$O(15) \otimes O(3)$ critical theories in $d = 3$: a multi-correlator conformal bootstrap study

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We study the space of conformal field theories with product-group symmetry $O(15) \otimes O(3)$ in $d = 3$ dimensions using the conformal bootstrap method. Our analysis is distinct from that of Nakayama and Ohtsuki [Phys. Rev. D 89, 126009 (2014)], because we enforce crossing symmetry on multiple four-point correlation functions. We show that this considerably increases the disallowed region in the plane of scaling dimensions $(\Delta_3, \Delta_3)$. The reduced ‘allowed’ region appears to have both the Heisenberg and chiral fixed points on its boundary, but the antichiral point is deep in the disallowed region.

I. INTRODUCTION

In a physical system undergoing a continuous phase transition, conformal symmetry emerges when the temperature (or other non-thermal tuning parameter) reaches its critical value. In the latter half of the twentieth century, it was realized that this dramatic enhancement of the applicable symmetry group opened the door to powerful and unifying theoretical treatments of such systems. In particular, the important concept of universality emerged: the idea that systems with quite different microscopic physics would be described at criticality by the same conformal field theory (CFT). As a consequence, the critical exponents describing the behavior of various physical observables in the approach to criticality would agree exactly. Indeed, in simple cases these critical exponents are expected to depend only on the dimensionality of space (or, in the case of quantum phase transitions, the effective dimensionality of spacetime) and the group describing the internal symmetry of the theory.

The critical exponents that describe the approach to criticality are related to the scaling dimensions, $\{\Delta_i\}$, that describe the spatial (or spatiotemporal) correlations in the system when it is precisely at the critical point. The idea that it might be possible to determine these purely from the requirement of conformal symmetry — the so-called ‘conformal bootstrap’ approach — is now several decades old. However, it did not become a practical method of placing strong bounds on the scaling dimensions until the recent realization [1, 2] that this requirement can be cast in the form of a semidefinite program, the solvability of which can then be determined computationally.

Since then, notable successes include its precise determination of the critical exponents of the 3d Ising model, and of the Heisenberg fixed points that appear in three-dimensional theories with $O(N)$ internal symmetry [3, 4]. These achievements have been rendered possible by various improvements of the original bootstrap technique, including its mixed-correlator extension [5], whereby it is now possible to enforce crossing symmetry on a wider range of four-point functions composed of non-identical primary scalar operators transforming in arbitrary representations of the global symmetry group.

In contrast to the considerable bootstrap literature concerning theories with simple internal symmetries such as $\mathbb{Z}_2$ and $O(N)$, relatively little attention has been paid to $d = 3$ theories with internal symmetry groups of direct-product form, such as $O(N) \otimes O(M)$. These are especially relevant in describing multicritical points in systems with competing ordered phases; improving our understanding of them could help to shed light on the possible phase diagrams of such systems, which include both the cuprate and iron-based families of high-temperature superconductors. There is a considerable literature on the physics of such product-group theories, including Monte Carlo treatments and large-$N$ calculations. However, the conformal bootstrap potentially has advantages over these methods, since (a) unlike Monte Carlo, it exploits from the beginning the fact that the critical theory is conformally invariant, and (b) unlike large-$N$ calculations, it is not dependent on a small-parameter expansion.

There have, to our knowledge, so far been only two applications of the conformal bootstrap method to $O(N) \otimes O(M)$ problems in $d = 3$: Nakayama and Ohtsuki’s 2014 paper [6], in which they use the single-correlator bootstrap technique [8] to explore the space of interacting CFTs in $O(15) \otimes O(3)$-symmetric critical theories, and their 2015 paper [7], in which they make a similar analysis for $O(3) \otimes O(2)$ and $O(4) \otimes O(2)$. They imposed crossing symmetry on the four-point function of four identical scalar fields transforming in the bifundamental representation of the global symmetry group, i.e. as a vector under $O(N)$ and as a vector under $O(M)$, and discovered that this divided various two-dimensional sections of the space of scaling dimensions into the familiar disallowed and ‘allowed’ regions.

In this paper, we take the conformal bootstrap analysis of $d = 3 O(15) \otimes O(3)$-symmetric CFTs further, by enforcing crossing symmetry on multiple four-point correlators, including some involving non-identical primary fields.

The remainder of this paper is structured as follows. In section [9] we derive the operator product expansions (OPEs) necessary to decompose the four-point correlation functions of interest into sums over conformal blocks, and we thus derive the bootstrap equations that encode...
the crossing symmetry of these correlators. In section III we briefly review the method by which these are turned into a semidefinite program susceptible of numerical treatment. In section IV we show the results of our computations, presenting new and more stringent bounds on the scaling dimensions of the most relevant bifundamental and the most relevant singlet operator in the critical theory. In section V we discuss the interpretation of our results, and indicate possible lines of future work.

II. BOOTSTRAP EQUATIONS

The conformal bootstrap technique starts from one or more four-point correlation functions of the CFT. Applying the operator product expansion (OPE), each such correlation function can be written as a sum of conformal blocks. However, since the OPE involves treating the operators in pairs, it can be applied to the four-point correlation function in more than one way, resulting in conformal-block decompositions that look superficially different. Crossing symmetry is the requirement that the expressions for the four-point correlation function derived by carrying out the OPE in these different channels should in fact agree with each other.

In the general case where the operators in the four-point correlation function are distinct from each other, it may be decomposed as

$$\langle a(x_1)b(x_2)c(x_3)f(x_4) \rangle = \frac{1}{x_{12}^{\Delta_a} + x_{34}^{\Delta_b} + x_{14}^{\Delta_c} + x_{24}^{\Delta_f}} \sum_{O} \lambda_{abO} \lambda_{c,O} \lambda_{f,O} g_{\Delta, \Delta} (u, v).$$

(1)

Here $a$, $b$, $c$, and $f$ are primary operators of the CFT; $\Delta_a$, $\Delta_b$, $\Delta_c$, and $\Delta_f$ are the scaling dimensions of those operators; $\Delta_{ij} \equiv \Delta_i - \Delta_j$; $x_1$, $x_2$, $x_3$, and $x_4$ are $d$-dimensional position vectors; $x_{ij} \equiv |x_i - x_j|$; the sum runs over primary operators $O$; $\Delta$ and $\ell$ denote respectively the scaling dimension and the spin of $O$; $\lambda_{abO}$ and $\lambda_{c,O}$ are OPE coefficients; $g_{\Delta, \Delta}(u, v)$ is the conformal block associated with the exchange of the operator $O$; and $u$ and $v$ are the conformal cross-ratios, defined by

$$u = \frac{x_{12} x_{34}^2}{x_{13} x_{24}^2}, \quad v = \frac{x_{14} x_{23}^2}{x_{13} x_{24}^2}.$$  

(2)

Aside from small notational changes, (1) is just equation (2.1) of ref. [8].

In a CFT with an internal symmetry group, the primary operators may be classified according to their transformation properties under that group. For the direct-product group $O(N) \otimes O(M)$, on which we focus in this work, we label the relevant representations $PQ$, where $P, Q \in \{S, V, T, A\}$. The letters in this list stand respectively for singlet, vector, traceless symmetric tensor, and antisymmetric tensor. $P$ and $Q$ respectively denote the transformation properties of the operator under $O(N)$ and $O(M)$.

For a given choice of representation for each of the external operators $a$, $b$, $c$, and $f$, we can use the fusion rules of the group to determine the allowed representations of the exchanged operator $O$. For $O(N) \otimes O(M)$, the fusion rules that we shall need in this work are

$$s \times s \sim \sum_{SS+} O;$$

$$\phi_{ia} \times s \sim \sum_{VV+} O_{ia};$$

$$\phi_{ia} \times \phi_{jβ} \sim \sum_{SS+} \delta_{ij} \delta_{αβ} O + \sum_{ST+} \delta_{ij} O_{(αβ)} + \sum_{SA-} \delta_{ij} O_{[αβ]} + \sum_{TS+} \delta_{αβ} O_{(ij)} + \sum_{TT+} O_{(ij)[αβ]} + \sum_{TA-} O_{(ij)αβ} + \sum_{AS-} \delta_{αβ} O_{[ij]} + \sum_{AA+} O_{[ij][αβ]}.$$  

(5)

Here $s$ denotes the most relevant primary operator in the $SS$ sector, and $\phi_{ia}$ the most relevant primary operator in the $VV$ sector. Roman indices are associated with the $O(N)$ subgroup and thus take values from 1 to $N$; Greek indices are associated with the the $O(M)$ subgroup and thus take values from 1 to $M$. $O$ denotes a primary operator in the sector indicated below the relevant summation sign. A $+$ superscript indicates that the sum runs over primary operators of even spin only; a $-$ superscript indicates odd spins only.

To derive our crossing symmetry equations, we begin with four four-point correlation functions:

$$C_{ijklαβγδ}^{(VV)^4} \equiv \langle \phi_{ia}(x_1) \phi_{jβ}(x_2) \phi_{kγ}(x_3) \phi_{lδ}(x_4) \rangle;$$

(6)

$$C_{ijklαβγδ}^{(VV)^2^2} \equiv \langle \phi_{ia}(x_1) \phi_{jβ}(x_2) s(x_3) s(x_4) \rangle;$$

(7)

$$C_{ijklαβ}^{(VV)(SS)(VV)} \equiv \langle \phi_{ia}(x_1) s(x_2) \phi_{jβ}(x_3) s(x_4) \rangle;$$

(8)

$$C_{ijklαβ}^{(SS)^4} \equiv \langle s(x_1) s(x_2) s(x_3) s(x_4) \rangle.$$  

(9)

For each of these four, we equate the results of two different conformal block decompositions of the correlator: one where the first operator is paired with the second, as in (6), and one where the first operators is paired with the fourth. In practice, the latter decomposition is obtained simply by making an exchange of labels such as $b \leftrightarrow f$ and $x_2 \leftrightarrow x_4$ in (6).

After separating the coefficients of different fundamental tensor structures, we obtain a total of thirteen bootstrap equations: nine from (6), one from (7), two from (8), and one from (9). If we make certain assumptions about the spectrum of scaling dimensions in each sector, our bootstrap equations may be regarded as constraints on the OPE coefficients $\lambda_{abO}$ that appear in the decompositions (6). By testing computationally the mutual consistency of these constraints, we can rule out certain possible ranges of scaling dimensions, and thus place
FIG. 1. The ‘allowed’ (i.e. the non-disallowed) region of scaling dimensions for $O(15) \otimes O(3)$ conformal field theories, as determined by the multi-correlator conformal bootstrap method described in the text. $\Delta_\phi$ is the scaling dimension of the most relevant operator in the vector-vector (VV) sector; $\Delta_s$ that of the most relevant operator in the singlet-singlet (SS) sector. Red crosses show the allowed points for derivative order $n_{\text{max}} = 6$; gray dots show the points that are allowed for derivative order $n_{\text{max}} = 5$ but disallowed for $n_{\text{max}} = 6$. The locations of the Heisenberg, anti-chiral, and chiral fixed points, determined from large-$N$ calculations [9, 10], are shown as black, blue, and green crosses respectively.

III. COMPUTATIONAL SOLUTION

To make these computational tests, we use a modified version of PyCFTBoot [11] to transform our bootstrap equations into a semidefinite program of the type expected by SDPB, the arbitrary-precision semidefinite program solver designed for conformal bootstrap calculations [12, 13]. We briefly summarise its operation here, taking the opportunity to give details of some of our chosen parameter settings; for full details, we refer the reader to the original references.

The bootstrap equations derived above should hold at any point $(u, v)$ in the two-dimensional space of conformal cross-ratios $u$ and $v$. For computational implementation, we in practice Taylor-expand the conformal blocks that appear in them around the crossing-symmetric point $(u, v) = (\frac{1}{2}, \frac{1}{2})$. This expansion can only be carried out to a finite order, which is specified by two parameters, $n_{\text{max}}$ and $m_{\text{max}}$. Combinations of these integers give the maximum number of derivatives to be taken in $a$ and $b$, where $a$ and $b$ are transformed versions of the co-ordinates $u$ and $v$ [11, 14]. For all results presented in this paper, $m_{\text{max}} = n_{\text{max}} - 2$; therefore we specify only $n_{\text{max}}$ in what follows.

Two further important approximations are made. First, the recursive way in which the rational approximations to the conformal blocks are generated is restricted to a certain finite order, denoted $k_{\text{max}}$; in all results shown here, $k_{\text{max}} = 40$. Second, the sums over spins in (3–5) are in principle unrestricted, except by symmetry requirements that may restrict $\ell$ to be odd or even in particular sectors. In practice, however, we must limit this sum to spins $0 \leq \ell \leq \ell_{\text{max}}$; all results shown here are for $\ell_{\text{max}} = 23$.

We then impose the following assumptions: first, there is only a single relevant scalar operator, of scaling dimension $\Delta_\phi$, in the VV sector; and second, there is only a single relevant scalar operator, of scaling dimension $\Delta_s$, appearing in the theory.
FIG. 2. The ‘allowed’ (i.e. the non-disallowed) region of scaling dimensions for $O(15) \otimes O(3)$ conformal field theories, with a wider field of view than that used in Fig. 1. $\Delta$ is the scaling dimension of the most relevant operator in the vector-vector (VV) sector; $\Delta_s$ that of the most relevant operator in the singlet-singlet (SS) sector. Red crosses show the allowed points for derivative order $n_{\text{max}} = 6$. The locations of the Heisenberg, anti-chiral, and chiral fixed points, determined from large-$N$ calculations [9, 10], are shown as black, blue, and green crosses respectively.

in the SS sector. The scaling dimensions of operators in all other sectors, and of spins $\ell > 0$ in the SS and VV sectors, are restricted only by the usual unitarity bounds.

We run SDPB for a fixed time using its default parameters, with the following exceptions: we set precision to 1024, set findPrimalFeasible and findDualFeasible to true, and set primalErrorThreshold to $10^{-30}$ and dualErrorThreshold to $10^{-15}$. Usually one of two things happens: either a primal feasible solution is returned relatively quickly, in which case we say that the point is ‘allowed’; or a dual feasible solution is eventually found, in which case we say that it is disallowed. The quotation marks around ‘allowed’ are deliberate: what this outcome really means is just that the point $(\Delta_\phi, \Delta_s)$ is not ruled out by crossing symmetry constraints at our chosen derivative order $n_{\text{max}}$. In a few cases, SDPB does not terminate for either of these reasons within its fixed run-time.

We compile many of these SDPB outcomes into parallelogram-shaped grids of points, which constitute our main results. We present and discuss these in the next section.

IV. RESULTS

Fig. 1 compares the result of the above-described procedure in the case of two successive derivative orders: $n_{\text{max}} = 5$ and $n_{\text{max}} = 6$. The red crosses mark those points that are ‘allowed’ at both derivative orders; the gray dots are the points that are ‘allowed’ at order $n_{\text{max}} = 5$ but disallowed at order $n_{\text{max}} = 6$. For reference, we have also marked on the diagram large-$N$ estimates of the locations of the Heisenberg, anti-chiral, and chiral fixed points.

As seen in previous work for the case of $O(N)$ with $N \gg 1$ [8], increasing the derivative order transforms the large ‘allowed’ region into a peninsula with the Heisenberg point at (or at least very near) its tip. Note also that, despite signatures of the anti-chiral point in an earlier single-correlator bootstrap analysis [6], we see no evidence for it here: the predicted scaling dimensions lie well within the disallowed region.

The results in Fig. 2 are also for $n_{\text{max}} = 6$, but with a wider field of view: the VV scaling dimension $\Delta_\phi$ now extends to 0.63, while the SS scaling dimension $\Delta_s$ extends all the way to $d = 3$. This confirms the formation
FIG. 3. The ‘allowed’ (i.e. the non-disallowed) region of scaling dimensions for $O(15) \otimes O(3)$ conformal field theories. Red crosses show the allowed points for derivative order $n_{\text{max}} = 7$, with the additional restriction that we enforce the relationship $\lambda_{\phi \phi s} = \lambda_{\phi s \phi}$ between two of the OPE coefficients. Blue dots are the points that were determined to be dual feasible (i.e. disallowed) with a dual error threshold of $10^{-15}$; in other cases (the missing points) the SDP solver failed to find either a primal or a dual feasible solution after 48 core hours. $\Delta_{\phi}$ is the scaling dimension of the most relevant operator in the vector-vector (VV) sector; $\Delta_{s}$ that of the most relevant operator in the singlet-singlet (SS) sector. The locations of the Heisenberg, anti-chiral, and chiral fixed points, determined from large-$N$ calculations [9, 10], are shown as black, blue, and green crosses respectively.

In Fig. 3 we present the results of a more constraining analysis, in which we set the derivative order $n_{\text{max}}$ to 7, and also enforce the relationship $\lambda_{\phi \phi s} = \lambda_{\phi s \phi}$ between two of the OPE coefficients. As before, the red crosses mark ‘allowed’ points. The blue dots mark points that were found to be disallowed. The missing points are those for which SDPB did not terminate for either of these reasons during the fixed time for which we ran it.

Unlike for $O(N)$ at small $N$, this increase in derivative order, even combined with the addition of the OPE relation, is not enough to split an island centred on the Heisenberg point away from the peninsula. It is noteworthy, however, that (a) the ‘allowed’ region in Fig. 3 is significantly reduced compared to that in Fig. 1, and (b) the chiral fixed point now also lies on (or very near) the boundary of this region. It is tempting to speculate that this is not coincidence, and that further increases in the derivative order will lead to the separation of the ‘allowed’ regions around the Heisenberg and chiral points; but that must for now be left as a subject of future investigation.

V. DISCUSSION

In this paper we have, to our knowledge for the first time, applied the multi-correlator conformal bootstrap method to the space of $d = 3$ conformal field theories with $O(15) \otimes O(3)$ internal symmetry. We have observed a significant reduction in the ‘allowed’ region compared to the single-correlator analysis of Nakayama and Ohtsuki [6]; in our most constraining study, the results of which are reported in Fig. 3, the Heisenberg and chiral fixed points both appear to be on (or very near) the boundary of the ‘allowed’ region, while the anti-chiral point is well outside it.

What are we to make of the invisibility of the anti-chiral point in this study? One obvious explanation would be that the assumptions of our multi-correlator are stricter than those of the earlier single-correlator work by
Nakayama and Ohtsuki [6]. For them, $\Delta_\phi$ and $\Delta_s$ were lower bounds on the scaling dimensions of operators in the VV and SS sectors respectively, but other relevant operators with scaling dimensions between $\Delta_\phi$ and $d$ (in the VV sector) or between $\Delta_s$ and $d$ (in the SS sector) were allowed. If the anti-chiral theory has a second relevant operator in either the VV or SS sectors, it would fall outside our assumptions, and thus would not be expected to be visible as an ‘allowed’ region. We note in passing that Nakayama and Ohtsuki did not find evidence of the anti-chiral point in their study of $O(3) \otimes O(2)$ and $O(4) \otimes O(2)$ [7].

To investigate this point further, it would be worthwhile to supplement this analysis with one in which additional relevant operators were included in the VV and SS sectors. This could be achieved by adding extra points in the specification of the semidefinite program, and then sweeping their scaling dimensions between $\Delta_\phi$ and $d$ (in the VV sector) and between $\Delta_s$ and $d$ (in the SS sector). Not least because of the large amount of computational resource that such a study would require, it must for now be left as a topic for future work.

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