REFLECTION GROUPS IN HYPERBOLIC SPACES 
AND THE DENOMINATOR FORMULA 
FOR LORENTZIAN KAC–MOODY LIE ALGEBRAS

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Abstract. This is a continuation of our "Lecture on Kac–Moody Lie algebras of the arithmetic type" [25].

We consider hyperbolic (i.e. signature \((n, 1)\)) integral symmetric bilinear form 
\(S : M \times M \rightarrow \mathbb{Z}\) (i.e. hyperbolic lattice), reflection group \(W \subset W(S)\), fundamental polyhedron \(\mathcal{M}\) of \(W\) and an acceptable (corresponding to twisting coefficients) set \(P(M) \subset M\) of vectors orthogonal to faces of \(\mathcal{M}\) (simple roots). One can construct the corresponding Lorentzian Kac–Moody Lie algebra \(g = g''(A(S, W, P(M)))\) which is graded by \(M\).

We show that \(g\) has good behavior of imaginary roots, its denominator formula is defined in a natural domain and has good automorphic properties if and only if \(g\) has so called restricted arithmetic type. We show that every finitely generated (i.e. \(P(M)\) is finite) algebra \(g''(A(S, W_1, P(M_1)))\) may be embedded to \(g''(A(S, W, P(M)))\) of the restricted arithmetic type. Thus, Lorentzian Kac–Moody Lie algebras of the restricted arithmetic type is a natural class to study.

Lorentzian Kac–Moody Lie algebras of the restricted arithmetic type have the best automorphic properties for the denominator function if they have a lattice Weyl vector \(\rho\). Lorentzian Kac–Moody Lie algebras of the restricted arithmetic type with generalized lattice Weyl vector \(\rho\) are called elliptic (if \(S(\rho, \rho) < 0\)) or parabolic (if \(S(\rho, \rho) = 0\)). We use and extend our and Vinberg’s results on reflection groups in hyperbolic spaces to show that the sets of elliptic and parabolic Kac–Moody Lie algebras with generalized lattice Weyl vector and lattice Weyl vector are essentially finite.

We also consider connection of these results with the recent results by R.Borcherds.

§0. Introduction

In [25], it was shown that a finitely generated symmetrizable Kac–Moody algebra has ”good behavior” of imaginary roots if and only if it is finite, affine, of rank two or of hyperbolic arithmetic type. Hyperbolic arithmetic type means that Weyl group is a group generated by reflections in a hyperbolic space of dimension \(\geq 2\) with a fundamental polyhedron of finite volume. This paper is a natural continuation of these studies.

We consider a hyperbolic (i.e. of signature \((n, 1)\)) integral symmetric bilinear form \(S : M \times M \rightarrow \mathbb{Z}\) (i.e. a hyperbolic lattice), a reflection group \(W \subset W(S)\), a fundamental polyhedron \(\mathcal{M}\) of \(W\) and an acceptable (corresponding to twisting
coefficients) set $P(M) \subset M$ of vectors orthogonal to faces of $M$ (simple real roots). Using these data, one can construct the corresponding Lorentzian Kac–Moody algebra $g = g''(A(S,W,P(M)))$ which is graded by $M$ (see Sects 2.1 and 2.2).

We show that $g = g''(A(S,W,P(M)))$ has "good behavior" of imaginary roots, its denominator formula is defined in a natural domain and has good automorphic properties if and only if this algebra has so called restricted arithmetic type (see Sect. 2.2). This means that the semi-direct product $W.A(P(M))$ has finite index in $O(S)$. Here

$$A(P(M)) = \{ g \in O_+(S) \mid g(P(M)) = P(M) \}$$

is the "group of symmetries" of the fundamental polyhedron.

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We show that every finitely generated (i.e. $P(M)$ is finite) Lorentzian Kac–Moody algebra $g''(A(S,W_1,P(M_1)))$ has an embedding to a Lorentzian Kac–Moody algebra $g''(A(S,W,P(M)))$ of restricted arithmetic type with the same lattice $S$ (see Sect. 2.2). Thus, it is natural to study Lorentzian Kac–Moody algebras of restricted arithmetic type.

The denominator formula of a Lorentzian Kac–Moody algebra of restricted arithmetic type has the best automorphic properties if this algebra has a lattice Weyl vector $\rho \in M \otimes \mathbb{Q}$ (see Sect. 2.3 and also 2.4). A lattice Weyl vector is an element $\rho \in M \otimes \mathbb{Q}$ such that

$$S(\rho, \alpha) = -S(\alpha, \alpha)/2 \text{ for any } \alpha \in P(M).$$

A Lorentzian Kac–Moody algebra with a generalized lattice Weyl vector $\rho$ (see Definition 1.4.9) is called elliptic if it has restricted arithmetic type and $S(\rho, \rho) < 0$; and is called parabolic if it has restricted arithmetic type and $S(\rho, \rho) = 0$, and there does not exist a generalized lattice Weyl vector with negative square. Ellipticity is equivalent to finiteness of the index $[O(S) : W]$. Parabolicity is equivalent to restricted arithmetic type and existence of $0 \neq c \in M$ such that $S(c, c) = 0$ and $g(c) = c$ for any $g \in A(M)$ where $A(M)$ is infinite. The corresponding lattice $S$ is called elliptic reflective and parabolic reflective respectively.

We use and extend our's and É.B. Vinberg's results on reflection groups in hyperbolic spaces to show that sets of primitive elliptic and parabolic reflective lattices $S$ of rk $S \geq 3$ are finite (see Sect. 1.1). For elliptic case this was known 15 years ago. Thus, we extend this finiteness result for the parabolic case. Surprisingly, exactly the same method which was used for the elliptic case is successful for the parabolic one. This shows that these two cases are very similar, and the method which had been used for the elliptic case is very natural.

We apply the main geometrical result using to obtain finiteness results above, to show that the set of elliptic Lorentzian Kac–Moody algebras $g''(A(S,W, P(M)))$ with a lattice Weyl vector $\rho$ is finite for rk $S \geq 3$. For the parabolic case, we obtain the same result if we additionally suppose that the index $[O(S)_\rho : A(P(M))] < D$ for a fixed constant $D > O$. Here $O(S)_\rho$ is the stabilizer subgroup of $\rho$. See Sect. 1.3. Example 1.3.4 demonstrates that finiteness may not be true for the parabolic case without this additional condition.

At last, in Sect. 2.4 we consider connection of these our results with the recent results by R. Borcherds.

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I am grateful to Professor É.B. Vinberg for very useful remarks. I am grateful to Professor I. R. Shafarevich for his interest to and support of these my studies. I plan to publish this paper in Math. Russian Izvest.

§1. SOME RESULTS ON REFLECTION GROUPS OF INTEGRAL HYPERBOLIC LATTICES

1.1. Elliptic and parabolic reflective lattices and reflection groups.

One can consider this section as a complement to our papers [19], [20] and to É.B. Vinberg [31]. In [19] and [20] we used signature (1, n) for a hyperbolic form and used one letter \( S \) to denote the form and the space. Here we use two letters \( S \) and \( M \) to denote a form. Also, here we use signature \((n, 1)\) for a hyperbolic form. These notations are standard for Lie algebras theory.

First, we extend results of the papers above to “reflection groups with a cusp”.

Let \( S : M \times M \to \mathbb{Z} \) (1.1.1)

be a hyperbolic (i.e. of signature \((n, 1)\)) integral symmetric bilinear form over \( \mathbb{Z} \). Here \( M \) is a free \( \mathbb{Z} \)-module of a finite rank. To be shorter, we call \( S \) as hyperbolic lattice. We consider the corresponding cone

\[
V(S) = \{ x \in M \otimes \mathbb{R} \mid S(x, x) < 0 \},
\]

choose its half-cone \( V^+(S) \) and consider the corresponding hyperbolic space \( L(S) = V^+(S)/\mathbb{R}^+ \) where \( \mathbb{R}^+ \) denote the set of positive numbers. Thus, a point of \( L(S) \) is a ray \( \mathbb{R}^+ x \) where \( x \in V^+(S) \). The distance \( \rho \) in \( L(S) \) is defined by the formula:

\[
\cosh \rho(\mathbb{R}^+ x, \mathbb{R}^+ y) = -S(x, y)/\sqrt{S(x, x)S(y, y)}, \; x, y \in V^+(S).
\]

With this distance the curvature is equal to \(-1\). Each element \( \delta \in M \otimes \mathbb{R} \) with \( S(\delta, \delta) > 0 \) defines the half-space

\[
\mathcal{H}^+_\delta = \{ \mathbb{R}^+ x \in L(S) \mid S(x, \delta) \leq 0 \}
\]
with the half-space

\[
\mathcal{H}_\delta = \{ \mathbb{R}^+ x \in L(S) \mid S(x, \delta) = 0 \}.
\]

The element \( \delta \in M \otimes \mathbb{R} \) is defined by the half-space \( \mathcal{H}^+_\delta \) (respectively, by the hyperplane \( \mathcal{H}_\delta \)) up to multiplication on elements of \( \mathbb{R}^+ \) (respectively, on elements of \( \mathbb{R}^+ \) of non-zero real numbers). The \( \delta \) is called orthogonal to the half-space \( \mathcal{H}^+_\delta \) (respectively, to the hyperplane \( \mathcal{H}_\delta \)).

Let \( O_+(S) \) be the subgroup of \( O(S) \) of the index 2 which fixes the half-cone \( V^+(S) \). It is well-known (this an easy corollary of the arithmetic groups theory) that \( O_+(S) \) is discrete in \( L(S) \) and has a fundamental domain of finite volume. If \( \phi \in O_+(S) \) defines a reflection in a hyperplane of \( L(S) \), then \( \phi = s_\delta \) for \( \delta \in M \) with \( S(\delta, \delta) > 0 \). Here

\[
s_\delta(x) = x - (2S(x, \delta)/S(\delta, \delta))\delta, \; x \in S.
\]

(1.1.6)
and \( s_\delta \in O(S) \) if and only if
\[
(2S(M, \delta)/S(\delta, \delta))\delta \subset M.
\]
(1.1.7)

In particular, if \( \delta \) is primitive in \( M \), this is equivalent to
\[
S(\delta, \delta)|2S(M, \delta).
\]
(1.1.8)

Obviously, \( s_\delta \) is the reflection in the hyperplane \( H_\delta \) which is orthogonal to \( \delta \). The reflection changes places half-spaces \( H_\delta^+ \) and \( H_{-\delta}^+ \). The automorphism \( s_\delta \in O(S) \) is called \textit{reflection} of the lattice \( S \). Any subgroup of \( O(S) \) (or the corresponding discrete group of motions of \( \mathcal{L}(S) \)) generated by a set of reflections is called \textit{a reflection group}.

We denote by \( W(S) \) the subgroup of \( O_+(S) \) generated by all reflections of \( S \). A lattice \( S \) is called \textit{reflective} if index \([O(S):W(S)]\) is finite; equivalently, \( W(S) \) has a fundamental polyhedron of finite volume in \( \mathcal{L}(S) \). É.B. Vinberg, in particular, proved in [29] that any arithmetic reflection group \( W \) in a hyperbolic space with the field of definition \( \mathbb{Q} \) is a subgroup \( W \subset W(S) \) of finite index for one of reflective hyperbolic lattices \( S \). For \( m \in \mathbb{Q} \) we denote by \( S(m) \) the lattice which one gets multiplying on \( m \) the form of \( S \). Obviously, \( S(m) \) is reflective if \( S \) does.

Almost 15 years ago there was proved

**Theorem 1.1.1** ([19, Appendix, Theorem 1] and [20, Theorem 5.2.1]). \textit{For a fixed \( \text{rk} \ S \geq 3 \), the set of reflective hyperbolic lattices \( S \) is finite up to similarity \( S \mapsto S(m) \) and isomorphism.}

The proof was based on a purely geometrical result on convex polyhedra of finite volume in hyperbolic spaces which we want to formulate.

A convex polyhedron \( M \) in a hyperbolic space \( \mathcal{L}(S) \) is an intersection
\[
\mathcal{M} = \bigcap_{\delta \in P(\mathcal{M})} H_\delta^+
\]
(1.1.9)
of several half-spaces orthogonal to elements \( \delta \in M \otimes \mathbb{R} \) with \( S(\delta, \delta) > 0 \). We suppose that \( \mathcal{M} \) is locally finite in \( \mathcal{L}(S) \). Then the minimal set \( P(\mathcal{M}) \) above is defined canonically up to multiplication of its elements by positive reals. Then it is called the set of vectors orthogonal to faces of \( \mathcal{M} \) (and directed outward) or shortly: orthogonal to \( \mathcal{M} \). We always suppose that \( P(\mathcal{M}) \) has this property. The polyhedron \( \mathcal{M} \) is called \textit{non-degenerate} if it contains a non-empty open subset of \( \mathcal{L}(S) \). A non-degenerate polyhedron \( \mathcal{M} \) is called \textit{elliptic} (equivalently, it has finite volume) if it is a convex envelope of a finite set of points in \( \mathcal{L}(S) \) or at infinity of \( \mathcal{L}(S) \). Then \( P(\mathcal{M}) \) is finite. The proof of Theorem 1.1.1 was based on the following result:

**Theorem 1.1.1’** ([19, appendix, Theorem 1]). \textit{Let} \( \mathcal{M} \text{ be an elliptic (equivalently, of finite volume) non-degenerate convex polyhedron in hyperbolic space } \mathcal{L}(S) \text{ of } \dim \mathcal{L}(S) = n \geq 2. \text{ Then there are elements } \delta_1, \ldots, \delta_{n+1} \in P(\mathcal{M}) \text{ with the following properties:}

(a) \( \text{rk} \ [\delta_1, \ldots, \delta_{n+1}] = n + 1; \)

(b) the Gram diagram of the elements \( \delta_1, \ldots, \delta_{n+1} \) is connected (i.e. one cannot divide the set \( \{\delta_1, \ldots, \delta_{n+1}\} \) on two non-empty subsets being orthogonal to one another).}
(c) \(-2 \leq -2S(\delta_i, \delta_j)/\sqrt{S(\delta_i, \delta_i)S(\delta_j, \delta_j)} < 62\) (other speaking, we have inequality \(-2 \leq -S(e_i, e_j) < 62\) if we normalize elements \(e_i \in P(M)\) by the condition \(S(e_i, e_i) = 2\), \(1 \leq i, j \leq n + 1\).

To prove Theorem 1.1.1, one should apply Theorem 1.1.1' to the fundamental polyhedron \(\mathcal{M}\) of \(W\) and elements \(P(\mathcal{M})\) which belong to the lattice \(M\). See [20, Theorem 5.2.1].

Further, we name reflection subgroups \(W \subset O(S)\) of finite index and the corresponding reflective hyperbolic lattices \(S\) also as elliptic reflection groups and elliptic reflective hyperbolic lattices respectively.

We want to extend results on elliptic reflection groups and elliptic reflective lattices above to the following situation.

Let \(S\) be a hyperbolic lattice, and \(W \subset W(S)\) a reflection subgroup. Let \(\mathcal{M}\) be a fundamental polyhedron of \(W\). Let

\[
A(\mathcal{M}) = \{\phi \in O_+(S) \mid \phi(\mathcal{M}) = \mathcal{M}\} \tag{1.1.10}
\]

be the group of symmetries of \(\mathcal{M}\). Clearly, then the semi-direct product \(W.A(\mathcal{M}) \subset O_+(S)\) where \(W\) is a normal subgroup in \(W.A(\mathcal{M})\).

**Definition 1.1.2.** A reflection group \(W \subset W(S)\) is parabolic if the group \(A(\mathcal{M})\) is infinite but it has finite index in \(O(S)/W\) (this means that \(W.A(\mathcal{M}) \subset O_+(S)\) has finite index) and there exists an element \(0 \neq c \in M\) with \(S(c, c) \leq 0\) such that \(\phi(c) = c\) for any \(\phi \in A(\mathcal{M})\). One can easily see (replacing \(c\) by \(-c\) if necessary) that \(S(c, c) = 0\) and \(\mathbb{R}_{++}c \in M\) (equivalently, \(S(c, P(M)) \leq 0\)). We call the primitive element \(c \in M\) (or the point \(\mathbb{R}_{++}c\) at infinity of \(L(S)\)) which satisfies this condition the cusp of \(W\). One can easily see that the cusp \(c\) (and the point \(\mathbb{R}_{++}c\)) is unique. A hyperbolic lattice \(S\) is called parabolic reflective if \(O_+(S)\) contains a parabolic reflection subgroup \(W\).

We want to prove that Theorem 1.1.1 is also valid for parabolic reflective hyperbolic lattices \(S\).

**Theorem 1.1.3.** For a fixed \(rk S \geq 3\), the set of parabolic reflective hyperbolic lattices \(S\) is finite up to similarity \(S \mapsto S(m)\) and isomorphism.

**Proof.** We prove an analog of Theorem 1.1.1' for appropriate "parabolic polyhedra". Let us fix a point \(O = \mathbb{R}_{++}c\) at infinity of \(L(S)\). Thus, \(c \in S \otimes \mathbb{R}\), \(S(c, c) = 0\) and \(S(c, V_+(S)) < 0\).

Recall that a horosphere \(\mathcal{E}_O\) with the center \(O\) is the set of all lines in \(L(S)\) containing the \(O\). The line \(l = O\mathbb{R}_{++}h \in \mathcal{E}_O\), \(\mathbb{R}_{++}h \in L(S)\), is the set \(l = \{t \mathbb{R}_{++}(tc + h) \mid t \in \mathbb{R} \\text{ and } S(tc + h, tc + h) < 0\}\). We fix a constant \(R > 0\). Then there exists a unique \(\mathbb{R}_{++}h \in l\) such that \(S(h, c) = -R\) and \(S(h, h) = -1\). Let \(l_1, l_2 \in \mathcal{E}_O\) and \(h_1, h_2\) the corresponding elements we have defined above. Let

\[
\rho(l_1, l_2) = \sqrt{S(h_1 - h_2, h_1 - h_2)}. \tag{1.1.11}
\]

The horosphere \(\mathcal{E}_O\) with this distance is an affine Euclidean space. If one changes the constant \(R\), the distance \(\rho\) is multiplying by a constant. The set

\[
\mathcal{E}_{--} = \{t \mathbb{R}_{++}h \in \mathcal{E}(S) \mid S(h, c) = -R\ \text{and } S(h, h) = -1\} \cup \{O\}. \tag{1.1.12}
\]
is a sphere in $L(S)$ touching $L(S)_\infty$ at the $O$. Moreover, the sphere $E_{O,R}$ is orthogonal to every line $l \in E_O$ at the corresponding to the $l$ point $R_{++h}, h \in E_{O,R}$. The distance of $L(S)$ induces an Euclidean distance in $E_{O,R}$ which is similar to the distance (1.1.11). The set $E_{O,R}$ is identified with $E_O$ and is also called horosphere.

Let $K \subset E_O$. The set

$$C_K = \bigcup_{l \in K} l$$

(1.1.13)

is called the cone with the vertex $O$ and the base $K$.

A non-degenerate locally finite polyhedron $M$ in $L(S)$ is called parabolic (relative to the point $O \in L(S)_\infty$), if the conditions 1) and 2) below are valid:

1) $M$ is finite at the point $O$, i.e. the set $\{ \delta \in P(M) | S(c, \delta) = 0 \}$ is finite.

2) For any elliptic polyhedron $N \subset E_O$ (i.e. $N$ is a convex envelope of a finite set of points in $E_O$), the polyhedron $M \cap C_N$ is elliptic.

A parabolic polyhedron $M$ is called restricted parabolic if the set

$$r(M) = \{-S(c, \delta/\sqrt{S(\delta, \delta)}) | \delta \in P(M)\}$$

(1.1.14)

is finite.

Geometrically this means that all hyperplanes $H_\delta, \delta \in P(M)$, are touching of a finite set of horospheres with the center $O$.

**Theorem 1.1.3’.** Theorem 1.1.1’ is also valid for any restricted parabolic polyhedron $M$ in hyperbolic space $L(S)$ of dim $L(S) = n \geq 2$. Thus, there are elements $\delta_1, ..., \delta_{n+1} \in P(M)$ with the following properties:

(a) $rk[\delta_1, ..., \delta_{n+1}] = n + 1$;

(b) the Gram diagram of the elements $\delta_1, ..., \delta_{n+1}$ is connected (i.e. one cannot divide the set $\{\delta_1, ..., \delta_{n+1}\}$ on two non-empty subsets orthogonal to one another).

(c) $-2 \leq -2S(\delta_i, \delta_j)/\sqrt{S(\delta_i, \delta_i)S(\delta_j, \delta_j)} \leq 62$ (other speaking, we have inequality $-2 \leq -S(e_i, e_j) \leq 62$ if we normalize elements $e_i \in P(M)$ by the condition $S(e_i, e_i) = 2), 1 \leq i, j \leq n + 1. \ (Unlike \ Theorem \ 1.1.1’, \ one \ has \ a \ non-strict \ inequality \ \leq 62 \ to \ the \ right.)$

A fundamental polyhedron $M(W)$ of a parabolic reflection group $W \subset W(S)$ of a hyperbolic lattice $S$ is restricted parabolic with respect to the cusp $R_{++c}$ of the group $W$, and the above statement holds for $M(W)$.

**Proof.** For the proof, we normalize orthogonal vectors $\delta \in P(M)$ of a polyhedron $M$ by the condition $S(\delta, \delta) = 2$. We normalize by this condition orthogonal vectors to all hyperplanes and half-spaces below.

To prove Theorem 1.1.1’ in [19, Appendix], we had fixed a point $O$ inside the polyhedron $M$ and had used the following formula for $S(\delta_1, \delta_2)$ of two elements $\delta_1, \delta_2 \in P(M)$ with $S(\delta_1, \delta_1) = S(\delta_2, \delta_2) = 2$ (see [19, Appendix, formula (2.1)]):

$$-S(\delta_1, \delta_2) = 4 \frac{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}}{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}} - 2.$$  (1.1.15)

Here $\theta_1$ and $\theta_2$ are angular openings of cones with a vertex $O$ and bases $H_{\delta_1}$ and $H_{\delta_2}$, respectively. For non-intersecting hyperplanes $H_{\delta_1}$ and $H_{\delta_2}$, the angle $\theta_{12}$ is the angular opening of the cone with the vertex $O$ which intersects the previous cones and has minimal angular opening (this is the minimal touching cone of two cones.
above). Here the cones and angles are taken in the sense of hyperbolic geometry. Using analytic continuation, one can obviously generalize this formula for arbitrary elements \( \delta_1, \delta_2 \in M \otimes \mathbb{R} \) with \( S(\delta_1, \delta_1) = S(\delta_2, \delta_2) = 2 \). One should consider the plane section containing \( O \) and orthogonal to the hyperplanes \( \mathcal{H}_{\delta_1} \) and \( \mathcal{H}_{\delta_2} \), and define an appropriate orientation of all plane angles of the section.

To prove Theorem 1.1.3', we use similar formula. Assume that the polyhedron \( \mathcal{M} \) is restricted parabolic with respect to the point \( O = \mathbb{R}_{++} c \) at infinity. For an element \( \delta \in M \otimes \mathbb{R} \) with \( S(\delta, \delta) = 2 \), we consider the "angle" \( \theta(\delta) = -1/S(c, \delta) \). Then one has the following analog of the formula (1.1.15) above.

\[
-S(\delta_1, \delta_2) = 4 \frac{(\theta(\delta_1) + \theta(\delta_{12})) \theta(\delta_2) + \theta(\delta_{12})}{\theta(\delta_1) \theta(\delta_2)} - 2. \tag{1.1.16}
\]

Here like above, for \( \delta_1, \delta_2 \in P(\mathcal{M}) \) with \( S(\delta_1, \delta_1) = S(\delta_2, \delta_2) = 2 \) and non-intersecting hyperplanes \( \mathcal{H}_{\delta_1}, \mathcal{H}_{\delta_2} \), the element \( \delta_{12} \in M \otimes \mathbb{R} \) is orthogonal to the hyperplane \( \mathcal{H}_{\delta_{12}} \) with \( S(\delta_{12}, \delta_{12}) = 2 \) and \( S(c, \delta_{12}) < 0 \), which intersects \( \mathcal{H}_{\delta_1} \) and \( \mathcal{H}_{\delta_2} \) only by infinite points and has minimal \( \theta(\delta_{12}) \).

Here \( \theta(\delta) \) behaves like an angular opening of the cone with vertex \( O \) and the base \( \mathcal{H}_\delta \). If \( A, B, C, O \) are 4 consecutive vertices at infinity of a convex polygon on a plane and \( e_1, e_2, e_3 \) are orthogonal to lines \( AB, BC \) and \( AC \) respectively, have \( S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = 2 \) and \( -S(c, e_1) > 0, -S(c, e_2) > 0 \) and \( -S(c, e_3) > 0 \), then

\[
\theta(e_3) = \theta(e_1) + \theta(e_2). \tag{1.1.17}
\]

Formulae (1.1.15), (1.1.16) and (1.1.17) are formulae of elementary analytic 2-dimensional hyperbolic geometry if one considers the plane which contains \( O \) and is orthogonal to \( \mathcal{H}_{\delta_1} \) and \( \mathcal{H}_{\delta_2} \).

Using the formula (1.1.1)) and "angles" \( \theta(\delta_1), \theta(\delta_2), \theta(\delta_{12}) \) instead of \( \theta_1, \theta_2, \theta_{12} \) for the formula (1.1.15), the proof of Theorem 1.1.3' for a restricted parabolic polyhedron \( \mathcal{M} \) is completely the same as for Theorem 1.1.1' (see the proof of Theorem 1 in [19, Appendix]).

Let us prove the last statement. Let \( K \) be a fundamental polyhedron for the action of \( A(\mathcal{M}) \) on the horosphere \( \mathcal{E}_O \). Let \( C_K \) be the cone with the vertex \( O \) and the base \( K \). Then \( C_K \cap \mathcal{M}(W) \) is a fundamental polyhedron for the semidirect product \( W.A(\mathcal{M}) \) which has finite index in \( O_+(S) \). Then \( C_K \cap \mathcal{M}(W) \) is an elliptic polyhedron. It follows that \( \mathcal{M}(W) \) is a parabolic polyhedron with respect to \( O \). The set of hyperplanes \( \mathcal{H}_\delta, \ delta \in P(\mathcal{M}) \), of faces of the polyhedron \( \mathcal{M}(W) \), which are also hyperplanes of faces of the polyhedron \( C_K \cap \mathcal{M}(W) \), is finite. It follows that \( r(\mathcal{M}(W)) \) is finite, and \( \mathcal{M}(W) \) is restricted parabolic.

This finishes the proof of Theorem 1.1.3'.

Now Theorem 1.1.3 follows from Theorem 1.1.3' (like Theorem 1.1.1 follows from Theorem 1.1.1'). See the proof of Theorem 5.2.1 in [20]. In [20] this is done over an arbitrary appropriate field. Over \( \mathbb{Q} \) the proof is very easy.

This finishes the proof of Theorem 1.1.3.

Ê.B. Vinberg [31] has shown that for an elliptic reflective hyperbolic lattice \( S \) the rank \( \text{rk} S \leq 30 \). F. Esselmann [8] improved this result and has shown that \( \text{rk} S \leq 22 \). This estimate for elliptic reflective hyperbolic lattices is exact. R. Borcherds [1] has proved that the maximal even sublattice of the odd unimodular hyperbolic lattice of the rank 22 is elliptic reflective. Using Vinberg's method one
can prove that the rank of parabolic reflective lattices $S$ is also absolutely bounded. An easy estimate one can get is $\text{rk } S \leq 43$ for parabolic reflective lattices $S$. Here we use existence of the Leech lattice and two different even unimodular positive lattices of the rank 16. But one can expect that the exact estimate here should be $\text{rk } S \leq 26$. J. Conway [7] proved that the even unimodular hyperbolic lattice of the rank 26 is parabolic reflective.

We remark that general results which bound dimension of arbitrary (not necessarily arithmetic) reflection groups of so called parabolic and hyperbolic type in hyperbolic spaces were obtained in [23] (see also [24]). They generalize results of author [20], É. B. Vinberg [31], M. N. Prokhorov [27] and A. G. Khovanskii [14] which bound dimension of elliptic (i.e. with a fundamental polyhedron of finite volume) reflection groups in hyperbolic spaces.

1.2. Twisting coefficients.

For Kac–Moody algebras and generalized Kac–Moody algebras which we consider later, the hyperbolic lattice $S : M \times M \to \mathbb{Z}$ up to similarity $S \mapsto S(m)$, $m \in \mathbb{Q}$, is the invariant of the algebra or of the corresponding generalized Cartan matrix. (We will consider only indecomposable generalized Cartan matrices.) Thus, it is natural to normalize $S$ to be primitive or even primitive.

Remind that a lattice $S$ is even if $S(x, x)$ is even for any $x \in M$. Otherwise, the lattice $S$ is called odd. The lattice $S$ is primitive (respectively, even primitive) if $S(1/m)$ is not a lattice (respectively even lattice) for any natural $m \in \mathbb{N}$ and $m \geq 2$. Difference between these two normalizations is that if $S$ is primitive and odd, then $S(2)$ will be primitive even. In most results below, it does not matter which of these two normalizations is chosen. Thus, below, ”primitive” (lattice) means primitive or even primitive if we don’t say exactly which normalization we choose.

Let $S : M \times M \to \mathbb{Z}$ be a primitive hyperbolic lattice. We fix a reflection group $W \subset W(S)$ and a fundamental polyhedron $\mathcal{M}$ of $W$. Let $P(\mathcal{M})_{pr}$ be the set of primitive elements of $\mathcal{M}$ orthogonal to faces of $\mathcal{M}$. Let $A(\mathcal{M}) \subset O_+(S)$ be the group of symmetries of $\mathcal{M}$ (see (1.1.10)).

**Definition 1.2.1.** A function $\lambda : P(\mathcal{M})_{pr} \to \mathbb{N}$ is called *twisting coefficients function* if

$$S(\lambda(\delta)\delta, \lambda(\delta)\delta)|2S(M, \lambda(\delta)\delta), \text{ for any } \delta \in P(\mathcal{M})_{pr}$$

(1.2.1)

and the subgroup

$$A(P(\mathcal{M})) = \{g \in A(\mathcal{M}) \mid \lambda(g(\delta)) = \lambda(\delta) \text{ for any } \delta \in P(\mathcal{M})_{pr}\} \quad (1.2.1')$$

has finite index in $A(\mathcal{M})$. Number $\lambda(\delta)$ is called a *twisting coefficient* of the $\delta \in P(\mathcal{M})_{pr}$.

Clearly, we can reformulate this definition as follows:

Let us consider a new set $P(\mathcal{M})$ of orthogonal vectors to $\mathcal{M}$:

$$P(\mathcal{M}) = \{\alpha = \lambda(\delta)\delta \mid \delta \in P(\mathcal{M})_{pr}\}.$$

This set $P(\mathcal{M}) \subset M$ is called *acceptable* if

$$S(\alpha, \alpha)|2S(M, \alpha), \text{ for any } \alpha \in P(\mathcal{M})$$

(1.2.2)
and the subgroup
\[ A(P(M)) = \{ g \in A(M) \mid g(P(M)) = P(M) \} \] (1.2.2')

has finite index in \( A(M) \). Obviously, (1.2.1) and (1.2.1') are equivalent to (1.2.2) and (1.2.2'). Thus, an acceptable set \( P(M) \) is equivalent to a twisting coefficient function. If \( P(M) \) is acceptable, then
\[ P(M)/P(M)_{pr} \] (1.2.3)
is a twisting coefficient function.

We need to estimate \( S(\delta, \delta), \lambda(\delta) \) for \( \delta \in P(M)_{pr} \), and \( S(\alpha, \alpha) \) for \( \alpha \in P(M) \) where \( P(M) \) is acceptable.

**Proposition 1.2.2.** Let \( a(S) \) be the exponent of the discriminant group \( A_S = M^*/M \). Let \( \delta \in P(M)_{pr} \), and \( a(\delta) \) be the maximal natural number such that \( \delta/a(\delta) \in M^* \). Then \( a(\delta)|a(S) \), and the \( \lambda(\delta) \) satisfies (1.2.1) if and only if \( \lambda(\delta)S(\delta, \delta)|2a(\delta) \). In particular, \( \lambda(\delta)S(\delta, \delta)|2a(S) \), and for any acceptable \( P(M) \) and \( \alpha \in P(M) \) one has \( S(\alpha, \alpha)|4a(S)^2 \).

**Proof.** This is trivial.

**Remark 1.2.3.** Let \( P(M) \) be an acceptable set above. We denote
\[ M^*_P(M) = \{ x \in M^* \mid S(\alpha, \alpha)|2S(x, \alpha) \text{ for any } \alpha \in P(M) \}. \] (1.2.4)

We consider elements \( h_j \in M^*_P(M) \cap \mathbb{R}_{++}M \) where \( J \) is a countable set. Then the Gram matrix of elements \( P(M) \cup \{ h_j \mid j \in J \} \) defines a generalized Kac–Moody algebra (see [2]) with the set of simple real roots \( P(M) \), set of imaginary simple roots \( h_j \in M^*_P(M), \ j \in J, \) and the Weyl group \( W \). One can consider \( M \) (respectively, \( M^*_P(M) \)) as an “extended root sublattice generated by real simple roots” (respectively, “extended root lattice” (generated by real and imaginary simple roots)) modulo kernel of the canonical symmetric bilinear form. Thus, Proposition 1.2.2 describes all possibilities for the part of this matrix connected with real simple roots.

**1.3. Elliptic and Parabolic reflection groups with the lattice Weyl vector.**

We fix a primitive even elliptic or parabolic reflective lattice \( S : M \times M \to \mathbb{Z} \) and elliptic or parabolic reflection group \( W \subset W(S) \). Let \( M \) be a fundamental polyhedron of \( W \) and \( P(M) \) an acceptable set of orthogonal vectors to \( M \). To be shorter, we name the pair \( (S, P(M)) \) elliptic or parabolic pair respectively.

**Definition 1.3.1.** An element \( \rho \in M \otimes \mathbb{Q} \) is called lattice Weyl vector of \( P(M) \) if
\[ S(\rho, \alpha) = -S(\alpha, \alpha)/2 \text{ for any } \alpha \in P(M). \] (1.3.1)

Evidently, the Weyl vector \( \rho \in [P(M)]^* \subset M \otimes \mathbb{Q} \) if \( \rho \) does exist. Evidently, \( \mathbb{R}_{++}^{++} \subset M \) and \( \rho \) is invariant with respect to the automorphism group \( A(P(M)) \). It follows that \( S(\rho, \rho) < 0 \) if \( W \) is elliptic, and \( \rho \in \mathbb{Q}_{++} \) and \( S(\rho, \rho) = 0 \) if \( W \) is parabolic where \( \rho \) is the cusp. Since \( P(M) \) generates \( M \otimes \mathbb{Q} \) (by Theorems 1.1.'
and 1.1.3'), the Weyl vector $\rho \in M \otimes \mathbb{Q}$ is evidently unique for the fixed subset $P(M) \subset M$.

We want to show that the set of elliptic or parabolic pairs $(S, P(M))$ with the lattice Weyl vector is essentially finite up to isomorphism. Here the pair $(S, P(M))$ is isomorphic to a pair $(S', P'(M'))$ if their exists an isometry $\phi : S \to S'$ of lattices such that $\phi(P(M)) = P'(M')$.

**Theorem 1.3.2.** For $\text{rk } S \geq 3$, the set of elliptic pairs $(S, P(M))$ with a lattice Weyl vector is finite up to isomorphism. Here $S$ is a primitive even elliptic reflective lattice, $W \subset W(S)$ an elliptic (i.e. of finite index in $O(S)$) reflection subgroup, $M$ a fundamental polyhedron of $W$, and $P(M) \subset M$ an acceptable set of all orthogonal vectors to $M$.

**Proof.** We fix one of primitive even elliptic reflective hyperbolic lattices $S : M \times M \to \mathbb{Z}$ of the $\text{rk } S = n + 1 \geq 3$ (we already know that their set is finite). Let $(S, P(M))$ be an acceptable pair with a lattice Weyl vector.

By Theorem 1.1.1' and Proposition 1.2.2, there are elements $\alpha_1, \ldots, \alpha_n \in P(M)$ such that they generate a sublattice $M' \subset M$ of finite index and Gram matrix of these elements has bounded integral coefficients. Thus, there exists only a finite set of possibilities for these Gram matrices.

Let us fix one of possibilities above for the Gram matrix $A = (S(\alpha_i, \alpha_j)), \ 1 \leq i, j \leq n + 1$. We have: $M' \subset M \subset (M')^*$ where the embedding $M' \subset (M')^*$ is defined by the Gram matrix above. Thus, there exists only a finite set of possibilities for the intermediate lattice $M$.

Let us fix one of possibilities above for $M$. Since the lattice $S$ is non-degenerate, there exists the unique element $\rho \in M \otimes \mathbb{Q}$ such that $\rho$ satisfies Definition 1.3.1 for the subset $\{\alpha_1, \ldots, \alpha_n\} \subset P(M)$.

By Proposition 1.2.2, $0 < S(\alpha, \alpha) \leq 4a(S)^2$ is bounded for $\alpha \in P(M)$. It follows that the set

$$\{\alpha \in M \mid 0 > S(\rho, \alpha) = -S(\alpha, \alpha)/2 \geq -2a(S)^2\} \quad (1.3.2)$$

is finite. It follows that we have only finite set of possibilities for the subset $P(M) \subset M$.

It follows Theorem.

For the parabolic case, the finiteness result will be the following.

**Theorem 1.3.3.** For any parabolic pair $(S, P(M))$ with a lattice Weyl vector $\rho$ the group $A(P(M)) \subset O(S)_\rho = \{\phi \in O(S) \mid \phi(\rho) = \rho\}$ has finite index in the Euclidean crystallographic group $O(S)_\rho$.

We fix a constant $D > 0$. Then for $\text{rk } S \geq 3$, the set of parabolic pairs $(S, P(M))$ with a lattice Weyl vector $\rho$ is finite up to isomorphism if index $[O(S)_\rho : A(P(M))] < D$. Here $S$ is a primitive even parabolic reflective lattice, $W \subset W(S)$ a parabolic reflection subgroup with the cusp $\mathbb{R}_{++}c = \mathbb{R}_{++}\rho$, $M$ a fundamental polyhedron of $W$, and $P(M) \subset M$ an acceptable set of orthogonal vectors to $M$.

**Proof.** Let us prove the first statement.

Let $(S, P(M))$ be the parabolic pair corresponding to a parabolic reflection group $W$ with a fundamental polyhedron $M$ and an acceptable set $P(M)$ of vectors orthogonal to $M$, and with a Weyl vector $\rho \in \mathbb{Q}$ where $\rho$ is the cusp of $W$.
Since $\rho \in \mathbb{Q}_{++} c$, we have $A(P(M)) \subset O(S)_c = O(S)_\rho$ and $A(P(M))$ has finite index in $O(S)/W$. Let $K$ be a fundamental domain of $A(P(M))$ on the horosphere $\mathcal{E}_O$ where $O = \mathbb{R}_{++} c$, and $C_K$ is the cone with the vertex $O$ and the base $K$. Then $C_K \cap M$ is a fundamental domain of finite volume for $W.A(P(M))$ in $\mathcal{E}(S)$. Since $\rho \in \mathbb{Q}_{++} c$ is Weyl vector, then $S(\alpha, \alpha) < 0$ for any $\alpha \in P(M)$. Thus, there does not exist a face of $M$ which contains the cusp $O$; on the other hand, $O$ belongs to $M$. It follows that the fundamental domain $K$ on the horosphere $\mathcal{E}_O$ has finite volume. Since $O(S)_c$ is discrete in $\mathcal{E}_O$, it follows that $A((P(M))$ has finite index in $O(S)_c$. This proves the first statement.

Let us prove second statement. We are arguing like for the proof of Theorem 1.3.2 (using Theorems 1.1.3 and 1.1.3' instead of 1.1.1 and 1.1.1'). But at the end of the proof we should use that number of subgroups $A \subset O(S)_\rho$ is finite if index $[O(S)_\rho : A] < D$, and replace the set (1.3.2) by the set

$$\{\alpha \in M \mid 0 > S(\rho, \alpha) = -S(\alpha, \alpha)/2 \geq -2a(S)^2 \text{ and } \mathcal{H}_{\alpha} \cap C_K \neq \emptyset\}. \quad (1.3.3)$$

Here (like above) $K$ is a fundamental domain for $A(P(M))$ on the horosphere $\mathcal{E}_O$, $O = \mathbb{R}_{++} \rho$, and $\mathcal{H}_{\alpha}$ is the hyperplane which is orthogonal to $\alpha \in M$.

This finishes the proof.

The next example demonstrates that the condition $[O(S)_\rho : A(P(M))] < D$ for index is essential in Theorem 1.3.3.

**Example 1.3.4.** Let us consider an even primitive hyperbolic lattice $M = \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3$ of the rank 3 where

$$S(\delta_i, \delta_j) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

Here $\{\delta_1, \delta_2, \delta_3\} = P(\Delta)_{pr}$ where $\Delta$ is a fundamental triangle with zero angles (i.e, it has 3 vertices at infinity) of the reflection group $W(2)(S)$ generated by reflections in all elements $\delta \in M$ such that $S(\delta, \delta) = 2$. This is one of examples of 2-reflective lattices of the rank 3 which were described in [18].

Let $O = \mathcal{H}_{\delta_2} \cap \mathcal{H}_{\delta_3}$. Here $O = \mathbb{R}_{++} c$ where $c = \delta_2 + \delta_3 \in M$. We have $c^2 = 0$, $S(c, \delta_2) = S(c, \delta_3) = S(c, \delta_1) = -4$.

Denote

$$\rho = c/4, \ e_0 = \delta_1, \ f_{01} = 2s_{\delta_1}(\delta_2), \ f_{02} = 2s_{\delta_1}(\delta_3).$$

One can easily check that

$$S(\alpha, \alpha) \mid 2S(M, \alpha), \ S(\rho, \alpha) = -S(\alpha, \alpha)/2 \quad (1.3.4)$$

for $\alpha = e_0, \ f_{01}, \ f_{02}$. Here $S(e_0, e_0) = 2$, and $S(f_{01}, f_{01}) = S(f_{02}, f_{02}) = 8$.

Let us consider $\phi \in O(S)_{\rho}$ such that

$$\phi(\delta_2) = \delta_3, \phi(\delta_3) = s_{\delta_3}(\delta_2), \phi(\delta_1) = s_{\delta_1}(\delta_1).$$

The $\phi$ is the parallel translation on the horosphere $\mathcal{E}_O$ which sends the line $\mathcal{H}_{\delta_2}$ to the line $\mathcal{H}_{\delta_3}$ and the triangle $\Delta$ to the triangle $\phi(\Delta)$ which has the same vertex $O$ and the same side $\mathcal{H}_{\delta_1}$ with the triangle $\Delta$. 

For \( k \in \mathbb{N} \), we define an infinite fundamental polygon \( \mathcal{M}_k \) for a reflection subgroup \( W_k \subset W^{(2)}(S) \) with the acceptable set of vectors \( P(\mathcal{M}_k) \) as follows:

\[
P(\mathcal{M}_k) = \{ \phi^t(e_0) | t \in \mathbb{Z} \text{ and } t \not\equiv 0 \mod k \} \cup \{ \phi^t(f_{01}), \phi^t(f_{02}) | t \equiv 0 \mod k \}.
\]

Clearly, \( P(\mathcal{M}_k) \) is an infinite polygon with zero angles. It follows that \( \mathcal{M}_k \) is a fundamental polygon for a subgroup \( W_k \subset W(S) \) generated by reflections in all elements of \( P(\mathcal{M}_k) \). By our construction, \( \phi^k \in A(P(\mathcal{M}_k)) \) generates a subgroup of finite index in \( O_\rho(S) \). Since (1.3.4), the set \( P(\mathcal{M}_k) \) is acceptable and \( \rho \) is the Weyl vector for \( P(\mathcal{M}_k) \). It follows that \( W_k \) is a parabolic reflection group with the cusp \( c \).

Obviously, all pairs \( (S, P(\mathcal{M}_k)) \) are different because elements \( \alpha \in P(\mathcal{M}_k) \) have square \( S(\alpha, \alpha) = 2 \) or \( S(\alpha, \alpha) = 8 \), and exactly \( k - 1 \) consecutive sides of \( \mathcal{M}_k \) have orthogonal vectors from \( P(\mathcal{M}_k) \) with the square 2.

### 1.4. Reflection groups of arithmetic type.

We consider hyperbolic lattices \( S : M \times M \to \mathbb{Z} \) and reflection groups \( W \subset W(S) \). Let \( \mathcal{M} \) be a fundamental polyhedron of \( W \) and \( P(\mathcal{M}) \) an acceptable set of orthogonal vectors to \( \mathcal{M} \). We want to define a class of these groups which is interesting from the viewpoint of Kac–Moody algebras. From the point of view of corresponding Kac–Moody algebras, the next definition means that imaginary roots "behave very nice" (see Sect. 2.2 below).

**Definition 1.4.1.** Consider an integral cone (semi-group)

\[
Q_+ = \sum_{\alpha \in P(\mathcal{M})} \mathbb{Z}_+ \alpha \subset M
\]

and the corresponding integral dual cone

\[
Q_+^* = \{ x \in M | S(x, P(\mathcal{M})) \leq 0 \} \subset M.
\]

The group \( W \) has **arithmetic type** if

\[
V_+^+(S) \cap (M \otimes \mathbb{Q}) \subset \mathbb{Q}_+Q_+.
\]

Equivalently, this means that for any \( x \in M \) with \( S(x, x) < 0 \) there exist \( n \in \mathbb{N} \) and \( a(\delta) \in \mathbb{Z}_+ \), \( \delta \in P(\mathcal{M}) \), which are almost all equal to 0 (i.e., only finite set is not equal to zero), such that

\[
nx = \pm \sum_{\alpha \in P(\mathcal{M})} a(\alpha) \alpha.
\]

Since the cone \( \overline{V_+^+(S)} \) is self-dual, i.e., \( \overline{V_+^+(S)}^* = \overline{V_+^+(S)} \), the reflection group \( W \subset W(S) \) has arithmetic type if and only if

\[
Q_+^* = \{ x \in M | S(x, P(\mathcal{M})) \leq 0 \} \subset M \cap \overline{V_+^+(S)}.
\]

Since the fundamental polyhedron of \( W \)

\[
\mathcal{M} = (\mathbb{R} - Q_+^* \cap \overline{V_+^+(S)}) / \mathbb{R} \quad \subset \quad \mathcal{F}(S)
\]

(1.4.6)
is locally finitely generated and \( M \otimes \mathbb{Q} \) is everywhere dense in \( M \otimes \mathbb{R} \), we can reformulate this definitions using real cones.

Consider the cone
\[
\mathbb{R}_+ \mathbb{Q}_+ = \sum_{\alpha \in P(M)} \mathbb{R}_+ \alpha \subset M \otimes \mathbb{R},
\]
and the corresponding dual cone
\[
(\mathbb{R}_+ \mathbb{Q}_+)^* = \{ x \in M \otimes \mathbb{R} \mid S(x, P(M)) \leq 0 \}.
\]
Then \( W \subset W(S) \) has arithmetic type if and only if
\[
\overline{V^+(S)} \subset \mathbb{R}_+ \mathbb{Q}_+,
\]
equivalently,
\[
(\mathbb{R}_+ \mathbb{Q}_+)^* \subset \overline{V^+(S)}.
\]
Thus, (1.4.3), (1.4.4), (1.4.5), (1.4.9), (1.4.10) are equivalent definitions for \( W \) to have arithmetic type. Obviously, this definition does not depend on a choice of an acceptable set \( P(M) \) of orthogonal vectors to \( M \).

One should consider "arithmetic type" as a very weakened condition of finiteness of volume for fundamental polyhedron of a reflection group. Another explanation why reflection groups of arithmetic type are important is that they are maximal: There does not exist a reflection group \( W' \subset W(S) \) with a fundamental polyhedron \( \mathcal{M}' \) such that \( W \subset W' \) and \( P(M) \subset P(M') \) and \( W \neq W' \) (equivalently, \( P(M) \neq P(M') \)). This follows from (1.4.5).

In spite of Definition 1.4.1 is very natural, it seems that it is too general (gives too "wild" groups in general). Therefore, we define more narrow class of reflection groups \( W \) which have "better automorphic properties" for the denominator formula of the corresponding Kac–Moody algebras. See \S 2.

**Definition 1.4.2.** A reflection group \( W \) has restricted arithmetic type if \( W \) has one reflection at least and for an acceptable set \( P(M) \) of orthogonal vectors to \( M \) the group of symmetries \( A(P(M)) \) has finite index in \( O(S)/W(S) \) (more exactly, this means that the corresponding semi-direct product \( W.A(P(M)) \) has finite index in \( O(S) \)).

We have the following result which shows that restricted arithmetic type implies arithmetic type and gives many examples of groups \( W \) of arithmetic type.

**Theorem 1.4.3.** Let \( S \) be a hyperbolic lattice, \( W \subset W(S) \) reflection group which contains one reflection at least, \( \mathcal{M} \) a fundamental polyhedron of \( W \) and \( P(M) \) an acceptable set of orthogonal vectors to \( M \) (e.g. one can take \( P(M)_{pr} \)). Assume that the group of symmetries \( A(P(M)) \) has finite index in \( O(S)/W \) (i.e. \( W.A(P(M)) \) has finite index in \( O(S) \)). In other words, \( W \) has restricted arithmetic type. Then \( W \) has arithmetic type (i.e. equivalent properties (1.4.3), (1.4.4), (1.4.5), (1.4.9), (1.4.10) are valid).

**Proof.** Assume that \( W \) does not have arithmetic type. Then (1.4.10) is not valid. Thus, there exists an element \( q \in \mathbb{R}_+ \mathbb{Q}^* \) such that \( S(q, q) > 0 \). We have \( \mathbb{R}_+ \mathcal{M} \subset \mathbb{R}_+ \mathbb{Q}_+ \), but
\[
(\mathbb{R}_+ \mathbb{Q}_+)^* \not\subset \overline{V^+(S)}.
\]
$\mathbb{R}_+Q_+^*$ where $\mathcal{M}$ is the fundamental polyhedron of $W$ in $\mathcal{L}(S)$ which is locally finite in $\mathcal{L}(S)$ and is non-degenerate. Here, $\mathbb{R}_+\mathcal{M} = (\mathbb{R}_+Q_+^*) \cap V^+(S)$. Considering intersection of the convex envelope of $\mathbb{R}_+\mathcal{M}$ and $\mathbb{R}_+q$ with $V^+(S)_\infty$, one can see that there exists a non-empty open in $L(S)$ subset $U \subset \mathcal{M}$ such that $U \cap \mathcal{L}(S)_\infty$ is a non-empty open subset of $\mathcal{L}(S)_\infty$. In particular, for any $\alpha \in P(\mathcal{M})$ the intersection $\mathcal{H}_\alpha \cap U = \emptyset$.

Since $W.A(P(\mathcal{M}))$ has finite index in $O(S)$, a fundamental domain $\mathcal{D}$ for the action of $A(P(\mathcal{M}))$ in $\mathcal{M}$ is a fundamental domain for $W.A(P(\mathcal{M}))$ in $\mathcal{L}(S)$. It follows that $\mathcal{D}$ is a convex polyhedron in $\mathcal{L}(S)$ which is a convex envelope of a finite set of points in $\mathcal{L}(S)$ and at infinity of $\mathcal{L}(S)$. Since $W$ has one reflection at least, there exists a codimension one face of $\mathcal{D}$ which is bounded by a hyperplane $\mathcal{H}_\alpha$, $\alpha \in P(\mathcal{M})$. We consider two cases.

Suppose that $\mathcal{D}$ is bounded. One can easily see that there exists $g \in A(P(\mathcal{M}))$ such that $g(\mathcal{D}) \subset U$. Then $\mathcal{H}_{g(\alpha)} \cap U$ is non-empty where $g(\alpha) \in P(\mathcal{M})$. We get the contradiction.

Now assume that $\mathcal{D}$ has an infinite vertex. It is well-known that this vertex is $\mathbb{R}_+c$ where $0 \neq c \in M \cap V^+(S)$ and $S(c, c) = 0$. Besides, it is known that infinite points $\mathbb{R}_+g(c)$, $g \in W.A(\mathcal{M})$, are everywhere dense in $\mathcal{L}(S)_\infty$. In particular, there exists $g \in A(\mathcal{M})$ such that $O = \mathbb{R}_+g(c) \subset U \cap \mathcal{L}(S)_\infty$, because $U \subset \mathcal{M}$. Then $g(\mathcal{D})$ is a fundamental domain of $A(P(\mathcal{M}))$ in $\mathcal{M}$ too. By our construction, hyperplanes of all codimension one faces of $g(\mathcal{D})$ which contain $O = \mathbb{R}_+g(c)$, are different from $\mathcal{H}_\alpha$, $\alpha \in P(\mathcal{M})$. It follows that the stabilizer subgroup $A(P(\mathcal{M}))_{g(c)}$ of $g(c)$ has finite index in $O(S)_{g(c)}$ and has a fundamental domain of finite volume on the horosphere $\mathcal{E}_O$. Then hyperplanes $\mathcal{H}_{f(g(\alpha))}$ where $f \in A(P(\mathcal{M}))_{g(c)}$, are everywhere dense in $U \cap \mathcal{L}(S)_\infty$. We again get a contradiction.

This finishes the proof.

Remark. Theorem 1.4.3 may be considered as a particular case of some general results about limit sets of discrete groups in hyperbolic spaces and their normal subgroups. See L. Greenberg [9] and review [34].

One easily can construct many examples of reflection groups of restricted arithmetic type and hence, by Theorem 1.4.3, of arithmetic type. The next construction is inspired by studying of automorphism groups of Enriques surfaces in [21], [22].

Definition 1.4.4. Let us consider a finite symmetric bilinear (or an appropriate quadratic) form $f : A \times A \to \mathbb{Z}/d$ where $A$ is a finite abelian group and $d \in \mathbb{N}$. A subset $\overline{A} \subset A$ is called a finite root system in $(f, A)$ if there exists an integral lattice $T : N \times N \to \mathbb{Z}$, its sublattice $N_1 \subset N$ of finite index, its reflection subgroup $W \subset W(T)$ and an identification

$$\left(\frac{N/N_1}{T \mod N_1}\right) = (A, f) \quad (1.4.11)$$

such that

$$\overline{A} = \{\delta \mod N_1 \mid s_\delta \in W \text{ and } \delta \text{ is primitive in } N\}. \quad (1.4.12)$$

It would be interesting to give a reasonable description of finite root systems.

Now let us consider a hyperbolic lattice $S : M \times M \to \mathbb{Z}$, its sublattice $M_1 \subset M$ of finite index and a finite root system $\overline{A}$ in $(S \mod M_1, M/M_1)$. Let

$$W(\overline{A}) \subset W(S) \quad (1.4.13)$$
be a reflection subgroup generated by all reflections $s_\delta$, where $\delta \in M$ is primitive and $\delta \mod M_1 \in \Delta$.

Since $\Delta$ is finite, using Theorem 1.4.3, we have

**Corollary 1.4.5.** The reflection subgroup $W(\Delta)$ has restricted arithmetic type (and hence, arithmetic type) if $W(\Delta)$ has one reflection at least.

By Proposition 1.2.2, for a reflection $s_\delta$ with a primitive $\delta$ the square $S(\delta, \delta)$ is bounded. Thus, using Corollary 1.4.5, for an appropriate finite root system $\Delta$, we formally get the most useful for applications following statement 1.4.6 below. On the other hand, this statement follows from Theorem 1.4.3 directly.

**Corollary 1.4.6.** Let us fix a subset $\{d_1, ..., d_k\} \subset N$, and a subgroup $W \subset W(S)$ is generated by all reflections in primitive elements $\delta \in M$ such that $S(\delta, \delta) \in \{d_1, ..., d_k\}$. Then $W$ has restricted arithmetic type (and hence, arithmetic type) if it contains one reflection at least.

For $S(\delta, \delta) = 2$, this statement was proved in [15].

The next statement shows that any finitely generated reflection group of a hyperbolic lattice is contained in a reflection group of restricted arithmetic type of the same lattice. Its interpretation for Kac–Moody algebras see in §2.

**Theorem 1.4.7.** Let $S : M \times M \to \mathbb{Z}$ be a hyperbolic lattice and $W \subset W(S)$ a finitely generated reflection subgroup with a fundamental polyhedron $M$ and finite acceptable set $P(M)$ of vectors orthogonal to $M$. Then $W$ is a reflection subgroup $W \subset W' \subset W(S)$ of a reflection group $W'$ of restricted arithmetic type with a fundamental polyhedron $M' \subset M$ and an acceptable set $P(M')$ containing $P(M)$.

**Proof.** Using Proposition 1.2.2, it is sufficient to prove Theorem for primitive orthogonal vectors $P(M)_{pr}$ and $P(M')_{pr}$.

For a reflection subgroup $\widetilde{W} \subset W(S)$ we denote by

$$\Delta(\widetilde{W}) = \{\text{primitive } \delta \in M \mid s_\delta \in \widetilde{W}\} \quad (1.4.14)$$

the set of all primitive roots of the reflection group $\widetilde{W}$.

Let $h \in M$, $\mathbb{R}_{++} h \in V^+(S)$ and $0 \not\in S(h, \Delta(\widetilde{W}))$. One has the following Vinberg’s algorithm [30] for calculation of the fundamental polyhedron $\widetilde{M}$ of $\widetilde{W}$ (equivalently, the set $P(\widetilde{M})_{pr}$) which contains $\mathbb{R}_{++} h$ (equivalently, $S(h, P(\widetilde{M})_{pr}) < 0$).

One defines the height

$$\text{height}(\delta) = -2S(h, \delta)/\sqrt{S(\delta, \delta)}, \quad \delta \in \Delta(\widetilde{W}). \quad (1.4.15)$$

Geometrically, the height is equivalent to the distance between the point $\mathbb{R}_{++} h$ and the hyperplane $H_\delta$ in the hyperbolic space $L(S)$. According to É.B. Vinberg,

$$P(\widetilde{M})_{pr} = \bigcup_{n \in \mathbb{N}} P(\widetilde{M})_{pr}^{(n)} \quad (1.4.16)$$

where

$$P(\widetilde{M})_{pr}^{(n)} = \{\delta \in \Delta(\widetilde{W}) \mid \text{height}(\delta) = \sqrt{n} \}$$

and $S(\delta, P(\widetilde{M})_{pr}^{(m)}) \leq 0$ for all $1 \leq m < n$. \quad (1.4.17)
It is true also that the set
\[
\{ \delta \in \Delta(W(S)) \mid 0 \leq \text{height}(\delta) \leq C \}
\]
is finite for the fixed \( C > 0 \).

Now let us prove Theorem. We fix \( h \in M \) such that \( \mathbb{R}_{++} h \in M \) and \( S(h, P(M)_{\text{pr}}) < 0 \) (i.e. \( \mathbb{R}_{++} h \) is inside \( M \)). We use this \( h \) to define the height. Let \( N \) be the largest height of elements from the finite set \( P(M)_{\text{pr}} \).

We consider a sublattice \( M_1 \subset M \) of finite index and the finite symmetric bilinear form
\[
(S \mod M_1, M/M_1)
\]
with the finite root system
\[
\overline{\Delta(W)} = \Delta(W) \mod M_1
\]
such that the set
\[
\{ \delta \in \Delta(W(S)) \mid 0 \leq \text{height}(\delta) \leq N \text{ and } \delta \mod M_1 \in \overline{\Delta(W)} \}
\]
is equal to the set
\[
\{ \delta \in \Delta(W) \mid 0 \leq \text{height}(\delta) \leq N \}.\]

Since the set (1.4.18) is finite, we always can satisfy this equality for a sufficiently deep sublattice \( M_1 \subset M \). Let us apply Vinberg’s algorithm for calculation of \( P(M)_{\text{pr}} \) and the fundamental polyhedron \( M_1 \) (equivalently, \( P(M_1)_{\text{pr}} \)) of the reflection group \( W(\overline{\Delta(W)}) \). Since the equality above, we evidently have that \( P(M_1)_{\text{pr}}(n) = P(M)_{\text{pr}}(n) \) for all \( 0 \leq n \leq N \). It follows that \( P(M)_{\text{pr}} \subset P(M_1)_{\text{pr}} \).

By Corollary 1.4.5, the reflection group \( W(\overline{\Delta(W)}) \) has restricted arithmetic type because it contains the non-trivial reflection subgroup \( W \).

This finishes the proof.

Using the same idea, one can prove the following more general statement:

**Theorem 1.4.8.** Let \( S : M \times M \to \mathbb{Z} \) be a hyperbolic lattice and \( W \subset W(S) \) a finitely generated reflection subgroup with a fundamental polyhedron \( M \) and finite acceptable set \( P(M) \) of vectors orthogonal to \( M \). Let \( h_1, \ldots, h_k \in (M - \{0\}) \cap \mathbb{R}_{++} M \) (equivalently, \( h_i \in (M - \{0\}) \cap V^+(S) \)) and \( S(h_i, P(M)) \leq 0 \), \( i = 1, \ldots, k \). Then \( W \) is a reflection subgroup \( W \subset W' \subset W(S) \) of a reflection group \( W' \) of restricted arithmetic type with a fundamental polyhedron \( M' \subset M \) and an acceptable set \( P(M') \) which contains \( P(M) \). Moreover, \( h_1, \ldots, h_k \in (M - \{0\}) \cap \mathbb{R}_{++} M' \).

At the end of this section, we consider relation between elliptic and parabolic reflection groups and reflection groups of arithmetic and restricted arithmetic type.

We have the following natural generalization of Definition 1.3.1.

**Definition 1.4.9.** Let \( S : M \times M \to \mathbb{Z} \) be a hyperbolic lattice and \( W \subset W(S) \) a reflection group with a fundamental polyhedron \( M \) and an acceptable set \( P(M) \) of vectors orthogonal to \( M \).

An element \( 0 \neq \rho \in M \) is called a **generalized lattice Weyl vector** of \( P(M) \) if there exists \( C > 0 \) such that
\[
0 \leq S(\rho, \omega) \leq C \text{ for any } \omega \in P(M).
\]
An element $\rho \in M \otimes \mathbb{Q}$ is called a \textit{lattice Weyl vector} of $P(M)$ if

$$S(\rho, \alpha) = -S(\alpha, \alpha)/2 \text{ for any } \alpha \in P(M). \quad (1.4.21)$$

Clearly, some multiple of a lattice Weyl vector gives a generalized lattice Weyl vector. Clearly, existence of a generalized lattice Weyl vector does not depend from a choice of an acceptable set $P(M)$ of vectors orthogonal to $M$.

We have the following statements which mean that elliptic reflection groups are exactly reflection groups of arithmetic type with a generalized lattice Weyl vector $\rho$ which has negative square $S(\rho, \rho)$. And parabolic reflection groups are exactly reflection groups which have restricted arithmetic type and have a generalized lattice Weyl vector with zero square $S(\rho, \rho) = 0$ and do not have a generalized lattice Weyl vector with negative square.

\textbf{Proposition 1.4.10.} Any elliptic or parabolic reflection group $W \subset W(S)$ of a hyperbolic lattice $S$ has restricted arithmetic type (and by Theorem 1.4.3, it has arithmetic type). If $W$ is elliptic, it has a generalized lattice Weyl vector $\rho$ with $S(\rho, \rho) < 0$. If $W$ is parabolic, it has a generalized lattice Weyl vector $\rho$ with $S(\rho, \rho) = 0$ and does not have a generalized lattice Weyl vector with negative square.

\textit{Proof.} This group has restricted arithmetic type by definition. By Theorem 1.4.3, it has arithmetic type. Let $M$ be a fundamental polyhedron of $W$. If $W$ is elliptic, every element $0 \neq \rho \in M$ such that $\mathbb{R}_{++} \rho \in \text{Int } M$ (equivalently, $S(\rho, P(M)) < 0$) is a generalized lattice Weyl vector. If $W$ is parabolic, the cusp $c$ is a generalized lattice Weyl vector with $S(c, c) = 0$. Below we prove that $W$ does not have a generalized lattice Weyl vector with negative square.

\textbf{Proposition 1.4.11.} Let $S$ be a hyperbolic lattice and $W \subset W(S)$ a reflection group of arithmetic type, $M$ a fundamental polyhedron of $W$, and $P(M)$ an acceptable set of orthogonal vectors to $M$. Assume that $P(M)$ has a generalized lattice Weyl vector $\rho$.

Then $S(\rho, \rho) \leq 0$. If additionally $S(\rho, \rho) < 0$, then $W$ is elliptic; if $S(\rho, \rho) = 0$ and $W$ has restricted arithmetic type, then $W$ is either elliptic or parabolic.

\textit{Proof.} By definition of generalized lattice Weyl vector $\rho$, we have $S(\rho, P(M)) \leq 0$. Thus, $\rho \in Q_+^*$ (see Definition 1.4.1). By (1.4.5), $\rho \in \hat{V}^+(S)$. It follows that $S(\rho, \rho) \leq 0$ and $\mathbb{R}_{++} \rho \in M$.

Assume that $S(\rho, \rho) < 0$. By definition of generalized lattice Weyl vector, it then follows that $P(M)$ is finite (since the lattice $S$ is hyperbolic). By (1.4.6) and (1.4.10), the fundamental polyhedron $M = (\mathbb{R}^+Q_+^*)/\mathbb{R}_{++}$ is a convex envelope of a finite set of points of $L(S)$ and $L(S)_\infty$. Thus, $M$ has finite volume and $W$ is elliptic.

Now assume that $S(\rho, \rho) = 0$ and $W$ has restricted arithmetic type. If $A(M)$ is finite, then $W \subset O(S)$ has finite index and $W$ is elliptic. Assume that $A(M)$ is infinite. Obviously, $g(\rho)$ is also a generalized lattice Weyl vector for any $g \in A(M)$. If $g(\rho) \neq \rho$, it follows that $P(M)$ is finite. Since $W$ has arithmetic type, like above, one can prove that $W$ is elliptic and $A(M)$ is infinite. We get a contradiction. Thus, we have proven that $g(\rho) = \rho$ for any $g \in A(M)$. Thus, $\rho$ is the cusp for $A(M)$. Since $W$ has restricted arithmetic type, $WA(M)$ has finite index in $O(S)$. It follows that $W$ is parabolic.
Remark 1.4.12. Modifying Example 1.3.4, one can construct an example of a reflexion group $W$ of arithmetic type and with a lattice Weyl vector $\rho$ with $S(\rho, \rho) = 0$ such that this group does not have restricted arithmetic type and is not then parabolic.

§2. Denominator formula for Lorentzian Kac–Moody algebras

We restrict considering Kac–Moody algebras, but similarly one can consider generalized Kac–Moody algebras (see Remark 1.2.3 and Theorem 1.4.8) and, it seems, appropriate generalized Kac–Moody superalgebras.

2.1. Basic definitions on Lorentzian Kac–Moody algebras.

We refer to V. Kac [13] and R. Borcherds [2] for basic definitions on Kac–Moody algebras.

We fix an even primitive hyperbolic lattice $S : M \times M \to \mathbb{Z}$, a finitely generated reflection group $W \subset W(S)$, its fundamental polyhedron $\mathcal{M}$ and an acceptable set $P(\mathcal{M})$ of vectors orthogonal to $\mathcal{M}$. Since $W$ is finitely generated, the set $P(\mathcal{M})$ is finite. We assume that $P(\mathcal{M})$ generates a sublattice of $M$ of finite index and Gram graph of $P(\mathcal{M})$ is connected. The last means that one cannot divide $P(\mathcal{M})$ on two orthogonal non-empty subsets. We require this for not to consider finite-dimensional and affine Lie algebras.

Let us consider a finite matrix

$$A = (a_{ij}) = \left( \frac{2S(\alpha_i, \alpha_j)}{S(\alpha_i, \alpha_i)} \right), \quad \alpha_i, \alpha_j \in P(\mathcal{M}). \quad (2.1.1)$$

Since $\mathcal{M}$ is a fundamental polyhedron of a reflection group and $P(\mathcal{M})$ is acceptable, one has that $a_{ii} = 2$, $a_{ij} \leq 0$ are integral for $i \neq j$, and $a_{ij}, a_{ji}$ are equal to 0 simultaneously. Matrices $(a_{ij})$ with these properties are called generalized Cartan matrices. The generalized Cartan matrix $(a_{ij})$ defined in (2.1.1) has special type. First, it is symmetrizable which means that

$$A = DB \quad (2.1.2)$$

where $D$ is diagonal with positive rational coefficients and $B$ is symmetric. Evidently, for our case,

$$D = \text{diag} \ (..., 2/S(\alpha_i, \alpha_i), ...) \quad (2.1.3)$$

and

$$B = (b_{ij}) = (S(\alpha_i, \alpha_j)) \quad (2.1.4)$$

is the Gram matrix of $P(\mathcal{M})$. Since the Gram graph defined by $B$ is connected, the matrix $A$ is indecomposable. Besides, for our case, the matrix $B$ is hyperbolic which means that it has exactly one negative square (and several positive and zero squares).

A generalized Cartan matrices with properties above is called indecomposable Lorentzian symmetrizable. One can easily see that by the construction above, any indecomposable Lorentzian symmetrizable generalized Cartan matrix $A$ defines the primitive even hyperbolic lattice $S, W \subset W(S)$ and $P(\mathcal{M})$ canonically up to isomorphism and a finite set of possibilities. If one requires that $P(\mathcal{M})$ generates $M$, then $S, W, P(\mathcal{M})$ are defined uniquely up to isomorphism.
Let $P(M) = \{\alpha_1, ..., \alpha_N\}$. The generalized Cartan matrix $A$ above defines the Kac–Moody algebra $\mathfrak{g}'(A)$ over $\mathbb{C}$. Lie algebra $\mathfrak{g}'(A)$ is defined by $3N$ generators $e_1, ..., e_N, f_1, ..., f_N, h_1, ..., h_N$ and defining relations

\[
[h_i, h_j] = 0, \ [e_i, f_j] = \delta_{ij} h_j, \ [h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j,
\]

\[(\text{ad } e_i)^{1-a_{ij}} e_j = 0, \ (\text{ad } f_i)^{1-a_{ij}} f_j = 0, \text{ if } i \neq j,
\]

for all $1 \leq i, j \leq N$.

The basic property of this Kac–Moody algebra is that $\mathfrak{g}'(A)$ is simple and graded by the lattice $M$. Here $\mathfrak{t}$ is the center of $\mathfrak{g}'(A)$; $\dim \mathfrak{g}_\alpha < \infty$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha + \beta}$; $\mathfrak{g}_0 = \mathbb{C} h_1 + ... + \mathbb{C} h_N = M^* \otimes \mathbb{C}$ where $< h_i, \alpha_j > = a_{ij}$; an element $X \in \mathfrak{g}_\alpha$ iff $[h, X] = < h, \alpha > X$, for any $h \in \mathfrak{g}_0$. The form $S$ may be extended canonically to $\mathfrak{g}''(A)$ to be an invariant symmetric bilinear form. It is called canonical symmetric bilinear form.

The part $\mathfrak{g}_0$ is called Cartan subalgebra of $\mathfrak{g}''(A)$. An element $0 \neq \alpha \in M$ is called root if the multiplicity $\text{mult}(\alpha) = \dim \mathfrak{g}_\alpha > 0$. Let $\Delta$ be the set of all roots. Roots and their multiplicities are invariant with respect to $W$ which is called Weyl group.

One has the following description of the set of roots $\Delta$ according to V. Kac [13]. A root $\alpha \in M$ is called real if $S(\alpha, \alpha) > 0$. A root $\alpha \in M$ is called imaginary if $S(\alpha, \alpha) \leq 0$. The set of real roots is denoted by $\Delta^r$, the set of imaginary roots is denoted by $\Delta^i$. Let $Q_+ = \mathbb{Z}_+ \alpha_1 + ... + \mathbb{Z}_+ \alpha_N$ be an integral cone generated by simple real roots (we have defined and used this cone in (1.4.1)). We have $\Delta = \Delta_+ \cup \Delta_-$ where $\Delta_+ = \Delta \cap Q_+$, $\Delta_- = -\Delta_+$. Similarly one defines $\Delta^r_+, \Delta^i_+$ and $\Delta^r_-, \Delta^i_-$. Obviously, a root $\alpha \in \Delta_+$ if and only if $S(\alpha, h) \leq 0$ for any $\mathbb{R}^+ h \in M$. An imaginary root $\alpha \in \Delta_+$ if and only if $\alpha \in V^+(S)$. Elements $\{\alpha_1, ..., \alpha_N\} = P(M)$ are real roots. They are called simple real roots and have multiplicity one. We have

\[
\Delta^r = W(P(M)).
\]

Let

\[
Q_+^\ast = \{x \in M \mid S(x, P(M)) \leq 0\}
\]

be the integral cone which is dual to $Q_+$, and

\[
K = Q_+ \cap Q_+^\ast.
\]

By V. Kac [13] (see also considerations in [25]), one has

\[
\Delta^i_+ = W(K).
\]

Let us consider the corresponding real cones

\[
\mathbb{R}_+ Q_+, (\mathbb{R}_+ Q_+)^\ast \text{ and } \mathbb{R}_+ K = (\mathbb{R}_+ Q_+) \cap (\mathbb{R}_+ Q_+)^\ast.
\]

We have

\[
\mathbb{R}_+ K \subset \mathbb{R}_+ M.
\]
The cone
\[ T^* = W(\mathbb{R}^+K) = W(\mathbb{R}^+Q_+ \cap (\mathbb{R}^+Q_+)^*) \subset \overline{V^+(S)} \] (2.1.13)
is called *dual Tits cone*. The cone \( T = (T^*)^* \) is called *Tits cone*. Obviously,
\[ \Delta^\text{im}_+ = Q_+ \cap T^*. \] (2.1.14)

We have the following Weyl–Kac denominator formula:
\[
\Phi(z) =: \sum_{w \in W} \det(w)e^{-2\pi i S(w(\rho),z)} = e^{-2\pi i S(\rho,z)} \prod_{\alpha \in \Delta_+} (1 - e^{-2\pi i S(\alpha,z)})^\text{mult}(\alpha). \] (2.1.15)

Here \( \rho \in (\oplus \mathbb{Z} \alpha_i)^* \) is any element which satisfies the equality
\[ <\rho, \alpha_i> = -S(\alpha_i, \alpha_i)/2 \text{ for any } \alpha_i \in P(M). \] (2.1.16)

It is called *Weyl vector*. The variable
\[ z \in M \otimes \mathbb{C}. \] (2.1.17)

The domain \( \Omega(S) = M \otimes \mathbb{R} + iV^+(S) \subset M \otimes \mathbb{C} \) is called the *complexified cone* \( V^+(S) \).

It is known (by V. Kac [13]) that \( \Phi(z) \) converges absolutely in the complexified Tits cone
\[ \Omega(S,W) = M \otimes \mathbb{R} + i\text{Int} \, T \supset \Omega(S) \] (2.1.18)
if \( \text{im } z > 0 \), and diverges absolutely in \( S \otimes \mathbb{C} - \overline{\Omega(S,W)} \). Thus, the domain \( \Omega(S,W) \) where the function \( \Phi(z) \) converges always contains the natural domain \( \Omega(S) \).

We denote the generalized Cartan matrix \( A \) in (2.1.1) as \( A(S,W,P(M)) \) and corresponding Kac–Moody algebras \( g'(A) \) and \( g''(A) \) as \( g'(A(S,W,P(M))) \) and \( g''(A(S,W,P(M))) \). They are also called Lorentzian Kac–Moody algebras.

One can see that Lorentzian Kac–Moody algebras \( g''(A(S,W,P(M))) \) have bad properties in general.

(-a) Rational cone \( Q_+ \Delta^\text{im}_+ \) generated by imaginary roots gives only a part of the natural cone \( (M \otimes \mathbb{Q}) \cap V^+(S) \).

(-b) The denominator function \( \Phi(z) \) converges in the bigger domain \( \Omega(S,W) \) than the natural domain \( \Omega(S) \).

(-c) The denominator function \( \Phi(z) \) is anti-invariant with respect to the Weyl group \( W \) and is invariant with respect to a finite group \( A(M) \) of symmetries of \( M \). The corresponding semi-direct product \( W.A(S) \) may be very small in the natural arithmetic group \( O(S) \).

(-d) The expressions \( S(\rho, z) \) and \( S(w(\rho), z) \) are not correctly defined if there does not exist \( \rho \in M \otimes \mathbb{Q} \) such that
\[ S(\rho, \alpha_i) = -S(\alpha_i, \alpha_i)/2. \]
If such a \( \rho \in M \otimes \mathbb{Q} \) does exist, we say that \( S \) has a lattice Weyl vector. If a lattice Weyl vector does not exist, one should rewrite (2.1.15) as
\[
\sum_{w \in W} \det(w)e^{-2\pi i S(w(\rho) - \rho, z)} = \prod_{\alpha \in \Delta_+} (1 - e^{-2\pi i S(\alpha, z)})^{\text{mult}(\alpha)}.
\]
(2.1.19)
Then it is defined correctly.

The only way to improve bad properties (-a), (-b) and (-c) is to consider bigger reflection groups
\[
W = W_1 \subset W_2 \subset \cdots \subset W_k \subset \cdots \subset W(S)
\]
(2.1.20)
with fundamental polyhedra \( M_k \) and acceptable sets \( P(M_k) \) such that
\[
P(M) = P(M_1) \subset P(M_2) \subset \cdots \subset P(M_k) \subset \cdots.
\]
(2.1.21)
These gives an increasing sequence of Lorentzian Kac–Moody algebras
\[
g''(A(S, W_1, P(M_1))) \subset g''(A(S, W_2, P(M_2))) \subset \cdots \subset g''(A(S, W_k, P(M_k))) \subset \cdots
\]
(2.1.22)
with the same Cartan subalgebra \( M^* \otimes \mathbb{C} \). One can see that this procedure gives an increasing sequence of dual Tits cones
\[
T_1^* \subset T_2^* \subset \cdots \subset T_k^* \subset \cdots \subset V^+(S)
\]
(2.1.23)
and then a decreasing sequence of Tits cones \( T_k \) and domains \( \Omega(S, W_k) \). Thus, at least properties (-a) and (-b) will be improving. Unfortunately, this procedure is infinite in general and gives an infinite generated Lorentzian Kac–Moody algebra.

In the next Section, we will show that using this may be infinite procedure we can always reverse bad properties (-a), (-b) and (-c).

### 2.2. Lorentzian Kac–Moody algebras of arithmetic and restricted arithmetic type.

Let \( S : M \times M \to \mathbb{Z} \) be a hyperbolic lattice, \( W \subset W(S) \) an arbitrary reflection subgroup, \( M \) a fundamental polyhedron of \( W \) and \( P(M) \) an acceptable set of orthogonal vectors to \( M \). Like above, we suppose that \( P(M) \) generates a sublattice of finite index in \( M \) and has connected Gram graph.

Using the same definition as in Sect. 2.1, one can define an indecomposable hyperbolic symmetrizable generalized Cartan matrix \( A = A(S, W, P(M)) \), and Lorentzian Kac–Moody algebras
\[
g'(A) = g'(A(S, W, P(M))), \quad g''(A) = g''(A(S, W, P(M))).
\]
They have the same properties as in Sect. 2.1. The only difference is that the set \( P(M) \) is infinite, the matrix \( A \) is infinite and the set of generators \( e_k, f_k, h_k \), where \( k \in P(M) \), is infinite. This Lie algebra is a union of its Kac–Moody subalgebras which are defined by finite sets \( P(M)^{(\leq n)} \subset P(M) \) such that their elements have height \( \leq \sqrt{n} \) for some fixed element \( h \in M \) such that \( \mathbb{R}_{++} h \in M \) and \( S(h, h) < 0 \).

In Definition 1.4.1 above we gave a definition for \( W \) to have arithmetic type. We can apply this definition to Weyl group \( W \) and the set of simple real roots \( P(M) \) for the Kac–Moody algebra \( g''(A(S, W, P(M))) \) to have arithmetic type. But we think that for Kac–Moody algebras \( g''(A(S, W, P(M))) \) it is more natural to give definition using imaginary roots. We will see that these two definitions are equivalent.

The next definition is a generalization of our definition in [25] which was given for finitely generated Kac–Moody algebras.
**Definition 2.2.1.** A Lorentzian Kac–Moody algebra $\mathfrak{g}''(A(S, W, P(M)))$ has arithmetic type if for any $x \in M$ with $S(x, x) < 0$ there exists $n \in \mathbb{N}$ such that

$$nx \in \Delta^\text{im}. \quad (2.2.1)$$

Obviously, this definition is equivalent to the equality for open rational cones

$$Q_+ \Delta^\text{im} \cap V^+(S) = (M \otimes \mathbb{Q}) \cap V^+(S). \quad (2.2.2)$$

We have

**Theorem 2.2.2.** A Lorentzian Kac–Moody algebra $\mathfrak{g}''(A(S, W, P(M)))$ has arithmetic type if and only if the Weyl group $W$ has arithmetic type (thus, we have equivalent properties (1.4.3), (1.4.4), (1.4.5), (1.4.9), (1.4.10)). Moreover, the $\mathfrak{g}''(A(S, W, P(M)))$ has arithmetic type if and only if the Tits cone essentially coincides with $V^+(S)$ which means

$$\text{Int } T = V^+(S), \text{ equivalently, } T = V^+(S). \quad (2.2.3)$$

**Proof.** Assume that $\mathfrak{g}''(A(S, W, P(M)))$ has arithmetic type. By (2.2.2) and (2.1.14), we then have

$$(M \otimes \mathbb{Q}) \cap V^+(S) \subset (Q_+ \Delta^\text{im}^+ \subset T^*). \quad (2.2.4)$$

Since $M \otimes \mathbb{Q}$ is everywhere dense in $V^+(S)$, it follows that $V^+(S) \subset T^*$. Clearly, $T^* \subset V^+(S)$. It follows the equality $T^* = V^+(S)$ and $T = \overline{V^+(S)}$ because $V^+(S)^* = \overline{V^+(S)}$. It follows (2.2.3). By (2.2.2), (2.2.3) and (2.1.14), we obviously get that $(M \otimes \mathbb{Q}) \cap V^+(S) \subset Q_+Q_+$ which is an equivalent definition (1.4.3) for $W$ to have arithmetic type.

Now suppose that $W$ has arithmetic type. By (1.4.9) and (1.4.10), we then have

$$(R_+Q_+)^* \subset \overline{V^+(S)} \subset R_+Q_+. \quad (2.2.5)$$

It follows that $R_+K = (R_+Q_+)^* \cap R_+Q_+ = (R_+Q_+)^* \cap \overline{V^+(S)} = (R_+Q_+)^* = R_+M$ where $M$ is the fundamental polyhedron of the Weyl group $W$ in $\mathcal{L}(S)$. It follows that the dual Tits cone $T^* = W(R_+K)$ contains $V^+(S)$, and we get the equality (2.2.3). By (2.2.3) and (2.1.14), we then get

$$Q_+ \Delta^\text{im} \cap V^+(S) = Q_+Q_+ \cap V^+(S).$$

By (1.4.3), $V^+(S) \cap (M \otimes \mathbb{Q}) \subset Q_+Q_+. \text{ Thus, by the equality above, } V^+(S) \cap (M \otimes \mathbb{Q}) \subset Q_+ \Delta^\text{im}^+ \subset V^+(S). \text{ By definition, } V^+(S) \cap (Q_+ \Delta^\text{im}^+) \subset V^+(S) \cap (M \otimes \mathbb{Q}). \text{ Thus, we get the equality (2.2.2), and the Lie algebra has arithmetic type.}$

As a corollary of equivalent definitions (2.2.1), (2.2.2) and (2.2.3), we get
Corollary 2.2.3. Lorentzian Kac–Moody algebras $g''(A(S,W,P(M)))$ of arithmetic type are exactly Lorentzian Kac–Moody algebras which have the following two nice properties (a) and (b):

(a) Rational cone $Q_+^{\Delta_{im}^\text{re}}$ generated by imaginary roots contains $(M \otimes \mathbb{Q}) \cap V^+(S)$.

(b) The denominator function $\Phi(z)$ converges in the natural complexified cone $\Omega(S) = M \otimes \mathbb{R} + iV^+(S)$ for large $i m z$ (here we understand the convergence formally as the equality of the complexified cones $\Omega(S,W) = \Omega(S)$; for concrete cases one should support this convergence by the appropriate estimates).

In fact, (a) and (b) are equivalent.

Now let us consider Lorentzian Kac–Moody algebras of restricted arithmetic type. One can define them like for Definition 1.4.2.

Definition 2.2.4. A Lorentzian Kac–Moody algebra $g''(A(S,W,P(M)))$ has restricted arithmetic type if it has arithmetic type and the symmetry group $A(P(M))$ has finite index in $O(S)/W$ (it means that the corresponding semi-direct product $W.A(P(M))$ has finite index in $O(S)$).

Using Theorems 1.4.3 and 2.2.2, we get

Theorem 2.2.5. Let $S : M \times M \to \mathbb{Z}$ be a hyperbolic lattice, $W \subset W(S)$ an arbitrary reflection subgroup, $M$ a fundamental polyhedron of $W$, $P(M)$ an acceptable set of orthogonal vectors to $M$ and

$$A(P(M)) = \{ g \in O_+(S) \mid g(P(M)) = P(M) \}$$

the corresponding group of symmetries. Then the Lorentzian Kac–Moody algebra $g''(A(S,W,P(M)))$ has restricted arithmetic type if and only if the group $A(P(M))$ of symmetries has finite index in $O(S)/W$ (equivalently, $W.A(P(M))$ has finite index in $O(S)$).

We can rewrite the denominator formula (2.1.19) as follows:

$$\prod_{\alpha \in \Delta_{im}^\text{re}} (1 - e^{-2\pi i S(\alpha,z)})^{\text{mult} (\alpha)} = \frac{\sum_{w \in W} \det(w)e^{-2\pi i S(w(\rho) - \rho, z)}}{\prod_{\alpha \in \Delta_{im}^\text{re}} (1 - e^{-2\pi i S(\alpha,z)})^{\text{mult} (\alpha)}}. \quad (2.2.5)$$

Clearly, both sides of this equality are invariant with respect to the subgroup $W.A(P(M))$. If the Kac–Moody algebra $g''(A(S,W,P(M)))$ has restricted arithmetic type, this subgroup has finite index in the arithmetic group $O(S)$.

Thus, from considerations above, we get

Corollary 2.2.6. Lorentzian Kac–Moody algebras $g''(A(S,W,P(M)))$ of restricted arithmetic type are exactly Lorentzian Kac–Moody algebras which have three nice properties (a), (b) and (c) below:

(a) Rational cone $Q_+^{\Delta_{im}^\text{re}}$ generated by imaginary roots contains $(M \otimes \mathbb{Q}) \cap V^+(S)$.

(b) The denominator function $\Phi(z)$ converges in the natural complexified cone $\Omega(S) = M \otimes \mathbb{R} + iV^+(S)$ for large $i m z$ (here we understand the convergence formally as the equality of the complexified cones $\Omega(S,W) = \Omega(S)$; for concrete cases one should support this convergence by the appropriate estimates).
(c) The denominator formula (2.2.5) is invariant with respect to the subgroup \( W.A(P(M)) \) of finite index in the arithmetic group \( O(S) \).

In fact, the property (c) implies (a) and (b).

At last, using results above and Theorem 1.4.7, we get that one can always find a right sequence of extensions (2.1.20)—(2.1.23) of finitely generated Lorentzian Kac–Moody algebras \( \mathfrak{g}''(A(S,W_k,P(M_k))) \) which improves bad properties (-a), (-b) and (-c) of finitely generated Lorentzian Kac–Moody algebras \( \mathfrak{g}''(A(S,W_k,P(M_k))) \).

**Theorem 2.2.7.** Let \( S : M \times M \to \mathbb{Z} \) be a hyperbolic lattice, \( W_1 \subset W(S) \) a finitely generated reflection subgroup, \( M_1 \) a fundamental polyhedron of \( W_1 \) with a finite acceptable set \( P(M_1) \) of vectors orthogonal to \( M_1 \). Let \( \mathfrak{g}''(A(S,W_1,P(M_1))) \) be the corresponding Lorentzian Kac–Moody algebra. Then \( W_1 \) is a reflection subgroup \( W \subset W(S) \) of a reflection group \( W \) of restricted arithmetic type with a fundamental polyhedron \( M \subset M_1 \) and an acceptable set \( P(M) \) such that \( P(M_1) \subset P(M) \). This gives the embedding

\[
\mathfrak{g}''(A(S,W_1,P(M_1))) \subset \mathfrak{g}''(A(S,W,P(M)))
\]

of the finitely generated Lorentzian Kac–Moody algebra \( \mathfrak{g}''(A(S,W_1,P(M_1))) \) to the Lorentzian Kac–Moody algebra \( \mathfrak{g}''(A(S,W,P(M))) \) of restricted arithmetic type (which has nice properties (a), (b) and (c) above).

2.3. Lorentzian Kac–Moody algebras of restricted arithmetic type with a lattice Weyl vector.

Results above show that Lorentzian Kac–Moody algebras of restricted arithmetic type give the most natural class of Lorentzian Kac–Moody algebras to study. But this class is huge. Every hyperbolic lattice \( S \) of rank \( \geq 3 \) with not-trivial reflection group \( W(S) \) has infinitely many reflection subgroups \( W \subset W(S) \) of restricted arithmetic type, which have infinite index between one another. Each this group defines its own Lorentzian Kac–Moody algebras of restricted arithmetic type \( \mathfrak{g}''(A(S,W,P(M))) \).

Fortunately, there is another very natural restriction on Lorentzian Kac–Moody algebras.

The denominator formula has the best and the most interesting automorphic properties only if it can be written in the form (2.1.15) when one can consider real and imaginary roots together. This may be done only if there exists a lattice Weyl vector \( \rho \).

**Definition 2.3.1.** Lorentzian Kac–Moody algebra \( \mathfrak{g}''(A(S,W,P(M))) \) has a lattice Weyl vector if there exists \( \rho \in M \otimes \mathbb{Q} \) such that

\[
S(\rho, \alpha) = -(1/2)S(\alpha, \alpha) \text{ for any } \alpha \in P(M). \tag{2.3.1}
\]

If the lattice Weyl vector \( \rho \) does exist, we can write the denominator formula in the form (2.1.15) which we rewrite here

\[
\Phi(z) = e^{-2\pi i S(\rho, z)} \prod_{\alpha \in \Delta^+} (1 - e^{-2\pi i S(\alpha, z)})^{\text{mult } (\alpha)} = \sum_{w \in W} \det (w)e^{-2\pi i S(w(\rho), z)}. \tag{2.3.2}
\]

It is anti-invariant with respect to \( W.A(P(M)) \) which means that

\[
\Phi(w.a(z)) = \det(w)\Phi(z) \text{ for any } w \in W, a \in A(P(M)). \tag{2.3.3}
\]

Using results above, we get...
Theorem 2.3.2. Lorentzian Kac–Moody algebras $\mathfrak{g}''(A(S,W,P(M)))$ of restricted arithmetic type and with lattice Weyl vector are exactly Lorentzian Kac–Moody algebras which have three nice properties (a), (b) and (c)+(d) below:

(a) Rational cone $\mathbb{Q}_+\Delta_{pm}$ generated by imaginary roots contains $(M \otimes \mathbb{Q}) \cap V^+(S)$.

(b) The denominator function $\Phi(z)$ converges in the complexified cone $\Omega(S) = M \otimes \mathbb{R} + iV^+(S)$ for big $\text{im } z$ (here we understand the convergence formally as the equality of the complexified cones $\Omega(S,W) = \Omega(S)$; for concrete cases one should support this equality by the appropriate estimates).

(c)+(d) The denominator formula (2.3.2) is anti-invariant with respect to the subgroup $W.A(P(M))$ of finite index in the arithmetic group $O(S)$.

We apply results of Sect. 1.1 and Sect. 1.4 to describe these the most interesting Lorentzian Kac–Moody algebras.

Like in Sect. 1.1, we have

**Definition 2.3.2.** Let $S : M \times M \to \mathbb{Z}$ be a hyperbolic lattice, $W \subset W(S)$ a reflection subgroup, $M$ a fundamental polyhedron of $W$, $P(M)$ an acceptable set of orthogonal vectors to $M$ and

$$A(P(M)) = \{g \in O_+(S) \mid g(P(M)) = P(M)\}$$

the corresponding group of symmetries. Lorentzian Kac–Moody algebra $\mathfrak{g}''(A(S,W,P(M)))$ is **elliptic** if $W \subset O(S)$ has finite index. In particular, the lattice $S$ is **elliptic reflective**. Lorentzian Kac–Moody algebra $\mathfrak{g}''(A(S,W,P(M)))$ is **parabolic** if $A(P(M))$ is infinite but $W.A(P(M))$ has finite index in $O(S)$ and there exists $c \in M$ such that $S(c,c) = 0$ and $g(c) = c$ for any $g \in A(P(M))$ (i.e. $A(P(M))$ has a cusp $c$). In particular, the lattice $S$ is **parabolic reflective**.

By Theorems 1.3.2 and 1.3.3, Propositions 1.4.10 and 1.4.11 and results of Sect. 2.2, we get

**Theorem 2.3.3.** Lorentzian Kac–Moody algebras $\mathfrak{g}''(A(S,W,P(M)))$ of restricted arithmetic type and with a lattice Weyl vector $\rho$ have $S(\rho, \rho) \leq 0$ if $\text{rk } S \geq 3$.

For $S(\rho, \rho) < 0$, these algebras are exactly elliptic Lorentzian Kac–Moody algebras with a lattice Weyl vector. Their set is finite.

For $S(\rho, \rho) = 0$, these algebras are exactly parabolic Lorentzian Kac–Moody algebras with a lattice Weyl vector. For a constant $D > 0$, their set is finite if

$$[O(S)_\rho : A(P(M))] < D.$$
Theorem 2.3.4. Lorentzian Kac–Moody algebras $g''(A(S, W, P(M)))$ of restricted arithmetic type and with a generalized lattice Weyl vector $\rho$ have $S(\rho, \rho) \leq 0$ if $\text{rk} S \geq 3$.

For $S(\rho, \rho) < 0$, these algebras are exactly elliptic Lorentzian Kac–Moody algebras. The set of their lattices $S : M \times M \to \mathbb{Z}$ is finite.

For $S(\rho, \rho) = 0$, these algebras are exactly elliptic or parabolic Lorentzian Kac–Moody algebras. The set of their lattices $S : M \times M \to \mathbb{Z}$ is also finite.

Remark 2.3.5. These results show that classification of elliptic and parabolic reflective lattices is of the extremal importance for the theory of Lorentzian Kac–Moody algebras. Many elliptic reflective lattices were constructed by É.B. Vinberg, see [29] — [34]. In [16], [17], [18], [22] all elliptic 2-reflective even hyperbolic lattices $S$ were classified. Here $S$ is elliptic 2-reflective if the subgroup $W^{(2)}(S)$ generated by reflections in all $\delta \in M$ with $S(\delta, \delta) = 2$ has finite index in $O(S)$. In [17] there is a series of twelve even 2-elementary (i.e. with 2-elementary discriminant group $M^*/M$) parabolic 2-reflective even hyperbolic lattices. The even unimodular hyperbolic lattice of the rank 26 which is parabolic by J. Conway [7] continues this series. In R. Borcherds [1] there are some examples of elliptic and parabolic reflective lattices. See reviews of É.B. Vinberg, O.V. Shvartsman [34] and R. Scharlau, C. Walhorn [28] for further results on reflective lattices.

We should say that in spite of finiteness results of §1 above, complete description of reflective hyperbolic lattices is the very difficult problem. On the other hand, now there are plenty of reflective lattices known, and because of these finiteness results, all these examples are very interesting.

2.4. A comment on the recent results due to R. Borcherds.

Here we want to write down how we understand the recent results of R. Borcherds [2] — [6] in connection with the results above.

Let us consider a Lorentzian Kac–Moody algebra $g''(A(S, W, P(M)))$ of restricted arithmetic type and with a lattice Weyl vector $\rho$ (thus, by Sect. 2.3, this Lorentzian Lie algebra is elliptic or parabolic). Then its denominator function $\Phi(z)$ is defined in the complexified cone $\Omega(S)$ which is

$$\Omega(S) = M \otimes \mathbb{R} + iV^+(S),$$

and converges for $\text{Im} \, z >> 0$.

We denote by $U : N \times N \to \mathbb{Z}$ an even unimodular lattice of the signature $(1, 1)$ where $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $U(e_1, e_1) = U(e_2, e_2) = 0$, $U(e_1, e_2) = -1$.

We consider a lattice

$$S' = S \oplus U(k) : M' \times M' \to \mathbb{Z}, \; k \in \mathbb{N},$$

where $M' = M \oplus N$. The lattice $S'$ defines the type IV domain which is one of two connected components of the domain

$$\Omega(S') = \{ \mathbb{C}\omega \subset M' \otimes \mathbb{C} \mid S'(\omega, \omega) = 0, \; S'(\omega, \overline{\omega}) < 0 \}.$$  

(2.4.3)

We normalize $\omega \in \mathbb{C}\omega \in \Omega(S')$ by the condition $S'(\omega, e_1) = -1$. Then

$$\omega = z \oplus (S(z, z)/2)e_1 \oplus (1/k)e_2, \; z \in \Omega(S).$$

(2.4.4)
This defines an embedding
\[ \Omega(S) \subset \Omega(S') \]
which is called the cusp \( e_1 \) embedding.

We comment the recent results by R. Borcherds as follows:

*The most interesting Lorentzian Kac–Moody algebras \( \mathfrak{g}''(A(S, W, P(M))) \) of restricted arithmetic type and with a lattice Weyl vector \( \rho \) (thus, they are elliptic or parabolic) have a generalized Kac–Moody algebra (or may be generalized Kac–Moody superalgebra) correction \( \mathfrak{g}''(A(S, W, P(M))) \subset \mathfrak{g}''(A(S, W, P(M) \cup H)) \) such that the denominator function \( \Phi'(z) \) of the correcting generalized Kac–Moody algebra \( \mathfrak{g}''(A(S, W, P(M) \cup H)) \) is an automorphic form with respect to an appropriate subgroup of finite index in \( O(S') \) for some \( k \). It is natural to name this correction as automorphic form correction (or automorphic correction).

We remark that a priori the denominator function \( \Phi(z) \) is anti-invariant with respect to the subgroup of finite index \( M_\rho^*(W.A(P(M)) \) of the stabilizer subgroup \( O(S')_{e_1} \) of the cusp. Here \( M_\rho^* = \{ x \in M^* | S(\rho, x) \in \mathbb{Z} \} \).

More precisely this means that there exists an automorphic form \( \tilde{\Phi}(z) \) which is automorphic with respect to a subgroup of finite index \( G \subset O(S')^+, \) containing \( W.A(P(M)) \), and is anti-invariant with respect to \( W.A(P(M)) \). Moreover, \( \tilde{\Phi}(z) \) has Fourier expansion with the properties which we describe below. This Fourier expansion is

\[
\tilde{\Phi}(z) = \sum_{w \in W} \det(w) \left( e^{-2\pi i S(w(\rho), z)} - \sum_{a \in M_\rho^* \cap \mathbb{R}_{++} M} m(a)e^{-2\pi i S(w(\rho + a), z)} \right),
\]

where all \( m(a) \in \mathbb{Z} \) and \( m(a) \in \mathbb{Z}_+ \) (for superalgebras case \( m(a) \in \mathbb{Z} \) if \( S(a, a) < 0 \). See (1.2.4) for definition of \( M_\rho^* \)). Let us consider \( a \in M_\rho^* \cap \mathbb{R}_{++} M \) such that \( S(a, a) = 0 \). These \( a \) define several isotropic rays which correspond to infinite vertices of \( M \), and the cusp of \( M \) if \( M \) is parabolic. We consider any of these rays \( t a_0 \), where \( a_0 \) is a primitive element of \( M_\rho^* \) with \( S(a_0, a_0) = 0 \) and \( t \in \mathbb{N} \). For any of these isotropic rays one has the equality of formal power series of one variable \( q \)

\[
1 - \sum_{t \in \mathbb{N}} m(ta_0)q^t = \prod_{k \in \mathbb{N}} (1 - q^k)^{\tau(ka_0)},
\]

where \( \tau(ka_0) \in \mathbb{Z}_+ \) (for superalgebras case \( \tau(ka_0) \in \mathbb{Z} \)).

Let us consider

\[
H = \{ m(a) a \mid a \in M_\rho^* \cap \mathbb{R}_{++} M \text{ and } S(a, a) < 0 \} \cup \\
\cup \{ \tau(a) a \mid a \in M_\rho^* \cap \mathbb{R}_{++} M \text{ and } S(a, a) = 0 \},
\]

where \( ka \) means that we repeat an element \( a \) exactly \( k \) times (consider with the multiplicity \( k \)). We consider the Gram matrix

\[
G(P(M) \cup H)
\]

of elements \( G(P(M) \cup H) \). One can easily see that \( G(P(M) \cup H) \) has all necessary properties to define a generalized Kac–Moody algebra

\[
\mathfrak{g}''(G(S, W, P(M) \cup H))
\]
corresponding to this matrix (see R. Borcherds [2] for definition). It has properties similar to Kac–Moody algebras we considered in Sects 2.1 and 2.2. The only difference is that its simple roots are \( P(\mathcal{M}) \cup H \). It is graded by the lattice \( M'_*(\mathcal{M}) \supset M \), has roots, multiplicities of roots and the same Weyl group \( W \). The Kac–Moody algebra \( g''(A(S, W, P(\mathcal{M}))) \) is defined by Gram matrix \( G(P(\mathcal{M})) \) of elements \( P(\mathcal{M}) \) and is a subalgebra of \( g''(G(S, W, P(\mathcal{M}) \cup H)) \). The denominator formula for \( g''(G(S, W, P(\mathcal{M}) \cup H)) \) equals to

\[
e^{-2\pi i S(\rho, z)} \prod_{\alpha \in \Delta^+} (1 - e^{-2\pi i S(\alpha, z)})^{\text{mult} (\alpha)} = \tilde{\Phi}(z) \tag{2.4.10}
\]

where \( \tilde{\Phi}(z) \) is the function described in (2.4.5)—(2.4.7) (see [3] where in fact necessary calculations were done). In particular, if one can find the automorphic function \( \tilde{\Phi}(z) \) which we described above, it automatically has the product formula (2.4.10).

The next the most amazing example of the “correction” above was found by R. Borcherds [3], [5]. Consider an even unimodular hyperbolic lattice \( S \) of signature \((25,1)\) and \( W = W(S) \). This reflection group is parabolic and has the lattice Weyl vector \( \rho \) with \( S(\rho, \rho) = 0 \). This was proved by J. Conway [7]. Take \( P(\mathcal{M}) = P(\mathcal{M})_{\text{pr}} \) for a fundamental polyhedron \( \mathcal{M} \) of \( W \). The corrected denominator formula for this algebra equals

\[
\tilde{\Phi}(z) = e^{-2\pi i S(\rho, z)} \prod_{\alpha \in \Delta^+} (1 - e^{-2\pi i S(\alpha, z)})^{p_{24}(1 - S(\alpha, \alpha)/2)} = \sum_w \det(w) \sum_{n>0} \tau(n) e^{-2\pi i n(w(\rho), z)}. \tag{2.4.11}
\]

Here

\[
\sum_{n \geq 0} p_{24}(n) q^n = \prod_{n>0} (1 - q^n)^{-24} \tag{2.4.12}
\]

and

\[
\sum_{n \geq 0} \tau(n) q^n = q \prod_{n>0} (1 - q^n)^{24}. \tag{2.4.13}
\]

Thus, for this case \( H = \rho, 2\rho, ..., t\rho, ... \), where each element \( t\rho \) is taken with multiplicity 24. The function \( \tilde{\Phi}(z) \) is an automorphic form of weight 12 with respect to the subgroup \( O(S \oplus U)^+ \subset O(S \oplus U) \) which fixes connected components of the domain \( \Omega(S \oplus U) \). See some other examples in [5], [6] (and also [3], [4]) and [12].

It seems that the bases for a construction of similar examples and many others is the arithmetic lifting of Jacobi forms on IV type domains which was constructed by V.A. Gritsenko [10], [11]. Some its multiplicative analog was constructed by R. Borcherds [6].

Theory of automorphic forms is a very delicate domain. Automorphic forms like (2.4.11) are probably very rare. On the other hand, as we have shown above, essentially one has only a finite number of elliptic and parabolic Kac–Moody algebras with a lattice Weyl vector. Thus, essentially one has to “correct” only a finite set of Kac–Moody algebras. May be it is possible like for the example above.
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