The Urban Skyline as a Moving Interface: An Indication of KPZ-EW Cross-over

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In this Letter we follow the asymptotic spatial correlation of buildings’ heights \( G_\infty(r) \) in the whole of the Netherlands \( \Omega \), which comprises \( \approx 10,000,000 \) buildings, for the purpose of recovering its scaling with respect to space and time given respectively by the exponents \( r^{2\alpha} \) and \( r^\beta \). This allows us to identify the universality class of the evolution of the urban skyline seen as a dynamically evolving interface. Two major classes of cities were identified based on the recovered value of \( \alpha = 0.4 \) and \( \beta = 0 \), which correspond respectively to the KPZ and the EW universality classes. Picking a discrete model from each of these classes and mapping it to physical rules for constructions in cities we conclude that imposed restrictions on buildings’ heights are reflected in the exponent \( \alpha \) and thus have implications on how the skyline evolves.

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There are three major strands of urban morphological analysis all of which give insight into the dynamical process of urban change \cite{2}. These respectively study the micro-morphology, the relation between morphological periods and the typological process, and finally the relation between decision making and urban form. The first follows the intra-city spatial change, which was shown to be clustered over time and diffuses spatially \cite{2}. The second strand lays out the typological process of change in buildings types where the latter are viewed as resulting from a process of learning from adaptations of the previous building types. It also follows how morphological characteristics are superseded by those of the next. Lastly, the interplay between decisions has been shown to lead to either intended or unintended fringe belts around preserved historical zones as well as to delineating different morphological periods \cite{3,4}.

The amount of change in urban form was linked to the neighborhood effect, which is itself dependent on to the dwelling density; that is in high density environments the changes tend to be initiative \cite{2}. Moreover, the typological process was investigated by H. Bernoulli who came up with the notion of property cycles, which follows the gradual filling of a lot of land by buildings which in due course are replaced with newer ones when their life cycle comes to an end \cite{5}. The cycle is divided into Boom, Slump and Recovery phases \cite{2}. The high density housing prevails in the booming phase while the interplay between low land values and the geographical constraints leads to fringe belts, which include vegetation areas, landmark, buildings of architectural importance. The belt thus forms a boundary zone between historically and morphologically distinct housing areas \cite{3,4}.

The city has been studied through the lens of statistical physics as the dynamical processes at play in urban allometry, mobility, urban form and social segregation, to list a few, have parallels in the study of magnetic materials, phase-transitions, the Ising model and many others \cite{9,10}. The urban skyline, being an important city metric in the assessment of the city’s solar energy and visual complexity \cite{17,18}, has been as well followed empirically however no dynamical description was provided to explain its evolution \cite{19}.

Under the effect of the dynamical processes described above, such as property cycles and the spatial diffusion of morphological changes, the buildings’ heights are constantly varying with alternating growth and decay linked to construction and destruction. Thus their heights \( h(r,t) \) can be thought of as a dynamic spatiotemporally evolving quantity describing a growth phenomena, where \( r \) and \( t \) denote respectively location and time as shown in Figure 1. This is reminiscent of interface dynamics problems commonly studied in contexts of film growth, flame propagation, random deposition, ballistic deposition, turbulent liquid crystals, growth of bacterial colonies, and directed polymers in random media just to list a few \cite{10,17,18,20–25}. In these systems, macroscopic observables, such as height fluctuations and free-energies, exhibit power-law dependence on system size \( L \) and on time \( t \) \cite{20,28}. These powers define the systems’ corresponding universality classes namely in this case the Kardar-Parizi-Zhang (KPZ) and the Edwards-Wilkinson (EW) classes \cite{29,30}.

The KPZ equation accounts for a restoring term playing the role of surface tension, a non-linear driving force, and a stochastic term respectively given by the right hand terms of the below equation:

\[
\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{1}{2} (\nabla h)^2 + \eta(r,t),
\]

where \( \eta \) is an uncorrelated Gaussian noise whose Fourier transformation satisfies \( \langle \eta(k,t)\eta(k',t') \rangle = \)
correlation function of the heights denoted by \( \beta \) is common to resort to the computation of the pair \( \lambda \) in the case, that is when \( \lambda = \alpha \), which is given by the following equation: \[ w(r, t) = \sum [h(r, t) - \bar{h}(t)]^2 \] exhibit the following dependencies on \( t \) and \( L \):

\[ w(L, t) \propto L^\alpha f \left( \frac{t}{L^z} \right), \]

with:

\[ f(x) \propto \begin{cases} x^\beta, & \text{if } x \ll 1 \\ \text{constant}, & \text{if } x \gg 1 \end{cases} \]

This translates into \( w(t) \propto t^\beta \) and \( w(t) \propto L^\alpha \), where \( \alpha = \frac{2}{\beta + 2} \), \( \beta = \frac{1}{\beta + 1} \), and \( \alpha + z = 2 \). As for the EW case, that is when \( \lambda = 0 \), the exponents are given by: \( \alpha = 0 \), \( \beta = 0 \), \( z = 2 \). In order to determine both \( \alpha \) and \( \beta \) it is common to resort to the computation of the pair correlation function of the heights denoted by \( G(r, t) \) \[ G(r, t) = \langle [h(r + \delta r, t) - h(r, t)]^2 \rangle_r, \]

where \( \langle > \rangle_r \) denotes averaging over all space. The asymptotic limit of \( G(r, t) \) is given by:

\[ G_\infty(r, t) = \begin{cases} r^{2\alpha}, & \text{if } r_c \ll t^{-1/z} \\ r^{2\beta}, & \text{if } r_c \gg t^{-1/z} \end{cases}, \]

where \( t_\infty \) is the saturation time.

The data represent the cities’ current configurations and age distributions, therefore their \( G(r, t) \) cannot be computed for all \( t \). Instead, \( G_\infty \), which is the heights correlations at year \( t = 2020 \), that is near \( t \to \infty \), can be solely computed. More precisely, using the information about the current building stock, or equivalently \( G(r, \infty) \), we can retrieve the value of \( \alpha \) according to Equation 5.

The algorithm was parallelized and ran on the GPU and \( G_\infty(r) \) was computed for each city. The cities are then clustered according to their values of \( \alpha \) and the goodness of the linear fit, \( R \)-squared, of the log \( G_\infty(r) \) versus log \( r \). The results are shown in Figure 2. We note that clusters 1,4,5 have a low \( R \)-squared and thus we neglect them from our analysis. We focus on the cities belonging to clusters 2 and 3, which comprise 117 and 15 cities respectively. Their corresponding cluster centers are given by the following: \( \alpha = 0.05 \pm 0.05 \), and \( R \)-squared = 0.76 ± 0.13, and \( \bar{\alpha} = 0.47 \pm 0.09 \), and \( R \)-squared = 0.79 ± 0.18.
\[ \text{3(b) and 3(a) respectively. They correspond to cities whose } G_\infty \text{ consists of a saturation phase preceded by linear growth with an } R\text{-squared } > 0.5. \text{ The linear curve had to also fit more than 10 data points per city; that is if before reaching the plateau if the city has less than 10 data points we neglected it from the analysis.} \]

FIG. 3: In the Fig. 3(a) \( 2\alpha = 0.95 \pm 0.19 \), or equivalently \( \alpha = 0.47 \pm 0.09 \), while in Fig. 3(b) \( 2\alpha = 0.11 \pm 0.06 \); that is \( \alpha = 0.05 \pm 0.05 \). 117 cities fell in the EW class while 14 in the KPZ. The rest of the cities belonged to the clusters with \( \bar{\alpha} \approx 0 \) but their respective \( R\)-squared was less than 0.5.

We note that the slope of \( G_\infty(r) \) in 3(a) \( 2\alpha \), was found to be \( 0.95 \pm 0.19 \), thus \( \alpha = 0.47 \pm 0.09 \), while the slope of the linear part of \( G_\infty(r) \) in Fig. 3(b) is given by \( 2\alpha = 0.31 \pm 0.14 \) or equivalently \( \alpha = 0.15 \pm 0.07 \). These correspond respectively to the \( \alpha \) values of the KPZ and EW. However, this parameter alone is not sufficient to characterize the universality class.

Next, the computation of \( G(r)_\infty \) for \( r_c \gg r_c^\infty \) requires knowing \( t_\infty \), which is indirectly available through \( r_c \); that is \( t_\infty \propto r_c^z \). Thus, in the saturation regime \( \log G_\infty / \log r_c = \text{constant} \). This is confirmed in the result shown of Figure 4 where the ratio \( \log G_\infty / \log r_c \approx 0.76 \). Moreover, we show the cities falling under the KPZ universality class and those belonging to EW in blue and red respectively on the Netherland’s map in Fig. 5.

FIG. 4: The ratio \( \log G_\infty / \log r_c \) of cities falling in the KPZ universality class is shown. The red line is its average.

FIG. 5: The cities which fall in the KPZ universality class are shown in blue while those to EW are shown in red. The rest belong to cluster 1,4, 5 whose \( \alpha \approx 0 \) however \( R\)-squared 0.5.

In trying to understand the cities’ characteristics in relation to their universality classes we mapped the continuum models to their discrete counterparts: the restricted solid on solid (RSOS) and the random deposition (RD), which correspond respectively to the KPZ and the EW universality classes [21, 32, 33]. This will help us gain more insight into the potentially imposed planning con-
straints on buildings constructions in the cities. In RD, one picks a random site and deposits a block on it. The site’s height is defined to be the number of deposited blocks at the location. A simple variation of this model is to restrict the height difference between neighbors; that is when a deposition violates a given height difference it would be rejected: this is the RSOS model. Their underlying mechanisms are shown in Figures 6(a) and 6(b) respectively.

Based on the above we suspect that the difference we observed in the behavior of the asymptotic correlation function $G_\infty(r)$ is solely related to urban planning constraints as we have explored the dependence of the parameter $\alpha$ on a handful of geospatial covariates: the area of the city, its perimeter, and the density of the built environment all of which proved to be statistically insignificant. More precisely, imposing a height limit in the city is equivalent to the restricted height constraint in the RSOS model, while leaving it uncontrolled is commensurate to the RD.

Finally, in this paper we explored an analogy between the city’s skyline and interface dynamics through the computation of the scaling exponent of the asymptotic height correlation function $G_\infty(r)$ with the buildings’ pairwise distance $r$. Based on the value of this exponent $\alpha$ we were able to classify cities into two universality classes: the KPZ and the EW. The cross-over between the two is observed when the parameter $\lambda$ is set to 0, which is a restoring force imposing restriction on interface height. To understand its effect we point out to two discrete models: RD and RSOS whose difference in update rules also sets them apart into the EW and KPZ classes. The first, whose interface grows with no constraints on $h$ is equivalent to cities with no imposed restriction buildings’ height as opposed to the RSOS which is analogous to cities with enforced limit on buildings’ heights.

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[1] Dukai, B. (Balzs), 3d registration of buildings and addresses (bag). 4tu.centre for research data. (2018), Datasets. https://doi.org/10.4121/uuid:ff9759d-024a-492a-b821-07014dd6131c
[2] J. Whitehand, Urban Morphology 5, 103 (2001).
[3] J. Whitehand, Planning Perspectives 3, 47 (1988).
[4] J. W. R. Whitehand, Transactions of the Institute of British Geographers 2, 400 (1977).
[5] B. Gauthiez, Urban Morphology 8, 71 (2004).
[6] L. M. Bettencourt, science 340, 1438 (2013).
[7] J. Krug, Advances in Physics 46, 139 (1997).
[8] C. Castellano, S. Fortunato, and V. Loreto, Reviews of modern physics 81, 591 (2009).
[9] F. Simini, M. C. González, A. Maritan, and A.-L. Barabási, Nature 484, 96 (2012).
[10] M. Barthelemy, Nature Reviews Physics 1, 406 (2019).
[11] M. Barthelemy, The structure and dynamics of cities (Cambridge University Press, 2016).
[12] M. Batty, R. Carvalho, A. Hudson-Smith, R. Milton, D. Smith, and P. Steadman, The European Physical Journal B 63, 303 (2008).
[13] M. Batty, science 319, 769 (2008).
[14] R. Louf and M. Barthelemy, Physical review letters 111, 198702 (2013).
[15] L. M. Bettencourt, J. Lobo, D. Helbing, C. Kühnert, and G. B. West, Proceedings of the national academy of sciences 104, 7301 (2007).
[16] S. Atis, A. K. Dubey, D. Salin, L. Talon, P. Le Doussal, and K. J. Wiese, Physical review letters 114, 234502 (2015).
[17] T. Heath, S. G. Smith, and B. Lim, Environment and behavior 32, 541 (2000).
[18] A. Calcabrini, H. Ziar, O. Isabella, and M. Zeman, Nature Energy 4, 206 (2019).
[19] M. Schlüfer, J. Lee, and L. Bettencourt, arXiv preprint arXiv:1512.00946 (2015).
[20] T. Halpin-Healy and Y.-C. Zhang, Physics reports 254, 215 (1995).
[21] I. Corwin, Random matrices: Theory and applications 1, 1130001 (2012).
[22] J. M. Kim and J. Kosterlitz, Physical review letters 62, 2280 (1989).
[23] R. Almeida, S. Ferreira, T. Oliveira, and F. A. Reis, Physical Review B 89, 045309 (2014).
[24] T. Halpin-Healy, Physical review letters 109, 170602 (2012).
[25] J. De Nardis, P. Le Doussal, and K. A. Takeuchi, Physical review letters 118, 125701 (2017).
[26] T. Halpin-Healy and Y. Lin, Physical Review E 89, 010103 (2014).
[27] T. Ala-Nissila, T. Hjelt, J. Kosterlitz, and O. Venäläinen, Journal of statistical physics 72, 207 (1993).
[28] B. Meerson, E. Katzav, and A. Vilenkin, Physical review letters 116, 070601 (2016).
[29] M. Kardar, G. Parisi, and Y.-C. Zhang, Physical Review A 89, 234502 (2015).
[30] S. F. Edwards and D. Wilkinson, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 381, 17 (1982).
[31] J. L. Nasar, T. Imeokparia, and R. Tiwari, in 33rd Annual Conference of the Environmental Design Research Association Conference, Philadelphia, PA, May (2002), pp. 22–26.
[32] E. Katzav and M. Schwartz, Physical Review E 70,
061608 (2004).

[33] K. Park and B. Kahng, Physical Review E 51, 796 (1995).