Holomorphic Koszul–Brylinski homology via Dolbeault cohomology

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Abstract
We use the Dolbeault cohomology to investigate the Koszul–Brylinski homology on holomorphic Poisson manifolds. We obtain the Leray–Hirsch theorem for Hochschild homology and the Mayer–Vietoris sequence, Künneth theorem for holomorphic Koszul–Brylinski homology. In particular, we show some relations of holomorphic Koszul–Brylinski homologies around a blow-up transformation for the general case (not necessarily compact) by our previous works on the Dolbeault cohomology.

Keywords Holomorphic Poisson manifold · Koszul–Brylinski homology · Dolbeault cohomology · Mayer–Vietoris sequence · Künneth theorem · Blow-up formula · Leray–Hirsch theorem

Mathematics Subject Classification Primary 53D17; Secondary 32C35 · 32Q99

1 Introduction

Holomorphic Poisson structures are a special class of Poisson structures, which naturally appear in various fields [8, 9, 16–18]. They have a close relationship with generalized complex geometry. In particular, the local model of generalized complex manifolds is the product of a holomorphic Poisson manifold and a symplectic manifold [1]. We refer the readers to [10–13, 19, 27] and references therein for more results on holomorphic Poisson structures.

For a holomorphic Poisson manifold \((X, \pi)\), the Lichnerowicz-Poisson cohomology \(H^k(X, \pi)\) and the Koszul–Brylinski homology \(H_k(X, \pi)\) are two kinds of important invariants. There are fruitful works on the former one [5, 6, 14, 15, 20, 25, 26], etc., but few results on the later one until now. Stiénon [28] used Lie algebroids to study holomorphic Koszul–Brylinski homology. In particular, he proved that Evens-Lu-Weinstein pairing on holomorphic Koszul–Brylinski homology is nondegenerate [28, Theorem 4.4] and \(H_k(X, \pi) \cong H^{n-k}(X, \pi)\) for the unimodular case [28, Proposition 4.7]. He also obtained that the Euler characteristic of holomorphic Koszul–Brylinski homology coincides with the signed Euler characteristic in the usual sense. X. Chen, Y. Chen, S. Yang and X. Yang gave a
studied in noncommutative geometry and K-theory for the trivial holomorphic Poisson structure [4, Theorem 1.1] and computed the Koszul–Brylinski homology of del Pezzo surfaces and some complex parallelizable manifolds [4, Section 6].

Hochschild homology is a significant invariant of complex manifolds, which is widely studied in noncommutative geometry and K-theory. By the Hochschild-Kostant-Rosenberg theorem, Hochschild homology is isomorphic to the holomorphic Koszul–Brylinski homology for the trivial holomorphic Poisson structure [4, 28]. Hence, the Hochschild homology can be investigated through the approach of holomorphic Koszul–Brylinski homology.

The present paper aims to generalize several classical theorems in topology, such as Leray–Hirsch theorem, Mayer–Vietoris sequence, Künneth theorem, to holomorphic Koszul–Brylinski homology. In particular, we show some relations of Koszul–Brylinski homologies under the blow-up transformation of general (not necessarily compact) holomorphic Poisson manifolds by the self-intersection and the explicit expression of the blow-up formula of Dolbeault cohomology in our previous works [21].

2 Preliminaries

2.1 Notations

For a double complex $K^{\bullet,\bullet}$ (resp. a complex $K^\bullet$), $K^{\bullet,\bullet}[m, n]$ (resp. $K^\bullet[m]$) means the shifted double complex (resp. shifted complex) by bidegree $(m, n)$ (resp. degree $m$), where $m, n \in \mathbb{Z}$. Denote by $sK^{\bullet,\bullet}$ the (simple) complex associated to the double complex $K^{\bullet,\bullet}$ and by $ss(K^{\bullet,\bullet} \otimes L^{\bullet,\bullet})$ the double complex associated to the tensor of double complexes $K^{\bullet,\bullet}$ and $L^{\bullet,\bullet}$. See [22, Sections 2.1, 2.2, 2.4] for more details.

For a complex manifold $X$, denote by $H^{p,q}(X)$ the Dolbeault cohomology of $X$ and by $\mathcal{A}^k_X$ (resp. $\mathcal{D}^k_X$, $\mathcal{A}^{p,q}_X$, $\mathcal{D}^{p,q}_X$, $\mathcal{O}_X$, $\Omega^p_X$) the sheaf of germs of complex-valued smooth $k$-forms (resp. complex-valued $k$-currents, smooth $(p, q)$-forms, $(p, q)$-currents, holomorphic $p$-forms) on $X$ for any $p, q, k \in \mathbb{Z}$. Notice that, if dim$_{\mathbb{C}}X = n$, we tacitly approve that $H^{p,q}(X) = 0$, $\mathcal{A}^{p,q}_X = 0$, $\mathcal{D}^{p,q}_X = 0$ for $p < 0$ or $n$, or $q < 0$ or $n$.

For complex vector spaces $U_1, \ldots, U_n$, the vector in $\bigoplus_{i=1}^n U_i$ is written as $(u_1, \ldots, u_n)^T$, where $u_i \in U_i$ for $1 \leq i \leq n$. The transposition of any matrix is defined for example as $\begin{pmatrix} f_{i1} & \cdots & f_{in} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix} : \bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m V_i$ by

$$(u_1, \ldots, u_n)^T \mapsto \begin{pmatrix} n \\ \vdots \\ 1 \end{pmatrix} \sum_{i=1}^n f_{i1}(u_i), \ldots, \sum_{i=1}^n f_{mi}(u_i)$$

In particular, $(f, g) : U \oplus V \to W$ means $(u, v)^T \mapsto f(u) + g(v)$ for $f : U \to W$ and $g : V \to W$ and $(f, g)^T : W \to U \oplus V$ means $w \mapsto (f(w), g(w))^T$ for $f : W \to U$ and $g : W \to V$.    

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2.2 Holomorphic Poisson manifolds

Suppose that $X$ is a complex manifold and $\mathcal{O}_X(U)$ is endowed with a Poisson bracket $\{\cdot, \cdot\}$ for any open subset $U \subseteq X$ satisfying that the restriction map $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ is a morphism of Poisson algebras for open sets $V \subseteq U$. Then $(X, \{-, -\})$ is said to be a holomorphic Poisson manifold. In this case, there exists a unique holomorphic bivector field $\pi \in H^0(X, \wedge^2 TX)$ such that $\pi(df \wedge dg) = \{f, g\}$ for any $f, g \in \mathcal{O}_X(U)$, where $TX$ is the holomorphic tangent bundle of $X$. We also write $(X, \pi)$ for this holomorphic Poisson manifold and say that $\pi$ is a holomorphic Poisson structure on $X$.

A holomorphic map $\rho : X \rightarrow Y$ between holomorphic Poisson manifolds $(X, \{-, -\})$ and $(Y, \{-, -\})$ is called a morphism if the pullback $\rho^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\rho^{-1}(U))$ is a morphism of Poisson algebras for each open set $U \subseteq Y$. A holomorphic map $\rho : X \rightarrow Y$ between holomorphic Poisson manifolds $(X, \pi)$ and $(Y, \sigma)$ is a morphism if and only if $\rho_* \pi_x = \sigma_{\rho(x)}$ for all $x \in X$.

Let $(X, \pi)$, $(Y, \sigma)$ be holomorphic Poisson manifolds and let $Y$ be also a closed complex submanifold of $X$. We say $(Y, \sigma)$ is a closed holomorphic Poisson submanifold of $(X, \pi)$, if the natural inclusion $i : (Y, \sigma) \rightarrow (X, \pi)$ is a morphism, i.e., $i_* \sigma_y = \pi_y$ for all $y \in Y$. Evidently, there exists at most one holomorphic Poisson structure $\sigma$ on $Y$ such that $(Y, \sigma)$ is a closed holomorphic Poisson submanifold of $(X, \pi)$. In such case, denote $\sigma$ by $\pi|_Y$.

Let $(X, \pi)$ be a holomorphic Poisson manifold. Denote by $l_{\pi} : \mathcal{A}^k(X) \rightarrow \mathcal{A}^{k-2}(X)$ the contraction by $\pi$ and set $\partial_{\pi} := l_{\pi} \circ \partial - \partial \circ l_{\pi} : \mathcal{A}^k_X \rightarrow \mathcal{A}^{k-1}_X$ for any $k \in \mathbb{Z}$. Then $\partial_{\pi}(\mathcal{O}^p_X) \subseteq \mathcal{O}^{p-1}_X$, $\partial_{\pi}(\mathcal{A}^{p,q}_X) \subseteq \mathcal{A}^{p-1,q}_X$, $\partial_{\pi}^2 = 0$ and $\partial_{\pi} \partial_{\pi} + \partial_{\pi} \partial_{\pi} \partial = 0$. If $\rho : (X, \pi) \rightarrow (Y, \sigma)$ is a morphism, then $\partial_{\pi} \rho^* = \rho^* \partial_{\sigma}$.

2.3 Holomorphic Koszul–Brylinski homology

In this subsection, we collect some element knowledge on holomorphic Koszul–Brylinski homology; see also [28, Section 4] and [4, Section 2.2, 3.1]. Let $(X, \pi)$ be a holomorphic Poisson manifold with complex dimension $n$.

2.3.1 Definition

The holomorphic Koszul–Brylinski complex $\mathcal{M}_X^*(\pi)$ of $X$ is the complex of sheaves

$$
0 \longrightarrow \Omega^n_X \xrightarrow{\partial_{\pi}} \Omega^{n-1}_X \xrightarrow{\partial_{\pi}} \cdots \xrightarrow{\partial_{\pi}} \Omega^1_X \xrightarrow{\partial_{\pi}} \mathcal{O}_X \xrightarrow{\partial_{\pi}} 0.
$$

More precisely, $\mathcal{M}_X^k(\pi) = \Omega^{n-k}_X$, $d^k = \partial_{\pi}$. The $k$-th Koszul–Brylinski homology of $(X, \pi)$ is defined as $H_k(X, \pi) := \mathbb{H}^k(X, \mathcal{M}_X^*(\pi))$.

2.3.2 Computation via smooth forms

Set $\mathcal{K}^{p,q}_X(\pi) := \mathcal{A}_X^{-p,q}$, $d_1^{p,q} := \partial_{\pi}$, $d_2^{p,q} := \partial_{\pi}$. Then $(\mathcal{K}^{*,*}_X(\pi), d_1, d_2)$ is a double complex, shortly denoted by $\mathcal{K}^{*,*}_X(\pi)$. Let $\mathcal{K}^{*,*}_X(\pi)$ be the complex associated to $\mathcal{K}^{*,*}_X(\pi)$. Set $\mathcal{K}^{p,q}_X(\pi) := \Gamma(X, \mathcal{K}^{p,q}_X(\pi))$ and $\mathcal{K}^p(\pi) := \Gamma(X, \mathcal{K}^p_X(\pi))$. For any $p \in \mathbb{Z}$, $\mathcal{M}^{p+n}_X(\pi) \rightarrow (\mathcal{K}^{p,n}_X(\pi), d_2^{p,n})$ given by the inclusion is a resolution of $\mathcal{M}^{p+n}_X(\pi)$. By [29, Lemma 8.5], the inclusion gives a quasi-isomorphism $\mathcal{M}_X^*(\pi)[n] \rightarrow \mathcal{K}_X^*(\pi)$ of complexes.
of sheaves. For any \( p \in \mathbb{Z}, \mathcal{K}_X^p(\pi) \) is a soft sheaf, so it is \( \Gamma \)-acyclic. By [29, Proposition 8.12],

\[
H_k(X, \pi) \cong H^{k-n}(K^\bullet(X, \pi))
\]

for any \( k \in \mathbb{Z} \). Associated to \( K^\bullet\bullet_1(X, \pi) \), there is a spectral sequence \( K E^{p,q}_r(X, \pi) \Rightarrow H_{p+q+n}(X, \pi) \), where

\[
K E^{p,q}_1(X, \pi) = H^q(K^{p,\bullet}(X, \pi)) = H^{-p,q}(X).
\]

2.3.3 Computation via currents

For \( T \in \mathfrak{D}'_X, \alpha \in \mathcal{A}^n, \beta \in \mathcal{A}^{n-1}, \) set \( (\partial \pi T)(\alpha) := (-1)^k T(\partial \pi \alpha) \) and \( (\partial \bar{T})(\beta) := (-1)^{k+1} T(\partial \bar{T}) \). Clearly, \( \partial \pi (\mathfrak{D}_X^{p,q}) \subseteq \mathfrak{D}_X^{p-1,q} \), \( \partial^2 = 0 \) and \( \bar{\partial} \partial \pi + \partial \bar{\partial} = 0 \). Let \( \rho : (X, \pi) \rightarrow (Y, \sigma) \) be a morphism of holomorphic Poisson manifolds and denote by \( \rho_* \) the pushforward of currents. Then \( \bar{\partial} \rho_* = \rho_* \bar{\partial} \).

As those of \( \mathcal{K}(\pi) \), we can define \( \mathcal{P}^{\bullet\bullet}_X(\pi), \mathcal{P}^{\bullet\bullet}_X(\pi), \mathcal{P}^{\bullet\bullet}(X, \pi), \mathcal{P}^\bullet(X, \pi) \) and \( \rho E^{p,q}_r(X, \pi) \), where \( \mathcal{P}_X^{p,q}(\pi) := \mathfrak{D}_X^{-p,q}, \mathcal{P}^{p,q}_X \subseteq \partial_\pi, d_2^{p,q} := \bar{\partial} \). Similarly, the inclusion gives a quasi-isomorphism \( \mathcal{M}^\bullet_X(\pi)[n] \rightarrow \mathcal{P}^\bullet_X(\pi) \) of complexes of sheaves.

The inclusion naturally gives the morphism \( K^{p,\bullet}(X, \pi) \hookrightarrow \mathcal{P}^{\bullet\bullet}(X, \pi) \) of double complexes. It induces isomorphisms \( K E^{p,q}_1(X, \pi) = H^q(A^{-p,\bullet}(X)) \rightarrow \rho E^{p,q}_r(X, \pi) = H^q(D^{p,\bullet}(X)) \) for all \( p, q \in \mathbb{Z} \), hence induces isomorphisms \( H^k(K^{\bullet\bullet}(X, \pi)) \rightarrow H^k(P^{\bullet\bullet}(X, \pi)) \) for all \( k \in \mathbb{Z} \). Both \( H^k(K^{\bullet\bullet}(X, \pi)) \) and \( H^k(P^{\bullet\bullet}(X, \pi)) \) will be written as \( H_k(X, \pi) \).

2.3.4 Pullback and pushforward

Let \( \rho : (X, \pi) \rightarrow (Y, \sigma) \) be a morphism of holomorphic Poisson manifolds and set \( r = \dim_X X - \dim_Y Y \). Then \( \rho \) induces the pullback \( \rho^* : K^{\bullet\bullet}(Y, \sigma) \rightarrow K^{\bullet\bullet}(X, \pi) \), hence induces \( \rho^* : H_k(Y, \sigma) \rightarrow H_{k+r}(X, \pi) \). In addition, if \( \rho \) is proper, it induces the pushforward \( \rho_* : K^{\bullet\bullet}(X, \pi) \rightarrow K^{\bullet\bullet}(Y, \sigma)[r, -r] \), hence induces \( \rho_* : H_k(X, \pi) \rightarrow H_{k-r}(Y, \sigma) \).

3 Hochschild homology

Assume that \( X \) is a complex manifold with complex dimension \( n \). Denote by \( \Delta : X \rightarrow X \times X \) the diagonal embedding. Its image is a complex submanifold of \( X \times X \) isomorphic to \( X \), still denoted by \( \Delta \). The Hochschild homology of \( X \) is defined as \( HH_k(X) := \text{Tor}_k(X \times X, \Omega_X) \) for each \( k \in \mathbb{Z} \). By the Hochschild-Kostant-Rosenberg theorem [3, Corollary 3.1.4],

\[
HH_k(X) \cong \bigoplus_{p-q=k} H^q(X, \Omega^p_X).
\]

Set \( \pi = 0 \). In such case, \( (X, \pi) \) is a holomorphic Poisson manifold and \( \partial \pi = 0 \). Moreover,

\[
\mathcal{M}_X^\bullet(0) = \bigoplus_{i=0}^n \Omega_X^{n-i}[-i],
\]
where $\Omega^j_X$ means the complex of sheaves concentrated on 0-th term with the sheaf $\Omega^j_X$. By (3.2), we have
\[ H_k(X, 0) = \bigoplus_{p-q=n-k} H^q(X, \Omega^p_X). \]
Combining (3.1) and (3.3), we have
\[ H_k(X, 0) \cong \text{HH}_{n-k}(X). \]  
See also [28, Remark 4.2] [4, p.17] for other discussions. Hence, we can study the Hochschild homology via the holomorphic Koszul–Brylinski homology.

**Theorem 3.1** Let $\rho : E \to X$ be a holomorphic fiber bundle over a complex manifold $X$. Assume that there exist $d$-closed forms $t_1, \ldots, t_r$ of pure degrees on $E$ such that the restrictions of their Dolbeault classes $[t_1]_\partial, \ldots, [t_r]_\partial$ to $E_x$ is a basis of $H^{*,*}(E_x) = \bigoplus H^{p,q}(E_x)$ for every $x \in X$. Then there exists an isomorphism
\[ \bigoplus_{i=1}^r \text{HH}_{k+v_i-u_i}(X) \cong \text{HH}_k(E) \]
for any $k \in \mathbb{Z}$, where $(u_i, v_i)$ is the bidegree of $t_i$ for $1 \leq i \leq r$.

**Proof** Set $n = \dim \mathbb{C}X$ and $m = \dim \mathbb{C}E - \dim \mathbb{C}X$. Set
\[ S^{*,*} := \bigoplus_{i=1}^r K^{*,*}(X, 0)[u_i, -v_i] \quad \text{and} \quad T^{*,*} := K^{*,*}(E, 0). \]
By (2.2), we get the first pages
\[ S E_1^{p,q} = \bigoplus_{i=1}^r H^{-(p+u_i), q-v_i}(X) \quad \text{and} \quad T E_1^{p,q} = H^{-p,q}(E) \]
of the spectral sequences associated to $S^{*,*}$ and $T^{*,*}$ respectively. By [23, Theorem 4.2], the morphism $\rho^* (\bullet) \wedge t_1, \ldots, \rho^* (\bullet) \wedge t_r : S^{*,*} \to T^{*,*}$ of double complexes induces an isomorphism $s E_1^{iq} \cong T E_1^{iq}$ at $E_1$-pages, hence induces an isomorphism $H^{k}(s S^{*,*}) \to H^{k}(s T^{*,*})$ for any $k \in \mathbb{Z}$. Notice that $s S^{*,*} = \bigoplus_{i=1}^r K^*(X, 0)[u_i - v_i]$ and $s T^{*,*} = K^*(E, 0)$. By (2.1) and (3.4),
\[ H^{-k}(s S^{*,*}) \cong \bigoplus_{i=1}^r H_{-k+u_i-v_i}(X, 0) \cong \bigoplus_{i=1}^r \text{HH}_{k+v_i-u_i}(X) \]
and
\[ H^{-k}(s T^{*,*}) \cong H_{-k+n+m}(E, 0) \cong \text{HH}_k(E). \]
We complete the proof. \(\square\)

**Corollary 3.2** Let $E$ be the flag bundle associated to a holomorphic vector bundle over a complex manifold $X$. Denote by $F$ the general fiber of $E$ over $X$. Then there exists an isomorphism
\[ \text{HH}_k(X)^{\oplus b(F)} \to \text{HH}_k(E) \]
for any $k \in \mathbb{Z}$, where $b(F)$ is the sum of all betti numbers of $F$. 

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Proof Assume that $V$ is a holomorphic vector bundle with rank $n$ over $X$ and $(n_1, \ldots, n_r)$ is a sequence of positive integers with $\sum_{i=1}^r n_i = n$, such that the fiber $E_x$ of $E$ over $x \in X$ is the flag manifold $Fl(n_1, \ldots, n_r)(V_x) = \{(W_0, W_1, \ldots, W_{r-1}, W_r) | 0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{r-1} \subseteq W_r = V_x\}$, where $W_i$ is a complex vector space with dimension $\sum_{j=1}^i n_j$ for $1 \leq i \leq r$.

For $0 \leq i \leq r$, assume that $V_i$ is the universal subbundle over $E$ whose fiber over the point $(W_0, W_1, \ldots, W_{r-1}, W_r)$ is $W_i$. Notice that $V_0 = 0$ and $V_r = \rho^*V$, where $\rho$ is the projection from $E$ onto $X$. For $1 \leq i \leq r$, set $V^{(i)} = V_i/V_{i-1}$ and denote by $t_j^{(i)} \in A^{j,i}(E)$ a $j$-th Chern form of $V^{(i)}$. For any $x \in X$, the restrictions $t_j^{(i)}|_{E_x}$ $(1 \leq i \leq r, 1 \leq j_i \leq n_i)$ to $E_x$ are Chern forms of successive universal quotient bundles of the flag manifold $E_x$. As we know, there exists the monomials $P_i(T_1^{(1)}, \ldots, T_n^{(1)}, \ldots, T_1^{(r)}, \ldots, T_n^{(r)})$ for $1 \leq i \leq l$ such that

$$t_i := P_i([t_1^{(1)}|_{E_x}] \bar{\alpha}, \ldots, [t_{n_1}^{(1)}|_{E_x}] \bar{\alpha}, \ldots, [t_1^{(r)}|_{E_x}] \bar{\alpha}, \ldots, [t_{n_r}^{(r)}|_{E_x}] \bar{\alpha}), \quad 1 \leq i \leq l,$$

is a basis of $H^{\bullet, \bullet}(E_x)$. Clearly, all $t_i$ are $d$-closed on $E$ and $l = \dim_\C H^{\bullet, \bullet}(E_x)$. By the Hodge decomposition theorem, $l = b(F)$, since the flag manifold $F = E_x$ is projective. By Theorem 3.1, the corollary follows.

Remark 3.3 A flag manifold can be viewed as a flag bundle over a single point. Its Hochschild homology can be obtained by Corollary 3.2, which is a special case of the following Sect. 3.2.1.

4 Holomorphic Koszul–Brylinski homology

4.1 Stein manifolds and flag manifolds

Let $(X, \pi)$ be a holomorphic Poisson manifold of complex dimension $n$.

Assume that $X$ is a Stein manifold. Then

$$\kappa E_1^{p,0}(X, \pi) = \Gamma(X, \Omega_X^{-p}), \quad \kappa E_1^{p,q}(X, \pi) = 0 \text{ for } q \neq 0,$$

and $d_1^{p,0} = \partial_\pi, \quad a_1^{p,q} = 0 \text{ for } q \neq 0$. We obtained that $\kappa E_2^{p,0}(X, \pi) = H^{-p}(\Gamma(X, \Omega_X^\bullet), \partial_\pi)$ and $E_2^{p,q}(X, \pi) = 0 \text{ for } q \neq 0$. Hence the spectral sequence $\kappa E_1^{\bullet, \bullet}(X, \pi)$ degenerates at $E_2$-pages and

$$H_k(X, \pi) = H^{-k+n}(\Gamma(X, \Omega_X^\bullet), \partial_\pi) = H^k(\Gamma(X, \mathcal{M}_X^\bullet(\pi)))$$

for any $k \in \Z$.

Assume that $X$ is a flag manifold. Then $\kappa E_1^{p,q}(X, \pi) = H^{-p,q}(X) = 0$ for $p + q \neq 0$ and hence all $d_1^{p,q} = 0$. The spectral sequence $\kappa E_r^{\bullet, \bullet}(X, \pi)$ degenerates at $E_1$-pages. We have

$$H_k(X, \pi) = \bigoplus_{p+q=k-n} H^{-p,q}(X) = \begin{cases} \mathbb{C}^b(X), & k = n \\ 0, & \text{others} \end{cases} \quad (4.1)$$

for any $k \in \Z$, where $b(X)$ is the sum of all betti numbers of $X$. 
4.2 Mayer–Vietoris sequence

Suppose that \((X, \pi)\) is a holomorphic Poisson manifold. For each open subset \(U \subseteq X\), \((U, \pi|_U)\) is naturally a holomorphic Poisson manifold. Shortly write \(H_k(U, \pi|_U)\) as \(H_k(U, \pi)\). The inclusion \(j : U \hookrightarrow X\) induces the pullback \(j^* : H_k(X, \pi) \rightarrow H_k(U, \pi)\).

We have the Mayer–Vietoris type sequence for holomorphic Koszul-Brylinski homology as follows.

**Proposition 4.1** Suppose that \((X, \pi)\) is a holomorphic Poisson manifold. For open subsets \(U, V \subseteq X\), there is a long exact sequence

\[
\cdots \rightarrow H_k(U \cup V, \pi) \xrightarrow{(j_1^*, j_2^*)^T} H_k(U, \pi) \oplus H_k(V, \pi) \xrightarrow{(j_1^*, -j_2^*)} H_k(U \cap V, \pi) \rightarrow g \rightarrow H_{k+1}(U \cup V, \pi) \rightarrow \cdots
\]

where \(j_1 : U \cap V \rightarrow U\) and \(j_2 : U \cap V \rightarrow V\) are inclusions.

**Proof** Denote \(f = (j_1^*, j_2^*)^T\) and \(g = (j_1^*, -j_2^*)\). For any \(p, q \in \mathbb{Z}\), consider the sequence

\[
0 \rightarrow A^{-p,q}(U \cup V) \xrightarrow{f} A^{-p,q}(U) \oplus A^{-p,q}(V) \xrightarrow{g} A^{-p,q}(U \cap V) \rightarrow 0.
\]

(4.2)

Evidently, \(f\) is injective and \(\ker g = \text{im } f\). Let \(\{\rho_U, \rho_V\}\) be a partition of unity subordinate to the open covering \([U, V]\) of \(U \cup V\). That is to say, \(\rho_U, \rho_V \in C^\infty(U \cup V)\) satisfy that \(\rho_U + \rho_V = 1\) and \(\text{supp } \rho_U \subseteq U\), \(\text{supp } \rho_V \subseteq V\). For any \(\eta \in A^{-p,q}(U \cap V)\), \(\rho_V \eta \in A^{-p,q}(U)\) and \(-\rho_U \eta \in A^{-p,q}(V)\). Clearly, \(g(\rho_V \eta, -\rho_U \eta)^T = \eta\). So \(g\) is surjective. We proved that (4.2) is exact. Hence, we easily obtain the short exact sequence of complexes

\[
0 \rightarrow K^*(U \cup V, \pi) \rightarrow K^*(U, \pi) \oplus K^*(V, \pi) \rightarrow K^*(U \cap V, \pi) \rightarrow 0,
\]

which induces the long exact sequence in this proposition. \(\square\)

4.3 Künneth theorem

Suppose that \((X, \pi)\) and \((Y, \sigma)\) are holomorphic Poisson manifolds. Define the bivector filed \(\pi \oplus \sigma\) as follows:

Let \((U, z_1, \ldots, z_n)\) and \((V, w_1, \ldots, w_m)\) be the charts of local coordinates of \(X\) and \(Y\) respectively. If \(\pi|_U = \sum_{1 \leq i, j \leq n} \pi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}\) and \(\sigma|_U = \sum_{1 \leq k, l \leq m} \sigma_{kl}(w) \frac{\partial}{\partial w_k} \wedge \frac{\partial}{\partial w_l}\), then \(\pi \oplus \sigma\) is defined as

\[
\sum_{1 \leq i, j \leq n} \pi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{1 \leq k, l \leq m} \sigma_{kl}(w) \frac{\partial}{\partial w_k} \wedge \frac{\partial}{\partial w_l}
\]
on \(U \times V\).

**Theorem 4.2** Assume that \((X, \pi)\) and \((Y, \sigma)\) are holomorphic Poisson manifolds. If \(X\) or \(Y\) is compact, then there is an isomorphism

\[
\bigoplus_{p+q=k} H_p(X, \pi) \otimes_{\mathbb{C}} H_q(Y, \sigma) \cong H_k(X \times Y, \pi \oplus \sigma)
\]

for any \(k\).

**Proof** Set \(n = \dim_{\mathbb{C}} X, m = \dim_{\mathbb{C}} Y\) and let \(pr_1, pr_2\) be the projections from \(X \times Y\) onto \(X, Y\), respectively.
Suppose that $\alpha \in \mathcal{A}^*(X)$ and $\beta \in \mathcal{A}^*(Y)$. By the definition of $\pi \oplus \sigma$, we get

$$l_{\pi \oplus \sigma} [pr_1^*(\alpha) \wedge pr_2^*(\beta)] = l_{\pi \oplus 0} pr_1^*(\alpha) \wedge pr_2^*(\beta) + l_{0 \oplus \sigma} pr_2^*(\beta) = pr_1^*(l_\pi \alpha) \wedge pr_2^*(\beta) + pr_2^*(l_\sigma \beta),$$

where the second equality uses the fact that $pr_1(\pi \oplus 0) = \pi$ and $pr_2(0 \oplus \sigma) = \sigma$. Hence,

$$\partial_{\pi \oplus \sigma} [pr_1^*(\alpha) \wedge pr_2^*(\beta)] = pr_1^*(\partial_\pi \alpha) \wedge pr_2^*(\beta) + (-1)^{\deg_a} pr_1^*(\alpha) \wedge pr_2^*(\partial_\sigma \beta). \quad (4.3)$$

Consider the two double complexes

$$S^{*,*} := \text{ss} \left(K^{*,*}(X, \pi) \otimes_{\mathbb{C}} K^{*,*}(Y, \sigma)\right) \quad \text{and} \quad T^{*,*} := K^{*,*}(X \times Y, \pi \oplus \sigma).$$

By [22, Proposition 2.7 (2)] and (2.2), we have the first pages

$$E^{p,q}_1 = \bigoplus_{a+b=p, \ r+s=q} H^r(K^{a,*}(X, \pi)) \otimes_{\mathbb{C}} H^s(K^{b,*}(Y, \sigma))$$

and $T E^{p,q}_1 = H^{-p,q}(X \times Y)$ of the spectral sequences associated to $S^{*,*}$ and $T^{*,*}$ respectively. By (4.3), $f = pr_1^*(\bullet) \wedge pr_2^*(\bullet) : S^{*,*} \rightarrow T^{*,*}$ is a morphism between double complexes. The morphism $E^{p,q}_1(f) : E^{p,q}_1 \rightarrow T E^{p,q}_1$ at $E_1$-pages induced by $f$ is just the cartesian product

$$\bigoplus_{a+b=p, \ r+s=q} H^{-a,r}(X) \otimes_{\mathbb{C}} H^{-b,s}(Y) \rightarrow H^{-p,q}(X \times Y).$$

Notice that $\Omega^{-p}_{X \times Y} = \bigoplus_{a+b=p} \left(\Omega^{-a}_X \boxtimes \Omega^{-b}_Y\right)$, where $\boxtimes$ means the analytic external tensor product of coherent analytic sheaves. By [7, IX, (5.23) (5.24)], $E^{p,q}_1(f)$ is an isomorphism for any $p, q \in \mathbb{Z}$, so is the morphism $H^{k-n-m}(sS^{*,*}) \rightarrow H^{k-n-m}(sT^{*,*})$ induced by $f$ for any $k \in \mathbb{Z}$. By [22, Proposition 2.7 (1)] and (2.1),

$$H^{k-n-m}(sS^{*,*}) \cong \bigoplus_{p+q=k-n-m} H^p(K^*(X, \pi)) \otimes_{\mathbb{C}} H^q(K^*(Y, \sigma))$$

and $H^{k-n-m}(sT^{*,*}) \cong H_k(X \times Y, \pi \oplus \sigma)$. We conclude this theorem. \qed

### 4.4 Blow-up formulae

Suppose that $\rho : X \rightarrow Y$ is a proper holomorphic map of complex manifolds. We have the projection formula

$$\rho^*_s(T \wedge \rho^* u) = \rho^* s \wedge u$$

for any $T \in \mathcal{D}^{*,*}(X)$ and $u \in \mathcal{A}^{*,*}(Y)$. Hence

$$\rho^*_s(\varphi \cup \rho^* \eta) = \rho^*_s \varphi \cup \eta \quad (4.4)$$
for $\varphi \in H^{\bullet, \bullet}(X)$ and $\eta \in H^{\bullet, \bullet}(Y)$. Let $\rho : (X, \pi) \to (Y, \sigma)$ be a proper surjective morphism of holomorphic Poisson manifolds with the same dimensions. Then $\rho_*(1_X) = \deg \rho \cdot 1_Y$, where $1_X, 1_Y$ mean the currents defined by the constant 1 on $X, Y$ respectively and $\deg \rho$ denotes the degree of $\rho$. So

$$\rho_\ast \rho^* = \deg \rho \cdot \text{id} : H_k(Y, \sigma) \to H_k(Y, \sigma). \quad (4.5)$$

See also [4, Theorem 3.6] for a Poisson modification $\rho$, where a Poisson modification means a surjective morphism $\rho : (X, \pi) \to (Y, \sigma)$ of compact holomorphic Poisson manifolds with the same dimensions satisfying that there is an analytic subset $S \subseteq Y$ of codimension $\geq 2$ such that the restriction $\rho : X - \rho^{-1}(S) \to Y - S$ is biholomorphic.

Let $\rho : \widetilde{X} \to X$ be the blow-up of a complex manifold $X$ along a complex submanifold $Y$ with the exceptional divisor $E$. Assume that $i_Y : Y \to X$ is the inclusion. As we know, $\rho|_E : E \to Y$ can be naturally viewed as the projective bundle $\mathbb{P}(N_Y/X)$ associated to the normal bundle $N_{Y/X}$ of $Y$ in $X$. Let $t \in \mathcal{A}^{1,1}(E)$ be a first Chern form of the universal line bundle $\mathcal{O}_E(-1)$ on $E \cong \mathbb{P}(N_Y/X)$ and let $h$ be the Dolbeault cohomology class of $t$. Suppose that $i_E : E \to \widetilde{X}$ is the inclusion and $r = \text{codim}_C Y \geq 2$. Notice that

$$(\rho|_E)_* h^i = 0, \quad 0 \leq i \leq r - 2, \quad (\rho|_E)_* h^{r-1} = (-1)^{r-1}, \quad (4.6)$$

see [23, p.20].

**Lemma 4.3** For any $p, q \in \mathbb{Z}$,

$$F : H^{p,q}(X) \oplus H^{p-1,q-1}(E) \to H^{p,q}(\widetilde{X}) \oplus H^{p-r,q-r}(Y)$$

$$(\alpha, \beta)^T \mapsto (\rho^* \alpha + i_{E*} \beta, (\rho|_E)^* \beta)^T$$

and

$$G : H^{p,q}(\widetilde{X}) \oplus H^{p,q}(Y) \to H^{p,q}(X) \oplus H^{p,q}(E)$$

$$(\alpha, \beta)^T \mapsto (\rho_* \alpha, i_E^* \alpha - (\rho|_E)^* \beta)^T$$

are isomorphisms.

**Proof** Firstly, we prove that $F$ is an isomorphism.

Suppose that $F(\alpha, \beta) = 0$ for $\alpha \in H^{p,q}(X), \beta \in H^{p-1,q-1}(E)$. Then $\rho^* \alpha + i_{E*} \beta = 0$ and $(\rho|_E)^* \beta = 0$. By [21, Corollary 3.2], $\beta = \sum_{i=0}^{r-1} h^i \cup (\rho|_E)^* \theta_i$ for some $\theta_i \in H^{p-1,q-1}(Y), 0 \leq i \leq r - 1$. By (4.4) and (4.6), $(\rho|_E)^* \beta = (-1)^{r-1} \theta_{r-1}$, which implies that $\theta_{r-1} = 0$. Therefore, $\beta = \sum_{i=0}^{r-2} h^i \cup (\rho|_E)^* \theta_i$, and then

$$\rho^* \alpha + \sum_{i=1}^{r-1} i_{E*}[h^{i-1} \cup (\rho|_E)^* \theta_{i-1}] = 0.$$ 

By [21, Theorem 1.2], $\alpha = 0$ and $\theta_i = 0$ for $0 \leq i \leq r - 2$. So $\beta = 0$. Thus, $F$ is injective. Give any $(\eta, \omega) \in H^{p,q}(\widetilde{X}) \oplus H^{p-r,q-r}(Y)$. Then $(-1)^{r-1} i_{E*}[h^{r-1} \cup (\rho|_E)^* \omega] - \rho^* i_Y \omega \in H^{p,q}(\widetilde{X})$. By [21, Theorem 1.2], there exist $\xi \in H^{p,q}(X)$ and $\xi_i \in H^{p-i,q-i}(Y), 1 \leq i \leq r - 1$ such that

$$(-1)^{r-1} i_{E*}[h^{r-1} \cup (\rho|_E)^* \omega] - \rho^* i_Y \omega = \rho^* \xi + \sum_{i=1}^{r-1} i_{E*}[h^{i-1} \cup (\rho|_E)^* \xi_i]. \quad (4.7)$$
Pushforward (4.7) by $\rho_*$, we have $\zeta = 0$ by (4.4). So

$$(-1)^{r-1} i_{E*}[h^{r-1} \cup (\rho|_E)^*\omega] - \rho^* i_{Y*}\omega = \sum_{i=1}^{r-1} i_{E*}[h^{i-1} \cup (\rho|_E)^*\xi_i].$$

(4.8)

By [21, Theorem 1.2], there exist $\gamma \in H^{p,q}(X)$ and $\delta_i \in H^{p-i,q-i}(Y)$, $1 \leq i \leq r - 1$ such that

$$\eta = \rho^*\gamma + \sum_{i=1}^{r-1} i_{E*}[h^{i-1} \cup (\rho|_E)^*\delta_i].$$

(4.9)

Set $\alpha := \gamma - i_{Y*}\omega$ and $\beta := \sum_{i=0}^{r-2} h^i \cup (\rho|_E)^*(\delta_{i+1} - \xi_{i+1}) + (-1)^{r-1} h^{r-1} \cup (\rho|_E)^*\omega$. We easily check that, $\rho^*\alpha + i_{E*}\beta = \eta$ by (4.8), (4.9) and $(\rho|_E)_*\beta = \omega$ by (4.4), (4.6). Hence $F$ is surjective. Up to now, we proved that $F$ is an isomorphism.

Secondly, we prove that $G$ is an isomorphism.

Assume that $Y$ is a Stein manifold. Suppose that $G(\alpha, \beta)^T = 0$ for $\alpha \in H^{p,q}(\tilde{X})$, $\beta \in H^{p,q}(Y)$. Then $\rho_\alpha = 0$ and $i^*_E\alpha - (\rho|_E)_*\beta = 0$. By [21, Theorem 1.2],

$$\alpha = \rho^*\gamma + \sum_{i=1}^{r-1} i_{E*}[h^{i-1} \cup (\rho|_E)^*\beta_i]$$

for some $\gamma \in H^{p,q}(X)$, $\beta_i \in H^{p-i,q-i}(Y)$, $1 \leq i \leq r - 1$. Then

$$(\rho|_E)_*\beta = i^*_E\rho^*\gamma + \sum_{i=1}^{r-1} i^*_E i_{E*}[h^{i-1} \cup (\rho|_E)^*\beta_i] = (\rho|_E)^*i^*_Y\gamma + \sum_{i=1}^{r-1} h^i \cup (\rho|_E)^*\beta_i,$$

where the second equality used [21, Lemma 4.4]. By [21, Corollary 3.2], $\beta = i^*_Y\gamma$ and $\beta_i = 0$ for $1 \leq i \leq r - 1$. So $\alpha = \rho^*\gamma$. By (4.4), $\gamma = \rho_\alpha \rho^*\gamma = \rho_\alpha 0 = 0$ and then, $\alpha = 0$, $\beta = 0$. Thus, $F$ is injective. For any $(\eta, \omega)^T \in H^{p,q}(X) \oplus H^{p,q}(E)$, $\omega = \sum_{i=0}^{r-1} h^i \cup (\rho|_E)^*\theta_i$ for some $\theta_i \in H^{p-i,q-i}(Y)$ by [21, Corollary 3.2]. Set $\alpha := \rho^*\eta + \sum_{i=1}^{r-1} i_{E*}(h^{i-1} \cup (\rho|_E)^*\theta_i)$ and $\beta := i^*_Y\eta - \theta_0$. By (4.4) and (4.6),

$$\rho_\alpha = \rho_\alpha \rho^*\eta + \sum_{i=1}^{r-1} i_{Y*}[(\rho|_E)_*(h^{i-1} \cup (\rho|_E)^*\theta_i)] = \eta,$$

and by [21, Lemma 4.4],

$$i^*_E\alpha - (\rho|_E)_*\beta = \left[i^*_E\rho^*\eta + \sum_{i=1}^{r-1} i^*_E i_{E*}[h^{i-1} \cup (\rho|_E)^*\theta_i] \right] - [(\rho|_E)^*i^*_Y\eta - (\rho|_E)^*\theta_0]$$

$$= \sum_{i=0}^{r-1} h^i \cup (\rho|_E)_*\theta_i = \omega.$$n

Thus, $G(\alpha, \beta)^T = (\eta, \omega)^T$. Hence $G$ is surjective. We proved that $G$ is an isomorphism if $Y$ is Stein.
Go back to general cases. Set $\tilde{U} := \rho^{-1}(U)$ and

$$F^{p,q}(U) := A^{p,q}(\tilde{U}) \oplus A^{p,q}(Y \cap U),$$

$$G^{p,q}(U) := D^{p,q}(U) \oplus A^{p,q}(E \cap \tilde{U}).$$

for any open set $U \subseteq X$ and any $p, q \in \mathbb{Z}$. Then $g^{p,q}_U$ gives a morphism $(F^{p,q}(U), \partial, \tilde{\partial}) \rightarrow (G^{p,q}(U), \partial, \tilde{\partial})$ of double complexes, and furthermore induces a morphism

$$G^{p,q}_U : H^{p,q}(\tilde{U}) \oplus H^{p,q}(Y \cap U) \rightarrow H^{p,q}(U) \oplus H^{p,q}(E \cap \tilde{U}).$$

for any open set $U \subseteq X$ and any $p, q \in \mathbb{Z}$. Then $g^{p,q}_U$ gives a morphism $(F^{p,q}(U), \partial, \tilde{\partial}) \rightarrow (G^{p,q}(U), \partial, \tilde{\partial})$ of double complexes, and furthermore induces a morphism

Denote by $\mathcal{P}(U)$ the statement that $G^{p,q}_U$ are isomorphisms for all $p, q \in \mathbb{Z}$. This lemma is equivalent to say that $\mathcal{P}(X)$ holds.

Now, we check that $\mathcal{P}$ satisfies the three conditions in [21, Lemma 2.1]. Obviously, $\mathcal{P}$ satisfies the disjoint condition. For open sets $V \subseteq U$, denote by $\iota^{U}_V : F^{p,q}(U) \rightarrow F^{p,q}(V)$ and $\iota^{U}_V : G^{p,q}(U) \rightarrow G^{p,q}(V)$ the corresponding restrictions. Fix an integer $p$. For open subsets $U, V$ in $X$, there is a commutative diagram

$$0 \rightarrow F^{p,\bullet}(U \cup V) \xrightarrow{(\iota^{U\cap V}_U, \iota^{U\cap V}_V)^T} F^{p,\bullet}(U) \oplus F^{p,\bullet}(V) \xrightarrow{g^{p,\bullet}_U \oplus g^{p,\bullet}_V} F^{p,\bullet}(U \cap V) \rightarrow 0$$

and

$$0 \rightarrow G^{p,\bullet}(U \cup V) \xrightarrow{(\iota^{U\cap V}_U, \iota^{U\cap V}_V)^T} G^{p,\bullet}(U) \oplus G^{p,\bullet}(V) \xrightarrow{g^{p,\bullet}_U \oplus g^{p,\bullet}_V} G^{p,\bullet}(U \cap V) \rightarrow 0$$

of complexes. By the exactness of (4.2), the two rows in (4.10) are both exact sequences of complexes. For convenience, set

$$I^{p,q}(U) := H^{p,q}(\tilde{U}) \oplus H^{p,q}(Y \cap U), \quad J^{p,q}(U) := H^{p,q}(U) \oplus H^{p,q}(E \cap \tilde{U}).$$

Then (4.10) induces a commutative diagram

$$\begin{array}{ccccccc}
I^{p,q+1}(U \cap V) & \rightarrow & I^{p,q}(U \cup V) & \rightarrow & I^{p,q}(U) \oplus I^{p,q}(V) & \rightarrow & I^{p,q}(U \cap V) & \rightarrow & I^{p,q+1}(U \cup V) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
G^{p,q+1}_U & \rightarrow & G^{p,q}_U \oplus G^{p,q}_V & \rightarrow & G^{p,q}_U \oplus G^{p,q}_V & \rightarrow & G^{p,q}_U \oplus G^{p,q}_V \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
G^{p,q+1}_U & \rightarrow & G^{p,q}_U \oplus G^{p,q}_V & \rightarrow & G^{p,q}_U \oplus G^{p,q}_V & \rightarrow & G^{p,q}_U \oplus G^{p,q}_V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
J^{p,q-1}(U \cap V) & \rightarrow & J^{p,q}(U \cup V) & \rightarrow & J^{p,q}(U) \oplus J^{p,q}(V) & \rightarrow & J^{p,q}(U \cap V) & \rightarrow & J^{p,q+1}(U \cup V) \\
\end{array}$$

of long exact sequences. If $G^{p,q}_U, G^{p,q}_V$ and $G^{p,q}_{U \cap V}$ are isomorphisms for all $p, q \in \mathbb{Z}$, then so are $G^{p,q}_{U \cup V}$ for all $p, q \in \mathbb{Z}$ by the five-lemma. Thus $\mathcal{P}$ satisfies the Mayer–Vietoris condition.

Let $\mathcal{U}$ be a basis of the topology of $X$ such that each $U \in \mathcal{U}$ is Stein. Then $Y \cap \bigcap_{i=1}^{l} U_i$ is empty or Stein for any $U_1, \ldots, U_l \in \mathcal{U}$. As we have proved, $G^{p,q}_{U_1 \cap \ldots \cap U_l}$ is an isomorphism for every $p, q \in \mathbb{Z}$, so $\mathcal{P}$ satisfies the local condition. By [21, Lemma 2.1], $\mathcal{P}(X)$ holds, i.e., $G$ is an isomorphism.

Suppose that $(Y, \pi|_Y)$ is a closed holomorphic Poisson submanifold of a holomorphic Poisson manifold $(X, \pi)$. In this case, the conormal bundle $N^*_{Y/X}$ has a fiberwise Lie algebra structure given by the Poisson bracket. Let $\rho : \tilde{X} \rightarrow X$ be the blow-up of $X$ along $Y$ and let $E, r, i_Y$ and $i_E$ be the ones defined as above. We add the following assumption:

(*) $N^*_{Y/X, y}$ is an abelian Lie algebra over each $y \in Y$.

By [24, Propositions 8.2, 8.3] or [2, Proposition 3.15], there exists a unique holomorphic Poisson structure $\tilde{\pi}$ on $\tilde{X}$ such that $\tilde{\pi}|_E$ is a holomorphic Poisson structure on $E$ and $\rho : \tilde{X} \rightarrow X$ is the blow-up of $X$ along $Y$. The Poisson bracket $\{\cdot, \cdot\}_Y$ on $Y$ is the same as the one defined above.

Let $\tilde{U} := \rho^{-1}(U)$ and $\pi^{-1}(U)$ be the submanifolds of $\tilde{X}$ and $X$ respectively. The push-forward $\rho_\ast : \pi_\ast \rightarrow \tilde{\pi}_\ast$ is a morphism of Poisson structures. It follows from [24, Theorem 8.1] or [2, Theorem 3.15] that $\{\cdot, \cdot\}_Y$ is the same as the one defined above.
$(\tilde{X}, \tilde{\pi}) \to (X, \pi), i_E : (E, \tilde{\pi}|_E) \to (\tilde{X}, \tilde{\pi}), \rho|_E : (E, \tilde{\pi}|_E) \to (Y, \tilde{\pi}|_Y)$ are all morphisms between holomorphic Poisson manifolds.

Under the assumption $(\star)$, we have the following blow-up formulae.

**Theorem 4.4** Fix an integer $k$. Then

$$(1) \quad \left( \begin{array}{ll} \rho^* & i_{E^*} \\ 0 & (\rho|_E)^* \end{array} \right) : H_k(X, \pi) \oplus H_{k-1}(E, \tilde{\pi}|_E) \to H_k(\tilde{X}, \tilde{\pi}) \oplus H_{k-r}(Y, \pi|_Y)$$

(4.11)

is an isomorphism, and moreover;

$$
\begin{array}{ccccccccc}
0 & \to & H_{k-1}(E, \tilde{\pi}|_E) & (i_{E^*}, (\rho|_E)^*)^T & H_k(\tilde{X}, \tilde{\pi}) & \oplus & H_{k-r}(Y, \pi|_Y) & (\rho^*, -i_Y^*) & H_k(X, \pi) & \to & 0 \\
\end{array}
$$

(4.12)

is a split exact sequence, where $(\rho^*, 0)^T$ is a right inverse of $(\rho_*, -i_Y^*)$;

$$(2) \quad \left( \begin{array}{cc} \rho^* & 0 \\ i_E^* - (\rho|_E)^* \end{array} \right) : H_k(\tilde{X}, \tilde{\pi}) \oplus H_{k-r}(Y, \pi|_Y) \to H_k(X, \pi) \oplus H_{k-1}(E, \tilde{\pi}|_E)$$

(4.13)

is an isomorphism, and moreover;

$$
\begin{array}{ccccccccc}
0 & \to & H_k(X, \pi) & (\rho^*, i_Y^*)^T & H_k(\tilde{X}, \tilde{\pi}) & \oplus & H_{k-r}(Y, \pi|_Y) & (i_E^*, - (\rho|_E)^*) & H_{k-1}(E, \tilde{\pi}|_E) & \to & 0 \\
\end{array}
$$

(4.14)

is a split exact sequence, where $(\rho_*, 0)$ is a left inverse of $(\rho^*, i_Y^*)^T$.

**Proof** (1) Consider the double complexes

$$S^{*, *} := K^{*, *}(X, \pi) \oplus K^{*, *}(E, \tilde{\pi}|_E)[1, -1],$$

$$T^{*, *} := P^{*, *}(\tilde{X}, \tilde{\pi}) \oplus P^{*, *}(Y, \pi|_Y)[r, -r].$$

By (2.2), we get the first pages

$$sE^{p,q}_1 = H^{-p,q} (X) \oplus H^{-p,q} (E) \quad \text{and} \quad T E^{p,q}_1 = H^{-p,q} (\tilde{X}) \oplus H^{-p,q} (Y),$$

of the spectral sequences associated to $S^{*, *}$ and $T^{*, *}$ respectively. Define $f : S^{*, *} \to T^{*, *}$ as $(a, b) \mapsto (\rho^* a + i_{E^*} b, (\rho|_E)^* b)$. It is easy to check that $f$ is a morphism of double complexes. By Lemma 4.3, $f$ induces an isomorphism $sE^{p,q}_1 \to T E^{p,q}_1$ at $E_1$-pages, hence induces an isomorphism $H^{k-n}(sS^{*, *}) \to H^{k-n}(sT^{*, *})$ for any $k \in \mathbb{Z}$, where $n = \dim_X X$. By (2.1),

$$H^{k-n}(sS^{*, *}) = H^{k-n}(K^{*, *}(X, \pi)) \oplus H^{k-n}(K^{*, *}(E, \tilde{\pi}|_E))$$

$$\cong H_k(X, \pi) \oplus H_{k-1}(E, \tilde{\pi}|_E),$$

$$H^{k-n}(sT^{*, *}) = H^{k-n}(P^{*, *}(\tilde{X}, \tilde{\pi})) \oplus H^{k-n}(P^{*, *}(Y, \pi|_Y))$$

$$\cong H_k(\tilde{X}, \tilde{\pi}) \oplus H_{k-r}(Y, \pi|_Y).$$

Thus, (4.11) is an isomorphism.

Suppose that $\left(i_{E^*}, (\rho|_E)^*\right)^T(\gamma) = 0$ for $\gamma \in H_{k-1}(E, \tilde{\pi}|_E)$. Then

$$\left( \begin{array}{cc} \rho^* & i_{E^*} \\ 0 & (\rho|_E)^* \end{array} \right) \left( \begin{array}{c} 0 \\ \gamma \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right).$$
which implies that $\gamma = 0$ by (4.11). So $(i_{E*}, (\rho|_{E})_*)^T$ is injective. Evidently, $(\rho_*, -i_{Y*}) \circ (i_{E*}, (\rho|_{E})_*)^T = 0$. Assume that $(\rho_*, -i_{Y*})(\alpha, \beta)^T = 0$ for $\alpha \in H_k(\tilde{X}, \tilde{\pi})$ and $\beta \in H_{k-q}(Y, \pi|_Y)$. By (4.11), there exist $\xi \in H_k(X, \pi)$ and $\eta \in H_{k-q}(E, \tilde{\pi}|_E)$ such that $\alpha = \rho^*\xi + i_{E*}\eta$ and $\beta = (\rho|_{E}_*)\eta$. Then $\xi = \rho_*\alpha - i_{Y*}\beta = 0$ by (4.5), which implies that $(\alpha, \beta)^T = (i_{E*}, (\rho|_{E})_*)^T(\eta)$. Hence $\text{im}(i_{E*}, (\rho|_{E})_*)^T = \ker(\rho_*, -i_{Y*})$. For any $\theta \in H_k(X, \pi)$, $(\rho_*, -i_{Y*})(\rho^*\theta, 0)^T = 0$ by (4.5). Thus, $(\rho_*, -i_{Y*})$ is surjective. We proved that (4.12) is exact. By (4.5), $(\rho_*, -i_{Y*}) \circ (\rho^*, 0)^T = \text{id}$, hence (4.12) is split and $(\rho^*, 0)^T$ is a right inverse of $(\rho_*, -i_{Y*})$.

(2) Consider the morphism \[ g : K^{**}(\tilde{X}, \tilde{\pi}) \oplus K^{**}(Y, \pi|_Y) \to P^{**}(X, \pi) \oplus K^{**}(E, \tilde{\pi}|_E) \]
\[ (\alpha, \beta) \mapsto (\rho_\ast \alpha, i_{E*}\alpha - (\rho|_{E})_\ast \beta) \]
of double complexes. By Lemma 4.3, we easily get the isomorphism (4.13) with the similar proof of (1).

Suppose that $(\rho^*, i^*_Y)^T(\gamma) = 0$ for $\gamma \in H_k(X, \pi)$. Then $\rho^*\gamma = 0$. By (4.5), $\gamma = \rho_\ast \rho^* \gamma = 0$. So $(\rho^*, i^*_Y)^T$ is injective. Clearly, $(i^*_E, -\rho|_{E}_*) \circ (\rho^*, i^*_Y)^T = 0$. Assume that $(i^*_E, -\rho|_{E}_*)(\alpha, \beta)^T = 0$ for $\alpha \in H_k(\tilde{X}, \tilde{\pi})$ and $\beta \in H_{k-q}(Y, \pi|_Y)$. Set $\gamma := \rho_\ast \alpha$. Then
\[
\begin{pmatrix}
\rho_\ast & 0 \\
-i^*_E & -\rho|_{E}_\ast
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= \begin{pmatrix}
\gamma \\
0
\end{pmatrix} = \begin{pmatrix}
\rho_\ast & 0 \\
-i^*_E & -\rho|_{E}_\ast
\end{pmatrix}
\begin{pmatrix}
\rho^* \gamma \\
i^*_Y 
\end{pmatrix},
\]
where the first equality use (4.5). By (4.13), $(\alpha, \beta)^T = (\rho^* \gamma, i^*_Y \gamma)^T = (\rho^*, i^*_Y)^T(\gamma)$. Thus $\text{im}(\rho^*, i^*_Y)^T = \ker(\rho^*, i^*_Y)^T$. Moreover, $(i^*_E, -\rho|_{E}_\ast)$ is surjective by (4.13). Up to present, we proved that (4.14) is exact. By (4.5), $(\rho_\ast, 0) \circ (\rho^*, i^*_Y)^T = \rho_\ast \circ \rho^* = \text{id}$, so (4.14) is split and $(\rho_\ast, 0)$ is a left inverse of $(\rho^*, i^*_Y)^T$.

**Remark 4.5** Using Lemma 4.3 instead of (4.11) and (4.13), we can prove the following results through almost the same procedures:

(1) \[ 0 \to H^{p-1,q-1}((\rho|_{E})_*)^T \to H^{p,q}(\tilde{X}) \oplus H^{p-r,q-r}(Y) \to H^{p,q}(X) \to 0 \]
is a split exact sequence, where $(\rho^*, 0)^T$ is a right inverse of $(\rho_*, -i_{Y*})$;

(2) \[ 0 \to H^{p,q}(X) \to H^{p,q}(\tilde{X}) \oplus H^{p,q}(Y) \to H^{p,q}(E) \to 0 \]
is a split exact sequence, where $(\rho_*, 0)$ is a left inverse of $(\rho^*, i^*_Y)^T$.

**Corollary 4.6** For any $k \in \mathbb{Z}$,

(1) $(\rho|_{E})_* : H_{k-q}(E, \tilde{\pi}|_E) \to H_{k-q}(Y, \pi|_Y)$ is surjective,

(2) $(\rho|_{E})_* : H_{k-r}(Y, \pi|_Y) \to H_{k-r}(E, \tilde{\pi}|_E)$ is injective,

(3) $(\rho^*, i_{E*}) : H_k(X, \pi) \oplus H_{k-1}(E, \tilde{\pi}|_E) \to H_k(\tilde{X}, \tilde{\pi})$ is surjective,

(4) $(\rho_*, i^*_E)^T : H_k(\tilde{X}, \tilde{\pi}) \to H_k(X, \pi) \oplus H_{k-1}(E, \tilde{\pi}|_E)$ is injective,

(5) $(\rho_*, i^*_E)^T : H_k(\tilde{X}, \tilde{\pi}) \to H_k(X, \pi) \oplus [H_{k-1}(E, \tilde{\pi}|_E)/(\rho|_{E}_*)H_{k-r}(Y, \pi|_Y)]$ is an isomorphism.

**Proof** Immediately, Theorem 4.4 (1) implies (3) and Theorem 4.4 (2) implies (4).

Let $\gamma$ be any element in $H_{k-r}(Y, \pi|_Y)$. By Theorem 4.4 (1), there exist $\alpha \in H_k(X, \pi)$ and $\beta \in H_{k-q}(E, \tilde{\pi}|_E)$ such that
\[
\begin{pmatrix}
0 \\
\gamma
\end{pmatrix} = \begin{pmatrix}
\rho_\ast & i_{E*} \\
0 & (\rho|_{E}_*)
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}.
\]
which implies that $\gamma = (\rho|_E)_* \beta$. Thus, $(\rho|_E)_*$ is surjective, i.e., (1) follows.

Suppose that $(\rho|_E)^*(\beta) = 0$ for $\beta \in H_{k-r}(Y, \pi|_Y)$. Then

$$\begin{pmatrix} \rho_* & 0 \\ i^*_E - (\rho|_E)^* \end{pmatrix} \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

By Theorem 4.4 (2), $\beta = 0$. We proved (2).

Notice that

$$\begin{pmatrix} \rho_* & 0 \\ i^*_E - (\rho|_E)^* \end{pmatrix} \begin{pmatrix} 0 \\ H_{k-r}(Y, \pi|_Y) \end{pmatrix} = 0 \oplus (\rho|_E)^* H_{k-r}(Y, \pi|_Y),$$

which implies (5) by Theorem 4.4 (2).

Remark 4.7 For compact cases, X. Chen, Y. Chen, S. Yang and X. Yang [4, Theorem 1.1] first proved Corollary 4.6 (5) by the relative Koszul–Brylinski homology and the finiteness of dimensions of holomorphic Koszul–Brylinski homologies.

Corollary 4.8 Let $\rho : \tilde{X} \rightarrow X$ be the blow-up of $X$ along a single point set $\{x_0\}$. Assume that $T^*_{X,x_0}$ is an abelian Lie algebra. Then

$$H_k(\tilde{X}, \tilde{\pi}) \cong \begin{cases} H_k(X, \pi) \oplus \mathbb{C}^{n-1}, & k = n \\ H_k(X, \pi), & \text{others,} \end{cases}$$

where $n = \dim_{\mathbb{C}} X$.

Proof Clearly, $\pi|_{\{x_0\}} = 0$, $E = \mathbb{C} P^{n-1}$ and $N_{\{x_0\}/X,x_0} = T_{X,x_0}$. By (3.3) and (4.1), we have

$$H_{k-n}(\{x_0\}, 0) = \begin{cases} \mathbb{C}, & k = n \\ 0, & \text{others} \end{cases} \quad \text{and} \quad H_{k-1}(E, \tilde{\pi}|_E) = \begin{cases} \mathbb{C}^n, & k = n \\ 0, & \text{others}. \end{cases}$$

By Corollary 4.6 (2), $(\rho|_E)^*(H_{k-n}(\{x_0\}, 0))$ is a one-dimensional subspace of $H_{k-1}(E, \tilde{\pi}|_E)$. Thus, the corollary follows by Corollary 4.6 (5).

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