Let \( L/F \) be a finite Galois extension of number fields. E. Artin conjectured that the \( L \)-series of a nontrivial irreducible complex representation of \( \text{Gal}(L/F) \) is entire, and proved this for monomial representations. The nonmonomial two-dimensional representations are those with image in \( \text{PGL}(2, \mathbb{C}) \) isomorphic to the group of rigid motions of the tetrahedron, octahedron or icosahedron. In [5] Langlands proved Artin’s conjecture for all two-dimensional representations of tetrahedral type and certain octahedral representations when \( F = \mathbb{Q} \). The purpose of this note is to prove the conjecture for all octahedral representations by using the methods of Langlands and an analytic result of Jacquet, Piatetski-Shapiro and Shalika.

Let \( \rho \) be an irreducible two-dimensional complex representation of \( \text{Gal}(L/F) \). We say that a cuspidal automorphic representation \( \pi \) of \( \text{GL}(2, \mathbb{A}_F) \) equals \( \pi(\rho) \) if \( \pi = \bigotimes \pi_v \) with \( \pi_v = \pi(\rho_v) \) in the sense of [2, §12] for almost all places \( v \) of \( F \). When \( \pi = \pi(\rho) \) the \( L \)-series of \( \pi \) and \( \rho \) agree, and since cuspidal representations have entire \( L \)-series, Artin’s conjecture follows. In [5, §3] Langlands used base change for \( \text{GL}(2) \) to produce candidates for \( \pi(\rho) \). When \( \rho \) is octahedral we will use the following result to show that one of Langlands’ candidates is in fact \( \pi(\rho) \).

**Theorem** [4]. Let \( K \) be a cubic extension of \( F \) (not necessarily Galois). For each automorphic cuspidal representation \( \pi \) of \( \text{GL}(2, \mathbb{A}_F) \) there exists an automorphic representation \( \Pi = \text{BC}_{K/F}(\pi) \) of \( \text{GL}(2, \mathbb{A}_K) \) such that for almost all places \( v \) of \( F \), and each place \( w \) of \( K \) dividing \( v \), \( \pi_v = \pi(\sigma_v) \) implies that \( \Pi_w = \pi(\text{Res}_W^{K/F} \sigma_w) \).

This theorem is proved using the theory of automorphic forms on \( \text{GL}(3) \) and \( \text{GL}(2) \times \text{GL}(3) \). The basic concept is similar to that of the example of quadratic base change given in [3, §20]. We recall that the theory of base change developed in [5] treats the case of Galois cyclic change of base of prime degree,
together with a characterization of the image and descent properties. We will denote by the symbol $\text{BC}$ the base change lifting.

Let $\rho$ be a two-dimensional representation of $\text{Gal}(L/F)$ of octahedral type. Let $E/F$ be the quadratic subextension of $L/F$ fixed by all elements of $\text{Gal}(L/F)$ mapping to the unique index two subgroup of the octahedral group $S_4$. Choose a 2-Sylow subgroup of the octahedral group and let $K/F$ be the cubic subextension fixed by all elements of $\text{Gal}(L/F)$ mapping to this chosen Sylow subgroup.

Let $M$ be the compositium of $E$ and $K$ in $L$, so that we have the diagram of fields and Galois groups above. For any subextension $T/F$ of $L/F$, let $\rho_T$ be the restriction of $\rho$ to $\text{Gal}(L/T)$.

In [5, §3] Langlands showed, using base change and results of Gelbart, Jacquet, Piatetskii-Shapiro and Shalika, that $\pi(\rho_E)$ exists. There are exactly two cuspidal representations $\pi_1$ and $\pi_2$ of $GL(2, A_F)$ such that $BC_{E/F}(\pi_i) = \pi(\rho_E)$. They are related by $\pi \approx \pi_2 \otimes \omega_{E/F}$, where $\omega_{E/F}$ is the idele class character corresponding to the quadratic extension $E/F$. Notice that $\rho_K$ is monomial, so $\pi(\rho_K)$ exists.

**Lemma.** There exists a unique index $i$ such that $BC_{K/F}(\pi_i) = \pi(\rho_K)$.

**Proof.** The theorem quoted above shows that $BC_{K/F}(\pi_i)$ exists for $i = 1, 2$. By transitivity of base change, $BC_{E/K}(BC_{K/F}(\pi_i)) = \pi(\rho_M)$ for $i = 1, 2$. Notice that $BC_{K/F}(\pi_1) \approx BC_{K/F}(\pi_2) \otimes \omega_{M/K}$. The representations $BC_{K/F}(\pi_i)$ are distinct for $i = 1, 2$, for if $BC_{K/F}(\pi_1) \approx BC_{K/F}(\pi_1) \otimes \omega_{M/K}$ then $\pi(\rho_M)$ would not be cuspidal. But $\rho_M$ is irreducible, so this does not occur. By the descent theory for base change, $BC_{K/F}(\pi_1)$ and $BC_{K/F}(\pi_2)$ are the two automorphic representations of $GL(2, A_K)$ which yield $\pi(\rho_M)$ after base change. Since $\pi(\rho_K)$ also has this property, it must be $BC_{K/F}(\pi_i)$ for a unique choice of $i$.

Let $\pi$ be the automorphic representation $\pi_i$ of the lemma which satisfies $BC_{K/F}(\pi) = \pi(\rho_K)$ and $BC_{E/F}(\pi) = \pi(\rho_E)$.

**Theorem.** Let $\rho$ be an octahedral representation of $\text{Gal}(L/F)$. Then $\pi(\rho)$ exists, and hence $L(\rho, s)$ is entire.

**Proof.** The proof is similar to [5, §3]. We show that the automorphic representation $\pi$ constructed by Langlands is equal to $\pi(\rho)$. For each place $v$ of...
$F$ such that $\rho_v$ is unramified we obtain a diagonal conjugacy class $\text{diag}(a_v, b_v)$ in $GL(2, \mathbb{C})$. For each place $v$ of $F$ such that $\pi_v$ is an unramified principal series we obtain a conjugacy class $\text{diag}(a'_v, b'_v)$. Let $v$ be such that both $\rho_v$ and $\pi_v$ are unramified. Since $BC_{E/F}(\pi) = \pi(\rho_E)$ we see that $\text{diag}(a'_v, b'_v)$ is conjugate to $\text{diag}(a_v\omega, b_v\omega)$ with $\omega^2 = 1$. We must show that $\text{diag}(a_v, b_v)$ and $\text{diag}(a_v\omega, b_v\omega)$ are conjugate.

Let $w$ be a place of $K$ dividing $v$, with $[K_w : F_v] = d(v)$. Since $BC_{K/F}(\pi) = \pi(\rho_K)$, $\text{diag}(a^d(w), b^d(w))$ and $\text{diag}(a_v\omega^d(w), b_v\omega^d(w))$ are conjugate. If $d(w) = 1$, the desired conjugacy results. If $d(w) = 3$, either $a_v^3 = a_v^3\omega$ or $a_v^3 = b_v^3\omega$. In the first case $\omega = 1$, while in the second $a_v = b_v\eta\omega$ with $\eta^3 = 1$. If $\eta = 1$, $\text{diag}(a_v, b_v)$ is conjugate to $\text{diag}(a_v\omega, b_v\omega)$. When $\eta$ is nontrivial, $\text{diag}(a_v, b_v) = \text{diag}(b_v\eta\omega, b_v)$ gives an element of order 6 in the projective image of $\rho$. But the octahedral group contains no elements of order 6, so this is impossible.

Therefore, in all cases $\text{diag}(a_v, b_v)$ is conjugate to $\text{diag}(a_v\omega, b_v\omega)$. Since this holds for almost all places of $F$, we have $\pi = \pi(\rho)$, proving the theorem.

The icosahedral representations are not susceptible to these base change methods. Examples of icosahedral representations for $F = \mathbb{Q}$ which have entire $L$-series are given in [1].

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