Korn type Inequalities for Objective Structures

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Abstract

We establish discrete Korn type inequalities for particle systems within the general class of objective structures that represents a far reaching generalization of crystal lattice structures. For space filling configurations whose symmetry group is a general space group we obtain a full discrete Korn inequality. For systems with non-trivial codimension our results provide an intrinsic rigidity estimate within the extended dimensions of the structure. As their continuum counterparts in elasticity theory, such estimates are at the core of energy estimates and, hence, a stability analysis for a wide class of atomistic particle systems.

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1 Introduction

The classical Korn inequality provides a quantitative rigidity estimate for $H^1$ functions in terms of their symmetrized gradient: If $\Omega \subset \mathbb{R}^d$ is bounded, connected and sufficiently regular (e.g., Lipschitz), then for all $u \in H^1(\Omega, \mathbb{R}^d)$

$$\min \{ \| \nabla u - A \|_{L^2(\Omega)} \mid A \in \text{Skew}(d) \} \leq C \| (\nabla u)^T + \nabla u \|_{L^2(\Omega)},$$

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induced seminorms as follows. Denoting by \( \pi \| \nabla u \| \) and hence \( \| u \| \)

An immediate corollary is the corresponding qualitative rigidity result which states that \( (\nabla u)^T + \nabla u = 0 \) a.e. on \( \Omega \) implies that \( u(x) = Ax + c \) for some \( A \in \text{Skew}(d) \), \( c \in \mathbb{R}^d \).

For our purposes it turns out useful to re-write Korn’s inequality in terms of projection-induced seminorms as follows. Denoting by \( \pi_{\text{rot}} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \), \( \pi_{\text{rot}} M = \frac{1}{2}(M^T + M) \) the orthogonal projection of \( d \times d \) matrices onto their symmetric part (whose kernel is the set of infinitesimal rotations \( \text{Skew}(d) \)) and by \( \Pi_{\text{rot}} : L^2(\Omega, \mathbb{R}^{d \times d}) \to L^2(\Omega, \mathbb{R}^{d \times d}) \), \( F \mapsto \Pi_{\text{rot}} F \) the orthogonal projection whose kernel is the set of constant linearized rotations \( \{ x \mapsto A | A \in \text{Skew}(d) \} \), Korn’s inequality reads

\[
\| \Pi_{\text{rot}} \nabla u \|_{L^2(\Omega)} \leq C \| \pi_{\text{rot}} \nabla u \|_{L^2(\Omega)}.
\]

In terms of \( \Pi_{\text{iso}} : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d) \), \( u \mapsto \Pi_{\text{iso}} u \), the orthogonal projection whose kernel is the set of linearized isometries \( \{ x \mapsto Ax + c | A \in \text{Skew}(d), c \in \mathbb{R}^d \} \), it can also be rephrased as

\[
\| \nabla \Pi_{\text{iso}} u \|_{L^2(\Omega)} \leq C \| \pi_{\text{rot}} \nabla u \|_{L^2(\Omega)}.
\]

In particular, on \( H_0^1(\Omega) \) or \( H^1_{\text{per}}(\Omega) \) (in case \( \Omega \) is a cuboid) one even has

\[
\| \nabla u \|_{L^2(\Omega)} \leq C \| \pi_{\text{rot}} \nabla u \|_{L^2(\Omega)}.
\]

The reverse estimates being trivial, an equivalent form is to say that the seminorms \( \| \nabla \Pi_{\text{iso}} \cdot \|_{L^2(\Omega)} \) and \( \| \pi_{\text{rot}} \nabla \cdot \|_{L^2(\Omega)} \), respectively, \( \| \nabla \cdot \|_{L^2(\Omega)} \) and \( \| \pi_{\text{rot}} \nabla \cdot \|_{L^2(\Omega)} \) are equivalent.

More recently, discrete versions of the Korn inequality have been developed that apply to systems of interacting particles and provide rigidity estimates for crystals in terms of their configurational energy. Such estimates are at the basis of the stability analysis of lattice systems: If a configuration is a critical point of the configurational energy, i.e., the forces within the particle system are in balance, one is interested in criteria that guarantee that such a configuration is stable, see, e.g., [7, 11, 18, 9].

There are two principal features that are at the core of a discrete Korn inequality for a lattice system (c.f. [13, 3]): 1. Periodicity: The periodic arrangement of particles allows for the application of Fourier transform methods to establish ‘phonon stability’; and 2. Exhaustion of the full space: In bulk systems there are no soft modes due to buckling type deformations.

The central aim of the present contribution is to investigate the validity of Korn type inequalities beyond the periodic setting and, to some extent, also beyond the bulk regime. It lies at the heart of our endeavor to examine the stability behavior of such generalized structures, cf. [19, 20, 21]. The main motivation for such an analysis is possible applications to objective structures. These particle systems, introduced by James in [15], constitute a far reaching generalization of lattice systems and have been successfully applied to a remarkable number of important structures, ranging from biology (to describe parts of viruses) to nanoscience (to model carbon nanotubes), see, e.g., [8, 6, 5, 9]. They are characterized by the fact that, up to rigid motions of the surrounding space, any two points

\[
\text{It is clear that } \| \Pi_{\text{rot}} \nabla u \|_{L^2} \leq \| \Pi_{\text{iso}} u \|_{L^2}. \text{ Conversely, if } \Pi_{\text{iso}} u = u - A \cdot c \text{ and } \Pi_{\text{rot}} \nabla u = \nabla u - A', \text{ then Poincaré’s inequality gives } \| u - A \cdot c \|_{L^2} \leq \| u - A' \cdot c' \|_{L^2} \leq C\| \nabla u - A' \|_{L^2} \text{ for some } c' \in \mathbb{R}^d \text{ and hence } \| (A' - A) \cdot c' \|_{L^2} \leq C\| \nabla u - A' \|_{L^2}, \text{ which implies } \| A' - A \| \leq C\| \nabla u - A' \|_{L^2}. \text{ Thus, } \\
\| \nabla u - A \|_{L^2} \leq \| A' - A \|_{L^2} + \| \nabla u - A' \|_{L^2} \leq C\| \nabla u - A' \|_{L^2}.
\]
\textit{see} an identical environment of other points. (In a lattice this would be true even up to translations.) As a consequence, objective structures correspond to orbits of a single point under the action of a general discrete group of Euclidean isometries, cf. \cite{13,16}. As the symmetry of these objects in general is considerably more complex than that of a lattice, the adaption of methods and results on lattices has only been achieved in a few cases so far. As notable examples we mention an algorithm for solving the Kohn-Sham equations for clusters \cite{1} and the X-ray analysis of helical structures set forth in \cite{10}.

Within an appropriate coordinate system for an objective structure, such a group might be assumed to embed into a subgroup of $O(d) \oplus S$ for a crystallographic spacegroup $S$ acting on $\mathbb{R}^{d_2}$, where $d_1 + d_2 = d$, with surjective projection onto $S$. In particular, for bulk structures with $d_2 = d$ the particles invade the whole space $\mathbb{R}^d$, whereas lower dimensional structures invade a tubular neighborhood of $\{0\} \times \mathbb{R}^{d_2}$.

A major difficulty in obtaining Korn type inequalities then results from the general structure and the non-commutativity of these groups. Whereas in principle a Fourier transform is defined on their dual spaces, the consideration of periodic mappings with significant “long wave-length” contributions turns out non-trivial. Yet, uniform estimates on such quantities that are stable in the limit of infinitely large periodicity (corresponding to infinitely many particles, respectively, vanishing interparticle distances in a rescaled set-up) are essential for a discrete Korn inequality to hold. However, as objective structures need not be periodic, even the definition of quantities that can serve the role of a wave vector is not obvious.

In \cite{19}, by exploiting the special structure of discrete subgroups of $E(d)$, we provided an efficient and extensive description of the dual space of a general discrete group of Euclidean isometries. In particular, we identified a finite union of convex ‘wave vector domains’ reflecting the existence of an underlying part of translational type of finite index. This structure is indeed tailor-made for our investigations on Korn inequalities. Due to the discrete nature of the underlying particle system, we consider finite difference stencils of (finite) interaction range and associate to them suitable seminorms measuring the (local) distances to the set of infinitesimal rigid motions and certain subsets thereof, respectively, in terms of $\ell^2$ norms of projections onto these sets. Our main results are then formulated in terms of such seminorms and state generic conditions for their equivalence, the main result being Theorem \ref{thm:main}. At the core of our proof lies the technical Lemma \ref{lem:technical} in which we utilize a classical minimax theorem of Turán on generalized power sums in order to obtain control on a general skew symmetric matrix in terms of certain oscillatory perturbations.

More in detail, for a given interaction range $\mathcal{R}$ we consider the three seminorms $\| \cdot \|_{\mathcal{R}}$, $\| \cdot \|_{\mathcal{R},0}$, and $\| \cdot \|_{\mathcal{R},0,0}$. Roughly speaking, $\| \cdot \|_{\mathcal{R}}$ measures the local distances from the set of all infinitesimal rigid motions, characterized by generic skew symmetric matrices

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3^T & S_3 \end{pmatrix} \in \text{Skew}(d),$$

where $S_1 \in \text{Skew}(d_1), S_2 \in \mathbb{R}^{d_1 \times d_2}, S_3 \in \text{Skew}(d_2)$. $\| \cdot \|_{\mathcal{R},0}$ measures the local distances to those rigid motions that fix $\{0\} \times \mathbb{R}^{d_2}$ intrinsically, corresponding to $S \in \text{Skew}(d)$ as above with $S_3 = 0$, and $\| \cdot \|_{\mathcal{R},0,0}$ measures the local distances to those rigid motions that fix $\{0\} \times \mathbb{R}^{d_2}$ in $\mathbb{R}^d$, corresponding to $S \in \text{Skew}(d)$ with $S_2 = 0$ and $S_3 = 0$. In particular, $\| \cdot \|_{\mathcal{R}} \leq \| \cdot \|_{\mathcal{R},0,0} \leq \| \cdot \|_{\mathcal{R},0} \leq \| \cdot \|_{\mathcal{R},0,0}$.

In Theorems \ref{thm:main} and \ref{thm:technical} we observe that each of these seminorms does – up to equivalence – not depend on the particular choice of $\mathcal{R}$ as long as $\mathcal{R}$ is rich enough. Our main Theorem \ref{thm:main} then states that indeed $\| \cdot \|_{\mathcal{R}}$ and $\| \cdot \|_{\mathcal{R},0}$ are equivalent. In particular, for bulk structures with $d_1 = 0$ we thereby obtain a full Korn inequality for
objective structures generated by a general space group. For \( d_1 \geq 1 \) it can be interpreted as an ‘intrinsic rigidity’ estimate within the extended dimensions of the structure. We summarize these findings in Theorem 3.31. In Propositions 5.1 and 5.2 we will also see that in general \( \| \cdot \|_{\mathbb{R}, 0} \) and \( \| \cdot \|_{\mathbb{R}, 0, 0} \) are not equivalent. In view of possible buckling modes, this is in fact not to be expected and indeed long wave-length modulations of the extended dimensions within the surrounding space impede a strong Korn type inequality.

In fact, in applications to the stability analysis of objective structures both of these seminorms will be of relevance. There the question is addressed if an objective structure is a stable configuration when the particles at different sites are assumed to interact. Despite its importance, little appears to be known beyond bulk lattice systems. (See, e.g., [14, 3] for lattice systems subject to very generic interaction potentials.) In the forthcoming contribution [20] we will provide such a stability analysis in the general framework of objective structures and, in particular, establish characterizations of stability constants for objective structures in terms of the seminorms \( \| \cdot \|_{\mathbb{R}} \) and \( \| \cdot \|_{\mathbb{R}, 0, 0} \). Here \( \| \cdot \|_{\mathbb{R}, 0, 0} \) applies to bulk systems and might also be used in lower dimensional tensile regimes in which pre-stresses have a stabilizing effect. The weaker seminorm \( \| \cdot \|_{\mathbb{R}} \) appropriately describes lower dimensional systems in their ground state even at the onset of (buckling type) instabilities. Based on these results, we will be able to provide a numerical algorithm for determining the stability of a given structure. By way of example we will also show that indeed novel stability results for nanotubes can be obtained.

Notation

We denote by \( e_i \) the \( i^{th} \) standard coordinate vector in \( \mathbb{R}^d \) and by \( I_d \in \mathbb{R}^{d \times d} \) the identity matrix of size \( d \). For \( A = (a_{ij}) \in \mathbb{C}^{m \times n} \) and \( B = (b_{ij}) \in \mathbb{C}^{p \times q} \), their direct sum and their Kronecker product are

\[
A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbb{C}^{(m+p) \times (n+q)}, \quad A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp \times nq},
\]

respectively.

For a group \( G \) and subsets \( S_1, S_2 \subset G \) we write

\[
S_1 S_2 := \{ s_1 s_2 \mid s_1 \in S_1, s_2 \in S_2 \} \subset G
\]

for the product of group subsets. For all \( S \subset G, n \in \mathbb{Z} \) and \( g \in G \) we set

\[
S^n := \{ s^n \mid s \in S \} \subset G \quad \text{and} \quad gS := \{ gs \mid s \in S \} \subset G.
\]

We write \( \langle S \rangle \) for the subgroup generated by \( S \).

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2 Objective structures

Objective structures are orbits of a point under the action of a discrete subgroup of the Euclidean group. For an efficient description we first describe the structure of these groups in some detail.
2.1 Discrete subgroups of the Euclidean group

We collect some basic material on discrete subgroups of the Euclidean group acting on \( \mathbb{R}^d \) from [19].

The Euclidean group \( \text{E}(d) \) in dimension \( d \in \mathbb{N} \) is the set of all Euclidean distance preserving transformations of \( \mathbb{R}^d \) into itself, their elements are called Euclidean isometries. It may be described as \( \text{E}(d) = \text{O}(d) \rtimes \mathbb{R}^d \), the outer semidirect product of \( \mathbb{R}^d \) and the orthogonal group \( \text{O}(d) \) in dimension \( d \) with group operation given by

\[
(A_1, b_1)(A_2, b_2) = (A_1A_2, b_1 + A_1b_2)
\]

for \((A_1, b_1), (A_2, b_2) \in \text{E}(d)\). We set

\[
L : \text{E}(d) \to \text{O}(d), \quad (A, b) \mapsto A \\
\tau : \text{E}(d) \to \mathbb{R}^d, \quad (A, b) \mapsto b
\]

and for \((A, b) \in \text{E}(d)\) we call \( L((A, b)) \) the linear component and \( \tau((A, b)) \) the translation component of \((A, b)\) so that

\[
g = (I_d, \tau(g))(L(g), 0)
\]

for each \( g \in \text{E}(d)\). An Euclidean isometry \((A, b)\) is called a translation if \( A = I_d \). The set \( \text{Trans}(d) := \{I_d\} \times \mathbb{R}^d \) of translations forms an abelian subgroup of \( \text{E}(d) \). \( \text{E}(d) \) acts on \( \mathbb{R}^d \) via

\[
(A, b) \cdot x := Ax + b \quad \text{for all } (A, b) \in \text{E}(d) \text{ and } x \in \mathbb{R}^d.
\]

For a group \( \mathcal{G} \subset \text{E}(d) \) the orbit of a point \( x \in \mathbb{R}^d \) under the action of the group is

\[
\mathcal{G} \cdot x := \{ g \cdot x \mid g \in \mathcal{G} \}.
\]

In the following we will consider discrete subgroups of the Euclidean group, which are those \( \mathcal{G} \subset \text{E}(d) \) for which every orbit \( \mathcal{G} \cdot x, x \in \mathbb{R}^d \), is discrete.

Particular examples of discrete subgroups of \( \text{E}(d) \) are the so-called space groups. These are those discrete groups \( \mathcal{G} \subset \text{E}(d) \) that contain \( d \) translations whose translation components are linearly independent. Their subgroup of translations is generated by \( d \) such linearly independent translations and forms a normal subgroup of \( \mathcal{G} \) which is isomorphic to \( \mathbb{Z}^d \).

In general, discrete subgroups of \( \text{E}(d) \) can be characterized as follows. Recall that two subgroups \( \mathcal{G}_1, \mathcal{G}_2 \subset \text{E}(d) \) are conjugate in \( \text{E}(d) \) if there exists some \( g \in \text{E}(d) \) such that \( g^{-1}\mathcal{G}_1g = \mathcal{G}_2 \). (This corresponds to a rigid coordinate transformation in \( \mathbb{R}^d \).)

**Theorem 2.1.** Let \( \mathcal{G} \subset \text{E}(d) \) be discrete, \( d \in \mathbb{N} \). There exist \( d_1, d_2 \in \mathbb{N}_0 \) such that \( d = d_1 + d_2 \), a \( d_2 \)-dimensional space group \( \mathcal{S} \) and a discrete group \( \mathcal{G}' \subset \text{O}(d_1) \oplus \mathcal{S} \) such that \( \mathcal{G} \) is conjugate under \( \text{E}(d) \) to \( \mathcal{G}' \) and \( \pi(\mathcal{G}') = \mathcal{S} \), where \( \pi \) is the natural surjective homomorphism \( \text{O}(d_1) \oplus \text{E}(d_2) \to \text{E}(d_2), A \oplus g \mapsto g \).

Here \( \oplus \) is the group homomorphism

\[
\oplus : \text{O}(d_1) \times \text{E}(d_2) \to \text{E}(d_1 + d_2) \\
(A_1, (A_2, b_2)) \mapsto A_1 \oplus (A_2, b_2) := \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right), \left( \begin{array}{c} 0 \\ b_2 \end{array} \right)
\]

and \( \text{O}(d_1) \oplus \mathcal{S} \) is understood to be \( \text{O}(d) \) if \( d_1 = d \) and to be \( \mathcal{S} \) if \( d_1 = 0 \). The theorem allows us to assume that \( \mathcal{G} \) from now on is of the form \( \mathcal{G}' \) with no loss of generality.
Throughout this period, it is called periodic if there exists some $m \in \mathbb{N}$ such that $T^m \to T$ is bijective. We remark that the quantities $d, d_1, d_2, F, S$ and $T_S$ are uniquely defined by $G$. However, in general there is no canonical choice for $T$, it might not be a group and the elements of $T$ might not commute. Yet, a main result of [19] states that there is an $m_0 \in \mathbb{N}$ such that $T^N = \{t^N \mid t \in T\}$ is a normal subgroup of $G$ if and only if $N$ is a multiple of $m_0$:

$$T^N \triangleleft G \iff N = m_0 \in \mathbb{N}.$$  

For each $N \in M_0$, $T^N$ is isomorphic to $\mathbb{Z}^{d^2}$ and of finite index in $G$.

The set $T$ allows to introduce a notion of periodicity for functions defined on $G$. For a set $S$ and $N \in M_0$ we say that a function $u : G \to S$ is $T^N$-periodic if

$$u(g) = u(gt) \quad \text{for all } g \in G \text{ and } t \in T^N.$$  

It is called periodic if there exists some $N \in M_0$ such that $u$ is $T^N$-periodic. We also set

$$L^\infty_{\text{per}}(G, C^{m \times n}) := \{ u : G \to C^{m \times n} \mid u \text{ is periodic} \}.$$  

(Throughout $C^{m \times n}$ is equipped with the usual Frobenius inner product and induced norm $\| \cdot \| \). We notice that the above definition of periodicity is independent of the choice of $T$ and that $L^\infty_{\text{per}}(G, C^{m \times n})$ is a vector space. In fact, one has

$$L^\infty_{\text{per}}(G, C^{m \times n}) = \left\{ \left. G \to C^{m \times n}, g \mapsto u(gT^N) \right\} \mid N \in M_0, u : G/T^N \to C^{m \times n} \right\}.$$  

For each $N \in M_0$ we now fix a representation set $C_N$ of $G/T^N$ and we equip $L^\infty_{\text{per}}(G, C^{m \times n})$ with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle u, v \rangle := \frac{1}{|C_N|} \sum_{g \in C_N} \langle u(g), v(g) \rangle \quad \text{if } u \text{ and } v \text{ are } T^N\text{-periodic}$$

for all $u, v \in L^\infty_{\text{per}}(G, C^{m \times n})$. The induced norm is denoted by $\| \cdot \|_2$.

We denote by $T_{m_0}$ the dual space of the abelian group $T^{m_0}$, which consists of all homomorphisms from $T^{m_0}$ to the complex unit circle. Observe that a homomorphism $\chi \in T^{m_0}$ is $T^N$-periodic, $N \in M_0$, if and only if $\chi|_{T^N} = 1$. Let $E$ be the set $\{ \chi \in T^{m_0} \mid \chi \text{ is periodic} \}$. Note that $T^{m_0} \cap C_N$ is a representation set of $T^{m_0}/T^N$ for all $N \in M_0$. We define the Fourier transform as follows.

**Definition 2.2.** If $u \in L^\infty_{\text{per}}(T^{m_0}, C^{m \times n})$ and $\chi \in E$, we set

$$\hat{u}(\chi) := \frac{1}{|T^{m_0} \cap C_N|} \sum_{g \in T^{m_0} \cap C_N} \chi(g)u(g) \in C^{m \times n},$$

where $N \in M_0$ is such that $u$ and $\chi$ are $T^N$-periodic.

**Proposition 2.3** (The Plancherel formula). The Fourier transformation

$$\hat{\cdot} : L^\infty_{\text{per}}(T^{m_0}, C^{m \times n}) \to \bigoplus_{\chi \in E} C^{m \times n}, \quad u \mapsto (\hat{u}(\chi))_{\chi \in E}$$

is well-defined and bijective. Moreover, the Plancherel formula

$$\langle u, v \rangle = \sum_{\chi \in E} \langle \hat{u}(\chi), \hat{v}(\chi) \rangle \quad \text{for all } u, v \in L^\infty_{\text{per}}(T^{m_0}, C^{m \times n})$$

holds true.
We remark that for all \( u : T^{m_0} \rightarrow \mathbb{C}^{m \times n} \) and \( N \in M_0 \) such that \( u \) is \( T^N \)-periodic, one gets
\[
\{ \chi \in \mathcal{E} \mid \hat{u}(\chi) \neq 0 \} \subset \{ \chi \in \mathcal{E} \mid \chi \text{ is } T^N \text{-periodic} \}.
\]

The following lemma provides the Fourier transform of a translated function.

**Lemma 2.4.** Let \( f \in L^\infty_{\text{per}}(T^{m_0} \times \mathbb{C}^{m \times n}) \), \( g \in \mathcal{G} \) and \( \tau_g f \) denote the translated function \( f(\cdot, g) \). Then we have \( \tau_g f \in L^\infty_{\text{per}}(T^{m_0} \times \mathbb{C}^{m \times n}) \) and
\[
\tau_g f(\chi) = \chi(g^{-1}) \hat{f}(\chi)
\]
for all \( \chi \in \mathcal{E} \).

### 2.2 Orbits of discrete subgroups of the Euclidean group

As a far reaching generalization of a lattice, James [15] defines an **objective (atomic) structure** as a discrete point set \( S \) in \( \mathbb{R}^d \) such that for any \( x_1, x_2 \in S \) there is an Euclidean isometry \( g \in \text{E}(d) \) with \( g \cdot S = S \) and \( g \cdot x_1 = x_2 \). Equivalently, \( S \) is an orbit of a point under the action of a discrete subgroup of \( \text{E}(d) \), see, e.g., [16, Proposition 3.14]. With only minor loss of generality in the following we restrict our attention to orbits where the stabilizer subgroup is trivial and thus we have a natural bijection between the discrete group and the atoms.

**Definition 2.5.** We call a subset \( S \) of \( \mathbb{R}^d \) a **general configuration** if there exist a discrete group \( \mathcal{G} < \text{E}(d) \) and a point \( x \in \mathbb{R}^d \) such that the map \( \mathcal{G} \rightarrow S, g \mapsto g \cdot x \) is bijective.

**Remark 2.6.**

(i) The representation of a general configuration by a discrete subgroup of \( \text{E}(d) \) and a point in \( \mathbb{R}^d \) is not unique, see Example 2.7.

(ii) For each discrete group \( \mathcal{G} < \text{E}(d) \), a.e. \( x \in \mathbb{R}^d \) is such that the map \( \mathcal{G} \rightarrow \mathbb{R}^d, g \mapsto g \cdot x \) is injective. Indeed, if \( g, h \in \mathcal{G}, g \neq h \), then the affine space \( \{ x \in \mathbb{R}^d \mid g \cdot x = h \cdot x \} \) has codimension at least 1. Since \( \mathcal{G} \) is at most countable, the claim follows. In particular, the set \( \mathcal{G} \cdot x \) is a general configuration for a.e. \( x \in \mathbb{R}^d \).

(iii) The orbit of a point in \( \mathbb{R}^d \) under the action of a discrete subgroup of \( \text{E}(d) \) need not be a general configuration, see Example 2.8.

**Example 2.7.** We present an example showing that in general for a given general configuration \( S \subset \mathbb{R}^d \) there exist discrete groups \( \mathcal{G}_1, \mathcal{G}_2 < \text{E}(d) \) and a point \( x \in \mathbb{R}^d \) such that the maps \( \mathcal{G}_1 \rightarrow S, g \mapsto g \cdot x \) and \( \mathcal{G}_2 \rightarrow S, g \mapsto g \cdot x \) are bijective but \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are not isomorphic. Let \( S = \{ \pm e_1, \pm e_2 \} \subset \mathbb{R}^2, R(\pi/2) \) be the rotation matrix by the angle \( \pi/2 \), \( \mathcal{G}_1 = \langle (R(\pi/2), 0) \rangle < \text{E}(2) \),
\[
\mathcal{G}_2 = \left\{ \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \right\} < \text{E}(2)
\]
and \( x = e_1 \in \mathbb{R}^2 \). The group \( \mathcal{G}_2 \) is the Klein four-group and thus, \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are not isomorphic. However, the maps \( \mathcal{G}_1 \rightarrow S, g \mapsto g \cdot x \) and \( \mathcal{G}_2 \rightarrow S, g \mapsto g \cdot x \) are bijective.

**Example 2.8.** In this example we present an orbit \( S \) of a point in \( \mathbb{R}^3 \) under the action of a discrete subgroup of \( \text{E}(3) \) which is not a general configuration.

Let be given a regular icosahedron centered at the origin. Let \( S \) be the set of the 30 centers of the edges of the icosahedron (i.e., \( S \) is the set of the vertices of the rectified icosahedron and moreover, \( S \) is the set of the vertices of a icosidodecahedron). The rotation
Thus we have $\{x \in E(\mathbb{R}^3) \mid x \in S\}$ that $A \cdot V = \text{aff} (\{x \in \mathbb{R}^3 \mid x \in S\})$. Thus, the set $S$ contains 15 points which lie in the same plane. This implies that $S$ cannot be the orbit of $G$, and we have a contradiction.

We proceed with a couple of lemmas implying that without loss of generality general configurations lie in $\{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}}$ where $d_{\text{aff}}$ is their affine dimension and, moreover, the associated discrete group of isometries acts trivially on $\mathbb{R}^{d-d_{\text{aff}}} \times \{0_{d_{\text{aff}}} \}$.

**Lemma 2.9.** Let $S \subset \mathbb{R}^d$ be a general configuration. Then for every $a \in E(d)$ the set $\{a \cdot x \mid x \in S\}$ is also a general configuration.

**Proof.** This follows directly from the observation that, if for a subgroup $G < E(d)$ and $x_0 \in \mathbb{R}^d$ the map $G \to S$, $g \mapsto g \cdot x_0$ is injective, then, for every $a \in E(d)$ the map $aG a^{-1} \to \{a \cdot x \mid x \in S\}$, $g \mapsto g \cdot (a \cdot x_0)$ is injective. \hfill $\square$

We denote the affine hull of a set $A \subset \mathbb{R}^d$ by $\text{aff}(A)$ and write $\dim(A) := \dim(\text{aff}(A))$ for its affine dimension. Recall that this is the dimension of the vector space span($\{x - x_0 \mid x \in A\}$) for any $x_0 \in A$.

**Lemma 2.10.** Let $G < E(d)$ be discrete and $x_0 \in \mathbb{R}^d$ such that the map $G \to \mathbb{R}^d$, $g \mapsto g \cdot x_0$ is injective. Let $d_{\text{aff}} = \dim(\text{aff}(x_0))$. Then there exists some $a \in E(d)$ such that for the discrete group $G' = aG a^{-1}$ and $x'_0 = a \cdot x_0$ it holds

$$\text{aff}(G' \cdot x'_0) = \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}}. $$

The map $G' \to \mathbb{R}^d$, $g \mapsto g \cdot x'_0$ is injective and we have $G' \cdot x'_0 = a \cdot (G \cdot x_0)$.

**Proof.** There exists some $d_{\text{aff}}$-dimensional vector space $V$ such that $\text{aff}(G \cdot x_0) = x_0 + V$. Choosing $A \in O(d)$ such that $\{Ax \mid x \in V\} = \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}}$ and setting $a = (A, -Ax_0) \in E(d)$ implies the assertion. \hfill $\square$

**Lemma 2.11.** Let $G < E(d)$ be discrete and $x_0 \in \mathbb{R}^d$ such that $\text{aff}(G \cdot x_0) = \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}}$, where $d_{\text{aff}} = \dim(G \cdot x_0)$. Then we have $G < O(d - d_{\text{aff}}) \oplus E(d_{\text{aff}})$.

**Proof.** Set $V = \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}} = \text{aff}(G \cdot x_0)$. For given $g \in G$ we define the map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto L(g)x$. First we show that $V$ is invariant under $\varphi$. Let $x \in V$. Since $V = \text{aff}(G \cdot x_0) - x_0$, there exist some $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G \cdot x_0$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n \alpha_i x_i$ and $\sum_{i=1}^n \alpha_i = 0$. It holds

$$L(g)x = \sum_{i=1}^n \alpha_i L(g)x_i = \sum_{i=1}^n \alpha_i (g \cdot x_i) \in V. $$

Thus we have $\{L(g)x \mid x \in V\} \subset V$. Since $L(g)$ is invertible, it holds $\{L(g)x \mid x \in V\} = V$. Since $L(g)$ is orthogonal, also the complement $V^\perp = \mathbb{R}^{d-d_{\text{aff}}} \times \{0_{d_{\text{aff}}} \}$ is invariant under $\varphi$. This implies $L(g) \in O(d - d_{\text{aff}}) \oplus O(d_{\text{aff}})$. It holds $\tau(g) = g \cdot x_0 - L(g)x_0 \in V$ and thus, $g \in O(d - d_{\text{aff}}) \oplus E(d_{\text{aff}})$. \hfill $\square$
Lemma 2.12. Let \( \mathcal{G} < E(d) \) be discrete and \( x_0 \in \mathbb{R}^d \) such that the map \( \mathcal{G} \to \mathbb{R}^d, g \mapsto g \cdot x_0 \) is injective and \( \text{aff}(\mathcal{G} \cdot x_0) = \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}}, \) where \( d_{\text{aff}} = \dim(\mathcal{G} \cdot x_0) \). Let \( \mathcal{G}' = \{I_{d-d_{\text{aff}}} \oplus g \mid g \in E(d_{\text{aff}}), \exists A \in O(d-d_{\text{aff}}) : A \oplus g \in \mathcal{G} \} \) and

\[
\varphi : \mathcal{G} \to \mathcal{G}'
\]

\[
A \oplus g \mapsto I_{d-d_{\text{aff}}} \oplus g \quad \text{if } A \in O(d-d_{\text{aff}}), g \in E(d_{\text{aff}}) \text{ and } A \oplus g \in \mathcal{G}.
\]

Then \( \mathcal{G}' \) is a discrete subgroup of \( E(d) \), \( \varphi \) is an isomorphism, \( \mathcal{G} \cdot x_0 = \mathcal{G}' \cdot x_0 \) and the map \( \mathcal{G}' \to \mathbb{R}^d, g \mapsto g \cdot x_0 \) is injective.

Proof. By Lemma 2.11 we have \( \mathcal{G} < O(d-d_{\text{aff}}) \oplus E(d_{\text{aff}}) \). It is clear that \( \varphi \) is a surjective homomorphism. Since \( x_0 \in \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}} \), for all \( g \in \mathcal{G} \) it holds \( g \cdot x_0 = \varphi(g) \cdot x_0 \). Particularly, we have \( \mathcal{G} \cdot x_0 = \mathcal{G}' \cdot x_0 \). Since the map \( \mathcal{G} \to \mathbb{R}^d, g \mapsto g \cdot x_0 \) is injective, the map \( \varphi \) is injective and thus, an isomorphism. Since the map \( \mathcal{G}' \to \mathcal{G}' \cdot x_0, g \mapsto g \cdot x_0 \) is a homeomorphism and \( \mathcal{G}' \cdot x_0 \) is discrete, \( \mathcal{G}' \) is discrete.

Remark 2.13. (i) Let \( \mathcal{G} < E(d) \) be discrete, \( x_0 \in \mathbb{R}^d \) and \( A = \text{aff}(\mathcal{G} \cdot x_0) \). For all \( g \in \mathcal{G} \) it holds \( \{g \cdot x \mid x \in A\} = A \).

(ii) Let \( \mathcal{G} < E(d) \) be discrete and \( x_0 \in \mathbb{R}^d \). Let \( V \) be the vector space such that \( \text{aff}(\mathcal{G} \cdot x_0) = x_0 + V \). Then for all \( g \in \mathcal{G} \) it holds \( \{L(g)x \mid x \in V\} = V \).

3 A discrete Korn type inequality

In this core section we introduce two seminorms on deformations of general configurations that measure the distance of a deformation of a general configuration to the set of (infinitesimally) rigid motions. More precisely, given a finite interaction range, one considers finite patches of a configuration by restricting to suitable neighborhoods of particles and averages the deviations from the set of rigid body motions (or a subclass thereof) over all such patches. The first seminorm is local in the sense that the full set of rigid motions is considered and so different finite patches can be close to completely different rigid motions, see Definition 3.1. The second seminorm is ‘intrinsically global’ as the set of rigid motions is restricted to those that vanish when both preimage and target space are projected to the subspace that is invaded by the objective structure, see Definition 3.13 for a precise statement. (For bulk structures defined in terms of a space group this is the whole space and the kernel of the resulting seminorm consists of translations only.)

Our main result is Theorem 3.21 (see also Theorem 3.31) which states that these two seminorms are equivalent as long as the interaction range is sufficiently rich. We thus establish a Korn-type estimate for objective structures. For bulk structures we indeed obtain a full discrete Korn inequality. For lower dimensional structures this is in fact not to be expected as the structure might show buckling exploring the ambient space. Still, Theorem 3.21 shows that intrinsically also such structures are rigid.

We close this section by explicitly computing the kernel of the relevant seminorms.

3.1 Deformations and local rigidity seminorms

Let \( \mathcal{G} < E(d) \) be a discrete group of Euclidean isometries and \( x_0 \in \mathbb{R}^d \) such that \( \mathcal{G} \cdot x_0 \) is a general configuration. Without loss of generality we assume in the following that the mapping \( \mathcal{G} \to \mathcal{G} \cdot x_0, g \mapsto g \cdot x_0 \) is bijective, \( \mathcal{G} < O(d_1) \oplus \mathcal{S}, d = d_1 + d_2, \) that \( \mathcal{T} \subset \mathcal{G} \) and \( \mathcal{C}_N (\text{for } N \in \mathcal{M}_0) \) have been chosen and that \( \mathcal{G} \cdot x_0 \subset \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^{d_{\text{aff}}} \) and \( \mathcal{G} \) acts trivially on \( \mathbb{R}^{d-d_{\text{aff}}} \times \{0_{d_{\text{aff}}} \} \), \( d_{\text{aff}} = \dim(\mathcal{G} \cdot x_0) \).
We consider deformation mappings \( y : \mathcal{G} \cdot x_0 \to \mathbb{R}^d \). One can alternatively describe such a mapping by the ‘group deformation’ \( v : \mathcal{G} \to \mathbb{R}^d \) given by \( v(g) = y(g \cdot x_0) \). It turns out useful to consider the ‘pulled back’ quantities \( g^{-1}v(g) \) and to define an associated ‘group displacement mapping’ \( u : \mathcal{G} \to \mathbb{R}^d \) by setting \( u(g) = g^{-1} \cdot v(g) - x_0 \), i.e.,

\[
v(g) = g \cdot (x_0 + u(g)) \quad \text{for all } g \in \mathcal{G}.
\]

In particular, \( v \) is the translation \( v(g) = g \cdot x_0 + a \) for all \( g \in \mathcal{G} \) and an \( a \in \mathbb{R}^d \) if and only if \( L(g)u(g) = a \) for all \( g \in \mathcal{G} \) and \( v \) is the rotation \( v(g) = R(g \cdot x_0) \) for all \( g \in \mathcal{G} \) and an \( R \in \text{SO}(d) \) if and only if \( L(g)u(g) = (R - I_d)(g \cdot x_0) \) for all \( g \in \mathcal{G} \).

As \( \mathcal{G} \cdot x_0 \) is typically infinite and we want to allow deformations of long wave-length, we consider deformations \( v \) corresponding to a periodic displacement \( u \). A crucial point in the following is then to provide estimates that do not depend on the characteristics of the periodicity.

Let \( \mathcal{R} \) be a finite subset of \( \mathcal{G} \). Suppose \( \nu : \mathcal{G} \to \mathbb{R}^d \) is \( T^N \)-periodic for some \( N \in M_0 \). A natural quantity to measure the size of the associated deformation \( v \) locally ‘modulo isometries’ is

\[
\left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \text{dist}^2(v|_{g\mathcal{R}}, \{ (a \cdot (h \cdot x_0))_{h \in \mathcal{G}} | a \in E(d) \}) \right)^{\frac{1}{2}},
\]

where \( \text{dist} \) is the induced metric of the Euclidean norm on \((\mathbb{R}^d)\mathcal{R}\). Here for every \( g \in \mathcal{C}_N \) we have

\[
\text{dist}(v|_{g\mathcal{R}}, \{ (a \cdot (h \cdot x_0))_{h \in \mathcal{G}} | a \in E(d) \}) = \text{dist}(\{(gh) \cdot (x_0 + u(gh))\}_{h \in \mathcal{R}}, \{(ah) \cdot x_0\}_{h \in \mathcal{R}} | a \in E(d) \}) = \text{dist}(\{u(gh)\}_{h \in \mathcal{R}}, \{(L(h)^T(ah) \cdot x_0 - h \cdot x_0)\}_{h \in \mathcal{R}} | a \in E(d) \})).
\]

With the aim to consider small displacements \( u \approx 0 \) we linearize by observing that, for \( U \subset E(d) \) a sufficiently small open neighborhood of \( id \), the set

\[
\{ (L(h)^T((ah) \cdot x_0 - h \cdot x_0))_{h \in \mathcal{R}} | a \in U \}
\]

is a manifold whose tangent space at the point \( 0 \in (\mathbb{R}^d)\mathcal{R} \) is

\[
U_{\text{iso}}(\mathcal{R}) = \{ (L(h)^T(b + S(h \cdot x_0)))_{h \in \mathcal{R}} | b \in \mathbb{R}^d, S \in \text{Skew}(d) \}.
\]

(This follows from the fact that the tangent space of \( E(d) \) at \( id \) is given by \( \text{Skew}(d) \times \mathbb{R}^d \).) A Taylor expansion shows

\[
\text{dist}(v|_{g\mathcal{R}}, \{ (a \cdot (h \cdot x_0))_{h \in \mathcal{G}} | a \in E(d) \}) \approx \text{dist}(u(g \cdot \cdot |\mathcal{R}, U_{\text{iso}}(\mathcal{R})) \quad (1)
\]

and we are led to introduce the seminorm \( \| \cdot \|_{\mathcal{R}} \) by

\[
\| u \|_{\mathcal{R}} = \left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \text{dist}^2(u(g \cdot \cdot |\mathcal{R}, U_{\text{iso}}(\mathcal{R})) \right)^{\frac{1}{2}}.
\]

More precisely and in agreement with these definitions we have the following general definition.
Definition 3.1. We define the vector spaces

\[ U_{\text{per}}, C := L^\infty_{\text{per}}(\mathcal{G}, \mathbb{C}^{d \times 1}) = \{ u : \mathcal{G} \to \mathbb{C}^d \mid u \text{ is periodic} \} \]

and

\[ U_{\text{per}} := \{ u : \mathcal{G} \to \mathbb{R}^d \mid u \text{ is periodic} \} \subset U_{\text{per}}, C. \]

For all \( R \subset \mathcal{G} \) we define the vector spaces

\[ U_{\text{trans}}(R) := \{ u : R \to \mathbb{R}^d \mid \exists a \in \mathbb{R}^d \forall g \in R : L(g)u(g) = a \}, \]

\[ U_{\text{rot}}(R) := \{ u : R \to \mathbb{R}^d \mid \exists S \in \text{Skew}(d) \forall g \in R : L(g)u(g) = S(g \cdot x_0 - x_0) \} \]

and

\[ U_{\text{iso}}(R) := U_{\text{trans}}(R) + U_{\text{rot}}(R). \]

For all finite sets \( R \subset \mathcal{G} \) we define the norm

\[ \| \cdot \| : \{ u : R \to \mathbb{R}^d \} \to [0, \infty), \quad u \mapsto \left( \sum_{g \in R} \| u(g) \|^2 \right)^{1/2} \]

and the seminorm

\[ \| \cdot \|_R : U_{\text{per}} \to [0, \infty), \quad u \mapsto \left( \frac{1}{|C_N|} \sum_{g \in C_N} \| \pi_{U_{\text{iso}}(R)}(u(g \cdot \cdot R)) \|^2 \right)^{1/2} \text{ if } u \text{ is } T^N\text{-periodic}, \]

where \( \pi_{U_{\text{iso}}(R)} \) is the orthogonal projection on \( \{ u : R \to \mathbb{R}^d \} \) with respect to the norm \( \| \cdot \| \) with kernel \( U_{\text{iso}}(R) \).

Remark 3.2. (i) The definition of \( \| \cdot \|_R \) is independent of the choice of \( C_N \).

(ii) Instead of \( U_{\text{rot}}(R) \) one could alternatively consider the vector space

\[ \left\{ u : R \to \mathbb{R}^d \mid \exists S \in \text{Skew}(d) \forall g \in R : L(g)u(g) = S(g \cdot x_0 - x_0) \right\}, \]

whose sum with \( U_{\text{trans}}(R) \) is also \( U_{\text{iso}}(R) \). Due to technical reasons we prefer \( U_{\text{rot}}(R) \).

It is worth noticing that, in view of the discrete nature of the underlying objective structure, the seminorm \( \| \cdot \|_R \) is equivalent to a seminorm acting on a ‘discrete derivative’ in form of a suitable finite difference stencil of \( u \).

Definition 3.3. For all \( u \in U_{\text{per}} \) and finite sets \( R \subset \mathcal{G} \) we define the discrete derivative

\[ \nabla_R u : \mathcal{G} \to \{ v : R \to \mathbb{R}^d \} \]

\[ g \mapsto (\nabla_R u(g) : R \to \mathbb{R}^d, h \mapsto u(gh) - L(h)^T u(g)). \]

Remark 3.4. Let \( R \subset \mathcal{G} \) be finite, \( u \in U_{\text{per}} \) and \( v : \mathcal{G} \to \mathbb{R}^d, g \mapsto g \cdot (x_0 + u(g)) \) the associated deformation mapping, see above. Then the relation between the discrete derivative of \( v \) and the discrete derivative \( \nabla_R u \) is given by

\[ (v(gh) - v(g))_{h \in R} = \left( (gh) \cdot x_0 - g \cdot x_0 + L(gh)((\nabla_R u(g))(h)) \right)_{h \in R} \quad \text{for all } g \in \mathcal{G}. \]
If \( u \in U_{\text{per}} \) is \( T^N \)-periodic for some \( N \in M_0 \) and \( \mathcal{R} \subset \mathcal{G} \) is finite, then also the discrete derivative \( \nabla_R u \) is \( T^N \)-periodic.

**Definition 3.5.** For each finite set \( \mathcal{R} \subset \mathcal{G} \) we define the seminorm
\[
\| \cdot \|_{\mathcal{R}, \nabla} : U_{\text{per}} \to [0, \infty) \\
u \mapsto \left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \| \pi_{U_{\text{rot}}(\mathcal{R})}(\nabla_R u(g)) \|_2^2 \right)^{\frac{1}{2}}
\]
where \( \pi_{U_{\text{rot}}(\mathcal{R})} \) is the orthogonal projection on \( \{ u : \mathcal{R} \to \mathbb{R}^d \} \) with respect to the norm \( \| \cdot \| \) with kernel \( U_{\text{rot}}(\mathcal{R}) \).

**Remark 3.6.**
(i) We have \( \| \cdot \|_{\mathcal{R}, \nabla} = \| \cdot \|_{\mathcal{R} \setminus \{id\}, \nabla} \) for all finite sets \( \mathcal{R} \subset \mathcal{G} \).

(ii) Let \( t_i = (I_d, e_i) \) for \( i = 1, \ldots, d \). If \( \mathcal{G} = \langle t_1, \ldots, t_d \rangle \) and \( \mathcal{R} = \{ t_1, \ldots, t_d \} \), then
\[
\| \pi_{U_{\text{rot}}(\mathcal{R})}(\nabla_R u(g)) \| \geq \| \nabla_R u(g) \|_2 \text{ for all } u \in U_{\text{per}} \text{ and } g \in \mathcal{G}.
\]

**Proposition 3.7.** Let \( \mathcal{R} \subset \mathcal{G} \) be finite and \( id \in \mathcal{R} \). Then the seminorms \( \| \cdot \|_R \) and \( \| \cdot \|_{\mathcal{R}, \nabla} \) are equivalent.

**Proof.** Let \( \mathcal{R} \subset \mathcal{G} \) be finite, \( id \in \mathcal{R} \) and without loss of generality \( \mathcal{R} \neq \{ id \} \). Let \( u \in U_{\text{per}} \). There exists some \( N \in M_0 \) such that \( u \) is \( T^N \)-periodic. We have
\[
\| u \|_{R, \nabla}^2 = \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \| \pi_{U_{\text{rot}}(\mathcal{R})}(\nabla_R u(g)) \|_2^2
\]
and thus, \( \| \cdot \|_{\mathcal{R}, \nabla} \geq \| \cdot \|_{\mathcal{R}} \). Let \( \mathcal{R}' = \mathcal{R} \setminus \{ id \} \). For all \( g \in \mathcal{C}_N \) it holds
\[
\| \pi_{U_{\text{rot}}(\mathcal{R})}(u(g \cdot |\mathcal{R})) \|_2^2
= \inf_{b \in \mathbb{R}^d} \inf_{S \in \text{Skew}(d)} \| (L(h)u(gh) - b - S(h \cdot x_0 - x_0))_{h \in \mathcal{R}} \|_2^2
= \inf_{b \in \mathbb{R}^d} \left( \| u(g) - b \|_2^2 + \inf_{S \in \text{Skew}(d)} \| (L(h)u(gh) - b - S(h \cdot x_0 - x_0))_{h \in \mathcal{R}} \|_2^2 \right)
= \inf_{b \in \mathbb{R}^d} \left( \| b \|_2^2 + \inf_{S \in \text{Skew}(d)} \| (L(h)u(gh) - u(g) - b - S(h \cdot x_0 - x_0))_{h \in \mathcal{R}} \|_2^2 \right)
\geq \inf_{b \in \mathbb{R}^d} \left( \| b \|_2^2 + \frac{1}{|\mathcal{R}'|} \inf_{S \in \text{Skew}(d)} \| (L(h)u(gh) - u(g) - S(h \cdot x_0 - x_0))_{h \in \mathcal{R}} \|_2^2 \right)
\geq \inf_{b \in \mathbb{R}^d} \left( \| b \|_2^2 + \frac{1}{|\mathcal{R}'|} \left( \frac{1}{2} \inf_{S \in \text{Skew}(d)} \| (L(h)u(gh) - u(g) - S(h \cdot x_0 - x_0))_{h \in \mathcal{R}} \|_2^2 - \| (b)_{h \in \mathcal{R}'} \|_2^2 \right) \right)
= \frac{1}{2|\mathcal{R}'|} \| \pi_{U_{\text{rot}}(\mathcal{R})}(\nabla_R u(g)) \|_2^2,
\]
where in the second to last step we used that $\|v + w\|^2 \geq \|v\|^2/2 - \|w\|^2$ for all $v, w : \mathbb{R}' \to \mathbb{R}^d$. Thus, we have

$$\|u\|^2_{\mathbb{R}} = \frac{1}{|C_N|} \sum_{g \in C_N} \|\pi_{U_{iso}(\mathbb{R})}(u(g \cdot)_{\mathbb{R}})\|^2$$

$$\geq \frac{1}{2|\mathbb{R}'||C_N|} \sum_{g \in C_N} \|\pi_{U_{rot}(\mathbb{R})}(\nabla_R u(g))\|^2$$

$$= \frac{1}{2|\mathbb{R}'|} \|u\|^2_{\mathbb{R}, \nabla}.$$  

Hence, we have $\| \cdot \|_{\mathbb{R}, \nabla} \leq \sqrt{2|\mathbb{R}'|} \| \cdot \|_{\mathbb{R}}$. \qed

### 3.2 Equivalence of local rigidity seminorms

Our aim is to show that, up to equivalence, $\| \cdot \|_{\mathbb{R}}$ does not depend on the particular choice of $\mathbb{R}$ as long as $\mathbb{R}$ is rich enough. We begin with some elementary preliminaries.

**Definition 3.8.** We say $\mathbb{R} \subset G$ has Property 1 if $\mathbb{R}$ is finite, $id \in \mathbb{R}$ and $aff(\mathbb{R} \cdot x_0) = aff(G \cdot x_0)$.

We say $\mathbb{R} \subset G$ has Property 2 if $\mathbb{R}$ is finite and there exist two sets $\mathbb{R}', \mathbb{R}'' \subset G$ such that $id \in \mathbb{R}'$, $\mathbb{R}'$ generates $G$, $\mathbb{R}''$ has Property 1 and $\mathbb{R}' \mathbb{R}'' \subset \mathbb{R}$.

Clearly, if $\mathbb{R} \subset G$ has Property 2, then $\mathbb{R}$ has also Property 1.

**Lemma 3.9.** Suppose that $\mathbb{R} \subset G$ has Property 1. Then there exists some $A \in \mathbb{R}^{d_{aff} \times |\mathbb{R}|}$ of rank $d_{aff}$ such that

$$(g \cdot x_0 - x_0)_{g \in \mathbb{R}} = \begin{pmatrix} 0_{d-d_{aff}, |\mathbb{R}|} \\ A \end{pmatrix}.$$  

**Proof.** Suppose that $\mathbb{R} \subset G$ has Property 1. Since $G \cdot x_0 \subset \{0_{d-d_{aff}} \} \times \mathbb{R}^{d_{aff}}$, there exists some $A \in \mathbb{R}^{d_{aff} \times |\mathbb{R}|}$ such that

$$(g \cdot x_0 - x_0)_{g \in \mathbb{R}} = \begin{pmatrix} 0 \\ A \end{pmatrix}.$$  

It holds

$$\dim(\text{span}(\{g \cdot x_0 - x_0 | g \in \mathbb{R}\})) = \dim(aff(\mathbb{R} \cdot x_0)) = \dim(aff(\mathbb{G} \cdot x_0)) = d_{aff}$$

and thus, $\text{rank}(A) = d_{aff}$.

**Definition 3.10.** For all finite sets $\mathbb{R} \subset G$ we define the seminorm

$$p_\mathbb{R} : \{u : \mathbb{R} \to \mathbb{R}^d \} \to [0, \infty), \quad u \mapsto \|\pi_{U_{iso}(\mathbb{R})}(u)\|$$

on $(\mathbb{R}^d)^{\mathbb{R}}$ whose kernel is $U_{iso}(\mathbb{R})$. Moreover, for all finite sets $\mathbb{R}_1, \mathbb{R}_2 \subset G$ we define the seminorm

$$q_{\mathbb{R}_1, \mathbb{R}_2} : \{u : \mathbb{R}_1 \mathbb{R}_2 \to \mathbb{R}^d \} \to [0, \infty), \quad u \mapsto \left( \sum_{g \in \mathbb{R}_1} p_{\mathbb{R}_2}^2(u(g \cdot)_{\mathbb{R}_2}) \right)^{\frac{1}{2}}$$

on $(\mathbb{R}^d)^{\mathbb{R}_1 \mathbb{R}_2}$.
Lemma 3.11. Suppose that $\mathcal{R}_1 \subset \mathcal{G}$ is finite and $\mathcal{R}_2 \subset \mathcal{G}$ has Property 2. Then there exists a finite set $\mathcal{R}_3 \subset \mathcal{G}$ such that $\mathcal{R}_1 \subset \mathcal{R}_3 \mathcal{R}_2$ and the seminorms $p_{\mathcal{R}_3 \mathcal{R}_2}$ and $q_{\mathcal{R}_3 \mathcal{R}_2}$ are equivalent.

Proof. Since $(\mathbb{R}^d)^{\mathcal{R}_3 \mathcal{R}_2}$ is finite dimensional, it suffices to show that there exists a finite set $\mathcal{R}_3 \subset \mathcal{G}$ with $\mathcal{R}_1 \subset \mathcal{R}_3 \mathcal{R}_2$ and

$$\ker(q_{\mathcal{R}_3 \mathcal{R}_2}) = U_{iso}(\mathcal{R}_3 \mathcal{R}_2).$$

First we show that $U_{iso}(\mathcal{R}_3 \mathcal{R}_2) \subset \ker(q_{\mathcal{R}_3 \mathcal{R}_2})$ for all finite sets $\mathcal{R}_3 \subset \mathcal{G}$ with $\mathcal{R}_1 \subset \mathcal{R}_3 \mathcal{R}_2$. Let $u \in U_{iso}(\mathcal{R}_3 \mathcal{R}_2)$. As there are $a \in \mathbb{R}^d$ and $S \in \text{Skew}(d)$ such that for all $h \in \mathcal{R}_2$ and $g \in \mathcal{R}_3$

$$L(h)u(gh) = L(g)^T a + L(g)^T S((gh) \cdot x_0 - x_0)$$

$$= L(g)^T a + L(g)^T S(g \cdot x_0 - x_0) + L(g)^T SL(g)(h \cdot x_0 - x_0),$$

we see that $u(g \cdot \mathcal{R}_3) \subset U_{iso}(\mathcal{R}_2)$ for every $g \in \mathcal{R}_3$. Since $p_{\mathcal{R}_2}$ vanishes on $U_{iso}(\mathcal{R}_2)$, it follows

$$q_{\mathcal{R}_3 \mathcal{R}_2}^2(u) = \sum_{g \in \mathcal{R}_3} p_{\mathcal{R}_2}^2(u(g \cdot \mathcal{R}_3)) = 0.$$

Hence, we have $U_{iso}(\mathcal{R}_3 \mathcal{R}_2) \subset \ker(q_{\mathcal{R}_3 \mathcal{R}_2})$.

Now we show that there exists some finite set $\mathcal{R}_3 \subset \mathcal{G}$ such that $\ker(q_{\mathcal{R}_3 \mathcal{R}_2}) \subset U_{iso}(\mathcal{R}_3 \mathcal{R}_2)$. By Property 2 of $\mathcal{R}_2$ there exist finite sets $\mathcal{R}_2', \mathcal{R}_2'' \subset \mathcal{G}$ such that $\text{id} \in \mathcal{R}_2'$, $\mathcal{R}_2'$ generates $\mathcal{G}$, $\mathcal{R}_2''$ has Property 1 and

$$\mathcal{R}_2' \mathcal{R}_2'' \subset \mathcal{R}_2.$$

Since $\mathcal{R}_2'$ generates $\mathcal{G}$, there exists some $n_0 \in \mathbb{N}$ such that

$$\mathcal{R}_1 \subset \{\text{id}\} \cup \bigcup_{k=1}^{n_0} \{g_1 \ldots g_k \mid g_1, \ldots, g_k \in \mathcal{R}_2' \cup (\mathcal{R}_2')^{-1}\}.$$

Let

$$\mathcal{R}_4 = \{\text{id}\} \cup \bigcup_{k=1}^{n_0} \{g_1 \ldots g_k \mid g_1, \ldots, g_k \in \mathcal{R}_2' \cup (\mathcal{R}_2')^{-1}\}.$$  

Let $u \in \ker(q_{\mathcal{R}_3 \mathcal{R}_2})$. By Definition 3.10 and Definition 3.11 for all $g \in \mathcal{R}_3$ there exist some $a(g) \in \mathbb{R}^d$ and $S(g) \in \text{Skew}(d)$ such that

$$L(h)u(gh) = a(g) + S(g)(h \cdot x_0 - x_0) \quad \text{for all } h \in \mathcal{R}_2. \quad (2)$$

Since $\mathcal{G} \cdot x_0 \subset \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^d_{\text{aff}}$, we have $h \cdot x_0 - x_0 \in \{0_{d-d_{\text{aff}}} \} \times \mathbb{R}^d_{\text{aff}}$ for all $h \in \mathcal{R}_2$. Hence, for all $g \in \mathcal{R}_3$ we may assume

$$S(g) = \begin{pmatrix} 0 & S_1(g) \\ -S_1(g)^T & S_2(g) \end{pmatrix}$$

for some $S_1(g) \in \mathbb{R}^{(d-d_{\text{aff}}) \times d_{\text{aff}}}$ and $S_2(g) \in \text{Skew}(d_{\text{aff}})$. We prove inductively that for $n = 0, 1, \ldots, n_0$ for all $g \in \{\text{id}\} \cup \bigcup_{k=1}^{n} \{g_1 \ldots g_k \mid g_1, \ldots, g_k \in \mathcal{R}_2' \cup (\mathcal{R}_2')^{-1}\}$ it holds

$$L(g)a(g) = a(id) + S(id)(g \cdot x_0 - x_0) \quad \text{and} \quad S(g) = L(g)^T S(id) L(g). \quad (3)$$

For $n = 0$ the induction hypothesis is true.

We assume the induction hypothesis holds for arbitrary but fixed $0 \leq n < n_0$. Let $g \in \{\text{id}\} \cup \bigcup_{k=1}^{n} \{g_1 \ldots g_k \mid g_1, \ldots, g_k \in \mathcal{R}_2' \cup (\mathcal{R}_2')^{-1}\}$ and $r \in \mathcal{R}_2' \cup (\mathcal{R}_2')^{-1}$.
Case 1: $r \in R'$. 
Since $g \in \mathcal{R}_3$ and $r\mathcal{R}_2^G \subseteq \mathcal{R}_2$, by (2) we have
\[
L(rh)u(grh) = a(g) + S(g)((rh) \cdot x_0 - x_0)
\quad \text{for all } h \in \mathcal{R}_2^G. \tag{4}
\]

Since $gr \in \mathcal{R}_3$ and $\mathcal{R}_2^G \subseteq \mathcal{R}_2$, by (2) we have
\[
L(h)u(grh) = a(gr) + S(gr)(h \cdot x_0 - x_0)
\quad \text{for all } h \in \mathcal{R}_2^G. \tag{5}
\]

By (4) and (5) we have
\[
L(r)u(gr) + L(r)S(gr)(h \cdot x_0 - x_0) = a(g) + S(g)((rh) \cdot x_0 - x_0)
\quad \text{for all } h \in \mathcal{R}_2^G. \tag{6}
\]

By Lemma 2.11 there exist some $\mathcal{B} \in \mathbb{R}^{d \times |\mathcal{R}_2^G|}$ of rank $d_{\text{aff}}$ such that
\[
(h \cdot x_0 - x_0)_{h \in \mathcal{R}_2^G} = \left(\begin{array}{c}
0_{d_{\text{aff}}, |\mathcal{R}_2^G|} \\
\mathcal{A}
\end{array} \right).
\]

By (6) and (7) we have
\[
L(r)S(gr)(h \cdot x_0 - x_0) = S(g)((rh) \cdot x_0 - r \cdot x_0)
= S(g)L(r)(h \cdot x_0 - x_0) \tag{8}
\]

By Lemma 3.9 there exists some $\mathcal{A} \in \mathbb{R}^{d \times |\mathcal{R}_2^G|}$ of rank $d_{\text{aff}}$ such that
\[
(S(gr) - L(gr)^T S(id)L(gr)) \left(\begin{array}{c}
0 \\
\mathcal{A}
\end{array} \right) = 0. \tag{9}
\]

By Lemma 2.11 there exist some $B_{gr} \in \mathbb{O}(d - d_{\text{aff}})$ and $C_{gr} \in \mathbb{O}(d_{\text{aff}})$ such that
$L(gr) = B_{gr} \oplus C_{gr}$. Equation (9) is equivalent to
\[
\left(\begin{array}{c}
(S_1(gr) - B_{gr}^T S_1(id)C_{gr}) \mathcal{A} \\
(S_2(gr) - C_{gr}^T S_2(id)C_{gr}) \mathcal{A}
\end{array} \right) = 0.
\]

Case 2: $r^{-1} \in \mathcal{R}_2^G$. 
Since $g \in \mathcal{R}_3$ and $\mathcal{R}_2^G \subseteq \mathcal{R}_2$, by (2) we have
\[
L(h)u(gh) = a(g) + S(g)(h \cdot x_0 - x_0)
\quad \text{for all } h \in \mathcal{R}_2^G. \tag{10}
\]

Since $gr \in \mathcal{R}_3$ and $r^{-1}\mathcal{R}_2^G \subseteq \mathcal{R}_2$, by (2) we have
\[
L(r^{-1}h)u(gh) = a(gr) + S(gr)((r^{-1}h) \cdot x_0 - x_0)
\quad \text{for all } h \in \mathcal{R}_2^G. \tag{11}
\]
By \((10)\) and \((11)\) we have

\[
a(gr) + S(gr)((r^{-1}h) \cdot x_0 - x_0) = L(r)^T a(g) + L(r)^T S(g)(h \cdot x_0 - x_0)
\]  

(12)
for all \(h \in \mathcal{R}_2'\). Since \(id \in \mathcal{R}_2'\), by \((12)\) we have

\[
a(gr) + S(gr)(r^{-1} \cdot x_0 - x_0) = L(r)^T a(g).
\]  

(13)

By \((12)\) and \((13)\) we have

\[
S(gr)((r^{-1}h) \cdot x_0 - x_0) = S(gr)(r^{-1} \cdot x_0 - x_0) + L(r)^T S(g)(h \cdot x_0 - x_0)
\]

for all \(h \in \mathcal{R}_2'\). This is equivalent to

\[
S(gr)L(r)^T (h \cdot x_0 - x_0) = L(r)^T S(g)(h \cdot x_0 - x_0)
\]

(14)
for all \(h \in \mathcal{R}_2'\). By Lemma 3.9 there exists some \(A \in \mathbb{R}^{d_{aff} \times |\mathcal{R}_2'|}\) of rank \(d_{aff}\) such that

\[
(h \cdot x_0 - x_0)_{h \in \mathcal{R}_2'} = \begin{pmatrix} 0_{d_{aff} \times |\mathcal{R}_2'|} \\ A \end{pmatrix}.
\]

By \((14)\) and the induction hypothesis we have

\[
(S(gr) - L(gr)^T S(id)L(gr))L(r)^T \begin{pmatrix} 0 \\ A \end{pmatrix} = 0.
\]  

(15)

By Lemma 2.11 there exist \(B_r, B_{gr} \in O(d - d_{aff})\) and \(C_r, C_{gr} \in O(d_{aff})\) such that \(L(r) = B_r \oplus C_r\) and \(L(gr) = B_{gr} \oplus C_{gr}\). Equation \((15)\) is equivalent to

\[
\begin{pmatrix} (S_1(gr) - B_{gr}^T S_1(id)C_{gr}C_r^T A) \\ (S_2(gr) - C_{gr}^T S_2(id)C_{gr}C_r^T A) \end{pmatrix} = 0.
\]

Since \(C_r\) is invertible and the rank of \(A\) is equal to the number of its rows, we have \(S_1(gr) = B_{gr}^T S_1(id)C_{gr}\) and \(S_2(gr) = C_{gr}^T S_2(id)C_{gr}\) which is equivalent to \(S(gr) = L(gr)^T S(id)L(gr)\). Since \(S(gr) = L(gr)^T S(id)L(gr)\), we have by \((13)\) and the induction hypothesis that

\[
L(gr)a(gr) = L(g)a(g) - L(gr)S(gr)(r^{-1} \cdot x_0 - x_0)
\]

\[
= a(id) + S(id)(g \cdot x_0 - x_0) - S(id)L(gr)(r^{-1} \cdot x_0 - x_0)
\]

\[
= a(id) + S(id)(g \cdot x_0 - x_0).
\]

By \((2)\) and \((3)\) we have that

\[
L(g)u(g) = L(g)a(g) = a(id) + S(id)(g \cdot x_0 - x_0) \quad \text{for all } g \in \mathcal{R}_3 \mathcal{R}_2
\]

and thus, \(u \in U_{iso}(\mathcal{R}_3 \mathcal{R}_2)\).

\[\square\]

**Theorem 3.12.** Suppose that \(\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{G}\) have Property 2. Then the two seminorms \(\| \cdot \|_{\mathcal{R}_1}\) and \(\| \cdot \|_{\mathcal{R}_2}\) are equivalent.

**Proof.** Suppose that \(\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{G}\) have Property 2. It is sufficient to show that there exists a constant \(C > 0\) such that \(\| \cdot \|_{\mathcal{R}_1} \leq C \| \cdot \|_{\mathcal{R}_2}\). Since \(\mathcal{R}_1\) is finite, by Lemma 3.11 there
exists a finite set \( \mathcal{R}_3 \subset \mathcal{G} \) such that \( \mathcal{R}_1 \subset \mathcal{R}_3 \mathcal{R}_2 \) and some \( C > 0 \) with \( p_{\mathcal{R}_3 \mathcal{R}_2} \leq C \eta_{\mathcal{R}_3, \mathcal{R}_2} \).

Let \( u \in U_{\text{per}} \). There exists some \( N \in M_0 \) such that \( u \) is \( T^N \)-periodic. We have

\[
\| u \|_{\mathcal{R}_1}^2 \leq \| u \|_{\mathcal{R}_3 \mathcal{R}_2}^2
\]

\[
= \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} p_{\mathcal{R}_3 \mathcal{R}_2}^2(u(g \cdot))|_{\mathcal{R}_3 \mathcal{R}_2}
\]

\[
\leq \frac{C^2}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} q_{\mathcal{R}_3, \mathcal{R}_2}^2(u(g \cdot))|_{\mathcal{R}_3 \mathcal{R}_2}
\]

\[
= \frac{C^2}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \sum_{g \in \mathcal{C}_N} p_{\mathcal{R}_2}^2(u(g \cdot)|_{\mathcal{R}_2})
\]

\[
= \frac{C^2}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \sum_{g \in \mathcal{C}_N} p_{\mathcal{R}_2}^2(u(g \cdot)|_{\mathcal{R}_2})
\]

where we used that \( \mathcal{C}_N \tilde{g} \) is a representation set of \( \mathcal{G} / T^N \) for all \( \tilde{g} \in \mathcal{R}_3 \) in the last step. Hence, we have \( \| \cdot \|_{\mathcal{R}_1} \leq C|\mathcal{R}_3|^\frac{1}{2} \| \cdot \|_{\mathcal{R}_2} \). \( \square \)

**Remark 3.13.** In Theorem 3.12 the premise that \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) have Property 2 cannot be weakened to the premise that \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are generating sets of \( \mathcal{G} \) and have Property 1, see Example 3.30.

### 3.3 Intrinsic seminorms and their equivalence to local seminorms

We now define the seminorm \( \| \cdot \|_{\mathcal{R},0} \) which measure the local distance of a deformation to the subset of those isometries that vanish if both the preimage and target space are projected to \( \mathbb{R}^{d_2} \). Thus \( \| u \|_{\mathcal{R},0} \) controls the size of the corresponding part of the discrete gradient of the displacement \( u \) globally.

**Definition 3.14.** For all \( \mathcal{R} \subset \mathcal{G} \) we define the vector spaces

\[
U_{\text{rot},0}(\mathcal{R}) := \left\{ u : \mathcal{R} \to \mathbb{R}^d \mid \exists S \in \text{Skew}_{0,d_2}(d) \forall g \in \mathcal{R} : L(g)u(g) = S(g \cdot x_0 - x_0) \right\}
\]

\[
\subset U_{\text{rot}}(\mathcal{R})
\]

and

\[
U_{\text{iso},0}(\mathcal{R}) := U_{\text{trans}}(\mathcal{R}) + U_{\text{rot},0}(\mathcal{R}) \subset U_{\text{iso}}(\mathcal{R}),
\]

where

\[
\text{Skew}_{0,d_2}(d) := \left\{ \begin{pmatrix} S_1 & S_2 \\ -S_2^T & 0 \end{pmatrix} \mid S_1 \in \text{Skew}(d_1), S_2 \in \mathbb{R}^{d_1 \times d_2} \right\} \subset \text{Skew}(d).
\]

**Definition 3.15.** For all finite sets \( \mathcal{R} \subset \mathcal{G} \) we define the seminorms

\[
\| \cdot \|_{\mathcal{R},0} : U_{\text{per}} \to [0, \infty)
\]

\[
\left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \| \pi_{U_{\text{iso},0}(\mathcal{R})}(u(g \cdot)|_{\mathcal{R}}) \|_{\mathcal{R}}^2 \right)^{\frac{1}{2}} \text{ if } u \text{ is } T^N\text{-periodic},
\]

and

\[
\| \cdot \|_{\mathcal{R},\nabla,0} : U_{\text{per}} \to [0, \infty)
\]

\[
\left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \| \pi_{U_{\text{rot},0}(\mathcal{R})}((\nabla_{\mathcal{R}} u(g))|_{\mathcal{R}}) \|_{\mathcal{R}}^2 \right)^{\frac{1}{2}} \text{ if } u \text{ is } T^N\text{-periodic},
\]

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where \( \pi_{\text{id}}(\mathcal{R}) \) and \( \pi_{\text{id}}(\mathcal{R}) \) are the orthogonal projections on \( \{u: \mathcal{R} \to \mathbb{R}^d\} \) with respect to the norm \( \| \cdot \| \) with kernels \( U_{\text{id}}(\mathcal{R}) \) and \( U_{\text{id}}(\mathcal{R}) \), respectively.

**Remark 3.16.** We have \( \| \cdot \|_{\mathcal{R},\mathcal{V},0} = \| \cdot \|_{\mathcal{R}\backslash \{\text{id}\},\mathcal{V},0} \) for all finite sets \( \mathcal{R} \subset \mathcal{G} \).

**Proposition 3.17.** Let \( \mathcal{R} \subset \mathcal{G} \) be finite and \( \text{id} \in \mathcal{R} \). Then the seminorms \( \| \cdot \|_{\mathcal{R},0} \) and \( \| \cdot \|_{\mathcal{R},\mathcal{V},0} \) are equivalent.

**Proof.** The proof is analogous to the proof of Proposition 3.7.

As a final preparation we state the following elementary lemma, which is well-known, and include its short proof.

**Lemma 3.18.** There exists a constant \( c > 0 \) such that for every \( n \in \mathbb{N} \) it holds

\[
\| x \otimes y^T + A \| \geq c(\| x \otimes y^T \| + \| A \|) \quad \text{for all } x, y \in \mathbb{C}^n, \ A \in \text{Skew}(n, \mathbb{C}).
\]

**Proof.** Let \( x, y \in \mathbb{C}^n \) and \( A \in \text{Skew}(n, \mathbb{C}) \). Since \( \mathbb{C}^{n \times n} = \text{Sym}(n, \mathbb{C}) \oplus \text{Skew}(n, \mathbb{C}) \) we have

\[
\| x \otimes y^T + A \|^2 \geq \left\| \frac{1}{2}(x \otimes y^T + y \otimes x^T) \right\|^2 = \frac{1}{2} \| x \otimes y \|^2 + \frac{1}{2} \left( \sum_{i=1}^{n} x_i y_i \right)^2 \geq \frac{1}{2} \| x \otimes y \|^2.
\]

If \( \| A \| \leq 2 \| x \otimes y^T \| \), then

\[
\| x \otimes y^T + A \| \geq \frac{1}{\sqrt{2}} \| x \otimes y^T \| \geq \frac{1}{3\sqrt{2}} (\| x \otimes y^T \| + \| A \|).
\]

If \( \| A \| \geq 2 \| x \otimes y^T \| \), then

\[
\| x \otimes y^T + A \| \geq \| A \| - \| x \otimes y^T \| \geq \frac{1}{3} (\| x \otimes y^T \| + \| A \|).
\]

The following lemma provides a technical core estimate on which the proof of our main Theorem 3.21 hinges.

**Lemma 3.19.** Let \( n \in \mathbb{N}, \ q \in \mathbb{N}_0 \) and \( \beta_1, \ldots, \beta_q \in \mathbb{R} \). Then there exists an integer \( N \in \mathbb{N} \) such that

\[
\max_{m \in \{1, \ldots, N\}} \left\| a \otimes (\sin(m\alpha_1), \ldots, \sin(m\alpha_n)) + \sum_{k=1}^{q} \sin(m\beta_k) B_k + mS \right\| \geq \| S \|
\]

for all \( a \in \mathbb{C}^n \), \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), \( B_1, \ldots, B_q \in \mathbb{C}^{n \times n} \) and \( S \in \text{Skew}(n, \mathbb{C}) \).

**Remark 3.20.** If \( q = 0 \), then the term \( \sum_{k=1}^{q} \sin(m\beta_k) B_k \) is the empty sum.

**Proof.** It suffices to prove that there exists a constant \( c > 0 \) such that for all \( n \in \mathbb{N}, q \in \mathbb{N}_0 \) and \( \beta_1, \ldots, \beta_q \in \mathbb{R} \) there exists an integer \( N \in \mathbb{N} \) such that

\[
\max_{m \in \{1, \ldots, N\}} \left\| a \otimes (\sin(m\alpha_1), \ldots, \sin(m\alpha_n)) + \sum_{k=1}^{q} \sin(m\beta_k) B_k + mS \right\| \geq c \| S \|
\]
for all $a \in \mathbb{C}^n$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $B_1, \ldots, B_q \in \mathbb{C}^{n \times n}$ and $S \in \text{Skew}(n, \mathbb{C})$ due to the fact that for $N = \lceil \frac{1}{c} \rceil N$ we have

$$
\max_{m \in \{1, \ldots, N\}} \left\| a \otimes (\sin(m\alpha_1), \ldots, \sin(m\alpha_n)) + \sum_{k=1}^q \sin(m\beta_k)B_k + mS \right\|
\geq \max_{m \in \{1, \ldots, N\}} \left\| a \otimes (\sin(\lceil \frac{1}{c} \rceil \alpha_1), \ldots, \sin(\lceil \frac{1}{c} \rceil \alpha_n))
+ \sum_{k=1}^q \sin(\lceil \frac{1}{c} \rceil \beta_k)B_k + m(\lceil \frac{1}{c} \rceil )S \right\|.
$$

Since

$$\|M\| \geq \frac{1}{\eta^2} \sum_{i<j} \| (m_{ij}, m_{ji} ) \|$$

for all $M = (m_{ij}) \in \mathbb{C}^{n \times n}$, it suffices to prove the assertion for $n = 2$.

Let $q \in \mathbb{N}_0$ and $\beta_1, \ldots, \beta_q \in \mathbb{R}$. Without loss of generality we assume $\beta_1, \ldots, \beta_q \in \mathbb{R} \setminus (\pi\mathbb{Q})$.

Let $n_0 \in \mathbb{N}$ be such that $n_0\beta_k \in \pi\mathbb{Z}$ for all $k \in \{1, \ldots, q\}$ with $\beta_k \in \pi\mathbb{Q}$. Then we have

$$\max_{m \in \{1, \ldots, n_0N\}} \left\| a \otimes (\sin(m\alpha_1), \sin(m\alpha_2)) + \sum_{k=1}^q \sin(m\beta_k)B_k + mS \right\| \geq \max_{m \in \{1, \ldots, n_0N\}} \left\| a \otimes (\sin(mn_0\alpha_1), \sin(mn_0\alpha_2)) + \sum_{k=1}^q \sin(mn_0\beta_k)B_k + mn_0S \right\|$$

for all $N \in \mathbb{N}$, $a \in \mathbb{C}^2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $B_1, \ldots, B_q \in \mathbb{C}^{2 \times 2}$ and $S \in \text{Skew}(2, \mathbb{C})$.

For all $a > 0$ we define the function

$$| \cdot |_a : \mathbb{R} \to [0, \infty), \quad x \mapsto \text{dist}(x, a\mathbb{Z}).$$

Moreover, without loss of generality we may assume $|\beta_k - \beta_l| 2\pi > 0$ for all $k \neq l$ and since

$$\sin(m\beta) = -\sin(m(2\pi - \beta))$$

also $|\beta_k - \beta_l| 2\pi \neq 0$ for all $k \neq l$. For the definition of a suitable integer $N \in \mathbb{N}$ and the following proof we define some positive constants. By Lemma 3.18 there exists a constant $c_L > 0$ such that

$$\|x \otimes y^T + S\| \geq c_L \|x\|(|y_1| + |y_2|) + c_L \|S\|$$

for all $x, y \in \mathbb{C}^2$ and $S \in \text{Skew}(2, \mathbb{C})$. In particular, this inequality implies the assertion for $q = 0$. Hence we may assume $q \neq 0$, i.e. $q \in \mathbb{N}$. Let

$$\delta_1 = \min_{\gamma_1, \gamma_2 \in \{\pm \beta_1, \ldots, \pm \beta_q\}, \gamma_1 \neq \gamma_2} |\gamma_1 - \gamma_2| 2\pi, \quad \mu_1 = \frac{1}{2q} \left( \frac{\delta_1}{2\pi} \right)^{2q-1}, \quad C_1 = \frac{4(2q+1)}{\mu_1}, \quad C_2 = \frac{6q}{\mu_1} \quad \text{and} \quad C_3 = \max \left\{ \frac{4q + 2}{\mu_1}, \frac{32qC_2}{\delta_1} \right\}.$$
For all $\alpha \in \mathbb{R}$ we define $(\alpha)_{2\pi} \in \mathbb{R}$ by $(\alpha)_{2\pi} = \lfloor -\pi, \pi \rfloor (\alpha + 2\pi \mathbb{Z})$. We have $|(\alpha)_{2\pi}| = |\alpha|_{2\pi}$. By Taylor’s Theorem we have for all $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\sin(n\alpha) = \sin(n(\alpha - (\alpha - \beta)_{2\pi})) = \sin(n\beta) + n(\alpha - \beta)_{2\pi} \cos(n\beta) + R(n, \alpha, \beta)$$

where $R(n, \alpha, \beta)$ is the remainder term. Let $\delta_2 > 0$ be so small that

$$|R(n, \alpha, \beta)| \leq \frac{1}{2} n |\alpha - \beta|_{2\pi} \cos(n\beta)$$

for all $n \in \{1, \ldots, N_1\}$, $\alpha \in \mathbb{R}$ with $|\alpha - \beta|_{2\pi} < \delta_2$ and $\beta \in \{0, \pi, \beta_1, \ldots, \beta_q\}$. Let

$$\delta_2 = \min \{\delta_1, \delta_2\}, \quad \mu_2 = \frac{1}{2q + 2} \left( \frac{\delta_1}{2\pi} \right)^{2q+1} \quad \text{and} \quad C_4 = \frac{2q + 3}{\mu_2}.$$ 

Let

$$N = \max\{N_1, 1 + [C_4]\} \in \mathbb{N}.$$ 

Now, let $a = (a_1, a_2)^T \in \mathbb{C}^2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, $B_k = \left( \begin{smallmatrix} b_{11}^{(k)} & b_{12}^{(k)} \\ b_{21}^{(k)} & b_{22}^{(k)} \end{smallmatrix} \right) \in \mathbb{C}^{2 \times 2}$ for all $k \in \{1, \ldots, q\}$ and $S = (0_{q \times q}) \in \text{Skew}(2, \mathbb{C})$. We denote

$$\text{LHS} = \max_{m \in \{1, \ldots, N\}} \left\| a \otimes (\sin(m\alpha_1), \sin(m\alpha_2)) + \sum_{k=1}^{q} \sin(m\beta_k)B_k + mS \right\|.$$ 

**Case 1:** $\forall i \in \{1, 2\} : (|\alpha_i|_{2\pi} < \delta_2) \lor (|\alpha_i - \pi|_{2\pi} < \delta_2)$.

**Case 1.1:** $\sum_{k=1}^{q} \|B_k\| \geq C_1(\|a\|(|\alpha_1|_{2\pi} + |\alpha_2|_{2\pi}) + \|S\|)$.

Let $i, j \in \{1, 2\}$ with $\sum_{k=1}^{q} |b_{ij}^{(k)}| \geq \frac{1}{4} \sum_{k=1}^{q} \|B_k\|$. By the definition of $\delta_1$ we have

$$\min_{\gamma_1, \gamma_2 \in \{\pm \beta_1, \ldots, \pm \beta_q\}} |e^{i\gamma_1} - e^{i\gamma_2}| \geq \min_{\gamma_1, \gamma_2 \in \{\pm \beta_1, \ldots, \pm \beta_q\}} \frac{|\gamma_1 - \gamma_2|_{2\pi}}{\pi} \geq \frac{\delta_1}{\pi}.$$ 

By Turán’s Minimax Theorem [A.4] there exist an integer $\nu \in \{1, \ldots, 2q\}$ such that

$$\left\| \sum_{k=1}^{q} \sin(\nu\beta_k)B_k \right\| \geq \left\| \sum_{k=1}^{q} b_{ij}^{(k)} \sin(\nu\beta_k) \right\|$$

$$= \sum_{k=1}^{q} \left| \frac{ib_{ij}^{(k)}}{2} \mathrm{e}^{-i\nu\beta_k} - \frac{ib_{ij}^{(k)}}{2} \mathrm{e}^{i\nu\beta_k} \right|$$

$$\geq \mu_1 \sum_{k=1}^{q} |b_{ij}^{(k)}|$$

$$\geq \frac{\mu_1}{4} \sum_{k=1}^{q} \|B_k\|.$$ 

We have

$$\text{LHS} \geq \left| \sum_{k=1}^{q} \sin(\nu\beta_k)B_k \right| - \|a \otimes (\sin(\nu\alpha_1), \sin(\nu\alpha_2))\| - \|\nu S\|$$

$$\geq \frac{\mu_1}{4} \sum_{k=1}^{q} \|B_k\| - 2q\|a\|(|\alpha_1|_{2\pi} + |\alpha_2|_{2\pi}) - 2q\|S\|$$

$$\geq \|S\|.$$
Case 2: \[ \sum_{k=1}^{q} \|B_k\| \leq C_1(\|a\|(\|a_1\|_\pi + |a_2|_\pi) + \|S\|). \]

We have

\[
\text{LHS} \geq \|a \otimes (\sin(N_1\alpha_1), \sin(N_1\alpha_2)) + N_1S\| - \left\| \sum_{k=1}^{q} \sin(N_1\beta_k)B_k \right\|
\]

\[
\geq c_L\|a\|(\|\sin(N_1\alpha_1)\| + |\sin(N_1\alpha_2)|) + c_L\|N_1S\| - \sum_{k=1}^{q} \|B_k\|
\]

\[
\geq \frac{c_LN_1}{2}\|a\|(\|a_1\|_\pi + |a_2|_\pi) + c_L\|N_1S\| - \sum_{k=1}^{q} \|B_k\|
\]

\[
\geq \frac{c_LN_1}{2}(\|a\|(\|a_1\|_\pi + |a_2|_\pi) + \|S\|) + \frac{c_L}{2}\|S\| - \sum_{k=1}^{q} \|B_k\|
\]

\[
\geq \frac{c_L}{2}\|S\|.
\]

Case 2: \( \exists i \in \{1, 2\}, \exists k \in \{1, \ldots, q\} : (|\alpha_i - \beta_k|_\pi < \delta_2) \lor (|\alpha_i + \beta_k|_\pi < \delta_2) \).

Without loss of generality let \( i = 1 \) and \( k = 1 \). Without loss of generality we may assume \( |\alpha_1 - \beta_1|_\pi < \delta_2 \) since

\[
a \otimes (\sin(m(\alpha_1)), \sin(m(\alpha_2))) = (-a) \otimes (\sin(m(-\alpha_1)), \sin(m(-\alpha_2))) \text{ for all } m \in \mathbb{N}.
\]

Let \( \delta_k \) be equal to 1 if \( k = 0 \) and equal to 0 otherwise.

Case 2.1: \( \sum_{k=1}^{q} |a_2\delta_{k-1} + b_{21}^{(k)}| \geq C_2|a_2||\alpha_1 - \beta_1|_\pi \) and

\[
\max\{ |a_2|\|\alpha_1 - \beta_1|_\pi, \sum_{k=1}^{q} |a_2\delta_{k-1} + b_{21}^{(k)}| \} \geq C_3|s|.
\]

Since \( C_2 \geq 1 \) the condition is equivalent to

\[
\sum_{k=1}^{q} |a_2\delta_{k-1} + b_{21}^{(k)}| \geq C_2|a_2||\alpha_1 - \beta_1|_\pi \text{ and } \sum_{k=1}^{q} |a_2\delta_{k-1} + b_{21}^{(k)}| \geq C_3|s|.
\]

By Turán’s minimax theorem (analogously to Case 1.1) there exists an integer \( \nu \in \{1, \ldots, 2q\} \) such that

\[
\left| \sum_{k=1}^{q} (a_2\delta_{k-1} + b_{21}^{(k)}) \sin(\nu\beta_k) \right|
\]

\[
= \left| \sum_{k=1}^{q} \left( i(a_2\delta_{k-1} + b_{21}^{(k)}) e^{-i\nu\beta_k} + \frac{-i(a_2\delta_{k-1} + b_{21}^{(k)}) e^{i\nu\beta_k}}{2} \right) \right|
\]

\[
\geq \mu_1 \sum_{k=1}^{q} |a_2\delta_{k-1} + b_{21}^{(k)}|.
\]

We have

\[
\text{LHS} \geq \left| \sum_{k=1}^{q} (a_2\delta_{k-1} + b_{21}^{(k)}) \sin(\nu\beta_k) \right| - \frac{3}{2} |a_2||\nu|\|\alpha_1 - \beta_1|_\pi \|\cos(\nu\beta_1)\| - \nu|s|
\]

\[
\geq \left( \frac{\mu_1}{2} + \frac{\mu_1}{2} \right) \sum_{k=1}^{q} |a_2\delta_{k-1} + b_{21}^{(k)}| - 3q |a_2||\alpha_1 - \beta_1|_\pi - 2q|s|
\]

\[
\geq |s|
\]

\[
eq \frac{1}{\sqrt{2}} \|S\|.
\]
Case 2.2: \( \sum_{k=1}^{q} |a_2 \delta_{k-1} + b_{21}^{(k)}| \leq C_2|a_2| |\alpha_1 - \beta_1|_{2\pi} \) and
\[ \max\{|a_2| |\alpha_1 - \beta_1|_{2\pi}, \sum_{k=1}^{q} |a_2 \delta_{k-1} + b_{21}^{(k)}|\} \leq C_3|s|. \]
By Turán’s minimax theorem there exists an integer \( \nu \in \{N_1 - 1, N_1\} \) such that
\[ |\cos(\nu \beta_1)| = \left| \frac{1}{2}e^{i\nu \beta_1} + \frac{1}{2}e^{-i\nu \beta_1} \right| \geq \frac{\delta_1}{4\pi}. \]
We have
\[ \text{LHS} \geq \frac{1}{2} |a_2| |\cos(\nu \beta_1)| |\nu| |\alpha_1 - \beta_1|_{2\pi} - \sum_{k=1}^{q} (a_2 \delta_{k-1} + b_{21}^{(k)}) \sin(\nu \beta_k)\right| - |\nu|s| \]
\[ \geq \left( \frac{\delta_1(N_1 - 1)}{16\pi} + \frac{\delta_1\nu}{10\pi}\right) |a_2| |\alpha_1 - \beta_1|_{2\pi} - \sum_{k=1}^{q} |a_2 \delta_{k-1} + b_{21}^{(k)}| - |\nu|s| \]
\[ \geq \nu|s| \]
\[ \geq \frac{1}{\sqrt{2}}\|S\|. \]

Case 2.3: \( \max\{|a_2| |\alpha_1 - \beta_1|_{2\pi}, \sum_{k=1}^{q} |a_2 \delta_{k-1} + b_{21}^{(k)}|\} \leq C_3|s|. \]
By Definition of \( q_1 \) we have
\[ |\cos(q_1 \beta_1)| = |\sin(q_1 \beta_1 + \frac{\pi}{2})| \leq |q_1 \beta_1 + \frac{\pi}{2}|_{\pi} = \pi|q_1 \beta_1 + \frac{\pi}{2}|_{\pi} \leq \frac{1}{3\pi}. \]
So we have
\[ \text{LHS} \geq q_1|s| - \frac{q_1}{2} |a_2| |\cos(q_1 \beta_1)| q_1 |\alpha_1 - \beta_1|_{2\pi} - \sum_{k=1}^{q} (a_2 \delta_{k-1} + b_{21}^{(k)}) \sin(q_1 \beta_k)\right| \]
\[ \geq \left(1 + \frac{q_1}{2} + C_3\right)|s| - \frac{q_1}{2C_3} |a_2| |\alpha_1 - \beta_1|_{2\pi} - \sum_{k=1}^{q} |a_2 \delta_{k-1} + b_{21}^{(k)}| \]
\[ \geq \frac{1}{\sqrt{2}}\|S\|. \]

Case 3: \( \exists i \in \{1, 2\} : (|\alpha_i - \beta|_{2\pi} \geq \delta_2 \forall \beta \in \{0, \pi, \pm \beta_1, \ldots, \pm \beta_q\}). \)
Without loss of generality let \( i = 1. \)

Case 3.1: \( |a_2| + \sum_{k=1}^{q} |b_{21}^{(k)}| \geq C_4|s|. \)
By Definition of \( \delta_3 \) we have
\[ \min_{\gamma_1, \gamma_2 \in \{\pm \alpha_1, \pm \beta_1, \ldots, \pm \beta_q\}} |e^{i\gamma_1} - e^{i\gamma_2}| \geq \min_{\gamma_1, \gamma_2 \in \{\pm \alpha_1, \pm \beta_1, \ldots, \pm \beta_q\}} \frac{|\gamma_1 - \gamma_2|_{2\pi}}{\pi} \]
\[ \geq \frac{\min\{\delta_1, \delta_2\}}{\pi} \]
\[ = \frac{\delta_3}{\pi}. \]
By Turán’s minimax theorem there exists an integer \( \nu \in \{1, \ldots, 2q + 2\} \) such that
\[ |a_2 \sin(\nu \alpha_1) + \sum_{k=1}^{q} b_{21}^{(k)} \sin(\nu \beta_k)| \]
\[ = \left| \frac{ia_2}{2}e^{-i\nu \alpha_1} - \frac{ia_2}{2}e^{i\nu \alpha_1} + \sum_{k=1}^{q} \left( \frac{ib_{21}^{(k)}}{2}e^{-i\nu \beta_k} + \frac{ib_{21}^{(k)}}{2}e^{i\nu \beta_k}\right) \right| \]
\[ \geq \mu_2 \left( |a_2| + \sum_{k=1}^{q} |b_{21}^{(k)}| \right). \]
We have
\[
\text{LHS} \geq |a_2 \sin(\nu \alpha_1) + \sum_{k=1}^{q} b_{21}^{(k)} \sin(\nu \beta_k)| - \nu|s|
\]
\[
\geq \mu_2 \left( |a_2| + \sum_{k=1}^{q} |b_{21}^{(k)}| \right) - (2q + 2)|s|
\]
\[
\geq |s| - \frac{1}{\sqrt{2}} \|S\|
\]

Case 3.2: \( |a_2| + \sum_{k=1}^{q} |b_{21}^{(k)}| \leq C_4|s| \).
We have
\[
\text{LHS} \geq N|s| - |a_2 \sin(N \alpha_1)| - \left| \sum_{k=1}^{q} b_{21}^{(k)} \sin(N \beta_k) \right|
\]
\[
\geq N|s| - \left( |a_2| + \sum_{k=1}^{q} |b_{21}^{(k)}| \right)
\]
\[
\geq |s| - \frac{1}{\sqrt{2}} \|S\|
\]

Since Case 2 and Case 3 include the case that
\[\exists i \in \{1, 2\} : (|\alpha_i|_{2\pi} \geq \delta_2) \land (|\alpha_i - \pi|_{2\pi} \geq \delta_2),\]
the assertion is proven.

\[\square\]

**Theorem 3.21** (A discrete Korn’s inequality). Suppose that \( \mathcal{R} \subset \mathcal{G} \) has Property 2. Then the two seminorms \( \| \cdot \|_{\mathcal{R}} \) and \( \| \cdot \|_{\mathcal{R},0} \) are equivalent.

**Proof.** Suppose that \( \mathcal{R} \subset \mathcal{G} \) has Property 2.

First we show the trivial inequality \( \| \cdot \|_{\mathcal{R}} \leq \| \cdot \|_{\mathcal{R},0} \):

Let \( u \in U_{\text{per}} \). Let \( N \in M_0 \) be such that \( u \) is \( \mathcal{T}^N \)-periodic. Since \( U_{\text{iso,0}}(\mathcal{R}) \subset U_{\text{iso}}(\mathcal{R}) \), we have
\[
\|u\|_{\mathcal{R}}^2 = \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \|\pi_{U_{\text{iso}}(\mathcal{R})}(u(g \cdot) | \mathcal{R})\|^2
\]
\[
\leq \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \|\pi_{U_{\text{iso,0}}(\mathcal{R})}(u(g \cdot) | \mathcal{R})\|^2
\]
\[
= \|u\|_{\mathcal{R},0}^2.
\]

Now we show with the aid of the Plancherel formula that there exists a constant \( c > 0 \) such that \( \| \cdot \|_{\mathcal{R}} \geq c \| \cdot \|_{\mathcal{R},0} \):

We choose \( m = m_0 \) such that \( M_0 = m \mathbb{N} \) and the group \( \mathcal{T}^m \) is isomorphic to \( \mathbb{Z}^{d_2} \), see Section 2.1. In particular, there exist \( t_1, \ldots, t_{d_2} \in \mathcal{T}^m \) such that \( \{t_1, \ldots, t_{d_2}\} \) generates \( \mathcal{T}^m \). Since \( L(\mathcal{T}^m) \) is a subgroup of \( \oplus \{I_{d_1-d_2} \} \times O(d_{\text{aff}}-d_2) \times \{I_{d_2}\} \) and the elements \( t_1, \ldots, t_{d_2} \) commute, by Theorem A.1 we may without loss of generality (by a coordinate transformation) assume that for all \( i \in \{1, \ldots, d_2\} \) there exist an integer \( q_i \in \{0, \ldots, [(d_{\text{aff}} - d_2)/2]\} \), a vector \( v_i \in \{-1\}^{d_{\text{aff}}-d_2-2q_i} \) and angles \( \theta_{i,1}, \ldots, \theta_{i,q_i} \in [0, 2\pi) \) such that
\[
L(t_i) = I_{d_{\text{aff}}} + \text{diag}(v_i) + R(\theta_{i,q_i}) \oplus \cdots \oplus R(\theta_{i,1}) \oplus I_{d_2}.
\]
By Lemma 3.19 there exists an integer $N_0 \in \mathbb{N}$ such that
\[
\max_{n \in \{1, \ldots, N_0\}} \left\| a \otimes (\sin(n\alpha_1), \ldots, \sin(n\alpha_d)) - \sum_{i=1}^{d_2} \sum_{j=1}^{q_i} \sin(n\theta_{i,j}) B_{i,j} - nS \right\| \geq \|S\| \tag{17}
\]
for all $a \in \mathbb{C}^{d_2}$, $\alpha_1, \ldots, \alpha_d \in [0, 2\pi)$, $B_{1,1}, \ldots, B_{d_2,d_2} \in \mathbb{C}^{d_2 \times d_2}$, and $S \in \text{Skew}(d_2, \mathbb{C})$. Let $\mathcal{R}_0 = \{t^n_i \mid i \in \{1, \ldots, d_2\}, n \in \{\pm 1, \ldots, \pm N_0\}\} \subset \mathcal{T}^m$. Since $\|\cdot\|_{\mathcal{R} \cup \mathcal{R}_0} \geq \|\cdot\|_{\mathcal{R}_0}$ and by Theorem 3.12 we may without loss of generality assume that $\mathcal{R}_0 \subset \mathcal{R}$. For all finite sets $\mathcal{R}' \subset \mathcal{G}$ we define the map
\[
g_{\mathcal{R}'} : \text{Skew}(d, \mathbb{C}) \to \mathbb{C}^{d^2 |\mathcal{R}'|} \quad S \mapsto (L(h)^T S(h \cdot x_0 - x_0))_{h \in \mathcal{R}'}.
\]
Recall the definition of the dual space $\overline{\mathcal{T}^m}$ from Section 2.1. Now we show that there exists a constant $c_0 > 0$ such that
\[
\| (\chi(h)v - L(h)^T v)_{h \in \mathcal{R}_0} - g_{\mathcal{R}_0}(S) \| \geq c_0 \|S\| \tag{18}
\]
for all $\chi \in \overline{\mathcal{T}^m}$, $v \in \mathbb{C}^d$ and $S = \left( \frac{s_1}{s_2}, \frac{s_2}{s_3} \right) \in \text{Skew}(d_1 + d_2, \mathbb{C})$.

Writing $v = \left( \begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix} \right) \in \mathbb{C}^{d_1 + d_2}$ we have
\[
\text{LHS} := \| (\chi(h)v - L(h)^T v)_{h \in \mathcal{R}_0} - g_{\mathcal{R}_0}(S) \| \\
\geq \| (\chi(h)v_2 - v_2 - (S_2, S_3)(h \cdot x_0 - x_0))_{h \in \mathcal{R}_0} \| \\
\geq \frac{1}{\sqrt{2}} \left( \| (\chi(t_1^n)v_2 - v_2 - (S_2, S_3)(t_1^n \cdot x_0 - x_0))_{i \in \{1, \ldots, d_2\}} \| + \| (\chi(t_1^{-n})v_2 - v_2 - (S_2, S_3)(t_1^{-n} \cdot x_0 - x_0))_{i \in \{1, \ldots, d_2\}} \| \right) \\
\geq \frac{1}{\sqrt{2}} \| (\chi(t_1^n) - \chi(t_1^{-n}))v_2 - (S_2, S_3)(t_1^n \cdot x_0 - t_1^{-n} \cdot x_0) \|_{i \in \{1, \ldots, d_2\}} \tag{19}
\]
for all $n \in \{1, \ldots, N_0\}$. For all $j \in \{1, \ldots, d_2\}$ we define $\alpha_j \in [0, 2\pi)$ by $e^{i\alpha_j} = \chi(t_j)$. Let $x_{0,1} \in \mathbb{R}^{d_1}$ and $x_{0,2} \in \mathbb{R}^{d_2}$ be such that $x_0 = \left( \begin{smallmatrix} x_{0,1} \\ x_{0,2} \end{smallmatrix} \right)$. For all $j \in \{1, \ldots, \max\{q_1, \ldots, q_{d_2}\}\}$ we define $n_j = d_1 - 2j$, $m_j = 2j - 2$ and
\[
b_j = S_2(0_{n_j, n_j} \oplus (\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}) \oplus 0_{m_j, m_j})x_{0,1} \in \mathbb{C}^{d_2}.
\]
Let $T_2 : \mathcal{T}^m \to \mathbb{R}^{d_2}$ be uniquely defined by the condition $T_2 = \left( \begin{smallmatrix} 0_{d_1} \\ T_2(t) \end{smallmatrix} \right)$ for all $t \in \mathcal{T}^m$.

Then for all $i \in \{1, \ldots, d_2\}$ and $n \in \{1, \ldots, N_0\}$ we have
\[
(S_2, S_3)(t_1^n \cdot x_0 - t_1^{-n} \cdot x_0) = S_2(0_{d_1-2q_i, d_1-2q_i} \oplus (R(n\theta_{i,q_i}) - R(-n\theta_{i,q_i})) \oplus \cdots \oplus (R(n\theta_{i,1}) - R(-n\theta_{i,1})))x_{0,1} + 2nS_3T_2(t_i) \\
= \sum_{j=1}^{q_i} \sin(n\theta_{i,j})S_2(0_{n_j, n_j} \oplus (\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}) \oplus 0_{m_j, m_j})x_{0,1} + 2nS_3T_2(t_i) \\
= \sum_{j=1}^{q_i} \sin(n\theta_{i,j})b_j + 2nS_3T_2(t_i).
\]
For all \( i \in \{1, \ldots, d_2 \} \) and \( j \in \{1, \ldots, q_i \} \) we define \( B_{i,j} = b_j \otimes e_i^T \in \mathbb{C}^{d_2 \times d_2} \). Let 
\( T = 2(\tau_2(t_1), \ldots, \tau_2(t_{d_2})) \in \text{GL}(d_2) \). By equation \((19)\) for all \( n \in \{1, \ldots, N_0 \} \) we have

\[
\text{LHS} \geq \frac{1}{\sqrt{2}} \left\| 2v_2 \otimes (\sin(n\alpha_1), \ldots, \sin(n\alpha_{d_2})) - \sum_{i=1}^{d_2} \sum_{j=1}^{q_i} \sin(n\theta_{i,j})B_{i,j} - nS_3T \right\|
\]

\[ 
\geq c_1 \left\| (2T^T v_2) \otimes (\sin(n\alpha_1), \ldots, \sin(n\alpha_{d_2})) - \sum_{i=1}^{d_2} \sum_{j=1}^{q_i} \sin(n\theta_{i,j})T^T B_{i,j} - nT^T S_3T \right\|, 
\]

where \( c_1 = \sigma_{\text{min}}(T^{-T})/\sqrt{2} > 0 \), \( \sigma_{\text{min}}(M) \) denotes the minimum singular value of a matrix \( M \) and we used Theorem A.2 in the last step. With equation \((17)\) it follows

\[
\text{LHS} \geq c_1 \| T^T S_3 T \| \geq c_0 \| S_3 \|, 
\]

where \( c_0 = \sigma_{\text{min}}(T)^2 c_1 > 0 \).

By Propositions 3.7 and 3.17 it suffices to show that there exists a constant \( c > 0 \) such that \( \| \cdot \|_{R, V} \geq c \| \cdot \|_{R, V, 0} \). Let \( u \in U_{\text{per}} \). Let \( N \in M_0 \) be such that \( u \) is \( T^N \)-periodic. In particular, \( m \) divides \( N \). Let \( v : G \to \text{Skew}(d) \) be \( T^N \)-periodic such that \( \pi_{U_{\text{rot}(R)}}(\nabla_R u(g)) = \nabla_R u(g) - g_R \circ v(g) \) for all \( g \in G \). Let

\[
v_1 : G \to \left\{ \begin{pmatrix} S_1 & S_2 \\ -S_2^T & 0 \end{pmatrix} \right| S_1 \in \text{Skew}(d_1), S_2 \in \mathbb{R}^{d_1 \times d_2} \}
\]

and

\[
v_2 : G \to \{0_{d_1, d_1} \oplus S \mid S \in \text{Skew}(d_2)\}
\]

such that \( v = v_1 + v_2 \). For all \( g \in C_m \) we define the functions

\[
\begin{align*}
ug : T^m &\to \mathbb{C}^d, t \mapsto u(gt) \\
v_g : T^m &\to \text{Skew}(d, \mathbb{C}), t \mapsto v(gt) \\
v_{1,g} : T^m &\to \text{Skew}(d, \mathbb{C}), t \mapsto v_1(gt) \\
v_{2,g} : T^m &\to \text{Skew}(d, \mathbb{C}), t \mapsto v_2(gt).
\end{align*}
\]

Let \( \mathcal{E} = \{ \chi \in \overline{T^m} \mid \chi \text{ is periodic} \} \). For all \( g \in C_m \) and \( \chi \in \mathcal{E} \) it holds

\[
\tilde{v}_g(\chi) = \tilde{v}_{1,g}(\chi) + \tilde{v}_{2,g}(\chi),
\]

\[
\tilde{v}_{1,g}(\chi) \in \left\{ \begin{pmatrix} S_1 & S_2 \\ -S_2^T & 0 \end{pmatrix} \right| S_1 \in \text{Skew}(d_1, \mathbb{C}), S_2 \in \mathbb{C}^{d_1 \times d_2} \}
\]

and

\[
\tilde{v}_{2,g}(\chi) \in \{0_{d_1, d_1} \oplus S \mid S \in \text{Skew}(d_2, \mathbb{C})\}.
\]

We have

\[
\| u \|_{R, V}^2 = \frac{1}{|C_N|} \sum_{(g,t) \in \mathcal{C}_m \times (T^m \cap \mathcal{C}_N)} \| \pi_{U_{\text{rot}(R)}}(\nabla_R u(gt)) \|^2
\]

\[ 
= \frac{1}{|C_N|} \sum_{g \in \mathcal{C}_m} \sum_{t \in T^m \cap \mathcal{C}_N} \| \nabla_R u(gt) - g_R \circ v(gt) \|^2
\]

\[ 
\geq \frac{1}{|C_N|} \sum_{g \in \mathcal{C}_m} \sum_{t \in T^m \cap \mathcal{C}_N} \| \nabla_R u(gt) - g_R \circ v(gt) \|^2
\]

\[
\geq \frac{1}{|C_N|} \sum_{g \in \mathcal{C}_m} \sum_{t \in T^m \cap \mathcal{C}_N} \| \nabla_R u(gt) - g_R \circ v(gt) \|^2.
\]
\[
\begin{align*}
\|u\|_{R,N}^2 &= \frac{1}{|C_N|} \sum_{g \in C_N} \sum_{t \in T^m \cap C_N} \left\| (u_g(th) - L(h)^T u_g(t))_{h \in R_0} - g_{R_0} \circ v_g(t) \right\|^2 \\
&= \frac{1}{|C_N|} \sum_{g \in C_N} |T^m \cap C_N| \sum_{\chi \in \mathcal{E}} \left\| (\chi(h)^{-1} u_g(\chi) - L(h)^T u_g(\chi))_{h \in R_0} - g_{R_0} \circ \tilde{v}_g(\chi) \right\|^2 \\
&\geq \frac{c_0^2}{|C_N|} \sum_{g \in C_N} |T^m \cap C_N| \sum_{\chi \in \mathcal{E}} \left\| \tilde{v}_{2,g}(\chi) \right\|^2 \\
&= \frac{c_0^2}{|C_N|} \sum_{g \in C_N} \sum_{t \in T^m \cap C_N} \left\| v_{2,g}(t) \right\|^2 \\
&= \frac{c_0^2}{|C_N|} \sum_{(g,t) \in C_N \times (T^m \cap C_N)} \left\| v_{2}(gt) \right\|^2 \\
&= c_0^2 \| v_2 \|_2^2. \quad (20)
\end{align*}
\]

In the first and last step we used that the set \( \bigcup_{(g,t) \in C_N \times (T^m \cap C_N)} \{gt\} \) is a representation set of \( \mathcal{G}/\mathcal{T}^N \). In the fifth and seventh step we used Proposition 2.3 for the group \( T^m \) and \( \mathcal{T}^N \)-periodic functions and Lemma 2.4. Note that \( T^m \cap C_N \) is a representation set of \( T^m/\mathcal{T}^N \). In the sixth step we used (18). Let \( C = |R| \max\{||h \cdot x_0 - x_0|| | h \in R\} \). We have

\[
\|u\|_{R,N}^2 = \frac{1}{|C_N|} \sum_{g \in C_N} \| \nabla R u(g) - g \circ v(g) \|^2 \\
&\geq \frac{1}{|C_N|} \sum_{g \in C_N} \left( \frac{1}{2} \| \nabla R u(g) - g \circ v_1(g) \|^2 - \| g \circ v_2(g) \|^2 \right) \\
&\geq \frac{1}{|C_N|} \sum_{g \in C_N} \left( \frac{1}{2} \| \pi_{U_{0,0,0}(R)}(\nabla R u(g)) \|^2 - C \| v_2(g) \|^2 \right) \\
&= \frac{1}{2} \| u \|_{R,N,0}^2 - C \| v_2 \|_2^2, \quad (21)
\]

where in the second step we used that \((a-b)^2 \geq a^2/2 - b^2\) for all \(a, b \geq 0\). Let \( c_2 = \min\{1/2, c_0^2/(2C)\} \). By (20) and (21) we have

\[
\|u\|_{R,N}^2 \geq \frac{1}{2} \| u \|_{R,N}^2 + c_2 \| u \|_{R,N}^2 \\
\geq \frac{c_0^2}{2} \| v_2 \|_2^2 + c_2 \left( \frac{1}{2} \| u \|_{R,N,0}^2 - C \| v_2 \|_2^2 \right) \\
\geq \frac{c_0^2}{2} \| u \|_{R,N,0}^2.
\]

Thus, we have \( \| \cdot \|_{R,N} \geq \sqrt{c_0^2/2} \| \cdot \|_{R,N,0} \). \hfill \Box

### 3.4 Seminorm kernels

It is interesting to explicitly describe the kernel of the seminorms that measure the rigidity of deformations as this entails a characterization of fully rigid deformations. Recall from Definition 3.1 that \( U_{\text{trans}} \) is the vector space of displacements corresponding to translations. We now introduce vector space \( U_{\text{rot},0,0} \) which corresponds infinitesimal rotations of \( \mathcal{G} \cdot x_0 \) about the subspace \( \{0_{d_1}\} \times \mathbb{R}^{d_2} \).
Definition 3.22. For all $\mathcal{R} \subset \mathcal{G}$ we define the vector spaces

$$U_{\text{rot},0,0}(\mathcal{R})$$

$$:= \left\{ u : \mathcal{R} \to \mathbb{R}^d \mid \exists S \in \text{Skew}(d_1) \forall g \in \mathcal{R} : L(g)u(g) = (S \oplus 0_{d_2,d_2})(g \cdot x_0 - x_0) \right\}$$

$$\subset U_{\text{rot},0}(\mathcal{R}) \cap L^\infty(\mathcal{G}, \mathbb{R}^d_1 \times \{0_{d_2}\})$$

and

$$U_{\text{iso},0,0}(\mathcal{R}) := U_{\text{trans}}(\mathcal{R}) + U_{\text{rot},0,0}(\mathcal{R}) \subset U_{\text{iso},0}(\mathcal{R}) \cap L^\infty(\mathcal{G}, \mathbb{R}^d).$$

with $U_{\text{trans}}(\mathcal{R})$ as in Definition 3.1. In case $\mathcal{R} = \mathcal{G}$ we suppress the argument $\mathcal{R}$ for brevity and simply write $U_{\text{trans}}$, $U_{\text{rot},0,0}$ and $U_{\text{iso},0,0}$, respectively.

Remark 3.23. We have $U_{\text{rot},0,0} \subset U_{\text{rot},0}(\mathcal{G})$. If $d_1 \geq 1$ and $d_2 \geq 1$, then we have $U_{\text{rot},0,0} \subsetneq U_{\text{rot},0}(\mathcal{G})$. Moreover, in general we have $U_{\text{trans}} \not\subset U_{\text{per}}$ and $U_{\text{rot},0,0} \not\subset U_{\text{per}}$. For example let $\alpha \in \mathbb{R} \setminus (2\pi\mathbb{Q})$, $R(\alpha)$ be the rotation matrix by the angle $\alpha$, $\mathcal{G} = (R(\alpha) \oplus (I_1, 1)) < E(3)$ and $x_0 = e_1$. Then we have $\dim(U_{\text{rot},0}(\mathcal{G})) = 3$, $\dim(U_{\text{rot},0,0}) = 1$ and $\dim(U_{\text{rot},0,0} \cap U_{\text{per}}) = 0$. Moreover, we have $\dim(U_{\text{trans}}) = 3$ and $\dim(U_{\text{trans}} \cap U_{\text{per}}) = 1$.

Example 3.24. If $d_1 = 1$ or $d_{\text{aff}} = d_2$, then we have $U_{\text{rot},0,0} = \{0\}$. In particular, if $\mathcal{G}$ is a space group, then we have $U_{\text{rot},0,0} = \{0\}$.

The following proposition characterizes the vector spaces $U_{\text{trans}}(\mathcal{R})$, $U_{\text{rot}}(\mathcal{R})$, $U_{\text{rot},0}(\mathcal{R})$, $U_{\text{rot},0,0}(\mathcal{R})$, $U_{\text{iso}}(\mathcal{R})$, $U_{\text{iso},0}(\mathcal{R})$ and $U_{\text{iso},0,0}(\mathcal{R})$ for appropriate $\mathcal{R} \subset \mathcal{G}$. In particular, the proposition characterizes $U_{\text{trans}}$, $U_{\text{rot},0,0}$ and $U_{\text{iso},0,0}$ since $\mathcal{G}$ has a subset with Property 1.

Proposition 3.25. Suppose that $\mathcal{R} \subset \mathcal{G}$ is such that $id \in \mathcal{R}$ and $\text{aff}(\mathcal{R} \cdot x_0) = \text{aff}(\mathcal{G} \cdot x_0)$. Then the maps

$$\varphi_1 : \mathbb{R}^d \to U_{\text{trans}}(\mathcal{R})$$

$$a \mapsto (\mathcal{R} \to \mathbb{R}^d, g \mapsto L(g)^T a),$$

$$\varphi_2 : \mathbb{R}^{d_3 \times d_{\text{aff}}} \times \text{Skew}(d_{\text{aff}}) \to U_{\text{rot}}(\mathcal{R})$$

$$(A_1, A_2) \mapsto \left( \mathcal{R} \to \mathbb{R}^d, g \mapsto L(g)^T \begin{pmatrix} 0 & A_1 \\ -A_1^T & A_2 \end{pmatrix} (g \cdot x_0 - x_0) \right),$$

$$\varphi_3 : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to U_{\text{rot},0}(\mathcal{R})$$

$$(A_1, A_2, A_3, A_4) \mapsto \left( \mathcal{R} \to \mathbb{R}^d, g \mapsto L(g)^T \begin{pmatrix} 0 & A_1 & A_2 \\ -A_1^T & A_3 & A_4 \end{pmatrix} (g \cdot x_0 - x_0) \right),$$

and

$$\varphi_4 : \mathbb{R}^{d_3 \times d_4} \times \text{Skew}(d_4) \to U_{\text{rot},0,0}(\mathcal{R})$$

$$(A_1, A_2) \mapsto \left( \mathcal{R} \to \mathbb{R}^d, g \mapsto L(g)^T \left( \begin{pmatrix} 0 & A_1 \\ -A_1^T & A_2 \end{pmatrix} \oplus 0_{d_2,d_2} \right) (g \cdot x_0 - x_0) \right)$$

are isomorphisms, where $d_3 = d - d_{\text{aff}}$ and $d_4 = d_{\text{aff}} - d_2$. In particular, we have

$$\dim(U_{\text{trans}}(\mathcal{R})) = d$$

$$\dim(U_{\text{rot}}(\mathcal{R})) = d_{\text{aff}}(d - \frac{1}{2}d_{\text{aff}} - \frac{1}{2}),$$

$$\dim(U_{\text{rot},0}(\mathcal{R})) = d_{3_{\text{aff}}} + \frac{1}{2}(d_{\text{aff}} + d_2 - 1)$$

and

$$\dim(U_{\text{rot},0,0}(\mathcal{R})) = d_3(d_3 + d_1 - 1)/2.$$
We include the elementary proof for the sake of completeness.

**Proof.** Since $L(id) = I_d$, the map $\varphi_1$ is injective and thus, an isomorphism.

Now we prove that $\varphi_3$ is an isomorphism. The map $\varphi_3$ is well-defined and linear. First we show that $\varphi_3$ is surjective. Let $u \in U_{\text{rot},0}(R)$. There exist some $A_1 \in \text{Skew}(d_1)$ and $A_2 \in \mathbb{R}^{d_1 \times d_2}$ such that
\[ L(g)u(g) = \left( \begin{array}{cc} A_1 & A_2 \\ -A_2^T & 0 \end{array} \right)(g \cdot x_0 - x_0) \quad \text{for all } g \in G. \]

Let $A_3 \in \text{Skew}(d_3)$, $A_4 \in \mathbb{R}^{d_3 \times d_4}$, $A_5 \in \text{Skew}(d_4)$, $A_6 \in \mathbb{R}^{d_4 \times d_2}$ and $A_7 \in \mathbb{R}^{d_4 \times d_2}$ be such that
\[ A_1 = \left( \begin{array}{cc} A_3 & A_4 \\ -A_4^T & A_5 \end{array} \right) \quad \text{and} \quad A_2 = \left( \begin{array}{c} A_6 \\ A_7 \end{array} \right). \]

Since $G \cdot x_0 \subset \{0_{d_5}\} \times \mathbb{R}^{d_{\text{aff}}}$, we have $\varphi_3(A_4, A_5, A_7) = u$.

Now we show that $\varphi_3$ is injective. Let $A_1, B_1 \in \mathbb{R}^{d_1 \times d_4}$, $A_2, B_2 \in \mathbb{R}^{d_3 \times d_4}$, $A_3, B_3 \in \text{Skew}(d_4)$ and $A_4, B_4 \in \mathbb{R}^{d_4 \times d_2}$ be such that $\varphi_3(A_1, A_2, A_3, A_4) = \varphi_3(B_1, B_2, B_3, B_4)$. Let $\mathcal{R}' \subset \mathcal{R}$ be such that $\mathcal{R}'$ has Property 1. By Lemma 3.30 there exists some $C \in \mathbb{R}^{d_{\text{aff}} \times |\mathcal{R}'|}$ of rank $d_{\text{aff}}$ such that
\[ (g \cdot x_0 - x_0)_{g \in \mathcal{R}'} = \left( \begin{array}{c} 0 \\ C \end{array} \right). \]

The identity $\varphi_3(A_1, A_2, A_3, A_4) = \varphi_3(B_1, B_2, B_3, B_4)$ implies
\[ \left( \begin{array}{ccc} 0 & A_1 & A_2 \\ -A_1^T & A_3 & A_4 \\ -A_2^T & -A_3^T & 0 \end{array} \right)(g \cdot x_0 - x_0) = \left( \begin{array}{ccc} 0 & B_1 & B_2 \\ -B_1^T & B_3 & B_4 \\ -B_2^T & -B_3^T & 0 \end{array} \right)(g \cdot x_0 - x_0) \]
for all $g \in \mathcal{R}$ and in particular, we have
\[ \left( \begin{array}{c} (A_1, A_2)_C \\ (A_3, A_4)_C \end{array} \right) = \left( \begin{array}{c} (B_1, B_2)_C \\ (B_3, B_4)_C \end{array} \right). \]

Since the rank of $C$ is equal to the number of its rows, we have $A_i = B_i$ for all $i \in \{1, \ldots, 4\}$.

The proofs that $\varphi_2$ and $\varphi_4$ are isomorphisms are analogous.

For all $u \in U_{\text{rot}}(\mathcal{R})$ we have $u(id) = 0$ and for all $u \in U_{\text{trans}}(\mathcal{R})$ and $g \in \mathcal{R}$ we have $L(g)u(g) = u(id)$. This implies $U_{\text{trans}}(\mathcal{R}) \cap U_{\text{rot}}(\mathcal{R}) = \{0\}$ and thus $U_{\text{iso}}(\mathcal{R}) = U_{\text{trans}}(\mathcal{R}) \oplus U_{\text{rot}}(\mathcal{R})$. Analogously, we have $U_{\text{iso},0}(\mathcal{R}) = U_{\text{trans}}(\mathcal{R}) \oplus U_{\text{rot},0}(\mathcal{R})$ and $U_{\text{iso},0,0}(\mathcal{R}) = U_{\text{trans}}(\mathcal{R}) \oplus U_{\text{rot},0,0}(\mathcal{R})$. \hfill $\square$

**Lemma 3.26.** If the group $L(\mathcal{G})$ is finite, then we have $U_{\text{iso},0,0} \subset U_{\text{per}}$.

**Proof.** Suppose that $L(\mathcal{G})$ is finite. Let $n = |L(\mathcal{G})|$. For all $g \in \mathcal{G}$ we have
\[ L(g)^n = I_d. \tag{22} \]

Choose $N = m_0n$. Let $u \in U_{\text{iso},0,0}$. By definition there exist some $a \in \mathbb{R}^{d_1}$ and $S \in \text{Skew}(d_1)$ such that
\[ L(g)u(g) = a + (S \oplus 0)(g \cdot x_0 - x_0) \quad \text{for all } g \in \mathcal{G}. \]

For all $g \in \mathcal{G}$ and $t \in \mathcal{T}$ we have
\[
u gtN= \frac{1}{L(g)^{-1}(a + (S \oplus 0)((gtN) \cdot x_0 - x_0))} \\
= L(t)^{-N} L(g)^{-1} (a + (S \oplus 0)(L(t)^N x_0 - x_0) + (S \oplus 0)L(g) \tau(t^N)) \\
= L(g)^{-1} (a + (S \oplus 0)(g \cdot x_0 - x_0)) \\
= u(g),
\]
where we used (22), that $L(\mathcal{G}) < \oplus (O(d_1) \times O(d_2))$ and that $\tau(\mathcal{G}) \subset \{0_{d_2}\} \times \mathbb{R}^{d_2}$ in the second to last step. Thus, $u$ is $T^N$-periodic and we have $u \in U_{\text{per}}$. \hfill $\square$

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The following theorem characterizes the kernel of the seminorm \( \| \cdot \|_\mathcal{R} \).

**Theorem 3.27.** Suppose that \( \mathcal{R} \subset \mathcal{G} \) has Property 2. Then we have

\[
\ker(\| \cdot \|_\mathcal{R}) = U_{iso,0,0} \cap U_{per}.
\]

**Proof.** Suppose that \( \mathcal{R} \subset \mathcal{G} \) has Property 2.

First we show that \( U_{iso,0,0} \cap U_{per} \subset \ker(\| \cdot \|_\mathcal{R}) \):

Let \( u \in U_{iso,0,0} \cap U_{per} \). There exist some \( a \in \mathbb{R}^d \) and \( S \in \text{Skew}(d) \) such that

\[
L(g)u(g) = a + S(g \cdot x_0 - x_0) \quad \text{for all } g \in \mathcal{G}.
\]

Let \( g \in \mathcal{G} \). For all \( h \in \mathcal{R} \) it holds

\[
L(h)u(gh) = L(g)^T a + L(g)^T S((gh) \cdot x_0 - x_0)
= L(g)^T a + L(g)^T S(g \cdot x_0 - x_0) + L(g)^T S L(g)(h \cdot x_0 - x_0).
\]

Since \( L(g)^T S L(g) \in \text{Skew}(d) \), we have \( u(g \cdot)|_\mathcal{R} \in U_{iso}(\mathcal{R}) \).

Let \( N \in M_0 \) be such that \( u \) is \( T^N \)-periodic. Since \( g \in \mathcal{G} \) was arbitrary, we have

\[
\| u \|^2_R = \frac{1}{|C_N|} \sum_{g \in C_N} \| \pi_{U_{iso}(\mathcal{R})}(u(g \cdot)|_\mathcal{R}) \|^2 = 0.
\]

Thus, we have \( u \in \ker(\| \cdot \|_\mathcal{R}) \).

Now we show that \( \ker(\| \cdot \|_\mathcal{R}) \subset U_{iso,0,0} \cap U_{per} \):

Let \( u \in \ker(\| \cdot \|_\mathcal{R}) \). By definition of \( \| \cdot \|_\mathcal{R} \) we have \( u \in U_{per} \). Let \( g \in \mathcal{G} \). By Theorem 3.12 we have \( u \in \ker(\| \cdot \|_{\mathcal{R} \cup \{g\}}) \) and thus \( u_{|\mathcal{R} \cup \{g\}} \in U_{iso}(\mathcal{R} \cup \{g\}) \). There exist some \( a \in \mathbb{R}^d \) and \( S \in \text{Skew}(d) \) such that

\[
L(h)u(h) = a + S(h \cdot x_0 - x_0) \quad \text{for all } h \in \mathcal{R} \cup \{g\}. \tag{23}
\]

Since \( \mathcal{R} \) has Property 2, it holds id \( \in \mathcal{R} \) and thus, \( a = u(id) \). In particular, the vector \( a \) is independent of \( g \).

Since \( \mathcal{R} \) has Property 2, by Lemma 3.9 there exists some \( A \in \mathbb{R}^{d_{aff} \times |\mathcal{R}|} \) of rank \( d_{aff} \) such that

\[
(g \cdot x_0 - x_0)_{g \in \mathcal{R}} = \begin{pmatrix} 0_{d_{aff} \times |\mathcal{R}|} \end{pmatrix} A.
\]

Since \( \mathcal{G} \cdot x_0 \subset \{0_{d_{aff}}\} \times \mathbb{R}^{d_{aff}} \), without loss of generality we may assume that

\[
S = \begin{pmatrix} 0 \\ -S_1^T \\ S_2 \end{pmatrix}
\]

for some \( S_1 \in \mathbb{R}^{(d_{aff} \times d_{aff})} \) and \( S_2 \in \text{Skew}(d_{aff}) \). By equation (23) we have

\[
(L(h)u(h) - a)_{h \in \mathcal{R}} = \begin{pmatrix} 0 \\ -S_1^T \\ S_2 \end{pmatrix} \begin{pmatrix} S_1 A \\ S_2 A \end{pmatrix}. \tag{24}
\]

Since the rank of \( A \) is equal to the number of its rows, by (24) the matrix \( S \) is independent of \( g \).

Since \( g \in \mathcal{G} \) was arbitrary, we have

\[
L(g)u(g) = a + S(g \cdot x_0 - x_0) \quad \text{for all } g \in \mathcal{G}. \tag{25}
\]
Let $C = \sup \{ \| u(g) \| \mid g \in \mathcal{G} \}$. Since $u$ is periodic, we have $C < \infty$. Let $t \in T$. By (25) for all $n \in \mathbb{N}$ we have
\[ n \| S \tau(t) \| = \| S \tau(t^n) \| = \| L(t^n)u(t^n) - a - SL(t^n)x_0 + Sx_0 \| \leq 2C + 2\| S \| \| x_0 \| \]
and thus, $S \tau(t) = 0$. Since $t \in T$ was arbitrary, we have
\[ Sx = 0 \quad \text{for all } x \in \text{span}\{ \tau(t) \mid t \in T \} = \{0_{d_1} \} \times \mathbb{R}^{d_2}, \]
and thus, $S \in \oplus(\text{Skew}(d_1) \times \{0_{d_2,d_2}\})$. By (25) we have $u \in U_{\text{iso},0,0}$.

**Corollary 3.28.** Suppose that $L(\mathcal{G})$ is finite and $\mathcal{R} \subset \mathcal{G}$ has Property 2. Then we have
\[ \ker(\| \cdot \|_{\mathcal{R}}) = U_{\text{iso},0,0}. \]
Moreover, the map
\[ \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d_1 \times d_1} \times \text{Skew}(d_4) \rightarrow \ker(\| \cdot \|_{\mathcal{R}}) \]
\[ (a, A_1, A_2) \mapsto \left( \mathcal{G} \rightarrow \mathbb{R}^d, g \mapsto L(g)^T \left( a + \left( \begin{array}{c} 0 \\ -A_1^T \ A_2 \end{array} \right) \oplus 0_{d_2,d_2} \right) (g \cdot x_0 - x_0) \right) \]
is an isomorphism and in particular we have
\[ \dim(\ker(\| \cdot \|_{\mathcal{R}})) = d + d_4(d_3 + d_1 - 1)/2, \]
where $d_3 = d - d_{\text{aff}}$ and $d_4 = d_{\text{aff}} - d_2$.

**Proof.** The assertion is clear by Theorem 3.27, Lemma 3.26 and Proposition 3.25.

**Corollary 3.29.** Suppose that $\mathcal{G}$ is a space group and $\mathcal{R} \subset \mathcal{G}$ has Property 2. Then we have
\[ \ker(\| \cdot \|_{\mathcal{R}}) = U_{\text{trans}}. \]

**Proof.** This is clear by Corollary 3.28 and Example 3.24.

**Example 3.30.** We present an example which shows that in Theorem 3.12 the premise that $\mathcal{R}_1$ and $\mathcal{R}_2$ have Property 2 cannot be weakened to the premise that $\mathcal{R}_1$ and $\mathcal{R}_2$ are generating sets of $\mathcal{G}$ and have Property 1.

Suppose that $d = 2$, $d_1 = 1$, $d_2 = 1$, $t = (l_2,e_2)$, $\mathcal{G} = \langle t \rangle$, $x_0 = 0$, $\mathcal{R}_1 = \{id, t\}$ and $\mathcal{R}_2 = \{id, t, t^2\}$. The set $\mathcal{R}_1$ generates $\mathcal{G}$ and has Property 1 but does not have Property 2. The set $\mathcal{R}_2$ has Property 2. Using that the seminorms $\| \cdot \|_{\mathcal{R}}$ and $\| \cdot \|_{\mathcal{R}\setminus{id},\nabla}$ are equivalent by Proposition 3.7, it follows
\[ \ker(\| \cdot \|_{R_1}) = \{ u \in U_{\text{per}} \mid \exists a \in \mathbb{R} \forall g \in \mathcal{G} : u_2(g) = a \}. \]
By Corollary 3.28 and Example 3.24 we have
\[ \ker(\| \cdot \|_{R_2}) = U_{\text{iso},0,0} = U_{\text{trans}}. \]
Since the kernels of $\| \cdot \|_{R_1}$ and $\| \cdot \|_{R_2}$ are not equal, the seminorms $\| \cdot \|_{R_1}$ and $\| \cdot \|_{R_2}$ are not equivalent.

The following theorem summarizes the main results of this section.

**Theorem 3.31.** Suppose that $\mathcal{R}_1$, $\mathcal{R}_2 \subset \mathcal{G}$ have Property 2. Then the seminorms $\| \cdot \|_{\mathcal{R}_1}$, $\| \cdot \|_{\mathcal{R}_2}$, $\| \cdot \|_{\mathcal{R}_1,0}$, $\| \cdot \|_{\mathcal{R}_2,0}$, $\| \cdot \|_{\mathcal{R}_1,\nabla,0}$, $\| \cdot \|_{\mathcal{R}_2,\nabla}$, $\| \cdot \|_{\mathcal{R}_1,\nabla,0}$ and $\| \cdot \|_{\mathcal{R}_2,\nabla,0}$ are equivalent and their kernel is $U_{\text{iso},0,0} \cap U_{\text{per}}$.

**Proof.** This is clear by Theorem 3.12, Proposition 3.7, Proposition 3.17, Theorem 3.21 and Theorem 3.27.

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4 Stronger seminorms

In this section introduce two stronger seminorms. First we consider \( \| \cdot \|_{R,0,0} \) and its variant \( \| \cdot \|_{\mathcal{R},\nabla,0,0} \) that are defined as the averaged local distance to \( U_{iso,0,0} \) respectively, \( U_{rot,0,0} \), cf. Definition 3.22. They thus measure rigidity up to local rotations about \( \{0_d\} \times \mathbb{R}^d \). Then we define the even stronger seminorm \( \| \nabla_R \cdot \|_2 \) as a discrete \( H^1 \) norm.

**Definition 4.1.** For all finite sets \( \mathcal{R} \subset \mathcal{G} \) we define the seminorms

\[
\| \cdot \|_{R,0,0}: U_{per} \to [0, \infty),
\]

\[
u \mapsto \left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \| \pi_{U_{iso,0,0}(\mathcal{R})}(u(g \cdot)|\mathcal{R}) \|^2 \right)^{\frac{1}{2}} \text{ if } u \text{ is } \mathcal{T}^N\text{-periodic},
\]

and

\[
\| \cdot \|_{\mathcal{R},\nabla,0,0}: U_{per} \to [0, \infty),
\]

\[
u \mapsto \left( \frac{1}{|\mathcal{C}_N|} \sum_{g \in \mathcal{C}_N} \| \pi_{U_{rot,0,0}(\mathcal{R})}(\nabla_R u(g)) \|^2 \right)^{\frac{1}{2}} \text{ if } u \text{ is } \mathcal{T}^N\text{-periodic},
\]

where \( \pi_{U_{iso,0,0}(\mathcal{R})} \) and \( \pi_{U_{rot,0,0}(\mathcal{R})} \) are the orthogonal projections on \( \{u: \mathcal{R} \to \mathbb{R}^d\} \) with respect to the norm \( \| \cdot \| \) with kernels \( U_{iso,0,0}(\mathcal{R}) \) and \( U_{rot,0,0}(\mathcal{R}) \), respectively.

**Remark 4.2.** For all finite sets \( \mathcal{R} \subset \mathcal{G} \) we have \( \| \cdot \|_\mathcal{R} \leq \| \cdot \|_{\mathcal{R},0,0} \), but the seminorms \( \| \cdot \|_\mathcal{R} \) and \( \| \cdot \|_{\mathcal{R},0,0} \) need not be equivalent, see Proposition 5.1.

**Theorem 4.3.** Suppose that \( \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{G} \) have Property 2. Then the seminorms \( \| \cdot \|_{\mathcal{R}_1,0,0}, \| \cdot \|_{\mathcal{R}_1,0,0}, \| \cdot \|_{\mathcal{R}_2,0,0}, \| \cdot \|_{\mathcal{R},\nabla,0,0} \) and \( \| \cdot \|_{\mathcal{R}_2,0,0} \) are equivalent and their kernel is \( U_{iso,0,0} \cap U_{per} \).

**Proof.** Suppose that \( \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{G} \) have Property 2. The proof that the seminorms \( \| \cdot \|_{\mathcal{R}_1,0,0} \) and \( \| \cdot \|_{\mathcal{R}_2,0,0} \) are equivalent is analogous to the proof of Theorem 3.12. For all finite sets \( \mathcal{R} \subset \mathcal{G} \) we define the seminorm

\[
p_{0,\mathcal{R}}: \{u: \mathcal{R} \to \mathbb{R}^d\} \to [0, \infty),
\]

\[
u \mapsto \| \pi_{U_{iso,0,0}(\mathcal{R})}(u) \|
\]

on \( (\mathbb{R}^d)^\mathcal{R} \) whose kernel is \( U_{iso,0,0}(\mathcal{R}) \). Moreover, for all finite sets \( \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{G} \) we define the seminorm

\[
g_{0,\mathcal{R}_1,\mathcal{R}_2}: \{u: \mathcal{R}_1 \mathcal{R}_2 \to \mathbb{R}^d\} \to [0, \infty),
\]

\[
u \mapsto \left( \sum_{g \in \mathcal{R}_1} P_{0,\mathcal{R}_2}^2 (u(g \cdot)|\mathcal{R}_2) \right)^{\frac{1}{2}}
\]

on \( (\mathbb{R}^d)^{\mathcal{R}_1,\mathcal{R}_2} \). Analogously to Lemma 3.11 for all \( \mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{G} \) such that \( \mathcal{R}_1 \) is finite and \( \mathcal{R}_2 \) has Property 2 there exists a finite set \( \mathcal{R}_3 \subset \mathcal{G} \) such that \( \mathcal{R}_1 \subset \mathcal{R}_3 \mathcal{R}_2 \) and the seminorms \( p_{0,\mathcal{R}_1,\mathcal{R}_2} \) and \( q_{0,\mathcal{R}_1,\mathcal{R}_2} \) are equivalent. As in the proof of Theorem 3.12 this implies that the seminorms \( \| \cdot \|_{\mathcal{R}_1,0,0} \) and \( \| \cdot \|_{\mathcal{R}_2,0,0} \) are equivalent.

Analogously to the proof of Proposition 3.7, the seminorms \( \| \cdot \|_{\mathcal{R},0,0} \) and \( \| \cdot \|_{\mathcal{R},\nabla,0,0} \) are equivalent for all finite sets \( \mathcal{R} \subset \mathcal{G} \) such that \( id \in \mathcal{R} \). In particular, if \( \mathcal{R} \subset \mathcal{G} \) has Property 2, then \( \| \cdot \|_{\mathcal{R},0,0} \) and \( \| \cdot \|_{\mathcal{R},\nabla,0,0} \) are equivalent.

Suppose that \( \mathcal{R} \subset \mathcal{G} \) has Property 2. Analogously to the proof of Theorem 3.22 we have \( U_{iso,0,0} \cap U_{per} \subset \ker(\| \cdot \|_{\mathcal{R},0,0}) \). Since \( \| \cdot \|_{\mathcal{R}} \leq \| \cdot \|_{\mathcal{R},0,0} \), by Theorem 3.27 we have \( \ker(\| \cdot \|_{\mathcal{R},0,0}) \subset U_{iso,0,0} \cap U_{per} \).

For the second seminorm to be discussed in this section we first slightly extend our notion of the \( \ell^2 \) norm \( \| \cdot \|_2 \).
Definition 4.4. For all finite sets \( R \subset G \) we define the norm
\[
\| \cdot \|_2 : \{ u : G \rightarrow \{ v : R \rightarrow \mathbb{R}^d \} \mid u \text{ is periodic} \} \rightarrow [0, \infty)
\]
\[
uu = \left( \frac{1}{|C_N|} \sum_{g \in C_N} \| u(g) \|_2^2 \right)^{1/2}
\]
if \( u \) is \( T_N \)-periodic.

Theorem 4.5. Let \( R_1, R_2 \subset G \) be finite generating sets of \( G \). Then the seminorms
\[
\| \nabla_{R_1} \cdot \|_2 \text{ and } \| \nabla_{R_2} \cdot \|_2 \text{ on } U_{\text{per}} \text{ are equivalent and their kernel is } U_{\text{trans}} \cap U_{\text{per}}.
\]

Proof. Let \( R_1, R_2 \subset G \) be finite generating sets of \( G \). First we show that the seminorms
\[
\| \nabla_{R_1} \cdot \|_2 \text{ and } \| \nabla_{R_2} \cdot \|_2
\]
are equivalent. It suffices to show that there exists a constant \( C > 0 \) such that
\[
\| \nabla_{R_1} \cdot \|_2 \leq C \| \nabla_{R_2} \cdot \|_2.
\]
Since \( R_2 \) generates \( G \), for every \( g \in R_1 \) there exist some \( n_r \in N \) and \( s_{r,1}, \ldots, s_{r,n_r} \in R_2 \cup R_2^{-1} \) such that
\[
r = s_{r,1} \ldots s_{r,n_r}. \text{ Let } u \in U_{\text{per}}.
\]
Let \( N \in M_0 \) be such that \( u \) is \( T_N \)-periodic. Then we have
\[
\| \nabla_{R_1} u \|_2^2 = \frac{1}{|C_N|} \sum_{g \in C_N} \| \nabla_{R_1} u(g) \|_2^2
\]
\[
= \frac{1}{|C_N|} \sum_{g \in C_N} \sum_{r \in R_1} \| L(r)u(gr) - u(g) \|_2^2
\]
\[
= \frac{1}{|C_N|} \sum_{g \in C_N} \sum_{r \in R_1} \left( \sum_{i=1}^{n_r} L(s_{r,1} \ldots s_{r,i-1})(L(s_{r,i})u(gs_{r,1} \ldots s_{r,i}) - u(gs_{r,1} \ldots s_{r,i-1})) \right)^2
\]
\[
\leq \frac{1}{|C_N|} \sum_{g \in C_N} \sum_{r \in R_1} \left( \sum_{i=1}^{n_r} |L(s_{r,i})u(gs_{r,1} \ldots s_{r,i}) - u(gs_{r,1} \ldots s_{r,i-1})| \right)^2
\]
\[
\leq \frac{1}{|C_N|} \sum_{g \in C_N} \sum_{s \in R_2} \| L(s)u(\tilde{g}s) - u(\tilde{g}) \|_2^2
\]
\[
= C \| \nabla_{R_2} u \|_2^2,
\]
where \( C = \sum_{r \in R_2} n_r^2 \). In the fifth step we used that the arithmetic mean is lower or equal than the root mean square. In the sixth step, if \( s_{r,i} \in R_2 \), we substituted \( gs_{r,1} \ldots s_{r,i-1} \) by \( \tilde{g} \), and if \( s_{r,i} \in R_2^{-1} \), we substituted \( gs_{r,1} \ldots s_{r,i} \) by \( \tilde{g} \).

Let \( R = R_1 \). Now we show that \( \ker(\| \nabla_{R} \cdot \|_2) = U_{\text{trans}} \cap U_{\text{per}} \). It is clear that \( U_{\text{trans}} \cap U_{\text{per}} \subset \ker(\| \nabla_{R} \cdot \|_2) \). If \( u \in \ker(\| \nabla_{R} \cdot \|_2) \), then for all \( g \in G \) we have
\[
0 = \| \nabla_{R \cup \{g\}} u \|_2 \geq \| L(g)u(g) - u(id) \|.
\]
where we used that the seminorms \( \| \nabla_{R} \cdot \|_2 \) and \( \| \nabla_{R \cup \{g\}} \cdot \|_2 \) are equivalent. By (26) we have \( L(g)u(g) = u(id) \) for all \( g \in G \) and thus \( u \in U_{\text{trans}} \).

Remark 4.6. For all finite sets \( R \subset G \) we have \( \| \cdot \|_{R,0,0} \leq \| \nabla_{R} \cdot \|_2 \), but the seminorms \( \| \cdot \|_{R,0,0} \) and \( \| \nabla_{R} \cdot \|_2 \) need not be equivalent since their kernels are not equal, see Theorem 4.3 and Theorem 4.5.

Theorem 3.21 yields the following corollary.

Corollary 4.7. (A discrete Korn’s inequality for space groups) Suppose that \( G \) is a space group and \( R \subset G \) has Property 2. Then the seminorms \( \| \cdot \|_{R}, \| \cdot \|_{R,0,0} \) and \( \| \nabla_{R} \cdot \|_2 \) are equivalent.

Proof. Suppose that \( G \) is a space group and \( R \subset G \) has Property 2. Then we have \( U_{\text{rot,0}}(R) = U_{\text{rot,0,0}}(R) = \{0\} \) and \( \| \cdot \|_{R, \nabla,0} = \| \cdot \|_{R, \nabla,0,0} = \| \nabla_{R} \cdot \|_2 \). With Theorem 3.31 and Theorem 4.3 follows the assertion. □
5 Two basic examples in real and Fourier space

We finally work out explicitly equivalent descriptions of the seminorms \( \| \cdot \|_R \) and \( \| \nabla R \cdot \|_2 \) (respectively, \( \| \cdot \|_{R,0,0} \)) in terms of their Fourier transform in two basic examples: a simple one-dimensional atomic chain in \( \mathbb{R}^2 \) with \( d_{\text{aff}} = d_2 = d_1 = 1 \) in Proposition 5.1 and an atomic chain with non-trivial bond angles and \( d_{\text{aff}} = 2, d_2 = d_1 = 1 \) in Proposition 5.2. While the seminorms \( \| \cdot \|_{R,0,0} \) and \( \| \nabla R \cdot \|_2 \) will be equivalent as \( d_1 = 1 \), in both examples we will see that \( \| \cdot \|_R \) and \( \| \cdot \|_{R,0,0} \) are not equivalent.

**Proposition 5.1.** Suppose that \( t = (I_2,e_2) \in \mathbb{E}(2), \mathcal{G} = \langle t \rangle < \mathbb{E}(2), x_0 = 0 \in \mathbb{R}^2 \) and \( \mathcal{R} \subset \mathcal{G} \) has Property 2, e.g. \( \mathcal{R} = \{id,t,t^2\} \). Then the seminorms \( \| \cdot \|_{R,0,0} \) and \( \| \nabla R \cdot \|_2 \) are equivalent and there exist constants \( C,C > 0 \) such that for all \( u \in U_{\text{per}} \) we have

\[
c\|\nabla R u\|_2^2 \leq \sum_{k \in \{0,1\} \cap \mathbb{Q}} |k|^2_1 \|\hat{u}(\chi_k)\|^2 \leq C\|\nabla R u\|_2^2
\]

and

\[
c\|u\|_R^2 \leq \sum_{k \in \{0,1\} \cap \mathbb{Q}} \left( |k|^4_1 \|\hat{u}_1(\chi_k)\|^2 + |k|^2_1 \|\hat{u}_2(\chi_k)\|^2 \right) \leq C\|u\|_R^2,
\]

where \( | \cdot |_1 : \mathbb{R} \to [0, \infty) \), \( k \mapsto \text{dist}(k,\mathbb{Z}) \) is the distance to nearest integer function.

**Proof.** Suppose that \( t = (I_2,e_2), \mathcal{G} = \langle t \rangle \) and \( x_0 = 0 \). We have \( d = 2 \) and \( d_1 = d_2 = 1 \). The set \( \{id,t,t^2\} \) has Property 2 and by Theorem 4.3 and Theorem 5.3, without loss of generality, let \( \mathcal{R} = \{id,t,t^2\} \). Since \( U_{\text{rot},0,0}(\mathcal{R}) = \{0\} \), we will \( \| \cdot \|_{R,0,0} = \| \nabla R \cdot \|_2 \) and thus the seminorms \( \| \cdot \|_{R,0,0} \) and \( \| \nabla R \cdot \|_2 \) are equivalent by Theorem 4.3.

We observe that \( \hat{\mathcal{G}} = \{\chi_k \mid k \in [0,1)\} \), where \( \chi_k : \mathcal{G} \to \mathbb{C} \) is given by \( \chi_k(t^n) = e^{2\pi i nk} \) for all \( n \in \mathbb{Z} \). (Here \( k \) is determined by the condition \( \chi(t) = e^{2\pi i k} \).) Since \( \chi_k \) is periodic if and only if \( k \in [0,1) \cap \mathbb{Q} \), we set \( \mathcal{E} = \{\chi_k \mid k \in [0,1) \cap \mathbb{Q}\} \), cp. the paragraph above and Definition 2.22.

Noting \( \{k \in [0,1) \mid e^{-2\pi i k} = 1\} = \{0\} \) and invoking Taylor’s theorem, we find a constant \( c_T \in (0,1) \) such that for all \( k \in [0,1) \) and \( n \in \{1,2\} \) we have

\[
c_T |k|_1 \leq |e^{-2\pi i k} - 1|,
\]

\[
c_T |e^{-2\pi i k} - 1| \leq |k|_1,
\]

and

\[
c_T |e^{-2\pi i k} - 1 + 2\pi i nk| \leq |k|_1^2.
\]

For all \( u \in U_{\text{per}} \) we have

\[
\|\nabla R u\|_2^2 = \sum_{\chi \in \mathcal{E}} \|\nabla R u(\chi)\|^2
= \sum_{k \in \{0,1\} \cap \mathbb{Q}} \| (\chi_k(h)\bigtimes\hat{u}(\chi_k) - \hat{u}(\chi_k))_{h \in \mathcal{R}} \|^2
= \sum_{k \in \{0,1\} \cap \mathbb{Q}} \sum_{n=1}^2 |e^{-2\pi i k} - 1|^2 \|\hat{u}(\chi_k)\|^2,
\]

where we used Proposition 2.23 in the first step and Lemma 2.4 in the second step. Equations (27), (28) and (30) imply the first assertion.
Now we show the second assertion. Let $\mathcal{R}' = \{t, t^2\}$. By Proposition 3.7 the seminorms $\| \cdot \|_\mathcal{R}$ and $\| \cdot \|_{\mathcal{R}'}$ are equivalent, i.e. there exist some constants $C, c > 0$ such that

$$c \| \cdot \|_\mathcal{R} \leq \| \cdot \|_{\mathcal{R}'} \leq C \| \cdot \|_\mathcal{R}.$$  \hfill (31)

We define the linear map

$$g_{\mathcal{R}'}: \text{Skew}(2, \mathbb{C}) \to \mathbb{C}^{2 \times |\mathcal{R}'|}\quad S \mapsto (S(h \cdot x_0 - x_0))_{h \in \mathcal{R}'}.$$

For all $u \in U_{\text{per}}$ we have

$$\|u\|^2_{\mathcal{R}', \mathcal{V}} = \inf \left\{ \|\nabla_{\mathcal{R}'} u - g_{\mathcal{R}'} \circ v\|^2 \middle| v \in L^\infty_{\text{per}}(\mathcal{G}, \text{Skew}(2, \mathbb{C})) \right\}$$

$$= \inf \left\{ \sum_{\chi \in \mathcal{E}} \left| \nabla_{\mathcal{R}'} \tilde{u}(\chi) - g_{\mathcal{R}'}(\chi) \right|^2 \middle| \tilde{v} \in \bigoplus_{\chi \in \mathcal{E}} \text{Skew}(2, \mathbb{C}) \right\}$$

$$= \sum_{\chi \in \mathcal{E}} \inf \left\{ \left| \nabla_{\mathcal{R}'} \tilde{u}(\chi) - g_{\mathcal{R}'}(S) \right|^2 \middle| S \in \text{Skew}(2, \mathbb{C}) \right\}$$

$$= \sum_{k \in \{0, 1\} \cap \mathbb{Q}} \inf \left\{ \left| (\chi k(h)^{-1} \tilde{u}(\chi k) - \tilde{u}(\chi k) - (k_1)^2 (h \cdot x_0 - x_0))_{h \in \mathcal{R}'} \right|^2 \middle| s \in \mathbb{C} \right\}$$

$$= \sum_{k \in \{0, 1\} \cap \mathbb{Q}} \inf \left\{ \sum_{n=1}^{2} \left| (e^{-2\pi i nk} - 1) \tilde{u}(\chi k) + nse_1 \right|^2 \middle| s \in \mathbb{C} \right\},$$  \hfill (32)

where we used Proposition 2.20 in the second step and Lemma 2.4 in the fourth step.

It holds

$$\sum_{i=1}^{n} a_i^2 \leq \left( \sum_{i=1}^{n} a_i \right)^2 \leq n \sum_{i=1}^{n} a_i^2$$  \hfill (33)

for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n \geq 0$.

We define the functions

$$f_1: [0, 1) \times \mathbb{C}^2 \times \mathbb{C} \to [0, \infty), \quad (k, v, s) \mapsto \sum_{n=1}^{2} \left| (e^{-2\pi i nk} - 1)v + nse_1 \right|$$

and

$$f_2: [0, 1) \times \mathbb{C}^2 \to [0, \infty), \quad (k, v) \mapsto |k|^2 |v_1| + |k| |v_2|.$$  

By (31), (32) and (33) it suffices so show that there exist some constants $C, c > 0$ such that for all $(k, v) \in [0, 1) \times \mathbb{C}^2$ we have

$$c \inf_{s \in \mathbb{C}} f_1(k, v, s) \leq f_2(k, v) \leq C \inf_{s \in \mathbb{C}} f_1(k, v, s).$$  \hfill (34)

First we show the left inequality of (34). By (28) and (29) for all $(k, v) \in [0, 1) \times \mathbb{C}^2$ we have

$$\inf_{s \in \mathbb{C}} f_1(k, v, s) \leq f_1(k, v, 2\pi kv_1)$$

$$\leq \sum_{n=1}^{2} \left| (e^{-2\pi i nk} - 1 + 2\pi i nk |v_1| + |e^{-2\pi i nk} - 1||v_2|) \right|$$

$$\leq \frac{2}{cT} f_2(k, v).$$
Now we show the right inequality of (35). Let \((k, v, s) \in [0, 1) \times \mathbb{C}^2 \times \mathbb{C}\). By (27) we have
\[
f_1(k, v, s) \geq |e^{-2\pi ik}v_1 - v_1 + s| + \frac{1}{2}|e^{-4\pi ik}v_1 - v_1 + 2s|
\geq \frac{1}{2}(2|e^{-2\pi ik}v_1 - v_1 + s| - (e^{-4\pi ik}v_1 - v_1 + 2s))
= \frac{1}{2}|e^{-2\pi ik} - 1|^2|v_1|
\geq \frac{c_2^2}{2}|k|^2|v_1|
\] (35)
and
\[
f_1(k, v, s) \geq |e^{-2\pi ik} - 1||v_2| \geq c_T|k||v_2|.
\] (36)
By (35) and (36) we have
\[
f_1(k, v, s) \geq \frac{c_2^2}{2}f_2(k, v).
\]

**Proposition 5.2.** Suppose that \(t = ((-1, 0), e_2) \in E(2), \mathcal{G} = \langle t \rangle \in E(2), x_0 = e_1 \in \mathbb{R}^2\) and \(R \subset \mathcal{G}\) has Property 2, e.g. \(R = \{t^0, \ldots, t^5\}\). Then the seminorms \(\|\cdot\|_{\mathcal{R}, 0, 0}\) and \(\|\nabla R \cdot \|_2\) are equivalent and there exist constants \(C, c > 0\) such that for all \(u \in U_{\text{per}}\) we have
\[
c\|\nabla u\|_{\mathcal{R}}^2 \leq \sum_{k \in [0, 1) \cap \mathbb{Q}} \left(|k - \frac{1}{2}|^2|\hat{u}_1(\chi_k)|^2 + |k|^2|\hat{u}_2(\chi_k)|^2\right) \leq C\|\nabla u\|_2^2
\]
and
\[
c\|u\|_{\mathcal{R}}^2 \leq \sum_{k \in [0, 1) \cap \mathbb{Q}} \left(|k - \frac{1}{2}|^2|\hat{u}_1(\chi_k)|^2 + |k|^2|2\pi i(k - \frac{1}{2})\hat{u}_1(\chi_k) - \hat{u}_2(\chi_k)|^2\right) \leq C\|u\|_2^2,
\]
where \(|\cdot| : \mathbb{R} \to [0, \infty), k \mapsto \text{dist}(k, \mathbb{Z})\) is the distance to nearest integer function.

**Proof.** Suppose that \(t = ((-1, 0), e_2), \mathcal{G} = \langle t \rangle\) and \(x_0 = e_1\). We have \(d = 2\) and \(d_1 = d_2 = 1\). The set \(\{t^0, \ldots, t^5\}\) has Property 2 and by Theorem 4.3 and Theorem 8.31 without loss of generality, let \(R = \{t^0, \ldots, t^5\}\) and thus the seminorms \(\|\cdot\|_{\mathcal{R}, 0, 0}\) are equivalent by Theorem 4.3.

As in the previous example we let \(E = \{\chi_k | k \in [0, 1) \cap \mathbb{Q}\}\), where \(\chi_k \in \mathcal{G}\) is given by \(\chi_k(t^n) = e^{2\pi ink}\) for all \(n \in \mathbb{Z}\). Since \(\{k \in [0, 1) | e^{-2\pi ik} = 1\} = \{0\}, \{k \in [0, 1) | e^{-2\pi ik} = -1\} = \{\frac{1}{2}\}\) and by Taylor’s theorem, there exists a constant \(c_T \in (0, 1)\) such that for all \(k \in [0, 1)\) and \(n \in \{1, 2, 3\}\) we have
\[
c_T|k| \leq |e^{-2\pi ik} - 1|,
\]
\[
c_T|k - \frac{1}{2}| \leq |e^{-2\pi ik} + 1|,
\]
\[
c_T|e^{-2\pi ink} - 1| \leq |k|,
\]
\[
c_T|e^{-2\pi ink} - (-1)^n| \leq |k - \frac{1}{2}|,
\]
and
\[
c_T|e^{-2\pi ink} - (-1)^n| \leq |k - \frac{1}{2}|^2.
\]

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We define the linear map

\[ \|\nabla_R u\|_2^2 = \sum_{\chi \in \mathcal{E}} \|\nabla_R u(\chi)\|^2 \]

\[ = \sum_{k \in (0,1)\cap \mathbb{Q}} \big\| (\chi_k(h)^{-1}\hat{u}(\chi_k) - L(h)^T\hat{u}(\chi_k))_{h \in \mathcal{R}} \big\|^2 \]

\[ = \sum_{k \in (0,1)\cap \mathbb{Q}} \sum_{n=1}^3 \left\| e^{-2\pi i nk} \hat{u}(\chi_k) - \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) e^{2\pi i nk} \hat{u}(\chi_k) \right\|^2 \]

\[ = \sum_{k \in (0,1)\cap \mathbb{Q}} \sum_{n=1}^3 \left( \left| e^{-2\pi i nk} - (-1)^n \right|^2 \left| \hat{u}_1(\chi_k) \right|^2 + \left| e^{-2\pi i nk} - 1 \right|^2 \left| \hat{u}_2(\chi_k) \right|^2 \right), \quad (42) \]

where we used Proposition 2.3 in the first step and Lemma 2.4 in the second step. Equations (37), (38), (39), (40) and (42) imply the first assertion.

Now we show the second assertion. Let \( \mathcal{R}' = \{ t^1, t^2, t^3 \} \). By Proposition 2.1 the seminorms \( \| \cdot \|_\mathcal{R} \) and \( \| \cdot \|_{\mathcal{R}', \mathcal{V}} \) are equivalent, i.e. there exist some constants \( C, c > 0 \) such that

\[ c \| \cdot \|_\mathcal{R} \leq \| \cdot \|_{\mathcal{R}', \mathcal{V}} \leq C \| \cdot \|_\mathcal{R}. \quad (43) \]

We define the linear map

\[ g_{\mathcal{R}'} : \text{Skew}(2, \mathbb{C}) \to \mathbb{C}^{2 \times |\mathcal{R}'|} \]

\[ S \mapsto (L(h)^T S(h \cdot x_0 - x_0))_{h \in \mathcal{R}'} \]

For all \( u \in U_{\text{per}} \) we have

\[ \|u\|_{\mathcal{R}', \mathcal{V}}^2 = \inf \left\{ \|\nabla_{\mathcal{R}'} u - g_{\mathcal{R}'} \circ v\|_2^2 \middle| v \in L_{\text{per}}^{\infty}(\mathcal{G}, \text{Skew}(2, \mathbb{C})) \right\} \]

\[ = \inf \left\{ \sum_{\chi \in \mathcal{E}} \left\| \nabla_{\mathcal{R}'} u(\chi) - g_{\mathcal{R}'} \circ \hat{v}(\chi) \right\|^2 \middle| \hat{v} \in \bigoplus_{\chi \in \mathcal{E}} \text{Skew}(2, \mathbb{C}) \right\} \]

\[ = \sum_{\chi \in \mathcal{E}} \inf \left\{ \left\| \nabla_{\mathcal{R}'} u(\chi) - g_{\mathcal{R}'}(S) \right\|^2 \middle| S \in \text{Skew}(2, \mathbb{C}) \right\} \]

\[ = \sum_{k \in (0,1)\cap \mathbb{Q}} \inf \left\{ \sum_{s \in \mathbb{C}} \left\| (\chi_k(h)^{-1}\hat{u}(\chi_k) - L(h)^T\hat{u}(\chi_k) - L(h)^T \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) (h \cdot x_0 - x_0))_{h \in \mathcal{R}'} \right\|^2 \middle| s \in \mathbb{C} \right\}, \quad (44) \]

where we used Proposition 2.3 in the second step and Lemma 2.4 in the fourth step.

It holds

\[ \sum_{i=1}^n a_i^2 \leq \left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2 \quad (45) \]

for all \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \geq 0 \).

We define the functions

\[ f_1 : [0, 1) \times \mathbb{C}^2 \times \mathbb{C} \to [0, \infty) \]

\[ (k, v, s) \mapsto \sum_{n=1}^3 \left\| e^{-2\pi i nk} v - \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \right\|_2^n v - \left( \begin{array}{c} (-1)^{n+1} s \\ \cdot \\ (-1)^n s \end{array} \right) \]
and

\[ f_2: [0, 1) \times \mathbb{C}^2 \to [0, \infty) \]

\[ (k, v) \mapsto |k - \frac{1}{2}(1)|v_1| + |k|1|2\pi|k - \frac{1}{2}|v_1 - v_2|. \]

By (43), (44) and (45) it suffices so show that there exist some constants \(c, v\) such that for all \((k, v) \in [0, 1) \times \mathbb{C}^2\) we have

\[ c \inf_{s \in \mathbb{C}} f_1(k, v, s) \leq f_2(k, v) \leq C \inf_{s \in \mathbb{C}} f_1(k, v, s). \]  

(46)

First we show the right inequality of (46). Let \(c_R > 0\) be small enough, e.g. \(c_R = \frac{\varepsilon}{300}\). Let \((k, v, s) \in [0, 1) \times \mathbb{C}^2 \times \mathbb{C}\). By (37) and (38) we have

\[ f_1(k, v, s) \geq |e^{2\pi ik}v_1 + v_1 - s| + \frac{1}{2}|e^{-4\pi ik}v_1 - v_1 + 2s| \]

\[ \geq \frac{1}{2}|2(e^{-2\pi ik}v_1 + v_1 - s) + e^{-4\pi ik}v_1 - v_1 + 2s| \]

\[ = \frac{1}{2}|e^{-2\pi ik} + 1|^2|v_1| \]

\[ \geq \frac{c^2}{2}|k - \frac{1}{2}|v_1|. \]  

(47)

and

\[ f_1(k, v, s) \geq \sum_{n \in \{1, 3\}} |e^{-2\pi ink}v_2 - v_2 + 2s| \]

\[ \geq |e^{-2\pi ink}v_2 - v_2 + 2s - (e^{-2\pi ik}v_2 - v_2 + 2s)| \]

\[ = |e^{-2\pi ik} + 1||e^{-2\pi ink} - 1||v_2| \]

\[ \geq \frac{c^2}{2}|k|1|k - \frac{1}{2}|v_2|. \]  

(48)

Case 1: \(k \in [0, \frac{1}{4}) \cup \left[\frac{3}{4}, 1\right)\).

Since \(k \in [0, \frac{1}{4}) \cup \left[\frac{3}{4}, 1\right)\), we have \(|k - \frac{1}{2}| \geq \frac{1}{4}\). By (37) and (48) we have

\[ f_1(k, v, s) \geq c_R|k - \frac{1}{2}|v_1| + \pi c_R|v_1| + c_R|k|1|v_2| \geq c_R f_2(k, v), \]

where in the last step we used the triangle inequality.

Case 2: \(k \in \left(\frac{1}{4}, \frac{3}{4}\right)\).

Since \(k \in \left(\frac{1}{4}, \frac{3}{4}\right)\), we have \(|k| \geq \frac{1}{4}\). By (37) and (41) we have

\[ f_1(k, v, s) \geq |(e^{-4\pi ik} - 1)v_1 + 2s| + |(e^{-2\pi ik} - 1)v_2 + 2s| \]

\[ \geq |(e^{-4\pi ik} - 1)v_1 + 2s - ((e^{-2\pi ik} - 1)v_2 + 2s)| \]

\[ = |e^{-2\pi ik} - 1||e^{-2\pi ik} + 1||v_1| \]

\[ \geq \frac{c^2}{2}|k - \frac{1}{2}|v_1| - v_2| \]

\[ \geq \frac{c^2}{2}|2\pi i(k - \frac{1}{2})v_1 - v_2| - \frac{c^2}{4}|e^{-2\pi ik} + 1 - 2\pi i(k - \frac{1}{2})||v_1| \]

\[ \geq \frac{c^2}{2}|2\pi i(k - \frac{1}{2})v_1 - v_2| - \frac{1}{4}|k - \frac{1}{2}|v_1|. \]  

(49)

By (17) and (19) we have \(f_3(k, v, s) \geq c_R f_2(k, v)\).

Now we show the left inequality of (46). Let \(C_L > 0\) be large enough, e.g. \(C_L = \frac{120}{c^2}\). Let \((k, v) \in [0, 1) \times \mathbb{C}^2\). We have

\[ f_2(k, v) \geq |k|1|k - \frac{1}{2}|1|2\pi i(k - \frac{1}{2})v_1 - v_2| \geq |k|1|k - \frac{1}{2}|1|v_2| - \pi|k - \frac{1}{2}|^2|v_1|. \]  

(50)

By (50) and the definition of \(f_2\), we have

\[ f_2(k, v) \geq \frac{1}{2}|k|1|k - \frac{1}{2}|1|v_2|. \]  

(51)
Case 1: $k \in [0, \frac{1}{4}] \cup \left[\frac{3}{4}, 1\right)$.

Since $k \in [0, \frac{1}{4}] \cup \left[\frac{3}{4}, 1\right)$, we have $|k - \frac{1}{2}|_1 \geq \frac{1}{4}$. We have

\[
\inf_{s \in \mathbb{C}} f_1(k, v, s) \leq f_1(k, v, 0) \leq 6|v_1| + |v_2| \sum_{n=1}^{3} |e^{-2\pi i nk} - 1| \leq 6|v_1| + |v_2| \sum_{n=1}^{3} n^{-1} |e^{-2\pi i nk}| \leq 6|v_1| + \frac{6}{cT} |k|_1 |v_2| \leq C_L f_2(k, v),
\]

where we used (39) in the second to last step and (51) in the last step.

Case 2: $k \in \left(\frac{1}{4}, \frac{3}{4}\right)$.

Since $k \in \left(\frac{1}{4}, \frac{3}{4}\right)$, we have $|k|_1 \geq \frac{1}{4}$. By (41) and (40) we have

\[
\inf_{s \in \mathbb{C}} f_1(k, v, s) \leq f_1(k, v, v_2) \leq \sum_{n=1}^{3} \left( |e^{-2\pi i nk} - (-1)^n v_1 + (-1)^n n v_2| + |e^{-2\pi i nk} - (-1)^n||v_2| \right) \leq \sum_{n=1}^{3} \left( |e^{-2\pi i nk} - (-1)^n + (-1)^n 2\pi i (k - \frac{1}{2}) v_1 - v_2| + |e^{-2\pi i nk} - (-1)^n||v_2| \right) \leq \frac{6}{cT} \left( |k - \frac{1}{2}|^2 |v_1| + |2\pi i (k - \frac{1}{2}) v_1 - v_2| + |k - \frac{1}{2}|_1 |v_2| \right). \tag{52}
\]

By (51) and (52) we have

\[
\inf_{s \in \mathbb{C}} f_1(k, v, s) \leq C_L f_2(k, v). \quad \square
\]

A Selected auxiliary results

For easy reference we collect a couple of auxiliary results in this appendix. It is well-known that commuting orthogonal matrices are simultaneously quasidiagonalisable, see e.g. [13, Corollary 2.5.11.(c), Theorem 2.5.15]:

**Theorem A.1.** Let $S \subset O(n)$ be a nonempty commuting family of real orthogonal matrices. Then there exist a real orthogonal matrix $Q$ and a nonnegative integer $q$ such that, for each $A \in S$, $Q^T AQ$ is a real quasidiagonal matrix of the form

\[
\Lambda(A) \oplus R(\theta_1(A)) \oplus \cdots \oplus R(\theta_q(A))
\]

in which each $\Lambda(A) = \text{diag}(\pm 1, \ldots, \pm 1) \in \mathbb{R}^{(n-2q) \times (n-2q)}$, $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is the rotation matrix and each $\theta_j(A) \in [0, 2\pi)$.

Let $\sigma_{\min}(M)$ and $\|M\|$ denote the minimum singular value and the Frobenius norm of a matrix $M$, respectively. We have the following singular value inequality, see Corollary 9.6.7 in [2].
Theorem A.2. Suppose $A, B \in \mathbb{C}^{d \times d}$. Then

$$\|AB\| \geq \sigma_{\min}(A)\|B\| \quad \text{and} \quad \|AB\| \geq \|A\|\sigma_{\min}(B).$$

Kronecker’s approximation theorem, see e.g. Corollary 2 on page 20 in [12], reads:

Theorem A.3. For each irrational number $\alpha$ the set of numbers $\{\alpha n \, \text{reduced modulo} \, 1 \mid n \in \mathbb{N}\}$ is dense in the whole interval $[0,1)$.

We also need the following minimax theorem of Turán on generalized power sums, see Theorem 11.1 on page 126 in [22].

Theorem A.4. Let $b_1, \ldots, b_n, z_1, \ldots, z_n \in \mathbb{C}$. If $m$ is a nonnegative integer and the $z_j$ are restricted by

$$\frac{\min_{\mu \neq \nu}|z_\mu - z_\nu|}{\max_j|z_j|} \geq \delta (>0), \quad z_j \neq 0$$

then the inequality

$$\max_{\nu=m+1,\ldots,m+n} \frac{|\sum_{j=1}^n b_j z_\nu^j|}{\sum_{j=1}^n |b_j||z_j|^\nu} \geq \frac{1}{n} \left(\frac{\delta}{2}\right)^{n-1}$$

holds.

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