Strongly Adaptive Regret Implies Optimally Dynamic Regret

Lijun Zhang
National Key Laboratory for Novel Software Technology
Nanjing University, Nanjing 210023, China

Tianbao Yang
Department of Computer Science
the University of Iowa, Iowa City, IA 52242, USA

Rong Jin
Alibaba Group, Seattle, USA

Zhi-Hua Zhou
National Key Laboratory for Novel Software Technology
Nanjing University, Nanjing 210023, China

Abstract
To cope with changing environments, recent developments in online learning have introduced the concepts of adaptive regret and dynamic regret independently. In this paper, we illustrate an intrinsic connection between these two concepts by showing that the dynamic regret can be expressed in terms of the adaptive regret and the functional variation. This observation implies that strongly adaptive algorithms can be directly leveraged to minimize the dynamic regret. As a result, we present a series of strongly adaptive algorithms whose dynamic regrets are minimax optimal for convex functions, exponentially concave functions, and strongly convex functions, respectively. To the best of our knowledge, this is the first time that such kind of dynamic regret bound is established for exponentially concave functions. Moreover, all of those adaptive algorithms do not need any prior knowledge of the functional variation, which is a significant advantage over previous specialized methods for minimizing dynamic regret.

1. Introduction
Online convex optimization is a powerful paradigm for sequential decision making (Shalev-Shwartz, 2011). It can be viewed as a game between a learner and an adversary: In the $t$-th round, the learner selects a decision $w_t \in \Omega$, simultaneously the adversary chooses a function $f_t(\cdot) : \Omega \rightarrow \mathbb{R}$, and then the learner suffers an instantaneous loss $f_t(w_t)$. This study focuses on the full-information setting (Cesa-Bianchi and Lugosi, 2006), where the function $f_t(\cdot)$ is revealed to the learner at the end of each round. The goal of the learner is to minimize the cumulative loss over $T$ periods. The standard performance measure is regret, which is the difference between the loss incurred by the learner and that of the best fixed decision in hindsight, i.e.,

\[ \text{Regret}(T) = \sum_{t=1}^{T} f_t(w_t) - \min_{w \in \Omega} \sum_{t=1}^{T} f_t(w). \]
The above regret is typically referred to as *static* regret in the sense that the comparator is time-invariant. The rationale behind this evaluation metric is that one of the decision in $\Omega$ is reasonably good over the $T$ rounds. However, when the underlying distribution of loss functions changes, the static regret may be too optimistic and fails to capture the hardness of the problem.

To address this limitation, new forms of performance measure, including *adaptive* regret ([Hazan and Seshadhri, 2007, 2009]) and *dynamic* regret ([Zinkevich, 2003]), were proposed and received significant interest recently. Given a parameter $\tau$, which is the length of the interval, the strong version of adaptive regret is defined as

$$\text{SA-Regret}(T, \tau) = \max_{[s, s+\tau-1] \subseteq [T]} \left( \sum_{t=s}^{s+\tau-1} f_t(w_t) - \min_{w \in \Omega} \sum_{t=s}^{s+\tau-1} f_t(w) \right).$$

From the definition, we observe that minimizing the adaptive regret enforces the learner has small static regret over any interval of length $\tau$. Since the best decision for different intervals could be different, the learner is essentially competing with a changing comparator.

A parallel line of research introduces the concept of dynamic regret, where the cumulative loss of the learner is compared against a comparator sequence $u_1, \ldots, u_T \in \Omega$, i.e.,

$$\text{D-Regret}(u_1, \ldots, u_T) = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(u_t).$$

It is well-known that in the worst case, a sublinear dynamic regret is impossible unless we impose some regularities on the comparator sequence or the function sequence ([Jadbabaie et al., 2015]). A representative example is the functional variation defined below

$$V_T = \sum_{t=2}^{T} \max_{w \in \Omega} |f_t(w) - f_{t-1}(w)|.$$ 

[Besbes et al. (2015)] have proved that as long as $V_T$ is sublinear in $T$, there exists an algorithm that achieves a sublinear dynamic regret. Furthermore, under the noisy gradient feedback, a general restarting procedure is developed, and it enjoys $O(T^{2/3}V_T^{1/3})$ and $O(\log T \sqrt{T V_T})$ rates for convex functions and strongly convex functions, respectively. This result is very strong in the sense that these rates are (almost) minimax optimal. However, the restarting procedure can only be applied when an upper bound of $V_T$ is known beforehand, thus limiting its application in practice.

While both the adaptive and dynamic regrets aim at coping with changing environments, little is known about their relationship. This paper makes a step towards understanding their connections. Specifically, we show that the strongly adaptive regret in (1), together with the functional variation, can be used to upper bound the dynamic regret in (2). Thus, an algorithm with a small strongly adaptive regret is automatically equipped with a tight dynamic regret. As a result, we obtain a series of algorithms for minimizing the dynamic regret that do not need any prior knowledge of the functional variation. The main contributions of this work are summarized below.

- We provide a general theorem that upper bounds the dynamic regret in terms of the strongly adaptive regret and the functional variation.
• For convex functions, we show that the strongly adaptive algorithm of Jun et al. (2016) has a dynamic regret of $O(T^{2/3}V_T^{1/3} \log^{1/3} T)$, which matches the minimax rate, up to a polylogarithmic factor.

• For exponentially concave functions, we propose a strongly adaptive algorithm that allows us to control the tradeoff between the adaptive regret and the computational cost explicitly. Furthermore, we demonstrate that its dynamic regret is $O(\sqrt{TV_T \log T})$, and this is the first time such kind of dynamic regret bound is established for exponentially concave functions.

• Since strongly convex functions with bounded gradients are also exponentially concave, our previous result immediately implies a dynamic regret of $O(\sqrt{TV_T \log T})$, which is also minimax optimal up to a polylogarithmic factor. It also indicates our bound for exponentially concave functions is almost optimal.

2. Related Work

In this section, we give a brief introduction to previous work on static, adaptive, and dynamic regrets in the context of online convex optimization.

2.1 Static Regret

The majority of studies in online learning are focused on static regret Shalev-Shwartz and Singer (2007); Langford et al. (2009). For general convex functions, the classical online gradient descent achieves $O(\sqrt{T})$ and $O(\log T)$ regret bounds for convex and strongly convex functions, respectively (Zinkevich, 2003; Hazan et al., 2007; Shalev-Shwartz et al., 2007). Both the $O(\sqrt{T})$ and $O(\log T)$ rates are known to be minimax optimal (Abernethy et al., 2009).

When functions are exponentially concave, a different algorithm, named online Newton step, is developed and enjoys an $O(\log T)$ regret bound (Hazan et al., 2007).

2.2 Adaptive Regret

The concept of adaptive regret is introduced by Hazan and Seshadhri (2007), and later strengthened by Daniely et al. (2015). To distinguish between them, we refer to the definition of Hazan and Seshadhri (2007) as weakly adaptive regret and the one of Daniely et al. (2015) as strongly adaptive regret. The weak version is given by

$$\text{WA-Regret}(T) = \max_{[s,q] \subseteq [T]} \sum_{t=s}^{q} f_t(w_t) - \min_{w \in \Omega} \sum_{t=s}^{q} f_t(w).$$

To minimize the adaptive regret, Hazan and Seshadhri (2007) have developed two meta-algorithms: an efficient algorithm with $O(\log T)$ computational complexity per iteration and an inefficient one with $O(T)$ computational complexity per iteration. These meta-algorithms use an existing online method (that was possibly designed to have small static regret) as a subroutine. For convex functions, the efficient and inefficient meta-algorithms

---

1. For brevity, we ignored the factor of subroutine in the statements of computational complexities. The $O(\cdot)$ computational complexity should be interpreted as $O(\cdot) \times s$ space complexity and $O(\cdot) \times t$ time complexity, where $s$ and $t$ are space and time complexities of the subroutine per iteration, respectively.
have $O(\sqrt{T \log^3 T})$ and $O(\sqrt{T \log T})$ regret bounds, respectively. For exponentially concave functions, those rates are improved to $O(\log^2 T)$ and $O(\log T)$, respectively. We can see that the price paid for the adaptivity is very small: The rates of weakly adaptive regret differ from those of static regret only by logarithmic factors.

A major limitation of weakly adaptive regret is that it does not respect short intervals well. Taking convex functions as an example, the $O(\sqrt{T \log^3 T})$ and $O(\sqrt{T \log T})$ bounds are meaningless for intervals of length $O(\sqrt{T})$. To overcome this limitation, Daniely et al. (2015) proposed a refined adaptive regret that takes the length of the interval as a parameter $\tau$, as indicated in (1). If the strongly adaptive regret is small for all $\tau < T$, we can guarantee the learner has small regret over any interval of any length. In particular, Daniely et al. (2015) introduced the following definition.

**Definition 1** Let $R(\tau)$ be the minimax static regret bound of the learning problem over $\tau$ periods. An algorithm is strongly adaptive, if

$$\text{SA-Regret}(T, \tau) = O(\text{poly}(\log T) \cdot R(\tau)).$$

It is easy to verify that the meta-algorithms of Hazan and Seshadhri (2007) are strongly adaptive for exponentially concave functions but not for convex functions. Thus, Daniely et al. (2015) developed a new meta-algorithm that satisfies $\text{SA-Regret}(T, \tau) = O(\sqrt{T \log T})$ for convex functions, and thus is strongly adaptive. The algorithm is also efficient and the computational complexity per iteration is $O(\log T)$. Later, the strongly adaptive regret of convex functions was improved to $O(\sqrt{\tau \log T})$ by Jun et al. (2016).

### 2.3 Dynamic Regret

In a seminal work, Zinkevich (2003) proposed to use the *path-length* defined as

$$\mathcal{P}(u_1, \ldots, u_T) = \sum_{t=2}^{T} \|u_t - u_{t-1}\|_2$$

to upper bound the dynamic regret. Specifically, Zinkevich (2003) proved that for any sequence of convex functions, the dynamic regret of online gradient descent can be upper bounded by $O(\sqrt{T \mathcal{P}(u_1, \ldots, u_T)})$. Another regularity of the comparator sequence, which is similar to the path-length, is defined as

$$\mathcal{P}'(u_1, \ldots, u_T) = \sum_{t=2}^{T} \|u_t - \Phi_t(u_{t-1})\|_2$$

where $\Phi_t(\cdot)$ is a dynamic model that predicts a reference point for the $t$-th round. Hall and Willett (2013) developed a novel algorithm named dynamic mirror descent and proved that its dynamic regret is on the order of $\sqrt{T \mathcal{P}'(u_1, \ldots, u_T)}$. The advantage of $\mathcal{P}'(u_1, \ldots, u_T)$ is that when the comparator sequence follows the dynamical model closely, it can be much smaller than the path-length $\mathcal{P}(u_1, \ldots, u_T)$.

---

2. That is because (i) $\text{SA-Regret}(T, \tau) \leq \text{WA-Regret}(T)$, and (ii) there is a $\text{poly}(\log T)$ factor in the definition of strong adaptivity.
Let \( w_t^* \in \arg\min_{w \in \Omega} f_t(w) \) be a local minimizer of \( f_t(\cdot) \). For any sequence of \( u_1, \ldots, u_T \in \Omega \), we have

\[
D\text{-Regret}(u_1, \ldots, u_T) = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(u_t)
\leq D\text{-Regret}(w_1^*, \ldots, w_T^*) = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} \min_{w \in \Omega} f_t(w).
\]

Thus, \( D\text{-Regret}(w_1^*, \ldots, w_T^*) \) can be treated as the worst case of the dynamic regret, and there are many work that were devoted to minimizing \( D\text{-Regret}(w_1^*, \ldots, w_T^*) \) \cite{Jadbabaie2015, Mokhtari2016, Yang2016, Zhang2016}.

When a prior knowledge of \( \mathcal{P}(w_1^*, \ldots, w_T^*) \) is available, \( D\text{-Regret}(w_1^*, \ldots, w_T^*) \) can be upper bounded by \( O(\sqrt{TP}(w_1^*, \ldots, w_T^*)) \) \cite{Yang2016}. If all the functions are strongly convex and smooth, the upper bound can be improved to \( O(\mathcal{P}(w_1^*, \ldots, w_T^*)) \) \cite{Mokhtari2016}. The \( O(\mathcal{P}(w_1^*, \ldots, w_T^*)) \) rate is also achievable when all the functions are convex and smooth, and all the minimizers \( w_t^* \)'s lie in the interior of \( \Omega \) \cite{Yang2016}. In a recent study, \cite{Zhang2016} introduced a new regularity—\emph{squared} path-length

\[
S(w_1^*, \ldots, w_T^*) = \sum_{t=2}^{T} \|w_t^* - w_{t-1}^*\|_2^2
\]

which could be much smaller than the path-length \( \mathcal{P}(w_1^*, \ldots, w_T^*) \) when the difference between successive local minimizers is small. \cite{Zhang2016} developed a novel algorithm named online multiple gradient descent, and proved that \( D\text{-Regret}(w_1^*, \ldots, w_T^*) \) is on the order of \( \min(\mathcal{P}(w_1^*, \ldots, w_T^*), S(w_1^*, \ldots, w_T^*)) \) for (semi-)strongly convex and smooth functions.

Although closely related, adaptive regret and dynamic regret are studied independently and there are few discussions of their relationships. In the literature, dynamic regret is also referred to as tracking regret or shifting regret \cite{Littlestone1994, Herbster1998, Herbster2001}. In the setting of “prediction with expert advice”, \cite{Adamskiy2012} have shown that the tracking regret can be derived from the adaptive regret. In the setting of “online linear optimization in the simplex”, \cite{Cesa-Bianchi2012} introduced a generalized notion of shifting regret which unifies adaptive regret and shifting regret. Different from previous work, this paper considers the setting of online convex optimization, and illustrates that the dynamic regret can be upper bounded by the adaptive regret and the functional variation.

3. From Adaptive to Dynamic

In this section, we first introduce a general theorem that bounds the dynamic regret by the adaptive regret, and then derive specific regret bounds for convex functions, exponentially concave functions, and strongly convex functions.
3.1 Adaptive-to-Dynamic Conversion

Let \( I_1 = [s_1, q_1], I_2 = [s_2, q_2], \ldots, I_k = [s_k, q_k] \) be a partition of \([1, T]\). That is, they are successive intervals such that

\[
s_1 = 1, \quad q_i + 1 = s_{i+1}, \quad i \in [k - 1], \quad \text{and} \quad q_k = T.
\]

(4)

Define the local functional variation of the \( i \)-th interval as

\[
V_T(i) = \sum_{t=s_i+1}^{q_i} \max_{w \in \Omega} |f_t(w) - f_{t-1}(w)|
\]

and it is obvious that \( \sum_{i=1}^{k} V_T(i) \leq V_T \). Then, we have the following theorem for bounding the dynamic regret in terms of the strongly adaptive regret and the functional variation.

**Theorem 1** Let \( w^*_t \in \arg\min_{w \in \Omega} f_t(w) \). We have

\[
D-\text{Regret}(w^*_1, \ldots, w^*_T) \leq \min_{I_1, \ldots, I_k} \sum_{i=1}^{k} \left( \text{SA-Regret}(T, |I_i|) + 2|I_i| \cdot V_T(i) \right)
\]

where the minimization is taken over any sequence of intervals that satisfy (4).

The above theorem is analogous to Proposition 2 of Besbes et al. (2015), which provides an upper bound for a special choice of the interval sequence. The main difference is that there is a minimization operation in our bound, which allows us to get rid of the issue of parameter selection. For a specific type of problems, we can plug in the corresponding upper bound of strongly adaptive regret, and then choose any sequence of intervals to obtain a concrete upper bound. In particular, the choice of the intervals may depend on the (possibly unknown) functional variation.

Before proceeding to specific bounds, we introduce the following common assumption.

**Assumption 1** Both the gradient and the domain are bounded.

- The gradients of all the online functions are bounded by \( G \), i.e., \( \max_{w \in \Omega} \|\nabla f_t(w)\| \leq G \) for all \( f_t \).
- The diameter of the domain \( \Omega \) is bounded by \( B \), i.e., \( \max_{w, w' \in \Omega} \|w - w'\| \leq B \).

3.2 Convex Functions

For convex functions, we choose the meta-algorithm of Jun et al. (2016) and take the online gradient descent as its subroutine. The following theorem regarding the adaptive regret can be obtained from that paper.

**Theorem 2** Under Assumption 1, the meta-algorithm of Jun et al. (2016) is strongly adaptive with

\[
\text{SA-Regret}(T, \tau) \leq \left( \frac{12BG}{\sqrt{2} - 1} + 8\sqrt{7 \log T + 5} \right) \sqrt{\tau} = O(\sqrt{\tau \log T}).
\]

3. Note that in certain cases, the sum of local functional variation \( \sum_{i=1}^{k} V_T(i) \) can be much smaller than the total functional variation \( V_T \). For example, when the sequence of functions only changes \( k \) times, we can construct the intervals based on the changing rounds such that \( \sum_{i=1}^{k} V_T(i) = 0 \).
From Theorems 1 and 2, we derive the following bound for the dynamic regret.

**Corollary 3** Under Assumption 1, the meta-algorithm of Jun et al. (2016) satisfies

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \max \left\{ \frac{(c + 9\sqrt{7\log T + 5})\sqrt{T}}{\log^{1/6} T}, \frac{(c + 8\sqrt{5})T^{2/3}V_T^{1/3}}{\log^{1/6} T} + 24T^{2/3}V_T^{1/3} \log^{1/3} T \right\} = O\left( \max \left\{ \sqrt{T\log T}, T^{2/3}V_T^{1/3} \log^{1/3} T \right\} \right)
\]

where \( c = 12BG/(\sqrt{2} - 1) \).

According to Theorem 2 of Besbes et al. (2015), we know that the minimax dynamic regret of convex functions is \( O(T^{2/3}V_T^{1/3}) \). Thus, our upper bound is minimax optimal up to a polylogarithmic factor. The key advantage of the meta-algorithm of Jun et al. (2016) over the restarted online gradient descent of Besbes et al. (2015) is that the former one does not need any prior knowledge of the functional variation \( V_T \). Notice that the meta-algorithm of Daniely et al. (2015) can also be used here, and its dynamic regret is on the order of \( \max \left\{ \sqrt{T\log T}, T^{2/3}V_T^{1/3} \log^{2/3} T \right\} \).

### 3.3 Exponentially Concave Functions

We first provide the definition of exponentially concave (abbr. exp-concave) functions (Cesa-Bianchi and Lugosi, 2006).

**Definition 2** A function \( f(\cdot) : \Omega \to \mathbb{R} \) is \( \alpha \)-exp-concave if \( \exp(-\alpha f(\cdot)) \) is concave over domain \( \Omega \).

Exponential concavity is stronger than convexity but weaker than strong convexity. It can be used to model many popular losses used in machine learning, such as the square loss in regression, logistic loss in classification and negative logarithm loss in portfolio management (Koren, 2013).

For exp-concave functions, Hazan and Seshadhri (2007) have developed two meta-algorithms that take the online Newton step as its subroutine, and proved the following properties.

- The inefficient one has \( O(T) \) computational complexity per iteration, and its weakly adaptive regret is \( O(\log T) \).
- The efficient one has \( O(\log T) \) computational complexity per iteration, and its weakly adaptive regret is \( O(\log^2 T) \).

As can be seen, there is a tradeoff between the computational complexity and the weakly adaptive regret: A lighter computation incurs a looser bound and a tighter bound requires a higher computation. In Section 4 we develop a unified approach, i.e., Algorithm 1 that allows us to trade effectiveness for efficiency explicitly. Lemma 6 indicates the proposed algorithm has

\[
([\log_K T] + 1)(K - 1) = O\left( \frac{K \log T}{\log K} \right)
\]
computational complexity per iteration, where $K$ is a tunable parameter. On the other hand, Theorem 6 implies that for $\alpha$-exp-concave functions that satisfy Assumption 1, the strongly adaptive regret of Algorithm 1 is

$$\left(\frac{(5d+1)\bar{m} + 2 \alpha}{\alpha} + 5d\bar{m}GB\right) \log T = O\left(\frac{\log^2 T}{\log K}\right)$$

where $d$ is the dimensionality and $\bar{m} = (\lceil \log_K T \rceil + 1)$.

We list several choices of $K$ and the resulting theoretical guarantees in Table 1 and have the following observations.

- When $K = 2$, we recover the guarantee of the efficient algorithm of Hazan and Seshadhri (2007), and when $K = T$, we obtain the inefficient one.
- By setting $K = T^{1/\gamma}$ where $\gamma > 1$ is a small constant, such as 10, the strongly adaptive regret can be viewed as $O(\log T)$, and at the same time, the computational complexity is also very low for a large range of $T$.

According to Definition 1, Algorithm 1 in this paper, as well as the two meta-algorithms of Hazan and Seshadhri (2007), is strongly adaptive. Based on Theorem 1, we derive the dynamic regret of the proposed algorithm.

**Corollary 4** Let $K = T^{1/\gamma}$, where $\gamma > 1$ is a small constant. Suppose Assumption 1 holds, $\Omega \subset \mathbb{R}^d$, all the functions are $\alpha$-exp-concave. Algorithm 1, with online Newton step as its subroutine, is strongly adaptive with

$$\text{SA-Regret}(T, \tau) \leq \left(\frac{(5d+1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB\right) \log T = O(\gamma \log T) = O(\log T)$$

and its dynamic regret satisfies

$$\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \left(\frac{(5d+1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB + 2\right) \max\left\{ \log T, \sqrt{TV_T \log T} \right\}

= O\left(\max\left\{ \log T, \sqrt{TV_T \log T} \right\} \right).$$

To the best of our knowledge, this is the first time such kind of dynamic regret bound is established for exp-concave functions. Furthermore, the discussions in Section 3.4 implies our upper bound is minimax optimal, up to a polylogarithmic factor.
3.4 Strongly Convex Functions

In the following, we study strongly convex functions, defined below.

**Definition 3** A function \( f(\cdot) : \Omega \mapsto \mathbb{R} \) is \( \lambda \)-strongly convex if

\[
f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\lambda}{2} \|y - x\|^2, \ \forall x, y \in \Omega.
\]

It is easy to verify that strongly convex functions with bounded gradients are also exp-concave (Hazan et al., 2007).

**Lemma 4** Suppose \( f(\cdot) : \Omega \mapsto \mathbb{R} \) is \( \lambda \)-strongly convex and \( \nabla f(w) \leq G \) for all \( w \in \Omega \). Then, \( f(\cdot) \) is \( \frac{\lambda G^2}{\gamma^2} \)-exp-concave.

Thus, Corollary 4 can be directly applied to strongly convex functions, and yields a dynamic regret of \( O(\sqrt{TV_T \log T}) \). According to Theorem 4 of Besbes et al. (2015), the minimax dynamic regret of strongly convex functions is \( O(\sqrt{TV_T}) \), which implies our upper bound is almost minimax optimal.

A limitation of Corollary 4 is that the constant in the upper bound depends on the dimensionality \( d \). In the following, we show that when the functions are strongly convex and online gradient descent is used as the subroutine of Algorithm 1, both the adaptive and dynamic regrets are independent from \( d \).

**Corollary 5** Let \( K = T^{1/\gamma} \), where \( \gamma > 1 \) is a small constant. Suppose Assumption 1 holds, and all the functions are \( \lambda \)-strongly convex. Algorithm 1, with online gradient descent as its subroutine, is strongly adaptive with

\[
SA\text{-Regret}(T, \tau) \leq \frac{C^2}{2\lambda} (\gamma + 1 + (3\gamma + 7) \log T) = O(\gamma \log T) = O(\log T)
\]

and its dynamic regret satisfies

\[
D\text{-Regret}(w_1^*, \ldots, w_T^*) \leq \max \left\{ \frac{\gamma C^2}{\lambda} + \left( \frac{5\gamma C^2}{\lambda} + 2 \right) \log T \right\}
\]

\[
= O \left( \max \left\{ \log T, \sqrt{TV_T \log T} \right\} \right).
\]

4. An Unified Adaptive Algorithm

In this section, we introduce a unified approach for minimizing the adaptive regret of exp-concave functions, as well as strongly convex functions.

Let \( E \) be an online learning algorithm that is designed to minimize the static regret of exp-concave functions or strongly convex functions, e.g., online Newton step (Hazan et al., 2007) or online gradient descent (Zinkevich, 2003). Similar to the approach of following the leading history (FLH) (Hazan and Seshadhri, 2007), at any time \( t \), we will instantiate an expert by applying the online learning algorithm \( E \) to the sequence of loss functions
\(f_t, f_{t+1}, \ldots\), and utilize the strategy of learning from expert advice to combine solutions of different experts [Herbster and Warmuth, 1998]. Our method is named as improved following the leading history (IFLH), and is summarized in Algorithm 1.

Let \(E_t\) be the expert that starts to work at time \(t\). To control the computational complexity, we will associate an ending time \(e_t\) for each \(E_t\). The expert \(E_t\) is alive during the period \([t, e_t - 1]\). In each round \(t\), we maintain a working set of experts \(S_t\), which contains all the alive experts, and assign a probability \(p^j_t\) for each \(E^j \in S_t\). In Steps 6 and 7, we remove all the experts whose ending times are no larger than \(t\). Since the number of alive experts has changed, we need to update the probability assigned to them, which is performed in Steps 12 to 14. In Steps 15 and 16, we add a new expert \(E_t\) to \(S_t\), calculate its ending time according to Definition 5 introduced below, and set \(p^t_t = \frac{1}{t}\). It is easy to verify \(\sum_{j \in S_t} p^j_t = 1\).

Let \(w^j_t\) be the output of \(E^j\) at the \(t\)-th round, where \(t \geq j\). In Step 17, we submit the weighted average of \(w^j_t\) with coefficient \(p^j_t\) as the output \(w_t\), and suffer the loss \(f_t(w_t)\). From Steps 18 to 25, we use the exponential weighting scheme to update the weight for each expert \(E^j\) based on its loss \(f_t(w^j_t)\). In Step 21, we pass the loss function to all the alive experts such that they can update their predictions for the next round.

The difference between our IFLH and the original FLH is how to decide the ending time \(e_t\) of expert \(E_t\). In this paper, we propose the following base-\(K\) ending time.

**Definition 5 (Base-\(K\) Ending Time)** Let \(K\) be an integer, and the representation of \(t\) in the base-\(K\) number system as

\[ t = \sum_{\tau \geq 0} \alpha_\tau K^\tau \]

where \(0 \leq \alpha_\tau < K\), for all \(\tau \geq 0\). Let \(k\) be the smallest integer such that \(\alpha_\tau > 0\), i.e.,

\[ k = \min\{\tau : \alpha_\tau > 0\}. \]

Then, the base-\(K\) ending time of \(t\) is defined as

\[ E_K(t) = \sum_{\tau \geq k+1} \alpha_\tau K^\tau + K^{k+1}. \]

In other words, the ending time is the number represented by the new sequence obtained by setting the first nonzero elements in the sequence \(\alpha_0, \alpha_1, \ldots\) to be 0 and adding 1 to the element after it.

Let’s take the decimal system as an example (i.e., \(K = 10\)). Then,

\[
\begin{align*}
E_{10}(1) &= E_{10}(2) = \cdots = E_{10}(9) = 10, \\
E_{10}(11) &= E_{10}(12) = \cdots = E_{10}(19) = 20, \\
E_{10}(10) &= E_{10}(20) = \cdots = E_{10}(90) = 100.
\end{align*}
\]

We note that a similar strategy for deciding the ending time was proposed by György et al. (2012), and a discussion about the difference is given in the supplementary.

When the base-\(K\) ending time is used in Algorithm 1, we have the following properties.

**Lemma 6** Suppose we use the base-\(K\) ending time in Algorithm 1.
Algorithm 1 Improved Following the Leading History (IFLH)

1. Input: An integer $K$  
2. Initialize $S_0 = \emptyset$.  
3. for $t = 1, \ldots, T$ do  
   4. Set $Z_t = 0$  
      {Remove some existing experts}  
   5. for $E^j \in S_{t-1}$ do  
      6. if $e^j \leq t$ then  
         7. Update $S_{t-1} \leftarrow S_{t-1} \setminus \{E^j\}$  
      8. else  
         9. Set $Z_t = Z_t + \hat{p}^j_t$  
   10. end if  
   11. end for  
      {Normalize the probability}  
   12. for $E^j \in S_{t-1}$ do  
      13. Set $p^j_t = \frac{\hat{p}^j_t}{Z_t} (1 - \frac{1}{t})$  
   14. end for  
      {Add a new expert $E^t$}  
   15. Set $S_t = S_{t-1} \cup \{E^t\}$  
   16. Compute the ending time $e^t = \mathcal{E}_K(t)$ according to Definition 5 and set $p^t_t = \frac{1}{t}$  
      {Compute the final predicted model}  
   17. Submit the solution  
      $$w_t = \sum_{E^j \in S_t} p^j_t w^j_t$$  
      and suffer loss $f_t(w_t)$  
      {Update weights and expert}  
   18. Set $Z_{t+1} = 0$  
   19. for $E^j \in S_t$ do  
      20. Compute $\hat{p}^j_{t+1} = p^j_t \exp(-\alpha f_t(w^j_t))$ and $Z_{t+1} = Z_{t+1} + \hat{p}^j_{t+1}$  
      21. Pass the function $f_t(\cdot)$ to $E^j$  
   22. end for  
   23. for $E^j \in S_t$ do  
      24. Set $\hat{p}^j_{t+1} = \frac{\hat{p}^j_{t+1}}{Z_{t+1}}$  
   25. end for  
   26. end for

1. For any $t \geq 1$, we have  
   $$|S_t| \leq ([\log_K t] + 1) (K - 1) = O\left(K \frac{\log t}{\log K}\right).$$

2. For any interval $I = [r, s] \subseteq [T]$, we can always find $m$ segments  
   $$I_j = [t_j, e^j_t - 1], \quad j \in [m]$$

11
\[ m \leq \lfloor \log_K s \rfloor + 1, \text{ such that } \]
\[ t_1 = r, \ e^j = t_{j+1}, \ j \in [m-1], \text{ and } e^m > s. \]

The first part of Lemma 6 implies that the size of \( S_t \) is \( O(K \log t / \log K) \). An example of \( S_t \) in the decimal system is given below.
\[
S_{486} = \begin{cases} 
481, 482, \ldots, 486, \\
410, 420, \ldots, 480, \\
100, 200, \ldots, 400 
\end{cases}
\]

The second part of Lemma 6 implies that for any interval \( I = [r, s] \), we can find \( O(\log s / \log K) \) experts such that their survival periods cover \( I \). Again, we present an example in the decimal system: The interval \([111, 832]\) can be covered by \([111, 119]\), \([120, 199]\), and \([200, 999]\) which are the survival periods of experts \( E_{111} \), \( E_{120} \), and \( E_{200} \), respectively. Recall that \( E_{10}(111) = 120, E_{10}(120) = 200, \) and \( E_{10}(200) = 1000. \)

Based on Lemma 6, we have the following theorem regarding the adaptive regret of exp-concave functions.

**Theorem 6** Suppose Assumption 1 holds, \( \Omega \subset \mathbb{R}^d \), all the functions are \( \alpha \)-exp-concave. If online Newton step is used as the subroutine in Algorithm 1, we have
\[
\sum_{t=r}^s f_t(w_t) - \min_{w \in \Omega} \sum_{t=r}^s f_t(w) \leq \left(\frac{(5d + 1)m + 2}{\alpha} + 5dmGB\right) \log T
\]
where \( m \leq \lfloor \log_K s \rfloor + 1 \). And thus,
\[
\text{SA-Regret}(T, \tau) \leq \left(\frac{(5d + 1)\bar{m} + 2}{\alpha} + 5d\bar{m}GB\right) \log T = O\left(\frac{\log^2 T}{\log K}\right)
\]

where \( \bar{m} = (\lfloor \log_K T \rfloor + 1) \).

From Lemma 6 and Theorem 6, we observe that the adaptive regret is a decreasing function of \( K \), while the computational cost is an increasing function of \( K \). Thus, we can control the tradeoff by tuning the value of \( K \).

For strongly convex functions, we have a similar guarantee but without any dependence on the dimensionality \( d \), as indicated below.

**Theorem 7** Suppose Assumption 1 holds, and all the functions are \( \lambda \)-strongly convex. If online gradient descent is used as the subroutine in Algorithm 1, we have
\[
\sum_{t=r}^s f_t(w_t) - \min_{w \in \Omega} \sum_{t=r}^s f_t(w) \leq \frac{G^2}{2\lambda} \left( m + (3m + 4) \log T \right)
\]
where \( m \leq \lfloor \log_K s \rfloor + 1 \). And thus
\[
\text{SA-Regret}(T, \tau) \leq \frac{G^2}{2\lambda} \left( \bar{m} + (3\bar{m} + 4) \log T \right) = O\left(\frac{\log^2 T}{\log K}\right)
\]
where \( \bar{m} = (\lfloor \log_K T \rfloor + 1) \).
5. Analysis

We here present the proofs of main theorems. The omitted proofs are provided in the supplementary.

5.1 Proof of Theorem \[1\]

First, we upper bound the dynamic regret in the following way

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) = \sum_{i=1}^{k} \left( \sum_{t=1}^{q_i} f_t(w_t) - \min_{w \in \Omega} \sum_{t=1}^{q_i} f_t(w) \right)
\]

(5)

Furthermore, for any \( t \in [s_i, q_i] \), we have

\[
f_t(w_{s_i}^*) - f_t(w_t^*) = f_t(w_{s_i}^*) - f_t(w_t^*) + f_t(w_t^*) - f_t(w_t^*) \leq 2V_T(i).
\]

(7)

Combining (6) with (7), we have

\[
\min_{w \in \Omega} \sum_{t=s_i}^{q_i} f_t(w) - \sum_{t=s_i}^{q_i} f_t(w) \leq 2|I_i| \cdot V_T(i).
\]

Substituting the upper bounds of \( a_i \) and \( b_i \) into (5), we arrive at

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \sum_{i=1}^{k} (\text{SA-Regret}(T, |I_i|) + 2|I_i| \cdot V_T(i)).
\]

Since the above inequality holds for any partition of \([1, T]\), we can take minimization to get a tight bound.
5.2 Proof of Corollary 3

To simplify the upper bound in Theorem 1, we restrict to intervals of the same length \( \tau \), and in this case \( k = T/\tau \). Then, we have

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \min_{1 \leq \tau \leq T} \left( \text{SA-Regret}(T, \tau) + 2\tau V_T(i) \right) = \min_{1 \leq \tau \leq T} \left( \frac{\text{SA-Regret}(T, \tau) T}{\tau} + 2\tau \sum_{i=1}^{k} V_T(i) \right)
\]

Combining with Theorem 2, we have

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \min_{1 \leq \tau \leq T} \left( \frac{c + 8\sqrt{5} T \log T + 5}{\sqrt{\tau}} + 2\tau V_T \right),
\]

where \( c = 12BG/(\sqrt{2} - 1) \).

In the following, we consider two cases. If \( V_T \geq \sqrt{\log T/T} \), we choose

\[
\tau = \left( \frac{T \log T}{V_T} \right)^{2/3} \leq T
\]

and have

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \frac{(c + 8\sqrt{5} T \log T + 5) T^{2/3} V_T^{1/3}}{\log^{1/6} T} + 2T^{2/3} V_T^{1/3} \log^{1/3} T
\]

Otherwise, we choose \( \tau = T \), and have

\[
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq (c + 8\sqrt{7} \log T + 5) \sqrt{T} + 2TV_T
\]

\[
\leq (c + 8\sqrt{7} \log T + 5) \sqrt{T} + 2T \sqrt{\frac{\log T}{T}}
\]

\[
\leq (c + 9\sqrt{7} \log T + 5) \sqrt{T}.
\]

6. Proof of Theorem 6

From the second part of Lemma 6, we know that there exist \( m \) segments

\[
I_j = [t_j, e^{t_j} - 1], \ j \in [m]
\]

with \( m \leq [\log_K s] + 1 \), such that

\[
t_1 = r, \ e^{t_j} = t_{j+1}, \ j \in [m - 1], \ \text{and} \ e^m > s.
\]
Furthermore, the expert $E^{t_j}$ is alive during the period $[t_j, e_j - 1]$.

Using Claim 3.1 of Hazan and Seshadhri (2009), we have

$$
\sum_{t=t_j}^{e_j-1} f_t(w_t) - f_t(w_t^{t_j}) \leq \frac{1}{\alpha} \left( \log t_j + 2 \sum_{t=t_j+1}^{e_j-1} \frac{1}{t} \right), \quad \forall j \in [m - 1]
$$

where $w_t^{t_j}, \ldots, w_t^{e_j-1}$ is the sequence of solutions generated by the expert $E^{t_j}$. Similarly, for the last segment, we have

$$
\sum_{t=t_m}^{s} f_t(w_t) - f_t(w_t^{t_m}) \leq \frac{1}{\alpha} \left( \log t_m + 2 \sum_{t=t_m+1}^{s} \frac{1}{t} \right).
$$

By adding things together, we have

$$
\sum_{j=1}^{m-1} \left( \sum_{t=t_j}^{e_j-1} f_t(w_t) - f_t(w_t^{t_j}) \right) + \sum_{t=t_m}^{s} f_t(w_t) - f_t(w_t^{t_m}) \leq \frac{1}{\alpha} \sum_{j=1}^{m} \log t_j + \frac{2}{\alpha} \sum_{t=r+1}^{s} \frac{1}{t} \leq \frac{m+2}{\alpha} \log T. \tag{8}
$$

According to the property of online Newton step (Hazan et al., 2007, Theorem 2), we have, for any $w \in \Omega$,

$$
\sum_{t=t_j}^{e_j-1} f_t(w_t^{t_j}) - f_t(w) \leq 5d \left( \frac{1}{\alpha} + GB \right) \log T, \quad \forall j \in [m - 1] \tag{9}
$$

and

$$
\sum_{t=t_m}^{s} f_t(w_t^{t_m}) - f_t(w) \leq 5d \left( \frac{1}{\alpha} + GB \right) \log T. \tag{10}
$$

Combining (8), (9), and (10), we have,

$$
\sum_{t=r}^{s} f_t(w_t) - \sum_{t=r}^{s} f_t(w) \leq \left( \frac{(5d + 1)m + 2}{\alpha} + 5dmGB \right) \log T
$$

for any $w \in \Omega$.

7. Proof of Corollary 4

The first part of Corollary 4 is a direct consequence of Theorem 6 by setting $K = T^{1/\gamma}$.

Now, we prove the second part. Following similar analysis of Corollary 3, we have

$$
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \min_{1 \leq \tau \leq T} \left\{ \left( \frac{(5d + 1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB \right) \frac{T \log T}{\tau} + 2\tau V_T \right\}.
$$
Then, we consider two cases. If $V_T \geq \log T/T$, we choose
$$
\tau = \sqrt{\frac{T \log T}{V_T}} \leq T
$$
and have
$$
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \left(\frac{(5d + 1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB + 2\right) \sqrt{TV_T \log T}.
$$
Otherwise, we choose $\tau = T$, and have
$$
\text{D-Regret}(w_1^*, \ldots, w_T^*) \leq \left(\frac{(5d + 1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB \right) \log T + 2TV_T
$$
$$
\leq \left(\frac{(5d + 1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB \right) \log T + 2T \frac{\log T}{T}
$$
$$
= \left(\frac{(5d + 1)(\gamma + 1) + 2}{\alpha} + 5d(\gamma + 1)GB + 2\right) \log T.
$$

8. Proof of Theorem 7

Lemma 4 implies that all the $\lambda$-strongly convex functions are also $\frac{\lambda}{G^2}$-exp-concave. As a result, we can reuse the proof of Theorem 6. Specifically, (8) with $\alpha = \frac{\lambda}{G^2}$ becomes
$$
\sum_{j=1}^{m-1} \left(\sum_{t=t_j}^{e^{j-1}} f_t(w) - f_t(w_t^j)\right) + \sum_{t=t_m}^{s} f_t(w) - f_t(w_t^m) \leq \frac{(m + 2)G^2}{\lambda} \log T. \quad (11)
$$

According to the property of online gradient descent [Hazan et al., 2007, Theorem 1], we have, for any $w \in \Omega$,
$$
\sum_{t=t_j}^{e^{j-1}} f_t(w_t^j) - f_t(w) \leq \frac{G^2}{2\lambda}(1 + \log T), \ \forall j \in [m - 1] \quad (12)
$$

and
$$
\sum_{t=t_m}^{s} f_t(w_t^m) - f_t(w) \leq \frac{G^2}{2\lambda}(1 + \log T). \quad (13)
$$

Combining (11), (12), and (13), we have,
$$
\sum_{t=r}^{s} f_t(w_t) - \sum_{t=r}^{s} f_t(w) \leq \frac{G^2}{2\lambda} (m + (3m + 4) \log T)
$$
for any $w \in \Omega$. 

16
9. Proof of Corollary 5

The first part of Corollary 5 is a direct consequence of Theorem 7 by setting \( K = T^{1/\gamma} \).

The proof of the second part is similar to that of Corollary 4. First, we have

\[
D\text{-Regret}(w^*_1, \ldots, w^*_T) \leq \min_{1 \leq \tau \leq T} \left\{ \frac{G^2}{2\lambda} \left( \gamma + 1 + (3\gamma + 7) \log T \right) \frac{T}{\tau} + 2\tau V_T \right\}
\]

where the last inequality is due to the condition \( \gamma > 1 \).

Then, we consider two cases. If \( V_T \geq \log T/T \), we choose\( \tau = \sqrt{T \log T} \leq T \)

and have

\[
D\text{-Regret}(w^*_1, \ldots, w^*_T) \leq \frac{\gamma G^2}{\lambda} \sqrt{TV_T \log T} + \frac{5\gamma G^2}{\lambda} \sqrt{TV_T \log T} + 2\sqrt{TV_T \log T}
\]

Otherwise, we choose \( \tau = T \), and have

\[
D\text{-Regret}(w^*_1, \ldots, w^*_T) \leq \frac{(\gamma + 5\gamma \log T) G^2}{\lambda} + 2TV_T
\]

\[
\leq \frac{(\gamma + 5\gamma \log T) G^2}{\lambda} + 2T \log T
\]

\[
= \frac{\gamma G^2}{\lambda} + \left( \frac{5\gamma G^2}{\lambda} + 2 \right) \log T.
\]

10. Conclusions and Future Work

In this paper, we demonstrate that the dynamic regret can be upper bounded by the adaptive regret and the functional variation, which implies strongly adaptive algorithms are automatically equipped with tight dynamic regret bounds. As a result, we are able to derive dynamic regret bounds for convex functions, exponentially concave functions, and strongly convex functions. All of these upper bounds are almost minimax optimal and this is the first time that such kind of dynamic regret bound is proved for exponentially concave functions.

The adaptive-to-dynamic conversion leads to a series of dynamic regret bounds in terms of the functional variation. As we mentioned in Section 2.3, dynamic regret can also be upper bounded by other regularities such as the path-length. It is interesting to investigate whether those kinds of upper bounds can also be established for strongly adaptive algorithms. Since we derive dynamic regret from adaptive regret, we conjecture that adaptive regret is more fundamental, and will try to give a rigorous proof in the future.
References

Jacob Abernethy, Alekh Agarwal, Peter L. Bartlett, and Alexander Rakhlin. A stochastic view of optimal regret through minimax duality. In Proceedings of the 22nd Annual Conference on Learning Theory, 2009.

Dmitry Adamskiy, Wouter M. Koolen, Alexey Chernov, and Vladimir Vovk. A closer look at adaptive regret. In Proceedings of the 23rd International Conference on Algorithmic Learning Theory, pages 290–304, 2012.

Omar Besbes, Yonatan Gur, and Assaf Zeevi. Non-stationary stochastic optimization. Operations Research, 63(5):1227–1244, 2015.

Nicolò Cesa-Bianchi and Gábor Lugosi. Prediction, Learning, and Games. Cambridge University Press, 2006.

Nicolò Cesa-bianchi, Pierre Gaillard, Gabor Lugosi, and Gilles Stoltz. Mirror descent meets fixed share (and feels no regret). In Advances in Neural Information Processing Systems 25, pages 980–988, 2012.

Amit Daniely, Alon Gonen, and Shai Shalev-Shwartz. Strongly adaptive online learning. In Proceedings of The 32nd International Conference on Machine Learning, 2015.

András György, Tamás Linder, and Gábor Lugosi. Efficient tracking of large classes of experts. IEEE Transactions on Information Theory, 58(11):6709–6725, 2012.

Eric C. Hall and Rebecca M. Willett. Dynamical models and tracking regret in online convex programming. In Proceedings of the 30th International Conference on Machine Learning, pages 579–587, 2013.

Elad Hazan and C. Seshadhri. Adaptive algorithms for online decision problems. Electronic Colloquium on Computational Complexity, 88, 2007.

Elad Hazan and C. Seshadhri. Efficient learning algorithms for changing environments. In Proceedings of the 26th Annual International Conference on Machine Learning, pages 393–400, 2009.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. Machine Learning, 69(2-3):169–192, 2007.

Mark Herbster and Manfred K. Warmuth. Tracking the best expert. Machine Learning, 32(2):151–178, 1998.

Mark Herbster and Manfred K. Warmuth. Tracking the best linear predictor. Journal of Machine Learning Research, 1:281–309, 2001.

Ali Jadbabaie, Alexander Rakhlin, Shahin Shahrampour, and Karthik Sridharan. Online optimization : Competing with dynamic comparators. In Proceedings of the 18th International Conference on Artificial Intelligence and Statistics, 2015.
Kwang-Sung Jun, Francesco Orabona, Rebecca Willett, and Stephen Wright. Improved strongly adaptive online learning using coin betting. *ArXiv e-prints*, arXiv:1610.04578, 2016.

Tomer Koren. Open problem: Fast stochastic exp-concave optimization. In *Proceedings of the 26th Annual Conference on Learning Theory*, pages 1073–1075, 2013.

John Langford, Lihong Li, and Tong Zhang. Sparse online learning via truncated gradient. In *Advances in Neural Information Processing Systems 21*, pages 905–912. 2009.

Nick Littlestone and Manfred K. Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2):212–261, 1994.

Aryan Mokhtari, Shahin Shahrampour, Ali Jadbabaie, and Alejandro Ribeiro. Online optimization in dynamic environments: Improved regret rates for strongly convex problems. *ArXiv e-prints*, arXiv:1603.04954, 2016.

Shai Shalev-Shwartz. Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194, 2011.

Shai Shalev-Shwartz and Yoram Singer. A primal-dual perspective of online learning algorithms. *Machine Learning*, 69(2):115–142, 2007.

Shai Shalev-Shwartz, Yoram Singer, and Nathan Srebro. Pegasos: primal estimated subgradient solver for SVM. In *Proceedings of the 24th International Conference on Machine Learning*, pages 807–814, 2007.

Tianbao Yang, Lijun Zhang, Rong Jin, and Jinfeng Yi. Tracking slowly moving clairvoyant: Optimal dynamic regret of online learning with true and noisy gradient. In *Proceedings of the 33rd International Conference on Machine Learning*, 2016.

Lijun Zhang, Tianbao Yang, Jinfeng Yi, Rong Jin, and Zhi-Hua Zhou. Improved dynamic regret for non-degeneracy functions. *ArXiv e-prints*, arXiv:1608.03933, 2016.

Martin Zinkevich. Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the 20th International Conference on Machine Learning*, pages 928–936, 2003.

**Appendix A. Proof of Theorem 2**

As pointed out by [Daniely et al. (2015)](https://arxiv.org/abs/1503.02859), the static regret of online gradient descent ([Zinkevich, 2003](https://arxiv.org/abs/0302182)) over any interval of length $\tau$ is upper bounded by $3BG\sqrt{\tau}$. Combining this fact with Theorem 2 of [Jun et al. (2016)](https://arxiv.org/abs/1610.04578), we get Theorem 2 in this paper.

**Appendix B. Proof of Lemma 4**

The gradient of $\exp(-\alpha f(w))$ is

$$\nabla \exp(-\alpha f(w)) = \exp(-\alpha f(w)) - \alpha \nabla f(w) = -\alpha \exp(-\alpha f(w)) \nabla f(w).$$
and the Hessian is
\[
\nabla^2 \exp(-\alpha f(w)) = -\alpha \exp(-\alpha f(w))\nabla f(w)\nabla^\top f(w) - \alpha \exp(-\alpha f(w))\nabla^2 f(w)
\]
\[
= \alpha \exp(-\alpha f(w)) \left( \alpha \nabla f(w)\nabla^\top f(w) - \nabla^2 f(w) \right).
\]

Thus, \( f(\cdot) \) is \( \alpha \)-exp-concave if
\[
\alpha \nabla f(w)\nabla^\top f(w) \preceq \nabla^2 f(w).
\]

We complete the proof by noticing
\[
\frac{\lambda}{G^2} \nabla f(w)\nabla^\top f(w) \preceq \lambda I \preceq \nabla^2 f(w).
\]

Appendix C. Proof of Lemma 6

We first prove the first part of Lemma 6. Let \( m = \lfloor \log_K t \rfloor \). Then, integer \( t \) can be represented in the base-\( K \) number system as
\[
t = \sum_{j=0}^{m} \alpha_j K^j.
\]

From the definition of base-\( K \) ending time, integers that are no larger than \( t \) and alive at \( t \) are
\[
\begin{align*}
\{ & 1 \cdot K^0 + \sum_{j=1}^{m} \alpha_j K^j, & 2 \cdot K^0 + \sum_{j=1}^{m} \alpha_j K^j, & \ldots, & \alpha_0 \cdot K^0 + \sum_{j=1}^{m} \alpha_j K^j \\
& 1 \cdot K^1 + \sum_{j=2}^{m} \alpha_j K^j, & 2 \cdot K^1 + \sum_{j=2}^{m} \alpha_j K^j, & \ldots, & \alpha_0 \cdot K^1 + \sum_{j=2}^{m} \alpha_j K^j \\
& \ldots \}
\end{align*}
\]
\[
\begin{align*}
& 1 \cdot K^{m-1} + \alpha_m K^m, & 1 \cdot K^{m-1} + \alpha_m K^m, & \ldots, & \alpha_{m-1} \cdot K^{m-1} + \alpha_m K^m \\
& 1 \cdot K^m, & 2 \cdot K^m, & \ldots, & \alpha_m K^m
\end{align*}
\]

The total number of alive integers are upper bounded by
\[
\sum_{i=0}^{m} \alpha_i \leq (m + 1)(K - 1).
\]

We proceed to prove the second part of Lemma 6. Let \( m = \lfloor \log_K r \rfloor \), and the representation of \( r \) in the base-\( K \) number system be
\[
r = \sum_{j=0}^{m} \alpha_j K^j.
\]
We generate a sequence of segments as

\[
\begin{align*}
\sum_{j=0}^{m} \alpha_j K^j, (\alpha_1 + 1)K^1 + \sum_{j=2}^{m} \alpha_j K^j - 1, \\
(\alpha_1 + 1)K^1 + \sum_{j=2}^{m} \alpha_j K^j, (\alpha_2 + 1)K^2 + \sum_{j=3}^{m} \alpha_j K^j - 1, \\
\quad \ldots\\n[(\alpha^{m-1} + 1)K^{m-1} + \alpha_m K^m, (\alpha_m + 1)K^m - 1], \\
[(\alpha_m + 1)K^m, K^{m+1} - 1], \\
[K^{m+1}, K^{m+2} - 1], \\
\quad \ldots
\end{align*}
\]

until \( s \) is covered. It is easy to verify that the number of segments is at most \( \lfloor \log_K s \rfloor + 1 \).

**Appendix D. More Discussions about the Base-\( K \) Ending Time**

In a study of “prediction with expert advice”, György et al. (2012) have introduced a similar strategy for deciding the ending time of experts. The main difference is that their strategy is built upon base-2 number system and introduces an additional parameter \( g \) to compromise between the computational complexity and the regret, in contrast our method relies on base-\( K \) number system and uses \( K \) to control the tradeoff. Lemma 2 of György et al. (2012) indicates an \( O(g \log t) \) bound on the number of alive experts, which is worse than our \( O(K \log t/ \log K) \) bound by a logarithmic factor.