Abstract

A recent development in the derivation of soliton solutions for initial-boundary value problems through Darboux transformations, motivated to reconsider solutions to the nonlinear Schrödinger (NLS) equation on two half-lines connected via integrable defect conditions. Thereby, the Darboux transformation to construct soliton solutions is applied, while preserving the spectral boundary constraint with a time-dependent defect matrix. In this particular model, $N$-soliton solutions vanishing at infinity are constructed. Further, it is proven that solitons are transmitted through the defect independently of one another.

Keywords: NLS equation, integrable boundary conditions, star-graph, initial-boundary value problems, soliton solutions, dressing transformation, inverse scattering method.

1 Introduction

As an important physical equation the NLS equation was subject to a great number of research works. Over time various methods to deal with integrable nonlinear PDEs in different settings have been formulated. One of these methods, the Unified Transform, announced in [8] was successfully applied to initial-boundary value problems of linear and integrable nonlinear PDEs of one space and one time variable. To this end, the Unified Transform was used to yield results for the NLS equation regarding various spatial domains like the half-line, a finite interval and even a star-graph [4]. As in the case for initial value problems, it is based on the representation of the equation through a Lax pair which consists of two matrices usually referred to as the $t$ part and the $x$ part.

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However, the structural innovation of the Unified Transform is the simultaneous use of $t$ and $x$ part in the direct scattering process.

In some cases which mainly depend on the boundary condition, the Unified Transform is for initial-boundary value problems as efficient as the inverse scattering transform [1] for initial value problems. These so-called linearizable boundary conditions make use of a natural symmetry relation to linearize the problem on the spectral side. Having identified linearizable boundary conditions, it is a priori not clear that they are also integrable boundary conditions. Though, most of the known examples conveniently fit both classes. Finding formulae for long-time asymptotics [7] and for explicit solutions [2, 13] for the NLS equation on the half-line with certain and linearizable boundary conditions, respectively, has been well addressed in the literature.

Nevertheless, the study of a defect or impurity at a fixed point which preserves integrability is still of interest in rather recent studies by several authors, not only for the NLS equation, but also for other PDEs. In one of these studies [5], the authors illuminate the Lagrangian description of “jump-defects”, integrability preserving discontinuities with two fields $u$, $v$, where the conditions relating the fields on the sides of the defect are Bäcklund transformations frozen at the defect location. For the NLS equation on the two half-lines they established the following defect at $x = 0$:

$$
(u - v)_x = i\alpha (u - v) + \Omega (u + v),
$$

$$
(u - v)_t = -\alpha (u - v)_x + i\Omega (u + v) + i(u - v)(|u|^2 + |v|^2),
$$

where $\Omega = \sqrt{\beta^2 - |v - u|^2}$, $\alpha$ and $\beta$ real parameters ($\alpha$ was added in [3]). Moreover, this jump-defect was also used by one of the authors to obtain new boundary condition for the NLS equation on the half-line by combining them with Dirichlet boundary condition, see [12]. It was shown that the defect condition [3] and also the new boundary condition [12] have infinitely many conserved quantities and hence, they are integrable. Moreover, the authors of [3] conjectured that in the model of the NLS equation with defect conditions, an arbitrary number of solitons are transmitted through the defect independently of one another. However, they have only proven this for particular cases of one- and two-soliton solutions.

Using the aforementioned natural symmetry, a method called mirror-image technique was developed to tackle initial-boundary value problems on the half-line by extending it to the whole axis, which may seem like an unnatural approach. On the other hand, there was recently a development for the Unified Transform [13] incorporating the Darboux transformation and hence the construction of exact solutions. The method is, as it uses the Darboux transformation, highly reliant on the integrability of the model. However given that it is, the idea of the method consists of the construction of solutions while preserving the integrability. For the NLS equation with Robin boundary conditions both methods were successfully applied, see [2] for the mirror-image technique.

For integrable PDEs, the Darboux transformation [10, 11] is a powerful method for constructing solutions. In particular, the well-known soliton solution appearing in many physical motivated PDEs like the NLS equation can be computed thereby. The crucial part of the new approach is to supplement the Darboux transformation with the boundary
conditions without destroying the integrability of the system, which was realized in [13] and called “dressing the boundary”.

In this paper, our objective is to take up the described model of the NLS equation on two half-lines together with defect conditions and compute exact solutions through the dressing the boundary method, which already yielded results for a similar integrable model. Then, only considering pure soliton solutions, we want to prove the conjecture formulated in [5], i.e. each soliton in the pure soliton solution is transmitted through the defect independently. To the best knowledge of the author, combining boundary conditions corresponding to a time-dependent boundary matrix with the latest method of computing exact solutions of initial-boundary value problems [13] extended to a star-graph is a novel approach.

In Section 2, we introduce the NLS equation and its equivalent spectral part for which the inverse scattering transform is discussed. In particular, the analysis for the Jost solutions and an understanding of the influence of parameter in the construction of soliton solutions is crucial. We present the methods of the Bäcklund transformations and Darboux transformations in Section 3 and 3.1 respectively and discuss briefly the idea of their connection. In preparation for dressing the boundary in the case of defect conditions, we present their analogous spectral expression in Section 3.2 and prove some helpful properties. Then, in Section 4 of this paper, we specify the model we want to solve: the NLS equation on two half-lines connected via defect conditions at $x = 0$ and realize the dressing the boundary in Proposition 4.2. Thereby, dressing the boundary lets us compute and visualize $N$-soliton solutions in Section 5. Moreover, we discuss these solutions and prove that the solitons are transmitted through the defect independently. Finally, we gather further information and directions in the Conclusion.

## 2 Initial value problem for the NLS

In the following, we give a brief summary of the inverse scattering transform of the focusing NLS equation. As in [2] and [9], it will serve as a guideline in order to implement additional results. Therefore, following the analysis given in [1], we introduce the NLS equation

$$iu_t + u_{xx} + 2|u|^2u = 0,$$
$$u(0, x) = u_0(x) \quad (2.1)$$

for $u(t, x) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ and the initial condition $u_0(x)$. The equation can be expressed in an equivalent compatibility condition of the following linear spectral problems

$$\psi_x = U \psi, \quad \psi_t = V \psi, \quad (2.2)$$

where $\psi(t, x, \lambda)$ and the matrix operators

$$U = -i\lambda \sigma_3 + Q, \quad V = -2i\lambda^2 \sigma_3 + \tilde{Q} \quad (2.3)$$
are 2 × 2 matrices. The potentials \( Q \) and \( \tilde{Q} \) of \( U \) and \( V \) are defined by

\[
Q(t, x) = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad \tilde{Q}(t, x, \lambda) = \begin{pmatrix} i|u|^2 & 2\lambda u + iu_\lambda \\ -2\lambda u^* + iu_x & -i|u|^2 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

In this context, the matrices \( U \) and \( V \) form a so-called Lax pair, depending not only on \( t \) and \( x \), but also on a spectral parameter \( \lambda \). Hereafter, the asterix denotes the complex conjugate, \( \mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \Im(\lambda) > 0 \} \) as well as \( \mathbb{C}_- = \{ \lambda \in \mathbb{C} : \Im(\lambda) < 0 \} \) and \( \psi^\dagger \) is the transpose of \( \psi \). For a solution \( \psi(t, x, \lambda) \) of the Lax system (2.2), the compatibility condition \( \psi_{tx} = \psi_{xt} \) for all \( \lambda \in \mathbb{C} \) is equivalent to \( u(t, x) \) satisfying the NLS equation (2.1).

Moreover, we will refer to \( U \) and \( V \) as the \( x \) and \( t \) part of the Lax pair, respectively. In that regard, given a sufficiently fast decaying function \( u(t, x) \to 0 \) and derivative \( u_x(t, x) \to 0 \) as \( |x| \to \infty \), it is reasonable to assume that there exist 2 × 2-matrix-valued solutions, we call modified Jost solutions under time evolution, \( \psi(t, x, \lambda) = \psi(t, x, \lambda)e^{i\theta(t, x, \lambda)\sigma_3} \), where \( \theta(t, x, \lambda) = \lambda x + 2\lambda^2 t \), of the modified Lax system

\[
\dot{\psi}_x + i\lambda[\sigma_3, \psi] = Q\psi, \quad \dot{\psi}_t + 2i\lambda^2[\sigma_3, \psi] = \tilde{Q}\psi
\]

with constant limits as \( x \to \pm \infty \) and for all \( \lambda \in \mathbb{R} \),

\[
\hat{\psi}_\pm(t, x, \lambda) \to \mathbb{I}, \quad \text{as} \quad x \to \pm \infty.
\]

They are solutions to the following Volterra integral equations:

\[
\hat{\psi}_-(t, x, \lambda) = \mathbb{I} + \int_{-\infty}^{x} e^{-i\theta(0, x-y, \lambda)\sigma_3} Q(y, x, \lambda)\hat{\psi}_-(y, x, \lambda)e^{i\theta(0, x-y, \lambda)\sigma_3} \, dy, \quad \text{(2.4)}
\]

\[
\hat{\psi}_+(t, x, \lambda) = \mathbb{I} - \int_{x}^{\infty} e^{-i\theta(0, x-y, \lambda)\sigma_3} Q(y, x, \lambda)\hat{\psi}_+(y, x, \lambda)e^{i\theta(0, x-y, \lambda)\sigma_3} \, dy.
\]

**Lemma 2.1.** Let \( u(t, \cdot) \in H^{1,1}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : xf, f_x \in L^2(\mathbb{R}) \} \). Then, for every \( \lambda \in \mathbb{R} \), there exist unique solutions \( \hat{\psi}_\pm(t, \cdot, \lambda) \in L^\infty(\mathbb{R}) \) satisfying the integral equations (2.4). Thereby, the second column vector of \( \hat{\psi}_-(t, x, \lambda) \) and the first column vector of \( \hat{\psi}_+(t, x, \lambda) \) can be continued analytically in \( \lambda \in \mathbb{C}_- \) and continuously in \( \lambda \in \mathbb{C}_- \cup \mathbb{R} \), while the first column vector of \( \hat{\psi}_-(t, x, \lambda) \) and the second column vector of \( \hat{\psi}_+(t, x, \lambda) \) can be continued analytically in \( \lambda \in \mathbb{C}_+ \) and continuously in \( \lambda \in \mathbb{C}_+ \cup \mathbb{R} \).

Analogously, the columns of \( \psi_\pm(t, x, \lambda) \) can be continued analytically and continuously into the complex \( \lambda \)-plane, \( \psi_\pm^{(2)} \) and \( \psi_\pm^{(1)} \) can be continued analytically in \( \lambda \in \mathbb{C}_- \) and continuously in \( \lambda \in \mathbb{C}_- \cup \mathbb{R} \), while \( \psi_\pm^{(1)} \) and \( \psi_\pm^{(2)} \) can be continued analytically in \( \lambda \in \mathbb{C}_+ \) and continuously in \( \lambda \in \mathbb{C}_+ \cup \mathbb{R} \).

The limits of the Jost solutions and the zero trace of the matrix \( U \) gives det \( \psi_\pm = 1 \) for all \( x \in \mathbb{R} \). Further, \( \psi_\pm \) are both fundamental matrix solutions to the Lax system (2.2), so there exists an \( x \) and \( t \) independent matrix \( A(\lambda) \) such that

\[
\psi_-(t, x, \lambda) = \psi_+(t, x, \lambda)A(\lambda), \quad \lambda \in \mathbb{R}.
\]
The scattering matrix $A$ is determined by this system and therefore we can also write $A(\lambda) = (\psi_+(t, x, \lambda))^{-1}\psi_-(t, x, \lambda)$, whereas its entries can be written in terms of Wronskians. In particular, $a_{11}(\lambda) = \det[\psi_+^{(1)} | \psi_-^{(2)}]$ and $a_{22}(\lambda) = -\det[\psi_-^{(1)} | \psi_+^{(2)}]$ implying that they can respectively be continued in $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$. The eigenfunction inherit the symmetry relation of the Lax pair

$$\psi_{\pm}(t, x, \lambda) = -\sigma(\psi_{\pm}(t, x, \lambda^*))^\star \sigma,$$  \hspace{1cm} (2.5)

which directly gives $a_{22}(\lambda) = a_{11}^*(\lambda^*)$ and $a_{21}(\lambda) = -a_{12}^*(\lambda)$. The asymptotic behavior of the modified Jost functions and scattering matrix as $\lambda \to \infty$ is

$$\hat{\psi}_- = \mathbb{1} + \frac{1}{2i\lambda}\sigma_3Q + \frac{1}{2i\lambda}\sigma_3\int_x^{\infty} |u(t, y)|^2\,dy + O(1/\lambda^2),$$

$$\hat{\psi}_+ = \mathbb{1} + \frac{1}{2i\lambda}\sigma_3Q - \frac{1}{2i\lambda}\sigma_3\int_{-\infty}^x |u(t, y)|^2\,dy + O(1/\lambda^2)$$

and $A(\lambda) = \mathbb{1} + O(1/\lambda)$.

Let $u(t, \cdot) \in H^{1,1}(\mathbb{R})$ be generic. That is, $a_{11}(\lambda)$ is nonzero in $\mathbb{C}_+$ except at a finite number of points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}_+$, where it has simple zeros $a_{11}(\lambda_j) = 0$, $a_{11}^*(\lambda_j) \neq 0$, $j = 1, \ldots, N$. This set of generic functions $u(t, \cdot)$ is an open dense subset of $H^{1,1}(\mathbb{R})$ usually denoted by $\mathcal{G}$. By the symmetry mentioned above, $a_{11}(\lambda_j) = 0$ if and only if $a_{22}(\lambda_j^*) = 0$ for all $j = 1, \ldots, N$. At these zeros of $a_{11}$ and $a_{22}$, we obtain for the Wronskians the following relation for $j = 1, \ldots, N$,

$$\psi_-^{(1)}(t, x, \lambda_j) = b_j\psi_-^{(2)}(t, x, \lambda_j), \quad \psi_-^{(2)}(t, x, \bar{\lambda}_j) = \bar{b}_j\psi_-^{(1)}(t, x, \bar{\lambda}_j),$$  \hspace{1cm} (2.6)

where we defined $\bar{\lambda}_j = \lambda_j^*$. Whereas for $j = 1, \ldots, N$, the relations then provide residue relations used in the inverse scattering method

$$\text{Res}_{\lambda = \lambda_j} \left( \frac{\hat{\psi}_-^{(1)}}{a_{11}} \right) = C_j e^{2i\theta(t, x, \lambda_j)}\hat{\psi}_+^{(2)}(t, x, \lambda_j),$$

$$\text{Res}_{\lambda = \lambda_j} \left( \frac{\hat{\psi}_-^{(2)}}{a_{22}} \right) = C_j e^{-2i\theta(t, x, \lambda_j)}\hat{\psi}_+^{(1)}(t, x, \bar{\lambda}_j),$$

where the weights are $C_j = b_j/a_{11}(\lambda_j)$ and $\bar{C}_j = \bar{b}_j/a_{22}(\lambda_j)$, and they satisfy the symmetry relations $\bar{b}_j = -b_j^*$ and $\bar{C}_j = -C_j^*$.

The inverse problem can be formulated using the jump matrix

$$J(t, x, \lambda) = \begin{pmatrix} \rho(\lambda) & e^{-2i\theta(t, x, \lambda)} \rho^*(\lambda) \\ e^{-2i\theta(t, x, \lambda)} \rho^*(\lambda) & 0 \end{pmatrix},$$

where the reflection coefficient is $\rho(\lambda) = a_{12}(\lambda)/a_{11}(\lambda)$ for $\lambda \in \mathbb{R}$. Defining sectionally meromorphic functions

$$M_- = (\hat{\psi}_+^{(1)} / \hat{\psi}_-^{(2)} / a_{22}), \quad M_+ = (\hat{\psi}_+^{(1)} / a_{11}, \hat{\psi}_+^{(2)}),$$

we can give the method of recovering the solution $u(t, x)$ from the scattering data.
Riemann–Hilbert problem 1. For given scattering data \((\rho, \{\lambda_j, C_j\}_{j=1}^N)\) as well as \(t, x \in \mathbb{R}\), find a \(2 \times 2\)-matrix-valued function \(C \setminus \mathbb{R} \ni \lambda \mapsto M(t, x, \lambda)\) satisfying

1. \(M(t, x, \cdot)\) is meromorphic in \(C \setminus \mathbb{R}\).
2. \(M(t, x, \lambda) = 1 + \mathcal{O}(1/\lambda)\) as \(|\lambda| \to \infty\).
3. Non-tangential boundary values \(M_+(t, x, \lambda)\) exist, satisfying the jump condition \(M_+(t, x, \lambda) = M_-(t, x, \lambda)(1 + J(t, x, \lambda))\) for \(\lambda \in \mathbb{R}\).
4. \(M(t, x, \lambda)\) has simple poles at \(\lambda_1, \ldots, \lambda_N, \bar{\lambda}_1, \ldots, \bar{\lambda}_N\) with

\[
\text{Res} \ M(t, x, \lambda) = \lim_{\lambda \to \lambda_j} M(t, x, \lambda) \begin{pmatrix} 0 & 0 \\ C_j e^{2i\theta(t, x, \lambda_j)} & 0 \end{pmatrix}
\]

\[
\text{Res} \ M(t, x, \lambda) = \lim_{\lambda \to \lambda_j} M(t, x, \lambda) \begin{pmatrix} 0 & C_j e^{-2i\theta(t, x, \lambda_j)} \\ 0 & 0 \end{pmatrix}
\]

After regularization, the Riemann–Hilbert problem can be solved via Cauchy projectors, and the asymptotic behavior of \(M_+(t, x, \lambda)\) as \(\lambda \to \infty\) yields the reconstruction formula

\[
u(t, x) = -2i \sum_{j=1}^N C_j^* e^{-2i\theta(t, x, \lambda_j^*)}\tilde{\psi}_+^{(2)}(t, x, \lambda_j^*)
- \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2i\theta(t, x, \lambda)} \rho^*(\lambda) |\tilde{\psi}_+^{(2)}(t, x, \lambda)|^2 d\lambda.
\]

In the reflectionless case, we have \(\rho(\lambda) = 0\) for \(\lambda \in \mathbb{R}\) and the Riemann–Hilbert problem can be reduced to an algebraic system

\[
\tilde{\psi}_+^{(1)}(t, x, \lambda_\ell) = e_1 + \sum_{j=1}^N C_j e^{2i\theta(t, x, \lambda_j)} \tilde{\psi}_+^{(2)}(t, x, \lambda_j)
\]

\[
\tilde{\psi}_+^{(2)}(t, x, \lambda_j) = e_2 + \sum_{m=1}^N C_m e^{-2i\theta(t, x, \lambda_j)} \tilde{\psi}_+^{(1)}(t, x, \lambda_m)
\]

for \(\ell, j = 1, \ldots, N\). The one-soliton solution is obtained for \(N = 1\), we obtain

\[
[\tilde{\psi}_+]_{21}(t, x, \lambda_1) = -\frac{C_1}{\lambda_1 - \lambda_1^*} e^{-2i\theta(t, x, \lambda_1^*)} \left[1 - \frac{|C_1|^2 e^{2i(\theta(t, x, \lambda_1) - \theta(t, x, \lambda_1^*))}}{(\lambda_1 - \lambda_1^*)^2}\right]^{-1},
\]

\[
[\tilde{\psi}_+]_{22}(t, x, \lambda_1) = \left[1 - \frac{|C_1|^2 e^{2i(\theta(t, x, \lambda_1) - \theta(t, x, \lambda_1^*))}}{(\lambda_1 - \lambda_1^*)^2}\right]^{-1}
\]

such that the one-soliton solution with \(\lambda_1 = \xi + i\eta\) can be written as

\[
u(t, x) = -2i \eta \frac{C_1}{|C_1|} e^{-i(2\xi x + 4(\xi^2 - \eta^2)t)} \text{sech}(2\eta(x + 4\xi t) - \log \frac{|C_1|}{2\eta}).
\]
We change the notation so that \( u(t, x) = u_{1s}(t, x; \xi, \eta, x_1, \varphi_1) \) has the following expression
\[
u_{1s}(t, x; \xi, \eta, x_1, \varphi_1) = 2\eta e^{-i(2\xi x + 4(\xi^2 - \eta^2) t + (\varphi_1 + \pi/2))} \text{sech}(2\eta(x + 4\xi t - x_1)),
\] (2.7)
where \( \varphi_1 = \text{arg}(C_1) \) and \( x_1 = \frac{1}{2\eta} \log \frac{|C_1|^2}{2\eta} \).

3 Bäcklund transformation

Obtaining solutions for nonlinear partial differential equations is usually not as easy as it may seem, given, we just constructed a one-soliton solution for the NLS equation by the inverse scattering method. Apart from this method, there is also the so-called Bäcklund transformation, which can be used to obtain new solutions from a known solution by solving a system of integrable PDEs. In the paper [3], the author looked more generally at transformations as the Bäcklund transformations and their implementation as boundary condition at a given point on the line. In that regard, consider the Lax system (2.2) for \( U \) and \( V \) as in (2.3). By defining
\[
\tilde{\psi}(t,x,\lambda) = B(t,x,\lambda)\psi(t,x,\lambda),
\]
and also consider the analogous system
\[
\begin{align*}
\tilde{\psi}_x &= \tilde{U}\tilde{\psi}, \\
\tilde{\psi}_t &= \tilde{V}\tilde{\psi},
\end{align*}
\]
for \( \tilde{U} \) and \( \tilde{V} \) as in (2.3) with \( u \) replaced by \( \tilde{u} \). Then, this definition gives us partial differential equations for the so-called defect matrix \( B \) for any \( t \) and \( x \),
\[
\begin{align*}
B_x &= \tilde{U}B - BU, \\
B_t &= \tilde{V}B - BV.
\end{align*}
\] (3.1)

Assuming that the defect matrix is linear in \( \lambda \), one can show that the matrix is of a particular form, see Proposition 2.2 in [3].

Proposition 3.1. The defect matrix \( B = \lambda B^{(1)} + B^{(0)} \), relating Lax systems corresponding to \( \tilde{u} \) and \( u \), has the following general form in terms of NLS class equations
\[
B(t,x,\lambda) = 2\lambda I + \begin{pmatrix}
\alpha \pm i\sqrt{\beta^2 - |\tilde{u} - u|^2} & -i(\tilde{u} - u) \\
-i(\tilde{u} - u)^* & \alpha \mp i\sqrt{\beta^2 - |\tilde{u} - u|^2}
\end{pmatrix},
\] (3.2)
where \( \alpha \in \mathbb{R}, \beta \in \mathbb{R} \) are the parameter of the defect and in particular, independent of \( t \) and \( x \).

Here, it is important to note that the root \( \sqrt{\beta^2 - |\tilde{u} - u|^2} \) is real. This fact follows from the symmetry \( U(t,x,\lambda^*)^* = \sigma U(t,x,\lambda)\sigma^{-1} \) which transfers to \( B \), where
\[
\sigma = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]
3.1 Darboux transformation

The structure of the Lax system permits the application of a, as we will see, very close related class of transformations, the Darboux transformations. In this section, we will outline the utilization of this method to directly obtain soliton solutions of the NLS equation. Darboux transformations are known to provide an algebraic procedure to derive soliton solutions of various integrable PDEs. In particular, they can be viewed as gauge transformation acting on forms of the Lax pair \( U, V \). Here, it is meant to be applied while preserving certain constraints to transform an “old” solution into a “new” solution. For that, the undressed Lax system (2.2) will be denoted as \( U[0], V[0] \) and \( \psi[0] \) and the transformed system as \( U[N], V[N] \) and \( \psi[N] \), whereby the solutions \( \psi[0] \) and \( \psi[N] \) are 2 \( \times \) 1 column solutions. In terms of the Bäcklund transformation, \( U[0], V[0] \) and \( \psi[0] \) can be seen as the standard Lax system and \( U[N], V[N] \) and \( \psi[N] \) as the analog system with \( u \) replaced by \( \tilde{u} \).

Suppose that it is possible to construct a gauge-like transformation

\[
\psi[1] = D[1]\psi[0]
\]

such that the structure of matrices

\[
\begin{align*}
U[1] &= (D[1]_x + D[1]U[0])D[1]^{-1}, \\
V[1] &= (D[1]_t + D[1]V[0])D[1]^{-1}
\end{align*}
\]  

is identical with the structure of \( U[0], V[0] \), i.e. \( Q[0] \) becomes \( Q[1] \) with updated off-diagonal entries. Indeed, if \( U[1] \) and \( V[1] \) satisfy (3.3), then the undressed Lax system (2.2) can be transformed into

\[
\begin{align*}
\psi[1]_x &= U[1]\psi[1], \\
\psi[1]_t &= V[1]\psi[1].
\end{align*}
\]

At this point, it seems that a pair of solutions \( \psi[0] \) and \( \psi[1] \) is needed to determine \( D[1] \). However, if and only if we are able to compute \( D[1] \) solely by a solution \( \psi[0] \) of the undressed Lax system (2.2), we can construct new solutions and then we call \( D[1] \) dressing matrix. Indeed, given a column solution \( \psi_1 = (\mu_1, \nu_1)^T \) of the undressed Lax system at \( \lambda = \lambda_1 \), we write \( D[1] \) in the following form, which satisfies the requirement,

\[
D[1] = (\lambda - \lambda_1^*)1 + (\lambda_1^* - \lambda_1)P[1], \quad P[1] = \frac{\psi_1\psi_1^\dagger}{\psi_1^\dagger\psi_1},
\]

where \( 1 \) is the identity and \( P[1] \) is a projector matrix. Here, \( \psi_1^\dagger \) denotes the transpose complex conjugate of \( \psi_1 \). The important point of this method is that the solution \( u[1] \) can be reconstructed through the first line of (3.3) or in terms of matrices

\[
Q[1] = Q[0] - i(\lambda_1 - \lambda_1^*)[\sigma_3, P[1]],
\]

which is called reconstruction formula. Technically, the Darboux transformation can be summarized in the following way: Suppose we have a system, of which we know the
solution. Then, transforming the system via the dressing matrix allows to construct the solution to a different system.

Especially, if both systems correspond to the same PDE, the reconstruction formula lets us obtain a new solution of the PDE. Therefore, in advance a good understanding of the set of solutions of the NLS equation is instrumental, since they are decisive when it comes to solutions of the Lax system. However, there is only a limited number of significant cases known, e.g. the zero solution. In this regard, using the zero solution as seed solution, i.e. \( u[0] = 0 \), one can construct among other solutions a one-soliton solution \( u[1] \), see (2.7). This will be of interest in the following studies.

Given \( N \) linear independent column solutions \( \psi_j = (\mu_j, \nu_j)^T \) of the undressed Lax system (2.2) evaluated at \( \lambda = \lambda_j \), \( j = 1 \ldots N \), the basic dressing matrix \( D[1] \) may be iterated in the following sense

\[
D[N] = ((\lambda - \lambda_N^N)I + (\lambda_N^N - \lambda_N)P[N]) \cdots ((\lambda - \lambda_1^1)I + (\lambda_1^1 - \lambda_1)P[1]),
\]

where \( P[j] \) are projector matrices defined by

\[
P[j] = \frac{\psi_j[j-1]\psi_j[j-1]^T}{\psi_j[j-1]\psi_j[j-1]^T}, \quad \psi_j[j-1] = D[j-1]|_{\lambda = \lambda_j}\psi_j.
\]

Note that it is sufficient for the \( \lambda = \lambda_j \), \( j = 1 \ldots N \), to be distinct in order for the solutions to be linearly independent. Analogously to \( N = 1 \), for the reconstruction formula we need to insert \( \psi[N] = D[N]\psi[0] \) into the transformed Lax system

\[
\psi[N]_x = U[N]\psi[N], \quad \psi[N]_t = V[N]\psi[N],
\]

and extract the information of the coefficient of \( \lambda^{N-1} \) of the first line. Then, the reconstruction formula can be computed as

\[
Q[N] = Q[0] - i \sum_{j=1}^{N} (\lambda_j - \lambda_j^*)[\sigma_3, P[j]].
\]

In the course of this paper we will also work with the half-line as domain, for which this construction can be done in the same way, resulting in a solution \( u[N] \) on the half-line given through (3.5).

Since the Darboux transformation is a matrix transforming undressed Lax systems of the NLS equation to Lax systems of the NLS equation, there is a correspondence between matrices from the Darboux transformation and of defect form. We will address this idea in the next remark.

**Remark 3.2.** (i) The one-fold dressing matrix \( D[1] \), constructed by \( C \setminus R \ni \lambda_1 = \xi + i\eta \) and \( \psi_1 \in C^2 \), satisfies (3.1) with \( \tilde{U} = U[1] \) and \( U = U[0] \). Given certain
information, we can, up to a function of \( \lambda \), write \( D[1] \) in the form of a defect matrix with \( \gamma \in \mathbb{R} \), \( \delta \in \mathbb{R} \setminus \{0\} \), that is

\[
D[1] = \lambda I + \frac{1}{2} \begin{pmatrix}
-2\gamma \pm i\sqrt{4\delta^2 - |u[1] - u[0]|^2} & -i(u[1] - u[0])^* \\
-i(u[1] - u[0]) & -2\gamma \mp i\sqrt{4\delta^2 - |u[1] - u[0]|^2}
\end{pmatrix}.
\]

(ii) The defect matrix \( B = 2\lambda I + B(0) \) in general form (3.8) with \( \alpha, \beta \in \mathbb{R} \), applied in the context of NLS class equations, admits a projector matrix \( P_0 \) if \( \beta \neq 0 \). Take \( \lambda_0 = -\frac{\alpha - i\beta}{2} \) and define

\[
P_0 = \frac{1}{2(\lambda_0^* - \lambda_0)} (B(0) + \lambda_0^* I).
\]

Then, we can write \( B \), up to a function of \( \lambda \), in the form of a dressing matrix

\[
B = 2(\lambda - \lambda_0^*) I + 2(\lambda_0^* - \lambda_0) P_0.
\]

This remark states that: if \( \beta \neq 0 \) (or \( \eta \neq 0 \)), the general form of a defect matrix and a one-fold dressing matrix are in certain cases interchangeable. In particular, this means that for a defect matrix \( B(t, x, \lambda) \) with a parameter \( \beta \neq 0 \), we know that there exists a vector \( \upsilon_0 \) which is in the kernel of \( B(t, x, \lambda_0) \). Conversely, taking a matrix polynomial of order one with a kernel vector \( \upsilon_0 \) at \( \lambda = \lambda_0 \) corresponding to a one-fold dressing matrix, i.e. it solves (3.3), we might be able to write it in the form of a defect matrix (3.1). However, it should be noted that coming from a dressing matrix \( D[1] \), it is also a priori not clear what the corresponding sign in front of the root in the (11)-entry and accordingly the (22)-entry has to be. In some cases, as we will see, this information can be extracted from the kernel vectors of the dressing matrix. On the other hand, coming from a defect matrix \( B \), we only have the projector matrix in terms of expressions of the solution side without knowledge of how the kernel vectors look like in terms of the spectral side. Again, in some cases, it is possible to obtain information on the kernel vectors from the signs in front of the root.

### 3.2 Localized Bäcklund transformation

The Bäcklund transformation has also been investigated as frozen at a specific point \( x_f \) and with that in mind as a means to generate integrable boundary value systems. We will introduce the idea of this method in this section. Restricting \( \tilde{u} \) and \( u \) to solutions of the NLS equation on different half-lines and therefore also their Lax systems respectively to \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}_- \) and \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \), we simultaneously restrict (3.1) to \( x_f = 0 \) on the line. To distinguish between the Bäcklund transformation and the localized Bäcklund transformation, we denote the denote the defect matrix by \( B(t, x, \lambda) \) as before and the localized defect matrix by \( G(t, 0, \lambda) \). For any \( t \in \mathbb{R}_+ \) and \( x = 0 \), we call the relations

\[
G_x = \tilde{U}G - GU, \\
G_t = \tilde{V}G - GV,
\]

(3.7)
From Proposition 3.1 it follows that assuming \( G(t, 0, \lambda) \) is linear in \( \lambda \), it admits the form
\[
G(t, 0, \lambda) = 2\lambda I + \left( \frac{\alpha \pm i\sqrt{\beta^2 - |\tilde{u} - u|^2}}{\mp i(\tilde{u} - u)} \right) \left( \frac{-i(\tilde{u} - u)}{\alpha \mp i\sqrt{\beta^2 - |\tilde{u} - u|^2}} \right)
\] (3.8)
with \( \alpha, \beta \in \mathbb{R} \). In [3], the defect matrix \( B(t, x, \lambda) \) has been discussed not only for equations of NLS type for \( u \) and \( \tilde{u} \), but as an universal approach to Lax pair systems. Hence, different defect matrices and consequently different localized defect matrices could be identified corresponding to the Lax systems of various PDEs.

Also note, that (3.7) at \( x = 0 \) has a structural difference to the boundary constraint for the half-line, see [13]. For the model on two half-lines, we are not relating \( V(t, 0, \lambda) \) and \( V(t, 0, -\lambda) \) as for the half-line, but \( V(t, 0, \lambda) \) and \( \tilde{V}(t, 0, \lambda) \). Furthermore, the relation of \( V(t, 0, \lambda) \) and \( \tilde{V}(t, 0, \lambda) \) immediately implies the relation for \( U(t, 0, \lambda) \) and \( \tilde{U}(t, 0, \lambda) \).

Let us discuss, which information is needed in order to determine the sign in front of the root in the \((11)\)-entry, considering we already constructed the localized defect form at \( x = 0 \). In that regard, important properties of a Bäcklund transformation with respect to \( c \) have been in detail discussed in the paper [6].

In particular, it was shown that the transformation \( B_{3(\lambda_1)}^\pm(\psi_1) : u \mapsto \tilde{u} = B_{3(\lambda_1)}^\pm(\psi_1)u \), the Bäcklund transformation of \( u(t, \cdot) \) with respect to \( \{3(\lambda_1), \psi_1\} \) on \( \mathbb{R}_+ \), is a bijection from \( H^{1,1}(\mathbb{R}_+) \) onto \( H^{1,1}(\mathbb{R}_+) \). Similarly, we want to analyze the localized defect matrix as Bäcklund transformation with \( \beta \neq 0 \) and with respect to \( t \). For functions \( f(\cdot, 0, \lambda) \), we introduce the function spaces
\[
H^{0,1}_t(\mathbb{R}_+) = \{ f \in L^2(\mathbb{R}_+) : tf \in L^2(\mathbb{R}_+) \},
\]
\[
H^{1,1}_t(\mathbb{R}_+) = \{ f \in L^2(\mathbb{R}_+) : \partial_t f, tf \in L^2(\mathbb{R}_+) \}
\]
and state the following lemma, which will be essential in the proof.

**Lemma 3.3.** Let \( f(\cdot, 0, \lambda) \in H^{0,1}_t(\mathbb{R}_+) \), \( g(\cdot, 0, \lambda) \in H^{1,1}_t(\mathbb{R}_+) \) and \( \Im(\lambda^2) < 0 \). Then,
\[
\left\| \int_{(t)} f(\tau, 0, \lambda)g(\tau, 0, \lambda) \, d\tau \right\|_{H_t^{1,1}(\mathbb{R}_+)} \leq c \| f(\cdot, 0, \lambda) \|_{H^{0,1}_t(\mathbb{R}_+)} \| g(\cdot, 0, \lambda) \|_{H^{1,1}_t(\mathbb{R}_+)},
\]
\[
\left\| \int_{(t)} f(\tau, 0, \lambda)e^{-4i\lambda^2(\cdot - \tau)} \, d\tau \right\|_{H_t^{1,1}(\mathbb{R}_+)} \leq c \| f(\cdot, 0, \lambda) \|_{H^{0,1}_t(\mathbb{R}_+)},
\]
where \( c \) depends on \( \lambda \).

**Proof.** Analogously to the proof in [6]. \( \square \)

We skip the part in [6], where the defect matrix is shown to be of the form of a dressing matrix and immediately assume we are given a spectral parameter \( \lambda_1 = \xi + i\eta \)

\[\text{together with a vector} \ \psi_1 = (\mu, \nu)^T, \ \text{which is a solution of the Lax system (2.2) at} \ \lambda = \lambda_1, \ \text{from which we can construct a dressing matrix} \ D[1] \ \text{such that} \]
\[
D[1] = \lambda I + \begin{pmatrix}
-\xi - i\eta & -2i\eta\mu^* \\
-2i\mu^* \nu & -\xi + i\eta
\end{pmatrix}.
\]
Fixing $x = 0$, we can see the dressing matrix as a connection of two NLS equations on the respective half-lines $\mathbb{R}_-$ and $\mathbb{R}_+$ and hence, it satisfies the boundary constraint (3.7) for some solutions $u_1(t, 0)$ and $u(t, 0)$ of the NLS equations, whereas

$$
u = 0,$$

$$(u_1)_x(t, 0) = u_x(t, 0) - 4u(t, 0)\eta \frac{\mu \nu^*}{|\mu|^2 + |\nu|^2},$$

$$(u_1)_{xx}(t, 0) = 4u(t, 0)\eta \frac{\mu \nu^*}{|\mu|^2 + |\nu|^2} - 2\eta \frac{\mu \nu^*}{|\mu|^2 + |\nu|^2} \left(-\frac{\xi}{\mu} + \frac{\nu}{\nu}ight),$$

where these can be derived by (3.3). So that we have that a transformation $B_\lambda^i(\psi_1): u \mapsto u_1 = B_\lambda^i(\psi_1)u$ mapping $u(\cdot, 0) \in L^1_t(\mathbb{R}_+) \rightarrow L^1_t(\mathbb{R}_+) \ni u_1(\cdot, 0)$. The denominator $|\mu|^2 + |\nu|^2$ can not be zero, since $\psi_1$ is a solution of $\psi_t = (-2i\lambda^2 \sigma_3 + \hat{Q})\psi$ at $\lambda = \lambda_1$. If there exists a $t_0 \in \mathbb{R}_+$ such that $\psi_1 = 0$, then $\psi_1|_{t = t_0} = 0$ at $t_0 \in \mathbb{R}_+$ and thereby $\psi_1 = 0$ for every $t \in \mathbb{R}_+$. Assuming a nonzero asymptotic limit of $\psi_1$ gives the contradiction. In particular, the transformation has a left inverse. Take $B_2(\psi_2)$, where $\lambda_2 = \lambda_1$, $\psi_2 = (-k_1 \psi^*, k_1 \mu^*)^T$, $k_1 \in \mathbb{C}$, we have

$$u_2(t, 0) = u_1(t, 0) + 2\eta \frac{\mu \nu^*}{|\mu|^2 + |\nu|^2} = u(t, 0).$$

We define

$$X = \{ f \in H^1_t(\mathbb{R}_+), f_x \in H^1_t(\mathbb{R}_+) \}$$

with $u(\cdot, 0, \lambda) \in \{ f \in H^1_t(\mathbb{R}_+), f_x \in H^1_t(\mathbb{R}_+) \} \subset X$, we can show that

$$\|\hat{Q}(\cdot, 0, \lambda)\|_{L^1(\mathbb{R}_+)} \leq \left( \begin{array}{c} \|u_2\|_{L^1(\mathbb{R}_+)} \|2\lambda u + iu_x\|_{L^1(\mathbb{R}_+)} \\ \|u_2\|_{L^1(\mathbb{R}_+)} \|2\lambda u + iu_x\|_{L^1(\mathbb{R}_+)} \end{array} \right) \leq \left( \begin{array}{c} \|u\|_{L^1(\mathbb{R}_+)} \|2\lambda u + iu_x\|_{L^1(\mathbb{R}_+)} \\ \|u\|_{L^1(\mathbb{R}_+)} \|2\lambda u + iu_x\|_{L^1(\mathbb{R}_+)} \end{array} \right).$$

So that each component is bounded by a constant depending on $c(\lambda)$ multiplied by $\|u\|_X$. Then, we can prove that

**Proposition 3.4.** $B_\lambda^i(\psi_1)$, where $\lambda_1 \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, maps functions $u(\cdot, 0) \in X$ onto $\hat{u}(\cdot, 0) \in X$.

**Proof.** Following the proof for the Bäcklund transformation with respect to $x$, see Proposition 4.7 in [3], we want to introduce a $t$ dependent (Jost) function. In that regard, we freeze the space variable $x$, whereas we need it particularly at $x = 0$. Then, given the limit behaviors $|u(t, 0)| \rightarrow 0$ and $|u_x(t, 0)| \rightarrow 0$ as $t \rightarrow \infty$, it is reasonable to assume that there exists a $2 \times 1$-vector-valued solution $m$ to the spectral problem

$$\psi_t = (-2i\lambda^2 \sigma_3 + \hat{Q})\psi$$

admitting the asymptotic behavior $m(t, 0, \lambda) \sim e_t e^{-2i\lambda^2 t}$ as $t \rightarrow \infty$. Then, we also define the normalized $t$ dependent (Jost) function by

$$\hat{m}(t, 0, \lambda) = m(t, 0, \lambda) e^{2i\lambda^2 t},$$

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whereas it admits the normalization \( \lim_{t \to \infty} \hat{m}(t, 0, \lambda) = e_1 \). The solution \( m(t, 0, \lambda) = \hat{m}(t, 0, \lambda) e^{-2i\lambda^2t} \) is uniquely specified by the asymptotic behavior \( \hat{m}(t, 0, \lambda) \to e_1 \) as \( t \to \infty \). The normalized (Jost) function is a solution to the following Volterra integral equation

\[
\hat{m}(t, 0, \lambda) = e_1 - \int_t^\infty \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) e^{4i\lambda^2(t-\tau)} \tilde{Q}(\tau, 0, \lambda) \hat{m}(\tau, 0, \lambda) \, d\tau.
\]

This, we will show by defining the operator

\[
\mathcal{M}\hat{m}(t, 0, \lambda) = -\int_t^\infty \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) e^{4i\lambda^2(t-\tau)} \tilde{Q}(\tau, 0, \lambda) \hat{m}(\tau, 0, \lambda) \, d\tau,
\]

which is a bounded operator mapping from \( L^\infty(\mathbb{R}_+) \) to \( L^\infty(\mathbb{R}_+) \) for any fixed \( \lambda \) such that \( \Im(\lambda^2) < 0 \), since \( t - \tau \leq 0 \). Also, we define

\[
\mathcal{M}_j\hat{m}(t, 0, \lambda) = -\int_t^{t_{j-1}} \left( \begin{array}{c} 1 \\ 0 \\ \end{array} \right) e^{4i\lambda^2(t-\tau)} \tilde{Q}(\tau, 0, \lambda) \hat{m}(\tau, 0, \lambda) \, d\tau,
\]

where we fix \( \lambda \) such that \( \Im(\lambda^2) = 0 \). For an arbitrary interval \((t_{j-1}, t_j) \subset \mathbb{R}_+\), we obtain the estimate

\[
||\mathcal{M}_j\hat{m}(\cdot, 0, \lambda)||_{L^\infty(t_{j-1}, t_j)} \leq ||\tilde{Q}(\cdot, 0, \lambda)||_{L^1(t_{j-1}, t_j)}||\hat{m}(\cdot, 0, \lambda)||_{L^\infty(t_{j-1}, t_j)}.
\]

Then, we can choose \( t_j \) in such a way that the operator \( \mathcal{M}_j \) is a contraction from \( L^\infty(t_{j-1}, t_j) \) to \( L^\infty(t_{j-1}, t_j) \). Repeating this argument starting from \( t_0 = 0 \) and appropriately chosen \( t_1, \ldots, t_{\ell-1} \) and \( t_\ell = \infty \), we can obtain finitely many intervals such that \( \mathcal{M}_j \) is contraction from \( L^\infty(t_{j-1}, t_j) \) to \( L^\infty(t_{j-1}, t_j) \), \( j = 1, \ldots, \ell \). Setting \( \hat{m}_0(t, 0, \lambda) \equiv e_1 \) on \((t_0, t_1)\), we can find a function \( \hat{m}_j(\cdot, 0, \lambda) \in L^\infty(t_{j-1}, t_j) \) by the Banach Fixed Point Theorem such that it solves the equation

\[
\hat{m}_j(t, 0, \lambda) = \hat{m}_{j-1}(t, 0, \lambda) + \mathcal{M}_j\hat{m}_j(t, 0, \lambda), \quad t \in (t_{j-1}, t_j)
\]

for every \( j = 2, \ldots, \ell \). Combining these functions, we find a continuous function in \( L^\infty(\mathbb{R}_+) \) satisfying the Volterra integral equation (3.10), which covers the existence of \( \hat{m}(t, 0, \lambda) \). Having two solutions \( \hat{m}(t, 0, \lambda) \) and \( \hat{m}(t, 0, \lambda) \) to the Volterra integral equation (3.10), we can deduce

\[
|\hat{m}(t, 0, \lambda) - \hat{m}(t, 0, \lambda)| \leq \int_t^\infty |\tilde{Q}(\tau, 0, \lambda)| \, |\hat{m}(\tau, 0, \lambda) - \hat{m}(\tau, 0, \lambda)| \, d\tau.
\]

Then by Grönwall’s lemma, we obtain uniqueness.

Now, for the claims regarding the continuation of \( \hat{m}(t, 0, \lambda) \) to \( \Im(\lambda^2) \leq 0 \). Analogously to the \( x \) dependent Jost solution \( \hat{\psi}_j^{-1}(t, x, \lambda) \), we introduce for \( \hat{m}(t, 0, \lambda) \) the Neumann series \( \sum_{j=0}^{\infty} \mathcal{M}^j[m_0](t, 0, \lambda) \), where \( m_0(t, 0, \lambda) \equiv e_1 \), which is formally a solution to the Volterra integral equation (3.10). Then, it is possible to derive a bound of the iterated operator \( \mathcal{M} \). We define \( T(t, \lambda) \) by

\[
T(t, \lambda) = \int_t^\infty |\tilde{Q}(\tau, 0, \lambda)| \, d\tau \leq \int_0^\infty |\tilde{Q}(\tau, 0, \lambda)| \, d\tau \leq ||\tilde{Q}(\cdot, 0, \lambda)||_{L^1(\mathbb{R}_+)}.
\]
And then show that by induction, we have
\[
|\mathcal{M}^{j+1}[\hat{m}](t,0,\lambda)| \leq c \frac{||\hat{m}(\cdot,0,\lambda)||_{L^\infty(\mathbb{R}_+)} j!}{j!} \int_t^\infty |\hat{Q}(\tau,0,\lambda)| (T(\tau,\lambda))^{j+1} d\tau \\
\leq c \frac{||\hat{m}(\cdot,0,\lambda)||_{L^\infty(\mathbb{R}_+)} T(t,\lambda)}{j!} \int_0^{T(t,\lambda)} s^j ds \\
= c||\hat{m}(\cdot,0,\lambda)||_{L^\infty(\mathbb{R}_+)} (T(t,\lambda))^{j+1} (j+1)!
\]
where we put \( s = T(\tau,\lambda) \). Thus, we have that \( \sum_{j=0}^{\infty} \mathcal{M}^j[m_0](t,0,\lambda) \) is majorized in norm by a uniformly convergent power series and is therefore itself uniformly convergent for \( \Im(\lambda) \leq 0 \). The analyticity and continuity continuation for \( \hat{m}(t,0,\lambda) \) respectively in \( \{\lambda \in \mathbb{C} \setminus \{0\} : \Im(\lambda^2) \leq 0 \} \) and in \( \{\lambda \in \mathbb{C} \setminus \{0\} : \Im(\lambda^2) < 0 \} \) holds also, as before, for the function \( m(t,0,\lambda) \). It is left, to show that the entries of \( \hat{m}(\cdot,0,\lambda) - e_1 \) are in \( H^{1,1}_t(\mathbb{R}_+) \).

Since \( \mathcal{M} \) maps \( L^\infty(\mathbb{R}_+) \) to \( L^\infty(\mathbb{R}_+) \) and writing \( \hat{m}(t,0,\lambda) = (\hat{m}_1,\hat{m}_2) \), we can estimate using Lemma 3.3
\[
||\hat{m}_2(\cdot,0,\lambda)||_{H^{1,1}_t(\mathbb{R}_+)} \leq c((\hat{Q}_{21}\hat{m}_1)(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} + c((\hat{Q}_{22}\hat{m}_2)(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} \\
\leq ||\hat{m}_1(\cdot,0,\lambda)||_{L^\infty(\mathbb{R}_+)} ||\hat{Q}_{21}(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} \\
+ ||\hat{m}_2(\cdot,0,\lambda)||_{L^\infty(\mathbb{R}_+)} ||\hat{Q}_{22}(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)}
\]
and
\[
||\hat{m}_1(\cdot,0,\lambda) - 1||_{H^{1,1}_t(\mathbb{R}_+)} \leq c((\hat{Q}_{11}\hat{m}_1)(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} + c((\hat{Q}_{12}\hat{m}_2)(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} \\
\leq ||\hat{m}_1(\cdot,0,\lambda)||_{L^\infty(\mathbb{R}_+)} ||\hat{Q}_{11}(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} \\
+ ||\hat{m}_2(\cdot,0,\lambda)||_{H^{1,1}_t(\mathbb{R}_+)} ||\hat{Q}_{12}(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)}
\]

And for the entries of \( \hat{Q}(t,0,\lambda) \), we find
\[
||\hat{Q}_{11}(\cdot,0,\lambda))_{H^{1,1}_t(\mathbb{R}_+)} \leq ||u(\cdot,0)||_{L^\infty(\mathbb{R}_+)} ||u(\cdot,0)||_{H^{1,1}_t(\mathbb{R}_+)} \\
||\hat{Q}_{12}(\cdot,0,\lambda))_{H^{0,1}_t(\mathbb{R}_+)} \leq 2||u(\cdot,0)||_{H^{0,1}_t(\mathbb{R}_+)} + ||u(\cdot,0)||_{H^{0,1}_t(\mathbb{R}_+)} \\
||\hat{Q}_{21}(\cdot,0,\lambda))_{H^{0,1}_t(\mathbb{R}_+)} \leq 2||u(\cdot,0)||_{H^{0,1}_t(\mathbb{R}_+)} + ||u(\cdot,0)||_{H^{0,1}_t(\mathbb{R}_+)} \\
||\hat{Q}_{22}(\cdot,0,\lambda))_{H^{0,1}_t(\mathbb{R}_+)} \leq ||u(\cdot,0)||_{L^\infty(\mathbb{R}_+)} ||u(\cdot,0)||_{H^{0,1}_t(\mathbb{R}_+)}. \tag{3.11}
\]

Thereby, if \( u(\cdot,0) \in X \), then \( \hat{m}(\cdot,0,\lambda) - e_1 \in H^{1,1}_t(\mathbb{R}_+) \). Next, we consider a solution \( n(t,0,\lambda) \) of the \( t \) part of the Lax pair defined on \( \Im(\lambda^2) \leq 0 \) and \( t \in \mathbb{R}_+ \) with the property
\[
n(t,0,\lambda) = (e_2 + r_1(t))e^{2i\lambda t}, \quad r_1 \in H^{1,1}_t(\mathbb{R}_+).
\]
Here, \( e_2 = (0,1)^T \) and \( n(t,0,\lambda) \) is a non-unique solution to the differential equation defined on the same domain as \( m(t,0,\lambda) \) and, as we will show, is linear independent of \( m(t,0,\lambda) \) for all \( t \in \mathbb{R}_+ \).
For given \( u(\cdot, x) \in X \) and \( \lambda \in \{ \lambda \in \mathbb{C} \setminus \{0\} : \Im(\lambda^2) \leq 0 \} \), we fix \( t_0 > 0 \) such that each entry of \( \int_{t_0}^{\infty} |Q(\tau, 0, \lambda)| \, d\tau \) is bounded by a constant \( c_{t_0} < 1 \) and by the arbitrary choice of \( t_0 \), the non-uniqueness is apparent. For \( \Im(\lambda^2) \leq 0 \), we consider the following integral equation for \( n(t, 0, \lambda) \),

\[
n(t, 0, \lambda) = e^{2i\lambda^2 t} e_2 + \int_{t_0}^{t} \begin{pmatrix} e^{-2i\lambda^2(t-\tau)} & 0 \\ 0 & 0 \end{pmatrix} \tilde{Q}(\tau, 0, \lambda) \, n(\tau, 0, \lambda) \, d\tau - \int_{t}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tilde{Q}(\tau, 0, \lambda) \, n(\tau, 0, \lambda) \, d\tau, \quad t \geq t_0.
\]

Set \( \hat{n}(t, 0, \lambda) = n(t, 0, \lambda)e^{-2i\lambda^2 t} \), then the integral equation becomes

\[
\hat{n}(t, 0, \lambda) = e_2 + (\mathcal{N}\hat{n})(t, 0, \lambda), \quad t \geq t_0, \tag{3.12}
\]

where \( \mathcal{N} \) is an integral operator defined by

\[
(\mathcal{N}\hat{n})(t, 0, \lambda) = \int_{t_0}^{t} \begin{pmatrix} e^{-4\lambda^2(t-\tau)} & 0 \\ 0 & 0 \end{pmatrix} \tilde{Q}(\tau, 0, \lambda) \, \hat{n}(\tau, 0, \lambda) \, d\tau - \int_{t}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tilde{Q}(\tau, 0, \lambda) \, \hat{n}(\tau, 0, \lambda) \, d\tau, \quad \hat{n}(\cdot, 0, \lambda) \in L^\infty[t_0, \infty).
\]

By the same argument as for \( \hat{n}(t, 0, \lambda) \), we have existence of \( \hat{n}(t, 0, \lambda) \) for \( t \in (t_0, \infty) \). As \( \Im(\lambda^2) \leq 0 \) and each entry of \( \tilde{Q}(\cdot, 0, \lambda) \) being in \( L^1[t_0, \infty) \), \( \mathcal{N} \) is a bounded operator from \( L^\infty[t_0, \infty) \) to \( L^\infty[t_0, \infty) \). Similar to before, put \( \hat{n}_0(t, 0, \lambda) = e_2 \) and define \( \hat{n}_{j+1}(t, 0, \lambda) = e_2 + (\mathcal{N}\hat{n}_j)(t, 0, \lambda) \), inductively. Then,

\[
\|\hat{n}_{j+1} - \hat{n}_j\|_{L^\infty[t_0, \infty)} \leq c_{t_0}^j, \quad j \geq 0.
\]

Indeed \( \|\hat{n}_1 \cdot, 0, \lambda) - \hat{n}_0 \cdot, 0, \lambda)\|_{L^\infty[t_0, \infty)} \leq c_{t_0} \) and for \( j \geq 1, \)

\[
\|\hat{n}_{j+1} - \hat{n}_j\|_{L^\infty[t_0, \infty)} = \|(\mathcal{N}(\hat{n}_j - \hat{n}_{j-1}))\|_{L^\infty[t_0, \infty)} \leq \|\tilde{Q}(\cdot, 0, \lambda)\|_{L^\infty[t_0, \infty)} \int_{t_0}^{\infty} |\tilde{Q}(\tau, 0, \lambda)| \, d\tau = c_{t_0} \|\tilde{Q}(\cdot, 0, \lambda)\|_{L^\infty[t_0, \infty)}
\]

Therefore, \( \hat{n}(t, 0, \lambda) = \hat{n}_0(t, 0, \lambda) + \sum_{j=1}^{\infty} \hat{n}_j(t, 0, \lambda) - \hat{n}_{j-1}(t, 0, \lambda) \) converges in \( L^\infty[t_0, \infty) \) and solves the integral equation (3.12). Writing \( \hat{n}(t, 0, \lambda) = (\hat{n}_1, \hat{n}_2)^\top, \) (3.12) becomes

\[
\hat{n}_1(t, 0, \lambda) = \int_{t_0}^{t} e^{-4i\lambda^2(t-\tau)}(\tilde{Q}_{11}(\tau, 0, \lambda)\hat{n}_1(\tau, 0, \lambda) + \tilde{Q}_{12}(\tau, 0, \lambda)\hat{n}_2(\tau, 0, \lambda)) \, d\tau
\]

\[
\hat{n}_2(t, 0, \lambda) = 1 - \int_{t}^{\infty} \tilde{Q}_{21}(\tau, 0, \lambda)\hat{n}_1(\tau, 0, \lambda) + \tilde{Q}_{22}(\tau, 0, \lambda)\hat{n}_2(\tau, 0, \lambda) \, d\tau
\]
As for $\hat{n}(t,0,\lambda)$, we can prove, if $u(\cdot,0) \in X$, then $\hat{n}_1(\cdot,0,\lambda) \in H^{1,1}_t(t_0,\infty)$. Therefore, we consider with Lemma 3.3 the estimate

$$
\|\hat{n}_1(\cdot,0,\lambda)\|_{H^{1,1}_t(t_0,\infty)} \leq c\|\hat{Q}_{11}\hat{n}_1(\cdot,0,\lambda)\|_{H^{0,1}_t(t_0,\infty)} + c\|\hat{Q}_{12}\hat{n}_2(\cdot,0,\lambda)\|_{H^{0,1}_t(t_0,\infty)} \\
\leq |\hat{n}_1(\cdot,0,\lambda)|_{L^\infty(t_0,\infty)}|\hat{Q}_{11}(\cdot,0,\lambda)|_{H^{0,1}_t(\mathbb{R}^+)} \\
\quad + |\hat{n}_2(\cdot,0,\lambda)|_{L^\infty(t_0,\infty)}|\hat{Q}_{12}(\cdot,0,\lambda)|_{H^{0,1}_t(\mathbb{R}^+)}.
$$

A similar reasoning involving Lemma 3.3 implies that $\hat{n}_2(\cdot,0,\lambda) - 1 \in H^{1,1}_t(t_0,\infty)$. We have

$$
\|\hat{n}_2(\cdot,0,\lambda) - 1\|_{H^{1,1}_t(t_0,\infty)} \leq c\|\hat{Q}_{21}\hat{n}_1(\cdot,0,\lambda)\|_{H^{1,1}_t(t_0,\infty)} + c\|\hat{Q}_{22}\hat{n}_2(\cdot,0,\lambda)\|_{H^{1,1}_t(t_0,\infty)} \\
\leq |\hat{n}_1(\cdot,0,\lambda)|_{H^{1,1}_t(t_0,\infty)}|\hat{Q}_{21}(\cdot,0,\lambda)|_{H^{0,1}_t(\mathbb{R}^+)} \\
\quad + |\hat{n}_2(\cdot,0,\lambda)|_{L^\infty(t_0,\infty)}|\hat{Q}_{22}(\cdot,0,\lambda)|_{H^{0,1}_t(\mathbb{R}^+)}.
$$

Except for

$$
|\hat{Q}_{22}(\cdot,0,\lambda)|_{H^{1,1}_t(\mathbb{R}^+)} \leq \|u(\cdot,0)\|_{L^\infty(\mathbb{R}^+)}\|u(\cdot,0)\|_{H^{1,1}_t(\mathbb{R}^+)},
$$

all estimates on the entries of $\hat{Q}(t,0,\lambda)$ are already done in (3.11). Therefore, we indeed have that $\hat{n}(\cdot,0,\lambda) - e_2 \in H^{1,1}_t(t_0,\infty)$ if $u(\cdot,0) \in X$. We know that $n(t,0,\lambda)$ defined through $\hat{n}(t,0,\lambda)$ solves the integral equation (3.10) for $t \in \mathbb{R}^+$ and we have its existence in $t \geq t_0$, it follows that, given $t_0$, $n(t,0,\lambda)$ can be uniquely extended to a solution of the $t$ part of the Lax pair for $\Im(\lambda^2) < 0$.

The linear independence of $m(t,0,\lambda_1)$ and $n(t,0,\lambda_1)$, $\lambda_1 \in \{\lambda \in \mathbb{C} : \Im(\lambda^2) < 0\}$, can be shown by

$$
\lim_{t \to \infty} \det(m(t,0,\lambda_1), n(t,0,\lambda_1)) = 1.
$$

Since $V$ has zero trace, we conclude that

$$
\det(m(t,0,\lambda_1), n(t,0,\lambda_1)) = 1, \quad t \geq 0.
$$

Then, for $t \geq 0$, we can write $\psi_1$ as a linear combination of $m(t,0,\lambda_1)$ and $n(t,0,\lambda_1)$ such that

$$
\psi_1(t) = c_1m(t,0,\lambda_1) + c_2n(t,0,\lambda_1)
$$

for some constants $c_1$, $c_2$. If $c_2 = 0$, then as $t \to \infty$,

$$
\psi_0(t) = c_1e^{-2i\lambda_1^2t} \left( \frac{1 + r_2(t)}{r_3(t)} \right), \quad r_j \in H^{1,1}_t(\mathbb{R}^+), \quad j = 2,3.
$$

Hence,

$$
(P[1])_{12} = \frac{(1 + r_2(t)r_3(t))^*}{|1 + r_2(t)|^2 + |r_3(t)|^2} \in H^{1,1}_t(\mathbb{R}^+)
$$

As in the argumentation for the Bäcklund matrix being a map from $u(\cdot,0) \in L^1_{loc}(\mathbb{R}^+) \to L^1_{loc}(\mathbb{R}^+) \supset u_1(\cdot,0)$, the denominator $|1 + r_2(t)|^2 + |r_3(t)|^2$ can not be zero, due to $m(t,0,\lambda)$
being a solution to the spectral problem \( \psi_t = (-2i\lambda^2 \sigma_3 + \tilde{Q}) \psi \) and given its asymptotic behavior as \( t \) goes to infinity. If \( c_2 \neq 0 \), then as \( t \to \infty \),

\[
\psi(t) = c_2 e^{2i\lambda_1^2 t} \begin{pmatrix} r_4(t) \\ 1 + r_5(t) \end{pmatrix}, \quad r_j \in H_t^{1,1}(\mathbb{R}_+), \quad j = 4, 5.
\]

The same reasoning makes sure that the denominator can not be zero and hence,

\[
(P[1])_{12} = \frac{(1 + r_5(t))^* r_4(t)}{|1 + r_4(t)|^2 + |r_5(t)|^2} \in H_t^{1,1}(\mathbb{R}_+)
\]

Thus,

\[
u_1 = u + (P[1])_{12} \in H_t^{1,1}(\mathbb{R}_+).
\]

By the second line of equation (3.9), it can also be shown that \((u_1)x(\cdot, 0) \in H_t^{0,1}(\mathbb{R}_+)\) in both cases, which implies \( u_1(\cdot, 0) \in X \). For \( \Im(\lambda^2) \geq 0 \), the choice of the normalization of \( m(t, 0, \lambda) \) and \( n(t, 0, \lambda) \) is reversed.

**Lemma 3.5.** Let \( u(\cdot, 0) \in X \), and \( D[1] \) be a dressing matrix constructed by \( \lambda_1 = \xi + i\eta \) and \( \psi_1 = (\mu, \nu)^T \) evaluated at \( x = 0 \). Then, \( D[1]|_{x=0} \) goes to either \( \text{diag}(\lambda - \lambda^*_1, \lambda - \lambda_1) \) or \( \text{diag}(\lambda - \lambda_1, \lambda - \lambda^*_1) \) as \( t \to \infty \), depending on the limit behavior of \( \psi_1 \).

**Proof.** At \( t = 0 \) and \( x = 0 \), \( \psi_1 \) is either being produced by \( (1, c)^T \), \( c \in \mathbb{C} \), or \( (0, 1)^T \). In the first case, \( \psi_1 = c_1 m(t, 0, \lambda_1) + c_2 n(t, 0, \lambda_1) \) for some constants \( c_1, c_2 \), where \( m(t, 0, \lambda) \) and \( n(t, 0, \lambda) \) are the linear independent solutions of the \( t \) part of the Lax system as constructed in the proof of Proposition 3.4. If \( \psi_1 \) is proportional to \( m(t, 0, \lambda_1) \), then necessarily \( c_2 = 0 \). As a consequence \( \frac{\nu}{\mu} = \frac{m(t, 0, \lambda_1)}{m_1(t, 0, \lambda_1)} \to 0 \) as \( t \to \infty \) and so \( D[1]|_{x=0} \) goes to \( \text{diag}(\lambda - \lambda_1, \lambda - \lambda^*_1) \) as \( t \to \infty \). If \( c_2 \neq 0 \), then, as \( t \to \infty \), \( \psi_1 = c_2 e^{2i\lambda_1^2 t} \begin{pmatrix} r_4(t) \\ 1 + r_5(t) \end{pmatrix}, \)

where \( r_4, r_5 \in H_t^{1,1}(\mathbb{R}_+) \) as before. Therefore, \( \frac{\nu}{\mu} \to 0 \) as \( t \to \infty \) and so \( D[1]|_{x=0} \) goes to \( \text{diag}(\lambda - \lambda^*_1, \lambda - \lambda_1) \) as \( t \to \infty \). In the second case, we necessarily have \( c_1 = 0 \) and again by \( n(t, 0, \lambda_1) \), we have that \( D[1]|_{x=0} \) goes to \( \text{diag}(\lambda - \lambda^*_1, \lambda - \lambda_1) \) as \( t \to \infty \).

4 DRESSING ON TWO HALF-LINES

In this section, we want to use the introduced Darboux transformation to construct soliton solutions for a model PDE on two half-lines which are connected via boundary conditions established through the localized Bäcklund transformation. Based on the ideas of [15], we will incorporate this boundary condition into the dressing process. In this way, we also show that this method can be, without further complications, generalized to a simple graph structure and to a time-dependent gauge transformation.
4.1 NLS equation with defect conditions

For the convenience of the reader, we will explicitly state the model as a generalization of the NLS equation (2.1) to the NLS equation on two half-lines

\[ iu_t + u_{xx} + 2|u|^2u = 0, \]
\[ u(0, x) = u_0(x) \]  
(4.1)

for \( u(t, x): \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{C} \) and initial condition \( u_0(x) \) for \( x \in \mathbb{R}_+ \) and

\[ i\tilde{u}_t + \tilde{u}_{xx} + 2|\tilde{u}|^2\tilde{u} = 0, \]
\[ \tilde{u}(0, x) = \tilde{u}_0(x) \]  
(4.2)

for \( \tilde{u}(t, x): \mathbb{R}_- \times \mathbb{R}_- \mapsto \mathbb{C} \) and initial condition \( \tilde{u}_0(x) \) for \( x \in \mathbb{R}_- \). In that context, taking for example \( u(t,0) = \tilde{u}(t,0) \) and \( u_x(t,0) = \tilde{u}_x(t,0) \) as boundary conditions, the two half-lines are connected such that there is no reflection and trivial transmission and by redefining the initial condition accordingly, we end up with the NLS equation as in (2.1). However, the model we are interested in arises with so-called defect conditions

\[ (\tilde{u} - u)_x = i\alpha(\tilde{u} - u) \pm \Omega(\tilde{u} + u), \]
\[ (\tilde{u} - u)_t = -\alpha(\tilde{u} - u)_x \pm i\Omega(\tilde{u} + u)_x + i(\tilde{u} - u)(|u|^2 + |\tilde{u}|^2) \]  
(4.3)

at \( x = 0 \). In particular, we have \( \Omega = \sqrt{\beta^2 - |\tilde{u} - u|} \) and defect parameter \( \alpha, \beta \in \mathbb{R} \).

It can be shown that the defect conditions (4.3) are equivalent to the general localized Bäcklund transformation (3.7) with \( u, \tilde{u}, \alpha, \beta \) and the respective sign as we have already matched. Since we want to incorporate the defect into the dressing method, we define the localized defect matrix corresponding to seed solutions \( u[0](t, x) \) and \( \tilde{u}[0](t, x) \) to the NLS equations on the respective half-line \( \mathbb{R}_+ \times \mathbb{R}_- \) and \( \mathbb{R}_- \times \mathbb{R}_+ \) and two spectral parameter \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \) \{0\} such that

\[ G_0(t, 0, \lambda) = 2\lambda I + \begin{pmatrix}
\alpha \pm i\sqrt{\beta^2 - |u[0] - u[0]|^2} & -i(\tilde{u}[0] - u[0]) \\
-i(\tilde{u}[0] - u[0])^* & \alpha \pm i\sqrt{\beta^2 - |\tilde{u}[0] - u[0]|^2}
\end{pmatrix} \]  
(4.4)

satisfies (3.7). Here, the sign in the (11)-entry and accordingly the (22)-entry can still be chosen freely. The goal in this section is to successfully apply the properties we have developed in order to dress the solutions \( u[0](t, x) \) and \( \tilde{u}[0](t, x) \) in such a way that we are able to find a localized defect matrix preserving the boundary constraint. In that regard, we talk about a similar form if the boundary constraint for the dressed solutions holds with the same spectral parameter \( \alpha \) and \( \beta \) as well as the same sign in front of the root of the (11)-entry. It is an important step in dressing the boundary to handle the sign in the entries of the matrix for the dressed localized defect matrix. Especially, we will use Proposition 3.4 and Lemma 3.5 to ensure that the signs match, when the solution is in a particular function space.
Remark 4.1. The connection of the defect conditions (4.3) to the boundary constraint (3.7) has been discussed, among other publications, in [3] and [5]. Therein, the authors additionally prove the existence of an infinite set of modified conservation laws, which means that the defect conditions are indeed integrable boundary conditions.

The defect conditions being integrable also establishes the possibility to apply the Darboux transformation to this model in order to produce soliton solutions.

4.2 Dressing the boundary

The results of this section are inspired by [13]. However, due to the differences in the model, we want to apply the dressing the boundary to, the implementation of the method differs from the original approach. In particular, the pairing of zeros $\lambda_1$ and $-\lambda_1$ in order to respect the relation of $V(t,0,\lambda)$ and $V(t,0,-\lambda)$ is not required in the presented model. There will be, nonetheless, a connection of the zeros $\lambda_1$ introduced for the positive half-line and the ones $\tilde{\lambda}_1$ introduced for the negative half-line. With that in mind, we present the main result of this paper.

Proposition 4.2. Consider solutions $u[0]$ and $\tilde{u}[0]$ to the NLS equation (4.1) and (4.2), which at $x = 0$ are in the function space $X$. In addition, let $u[0]$ and $\tilde{u}[0]$ be subject to the defect condition (4.3) with parameter $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$. Further, take $N + 1$ solutions $\psi_j$, $j = 0, \ldots, N$, of the undressed Lax system corresponding to $u[0]$ for $\lambda = \lambda_0 = -\frac{\alpha + i\beta}{2}$ and distinct $\lambda = \lambda_j \in \mathbb{C} \setminus (\mathbb{R} \cup \{\lambda_0, \lambda_0^*\})$, $j = 1, \ldots, N$. Constructing $G_0$ of localized defect form as in (4.4) with $u[0]$, $\tilde{u}[0]$, $\alpha$, $\beta$ and chosen sign, we assume that there exist paired solutions $\tilde{\psi}_j$ of the undressed Lax system corresponding to $\tilde{u}[0]$ for $\lambda = \lambda_j$, $j = 1, \ldots, N$, satisfying

$$\tilde{\psi}_j|_{x=0} = G_0(t,0,\lambda_j)\psi_j|_{x=0}, \quad j = 1, \ldots, N. \quad (4.5)$$

Then, two $N$-fold Darboux transformations $D[N]$, $\tilde{D}[N]$ using the corresponding solutions lead to solutions $u[N]$ and $\tilde{u}[N]$ to the NLS equations (4.1) and (4.2). In particular, the defect conditions are preserved.

To this end, we shall show that the functions $u[N]$ and $\tilde{u}[N]$ (a) satisfy the NLS equation on the respective half-line, (b) are in their Lax systems subject to the boundary constraint with an, for the time being, unspecified matrix $G_N$, and further, that (c) $G_N$ is of a similar localized defect form as $G_0$.

Proof. (a) The $N$-fold Darboux transformations $D[N]$, $\tilde{D}[N]$ construct, as presented in Section 3.1, solutions $u[N]$, $\tilde{u}[N]$ from seed solutions $u[0]$, $\tilde{u}[0]$, which satisfy the same partial differential equations. Therefore, having $N$ linearly independent solutions is enough to ensure that the transformed solutions satisfy the respective NLS equation. In that regard, as already mentioned the linear independence of $\psi_j$ and $\tilde{\psi}_j$ is implied by choosing distinct $\lambda_j$, $j = 1, \ldots, N$.

For (b) and (c) the existence of $\psi_0$ and a kernel vector, we call $\psi_0$, corresponding to the defect matrix $G_0$ is crucial. Given the generality of the statement, we have either
linear independence of \( v_0 \) in terms of \( \{ \psi_0, \varphi_0 \} \) or linear dependence, where \( \varphi_0 = \sigma_2 \psi_0^* \), 
\( \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) is an orthogonal vector of \( \psi_0 \). Moreover, we have that for a matrix \( G_N \), 
of the form \( G_N = \lambda \mathbb{I} + G^{(0)} \), the equality
\[
\tilde{D}[N]G_0 = G_N D[N],
\]
where \( G_0 \) is divided by 2 (to be similar to a dressing matrix), is sufficient for \( G_N \) to satisfy
\[
(G_N)_t = \tilde{V}[N]G_N - G_N V[N]
\]
at \( x = 0 \), except for the zeros of the \( t \) part of the Lax systems \( \lambda = \lambda_j, \ j = 1, \ldots, N \).
To show that, we multiply the first equation, evaluated \( x = 0 \), with \((D[N])^{-1}\) and 
differentiate the resulting equation to obtain
\[
(G_N)_t = (\tilde{D}[N]G_0(D[N])^{-1})_t 
= \tilde{D}_t[N]G_0(D[N])^{-1} + \tilde{D}[N](G_0)_t(D[N])^{-1} + \tilde{D}[N]G_0((D[N])^{-1})_t.
\]
Whereas, the first two summands can be simplified using (3.3) and (3.7) such that
\[
\tilde{D}_t[N]G_0 + \tilde{D}[N](G_0)_t = (\tilde{V}[N]\tilde{D}[N] - \tilde{D}[N]\tilde{V}[0])G_0 + \tilde{D}[N](\tilde{V}[0]G_0 - G_0 V[0])
= \tilde{V}[N]\tilde{D}[N]G_0 - \tilde{D}[N]G_0 V[0].
\]
In addition, it can be shown with (3.3) that for the third summand
\[
((D[N])^{-1})_t = -(D[N])^{-1}D_t[N](D[N])^{-1}
= -(D[N])^{-1}V[N] + V[0](D[N])^{-1}
\]
holds. Put together and notice that the expressions \( \tilde{D}[N]G_0(D[N])^{-1} \) are in fact again
\( G_N \), we obtain
\[
(G_N)_t = \tilde{V}[N]\tilde{D}[N]G_0(D[N])^{-1} - \tilde{D}[N]G_0(D[N])^{-1}V[N]
= \tilde{V}[N]G_N - G_N V[N].
\]
Analogously, the \( x \) part of the defect constraint is implied. Thereby, we know that if \( G_N \)
is of similar form as \( G_0 \), see (4.4), with \( \tilde{u}[0] - u[0] \) replaced by \( \tilde{u}[N] - u[N] \), then \( u[N] \) 
and \( \tilde{u}[N] \) satisfy the defect conditions. However, let us first prove that we can construct 
a matrix \( G_N \) of first order in \( \lambda \) such that \( \tilde{D}[N]G_0 = G_N D[N] \) at \( x = 0 \), as above.

(b) As mentioned already, the defect matrix \( G_0 \) admits, see Remark 3.2 a kernel 
vector \( v_0 \) at \( \lambda_0 \), since \( \Im(\lambda_0) \neq 0 \). Hence,
\[
G_0(t, 0, \lambda_0)v_0 = 0.
\]
ψ_0 solves \[
\begin{align*}
\psi_x &= U[0]\psi \\
\psi_t &= V[0]\psi 
\end{align*}
\]
\[\rightarrow\]
\[
\psi'_0 = D[N]\psi_0\]

\[
\tilde{\psi}_0 = G_0\psi_0\]
\[
\tilde{\psi}_0 = \tilde{G}_0\tilde{\psi}_0\]
\[
\tilde{\psi}_0 = \tilde{G}_N\tilde{\psi}_0\]

Fig. 1. Properties of \(\psi_0\) at \(\lambda = \lambda_0\) and \(x = 0\) if \(G_0\psi_0 \neq 0\).

If this vector \(\psi_0\) differs from a linear combination of \(\{\psi_0, \varphi_0\}\), we have that the diagram of Figure 1 holds. Thus, as \(\psi_0\) is linearly independent of \(\psi_1, \ldots, \psi_N\), \(D[N]\) is invertible at \(\lambda = \lambda_0\). Therefore, we can transform \(\psi_0\), the solution to the Lax system corresponding to \(u[N]\) at \(\lambda = \lambda_0\), to \(\tilde{\psi}_0 = \tilde{D}[N]G_0[D[N]]^{-1}\psi_0\). In turn, this vector \(\tilde{\psi}_0\) is a solution to the Lax system corresponding to \(\tilde{u}[N]\) at \(\lambda = \lambda_0\). Consequently, the matrix, we call \(G_{N_1}\), given by the product \(\tilde{D}[N]G_0(D[N])^{-1}\), satisfies the equations
\[
((G_{N_1})_x - \tilde{U}[N]G_{N_1} + G_{N_1}U[N])\psi'_0 = 0,
\]
\[
((G_{N_1})_t - \tilde{V}[N]G_{N_1} + G_{N_1}V[N])\psi'_0 = 0
\]
at \(\lambda = \lambda_0\) and \(x = 0\). Then, we have with the equivalence above at \(x = 0\) the following
\[
\tilde{D}[N]G_0\psi_0 = G_{N_1}D[N]\psi_0 \neq 0, \quad \lambda = \lambda_0,
\]
\[
\tilde{D}[N]G_0\varphi_0 = G_{N_1}D[N]\varphi_0 \neq 0, \quad \lambda = \lambda_0'.
\]
\[\text{(4.6)}\]

It is reasonable to assume that \(G_{N_1}\) is a polynomial matrix of order 1, due to the product \(\tilde{D}[N]G_0(D[N])^{-1}\). Indeed, the dressing matrices can be written as \(\lambda^N(\mathbb{I} + O(\frac{1}{\lambda}))\) and therefore,
\[
\tilde{D}[N]G_0(D[N])^{-1} = (\mathbb{I} + O(\frac{1}{\lambda}))G_0(\mathbb{I} + O(\frac{1}{\lambda})) = \lambda\mathbb{I} + \tilde{G}(0) + O(\frac{1}{\lambda}).
\]

Whereas the term \(O(\frac{1}{\lambda})\) on the right hand side needs to be identically zero, since we have a product of polynomials. Further, evaluating the determinant of \(G_{N_1}\) at \(\lambda = \lambda_0\) and \(\lambda = \lambda_0'\), we obtain \(\det(G_{N_1}) = 0\). This is implying that there is a kernel vector \(\tilde{v}_0\) such that \(G_{N_1}(t, 0, \lambda_0)\tilde{v}'_0 = 0\). Constructing a one-fold dressing matrix with \(\tilde{v}_0\) at \(\lambda = \lambda_0\), we obtain
\[
G_{N_1} = (\lambda - \lambda_0^*\mathbb{I} + (\lambda_0^* - \lambda_0)\tilde{P}_0', \quad \tilde{P}_0' = \frac{\tilde{v}'_0(\tilde{v}_0)}{(\tilde{v}_0)}\]
\[\text{(4.7)}\]

where \(G_{N_1}\) satisfies the property \(\text{(4.6)}\).

On the other hand, if w.l.o.g. \(v_0 = \psi_0\), we construct \(G_{N_2}\) with a different vector. Remember that \(\psi_0\) is linearly independent of \(\psi_1, \ldots, \psi_N\). Define a new vector
\[
\nu'_0 = D[N](t, 0, \lambda_0)v_0.
\]

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Fig. 2. Permutability of defect matrices at $x = 0$.

In this case, the strategy is to construct a one-fold dressing matrix with the defined vector $v_0'$ at $\lambda = \lambda_0$ by

$$G_{N_2} = (\lambda - \lambda_0^*)I + (\lambda_0^* - \lambda_0)P_0', \quad P_0' = \frac{v_0'(v_0')^\dagger}{(v_0')^\dagger v_0'},$$

(4.8)
such that $G_{N_2}(t, 0, \lambda_0)v_0' = 0$. This results in the property

$$\tilde{D}[N]G_0\psi_0 = G_{N_2}D[N]\psi_0 = 0, \quad \lambda = \lambda_0,$$

$$\tilde{D}[N]G_0\varphi_0 = G_{N_2}D[N]\varphi_0 = 0, \quad \lambda = \lambda_0^*$$

(4.9)
at $x = 0$.

Constructing $G_N$ as in one of the two cases $G_{N_1}$ or $G_{N_2}$ will give us commutating matrices at the point $x = 0$ of the defect conditions. In particular, we can now show that

$$(\tilde{D}[N]G_0)|_{x=0} = (G_ND[N])|_{x=0}.$$  (4.10)

To prove (4.10), we write each side as a matrix polynomial, by dividing $G_0$ by 2 and denoting the left and right hand side respectively as $L(\lambda)$ and $R(\lambda)$, we obtain in both cases, $N = N_1$ or $N = N_2$, the following

$$L(\lambda) = \tilde{D}[N]G_0 = \lambda^{N+1}L_{N+1} + \lambda^NL_N + \cdots + \lambda L_1 + L_0,$$

$$R(\lambda) = G_ND[N] = \lambda^{N+1}R_{N+1} + \lambda^NR_N + \cdots + \lambda R_1 + R_0.$$

Since $L_{N+1} = I = R_{N+1}$, only $L_N, R_N, \ldots, L_1, R_1, L_0$ and $R_0$ need to be determined. In that regard, we consider the zeros and associated kernel vectors of $L(\lambda)$ and $R(\lambda)$.

By construction of the dressing matrices $D[N], \tilde{D}[N]$, we have that $D[N](t, x, \lambda_j)\psi_j = 0$ and $\tilde{D}[N](t, x, \lambda_j)\tilde{\psi}_j = 0, j = 1, \ldots, N$, which we will combine with the assumed relation between $\psi_j$ and $\tilde{\psi}_j$. Thus, for the $N$ linearly independent $\psi_1, \ldots, \psi_N$, we have

$$L(\lambda)|_{\lambda = \lambda_j} \psi_j = 0,$$

$$R(\lambda)|_{\lambda = \lambda_j} \psi_j = 0,$$

$$j = 1, \ldots, N,$$

whereby these equalities hold for $x = 0$. Here, the symmetry of the Lax pair provides another vector $\varphi_j = \sigma_2\psi_j^*$, which is orthogonal to $\psi_j$. Analogously, let $\tilde{\varphi}_j = \sigma_2\tilde{\psi}_j^*$ and it follows that

$$L(\lambda)|_{\lambda = \lambda_j^*} \varphi_j = 0,$$

$$R(\lambda)|_{\lambda = \lambda_j^*} \varphi_j = 0$$
for $j = 1, \ldots, N$ and $x = 0$. For a defect matrix of order one, this is not enough to ensure equality in (4.10). However, we constructed $G_N$ in a way such that there is an additional vector pair for which the two sides are equal. For $N = N_2$, it should be noted that even if the kernel vector is a linear combination of $\psi_0$ and $\varphi_0$, it is possible to repeat the following steps, but the notation becomes unhandy without giving more insight. Hence for $N = N_1$ and $N = N_2$, we consider

$$L(\lambda)|_{\lambda = \lambda_0} \psi_0 = R(\lambda)|_{\lambda = \lambda_0} \psi_0,$$

whereby this equality is either nonzero for $N = N_1$ or zero for $N = N_2$ and holds for $x = 0$. As before, the symmetry of the Lax pair provides another vector $\varphi_0 = \sigma_2 \psi_0^*$, which is orthogonal to $\psi_0$ and for $x = 0$, it satisfies

$$L(\lambda)|_{\lambda = \lambda_0} \varphi_0 = R(\lambda)|_{\lambda = \lambda_0} \varphi_0.$$

This additional pair of vectors determines $L(\lambda) - R(\lambda) = C(\lambda) = \lambda^N C_N + \cdots + \lambda C_1 + C_0$. Together with the zeros and associated kernel vectors of the Darboux matrices $D[N]$, $\bar{D}[N]$, it can be written as a set of algebraic equations

$$\begin{pmatrix}
\lambda_0^N C_N + \cdots + \lambda_0 C_1 + C_0 \\
\vdots \\
\lambda_N^N C_N + \cdots + \lambda_N C_1 + C_0
\end{pmatrix} \psi_0 = 0, 
\begin{pmatrix}
(\lambda_0^*)^N C_N + \cdots + \lambda_0^* C_1 + C_0 \\
\vdots \\
(\lambda_N^*)^N C_N + \cdots + \lambda_N^* C_1 + C_0
\end{pmatrix} \varphi_0 = 0.$$

In matrix form, we have

$$\begin{pmatrix}
\psi_0 \\
\vdots \\
\psi_N
\end{pmatrix} \begin{pmatrix}
\lambda_0^N & (\lambda_0^*)^N & \cdots & \lambda_N^N & (\lambda_N^*)^N \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\psi_0 & \varphi_0 & \cdots & \psi_N & \varphi_N
\end{pmatrix} = 0.$$

The $(2N + 2) \times (2N + 2)$ matrix filled with $\{\psi_0, \varphi_0, \ldots, \psi_N, \varphi_N\}$ is invertible. If the determinant was zero, we could find coefficients in $\mathbb{C}$ such that a linear combination of $\{\psi_0, \varphi_0, \ldots, \psi_N, \varphi_N\}$ would be zero, which is a contradiction to their linear independence. If, for $N = N_2$, $\psi_0$ was a linear combination of $\psi_0$ and $\varphi_0$, the matrix would still be invertible with this linear combination and its orthogonal in the first and second column, respectively. Thereby, $L(\lambda) = R(\lambda)$ holds in both cases $N = N_1$ as well as $N = N_2$, which, in turn, implies that (b) is satisfied.

By (b), we have matrices $G_{N_1}$ and $G_{N_2}$ which satisfy the boundary constraint and are of the form of a dressing matrix at $x = 0$. Further, the equality (4.10) ensures that $\psi'_0$ is also for $G_{N_1}$ equal to $D[N](t, 0, \lambda_0)\psi_0$. It is an important fact that in both cases the kernel vector $\psi'_0$ of $G_0(t, 0, \lambda_0)$ takes the role of the kernel vector $\psi'_0$ and $\psi'_0$ of respectively $G_{N_1}$ and $G_{N_2}$. Only then, we can prove that the explicit forms of $G_{N_1}$ and $G_{N_2}$ are consistent with $G_0$ through Proposition 3.4 and Lemma 3.5. In general, we can not think of them being the same, since we assumed for $N = N_2$ that w.l.o.g.
\[ v_0 = \psi_0. \] Nevertheless, the equality \[4.10\] already provides the localized defect form of \( G_{N_1} \) and \( G_{N_2} \), which was commented on in Remark \[3.2\] except for the sign in front of the \((11)\)-entry.

(c) With the given information we are able to find the localized defect form of \( G_N = \lambda I + \tilde{G}^{(0)} \), where the proof is similar in both cases \( G_{N_1} \) and \( G_{N_2} \). That is, from the off-diagonal of \( L_N = R_N \), it can be seen that \( \tilde{G}_{12}^{(0)} \) and \( \tilde{G}_{21}^{(0)} \) can respectively be written as \(-i(\tilde{u}[N] - u[N])/2\) and \(-i(\tilde{u}[N] - u[N])^*/2\) at \( x = 0 \), which gives

\[
G_N(t, 0, \lambda) = \lambda I + \frac{1}{2} \begin{pmatrix}
2\tilde{G}_{11}^{(0)} & -i(\tilde{u}[N] - u[N])^* \\
-i(\tilde{u}[N] - u[N]) & 2\tilde{G}_{22}^{(0)}
\end{pmatrix}, \tag{4.11}
\]

We can compute the determinant of \( G_N \), where we use the property of determinants of Darboux transformations, so that at \( x = 0 \),

\[
det(G_N) = det(\tilde{D}[N]) det(G_0) det((D[N])^{-1}) = det(G_0)
\]

\[
= (\lambda - \lambda_0)(\lambda - \lambda_0^*) = \lambda^2 + \alpha \lambda + \frac{\alpha^2 + \beta^2}{4}
\]

and in particular, the determinant is independent of \( t \) and \( x \). Comparing with \[4.11\], we obtain

\[
\alpha = \tilde{G}_{11}^{(0)} + \tilde{G}_{22}^{(0)},
\]

\[
\alpha^2 + \beta^2 = 4\tilde{G}_{11}^{(0)} \cdot \tilde{G}_{22}^{(0)} + |\tilde{u}[N] - u[N]|^2.
\]

Solving for \( \tilde{G}_{11}^{(0)} \) and \( \tilde{G}_{22}^{(0)} \) at \( x = 0 \), we have

\[
2\tilde{G}_{11}^{(0)} = \alpha \pm i\sqrt{\beta^2 - |\tilde{u}[N] - u[N]|^2},
\]

\[
2\tilde{G}_{22}^{(0)} = \alpha \mp i\sqrt{\beta^2 - |\tilde{u}[N] - u[N]|^2}.
\]

However, at this point the signs of \( \tilde{G}_{11}^{(0)} \) and \( \tilde{G}_{22}^{(0)} \) are not necessarily the same as the signs of the defect matrix \( G_0 \). In that regard, we know that from solutions \( u[0], \tilde{u}[0] \) to the defect conditions with a selected sign, we can construct solutions \( u[N], \tilde{u}[N] \) which satisfy the defect conditions with either the plus or the minus sign. A particular case can be determined for which we are able to prove that the signs stay the same, ultimately restricting the solution space.

Assuming that we have

\[
u[0](\cdot, 0), \tilde{u}[0](\cdot, 0) \in X, \tag{4.12}\]

then \( u[N](\cdot, 0), \tilde{u}[N](\cdot, 0) \in X \) by Proposition \[3.4\] since \( D[N], \tilde{D}[N] \) are \( N \) transformations of the form \( B_{\lambda_j}(\psi_j) \) for \( j = 1, \ldots, N \). In that class of solutions, we can identify the signs for matrices \( G_0 \) and \( G_N \) of localized defect form through the kernel vectors respective to their form as Darboux transformation. We know that in both cases \( v_0 \) is the kernel vector for \( G_0 \) at \( \lambda = \lambda_0 \) and by construction, we have that \( \omega_0 = D[N](t, 0, \lambda_0)v_0 \) is the kernel vector of \( G_N \) at \( \lambda = \lambda_0 \) and \( x = 0 \). On the other hand, as \( t \) goes to infinity \( G_0 \)
becomes a diagonal matrix and as a consequence, the limit behaviors of \( \psi_j, \tilde{\psi}_j \) are the same for \( j = 1, \ldots, N \), since they are connected through \( G_0 \). Consequently, the dressing matrices \( \tilde{D}[N] \) and \( D[N] \) have the same distribution of \( \lambda - \lambda_j \) and \( \lambda - \lambda_j^* \) in their diagonal form as \( t \to \infty \). Thus,

\[
\lim_{t \to \infty} G_N = \lim_{t \to \infty} \tilde{D}[N]G_0(D[N])^{-1} = \lim_{t \to \infty} G_0. \tag{4.13}
\]

Also the vectors \( \upsilon_0 \) and \( \omega_0 \), respectively the kernel vectors of \( G_0 \) and \( G_N \) at \( \lambda = \lambda_0 \), admit the same limit behavior as \( t \to \infty \), since they are connected by \( D[N] \) which admits a diagonal structure as \( t \to \infty \). Starting with a plus (minus) sign in the \((11)\)-entry of the defect matrix \( G_0 \), we can then conclude by the limit behavior of \( \upsilon_0 \) and \( \omega_0 \) as well as (4.13) that the sign in the \((11)\)-entry of the defect matrix \( G_N \) needs to plus (minus). Therefore, they satisfy similar defect conditions on the spectral side with \( V[0] \), \( \tilde{V}[0] \) replaced by \( V[N] \) and \( \tilde{V}[N] \), which gives the result in the solution class \( X \).

With \( G_N \) of similar localized defect form and satisfying \((G_N)_t = \tilde{V}[N]G_N - G_NV[N] \) at \( x = 0 \), we can conclude that the defect conditions are preserved for \( u[N] \) and \( \tilde{u}[N] \).

With Proposition 4.2 we proved that dressing the boundary can be applied to the NLS equation on a simple star-graph with a non trivial boundary condition. Therefore, extending the method, presented in [13], to more than one half-line and also considering time-dependent boundary matrices. Thereby, we have given a way to construct \( N \)-soliton solutions for particular seed solutions \( u[0] \) and \( \tilde{u}[0] \). It should be mentioned that, since the seed solutions are a part of the localized defect matrix \( G_0 \), their influence on the construction of \( \tilde{u}[N] \) is decisive. In fact, it is a priori not clear, whether there exists a solution \( \psi_1 \) at \( \lambda = \lambda_1 \) to the undressed Lax system of (4.2) satisfying (4.5).

The important feature of the proof is that the localized defect matrix \( G_0 \) is interchangeable with the Darboux matrices \( D[N] \) and \( \tilde{D}[N] \) in the sense of Figure 2. In turn, this is realized with the transformation of the localized defect matrix \( G_0 \) to a Darboux transformation and vice versa the Darboux matrix \( G_N \) to a localized defect matrix, which has been mentioned in Remark 3.2. Apparently, this transformation process is not necessary until applying a time-dependent boundary matrix.

**Remark 4.3.** In [5] similar results of a two-soliton solution subject to the defect conditions have been presented without utilizing the spectral side of the model. With the background of a Bäcklund transformation, the solution was assumed to be an individual soliton on each side of the defect, it was checked with an algebra program that these functions indeed solve the defect conditions, however only with \( \alpha = 0 \).

In the following section, we want to give an application of Proposition 4.2.
5 Dressing soliton solutions

5.1 N-soliton solutions

Consider the zero seed solutions $u[0] = \tilde{u}[0] = 0$. Particularly, $u[0](\cdot, 0), \tilde{u}[0](\cdot, 0) \in X$ and $u[0](t, \cdot) \in H^{1,1}(\mathbb{R}_+), \tilde{u}[0](t, \cdot) \in H^{1,1}(\mathbb{R}_-) \text{ and } \mathbb{C} \setminus \mathbb{R} \ni \lambda_0 = -\frac{\alpha + i\beta}{2}$. Hence,

$$G_0(t, 0, \lambda) = 2\lambda I + \begin{pmatrix} \alpha \pm i\beta & 0 \\ 0 & \alpha \mp i\beta \end{pmatrix}.$$  

So for solutions $\psi_j$ to the Lax pair corresponding to $u[0]$ at $\lambda = \lambda_j \in \mathbb{C} \setminus \{\mathbb{R} \cup \{\lambda_0, \lambda_0^*\}\}$, $j = 0, \ldots, N$, we have

$$\psi_j = \begin{pmatrix} \mu_j \\ \nu_j \end{pmatrix} = e^{(-i\lambda_j x - 2i\lambda_j^3 t)\sigma_3} \begin{pmatrix} u_j \\ v_j \end{pmatrix}$$

with $(u_j, v_j) \in \mathbb{C}^2$ and since the relation

$$\lim_{x \to 0} \tilde{\psi}_j(x) = G_0(t, 0, \lambda_j)\psi_j \bigg|_{x=0}$$

should hold for $j = 1, \ldots, N$ and solutions defined by $\tilde{\psi}_j = e^{(-i\lambda_j x - 2i\lambda_j^3 t)\sigma_3}(\tilde{u}_j, \tilde{v}_j)^\dagger$, $(\tilde{u}_j, \tilde{v}_j) \in \mathbb{C}^2$, of the Lax system corresponding to $\tilde{u}[0]$ at $\lambda = \lambda_j$, we obtain the following relation for the spectral parameter $\tilde{u}_j, \tilde{v}_j, u_j$ and $v_j$,

$$\tilde{u}_j \nu_j = \frac{2\lambda_j + \alpha \pm i\beta}{2\lambda_j + \alpha \mp i\beta} u_j \nu_j, \quad j = 1, \ldots, N.$$  

This is enough to apply Proposition 4.2. Note that changing the sign of $\beta$ is the same as changing the sign in the defect conditions. We also know that the N-soliton solution constructed with Proposition 4.2 satisfy $u[N](\cdot, 0), \tilde{u}[N](\cdot, 0) \in X$ and $u[N](t, \cdot) \in H^{1,1}(\mathbb{R}_+)$, $\tilde{u}[N](t, \cdot) \in H^{1,1}(\mathbb{R}_-)$, due to Proposition 3.4 and Proposition 4.7 in [6], which can easily be extended to Darboux transformations where $\lambda_1$ has a real part. Moreover, similar analysis holds true for $u[0](t, \cdot) = 0$ in $H^{1,1}(\mathbb{R})$, then $u[N](t, \cdot) \in H^{1,1}(\mathbb{R})$. As in the proof of Proposition 4.2, we can use this fact to make sure that, after finding the defect form $B_N(t, x, \lambda)$ for $x \in \mathbb{R}$ of the localized defect matrix $G_N(t, 0, \lambda)$, the sign in front of the root in the (11)-entry is consistent with the sign of the defect form $B_0(t, x, \lambda) = G_0(t, 0, \lambda)$ for $x \in \mathbb{R}$ of the localized defect matrix $G_0(t, 0, \lambda)$. Ultimately, we can use this extension to show that each soliton interacts with the defect individually.

Taking the same Darboux transformations, however, applying them to zero seed solutions $u[0]$ and $\tilde{u}[0]$ on the whole line $x \in \mathbb{R}$, we obtain two N-soliton solutions $u_N(t, x)$ and $\tilde{u}_N(t, x)$ for the NLS equation for $x \in \mathbb{R}$. Suppose their corresponding solutions to the Lax system are related by the dressing transformation

$$\tilde{\psi}(t, x, \lambda) = B_N(t, x, \lambda)\psi(t, x, \lambda).$$

Then the matrix $B_N(t, x, \lambda)$ solves the system [3.1]. As explained before, assuming this matrix is linear in $\lambda$, it can only be of the form described in Proposition 3.1 which means
there exist real parameter \( \delta, \gamma \in \mathbb{R} \) and a \( \pm \) sign to be determined such that
\[
B_N(t, x, \lambda) = 2\lambda I + \begin{pmatrix}
\delta + i\sqrt{\gamma^2 - |\tilde{u}_N - u_N|^2} & -i(\tilde{u}_N - u_N) \\
-i(\tilde{u}_N - u_N)^* & \delta + i\sqrt{\gamma^2 - |\tilde{u}_N - u_N|^2}
\end{pmatrix}.
\]
At their respective half-line, the full line solutions \( u_N(t, x) \) and \( \tilde{u}_N(t, x) \) can be reduced to their half-line counterpart \( u[N](t, x) \), \( \tilde{u}[N](t, x) \). Hence,
\[
B_N(t, 0, \lambda) = 2\lambda I + \begin{pmatrix}
\delta + i\sqrt{\gamma^2 - |\tilde{u}[N] - u[N]|^2} & -i(\tilde{u}[N] - u[N]) \\
-i(\tilde{u}[N] - u[N])^* & \delta + i\sqrt{\gamma^2 - |\tilde{u}[N] - u[N]|^2}
\end{pmatrix}.
\]
However, at \( x = 0 \), we know that the two solutions \( u[N](t, x) \) and \( \tilde{u}[N](t, x) \) can be connected with the defect matrix \( G_N(t, 0, \lambda) \) used in the proof of Proposition 4.2, i.e. \( B_N(t, 0, \lambda) = 2G_N(t, 0, \lambda) \). Therefore, we can deduce that \( \delta = \alpha, \gamma^2 = \beta^2 \). This means that the matrix \( G_N(t, 0, \lambda) \), constructed in the proof in order to show that the boundary condition is preserved, has in fact a continuation \( B_N(t, x, \lambda) \) for \( x \in \mathbb{R} \). Due to \( u_N(t, \cdot), \tilde{u}_N(t, \cdot) \in H^{1, 1}(\mathbb{R}) \), we have that as \( x \) goes to plus or minus infinity:
\[
\lim_{|x| \to \infty} B_N(t, x, \lambda) = 2\lambda I + \begin{pmatrix}
\alpha + i|\beta| & 0 \\
0 & \alpha - i|\beta|
\end{pmatrix}.
\]
As before, we can make out the exact sign through the kernel vectors. For the \( N \)-soliton solution, we have that the kernel vector \( \psi_0 \) can be connected with the localized defect matrix \( G_0 \) in the following equation (5.1), where \( \lambda_0 = -\frac{\alpha \pm i\beta}{2} \), \( u_0 \in \mathbb{C} \setminus \{0\} \) arbitrary and \( v_0 = 0 \). Here, the \( \pm \) sign in \( \lambda_0 \) is the same as in the localized defect matrix \( G_0 \). Continuing \( G_0 \) to a defect matrix \( B_0(t, x, \lambda) \) for both zero seed solutions on the full line, we see that the kernel vector \( \psi_0 \) carries the information of the signs as \( |x| \to \infty \).

Now, we know that the Darboux transformed kernel vector \( \tilde{\psi}_0 = D[N](t, 0, \lambda_0)\psi_0 \) is the kernel vector for \( G_N(t, 0, \lambda) \) and hence for \( B_N(t, 0, \lambda_0) \). However, since in this case the kernel vector is at the same time a solution to the \( x \) part of the Lax system, we obtain at \( \lambda = \lambda_0 \) the following
\[
(B_N)_{x}\tilde{\psi}_0 = \tilde{U}B_N\tilde{\psi}_0 - B_NU\tilde{\psi}_0 = \tilde{U}B_N\tilde{\psi}_0 - B_N(\tilde{\psi}_0)_x.
\]
Thus, \( (B_N\tilde{\psi}_0)_x = \tilde{U}B_N\tilde{\psi}_0 \) at \( \lambda = \lambda_0 \) and every \( x \in \mathbb{R} \). In turn, this implies, given \( B_N(t, 0, \lambda_0)\tilde{\psi}_0 = 0 \), that \( B_N(t, x, \lambda_0)\tilde{\psi}_0 = 0 \) for every \( x \in \mathbb{R} \). Then, notice that \( \tilde{\psi}_0 \) has the same asymptotic behavior as \( \psi_0 \), since the dressing matrix goes to a diagonal matrix for \( |x| \to \infty \). As a consequence, the signs in the entries of \( \lim_{x \to \pm \infty} B_N(t, x, \lambda) \) are completely determined by the signs of the limits from the defect matrix \( B_0(t, x, \lambda) \). Which amounts in the problem presented to
\[
B_\infty(\lambda) = \lim_{|x| \to \infty} B_N(t, x, \lambda) = 2\lambda I + \begin{pmatrix}
\alpha + i\beta & 0 \\
0 & \alpha - i\beta
\end{pmatrix}.
\]
Knowing that, we see that the Jost solutions have relations induced by the Bäcklund transformation and the same normalization factor \( B_\infty^{-1}(\lambda) \),
\[
\tilde{\psi}_\pm(t, x, \lambda) = B_N(t, x, \lambda)\psi_\pm(t, x, \lambda)B_\infty^{-1}(\lambda).
\]

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In turn, this relation implies the following relation for the corresponding scattering matrices:

\[ \tilde{A}(\lambda) = B_\infty(\lambda)A(\lambda)B_\infty^{-1}(\lambda), \quad \lambda \in \mathbb{R}. \]  
(5.2)

**Corollary 5.1.** Let \( u(t, x) \) and \( \tilde{u}(t, x) \) be two pure N-soliton solutions of the NLS equation on \( \mathbb{R} \) constructed by the corresponding vectors used in Proposition 4.2 and let their restrictions to respectively the positive and negative half-line be subject to the defect conditions \((4.3)\) at \( x = 0 \). Then, it follows for \( \lambda_j \in \mathbb{C}_+ \) that solitons are transmitted through the defect independently of one another, i.e. for all \( j = 1, \ldots, N \) the following holds

\[ \tilde{x}_j - x_j = \frac{1}{2\eta_j} \log \left( \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta} \right), \]

\[ \tilde{\varphi}_j - \varphi_j = \arg \left( \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta} \right). \]

**Proof.** By the analysis above, we know in this case that

\[ B_\infty(\lambda) = \lim_{|x| \to \infty} B_N(t, x, \lambda) = (2\lambda + \alpha) \mathbb{I} \pm i\beta \sigma_3, \]

where the sign in front of \( \beta \) matches the sign of the defect. The relation of the Jost solutions gives

\[ \tilde{\psi}_-(1) = B_N(t, x, \lambda)\psi_-(1)(2\lambda + \alpha \pm i\beta)^{-1}, \]

\[ \tilde{\psi}_+(2) = B_N(t, x, \lambda)\psi_+(2)(2\lambda + \alpha \mp i\beta)^{-1}. \]

We can deduce using \((5.3)\) and \((2.6)\) that

\[ \tilde{\psi}_-(1)(t, x, \lambda_j) = \frac{b_j}{2\lambda_j + \alpha \pm i\beta} B_N(t, x, \lambda_j)\psi_+(2)(t, x, \lambda_j) \]

and with \((5.3)\) and the corresponding weight relations to \((2.6)\) for the extended solution \( \tilde{u}(t, x) \), we obtain

\[ = \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta} \tilde{b}_j \tilde{\psi}_-(1)(t, x, \lambda_j). \]

Therefore, the constants \( \tilde{b}_j \) and \( b_j \) can be related by

\[ \frac{\tilde{b}_j}{b_j} = \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta}. \]  
(5.4)

Moreover, the relation \((5.2)\) for the scattering matrices implies

\[ \tilde{a}_{22}(\lambda) = a_{22}(\lambda), \]  
(5.5)

\[ \tilde{a}_{12}(\lambda) = \frac{2\lambda + \alpha \pm i\beta}{2\lambda + \alpha \mp i\beta} a_{12}(\lambda). \]
These two relation (5.4) and (5.5) can be combined to relate the weights $\tilde{C}_j$ and $C_j$ in the following way

$\tilde{C}_j = \frac{\tilde{b}_j a_{22}'(\lambda_j)}{b_j a_{22}'(\lambda_j)} = \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta},$

from where we can see the influence on the $N$-soliton solution. Thereby, writing the norming constants as

$C_j = 2\eta_j e^{2\eta_j x_j + i\varphi_j}, \quad \tilde{C}_j = 2\eta_j e^{2\eta_j \tilde{x}_j + i\tilde{\varphi}_j}$

for $j = 1, \ldots, N$ as motivated for the one-soliton solution in Section 2, we obtain for the spatial shift $\tilde{x}_j - x_j$ and the phase shift $\tilde{\varphi}_j - \varphi_j$ the following

$\tilde{x}_j - x_j = \frac{1}{2\eta_j} \log \left( \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta} \right),$

$\tilde{\varphi}_j - \varphi_j = \arg \left( \frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta} \right),$

which implies that solitons experience independently of one another.

Remark 5.2. Another way of proving Corollary 5.1 is to use Theorem 1.22 of [10], where it is shown that the scattering data is, after successive iteration of the Darboux transformation, in each step, which we indicate by $[j]$, given by

$a_{11}^{[j]}(\lambda) = \frac{\lambda - \lambda_j}{\lambda - \lambda_j^*} a_{11}^{[j-1]}(\lambda), \quad \tilde{a}_{11}^{[j]}(\lambda) = \frac{\lambda - \lambda_j}{\lambda - \lambda_j^*} \tilde{a}_{11}^{[j-1]}(\lambda), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R},$

$a_{21}^{[j]}(\lambda) = a_{21}^{[j-1]}(\lambda), \quad \tilde{a}_{21}^{[j]}(\lambda) = \tilde{a}_{21}^{[j-1]}(\lambda), \quad \lambda \in \mathbb{R},$

$C_k^{[j]} = \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j^*} C_k^{[j-1]}, \quad \tilde{C}_k^{[j]} = \frac{\lambda_k - \lambda_j}{\lambda_k - \lambda_j^*} \tilde{C}_k^{[j-1]}, \quad k = 1, \ldots, j - 1,$

$C_j^{[j]} = \frac{\lambda_j - \lambda_j^*}{a_{11}^{[j-1]}(\lambda_j)}, \quad \tilde{C}_j^{[j]} = \frac{\lambda_j - \lambda_j^*}{\tilde{a}_{11}^{[j-1]}(\lambda_j)}.$

Therefore, given that $a_{11}^{[0]}(\lambda) = 1$ and $a_{11}^{[0]}(\lambda) = 0$ for the zero seed solution $u[0](t, x) = 0$, we have that

$\frac{\tilde{C}_j^{[N]} C_j^{[N]}}{C_j^{[N]}} = 2\lambda_j + \alpha \mp i\beta \quad j = 1, \ldots, N.$

The complex conjugation of the quotients $\tilde{u}_j^* + i\tilde{r}_j^*$ is due to the fact that in the referenced book, the dressing is done with the Jost functions which effectively go to $e_1$ and $e_2$ as $x$ respectively goes to $-\infty$ and $\infty$. In order for the $\psi_j$ to comply with that requirement, we need to change their asymptotic behavior while making sure that the $\lambda_j$ are chosen correctly. In summary, the theorem affirms Corollary 5.1.

The expression $\frac{2\lambda_j + \alpha \mp i\beta}{2\lambda_j + \alpha \pm i\beta}$ lets us state some facts about the behavior of the spatial and phase shift of the $N$-soliton after interacting with the defect. Letting $\beta$ go to zero,
Fig. 3. Contour plot of a one-soliton solution satisfying the NLS equation on each half-line and the defect conditions (4.3) with defect parameter $\alpha = 0$ and $\beta = 1$ (left) as well as $\beta = 3$ (right).

Fig. 4. Plot of a three-soliton solution (left) and its contour (right) satisfying the NLS equation on each half-line and the defect conditions (4.3) with defect parameter $\alpha = 0$ and $\beta = 1$. 
the quotient goes to 1, which indicates the discontinuity at \( x = 0 \) disappears, suggesting that \( \alpha \) by itself can not maintain it. Whereas letting \(|\beta|\) go to infinity, the quotient goes to \(-1\), which means no considerable spatial shift as \( \tilde{x}_j - x_j \) goes to zero and essentially a shape inversion as \( \tilde{\varphi}_j - \varphi_j \) goes to \( \pi \) for all \( j = 1, \ldots, N \). However, if we take \( \beta \in \mathbb{R} \setminus \{0\} \) and let \(|\alpha|\) go to infinity, the effect of the discontinuity also disappears, i.e. \( \tilde{x}_j - x_j \) and \( \tilde{\varphi}_j - \varphi_j \) both go to zero for all \( j = 1, \ldots, N \). Hence, the second defect parameter may be understood as a means to smooth out the discontinuity in the presence of the defect condition \((\beta \neq 0)\). Therefore, the discontinuity reaches its full potential, when \( \alpha = 0 \).

In this regard, we plotted the absolute value of the one-soliton solutions, \( u[1] \) and \( \tilde{u}[1] \) satisfying the NLS equation on \( \mathbb{R}_+ \) and \( \mathbb{R}_- \) and being subject to the defect condition, in Figure 3 and thereby showing the effect of an increasing defect parameter \( \beta \). In Figure 4, we plot the absolute value of a three-soliton solution \( u[3] \) and \( \tilde{u}[3] \) satisfying the presented model with defect parameter \( \alpha = 0 \) and \( \beta = 1 \) and also its contour. All of the three solitons have the same amplitude, two of them have opposite velocity and the velocity of the third soliton is chosen to be slow in order to show the discontinuity. Again, one can observe a smoothing effect when choosing either \( \alpha \) not equal to zero or \( \beta \) large enough. Similarly, a three-soliton solution, where the slow moving soliton is replaced by a fast moving soliton with the same amplitude, is shown in Figure 5. Conceptually, higher order soliton solutions could be computed and plotted.

The authors of [5] have been investigating the construction of soliton solutions by confirming through direct calculation that the one- and two-soliton solutions satisfy the defect conditions. For the convenience of the reader, we give the connection to the notation therein for the one-soliton solution.
Remark 5.3. To translate the expression into the notation used in [5], first off we need to take $\beta = 0$ and additionally $\Omega = \sqrt{\alpha^2 - |\tilde{u} - u|^2}$. Then, for the one-soliton solution consider $\frac{u_1}{u_1} = 1$, $a = 2\eta$, $c = -2\xi$, $p = e^{-2\eta \tilde{\xi}}$ and finally $q = e^{-i\tilde{\xi}}$ to recover the same result.

5.2 Destructive soliton solution

There is also a particular solution known for which we can avoid restricting the solution space to $X$. Beginning again with zero seed solutions $u[0] = 0$, $\tilde{u}[0] = 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{0\}$ and a choice in the sign in front of the root in the (11)-entry of the localized defect matrix $G_0(t, 0, \lambda)$, we can construct

$$G_0(t, 0, \lambda) = 2 \text{diag}(\lambda - \lambda_0, \lambda - \lambda_0^*),$$

where $\lambda_0 = -\frac{\alpha + \beta}{2}$. Taking the same spectral parameter $\lambda_0$ to construct a dressing matrix on one side, we take $\mathbb{R}$, of the defect together with a solution $\tilde{\psi}_0 = (\mu, \nu)^T$ to the undressed Lax system (2.2) at $\lambda = \lambda_0$ corresponding to $\tilde{u}[0]$, we obtain

$$\tilde{D}[1] = (\lambda - \lambda_0^*) \mathbb{I} + (\lambda_0^* - \lambda_0) P[1], \quad P[1] = \frac{\tilde{\psi}_0 \tilde{\psi}_0^†}{\psi_0 \psi_0^†}.$$ 

For the sign in front of the root in the (11)-entry of the localized defect matrix $G_0(t, 0, \lambda)$ to be plus or minus, the solution $\tilde{\psi}_0 = (\mu, \nu)^T$ respectively needs to have the limit value $e_1$ or $e_2$ as $x \to -\infty$. On the other half-line $\mathbb{R}_+$, we assume that the solution stays the same $u[1] = 0$. Then, constructing $G_1(t, 0, \lambda)$ as dressing matrix with $\lambda_0$ and corresponding vector $\tilde{\psi}_0 = (\mu, \nu)^T$ such that at $x = 0$ this vector is the kernel vector of $G_1$, we have everything we need in order to prove that the boundary constraint is preserved. Remark 3.2 suggests, that $G_1$ can be written in localized defect form connecting $u[1]$ and $\tilde{u}[1]$, in other words

$$(G_1)_x = \tilde{U}[1] G_1 - G_1 U[1] = \tilde{U}[1] G_1 - G_1 (-i\lambda \sigma_3),$$

$$(G_1)_t = \tilde{V}[1] G_1 - G_1 V[1] = \tilde{V}[1] G_1 - G_1 (-2i\lambda^2 \sigma_3),$$

which follows directly from the property of $\tilde{D}[1]$. Further, the assumed limit behavior of the solution $\tilde{\psi}_0$ makes sure that—after extending $G_1$ to a defect matrix on $x \in \mathbb{R}$—the form is similar to the form of $G_0$ in terms of the sign in front of the root.

Remark 5.4. It is possible to switch the roles of $u[1]$ and $\tilde{u}[1]$, keeping the zero solution on $\mathbb{R}_-$ and dressing at $\lambda_0$ for $\mathbb{R}_+$ with predetermined limit behavior of the corresponding solution $\psi_0$ of the Lax system corresponding to $u[0]$. However, in that case $G_1$ needs to be constructed as $(D[1])^{-1}$ in order for the localized defect form to connect $u[1]$ and $\tilde{u}[1]$, since $D[1]$ has different partial differential equations than $\tilde{D}[1]$.

In Figure 6 we plotted two examples of one-soliton solutions interacting destructively with the defect condition. As mentioned before, the amplitude and velocity of the soliton
Fig. 6. Plot of a boundary-bound and a one-soliton solution satisfying the NLS equation on each half-line and interacting destructively with the defect conditions (4.3) with defect parameter $\alpha = 0$ (left) as well as $\alpha = 0.5$ (right) and $\beta = 1$.

is prescribed by the strength of the defect parameter $\alpha$ and $\beta$. The spatial and phase shift however can be chosen arbitrarily. The idea of these solutions emerged as a special case, when working with $\alpha = 0$, in order to find nonlinear counterparts of bound states which are solutions to the linear, potential-free, Schrödinger equation with a defect in [5]. Here, this idea takes the form of a boundary-bound soliton solution and a one-soliton solution on one of the half-lines.

**Remark 5.5.** Taking $\alpha = 0$, the destructive soliton solution is in fact a boundary-bound soliton solution, which especially is not covered by Proposition 4.2.

**Conclusion**

The defect conditions are subject to some interesting properties as classical systems are, due to the fact that they stem from a localized Bäcklund transformation. In combination with Darboux transformations, we provide a direct method in order to compute exact solutions of the focusing NLS equation on two half-lines connected via the defect conditions. By carefully reviewing the properties we need, we give a reduction of the class of solutions which is needed to determine the localized defect matrix for the Darboux transformation of the solutions. Thereby, we introduce the method of dressing the boundary to a system consisting of two half-lines connected through a boundary condition induced by the localized Bäcklund transformation. Hence, not only readjusting the method to encompass a simple star-graph, but also putting forward an application on
time-dependent gauge transformations, ultimately generalizing the method presented in [13] in two ways.

To discuss soliton behavior, we provided additionally the proof of the conjecture formulated in [5]: In the model of the NLS equations on two half-lines connected via defect conditions, an arbitrary number of solitons are transmitted through the defect independently of one another. Through this means, we simultaneously made it more relatable to the works where the mirror-image technique was employed [2]. Thereby, the question arises whether it is possible to use the mirror-image technique in the model presented to arrive the same results.

The analysis we carried out for a Darboux transformation with respect to \( t \) is an analogous result to the one with respect to \( x \) given in [6]. That being said, it is also possible to apply the same analysis at an arbitrary point \( x_f \in \mathbb{R} \). Since the defect conditions can simultaneously be shifted, the results we have are easily applicable for a defect condition at an arbitrary point \( x_f \in \mathbb{R} \) connecting two semi-infinite sets in \( \mathbb{R} \).

It is reasonable to assume that the method of dressing the boundary can be applied to a wide range of systems on which integrable boundary structures exist. The closest application would be to combine the results in this paper with the results of [13] to obtain soliton solutions for the new boundary conditions [12] for the NLS equation on the half-line. Nonetheless, it could also be used to extend the results [14] on the sine-Gordon equation with integrable boundary to include time-dependent transformations on the half-line. Other systems with integrable boundaries are presented in [3, 5] and it can be investigated if the method extends to these systems.

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