MAXIMUM VIOLATION OF WIGNER INEQUALITY FOR TWO-SPIN ENTANGLED STATES WITH PARALLEL AND ANTIPARALLEL POLARIZATIONS

YAN GU, HAIFENG ZHANG, ZHIGANG SONG
Institute of Theoretical Physics and Department of Physics, State Key Laboratory of Quantum Optics and Quantum Optics Devices, Shanxi University, Taiyuan, Shanxi 030006, China

J. -Q. LIANG
Institute of Theoretical Physics and Department of Physics, State Key Laboratory of Quantum Optics and Quantum Optics Devices, Shanxi University, Taiyuan, Shanxi 030006, China
*jqliang@sxu.edu.cn

L. -F. WEI
State Key Laboratory of Optoelectronic Materials and Technologies, School of Physics and Engineering, Sun Yat-Sen University, Guangzhou 510275, China and Quantum Optoelectronics Laboratory, School of Physics and Technology, Southwest Jiaotong University, Chengdu 610031, China
(Received 4 March 2018; Revised 24 May 2018)

The experimental test of Bell’s inequality is mainly focused on Clauser-Horne-Shimony-Holt (CHSH) form, which provides a quantitative bound, while little attention has been paid on the violation of Wigner inequality (WI). Based on the spin coherent state quantum probability statistics we in the present paper extend the WI and its violation to arbitrary two-spin entangled states with antiparallel and parallel spin-polarizations. The local part of density operator gives rise to the WI while the violation is a direct result of non-local interference between two components of the entangled states. The Wigner measuring outcome correlation denoted by $W$ is always less than or at most equal to zero for the local realist model ($W_{lc} \leq 0$) regardless of the specific initial state. On the other hand the violation of WI is characterized by any positive value of $W$, which possesses a maximum violation bound $W_{max} = 1/2$. We conclude that the WI is equally convenient for the experimental test of violation by the quantum entanglement.

PACS numbers: 03.65.Ud; 03.65.Vf; 03.67.Bg; 42.50.Xa

Keywords: Wigner inequality; entanglement; non-locality; spin coherent state.

1. Introduction

The non-locality as one of the most striking characteristic of quantum mechanics does not have classical correspondence within our intuition of space and time in the classical field theory. Quantum entangled-state, which originally was introduced by Einstein-Podolsky-Rosen to question the completeness of quantum mechanics, has become a key concept of quantum information and computation. From a two-spin entangled state proposed by Bohm, Bell proved a quantitative criteria between the quantum and classical measuring-outcome correlations known as Bell’s inequality (BI). It was established by means of classical statistics with the assumption of hidden variable. The BI, which plays a fundamental role in quantum entanglement, has attracted great attentions both theoretically and experimentally. Bell nonlocality and quantum entanglement in two-qubit spin model are also measured by use of measurement induced disturbance and quantum discord. The experimental evidences confirming the violation of BI provides an overwhelming superiority for the non-locality in quantum mechanics against the proposition of local realism. In various modified forms of the BI, the Clauser-Horne-Shimony-Holt (CHSH) inequality is of particular interesting to the experimental test, since it provides a quantitative bound of the four-direction measuring-outcome correlation $P_{CHSH}^{lc} \leq 2$ for the local realist theory. The inequality can be violated by the two-spin entangled state with a maximum violation known as $P_{max}^{CHSH} = 2\sqrt{2}$.

Recently a quantum mechanical framework was presented to formulate the various forms of BI and their violation in a unified formalism. The density operator of a bipartite entangled-state is separated into the local and non-local (interference) parts. The measuring outcome correlation is then evaluated by the quantum probability statistics in the spin coherent-state base vectors along the measuring directions. The local part gives rise to the local-realistic correlation, which results in the BI, while the non-local one is responsible for the violation. For the arbitrary high
spins a spin-parity effect in the violation of BI is found as a result of Berry phase interference of spin coherent states.

The Wigner inequality (WI) is a simpler form, in which the particle number probability of positive spin is measured. We in the present work reformulate the WI and its violation for arbitrary two-spin entangled states with both antiparallel and parallel polarizations. The measuring outcome correlation is evaluated by the spin coherent-state quantum probability statistics. Although the WI is simple it attracts a little attention of experimenters. The reason may be that it lacks a quantitative bound, which is convenient for the experimental verification. Following CHSH we propose a Wigner correlation $W$, which is less than or at most equal to zero (equivalent to the WI) according to the local realism. The maximum violation of WI is also found for the arbitrary two-spin entangled states with both antiparallel and parallel polarizations.

2. Spin coherent-state quantum probability statistics for measuring outcome correlation

The original BI and the modified form CHSH inequality are derived based on classical statistics with hidden variable assumption. In previous publications the Bell-type inequalities and their violation are formulated in a unified manner by means of the spin coherent-state quantum probability statistics. The density operator of an entangled state for a bipartite system can be separated to the local (or classical) and non-local (or quantum coherent) parts. The former part gives rise to the local realist bound of measuring outcome correlation, namely the BIs, while the latter part leads to the violation of the inequalities. We begin with an arbitrary two-spin entangled state of antiparallel polarization in the bases $\hat{\sigma}_z |\pm\rangle = \pm |\pm\rangle$ that

$$|\psi\rangle = c_1 |+\rangle + c_2 |-\rangle,$$

where the normalized coefficients can be generally parameterized as $c_1 = e^{i\eta} \sin \xi$, $c_2 = e^{-i\eta} \cos \xi$. We assume that two spins are separated to a space-like distance when the entangled state is prepared. The density-operator $\hat{\rho}$ of entangled state can be divided into two parts

$$\hat{\rho} = \hat{\rho}_{lc} + \hat{\rho}_{nlc}.$$

The local part

$$\hat{\rho}_{lc} = \sin^2 \xi |+\rangle \langle +| + \cos^2 \xi |-\rangle \langle -|,$$

which is the classical two-particle probability-density operator, describes the individual spin of the bipartite system separated remotely. While what we called the non-local part

$$\hat{\rho}_{nlc} = \sin \xi \cos \xi (e^{2i\eta} |+\rangle \langle -| + e^{-2i\eta} |+\rangle \langle -|)$$

is the quantum coherence density-operator between two remote spins.

2.1 Spin measuring outcome correlation and violation of BI

The measurements of two spins are performed independently along two arbitrary directions, say $a$ and $b$. The measuring outcomes fall into the eigenvalues of projection spin-operators $\hat{\sigma} \cdot a$ and $\hat{\sigma} \cdot b$, i.e.

$$\hat{\sigma} \cdot a |\pm a\rangle = \pm |\pm a\rangle, \quad \hat{\sigma} \cdot b |\pm b\rangle = \pm |\pm b\rangle,$$

according to the quantum measurement theory. Solving the eigenvalue equation for each direction denoted by $r = a, b$, we have two orthogonal eigenstates given by

$$|+r\rangle = \cos \frac{\theta_r}{2} |+\rangle + \sin \frac{\theta_r}{2} e^{i\phi_r} |-\rangle,$n

$$|-r\rangle = \sin \frac{\theta_r}{2} |+\rangle - \cos \frac{\theta_r}{2} e^{i\phi_r} |-\rangle.$$

In the above solutions the general unit vector $r = (\sin \theta_r \cos \phi_r, \sin \theta_r \sin \phi_r, \cos \theta_r)$ is parameterized by the polar and azimuthal angles $\theta_r, \phi_r$ in the coordinate frame with $z$-axis along the direction of the initial spin-polarization. The
two orthogonal states $|\pm r\rangle$ are known as spin coherent states of north- and south- pole gauges. The eigenstate product of operators $\hat{\sigma} \cdot \hat{a}$ and $\hat{\sigma} \cdot \hat{b}$ forms an outcome-independent vector base for measuring two spins respectively along the $a$, $b$ directions. We label the four base vectors as

$$
|1\rangle = |+a, +b\rangle, |2\rangle = |+a, -b\rangle, |3\rangle = |-a, +b\rangle, |4\rangle = |-a, -b\rangle
$$

for the sake of simplicity. The measurement correlation operator is denoted by

$$
\hat{\Omega}(ab) = (\hat{\sigma} \cdot \hat{a})(\hat{\sigma} \cdot \hat{b}).
$$

The correlation probability is obtained as

$$
P(a, b) = Tr[\hat{\Omega}(a,b)\hat{\rho}] \tag{5}
$$

which can be also separated to local and non-local parts

$$
P(a, b) = P_{lc}(a, b) + P_{nlc}(a, b)
$$

with

$$
P_{lc}(a, b) = Tr[\hat{\Omega}(a,b)\hat{\rho}_{lc}]
$$

and

$$
P_{nlc}(a, b) = Tr[\hat{\Omega}(a,b)\hat{\rho}_{nlc}].
$$

In terms of the outcome-independent base vectors given by Eq.(4) we derive the well known measurement correlation for the local realist model

$$
P_{lc}(a, b) = \rho_{11}^{lc} - \rho_{22}^{lc} - \rho_{33}^{lc} + \rho_{44}^{lc} = -\cos \theta_a \cos \theta_b,
$$

which independent of the state parameters $\xi$, $\eta$ is valid for arbitrary normalized entangled states Eq.(1). The BI and CHSH inequalities are recovered with this correlation. The non-local part found as

$$
P_{nlc}(a, b) = 2\sin \xi \cos \xi \sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b + 2\eta)
$$

however depends on the specific states. The violation of BI is seen to be a direct result of the non-local correlation. Particularly when the initial entangled state is the two-spin singlet

$$
|\psi_s\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle),
$$

with the state parameters $\xi = (3\pi/4) \text{ mod } 2\pi$ and $\eta = 0 \text{ mod } 2\pi$, the total correlation $P(a, b)$ becomes a scaler product of the two unit vectors

$$
P(a, b) = -a \cdot b,
$$

from which the BI s are violated. A maximum violation value for the CHSH correlation is found as

$$
P_{CHSH}^{\text{max}} = |P(a, b) + P(a, c) + P(d, b) - P(d, c)| = 2\sqrt{2}.
$$

### 2.2 Particle-number correlation probability

The particle-number correlation probability is considered in the Wigner formalism instead of the spin measuring outcome correlation. For example

$$
N(+a, +b) = |\langle +a, +b|\psi\rangle|^2 = \langle +a, +b|\hat{\rho}| +a, +b \rangle = \rho_{11}
$$

(6)
describes the particle number correlation probability for two positive-spin particles respectively along $a$, $b$ directions. The WI can be recovered in terms of the quantum probability statistics with the particle number correlation $N_{lc}(+a,+b) = \rho_{11}^{lc}$ of local realist model. Correspondingly three more correlations are related to the elements of density operator by

$$N(+a,-b) = \rho_{22}, N(-a,+b) = \rho_{33}, N(-a,-b) = \rho_{44},$$

which are all positive quantities different from the spin measuring-outcome correlations.

3. Wigner inequality and upper-bound of violation for two-spin entangled state with antiparallel spin-polarization

We consider the two-spin entangled state with antiparallel spin-polarization in Eq.(1). The WI is given by

$$N_{lc}(+a,+b) \leq N_{lc}(+a,+c) + N_{lc}(+c,+b)$$

in which only the number probability of positive spin is assumed to be detected along all three directions ($a$, $b$, and $c$) for both particles. From the viewpoint of symmetry we extend the original inequality Eq.(8) to that including also the number probability of the negative spin. The particle number correlation-probability along two directions $a$, $b$ can be obtained in terms of Eq.(6) and Eqs.(7)

$$N(\pm a, \pm b) = N_{lc}(\pm a, \pm b) + N_{nlc}(\pm a, \pm b).$$

for the arbitrary entangled state Eq.(1) with two particles of both positive and negative spins respectively. Following the same procedure in the above section we have $N_{lc}(+a,+b) = \rho_{11}^{lc}$, $N_{lc}(-a,-b) = \rho_{44}^{lc}$. It is a simple algebra to find

$$N_{lc}(+a,+b) = \sin^2 \xi \cos^2 \frac{\theta_a}{2} \sin^2 \frac{\theta_b}{2} + \cos^2 \xi \sin^2 \frac{\theta_a}{2} \cos^2 \frac{\theta_b}{2}$$

$$N_{lc}(-a,-b) = \sin^2 \xi \sin^2 \frac{\theta_a}{2} \cos^2 \frac{\theta_b}{2} + \cos^2 \xi \cos^2 \frac{\theta_a}{2} \sin^2 \frac{\theta_b}{2}$$

which depend on the state parameter $\xi$. The correlation probabilities of two-direction measurements are different for the positive and negative spin particles. However, we are going to show an interesting fact that the WI itself is independent of the state parameter $\xi$. It is also the same no matter whether the positive or negative spin particles are measured.

To have a quantitative bound for the violation of WI, we following CHSH define a correlation probability for the three-direction measurement

$$W_{lc} = N_{lc}(\pm a, \pm b) - N_{lc}(\pm a, \pm c) - N_{lc}(\pm c, \pm b).$$

Then the original form of WI is equivalent to

$$0 \geq W_{lc}.$$ 

Substitution of the corresponding correlations $N_{lc}(\pm a, \pm b)$, $N_{lc}(\pm a, \pm c)$ and $N_{lc}(\pm c, \pm b)$ into $W_{lc}$ yields

$$W_{lc} = -\left( \cos^2 \frac{\theta_a}{2} - \cos^2 \frac{\theta_c}{2} \right) \cos^2 \frac{\theta_b}{2} - \cos^2 \frac{\theta_a}{2} \sin^2 \frac{\theta_b}{2}$$

for both positive and negative spins. From Eq.(11) we can verify after a simple algebra the inequality that

$$0 \geq - \sin^2 \frac{\theta_c}{2} \cos^2 \frac{\theta_a}{2} \geq W_{lc}.$$ 

Therefore the original form of WI is satisfied not only for detection of the positive but also the negative spin particle-numbers.
The non-local part of correlation is \( N_{nlc}(+a, +b) = \rho^{nlc}_{11}, N_{nlc}(-a, -b) = \rho^{nlc}_{44} \). Since \( \rho^{nlc}_{11} = \rho^{nlc}_{44} \), we have

\[
N_{nlc}(\pm a, \pm b) = \frac{1}{2} \sin \xi \cos \xi \sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b + 2\eta),
\]

which depends on the state parameters \( \xi, \eta \). Including the non-local part the three-direction correlation probability becomes

\[
W = N(\pm a, \pm b) - N(\pm a, \pm c) - N(\pm c, \pm b) = W_{lc} + W_{nlc},
\]

where the non-local part

\[
W_{nlc} = \frac{1}{4} \sin(2\xi) [\sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b + 2\eta) - \sin \theta_a \sin \theta_c \cos (\phi_a - \phi_c + 2\eta) - \sin \theta_c \sin \theta_b \cos (\phi_c - \phi_b + 2\eta)],
\]

depends also on the state parameters.

We now analyze the violation of WI by the non-local part of correlation \( W_{nlc} \). Since the polar angles are restricted by \( 0 \leq \theta < \pi \), the non-local probability of Eq. (13) obeys the following inequality

\[
W_{nlc} \leq \frac{1}{4} (\sin \theta_a \sin \theta_b + \sin \theta_a \sin \theta_c + \sin \theta_c \sin \theta_b).
\]

On other hand the local part Eq. (11) can be rewritten as

\[
W_{lc} = \frac{1}{4} (-1 - \cos \theta_a \cos \theta_b + \cos \theta_c \cos \theta_b + \cos \theta_a \cos \theta_c).
\]

Adding them together we obtain an inequality obeyed by the Wigner correlation that

\[
W \leq F(\theta_a, \theta_b, \theta_c),
\]

where

\[
F(\theta_a, \theta_b, \theta_c) = \frac{1}{4} [-1 - \cos (\theta_a + \theta_b) + \cos (\theta_c - \theta_b) + \cos (\theta_a - \theta_c)].
\]

Since the function \( F(\theta_a, \theta_b, \theta_c) \) can be greater than zero, the WI is then violated. It is easy to prove that

\[
F(\theta_a, \theta_b, \theta_c) \leq \frac{1}{2}.
\]

We thus derive a maximum violation bound

\[
W_{\text{max}} = \frac{1}{2},
\]

which is universal for arbitrary entangled state and any three-direction measurements.

As a matter of fact, for the state-parameter angles \( \xi = \pi/4 \mod 2\pi \) and \( \eta = 0 \mod 2\pi \), the two-spin entangled state becomes the two-spin triplet with the vanishing magnetic-eigenvalue \( m = 0 \) that

\[
|\psi_t\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle).
\]

Corresponding non-local part of measuring outcome probability becomes

\[
W_{nlc} = \frac{1}{4} [\sin \theta_a \sin \theta_b \cos (\phi_a - \phi_b) - \sin \theta_a \sin \theta_c \cos (\phi_a - \phi_c) - \sin \theta_c \sin \theta_b \cos (\phi_c - \phi_b)].
\]

If \( \theta_a = \theta_c = \theta_b = \pi/2, \phi_a = \phi_b = \pi, \phi_c = 0 \), namely \( a, b, c \) are perpendicular to the original spin polarization with \( a, b \) along \(-x\)-directions \( c \) along \( x\)-direction, the Wigner correlation reaches, in this case, the maximum violation bound, \( W_{\text{max}} = 1/2 \).
4. Parallel spin polarization

We now consider the two-spin entangled state with parallel polarization

$$|\psi_{pl}\rangle = c_1 |+, +\rangle + c_2 |-, -\rangle$$

in which two arbitrary coefficients are parameterized as before. The density operator $\hat{\rho}_{pl}$ is also separated to local part

$$\hat{\rho}_{lc}^{pl} = \sin^2 \xi |+, +\rangle \langle +, +| + \cos^2 \xi |-, -\rangle \langle -, -|$$

and the non-local part

$$\hat{\rho}_{nlc}^{pl} = \sin \xi \cos \xi (e^{2i\eta}|+, +\rangle \langle -, -| + e^{-2i\eta}|-, -\rangle \langle +, +|).$$

We find that particles with opposite spins have to be detected respectively for the two directions, namely

$$N_{pl}(\pm a, \mp b) = N_{lc}^{pl}(\pm a, \mp b) + N_{nlc}^{pl}(\pm a, \mp b).$$

The local part of particle-number correlation with positive-spin particle detected in $a$-direction and negative-spin in $b$-direction is evaluated as

$$N_{lc}^{pl}(+a, -b) = \left(\rho_{lc}^{pl}\right)_{22} = N_{lc}(+a, +b),$$

which equals exactly the correlation probability $N_{lc}(+a, +b)$ in Eq.(9) for the antiparallel case. While the local correlation-probability for $a$-direction negative and $b$-direction positive is

$$N_{lc}^{pl}(-a, +b) = \left(\rho_{lc}^{pl}\right)_{33} = N_{lc}(-a, +b),$$

which equals $N_{lc}(-a, -b)$ in Eq.(11) for the antiparallel case. Thus, the Wigner correlation probability for the entangled state with parallel spin-polarizations is the same as the antiparallel case given in Eq.(11)

$$N_{pl}(\pm a, \mp b) - N_{pl}(\pm a, \mp c) - N_{pl}(\pm c, \mp b) = W_{lc} \leq 0.$$ 

The validity of WI for parallel spin-polarizations is also verified in Appendix by means of classical statistics following the original work of Wigner.

The non-local parts, which result in the violation of WI, are evaluated from the density operator elements of entangled state with parallel spin-polarization, such that

$$N_{nlc}^{pl}(+a, -b) = \left(\rho_{nlc}^{pl}\right)_{22}, \quad N_{nlc}^{pl}(-a, +b) = \left(\rho_{nlc}^{pl}\right)_{33}.$$

We find that the interchange of detecting positive and negative spin particles in the two directions gives rise to the same result that

$$N_{nlc}^{pl}(\pm a, \mp b) = -\frac{1}{2} \sin \xi \cos \xi \sin \theta_a \sin \theta_b \cos (\phi_a + \phi_b + 2\eta).$$

The total non-local part for three-direction measurements is seen to be

$$W_{nlc}^{pl} = \frac{1}{4} \sin 2\xi \left[\sin \theta_a \sin \theta_b \cos (\phi_a + \phi_b + 2\eta) - \sin \theta_a \sin \theta_c \cos (\phi_a + \phi_c + 2\eta) - \sin \theta_c \sin \theta_b \cos (\phi_c + \phi_b + 2\eta)\right].$$

Following the same procedure of analyses as in the antiparallel case we again have the maximum violation bound $W_{max} = 1/2$. Thus we conclude that WI and its violation are universal for arbitrary two-spin entangled states with both antiparallel and parallel spin-polarizations. As an example we consider a particular entangled state

$$|\psi_{pl}\rangle = \frac{1}{\sqrt{2}} (|+, +\rangle + |-, -\rangle),$$
resulted by the parameter angles $\xi = \pi/4 \mod 2\pi$ and $\eta = 0 \mod 2\pi$. Corresponding non-local part of correlation probability in Eq. (17) becomes

$$W_{nlc}^{pl} = -\frac{1}{4}[\sin \theta_a \sin \theta_b \cos (\phi_a + \phi_b) - \sin \theta_a \sin \theta_c \cos (\phi_a + \phi_c)$$
$$- \sin \theta_c \sin \theta_b \cos (\phi_c + \phi_b)].$$

The maximum violation $W_{\text{max}} = 1/2$ can be approached when $\theta_a = \theta_c = \theta_b = \pi/2$, $\phi_a = \phi_b = \pi/2$, $\phi_c = 3\pi/2$, namely the three measuring directions are colinear with $\mathbf{a}$, $\mathbf{b}$ along $y$-direction and $\mathbf{c}$ along $-y$-direction in the chosen coordinate frame with initial spin-polarization in $z$-axis.

5. Conclusion and Discussion

By means of the spin coherent-state quantum probability statistics, the original WI is extended to arbitrary two-spin entangled states with antiparallel and parallel spin polarizations. For the antiparallel case both positive or both negative spin particles ought to be detected respectively in three directions. While opposite spin measurements are necessary for the parallel case. The Wigner correlation $W_{lc}$ is always less than or at most equal to zero by the local realist-theory. A maximum violation $W_{\text{max}} = 1/2$ is found for arbitrary two-spin entangled states with both parallel and antiparallel spin-polarizations. The measured violation-value depends on specific state, which is parameterized by parameters $\xi$ and $\eta$, and also on the measuring directions. The positive and negative spin particles can be detected by the Stern-Gerlach experiment with a gradient magnetic-field. A loophole-free experimental verification of the violation of CHSH inequality was reported recently by means of electronic spin associated with a single nitrogen-vacancy defect centre in a diamond chip.\(^{37}\) The experimental verification of the WI violation is also expected.

We conclude that the WI is equally convenient for the experimental verification of its violation. It may be more suitable for any bipartite systems besides the two spins since only particle number probabilities are needed for detection but not the spin variables. Although our formalism is based on two-spin entangled state, the result can be used for the two-photon entangled states with perpendicular polarizations.\(^{38}\)

### Acknowledge

This work was supported in part by National Natural Science Foundation of China, under Grants No. 11275118, U1330201.

### Appendix

The extended WI $N_{lc}(\pm a, \mp b) \leq N_{lc}(\pm a, \pm c) + N_{lc}(\pm c, \mp b)$ can be proved in terms of classical statistics for two-spin entangled state with parallel polarizations. Eight independent particle-number probabilities are denoted by\(^{39}\).

#### Table. Spin-correlation Measurements

| population | particle1 | particle2 |
|------------|-----------|-----------|
| $N_1$      | $+a, +b, +c$ | $+a, +b, +c$ |
| $N_2$      | $+a, +b, -c$ | $+a, +b, -c$ |
| $N_3$      | $+a, -b, +c$ | $+a, -b, +c$ |
| $N_4$      | $+a, -b, -c$ | $+a, -b, -c$ |
| $N_5$      | $-a, +b, +c$ | $-a, +b, +c$ |
| $N_6$      | $-a, +b, -c$ | $-a, +b, -c$ |
| $N_7$      | $-a, -b, +c$ | $-a, -b, +c$ |
| $N_8$      | $-a, -b, -c$ | $-a, -b, -c$ |

for measurement of two spin-particles along unit-vector directions $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The measuring outcome correlation probabilities among the three directions are represented in terms of the population probabilities such that
\[ N_{lc} (a, -b) = \frac{(N_3 + N_4)}{\sum_i N_i}, \]
\[ N_{lc} (a, -c) = \frac{(N_2 + N_4)}{\sum_i N_i}, \]
and
\[ N_{lc} (c, -b) = \frac{(N_3 + N_7)}{\sum_i N_i}. \]

Since
\[ N_3 + N_4 \leq (N_2 + N_4) + (N_3 + N_7), \]
thus we have the WI
\[ N_{lc} (a, -b) \leq N_{lc} (a, -c) + N_{lc} (c, -b). \]

Alternatively interchange of the measuring positive and negative spin-particles leads to
\[ N_{lc} (-a, +b) = \frac{(N_5 + N_6)}{\sum_i N_i}, \]
\[ N_{lc} (-a, +c) = \frac{(N_5 + N_7)}{\sum_i N_i}, \]
and
\[ N_{lc} (-c, +b) = \frac{(N_2 + N_6)}{\sum_i N_i}. \]

Since
\[ N_5 + N_6 \leq (N_5 + N_7) + (N_2 + N_6), \]
we have again the WI
\[ N_{lc} (-a, +b) \leq N_{lc} (-a, +c) + N_{lc} (-c, +b). \]

References

[1] S. Groblacher, T. Paterek, R. Kaltenbaek, C. Brukner, M. Z. Ukowski, M. Aspelmeyer, and A. Zeilinger, Nature 446 (2007) 871.
[2] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Rev. Mod. Phys. 82 (2010) 665.
[3] C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, and V. Scarani, Phys. Rev. Lett. 99 (2007) 210407.
[4] M. D. Eisaman, E. A. Goldschmidt, J. Chen, J. Fan, and A. Migdall, Phys. Rev. A 77 (2008) 032339.
[5] M. Paternostro and H. Jeong, Phys. Rev. A 81 (2010) 032115.
[6] C.-W. Lee, M. Paternostro, and H. Jeong, Phys. Rev. A 83 (2011) 022102.
[7] S. Pironio et al., Nature 464 (2010) 1021.
[8] D. Bohm, Phys. Rev. 108 (1957) 1070.
[9] J. S. Bell, *Physics* 1 (1964) 195.
[10] L. F. Wei, Y. X. Liu, and F. Nori, *Phys. Rev. B* 72 (2005) 104516; M. Ansmann, *et al.*, *Nature* 461 (2009) 504.
[11] S. Groblacher, T. Paterek, R. Kaltenbaek, C. Brukner, M. Z. Uckowski, M. Aspelmeyer, and A. Zeilinger, *Nature* 446 (2007) 871.
[12] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, *Rev. Mod. Phys.* 82 (2010) 665.
[13] L. F. Wei, Y. X. Liu, and F. Nori, *Phys. Rev. B* 72 (2005) 104516; M. Ansmann, *et al.*, *Nature* 461 (2009) 504.
[14] A. Cabello and F. Sciarrino, *Phys. Rev. X* 2 (2012) 021010.
[15] G.-F. Zhang, *et al.*, *Annals of Physics* 326(10) (2011) 2694-2701.
[16] A. Aspect, *Nature* 398 (1999) 189.
[17] A. C. Dada, J. Leach, G. S. Buller, M. J. Padgett, E. Andersson, *Nature Phys.* 7 (2011) 677.
[18] W. Tittel, J. Brendel, B. Gisin, T. Herzog, H. Zbinden, and N. Gisin, *Phys. Rev. A* 57 (1998) 3229; W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, *Phys. Rev. Lett.* 81 (1998) 3563.
[19] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, *Phys. Rev. Lett.* 81 (1998) 5039.
[20] M. Rowe, *et al.*, *Nature* 409 (2001) 791.
[21] H. Sakai, *et al.*, *Phys. Rev. Lett.* 97 (2006) 150405.
[22] D. Kaszlikowski, P. Guacciński, M. Żukowski, W. Miklaszewski, and A. Zeilinger, *Phys. Rev. Lett.* 85 (2000) 4418.
[23] J. R. Torgerson, D. Branning, C. H. Monken, L. Mandel, *Phys. Lett. A* 204(5-6) (1995) 323-328.
[24] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, *Phys. Rev. Lett.* 23 (1969) 880.
[25] Z. Song, J.-Q. Liang, and L.-F. Wei, *Mod. Phys. Lett. B* 28(1) (2014) 1450004.
[26] H. Zhang, J. Wang, Z. Song, J.-Q. Liang, L.-F. Wei, *Mod. Phys. Lett. B* 31(4) (2017) 1750032.
[27] J.-Q. Liang and L. F. Wei, *New Advances in Quantum Physics*, (Science Press, Beijing, 2011).
[28] X.-M. Bai, C.-P. Gao, J.-Q. Li, J.-Q. Liang, *Optics express*, 25 (2017) 17051-17065.
[29] E. P. Wigner, *Am. J. Phys.* 38 (1970) 1005.
[30] J. J. Sakurai, S. F. Tuan, and E. D. Commins, *Modern Quantum Mechanics*, revised edition, *Am. J. Phys.* 63 (1995) 93.
[31] N. Gisin, *Phys. Lett. A* 154 (1991) 201.
[32] N. Gisin and A. Peres, *Phys. Lett. A* 162 (1992) 15.
[33] F. A. Bovino, I. P. Degiovanni, *Phys. Rev. A* 77 (2008) 052110.
[34] D. Home, D. Saha, S. Das, *Phys. Rev. A* 91(1) (2015) 012102.
[35] D. Das, S. Datta, S. Goswami, A. S. Majumdar, D. Home, *Phys. Lett. A* 381(39) (2017) 3396-3404.
[36] X.-Q. Zhao, N. Liu, J. -Q. Liang, *Phys. Rev. A* 90 (2014) 023622.
[37] B. Hensen, *et al.*, *Nature* 526 (2015) 682; *Scientific Reports* 6 (2016) 30289.
[38] J. Yin *et al.*, *Science* 356 (2017) 1140-1144.