LINE BUNDLES ON THE MODULI SPACE OF PARABOLIC CONNECTIONS OVER A COMPACT RIEMANN SURFACE

ANOOP SINGH

Abstract. Let $X$ be a compact Riemann surface of genus $g \geq 3$ and $S$ a finite subset of $X$. Let $\xi$ be fixed a holomorphic line bundle over $X$ of degree $d$. Let $M_{pc}(r, d, \alpha)$ (respectively, $M_{pc}(r, \alpha, \xi)$) denote the moduli space of parabolic connections of rank $r$, degree $d$ and full flag rational generic weight system $\alpha$, (respectively, with the fixed determinant $\xi$) singular over the parabolic points $S \subset X$. Let $M'_{pc}(r, d, \alpha)$ (respectively, $M'_{pc}(r, \alpha, \xi)$) be the Zariski dense open subset of $M_{pc}(r, d, \alpha)$ (respectively, $M_{pc}(r, \alpha, \xi)$) parametrizing all parabolic connections such that the underlying parabolic bundle is stable. We show that there is a natural compactification of the moduli spaces $M'_{pc}(r, d, \alpha)$, and $M'_{pc}(r, \alpha, \xi)$ by smooth divisors. We describe the numerically effectiveness of these divisors at infinity. We determine the Picard group of the moduli spaces $M_{pc}(r, d, \alpha)$, and $M_{pc}(r, \alpha, \xi)$. Let $C(L)$ denote the space of holomorphic connections on an ample line bundle $L$ over the moduli space $M(r, d, \alpha)$ of parabolic bundles. We show that $C(L)$ does not admit any non-constant algebraic function.

1. Introduction and statements of the results

The moduli space of parabolic $\Lambda$-modules over a smooth projective curve has been constructed in [1], which is a generalization of the moduli space of $\Lambda$-modules constructed in [23], [24] by Simpson. The moduli space of parabolic connections over a smooth projective curve is the moduli space of parabolic $\Lambda$-modules where $\Lambda$ is the sheaf of rings of differential operators. The moduli space of parabolic connections is also constructed in [14].

Let $X$ be a compact Riemann surface of genus $g \geq 3$, and $S = \{x_1, \ldots, x_m\}$ the finite subset of $X$, which we call the set of parabolic points. Let $E$ be a holomorphic vector bundle over $X$ of rank $r$ and degree $d$. See section 2 for the definition of parabolic weights $\alpha$ (parabolic structure) on $E$, parabolic connections on a parabolic vector bundle $E_\alpha$, parabolic Higgs bundles and their moduli spaces. We consider the full flag rational generic parabolic weights. Let

$$\alpha = \{\alpha_{x_1}^1, \ldots, \alpha_{x_r}^r\}_{x \in S}$$

be the fixed generic system of rational parabolic weights corresponding to the full flag filtration such that

$$\sum_{x \in S} \sum_{i=1}^{r} \alpha_{x}^i \in \mathbb{Z}.$$
Let $\xi$ be a fixed holomorphic line bundle over $X$ of degree $d$ such that
\[ d = \deg \xi = -\sum_{x \in S} \sum_{i=1}^{r} \alpha_i x_i. \tag{1.1} \]

Let $\mathcal{M}(r, d, \alpha)$ (respectively, $\mathcal{M}(r, \alpha, \xi)$) denote the moduli space of stable parabolic vector bundles of rank $r$, degree $d$ and full flag rational generic system of weights $\alpha$ (respectively, with determinant $\xi$).

Let $\xi$ be given a trivial filtration at each $x \in S$ with parabolic weight $\beta_x := r \sum_{i=1}^{r} \alpha_i x_i$.

We denote this parabolic line bundle by $\xi_\ast$. Now, since $p \deg(\xi_\ast) = 0$, from [7, Theorem 3.1], $\xi_\ast$ admits a parabolic connection $D_{\xi_\ast}$ such that $\text{Res}(D_{\xi_\ast}, x) = \beta_x$ for every $x \in S$.

Let $\mathcal{M}_{pc}(r, d, \alpha)$ (respectively, $\mathcal{M}_{pc}(r, \alpha, \xi)$) be the moduli space of stable parabolic connections $(E_\ast, D)$ of rank $r$, degree $d$ and parabolic weight $\alpha$ (respectively, with $(\Lambda^r E_\ast, \tilde{D}) \cong (\xi_\ast, D_{\xi_\ast})$, where $\tilde{D}$ is the parabolic connection on the parabolic vector bundle $\Lambda^r E_\ast$ induced by $D$).

Let $\mathcal{M}'_{pc}(r, d, \alpha)$ (resp. $\mathcal{M}'_{pc}(r, \alpha, \xi)$) be the open dense subset of $\mathcal{M}_{pc}(r, d, \alpha)$ (resp. $\mathcal{M}_{pc}(r, \alpha, \xi)$) consists of those parabolic connections whose underlying vector bundle is stable.

In [15], the Picard group of the moduli space of parabolic vector bundles has been computed. Now using the techniques from [8], and [21], we show the following (see Theorem 3.3)
\[ \text{Pic}(\mathcal{M}_{pc}(r, d, \alpha)) \cong \text{Pic}(\mathcal{M}(r, d, \alpha)). \]

While proving the isomorphism between Picard groups, we show that there is a natural compactification of the moduli space $\mathcal{M}'_{pc}(r, d, \alpha)$ by a smooth divisor.

Let $L$ be an ample line bundle over $\mathcal{M}(r, d, \alpha)$, and $\mathcal{C}(L)$ denote the space of all holomorphic connections on $L$. Then, $\mathcal{C}(L)$ is a $T^\ast \mathcal{M}(r, d, \alpha)$-torsor, where $T^\ast \mathcal{M}(r, d, \alpha)$ is the cotangent bundle of $\mathcal{M}(r, d, \alpha)$. Then, we have the following (see Proposition 4.1)
\[ \text{Pic}(\mathcal{C}(L)) \cong \text{Pic}(\mathcal{M}(r, d, \alpha)). \]

Also, we show that the global regular functions on the variety $\mathcal{C}(L)$ are constant functions (see Theorem 4.2), that is,
\[ H^0(\mathcal{C}(L), \mathcal{O}_{\mathcal{C}(L)}) = \mathbb{C}. \]

Let $\text{At}(L)$ denote the Atiyah bundle of $L$ (see (4.5)). In order to prove the Theorem 4.2 we have shown the following
\[ H^0(\mathcal{M}, \text{Sym}^k \text{At}(L)) = \mathbb{C}, \]
for every $k \geq 0$, where $\text{Sym}^k \text{At}(L)$ denote the symmetric powers of the Atiyah bundle $\text{At}(L)$. 
Let $P(V)$ be the compactification of the moduli space $M_{pc}'(r, \alpha, \xi)$ (see Proposition \[5.1\]) with the complement

$$H_0 = P(V) \setminus M_{pc}'(r, \alpha, \xi)$$

a smooth divisor at infinity. Then, we prove that $H_0$ is numerically effective if and only if the tangent bundle $T_M(r, \alpha, \xi)$ is numerically effective (see Proposition \[5.2\]).

2. Preliminaries

We recall the notion of parabolic vector bundles and parabolic connections on a parabolic vector bundle over a compact Riemann surface. Let $X$ be a compact Riemann surface of genus $g \geq 3$. Let $S = \{x_1, \ldots, x_m\}$ be a finite subset of $X$, which we call the set of parabolic points. Let $E$ be a holomorphic vector bundle over $X$.

A **quasi-parabolic structure** on $E$ at a point $x \in X$ is a strictly decreasing flag $E_x = E_1^x \supseteq E_2^x \supseteq \cdots \supseteq E_k^x \supseteq E_{k+1}^x = 0$ of linear subspaces in the fibre $E_x$. We set

$$m_j^x = \dim_{\mathbb{C}}(E_j^x) - \dim_{\mathbb{C}}(E_{j+1}^x).$$

The integer $k$ is called the **length** of the flag and the $k$-tuple $(m_1^x, \ldots, m_k^x)$ is called the **type** of the flag. We say that the flag is a full flag if $m_j^x = 1$ for all $1 \leq j \leq k$. A **parabolic structure** in $E$ at $x$ is just a quasi-parabolic structure at $x$ together with a sequence of real numbers $0 \leq \alpha_1^x < \cdots < \alpha_k^x < 1$.

The real numbers $\alpha_j^x$ are called the **weights**. We denote by $\alpha = \{(\alpha_1^x, \ldots, \alpha_k^x)\}_{x \in S}$ the system of real weights corresponding to a fixed parabolic structure.

A **parabolic vector bundle** with parabolic structure on $S$ is a holomorphic vector bundle $E$ together with a parabolic structure in $E$ at each point $x \in S$. We shall write $E_*$ to denote the parabolic vector bundle with underlying vector bundle $E$.

For a parabolic bundle $E_*$ with system of weights $\alpha$ the **parabolic degree** is defined to be the real number

$$p \deg(E_*) = \deg(E) + \sum_{x \in S} \sum_{j=1}^{k} m_j^x \alpha_j^x,$$

where $\deg(E)$ denotes the degree of $E$, and we put

$$p\mu(E_*) = \frac{p \deg(E_*)}{\text{rk}(E)},$$

where $\text{rk}(E)$ is the rank of $E$. The real number $p\mu(E_*)$ is called the parabolic slope of $E_*$.

Let $E_*$ be a parabolic bundle and let $F$ be a vector subbundle of $E$. Then the parabolic structure on $E$ induces a parabolic structure on $F$ as follows: first we take the induced filtration on $F$. Next, for each $x \in S$ and for every $j \in \{1, \ldots, k_x\}$, we set $\alpha_j^x(F) := \alpha_j^x(E)$, where $i$ is the largest integer such that $F_i^x \subset E_i^x$ and $k_x$ is the length of the flag in $F_x$. The vector bundle $F$ together with this parabolic structure is denoted by $F_*$ and is called a parabolic subbundle of $E$. 


A parabolic bundle $E_*$ is said to be \textit{parabolic semistable} if for every non-zero proper parabolic subbundle $F_*$ we have

$$p\mu(F_*) \leq p\mu(E_*).$$ \hfill (2.1)

The parabolic bundle $E_*$ is said to be \textit{parabolic stable} if all the inequalities in (2.1) are strict.

In what follows, we consider the full flag filtration. We fix the integer $s \geq 1$ and $d \in \mathbb{Z}$. We can represent the system of weights $\alpha = \{\alpha^x_1, \ldots, \alpha^x_r\}_{x \in S}$ in matrix form $(\alpha^x_1)_{1 \leq i \leq m}$ \hfill (2.2)

In [14, Definition 2.2], Inaba defined the system of special weights, and the system of weights which is not special is called generic. See [5, section 2.2] for the non-emptiness of the space $W(m,d)$ of admissible parabolic weights.

Let

$$\alpha = (\alpha^x_1, \ldots, \alpha^x_r)_{x \in S} \in W(m,d) \quad \text{be the fixed system of generic rational parabolic weights corresponding to the full flag filtration.}$$

Let

$$\mathcal{M}^{ss}(r,d,\alpha)$$

be the moduli space of semi-stable parabolic vector bundles of rank $r$, degree $d$ and weight system $\alpha$ (see [16]). The moduli space $\mathcal{M}^{ss}(r,d,\alpha)$ is a normal projective variety of dimension

$$\dim(\mathcal{M}^{ss}(r,d,\alpha)) = r^2(g-1) + 1 + \frac{m(r^2-r)}{2}.$$  

Moreover, the moduli space $\mathcal{M}(r,d,\alpha)$ of stable parabolic bundles is an smooth and open subset of $\mathcal{M}^{ss}(r,d,\alpha)$.

Let $\xi$ be a fixed holomorphic line bundle over $X$ of degree $d$ such that

$$d = \deg \xi = -\sum_{x \in S} \sum_{i=1}^r \alpha^x_i.$$ \hfill (2.2)

Let $\mathcal{M}^{ss}(r,\alpha,\xi)$ be the moduli space of semi-stable parabolic vector bundles on $X$ of rank $r$ and determinant $\xi$, that is, $\bigwedge^r E \cong \xi$, with weight system $\alpha$. Then $\mathcal{M}^{ss}(r,\alpha,\xi)$ is a projective variety of dimension (see [3, p.n. 557])

$$(r^2 - 1)(g-1) + \frac{m(r^2-r)}{2}.$$  

Let $\mathcal{M}(r,\alpha,\xi) \subset \mathcal{M}^{ss}(r,\alpha,\xi)$ be the open subset parametrizing the stable parabolic bundles. This is an irreducible smooth variety.

Since the full flag weight system $\alpha$ is generic (see [2, Definition 2.2]), we have

$$\mathcal{M}(r,d,\alpha) := \mathcal{M}^{ss}(r,d,\alpha) = \mathcal{M}(r,d,\alpha),$$

and

$$\mathcal{M}(r,\alpha,\xi) := \mathcal{M}^{ss}(r,\alpha,\xi) = \mathcal{M}(r,\alpha,\xi).$$
Let $E_*$ be a parabolic bundle. We say that an endomorphism $\phi : E \rightarrow E$ is strongly parabolic if for every $x \in S$, we have

$$\phi(E^i_x) \subset E^{i+1}_x.$$ 

Similarly, $\phi : E \rightarrow E$ is said to be weakly parabolic or just parabolic if it satisfies

$$\phi(E^i_x) \subset E^i_x.$$ 

The sheaf of strongly parabolic endomorphism on $E_*$ is denoted by $\text{SParEnd}(E_*)$ and the sheaf of parabolic endomorphism on $E_*$ is denoted by $\text{ParEnd}(E_*)$.

Now, we recall the notion of logarithmic connections (see [6] and [11]) and its residues, and using these notions we define parabolic connections. Let $E$ be a holomorphic vector bundle over $X$. Let $S = x_1 + \cdots + x_m$ be the reduced effective divisor associated with $S$. A logarithmic connection on $E$ singular over $S$ is a $\mathbb{C}$-linear map

$$D : E \rightarrow E \otimes \Omega^1_X(S) = E \otimes \Omega^1_X \otimes \mathcal{O}_X(S)$$

which satisfies the Leibniz identity

$$D(fs) = fD(s) + df \otimes s,$$

where $f$ is a local section of $\mathcal{O}_X$ and $s$ is a local section of $E$.

We next describe the notion of residues of a logarithmic connection $D$ in $E$ singular over $S$.

Let $v \in E_{x_\beta}$ be any vector in the fiber of $E$ over $x_\beta \in S$. Let $U$ be an open set around $x_\beta$ and $s : U \rightarrow E$ be a holomorphic section of $E$ over $U$ such that $s(x_\beta) = v$. Consider the following composition

$$\Gamma(U, E) \rightarrow \Gamma(U, E \otimes \Omega^1_X \otimes \mathcal{O}_X(S)) \rightarrow (E \otimes \Omega^1_X \otimes \mathcal{O}_X(S))_{x_\beta} = E_{x_\beta},$$

where the equality is given because for any $x_\beta \in S$, the fibre $(\Omega^1_X \otimes \mathcal{O}_X(S))_{x_\beta}$ is canonically identified with $\mathbb{C}$ by sending a meromorphic form to its residue at $x_\beta$. Then, we have an endomorphism on $E_{x_\beta}$ sending $v$ to $D(s)(x_\beta)$. We need to check that this endomorphism is well defined. Let $s' : U \rightarrow E$ be another holomorphic section such that $s'(x_\beta) = v$. Then

$$(s - s')(x_\beta) = v - v = 0.$$ 

Let $t$ be a local coordinate at $x_\beta$ on $U$ such that $t(x_\beta) = 0$, that is, the coordinate system $(U, t)$ is centered at $x_\beta$. Since $s - s' \in \Gamma(U, E)$ and $(s - s')(x_\beta) = 0$, $s - s' = t\sigma$ for some $\sigma \in \Gamma(U, E)$. Now,

$$D(s - s') = D(t\sigma) = tD(\sigma) + dt \otimes \sigma = tD(\sigma) + t\left(\frac{dt}{t} \otimes \sigma\right),$$

and hence $D(s - s')(x_\beta) = 0$, that is, $D(s)(x_\beta) = D(s')(x_\beta)$.

Thus, we have a well defined endomorphism, denoted by

$$\text{Res}(D, x_\beta) \in \text{End}(E)_{x_\beta} = \text{End}(E_{x_\beta})$$

(2.5)
that sends \( v \) to \( D(s)(x_\beta) \). This endomorphism \( \text{Res}(D, x_\beta) \) is called the residue of the logarithmic connection \( D \) at the point \( x_\beta \in S \) (see [11] for the details).

From [18, Theorem 3], for a logarithmic connection \( D \) singular over \( S \), we have

\[
\deg E + \sum_{j=1}^{m} \text{Tr}(\text{Res}(D, x_j)) = 0,
\]

where, \( \deg E \) denotes the degree of \( E \), and \( \text{Tr}(\text{Res}(D, x_j)) \) denote the trace of the endomorphism \( \text{Res}(D, x_j) \in \text{End}(E_{x_j}) \), for all \( j = 1, \ldots, m \).

Let \( E_* \) be a parabolic vector bundle over \( X \) with a fixed system of weights \( \alpha \) for full flag filtration. A parabolic connection (for the group GL(\( r, \mathbb{C} \))) on \( E_* \) is a logarithmic connection \( D \) on the underlying vector bundle \( E \) satisfying following conditions:

1. For each \( x \in S \) the homomorphism induced in the filtration over the fibre \( E_x \) satisfies
   \[
   D(E^i_x) \subset E^i_x \otimes \Omega^1_X(S)|_x
   \]
   for every \( i = 1, \ldots, r \).

2. For every \( x \in S \) and for every \( i = 1, \ldots, r \) the action of \( \text{Res}(D, x) \) on the quotient \( E^i_x/E^{i+1}_x \) is the multiplication by \( \alpha^x_i \). Since \( \text{Res}(D, x) \) preserves the filtration it acts on each quotient.

We denote the parabolic connections by the pair \((E_*, D)\).

Let \( \xi \) be the fixed line bundle of degree \( d \) and weight \( \alpha \) such that

\[
d = \deg \xi = - \sum_{x \in S} \sum_{i=1}^{r} \alpha^x_i.
\]

We want to fix a parabolic connection on \( \xi \). We first equip \( \xi \) with a parabolic structure as follows. Let \( \xi \) be given a trivial filtration at each \( x \in S \) with parabolic weight

\[
\beta^x := \sum_{i=1}^{r} \alpha^x_i.
\]

We denote \( \xi \) with a parabolic structure by \( \xi_* \). Then, from (2.8), the parabolic degree of \( \xi_* \) is

\[
p \deg(\xi_*) = \deg(\xi) + \sum_{x \in S} \beta^x = \deg(\xi) + \sum_{x \in S} \sum_{i=1}^{r} \alpha^x_i = 0.
\]

Since \( p \deg(\xi_*) = 0 \), from [7, Theorem 3.1], \( \xi_* \) admits a parabolic connection \( D_{\xi_*} \) such that \( \text{Res}(D_{\xi_*}, x) = \beta(x) \) for every \( x \in S \). Now, we fix a pair \((\xi_*, D_{\xi_*})\) as described above.

Next, given a pair \((E_*, D)\), we say that \( D \) is a parabolic connection (for the group SL(\( r, \mathbb{C} \))) if the logarithmic connection

\[
\text{Tr}(D) : \bigwedge^r E \longrightarrow \bigwedge^r E \otimes \Omega^1_X(S)
\]

induced from \( D \) coincides with \( D_{\xi_*} \), that is, we have an isomorphism

\[
(\bigwedge^r E_*, \text{Tr}(D)) \cong (\xi_*, D_{\xi_*}).
\]

(2.9)
A parabolic connection \((E_*, D)\) (for the group \(\text{GL}(r, \mathbb{C})\) or \(\text{SL}(r, \mathbb{C})\)) is said to be semi-stable (respectively, stable) if for every non-zero proper parabolic subbundle \(F_*\) of \(E_*\), which is invariant under \(D\), that is, \(D(F) \subset F \otimes \Omega^1_X(S)\), we have
\[
p\mu(F_*) \leq p\mu(E_*) \quad \text{(respectively,}<)\.
\]
Note that \((E_*, D)\) is stable does not imply that \(E_*\) is stable.

Recall that a parabolic connection \((E_*, D)\) is said to be irreducible if there does not exist a non-zero proper parabolic subbundle of \(E_*\) which is invariant under \(D\).

We have an useful result as follows.

**Lemma 2.1.** Suppose that a parabolic connection \((E_*, D)\) is irreducible. Then \((E_*, D)\) is stable.

Let \((E_*, D)\) and \((E'_*, D')\) be two full flag parabolic connections with same generic weight \(\alpha\). A morphism between parabolic connections \((E_*, D)\) and \((E'_*, D)\) is a parabolic morphism (already defined)
\[
\phi: E_* \longrightarrow E'_*
\]
of parabolic vector bundles such that the following diagram involving logarithmic connections
\[
\begin{array}{c}
E \xrightarrow{D} E \otimes \Omega^1_X(S) \\
\phi \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \phi \otimes 1 \Omega^1_X(S) \\
E' \xrightarrow{D'} E' \otimes \Omega^1_X(S)
\end{array}
\tag{2.10}
\]
commutes. Moreover, we say that \((E_*, D)\) and \((E'_*, D')\) are isomorphic if \(\phi\) is an isomorphism.

**Lemma 2.2.** Let \((E_*, D)\) and \((E'_*, D')\) be semi-stable full flag parabolic connections over \(X\) with same generic weight \(\alpha\). Then we have the followings.

1. Suppose \((E_*, D)\) and \((E'_*, D')\) are stable and \(p\mu(E_*) = p\mu(E'_*)\). If
   \[
   \phi: (E_*, D) \rightarrow (E'_*, D')
   \]
is a non-zero morphism of parabolic connections, then it is an isomorphism.

2. If \((E_*, D)\) is stable, then the only parabolic endomorphisms of the pair \((E_*, D)\) are scalars.

*Proof.*

(1) First note that a subbundle and its quotient bundle of a parabolic vector bundle admits a parabolic structure from its original parabolic bundle (see the remark after Definition 1.8 of [14]). We shall use usual technique of Kernel-Image and Image-Coimage sequences to prove the first part.

Let \(\text{Ker}(\phi)_* \subset E_*\) is a parabolic subbundle. Then, \(\text{Ker}(\phi)_*\) is \(D\)-invariant parabolic subbundle of \(E\). Since \((E_*, D)\) is stable, we have \(p\mu(\text{Ker}(\phi)_*) < p\mu(E_*)\). Since \(\phi \neq 0\), we have \(\text{Im}(\phi) \neq 0\). Now, \(\text{Im}(\phi)\) inherits a parabolic structure from \(E'_*\).

Consider the Kernel-Image short exact sequence
\[
0 \rightarrow \text{Ker}(\phi)_* \rightarrow E_* \rightarrow \text{Im}(\phi)_* \rightarrow 0.
\]
Then, $p\mu(E_\ast) < p\mu(\text{Im}(\phi)_\ast)$. Next, consider the Image-Coimage short exact sequence

$$0 \to \text{Im}(\phi)_\ast \to E'_\ast \to (E'/\text{Im}(\phi))_\ast \to 0.$$ 

Note that $\text{Im}(\phi)_\ast$ is a $D'$-invariant parabolic subbundle of $E'$. Since $(E'_\ast, D')$ is stable, we have $p\mu(\text{Im}(\phi)_\ast) < p\mu(E'_\ast)$. Thus, we get that $p\mu(E_\ast) < p\mu(E'_\ast)$ which contradicts the assumption that $p\mu(E_\ast) = p\mu(E'_\ast)$. Therefore, on underlying vector bundles, we have $\text{Ker}(\phi) = 0$ and $\text{Im}(\phi) = E'$.

(2) Let $\psi : (E_\ast, D) \to (E_\ast, D)$ be the given endomorphism. Let $x \in X$, and let $\lambda \in \mathbb{C}$ be an eigen value of the linear map $\psi(x) : E_x \to E_x$. Note that $X$ is a compact Riemann surface, it does not admit any non-constant holomorphic function, therefore the eigen values and their multiplicities are independent of $x \in X$. Since $D$ is $\mathbb{C}$-linear, $\psi - \lambda \mathbf{1}_{E_\ast}$ is an endomorphism of $(E_\ast, D)$, that is,

$$D \circ (\psi - \lambda \mathbf{1}_{E_\ast}) = (\psi - \lambda \mathbf{1}_{E_\ast}) \otimes 1_{\Omega^1_X(S)} \circ D.$$ 

Since $(E_\ast, D)$ is stable, from [1], $\psi - \lambda \mathbf{1}_{E_\ast}$ is either a zero morphism or an isomorphism. Since $\lambda$ is an eigen value of $\psi(x)$, kernel of $\psi - \lambda \mathbf{1}_{E_\ast}$ is non-trivial. Thus, $\psi = \lambda \mathbf{1}_{E_\ast}$, where $\lambda \in \mathbb{C}$.

Let $\mathcal{M}_{\text{pc}}^{ss}(r, d, \alpha)$ denote the moduli space of semi-stable parabolic connections for the group $\text{GL}(r, \mathbb{C})$ of rank $r$, degree $d$ and weight system $\alpha$. Let

$$\mathcal{M}_{\text{pc}}^{sm}(r, d, \alpha) \subset \mathcal{M}_{\text{pc}}^{ss}(r, d, \alpha)$$

be the smooth locus of $\mathcal{M}_{\text{pc}}^{ss}(r, d, \alpha)$. Let $\mathcal{M}_{\text{pc}}(r, d, \alpha)$ be the moduli space of stable parabolic connections. Then, from [14, Theorem 2.1], $\mathcal{M}_{\text{pc}}(r, d, \alpha)$ is a smooth irreducible quasi-projective variety of dimension $2r^2(g - 1) + 2 + m(r^2 - r)$, and hence $\mathcal{M}_{\text{pc}}(r, d, \alpha) \subset \mathcal{M}_{\text{pc}}^{sm}(r, d, \alpha)$.

Let

$$\iota : \mathcal{M}_{\text{pc}}'(r, d, \alpha) \hookrightarrow \mathcal{M}_{\text{pc}}(r, d, \alpha) \quad (2.11)$$

be the natural inclusion, where $\mathcal{M}_{\text{pc}}'(r, d, \alpha)$ is the open subvariety consists of the pairs $(E_\ast, D)$ whose underlying vector bundle $E_\ast$ is stable.

Let

$$\pi : \mathcal{M}_{\text{pc}}'(r, d, \alpha) \longrightarrow \mathcal{M}(r, d, \alpha) \quad (2.12)$$

be a map defined by sending $(E_\ast, D)$ to $E_\ast$. In other words, $\pi$ is the forgetful map which forgets the parabolic connection. Note that $\pi$ is a surjective morphism follows from [7, Theorem 3.1]. Now, for every $E_\ast \in \mathcal{M}(r, d, \alpha)$, $\pi^{-1}(E_\ast)$ is an affine space modelled over $H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}(E_\ast))$ which is described as follows. Given two parabolic connections $D$ and $D'$ on a parabolic vector bundle $E_\ast$, the difference $D - D'$ is an $\mathcal{O}_X$-module homomorphism from $E$ to $E \otimes \Omega^1_X(S)$ such that the residues

$$\text{Res}(D, x) - \text{Res}(D', x) = \text{Res}(D - D', x)$$

acts as the zero morphism on each successive quotients $E^i_x / E^{i+1}_x$. Therefore, for every $x \in S$, we have

$$(D - D')(E^i_x) \subset E^{i+1}_x \otimes \Omega^1_X(S)|_x.$$
Conversely, given any parabolic connection \((E, D)\) and
\[
\Phi \in H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}(E_*)),
\]
\(D + \Phi\) is again a parabolic connection on \(E_*\). Thus, the space \(\pi^{-1}(E_*)\) of parabolic connections on \(E_*\) forms an affine space modelled over the vector space \(H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}(E_*))\).

Now, by deformation theory, the tangent space \(T_{E_*} \mathcal{M}(r, d, \alpha)\) at \(E_*\) is isomorphic to \(H^1(X, \text{ParEnd}(E_*))\). By parabolic version of Serre duality, we have
\[
H^1(X, \text{ParEnd}(E_*))^\vee \cong H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}(E_*)),
\] (2.13)
therefore the cotangent space \(T^*_{E_*} \mathcal{M}(r, d, \alpha)\) at \(E_*\) is isomorphic to \(H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}(E_*))\).

Thus, \(T^*_{E_*} \mathcal{M}(r, d, \alpha)\) acts on the fibre \(\pi^{-1}(E_*)\) faithfully and transitively, which proves the following Lemma.

**Lemma 2.3.** The moduli space \(\mathcal{M}'_{pc}(r, d, \alpha)\) is a \(T^* \mathcal{M}(r, d, \alpha)\)-torsor.

Let \(\mathcal{M}_{pc}(r, d, \xi)\) be the moduli space of stable parabolic connections (for the group \(\text{SL}(r, \mathbb{C})\)) of rank \(r\), degree \(d\), generic weight system \(\alpha\), and with fixed determinant \((\xi, D_\xi)\).

The moduli space \(\mathcal{M}_{pc}(r, d, \xi)\) is a smooth irreducible quasi-projective variety of dimension \(2(g - 1)(r^2 - 1) + m(r^2 - r)\) (see [14, Proposition 5.1]).

Let
\[
\mathcal{M}'_{pc}(r, \alpha, \xi) \subset \mathcal{M}_{pc}(r, \alpha, \xi)
\] (2.14)
be the subset containing \((E_*, D)\) whose underlying parabolic vector bundle \(E_*\) is stable. Then, from [17, Theorem 2.8(A)], \(\mathcal{M}'_{pc}(r, \alpha, \xi)\) is a Zariski open subset of \(\mathcal{M}_{pc}(r, \alpha, \xi)\).

Let \(\text{SParEnd}'(E_*)\) and \(\text{ParEnd}'(E_*)\) denote respectively the sheaves of trace zero strongly and weakly parabolic endomorphisms on \(E_*\).

Let
\[
\pi_\xi : \mathcal{M}'_{pc}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \alpha, \xi)
\] (2.15)
be the map defined by sending \((E_*, D)\) to \(E_*\), that is, \(\pi_\xi\) is the forgetful map. Again from [7, Theorem 3.1], \(\pi_\xi\) is a surjective morphism. Now, for every \(E_* \in \mathcal{M}(r, \alpha, \xi)\), \(\pi^{-1}_\xi(E_*)\) is an affine space modelled over \(H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}'(E_*))\) which is described as follows. Given two parabolic connections \(D\) and \(D'\) on a parabolic vector bundle \(E_*\), the difference \(D - D'\) is an \(O_X\)-module homomorphism from \(E\) to \(E \otimes \Omega^1_X(S)\) such that the residues
\[
\text{Res}(D, x) - \text{Res}(D', x) = \text{Res}(D - D', x)
\]
acts as the zero morphism on each successive quotients \(E^i_x/E^{i+1}_x\). Therefore, for every \(x \in S\), we have
\[
(D - D')(E^i_x) \subset E^{i+1}_x \otimes \Omega^1_X(S)|_x.
\]
Moreover, since the determinant of parabolic connections are fixed, we have
\[
\text{Tr}(D - D') = 0.
\]
Conversely, given any parabolic connection \((E_*, D)\) and 
\[ \Phi \in H^0(X, \Omega_X^1(S) \otimes \text{SParEnd}'(E_*)), \]
that is, \(\Phi\) is strongly parabolic morphism with \(\text{Tr}(\Phi) = 0\). Then, \(D + \Phi\) is again a parabolic connection on \(E_*\). Thus, the space \(\pi^{-1}(E_*)\) of parabolic connections on \(E_*\) with fixed determinant forms an affine space modelled over the vector space \(H^0(X, \Omega_X^1(S) \otimes \text{SParEnd}'(E_*))\).

Now, by deformation theory, the tangent space \(T_{E_*} M(r, \alpha, \xi)\) at \(E_*\) is isomorphic to \(H^1(X, \text{ParEnd}'(E_*))\). By parabolic version of Serre duality, we have
\[ H^1(X, \text{ParEnd}'(E_*))^\vee \cong H^0(X, \Omega_X^1(S) \otimes \text{SParEnd}'(E_*)), \tag{2.16} \]
therefore the cotangent space \(T_{E_*}^* M(r, \alpha, \xi)\) at \(E_*\) is isomorphic to \(H^0(X, \Omega_X^1(S) \otimes \text{SParEnd}'(E_*))\).

Thus, \(T_{E_*}^* M(r, \alpha, \xi)\) acts on the fibre \(\pi^{-1}_x(E_*)\) faithfully and transitively, which proves the following Lemma.

**Lemma 2.4.** The moduli space \(M_{\text{par}}(r, \alpha, \xi)\) is a \(T^* M(r, \alpha, \xi)\)-torsor.

A **strongly** parabolic Higgs bundle \((E_*, \Phi)\) is a parabolic vector bundle \(E_*\) together with an \(\mathcal{O}_X\)-module homomorphism 
\[ \Phi : E \longrightarrow E \otimes \Omega_X^1(S) \]
such that for every \(x \in S\) the homomorphism induced in the filtration over the fibre \(E_x\) satisfies
\[ \Phi(E_x^i) \subset E_x^{i+1} \otimes \Omega_X^1(S)|_x, \]
where \(\Omega_X^1(S)\) is the line bundle \(\Omega_X^1 \otimes \mathcal{O}_X(S)\). \(\Phi\) is called parabolic Higgs field on \(E_*\).

Similarly, we can define **weakly** parabolic Higgs bundle to be a pair \((E_*, \Phi)\) where \(E_*\) is a parabolic vector bundle and
\[ \Phi : E \longrightarrow E \otimes \Omega_X^1(S) \]
is an \(\mathcal{O}_X\)-module homomorphism such that for every \(x \in S\), we have
\[ \Phi(E_x^i) \subset E_x^i \otimes \Omega_X^1(S)|_x. \]

A parabolic Higgs bundle \((E_*, \Phi)\) (either strongly or weakly), is said to be semi-stable (respectively, stable) if for every non-zero proper parabolic subbundle \(F_*\) of \(E_*\), which is invariant under \(\Phi\), that is, \(\Phi(F) \subset F \otimes \Omega_X^1(S)\), we have
\[ p\mu(F) \leq p\mu(E_*) \] (respectively, <).

If we do not say explicitly if a parabolic Higgs bundle is strongly or weakly parabolic, it will be understood that it is strongly parabolic.

Let \(M_{\text{Higgs}}(r, d, \alpha)\) be the moduli space of semi-stable (strongly) parabolic Higgs bundles of rank \(r\), degree \(d\), weight system \(\alpha\) [23], [12]. Then, \(M_{\text{Higgs}}(r, d, \alpha)\) is an irreducible normal quasi-projective variety of dimension \(2r(g - 1) + 2 + m(r^2 - r)\) (see [12] p.n. 432). If \(\alpha\) is a generic system of weights, then every semi-stable parabolic Higgs bundle is stable, and the moduli space of stable Higgs bundle lies in the smooth locus of \(M_{\text{Higgs}}(r, d, \alpha)\). As we have assumed \(\alpha\) is generic system of weights, the moduli space \(M_{\text{Higgs}}(r, d, \alpha)\) is smooth.
Let $\mathcal{M}'_{\text{Higgs}}(r, d, \alpha) \subset \mathcal{M}_{\text{Higgs}}(r, d, \alpha)$ be the subset consists of those parabolic Higgs bundles whose underlying parabolic bundle is stable. As described above and from \cite{13},

$$T^* \mathcal{M}(r, d, \alpha) \cong \mathcal{M}'_{\text{Higgs}}(r, d, \alpha). \tag{2.17}$$

Let $Z := \mathcal{M}_{\text{Higgs}}(r, d, \alpha) \setminus \mathcal{M}'_{\text{Higgs}}(r, d, \alpha)$. Then, in view of \cite{10} Proposition 5.10 for $r \geq 3$, we have

$$\text{codim}(Z, \mathcal{M}_{\text{Higgs}}(r, d, \alpha)) \geq 2. \tag{2.18}$$

Now, we recall the parabolic Hitchin map on the moduli space of strongly parabolic Higgs bundles \cite{12}, \cite{19}.

Set $(\Omega^1_X)^i S^{(i-1)} := (\Omega^1_X)^i \otimes \mathcal{O}_X(S)^{\otimes (i-1)}$, where $i = 1, \ldots, r$, and consider the vector space

$$\mathcal{H}_P = \bigoplus_{i=1}^r \mathcal{H}^0(X, (\Omega^1_X)^i S^{(i-1)}).$$

Then the dimension $\dim \mathcal{H}_P$ is half of the dimension of the moduli space $\mathcal{M}_{\text{Higgs}}(r, d, \alpha)$ \cite{12} Section 3, p.n. 433, that is, $r^2(g-1) + 1 + \frac{1}{2}m(r^2 - r)$.

Define the parabolic Hitchin map

$$h_P : \mathcal{M}_{\text{Higgs}}(r, d, \alpha) \longrightarrow \mathcal{H}_P \tag{2.19}$$

by sending each (strongly) parabolic Higgs bundle $(E_\ast, \Phi)$ to the characteristic polynomial of $\Phi$. Then, the $h_P$ is a proper morphism \cite{25} Corollary 5.12, and for any generic point $a \in \mathcal{H}_P$, the fibre $h_P^{-1}(a)$ is an abelian variety (see \cite{12} Lemma 3.2 and \cite{19} Theorem 6).

### 3. Picard group of the moduli spaces

The Picard group of moduli spaces is a very important invariant while studying the classification problems in algebraic geometry. In this section, we compute the Picard group of the moduli spaces $\mathcal{M}_{\text{pc}}(r, d, \alpha)$.

We first compute the dimension of the space of isomorphic stable parabolic connections on a parabolic vector bundle. Let $E_\ast$ be a full flag parabolic vector bundle over $X$ with a fixed $\alpha \in W_r^{(m)}(d)$.

Let $\text{Conn}_\alpha(E_\ast)$ denote the space of all parabolic connections $D$ on $E_\ast$ such that $(E_\ast, D)$ is stable. Notice that $\text{Conn}_\alpha(E_\ast)$ is an affine space modelled over the vector space $H^0(X, \Omega^1_X(S) \otimes \text{ParEnd}(E_\ast))$.

Given a parabolic automorphism $\Phi$ of $E_\ast$ and a parabolic connection $D$ on $E_\ast$, the $\mathbb{C}$-linear morphism $\Phi \otimes 1_{\Omega^1_X(S)} \circ D \circ \Phi^{-1}$ defines a parabolic connection on $E_\ast$. In fact,

$$(D, \Phi) \mapsto \Phi \otimes 1_{\Omega^1_X(S)} \circ D \circ \Phi^{-1}$$

defines a natural action of $\text{Aut}(E_\ast)$ on $\text{Conn}_\alpha(E_\ast)$, called gauge transformation. We would like to compute the dimension of the quotient space $\text{Conn}_\alpha(E_\ast)/\text{Aut}(E_\ast)$, which parametrizes all isomorphic parabolic connections on $E_\ast$. The Lie algebra of the holomorphic automorphism group $\text{Aut}(E_\ast)$ is $H^0(X, \text{ParEnd}(E_\ast))$. Therefore,

$$\dim \text{Aut}(E_\ast) = \dim H^0(X, \text{ParEnd}(E_\ast)).$$
Choose any $D \in \text{Conn}_\alpha(E_\ast)$. Since the pair $(E_\ast, D)$ is stable, from Lemma 2.2 the isotropy subgroup

$$\text{Aut}(E_\ast)_D = \{ \Phi \in \text{Aut}(E_\ast) \mid \Phi \otimes \mathbf{1}_{\Omega_X(S)} \circ D \circ \Phi^{-1} = D \}$$

consists of the scalar automorphisms of $E_\ast$. Then, the dimension of the space $\text{Conn}_\alpha(E_\ast)/\text{Aut}(E_\ast)$ is

$$\dim H^0(X, \Omega_X^1(S) \otimes \text{SParEnd}(E_\ast)) - \dim H^0(X, \text{ParEnd}(E_\ast)) + 1$$

$$= \dim H^1(X, \text{ParEnd}(E_\ast)) - \dim H^0(X, \text{ParEnd}(E_\ast)) + 1$$

$$= -\chi(\text{ParEnd}(E_\ast)) + 1,$$

(3.1)

where the first equality is due to Parabolic Serre duality and $\chi(\text{ParEnd}(E_\ast))$ denotes the Euler-Poincaré characteristic of $\text{ParEnd}(E_\ast)$ over $X$.

Clearly $\text{ParEnd}(E_\ast)$ is a subsheaf of $\text{End}(E)$. Then, there is a natural skyscraper sheaf $\mathcal{K}_S$ supported on parabolic points $S$ such that

$$0 \to \text{ParEnd}(E_\ast) \to \text{End}(E) \to \mathcal{K}_S \to 0$$

is a short exact sequence of sheaves on $X$. Then, we get

$$\chi(\text{End}(E)) = \chi(\text{ParEnd}(E_\ast)) + \chi(\mathcal{K}_S)$$

(3.2)

Since we are considering full flag filtrations, from [5, Lemma 2.4], we have

$$\chi(\mathcal{K}_S) = \frac{m(r^2 - r)}{2}. $$

(3.3)

Moreover, from Riemann-Roch Theorem, we have

$$\chi(\text{End}(E)) = r^2(1 - g).$$

(3.4)

Therefore, from (3.1), (3.2), (3.3) and (3.4), we get that the dimension of the space $\text{Conn}_\alpha(E_\ast)/\text{Aut}(E_\ast)$ is

$$r^2(g - 1) + 1 + \frac{m(r^2 - r)}{2}.$$

Lemma 3.1. Let $E_\ast$ be a full flag stable parabolic vector bundle of rank $r$ and $d$ with a fixed $\alpha \in W_r^{(m)}(d)$. Then

$$\text{Conn}_\alpha(E_\ast)/\text{Aut}(E_\ast) = \text{affine space over the vector space } H^0(X, \Omega_X^1(S) \otimes \text{SParEnd}(E_\ast)),$$

and dimension of the space is equal to $r^2(g - 1) + 1 + \frac{m(r^2 - r)}{2}$.

Proof. Since $E_\ast$ is stable parabolic bundle, we have $H^0(X, \text{ParEnd}(E_\ast)) = \mathbb{C} \cdot \mathbf{1}_{E_\ast}$. Now, Lemma follows from above discussion. \qed

Let

$$Z_{pc} := \mathcal{M}_{pc}(r, d, \alpha) \setminus \mathcal{M}'_{pc}(r, d, \alpha).$$

Then, using the similar steps as in [9, Lemma 3.1], we can show the following lemma.

Lemma 3.2. Let $\alpha \in W_r^{(m)}(d)$. Then, for $r \geq 2$, and $g \geq 3$, we have

$$\text{codim}(Z_{pc}, \mathcal{M}_{pc}(r, d, \alpha)) \geq 2.$$
Proof. Let $(E_*, D) \in Z_{pc}$. Then, $E_*$ is not stable. Since $\alpha$ is generic, $E_*$ is not semi-stable. Let

$$0 = E^0_* \subset E^1_* \subset E^2_* \subset \cdots \subset E^{l-1}_* \subset E^l_* = E_*$$

be the Harder-Narasimhan filtration of the parabolic vector bundle $E_*$. The collection of pairs of integers $\{(\text{rk}(E^i_*), p\deg E^i_*)\}_{i=1}^l$ is called the Harder-Narasimhan polygon of $E_*$ (see [20, p.n.178]).

Now, analogous to the techniques in [9, Lemma 3.1] (p.n. 303), the space of all isomorphism classes of parabolic vector bundles over $X$ whose Harder-Narasimhan polygon coincides with the given parabolic vector bundle $E_*$ is of dimension at most

$$r^2(g-1) - (r-1)(g-2) + \frac{m(r^2 - r)}{2}.$$

We have already computed that the dimension of the space of all isomorphism classes of parabolic connections, lying in $\mathcal{M}_{pc}(r, d, \alpha)$, on any given parabolic vector bundle $E'_*$ (assuming that parabolic connection exists on $E'_*$) which is

$$r^2(g-1) + 1 + \frac{m(r^2 - r)}{2}.$$

The subvariety of $\mathcal{M}_{pc}(r, d, \alpha)$ parametrizing all pairs of the form $(E'_*, D') \in \mathcal{M}_{pc}(r, d, \alpha)$ such that the Harder-Narasimhan polygon of the parabolic vector bundle $E'_*$ coincides with that of $E_*$ is at most of dimension

$$r^2(g-1) - (r-1)(g-1) + 1 + m(r^2 - r)$$

$$= 2r^2(g-1) - (r-1)(g-1) + 1 + m(r^2 - r).$$

Since $\dim \mathcal{M}_{pc}(r, d, \alpha) = 2r^2(g-1) + 2 + m(r^2 - r)$, we have

$$\dim \mathcal{M}_{pc}(r, d, \alpha) - \left[ 2r^2(g-1) - (r-1)(g-1) + 1 + m(r^2 - r) \right] = (r-1)(g-2) + 1.$$ 

Thus, $\operatorname{codim}(Z_{pc}, \mathcal{M}_{pc}(r, d, \alpha)) \geq (r-1)(g-2) + 1$. \hfill $\square$

The morphism in (2.11) induces a morphism of Picard groups

$$\iota^* : \operatorname{Pic}(\mathcal{M}_{pc}(r, d, \alpha)) \longrightarrow \operatorname{Pic}(\mathcal{M}'_{pc}(r, d, \alpha))$$

defined by restricting any line bundle $\eta$ over $\mathcal{M}_{pc}(r, d, \alpha)$ to $\mathcal{M}'_{pc}(r, d, \alpha)$.

Also, the morphism $\pi$ in (2.12) induces a morphism of Picard groups

$$\pi^* : \operatorname{Pic}(\mathcal{M}(r, d, \alpha)) \longrightarrow \operatorname{Pic}(\mathcal{M}'_{pc}(r, d, \alpha))$$

defined by pulling back of line bundles, that is, $\eta \mapsto \pi^*\eta$, where $\eta$ is a line bundle over $\mathcal{M}'_{pc}(r, d, \alpha)$.

Theorem 3.3. The morphisms defined in (3.5) and (3.6)

$$\operatorname{Pic}(\mathcal{M}(r, d, \alpha)) \xrightarrow{\pi^*} \operatorname{Pic}(\mathcal{M}'_{pc}(r, d, \alpha)) \xleftarrow{\iota^*} \operatorname{Pic}(\mathcal{M}_{pc}(r, d, \alpha))$$

are isomorphisms.
Proof. The morphism $\iota^*$ defined in (3.5) is an isomorphism follows from the fact that $\text{codim}(Z_{pc}, \mathcal{M}_{pc}(r, d, \alpha)) \geq 2$ as proved in Lemma 3.2.

Now, we show that $\pi^*$ is an isomorphism. From Lemma 2.3, the moduli space $\mathcal{M}_{pc}'(r, d, \alpha)$ is $T^*\mathcal{M}(r, d, \alpha)$-torsor. We use this fact to compactify the moduli space $\mathcal{M}_{pc}'(r, d, \alpha)$, and using this compactification, we show that the Picard group $\text{Pic}(\mathcal{M}_{pc}'(r, d, \alpha))$ is isomorphic to $\text{Pic}(\mathcal{M}(r, d, \alpha))$.

We have seen that for any $E_* \in \mathcal{M}(r, d, \alpha)$, the fibre $\pi^{-1}(E_*)$ is an affine space modelled on $H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}(E_*))$. We know that the dual of an affine space is a vector space, therefore the dual

$$
\pi^{-1}(E_*)^\vee = \{ \varphi : \pi^{-1}(E_*) \rightarrow \mathbb{C} | \varphi \text{ is an affine linear map} \}
$$

is a vector space over $\mathbb{C}$.

Let

$$
\psi : \mathcal{W} \rightarrow \mathcal{M}(r, d, \alpha)
$$

be the algebraic vector bundle such that for every Zariski open subset $U$ of $\mathcal{M}(r, d, \alpha)$, a section of $\mathcal{W}$ over $U$ is an algebraic function $f : \pi^{-1}(U) \rightarrow \mathbb{C}$ whose restriction to each fiber $\pi^{-1}(E_*)$, is an element of $\pi^{-1}(E_*)^\vee$.

Thus, a fibre $\mathcal{W}(E_*) = \psi^{-1}(E_*)$ of $\mathcal{W}$ at $E_* \in \mathcal{M}(r, d, \alpha)$ is $\pi^{-1}(E_*)^\vee$. The dimension of $\pi^{-1}(E_*)^\vee$ is equal to $\dim_{\mathbb{C}} \pi^{-1}(E_*) + 1$, and since $E_*$ is stable, the dimension of $\pi^{-1}(E_*)^\vee$ is equal to

$$
r^2(g - 1) + \frac{m(r^2 - r)}{2} + 2,
$$

follows from Lemma 3.1.

Let $(E_*, D) \in \mathcal{M}_{pc}'(r, d, \alpha)$, and define a map

$$
\Psi_{(E_*, D)} : \pi^{-1}(E_*)^\vee \rightarrow \mathbb{C},
$$

by $\Psi_{(E_*, D)}(\varphi) = \varphi[(E_*, D)]$, that is in fact the evaluation map. Now, the kernel $\text{Ker}(\Psi_{(E_*, D)})$ defines a hyperplane in $\pi^{-1}(E_*)^\vee$ denoted by $H_{(E_*, D)}$.

Let $\mathbf{P}(\mathcal{W})$ be the projective bundle defined by hyperplanes in the fibre $\pi^{-1}(E_*)^\vee$, that is, we have

$$
\tilde{\psi} : \mathbf{P}(\mathcal{W}) \rightarrow \mathcal{M}(r, d, \alpha) \tag{3.7}
$$

induced from $\psi$.

Next, define a map

$$
i : \mathcal{M}_{pc}'(r, d, \alpha) \rightarrow \mathbf{P}(\mathcal{W}) \tag{3.8}
$$

by sending $(E_*, D)$ to the equivalence class of $H_{(E_*, D)}$, which is an open embedding. Set

$$
\mathbf{H} = \mathbf{P}(\mathcal{W}) \setminus \mathcal{M}_{pc}'(r, d, \alpha). \tag{3.9}
$$

Then $\tilde{\psi}^{-1}(E_*) \cap \mathbf{H}$ is a projective hyperplane in $\tilde{\psi}^{-1}(E_*)$ for every $E_* \in \mathcal{M}(r, d, \alpha)$, and hence $\mathbf{H}$ is a hyperplane at infinity.

We first show that $\pi^*$ is injective. Let $\eta \rightarrow \mathcal{M}(r, d, \alpha)$ be a line bundle such that $\pi^* \eta$ is a trivial line bundle over $\mathcal{M}_{pc}'(r, d, \alpha)$. A trivialization of $\pi^* \eta$ is equivalent to have a nowhere vanishing section of $\pi^* \eta$ over $\mathcal{M}_{pc}'(r, d, \alpha)$. Fix $s \in H^0(\mathcal{M}_{pc}'(r, d, \alpha), \pi^* \eta)$
Choose a point $E_* \in \mathcal{M}(r, d, \alpha)$. Then, from the following commutative diagram

$$
\begin{array}{ccc}
\pi^* \eta & \xrightarrow{\tilde{\pi}} & \eta \\
\downarrow & & \downarrow \\
\mathcal{M}'_{\text{pc}}(r, d, \alpha) & \xrightarrow{\pi} & \mathcal{M}(r, d, \alpha)
\end{array}
$$

we get

$$ s|_{\pi^{-1}(E_*)} : \pi^{-1}(E_*) \rightarrow \eta(E_*) $$

da nowhere vanishing map. Notice that $\pi^{-1}(E_*) \cong \mathbb{C}^N$ and $\eta(E_*) \cong \mathbb{C}$, where $N = r^2(g-1) + \frac{m(r^2-r)}{2} + 1$. Now, any nowhere vanishing algebraic function on an affine space $\mathbb{C}^N$ is a constant function, that is, $s|_{\pi^{-1}(E_*)}$ is a constant function and hence corresponds to a non-zero vector $\alpha_{E_*} \in \eta(E_*)$. Since $s$ is constant on each fiber of $\pi$, the trivialization $s$ of $\pi^* \eta$ descends to a trivialization of the line bundle $\eta$ over $\mathcal{M}(r, d, \alpha)$, and hence giving a nowhere vanishing section of $\eta$ over $\mathcal{M}(r, d, \alpha)$. Thus, $\eta$ is a trivial line bundle over $\mathcal{M}(r, d, \alpha)$.

It remains to show that $\pi^*$ is surjective. Let $\vartheta \rightarrow \mathcal{M}'_{\text{pc}}(r, d, \alpha)$ be an algebraic line bundle. Since $\mathbb{P}(W)$ is a smooth compactification of $\mathcal{M}'_{\text{pc}}(r, d, \alpha)$ follows from the embedding $i : \mathcal{M}'_{\text{pc}}(r, d, \alpha) \hookrightarrow \mathbb{P}(W)$ in (3.8), the homomorphism of Picard groups

$$ i^* : \text{Pic}(\mathbb{P}(W)) \rightarrow \text{Pic}(\mathcal{M}'_{\text{pc}}(r, d, \alpha)) $$

defined by pull back of line bundles via $i$ in (3.8), is a surjective homomorphism. Therefore, we can extend $\vartheta$ to a line bundle $\vartheta'$ over $\mathbb{P}(W)$. Again from the morphism $\tilde{\psi} : \mathbb{P}(W) \rightarrow \mathcal{M}(r, d, \alpha)$ in (3.7) and from [13], Chapter III, Exercise 12.5, p.n. 291, we have

$$ \text{Pic}(\mathbb{P}(W)) \cong \tilde{\psi}^* \text{Pic}(\mathcal{M}(r, d, \alpha)) \oplus \mathbb{Z} \mathcal{O}_{\mathbb{P}(W)}(1). \quad (3.11) $$

Therefore,

$$ \vartheta' = \tilde{\psi}^* \Lambda \otimes \mathcal{O}_{\mathbb{P}(W)}(l) \quad (3.12) $$

where $\Lambda$ is a line bundle over $\mathcal{M}$ and $l \in \mathbb{Z}$. Since $H = \mathbb{P}(W) \setminus \mathcal{M}'_{\text{pc}}(r, d, \alpha)$ is the hyperplane at infinity, using (3.11) the line bundle $\mathcal{O}_{\mathbb{P}(W)}(H)$ associated to the divisor $H$ can be expressed as

$$ \mathcal{O}_{\mathbb{P}(W)}(H) = \tilde{\psi}^* \Gamma \otimes \mathcal{O}_{\mathbb{P}(W)}(1) \quad (3.13) $$

for some line bundle $\Gamma$ over $\mathcal{M}(r, d, \alpha)$. Now, from (3.12) and (3.13), we get

$$ \vartheta' = \tilde{\psi}^* (\Lambda \otimes (\Gamma^\vee)^{\otimes l}) \otimes \mathcal{O}_{\mathbb{P}(W)}((lH)). $$

Since, the restriction of the line bundle $\mathcal{O}_{\mathbb{P}(W)}(H)$ to the complement $\mathbb{P}(W) \setminus H = \mathcal{M}'_{\text{pc}}(r, d, \alpha)$

is the trivial line bundle and restriction of $\tilde{\psi}$ to $\mathcal{M}'_{\text{pc}}(r, d, \alpha)$ is the map $\pi$ defined in (2.15), therefore, we have

$$ \vartheta = \pi^* (\Lambda \otimes (\Gamma^\vee)^{\otimes l}). $$

This completes the proof of the theorem.
Remark 3.4. There is an alternative way of proving that $\pi^*$ as defined in (3.6) is an isomorphism, without using the compactification $\mathbf{P}(W)$ of $\mathcal{M}_{pc}'(r, d, \alpha)$. Recall that if we have an algebraic morphism between varieties $f : \mathcal{X} \to \mathcal{S}$ such that the fibres are geometrically connected, then there exists an exact sequence

$$1 \to \text{Pic}(\mathcal{S}) \to \text{Pic}(\mathcal{X}) \to \text{Pic}_{\mathcal{X}/\mathcal{S}}(\mathcal{S}) \to \text{Br}(\mathcal{S}),$$

(3.14)

of groups, where $\text{Pic}_{\mathcal{X}/\mathcal{S}}(\mathcal{S})$ denote the group of section of relative Picard scheme $\text{Pic}_{\mathcal{X}/\mathcal{S}}$ over $\mathcal{S}$, and $\text{Br}(\mathcal{S})$ denote the Brauer group of $\mathcal{S}$. In our set up, we have following exact sequence corresponding to the morphism $\pi$ defined in (2.15)

$$1 \to \text{Pic}(\mathcal{M}(r, d, \alpha)) \to \text{Pic}(\mathcal{M}_{pc}'(r, d, \alpha)) \to \text{Pic}_{\mathcal{M}_{pc}'(r, d, \alpha)/\mathcal{M}(r, d, \alpha)}(\mathcal{M}(r, d, \alpha))$$

(3.15)

Since, each fibre of the morphism $\pi$ (see (2.15)) is an affine space, the group $\text{Pic}_{\mathcal{M}_{pc}'(r, d, \alpha)/\mathcal{M}(r, d, \alpha)}(\mathcal{M}(r, d, \alpha))$ is trivial.

Similarly, we can compactify (see Proposition 5.1) the moduli space $\mathcal{M}_{pc}'(r, \alpha, \xi)$ and we can show the following.

Proposition 3.5. $\text{Pic}(\mathcal{M}_{pc}'(r, \alpha, \xi)) \cong \text{Pic}(\mathcal{M}(r, \alpha, \xi))$.

4. Space of holomorphic connections on an ample line bundle

In this section, we assume that $r \geq 3$, and consider the space of holomorphic connections on an ample line bundle $L$ over the moduli space $\mathcal{M}(r, d, \alpha)$. For the simplicity of the notation, we just write $\mathcal{M}$ for the moduli space $\mathcal{M}(r, d, \alpha)$.

Let $L$ be an ample holomorphic vector bundle over $\mathcal{M}$. Let $\Omega^1_{\mathcal{M}}$ denote the sheaf of holomorphic 1-forms on $\mathcal{M}$. Consider the space $\mathcal{C}(L)$ of holomorphic connections on $L$, that is, for every analytic open subset $U$ of $\mathcal{M}$, $\mathcal{C}(L)|_U$ consists of following operators

$$\nabla : L|_U \to L|_U \otimes \Omega^1_{\mathcal{M}}|_U$$

satisfying the Leibniz rule

$$\nabla(fs) = f\nabla(s) + df \otimes s,$$

where $f$ is a holomorphic function on $U \subset \mathcal{M}$, and $s$ is a holomorphic section of $L$ over $U$. Then, there is a natural projection morphism

$$\Psi : \mathcal{C}(L) \to \mathcal{M} := \mathcal{M}(r, d, \alpha).$$

(4.1)

For any analytic open subset $U$ of $\mathcal{M}$, $\Psi^{-1}(U)$ is an affine space modelled over the vector space $H^0(U, \Omega^1_{\mathcal{M}})$. Thus, the space $\mathcal{C}(L)$ is a $T^*\mathcal{M}$-torsor. Now, using the similar technique as above we can show the following.

Proposition 4.1. $\text{Pic}(\mathcal{C}(L)) \cong \text{Pic}(\mathcal{M})$.

Next, we want to study the space of regular function on $\mathcal{C}(L)$. We recall the definition of differential operators of finite order on $L$. Let $k \geq 0$ be an integer. A differential operator of order $k$ on $L$ is a $\mathbb{C}$-linear map

$$Q : L \to L$$

(4.2)
such that for every open subset $U$ of $\mathcal{M}$ and for every $f \in \mathcal{O}_\mathcal{M}(U)$, the bracket
\[ [Q|_U, f] : L|_U \to L|_U \]
defined as
\[ [Q|_U, f]|_V(s) = Q_V(f|_V s) - f|_V Q_V(s) \]
is a differential operator of order $k - 1$, for every open subset $V$ of $U$, and for all $s \in L(V)$, where differential operator of order zero from $L$ to $L$ is just $\mathcal{O}_\mathcal{M}$-module homomorphism.

Let $\text{Diff}^k_\mathcal{M}(L, L)$ denote the set of all differential operators of order $k$. Then $\text{Diff}^k_\mathcal{M}(L, L)$ is an $\mathcal{O}_\mathcal{M}(\mathcal{M})$-module. For every open subset $U$ of $\mathcal{M}$,
\[ U \mapsto \text{Diff}^k_\mathcal{M}(L|_U, L|_U) \]
is a sheaf of differential operator of order $k$ from $L|_U$ to $L|_U$. This sheaf is denoted by $\mathcal{D}^k_\mathcal{M}(L, L)$, which is locally free. For $k \geq 0$, we denote by $\mathcal{D}^k(L)$ the vector bundle over $\mathcal{M}$ defined by the sheaf $\mathcal{D}^k_\mathcal{M}(L, L)$. Note that $\mathcal{D}^0(L) = \mathcal{O}_\mathcal{M}$ and we have following inclusion of vector bundles
\[ \mathcal{O}_\mathcal{M} = \mathcal{D}^0(L) \subset \cdots \subset \mathcal{D}^k(L) \subset \mathcal{D}^{k+1}(L) \subset \cdots \] (4.3)

Let $Q$ be a first order differential operator on $L$. Define a map
\[ \sigma(Q) : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M} = \text{End}(L) \] (4.4)
by
\[ \sigma(Q)(f) = [Q|_U, f] \]
for every open subset $U \subset \mathcal{M}$, and $f \in \mathcal{O}_\mathcal{M}(U)$. Then $\sigma(Q)$ is a $\mathbb{C}$-derivation, that is, $\sigma(Q)$ gives a section of $T\mathcal{M}$. Thus, we get a short exact sequence of vector bundles, called the Atiyah exact sequence [3]
\[ 0 \to \mathcal{O}_\mathcal{M} \xrightarrow{i} \text{At}(L) := \mathcal{D}^1(L) \xrightarrow{\sigma} T\mathcal{M} \to 0, \] (4.5)
where $\text{At}(L)$ is called the Atiyah bundle of $L$.

**Theorem 4.2.** Let $L$ be an ample line bundle over $\mathcal{M}$, and $\mathcal{O}_{\mathcal{C}(L)}$ denote the sheaf of regular functions on $\mathcal{C}(L)$. Then
\[ H^0(\mathcal{C}(L), \mathcal{O}_{\mathcal{C}(L)}) = \mathbb{C} \] (4.6)

**Proof.** Consider the dual of the Atiyah exact sequence in (4.5), that is,
\[ 0 \to T^*\mathcal{M} \xrightarrow{\sigma^*} \text{At}(L)^* \xrightarrow{\iota^*} \mathcal{O}_\mathcal{M} \to 0. \] (4.7)

Consider $\mathcal{O}_\mathcal{M}$ as the trivial line bundle $\mathcal{M} \times \mathbb{C}$. Let
\[ \alpha : \mathcal{M} \to \mathcal{M} \times \mathbb{C} \]
be a holomorphic section of the trivial line bundle defined by $x \mapsto (x, 1)$. Let $Y = \text{Im}(\alpha) \subset \mathcal{M} \times \mathbb{C}$ be the image of $\alpha$. Then
\[ \mathcal{C}(L) = \iota^*-1 Y \subset \text{At}(L)^*. \]

In fact, for every open subset $U \subset \mathcal{M}$, a holomorphic section of $\mathcal{C}(L)|_U$ over $U$ gives a holomorphic splitting of the Atiyah exact sequence (4.5), associated to the holomorphic vector bundle $L|_U \to U$. For instance, suppose $\tau : U \to \mathcal{C}(L)|_U$ is
a holomorphic section. Then \( \tau \) will be a holomorphic section of \( \text{At}(L)^*|_U \) over \( U \), because \( \mathcal{C}(L) = \iota^* \gamma \subset \text{At}(L)^* \). Since
\[
\tau \circ \iota = \iota^*(\tau) = 1_U,
\]
so we get a holomorphic splitting \( \tau \) of the Atiyah exact sequence \( (4.5) \) associated to \( L|_U \). Thus, \( L|_U \) admits a holomorphic connection. Conversely, given any splitting of Atiyah exact sequence \( (4.5) \) over an open subset \( U \subset \mathcal{M} \), we get a holomorphic section of \( \mathcal{C}(L)|_U \) over \( U \).

Let \( \mathbf{P}(\text{At}(L)) \) be the projectivization of \( \text{At}(L) \), that is, \( \mathbf{P}(\text{At}(L)) \) parametrises hyperplanes in \( \text{At}(L) \). Let \( \mathbf{P}(T\mathcal{M}) \) be the projectivization of the tangent bundle \( T\mathcal{M} \). Notice that \( \mathbf{P}(T\mathcal{M}) \) is a subvariety of \( \mathbf{P}(\text{At}(L)) \), and \( \mathbf{P}(T\mathcal{M}) \) is the zero locus of the of a section of the tautological line bundle \( \mathbf{O}_{\mathbf{P}(\text{At}(L))}(1) \). Now, observe that
\[
\mathcal{C}(L) = \mathbf{P}(\text{At}(L)) \setminus \mathbf{P}(T\mathcal{M}).
\]
So we have
\[
H^0(\mathcal{C}(L), \mathbf{O}_{\mathcal{C}(L)}) = \varinjlim_{k \geq 0} H^0(\mathbf{P}(\text{At}(L)), \mathbf{O}_{\mathbf{P}(\text{At}(L))}(k)). \tag{4.8}
\]

Since for any finite dimensional vector space \( V \) over \( \mathbb{C} \) and for every \( k \geq 0 \), we have \( H^0(\mathbf{P}(V), \mathbf{O}_{\mathbf{P}(V)}(k)) = \text{Sym}^k(V) \), where \( \text{Sym}^k(V) \) denote the \( k \)-th symmetric powers of \( V \). We get a natural isomorphism
\[
H^0(\mathbf{P}(\text{At}(L)), \mathbf{O}_{\mathbf{P}(\text{At}(L))}(k)) \cong H^0(\mathcal{M}, \text{Sym}^k \text{At}(L)),
\]
where \( \text{Sym}^k \text{At}(L) \) denote the \( k \)-the symmetric powers of \( \text{At}(L) \), and hence from \( (4.8) \), we get
\[
H^0(\mathcal{C}(L), \mathbf{O}_{\mathcal{C}(L)}) = \varinjlim_{k \geq 0} H^0(\mathcal{M}, \text{Sym}^k \text{At}(L)). \tag{4.9}
\]

The symbol operator
\[
\sigma : \text{At}(L) \to T\mathcal{M}
\]
as described in \( (4.3) \), induces a morphism
\[
\text{Sym}^k \sigma : \text{Sym}^k \text{At}(L) \to \text{Sym}^k T\mathcal{M}
\]
on \( k \)-th symmetric powers of bundles. In view of the following composition
\[
\text{Sym}^{k-1} \text{At}(L) = \mathbf{O}_\mathcal{M} \otimes \text{Sym}^{k-1} \text{At}(L) \hookrightarrow \text{At}(L) \otimes \text{Sym}^{k-1} \text{At}(L) \to \text{Sym}^k \text{At}(L),
\]
we have
\[
\text{Sym}^{k-1} \text{At}(L) \subset \text{Sym}^k \text{At}(L) \text{ for all } k \geq 1.
\]
In fact, we have the symbol exact sequence associated to \( L \) over \( \mathcal{M} \),
\[
0 \to \text{Sym}^{k-1} \text{At}(L) \to \text{Sym}^k \text{At}(L) \xrightarrow{\text{Sym}^k \sigma} \text{Sym}^k T\mathcal{M} \to 0. \tag{4.10}
\]
In other words, we get a filtration
\[
0 \subset \text{Sym}^0 \text{At}(L) \subset \text{Sym}^1 \text{At}(L) \subset \ldots \subset \text{Sym}^{k-1} \text{At}(L) \subset \text{Sym}^k \text{At}(L) \subset \ldots
\]
such that
\[
\text{Sym}^k \text{At}(L)/\text{Sym}^{k-1} \text{At}(L) \cong \text{Sym}^k T\mathcal{M} \text{ for all } k \geq 1. \tag{4.11}
\]
To prove (4.6), it is enough to show that
\[ H^0(M, \text{Sym}^{k-1} \text{At}(L)) \cong H^0(M, \text{Sym}^k \text{At}(L)) \text{ for all } k \geq 1. \] (4.12)

We have the following commutative diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Sym}^{k-1} \text{At}(L) & \longrightarrow & \text{Sym}^k \text{At}(L) & \longrightarrow & \text{Sym}^k TM & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \sigma & & \downarrow & & \\
0 & \longrightarrow & \text{Sym}^{k-1} TM & \longrightarrow & \frac{\text{Sym}^k \text{At}(L)}{\text{Sym}^{k-2} \text{At}(L)} & \longrightarrow & \text{Sym}^k TM & \longrightarrow & 0
\end{array}
\] (4.13)

which gives the following commutative diagram of long exact sequences
\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & H^0(M, \text{Sym}^k TM) & \longrightarrow & H^1(M, \text{Sym}^{k-1} \text{At}(L)) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^0(M, \text{Sym}^k TM) & \longrightarrow & H^1(M, \text{Sym}^{k-1} TM) & \longrightarrow & \cdots
\end{array}
\] (4.14)

In order to show (4.6), in view of (4.9), it is enough to prove that the boundary operator \( \delta'_k \) is injective for all \( k \geq 1 \) and which is equivalent to showing that the boundary operator
\[ \delta_k : H^0(M, \text{Sym}^k TM) \rightarrow H^1(M, \text{Sym}^{k-1} TM) \] (4.15)
is injective for every \( k \geq 1 \).

Further a connecting homomorphism can be expressed as the cup product by the extension class of the corresponding short exact sequence. Denote the extension class of the following short exact sequence
\[ 0 \rightarrow \text{Sym}^{k-1} (TM) \rightarrow \frac{\text{Sym}^k \text{At}(L)}{\text{Sym}^{k-2} \text{At}(L)} \rightarrow \text{Sym}^k (TM) \rightarrow 0 \] (4.16)
by \( \gamma_k \).

Moreover, \( \gamma_k \) can be expressed in terms of the first Chern class \( c_1(L) \), because the first Chern class of \( c_1(L) \) is nothing but the extension class of the following Atiyah exact sequence (see [3])
\[ 0 \rightarrow \mathcal{O}_M \rightarrow \text{At}(L) \rightarrow TM \rightarrow 0, \] (4.17)
and the short exact sequence (4.10) is the \( k \)-th symmetric power of the Atiyah exact sequence (4.17).

Thus, the connecting homomorphism \( \delta_k \) can be described using the first Chern class \( c_1(L) \in H^1(M, T^*M) \) of the line bundle \( L \).

The cup product with \( kc_1(L) \) gives rise to a homomorphism
\[ \mu : H^0(M, \text{Sym}^k TM) \rightarrow H^1(M, \text{Sym}^k TM \otimes T^*M) \]
Also, we have a canonical homomorphism of vector bundles
\[ \beta : \text{Sym}^k TM \otimes T^*M \rightarrow \text{Sym}^{k-1} TM \]
which induces a morphism of $\mathbb{C}$-vector spaces

$$\beta^*: H^1(M, Sym^k TM \otimes T^* M) \to H^1(M, Sym^{k-1} TM).$$

So, we get a morphism

$$\tilde{\mu} = \beta^* \circ \mu : H^0(M, Sym^k TM) \to H^1(M, Sym^{k-1} TM).$$

Then $\tilde{\mu} = \delta_k$. Now, it is enough to show that $\tilde{\mu}$ is injective.

Consider the natural projection

$$p : T^* M \to M,$$

where $T^* M$ denote the cotangent bundle of $M$. Then, we have

$$H^0(T^* M, O_{T^* M}) = H^0(M, p_* O_{T^* M}).$$

(4.18)

Moreover, using the projection formula we have

$$p_* O_{T^* M} = \bigoplus_{k \geq 0} Sym^k TM. \quad (4.19)$$

Using (4.18) and (4.19) we get

$$H^0(T^* M, O_{T^* M}) = \bigoplus_{k \geq 0} H^0(M, Sym^k TM) \quad (4.20)$$

Now, to compute $H^0(T^* M, O_{T^* M})$ we use the Hitchin fibration for the moduli space of strongly parabolic Higgs bundles as defined in (2.19). Recall that for any generic point $a \in H^P$, we have $h^{-1}_P(a) = A$, where $A$ is some abelian variety (for more details see [19], [10]), and we will be using this fact showing that $\tilde{\mu}$ is injective.

Let $g : T^* M \to \mathbb{C}$ be an algebraic function. Since $T^* M$ is an open subset of $M_{Higgs}(r, d, \alpha)$ such that the complement has codimension at least 2 (see (2.18)), from Hartog’s theorem, the algebraic function $g$ is extended to an algebraic function

$$\tilde{g} : M_{Higgs}(r, d, \alpha) \to \mathbb{C}.$$

Then its restriction $\tilde{g}|_{h^{-1}_P(a)} : h^{-1}_P(a) \to \mathbb{C}$ to $h^{-1}_P(a)$ is an algebraic function. Since $h^{-1}_P(a)$ is an abelian variety, $\tilde{g}$ is a constant function. Therefore, on generic fibre $h^{-1}(a)$, $\tilde{g}$ is constant, and $h_P$ is proper, hence gives an algebraic function on $H_P$. Thus, any algebraic function on $T^* M$ descends to an algebraic function on $H_P$.

Set $\mathcal{P} = d(H^0(H_P, O_{H_P})) \subset H^0(H_P, \Omega^1_{H_P})$ the space of all exact algebraic 1-form. Define a map

$$\theta : H^0(T^* M, O_{T^* M}) \to \mathcal{P} \quad (4.21)$$

by $g \mapsto d\tilde{g}$, where $\tilde{g}$ is the function which is defined by descent of $g$. Then $\theta$ is an isomorphism.

From (4.20) and (4.21), we have

$$\theta : \bigoplus_{k \geq 0} H^0(M, Sym^k TM) \to \mathcal{P} \quad (4.22)$$

which is an isomorphism. Let us denote the restriction of $h_P$ on $T^* M$ by $h'_P$.

Let $T_{h'_P} = T_{T^* M/H_P} = Ker(dh'_P)$ be the relative tangent sheaf on $T^* M$, where

$$dh'_P : T(T^* M) \to h^*_P T H_P$$
morphism of bundles.

Note that \(H^0(\mathcal{H}_P, \Omega^1_{\mathcal{H}_P}) \subset H^0(T^*\mathcal{M}, T_{h'_P}),\) and hence from (4.22), we have an injective homomorphism

\[\nu: \mathcal{P} = \bigoplus_{k \geq 0} \theta(H^0(\mathcal{M}, \text{Sym}^k T\mathcal{M})) \to H^0(T^*\mathcal{M}, T_{h'_P}).\]

Consider the morphism

\[H^0(T^*\mathcal{M}, T_{h'_P}) \to H^1(T^*\mathcal{M}, T_{h'_P} \otimes T^*T^*\mathcal{M})\]

defined by taking cup product with the first Chern class \(c_1(p^*L) \in H^1(T^*\mathcal{M}, T^*T^*\mathcal{M}).\)

Using the pairing

\[T_{h'_P} \otimes T^*T^*\mathcal{M} \to O_{T^*\mathcal{M}},\]

we get a homomorphism

\[\eta: H^0(T^*\mathcal{M}, T_{h'_P}) \to H^1(T^*\mathcal{M}, O_{T^*\mathcal{M}})\]

Since \(c_1(p^*L) = p^*c_1(L),\) we have

\[k\eta \circ \nu \circ \theta(\omega_k) = \tilde{\mu}(\omega_k), \quad (4.23)\]

for all \(\omega_k \in H^0(\mathcal{M}, \text{Sym}^k T\mathcal{M})).\) Since \(\nu\) and \(\theta\) are injective homomorphisms, it is enough to show that \(\eta|_{\nu(\mathcal{P})}\) is injective homomorphism.

Let \(\omega \in \mathcal{P} \setminus \{0\}\) be a non-zero exact 1-form. Choose \(a \in \mathcal{H}_P\) such that \(\omega(a) \neq 0.\) In view of (2.17) and (2.18), the generic fibre has the form

\[h^{-1}_P(a) = A \setminus F,\]

where \(A\) is an abelian variety and \(F\) is a subvariety of \(A\) such that \(\text{codim}(F, A) \geq 2.\)

Now, \(\eta(\nu(\omega)) \in H^1(T^*\mathcal{M}, O_{T^*\mathcal{M}})\) and we have restriction map

\[H^1(T^*\mathcal{M}, O_{T^*\mathcal{M}}) \to H^1(h^{-1}_P(a), O_{h^{-1}_P(a)}).\]

Since \(\omega(a) \neq 0,\) \(\eta(\nu(\omega)) \in H^1(h^{-1}_P(a), O_{h^{-1}_P(a)}).\) Because of the following isomorphisms

\[H^1(h^{-1}_P(a), O_{h^{-1}_P(a)}) \cong H^1(A, O_A) \cong H^0(A, TA),\]

it follows that \(\eta(\nu(\omega)) \neq 0.\) This completes the proof. \(\square\)

Similar steps involved in the above Theorem 4.2, and using (4.3), we have generalisation of [21 Corollary 2.3] in parabolic set up.

**Corollary 4.3.** Let \(L\) be an ample line bundle over \(\mathcal{M}(r, d, \alpha).\) Then,

\[H^0(\mathcal{M}, D^k(L)) = \mathbb{C}\]

for every \(k \geq 0.\)

**Remark 4.4.**

(1) Note that the above Theorem 4.2 is equivalent to proving

\[H^0(\mathcal{M}, \text{Sym}^k \text{At}(L)) = \mathbb{C}\]

for every \(k \geq 0.\)
(2) Since $\mathcal{M}'_{pc}(r, d, \alpha)$ and $\mathcal{C}(L)$ are $T^*\mathcal{M}(r, d, \alpha)$-torsors, there is a natural question to ask,

**Question 4.5.** Does there exist an ample line bundle $L$ over $\mathcal{M}(r, d, \alpha)$ and an isomorphism

$$\varpi : \mathcal{M}'_{pc}(r, d, \alpha) \rightarrow \mathcal{C}(L)$$

of varieties such that the following diagram

$$\begin{array}{ccc}
\mathcal{M}'_{pc}(r, d, \alpha) & \xrightarrow{\varpi} & \mathcal{C}(L) \\
\downarrow{\pi} & & \downarrow{\Psi} \\
\mathcal{M}(r, d, \alpha) & & \\
\end{array}$$

commutes?

If the answer of the Question 4.5 is in affirmative, then from the Theorem 4.2, the moduli space $\mathcal{M}'_{pc}(r, d, \alpha)$ does not admit any non-constant algebraic function.

5. **Divisor at infinity**

In [8], compactification of the moduli space of logarithmic connections has been described, and authors have studied some important properties, like numerically effectiveness, of the smooth divisor at infinity.

From the proof of the Theorem 3.3 in section 3, we have a natural compactification $P(W)$ of the moduli space $\mathcal{M}'_{pc}(r, d, \alpha)$ such that the complement $P(W)\setminus\mathcal{M}'_{pc}(r, d, \alpha)$ is a smooth divisor $H$ (see (3.9)) at infinity.

In present section, we compactify the moduli space $\mathcal{M}'_{pc}(r, \alpha, \xi)$ as described in (2.14), and study the properties of the smooth divisor at infinity.

**Proposition 5.1.** There exists an algebraic vector bundle

$$\Psi : \mathcal{V} \rightarrow \mathcal{M}(r, \alpha, \xi) \tag{5.1}$$

such that $\mathcal{M}'_{pc}(r, \alpha, \xi)$ is embedded in $P(\mathcal{V})$ with

$$H_0 = P(\mathcal{V}) \setminus \mathcal{M}'_{pc}(r, \alpha, \xi)$$

as a smooth divisor at infinity.

**Proof.** Recall that the map $\pi_\xi$ defined in (2.15) is a $T^*\mathcal{M}(r, \alpha, \xi)$-torsor. Let

$$\Psi : \mathcal{V} \rightarrow \mathcal{M}(r, \alpha, \xi)$$

be the algebraic vector bundle such that for every Zariski open subset $U$ of $\mathcal{M}(r, \alpha, \xi)$, a section of $\mathcal{V}$ over $U$ is an algebraic function $f : \pi_\xi^{-1}(U) \rightarrow \mathbb{C}$ whose restriction to each fiber $\pi_\xi^{-1}(E_\xi)$ is an element of $\pi_\xi^{-1}(E_\xi)^\vee$. Now, the rest of the proof is exactly similar to a part of the proof of the Theorem 3.3. \qed

Now, we generalise [8, proposition 5.1] in parabolic set up.
**Proposition 5.2.** Consider the smooth divisor $H_0$ defined in Proposition 5.1. Then, the divisor $H_0$ is numerically effective if and only if the tangent bundle $T_M^\xi(r,\alpha,\xi)$ is numerically effective.

**Proof.** Let $\mathcal{N}_{\mathcal{P}(\mathcal{V})/H_0}$ denote the normal bundle of the divisor $H_0 \subset \mathcal{P}(\mathcal{V})$, where

$$
\Psi : \mathcal{V} \rightarrow \mathcal{M}_\xi := \mathcal{M}(r,\alpha,\xi)
$$

is the vector bundle (see (5.1)) in Proposition 5.1.

Recall that the effective divisor $H_0$ is numerically effective if and only if the restriction of the line bundle $\mathcal{O}_{\mathcal{P}(\mathcal{V})}(H_0)$ to $H_0$ is numerically effective. From Poincaré adjunction formula we have the following

$$
\mathcal{O}_{\mathcal{P}(\mathcal{V})}(H_0)|_{H_0} \cong \mathcal{N}_{\mathcal{P}(\mathcal{V})/H_0}.
$$

(5.2)

Therefore, $H_0$ is numerically effective if and only if the normal bundle $\mathcal{N}_{\mathcal{P}(\mathcal{V})/H_0}$ is numerically effective. Recall that the tangent bundle $T_M^\xi$ is numerically effective if and only if the tautological line bundle $\mathcal{O}_{\mathcal{P}(T_M^\xi)}(1)$ is numerically effective.

Thus, to prove the proposition it is enough to show that the normal bundle $\mathcal{N}_{\mathcal{P}(\mathcal{V})/H_0}$ is canonically isomorphic to the tautological line bundle $\mathcal{O}_{\mathcal{P}(T_M^\xi)}(1)$.

First we show that the divisor $H_0$ is canonically isomorphic to projective bundle $\mathcal{P}(T_M^\xi)$. Let

$$
\Psi : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{M}_\xi
$$

be the projective bundle. Let $E_\ast \in \mathcal{M}_\xi$, and

$$
\theta \in \Psi^{-1}(E_\ast) \cap H_0 \subset \mathcal{P}(\mathcal{V}).
$$

(5.3)

Then, $\theta$ represents a hyperplane in the fibre $\mathcal{V}(E_\ast) = \pi_\xi^{-1}(E_\ast)\nu$ of the vector bundle $\mathcal{V}$, where $\pi_\xi$ is defined in (2.15). Let $H_\theta$ denote this hyperplane represented by $\theta$. Note that $H_\theta \subset \pi_\xi^{-1}(E_\ast)\nu$ is the affine space modelled over the vector space $H^0(X, \Omega^1_X(S) \otimes \text{SParEnd}^d(E_\ast))$, therefore for any $f \in H_\theta$, we have

$$
df \in (T^\ast_{E_\ast} \mathcal{M}_\xi)^* = T^\ast_{E_\ast} \mathcal{M}_\xi.
$$

Since $\theta \in H_0$, and $H_\theta$ is a hyperplane, the subspace of $T_{E_\ast} \mathcal{M}_\xi$ generated by $\{df\}_{f \in H_\theta}$ is a hyperplane in $T_{E_\ast} \mathcal{M}_\xi$. Let $\tilde{\theta} \in \mathcal{P}(T_{E_\ast} \mathcal{M}_\xi)$ denote this hyperplane. Thus, we get a canonical isomorphism

$$
H_0 \cong \mathcal{P}(T_{E_\ast} \mathcal{M}_\xi)
$$

(5.4)

by sending $\theta$ to $\tilde{\theta}$.

Next, note that the fibre $\mathcal{N}_{\mathcal{P}(\mathcal{V})/H_0}(\theta)$ of the normal bundle $\mathcal{N}_{\mathcal{P}(\mathcal{V})/H_0}$ is canonically isomorphic to the quotient $\mathcal{V}(E_\ast)/H_\theta$. Consider the morphism

$$
\mathcal{V}(E_\ast) \rightarrow T_{E_\ast} \mathcal{M}_\xi
$$

of vector spaces defined by sending $f \mapsto df$. Since image of the hyperplane $H_\theta$ is contained in $\tilde{\theta}$, we have well defined morphism on quotients

$$
\mathcal{V}(E_\ast)/H_\theta \rightarrow T_{E_\ast} \mathcal{M}_\xi/\tilde{\theta},
$$

which is an isomorphism. Recall that the fibre of the tautological line bundle $\mathcal{O}_{\mathcal{P}(T_M^\xi)}(1)$ at $E_\ast$ is canonically identified with the quotient $T_{E_\ast} \mathcal{M}_\xi/\tilde{\theta}$. This completes the proof. □
Now consider the moduli space $\mathcal{M}'_{pc}(r,d,\alpha)$, and the smooth divisor

$$H = \mathcal{P}(W) \setminus \mathcal{M}'_{pc}(r,d,\alpha)$$

as defined in (3.9). Then using the same steps as in Proposition 5.2, we can show the following.

**Proposition 5.3.** The smooth divisor $H$ is numerically effective if and only if the tangent bundle $T\mathcal{M}(r,d,\alpha)$ is numerically effective.

**Acknowledgements**

The author would like to thank the referee for comprehensive comments, pointing out some issues to be clarified in the first version of the paper and suggesting some appropriate references. The author is deeply grateful to Prof. Indranil Biswas for useful discussions.

**References**

[1] D. Alfaya, Moduli space of parabolic $\Lambda$-modules over a curve, arXiv:1710.02080.
[2] D. Alfaya, and Tomás L. Gómez, Torelli theorem for the parabolic Deligne-Hitchin moduli space. *J. Geom. Phys.* **123** (2018), 448–462.
[3] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181-207.
[4] Hans U. Boden and Yoji Yokogawa, Rationality of the moduli spaces of parabolic bundles. *J. London Math. Soc.*, **59** (2): 461–478, 1999.
[5] Hans U. Boden and Yi Hu, Variations of moduli of parabolic bundles. *Math. Ann.* **301** (1995), no. 3, 539–559.
[6] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers. *Algebraic D-modules*. Perspectives in Mathematics, 2. Academic Press, Inc., Boston, MA, xii+355 pp, 1987.
[7] I. Biswas, A criterion for the existence of a flat connection on a parabolic vector bundle, *Advances in Geometry*, **2**, 231-241, 2002.
[8] I. Biswas, N. Raghavendra, Line bundles over a moduli space of logarithmic connections on a Riemann surface. *Geom. Funct. Anal.*, **15** (2005), 780-808.
[9] I. Biswas, and V. Muñoz. The Torelli theorem for the moduli spaces of connections on a Riemann surface. *Topology* **46** (2007), no. 3, 295–317.
[10] I. Biswas, Tomás L. Gómez, M. Logares, Integrable systems and Torelli theorems for the moduli spaces of parabolic bundles and parabolic Higgs bundles. *Canad. J. Math.* **68** (2016), no. 3, 504–520.
[11] P. Deligne, *Equations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, vol. 163. Springer, Berlin(1970).
[12] T. Gomez and M. Logares, Torelli theorem for the moduli space of parabolic Higgs bundles. *Adv. Geom.* **11**(2011), 429-444.
[13] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Vol. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp. ISBN: 0-387-90244-9
[14] Michi-Aki Inaba, Moduli of parabolic connections on curves and the Riemann-Hilbert correspondence. *J. Algebraic Geom.* **22** (2013), no. 3, 407–480.
[15] Yves Laszlo, and Christoph Sorger. The line bundles on the moduli of parabolic $G$-bundles over curves and their sections. *Ann. Sci. École Norm. Sup.* (4) **30** (1997), no. 4, 499–525.
[16] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles with parabolic structures, *Math. Ann.* **248** (1980), no. 3, 205–239.
[17] M. Maruyama, Openness of a family of torsion free sheaves, *Jour. Math. Kyoto Univ.* **16**(1976), 627-637.

[18] M. Ohtsuki, A residue formula for Chern classes associated with logarithmic connections, *Tokyo J. Math.* Vol. **5**, No. 1, 1982.

[19] Xiaoyu Su, Bin Wang, Xueqing Wen, Parabolic Hitchin maps and their generic fibers *Math. Z.* (2022). https://doi.org/10.1007/s00209-021-02896-3

[20] S. S. Shatz, The decomposition and specialization of algebraic families of vector bundles, *Compos. Math.* **35** (1997), 163-187.

[21] A. Singh, Moduli space of logarithmic connections singular over a finite subset of a compact Riemann surface, *Math. Res. Lett.* **28** (2021), no. 4, 863–887.

[22] A. Singh, Differential operators on Hitchin variety. *J. Algebra** 566** (2021), 361–373.

[23] C. T. Simpson, Moduli of representations of fundamental group of a smooth projective variety, *I, Inst. Hautes Études Sci. Publ. Math.* **79**(1994), 47-129.

[24] C. T. Simpson, Moduli of representations of fundamental group of a smooth projective variety, *II, Inst. Hautes Études Sci. Publ. Math.* **80**(1994), 5-79.

[25] K. Yokogawa, Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves. *J. Math. Kyoto Univ.* **33** (1993), no. 2, 451–504.

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India, 400005

Email: anoops@math.tifr.res.in