DIFFERENTIAL ALGEBRAS ON $\kappa$-MINKOWSKI SPACE
AND ACTION OF THE LORENTZ ALGEBRA

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ABSTRACT. We propose two families of differential algebras of classical dimension on
$\kappa$-Minkowski space. The algebras are constructed using realizations of the generators as
formal power series in a Weyl super-algebra. We also propose a novel realization of the
Lorentz algebra $\mathfrak{so}(1, n - 1)$ in terms of Grassmann-type variables. Using this realization
we construct an action of $\mathfrak{so}(1, n - 1)$ on the two families of algebras. Restriction of
the action to $\kappa$-Minkowski space is covariant. In contrast to the standard approach the
action is not Lorentz covariant except on constant one-forms, but it does not require an
extra cotangent direction.

Keywords: $\kappa$-Minkowski space, Lorentz algebra, realizations, differential algebra,
covariance

PACS numbers: 02.20.Sv, 02.20.Uw, 02.40.Gh

1. INTRODUCTION

Noncommutative (NC) geometry has been proposed for many years as a suitable model
for unification of quantum field theory and gravity. Noncommutative spaces have been
studied from many different points of view, including operator theory [1] and Hopf algebras
[2]–[24]. In particular, the notion of differential calculus on NC spaces has been studied in
Refs. [4]–[14]. It is known that many classes of NC spaces do not admit differential calculi
of classical dimensions which are fully covariant under the expected group of symmetries
[25]. This quantum anomaly for differential structures is usually fixed by introducing extra
cotangent directions.

In this paper we focus our attention to $\kappa$-Minkowski space. This is a Lie algebra type
NC space which appears as a deformation of ordinary Minkowski space-time within the
framework of doubly special relativity (DSR) [26]–[33]. The symmetry algebra for DSR
is obtained by deforming the ordinary Poincaré algebra into a Hopf algebra known as
$\kappa$-Poincaré algebra [29]–[32]. Different bases of $\kappa$-Poincaré algebra correspond to different
versions of DSR theory [29]. The $\kappa$-deformed Poincaré algebra as deformed symmetry of
the $\kappa$-Minkowski space-time inspired many authors to construct quantum field theories (see e.g. Refs. [33]–[38]) and electrodynamics on $\kappa$-Minkowski space-time [39]–[41], or to modify particle statistics [42], [43]. Bicovariant differential calculus on $\kappa$-Minkowski space-time was considered by Sitarz in Ref. [7]. He has shown that if the bicovariant calculus is required to be Lorentz covariant, then one obtains a contradiction with a Jacobi identity for the generators of the differential algebra. This contradiction is resolved by adding an extra cotangent direction (one-form) which has no classical analogue. Thus, the differential calculus in 3+1 dimensions developed in Ref. [7] is five-dimensional. This work was generalized to $n$ dimensions by Gonera et al. in Ref. [8]. There have been several attempts to deal with this issue in $\kappa$-Euclidean and $\kappa$-Minkowski spaces [9]–[14]. In Ref. [12] Bu et al. constructed a differential algebra on $\kappa$-Minkowski space from Jordanian twist of the Weyl algebra and showed that the algebra is closed in four dimensions. In their approach they extended the $\kappa$-Poincaré algebra with a dilatation operator and used a coproduct of the Lorentz generators which is different from the one used in Ref. [7]. In Refs. [13] and [14] differential algebras of classical dimension on $\kappa$-Euclidean and $\kappa$-Minkowski spaces are constructed. In this approach one-forms are obtained from an action of a deformed exterior derivative on NC coordinates. Different deformations of exterior derivative and NC coordinates lead to different versions of differential calculus compatible with $\kappa$-deformation.

In the present work we propose new families of differential algebras (denoted $D_1$ and $D_2$) on $\kappa$-Minkowski space $M_\kappa$ using realizations of the generators as formal power series in a Weyl super-algebra. We also present a novel realization of the Lorentz algebra $\mathfrak{so}(1, n - 1)$ in terms of Grassmann-type variables. This realization is used to define an action of $\mathfrak{so}(1, n - 1)$ on $D_1$ and $D_2$ which is consistent with commutation relations in $D_1$ and $D_2$ and Lorentz covariant on constant one-forms. Restriction of the action to $M_\kappa$ is covariant, thus $M_\kappa$ is an $\mathfrak{so}(1, n - 1)$-module algebra.

The paper is organized as follows. In Sec. 2 we discuss briefly a method for constructing differential algebras based on realizations of NC coordinates and exterior derivative as formal power series in a Weyl super-algebra. We construct two families of differential algebras $D_1$ and $D_2$ of classical dimension on $\kappa$-Minkowski space, and discuss their properties. We also show that by the same method the differential algebras in Refs. [7] and [15] can be constructed. In Sec. 3 we propose an action of $\mathfrak{so}(1, n - 1)$ on $D_1$ and
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$\mathcal{D}_2$ using a realization of $\mathfrak{so}(1, n - 1)$ in terms of Grassmann-type variables. The action is not Lorentz covariant except on one-forms, but it does not require an extra cotangent direction as in Ref. [7]. When the action is restricted to $\kappa$-Minkowski space (which is a subalgebra of $\mathcal{D}_1$ and $\mathcal{D}_2$), then the Lorentz algebra acts covariantly on products of space-time coordinates which reproduces the well-know result of Majid and Ruegg [3]. In Sec. 4 a short conclusion and future outlook is given.

2. Differential algebras on $\kappa$-Minkowski space

In this section we present the main points of the construction of differential algebras on $\kappa$-Minkowski space using realizations. First, we consider the differential algebra introduced by Sitarz [7] (see also Ref. [15]).

The $\kappa$-Minkowski space $\mathcal{M}_\kappa$ is an associative algebra generated by space-time coordinates $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$ satisfying the Lie algebra type commutation relations

\[
[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_j] = i a_0 \hat{x}_j, \quad a_0 \in \mathbb{R},
\]  

(1)

By convention Latin indices run from 1 to $n - 1$, and Greek indices run from 0 to $n - 1$. A bicovariant differential algebra compatible with relations (1) was constructed by Sitarz in Ref. [7]. He has shown that if the differential algebra is also Lorentz covariant, then the smallest such algebra in 3+1 dimensions is five-dimensional. One of its equivalent forms is given by

\[
[\hat{\xi}_0, \hat{x}_0] = -ia_0 \theta' + ia_0 \hat{\xi}_0, \quad [\hat{\xi}_0, \hat{x}_j] = ia_0 \hat{\xi}_j,
\]  

(2)

\[
[\hat{\xi}_i, \hat{x}_0] = 0, \quad [\hat{\xi}_i, \hat{x}_j] = -ia_0 \delta_{ij} \theta',
\]  

(3)

\[
[\theta', \hat{x}_0] = -ia_0 \theta', \quad [\theta', \hat{x}_j] = 0,
\]  

(4)

where $\hat{\xi}_\mu$ is the one-form corresponding to $\hat{x}_\mu$, and $\theta'$ is a one-form representing an extra cotangent direction that has no classical analogue. The one-forms $\hat{\xi}_\mu$ and $\theta'$ anticommute.

Let $\mathfrak{D}$ denote the algebra [2]–[4]. The algebra $\mathfrak{D}$ was considered in Ref. [15] where it is shown that by gauging a coefficient of $\theta'$ one can introduce gravity in the model. If we make a change of basis $\theta = \hat{\xi}_0 - \theta'$, we recover the original algebra introduced in Ref. [7]. As stated above, $\mathfrak{D}$ is constructed by postulating both bicovariance and Lorentz covariance of the differential calculus on $\kappa$-Minkowski space. These conditions imply that

(1) $[\hat{x}_\mu, \hat{\xi}_\nu]$ and $[\hat{x}_\mu, \theta']$ are closed in the vector space spanned by one-forms alone,

(2) all graded Jacobi identities in $\mathfrak{D}$ hold,
(3) the action of the Lorentz algebra $\mathfrak{so}(1, n - 1)$ is covariant:

$$M \triangleright (\tilde{x}_\mu \hat{x}_\nu) = (M(1) \triangleright \hat{x}_\mu) \hat{d}(M(2) \triangleright \hat{x}_\nu),$$

$$M \triangleright (\hat{x}_\mu \tilde{x}_\nu) = \hat{d}(M(1) \triangleright \hat{x}_\mu)(M(2) \triangleright \hat{x}_\nu), \quad M \triangleright \theta' = 0,$$

where $\hat{d}$ is the exterior derivative and $M$ is a generator of $\mathfrak{so}(1, n - 1)$. Here, the commutation relations in $\mathfrak{so}(1, n - 1)$ are undeformed,

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda},$$

and $\Delta M = M(1) \otimes M(2)$ is the coproduct of $M$ in Sweedler notation:

$$\Delta M_{i0} = M_{i0} \otimes 1 + e^{a_0 p_0} \otimes M_{i0} - a_0 \sum_{j=1}^{n-1} p_j \otimes M_{ij},$$

$$\Delta M_{ij} = M_{ij} \otimes 1 + 1 \otimes M_{ij},$$

where $p_\mu$ is the momentum generator. The coproduct of the momentum generators is given by [7], [11]

$$\Delta p_0 = p_0 \otimes 1 + 1 \otimes p_0,$$

$$\Delta p_i = p_i \otimes 1 + e^{a_0 p_0} \otimes p_i.$$  

Relations [8]-[11] describe the coalgebra structure of the $\kappa$-Poincaré algebra generated by $M_{\mu\nu}$ and $p_\mu$. Note that Eq. 5 implies that the Lorentz generators act on a constant one-form by $M \triangleright \hat{x}_\mu = \hat{d}(M \triangleright \hat{x}_\mu)$.

In the following we shall briefly outline the construction of the algebra $\mathfrak{D}$ using realizations of $\hat{x}_\mu$, $\hat{\xi}_\mu$ and $\theta'$ as formal power series in a Weyl super-algebra. Let $\mathcal{A}$ denote the unital associative algebra generated by commutative coordinates $x_\mu$, differential operators $\partial_\mu = \frac{\partial}{\partial x_\mu}$ and ordinary one-forms $dx_\mu$ satisfying the defining relations

$$[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0, \quad [\partial_\mu, x_\nu] = \eta_{\mu\nu},$$

$$[dx_\mu, x_\nu] = [dx_\mu, \partial_\nu] = 0, \quad \{dx_\mu, dx_\nu\} = 0.$$  

Here, $\{ , \}$ denotes the anticommutator and $\eta = \text{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric. $\mathcal{A}$ becomes a Weyl super-algebra if we define a graded commutator

$$[[u, v]] = uv - (-1)^{|u||v|} vu$$  

(14)
where \(|u|\) denotes the degree of a homogeneous element \(u \in \mathcal{A}\). The degrees of the generators are defined by \(|x_\mu| = |\partial_\mu| = 0\) and \(|dx_\mu| = 1\). In this paper we consider two types of realizations of \(\hat{x}_\mu\), the natural \([44]\) and noncovariant \([45], [46]\). Following the notation in Ref. \([14]\) the variables used in the natural and noncovariant realizations are denoted by \((X_\mu, D_\mu)\) and \((x_\mu, \partial_\mu)\), respectively. The reason for using different notation for the generators of \(\mathcal{A}\) is that there exists an invertible transformation \((x_\mu, \partial_\mu) \mapsto (X_\mu, D_\mu)\) mapping the noncovariant into natural realization. The natural realization is defined by

\[
\hat{x}_\mu = X_\mu Z^{-1} - i a_0 X_0 D_\mu
\]  

(15)

where \(Z\) is invertible operator given by

\[
Z^{-1} = i a_0 D_0 + \sqrt{1 + a_0^2} D_0^2.
\]  

(16)

The scalar product in (16) is taken with respect to the Minkowski metric, i.e. \(D^2 = -D_0^2 + \sum_{k=1}^{n-1} D_k^2\). \(Z\) is called the shift operator because conjugation of \(\hat{x}_\mu\) by \(Z\) yields \(Z\hat{x}_\mu Z^{-1} = \hat{x}_\mu + i a_0 \delta_0 \mu\). One easily checks that the space-time coordinates represented by (15) satisfy the commutation relations (1). The realization (15) is a special case of covariant realizations of \(\kappa\)-Minkowski space introduced in Ref. \([44]\).

Exterior derivative \(\hat{d}\) is defined by \(\hat{d} = \sum_{\alpha,\beta=0}^{n-1} k_{\alpha\beta}(D) dX_\alpha D_\beta\) where \(k_{\alpha\beta}(D)\) is a formal power series in \(a_0\) with coefficients in the ring of differential operators \(D_\mu\). We require that \(\lim_{a_0 \to 0} \hat{d} = d\) where \(d = -dX_0 D_0 + \sum_{k=1}^{n-1} dX_k D_k\) is the classical exterior derivative. The exterior derivative acts on space-time coordinates by \(\hat{d} \cdot \hat{x}_\mu = [[\hat{d}, \hat{x}_\mu]]\). We define a noncommutative version of one-forms by \(\hat{\xi}_\mu = \hat{d} \cdot \hat{x}_\mu\). Using relations (12)-(13) we find

\[
\lim_{a_0 \to 0} \hat{\xi}_\mu = [d, X_\mu] = dX_\mu,
\]  

(17)

hence \(\hat{\xi}_\mu\) is a deformation of ordinary one-form \(dX_\mu\). Before proceeding further let us point out some general properties of a differential algebra constructed in this way:

1. \(\hat{d}\) satisfies the undeformed Leibniz rule

\[
\hat{d} \cdot (f(\hat{x})g(\hat{x})) = (\hat{d} \cdot f(\hat{x}))g(\hat{x}) + f(\hat{x})(\hat{d} \cdot g(\hat{x}))
\]  

(18)

where \(f(\hat{x})\) and \(g(\hat{x})\) are monomials in \(\hat{x}_\mu\).
2. one-forms are closed, i.e. \(\hat{d} \cdot \hat{\xi}_\mu = [[\hat{d}, \hat{\xi}_\mu]] = 0\),
3. one-forms anticommute, \(\{\hat{\xi}_\mu, \hat{\xi}_\nu\} = 0\),
(4) the commutator for \( \hat{\xi}_\mu \) and \( \hat{x}_\nu \) is given by

\[
[\hat{\xi}_\mu, \hat{x}_\nu] = \sum_{\alpha=0}^{n-1} K_{\mu\nu}^\alpha(D) \hat{\xi}_\alpha
\]

where \( K_{\mu\nu}^\alpha(D) \) generally depends on the differential operators \( D_\mu \). If \( K_{\mu\nu}^\alpha \) are constant for all values of \( \mu, \nu \) and \( \alpha \), then the differential algebra is closed.

We note that the Jacobi identity for \( \hat{d} \cdot [\hat{x}_\mu, \hat{x}_\nu] = [\hat{d}, [\hat{x}_\mu, \hat{x}_\nu]] \) together with commutation relations (1) implies that \( \hat{x}_\mu \) and \( \hat{\xi}_\mu \) satisfy the compatibility condition

\[
[\hat{\xi}_\mu, \hat{x}_\nu] - [\hat{\xi}_\nu, \hat{x}_\mu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{\xi}_\mu)
\]

where \( a_\mu = a_0 \delta_{0\mu} \). Extension of the above construction to higher order forms was presented in detail in Ref. [13].

Given the realization (15) we want to find a realization of \( \hat{d} \) such that the action of \( \hat{d} \) on \( \hat{x}_\mu \) generates one-forms \( \hat{\xi}_\mu \) and \( \theta' \) which close the algebra (2)-(4). Consider the following ansatz for \( \hat{d} \):

\[
\hat{d} = -dX_0 D_0 + \left( \sum_{k=1}^{n-1} dX_k D_k \right) Z. \tag{21}
\]

Substituting Eqs. (15) and (21) into \( \hat{\xi}_\mu = [[\hat{d}, \hat{x}_\mu]] \) one finds

\[
\hat{\xi}_0 = dX_0 (Z^{-1} - ia_0 D_0) + ia_0 \left( \sum_{k=1}^{n-1} dX_k D_k \right) Z, \tag{22}
\]

\[
\hat{\xi}_k = dX_k - ia_0 dX_0 D_k. \tag{23}
\]

The commutation relations for \( \hat{x}_\mu \) and \( \hat{\xi}_\mu \) are given by

\[
[\hat{\xi}_0, \hat{x}_0] = -ia_0 dX_0 Z^{-1} + ia_0 \hat{\xi}_0, \quad [\hat{\xi}_0, \hat{x}_j] = ia_0 \hat{\xi}_j, \tag{24}
\]

\[
[\hat{\xi}_i, \hat{x}_0] = 0, \quad [\hat{\xi}_i, \hat{x}_j] = -ia_0 \delta_{ij} dX_0 Z^{-1}. \tag{25}
\]

Note that the algebra (24)-(25) is not closed since the commutators involve an additional term \( dX_0 Z^{-1} \) which does not correspond to any one-form \( \hat{\xi}_\mu \). However, the algebra can be closed by defining an extra one-form by \( \theta' = dX_0 Z^{-1} \). Then one easily finds

\[
[\theta', \hat{x}_0] = -ia_0 \theta', \quad [\theta', \hat{x}_j] = 0. \tag{26}
\]

The commutation relations (24)-(26) agree with the differential algebra (2)-(4). Thus, in our approach the extra cotangent direction \( \theta' \) introduced in Refs. [7] and [15] appears as a deformation of one-form \( dX_0 \) associated with time coordinate. In fact, \( \hat{\xi}_0 \) and \( \theta' \) are both deformations of \( dX_0 \), albeit different.
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2.1. Differential algebras of classical dimension. Different realizations of $\hat{x}_\mu$ and $\hat{d}$ lead to different differential calculi on $\kappa$-Minkowski space $M_\kappa$. In the following we construct two families of differential algebras on $M_\kappa$ such that there is a one-to-one correspondence between deformed one-forms and space-time coordinates. In both cases the realization of $\hat{x}_\mu$ is the same, but we consider two different deformations of the exterior derivative $\hat{d}$.

Let

\[ \hat{x}_0 = x_0 + ia_0 \sum_{k=1}^{n-1} x_k \partial_k, \quad \hat{x}_k = x_k. \tag{27} \]

This is a special case of noncovariant realizations of the algebra (1) introduced in Ref. [45], [46]. The transformation of variables $(x_\mu, \partial_\mu) \mapsto (X_\mu, D_\mu)$ which maps the noncovariant into natural realization is given in Ref. [14]. The realization (27) corresponds to the bicrossproduct basis in Ref. [3]. The shift operator corresponding to realization (27) is given by

\[ Z = \exp(A), \quad A = -ia_0 \partial_0. \tag{28} \]

Let us define the exterior derivative

\[ \hat{d}_1 = dx_0 \frac{Z^c - 1}{ia_0 c} + \left( \sum_{k=1}^{n-1} dx_k \partial_k \right) Z^{-1}, \quad c \neq 0, \tag{29} \]

where for $c \to 0$ we have $\lim_{c \to 0} \hat{d}_1 = -dx_0 \partial_0 + \left( \sum_{k=1}^{n-1} dx_k \partial_k \right) Z^{-1}$. The one-forms $\hat{\xi}_\mu = [\hat{d}_1, \hat{x}_\mu]$ with space-time coordinates represented by (27) are given by

\[ \hat{\xi}_0 = dx_0 Z^c, \quad \hat{\xi}_k = dx_k Z^{-1}. \tag{30} \]

Using realizations (27) and (30) we find

\[ [\hat{\xi}_0, \hat{x}_0] = ia_0 c \hat{\xi}_0, \quad [\hat{\xi}_k, \hat{x}_0] = -ia_0 \hat{\xi}_k, \quad [\hat{\xi}_\mu, \hat{x}_j] = 0. \tag{31} \]

We denote the differential algebra (31) by $\mathcal{D}_1$. Similarly, if the exterior derivative is defined by

\[ \hat{d}_2 = dx_0 \frac{Z^c - 1}{ia_0 c} + \left( \sum_{k=1}^{n-1} dx_k \partial_k \right) Z^{c-1}, \tag{32} \]

then the one-forms $\hat{\xi}_\mu = [\hat{d}_2, \hat{x}_\mu]$ are found to be

\[ \hat{\xi}_0 = dx_0 Z^c + ia_0 c \left( \sum_{k=1}^{n-1} dx_k \partial_k \right) Z^{c-1}, \quad \hat{\xi}_k = dx_k Z^{c-1}. \tag{33} \]
Now Eqs. (27) and (33) imply
\[
\begin{align*}
\hat{\xi}_0, \hat{x}_\mu &= \text{i}a_0 \epsilon \hat{\xi}_\mu, \\
\hat{\xi}_k, \hat{x}_0 &= \text{i}a_0(\epsilon - 1)\hat{\xi}_k, \\
\hat{\xi}_k, \hat{x}_j &= 0.
\end{align*}
\]
(34)

The differential algebra (34) is denoted by \( D_2 \). Note that \( D_1 \) and \( D_2 \) are two families of differential algebras depending on a real parameter \( \epsilon \) obtained from a fixed realization of \( \hat{x}_\mu \). They are compatible with \( \kappa \)-Minkowski space since they satisfy the compatibility condition (20) and all graded Jacobi identities for the generators of \( D_1 \) and \( D_2 \) hold. The commutator \([ \hat{\xi}_\mu, \hat{x}_\nu ]\) in both algebras is closed in the vector space spanned by \( \hat{\xi}_0, \hat{\xi}_1, \ldots, \hat{\xi}_{n-1} \), hence \( D_1 \) and \( D_2 \) are differential algebras of classical dimension.

3. Action of the Lorentz algebra on \( D_1 \) and \( D_2 \)

The aim of this section is to construct an action of the Lorentz algebra so\((1, n-1)\) on the algebras \( D_1 \) and \( D_2 \). First, we define an action of so\((1, n-1)\) on the subalgebra \( M_\kappa \) and then extend it to \( D_1 \) and \( D_2 \).

It is natural to consider extension of the \( \kappa \)-Minkowski space (1) by momentum operators \( p_\mu \). If we take the realization (27) and define \( p_\mu = -\text{i}\partial_\mu \), then \( \hat{x}_\mu \) and \( \partial_\mu \) generate a deformed Heisenberg algebra given by the relations (1) and
\[
\begin{align*}
[p_\mu, p_\nu] &= 0, \\
[p_0, \hat{x}_\mu] &= \text{i}\delta_{\mu 0}, \\
[p_k, \hat{x}_0] &= \text{i}a_0p_k, \\
[p_k, \hat{x}_j] &= -\text{i}\delta_{kj}.
\end{align*}
\]
(35)

Note that the deformation of the algebra (35) depends on the realization of the Minkowski coordinates \( \hat{x}_\mu \). A large class of such deformations was found in Refs. [44], [45]. Similarly, the \( \kappa \)-Minkowski space can be extended by the Lorentz algebra such that the direct sum of vector spaces \( g_\kappa = M_\kappa \oplus \text{so}(1, n-1) \) is a Lie algebra. It can be shown that the cross commutator \([M_\mu\nu, \hat{x}_\lambda]\), which must be linear in \( M_\mu\nu \) and \( \hat{x}_\mu \), is uniquely given by [45], [46]
\[
\begin{align*}
[M_00, \hat{x}_0] &= -\hat{x}_0 + \text{i}a_0M_{00}, \\
[M_{ij}, \hat{x}_0] &= 0, \\
[M_{0k}, \hat{x}_0] &= -\delta_{ik}\hat{x}_0 + \text{i}a_0M_{0k}, \\
[M_{ij}, \hat{x}_k] &= \delta_{jk}\hat{x}_i - \delta_{ik}\hat{x}_j.
\end{align*}
\]
(36)

The algebra (36)-(37) is a subalgebra of the DSR algebra obtained as a cross product extension of \( \kappa \)-Minkowski and \( \kappa \)-Poincaré algebras [17], [48]. Since the commutation relations (36)-(37) are unique, the extension of \( M_\kappa \) by \( \text{so}(1, n-1) \) is independent of the
realization of \( \hat{x}_\mu \). If the coordinates \( \hat{x}_\mu \) are given by the noncovariant realization (27), then the Lorentz generators are represented by

\[
M_{i0} = x_i \left( 1 - \frac{Z}{\lambda a_0} + \frac{ia_0}{2} \Delta - \frac{2}{ia_0} \sinh \left( \frac{1}{2} A \right) Z \right) - \left( x_0 + ia_0 \sum_{k=1}^{n-1} x_k \partial_k \right) \partial_i, \quad (38)
\]

\[
M_{ij} = x_i \partial_j - x_j \partial_i, \quad (39)
\]

where \( Z \) is given by Eq. (28) and \( \Delta = \sum_{k=1}^{n-1} \partial_k^2 \) is the Laplace operator. The realization (38)-(39) is a special case of the noncovariant realizations of the Lorentz algebra found in Refs. [45, 46]. In the classical limit we have \( \lim_{\lambda a_0 \to 0} M_{\mu \nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \), as required.

Given the commutation relations (11) and (36)-(37) we want to define an action \( \lambda : \mathfrak{so}(1, n-1) \times \mathcal{M}_\kappa \to \mathcal{M}_\kappa \). Let \( U(\mathfrak{g}_\kappa) \) be the enveloping algebra of \( \mathfrak{g}_\kappa \) and let the generators of \( U(\mathfrak{g}_\kappa) \) act on \( 1 \in U(\mathfrak{g}_\kappa) \) by \( \hat{x}_\mu \mapsto 1 = \hat{x}_\mu \) and \( M_{\mu \nu} \mapsto 1 = 0 \). Now define

\[
M_{\mu \nu} \mapsto f(\hat{x}) = [M_{\mu \nu}, f(\hat{x})] \mapsto 1 \quad (40)
\]

where \( f(\hat{x}) \) is a monomial in \( \mathcal{M}_\kappa \). Using relations (33)-37 the commutator in (40) can be written as a linear combination of terms with \( M_{\mu \nu} \), if any, pushed to the far right. Thus, the action (41) is the projection of \([M_{\mu \nu}, f(\hat{x})]\) to the subalgebra \( \mathcal{M}_\kappa \). For example, the action on Minkowski coordinates yields

\[
M_{\mu \nu} \mapsto \hat{x}_\lambda = \eta_{0 \lambda} \hat{x}_\mu - \eta_{\mu \lambda} \hat{x}_\nu. \quad (41)
\]

For monomials of order two we find

\[
M_{i0} \mapsto (\hat{x}_0 \hat{x}_k) = -\hat{x}_i \hat{x}_k - ia_0 \delta_i k \hat{x}_0 - \delta_i k \hat{x}_0^2, \quad (42)
\]

\[
M_{i0} \mapsto (\hat{x}_k \hat{x}_0) = -\hat{x}_k \hat{x}_i - \delta_{i k} \hat{x}_0^2, \quad (43)
\]

\[
M_{i0} \mapsto (\hat{x}_i \hat{x}_0) = \delta_{i k} \hat{x}_0 \hat{x}_0 \hat{x}_i \hat{x}_0 - \delta_{i i} \hat{x}_k \hat{x}_0 + ia_0 (\delta_{i k} \hat{x}_i - \delta_{i i} \hat{x}_k), \quad (44)
\]

\[
M_{ij} \mapsto (\hat{x}_0 \hat{x}_k) = \delta_{j k} \hat{x}_0 \hat{x}_i + \delta_{j i} \hat{x}_k \hat{x}_0 - \delta_{i j} \hat{x}_k \hat{x}_j, \quad (45)
\]

\[
M_{ij} \mapsto (\hat{x}_k \hat{x}_0) = \delta_{k i} \hat{x}_0 \hat{x}_i - \delta_{k i} \hat{x}_j \hat{x}_0, \quad (46)
\]

\[
M_{ij} \mapsto (\hat{x}_k \hat{x}_j) = \delta_{i k} \hat{x}_j \hat{x}_i - \delta_{i k} \hat{x}_j \hat{x}_j + \delta_{i k} \hat{x}_k \hat{x}_j - \delta_{i l} \hat{x}_k \hat{x}_j. \quad (47)
\]

The above result is the same as that obtained by Majid and Ruegg [3] using the covariance condition \( M_{\mu \nu} \mapsto (ab) = (M_{\mu \nu (1)} \mapsto a) (M_{\mu \nu (2)} \mapsto b) \), \( a, b \in \mathcal{M}_\kappa \), where the coproduct \( \Delta M_{\mu \nu} = M_{\mu \nu (1)} \otimes M_{\mu \nu (2)} \) is given by Eqs. (33)-(39) and the momentum operator in (8) acts by \( p_\mu \mapsto \hat{x}_\mu = -i \eta_{\mu \nu} \). This makes \( \mathcal{M}_\kappa \) into an \( \mathfrak{so}(1, n-1) \)-module algebra.
Next, we want to extend the action of $\mathfrak{so}(1, n - 1)$ to the differential algebras $\mathfrak{D}_1$ and $\mathfrak{D}_2$ using the same prescription (40). For this purpose we need the commutator $[M_{\mu\nu}, \hat{\xi}_\lambda]$ where $\hat{\xi}_\lambda \in \mathfrak{D}_1$ or $\hat{\xi}_\lambda \in \mathfrak{D}_2$. One can show that the general form of this commutator is given by

$$[M_{\mu\nu}, \hat{\xi}_\lambda] = [\hat{x}_\mu, \hat{\xi}_\lambda] \Phi_\nu - [\hat{x}_\nu, \hat{\xi}_\lambda] \Phi_\mu$$  \hspace{1cm} (48)$$

where $\Phi_\mu$ is a power series in $\partial_\mu$ such that $\Phi(0) = 0$. The functions $\Phi_\mu$ depend on the realization of $M_{\mu\nu}$. If $M_{\mu\nu}$ is given by Eqs. (38)-(39), then

$$\Phi_0 = \frac{1 - Z}{ia_0} + \frac{i a_0}{2} \Delta - \frac{2}{ia_0} \sinh^2 \left(\frac{1}{2} A\right) Z,$$

$$\Phi_k = \partial_k,$$  \hspace{1cm} (50)$$

where $\Delta = \sum_{k=1}^{n-1} \partial_k^2$. Since $[\hat{x}_\mu, \hat{\xi}_\nu] \in \text{span}\{\hat{\xi}_\lambda | 0 \leq \mu \leq n - 1\}$, the commutator $[M_{\mu\nu}, \hat{\xi}_\lambda]$ depends only on momentum operators $p_\mu = -i \partial_\mu$ and one-forms $\hat{\xi}_\lambda$. This means that in order to extend the action of $\mathfrak{so}(1, n - 1)$ to $\mathfrak{D}_k$ we need to extend the algebra $U(\mathfrak{g}_\kappa)$ by the generators $p_\mu$ and $\hat{\xi}_\mu$ where the extension depends on whether $\hat{\xi}_\mu \in \mathfrak{D}_1$ or $\hat{\xi}_\mu \in \mathfrak{D}_2$. Denote the extended algebras by $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then $\mathfrak{H}_k$ contains $\mathfrak{D}_k$ and $\mathfrak{so}(1, n - 1)$ as Lie subalgebras, as well as the abelian algebra of translations generated by $p_\mu$. Define the action of $p_\mu$ and $\hat{\xi}_\mu$ on $1 \in \mathfrak{H}_k$ by $p_\mu \cdot 1 = 0$ and $\hat{\xi}_\mu \cdot 1 = \hat{\xi}_\mu$. Now we may define the action $\mapsto: \mathfrak{so}(1, n - 1) \times \mathfrak{D}_k \to \mathfrak{D}_k$ by

$$M_{\mu\nu} \mapsto f(\hat{x}, \hat{\xi}) = [M_{\mu\nu}, f(\hat{x}, \hat{\xi})] \mapsto 1$$  \hspace{1cm} (51)$$

where $f(\hat{x}, \hat{\xi})$ is a monomial in $\mathfrak{D}_k$, $k = 1, 2$. The action (51) is uniquely fixed by the commutation relations in $\mathfrak{D}_k$ and Eqs. (36)-(37) and (48). Since the generators of $\mathfrak{H}_k$ are constructed as elements of an associative algebra all Jacobi identities in $\mathfrak{H}_k$ hold.

The Jacobi relations for $M_{\mu\nu}$, $\hat{x}_\mu$ and $\hat{\xi}_\mu$ guarantee that the action (51) is compatible with the commutation relations in $\mathfrak{D}_k$. Any monomial in $\mathfrak{D}_k$ can be written as a finite sum $f(\hat{x}, \hat{\xi}) = \sum f_1(\hat{x})f_2(\hat{\xi})$, hence it suffices to consider the action (51) on the products $f_1(\hat{x})f_2(\hat{\xi})$. Since $\Phi_\mu \mapsto 1 = 0$, Eq. (48) implies that the action of $M_{\mu\nu}$ on one-forms is trivial, i.e. $M_{\mu\nu} \mapsto f(\hat{\xi}) = 0$ for any monomial $f(\hat{\xi})$. Consequently, the action of $M_{\mu\nu}$ on product of monomials $f_1(\hat{x})f_2(\hat{\xi})$ is given by

$$M_{\mu\nu} \mapsto f_1(\hat{x})f_2(\hat{\xi}) = (M_{\mu\nu} \mapsto f_1(\hat{x}))f_2(\hat{\xi}).$$  \hspace{1cm} (52)$$

The construction outlined here has the advantage that the action (51) is compatible with the algebra structure of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ without introducing the extra one-form $\theta$ as in Ref.
However, the action is not Lorentz covariant since the necessary condition $M_{\mu\nu} \triangleright \hat{\xi}_\lambda = \hat{d} \cdot (M_{\mu\nu} \triangleright \hat{x}_\lambda) = \eta_{\nu\lambda} \hat{\xi}_\mu - \eta_{\mu\lambda} \hat{\xi}_\nu$ while $M_{\mu\nu} \triangleright \hat{\xi}_\lambda = 0$. This problem can be partially resolved by modifying the realization of $M_{\mu\nu}$ such that $M_{\mu\nu} \triangleright \hat{\xi}_\lambda = \eta_{\nu\lambda} \hat{\xi}_\mu - \eta_{\mu\lambda} \hat{\xi}_\nu$ is satisfied. This modification of (51) is given as follows.

Consider extension $\tilde{A}$ of the algebra (12)-(13) by a set of generators $q_\mu$ subject to defining relations

$$[x_\mu, q_\nu] = [\partial_\mu, q_\nu] = 0, \quad \{q_\mu, q_\nu\} = 0, \quad \{dx_\mu, q_\nu\} = \eta_{\mu\nu}. \quad (53)$$

The degree of $q_\mu$ is defined to be $|q_\mu| = 1$. Note that the variables $q_\mu$ play the role of a Grassmann type derivative with respect to one-forms $dx_\mu$. The $\kappa$-deformed super-Heisenberg algebra, generated by $\hat{x}_\mu$, $\partial_\mu$, $\hat{\xi}_\mu$ and $q_\mu$, satisfies all graded Jacobi identities.

Let us define

$$M_{\mu\nu}^{(1)} = dx_\mu q_\nu - dx_\nu q_\mu, \quad (54)$$

and let

$$\tilde{M}_{\mu\nu} = M_{\mu\nu} + M_{\mu\nu}^{(1)} \quad (55)$$

where the Lorentz generators $M_{\mu\nu}$ are given by the realization (38)-(39). It is easily seen that $M_{\mu\nu}^{(1)}$ close the relations (7) and $[M_{\mu\nu}, M_{\lambda\rho}^{(1)}] = 0$. Consequently, $\tilde{M}_{\mu\nu}$ also satisfy the relations (17), hence Eq. (55) represents a new realization of $\mathfrak{so}(1, n-1)$ in terms of the extended algebra $\tilde{A}$. To this realization of $\mathfrak{so}(1, n-1)$ we associate the action

$$\tilde{M}_{\mu\nu} \triangleright f(\hat{x}, \hat{\xi}) = [\tilde{M}_{\mu\nu}, f(\hat{x}, \hat{\xi})] \triangleright 1. \quad (56)$$

The action is consistent with the commutation relations in $\mathcal{D}_1$ and $\mathcal{D}_2$ since all Jacobi identities for $\tilde{M}_{\mu\nu}$, $\hat{x}_\mu$ and $\hat{\xi}_\mu$ are satisfied. For products of monomials $f_1(\hat{x})f_2(\hat{\xi})$ the action (56) satisfies the Leibniz-like rule

$$\tilde{M}_{\mu\nu} \triangleright f_1(\hat{x})f_2(\hat{\xi}) = (M_{\mu\nu} \triangleright f_1(\hat{x}))f_2(\hat{\xi}) + f_1(\hat{x})(M_{\mu\nu}^{(1)} \triangleright f_2(\hat{\xi})) \quad (57)$$

where $M_{\mu\nu}$ acts only on coordinates $\hat{x}_\mu$ and $M_{\mu\nu}^{(1)}$ acts only on one-forms $\hat{\xi}_\mu$. It follows from Eq. (57) that $\tilde{M}_{\mu\nu} \triangleright f_1(\hat{x}) = M_{\mu\nu} \triangleright f_1(\hat{x})$, hence the actions (51) and (56) agree on the $\kappa$-Minkowski space. In particular, we have the vector-like transformation $\tilde{M}_{\mu\nu} \triangleright \hat{x}_\lambda = \eta_{\nu\lambda} \hat{x}_\mu - \eta_{\mu\lambda} \hat{x}_\nu$. On the other hand, the action of $\tilde{M}_{\mu\nu}$ on one-forms is
nontrivial since one-forms also transform vector-like, \( \tilde{M}_{\mu\nu} \mapsto \tilde{\xi}_\lambda = \eta_{\nu\lambda} \hat{\xi}_\mu - \eta_{\mu\lambda} \hat{\xi}_\nu \). This implies
\[
\tilde{M}_{\mu\nu} \mapsto \tilde{\xi}_\lambda = \hat{d} \cdot (\tilde{M}_{\mu\nu} \mapsto \hat{x}_\lambda),
\]
thus the action is Lorentz covariant on constant one-forms. Restriction of (56) to monomials in \( \hat{\xi}_\mu \) satisfies the ordinary Leibniz rule
\[
\tilde{M}_{\mu\nu} \mapsto f(\hat{\xi}) g(\hat{\xi}) = (\tilde{M}_{\mu\nu} \mapsto f(\hat{\xi})) g(\hat{\xi}) + f(\hat{\xi}) (\tilde{M}_{\mu\nu} \mapsto g(\hat{\xi})).
\]
(59)
Using Eq. (58) and the rules for computing \( M_{\mu\nu} \mapsto f_1(\hat{x}) \) and \( M_{\mu\nu}^{(1)} \mapsto f_2(\hat{\xi}) \) one can easily calculate the action of \( \tilde{M}_{\mu\nu} \) on arbitrary monomials \( f(\hat{x}, \hat{\xi}) \in \mathcal{D}_k, k = 1, 2 \). For example,
\[
\tilde{M}_{\mu\nu} \mapsto \hat{x}_\lambda \hat{\xi}_\rho = (\eta_{\nu\rho} \hat{x}_\mu - \eta_{\mu\rho} \hat{x}_\nu) \hat{\xi}_\lambda + \hat{x}_\lambda (\eta_{\nu\rho} \hat{\xi}_\mu - \eta_{\mu\rho} \hat{\xi}_\nu).
\]
(60)

The condition (58) does not extend by Lorentz covariance to entire algebras \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \). This is in accordance with the theory developed in Ref. [7] since otherwise this would be in contradiction with the Jacobi identity for \( \hat{x}_\mu, \hat{x}_\nu \text{ and } \hat{\xi}_\lambda \) (for more details see Ref. [7]).

4. Conclusion

In this paper we have constructed differential algebras on \( \kappa \)-Minkowski space-time using realizations of coordinates \( \hat{x}_\mu \) and one-forms \( \hat{\xi}_\mu \) as formal power series in a Weyl superalgebra. The algebras considered here are the well-known differential algebra introduced by Sitarz [7] as well as new families of differential algebras \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \). The algebras \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are obtained from a fixed realization of \( \hat{x}_\mu \) and using different realization of exterior derivative \( \hat{d} \). The resulting one-forms \( \hat{\xi}_\mu = [[\hat{d}, \hat{x}_\mu]] \) have the property that the commutator \( [\hat{\xi}_\mu, \hat{x}_\nu] \) is closed in the vector space spanned by \( \hat{\xi}_0, \hat{\xi}_1, \ldots, \hat{\xi}_{n-1} \) alone.

We have also presented a novel construction of an action of \( \mathfrak{so}(1, n-1) \) on \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) using realizations of the Lorentz generators in terms of Grassmann-type variables \( q_\mu \). The action does not require introduction of an extra cotangent direction \( \theta \) as in Ref. [7]. When restricted to Minkowski coordinates, \( \mathfrak{so}(1, n-1) \) acts covariantly on the \( \kappa \)-Minkowski space making it into an \( \mathfrak{so}(1, n-1) \)-module algebra. The Lorentz covariance is valid for constant one-forms but it does not extend to entire algebras \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \). In this approach there is a one-to-one correspondence between the Minkowski coordinates and one-forms. This provides a certain advantage since every variable in the noncommutative setting is for a given realization a unique deformation of the corresponding classical variable. In this
paper we have focused only on the action of the Lorentz algebra generated by $M_{\mu\nu}$ and $\tilde{M}_{\mu\nu}$ (Eqs. (38), (39), (54) and (55)), but there are also other implementations of Lorentz algebras compatible with the $\kappa$-Minkowski space-time (Refs. [11], [12], [49] and [50]). Further developments of this approach as well as its applications to field theory, statistics, twist operators (see Refs. [12], [42], [43] and [49]) and dispersion relations [51] will be presented elsewhere.

Acknowledgements

This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contract No. 098-000000-2865 and 177-0372794-2816.

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