Low-dimensional behavior of Kuramoto model with inertia in complex networks

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Low-dimensional behavior of large systems of globally coupled oscillators has been intensively investigated since the introduction of the Ott-Antonsen ansatz. In this report, we generalize the Ott-Antonsen ansatz to second-order Kuramoto models in complex networks. With an additional inertia term, we find a low-dimensional behavior similar to the first-order Kuramoto model, derive a self-consistent equation and seek the time-dependent derivation of the order parameter. Numerical simulations are also conducted to verify our analytical results.

Synchronization phenomena in large ensembles of coupled systems play a prominent role in many branches of natural and social sciences as well as in engineering1,2. The study of collective synchronization has many applications including the modeling of the flashing of groups of fireflies3, the collective oscillations of pancreatic beta cells4, the human cardiorespiratory system5, and the pedestrian induced oscillations in bridges6. A fundamental contribution to the mathematical aspects of collective synchronization was given by Kuramoto7. In 1975 Kuramoto proposed a model to describe the behavior of a population of coupled non-linear oscillators, employing three key simplifying assumptions, i.e., (i) the coupling strength was chosen to be homogeneous for all pairs of coupled oscillators; (ii) the coupling strength and the natural frequency become finite; and (iii) the number of oscillators was considered to be infinite. Diversity in the oscillators properties is usually incorporated by taking natural frequencies from a given probability distribution function. The phase transition to synchronization occurs when the coupling strength exceeds a threshold, which depends on this probability density function.

In 2008, Ott and Antonsen8 introduced an ansatz for studying the behavior of globally coupled oscillators. The Ott-Antonsen ansatz has been considered to investigate continuously time-dependent collective behavior9 and for the study of delay heterogeneity10. In addition, such ansatz has enabled to find nonuniversal transitions to synchrony in the model with a phase lag for certain unimodal frequency distributions11.

Although these works have provided important contributions to synchronization theory, only oscillators with global coupling have been taken into account10–13. Thus, a natural extension of these works can investigate how these results change when different coupling schemes are introduced. Barlev et al.14 studied the dynamics of coupled phase oscillators, but such approach involved integrating $N$ ordinary differential equations. To overcome this limitation, in this report we generalize the Ott-Antonsen ansatz to complex networks in the continuum limit to investigate a time-dependent phase transition to synchronization. We reduce the dimension of the system of equations from $N$ to the number of possible degrees in the network.

Motivated by the results of the first-order Kuramoto model, we substantially extend the theory to the second-order Kuramoto model. The Kuramoto model with inertia has been widely used for deepening the understanding of power grids15–18, superconducting Josephson Junctions19 and many other applications20–22. Therefore a theory that investigates the low-dimensional character of such systems giving access to their time-dependent behavior can bring important new insights into the study of the second-order Kuramoto model. We substantially address this problem for what is perhaps the simplest choice of inertia term. In this case, the Fourier series expansion, the key approach of the Ott-Antonsen ansatz, no longer applies directly. Thus, a generalized framework for the second derivative needs to be developed, as already pointed out in recent studies23–25. In order to fill this gap, we derive self-consistent equations and seek the time evolution of the order parameter. Comparison of analytical and simulation results shows a good agreement. Our results shed light on the impact of the topology on the global dynamics.
Results

We consider the first-order Kuramoto model on an unweighted and undirected complex network. The state of oscillator \( i \) is denoted by its phase \( \theta_i \), and the governing equation of the model is

\[
\frac{d\theta_i}{dt} = \Omega_i + K \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i),
\]

where \( \Omega_i \) stands for the natural frequency of oscillator \( i \), which is distributed according to some probability density \( g(\Omega) \), \( K \) specifies the homogeneous coupling strength between interconnected nodes, and \( A_{ij} \) is the element of the adjacency matrix \( A \), i.e., \( A_{ij} = 1 \) if nodes \( i \) and \( j \) are connected or \( A_{ij} = 0 \), otherwise.

In uncorrelated networks, if \( N \) approaches infinity (in thermodynamic limit), the probability of selecting an edge connected to a node with degree \( k \), natural frequency \( \Omega \), and phase \( \theta \) at time \( t \) is \( kP(k)p(\Omega; \theta, t) \), where we define \( P(k) \) as the degree distribution and \( \rho(k; \Omega, \theta, t) \) as the probability distribution function of nodes with degree \( k \) that have natural frequency \( \Omega \) and phase \( \theta \) at time \( t \) and \( k \) the average degree.

To characterize the macroscopic behavior of the oscillators, in the continuum limit, we consider the order parameter (see Methods for details)

\[
r e^{i\psi} = \int d\Omega \int d\Omega \sum_{k=1}^{N} A_{ij} \sin(\theta_j - \theta_i) e^{i\theta_j} / \int dkP(k)k
\]

\[
= \int dkP(k)k \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) e^{i\theta_j} / \int dkP(k)k,
\]

where \( r_k \) and \( \psi_k \) evolve according to

\[
r_k e^{i\psi_k} = \int d\Omega \int d\Omega \rho(k; \Omega, \theta, t) e^{i\theta}.
\]

For simplicity, we assume that the natural frequencies \( \Omega_i \) are distributed according to an unimodal and symmetric Cauchy-Lorentz distribution \( g(\Omega) \) (see Methods for details) with zero mean. We set \( \psi_k = 0 \) without loss of generality. The coupling term in Eq. (1) can be written as \( \sum_{k=1}^{N} A_{ij} \sin(\theta_j - \theta_i) = k_r r_k e^{i\psi_k} \). Thus the governing Eq. (1) can be rewritten as

\[
\frac{d\theta_i}{dt} = \Omega_i + K r_k e^{-i\theta_i} - e^{i\theta_i}.
\]

which shows that the oscillators are coupled via the mean-field order parameter \( r \). The restoring force tends to bring each oscillator towards equilibrium and the amount of forcing is proportional to its degree \( k \).

The evolution of \( \rho(k; \Omega, \theta, t) \) is governed by the continuity equation, i.e., \( \partial \rho / \partial t + \partial \rho v / \partial \theta = 0 \), where \( v(k; \Omega, \theta, t) = d\theta / dt \). We use the Ott-Antonsen ansatz and expand the density function in a Fourier series, i.e.,

\[
\rho(k; \Omega, \theta, t) = \frac{g(\Omega)}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} [a(k; \Omega, t)]^n e^{in\theta} + c.c. \right],
\]

where \( c.c \) stands for the complex conjugate. Substituting the expansion into Eqs. (3) and in the continuity equation, we get that \( r_k = a(k) \) and \( r_k \) evolve according to

\[
\frac{dr_k}{dt} = -r_k + K r_k^2 \left( 1 - r_k^2 \right) \text{ for } k \in \left[ k_{\text{min}}, k_{\text{max}} \right]
\]

where \( k_{\text{min}} \) and \( k_{\text{max}} \) are the minimum and the maximum degree, respectively. This method works efficiently compared to the reference especially when the power law behavior has some cutoff. The global order parameter \( r \) is a sum of different \( r_k \) multiplied by their degree and degree distribution (see Eq. (2)).

To verify the accuracy of the time evolution of the order parameter \( r_k \) (see Eq. (6)), we compare the time evolution of the order parameter \( r \) with numerical simulations. Fig. 1 shows the results. Initially, the values of oscillators are selected at random from \( -\pi \) to \( \pi \), which implies that the initial value of each \( r_k(0) \) tends to zero. In our simulations, we set \( r_k(0) = 0.001 \). As we can see in Fig. 1, the results obtained through the solution of the reduced system in Eq. 6 are in good agreement with the numerical simulations.

The analysis above shows the remarkable usefulness of the Ott-Antonsen ansatz of the first-order Kuramoto model in complex net-

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Figure 1 | The order parameter as a function of time. Numerical simulations of the Kuramoto model are conducted on a scale-free network (see Methods for details). The coupling strength \( K = 2.5 \) and \( \theta \) are randomly selected from \( -\pi \) to \( \pi \) at \( t = 0 \).
works, but what happens when we consider the Kuramoto model with inertia? The simplest and most straightforward way is to include the extra inertia term. This leads to the mean-field character of the second-order Kuramoto model 15–18 with inertia. This leads to the mean-field character of the second-order Kuramoto model 15,16, nodes are divided into three groups. One group is an oscillator that is synchronized. The boundary of bistable region are specified by \(|\Omega|\) within \([4\sqrt{Kr}/\pi, 5\sqrt{Kr}/\pi] = [3.57, 5.78]\). Melnikov’s method 26 is used to show that oscillators are within a stable fixed point area as \(|\Omega| \leq 4\beta/\pi\); only limit-cycle oscillators exist for \(|\Omega| > 4\beta/\pi\). Let us first investigate the stationary states of phases \(\theta\) and \(\dot{\theta}\) in terms of the natural frequencies \(\Omega\) separately. In Fig. 2, every single point represents the state of one oscillator at time \(T(T=1)\) using simulations with \(N=10000\) nodes and degree \(K=10\). It is interesting to find that instead of three different regions mentioned above, the oscillators fall into either of the following two groups. (i) If the natural frequencies of nodes are within the boundary of the phase synchronization regime \([\Omega_{lower}, \Omega_{upper}] \equiv [-4\sqrt{Kr}/\pi, 4\sqrt{Kr}/\pi]\), which is the same as the above stable fixed points area, these nodes converge to fixed points and the stationary state of phases are functions of \(\Omega\), which are equal to \(\arcsin(\Omega/(Kr))\). This boundary is smaller than that of the Kuramoto model, in which oscillators in the locked state for all \(|\Omega| \leq K\). (ii) In contrast, the oscillators with \(|\Omega| > 4\sqrt{Kr}/\pi\) are drift. Thus, in networks, instead of three different areas of single pendulum model, only two distinct areas could exist: fixed point and limit cycle. Nodes with the same natural frequency are either converging to single fixed points or oscillating periodically; and nodes always return to previous states even after large perturbations.

To investigate how the phase synchronization boundary changes with different coupling strengths, we project the Fig. 2 on the \(I-\beta\) parameter space and color the oscillators according to their stationary states in the parameter space. A comparison between the dynamics with average degree 10 and that with 30 is shown in Fig. 3. We can see that oscillators with the same coupling share the same \(\beta\) axis and the diversity of \(I\) is due to the distribution of the natural frequencies \(\Omega\). All synchronized nodes are inside the synchronized area, which is at the right side of the line \(I = 4\beta/\pi\). Therefore, after substituting the boundaries of the synchronized natural frequencies \([\Omega_{lower}, \Omega_{upper}]\) and the Cauchy-Lorentz distribution into the definition of the order parameter \(r\),
\[ r = \int_{\theta_{\text{lower}}}^{\theta_{\text{upper}}} \cos(\theta_{s}) g(\Omega) d\Omega, \]  

(10)

where \( \theta_{s} \) denotes the synchronized oscillator \( \sin(\theta_{s}) = \lambda \). Performing some mathematical manipulations, we get

\[ r = \frac{2}{\pi K r} \left[ \sqrt{1 + (K r)^2} \arctan \left( \frac{4 \sqrt{K r} (K r^3)^3}{(\pi K r)^2 - 16 K r} \right) - \arcsin \left( \frac{4}{\pi \sqrt{K r}} \right) \right]. \]  

(11)

Due to the difference of boundaries between the first-order Kuramoto model (proportional to \( K \)) and the second-order Kuramoto model (proportional to \( 4\sqrt{K} / \pi \)), we set \( \Lambda(K) = 4\sqrt{K} / \pi \). When \( \dot{r} = 0 \),

\[ f(K, r) = r - 4\sqrt{K r} (1 - r^2) / (2\pi), \]  

(12)

and this stationary solution should be met by the self-consistent Eq. (11). Here, we use numerical methods to calculate the values of \( a, b \) and \( c \). As shown in Fig. 4, after substituting the stationary solutions \( K \) and \( r \) of Eq. (11) into Eq. (12), \( f(K, r) \) is colored in red and we get the values \( a = 0.389, b = 1/4 \) and \( c = 3 \). When \( r \) is small, \( f(K, r) \) is close to 0 and cannot influence the time evolution of the order parameter \( r(t) \), or vary the stationary solution, otherwise.
Let us consider again the nonlinear evolution of the order parameter \( r \) in complex networks. From the above analysis, we get that 
\[
\Lambda(K_k) \approx 4\sqrt{K_k}/\pi.
\]
To check the validity of this assumption, we compare the stationary solution with simulation results in Fig. 5. The theoretical predictions (green lines derived from Eq. (8) with effective coupling and \( f(K_k, r) \)) are in agreement with red lines of numerical simulations.

The nonlinear evolution of \( r(t) \) is illustrated in Fig. 6, for a selection of coupling strengths \( K \). Initial values of \( \theta_i \) and \( \dot{\theta}_i \) are the same as in Fig. 5. For the order parameter formulation the initial value of \( r \) is set to a small value (\( r(0) \ll 1 \)). The \( r \) formulation of Eq. (8) does not only reproduce the stationary states in Fig. 5, but also matches the transition to synchrony. The analytic results and simulation results are in good agreement.

**Conclusions**

In conclusion, we proposed a generalization for the Ott-Antonsen ansatz to complex networks with a Cauchy-Lorentz distribution of the natural frequency for the Kuramoto model. Compared to the ensemble approach\(^\text{14}\), the dimension of ordinary differential equations was reduced from \( N \) to the number of possible degrees in the network. We have investigated the collective dynamics of the Kuramoto model with inertia and found the synchronization bound-
ary is \([-4\sqrt{Kr}/\pi, 4\sqrt{Kr}/\pi]\) instead of \([-Kr, Kr]\) as in the Kuramoto model without inertia. Based on these results, we analytically derived self-consistent equations for the order parameter and nonlinear time-dependent order parameter. The agreement between the analytical and simulation results is excellent.

**Methods**

The networks. The model has been implemented on undirected and unweighted scale-free networks with \(N = 10000\), \(P(k) \propto k^{-1}\) and \(k \geq 5\).

**Numerical integration.** Eqs. (6) and (12) are solved by a 4th order Runge-Kutta method with time step \(h = 0.01\) and with the Cauchy-Lorentz distribution \(g(\Omega) = \frac{1}{\pi(1 + \Omega^2)}\).

**Order parameter.** In complex networks, in order to understand the dynamics of the system, it is natural to use the definition of order parameter \(r^2\) as \(r^2 = \frac{1}{N} \sum_{i,j} \langle c_{ij}\rangle\), instead of the definition \(r^2 = \frac{1}{N} \sum_{i} c_i^2\), which accounts for the mean-field in the fully connected graph regime.

The magnitude \(r \in [0, 1]\) quantifies the phase coherence, while \(\psi\) denotes the average phase of the system. In particular, \(r = 0\) if the phases are randomly distributed over \([0, 2\pi]\) and all nodes oscillate at its natural frequency. On the other hand, if all oscillators run as a giant component, \(r \geq 1\). The system is known to exhibit a phase transition from the asynchronous state \((r = 0)\) to the synchronous one \((r > 1)\) at a certain critical value \(\lambda_c\), characterizing the onset of partial synchronization and, for unimodal and symmetric frequency distributions \(g(\Omega)\), the transition is continuous. It turns out that for uncorrelated networks, \(\lambda_c\) is given by \(\lambda_c = \frac{2}{Ng(\Omega)}\lambda_{\text{max}}^2\), where \(\lambda_{\text{max}}\) is the maximal eigenvalue of the adjacency matrix.

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**Author contributions**

P. Ji, T. Peron, F.A. Rodrigues. and J. Kurths designed and performed the research, analyzed the results and wrote the paper.

**Additional information**

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