Ordinary differential equations

The master Painlevé VI heat equation

*L'équation maîtresse de la chaleur associée à Painlevé VI*

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**Abstract**

Given the second-order scalar Lax pair of the sixth Painlevé equation, we build a generalized heat equation with rational coefficients which does not depend any more on the Painlevé variable.

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**Résumé**

Étant donné la paire de Lax scalaire de la sixième équation de Painlevé, nous donnons une construction directe de l’équation de la chaleur généralisée à coefficients rationnels qui ne dépend plus de la variable de Painlevé.

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**Version française abrégée**

Soit l’équation différentielle ordinaire (EDO)\(^1\), dotée de quatre singularités fuchsiennes \(x = x_0 = \infty, 0, 1, t\) et d’une singularité apparente \(t = u\). La condition d’isomonodromie (indépendance de la monodromie envers le birapport \(t\) des quatre singularités fuchsiennes) équivaut à une condition différentielle entre \(u\) et \(t\), qui a ainsi conduit R. Fuchs \(^6\) à la découverte de la sixième équation de Painlevé P6\(^2\).

Le processus d’isomonodromie conduit à adjoindre à l’EDO linéaire \(^1\) une deuxième équation linéaire, ce couple \(^3\), \(^4\) définissant en langage moderne une paire de Lax scalaire.

Le but de cet article est de donner, pour la première fois, une preuve constructive de l’existence d’une équation de la chaleur généralisée, voir \(^20\), dont les coefficients sont indépendants de la fonction de Painlevé \(u\) et ne dépendent, sous une forme rationnelle, que du birapport \(t\) et de la variable \(x\). Les deux démonstrations antérieures n’étaient valides, comme détaillé dans la Section \(^3\), que sous la condition \((\theta_x^2, \theta_x^1, \theta_t^1, \theta_t^2) \neq (1, 1, 1, 1)\), voir relation \(^10\).
1. Introduction. The scalar Lax pair of the sixth Painlevé equation

Let us first recall the 1905 classical result of R. Fuchs [6]. Consider a second-order linear ordinary differential equation (ODE) for a function $\psi = \psi(x)$ — the wave function, or wave vector — with four Fuchsian singularities of cross ratio $t$, put for convenience (but without loss of generality after a homographic transformation) at $x = x_0 = \infty, 0, 1, t$. As prescribed by Poincaré to have sufficient degrees of freedom for the isomonodromy problem to be non-trivial [14, pp. 217–220], one must in addition put one apparent singularity located at $x = u$, so that the ODE satisfied by $\psi$ writes [6, Eq. (1)]:

$$
\frac{d^2 \psi}{dx^2} - \left[ \frac{A}{x^2} + \frac{B}{(x-1)^2} + \frac{C}{(x-t)^2} + \frac{E}{(x-u)^2} + \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x-t} + \frac{e}{x-u} \right] \psi = 0.
$$

(1)

In Eq. (1), $A, B, C, E$ are constant parameters (independent of $t$ and $x$), while $a, b, c, e$ will ultimately depend on $t$ but not on $x$. The requirement that the monodromy matrix (which transforms two independent solutions $\psi_1, \psi_2$ when $x$ goes around a singularity $x_0$) be independent of the location of the nonapparent singularity $t$ — the isomonodromy condition — results in the constraint that $u$, as a function of the deformation parameter $t$, obeys the nonlinear second-order ordinary differential equation,

$$
\frac{d^2 u}{dt^2} = \frac{1}{2} \left[ \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t} \right] \left( \frac{du}{dt} \right)^2 - \left[ \frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t} \right] \frac{du}{dt} + \frac{u(u-1)(u-t)}{t^2(t-1)^2} \left[ \alpha + \beta \frac{t}{u} + \gamma \frac{t-1}{(u-1)^2} + \delta \frac{t(t-1)}{(u-t)^2} \right].
$$

(2)

Eq. (2) is the celebrated sixth Painlevé equation P6, the most general second-order nonlinear ODE without movable critical singularities, which lies at the heart of many problems of mathematics and theoretical physics of current active interest.

The set of four parameters $(\alpha, \beta, \gamma, \delta)$ is in one-to-one correspondence with the set $A, B, C, E$ defined in (1) and with the squares $\theta_i^2$ of the monodromy exponents, see relations (10) and (8) below.

Coming back to Eq. (1), what happens more precisely is that demanding the isomonodromy of Eq. (1) is tantamount to the existence of two linear equations for the wave vector (now a function $\psi = \psi(x, t)$). To endorse a modern terminology, the corresponding Fuchs–Garnier scalar Lax pair of equations writes [6–8]:

$$
\partial_x^2 \psi + \left( S/2 \right) \psi = 0,
$$

(3)

$$
\partial_t \psi + W \partial_x \psi - \left( 1/2 \right) W_x \psi = 0,
$$

(4)

and their commutativity (or compatibility) condition yields P6. In their most concise form (first exhibited by Garnier), the two scalar functions $S, W$ display a remarkable symmetry between $x$ and $u$:

$$
-\frac{S}{2} = \frac{3/4}{(x-u)^2} + \frac{g_1 u'}{x(x-1)} + \frac{(g_1 u')^2 - \frac{g_2}{u-u-1} + \frac{f_c(u)}{x(x-1) (x-t)}}{w_{uu-1}} + \frac{f_c(u)}{x(x-1) (x-t)} + \frac{f_c(x)}{x(x-1) (x-t)},
$$

(5)

$$
W = \left[ \frac{x(x-1) (u-t)}{(x-u) (x-t)} \right] g_1 = -\frac{t(t-1)}{2(u-t)}, \quad g_0 = -u + \frac{1}{2},
$$

(6)

$$
F_c(z) = \frac{A}{z^2} + \left[ \frac{B}{(z-1)^2} + \frac{C}{(z-t)^2} + \frac{E}{z(z-1)} \right] + 2 \alpha - 2 \beta, 2 \gamma, 1 - 2 \delta = 4(A + B + C + E + 1), 4A + 1, 4B + 1, 4C + 1.
$$

(7)

(8)

The purpose of this paper is to eliminate the dependent variable $u$ (and its derivative) between the two linear equations (3), (4) while preserving the linearity of the resulting single equation. This provides us with a heat equation for the wave vector whose coefficients are solely rational functions of $t$ and $x$ (and of the monodromy parameters $\theta_i$). This generalized heat equation had in fact appeared earlier in the literature. We shall discuss this in the final section of the present Note. The paper is organized as follows. In Section 2, we present the elimination procedure. In Section 3, we compare our findings with previous results, discussing in particular the underlying motivation.

2. From the scalar Lax pair to the generalized heat equation

The guideline of our procedure is the singularity structure of the two linear equations (3), (4) in the complex plane of $x$. To achieve our goal, it is necessary (but not sufficient) to eliminate the polar singularity $x = u$ between the two equations.

The first equation (3) is an ODE with five Fuchsian singularities in the complex plane of $x$ whose Riemann scheme is

$$
\begin{pmatrix}
\frac{\infty}{(1 - \theta_\infty)/2} & (1 - \theta_0)/2 & (1 - \theta_1)/2 & (1 - \theta_0)/2 & -1/2 \\
(1 + \theta_0)/2 & (1 + \theta_0)/2 & (1 + \theta_1)/2 & (1 + \theta_0)/2 & 3/2 \\
0 & 1 & t & u
\end{pmatrix},
$$

(9)
with the correspondence
\[(2 \alpha, -2 \beta, 2 \gamma, 1 - 2 \delta) = (\theta_0^2, \theta_0^2, \theta_1^2, \theta_1^2). \tag{10}\]
As to the second equation (4), its ODE reduction \(\partial_x = 0\) possesses one Fuchsian singularity at \(x = u\), with the Riemann scheme
\[
\begin{pmatrix}
u \\
-1/2
\end{pmatrix}.
\tag{11}\]

In a first step, we remove the four finite double poles in (3), via the change of wave function
\[
\psi = x^{(1-\theta_0)/2}(x-1)^{(1-\theta_1)/2}(x-t)^{(1-\theta_2)/2}(x-u)^{-1/2}e^{G(t)}\psi,
\tag{12}\]
in which the gauge \(G\) is a function of \(t\) that is left for the moment arbitrary. The change of wave function (12) is quite similar to the classical one for the Gauss hypergeometric equation. After decomposition of its coefficients in simple elements of \(x\), the Lax pair becomes
\[
\frac{\partial^2 \psi}{\partial x^2} + \left(\frac{1 - \theta_0}{x} + \frac{1 - \theta_1}{x-1} + \frac{1 - \theta_1}{x-t} - \frac{1}{x-u}\right) \partial_x \psi
\]
\[+ \frac{1}{4u(u-1)(u-t)} \left(\frac{R_0}{x} + \frac{R_1}{x-1} + \frac{R_t}{x-t} + \frac{2R_u}{x-u}\right) \psi = 0,
\tag{13}\]
\[
t(t-1)\partial_t \psi - \frac{x(x-1)(u-t)}{x-u} \partial_x \psi + \left(t(t-1)G' + \frac{R_u}{2(x-u)} + \frac{(\theta_0 + \theta_1 + \theta_2 - 1)(u-t)}{2}\right) \psi = 0,
\tag{14}\]
in which the residues \(R_j\) are best expressed in terms of the two relations defining the one-parameter classical Riccati solution of P6 in terms of the hypergeometric function [7].

\[
R(\theta_0, \theta_1, \theta_2) \equiv t(t-1)u' + u(u-1)(u-t)\left(\frac{\theta_0}{u} + \frac{\theta_1}{u-1} + \frac{\theta_2 - 1}{u-t}\right) = 0,
\tag{15}\]
\[
\rho \equiv (1 - \theta_0 - \theta_1 - \theta_2)^2 - \theta_2^2 = 0.
\tag{16}\]
These residues are
\[
R_u = R(\theta_0, \theta_1, \theta_2),
\]
\[
R_0 = -\frac{R(\theta_0, \theta_1, \theta_2)R(2-\theta_0, -\theta_1, -\theta_2) + \rho u(u-1)(u-t)}{(u-t)},
\]
\[
R_1 = -\frac{R(\theta_0, \theta_1, \theta_2)R(-\theta_0, 2-\theta_2, -\theta_2) + \rho u(u-1)(u-t)}{(u-1)(1-t)},
\]
\[
R_t = -\frac{R(\theta_0, \theta_1, \theta_2)R(-\theta_0, -\theta_1, 2-\theta_2) + \rho u(u-1)(u-t)}{t(t-1)}.
\tag{17}\]
It is remarkable that \(-R_j/(4u(u-1)(u-t))\) is precisely the polynomial Hamiltonian of P6 [11].

In a second step, we eliminate this simple pole. Since the quotient of the residues of (13) and (14) at the simple pole \(x = u\) does not depend on \(\psi\), the resulting equation remains linear in \(\psi\),
\[
-\frac{t(t-1)}{x(x-1)(x-t)} \partial_t \psi + \partial_x^2 \psi - \left(\frac{\theta_0 - 1}{x} + \frac{\theta_1 - 1}{x-1} + \frac{\theta_2}{x-t}\right) \partial_x \psi
\]
\[+ \frac{1}{x(x-1)(x-t)} \left[\frac{\rho}{4}(x-t) - t(t-1)G'(x) - F(t)\right] \psi = 0,
\tag{18}\]
and its dependence on \(u\) and \(u'\) is gathered in an expression independent of \(x\),
\[
F(t) = -\frac{R(\theta_0, \theta_1, \theta_2)R(-\theta_0, -\theta_1, -\theta_2) + \rho u(u-1)(u-t)}{4u(u-1)(u-t)} + \left(\theta_2^2 + 1 - (\theta_0 + \theta_1 + \theta_2)^2\right) \frac{u-t}{4}.
\tag{19}\]
The third and last step is to choose the arbitrary function \(G(t)\) so as to cancel this contribution of \(u\) and \(u'\). The final result is a generalized heat equation whose coefficients are rational functions of \(t\) and \(x\),
\[
-t(t-1)\partial_t \psi + x(x-1)(x-t) \left[\partial_x^2 \psi - \left(\frac{\theta_0 - 1}{x} + \frac{\theta_1 - 1}{x-1} + \frac{\theta_2}{x-t}\right) \partial_x \psi\right] + \left[\frac{\rho}{4}(x-t) - g(t)\right] \psi = 0,
\tag{20}\]
in which \(g(t)\) can be arbitrarily chosen. In the Picard case \(\theta_v = 0\), its reduction \(\partial_t = 0\), \(g(t) = 0\) is identical to the classical linear ODE of Legendre for the periods of the elliptic function.
3. Discussion

The heat equation we have obtained is in fact not new, but the present proof is the first one without any restriction. Indeed, previous occurrences of this heat equation are the following.

(i) By establishing a formal correspondence between the scalar Lax pair \((3), (4)\) and the time-dependent Schrödinger equation of quantum mechanics, Suleimanov [15] obtained this heat equation.

(ii) Starting from the second-order matrix Lax pair of \(P_6\) as given by Jimbo and Miwa [9], in which the monodromy matrix is just the sum of four simple poles, D.P. Novikov [12] assumed, like Ref. [9], that the residue at \(\infty\) is a constant matrix and finally proved that the first component of the two-dimensional wave vector obeys the heat equation \((20)\).

(iii) In a context of quantization of classical integrable systems (like Ref. [15]), Zabrodin and Zotov [17] also started from the matrix Lax pair of Ref. [9] and, after a suitable gauge transformation and change of variables, obtained a master \(P_6\) heat equation of the form

\[ \partial_t \Psi = (1/2) \partial_X^2 \Psi + V(X, T) \Psi. \tag{21} \]

This time-dependent Schrödinger (or Fokker–Planck) equation coincides with the rational heat equation \((20)\) after a point transformation \(t \to T = T(t), x \to X = X(x, t)\) involving elliptic functions totally analogous to the transformation \([6, 13]\) that maps \(P_6\) to a Hamiltonian which is the sum of a kinetic energy and a time-dependent potential energy.

The drawback with the derivations of Refs. [12] and [17] is that, because of the assumption made in Ref. [9] that the residue at \(\infty\) is a constant matrix, the matrix Lax pair as assumed by Jimbo and Miwa does not exist when all four \(\theta_i^2\) are unity, see details in \([10, 3]\).

The earliest occurrence of the heat equation \((20)\) which we are aware of is in conformal field theory \([1, Eq. (5.17)]\) in the particular case of a value \(c = 1\) of the central charge of a Virasoro algebra.

From the present results one deduces easily by the classical confluence of the four singularities \([13, 4]\) similar results for all the other Painlevé functions. In particular, the Tracy–Widom probability distribution in random matrix theory \([16]\) has been recently characterized by such a time-dependent Schrödinger equation \((2)\), associated with the second Painlevé function. Similar results hold for the sine-Gordon third Painlevé function \([5]\).

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