Incompatibility of the Dirac-like Field Operators with the Majorana Anzatzen

Valeriy V. Dvoeglazov
UAF, Universidad Autónoma de Zacatecas
Apartado Postal 636, Suc. 3, Zacatecas 98061 Zac., México
E-mail: valeri@fisica.uaz.edu.mx

Abstract. We show explicitly incompatibility of the Majorana anzatzen with the Dirac-like field operators in both the original Majorana theory and its generalizations. Several explicit examples are presented for higher spins too.

1. Introduction.
In Refs. [1]-[6] we considered the procedure of construction of the field operators ab initio (including for neutral particles). The Bogoliubov-Shirkov method has been used.

In the present article we investigate the spin-1/2 and spin-1 cases in different bases. We look for relations the Dirac-like field operator with the Majorana-like field operator. It seems that the calculations in the helicity basis only give mathematically and physically reasonable results.

2. The Spin-1/2.
The Dirac equation is:
\[ i\gamma^\mu \partial_\mu - mc/\hbar \Psi(x) = 0, \]
\( \mu = 0, 1, 2, 3. \) The most known methods of its derivation are [7, 8, 9]:
- the Dirac one (the Hamiltonian should be linear in \( \partial/\partial x^i \), and be compatible with \( E_p^2 - p^2c^2 = m^2c^4 \));
- the Sakurai one (based on the equation \( (E_p - c\sigma \cdot p)(E_p + c\sigma \cdot p)\phi = m^2c^4\phi \), \( \phi \) is the 2-component spinor);
- the Ryder one (the relation between 2-spinors at rest is \( \phi_R(0) = \pm \phi_L(0) \), and boosts).

The \( \gamma^\mu \) are the Clifford algebra matrices
\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \]
\( g^{\mu\nu} \) is the metric tensor. Usually, everybody uses the following definition of the field operator [10] in the pseudo-Euclidean metrics:
\[ \Psi(x) = \frac{1}{(2\pi)^3} \sum_h \int \frac{d^3p}{2E_p} [u_h(p)u_h(p)e^{-ip\cdot x}\gamma^\mu + v_h(p)v_h(p)e^{+ip\cdot x}\gamma^\mu]. \]
as given \textit{ab initio}. After actions of the Dirac operator on \( \exp(\mp ip_\mu x^\mu) \) the 4-spinors \((u-\) and \(v-\)) satisfy the momentum-space equations: \((\hat{p} - m)u_h(p) = 0\) and \((\hat{p} + m)v_h(p) = 0\), respectively; the \( h \) is the polarization index. It is easy to prove from the characteristic equations \( \text{Det}(\hat{p} \mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0 \) that the solutions should satisfy the energy-momentum relations \( p_0 = \pm E_p = \pm \sqrt{\mathbf{p}^2 + m^2} \) for both \( u- \) and \( v- \) solutions.

The general scheme of construction of the field operator has been presented in [11]. In the case of the \( (1/2, 0) \oplus (0, 1/2) \) representation we have:

\[
\Psi(x) = \frac{1}{(2\pi)^3} \int dp e^{ip_\mu x^\mu} \tilde{\Psi}(p). \tag{4}
\]

We know the condition of the mass shell: \((p^2 - m^2)\Psi(p) = 0\). Thus, \( \tilde{\Psi}(p) = \delta(p^2 - m^2)\Psi(p) \).

\[
\Psi(x) = \frac{1}{(2\pi)^3} \int dp e^{ip_\mu x^\mu} \delta(p^2 - m^2)(\theta(p_0) + \theta(-p_0))\Psi(p) = \frac{1}{(2\pi)^3} \int dp \left[ e^{ip_\mu x^\mu} \delta(p^2 - m^2)\Psi^+(p) + e^{-ip_\mu x^\mu} \delta(p^2 - m^2)\Psi^-(p) \right],
\]

where

\[
\Psi^+(p) = \theta(p_0)\Psi(p), \text{ and } \Psi^-(p) = \theta(p_0)\Psi(-p). \tag{6}
\]

Next, we adjust the notation to the modern one. We continue:

\[
\Psi(x) = \frac{1}{(2\pi)^3} \int d^4 p \delta(p^2 - m^2)e^{-ip x} \Psi(p) = \frac{1}{(2\pi)^3} \sum_h \int d^4 p \delta(p_0^2 - E_p^2)e^{-ip x} u_h(p_0, \mathbf{p}) a_h(p_0, \mathbf{p}) = \frac{1}{(2\pi)^3} \sum_h \int d^4 p \frac{\delta(p_0 - E_p) + \delta(p_0 + E_p)}{2E_p} \left[ \theta(p_0) + \theta(-p_0) \right] e^{-ip x} \times
\]

\[
\times \sum_h u_h(p_0) a_h(p) = \frac{1}{(2\pi)^3} \sum_h \int d^4 p \left[ \delta(p_0 - E_p) + \delta(p_0 + E_p) \right] \times
\]

\[
\times \left[ \theta(p_0)u_h(p_0)a_h(p)e^{-ip x} + \theta(p_0)u_h(-p_0)a_h(-p)e^{ip x} \right] = \frac{1}{(2\pi)^3} \sum_h \int d^3 \mathbf{p} \left[ u_h(p_0)a_h(p) \right] \left[ \theta(p_0)|p_0 = E_p, e^{-i(E_p(t - p \cdot x)} + u_h(-p_0)a_h(-p)|p_0 = E_p, e^{i(E_p(t - p \cdot x)} \right].
\]

During the calculations we had to represent \( 1 = \theta(p_0) + \theta(-p_0) \) above in order to get positive- and negative-frequency parts.\(^1\) Moreover, we did not yet assumed, which equation this field operator (namely, the \( u- \) spinor) satisfies, with negative- or positive- mass and/or \( p_0 = \pm E_p \).

In general we should transform \( u_h(-p) \) to the \( v_h(p) \). The procedure is the following one [1, 2].

\(^1\) See Ref. [5] for discussion.
2.1. The Standard Basis.
The explicit forms of the 4-spinors are:

\[ u_{\sigma}(p) = \frac{N_{\sigma}^+}{2\sqrt{m(E_p + m)}} \left( \begin{array}{c} E_p + m + \sigma \cdot p \phi_{\sigma}(0) \\ E_p + m - \sigma \cdot p \chi_{\sigma}(0) \end{array} \right), \quad v_{\sigma}(p) = \gamma^5 u_{\sigma}(p), \tag{10} \]

in the spinorial basis \( \phi_1(0) = \chi_1(0) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) and \( \phi_{\frac{1}{2}}(0) = \chi_{\frac{1}{2}}(0) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \). The transformation to the standard basis is produced with the \( (\gamma^5 + \gamma^0)/\sqrt{2} \) matrix. The normalizations, projection operators, propagators, dynamical invariants etc have been given in [9], for example.

In the Dirac case we should assume the following relation in the field operator (9):

\[ \sum_{h=\pm \frac{1}{2}} v_h(p)b_h^\dagger(p) = \sum_{h=\pm \frac{1}{2}} u_h(-p)a_h(-p), \tag{11} \]

which is compatible with the “hole” theory and the Feynman-Stueckelberg interpretation. We know that [9]

\[ \bar{u}_\mu(p)u_\lambda(p) = +m\delta_{\mu\lambda}, \tag{12} \]
\[ \bar{\mu}_\mu(p)u_\lambda(-p) = 0, \tag{13} \]
\[ \bar{v}_\mu(p)u_\lambda(p) = -m\delta_{\mu\lambda}, \tag{14} \]
\[ \bar{\nu}_\mu(p)u_\lambda(p) = 0, \tag{15} \]

\( \mu, \lambda \) are now the polarization indices. However, we need \( \Lambda_{\mu\lambda}(p) = \bar{v}_\mu(p)u_\lambda(-p) \). By direct calculations, we find

\[ -mb_{\mu}^\dagger(p) = \sum_{\lambda} \Lambda_{\mu\lambda}(p)a_\lambda(-p). \tag{16} \]

Hence, \( \Lambda_{\mu\lambda} = -im(\sigma \cdot n)_{\mu\lambda} \), \( n = p/|p| \), and

\[ b_{\lambda}^\dagger(p) = +i \sum_{\lambda}(\sigma \cdot n)_{\mu\lambda}a_\lambda(-p). \tag{17} \]

Multiplying (11) by \( \bar{u}_\mu(-p) \) we obtain

\[ a_\mu(-p) = -i \sum_{\lambda}(\sigma \cdot n)_{\mu\lambda}b_{\lambda}^\dagger(p). \tag{18} \]

The equations are self-consistent.

2.2. The Helicity Basis.
The 2-eigenspinors of the helicity operator

\[ \frac{1}{2}\sigma \cdot \hat{p} = \frac{1}{2} \left( \begin{array}{cc} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{+i\phi} & -\cos \theta \end{array} \right) \tag{19} \]

can be defined as follows [12, 13, 14]:

\[ \phi_{\frac{1}{2}}(0) = \left( \begin{array}{c} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{+i\phi/2} \end{array} \right), \quad \phi_{\frac{1}{2}}(0) = \left( \begin{array}{c} \sin \frac{\theta}{2} e^{-i\phi/2} \\ -\cos \frac{\theta}{2} e^{+i\phi/2} \end{array} \right). \tag{20} \]

for \( \pm 1/2 \) eigenvalues, respectively.
We start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i.e., two degrees of freedom); \( c = \hbar = 1 \):

\[
(p_0 + \sigma \cdot p)(p_0 - \sigma \cdot p)\phi = m^2 \phi.
\]

(21)

It can be re-written in the form of the set of two first-order equations for 2-spinors. We observe at the same time that they may be chosen as eigenstates of the helicity operator which present in (21):\(^2\)

\[
\begin{align*}
(E - (\sigma \cdot p))\phi_+ & = (E - p)\phi_+ = m\chi_+ , \\
(E + (\sigma \cdot p))\chi_+ & = (E + p)\chi_+ = m\phi_+ , \\
(E - (\sigma \cdot p))\phi_- & = (E + p)\phi_- = m\chi_-, \\
(E + (\sigma \cdot p))\chi_- & = (E - p)\chi_- = m\phi_-.
\end{align*}
\]

(22) - (25)

If the \( \phi_\pm \) spinors are defined in the equation (20) then we can construct the corresponding \( u- \) and \( v- \) 4-spinors:\(^3\)

\[
\begin{align*}
u_+^\dagger(p) & = N_+^\dagger \left( \frac{\phi_+}{E_{+} - p} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} E_{+} + p \\ E_{+} - p \end{array} \right) \phi_+ , \\
u_-^\dagger(p) & = N_-^\dagger \left( \frac{\phi_-}{E_{-} + p} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} E_{-} - p \\ E_{-} + p \end{array} \right) \phi_- , \\
v_+^\dagger(p) & = N_+^\dagger \left( \frac{\phi_+}{E_{+} - p} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} E_{+} - p \\ E_{+} + p \end{array} \right) \phi_+ , \\
v_-^\dagger(p) & = N_-^\dagger \left( \frac{\phi_-}{E_{-} + p} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} E_{-} + p \\ E_{-} - p \end{array} \right) \phi_- .
\end{align*}
\]

(26)

where the normalization to the unit \((\pm 1)\) was used:\(^4\)

\[
\begin{align*}
\bar{u}_h(p)u_{h'}(p) & = \delta_{hh'}, \bar{v}_h(p)v_{h'}(p) = -\delta_{hh'}, \\
\bar{u}_h(p)v_{h'}(p) & = 0 = \bar{v}_h(p)u_{h'}(p).
\end{align*}
\]

(28) - (29)

We define the field operator as follows:

\[
\Psi(x^\mu) = \sum_h \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sqrt{m} \left[ u_h(p) a_{h}^\dagger(p)e^{-ip_\mu x^\mu} + v_{h}(p) a_{h'}^\dagger(p)e^{+ip_\mu x^\mu} \right].
\]

(30)

The commutation relations are assumed to be the standard ones \([11, 10, 18, 19]\)\(^5\) (compare with \([17, 20]\))

\[
\begin{align*}
[a_h(p), a_{h'}^\dagger(k)]_+ & = 2E_p \delta^{(3)}(p - k)\delta_{hh'}, [a_h(p), a_{h'}(k)]_+ = 0 = \left[ a_{h'}^\dagger(p), a_{h'}^\dagger(k) \right]_+.
\end{align*}
\]

\(^2\) This opposes to the choice of the basis of the subsection (2.1), where 4-spinors are the eigenstates of the parity operator, cf. [15].

\(^3\) Alternatively, \( \uparrow \downarrow \) may refer to the chiral helicity eigenstates, e.g. \( u_\eta = \frac{1}{\sqrt{2}} \left( \begin{array}{c} N\phi_\eta \\ N^{-1}\phi_{-\eta} \end{array} \right) \), see next sections and cf. [16, 17].

\(^4\) Of course, there are no any mathematical difficulties to change it to the normalization to \( \pm m \), which may be more convenient for the study of the massless limit.

\(^5\) The only possible changes may be related to different forms of normalization of 4-spinors, which would have influence on the factor before \( \delta \)-function.
\[ [a_h(p), b^\dagger_{h'}(k)]_+ = 0 = [b_h(p), a^\dagger_{h'}(k)]_+ , \]
\[ [b_h(p), b^\dagger_{h'}(k)]_+ = 2E_p\delta^{(3)}(p-k)\delta_{hh'}, [b_h(p), b_{h'}(k)]_+ = 0 = [b^\dagger_h(p), b^\dagger_{h'}(k)]_+ \]

Other details of the helicity basis are given in Refs. [21, 14].

However, in this helicity case we have:
\[ \Lambda_{hh'}(p) = \bar{v}_h(p)u_{h'}(-p) = i\sigma_{hh'}^y. \]  

So, someone may advise that we should introduce the creation operators by hand in every basis.

2.3. Application of the Majorana anzatzen.

It is well known that “particle=antiparticle” in the Majorana theory [22]. So, in the language of the quantum field theory we should have

\[ b_\mu(E_p, p) = e^{i\varphi}a_\mu(E_p, p). \]  

Usually, different authors use \( \varphi = 0, \pm \pi/2 \) depending on the metrics and on the forms of the 4-spinors and commutation relations.

So, on using (17) and the above-mentioned postulate we come to:
\[ a^\dagger_\mu(p) = +ie^{i\varphi}(\sigma \cdot n)_{\mu\lambda}a_\lambda(-p). \]

On the other hand, on using (18) we make the substitutions \( E_p \rightarrow -E_p, \ p \rightarrow -p \) to obtain
\[ a_\mu(p) = +i(\sigma \cdot n)_{\mu\lambda}b^\dagger_\lambda(-p). \]

The totally reflected (35) is \( b_\mu(-E_p, -p) = e^{i\varphi}a_\mu(-E_p, -p) \). Thus,
\[ b^\dagger_\mu(-p) = e^{-i\varphi}a^\dagger_\mu(-p). \]

Combining with (37), we come to
\[ a_\mu(p) = +ie^{-i\varphi}(\sigma \cdot n)_{\mu\lambda}b^\dagger_\lambda(-p), \]
and
\[ a^\dagger_\mu(p) = -ie^{i\varphi}(\sigma^* \cdot n)_{\mu\lambda}a_\lambda(-p). \]

This contradicts with the equation (36) unless we have the preferred axis in every inertial system.

Next, we can use another Majorana anzatz \( \Psi = \pm e^{i\alpha}\Psi^c \) with usual definitions
\[ C = e^{i\theta_c} \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} \mathcal{K}, \quad \Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

Thus, on using \( Cu^\dagger_\uparrow(p) = iv_\uparrow(p), \ Cu^\dagger_\downarrow(p) = -iv_\downarrow(p) \) we come to other relations between creation/annihilation operators
\[ a^\dagger_\uparrow(p) = \mp ie^{-i\alpha}b^\dagger_\downarrow(p), \]
\[ a^\dagger_\downarrow(p) = \pm ie^{-i\alpha}b^\dagger_\uparrow(p). \]
which may be used instead of (35). Due to the possible signs ± the number of the corresponding
states is the same as in the Dirac case that permits us to have the complete system of the Fock
states over the (1/2, 0) ⊕ (0, 1/2) representation space in the mathematical sense.\(^6\) However,
in this case we deal with the self/anti-self charge conjugate quantum field operator instead of
the self/anti-self charge conjugate quantum states. Please remember that it is the latter that
answer for neutral particles; the quantum field operator contains operators for more than one
state, which may be either electrically neutral or charged.

We conclude that something is missed in the foundations of the original Majorana theory in
the (1/2, 0) ⊕ (0, 1/2) representation.

3. The Spin-1.

3.1. The Standard Basis.

We use the results of Refs. [23, 18, 24] in this Section. The polarization vectors of the standard
basis are defined [12]:

\[
e^{\mu}(0,+1) = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{\mu}(0,-1) = +\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

\[
e^{\mu}(0,0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e^{\mu}(0,0_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

The Lorentz transformations are \((\hat{p}_i = p_i/|p|)\):

\[
e^{\mu}(p,\sigma) = L_{\mu}^{\nu}(p)e^{\nu}(0,\sigma),
\]

\[
L_{0}^{0}(p) = \gamma, \quad L_{0}^{i}(p) = L_{i}^{0}(p) = \hat{p}_i \sqrt{\gamma^2 - 1}, \quad L_{i}^{k}(p) = \delta_{ik} + (\gamma - 1)\hat{p}_i\hat{p}_k.
\]

Hence, for the particles of the mass \(m\) we have:

\[
u^{\mu}(p,+1) = -\sqrt{2m} \begin{pmatrix} -p^i \\ m + \frac{p^1p^i}{E_p+m} \\ im + \frac{p^2p^i}{E_p+m} \end{pmatrix}, \quad \nu^{\mu}(p,-1) = \sqrt{2m} \begin{pmatrix} m + \frac{p^1p^i}{E_p+m} \\ -im + \frac{p^2p^i}{E_p+m} \\ \frac{p^3p^i}{E_p+m} \end{pmatrix},
\]

\[
u^{\mu}(p,0) = \frac{N}{m} \begin{pmatrix} -p^3 \\ \frac{p^2p^3}{E_p+m} \\ m + \frac{(p^3)^2}{E_p+m} \end{pmatrix}, \quad \nu^{\mu}(p,0_1) = \frac{N}{m} \begin{pmatrix} -p^3/m \\ -p^2/m \\ -p^3/m \end{pmatrix}.
\]

\(N\) is the normalization constant for \(\nu^{\mu}\). They are the eigenvectors of the parity operator
\((\gamma_{00} = \text{diag}(1 - 1 - 1 - 1)):\)

\[
\hat{P}\nu_{\mu}(\bar{p},\sigma) = -\nu_{\mu}(p,\sigma), \quad \hat{P}\nu_{\mu}(\bar{p},0_1) = +\nu_{\mu}(p,0_1).
\]

It is assumed that they form the complete orthonormalized system of the (1/2, 1/2) representational,
\(c_{\mu}^{*}(p,0_1)e^{\nu}(p,0_1) = 1, \quad c_{\mu}^{*}(p,\sigma)e^{\nu}(p,\sigma) = -\delta_{\sigma\sigma}6\).

\(^6\) Please note that the phase factors may have physical significance in quantum field theories as opposed to
the textbook nonrelativistic quantum mechanics, as it was discussed recently by several authors.
3.2. The Helicity Basis.

The helicity operator is:

\[
\frac{(S \cdot p)}{p} = \frac{1}{p} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -ip^3 & ip^2 \\ 0 & ip^3 & 0 & -ip^1 \\ 0 & -ip^2 & ip^1 & 0 \end{pmatrix}, \quad \frac{(S \cdot p)}{p} \epsilon^\mu_{\pm1} = \pm \epsilon^\mu_{\pm1}, \quad \frac{(S \cdot p)}{p} \epsilon^\mu_{00} = 0. \tag{51}
\]

The eigenvectors are:

\[
\epsilon^\mu_{+1} = \frac{1}{\sqrt{2}} \frac{e^{i\alpha}}{p} \left( \frac{0}{\sqrt{(p^1)^2 + (p^2)^2}} \right), \quad \epsilon^\mu_{-1} = \frac{1}{\sqrt{2}} \frac{e^{i\beta}}{p} \left( \frac{0}{\sqrt{(p^1)^2 + (p^2)^2}} \right)
\]

\[
\epsilon^\mu_0 = \frac{1}{m} \left( \frac{E_p}{p^1} \right), \quad \epsilon^\mu_0 = \frac{1}{m} \left( \frac{E_p}{p^2} \right).
\tag{52}
\]

The normalization is the same as in the standard basis. The eigenvectors \(\epsilon^\mu_{+1}\) are not the eigenvectors of the parity operator \((\gamma_0 R)\) of this representation. However, the \(\epsilon^\mu_{00}, \epsilon^\mu_{00}\) are. Surprisingly, the latter have no well-defined massless limit. In order to get the well-known massless limit one should use the basis of the light-front form representation, cf. [25].

3.3. The Field Operators.

Various-type field operators are possible in this representation. Let us remind the procedure to get them. Again, during the calculations below we have to present \(1 = \theta(p_0) + \theta(-p_0)\) in order to get positive- and negative-frequency parts. Meanwhile, the Heaviside \(\theta-\) function is not defined in \(p_0 = 0\). In general, due to integral theorems this presentation is possible even for distributions because we use the \(\theta-\) function in the integrand.\(^7\)

\[
A_\mu(x) = \frac{1}{(2\pi)^3} \int d^4p \delta(p^2 - m^2)e^{-ip \cdot x} A_\mu(p) =
\]

\[
= \frac{1}{(2\pi)^3} \sum_{\lambda} \int d^4p \delta(p_0^2 - E_p^2)e^{-ip \cdot x} \epsilon_\mu(p, \lambda) a_\lambda(p) =
\]

\[
= \frac{1}{(2\pi)^3} \int d^4p \frac{\delta(p_0 - E_p)}{2E_p} \left[ \theta(p_0) + \theta(-p_0) \right] e^{-ip \cdot x} A_\mu(p) =
\]

\[
= \frac{1}{(2\pi)^3} \int d^4p \frac{\delta(p_0 - E_p) + \delta(p_0 + E_p)}{2E_p} \left[ \theta(p_0) A_\mu(p) e^{-ip \cdot x} + \theta(-p_0) A_\mu(-p) e^{ip \cdot x} \right] =
\]

\[
= \frac{1}{(2\pi)^3} \sum_{\lambda} \int \frac{d^3p}{2E_p} \epsilon_\mu(p, \lambda) a_\lambda(p) e^{-ip \cdot x} + \epsilon_\mu(-p, \lambda) a_\lambda(-p) e^{ip \cdot x} \right]. \tag{54}
\]

\(^7\) However, remember, that we have the \(p_0 = 0\) solution of the Maxwell equations. It has the experimental confirmation (for instance, the stationary magnetic field \(\nabla \times \mathbf{B} = 0\)).
We should transform the second part to $\epsilon^*_\mu(p, \lambda)b^\dagger_\lambda(p)$ as usual. In such a way we obtain the states which are considered to be the charge-conjugate states. In this Lorentz group representation the charge conjugation operator is assumed to be the complex conjugation operator for 4-vectors. Of course, one can try to get $P$-conjugates or $CP$-conjugate states too. We posulate

$$\sum_\lambda \epsilon_\mu(-p, \lambda)a_\lambda(-p) = \sum_\lambda \epsilon^*_\mu(p, \lambda)b^\dagger_\lambda(p).$$

(55)

Then we multiply both parts by $\epsilon^\mu(p, \sigma)$, and use the normalization conditions for polarization vectors. In the $(\frac{1}{2}, \frac{1}{2})$ representation we can also expand (apart of the equation (55)) in a different way. For example,

$$\sum_\lambda \epsilon_\mu(-p, \lambda)c_\lambda(-p) = \sum_\lambda \epsilon_\mu(p, \lambda)d^\dagger_\lambda(p).$$

(56)

From the first definition we obtain:

$$\begin{pmatrix}
    b^\dagger_0(p)
    
    -b^\dagger_{+1}(p)
    
    -b^\dagger_0(p)
    
    -b^\dagger_{-1}(p)
\end{pmatrix} = \sum_{\mu\lambda} \epsilon^\mu(p, \sigma)\epsilon_\mu(-p, \lambda)a_\lambda(-p) = \sum_\lambda \Lambda^{(1a)}_{\sigma\lambda}a_\lambda(-p) =$$

$$= \begin{pmatrix}
    -1
    0
    0
    0
\end{pmatrix}
- \begin{pmatrix}
    \sqrt{2}p_zp_\perp
    0
    0
    \sqrt{2}p_zp_\perp
\end{pmatrix}
\begin{pmatrix}
    a_{00}(-p)
    a_{11}(-p)
    a_{10}(-p)
    a_{1-1}(-p)
\end{pmatrix}. $$

(57)

From the second definition $\Lambda^{(1b)}_{\sigma\lambda} = \sum_\mu \epsilon^\mu(p, \sigma)\epsilon_\mu(-p, \lambda)$ we have

$$\begin{pmatrix}
    d^\dagger_0(p)
    
    -d^\dagger_{+1}(p)
    
    -d^\dagger_0(p)
    
    -d^\dagger_{-1}(p)
\end{pmatrix} = \sum_{\mu\lambda} \epsilon^\mu(p, \sigma)\epsilon_\mu(-p, \lambda)c_\lambda(-p) = \sum_\lambda \Lambda^{(1b)}_{\sigma\lambda}c_\lambda(-p) =$$

$$= \begin{pmatrix}
    -1
    0
    0
    0
\end{pmatrix}
- \begin{pmatrix}
    -\sqrt{2}p_zp_\perp
    0
    0
    \sqrt{2}p_zp_\perp
\end{pmatrix}
\begin{pmatrix}
    c_{00}(-p)
    c_{11}(-p)
    c_{10}(-p)
    c_{1-1}(-p)
\end{pmatrix} = -g^{\sigma\lambda} + (S \cdot n)_{\sigma\lambda}^2. $$

(58)

Possibly, we should think about modifications of the Fock space in this case, or introduce several field operators for the $(\frac{1}{2}, \frac{1}{2})$ representation. The Majorana-like anzatz is compatible for the $0_t$ polarization state only\footnote{See $S = 0$ in other notation.} in this basis of this representation.

However, the corresponding matrices $\Lambda^2$ in the helicity basis are different. Here they are:

$$\begin{pmatrix}
    b^\dagger_0(p)
    
    -b^\dagger_{+1}(p)
    
    -b^\dagger_0(p)
    
    -b^\dagger_{-1}(p)
\end{pmatrix} = \sum_{\mu\lambda} \epsilon^\mu(p, \sigma)\epsilon_\mu(-p, \lambda)a_\lambda(-p) = \sum_\lambda \Lambda^{(2a)}_{\sigma\lambda}a_\lambda(-p) =$$

$$= \begin{pmatrix}
    -1
    0
    0
    0
\end{pmatrix}
- \begin{pmatrix}
    \sqrt{2}p_zp_\perp
    0
    0
    -\sqrt{2}p_zp_\perp
\end{pmatrix}
\begin{pmatrix}
    c_{00}(-p)
    c_{10}(-p)
    c_{10}(-p)
    c_{1-1}(-p)
\end{pmatrix}. $$
and
\[
\begin{pmatrix}
  d_0^\dagger(p) \\
  -d_{+1}^\dagger(p) \\
  -d_0(p) \\
  -d_{-1}(p)
\end{pmatrix} = \sum_{\mu\lambda} \epsilon^{\mu\nu}(p, \sigma) \epsilon_{\mu}(\sigma, -p, \lambda) c_\lambda(-p) = \sum_{\lambda} \Lambda^{(2b)}_{\sigma\lambda} c_\lambda(-p) = \\
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & e^{-i(\alpha - \beta)} & 0 \\
  0 & e^{i(\alpha - \beta)} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  c_00(-p) \\
  c_{11}(-p) \\
  c_{10}(-p) \\
  c_{11}(-p)
\end{pmatrix}.
\] (60)

This appears to be compatible with the Majorana-like anzatzen.

Of course, the same procedure can be applied in the construction of the quantum-field operator for \( F_{\mu\nu} \).

**3.4. The \((1, 0) \oplus (0, 1)\) Representation.**

The solutions of the Weinberg-like equation
\[
[\gamma^{\mu\nu}\partial_\mu \partial_\nu - \left(\frac{i\partial}{\partial t}\right) m^2] \Psi(x) = 0.
\] (61)

are found in Refs. [26, 23, 27, 28]. Here they are:
\[
u_\sigma(p) = \begin{pmatrix}
  D^S(\Lambda_R)\xi_\sigma(0) \\
  \frac{1}{2}(\Lambda_R \Theta)\xi_\sigma(0)
\end{pmatrix},
\] (62)
\[
\Gamma^5 = \begin{pmatrix}
  1_{3\times 3} & 0 \\
  0 & -1_{3\times 3}
\end{pmatrix},
\] (63)

where \( D^S \) is the matrix of the \((S, 0)\) representation of the spinor group \( SL(2, c) \).

In the \((1, 0) \oplus (0, 1)\) representation the procedure of derivation of the creation operators leads to somewhat different situation:
\[
\sum_{\sigma = 0, \pm 1} \nu_\sigma(p) b^\dagger_\sigma(p) = \sum_{\sigma = 0, \pm 1} \nu_\sigma(-p) a_\sigma(-p), \quad \text{hence } b^\dagger_\sigma(p) = 0.
\] (64)

However, if we return to the original Weinberg equations \([\gamma^{\mu\nu}\partial_\mu \partial_\nu \pm m^2] \Psi_{1,2}(x) = 0\) with the field operators:
\[
\Psi_1(x) = \frac{1}{(2\pi)^3} \sum_\mu \int \frac{d^3p}{2E_p} \left[ \nu_\mu(p)a_\mu(p)e^{-ip_{\mu}\cdot x_\mu} + \frac{1}{2}(\Lambda_R \Theta)\xi^\dagger_\mu(0)a_\mu(-p) \right],
\] (65)
\[
\Psi_2(x) = \frac{1}{(2\pi)^3} \sum_\mu \int \frac{d^3p}{2E_p} \left[ \nu_\mu(p)c_\mu(p)e^{-ip_{\mu}\cdot x_\mu} + \frac{1}{2}(\Lambda_R \Theta)\xi^\dagger_\mu(0)c_\mu(-p) \right],
\] (66)

we obtain
\[
b^\dagger_\mu(p) = \left[ 1 - 2(S \cdot n)^2 \right]_{\mu\lambda} a_\lambda(-p),
\] (67)
\[
d^\dagger_\mu(p) = \left[ 1 - 2(S \cdot n)^2 \right]_{\mu\lambda} c_\lambda(-p).
\] (68)
The application of \(\pi_{\mu}(-p)u_\lambda(-p) = \delta_{\mu\lambda}\) and \(\pi_{\mu}(-p)u_\lambda(p) = [1 - 2(S \cdot n)^2]_{\mu\lambda}\) prove that the equations are self-consistent (similarly to those of the subsection (2.1)). This situation signifies that in order to construct the Sankaranarayanan-Good field operator (which was used by Ahluwalia, Johnson and Goldman [27]) we need additional postulates. One can try to construct the left- and the right-hand sides of the field operator separately each other. In this case the commutation relations may also be more complicated.

Is it possible to apply the Majorana-like anzatz to the \((1,0) + (0,1)\) fields? Repeating the procedure of the Section 2.3, on using (67) and the Majorana posulate we come to:

\[
a^\dagger_\mu(p) = +e^{+i\phi}[1 - 2(S \cdot n)^2]_{\mu\lambda}a_\lambda(-p).
\]  

On the other hand, on using the inverse relation, namely, that for \(a_\mu(-p)\), we make the substitutions \(E_p \rightarrow -E_p\), \(p \rightarrow -p\) to obtain

\[
a_\mu(p) = +[1 - 2(S \cdot n)^2]_{\mu\lambda}b^\dagger_\lambda(-p).
\]  

The totally reflected Majorana nazatz is \(b_\mu(-E_p, -p) = e^{+i\phi}a_\mu(-E_p, -p)\). Thus,

\[
b^\dagger_\mu(-p) = e^{-i\phi}a^\dagger_\mu(-p).
\]

Combining with (70), we come to

\[
a_\mu(p) = +e^{-i\phi}[1 - 2(S \cdot n)^2]_{\mu\lambda}a^\dagger_\lambda(-p),
\]

and

\[
a^\dagger_\mu(p) = +e^{+i\phi}[1 - 2(S^* \cdot n)^2]_{\mu\lambda}a_\lambda(-p).
\]

In the basis where \(S_z\) is diagonal the matrix \(S_y\) is imaginary [12]. So, \((S^* \cdot n) = S_xn_x - S_yn_y + S_zn_z\), and \((S^* \cdot n)^2 \neq (S \cdot n)^2\) in the case of \(S = 1\). Similarly, one can proceed with (68). So, we conclude that there is the same problem in this point, in the application of the Majorana-like anzatz, as in the case of spin-1/2.

Meanwhile, the attempts of constructing the self/anti-self charge conjugate states failed in Ref. [17]. Instead, the \(\Gamma^5S^\dagger_{[1]}\) – self/anti-self conjugate states have been constructed therein.

4. Conclusions.

We conclude that something is missed in the foundations of both the original Majorana theory and its generalizations. Similar problems exist in the theories of higher spins.

Acknowledgements. I acknowledge discussions with colleagues at recent conferences. I am grateful to the Zacatecas University for professorship.

[1] Dvoeglazov V V 2003 Hadronic J. Suppl. 18 239.
[2] Dvoeglazov V V 2006 Int. J. Mod. Phys. B20 1317.
[3] Dvoeglazov V V 2015 Einstein and Others: Unification ed V V Dvoeglazov. (Hauppauge:Nova Sci. Pubs.), p. 211.
[4] Dvoeglazov V V 2016 Z. Naturforsch. A 71 345.
[5] Dvoeglazov V V 2011 J. Phys. Conf. Ser. 284 012024.
[6] Dvoeglazov V V 2013 Bled Workshops 14-2 199.
[7] Dirac P A M 1928 Proc. Roy. Soc. Lond. A117 610.
[8] Sakurai J J 1967 Advanced Quantum Mechanics. (Reading:Addison-Wesley).
[9] Ryder L H 1985 Quantum Field Theory. (Cambridge: University Press).
[10] Itzykson C and Zuber J B 1980 Quantum Field Theory (New-York:McGraw-Hill Book Co.).
[11] Bogoliubov N N and Shirkov D V 1973 Introduction to the Theory of Quantized Fields. 2nd Edition. (Moscow:Nauka).
[12] Varshalovich D A, Moskalev A N and Khersonskii V K 1988 Quantum Theory of Angular Momentum (Singapore:World Scientific), §6.2.5.

[13] Dvoeglazov V V 1997 Fizika B6 111.

[14] Dvoeglazov V V 2004 Int. J. Theor. Phys. 43 1287.

[15] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1982 Quantum Electrodynamics. (Oxford:Pergamon Press), §16.

[16] Dvoeglazov V V 1996 Nuovo Cim. B111 483.

[17] Ahluwalia D V 1996 Int. J. Mod. Phys. A11 1855.

[18] Weinberg S 1995 The Quantum Theory of Fields. Vol. I. Foundations. (Cambridge: University Press).

[19] Greiner W 1996 Field Quantization. (Berlin-Heidelberg:Springer).

[20] Tokuoka Z 1967 Prog. Theor. Phys. 37 603.

[21] Rück H M and Greiner W 1977 J. Phys. G: Nucl. Phys. 3, 657.

[22] Majorana E 1937 Nuovo Cim. 14 171.

[23] Novozhilov Yu V 1975 Introduction to Elementary Particle Physics (Oxford:Pergamon Press).

[24] Dvoeglazov V V 2000 Photon: Old Problems in Light of New Ideas ed V V Dvoeglazov (Huntington: Nova Sci. Pubs.).

[25] Ahluwalia D V and Sawicki M 1993 Phys. Rev. D47 5161; 1994 Phys. Lett. B335 24.

[26] Sankaranarayanan A and Good, Jr. R H 1965 Nuovo Cim. 36 1303.

[27] Ahluwalia D V, Johnson M B and Goldman T 1993 Phys. Lett. B316 102.

[28] Dvoeglazov V V 1998 Int. J. Theor. Phys. 37 1915.