Spectral Frank-Wolfe Algorithm: Strict Complementarity and Linear Convergence

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Abstract

We develop a novel variant of the classical Frank-Wolfe algorithm, which we call spectral Frank-Wolfe, for convex optimization over a spectrahedron. The spectral Frank-Wolfe algorithm has a novel ingredient: it computes a few eigenvectors of the gradient and solves a small-scale SDP in each iteration. Such procedure overcomes slow convergence of the classical Frank-Wolfe algorithm due to ignoring eigenvalue coalescence. We demonstrate that strict complementarity of the optimization problem is key to proving linear convergence of various algorithms, such as the spectral Frank-Wolfe algorithm as well as the projected gradient method and its accelerated version.

1 Introduction

We consider solving the following optimization problem with the decision variable $X \in \mathbb{R}^{n \times n}$:

$$\begin{align*}
\text{minimize} & \quad f(X) := g(AX) + \langle C, X \rangle \\
\text{subject to} & \quad \text{tr}(X) = 1, \quad X \succeq 0.
\end{align*}$$

Problem setup. The setup of Problem (1) is as follows. We assume $C \in \mathbb{R}^{n \times n}$ is a symmetric matrix. The constraint $X \succeq 0$ means that $X$ is symmetric and positive semidefinite. We assume that $A : S^n \to \mathbb{R}^m$ is a linear map from the set of symmetric matrices $S^n$ to the $m$-dimensional Euclidean space. We also assume that the function $g : \mathbb{R}^m \to \mathbb{R}$ is differentiable and its gradient $\nabla g$ is $L_g$-Lipschitz continuous. We use $\text{tr}(\cdot)$ to denote the standard trace operation, the sum of diagonal entries of the input matrix. We denote by $S_n$ the feasible
Algorithm 1 Frank-Wolfe with line search

Input: initialization \( X_0 \in S_n \)

for \( t = 1, 2, \ldots \) do

Eigenvalue computation: compute an eigenvector \( v \) of \( \nabla f(X_t) \) associated with smallest eigenvalue.

Line search: solve \( \hat{\eta} = \arg \min_{\eta \in [0, 1]} f(\eta X_t + (1 - \eta)vv^\top) \) and set \( X_{t+1} = \hat{\eta}X_t + (1 - \hat{\eta})vv^\top \).

end for

region of Problem (1). The set \( S_n \) is called the spectrahedron, which is nonempty and compact. Hence Problem (1) always has an optimal solution. In this paper, we assume Problem (1) admits a unique optimal solution \( X_\star \) with rank \( r_\star \) for the sake of simplicity. The main results, Theorem 3 and 6 below, can be adapted to the setting where multiple optimal solutions exist; see Section A in the Appendix for a further discussion. It is worth noting that for almost all matrix \( C \), the solution of Problem (1) is indeed unique [DL11, Corollary 3.5].

Applications. The optimization problem covers many low rank matrix recovery problems including matrix sensing [RFP10], matrix completion [CR09, JS10], phase retrieval [CESV15, YUTC17], and blind deconvolution [ARR13]. The constraints \( X \succeq 0 \) and \( \text{tr}(X) = 1 \) impose low-rankness on the solution. The rank \( r_\star \) of optimal solutions in these applications is expected to be small comparing to the problem dimension \( n \). We note that the following problem:

\[
\minimize_{||X||_* \leq \alpha} f(X),
\]

is sometimes a more direct optimization formulation for aforementioned low rank matrix recovery problems. Since Problem (2) can be re-formulated as Problem (1) [JS10], we consider Problem (1) as our main focus of study in this paper.

Background and related works. A natural but costly algorithm for solving (1) is using the projected gradient descent method (PGD) or its accelerated version (APGD) [Nes13]. Although the iteration complexity of PGD or APGD is considerably low for each of their iteration requires computing a full eigenvalue decomposition of an \( n \times n \) matrix, which scales as \( \mathcal{O}(n^3) \) [TB97]. The high per-iteration cost prevents their large-scale deployment. Hence, projection-free methods are sought, such as the Frank-Wolfe method (FW) [FW56, Jag13] presented in Algorithm 1. In the spectrahedron setting, each step only requires computing one eigenvector of the gradient of \( f \), which can be efficiently done using the Lanczos method [KW92] by taking advantage of the structure of \( \nabla f(X) = \mathcal{A}^* (\nabla g)(AX) + C \) as well as the sparsity of \( \mathcal{A} \) and \( C \). FW converges to

\[1\] PGD or APGD achieves an \( \epsilon \)-approximate solution in \( \mathcal{O}(\log(\frac{1}{\epsilon})) \) iterations for strongly convex \( f \). APGD achieves an \( \epsilon \)-approximate solution \( \mathcal{O}(\frac{1}{\sqrt{\epsilon}}) \) for general smooth \( f \).
Algorithm 2 Generalized BlockFW (G-BlockFW)

**Input:** initialization $X_0 \in S_n$, a step size $\eta \in [0, 1]$, a smooth parameter $\beta$, and an integer $k > 0$

for $t = 1, 2, \ldots, \text{do}$

Eigenvalue computation: compute top $k$ eigenvalues $(\lambda_1, \ldots, \lambda_r)$ and their eigenvectors $V = [v_1, \ldots, v_k]$ of $X_t - \frac{1}{\eta \beta} \nabla f(X_t)$.

Eigenvalue projection: project $(\lambda_1, \ldots, \lambda_k)$ to the $k$-dimensional probability simplex $\{x \in \mathbb{R}^k \mid \sum_{i=1}^k x_i = 1, x_i \geq 0\}$, and get the projected point $\Lambda$.

Forming a new iterates: set $X_{t+1} = (1 - \eta)X_t + \eta V \text{diag}(\Lambda) V^\top$.

end for

An $\epsilon$-approximate solution is within $O(\frac{1}{\epsilon})$ many iterations. However, the iteration complexity $O(\frac{1}{\epsilon})$ is tight as shown in [Gar16] even if $f$ is strongly convex and no structural assumption is posed on the solution of (1). Considerable recent research effort [Gar16, FGM17, AZHHL17, Gar19a] has focused on incorporating the low-rankness of solution $X_\star$. Of particular relevance to our work are [Gar19a] and [AZHHL17].

- [Gar19a] shows that Algorithm 1 converges linearly given that the solution is rank one, and an eigengap assumption on the gradient $\nabla f(X_\star)$ at the optimal solution is satisfied. We note that the rank-one assumption is crucial for the linear convergence of Algorithm 1 to hold. As we will demonstrate in Section 5, if the solution is not rank one, Algorithm 1 gets stagnant and behaves in the worst case as $O(\frac{1}{\epsilon})$.

- [AZHHL17] proposes an algorithm called BlockFW, which is re-formulated as Algorithm 2 for our setting and renamed as generalized BlockFW (G-BlockFW). It computes only $k$ eigenvectors in each step, and converges linearly so long as $k \geq r_\star = \text{rank}(X_\star)$ and $f$ is strongly convex. However, the method relies critically on the assumption $k \geq r_\star$: no convergence guarantees can be made if this assumption fails. Indeed, we will demonstrate in Section 5 that if $k < r_\star$, G-BlockFW gets stuck at moderate accuracy and cannot make further progress. Moreover, the method needs to store iterates explicitly to compute the eigenvectors. This not only incurs an extra $O(n^2)$ space complexity, but also increases the burden of computing eigenvectors as the iterates themselves have no structure to be exploited for fast eigenvector computation.

In summary, previous methods converge linearly only when the optimal solution

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2 A matrix $X$ is $\epsilon$-approximate solution to Problem [1] if $X$ is feasible and $f(X) - f(X_\star) \leq \epsilon$.

3 We note that BlockFW is not designed for (1), but rather for (2). Since (2) covers (1), we renamed the algorithm as G-BlockFW.

4 [AZHHL17] gives an adaptive $k$ selection procedure which works well in their experiments, but there is no theoretical guarantee for the procedure.

5 Actually [AZHHL17] provides a method to avoid the extra space and time costs. However, the method requires knowledge of the strong convexity parameter, which is unavailable in all experiments they perform.
| Algorithm          | Worst Convergence Rate | Linear Convergence Rate | Condition               |
|--------------------|------------------------|-------------------------|-------------------------|
| FW (Alg. 1)        | \( \frac{8L_f}{t} \) | \( (1 - \frac{\delta}{12L_f})^t \) | \( r_\star = 1 \) and strict comp. |
| G-BlockFW (Alg. 2) | \( \frac{k}{L_f} \)  | \( (1 - \frac{\gamma}{2L_f})^t \) | \( k \geq r_\star \) and QG |
| SpecFW (Alg. 3)    | \( \frac{8L_f}{t} \)  | \( (1 - \frac{\min(\delta,\gamma)}{12L_f})^t \) | \( k \geq r_\star \), QG, and strict comp. |

Table 1: Comparison of FW, G-BlockFW and SpecFW. Here, we assume \( f \) has gradients \( \nabla f \) that are \( L_f \)-Lipschitz. The optimal solution rank is \( r_\star = \text{rank}(X_\star) \). We let \( t \) be the number of iterations. Convergence rates are measured by \( f(X_t) - f(X_\star) \). We set \( \delta \) to be the difference between the smallest eigenvalue and the \((r_\star + 1)\)th-smallest eigenvalue of \( \nabla f(X_\star) \), that is, \( \delta = \lambda_{n-r_\star}(\nabla f(X_\star)) - \lambda_n(\nabla f(X_\star)) \). "Strict comp." means strict complementarity (Definition 1). "QG" means quadratic growth with parameter \( \gamma \) (Definition 2). Both FW and SpecFW have burn-in phases which are bounded by \( \frac{72L_f^2}{(\min\{\gamma,\delta\})^t} \). Here, the burn-in phase is the number of iterations in which the method converges with standard rate \( L_f/t \), before shifting to the faster rate (if linear convergence condition is satisfied). The convergence rate of G-BlockFW can be found in Lemma 14 in Section F of the Appendix.

is rank one, or the number of eigenvectors computed in each iteration is no smaller than the rank of the optimal solution.

**Our contributions.** The contribution of this work is two-fold. On the problem structure side:

- We show that the eigengap assumption in [Gar19a] is equivalent to the strict complementarity condition, a well-known regularity condition of semidefinite programming [AHO97]; see Section 2 for more detail.
- Based on the eigengap condition, or the equivalent strict complementarity condition, we show that Problem (1) satisfies the quadratic growth property (Definition 2 below) when the outer function \( g \) is strongly convex over the feasible region \( S_n \) of Problem (1), which is true for all the application being considered. This governs the linear convergence of many first order methods such as PGD, APGD, and our method, Spectral Frank Wolfe.

On the algorithm side, we propose a new algorithm called Spectral Frank-Wolfe (SpecFW) in Section 3, which has the following properties:

- In each of its iteration, it computes \( k \) eigenvectors using only the current gradient information.
- In each of its iteration, it solves a small-scale sub-problem efficiently by APGD for small \( k \).
- It always converges at the rate \( O(\frac{1}{t}) \) no matter what choice of \( k \) is.
• It converges linearly when $k \geq r_\star$, and the strict complementarity and quadratic growth condition are satisfied. In particular, we do not require $f$ to be strongly convex or the rank $r_\star$ to be 1.

• It can easily incorporate the matrix sketching idea from [TYUC17] and achieves the so-called storage optimality discussed in [YUTC17]. The sketching procedure obviates the need for storing the full decision matrix $X$ throughout iterations, thereby saving $\mathcal{O}(n^2)$ space.\(^6\)

**Organization.** The rest of the paper is organized as follows. In Section 2, we explain the concept of strict complementarity and the classical Frank-Wolfe algorithm, and how they motivate our Spectral Frank-Wolfe. In Section 3, we present the Spectral Frank-Wolfe and its convergence guarantees. In Section 4, we show that the strict complementarity enforces the quadratic growth condition whenever $g$ is strongly convex on $S_n$. Finally, we demonstrate numerically the effectiveness of the Spectral Frank-Wolfe in Section 5.

**Notation.** For a symmetric matrix $A \in \mathbb{S}^n$, we denote its $i$-th largest eigenvalue as $\lambda_i(A)$. The operator two norm, nuclear norm, and Frobenius norm are denoted as $\|A\|_{op}$, $\|A\|_*$, and $\|A\|_F$, respectively. The inner product $\langle \cdot, \cdot \rangle$ on symmetric matrices is the standard trace inner product. We also equip $\mathbb{R}^m$ with the dot product. For a linear map $B : \mathbb{S}^d \to \mathbb{R}^l$, the adjoint map of $B$ is denoted as $B^*$. We also define its largest and smallest singular values as $\|B\|_{op} = \sigma_{\text{max}}(B) = \max_{\|A\|_F=1} \|B(A)\|_2$ and $\sigma_{\text{min}}(B) = \min_{\|A\|_F=1} \|B(A)\|_2$. Given a matrix $V \in \mathbb{R}^{d \times r}$, we denote the restriction of $B$ to $V$ as $B_V : \mathbb{S}^r \to \mathbb{R}^l$ by $B_V(S) = B(VSV^T)$ for any $S \in \mathbb{S}^r$.

2 Motivating SpecFW from complementarity and Frank-Wolfe

In this section, we explain the motivations of the spectral Frank-Wolfe from strict complementarity and its relationship with the classical Frank-Wolfe.

2.1 Observation from complementarity

Let first introduce the KKT condition to see what complementarity means.

**KKT condition.** By Slater’s condition for (1) and the fact that the feasible region $S_n$ is compact, the following KKT condition of (1) always holds: there is

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\(^6\)Interested readers can find the procedure in Section D in the Appendix. We note the matrix sketching idea cannot be combined with G-BlockFW easily to avoid storing $X$, as G-BlockFW uses a sum of the current iterate and current gradient to compute the eigenvectors, which destroys the fast matrix-vector product property of the gradient.
some dual optimal solution \( Z_* \succeq 0 \) and \( s_* \in \mathbb{R} \) such that

\[
\nabla f(X_*) - Z_* - s_* I = 0, \quad \text{(First Order Condition)}
\]

\[
\langle Z_*, X_* \rangle = 0, \quad \text{(Complementarity)}
\]

\[
\text{tr}(X_*) = 1, \quad \text{(Linear Constraint Feasibility)}
\]

\[
Z_*, X_* \succeq 0. \quad \text{(PSD Feasibility)}
\]

Here \( I \) is the identity matrix in \( \mathbb{S}^n \). We prove in Lemma 7 in the Appendix that the dual solution \((Z_*, s_*)\) is actually unique.

**Complementarity: extract \( X_* \) from \( Z_* \).** We first note that using \( Z_*, X_* \succeq 0 \) and complementarity \( \langle Z_*, X_* \rangle = 0 \), we have \( Z_* X_* = 0 \). This equality implies that

\[
\text{range}(X_*) \subset \text{nullspace}(Z_*),
\]

and

\[
r_* = \text{rank}(X_*) \leq \dim(\text{nullspace}(Z_*)) =: k_*.
\]

Hence, if we can compute a matrix \( V_* \in \mathbb{R}^{n \times k_*} \) with orthonormal columns that span the null space of \( Z_* \), and solve for

\[
S_* = \arg \min_{S \in S_{k_*}} f(V_* S V_*^T),
\]

then we get the primal optimal solution \( X_* = V_* S_* V_*^T \).

We note that it is necessary to optimize over the \( k_* \)-spectrahedron \( S_{k_*} \) instead of just a \( k_* \)-dimensional probability simplex, as \( V_* \) may not be the eigenvectors of \( X_* \) for \( k_* > 1 \). Problem (6) can be solved by APGD rapidly so long as \( k_* \), the size of \( S_* \), is small.

This naturally leads to the following questions:

1. Problem (6) is easy to solve only if \( k_* \) is small; yet for now we only have \( k_* \geq r_* \). With \( r_* \) expected to be small, can we hope for \( k_* = r_* \) to hold, so that \( k_* \) is small as well?

2. Suppose we have \( k_* = r_* \), can we compute \( V_* \) exactly or approximate it well enough?

We answer the first question in the next section by defining strict complementarity and establishing its equivalence to an eigengap condition on \( \nabla f(X_*) \).

To answer the second question, we draw relationship between the first order condition in (3) and the classical Frank-Wolfe algorithm in Section 2.3.

\[\text{If there are multiple primal optimal solutions, then the KKT condition holds for any one of them.}\]
2.2 Strict complementarity

We answer why we expect \( r_\star = k_\star \) in this section. Using the rank-nullity theorem, we see that the equation \( r_\star = \text{rank}(X_\star) \leq \text{dim}(\text{nullspace}(Z_\star)) \) is equivalent to
\[
\text{rank}(X_\star) + \text{rank}(Z_\star) \leq n.
\]

Strict complementarity \cite{AHO97} assumes that we have equality instead of inequality.

**Definition 1.** (Strict Complementarity) Let \( X_\star, Z_\star \) and \( s_\star \) satisfy the KKT condition \cite{AV97}. We say that Problem \( (1) \) (or the pair \( (X_\star, Z_\star) \)) satisfies strict complementarity if
\[
\text{rank}(X_\star) + \text{rank}(Z_\star) = n.
\]

It is immediately clear that using the rank-nullity theorem again, we see that strict complementarity is equivalent to
\[
r_\star = k_\star,
\]
which is what we desire. By \cite{AV97} and given that the solution rank is \( r_\star \), strict complementarity is equivalent to
\[
\lambda_{n - r_\star}(Z_\star) > 0. \tag{7}
\]
Equation \cite{AV97} also implies that we always have for all \( i = 1, \ldots, r_\star \),
\[
\lambda_{n - r_\star + i}(Z_\star) = 0. \tag{8}
\]

**Relation with the eigengap assumption.** In \cite{Gar19b} \cite{Gar19a}, the author proposed an eigengap condition:
\[
\lambda_{n - r_\star} (\nabla f(X_\star)) - \lambda_n (\nabla f(X_\star)) > 0.
\]
This is in fact equivalent to strict complementarity: since \( \nabla f(X_\star) = Z_\star + s_\star I \), we have
\[
\lambda_{n - r_\star} (\nabla f(X_\star)) - \lambda_n (\nabla f(X_\star)) = \lambda_{n - r_\star} (Z_\star + s_\star I) - \lambda_n (Z_\star + s_\star I) = \lambda_{n - r_\star} (Z_\star) + s_\star - \lambda_n (Z_\star) - s_\star = \lambda_{n - r_\star} (Z_\star),
\]
where the last step is due to \cite{AV97}. Using \cite{AV97}, we deduce the equivalence.

**Why strict complementarity should hold.** Strict complementarity as shown in \cite{DL11} holds for almost all \( C \) (see Lemma \cite{DL11} for a more detailed derivation). We will also verify this assumption numerically in our experiments in Section \cite{DL11}. Moreover, as demonstrated in \cite{Gar19a} Lemmas 2 and 10], such assumption should hold if we expect the solution rank \( r_\star \) to be stable under small perturbations.
2.3 FW and approximation of nullspace($Z_*$)

We have just argued why we expect $r_\star = k_\star$ should hold for Problem (1). In this section, we draw relation of FW and approximation of nullspace($Z_*$).

Denote by $\text{EV}_{r}(A)$ the eigenspace of the smallest $r$ eigenvalues of a matrix $A \in S^n$. In view of the first order condition (3), we have

$$\text{EV}_{k_\star}(\nabla f(X_\star)) = \text{nullspace}(Z_\star).$$

Hence nullspace($Z_\star$) can be identified using the gradient of $f$ at $X_\star$.

Note that FW indeed uses the eigenvector corresponding to the smallest eigenvalue of $\nabla f(X_t)$ in each of its iteration, and therefore it tries to approximate $\text{EV}_{k_\star}(\nabla f(X_\star))$. This is the main intuition that linear convergence of FW can be established when $r_\star = 1$ as in \cite{gar2019}. It also reveals that FW fails to converge in a linear rate for $k_\star > 1$, as approximation using one eigenvector is not enough for a $k_\star$-dimensional space. Also, from (8) and the first order condition in the KKT condition, we see the smallest $k_\star$ eigenvalues of the gradient coalesce, and hence it is important to compute the $k_\star$-dimensional space to attain better numerical stability and accuracy. Hence, to overcome this issue, we need to compute at least $k_\star$ eigenvectors and solve a sub-problem like (6) in each iteration.

The above discussion motivates our algorithm, the Spectral Frank-Wolfe (Algorithm 3), described in the next section.

3 Spectral Frank-Wolfe and its Convergence guarantees

In this section, we describe the Spectral Frank-Wolfe algorithm and its theoretical guarantees.

3.1 The Spectral Frank-Wolfe algorithm

The Spectral Frank-Wolfe algorithm is presented in Algorithm 3. We highlight its key mechanism as follows.

Solving a small-scale SDP. The small-scale semidefinite programming (SDP)

$$\min_{\eta + \text{tr}(S) = 1, S \succeq 0, \eta \geq 0} f(\eta X_t + VSV^T).$$

(10)

can be solved easily using APGD since projection to the set $\{(\eta, S) \mid \eta + \text{tr}(S) = 1, S \succeq 0, \eta \geq 0\}$ only requires an eigenvalue decomposition of a symmetric matrix of size $k$ and a projection to the $(k+1)$-dimensional probability simplex. The correctness of the procedure for projection can be verified using arguments in \cite[Lemma 3.1]{azu2017}, and \cite[Lemma 6]{gar2019}. We note that when evaluating gradient is very expensive, instead of minimizing $f(\eta X_t + VSV^T)$, one can also minimize an upper bound of it (and the guarantees in the next section continue to hold). This is discussed in Section C in the Appendix.
Algorithm 3 Spectral Frank-Wolfe

Input: initialization $X_0 \in S_n$, an integer $k > 0$

for $t = 1, 2, \ldots,$ do

Eigenvalue computation: compute the $k$ eigenvectors, $v_1, \ldots, v_k$ of $\nabla f(X_t)$ associated with the $k$ smallest eigenvalues, and form the matrix $V = [v_1, \ldots, v_k] \in \mathbb{R}^{n \times k}$.

Solving a small-scale SDP: solve $\min_{\eta \geq 0} \{ \eta + \text{tr}(S) = 1, S \succeq 0, f(\eta X_t + VSV^T) \}$ and get an optimal solution $(\hat{S}, \hat{\eta})$.

Forming a new iterate: set $X_{t+1} = \hat{\eta}X_t + V\hat{S}V^T$.

end for

Averaging with current $X_t$. In addition to the eigenvectors from the current gradient, we also utilize the information of previous iterates when solving the small-scale SDP (10). This follows the same spirit as the classical Frank-Wolfe, which performs a line search over the current iterate and the new atom $vv^T$. This averaging scheme stabilizes the algorithm and facilitates the $O(\frac{1}{\epsilon})$ convergence rate.

The choice of $k$. From the proof of the convergence in the next section, it can be observed that so long as $k \geq k_\star$, Algorithm 4 converges linearly. Of course, one may not know $k_\star$ in advance. In this case, $k$ may be taken as the largest value subject to the user’s computational budget or the largest rank of the solution the user can afford in terms of storage. An adaptive strategy may also be employed based on the progress of objective value decay as in [AZHHL17, Section 6.2]. We do not further the discussion of this issue due to the space limit.

3.2 Theoretical guarantees

To state our result, we first define the notion of quadratic growth.

Definition 2 (Quadratic Growth (QG)). We say that the optimization problem (1) satisfies quadratic growth with parameter $\gamma > 0$, if for every feasible $X \in S_n$ there holds

$$f(X) - f(X_\star) \geq \gamma \| X - X_\star \|^2_F.$$  

The quadratic growth condition is necessary for linear convergence of gradient descent type methods as shown in [NNG19, Theorem 13]. Hence we should expect it to hold if we are to show linear convergence of Frank-Wolfe methods. The condition automatically holds for strongly convex $f$, and more broadly, it is satisfied for almost all $C$ so long as $g$ is semi-algebraic, as shown in [DIL16, Corollary 4.8]. In Section 4, we show that strict complementarity and strong convexity of the outer function $g$ (but not $f$) implies quadratic growth, as well as an explicit formula of $\gamma$ in terms of the solution $X_\star$, the map $A$, and smoothness and strong convexity parameters of $g$.

We now state the theoretical guarantees for our Algorithm 3.
Theorem 3. Suppose strict complementarity holds for Problem (1), the optimal solution $X_\star$ is unique with rank $r_\star$, the function $g$ has $L_g$-Lipschitz continuous gradients, Problem (1) satisfies quadratic growth with parameter $\gamma$, and the choice of $k$ satisfies $k \geq r_\star = k_\star$. Define $h_t = f(X_t) - f(X_\star)$ for each $t$, and $\beta = \|A\|_{op}^2 L_g$. Then for all $t$, we have

$$f(X_t) - f(X_\star) \leq \frac{8\beta}{t}. \quad (11)$$

For all $t \geq T_0 = \frac{72\beta^3}{\gamma \lambda_{n-r_\star}(Z_\star)}$, we have

$$h_{t+1} \leq \left( 1 - \min \left\{ \frac{\gamma}{4\beta}, \frac{\lambda_{n-r_\star}(Z_\star)}{12\beta} \right\} \right) h_t. \quad (12)$$

Discussion on the assumptions. As discussed before, these assumptions are expected to be necessary for linear convergence and robustness of the rank under small perturbations. The assumption of the unique optimal solution is only for the purpose of clear presentation.

Preparation of the proof. Let us first give the definition of the $r$-th spectral set.

Definition 4. For each $X \in \mathbb{S}^n$, let $V_X \in \mathbb{R}^{n \times k}$ having orthonormal eigenvectors as columns corresponding to the smallest $k$ eigenvalues of $X$. Define the spectral $k$-th set $C_k(X)$ of $X$ as

$$C_k(X) := \{V_X S V_X^\top \in \mathbb{S}^n \mid S \in S_k \}.$$  

We next present the following important lemma which is proved in Section E in the Appendix.

Lemma 5. Given $Y \in \mathbb{S}^n$ which satisfies $\lambda_{n-r_\star}(Y) - \lambda_{n-r_\star+1}(Y) \geq \delta$ for some $\delta > 0$, then for any $X \in \mathbb{S}^n$, $X \succeq 0$, and $\text{tr}(X) = 1$, there is some $W \in C_r(Y)$ such that

$$\langle X - W, Y \rangle \geq \frac{\delta}{2} \|X - W\|_F^2.$$  

We are now ready to start the proof.

Proof of Theorem 3. Using the Lipschitz smoothness of $f$, we have for any $t \geq 1$, $\eta \in [0,1]$, and any $W \in C_{r_\star}(\nabla f(X_t))$:

$$f(X_{t+1}) \leq f(X_t) + (1 - \eta) \langle W - X_t, \nabla f(X_t) \rangle + \frac{(1 - \eta)^2 \beta}{2} \|W - X_t\|_F^2. \quad (13)$$

Now choose $W = v_n v_n^\top$ where $v_n$ is the eigenvector of $\nabla f(X_t)$ with the smallest eigenvalue, we can then perform the analysis as normal Frank-Wolfe as is done in [Jag13] to reach the first part of the theorem, the inequality (11).
For the second part, we first note that by the discussion after the Definition 1 of strict complementarity, we have \( \lambda_{n-r_1}(\nabla f(X_*)) - \lambda_{n-r_1+1}(\nabla f(X_*)) = \lambda_{n-r_1}(Z_*), \) and \( \lambda_{n-r_1+1}(\nabla f(X_*)) = \cdots = \lambda_n(\nabla f(X_*)). \)

Using Lipschitz continuous gradient of \( f \) in step (a), the quadratic growth of \( f \) in step (b), and the choice of \( T_0 \) in step (c), we find that for all \( t \geq T_0, \)

\[
\|\nabla f(X_t) - \nabla f(X_*)\|_F \leq \beta \|X_t - X_*\|_F
\]

\[
\leq \beta \left( f(X_t) - f(X_*) \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{3} \lambda_{n-r_1}(Z_*).
\]

Using the inequality (14) and Weyl’s inequality, we find that

\[
\lambda_{n-r_1}(\nabla f(X_*)) - \lambda_{n-r_1+1}(\nabla f(X_*)) = \lambda_{n-r_1}(Z_*)
\]

\[
+ (\lambda_{n-r_1}(\nabla f(X_*)) - \lambda_{n-r_1+1}(\nabla f(X_*))) \geq \frac{1}{6} \lambda_{n-r_1}(Z_*)
\]

\[
+ (\lambda_{n-r_1+1}(\nabla f(X_*)) - \lambda_{n-r_1+1}(\nabla f(X_*))) \geq \frac{1}{6} \lambda_{n-r_1}(Z_*)
\]

\[
\geq \frac{1}{3} \lambda_{n-r_1}(Z_*).
\]

Now we subtract the inequality (13) both sides by \( f(X_*), \) and denote \( h_t = f(X_t) - f(X_*) \) for each \( t, \) we reach

\[
h_{t+1} \leq h_t + (1 - \eta) \frac{\langle W - X_t, \nabla f(X_t) \rangle}{R_1} + \frac{(1 - \eta)^2 \beta}{2} \|W - X_t\|_F. \]

Using Lemma 8 and the inequality (15), we can choose \( W \in C_{r_1}(\nabla f(X_*)) \) such that

\[
\langle W - X_*, \nabla f(X_t) \rangle \leq - \frac{\lambda_{n-r_1}(Z_*)}{6} \|X_* - W\|_F^2. \]

Let us now analyze the term \( R_1 = \langle W - X_t, \nabla f(X_t) \rangle \) using (16) and convexity of \( f: \)

\[
R_1 = \langle W - X_t, \nabla f(X_t) \rangle
\]

\[
= \langle W - X_*, \nabla f(X_t) \rangle + \langle X_* - X_t, \nabla f(X_t) \rangle
\]

\[
\leq - \frac{\lambda_{n-r_1}(Z_*)}{6} \|X_* - W\|_F^2 - h_t.
\]
The term $R_2 = \|X_t - W\|_F^2$ can be bounded by

$$R_2 = \|X_t - W\|_F^2 \overset{(a)}{\leq} 2 (\|X_t - X_*\|_F^2 + \|X_* - W\|_F^2)$$

$$\overset{(b)}{\leq} \frac{2}{\gamma} h_t + 2\|X_* - W\|_F^2,$$

where we use triangle inequality and the basic inequality $(a + b)^2 \leq 2a^2 + 2b^2$ in step (a), and the quadratic growth condition in step (b).

Now combining (15), and the bounds of $R_1$ and $R_2$, we reach that there is a $W \in C_{r_*}(\nabla f(X_t))$ such that for any $\xi = 1 - \eta \in [0, 1]$, we have

$$h_{t+1} \leq h_t + \xi \left( -\frac{\lambda n - r_* (Z_\star)}{6} \|X_* - W\|_F^2 - h_t \right) + \frac{\xi^2 \beta}{2} \left( \frac{2}{\gamma} h_t + 2\|X_* - W\|_F^2 \right)$$

$$= \left( 1 - \xi + \frac{\xi^2 \beta}{\gamma} \right) h_t + \left( \frac{\xi^2 \beta - \xi \lambda n - r_* (Z_\star)}{6} \right) \|X_* - W\|_F^2.$$

A detailed calculation and choice of $\xi$ in Section [15] in Appendix reveals that we can reach the second part of the theorem, the inequality (12).

4 Quadratic Growth and Linear Convergence of Algorithms

In this section, we show that when $g$ is $\alpha$-strongly convex and strict complementarity of (1) holds, then we have quadratic growth of Problem (1). We also demonstrate when the dual matrix $Z_\star$ has rank $n - 1$ then we do not require $g$ to be $\alpha$-strongly convex. An immediate consequence is the linear convergence of PGD and APGD, the generalized blockFW8 (Algorithm 2, and the spectral Frank-Wolfe (Algorithm 3) as shown in Theorem 3.

**Theorem 6.** Suppose strict complementarity of (1) and one of the following conditions hold:

(i) $g$ is $\alpha$-strongly convex, and the solution $X_\star$ is unique, or

(ii) the dual matrix $Z_\star$ in the KKT condition (3) has rank $n - 1$, then Problem (1) satisfies quadratic growth. The constant $\gamma$ takes the form of

$$\gamma = \min \left\{ \frac{\lambda n - r_* (Z_\star)}{4 + 8 \frac{\sigma_{\min}^2(\hat{A})}{\sigma_{\max}(\hat{A})}}, \frac{\alpha \sigma_{\min}^2(\hat{A})}{8} \right\}$$

in the first case, where $\hat{A}(X) = \begin{bmatrix} \text{tr}(X) \\ A(X) \end{bmatrix}$, and

8We show its convergence under quadratic growth in Lemma [14] in Section [15] in the Appendix.
(ii) $\gamma = \frac{\lambda_{n-r}(Z_\star)}{2}$ in the second case. In addition, the uniqueness of $X_\star$ is implied in the second case.

Proof. The second case has been verified in [Gar19b, Lemmas 1 and 2]. We provide a self-contained and different proof in Section F in Appendix.

Now consider the first case. For any feasible $X$ and the optimal solution $X_\star$, we have

$$f(X) - f(X_\star) = g(AX) - g(AX_\star) + (C, X - X_\star)$$

$$(a) \geq ((\nabla g)(AX_\star), A(X - X_\star)) + (C, X - X_\star) + \frac{\alpha}{2}||AX - AX_\star||_2^2$$

$$(b) = (A^*(\nabla g)(AX_\star) + C, X - X_\star) + \frac{\alpha}{2}||AX - AX_\star||_2^2$$

$$(c) = (Z_\star + s_\star I, X - X_\star) + \frac{\alpha}{2}||A(X - X_\star)||_2^2$$

$$(d) = (Z_\star, X) + \frac{\alpha}{2}||A(X - X_\star)||_2^2 \geq 0$$

Here step $(a)$ is due to the strong convexity of $g$. Step $(b)$ is because of the definition of $A^*$. For step $(c)$, we use the first order condition of KKT condition (3) in terms of $g$ and $A$: $A^*(\nabla g)(AX_\star) + C - Z_\star - s_\star I = 0$. The step $(d)$ is due to the complementarity in KKT condition (3) and feasibility of $X$ and $X_\star$. The last inequality $(e)$ is because $Z, X \succeq 0$.

We claim that a feasible matrix $X \in \mathbb{S}^n$ is optimal if and only if $X$ satisfies

$$\begin{align*}
\langle Z_\star, X \rangle &= 0, \\
AX - AX_\star &= 0, \\
\text{tr}(X) &= 1, \quad \text{and} \quad X \succeq 0.
\end{align*}$$

(18)

Indeed, if $X$ is optimal, then $(17)$ and feasibility of $X$ implies $(18)$. Conversely, if $X$ satisfies $(18)$, then it satisfies the KKT condition (3) and hence it is optimal because the problem (1) is convex. Since the optimal solution is unique by assumption, we know the system $(18)$ admits a unique solution. Using Lemma 12 in Section F in the Appendix, we have the relationship between $(\langle Z_\star, X \rangle, ||A(X - X_\star)||_2)$ and the distance to the solution $||X - X_\star||_F$:

$$||X - X_\star||_F^2 \leq \left( 4 + 8 \frac{\sigma^2_{\text{max}}(A)}{\sigma^2_{\text{min}}(A_V)} \right) \frac{\langle Z_\star, X \rangle}{\lambda_{n-r}(Z_\star)} + \frac{4}{\sigma^2_{\text{min}}(A_V)} ||A(X) - b||_2^2.$$ 

(19)

Combining $(17)$ and $(19)$, we see that

$$f(X) - f(X_\star) \geq \gamma ||X - X_\star||_F^2$$

13
\[ \gamma = \min \left\{ \lambda_{n-r^*} \left( Z^* \right), \frac{\alpha \sigma^2_2}{8} \frac{\lambda_{n} \left( H\sigma_2 \right)}{\min \left( \lambda_{r^*} \right)} \right\}. \]

5 Numerics

In this section, we verify numerically a few of our claims in the paper, and show the advantages of the Spectral Frank-Wolfe algorithm when strict complementarity is satisfied and the solution rank is larger than 1. We focus on the quadratic sensing problem [CCG15]. Given a random matrix
\[ U^* \in \mathbb{R}^{n \times r} \]
with \( r^* = 3 \) and Frobenius norm \( \|U^*\|_F = 1 \), we generate Gaussian vectors \( a_i \in \mathbb{R}^{n \times 1}, i = 1, \ldots, m \) and construct quadratic measurement vectors \( y_0(i) = \|U^*_i a_i\|_F^2, i = 1, \ldots, m \). We then add noise \( n = c \|y_0\|_2 v \), where \( c \) is the inverse signal-to-noise ratio and \( v \) is a random unit vector. Our observation is given by \( y = y_0 + n \) and we aim to recover \( U^* \) from \( y \). To this end, we solve the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f(X) := \frac{1}{2} \sum_{i=1}^{m} (a_i^T X a_i - y_i)^2 \\
\text{subject to} & \quad \text{tr}(X) = \tau, X \succeq 0.
\end{align*}
\]

We set \( m = 15nr^* \) in all our experiments.

**Low rankness and strict complementarity.** We verify the low rankness and strict complementarity for \( n = 100, 200, 400 \) and 600. We set \( c = 0.5 \) for the noise level. We also set \( \tau = 0.5 \), since otherwise, the optimal solution will fit the noise and results in a higher rank matrix. Problem (20) is solved via FASTA [GSB14, GSB15]. We found that every optimal solution rank in this case is \( r^* = 3 \), and there is indeed a significant gap between \( \lambda_{n-3} \left( \nabla f(X_i) \right) \) and \( \lambda_n \left( \nabla f(X_i) \right) \), which verifies strict complementarity. More details can be found in Table 5.

| Dimension n | Avg. gap | Avg. recovery error |
|-------------|----------|---------------------|
| 100         | 288.06   | 0.0013              |
| 200         | 505.16   | 0.00064             |
| 400         | 961.09   | 0.00031             |
| 600         | 1358.62  | 0.00021             |

Table 2: Verification of low rankness and strict complementarity. The recovery error is measured by \( \frac{\|\lambda_{n-3} U^*_i U^*_i^T \|_F}{\|U^*_i U^*_i^T \|_F} \). The gap is measured by \( \lambda_{n-3} \left( \nabla f(X_i) \right) - \lambda_n \left( \nabla f(X_i) \right) \). All the results is averaged over 20 iid trials.

**Comparison of algorithms.** We now compare the performance of FW, G-BlockFW, and SpecFW. We follow the setting as the previous paragraph for \( n = 100, 200, 400, \) and 600. We set \( k = 4 \) for both SpecFW and G-BlockFW,
which is larger than $r_\star = 3$. We also set $\eta = 0.4$ and $\beta = 2.5n^2$. The small-scale SDP (10) is solved via FASTA. We plot the relative objective value against both the time and iteration counter in Figure 1. We only present the plot for the case of $n = 600$ here and those for the other cases can be found in Section C in the Appendix. As can be seen from Figure 1(a), SpecFW converges faster in terms of both the iteration counter and the time. The oscillation in the end may be attributed to the sub-problem solver.

**Misspecification of $k$.** We adopt the same setting as before. In this experiment, we set $k = 2$ for both SpecFW and G-BlockFW, which is less than $r_\star = 3$. As can be seen from the Figure 1(b), SpecFW still converges as fast as FW (the two line coincide). G-BlockFW gets stuck around $10^{-1}$ and stop converging to the optimal solution.

6 Conclusion

In this paper, we propose the Spectral Frank-Wolfe algorithm, a novel variant of the classical Frank-Wolfe algorithm, which converges sublinearly for convex smooth optimization problems and converges linearly when strict complementary is satisfied for structural convex optimization problems. We also show that the quadratic growth condition, which is essential for linear convergence of first order methods, holds under strict complementarity.

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9This choice might appear conservative. But we note that $a_i$ has length around $\sqrt{n}$. Hence the operator norm of $A$ is around $\sqrt{mn}$ $\langle a_i a_i^\top, X \rangle \leq \|a_i\|_2^2 \|X\|_F \approx n \|X\|_F$, which suggests $L_f = \|A\|_{op}^2 = n^2m$ as a safe choice. We have already omitted one $m$ factor here for better algorithmic performance.
Figure 1: Comparison of algorithms under different setting. $f^*$ is obtained from the best value of the three methods and FASTA.

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A  Uniqueness assumption

Here we discuss how to adapt our results to a multiple solution setting. First of all, if there are multiple solutions, the strict complementarity condition means that there is a primal optimal solution $X_*$ such that

$$\text{rank}(X_*) + \text{rank}(Z_*) = n.$$  

Thus we should set $r_* = \text{maximal rank among all primal solutions.}$ Denote the set of primal optimal solution of Problem (1) as $X_*$. Quadratic growth in this situation is understood as

$$f(X) - f(X_*) \geq \gamma \inf_{X_* \in X_*} \|X - X_*\|_F =: \text{dist}(X, X_*),$$

for any $X \succeq 0$ and $\text{tr}(X) = 1$. Now due to strict complementarity, we still have $r_* = k_*$ (dual solution $Z_*$ is unique as shown in the next section). Theorem 3 can be now be proved in the exactly same way by considering the nearest $X_* \in X_*$ to $X$ without the uniqueness assumption. To prove Theorem 6, the argument follows exactly as the main proof by considering the nearest $X_* \in X_*$ to $X$, and replacing Lemma 12 by Lemma 13. In this case, the parameter $\gamma$ of quadratic growth is

$$\gamma = \min \left\{\frac{\lambda_{n-r_*(Z_*)}}{4+8\sigma_\infty \|A\|_{2\infty}} a \mu^2}{\lambda_{n-r_*(Z_*)}} \right\} \text{ where } \mu := \sup \{a \geq 0 | a \cdot \text{dist}(X, X_*) \leq \|\tilde{A}(X) - b\|_2 \text{ for all } X \in C_{r_*(Z_*)}\} \text{ and is indeed positive using Lemma 13.}$$

B  Lemmas for Section 2

Lemma 7. The dual solution $(Z_*, s_*)$ of Problem (1) is unique even if the primal solution is not unique.

Proof. We first show that for any primal solution $X_*$, its gradient $\nabla f(X_*)$ is the same. Using $\beta$-smoothness of $f$ (the constant $\beta$ can be taken to be $\|A\|_{2\infty}^2 L_g$), we have for any optimal $X_*$ and $X'_*$

$$\langle X_* - X'_*, \nabla f(X_*) - \nabla f(X'_*) \rangle \geq \frac{1}{\beta} \|\nabla f(X_*) - \nabla f(X'_*)\|_F^2. \quad (21)$$

Since $X_*$ and $X'_*$ are optimal solution, we have the following two inequalities using the optimality

$$\langle X_* - X'_*, \nabla f(X_*) \rangle \leq 0, \quad (22)$$

$$\langle X'_* - X_*, \nabla f(X'_*) \rangle \leq 0. \quad (23)$$

Combining the inequalities (21), (22), and (23), we have

$$\|\nabla f(X_*) - \nabla f(X'_*)\|_F \leq 0 \implies f(X_*) = f(X'_*). \quad (24)$$
This shows that $\nabla f(X_\star)$ is unique. Now for any $Z_\star, s_\star$ and $Z'_\star, s'_\star$ satisfying the KKT condition, we have

$$\nabla f(X_\star) + C = Z_\star + s_\star I$$

$$= Z'_\star + s'_\star I$$

$$\implies Z_\star - Z'_\star = (s'_\star - s_\star)I.$$  \hspace{1cm} (25)

Now using complementarity in step (a) and feasibility of $X_\star$ in step (b):

$$0 \overset{(a)}{=} \langle Z_\star - Z'_\star, X_\star \rangle = (s'_\star - s_\star)(I, X_\star)$$

$$\overset{(b)}{=} (s'_\star - s_\star)$$

$$\implies s_\star = s'_\star, \quad \text{and} \quad Z_\star = Z'_\star.$$  \hspace{1cm} (26)

Hence the dual solution $Z_\star$ and $s_\star$ is unique.

**Lemma 8.** For almost all $C$, the strict complementarity condition holds for (1).

**Proof.** Let us first define indicator function: for any given $D \subset \mathbb{R}^n$, we define

$$\chi_D(x) = \begin{cases} 0, & x \in D \\ +\infty, & x \notin D. \end{cases}$$

Also denote the relative interior of a set $D$ as $\text{relint}(D)$. We utilize the result in [DL11, Corollary 3.5], that for almost all $C$, we have

$$-C \in \text{relint}(\partial (g(AX))$$

$$+ \chi_{\{X | \text{tr}(X) = 1\}}(X) + \chi_{\{X \succeq 0\}}(X))(X_\star))$$

$$\overset{(a)}{=} \text{relint}(A^*(\nabla g)(AX_\star) + \{sI | s \in \mathbb{R}\}$$

$$+ \{-Z | Z \succeq 0, \text{range}(Z) \subset \text{nullspace}(X_\star)\})$$

$$\overset{(b)}{=} A^*(\nabla g)(AX_\star) + C + \{sI | s \in \mathbb{R}\}$$

$$+ \{-Z | Z \succeq 0, \text{range}(Z) = \text{nullspace}(X_\star)\}).$$  \hspace{1cm} (27)

Here we use the sum rule in step (a) as $\frac{1}{n}I$ is in $\{X | \text{tr}(X) = 1\}$ and the interior of $\{X | X \succeq 0\}$. In step (b), we use the sum rule of relative interior. Hence, there is some $s_\star$ and $Z_\star$ such that

$$\text{range}(Z_\star) = \text{nullspace}(X_\star)$$

$$\implies \langle Z_\star, X_\star \rangle = 0, \quad \text{and}$$

$$\text{rank}(Z_\star) + \text{rank}(X_\star) = n.$$  \hspace{1cm} (28)

and

$$A^*(\nabla g)(AX_\star) + C = Z_\star + s_\star I.$$  \hspace{1cm} (29)

We thus conclude $(Z_\star, s_\star)$ satisfies the KKT condition (3), and strict complementarity holds.  \hspace{1cm} $\square$

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C SpecFW: minimizing an upper bound of $f(\eta X_t + VSV^\top)$.

When the function $f$ is not fully known or gradient might be hard to query, we may consider the following subproblem instead: solve

$$
\begin{align*}
\text{minimize} & \quad g(A X_t) \\
& \quad + \langle A(\eta X_t + VSV^\top) - AX_t, (\nabla g)(AX_t) \rangle \\
& \quad + \frac{Lg}{2} \| A(\eta X_t + VSV^\top) - AX_t \|_2^2 \\
& \quad + \langle C, \eta X_t + VSV^\top \rangle \\
\text{subject to} & \quad \eta + \text{tr}(S) = 1, \ S \succeq 0, \text{ and } \eta \geq 0.
\end{align*}
$$

(29)

with decision variable $S$ and $\eta$. Then set $X_{t+1} = \eta X_t + VSV^\top$ for the optimal $\eta$ and $S$.

The above formulation enjoys the advantage of efficient computation in terms of time when $m$ is small and the linear map $A$ and $\langle C, \cdot \rangle$ are easy to apply to low rank matrices. One may also save $AX_t$ during the process to avoid forming $X_t$ and sketching $X_t$ using idea from [TYUC17] for storage purpose.

One could also consider solving

$$
\begin{align*}
\text{minimize} & \quad f(X_t) \\
& \quad + \langle \eta X_t + VSV^\top - X_t, \nabla f(X_t) \rangle \\
& \quad + \frac{L_f}{2} \| X_t - (\eta X_t + VSV^\top) \|_F \\
\text{subject to} & \quad \eta + \text{tr}(S) = 1, \ S \succeq 0, \text{ and } \eta \geq 0.
\end{align*}
$$

(30)

Then set $X_{t+1} = \eta X_t + VSV^\top$ for the optimal $\eta$ and $S$. Here $L_f$ is the Lipschitz constant of $\nabla f$. This method requires to store $X_t$ in each iteration though.

D Combination with matrix sketching idea in [TYUC17]

When $m$ is on the order $n$, we can employ the matrix sketching idea developed in [TYUC17] and [YUTC17] to achieve storage reduction. We note that if we store $A(X_t) = z_t$ and $c_t = \langle C, X_t \rangle$ at each iteration, then we have no problem in doing the small-scale SDP (10), as $f(\eta X_t + VSV^\top) = g(\eta (AX_t) + A(VSV^\top)) + \eta (C, X_t) + (C, VSV^\top)$. If $A$ and inner product with $C$ can be applied to low rank matrices efficiently, then updating $z_t$ and $c_t$ is not hard due to linearity of our updating scheme $X_{t+1} = \eta X_t + VSV^\top$.

Now we explain how to omit storing the iterate $X_t$. First, we draw two matrices with independent standard normal entries

$$
\begin{align*}
\Psi & \in \mathbb{R}^{n \times k} \quad \text{with} \quad k = 2r + 1; \\
\Phi & \in \mathbb{R}^{l \times n} \quad \text{with} \quad l = 4r + 3;
\end{align*}
$$

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Here $r$ is chosen by the user. It either represents the estimate of the true rank of the primal solution or the user’s computational budget in dealing with large matrices.

We use $Y^C_t$ and $Y^R_t$ to capture the column space and the row space of $X_t$:

$$Y^C_t = X_t \Psi \in \mathbb{R}^{n \times k}, \quad Y^R_t = \Phi X_t \in \mathbb{R}^{l \times n}. \quad (31)$$

Hence we initially have $Y^C_0 = 0$ and $Y^R_0 = 0$. Notice that SpecFW does not observe matrix $X_t$ directly. Rather, it observes a stream of rank $k$ updates

$$X_{t+1} = VS V^T + \eta X_t,$$

where $V \in \mathbb{R}^n \times k$ and $S \in S^k$.

In this setting, $Y^C_{t+1}$ and $Y^R_{t+1}$ can be directly computed as

$$Y^C_{t+1} = VS (V^T \Psi) + \eta Y^C_t \in \mathbb{R}^{n \times k}, \quad (32)$$

$$Y^R_{t+1} = (\Psi V) SV^T + \eta Y^R_t \in \mathbb{R}^{l \times n}. \quad (33)$$

This observation allows us to form the sketch $Y^C_t$ and $Y^R_t$ from the stream of updates.

We then reconstruct $X_t$ and get the reconstructed matrix $\hat{X}_t$ by

$$Y^C_t = Q_t R_t, \quad B_t = (\Phi Q_t) Y^R_t, \quad \hat{X}_t = Q_t [B_t]_r, \quad (34)$$

where $Q_t R_t$ is the QR factorization of $Y^C_t$ and $[\cdot]_r$ returns the best rank $r$ approximation in Frobenius norm. Specifically, the best rank $r$ approximation of a matrix $Z$ is $U \Sigma V^*$, where $U$ and $V$ are right and left singular vectors corresponding to the $r$ largest singular values of $Z$ and $\Sigma$ is a diagonal matrix with $r$ largest singular values of $Z$. In actual implementation, we may only produce the factors $(QU, \Sigma, V)$ defining $\hat{X}_t$ in the end instead of reconstructing $\hat{X}_t$ in every iteration. We refer the reader to [TYUC17, Theorem 5.1] for the theoretical guarantees on the reconstruction matrix $\hat{X}_t$.

Hence we can avoid the forming a new iterate procedure in SpecFW. We remark that the reconstructed matrix $\hat{X}_t$ is not necessarily positive semidefinite. However, this suffices for the purpose of finding a matrices close to $X_t$. More sophisticated procedure is available for producing a positive semidefinite approximation of $X_t$ [TYUC17, Section 7.3].

## E Proofs for Section 3

We first give the detailed calculation of the derivation for (12).

**Continuation of proof of Theorem 3.** We need to choose $\xi \in [0, 1]$ so that $1 - \xi + \frac{\xi^2 \beta}{\gamma}$ is minimized while keeping $\xi^2 \beta - \frac{\lambda_{n-r} (Z_\star)}{6} \leq 0$. For $\xi^2 \beta - \frac{\lambda_{n-r} (Z_\star)}{6} \leq 0$, we need $\xi \leq \frac{\lambda_{n-r} (Z_\star)}{6 \beta}$. The function $q(\xi) = 1 - \xi + \frac{\xi^2 \beta}{\gamma}$ is decreasing for $\xi \leq \frac{\gamma}{2 \beta}$ and increasing for $\xi \geq \frac{\gamma}{2 \beta}$. If $\frac{\gamma}{2 \beta} \leq \frac{\lambda_{n-r} (Z_\star)}{6 \beta}$, then we can pick $\xi = \frac{\gamma}{2 \beta}$, and $q(\xi) =$
Then for any eigenvalue. Also let \( v \) responding orthogonal projection note the eigenspace corresponding to the last \( r \). Let \( V \) and \( \text{tr} \) \( Y \) satisfies. Consider the eigenvalue decomposition of \( X \), we shall prove Lemma 5 in this section. We restate Lemma 5 in a self-contained way.

**Lemma 9.** Suppose \( Y \in \mathbb{S}^n \) with eigenvalues \( \lambda_1(Y) \geq \cdots \geq \lambda_n(Y) \), and \( \lambda_{n-r}(Y) - \lambda_{n-r+1}(Y) \geq \delta \). Here \( \lambda_i(\cdot) \) denote the operator of taking the \( i \)-th largest eigenvalue. Also let \( v_1, \ldots, v_n \) be the corresponding orthonormal eigenvectors. Denote the eigenspace corresponding to the last \( r \) eigenvalues of \( Y \) as \( V_{Y,r} \) and the corresponding orthonormal projection \( P_{Y,r} : \mathbb{R}^n \to \mathbb{R}^n \) which is also a matrix in \( \mathbb{R}^{n \times n} \). Let \( V_{Y,r} \in \mathbb{R}^{n \times r} \) formed by the last \( r \) many eigenvectors \( v_{n-r+1}, \ldots, v_n \) which represents the eigenspace \( V_{Y,r} \). Define \( C_r(Y) = \{ V_{Y,r} S V_{Y,r}^\top \mid S \succeq 0, \text{tr}(S) = 1 \} \).

Then for any \( X \in \mathbb{S}^n \) with \( \text{tr}(X) = 1, X \succeq 0 \), there is some \( W \in C_r(Y) \) such that

\[
\langle X - W, Y \rangle \geq \frac{\delta}{2} \|X - W\|_2^2.
\]

**Remark 10.** We note that as long as \( \text{range}(V) = \text{range}(V_{Y,r}) \) for some matrix \( V \in \mathbb{R}^{n \times r} \) with orthonormal columns, the set \( C_r(Y) \) is the same as \( \{ V S V^\top \mid S \succeq 0, \text{tr}(S) = 1 \} \).

**Proof of Lemma 5.** We first decompose \( X \) by

\[
X = (X - P_{Y,r}XP_{Y,r}) + P_{Y,r}XP_{Y,r}.
\]

Note that \( P_{Y,r} = P_{Y,r}^\top \), so \( X_2 = P_{Y,r}XP_{Y,r} \) is still symmetric. Let \( 1 - \epsilon = \text{tr}(P_{Y,r}XP_{Y,r}) \). Since \( \text{tr}(X) = 1 \), we have \( \epsilon = \text{tr}(X - P_{Y,r}XP_{Y,r}) \). We have \( \epsilon \in [0, 1] \) as \( \text{tr}(P_{Y,r}XP_{Y,r}) = \langle X, P_{Y,r}XP_{Y,r} \rangle \leq \|P_{Y,r}\|_{\text{op}} \text{tr}(X) \leq 1 \) where step \( (a) \) is due to Hölder’s inequality.

Consider the eigenvalue decomposition of \( X_2 = V_2 \Lambda_2 V_2^\top \), where \( V_2 \in \mathbb{R}^{n \times r} \) and \( \Lambda_2 \in \mathbb{S}^r \) with all diagonal nonnegative. Here the column space of \( V_2 \) satisfies range(\( V_2 \)) = \( V_{Y,r} \).

Because \( P_{Y,r}XP_{Y,r} = X_2 \) is a member in \( C_r(Y) \), we know there is an \( W \in C_r(Y) \) such that \( W = V_2 \Lambda W V_2^\top \) where \( \Lambda W \in \mathbb{S}^r \) has nonnegative diagonal with \( \text{tr}(\Lambda W) = 1 \) and the difference matrix \( \Delta = \Lambda W - \Lambda_2 \) has nonnegative entries. We also have \( \text{tr}(\Delta) = \epsilon \), as the trace of both \( \Lambda W \) and \( X \) are one.
With such choice of $W$, let us now analyze $\langle X - W, Y \rangle$:

$$
\langle X - W, Y \rangle = \langle X_1, Y \rangle + \langle X_2 - W, Y \rangle
$$

$$
= \langle X - P_{Y,r}X P_{Y,r}, \sum_{i=1}^{n} \lambda_i(Y)v_i v_i^T \rangle
$$

$$
- \langle V_2 \Delta V_2^T, \sum_{i=1}^{n} \lambda_i(Y)v_i v_i^T \rangle .
$$

(35)

The first term $R_1 = \langle X - P_{Y,r}X P_{Y,r}, \sum_{i=1}^{n} \lambda_i(Y)v_i v_i^T \rangle$ satisfies

$$
\langle X - P_{Y,r}X P_{Y,r}, \sum_{i=1}^{n} \lambda_i(Y)v_i v_i^T \rangle 
\equiv \sum_{i=1}^{n} \lambda_i(Y)v_i^T X v_i - \sum_{i=n-r+1}^{n} \lambda_i(Y)v_i^T X v_i 
\equiv \sum_{i=1}^{n-r} \lambda_i(Y)v_i^T X v_i
$$

$$
\geq (\lambda_{n-r+1}(Y) + \delta) \sum_{i=1}^{n-r} v_i^T X v_i.
$$

Here in step (a) we use the fact that $P_{Y,r}v_i = v_i$ for $i = n - r + 1, \ldots, n$ and is zero for other $v_i$. In step (b), we use the assumption that $\lambda_{n-r} - \lambda_{n-r+1} \geq \delta$ and each $v_i^T X v_i \geq 0$ as $X \succeq 0$. We note that $\sum_{i=1}^{n-r} v_i^T X v_i$ satisfies

$$
\sum_{i=1}^{n-r} v_i^T X v_i = \text{tr}(X \left( \sum_{i=1}^{n-r} v_i v_i^T \right)) 
\equiv \text{tr}(X(I - P_{Y,r})) 
\equiv \text{tr}(X) - \text{tr}(P_{Y,r}X P_{Y,r}) = \epsilon.
$$

Here step (a) uses the $P_{Y,r} = V_{Y,r}V_{Y,r}^T$ and we use $P_{Y,r}^2 = P_{Y,r}$ and cyclic property of trace in step (b).

Now let us analyze the second term $R_2$:

$$
R_2 = \langle V_2 \Delta V_2^T, \sum_{i=1}^{n} \lambda_i(Y)v_i v_i^T \rangle
$$

$$
\equiv \langle V_2 \Delta V_2^T, \sum_{i=n-r+1}^{n} \lambda_i(Y)v_i v_i^T \rangle.
$$

Here we use the fact that $V_2^T v_i = 0$ for all $v_i, i = 1, \ldots, n-r$. Since $V_{Y,r}$ and $V_2$ are both orthonormal representation of $V_{Y,r}$, we know there is an orthonormal matrix
\( O \in \mathbb{R}^{r \times r} \) such that \( V_{Y,r} = V_2 O \). Define the linear operator \( \text{diag} : \mathbb{S}^n \to \mathbb{R}^n \), which takes the diagonal of a matrix. Let \( \Lambda_{Y,r} = \text{diag}^\ast (\lambda_{n-r+1}(Y), \ldots, \lambda_n(Y)) \), we see \( R_2 \) further equals to

\[
R_2 = \text{tr} \left( V_2 \Delta V_2^T V_2 O \Lambda_{Y,r} O^T V_2^T \right)
\]

\[
\overset{(a)}{=} \text{tr} \left( \Delta O \Lambda_{Y,r} O^T \right)
\]

\[
\overset{(b)}{\leq} \epsilon \lambda_{n-r+1}(Y).
\]

Here we use the cyclic property in step \( (a) \) and the step \( (b) \) is an easy consequence of \( \Delta \) has nonnegative diagonal and Von Neumann’s trace inequality: for symmetric matrices \( A, B \in \mathbb{S}^r \), \( \langle A, B \rangle \leq \sum_{i=1}^{r} \lambda_i(A) \lambda_i(B) \). Combining pieces, we find that

\[
\langle X - W, Y \rangle \geq (\lambda_{n-r+1}(Y) + \delta) \epsilon - \epsilon \lambda_{n-r+1}(Y) = \delta \epsilon.
\]

Now we turn to analyzing the term \( \|X - W\|_F^2 \). Using \( \langle X_1, X_2 \rangle = 0, \langle X_1, W \rangle = 0 \), we find that

\[
\|X - W\|_F^2 = \|X_1\|_F^2 + \|X_2 - W\|_F^2.
\]

The second term \( \|X_2 - W\|_F^2 \) satisfies

\[
\|X_2 - W\|_F^2 = \|V_2 \Delta V_2^T\|_F^2 = \sum_{i=1}^{r} \Delta_i^2 \leq \left( \sum_{i=1}^{r} \Delta_i \right)^2 = \epsilon^2.
\]

If we write \( X \) in terms of the coordinates given by \( V_2 \) and its orthogonal compliment say \( V_1 \), then in this new coordinate \( V = [V_1, V_2] \):

\[
V^T X V = \begin{bmatrix} A & B \\ B^T & V_2^T X_2 V_2 \end{bmatrix}, \quad \text{and} \quad V^T X_1 V = \begin{bmatrix} A & B \\ B & 0 \end{bmatrix}.
\]

Then \( \text{tr}(X_1) = \text{tr}(A) \). Lemma 11 implies that

\[
\|B\|_F^2 \leq \text{tr}(X_2) \text{tr}(A) = \epsilon(1 - \epsilon) = \epsilon - \epsilon^2.
\]

Hence \( \|X_1\|_F^2 = \|A\|_F^2 + 2\|B\|_F^2 \leq (\text{tr}(A))^2 + 2\epsilon - 2\epsilon^2 = -\epsilon^2 + 2\epsilon \). Combining pieces and \( \epsilon \in [0, 1] \), we find that

\[
\|X - W\|_F^2 \leq 2\epsilon = \frac{2}{\delta} \delta \epsilon \leq \frac{2}{\delta} \langle X - W, Y \rangle
\]

\[
\implies \langle X - W, Y \rangle \geq \frac{\delta}{2} \|X - W\|_F^2.
\]

Lemma 11. Suppose \( Y = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \succeq 0 \). Then \( \|A\|_{op} \text{tr}(D) \geq \|BB^T\|_* = \text{tr}(BB^T) = \|B\|_F^2 \).
Proof. For any $\epsilon > 0$, denote $A_{\epsilon} = A + \epsilon I$ and $Y_{\epsilon} = \begin{bmatrix} A_{\epsilon} & B \\ B^* & D \end{bmatrix}$. We know $Y_{\epsilon}$ is psd, as is its Schur complement $D - B^TA_{\epsilon}^{-1}B \succeq 0$ with $	ext{tr}(D) - \text{tr}(A_{\epsilon}^{-1}BB^*) \geq 0$.

Von Neumann’s lemma for $A_{\epsilon}$, $BB^*$ shows $\text{tr}(A_{\epsilon}^{-1}BB^*) \geq \frac{1}{\|A_{\epsilon}\|_{\text{op}}} \|BB^*\|_*$. Use this with the previous inequality to see $\text{tr}(D) \geq \frac{1}{\|A_{\epsilon}\|_{\text{op}}} \|BB^*\|_*$. Multiply by $\|A_{\epsilon}\|_{\text{op}}$ and let $\epsilon \to 0$ to complete the proof. \hfill \Box

\section{F Lemmas for Section 4}

We first give a self-contained proof for the second case of Theorem 6.

Proof of second case of Theorem 6. For any feasible $X$ and the optimal solution $X_\star$, we have

\[ f(X) - f(X_\star) \geq \langle \nabla f(X_\star), X - X_\star \rangle \]

\[ \overset{(b)}{=} \langle Z_\star + s_\star I, X - X_\star \rangle \]

\[ \overset{(c)}{=} \langle Z_\star, X - X_\star \rangle. \]

Here step (a) is due to the convexity of $f$. For step (b), we uses the first order condition of KKT condition (3). The step (c) is due to feasibility of $X$ and $X_\star$.

Since $Z_\star$ has rank $n - 1$, using strict complementarity, we reach that any optimal solution $X_\star$ has rank 1 with $\text{range}(X_\star) = \text{nullspace}(Z_\star)$. Thus any optimal solution $X_\star$ is of the form $X_\star = \xi vv^T$, $v$ is the non-zero unit vector in the null space of $Z_\star$, and $\xi$ is a nonnegative scaler. Since $X_\star$ has to be feasible, the constraint $\text{tr}(X_\star) = 1$ implies that $\xi = 1$ and hence the solution $X_\star$ is unique. The same argument implies that the set $C_1(Z_\star) = \{X_\star\}$. Hence using Lemma 5 and $\lambda_n(Z_\star) = 0$, we see that

\[ f(X) - f(X_\star) \geq \langle Z_\star, X - X_\star \rangle \geq \frac{\lambda_{n-1}(Z_\star)}{2}\|X - X_\star\|_F^2. \]

Next, we establish the lemma that is core to the proof of Theorem 6 under the assumption of uniqueness.

Lemma 12. Suppose the following system admits a unique solution $X_\star$ with rank $r_\star$:

\[ \langle Z_\star, X_\star \rangle = 0, AX = b, \quad \text{and} \quad X \succeq 0, \]  

(36) for a $Z_\star \succeq 0$ such that $\text{rank}(Z_\star) + \text{rank}(X_\star) = n$, a linear map $A : \mathbb{S}^n \to \mathbb{R}^m$, and a vector $b \in \mathbb{R}^m$. Further suppose that $AX = b \implies \text{tr}(X) = 1$. Then for
any $X \succeq 0$ with $\text{tr}(X) = 1$, we have
\[
\|X - X^\star\|_F^2 \leq \left( 4 + 8 \frac{\sigma_{\max}(A)}{\sigma_{\min}(A_V)} \right) \frac{\langle Z^\star, X \rangle}{\lambda_{n-r^\star}(Z^\star)}
\tag{37}
+ 4 \frac{\sigma_{\min}^2(A_V)}{\sigma_{\min}(A_V)} \|A(X) - b\|_2^2.
\]

**Proof.** Let $V \in \mathbb{R}^{n \times r^\star}$ be a matrix with orthonormal columns corresponding to the eigenspace $V$ of $X^\star$ of positive eigenvalues. Then $X^\star$ can be written as $X^\star = VS^\star V^\top$ for some $S^\star \in \mathbb{S}^{r^\star}$ such that $S^\star \succ 0$. We claim that the linear map $A_V$ defined as follows is injective:
\[
A_V : \mathbb{S}^{r^\star} \to \mathbb{R}^m
\]
\[
S \mapsto A(VSV^\top).
\]
Suppose not, then there is some nonzero $S_0 \in \mathbb{S}^{r^\star}$ such that $A_V(S_0) = 0$. Then $V(\alpha S_0 + S^\star)V^\top$ also satisfies the system (36) for all small enough $\alpha$. Hence we see that for any $S \in \mathbb{S}^r$
\[
\|VSV^\top - X^\star\|_F \leq \frac{1}{\sigma_{\min}(A_V)} \|A(VSV^\top) - A(X^\star)\|_2
\tag{38}
= \frac{1}{\sigma_{\min}(A_V)} \|A(VSV^\top) - b\|_2.
\]
Here $\sigma_{\min}(A_V) = \min_{\|S\|_F = 1} \|A_V(S)\|_2 > 0$.

Using strict complementarity on $Z^\star$ and $X^\star$, we know $V$ is also a representation of the null space of the $Z^\star$. Using Lemma [3] we know there is some $W = VSV^\top \in C_r(Z^\star)$ such that
\[
\langle X, Z^\star \rangle \overset{(a)}{=} \langle X - W, Z^\star \rangle \geq \frac{\lambda_{n-r^\star}(Z^\star)}{2} \|X - W\|_F^2, \tag{39}
\]
where step (a) is because $\lambda_{n-r^\star+1}(Z^\star) = \cdots = \lambda_n(Z^\star) = 0$. We note if $r^\star = 1$, then $C_r(Z^\star)$ has $X^\star$ as its only element, as $\text{tr}(X) = 1$ and we are done.

We can bound $\|X - X^\star\|_F^2$ by
\[
\|X - X^\star\|_F^2 \overset{(a)}{\leq} 2\|X - W\|_F^2 + 2\|W - X^\star\|_F^2 \tag{40}
\overset{(b)}{\leq} 2\|X - W\|_F^2 + \frac{2}{\sigma_{\min}(A_V)} \|A(W) - b\|_2^2.
\]
Here we use triangle inequality and basic inequality $(a + c)^2 \leq 2a^2 + 2c^2$ for any real $a, c$ in step (a). In step (b), we use (38).

We can further bound the term $\|A(W) - b\|_2$ by
\[
\|A(W) - b\|_2 = \|A(W - X) + A(X) - b\|_2 \leq \|A(W - X)\|_2 + \|A(X) - b\|_2. \tag{41}
\]


Now combining (40), (41) and \((a + c)^2 \leq 2a^2 + 2c^2\) for any \(a, c \in \mathbb{R}\) in the following step (a), we see
\[
\|X - X^*\|_F^2 \leq 2\|X - W\|_F^2 + \frac{4\|A(W - X)\|_2^2}{\sigma_{\min}^2(\mathcal{A}_V)}
+ \frac{4}{\sigma_{\min}^2(\mathcal{A}_V)}\|A(X) - b\|_2^2
\leq \left(2 + \frac{4\sigma_{\max}^2(\mathcal{A})}{\sigma_{\min}^2(\mathcal{A}_V)}\right)\|X - W\|_F^2
+ \frac{4}{\sigma_{\min}^2(\mathcal{A}_V)}\|A(X) - b\|_2^2.
\]
Finally using (39) to bound \(\|X - W\|_F\), we reached the inequality we want to prove:
\[
\|X - X^*\|_F^2 \leq \left(4 + 8\frac{\sigma_{\max}^2(\mathcal{A})}{\sigma_{\min}^2(\mathcal{A}_V)}\right)\frac{\langle Z^*, X \rangle}{\lambda_{n-r^*}(Z^*)}
+ \frac{4}{\sigma_{\min}^2(\mathcal{A}_V)}\|A(X) - b\|_2^2.
\]

We now establish a lemma to handle the general case that the solution might not be unique. For a convex closed set \(\mathcal{X}_*\), we define the distance to it as
\[
\text{dist}(X, \mathcal{X}_*) := \inf_{X^* \in \mathcal{X}_*} \|X - X^*\|_F.
\]

**Lemma 13.** Denote the solution set of the following system as \(\mathcal{X}_*\):
\[
\langle Z^*, X \rangle = 0, \quad AX = b, \quad \text{and} \quad X \succeq 0,
\]
for a \(Z^* \succeq 0\), a linear map \(\mathcal{A} : \mathbb{S}^n \rightarrow \mathbb{R}^m\), and a vector \(b \in \mathbb{R}^m\). Suppose the system (36) admits a solution \(X^*_0\) with rank \(r^*_0 \geq 1\) such that rank\((Z^*) + \text{rank}(X^*_0) = n\). Further suppose that \(AX = b \implies \text{tr}(X) = 1\). Then the constant \(\mu := \sup\{a \geq 0 \mid a \cdot \text{dist}(X, \mathcal{X}_*) \leq \|A(X) - b\|_2 \text{ for all } X \in \mathcal{C}_{r^*_0}(Z^*)\}\) is positive, and for any \(X \succeq 0\) with \(\text{tr}(X) = 1\), we have
\[
\text{dist}(X, \mathcal{X}_*)^2 \leq \left(4 + 8\frac{\sigma_{\max}(\mathcal{A})}{\mu}\right)\frac{\langle Z^*, X \rangle}{\lambda_{n-r^*}(Z^*)}
+ \frac{4}{\mu^2}\|A(X) - b\|_2^2.
\]

**Proof.** Let \(V \in \mathbb{R}^{n \times r^*_0}\) be a matrix with orthonormal columns corresponding to the eigenspace \(V\) of \(r^*_0\) zero eigenvalues. Consider the linear map \(\mathcal{A}_V\):
\[
\mathcal{A}_V : \mathbb{S}^{r^*_0} \rightarrow \mathbb{R}^m
\]
\[
S \mapsto \mathcal{A}(VSV^T).
\]
The key replacement of multiple solution setting is to establish an inequality similar to (38), which depicts the injectivity of \( A_V \) for unique solution setting.

Define the solution set \( S \subset S^\ast \) of the following system:

\[
A_V(S) = b, \quad S \succeq 0.
\]

(44)

Note that any \( S \in S \) satisfies that \( VSV^\top \in X^\ast \). Conversely, for any \( X \in X^\ast \), it can be written as \( X = VSV^\top \) for some \( S \in S^\ast \) such that \( S \succeq 0 \) and \( A_V(S) = b \). Hence we have \( X^\ast = \{ X \mid X = VSV^\top, S \in S \} \).

Now if we take the \( X^0 \in X^\ast \) such that \( \text{rank}(Z^\ast) + \text{rank}(X^0) = n \), then \( X^0 = VSV^\top \) for some \( S^0 \in S^\ast \) such that \( S^0 \succeq 0 \). This means the system (44) satisfies the condition in Corollary 3 in \cite{BBL99}. By applying this corollary to (44), we know there is a \( \mu > 0 \) such that for all \( S \succeq 0 \) and \( \text{tr}(S) = 1 \),

\[
\text{dist}(S, S) \leq \frac{1}{\mu} \| A_V S - b \|_2.
\]

(45)

Translating the inequality to the space \( L = \{ X \in S \mid X = VSV^\top \text{ for some } S \in S^\ast \} \), we have for all \( X \succeq 0 \), \( \text{tr}(X) = 1 \), and \( X \in L \), i.e., \( X \in C_{r^\ast}(Z^\ast) \):

\[
\text{dist}(X, X^\ast) \leq \frac{1}{\mu} \| A(X) - b \|_2.
\]

(46)

This is our replacement of (38) in Lemma 12.

Following the proof of Lemma 12, we know there is some \( W = VSV^\top \in C_{r^\ast}(Z^\ast) \) such that

\[
\langle X, Z^\ast \rangle = \langle X - W, Z^\ast \rangle \geq \frac{\lambda_{n-r^\ast}(Z^\ast)}{2} \| X - W \|_F^2.
\]

(47)

To bound \( \text{dist}(X, X^\ast) \), we pick an \( X^\ast \in X^\ast \) such that it is nearest to \( W \) (note \( X^\ast \) is compact as \( A(X) = b \) implies \( \text{tr}(X) = 1 \)). Then we have

\[
\text{dist}(X, X^\ast)^2 \leq \| X - X^\ast \|_F^2.
\]

(a)

\[
\leq \frac{1}{2} \| X - W \|_F^2 + \| W - X^\ast \|_F^2.
\]

(48)

(b)

\[
\leq \| X - W \|_F^2 + \frac{2}{\mu^2} \| A(W) - b \|_2^2.
\]

(49)

Here we use triangle inequality and basic inequality \((a + c)^2 \leq 2a^2 + 2c^2\) for any real \( a, c \) in step (a). In step (b), we use (38). The rest of the proof is exactly the same as those in Lemma 12.

The following Lemma establishes the linear convergence of G-BlockFW under quadratic growth condition.

**Lemma 14.** Suppose \( f \) of Problem (1) is \( \beta \) smooth and Problem (1) satisfies quadratic growth with parameter \( \gamma \). If \( \eta = \frac{\gamma}{\beta} \) and \( k \geq r^\ast = \text{rank}(X^\ast) \), where \( X^\ast \),
is an optimal solution of Problem (1), then the generalized Block FW converges linearly:

\[ h_{t+1} \leq (1 - \frac{\gamma}{2\beta})h_t, \]

where \( h_t = f(X_t) - f(X_\star) \) for each \( t \).

**Proof.** Denote \( \hat{Y} = V \text{diag}(\Lambda)V^\top \). The Lipschitz smoothness of \( f \) shows that

\[
f(X_{t+1}) \leq f(X_t) + \eta(\hat{Y} - X_t, \nabla f(X_t)) + \frac{\eta^2 \beta}{2} \| \hat{Y} - X_t \|_F^2.
\]  

(50)

Using a similar argument as [AZHHL17, Lemma 3.1], we have

\[
\hat{Y} = \arg \min_{Y \in S_n, \text{rank}(Y) \leq r} \eta(\hat{Y} - X_t, \nabla f(X_t)) + \frac{\eta^2 \beta}{2} \| \hat{Y} - X_t \|_F^2.
\]

Hence, we can replace \( \hat{Y} \) in (50) by \( X_\star \) in the following step (a),

\[
f(X_{t+1}) \overset{(a)}{\leq} f(X_t) + \eta(\hat{Y} - X_t, \nabla f(X_t)) + \frac{\eta^2 \beta}{2} \| \hat{Y} - X_t \|_F^2
\]

\[
\overset{(b)}{\leq} f(X_t) - \eta(f(X_t) - f(X_\star)) + \frac{\eta^2 \beta}{2\gamma} (f(X_t) - f(X_\star)),
\]  

(51)

where step (b) is due to the quadratic growth of Problem (1). Now subtract both sides by \( f(X_\star) \), and let \( h_t = f(X_t) - f(X_\star) \) for each \( t \), we find that

\[ h_{t+1} \leq (1 - \eta + \frac{\eta^2 \beta}{2\gamma})h_t. \]

Our choice \( \eta = \frac{\gamma}{\beta} \) set \( (1 - \eta + \frac{\eta^2 \beta}{2\gamma}) = 1 - \frac{\gamma^2}{2\beta} \) which is what we desired. \( \square \)

**G Additional Numerics**

We include extra numerics for \( n = 100, 200, 400 \) in Figure 2. As can be seen, SpecFW in these cases are a bit slower than G-BlockFW when \( \tau = 0.5 \) and \( c = 0.5 \). SpecFW is as good as FW when \( k \) is miss specified.

**What if \( \nabla f(X_\star) = 0 \)?** Here we also discuss an interesting situation that \( c = 0 \), and \( \tau = 1 \), then we see \( X_\star = U U^\top \) is an optimal solution and gradient in this case is 0. Such situation means strict complementarity fails and the small perturbation to \( \tau \) will result in a higher-rank solution, meaning the convex relaxation (20) is ill-posed for the purpose of low-rank matrix recovery [Lemma 2 Gar19a]. Indeed, this is where SpecFW is not advantageous comparing to G-BlockFW as shown in Figure 4. \( \tau = 1 \) and \( c = 0 \).
Figure 2: Comparison of algorithms under $\tau = \frac{1}{2}$ and noise level $c = 0.5$. 
Figure 3: Comparison of algorithms under $\tau = \frac{1}{2}$, noise level $c = 0.5$, and $k = 2 < r_*$. 
Figure 4: Comparison of algorithms under $\tau = 1$, noise level $c = 0$, and $k = 4 > r^*$. 