Application of fuzzy finite difference scheme for the non-homogeneous fuzzy heat equation

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Abstract

A numerical framework based on fuzzy finite difference is presented for approximating fuzzy triangular solutions of fuzzy partial differential equations by considering the type of $gH$-differentiability. The fuzzy triangle functions are expanded using full fuzzy Taylor expansion to develop a new fuzzy finite difference method. By considering the type of $gH$-differentiability, we approximate the fuzzy derivatives with a new fuzzy finite difference. In particular, we propose using this method to solve non-homogeneous fuzzy heat equation with triangular initial-boundary conditions. We examine the truncation error and the convergence conditions of the proposed method. Several numerical examples are presented to demonstrate the performance of the methods. The final results demonstrate the efficiency and the ability of the new fuzzy finite difference method to produce triangular fuzzy numerical results which are more consistent with existing reality.

Keywords Forward fuzzy difference scheme · Backward fuzzy difference scheme · Centered fuzzy finite difference scheme · Generalized Hukuhara difference · Non-homogeneous initial-boundary fuzzy heat equation

1 Introduction

In recent decades, fuzzy set theory has been proven to be a useful tool for modeling systems with uncertainties, giving the models a more realistic look at reality and enabling them to express themselves with a more comprehensive outlook.

The fuzzy derivative concept first appeared in Chang and Zadeh’s (1972). Hukuhara’s paper Hukuhara (1967) is the starting point for the set-valued and fuzzy differential equations. Puri and Ralescu (1986) suggested the fuzzy differential equations modeling with uncertainty under the concept of H-differentiability. Further studies developed fuzzy differential equations based on the Hukuhara derivative, such as those presented by Kaleva in Kaleva (1987). There are some fuzzy differential equations in this framework, however, for which the diameter of the solution increases as the time $t$ increases (Diamond 2002).

To overcome this shortcoming, Bede and Gal introduced the weakly generalized differentiability and the strongly generalized differentiability for the fuzzy functions (Bede and Gal 2005). Moreover, they presented a more general definition of derivatives for the fuzzy functions and their applications for solving fuzzy differential equations (Bede and Gal 2005, 2006). Stefanini and Bede introduced generalized Hukuhara differentiability ($gH$-differentiability) Stefanini and Bede (2009) for interval-valued functions by using the concept of generalization of the Hukuhara difference of compact convex set. They showed that this concept of differentiability has relationships with weakly generalized differentiability and strongly generalized differentiability. The disadvantage of the strongly generalized differentiability of a function compared to H-differentiability is that in this case, the fuzzy differential equation has no unique solution (Bede and Gal 2005). Also, in Chalco-Cano et al. (2011), the authors studied relationships between the strongly generalized differentiability and the $gH$-differentiability, showing the equivalence between these two concepts when the set of switching points of the interval-valued function is finite. Recently, Chalco-Cano et al. (2020) provided a new characterization of the switching points for $gH$-differentiability and shown that the set of all switching points is at most countable.
Partial differential equations explain the majority of phenomena in the fields of mathematics, physics, and engineering. However, mathematical modeling of these phenomena requires a wide variety of data and information. Unfortunately, the measurement of these variables is inherently uncertain. Therefore, the fuzzy partial differential equation is a useful tool for modeling systems with uncertainties (Buckley and Feuring 1999; Allahviranloo 2002; Allahviranloo and Taheri 2009; Bertone et al. 2013; Moghaddam and Allahviranloo 2018; Allahviranloo et al. 2015b; Gouyandeh et al. 2017).

For many fuzzy partial differential equations, analytical solutions are challenging to obtain. Consequently, it is crucial to create some reliable and efficient methods for solving fuzzy partial differential equations. Numerous researchers are presently focusing on the numerical solution of fuzzy partial differential equations, such as difference method (Allahviranloo 2002; Allahviranloo and Afshar 2010), Adomian method (Allahviranloo and Taheri 2009; Pirzada and Vakaskar 2015), finite volume method (Mahmoud and Iman 2011).

In recent years, there has been an increase in interest in the use of the finite difference method to solve fuzzy partial differential equations. According to our knowledge, all papers that have used this method have rewritten the fuzzy partial differential equation as two crisp partial differential equations and solved them using the usual finite difference method. In comparison, this paper is devoted to developing a new fuzzy finite difference method through fuzzy arithmetic and fuzzy Taylor expansion. We approximate the fuzzy derivatives with a fuzzy finite difference by considering the type of $gH$-differentiability. The fuzzy numerical solution of the fuzzy partial differential equations can be obtained without implicitly embedding them into crisp equations through our method. Even though this paper deals with the fuzzy heat equation, our method can be used to find the numerical solution for a wide variety of fuzzy partial differential equations.

Now, let us take a quick look at the contents. Section 2 presents some concepts related to fuzzy numbers and generalized Hukuhara differentiability, as well as some theorems and lemmas used in the central part of the paper. The fuzzy finite difference method for one variable fuzzy functions is discussed in Section 3, and we obtain different formulas for forward, backward, and central difference depending on the type of $gH$-differentiability. Taking into account the type of $[gH−p]$-differentiability, we show corresponding formulas for the fuzzy finite difference method of the non-homogeneous heat equation in Section 4. Further, we describe and analyze in detail the convergence condition of the method, as well as truncation error. A full description is given for one of the three examples in Section 5. The last section of the paper discusses conclusions, applications, and future possibilities.

### 2 Preliminaries

The purpose of this section is to introduce the general terms and definitions used to describe fuzzy operations and the necessary notations.

The triangular fuzzy number $a \in \mathbb{R}_T$ is defined as an ordered triple $a = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$. Some properties of the triangular fuzzy number are discussed in Kaufmann and Gupta (1985), but we will describe some of the properties of this class of numbers here which are used in this paper.

**Definition 2.1** (See Bede 2013; Alikhani and Bahrami 2019) The generalized Hukuhara difference of two fuzzy numbers $a, b \in \mathbb{R}_F$ is the fuzzy number $c$, (if it exists), such that

$$a \circledminus gH b = c \iff \begin{cases} (i). \ a = b \oplus c; \\ or (ii). \ b = a \oplus (-1)c. \end{cases}$$

Now consider $a, b \in \mathbb{R}_T$, then

$$a \circledminus gH b = c \iff \begin{cases} (i). \ c = (a_1 - b_1, a_2 - b_2, a_3 - b_3); \\ or (ii). \ c = (a_3 - b_3, a_2 - b_2, a_1 - b_1). \end{cases}$$

provided that $c$ is a triangular fuzzy number.

**Remark 2.2** In the rest of this paper, all fuzzy numbers and fuzzy functions will be considered triangular. Additionally, all the lemmas and theorems will be proved on the assumption that the generalized Hukuhara difference exists.

**Proposition 2.3** Consider $a, b$ and $c$ are triangular fuzzy numbers and Hukuhara difference exists, then

1. $a \oplus (-1)b = (a_1 + b_3, a_2 + b_2, a_3 + b_1)$ provided that $a \oplus (-1)b$ is a triangular fuzzy number.
2. If $a = c \oplus (-1)b$, then $b = -1(c \oplus a)$.

**Proof**

Case 1. We have

$$a \oplus (-1)b = (a_1, a_2, a_3) \oplus (-b_3, -b_2, -b_1) = (a_1 + b_3, a_2 + b_2, a_3 + b_1).$$

Case 2. According to assumption $a = c \oplus (-1)b$ and Hukuhara difference exists, so

$$a = c \oplus (-1)b$$

$$(a_1, a_2, a_3) = (c_1, c_2, c_3) \oplus (-1)(b_1, b_2, b_3) = (c_1 + b_3, c_2 + b_2, c_3 + b_1).$$
So
\[ a_1 = c_1 + b_3, \quad a_1 - c_1 = b_3, \]
\[ a_2 = c_2 + b_2, \quad a_2 - c_2 = b_2, \]
\[ a_3 = c_3 + b_1, \quad a_3 - c_3 = b_1. \]

Then,
\[ b = (b_1, b_2, b_3) = (a_3 - c_3, a_2 - c_2, a_1 - c_1) \]
\[ = (-1)(c_1 - a_1, c_2 - a_2, c_3 - a_3) \]
\[ = (-1)(c \cap a). \]

Thus, the proof is complete. \(\square\)

**Proposition 2.4** Let \(\lambda_1\) and \(\lambda_2\) are two real constants such that \(\lambda_1, \lambda_2 \geq 0\) (or \(\lambda_1, \lambda_2 \leq 0\)). If \(y(t)\) is a triangular fuzzy function, then

\[
\lambda_1 y(t) \oplus y_2(t) \cap y_3(t) = (\lambda_1 - \lambda_2) y(t),
\]

**Proof** First consider \(\lambda_1\) and \(\lambda_2\) are positive constants, then

\[
\lambda_1 y(t) = \left(\lambda_1 y_1(t), \lambda_1 y_2(t), \lambda_1 y_3(t)\right),
\]
\[
\lambda_2 y(t) = \left(\lambda_2 y_1(t), \lambda_2 y_2(t), \lambda_2 y_3(t)\right).
\]

Now, we have two cases

i. If \(\lambda_1 \geq \lambda_2\), we have

\[
\lambda_1 y(t) \oplus y_2(t) \cap y_3(t) = \left(\lambda_1 - \lambda_2\right) y(t).
\]

ii. If \(\lambda_1 \leq \lambda_2\), therefore

\[
\lambda_1 y(t) \oplus y_2(t) \cap y_3(t) = \left(\lambda_1 - \lambda_2\right) y(t).
\]

Hence, we have Eq.(1). The other case \((\lambda_1\) and \(\lambda_2\) are negative constants) can be proved in a similar way and we omit the details. \(\square\)

**Definition 2.5** (See Bede 2013) Let \(y : (a, b) \rightarrow \mathbb{R}_T\) is a fuzzy-valued function such that \(y(t) = \left(y_1(t), y_2(t), y_3(t)\right)\), where \(y_1(t), y_2(t)\) and \(y_3(t)\) are real-valued differentiable functions on \((a, b)\). Then, \(y\) is a \([(t) - gH]\)-differentiable function at \(t_0 \in (a, b)\) if and only if

\[
y'_{gH}(t_0) = \left(y'_1(t), y'_2(t), y'_3(t)\right),
\]

defines a triangular fuzzy number. Similarly, \(y\) is a \([(ii) - gH]\)-differentiable function at \(t_0\) if and only if

\[
y'_{gH}(t_0) = \left(y'_1(t), y'_2(t), y'_1(t)\right),
\]

is a triangular fuzzy number. In general, if \(y(t)\) is a \([(i) - gH]\) or \([(ii) - gH]\)-differentiable for all \(t_0 \in (a, b)\), then \(y\) is generalized Hukuhara differentiable function on \((a, b)\).

**Remark 2.6** We assume that the notations \(C^k_{gH}(\mathbb{R}_T)\) stand for all triangular fuzzy function \(f\) and it’s first \(k\), gH-derivatives which are defined on \([a, b]\) and fuzzy continuous (Allahviranloo et al. 2015a). Throughout the rest of this paper, \(y(t) \in C^k_{gH}(\mathbb{R}_T)\) for \(j = 1, \ldots, n - 1\) and \(t \in [a, b]\) with no switching point on \([a, b]\). Moreover, for simplicity

- When \(y^{(j)}_{gH}(t) = \left(y^{(j)}_1(t), y^{(j)}_2(t), y^{(j)}_3(t)\right)\), we will use the notation denote \(y^{(j)}_{gH}(t)\) to show \(y^{(j)}_{gH}(t)\).
- When \(y^{(j)}_{gH}(t) = \left(y^{(j)}_1(t), y^{(j)}_2(t), y^{(j)}_1(t)\right)\), we will use the notation denote \(y^{(j)}_{gH}(t)\) to show \(y^{(j)}_{gH}(t)\).

(Notice the position of functions \(y^{(j)}_1(t)\) and \(y^{(j)}_3(t)\) in these triangular fuzzy functions.) In particular, we have the following cases to show the all kind of gH-differentiability for \(y^{(j)}_{gH}(t)\) of order \(j\), when \(j = 0, 1, 2\).

**Case 1.** If \(y(t), y'_g(t)\) and \(y''_{gH}(t)\) are \([i - gH]\)-differentiable, we have

\[
y'_{gH}(t) = \left(y'_1(t), y'_2(t), y'_3(t)\right),
y''_{gH}(t) = \left(y''_1(t), y''_2(t), y''_3(t)\right),\]

Case 2. If \(y(t)\) and \(y'_g(t)\) are \([i - gH]\)-differentiable and \(y''_{gH}(t)\) is \([i - gH]\)-differentiable,

\[
y'_{gH}(t) = \left(y'_1(t), y'_2(t), y'_3(t)\right),
y''_{gH}(t) = \left(y''_1(t), y''_2(t), y''_3(t)\right),\]

\[
y'''_{gH}(t) = \left(y'''_1(t), y'''_2(t), y'''_1(t)\right).\]
Case 3. If \( y(t) \) and \( y''_{gH}(t) \) are \([i - gH]\)-differentiable and
\( y'_{gH}(t) \) is \([ii - gH]\)-differentiable,
\[
y'_{i,gH}(t) = \left( y'_1(t), y'_2(t), y'_3(t) \right), \\
y''_{ii,gH}(t) = \left( y''_2(t), y''_1(t), y''_3(t) \right), \\
y'''_{ii,gH}(t) = \left( y'''_1(t), y'''_2(t), y'''_3(t) \right)
\]

Case 4. If \( y(t) \) is \([i - gH]\)-differentiable and \( y'_{gH}(t) \) and
\( y''_{gH}(t) \) are \([i - gH]\)-differentiable,
\[
y'_{i,gH}(t) = \left( y'_1(t), y'_2(t), y'_3(t) \right), \\
y''_{ii,gH}(t) = \left( y''_2(t), y''_1(t), y''_3(t) \right), \\
y'''_{ii,gH}(t) = \left( y'''_1(t), y'''_2(t), y'''_3(t) \right)
\]

Case 5. If \( y(t) \) is \([ii - gH]\)-differentiable and \( y'_{gH}(t) \) and
\( y''_{gH}(t) \) are \([i - gH]\)-differentiable,
\[
y'_{i,gH}(t) = \left( y'_1(t), y'_2(t), y'_3(t) \right), \\
y''_{ii,gH}(t) = \left( y''_2(t), y''_1(t), y''_3(t) \right), \\
y'''_{ii,gH}(t) = \left( y'''_1(t), y'''_2(t), y'''_3(t) \right)
\]

Case 6. If \( y(t) \) and \( y'_{gH}(t) \) are \([ii - gH]\)-differentiable and
\( y'_{gH}(t) \) is \([i - gH]\)-differentiable,
\[
y'_{i,gH}(t) = \left( y'_1(t), y'_2(t), y'_3(t) \right), \\
y''_{ii,gH}(t) = \left( y''_2(t), y''_1(t), y''_3(t) \right), \\
y'''_{ii,gH}(t) = \left( y'''_1(t), y'''_2(t), y'''_3(t) \right)
\]

Case 7. If \( y(t) \) and \( y'_{gH}(t) \) are \([ii - gH]\)-differentiable and
\( y''_{gH}(t) \) is \([i - gH]\)-differentiable,
\[
y'_{i,gH}(t) = \left( y'_1(t), y'_2(t), y'_3(t) \right), \\
y''_{ii,gH}(t) = \left( y''_2(t), y''_1(t), y''_3(t) \right), \\
y'''_{ii,gH}(t) = \left( y'''_1(t), y'''_2(t), y'''_3(t) \right)
\]

Case 8. If \( y(t) \), \( y'_{gH}(t) \) and \( y''_{gH}(t) \) are \([ii - gH]\)-differentiable,
\[
y'_{i,gH}(t) = \left( y'_1(t), y'_2(t), y'_3(t) \right), \\
y''_{ii,gH}(t) = \left( y''_2(t), y''_1(t), y''_3(t) \right), \\
y'''_{ii,gH}(t) = \left( y'''_1(t), y'''_2(t), y'''_3(t) \right)
\]

**Definition 2.7** (See Bede 2013) Let \( y : (a, b) \rightarrow \mathbb{R}_T \) is a triangular fuzzy-valued function and \( n \in (a, b) \) then
\[
\int_a^b y(t)dt = \left( \int_a^b y_1(t)dt, \int_a^b y_2(t)dt, \int_a^b y_3(t)dt \right)
\]

**Theorem 2.8** Let \( y : [a, b] \rightarrow \mathbb{R}_T \) be a triangular fuzzy function such that \( y \in C^n_{gH}([a, b], \mathbb{R}_T) \) with no switching points in \([a, b]\). Then, for \( j = 1, 2, ..., n \), there are the following different scenarios

i.
\[
y^{(j-1)}_{i,gH}(t + \Delta t) = y^{(j-1)}_{i,gH}(t) + \Delta t y^{(j)}_{i,gH}(t)dx
\]

ii.
\[
y^{(j-1)}_{ii,gH}(t - \Delta t) = y^{(j-1)}_{ii,gH}(t) + \Delta t y^{(j)}_{ii,gH}(t)dx
\]

iii.
\[
y^{(j-1)}_{ii,gH}(t + \Delta t) = y^{(j-1)}_{ii,gH}(t) + \Delta t y^{(j)}_{ii,gH}(t)dx
\]

iv.
\[
y^{(j-1)}_{ii,gH}(t - \Delta t) = y^{(j-1)}_{ii,gH}(t) + \Delta t y^{(j)}_{ii,gH}(t)dx
\]

**Proof** We have \( y \in C^n_{gH}([a, b], \mathbb{R}_F) \), therefore, \( f^{(j)}(t) \), \( j = 0, 1, ..., n \) are integrable. We will prove parts (i) and (iii); the other parts are similar, and we omit the details. By using Remark 2.6 and Definition 2.7, we get
\[
y^{(j-1)}_{i,gH}(t) + \int_t^{t+\Delta t} y^{(j)}_{i,gH}(x)dx
\]

\[
y^{(j-1)}_{ii,gH}(t) + \int_t^{t+\Delta t} y^{(j)}_{ii,gH}(x)dx
\]

\[
y^{(j-1)}_{ii,gH}(t) + \int_t^{t+\Delta t} y^{(j)}_{ii,gH}(x)dx
\]

\[
y^{(j-1)}_{ii,gH}(t) + \int_t^{t+\Delta t} y^{(j)}_{ii,gH}(x)dx
\]

\[
y^{(j-1)}_{ii,gH}(t) + \int_t^{t+\Delta t} y^{(j)}_{ii,gH}(x)dx
\]
\[
\begin{align*}
\oplus & \left( y_1^{(j-1)}(t + \Delta t) - y_1^{(j-1)}(t), y_2^{(j-1)}(t + \Delta t) \right) \\
& - y_2^{(j-1)}(t), y_3^{(j-1)}(t + \Delta t) - y_3^{(j-1)}(t) \\
= & \left( y_1^{(j-1)}(t + \Delta t), y_2^{(j-1)}(t + \Delta t), y_3^{(j-1)}(t + \Delta t) \right). \\
\end{align*}
\]

And
\[
\begin{align*}
y_i^{(j-1)}(t) \oplus \int_{t - \Delta t}^t y_i^{(j)}(x)dx \\
= & \left( y_1^{(j-1)}(t), y_2^{(j-1)}(t), y_3^{(j-1)}(t) \right) \oplus \left( \int_{t - \Delta t}^t y_1^{(j)}(x)dx, \int_{t - \Delta t}^t y_2^{(j)}(x)dx, \int_{t - \Delta t}^t y_3^{(j)}(x)dx \right) \\
= & \left( y_1^{(j-1)}(t), y_2^{(j-1)}(t), y_3^{(j-1)}(t) \right) \\
\oplus & \left( y_1^{(j-1)}(t) - y_1^{(j-1)}(t - \Delta t), y_2^{(j-1)}(t) - y_2^{(j-1)}(t - \Delta t), y_3^{(j-1)}(t) - y_3^{(j-1)}(t - \Delta t) \right) \\
= & \left( y_1^{(j-1)}(t - \Delta t), y_2^{(j-1)}(t - \Delta t), y_3^{(j-1)}(t - \Delta t) \right). \\
\end{align*}
\]

Now, we want to prove case (iii). We have
\[
\begin{align*}
y_i^{(j-1)}(t) \oplus (-1) \int_t^{t + \Delta t} y_i^{(j)}(x)dx \\
= & \left( y_1^{(j-1)}(t), y_2^{(j-1)}(t), y_3^{(j-1)}(t) \right) \\
\oplus & (-1) \left( \int_t^{t + \Delta t} y_1^{(j)}(x)dx, \int_t^{t + \Delta t} y_2^{(j)}(x)dx, \int_t^{t + \Delta t} y_3^{(j)}(x)dx \right) \\
= & \left( y_1^{(j-1)}(t), y_2^{(j-1)}(t), y_3^{(j-1)}(t) \right) \\
\oplus & \left( y_1^{(j-1)}(t) - y_1^{(j-1)}(t + \Delta t), y_2^{(j-1)}(t) - y_2^{(j-1)}(t + \Delta t), y_3^{(j-1)}(t) - y_3^{(j-1)}(t + \Delta t) \right) \\
= & \left( y_1^{(j-1)}(t + \Delta t), y_2^{(j-1)}(t + \Delta t), y_3^{(j-1)}(t + \Delta t) \right). \\
\end{align*}
\]

\[
\begin{align*}
\int_{t - \Delta t}^t y_i^{(j)}(x)dx \\
= & \left( y_1^{(j-1)}(t), y_2^{(j-1)}(t), y_3^{(j-1)}(t) \right) \\
\oplus & (-1) \left( y_3^{(j-1)}(t - \Delta t) - y_3^{(j-1)}(t), y_2^{(j-1)}(t - \Delta t) - y_2^{(j-1)}(t), y_1^{(j-1)}(t - \Delta t) - y_1^{(j-1)}(t) \right) \\
= & \left( y_1^{(j-1)}(t - \Delta t), y_2^{(j-1)}(t - \Delta t), y_3^{(j-1)}(t - \Delta t) \right), \\
\end{align*}
\]

which proves this case.

Next, we are going to prove a crucial theorem to all the different cases in Remark 2.6, which will be used in the following sections. Actually, we will obtain four terms of the fuzzy Taylor’s expansion about the point \( t_k \) for \( t_k \leq t \) and \( t \leq t_k \) by considering different type of \( gH \)-differentiability for \( y(t), y_i^{(j)}(t) \) and \( y_i^{(j)}(t) \).

**Theorem 2.9** Let \( \mathbb{T} = [a, b] \subset \mathbb{R} \), \( y \in C_{gH}^4([a, b], \mathbb{R}_\mathbb{T}) \). For \( t, t \pm \Delta t \in \mathbb{T} \), we have

**Case 1.** If \( y(t), y_i^{(j)}(t) \) and \( y_i^{(j)}(t) \) are \([i - gH]-differentiable: \)
\[
y(t + \Delta t) = y(t) \oplus y_i^{(j)}(t) \oplus \Delta t \oplus y_i^{(j)}(t) \\
\oplus & \frac{\Delta t^2}{2!} \oplus y_i^{(j)}(t) \oplus \frac{\Delta t^3}{3!} \oplus R(t + \Delta t), \\
y(t - \Delta t) = y(t) \oplus y_i^{(j)}(t) \oplus \Delta t \oplus y_i^{(j)}(t) \\
\oplus & \frac{\Delta t^2}{2!} \oplus (-1)y_i^{(j)}(t) \oplus \frac{\Delta t^3}{3!} \oplus R(t - \Delta t)
\]

**Case 2.** If \( y(t) \) and \( y_i^{(j)}(t) \) are \([i - gH]-differentiable and \( y_i^{(j)}(t) \) is \([ii - gH]-differentiable: \)
\[
y(t + \Delta t) = y(t) \oplus y_i^{(j)}(t) \oplus \Delta t \oplus y_i^{(j)}(t) \\
\oplus & \frac{\Delta t^2}{2!} \oplus (-1)y_i^{(j)}(t) \oplus \frac{\Delta t^3}{3!} \oplus R(t + \Delta t), \\
y(t - \Delta t) = y(t) \oplus y_i^{(j)}(t) \oplus \Delta t \oplus y_i^{(j)}(t) \\
\oplus & \frac{\Delta t^2}{2!} \oplus (-1)y_i^{(j)}(t) \oplus \frac{\Delta t^3}{3!} \oplus R(t - \Delta t)
\]
Case 3. If $y(t)$ and $y''_{gH}(t)$ are $[i - gH]$-differentiable and $y'_{gH}(t)$ is $[i - gH]$-differentiable:

\[
y(t + \Delta t) = y(t) \oplus y'_{i,gH}(t) \odot \Delta t \oplus (-1) y''_{i,gH}(t) \begin{array}{l}
\odot \frac{\Delta t^2}{2!} \oplus (-1) y''_{i,gH}(t) \\
\odot \frac{\Delta t^3}{3!} \odot \mathcal{R}(t + \Delta t),
\end{array}
\]

\[
y(t - \Delta t) = y(t) \ominus y'_{i,gH}(t) \odot \Delta t \ominus (-1) y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus (-1) y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t - \Delta t).
\end{array}
\]

Case 4. If $y(t)$ is $[i - gH]$-differentiable and $y'_{gH}(t)$ and $y''_{gH}(t)$ are $[i - gH]$-differentiable:

\[
y(t + \Delta t) = y(t) \oplus y'_{i,gH}(t) \odot \Delta t \oplus (-1) y''_{i,gH}(t) \begin{array}{l}
\odot \frac{\Delta t^2}{2!} \oplus (-1) y''_{i,gH}(t) \\
\odot \frac{\Delta t^3}{3!} \odot \mathcal{R}(t + \Delta t),
\end{array}
\]

\[
y(t - \Delta t) = y(t) \ominus y'_{i,gH}(t) \odot \Delta t \ominus (-1) y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus (-1) y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t - \Delta t).
\end{array}
\]

Case 5. If $y(t)$ is $[i - gH]$-differentiable and $y'_{gH}(t)$ and $y''_{gH}(t)$ are $[i - gH]$-differentiable:

\[
y(t + \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus (-1) y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus (-1) y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t + \Delta t),
\end{array}
\]

\[
y(t - \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus (-1) y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus (-1) y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t - \Delta t).
\end{array}
\]

Case 6. If $y(t)$ and $y'_{gH}(t)$ are $[i - gH]$-differentiable and $y''_{gH}(t)$ is $[i - gH]$-differentiable:

\[
y(t + \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus (-1) y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t + \Delta t),
\end{array}
\]

\[
y(t - \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus (-1) y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t - \Delta t).
\end{array}
\]

Case 7. If $y(t)$ and $y''_{gH}(t)$ are $[i - gH]$-differentiable and $y'_{gH}(t)$ is $[i - gH]$-differentiable:

\[
y(t + \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus (-1) y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t + \Delta t),
\end{array}
\]

\[
y(t - \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus (-1) y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t - \Delta t).
\end{array}
\]

Case 8. If $y(t)$ and $y''_{gH}(t)$ are $[i - gH]$-differentiable and $y'_{gH}(t)$ is $[i - gH]$-differentiable:

\[
y(t + \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t + \Delta t),
\end{array}
\]

\[
y(t - \Delta t) = y(t) \ominus (-1) y'_{i,gH}(t) \odot \Delta t \ominus y''_{i,gH}(t) \begin{array}{l}
\ominus \frac{\Delta t^2}{2!} \ominus y''_{i,gH}(t) \\
\ominus \frac{\Delta t^3}{3!} \ominus \mathcal{R}(t - \Delta t).
\end{array}
\]

where

\[
\mathcal{R}(t + \Delta t) = \int_{\xi_1}^{t + \Delta t} \left( \int_{\xi_2}^{\xi_3} \left( \int_{\xi_4}^{\xi_5} y_{gH}(\xi_4) d\xi_4 \right) d\xi_3 \right) d\xi_2 d\xi_1.
\]

and

\[
\mathcal{R}(t - \Delta t) = \int_{t - \Delta t}^{t} \left( \int_{\xi_2}^{\xi_3} \left( \int_{\xi_4}^{\xi_5} y_{gH}(\xi_4) d\xi_4 \right) d\xi_3 \right) d\xi_2 d\xi_1.
\]

and $\oplus$ can be one of the $\oplus$, $\ominus (-1)$ or $\ominus (-1)$.

**Proof** Since $y \in C^4_{gH}([a, b], \mathbb{R})$ with no switching points, so $y^{(k)}_{gH}, i = 0, 1, 2, 3, 4$ are integrable on $[a, b]$. We want to prove Case 1, therefore $y(t)$, $y'_{gH}(t)$ and $y''_{gH}(t)$ are $[i - gH]$-differentiable. According to Theorem 2.8, we can write

\[
y(t + \Delta t) = y(t) \oplus \int_{t}^{t + \Delta t} y'_{gH}(\xi_1) d\xi_1,
\]

and

\[
y_{i,gH}(\xi_1) = y'_{i,gH}(t) \oplus \int_{t}^{\xi_1} y''_{i,gH}(\xi_2) d\xi_2.
\]
By integration from each side of equation (3), we conclude that

\[ \int_t^{t+\Delta t} y_{i,g,H}^\prime(\xi) \, d\xi_1 = \int_t^{t+\Delta t} y_{i,g,H}^\prime(\xi) \, d\xi_1 \]

\[ + \int_t^{t+\Delta t} \left( \int_t^{\xi_1} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1 \]

\[ = y_{i,g,H}^\prime(t) \circ \Delta t \]

\[ + \int_t^{t+\Delta t} \left( \int_t^{\xi_1} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1. \]

By continuing this process

\[ y_{i,g,H}^{\prime\prime}(\xi_2) = y_{i,g,H}^\prime(t) \circ \int_t^{\xi_2} y_{i,g,H}^{\prime\prime}(\xi_3) \, d\xi_3. \]

Applying the integral operator to \( y_{i,g,H}^{\prime\prime}(\xi_2) \) gives

\[ \int_t^{\xi_1} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 = y_{i,g,H}^\prime(t) \circ (\xi_1 - t) \]

\[ + \int_t^{\xi_1} \left( \int_t^{\xi_2} y_{i,g,H}^{\prime\prime}(\xi_3) \, d\xi_3 \right) \, d\xi_2. \]

Furthermore,

\[ \int_t^{t+\Delta t} \left( \int_t^{\xi_1} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1 = y_{i,g,H}^\prime(t) \]

\[ + \int_t^{t+\Delta t} (\xi_1 - t) \, d\xi_1 \]

\[ + \int_t^{t+\Delta t} \left( \int_t^{\xi_1} \left( \int_t^{\xi_2} y_{i,g,H}^{\prime\prime}(\xi_3) \, d\xi_3 \right) \, d\xi_2 \right) \, d\xi_1. \]

And

\[ y(t + \Delta t) = y(t) \circ y_{i,g,H}^\prime(t) \circ \Delta t \circ y_{i,g,H}^{\prime\prime}(t) \circ \frac{\Delta t^2}{2!} \]

\[ + \int_t^{t+\Delta t} \left( \int_t^{\xi_1} \left( \int_t^{\xi_2} y_{i,g,H}^{\prime\prime}(\xi_3) \, d\xi_3 \right) \, d\xi_2 \right) \, d\xi_1. \]

With the similar manner,

\[ y(t + \Delta t) = y(t) \circ y_{i,g,H}^\prime(t) \circ \Delta t \circ y_{i,g,H}^{\prime\prime}(t) \circ \frac{\Delta t^2}{2!} \]

\[ + y_{i,g,H}^{\prime\prime\prime}(t) \circ \frac{\Delta t^3}{3!} \circ \mathcal{R}(t + \Delta t), \]

where

\[ \mathcal{R}(t + \Delta t) = \int_t^{t+\Delta t} \left( \int_t^{\xi_1} \left( \int_t^{\xi_2} y_{i,g,H}^{\prime\prime\prime}(\xi_4) \, d\xi_4 \right) \, d\xi_3 \right) \, d\xi_2 \, d\xi_1. \]

Now consider \( y(t) \) is a continuous and \([i - gH]\)-differentiable fuzzy function. By using Theorem 2.8 for \( j = 1 \), we can write

\[ y(t - \Delta t) = y(t) \circ y_{i,g,H}^\prime(\xi_1) \, d\xi_1. \]

According to Theorem 2.8

\[ y_{i,g,H}^{\prime\prime}(\xi_1) = y_{i,g,H}^\prime(t) \circ \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2. \]

(4)

Therefore, by integration of (4), we get that

\[ \int_{t-\Delta t}^{t} y_{i,g,H}^{\prime\prime}(\xi_1) \, d\xi_1 = \int_{t-\Delta t}^{t} y_{i,g,H}^{\prime\prime}(\xi_1) \, d\xi_1 \]

\[ \circ \int_{t-\Delta t}^{t} \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1 \]

\[ = y_{i,g,H}^\prime(t) \circ \Delta t \circ \int_{t-\Delta t}^{t} \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1 \]

\[ \times \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1. \]

Therefore,

\[ y(t - \Delta t) = y(t) \circ y_{i,g,H}^\prime(t) \circ \Delta t \circ \int_{t-\Delta t}^{t} \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1. \]

But we have

\[ y_{i,g,H}^{\prime\prime}(\xi_2) = y_{i,g,H}^\prime(t) \circ \int_{\xi_2}^{t} y_{i,g,H}^{\prime\prime}(\xi_3) \, d\xi_3. \]

With repeated integrals, we have

\[ \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 = y_{i,g,H}^{\prime\prime}(t) \circ (t - \xi_1) \]

\[ \circ \int_{\xi_1}^{t} \left( \int_{\xi_2}^{t} y_{i,g,H}^{\prime\prime}(\xi_3) \, d\xi_3 \right) \, d\xi_2, \]

\[ \Rightarrow \int_{t-\Delta t}^{t} \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1 = y_{i,g,H}^\prime(t) \]

\[ \circ \int_{t-\Delta t}^{t} (t - \xi_1) \, d\xi_1 \circ \int_{t-\Delta t}^{t} \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1 \]

\[ \times \left( \int_{\xi_1}^{t} y_{i,g,H}^{\prime\prime}(\xi_2) \, d\xi_2 \right) \, d\xi_1. \]

In this case, we can conclude
\[ y(t - \Delta t) = y(t) \odot y'_{i,gH}(t) \odot \Delta t \odot y''_{i,gH}(t) \odot \frac{\Delta t^2}{2!} \]
\[ \odot \int_{t-\Delta t}^{t} \left( \int_{\xi_1}^{t} y'''_{i,gH}(\xi) d\xi \right) d\xi_1. \]

In the same way, the other cases outlined in the theorem are also proven using Theorem 2.8.

**Definition 2.10** (See Allahviranloo et al. 2015b) The first generalized Hukuhara partial derivative ([gH-p]-derivative for short) of a fuzzy-valued function \( u(x,t) : \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}_F \) at \((x_0,t_0)\) with respect to \( x \) is defined by

\[ \partial_{x,gH} u(x_0,t_0) = \lim_{h \to 0} \frac{u(x_0 + h, t) \odot gH u(x_0, t_0)}{h}, \]

A triangular fuzzy function \( u(x,t) : \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}_T \) without any switching point on \( \mathbb{D} \) is called

- \([i \times -p]\)-differentiable w.r.t. \( x \) at \((x_0,t_0)\) if
  \[ u'_{x_{i,p}}(x_0,t_0) = \left( u_{1i}, (x_0,t_0), u_{2i}, (x_0,t_0), u_{3i}, (x_0,t_0) \right) \]
- \([(i+p)]\)-differentiable w.r.t. \( x \) at \((x_0,t_0)\) if
  \[ u''_{x_{i,p}}(x_0,t_0) = \left( u_{3i}, (x_0,t_0), u_{2i}, (x_0,t_0), u_{1i}, (x_0,t_0) \right) \]

Moreover, if \( u_{x_{i,p}}(x,t) \) is \([gH - p]\)-differentiable at \((x_0,t_0)\) with respect to \( x \) without any switching point on \( \mathbb{D} \)

- if the type of \([gH - p]\)-differentiability of both \( u(x,t) \) and \( u_{x_{i,p}}(x,t) \) are the same, then \( u_{x_{i,p}}(x,t) \) is \([(i+p)]\)-differentiable w.r.t. \( x \) and
  \[ u'_{x_{i,p}}(x_0,t_0) = \left( u_{1i}, (x_0,t_0), u_{2i}, (x_0,t_0), u_{3i}, (x_0,t_0) \right) \]
- if the type of \([gH-p]\)-differentiability \( u(x,t) \) and \( u_{x_{i,p}}(x,t) \) are different, then \( u_{x_{i,p}}(x,t) \) is \([(i+p)]\)-differentiable w.r.t. \( x \)
  \[ u'_{x_{i,p}}(x_0,t_0) = \left( u_{3i}, (x_0,t_0), u_{2i}, (x_0,t_0), u_{1i}, (x_0,t_0) \right) \]

### 3 Finite difference methods

Our goal here is to describe the fundamentals of the fuzzy finite difference method. To accomplish this, we will first show you how to obtain the finite difference formula for the first and second derivatives of a triangular fuzzy function \( y(t) \).

Now, we describe the essential details of finite difference methods. First, we select an integer \( N > 0 \) and divide the interval \([a, b]\) into \((N + 1)\) equal sub-intervals whose endpoints are the mesh points \( t_i = a + i \Delta t \), for \( i = 0, 1, ..., N + 1 \), where \( \Delta t = \frac{b-a}{N+1} \). Let \( y(t) \in C^4_{gH}([a, b], \mathbb{R}_T) \), so based on the different types of differentiability that mentioned in Remark 2.6, the first and second \( gH \)-derivative of this fuzzy function can be approximated by fuzzy finite difference method as follows

**Case 3.1** Consider \( y(t) \) and \( y'_{i,gH}(t) \) are \([i \times gH]\)-differentiable and \( y''_{i,gH}(t) \) is \([i - gH]\)-differentiable or \([i - gH]\)-differentiable.

**The first fuzzy forward difference.**

First Consider \( y''_{i,gH}(t) \) is \([i - gH]\)-differentiable. Hence, by using the Case 1 in Theorem 2.9, we have

\[ y(t + \Delta t) = y(t) \odot y'_{i,gH}(t) \odot \Delta t \odot y''_{i,gH}(t) \]
\[ \odot \Delta t^2 \odot y'''_{i,gH}(t) \odot \Delta t^3 \odot R(t + \Delta t), \]

solve for \( y'_{i,gH}(t) \) yields

\[ y'_{i,gH}(t) = \frac{y(t + \Delta t) \odot y(t)}{\Delta t} \odot \frac{\Delta t^2}{2} y''(t) \]
\[ \odot \frac{\Delta t^2}{3!} y'''(t) \odot \cdots, \]

By using the fuzzy mean value theorem in Allahviranloo et al. (2015b), for all \( i = 0, 1, ..., N \), there are \( \xi^+ \in (t, t + \Delta t) \) such that

\[ y'_{i,gH}(t) = \frac{y(t + \Delta t) \odot y(t)}{\Delta t} \odot \frac{\Delta t}{2} y''(\xi^+), \]

or

\[ y'_{i,gH}(t) \odot \frac{y(t + \Delta t) \odot y(t)}{\Delta t} = \odot \frac{\Delta t}{2} y''(\xi^+) \]

where the term \( \frac{\Delta t}{2} y''(\xi^+) \) is called truncation error of the forward fuzzy finite difference approximation. Moreover, the properties of the Hausdorff distance (Lakshmikantham et al. 2006) are implied that

\[ D\left(y'_{i,gH}(t), \frac{y(t + \Delta t) \odot y(t)}{\Delta t} \odot \frac{\Delta t}{2} y''(\xi^+)\right) \]
\[ \leq D\left(y'_{i,gH}(t), \frac{y(t + \Delta t) \odot y(t)}{\Delta t}\right) \]
\[ + D\left(0, \odot \frac{\Delta t}{2} y''(\xi^+)\right) \to 0, \]

as \( \Delta t \to 0 \). Therefore, \( \Delta t \) should be sufficiently small to get a good approximation. Finally, for sufficiently small \( \Delta t \), the first forward fuzzy finite
The first fuzzy backward difference.

To obtain the backward fuzzy finite difference, using Theorem 2.9 (case 1), we can write:

\[ y(t - \Delta t) = y(t) \oplus y'_{i,gH}(t) \odot \Delta t \oplus y''_{i,gH}(t) \odot \frac{\Delta t^2}{2!} \odot R(t - \Delta t), \]

(7)

Rearranging equation (7) gives:

\[ \frac{y(t) \oplus y(t - \Delta t)}{\Delta t} = y'_{i,gH}(t) \oplus y''_{i,gH}(t) \odot \frac{\Delta t}{2!} \oplus y'''_{i,gH}(t) \odot \frac{\Delta t^3}{3!} \oplus \ldots \]

For having a more useful approximation value for \( y'_{i,gH}(t) \), by using the fuzzy mean value theorem in Allahviranloo et al. (2015b), there are \( \xi^- \in (t - \Delta t, t) \) such that:

\[ \frac{y(t) \oplus y(t - \Delta t)}{\Delta t} = y'_{i,gH}(t) \oplus y''_{i,gH}(\xi^-) \odot \frac{\Delta t}{2!} \]

So by considering \( \Delta t \) is small enough, the approximation value obtained for the first-order gH-derivative is equal to:

\[ y'_{i,gH}(t) \approx \frac{y(t) \oplus y(t - \Delta t)}{\Delta t}. \]

The first fuzzy central difference.

We use Hukuhara to subtract Eq. (5) from Eq. (7) and divide by \( 2\Delta t \), then we obtain:

\[ \frac{y(t + \Delta t) \oplus y(t - \Delta t)}{2\Delta t} = y'_{i,gH}(t) \oplus \frac{\Delta t^2}{12} y''_{i,gH}(\xi). \]

On the other hand, given the Hausdorff distance properties, it can be seen that:

\[ D\left(\frac{y(t + \Delta t) \oplus y(t - \Delta t)}{2\Delta t}, y'_{i,gH}(t) \oplus \frac{\Delta t^2}{12} y''_{i,gH}(\xi)\right) \]

\[ \leq D\left(\frac{y(t + \Delta t) \oplus y(t - \Delta t)}{2\Delta t}, y'_{i,gH}(t)\right) \]

\[ + D\left(0, \frac{\Delta t^2}{12} y''_{i,gH}(\xi)\right) \rightarrow 0. \]

When \( \Delta t \rightarrow 0 \), we have the following equation:

\[ y'_{i,gH}(t) \approx \frac{y(t + \Delta t) \oplus y(t - \Delta t)}{2\Delta t}, \]

is the first fuzzy central difference approximation of \( y'_{i,gH}(t) \).

The second-order fuzzy central difference.

To obtain an appropriate approximation for the second-order derivative of the fuzzy function \( y(t) \), (5) is added to (7), then the equations are rearranged and divided by \( \Delta t^2 \):

\[ \frac{y(t + \Delta t) \oplus y(t - \Delta t) \ominus 2y(t)}{\Delta t^2} = y''_{i,gH}(t) \ominus \frac{1}{\Delta t^2} \left(R(t + \Delta t) \ominus R(t - \Delta t)\right). \]

Beside:

\[ D\left(\frac{y(t + \Delta t) \oplus y(t - \Delta t) \ominus 2y(t)}{\Delta t^2}, y''_{i,gH}(t) \ominus \frac{1}{\Delta t^2} \left(R(t + \Delta t) \ominus R(t - \Delta t)\right)\right) \]

\[ \leq D\left(\frac{y(t + \Delta t) \oplus y(t - \Delta t) \ominus 2y(t)}{\Delta t^2}, y''_{i,gH}(t)\right) \]

\[ + D\left(0, \frac{1}{\Delta t^2} \left(R(t + \Delta t) \ominus R(t - \Delta t)\right)\right) \rightarrow 0. \]

But by the definition of \( R(t + \Delta t) \) and \( R(t - \Delta t) \) when \( \Delta t \rightarrow 0 \):

\[ D\left(\frac{y(t + \Delta t) \oplus y(t - \Delta t) \ominus 2y(t)}{\Delta t^2}, y''_{i,gH}(t)\right) \rightarrow 0, \]

\[ D\left(0, \frac{1}{\Delta t^2} \left(R(t + \Delta t) \ominus R(t - \Delta t)\right)\right) \rightarrow 0. \]

Hence, for \( \Delta t \) sufficiently small, the appropriate approximation obtained for the second-order derivative \( y''_{i,gH}(t) \) is equal to:

\[ y''_{i,gH}(t) \approx \frac{y(t + \Delta t) \oplus y(t - \Delta t) \ominus 2y(t)}{\Delta t^2}. \]

Now, if \( y''_{i,gH}(t) \) is \([i - gH]-\)differentiable, we obtain the similar approximation value for \( y'_{i,gH}(t) \) and \( y''_{i,gH}(t) \) and there is no need to repeat the process of obtaining these approximate values. Accordingly, the type of gH-differentiability of the second-order derivative has no effect on the obtained values, and these approximation values all depend on the fuzzy function \( y(t) \) and its first-order derivative \( y_{gH}(t) \).
Here, for the first and second gH-derivatives, we present the relevant fuzzy finite difference formulas by considering the type of gH-differentiability. We will not elaborate on the proof in these cases since it is the same as Case 3.1.

Case 3.2. If \( y(t) \) is \([i - g H]\)–differentiable and \( y'_{gH}(t) \) is \([i i - g H]\)–differentiable. In this case, \( y''_{gH}(t) \) can be \([i - g H]\)–differentiable or \([i i - g H]\)–differentiable.

- The first fuzzy forward difference.
  \[
y'_{i, gH}(t) \approx \frac{y(t + \Delta t) \ominus y(t)}{\Delta t}
\]

- The first fuzzy backward difference.
  \[
y'_{ii, gH}(t) \approx \frac{y(t) \ominus y(t - \Delta t)}{\Delta t}
\]

- The first fuzzy central difference.
  \[
y'_{i, gH}(t) \approx \frac{y(t + \Delta t) \ominus y(t - \Delta t)}{2\Delta t}
\]

- The second-order fuzzy central difference.
  \[
y''_{i, gH}(t) \approx \frac{-2(y(t) \ominus y(t - \Delta t) \ominus y(t + \Delta t))}{\Delta t^2}
\]

Case 3.3. consider \( y(t) \) is \([ii - g H]\)–differentiable and \( y'_{gH}(t) \) is \([ii - g H]\)–differentiable. In this case, \( y''_{gH}(t) \) may be \([i - g H]\)–differentiable or \([ii - g H]\)–differentiable.

- The first fuzzy forward difference.
  \[
y'_{ii, gH}(t) \approx \frac{-1(y(t) \ominus y(t + \Delta t))}{\Delta t}
\]

- The first fuzzy backward difference.
  \[
y'_{ii, gH}(t) \approx \frac{-1(y(t - \Delta t) \ominus y(t))}{\Delta t}
\]

- The first fuzzy central difference.
  \[
y'_{i, gH}(t) \approx \frac{-1(y(t - \Delta t) \ominus y(t + \Delta t))}{2\Delta t}
\]

- The second-order fuzzy central difference.
  \[
y''_{i, gH}(t) \approx \frac{-2(y(t) \ominus y(t - \Delta t) \ominus y(t + \Delta t))}{\Delta t^2}
\]

Case 3.4. Let \( y(t) \) and \( y'_{gH}(t) \) are \([ii - g H]\)–differentiable. \( y''_{gH}(t) \) can be \([i - g H]\)–differentiable or \([ii - g H]\)–differentiable, the final formulas are obtained same.

- The first fuzzy forward difference.
  \[
y'_{ii, gH}(t) \approx -\frac{1(y(t) \ominus y(t + \Delta t))}{\Delta t}
\]

- The first fuzzy backward difference.
  \[
y'_{ii, gH}(t) \approx -\frac{1(y(t - \Delta t) \ominus y(t))}{\Delta t}
\]

- The first fuzzy central difference.
  \[
y'_{i, gH}(t) \approx -\frac{1(y(t - \Delta t) \ominus y(t + \Delta t))}{2\Delta t}
\]

- The second-order fuzzy central difference.
  \[
y''_{i, gH}(t) \approx \frac{(y(t + \Delta t) \ominus y(t - \Delta t) \ominus 2y(t))}{\Delta t^2}
\]

4 The non-homogeneous fuzzy heat equation

In mathematical physics, motion or transport of particles, i.e., ions, molecules, etc., from higher concentration to lower concentration is modeled by the diffusion equation with appropriate boundary and initial conditions. Heat conduction in a rod is a prototypical diffusion equation. Consider a uniform rod of length \( L \) which is insulated everywhere except at its two ends and the temperature is transmitted non-uniformly from beginning to end. This temperature is denoted by \( u(x, t) \), and \( x \) is a coordinate in space, \( t \) represents time. Measuring the temperature is an uncertain problem, and this vagueness may appear in the initial and boundary conditions. Suppose the temperature at the ends are kept at a fixed fuzzy temperature of \( u(0, t) \) and \( u(L, t) \), respectively. The problem is to find the future temperature along the rod by considering the given fuzzy initial temperature \( u(x, 0) \). In this case, the above problem is formulated as the following fuzzy non-homogeneous initial-boundary-value heat equation

\[
\begin{aligned}
&u_{xH}(x, t) = u_{xxxH}(x, t) \oplus F(x, t), \quad x \in [0, L], \ t \in [0, T]; \\
u(x, 0) = f(x), \quad x \in [0, L], \\
u(0, t) = g(t), \quad t > 0, \\
u(L, t) = h(t), \quad t > 0.
\end{aligned}
\]

where \( f(x), g(t), h(t) \) and \( F(x, t) \) are triangular fuzzy functions such that \( f(x), g(t), h(x) \) and \( F(x, t) \in C^3_{gH}(0, L) \times C^2_{gH}(0, T) \).
This equation has a unique solution in different states of \([gH-p]\)-differentiability (Allahviranloo et al. 2015b) and the main purpose of this section is to obtain an approximate fuzzy solution for the fuzzy heat equation using the fuzzy finite difference method. Suppose that \(u(x,t)\) is the exact fuzzy solution of equation (8) provided that the types of \([gH-p]\)—differentiability with respect to \(x\) and \(t\) are the same. The basic idea is to replace all the derivatives in equation (8) by corresponding difference approximation.

Let \(u(x,t) \in C_{gH}^{p}(0, L) \times [0, T], \mathbb{R}_{T}\), by considering the type of \([gH-p]\)—differentiability, the following different situations will be happen

Case 1. Let \(u(x,t)\) is \([(i) - p]\)—differentiable with respect to \(t\) and \(u_{x,gH}(x,t)\) is a \([(i) - p]\)—differentiable fuzzy function with respect to \(x\). In this case, the heat equation will be as follows

\[
u_{t,gH}(x,t) = u_{xx,gH}(x,t) \oplus F(x,t). \tag{9}\]

- **Forward Difference in time:** Since \(u(x,t)\) is \([(i) - p]\)—differentiable with respect to \(t\), then different cases 1, 2, 3 and 4 in Theorem 2.9 can be used, in which

\[
u(x, t + \Delta t) = u(x, t) \oplus u_{t,gH}(x, t) \ominus \Delta t \ominus u_{t,t,gH} \times (x, t) \ominus \Delta t^{2} + \ldots \tag{10}\]

Therefore, according to Section 3, we obtain

\[
u_{t,gH}(x,t) \approx \frac{u(x, t + \Delta t) \ominus u(x, t)}{\Delta t}, \tag{11}\]

- **Central Differences in Space:** Due to the fact that \(u_{x,gH}(x, t)\) is a \([(i) - p]\)—differentiable function, all cases 1, 2, 7 and 8, which are expressed in Theorem 2.9, can be used. So let us take case 1

\[
u(x + \Delta x, t) = u(x, t) \ominus u_{x,gH}(x, t) \ominus \Delta x \ominus u_{x,x,gH}(x, t) \ominus \Delta x^{2} + \ldots \]

Adding and re-arranging:

\[
u_{x,gH}(x,t) \approx \frac{u(x + \Delta x, t) \ominus u(x, t)}{\Delta x} \ominus \Delta x \ominus R(x + \Delta x, t), \tag{12}\]

\[
u(x, t + \Delta t) \ominus u(x, t) \ominus \Delta t \ominus \Delta x \ominus u_{x,gH}(x, t) \ominus \Delta x \ominus \Delta x \ominus u_{x,x,gH}(x, t) \ominus \Delta x^{2} + \ldots \]

Now, substitute equations (11) and (13) into the main equation (9), accordingly

\[
u(x, t + \Delta t) \ominus u(x, t) \ominus \frac{\Delta t}{\Delta x^{2}} \ominus \Delta x \ominus R(x + \Delta x, t) \tag{13}.

To obtain an approximation solution for equation (9) using the fuzzy finite difference method, we must divide the domain \([0, L] \times [0, T]\) into a set of mesh points. Here, we subdivide the domain \([0, L] \times [0, T]\) into \(N_{x} + 1\) and \(N_{t} + 1\) equally mesh points

\[
x_{k} = k \Delta x, \quad k = 0, \ldots, N_{x},
\]

\[
t_{n} = n \Delta t, \quad n = 0, \ldots, N_{t}.
\]

Now, consider \(U_{k}^{n}\) denotes the mesh function that approximates \(u(x_{k}, t_{n})\) for \(k = 0, \ldots, N_{x}\) and \(n = 0, \ldots, N_{t}\). By putting mesh point \((x_{k}, t_{n})\) into equation (9), the following formula is obtained

\[
U_{k+1}^{n} = U_{k}^{n} \ominus \mu \left( U_{k+1}^{n} \ominus 2U_{k}^{n} \ominus U_{k-1}^{n} \right) \ominus \Delta t E_{k}^{n}, \tag{14}
\]

where \(\mu = \frac{\Delta t}{\Delta x^{2}}.\)
Truncation error: Now, we want to investigate the truncation error of the scheme (14). The truncation error, \( T(x, t) \), is the difference between two side of equation when the exact solution \( u(x_k, t_n) \) is replaced with the approximation value \( U^a_k \), hence

\[
T(x, t) := \frac{\Delta u(x, t)}{\Delta t} \oplus \frac{\delta^2 u(x, t)}{\Delta x^2},
\]

where

\[
\Delta u(x, t) := u(x, t + \Delta t) \ominus u(x, t)
\]

\[
\delta^2 u(x, t) := u(x + \Delta x, t) \ominus 2u(x, t) \oplus u(x - \Delta x, t)
\]

By using the fuzzy mean value theorem in Allahviranloo et al. (2015b), there is \( \eta \in (t, t + \Delta t) \) such that equation (10) can be written as follows

\[
u(x, t + \Delta t) \ominus u(x, t) = u_{t, \eta}(x, t)
\]

\[
\ominus \Delta t \oplus u_{t, \eta}(x, \eta) \ominus \frac{\Delta t^2}{2!},
\]

we obtain

\[
\Delta u(x, t) = u_{t, \eta}(x, t) \ominus \Delta t \oplus u_{t, \eta}(x, \eta)
\]

\[
\ominus \frac{\Delta t^2}{2!}, \quad (15)
\]

On the other hand, equation (12) concludes that

\[
\delta^2 u(x, t) = u_{x, \eta} \ominus \Delta x^2 \ominus (R(x + \Delta x, t) \ominus R(x - \Delta x, t)).
\]

So

\[
T(x, t) = (u_{t, \eta} \ominus u_{x, \eta}) \oplus \left( \frac{1}{2} u_{t, \eta}(x, \eta) \Delta t \ominus \frac{1}{\Delta x^2} \left( R(x + \Delta x, t) \ominus R(x - \Delta x, t) \right) \right)
\]

\[
= \frac{1}{2} u_{t, \eta}(x, \eta) \Delta t \ominus \frac{1}{\Delta x^2} \left( R(x + \Delta x, t) \ominus R(x - \Delta x, t) \right).
\]

Since \( u(x, t) \in C^4_{\mathbb{R}}([0, L] \times [0, T], \mathbb{R}) \), we can consider

\[
D(u_{t, \eta}, 0) \leq M_{t},
\]

\[
D(R(x + \Delta x, t) \ominus R(x - \Delta x, t), 0) \leq M_{R}.
\]

Then, by the help of Hausdorff distance properties (Lakshmikantham et al. 2006), we get the following result

\[
D(T(x, t), 0) \leq \frac{1}{2} M_{t} \Delta t + \frac{1}{\Delta x^2} M_{R}
\]

\[
\leq \frac{1}{2} \Delta t \left( M_{t} + \frac{\mu}{2} M_{R} \right).
\]

So

\[
T(x, t) \to 0, \quad as \quad \Delta t \to 0, \forall (x, t) \in [0, 1] \times [0, T].
\]

The convergence of method: To check the convergence of the given method, suppose fixed point \((x^*, t^*)\) in domain \([0, L] \times [0, T]\). We say that the method is convergent if \(x_k \to x^*, \ t_n \to t^*\) implies that

\[
U^n_k \to u(x^*, t^*).
\]

Let \( e^n_k := U^n_k \ominus u(x_k, t_n) \) be the error function for finite difference method in this case. Putting the error function in equation (14) and using Proposition 2.4 results in

\[
e^{n+1}_k = e^n_k \oplus \mu \left( e^{n+1}_{k+1} \oplus 2e^n_k \oplus e^{n+1}_{k-1} \right) \ominus T^n_k \Delta t
\]

\[
= (1 - 2\mu)e^n_k \oplus \mu e^{n+1}_{k+1} \oplus \mu e^{n+1}_{k-1} \ominus T^n_k \Delta t,
\]

where \( T^n_k := T(x_k, t_n) \). Now, if \( \mu \leq \frac{1}{2} \), the coefficient of the three terms of \( e^n \) on the right-hand side of the above equation will be positive, and the result will be unity. Let us to consider

\[
E^n := \max \{ D(e^n_k, 0), \; k = 0, 1, ..., N_x \}.
\]

So

\[
D(e^{n+1}_k, 0) \leq E^n + D(T^n_k, 0) \Delta t.
\]

Since this inequality holds for all values of \( k \), then

\[
E^{n+1} \leq E^n + D(T^n_k, 0) \Delta t.
\]

On the other hand, by definition of \( e^n_k \), we know that \( E^0 = 0 \). In this case, \( E^n \leq n D(T^n_k, 0) \Delta t \), which is achieved with a simple induction. Therefore, from (16), we obtain

\[
E^{n+1} \leq \frac{1}{2} \Delta t \left( M_t + \frac{\mu}{2} M_R \right) T
\]

\[
\to 0 \quad as \quad \Delta t \to 0.
\]
In fact we showed that the approximate solution \( U^n_k \) obtained by finite difference method (14) converges to the exact solution \( u(x_k, t_n) \) provided that \( \mu \leq \frac{1}{2} \) for sufficiently large value of \( N_t \), besides \( D(u_{xx}, 0) \leq M_x \) and \( D \left( R(x + \Delta x, t) \bigcap R(x - \Delta x, t), 0 \right) \leq M_R \).

The following algorithm summarizes the proposed fuzzy finite difference.

Algorithm 4.1  
1. Choose \( N_t \) and \( N_x \) such that \( \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \).
2. Compute \( U^n_k = \left( f_1(x_k), f_2(x_k), f_3(x_k) \right) \) for \( k = 0, 1, ..., N_x \).
3. For \( n = 0, 1, ..., N_t \):
   i. apply \( U^{n+1}_k = U^n_k \oplus \mu \left( U^n_{k+1} \oplus 2U^n_k \oplus U^n_{k-1} \right) \oplus \Delta t F^n_k \) for all \( k = 1, ..., N_x - 1 \).
   ii. set the boundary value \( U^n_0 = \left( g_1(t_{n+1}), g_2(t_{n+1}), g_3(t_{n+1}) \right) \).
   iii. set the boundary value \( U^n_{N_x} = \left( h_1(t_{n+1}), h_2(t_{n+1}), h_3(t_{n+1}) \right) \).

In the following, we will briefly consider the other case of the \([gH - p]-\)differentiability for equation (8). The whole process of proof for the following situations is the same as in case 1, so we will not go into details and we just express the algorithm.

Case 2. Consider the following fuzzy heat equation

\[
 u_{(x,t)}(x,t) = u_{(x,t)}(x,t) \oplus F(x,t) \tag{17}
\]

in this equation, \( u(x,t) \) is \([ii] - p\)-differentiable with respect to \( t \) and \( u_{(x,t)}(x,t) \) is a \([ii] - p\)-differentiable fuzzy entiable function with respect to \( x \).

Algorithm 4.2  
1. Choose \( N_t \) and \( N_x \) such that \( \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \).
2. Compute \( U^n_k = \left( f_1(x_k), f_2(x_k), f_3(x_k) \right) \) for \( k = 0, 1, ..., N_x \).
3. For \( n = 0, 1, ..., N_t \):
   i. apply \( U^{n+1}_k = U^n_k \oplus \mu \left( U^n_{k+1} \oplus 2U^n_k \oplus U^n_{k-1} \right) \oplus \Delta t F^n_k \) for all \( k = 1, ..., N_x - 1 \).
   ii. set the boundary value \( U^n_0 = \left( g_1(t_{n+1}), g_2(t_{n+1}), g_3(t_{n+1}) \right) \).
   iii. set the boundary value \( U^n_{N_x} = \left( h_1(t_{n+1}), h_2(t_{n+1}), h_3(t_{n+1}) \right) \).

5 Numerical examples

We will solve a few examples of the fuzzy finite difference method in this section to illustrate its efficiency and accuracy in solving the fuzzy heat equation. All calculations were performed on a PC running Mathematica software.

Example 5.1 (Numerical illustration) Consider the following initial-boundary non-homogeneous fuzzy heat equation

\[
\begin{align*}
 u_{(x,t)}(x,t) &= u_{(x,t)}(x,t) \oplus (2x, 3x, 5x)x \in [0, 2], t \in [0, 1]; \\
 u(x, 0) &= (2x, 3x, 5x), \\
 u(0, t) &= 0, \quad u(2, t) = (4t + 4, 6t + 6, 10t + 10)
\end{align*}
\]

This equation has exact fuzzy solution \( u(x,t) = (2(t + 1)x, 3(t + 1)x, 5(t + 1)x) \).

Considering Algorithm 4.1, step-by-step procedure to solve the given example is as follows

1. Suppose \( N_x = 2 \) and \( N_t = 3 \), then \( \frac{\Delta t}{\Delta x} = \frac{1}{2} \) and
   \(t_0 = 0, \quad t_1 = \frac{1}{3}, \quad t_2 = \frac{2}{3}, \quad t_3 = 1 \),
   \(x_0 = 0, \quad x_1 = 1, \quad x_2 = 2 \).
2. For \( k = 0, 1, 2 \)
   \( U^n_0 = (2x_k, 3x_k, 5x_k) \).
   Therefore,
   \( U^n_0 = (0, 0, 0), \quad U^n_1 = (2, 3, 5), \quad U^n_2 = (4, 6, 10) \).
3. For \( n = 0, 1, 2 \)
   \( U^{n+1}_k = U^n_k \oplus \mu \left( U^n_{k+1} \oplus 2U^n_k \oplus U^n_{k-1} \right) \oplus \Delta t F^n_k, \quad k = 1 \)
   \( U^{n+1}_{N_x} = (4t_{n+1} + 4, 6t_{n+1} + 6, 10t_{n+1} + 10) \),
   \( U^{n+1}_0 = (0, 0, 0) \).

Hence,
\[ n = 0 \]
\[ U^1_1 = U^0_2 \oplus \mu \left( U^0_2 \oplus 2U^0_1 \oplus U^0_0 \right) \oplus \Delta t F^0_1 \]
\[ = (2, 3.5) \oplus \frac{1}{3} \left( (4, 6, 10) \oplus (2, 3, 5) \oplus (0, 0, 0) \right) \]
\[ \oplus \frac{1}{3} (2x_1, 3x_1, 5x_1) \]
\[ = (2, 3.5) \oplus \frac{1}{3} (2, 3, 5) = \left( \frac{8}{3}, \frac{4}{3} \right) \]
\[ U^1_2 = (4t_1 + 4, 6t_1 + 6, 10t_1 + 10) = \left( \frac{16}{3}, 8, \frac{40}{3} \right) \]
\[ U^1_0 = (0, 0, 0) \]
We consider the following fuzzy PDE
\[ u_{t_{x_{i+y_{j}}}} = u_{xx_{i+y_{j}}}, \quad x \in [0, 0.5], \quad t \in [0, 1]; \]

where \( u(x, 0) = \left( \sin(\pi x), 3 \sin(\pi x), 5 \sin(\pi x) \right) \), \( u(0, 0) = (0, 0, 0) \) and \( u(0.5, t) = \left( e^{-\pi^2 t}, 3e^{-\pi^2 t}, 5e^{-\pi^2 t} \right) \) with exact solution \( u(x, t) = \left( \sin(\pi x)e^{-\pi^2 t}, 3 \sin(\pi x)e^{-\pi^2 t}, 5 \sin(\pi x)e^{-\pi^2 t} \right) \).

According to the procedure outlined in Algorithm 4.2, \( \Delta t \) and \( \Delta x \) should be considered large such that \( \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \). We consider \( N_x = 2 \) and \( N_t = 50 \). So we have many sub-intervals and it is not possible to show the approximate numbers, \( U_{nx}^n \), and only the approximate and exact solutions are shown in Fig. 2. In addition, Fig. 3 represents the logarithm of the error for various \( N_t \).

By placing the values \( (x_1, t_n) \) in the exact solution, it is easy to verify \( U_k^n = u(x_k, t_n) \). Then, the exact solution of the fuzzy heat equation (18) is obtained by this method.

**Example 5.2** Consider the following initial-boundary fuzzy heat equation
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left( \alpha(x,t) \frac{\partial u}{\partial x} \right) \\
\text{subject to } & u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(0, t) = g(t), \quad \frac{\partial u}{\partial t}(L, t) = h(t), \quad x \in [0, L], \quad t > 0,
\end{align*}
\]

Note that, the exact solution of this equation is \( u(x, t) = e^{-\pi^2 t} \left( \sin(\pi x), 3 \sin(\pi x), 5 \sin(\pi x) \right) \). Let \( N_x = 2 \) and \( N_t = 10 \). The numerical results are shown in Table 1. In addition, the approximate solution, \( U_{n+1}^{n+1} \), and the exact solution, \( u(x_1, t_{n+1}) \), are shown in Fig. 1 when \( x_1 = \frac{1}{2}, n = 0, 1, \ldots, N_t \) and for all \( \alpha \in [0, 1] \).

We observe that the fuzzy finite difference method is an accurate method for solving the given fuzzy heat equation.

**Example 5.3** Consider the following fuzzy PDE
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left( \alpha(x,t) \frac{\partial u}{\partial x} \right) \\
\text{subject to } & u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(0, t) = g(t), \quad \frac{\partial u}{\partial t}(L, t) = h(t), \quad x \in [0, L], \quad t > 0,
\end{align*}
\]

6 Conclusion

We presented the new fuzzy finite difference method for approximating the fuzzy triangular solution of the fuzzy nonhomogeneous heat equation with triangular initial-boundary conditions. To do this, the fuzzy Taylor expansion was extended according to the type of \( gH \)–differentiability, and the finite difference formulas for the first and second derivatives of a triangular fuzzy function \( y(t) \) were obtained. Moreover, the convergence conditions for solving the fuzzy heat problem were also investigated. Several
Table 1 Exact and approximate values by finite difference method for Example 5.2

| $x_i$ | $u(x_i, t_n)$ | $U^+_n$ |
|-------|----------------|---------|
| $(\frac{1}{2}, 0.1)$ | (0.5525, 1.9340, 2.4866) | (0.55, 1.925, 2.475) |
| $(\frac{1}{2}, 0.2)$ | (0.6107, 2.1374, 2.7481) | (0.6103, 2.1356, 2.7429) |
| $(\frac{1}{2}, 0.3)$ | (0.67492, 2.3622, 3.0371) | (0.6710, 2.3688, 3.0399) |
| $(\frac{1}{2}, 0.4)$ | (0.7459, 2.6106, 3.3566) | (0.7416, 2.6158, 3.3574) |
| $(\frac{1}{2}, 0.5)$ | (0.8243, 2.8852, 3.7096) | (0.8196, 2.8857, 3.7884) |
| $(\frac{1}{2}, 0.6)$ | (0.9110, 3.1887, 4.0997) | (0.9158, 3.1804, 4.0863) |
| $(\frac{1}{2}, 0.7)$ | (1.0068, 3.5240, 4.5309) | (1.0051, 3.5339, 4.5309) |
| $(\frac{1}{2}, 0.8)$ | (1.1127, 3.8947, 5.0074) | (1.11641, 3.8824, 4.9788) |
| $(\frac{1}{2}, 0.9)$ | (1.2298, 4.3043, 5.5341) | (1.2227, 4.3097, 5.5024) |
| $(\frac{1}{2}, 1)$ | (1.3591, 4.7569, 6.1161) | (1.3513, 4.7498, 6.1811) |

Fig. 2 Graphs of $u(x_1, t^{n+1})$ (Right) and $U^+_n$ (Left) for Example 5.3

Fig. 3 Graph of the finite difference approximation error of Example 5.3

Numerical examples were presented to demonstrate the performance of the methods. The final results demonstrated the efficiency and the ability of the new fuzzy finite difference method to produce triangular fuzzy numerical results which are more consistent with existing reality. Even though this paper deals with the fuzzy non-homogeneous heat equation, our method can be used to find the numerical solution for a wide variety of fuzzy partial differential equations. The fuzzy
numerical solution of the fuzzy partial differential equations can be obtained without implicitly embedding them into crisp equations through our method.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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