The unified method: III. Nonlinearizable problems on the interval

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Abstract
Boundary value problems for integrable nonlinear evolution PDEs formulated on the finite interval can be analyzed by the unified method introduced by one of the authors and extensively used in the literature. The implementation of this general method to this particular class of problems yields the solution in terms of the unique solution of a matrix Riemann–Hilbert problem formulated in the complex $k$-plane (the Fourier plane), which has a jump matrix with explicit $(x,t)$-dependence involving six scalar functions of $k$, called the spectral functions. Two of these functions depend on the initial data, whereas the other four depend on all boundary values. The most difficult step of the new method is the characterization of the latter four spectral functions in terms of the given initial and boundary data, i.e. the elimination of the unknown boundary values. Here, we present an effective characterization of the spectral functions in terms of the given initial and boundary data. We present two different characterizations of this problem. One is based on the analysis of the so-called global relation, on the analysis of the equations obtained from the global relation via certain transformations leaving the dispersion relation of the associated linearized PDE invariant and on the computation of the large $k$ asymptotics of the eigenfunctions defining the relevant spectral functions. The other is based on the analysis of the global relation and on the introduction of the so-called Gelfand–Levitan–Marchenko representations of the eigenfunctions defining the relevant spectral functions. We also show that these two different characterizations are equivalent and that in the limit when the length of the interval tends to infinity, the relevant formulas reduce to the analogous formulas obtained recently for the case of boundary value problems formulated on the half-line.

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1. Introduction

This is the third paper in a series of papers addressing the most difficult problem in the analysis of integrable nonlinear evolution PDEs, namely the problem of expressing the so-called spectral functions in terms of the given initial and boundary conditions. In [6], this problem was analyzed for the case of the half-line. In [7], the same problem was also analyzed with the aid of the so-called Gelfand–Levitan–Marchenko (GLM) representations. Here, we analyze this problem for the case of the finite interval.

We refer the interested readers to [6] for an introduction to the unified method of [2, 3, 4] and for a discussion of the difference between linearizable versus nonlinearizable boundary value problems. Here, we only note that the unified method expresses the solution $q(x, t)$ of an integrable evolution PDE in terms of an integral formulated in the complex $k$-plane. This representation is similar to the integral representation obtained by the new method for the linearized version of the given nonlinear PDE, but it also contains the entries of a certain matrix-valued function $M(x, t, k)$, which is the solution of a matrix Riemann–Hilbert (RH) problem. The main advantage of the new method is the fact that this RH problem involves a jump matrix with explicit $(x, t)$-dependence, uniquely defined in terms of the spectral functions. For the problem on the interval, there exist six spectral functions denoted by $[a(k), b(k), A(k), B(k), \varphi(k, k), \Phi(k, k)]$. The functions $[a(k), b(k)]$ are defined in terms of the initial data $q_0(x) = q(x, 0)$ via a system of linear Volterra integral equations; the functions $[A(k), B(k)]$ and $[\varphi(k, k), \Phi(k, k)]$ are defined in terms of the boundary values at $x = 0$ and $x = L$, respectively, and also via systems of linear Volterra integral equations. However, the integral equations defining $[A(k), B(k)]$ and $[\varphi(k, k), \Phi(k, k)]$ involve all boundary values, whereas for a well-posed problem only a subset of these boundary values can be prescribed as boundary conditions. Thus, the complete solution of a concrete initial-boundary value problem requires the characterization of $[A(k), B(k), \varphi(k, k), \Phi(k, k)]$ in terms of the given initial and boundary conditions. For example, for the Dirichlet problem of the NLS on the interval $0 < x < L$, it is necessary to characterize $[A(k), B(k), \varphi(k, k), \Phi(k, k)]$ in terms of $q_0(x)$, $g_0(t) := q(0, t)$ and $h_0(t) = q(L, t)$.

A characterization of the spectral functions is called effective if it fulfills the following requirements.

(a) In the linear limit, it yields an effective solution of the linearized boundary value problem, i.e. it yields a solution in the form of an integral that involves the transforms of the given initial and boundary conditions.

(b) For ’small’ boundary conditions, it yields an effective perturbative scheme, i.e. it yields an expression in which each term can be computed uniquely in a well-defined recursive scheme.

The effective characterization presented here is based on the construction of the generalized Dirichlet to Neumann map, i.e. on the characterization of the unknown boundary values in terms of the given initial and boundary conditions. This characterization employs the same three ingredients introduced in [6] for the analogous problem on the half-line.

(a) The computation of the large $k$ asymptotics of the eigenfunctions $\Phi(t, k)$ and $\varphi(t, k)$ defining $[A(k), B(k)]$ and $[\varphi(k, k), \Phi(k, k)]$, respectively.

(b) The analysis of the so-called global relation and of the equations obtained from the global relation under the transformations that leave invariant the dispersion relation of the associated linearized equation.

(c) The construction of a perturbative scheme for establishing effectiveness.
2. Preliminaries

We consider the NLS equation on the interval \([0, L]\):

\[
\imath q_t + q_{xx} - 2\lambda |q|^2 q = 0, \quad \lambda = \pm 1, \quad 0 < x < L, \quad 0 < t < T,
\]

(2.1)

where \(L > 0\) is the length of the interval and \(T > 0\) is a fixed finite time. We let \(g_0(t)\) and \(h_0(t)\) denote the Dirichlet boundary values of \(q(x, t)\), whereas \(g_1(t)\) and \(h_1(t)\) denote the Neumann boundary values:

\[
q(0, t) = g_0(t), \quad q(L, t) = h_0(t), \quad q_x(0, t) = g_1(t), \quad q_x(L, t) = h_1(t).
\]

(2.2)

2.1. Bounded and analytic eigenfunctions

In what follows, we present a summary of the results obtained in [5].

The Lax pair of (2.1) can be written in the differential form as

\[
d(e^{i(kx + 2\lambda t)i\hat{\sigma}_3}) \mu(x, t, k) = W(x, t, k),
\]

(2.3)

where \(k \in \mathbb{C}\) is the spectral parameter, the closed 1-form \(W\) is defined by

\[
W(x, t, k) = e^{i(kx + 2\lambda t)i\hat{\sigma}_3} (Q \, dx + \hat{Q} \, dt) \mu, \quad \mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(2.4)

and \(\hat{\sigma}_3 A = [\sigma_3, A]\). We define four eigenfunctions \(\mu_j^I, j = 1, 4\) by integrating from the four corners of the domain \([0 < x < L, \quad 0 < t < T]\):

\[
\mu_j(x, t, k) = I + \int_{(x_j, t_j)}^{(x, t)} e^{-i(kx + 2\lambda t)i\hat{\sigma}_3} W_j(x', t'),
\]

(2.5)

where \(W_j\) is the differential form defined in (2.4) with \(\mu\) replaced by \(\mu_j\) and \((x_j, t_j)\) denote the points \((0, T), (0, 0), (L, 0)\) and \((L, T)\), respectively; see figure 1. The spectral functions \(a(k), b(k), A(k), B(k), \hat{A}(k), \hat{B}(k)\) are defined for \(k \in \mathbb{C}\) by

...
Indeed, for each $a \cdot \cdots \cdot \Phi_1$ such that $\lambda \in D_j$, there denote the four quadrants of the complex $4$-plane. Then, we have the following.

- $a(k)$ and $b(k)$ are entire functions of $k$ that are bounded in $D_1 \cup D_2$.
- $A(k), B(k), \lambda A(k)$ and $\lambda B(k)$ are entire functions of $k$ that are bounded for $k \in D_1 \cup D_3$.
- $\Phi_1(t, k), \Phi_2(t, k), \psi_1(t, k)$ and $\psi_2(t, k)$ are entire functions of $k$ that are bounded for $k \in D_2 \cup D_4$.

2.2. The global relation

The eigenfunctions $\{\Phi_j, \psi_j\}_j^2$ satisfy the following global relation:

$$c(t, k) = \Phi_2(t, \bar{k}) \Phi_1(t, k) e^{4ikt} - a(k) \psi_1(t, k) e^{2ikt}$$

$$+ \Phi_1(t, k) [a(k) \psi_2(t, k) - \lambda b(k) \psi_1(t, k) e^{2ikt}], \quad k \in \mathbb{C},$$

(2.7)

where $c(t, k)$ is an entire function of $k$ such that

$$c(t, k) = O \left( \frac{1 + e^{2ikL}}{k} \right), \quad k \to \infty, \quad k \in \mathbb{C}. \quad (2.8)$$

Indeed, for each $t \in (0, T)$, let $R(x, t, k)$ be the solution of the $x$-part of the Lax pair of (2.1), such that $R(L, t, k) = I$, i.e. $R$ is the unique solution of the Volterra integral equation

$$R(x, t, k) = I + \int_L^x e^{ik(x'-x)} \bar{\psi}_1(QR)(x', t, k) \, dx, \quad 0 < x < L. \quad (2.9)$$
It follows that $R$ is related to $\mu_3$ by
\begin{equation}
R(x, t, k) = \mu_3(x, t, k) e^{i k(L-x)\delta} \mu_3^{-1}(L, t, k).
\end{equation}
(2.10)

Since
\begin{equation}
\mu_3(x, t, k) = \mu_2(x, t, k) e^{-i k s^2 \delta^2} s(k),
\end{equation}
this yields
\begin{equation}
R(x, t, k) = \mu_2(x, t, k) e^{-i k s^2 \delta^2} s(k) e^{i k(L-x)\delta} \mu_3^{-1}(L, t, k).
\end{equation}
(2.11)
Evaluating the (12) entry of (2.11) at $x = 0$ and recalling the definitions (2.6) of $\{\Phi_j, \varphi_j\}_1^2$, we find (2.7) with $c(t, k) := R_{12}(t, k)$. It follows from (2.9) that $c(t, k)$ is an entire function satisfying (2.8).

In the case of vanishing initial data, the global relation (2.7) reduces to
\begin{equation}
c = \Phi_1 \bar{\varphi}_2 - \Phi_2 \varphi_1 e^{2i k L}.
\end{equation}
(2.12)

2.3. Asymptotics

Integration by parts in (2.5) shows that
\begin{equation}
\Phi_1(t, k) = \frac{\Phi_1^{(1)}(t)}{k} + \frac{\Phi_1^{(2)}(t)}{k^2} + O\left(\frac{1}{k^3}\right) + O\left(\frac{e^{-4\pi i}}{k}\right), \quad k \to \infty, \quad k \in D_2 \cup D_3,
\end{equation}
\begin{equation}
\Phi_2(t, k) = 1 + \frac{\Phi_2^{(1)}(t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \quad k \in D_2 \cup D_3,
\end{equation}
\begin{equation}
\varphi_1(t, k) = \frac{\varphi_1^{(1)}(t)}{k} + \frac{\varphi_1^{(2)}(t)}{k^2} + O\left(\frac{1}{k^3}\right) + O\left(\frac{e^{-4\pi i}}{k}\right), \quad k \to \infty, \quad k \in D_2 \cup D_3,
\end{equation}
\begin{equation}
\varphi_2(t, k) = 1 + \frac{\varphi_2^{(1)}(t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \quad k \in D_2 \cup D_3,
\end{equation}
where
\begin{equation}
\Phi_1^{(1)}(t) = \frac{g_0(t)}{2i}, \quad \Phi_1^{(2)}(t) = \frac{g_1(t)}{4} + \frac{g_0(t)}{2i} \int_{(0,0)}^{(0,t)} \omega, \quad \Phi_2^{(1)}(t) = \int_{(0,0)}^{(0,t)} \omega,
\end{equation}
\begin{equation}
\varphi_1^{(1)}(t) = \frac{h_0(t)}{2i}, \quad \varphi_1^{(2)}(t) = \frac{h_1(t)}{4} + \frac{h_0(t)}{2i} \int_{(0,0)}^{(L,t)} \omega, \quad \varphi_2^{(1)}(t) = \int_{(L,0)}^{(L,t)} \omega,
\end{equation}
and the closed 1-form $\omega$ is defined by
\begin{equation}
\omega = \frac{\lambda}{2} [-i q q^2 dx + (q q_\gamma - q q_\gamma) dt].
\end{equation}

In particular, we find the following expressions for the boundary values:
\begin{equation}
g_0(t) = 2i \Phi_1^{(1)}(t), \quad h_0(t) = 2i \Phi_1^{(1)}(t), \quad g_1(t) = 4 \Phi_1^{(2)}(t) + 2i g_0 \Phi_2^{(1)}(t), \quad h_1(t) = 4 \varphi_1^{(2)}(t) + 2i h_0 \varphi_2^{(1)}(t), \quad 0 < t < T.
\end{equation}
(2.15a)

(2.15b)

We will also need the asymptotics of $c$.

**Lemma 2.1.** The global relation (2.7) implies that
\begin{equation}
c(t, k) = \frac{\Phi_1^{(1)}(t)}{k} + \frac{\Phi_1^{(2)}(t) + \Phi_1^{(1)}(t) (a^{(1)} + \varphi_2^{(1)}(t))}{k^2} + O\left(\frac{1}{k^3}\right)
\end{equation}
\begin{equation}
- \left[ \frac{\varphi_1^{(1)}(t)}{k} + \frac{\varphi_1^{(2)}(t) + \varphi_1^{(1)}(t) (a^{(1)} + \Phi_2^{(1)}(t))}{k^2} + O\left(\frac{1}{k^3}\right) \right] e^{2i k L},
\end{equation}
k \to \infty, \quad k \in D_1 \cup D_3,
where $a^{(1)} = \int_{(L,0)}^{(L,t)} \omega$. 
(2.16)
Theorem 3.1. with the initial data. For the Neumann problem, it is assumed that the functions $g$ satisfies (2.7) and (2.8).

Remark 2.2. We emphasize that the proof of lemma 2.1 does not require the knowledge of the explicit form of $c(t, k)$; it only requires the existence of an entire function $c(t, k)$.

3. The Dirichlet and Neumann problems

We will use the following notations.

- For $j = 1, \ldots, 4$, $\partial D_j$ denotes the boundary of the $j$th quadrant $D_j$, oriented so that $D_j$ lies to the left of $\partial D_j$.
- $\partial D_3$ denotes the contour obtained by deforming the contour $\partial D_3$ so that it passes below the zeros of $\Delta(k)$ in $\mathbb{R} \setminus 0$, i.e. below the set $\{-\frac{n\pi}{2}, |n = 0, 1, 2, \ldots\}$. Moreover, we let $\partial D_3' = -\partial D_3$.
- The functions $f_+(k)$ and $f_-(k)$ denote the following even and odd combinations of the function $f(k)$:
  
  $$f_+(k) = f(k) + f(-k), \quad f_-(k) = f(k) - f(-k), \quad k \in \mathbb{C}.$$ 

- $\Delta(k)$ and $\Sigma(k)$ are defined by
  
  $$\Delta(k) = e^{2ikL} - e^{-2ikL}, \quad \Sigma(k) = e^{2ikL} + e^{-2ikL}.$$ 

The following theorem expresses the spectral functions $\{A(k), B(k), \Phi(k), \Theta(k)\}$ in terms of the given boundary data via the solution of a system of nonlinear integral equations. For simplicity, we assume that $q_0(x)$ vanishes identically.

**Theorem 3.1.** Let $T < \infty$ and let $q_0(x) = 0$ for $x \geqslant 0$. For the Dirichlet problem, it is assumed that the functions $g_0(t)$ and $h_0(t)$, $0 \leqslant t < T$, have sufficient smoothness and are compatible with the initial data. For the Neumann problem, it is assumed that the functions $g_1(t)$ and $h_1(t)$, $0 \leqslant t < T$, have sufficient smoothness and are compatible with the initial data.

Then, the spectral functions $\{A(k), B(k), \Phi(k), \Theta(k)\}$ are given by

$$\begin{align*}
A(k) &= \Phi_2(T, k), \quad B(k) = -\Phi_1(T, k) e^{4ikL}, \\
\Phi(k) &= \varphi_2(T, k), \quad \Theta(k) = -\varphi_1(T, k) e^{4ikL},
\end{align*}$$

where the complex-valued functions $\{\Phi_j(t, k), \varphi_j(t, k)\}_{ij}^2$ satisfy the following system of nonlinear integral equations:

$$\begin{align*}
\Phi_1(t, k) &= \int_0^t e^{4ikL(t-t')}[-i\lambda|g_0|^2\Phi_1 + (2kg_0 + ig_1)\Phi_2](t', k) \, dt', \\
\Phi_2(t, k) &= 1 + \lambda \int_0^t [(2kg_0 - ig_1)\Phi_1 + i|g_0|^2\Phi_2](t', k) \, dt', \\
\varphi_1(t, k) &= \int_0^t e^{4ikL(t-t')[-i\lambda|h_0|^2\varphi_1 + (2kh_0 + ih_1)\varphi_2](t', k) \, dt',} \\
\varphi_2(t, k) &= 1 + \lambda \int_0^t [(2kh_0 - ih_1)\varphi_1 + i|h_0|^2\varphi_2](t', k) \, dt', \quad 0 < t < T, \quad k \in \mathbb{C}.
\end{align*}$$

Proof. See appendix B. $\square$
Thus, Equations (3.1) and (3.2) follow from the definitions and (2.5).

(a) For the Dirichlet problem, the unknown Neumann boundary values \( g_1(t) \) and \( h_1(t) \) are given by the following expressions:

\[
g_1(t) = \frac{4}{\pi i} \int_{\partial D_2} \left( \frac{1}{2\Delta(k)} \left[ k\Phi_1^-(t, k) + i\phi(t) \right] - \frac{1}{\Delta(k)} [k\phi(t, k) + i\phi(t)] \right) \, dk
\]

and

\[
h_1(t) = \frac{4}{\pi i} \int_{\partial D_2} \left( \frac{1}{2\Delta(k)} [k\phi(t, k) + i\phi(t)] + \frac{1}{\Delta(k)} [k\phi(t, k) + i\phi(t)] \right) \, dk
\]

(b) For the Neumann problem, the unknown boundary values \( g_0(t) \) and \( h_0(t) \) are given by the following expressions:

\[
g_0(t) = \frac{2}{\pi} \int_{\partial D_2} \left( \frac{1}{\Delta(k)} \left[ k\phi(t, k) + i\phi(t) \right] - \frac{1}{2\Delta(k)} [k\phi(t, k) + i\phi(t)] \right) \, dk
\]

and

\[
h_0(t) = \frac{2}{\pi} \int_{\partial D_2} \left( \frac{1}{\Delta(k)} \left[ k\phi(t, k) + i\phi(t) \right] + \frac{1}{2\Delta(k)} [k\phi(t, k) + i\phi(t)] \right) \, dk
\]

Proof. Equations (3.1) and (3.2) follow from the definitions and (2.5).

(a) In order to derive (3.3a) we note that the first of equations (2.15b) expresses \( g_1(t) \) in terms of \( \Phi_2^{(1)} \) and \( \Phi_1^{(2)} \). Furthermore, equations (2.13) and Cauchy’s theorem imply

\[
\frac{i\pi}{2} \Phi_2^{(1)}(t) = \int_{\partial D_2} [\Phi_2 - 1] \, dk = \int_{\partial D_1} [\Phi_2 - 1] \, dk
\]

and

\[
\frac{i\pi}{2} \Phi_1^{(2)}(t) = \int_{\partial D_2} [k\Phi_1 - \Phi_1^{(1)}] \, dk = \int_{\partial D_1} [k\Phi_1 - \Phi_1^{(1)}] \, dk.
\]

Thus,

\[
\int_{\partial D_2} [\Phi_2(t, k) - 1] \, dk = \int_{\partial D_1} [\Phi_2(t, -k) - 1] \, dk
\]

and

\[
\int_{\partial D_2} [\Phi_2(t, k) - 1] \, dk = \int_{\partial D_1} [\Phi_2(t, -k) - 1] \, dk
\]

(3.7)
and

\[ i \pi \Phi_1^{(2)}(t) = \left( \int_{aD} + \int_{bD} \right) \left[ k \Phi_1 - \Phi_1^{(1)} \right] dk = \int_{aD} \left[ k \Phi_1 - \Phi_1^{(1)} \right] dk \]

\[ = \int_{aD} \left\{ k \Phi_1 - \Phi_1^{(1)} + \frac{2e^{-2itL}}{\Delta} [k \Phi_1 - \Phi_1^{(1)}] \right\} dk + I(t) \]

\[ = \int_{aD} \left\{ k \Sigma \Phi_1 \frac{\Delta}{k} - \frac{\Sigma}{\Delta} \Phi_1^{(1)} \right\} dk + I(t), \quad (3.8) \]

where the function \( I(t) \) is defined by

\[ I(t) = - \int_{aD} \left\{ \frac{2e^{-2itL}}{\Delta} [k \Phi_1 - \Phi_1^{(1)}] \right\} dk. \]

The last step involves using the global relation (2.12) to compute \( I(t) \):

\[ I(t) = \int_{aD} \left\{ \frac{2e^{-2itL}}{\Delta} \left[ kc - \Phi_1^{(1)} - \frac{\Phi_1^{(1)}}{k} \right] \right\} dk \]

\[ + \int_{aD} \left\{ \frac{2e^{-2itL}}{\Delta} \frac{\Phi_1^{(1)}}{k} \right\} dk \]

\[ + \int_{aD} \left\{ \frac{2k}{\Delta} \Phi_1 (\tilde{\varphi}_2 - 1) e^{-2itL} - (\tilde{\Phi}_2 - 1) \tilde{\varphi}_1 \right\} dk. \quad (3.9) \]

The asymptotics (2.16) of \( c(t, k) \) and Cauchy’s theorem imply that the first integral on the rhs of (3.9) equals \(-i \pi \Phi_1^{(2)}(t)\). Moreover, analogously to (3.7), we have the identity

\[ i \pi \varphi_2^{(1)}(t) = \int_{aD} \varphi_2^{(1)}(t, k) dk. \quad (3.10) \]

Using this identity, the fact that \( \varphi_2^{(1)} = -\varphi_2^{(1)} \), and expressions (2.14) for \( \Phi_1^{(1)} \) and \( \varphi_1^{(1)} \), we can write the second integral on the rhs of (3.9) as

\[ \int_{aD} \left\{ \frac{g_0}{2} + \frac{2}{\Delta} \left[ k \varphi_1^{(1)} - ih_0 \right] \right\} dk. \]

Therefore, equations (3.8) and (3.9) imply

\[ 2i \pi \Phi_1^{(2)}(t) = \int_{aD} \left\{ \frac{k \Sigma \Phi_1}{\Delta} - \frac{\Sigma}{\Delta} \Phi_1^{(1)} \right\} dk + \int_{aD} \left\{ \frac{g_0}{2} + \frac{2}{\Delta} \left[ k \varphi_1^{(1)} + ih_0 \right] \right\} dk \]

\[ + \int_{aD} \left\{ \frac{2k}{\Delta} \Phi_1 (\tilde{\varphi}_2 - 1) e^{-2itL} - (\tilde{\Phi}_2 - 1) \tilde{\varphi}_1 \right\} dk. \quad (3.11) \]

Using (3.7) and (3.11) in the first of equations (2.15b), we find (3.3a).

Expression (3.3b) for \( h_1(t) \) can be derived in a similar way. Indeed, the second of equations (2.15b) expresses \( h_1 \) in terms of \( \varphi_2^{(1)} \) and \( \varphi_1^{(2)} \). The coefficient \( \varphi_2^{(1)} \) is given by (3.10), whereas \( \varphi_1^{(2)} \) satisfies the following analog of equation (3.8):

\[ i \pi \varphi_1^{(2)}(t) = \int_{aD} \left[ k \varphi_1^{(1)} - \varphi_1^{(1)} \right] dk \]

\[ = \int_{aD} \left\{ k \varphi_1^{(1)} - \varphi_1^{(1)} + \frac{2e^{2itL}}{\Delta} [k \varphi_1^{(1)} - \varphi_1^{(1)}] \right\} dk + J(t) \]

\[ = \int_{aD} \left\{ -k \Sigma \varphi_1^{(1)} + \frac{\Sigma}{\Delta} \varphi_1^{(1)} \right\} dk + J(t). \quad (3.12) \]
where the function $J(t)$ is defined by

$$J(t) = \int_{aD^I} \left\{ \frac{2e^{2ik\ell}}{\Delta} \left( k\varphi_1 - \varphi_1^{(1)} \right) \right\} \mkern-1mu dk.$$

The last step involves using the global relation (2.12) to compute $J(t)$:

$$J(t) = \int_{aD^I} \left\{ -\frac{2}{\Delta} \left[ kc - \Phi_1^{(1)} + \frac{\varphi_1^{(1)}}{k} \Phi_2^{(1)} e^{2ik\ell} + \varphi_1^{(1)} e^{2ik\ell} \right] \right\} \ mkern-1mu dk$$

$$+ \int_{aD^I} \left\{ \frac{2}{\Delta} \frac{\varphi_1^{(1)}}{k} \Phi_2^{(1)} e^{2ik\ell} + \frac{2}{\Delta} \left[ k\Phi_1 - \Phi_1^{(1)} \right] \right\} \mkern-1mu dk$$

$$+ \int_{aD^I} \left\{ \frac{2k}{\Delta} \left[ \Phi_1 (\tilde{\varphi}_2 - 1) - (\Phi_2 - 1) \varphi_1 \right] e^{2ik\ell} \right\} \mkern-1mu dk. \quad (3.13)$$

The asymptotics (2.16) of $c(t, k)$ and Cauchy’s theorem imply that the first integral on the rhs of (3.13) equals $-i\pi \varphi_1^{(2)}(t)$. Moreover, we can write the second integral on the rhs of (3.13) as

$$\int_{aD^I} \left\{ \frac{h_0}{2i} \Phi_{2-} + \frac{2}{\Delta} \left[ k\Phi_{1-} + ig_0 \right] \right\} \mkern-1mu dk.$$

Therefore, equations (3.12) and (3.13) imply

$$2i\pi \varphi_1^{(2)}(t) = \int_{aD^I} \left\{ -\frac{k\varphi_1}{\Delta} + \frac{\varphi_1^{(1)}}{\Delta} \right\} \mkern-1mu dk + \int_{aD^I} \left\{ \frac{h_0}{2i} \Phi_{2-} + \frac{2}{\Delta} \left[ k\Phi_{1-} + ig_0 \right] \right\} \mkern-1mu dk$$

$$+ \int_{aD^I} \left\{ \frac{2k}{\Delta} \left[ \Phi_1 (\tilde{\varphi}_2 - 1) - (\Phi_2 - 1) \varphi_1 \right] e^{2ik\ell} \right\} \mkern-1mu dk. \quad (3.14)$$

Using (3.10) and (3.14) in the second of equations (2.15b), we find (3.3b).

(b) In order to derive (3.4a) we note that the first of equations (2.15a) expresses $g_0$ in terms of $\Phi_1^{(1)}$. Furthermore, equations (2.13) and Cauchy’s theorem imply

$$-\frac{i\pi}{2} \Phi_1^{(1)}(t) = \int_{aD^I} \Phi_1 \mkern-1mu dk = \int_{aD^I} \Phi_1 \mkern-1mu dk. \quad (3.15)$$

Thus,

$$i\pi \Phi_1^{(1)}(t) = \left( \int_{aD_1} + \int_{aD_n} \right) \Phi_1 \mkern-1mu dk = \int_{aD_1} \Phi_{1-} \mkern-1mu dk = \int_{aD_1} \frac{\varphi_1^{(1)}}{\Delta} \mkern-1mu dk + I(t), \quad (3.16)$$

where the function $I(t)$ is defined by

$$I(t) = -\int_{aD^I} \frac{2}{\Delta} (e^{-2ik\ell} \Phi_1)_+ \mkern-1mu dk.$$

The last step involves using the global relation (2.12) to compute $I(t)$:

$$I(t) = -\int_{aD^I} \frac{2}{\Delta} (e^{-2ik\ell})_+ \mkern-1mu dk + \int_{aD^I} \frac{2}{\Delta} \left[ \Phi_1 (\tilde{\varphi}_2 - 1) e^{-2ik\ell} - \Phi_2 \varphi_1 \right]_+ \mkern-1mu dk. \quad (3.17)$$

The asymptotics (2.16) of $c(t, k)$ and Cauchy’s theorem imply that the first term on the rhs equals $-i\pi \Phi_1^{(1)}$; equations (3.16), (3.17) and the first of equations (2.15a) yield (3.4a). The proof of (3.4b) is similar. □
3.1. Effective characterizations

The substitution of expressions (3.3) for \( g_1(t) \) and \( h_1(t) \) into (3.2) yields a system of quadratically nonlinear integral equations for \( \{ \Phi_j(t, k), \varphi_j(x, t) \}_{j=1}^2 \). This nonlinear system provides an effective characterization of the spectral functions for the Dirichlet problem. In particular, given the Dirichlet data \( g_0(t) \) and \( h_0(t) \), the system can be solved recursively to all orders in a well-defined perturbative scheme. Indeed, substituting into (3.3) the expansions

\[
\Phi_j = \Phi_j^0 + \epsilon \Phi_j^1 + \epsilon^2 \Phi_j^2 + \cdots, \quad \varphi_j = \varphi_j^0 + \epsilon \varphi_j^1 + \epsilon^2 \varphi_j^2 + \cdots, \quad j = 1, 2,
\]

\[
g_0 = \epsilon g_{01} + \epsilon^2 g_{02} + \cdots, \quad g_1 = \epsilon g_{11} + \epsilon^2 g_{12} + \cdots,
\]

\[
h_0 = \epsilon h_{01} + \epsilon^2 h_{02} + \cdots, \quad h_1 = \epsilon h_{11} + \epsilon^2 h_{12} + \cdots,
\]

where \( \epsilon > 0 \) is a small parameter, the terms of \( O(\epsilon^n) \) yield

\[
g_{1n}(t) = \frac{4}{\pi^2} \int_{\partial D^2} \left\{ \frac{\sum k \Phi_{1n-} + ig_{0n}}{\Delta} - \frac{1}{\Delta} \left[ k\varphi_{1n-} + ih_{0n} \right] \right\} dk + \text{lower order terms}, \quad (3.19a)
\]

\[
h_{1n}(t) = \frac{4}{\pi^2} \int_{\partial D^2} \left\{ \frac{\sum k \varphi_{1n-} + ih_{0n}}{\Delta} + \frac{1}{\Delta} \left[ k\Phi_{1n-} + ig_{0n} \right] \right\} dk + \text{lower order terms}. \quad (3.19b)
\]

The terms of \( O(\epsilon^n) \) of the first and third equations in (3.2) yield

\[
\Phi_{1n}(t, k) = \int_0^t e^{i4(k-\zeta)(2kg_{0n}(\zeta') + ig_{1n}(\zeta'))} d\zeta' + \text{lower order terms}, \quad (3.20a)
\]

\[
\varphi_{1n}(t, k) = \int_0^t e^{i4(k-\zeta)(2kh_{0n}(\zeta') + ih_{1n}(\zeta'))} d\zeta' + \text{lower order terms}. \quad (3.20b)
\]

The odd parts of the latter two equations yield

\[
\Phi_{1n-}(t, k) = 4k \int_0^t e^{i4(k-\zeta)(\varphi_{1n}(\zeta') - \Phi_{1n}(\zeta'))} d\zeta' + \text{lower order terms}, \quad (3.21a)
\]

\[
\varphi_{1n-}(t, k) = 4k \int_0^t e^{i4(k-\zeta)(\varphi_{1n}(\zeta') - \Phi_{1n}(\zeta'))} h_{0n}(\zeta') d\zeta' + \text{lower order terms}. \quad (3.21b)
\]

It follows from (3.21) that \( \Phi_{1n-} \) and \( \varphi_{1n-} \) can be determined at each step from the known Dirichlet boundary values \( g_{0n} \) and \( h_{0n}; g_{1n} \) and \( h_{1n} \) can then be determined from (3.19).

Similarly, the nonlinear system obtained by substituting expressions (3.3) for \( g_0(t), h_0(t) \) into (3.2) provides an effective characterization of the spectral functions for the Neumann problem. Indeed, the terms of \( O(\epsilon^n) \) of (3.4a) and (3.4b) yield

\[
g_{0n}(t) = \frac{2}{\pi} \int_{\partial D^2} \left\{ \frac{\sum 2 \Phi_{1n-} - \varphi_{1n-}}{4 \Delta} \right\} dk, \quad (3.22a)
\]

\[
h_{0n}(t) = \frac{2}{\pi} \int_{\partial D^2} \left\{ \frac{\sum -2 \varphi_{1n-} + \Phi_{1n-}}{4 \Delta} \right\} dk, \quad (3.22b)
\]

while the even parts of (3.20) yield

\[
\Phi_{1n+}(t, k) = 2i \int_0^t e^{i4(k-\zeta-)} g_{1n}(\zeta') d\zeta' + \text{lower order terms},
\]

\[
\varphi_{1n+}(t, k) = 2i \int_0^t e^{i4(k-\zeta-)} h_{1n}(\zeta') d\zeta' + \text{lower order terms}.
\]

At each step in the perturbative scheme the functions \( \Phi_{1n+} \) and \( \varphi_{1n+} \) can be determined from the latter two equations, and then, \( g_{0n} \) and \( h_{0n} \) can be found from (3.22).
3.2. The linear limit

The linear limit of (3.3a) yields \( g_1 = \epsilon g_{11} + O(\epsilon^2) \), where
\[
g_{11} = \frac{4}{\pi_1} \int_{\partial D_1^T} \left\{ \frac{\sum}{2\Delta} \left[ k\Phi_{11-} + ig_{01} \right] - \frac{1}{\Delta} \left[ k\varphi_{11-} + ih_{01} \right] \right\} \, dk.
\]
Equation (3.21) becomes
\[
\Phi_{11-} = 4k \int_0^t e^{4ik^2(s-t)}g_{01}(s) \, ds, \quad \varphi_{11-} = 4k \int_0^t e^{4ik^2(s-t)}h_{01}(s) \, ds.
\]
Thus,
\[
g_{11} = \frac{4}{\pi_1} \int_{\partial D_1^T} \left\{ \frac{\sum}{\Delta} \left( 2k^2 \int_0^t e^{4k^2(s-t)}g_{01} \, ds - \frac{g_{01}}{2i} \right) - \frac{4k^2}{\Delta} \int_0^t e^{4k^2(s-t)}h_{01} \, ds + \frac{h_{01}}{1\Delta} \right\} \, dk.
\]
(3.23)

This coincides with the formula of appendix A, where the linearized equation \( \varphi_t + q_{1a} = 0 \) is solved directly.

3.3. The large \( L \) limit

In the limit \( L \to \infty \), the representations for \( g_1 \) and \( g_0 \) of theorem 3.1 reduce to the corresponding representations on the half-line. Indeed, as \( L \to \infty \),
\[
h_0 \to 0, \quad h_1 \to 0, \quad \varphi_1 \to 0, \quad \varphi_2 \to 1, \quad \frac{\sum}{\Delta} \to 1 \text{ as } k \to \infty \text{ in } D_3, \quad \frac{\sum}{\Delta} \to -1 \text{ as } k \to \infty \text{ in } D_1.
\]
Thus, the \( L \to \infty \) limits of the representations (3.3a) and (3.4a) are
\[
g_1 = \frac{4}{\pi_1} \int_{\partial D_1^T} \left\{ \frac{k}{2} \Phi_{1-} - \frac{g_{01}}{2i} + \frac{i\varphi_{2-}}{2} \right\} \, dk \quad \text{and} \quad g_0 = \frac{1}{\pi} \int_{\partial D_1^T} \Phi_{1+} \, dk,
\]
respectively, and these formulas coincide with the corresponding half-line formulas, cf [6].

4. The GLM approach

In theorem 3.1, we derived effective representations for \( \{g_1, h_1, g_0, h_0\} \) in terms of the eigenfunctions \( \{\Phi_j, \varphi_j\}_{1}^{\infty} \). In what follows, we will express the above boundary values in terms of the GLM representations.

For a function \( f(t, s) \), we let \( \tilde{f}(t, k) \) denote the transform
\[
\tilde{f}(t, k) = \int_{-t}^t e^{\pm ik^2(s-t)}f(t, s) \, ds.
\]
The eigenfunctions \( \{\Phi_j, \varphi_j\}_{1}^{\infty} \) admit the following GLM representations:
\[
\Phi_1(t, k) = \hat{L}_1 - \frac{i}{2} \hat{g}_0(t) \hat{M}_2 + k\hat{M}_1, \quad \Phi_2(t, k) = 1 + \hat{L}_2 + \frac{i\lambda}{2} \hat{g}_0 \hat{M}_1 + k\hat{M}_2, \quad (4.1a)
\]
\[
\varphi_1(t, k) = \hat{L}_1 - \frac{i}{2} \hat{h}_0(t) \hat{M}_2 + k\hat{M}_1, \quad \varphi_2(t, k) = 1 + \hat{L}_2 + \frac{i\lambda}{2} \hat{h}_0 \hat{M}_1 + k\hat{M}_2, \quad (4.1b)
\]
where the functions \( \{L_j(t, s), M_j(t, s), \hat{L}_j(t, s), \hat{M}_j(t, s)\}_{1}^{\infty}, -t < s < t \), satisfy a nonlinear Goursat system (see [5]) together with the initial conditions
\[
L_1(t, t) = \frac{1}{2} \hat{L}_1(t), \quad M_1(t, t) = \hat{g}_0(t), \quad L_1(t, -t) = \frac{1}{2} \hat{h}_1(t), \quad M_1(t, -t) = \hat{h}_0(t), \quad (4.2)
\]
\[
L_2(t, -t) = \hat{M}_2(t, -t) = \hat{M}_2(t, -t) = 0.
\]
Theorem 4.1. Define the function $F(t, k)$ by
\[
F(t, k) = \frac{i}{2} h_0(t) e^{2ik\hat{L}} - \frac{i}{2} g_0(t) \hat{M}_2 \\
+ \left( \hat{L}_2 - i\lambda \frac{h_0(t)}{2} \hat{M}_1 + k \hat{M}_2 \right) \left( \hat{L}_1 - \frac{i}{2} g_0(t) \hat{M}_2 + k \hat{M}_1 \right) \\
- e^{2ikL} \left( \hat{L}_2 - i\lambda \frac{g_0(t)}{2} \hat{M}_1 + k \hat{M}_2 \right) \left( \hat{L}_1 - \frac{i}{2} h_0(t) \hat{M}_2 + k \hat{M}_1 \right),
\]
where $\hat{L}_2$ is short-hand notation for $\hat{L}_2(t, k)$, etc. Under the assumptions of theorem 3.1, the following formulas are valid.

(a) For the Dirichlet problem, the unknown boundary values $g_1$ and $h_1$ are given by
\[
g_1(t) = \frac{4}{i\pi} \int_{\partial D} \left\{ -\frac{2k^2}{\Delta} \left[ \hat{M}_1(t, k) h_0(t) \right] - \frac{2k^2}{2i\Delta^2} \right\} dk,
\]
which are valid for all $t \geq 0$.

(b) For the Neumann problem, the unknown boundary values $g_0$ and $h_0$ are given by
\[
h_0(t) = \frac{2}{\pi} \int_{\partial D} \left\{ -\frac{1}{\Delta(k)} \left[ \Sigma(k) \hat{L}_1(t, k) - 2\hat{L}_1(t, k) + (e^{2ikL} F(t, k))_+ \right] \right\} dk,
\]
where $\Sigma(k) = \frac{\hat{M}_1(t, k)}{\hat{M}_2(t, k)}$.

Remark 4.2. The representations in (4.4) coincide with the representations (4.7) and (4.8) in [5], except that the last two terms on the rhs of (4.4a), as well as the last two terms on the rhs of (4.4b), were missed in [5]. These terms, which arise from somewhat subtle boundary effects, are needed in order for equations (4.4) to be consistent with the representations of theorem 3.1 and are also required in order to obtain the correct large $L$ limit.

Proof. Let us first consider the Dirichlet problem. In view of the GLM representations (4.1), we may write the global relation (2.12a) as
\[
-\hat{L}_1 + e^{2ikL} \hat{L}_1 = k \hat{M}_1 - ke^{2ikL} \hat{M}_1 + F - c,
\]
which is valid for all $t \geq 0$.

The expression of $F$ can be expressed as in (4.3). Letting $k \to -k$ in (4.6), we find
\[
-\hat{L}_1 + e^{-2ikL} \hat{L}_1 = -k \hat{M}_1 + ke^{-2ikL} \hat{M}_1 + F(t, -k) - c(t, -k).
\]
Solving (4.6) and (4.8) for $\hat{L}_1$ and $\hat{L}_1$, we find
\[
\hat{L}_1 = \frac{2k}{\Delta} \hat{M}_1 - \frac{k\Sigma}{\Delta} \hat{M}_1 + \frac{1}{\Delta} (F - c)_-.
\]
Multiplying these equations by \( k e^{ikt(x-t')}, \) \(0 < t' < t,\) and integrating along \( \partial D_1^0 \) with respect to \( dk, \) we obtain
\[
\int_{\partial D_1^0} k e^{ikt(x-t')} \hat{L}_1 \, dk = \int_{\partial D_1^0} \frac{k^2}{\Delta} e^{ikt(x-t')} \hat{M}_1 \, dk - \int_{\partial D_1^0} \frac{k^2 \Sigma}{\Delta} e^{ikt(x-t')} \hat{M}_1 \, dk
+ \int_{\partial D_1^0} \frac{k}{\Delta} e^{ikt(x-t')} F_-(k) \, dk,
\]
(4.11a)
and
\[
-\int_{\partial D_1^0} k e^{ikt(x-t')} \hat{L}_1 \, dk = \int_{\partial D_1^0} \frac{k^2}{\Delta} e^{ikt(x-t')} \hat{M}_1 \, dk - \int_{\partial D_1^0} \frac{k^2 \Sigma}{\Delta} e^{ikt(x-t')} \hat{M}_1 \, dk
- \int_{\partial D_1^0} \frac{k}{\Delta} e^{ikt(x-t')} (e^{-2ikL} F)_- \, dk,
\]
(4.11b)
where we have used that the functions
\[
\frac{k}{\Delta} c(t, k)_- \quad \text{and} \quad \frac{k}{\Delta} (e^{-2ikL} c(t, k))_-,
\]
are bounded and analytic in \( D_1^0, \) so that their contributions vanish by Jordan’s lemma.

The next step is to take the limit \( t' \to t^{-} \) in (4.11) (the notation \( t' \to t^{-} \) indicates that \( t' \) approaches \( t \) from below). This can be achieved by using the identities
\[
\int_{\partial D_1} k e^{ikt(x-t')} \hat{f}(t, k) \, dk = \begin{cases} \frac{\pi}{2} f(t, 2t' - t), & 0 < t' < t, \\ \frac{\pi}{4} f(t, t), & 0 < t' = t, \end{cases}
\]
(4.12)
and
\[
\int_{\partial D_1^0} \frac{k^2}{\Delta} e^{ikt(x-t')} \hat{f}(t, k) \, dk = 2 \int_{\partial D_1^0} \frac{k^2}{\Delta} \left[ \int_0^t e^{ik\tau} \frac{f(t, 2\tau - t)}{4ik^2} \, d\tau \right] \, dk,
\]
\[
0 < t' < t.
\]
(4.13)
Identity (4.13) is also valid if \( \frac{k^2}{\Delta} \) is replaced by \( k^2 \) or \( \frac{k^2 \Sigma}{\Delta}. \) Utilizing these identities in (4.11), we find
\[
-\frac{\pi}{2} \hat{L}_1(t, 2t' - t) = 4 \int_{\partial D_1^0} \frac{k^2}{\Delta} \left[ \int_0^t e^{ik\tau} \hat{M}_1(t, 2\tau - t) \, d\tau - \frac{M_1(t, 2t' - t)}{4ik^2} \right] \, dk
- 2 \int_{\partial D_1^0} \frac{k^2 \Sigma}{\Delta} \left[ \int_0^t e^{ik\tau} \hat{M}_1(t, 2\tau - t) \, d\tau - \frac{M_1(t, 2t' - t)}{4ik^2} \right] \, dk
+ \int_{\partial D_1^0} \frac{k}{\Delta} e^{ikt(x-t')} F_-(t, k) \, dk
\]
and
\[
-\frac{\pi}{2} L_1(t, 2t' - t) = 4 \int_{\partial D_1^0} \frac{k^2}{\Delta} \left[ \int_0^t e^{ik\tau} \hat{M}_1(t, 2\tau - t) \, d\tau - \frac{M_1(t, 2t' - t)}{4ik^2} \right] \, dk
- 2 \int_{\partial D_1^0} \frac{k^2 \Sigma}{\Delta} \left[ \int_0^t e^{ik\tau} \hat{M}_1(t, 2\tau - t) \, d\tau - \frac{M_1(t, 2t' - t)}{4ik^2} \right] \, dk
- \int_{\partial D_1^0} \frac{k}{\Delta} e^{ikt(x-t')} (e^{-2ikL} F(t, k))_- \, dk.
\]
Letting \( t' \to t^{-} \) in these equations and using the initial conditions (4.2), as well as the following lemma, we find the representations in (4.4).
Lemma 4.3.

\[
\lim_{t' \to t} \int_{\partial D'_1} \frac{k}{\Delta} e^{i k (\tau - t')} \hat{F}_-(\tau, k) \, dk = \int_{\partial D'_1} \frac{k}{\Delta} \hat{F}_-(t, k) \, dk \\
+ \frac{i \hbar_0(t)}{2} \int_{\partial D'_1} k \hat{M}_2(t, k) \, dk + \int_{\partial D'_1} \frac{k \hbar_0(t)}{2i} \hat{M}_2(t, k) \, dk, \quad (4.14a)
\]

\[
\lim_{t' \to t} \int_{\partial D'_1} \frac{k}{\Delta} e^{i k (\tau - t')} (e^{-2ik\tau} F(t, k))_\tau \, dk = \int_{\partial D'_1} \frac{k}{\Delta} (e^{-2ik\tau} F(t, k))_\tau \, dk \\
+ \frac{i g_0(t)}{2} \int_{\partial D'_1} k \hat{M}_2(t, k) \, dk + \int_{\partial D'_1} \frac{k g_0(t)}{2i} \hat{M}_2(t, k) \, dk. \quad (4.14b)
\]

Proof. We prove (4.14b); the proof of (4.14a) is similar. If one naively takes the limit inside the integral in (4.14b), one finds the first term on the rhs of (4.14b). The other two terms on the rhs of (4.14b) arise from interchanging the limit and the integration. We will next describe how these terms arise in detail.

We write

\[
\int_{\partial D'_1} \frac{k}{\Delta} e^{i k (\tau - t')} (e^{-2ik\tau} F(t, k))_\tau \, dk = \int_{\partial D'_1} k e^{i k (\tau - t')} \frac{i g_0(t)}{2} \hat{M}_2 \, dk \\
- \int_{\partial D'_1} k e^{i k (\tau - t')} \left( \frac{\pi_2}{2} - i \frac{\hbar_0}{\Delta} \hat{M}_1 \right) \left( \hat{L}_1 - i \frac{\hbar_0}{2 \Delta} \hat{M}_2 \right) \, dk \\
- 2 \int_{\partial D'_1} \frac{k^2}{\Delta} e^{i k (\tau - t')} \left( \frac{\pi_2}{2} - i \frac{\hbar_0}{\Delta} \hat{M}_1 \right) \hat{M}_1 + \hat{M}_2 \left( \hat{L}_1 - i \frac{\hbar_0}{2 \Delta} \hat{M}_2 \right) \right) \, dk \\
+ \int_{\partial D'_1} \frac{k^2 \Sigma}{\Delta} e^{i k (\tau - t')} \left( \frac{\pi_2}{2} - i \frac{\hbar_0}{\Delta} \hat{M}_1 \right) \hat{M}_1 + \left( \hat{L}_1 - i \frac{\hbar_0}{2 \Delta} \hat{M}_2 \right) \hat{M}_2 \right) \, dk \\
- \int_{\partial D'_1} \frac{k^3}{\Delta} \hat{M}_2 \hat{M}_1 e^{i k (\tau - t')} \, dk. \quad (4.15)
\]

The first integral on the rhs of (4.15) yields the following contribution in the limit \( t' \to t \):

\[
\lim_{t' \to t} \frac{i g_0(t)}{2} \int_{\partial D'_1} k \hat{M}_2(t, k) \, dk = \lim_{t' \to t} \frac{i g_0(t)}{2} \pi \frac{\hbar_0}{2 \Delta} M_2(t, 2t' - t) = \frac{i \pi g_0(t)}{4} M_2(t, t).
\]

On the other hand, utilizing the second row of (4.12),

\[
\frac{i g_0(t)}{2} \int_{\partial D'_1} k \hat{M}_2(t, k) \, dk = \frac{i \pi g_0(t)}{8} M_2(t, t).
\]

Therefore,

\[
\lim_{t' \to t} \frac{i g_0(t)}{2} \int_{\partial D'_1} k e^{i k (\tau - t')} \hat{M}_2(t, k) \, dk = \frac{i g_0(t)}{2} \frac{\hbar_0}{2} \int_{\partial D'_1} k \hat{M}_2(t, k) \, dk \\
+ \frac{i g_0(t)}{2} \int_{\partial D'_1} k \hat{M}_2(t, k) \, dk. \quad (4.16)
\]

The first term on the rhs of (4.16) is the contribution obtained by taking the limit inside the integral; this term is included in the first term on the rhs of (4.14b). In addition to this term, there is also an additional term arising from the interchange of the limit and the integration; this is the second term on the rhs of (4.14b).

We now consider the last integral on the rhs of (4.15), which can be written as

\[
- \int_{\partial D'_1} \frac{k^3}{\Delta} \hat{M}_2 \hat{M}_1 e^{i k (\tau - t')} \, dk = -2 \int_{\partial D'_1} \frac{k^3}{\Delta} \hat{M}_2(t, k) \int_{0}^{t'} e^{i k \tau} M_1(t, 2\tau - t) \, d\tau \, dk. \quad (4.17)
\]
The rhs of (4.17) equals
\[ -2 \int_{aD^2_k} k^3 \hat{M}_2(t, k) \left\{ \int_0^t e^{4i\hat{z}(t-\tau)} M_1(t, 2\tau - t) d\tau - \frac{M_1(t, 2t' - t)}{4ik^2} \right\} dk. \] (4.18)

Indeed, the rhs of (4.17) equals this term plus the following expression:
\[ -2 \int_{aD^2_k} k^3 \hat{M}_2(t, k) \left\{ \int_0^t e^{4i\hat{z}(t-\tau)} M_1(t, 2\tau - t) d\tau + \frac{M_1(t, 2t' - t)}{4ik^2} \right\} dk. \]

Integration by parts shows that this integral vanishes by Jordan’s lemma, because \( \hat{M}_2(t, k) \) is of \( O(1/k^2) \) as \( k \to \infty \) within \( D_k \). Taking the limit \( t' \to t^- \) in (4.18) and using (4.2), we find that the contribution of the last integral in (4.15) is
\[ - \lim_{t' \to t^-} \int_{aD^2_k} k^3 \hat{M}_2 \hat{M}_1 e^{4i\hat{z}(t-\tau)} dk = - \int_{aD^2_k} k^3 \hat{M}_2 \hat{M}_1 dk + \int_{aD^2_k} \frac{kg_0(t)}{2i} \hat{M}_2 dk. \]

The first term on the rhs is the contribution obtained by taking the limit inside the integral. In addition to this term, there is also an additional term arising from the interchange of the limit and the integration; this is the third term on the rhs of (4.14b).

Finally, we claim that the limits of the second, third and fourth integrals on the rhs of (4.15) can be computed by simply taking the limit inside the integral, i.e. in these cases no additional terms arise. We show this for the term
\[ I := \int_{aD^2_k} k^2 \hat{M}_2 M_1(t, 2\tau - t) d\tau \]
the proofs for the other terms are similar. We have
\[ I = 2 \int_{aD^2_k} k^2 \hat{M}_2 \hat{M}_1(t, 2\tau - t) d\tau dk. \]

We can write this as
\[ I = 2 \int_{aD^2_k} k^2 \hat{M}_2 M_1(t, 2\tau - t) d\tau = \left[ \hat{M}_1(t, 2\tau - t) \right] \hat{M}_2 M_1. \]

Indeed, the difference between the previous two expressions,
\[ 2 \int_{aD^2_k} k^2 \hat{M}_2 M_1(t, 2\tau - t) d\tau = \left[ \hat{M}_1(t, 2\tau - t) \right] \hat{M}_2 M_1. \]

vanishes by integration by parts and Jordan’s lemma, since \( \hat{L}_2 \) and \( \hat{M}_1 \) are of \( O(1/k^2) \) as \( k \to \infty \) in \( D_k \). Taking the limit \( t' \to t^- \), we find
\[ \lim_{t' \to t^-} \int_{aD^2_k} k^2 \hat{M}_2 M_1(t, 2\tau - t) d\tau = \int_{aD^2_k} k^2 \hat{M}_2 M_1(t, k) \left[ \hat{M}_1(t, k) \right] \frac{g_0(t)}{2ik^2} dk. \]

However, in this case, the additional term
\[ - \int_{aD^2_k} k^2 \hat{M}_2 M_1(t, k) \frac{g_0(t)}{2ik^2} dk \]
vanishes because the integrand is analytic and of \( O(1/k^2) \) as \( k \to \infty \) in \( D_k \). This completes the proof of lemma 4.3. \( \square \)
4.1. Equivalence of the two representations

We now return to theorem 4.1 and consider the Neumann problem. Solving (4.6) and (4.8) for \( \dot{\mathcal{M}}_1 \) and \( \ddot{\mathcal{M}}_1 \), we find

\[
\begin{align*}
    k \dot{\mathcal{M}}_1 &= \frac{2}{\Delta} \dot{\mathcal{L}}_1 - \frac{\Sigma}{\Delta} \dot{\mathcal{L}}_1 + \frac{1}{\Delta} (F - c)_+, \\
    k \ddot{\mathcal{M}}_1 &= \frac{\Sigma}{\Delta} \dot{\mathcal{L}}_1 - \frac{2}{\Delta} \dot{\mathcal{L}}_1 + \frac{1}{\Delta} (e^{-2\Delta t} (F - c))_+.
\end{align*}
\]

Multiplying these equations by \( ke^{\Delta t (\tau - \tau')} \), \( 0 < \tau' < \tau \), and integrating along \( \partial D^1 \) with respect to \( dk \), we find

\[
\begin{align*}
    \frac{\pi}{2} \mathcal{M}_1 (t, 2\tau' - t) &= \int_{\partial D^1} e^{\Delta t (\tau - \tau')} \left\{ \frac{2}{\Delta} \dot{\mathcal{L}}_1 - \frac{\Sigma}{\Delta} \dot{\mathcal{L}}_1 + \frac{1}{\Delta} F_+ \right\} dk, \\
    \frac{\pi}{2} \mathcal{M}_1 (t, 2\tau' - t) &= \int_{\partial D^1} e^{\Delta t (\tau - \tau')} \left\{ \frac{\Sigma}{\Delta} \dot{\mathcal{L}}_1 dk - \frac{2}{\Delta} \dot{\mathcal{L}}_1 + \frac{1}{\Delta} (e^{-2\Delta t} F)_+ \right\} dk,
\end{align*}
\]

where we used that the functions

\[
\frac{1}{\Delta} (c(t, k))_+, \quad \frac{1}{\Delta} (e^{-2\Delta t} c(t, k))_+
\]

are bounded and analytic in the interior of \( \partial D^1 \) so that their contributions vanish by Jordan’s lemma. Letting \( \tau' \to \tau^- \) in these equations and using the initial conditions (4.2), as well as the following lemma, we find the representations in (4.5).

**Lemma 4.4.**

\[
\begin{align*}
    \lim_{\tau' \to \tau^-} \int_{\partial D^1} \frac{1}{\Delta} e^{\Delta t (\tau - \tau')} F_+ (t, k) \, dk &= \int_{\partial D^1} \frac{1}{\Delta} F_+ (t, k) \, dk, \\
    \lim_{\tau' \to \tau^-} \int_{\partial D^1} \frac{1}{\Delta} e^{\Delta t (\tau - \tau')} (e^{-2\Delta t} F(t, k))_+ \, dk &= \int_{\partial D^1} \frac{1}{\Delta} (e^{-2\Delta t} F(t, k))_+ \, dk.
\end{align*}
\]

**Proof.** Note that

\[
\begin{align*}
    \int_{\partial D^1} \frac{1}{\Delta} e^{\Delta t (\tau - \tau')} F_+ (t, k) \, dk &= \int_{\partial D^1} \frac{1}{\Delta} e^{\Delta t (\tau - \tau')} \left\{ \frac{i}{2} h_0 (t) \Sigma \dot{\mathcal{M}}_2 - ig_0 (t) \dot{\mathcal{M}}_2 \right. \\
    &\quad + 2 \left( \frac{i}{2} \dot{\mathcal{L}}_2 - i k \frac{h_0}{2} \dot{\mathcal{M}}_1 \right) \left( \dot{\mathcal{L}}_1 - i \frac{g_0 (t)}{2} \dot{\mathcal{M}}_2 \right) + 2k^2 \dot{\mathcal{M}}_2 \dot{\mathcal{M}}_1 \\
    &\quad - \Sigma \left( \frac{i}{2} \dot{\mathcal{L}}_2 - i \frac{g_0}{2} \dot{\mathcal{M}}_1 \right) \left( \dot{\mathcal{L}}_1 - i \frac{h_0 (t)}{2} \dot{\mathcal{M}}_2 \right) \\
    &\quad - \Delta k \dot{\mathcal{M}}_2 \left( \dot{\mathcal{L}}_1 - i \frac{h_0 (t)}{2} \dot{\mathcal{M}}_2 \right) - \Delta \left( \frac{i}{2} \dot{\mathcal{L}}_2 - i \frac{g_0}{2} \dot{\mathcal{M}}_1 \right) k \dot{\mathcal{M}}_1 - \Sigma k^2 \dot{\mathcal{M}}_2 \dot{\mathcal{M}}_1 \right\}.
\end{align*}
\]

Equation (4.21a) now follows by using arguments similar to those that led to lemma 4.3. The proof of (4.21b) is similar. This completes the proof of lemma 4.4 and hence of theorem 4.1.

**4.1. Equivalence of the two representations**

We will show that the representations derived using the GLM approach in theorem 4.1 coincide with those of theorem 3.1.
4.1.1. The representations for $g_1$ and $h_1$. Using expression (4.7) for $F$ as well as the formulas

\[
M_j = \frac{1}{2k} \Phi_j, \quad \hat{M}_j = \frac{1}{2k} \phi_j, \quad j = 1, 2,
\]

we can write the representation (4.4a) of $g_1$ as

\[
g_1(t) = \frac{4}{i\pi} \int_{\partial D_0^1} \left\{ -\frac{1}{\Delta} [k \psi_- + i \hat{h}_0(t)] + \sum_{j=1}^{2} [k \phi_1 - (k_0 + i \hat{g}_0(t))] \right. \\
+ \frac{k}{\Delta} (\Phi_2 - 1) e^{-2ikL} - (\hat{\Phi}_1 - 1) \psi_1, \\
- \frac{k}{\Delta} \left( e^{-2ikL} \hat{g}_0 \Phi_2 \right)_- + \frac{i \hat{g}_0(t)}{4} \Phi_2 - \frac{g_0(t)}{4i} \psi_2 \left\} \, dk.
\]

The identities

\[
\int_{\partial D_0^1} \varphi_2(t, k) \, dk = -\pi i \varphi_2^{(1)} = \pi i \varphi_2^{(1)} = \int_{\partial D_0^1} \varphi_2 \, dk
\]

imply that the function $\varphi_2$ in the above integrand can be replaced with $\varphi_2$. Moreover, since the integrand is an odd function of $k$, the contour $\partial D_0^1$ can be replaced with $\partial D_0^3$. In view of the identity

\[
- \frac{k}{\Delta} \left[ e^{-2ikL} \hat{g}_0 \Phi_2 \right]_- + \frac{i \hat{g}_0(t)}{4} \Phi_2 = - \frac{g_0(t)}{2} \Phi_2,
\]

we find the representation for $g_1$ in (3.3a). Similar computations show that the representations for $h_1$ are also equivalent.

4.1.2. The representations for $g_0$ and $h_0$. Using expression (4.7) for $F$ as well as the formulas

\[
\Phi_1 = 2 \hat{L}_1 - i \hat{g}_0 \hat{M}_2, \quad \phi_1 = 2 \hat{L}_1 - i \hat{h}_0 \hat{M}_2,
\]

a straightforward computation shows that the representations (3.4) and (4.5) are equivalent.

5. Conclusions

This is the third paper in a series of papers investigating the most difficult problem in the analysis of initial-boundary value problems for integrable evolution PDEs, namely the problem of characterizing the unknown boundary values in terms of the given initial and boundary conditions. In the first paper, this problem was analyzed for the NLS on the half-line, by employing the global relation, as well as the equation obtained from the global relation using the transformation $k \rightarrow -k$. In the second paper, the same problem was analyzed by employing the global relation, together with the so-called GLM representation. In this paper, we have analyzed the above problem for the NLS on the finite interval; in section 3, we have employed the former approach (see theorem 3.1), whereas in section 4, we have employed the latter approach (see theorem 4.1). Furthermore, we have shown that these two alternative approaches are equivalent (see section 4.1) and have also shown that in the $L \rightarrow \infty$ limit, the approaches in sections 3 and 4 yield the analogous results obtained in the first and the second paper, respectively.

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Appendix A. The linear limit

In this appendix, we analyze the linearized version of the NLS equation, \( i\eta_t + q_{xx} = 0 \), on the interval \([0, L]\) using the unified method of [2]. The global relation for this equation is (cf equation (2.8) in [4])

\[
\dot{q}_0(k) - \ddot{g}(ik^2) + e^{-ik}\dot{h}(ik^2) = e^{ikT}q_T(k), \quad k \in \mathbb{C},
\]  

(A.1)

where

\[
\dot{q}_0(k) = \int_0^L e^{-ikx}q_0(x) \, dx, \quad \dot{q}_T(k) = \int_0^L e^{-ikx}q(x, T) \, dx,
\]

\[
\ddot{g}(k) = -k\ddot{g}_0(ik^2) + i\dot{g}_1(ik^2), \quad \dot{h}(k) = -k\dot{h}_0(ik^2) + i\ddot{h}_1(ik^2),
\]

\[
\ddot{g}_1(ik^2) = \int_0^T e^{ik^2}g_j(s) \, ds, \quad \ddot{h}_j(ik^2) = \int_0^T e^{ik^2}h_j(s) \, ds, \quad j = 0, 1,
\]

and \( g_j(t) \) and \( h_j(t) \), \( j = 0, 1 \), denote the Dirichlet and Neumann boundary values as in (2.2). Equation (A.1) and the equation obtained by letting \( k \to -k \) in (A.1) are the following equations:

\[
\dot{q}_0(k) + k\ddot{g}_0(ik^2) - i\dot{g}_1(ik^2) - k\dot{h}_0(ik^2) e^{-ikT} = e^{ikT}q_T(k),
\]

\[
\dot{q}_0(-k) - k\ddot{g}_0(ik^2) + i\dot{g}_1(ik^2) + k\dot{h}_0(ik^2) e^{-ikT} = e^{ikT}\dot{q}_T(-k).
\]

Eliminating \( \dot{h}_1 \) from these equations and then solving for \( \ddot{g}_1 \), we find

\[
\ddot{g}_1(ik^2) = \frac{i}{\Delta(k/2)} \left[ e^{ikT}q_T(k) \right]_+ - \left( e^{ikT}q_0(k) \right)_- - k\Sigma(k/2)\ddot{g}_0(ik^2) + 2k\dot{h}_0(ik^2)].
\]

Multiplying this equation by \( ke^{-ik^2} \) and integrating along \( \partial D^1 \) with respect to \( dk \), the term involving \( \dot{q}_T \) is eliminated and we find

\[
\pi g_1 = \int_{\partial D^1} \frac{i}{\Delta(k/2)} \left[ -\left( e^{ikT}q_0(k) \right)_- - k\Sigma(k/2)\ddot{g}_0(ik^2) + 2k\dot{h}_0(ik^2)] \right. \]

Assuming that \( q_0 = 0 \) and performing the change of variables \( k = -2l \), we find

\[
g_1(t) = \frac{4}{\pi i} \int_{\partial D^1} \left[ \frac{2i\Sigma(l)}{\Delta(l)} \int_0^T e^{i4\pi(2l - 1)}g_0(s) \, ds - \frac{4l^2}{\Delta(l)} \int_0^T e^{i4\pi(2l - 1)}h_0(s) \, ds \right] \, dl.
\]

Using the identity

\[
\int_{\partial D^1} \frac{k^2}{\Delta(k)} \int_0^T e^{i4\pi(t - 1)}K(s, T) \, ds \, dk
\]

\[
= \int_{\partial D^1} \frac{k^2}{\Delta(k)} \left[ \int_0^T e^{i4\pi(2l - 1)}K(s, T) \, ds - \frac{K(t, T)}{4ik^2} \right] \, dk, \quad 0 < t < T,
\]

(A.2)

and a similar identity obtained by replacing \( k^2 \Delta(k) \) by \( \frac{k^2\Sigma(l)}{\Delta(l)} \) in (A.2), we arrive at

\[
g_1(t) = \frac{4}{\pi i} \int_{\partial D^1} \left[ \frac{2i\Sigma(l)}{\Delta(l)} \int_0^T e^{i4\pi(2l - 1)}g_0(s) \, ds - \frac{\Sigma(l)g_0}{2i\Delta(l)} \right.
\]

\[
- \frac{4l^2}{\Delta(l)} \int_0^T e^{i4\pi(2l - 1)}h_0(s) \, ds + \frac{J_0}{1\Delta(l)} \right] \, dl,
\]

which coincides with formula (3.23) for \( g_{11} \).
Appendix B. The asymptotics of $c(t, k)$

We will prove lemma 2.1. We will make no assumption on the function $c(t, k)$ except that it satisfies the global relation (2.7) and that it has the boundedness properties stated in (2.8). The functions $\{\Phi_j, \varphi_j\}_{j=1}^2$ satisfy the systems

\[
\begin{align*}
\Phi_{1t} &= -4ik^2 \Phi_1 - i\lambda |g_0|^2 \Phi_1 + (2k g_0 + ig_1) \Phi_2, \\
\Phi_{2t} &= \lambda (2k g_0 - ig_1) \Phi_1 + \lambda |g_0|^2 \Phi_2
\end{align*}
\]

(B.1)

and

\[
\begin{align*}
\varphi_{1t} &= -4ik^2 \varphi_1 - i\lambda |h_0|^2 \varphi_1 + (2k h_0 + ih_1) \varphi_2, \\
\varphi_{2t} &= \lambda (2k h_0 - ih_1) \varphi_1 + \lambda |h_0|^2 \varphi_2.
\end{align*}
\]

(B.2)

In view of the initial conditions

$$
\Phi_1(0, k) = \varphi_1(0, k) = 0, \quad \Phi_2(0, k) = \varphi_2(0, k) = 1,
$$

this leads to the asymptotic expansions

\[
\begin{align*}
\left( \Phi_1(t, k), \Phi_2(t, k) \right) &= \left( 0, 1 \right) + \left( \Phi_1^{(1)}(t) \right) \frac{1}{k} + \left( \Phi_2^{(1)}(t) \right) \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \\
&\quad + \left[ \left( -\Phi_1^{(1)}(0) \right) \frac{1}{k} + \left( -\Phi_2^{(1)}(0) + \Phi_1^{(1)}(0) f_{tL}(t) \omega \right) \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \right] e^{4ik^2t},
\end{align*}
\]

$$
k \to \infty, \ k \in \mathbb{C}.
$$

\[
\begin{align*}
\left( \varphi_1(t, k), \varphi_2(t, k) \right) &= \left( 0, 1 \right) + \left( \varphi_1^{(1)}(t) \right) \frac{1}{k} + \left( \varphi_2^{(1)}(t) \right) \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \\
&\quad + \left[ \left( -\varphi_1^{(1)}(0) \right) \frac{1}{k} + \left( -\varphi_2^{(1)}(0) + \varphi_1^{(1)}(0) f_{tL}(t) \omega \right) \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \right] e^{4ik^2t},
\end{align*}
\]

$$
k \to \infty, \ k \in \mathbb{C}.
$$

Similarly, we have

\[
\begin{align*}
\begin{bmatrix} b(k) \ a(k) \end{bmatrix} &= \begin{bmatrix} 0 \ 1 \end{bmatrix} + \begin{bmatrix} b_1^{(1)} \ a_1^{(1)} \end{bmatrix} \frac{1}{k} + \begin{bmatrix} b_1^{(2)} \ a_1^{(2)} \end{bmatrix} \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \\
&\quad + \left[ \begin{bmatrix} -q_{0L}(k) \ 0 \end{bmatrix} \frac{1}{k} + \begin{bmatrix} \chi_1^{(2)} \ \chi_1^{(1)} \end{bmatrix} \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \right] e^{2ik^2t},
\end{align*}
\]

$$
k \to \infty, \ k \in \mathbb{C}. \quad \text{(B.3)}
$$

Substituting these expansions into the global relation (2.7), we find

\[
\begin{align*}
c(t, k) &= \left[ O \left( \frac{1}{k^2} \right) + O \left( \frac{1}{k^3} \right) \right] e^{4ik^2t} \\
&\quad + \begin{bmatrix} b_1^{(1)} - \Phi_1^{(1)}(0) \ k \ 0 \ \ 0 \end{bmatrix} + \begin{bmatrix} \Phi_1^{(1)}(0) - \frac{q_{0L}(k)}{2i} \ \ 0 \ \ 0 \end{bmatrix} \frac{1}{k} + \begin{bmatrix} \Phi_2^{(1)}(t) + \Phi_1^{(1)}(t)(a_1^{(1)} + \varphi_1^{(1)}(t)) \ \ 0 \ \ 0 \end{bmatrix} \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \right] e^{2ik^2t} \\
&\quad - \begin{bmatrix} \Phi_1^{(1)}(t) \ \ 0 \ \ 0 \end{bmatrix} + \begin{bmatrix} \varphi_1^{(1)}(t) + \varphi_1^{(1)}(t)(\tilde{g}_1^{(1)} + \tilde{\Phi}_1^{(1)}(t)) \ \ 0 \ \ 0 \end{bmatrix} \frac{1}{k^2} + O \left( \frac{1}{k^3} \right) \right] e^{2ik^2t},
\end{align*}
\]

$$
k \to \infty, \ k \in \mathbb{C}. \quad \text{(B.4)}
$$

The assumption that $c(t, k)$ is of $O((1 + e^{2ik^2t})/k)$ as $k \to \infty$ implies that the terms in (B.4) involving $e^{-4ik^2t}$ and $e^{4ik^2t}$ must vanish, i.e. for consistency we require

\[
\begin{align*}
b_1^{(1)} &= \Phi_1^{(i)}(0), \quad \frac{q_{0L}(k)}{2i} = \varphi_1^{(1)}(0).
\end{align*}
\]

(B.5)
In appendix B, we made no assumption on the form of the spectral functions \( \{A(k), B(k), \tilde{A}(k), \tilde{B}(k)\} \) in the sense that the resulting nonlinear system can be solved uniquely at each step in a well-defined perturbative scheme.

In what follows, we present a set of non-effective formulas (cf equations (14)–(16) in [1]).

Let
\[
R(0, t, k) = \left( \begin{array}{c} d(t, k) \\ \lambda c(t, k) \\ d(t, k) \end{array} \right),
\]
where \( R \) is defined in (2.9). It follows from (2.10) that \( R \) satisfies
\[
R_t = -2i k^2 [\sigma_3, R] + \tilde{Q}(0, t, k) R - R e^{i k L} \tilde{Q}(L, t, k).
\]
This implies that \( \{c, d\} \) satisfy the system
\[
c_t = -4i k c - \lambda t(|g_0|^2 + |h_0|^2) c + (2 k g_0 + i g_1) d - (2 k h_0 + i h_1) e^{i k L} \tilde{d},
\]
\[
d_t = \lambda t(|g_0|^2 - |h_0|^2) d + \lambda (2 k \bar{g}_0 - i \bar{g}_1) c - \lambda (2 k h_0 + i h_1) e^{i k L} \tilde{c},
\]
\[
c(0, k) = b(k), \quad d(0, k) = a(k),
\]
where \( \tilde{c} \) and \( \tilde{d} \) are short-hand notations for \( c(t, k) \) and \( d(t, k) \), respectively. Let us consider the Neumann problem. Integration by parts in (2.9) shows that
\[
c(t, k) = \begin{cases} \frac{g_0}{2i k} + O(k^{-2}) + O(e^{2i k L} k^{-1}), & k \in \mathbb{C}^+, \\ e^{2i k L} \left( -\frac{h_0}{2i k} + O(k^{-2}) \right) + O(k^{-1}), & k \in \mathbb{C}^-, \end{cases}\]

This gives the representations
\[
g_0(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} c_+(t, k) \, dk, \quad (C.3a)
\]
\[
h_0(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} [\cos(2 k L) c_+(t, k) - i \sin(2 k L) c_-(t, k)] \, dk, \quad (C.3b)
\]
which can be used to eliminate \( \{g_0, h_0\} \) from (C.2). The resulting system for \( \{c, d\} \) is formulated only in terms of \( \{g_1, h_1\} \). However, this system is not effective. Indeed, substituting into (C.2) and (C.3) the expansions
\[
d = 1 + \epsilon^2 d_2 + \cdots, \quad c = \epsilon c_1 + \epsilon^2 c_2 + \cdots,
\]
the terms of \( O(\epsilon^n) \) yield
\[
e^{i k^2 c_n} = e^{4i k^2 [2 k g_{0n} + i g_{1n} - (2 k h_{0n} + i h_{1n}) e^{i k L}]} + \text{lower order terms}\]

In appendix B, we made no assumption on the form of \( c(t, k) \) except that it be an entire function satisfying (2.8). Here, since we are using the form of the function \( c(t, k) \) as defined in (C.1), it is much easier to determine its asymptotics.

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and
\[ g_{0n} = -\frac{1}{\pi} \int_{-\infty}^{\infty} c_{n+} dk, \quad h_{0n} = -\frac{1}{\pi} \int_{-\infty}^{\infty} [\cos(2kL)c_{n+} - i \sin(2kL)c_{n-}] dk. \]

Thus, in order to determine \( \{g_{0n}, h_{0n}\} \) at each step of the perturbative scheme, we need to know the combinations \( c_{n+} \) and \( \cos(2kL)c_{n+} - i \sin(2kL)c_{n-} \). Equation (C.4) shows that these combinations satisfy
\[
\begin{align*}
[e^{4ik^2}c_{n+}]_t &= e^{4ik^2}[2ig_{1n} - 2kh_{0n}\Delta(k) - ih_{1n}\Sigma(k)] + \text{lower order terms}, \quad (C.5a) \\
[e^{4ik^2}[-2k\rho_{0n}\Delta(k) + ig_{1n}\Sigma(k)] - 2ih_{1n}] + \text{lower order terms}. \quad (C.5b)
\end{align*}
\]

Although the function \( g_{0n} \) has been eliminated from the rhs of (C.5a), the function \( h_{0n} \) remains unknown. Similarly, although the function \( h_{0n} \) has been eliminated from the rhs of (C.5b), the function \( g_{0n} \) remains unknown. This shows that the solution is not effective.

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