Irreducibility of the set of field operators in Noncommutative Quantum Field Theory

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Abstract

Irreducibility of the set of quantum field operators has been proved in noncommutative quantum field theory in the general case when time does not commute with spatial variables.

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1 Introduction

Irreducibility of the set of quantum field operators $\varphi(x)$ is one of the principal results in axiomatic quantum field theory (QFT) [1], [2].

It implies that, if vacuum vector is cyclic, then the corresponding set of quantum field operators has to be irreducible one.

Let us recall that vacuum vector $\Psi_0$ is a cyclic one, if any vector in the space under consideration can be approximated by a finite linear combination of the vectors

$$\Psi_n = \varphi(x_1) \ldots \varphi(x_n) \Psi_0$$

with arbitrary accuracy.

In accordance with axiom of vacuum vector cyclicity any scalar product in the space in question can be approximated by the linear combination of Wightman functions

$$W(x_1, \ldots, x_n) \equiv \langle \Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0 \rangle.$$

Let us prove that the set of quantum field operators is irreducible one in noncommutative quantum field theory (NC QFT) as well.

Besides, we prove that in usual commutative QFT the irreducibility of a set of quantum field operators follows from assumptions weaker then standard.
Let us recall that NC QFT is defined by the Heisenberg-like commutation relations between coordinates:

\[ [\hat{x}_\mu, \hat{x}_\nu] = i \theta_{\mu\nu}, \]

where \( \theta_{\mu\nu} \) is a constant antisymmetric matrix.

It is very important that NC QFT can be also formulated in commutative space, if we replace the usual product of quantum field operators (strictly speaking, of the corresponding test functions) by the \( \star \) (Moyal-type) product (see [3], [4].

Let us remind that the \( \star \)-product is defined as

\[
\varphi(x) \star \varphi(y) = \exp \left( \frac{i}{2} \theta_{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \right) \varphi(x) \varphi(y)
\]

\[
\equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{2} \theta_{\mu\nu} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \right)^n \varphi(x) \varphi(y).
\]

Evidently the set in equation (2) has to be convergent. It was proved [5] that this set is a convergent one if \( f(x) \) belongs to one of the Gel'fand-Shilov spaces \( S^\beta \) with \( \beta < \frac{1}{2} \). The similar result was obtained also in paper [6].

Noncommutative theories defined by Heisenberg-like commutation relations (1) can be divided in two classes.

The first of them is the case of only space-space non-commutativity, that is \( \theta_{0i} = 0 \), time commutes with spatial coordinates.

It is known that this case is free from the problems with causality and unitarity [7] - [9] and in this case the main axiomatic results: CPT and spin-statistics theorems, Haag’s theorem remain valid [10] - [13]. Besides, this case can be obtained as low-energy limit from string theory [14].

Let us remind that if time commutes with spatial coordinates, then there exists one spatial coordinate, say \( x_3 \), which commutes with all others. Besides it easy to show that this result is valid in any space if its dimension is even. For simplicity we consider only four-dimensional case, thus in space-space NC QFT we have two commuting coordinates and two non-commuting coordinates.

In the second case all coordinates, including time, are non-commuting.

Let us recall that the set of quantum field operators is irreducible if from the condition

\[ [A, \varphi(x)] = 0, \]

where \( A \) is a bounded operator, follows that

\[ A = C \mathbb{I} \quad C \in \mathbb{C} \]
where $\mathbb{I}$ is an identity operator.

It is known that if condition (3) is fulfilled, then the space under consideration cannot contain nontrivial subspaces invariant under the action of the set of operators $\varphi(x)$.

Note that actually there is no field operator defined in a point. Only the smoothed operators written symbolically as

$$\varphi_f \equiv \int \varphi(x) f(x) \, dx,$$

where $f(x)$ are test functions, can be rigorously defined and be nontrivial operators in QFT as well as in NC QFT. As the proof is identical both for operators $\varphi_f$ and $\varphi(x)$, so we give the proof for $\varphi(x)$, for simplicity.

First we consider the case of space-space noncommutativity.

In this case test functions correspond to tempered distributions in respect with commuting coordinates.

This fact leads to the following spectral condition:

$$\int da \, e^{-ip \cdot a} \langle \Phi, U(a) \Psi \rangle = 0, \quad \text{if} \ p_0 < |p_3|,$$

where $a = \{a_0, a_3\}$ is a two-dimensional vector, $U(a)$ is a translation in the plane $x_0, x_3$, and $\Phi$ and $\Psi$ are arbitrary vectors. The equality (6) is similar to the corresponding equality in the standard case $[1]$. It implies that complete system of physical states (in gauge theories also nonphysical ones) does not contain tahyon states in momentum space in respect with commuting coordinates. It means that momentum $P_n$ for every state satisfies the condition:

$$P_n^0 \geq |P_n^3|.$$

2 The Proof

To prove irreducibility of the set of quantum field operators fulfilment of the condition

$$P_n^0 \geq 0$$

is sufficient.

Let us give the sketch of the proof omitting all technical details, which are similar to corresponding proof in $[1]$.

Let us consider

$$\langle A^* \Psi_0, U(a) \varphi(x_1) \ldots \varphi(x_n) \Psi_0 \rangle.$$
After simple calculations using the condition (3) and translation operator’s unitarity we come to the equality

$$\langle A^* \Psi_0, U(a) \varphi(x_1), \ldots \varphi(x_n) \Psi_0 \rangle =$$

$$\langle \varphi(x_n) \ldots \varphi(x_1), \Psi_0, U(-a) A \Psi_0 \rangle.$$  

(7)

In accordance with the spectral condition

$$\int da \ e^{-i p_0 a} \langle A^* \Psi_0, U(a) \varphi(x_1) \ldots \varphi(x_n) \Psi_0 \rangle \neq 0,$$

only if $p_0 \geq 0$. However,

$$\int da \ e^{-i p_0 a} \langle \varphi(x_n) \ldots \varphi(x_1) \Psi_0, U(-a) A \Psi_0 \rangle \neq 0,$$

only if $p_0 \leq 0$. Hence, the equality (7) can be fulfilled only when $p_0 = 0$.

It leads to the equality:

$$A \Psi_0 = C \Psi_0,$$

as $\varphi(x_1) \ldots \varphi(x_n) \Psi_0$ is an arbitrary vector.

From the last equality it follows that

$$A \varphi(x_1) \ldots \varphi(x_n) \Psi_0 = C \varphi(x_1) \ldots \varphi(x_n) \Psi_0.$$

In order to complete the proof it is necessary to take into account axiom of vacuum vector cyclicity and the boundedness of operator $A$.

Now let us show that our statement is true also in the general case, when time does not commute with spatial variables. The crucial point in the proof is that

$$\varphi(x) \ast \varphi(y) = \{ \varphi(x) \ast \varphi(y) \}_N + \varepsilon(N),$$

where

$$\{ \varphi(x) \ast \varphi(y) \}_N \equiv \sum_1^N \exp \left( \frac{i}{2} \theta_{\mu\nu} \frac{\partial}{\partial x_{\mu}} \frac{\partial}{\partial y_{\nu}} \right) \varphi(x) \varphi(y),$$

and $\varepsilon(N) \to 0$ at $N \to \infty$.

At arbitrary $N$ we can derive the statement of the irreducibility of the set of quantum field operators just as it has been done in the case of space-space noncommutativity.

As $\varepsilon(N) \to 0$ at $N \to \infty$, we can pass to the limit $N = \infty$. 

4
3 Conclusions

We see that irreducibility of the set of quantum field operators takes place in the very general theory and under weaker conditions than usual if axiom of cyclicity of vacuum vector is fulfilled. The important physical example of such a theory is NC QFT.

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