Cyclo-Dissipativity Revisited

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Abstract—Starting from a symmetrization and extension of the basic definitions and results of dissipativity theory, we obtain new results on cyclo-dissipativity, in particular on external characterization and description of the set of storage functions.

Index Terms—Cyclo-dissipativity, dissipation inequality, dissipativity, passivity, storage function, thermodynamics.

I. INTRODUCTION

Dissipativity theory originates from the seminal paper [10]. It unifies classical input–output stability theory, centered around the passivity and small-gain theorems, with Lyapunov function theory for autonomous dynamical systems. In particular, it aims at deriving Lyapunov functions for large-scale interconnected systems, based on the knowledge of the component systems, and the way they are coupled to each other. Furthermore, it directly relates to physical systems theory, network synthesis, and optimal control.

The more general notion of cyclo-dissipativity was first1 formulated in [11], aimed at extending the stability analysis based on dissipativity toward instability results. Implicitly the notion was already present in [9], motivated by infinite-horizon optimal control. In the technical report [3] cyclo-dissipativity was further explored, extending the fundamental results for ordinary dissipativity obtained in [10]. Since then the notion of cyclo-dissipativity has not received much detailed attention, although the concept regularly appears in passivity-based control (e.g., [7]) and stability analysis of interconnected systems [6].

In this article, we will revisit the notion of dissipativity, by unifying earlier definitions and developments. This will turn out to be instrumental for developing a more complete theory of cyclo-dissipativity, extending the results of [3]. Finally, the developed theory will be illustrated on the formulation of the Clausius inequality in thermodynamics.

II. DISSIPATIVITY REVISITED

In this section, we recall the basic definitions and results of dissipativity theory as developed in the groundbreaking paper [10], with extensions due to [4] and [8]. In particular, we put them into a more general and unifying context, as a preparation for the results on cyclo-dissipativity in the following section.

Consider a nonlinear input-state-output system

\[ \dot{x} = f(x,u), \quad x \in \mathcal{X}, u \in \mathbb{R}^m \]

\[ y = h(x,u), \quad y \in \mathbb{R}^p \]

on an n-dimensional state-space manifold \( \mathcal{X} \). Consider a supply rate

\[ s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}. \]

Throughout it will be assumed that for all solutions of \( \Sigma \), the integrals \( \int_{t_1}^{t_2} s(u(t),y(t))dt \) are well-defined for all \( t_1, t_2 \).

Given \( \Sigma \) and the supply rate \( s \) the, possibly extended, function \( S : \mathcal{X} \to -\infty \cup \mathbb{R} \cup \mathbb{R}^\infty \) satisfies the dissipation inequality if2

\[ S(x(t_2)) \leq S(x(t_1)) + \int_{t_1}^{t_2} s(u(t),y(t))dt \]

holds for all \( t_1 \leq t_2 \), all input functions \( u : [t_1, t_2] \to \mathbb{R}^m \), and all initial conditions \( x(t_1) \), where \( S(t) = h(x(t),u(t)) \), with \( x(t) \) denoting the solution of \( \dot{x} = f(x,u) \) for initial condition \( x(t_1) \) and input function \( u : [t_1, t_2] \to \mathbb{R}^m \). In particular this implies that if \( S(x(t_2)) \) equals \( \infty \), then so does \( S(x(t_1)) \), and if \( S(x(t_1)) \) equals \( -\infty \), then so does \( S(x(t_2)) \).

A nonextended function \( S : \mathcal{X} \to \mathbb{R} \) satisfying the dissipation inequality (3) is called a storage function.3 This leads to the following standard definition of dissipativity as pioneered in the seminal paper [10] (see also [4]).

Definition II.1: The system \( \Sigma \) is dissipative with respect to the supply rate \( s \) if there exists a nonnegative storage function \( S : \mathcal{X} \to \mathbb{R}^+ \). If the nonnegative storage function \( S : \mathcal{X} \to \mathbb{R}^+ \) satisfies (3) with equality, then the system is called lossless.

In case of the supply rate \( s(u,y) = y^T u, u, y \in \mathbb{R}^m \), “dissipativity” is usually referred to as “passivity.”

In order to characterize dissipativity (and subsequently the weaker property of cyclo-dissipativity), let us define the following, possibly extended, functions\(^4\) \( S_a : \mathcal{X} \to \mathbb{R} \cup \mathbb{R}^\infty \) and \( S_r : \mathcal{X} \to -\infty \cup \mathbb{R} \)

\[ S_a(x) = \sup_{u,T \geq 0,x(0)=x} \int_0^T s(u(t),y(t))dt \]

\[ S_r(x) = \inf_{u,T \geq 0,x(0)=x} \int_0^T s(u(t),y(t))dt. \]

Interpreting the supply rate \( s \) as the “power” supplied to the system, and the storage function \( S \) as the “energy” stored in the system, the

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1 It would like to thank Pablo Borja and Romeo Ortega for pointing out to me the somewhat forgotten references [3] and [11].

2 Assuming differentiability, the dissipation inequality (3) is easily seen [8] to be equivalent to the differential dissipation inequality \( \dot{S}(x,u) \leq s(u,h(x,u)) \), where \( \dot{S}(x,u) := \frac{d}{dt} \left( \frac{d}{dt} S \right)(x) \).

3 Note that we do not require \( S \) to be nonnegative or bounded from below.

4 Here, \( a \) refers to “available storage,” and \( r \) to “required storage.” Note that in the second case, we deviate from the standard notation, where \( S_r \) in fact refers to the function \( S_{rc} \) as defined next. In fact, in the standard treatments of dissipativity [3], [4], [8], [10], [11] only the functions \( S_a \) and \( S_r \) (here denoted as \( S_a \) and \( S_r \)) are used. The present set-up aims at “symmetrizing” the picture; also with a view on cyclo-dissipativity.
function $S_u(x)$ equals the maximally extractable “energy” from the system at state $x$ [10]. Similarly, $S_r$ equals the “energy” that needs to be minimally supplied to the system while bringing it to state $x$. Obviously $S_u, S_r$ satisfy

$$S_u(x) \geq 0, \quad S_r(x) \leq 0. \quad (5)$$

Furthermore, assuming reachability from a ground-state $x^*$ and controllability to this same state $x^*$, we define the, possibly extended, functions $S_{ac} : \mathcal{X} \to \mathbb{R} \cup \infty, S_{rc} : \mathcal{X} \to \mathbb{R} \cup \infty, \mathcal{X} = \mathbb{R}^n \cup \infty$ as

$$S_{ac}(x) = \sup_{u, t \geq 0} \left( \sup_{x(x(0)=x, x(T) = x^*)} \int_0^T s(u(t), y(t)) dt \right)$$

$$S_{rc}(x) = \inf_{u, t \geq 0} \left( \inf_{x(x(0)=x, x(T) = x^*)} \int_0^T s(u(t), y(t)) dt \right) \quad (6)$$

which again have an obvious interpretation in terms of “energy.” Clearly for all $x \in \mathcal{X}$

$$-\infty < S_{ac}(x) \leq S_u(x)$$

$$S_r(x) \leq S_{rc}(x) < \infty. \quad (7)$$

Furthermore, it is straightforward\(^5\) to check that the abovementioned four functions are related by

$$S_u(x^*) = \sup_x -S_{rc}(x)$$

$$S_r(x^*) = \inf_x S_{ac}(x)$$

$$\times (\sup_x \sup_{u, T \geq 0} s(u(t), y(t)) dt - \int_0^T s(u(t), y(t)) dt)$$

$$\times (\sup_x \inf_{u, T \geq 0} s(u(t), y(t)) dt - \int_0^T s(u(t), y(t)) dt). \quad (8)$$

In particular, it follows that

$$S_u(x^*) < \infty \iff \liminf_{x \to x^*} S_r(x) > -\infty$$

$$S_r(x^*) > -\infty \iff \limsup_{x \to x^*} S_{ac}(x) < \infty. \quad (9)$$

By using the definitions of infimum and supremum, it is easily verified (see [8], [9], and [10]) that all four (possibly extended) functions $S_u, S_r, S_{ac}, S_{rc}$ satisfy the dissipation inequality (3).

The following theorem summarizes some of the main findings of dissipativity theory as formulated in [8], extending the fundamental results of [10] (see also [4]).

**Theorem II.2:** $\Sigma$ is dissipative if and only if $S_u(x) < \infty$ for all $x \in \mathcal{X}$ (that is, $S_r : \mathcal{X} \to \mathbb{R}$). If $\Sigma$ is reachable from $x^*$, then $\Sigma$ is dissipative if and only if $S_u(x^*) < \infty$, or equivalently $\inf_x S_r(x) > -\infty$. Furthermore, if $\Sigma$ is dissipative, then $\Sigma$ is a nonnegative storage function satisfying $\inf_x S_u(x) = 0$, and all other nonnegative storage functions $S$ satisfy

$$S_u(x) \leq S(x) - \inf_x S(x), \quad x \in \mathcal{X}. \quad (10)$$

Also, if $\Sigma$ is dissipative, then $S_{rc} - \inf_x S_r(x)$ is a nonnegative storage function, and all other nonnegative storage functions $S$ satisfy

$$S(x) - S(x^*) \leq S_{rc}(x). \quad (11)$$

Moreover the set of nonnegative storage functions is convex.

\(^5\)Here, $c$ stands for “constrained,” since either $x(T) = x^*$ or $x(-T) = x^*$.

\(^6\)The first equality in (8) already figures in [8]; the second one is similar.

**Remark II.3:** Thus, $\Sigma$ is dissipative if and only if the maximally extractable “energy” $S_u(x)$ is finite for every initial state. In case the system is reachable from $x^*$, while furthermore $S_u(x^*) = 0$ (maximally extractable energy from ground-state $x^*$ is zero), then it directly follows from the dissipation inequality (3) that dissipativity is equivalent to

$$\int_0^T s(u(t), y(t)) dt \geq 0 \quad (12)$$

for all $u : [0, T] \to \mathbb{R}^n, T \geq 0$, where $y(t) = h(x(t), u(t))$ with $x(t)$ the solution of $\dot{x} = f(x(t), u(t))$ for initial condition $x(0) = x^*$. (Hence, “energy” always needs to be supplied to $\Sigma$ initialized at $x^*$.) This is sometimes taken as the definition of dissipativity, especially in the linear case with $x^* = 0$. Note, however, that in a nonlinear context there is often no natural ground-state (see also Example II.7).

In the next proposition, similar results will be derived for the newly defined functions $S_u$ and $S_{ac}$; however with the key difference that $S_r \leq 0$ and, thus, $\inf_S$ corresponding to dissipativity.

**Proposition II.4:** If $S_r(x) > -\infty$ for all $x \in \mathcal{X}$, then $\Sigma$ is a nonpositive storage function. If $\Sigma$ is controllable to $x^*$, then $S_r(x) > -\infty$ for all $x \in \mathcal{X}$ if and only if $S_r(x^*) > -\infty$, or equivalently $\sup_x S_{ac}(x) < \infty$. Furthermore, if $S_r(x) > -\infty$ for all $x \in \mathcal{X}$, then $\Sigma$ is a nonpositive storage function satisfying $\sup_x S_r(x) = 0$, and all other nonpositive storage functions $S$ satisfy

$$S_r(x) \geq S(x) - \sup_x S(x). \quad (13)$$

Moreover, if $S_r(x^*) > -\infty$, then $\inf_x S_{ac} - \sup_x S_{ac}(x)$ is a nonpositive storage function, and all nonpositive storage functions $S$ satisfy

$$S(x) - S(x^*) \geq S_{ac}(x). \quad (14)$$

**Remark II.5:** The reversed dissipation inequality

$$S(x(t_2)) \geq S(x(t_1)) + \int_{t_1}^{t_2} s(u(t), y(t)) dt \quad (15)$$

appears in optimal control theory, with $s$ the running cost (see, e.g., [8] and [9]). Obviously the reversed dissipation inequality is obtained from (3) by replacing $s$ by $-s$ and $\Sigma$ by $-\Sigma$.

**Remark II.6:** Under certain conditions on the supply rate $s$ and the system $\Sigma$, the definitions of $S_u$ and $S_r$ can be modified to the indefinite integrals

$$S_u(x) = \sup_{u(x(0)=x)} -\int_0^\infty s(u(t), y(t)) dt$$

$$S_r(x) = \inf_{u(x(0)=x)} \int_0^\infty s(u(t), y(t)) dt \quad (16)$$

with $s(x)$ approaching $x^*$ for $t \to \infty$ in the first case (see [9] and [8] for precise statements), and for $t \to -\infty$ in the second case. This is the context of the developments in [9], where however the system dynamics was assumed to be linear.

**Example II.7:** Consider the scalar system

$$\Sigma : \dot{x} = u, \quad y = e^x \quad (17)$$

with passivity supply rate $s(u, y) = uy$. Then

$$S_u(x) = \sup_{T>0} (e^x(T) - e^x) = e^x$$

$$S_r(x) = \inf_{T>0} (e^x - e^{-x(T)}) = -\infty. \quad (18)$$

In fact, $\Sigma$ is lossless with nonnegative storage function $e^x$, unique up to a constant. Obviously, $\Sigma$ is reachable from and controllable to, e.g.,
\[ x^* = 0, \]
\[ S_{ac}(x) = -(e^0 - e^*) = e^* - 1, \quad S_{rc}(x) = -(e^* - e^0) = e^* - 1 \]  
(i.e., \( S_{ac}(x) = S_{rc}(x) = e^* - 1 \)). Note that the example could be extended to, e.g., \( \dot{x} = -x^3 + u, \ y = e^x \), which is passive but not lossless.

### III. Cyclo-Dissipativity

Next, we come to the study of cyclo-dissipativity, as coined in [11] with instability theorems for interconnected systems in mind, and further developed in [3].

**Definition III.1:** \( \Sigma \) is cyclo-dissipative if
\[
\int_0^T s(u(t), y(t)) dt \geq 0
\]  
for all \( T \geq 0 \) and all \( u : [0, T] \rightarrow \mathbb{R}^m \) such that \( x(T) = x(0) \). Furthermore, \( \Sigma \) is cyclo-dissipative with respect to \( x^* \) if
\[
\int_0^T s(u(t), y(t)) dt \geq 0
\]  
for all \( T \geq 0 \) and all \( u : [0, T] \rightarrow \mathbb{R}^m \) such that \( x(T) = x(0) = x^* \). In case (20) or (21) holds with equality, we speak about cyclo-losslessness (with respect to \( x^* \)).

The following proposition is obvious, and follows from substituting \( x(T) = x(0) \) in (3).

**Proposition III.2:** If there exists a storage function for the system \( \Sigma \) then \( \Sigma \) is cyclo-dissipative.

The following main theorem extends the results in [3] in a number of directions.

**Theorem III.3:** Assume that \( \Sigma \) is reachable from \( x^* \) and controllable to \( x^* \). Then, \( \Sigma \) is cyclo-dissipative with respect to \( x^* \) if and only if
\[
S_{ac}(x) \leq S_{rc}(x), \quad \text{for all } x \in \mathcal{X}.
\]  
Furthermore, if \( \Sigma \) is cyclo-dissipative with respect to \( x^* \), then both \( S_{ac} \) and \( S_{rc} \) are storage functions for \( \Sigma \), implying that \( \Sigma \) admits storage functions and that \( \Sigma \) is cyclo-dissipative. Moreover, if \( \Sigma \) is cyclo-dissipative with respect to \( x^* \), then
\[
S_{ac}(x^*) = S_{rc}(x^*) = 0
\]  
and any other storage function \( S \) for \( \Sigma \) satisfies
\[
S_{ac}(x) \leq S(x) - S(x^*) \leq S_{rc}(x).
\]  
The set of storage functions is convex. If the system is cyclo-lossless from \( x^* \), then \( S_{ac} = S_{rc} \), and the storage function is unique up to a constant.

**Proof:** Suppose \( \Sigma \) is cyclo-dissipative with respect to \( x^* \). Pick any \( x \). Since \( \Sigma \) is reachable from and controllable to \( x^* \), there exist trajectories \( x(t) \) with \( x(-T') = x(0) = x \). For any of these
\[
\int_{-T'}^0 s(u(t), y(t)) dt + \int_0^{T'} s(u(t), y(t)) dt \geq 0
\]  
and thus
\[
\int_{-T'}^0 s(u(t), y(t)) dt \geq - \int_0^{T'} s(u(t), y(t)) dt.
\]  
Taking infimum on the left-hand side and supremum on the right-hand side, we obtain \( S_{ac}(x) \geq S_{rc}(x) \) for any \( x \). Conversely, let (22) hold. Since by definition \( S_{ac}(x) > -\infty, S_{rc}(x) < \infty \), it follows that both \( S_{ac} \) and \( S_{rc} \) are storage functions. Thus, by Proposition III.2, \( \Sigma \) is cyclo-dissipative.

Furthermore, let \( \Sigma \) be cyclo-dissipative with respect to \( x^* \). By definition, \( S_{ac}(x^*) \geq 0 \). On the other hand, in view of (21), we also have \( S_{ac}(x^*) \leq 0 \), thus, implying \( S_{ac}(x^*) = 0 \). Similarly, \( S_{rc}(x^*) \leq 0 \), and in view of (21) also \( S_{rc}(x^*) \geq 0 \), thus, implying \( S_{rc}(x^*) = 0 \).

Finally, let \( \Sigma \) be cyclo-dissipative with respect to \( x^* \). Then, as proved above, there exist storage functions for \( \Sigma \). Let \( S \) be any storage function. Then
\[
S(x) - S(x^*) \leq \int_0^T s(u(t), y(t)) dt
\]  
for any trajectory from \( x \) at \( t = 0 \) to \( x^* \) at \( t = T \). This is equivalent to
\[
S(x) - S(x^*) \geq \sup_{u, T \geq 0} x(x(0) = x(T) = x^*) - \int_0^T s(u(t), y(t)) dt = S_{ac}(x).
\]  
Similarly
\[
S(x) - S(x^*) \leq \int_0^T s(u(t), y(t)) dt
\]  
for any trajectory from \( x^* \) at \( t = T \) to \( x \) at \( t = 0 \). This is equivalent to
\[
S(x) - S(x^*) \leq \inf_{u, T \geq 0} x(x(0) = x(\infty) = x) - \int_0^T s(u(t), y(t)) dt = S_{rc}(x).
\]  

**Remark III.4:** It is immediately checked that if \( \Sigma \) is reachable from and controllable to \( x^* \), then the same holds for any other choice of the ground-state \( x^* \). Similarly, Theorem III.3 shows that the property of cyclo-dissipativity from \( x^* \) is independent of the choice of \( x^* \). Also, Theorem III.3 implies that cyclo-dissipativity is equivalent to the existence of an (indefinite) storage function.

**Remark III.5:** Cyclo-dissipativity, instead of dissipativity, is not uncommon in physical systems modeling. For example, the gravitational energy between two masses is proportional to \( -\frac{1}{2} \), with \( r \geq 0 \) the distance between the two masses. This function is not bounded from below and, thus, cannot be turned into a nonnegative storage function by addition of a constant.

**Remark III.6:** Interpreting \( s(u(t), y(t)) \) as power provided to the system at time \( t \), Theorem III.3 says that the system is cyclo-dissipative from any ground state \( x^* \) if and only if the maximal energy that can be extracted from the system while bringing the system from the initial state \( x \) to the ground state is less than or equal to the minimal energy that is needed to bring the system from the ground state to \( x \). On the other hand, dissipative systems are characterized by the fact that the maximally extractable energy is finite for every initial state. This property does not hold anymore in the case of cyclo-dissipativity. For example, see the previous remark, it is possible to extract an infinite amount of energy from the system consisting of an actuated mass in the gravitational field around a second fixed mass.

**Remark III.7:** Note that all results concerning interconnection of dissipative systems as developed, e.g., in [4], [8] and, [10] remain to hold for cyclo-dissipative systems, in the sense that the interconnected system will be cyclo-dissipative with (indefinite) storage function equal to the sum of the storage functions of the subsystems. Hence, instability results can be inferred; this was the motivation for cyclo-dissipativity in [11].

In general, the values of the maximal and minimal storage functions \( S_{ac} \) and \( S_{rc} \) are different from each other unless the system is cyclo-lossless. Thus, in general, the storage function of a cyclo-dissipative...
system is not unique. Interestingly, uniqueness can be guaranteed as follows.

**Proposition III.8:** Suppose $Σ$ is reachable from and controllable to $x^*$ and cyclo-dissipative with respect to $x^*$. Assume additionally that for every $x$ there exists a solution $(x(t), u(t), y(t))$ on some time-interval $[0, T]$ such that $x(T) = x(0) = x^*$ and $x(τ) = x$ for some $τ ∈ [0, T]$, and with the property that

$$\int_0^T s(u(t), y(t))dt = 0. \quad (31)$$

Then, $S_{ac}(x) = S_{rc}(x)$ for all $x ∈ X$, and thus any other storage function $S$ satisfies

$$S(x) - S(x^*) = S_{ac}(x) = S_{rc}(x), \quad x ∈ X. \quad (32)$$

**Proof:** Pick $x$. By assumption there exists a solution $(x(t), u(t), y(t))$ on $[0, T]$ such that $x(T) = x(0) = x^*$ and $x(τ) = x$ for some $τ ∈ [0, T]$ satisfying (31). Shift the time-axis and redefine the time-instants such that

$$x_1(-T^*) = x^*, \quad x_1(0) = x, \quad x_1(T^*) = x^*$$

$$\int_{-T^*}^0 s(u(t), y(t))dt = 0, \quad T^*, T'' ≥ 0. \quad (33)$$

Then consider any other solution $x(·), u(·), y(·)$ on $[-T, 0]$ with $x(-T) = x^*, x(0) = x, T ≥ 0$. By cyclo-dissipativity from $x^*$, it follows that

$$\int_{-T}^0 s(u(t), y(t))dt + \int_{-T}^0 s(u(t), y(t))dt ≥ 0 \quad (34)$$

which, in view of (31), is equivalent to

$$\int_{-T}^0 s(u(t), y(t))dt ≤ \int_{-T}^0 s(u(t), y(t))dt. \quad (35)$$

Taking the infimum of the right-hand side over all input functions $u(·)$ and times $T ≥ 0$ such that $x(-T) = x^*, x(0) = x$, and taking into account that $u(t)$ on $[-T, 0]$ is also such a function, this yields

$$\int_{-T}^0 s(u(t), y(t))dt = S_{rc}(x). \quad (36)$$

Similarly, consider any solution $x(·), u(·), y(·)$ on $[0, T]$ with $x(0) = x, x(T) = x^*$. Then

$$\int_{-T}^0 s(u(t), y(t))dt + \int_{0}^T s(u(t), y(t))dt ≥ 0 \quad (37)$$

or equivalently

$$\int_{-T}^0 s(u(t), y(t))dt ≥ - \int_{0}^T s(u(t), y(t))dt. \quad (38)$$

Taking the supremum of the right-hand side (over all input functions $u(·)$ and times $T ≥ 0$) this implies

$$\int_{-T}^0 s(u(t), y(t))dt = S_{rc}(x). \quad (39)$$

Together with (36) this proves $S_{ac}(x) = S_{rc}(x)$.

**Remark III.9:** As a consequence, the unique storage function is given by (36). Note furthermore that (31) means that the system is weakly cyclo-lossless with respect to $x^*$, in the sense that for every $x$ there exists a cyclic solution passing through $x$ and $x^*$ satisfying (21) with equality. (Although other such solutions will generally satisfy (21) with inequality.)

Finally, the functions $S_{ac}$ and $S_{rc}$ depend on the choice of the ground-state $x^*$, and one may wonder about the relation between these functions for different ground-states. Partial information is provided in the following proposition.

**Proposition III.10:** Denote $S_{ac}$ for ground-state $x^*$ by $S_{ac}^*$, and for ground-state $x^{**}$ by $S_{ac}^{**}$. Similarly define $S_{rc}^*$ and $S_{rc}^{**}$. Then

$$S_{ac}^*(x^{**}) + S_{rc}^*(x^*) ≤ 0, \quad S_{ac}^{**}(x^*) + S_{rc}^*(x^*) ≥ 0. \quad (40)$$

**Proof:** By the first inequality of (24)

$$S_{ac}^*(x) ≤ S_{ac}^{**}(x) - S_{ac}^{**}(x^*) \quad (41)$$

This implies

$$S_{ac}^*(x) + S_{ac}^{**}(x^*) ≤ S_{ac}^*(x) - S_{ac}^{**}(x^*)$$

and thus $S_{ac}^*(x) + S_{ac}^{**}(x^*) ≤ 0$. The second inequality follows analogously from the second inequality of (24).

**A. Noncontrollable and/or Nonreachable Case**

The definitions of $S_{ac}$ and $S_{rc}$ can be extended to the noncontrollable/nonreachable case by deferring (see already [3])

$$S_{ac}(x) = -∞, \quad S_{rc}(x) = ∞,$$

if $x$ is not controllable to $x^*$ or if $x$ is not reachable from $x^*$. (42)

Then, it is easily verified that these extended functions $S_{ac} : X → -∞ ∪ R$ and $S_{rc} : X → R ∪ ∞$ still satisfy the dissipation inequality (3), while also (9) remains to hold. On the other hand, (7) needs to be replaced by

$$-∞ ≤ S_{ac}(x) ≤ S_{ac}(x), \quad S_{rc}(x) ≤ S_{rc}(x), \quad x ∈ X. \quad (43)$$

With regard to the characterization of cyclo-dissipativity, we note that trivially $S_{ac}(x) ≤ S_{rc}(x)$ whenever $x$ is noncontrollable to $x^*$ or nonreachable from $x^*$. Thus, also in the noncontrollable/nonreachable case, $Σ$ is cyclo-dissipative with respect to $x^*$ if and only if (22) holds. On the other hand, if $Σ$ is not controllable from $X^*$ and not reachable to $x^*$, then $S_{ac}$ and $S_{rc}$ are extended functions, and hence, no cyclo-dissipativity can be concluded. We arrive at the following extension of Theorem III.3 with completely analogous proof.

**Theorem III.11:** $Σ$ is cyclo-dissipative with respect to $x^*$ if and only if

$$S_{ac}(x) ≤ S_{rc}(x), \quad x ∈ X. \quad (44)$$

Furthermore, if $Σ$ is cyclo-dissipative with respect to $x^*$, then

$$S_{ac}(x^*) = S_{rc}(x^*) = 0 \quad (45)$$

and any other (possibly extended) function $S$ satisfying the dissipation inequality (3) is such that

$$S_{ac}(x) ≤ S(x) - S(x^*) ≤ S_{rc}(x), \quad x ∈ X. \quad (46)$$

**B. Application to the Clausius Inequality**

An interesting application of cyclo-dissipativity concerns the formulation and implications of the Second Law of thermodynamics. The standard argumentation in classical thermodynamics (see, e.g., [1] and [5]), is to derive from the Second Law, by using the Carnot cycle, the inequality

$$\int_0^T q(t) \frac{dt}{T(t)} ≤ 0 \quad (47)$$

for all cyclic trajectories, where equality holds for so-called reversible7 cyclic trajectories. Here, $q$ denotes the heat flow supplied to the system.
and \( T \) is the temperature. Clearly this is the same as cyclo-dissipativity with respect to the supply rate \(-\frac{q}{T}\). Based on (47), one defines the entropy \( S \) as a function of the state of the thermodynamic system, and derives the Clausius inequality
\[
S(x(t_2)) - S(x(t_1)) \geq \int_{t_1}^{t_2} \frac{q(t)}{T(t)} dt \tag{48}
\]
with equality for reversible trajectories. This leads to the dissipativity formulation of the Second Law as given in [10] (see also [2]). In fact, if the entropy \( S \) is assumed to be bounded from above, then it follows that the thermodynamic system is dissipative with respect to the supply rate \(-\frac{q}{T}\) and the nonnegative storage function \( S = -S + c \), where \( c \) is a suitable constant. For general thermodynamic systems, however, there is no reason why the entropy \( S \) is bounded from above. In this case \( S = -S \) is truly indefinite, and the system is only cyclo-dissipative.

As indicated above, the standard definition [1], [5] of the entropy function \( S \) is based on the additional assumption that every state \( x \) is reachable from a ground-state \( x^* \) using a reversible trajectory. If this is the case, then \( S \) is uniquely defined as
\[
S(x) := S(x^*) + \int_0^T \frac{q(t)}{T(t)} dt \tag{49}
\]
where \( x(0) = x^*, x(\tau) = x \) are the initial and final states of this reversible trajectory. Without this assumption, Theorem III.3 can be invoked as follows. Assuming reachability from and controllability to a ground-state \( x^* \), it follows from Theorem III.3 that (47) implies the existence of storage functions \( S \) satisfying \( S_{ac}(x) \leq S(x) - S(x^*) \leq S_{rc}(x) \). Hence, for any such \( S \), the function \( S := -S \) qualifies as a possible entropy function satisfying the Clausius inequality (48). Nevertheless, from a thermodynamic point of view, it is desirable to have a unique entropy function (up to a constant). As stated in Theorem III.3, this is the case if the system is cyclo-lossless, i.e., (47) holds with equality. However, by invoking Proposition III.8, uniqueness is also guaranteed by assuming that for every \( x \) there exists a cyclic solution on \([0, T]\), starting from and ending at \( x^* \) and passing through \( x \), satisfying
\[
\int_0^T \frac{q(t)}{T(t)} dt = 0. \tag{50}
\]
This is much weaker than the standard reversibility assumption.

### IV. Conclusion

Cyclo-dissipativity or cyclo-passivity arises naturally in quite a few cases, from modeling to passivity-based control (see also [12] for additional motivation). By symmetrizing and extending classical definitions and results of dissipativity theory, we derived in a transparent way, some new results concerning cyclo-dissipativity, including external characterizations and description of the set of (indefinite) storage functions.

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