TOWARDS A GEOMETRIC INTERPRETATION OF
GENERALIZED FRACTIONAL INTEGRALS -
ERDÉLYI-KOBER TYPE INTEGRALS ON $\mathbb{R}^N$ AS AN EXAMPLE

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Abstract

A family of generalized Erdélyi-Kober type fractional integrals is interpreted geometrically as a distortion of the rotationally invariant integral kernel of the Riesz fractional integral in terms of generalized Cassini ovaloids on $\mathbb{R}^N$. Based on this geometric view, several extensions are discussed.

Key Words and Phrases: Fractional calculus, Riesz fractional integrals, Erdélyi-Kober fractional integrals, Cassini ovaloids.

1. Introduction

In the following we want to present a geometric approach for a deeper understanding of concepts and strategies used in generalized fractional calculus [Kiryakova (1994)].

We will collect arguments in support of the idea, that a generalization of fractional calculus may be considered from a geometrical point of view as a distortion of the isotropic kernel commonly used in standard fractional calculus, mediated by one or more additional fractional parameters.

E.g. a fractional integral $I^\alpha$ acting on a function $f(x)$ on $\mathbb{R}^N$ is therefore generalized to a multi-parameter fractional integral, where the additional parameters are a measure of distortion:

$$I^\alpha f(\vec{x}) \rightarrow I^{\alpha,\gamma,\ldots} f(\vec{x}) \quad (1)$$

According to [Gorenflo et al (2000)], fractional integrals are of convolution type and exhibit weakly singular kernels of power-law type.
Therefore as a first step we will investigate in this paper the specific geometric properties of kernels or weight-functions of a generalized set of multi-dimensional fractional integrals of Erdélyi-Kober type.

For this case, we will demonstrate, that a geometric approach allows a direct classification and interpretation of generalized multi-parameter fractional integrals in a straight forward manner in terms of Cassini and Maxwell ovaloids.

Furthermore, based on this viewpoint, we will present some generalizations of fractional operators of Erdélyi-Kober type, which allow a direct application in hadron physics.

2. Two examples as an illustration

In one dimensional space we start with some examples to illustrate the procedure: The Liouville definition of the left and right fractional integral \cite{Liouville1832} is given by:

\[
I_+^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} d\xi \ (x-\xi)^{-\alpha} f(\xi) \tag{2}
\]

\[
I_-^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x}^{\infty} d\xi \ (\xi-x)^{-\alpha} f(\xi) \tag{3}
\]

With the fractional parameter in the interval \(0 \leq \alpha \leq 1\). Consequently for the limiting case \(\alpha = 1\) \(I_+\) and \(I_-\) both coincide with the unit-operator and for \(\alpha = 0\) \(I_+\) and \(I_-\) both correspond to the standard integral operator.

If \(x\) is a time-like coordinate, the left Liouville integral is causal, the right Liouville integral is anti-causal. For space like coordinates, in order to preserve isotropy of space, both integrals must be combined.

The symmetric combination of \(I_+\) and \(I_-\) yields the Riesz integral \(RZ I^{\alpha}\):

\[
RZ I^{\alpha} f(x) = \frac{1}{2} \left( I_+^{\alpha} + I_-^{\alpha} \right) f(x) \tag{4}
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{\infty} d\xi \ |x-\xi|^{-\alpha} f(\xi) \tag{5}
\]

where \(|\cdot|\) denotes the absolute value. These integrals are examples of a one parameter convolution integral of power law type

\[
I^{\alpha} f(x) = c(x) \int_{-\infty}^{\infty} d\xi \ w(d) f(\xi) \tag{6}
\]

with the kernel

\[
w(d) = d^{-\alpha} \tag{7}
\]

and \(d\)

\[
d = |x-\xi| \tag{8}
\]

is a measure of distance on \(R^1\).
Erdélyi-Kober integrals are extensions of the Riemann-Liouville left and right fractional integrals, depending not only on the order \( \alpha > 0 \) but also on weight \( \gamma \in \mathbb{R} \) and an additional parameter \( \beta > 0 \) as follows:

\[
I_{+;\beta}^{\alpha,\gamma} f(x) = \frac{x^{\beta(\gamma+1-\alpha)}}{\Gamma(1-\alpha)} \int_0^x d\xi \beta(x^\beta - \xi^\beta)^{-\alpha} \xi^{-\beta\gamma} f(\xi)
\]

(9)

\[
= \frac{1}{\Gamma(1-\alpha)} \int_0^1 d\sigma (1-\sigma)^{-\alpha} \sigma^{-\gamma} f(x\sigma^{1/\beta})
\]

(10)

\[
I_{-;\beta}^{\alpha,\gamma} f(x) = \frac{x^{\beta\gamma}}{\Gamma(1-\alpha)} \int_x^\infty d\xi \beta(\xi^\beta - x^\beta)^{-\alpha} |\xi|^{-\beta(\gamma+1-\alpha)} f(\xi)
\]

(11)

\[
= \frac{1}{\Gamma(1-\alpha)} \int_1^\infty d\sigma (\sigma - 1)^{-\alpha} \sigma^{-(\gamma+1-\alpha)} f(x\sigma^{1/\beta})
\]

(12)

If we take the special case \( \beta = 1 \) and change the initial point 0 to \(-\infty\) in (9), the resulting modifications of the Erdélyi-Kober operators can be written as follows:

\[
\tilde{I}_{+;\beta}^{\alpha,\gamma} f(x) = \frac{x^{\gamma+1-\alpha}}{\Gamma(1-\alpha)} \int_{-\infty}^x d\xi (x-\xi)^{-\alpha} |\xi|^{-\gamma} f(\xi)
\]

(13)

\[
\tilde{I}_{-;\beta}^{\alpha,\gamma} f(x) = \frac{x^\gamma}{\Gamma(1-\alpha)} \int_x^\infty d\xi (\xi - x)^{-\alpha} |\xi|^{-\gamma+\alpha-1} f(\xi)
\]

(14)

which clearly shows the analogy to the fractional Liouville left and right integral definitions.

For space-like coordinates, both integrals may be combined, which yields a symmetric Erdélyi-Kober type generalized fractional integral of the form:

\[
\text{EK} I_{+;\beta}^{\alpha,\gamma} f(x) = \frac{1}{2} (\tilde{I}_{+;\beta}^{\alpha,\gamma} + \tilde{I}_{-;\beta}^{\alpha,\gamma}) f(x)
\]

(15)

\[
= c(x) \int_{-\infty}^\infty d\xi |x - \xi|^{-\alpha} |\xi|^{-\gamma} f(\xi)
\]

(16)

where \(||\) denotes the absolute value. This integral is an example for a two parameter convolution integral of power law type

\[
I_{+;\beta}^{\alpha,\gamma} f(x) = c(x) \int_{-\infty}^{+\infty} d\xi W(d_1, d_2) f(\xi)
\]

(17)

with the kernel \( W \), which now consists of two factors \( w_i \)

\[
W(d_1, d_2) = w_1(d_1)w_2(d_2)
\]

(18)

with

\[
w_1(d_1) = d_1^{-\alpha} \quad \text{and} \quad w_2(d_2) = d_2^{-\gamma}
\]

(19)
and the distances $d_i$

$$d_1 = |x - \xi| \quad \text{and} \quad d_2 = |\xi|$$

(20)

It should be mentioned, that for vanishing $\gamma$ the symmetric Erdélyi-Kober integral (15) smoothly reduces to the Riesz integral (5)

$$\lim_{\gamma \to 0} \text{EK} I_{\alpha, \gamma} = \text{RZ} I_{\alpha}$$

(21)

because the weight (18) is the product of two single power law factors as a function of a measure of distance. Values of equal weight are then determined by the equation:

$$d_1^{-\alpha} d_2^{-\gamma} = \text{const}$$

(22)

which is from a geometric point of view when extended to $\mathbb{R}^N$ nothing else, but the definition of Cassini ovaloids. Therefore in the next section, we will investigate the extension of the symmetric Erdélyi-Kober integral (15) to higher dimensions.

### 3. Symmetric Erdélyi-Kober integral on $\mathbb{R}^N$

On $\mathbb{R}^N$ which is spanned by the coordinate set $\{x_n, n = 1, ..., N\}$ we define a set of $M$ foci $F_m$ at positions $\{x_{nm}\}$ via:

$$F_m = \{x_{nm}\} \quad n = 1, ..., N \quad m = 1, ..., M$$

(23)

The distance $r_m$ between a given focus position $F_m$ and a given point $\vec{x}$ is the given by the Euclidean norm:

$$r_m = |\vec{x} - \vec{F}_m| \quad m = 1, ..., M$$

(24)

$$= \sqrt{\sum_{n=1}^{N} (x_n - x_{nm})^2}$$

(25)

Introducing a corresponding set of $M$ fractional parameters $\{\alpha_m, m = 1, ..., M\}$ we define the Cassini type weight $W$ as the product

$$W(\vec{x}) = \prod_{m=1}^{M} (r_m)^{-\alpha_m}$$

(26)

and the generalized symmetric Erdélyi-Kober integral on $\mathbb{R}^N$ follows as

$$\text{EK} I_{\{\alpha_m\}} f(\vec{x}) = c \int_{\mathbb{R}^N} d\xi^N W(\vec{\xi}, \{\vec{F}_m(\vec{x})\}) f(\vec{\xi}), \quad 0 < \sum_{m=1}^{M} \alpha_m < N$$

(27)

For the one dimensional case ($N = 1$) setting one focal point at $x_{11} = x$ we obtain the Riesz integral (5), for two focal points $x_{11} = x$ and $x_{12} = 0$ we obtain up to a constant the symmetric Erdélyi-Kober integral (15).
Figure 1. Contours of the weight $W$ in $R^2$ for two focal points $F_1 = \{4, 4\}$, $\alpha_1 = 0.6$ and $F_2 = \{0, 0\}$, $\alpha_2 = 0.4$. Thick lines indicate 0.25 steps.

In figure 1 for the two-dimensional case we have plotted contours of the weight $W$ for two foci with two different $\alpha$.

4. A physical interpretation and dynamic extensions

In [Herrmann(2011)] we already mentioned, that a left handed fractional integral is causal and therefore may be used to describe the dynamics
of a particle, while the right handed fractional integral is anti-causal and may be the appropriate tool to describe the dynamics of an anti-particle, which develops backwards in time. As a consequence, the symmetric integral may be used to describe particle-anti-particle pairs.

In a similar manner the generalized symmetric Erdélyi-Kober type integrals with $M$ different foci may be interpreted as operators, which describe multi-particle systems, which have a finite size. A typical example

**Figure 2.** Contours of the weight $W$ in $R^2$ for three focal points $F_1 = \{0, 4\}$, and $F_2, F_3$ rotated by $2\pi/3$ and $4\pi/3$ about the origin $\alpha_i = 0.4$. Thick lines indicate 0.25 steps.
for $M = 2$ within the framework of hadron physics are mesons, which are defined as quark anti-quark systems

$$m = q_1 \bar{q}_2$$

(28)

where different charge/mass ratios are modeled using different $\alpha_1$ and $\alpha_2$ values, a method, which has already been used in nuclear physics to describe asymmetric nuclear shapes [Pashkevich(1971)].

Excitations of such a system may then be described by a fractional differential equation of Klein-Gordon type:

$$\left( E_{K} I^{\alpha_1, \alpha_2} \Box - m^2(\alpha_1 + \alpha_2) \right) \Psi(\vec{x}) = 0$$

(29)

For $M = 3$ we may interpret the generalized symmetric Erdélyi-Kober type as an operator suitable for a description of baryons. In figure 2 we have sketched the weight for a symmetric configuration, which could be applied to symmetric 3-quark systems like $\Omega^-(sss)$. Excitations of such a system may then be described by a fractional differential equation of Klein-Gordon type:

$$\left( E_{K} I^{\alpha_1, \alpha_2, \alpha_3} \Box - m^2(\alpha_1 + \alpha_2 + \alpha_3) \right) \Psi(\vec{x}) = 0$$

(30)

The proposed physical interpretation also allows for an inclusion of vibrational and rotational degrees of freedom. Until now, the presented Erdélyi-Kober type operators were static. In fractional calculus, a possible time dependence of spatial operators has been discussed until now only in terms of variable order fractional parameters $\alpha \rightarrow \alpha(t)$ [Samko (1995)]. Within the framework of a geometric interpretation, a dynamic behavior may also be mediated introducing time dependent focus positions:

$$\vec{F}_m \rightarrow \vec{F}_m(t)$$

(31)

For example, the static weight shown in figure 2 may be extended to describe rotations

$$\vec{F}_m(t) = \hat{D}(t)\vec{F}_m(t = 0)$$

(32)

introducing the time-dependent rotation matrix $\hat{D}$.

Another generalization may realize the weight function $W$ not in terms of Cassini but Maxwell ovoids, which are defined using the sum rather than the product of focal distances [Maxwell(1846)]:

$$\tilde{W}(\vec{x}) = \sum_{m=1}^{M} (r_m)^{-\alpha_m} \quad 0 < \alpha_m < N$$

(33)
5. Conclusion

We have demonstrated that generalized classes of multi-parameter fractional integrals of power law type, which we defined as symmetric Erdélyi-Kober integrals indeed may be interpreted geometrically as distortions of the rotationally symmetric kernel of Riesz fractional integrals. This interpretation allows a direct classification of higher order fractional integrals and a physical interpretation in terms of multi-particle operators. Furthermore a new type of variable order fractional calculus in terms of space and time dependent focal position sets has been proposed.

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