Augmented Cucker-Smale Model for Distributed Optimization

QIAO ZHANG1, MING HE1, BING XU1, HONGYUE HE1, AND MINYANG LIU1,2
1College of Command and Control Engineering, Army Engineering University of PLA, Nanjing 210007, China
2School of Foreign Languages, Central China Normal University, Wuhan 430079, China
Corresponding author: Ming He (paper_review@126.com)

ABSTRACT The Cucker-Smale (C-S) model describes an interacting particle system in which the connection weights decrease with increasing distance. This model features emergent behaviors by which the velocities of the particles converge to a common value without a central command. However, the consensus value of the original C-S flocking model is restricted to the leader-following consensus or average consensus. Moreover, for the short-range communication-based C-S model, consensus can only be obtained for specific initial configurations. In this paper, the short-range communication-based C-S model is extended to achieve distributed optimization, where the consensus value optimizes the objective function of the group for any bounded initial configuration. Simulation examples are provided to demonstrate the effectiveness of our approach.

INDEX TERMS Cucker-Smale model, distributed optimization, multiagent systems.

I. INTRODUCTION
Inspired by the observation that remote neighbors have less influence than closer neighbors in flocks of birds or schools of fish, Cucker and Smale proposed the Cucker-Smale (C-S) model, which illustrates autonomous interacting agents with connectivity intensity that fades as the pairwise distance increases [1], [2]. The flocking behavior of the C-S model has been a hot research topic for decades due to its emergent collective behaviors and engineering applications, such as distributed sensor networks and flock control for unmanned aerial vehicles. References [3]–[5] extended the C-S model with hierarchical leadership, and the authors in [6], [7] considered more general interactions, nonlinear velocity couplings were examined in [8], and collision avoidance problems were studied in [9].

The emergent behaviors of the C-S flocking model lie in the fact that the velocities of all agents converge to a common consensus value without a central command. However, it is worth noting that the asymptotic flocking mechanism of the original C-S flocking model has two drawbacks. First, its flocking behavior depends on the decay rate of the communication weights between agents. More precisely, for the long-range communication-based C-S model, asymptotic flocking occurs independently of the initial configurations; however, when the communication weight has a short range, asymptotic flocking can only be achieved for specific initial configurations. Second, according to whether a leader is present, the consensus of the C-S flocking model either obeys leader-following consensus or average consensus. For the former, the velocities of the particles converge to the velocity of the leader [1]–[3]; for the latter, the velocities of the particles converge to the average value of the initial velocities of the agents [4], [5]. Based on the above observations, the original C-S flocking model cannot satisfy diverse demands.

When a consensus value is required to optimize the sum of local objective functions, the problem becomes an optimal consensus problem, i.e., the distributed optimization problem, which has been under extensive consideration due to its wide applications in sensor networks, task/resource allocation, and machine learning [10]–[27]. The aim is to design
proper control protocols to minimize the sum of the local cost functions that are known only to the individual agents.

In this paper, inspired by [21], we add optimization terms on top of the original C-S flock term to extend the short-range communication-based C-S model and achieve distributed optimization. It is worth mentioning that distributed optimization can be achieved even through short-range interaction ($\beta > 1$) for any bounded initial configuration. The main contributions of this note are summarized in two aspects. First, to the best of our knowledge, this is the first study to extend the C-S model to the field of distributed optimization, where the consensus value is determined by the optimization objective of the given group. Second, for the short-range communication-based C-S model, optimal consensus is achieved for any bounded initial configuration.

The remainder of the note is organized as follows. In the second section, we present the preliminaries of the C-S flocking model and the problem formulation. In the third section, the augmented C-S model for the described time-invariant distributed optimization problem is given. In the fourth section, the augmented C-S model for the time-varying distributed optimization problem is presented. To validate the effectiveness of our model, simulation examples are given in the fifth section. In the last section, the conclusions are presented.

Notations: For a real vector $v_i = [v_{i1}, v_{i2}, \ldots, v_{in}]^T \in \mathbb{R}^n$, let $\nabla f_i(v_i)$ denote the partial derivatives of the local cost function with respect to $v_i$; define $\text{sign}(v_i) = [\text{sign}(v_{i1}), \text{sign}(v_{i2}), \ldots, \text{sign}(v_{in})]^T$, where $\text{sign}(\cdot)$ is the sign function; and let $R, R_+, R^n$ represent the sets of real numbers, one-dimensional positive real vectors, and $n$-dimensional real vectors, respectively, where $\|\cdot\|$ denotes the Euclidean norm, $\|\cdot\|_1$ denotes the $1$-norm, and $\|\cdot\|_1 \leq \|\cdot\|_2$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. CUCKER-SMALE FLOCKING MODEL

Consider a multiagent system consisting of $N$ agents without a leader, where $x_i(t), v_i(t)$ represent the position and velocity, respectively. The acceleration of each agent equals a weighted average of the velocity differences of pairwise agents, and the weight is a positive and decreasing function of the pairwise distance. The dynamic of such a multiagent system is given by

$$
\frac{dx_i}{dt} = v_i \\
\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \frac{v_j - v_i}{(1 + \|x_j - x_i\|^2)^{\frac{3}{2}}} 
$$

subject to the initial configuration $(x_i(0), v_i(0)) = (x_{i0}, v_{i0})$, where $\lambda$ is a positive coupling constant.

The following lemma contains the most important structural properties of the C-S model, which show that for restricted initial configurations, the velocities of the particles converge to the average value of their initial velocities (average consensus).

**Lemma 1:** Let $\{(x_i, v_i)\}$ be a smooth solution of the above C-S model. Then, one obtains

$$
\frac{d}{dt} \sum_{i=1}^{N} v_i = 0 \\
\frac{d}{dt} \sum_{i=1}^{N} \|v_i\|^2 = -\frac{\lambda}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(1 + \|x_j - x_i\|^2)^{\frac{3}{2}}} \|v_j - v_i\|^2 
$$

**Remark 1 [28]:** From (2), we can conclude that for the long-range communication-based C-S model ($\beta \leq 1$), a consensus can be obtained for any bounded initial configuration. However, for the short-range communication-based C-S model ($\beta > 1$), consensus can be obtained only for specific initial configurations.

**Remark 2:** According to whether a leader is present, the consensus of the C-S model is categorized into the leader-following consensus (the common velocity equals the velocity of the leader) and the average consensus (the common velocity equals the average velocity of the group). However, optimal consensus cannot be obtained for the original C-S flocking model.

B. PROBLEM FORMULATION

To extend the short-range communication-based C-S model to achieve optimal consensus for any bounded initial configuration, we define distributed optimization problem as follows.

**Definition 1:** Consider a multiagent system with its dynamic described by (1). Our purpose is to design distributed protocols such that the velocities of all agents achieve the optimal value, which is the minimizer of a time-invariant convex optimization problem:

**Problem 1:**

minimize $\sum_{i=1}^{N} f_i(v_i)$ subject to $v_i = v_j \in \mathbb{R}^n$ or a time-varying convex optimization problem:

**Problem 2:**

minimize $\sum_{i=1}^{N} f_i(v_i, t)$ subject to $v_i(t) = v_j(t) \in \mathbb{R}^n$ where $f_i(v_i) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f_i(v_i, t) : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}$ represent the local cost functions assigned to individual agents.

**Lemma 2 [22]:** Let $f(v) : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable convex function; then, $f(v)$ is minimized if and only if $\nabla f(v) = 0$.

**Lemma 3 [8]:** Let $g(t) : [0, \infty) \rightarrow \mathbb{R}_+$ be a smooth positive function, and let its derivative satisfy the following differential inequality:

$$
\frac{dg(t)}{dt} \leq -kg(t)
$$

where $k$ is a positive constant; then, one obtains

$$
g(t) \leq g(0)e^{-kt}, \quad t \in [0, \infty)
$$
III. AUGMENTED C-S MODEL FOR TIME-IN Variant OPTIMIZATION

In this section, based on the C-S flocking model, we present the augmented C-S model for achieving time-invariant distributed optimization.

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i \\
\frac{dv_i}{dt} &= \frac{\lambda}{N} \sum_{j=1}^{N} \frac{1}{(1 + \|x_j - x_i\|^2)^{\frac{3}{2}}} (v_j - v_i) \\
&\quad + \gamma \sum_{j=1}^{N} \text{sign}(v_j - v_i) - k \nabla f_i(v_i)
\end{align*}
\]  

(3)

where \( \beta > 0, \gamma > 0, k > 0, \) the first term is from the C-S flocking model and is employed to achieve consensus, and the last two terms are the sign function term and objective gradient term for distributed optimization, respectively.

**Assumption 1:** Consider the optimization objective function \( \sum_{i=1}^{N} f_i(v) \) with the quadratic local cost function \( f_i(v) \), which is illustrated as \( f_i(v) = \alpha v^2 + b_i v + c_i \). The gradient of \( f_i(v) \) is \( \nabla f_i(v) = 2\alpha v + b_i \), where \( \alpha > 0, v \in \mathbb{R}^n, b_i \in \mathbb{R}^n \), and \( \|b_i\| < b_0 \) (\( b_0 \) is a given positive constant).

**Remark 3:** Different from the C-S flocking model, for the augmented C-S model for distributed optimization, the average velocity of the group is not constant. To facilitate the analysis, we decouple system (3) into a macrosystem and a microsystem as follows:

\[
\begin{align*}
x_c &= \frac{1}{N} \sum_{i=1}^{N} x_i, \quad v_c = \frac{1}{N} \sum_{i=1}^{N} v_i \\
\hat{x}_i &= x_i - x_c, \quad \hat{v}_i = v_i - v_c.
\end{align*}
\]

We have the dynamics of the macrosystem:

\[
\begin{align*}
\frac{dx_c}{dt} &= v_c, \quad \frac{dv_c}{dt} = -k \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(v_i) = -k \frac{1}{N} \sum_{i=1}^{N} (av_i + b_i)
\end{align*}
\]

(4)

and the dynamics of the microsystem:

\[
\begin{align*}
\frac{d\hat{x}_i}{dt} &= \hat{v}_i \\
\frac{d\hat{v}_i}{dt} &= \frac{\lambda}{N} \sum_{j=1}^{N} \frac{1}{(1 + \|\hat{x}_j - \hat{x}_i\|^2)^{\frac{3}{2}}} (\hat{v}_j - \hat{v}_i) \\
&\quad + \gamma \sum_{j=1}^{N} \text{sign}(\hat{v}_j - \hat{v}_i) - k \nabla f_i(v_i)
\end{align*}
\]

(5)

Then, from (5), we have the following theorem that shows for the short-range communication-based C-S model, distributed optimization with a time-invariant cost function is achieved for any bounded initial configuration.

**Theorem 1:** System (3) achieves optimal consensus for the quadratic cost function

\[
\sum_{i=1}^{N} f_i(v) = \sum_{i=1}^{N} av_i^2 + b_i v_i + c_i
\]

when \( \gamma > 2Nk\beta_0 \), i.e., all agents cooperatively solve a convex optimization problem with time-invariant cost functions:

\[
\text{minimize} \sum_{i=1}^{N} f_i(v_i) \quad \text{subject to} \quad v_i = v_j \in \mathbb{R}^n.
\]

**Proof:** i) Define the kinetic energy as: \( \dot{E}_k = \sum_{i=1}^{N} \|v_i(t)\|^2 \)

Let \( \hat{x}_i, \hat{v}_i \) be the solution of system (5). then, we have

\[
\frac{d\dot{E}_k}{dt} = \sum_{i=1}^{N} \left\langle \hat{v}_i, \hat{v}_i \right\rangle
\]

\[
\begin{align*}
&\leq -\frac{\lambda}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(1 + \|\hat{x}_j - \hat{x}_i\|^2)^{\frac{3}{2}}} \|\hat{v}_j - \hat{v}_i\|^2 \\
&\quad - k \sum_{i=1}^{N} \left\langle \hat{v}_i, a_i \hat{v}_i \right\rangle \\
&\quad + k \sum_{i=1}^{N} \left\langle \hat{v}_i, \frac{1}{N} \sum_{i=1}^{N} b_i - b_i \right\rangle
\end{align*}
\]

(6)

Note that for the fourth term in (6), one obtains

\[
k b_0 \sum_{i=1}^{N} \|v_i\| < N k b_0 \sum_{i=1}^{N} \|\hat{x}_i - \hat{x}_i\|.
\]
Then, for the last term in (6), we have

\[ k \sum_{i=1}^{N} \left( \frac{\hat{v}_i}{N} \sum_{j=1}^{N} b_{ij} \right) = k \left( \sum_{i=1}^{N} \frac{\hat{v}_i}{N} \sum_{j=1}^{N} b_{ij} \right) = 0 \]

where \( \sum_{i=1}^{N} \hat{v}_i = 0 \) is used.

Since \( \gamma > 2Nk \beta_0 \), we have

\[ \frac{dE_k}{dt} \leq -k \sum_{i=1}^{N} \| \hat{v}_i \|^2 = -k \hat{E}_k \] (7)

Then, from Lemma 3, we have

\[ \lim_{t \to \infty} \hat{E}_k = 0, \ \text{i.e.,} \ \lim_{t \to \infty} v_i(t) = v_c(t) \]

ii) From i., when \( t \to \infty \), the velocities of all agents reach the consensus value \( v_c \); in this situation, we have that \( \frac{dv_i}{dt} = -k \sum_{j=1}^{N} \nabla f_j(v_c) = -k v_c - k \sum_{j=1}^{N} b_j \). Then similar to the proof of theorem 1 in [21], one obtains

\[ \frac{d}{dt} \sum_{i=1}^{N} f_i(v_i) = \sum_{i=1}^{N} \nabla f_i(v_i) \frac{dv_i}{dt} = -k \left( \sum_{i=1}^{N} \nabla f_i(v_i) \right)^T \sum_{i=1}^{N} \nabla f_i(v_i) \]

We conclude that since \( \lim_{t \to \infty} \sum_{i=1}^{N} \nabla f_i(v_i) = 0 \) from Lemma 2, \( v = v_c \) minimizes the differentiable convex function \( \sum_{i=1}^{N} f_i(v_i) \) when \( t \) goes to infinity.

Combining i) and ii), the optimal consensus is achieved, and the proof is completed.

Remark 4: Compared to the original C-S flocking model (1), the augmented C-S model (3) can achieve distributed optimization for any bounded initial configuration even when \( \beta > 1 \) (short-range communication).

Remark 5: In the above proof, the local gradient term \( \nabla f_j(v_i) \) is used to obtain gradient information of local cost for optimization, and the sign function term \( \sum_{j=1}^{N} \text{sign}(v_j - v_i) \) is used to let Equation (7) hold for consensus. Note that compared with the node-based term \( \sum_{j=1}^{N} (v_j - v_i) \) in [29], \( \sum_{j=1}^{N} \text{sign}(v_j - v_i) \) represents the edge-based approach, which only needs the sign information of the velocity difference, not the accurate difference value.

IV. AUGMENTED C-S MODEL FOR TIME-VARYING OPTIMIZATION

Assumption 2 [21]: The time-varying cost function \( \sum_{i=1}^{N} f_i(v_i, t) \) is twice continuously differentiable with respect to \( v_i \), and the Hessian matrix is the same for all agents, i.e., \( H_i(v, t) = H_j(v, t) \) for all \( i, j \). In addition, the Hessian matrix is nonsingular, and the term \( \eta_i \) can be reformulated as \( \omega v_i + \varepsilon_i(t) \), where \( \omega \leq 0 \) is a constant and \( \varepsilon_i(t) \in \mathbb{R}^n, \| \varepsilon_i(t) \| \leq \varepsilon_0 \) (\( \varepsilon_0 \) is a given positive constant). Note that such an assumption is reasonable and common, and it can also be found in [21].

In this section, we address the time-varying distributed optimization problem (Problem 2), and a new augmented C-S model is given as follows:

\[ \frac{dx_i}{dt} = v_i \]
\[ \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \frac{1}{1 + \| x_j - x_i \|^2} (v_j - v_i) + \varepsilon_i(t) \]
\[ + \gamma \sum_{j=1}^{N} \text{sign}(v_j - v_i) + \omega v_i \] (8)

where \( \eta_i = -H_i^{-1}(v_i, t)(\nabla f_i(v_i, t) + \frac{dv_i}{dt}) \)

Similarly, by decomposing (8) into a macro subsystem and micro subsystem, we derive the dynamic equation of the microsystem:

\[ \frac{d\hat{v}_i}{dt} = \hat{v}_i \]
\[ \frac{d\hat{v}_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \frac{1}{1 + \| \hat{x}_j - \hat{x}_i \|^2} (\hat{v}_j - \hat{v}_i) \]
\[ + \varepsilon_i(t) \frac{1}{N} \sum_{j=1}^{N} \varepsilon_j \]
\[ + \varepsilon_i(t) \frac{1}{N} \sum_{j=1}^{N} \varepsilon_j \] (9)

Then, from (9), we have the following theorem that shows that for the short-range communication-based C-S model, distributed optimization with a time-varying cost function is achieved for any bounded initial configuration.

Theorem 2: System (8) exhibits optimal consensus in terms of the velocity for a quadratic time-varying cost function satisfying assumption 2 with \( \gamma > 2Nk \beta_0 \), i.e., all agents cooperatively solve a convex optimization problem with time-varying cost functions:

\[ \text{minimize} \sum_{i=1}^{N} f_i(v_i, t) \text{ subject to } v_i = v_j \in \mathbb{R}^n. \]

Proof: Define the kinetic energy for system (8) as:

\[ \hat{E}_k = \sum_{i=1}^{N} \| v_i(t) \|^2 \]
Letting $(\mathbf{\hat{x}}, \mathbf{\hat{v}})$ be the solution of system (9), we have
\[
\frac{dE_k}{dt} = \sum_{i=1}^{N} \left( v_i(\mathbf{t}) - \mathbf{\hat{v}}_i(\mathbf{t}) \right) = \frac{\lambda}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(1 + \|\mathbf{\hat{x}}_j - \mathbf{\hat{x}}_i\|^2)^{\frac{\beta}{2}}} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\|^2
\]
\[
- \frac{\lambda}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\| + \sum_{i=1}^{N} \|\mathbf{\hat{v}}_i\| + \sum_{i=1}^{N} (\mathbf{\hat{v}}_i, \epsilon_i)
\]
\[
- \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\| + \omega \sum_{i=1}^{N} \|\mathbf{\hat{v}}_i\| + N \epsilon_0 \sum_{i=1}^{N} \sum_{j=1}^{N} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\| \right)
\]
\[
\leq \frac{\lambda}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{(1 + \|\mathbf{\hat{x}}_j - \mathbf{\hat{x}}_i\|^2)^{\frac{\beta}{2}}} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\|^2
\]
\[
- \frac{\lambda}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\| + \sum_{i=1}^{N} \|\mathbf{\hat{v}}_i\| + N \epsilon_0 \sum_{i=1}^{N} \sum_{j=1}^{N} \|\mathbf{\hat{v}}_j - \mathbf{\hat{v}}_i\| \leq \omega \sum_{i=1}^{N} \|\mathbf{\hat{v}}_i\| = \omega \mathbf{\hat{E}}_k
\]

From Lemma 3, we have
\[
\lim_{t \to \infty} \mathbf{\hat{E}}_k = 0, \quad \text{i.e.,} \quad \lim_{t \to \infty} v_i(t) = v_c(t)
\]

Then, system (8) achieves optimal consensus when the time goes to infinity, and the proof is completed.

**Remark 6:** Different from the works in [10]–[27], where the authors focused on solving optimization problems for multiagent systems with general cost functions, this paper focuses on extending the short-range communication-based C-S model to achieve distributed optimization for all bounded initial configurations. Though the proposed method only considers quadratic optimization problems, it is simple and easy to implement.

**V. NUMERICAL SIMULATION**

To demonstrate the effectiveness of our approach, two numerical examples are presented to achieve optimal consensus for the short-range communication-based C-S model for any bounded initial condition.

**Example 1:** To confirm theorem 1, we select the time-invariant cost function as \( f_i(v_i) = (v_{i1} - i \cos t)^2 + (v_{i2} - i \sin t)^2 \). Through theoretical calculation, we obtain the minimum of the overall cost function as \(-181059.375\), and the optimal consensus value of velocity is \((-7.75, -77.5)\). The initial states of thirty autonomous agents are set randomly. For the augmented C-S model (3), we set \( \beta = 10 \), which corresponds to short-range communication. Figs. 1 and 2 are the trajectories of the velocity with respect to time, Fig. 3 is the evolutionary trajectory of the objective function, which achieves the optimal value of \(-181059.375\), and the theoretical results agree with the simulated values.

**Example 2:** To confirm theorem 2, we select the time-varying cost function as \( f_i(v_i) = (v_{i1} - i \cos t)^2 + (v_{i2} - i \sin t)^2 \). Through theoretical calculation, we obtain \( \nabla f_i(v_i, t) = [2(v_{i1} - i \cos t), 2(v_{i2} - i \sin t)] \), \( H_i^{-1}(v_i) = (0.5, 0; 0, 0.5) \cdot \frac{\partial \nabla f_i(v_i, t)}{\partial v_i} = (2i \sin t; -2i \cos t) \), and \( \eta_i = \langle v_{i1}; v_{i2} \rangle^T + i[\cos t - \sin t; \cos t + \sin t]^T \)
randomly. For the augmented C-S model (8), we set cost function.

Trajectories of the velocities of the agents with a time-varying cost function.

FIGURE 5. Trajectories of the velocities of the agents with a time-varying cost function.

The initial states of the thirty autonomous agents are set randomly. For the augmented C-S model (8), we set $\beta = 10$. Figs. 4 and Figs. 5 show that the velocities of all agents converge to a common consensus value and rotate around a circle centered at the origin with a radius of 15.5, which is caused by the time-varying optimization objective function $30 \sum_{i=1}^{30} f_i(v_i)$.

VI. CONCLUSION

In this paper, we extend the short-range communication-based C-S model to achieve distributed optimization for any bounded initial configuration. The proposed augmented C-S models achieve velocity agreement, while the sum of a class of quadratic cost functions assigned to individual agents is minimized. The effectiveness of the proposed augmented C-S models is verified with numerical simulations. Our work can be extended in the future to fixed-time distributed optimization problems with general cost functions.

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QIAO ZHANG was born in 1986. He received the M.Sc. degree in electronic science and technology from the PLA University of Science and Technology, in 2015. He is currently a Teaching Assistant with the College of Command and Control Engineering, Army Engineering University of PLA. His main research interests include robot swarm and multi-agent control.

MING HE received the B.Sc., M.Sc., and Ph.D. degrees from the PLA University of Science and Technology, in 2000, 2003, and 2007, respectively. He is currently a Professor with the Army Engineering University of PLA. His main research interests include emergency command, big data analytics, and multi-agent control.

BING XU received the B.E., M.E., and Ph.D. degrees in computer science and technology from the PLA University of Science and Technology, Nanjing, China, in 2010, 2013, and 2017, respectively. He is currently a Lectorate with the Army Engineering University of PLA. His research interests include wireless positioning and navigation, positioning in UAVs, and wireless sensing.

HONGYUE HE was born in 1985. He received the M.S. and Ph.D. degrees from the PLA University of Science and Technology, in 2007 and 2014, respectively. He is currently a Lecturer with the Army Engineering University of PLA. His research interests include system of systems engineering, focusing on specification.

MINYANG LIU was born in 2000. She is currently a junior majoring in English with the School of Foreign Languages, Central China Normal University. Her main interests include studying English literature and linguistics.