M-Theory and Topological Strings–I

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The $R^2 F^{2g-2}$ terms of Type IIA strings on Calabi-Yau 3-folds, which are given by the corresponding topological string amplitudes (a worldsheet instanton sum for all genera), are shown to have a simple M-theory interpretation. In particular, a Schwinger one-loop computation in M-theory with wrapped M2 branes and Kaluza-Klein modes going around the loop reproduces the all genus string contributions from constant maps and worldsheet instanton corrections. In the simplest case of an isolated M2 brane with the topology of the sphere, we obtain the contributions of small worldsheet instantons (sphere “bubblings”) which extends the results known or conjectured for low genera. Surprisingly, the ’t Hooft expansion of large $N$ Chern-Simons theory on $S^3$ can also be used in a novel way to compute these gravitational terms at least in special cases.

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1. Introduction

The relationship between eleven dimensional M-theory and type IIA string theory
has been a fruitful source of insight since its discovery. Specifically, since fundamental
strings and D-branes of type IIA are on a more unified footing in eleven dimensions, we
can often say something non-perturbative about string theory from simple considerations
in M-theory.

In this paper we concentrate on Type IIA strings on Calabi-Yau threefolds. The full
IIA theory compactified on a Calabi-Yau threefold contains terms in the low energy action
which are captured by the simpler $N = 2$ topological string theory. In this paper, we will
reinterpret the worldsheet instanton sum of the latter in terms of a one loop computation
in M-theory. This generalizes some of the observations made in [1] about obtaining the
genus 0 topological amplitudes (the prepotential) from M-theory on a Calabi-Yau times a
circle. It is also similar in spirit to the observations in [2] relating $R^4$ amplitudes of type
IIA to Kaluza-Klein contributions in M-theory.

The topological string theory computes certain F-terms in the low energy four dimen-
sional theory. We briefly review this in Sec.2. Since some of these quantities are coupling
constant independent, they can equally well be computed at strong coupling. M-theory
provides a simple, physical way of doing this. For a large Calabi-Yau, in this limit, the
lightest relevant objects are Kaluza-Klein modes (D0 branes). Integrating them out in a
Schwinger type computation gives us precisely the leading contribution at every genus from
constant maps (the whole worldsheat mapped to a point) in the topological string theory.
The next to leading contributions come from M2 branes wrapped on Riemann surfaces in
the Calabi-Yau. In this paper we will treat the simplest case when the surface is an isolated
$S^2$ in the Calabi-Yau. There can be additional Kaluza-Klein modes here too, corres-
ponding to bound states of $D2 - D0$ branes. Again, we can sum their one loop contributions
exactly. What we obtain is usually interpreted in the string language as contributions from
small worldsheet instantons. The one loop Schwinger computation has non-perturbative
information as well, allowing us to compute non-perturbative corrections to topological
string amplitudes. In fact, we can add together the perturbative and non-perturbative
contributions and write the full partition function for the topological string in a suggestive
way. All this will be the subject of Sec.3. In Sec.4, we consider open topological strings on

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3 The higher genus M2 branes (as well as families of them) will be treated in a subsequent
paper [3].
$T^*S^3$ with dirichlet boundary conditions on the $S^3$. We can compute the open topological string amplitudes using the large $N$ Chern-Simons theory, to which it is equivalent \[4\]. By extrapolating the expressions for $h$ boundaries, to the case $h = 0$, we surprisingly recover the leading topological closed string amplitude. This is a novel “$h \to 0$” way of recovering a gravity result from (open string) gauge theory. The open string expansion itself seems to also reproduce the subleading contribution from 2-branes on an $S^2$, though we do not have a deep understanding of this aspect of it.

After we had obtained the results of the present paper, we received a paper \[5\] which has some overlap with ours, but uses very different arguments.

2. Topological Strings and What They Compute

Consider type IIA strings compactified on a Calabi-Yau threefold to 4 dimensions. We obtain an $N = 2$ supersymmetric theory, with vector and hypermultiplet fields. The scalars in these multiplets are moduli belonging to the Kahler moduli space and (a jacobian variety over) the complex moduli space respectively. For type IIB strings it is much the same, except that the role of Kahler and complex moduli are exchanged. There are a special class of F-terms which do not mix the two types of moduli.

For example, in the IIA theory the F-terms of the low energy theory involving only Kahler moduli are of the form $F_g R^2 + F^2_{g} - 2$. Here, + denotes the self-dual parts of the curvature, $R$ denotes the Riemann tensor and $F$ the graviphoton field strength. (We will be considering the euclidean effective action.) \[6\] The coefficient, $F_g$ is purely a function of Kahler moduli. Rather remarkably, the $F_g$ can be computed using a much simpler string theory, an $N = 2$ topological string, with the Calabi-Yau threefold as its target space \[6\]. In fact, $F_g$ is the partition function of the perturbative A-model topological closed string theory at genus $g$.

One can show that the above superpotential terms receive contributions only from genus $g$ amplitudes in the physical string theory. Moreover, there are no non-perturbative string corrections. (The dilaton lies in a hypermultiplet and cannot mix with the vector multiplet moduli – the same statement would therefore not be true for superpotential terms involving hypermultiplets). Thus, the partition function of topological strings computes certain exact quantities in the full, physical string theory.

\[4\] The contractions of indices here are determined from the fact that this term comes from $\int d^4 \theta W^{2g}$ in superspace. Here $W$ is a Weyl superfield, $W_{\mu\nu} = F_{\mu\nu}^+ - R_{\mu\nu\lambda\rho}^+ \theta \sigma^{\lambda\rho} + \ldots$. 

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In the limit of large volume (radius) of the Calabi-Yau three-fold, the \( F_g \) admits a purely topological interpretation: It is roughly given by the worldsheet instanton sum

\[
F_g = \sum_{C_g} \exp(-A_C).
\] (2.1)

The sum is over Riemann surfaces (holomorphic curves) \( C_g \) of genus \( g \) embedded in the Calabi-Yau threefold – the target space images of the worldsheet. And \( A_C \) denotes the complex area (including the imaginary contribution from the B-field) of \( C \). This, however, is not the full story: Sometimes, there are whole families of holomorphic curves embedded in the three-fold. The sum is now really an integral and one has to weight it appropriately. The integral is over an appropriate moduli space and the relevant object that enters as the weight turns out to be an appropriate cohomology class over this moduli space. In addition, there can be further contributions from degenerate genus \( g \) curves. In fact, the leading large radius contribution to (2.1) comes from constant maps where the whole genus \( g \) worldsheet is mapped to a point in the Calabi-Yau. (This is the leading contribution in the large volume limit, since it is not suppressed by the area exponent.) In this case, the relevant moduli space is the product, \( \mathcal{M}_g \) times the Calabi-Yau threefold \( K \) itself (corresponding to the choice of the point in the target space). \( \mathcal{M}_g \) is the familiar moduli space of all Riemann surfaces of genus \( g \). Then the appropriately weighted contribution to \( F_g \) from constant maps (which, as we remarked, is the leading contribution for large volumes), was found by [6] to be

\[
F_g = \frac{1}{2} \chi_K \int_{\mathcal{M}_g} c_{g-1}^3 + O(\exp(-A))
\] (2.2)

Here \( \chi_K \) denotes the Euler characteristic of \( K \) and \( c_{g-1} \) denotes the \((g-1)\)-th chern class of the Hodge bundle over \( \mathcal{M}_g \) (The Hodge bundle is the \( g \)-dimensional holomorphic vector bundle over \( \mathcal{M}_g \) locally spanned by the \( g \) holomorphic 1-forms on the genus \( g \) Riemann surface \( g > 1 \)). \( \int_{\mathcal{M}_g} c_{g-1}^3 \) has only very recently been computed by mathematicians to be [8]

\[
\int_{\mathcal{M}_g} c_{g-1}^3 = \frac{B_g}{2g(2g-2)} \frac{B_{g-1}}{(2g-2)!} = (-1)^{g-1} \chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}}.
\] (2.3)

Here \( \chi_g = (-1)^{g-1} \frac{B_g}{2g(2g-2)} \) is the euler characteristic of \( \mathcal{M}_g \). \( (B_g \) are the Bernoulli numbers taken here to be all positive.) The second way of writing this expression in terms of \( \chi_g \) and the Riemann zeta function \( \zeta \) was motivated by the physical derivation we shall present in this paper for this leading contribution.
There are further kinds of degenerate contributions, known as “bubblings”. The simplest instance of this is when all of a genus $g$ worldsheet, except the infinitesimal neighborhood of a single point, is mapped to a point on an $S^2$ in the Calabi-Yau, while this neighborhood itself wraps the rest of the $S^2$. From the two-dimensional worldsheet point of view, this is what is sometimes called a “small” instanton. The general case similarly involves a genus $g$ curve which has a lower genus part which is mapped non-trivially to a genus $g' (g > g')$ curve in the Calabi-Yau, while the rest of the surface is mapped to a point. These bubblings were first encountered in computing $F_1$ which has a contribution from a sphere bubbling off of a torus [9] and was explained by Katz (see the appendix of [9] by Katz). Subsequently in [6], the contribution of genus 0 and genus 1 bubbling off of a genus 2 surface was conjectured (the genus 1 contribution was actually conjectured to be zero). Since the computation of higher genus $F_g$ for $g > 2$ has not been carried out, the structure of degenerate contributions has not even been conjectured. We will see that the one loop amplitude in M-theory involving Kaluza-Klein states of massless fields as well as Kaluza-Klein modes of wrapped M2 branes reproduces and extends these topological string amplitudes to all perturbative orders as well as imply further non-perturbative corrections. In other words, all these contributions to the $R^2F^{2g-2}$ amplitudes in the IIA theory can be alternatively viewed as coming from integrating out wrapped brane states.

3. M-theory and IIA amplitudes

M-theory compactified on a circle is conjectured to be equivalent to type IIA strings [10], where Kaluza-Klein modes of the circle become equivalent to bound states of D0 branes. In the limit of a small $S^1$ in the eleventh dimension, we recover the perturbative regime of type IIA strings. Again, the M2 branes of M-theory reduce to fundamental strings or D2 branes of type IIA depending on whether or not they wrap around the $S^1$. Thus, M-theory on a large radius Calabi-Yau threefold times a small $S^1$ is equivalent to perturbative type IIA strings on the same large Calabi-Yau. In particular, it is in this limit that the F-terms of Sec. 2 are given by a worldsheet instanton sum. Now consider the limit where the circle grows, corresponding to increasing the type IIA coupling constant. Since the superpotential computations (depending on Kahler moduli) are exact they are unmodified in this limit. But now, we have the perspective of M-theory and we can ask how the same computation would look from this vantage point. This is the basic strategy we follow in the next subsection.
3.1. \textit{M-theory} Computation of Topological String Amplitudes

In the limit of large string coupling of the IIA theory on a big Calabi-Yau threefold, the light objects from the IIA viewpoint are D0 branes and their bound states. As mentioned above, these are the Kaluza-Klein modes of massless fields of M-theory on the Calabi-Yau threefold.

First, let us determine from the type IIA perspective what the precise number of D0 brane bound states are, in this geometry. On $\mathbb{R}^{10}$ we have one bound state of $n$ D0-branes for each $n$. The D0-branes are point-like objects on the Calabi-Yau threefold times $\mathbb{R}^3$. Therefore the quantum mechanical supersymmetric ground states are in one to one correspondence to the compact cohomology elements of the Calabi-Yau (times a hypermultiplet). Thus for each cohomology element and each integer $n$ we obtain a state in type IIA with mass $\frac{2\pi n}{\lambda}$ where $\lambda$ is the string coupling. (It will be convenient in the following to work in units where $\alpha' = \frac{1}{4\pi}$. ) These are exactly the same as the $S^1$ Kaluza-Klein massless modes of M-theory compactified on the Calabi-Yau. Since they are charged under the $U(1)$ graviphoton, they will have a one loop contribution to $R^2 F^{2g-2}$, which we will now compute.

Let us consider the contribution of one 4 dimensional $N = 2$ hypermultiplet to $F_g$. Let $Z$ denote the central terms in the supersymmetry algebra for this hypermultiplet, where the mass $m = |Z|$. Then following the argument in [11] and the direct “Schwinger-type” computation of [12] one deduces that the contribution of this hypermultiplet to $F_g$ ($g > 1$ for most of the expressions in this section, though most statements hold more generally after an appropriate continuation) is

$$F_g = -\chi_g Z^{2-2g} \quad (3.1)$$

where $\chi_g = \chi(\mathcal{M}_g)$ as before. After taking account of all the multiplets for a fixed Kaluza-Klein momentum around the circle, the net contribution turns out to be $h^{2,1}(K) - h^{1,1}(K) = -\chi(K)/2$ times the contribution of a single hypermultiplet. Using (3.1), with $Z = \frac{2\pi in}{\lambda}$ and summing over the contribution of all Kaluza-Klein modes we find (for the coefficient of $\lambda^{2g-2}$)

$$F_g = \chi_g \frac{\chi K}{2} \sum_{n \in \mathbb{Z}, n \neq 0} (2\pi in)^{2-2g}$$

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5 This involves a somewhat subtle contribution from various massive vector, hyper and spin 2 multiplets, which will be explained in more detail in [3].
Comparing this with the expected large radius behaviour of topological amplitude in equation (2.2) gives us
\[ \int_{\mathcal{M}_g} c_{g-1}^3 = (-1)^{g-1} \chi_g \frac{2\zeta(2g-2)}{(2\pi)^{2g-2}}. \]
precisely agreeing with the recent mathematical results (2.3).

Actually, we can say more from M-theory. The Schwinger computation is famous for its prediction of non-perturbative pair creation. If we consider a constant self-dual graviphoton field, then in addition to the contribution to $R_+^2$ from the terms proportional to $F^{2g-2}$, there will be those from terms behaving like $e^{-\frac{1}{F}}$. The expression derived by Schwinger for the one-loop determinant captures both perturbative and non-perturbative parts. In our context, for a particle of BPS charge $Z$ in a constant self-dual field $F$, the expression is (see for example [13])
\[ F(Z) = \sum_{g} \mathcal{F}_g F^{2g-2} + O(e^{-\frac{Z}{F}}). \]
(3.2)

In our case with $Z = \frac{2\pi in}{\chi}$, we can easily carry out the sum over $n \in \mathbb{Z}, n \neq 0$, using
\[ \sum_{n=-\infty}^{\infty} e^{in\theta} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\theta - 2\pi m). \]
Also taking into account the factor of $\frac{\chi K}{2}$, we have the complete zero brane contribution to be
\[ F(Z) = \frac{\chi K}{8} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{(\sinh \frac{m\lambda F}{2})^2}, \]
after dropping an irrelevant field independent term. This sum can be more suggestively written after expanding out the denominator and carrying out the sum over $m$, as
\[ F(Z) = -\frac{\chi K}{2} \sum_{n=1}^{\infty} n \ln(1 - q^n); \quad q = e^{-\lambda F}. \]

Let us now consider the contribution of isolated M2 branes with the topology of $S^2$. This gives a hupermultiplet in 5 dimensions. Let $A$ denote the area of the sphere. Consider the Kaluza-Klein modes of these states (which in the type IIA description correspond to bound states of a single D2 brane with a number of D0 branes). Then the central term
in the susy algebra of the state with $n$ units of Kaluza-Klein momentum is given by $Z = 2\pi \frac{(A + in)}{\lambda}$ and so using (3.1) we get its contribution to $F_g$ to be

$$\frac{\chi_g}{(2\pi)^{2g-2}} \sum_{n \in \mathbb{Z}} (A + in)^{(2-2g)} = \frac{-\chi_g}{(2\pi)^{2g-2}(2g - 3)!} \left( \frac{d}{dA} \right)^{2g-2} \ln(1 - e^{-2\pi A})$$

$$= \chi_g \frac{1}{(2g - 3)!} \sum_n n^{2g-3} e^{-2\pi nA}. \quad (3.3)$$

(Here we have used the product formula $\frac{\sinh(\pi x)}{\pi x} = \prod_{n=1}^{\infty} (1 + \frac{x^2}{n^2}).$) This agrees, for $g = 1, 2$, with the contribution from degenerate genus zero instantons $[9][6]$ and generalizes it to spherical bubbling contributions to an arbitrary genus topological amplitude $[5].$

Once again, the Schwinger computation gives us non-perturbative information, now about pair creation of bound states of 2-branes with 0-branes in a graviphoton field. With $Z = 2\pi \frac{(A + in)}{\lambda}$, we can go over steps similar to before to obtain

$$\mathcal{F}(Z) = \frac{1}{4} \sum_{m=1}^{\infty} \frac{e^{-2\pi mA}}{m} \frac{1}{\left( \sinh \frac{mF}{2} \right)^2}$$

or alternatively

$$\mathcal{F}(Z) = -\sum_{n=1}^{\infty} n \ln(1 - z^n); \quad z = e^{-2\pi A}.$$  

Thus the combined contributions to the free energy can be written as

$$\mathcal{F}_{tot}(Z) = -\sum_{n=1,[n_i]}^{\infty} n \ln[1 - q^n] \frac{\Delta}{\pi} (1 - q^n \prod_i a_i n_i^{d_i[i]}],$$

where we are including the contribution of all isolated genus 0 M2 branes, assuming there are $d_{[n_i]}$ of them in a homology class labelled by $[n_i]$ and $-\log z_i$ denotes the area of the corresponding class. These formulae are very much reminiscent of some observations in $[13].$

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6 For the genus 0 and 1 the above expression should be interpreted in the regularized form $\chi_g/(2g - 3)! = 1, -1/12$ respectively.

7 One may wonder about the existence of bound states of multiple D2 branes wrapped over $S^2$, but using the methods in $[14]$ one can show that there are none. This is not the case for multiple D2 branes wrapped on higher genus surfaces and they do lead to contributions to topological amplitudes $[3].$
Alternatively, with $\lambda F \rightarrow \lambda$, this can be thought of as a portion of the non-perturbative topological string partition function. We also recall that a similar Schwinger computation for wrapped D3 branes had yielded non-perturbative contributions to the full B-model topological string in the vicinity of a conifold [16]. In that case, a connection with Chern-Simons theory had also been made. In the next section we will see that a connection to Chern-Simons theory exists for our A-model closed string amplitudes as well.

4. Chern-Simons theory in the ’t Hooft Expansion

Chern-Simons theory enjoys the advantage of being an exactly solvable gauge theory [17]. Therefore its large $N$ expansion offers an instance where one can study the correspondence to string theory explicitly. Periwal [18] has studied the $N \rightarrow \infty$ limit of this theory on $S^3$. In [16] it was shown that the free energy of the gauge theory in the $N = \infty$ limit matched that of the $B$-model topological closed string in the vicinity of the conifold (and some generalizations). The gauge theory, remarkably enough, seems to capture both the perturbative and non-perturbative features of the closed string theory.

But here, we will be concerned with the alternative interpretation by Witten of Chern-Simons theory as an open string theory. This goes as follows. In any large $N$ gauge theory with adjoint fields only, the ’t Hooft expansion for, say, the free energy can be written as

$$F = \sum_{g=0, h=1} C_{g,h} N^h \kappa^{2g-2+h} = \sum_{g=0, h=1} C_{g,h} N^{2-2g} \lambda^{2g-2+h}$$

(4.1)

Here $\kappa$ is the ordinary gauge coupling while $\lambda = \kappa N$ is the ’t Hooft coupling which is held fixed in the large $N$ limit. As usual $g$ is the genus of the Riemann surface built out of double index lines. While $h$ is the number of closed index loops (or faces of the triangulated Riemann surface) which accounts for the factor of $N^h$. It is not to be confused with the holes that appear in the presence of matter in the fundamental.

What Witten argued in the particular case of Chern-Simons theory on a three manifold $M$ was that $C_{g,h} = -Z_{g,h}$, where $Z_{g,h}$ is the partition function of the A-model topological open string theory at genus $g$ with $h$ boundaries on the 6-dimensional target space $T^*M$. Thus the ’t Hooft expansion tells us about the complete perturbative expansion of an open string theory.

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8 For another possible connection between Chern-Simons theory and type IIA string theory instantons see [19].
In the case when \( M = S^3 \), the coefficients \( C_{g,h} \) for the \( SU(N) \) level \( k \) theory are not too difficult to compute explicitly (see the appendix). The main subtlety to keep in mind is the fact that the bare Chern-Simons coupling undergoes a finite quantum renormalization. Therefore, the ’t Hooft expansion is performed with \( \lambda = \frac{2\pi N}{k+N} \). (To connect the ’t Hooft expansion in the appendix with the open string expansion, we must take \( \lambda \to i\lambda \), due to the \( i \) that is present in the definition of the Chern-Simons action. This introduces the extra factor of \((-1)^{g-1+p}\) in the answer below.) The final answer then is fairly simple:

\[
C_{g,2p} = (-1)^{g-1+p} \chi_{g,2p} \frac{\zeta(2g - 2 + 2p)}{(2\pi)^{2g-2+2p}}, \quad C_{g,2p+1} = 0
\]  

(4.2)

where

\[
\chi_{g,h} = (-1)^h \left( \frac{2g - 3 + h}{h} \right) \chi_g = (-1)^h \left( \frac{2g - 3 + h}{h} \right) (-1)^{g-1} \frac{B_g}{2g(2g-2)}
\]  

(4.3)

is the Euler characteristic of the moduli space of Riemann surfaces with genus \( g \) and \( h \) punctures (see [2[4] for a connection to the \( c = 1 \) string). Note that the ’t Hooft expansion starts off with at least one hole (actually in this case all the amplitudes with an odd number of holes vanish and the first term is one with 2 holes). Let us consider \( C_{g,2p} \) as an analytic function in \( h = 2p \). The case with \( h = 0 \) would have corresponded to closed topological strings. In this limit we should get

\[
Z_{g,h \to 0} \to \frac{\chi(T^*S^3)}{2} \int_{M_g} c^3_{g-1}.
\]

If we take \( \chi(T^*S^3) = -2 \) (based on the fact that it is non-compact and that there is one complex deformation) we obtain

\[
\int_{M_g} c^3_{g-1} = C_{g,h \to 0} = (-1)^{g-1} \chi_g \frac{\zeta(2g - 2)}{(2\pi)^{2g-2}}
\]

in agreement with the mathematical result Eq.(2.3)!

But can we also physically interpret the open string expansion \((h \neq 0)\) in terms of closed strings? *A priori*, we do not have any reason to anticipate any relation in this case. However, let us add the “closed string sector” \((p = 0)\) to the Chern-Simons free energy and weigh terms with 2\(p \) holes with a factor of \((2\pi A)^{2p}\). In other words, for \( g > 1 \), we write the genus \( g \) contribution to the free energy as (remembering also the factor of \((-1)^{g-1+p}\))

\[
F_g = \frac{(-1)^{g-1}B_g}{2g(2g-2)} \sum_{n=1}^{\infty} \left( \frac{\lambda}{2\pi n \zeta(2g-2)} \right)^n \sum_{p=0}^{g-1+p} (-1)^{g-1+p} \frac{1}{n^{2p}} \left( \frac{2g - 3 + 2p}{2p} \right) (2\pi A)^{2p}
\]
or on performing the sum over $p$, 

$$F_g = \frac{(-1)^{g-1} B_g}{2g(2g - 2)} \frac{\lambda^{2g-2}}{(2\pi)^{2g-2}} \sum_{n=1}^{\infty} \left[ \frac{1}{(A + in)^{2g-2}} + \frac{1}{(A - in)^{2g-2}} \right].$$

This is what we had seen in the previous section as the contribution to $F_g$ from membranes wrapped on $S^2$, with KK excitations of momentum $n$. It remains an open question to interpret this result. It is conceivable that the conifold transition from $S^3$ to $S^2$, which is geometrically understood, is relevant here.

5. Conclusions

In this paper we showed how the BPS content of type IIA strings on a Calabi-Yau threefold involving wrapped $D2$ branes and its bound states with $D0$ branes (or equivalently $M2$ branes with Kaluza-Klein momenta) can be used, through a one loop computation to reproduce all the $R^2 F^{2g-2}$ amplitudes that topological strings compute. In particular, a given BPS object affects all genus $g$ topological amplitudes in a simple and computable way. In this paper, we exclusively dealt with the case of plain $D0$ branes or their bound states with $D2$ branes with the topology of (an isolated) $S^2$ in the Calabi-Yau. This in particular allowed us to compute non-trivial quantities over the moduli space of genus $g$ curves, as well as the contribution of small worldsheet instantons in a physical way. The fact that they agree with known/conjectured results can be viewed as further confirmation of properties of D-branes in type IIA context, or equivalently as further evidence (or applications) of M-theory–type IIA duality. We also expect that the non-perturbative form of the topological string partition function that we arrived at, will yield future insights.

In a subsequent paper [3] we extend these results to the case where the $D2$ branes have higher genus (including also wrapped multi $D2$ branes on continuous families of curves.) The M-theory description in conjunction with the type IIA description provides a powerful way to compute these contributions, and thus allows one to reformulate the entire topological string amplitudes in terms of its BPS content.

It is natural to expect these results to also extend to the $N = 4$ topological strings [21]. In particular the $R^4 F^{4g-4}$ amplitudes of type IIA string in $R^{10}$ should have a similar $D0$ brane interpretation [22]. The same should be true for its compactification on tori [23] or $K3$. It would be extremely interesting to relate the BPS content of these theories with the topological $N = 4$ amplitudes.
One possible application of our results is to questions involving 4 dimensional black holes. In particular the F-terms we have computed correct the $R^2$ term in the effective 4 dimensional action, in the presence of a non-vanishing graviphoton field strength. Noting that the graviphoton field strength is non-vanishing in a black hole background we conclude that there are $R^2$ corrections which would affect the black hole geometry.

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Appendix

Here we explicitly perform the ’t Hooft expansion of the Chern-Simons free energy on $S^3$. The theory is defined on a three manifold $M$ as

$$Z[M, N, k] = \int [DA] \exp \left[ \frac{ik}{4\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right].$$

On the sphere with gauge group $SU(N)$, the exact partition function is

$$Z[S^3, N, k] = e^{i\pi N(N-1)} \frac{1}{(N + k)^{N/2}} \sqrt{\frac{N + k}{N}} \prod_{j=1}^{N-1} \{2 \sin \left( \frac{j\pi}{N + k} \right) \}^{N-j}. \quad (5.1)$$

We’d like to examine the $\frac{1}{N}$ expansion of the free energy in the form Eqn. ’t Hooft. The bare ’t Hooft coupling in the Chern-Simons theory is $\lambda_b = \frac{2\pi N}{k}$. But it is known that there is a finite renormalisation of the coupling such that $k \to k + N$. In other words, the renormalised ’t Hooft coupling $\lambda = \frac{2\pi N}{k + N}$. This is the parameter we should be expanding in a perturbative (large $k$ ) expansion.

Therefore, as in [18],

$$F(S^3, N, \lambda) = -\frac{N}{2} \ln N + \frac{N - 1}{2} \ln \lambda + \sum_{j=1}^{N-1} (N - j) [\ln j + \ln \left( \frac{\lambda}{2\pi N} \right) + \sum_{n=1} l_n (1 - \frac{j^2 \lambda^2}{4\pi^2 n^2 N^2})]$$

(Using $\sin(\pi x) = \pi x \prod_{n=1} (1 - \frac{x^2}{\pi^2})$) We will focus on the last term in the sum, i.e.

$$\tilde{F} \equiv \sum_{j=1}^{N-1} (N - j) \sum_{n=1} \ln \left( 1 - \frac{j^2 \lambda^2}{4\pi^2 n^2 N^2} \right).$$
(The term in the $F(S^3, N, \lambda)$ given by $\sum_{j=1}^{N-1} (N - j) \ln j = \ln G(N + 1)$, where the G-function is defined recursively as $G(z + 1) = \Gamma(z) G(z)$. This term is independent of $\lambda$ and will not play a role in the following, though the term related to it by $N, k$ duality is the one that survives in the $N = \infty$ limit and is related to IIB topological amplitudes [16].)

Expanding the logarithm in the second term and carrying out the sum over $n$ gives

$$\sum_{m=1}^{\infty} \frac{\zeta(2m)}{m} \left( \frac{\lambda}{2\pi N} \right)^{2m} \sum_{j=1}^{N-j} (N - j) j^{2m}.$$  

The sum over $j$ can be carried out so that the final answer is

$$\sum_{j=1}^{N-j} (N - j) j^{2m} = \frac{N^{2m+2}}{(2m+1)(2m+2)} + 2m \sum_{g=1}^{m} \left( \frac{2m-1}{2g-3} \right) (-1)^g \frac{B_g}{2g(2g-2)} N^{2m+2-2g}.$$  

Then making the replacement for $g > 0$, $(m = g - 1 + p)$, the relevant part of the free energy then reads as

$$\tilde{F} = \sum_{p=2}^{\infty} N^2 \frac{\zeta(2p-2)}{p-1} \frac{\lambda^{2p-2}}{2p(2p-1)} - \sum_{p=1}^{\infty} B_1 \frac{\zeta(2p)}{2p} \frac{\lambda^{2p}}{2\pi}^{2p}$$

$$+ \sum_{g=2}^{\infty} N^{2-2g} \frac{(-1)^g B_g}{2g(2g-2)} \sum_{p=1}^{\infty} \zeta(2g-2 + 2p) \left( \frac{2g-3 + 2p}{2p} \right) \frac{\lambda^{2g-2+2p}}{2\pi}.$$
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