CONTINUOUS COBOUNDARIES OF THE PRODUCT OF SMOOTH FUNCTIONS

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ABSTRACT. Let $X$ be a compact metric space and let $T_i$, $1 \leq i \leq H$ be surjective continuous maps on $X$. We give necessary and sufficient conditions for the product of $H$ smooth functions $f_1, f_2, \ldots, f_H$ to be a continuous coboundary. As a consequence, we provide a topological analog of a result by I. Assani [6].

1. INTRODUCTION

1.1. Coboundaries in topological dynamical systems. Let $(X, T)$ be a topological dynamical system, where $X$ is a compact metric space and $T : X \to X$ is a continuous map. We say a real-valued continuous function $f \in C(X) := C(X, \mathbb{R})$ is a coboundary if there exists $g \in C(X)$ such that $f = g - g \circ T$. There has been numerous studies in topological dynamics on the conditions under which a continuous function $f$ is a coboundary. There are two famous results: One of them is credited to A. Livšic, which can be stated as follows.

**Theorem 1.1** ([30,31]). Let $T : X \to X$ be a transitive, surjective continuous map that satisfies the closing property (see Definition 2.3). Let $f : X \to \mathbb{R}$ be a $\alpha$-Hölder map (see Definition 2.5) such that for any $p$ in $X$ with $T^k p = p$ for some $k$ in $\mathbb{N}$, the sum $\sum_{j=0}^{k-1} f(T^j p)$ is equal to zero. Then there exists $g : X \to \mathbb{R}$ such that it is $\alpha$-Hölder and $g - g \circ T = f$.

We refer to the inspiring monograph by W. Parry and M. Pollicott [34, Proposition 3.7] for a short proof in the setting of subshifts of finite type and to [33, Theorem 3] for a short proof in the setting of topologically transitive homeomorphisms of a compact metric space. The Livšic theorem can be interpreted as that the dynamical information of the system is stored in the periodic points. In thermodynamic formalism, for instance, Livšic theorem gives a criteria to decide when two functions have the same equilibrium states (see: [10, Theorem 1.28]).

The other result is credited to W. Gottschalk and G. Hedlund, which deals with a minimal topological system.

**Theorem 1.2** ([21]). Let $(X, T)$ be a minimal topological system and $f : X \to \mathbb{R}$ continuous. The following statements are equivalent.

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1.1 There exists $x$ in $X$ for which
\[
\sup_{n \geq 1} \left| \sum_{j=0}^{n-1} f(T^j x) \right| < \infty.
\]

(2) The function $f$ is a coboundary.

We remark that the Gottschalk-Hedlund theorem holds without the compactness assumption; see [29, Theorem 9 & Corollary 10].

1.2 Coboundaries in operator theory and ergodic theory. In both Theorems 1.1 and 1.2, one sees that the boundedness of the Birkhoff sums $\sum_{j=0}^{n-1} f(T^j x)$ on some point(s) $x \in X$ is enough and necessary. There are similar results in the settings of operator theory and ergodic theory. Given a Banach space $B$, $U$ a linear operator on $B$, and a given $y \in B$, one can ask whether there exists $x \in B$ for which $(I - U)x = y$ (where $I$ is the identity operator on $B$). It is well known that if $\|U\| < 1$, then there is a solution $x = (I - U)^{-1}y$ where $(I - U)^{-1}$ is the von Neumann series, so the interests in this problem rises when $\|U\| \geq 1$, and particularly when $\|U\| = 1$.

A particular instance of this problem can be solved using ergodic theory: Let $(X, \mu)$ be a measure space, and $T : X \to X$ be a measurable map. We set $B = L^p(\mu)$ for some $1 \leq p \leq \infty$, and $U_T$ be the Koopman operator on $L^p(\mu)$, defined by $U_T f := f \circ T$ for every $f \in L^p(\mu)$. For $f \in L^p(\mu)$, the equation $(I - U_T)g = f$ is equivalent to the equation $f(x) = g(x) - g(Tx)$ for $\mu$-a.e. $x \in X$, where $g \in L^p(\mu)$ is the unknown. We refer to [29], [16], and [2] for this avenue of study; as an example, we present a result of M. Lin and R. Sine here:

**Theorem 1.3** ([29, Corollary 6]). Let $(X, \mu)$ be a $\sigma$-finite measure space and $T$ be a non-singular measurable transformation on $X$. If $f \in L^{\infty}(\mu)$, then the equation $(I - U_T)g = f$ has solution with $g \in L^{\infty}(\mu)$ if and only if $\sup_{n \geq 1} \left\| \sum_{j=0}^{n-1} U_T f \right\|_{L^\infty(\mu)} < \infty$.

1.3 Nonconventional ergodic averages and sums. In the past few decades, nonconventional ergodic averages have received many spot lights in ergodic theory. One of the major problems in this field, often referred to as the convergence problem of multiple ergodic averages (a.k.a. multiple recurrence averages or the Furstenberg averages), is as follows: Given a probability measure-preserving system with several transformations $(X, \mathcal{F}, \mu, T_1, T_2, \ldots, T_H)$ (where $\mu$ preserves $T_i$ for all $1 \leq i \leq H$) and functions $f_i \in L^{p_i}(\mu)$ for some $1 \leq p_i \leq \infty$ for each $1 \leq i \leq H$, can we show that whether the averages
\[
\frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{H} f_i \circ T_i^n
\]
converge in $L^2$-norm or for $\mu$-a.e.? This type of averages originally appeared in the work of H. Furstenberg [18] in the 1970’s, where he provides a proof of Szemerédi’s theorem using ergodic theory. The problems of the $L^2$-norm convergence of the Furstenberg averages and its pointwise almost everywhere convergence are quite different. Regarding the $L^2$-norm convergence, the problem can be considered closed. The first work was by
J-P. Conze and E. Lesigne [12], then generalized by B. Host and B. Kra [22], and independently by T. Ziegler [39]. These results were extended by T. Tao [35], and further generalized by M. Walsh [36], where it is proved the $L^2$-norm convergence when the transformations generate a nilpotent group. On the negative direction, V. Bergelson and A. Leibman showed that the averages may not converge if the transformations generate a solvable group [8]. Regarding the pointwise almost everywhere convergence, it is less understood and it can be considered in most of the cases an open problem. The question was originally proposed by H. Furstenberg [19] and solved by J. Bourgain [9] for the case $H = 2$, $T_i$ distinct power of single ergodic transformation for each $i = 1, 2$, and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$. Under some assumptions on the space and/or transformations, some other partial results were obtained by J-M. Derrien and E. Lesigne [13], I. Assani [1], and S. Donoso and W. Sun [14]. On the negative direction, I. Assani and Z. Buczolich showed that Bourgain’s result does not hold when $p_1 = p_2 = 1$ [7]. Complete pointwise results have been obtained for cubic averages, even for not necessarily commuting transformations; firstly by I. Assani [3], and later by Q. Chu and N. Frantzikinakis [11]. This list is far from complete, and we refer to [17] for more details of the study on these averages.

Most of the study on nonconventional ergodic averages have been done in measure theoretic settings. In recent years, however, some interesting studies on them have been done in more topological settings, such as thermodynamic formalism and multifractal analysis (for instance: [26], [15], and [27]). Also, there have been some attempts to solve the pointwise convergence problem on nonconventional ergodic averages by using a strictly ergodic topological model that was obtained by B. Weiss [38]. Such attempt appeared first in a preprint by I. Assani [4], and shortly after in preprints by W. Huang, S. Shao, and X. Ye [23, 24].

In this note, we are interested on the sums that appear in the nonconventional ergodic average—which we refer to as the nonconventional ergodic sums.

1.4. Relation between coboundary results and nonconventional ergodic sums. We consider $(X^H, \Phi)$ to be the dynamical system, with $H \geq 2$ and $\Phi = T_1 \times T_2 \times \cdots \times T_H$. The main problem is finding conditions on $(X, \mu, T_1, \ldots, T_H)$ so that $F = \otimes_{i=1}^H f_i$ is a measure-theoretic coboundary, i.e., whether there exists a real-valued function $V$ on $X^H$ such that the map $x \mapsto V \circ \Phi(x, x, \ldots, x)$ is essentially bounded and $\mu$-measurable, and $F(x, x, \ldots, x) = V(x, x, \ldots, x) - V \circ \Phi(x, x, \ldots, x)$ for $\mu$-a.e $x \in X$. Recently, I. Assani [6] obtained a necessary and sufficient condition for this case.

**Theorem 1.4 ( [6, Corollary 1.6 and the remark]).** Let $(X, \mu, T_1, T_2, \ldots, T_H)$ be a measure preserving system, where $T_i$ is bi-measurable for each $1 \leq i \leq H$, and $f_1, f_2, \ldots, f_H \in L^\infty(\mu)$. The following statements are equivalent.

1. We have

$$\sup_N \left\| \sum_{n=1}^N \prod_{i=1}^H f_i \circ T_i^n \right\|_{L^\infty(\mu)} < \infty.$$
There exists a real-valued function \( V \) on \( X^H \) such that the map \( x \mapsto V \circ \Phi^j(x, x, \ldots, x) \) is essentially bounded and \( \mu \)-measurable for every \( j \in \mathbb{Z} \), and

\[
\prod_{i=1}^{H} f_i(x) = V(x, x, \ldots, x) - V \circ \Phi(x, x, \ldots, x) \quad \text{for } \mu\text{-a.e. } x \in X.
\]

The proof uses the diagonal-orbit measure of \( \mu \) (cf. [6, Definition 1.2]), which is a tool introduced in [5] to study the pointwise convergence of nonconventional ergodic averages. Diagonal-orbit measures were used to describe the behavior of the nonconventional ergodic sums and averages along the orbit of the diagonal space \( \Delta = \{(x, x, \ldots, x) \in X^H : x \in X\} \) iterated by the map \( \Phi \).

By contrast to Assani’s work, here we will focus on the topological aspect of the coboundary problem. The main goal is to show that for a certain class of topological system \((X, T_1, T_2, \ldots, T_H)\) and under certain smoothness assumption of the functions \( f_1, f_2, \ldots, f_H \), one can find necessary and sufficient condition so that \( \otimes_{i=1}^{H} f_i \) is a topologically smooth coboundary with respect to the map \( \Phi \).

1.5. Layout of the paper. There are two main sections: Section 2 that contains the statement of our results and Section 3 that contains the proofs. Section 2 has four subsections. In subsection 2.1 we introduce our setting. In subsection 2.2 we state the main result: Theorem 2.7, which corresponds to a nonconventional Livšic theorem, and Corollary 2.9, a topological analog of the main theorem in [6]. Subsection 2.3 contains a density result similar to [32, Theorem 3]. Finally, in subsection 2.4, we discuss a relationship between our main result and the Gottschalk-Hedlund theorem.

2. Results

2.1. Setting and some definitions. Let \((X, d)\) be a compact metric space, that we assume along these notes to be separable, complete, and without isolated points, and \( T : X \rightarrow X \) be a surjective and continuous transformation. The pair \((X, T)\) is called a dynamical system. We start by recalling some classical topological properties of dynamical systems, for this we require the following definitions.

If \( T : X \rightarrow X \) is a continuous transformation, we define the orbit of \( x \in X \) as the set \( \{T^n x : n \geq 0\} \), and given \( U, V \subset X \), we define

\[
N_T(U, V) = \{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}.
\]

Definition 2.1. A dynamical system \((X, T)\) is:

1. Minimal if every point has dense orbit.
2. Transitive if \( N_T(U, V) \neq \emptyset \) for every non empty open subsets \( U, V \subset X \).
3. Weakly-mixing if \( T \times T : X^2 \rightarrow X^2 \) is transitive.
4. Mixing if \( N_T(U, V) \) is cofinite for every non empty open subsets \( U, V \subset X \).

Under some extra assumptions transitivity is related to the existence of points with dense orbit. Indeed we have the following for a continuous transformation (see [37, Theorem 5.9]).
Proposition 2.2. If $T : X \to X$ is a continuous transformation and $TX = X$. Then the following are equivalent.

1. $(X, T)$ is transitive.
2. There exists some $x$ with dense orbit in $X$.
3. The set of points $x$ with dense orbit in $X$ is a dense $G_\delta$ set.

We recall the closing property.

Definition 2.3 (CP). We say that a dynamical systems $(X, T)$ satisfies the closing property (CP) if there exists $D, \delta, \delta_0 > 0$ such that for all $x$ in $X$ and $k$ in $\mathbb{N}$ with $d(x, T^kx) < \delta_0$ there exists $p$ in $X$ such that $T^kp = p$ and such that

$$d(T^ix, T^ip) \leq Dd(x, T^kx)e^{-\delta \min\{i,k-i\}} \text{ for all } 0 \leq i \leq k.$$  

We introduce a density condition (DC) that appears without name in [32]. This is a slightly stronger condition than mixing.

Definition 2.4 (DC). We say that a dynamical systems $(X, T)$ satisfies the density condition (DC) if for any non empty open subsets $U, V \subset X$, there exists a nonempty open set $W \subset V$ and $n_0 \in \mathbb{N}$ such that $W \subset T^n(U)$ for any $n \geq n_0$.

It is worth mentioning that both conditions are satisfied by many well studied dynamical systems, for example, subshifts of finite type (Definition 2.6) and hyperbolic diffeomorphisms on compact manifolds.

We define the space of $\alpha$-Hölder functions on $X$.

Definition 2.5 ($\alpha$-Hölder). We say that a function $f : X \to \mathbb{R}$ is $\alpha$-Hölder for $\alpha \in (0,1]$ if there exists a constant $C > 0$ such that for every pair of points $x, y$ in $X$,

$$|f(x) - f(y)| \leq Cd(x,y)^\alpha.$$  

We end this subsection with the definition of a subshift of finite type, as it is an important example of a dynamical system where we can trivially apply our results.

Definition 2.6. Let $m \geq 2$ and $A = (A_{i,j})_{1 \leq i,j \leq m}$ be a matrix of size $m \times m$ with coordinates in $\{0,1\}$. A subshift of finite type is the dynamical systems $(X, \sigma)$, where $X = \{(x_0, x_1, x_2, \ldots) : x_i \in \{1,2,\ldots,m\}, A_{x_i,x_{i+1}} = 1\}$ and $\sigma$ is the action defined by $\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$.

The action $\sigma$ is continuous with respect to the metric $d$ defined by $d(x,x) := 0$ and for $x \neq y, d(x,y) := \theta^N$, where $\theta \in (0,1)$ and $N = N(x,y)$ is largest integer such that $x_i = y_i$ for $0 \leq i < N$. There is a characterization of mixing subshifts of finite type given the matrix $A$. Indeed, let $G$ be the oriented graph associated with $A$, then $(X, \sigma)$ is mixing iff $A$ is irreducible and there exist cycles $\pi_1, \ldots, \pi_k$ of lengths $m_1, \ldots, m_k$ in $G$ such that the greatest common divisor of $\{m_1, \ldots, m_k\}$ is one.
2.2. Nonconventional Livšic Theorem. Given $H$ in $\mathbb{N}$ and the dynamical systems $(X, T_i)$ for $1 \leq i \leq H$, we obtain a Livšic Theorem for multiple ergodic averages, that is, we consider the metric space $(X^H, d_H)$ with

$$d_H(x, y) := \max_{1 \leq i \leq H} d(x_i, y_i),$$

and the dynamical system $(X^H, \Phi)$ with

$$\Phi^n(x) = \Phi^n(x_1, x_2, \ldots, x_H) := (T_1^n x_1, T_2^n x_2, \ldots, T_H^n x_H)$$

for $n \in \mathbb{N}$, and

$$\Phi^0(x) := x.$$

We recover a nonconventional Livšic theorem.

**Theorem 2.7** (Nonconventional Livšic). Let $f : X^H \to \mathbb{R}$ be an $\alpha$-Hölder map. For each $1 \leq i \leq H$, let $(X, T_i)$ satisfy the CP and the DC, where $T_i$ is a continuous, surjective, and open map such that $T_i T_j = T_j T_i$ for every $1 \leq j \leq H$. Then the following statements are equivalent.

1. There exists a non-empty open subset $U \subset X$ such that

$$\sup_{n} \sup_{x \in U} \left| \sum_{j=0}^{n} f(\Phi^j \hat{x}) \right| < \infty \text{ where } \hat{x} = (x, x, \ldots, x).$$

2. For any $p \in X^H$ such that $T^k p = p$, we have

$$\sum_{j=0}^{k-1} f(\Phi^j p) = 0.$$

3. There exists $V : X^H \to \mathbb{R}$ such that it is $\alpha$-Hölder and $V - V \circ \Phi = f$.

4. There exists $D > 0$ such that

$$\sup_{n} \sup_{x \in X^H} \left| \sum_{j=0}^{n} f(\Phi^j x) \right| < D.$$

We note that statement (i) only requires one to check the boundedness of the sums along the diagonal of $X^H$, as opposed to the entire space $X^H$. This may be an easier property to check than the others, and leaves a possibility for a simulation to find if such open set $U$ exists.

**Remark 2.8.** A particular important case of application of Theorem 2.7 is to the case that $T_i = \sigma^i$ for $1 \leq i \leq H$ and $(X, \sigma)$ is a mixing subshift of finite type.

Furthermore, we recover a topological analog of the main result in [6].

**Corollary 2.9.** Assume the same hypothesis as in Theorem 2.7. Then the following statements are equivalent.

1. There exists a non-empty open subset $U \subset X$ such that

$$\sup_{N} \sup_{x \in U} \left| \sum_{j=0}^{N} \prod_{i=1}^{H} f_i(T_i^j x) \right| < \infty.$$
(2) The product of the function is a $\alpha$-Hölder coboundary, i.e. if $\Phi = T_1 \times T_2 \times \cdots \times T_H$, there exists $V$ $\alpha$-Hölder such that

$$\bigotimes_{i=1}^{H} f_i = V - V \circ \Phi.$$ 

This corollary answers a question that was raised during a discussion between the second author and S. Donoso: When is the product function $\bigotimes_{i=1}^{H} f_i$ a continuous coboundary?

2.3. Denseness of the orbit of the diagonal space. In order to prove that (i) of Theorem 2.7 implies (ii) of the same theorem, we will need some density argument. Indeed, we require a bound for the sums along the iterations by $\Phi = T_1 \times T_2 \times \cdots \times T_n$ of points in the diagonal space of $X^H$ that have dense orbit. Therefore, in particular, we will require conditions on $(X, T_1, \ldots, T_n)$ such that those points exist.

We define the diagonal space of $X^H$ by $\Delta := \{(x, x, \ldots, x) \in X^H : x \in X\}$. If $x \in X$, we denote $\hat{x} := (x, x, \ldots, x) \in \Delta$. In particular, given $x \in X$ and $n \in \mathbb{N}$, we denote $\Phi^n \hat{x} := (T_1^n x, T_2^n x, \ldots, T_H^n x)$. It is of our interest to know sufficient conditions for the set

$$E_{\Delta} := \{x \in X : \{\Phi^n \hat{x}\}_{n=0}^{\infty} \text{ is dense in } X^H\}$$

to be non-empty. Such study was originally done by E. Glasner [20], and later by T. K. S. Moothathu [32], D. Kwietniak and P. Oprocha [28], and W. Huang, S. Shao, and X. Ye [25]. Also, I. Assani showed that if the maps $T_i$’s are weakly mixing homeomorphisms (in a measure-theoretic sense) preserving a common probability measure $\mu$ for which the measure of every non-empty open set is positive, then $\mu(E_{\Delta})$ equals to one, hence $E_{\Delta}$ is non-empty; see the proof of [4, Theorem 1].

Here we prove the following density result.

**Theorem 2.10.** Suppose that $T_1, T_2, \ldots, T_H$ are mixing, open and commuting, and satisfying the DC. Then for any infinite subset $A \subset \mathbb{N}$, there exists a dense $G_\delta$-set $Y \subset X$ such that for any $x \in Y$, the set $\{\Phi^n \hat{x} : n \in A\}$ is dense in $X^H$.

In an event $T_i = T^i$ for $1 \leq i \leq H$ for some continuous function $T : X \to X$, the result was proven in [32, Theorem 3]$^1$. In particular, the theorem holds if $T_i = \sigma^i$ for $1 \leq i \leq H$ and $(X, \sigma)$ is a mixing subshift of finite type [32, Proposition 5].

2.4. A relation to the Gottschalk-Hedlund theorem. We also obtain a nonconventional version of the Gottschalk-Hedlund theorem. For the case $H = 1$, we observe that Theorem 1.2 remains unchanged if we replace "there exists $x_0 \in X"$ in the first statement with "there exists $x_0 \in E_{\Delta} = \{x \in X : \{T^n x\}_{n=0}^{\infty} \text{ is dense in } X\},"$ since $(X, T)$ is minimal, which implies that $E_{\Delta} = X$. We obtain the following result that is related to the Gottschalk-Hedlund theorem, using an argument similar to the proof of Theorem 2.7.

$^1$In this case, the openness of $T$ was unnecessary.
**Theorem 2.11** (Nonconventional Gottschalk-Hedlund Theorem). For each $1 \leq i \leq H$, let $(X, T_i)$ be a minimal and weakly-mixing system that satisfies CP, and $T_i T_j = T_j T_i$ for each $1 \leq j \leq H$. Suppose $f : X^H \to \mathbb{R}$ is $\alpha$-Hölder. Then the following statements are equivalent.

(a) There exists $x_0 \in E_\Delta$ such that $\sup_n \left| \sum_{j=0}^n f(T_i^j x_0) \right| < \infty$.

(b) There exists $g \in C(X^H)$ for which $f = g - g \circ \Phi$.

We remark that the set $E_\Delta$ in the theorem is non-empty, due to a result obtained by W. Huang, S. Shao, and X. Ye. We present a simplified version of their result here (the original statement concerns nilpotent group actions).

**Proposition 2.12** (See: [25, Theorem 1.1]). Let $(X, T_i)$ be a minimal and weakly-mixing system, and $T_i T_j = T_j T_i$ for $1 \leq i, j \leq H$. Then the set $E_\Delta$ is $G_\delta$-dense in $X$.

### 3. Proofs

We start by a generalization of [32, Proposition 1] that guarantees the set $E_\Delta$ to be non-empty. Recall that $\Delta := \{(x, x, \ldots, x) \in X^H : x \in X\}$, $\hat{x} := (x, x, \ldots, x)$ for $x \in X$ and that $\Phi^n \hat{x} := (T_1^i x, T_2^i x, \ldots, T_H^i x)$ for $x \in X, \hat{x} \in \Delta$ and $n \in \mathbb{N}$.

**Proposition 3.1.** Let $H \in \mathbb{N}$ and $A \subset \mathbb{N}$ be an infinite set. Then the following statements are equivalent.

1. If $U_0, U_1, \ldots, U_H \subset X$ are nonempty open sets, there exists $n \in A$ such that
   $$\bigcap_{i=0}^H T_i^{-n}(U_i) \neq \emptyset.$$

2. There exists a dense $G_\delta$ subset $Y \subset X$ such that for every $x \in Y$, the set $\{\Phi^n \hat{x} : n \in A\}$ is dense in $X^H$. Moreover, if there exists $i$ such that $T_i$ commutes with all $T_j$ for $1 \leq j \leq H$ and $A = \mathbb{N}$, statements (1) and (2) are equivalent to

3. There exists $x$ in $X$ such that the set $\{\Phi^n \hat{x} : n \in \mathbb{N}\}$ is dense in $X^H$.

The argument follows directly from the proof of [32, Proposition 1].

**Proof.** Assume that (1) is satisfied. Let $\{B_k : k \in \mathbb{N}\}$ be a countable base of open balls of $X$. If
   $$Y = \bigcap_{(k_1, \ldots, k_H) \in \mathbb{N}^H} \bigcup_{n \in A} \bigcup_{i=1}^H T_i^{-n}(B_k).$$

Then, by the Baire Category Theorem, $Y$ is a dense $G_\delta$ subset of $X$, and by construction, every $x \in Y$ satisfies (2).

Now assume that (2) is satisfied. Then, for every $x \in Y \cap U_0$ there exists $n \in A$ such that $(T_1^i x, T_2^i x, \ldots, T_H^i x) \in U_1 \times U_2 \times \cdots \times U_H$, then $\bigcap_{i=0}^H T_i^{-n}(U_i) \neq \emptyset$.

Now let $A = \mathbb{N}$. To show that (3) implies (1), suppose that there exists $i$ such that $T_i$ commutes with all $T_j$ for $1 \leq j \leq H$ and $A = \mathbb{N}$. Choose $k \in \mathbb{N}$ such that $y = T_k^i(x) \in U_0$, then
   $$\{\Phi^n \hat{x} : n \in \mathbb{N}\}$$

This completes the proof.
is dense in $X^H$, because $T_i$ commutes with all $T_j$ for $1 \leq j \leq H$. In particular, there exists $n \in \mathbb{N}$ such that $(T_1^n y, T_2^n y, \cdots, T_H^n y) \in U_1 \times U_2 \times \cdots \times U_H$, then $\bigcap_{i=0}^HT_i^{-n}(U_i) \neq \emptyset$. \hfill $\square$

We state and prove a lemma used in the proof Theorem 2.10.

**Lemma 3.2.** Let $T_1, T_2$ be continuous maps on a compact metrizable space $X$ that commute with each other. Then for any $U \subset X$, we have

$$T_2^{-1}[T_1(U)] = T_1[T_2^{-1}(U)].$$

**Proof.** We simply observe that

$$T_1[T_2^{-1}(U)] = T_1\{y \in X : T_2(y) \in U\} = \{z \in X : T_1(T_2(y)) = z \text{ for some } y \in U\} = \{z \in X : T_2(T_1(y)) = z \text{ for some } y \in U\} = T_2^{-1}\{w \in X : T_1(y) = w \text{ for some } y \in U\} = T_2^{-1}[T_1(U)].$$

We are ready to prove Theorem 2.10. The proof is motivated by [32, Theorem 3].

**Proof of Theorem 2.10.** We will show the following: Given $k \in \mathbb{N}$, and nonempty open sets $U_0, U_1, \ldots, U_k \subset X$ and infinite $A \subset \mathbb{N}$, the set $U_0 \cap \left(\bigcap_{i=1}^k T_i^{-n}(U_i)\right)$ is nonempty for some $n \in A$, which would satisfy (1) of Proposition 3.1. The base case $k = 1$ follows from the assumption. Now assume the claim holds for $k = m$. From the assumption, there exists non-empty and open $W \subset U_1$ and $n_0 \in \mathbb{N}$ such that $W \subset T_1^n(U_0)$ for any $n \geq n_0$. By the inductive hypothesis, there exists $n \in A$ with $n \geq n_0$ so that

$$W' = W \cap T_2^{-n}[T_1^n(U_2)] \cap T_3^{-n}[T_1^n(U_3)] \cap \cdots \cap T_{m+1}^{-n}[T_1^n(U_{m+1})] \neq \emptyset$$

(since $T_1$ is an open map). Since $W \subset U_1$ and $W' \subset T_1^n(U_0)$, we get from applying Lemma 3.2 that

$$\emptyset \neq W' \subset T_1^n(U_0) \cap U_1 \cap T_2^{-n}[T_1^n(U_2)] \cap T_3^{-n}[T_1^n(U_3)] \cap \cdots \cap T_{m+1}^{-n}[T_1^n(U_{m+1})] = T_1^n(U_0) \cap U_1 \cap T_1^n[T_2^{-n}(U_2)] \cap T_1^n[T_3^{-n}(U_3)] \cap \cdots \cap T_1^n[T_{m+1}^{-n}(U_{m+1})].$$

Finally, taking the pre-image by $T_1^n$, and knowing that $T_1$ is surjective, we obtain that

$$U_0 \cap \left(\bigcap_{i=1}^m T_i^{-n}(U_i)\right) \neq \emptyset.$$ 

\hfill $\square$

**Lemma 3.3.** If $\Phi = T_1 \times \cdots \times T_H$ is transitive for some $x_0 \in X^H$ (i.e. $\{\Phi^n x_0\}_{n \geq 0}$ is dense in $X^H$), and $f : X^H \to \mathbb{R}$ is a $\alpha$-Hölder function such that

$$\sup \left\{ \sum_{j=0}^{n-1} f(\Phi^{j+k}x_0) : n, k \in \mathbb{N} \right\} < C \text{ for some } C > 0.$$
Then, there exists $c > 0$ such that for every $x$ in $X^H$, for every $n$ in $\mathbb{N}$, the inequality
\[
\left| \sum_{j=0}^{n-1} f(\Phi^j x) \right| \leq c
\]
holds.

Proof. Suppose the hypotheses in the statement are valid. Let $x$ in $X^H$ and $n \in \mathbb{N}$. Given any $\delta > 0$ we can find $k_0, k_1, \ldots, k_{n-1} \in \mathbb{N}$ such that $d(\Phi^j x, \Phi^{k_j}(\Phi^i x_0)) < \delta$ for every $0 \leq j \leq n - 1$, because $\Phi^i x_0$ has dense orbit for every $0 \leq j \leq n - 1$. We have that
\[
\left| \sum_{j=0}^{n-1} f(\Phi^j x) \right| \leq \sum_{j=0}^{n-1} |f(\Phi^j x) - f(\Phi^{k_j}(\Phi^i x_0))| + \sum_{j=0}^{n-1} f(\Phi^{k_j}(\Phi^i x_0))
\]
\[
\leq \sum_{j=0}^{n-1} d(\Phi^j x, \Phi^{k_j}(\Phi^i x_0))^{\alpha} + \sum_{j=0}^{n-1} f(\Phi^{k_j}(\Phi^i x_0))
\]
\[
\leq n \delta^{\alpha} + C.
\]
Therefore, choosing $0 < \delta < n^{-\frac{1}{\alpha}}$, we obtain from the last inequality that
\[
\left| \sum_{j=0}^{n-1} f(\Phi^j x) \right| < 1 + C.
\]

\[\square\]

Lemma 3.4. If $\Phi = T_1 \times \cdots \times T_H$, $f : X \to \mathbb{R}$ is an $\alpha$-Hölder function and there exists $c > 0$ such that for every $x$ in $X^H$, for every $n$ in $\mathbb{N}$, the inequality
\[
\left| \sum_{j=0}^{n-1} f(\Phi^j x) \right| \leq c
\]
holds. Then for every $p$ in $X^H$ such that $\Phi^k p = p$ for some $k \in \mathbb{N}$, the equality
\[
\sum_{j=0}^{k-1} f(\Phi^j p) = 0
\]
holds.

Proof. Suppose $\Phi^k p = p$ for some $k \in \mathbb{N}$ and assume that $\sum_{j=0}^{k-1} f(\Phi^j p) = b$. Given any $n \in \mathbb{N}$ for which $n \geq k$, one writes $n = qk + r$ for some $q \in \mathbb{N}$ and $0 \leq r < k$, we have
\[
\sum_{j=0}^{k-1} f(\Phi^j p) = qb + \sum_{j=0}^{r-1} f(\Phi^j p),
\]
therefore, the inequality $|\sum_{j=0}^{n-1} f(\Phi^j p)| \leq c$ holds for every $n \in \mathbb{N}$ iff $b = 0$, which finishes the proof. \[\square\]

Lemma 3.5. If $(X, T_i)$ satisfies the CP for every $1 \leq i \leq H$. Then $(X^H, \Phi = T_1 \times \cdots \times T_H)$ also satisfies the CP.
Proof. Assume that \((X, T_i)\) satisfies the CP for every \(1 \leq i \leq H\). By the definition, for every \(1 \leq i \leq H\), there exists \(D_i, \delta_i, \delta_0(i) > 0\) such that for all \(x\) in \(X\) and \(k\) in \(\mathbb{N}\) with \(d(x, T_i^k x) < \delta_0(i)\) there exists \(p_i\) in \(X\) such that \(T_i^k p_i = p_i\) and such that \(d(T_i^k x, T_i^k p_i) \leq D_j d(x, T_i^k x)e^{-\delta_i \min\{j, k - j\}}\) for all \(j = 0, \ldots, k\). Therefore, choosing \(D = \max_{1 \leq i \leq H} D_i, \delta = \min_{1 \leq i \leq H} \delta_i\) and \(\delta_0 = \min_{1 \leq i \leq H} \delta_0(i)\), we obtain that for all \(x = (x_1, \ldots, x_H)\) in \(X^H\) and \(k\) in \(\mathbb{N}\) with \(d_H(x, \Phi^k x) < \delta_0\) there exists \(p = (p_1, \ldots, p_H)\) in \(X^H\) such that \(\Phi^k p = p\) and such that \(d_H(\Phi^j x, \Phi^j p) \leq D d_H(x, \Phi^j x)e^{-\delta \min\{j, k - j\}}\) for all \(j = 0, \ldots, k\). \(\square\)

Proof of Theorem 2.7. We start by proving that (i) implies (ii). Assume that (i) is satisfied for some non empty open set \(U \subset X\). As a consequence of Theorem 2.10 with \(A = \mathbb{N}\), we have that there exists a dense \(G_\sigma\)-set \(Y\) such that for every \(x \in Y\) the set \(A_x := \{\Phi^n x : n \in \mathbb{N}\}\) is dense in \(X^H\), in particular, there exists \(x_0 \in U \cap Y\) such that \(A_{x_0}\) is dense. We can use Lemma 3.3 and Lemma 3.4 to conclude that (ii) is satisfied. The proof that (ii) implies (iii) follows from Theorem 1.1, because \((X^H, \Phi)\) is a dynamical systems is transitive, by Theorem 2.10, and that satisfies CP, by Lemma 3.5. To prove that (iii) implies (iv) we note that for any \(n > 0\)

\[
\sup_{x \in X^H} \left| \sum_{j=0}^{n-1} f(\Phi^j x) \right| = \sup_{x \in X^H} |f(x) - f(\Phi^n x)| \leq 2 \sup_{x \in X^H} |f(x)|,
\]

and since \(f\) is a continuous function on a compact set \(X^H\), the right-hand side of the inequality is finite. Finally, (iv) trivially implies (i). \(\square\)

We now turn to the proof of Theorem 2.9. Before we do so, we prove the following lemma regarding when a product of functions is \(\alpha\)-Hölder.

**Lemma 3.6.** If \(f_1, \ldots, f_H\) are \(\alpha\)-Hölder. Then \(\bigotimes_{i=1}^H f_i\) is \(\alpha\)-Hölder.

In the proof we use the following notation. Given \(H \geq 2, x = (x_1, x_2, \ldots, x_H)\) and \(f := \bigotimes_{i=1}^H f_i\), we define \(f^* := \bigotimes_{i=1}^{H-1} f_i\) and \(x^* := (x_1, x_2, \ldots, x_{H-1})\).

**Proof.** We proceed by an induction on \(H\). The base case \(H = 1\) is trivial. Assume that \(f_1, \ldots, f_H\) are \(\alpha\)-Hölder and the property is valid for \(H - 1\) (i.e. \(f^*\) is \(\alpha\)-Hölder). Let \(x, y \in X^H\). We have that

\[
|f(x) - f(y)| = |f^*(x^*)f_H(x_H) - f^*(y^*)f_H(y_H)|
\]

\[
\leq |f^*(x^*)||f_H(x_H) - f_H(y_H)| + |f_H(y_H)||f^*(x^*) - f^*(y^*)|.
\]

Using that \(f_H\) is \(\alpha\)-Hölder we obtain that \(|f_H(x_H) - f_H(y_H)| \leq C_1 d(x_H, y_H)^\alpha\) for some \(C_1 > 0\). Using that \(f^*\) is \(\alpha\)-Hölder we obtain that \(|f^*(x^*) - f^*(y^*)| \leq C_2 d_{m-1}(x^*, y^*)^\alpha\) for some \(C_2 > 0\). Using the compactness of the space \(X\) (therefore also of \(X^{H-1}\)) and the continuity of \(f_H\) and \(f^*\), we have that there exist constants \(C_3, C_4 > 0\) such that \(C_3 := \sup\{|f_H(x)| : x \in X\}\) and \(C_4 := \sup\{|f^*(x)| : x \in X^{H-1}\}\). Therefore, for \(C := \max\{C_4 C_1, C_3 C_2\}\), we have that

\[
|f(x) - f(y)| \leq C_4 C_1 d(x_H, y_H)^\alpha + C_3 C_2 d_{H-1}(x^*, y^*)^\alpha \leq 2 C d_H(x, y)^\alpha.
\]

\(\square\)
Proof of Theorem 2.9. We use Lemma 3.6 to obtain that $\bigotimes_{i=1}^{H} f_i$ is $\alpha$-Hölder. If (1) is satisfied, then the hypotheses and the condition (i) of Theorem 2.7 are satisfied. Then the condition (iii) of Theorem 2.7 is satisfied, in particular we obtain (2). And of course, (2) easily implies (1). □

Finally, we prove our version of a nonconventional Gottschalk-Hedlund theorem.

Proof of Theorem 2.11. It is clear that (b) implies (a). Now suppose that (a) holds. By Proposition 2.12, we can apply Lemmas 3.3 and 3.4 to show that the hypothesis of Theorem 1.1 is satisfied (for the dynamical system $(X^H, \Phi)$, which also satisfies CP by Lemma 3.5). Hence, (b) holds. □

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