ON TOPOLOGICAL COMPLEXITY AND LS-CATEGORY

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Abstract. We present some results supporting the Iwase-Sakai conjecture about coincidence of the topological complexity $TC(X)$ and monoidal topological complexity $TC^M(X)$. Using these results we provide lower and upper bounds for the topological complexity of the wedge $X \vee Y$. We use these bounds to give a counterexample to the conjecture asserting that $TC(X') \leq TC(X)$ for any covering map $p : X' \to X$.

We discuss a possible reduction of the monoidal topological complexity to the LS-category. Also we apply the LS-category to give a short proof of the Arnold-Kuiper theorem.

1. Introduction

Let $PX = X^{[0,1]}$ denote the space of all paths in $X$. Let $i_X : X \to PX$ be the inclusion of $X$ into $PX$ as a subspace of constant paths. There is a natural fibration $\pi : PX \to X \times X$ defined as $\pi(f) = (f(0), f(1))$ for $f \in PX$, $f : [0, 1] \to X$.

Let $X$ be an ENR. A section $s : X \times X \to PX$ of $\pi$ is called a motion planning algorithm. We say that a motion planning algorithm $s$ has complexity $k$ if $X \times X$ can be presented as a disjoint union $F_1 \cup \cdots \cup F_k$ of ENRs such that $s$ is continuous on each $F_i$. The topological complexity $TC(X)$ of a space $X$ was defined by Farber as the minimum of $k$ such that there is a motion planning algorithm of complexity $k$ [F1]. Equivalently, $TC(X)$ is the minimal number $k$ such that $X \times X$ admits an open cover $U_1, \ldots, U_k$ such that over each $U_i$ there is a continuous section of $\pi$.

We say that a motion planning algorithm $s : X \times X \to PX$ is reserved if $s|_{\Delta X} = i_X$ where $\Delta X \subset X \times X$ is the diagonal. In other words, if the initial position of a robot in the configuration space $X$ coincides with the terminal position, then the algorithm keeps the robot still. This

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condition on the motion planning algorithms seems to be very natural. The corresponding complexity of a space $X$ was denoted by Iwase and Sakai as $TC^M(X)$ and was called the monoidal topological complexity of $X$ \cite{IS1}. In the original definition they additionally assumed that all sets $U_i$ contain the diagonal. Their definition agrees with the above since their condition always can be achieved by reduction of an open cover $U_1, \ldots, U_k$ with reserved sections $s_i$ to a closed cover $F_1, \ldots, F_k$, $F_i \subseteq U_i$, then by adding the diagonal to each $F_i$ with the natural extension of the sections $\bar{s}_i$, and then by taking open enlargement $V_i$ of the sets $F_i \cup \Delta X$ that admit extensions of the sections $\bar{s}_i$.

Iwase and Saki conjectured that $TC^M(X) = TC(X)$. In fact, first they gave a proof to the conjecture in \cite{IS1} and then withdrew it in \cite{IS2}. We prove this conjecture under the assumption $TC(X) > \dim X + 1$. Also, using the Weinberger Lemma from \cite{F3} we show that the conjecture holds true when $X$ is a Lie group.

The topological complexity is closely related to the Lusternik-Schnirelmann category $\text{cat}(X)$ of a space which is defined as the minimal number $k$ such that $X$ can be covered by $k$ open sets $U_1, \ldots, U_k$ all contractible to a point in $X$. We denote by

$$\text{Cat}(X) = \text{cat}(X) - 1,$$

the reduced LS-category. The reduced category appears naturally in several inequalities in the theory \cite{CLOT}:

$$\text{cup-length}(X) \leq \text{Cat}(X) \leq \dim(X)$$

and

$$\text{Cat}(X \times Y) \leq \text{Cat}(X) + \text{Cat}(Y).$$

In the first inequality the cup-length is taken for any reduced cohomology (possibly twisted).

Some of the formulas for cat translate to similar statements for TC. For example for $TC$ there is an inequality similar to the above for the product of two spaces \cite{F4}. Also there are analogous estimates of TC in terms of the cup product and dimension \cite{F4}. On the other hand, the simple cat formula for the wedge $\text{cat}(X \vee Y) = \max\{\text{cat} X, \text{cat} Y\}$ does not hold for $TC$. So far there is no nice analog of it for TC. The best that we can prove here is Theorem 3.6 from this paper. Another example is the formula $\text{cat}(Y) \geq \text{cat}(X)$ for a covering map $p : X \rightarrow Y$ which supports an intuitive idea that a covering space is always simpler than the base. So it was natural to assume that the same holds true for TC. I've learned about this problem from Yuli Rudyak. In this paper Theorem 3.8 gives a negative answer to this question.
There have been several attempts to reformulate the topological complexity of $X$ as some modified category of a related space. In this paper we discuss a possible characterization of the monoidal topological complexity in terms of the category. We define a $\rel\infty$ category $\rel\infty\cat(Y)$ of non-compact spaces $Y$ and discuss the problem of coincidence between $\cat(X/A)$ and $\rel\infty\cat(X \setminus A)$ for a subcomplex $A \subseteq X$ of a finite complex $X$. Then we show that $TC^M(X)$ is always between $\cat(X \times X)/\Delta(X)$ and $\rel\infty\cat(X \times X \setminus \Delta X)$.

Note that both $\cat(X)$ and $TC(X)$ are partial case of the Schwarz genus $[\text{Sch}]$: $\cat(X) = \sg(\pi_0 : P_0 X \to X)$ and $TC(X) = \sg(\pi : PX \to X \times X)$ where $P_0 X \subseteq PX$ is the subspace of paths $f : [0,1] \to X$ that start in a base point $x_0 \in X$, $f(0) = x_0$, and $\pi_0(f) = f(1)$. We recall the Schwarz genus $[\text{Sch}]$ of a fibration $p : X \to Y$ is the minimal number of open sets $U_1, \ldots, U_k$ that cover $Y$ and admit sections $s_i : U_i \to X$ of $p$. In the paper we estimate the Schwarz genus $[\text{Sch}]$ of arbitrary fibration $p : X \to Y$ in terms the category of its mapping cone $C_p$.

Finally, we apply the LS-category to give a short proof of the Arnold-Kuiper theorem which states that the orbit space of the action of $\mathbb{Z}_2$ on the complex projective plane $\mathbb{C}P^2$ by the conjugation is the 4-sphere. Note that this theorem was discovered by Arnold [Ar1] who published his proof much later [Ar2]. It was proven independently by Kuiper [K] and by Massey [M].

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2. Monoidal topological complexity

2.1. Theorem. For ENR spaces,

$$TC(X) \leq TC^M(X) \leq TC(X) + 1.$$ 

This theorem was proved in [IS2]. Since the proof there is too technical we give an alternative proof.

Proof. The first inequality is obvious. Since $X$ is ANR, there is an open neighborhood $W$ of the diagonal $\Delta X$ in $X \times X$ and a continuous map $\phi : W \times [0,1] \to X$ such that $\phi(x,x',0) = x$, $\phi(x,x',1) = x'$, and $\phi(x,x,t) = x$ for all $t \in [0,1]$, $x,x' \in X$. Let $U_1, \ldots, U_n$ be an open cover of $X \times X$ by sets that admit sections $s_i : U_i \to PX$ of $\pi$. Let $F$ be a closed neighborhood of $\Delta X$ that lies in $W$. Then all sets in the open cover $U_1 \setminus F, \ldots, U_n \setminus F, W$ of $X \times X$ admit reserved sections. Hence $TC^M(X) \leq n + 1$. \hfill $\square$

Note that the path fibration $\pi : PX \to X \times X$ restricted over the diagonal defines the free loop fibration $p : LX \to X$. A canonical
section \( \bar{s} : \Delta X \to LX \) of \( p \) is defined as \( \bar{s}(x) = c_x \), where \( c_x : I \to X \) is the constant map to \( x \).

We use the standard convention to denote the elements of the iterated join product \( X_1 \ast X_2 \ast \cdots \ast X_n \) as formal linear combinations \( t_1x_1 + t_2x_2 + \cdots + t_nx_n \), \( \sum t_i = 1, t_i \geq 0, x_i \in X_i \) where all summands of the type \( 0x_i \) are dropped. We use the notation \( \ast^nX \) for the iterated join product of \( n \) copies of \( X \) with itself.

We recall that a fiber-wise join of maps \( f_i : X_i \to Y, i = 1, \ldots, n \) is the map

\[
    f_1 \tilde{\ast} \cdots \tilde{\ast} f_n : X_1 \tilde{\ast} Y \cdots \tilde{\ast} Y X_n \to Y
\]

where

\[
    X_1 \tilde{\ast} Y \cdots \tilde{\ast} Y X_n = \{ t_1x_1 + \cdots + t_nx_n \in X_1 \ast \cdots \ast X_n \mid f_1(x_1) = \cdots = f_n(x_2) \}
\]

is the fiber-wise join of spaces \( X_1, \ldots, X_n \) and

\[
    (f_1 \tilde{\ast} \cdots \tilde{\ast} f_n)(t_1x_1 + \cdots + t_nx_n) = f_i(x_i).
\]

Thus, the preimage \((f_1 \tilde{\ast} \cdots \tilde{\ast} f_n)^{-1}(y)\) of a point \( y \in Y \) is the join product of the preimages \( f_1^{-1}(y) \ast \cdots \ast f_n^{-1}(y) \).

We define \( P_nX = PX \tilde{\ast} X \ast \cdots \ast X \ast PX \) and

\[
    \pi_n = \pi \tilde{\ast} \cdots \tilde{\ast} \pi : P_nX \to X \times X
\]

to be the fiber-wise join product of \( n \) copies of \( \pi \). Note that there are imbeddings \( P_1X \subset P_2X \subset \cdots \subset P_nX \) such that \( \pi_i|_{P_{i-1}} = \pi_{i-1} \). Then the section \( \bar{s} : X \times X \to P_1X \) of \( \pi_1 \) can be regarded as a section of \( \pi_n \). Also we define \( p_1 = p : LX \to X \), \( L_nX = L_{n-1}X \ast LX \), and \( p_n = p_{n-1} \ast p : L_n \to X \). Note that \( \pi_n^{-1}(\Delta X) \cong L_nX \) and \( p_n \) is the restriction of \( \pi_n \) to \( \pi_n^{-1}(\Delta X) \). Note also that the canonical section \( \bar{s} \) defines a trivial subbundle \( p'_n : E \to X \) of \( p_n \) with the fiber the \((n - 1)\)-simplex \( \Delta^{n-1} \).

We recall that a map \( p : E \to B \) satisfies the Homotopy Lifting Property for a pair \((X, A)\) if for any homotopy \( H : X \times I \to B \) with a lift \( H' : A \times I \to E \) of the restriction \( H|_{A \times I} \) and a lift \( H_0 \) of \( H|_{X \times 0} \) which agrees with \( H' \), there is a lift \( \tilde{H} : X \times I \to E \) of \( H \) which agrees with \( H_0 \) and \( H' \). The following is well-known [H]:

2.2. Theorem. Any Hurewicz fibration \( p : E \to B \) satisfies the Homotopy Lifting Property for CW complex pairs \((X, A)\).

2.3. Corollary. Let \( p : E \to X \) be a Hurewicz fibration with a section \( s : X \to E \). A fiber-wise homotopy \( G : A \times I \to E \) of the restriction \( s|_A \) to a closed subset \( A \subset X \) can be extended to a fiber-wise homotopy \( G : X \to E \) of \( s \) provided \((X, A)\) is a CW complex pair.
2.4. Proposition. For CW complexes $X$,

1. $TC_1(M) \leq n \iff \pi_n : P_n X \to X \times X$ admits a section.
2. $TC_1^M(M) \leq n \iff \pi_n : P_n X \to X \times X$ admits a section $s$ which agrees with the canonical section over the diagonal $s|{\Delta X} = \bar{s}$.

Proof. The statement (1) is a part of a general theorem proven by Schwartz [Sch] for fibrations $q : X \to Y$: $sg(q) \leq n$ if and only if the $n$-fold iterated fiber-wise join product $\bar{s}^n q : \bar{s}_Y^n X \to Y$ admits a section.

The implication $\Leftarrow$ in (2) is obvious. For the other direction we note that $n$ reserved sections $s_i : U_i \to P X$ defined for an open cover $U_1, \ldots, U_n$ of $X \times X$ define a section $s$ of $\pi_n$ with the image $s(\Delta X)$ lying in $E$. Therefore over $\Delta X$ it could be fiber-wise deformed to $\bar{s}$. By Proposition 2.2 that deformation can be extended to a fiber-wise deformation of $s$. \hfill $\Box$

2.5. Theorem. The equality

$$TC(X) = TC_1^M(X)$$

holds true for $k$-connected simplicial complexes $X$ such that

$$(k + 1)TC(X) > \dim X + 1.$$ 

Proof. Let $TC(X) = n$. Note that the fiber $\pi^{-1}(x, x')$ is homotopy equivalent to the loop space $\Omega(X)$. Since $\Omega(X)$ is $(k - 1)$-connected, the iterated join product $*^n \Omega(X)$ is $((k + 1)n - 2)$-connected. We show that any section $s : \Delta X \to L_n X$ can be fiber-wise joined by a homotopy with a canonical section $\bar{s} : \Delta X \to L_n X$. By induction on $i$ we construct a section $s_i : X \to L_n X$, that coincides with $\bar{s}$ on the $i$-skeleton $X^{(i)}$, together with a fiber-wise homotopy joining $s$ and $s_i$. Here we use the identification $\Delta X = X$. For $i = 0$ we take paths in the fibers $p_n^{-1}(v)$ joining $s(v)$ and $\bar{s}(v)$ for all $v \in X^{(0)}$. Then we extend them to a fiber-wise homotopy of $s$ to a section $s_0$. Assume that $s_{i-1}$ is already constructed and $i \leq \dim X \leq (k + 1)n - 2$. Independently for every $i$-simplex $\sigma \subset X$ we consider the problem of joining $s_{i-1}$ with $\bar{s}$ over $\sigma$ by a fiber-wise homotopy. Since the fiber bundle $p_n$ is trivial over $\sigma$ with a $i$-connected fiber, the identity homotopy on the boundary $\partial \sigma$ can be extended to a homotopy between $\bar{s}|_{\sigma}$ and $s_{i-1}|_{\sigma}$. This extension can be deformed to a fiber-wise homotopy. All these homotopies together define a fiber-wise homotopy between $s_{i-1}$ and $\bar{s}$ over $X^{(i)}$. Since $(X, X^{(0)})$ is a CW pair, by Proposition 2.2 we can extend it to a fiber-wise homotopy over $X$.

Let $s : X \times X \to P_n X$ be a section. On $\Delta X$ it can be deformed to a canonical section $\bar{s}$. Since $(X \times X, \Delta X)$ is a CW pair, by Proposition 2.2 there is a fiber-wise homotopy of $s$ to a section $s'$ that coincides with $\bar{s}$ on $\Delta X$. Therefore, $TC_1^M(X) \leq n$. \hfill $\Box$
2.6. **Corollary.** $TC(S^m) = TC^M(S^m)$ for all $m$.

The following is an extension of Weinberger’s Lemma from [F3] to the case of monoidal topological complexity.

2.7. **Lemma.** For a connected Lie group $G$,

$$TC(G) = TC^M(G) = \text{cat}(G).$$

**Proof.** In view of what is already known [F3], it suffices to show the inequality $TC^M(G) \leq \text{cat}(G)$. Let $\text{cat}(G) = n$ and let $U_1, \ldots, U_n$ be an open cover of $G$ together with homotopies $H_i : U_i \times [0,1] \to G$ contracting $U_i$ to the unit $e \in G$. Clearly, we may assume that $e \notin U_i$ for $i > 1$. Since the inclusion $e \in G$ is a cofibration, we may assume that $H_i(e,t) = e$ for all $t$. Then for the open cover of $G \times G$ as defined

$$W_i = \{(a,b) \in G \times G \mid a^{-1}b \in U_i\}$$

the sections $s_i : W_i \to PG$ defined as

$$s_i(a,b)(t) = ah_i(a^{-1}b, t) \in G, \quad (a,b) \in W_i$$

are reserved. Indeed, $\Delta G \cap W_i = \emptyset$ for $i > 1$ and

$$s_1(a,a)(t) = ah_1(a^{-1}a, t) = ah_1(e, t) = ae = a$$

for all $(a,a) \in \Delta G$. □

3. **Topological complexity of wedge and covering maps**

A deformation of $U \subset Z$ in $Z$ to a subset $A \subset Z$ is a continuous map $D : U \times I \to Z$ such that: $D(u,0) = u$, $D(u,1) \in A$ for all $u \in U$. A strict deformation of $U \subset Z$ in $Z$ to $A \subset Z$ is a deformation $D : U \times I \to Z$ such that $D(u,t) = u$ for all $t \in I$ whenever $u \in A$.

3.1. **Proposition.** Let $X$ be a metric space. For an open set $U \subset X \times X$ the following are equivalent:

1. There is a reserved section $s : U \to PX$ over $U$ of the fibration $\pi : PX \to X \times X$.

2. There is a strict deformation $D : U \times I \to X \times X$ to the diagonal $\Delta X = \{(x,x) \in X \times X \mid x \in X\}$

3. For any choice of a base point $x_0 \in X$ there is a strict deformation $D$ of $U$ to $\Delta X$ which preserves faces $X \times x_0$ and $x_0 \times X$, i.e., for all $t \in I$,

$$D((x,x_0),t) \in X \times x_0 \quad \text{and} \quad D((x_0,x),t) \in x_0 \times X.$$
Proof. (1) $\Rightarrow$ (3). Let $\|x\| = d(x, x_0)$. We define
\[
D((x, y), t) = (s(x, y)(\frac{\|x\|}{\|x\| + \|y\|}t), s(x, y)(1 - \frac{\|y\|}{\|x\| + \|y\|}t))
\]
if $(x, y) \neq (x_0, x_0)$ and define $D((x_0, x_0), t) = (x_0, x_0)$. Since $s(x, y)(0) = x$ and $s(x, y)(1) = y$, we obtain that $D((x, y), 0) = (x, y)$. Note that
\[
D((x, y), 1) = (s(x, y)(\frac{\|x\|}{\|x\| + \|y\|}), s(x, y)(\frac{\|x\|}{\|x\| + \|y\|})) \in \Delta X.
\]
Since the section $s$ is reserved, $D((x, x), t) = (s(x, x)(t/2), s(x, x)(t/2)) = (x, x)$. Note that
\[
D((x, x_0), t) = (s(x, x_0)(t), s(x, x_0)(1)) = (s(x, x_0)(t), x_0) \in X \times x_0
\]
and
\[
D((x_0, y), t) = (s(x_0, y)(0), s(x_0, y)(1-t)) = (x_0, s(x_0, y)(1-t)) \in x_0 \times X.
\]
The deformation $D$ is continuous at $(x_0, x_0)$ (if defined) since the section $s(x_0, x_0)$ is stationary at $(x_0, x_0)$.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $p_r : X \times X \to X$ denote the projection to the first factor and $p_r : X \times X \to X$ to the second. Given a strict deformation $D$ we define a section $s : U \times I \to PX$ as follows:
\[
s(x, y)(t) = \begin{cases} 
p_rD((x, y), 2t) & \text{if } t \leq 1/2 
p_rD((x, y), 2 - 2t) & \text{if } t \geq 1/2.
\end{cases}
\]
This path is well-defined since $D((x, y), 1) \in \Delta X$. Clearly it is a path from $x$ to $y$. If $x = y$, the path is stationary. Thus $s$ is a reserved section.

3.2. Proposition. Let $A$ be a retract of an ENR space $X$. Then $TC(X) \geq TC(A)$.

Proof. Let $r : X \to A$ be a retraction. Let $TC(X) = k$ and let $X \times X = U_1 \cup \cdots \cup U_k$ be an open cover together with continuous sections $s_i : U_i \to PX$. We define sections $\sigma_i : U_i \cap (A \times A) \to PA$ by the formula $\sigma_i(a_1, a_2)(t) = r(s(a_1, a_2)(t))$.

We recall that a family $\mathcal{U}$ of subsets of $X$ is called a $k$-cover, $k \in \mathbb{N}$ if every subfamily that consists of $k$ elements forms a cover of $X$. We use the following theorem [Dr1].

3.3. Theorem. Let $\{U'_0, \ldots, U'_n\}$ be an open cover of a normal topological space $X$. Then for any $m = n, n+1, \ldots, \infty$ there is an open $(n+1)$-cover of $X$, $\{U'_k\}_{k=0}^m$ such that $U_k = U'_k$ for $k \leq n$ and $U_k = \bigcup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$ for $k > n$. 

3.4. **Corollary.** Suppose that all sets $U'_i$, $i = 0, \ldots, n$, in the theorem are (strictly) deformable in $X$ to a subspace $A \subset X$. Then the sets $U_k$ for all $k$ are (strictly) deformable in $X$ to $A$.

The following proposition is well-known. The trick presented there can be traced back to the work of Kolmogorov on 13th Hilbert’s problem [Os].

3.5. **Proposition.** Let $U_0, \ldots, U_{n+m}$ be an $(n+1)$-cover of $X$ and let $V_0, \ldots, V_{m+n}$ be an $(m+1)$-cover of $Y$. Then the sets $W_k = U_k \times V_k$, $k = 0, \ldots n + m$, cover $X \times Y$.

**Proof.** Let $(x, y) \in X \times Y$. A point $x$ is covered at least by $m + 1$ elements. Otherwise $n+1$ elements that do not cover $x$ would not form a cover of $X$. That would give a contradiction with the assumption that $U_0, \ldots, U_{n+m}$ is an $(n+1)$-cover of $X$. Let $x \in U_{i_0} \cap \cdots \cap U_{i_m}$. By the assumption, the family $V_{i_0}, \ldots, V_{i_m}$ covers $Y$. Hence $y \in V_{i_s}$ for some $s$. Then $(x, y) \in W_{i_s}$. \hfill \Box

3.6. **Theorem.** For all ENR spaces $X$ and $Y$,

$$\max \{TC(X), TC(Y), \text{cat}(X \times Y)\} \leq TC(X \vee Y) \leq TC^M(X \vee Y) \leq TC^M(X) + TC^M(Y) - 1$$

**Proof.** Note that $TC(X \vee Y) \geq TC(X), TC(Y)$ by Proposition 3.2. Let $r_X : X \vee Y \to X$ and $r_Y : X \vee Y \to Y$ be the retraction collapsing the wedge onto $X$ and $Y$ respectively. The subset $X \times Y \subset (X \vee Y) \times (X \vee Y)$ is covered by $\leq TC(X \vee Y)$ open sets $U$ supplied with a homotopy $H_U : U \times I \to X \vee Y$ such that $H(x, y, 0) = x$ and $H(x, y, 1) = y$. For each $U$ we define a homotopy $G : U \times I \to X \times Y$ by the formula

$$G(x, y, t) = (r_X H_U(x, y, t), r_Y H_U(x, y, 1 - t)).$$

Then

$$G(x, y, 0) = (r_X H_U(x, y, 0), r_Y H_U(x, y, 1)) = (r_X(x), r_Y(y)) = (x, y)$$

and

$$G(x, y, 1) = (r_X H_U(x, y, 1), r_Y H_U(x, y, 0)) = (r_X(y), r_Y(x)) = (v_0, v_0)$$

where $v_0$ is the wedge point in $X \vee Y$. Thus, $G$ contracts $U$ to a point in $X \times Y$.

Let $TC^M(X) = n + 1$ and $TC^M(Y) = m + 1$. Then there is an open cover $\tilde{U}_0, \ldots, \tilde{U}_n$ of $X \times X$ with reserved sections $s_i : \tilde{U}_i \to PX,$
$i = 0, \ldots, n$. Similarly, let $\tilde{V}_0, \ldots, \tilde{V}_m$ be an open covering of $Y \times Y$ with reserved sections $\sigma_j : \tilde{V}_j \to PY$, $j = 0, \ldots, m$. By Proposition 3.4 all these sets are strictly deformable to the diagonal in $X \times X$ and $Y \times Y$ respectively. By Corollary 3.4 there is an open $(m+1)$-cover $\tilde{U}_0, \ldots, \tilde{U}_n, \ldots, \tilde{U}_{m+n}$ of $X \times X$ by sets strictly deformable to the diagonal. By Proposition 3.1 there are strict deformations

$$D^k_X : \tilde{U}_k \times I \to X \times X$$

of $\tilde{U}_k$ to $\Delta X$ that preserves faces $X \times v_0$ and $v_0 \times X$. Similarly, there is an open $(m+1)$-cover $\tilde{V}_0, \ldots, \tilde{V}_m, \ldots, \tilde{V}_{m+n}$ of $Y \times Y$ and there are strict deformations $D^k_Y$ of $\tilde{V}_k$ in $Y \times Y$ to the diagonal $\Delta Y$ that preserves faces.

We use notations

$$U_k = \tilde{U}_k \cap (X \times v_0) \quad \text{and} \quad V_k = \tilde{V}_k \cap (v_0 \times Y), \quad k = 0, \ldots, m+n.$$ 

Note that $U_0, \ldots, U_{m+n}$ is an $(n+1)$-cover of $X \times v_0 = X$ and $V_0, \ldots, V_{m+n}$ is an $(m+1)$-cover of $v_0 \times Y = Y$. Let $W_k = U_k \times V_k$. By Proposition 3.5 $W_0, \ldots, W_{m+n}$ is an open cover of $X \times Y$.

The deformations $D^k_X$ define the deformations $H^k : U_k \times I \to X \times v_0$ to the point $v_0 \in X$ and the deformations $D^k_Y$ define the deformations $G^k : V_k \times I \to v_0 \times Y$ to the point $v_0 \in Y$. These deformations define the deformations

$$T^k : W_k \times I \to X \times Y$$

to the point $(v_0, v_0)$ such that if $W_k \cap (X \times v_0) \neq \emptyset$ then $W_k \cap (X \times v_0) = U_k$ and $T^k|_{U_k \times I} = H^k$ and if $W_k \cap (v_0 \times Y) \neq \emptyset$ then $W_k \cap (v_0 \times Y) = V_k$ and $T^k|_{V_k \times I} = G^k$ for $k = 0, \ldots, m+n$.

Symmetrically, define

$$U'_k = \tilde{U}_k \cap (v_0 \times X) \quad \text{and} \quad V'_k = \tilde{V}_k \cap (Y \times v_0), \quad k = 0, \ldots, m+n,$$

and corresponding deformations

$$H'_k : U'_k \times I \to X \quad \text{and} \quad G'_k : V'_k \times I \to Y$$

to the base points. Define $W'_k = U'_k \times V'_k$. By Proposition 3.5 the family $W'_0, \ldots, W'_{n+m}$ is an open cover of $Y \times X$. As before there are deformations

$$T'_k : W'_k \times I \to Y \times X$$

to the point $(v_0, v_0)$ such that if $W'_k \cap (v_0 \times X) \neq \emptyset$, then $W'_k \cap (v_0 \times X) = U'_k$ and $T'_k|_{U'_k \times I} = H'_k$ and if $W'_k \cap (Y \times v_0) \neq \emptyset$, then $W'_k \cap (Y \times v_0) = V'_k$, $T'_k|_{V'_k \times I} = G'_k$ for $k = 0, \ldots, m+n$.

We define open sets

$$O_k = W_k \cup W'_k \cup \tilde{U}_k \cup \tilde{V}_k \subset (X \lor Y) \times (X \lor Y), \quad k = 0, \ldots, n+m$$
and note that $\mathcal{O} = \{O_k\}$ covers $(X \vee Y) \times (X \vee Y)$. Note that the set
$$C = (X \vee Y) \times v_0 \cup v_0 \times (X \vee Y)$$
defines a partition of $(X \vee Y) \times (X \vee Y)$ in four pieces $X \times X$, $X \times Y$, $Y \times X$, and $Y \times Y$. Also note that the intersection $O_k \cap C \subset U_k \cup V_k \cup U'_k \cup V'_k$. By the construction the deformations $D^k_X, D^k_Y, T_k, T_k$ all agrees on $O_k \cap C$. Therefore the union of deformations $T_k \cup T'_k \cup D^k_X \cup D^k_Y : O_k \times I \to (X \vee Y) \times (X \vee Y)$
is a well defined deformation $Q_k$ of $O_k$ to the diagonal $\Delta (X \vee Y)$. Note that for all $k$, $Q_k$ are strict deformations. By Proposition 3.1 each $Q_k$ defines a reserved section $\alpha_k : O_k \rightarrow P(X \vee Y)$. Therefore,
$$TX^M (X \vee Y) \leq n + m + 1 = TC(X) + T(Y) - 1.$$  

\[\square\]

3.7. Remark. A stronger version of the upper bound of Theorem 3.6 was proposed in [F2], (Theorem 19.1):
$$TC(X \vee Y) \leq \max\{TC(X), TC(Y), \text{cat}(X) + \text{cat}(Y) - 1\}.$$ Since the proof in [F2] contains a gap, we call this inequality Farber’s Conjecture. Note that Farber’s inequality in view of Theorem 3.6 would turns into the equality for spaces $X$ and $Y$ with $\text{Cat}(X \times Y) = \text{Cat}(X) + \text{Cat}(Y)$.

3.8. Theorem. (1) There is a 2-to-1 covering map $p : E \to B$ with $TC(E) > TC(B)$.
(2) There is a finite complex $X$ with $TC(X) < TC(\hat{X})$ where $\hat{X}$ is the universal covering of $X$.

Proof. (1) We take $B = T \vee S^1$ where $T = S^1 \times S^1$ is a 2-torus. Let $E$ to be the covering space defined by the 2-fold covering of $S^1$. Note that $E$ is homeomorphic to the circle with two tori $T$ attached at antipodal points. Thus, $E$ is homotopy equivalent to $T \vee T \vee S^1$. By Theorem 3.6 and Lemma 2.7
$$TC(B) \leq TC^M(T) + TC^M(S^1) - 1 = \text{cat}(T) + \text{cat}(S^1) - 1 = 3 + 2 - 1 = 4.$$ On the other hand by Proposition 3.6
$$TC(E) \geq \text{cat}((T \vee S^1) \times T)) = 3 + 3 - 1 = 5.$$ (2) Consider $X = (S^3 \times S^3) \vee S^1$. Since $S^3 \times S^3$ is a connected Lie group, by Lemma 2.7, $TC^M(S^3 \times S^3) = \text{cat}(S^3 \times S^3) = 3$. By Theorem 3.6
$$TC(X) \leq TC^M(S^3 \times S^3) + TC^M(S^1) - 1 = 3 + 2 - 1 = 4.$$
Note that the universal cover $\tilde{X}$ is homotopy equivalent to an infinite wedge $Y = \bigvee_{\infty} (S^3 \times S^3)$. Then $Y$ admits a retraction onto $(S^3 \times S^3) \vee (S^3 \times S^3)$. By Proposition 3.2, Theorem 3.6, and the cup-length lower bound on cat,

$$TC(\tilde{X}) \geq TC((S^3 \times S^3) \vee (S^3 \times S^3)) \geq \text{cat}(S^3 \times S^3 \times S^3 \times S^3) \geq 5.$$

□

4. Topological complexity, LS-category, and Schwartz genus

We say a subset $A \subset X$ can be rel $\infty$ contracted to infinity if for every compact subset $F \subset X$ there is a larger compact set $F \subset C$ and a homotopy $h_t : A \to X$ with $h_0 = 1_A$, $h_1(A) \cap F = \emptyset$ and $h_t(a) = a$ for $a \in A \setminus C$.

4.1. Definition. We define the rel $\infty$ category $\infty$-cat($X$) of a locally compact space $X$ as the minimal $k$ such that there is a cover $X = V_1 \cup \cdots \cup V_k$ by closed subsets where each $V_i$ can be rel $\infty$ contracted to infinity.

4.2. Remark. It follows from the definition that for every locally compact space $X$,

$$\text{cat}(\alpha X) \leq \infty$-cat($X$)

where $\alpha X$ is the one-point compactification.

4.3. Question. Does the equality $\text{cat}(\alpha X) = \infty$-cat($X$) hold for all locally finite complexes with tame ends?

We recall that $X$ has a tame end if there is a compactum $C \subset X$ such that $X \setminus \text{Int}(C) \cong \partial C \times [0, 1]$.

In the case when $\alpha X$ is a closed manifold this question could be related to the difference between the category and the ball-category for manifolds. We recall that for a closed $n$-manifold $M$, $\text{ballcat}(M) \leq k$ is there is a cover of $M$ by $k$ closed topological $n$-dimensional balls.

4.4. Proposition. For any closed $n$-manifold $M$ and any $x_0 \in M$,

$$\text{cat}(M) \leq \infty$-cat$(M \setminus \{x_0\}) \leq \text{ballcat}(M) \leq \text{cat}(M) + 1.$$

Proof. In view of Remark 4.3 and some known fact about the ball-category [CLOT], only the second inequality needs a proof. Let $\text{ballcat}(M) = m$ and let $B_1, \ldots, B_m$ be a cover of $M$ by topological closed $n$-balls such that $x_0 \notin \partial B_i$ for all $i$. Then all $B_i \setminus \{x_0\}$ can be rel $\infty$ contracted in $M \setminus \{x_0\}$ to $x_0$. □
Since the one-point compactification of \(X \times X\) with the diagonal \(\Delta X\) removed is the quotient space \((X \times X)/\Delta X\), the following theorem shows that Question 4.3 is closely related to characterization of the topological complexity \(TC^M\) by means of the LS-category.

4.5. **Theorem.** For any compact ENR \(X\),

\[
\text{cat}((X \times X)/\Delta X) \leq TC^M(X) \leq \infty - \text{cat}((X \times X) \setminus \Delta X).
\]

**Proof.** Suppose that \(TC^M(X) = k\). Then by the definition there is an open cover \(U_1, \ldots, U_k\) of \(X \times X\) with continuous reserved sections \(s_i : U_i \to PX\) of \(\pi : PX \to X \times X\). By Proposition 3.1 there are strict deformations of \(U_i\) in \(X \times X\) to the diagonal \(\Delta X\). They define the deformations of \(U_i/(U_i \cap \Delta X)\) to the point \(\{\Delta X\}\) in \((X \times X)/\Delta X\). Thus, \(\text{cat}((X \times X)/\Delta X) \leq k\).

Let \(\infty - \text{cat}((X \times X) \setminus \Delta X) = k\) and let \((X \times X) \setminus \Delta X = F_1 \cup \cdots \cup F_k\) be the union of \(k\) closed sets rel \(\infty\) contractible to infinity. Let \(W\) be a neighborhood of the diagonal \(\Delta X\) in \(X \times X\) that admits a deformation retraction \(r_t\) to \(\Delta X\). Let \(h_i^t\) be a deformation of \(F_i\) into \(W\). Then the concatenation of \(h_i^t\) and \(r_t\) defines a deformation \(H_i\) of \(F_i\) to the diagonal. Let \(\tilde{F}_i = F_i \cup \Delta X\). Note that \(H_i\) together with identity on \(\Delta X\) define a strict deformation of \(\tilde{F}_i\) to the diagonal. \(\square\)

4.6. **Remark.** For the topological complexity \(TC(X)\) a weaker version of the first inequality from Theorem 4.5 was proven in [12], Lemma 18.3.

\[
\text{cat}((X \times X)/\Delta X) - 1 \leq TC(X).
\]

The topological complexity of \(X\) equals the Schwarz genus of a certain fibration. It turns out that for general fibrations we still have the inequalities similar to Theorem 4.5.

4.7. **Theorem.** For any fibration of compact spaces \(p : X \to Y\),

\[
\text{cat}(C_p) - 1 \leq sg(p) \leq \infty - \text{cat}(C_p \setminus \{\ast\}).
\]

**Proof.** We claim that if a subset \(U \subset Y\) admits a section \(s : U \to X\), then \(U\) is contractible in \(C_p\). Indeed, it can be moved to \(X\) in the mapping cylinder \(M_p\). Since the cone \(\text{Con}(X)\) is contained in \(C_p\), it could be further contracted to a point. Moreover, the mapping cylinder \(\tilde{U} = M_p|_{p^{-1}(U)}\) of the restriction of \(p\) to the preimage \(p^{-1}(U)\) is contractible in \(C_p\), since it can be pushed to \(U\) first. If \(Y\) is covered by \(n\) open sets \(U_1, \ldots, U_n\) each of which admits a section of \(p\), then the mapping cylinder \(M_p\) can be covered by \(n\) sets \(\tilde{U}_1, \ldots, \tilde{U}_n\) all contractible in the mapping cylinder \(C_p\). Since \(C_p = M_p \cup \text{Con}(X)\), the open enlargements of the sets \(\tilde{U}_1, \ldots, \tilde{U}_n\) and \(\text{Con}(X)\) define an open cover of \(C_p\) by \(n + 1\) elements all contractible in \(C_p\). Hence \(\text{cat}(C_p) - 1 \leq sg(p)\).
Suppose that $\infty \cdot \text{cat}(C_p \setminus \{\ast\}) \leq n$. Let $V_1, \ldots, V_n$ be a closed cover of $C_p \setminus \{\ast\}$ by sets that can be rel $\infty$ contracted to infinity. Let $H_i : V_i \times I \to C_p \setminus \{\ast\}$ be a contraction such that $H_i(V_i \times 1) \subset \text{Con}(X) \setminus \{\ast\} \subset C_p \setminus \{\ast\}$.

We define $F_i = V_i \cap Y \subset C_p$. Let $\pi : \text{Con}(X) \setminus \{\ast\} \to X$ be the projection. By the Homotopy Lifting Property, the homotopy $p \circ H_i|_{F_i \times [0,1]} : F_i \times [0,1] \to Y$ has a lift $H'_i : F_i \times [0,1] \to X$ which coincides with $\pi \circ H_i$ on $F_i \times 1$. Then $H'_i$ restricted to $F_i \times 0$ is a section of $p$ over $F_i$. Thus, $\text{sg}(p) \leq \infty \cdot \text{cat}(C_p \setminus \{\ast\})$. □

The following example shows that neither of the two inequalities of Theorem 4.7 can be improved.

4.8. Example. (1) For the identity map $1_X : X \to X$ in view of the equality $C_{1_X} = \text{Con}(X)$ we obtain:
\[
\text{cat}(C_{1_X}) - 1 = 0 < \text{sg}(1_X) = 1 = \text{cat}(\text{Con}(X)) = \infty \cdot \text{cat}(C_{1_X} \setminus \{\ast\}).
\]

For the square map $p : S^1 \to S^1$, $p(z) = z^2$,
\[
\text{cat}(C_p) - 1 = 2 = \text{sg}(p) < 3 = \text{cat}(C_p) \leq \infty \cdot \text{cat}(C_p \setminus \{\ast\}),
\]

since $C_p = \R P^2$ and $\text{cat}(\R P^2) = 3$.

5. On the Arnold-Kuiper theorem

5.1. Theorem. The non-reduced Lusternik-Schnirelmann category of the orbit space $\C P^2 / \Z_2$ of the action of $\Z_2$ on the complex projective plane $\C P^2$ by the conjugation is 2,
\[
\text{cat}(\C P^2 / \Z_2) = 2.
\]

5.2. Corollary (Arnold, Kuiper). The orbit space $\C P^2 / \Z_2$ of the action of $\Z_2$ on the complex projective plane $\C P^2$ by the conjugation is a 4-sphere.

Proof. Clearly, the fixed point set of this action is a real projective plane
\[
\R P^2 = \{[a:b:c] | a, b, c \in \R, |a| + |b| + |c| \neq 0\} \subset \{[a:b:c] | a, b, c \in \C, |a| + |b| + |c| \neq 0\} = \C P^2.
\]

Moreover, the action preserves the normal bundle to $\R P^2$. Therefore, the orbit space $\C P^2 / \Z_2$ is a 4-manifold. A closed $n$-manifold of the category 2 is homotopy equivalent to the $n$-sphere (see [CLOT]). Then by Freedman’s theorem [Fr], $\C P^2 / \Z_2$ is homeomorphic to the 4-sphere. □
5.3. Remark. We note that Arnold and Kuiper proved a diffeomorphism theorem. Since the smooth 4-dimensional Poincare conjecture is still a conjecture, here we can provide only a homeomorphism.

We identify the 2-sphere \( S^2 \) with the one-point compactification \( \mathbb{C} \cup \infty \) of the complex plane. Then \( \mathbb{Z}_2 \)-action on \( \mathbb{C} \) by the conjugation extends to an action on \( S^2 \). Clearly, a \( \mathbb{Z}_2 \)-action on \( S^2 \) extends to an action on the symmetric \( n \)th power \( SP^n(S^2) \) of \( S^2 \). We recall that \( SP^nX = X^n/\Sigma_n \) is the orbit space on the \( n \)th power \( X^n \) under the action of the symmetric group \( \Sigma_n \) by permutation of coordinates.

5.4. Proposition. There is a \( \mathbb{Z}_2 \)-equivariant homeomorphism between complex projective space \( \mathbb{C}P^2 \) and the symmetric square \( SP^2(S^2) \).

Proof. The points \([a : b : c] \in \mathbb{C}P^2\) are in bijection with non-degenerate quadratics \( ax^2 + bxy + cy^2 \). Any factorization of this quadratic

\[
ax^2 + bxy + cy^2 = (a_1x + b_1y)(a_2x + b_2y)
\]

defines the same non-ordered (perhaps repeated) pairs of points \( \frac{a_1}{b_1}, \frac{a_2}{b_2} \in \mathbb{C} \cup \infty = S^2 \).

Note that the non-degeneration condition \(|a| + |b| + |c| \neq 0\) implies that \( a_i \) and \( b_i \) cannot be all equal zero for \( i = 1, 2 \). Also we use the standard convention \( \frac{z}{0} = \infty \) for any \( z \in \mathbb{C} \).

This correspondence is the required homeomorphism.

5.5. Remark. The above proposition is an equivariant version of the well-known fact: \( \mathbb{C}P^n \cong SP^n(S^2) \).

Proof of Theorem 5.1. We present \( M = SP^2(S^2)/\mathbb{Z}_2 = F \cup U \) as a union of two contractible sets one closed and one open. Note that the set \( U = SP^2(\mathbb{C})/\mathbb{Z}_2 \) is open and contractible, since \( \mathbb{C} \) is contractible to a point equivariantly. The equator \( S^1 = \mathbb{R} \cup \infty \subset S^2 \) separates \( S^2 \) in two hemispheres \( D_- \) and \( D_+ \). We show that the complement \( F = M \setminus U \) admits a continuous bijection onto the closed upper hemisphere \( \bar{D}_+ \). Indeed, it consists of non-ordered pairs of pairs \( \{\infty, z\}, \{\infty, \bar{z}\} \) where \( z \in \bar{D}_+ \). This defines the bijection which is clearly continuous. Since \( F \) is compact, it is homeomorphic to \( \bar{D}_+ \) and hence is contractible. Since \( F \) is an absolute retract and \( M \) is absolute neighborhood retract, there is an open neighborhood \( V \) of \( F \) in \( M \) that contracts to \( F \) in \( M \) and, hence, to a point. Thus, \( M \) is covered by two open sets \( U \) and \( V \), both contractible in \( M \).

\[\square\]
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