Four String Amplitudes for the Generalized Protostring.

Charles B. Thorn ¹

Institute for Fundamental Theory,
Department of Physics, University of Florida Gainesville FL 32611

Abstract

We specialize the N string scattering amplitudes for the generalized protostring to N = 4. This allows for a much more detailed and explicit study of their basic physical and mathematical properties, such as singularity structure and high energy behavior. Since this class of models does not enjoy full Poincaré invariance, the high energy behavior depends on the Lorentz frame. The relative simplicity of the four string amplitudes allows a direct understanding of the complications due to this non-covariance.

¹E-mail address: thorn@phys.ufl.edu
1 Introduction

Several years ago, motivated by string bit models [1–3], some novel string theories, including ones which lacked full Poincaré invariance, were proposed [4]. Despite the string bit motivation, these models can be formulated directly on the continuous lightcone worldsheet [5, 6]. In particular, Mandelstam’s interacting string formalism [7, 8] could be used to obtain multi-string scattering amplitudes [9]. Although formulas for these amplitudes were obtained, their physical properties were not studied extensively. This paper is an addendum to [9] focusing exclusively on the mathematical and physical properties of four open string amplitudes in these models.

The protostring model can be defined as the string model in which each of the 24 transverse coordinate worldsheet fields of the lightcone bosonic string is replaced by a spinor valued left-right pair of integer moded Grassmann worldsheet fields $\theta_L, \theta_R$. In [9] we obtained amplitudes for a generalization of the protostring in which only $s$ bosonic dimensions are so replaced, with the remaining $d = 24 - s$ left as transverse coordinates. In all these models, the string interaction is a simple overlap without operator insertions at the join/break point. The condition $s + d = 24$ ensures the finite continuum limit of the string bit overlap.

With the continuum scattering amplitude written in the form $\mathcal{M} \prod_k |p^+|^{-1/2}$, this finiteness condition means that $\mathcal{M}$ is invariant under the scale transformation $p^+_r \to \lambda p^+_r$. In other words, invariance under the subgroup $SO(1,1)$ of the Lorentz group in $d + 1$ space dimensions ($SO(d + 1,1)$) is maintained. For the protostring ($s = 24$ or $d = 0$), this is the entire Lorentz group, but for $s < 24$, with the exception of the bosonic string ($s = 0$), the Lorentz group is broken to $SO(1,1) \times SO(d)$.

We assume here, as in [9], that $s$ is even so the Grassmann worldsheet fields may be replaced by $s/2$ compactified bosonic worldsheet fields $\phi_r$. Integer moded Grassmann fields are equivalent to compactified bosonic fields for which the components of the momenta $\pi_k$ are quantized in odd multiples of a fixed number $\gamma$ with $\gamma^2 = 1/8$ for the open string and $\gamma^2 = 1/2$ for the closed string. By comparing the overlap in the original spinor formulation to the bosonized formulation, Sun showed that the three string vertex violates momentum conservation by $\pm \gamma$ for each of the $s/2$ components [10]. The bosonized overlap therefore contains operator factors $\prod_r \cos(\gamma \phi_r)$, which insert, for each component, momentum $\pm \gamma$ into the process. For $N$ string amplitudes, there are $N - 2$ such factors, which are described by $N - 2$ $s/2$-vectors $\gamma_r$ which satisfy the constraint

$$\sum_k \pi_k + \sum_r \gamma_r = 0. \quad (1)$$

For even $N$ it is possible that $\sum_r \gamma_r = 0$, allowing nonzero momentum-conserving scattering amplitudes.

There are also the momenta $p_k$ of the $d = 24 - s$ uncompactified transverse coordinates and their Minkowski extensions $p^\mu = (p_k, p^+_k, p^-_k)$. Finally it is convenient to append to $p^\mu$ $\text{SO}(d)$ factor can be extended to the Galilei group in $d$ space dimensions by including the Galilei boosts $M^i$ with $i = 1, \ldots, d$, of lightcone quantization. The broken generators are $M^i$ when $d < 24$ or $s > 0$. 

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the \( \pi \) and denote the resulting \( 2 + d + s/2 \) Minkowski vector by \( P^\mu \). In this notation the mass shell condition is, in units where \( \alpha' = 1 \), \( P \cdot P \equiv p^2 + \pi^2 - 2p^+p^- = 1 \) for each external string.

The scattering amplitudes of \([9]\), quoted for the reader’s convenience in the appendix, involve \( N \) external strings, all with no oscillator excitations and, in the closed string case, with zero winding number. However, their compactified momenta \( \pi_k \) could have components any odd multiple of \( \pm \gamma \). In this paper, we specialize further to external strings of minimal mass. Then each component of each \( \pi_k \) should be \( \pm \gamma \). The \( d + 2 \) dimensional mass squared of each string is then \( -p \cdot p = -(P^2 - \pi^2) = -1 + \pi^2 = s/12 - 1 \) for the open string, or \( s/3 - 4 \) for the closed string. There are all together \( N + N - 2 = 2(N - 1) \gamma \)'s and \( \pi \)'s, so to satisfy the conservation law \( N - 1 \) of them should have value \( +\gamma \) and the remaining ones should have value \( -\gamma \). There are \( (2^{(N-1)}) \) ways to do this for each of the \( s/2 \) compactified bosonic fields. For \( N = 4 \) this means that there are \( 20s/2 \) ways to assign \( \pm \gamma \) to the \( \pi \)'s and \( \gamma \)'s.

In the next Section 2 we discuss the conformal mappings required to evaluate any 4 string amplitude. Section 3 studies the amplitudes for several simple choices for the \( \pi_k \)'s. Section 4 analyzes high energy scattering in the 2 to 2 case. Concluding comments are in Section 5.

An appendix collects formulas for the \( N \)-string scattering amplitudes obtained in \([9]\).

### 2 Conformal maps for four string amplitudes

First specialize the conformal mapping from the complex \( z \)-plane to the lightcone worldsheet \( \rho \) to four external strings \([7]\):

\[
\rho = \alpha_1 \ln z + \alpha_2 \ln(z - Z) + \alpha_3 \ln(z - 1).
\]  

(2)

Then the two interaction points are determined by \( d\rho/dz = 0 \), which implies

\[
x_\pm = \frac{\alpha_1(Z + 1) + \alpha_2 + \alpha_3Z \pm \sqrt{(\alpha_1(Z + 1) + \alpha_2 + \alpha_3Z)^2 + 4\alpha_1\alpha_4Z}}{2(\alpha_1 + \alpha_2 + \alpha_3)}
\]  

(3)

\[
= \frac{\alpha_{12} + \alpha_{13}Z \pm \sqrt{\alpha_{12}^2 + \alpha_{13}^2Z^2 + 2(\alpha_1\alpha_4 + \alpha_2\alpha_3)Z}}{-2\alpha_4}
\]  

(4)

\[
= 1 + \frac{\alpha_{14} - \alpha_{13}(1 - Z) \pm \sqrt{\alpha_{14}^2 + \alpha_{13}^2(1 - Z)^2 + 2(\alpha_1\alpha_2 + \alpha_3\alpha_4)(1 - Z)}}{-2\alpha_4}.
\]  

(5)

The alternative forms follow from the identities

\[
\alpha_4^2|x_+ - x_-|^2 = \alpha_{12}^2(1 - Z) + \alpha_{23}^2Z - \alpha_{13}^2(1 - Z)
\]  

(6)

\[
= \alpha_{12}^2 + \alpha_{13}^2Z^2 - 2(\alpha_1\alpha_4 + \alpha_2\alpha_3)Z
\]  

(7)

\[
= \alpha_{14}^2 + \alpha_{13}^2(1 - Z)^2 - 2(\alpha_1\alpha_2 + \alpha_3\alpha_4)(1 - Z),
\]  

(8)

where \( \alpha_{kl} \equiv \alpha_k + \alpha_l \). The last two lines \((7)\) and \((8)\), give forms convenient for the study of the \( Z \sim 0 \) and \( Z \sim 1 \) limits respectively.
The integrand for the 4 string amplitude requires the factors

\[
\prod_{r<s} \frac{|x_r - x_s|^2 |\gamma_r - \gamma_s - s/24|}{s/24} = \left| x_1 - x_2 \right| (\gamma_1 + \gamma_2)^2 - s/6 \tag{9}
\]

\[
\prod_{k<l<N} |Z_k - Z_l|^2 P_k P_l - s/24 = Z^{-S + (\pi_1 + \pi_2)^2 - 2} (1 - Z)^{-t + (\pi_2 + \pi_3)^2 - 2} \tag{10}
\]

\[
\prod_{r,k<N} |x_r - Z_k|^4 = \left[ \frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{3} Z^2 (1 - Z)^2 \right]^{s/24} \tag{11}
\]

\[
\prod_{r,k<N} |x_r - Z_k|^{2\pi_k \gamma_r} = \prod_{k=1}^{3} \frac{|x_1 - Z_k|}{x_2 - Z_k} \prod_{k=1}^{3} \left| (x_1 - Z_k)(x_2 - Z_k) \right|^{\pi_k (\gamma_1 + \gamma_2)} 
= |Z \frac{\alpha_1}{\alpha_4}|^{1 - (\gamma_1 + \gamma_2)} \left| Z (1 - Z) \frac{\alpha_2}{\alpha_4} \right|^{\pi_2 (\gamma_1 + \gamma_2)} \left| (1 - Z) \frac{\alpha_3}{\alpha_4} \right|^{\pi_3 (\gamma_1 + \gamma_2)} \tag{12}
\]

where we have chosen \( Z_1 = 0 \), \( Z_2 = Z \), and \( Z_3 = 1 \), and we have introduced the Mandelstam invariants \( S = -(p_1 + p_2)^2 \) and \( t = -(p_2 + p_3)^2 \). To construct the open 4-string amplitude one integrates the product of all these factors over \( Z \) from 0 to 1, and sums over all choices for \( \gamma_{1,2} \) consistent with the constraint (1). Note however that for a given component, if \( \gamma_1 + \gamma_2 = \pm 2 \), then \( \gamma_1 - \gamma_2 = 0 \), and vice versa.

### 2.1 Crossing Symmetry

Because there is explicit dependence of the integrand of the amplitude on the \( x_\pm \) when \( s \neq 0 \), crossing symmetry \( (S \leftrightarrow t) \) for the open string is not as transparent as for the bosonic string. However, it still follows from the simple change of variables \( Z \rightarrow 1 - Z \). This is because of the identities

\[
x_\pm (1 - Z; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1 - x_\mp \left( Z; \alpha_3, \alpha_2, \alpha_1, \alpha_4 \right)
\]

\[
1 - x_\pm (1 - Z; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = x_\mp \left( Z; \alpha_3, \alpha_2, \alpha_1, \alpha_4 \right) \tag{13}
\]

\[
1 - Z - x_\pm (1 - Z; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = -(Z - x_\mp \left( Z; \alpha_3, \alpha_2, \alpha_1, \alpha_4 \right)).
\]

When we construct the amplitudes in the next section, we shall see that these identities enable one to show that the amplitudes are symmetric under the combined transformation \( S \leftrightarrow t, \pi_1 \leftrightarrow \pi_3 \), and \( \alpha_1 \leftrightarrow \alpha_3 \). Or more simply under \( P_1 \leftrightarrow P_3 \).
2.2 Singularities of Amplitudes

The poles in $-S = (p_1 + p_2)^2$ are controlled by $Z \sim 0$ and those in $-t = (p_2 + p_3)^2$ by $Z \sim 1$. We therefore need the behavior of $x_\pm$ in these regions of integration:

$$x_+ \sim -\frac{\alpha_{12}}{\alpha_4} - \frac{\alpha_2 \alpha_3}{\alpha_{12} \alpha_4} Z + O(Z^2)$$

$$x_- \sim \frac{\alpha_1}{\alpha_{12}} Z + O(Z^2)$$

for $Z \sim 0$ at fixed $\alpha_k$, and

$$x_+ \sim 1 - \frac{\alpha_{14}}{\alpha_4} - \frac{\alpha_1 \alpha_2}{\alpha_{14} \alpha_4} (1 - Z) + O((1 - Z)^2)$$

$$x_- \sim 1 + \frac{\alpha_3}{\alpha_{14}} (1 - Z) + O((1 - Z)^2)$$

for $Z \sim 1$ at fixed $\alpha_k$, and assuming for definiteness, that $\alpha_{14} = -\alpha_{23} > 0$. With the opposite signs the roles of $x_+, x_-$ are switched. As long as $\alpha_{12}$ and $\alpha_{23}$ are both non zero, the lowest lying pole locations are determined by the factors $x_-$ and $Z - x_-$ when analyzing $Z \sim 0$, and by the factors $1 - x_-$ and $Z - x_-$ when $Z \sim 1$.

2.3 High Energy

The high energy Regge behavior (large $-S$ at fixed $t$) is also controlled by $Z \sim 1$, more precisely by $1 - Z \sim 1/(-S)$. However, in order to confirm Mueller's argument [11] for constant cross sections, we should also like to consider large $-S$ at fixed $\alpha_1/(-S)$ and fixed $\alpha_2, \alpha_3$. This means that factors like $\alpha_1 (1 - Z)$ will be of order 1. Then in leading order, we can approximate the $x_\pm$ by

$$x_+ \approx 1 + \frac{\alpha_{14} - \alpha_1 (1 - Z) \pm \sqrt{\alpha_{14}^2 + \alpha_1^2 (1 - Z)^2 + 2 \alpha_1 (\alpha_2 - \alpha_3) (1 - Z)}}{-2 \alpha_4}.$$  

To analyze the amplitudes in this limit, it is convenient to change variables $Z = e^{-(u/(-S))}$ and replace $1 - Z \approx u/(-S)$. Then $dZ = -Z du/(-S)$. Also, to reduce clutter we introduce the ratio $\eta \equiv \alpha_1/(-S)$. Then the approximate form for $x_\pm$ can be written

$$x_+ \approx 1 + \frac{\alpha_{14} - \eta u \pm \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2 \eta u (\alpha_2 - \alpha_3)}}{-2 \alpha_4}.$$  

We note that, because the numerator of the second term on the right is of order 1 in the limit, the second term is of order $1/\alpha_1$ and can be neglected except in factors such as $x_+ - 1$ or $x_- - Z$ which are of order $1/\alpha_1$ in the limit. Both amplitudes depend on the $x_\pm$ through
the ratios
\[
\frac{x_-}{x_+} \approx 1
\]  
(20)  
\[
\frac{x_- - 1}{x_+ - 1} \approx \frac{\alpha_{14} - \eta u - \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\alpha_{14} - \eta u + \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}
\]  
(21)  
\[
\frac{x_- - Z}{x_+ - Z} \approx \frac{\alpha_{14} + \eta u - \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\alpha_{14} + \eta u + \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}
\]  
(22)  
all of which are of order 1 in the limit, provided \( \eta \) is nonzero and finite.

3 Amplitudes for four strings with minimal mass

Each of the \( s/2 \) components of the four compactified momenta \( \pi_k \) is constrained by
\[
\pi_1^a + \pi_2^a + \pi_3^a + \pi_4^a + \gamma_1^a + \gamma_2^a = 0.
\]  
(23)

Since we are specializing to minimal mass, these constraints must be satisfied with each component of \( \pi_k \) and \( \gamma_k \) either \(+\gamma\) or \(-\gamma\): three of the six one sign and the other three the opposite sign. We can categorize the choices by the values of \( \gamma_1^a + \gamma_2^a \): \(+2\gamma\), \(0\), or \(-2\gamma\). The multiplicities of each choice are 1, 2, and 1 respectively. But these choices can be made independently for each component. For each external string, we can write \( \pi^a = h^a \gamma \) and refer to \( h^a \) as the \( a \)th component of the helicity of that string. Thus we can refer to the three categories as \( \Delta h = +2 \), \( \Delta h = 0 \), and \( \Delta h = -2 \). In the following we shall concentrate on the cases where all components are in the same category. But of course there are many “mixed” possibilities, which will not be discussed explicitly.

3.1 \( \Delta h = \pm 2 \)

Putting \( N = 4 \) in (57) we find the four open string amplitude in the special case \( \gamma_r = \gamma = 1/\sqrt{8} \), \( \pi_k = -\gamma = -1/\sqrt{8} \) for \( k < 4 \):
\[
M_{\Delta h=\pm 2} = \int_0^1 dZ \frac{|\alpha_4|^{s/4}|x_2 - x_1|^{s/12}}{|\alpha_1 \alpha_2 \alpha_3|^{s/12}} Z^{(p_1 + p_2)^2 - 2s/12} \frac{(1 - Z)^{(p_2 + p_3)^2 - 2s/12}}{\alpha_{12}^2 + \alpha_{13}^2 Z^2 + 2(\alpha_1 \alpha_4 + \alpha_2 \alpha_3)Z}^{s/24}
\]  
(24)

The expression inside the absolute value signs on the last line stays positive throughout the integration region \( 0 < Z < 1 \), so that it is safe to drop them. It is noteworthy that when \( s = 24 \) (the protostring case) this factor is a second order polynomial in \( Z \), so the
amplitude reduces to a linear combination of three Euler beta functions. In addition to this simplification, the kinematics of scattering is limited to forward and backward scattering since there is only 1 space dimension when \( d = 0 \). The consequences of these simplifications were already discussed in [9] and will not be elaborated further here.

It is not hard to check that the pole singularities in \( s \) and \( t \) are where they should be as long as \( \alpha_{12} \) and \( \alpha_{23} \) are non zero. This is reasonable since excluding these values of the \( \alpha \)'s guarantees that the dynamical singularities are all due to the long time propagation of protostring mass eigenstates. On the other hand, if \( \alpha_{23} = 0 \), \( \alpha_4^2(x_2 - x_1)^2 \sim 4\alpha_1\alpha_2(1 - Z) \) as \( Z \to 1 \) so the poles in \( t \) are shifted by an amount \( s/24 \). When \( \alpha_{23} = 0 \) these singularities are due to the collision of the interaction points on the worldsheet and not the long time propagation of a particle state. This nonuniformity of singularity structure is absent for the bosonic and superstring because the amplitude integrands do not depend explicitly on the \( x_r \).

### 3.2 \( \Delta h^a = 0 \)

In this case the \( \pi \) are conserved, so that the \( \gamma_r \) must sum to zero. In the 4-string case this requires \( \gamma_2 = -\gamma_1 \). Then the various factors in the integrand, for each choice for \( \gamma_1 \), are

\[
\prod_{r < s} |x_r - x_s|^{2\gamma_r - s/24} = |x_1 - x_2|^{-2\gamma_1^2 - s/24} = |x_1 - x_2|^{-s/6} \quad (25)
\]

\[
\prod_{k < l < N} |Z_k - Z_l|^{2\pi_k - s/24} = Z^{-S + (\pi_1 + \pi_2)^2 - 2} (1 - Z)^{-t + (\pi_2 + \pi_3)^2 - 2} \quad (26)
\]

\[
\prod_{r, k < N} |x_r - Z_k|^{s/24} = \left[ \frac{\alpha_1\alpha_2\alpha_3}{\alpha_4^2} Z^2 (1 - Z)^2 \right]^{s/24} \quad (27)
\]

\[
\prod_{r, k < N} |x_r - Z_k|^{2\pi_k \gamma} = \prod_{k=1}^3 \left| \frac{x_1 - Z_k}{x_2 - Z_k} \right|^{2\pi_k \gamma_1} = \left| \frac{x_1}{x_2} \right|^{2\pi_1 \gamma_1} \left| \frac{x_1 - Z}{x_2 - Z} \right|^{2\pi_2 \gamma_1} \left| \frac{x_1 - 1}{x_2 - 1} \right|^{2\pi_3 \gamma_1} \quad (28)
\]

where we have again chosen \( Z_1 = 0 \), \( Z_2 = Z \), and \( Z_3 = 1 \). The complete (tree) amplitude is constructed by taking the product of all these factors, summing over all \( 2^{s/2} \) possibilities for \( \gamma_1 \) and integrating \( Z \) from 0 to 1. The only factors that depend on \( \gamma_1 \) are those on the last line. Since the components of \( \gamma_1 \) are \( \pm \gamma \), this sum yields the product of factors

\[
\prod_{i=1}^{s/2} \left( \left| \frac{x_1}{x_2} \right|^{2\pi_i^1 \gamma} \left| \frac{x_1 - Z}{x_2 - Z} \right|^{2\pi_i^2 \gamma} \left| \frac{x_1 - 1}{x_2 - 1} \right|^{2\pi_i^3 \gamma} + (x_1 \leftrightarrow x_2) \right) \quad (29)
\]

where \( i \) labels the component of \( \pi_k \).

We analyze two simple examples of fully elastic scattering amplitudes. First, we choose \( \pi_4 = -\pi_1 \) and \( \pi_3 = -\pi_2 \) so that the outgoing strings are in the same internal states as the
incoming ones. Necessarily then we must have \( \gamma_2 = -\gamma_1 \), which we are assuming in this subsection anyway. Let \( \gamma \) be the \( s/2 \) vector with each component equal to \( \gamma = 1/\sqrt{8} \) for the open string. Then our first simple example of elastic scattering is

\[
\pi_1 = \pi_2 = \gamma, \quad \pi_3 = \pi_4 = -\gamma, \quad \gamma_1 = -\gamma_2
\]  

As already mentioned, there are \( 2^{s/2} \) choices for \( \gamma_1 \) since each component can be \( \pm \gamma \) independently. The total amplitude should include the sum over all such choices. After summing over both signs for each component of \( \gamma_1 \), the total amplitude becomes

\[
\mathcal{M}_1^{\Delta h = 0} = \left[ \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_4^3} \right]^{s/24} \int_0^1 dZZ^{-s-2+s/3}(1-Z)^{-t-2+s/12} \left( \frac{\alpha_{12}^2(1-Z) + \alpha_{23}^2 Z - \alpha_{13}^2 Z(1-Z)}{\alpha_4^2} \right)^{-s/12} \left[ \frac{x_1(x_1 - Z)(x_2 - 1)}{x_2(x_2 - Z)(x_1 - 1)} \right]^{1/4} + \left[ \frac{x_2(x_2 - Z)(x_1 - 1)}{x_1(x_1 - Z)(x_2 - 1)} \right]^{1/4} \right]^{s/2}
\]

The second simple example is

\[
\pi_1 = -\pi_2 = \gamma, \quad \pi_4 = -\pi_3 = -\gamma, \quad \gamma_1 = -\gamma_2
\]  

for which the scattering amplitude is

\[
\mathcal{M}_2^{\Delta h = 0} = \left[ \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_4^3} \right]^{s/24} \int_0^1 dZZ^{-s-2+s/3}(1-Z)^{-t-2+s/12} \left( \frac{\alpha_{12}^2(1-Z) + \alpha_{23}^2 Z - \alpha_{13}^2 Z(1-Z)}{\alpha_4^2} \right)^{-s/12} \left[ \frac{x_1(x_2 - Z)(x_1 - 1)}{x_2(x_1 - Z)(x_2 - 1)} \right]^{1/4} + \left[ \frac{x_2(x_2 - Z)(x_1 - 1)}{x_1(x_2 - Z)(x_2 - 1)} \right]^{1/4} \right]^{s/2}
\]

Both of these examples assume the individual \( \pi \)'s have uniform signs for their individual components. The full range of \( \pi \) conserving amplitudes for minimal mass external strings is considerably larger.

For this second example, the total compactified momentum in the \( S \) channel and in the \( t \) channel are both zero. Thus \( S \leftrightarrow t \) crossing symmetry reads \( \mathcal{M}_2(S, t; \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathcal{M}_2(t, S; \alpha_3, \alpha_2, \alpha_1, \alpha_4) \). This can be proved by changing integration variables \( Z \rightarrow 1 - Z \) and using the identities (13).

### 4 High energy four string scattering

In this section we discuss the Regge high energy behavior of some of the four string amplitudes. In a Poincaré invariant theory the Regge limit is \( -S \rightarrow \infty \) at fixed \( t \) and the
amplitudes have the typical behavior

\[ \mathcal{M} \sim \beta(t)(-S)^{\alpha(t)}. \] (34)

The Regge trajectory \( \alpha(t) \) passes through nonnegative integers \( J \) when \( t \) passes through the squared mass eigenvalues of the string: \( J \) is then the spin of the string mass eigenstate. Physical scattering requires \( t < 0 \), whereas the masses squared are nonnegative. Thus \( \alpha(t) \) is an extrapolation of the relation between angular momentum and string mass squared. In string theory in tree approximation this relationship is linear \( \alpha(t) = \alpha't + \alpha_0 \). We have chosen units where \( \alpha' = 1 \).

For \( s > 0 \), the generalized protostring does not enjoy Poincaré invariance. In particular the four string amplitudes depend on the \( \alpha_k = 2p_k^+ \) in addition to \( S \) and \( t \), and one can consider various Regge limits, depending on whether some of the \( \alpha_k \)’s are also getting large with \(-S\). We shall be particularly interested in limits where \( \alpha_1 \) and \(-\alpha_4 \) tend to infinity linearly with \(-S\), with \( \alpha_{2,3} \) fixed. With Poincaré invariance, one can always choose a frame where this is true. This is the frame that Mueller chose to show that the physics of the lightcone worldsheet for open strings implies that open string scattering amplitudes in the forward direction must grow linearly with \(-S\) at fixed \( \eta = \alpha_1/(-S) \) [11]. This behavior corresponds to constant cross sections by the optical theorem. In a Poincaré invariant theory this implies \( \alpha(0) = 1 \). But in the protostring models this last conclusion does not necessarily hold, as we shall see.

For the Lorentz covariant bosonic open string scattering amplitude \((s = 0)\), the limit \(-S = (p_1 + p_2)^2 \to \infty\) is evaluated by changing variables \( Z = e^{-u/(-S)} \)

\[ \mathcal{M} \approx (-S)^{t+1} \int_0^{-u} du e^{-u} (-S)^{t-2} = (-S)^{t+1} \Gamma(-t-1), \quad S \to -\infty. \] (35)

where \(-t = (p_2 + p_3)^2\) is the momentum transfer \((= 0\) in the forward direction\). Here \( \alpha(t) = t + 1 \) so indeed \( \alpha(0) = 1 \).

When the Grassmann dimension \( s > 0 \), the analysis is complicated by the explicit dependence of the integrand on \( x_\pm \). In the following we obtain the high energy behavior of the scattering amplitudes obtained in Section 3.

### 4.1 \( \Delta h^a = \pm 2 \)

Applying the change of variables \( Z = e^{-u/(-S)} \) developed in Section 2.3 to (24) using (19), we find the leading high energy behavior

\[ \mathcal{M}^{\Delta h=2} \approx (-S)^{t+1-s/12} \frac{|\alpha_4|^{s/4}}{|\alpha_1\alpha_2\alpha_3|^{s/12}} \int_0^\infty du e^{-u} u^{-2+s/12} \left| \frac{\alpha_1^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}{\alpha_4^2} \right|^{s/24}. \] (36)
In the limit we are taking $\alpha_4 \approx -\alpha_1$ and $\alpha_{14}$, $\alpha_2$, $\alpha_3$ are fixed. Making these replacements then gives
\[
\mathcal{M}^{\Delta h=2} \approx (-S)^{t+1-s/12} \alpha_1^{s/12} \alpha_3 \frac{1}{\alpha_3^{s/12}} \int_0^\infty du e^{-u} u^{-t-2+s/12} |\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)|^{s/24}
\]  
(37)

Here we see that if $\alpha_1$ and $-S$ go to infinity at fixed ratio, the result is in accord with Mueller’s argument. Note however, that the power of $(-S)$, $\alpha(t) = t + 1 - s/12$, so $\alpha(0) = 1 - s/12$ is not unity, as must be the case if the theory were Poincaré covariant.

4.2 $\Delta h^a = 0$

We next turn to the high energy behavior of the amplitudes $\mathcal{M}_1^{\Delta h=0}$ and $\mathcal{M}_2^{\Delta h=0}$. Making the approximation for high energy
\[
\mathcal{M}_1^{\Delta h=0} \approx \left[ \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_4^3} \right]^{s/24} (-S)^{t+1-s/12} \int_0^\infty du e^{-u(-S-1+s/3)/(-S)} u^{-t-2+s/12} \left( \frac{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}{\alpha_4^2} \right)^{-s/12} \left[ \frac{x_1(x_1 - Z)(x_2 - 1)}{x_2(x_2 - Z)(x_1 - 1)} \right]^{1/4} + \left[ \frac{x_2(x_2 - Z)(x_1 - 1)}{x_1(x_1 - Z)(x_2 - 1)} \right]^{1/4} \right]^{s/2}
\]  
(38)

where, identifying $x_1 = x_-$ and $x_2 = x_+$,
\[
\left| \frac{x_1(x_1 - Z)(x_2 - 1)}{x_2(x_2 - Z)(x_1 - 1)} \right| \approx \frac{\alpha_{14} - \eta u - \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\alpha_{14} - \eta u + \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}} \approx \frac{\alpha_{14} + \eta u + \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\alpha_{14} + \eta u - \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}} \approx \frac{\eta u + \alpha_2 - \alpha_3 + \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\eta u + \alpha_2 - \alpha_3 - \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}
\]  
(39) \hspace{1cm} (40)

The second example in the high energy limit becomes
\[
\mathcal{M}_2^{\Delta h=0} \approx \left[ \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_4^3} \right]^{s/24} (-S)^{t+1-s/12} \int_0^\infty du e^{-u(-S-1+s/3)/(-S)} u^{-t-2+s/12} \left( \frac{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}{\alpha_4^2} \right)^{-s/12} \left[ \frac{x_2(x_1 - Z)(x_2 - 1)}{x_1(x_2 - Z)(x_1 - 1)} \right]^{1/4} + \left[ \frac{x_1(x_2 - Z)(x_1 - 1)}{x_2(x_1 - Z)(x_2 - 1)} \right]^{1/4} \right]^{s/2}
\]  
(42)
where
\[
\frac{|x_2(x_1 - Z)(x_2 - 1)|}{|x_1(x_2 - Z)(x_1 - 1)|} \approx \left| \frac{\eta u + \alpha_2 - \alpha_3 + \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)} - \eta u + \alpha_2 - \alpha_3 - \sqrt{\alpha_{14}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\eta u + \alpha_2 - \alpha_3 + \sqrt{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)} + \eta u + \alpha_2 - \alpha_3 - \sqrt{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}} \right|^{s/12} \left[ \frac{\alpha_{23}}{\alpha_1} \right]^{s/12} (43)
\]

because \( \frac{x_2}{x_1} \approx \frac{x_1}{x_2} \) in the high energy limit we are discussing. Indeed the differences between \( \mathcal{M}_2 \) and \( \mathcal{M}_1 \) in this high energy limit are subleading so, in leading order we can write

\[
\mathcal{M}_{1}^{\Delta h=0} \approx \mathcal{M}_{2}^{\Delta h=0}
\]

\[
\approx \left| \frac{\alpha_2 \alpha_3}{\alpha_{23}} \right|^{s/24} \left| \frac{\alpha_{23}}{\alpha_1} \right|^{-s/12} \left[ \frac{\alpha_2}{\alpha_3} \right]^{1/4} \left[ \frac{\alpha_3}{\alpha_2} \right]^{1/4} (-S)^{t+1-s/12} \int_{0}^{\infty} du e^{-u(t-2+s/12)} \left( \frac{\alpha_{23} + \eta^2 u^2 + 2(\alpha_2 - \alpha_3)\eta u}{\alpha_{23}^2} \right)^{-s/12} \left( \frac{\eta u + \alpha_2 - \alpha_3 + \sqrt{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)} - \eta u + \alpha_2 - \alpha_3 - \sqrt{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}}{\eta u + \alpha_2 - \alpha_3 + \sqrt{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)} + \eta u + \alpha_2 - \alpha_3 - \sqrt{\alpha_{23}^2 + \eta^2 u^2 + 2\eta u(\alpha_2 - \alpha_3)}} \right|^{s/12} (44)
\]

where we have rearranged factors and dropped some terms, which are subleading in the high energy limit we are discussing. Here we see how the argument for linear growth in \( \alpha_1 \) is satisfied by this result: it is a product of the factors \( \alpha_1^{s/12} \) and \( (-S)^{t+1-s/12} \), which in the forward direction scales like \( \alpha_1 \) provided \( \eta \) is fixed.

If \( |\eta| \ll |\alpha_{23}| \) the complicated factors simplify considerably:

\[
\mathcal{M}_{1}^{\Delta h=0} \approx \mathcal{M}_{2}^{\Delta h=0}
\]

\[
\approx \left| \frac{\alpha_2 \alpha_3}{\alpha_{23}} \right|^{s/24} \left| \frac{\alpha_{23}}{\alpha_1} \right|^{-s/12} \left[ \frac{\alpha_2}{\alpha_3} \right]^{1/4} \left[ \frac{\alpha_3}{\alpha_2} \right]^{1/4} (-S)^{t+1-s/12} \Gamma(-t - 1 + s/12) (45)
\]

which is in accord with the high energy limit obtained in [9], which implicitly assumed very small \( \eta \).

Here we see that, with \( \alpha_{23} \neq 0 \) and fixed, the coefficient of \( \alpha_1^{t+1} \) has poles at \( t = n + s/12 - 1 \) which are the mass squared eigenvalues of the open generalized protostring. The linear high energy behavior, when \( |\eta| \ll |\alpha_{23}| \), at \( t = 0 \) is the product of \( (-S)^{1-s/12} \) and \( \alpha_1^{s/12} \) netting precisely linear growth in the forward direction.
Contrast this with the high energy limit taken with $\alpha_{23} = 0$ from the beginning:

$$\mathcal{M}^{\Delta h=0}_1 \approx \mathcal{M}^{\Delta h=0}_2 \approx |\alpha_2|^{s/12} \frac{(-S)^{t+1}}{(\eta u + 4\alpha_2)^{-s/12}} \int_0^\infty du e^{-u} u^{-t-2} \left( \frac{\eta u + 2\alpha_2 + \sqrt{\eta^2 u^2 + 4\eta u\alpha_2}}{\eta u + 2\alpha_2 - \sqrt{\eta^2 u^2 + 4\eta u\alpha_2}} \right)^{1/4} + \left( \frac{\eta u + 2\alpha_2 + \sqrt{\eta^2 u^2 + 4\eta u\alpha_2}}{\eta u + 2\alpha_2 - \sqrt{\eta^2 u^2 + 4\eta u\alpha_2}} \right)^{-1/4}$$

(46)

For $\eta \ll \alpha_2$ this result simplifies to

$$\mathcal{M}^{\Delta h=0}_1 \approx \mathcal{M}^{\Delta h=0}_2 \approx 4^{-s/12} \frac{(-S)^{t+1}}{(\eta u + 4\alpha_2)^{-s/12}} \int_0^\infty du e^{-u} u^{-t-2} = 4^{-s/12} (-S)^{t+1} \Gamma(-t - 1)$$

(47)

which is just the Regge behavior of the bosonic string amplitude. We stress that the limit taken here is $\alpha_1, -S \to \infty$ at fixed ratio. The coefficient of the Regge behavior is a function of $t$. Its pole locations are not those of the particles of the theory: they correspond to a linear Regge trajectory of intercept 1. Because the formula was obtained assuming $\alpha_3 = -\alpha_2$, the high energy behavior comes from the collision of two interaction points on the lightcone worldsheet, and not from the long time propagation of a protostring mass eigenstate as in the $\alpha_{23} \neq 0$ case. This mismatch can occur because the Lorentz boost symmetry generated by $M^{-k}$ is absent in the generalized protostring: for the protostring because there is no transverse space and for $0 < s < 24$ because this part of the Lorentz symmetry is broken. For the bosonic string ($s = 0$), of course, there is no such mismatch.

5 Concluding Remarks

In [9] $N$ string scattering amplitudes were obtained for the generalized protostring. The present article focused on the case $N = 4$, which was treated only lightly in [9]. In this case the conformal mapping used in Mandelstam’s interacting string formalism can be inverted explicitly, which is not possible for higher $N$. In particular the images of the joining/breaking points on the lightcone worldsheet are explicit functions of the Koba-Nielsen variables. We could then use these explicit formulas to obtain explicit integral representations of four string tree amplitudes.

We interpreted the zero mode of the worldsheet momentum density for the compactified coordinates representing the spinor world sheet fields as “helicity” $h^a$, $a = 1, 2, \ldots, s/2$. and then studied the 4-string amplitudes for $\Delta h^a = 2$ and $\Delta h^a = 0$. we calculated the pole locations and the Regge high energy behavior of the amplitudes.

The high energy behavior of these models is compatible with Mueller’s argument that total cross sections for open string scattering approach constants in the limit. In Lorentz covariant theories, this would imply a Regge trajectory with intercept $\alpha(0) = 1$, and the existence of massless spin one particles. In the amplitudes studied in the present article
the mass spectrum implies \( \alpha(0) = 1 - s/12 \), but for \( s > 0 \) Lorentz covariance is lost. Mueller’s argument chooses a frame in which \((-S) \propto p_1^+\) and uses the physics of the lightcone worldsheet to show that as \( p_1^+ \to \infty \) the cross section approaches a nonzero constant. Since the models studied here are not Lorentz covariant, the constant cross sections implied by Mueller’s argument are realized by a behavior \((p_1^+)^{1-\alpha(0)}(-S)^{\alpha(0)}\). If \((-S)\) tends to \( \infty \) at fixed \( p_1^+ \), cross sections need not tend to constants.

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A The Scattering Amplitudes

We quote the formulas for the general \( N \)-string scattering amplitudes obtained in [9]. They are expressed as integrals over Koba-Nielsen variables \( Z_k, k = 1, \ldots, N \) and the mapping function from the Koba-Nielsen plane (or half plane) \( z \) to the lightcone worldsheet [7].

\[
\rho = \sum_{k=1}^{N-1} \alpha_k \ln(z - Z_k),
\]

(48)

where \( \alpha_k = 2p_k^+ \equiv \sqrt{2(p^0 + p^{d+1})} \) is positive (negative) if the external string is incoming (outgoing). We also introduce the quantities \( x_r \), which are the zeros of \( d\rho/dz \).

\[
\left. \frac{d\rho}{dz} \right|_{z=x_r} = 0
\]

(49)

and depend on the \( Z \)'s and \( \alpha \)'s. For convenience we choose \( Z_1 = 0, Z_{N-1} = 1, \) and \( Z_N = \infty \). The \( x_r \) are roots of a polynomial of order \( N - 2 \) so we can take \( r = 1, 2, \ldots, N - 2 \).

The kinematics of the scattering process are determined by the incoming momenta of the external strings. Of these there are the momenta of the \( d + 2 \) uncompactified coordinates \( p_k^\mu \), which are continuous and conserved \((\sum_k p_k^\mu = 0)\). After bosonizing the Grassmann worldsheet fields there are also the \( s/2 \) dimensional discretized momenta \( \pi_k \) which are not conserved: Their components are positive or negative odd multiples of a number \( \gamma \) which is determined by insisting that the three string vertex, initially defined in terms of string bits, has a finite continuum limit.

The \( x_r \) are the images in the Koba-Nielsen plane of the breaking or joining points on the lightcone worldsheet. The non conservation of the \( \pi_k \) is specified by assigning a spurious discretized momentum \( \gamma_r \) to each vertex. Each of the \( s/2 \) components of \( \gamma_r \) are \( \pm \gamma_i \), and the full vertex is obtained by summing over all choices of signs. For each choice, the non-conservation of discretized momenta reads.

\[
\sum_k \pi_k + \sum_r \gamma_r = 0
\]

(50)
Note that for even $N$ some of the choices satisfy $\sum_r \gamma_r = 0$ for which $\sum_k \pi_k = 0$. Mandelstam’s interacting string formalism [7] works for any value of $\gamma$ and gives for the Koba-Nielsen integrand of the open string amplitudes

$$I_O = \prod_{k=1}^{N} \frac{1}{|\alpha_k|} \left[ \prod_{k<N} \frac{|\alpha_k|}{|\alpha_N|} \right]^{s(1-8\gamma^2)/32} \left[ \prod_{r<t} \frac{|x_t - x_r|}{|l_t|} \prod_{m<l} \frac{|Z_l - Z_m|}{|Z_l - x_r|} \right]^{s/48}$$

$$(2\delta)^{(N-2)\gamma^2/4} \left[ \prod_{r<s} \frac{|x_r - x_s|^{2\gamma_r\gamma_s - \gamma^2/2}}{\prod_{r,k<N} |x_r - Z_k|^{-2\pi_k\gamma_r - \gamma^2/2}} \prod_{k<l<N} |Z_k - Z_l|^{2P_k\cdot P_l - \gamma^2/2} \right]$$

(51)

In string bit models, the factor within the first set of square brackets would scale as $M^{N-2}$ where $M$ is the bit number. So a finite continuum limit requires $\gamma^2 = 1/8$, in which case

$$I_O = (2\delta)^{(N-2)/32} \prod_{k=1}^{N} \frac{1}{|\alpha_k|} \left[ \prod_{k<N} \frac{|\alpha_k|}{|\alpha_N|} \right]^{s(1-8\gamma^2)/32} \left[ \prod_{r<t} \frac{|x_t - x_r|}{|l_t|} \prod_{m<l} \frac{|Z_l - Z_m|}{|Z_l - x_r|} \right]^{s/24}$$

$$(2\delta)^{(N-2)\gamma^2/8} \left[ \prod_{r<s} \frac{|x_r - x_s|^{2\gamma_r\gamma_s - \gamma^2/4}}{\prod_{r,k<N} |x_r - Z_k|^{-2\pi_k\gamma_r - \gamma^2/4}} \prod_{k<l<N} |Z_k - Z_l|^{2P_k\cdot P_l - \gamma^2/4} \right]$$

(52)

and a smooth continuum limit in the closed case requires $\gamma^2 = 1/2$:

$$I_C = (4\delta)^{(N-2)/16} \prod_{k=1}^{N} \frac{1}{|\alpha_k|} \left[ \prod_{k<N} \frac{|\alpha_k|}{|\alpha_N|} \right]^{s(1-8\gamma^2)/32} \left[ \prod_{r<t} \frac{|x_t - x_r|}{|l_t|} \prod_{m<l} \frac{|Z_l - Z_m|}{|Z_l - x_r|} \right]^{s/24}$$

$$(2\delta)^{(N-2)\gamma^2/8} \left[ \prod_{r<s} \frac{|x_r - x_s|^{2\gamma_r\gamma_s - \gamma^2/4}}{\prod_{r,k<N} |x_r - Z_k|^{-2\pi_k\gamma_r - \gamma^2/4}} \prod_{k<l<N} |Z_k - Z_l|^{2P_k\cdot P_l - \gamma^2/4} \right]$$

(53)

(54)

In these formulas each component of $\gamma_r$ is $\pm \gamma = \pm 1/(2\sqrt{2})$ for the open string and $\pm \gamma = \pm 1/\sqrt{2}$ for the closed string. And of course each component of $\pi_k$ is an odd integer multiple of $\gamma$.

The scattering amplitudes are obtained by integrating the expression (52) or (54) over the unfixed $Z_k$. In the case of the open string, the $Z_k$ are on the real axis satisfying $Z_1 = 0 < Z_2 < \ldots < Z_{N-2} < Z_{N-1} = 1$. In the case of the closed string the $Z_k$ for $k = 2, \ldots , N-2$ are integrated over the whole complex plane. In both cases $Z_1 = 0, Z_{N-1} = 1, Z_N = \infty$. We remind the reader that for physical values of the momenta the resulting integrals are formally divergent. To handle these divergences, one starts with (unphysical) values of the momenta for which the integrals converge, and then one analytically continues to the physical values. For open string amplitudes one can do this keeping the range of the $Z$ integrations complete. But for closed string amplitudes one is forced to divide the integration region up into cells, with separate analytic continuations in each cell.
A.1 Maximal helicity violation

A dramatic simplification occurs when there is maximal helicity violation. For instance, choose all components of the first $N-1$ $\pi_k$ to have the value $-\gamma$. Then necessarily all components of $\pi_N$ and of each $\gamma_r$ have the value $+\gamma$. In this case $\gamma_r \cdot \gamma_s = \pi_k \cdot \pi_l = -\gamma_r \cdot \pi_l = s \gamma^2/2$ for $k, l \neq N$ and $\pi_N$ doesn’t appear in the formula. Then for the open case ($\gamma^2 = 1/8$) the contribution to the integrand of the scattering amplitude is

$$I_O \left| \frac{\partial T}{\partial Z} \right| \det^{- (24-s)/2} (-\nabla^2)_{\text{open}}$$

$$\to (2\delta)^s (N-2)/32 \prod_{k=1}^{N} \frac{1}{\sqrt{\alpha_k}} \left[ \prod_{r<s} |x_r - x_s|^2 \prod_{k<l<N} |Z_k - Z_l|^2 \right]^{s/24} |Z_k - Z_l|^{p_k \cdot p_l}$$

$$\to \epsilon^s (N-2)/32 \prod_{k=1}^{N} \frac{1}{\sqrt{\alpha_k}} |\alpha_N|^{s(N-1)/12} \left[ \prod_{r<s} |x_r - x_s|^{s/12} \prod_{k<l<N} |Z_k - Z_l|^{s/12} \right] |Z_k - Z_l|^{p_k \cdot p_l}$$

where we made use of Eq.(128) (Eq(B3) of the PRD version) of [9] to arrive at the last line. With this simple choice the scattering amplitude is then

$$A_{\text{Open}}^N = g^{N-2} \prod_{k=1}^{N} \frac{1}{\sqrt{\alpha_k}} |\alpha_N|^{s(N-1)/12} \int dZ_2 \cdots dZ_{N-1}$$

$$\prod_{r<s} |x_r - x_s|^{s/12} \prod_{k<l<N} |Z_k - Z_l|^{2p_k \cdot p_l - s/12}$$

(57)

where we identified $g = [2\delta]^s/32$. Making the same simplifications for the case of the closed string leads to

$$A_{\text{Closed}}^N = G^{(N-2)} \prod_{k=1}^{N} \frac{1}{\sqrt{\alpha_k}} |\alpha_N|^{s(N-1)/6} \int d^2 Z_2 \cdots d^2 Z_{N-1}$$

$$\prod_{r<s} |x_r - x_s|^{s/6} \prod_{k<l<N} |Z_k - Z_l|^{p_k \cdot p_l - s/6}$$

(58)

where $G = 2^{s/16} g^2$. If desired, one can replace $2p_k \cdot p_l$ by $(p_k + p_l)^2 - 2 + s/6$ in the open case and by $(p_k + p_l)^2 - 8 + 2s/3$ in the closed case. Keep in mind that this simpler expression only applies for a very special choice for the $\pi$’s and $\gamma$’s. In particular, even for a particular set of the $\pi_k$, the full insertion factor at each vertex is $2 \cos(\gamma \phi)$, which can be implemented by summing the amplitudes over each component of each $\gamma_r$ assuming both possible values $\pm \gamma$.

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