IDENTIFICATION OF PHOTON SOURCES, STOCHastically EMBEDDED IN AN INTERSTELLAR CLOUD

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Abstract. Photon transport is considered in an interstellar cloud containing one or several photon sources (stars), defined by \( q_i \delta(x - x_i) \), \( i = 1, 2, \ldots \), where the locations \( x_i \)'s are given in a stochastic way. First, the case is examined of a single source of intensity \( q_1 \) and located at \( x_1 \) with a probability density \( p_1 = p(x_1) \), such that \( p(x_1) \geq 0 \) and \( \int_V p(x_1) \, dx_1 = 1 \), where \( V \subset \mathbb{R}^3 \) is the region occupied by the cloud. Then, a Boltzmann-like equation for the average photon distribution function \( \langle n \rangle(x, u; x_1) \) is derived and it is shown that \( p(x_1) \) can be evaluated starting from a far-field measurement of \( \langle n \rangle \).

Finally, the case of two or more photon sources is discussed: the corresponding results are reasonably simple if \( p(x_1, x_2) = p_1(x_1)p_2(x_2) \), i.e. if the locations of the two photon source are “independent”.

1. Introduction. Stars form from clouds of gas (mainly in molecular form) and dust (tiny solid particles a few hundred Angstroms in size), that inhabit the interstellar space. Stars tend not to form in isolation. In fact, most star-forming regions seem to harbor clusters associations of stars with varying degrees of richness, from a few to several hundreds of stars. [5], [6].

Clusters are important laboratories to study stellar evolution because stars in clusters share the common heritage of being formed from the same progenitor molecular cloud. Embedded clusters are particularly important in this regard since their study provides insights into the phenomena of stellar creation and early life. Since embedded clusters of young stellar objects are heavily obscured by dust, they are invisible in the optical and UV ranges and IR observations are necessary for their detection and investigation. To an observer they appear as a collection of IR point sources physically related. Observations allow to evaluate distances and surface densities of a star cluster but not the mutual location of its components, a factor that can affect the evolution of the cluster and also of the whole cloud.

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This paper is dedicated to the memory of our dear friend and colleague Aldo (S. A. and L. B.).
The simplest sites of star formation are the Bok globules, small dark clouds of gas and dust, dense and very cold. The stars born in Bok globules are likely to be low-mass stars, like our Sun, since the globules themselves have small mass. Even though Bok globules have physical conditions and chemical composition that are similar in many ways to those of larger dark clouds, they are, unlike these, isolated. Their simple geometry and environment make them good subjects for theoretical studies and modelling.

For simplicity, in this paper we shall investigate photon transport in a small interstellar cloud (like a Bok globule), containing one or several photon sources (stars), whose locations are known only in a stochastic way and should be evaluated by using some kind of observation. However, the procedures that follow can be extended to the case of clusters partially embedded or located “behind” a cloud.

2. The case of a single point source. Assume for simplicity that a single photon source is located at \( \mathbf{x}_1 \in V_i \), where \( V_i \) is the interior of the region \( V \). The case of two or more sources will be discussed later on.

Let \( n(\mathbf{x}, \mathbf{u}; \mathbf{x}_1) \) be the distribution function of photons that are at \( \mathbf{x} \in V \) and have velocity \( \mathbf{v} = c \mathbf{u} \) \( (c = \text{speed of light}, \mathbf{u} = \text{a unit vector}) \). Then, the radiation transfer phenomenon in the cloud is governed by the following stationary Boltzmann-like equation

\[
0 = -\mathbf{u} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \mathbf{u}; \mathbf{x}_1) - \sigma n(\mathbf{x}, \mathbf{u}; \mathbf{x}_1) + \frac{\sigma_s}{4\pi} \int_S n(\mathbf{x}, \mathbf{u}'; \mathbf{x}_1) \, d\mathbf{u}' + q_1 \delta(\mathbf{x} - \mathbf{x}_1), \quad (\mathbf{x}, \mathbf{u}) \in V_i \cap S. \tag{1}
\]

where \( \sigma \) and \( \sigma_s \) are the total and the scattering cross sections of the dust grains, \( q_1 \) is the source strength, \( \delta \) is Dirac’s delta functional, and \( S \) is the surface of the unit sphere.\(^1\)

Equation (1) is supplemented by the following non re-entry boundary condition

\[
n(\mathbf{y}, \mathbf{u}; \mathbf{x}_1) = 0 \quad \forall \mathbf{y} \in \partial V, \quad \mathbf{u} \cdot \nu(\mathbf{y}) < 0, \tag{2}
\]

where \( \nu(\mathbf{y}) \) is the outward directed normal to the boundary surface \( \partial V \) at \( \mathbf{y} \in \partial V \), \( [1], [2], [3] \).

Existence and uniqueness of a distributional solution of system (1)+(2) are discussed in [4].

Note that the photon distribution function \( n \) obviously depends on the “parameter” \( \mathbf{x}_1 \in V_i \), which (together with \( q_1 \)) identifies the photon source.

Let now \( p_1 = p(\mathbf{x}_1) \) be a probability density, such that \( p(\mathbf{x}_1) \, d\mathbf{x}_1 \) is the probability that our photon source is located within the volume element \( d\mathbf{x}_1 \), centered at \( \mathbf{x}_1 \). Hence

\[
p(\mathbf{x}_1) \geq 0, \quad \int_V p(\mathbf{x}_1) \, d\mathbf{x}_1 = 1. \tag{3}
\]

\(^1\) For simplicity, we assume that \( \sigma \) and \( \sigma_s \) are (positive) constants; if the two cross sections depend on \( \mathbf{x} \in V \), things still work. In general, \( \sigma, \sigma_s \) (and \( q_1 \)) also depend on the frequency \( f \) of the photons under consideration. Correspondingly, \( n = n(\mathbf{x}, \mathbf{u}, f; \mathbf{x}_1) \) and the scattering term in (1) becomes a little more involved.
Multiplication of (1) by \( p(x_1) \) and integration with respect to \( x_1 \) over the whole \( V \) gives

\[
0 = -\mathbf{u} \cdot \nabla_x <n>(x, u) - \sigma <n>(x, u) + \frac{\sigma_s}{4\pi} \int_S <n>(x, u') \, du' + q_1 p(x),
\]

(4)

where

\[
<n>(x, u) = \int_V p(x_1) n(x, u; x_1) \, dx_1.
\]

(5)

**Remark 1.** The average photon distribution function \( <n> \) satisfies the Boltzmann-like equation (4) with the source term \( q_1 p(x) \) (i.e., as if the source were spread over the whole \( V \)). On the other hand, \( n \) satisfies equation (1) with a “more complicated” source term containing a delta functional.

The problem is now the following: is it possible to identify the probability density \( p(x) \), starting from the knowledge of the value \( <n>(\tilde{x}, \tilde{u}) = <n>(\tilde{z}, \tilde{u}), \) see Figure 1, i.e. from a “far-field” measurement?

By a standard procedure, equation (4) can be put into its integral form:

\[
<n>(x, u) = \int_0^{R(x, u)} dr \exp (-\sigma r) \cdot \left\{ \frac{\sigma_s}{4\pi} \int_S <n>(x - ru, u') \, du' + q_1 p(x - ru) \right\},
\]

(6)
where $R(x, u)$ is such that $x - R(x, u)u \in \partial V$, see Figure 1.

If we put

$$
(Bf)(x, u) = \int_0^{R(x, u)} \exp(-\sigma r)f(x - ru, u)\,dr
$$

$$
(Kf)(x, u) = \frac{\sigma_s}{4\pi} \int_S f(x, u')\,du',
$$

with $f \in D(B) = D(K) = X = L^1(V \times S)$ and

$$
\|f\| = \int_V dx \int_S |f(x, u)|\,du,
$$

(6) becomes

$$
<n>(x, u) = (BK<n>)(x, u) + q_1(Bp)(x, u), \quad (x, u) \in V \times S.
$$

Equation (9) gives

$$
<n>(x, u) = q_1(Hp)(x, u)
$$

where

$$
(Hp)(x, u) = ((I - BK)^{-1}Bp)(x, u)
$$

and

$$
\|B\| \leq \frac{1}{\sigma} \left[\exp(-\sigma \bar{R})\right]
$$

$$
\|K\| \leq \sigma_s
$$

$$
\|BK\| \leq \frac{\sigma_s}{\sigma} [1 - \exp(-\sigma \bar{R})] < 1
$$

(12a)

(12b)

(12c)

with $\bar{R} = \sup \{R(x, u), (x, u) \in V \times S\}$ is the diameter of $V$.

Assume now that experimental evidence suggests that the probability density should be properly approximated by an element of the family

$$
\Phi = \Phi_{[0,1]} = \{\varphi(h): \varphi(h) = (1 - h)\varphi_m + h\varphi_M, \quad h \in [0, 1]\}
$$

where $\varphi_m$ and $\varphi_M$ are suitable probability densities (hence, $\int_V \varphi(h)(x)\,dx = (1 - h) \int_V \varphi_m(x)\,dx + h \int_V \varphi_M(x)\,dx = (1 - h) + h = 1$).

Consider then (10) at $x = \hat{z}, u = \hat{u}$:

$$
<n>(\hat{z}, \hat{u}) = q_1(Hp)(\hat{z}, \hat{u})
$$

(14)

i.e.

$$
<n>(\hat{z}, \hat{u}) = q_1\hat{H}p,
$$

(15)

where

$$
\hat{H}p = (Hp)(\hat{z}, \hat{u}) = ((I - BK)^{-1}Bp)(\hat{z}, \hat{u})
$$

(16)

and where we recall that $<n>(\hat{z}, \hat{u})$ is known from a far-field measurement.

In order to approximate $p$ in (15) by means of a suitable element of the family $\Phi_{[0,1]}$, we observe that

$$
\hat{H}\varphi(h) = (1 - h)\hat{H}\varphi_m + h\hat{H}\varphi_M =
$$

$$
= \hat{H}\varphi_m + h\hat{H}(\varphi_M - \varphi_m), \quad \varphi(h) \in \Phi_{[0,1]},
$$

(17)

with

$$
\frac{d}{dh}\hat{H}\varphi(h) = \hat{H}(\varphi_M - \varphi_m) = \hat{H}\varphi_M - \hat{H}\varphi_m, \quad \forall h \in [0, 1].
$$

(18)
(18) shows that \( \hat{H}\phi_{(h)} \) is an increasing function of \( h \in [0, 1] \) if \( \hat{H}\phi_M > \hat{H}\phi_m \) and a decreasing function if \( \hat{H}\phi_M < \hat{H}\phi_m \). Thus, if for instance \( \hat{H}\phi_M > \hat{H}\phi_m \), we have that \( \hat{H}\phi_m = \min_{h \in [0, 1]} \hat{H}\phi_{(h)} \), and \( \hat{H}\phi_M = \max_{h \in [0, 1]} \hat{H}\phi_{(h)} \).

Assume now that the family \( \Phi_{[0,1]} \) is such that

\[
q_1\hat{H}\phi_m \leq <n>(\hat{z}, \hat{u}) \leq q_1\hat{H}\phi_M.
\]

(Of course, it may happen that \( <n>(\hat{z}, \hat{u}) < q_1\hat{H}\phi_m \) or \( q_1\hat{H}\phi_M < <n>(\hat{z}, \hat{u}) \): in this case the family \( \Phi_{[0,1]} \) must be changed, see also Remark 1).

Relation (18) and inequalities (19) imply that a unique \( \hat{h} \in [0, 1] \) exists, such that

\[
<n>(\hat{z}, \hat{u}) = q_1\hat{H}\phi_{(\hat{h})} = q_1 \left[ \hat{H}\phi_m - \hat{h}\hat{H}(\phi_M - \phi_m) \right],
\]

i.e.

\[
\hat{h} = \frac{[<n>(\hat{z}, \hat{u}) - q_1\hat{H}\phi_m]}{[q_1\hat{H}(\phi_M - \phi_m)]}
\]

and \( \hat{h} \in [0, 1] \) because of assumptions (19).

Hence,

\[
q_1\phi_{(\hat{h})} = q_1[\phi_m + \hat{h}(\phi_M - \phi_m)]
\]

\[
= q_1\phi_m + \frac{<n>(\hat{z}, \hat{u}) - q_1\hat{H}\phi_m}{\hat{H}(\phi_M - \phi_m)}(\phi_M - \phi_m)
\]

(20)

is, in some sense, the best approximation to \( q_1p \) within \( \Phi_{[0,1]} \).

Remark 2. (i) Most likely, inequalities (19) should be satisfied if we choose \( \phi_m(x) > 0 \) if \( x \) is close to \( \hat{y} \) and small otherwise, and \( \phi_M(x) > 0 \) if \( x \) is close to \( \hat{z} \) and small otherwise, see Figure 1. (ii) In principle, \( <n>(\hat{z}, \hat{u}) \) might be evaluated by measuring \( n(\hat{z}, \hat{u}) \) for several clouds, similar to the one under consideration. More directly, \( <n>(\hat{z}, \hat{u}) \) might be approximated by the value of \( n(\hat{z}, \hat{u}) \) for the cloud under consideration, assuming that the standard deviation \( \sqrt{<n>^2 - <n>^2} \) is small (with respect to \( <n> \)). Since the equation for \( <n>^2 \) contains the square of a delta functional, it is not easy to check whether this assumption is satisfied. In any case, a further paper will be devoted to such a problem, by using the procedures introduced in [4].

3. Two (or more) point sources. Assume that two photon sources are located at \( x_1 \in V_i \) and at \( x_2 \in V_i \) (the case of a finite number of point sources may be studied by similar procedures).

If \( n(x, u; x_1, x_2) \) is the distribution function of photons that are at \( x \in V \) and have velocity \( v = c u \), then we have

\[
0 = -u \cdot \nabla_x n(x, u; x_1, x_2) - \sigma n(x, u; x_1, x_2) + \frac{\sigma_s}{4\pi} \int_S n(x, u'; x_1, x_2) \, du' + q_1 \delta(x - x_1) + q_2 \delta(x - x_2), \quad (x, u) \in V_i \cap S.
\]

(21)

As usual, equation (21) is supplemented by the boundary condition

\[
n(y, u; x_1, x_2) = 0 \quad \forall y \in \partial V, \ u \cdot v(y) < 0,
\]

see (1), and (2).
Note that the photon distribution function \( n \) now depends on the “parameters” \( x_1, x_2 \in V \), which identify the photon source (together with \( q_1 \) and \( q_2 \)).

Let now \( p(x_1, x_2) \) be a probability density, such that \( p(x_1, x_2) \, dx_1 \, dx_2 \) is the probability that the photon source “1” is located within the volume element \( dx_1 \) around \( x_1 \) and the source “2” is within \( dx_2 \) around \( x_2 \). Multiplication of (21) by \( p(x_1, x_2) \) and integration with respect to \( x_1 \) and \( x_2 \) over \( V \times V \) gives

\[
0 = -u \cdot \nabla_x <n>(x, u) - \sigma <n>(x, u) +
+ \frac{\sigma_s}{4\pi} \int_S <n>(x, u') \, du' +
+ q_1 \int_V p(x_2) \, dx_2 + q_2 \int_V p(x_1) \, dx_1
\]

where

\[
<n>(x, u) = \int_V dx_1 \int_V dx_2 p(x_1, x_2) n(x, u; x_1, x_2).
\]

Note that, if

\[
p(x_1, x_2) = p_1(x_1)p_2(x_2)
\]

(\( \int_V p_1(x_1) \, dx_1 = \int_V p_2(x_2) \, dx_2 = 1 \)), then (23) becomes

\[
0 = -u \cdot \nabla_x <n>(x, u) - \sigma <n>(x, u) +
+ \frac{\sigma_s}{4\pi} \int_S <n>(x, u') \, du' + q_1p_1(x) + q_2p_2(x).
\]

**Remark 3.** Assumption (25) is “reasonable” from a physical viewpoint, because the location of the photon source “1” is usually independent of the location of the source “2”.

If we put

\[
\Pi_1(x) = \int_V p(x, x_2) \, dx_2, \quad \Pi_2(x) = \int_V p(x_1, x) \, dx_1,
\]

then, for instance, \( \Pi_1(x) \, dx \) is the probability that the source “1” is within \( dx \) around \( x \), independently of where the source “2” is located (of course, \( \Pi_1(x) = p_1(x) \) and \( \Pi_2(x) = p_2(x) \) if (25) holds).

Correspondingly, (23) becomes

\[
0 = -u \cdot \nabla_x n(x, u; x_1, x_2) - \sigma n(x, u; x_1, x_2) +
+ \frac{\sigma_s}{4\pi} \int_S n(x, u'; x_1, x_2) \, du' +
+ q_1\Pi_1(x) + q_2\Pi_2(x), \quad (x, u) \in V \times S.
\]

As in Section 2, the integral version of (24) has the form

\[
<n>(x, u) = \int_0^{\mathcal{R}(x,u)} dr \exp(-\sigma r) \cdot \left\{ \frac{\sigma_s}{4\pi} \int_S <n>(x - ru, u') \, du' +
+ q_1\Pi_1(x - ru) + q_2\Pi_2(x - ru) \right\},
\]
"unknown" is then the probability density $p$.

where experimentally. Such a single value is obviously not enough to find the "function"

are the "best approximation within $\Phi_{[0]}$" of system (32), we introduce the two families

In order to approximate $\Pi_1$ and $\Pi_2$ in (32), we introduce the two families

where $\varphi_m$, $\varphi_M$ and $\psi_m$, $\psi_M$ are suitably chosen.

The family $\Phi_{[0]}$ will be used to approximate $\Pi_1$ and $\Psi_{[0]}$ to approximate $\Pi_2$. In fact, substitution of $\Pi_1$ by $\varphi(h) \in \Phi_{[0]}$ and $\Pi_2$ by $\psi(k) \in \Psi_{[0]}$ in system (32) gives

or, equivalently (see (34)),

Solution of the linear system (35) leads to the couple $(\hat{h}, \hat{k})$, such that

Relations (36) show that the couple $(\hat{h}, \hat{k})$ is the solution of system (35) (or (36)) which is the "approximate version within $\Phi_{[0]} \times \Psi_{[0]}$" of system (32). Hence

are the "best approximation within $\Phi_{[0]} \times \Psi_{[0]}$" to $\Pi_1$ and $\Pi_2$ respectively.

4. Concluding remarks. For simplicity, consider the case of a single source: the "unknown" is then the probability density $p(x_1)$ as a function of $x_1 \in V$, see (3). On the other hand, $< n > (z, \hat{u})$ is the only value of $< n > (x, u)$ which is known experimentally. Such a single value is obviously not enough to find the "function" $p(x_1)$.

This justifies the choice of the one-parameter family

\[ \Phi_{[0]} = \{ \varphi(h)(x_1) = (1 - h)\varphi_m(x_1) + h\varphi_M(x_1), \; h \in [0, 1] \} \]
to approximate \( p(x_1) \). Correspondingly, the measured value \( \langle n \rangle (\hat{z}, \hat{u}) \) is sufficient to evaluate the “best value” \( \hat{h} \) of \( h \in [0,1] \), and the best approximation \( \varphi(\hat{h})(x_1) \) to \( p(x_1) \) within \( \Phi_{[0,1]} \).

In any case, \( \varphi_m \) and \( \varphi_M \) (which define \( \Phi_{[0,1]} \)) must be found keeping in mind the particular experiment which is being made. For instance, if \( x_1 = (x_{11}, x_{12}, x_{13}) \) and \( x_{11} \) is the distance between the star and the “observer”, it may happen that \( x_{12} \) and \( x_{13} \) are known (with a reasonably small error) whereas \( x_{11} \) is to be found. Correspondingly, \( \varphi_m(x_1) \) and \( \varphi_M(x_2) \) should be functions that depend on \( x_{12} \) and \( x_{13} \) in a given (experimentally inferred) way and \( x_{11} \) in some “reasonable” way (suggested by the physical characteristics of the cloud under consideration and by other similar clouds).

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