HOMOTOPY PATH ALGEBRAS

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Abstract. We define a basic class of algebras which we call homotopy path algebras. We find that such algebras always admit a cellular resolution and detail the intimate relationship between these algebras, stratifications of topological spaces, and entrance/exit paths. As examples, we prove versions of homological mirror symmetry due to Bondal-Ruan for toric varieties and due to Berglund-Hübsch-Krawitz for hypersurfaces with maximal symmetry. We also demonstrate that a form of shellability implies Koszulity and the existence of a minimal cellular resolution. In particular, when the algebra determined by the image of the toric Frobenius morphism is directable, then it is Koszul and admits a minimal cellular resolution.

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In 2006, Bondal and Ruan introduced a completely novel approach to homological mirror symmetry (HMS) \[6\]. Their description connected sections of line bundles on toric varieties to entrance paths on a “mirror” torus. The mantle for this approach was soon taken up by Fang-Liu-Treumann-Zaslow in a series of articles \[16, 17, 18\] which included proofs of HMS for smooth, proper toric varieties and maximally equivariant toric stacks. Many great works have followed which we will not attempt to catalogue, however, let us mention that a full proof of HMS for all toric DM stacks was provided by Kuwagaki \[25\].

The work herein began as an attempt to codify the original ideas of Bondal-Ruan with the hope of extending their approach beyond the toric framework. The resulting theory can be thought of as nothing more than the study of a simple class of \(k\)-algebras which we would like to call “homotopy path algebras” (HPAs) (where \(k\) is any unital commutative Noetherian base ring).

The name HPA is descriptive. If we embed a quiver in a topological space, then the corresponding HPA is just the path algebra of the quiver quotiented by the ideal generated by path homotopy. On the other hand, these HPAs have a completely algebraic description (see Definition 3.1) and hence can be studied as a class of algebras in their own right. Alternatively, bases of HPAs also arise exactly as entrance paths for a topological space \(Y\) stratified by a poset \(S\) of contractible strata: picture the vertices as chosen base points in \(S\) and arrows as a basis of homotopy classes of indecomposable directed paths in \(Y\).

The study of entrance paths on a space stratified by a poset of subspaces really goes back to MacPherson, who famously conjectured an equivalence between constructible (co)sheaves and representations of entrance paths. The strongest incarnation of this conjecture was handled by Curry-Patel \[11\] for conically stratified spaces. A higher categorical version is given in Lurie’s Derived Algebraic Geometry \[26\], Appendix A, (see also \[31\]).

Our motivating examples were both the stratification of a torus considered by Bondal-Ruan and the tree stratification (see Definition 4.27) on the classifying space of entrance paths. These stratifications are not conical. Instead, they are what we call block stratifications (see Definition 4.22), which are in particular simple stratifications (see Definition 4.8). Our first result is a version of MacPherson’s conjecture for simple stratifications.

**Theorem 1.1.** Let \(S\) be a simple stratification of \(Y\). The following categories are equivalent:

- \( \text{Sh}_S(Y) \), the category of \(S\)-constructible sheaves on \(Y\);
- \( A\text{-mod} \), the category of modules over the HPA described above;
- \( \text{Fun}(\text{Ent}_S(Y), k\text{-mod})^\text{op} \), the opposite category of representations of \(S\)-entrance paths.

Entrance paths themselves may be thought of as a type of directed fundamental group or as similar to flow lines in Morse theory. When \(S\) is sufficiently nice, entrance paths should recover the homotopy type of \(Y\), reminiscent of the seminal work \[10\] where gradient flow lines of a Morse function are used to reconstruct the homotopy type of a smooth manifold. This direction has recently been explored by Nanda \[28\] in the context of discrete Morse theory \[19\], who proves the classifying space of the discrete Morse flow category recovers the homotopy type of a regular CW complex. Following Nanda’s approach, we prove:

**Theorem 1.2.** Let \(S\) be a block stratification of \(Y\). Then \(Y\) is homotopic to the classifying space of \(S\)-entrance paths. That is, the homotopy type of \(Y\) is encoded in the category of \(S\)-entrance paths.
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The topological take on HPAs also provides some pleasant algebraic consequences. For example, Table 1 relates fundamental $A$-modules to $S$-constructible sheaves described by strata, exit paths, and entrance paths. Furthermore, projective resolutions of $A$-modules have topological interpretations similar to CW and Morse homology as in the following theorem.

**Theorem 1.3.** Let $A$ be any HPA. The geometric realization of $A$ (see Definition 3.4) $X_A$ together with the tree stratification $S^{tr}$ recovers $A$ as the algebra of $S^{tr}$-entrance paths. In addition, $X_A$ is a projective cellular resolution (see Definition 6.4) of the diagonal bimodule. Furthermore, for any internal acyclic matching (see Definition 6.15), the simplicial collapse $X_A$ is a projective cellular subresolution. Finally, if $X_A$ is shellable in a certain sense (see Proposition 6.29), then $A$ is Koszul, and there exists an internal acyclic matching such that the projective cellular resolution $X_A$ is minimal.

For quiver algebras with relations, the existence and uniqueness of a minimal resolution of the diagonal bimodule is well-known [14, 5] (though not always explicit). Explicit minimal resolutions of the diagonal bimodule are of both independent interest and structural importance. They give interesting numerical invariants called Betti-numbers, allow for efficient calculations of Hochschild homology and cohomology, and provide functorial resolutions of all modules. The close similarity to CW homology, called cellular resolutions, is more recent (see [7]).

Our approach to constructing minimal cellular resolutions is similar to e.g. [24]. We use discrete Morse theory to construct minimal cellular resolutions analogous to the discrete Morse complex of Forman [19]. We comment that the resulting resolutions are closely related to the minimal cellular bimodule resolutions of abelian skew group algebras provided by Craw-Quintero Vélez [12]. While their construction uses a different approach, the algebras they consider are nothing more than HPAs which admit cycles from our perspective.

Stringing our results together, we now return to the inspiration for this work: Bondal-Ruan mirror symmetry. As a consequence of the theorems above we recover, generalize, and extend Bondal-Ruan’s original result:

**Theorem 1.4** (Bondal-Ruan Homological Mirror Symmetry). Let $\mathcal{X}$ be a toric DM stack of Bondal-Ruan type (see Definition 5.11). There is a stratification $S$ of the torus $\mathbb{T}^n$ and an associated HPA, $A$, such that the following derived categories are equivalent

$$D(\mathcal{X}) \cong D(A) \cong DSh_{S^{tr}}(X_A) \cong DSh_S(\mathbb{T}^n).$$

Furthermore, $X_A$ is homotopic to a torus and when $A$ is directable, it is Koszul and there is an explicit minimal resolution of the diagonal bimodule $\Delta_X$ which corresponds to a resolution of the diagonal $\Delta_X$ by line bundles.

We conclude by noting that our approach may, in some cases, be applicable to homological mirror symmetry beyond the toric context. For example, our methods can be deployed on Berglund-Hübsch-Krawitz hypersurfaces with maximal symmetry (see Example 6.40).

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2. Notation and conventions

2.1. The base ring. We work over an arbitrary unital commutative Noetherian base ring \( k \) e.g. \( k = \mathbb{Z} \) or \( \mathbb{C} \).

2.2. Quivers. A quiver is denoted by \( Q \) and its path algebra by \( kQ \). The vertex set of \( Q \) is denoted by \( Q_0 \) and the set of arrows is denoted by \( Q_1 \). We also want to emphasize our convention, which may not be standard: the direction of the arrows and their multiplication in the quiver agrees with the entrance path direction and the concatenation of entrance paths. We use \textit{increasing} order of the strata to define entrance paths. For a regular CW complex with cell strata, it means our indexing poset is the usual face poset with opposite ordering.

2.3. CW complexes. In this paper, a CW complex is a Hausdorff topological space \( X \) filtered by a sequence of closed subspaces

\[
X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,
\]

where \( X_k = \coprod_{\alpha \in I, i \leq k} B^i_\alpha \) with \( B^i_\alpha \) homeomorphic to \( \mathbb{R}^n \) and \( \overline{B}^n_i \setminus B^n_i \subset X_{n-1} \), satisfying the CW axioms

- \{ \( B^k_j : B^k_j \cap \overline{B}^n_i \) is finite for all \( i \in I \) and \( n \geq 0 \) \}.
- \( A \subset X \) is closed iff \( A \cap \overline{B}^n_i \) is closed for all \( i \in I \) and \( n \geq 0 \).

\textit{Importantly}, all CW complexes we consider satisfy the \textbf{axiom of frontier}:

- If \( B^k_i \cap \overline{B}^n_j \neq \emptyset \), then \( B^k_i \subset \overline{B}^n_j \).

A CW complex is \textbf{finite} if the index set \( I \) is finite. A CW complex is \textbf{regular} if in addition \( \overline{B}^n_i \) is homeomorphic to the closed unit ball in \( \mathbb{R}^n \).

A semi-simplicial complex (or \( \Delta \)-complex [22]) is the quotient space

\[
X = \coprod_{k, \alpha} \Delta_{k, \alpha} / \sim
\]

where \( \Delta_{k, \alpha} \) are closed simplices, and \( \sim \) identifies in each dimension a set of faces in that dimension from distinct simplices. By construction, a semi-simplicial complex \( X \) is a CW complex where each closed \( k \)-cell \( \overline{B}^k_i \) is homeomorphic to a closed \( k \)-simplex with a set of faces identified. A semi-simplicial complex is \textbf{regular} if every \( \overline{B}^k_i \) is homeomorphic to a \( k \)-simplex. A regular semi-simplicial complex is \textbf{simplicial} if any finite set of \( 0 \)-cells of cardinality \( k + 1 \) is the vertex set of a unique \( k \)-cell.

For classifying spaces, we follow the convention in [30]. Namely, we define the \textbf{classifying space} \( \mathcal{B} \mathcal{C} \) of a category \( \mathcal{C} \) to be the geometric realization of its nerve \( N(\mathcal{C}) \). By \textbf{geometric realization} of a (semi-)simplicial set, we mean the CW complex with one \( n \)-cell for each non-degenerate \( n \)-simplex as in [27].
2.4. Categories. All categories considered are small categories. All categories and functors are equipped with the derived notation if derived and usual notation if not. For functors which are already exact (e.g. the proper pushforward for a locally-closed embedding) we do not use the derived notation.

For constructible sheaves, $\mathcal{S}\mathfrak{h}_S(X)$ is the abelian category of $S$-constructible sheaves on $X$, and $D\mathcal{S}\mathfrak{h}_S(X)$ is the bounded derived category of $\mathcal{S}\mathfrak{h}_S(X)$. For a continuous map $f : X \to Y$, the right adjoint to the derived proper pushforward in $D\mathcal{S}\mathfrak{h}_S(X)$ is denoted by $f^!$ since this functor is taken to always live in the derived category.

For a finite dimensional $k$-algebra $A$, $A$-mod means the category of finitely generated left $A$-modules, and $D(A)$ means the bounded derived category of finitely generated left $A$-modules.

For an algebraic stack $\mathfrak{X}$, $D(\mathfrak{X})$ denotes the bounded derived category of coherent sheaves on $\mathfrak{X}$.

3. Homotopy Path Algebras

3.1. HPAs and classifying spaces. Consider a finite acyclic quiver $Q$. We denote by $\text{Path}_Q = \text{the path poset}$ of $Q$ i.e. the set of all paths in $Q$ equipped with the partial order

\[ p < q \iff \exists r \in \text{Path}_Q \text{ s.t. } q = pr \in kQ \]

For each pair of vertices $(v, v')$, choose a (possibly empty) collection $C_{v,v'}$ of disjoint subsets $S^\alpha_{v,v'}$, $\alpha \in C_{v,v'}$ of paths from $v$ to $v'$, and let $S = \bigsqcup_{v,v' \in Q_0, \alpha \in C_{v,v'}} S^\alpha_{v,v'}$. From $S$ we can form the following ideal

\[ I_S := \langle p - p' : p, p' \in S^\alpha_{t(p), h(p)} \text{ for some } \alpha \in C_{t(p), h(p)} \rangle \subset kQ. \]

Definition 3.1. The quiver algebra $A = kQ/I_S$ is a homotopy path algebra (HPA), if

- For any paths $r, p, p'$ and nonzero $rp - rp' \in I_S$, $p - p' \in I_S$.
- For any paths $r, p, p'$ and nonzero $pr - p'r \in I_S$, $p - p' \in I_S$.

Example 3.2. The path algebra of the quiver below, with relations $I = \langle a_1b_1 - a_2b_2 \rangle$, is a homotopy path algebra.

Example 3.3. The path algebras of the two quivers below, with relations $\langle ab - ac \rangle$ (left) and $\langle ac = bc \rangle$ (right), are not homotopy path algebras. The quiver on the left fails the first condition, and the quiver on the right fails the second condition.

For the remainder of the paper, $A = kQ/I_S$ is always an HPA. Given such an $A$, we can define the path poset of $A$ as $\text{Path}_A := \text{Path}_Q / \sim$. 
where \( p \sim q \iff p - q \in I \). So, given an HPA, \( A = kQ/I_S \), we can define a category \( C_A \) with

- **Objects of** \( C_A := Q_0 \)
- \( \text{Hom}_{C_A}(v, v') := \{ p \in \text{Path}_A : t(p) = v \text{ and } h(p) = v' \} \)

where composition is given by concatenation of paths.

**Definition 3.4.** The geometric realization of \( A \) is the classifying space of \( C_A \). We denote this by

\[
X_A := B(C_A) = B(C_A^{op}).
\]

The geometric realization of \( A \) also has the following concrete description. For a poset \( P \), let \( K(P) \) denote the (unbounded) order complex of \( P \).

**Proposition 3.5.** We have the following explicit realization of \( X_A \):

\[
X_A = K(\text{Path}_A)/\sim, \quad \text{where}
\]

\[
[p_0 < \cdots < p_n] \sim [q_0 < \cdots < q_n] \text{ iff } \frac{p_i}{p_0} - \frac{q_i}{q_0} \in I_S, \forall 1 \leq i \leq n.
\]

**Proof.** Starting with the definition of the classifying space, we can rewrite

\[
X_A = \left( \coprod_{v \in Q_0} \left[ [p_0 < \cdots < p_n] \times \Delta_n \right] / \sim \right) / \sim
\]

\[
= \left( \coprod_{v \in Q_0} \left( \coprod_{e_v} \left[ [p_0 < \cdots < p_n] \times \Delta_n \right] / \sim \right) / \sim \right)
\]

\[
= \left( \coprod_{v \in Q_0} \left( K(\text{Path}_{Q,e_v}) / \sim \right) \right) / \sim
\]

\[
= \left( \coprod_{v \in Q_0} K(\text{Path}_{A,e_v}) \right) / \sim
\]

\[
= K(\text{Path}_A) / \sim.
\]

**Remark 3.6.** Notice that \( \sim \) identifies all simplices \([p_0 < \cdots < p_n] \) with the simplex \([e_{t(p_0)} < p_1/p_0 < \cdots < p_n/p_0] \) and this provides a distinguished representative in each equivalence class.

**Lemma 3.7.** The geometric realization \( X_A \) is a regular semi-simplicial complex (\( \Delta \)-complex).

**Proof.** By construction, \( X_A \) is a semi-simplicial complex. Since \( \sim \) respects path ordering, each closed simplex in \( X_A \) must have distinct vertices i.e. every closed simplex is embedded homeomorphically into \( X_A \). Hence \( X_A \) is regular. \( \square \)

Let \( \text{Cell}(X_A) \) be the facet poset of \( X_A \) graded by the dimension of the cells. By construction, we have \( Q_0 = \text{Cell}(X_A)_0, Q_1 \subset \text{Cell}(X_A)_1 \). Hence there is an embedding \( \iota_Q : \Gamma_Q \hookrightarrow X_A \), under which any path in the quiver becomes a path in \( X_A \). Additionally, for a path \( p \) in \( Q \), its saturated sequence of subpaths starting at \( t(p) \), denoted \( p \), corresponds to a cell \([p] \) in \( X_A \).

**Remark 3.8.** The CW structure of \( X_A \) is, in fact, quite redundant and can be reduced to one such that \( \Gamma_Q = (X_A)_1 \). This can be done using a Morse matching as discussed in §6.2.
Proposition 3.9. Suppose $A$ is a homotopy path algebra. Given two paths $p, q$ in the quiver, $p - q \in I$ iff $\iota_Q(p)$ is path homotopic to $\iota_Q(q)$ in $X_A$.

Proof. ($\Rightarrow$) First, consider the case where $p$ and $q$ have no common factor on the left or right. This implies $\iota_Q(p) \cap \iota_Q(q)$ is the union of the two endpoints. Since $p - q \in I$, $[e_{\iota_Q(p)}] < p \sim [e_{\iota_Q(q)}] < q$, and they become a single 1-cell in $X_A$ contained in both $[p]$ and $[q]$; its closure union with either path forms a circle in the boundary of the simplex $[p]$. This means the closure of $[e_{\iota_Q(p)}] < p = [e_{\iota_Q(q)}] < q$ is homotopic to both $\iota_Q(p)$ and $\iota_Q(q)$.

Now consider the general case where $p = lp'r$ and $q = lq'\bar{r}$ where $l$ and $r$ are maximal paths from the left and right that factor both. Since $A$ is a homotopy path algebra, $p' - q' \in I$. The above argument shows $\iota_Q(p')$ is homotopic to $\iota_Q(q')$. We also have $\iota_Q(p) = \iota_Q(l)*\iota_Q(p')*\iota_Q(r)$ and $\iota_Q(q) = \iota_Q(l) * \iota_Q(q') * \iota_Q(r)$, hence $\iota_Q(p)$ is homotopic to $\iota_Q(q)$ as well.

($\Leftarrow$) If $\iota_Q(p)$ and $\iota_Q(q)$ are path homotopic in $X_A$, they share the same endpoints in $X_A$. The 1-cell in $[p]$ connecting the endpoints is represented by $[e_{\iota_Q(p)}] < p$ in $K(Path_Q)$, hence $[e_{\iota_Q(p)}] < p = [e_{\iota_Q(q)}] < q$ is the same 1-cell in $X_A$. This means $[e_{\iota_Q(p)}] < p \simeq [e_{\iota_Q(q)}] < q$ in $K(Path_Q)$ i.e. $p - q \in I$.

Remark 3.10. Conversely, if one begins with a topological space $Y$ and an embedding $\Gamma_Q \subseteq Y$, we can define an HPA, $A(Y) = kQ/I_Y$, where

$$I = \langle p - q : p \text{ is path homotopic to } q \text{ in } Y \rangle,$$

3.2. Examples.

Example 3.11 (No relation). For any acyclic quiver $Q$ (not necessarily finite) without relations, one can construct a deformation retraction from $X_{kQ}$ onto $\Gamma_Q$ by collapsing each simplex $[p_0 < \cdots < p_n]$ to the composition of arrows from $t(p_0)$ to $h(p_n)$.

Example 3.12 ($\mathbb{P}^2$ quiver). Consider the Beilinson quiver with relations

$$I = \langle xy' - yx', xz' - zz', yz' - zy' \rangle$$

$\begin{tikzpicture}
\draw[->] (0,0) -- (1,0) node[midway, above] {\scriptsize $x$};
\draw[->] (1,0) -- (1.5,0) node[midway, above] {\scriptsize $y$};
\draw[->] (1.5,0) -- (2,0) node[midway, above] {\scriptsize $z$};
\draw[->] (0,0) -- (1,1) node[midway, above] {\scriptsize $x'$};
\draw[->] (1,1) -- (1.5,1) node[midway, above] {\scriptsize $y'$};
\draw[->] (1.5,1) -- (2,1) node[midway, above] {\scriptsize $z'$};
\draw[->] (0.5,-0.5) node[below] {$e_0$} -- (0.5,0.5) node[above] {$e_1$};
\draw[->] (1.25,-0.5) node[below] {$e_1$} -- (1.25,0.5) node[above] {$e_2$};
\end{tikzpicture}$

Let $Path_{Q,e_0} := \{ e_0, x, y, z, xx', xx', xy', xz', yx', yz', yz', zy', zz' \}$, $Path_{Q,e_1} := \{ e_1, x', y', z' \}$ and $Path_{Q,e_2} := \{ e_3 \}$. and let $f : Path_Q \to Path_A$ be the natural map

$$f(e_i) = e_i, \quad f(x) = f(x') = x, \quad f(y) = f(y') = y, \quad f(z) = f(z') = z, \ldots$$

The two dimensional simplicial complex $K(Path_A)$ has three connected components: $K(Path_{A,e_0})$ (left), $K(Path_{A,e_1})$ (middle), and $K(Path_{A,e_2})$ (right).

To obtain the geometric realization $X_A$, simply identify the following cells in $K(Path_A)$: 1) all vertices of the same color, 2) all light blue edges, 3) all light green edges, and 4) all light red edges.
One can observe from the picture that the CW structure of $X_A$ can be reduced by collapsing all gray edges to an adjacent 2-cell. One can do this at the level of $K(\text{Path}_A)$, and get

$$
\text{with cells being glued the same way. This CW complex is indeed a 2-torus } \mathbb{T}^2. \text{ We draw it in three different fundamental domains.}
$$

The 1-skeleton of the CW complex is the quiver itself, and the three 2-cells correspond to the generators of the ideal $I$. Call this CW complex $X_{A_{\min}}$.

We note there is a K"unneth-type formula for our construction

**Proposition 3.13.** Let $A, A'$ be homotopy path algebras. Let $A \otimes A'$ be the homotopy path algebra for the product quiver with relations. Then $X_{A \otimes A'} \simeq X_A \times X_A'$.

**Proof.** This follows from

$$
B(C_{A \otimes A'}) = B(C_A \times C_A')
\simeq B(C_A) \times B(C_A')
$$

by [30], Eq. (4).

\[\square\]

**Example 3.14.** Let $A_k$ be the $A_k$-quiver with no relations. For $k_1 \geq 1$, $X_{A_{k_1} \otimes \cdots \otimes A_{k_n}}$ can be embedded into $\mathbb{R}^n$ as the union of $k_1 \times \cdots k_n$ hypercubes forming an $n$-rectangle, with a simplicial subdivision on each hypercube. In this case one can also collapse the CW structure on $X_{A_{k_1} \otimes A_{k_n}}$ to one with cubical cells. Call this cubical CW complex $X_{A_{k_1} \otimes A_{k_n}}^{\min}$.

The reduction of the CW structures in these examples can be formulated rigorously using discrete Morse theory [19], as discussed in §6.2. The CW complexes $\Gamma_Q, X_{P^2}^{\min}$, and $X_{A_{k_1} \otimes A_{k_n}}^{\min}$, are examples of a minimal cellular resolution of the diagonal bimodule of the quiver algebra, which can sometimes be obtained from a Morse matching.

4. Stratifications and Entrance Paths

4.1. Entrance Paths. Let $Y$ be a locally-compact, Hausdorff, path-connected, locally-path connected, semilocally simply-connected\(^1\) topological space and $I$ be a poset. Consider a

\[\text{Locally-compact ensures the existence of a Verdier dual [23] and path-connected, locally-path connected, semilocally simply-connectedness provides the existence of a universal cover [22], §1.3.}\]
stratification indexed by \( I \),
\[
Y = \prod_{\alpha \in I} S_{\alpha}.
\]

**Definition 4.1.** An entrance path is a continuous map
\[
\gamma : [0, 1] \to Y
\]
such that there exists a sequence of increasing strata \( S_1 < ... < S_s \) such that \( \gamma^{-1}(S_1) < ... < \gamma^{-1}(S_s) \) is an increasing sequence of intervals stratifying \([0, 1]\).

Choose a collection of base points \( y_i \in S_i \) (indexed by \( i \in I \)). The entrance path category with respect to \( S \) is the category whose objects are the base points \( y_i \) and whose morphisms from \( y_i \) to \( y_j \) are homotopy classes of entrance paths \( \gamma \) with \( \gamma(0) = y_i \) and \( \gamma(1) = y_j \). We denote this category by \( \text{Ent}_S(Y) \).

Similarly, an exit path is a continuous map
\[
\gamma : [0, 1] \to Y
\]
such that there exists a sequence of decreasing strata \( S_1 > ... > S_s \) such that \( \gamma^{-1}(S_1) < ... < \gamma^{-1}(S_s) \) is an increasing sequence of intervals stratifying \([0, 1]\). The exit path category is defined similarly and denoted \( \text{Exit}_S(Y) \).

We recall the following standard fact about regular CW complexes:

**Proposition 4.2.** Given a regular CW complex \( Y \), let \( S_{\text{CW}} \) be the stratification by open cells regarded as a poset under the face relations. The classifying space of entrance paths, \( B(\text{Ent}_{S_{\text{CW}}}(Y)) \), is the barycentric subdivision of \( Y \).

**Example 4.3.** A nice consequence of the above is that any regular CW complex is homeomorphic to the geometric realization of an HPA. Given a regular CW complex \( Y \), we can construct a quiver \( Q_Y \) together with an ideal \( I_Y \subseteq kQ_Y \) as follows:
\[
\begin{align*}
(Q_Y)_0 &:= \{ \text{cells of } Y \} \\
(Q_Y)_1 &:= \{ \text{boundary relations} \} \\
I_Y &:= \langle p - q : p \sim_Y q \rangle
\end{align*}
\]

The HPA associated to \( Y \) is then defined as \( A_Y := kQ_Y/I_Y \). Notice that by Proposition 4.2, \( X_{A_Y} \) is homeomorphic to \( Y \) since \( \mathcal{C}_{A_Y} = \text{Ent}_{S_{\text{CW}}}Y \).

**4.2. Exceptional stratifications.**

**Definition 4.4.** A stratification \( Y = \coprod_{i \in I} S_i \) indexed by a finite poset \( I \) is called exceptional if each of the \( S_i \) is contractible and the transitive closure of the condition \( \overline{S_i} \cap S_j \neq \emptyset \Leftrightarrow i \leq j \) recovers the poset ordering.

**Lemma 4.5.** Let \( S \) be an exceptional stratification. Then, the set
\[
S_{\geq i_0} := \bigcup_{i \geq i_0} S_i
\]
is closed and
\[
S_{\leq i_0} := \bigcup_{i \leq i_0} S_i
\]
is open.
Proof. Notice that
\[ S_{\geq i_0} = S_{\geq i_0} \cap Y = S_{\geq i_0} \cap \bigcup S_j = \bigcup S_{\geq i_0} \cap S_j = \bigcup_{i \geq i_0, j} S_{i_0} \cap S_j \] because \( I \) is finite
\[ = S_{\geq i_0} \] because \( S_{i_0} \cap S_j \subseteq \begin{cases} S_j & \text{if } j \geq i_0 \\ \emptyset & \text{if } j < i_0 \end{cases} \)
i.e. \( S_{\geq i_0} \) is closed.

Now
\[ Y \setminus S_{\leq i_0} = Y \setminus \bigcup_{i \leq i_0} S_i = \bigcup_{j \geq i_0} S_j = \bigcup_{j \geq i_0} S_{\geq j} \]
is a finite union of closed sets by the above (hence closed).

Proposition 4.6. If \( Y = \bigsqcup_{i \in I} S_i \) is an exceptional stratification, then the set
\[ \{(\iota_i)_! k S_i : i \in I\} \]
forms a full exceptional collection in \( D \text{Sh}_S(Y) \).

Proof. First we prove the objects are exceptional. Namely, we calculate:
\[ R \text{Hom}((\iota_i)_! k S_i, (\iota_i)_! k S_i) = k. \]
We have
\[ R \text{Hom}((\iota_i)_! k S_i, (\iota_i)_! k S_i) = R \text{Hom}(k S_i, (\iota_i)_!(\iota_i)_! k S_i) = R \text{Hom}(k S_i, k S_i) \]
by adjunction
\[ = k \]
by Proposition 6.6 of [23]
since \( S_i \) is contractible.

Now we check that morphisms give a partial ordering on the exceptional objects, \((\iota_i)_! k S_i\). For this, suppose \( S_j \cap S_i = \emptyset \). Then
\[ R \text{Hom}((\iota_i)_! k S_i, (\iota_j)_! k S_j) = R \text{Hom}(k S_i, (\iota_i)_!(\iota_j)_! k S_j) = R \text{Hom}(k S_i, \mathbb{D}_{\iota_i^* \iota_j*} \mathbb{D} k S_j) = 0 \]
since \( \iota_i^* \iota_j* \mathcal{F} \) is supported on \( S_j \cap S_i = \emptyset \). Now, by the assumption that the stratification is exceptional, the partial ordering on the strata is the opposite partial ordering on the exceptional objects.
Now we prove that the strata generate. Let \( \kappa_i : M \setminus S_i \to M \) be the inclusions and \( i_0 \) be a maximal element of \( I \). Then, for any \( \mathcal{F} \in DSh_S(Y) \) we have an exact triangle

\[
\kappa_{i_0} ! : \mathcal{F} \to \mathcal{F} \to \iota_{i_0} ! \iota_{i_0} ^* \mathcal{F} \xrightarrow{[1]} \]

Notice that since \( \mathcal{F} \) is constructible with respect to the stratification, \( (\iota_{i_0})^* \mathcal{F} \) is locally constant, hence generated by \( k_{S_{i_0}} \) since \( S_{i_0} \) is constructible. Since \( S_{i_0} \) is closed by Lemma 4.5, it follows that \( (\iota_{i_0})_!(\iota_{i_0})^* \mathcal{F} = (\iota_{i_0})_! (\iota_{i_0})^* \mathcal{F} \) is generated by \( (\iota_{i_0})_! k_{S_{i_0}} \). Now we are done by induction on the number of strata. \( \square \)

**Definition 4.7.** Let \( S \) be an exceptional stratification of \( Y \) with universal cover \( \pi : \tilde{Y} \to Y \).

This induces a stratification \( \tilde{S} \) of \( \tilde{Y} \). For a point \( \tilde{y} \in \tilde{Y} \) we define the **entrance space** at \( \tilde{y} \) to be the subspace

\[
\tilde{Y}_{\text{Ent}}(\tilde{y}) := \{ x \in \tilde{Y} : \exists \tilde{\gamma} \in \text{Ent}_{\tilde{S}}(\tilde{Y}) \text{ with } \tilde{\gamma}(0) = \tilde{y}, \tilde{\gamma}(1) = x \} \xrightarrow{i_{\tilde{y}}} \tilde{Y}.
\]

Similarly, we define the **exit space** at \( \tilde{y} \) to be the subspace

\[
\tilde{Y}_{\text{Exit}}(\tilde{y}) := \{ x \in \tilde{Y} : \exists \tilde{\gamma} \in \text{Exit}_{\tilde{S}}(\tilde{Y}) \text{ with } \tilde{\gamma}(0) = \tilde{y}, \tilde{\gamma}(1) = x \} \xrightarrow{j_{\tilde{y}}} \tilde{Y}.
\]

**Definition 4.8.** An exceptional stratification of \( Y \) is called **simple** if for all \( \tilde{y} \in \tilde{Y} \) the entrance space at \( \tilde{y} \) is contractible and for all \( \tilde{y}, \tilde{y}' \) the difference \( \tilde{Y}_{\text{Ent}}(\tilde{y}) \setminus \tilde{Y}_{\text{Ent}}(\tilde{y}') \) is contractible whenever it is non-empty.

**Definition 4.9.** Given \( y \in Y \) choose a lift \( \tilde{y} \in \tilde{Y} \). The **entrance sheaf** at \( y \) is defined as follows

\[
\mathcal{I}_y := (\pi \circ i_{\tilde{y}})_! k_{\tilde{Y}_{\text{Ent}}(\tilde{y})}.
\]

Similarly, the **exit sheaf** at \( y \) is defined as follows

\[
\mathcal{P}_y := (\pi \circ j_{\tilde{y}})_! k_{\tilde{Y}_{\text{Exit}}(\tilde{y})}.
\]

**Definition 4.10.** Let \( A := \text{End}(\bigoplus_{w \in I} \mathcal{I}_w)^{\text{op}} \) and \( e_w \in A \) be the idempotent corresponding to the summand \( \mathcal{I}_w \). We define

\[
P_w := A e_w \quad \text{and} \quad I_w := (e_w A)^* \quad \text{and} \quad k_w := e_w A e_w.
\]

**Remark 4.11.** By Serre duality for finite dimensional algebras, \( P_w \) is a projective left \( A \)-module and \( I_w \) is an injective left \( A \)-module. Also note that \( P_w^* = (A e_w)^* = A^{\text{op}} e_w \) is a projective left \( A^{\text{op}} \)-module or right \( A \)-module.

**Lemma 4.12.** Let \( S \) be a simple stratification of \( Y \). Then,

\[
R \text{Hom}((i_v)_! k_{S_i}, \mathcal{I}_j) = \begin{cases} k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** We proceed by induction on the number of strata. For the base case with \( I = \{ i \} \), the assertion follows from that fact that \( S_i \) is contractible.

Now for the inductive step, consider a minimal element \( i_0 \in I \). The induction hypothesis says that the statement holds for \( I \setminus \{ i_0 \} \), hence it remains to compute \( \text{Hom}((i_{i_0})_! k_{S_{i_0}}, \mathcal{I}_j) \) and \( \text{Hom}((i_0)_! k_{S_i}, \mathcal{I}_{i_0}) \).

Notice that \( S_{i_0} \) is open by Lemma 4.5, hence, for \( j \neq i_0 \)

\[
R \text{Hom}((i_{i_0})_! k_{S_{i_0}}, \mathcal{I}_j) = R \text{Hom}(k_{S_{i_0}}, \mathcal{I}_j|_{S_{i_0}}) = 0
\]
since \( I_j|_{S_{i_0}} = 0 \).

Now we compute
\[
R \text{Hom}( (i_i)_* k_{S_i}, \mathcal{I}_{i_0} ) = R \text{Hom}( k_{S_i}, (i_i)^! (\pi \circ i_{i_0})_* \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})} )
\]
by adjunction
\[
= R \text{Hom}( k_{S_i}, \pi_* (\widetilde{i}_i)^! \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})} )
\]
by base change
\[
= R \text{Hom}( k_{\pi^{-1}(S_i) \cap \widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})}, (\widetilde{i}_i)^! \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})} )
\]
by adjunction
\[
= R \text{Hom}( k_{\pi^{-1}(S_{j>i}) \cap \widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})}, (\widetilde{i}_i)^! \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})} )
\]
by induction for \( j > i \)
\[
= R \text{Hom}( (\widetilde{i}_i)_* \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})}, (\widetilde{i}_i)^! \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})} )
\]
by adjunction
\[
= R \text{Hom}( (\widetilde{i}_i)_* \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})}, (\widetilde{i}_i)^! \kappa_{\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0})} )
\]
by induction for \( j > i \)
\[
= H^* (\widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0}), \widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i_0}) \setminus \widetilde{Y}_{\text{Ent}}(\widetilde{y}_{i}) )
\]
by Lemma [4.15]
\[
= \begin{cases} 
  k & \text{if } i = i_0 \\
  0 & \text{if } i \neq i_0 
\end{cases}
\]
since \( S \) is simple.

\[ \square \]

**Proposition 4.13.** Let \( S \) be a simple stratification of \( Y \). Then, the functor
\[
F_w : Sh_S(Y) \to k\text{-mod}
\]
\[
\mathcal{F} \to (\mathcal{F}_{y_w})^*
\]
is corepresented by \( \mathcal{I}_w \).

**Proof.** Let \( v \) be a maximal element of the poset \( I \). Then we observe
\[
(i_v)_* \mathcal{F} = k_{S_v} \otimes_k \mathcal{F}_{x_v}
\]
since \( \mathcal{F} \) is \( S \)-constructible, \( S_v \) is contractible, and closed by Lemma [4.5]

Again, we proceed by induction on the number of strata. Fix consider the (base) case where \( w \) is a maximal element of the poset \( I \). Then \( \mathcal{I}_w = (i_w)_* k_{S_w} \) so
\[
R \text{Hom}( \mathcal{F}, \mathcal{I}_w ) = R \text{Hom}( \mathcal{F}, (i_w)_* k_{S_w} )
\]
\[
= R \text{Hom}( (i_w)_* \mathcal{F}, k_{S_w} )
\]
by adjunction
\[
= R \text{Hom}( k_{S_w} \otimes_k \mathcal{F}_{x_w}, k_{S_w} )
\]
by Eq. (4)
\[
= (\mathcal{F}_{x_w})^*
\]
since \( S_w \) is contractible.

Now, suppose \( w \) is not maximal and let \( i_{i_0} \) be a maximal element. Let \( j_{i_0} : Y \setminus S_{i_0} \to Y \) be the inclusion of the open set. Consider the exact triangle
\[
j_{i_0} : j_{i_0}^! \mathcal{F} \to \mathcal{F} \to i_{i_0*} i_{i_0}^* \mathcal{F} \xrightarrow{[1]} 
\]
and apply \( R \text{Hom}( -, \mathcal{I}_w ) \) to get
\[
R \text{Hom}( j_{i_0}^! \mathcal{F}, \mathcal{I}_w ) \to R \text{Hom}( \mathcal{F}, \mathcal{I}_w ) \to R \text{Hom}( j_{i_0*} i_{i_0}^* \mathcal{F}, \mathcal{I}_w ) \xrightarrow{[1]} 
\]
which by adjunction, Eq. (4), and the fact that \( i_{i_0} \) is proper (Lemma [4.5], is the same as
\[
R \text{Hom}( i_{i_0*} (k_{S_{i_0}} \otimes_k \mathcal{F}_{x_{i_0}}), \mathcal{I}_w ) \to R \text{Hom}( \mathcal{F}, \mathcal{I}_w ) \to R \text{Hom}( \mathcal{F} \otimes_{Y \setminus S_{i_0}} \mathcal{I}_w \otimes_{Y \setminus S_{i_0}} ) \xrightarrow{[1]} .
\]
Hence by Lemma 4.12 we get an isomorphism
\[ R \text{Hom}(\mathcal{F}, \mathcal{I}_w) \cong R \text{Hom}(\mathcal{F}|_{Y \setminus S_{i_0}}, \mathcal{I}_w|_{Y \setminus S_{i_0}}) \]
and we are done by the induction hypothesis.

\[ \square \]

**Proposition 4.14.** Let \( S \) be a simple stratification of \( Y \). Then, the stalk functor
\[ F : Sh_S(Y) \to \text{mod-End}(\bigoplus I_i) = A\text{-mod} \]
\[ F \to \bigoplus_i \text{Hom}(F, I_i)^* = \bigoplus_i F_{x_i} \]
has an inverse
\[ G : \text{mod-End}(\bigoplus I_i) = A\text{-mod} \to Sh_S(Y) \]
\[ M \mapsto \bigoplus_i I_i \otimes_{k_A} k_M \]

**Proof.** We have a coevaluation map
\[ F \xrightarrow{ev^*} \bigoplus_i I_i \otimes_{k_A} \bigoplus_j k_{\text{Hom}(F, I_j)^*} = G \circ F(F) \]

It is enough to check that \( ev^* \) is an isomorphism on stalks and since the sheaves are constructible we can check this on the base points of the strata. For \( y_i \) we compute the stalk as
\begin{align*}
(\bigoplus_i I_i \otimes_{k_A} \bigoplus_j k_{\text{Hom}(F, I_j)^*})_{y_i} \\
= \bigoplus_i (I_i)_{y_i} \otimes_A \bigoplus_j \text{Hom}(F, I_j)^* \\
= \bigoplus_i \text{Hom}(I_i, I_i)^* \otimes_A \bigoplus_j \text{Hom}(F, I_j)^* \quad \text{by Proposition 4.13} \\
= \bigoplus_i \text{Hom}(I_i, I_i)^* \otimes_A \text{Hom}(F, I_i)^* \quad \text{since } e_i \text{ is the identity on the left of the tensor product} \\
= P_i^* \otimes_A \text{Hom}(F, I_i)^* \quad \text{since } \bigoplus_i \text{Hom}(I_i, I_i) = P_i \text{ by definition} \\
= \text{Hom}(F, I_i)^* \quad \text{since } e_i \text{ is the identity on the right of the tensor product} \\
= F_{y_i} \quad \text{by Proposition 4.13}
\end{align*}

and the coevaluation map induces this isomorphism on stalks.

Next we observe that there are natural isomorphisms
\[ G \circ F(M) = \bigoplus_{i,j} (I_i \otimes_{k_A} k_M)_{y_j}^* \\
= \bigoplus_{i,j} (I_i)_{y_j}^* \otimes_A M \\
= A \otimes_A M \\
= M \]
Lemma 4.15. Let $X$ be a finite CW complex. For closed subcomplexes $F_1 \xrightarrow{i} X$ and $F_2 \xrightarrow{j} X$,

$$R \text{Hom}(i_* k_{F_1}, j_* k_{F_2}) \simeq H^*(F_2, F_2 \setminus F_1; k)$$

(5)

Proof. Let $s : F_1 \setminus F_2 \to F_1$ and $t : F_1 \cap F_2 \to F_1$ and be the inclusions. Consider the exact sequence

$$0 \to s! k_{F_1 \setminus F_2} \to k_{F_1} \to t! k_{F_1 \cap F_2} \to 0$$

Applying $R \text{Hom}(\cdot, k_{F_1})$, we get an exact triangle

$$R \text{Hom}(t! k_{F_1 \cap F_2}, k_{F_1}) \to R \text{Hom}(k_{F_1}, k_{F_1}) \to R \text{Hom}(s! k_{F_1 \setminus F_2}, k_{F_1}),$$

which is equal to

$$R \text{Hom}(t! k_{F_1 \cap F_2}, k_{F_1}) \to H^*(F_1; k) \to H^*(F_1 \setminus F_2; k).$$

Hence we identify $R \text{Hom}(t! k_{F_1 \cap F_2}, k_{F_1})$ with the relative cohomology of the pair $(F_1, F_1 \setminus F_2)$ i.e.

$$R \text{Hom}(t! k_{F_1 \cap F_2}, k_{F_1}) \cong H^*(F_1, F_1 \setminus F_2; k).$$

(6)

The following cartesian square comes from the fiber product of two closed subsets:

$$
\begin{array}{ccc}
F_1 \cap F_2 & \xrightarrow{f} & F_1 \\
\downarrow{g} & & \downarrow{\beta} \\
F_2 & \xrightarrow{i} & X
\end{array}
$$

(7)

Now we compute

$$R \text{Hom}(i_* k_{F_1}, j_* k_{F_2}) = R \text{Hom}(i_* k_{F_1}, j_* k_{F_2})$$

since $F_2$ is closed

$$= R \text{Hom}(j^* i_* k_{F_1}, k_{F_2})$$

by adjunction

$$= R \text{Hom}(g_* f_* k_{F_1}, k_{F_2})$$

by proper base change of (7)

$$= R \text{Hom}(g_* k_{F_1 \cap F_2}, k_{F_2})$$

since $F_1 \cap F_2$ is closed

$$= R \text{Hom}(g_* k_{F_1 \cap F_2}, k_{F_2})$$

by (6).

□

Proposition 4.16. Let $A$ be the HPA whose vertices are base points of the stratification $S$ and whose paths are homotopy classes of entrance paths between the base points. We have the following identification

$$A = \text{End}(\bigoplus_w \mathcal{I}_w)^{op}.$$
Since there are finitely many arrows between any two vertices, \( \alpha \) and \( \beta \) are both proper. By proper base change we get

\[
R \text{Hom}(I_v, I_w) = R \text{Hom}(\alpha \ast k_{\tilde{Y}_{\text{Ent}}(\tilde{v})}, \beta \ast k_{\tilde{Y}_{\text{Ent}}(\tilde{w})})
\]

\[
= R \text{Hom}(\beta^\ast \alpha \ast k_{\tilde{Y}_{\text{Ent}}(\tilde{v})}, k_{\tilde{Y}_{\text{Ent}}(\tilde{w})})
\]

\[
= R \text{Hom}(f \ast k_{\tilde{Y}_{\text{Ent}}(\tilde{v})} \times_Y \tilde{Y}_{\text{Ent}}(\tilde{w}) \times \tilde{Y}_{\text{Ent}}(\tilde{q}(1)), k_{\tilde{Y}_{\text{Ent}}(\tilde{w})})
\]

Now we compute

\[
\tilde{Y}_{\text{Ent}}(\tilde{v}) \times_Y \tilde{Y}_{\text{Ent}}(\tilde{w})
\]

\[
= \bigsqcup \{ (p,q) | p(0) = v, q(0) = w, p(1) = q(1) \}
\]

Hence

\[
R \text{Hom}(f \ast k_{\tilde{Y}_{\text{Ent}}(\tilde{v})} \times_Y \tilde{Y}_{\text{Ent}}(\tilde{w}), k_{\tilde{Y}_{\text{Ent}}(\tilde{w})}) = R \text{Hom}(\bigsqcup \{ (p,q) | p(0) = v, q(0) = w, p(1) = q(1) \}, k_{\tilde{Y}_{\text{Ent}}(\tilde{w})})
\]

\[
= \bigsqcup \{ (p,q) | p(0) = v, q(0) = w, p(1) = q(1) \} \text{Hom}(k_{\tilde{Y}_{\text{Ent}}(\tilde{q}(1))}, k_{\tilde{Y}_{\text{Ent}}(\tilde{w})})
\]

By Lemma 4.15, this amounts to computing relative cohomology. Hence, all terms vanish when \( \tilde{q}(1) \neq \tilde{w} \) since \( H^\ast(\tilde{Y}_{\text{Ent}}(\tilde{w}) \setminus Y_{\text{Ent}}(\tilde{q}(1))) = 0 \) as both spaces are contractible (by assumption). When \( \tilde{q}(1) = \tilde{w} \) this contributes a copy of \( k \) in degree 0 since \( R \text{Hom}(k_{\tilde{Y}_{\text{Ent}}(\tilde{w})}, k_{\tilde{Y}_{\text{Ent}}(\tilde{w})}) = k \) since \( \tilde{Y}_{\text{Ent}}(\tilde{w}) \) is contractible.

This gives

\[
R \text{Hom}(I_v, I_w) = \text{Hom}(I_v, I_w)
\]

\[
= \bigsqcup \{ p : p(0) = v, p(1) = w \} \times_Y \tilde{Y}_{\text{Ent}}(\tilde{q}(1)) \times \tilde{Y}_{\text{Ent}}(\tilde{w})
\]

On the other hand, we have injective left modules \( I_v = (e_v A)^\ast \), and

\[
\text{Hom}_A(I_v, I_w) = \bigsqcup \{ p : p(0) = v, p(1) = w \} \times_Y \tilde{Y}_{\text{Ent}}(\tilde{q}(1)) \times \tilde{Y}_{\text{Ent}}(\tilde{w})
\]

**Theorem 4.17.** Let \( S \) be a simple stratification. There are equivalences of categories

\[
Sh_S(Y) \cong A^{op}\text{-mod} \cong \text{Fun}(\text{Ent}_S(Y), k\text{-mod})^{op}
\]

**Proof.** We have

\[
Sh_S(Y) \cong \text{End}(\bigoplus_{w} I_w)\text{-mod}
\]

\[
\cong A^{op}\text{-mod}
\]

\[
\cong \text{Fun}(\text{Ent}_S(Y), k\text{-mod})^{op}
\]

because \( A \) is, by definition, the HPA of entrance paths.

**Remark 4.18.** A version of the above theorem was proven in [11], Theorem 6.1, for conical stratifications. Unfortunately, we were unable to apply this result as the stratifications of primary interest herein are not conical.
4.3. Localization and homotopy. In this section, we introduce block stratifications and demonstrate that for a block stratification $S$ of $Y$ there is a homotopy between $Y$ and $B\operatorname{Ent}_S(Y)$. For this purpose, we recall the following from Quillen’s Higher Algebraic K-Theory [30], §1.

**Definition 4.19** (Quillen, [30] pg 6). Consider a functor
\[ F : \mathcal{C} \to \mathcal{C}' \]
Given $v' \in \mathcal{C}'$, we denote by $v' \setminus F$ the category whose objects are pairs $(v, \alpha)$ where $v$ is an object of $\mathcal{C}$ and $\alpha$ is a morphism from $v'$ to $F(v)$. A morphism in $v' \setminus F$ from $(v_1, \alpha_1)$ to $(v_2, \alpha_2)$ is a morphism $\phi$ in $\mathcal{C}$ from $v_1$ to $v_2$ such that $F(\phi) \circ \alpha_1 = \alpha_2$.

**Theorem 4.20** (Quillen, [30] Theorem A). Assume that $F$ is essentially surjective and $B(v' \setminus F)$ is contractible for all $v' \in \mathcal{C}'$. Then $BC$ is homotopic to $BC'$.

**Lemma 4.21** (Quillen, [30] Corollary 1). If a functor has a left or right adjoint then it induces a homotopy equivalence on classifying spaces.

Our goal now is to combine Quillen’s work with Proposition 4.2. This will allow us to understand the homotopy type of the classifying space of entrance path categories for a more general class of stratifications which we now define.

**Definition 4.22.** Let $Y$ be a regular CW complex. We say that a simple stratification $Y = \bigsqcup S_i$ is a block stratification if each stratum is a union of cell interiors
\[ S_i = \bigcup_{e \in I_i} \operatorname{int}(e). \]

**Proposition 4.23.** Let $S$ be a block stratification of a regular CW complex $Y$. Then the natural map
\[ F : \operatorname{Ent}_{SCW}(Y) \to \operatorname{Ent}_S(Y) \]
induces an equivalence
\[ \overline{F} : \operatorname{Ent}_{SCW}(Y)/L_S \cong \operatorname{Ent}_S(Y) \]
where $L_S$ is the full subcategory of paths which lies entirely in a single stratum of $S$.

**Proof.** The functor $F$ simply takes a point $y \in Y$ to the same point in $Y$ and a path $\gamma \in \operatorname{Ent}_{SCW}(Y)$ to the same path viewed as a path in $\operatorname{Ent}_S(Y)$. Clearly, if $\gamma$ is a morphism in $L_S$, then $F(\gamma)$ is invertible with inverse the reverse path.

On the other hand, if $\gamma$ is a path in $\operatorname{Ent}_S(Y)$, then we can write $\gamma$ as a concatenation
\[ \gamma = \gamma_1 \ast \ldots \ast \gamma_n \]
where each $\gamma_j([0,1))$ lies in a single stratum of $SCW$. Notice that $\gamma_j$ is an entrance path or $\gamma^{-1}_j$ is an entrance path for all $j$. Hence, we can construct an inverse functor
\[ \overline{G} : \operatorname{Ent}_S(Y) \to \operatorname{Ent}_{SCW}(Y)/L_S \]
where $\overline{G}(\gamma)$ formally reverses all $\gamma_j$ which are not entrance paths. Clearly $\overline{F} \circ \overline{G} = \operatorname{Id}_{\operatorname{Ent}_S(Y)}$ and $\overline{G} \circ \overline{F} = \operatorname{Id}_{\operatorname{Ent}_{SCW}(Y)}$ i.e. $\overline{F}$ is an equivalence.

\[ \square \]
Lemma 4.24. Let $S$ be a block stratification of $Y$. For all $y \in Y$ and all lifts $\tilde{y}$, there is an equivalence of categories

$$L : y \setminus F \to \text{Ent}_S(\tilde{Y}_{\text{Ent}}(\tilde{y})).$$

Proof. By definition, objects of $y \setminus F$ are pairs $(y_0, \gamma_0)$ where $\gamma_0$ is an entrance path from $y$ to $y_0$ in $\text{Ent}_S(Y)$. We define $L$ on objects by $L(y_0, \gamma_0) = \tilde{\gamma}_0(1)$ where $\tilde{\gamma}_0$ is the unique lift of $\gamma_0$ starting at $\tilde{y}$. By definition, a morphism from $(y_1, \gamma_1)$ to $(y_2, \gamma_2)$ is an entrance path $\alpha$ from $y_1$ to $y_2$ in $\text{Ent}_S(Y)$ with

$$\gamma_2 \sim_p \alpha \circ \gamma_1$$

We define $L(\alpha)$ to be the unique lift of $\alpha$ starting at $\tilde{\gamma}(1)$.

Conversely, we can define $L^{-1}$ which takes objects $\tilde{y}_0$ to $(y_0, \pi(\tilde{\gamma}_0))$ where $\tilde{\gamma}_0$ is the unique homotopy class of path from $\tilde{y}$ to $\tilde{y}_0$. Similarly, $L^{-1}$ takes morphisms $\tilde{\alpha} : \tilde{y}_1 \to \tilde{y}_2$ to $\pi(\tilde{\alpha})$. These functors are mutually inverse. □

The following is now a consequence of the cited result of Quillen.

Theorem 4.25. Let $S$ be a block stratification of a regular CW complex $Y$. Then $B(\text{Ent}_S(Y))$ is homotopic to $Y$.

Proof. Consider the functor $F : \text{Ent}_{SCW}(Y) \to \text{Ent}_S(Y)$. We have

$$B(y \setminus F) \simeq B(\text{Ent}_{SCW}(\tilde{Y}_{\text{Ent}}(\tilde{y}))) \quad \text{by Lemma 4.21 and Lemma 4.24}$$

$$\simeq \tilde{Y}_{\text{Ent}}(\tilde{y}) \quad \text{by Proposition 4.2}$$

Hence, since $S$ is simple, $B(y \setminus F)$ is contractible. Therefore

$$Y \simeq B(\text{Ent}_{SCW}(Y)) \quad \text{by Proposition 4.2}$$

$$\simeq B(\text{Ent}_S(Y)) \quad \text{by Theorem 4.20 and Proposition 4.23}$$

□

Example 4.26. There are easy counterexamples to Theorem 4.25 when $S$ is not block (that is, when $\tilde{Y}_{\text{Ent}}(\tilde{y})$ is not contractible). For example, consider the stratification of the sphere into the closed northern hemisphere and open southern hemisphere. The geometric realization of the entrance path category in this case is an interval (which is not homotopic to a sphere).

4.4. The tree stratification. Let $A$ be an HPA with vertices the poset $I$ and $C := C_A$. For each $i \in I$ we may define the full subcategory $C_{\geq i}$ consisting of all objects greater than or equal to $i$. We view $B(C_{\geq i})$ as a subset of $B(C)$.

Definition 4.27. The tree stratification, denoted by $S^{tr}$ of $BC$ is the stratification given by

$$X_A = \coprod_{i \in I} S^{tr}_i$$

where

$$S^{tr}_i := B(C_{\geq i}) \setminus \bigcup_{j > i} B(C_{\geq j}) = B(C_{\leq i}^{op}) \setminus \bigcup_{j < i} B(C_{\leq j}^{op}).$$

We denote the exit sheaves for the tree stratification by $Q_w$ and the entrance sheaves by $T_w$. 
Proposition 4.28. For each $v \in I$, choose a lift $\tilde{v}$ to the universal cover $X_A$. Then
\[(\tilde{X}_A)_{\text{Ent}}(\tilde{v}) = K(\text{Path}_{A,\tilde{v}}).\]
Hence, the tree stratification is a block stratification.

Proof. Let $\tilde{C}$ be the category whose objects are vertices of $\tilde{X}_A$, and morphisms are the edges of $\tilde{X}_A$. Then,
\[K(\text{Path}_{A,\tilde{v}}) = B\tilde{C}_{\geq \tilde{v}} = (\tilde{X}_A)_{\text{Ent}}(\tilde{v}).\]

To justify that the tree stratification is a block stratification, we regard $B\tilde{C}$ as a regular CW complex and observe that the strata are unions of open cells by definition. Furthermore, since $K(\text{Path}_{A,\tilde{v}})$ is the order complex of a bounded below poset, it is contractible and the contraction (to the vertex $\tilde{v}$) also contracts the subspace $K(\text{Path}_{A,\tilde{v}}) \setminus K(\text{Path}_{A,\tilde{v}'}).$ $\square$

Consider a point $y \in Y$. The trivial path $e_y$ is an entrance path hence becomes a vertex in the simplicial complex $B(\text{Ent}_S(Y))$. Similarly, for an entrance path $\gamma \in \text{Ent}_S(Y)$. As a morphism in $\text{Ent}_S(Y)$ this naturally corresponds to the edge in $B(\text{Ent}_S(Y))$ which we denote by $F(\gamma)$.

Lemma 4.29. Let $S$ be a block stratification of a regular CW complex, $Y$. The functor
\[
F : \text{Ent}_S(Y) \to \text{Ent}_{S'}(B(\text{Ent}_S(Y)))
\]
\[
y \mapsto e_y \in B(\text{Ent}_S(Y))
\]
\[
\gamma \mapsto F(\gamma)
\]
is an equivalence.

Proof. Notice that the tree stratification of $B(\text{Ent}_S(Y))$ has a canonical set of base points $e_y$, and the functor is, by definition, a bijection on base-points, hence essentially surjective by definition.

We may choose $\gamma$ so that the homotopy in Theorem 4.25 takes $\gamma$ to $F(\gamma)$. This implies $F$ is faithful.

On the other hand since $S'$ is a block stratification by Proposition 4.28, any entrance map in $\text{Ent}_{S'}(B(\text{Ent}_S(Y)))$ can be moved homotopically to an edge. This implies that $F$ is full. $\square$

Corollary 4.30. We have an equivalence of categories given by the composition
\[
\begin{array}{c}
\text{Fun}(\text{Ent}_S(Y), k\text{-mod})^{\text{op}} \xrightarrow{\text{Lemma 4.29}} \text{Fun}(\text{Ent}_{S'}(B(\text{Ent}_S(Y))), k\text{-mod})^{\text{op}} \\
\text{Sh}_S(Y) \xrightarrow{\text{Theorem 4.17}} \text{Sh}_{S'}B(\text{Ent}_S(Y))
\end{array}
\]

Note that $\mathcal{T}_w$ is the entrance sheaf at $w$ on $B\text{Ent}_S(Y)$ for the tree stratification. We define $Q_w$ as the exit sheaf at $w$ on $B\text{Ent}_S(Y)$ for the tree stratification. Corollary 4.30 says that the equivalence in Theorem 4.17 and the homotopy in Theorem 4.25 connect objects in the following table.
Remark 4.31. Notice that in Table 1 as A is a finite dimensional algebra, the Serre functor for $\text{D}(A)$ is $\text{Hom}_A(-,A)^\ast$. This takes $P_w$ to $I_w$. Hence, the Serre functor for $\text{DSh}_S(Y)$ takes $P_w$ to $I_w$ and similarly the Serre functor for $\text{DSh}_{Str}(B\text{Ent}_S(Y))$ takes $Q_w$ to $T_w$.

Remark 4.32. Given a block stratification, we know that $Y$ is homotopic to $B\text{Ent}_S(Y)$ by Theorem 4.25. However, the homotopy need not preserve the strata (although it preserves the entrance spaces), see Example 5.16. On the other hand, there is a correspondence (in the proof of Quillen’s Theorem A) which should induce the equivalence in Corollary 4.30 by Fourier-Mukai transform.

5. Toric HPAs

5.1. HPAs from line bundles. Let $G \subseteq \text{Gl}_{n+k}$ be an abelian group acting diagonally on a vector space $V = \mathbb{A}^{n+k}$. We denote by $\mathfrak{X} := [V/G]$ the associated Artin stack.

Applying the exact functor $\hat{\cdot} = \text{Hom}(-,\mathbb{G}_m)$ we obtain an exact sequence

$$0 \to M \to \mathbb{Z}^{n+k} \xrightarrow{\mu} \hat{\mathbb{G}} \to 0.$$ 

where $M$ is defined to be the kernel of $\mu$.

Definition 5.1. We say $\mathfrak{X}$ is cohomologically proper if $\mu(\mathbb{R}_{\geq 0}^{n+k})$ is a strongly convex cone in $\hat{\mathbb{G}}_\mathbb{R}$.

Lemma 5.2. The (underived) endomorphism algebra of a collection of line bundles on a cohomologically proper toric stack is an HPA.

Proof. The vertices of the quiver can be identified with elements in the Picard lattice. Since the stack is cohomologically proper, this guarantees that the endomorphism algebra is finite. Morphisms between line bundles are generated by monomial sections (realized as graded pieces of the Cox ring). Relations among the sections are generated by equalities of monomials $m_1\ldots m_t = n_1\ldots n_s$. The corresponding ideal satisfies the conditions of Definition 3.1.

Definition 5.3. Let $L_1,\ldots, L_t$ be a collection of line bundles on a toric stack. The HPA $A = \text{End}(\oplus L_i)$ is called a toric HPA. If $L_1,\ldots, L_t$ is a full strong exceptional collection then $A$ is called a toric FSEC HPA.

5.2. Bondal-Ruan type HPAs. In this section, we study a stratification on $\mathbb{T}^n$ defined by Bondal-Ruan [6].

Let $D_i$ for $1 \leq i \leq n+k$ be the standard basis of $\mathbb{Z}^{n+k}$. We consider the Bondal-Ruan map

$$\Phi : \mathbb{T}^n \to \hat{\mathbb{G}}$$

(12)

$$\sum_i a_i D_i + M \mapsto \mu(-\sum_i |a_i| D_i)$$

(13)
naturally defined on the fiber $\mu_{-1}(0)/M = \mathbb{R}^n$, where $[a_i]$ is the floor of $a_i$. Let

$$\tilde{\Phi} : \mu_{-1}(0) \to \hat{G}$$

be the natural lift of $\Phi$ to an $M$-periodic map.

We take the associated stratification $S$ on $\mathbb{T}^n$ whose strata are given by the level sets of $S_D := \Phi^{-1}(D)$, equipped with the Picard order:

$$S_E \leq S_D \text{ iff } E - D \text{ is effective.} \quad (16)$$

For effective $D = \sum \rho_i b_i D_i \in \mathbb{Z}^{n+k}$, we consider the following regions

$$\text{Poly}_D = \mathbb{R}_{\geq 0}^{n+k} \cap \mu_{-1}(0) = \{ m \in M_{\mathbb{R}} : \langle m, D \rangle \geq -b \rho \}$$

$$\tilde{S}_D = (-D + [0,1)_{n+k}) \cap \mu_{-1}(0)$$

Lemma 5.4. There exists a canonical homeomorphism $f : \tilde{S}_D \to S_D$. Furthermore, \[ \text{Im } \Phi = \mu_{\mathbb{R}}([0,1)^{n+k}) \cap \hat{G} = \{ \mu(D) : S_D \neq \emptyset \}. \]

Proof. For any $D = \sum b_i D_i \in \mathbb{Z}^{n+k}$,

$$\tilde{S}_D = \mu_{-1}(0) \cap [-b_i, -b_i + 1)_{i=1}^{n+k}$$

$$= \{ \sum a_i D_i : -b_i \leq a_i < -b_i + 1, \mu_{\mathbb{R}}(\sum a_i D_i) = 0 \} \subset M_{\mathbb{R}}$$

$$= \{ \sum a_i D_i + M : \Phi(\sum a_i D_i + M) = \mu(D) \} \subset \mathbb{T}^n = M_{\mathbb{R}}/M$$

(since $\mu_{-1}(0) \cap [-b_i, -b_i + 1)_{i=1}^{n+k}$ is already in a fundamental domain)

$$= \Phi^{-1}(D) = S_D \quad \text{by definition of } S_D$$

In other words, the composition of the above gives a canonical homeomorphism

$$f : \tilde{S}_D \to S_D$$

$$\sum a_i D_i \mapsto \sum a_i D_i + M$$

In particular, this implies $\tilde{S}_D + D = \mu_{-1}(0) \cap [0,1)^{n+k} \neq \emptyset \iff S_D = \Phi^{-1}(D) \neq \emptyset$. $\Box$

Proposition 5.5. We have an equality

$$\mathbb{R}^n_{\text{Ent}}(D) = \text{Poly}_D.\]$$

In particular, if we fix a base-point $x_D \in S_D$ for each stratum. Then, there is a 1-1 correspondence between the monomial basis

$$m \in \text{Hom}_{D(x)}(\mathcal{O}(-D), \mathcal{O}(-E))$$

and entrance paths starting at $x_D$ and ending at $x_E$. Similarly, if $\gamma$ is an entrance path originating in $S_D$, then the unique lift $\tilde{\gamma}$ to $M_{\mathbb{R}}$ starting in $\tilde{S}_D$ lies inside $\text{Poly}_D$. Finally, $\text{Poly}_D \setminus \text{Poly}_E$ is star-shaped with center $x_D$ for any $\mu(D), \mu(E) \in \text{Im } \Phi$. 

**Proof.** Consider the floor of the projection onto the $\rho^{th}$ factor $[(\tilde{\gamma}, D_\rho)] : [0, 1] \rightarrow \mathbb{Z}$ for each $\rho$. If $\tilde{\gamma}$ is not entirely inside $\text{Poly}_D$, then there exists $t \in [0, 1]$ and $\epsilon > 0$ such that $\tilde{\gamma}(t) \in \text{Poly}_D$ and $\tilde{\gamma}|_{(t,t+\epsilon]} \cap \text{Poly}_D = \emptyset$. This means there exists $\rho_0$ and $\epsilon' \leq \epsilon$, s.t.

1. $[(\gamma(t), D_{\rho_0})] > [(\gamma(t + \epsilon'), D_{\rho_0})]$

2. $[(\gamma|(t,t+\epsilon'), D_{\rho_0})]$ does not increase.

Hence $\tilde{\gamma}(t)$ and $\tilde{\gamma}|_{(t,t+\epsilon')} \in$ separate strata, $S_{D_{\rho}}$ and $S_{D_{t+\epsilon'}}$, but then $-D_{t+\epsilon'} + D_{\epsilon}$ is anti-effective and hence not effective since $X$ is homologically proper. This is a contradiction as $\tilde{\gamma}$ is therefore not an entrance path. Hence

$$\mathbb{P}^n_{\text{Ent}}(D) \subseteq \text{Poly}_D.$$

Now suppose $m \in \text{Hom}_D(X)(\mathcal{O}(-D), \mathcal{O}(-E))$ is an element of the monomial basis. This is equivalent to having $m + \text{Poly}_E \subseteq \text{Poly}_D$. Let $\tilde{x}_D$ be the lift of $x_D$ in $\tilde{S}_D$. Consider the straight line path $l_m : [0, 1] \rightarrow \mathbb{R}$ from $\tilde{x}_D$ to $m + \tilde{x}_E$. Since for each $\rho$, $(\gamma(D_{\rho})|_{l_m} : [0, 1] \rightarrow \mathbb{R}$ is a linear function, it is either increasing or decreasing. This means the discretization $[(\gamma(D_{\rho})|_{l_m} : [0, 1] \rightarrow \mathbb{Z}$ is also either increasing or decreasing, but since $l_m$ starts at $\tilde{S}_D$ and never leaves $\text{Poly}_D$, the discretization can only increase. This implies that $l_m$ is an entrance path. Hence

$$\mathbb{P}^n_{\text{Ent}}(D) \supseteq \text{Poly}_D.$$

Finally, we check that $\text{Poly}_D \setminus \text{Poly}_E$ is star-shaped. To the contrary, suppose there is a point $x \in \text{Poly}_D \setminus \text{Poly}_E$ such that the line $l_x$ from $x$ to $\tilde{x}_D$ passes through $\text{Poly}_E$. Choose $x' \in l_x \cap \text{Poly}_E$. Then, from the above, $l_x$ is an entrance path and hence, the line from $x'$ to $x$ is also an entrance path. However, we have already proven that all entrance paths starting at $x'$ lie entirely in $\text{Poly}_E$, which is a contradiction since $x \notin \text{Poly}_E$.

\[ \square \]

**Lemma 5.6.** For any $D \in \text{Im } \Phi$, $S_D = \bigcap_i (\text{Poly}_D \setminus \text{Poly}_{D-D_i}) / M$, in particular all strata in $S$ are contractible.

**Proof.** Since

$$-D + [0,1]^{n+k} = \bigcap_i (\mathbb{R}^{n+k}_{\geq -D} \setminus \mathbb{R}^{n+k}_{\geq -D+D_i}),$$

$\tilde{S}_D = \bigcap_i \text{Poly}_D \setminus \text{Poly}_{D-D_i}$ is by definition. Hence, $\tilde{S}_D$ is contractible because it is the intersection of star-shaped regions with center $\tilde{x}_D$ (hence star-shaped) by Proposition 5.5. Hence $S_D$ is contractible by Lemma 5.4. \[ \square \]

**Corollary 5.7.** The stratification $S$ of $\mathbb{T}^n$ is a block stratification.

**Proof.** This follows from Proposition 5.5 and Lemma 5.6. \[ \square \]

**Definition 5.8.** The Bondal-Ruan HPA is defined as the endomorphism algebra

$$A_\Phi := \text{End}(\bigoplus_{\mu(D) \in \text{Im } \Phi} \mathcal{O}_X(-D)) = R \text{End}(\bigoplus_{\mu(D) \in \text{Im } \Phi} \mathcal{O}_X(-D)).$$

**Corollary 5.9.** There is an equivalence of categories $\mathcal{C}_{A_\Phi} \cong \text{Ent}_S(\mathbb{T}^n)$.

**Proof.** The 1-1 correspondence from Proposition 5.5 induces an equivalence

$$\mathcal{C}_{A_\Phi} \rightarrow \text{Ent}_S(\mathbb{T}^n).$$
which takes a vertex \( D \in (A_\Phi)_0 \) to the base point \( x_D \) and a path in \( m \in A_\Phi \) to the corresponding entrance path \( l_m \).

**Theorem 5.10.** Let \( A_\Phi \) be the Bondal-Ruan HPA. Then \( X_{A_\Phi} \) is homotopic to \( \mathbb{T}^n = M_\mathbb{R}/M \).

**Proof.** The condition of Theorem 4.25 is satisfied by Corollary 5.7. Hence,

\[
X_{A_\Phi} = B(C_{A_\Phi}) \quad \text{by definition}
\]

\[
\simeq B(\text{Ent}_S(\mathbb{T}^n)) \quad \text{by Corollary 5.9}
\]

\[
\simeq \mathbb{T}^n \quad \text{by Theorem 4.25}
\]

\( \square \)

Consider a toric DM stack \( X \) coming from the GIT construction for the action of \( G \) on \( V \). That is, consider any character \( \theta \in \hat{G} \), and define

\[
\mathcal{X} := [(\mathbb{A}^{n+k})_{ss}/G].
\]

This is an open substack of \( \mathcal{X} \). Furthermore, for generic \( \theta \), this is a DM stack.

**Definition 5.11.** If \( \text{Im } \Phi \) forms a full strong exceptional collection of line bundles on \( X \), we say \( X \) is of Bondal-Ruan type.

When \( X \) is of Bondal-Ruan type then, by assumption,

\[
A_\Phi = \text{End}( \bigoplus_{D \in \text{Im } \Phi} O_X(-D)) = R \text{End}( \bigoplus_{D \in \text{Im } \Phi} O_X(-D))
\]

and

\[
D(X) = D(A_\Phi).
\]

**Theorem 5.12** (Bondal-Ruan Homological Mirror Symmetry). Let \( X \) be a toric DM stack of Bondal-Ruan type. Then there are equivalences of categories

\[
D(X) = D(A_\Phi) \quad \text{by the assumption that } X \text{ is of Bondal-Ruan type}
\]

\[
= \text{DSh}_S(\mathbb{T}^n) \quad \text{by Corollary 5.9 and Corollary 4.30}
\]

As a consequence, our construction of \( X_A \), which only depends on knowing the HPA for \( D(X) \), recovers the FLTZ mirror up to homotopy in the Bondal-Ruan case.

We finish this section by discussing the class of toric varieties of Bondal-Ruan type originally given in [6].

**Lemma 5.13.** Assume that \( C \cap \mu(D_\rho) \geq -1 \) for all torus-invariant complete irreducible curves \( C \subseteq X_{\Sigma} \) and all \( \rho \in \Sigma(1) \), and \( C \cap D_\rho = -1 \) for no more than one \( \rho \). Then every \( \mu(D) \in \text{Im } \Phi \) is nef.

**Proof.** Let \( \mu(D) \in \text{Im } \Phi \). Then by Lemma 5.4

\[
\mu(D) = \mu(\sum a_\rho D_\rho)
\]

with \( 0 \leq a_\rho < 1 \). Hence,

\[
\langle \mu(D), C \rangle = \sum a_\rho \langle \mu(D_\rho), C \rangle
\]

is greater than \(-1\) by the assumption. Now since \( \langle \mu(D), C \rangle \in \mathbb{Z} \) it follows that \( \langle \mu(D), C \rangle \geq 0 \), as desired. \( \square \)
Proposition 5.14 (Bondal-Ruan [6]). Let $X_\Sigma$ be a smooth proper toric variety. Assume that $C \cap D_\rho \geq -1$ for all irreducible curves $C \subseteq X_\Sigma$ and all $\rho \in \Sigma(1)$, and $C \cap D_\rho = -1$ for no more than one $\rho$. Then $\text{Im } \Phi$ forms a full strong exceptional collection of line bundles.

Proof. By [2] (see also [32]), $\text{Im } \Phi$ generates $D(X_\Sigma)$. It remains to show that

$$\text{Ext}^i \left( \bigoplus_{\mu(D) \in \text{Im}(\Phi)} \mathcal{O}_X(-D), \bigoplus_{\mu(D) \in \text{Im}(\Phi)} \mathcal{O}_X(-D) \right) = 0$$

for all $i > 0$.

For this, we follow the proof in [6]. For $l$ sufficiently divisible we have

$$\text{Ext}^i \left( \bigoplus \mathcal{O}_X(-D), \bigoplus \mathcal{O}_X(-D) \right) = \text{Ext}^i((F_l)_* \mathcal{O}_X, (F_l)_* \mathcal{O}_X)$$

by [29], Theorem 4.5

$$= \text{Ext}^i((F_l)_* \mathcal{O}_X, (F_l)_* \mathcal{O}_X)$$

by adjunction

$$= \bigoplus \text{Ext}^i((F_l)_* \mathcal{O}_X(-D), \mathcal{O}_X)$$

by [29], Theorem 4.5

$$= H^i(F_l^* \mathcal{O}_X(D))$$

$$= H^i(\mathcal{O}_X(lD)).$$

By Demazure vanishing, it enough to know that $\mu(D)$ is nef for all $\mu(D) \in \text{Im } \Phi$. This is precisely Lemma 5.13. □

Remark 5.15. As pointed out in [6], all 18 smooth toric Fano threefolds except for (II)(d) and (III)(k) in [33] are of Bondal-Ruan type.

Example 5.16. We go back to Example 3.12 where the poset is $I = \{-2, -1, 0\}$. Below are the tree strata $S_{tr}^\bullet$ of $X_A$ (with $S_{tr}^2 < S_{tr}^1 < S_{tr}^0$).
We compare $S^{tr}$ with the stratification $S$ of the torus $T^2$ (with $S_2 < S_1 < S_0$, see Eq. 16):

Specifically, gray, purple, and cyan are the three strata in $S$, corresponding to the black, purple and cyan vertices in the quiver. The gray stratum $S_2$ is the minimal ($\Rightarrow$ open) stratum. The cyan stratum $S_0$ is the maximal ($\Rightarrow$ closed) stratum. The purple stratum $S_1$ is a closed triangle minus 3 points and is locally closed. The $S$-entrance and $S$-exit sheaves are as follows:

Although the homotopy $X_A \simeq T^2$ does not preserve the strata, it preserves the upward closures of the strata and classes of entrance paths. By Corollary 4.30, the sheaf categories are equivalent. The following table summarizes notable equivalent objects in this example.
In this section, we study projective cellular resolutions of the diagonal bimodule of an HPA. We observe that $X_A$ itself always gives a projective cellular resolution. Furthermore, we provide a sequence of simplicial collapses of $X_A$ which, at least in a large class of examples, provide a projective cellular resolution which is equal to the minimal resolution. As an application, we show that the Bondal-Ruan HPA is always Koszul and admits a minimal cellular resolution.

6. Cellular resolution. For a finite CW complex $X$ satisfying the axiom of the frontier, let $\text{Cell}(X)^{\text{op}}$ be the set of cells, with partial ordering

\[(17) \quad \sigma \leq \tau \text{ iff } \tau \subset \overline{\sigma}.\]

Let $\text{Cell}(X)^{\text{op}}$ be the category of cells on $X$ associated to the poset $\text{Cell}(X)^{\text{op}}$, whose objects are cells and morphisms be such that

\[(18) \quad \exists! \text{ morphism } \sigma \to \tau \text{ if } \sigma \leq \tau \text{ in } \text{Cell}(X)^{\text{op}} \quad \text{no morphism otherwise.}\]

Let $R$ be a ring and $R$-mod be the category of left $R$-modules. We first define a cellular resolution, using the cosheaf language in [9] for convenience.

Definition 6.1. Let $P$ be a poset. The Alexandrov topology on $P$ is a topology on $P$ whose open sets are upwardly closed sets

\[(19) \quad U \subset P \text{ is open iff } \forall x \in U, \, y > x \Rightarrow y \in U.\]

A cellular cosheaf of $R$-modules on $X$ is a cosheaf of $R$-modules on $\text{Cell}(X)$ equipped with the Alexandrov topology.

Theorem 6.2 ([9], Theorem 4.2.10). For any poset $P$, and category $D$ that is both complete and co-complete, the left Kan extension $\text{Lan}_{iX}(\text{−}) : \text{Fun}(P^{\text{op}}, D) \to \text{CoSh}(P, D)$ is an isomorphism.
The theorem says that a cellular cosheaf of $R$-modules on $X$ is simply determined by a functor

$$F : \text{Cell}(X)^{\text{op}} \to R\text{-mod},$$

which assigns each $k$-cell $\eta_k$ an $R$-module $F(\eta_k)$, and a module map $F^\eta_{k+1} := F(\eta_{k+1}) \to F(\eta_k)$ if $\eta_k > \eta_{k+1}$.

For each facet relation $\sigma > \tau$, write $[\tau : \sigma]$ for its degree. There is a chain complex associated to a cellular cosheaf $F$, namely

$$C_\bullet(F) := \cdots \bigoplus_{\eta_{k+1} \in \text{Cell}_{k+1}(X)} F(\eta_{k+1}) \xrightarrow{\delta_{k+1}} \bigoplus_{\eta_k \in \text{Cell}_k(X)} F(\eta_k) \to \cdots \bigoplus_{\eta_0 \in \text{Cell}_0(X)} F(\eta_0), \text{ where}$$

$$\delta_{k+1} := [\eta_{k+1} : \eta_k]F^\eta_{k+1}.$$

**Example 6.3.** The chain complex associated to the cellular cosheaf

$$\text{CW} : \text{Cell}^{\text{op}}(X) \to \text{k-mod}$$

$$\sigma \mapsto k$$

$$\sigma \geq \tau \mapsto \text{Id} : k \to k$$

is the CW-chain complex computing the CW-homology of $X$ with coefficients in $k$.

**Definition 6.4.** Let $M$ be an $R$-module. A cellular resolution of $M$ supported on $X$ is a cellular cosheaf $F : \text{Cell}(X)^{\text{op}} \to R\text{-mod}$ such that $C_\bullet(F)$ is a resolution of $M$. The cellular resolution is projective if $C_\bullet(F)$ is a projective resolution of $M$.

**Remark 6.5.** When $X$ is regular, the degree is 0 or $\pm 1$; an incidence function $\epsilon : \text{Cell}(X)^{\text{op}} \times \text{Cell}(X)^{\text{op}} \to \{-1, 0, 1\}$ can be chosen such that the value of the incidence function equals the degree. When $X$ is a regular semi-simplicial complex, one can pick a canonical incidence function, and a cellular cosheaf of $R$-modules in this case is simply a semi-simplicial object in $R\text{-mod}$.

We now describe a cellular resolution of the diagonal bimodule $A$ supported on $X_A$. For a vertex $v \in Q_0$, consider the projective modules

$$P_v := Ae_v \text{ and } P^{\text{op}}_v := e_v A.$$

We view $P_v$ as a left $A$-module, and $P^{\text{op}}_v$ as a right $A$-module. Consider the $k$-algebra

$$A_0 := \bigoplus_{v \in Q_0} k \to A$$

$$1_v \mapsto e_v.$$

We also consider, for each $k$-cell $\eta_k$ in $X_A$, the projective $(A, A)$-bimodule

$$P_{\eta_k} := P_{t(\eta_k)} \otimes P^{\text{op}}_{h(\eta_k)} = P_{t(\eta_k)} \otimes_{A_0} k_{\eta_k} \otimes_{A_0} P^{\text{op}}_{h(\eta_k)} \in A \otimes_k A^{\text{op}}\text{-mod},$$

where the left action of $A$ is given by concatenation with paths ending at $t(\eta_k)$, and the right action of $A$ is given by concatenation with paths starting at $h(\eta_k)$.

Recall that each $k$-simplex $\eta_k$ can be represented by a sequence of paths $[p_0, \ldots, p_k]$ in $K(\text{Path})$ up to the equivalence relation $\sim$. We say $[p_0, \ldots, p_k]$ is a canonical representative of $\eta_k$ if its image under the quotient / $\sim$ is the cell $\eta_k$ and $p_0 = e_{t(\eta_k)}$. By construction, every
cell in $X_A$ has a unique canonical representative. We denote the canonical representative of $\eta_k$ by $[\eta_k]$.

We first show that the projectives

$$C_k := \bigoplus_{\eta_k \in Cell_k(X_A)} P_{\ell(\eta_k)} \boxtimes P^{op}_{h(\eta_k)}$$

form a semi-simplicial object in $A \otimes_k A^{op}$-mod i.e. $C : Cell(X_A) \to A \otimes_k A^{op}$-mod forms a cellular cosheaf. For each cell $\eta_k$, we define the semi-simplicial maps $\partial_i$ on the canonical representative $[\eta_k] = [e_{\ell(\eta_k)} < p_1 < \cdots < p_k]$ as follows.

$$\partial_i := \sum_{\eta_k \in Cell_k(X_A)} \partial_{i,\eta_k} : C_k \to C_{k-1}$$

(24)

$$\partial_{i,\eta_k}(1 \otimes [\eta_k] \otimes 1) := \begin{cases} p_1 \otimes [e_{h(p_1)}] < \cdots < p_k] \otimes 1 & i = 0 \\ 1 \otimes [e_{\ell(\eta_k)}] < \cdots < p_k] \otimes 1 & 0 < i < k \\ 1 \otimes [e_{\ell(\eta_k)}] < \cdots < p_{k-1}] \otimes p_k/p_{k-1} & i = k \end{cases}$$

Lemma 6.6. The data $(C_k, \partial_i)$ forms a semi-simplicial $(A, A)$-bimodule. In particular, there is an associated chain complex $C_\bullet$ whose $k^{th}$ component is $C_k$ and with differential

$$d_k := \sum_{i=0}^{k} (-1)^i \partial_i$$

Proof. One only needs to check the semi-simplicial identities $\partial_i \partial_j = \partial_{j-1} \partial_i$ for all $i < j$. For a canonical representative $[e_{\ell(p_i)}] < \cdots < p_k]$, we compute this in 5 cases:

1. For $0 < i < j < k$:

$$\partial_i \partial_j(1 \otimes [e_{\ell(p_i)}] < \cdots < p_k] \otimes 1) = \partial_{j-1} \partial_i(1 \otimes [e_{\ell(p_i)}] < \cdots < p_k] \otimes 1) = 1 \otimes [e_{\ell(p_i)}] < \cdots < p_k] \otimes 1$$

2. For $i = 1, j = 1$:

$$\partial_0 \partial_1(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1) = p_2 \otimes [e_{h(p_2)}] < p_3/p_2 < \cdots < p_k/p_2] \otimes 1 = \partial_0 \partial_0(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1)$$

3. For $1 < j < k$:

$$\partial_0 \partial_j(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1) = p_1 \otimes [e_{h(p_1)}] < \cdots < p_{k-1}] \otimes p_k/p_{k-1} = \partial_{j-1} \partial_0(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1)$$

4. For $0 < i < k - 1$:

$$\partial_i \partial_k(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1) = 1 \otimes [e_{\ell(p_1)}] < \cdots < p_{k-1}] \otimes p_k/p_k - 1 \partial_{k-1} \partial_i(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1)$$

5. For $i = k - 1, j = k$:

$$\partial_{k-1} \partial_k(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1) = 1 \otimes [e_{\ell(p_1)}] < \cdots < p_{k-2}] \otimes p_k/p_k - 2 \partial_{k-1} \partial(1 \otimes [e_{\ell(p_1)}] < \cdots < p_k] \otimes 1).$$

$\square$
In particular, we have \( C_0 = \bigoplus_{v \in Q_0} P_v \otimes P_v^{op} \). Let

\[
\tilde{C}_* = \cdots C_1 \to C_0 \xrightarrow{m} A \to 0
\]

be the augmented chain complex where \( m \) is the multiplication map. Consider the collection of right \( A \)-module morphisms

\[
h_k : C_k \to C_{k+1}
\]

\[
h_k(a \otimes [e_v < p_1 < \cdots < p_k] \otimes b) := \begin{cases} 
1 \otimes [e_{t(a)} < a < ap_1 \cdots < ap_k] \otimes b & v = h(a) \\
0 & \text{else}
\end{cases}
\]

(25)

for every path \( a \), extended \( k \)-linearly on the left

\[
h_{-1} : A \to C_0
\]

\[
h_{-1}(b) := 1 \otimes [e_{t(b)}] \otimes b, \text{ extended } k \text{-linearly on the left}
\]

**Corollary 6.7.** The collection of maps \( \{h_k\} \) form a contracting homotopy of \( \tilde{C}_* \). In particular, \( C_* \) is a projective cellular resolution of \( A \) as as a \( (A, A) \)-bimodule.

**Proof.** For convenience, set \( p_0 := a \). We show \( d_{k+1}h_k + h_{k-1}d_k = \text{Id} \) by computation.

(1) For \( k \geq 1 \), we compute \( d_{k+1}h_k(a \otimes [e_{h(a)} < p_1 \cdots < p_k] \otimes b) \)

\[
d_{k+1}(1 \otimes [e_{t(a)} < a < ap_1 \cdots < ap_k] \otimes b)
\]

\[
a \otimes [e_{h(a)} < p_1 \cdots < p_k] \otimes b + \sum_{i=0}^{k-1} (-1)^{i+1} \otimes [e_{t(a)} < a \cdot p_i \cdots < ap_k] \otimes b
\]

\[
+(-1)^{k+1} \otimes [e_{h(a)} < \cdots < ap_{k-1}] \otimes pk/b/pk-1.
\]

(2) and from the other direction we compute \( h_{k-1}d_k(a \otimes [e_{h(a)} < p_1 \cdots < p_k] \otimes b) \)

\[
h_{k-1}(ap_1 \otimes [e_{h(p_1)} < \cdots < p_k/p_1] \otimes b) + \sum_{i=1}^{k-1} (-1)^i a \otimes [e_{h(a)} < \cdots \hat{p}_i < ap_1] \otimes b
\]

\[
+(-1)^k a \otimes [e_{h(a)} < \cdots < p_{k-1}] \otimes pk/b/pk-1
\]

\[
=1 \otimes [e_{t(ap_1)} < ap_1 \cdots < ap_k] \otimes b + \sum_{i=1}^{k-1} (-1)^i 1 \otimes [e_{t(a)} < a \cdots \hat{p}_i \cdots < ap_k] \otimes b
\]

\[
+(-1)^{k-1} \otimes [e_{t(a)} < a \cdots < ap_{k-1}] \otimes pk/b/pk-1
\]

\[
= \sum_{i=0}^{k-1} (-1)^i \otimes [e_{t(a)} < \cdots \hat{p}_i \cdots < ap_k] \otimes b + (-1)^k \otimes [e_{t(a)} < a \cdots < p_{k-1}] \otimes pk/b/pk-1
\]

\[
= -d_{k+1}h_k(a \otimes [e_{h(a)} < p_1 \cdots < p_k] \otimes b) + \text{Id}(a \otimes [e_{h(a)} < p_1 \cdots < p_k] \otimes b).
\]

(3) Similarly for \( k = 0 \) we compute,

\[
d_1h_0(a \otimes [e_{h(a)}] \otimes b) = d_1(1 \otimes [e_{t(a)} < a] \otimes b)
\]

\[
= a \otimes [e_{h(a)}] \otimes b - 1 \otimes [e_{t(a)}] \otimes ab
\]

(4) and

\[
h_{-1}m(a \otimes [e_{h(a)}] \otimes b) = h_{-1}(ab)
\]

\[
= 1 \otimes [e_{t(ab)}] \otimes ab
\]

\[
= 1 \otimes [e_{t(a)}] \otimes ab.
\]
Finally
\[ mh_{-1}(b) = m(1 \otimes [e_t(b)] \otimes b) = b. \]

**Corollary 6.8.** Let \( A \) be an HPA and \( Y \) be a projective cellular resolution of \( A \) as a bimodule. Then, there are isomorphisms of CW homology groups
\[ H_i(Y, k) \cong H_i(X_A, k) \]
for all \( i \).

**Proof.** Let \( P \cong A \) be the projective resolution coming from \( Y \) and \( F : C_A \rightarrow mod - k \)
\[ v \mapsto k \]
\[ a \mapsto id \]
be the trivial representation.

We have
\[ (F \otimes_k F^{op}) \otimes_{A \otimes_k A^{op}} C_\bullet = (F \otimes_k F^{op}) \otimes_{A \otimes_k A^{op}} A \]
by Corollary 6.7
\[ = (F \otimes_k F^{op}) \otimes_{A \otimes_k A^{op}} P_\bullet \]
by assumption
but \((F \otimes_k F^{op}) \otimes_{A \otimes_k A^{op}} C_\bullet\) is exactly the CW homology of \( X_A \) and \((F \otimes_k F^{op}) \otimes_{A \otimes_k A^{op}} P_\bullet\) is exactly the CW homology of \( Y \).

We make the following conjecture

**Conjecture 6.9.** If \( Y \) is a projective cellular resolution of \( A \) as an \( A \otimes_k A^{op} \)-module, then \( Y \) is homotopic to \( X_A \).

**Remark 6.10.** In the next section, we produce smaller (and sometimes minimal) projective cellular resolutions using discrete Morse theory. In particular, all examples produced in the next section satisfy the above conjecture.

### 6.2. Morse Matching

Our next goal is to reduce \( C_\bullet \) to a cellular resolution supported on a CW complex with fewer cells. The resulting complex is an analog of the Morse complex. Throughout this section, we use \( C_\bullet \) to denote CW chain complexes of \( k \)-modules, and use \( C_\bullet \) to denote chain complexes of \((A, A)\)-bimodules.

We first recall the main theorem in discrete Morse theory. Fix a regular CW complex \( X \).

**Definition 6.11.** A matching \( M \) is a collection of subsets \( M_k^{top} \sqcup M_k^{bottom} \subset Cell_k(X) \), together with a bijection \( m_k : M_k^{top} \rightarrow M_k^{bottom} \) for every \( k \) which takes a \( k \)-cell to one of its facets.

Write \( M_k^{top} := \bigsqcup_k M_k^{top} \) and \( M_k^{bottom} := \bigsqcup_k M_k^{bottom} \). A matching gives a bijection \( m : M^{top} \rightarrow M^{bottom} \). We think of a matching as \( M = \text{Graph}(m) \subset M^{top} \times M^{bottom} \), where an element in \( M \) is a pair of matched cells in consecutive dimensions. A cell is called **critical** if it is unmatched.

Consider the Hasse diagram \( \Gamma_X \) of the cell poset of \( X \). A matching \( M \) is simply a graph-theoretic matching on \( \Gamma_X \). Now orient each matched edge towards the vertex whose corresponding cell has smaller dimension, and each unmatched edge towards the vertex whose
corresponding cell has larger dimension. Denote the Hasse diagram with this orientation \( \Gamma^M_X \).

**Definition 6.12.** A matching \( M \) is acyclic if \( \Gamma^M_X \) has no oriented cycles.

**Theorem 6.13** (Fundamental Theorem of Discrete Morse Theory, [19]). An acyclic matching of cells in \( X \) gives a homotopy from \( X \) to a CW complex \( X^{\text{crit}} \) consisting of critical cells.

**Remark 6.14.** Writing \( C_\bullet(X) \) for the CW-homology of a CW complex, the above implies there is a short exact sequence of chain complexes of \( k \)-modules

\[
0 \to C^M_\bullet(X) \to C_\bullet(X) \to C_\bullet(X^{\text{crit}}) \to 0,
\]

where \( C^M_\bullet(X) \) is defined as the kernel and \( M \) is the matching which defines \( X^{\text{crit}} \). Theorem 6.13 says that \( M \) is acyclic iff \( C^M_\bullet(X) \) is acyclic.

Ideally, one would like the critical cells in the minimal resolution to describe vertices and arrows in the quiver, the relations and higher syzygies. Motivated by such consideration, we do not consider all matchings on \( X_A \) but only those internal to the path algebra:

**Definition 6.15.** A matching \( M \) on \( X_A \) is internal to \( A \) if it satisfies

1. Cells corresponding to \( Q_0 \) and \( Q_1 \) are unmatched;
2. For each \((\eta_k, \eta_{k-1}) \in M\), \( h(\eta_k) = h(\eta_{k-1}) \) and \( t(\eta_k) = t(\eta_{k-1}) \).

An internal matching can be written as \( M = \coprod_{v_1, v_2} M_{v_1, v_2} \), where

\[
M_{v_1, v_2} := \{ (\eta_k, \eta_{k-1}) : h(\eta_k) = h(\eta_{k-1}) = v_1, t(\eta_k) = t(\eta_{k-1}) = v_2 \}
\]

For each \((v_1, v_2)\), take \( I_{\text{max}}(v_1, v_2) \) to be the set of saturated cells from \( v_1 \) to \( v_2 \), and define the subcomplex

\[
X_{v_1, v_2} := \bigcup_{\eta \in I_{\text{max}}(v_1, v_2)} \bigcup_{\sigma \leq \eta} \sigma \subset X_A.
\]

**Definition 6.16.** The matching complex associated to an internal matching \( M \) is defined to be the complex \( C^M_\bullet := \bigoplus_{v_1, v_2} C^M_{v_1, v_2} \), where

\[
C^M_{v_1, v_2} := C^M_{v_1, v_2}(X_{v_1, v_2}) \otimes (P_{v_1} \boxtimes P_{v_2}^{\text{op}}).
\]

Algebraically, a matching peels off pairs of terms from the resolution, so that the quotient complex is quasi-isomorphic.

**Proposition 6.17.** Let \( M \) be an internal matching. The following are equivalent

1. \( M \) is acyclic.
2. \( C^M_\bullet \) is acyclic.
3. \( C^M_\bullet \simeq \bigoplus_{v_1, v_2} \bigoplus_{(\eta_k, \eta_{k-1}) \in M_{v_1, v_2}} [k_{\eta_k} \xrightarrow{\text{id}} k_{\eta_{k-1}}] \otimes (P_{v_1} \boxtimes P_{v_2}^{\text{op}}) = \bigoplus_{(\eta_k, \eta_{k-1}) \in M} [k_{\eta_k} \xrightarrow{\text{id}} k_{\eta_{k-1}}] \otimes (P_{v_1} \boxtimes P_{v_2}^{\text{op}}).

**Proof.** (1) \( \Leftrightarrow \) (2): Notice that any oriented cycle \( \gamma \) in \( \Gamma^M_{X_A} \) is a composition of alternating upward and downward arrows, therefore one can write \( \gamma = a_{i_1}^u a_{i_2}^d \cdots a_{i_n}^u a_{i_{n+1}}^d \), where \( a_{i_j}^u \) and \( a_{i_j}^d \) are upward and downward arrows in \( \Gamma^M_{X_A} \) such that \( h(a_{i_j}^u) = t(a_{i_j}^d) \). Furthermore, since \( M \) is internal, once the head or tail is cut off by a downward arrow, it can never be restored by
the subsequent upward arrows. Therefore, each \( a_i^d \) in \( \gamma \) has to fix the head and the tail of the cell, which means a cycle can only occur in \( \Gamma_{X_{\mathcal{M}_v}^2} \) for some \( v_1 \) and \( v_2 \).

The fundamental theorem of discrete Morse theory demonstrates \( \mathcal{M}_{X_{\mathcal{M}_v}^2}(X_{v_1,v_2}) \) is acyclic iff \( M_{v_1,v_2} \) is acyclic. Now viewing \( P_{v_1} \otimes P_{v_2}^{op} \) as a free \( k \)-module we have,

\[
H_\ast(\mathcal{M}_{X_{\mathcal{M}_v}^2}) \simeq H_\ast(\mathcal{M}_{X_{\mathcal{M}_v}^2}(X_{v_1,v_2})) \otimes_k (P_{v_1} \otimes P_{v_2}^{op}).
\]

Therefore, \( \mathcal{M}_{X_{\mathcal{M}_v}^2}(X_{v_1,v_2}) \) is acyclic iff \( M_{v_1,v_2} \) is acyclic. Now by definition of \( \mathcal{M} \) as a direct sum, hence \( \mathcal{M} \) is acyclic iff \( M_{v_1,v_2} \) is acyclic for all \( v_1 \) and \( v_2 \).

(3) \( \Rightarrow \) (2): This is immediate since each direct summand \([k_{\eta_k} \overset{Id}{\to} k_{\eta_{k-1}}]\) is acyclic.

(2) \( \Rightarrow \) (3): Suppose \( \mathcal{M}_{X_{\mathcal{M}_v}^2}(X_{v_1,v_2}) \) is acyclic for all \( v_1 \) and \( v_2 \). The homotopy equivalence between \( X_{v_1,v_2} \) and \( X'_{\mathcal{M}_v} \) is a sequence of simplicial collapses where each step collapses a matched cell to the union of its unmatched boundary. Let \( X_{v_1,v_2}(n) \) be the remaining CW complex after the \((n-1)\)-th step and let \( M_{v_1,v_2}(n) \) be the remaining matching on \( X_{v_1,v_2}(n) \).

The snake lemma gives the following diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & [k_{\eta_k} \overset{Id}{\to} k_{\eta_{k-1}}] & \longrightarrow & C_{\mathcal{M}_{X_{\mathcal{M}_v}^2}(n)} & \longrightarrow & C_{\mathcal{M}_{X_{\mathcal{M}_v}^2}(n+1)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & [k_{\eta_k} \overset{Id}{\to} k_{\eta_{k-1}}] & \longrightarrow & C_{X_{\mathcal{M}_v}^2(n)} & \longrightarrow & C_{X_{\mathcal{M}_v}^2(n+1)} & \longrightarrow & 0 \\
\end{array}
\]

Since the top line is a short exact sequence of acyclic complexes with projective components, the top exact sequence of complexes splits giving as isomorphism

\[
C_{\mathcal{M}_{X_{\mathcal{M}_v}^2}(n)} \simeq C_{\mathcal{M}_{X_{\mathcal{M}_v}^2}(n+1)} \oplus [k_{\eta_k} \overset{Id}{\to} k_{\eta_{k-1}}].
\]

By induction we obtain

\[
C_{\mathcal{M}_{X_{\mathcal{M}_v}^2}} \simeq \bigoplus_{(\eta_k, \eta_{k-1}) \in M_{v_1,v_2}} [k_{\eta_k} \overset{Id}{\to} k_{\eta_{k-1}}].
\]

Tensoring both sides by \( P_{v_1} \otimes P_{v_2}^{op} \) gives the desired isomorphism. \( \square \)

We now describe the \textbf{Morse complex of projective bimodules} for an acyclic internal matching \( M \). A discrete gradient path \( \gamma = a_0^d a_1^d \ldots a_n^d a_{n+1}^d \) from \( \eta_k \) to \( \eta_{k-1} \) is a composition of downward and upward arrows in the Hasse diagram such that \( t(a_0^d) = \eta_k \) and \( h(a_n^d) = \eta_{k-1} \). Let \( \Gamma(\eta_k, \eta_{k-1}) \) be the set of gradient paths from \( \eta_k \) to \( \eta_{k-1} \). We will also use

\[
Z_+(\eta_k, \eta'_k) := \{ w = a_0^u a_1^d \ldots a_n^u a_{n+1}^u : t(w) = \eta_k, \ h(w) = \eta'_k, \ n \in \mathbb{Z}_+ \}
\]

\[
Z_- (\eta_k, \eta'_k) := \{ v = a_0^d a_1^u \ldots a_n^d a_{n+1}^u : t(v) = \eta_k, \ h(v) = \eta'_k, \ n \in \mathbb{Z}_- \}
\]

We set the Morse complex of bimodules to be

\[
\mathcal{C} \cdot := 0 \to \cdots \to \mathcal{C}_k \overset{\delta_k}{\to} \mathcal{C}_{k-1} \to \cdots \to 0,
\]
where \( \mathfrak{C}_k = \bigoplus_{\eta_k \in I_k \cap M_{\text{rev}}} P_{\eta_k} \), and

\[
\delta_k(1 \otimes [\eta_k] \otimes 1) = \sum_{\eta_{k-1} \in I_{k-1} \cap M^{\text{rev}}} \sum_{\gamma \in \Gamma(\eta_k, \eta_{k-1})} \epsilon(\gamma) l(\gamma) \otimes [\eta_{k-1}] \otimes r(\gamma).
\]

The multiplicity \( \epsilon(\gamma) = \pm 1 \) is the same as the one picked to define the topological discrete Morse complex. The left and right coefficients are given by

\[
l(a_0^d a_1^u a_2^d \ldots a_n^u a_n^d) = \prod_{h(a_i^d) \text{ is an initial facet of } t(a_i^d)} [t(a_i^d)/h(a_i^d)]
\]

\[
r(a_0^d a_1^u a_2^d \ldots a_n^u a_n^d) = \prod_{h(a_i^d) \text{ is a terminal facet of } t(a_i^d)} [t(a_i^d)/h(a_i^d)]
\]

These coefficients keep track of the initial or terminal path being factored out by a non-internal facet relation and make \( \delta_k \) a bimodule map.

**Lemma 6.18.** \( \mathfrak{C}_* \) is a chain complex.

**Proof.** By definition,

\[
\delta_{k-1}\delta_k([\eta_k]) = \sum_{[\eta_{k-2}] \gamma_k \gamma_{k-1}} \sum \epsilon(\gamma_k)\epsilon(\gamma_{k-1}) l(\gamma_{k-1}) l(\gamma_k) \otimes [\eta_{k-2}] \otimes r(\gamma_k) r(\gamma_{k-1}),
\]

where \( \gamma_k \gamma_{k-1} \) is the concatenation of the two paths through \( h(\gamma_k) = t(\gamma_{k-1}) \). This means one can write \( \gamma_k \gamma_{k-1} = \gamma_k' \gamma_{k-1} \gamma_{k-1}' \), with \( h(\gamma_k') = h(\gamma_k) = t(\gamma_{k-1}) = t(\gamma_{k-1}') \).

For a fixed codimension two face \( \rho_{k-2} \subset \rho_k \) in a \( k \)-simplex \( \rho_k \), there are precisely two facets of \( \rho_k \) containing \( \rho_{k-2} \). In other words, \( \rho_{k-2} \) and \( \rho_k \) are connected by precisely two configurations of consecutive downward arrows, \( \rho_k \xrightarrow{\rho_k^d} \rho_{k-1} \xrightarrow{\rho_{k-1}^d} \rho_{k-2} \) and \( \rho_k \xrightarrow{\rho_k^b} \rho_{k-1} \xrightarrow{\rho_{k-1}^b} \rho_{k-2} \). We also have \( l(a_k^d a_{k-1}^d) = l(b_k^d b_{k-1}^d) \) and \( r(a_k^d a_{k-1}^d) = r(b_k^d b_{k-1}^d) \), since the left and right coefficients only depend on the endpoints. For \( u \in Z_+(\eta_k, \rho_k) \), \( v \in Z_- (\rho_{k-2}, \eta_k) \), this gives

\[
\gamma_k^a := u a_k^d, \quad \gamma_{k-1}^a := a_{k-1}^d v
\]

\[
\gamma_k^b := u b_k^d, \quad \gamma_{k-1}^b := b_{k-1}^d v
\]

Hence, we can rewrite

\[
\delta_{k-1}\delta_k([\eta_k]) = \sum_{\rho_k \rho_{k-2} \eta_{k-2}} \sum_{u \in Z_+(\eta_k, \rho_k), v \in Z_- (\rho_{k-2}, \eta_{k-2})} \epsilon(\gamma_k^a)\epsilon(\gamma_{k-1}^a) l(\gamma_{k-1}) l(\gamma_k) \otimes [\eta_{k-2}] \otimes r(\gamma_k^a) r(\gamma_{k-1}^a)
\]

\[
+ \epsilon(\gamma_k^b)\epsilon(\gamma_{k-1}^b) l(\gamma_{k-1}) l(\gamma_k) \otimes [\eta_{k-2}] \otimes r(\gamma_k^b) r(\gamma_{k-1}^b).
\]

The two summands have the same left and right coefficients, and \( \epsilon(\gamma_k^a)^2 + \epsilon(\gamma_k^b)^2 = 0 \) because the corresponding two summands in the topological Morse complex have the same multiplicity, whose sum has to vanish.

**Lemma 6.19.** For any acyclic matching, the corresponding Morse complex \( \mathfrak{C}_* \) is quasi-isomorphic to \( \mathfrak{C}_* \). That is the Morse complex gives a projective cellular resolution of \( A \).

**Proof.** Following [19], §6, we get an exact sequence

\[
0 \rightarrow C^M \xrightarrow{f} \mathfrak{C} \xrightarrow{g} \mathfrak{C} \rightarrow 0
\]
Lemma 6.20. There is a bijection of sets

\[ \prod_{p \neq e_v} \text{Cell}(K(e_t(p), p) \rightarrow \text{Cell}_{\geq 2}(X_A)) \]

\[ \alpha \mapsto [e_t(p) < \alpha < p]. \]

Proof. This map is surjective since any cell of dimension at least 2 has a canonical representation of this form. The map is also injective since two canonical representations are never equivalent. \quad \square

Now, for each \([p]\), choose a lexicographical ordering on the maximal chains of the poset \((e_t(p), [p])\). Let \(M_{[p]}\) denote the corresponding matching on order complex of the poset \((e_t(p), [p])\) defined by Babson-Hersh [4].

We extend this to an internal matching on \(\overline{M_{[p]}}\) on \([e_t(p), [p]]\) as follows. For each matched pair of cells \([p_1 < \ldots < p_n] \leftrightarrow [q_1 < \ldots < q_n]\), we match the pair \([e_t(p) < p_1 < \ldots < p_n < p] \leftrightarrow [e_t(p) < q_1 < \ldots < q_n < p]\). This is well-defined by Lemma 6.20.

Furthermore, we choose any unmatched 0-cell \([t]\) and match \([e_t(p) < p]\) with \([e_t(p) < t < p]\). Observe that, by definition, all matched cells in \(\overline{M_{[p]}}\) contain \(p\). Hence for \([p] \neq [p']\), we have \(\overline{M_{[p]}} \cap \overline{M_{[p']}} = \emptyset\). It follows that the union

\[ \overline{M} = \bigcup_{[p]} \overline{M_{[p]}} \]

gives a well-defined global matching on \(\text{Cell}(X_A)\).

Definition 6.21. We call the matching \(\overline{M}\) the Babson-Hersh matching.

Lemma 6.22. The Babson-Hersh matching is acyclic.

Proof. Suppose to the contrary that we have a cycle

\[ [e_t(p) < \ldots < p] \xrightarrow{\text{down}} \partial_t[e_t(p) < \ldots < p] \xrightarrow{\text{up}} m_k(\partial_t[e_t(p) < \ldots < p]) \xrightarrow{\text{down}} \ldots \xrightarrow{\text{up}} [e_t(p) < \ldots < p]. \]

If at any step \(i = 0\) or \(i = k\), then this is impossible since all upward arrows do not change the end points (as the matching is internal). If \(0 < i < k\) for all steps, then this induces a cycle in \(M_{[p]}\) which is acyclic, a contradiction. \quad \square

Lemma 6.23. For \(i \geq 2\), there is an isomorphism of \(k\)-modules

\[ \text{Tor}_i(S_v, S_w) = \bigoplus_{t(p) = v, h(p) = w} \tilde{H}_{i-2}(K((e_v, p))). \]

where for \((\eta_k, \eta_{k+1}) \in M\), \(f(1_{\eta_k}) = d(1_{\eta_{k+1}})\) and \(f(1_{\eta_{k+1}}) = 1_{\eta_k + 1}\) and \(g = 1 + dm + md + (dm)^2 + (md)^2 + \ldots\) where \(m(\eta_k) = \eta_{k+1}\). The result now follows from the fact that \(C^M\) is acyclic by Proposition 6.17. \quad \square
where $\tilde{H}_i(K((e_v,p)))$ is the reduced homology of the order complex on the poset of paths between $e_v$ and $p$ with coefficients in $k$.

**Proof.** Consider the projective resolution $C_\bullet$ of $A$.

Then

$$\text{Tor}_i(S_v, S_w) = H_i(S_v \otimes_A C_\bullet \otimes_A S_w)$$

Now let us consider $S_v \otimes_A C_i \otimes_A S_w$. First notice that

$$S_v \otimes_A A e_v \otimes_k e_w A \otimes_A S_w = k$$

and is zero otherwise. Hence

$$S_v \otimes_A C_i \otimes_A S_w = \bigoplus_{\eta_i \in \text{Cell}_i(X_A), t(\eta_i) = v, h(\eta_i) = w} k$$

Now, notice that the “non-internal” differentials in the above complex vanish since tensoring with the simples kills all arrows. Hence

$$S_v \otimes_A C_i \otimes_A S_w = \bigoplus_{p | t(p) = v, h(p) = w} S^p_i$$

where $S^p_i$ is the summand consisting of all cells equivalent to cells of the form $[e_{t(p)} < \ldots < p]$.

By Lemma 6.20 the $i$-cells in $S^p_i$ are just the $i - 2$-cells of the order complex on $(e_{t(p)}, p)$ except for $S^p_1$ which is the cell $[e_t(p) < p]$. Furthermore, the differential agrees, by definition, with the simplicial differential. In summary, $S^p_i$ is nothing more than the reduced simplicial homology complex for the order complex on $(e_{t(p)}, p)$ shifted by 2.

$\square$

**Definition 6.24.** We say that a resolution $P_\bullet$ of an $A$-module is **minimal** if all components of the differential lie in the ideal generated by the arrows.

**Theorem 6.25.** Assume that for all paths $p$ all the homologies of $K((e_{t(p)}, p))$ are free $k$-modules and the critical cells for the Babson-Hersh matching form a basis of these homologies. Then, the Morse complex associated to the Babson-Hersh matching is a minimal projective resolution of $A$ as an $A \otimes A^{op}$-module (which is cellular). The converse holds when $A$ is a graded with respect to path length.

**Proof.** By Lemma 6.19 the Morse complex is a resolution. Now,

$$C_i = \bigoplus_{t(p) = v, h(p) = w} H_{i-2}(K((e_v,p))) \otimes_k P_v \otimes_k P_{w}^{op} \text{ by assumption}$$

$$= \bigoplus_{v \leq w} \text{Tor}_i(S_v, S_w) \otimes_k P_v \otimes_k P_{w}^{op} \text{ by Lemma 6.23}$$

Since for all $v, w$

$$\text{Tor}_i(S_v, S_w) = H_i(S_v \otimes_A C_\bullet \otimes_A S_w)$$

is a free $k$-module, this forces the differential on $S_v \otimes_A C_\bullet \otimes_A S_w$ to vanish for all $v, w$. Hence the differential on $C_\bullet$ is contained in the ideal generated by the arrows i.e. $C_\bullet$ is minimal.

---

2 Our convention is that the empty set is an empty simplicial complex so that reduced simplicial chains for the empty set are $k$ in degree -1 and $\tilde{H}_{-1}(\emptyset) = k$. 
Conversely, when $A$ is a graded ring, any minimal resolution $\mathfrak{P}_i$ is a summand of the Morse complex $\mathfrak{C}_i$ [14] (since the Morse complex gives a projective resolution of $A$ by Lemma [6.19]). On the other hand, the $i$th component of the minimal resolution is

$$\mathfrak{P}_i = \bigoplus_{v \leq w} Tor_i(S_v, S_w) \otimes_k P_v \otimes_k P_{w}^{op}$$

see e.g. [5]

$$\bigoplus_{t(p)=v, h(p)=w} \tilde{H}_{i-2}(K((e_v, p))) \otimes_k P_v \otimes_k P_{w}^{op}$$

by Lemma [6.23]

Hence, the inclusion of $\mathfrak{P}_i$ into $\mathfrak{C}_i$ is an isomorphism if and only if for all $p$ the critical $i$-cells of the Babson-Hersh matching form a basis of $\tilde{H}_{i-2}(K((e_v, p)))$.

**Remark 6.26.** The existence of a minimal resolution forces $Tor_i(S_v, S_w)$ to be torsion-free. Hence by Lemma [6.23] a minimal resolution cannot exist if there exists a path $p$ such that the reduced homology of the order complex of $(e_t(p), p)$ with coefficients in $k$ has torsion.

**Remark 6.27.** The assumption that $A$ is graded above was only used to verify that a minimal resolution exists and is a summand of any other resolution. The converse above holds whenever this is the case.

**Remark 6.28.** We are unsure as to whether or not the Babson-Hersh matching produces the minimal celluler resolution in general.

**Proposition 6.29.** Assume that for all $[p]$, the order complex on the poset $(e_t(p), p)$ is either shellable or equal to a finite set of points. Then, $A$ is Koszul and the Morse complex associated to the Babson-Hersch matching is the minimal projective resolution of $A$ as an $A \otimes_k A^{op}$-module.

**Proof.** Suppose $(e_{t(p)} < p)$ is a finite set of points. Match nothing. Then all cells $[e_{t(p)} < q < p]$ are all saturated by assumption.

Next, suppose $(e_{t(p)} < p)$ is shellable. Then by [4], Proposition 4.1 all critical cells besides the 0-cell correspond to facets (i.e. saturated chains).

Next, we claim that the minimal resolutions have linear differentials. Since any internal downward path drops an intermediate path, a downward path can not go to a saturated cell at the last step, hence there is no internal gradient paths i.e. all differentials lie in the radical (therefore, the Morse complex is the minimal resolution [14], Proposition 3, §3).

To see there are no higher order terms in the differentials, let $[p_0 = e_{t(p)} < p_1 \cdots < p_n = p]$ be a critical cell. A downward path will either drop $p_0$ or $p_n$, and we get a saturated cell in either case. But since the matching has to be internal, there is no upward path originating from this cell. Hence all gradient paths are linear. This means $A$ is Koszul.

Now we relate our minimal resolution to our results in the previous section.

**Lemma 6.30.** If $A$ is of Bondal-Ruan type, then all arrows in $A$ are linear monomials.

**Proof.** By Proposition [5.5] $\text{Hom}(\mathcal{O}(-D), \mathcal{O}(-E))$ correspond to entrance paths $\gamma : [0, 1] \rightarrow \text{Poly}_D$ with $\gamma(0) \in S_D$ and $\gamma(1) \in S_E$. Since $\text{Poly}_D$ is simply connected, we can homotopically perturb $\gamma$ to a path $\gamma'$ so that $\gamma'$ passes through the interior of a facet of $S_D$ when it leaves $S_D$ at $0 < t_0 \leq 1$. Then $\gamma = \gamma'|_{[0,1]} \ast \gamma'|_{[0,t_0]}$ and $\gamma'|_{[0,t_0]}$ corresponds to a linear monomial. By induction all arrows in $A$ correspond to linear monomials. 

\[ \square \]
Definition 6.31. Let $A$ be an HPA of Bondal-Ruan type. We say that $A$ is directable if there exists a relabeling of the variables $x_1 < \cdots < x_n$ such that any concatenation of arrows $x_i \cdots x_k$ is equal in $A$ to the concatenation of arrows $x_{\sigma(i_1)} \cdots x_{\sigma(i_k)}$ with $\sigma(i_1) \leq \cdots \leq \sigma(i_k)$ for some permutation $\sigma$.

Remark 6.32. The directability condition is equivalent to the quadratic condition that there exists a relabeling of the variables such that whenever $x_ix_j$ is a path with $j < i$, $x_ix_j$ is also a path.

Example 6.33 (Non-example). We notice that the Hirzebruch surface $\mathbb{F}_1$ is of Bondal-Ruan type.

However, this HPA is not directable, since either $x_2x_3$ cannot be changed to $x_3x_2$ (if $x_3 < x_2$), or $x_3x_2$ cannot be changed to $x_2x_3$ (if $x_2 < x_3$). As can be easily observed, this HPA is not quadratic due to the relation $x_1x_2x_3 = x_3x_2x_1$.

Corollary 6.34. If $A$ is a directable, then $A$ is Koszul.

Proof. Without loss of generality, one can assume the total ordering on the variables to be $x_1 < \cdots < x_n$. This induces an edge labeling on the intervals [1], Definition 2.1.

Then, since EL shellable $\Rightarrow$ CL shellable $\Rightarrow$ shellable [1], Proposition 2.3, each interval is shellable (unless it has length 2, in which case it is a finite set of points). By Proposition 6.29 $A_\Phi$ is Koszul.

We provide a few more toric examples that are not necessarily of Bondal-Ruan type. Nevertheless, the acyclic matching above agrees with the minimal resolution.

Example 6.35. Let $A$ be the toric FSEC HPA of the quiver of line bundles for the Hirzebruch surface $\mathbb{F}_3$.

Below is an example of maximal internal acyclic matching on $X_A$.

(1) $M_{1\text{bottom}}$: 1) $[1 < x_1^3]$ 2) $[1 < x_1^2x_3]$ 3) $[1 < x_1x_3^2]$ 4) $[1 < x_1x_4]$ 5) $[1 < x_3x_4]$ 6) $[1 < x_3x_4]$ 7) $[1 < x_4^2]$ 8) $[1 < x_3x_4]$ 9) $[1 < x_2x_3^2]$ 10) $[1 < x_1x_3^3]$ 11) $[1 < x_4]$ 12) $[x_1 < x_4]$ 13) $[x_1 < x_1^2x_3]$ 14) $[x_1 < x_1^2x_3^2]$ 15) $[x_1 < x_1x_3^3]$ 16) $[x_3 < x_3x_3]$ 17) $[x_3 < x_3x_3]$ 18) $[x_3 < x_3x_3]$ 19) $[x_3 < x_3]$.  

$M_{2\text{top}}$: 1) $[1 < x_1 < x_3]$ 2) $[1 < x_1 < x_1^2x_3]$ 3) $[1 < x_1 < x_1x_3]$ 4) $[1 < x_1 < x_1x_3]$ 5) $[1 < x_1 < x_1x_4]$ 6) $[1 < x_1 < x_3x_4]$ 7) $[1 < x_1 < x_1^2]$ 8) $[1 < x_1 < x_1^2x_3]$ 9) $[1 < x_1 < x_1^2x_3^2]$ 10) $[1 < x_1 < x_1x_3]$ 11) $[1 < x_3 < x_4]$ 12) $[x_1 < x_1^3 < x_4]$ 13) $[x_1 < x_1^3x_3]$ 14) $[x_1 < x_1^3x_3 < x_1^2x_3]$ 15) $[x_1 < x_1x_3^2 < x_1x_3^3]$ 16) $[x_3 < x_3x_3]$ 17) $[x_3 < x_3x_3]$ 18) $[x_3 < x_3x_3]$ 19) $[x_3 < x_3x_3]$.
(2) $M_{\text{bottom}}$: a) $[1 < x_3 < x_1^3x_3]$ b) $[1 < x_3 < x_1^2x_3^2]$ c) $[1 < x_3 < x_1x_3^2]$ d) $[1 < x_1^3 < x_1^2]$ e) $[1 < x_1^3 < x_1^2x_3^2]$ f) $[1 < x_1^2x_3 < x_1^3x_3]$ g) $[1 < x_1^2x_3 < x_1^2x_3^2]$ h) $[1 < x_1x_3 < x_1^2x_3^2]$ i) $[1 < x_1x_3^2 < x_1^2x_3^2]$ j) $[1 < x_1x_3 < x_1^2x_3^2]$ k) $[1 < x_1x_3 < x_1^2x_3^2]$ l) $[1 < x_1x_3 < x_1^2x_3^2]$ m) $[1 < x_1x_3 < x_1^2x_3^2]$ n) $[1 < x_1x_3 < x_1^2x_3^2]$ o) $[1 < x_1x_3 < x_1^2x_3^2]$ p) $[1 < x_1x_3 < x_1^2x_3^2]$ q) $[1 < x_1x_3 < x_1^2x_3^2]

The following critical cells of $X_A$ remains after the matching:

(1) All 4 0-cells and all 9 consecutive 1-cells.
(2) Six 2-cells: $[1 < x_3 < x_1x_3^2], [1 < x_3 < x_1x_3^2], [1 < x_4 < x_1x_3^2], [1 < x_4 < x_3x_4], [x_1 < x_1x_3 < x_1x_3^2] \sim [x_3 < x_1x_2 < x_2^2x_3], [x_1 < x_1x_3 < x_1^2x_3^2] \sim [x_3 < x_3^3 < x_1x_3^3].$
(3) One 3-cell: $[1 < x_3 < x_1x_3 < x_1x_3^2].$

We get the 3 dimensional minimal cellular resolution $X_A^{\text{min}}$.

This is a homotopy torus, where the lower half of the fundamental domain was thickened to a tetrahedron.

This example shows that the minimal bimodule resolution of a toric FSEC HPA, if cellular, can only possibly be a homotopy torus. This is expected from $\mathbb{F}_3$, since the Hochschild dimension($=3$) of $\mathbb{F}_3$ is greater than its dimension($=2$). It is possible to obtain a $\mathbb{T}^2$ by matching more. However, such a matching will not be an internal matching, and will no longer provide a bimodule resolution.

**Example 6.36.** We compare $\mathbb{F}_3$ to the weighted projective stack $\mathbb{P}(1 : 1 : 3)$, with the usual FSEC of line bundles:
The minimal resolution turns out to be homeomorphic to $\mathbb{T}^2$:

![Diagram](image)

Note that if one thinks of $\mathbb{P}(1 : 1 : 3)$ as obtained from quotient by $\mathbb{C}^*$, the HPA is of Bondal-Ruan type. But if instead it is considered as a VGIT of $\mathbb{F}_3$, thus a quotient by $(\mathbb{C}^*)^2$, the HPA is not of Bondal-Ruan type. There are 6 lattice points in $\text{Im}(\Phi)$, but the $\mathbb{P}(1 : 1 : 3)$ quiver only has 5 vertices and $\mathbb{F}_3$ quiver has 4.

In fact, the minimal resolutions for $\mathbb{P}(1 : 1 : n)$ are all homeomorphic to $\mathbb{T}^2$.

**Proposition 6.37.** Suppose $A$ is a HPA corresponding to an FSEC of line bundles on a smooth variety $X$. Then $X_A$ has Euler characteristic zero.

**Proof.** Consider the cellular resolution $C^\bullet$ of $A$. The equivalence $D(A \otimes_k A^{\text{op}}) \cong D(X \times X)$ sends $C_\bullet$ to a locally-free resolution of the diagonal $C_\bullet \cong \Delta_X$. Notice that

$$\text{rk}(C_i) = \text{rk}(C_i) = \text{# of i-cells in } X_A$$

by definition of $C_\bullet$.

Hence $\chi(\Delta_X) = \chi(X_A)$. On the other hand, by the HKR theorem $\chi(\Delta_X) = 0$. □

We end this paper with the following conjectures.

**Conjecture 6.38.** For a toric FSEC HPA $A_X$ of a proper DM toric stack $X$, $X_A$ is homotopic to a real torus $\mathbb{T}$.

**Conjecture 6.39.** Suppose there exists a full strong exceptional collection on a proper DM stack $X$ of dimension $n$ whose endomorphism algebra $A$ is an HPA. Then there exists a subcomplex $Y \subseteq X_A$ which is homotopic to $\mathbb{T}^n$. Furthermore, $X$ is rational with local systems on $Y$ corresponding to points on $X$. Finally, $X_A$ is homotopic to a torus if and only if $X$ is toric.

We end with a non-toric example which illustrates the conjecture above.

**Example 6.40** (Homological Berglund-Hübsch-Krawitz Mirror Symmetry). Let $Z$ be a Calabi-Yau hypersurface $Z$ in a weighted projective space $\mathbb{P}(a_0 : \ldots : a_n)$ defined by an invertible polynomial $w$ and $G_w$ (resp. $\Gamma_w$) be its (resp. extended) maximal symmetry group. Assume that the mirror is Gorenstein (i.e. the dual polynomial $w^T$ has weights $r_i$ such that $r_i$ divides the degree $d^T$) and recall (see Example 3.14) that $kA_i$ is the path algebra of the Dynkin quiver. Now consider the following HPA

$$A_w := kA_{d^T_{r_0}-1} \otimes_k \ldots \otimes_k kA_{d^T_{r_n}-1}.$$
Then we have the following equivalences

\[
D([Z/G_{\max}]) \cong D(A^{n+1}, \Gamma_{\max}, W) \quad \text{by } [3], \text{ Theorem 4.2.1, since } Z \text{ is Calabi-Yau}
\]
\[
\cong D(A_w) \quad \text{by [15], Theorem 1.1}
\]
\[
\cong D(Sh_{Str}(X_A)) \quad \text{by Theorem 4.17}
\]
\[
\cong D(Sh_S(X_A^{min})) \quad \text{by Corollary 4.30 and Example 3.14}
\]
\[
\cong D(Sh_{sm}(X_A^{min, sm})) \quad \text{since the smoothing is a homeomorphism}
\]
\[
\cong DW(T^* X_A^{min, sm}, \Lambda_{sm}) \quad \text{by [20], Theorem 1.1.}
\]

In the last two lines, \(X_A^{min, sm} \cong (D, \partial D)\) is a disk with smooth boundary obtained by smoothing the corners, so that \(T^* X_A^{min, sm}\) becomes a Liouville sector by [21], §2.5, and \(DW\) refers to the homotopy category of twisted modules over the (partially) wrapped Fukaya category.

**References**

[1] A. Björner and M. Wachs, On lexicographically shellable posets. Trans. Amer. Math. Soc., 277 (1983), 323-341.

[2] M. Ballard, A. Duncan and P. McFaddin, Generation and the toric Frobenius, in preparation.

[3] M. Ballard, D. Favero and L. Katzarkov, Variation of geometric invariant theory quotients and derived categories. J. Reine Angew. Math., 746 (2019), 235-303.

[4] E. Babson and P. Hersh, Discrete Morse functions from lexicographic orders. Trans. Amer. Math. Soc., 357 (2005), 509-534.

[5] M.C.R. Butler and A.D. King, Minimal Morse functions on algebraic varieties. J. Algebra, 212 (1999), 323-362.

[6] A. Bondal, Derived categories of toric varieties. Oberwolfach Rep. 3 (2006), 284-286.

[7] D. Bayer and B. Sturmfels, Cellular resolutions of monomial modules. J. Reine Angew. Math., 502 (1998), 123-140.

[8] A. Björner and M.L. Wachs, Shellable nonpure complexes and posets. II. Trans. Amer. Math. Soc., 357 (1997), 3945-3975.

[9] J.M. Curry, Sheaves, cosheaves and applications. University of Pennsylvania PhD thesis (2014).

[10] R.L. Cohen, J.D. Jones and G.B. Segal, Morse theory and classifying spaces. (1995)

[11] J. Curry and A. Patel, Classification of constructible cosheaves. Theory Appl. Categ. 35 (2020), 1012–1047.

[12] A. Craw, and A. Quintero Vélez, Cellular resolutions of noncommutative toric algebras from superpotentials. Adv. Math., 229 (2012), 1516-1554.

[13] E. Delucchi, Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex. IMRN. 9 (2008), rnm167.

[14] S. Eilenberg, Homological dimension and syzygies. Ann. Math. (1956), 328-336.

[15] D. Favero, D. Kaplan, and T. Kelly, Exceptional collections for mirrors of invertible polynomials. (2020) arXiv preprint arXiv:2001.06500.

[16] B. Fang, C.C.M. Liu, D. Treumann, and E. Zaslow, A categorification of Morelli’s theorem. Inventiones mathematicae. 186 (2011), 79-114.

[17] B. Fang, C.C.M. Liu, D. Treumann, and E. Zaslow, T-duality and homological mirror symmetry for toric varieties. Adv. Math. 229 (2012), 1873-1911.

[18] B. Fang, C.C.M. Liu, D. Treumann, and E. Zaslow, The coherent–constructible correspondence for toric Deligne–Mumford stacks. International Mathematics Research Notices. 4 (2014), 914-954.

[19] R. Forman, Morse theory for cell complexes. Adv Math. 134 (1998), pp.90-145.

[20] S. Ganatra, J. Pardon, and V. Shende, Microlocal Morse theory of wrapped Fukaya categories. (2018) arXiv preprint arXiv:1809.08807.

[21] S. Ganatra, J. Pardon, and V. Shende, Covariant functorial wrapped Floer theory on Liouville sectors. Publ. Math. Inst. Hautes Etudes Sci. 131 (2020), 73-200.

[22] A. Hatcher, Algebraic topology. Cambridge University Press. (2002)

[23] B. Iversen, Cohomology of Sheaves. Springer, Berlin, Heidelberg. (1986)
[24] M. Jöllenbeck and V. Welker, Minimal resolutions via algebraic discrete Morse theory. American Mathematical Soc. (2009)

[25] T. Kuwagaki, The nonequivariant coherent-constructible correspondence for toric stacks. Duke Math. J. 169 (2020), 2125-2197.

[26] J. Lurie, Derived Algebraic Geometry VI: $E[k]$-Algebras. (2009), Available at http://people.math.harvard.edu/~lurie/papers/DAG-VI.pdf.

[27] J. Milnor, The geometric realization of a semi-simplicial complex. Ann. Math. (1957) 357-362.

[28] V. Nanda, Discrete Morse theory and localization. J Pure Appl Algebra. 223 (2019), 459-488.

[29] R. Ohkawa and H. Uehara, Frobenius morphisms and derived categories on two dimensional toric Deligne–Mumford stacks. Adv Math. 244 (2013), 241-267.

[30] D. Quillen, Higher algebraic K-theory: I. Higher K-theories. Springer, Berlin, Heidelberg. (1973), 85-147

[31] D. Treumann, Exit paths and constructible stacks. Compos. Math. 145 (2009), 1504-1532.

[32] H. Uehara, Exceptional collections on toric Fano threefolds and birational geometry. Int. J. Math. 25 (2014), 1450072.

[33] K. Watanabe and M. Watanabe. The classification of Fano 3-folds with torus embeddings. Tokyo J. Math. 5 (1982), 37-48.

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