TRANSIENT ANALYSIS OF A SINGLE SERVER QUEUE WITH DISASTERS AND REPAIRS UNDER BERNOULLI WORKING VACATION SCHEDULE

M. LAKSHMI PRIYA, B. JANANI*

Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai 600062, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract: An $M/M/1$ queueing model with disasters and repairs under Bernoulli working vacation schedule is considered. In this model, after every completion of service the server may take vacation with probability $q$ or the server may render service to the next customer with probability $p$. By considering the disaster to occur, only when the server is in busy state, the explicit analytical expressions for time dependent probabilities are derived using Laplace transform and generating function technique.

Keywords: disaster; repair; Bernoulli working vacation.

2010 AMS Subject Classification: 90B15.

1. INTRODUCTION

Queues with disasters are extensively discussed by various researchers. As disaster occurs all customers in the system are removed. This type of situations is seen to prevail in the computer networks (where arrival of virus can be considered as disaster), ATM in a bank, manufacturing systems and so on.

*Corresponding author
E-mail address: jananisrini2009@gmail.com
Received October 4, 2020
Gelenbe [4] was the first to introduce the concept of arrival of negative customers in the queue. For better understanding the reader may refer to Gelenbe [5], Harrison and Pitel [6], Chao[2], Atencia and Bocharov [1], Kumar and Arivudainambi [10], Kumar and Madheswari [11], Yang et al [15].

Yechiali [16] analysed queue with disaster and impatience. Sudesh [13], Dimou and Economou [3] were some of the remarkable papers in queue with disasters and impatience.

Queue with vacations were studied by many researchers since the late 70’s. Reader may look into the survey paper of Ke et al [9] for recent developments in vacation queuing models. But there are only few articles related to queue with disasters and vacations. Queue with disasters and vacations were first introduced by Mytalas and Zazanis [12]. Also reader may refer Ye et al [7], Kalidass et al [8], Suranga Sampath [14], for better understanding of queues with disasters and vacations.

Due to wide spread applications as well as due to flexibility, Bernoulli vacation was analyzed by many researchers. Practically, the server may opt working vacation after every completion of service depending upon his physical condition. More elaborately, a driver can opt long trip or short trip depending upon his physical condition. Motivated by the above example, in this paper we derived transient probabilities of an $M/M/1$ queue with disasters and repairs under Bernoulli working vacation schedule.

The contents of the paper are arranged as follows.

- Section 2 – Description of the model
- Section 3 – Transient Probabilities
- Section 4 - Conclusion and Future scope of the model

2. MODEL DESCRIPTION

A single server queue with disasters and repairs under Bernoulli vacation schedule is considered. Customers are allowed to join the system according to a Poisson process with the rate $\lambda$ and service takes place exponentially with the rate $\mu$. Whenever the server completes the service to a customer, the server may choose a working vacation with probability $q$ or the server may continue the service to the next waiting customer with probability $p$. Also the
duration of vacation times follow exponential distribution with parameter $\eta$. The disaster occurs during the busy period. After the occurrence of disaster all customers in the system are flushed out and system becomes empty. Meanwhile the repair period starts. Both disaster and repair times are exponentially distributed with parameter $\alpha$ and $r$ respectively.

Number of customers in the system and system states are represented by $X(t)$ and $J(t)$ respectively. Mathematically,

$$J(t) = \begin{cases} 
1; \text{ server is in busy state} \\
0; \text{ server is in working vacation state} \\
2; \text{ server is in repair state} \\
3; \text{ server being idle}
\end{cases}$$

Hence $(J(t), \chi(t))$ is a Markov process with state space

$$\Omega = \{(0,0) \cup (3,0) \cup (2,0) \cup (j,n); j = 0,1,2,3 \; n = 1,2,\ldots\}.$$

Figure 2.1: State transition diagram of a Single Server Queue with Disasters and Repair under Bernoulli Working Vacation Schedule
Let $P_{jn}(t)$ denote the time dependent probability for the system to be in state $j$ with $n$ customers at time $t$. Assume that initially the system is empty and the server is being idle i.e., $P_{3,0}(0) = 1$. By standard methods, the system of Kolmogorov differential difference equations governing the process are given by

$$P_{0,0}'(t) = -(\lambda + \eta)P_{0,0}(t) + \mu q P_{1,1}(t) + \mu v P_{0,1}(t),$$  \hspace{1cm} (1)

$$P_{0,n}'(t) = -(\lambda + \eta + \mu v)P_{0,n}(t) + \mu q P_{1,n+1}(t) + \lambda P_{0,n-1}(t) + \mu v P_{0,n+1}(t), \quad n = 1, 2, \ldots$$ \hspace{1cm} (2)

$$P_{1,1}'(t) = -(\lambda + \mu + \alpha)P_{1,1}(t) + \lambda P_{3,0}(t) + \mu p P_{1,2}(t) + \eta P_{0,1}(t) + r P_{2,1}(t),$$ \hspace{1cm} (3)

$$P_{1,n}'(t) = -(\lambda + \mu + \alpha)P_{1,n}(t) + \lambda P_{1,n-1}(t) + \mu p P_{1,n+1}(t) + r P_{2,n}(t) + \eta P_{0,n}(t),$$ \hspace{1cm} \quad n = 2, 3, \ldots$$ \hspace{1cm} (4)

$$P_{3,0}'(t) = -\lambda P_{3,0}(t) + \mu p P_{1,1}(t) + \eta P_{0,0}(t) + r P_{2,0}(t),$$ \hspace{1cm} (5)

$$P_{2,0}'(t) = (\lambda + r)P_{2,0}(t) + \alpha \sum_{n=1}^{\infty} P_{1,n}(t),$$ \hspace{1cm} (6)

and

$$P_{2,n}'(t) = -(\lambda + r)P_{2,n}(t) + \lambda P_{2,n-1}(t), \quad n = 1, 2, \ldots$$ \hspace{1cm} (7)

### 3. Transient Probabilities

Define

$$G(z, t) = \sum_{n=1}^{\infty} P_{0,n}(t) z^n.$$ 

Then,

$$G'(z, t) = \sum_{n=1}^{\infty} P_{0,n}'(t) z^n.$$ 

By substituting equation (2) gives,

$$G'(z, t) - \left( -(\lambda + \eta + \mu v) + \lambda z + \frac{\mu q}{z} \right) G(z, t) = \lambda z P_{0,0}(t) - \mu v P_{0,1}(t) + \frac{\mu q}{z} \sum_{n=1}^{\infty} P_{1,n+1}(t) z^{n+1}.$$ \hspace{1cm} (8)
Integrating equation (8) with respect to time ‘t’ we get,

\[ G(z, t) = \lambda \int_0^t z P_{0,0}(y) e^{\left(-\left(\lambda + \eta + \mu_v + \lambda z + \frac{\mu_v}{z}\right)(t-y)\right)} dy \]

\[ - \mu_v \int_0^t P_{0,1}(y) e^{\left(-\left(\lambda + \eta + \mu_v + \lambda z + \frac{\mu_v}{z}\right)(t-y)\right)} dy \]

\[ + \mu q \int_0^t \frac{1}{z} \left( \sum_{n=1}^{\infty} P_{1,n+1}(y) z^{n+1} \right) e^{\left(-\left(\lambda + \eta + \mu_v + \lambda z + \frac{\mu_v}{z}\right)(t-y)\right)} dy. \]

(9)

If \( \alpha_1 = 2\sqrt{\lambda \mu_v} \) and \( \beta_1 = \frac{\lambda}{\sqrt{\mu_v}} \) then

\[ e^{\left(\lambda z + \frac{\mu_v}{z}\right)t} = \sum_{n=-\infty}^{\infty} (\beta_1 z)^n l_n(\alpha_1 t). \]

Therefore equation (9) can be written as,

\[ G(z, t) = \lambda \int_0^t z P_{0,0}(y) e^{-\left(\lambda + \eta + \mu_v\right)(t-y)} \sum_{n=-\infty}^{\infty} (\beta_1 z)^n l_n(\alpha_1 (t - y)) dy \]

\[ - \mu_v \int_0^t P_{0,1}(y) e^{-\left(\lambda + \eta + \mu_v\right)(t-y)} \sum_{n=-\infty}^{\infty} (\beta_1 z)^n l_n(\alpha_1 (t - y)) dy \]

\[ + \mu q \int_0^t \frac{1}{z} \left( \sum_{n=1}^{\infty} P_{1,n+1}(y) z^{n+1} \right) e^{-\left(\lambda + \eta + \mu_v\right)(t-y)} \sum_{n=-\infty}^{\infty} (\beta_1 z)^n l_n(\alpha_1 (t - y)) dy \]

(10)

Equating the coefficient of \( z^n \) in equation (10) gives,

\[ P_{0,n}(t) = \lambda \int_0^t P_{0,0}(y) e^{-\left(\lambda + \eta + \mu_v\right)(t-y)} \beta_1^{n-1} l_{n-1}(\alpha_1 (t - y)) dy \]

\[ - \mu_v \int_0^t P_{0,1}(y) e^{-\left(\lambda + \eta + \mu_v\right)(t-y)} \beta_1^n l_n(\alpha_1 (t - y)) dy \]

\[ + \mu q \int_0^t \sum_{m=2}^{\infty} P_{1,m}(y) e^{-\left(\lambda + \eta + \mu_v\right)(t-y)} \beta_1^{n-m+1} l_{n-m+1}(\alpha_1 (t - y)) dy. \]

(11)
Similarly equating the coefficient of \( z^n \) in equation (10) yields,

\[
0 = \lambda \int_0^t P_{0,0}(y) e^{-(\lambda + \eta + \mu_v)(t-y)} \beta_1^n l_{n+1}(\alpha_1(t-y)) dy
\]

\[ - \mu_v \int_0^t P_{0,1}(y) e^{-(\lambda + \eta + \mu_v)(t-y)} \beta_1^n l_n(\alpha_1(t-y)) dy \]

\[ + \mu q \int_0^t \sum_{m=2}^\infty P_{1,m}(y) e^{-(\lambda + \eta + \mu_v)(t-y)} \beta_1^n l_{n-m+1}(\alpha_1(t-y)) dy. \]

Subtracting equation (12) from (11) gives,

\[
P_{0,n}(t) = \lambda \int_0^t P_{0,0}(y) e^{-(\lambda + \eta + \mu_v)(t-y)} \beta_1^n (I_{n-1}(\alpha_1(t-y)) - I_{n+1}(\alpha_1(t-y))) dy
\]

\[ + \mu q \int_0^t \sum_{m=2}^\infty P_{1,m}(y) e^{-(\lambda + \eta + \mu_v)(t-y)} \beta_1^n (I_{n-m+1}(\alpha_1(t-y)) - I_{n+1}(\alpha_1(t-y))) dy. \]

Taking Laplace transform for the equation (13) gives,

\[
\hat{P}_{0,n}(s) = \lambda \beta_1^n \hat{P}_{0,0}(s) \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^n \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}
\]

\[ - \lambda \beta_1^n \hat{P}_{0,0}(s) \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n+1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}} \]

\[ + \mu q \sum_{m=2}^\infty \hat{P}_{1,m}(s) \beta_1^n \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n-m+1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}} \]

\[ - \mu q \sum_{m=2}^\infty \hat{P}_{1,m}(s) \beta_1^n \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n+m-1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}} ; n = 1, 2, \ldots \]

where \( P_1 = s + \lambda + \eta + \mu_v \).

Substitute \( n = 1 \) in equation (14) gives,
\[ P_{0,1}(s) = \lambda P_{0,0}(s) \frac{1}{\sqrt{P_1^2 - \alpha_1^2}} - \lambda P_{0,0}(s) \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^2 \frac{1}{\sqrt{P_1^2 - \alpha_1^2}} + \mu q \sum_{m=2}^{\infty} P_{1,m}(s) \beta_1^{2-m} \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{2-m} - \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^m \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}. \] (15)

Taking Laplace transform for the equation (1) we get,
\[(s + \lambda + \eta)\hat{P}_{0,0}(s) = \mu q \hat{P}_{1,1}(s) + \mu_v \hat{P}_{0,1}(s) \] (16)

By substituting equation (15) in equation (16) yields,
\[
\hat{P}_{0,0}(s)(s + \lambda + \eta) \left\{ 1 - \frac{\lambda \mu_v}{(s + \lambda + \eta)} \left( \frac{1}{\sqrt{P_1^2 - \alpha_1^2}} - \hat{\Omega}_2(s) \right) \right\} = \mu q \hat{P}_{1,1}(s) + \mu q \sum_{m=2}^{\infty} P_{1,m}(s) \beta_1^{2-m} (\hat{\Omega}_{2-m}(s) - \hat{\Omega}_m(s)),
\] (17)

where
\[
\hat{\Omega}_i(s) = \left( \frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^i \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}.
\]

On further simplification of equation (17) gives,
\[
\hat{P}_{0,0}(s) = \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} (F(s))^j \left\{ \mu q \hat{P}_{1,1}(s) + \mu q \sum_{m=2}^{\infty} P_{1,m}(s) \beta_1^{2-m} (\hat{\Omega}_{2-m}(s) - \hat{\Omega}_m(s)) \right\}.
\] (18)

Substituting equation (18) in equation (14) gives,
\[
\hat{P}_{0,n}(s) = \lambda \beta_1^{n-1} (\hat{\Omega}_{n-1}(s) - \hat{\Omega}_{n+1}(s)) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} (F(s))^j \left\{ \mu q \hat{P}_{1,1}(s) + \mu q \sum_{m=2}^{\infty} P_{1,m}(s) \beta_1^{2-m} (\hat{\Omega}_{2-m}(s) - \hat{\Omega}_m(s)) \right\} + \mu q \sum_{m=2}^{\infty} \beta_1^{n-m+1} (\hat{\Omega}_{n-m+1}(s) - \hat{\Omega}_{n+m-1}(s)) \hat{P}_{1,m}(s).
\] (19)

Equation (19) can also be written as,
where

\[ \hat{\psi}_{l-1,l+1}(s) = \hat{\Omega}_{l-1}(s) - \hat{\Omega}_{l+1}(s). \]

Taking inverse Laplace transform for the equation (20) gives,

\[
P_{0,n}(t) = \lambda \beta_1^{n-1} * \psi_{n-1,n+1}(t) * e^{-(\lambda+\eta)t} * \sum_{j=0}^{\infty} (F(t))^j * \left( \mu q P_{1,1}(t) + \mu q \sum_{m=2}^{\infty} \beta_1^{2-m} P_{1,m}(t) * \psi_{2-m,2-m}(t) \right) + \mu q \sum_{m=2}^{\infty} \beta_1^{n-m+1} \psi_{n-m+1,n+m-1}(t) * P_{1,m}(t).
\]

(21)

Hence \( P_{0,n}(t) \) is expressed explicitly in terms of \( P_{1,1}(t) \) and \( P_{1,n}(t) \).

**Evaluation of \( P_{2,0}(t) \)**

Taking Laplace transform for the equation (6) we get,

\[ \hat{P}_{2,0}(s) = \frac{\alpha}{s + \lambda + r} \sum_{n=1}^{\infty} \hat{P}_{1,n}(s) \]

(22)

**Evaluation of \( P_{2,n}(t) \)**

Taking Laplace transform for the equation (7) we get,

\[ \hat{P}_{2,n}(s) = \left( \frac{\lambda}{s + \lambda + r} \right)^n \hat{P}_{2,0}(s) \]

(23)

By substituting the equation (22) in equation (23) yields,

\[ \hat{P}_{2,n}(s) = \frac{\lambda^n \alpha}{(s + \lambda + r)^{n+1}} \sum_{n=1}^{\infty} \hat{P}_{1,n}(s) \]

(24)

By inverting equation (24) we get,

\[ P_{2,n}(t) = \lambda^n \alpha \frac{e^{-(\lambda+r)t}}{n!} t^n \sum_{n=1}^{\infty} P_{1,n}(t); n = 1,2 \ldots \]

(25)
Evaluation of $P_{3,0}(t)$

Taking Laplace transform for the equation (5) we get,

$$\bar{P}_{3,0}(s) = \frac{1}{s + \lambda} + \frac{\mu p}{s + \lambda} \bar{P}_{1,1}(s) + \frac{\eta}{s + \lambda} \bar{P}_{0,0}(s) + \frac{r \alpha}{(s + \lambda)(s + \lambda + r)} \sum_{n=1}^{\infty} \bar{P}_{1,n}(s) \tag{26}$$

Inverting the equation (26) we get

$$P_{3,0}(t) = e^{-\lambda t} + \mu p e^{-\lambda t} \ast P_{1,1}(t) + \eta e^{-\lambda t} \ast P_{0,0}(t) + \alpha (e^{-\lambda t} - e^{-(\lambda + r)t}) \ast \sum_{n=1}^{\infty} P_{1,n}(t) \tag{27}$$

Evaluation of $P_{1,n}(t)$

$$H(z, t) = \sum_{n=2}^{\infty} P_{1,n}(t) z^n.$$  

Then,

$$H'(z, t) = \sum_{n=2}^{\infty} P'_{1,n}(t) z^n.$$  

By substituting equation (4) gives,

$$H'(z, t) = \left(-1 (\lambda + \mu + \alpha) + \lambda z + \frac{\mu p}{z}\right) H(z, t)$$

$$= \lambda z^2 P_{1,1}(t) - \mu p P_{1,2}(t) z + r \sum_{n=2}^{\infty} P_{2,n}(t) z^n + \eta \sum_{n=2}^{\infty} P_{0,n}(t) z^n.$$  

Integrating equation (28) with respect to time ‘$t$’ we get,

$$H(z, t) = \int_{0}^{t} \left( \lambda z^2 P_{1,1}(y) - \mu p P_{1,2}(y) z \right) e^{-\lambda (\lambda + \mu + \alpha) + \lambda z + \frac{\mu p}{z} (t-y)} dy$$

$$+ r \int_{0}^{t} e^{-\lambda (\lambda + \mu + \alpha) + \lambda z + \frac{\mu p}{z} (t-y)} \sum_{n=2}^{\infty} P_{2,n}(y) z^n dy$$

$$+ \eta \int_{0}^{t} \left( \sum_{n=2}^{\infty} P_{0,n}(y) z^n \right) e^{-\lambda (\lambda + \mu + \alpha) + \lambda z + \frac{\mu p}{z} (t-y)} dy.$$  

If $\alpha = 2\sqrt{\lambda \mu}$ and $\beta = \frac{\lambda}{\sqrt{\mu}}$ then
\[ e^{(\lambda z + \mu z) t} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t). \]

Therefore equation (29) can be written as,

\[
H(z, t) = \int_0^t (\lambda z^2 P_{1,1}(y) - \mu P_{1,2}(y) z) e^{-(\lambda + \mu + \alpha)(t-y)} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-y)) \, dy \\
+ r \int_0^t e^{-(\lambda + \mu + \alpha)(t-y)} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-y)) \sum_{n=2}^{\infty} P_{2,n}(y) z^n \, dy \\
+ \eta \int_0^t \left( \sum_{n=2}^{\infty} P_{0,n}(y) z^n \right) e^{-(\lambda + \mu + \alpha)(t-y)} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-y)) \, dy
\]

Equating the coefficient of \( z^n \) in equation (30) gives,

\[
P_{1,n}(t) = \lambda \int_0^t P_{1,1}(y) e^{-(\lambda + \mu + \alpha)(t-y)} \beta^{n-2} I_{n-2}(\alpha(t-y)) dy \\
- \mu \int_0^t P_{1,2}(y) e^{-(\lambda + \mu + \alpha)(t-y)} \beta^{n-1} I_{n-1}(\alpha(t-y)) dy \\
+ r \int_0^t e^{-(\lambda + \mu + \alpha)(t-y)} \sum_{k=2}^{\infty} P_{2,k}(y) \beta^{n-k} I_{n-k}(\alpha(t-y)) dy \\
+ \eta \int_0^t \left( \sum_{k=2}^{\infty} P_{0,k}(y) \beta^{n-k} I_{n-k}(\alpha(t-y)) \right) e^{-(\lambda + \mu + \alpha)(t-y)} dy
\]

The equation (31) can also be written as

\[
P_{1,n}(t) = \lambda \beta^{n-2} \left[ P_{1,1}(t) * e^{-(\lambda + \mu + \alpha)t} I_{n-2}(\alpha t) \right] - \mu \beta^{n-1} \left[ P_{1,2}(t) * e^{-(\lambda + \mu + \alpha)t} I_{n-1}(\alpha t) \right] \\
+ r \left[ \sum_{k=2}^{\infty} P_{2,k}(t) * e^{-(\lambda + \mu + \alpha)t} \beta^{n-k} I_{n-k}(\alpha t) \right] \\
+ \eta \left[ \sum_{k=2}^{\infty} P_{0,k}(t) * e^{-(\lambda + \mu + \alpha)t} \beta^{n-k} I_{n-k}(\alpha t) \right]; n = 2,3,..
\]
Evaluation of \( P_{1,2}(t) \)

Substituting \( n = 2 \) in the equation (32) yields,

\[
P_{1,2}(t) = \lambda P_{1,1}(t) * e^{-(\lambda + \mu + \alpha)t} I_0(\alpha t) - \mu p \beta P_{1,2}(t) * e^{-(\lambda + \mu + \alpha)t} I_1(\alpha t)
\]

\[
+ r \left[ \sum_{k=2}^{\infty} P_{2,k}(t) * e^{-(\lambda + \mu + \alpha)t} \beta^{2-k} I_{2-k}(\alpha t) \right]
\]

\[
+ \eta \left[ \sum_{k=2}^{\infty} P_{0,k}(t) * e^{-(\lambda + \mu + \alpha)t} \beta^{2-k} I_{2-k}(\alpha t) \right]
\]

Taking Laplace transform for the equation (33) yields

\[
\tilde{P}_{1,2}(s) \left(1 - \tilde{F}_1(s)\right) = \lambda \tilde{P}_{1,1}(s) \frac{1}{\sqrt{P^2 - \alpha^2}} + \sum_{k=2}^{\infty} r \tilde{P}_{2,k}(s) \tilde{u}_{2-k}(s) + \sum_{k=2}^{\infty} \eta \tilde{P}_{0,k}(s) \tilde{u}_{2-k}(s)
\]

where

\[
\tilde{F}_1(s) = -\mu p \tilde{u}_1(s),
\]

and

\[
\tilde{u}_i(s) = \beta^i \left( \frac{P - \sqrt{P^2 - \alpha^2}}{\alpha} \right) \frac{1}{\sqrt{P^2 - \alpha^2}}.
\]

By substituting the equation (20) and (24) in equation (34) yields,

\[
\tilde{P}_{1,2}(s) \left(1 - \tilde{F}_1(s)\right)
\]

\[
= \lambda \tilde{P}_{1,1}(s) \frac{1}{\sqrt{P^2 - \alpha^2}} + \frac{\alpha}{s + \lambda + r} \sum_{k=2}^{\infty} \left( \frac{\lambda}{s + \lambda + r} \right)^k \left[ \sum_{n=1}^{\infty} \tilde{P}_{1,n}(s) \tilde{u}_{2-k}(s) \left( \sum_{n=1}^{\infty} \tilde{P}_{1,n}(s) \right) \right]
\]

\[
+ \sum_{k=2}^{\infty} \left[ \lambda \beta_1^{k-1} \tilde{P}_{k-1,k+1}(s) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} \left( P(s) \right)^j \mu q \tilde{P}_{1,1}(s) 
\]

\[
+ \lambda \beta_1^{k-1} \tilde{P}_{k-1,k+1}(s) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} \left( P(s) \right)^j \mu q \sum_{m=2}^{\infty} \tilde{P}_{1,m}(s) \beta_1^{2-m} \tilde{u}_{2-m,m}(s)
\]

\[
+ \mu q \sum_{m=2}^{\infty} \tilde{P}_{1,m}(s) \beta_1^{k-m+1} \tilde{u}_{k-m+1,k+m-1}(s) \right] \eta \tilde{u}_{2-k}(s)
\]

Rewriting the above terms gives,
TRANSIENT ANALYSIS OF A SINGLE SERVER QUEUE

\[ \hat{P}_{1,2}(s) \]

\[ = \sum_{j=0}^{\infty} \left( \hat{F}_1(s) \right)^j \left\{ \hat{P}_{1,1}(s) \left[ \frac{\lambda}{\sqrt{P^2 - \alpha^2}} + \frac{\lambda \mu \eta}{\beta_1(s + \lambda + \eta)} \sum_{k=2}^{\infty} \beta_1^k \hat{\psi}_{k-1,k+1}(s) \hat{u}_{2-k}(s) \sum_{j=0}^{\infty} (F(s))^j \right] \right. \]

\[ + \frac{\alpha}{s + \lambda + r} \sum_{k=2}^{\infty} \left( \frac{\lambda}{s + \lambda + r} \right)^k r \hat{u}_{2-k}(s) \left( \sum_{n=1}^{\infty} \hat{P}_{1,n}(s) \right) \]

\[ + \frac{\lambda \mu \eta}{\beta_1(s + \lambda + \eta)} \sum_{k=2}^{\infty} \beta_1^k \hat{\psi}_{k-1,k+1}(s) \hat{u}_{2-k}(s) \sum_{j=0}^{\infty} (F(s))^j \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^2 \psi_{2-m}(s) \]

\[ + \mu \eta \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{k-m+1} \hat{\psi}_{k-m+1}(s) \hat{u}_{2-k}(s) \left\} \right. \]

\[ (35) \]

The equation (35) can be written as

\[ \hat{P}_{1,2}(s) \]

\[ = \sum_{j=0}^{\infty} \left( \hat{F}_1(s) \right)^j \left\{ \hat{R}(s) \hat{P}_{1,1}(s) + \hat{Q}(s) \sum_{n=1}^{\infty} \hat{P}_{1,n}(s) \right. \]

\[ + \hat{\omega}(s) \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^2 \psi_{2-m}(s) \]

\[ + \mu \eta \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{k-m+1} \hat{\psi}_{k-m+1}(s) \hat{u}_{2-k}(s) \left\} \right. \]

\[ (36) \]

where

\[ \hat{R}(s) = \frac{\lambda}{\sqrt{P^2 - \alpha^2}} + \frac{\lambda \mu \eta}{\beta_1(s + \lambda + \eta)} \sum_{k=2}^{\infty} \beta_1^k \hat{\psi}_{k-1,k+1}(s) \hat{u}_{2-k}(s) \sum_{j=0}^{\infty} (F(s))^j, \]

\[ \hat{Q}(s) = \frac{\alpha}{s + \lambda + r} \sum_{k=2}^{\infty} \left( \frac{\lambda}{s + \lambda + r} \right)^k r \hat{u}_{2-k}(s), \]

and

\[ \hat{\omega}(s) = \hat{R}(s) - \frac{\lambda}{\sqrt{P^2 - \alpha^2}}. \]

Inverting the equation (36) we get,
\[ P_{1,2}(t) = \sum_{j=0}^{\infty} (F_1(t))^j \]

\[
* \left[ R(t) * P_{1,1}(t) + Q(t) \\
* \sum_{n=1}^{\infty} P_{1,n}(t) + w(t) \\
* \sum_{m=2}^{\infty} \beta_1^{2-m} P_{1,m}(t) * \psi_{2-m,m}(t) \\
+ \mu \eta \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} P_{1,m}(t) * \beta_1^{k+m+1} \psi_{k-m+1,k+m-1}(t) * u_{2-k}(t) \right]
\]

where

\[
R(t) = \lambda e^{-(\lambda + \mu + \alpha)t} I_0(\alpha t) + \frac{\mu \eta \lambda}{\beta_1} e^{-(\lambda + \eta)t} * \sum_{k=2}^{\infty} \beta_1^k u_{2-k}(t) * \psi_{k-1,k+1}(t) * \sum_{j=0}^{\infty} (F(t))^j,
\]

\[
Q(t) = \alpha r e^{-(\lambda + r)t} * \sum_{k=2}^{\infty} \lambda^k e^{-(\lambda+r)t} * \frac{t^{k-1}}{(k-1)!} * u_{2-k}(t),
\]

and

\[
w(t) = R(t) - \lambda e^{-(\lambda + \mu + \alpha)t} I_0(\alpha t).
\]

Hence \( P_{1,2}(t) \) is expressed in terms of \( P_{1,1}(t) \) and \( P_{1,n}(t) \).

**Evaluation of \( P_{1,1}(t) \)**

Taking Laplace transform for the equation (3) we get,

\[
\hat{P}_{1,1}(s) = \frac{\lambda}{s + \lambda + \mu + \alpha} \hat{P}_{3,0}(s) + \frac{\mu \eta}{s + \lambda + \mu + \alpha} \hat{P}_{1,2}(s) + \frac{\eta}{s + \lambda + \mu + \alpha} \hat{P}_{0,1}(s) + \frac{r}{s + \lambda + \mu + \alpha} \hat{P}_{2,1}(s)
\]

Using the equations (18),(20),(24),(26) and (36) in equation (38) gives,
TRANSIENT ANALYSIS OF A SINGLE SERVER QUEUE

\[
\hat{P}_{1,1}(s) = \frac{\lambda}{s + \lambda + \mu + \alpha} \left[ \frac{1}{s + \lambda} + \frac{\mu p}{s + \lambda} \hat{P}_{1,1}(s) \right]
\]

\[
+ \frac{\eta}{(s + \lambda)(s + \lambda + \eta)} \sum_{j=0}^{\infty} (F(s))^j \left\{ \mu q \hat{P}_{1,1}(s) + \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{P}_{2-m,m}(s) \right\}
\]

\[
+ \frac{r \alpha}{(s + \lambda)(s + \lambda + r)} \sum_{n=1}^{\infty} \hat{P}_{1,n}(s)
\]

\[
+ \frac{\mu p}{s + \lambda + \mu + \alpha} \left[ \sum_{j=0}^{\infty} \left( \hat{F}_1(s) \right)^j \right] \left[ \hat{R}(s) \hat{P}_{1,1}(s) + \hat{Q}(s) \sum_{n=1}^{\infty} \hat{P}_{1,n}(s) \right]
\]

\[
+ \hat{W}(s) \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{P}_{2-m,m}(s)
\]

\[
+ \mu q \eta \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{k-m+1} \hat{P}_{k-m+1,k+m-1}(s) \hat{U}_{2-k}(s)
\]

\[
+ \frac{\eta}{s + \lambda + \mu + \alpha} \left[ \lambda \hat{P}_{0,2}(s) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} (F(s))^j \mu q \hat{P}_{1,1}(s) \right]
\]

\[
+ \lambda \hat{P}_{0,2}(s) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} (F(s))^j \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{P}_{2-m,m}(s)
\]

\[
+ \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{P}_{2-m,2}(s) \right] + \frac{r}{(s + \lambda + \mu + \alpha)(s + \lambda + r)^2} \sum_{n=1}^{\infty} \hat{P}_{1,n}(s)
\]

Bringing all \( \hat{P}_{1,1}(s) \) to one side, the above equation can be written as
\[ \hat{P}_{1,1}(s) = \sum_{j=0}^{\infty} \left( \hat{F}_2(s) \right)^j \left[ \frac{\lambda}{(s + \lambda)(s + \lambda + \mu + \alpha)} \right] \]
\[ + \left[ \frac{\lambda}{(s + \lambda + \mu + \alpha)(s + \lambda)(s + \lambda + r)} + \frac{\mu p}{(s + \lambda + \mu + \alpha)(s + \lambda)} \sum_{j=0}^{\infty} (\hat{F}_1(s))^j \hat{Q}(s) \right] \]
\[ + \frac{\lambda}{(s + \lambda + \mu + \alpha)(s + \lambda + r)^2} \sum_{n=2}^{\infty} \hat{P}_{1,n}(s) \]
\[ + \left[ \frac{\lambda \eta}{(s + \lambda + \mu + \alpha)(s + \lambda + \eta)} \sum_{j=0}^{\infty} (\hat{F}(s))^j + \frac{\mu p}{(s + \lambda + \mu + \alpha)} \sum_{j=0}^{\infty} (\hat{F}_1(s))^j \hat{\psi}(s) \right] \]
\[ + \left[ \frac{\lambda \eta q}{(s + \lambda + \mu + \alpha)(s + \lambda + \eta)} \sum_{j=0}^{\infty} (\hat{F}(s))^j + 1 \right] \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{\psi}_{2-m,m}(s) \]
\[ + \sum_{j=0}^{\infty} \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{k-m+1} \hat{\psi}_{k-m+1,k+m-1} \hat{u}_{2-k}(s) \]
(39)

where
\[ \hat{F}_2(s) = \frac{\mu p \lambda}{(s + \lambda)(s + \lambda + \mu + \alpha)} + \frac{r \alpha \lambda}{(s + \lambda)(s + \lambda + \mu + \alpha)(s + \lambda + r)} \]
\[ + \frac{\mu p}{(s + \lambda + \mu + \alpha)} \sum_{j=0}^{\infty} (\hat{F}_1(s))^j (\hat{R}(s) + \hat{Q}(s)) \]
\[ + \frac{\lambda \eta q}{(s + \lambda + \mu + \alpha)(s + \lambda + \eta)} \sum_{j=0}^{\infty} (\hat{F}(s))^j \hat{\psi}_{0,2}(s) \]
\[ + \frac{r \lambda \alpha}{(s + \lambda + \mu + \alpha)(s + \lambda + r)^2} + \frac{\eta q}{(s + \lambda)(s + \lambda + \eta)} \sum_{j=0}^{\infty} (\hat{F}(s))^j. \]

Inverting the equation (39) we get
TRANSIENT ANALYSIS OF A SINGLE SERVER QUEUE

\[ P_{1,1}(t) = \sum_{j=0}^{\infty} (F_2(t))^{j} \left[ \lambda e^{-\lambda t} * e^{-(\lambda + \mu + \alpha)t} \right. \\
+ \left. \lambda e^{-(\lambda + \mu + \alpha)t} * r \alpha e^{-\lambda t} * e^{-(\lambda + r)t} + \mu pe^{-(\lambda + \mu + \alpha)t} \sum_{j=0}^{\infty} \left( F_1(t) \right)^{s_j} Q(t) \right] \\
+ \left. \lambda he^{-(\lambda + \mu + \alpha)t} * \mu q e^{-\lambda t} \right. \\
\times e^{-(\lambda + r)t} \sum_{j=0}^{\infty} (F(t))^{s_j} + \mu pe^{-(\lambda + \mu + \alpha)t} \\
\times \left( \lambda e^{-(\lambda + \mu + \alpha)t} \psi_{0,2}(t) + \sum_{j=0}^{\infty} (F(t))^{s_j} + \delta(t) \right) \\
\times \sum_{m=2}^{\infty} P_{1,m}(t) \beta_2^{2-m} \psi_{2-m,m}(t) \\
\left. + \sum_{m=2}^{\infty} (F(t))^{s_j} \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} P_{1,m}(t) \beta_1^{k-m+1} \psi_{k-m+1,k+m-1}(t) u_{2-k}(t) \right] \]

where

\[ F_2(t) = \mu p \lambda \alpha e^{-\lambda t} * e^{-(\lambda + \mu + \alpha)t} + r \alpha \lambda e^{-\lambda t} * e^{-(\lambda + \mu + \alpha)t} * e^{-(\lambda + r)t} + \mu pe^{-(\lambda + \mu + \alpha)t} \]

\times \sum_{j=0}^{\infty} (F_1(t))^{s_j} \left( R(t) + Q(t) \right) + \lambda \eta pe^{-(\lambda + \mu + \alpha)t} * e^{-(\lambda + \eta)t} \psi_{0,2}(t) \\
\times \sum_{j=0}^{\infty} (F(t))^{s_j} + r \lambda e^{-(\lambda + \mu + \alpha)t} * e^{-(\lambda + r)t} \psi_{0,2}(t) \]

Hence all probabilities are explicitly expressed in terms of \( P_{1,n}(t) \). Therefore by using normalization condition \( P_{1,n}(t) \) can be found explicitly.
4. CONCLUSION AND FUTURE SCOPE

Single server queue with system disaster and repair under Bernoulli working vacation schedule is considered. The transient probabilities of the system are derived explicitly. This model can also be extended by allowing disaster to occur in working vacation state.

ACKNOWLEDGEMENT

This work was supported by Vel Tech Seed Grant, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Avadi, Chennai, India.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] I. Atencia, G. Ayuillera, P.P. Bocharov, On the M/G/1/0 queueing system under LCFSPR discipline with repeated and negative customers with batch arrivals, Ann. Conf. Univ. Swansea, 2000.

[2] X. Chao, Networks of queues with customers, signals and arbitrary service time distributions, Oper. Res. 43 (1995) 537–544.

[3] S. Dimou, A. Economou, The single server queue with catastrophes and geometric reneging, Meth. Comput. Appl. Probab. 15 (2013), 595-621.

[4] E. Gelenbe, Random neural networks with negative and positive signals and product form solution, Neural Comput. 1(1989), 502-510.

[5] E. Gelenbe, Stability of the random neural network model, Neural Comput. 2 (1990), 239-247.

[6] P.G. Harrison, E. Pitel, Sojourn times in single-server queues with negative Customers, J. Appl. Probab. 30 (1993), 943-963.

[7] J. Ye, L. Liu, T. Jiang, Analysis of a single-sever queue with disasters and repairs under Bernoulli vacation schedule, J. Syst. Sci. Inform. 4 (2016), 547-559.
[8] K. Kalidass, J. Gnanaraj, S. Gopinath, R. Kasturi, Transient analysis of an M/M/1 queue with a repairable server and multiple vacations, Int. J. Math. Oper. Res. 6 (2014), 193-216.

[9] J.C. Ke, C.H. Wu, Z.G. Zhang, Recent developments in vacation queueing models: a short survey, Int. J. Oper. Res. 7 (2010), 3-8.

[10] B.K. Kumar, D. Arivudainambi, Transient solution of an M/M/1 queue with catastrophes, Comput. Math. Appl. 40 (2000), 1233-1240.

[11] B.K. Kumar, S.P. Madheswari, Transient analysis of an M/M/1 queue subject to catastrophes and server failures, Stochastic Anal. Appl. 23 (2005), 329-340.

[12] G.C. Mytalas, M.A. Zazanis, An M^X/G/1 queueing system with disasters and repairs under a multiple adapted vacation policy, Nav. Res. Logist. 62 (2015), 171-189.

[13] R. Sudhesh, Transient analysis of a queue with system disasters and customer Impatience, Queueing Syst. 66 (2010), 95-105.

[14] Sampath, M. S., Steady state expression for a repairable single server queue with working vacations and system disasters, 11th Triennial conference of association and of Asia Pacific Operational Research Societies (APORS), Kathmandu, Nepal, (2018).

[15] W.S. Yang, J.D. Kim, K.C. Chae, Analysis of M/G/1 stochastic clearing systems. Stochastic Anal. Appl. 20 (2002), 1083-1100.

[16] U. Yechiali, Queues with system disasters and impatient customers when system is down, Queueing Syst. 56 (2007), 195-202.