

\[
(Q_h^k - Q_h)^2 = \alpha_t^0(H - Q_h^*(s, a)) + \sum_{i=1}^{t} \alpha_i^t \cdot (W_h^{k_i} - W_h^*(s_h^{k_i}))(s_h^{k_i} + 1) + ([\bar{P}_h^{k_i} - P_h]W^{*}_{h+1})(s, a) + b_i
\]

(a) \geq \sum_{i=1}^{t} \alpha_i^t(W_h^{k_i} - W_h^*(s_h^{k_i}))(s_h^{k_i} + 1) - 4(H + 1)\sqrt{\frac{H^3\ell}{t}} + \sum_{i=1}^{t} \alpha_i^t b_i

(b) \geq \sum_{i=1}^{t} \alpha_i^t \left( \min_{a' \in A} Q_{h+1}^k(s_{h+1}, a') - \max_{a' \in A} Q_h^*(s_{h+1}, a') \right)

(35)

where step (a) is true because \(Q_h^*(s, a) \leq H\) and Eq. (33), and step (b) follows from Lemma 8(c). For \(h = H\), right hand side of Eq. (35) is always larger than or equal to 0. When \(h = H - 1\), by the induction assumption, \(Q_h^*(s, a) \leq Q_h^k(s, a)\) holds for any \((s, a, k) \in S \times A \times [K]\) and \(Q_h^*(s, a) \leq H\) is satisfied. Thus, \((Q_h^k - Q_h^k)(s, a) \geq 0\) holds for \(h = H - 1\). Then, by using induction on \(h\), we see that the first inequality in the statement of Lemma 5 holds. 

}\n
\appendix

\section{Proof of Theorem 6}

\textbf{Proof} Let

\[
\delta_h^k := \sum_{k=1}^{K}(W_h^{k} - W_h^*(s_h^{k}))(s_h^{k}) \quad \text{and} \quad \varphi_h^k := (W_h^k - W_h^*(s_h^k))(s_h^k)
\]

(36)

By Lemma 5, we have with probability \(1 - p\), \((Q_h^k - Q_h^k)(s, a) \geq 0\) for all \((s, a) \in S \times A\), thus \(W_h^k(s) \geq W_h^*(s)\). The total regret can be bounded as

\[
\text{Regret}(K) = \sum_{h=1}^{K}(W_1^k - W_1^*(s_1^k))(s_1^k) \leq \sum_{h=1}^{K}(W_h^k - W_h^*(s_h^k))(s_h^k) = \sum_{h=1}^{K} \delta_h^k
\]

(37)
For any fixed \((k, h) \in [K] \times [H]\), let \(t = N^k_h(s^k_h, a^k_h)\), and suppose \((s^k_h, a^k_h)\) was previously taken at step \(h\) of episodes \(k_1, \ldots, k_t < k\), then we have,

\[
\delta^k_h = (W^k_h - W^\pi_h)^{(a)} \leq (Q^k_h - Q^\pi_h)(s^k_h, a^k_h)
\]

\[
= (Q^k_h - Q^t_h)(s^k_h, a^k_h) + (Q^t_h - Q^\pi_h)(s^k_h, a^k_h)
\]

\[
\leq \alpha^0_t(H + 1)^2 + \sum_{i=1}^t \alpha^i_t \phi^k_i + \beta_t + [\mathbb{P}(W_{h+1} - W^\pi_{h+1})](s^k_h, a^k_h)
\]

\[
= \alpha^0_t(H + 1)^2 + \sum_{i=1}^t \alpha^i_t \phi^k_i + \beta_t - \phi^k_{h+1} + \delta^k_{h+1} + \xi^k_{h+1}
\]

where \(\beta_t = 12(H + 1)\sqrt{H^3t} / t\) and \(\xi^k_{h+1} := [(\mathbb{P}^t_h - \mathbb{P}^k_h)(W^*_{h+1} - W^\pi_{h+1})](s^k_h, a^k_h)\) is also a martingale difference sequence. Inequality \((a)\) holds due to the update rule in line 9 of algorithm 1 that \(W^k_h(s^k_h) = \min\{H, \max_{a \in A} Q_h(s^k_h, a')\} \leq \max_{a \in A} Q_h(s^k_h, a') = Q^k_h(s^k_h, a^k_h)\) and \(W^\pi_h = Q^\pi_h(s^k_h, a^k_h)\) because the policy in step \(h\) episode \(k\) select the action \(a^k_h\). Step \((b)\) holds due to the Lemma 5 and the Bellman equation. Inequality \((c)\) holds due to the definition of \(\delta, \phi, \) and \(\xi\). Denote \(n^k_h = N^k_h(s^k_h, a^k_h) = t\). It’s easy to bound the first term as

\[
\sum_{k=1}^K \alpha^0_{n^k_h}(H + 1)^2 = (H + 1)^2 \sum_{k=1}^K \mathbb{I}[n^k_h = 0] \leq SA(H + 1)^2
\]

To bound the second term, we rearrange the summation as

\[
\sum_{k=1}^K \sum_{i=1}^{n^k_h} \alpha^i_{n^k_h} \phi^k_i(\theta^k_{h+1}) \leq \sum_{k'^{k'}_{h+1}} \sum_{t=n^k_{h+1}}^\infty \alpha^k_{n^k_{h+1}} \leq \left(1 + \frac{1}{H^3}\right) \sum_{k=1}^K \phi^k_{h+1}
\]

where the last inequality uses Lemma 8(c). Substituting Eq. \((39)\) and Eq. \((40)\) into Eq. \((38)\), we have

\[
\sum_{k=1}^K \delta^k_h \leq SA(H + 1)^2 + \left(1 + \frac{1}{H}\right) \sum_{k=1}^K \phi^k_{h+1} - \sum_{k=1}^K \phi^k_{h+1} + \sum_{k=1}^K \delta^k_{h+1} + \sum_{k=1}^K \beta_{n^k_{h+1}} + \xi^k_{h+1}
\]

\[
\leq SA(H + 1)^2 + \left(1 + \frac{1}{H}\right) \sum_{k=1}^K \delta^k_{h+1} + \sum_{k=1}^K \beta_{n^k_{h+1}} + \xi^k_{h+1}
\]

where the last inequality uses \(\phi^k_{h+1} \leq \delta^k_{h+1}\) (the fact that \(W^* \geq W^\pi_h\)). Recursing the result for \(h = 1, 2, \ldots, H\) and using the fact \(\delta^k_{H+1} = 0\), we have

\[
\sum_{k=1}^K \delta^k_{h} \leq O \left(H^3SA + \sum_{h=1}^H \sum_{k=1}^K \beta_{n^k_{h+1}} + \xi^k_{h+1}\right)
\]
Finally, using pigeonhole principle, for any \( h \in [H] \), we have
\[
\sum_{k=1}^{K} \beta_{n_h^k} \leq O(H) \cdot \sum_{k=1}^{K} \sqrt{\frac{H^3 \ell}{n_h^k}} = O(H) \cdot \sum_{s,a} \sum_{n=1}^{N_h^k(s,a)} \sqrt{\frac{H^3 \ell}{n}}
\]
\[
\leq O(HSA \sqrt{\frac{H^3 K \ell}{SA}}) = O(\sqrt{H^4 SAT \ell})
\]
where inequality (a) holds because \( \sum_{s,a} N_h^K(s,a) = K \) and the left hand side of (a) is maximized when \( N_h^K(s,a) = \frac{K}{SA} \). Moreover, using Azuma-Hoeffding inequality again, with probability \( 1 - p \), we have,
\[
\left| \sum_{h=1}^{H} \sum_{k=1}^{K} \xi_{h+1}^k \right| = \left| \sum_{h=1}^{H} \sum_{k=1}^{K} (\mathbb{P}_h - \hat{\mathbb{P}}_h^k) \cdot (W_{h+1}^* - W_{h+1}^\pi) (s_h^k, a_h^k) \right| \leq O(H^2 \sqrt{T \ell})
\]
Thus, \( \sum_{k=1}^{K} \delta_1^k \leq O(H^2 SA + \sqrt{H^6 SAT \ell}) \). When \( T \geq \sqrt{H^6 SAT \ell} \), then \( \sqrt{H^6 SAT \ell} \geq H^6 SA \ell \geq H^3 SA \) since \( H \geq 1 \) and \( \ell = \log_e (2SAT/p) \geq \log_e (4ST/p) > 1 \) for \( A > 1 \). When \( T \leq \sqrt{H^6 SAT \ell} \), we have \( \sum_{k=1}^{K} \delta_1^k \leq 2KH(H+1) = 2T(H+1) \leq O(\sqrt{H^6 SAT \ell}) \). Therefore, we may remove the term \( H^3 SA \) in the regret bound. Thus, we have \( \sum_{k=1}^{K} \delta_1^k \leq O(\sqrt{H^6 SAT \ell}) \) with probability at least \( 1 - 2p \).