BIAS COMPENSATION IN ITERATIVE SOFT-FEEDBACK ALGORITHMS WITH APPLICATION TO (DISCRETE) COMPRESSED SENSING

Susanne Sparrer, Robert F.H. Fischer

Institute of Communications Engineering, Ulm University, Germany

ABSTRACT
In all applications in digital communications, it is crucial for an estimator to be unbiased. Although so-called soft feedback is widely employed in many different fields of engineering, typically the biased estimate is used. In this paper, we contrast the fundamental unbiasing principles, which can be directly applied whenever soft feedback is required. To this end, the problem is treated from a signal-based perspective, as direct application whenever soft feedback is required. To this end, the problem is treated from a signal-based perspective, as well as from the approach of estimating the signal based on an estimate of the noise. Numerical results show that when employed in iterative reconstruction algorithms for Compressed Sensing, a gain of 1.2 dB due to proper unbiasing is possible.

Index Terms— MMSE estimation, Unbiasing, Soft Feedback, Compressed Sensing

1. INTRODUCTION
In communications engineering, estimation is very often based on the minimum mean-squared error (MMSE) criterion, which is also justified by the relation between MMSE and mutual information [1]. Independent of the problem to be solved and of the application at hand, the used estimators need to be unbiased, i.e., no systematic misalignment has to be present. While the unbiasing is state of the art for some estimators such as linear MMSE estimation [2, 3], for the so-called soft feedback [4], which is used in many different fields such as successive interference cancellation (SIC), a.k.a. decision-feedback equalization (DFE) in multiuser detection [5, 6, 7], unbiasing is generally ignored. In the following, we give a review of general unbiasing principles for soft feedback or nonlinear MMSE estimates in general, and apply them to iterative algorithms in Compressed Sensing.

2. UNBIASING OF NONLINEAR MMSE ESTIMATORS
In many problems, an observation\footnote{This work was supported by Deutsche Forschungsgemeinschaft (DFG) under grant FI 982/8-1.} is available, which can be assumed to be a noisy variant of the true value \( x \), with the measurement noise \( n \) (variance \( \sigma_n^2 \)) which is independent of \( x \). For brevity, all variables are assumed to be real-valued and zero-mean; a generalization is straightforward. The goal is to estimate \( x \) given the knowledge of the distributions of \( x \) and \( n \), such that the mean-squared error is minimized. To this end, the conditional mean estimator is the optimum solution [8, 9]. In digital communications, the corresponding estimate is often denoted as soft value\footnote{Random variables are denoted by capital letters, actual realizations by lower-case letters. For brevity, all variables are assumed to be real-valued.} \( x_B \); it is calculated by [8, 4] (\( \eta(\cdot) \): real-valued estimator function, \( E[\cdot] \): expectation, \( \text{var} \{\cdot\} \): variance)

\[
x_B = \min_{\eta} \mathbb{E}\{\|\eta(z) - X\|_2^2\} = \mathbb{E}\{X|z\} = \eta X(z) .
\]

The estimate can be written as the sum of the true value \( x \) and the estimation error \( e_B \), i.e.,

\[
x_B = x + e_B ,
\]

with variance

\[
\sigma_B^2(z) = \text{var}_X\{X|z\} = \mathbb{E}_{X}\{E_{B}^2\},
\]

and mean-squared error [1]

\[
\sigma_B^2 = \mathbb{E}_Z\{\sigma_B^2(Z)\} .
\]

Note that \( \sigma_B^2 \) is the squared error averaged over the distribution of \( X \) and the noise \( N \), thereby keeping the sum of them, i.e., the observation \( z \), fixed. \( \sigma_B^2 \), on the other hand, averages \( \sigma_B^2 \) over all possible values of \( Z \).

This estimate, as any MMSE solution, is multiplicatively biased (index \( -B \)), i.e., a part of the useful signal is accounted to the error, as for any MMSE solution the error is orthogonal to the estimate \( \mathbb{E}_Z\{Z E_B\} = 0 \), not to the true value.

2.1. Signal-Based Unbiasing
In case of (scalar) linear MMSE estimation, the biased estimate is a scaled version of the observation, i.e., \( x_B = k \cdot z \), with the scaling factor \( k \). In order for an estimate to be unbiased (index \( \cdot_U \)), this scaling has to be compensated for, i.e.,

\[
x_U = h \cdot x_B ,
\]

where the unbiasing factor \( h \) has to be adjusted such that the error \( E_U = X_U - X \) present in the unbiased estimate is orthogonal to \( X \). In the linear case, it follows immediately \( h = 1/k \).
While the obvious scaling factor $k$ is universal for linear estimators, in the case of non-linear estimators an average scaling has to be calculated. To this end, the nonlinear estimate $\eta_X(Z)$ is represented as the sum of a linear estimate $k_X X$ and noise $W_X$

$$X_B = \eta_X(Z) = k_X X + W_X,$$
(7)
as visualized in the block diagram given in Fig. 1, and in the signal-space representation Fig. 4. Adjusting $k_X$ according to the MMSE criterion, i.e., such that

$$\text{E}_X \{ (X_B - k_X X)^2 \} = \text{E}_X \{ W_X^2 \} \rightarrow \min,$$
(8)
leads to [11]

$$k_X = \frac{\text{E}_X \{ \eta_X(X + N) X \}}{\text{E}_X \{ X^2 \}} = \frac{\text{E}_X \{ X_B X \}}{\text{E}_X \{ X^2 \}}. \tag{9}$$

In this case, $W_X$ is uncorrelated to the linear estimate $k_X X$ [10, 11]. With $\text{E}_X \{ X_B N \} = \sigma_{\varepsilon \eta}^2$ and $\text{E}_X \{ X_B Z \} = \sigma_Z^2$ [14], it calculates to

$$k_X = \frac{\text{E}_X \{ X_B Z \} - \text{E}_X \{ X_B N \}}{\text{E}_X \{ X^2 \}} = \frac{\sigma_Z^2 - \sigma_{\varepsilon \eta}^2}{\sigma_Z^2}. \tag{10}$$

Hence, plugging $h = 1/k_X$ into (6) gives

$$x_U = \frac{\sigma_Z^2}{\sigma_Z^2 - \sigma_{\varepsilon \eta}^2} \cdot x_B = (1 - C_X) \cdot x_B, \tag{11}$$

where we defined

$$C_X \equiv \frac{\sigma_{\varepsilon \eta}^2}{\sigma_Z^2 - \sigma_{\varepsilon \eta}^2} = \frac{1}{\sigma_Z^2} \left( \frac{1}{\sigma_{\varepsilon \eta}^2} - 1 \right)^{-1}. \tag{12}$$

The estimate can be written as noisy variant of the true value, i.e., $x_U = x + \varepsilon_U$. As for any unbiased estimate, the (zero-mean) error $\varepsilon_U$ is the one with minimum mean-squared error $\sigma_{\varepsilon_U}^2 = \text{E}_X \{ E_U^2 \}$, which is orthogonal to the signal to be estimated

$$\text{E}_X \{ X \cdot E_U \} = 0. \tag{13}$$

With straightforward reformulations, the variance of $E_U$ reads

$$\varepsilon_U^2 = \text{E}_X \{ E_U^2 \} = \text{E}_X \{ (x_U - X)^2 \} = (1 - C_X^2) \cdot \varepsilon_{\varepsilon \eta}^2 + C_X^2 \cdot \sigma_X^2. \tag{14}$$

Thus, instead of the variance $\varepsilon_{\varepsilon \eta}^2$ if no unbiasing is applied, a tradeoff between $\varepsilon_{\varepsilon \eta}^2$ and $\sigma_X^2$ is active.

### 2.2. Noise-Based Unbiasing

In the previous section, $X$ has been estimated and unbiased directly. However, since $Z = X + N$, we can also estimate $N$, from which, in turn, $X$ can be calculated. This approach is visualized in Fig. 2 [12].

Note that the nonlinear estimator of the noise is simply connected to the estimator of the signal by

$$\eta_N(z) \equiv \text{E}_N \{ N | z \} = \text{E}_N \{ (Z - X) | z \} = z - \eta_X(z). \tag{15}$$

Hence, the output of the estimator can be written as $N_B = Z - X_B$. Analogous to the signal-based case, we linearize the estimator $\eta_N(z)$ by

$$N_B = k_N N + W_N, \tag{16}$$

where the scaling factor is calculated by [12, 14]

$$k_N = \frac{\text{E}_X \{ N_B N \}}{\text{E}_N \{ N^2 \}} = \frac{\text{E}_X \{ (Z - X_B)N \}}{\text{E}_N \{ N^2 \}} = \frac{\sigma^2_N - \sigma_{\varepsilon \eta}^2}{\sigma^2_N}. \tag{17}$$

Plugging $n_U = 1/k_N \cdot n_B$ into (15) gives the unbiased estimate for $x$

$$x_U = z - \frac{1}{k_N} \cdot n_B = (1 - C_N) \cdot x_B + C_N \cdot z, \tag{18}$$

with

$$C_N \equiv \frac{\sigma_{\varepsilon \eta}^2}{\sigma_{\varepsilon \eta}^2 - \sigma^2_n} = \frac{1}{\sigma^2_n} \left( \frac{1}{\sigma_{\varepsilon \eta}^2} - 1 \right)^{-1}. \tag{19}$$

Due to construction, in this case the estimation error $E_U$ is not orthogonal to $X$, but to $N$. Noteworthy, in contrast to the signal-based estimation where the unbiased estimate was a scaled version of $x_B$ only, $x_U$ depends on $x_B$ as well as on $z$ if noise-based unbiasing is applied.

Finally, the unbiased error variance (w.r.t. $X$, $\varepsilon_U = x_U - x$) can be calculated by

$$\varepsilon_{\varepsilon_U}^2 = \text{E}_X \{ E_U^2 \} = (1 - C_N^2) \cdot \varepsilon_{\varepsilon \eta}^2 + C_N^2 \cdot \sigma_n^2. \tag{20}$$

### 2.3. Discussion

Examples of the characteristic curves (top) of the biased and unbiased soft values, as well as the corresponding error variances (bottom), are given in Fig. 3 [12]. While the characteristic curves of the biased (blue) and the signal-based unbiased (green) estimates are strictly monotonically increasing, the noise-based unbiased curves (red) do not fulfill this property. As for all MMSE solutions, the unbiased estimates converge
to the biased values if $\sigma_n^2$ tends to zero, since the deviation of the (biased) estimates from the correct value is negligible in this case.

The variables present in the estimation process are visualized in Fig. 4, where the squared length of a vector corresponds to the variance of the corresponding variable [8]. The given variables $X$ and $N$ are orthogonal, and $Z = X + N$. In case of a linear estimator, $X_B$ and $X_U$ are scaled versions of $Z$, and $E_B \perp Z$, cf. Fig. 4, left part. In case of signal-based unbiasing (red), $E_U$ has to be orthogonal to $Z$, and hence $X_U = Z$. For noise-based unbiasing (blue), $E_U$ has to be orthogonal to $N$, and hence $X_U = 0$.

The nonlinearity of soft feedback calculation introduces an additional degree of freedom, i.e., a three-dimensional signal space is needed to represent the variables. In particular, $X_B$ will point out of the $X$-$N$-plane; this additional dimension, orthogonal to $X$ and $N$, is denoted as vertical dimension $V$ in the following. The visualization in the right part of Fig. 4 is a two-dimensional projection of the three-dimensional graph. Due to the orthogonality conditions, $X_U$ is an elongation of $X_B$ on the $N$-$V$-plane for signal-based unbiasing; for noise-based unbiasing, $X_U$ is determined by the intersection of an elongation of $N_B$ with the $X$-$V$-plane.

Hence, while the unbiased error $E_U$ remains in the $N$-$V$-plane in case of signal-based unbiasing, if noise-based unbiasing is applied it lays in the $X$-$V$-plane, i.e., $E_U$ is orthogonal to the noise. If applied in an algorithm where linear estimation and soft-feedback calculation are iterated, in case of signal-based unbiasing the error is constantly in the $N$-$V$-plane. For noise-based unbiasing the error alternates between the $N$-$V$- and $X$-$V$-plane every other iteration, which may be beneficial for convergence and accuracy.

![Fig. 4. Visualization of the variables present in the estimation process [12]. Linear estimation (left), nonlinear estimation (right). Red: Signal-based unbiasing. Blue: Noise-based unbiasing.](image)

### 2.4. Connection to Average Variances

Often, the soft values of a vector $z$ have to be calculated. In this case, the unbiased equations ((11) and (14) in the signal-based case, (18) and (20) for noise-based unbiasing) are applied for each element individually, leading to individual estimates and individual error variances.

However, in some situations, instead of individual variances, an average variance (averaged over the entire vector) should characterize the reliability. To this end, all elements of $z$ are assumed to have the same average noise variance $\sigma_n^2$.

Then, $\sigma_z^2$ of the individual elements (Eq. (4)) has to be replaced by $\sigma_n^2$ (in practice, the expectation according to (5) is replaced by $\sigma_n^2$ which is the average of $c^2$ over the vector elements), leading to the unbiased equations (for the $l$th element of the vector)

$$x_{U,l} = \frac{\sigma_{eU}}{\sigma_{eB}} \cdot \frac{x_{B,l}}{\sigma_{eB}} \quad \text{and} \quad \sigma_{eU}^2 = \left( \frac{1}{\sigma_{eB}} - \frac{1}{\sigma_{eB}^2} \right)^{-1} \sigma_n^2$$

for signal-based unbiasing, and

$$x_{U,l} = \frac{\sigma_{eU}}{\sigma_{eB}^2} \cdot \left( \frac{x_{B,l}}{\sigma_{eB}^2} \frac{z_l}{\sigma_n^2} \right) \quad \text{and} \quad \sigma_{eU}^2 = \left( \frac{1}{\sigma_{eB}^2} - \frac{1}{\sigma_n^2} \right)^{-1} \sigma_n^2$$

in the noise-based case.³ Note that the latter equations equal the ones used in [13, 14, 15] without any justification, in particular not the above given interpretation.

### 3. APPLICATION TO COMPRESSED SENSING

In Compressed Sensing, a sparse vector $x$ has to be estimated from an underdetermined system of linear equations, which is given by [16]

$$y = Ax + w,$$  \hspace{1cm} (21)

where the received vector $y \in \mathbb{R}^K$ depends on the measurement (channel) matrix $A \in \mathbb{R}^{K \times L}$, $L > K$, and on the sparse vector $x \in \mathbb{C}^L$ (with sparsity $s$), where $C \subseteq \mathbb{R}$. Furthermore, the measurements are corrupted by i.i.d. zero-mean Gaussian noise $w$ with variance $\sigma_w^2$ per component, which is independent of $x$. In discrete Compressed Sensing, the elements of $x$ are drawn from a finite set, i.e., $C = \{0, c_1, \ldots, c_{|C|-1}\}$.

³In [15], the unbiased estimator without the correct scaling by $\sigma_{eU}^2$ is denoted as divergence-free.
Due to the sparsity constraint and the discrete alphabet, the problem of estimating \( \mathbf{x} \) based on (21) is non-convex. Different algorithms for the approximate solution of the problem are available in the literature; for a detailed discussion thereon, cf., e.g., [13, 14, 17, 18].

3.1. Algorithm

In [13], an iterative algorithm denoted as IMS has been proposed which splits the estimation problem into two parts, alternatingly estimating \( \mathbf{x} \) w.r.t. the Gaussian noise (Linear MMSE estimation, index \( \cdot_L \)), and w.r.t. \( \mathbf{s} \) and \( \mathbf{C} \) (Nonlinear MMSE estimation (soft feedback), index \( \cdot_N \)), with a final quantization w.r.t. the alphabet (\( \mathbb{Q}_c() \)). The pseudocode of this algorithm is given in Alg. 1. See [13] for further details.

While the MMSE estimate in the first step is unbiased, the soft values in the second step are not unbiased in the original algorithm (i.e., biased, Line 5B). Thus, we apply the equations for individual signal- or noise-based unbiasing derived in this paper and denote the new algorithm, including the unbiasing, as xuIMS and nuIMS, respectively (cf. Alg. 1, Line 5U).

Note that the TMS algorithm [14] (strongly related to OAMP or VAMP [15, 17]) is similar to nuIMS, however using average variances instead of individual ones; thus, it does not benefit from the information about the reliability of the particular elements as does uIMS.\(^4\)

\[
\text{Alg. 1 } \hat{\mathbf{x}} = \text{recover } (\mathbf{y}, \mathbf{A}, \sigma_\varepsilon^2, \mathbf{s}, \mathbf{C})
\]

\[\begin{align*}
&\text{Variants: B: IMS, U: (\cdot)uIMS} \\
&1: \mathbf{x}_{N,U} = \mathbf{0}, \mathbf{\bar{c}}_{N,U,l} = \mathbf{s}/L \forall l \\
&2: \text{while stopping criterion not met} \rightarrow \\
&3: (\mathbf{x}_{L,U}^*, \mathbf{x}_{L,U,l}^*, \mathbf{c}_{N,U}^*) = \text{unbiased linear MMSE estimate } (\mathbf{A}, \mathbf{y}, \mathbf{x}_{N,U}, \mathbf{c}_{N,U}^*) \\
&4: (\mathbf{x}_{N,B,l}^*, \mathbf{c}_{N,B,l}^*) = \text{biased soft feedback } (\mathbf{x}_{L,U,l}^*, \mathbf{x}_{L,U,l}^*, \mathbf{c}_{N,B,l}^*) \\
&5B: (\mathbf{x}_{N,U,l}^*, \mathbf{c}_{N,U,l}^*) = (\mathbf{x}_{N,B,l}^*, \mathbf{c}_{N,B,l}^*); \mathbf{c}_{N,U,l}^* = \mathbb{Q}_c(\mathbf{x}_{N,B,l}^*) \\
&6: \rightarrow \\
&7: \hat{\mathbf{x}} = \mathbb{Q}_c(\mathbf{x}_{N,B}) \\
\end{align*}
\]

3.2. Simulation Results

The performance of IMS with unbiased feedback is shown in Fig. 5 for \( L = 258, K = 129, s = 15, \mathbf{C} = \{-1, 0, +1\} \). The measurement matrix is a random Gaussian matrix, with the columns normalized to unit norm. In the upper part, the symbol error rate (SER) over the noise variance is shown. In order to ensure convergence, all algorithms perform 50 iterations. Besides IMS and (\cdot)uIMS, also the results for noise-based unbiaseding with average variances (TMS [14]) and for the BAMP algorithm [18] are shown.

While IMS without unbiasedness of the soft feedback (green) performs even worse than TMS (blue) which tracks only average instead of individual variances, the performance can be improved if individual signal-based unbiasedness (xuIMS, red dashed) is applied. Individual noise-based unbiasedness (nuIMS, red solid), however, clearly outperforms the other algorithms by 0.7 dB and 0.5 dB, respectively. Hence, the fact that the unbiased error is not orthogonal to the signal, is less important than the orthogonality of the noise/input error and the estimation error (cf. Sec. 2.3). Furthermore, the BAMP algorithm (black), which is state of the art in Compressed Sensing, is also clearly outperformed.

In the lower part of Fig. 5, the SER over the number of iterations is shown for 10 \( \log_{10}(1/\sigma_n^2) = 18 \) dB. Noteworthy, IMS and both variants of uIMS converge significantly faster than TMS, i.e., the tracking of individual instead of average variances does not only improve the performance, but accelerates also the convergence. Furthermore, BAMP performs again worst.

4. CONCLUSION

In this paper, we have discussed the unbiasedness for soft feedback, treating the estimation problem from the signal as well as from the noise perspective. Both approaches have been compared, and the connection to solutions with average variances (widely used in the literature) has been pointed out. Furthermore, both derived unbiasedness variants have been employed in an iterative algorithm, and the gains due to the unbiasedness have been investigated by numerical results.

\(^4\)The computational complexity of (\cdot)uIMS and TMS is comparable; the one of unbiasedness is negligible if the mean-squared error (5) as a function of \( \sigma_n^2 (\mathbf{c}_{N,U,l}^2 \text{ in Alg. 1}) \) is precalculated.
5. REFERENCES

[1] D. Guo, Y. Wu, S. Shamai, S. Verdú. “Estimation in Gaussian Noise: Properties of the Minimum Mean-Square Error.” *IEEE Transactions on Information Theory*, vol. 57, no. 4, pp. 2371–2385, Apr. 2011.

[2] G. D. Forney Jr. “On the Role of MMSE Estimation in Approaching the Information-Theoretic Limits of Linear Gaussian Channels: Shannon Meets Wiener.” *Proc. Allerton Conference*, pp. 430–439, Oct. 2003.

[3] R.F.H. Fischer. *Precoding and Signal Shaping for Digital Transmission*, John Wiley & Sons, New York, 2002.

[4] F. Tarköy. “MMSE-Optimal Feedback and its Applications.” *Proc. IEEE International Symposium on Information Theory (ISIT)*, p. 334, Sep. 1995.

[5] T. Frey, M. Reinhardt. “Signal Estimation for Interference Cancellation and Decision Feedback Equalization.” *Proc. IEEE Vehicular Technology Conference (VTC)*, pp. 113–121, May 1997.

[6] A. Lampe, J.B. Huber. “On Improved Multiuser Detection with Iterated Soft Decision Interference Cancellation.” *Proc. Communications Theory Mini-Conference at GLOBECOM*, pp. 172–176, June 1999.

[7] R.R. Müller, J.B. Huber. “Iterated Soft-Decision Interference Cancellation for CDMA.” *Broadband Wireless Communications*, eds. M. Luise and S. Pupolin, pp. 110–115, Springer, London, 1998.

[8] A. Papoulis. *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, Inc., Tokyo, Japan, 1965.

[9] S.M. Kay. *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice-Hall Inc., Upper Saddle River, NJ, USA, 1993.

[10] P. Banelli. “Non-Linear Transformations of Gaussians and Gaussian-Mixtures with Implications on Estimation and Information Theory.” *arXiv*, 1111.5950v3, 10. May 2013.

[11] H.E. Rowe. “Memoryless Nonlinearities With Gaussian Inputs: Elementary Results.” *The Bell System Technical Journal*, vol. 61, no. 7, pp. 1519–1525, Sep. 1982.

[12] R.F.H. Fischer. Notes on MMSE Estimation and Bias Compensation. *Unpublished Manuscript*, 2016.

[13] S. Sparrer, R.F.H. Fischer. “Algorithms for the Iterative Estimation of Discrete-Valued Sparse Vectors.” *Proc. ITG Conference on Systems, Communications and Coding (SCC)*, Feb. 2017.

[14] S. Sparrer, R.F.H. Fischer. “Unveiling Bias Compensation in Turbo-Based Algorithms for (Discrete) Compressed Sensing.” *Proc. European Signal Processing Conference (EUSIPCO)*, Aug. 2017.

[15] J. Ma, L. Ping. “Orthogonal AMP.” *IEEE Access*, vol. 5, pp. 2020–2033, Jan. 2017.

[16] D.L. Donoho. “Compressed Sensing.” *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1289–1306, Apr. 2006.

[17] S. Rangan, P. Schniter, A.K. Fletcher. “Vector Approximate Message Passing.” *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 1588–1592, June 2017.

[18] D.L. Donoho, A. Maleki, A. Montanari. “Message Passing Algorithms for Compressed Sensing: I. Motivation and Construction.” *Proc. Information Theory Workshop (ITW)*, Jan. 2010.