Periodic structure with electrostatic forces: Interactions beyond the nearest neighbor

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Abstract

Periodic structures are a type of metamaterial in which the physical properties depend not only on the details of the unit cell but also on how unit cells are arranged and interact with each other. In conventional engineering structures, each unit cell interacts with adjacent cells. Methods developed for vibrational and wave propagation analysis in periodic engineering structures consider only nearest-neighbor interactions. The dispersion curves of such systems, in which only adjacent cells interact, have been extensively studied. Metamaterial properties depend on the interactions of a unit cell with other cells. Further interactions, and specifically, interactions beyond the closest neighbors, imply a more complex band structure and wave behavior. In this paper, an example class of such structures, in which electrostatic forces are the driving force, has been investigated. In this paper, properties affecting these periodic structures, such as elastic forces, have been investigated. An attractive property of such structures is that the band structures of such metamaterials can be tuned by changing electric voltages.

Keywords

Metamaterial, dispersion curves, nonlocal interaction, electrostatic forces, tunable band gaps

Introduction

Periodic structures have the interesting property that their vibrational characteristics depend not only on the materials and properties of their unit cell but also on how unit cells interact. Bloch’s theorem [1] is commonly used in the vibrational analysis of these materials. Bloch’s work was followed by Mead et al. (Mead and Markus, 1969; Mead, 1975), who developed a method for investigating harmonic wave propagation in periodic structures. Research has since taken place into a wide range of structures, such as mono-coupled periodic systems (Faulkner and Hong, 1985), beam grillage under harmonic point loading (Langley et al., 1997), a periodic Bernoulli–Euler beam (Yu et al., 2009), beam grids (Bordiga et al., 2019; Botshkan et al., 2019; Bacigalupo and Lepidi, 2018), and metamaterials with locally resonant components (Nouh et al., 2015; Ragonese and Nouh, 2021). Many engineering applications of periodic structures exist, including wave-beaming devices (Vonfotow, 1986; Yong and Lin, 1989; Ma et al., 2013) and filters (Mead and Markus, 1969; Mead, 1975; Brillouin, 2003). As well as linear periodic structures, nonlinear materials (Manktelow et al., 2013, 2014a; Manktelow et al., 2014b; Bernard et al., 2013, 2014), and materials with dissipation (Andreasen and Jensen, 2013; Farzbod and Leamy, 2011; Al Ba’ba’a and Nouh, 2017; Manimala and Sun, 2014; Krushynska et al., 2016; Scarpa et al., 2013; Palermo and Marzani, 2015; Aladwani and Nouh, 2020; Farzbod, 2010) have been studied theoretically. Others employ techniques inspired by topological electronic insulators, which exhibit a bulk band gap similar to a regular insulator but possess conductive states on their edges or surfaces. Torrent et al. (2013) used multiple scattering theory to show the presence of edge states in a structure with periodically arranged resonators. Chaunsali et al. (2018) investigated the directional control of flexural waves, while Zhou et al. (2022) studied topological phases for nonlinear bulk waves. In such work, the interaction between unit cells is limited to the nearest neighbor; each unit cell exerts forces on other unit cells with some boundary coordinates in common. This is because, in most engineering structures, the forces and

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interactions are limited to mechanical forces through contacts and joints. However, that is not the case for electromagnetic forces, such as the interatomic forces in a crystal. In crystals, for example, atoms interact with each other beyond the nearest neighbor. Magnetic and electrostatic forces have been used in periodic structures for modulation and tuning (Chen et al., 2018b), but they have not been used to design metamaterials with beyond-nearest-neighbor interaction. Others investigated the winding number of the phase diagram in a 1-D chain when interactions extend to the second nearest neighbor. The present work is concerned with other aspects of electrostatic forces; the way dispersion curves are affected when the interactions reach beyond the nearest neighbor. Interactions beyond the nearest neighbor and their effects on the dispersion curves were first observed by Brillouin in a 1-D chain and have been further studied by one of the authors (Farzbod, 2017; Farzbod and Scott-Emuakpor, 2020). In a periodic structure, interactions running beyond the nearest neighbor have an effect on its dispersion curves. For example, the maximum number of wavevectors at each frequency in periodic materials depends on the various interacting layers (Farzbod, 2017). It was also shown (Farzbod, 2017; Farzbod and Scott-Emuakpor, 2020) that dispersion curves are affected topologically when interactions are not limited to the nearest neighbor. In the following, we show that by using electrostatic forces, it is possible to fabricate such structures.

**Structures with electrostatic forces**

Consider a structure similar to the one depicted in Figure 1. The slab and the columns are made out of a material that can be conductive such as copper, or non-conductive, such as glass or flexible polyvinyl chloride (PVC), with a conductive coating. It can also be made by etching out a substrate to make rods and then coating them with conducting materials. It is also possible to fabricate a similar structure with additive manufacturing methods. For example, we can connect all the rods to the same electrical voltage or columns with matching colors to the same electrical voltage, making the forces vary from repellant to attraction with various amounts. Due to the nature of the electrostatic forces, the interactions in these structures run beyond the nearest neighbor. Also, the forces involved are nonlinear. However, suppose the structure is designed in such a way that the neutral position is the equilibrium position. In that case, we can linearize the equations around the equilibrium position and use Bloch analysis. We first do force analysis and then investigate dispersion curves in the following subsections.

**Force analysis**

There are many possible configurations for a unit cell and how it is repeated to make the periodic structure. They can be arranged in a non-uniform shape or a circular one such as the ones in Figure 2. However, we are considering a subset of these possibilities, in which columns are placed in a rectangular arrangement. To avoid confusion with matrix columns, we call the material columns masses from now on. It is also assumed that a unit cell has m masses along the x-axis and n masses along the y-axis. To simplify the formulation, we took that these masses are separated by distances a and b in the x and y directions (see Figure 3). The formulation is derived for the general case where we have m by n number of columns in a unit cell. For the example structure of Figure 3, m = 3 and n = 2. The reason for deriving equations for the general case is that we can then write a code for the general case and vary the design parameters to see their effect on the dispersion curves. This formulation is beneficial when considering hundreds of configurations and geometries for possible neural network training.

We linearize the force equation for a generic unit cell located at k and l, such as the one indicated by the double-dashed line in Figure 3. For the structure in Figure 3, the box with double-dashed lines is at (k,l) = (2,1), and for the reference unit cell with a solid dashed line (k,l) = (0,0). Each mass in a unit cell is defined by two numbers such as (r,s) where 1 \leq r \leq m and 1 \leq s \leq n. Other geometrical parameters are depicted in Figure 3. The distance between the masses at (r,s) and (u,w) in the x and y directions are

$$d_x = k(m - 1) \times a + (u - r) \times a + k \times c + x_{rs} - x_{uw} \quad (1)$$

$$d_y = l(n - 1) \times b + (w - s) \times b + l \times d + y_{rs} - y_{uw} \quad (2)$$

in which $x_{rs}$ is the displacement of the mass (r,s) of the reference unit cell from the equilibrium position in the x direction. Other displacements in the x and y directions are named similarly. All $x_{rs}$, $x_{uw}$, $y_{rs}$, and $y_{uw}$ can assume both positive and negative values. The force between masses indexed by (r,s) and (u,w) in x and y directions are

$$F_x(x,y) = \frac{K \times Q \times q}{d_x^2 + d_y^2} \cos \theta, \quad F_y(x,y) = \frac{K \times Q \times q}{d_x^2 + d_y^2} \sin \theta \quad (3)$$

in which Q and q are the electric charges of the masses (i.e., columns), K is the Coulomb’s constant, and $\theta$ is the angle
Figure 2. An example structure is arranged in a non-rectangular geometry; similar to the elastic metamaterials, there are abundant possible configurations for a periodic structure with electrostatic forces.

Figure 3. There are three columns and two rows in the depicted structure here. Distances between masses and unit cells in x and y directions are marked. Electrostatic forces between the reference unit cell and the one marked by a double-dashed line are formulated.
between the line connecting the centers of the columns and the x-axis. The vacuum’s Coulomb constant equals \(1/(4\pi\varepsilon_0)\), where \(\varepsilon_0\) is the vacuum permittivity. For the air, it is very close to the one of vacuum. While we can use materials between masses with complex permittivity, we consider the air as the medium between masses. Using geometric values of sine and cosine, we can write

\[
F_x(x,y) = \frac{K \times Q \times q \times x}{(x_1^2 + y_1^2)^2}, \quad F_y(x,y) = \frac{K \times Q \times q \times y}{(x_1^2 + y_1^2)^2}
\]

(4)

We want to linearize forces and stiffness relations such that

\[
F_x(x,y) = K_{x1}d_1 + K_{x2}d_2, \quad F_y(x,y) = K_{y1}d_1 + K_{y2}d_2
\]

(5)

To simplify the process of linearization, we can denote \(x_{rs} - x_{uw}\) and \(y_{rs} - y_{uw}\) by \(x\) and \(y\) in both \(d_1\) and \(d_2\). Then, find the partial derivative with respect to \(x\) and \(y\). We evaluate the partial derivatives at equilibrium; in other words, when \((x,y) = (0,0)\)

\[
F_{x,x}(x,y) = F_{x,x}(x,y) \times x_{uw} + F_{x,y}(x,y) \times y_{uw}
\]

\[
F_{y,y}(x,y) = F_{y,y}(x,y) \times x_{uw} + F_{y,y}(x,y) \times y_{uw}
\]

(6)

For an example, consider the structure in Figure 3 in which \(m = 3\) and \(n = 2\). Forces along the \(x\) and \(y\) directions between the masses located at \((r,s) = (1,1), (u,w) = (3,2),\) and \((k,l) = (2,1)\) can be found as

\[
F_x(x,y) = K_{x1}d_1 + K_{x2}d_2, \quad F_y(x,y) = K_{y1}d_1 + K_{y2}d_2
\]

(11)

Using equations (1) and (2) to evaluate \(d_{1,x}\) and \(d_{2,y}\) at \((x,y) = (0,0)\), equation (6) takes the form of

\[
F_{x,x}(x,y) = \frac{1}{(d_1^2 + d_2^2)^2} \times x_{uw}
\]

\[
F_{y,y}(x,y) = \frac{1}{(d_1^2 + d_2^2)^2} \times y_{uw}
\]

(7)

As such, we get

\[
K_{xx} = K_{Q1} \left[ \frac{1}{((6a + 2c)^2 + (2b + d)^2)^2} \right]
\]

\[
- \frac{3 \times (6a + 2c)^2}{((6a + 2c)^2 + (2b + d)^2)^2} \right]
\]

(13)

\[
K_{xy} = K_{yx} \left[ \frac{3 \times (6a + 2c) \times (2b + d)}{((6a + 2c)^2 + (2b + d)^2)^2} \right]
\]

(14)

\[
K_{yy} = K_{Q1} \left[ \frac{1}{((6a + 2c)^2 + (2b + d)^2)^2} \right]
\]

\[
- \frac{3 \times (2b + d)^2}{((6a + 2c)^2 + (2b + d)^2)^2} \right]
\]

(15)
Coefficient values obtained for the interaction of each mass of the reference unit cell with each mass of the interacting unit cell yield the stiffness matrix $\mathbf{K}$. To find the eigenvalue problem, we follow the procedure explained in Farzbod and Scott-Emuakpor (2020). Each stiffness matrix is multiplied with the respective $\mathbf{T}$ and $\mathbf{T}^T$ as linear push forward and backward operator, and then the summation of these products along with the addition of $-\omega^2 \mathbf{M}$ will yield an eigenvalue problem:

$$
\left(-\omega^2 \mathbf{M} + \sum_{k=0}^{p} \sum_{l=0}^{q} \mathbf{T}_{k,l}^T \mathbf{K}_{k,l} \mathbf{T}_{k,l}\right) \hat{\mathbf{q}} = \mathbf{0} \quad (16)
$$

In this eigenvalue equation, $(k,l)$ represents the position expressed as coordinates of the interacting unit cell with the reference unit cell, and $\hat{\mathbf{q}}$ is the minimum set of displacements for the reference unit cell. We separate the $x$ and $y$ coordinates of the masses and stack them up on each other. This is a general equation for up to the $p^\text{th}$ and $q^\text{th}$ nearest neighbors in the $x$ and $y$ directions. In order to consider only the adjacent cells, $p$ and $q$ are bounded by 1. In this case then, equation (16) becomes the familiar equation published in previous works such as equation (22) in Ref. [34] or equation (16) in Ref. [35]. Following this practice, for the example structure of Figure 3, we get

$$
\hat{\mathbf{q}} = \begin{bmatrix}
X(0,0) \\
X(0,1) \\
X(0,2) \\
X(0,3) \\
X(0,4) \\
X(0,5) \\
X(0,6) \\
Y(0,0) \\
Y(0,1) \\
Y(0,2) \\
Y(0,3) \\
Y(0,4) \\
Y(0,5) \\
Y(0,6)
\end{bmatrix}, \mathbf{T}_{1,2} = \begin{bmatrix} 1_2 \\ e^{i(2\pi x + \mu_x)} \\ e^{i(2\pi y + \mu_y)} \end{bmatrix}, \mathbf{T}_{2,0} = \begin{bmatrix} 1_2 \\ e^{-i(2\pi x + \mu_x)} \\ e^{-i(2\pi y + \mu_y)} \end{bmatrix}
$$

in which $1_2$ is the identity matrix with the dimension of 12. In this equation and all the following dispersion plots, similar to our previous works, $\mu_x$ and $\mu_y$ are used for the phase shifts. To explain phase shift coefficients, consider vectors $\mathbf{a}_1 = \mathbf{a}^T$ and $\mathbf{a}_2 = \mathbf{b}^T$ in Figure 3. The periodic structure is made by repeating a unit cell along these lattice vectors. We can then write the wavevector as $\mathbf{k}_\Gamma = k_1 \mathbf{B}_1 + k_2 \mathbf{B}_2$, where the inner product $(\mathbf{B}_i, \mathbf{a}_j)$ equals to $\delta_{ij}$. In other words, $\mathbf{B}_1$ and $\mathbf{B}_2$ are reciprocal lattice vectors. The phase shift between a point on the reference unit cell and the corresponding point on the unit cell located at, for example, $2\mathbf{a}_1 + 1\mathbf{a}_2$, can be stated as $e^{2\pi i(2k_1 + 1k_2)}$. In this manuscript, rather than changing the wavevectors and working with reciprocal lattice vectors, we denote $2\mathbf{k}_1$ and $2\mathbf{k}_2$ by $\mu_x$ and $\mu_y$. So, for the example above, the phase shift coefficient becomes $e^{i(2\mu_x + 1\mu_y)}$.

In equation (17), since all the masses interact with each other in both the $x$ and $y$ directions, $\mathbf{K}_{(2,1)}$, the stiffness matrix for $(k,l)=(2,1)$ is a $24 \times 24$ matrix with no element being zero. Four of those elements are the ones in equations (13)–(17).

**Dispersion curves**

A code was written to vary various physical properties and design elements, and plot the dispersion curves. A rectangular arrangement of masses similar to the one depicted in Figure 3 was assumed. Among possible configurations, we envisioned two-by-two mass columns in each unit cell for the ensuing investigation. The columns are with a radius and height of 0.35 and 2 mm. We envisioned these columns to be made out of silicon, so the mass of each column is 1.793 mg. The electric charge on each column is assumed to be 500 picocoulomb. In other words, all $Q$’s and $q$’s are the same and equal to 500 picocoulomb. Using notation similar to the ones in Figure 3, we have distances $a$, $b$, $c$, and $d$ equal to 1.5, 2.5, 2, and 3 mm for the current structure. In this example, we considered interactions up to the eighth nearest neighbor. The following subsections further discuss the logic for a cut-off number of eight. We also investigate the effect of other parameters on the dispersion curves in these electrostatic structures. For the explained structure, dispersion curves are depicted in Figure 4. It should be mentioned that points $(\mu_x, \mu_y)$ of $(0,0)$, $(\pi, 0)$, and $(\pi, \pi)$ correspond to $\Gamma$, $X$, and $M$ of the irreducible Brillouin zone. It can be seen that there are eight branches representing two degrees of freedom for each of the four masses. There is a band gap in the upper range of the band structure.

**Electric charges.** The spring constants in equations (13)–(15) are linearly proportional to the electric charges. Consequently, if the electric charges are multiplied by, for example, 10, then all the dispersion curves are scaled up by a factor of 10 in frequency. This property is, in fact, a significant advantage of such structures. Because electric charges can be changed by applying different voltages to the structure, as an example, the band gap in Figure 4 can be tuned and changed easily. Also, the wave beaming can be adjusted quickly. This paper has discussed and analyzed only similar charges, positive or negative, on each mass.
However, it is possible to design the structure with positive and negative charges. Different charges should be arranged so that we have a stable equilibrium position.

**Elastic stiffness effect.** In the stiffness calculations so far, only electrostatic forces have been considered. The inclusion of elastic forces due to the deflection of the beams is also required to model the stiffness completely. The inclusion of the elastic stiffness is not a difficult task, however. Beams of the proposed structures are not mechanically connected and behave similarly to a cantilever beam. As such, only the diagonal terms on the stiffness matrix have to be updated, namely, the diagonal terms on the $K_{0,0}$ matrix. The stiffness term that should be added due to the distributed load can be stated as $(Gere \text{ and Goodno, 2012})$ $8EI/L^3$, in which $E$ is the Young’s modulus, $I$ is the area moment of inertia of cross section, and $L$ is the length of the beam. For a material such as copper with an elastic modulus of 112 GPa, the mechanical stiffness relative to the electrical stiffness terms in $K_{0,0}$ becomes a significantly bigger number. For the example structure that was used with columns of radius, height, and electric charge of 0.35 mm, 2 mm, and 500 picocoulomb, the elastic stiffness is about one hundred thousand times bigger than the terms due to electrostatic forces. In this case, the dispersion curves look similar if scaled accordingly (see Figure 5). For the stiff material, the change in electrostatic forces around the equilibrium point is relevant in the proposed structure, not the forces themselves. The deviation from the equilibrium position relative to the distances between masses is slight, so we could linearize the forces and treat them like a linear spring. For the sake of argument, we consider forces along the vector connecting two masses separated by distance $r$ in the $x$ direction. Linearizing this force means that the electrostatic interaction is replaced by a linear spring. This spring constant would be in the form of $\alpha \times 1/r^3$ for some constant $\alpha$. In other words, the stiffness goes down proportional to the reciprocals of the cubes of distances. Adding all spring constants along the $x$ direction, assuming masses are distanced equally, would amount to

![Figure 4. Dispersion surfaces (left) and dispersion curves (middle) of a structure with a unit cell of four columns arranged in a two-by-two configuration. The columns are with a radius and height of 0.35 and 2 mm and a mass of 1.793 milligrams. The electric charge on each column is 500 picocoulomb. Following Figure 3 notation, a, b, c, and d are 1.5, 2.5, 2, and 3 mm. The right figure depicts mode shapes for the eight frequencies corresponding to $(\mu_x, \mu_y)$ of $(\pi/4, 0)$, $(0, \pi/3)$, and $(\pi/6, \pi/5)$. Dashed blue circles indicate displaced masses (not to scale).](image)
Figure 5. Dispersion curves of two geometrically similar structures with one made with copper (depicted on the left) and the other with PVC. Frequencies on the left plot are subtracted by about 860 kHz.

Figure 6. Dispersion curves for the structure of Figure 4, when interactions up to the 1st (green dash-dotted line), 6th (solid black lines), 8th (dotted red lines), and 10th (dashed blue lines) nearest neighbor are considered. Except for the 1st nearest neighbor, all the other cases of dispersion curves completely overlap for low frequencies, and for high frequencies, they are close. A zoom-out segment of the band structure at frequencies of about 1500 is indicated by a green dashed box.
\[
\beta \lim_{n \to \infty} \left( \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{n^3} \right) .
\]

in which \( \beta \) is a constant encompassing all coefficients such as electric charges, Coulomb constant, and distances between masses. This sum is in fact \( \zeta(3) \) where \( \zeta \) is the Riemann zeta function. The value of \( \zeta(3) \) has been calculated up to more than a million digits, so verifying that the sum in equation (18) converges fast is simple. For example, summing up to the \( n = 8 \), the error goes down to about 0.5%. It should be mentioned that equation (18) only represents summation over one direction. The structure is considered to be stretched to infinity in two dimensions. As such, there will be infinite directions, and more summations similar to equation (18) must be considered, so we have infinitely many summations. On the other hand, the sum starts from higher values of \( n \) and with increments of more than one for most directions. For this reason, even though not mathematically proven, we hypothesized that the convergence is still fast. We have investigated convergence with a couple of examples. In Figure 4, dispersion curves are depicted for a sample structure up to the eighth nearest neighbor, meaning \( p \) and \( q \) in equation (16) are bounded by 8. In this case, \((8 + 1) \times (8 + 1) - 1\) neighbors exist because \( p \) and \( q \) can assume any number between zero and eight, minus one when both \( p \) and \( q \) are zero representing internal forces within the unit cell. For the same structure, interactions up to the first, sixth, eighth, and tenth nearest neighbor are depicted in Figure 6. As shown in Figure 6, dispersion curves are different for the first nearest neighbor, both in terms of value and slope in various places. For the other three cases, dispersion curves overlap for low frequencies, with only some slight differences for high frequencies.

**Concluding remarks**

In this article, we investigated an example structure in which electrostatic forces are used as the means of energy transfer through mechanical vibrations. The difference between electrostatic and elastic forces is how they can reach beyond the nearest neighbor of each unit cell. Dispersion curves of such structures can be changed by varying electric voltage, making them a suitable candidate for tunable metamaterials. Dispersion curves of a couple of sample structures were investigated in this work. Such structures can be made by surface machining methods on the microscale or by additive manufacturing methods on a macro scale. It can be made from materials with low elastic coefficients (such as flexible PVC) and then coated with conductive materials.

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