CLASSIFYING FANO COMPLEXITY-ONE $T$-VARIETIES VIA DIVISORIAL POLYTOPES

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Abstract. The correspondence between Gorenstein Fano toric varieties and reflexive polytopes has been generalized by Ilten and Süß to a correspondence between Gorenstein Fano complexity-one $T$-varieties and Fano divisorial polytopes. Motivated by the finiteness of reflexive polytopes in fixed dimension, we show that over a fixed base polytope, there are only finitely many Fano divisorial polytopes, up to equivalence. We classify two-dimensional Fano divisorial polytopes, recovering Huggenberger’s classification of Gorenstein del Pezzo $K^*$-surfaces. Furthermore, we show that any three-dimensional Fano divisorial polytope is equivalent to one involving only eight functions.

1. Introduction

It is well-known that Fano toric varieties with at worst Gorenstein singularities correspond to so-called reflexive polytopes, see e.g. [CLS11] §8.3. Furthermore, in any fixed dimension, there are only a finite number of reflexive polytopes up to equivalence [LZ91], and there is an algorithm for classifying them [KS97]. It follows that there are only finitely many Gorenstein Fano toric varieties in a fixed dimension, and they may be classified algorithmically.

A natural generalization of toric varieties are complexity-one $T$-varieties: normal varieties $X$ equipped with the effective action of an algebraic torus $T$ whose generic orbit has codimension one in $X$. Any toric variety $X$ may be considered as a complexity-one $T$-variety by restricting the action of the big torus on $X$ to a codimension-one subtorus $T$.

In a manner similar to the case of toric varieties, these varieties can also be encoded using quasi-combinatorial data, specifically, a generalization of polytopes. Throughout, suppose $M$ is a lattice.

Definition 1.1. A combinatorial divisorial polytope (CDP) with respect to the lattice $M$ consists of a full-dimensional lattice polytope $\Box \subset M \otimes \mathbb{R}$, along with an $n$-tuple $\Psi = (\Psi_1, \ldots, \Psi_n)$ (for some $n \in \mathbb{N}$) of piecewise-affine concave functions $\Psi_i : \Box \to \mathbb{R}$ such that

1. For each $i$, the graph of $\Psi_i$ is a polyhedral complex with integral vertices;
2. For each $u \in \Box^\circ$, $\sum \Psi_i(u) > 0$.

We call $\Box$ the base of $\Psi$. The dimension of $\Psi$ is $\dim \Box + 1$.

Attaching each of the functions $\Psi_i$ to a point $P_i$ in the curve $\mathbb{P}^1$ gives rise to a divisorial polytope on $\mathbb{P}^1$; these correspond to rational polarized complexity-one $T$-varieties [IS11].

Just as there is a special subclass of lattice polytopes corresponding to Gorenstein Fano toric varieties, there is a special subclass of CDPs corresponding to Fano complexity-one $T$-varieties with at worst canonical Gorenstein singularities [IS17, Definition 3.3 and Theorem 3.5]. We call such CDPs Fano, and recall the details in Definition 2.3.

Considering the finiteness result for reflexive polytopes, one might ask if a similar result holds for Fano CDPs. To pose this question we define a natural notion of equivalence of CDPs in §2.1. We say that a CDP is toric if it is equivalent to a CDP consisting of at most two functions $\Psi_1, \Psi_2$. Our conjecture for Fano CDPs is the following:

Conjecture 1.2. In any fixed dimension $d$, there are only finitely many equivalence classes of non-toric Fano CDPs.
Geometrically, this conjecture is equivalent to stating that there are only finitely many families of canonical Gorenstein Fano complexity-one $T$-varieties in any given dimension. It should be noted that any Fano toric variety can be considered as a complexity-one $T$-variety in infinitely many ways, which is why we exclude the toric case from the above conjecture.

1.1. **Main results.** Our first main result is the following:

**Main Theorem 1** (Theorem 4.2). *Over any fixed base $\square$, there are at most finitely many equivalence classes of Fano CDPs.*

While we do not present it as such, this result can be made effective. The proof of Main Theorem 1 suggests an algorithm to enumerate all equivalence classes of Fano CDPs over a fixed base.

Given Main Theorem 1 to prove Conjecture 1.2 it is sufficient to prove that only finitely many base polytopes $\square$ occur for non-toric Fano CDPs in any given dimension. Equivalently, a bound on the volume of the bases which occur (or the number of interior lattice points $[LZ91]$) would also prove the conjecture.

We then apply the tools we develop in the proof of Main Theorem 1 to analyze low-dimensional cases:

**Main Theorem 2** (Theorem 5.1). *There are exactly 34 equivalence classes of two-dimensional non-toric Fano CDPs.*

**Main Theorem 3** (Theorem 6.1). *Any three-dimensional Fano CDP is equivalent to one with at most eight functions.*

Our analysis in these cases leads us to conjecture (Conjecture 6.6) that any $d$-dimensional Fano CDP is equivalent to one with at most $2^d$ functions.

1.2. **Geometric interpretations of the main results.** Our results have an interesting geometric interpretation. Given a Fano complexity-one $T$-variety $X$ with at worst canonical Gorenstein singularities, let $\pi: \widetilde{X} \to \mathbb{P}^1$ be the resolution of the rational quotient map $X \dashrightarrow \mathbb{P}^1$.

**Corollary 1.3.** *For a fixed polarized toric variety $Y$, there are finitely many families of canonical Gorenstein Fano complexity-one $T$-varieties $X$ such that the general fiber of $\pi: \widetilde{X} \to \mathbb{P}^1$ is isomorphic to $Y$, polarized with regards to the pullback of $-K_X$.*

**Corollary 1.4.** *There are exactly 34 families of non-toric Gorenstein del Pezzo $K^*$-surfaces $X$.*

**Corollary 1.5.** *For any canonical Gorenstein Fano complexity-one threefold, the resolved quotient map $\pi$ has at most 8 non-integral fibers.*

These corollaries are direct translations of Main Theorems 1, 2, and 3 respectively.

An alternate approach to the classification of Fano complexity-one $T$-varieties is via a classification of their Cox rings. This has been employed by Huggenberger [Hug13, Hau13] to classify Gorenstein (log) del Pezzo surfaces with a $K^*$-action; our Main Theorem 2 and Corollary 1.4 recover this classification in the del Pezzo case. A translation of this classification into the language of divisorial polytopes used here is found in [CS17].

This approach via Cox rings has also been employed in [HHS11, BHNN16, FHN17] to classify higher-dimensional Fano complexity-one $T$-varieties of Picard ranks one and two. These techniques should work more generally, as long as one has a bound on the Picard rank of the varieties in question. Interestingly, while our Main Theorem 2 does not bound the Picard rank of Fano complexity-one threefolds, it does limit the number of equations which may appear in the defining ideal of the Cox ring. We hope that by combining this bound with the Cox ring approach, one may obtain a complete classification of canonical Gorenstein Fano complexity-one threefolds.

1.3. **Outline of the article.** First we provide some background information on CDPs in §2 including a description of operations preserving equivalence, as well as establishing the relationship between lattice polytopes and toric CDPs. We recall the definition of a Fano CDP given by Ilten and Süss in [IS17], and use the notion of equivalence to make some key simplifying assumptions about the properties of Fano CDPs. In §3 we provide, for fixed base polytope $\square$, an upper bound on the number of functions in a Fano CDP. Main Theorem 1 stating the finiteness of the number of equivalence classes of Fano CDPs over any fixed base, is...
Figure 1. Example of equivalent CDPs. The base polytope of the leftmost CDP is $\square = [-1, 1]$. First we shear one function by a factor of $-1$ and the other by a factor of $1$; next we translate the two functions; finally we transform the base by reflecting through the origin.

established in §4. In §5, we use the tools developed throughout this paper to provide a classification of all equivalence classes of two-dimensional Fano CDPs, proving Main Theorem 2. Finally, in §6, we prove Main Theorem 3.

2. Generalizing reflexive polytopes to Fano CDPs

In this section, we introduce the notion of equivalence of CDPs, discuss how CDPs can be viewed as a generalization of lattice polytopes, and introduce the Fano property of CDPs.

2.1. Equivalence of CDPs. Let $\Psi$ be a $d$-dimensional CDP with base $\square$ and functions $\Psi_1, \ldots, \Psi_n$. We may perform any combination of the following actions or their inverses to obtain an equivalent CDP:

- **Addition of the zero function.** If $0 = \Psi_{n+1}$, then $(\Psi_1, \ldots, \Psi_n, \Psi_{n+1})$ is equivalent to $\Psi$.
- **Permutation/Relabelling.** For $\sigma \in S_n$, the CDP with base $\square$ and functions $(\Psi_{\sigma(1)}, \ldots, \Psi_{\sigma(n)})$ is equivalent to $\Psi$.
- **Transformation of the base.** If $\phi$ is an invertible affine linear transformation of the lattice $M$, the CDP with base $\phi(\square)$ and functions $\Psi_i \circ \phi^{-1}$ is equivalent to $\Psi$.
- **Translation.** For $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ with $\sum_{i=1}^n \alpha_i = 0$, the CDP with base $\square$ and functions $\Psi_i + \alpha_i$ is equivalent to $\Psi$.
- **Shearing Action.** For $v \in M^*$ and $\beta_1, \ldots, \beta_n \in \mathbb{Z}$ with $\sum_{i=1}^n \beta_i = 0$, the CDP with base $\square$ and functions $u \mapsto \Psi_i(u) + \beta_i \cdot \langle u, v \rangle$ for all $u \in \square$, is equivalent to $\Psi$.

In Figure 1 we illustrate the equivalence operations of shearing, translating, and transforming the base.

Remark 2.1. The equivalences are motivated geometrically as follows. Given a CDP $\Psi$, attaching points $P_i \in \mathbb{P}^1$ to each function $\Psi_i$ gives rise to a divisorial polytope on $\mathbb{P}^1$; these correspond to rational polarized complexity-one $T$-varieties [IS11]. Any two equivalent CDPs will give rise to complexity-one $T$-varieties which are equivariantly isomorphic, after appropriate choice of the points $P_i$.

2.2. From polytopes to toric CDPs. Consider a polytope $P$ in $(M \times \mathbb{Z}) \otimes \mathbb{R}$, with vertices in the lattice $M \times \mathbb{Z}$. This gives rise to a CDP with two functions as follows. Let $\pi_1$ be the projection to $M \otimes \mathbb{R}$ and $\pi_2$ be the projection to $\mathbb{R}$. Set $\square = \pi_1(P)$ and define $\Psi_1, \Psi_2 : \square \to \mathbb{R}$ by

$$\Psi_1(u) = \max(\pi_2(\pi_1^{-1}(u) \cap P)) \quad \text{and} \quad \Psi_2(u) = -\min(\pi_2(\pi_1^{-1}(u) \cap P)).$$

Then the base $\square$ with the functions $\Psi_1, \Psi_2$ is a CDP. This process is illustrated with an example in Figure 2.

Conversely, this process can be inverted to obtain a lattice polytope from a CDP with two functions by reflecting one of the functions and taking the convex hull of the vertices of the graph of the CDP.

A CDP $\Psi$ is toric if it is equivalent to a CDP with at most two functions. Since we may add a constant zero function to obtain an equivalent CDP, we have that toric CDPs are those equivalent to a CDP with
Figure 2. Example of correspondence between polytopes and CDPs with two functions: (A) the lattice polytope; (B) the lattice polytope projected down one dimension; and (C) the resulting CDP.

Figure 3. Equivalent reflexive polytopes can correspond to inequivalent Fano CDPs.

exactly two functions. Thus, toric CDPs are exactly those which arise from a lattice polytope via the above construction.

Remark 2.2. If two toric CDPs are equivalent, then their corresponding polytopes are equivalent as lattice polytopes.

The converse of Remark 2.2 does not hold; that is, equivalent polytopes do not necessarily correspond to equivalent CDPs. Examples of this fact can be found by shearing a polytope in a direction other than the direction that we project along to obtain the polytope’s corresponding CDP. Such an example is given in Figure 3; the CDPs depicted there are inequivalent because their base polytopes are inequivalent. From the point of view of algebraic geometry, this is saying that a toric variety may be given the structure of a complexity-one $T$-variety by restricting to a codimension-one subtorus in infinitely many different ways.

2.3. Fano Divisorial Polytopes. The Fano property of a CDP, as defined by Ilten and Süß in [IS17], generalizes the reflexive property of polytopes. Under the correspondence between divisorial polytopes and polarized rational complexity-one $T$-varieties, it corresponds exactly to canonical Gorenstein Fano $T$-varieties with anti-canonical polarization. We recall this property below in the context of CDPs.

For a polytope $\square$, we denote its interior by $\square^\circ$ and its boundary by $\partial \square$. For a function $\Psi_i$, we denote the graph of $\Psi_i$ by $\Gamma(\Psi_i)$. A facet $F$ of a lattice polytope $P$ in $(M \times \mathbb{Z}) \otimes \mathbb{R}$ is in height one if there exists some $u \in M^* \times \mathbb{Z}$ such that $\langle v, u \rangle = 1$ for all $v \in F$.

**Definition 2.3.** A CDP is Fano if it is equivalent to a CDP $\Psi$ with base polytope $\square$ and functions $\Psi_1, \ldots, \Psi_n$ for which there are integers $a_1, \ldots, a_n$ such that

1. $0 \in \square^\circ$;
2. $\sum_{i=1}^n a_i = -2$;
3. For all $i = 1 \ldots n$, $\Psi_i(0) + a_i + 1 > 0$, and each facet of $\Gamma(\Psi_i + a_i + 1)$ is at height one;
4. For any facet $F$ of $\square$ not at height one, $\sum_{i=1}^n \Psi_i \equiv 0$ on $F$. 


When each facet of $\Gamma(\Psi_i + a_i + 1)$ is at height one, we say that $\Psi_i + a_i + 1$ is at height one.

**Remark 2.4.** The four properties of Definition 2.3 are preserved for equivalent CDPs, provided that any transformation of the base preserves the origin.

**Remark 2.5.** It is straightforward to check that under the construction producing a toric CDP from a lattice polytope in $M \times \mathbb{Z}$, the CDP is Fano if and only if its corresponding polytope is isomorphic to a reflexive polytope.

A consequence of Remark 2.4 along with the fact that the converse of Remark 2.2 does not hold in general, is that infinite families of inequivalent toric Fano CDPs can be constructed from a single isomorphism class of reflexive polytopes. In fact, Figure 3 gives an example of how to construct such a family. Such examples explain the restriction of the statement of Conjecture 1.2 to non-toric CDPs.

### 2.4. Normalization

For the purposes of classifying Fano CDPs, it would be useful to have some kind of normal form, that is, a distinguished representative in each equivalence class of Fano CDPs. While we have yet to find a natural normal form for Fano CDPs, in the following we describe a number of normalizing assumptions we will be making whenever considering a Fano CDP. This will be key to our strategy for obtaining bounds on the structure of Fano CDPs.

**Definition 2.6.** A Fano CDP $\Psi = (\Psi_1, \ldots, \Psi_n)$ is normalized if

1. $\Psi$ satisfies the four criteria of Definition 1.1;
2. If $\Psi_i$ is linear, then it does not have integral slope.

Note that any non-toric Fano CDP $\Psi$ is equivalent to a normalized Fano CDP. Indeed, by virtue of being Fano, we can certainly satisfy the first property above by replacing $\Psi$ with an equivalent CDP. Furthermore, given any linear $\Psi_i$ with integral slope, we can shear and translate so that the corresponding function becomes zero, and then eliminate it. We repeat this until we have only two functions remaining (and have a toric CDP), or there are no more linear functions with integral slope.

In the remainder of this paper, we will always be dealing with a normalized Fano CDP $\Psi$ with base polytope $\square$ and functions $\Psi_1, \ldots, \Psi_n$. In particular, there are integers $a_i$ satisfying properties 2 and 3 of Definition 1.1. It is often more convenient to consider the translated functions $\Psi'_i := \Psi_i + a_i + 1$, which are at height one. Applying Property 2 of Definition 1.1 to the translated functions $\Psi'_i$ yields

\[
\sum_{i=1}^{n} \Psi'_i > n - 2 \quad \text{on } \square^o
\]

and

\[
\sum_{i=1}^{n} \Psi'_i \geq n - 2 \quad \text{on } \partial \square.
\]

Furthermore, applying Property 3 of Definition 2.3 to the translated functions $\Psi'_i$ yields

\[
\sum_{i=1}^{n} \Psi'_i \equiv n - 2
\]

on any facet of $\square$ that is not at height one. We almost exclusively work with the translated functions $\Psi'_i$ in order to most easily exploit the property that the facets of their graphs are at height one.

### 3. Bounds on Number of Functions

#### 3.1. Overview

In this section we establish a bound on the number of functions in a normalized Fano CDP that is dependent on the base of the CDP (Theorem 3.9). This bound is established as follows: after assuming $M = \mathbb{Z}^d$, we pick a point $v_j$ so that $v_j$ and its reflection through the origin $-v_j$ both lie on the $j^{th}$ coordinate axis and in the base polytope. We give both lower and upper bounds on the sum

\[
\sum_{i=1}^{n} (\Psi'_i(-v_j) + \Psi'_i(v_j)),
\]
where $\Psi_1, \ldots, \Psi_n$ are the functions in a Fano CDP. The lower bound on (1) follows from Inequality (2). The upper bound on (1) is provided by Lemma 3.7, which uses the concavity of the functions $\Psi'_i$ to provide an upper bound for the sum $\Psi'_i(-v_j) + \Psi'_i(v_j)$. We sum (1) over all $j = 1, \ldots, d$. By arguing that the upper bound on $\Psi'_i(-v_j) + \Psi'_i(v_j)$ can only be achieved $d - 1$ times for fixed $i$, as otherwise $\Psi'_i$ would be linear with integral slope (see Lemmas 3.5 and 3.7), the bound given in the theorem is obtained.

3.2. Preliminaries. We establish some straightforward results on Fano CDPs. We let $\Psi$ be a $(d + 1)$-dimensional normalized Fano CDP with base polytope $\square$ and translated functions $\Psi'_1, \ldots, \Psi'_n$, which are at height one. For simplicity, we assume that $M = \mathbb{Z}^d$. First we introduce the notion of integral and non-integral CDPs:

**Definition 3.1.** The function $\Psi_i$ is said to be integral if $\Psi_i(v) \in \frac{1}{\lambda} \mathbb{Z}$ for all $\lambda \in \mathbb{N}$, all points $v \in \partial \square \cap \frac{1}{\lambda} M$. Otherwise we say that $\Psi_i$ is non-integral.

**Remark 3.2.** If $\Psi_i$ is linear, then it is integral in the above sense if and only if it has integral slope.

**Lemma 3.3.** For any $i = 1, \ldots, n$, the function $\Psi_i$ is integral if and only if $\Psi'_i(0) = 1$.

**Proof.** See [IS17, Remark 3.7] □

**Lemma 3.4.** Suppose $\Psi'_i(0) = 1$. Then $\Psi'_i$ is linear along the line segment from 0 to $v$ for all $v \in \partial \square$.

**Proof.** Restrict the base polytope $\square$ to the segment, which we denote by $L$, and let $F$ be a facet of $\Gamma(\Psi'_i)$ containing the point above the origin and some other point of $L$. Suppose $F$ meets another facet, say $F''$, along $L$. Extending $F'$, its value over the origin would be larger than 1 by the concavity of $\Psi'_i$. Then the point $(0, \ldots, 0, 1)$ lies between the extension of $F''$ and the origin, contradicting $F''$ being at height one. ■

**Lemma 3.5.** If $\Psi'_i$ is identically one along every coordinate axis, then $\Psi'_i \equiv 1$.

**Proof.** By concavity of $\Psi'_i$, $\Psi'_i \geq 1$ on the convex hull of the intersection of $\square$ with the coordinate axes. Thus by Lemma 3.4, $\Psi'_i \geq 1$ along the line from the origin to any point $v \in \square$, and hence $\Psi'_i \geq 1$ on $\square$.

Suppose that there is some point $v \in \square$ such that $\Psi'_i(v) > 1$. Choosing $\alpha > 0$ sufficiently small so that $-\alpha v \in \square$, the concavity of $\Psi'_i$ would imply that $\Psi'_i(-\alpha v) < 1$, a contradiction. Hence $\Psi'_i \equiv 1$. ■

**Lemma 3.6.** If $\Psi'_i$ is non-integral, then $\Psi'_i(0) = 1/\lambda$ for some $\lambda \in \mathbb{Z}_{>1}$. In particular, $\Psi'_i(0) \leq \frac{1}{2}$.

**Proof.** The point $p = (0, \ldots, 0, \Psi'_i(0))$ is in $\Gamma(\Psi'_i)$. Since $\Gamma(\Psi'_i)$ at height one, there is some $v \in \mathbb{Z}^{d+1}$ such that $(p, v) = -1$, that is,

$$\Psi'_i(0) = -\frac{1}{\lambda},$$

where $\lambda = v_{d+1}$ is the $(d + 1)^{th}$ component of $v$. Since $0 < \Psi'_i(0) < 1$ by assumption and Lemma 3.3 and $v_{d+1} \in \mathbb{Z}$, we have the desired result. ■

**Lemma 3.7.** Let $v \in \mathbb{R}^d$ lie on one of the coordinate axes, and suppose that $\pm v \in \square$.

1. If $\Psi'_i$ is non-integral, then $\Psi'_i(-v) + \Psi'_i(v) \leq 1$.

2. If $\Psi'_i$ is integral but non-linear along the line segment from $-v$ to $v$, then $\Psi'_i(-v) + \Psi'_i(v) \leq 2 - |v|$.

3. If $\Psi'_i$ is integral and linear along the line segment from $-v$ to $v$, then $\Psi'_i(-v) + \Psi'_i(v) = 2$.

**Proof.** First assume that $\Psi'_i$ is non-integral. Then, by the concavity of $\Psi'_i$ and by Lemma 3.6 we have

$$\frac{\Psi'_i(-v) + \Psi'_i(v)}{2} \leq \Psi'_i(0) \leq \frac{1}{2},$$

and the result follows.

Suppose $\Psi'_i$ is integral. By shearing we may assume that $\Psi'_i(-v) = 1$. If $\Psi'_i$ is linear along the line segment from $-v$ to $v$, then $\Psi'_i(v) = 1$, and the result in this case holds. Otherwise suppose $\Psi'_i$ is non-linear along this line segment. Then the slope of $\Psi'_i$ along the line segment from the origin to $v$ is at most $-1$, which
gives $\Psi'_i(v) \leq 1 - |v|$. Since the bound on the sum $\Psi'_i(-v) + \Psi'_i(v)$ is invariant under shearing, the result follows.

### 3.3. Main Result

We now show that the number of functions in a normalized Fano CDP is bounded by a value determined only by the base of the CDP.

Fix a lattice polytope $\square$ containing the origin in its interior. For any primitive element $v$ of the lattice $M$, define

$$\alpha_v = \min\{1, \max\{\alpha \in \mathbb{R}_{\geq 0} \mid \pm \alpha v \in \square\}\}.$$  

**Lemma 3.8.** Let $v \in M$ be primitive. Any normalized Fano CDP has at most $4/\alpha_v$ functions $\Psi_i$ which are either non-integral, or non-linear along the line spanned by $v$.

**Proof.** Set $v' = \alpha_v v$. Define

$$R := \{i \mid 1 \leq i \leq n, \text{ and } \Psi'_i(-v') + \Psi'_i(v') < 2\}$$

and $r := \#R$. Then by Lemmas 3.7 and 3.4, $r$ is exactly the number of functions which are either non-integral, or non-linear along the line spanned by $v$. Note that, for each $i \in R$, we have by Lemma 3.7 that $\Psi'_i(-v') + \Psi'_i(v') \leq 2 - \alpha_v$. This bound works for non-integral $\Psi'_i$ because $\alpha_v \leq 1$. Using this and the lower bound given in Inequality (2), we have

$$2n - 4 \leq \sum_{i=1}^{n} (\Psi'_i(-v') + \Psi'_i(v')) \leq 2(n - r) + r(2 - \alpha_v) = 2n - \alpha_v r$$

and hence $r \leq \frac{1}{\alpha_v}$.

Now, fix a basis $e_j$ of the lattice $M$. We define the constant $c_\square$ by

$$c_\square = \sum_{j \leq d} \frac{4}{\alpha_v e_j}.$$  

**Theorem 3.9.** Let $\Psi$ be a $(d + 1)$-dimensional normalized Fano CDP with base polytope $\square$. Then $\Psi$ has at most $c_\square$ functions.

**Proof.** Let $\Psi$ and $\square$ be as above. By Lemma 3.5 any integral function that is linear along each coordinate axis is linear; since $\Psi$ is normalized, no such functions exist. Hence, each $\Psi_i$ must be either non-integral, or non-linear along one of the coordinate axes. Applying Lemma 3.8 for $v = e_1, \ldots, e_d$, we obtain

$$n \leq \sum_{j=1}^{d} 4/\alpha_v e_j = c_\square.$$  

**Example 3.10.** Let $\square$ be the $d$-dimensional cross-polytope, that is, $\square$ is the convex hull in $\mathbb{R}^d$ of $\pm e_1, \ldots, \pm e_d$, where $e_i$ are the standard basis vectors. Then using the standard basis, $c_\square = 4d$, so by Theorem 3.9 any normalized Fano CDP with base $\square$ has at most $4d$ functions.

In fact, this bound is sharp. For $j = 0, \ldots, d - 2$, set

$$\Psi'_{4j+1}(x) = \Psi'_{4j+2}(x) = \Psi'_{4j+3}(x) = \Psi'_{4j+4}(x) = \min\{1, x_{j+1} + 1\}$$

and also set

$$\Psi'_{4d-3}(x) = \Psi'_{4d-2}(x) = \Psi'_{4d-1}(x) = \min\{1, x_d + 1\},$$

$$\Psi'_{4d} = \min\{1, x_d + 1\} - \sum_{k=1}^{d} 2x_k.$$  

These functions are all at height one, and satisfy Inequality (2), hence they come from a normalized Fano CDP $\Psi$ with $4d$ non-linear functions.

$\square$
4. Finiteness over Fixed Base

In this section, we establish Main Theorem 1 which says that there are only finitely many equivalence classes of Fano CDPs over a fixed base. We do this by showing in Theorem 3.9 that if we first fix the number of functions in our CDP, there are only finitely many possibilities over the given base. Main Theorem 1 then follows immediately, given Theorem 3.9.

The proof of Theorem 3.9 is an argument which reduces possibilities by considering the regions of linearity of a function. Let \( \Psi \) be a CDP with base \( \square \) and functions \( \Psi_1, \ldots, \Psi_n \). Each of the functions \( \Psi_i \) is piecewise linear and concave. Consequently, the facets of \( \Gamma(\Psi_i) \) are convex, and so project down to a subdivision of \( \square \) into polytopes. Moreover, as \( \Gamma(\Psi_i) \) has integral vertices, the vertices of the polytopes in the decomposition are integral. Hence \( \Gamma(\Psi_i) \) induces a subdivision of the base polytope into a union of finitely many lattice polytopes. We say that a \( d \)-dimensional polytope in this subdivision is a region of linearity of the function \( \Psi_i \), where \( d \) is the dimension of the base polytope \( \square \).

**Theorem 4.1.** Let \( \square \) be a fixed lattice polytope and \( n \) a fixed positive integer. Up to equivalence, there are only finitely many Fano CDPs with base polytope \( \square \) and \( n \) functions.

**Proof.** Suppose \( \Psi \) is a Fano CDP with base polytope \( \square \) and translated functions \( \Psi'_1, \ldots, \Psi'_n \). By shearing and by using Inequality (2) we can bound \( \Psi'_n(u) \) at each point \( u \in \square \) as follows:

**Step 1.** Fix regions of linearity.

As discussed above, each \( \Psi'_i \) induces a subdivision of \( \square \) into its regions of linearity: the set of these regions we denote by \( R_i \). Since \( \square \) contains only finitely many lattice points, there are only finitely many possibilities for the sets \( R_i \), so we may assume that we have fixed them once and for all.

**Step 2.** Give upper bounds for \( \Psi'_1, \ldots, \Psi'_{n-1} \).

Consider the polytopes given by intersections of the form \( P_1 \cap \cdots \cap P_{n-1} \), where each \( P_i \in R_i \). Let \( P \) be one of the \( d \)-dimensional polytopes containing the origin obtained through this process. Note that the restriction of each \( \Psi'_i \) to \( P \) is a linear function for \( i = 1, \ldots, n-1 \). Let \( C \) be the cone generated by the elements of \( P \). By [CIS11] Theorem 11.1.9, it contains a lattice basis \( v_1, \ldots, v_d \in M \). Any element of \( C \) is a scalar multiple of a point in \( P \), and hence \( P \) contains points \( p_1, \ldots, p_d \) where \( p_i = \alpha_i v_i \) for some \( \alpha_i \in \mathbb{R} \). As \( P \) is convex, we can assume that \( |\alpha_i| \leq 1 \).

The unimodular basis \( v_1, \ldots, v_d \) corresponds to a basis \( v_1^*, \ldots, v_d^* \) in \( M^* \) so that \( v_i^*(v_j) = \delta_{ij} \). Thus, if we shear \( \Psi'_i \) using \( v_j^* \), then \( \Psi'_i(p_k) \) stays fixed for \( k \neq j \). Hence we may assume that \( 0 \leq \Psi'_i(p_j) < 1 \) for all \( i = 1, \ldots, n-1 \) and \( j = 1, \ldots, d \).

As the restriction of \( \Psi'_i \) to \( P \) is a linear function, it is determined by its values at the points in the set \( \{0, p_1, \ldots, p_d\} \). Let \( L_i \) be the extension of this function to all of \( \square \). By the concavity of \( \Psi'_i \), the maximum of \( L_i \) bounds \( \Psi'_i \) by above. Consider the set of linear functions \( \phi \) defined on \( \square \) such that for each point \( p \in \{0, p_1, \ldots, p_d\} \) either \( \phi(p) = 0 \) or \( \phi(p) = 1 \). There are only finitely many such functions, and the maximum value obtained by them provides an upper bound for \( L_i \) and hence for \( \Psi'_i \).

**Step 3.** Give lower bound for \( \Psi''_n \).

The upper bounds for \( \Psi'_1, \ldots, \Psi'_{n-1} \) give a lower bound for \( \Psi''_n \), through use of Inequality (2), that is, \( n - 2 \geq \sum \Psi'_i \).

**Step 4.** Give upper bound for \( \Psi''_n \).

Let \( u \in \square \). Let \( \alpha > 0 \) be sufficiently small so that \( v = -\alpha u \) and the origin are in the same region of linearity of \( \Psi''_n \). By the concavity of \( \Psi''_n \), the line through the points \( (0, \Psi''_n(0)) \) and \( (v, \Psi''_n(v)) \) provides an upper bound for \( \Psi''_n(u) \). Moreover, this upper bound is maximized by increasing the value for \( \Psi''_n(0) \) and decreasing the value for \( \Psi''_n(v) \). Thus the upper bound \( \Psi''_n(0) \leq 1 \) and the lower bound for \( \Psi''_n(v) \) given in Step 3 provides an upper bound for \( \Psi''_n(u) \).

**Step 5.** Give lower bounds for \( \Psi'_1, \ldots, \Psi'_{n-1} \).

The upper bounds for \( \Psi'_1, \ldots, \Psi'_{n-1} \) yield lower bounds for \( \Psi'_1, \ldots, \Psi'_{n-1} \), again by Inequality (2).

Having bounded the functions \( \Psi'_i \) both from above and below, we may now conclude the proof of the theorem.
Suppose Lemma 5.3.

We can argue about the final function by using the condition of Inequality (1), namely \( \sum_{i=1}^{n} \Psi_{i} > n - 2 \) on \( \square \). Consequently,

\[
-1 < \Psi_{n}(-1) \quad \text{and} \quad -1 \leq \Psi_{n}(1).
\]

The latter inequality is strict if 1 is in the interior of \( \square \). Finally, from this, and Lemma 5.3 we conclude that if \( 1 \in \square \),

\[
\Psi_{n}(-1) \leq 1 \quad \text{and} \quad \Psi_{n}(1) \leq 1.
\]

These bounds severely restrict the possible base polytope for our representatives.
Proposition 5.4. Under the above restrictions on the form of the functions, there are no normalized two-dimensional non-toric Fano CDPs with a base \( \square \) such that \(-1,1 \in \square^0\).

Proof. Suppose \( \Psi'_1, \ldots, \Psi'_n \) are the translated functions of a Fano CDP \( \Psi \) with such a base \( \square \). Assume \( |E_1|, |E_2|, |E_3|, \) and \( |E_4| \). We can also assume \( \Psi'_i(u) \leq 0 \) for either \( u = -1 \) or \( u = 1 \), given Lemma 5.2. By \( |E_1| \) or \( |E_2| \), if there is some \( j < n \) such that \( \Psi'_j(u) \leq 0 \), then inequality \( |E_1| \), \( n - 2 < \sum_{i=1}^{n-1} \Psi'_i(u) \), cannot be satisfied. Hence \( \Psi'_i(u) > 0 \) for all \( i = 1 \ldots n - 1 \), and so by Lemma 5.2 \( \Psi'_i(-u) \leq 0 \) for all \( i = 1 \ldots n - 1 \). Combining inequalities \( |E_1| \) and \( |E_4| \) gives \( n - 2 < \Psi'_n(-u) \leq 1 \) from which we deduce that \( n < 3 \). Thus \( \Psi \) is toric.

We now fix a normalized two-dimensional non-toric Fano CDP \( \Psi = (\Psi_1, \ldots, \Psi_n) \) with base \( \square \). The base polytope \( \square \) thus has at least one of \(-1 \) or \( 1 \) as boundary points. Without loss of generality, in our choice representative we assume that \(-1 \) is a boundary point of \( \square \). In the following, we further assume that \( \Psi'_1(-1) = \cdots = \Psi'_{n-1}(-1) = 0 \) (again, shearing if necessary).

The next parameter that we consider is the length of the base polytope, which we denote by \( m \). If \( m = 2 \), then \( \square = [-1,1] \). As mentioned above, Theorem 3.9 gives the bound \( n \leq 4 \). We have listed all two-dimensional Fano CDPs over \( \square = [-1,1] \) with three and four functions in Tables 1 and 2, respectively. If \( m > 2 \), then \( 1 \) is an interior point of \( \square \). The inequality computed in Theorem 3.9 is strict, which yields a bound \( n < 4 \). In this case \( m > 2 \), our normalized CDP has exactly three functions \( \Psi_1, \Psi_2, \Psi_3 \).

Lemma 5.5. Suppose that \( \square \) has length larger than two. If \( \Psi'_i(0) \) is integral, then \( \Psi'_i \) has at most one interior vertex.

Proof. It follows directly from Lemma 3.4 that the only vertex must live above 0.

Lemma 5.6. Suppose that \( \square = [-1,m-1] \). For \( i = 1,2 \), if \( \Psi'_i(0) \) is non-integral, then up to shearing, either

(1) \( \Psi'_i \) has a single vertex at some integer \( \lambda - 1 \) satisfying \( 1 < \lambda < m \):

\[
\Psi'_i(x) = \begin{cases} (x+1)/\lambda & x+1 \leq \lambda \\ 1 & \lambda \leq x+1 \leq m; \end{cases}
\]

(2) or \( \Psi'_i \) is simply a line with slope \( \lambda \) \( \mid m \):

\[
\Psi'_i(x) = (x+1)/\lambda.
\]

Proof. We have determined that \( \Psi'_i(-1) = 0 \) and by Lemma 3.5 there is a positive integer \( \lambda \) so that \( \Psi'_i(0) = 1/\lambda \). Either \( \Psi'_i \) is a line, or the point \( (\lambda - 1,1) \) is a vertex. If it is not a line, then \( \Psi'_i \equiv 1 \) on \( [\lambda - 1,m-1] \) to ensure that the facet is at height one.

The shape of \( \Psi'_i \) can similarly be described. The normalization of the other functions gives that \( \Psi'_3(1) = 1 \). Furthermore, recall that \( \Psi'_3(1) > -1 \) (by \( \square^0 \)).

Next we consider the possible values at the boundary of the polytope, \( m - 1 \). The major restricting factor is that the regions of linearity start and end at lattice points, and the functions remain at height one. We next show that the nature of the equations limits \( m \) to be 3, 4 or 6, and with this information we can construct all cases by iterating through possible choices of shape, and \( \lambda \) for the three functions.

To bound \( m \) we look closer at the inequalities. These lemmas determine the possible values for \( \Psi'_1(m-1), \Psi'_2(m-1), \) and \( \Psi'_3(m-1) \). For \( i = 1,2 \), either \( \Psi'_i(m-1) = \frac{m}{\lambda} \) for some \( 1 < \lambda \) which divides \( m \), or \( \Psi'_i(m-1) = \{ -k(m-1) + 1 \} \) for some non-negative integer \( k \).

The permissible values for \( \Psi'_3(m-1) \) can be found by subtracting \( m - 1 \) from the permissible values of \( \Psi'_1(m-1) \). Moreover, we only keep the values that imply \( \Psi'_3(1) > -1 \) and deduce

\[
\Psi'_3(m-1) \in \left\{ -m + 2, \frac{m}{\mu + 1}m + 1 \right\}
\]

for some \( 0 \leq \mu \) and \( \mu + 1 \mid m \). We substitute these possibilities into Equation 3, and search for integer solutions to

\[
\Psi'_1(m-1) + \Psi'_2(m-1) + \Psi'_3(m-1) = 1.
\]
### Table 1. Base polytope length $m = 2$, number of functions $n = 3$

| Number | $\Psi'_1$ | $\Psi'_2$ | $\Psi'_3$ |
|--------|-----------|-----------|-----------|
| 1      |           |           |           |
| 2      |           |           |           |
| 3      |           |           |           |
| 4      |           |           |           |
| 5      |           |           |           |
| 6      |           |           |           |
| 7      |           |           |           |

We find a finite number of integral solutions to this equation, and find that in these solutions, $m \in \{3, 4, 6\}$. We illustrate one of these computations in the next example. The equivalence classes of Fano CDPs with three functions and base of length 3, 4, and 6 are given in Tables 3, 4, and 5 respectively. This completes the proof of Theorem 5.1 (and Main Theorem 2).

**Example 5.7.** Suppose that $\Psi'_1(m - 1) = \frac{m}{\lambda_1}$, $\Psi'_2(m - 1) = \frac{m}{\lambda_2}$, and $\Psi'_3(m - 1) = -m + 2$. We seek integer solutions to the equation

$$\frac{m}{\lambda_1} + \frac{m}{\lambda_2} - m + 2 = 1.$$  

Simplifying to $\frac{m}{\lambda_1} + \frac{m}{\lambda_2} - m + 2 = 1$, it is easy to see that either $\lambda_1 = 2$ or $\lambda_2 = 2$. Without loss of generality, assume $\lambda_1 = 2$. It follows that $m \mid 2\lambda_2$. Thus $\frac{m}{\lambda_2} = 1$ or $\frac{m}{\lambda_2} = 2$. With the values for $\frac{m}{\lambda_2}$ and $\lambda_1$ fixed, we are able to solve for $m$. The resulting solutions are $(m, \lambda_1, \lambda_2) \in \{(6, 2, 3), (4, 2, 4)\}$.

Enumerating all Fano CDPs satisfying $(m, \lambda_1, \lambda_2) = (6, 2, 3)$ leads to all equivalence classes of Fano CDPs given in Table 5. Enumerating all Fano CDPs satisfying $(m, \lambda_1, \lambda_2) = (4, 2, 4)$ leads to all equivalence classes of Fano CDPs given in Table 4.

### 6. Three Dimensional Fano T-Varieties

For a fixed polytope, the possible functions in a CDP are quite restricted by the regions of linearity that the polytope supports, and conditions of integrality at lattice points. In the three dimension case these ideas are sufficient to determine a global bound on the number of functions from Theorem 3.9.

**Theorem 6.1.** Every normalized Fano CDP with two-dimensional base has at most 8 functions. Furthermore, this bound is sharp in the sense that there exist normalized Fano CDPs with two-dimensional base and 8 functions.

Recall, Example 3.10 is a Fano CDP with 8 non-linear functions supported on the two-dimensional cross polytope. This shows that the bound in the theorem is sharp.

To prove the remainder of the theorem we divide the set of base polytopes into three cases depending on the lattice basis elements they contain. The intuition, and some of the tools are best developed through a few examples.

**Example 6.2.** We prove Theorem 6.1 in the case of the triangular polytope $\square$ pictured in Figure 4. Our convention throughout this section is to indicate the origin by a dot.
Table 2. Base polytope length $m = 2$, number of functions $n = 4$

Table 3. Base polytope length $m = 3$, number of functions $n = 3$

Suppose that $Ψ$ is a normalized Fano CDP over $□$. The dotted lines indicate the unique maximal triangulation of $□$ using lattice points. A region of linearity for any of the functions $Ψ'$ is necessarily a union of neighboring triangles in the triangulation.

Taking $v$ to be the vector $(1, 0)$, we have $α_v = 1/2$ and by Lemma 3.8 there are at most 8 functions $Ψ'_i$ which are either non-integral, or non-linear along the horizontal axis. On the other hand, consider any function $Ψ'_i$ which is integral, and linear along the horizontal axis. By Lemma 3.4 this function is linear along the segment from the origin to $(0, -1)$, which implies that it is linear on the union of the triangles $A$
Table 4. Base polytope length $m = 4$, number of functions $n = 3$

| Number | $\Psi'_1$ | $\Psi'_2$ | $\Psi'_3$ |
|--------|-----------|-----------|-----------|
| 21     |           |           |           |
| 22     |           |           |           |
| 23     |           |           |           |
| 24     |           |           |           |
| 25     |           |           |           |
| 26     |           |           |           |
| 27     |           |           |           |
| 28     |           |           |           |
| 29     |           |           |           |

Table 5. Base polytope length $m = 6$, number of functions $n = 3$

| Number | $\Psi'_1$ | $\Psi'_2$ | $\Psi'_3$ |
|--------|-----------|-----------|-----------|
| 30     |           |           |           |
| 31     |           |           |           |
| 32     |           |           |           |
| 33     |           |           |           |
| 34     |           |           |           |
and $B$ in the figure. By concavity it is thus linear across all of $\square$, but since $\Psi$ is normalized, it has no linear integral functions.

Hence, $\Psi$ has at most 8 functions.

This example can be generalized to the following lemma:

**Lemma 6.3.** Let $\Psi$ be a normalized Fano CDP with two-dimensional base $\square \subset M \otimes \mathbb{R}$. Consider any $v \in M$ such that the line $\langle v \rangle$ spanned by $v$ intersects the boundary of $\square$ in a non-lattice point. Then $\Psi$ has at most $4/\alpha_v$ functions, where

$$\alpha_v = \min \{1, \max \{\alpha \in \mathbb{R}_{\geq 0} \mid \pm \alpha v \in \square\}\}.$$

**Proof.** By Lemma 3.8, $\Psi$ has at most $4/\alpha_v$ functions $\Psi'_i$ which are either non-integral, or non-linear along the line $\langle v \rangle$. Consider instead an integral function $\Psi'_i$ which is linear along $\langle v \rangle$. Then by concavity and the fact that $\Psi'_i$ is linear along lines from the origin (Lemma 3.4), $\Psi'_i$ has at most two regions of linearity, divided by exactly the line $\langle v \rangle$.

We have assumed that this line intersects the boundary of $\square$ in some point $y \notin M$, which thus cannot lie under a vertex of the graph of $\Psi'_i$ (since it has integral vertices). If $F$ is the facet of $\square$ containing $y$ in its interior, it follows that $\Psi'_i$ must be linear along $F$ in a neighborhood of $y$. Since $\Psi'_i$ is also linear along lines from the origin, it is linear in a neighborhood of $y$. But by the descriptions of the regions of linearity of $\Psi'_i$ in the preceding paragraph, it follows that $\Psi'_i$ is linear on all of $\square$. But since $\Psi$ is normalized, it has no integral linear functions.

Next we prove Theorem 6.1 for a second type of polytope.

**Example 6.4.** Let $\square$ be a base polytope which contains the gray polytope pictured in Figure 5 as a (possibly non-proper) subset, where the origin is the marked lattice point.

We claim that any normalized Fano CDP $\Psi$ supported on $\square$ consists of at most 8 functions.

There are two cases to consider depending on how the dotted line spanned by $v = (2, 1)$ intersects the polytope. If the intersection of the dotted line spanned by $v$ with the boundary of $\square$ contains only lattice points, then $(-1, -2)$ must be in $\square$ since that is the lattice point closest to the origin that it will pass through. Taking a lattice basis given by $(1, 2)$ and $(0, 1)$, Theorem 3.9 yields that $\Psi$ has at most $c(\square) = 8$ functions.

If instead the line intersects the boundary at a non-lattice point, we again use Lemma 6.3 to bound the number of functions by $4/\alpha_v$. By visual inspection, $\alpha_v \geq 1/2$, hence $4/\alpha_v \leq 8$.

In either case, the number of non-trivial functions is bounded by 8.

**Lemma 6.5.** Let $\square$ be a two-dimensional lattice polytope in $M \otimes \mathbb{R}$ containing the origin in its interior. Then either $\square$ is equivalent to the two-dimensional cross polytope, or there is a basis $e_1, e_2$ of $M$ such that $-e_1, -e_2 \in \square$, along with some $ae_1 + be_2 \in \square$ for $a, b \in \mathbb{Z}_{\geq 0}$.

**Proof.** Let $\rho_1, \ldots, \rho_m$ be the rays through all non-zero lattice points of $\square$, ordered consecutively; let $v_i$ denote the primitive lattice generator of $\rho_i$. Then $v_i$ and $v_{i+1}$ form basis of $M$ for all $i$, see e.g. [Ful93 §2.6]. Here indices are taken modulo $m$. Furthermore, $v_i \in \square$ for all $i$.

Now, $-\rho_1$ is contained in some cone $\sigma$ spanned by $\rho_1$ and $\rho_{i+1}$. If $-\rho_i$ is in the interior of this cone, we take $e_1 = -v_i, e_2 = -v_{i+1}$, implying that $v_1 = ae_1 + be_2$ for $a, b > 0$. 

---

**Figure 4.** The base polytope in Example 6.2 and its regions of linearity
Suppose instead that $-\rho_1$ is on the boundary; without loss of generality $-\rho_1 = \rho_i$. If $-\rho_2$ is in the interior of the cone $\sigma$, then as above taking $e_1 = -v_i$, $e_2 = -v_{i+1}$ implies that $v_2 = ae_1 + be_2$ for $a, b > 0$. If $-\rho_2$ is in the boundary of $\sigma$, then $-v_2 = v_{i+1}$. By assumption we already had $-v_1 = v_i$, so $\square$ contains $\pm e_1, \pm e_2$, that is, the two-dimensional cross polytope. If $\square$ contains any other lattice points, then clearly the above criterion is satisfied, otherwise $\square$ is equivalent to the cross polytope.

Finally, if $-\rho_2$ is not in $\sigma$ at all, then $-\rho_{i+1}$ is in the interior of the cone generated by $\rho_1$ and $\rho_2$, so taking $e_1 = -v_1$, $e_2 = -v_2$ implies that $v_{i+1} = ae_1 + be_2$ for $a, b > 0$.

The proof of Theorem 6.1 follows the spirit of the above examples.

Proof of Theorem 6.1. Let $\Psi$ consist of functions $\Psi_1, \ldots, \Psi_n$, which we may assume to be non-linear or non-integral. There are three main possibilities for the base polytope $\square$.

Case 1. $\square$ contains a cross polytope.

In this case, Theorem 3.9 directly implies that $n \leq 8$.

If, rather, $\square$ does not contain a cross polytope, Lemma 6.5 implies that, we may choose a basis $e_1, e_2$ of $M$ such that $-e_1, -e_2 \in \square$, as well as some lattice point in the interior of the positive orthant. Fix such a basis. There are only two possibilities.

Case 2. Neither $e_1$ nor $e_2$ is in $\square$. 

Figure 7. Case 3 in proof of Theorem 6.1

Figure 6 depicts this case, where the gray region is contained in □, but the marked crosses in the figure are not. Since □ contains a point in the interior of the positive orthant, in fact the point $e_1 + e_2$ must be in □. But then $\pm e_1/2 \in □$, and the line spanned by $e_1$ intersects the boundary of □ in a non-lattice point, so Lemma 6.3 implies that $n \leq 8$ as desired.

**Case 3.** The point $e_2$ is in □, but $e_1$ is not.

Since the argument of Case 2 applies if $e_1 + e_2 \in □$, we may thus assume that every $(x_1, x_2) \in □$ located in the positive orthant satisfies $x_2 > 2x_1 - 1$, see the black dotted line in Figure 7. If $e_2$ is on the boundary of □, then the only lattice point of □ in the positive orthant must be $e_1 + 2e_2$. But then every lattice point $(x_1, x_2)$ of □ must satisfy $x_2 > 4x_1 - 2$, otherwise $e_1/2 \in □$ and again the argument of Case 2 applies, see the dashed gray line in Figure 7. It follows that either □ is the convex hull of $e_1 + 2e_2, -e_1$, and $-e_2$, or, that □ contains the gray region pictured in Figure 5. In the former situation, □ is equivalent to the polytope of Example 6.2 guaranteeing $n \leq 8$. In the latter situation, Example 6.3 also guarantees $n \leq 8$.

We may thus continue with Case 3 assuming that $e_2$ is in the interior of □. Let $R$ be the set of those $i$ such that $\Psi'_i$ is non-integral, or non-linear along the $e_2$-axis, with $r = |R|$. By Lemma 6.3 we have $r \leq 4$.

For any $\Psi'_i$ with $i \notin R$, the concavity of $\Psi'_i$ and linearity along the $e_2$-axis implies that $\Psi'_i$ has exactly two domains of linearity, divided by the $e_2$-axis. For the moment, assume that $R \neq \emptyset$. For all $i \notin R$ we normalize $\Psi$ so that $\Psi'_i \equiv 1$ on the region of those points $(x_1, x_2)$ with $x_1 \leq 0$. Consider $y = \lambda e_2$ for $\lambda$ maximal such that $y \in □$. By our assumptions above, $\lambda > 1$, which implies that it lies on a facet of □ which is not at height one. We thus have that $n - 2 = \sum_{i \notin R} \Psi'_i(y) = (n - r) + \sum_{i \in R} \Psi'_i(y)$, or equivalently that

$$\sum_{i \in R} \Psi'_i(y) = r - 2. \tag{5}$$

Likewise, consider some $z \in □$ with first coordinate equal to 1. Our normalization implies that for $i \notin R$, $\Psi'_i(z) \leq 0$. Set $z' = \frac{x_1 - e_1}{x_1 - e_2}$. This can be written as $sy + (1 - s)0$ for some $s \in (0, 1)$. See Figure 8 for a depiction of $y, z,$ and $z'$.

For $i \in R$, $\Psi'_i$ restricted to the $e_2$ axis is bounded above by the linear function taking values 1 at the origin, and $\Psi'_i(y)$ at $y$. Indeed, if $\Psi'_i$ is integral, this follows from Lemma 6.3. If $\Psi'_i$ is not integral, the fact that each facet of the graph of $\Psi'_i$ (and hence its restriction to the $e_2$-axis) is at height one implies the bound on the domain of linearity closest to $y$; concavity implies the bound holds for the entire axis.

We thus obtain

$$2n - 4 \leq \sum_{i \in R} \Psi'_i(z) + \sum_{i \notin R} \Psi'_i(e_1) \leq \sum_{i \notin R} 1 + 2 \sum_{i \in R} \Psi'_i(z')$$

$$\leq (n - r) + 2 \sum_{i \in R} s \Psi'_i(y) + (1 - s) = n + r - 4s,$$
where the first inequality follows from Equation (2), the second from the concavity of $\Psi_i'$, the third from the bound on $\Psi_i'$ along the $e_2$ axis, and the final equality follows from Equation (5). This in turn implies

$$n \leq 4 + r - 4s.$$ 

Using that $r \leq 4$ and $s > 0$, we obtain $n \leq 7$.

If instead $R = \emptyset$, then we can only normalize all but one of the functions $\Psi_i'$ not in $R$; say that $\Psi_1'$ is the function we don’t normalize. Then as in Equation (5) we obtain $\Psi_1'(y) = -1$ with $y$ as above, and arguing as above (with $R$ replaced by $\{1\}$) we obtain $2n - 4 \leq (n - 1) - 2s + 2(1 - s) = n + 1 - 4s$. Hence, $n \leq 4$.

We have shown that in all cases, $n \leq 8$. On the other hand, Example 3.10 shows that this bound is sharp, as the two-dimensional cross polytope supports a Fano CDP with 8 non-linear functions.

Based on our results for two- and three-dimensional Fano CDPs, we conjecture the following:

**Conjecture 6.6.** Any normalized Fano CDP of dimension $d$ has at most $2^d$ functions.

As Example 3.10 shows, there is a Fano CDP supported on the $(d - 1)$-dimensional cross polytope which achieves this conjectural bound.

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**References**

[BHHN16] Benjamin Bechtold, Jürgen Hausen, Elaine Huggenberger, and Michele Nicolussi. On terminal Fano 3-folds with 2-torus action. *Int. Math. Res. Not. IMRN*, (5):1563–1602, 2016.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[CS17] Jacob Cable and Hendrik Süß. On the classification of Kähler-Ricci solitons on Gorenstein del Pezzo surfaces. arXiv:1705.02920v1, 2017.

[FHN17] Anne Fahrner, Jürgen Hausen, and Michele Nicolussi. Smooth projective varieties with a torus action of complexity 1 and Picard number 2. arXiv:1602.04360v3, 2017.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Hau13] Jürgen Hausen. Three lectures on Cox rings. In *Torsors, étale homotopy and applications to rational points*, volume 405 of *London Math. Soc. Lecture Note Ser.*, pages 3–60. Cambridge Univ. Press, Cambridge, 2013.

[HHS11] Jürgen Hausen, Elaine Herppich, and Hendrik Süß. Multigraded factorial rings and Fano varieties with torus action. *Doc. Math.*, 16:71–109, 2011.

[Hug13] Elaine Huggenberger. *Fano varieties with torus action of complexity one*. PhD thesis, Doctoral Thesis http://ubn-resolving.de/urn:nb:nb:de:bsz:21-opus-69570, 2013.

[IS11] Nathan Owen Ilden and Hendrik Süß. Polarized complexity-1 $T$-varieties. *Michigan Math. J.*, 60(3):561–578, 2011.

[IS17] Nathan Ilden and Hendrik Süß. K-stability for Fano manifolds with torus action of complexity 1. *Duke Math. J.*, 166(1):177–204, 2017.
[KS97] M. Kreuzer and H. Skarke. On the Classification of Reflexive Polyhedra. *Communications in Mathematical Physics*, 185:495–508, 1997.

[LZ91] Jeffrey C. Lagarias and Günter M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canad. J. Math.*, 43(5):1022–1035, 1991.