Long-wave equation for a confined ferrofluid interface: periodic interfacial waves as dissipative solitons

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We study the dynamics of a ferrofluid thin film confined in a Hele-Shaw cell, and subjected to a tilted non-uniform magnetic field. It is shown that the interface between the ferrofluid and an inviscid outer fluid (air) supports travelling waves, governed by a novel modified Kuramoto–Sivashinsky-type equation derived under the long-wave approximation. The balance between energy production and dissipation in this long-wave equation allows for the existence of dissipative solitons. These permanent travelling waves’ propagation velocity and profile shape are shown to be tunable via the external magnetic field. A multiple-scale analysis is performed to obtain the correction to the linear prediction of the propagation velocity, and to reveal how the nonlinearity arrests the linear instability. The travelling periodic interfacial waves discovered are identified as fixed points in an energy phase plane. It is shown that transitions between states (wave profiles) occur. These transitions are explained via the spectral stability of the travelling waves. Interestingly, multi-periodic waves, which are a non-integrable analogue of the double cnoidal wave, are also found to propagate under the model long-wave equation. These multi-periodic solutions are investigated numerically, and they are found to be long-lived transients, but ultimately abruptly transition to one of the stable periodic states identified.

1. Introduction

Immiscible fluid flows confined in Hele-Shaw cells have been investigated extensively during the past several decades [1]. Going back to the classical work by Saffman & Taylor [2], interest has focused on the
dynamics of the sharp interface between the fluids [3]. The interface’s displacement, when the motion of the fluids is normal to the unperturbed interface, has been of particular interest to most studies, specifically viscous fingering instabilities and finger growth [4]. By contrast, Hele-Shaw flows in which the main flow direction is parallel to the fluid interface have received less attention. Early work by Zeybek & Yortsos [5,6] considered such a parallel flow in a horizontal Hele-Shaw cell, both theoretically and experimentally. They found that, in the limit of large capillary number and under the long-wave assumption, interfacial waves between the two viscous fluids in this set-up are governed by a set of coupled Korteweg–de Vries (KdV) and Airy equations. Similarly, Charru & Fabre [7] investigated periodic interfacial waves between two viscous fluid layers in a Couette flow, in which case the long-wave equation was found to be of Kuramoto–Sivashinsky (KS) type. Subsequently, experimental work by Gondret and co-workers [8,9] demonstrated travelling waves in a parallel flow in a vertical Hele-Shaw cell. In this case, the phenomenon is well-described by a modified Darcy equation accounting for inertial effects, in which context a Kelvin–Helmholtz instability for inviscid fluids was found [10,11]. These prior studies considered fluids that are not responsive to external stimuli.

Ferrofluids (also known as ‘magnetic fluids’ [12,13]), on the other hand, are colloidal suspensions of nanometer-sized magnetic particles dispersed in a non-magnetic carrier fluid. These fluids are typically Newtonian but respond to applied magnetic fields, which is of particular interest in the present work. The linear theory of the Kelvin–Helmholtz instability for unconfined ferrofluids was developed by Rosensweig [14], which revealed how the strength of the applied magnetic field (on top of the velocity difference and viscosity contrast between the fluid) enters the threshold for instability. Miranda & Widom [15] extended this result to a parallel ferrofluid flow in a vertical Hele-Shaw cell under an external non-tilted magnetic field and deduced that the magnetic field does not affect the propagation speed of waves. Using a perturbative weakly nonlinear analysis, Lira & Miranda [16] further extended the latter analysis by adopting an in-plane tilted applied magnetic field, showing that the wave speed is sensitive to the angle. Such a field was shown to generate nonlinear travelling surface waves between a ferrofluid and an inviscid fluid (such as air). Jackson & Miranda [17] introduced a ‘crossed’ magnetic field (with perpendicular and azimuthal components) to influence the mode selection for a ferrofluid drop confined in a horizontal Hele-Shaw cell. Beyond Hele-Shaw configurations, Seric et al. [18] derived a long-wave equation to model dewetting of a two-dimensional thin film resulting from the interaction between a uniform applied magnetic field and disjoining pressure. More recently, Yu & Christov [19] conducted fully nonlinear simulations, using a vortex sheet Lagrangian method, of ferrofluid droplets in a horizontal Hele-Shaw cell. They showed that nonlinear periodic waves can be generated on the ferrofluid interface by tuning an external magnetic field’s orientation. In their analysis, the nonlinear wave propagation speed was well predicted by perturbation theory, showing that the magnetic field can set the wave speed and induce rotation of the droplet. Despite the recent work and interest on how tilted magnetic fields generate nonlinear waves on ferrofluid interfaces, a model long-wave equation to describe these phenomena is still lacking. Such a reduced-order (low-dimensional) model would provide deeper insight into the nonlinear wave dynamics and the mechanisms that sustain them [20].

To this end, in this work, our goal is to derive a novel model long-wave equation [21–23] to describe the nonlinear wave dynamics on a confined ferrofluid interface. First, we re-examine the problem proposed in [19] by considering a thin ferrofluid film in Cartesian coordinates (as shown in figure 1), subjected to an in-plane tilted magnetic field, which makes an arbitrary angle with the unperturbed (flat, horizontal) interface. Next, a perturbation analysis similar to that for shallow water waves, valid for small wave amplitudes and long wavelengths, is conducted. We show that the interfacial waves are governed by a modified equation of the KS type. Although the KS equation is usually mentioned in the context of the work by Kuramoto & Tsuzuki [24], on phase turbulence in reaction–diffusion systems, and the work by Sivashinsky [25], on wrinkled flame front propagation, the equation was first derived by Homsy [21] for thin liquid films (see also the discussions in [26,27]). On the other hand, the generalized KS equation, which additionally contains a dispersion term, has been derived in the context of a wide variety of falling thin film
problems [28], for which the driving force is typically gravity. In the present work, the novel feature is the non-invasive forcing of the ferrofluid by a magnetic field, which leads to a new type of generalized KS equation that captures the myriad of nonlinear effects (interestingly, in the absence of the Hopf-like convective nonlinearity found in the traditional KS-type equations) on long-wave evolution in one dimension.

This paper is organized as follows. Section 2 introduces the governing equations of the parallel ferrofluid thin film flow confined to a horizontal Hele-Shaw cell. In §3, the long-wave equation for the interface dynamics is derived, exposing the key parameters governing the physics. In §4, linear and weakly nonlinear analyses are conducted to understand the wave dynamics. An energy budget for the nonlinear travelling wave solution of the long-wave equation is obtained, showing that a dissipative soliton can propagate under the novel balance of surface tension and magnetic forces in this system. The effects of the key parameters on the wave profile and its propagation are discussed in §5. Then §6 considers the transition between different nonlinear states, and their spectral stability, to address the pattern selection problem. Additionally, propagating multi-periodic waves are uncovered numerically, and their persistence is investigated. Finally, conclusions and avenues for further work are summarized in §7.

2. Mathematical model and governing equations

Building on our previous work [19], we study the dynamics of interfacial waves on a thin ferrofluid film, confined in the transverse direction within a Hele-Shaw cell with gap thickness $b$, as shown in figure 1. In the reference configuration, the unperturbed interface is at $r = R_0 + h_0$. The entire cell is subjected to a radially varying external magnetic field via a long wire carrying an electric current $I$ through the origin. This current produces an azimuthal magnetic field component $H_a = (I/2\pi)(1/r)\hat{e}_\theta$. Then, anti-Helmholtz coils can be used to produce a radial magnetic field component $H_r = (H_0/R_0)r\hat{e}_r$, where $H_0$ is the strength of the magnetic field at $r = R_0$ [19,29,30]. Now assume that $R_0 \gg h_0$, where $h_0$ is a characteristic ‘depth’ of the ferrofluid film at rest. Under this ‘small film curvature’ assumption [31], the non-uniform magnetic field $H = H_a + H_r$ can be approximated in locally Cartesian coordinates as

$$H \simeq \frac{I}{2\pi} \frac{1}{R_0 + y} \hat{e}_x + \frac{H_0}{R_0} (R_0 + y)\hat{e}_y. \quad (2.1)$$

From equation (2.1), we understand that a magnetic body force $\propto |\mathbf{M}| \nabla |\mathbf{H}|$ acts on the thin film, where $\mathbf{M}$ is the magnetization vector of the ferrofluid. For the purposes of studying the interface
and shape dynamics [29,32–34], we assume that the ferrofluid is uniformly magnetized, and the magnetization is collinear with the external field, i.e. $\mathbf{M} = \chi \mathbf{H}$, where $\chi$ is the constant magnetic susceptibility. Since the applied field is spatially varying, i.e. $\nabla |\mathbf{H}| \neq 0$, then $\nabla |\mathbf{H}|$ becomes the main contribution to the magnetic body force. According to the prior literature, this observation leads us to neglect the effect of the demagnetizing field in comparison.

It is straightforward to show by standard methods (e.g. [29] and the references therein) that neglecting inertial hydrodynamic terms, enforcing the no-slip condition on the confining boundaries (transverse to the flow) of the Hele-Shaw cell, and averaging across the gap (i.e. over $z$) yields a modified ‘Darcy’s Law’ that governs this flow [29]

$$\bar{v} = -\frac{b^2}{12\mu_f} \nabla (p - \Psi), \quad \nabla \cdot \bar{v} = 0, \quad -\infty < x < \infty, \quad 0 \leq y \leq f(x,t). \quad (2.2)$$

Here, $p$ is the hydrodynamic pressure in the film, $\mu_f$ is the ferrofluid’s dynamic viscosity, $\Psi = \mu_0 \chi |\mathbf{H}|^2/2$ is a scalar potential accounting for the magnetic body force (such that $p - \Psi$ is a modified pressure) and $\mu_0$ is the free-space permeability. Gravity acts in the $-z$-direction, but it is neglected due to the narrow confinement. Both fluids are considered incompressible. The viscosity of the ‘upper’ fluid is considered negligible (i.e. it is considered inviscid, as would be the case with air), so the flow outside the ferrofluid film is not considered. We denote by $\bar{v} = u(x,y,t) \hat{e}_x + v(x,y,t) \hat{e}_y$ the $z$-averaged velocity field in the ‘lower’ fluid (the ferrofluid).

At the interface, having neglected the dynamics of the upper fluid, the pressure is given by a modified Young–Laplace Law [14,35]:

$$p = \sigma \kappa - \frac{\mu_0}{2} (\mathbf{M} \cdot \mathbf{n})^2 \quad \text{on} \quad y = f(x,t), \quad (2.3)$$

where $\sigma$ is the constant surface tension, and $\kappa \equiv -f_x^2/(1 + f_x^2)^{3/2}$ is the curvature of the surface $y = f(x,t)$ ($x$ and $t$ subscripts denote partial derivatives). The second term on the right-hand side of equation (2.3) is the magnetic normal traction [14,35], where $\mathbf{n} = (-f_x,1)/\sqrt{1 + f_x^2}$ denotes the upward unit normal vector to the interface. This contribution, due to the projection of $\mathbf{M}$ onto $\mathbf{n}$, induces unequal normal stress on either side of the profile’s peaks on the perturbed interface, thus breaking the initial equilibrium and leading to wave propagation [19].

A kinematic boundary condition is also imposed at the interface

$$v = f_t + uf_x \quad \text{on} \quad y = f(x,t), \quad (2.4)$$

which requires that the film boundary is a material surface. The no-penetration condition

$$v = 0 \quad \text{on} \quad y = 0 \quad (2.5)$$

is imposed at the ‘bottom’ of the layer, which is the material surface at $r = R_0$ in the original radial coordinates (figure 1) that maps to $y = 0$.

Introducing the potential $\phi = p - \Psi - \Psi_0$, where the constant

$$\Psi_0 = -\frac{\mu_0 \chi H_0^2}{R_0^2} \frac{1}{(R_0 + h_0)^2(1 + \chi)} - \frac{\mu_0}{2} \frac{\chi I^2}{4\pi^2} \frac{1}{(R_0 + h_0)^2} \quad (2.6)$$

accommodates the trivial solution, and combining the two equations in (2.2) together, the governing equation becomes Laplace’s equation

$$\nabla^2 \phi = 0. \quad (2.7)$$

From equations (2.3) and (2.4), equation (2.7) is subject to the following boundary conditions on $y = f(x,t)$:

$$v = f_t + uf_x \quad (2.8a)$$
and
\[
\phi + \frac{\mu_0 \chi}{2} \frac{H_0^2(R_0 + y)^2}{R_0^2} + \frac{\mu_0 \chi}{2} \frac{i^2}{4\pi^2(R_0 + y)^2} + \Psi_0 = \sigma \kappa - \frac{\mu_0 \chi^2}{2} \left[ \frac{i^2}{4\pi^2(R_0 + y)^2} \frac{f_x^2}{1 + f_x^2} + \frac{H_0^2(R_0 + y)^2}{R_0^2} \frac{1}{1 + f_x^2} - \frac{1}{\pi R_0} \frac{f_x}{1 + f_x^2} \right].
\] (2.8b)

3. Derivation of the long-wave equation

(a) Expansion of the potential and non-dimensionalization

To reduce the governing equations to a single partial differential equation (PDE) for the surface deformation \( \eta \), we expand \( \phi \) in a power series in \( y \), a standard approach for small amplitude surface deformations (e.g. [36]):
\[
\phi(x, y, t) = \sum_{n=0}^{\infty} y^n \phi_n(x, t).
\] (3.1)

Substituting this expansion into Laplace’s equation (2.7) generates a recursion relation \( \phi_{n,xx} + (n + 2)(n + 1) \phi_{n+2} = 0 \). On the other hand, since \( \phi_0 = \sum_{n=0}^{\infty} y^n \phi_n(x, t) \), the constraint at the bottom (i.e. equation (2.5)) requires that \( \phi_0 = 0 \), which eliminates the odd terms from the expansion. Hence, we can simplify equation (3.1) as
\[
\phi(x, y, t) = \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} g^{(2m)}(x, t), \quad g^{(2m)}(x, t) \equiv \frac{\partial^{2m}}{\partial x^{2m}} \phi_0(x, t).
\] (3.2)

Let \( a \) be the typical amplitude scale for the surface deformation \( \eta(x, t) \). Now, we introduce the following non-dimensionalization:
\[
\begin{align*}
    x &\mapsto \ell x, & y &\mapsto h_0 y, & t &\mapsto \frac{12 \mu_f \ell^3}{\sigma b^2} t, & \eta &\mapsto a \eta,
\end{align*}
\] (3.3)

and
\[
\begin{align*}
    u &\mapsto \left( \frac{a}{h_0} \right) \frac{\sigma b^2}{12 \mu_f \ell^2} u, & v &\mapsto \left( \frac{a}{h_0} \right) \left( \frac{h_0}{\ell} \right) \frac{\sigma b^2}{12 \mu_f \ell^2} v, & \phi &\mapsto \left( \frac{a}{h_0} \right) \frac{\sigma}{\ell} \phi, & g &\mapsto \left( \frac{a}{h_0} \right) \frac{\sigma}{\ell} g,
\end{align*}
\]

where \( \ell \) is the horizontal length scale. Next, we define the small parameters of the model
\[
\delta := \frac{h_0}{\ell}, \quad \epsilon := \frac{a}{h_0} \quad \text{and} \quad \varepsilon := \frac{h_0}{R_0},
\] (3.4)

corresponding to a wavelength parameter, an amplitude parameter and a magnetic field gradient parameter, respectively. To implement the upcoming asymptotic expansion, a long wavelength \( \delta \ll 1 \) and small amplitude \( \epsilon \ll 1 \) approximation is made [37]. (Although it is possible to also derive arbitrary-amplitude long-wave equations [22,37], Homsy [21] argued that the distinguished limit of \( \epsilon \ll 1 \) leads to the model equations capturing the essential physics.) Note that \( \varepsilon \ll 1 \) is determined by the geometric configuration; specifically, \( R_0 \) is chosen sufficiently large to allow the Cartesian approximation, but small enough to ensure that \( \nabla \mathbf{H} \) is still the dominant term in the magnetic body force [29,32,33]. Note that demagnetization can still be neglected because it can be made arbitrarily small via the thickness \( b \) [34].

The scaled potential obeys
\[
\phi_{xx} + \delta^{-2} \phi_{yy} = 0, \quad u = -\phi_x \quad \text{and} \quad v = -\delta^{-2} \phi_y,
\] (3.5)

and, to \( O(\delta^2) \), the scaled and truncated equation (3.2) yields
\[
\phi = g - \frac{1}{2} \delta^2 y^2 g_{xx}, \quad u = -g_x + \frac{1}{2} \delta^2 y^2 g_{xxx} \quad \text{and} \quad v = y g_{xx} - \frac{1}{6} \delta^2 y^3 g_{xxxx},
\] (3.6)

consistent with equation (3.5).
(b) Boundary conditions and reduction of the governing equations

Under the above assumptions on the small parameters, the leading-order terms involve $\epsilon$, $\epsilon^2$ and $\delta^2$. Without making further assumption on their relative scalings (thus, keeping cross-terms as well), the corresponding kinematic and dynamic boundary conditions (2.8) on the fluid–fluid interface become

\begin{align}
v &= \eta_l + \epsilon u \eta_x \quad \text{on} \quad y = 1 + \epsilon \eta(x,t) \quad (3.7a) \\
\phi &= B_1 \eta - \delta \eta_{xx} + \delta B_2 \eta_x + \epsilon \delta^2 B_3 \eta_x^2 - B_4 \epsilon \eta^2 \quad \text{on} \quad y = 1 + \epsilon \eta(x,t), \quad (3.7b)
\end{align}

where $B_n$ are constants (see the electronic supplementary material, §A for their expressions). Importantly, the constants are functions of the magnetic bond numbers:

\begin{align}
N_{Bx} &= \frac{\mu_0 \chi}{2} \frac{I^2}{4\pi^2 R_0^2} \frac{\ell}{\sigma} \quad \text{and} \quad N_{By} = \frac{\mu_0 \chi}{2} \frac{H_0^2 \ell}{\sigma}, \quad (3.8)
\end{align}

which quantify the ratios of the magnitudes of the $x$ and $y$ components of the magnetic body force to the surface tension force.

Before proceeding further in the analysis, we rewrite the boundary conditions from equations (3.7) to hold at $y = 1$ through Taylor series expansions of $u$, $v$ and $\phi$:

\begin{align}
v + v_y \epsilon \eta &= \eta_l + \epsilon (u + u_y \epsilon \eta) \eta_x \quad \text{on} \quad y = 1 \quad (3.9a) \\
\phi + \phi_y \epsilon \eta &= B_1 \eta - \delta \eta_{xx} + \delta B_2 \eta_x + \epsilon \delta^2 B_3 \eta_x^2 - B_4 \epsilon \eta^2 \quad \text{on} \quad y = 1. \quad (3.9b)
\end{align}

With the relations in equation (3.5), equations (3.9) can be rewritten, within the assumed order, as

\begin{align}
v &= \eta_l - \epsilon \{[B_1 \eta_x + \delta (B_2 \eta_{xx} - \eta_{xxx})] \eta_x \} \quad \text{on} \quad y = 1 \quad (3.10a) \\
\phi &= B_1 \eta + \delta (B_2 \eta_x - \eta_{xx}) - B_4 \epsilon \eta^2 + \epsilon \delta^2 (B_3 \eta_x^2 + \eta_x) \quad \text{on} \quad y = 1. \quad (3.10b)
\end{align}

Combining equations (3.6), evaluated at $y = 1$, and equations (3.10) allows us to eliminate $g(x,t)$, and the dynamics of the interface $\eta(x,t)$ is governed by

\begin{align}
\eta_l &= (-\epsilon \alpha - \epsilon^2 \vartheta) \eta_{xx} + \delta (\beta \eta_{xxx} - \eta_{xxxx}) + \epsilon \{(-\epsilon \alpha - \epsilon^2 \vartheta) \eta_x + \delta (\beta \eta_{xx} - \eta_{xxx}) \} \eta_x \\
&\quad - \frac{1}{2} \epsilon \delta^2 \eta_{xx}^2 + \frac{1}{2} \delta^2 \left[ \eta_{xtt} - \frac{1}{3} (-\epsilon \alpha - \epsilon^2 \vartheta) \eta_{xxxx} \right] \\
&\quad + \epsilon \delta^2 \left[ (\gamma \eta_x^2 + \eta_{xx}) \eta_{xx} - \frac{1}{4} (-\epsilon \alpha - \epsilon^2 \vartheta) (\eta^2)_{xxxx} + \frac{1}{12} \epsilon \delta^2 \eta_x^2 \right], \quad (3.11)
\end{align}

where

\begin{align}
\alpha &= 2 [N_{By}(1 + \chi) - N_{Bx}], \quad (3.12a) \\
\beta &= 2 \chi \sqrt{N_{Bx}N_{By}}, \quad (3.12b) \\
\gamma &= \chi (N_{By} - N_{Bx}), \quad (3.12c) \\
\vartheta &= 2 [(1 + \chi) N_{By} + 3 N_{Bx}], \quad (3.12d)
\end{align}

are now the governing dimensionless parameters of the model, beyond the previously defined small quantities in equation (3.4). Note that equation (3.11) is a general expression of the interface dynamics without any assumption about the relation between the (three) small parameters $\epsilon$, $\delta$ and $\epsilon$. To obtain a model equation, in the sense of [21], we must consider the relevant distinguished limit.
(c) The model long-wave equation

Next, we seek to simplify the governing equation (3.11) in the distinguished asymptotic limit(s) of interest. For \( \varepsilon = O(\delta^2) \), without loss of generality, we let \( \varepsilon = \delta^2 \) and conduct another rescaling

\[
\eta \mapsto \frac{\eta}{\varepsilon} \quad \text{and} \quad t \mapsto \frac{t}{\delta},
\]

(3.13)

to describe the long-time evolution (as expected, since we focus on travelling wave solutions). From equation (3.11), the interface evolution equation for \( \varepsilon = O(\delta^2) \) can be written as

\[
\eta_t = -\delta \alpha \eta_{xx} + \beta \eta_{xxx} - \eta_{xxxx} + [(\delta \alpha \eta_x + \beta \eta_{xx} - \eta_{xxx}) \eta]_x + \delta(\gamma \eta_x^2)_{xx}.
\]

(3.14)

The long-wave equation for \( \varepsilon = O(\delta) \) has a similar structure as equation (3.14) (see the electronic supplementary material, §B for details), so that in this study we will focus on equation (3.14), which is a modified generalized KS equation. The main difference lies in the dispersion and nonlinear terms. Whereas the KS equation features the Hopf nonlinearity \( \eta \eta_x \) and \( \eta \eta_{xx} \), equation (3.14) does not. Instead, the last two terms on the right-hand side of equation (3.14) depict a more complicated nonlinearity introduced almost entirely by the magnetic forces. One of the latter terms, \( \alpha \eta_x \eta_x \), is similar to the term due to the Maragoni effect in the so-called KdV–Kuramoto–Sivashinsky–Velarde equation [26, equation (6)]. We note in passing that this term, together with the term \( \alpha \eta_x \eta_x \), also appear in the nonlinear terms of the model equation for interfacial periodic waves in [7, equation (9)]. As in the present study, the \( \eta_{xxx} \) nonlinearity arises from surface tension. However, while \( \eta_{xx} \eta_x \) in [7, equation (9)] comes about from inertia, in our model equation this term arises from magnetic forces. Meanwhile, the role of the linear terms is well known, as in KS: \( \eta_x \) is responsible for the instability at large scales, while \( \eta_{xxx} \) provides dissipation at small scales. As in the generalized KS equation, the KdV-like term \( \eta_{xxx} \) in equation (3.14) leads to dispersion.

4. Stability of the flat state and nonlinear energy budget

(a) Linear growth rate and weakly nonlinear mode coupling

Let \( \eta(x, t) = \sum_{k=0}^{\infty} \eta_k(t)e^{ikx} \) be the Fourier decomposition of the surface elevation on the periodic domain \( x \in [0, 2\pi] \). Then, substituting the Fourier series into equation (3.14), we immediately obtain

\[
\dot{\eta}_k = \Lambda(k) \eta_k + \sum_{k'} F(k, k') \eta_{k'} \eta_{k-k'},
\]

(4.1)

where the overdot denotes a time derivative, \( k \neq 0, k' \neq 0, \eta_{k=0} = 0, i = \sqrt{-1} \), and

\[
\Lambda(k) = \delta \alpha k^2 - k^4 - i \beta k^3
\]

(4.2a)

and

\[
F(k, k') = \delta \alpha kk' - i \beta kk'^2 - kk'^3 + 2\delta \gamma (k^2 k'^2 - kk'^3).
\]

(4.2b)

Recalling the definition of \( \alpha \) from equation (3.12a), the real part of the linear growth rate \( \Re[\Lambda(k)] \) indicates that the \( y \)-component of the magnetic field \( \propto (1 + \chi)N_{Bz} \) is destabilizing, while the \( x \)-component \( \propto N_{Bz} \) and surface tension are stabilizing. Weakly nonlinear mode coupling at the second-order is accounted for by the function \( F \). Note that the terms in \( \Re[\Lambda(k)] \) from equation (4.2a) above are quite similar to the ones in [19] (for a radial geometry), apart from being multiplied by an additional power of \( k \).

The most unstable mode \( k_m \) satisfies

\[
\left. \frac{\text{d} \Re[\Lambda(k)]}{\text{d}k} \right|_{k=k_m} = 0 \quad \iff \quad 2k_m^2 = \delta \alpha,
\]

(4.3)

which implies the important role of \( \delta \alpha \) on stability. Figure 2a shows examples of how \( \delta \alpha \) controls the most unstable mode and determines the range of linearly unstable modes (for which
Figure 2. (a) Real part of the linear growth rate $\Re[\Lambda(k)]$ as a function of the wavenumber $k$ for $\delta\alpha = 8, 32, 72$ and $128$; the markers denote the most unstable mode $k_m$. (b) The nonlinear evolution of the interface from a small perturbation of the flat base state [$\eta(x, 0) = 0.01 \sin(4x)$] into a permanent travelling wave with $\delta\alpha = 32$, $\beta = 16$ and $\gamma$ determined by equation (3.12) accordingly. (c) Energy budget of the nonlinear travelling generation process shown in (b); the red curve represents the $\delta\alpha$ term, the green curve represents the surface tension term and the blue curve represents the $\beta$ term from the PDE (3.14). The contribution of the linear term is denoted by the solid curves while the dashed curves represent the nonlinear term(s). The black curve in (c) shows the sum of these components, which is seen to approach zero as the wave evolves into a dissipative soliton. (Online version in colour.)

$\Re[\Lambda(k)] > 0$. We will show that $k_m$ (and $\delta\alpha$) can be used to predict the possible states (period of the nonlinear interfacial wave), and it is helpful for selecting suitable initial conditions that evolve into (nonlinear) travelling wave solutions.

The imaginary part of the linear growth $\Im[\Lambda(k)]$ rate reveals the phase velocity of each mode

$$v_p(k) = -\Im[\Lambda(k)]/k = \beta k^2.$$  

Perturbations to the flat base state of the film can propagate with velocity controlled by the coupling term $\beta = x \sqrt{N_{Bx} N_{By}}$ (and, since $v_p = v_p(k)$, they also experience dispersion). Here, $\beta$ results from the magnetic normal stress due to the asymmetric projection of the $x$- and $y$-components of the magnetic force onto the interface. Changing the direction of the $x$-component of $H$ will reverse the sign of these terms, i.e. $\beta \to -\beta$. The linear analysis indicates that such wavepackets will either decay or blow up exponentially according to the sign of $\Re[\Lambda(k)]$. However, below we will show, through simulations of the governing PDE, that this linear instability is arrested by nonlinearity.

(b) Nonlinear energy balance and the dissipative soliton concept

The energy method [38] can be applied to any PDE to understand the stability of its solutions. For example, the energy method was used to establish the stability and uniqueness of generic ferrofluid flows [39]. Here, we employ this approach to understand the stability of the travelling wave in our model long-wave equation, which features both damping and gain. Multiplying equation (3.14) by $\eta$, and integrating by parts over $x \in [0, 2\pi]$, yields an energy balance

$$\dot{\mathcal{E}} = \int_0^{2\pi} \left[ \delta\alpha \eta_x^2 - \eta_{xx}^2 + \delta\alpha \eta_x^2 \eta + \frac{1}{2} \beta \eta_x^2 - \eta_{xx}^2 \right] dx,$$  

where $\mathcal{E}(t) = \frac{1}{2} \int_0^{2\pi} \eta(x, t)^2 dx$ denotes the total energy of the wave field. The $\delta\alpha \eta_x^2$ term on the right-hand side of equation (4.5) produces energy, while the surface tension term $-\eta_{xx}^2$ acts as a sink. This result matches well with the observation regarding the linear growth rate, i.e. that the destabilizing $\delta\alpha$ term is balanced by the (stabilizing) surface tension ($\delta\alpha k^2 > 0$ and $-k^4 < 0$ in equation (4.2a)). The linear dispersion term conserves energy and thus drops out of equation (4.5). Meanwhile the sign of the three remaining terms is indeterminate a priori. Figure 2c shows
the evolution of the various terms on the right-hand side of equation (4.5) for the solution $\eta(x, t)$ shown in figure 2b.

Eventually, all curves in figure 2c become independent of time. In general, we expect that, for some distinguished solutions $\eta(x, t)$, $\dot{\mathcal{E}} = 0$ holds exactly. If this is the case for one of the travelling wave solutions, then they are classified as dissipative solitons in the sense of [26]. Dissipative solitons are expected to be long-lived stable structures. We wish to address if such structures arise in our model of a ferrofluid interface subjected to a magnetic field.

From the energy analysis in equation (4.5), we can conclude that $\alpha$, $\beta$ and $\gamma$ are the three key parameters controlling the wave propagation and existence of the dissipative soliton. Recall that these three parameters, which show up in equation (3.14), are given as combinations of the physical parameters (i.e. $\chi, N_{Bx}, N_{By}$), as per equation (3.12). In particular, $\alpha$ and $\beta$ in the linear terms of equation (3.14) are expected to strongly affect the stability and the characteristics of the travelling wave profile. We explore this issue next through numerical simulations.

(c) Numerical simulation strategy for the governing long-wave PDE

To understand the nonlinear interfacial wave dynamics, in the upcoming sections below, we solve equation (3.14) numerically using the pseudospectral method [40]. For the linear terms, the spatial derivatives are evaluated using the fast Fourier transform (FFT) with $N = 512$, while the nonlinear terms are inverted back to the physical domain (via the inverse FFT), evaluated, and then transformed back to Fourier space. The modified exponential time-differencing fourth-order Runge–Kutta (ETDRK4) scheme [41], which is stable and accurate for stiff systems [40], is adopted for the time advancement. Grid and time-step convergence of the numerical scheme was established (see the electronic supplementary material, §C). Figure 2b shows an example evolution from the infinitesimal perturbation of the flat state, to the formation of a nonlinear travelling wave.

5. Nonlinear periodic interfacial waves: propagation velocity and shape

As discussed in §4b, $\alpha$ and $\beta$ play an important role in the energy balance. In this section, we will investigate their effects on the travelling wave’s propagation velocity and the wave profile (shape). Before we start, it is helpful to discuss the physical meaning of these parameters, which can be useful in designing control strategies in practice.

First, as explained above, $\beta$ is the coupling term resulting from the asymmetry of the surface force on the perturbed interface. This parameter is also closely related to the orientation of the magnetic field. To understand this point better, let

$$\rho = N_{Bx} + N_{By} \quad \text{and} \quad q = \frac{N_{Bx}}{\rho}. \quad (5.1)$$

Here, $\rho \propto |\mathbf{H}|^2$ relates to the magnitude of the magnetic field at $R_0$, and $q = \cos^2 \varphi$, where $\varphi$ is the angle of $\mathbf{H}$ with respect to the flat interface (recall figure 1). With $\chi = 1$, the main parameters can be rewritten as

$$\alpha = 2\rho(2 - 3q), \quad \beta = 2\rho\sqrt{q(1 - q)} \quad \text{and} \quad \gamma = \rho(1 - 2q). \quad (5.2)$$

In this study, we restrict ourselves to magnetic fields with small $x$-component magnitude, with $q \in [0, 0.06]$, i.e. $\varphi \in [0.42\pi, \pi/2]$. For this choice, $\alpha \approx 4\rho$, $\beta \approx 2\rho\sqrt{q}$ and $\gamma \approx \rho$. Hence, controlling $\alpha$ is equivalent to controlling the magnitude of the magnetic field, while $\beta$ is sensitive to the orientation. Note that two independent variables will set the dynamics, and in this section, we will control $\alpha$ and $\beta$, with $\gamma$ determined by equation (5.2). Furthermore, in the numerical studies below, we will use one initial condition, $\eta(x, t = 0) = 0.01 \sin(k_0 x)$, with an initial perturbation wavenumber $k_0 = 4$, and we will only consider $\delta = 0.1$. These perturbations will first grow, then become arrested by saturating nonlinearity [42], and finally lead to a permanent travelling wave. The latter is of interest in this section.
(a) Propagation velocity

(i) Linear prediction and nonlinear expression

Lira & Miranda [16] reported that the propagation velocity of interfacial ferrofluid waves in a Cartesian configuration in a vertical Hele-Shaw cell is sensitive to the magnetic field’s angle. The fully nonlinear simulations of a radial configuration in a horizontal Hele-Shaw cell in [19] further showed that this velocity can be well predicted by the linear phase velocity, which is determined by the coupling term of azimuthal and radial magnetic field components in that work. In this study, we examine how this coupling term, which is captured by our parameter $\beta$ (and closely related to the angle $\varphi$ of the magnetic field), controls the nonlinear wave propagation velocity.

A permanent travelling wave profile takes the form $\eta(x,t) = \Theta(kx - \omega t)$, where $\nu = \omega/k$ is its propagation (phase) velocity. The modes’ complex amplitudes can be expressed as $\eta_k(t) = c_k e^{-i\omega(t)}$, with constant $c_k \in \mathbb{C}$ accounting for their relative phases. A nonlinear travelling wave profile would consist of a fundamental mode $k_0$ and its harmonics $nk_0$ ($n \in \mathbb{Z}^+$), with $\omega(nk_0) = n\omega(k_0)$, so that the phase velocity can be evaluated as $v_p(nk_0, t) = n\omega(k_0)/nk_0 = \nu$. The average $v_p$ of the first five harmonics is used to calculate $v_p^N$ for the nonlinear simulation. Meanwhile, the linear phase velocity $v_l^N = \nu_p$ is given by equation (4.4).

Figure 3a shows the comparison of the nonlinear propagation velocity and the linear prediction for $\delta \alpha = 32$. The fundamental mode, computed as $k_f = 4$ from the simulation, sets the linear propagation velocity as $v_l^N = 4 \beta k^2 = 16 \beta$. It is surprising to see that the actual nonlinear propagation velocity can be well fit by the straight line $v_l^N = 14.05 \beta$ with small variance $\sigma^2 = 0.005$, even if the wave profile changes with $\beta$ dramatically, as shown in figure 3c. This curious correction is not as trivial as it looks, as the nonlinear phase velocity can be evaluated a posteriori through equations (4.2) as

$$v_l^N = \beta \left( k^2 + \sum_{k'} k^2 \Re \left[ \frac{\eta_{k'} \eta_k - k^2}{\eta_k} \right] - \frac{k}{\beta} \sum_{k'} [\delta \alpha k' - k^3 + 2\delta \gamma (kk^2 - k^3)] \Im \left[ \frac{\eta_{k'} \eta_k - k^2}{\eta_k} \right] \right),$$

(5.3)

where $v_l^N = \Im[N(k,k')/k]$ is derived from the propagator operator $N(k,k') = \lambda(k) + \sum_{k'} F(k,k') \eta_k \eta_{k'-k'/\eta_k}$ derived in equation (4.1), such that $\dot{\eta}_k = \lambda \eta(k',k') \eta_k$. The terms arising from the summation over $k'$ represent the nonlinear effects. When a travelling wave solution is obtained, $\eta_{k',k'-k}/\eta_k = c_k c_{k-k}/c_k$ becomes independent of time, and the nonlinear phase velocity can be evaluated from equation (5.3), knowing $c_k$ from the propagation profile’s Fourier decomposition. (This is equivalent to our approach in the electronic supplementary material, §D. That approach is simpler, therefore the results hereafter follow the approach from the electronic supplementary material, §D, for simplicity and clarity.)

The correction in equation (5.3) is an a posteriori result, and it is accurate but not obvious how it changes the pre-factor $k^2 = 16$ into 14.05. Nevertheless, the strong, linear correlation between $v_l^N$ and $\beta$ for the chosen parameters of interest is the key point.

(ii) Multiple-scale analysis and velocity correction

To better understand the linear correlation between $v_l^N$ and $\beta$, an analytical approximation can be obtained via a multiple-scale analysis of the harmonic wave [43]. However, when subject to the current parameters (i.e. $k_m = 4$ as the most unstable mode), the linear instability poses difficulties when using a standard travelling wave ansatz. We introduce the critical wave number $k_c$ so that $\Re[\mathcal{A}(k_c)] = 0 \Rightarrow k_c^2 = \delta \alpha$. The linear theory predicts that all $k < k_c$ are unstable. Thus, we assume that the $\delta \alpha$ in the linear term is slightly larger than $k_f^2$, thereby making $k_f = 4$ marginally unstable, and also the unique unstable mode. In other words

$$\delta \alpha = k_f^2 + \epsilon^2 \kappa,$$

(5.4)
where $\epsilon \ll 1$ is a small perturbation parameter and $\kappa > 0$ is independent of $\epsilon$. We first scale equation (3.14) to a weakly nonlinear problem by introducing $\eta = \epsilon Y$

$$Y_t + (k_f^2 + \epsilon^2 \kappa)Y_{xx} - \beta Y_{xxx} + \gamma_{xxxx} = \epsilon[(\delta \alpha Y_1 + \beta Y_{xx} - Y_{xxx})Y_t + (\delta \gamma Y_x^2)_{xx}]. \tag{5.5}$$

Next, we introduce the travelling wave coordinate $\xi = kx - \omega_p t$ of a harmonic wave, where $\omega_p = k^2 \beta$ by the linear dispersion relation. We assume that $Y$ has a multiple-scale expansion of the form

$$Y = Y_0(\xi, t_2) + \epsilon Y_1(\xi, t_2) + \epsilon^2 Y_2(\xi, t_2) + O(\epsilon^3), \tag{5.6}$$

where the slow time is $t_2 = \epsilon^3 t$. By eliminating the secular term at $O(\epsilon^2)$ (see the electronic supplementary material, §E, for details), we obtain the leading-order solution

$$Y_0 = 2a \cos \left( kx - \omega_p t - k^2 \frac{\Re[Q]}{\Im[Q]} \epsilon^2 \kappa t + b_0 \right), \tag{5.7}$$

which gives the phase velocity with the multiple-scales correction as

$$v_f^{MS} = \beta k^2 + \frac{\Re[Q]}{\Im[Q]} \kappa \epsilon^2 \kappa, \tag{5.8}$$

where $a = \sqrt{\alpha k^2 / \Im[Q]}$ is the equilibrium amplitude, $b_0$ is an integration constant, and

$$Q = \frac{[\delta \alpha - i\beta k + (2\delta \gamma - 1)k^2]}{6k^2 + 3i\beta k} [-\delta \alpha k^2 + 5i\beta k^3 + (7 + 4i\gamma)k^4]. \tag{5.9}$$

Equation (5.8) predicts the propagation velocity of the travelling wave solution when $k_f = 4$ is subjected to weak linear instability. The weak linear instability is important to emphasize in this multiple-scales derivation because we assumed $\epsilon^2 \kappa \ll 1$. However, the results of this analysis appear to hold even for stronger linear instability. As $\delta \alpha$ in the original formulation increases from $k_f^2 = 16$, $\epsilon^2 \kappa$ increases correspondingly. In figure 3a, we show two cases with $\delta \alpha = 18$ ($\epsilon^2 \kappa = 2$) and $\delta \alpha = 32$ ($\epsilon^2 \kappa = 16$). For the weakly linearly unstable case ($\delta \alpha = 18$), the propagation velocity $v_f^{MS}$ predicted by the multi-scale expansion matches well with the nonlinear velocity $v_f^N$, with error less than 0.5%. For stronger linear instability, i.e. $\delta \alpha = 32$, $v_f^{MS}$ is still qualitatively corrected, but the error is now less than 26% for smaller $\beta$, while the agreement improves for larger $\beta$, with the error reducing to about 5%.
Another message obtained from figure 3 is that, even if the nonlinear propagation velocity \( v_n \) shows a linear correlation with \( \beta \), it is not necessarily linearly related to \( \beta \) due to the nonlinearities of the PDE. However, this observation will not change the fact that such linear correlation enables both \( v_n \) and \( v_{NS} \) to be good predictors for the wave dynamics (and their possible control via the imposed magnetic field). In this respect, another reason that \( v_n \) is a good quantitative prediction is the lack of ‘inertia’ in this system. In the classical model equations, such as the KS, KdV and Burgers, the \( \eta \) term accounts for nonlinear advection, and thus the initial ‘mass’ \( \int_0^{2\pi} \eta \, dx = 0 \) due to the definition of \( \eta \) as a periodic perturbation. Thus, the propagation velocity is well predicted directly by the dispersion parameter \( \beta \).

(b) Travelling wave profile

In addition to setting the propagation velocity, the coupling term \( \beta \) (as the source of asymmetry of the magnetic traction force) also strongly affects the shape of the travelling wave. To explore the shape change, as in [19], we introduce the skewness \( Sk \) and asymmetry \( As \):

\[
Sk(t) = \frac{\langle \eta(x,t)^3 \rangle}{\langle \eta(x,t)^2 \rangle^{3/2}} \quad \text{and} \quad As(t) = \frac{\langle H[\eta(x,t)]^3 \rangle}{\langle \eta(x,t)^2 \rangle^{3/2}},
\]

where \( \langle \cdot \rangle = (1/2\pi) \int_0^{2\pi} \cdot \, dx \), and \( H[\cdot] \) is the Hilbert transform. \( Sk(t) \) quantifies the vertical asymmetry of nonlinear surface water waves [44,45], about the unperturbed interface, with \( Sk > 0 \) corresponding to narrow crests and flat troughs (and vice versa for \( Sk < 0 \)). Meanwhile, \( As(t) \) quantifies the fore-aft asymmetry of a wave profile [44,45], with \( As > 0 \) corresponding to waves that ‘tilt forward’ (in the direction of propagation).

Figure 3b shows the effect of \( \beta \) on the wave profile for \( km = 4 \) (i.e. for \( \delta \alpha = 32 \)). The asymmetry is a concave function of \( \beta \) with a maximum around \( \beta \approx 20 \). This implies the existence of an ‘optimum’ magnetic field angle that allows tuning of the wave profile shape. For the parameters used in figure 3c, i.e. \( \delta \alpha = 32 \) and \( \beta = 0, 10, 20, 40 \), correspondingly we have \( q = 0, 3.9 \times 10^{-3}, 0.015, 0.056 \), spanning two orders of magnitude of the magnetic field angle parameter. For \( \beta = 0 \) (\( q = 0 \)), the profile is symmetric, and we observe that even a small angle of the magnetic field breaks the fore-aft asymmetry of the wave profile. The skewness, on the other hand, monotonically decreases with the angle, becoming negative beyond \( \beta \approx 30 \).

Figure 4 shows how \( km \) (or, equivalently, \( \alpha \) since \( \delta \) is fixed) affects the wave profile. When the initial perturbation wavenumber \( k_0 = 4 \) is close to the most unstable mode \( km \), the travelling wave profile maintains the same period as the initial condition. For larger \( km \), the wave profile exhibits a sharper peak. This sharpening was also observed in [19, fig. 3b,c], wherein the skewness increases with \( km \), and the profiles saturate for large values of \( km \). In [19], the possibly unstable evolution for \( km \) was not discussed, while the wave studied therein shows ‘wave breaking’ for large values of the dispersion parameter. Therefore, an open problem that can be addressed with the present long-wave model is the ‘asymptotic’ behaviour of the steepening wave profile (As and Sk) with \( km \). This leads us to a new question: Is the range of \( km \) that allows such period-four waves bounded, or will the shape eventually become unstable (and/or ‘break’)? Or, we can reframe the question as: Given \( km \), which states (period of the travelling wave) exist in this system? What about their stability? In the next section, we perform numerical investigations to shed light on these questions.

6. State transition and stability of travelling waves

The KS equation is well known for the chaotic behaviour of its solutions. It has been thoroughly investigated within the scope of instability and bifurcation theory [27,46–48], yielding a wealth of results on how different dynamical states can be ‘reached’ from given initial data, and the
transition between such states. Specifically, as the ratio of coefficients of the second- and fourth-order derivative terms (i.e. their relative importance, quantified by $\delta \alpha$ in our model (3.14)) increases, the KS equation’s steady profile exhibits more complexity and dynamical possibilities, and finally the dynamics becomes chaotic. This feature can be understood intuitively from figure 2a, wherein higher $\delta \alpha$ allows a wider unstable band for the long waves in the system.

Considering some of the similarities between our long-wave equation (3.14) and the generalized KS equation, a thorough examination of all the parametric dependencies of the wave (including chaotic) dynamics is not of interest herein. Instead, we focus on showing that the dissipative solitons emerging from perturbations in the linearly unstable band are fixed points in an energy phase plane. Then, we analyse the state transitions via this phase plane, and explain the stability of fixed points via the spectral stability of the wave profiles. It is noteworthy that this state transition process is a feature of systems having multi-mode wave solutions. The multi-mode transition process is a generalization of the dynamics studied in the last section, which focused on single-mode evolution. Finally, we highlight multi-periodic profiles analogous to ‘double cnoidal waves’ of the KdV equation.

(a) Fixed points in the energy phase plane

To reduce the parameter space exploration, in this section we fix $q = 0.01$, with $q$ defined in equation (5.1), and focus on the dynamics for different $k_m$ only, by controlling the magnitude $\rho = 5, 20, 45, 80$. Recall that (as discussed at the beginning of §5) there are two independent physical dimensionless groups (i.e. $N_{Bx}$ and $N_{By}$), so that fixing $q$ and $\rho$ determines all other parameters (i.e. $\alpha$, which sets $k_m$, $\beta$ and $\gamma$). In this subsection, the energy phase plane ($E, \dot{E}$) (e.g. [27]) will be used to identify the travelling wave solutions, which emerge as fixed points with finite $E$ and $\dot{E} = 0$ (i.e. they are dissipative solitons).

The wavenumber range, $k \in (0, k_c]$, of linearly unstable modes can be obtained by solving $\Re[\Lambda(k_c)] = 0$ to obtain $k_c = \sqrt{2} k_m$. Figure 5 shows four slices of the energy phase plane at $k_m = 1, 2, 3, 4$. The corresponding maximal linearly unstable modes have wavenumbers $|k_c| = 1, 2, 4, 5$. As before, the initial condition is selected as a small-amplitude single-mode perturbation: $\eta(x, t = 0) = 0.01 \sin(k_0 x)$ with $k_0 = 1, 2, 3, 4, 5$ as the initial wavenumbers. From the nonlinear growth rate in equation (4.2b), we know that a nonlinear interaction exists only between harmonic modes $n k_0$ ($n \in \mathbb{Z}$) when initializing with the single mode $k_0$. These interacting modes will grow or decay and finally become balanced harmonic components of the permanent travelling wave profile that emerges.

The fundamental mode $k_f$ contains the highest energy, $\eta_{k_f} \eta^*_{k_f}$, in the system. Physically, the wave will exhibit a period-$k_f$ profile, or a ‘$k_f$-state.’ For a period-$k_f$ travelling profile, only the harmonic modes $n k_f$ exist in the system. Therefore, the fixed points identified in figure 5 are of different periods for a given $k_m$. In figure 5, $k_0 \neq k_f$ when $k_0 = 1, k_m = 3, 4$.

For $k_m = 1$, only one initial mode, $k_0 = 1$, is linearly unstable, so that an initial perturbation with $k_0 > 1$ will decay exponentially back to base state (flat interface). On the other hand, the
linearly unstable mode $k_0 = 1$ will first grow, then saturate to a travelling wave profile, and thus one fixed point can be identified in the $(\mathcal{E}, \dot{\mathcal{E}})$ phase plane. Similarly, picking $k_m = 2$ allows two linearly unstable modes, thus two fixed points in the energy phase plane. One fixed point is a period-one state, and the other is a period-two state.

However, while four unstable modes exists for $k_m = 3$, only three fixed points are identified with periods two, three and four. When initialized with $k_0 = 1$, the period-one perturbation evolves and converges to a period-two travelling wave, as can be seen from figure 5. Note that $k_0 = 1$ is a special case in terms of the nonlinear interaction. For $k_0 = 1$, all normal modes in the system are harmonic components, so that $k = 2, 3, 4$ will gain energy from $k_0 = 1$ also. A similar phenomenon can be observed in the $k_m = 4$ slice of the energy phase plane. The initial perturbations with modes $k_0 = 2, 3, 4, 5$ will evolve into states with corresponding $k_f = k_0$, while $k_0 = 1$ evolves into the period-three state.

Note that figure 5 shows only four slices at integer $k_m$, but $k_m$ does not necessarily have to be an integer (because it is set by the non-integer system parameter $\alpha$ via equation (4.3)). Thus, our discussion only provides a representative view of the rich higher-dimensional dynamics. It is evident, from the four slices in figure 5, that bifurcations of fixed points occur as the parameter $k_m$ is varied. The number of fixed points increases with $k_m$, or more accurately, with the number of linearly unstable modes. Some fixed points move along the $\mathcal{E}$ axis with increasing $k_m$, such as the period-two and period-three states, while some disappear, like the period-one state. This observation partially answers the question of whether the number of travelling wave states will increase with $k_m$, and whether for certain states there is a possibly bounded ranged of $k_m$ allowing them. However, what exactly is this bound for each state, or each $k_m$, is beyond the scope of this study. This question would be challenging, since as $k_m$ increases, more and more linearly unstable modes participate in the competition for setting the fundamental mode.

(b) Spectral stability of the travelling wave

The tendency of a system to prefer a narrow set of states out of many possible ones is known as wavenumber selection [49,50]. In this section, we study this phenomenon by addressing the stability of these fixed points in the energy phase plane, focusing on the case of $k_m = 4$.

To this end, we perturb the travelling wave profile, and numerically track the evolution of the perturbation via direct simulation of the PDE. We find that period-two and period-three states
behave like local attractors, while period-four and period-five states are saddle points. We verify
the type of the fixed points through spectral (in)stability analysis [51,52]. Specifically, we rewrite
equation (3.14) in the moving frame with \( \zeta = x - \nu ft, \tau = t \) as:

\[
\eta_t - \nu f \eta_\zeta = -\delta \alpha \eta_{\zeta \zeta} + \beta \eta_{\zeta \zeta \zeta} - \eta_{\zeta \zeta \zeta \zeta} + \left[ (-\delta \alpha \eta_\zeta + \beta \eta_{\zeta \zeta} - \eta_{\zeta \zeta \zeta}) \eta_\zeta \right] + \delta (\gamma \eta^2)_{\zeta \zeta}, \quad (6.1)
\]

with the propagation velocity \( \nu f \) calculated numerically. The perturbed travelling wave solution
is written as \( \eta(\zeta, \tau) = \Xi(\zeta) + d W(\zeta) e^{\lambda \tau} \), where \( \Xi(\zeta) \) is the stationary solution of equation (6.1)
(hence, the travelling wave solution of equation (3.14)), and \( d \ll 1 \) is an arbitrary perturbation
parameter. Substituting the perturbed \( \eta(\zeta, \tau) \) into equation (6.1), and neglecting nonlinear terms,
we obtain a linear eigenvalue problem

\[
\lambda W = \mathcal{L}W, \quad \mathcal{L} := \sum_{n=0}^{4} C_n D_n. \quad (6.2)
\]

Here, the \( C_n = C_n(D_0 \Xi, \ldots, D_4 \Xi, \nu f) \) are vector-valued functions (see the electronic supplementary
material §F for their expressions) of the travelling wave profile \( \Xi(\zeta) \) and its gradients, and the
differentiation matrices \( D_n \) are discretizations of \( \partial^n / \partial \zeta^n \) (\( D_0 = I \) is the \( N \times N \) identity matrix)
evaluated by the Fourier spectral approach [40]. The eigenvalue problem in equation (6.2) is
solved numerically with \texttt{linalg.eig} from the NumPy stack in Python [53]. The spectrum was
validated via a grid-independence study using grids with \( N = 256, 512 \) and 1024 points.

Next, we use this numerical spectral stability approach to understand the state transitions and
the stability of fixed points in the energy phase plane introduced in §6a.

(c) The state transition process

Figure 6a,b shows that the period-three and period-two fixed points, respectively, in the \((E, \dot{E})\)
phase plane are attractors. Small perturbations about them will decay, and the evolution will
converge back to the corresponding periodic travelling wave profiles. This observation can be
confirmed by the spectral stability calculation, its results shown in figure 6f, which shows that
all eigenvalues have a negative real part, except for the zero eigenvalue, which represents the
translational invariance of the travelling wave solution.

On the other hand, figure 6c,d show that the period-five and period-four fixed points,
respectively, are saddles. A small perturbation around the period-four fixed point will grow
an \( +d \) oscillate away, till the evolution converges to the period-three fixed point (an attractor),
the black curve in figure 6d. For a different perturbation, the grey curve in figure 6d, this
process can lead to convergence to the period-two attractor (see output_f4_p2.mp4 in
the electronic supplementary material for a video of this process). The perturbation evolution around
the period-five fixed point, the black curve in figure 6c, is more interesting. It is featured
by a two-stage transition process. First, the perturbation will first oscillate and grow rapidly,
attached to the neighbourhood of the period-four fixed point. Then, it will oscillate away again,
until finally converging to the period-three attractor (see output_f5_p4.mp4 in the electronic
supplementary material for a video of this process). These saddle point behaviours can be
confirmed from the linear eigenspectra shown in figure 6f as well. The period-four profile has
two pairs of conjugate eigenvalues with a positive real part, while the period-five profile has four
pairs.

A closer examination of the state transition process is shown in figure 7 for three representative
perturbations around the period-five fixed point. Rapid oscillation of the modes’ energies can
be observed during the transition process, indicating intense nonlinear interactions. The space–
time plot shows a similar phase shift feature as seen during the collision of solitons [54], but the
wave profile is completely modified here. Figure 7b shows a one-stage transition due to a single-
mode perturbation \( \partial W(\zeta) = 0.02 \sin(k_p \zeta), k_p = 3 \). This mode’s energy \(|\eta_3|\) increases exponentially,
overlaps the initial \(|\eta_5|\) value and converges to the period-three attractor. Figure 7a,c show a two-
stage transition with single-mode perturbations \( k_p = 1, 4 \), respectively. Figure 7a shows a higher
Figure 6. Stability diagram based on the energy phase plane. Perturbations around the attractors, corresponding to (a) the period-three and (b) the period-two travelling wave solutions, converge. State transitions are observed near the saddle points corresponding to (c) the period-five and (d) the period-four travelling wave profiles. The solid curves’ colours represent initial perturbations with different wavenumbers, which lead to different dynamics (and outcomes). The wave profiles are shown in (e), with the symbols in the corners of the plots denoting the corresponding fixed points in the phase planes in (a–d). In (f), the leading eigenvalues of the linearization about the corresponding wave profile in (e) are shown. (Online version in colour.)

While such transition paths are complex and intriguing, we would like to emphasize that the existence of the transition depends on the spectral stability of the travelling wave profile itself, which is interpreted as a saddle point or an attractor in the energy phase plane, and the transition direction is determined by the perturbation $W(\xi)$. After an immediate targeted transition, whether another transition happens or not depends on the spectral stability of the subsequent wave profile attained.

Another intriguing aspect of this topic is multi-mode perturbations to the unperturbed flat interface, which is a more realistic situation that might arise in experiments, where the mode of the ambient noise is hard to control in an experiment. The competition between all possible states will finally select the observable pattern. Next, we analyse this multi-mode case and provide an explanation of the selection process leading to multi-periodic nonlinear travelling waves.

(d) Multi-periodic waves

An interesting observation from figure 7a is the coexistence of mode 1 and mode 4 during the transition, exemplified by the oscillations about the period-four fixed point in the energy phase plane shown in figure 6c. The energy components of the wave profile are harmonics of
Figure 7. Fourier mode energy evolution (mode competition and nonlinear interaction) for a perturbed period-five travelling wave subjected to harmonic perturbation with (a) $k_p = 1$, (b) $k_p = 3$ and (c) $k_p = 4$. The top row shows the corresponding space–time plot of the transition process with colour representing the amplitude of the wave profile $\eta$. (Online version in colour.)

$k_f = 4$, except the non-trivial $|\eta_1| \approx |\eta_8|$. During the time interval $t \in [0.05, 0.2]$, the space–time plot of wave profile evolution shows that a period-four wave is modulated by mode 1. This coexistence lasts for a relatively long time (compared with the total transition time) until mode 3 ultimately becomes dominant. An even longer coexistence is found when perturbing the period-four travelling profile with mode 2, as shown by the grey curve in figure 6d, leading to a period-four wave modulated by mode 2, as in figure 8a (see output_f4_p2.mp4 in the electronic supplementary material for a video of this process). The interaction between mode 2 and mode 4 occurs for $t \in [0, 0.6]$, an interval twice longer than any complete transitions in figure 7.

These long-lived multi-periodic waves states, which we have identified numerically, can be considered analogous to double cnoidal waves of the KdV equation. Double cnoidal waves are the spatially periodic generalization of the well-known two-soliton solution of KdV [55]. They can be considered as exact solutions with two independent phase velocities [56]. The evolution of the phase velocities $v_p(k)$ of modes $k = 2$ and 4 (of the Fourier decomposition of $\eta$) are shown in figure 8b. The phase velocity of mode 2 experiences more intense oscillations than mode 4, which
can be seen from figure 8a. These oscillations are caused by the energy interaction between even modes, and a low pass filter can be applied to evaluate a time-averaged phase velocity for mode 2, shown as the black curve (the jump around $t = 0$ is a windowing effect). It is surprising to see that while $|\eta_2|$, the amplitude of mode 2, is growing slowly, its phase velocity maintains around $v_p(k = 2) \approx 53.5$, which is independent of $v_p(k = 4) \approx 218.1$. Haupt & Boyd [56] constructed double cnoidal solutions of KdV through a harmonic balance of lower modes. On the other hand, the sharper peak of the quasi-double-cnoidal-waves in figure 8c shows the importance of the balance among higher harmonic modes in our model KS-type long-wave equation.

The rapid transition during $t \in [0.65, 0.8]$ is characterized by ‘wave chasing’ in the physical domain. Mode 2 and mode 4 become comparable in Fourier energy, with mode 4 propagating faster than mode 2. Visually, this observation is similar to soliton collisions: when the peak of mode 2 is caught by that of mode 4, an elevation can be observed and then a depression as they separate, shown in the wave profiles in figure 8c for $t \in [0.713, 0.72]$. However, while soliton collision (in the sense of Zabusky & Kruskal [54]) leave the wave profile and propagation velocity unchanged, the ‘chasing’ (and interaction) in the current study results in the waves ultimately merging into the period-two nonlinear travelling wave. The phase velocity of mode 4 dramatically decreases, and all modes in the system merge into a phase velocity $v_p \approx 55.8$, which becomes the propagation velocity $v_f$ of the period-two travelling wave. It is interesting to note that, in this process, the time-averaged propagation velocity of mode 2 barely changes, except for the mitigation of the oscillations. This can be intuitively understood from the strong stability of the period-two travelling wave profile, while a mathematical reason might emerge from the singular limit of a double cnoidal wave (if it exists in this system).

In the end, this study answers one question posed in [19]: when the energy of higher modes is dominant, this confined ferrofluid system can accommodate multi-periodic travelling waves, resembling a long-lasting, but non-integrable, double cnoidal wave field. When the energy of the two component modes becomes comparable, a rapid transition happens and the modulated propagating wave profile saturates to its envelope. In a sense, this means that these periodic nonlinear waves lose their shapes upon ‘collision’. However, it would be interesting to ask if a localized solitary wave also exists for our model equation, and to address what would happen during the localized waves’ collisions.

7. Conclusion

The dynamics of long, small-amplitude nonlinear waves on the interface of a thin ferrofluid film was analysed for the configuration of a horizontal Hele-Shaw flow subjected to a tilted magnetic field. We showed that such ferrofluid interfaces support periodic travelling waves governed by a modified KS-type equation, which we derived. A linear stability analysis and a nonlinear energy budget were employed to reveal that the balance between stabilizing surface tension forces (energy sink/loss) and destabilizing magnetic forces (energy source/gain) leads to the generation of dissipative solitons on the ferrofluid interface. The effect of key parameters was investigated, and the corresponding magnetic field configurations were discussed. Our results lead to quantitative understanding of these nonlinear periodic travelling wave profiles, and how interfacial waves can be generated and controlled (specifically, their propagation velocity and shape) non-invasively by an external magnetic field. A multiple-scale analysis provides a weakly nonlinear correction to the propagation velocity of harmonic waves. This calculation also reveals how the marginally unstable linear solution is equilibrated by the weak nonlinearity and tends to the permanent travelling wave solution. At the same time, the model equation (3.14) features a variety of interesting novel nonlinearities that could open avenues of future mathematical research.

In this respect, we identified the allowed wave states (specifically, their spatial periods), which bifurcate as the most unstable linear mode $k_m$ is varied, as fixed points in an energy phase plane, using the dissipative soliton concept [26]. State transitions are observed when some travelling wave profiles are perturbed, depending on their spectral stability, and the transition ‘direction’
(towards another fixed point in the energy phase plane) is determined by the perturbation. It would be of interest to realize the obtained travelling wave profiles (and their transition dynamics) in laboratory experiments. The wave selection process with multi-mode perturbations poses a challenge in that the initial perturbation must be carefully controlled, especially for the spectrally unstable profiles.

Another novel feature of this study is that multi-periodic nonlinear waves (akin to the double cnoidal wave of the KdV equation) were found numerically in the context of a (non-integrable) long-wave equation of the modified KS type. Perturbations of spectrally stable modes interact intensely with their harmonics, which are already present as part of the original spectrally unstable travelling wave profile. Such interactions are long-lived, until an abrupt transition to a final stable travelling wave occurs. As mentioned in §6d, we were unable to construct perturbative solutions in the sense of the double cnoidal waves [56], therefore a complete mathematical explanation of these multi-periodic nonlinear wave dynamics (and the transitions between them) remains an open problem to be addressed in future work. Finally, it would also be of interest to derive a two-dimensional version of our model long-wave equation, and the dynamics it governs could be compared with and contrasted to recent work on the two-dimensional KS equation [27].

Data accessibility. Python script required to run and analyse the numerical simulations of equation (3.14), and resulting data files, are available at https://github.com/zongxin/long_wave_eq_ferrofluid_thin_film.

Authors’ contributions. Both authors contributed equally to the analysis of the problem and the derivation of the mathematical model. I.C.C. initiated the project and supervised the analysis. Z.Y. conducted all the case studies, simulations and data analysis. Both authors discussed the results and contributed equally to the final version of the manuscript.

Competing interests. We declare that we have no competing interests.

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