A three shuffle case of the compositional parking function conjecture

by

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ABSTRACT.

We prove here that the polynomial \( \langle \nabla C_{p}, e_{a}h_{b}h_{c} \rangle q, t \)-enumerates, by the statistics dinv and area, the parking functions whose supporting Dyck path touches the main diagonal according to the composition \( p = a + b + c \) and have a reading word which is a shuffle of one decreasing word and two increasing words of respective sizes \( a, b, c \). Here \( C_{p} \) is a rescaled Hall-Littlewood polynomial and \( \nabla \) is the Macdonald eigenoperator introduced in [1]. This is our latest progress in a continued effort to settle the decade old shuffle conjecture of [14]. It includes as special cases all previous results connected with this conjecture such as the \( q, t \)-Catalan [3] and the Schröder and \( h, h \) results of Haglund in [12] as well as their compositional refinements recently obtained in [9] and [10]. It also confirms the possibility that the approach adopted in [9] and [10] has the potential to yield a resolution of the shuffle parking function conjecture as well as its compositional refinement more recently proposed by Haglund, Morse and Zabrocki in [15].

Introduction

A parking function may be visualized as a Dyck path in an \( n \times n \) lattice square with the cells adjacent to the vertical edges of the path labelled with a permutation of the integers \( \{1, 2, \ldots, n\} \) in a column increasing way (see attached figure). We will borrow from parking function language by calling these labels cars. The corresponding preference function is simply obtained by specifying that \( \text{car } i \) prefers to park at the bottom of its column. This visual representation, which has its origins in [4], uses the Dyck path to assure that the resulting preference function parks the cars.

The sequence of cells that joins the SW corner of the lattice square to the NE corner will be called the main diagonal or the 0-diagonal of the parking function. The successive diagonals above the main diagonal will be referred to as diagonals 1, 2, \ldots, \( n-1 \) respectively. On the left of the adjacent display we have listed the diagonal numbers of the corresponding cars. It is also convenient to represent a parking function as a two line array

\[
\begin{bmatrix}
  v_{1} & v_{2} & \cdots & v_{n} \\
  u_{1} & u_{2} & \cdots & u_{n}
\end{bmatrix}
\]

Where \( v_{1}, v_{2}, \ldots, v_{n} \) are the cars as we read them by rows from bottom to top and \( u_{1}, u_{2}, \ldots, u_{n} \) are their corresponding diagonal numbers. From the geometry of the above display we can immediately see that for a two line array to represent a parking function it is necessary and sufficient that we have

\[
u_{1} = 0 \quad \text{and} \quad 0 \leq u_{i} \leq u_{i-1} + 1
\]

with \( V = (v_{1}, v_{2}, \ldots, v_{n}) \) a permutation satisfying the condition

\[
u_{i} = u_{i-1} + 1 \implies v_{i} > v_{i-1}.
\]

For instance the parking function in the above display corresponds to the following two line array

\[
\begin{bmatrix}
  4 & 6 & 8 & 1 & 3 & 2 & 7 & 5 \\
  0 & 1 & 2 & 2 & 3 & 0 & 1 & 1
\end{bmatrix}
\]
Parking functions will be enumerated here by means of a weight that is easily defined in terms of their two line arrays. To this end, let us denote by $\sigma(PF)$ the permutation obtained by successive right to left readings of the components of $V = (v_1, v_2, \ldots, v_n)$ according to decreasing values of $u_1, u_2, \ldots, u_n$. We will call $\sigma(PF)$ the diagonal word of $PF$. We will also let $ides(PF)$ denote the descent set of the inverse of $\sigma(PF)$. This given, each parking function is assigned the weight

$$w(PF) = t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X]$$

where

$$area(PF) = \sum_{i=1}^{n} u_i$$

$$dinv(PF) = \sum_{1 \leq i < j \leq n} \chi(u_i = u_j & \& v_i < v_j) + \sum_{1 \leq i < j \leq n} \chi(u_i = u_j + 1 & \& v_i > v_j)$$

and for a subset $S \subseteq \{1, 2, \ldots, n - 1\}$, $Q_S[X]$ denotes Gessel’s [11] fundamental quasi-symmetric function.

These statistics also have a geometrical meaning and can be directly obtained from the visual representation. In fact we can easily see that the diagonal numbers give the number of lattice cells between the Dyck path and the main diagonal in the row of each corresponding car. Thus the sum in I.5 gives the total number of cells between the supporting Dyck path and the main diagonal. It is also easily seen that two cars in the same diagonal with the car on the left smaller than the car on the right will contribute a unit to $dinv(PF)$ called a primary diagonal inversion. Likewise, a car on the left that is bigger than a car on the right with the latter in the adjacent lower diagonal contributes a unit to $dinv(PF)$ called a secondary diagonal inversion. Thus the sum in I.6 gives the total number of diagonal inversions of the parking function.

Note that reading the cars by diagonals from right to left starting with the highest diagonal we see that car 3 is in the third diagonal, 1 and 8 are in the second diagonal, 5, 7 and 6 are in the first diagonal and 2 and 4 are in the main diagonal. This gives

$$\sigma(PF) = 31857624 \quad \text{and} \quad ides(PF) = \{2, 4, 6, 7\}$$

Thus for the parking function given above we have

$$area(PF) = 10, \quad dinv(PF) = 4,$$

which together with I.7 give

$$w(PF) = t^{10} q^4 Q_{\{2, 4, 6, 7\}}[X].$$

In [15], Haglund, Morse and Zabrocki introduce an additional statistic, the diagonal composition of a parking function, which we denote by $p(PF)$. This is the composition whose parts determine the position of the zeros in the vector $U = (u_1, u_2, \ldots, u_n)$, or equivalently give the lengths of the segments between successive diagonal touches of its Dyck path. Thus $p(PF) = (5, 3)$ for the above example.

Denoting by $\mathcal{PF}_n$ the collection of parking functions in the $n \times n$ lattice square one of the conjectures in [15] states that for any $p = (p_1, p_2, \ldots, p_k) \mid n$ we have

$$\nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{PF \in \mathcal{PF}_n, \ p(PF) = (p_1, p_2, \ldots, p_k)} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X]$$

![Diagram](image.png)
where $\nabla$ is the Bergeron-Garsia operator introduced in [1] and, for each integer $a$, $C_a$ is the operator plethystically defined by setting for any symmetric function $P[X]$

$$C_a P[X] = \left( \frac{1}{q} \right)^{a-1} \sum_{k \geq 0} P \left[ X - \frac{1-1/q}{z} \right] z^k h_k[X] \bigg|_{z^n}.$$  \hspace{1cm} I.9

It follows from a theorem of Gessel [11] that the identity in I.9 is equivalent to the statement that for any composition $(p_1, p_2, \ldots, p_k) \models n$ and any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \vdash n$ we have

$$\langle \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1, h_{\mu_1} h_{\mu_2} \cdots h_{\mu_\ell} \rangle = \sum_{\sigma \in \mathcal{P} F_n} t^{\text{area}(\sigma)} q^{dinv(\sigma)} \chi(\sigma(\mathcal{P} F) \in \downarrow E_1 \cup \cdots \cup \downarrow E_{\ell})$$

where $E_1, E_2, \ldots, E_\ell$ are successive segments of the word $1234 \cdots n$ of respective lengths $\mu_1, \mu_2, \ldots, \mu_\ell$ and the symbol $\chi(\sigma(\mathcal{P} F) \in \downarrow E_1 \cup E_2 \cup \cdots \cup E_\ell)$ is to indicate that the sum is to be carried out over parking functions in $\mathcal{P} F_n$ whose diagonal word is a shuffle of the words $E_1, E_2, \ldots, E_\ell$. In this paper we show that the symmetric function methods developed in [2] and [6] can also be used to obtain the following identity

**Theorem I.1**

For any triplet of integers $a, b, c \geq 0$ and compositions $p = (p_1, p_2, \ldots, p_k) \models a + b + c$ we have

$$\langle \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1, e_a h_b h_c \rangle = \sum_{\sigma \in \mathcal{P} F_n} t^{\text{area}(\sigma)} q^{dinv(\sigma)} \chi(\sigma(\mathcal{P} F) \in \downarrow E_1 \cup E_2 \cup \downarrow E_3)$$

where $E_1, E_2, E_3$ are successive segments of the word $1234 \cdots n$ of respective lengths $a, b, c$ and $\downarrow E_1$ denotes the reverse of the word $E_1$.

Since in [15] it is shown that

$$\sum_{p \models n} C_{p_1} C_{p_2} \cdots C_{p_k} 1 = e_n,$$

summing I.11 over all compositions of $n$ we obtain that

$$\langle \nabla e_n, e_a h_b h_c \rangle = \sum_{\sigma \in \mathcal{P} F_n} t^{\text{area}(\sigma)} q^{dinv(\sigma)} \chi(\sigma(\mathcal{P} F) \in \downarrow E_1 \cup E_2 \cup \downarrow E_3).$$

I.13

Setting $b = c = 0$ in I.13 gives

$$\langle \nabla e_n, e_n \rangle = \sum_{\sigma \in \mathcal{P} F_n} t^{\text{area}(\sigma)} q^{dinv(\sigma)} \chi(\sigma(\mathcal{P} F) = n \cdots 321),$$

I.14

which is the $q, t$-Catalan result of [3]. Setting $c = 0$ or $a = 0$ gives the following two identities proved by Haglund in [12], Namely the Schröder result

$$\langle \nabla e_n, e_k h_{n-k} \rangle = \sum_{\sigma \in \mathcal{P} F_n} t^{\text{area}(\sigma)} q^{dinv(\sigma)} \chi(\sigma(\mathcal{P} F) \in \downarrow A \cup B)$$

I.15

and the shuffle of two segments result

$$\langle \nabla e_n, h_k h_{n-k} \rangle = \sum_{\sigma \in \mathcal{P} F_n} t^{\text{area}(\sigma)} q^{dinv(\sigma)} \chi(\sigma(\mathcal{P} F) \in A \cup B)$$

I.16

with $A$ and $B$ successive segments of respective lengths $k$ and $n - k$. We should also note that setting $c = 0$ and $a = 0$ in I.11 gives the two identities proved in [9] and [10].
It will be good at this point to exhibit at least an instance of the identity in I.11. Below we have the six parking functions with diagonal composition $(3, 2)$ whose diagonal word is in the shuffle $112, 1, 23, 1, 23, 1, 24, 1, 24, 1, 24, 1$.

Reading the display by rows, starting from the top row, we see that the first PF has area 4 and the remaining ones have area 3. Their dinvs are respectively created by the pairs of cars listed below each parking function.

\[
\begin{align*}
\{(1, 3), (2, 5)\} & \quad \{(1, 3), (2, 5), (5, 3), (2, 4)\} & \quad \{(5, 2), (1, 4)\} \\
\{(5, 1), (2, 1), (2, 4)\} & \quad \{(5, 1), (4, 1)\} & \quad \{(1, 3), (5, 3), (2, 4)\}
\end{align*}
\]

Thus Theorem I.1 gives the equality

\[
\langle \nabla C_3 C_2 1, e_1 h_2, h_2 \rangle = t^4 q^2 + t^3 (q^4 + 2q^3 + 2q^2).
\]

Our proof of Theorem I.1, follows a similar path we used in [9] and [10]. By means of a small collection of Macdonald polynomial identities (established much earlier in [2] and [6]) we prove a recursion satisfied by the left hand side of I.11. We then show that the right hand side satisfies the same recursion, with equality in the base cases.

Setting, for a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$

\[
C_\alpha = C_{\alpha_1} C_{\alpha_2} \cdots C_{\alpha_l}
\]

this recursion, which is the crucial result of this paper, may be stated as follows.

**Theorem I.2**

Let and $a, b, c, m, n \in \mathbb{Z}$ such that $a + b + c = m + n$ and $\alpha \vdash n$. If $m > 1$, then

\[
\langle \nabla C_m C_\alpha 1, e_a h_b h_c \rangle = t^{m-1} q^{\ell(\alpha)} \sum_{\beta \vdash m-1} \langle \nabla C_\alpha C_\beta 1, e_a h_b h_c \rangle + t^{m-1} q^{\ell(\alpha)} \sum_{\beta \vdash m-2} \langle \nabla C_\alpha C_\beta 1, e_a h_b h_{c-1} \rangle.
\] I.17

If $m = 1$, then

\[
\langle \nabla C_1 C_\alpha, e_a h_b h_c \rangle = q^{\ell(\alpha)} \langle \nabla C_\alpha 1, e_a h_b h_c \rangle + \langle \nabla C_\alpha 1, e_a h_{b-1} h_c + e_a h_b h_{c-1} \rangle + (q - 1) \sum_{i: \alpha_i = 1} q^{i-1} \langle \nabla C_{\tilde{\alpha}(i)} 1, e_a h_{b-1} h_{c-1} \rangle
\] I.18

where $\tilde{\alpha}(i) = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{\ell(\alpha)})$. 
What is different in this case in contrast with the developments in [3], [9] and [12], is that there the combinatorial side suggested the recursion that both sides had to satisfy. By contrast in the present case we could not have even remotely come up with the combinatorics that unravels out of the above recursion. In fact we shall see that I.17 will guide us, in a totally unexpected manner, to carry out some remarkable inclusion-exclusions sieving of parking functions to prove the right hand side of I.11 satisfies this recursion.

We must also mention that Theorem I.1 may also be viewed as a path result with the same flavor as Haglund’s Schröder result [12]. This is simply obtained by converting each of our parking functions into a lattice path with the following 5 steps

always remaining weakly above the main diagonal and touching it according to the corresponding composition. Areas and dinvs of our parking functions can easily be converted into geometric properties of the corresponding path. This correspondence, which is dictated by the recursion in I.17, is obtained as follows. For convenience we will refer to cars in the words ↓$E_1$, $E_2$ and $E_3$ as small, middle and big, or briefly as $S, M, B$. This given, the path corresponding to one of our parking functions is obtained by deforming its supporting Dyck path according to the following rules

1. Every East step remains an east step.
2. Every North step adjacent to an $S$ remains an North step.
3. Every pair of successive North steps adjacent to a pair $B \ M$ is replaced by the slope 2 step in I.19.
4. The remaining North steps are replaced by the two slope 1 steps in I.19, the red one if adjacent to an $M$ and the blue one if adjacent to a $B$.

In our example above the only $S$ is 1 the $M$‘s are 2, 3 and the $B$‘s are 4, 5. Using the above rules these parking functions convert into the following six paths.

We have divided our presentation into three sections. In the first section we list the Macdonald polynomials identities we plan to use, referring to previous publications for their proofs. Next we give proofs of some identities specifically derived for our present needs. This section also includes an outline of the symmetric function manipulations we plan to use to prove Theorem I.2.

The second section is totally devoted to the proof of Theorem I.2. This is the most technical part of the paper, and perhaps the most suggestive of directions for future work in this subject. We tried whenever possible to motivate some of the steps but in ultimate analysis the majority of them was forced on us by the complexities of the problem.

In the third and final section we prove that the combinatorial side satisfies the same recursion and verify the identities that cover the base cases. This section makes lighter reading, with only prerequisite the statements of Theorems I.1 and I.2. It can be read before embarking in section 2.
1. Auxiliary identities and results.

In this section, after a few necessary definitions, we will list the identities and prove a few preliminary results that are needed in our arguments. The identities have been stated and proved elsewhere but we include them here as a sequence of propositions without proofs for the benefit of the reader.

The space of symmetric polynomials will be denoted Λ. The subspace of homogeneous symmetric polynomials of degree \(d\) will be denoted by \(\Lambda^d\). We must refer to Macdonald’s exposition on symmetric functions [18] for the basic identities that we will use here. We will seldom work with symmetric polynomials expressed in terms of variables but rather express them in terms of one of the classical symmetric function bases: \(\text{power} \{p_\mu\}_\mu, \text{monomial} \{m_\mu\}_\mu, \text{homogeneous} \{h_\mu\}_\mu, \text{elementary} \{e_\mu\}_\mu, \text{and Schur} \{s_\mu\}_\mu\).

Let us recall that the fundamental involution \(\omega\) may be defined by setting for the power basis indexed by \(\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n\)

\[
\omega p_\mu = (-1)^{n-k} p_\mu = (-1)^{|\mu| - l(\mu)} p_\mu
\]

where for any vector \(v = (v_1, v_2, \ldots, v_k)\) we set \(|v| = \sum_{i=1}^{k} v_i\) and \(l(v) = k\).

In dealing with symmetric function identities, especially with those arising in the theory of Macdonald polynomials, we find it convenient and often indispensable to use plethystic notation. This device has a straightforward definition. We simply set for any expression \(E = E(t_1, t_2, \ldots)\) and any power symmetric function \(p_k\),

\[
p_k[E] = E(t_1^k, t_2^k, \ldots).
\]

In particular, if \(E\) is the sum of a set of variables \(E = X = x_1 + x_2 + x_3 + \cdots\), then

\[
p_k[X] = x_1^k + x_2^k + x_3^k + \cdots
\]

is the power sum symmetric function evaluated at those variables.

This given, for any symmetric function \(F\) we set

\[
F[E] = Q_F(p_1, p_2, \ldots) \bigg|_{p_k \rightarrow E(t_1^k, t_2^k, \ldots)}
\]

where \(Q_F\) is the polynomial yielding the expansion of \(F\) in terms of the power basis.

In this notation, \(-E\) does not, as one might expect, correspond to replacing each variable in an expression \(E\) with the negative of that variable. Instead, we see that \(1.3\) gives

\[
p_k[-E] = -p_k[E]. \tag{1.4}
\]

However, we will still need to carry out ordinary changes of signs of the variables, and we will achieve this by the introduction of an additional symbol \(\epsilon\) that this to be acted upon just like any other variable which however, outside of the plethystic bracket, is simply replaced by \(-1\). For instance, these conventions give for \(X_k = x_1 + x_2 + \cdots + x_n\),

\[
p_k[-\epsilon X_n] = -\epsilon^k \sum_{i=1}^{n} x_i^k = (-1)^{k-1} \sum_{i=1}^{n} x_i^k. \tag{1.5}
\]

As a consequence, for any symmetric function \(F \in \Lambda\) and any expression \(E\) we have \(\omega F[E] = F[-\epsilon E]\). In particular, if \(F \in \Lambda^{-k}\) we may also write

\[
F[-E] = \epsilon^{-k} F[-\epsilon E] = (-1)^k \omega F[E]. \tag{1.7}
\]
We must also mention that the formal power series
\[ \Omega = \exp \left( \sum_{k \geq 1} \frac{p_k}{k} \right) \]
combined with plethysic substitutions will provide a powerful way of dealing with the many generating functions occurring in our manipulations.

Here and after it will be convenient to identify partitions with their (french) Ferrers diagram. Given a partition \( \mu \) and a cell \( c \in \mu \), Macdonald introduces four parameters
\[ l_\mu(c), l'_\mu(c), a_\mu(c), a'_\mu(c) \]
called leg, coleg, arm and coarm which give the number of lattice cells of \( \mu \) strictly NORTH, SOUTH, EAST and WEST of \( c \) (see adjacent figure).

Following Macdonald we will set
\[ n(\mu) = \sum_{c \in \mu} l_\mu(c) = \sum_{c \in \mu} l'_\mu(c) = \sum_{i=1}^{l(\mu)} (i-1) \mu_i. \]

Denoting by \( \mu' \) the conjugate of \( \mu \), the notational ingredients playing a role in the theory of Macdonald polynomials are
\[ T_{\mu} = t^{n(\mu)}q^{n(\mu')}, \quad B_{\mu}(q,t) = \sum_{c \in \mu} t'_\mu(c) q^{a'_\mu(c)}, \]
\[ \Pi_{\mu}(q,t) = \prod_{c \in \mu: c \neq (0,0)} (1-t'_\mu(c)q^{a'_\mu(c)}), \quad M = (1-t)(1-q), \]
\[ D_{\mu}(q,t) = MB_{\mu}(q,t) - 1, \quad w_{\mu}(q,t) = \prod_{c \in \mu} (q^{a_\mu(c)} - t'_\mu(c)+1)(t'_\mu(c) - q^{a_\mu(c)}+1). \]

We will also use a deformation of the Hall scalar product, which we call the star scalar product, defined by setting for the power basis
\[ \langle p_\lambda, p_\mu \rangle_* = (-1)^{|\mu|-l(\mu)} \prod_i (1-t^{\mu_i})(1-q^{\mu_i}) z_\mu(\lambda = \mu), \]
where \( z_\mu \) gives the order of the stabilizer of a permutation with cycle structure \( \mu \).

This given, the modified Macdonald Polynomials we will deal with here are the unique symmetric function basis \( \{ \tilde{H}_\mu(X;q,t) \}_{\mu} \) which is upper triangularly related to the basis \( \{ s_\lambda(X_t) \}_{\lambda} \) and satisfies the orthogonality condition
\[ \langle \tilde{H}_\lambda, \tilde{H}_\mu \rangle_* = \chi(\lambda = \mu)w_{\mu}(q,t). \]

In this writing we will make intensive use of the operator \( \nabla \) defined by setting for all partitions \( \mu \)
\[ \nabla \tilde{H}_\mu = T_{\mu} \tilde{H}_\mu. \]

The following identities will play a crucial role in our present developments. Their proofs can be found in [2], [4] and [6].
Proposition 1.1 (Macdonald’s Reproducing Kernel)

The orthogonality relations in 1.11 yield the Cauchy identity for our Macdonald polynomials in the form

$$
\Omega \left[ -\varepsilon_{XY} \right] = \sum_{\mu} \frac{\tilde{H}_\mu[X] \tilde{H}_\mu[Y]}{w_\mu}
$$

which restricted to its homogeneous component of degree \( n \) in \( X \) and \( Y \) reduces to

$$
e_n \left[ \frac{XY}{M} \right] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X] \tilde{H}_\mu[Y]}{w_\mu}.
$$

Proposition 1.2 (Macdonald Reciprocity)

For all pairs of partitions \( \alpha, \beta \) we have

\[
a) \quad \frac{\tilde{H}_\alpha[MB_\beta]}{\Pi_\alpha} = \frac{\tilde{H}_\beta[MB_\alpha]}{\Pi_\beta}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
where it has been customary to let \( \phi \) be the operator defined by setting for any symmetric function \( f \)
\[
\phi f[X] = f[MX].
\]

Note that the inverse of \( \phi \) is usually written in the form
\[
f^*[X] = f[X/M].
\]

In particular we also have for all symmetric functions \( f, g \)
\[
\langle f, g \rangle = \langle f, \omega g^* \rangle.
\]

Note that the orthogonality relations in 1.11 yield us the following Macdonald polynomial expansions

**Proposition 1.4**

For all \( n \geq 1 \), we have

\[
a) \quad e_n[X_M] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X]}{w_\mu}, \quad b) \quad h_n[X_M] = \sum_{\mu \vdash n} \frac{T_\mu H_\mu[X]}{w_\mu},
\]

\[
c) \quad h_k[X_M]e_{n-k}[X_M] = \sum_{\mu \vdash n} \frac{e_k[B_\mu] \tilde{H}_\mu[X]}{w_\mu}.
\]

The following identity (proved in [6]) may sometimes provide an explicit expression for the scalar product of a Macdonald polynomial with a symmetric polynomial which contains an \( h \) factor. More precisely, we have the following proposition.

**Proposition 1.5**

For all \( f \in \Lambda \) and \( \mu \vdash n \) we have
\[
\langle f h_{n-r}, \tilde{H}_\mu \rangle = \nabla^{-1}(\omega f[X])_{X \rightarrow MB_\mu - 1}.
\]

For instance for \( f = e_r \) this identity combined with a) and b) of Proposition 1.4 gives
\[
\langle \tilde{H}_\mu, e_r h_{n-r} \rangle = \nabla^{-1} h_r[X]_{X \rightarrow MB_\mu - 1} = \sum_{i=0}^{r} e_{r-i} \frac{\tilde{H}_i[X]}{w_i} = \sum_{i=0}^{r} e_{r-i} \frac{MB_{\mu-1}}{w_i} = e_r[B_\mu].
\]

We thus have
\[
\langle \tilde{H}_\mu, e_r h_{n-r} \rangle = e_r[B_\mu].
\]

This is an identity which will provide an important step in the proof of Theorem I.2.

In addition to the \( \mathbf{C}_a \) operator given in I.9 which we recall acts on a symmetric polynomial according to the plethystic formula
\[
\mathbf{C}_a P[X] = (-1)^{a-1} P[X - \frac{1-1/q}{z}] \sum_{m \geq 0} z^m h_m[X] \bigg|_{z^a},
\]
Haglund-Morse-Zabrocki in [15] introduce also the $B_a$ operator obtained by setting

$$B_a P[X] = P[X + \epsilon\frac{1}{z^2}] \sum_{m \geq 0} z^m e_m[X] \bigg|_{z^a}. \quad 1.26$$

They showed there that these operators, for $a + b > 0$ have the commutativity relation

$$B_a C_b = q C_b B_a. \quad 1.27$$

However, we will need here the more refined identity that yields the interaction of the $B_{-1}$ and $C_1$ operators. The following identity also explains what happens when $a + b < 0$.

**Proposition 1.6**

*For all pairs of integers $a, b$ we have*

$$(q C_b B_a - B_a C_b) P[X] = (q - 1)(-1)^{a+b-1}/q^{b-1} \times \begin{cases} 0 & \text{if } a + b > 0 \\ P[X] & \text{if } a + b = 0 \\ P[X + (\frac{1}{q} - q)/z] \bigg|_{z^{a+b}} & \text{if } a + b < 0. \end{cases} \quad 1.28$$

**Proof**

Using 1.25 we get for $P \in \Lambda^{d}$,

$$(-q)^{b-1} C_b P[X] = \sum_{r_1=0}^{d} P[X - \frac{1-1/q}{z_1}] \bigg|_{z_1^{-r_1}} \sum_{m \geq 0} z^m h_m[X] \bigg|_{z^{b+r_1}} = \sum_{r_1=0}^{d} P[X - \frac{1-1/q}{z_1}] \bigg|_{z_1^{-r_1}} h_{b+r_1}[X],$$

Now it will be convenient go set for symmetric polynomial $P[X]$

$$P^{(r_1, r_2)}[X] = P[X - \frac{1-1/q}{z_1} + \epsilon\frac{1-q}{z_2}] \bigg|_{z_1^{-r_1} z_2^{-r_2}}$$

This given, 1.26 gives

$$(-q)^{b-1} B_a C_b P[X] = \sum_{r_1=0}^{d} P[X - \frac{1-1/q}{z_1} + \epsilon\frac{1-q}{z_2}] \bigg|_{z_1^{-r_1} z_2^{-r_2}} h_{b+r_1}[X + \epsilon\frac{1-q}{z_2}] \sum_{m \geq 0} z^m e_m[X] \bigg|_{z^{a+r_2}}$$

$$= \sum_{r_1,r_2=0}^{d} P^{(r_1, r_2)}[X] \sum_{s=0}^{b+r_1} h_{b+r_1-s}[X](-1)^s h_s[(1-q)] \sum_{m \geq 0} z^m e_m[X] \bigg|_{z^{a+r_2+s}}$$

$$= \sum_{r_1,r_2=0}^{d} \sum_{s=0}^{b+r_1} P^{(r_1, r_2)}[X] h_{b+r_1-s}[X](-1)^s h_s[(1-q)] e_{a+r_2+s}[X].$$

Since it is easily shown that

$$h_s[(1-q)] = \begin{cases} 1 & \text{if } s = 0 \\ 1 - q & \text{if } s > 0, \end{cases} \quad 1.29$$

we can write

$$(-q)^{b-1} B_a C_b P[X] = \sum_{r_1,r_2=0}^{d} P^{(r_1, r_2)}[X] h_{b+r_1}[X] e_{a+r_2}[X] +$$

$$+ (1-q) \sum_{r_1,r_2=0}^{d} \sum_{s=1}^{b+r_1} P^{(r_1, r_2)}[X] h_{b+r_1-s}[X](-1)^s e_{a+r_2+s}[X]$$
and the change of summation index \( u = a + r_2 + s \) gives

\[
(-q)^{b-1} B_a C_b P[X] = \sum_{r_1, r_2=0}^{d} P^{(r_1, r_2)}[X] h_{b+r_1}[X] e_{a+r_2}[X] +
\]

\[
+ (-1)^a (1 - q) \sum_{r_1, r_2=0}^{d} \sum_{u=0}^{a+b+r_1+r_2+1} P^{(r_1, r_2)}[X] h_{a+b+r_1+r_2-u}[X] (-1)^{-r_2} e_u[X].
\]

An entirely similar manipulation gives

\[
(-q)^{b-1} q C_b B_a P[X] = \sum_{r_1, r_2=0}^{d} P^{(r_1, r_2)}[X] e_{r_2+a}[X] h_{r_1+b}[X] - (-1)^a (1 - q) \sum_{r_1, r_2=0}^{d} \sum_{u=0}^{a+r_2} e_u[X] (-1)^{u+r_2} h_{a+b+r_1+r_2-u}[X],
\]

and thus by subtraction we get

\[
(-q)^{b-1} \left( q C_b B_a P[X] - B_a C_b P[X] \right) = (q - 1)(-1)^a \sum_{r_1, r_2=0}^{d} P^{(r_1, r_2)}[X] h_a[X] (-1)^r h_{a+b+r_1+r_2-u}[X]
\]

\[
= (q - 1)(-1)^a \sum_{r_1, r_2=0}^{d} P^{(r_1, r_2)}[X] (-1)^r h_{a+b+r_1+r_2}[X] - X
\]

Since the homogeneous symmetric function \( h_a \) on an empty alphabet vanishes unless \( a = 0 \), using the definition of \( P^{(r_1, r_2)}[X] \), this reduces to

\[
(q C_b B_a P[X] - B_a C_b) P[X] = \left( q - 1 \right)(-1)^{a+b-1} \sum_{r_1, r_2=0}^{d} (-1)^r (X - \frac{1-\frac{1}{q}}{z_1} + \frac{\epsilon \frac{1}{z_2}}{z_1}) P[X - \frac{1-\frac{1}{q}}{z_1} + \frac{\frac{1}{z_2}}{z_1}] \mid_{z_1+z_2} = \epsilon^r a_{r_1, r_2}
\]

which immediately gives the first two cases of 1.28. For the remaining case, notice that for suitable coefficients \( a_{r_1, r_2} \) we may write

\[
P[X - \frac{1-\frac{1}{q}}{z_1} + \frac{\frac{1}{z_2}}{z_1}] \mid_{z_1+z_2} = \epsilon^r a_{r_1, r_2}
\]

and the sum on the right hand side of 1.30 may be rewritten as

\[
\sum_{r_1, r_2=0}^{d} (-1)^r \epsilon^r a_{r_1, r_2} = \sum_{r_1, r_2=0}^{d} a_{r_1, r_2} \frac{1}{z_1} \frac{1}{z_2} \mid_{z_1+z_2} = \epsilon^r a_{r_1, r_2} \mid_{z_1+z_2} = \epsilon^r a_{r_1, r_2}
\]

which is precisely as asserted in 1.28.

We are now finally in a position to give an outline of the manipulations that we will carry out to prove the recursion of Theorem I.2.

A close examination of our previous work, most particularly our result in [9] and [10], might suggest that also in the present case we should also have (for \( m > 1 \)) an identity of the form

\[
\langle \nabla C_\alpha C_\beta 1, e_{a-1} h_b h_c \rangle = \sum_{\beta = 0}^{m-1} q^{\ell(\alpha)} \langle \nabla C_\beta 1, e_{a-1} h_b h_c \rangle.
\]
Now we can easily see from Theorem 1.2 that an additional term is need to obtain a true identity. But to give an idea of how one may end up discovering the correct recursion, let us pretend that 1.31 is true and try to derive from it what is needed to prove it.

Note first that using 1.12 for \( n = m - 1 \) and 1.26 with \( P[X] = 1 \) we obtain that

\[
\sum_{\beta \models m-1} C_\beta \alpha = e_{m-1}[X] = B_{m-1} 1.
\]

This allows us to rewrite 1.31 in the more compact form

\[
\langle \nabla C_m C_\alpha 1, e_a h_b h_c \rangle = t^{m-1} q^{\ell(A)} \langle \nabla C_{\alpha} B_{m-1} 1, e_{a-1} h_b h_c \rangle
\]

and, by a multiple use of the commutativity relation in 1.27, we can further simplify this to

\[
\langle \nabla C_m C_\alpha 1, e_a h_b h_c \rangle = t^{m-1} \langle \nabla B_{m-1} C_\alpha 1, e_{a-1} h_b h_c \rangle.
\]

Following the manipulations carried out in [9] our next step is to pass to *-scalar products (using 1.23) and obtain

\[
\langle \nabla C_m C_\alpha 1, h_a^* e_b^* e_c^* \rangle_* = t^{m-1} \langle \nabla B_{m-1} C_\alpha 1, h_a^* e_b^* e_c^* \rangle_*.
\]

Since the \( \nabla \) operator is self adjoint with respect to the *-scalar product, this in turn can be rewritten as

\[
\langle C_\alpha 1, C_m^* \nabla h_a^* e_b^* e_c^* \rangle_* = t^{m-1} \langle C_\alpha 1, B_{m-1}^* \nabla h_a^* e_b^* e_c^* \rangle_*,
\]

where \( C_m^* \) and \( B_{m-1}^* \) denote the *-scalar product adjoints of \( C_m \) and \( B_{m-1} \).

Now, except for a minor rescaling, the \( C_\alpha 1 \) are essentially Hall-Littlewood polynomials and the latter, for \( \alpha \) a partition, are a well known basis. Thus 1.32 can be true if and only if we have the symmetric function identity

\[
C_m^* \nabla h_a^* e_b^* e_c^* = t^{m-1} B_{m-1}^* \nabla h_a^* e_b^* e_c^*.
\]

Starting from the left hand side of 1.33, a remarkable sequence of manipulations, step by step guided by our previous work, reveals that 1.33 is but a tip of an iceberg. In fact, as we will show in the next section that the correct form of 1.33 may be stated as follows.

**Theorem 1.1**

Let and \( a, b, c, m, n \in \mathbb{Z} \) be such that \( a + b + c = m + n \) and \( \alpha \models n \). Then

\[
C_m^* \nabla h_a^* e_b^* e_c^* = t^{m-1} B_{m-1}^* \nabla h_a^* e_b^* e_c^* + t^{m-2} B_{m-2}^* \nabla h_a^* e_b^* e_c^* + \chi(m = 1)(\nabla h_a^* e_b^* e_c^* + \nabla h_a^* e_b^* e_c^*).
\]

Postponing to next section the proof of this result, we will terminate this section by showing the following connection with the results stated in the introduction.

**Proposition 1.7**

Theorems 1.2 and 1.1 are equivalent.

**Proof**

Note first that for \( m \geq 2 \) the identity in 1.34 reduces to

\[
C_m^* \nabla h_a^* e_b^* e_c^* = t^{m-1} B_{m-1}^* \nabla h_a^* e_b^* e_c^* + t^{m-2} B_{m-2}^* \nabla h_a^* e_b^* e_c^*.
\]
and the recursion in I.17 is simply obtained by reversing the steps that got us from 1.31 to 1.33. We will carry them out for sake of completeness. To begin we take the *-scalar product with by $C_\alpha 1$ on both sides of this equation to obtain

$$\langle C_\alpha 1, C^*_m \nabla h_* a e_b c^*_e \rangle_\ast = t^{m-1} \langle C_\alpha 1, B^*_{m-1} \nabla h_* a e_b c^*_e \rangle_\ast + t^{m-1} \langle C_\alpha 1, B^*_{m-2} \nabla h_* a e_b c^*_e \rangle_\ast$$

and moving all the operators from the right to the left of the *-scalar product gives

$$\langle \nabla C_m C_\alpha, h_* a e_b c^*_e \rangle_\ast = t^{m-1} \langle \nabla B_{m-1} C_\alpha, h_* a e_b c^*_e \rangle_\ast + t^{m-1} \langle \nabla B_{m-2} C_\alpha, h_* a e_b c^*_e \rangle_\ast.$$ 

In terms of the ordinary Hall scalar product this identity becomes

$$\langle \nabla C_m C_\alpha 1, e_a h_b h_c \rangle = t^{m-1} \langle \nabla B_{m-1} C_\alpha 1, e_a h_b h_c \rangle + t^{m-1} \langle \nabla B_{m-2} C_\alpha 1, e_a h_b h_c \rangle.$$ 

The commutativity relation in 1.27 then gives

$$\langle \nabla C_m C_\alpha 1, e_a h_b h_c \rangle = t^{m-1} q^{l(\alpha)} \langle \nabla C_\alpha B_{m-1} 1, e_a h_b h_c \rangle + t^{m-1} q^{l(\alpha)} \langle \nabla C_\alpha B_{m-2} 1, e_a h_b h_c \rangle.$$ 

Finally the two expansions

$$B_{m-1} 1 = \sum_{\beta=m-1} C_\beta 1 \quad \text{and} \quad B_{m-2} 1 = \sum_{\beta=m-2} C_\beta 1$$

give that

$$\langle \nabla C_m C_\alpha 1, e_a h_b h_c \rangle = t^{m-1} q^{l(\alpha)} \sum_{\beta=m-1} \langle \nabla C_\alpha C_\beta 1, e_a h_b h_c \rangle + t^{m-1} q^{l(\alpha)} \sum_{\beta=m-2} \langle \nabla C_\alpha C_\beta 1, e_a h_b h_c \rangle.$$ 

This shows that, for $m > 1$, 1.34 implies I.17.

Next note that for $m = 1, 1.34$ reduces to

$$C^*_1 \nabla h_* a e_b c^*_e = B^*_0 \nabla h_* a e_b c^*_e + B^*_{-1} \nabla h_* a e_b c^*_e + (\nabla h_* a e_b c^*_e + \nabla h_* a e_b c^*_e)$$

and the same steps that brought us from 1.35 to 1.36 yield us the identity

$$\langle \nabla C_1 C_\alpha 1, e_a h_b h_c \rangle = \langle \nabla B_0 C_\alpha 1, e_a h_b h_c \rangle + \langle \nabla B^{-1} C_\alpha 1, e_a h_b h_c \rangle + \langle \nabla C_\alpha 1, e_a h_b h_c \rangle + \langle \nabla C_\alpha 1, e_a h_b h_c \rangle.$$ 

Here the only thing that remains to be done is moving $B_0$ and $B_{-1}$ to the right past all components of $C_\alpha$.

This can be achieved by the following two cases of 1.28

a) $B_0 C_b = q C_b B_0$ \quad and \quad b) $B_{-1} C_b = q C_b B_{-1} + \chi(b = 1)(q-1)I$ \quad (for all $b \geq 1$)

where $I$ denotes the identity operator. Now using the first we immediately obtain

$$\langle \nabla B_0 C_\alpha 1, e_a h_b h_c \rangle = q^{l(\alpha)} \langle \nabla C_\alpha 1, e_a h_b h_c \rangle,$$

since 1.26 gives $B_0 1 = 1$. 

The effect of 1.38 b) on the polynomial \( \langle \nabla \mathbf{B}_{-1} \mathbf{C}_\alpha \mathbf{1} , e_a h_{b-1} h_{c-1} \rangle \) is best understood by working out an example. Say \( \alpha = \beta_1 \gamma_1 \delta_1 \tau \) with \( \beta, \gamma, \delta, \tau \) compositions, the 1’s in positions \( i_1, i_2, i_3 \) and \( l(\alpha) = k \). In this case multiple applications of 1.38 b) yield the sum of the three terms on the right.

\[
B_{-1} \mathbf{C}_\alpha \mathbf{1} = B_{-1} \mathbf{C}_\beta \mathbf{C}_\gamma \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \\
q^{i_1 - 1} \mathbf{C}_\beta B_{-1} \mathbf{C}_\gamma \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \Rightarrow (q - 1) q^{i_1 - 1} \mathbf{C}_\beta \bullet \mathbf{C}_\gamma \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \\
q^{i_2 - 1} \mathbf{C}_\beta \mathbf{C}_\gamma B_{-1} \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \Rightarrow (q - 1) q^{i_2 - 1} \mathbf{C}_\beta \mathbf{C}_\gamma \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \\
q^{i_3 - 1} \mathbf{C}_\beta \mathbf{C}_\gamma \mathbf{C}_1 \mathbf{C}_\delta B_{-1} \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \Rightarrow (q - 1) q^{i_3 - 1} \mathbf{C}_\beta \mathbf{C}_\gamma \mathbf{C}_1 \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau \mathbf{1} \\
q^k \mathbf{C}_\beta \mathbf{C}_\gamma \mathbf{C}_1 \mathbf{C}_\delta \mathbf{C}_1 \mathbf{C}_\tau B_{-1} \mathbf{1} = 0 \text{ (since } B_{-1} \mathbf{1} = 0) \\
\]

and we can easily see that in general we will have

\[
B_{-1} \mathbf{C}_\alpha \mathbf{1} = (q - 1) \sum_{i: \alpha_i = 1} q^{i - 1} \mathbf{C}_{\hat{\alpha}^{(i)}} \mathbf{1}. \\
\]

This gives that

\[
\langle \nabla B_{-1} \mathbf{C}_\alpha \mathbf{1} , e_a h_{b-1} h_{c-1} \rangle = (q - 1) \sum_{i: \alpha_i = 1} q^{i - 1} \langle \nabla \mathbf{C}_{\hat{\alpha}^{(i)}} \mathbf{1} , e_a h_{b-1} h_{c-1} \rangle \\
\]

where, as before we have set \( \hat{\alpha}^{(i)} = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{l(\alpha)}) \).

Thus combining 1.37, 1.39 and 1.40 we obtain

\[
\langle \nabla \mathbf{C}_\alpha \mathbf{1} , e_a h_b h_c \rangle = q^{l(\alpha)} \langle \nabla \mathbf{C}_\alpha \mathbf{1} , e_a h_{b-1} h_c + e_a h_b h_{c-1} \rangle \\
+ (q - 1) \sum_{i: \alpha_i = 1} q^{i - 1} \langle \nabla \mathbf{C}_{\hat{\alpha}^{(i)}} \mathbf{1} , e_a h_{b-1} h_{c-1} \rangle \\
\]

proving that 1.34 for \( m = 1 \) implies 1.18. Since all the steps that yielded 1.17 and 1.18 from 1.34 are reversible our proof of the equivalence of Theorems 1.1 and 1.2 is now complete.

\[ \diamond \]

2. Proof of Theorem 1.1

It was shown in \([9]\) and will not be repeated here that the \(*\)-duals of the \( C \) and \( B \) operators can be computed by means of the following plethysm formulas

\[
C^*_a P[X] = (\frac{1}{q})^{a - 1} P[X - \frac{M}{z}] \Omega[\frac{e^X}{q(1-t)}] \big|_{z = -a}, \\
B^*_a P[X] = P[X + \frac{M}{z}] \Omega[\frac{e^X}{1-t}] \big|_{z = -a} \\
\]

where \( P[X] \) denotes a generic symmetric polynomial and we must recall, for the benefit of the reader, that

\[
\Omega[\frac{e^X}{q(1-t)}] = \sum_{m \geq 0} z^m e_m \left[ \frac{X}{q(1-t)} \right] \quad \text{and} \quad \Omega[\frac{e^X}{1-t}] = \sum_{m \geq 0}z^m e_m \left[ \frac{X}{1-t} \right] \\
\]

In the next few pages we will prove a collection of lemmas and propositions which, when combined, yield the identity in 1.34. We will try whenever possible to motivate the sequence of steps converging to the final result.
To start computing the left hand side of 1.33 as well as for later purposes we need the following auxiliary identity.

Lemma 2.1

For any $a, b, c \geq 0$, we have

$$\nabla h_a^* e_b^* e_c^* = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} [\frac{1}{M}] h_{b-s} [\frac{1}{M}] (-1)^{n-r-s} \sum_{\nu^r r + s} e_r [B_\nu] e_{a+b+c} [\frac{X D_n}{M}] \frac{h_\nu}{w_\nu}. \quad 2.4$$

Proof

Setting $n = a + b + c$ and using 1.23 we obtain the expansion

$$\nabla h_a^* e_b^* e_{n-a-b} = \sum_{\mu^r n} T_\mu \tilde{H}_\mu [X: q, t] \langle \tilde{H}_\mu, e_a h_b h_{n-a-b} \rangle. \quad 2.5$$

Now Proposition 1.5 gives

$$\langle \tilde{H}_\mu, e_a h_b h_{n-a-b} \rangle = \nabla^{-1} h_a [\frac{X-c}{M}] e_b [\frac{X-e}{M}] \big|_{X \rightarrow D_\mu} \quad 2.6$$

and again Proposition 1.5 gives

$$\nabla^{-1} h_a [\frac{X-c}{M}] e_b [\frac{X-e}{M}] \big|_{X \rightarrow D_\mu} = \sum_{\nu^r r + s} T_\nu^{-1} \tilde{H}_\nu [D_\nu; q, t] \langle \tilde{H}_\nu, e_r h_s \rangle \quad 2.7$$

(by 1.13 a) and (1.24) = $(-1)^{n-r-s} \sum_{\nu^r r + s} T_\nu^{-1} \tilde{H}_\nu [D_\nu; q, t] e_r [B_\nu].$

Combining 2.5, 2.6 and 2.7 we can thus write

$$\nabla h_a^* e_b^* e_{n-a-b} = \sum_{\mu^r n} T_\mu \tilde{H}_\mu [X: q, t] \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} [\frac{1}{M}] h_{b-s} [\frac{1}{M}] (-1)^{n-r-s} \sum_{\nu^r r + s} T_\nu^{-1} \tilde{H}_\nu [D_\nu; q, t] e_r [B_\nu] \quad 2.8$$

or better

$$\nabla h_a^* e_b^* e_{n-a-b} = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} [\frac{1}{M}] h_{b-s} [\frac{1}{M}] (-1)^{n-r-s} \sum_{\nu^r r + s} e_r [B_\nu] e_{a+b+c} [\frac{X D_n}{M}] \frac{h_\nu}{w_\nu}. \quad 2.9$$

This completes our proof.

Our next task is to compute the following expression

$$\Gamma_{m,a,b,n}[X, q, t] = C_m \nabla h_a^* e_b^* e_{n-a-b}. \quad 2.8$$

In view of 2.4 we need the following auxiliary result.
Lemma 2.2

\[ C_m^* e_n \left( \frac{XD_n}{M} \right) = (-1)^{m-1} \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_n}{M} \right] e_{u-m} \left( \frac{X}{1-t} \right)^{u-1} M \sum_{\tau \to \nu} c_{\nu\tau} \left( \frac{T_{\tau\nu}}{T_{\nu}} \right)^{u-1} - \chi(m = 1) e_{n-1} \left( \frac{XD_{n-1}}{M} \right) \] 2.9

Proof

Recalling that

\[ C^*_a P[X] = \left( \frac{-1}{q} \right)^{n-1} P\left[ X - \frac{tM}{q} \Omega \left[ \frac{-etX}{q(1-t)} \right] \right]_{z-a}, \]

we get

\[ C_m^* e_n \left( \frac{XD_n}{M} \right) = \left( \frac{-1}{q} \right)^{m-1} e_n \left( \frac{X - \frac{tM}{q} \Omega \left[ \frac{-etX}{q(1-t)} \right]}{M} \right)_{z-m} = \left( \frac{-1}{q} \right)^{m-1} e_n \left( \frac{X - \frac{tM}{q} \Omega \left[ \frac{-etX}{q(1-t)} \right]}{M} \right)_{z-u-m} \]

(by 1.7, 1.10 and 2.3) = \left( \frac{-1}{q} \right)^{m-1} \sum_{u=0}^{n} e_{n-u} \left[ \frac{XD_n}{M} \right] e_{u-m} \left[ \frac{X}{q(1-t)} \right] h_u \left[ MB_{\nu} - 1 \right] \]

(by 1.17) = \left( \frac{-1}{q} \right)^{m-1} \sum_{u=0}^{n} e_{n-u} \left[ \frac{XD_n}{M} \right] e_{u-m} \left[ \frac{X}{q(1-t)} \right] \left( (tq)^{u-1} M \sum_{\tau \to \nu} c_{\nu,\tau} \left( \frac{T_{\tau\nu}}{T_{\nu}} \right)^{u-1} - \chi(u = 1) \right) \]

\[ = (-1)^{m-1} \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_n}{M} \right] e_{u-m} \left[ \frac{X}{q(1-t)} \right] \left( (tq)^{u-1} M \sum_{\tau \to \nu} c_{\nu,\tau} \left( \frac{T_{\tau\nu}}{T_{\nu}} \right)^{u-1} - \chi(u = 1) \right) \]

This equality proves 2.9.

And now the first special case peals off.

Proposition 2.1

\[ \Gamma_{m,a,b,n}[X,q,t] = C_m^* \nabla h^*_a e^*_b e^*_n-a-b = \Gamma^{(1)}_{m,a,b,n}[X,q,t] + \chi(m = 1) \nabla h^*_a e^*_b e^*_n-1-a-b \] 2.10

with

\[ \Gamma^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left[ \frac{1}{M} \right] h_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \times \]

\[ \times \sum_{\nu \uparrow r+s} \frac{e_r \left[ B_{\nu} \right]}{w_{\nu}} \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_n}{M} \right] e_{u-m} \left[ \frac{X}{1-t} \right] (tq)^{u-1} M \sum_{\tau \to \nu} c_{\nu,\tau} \left( \frac{T_{\tau\nu}}{T_{\nu}} \right)^{u-1}. \] 2.11

Proof

By Lemma 2.1 we get

\[ \nabla h^*_a e^*_b e^*_n-a-b = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left[ \frac{1}{M} \right] h_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{\nu \uparrow r+s} \frac{e_r \left[ B_{\nu} \right]}{w_{\nu}} e_n \left[ \frac{XD_n}{M} \right]. \]
Now applying $C_m^*$ on both sides of the equation, 2.8 becomes

$$
\Gamma_{m,a,b,n}[X,q,t] = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left(\begin{array}{c} \frac{1}{M} \\ r \end{array}\right) \frac{h_{s}}{\frac{1}{M}} (-1)^{n-r-s} \sum_{\nu \to \tau + s} \frac{e_{\nu} \left(B_{\nu} \right)}{w_{\nu}} C_{m}^{*} \left[ \frac{X D_{\nu}}{M} \right].
$$

Since Lemma 2.2 gives

$$
C_{m}^{*} \left[ \frac{X D_{a}}{M} \right] = (-1)^{m-1} \sum_{u=m}^{n} e_{n-u} \left[ \frac{X D_{a}}{M} \right] e_{u-m} \left[ \frac{X}{(1-t)} \right] t^{u-1} M \sum_{\tau \to \nu} e_{\tau} \left(\frac{T_{\nu}}{T_{\tau}} \right)^{u-1} - \chi(m=1) e_{n-1} \left[ \frac{X D_{a}}{M} \right],
$$

we obtain

$$
\Gamma_{m,a,b,n}[X,q,t] = \Gamma_{(1)}^{(1)}_{m,a,b,n}[X,q,t] + \Gamma_{(2)}^{(2)}_{m,a,b,n}[X,q,t] \tag{2.12}
$$

with

$$
\Gamma_{(1)}^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left(\begin{array}{c} \frac{1}{M} \\ r \end{array}\right) \frac{h_{s}}{\frac{1}{M}} (-1)^{n-r-s} \times
$$

$$
\sum_{\nu \to \tau + s} \frac{e_{\nu} \left(B_{\nu} \right)}{w_{\nu}} \sum_{u=m}^{n} e_{n-u} \left[ \frac{X D_{a}}{M} \right] e_{u-m} \left[ \frac{X}{(1-t)} \right] t^{u-1} M \sum_{\tau \to \nu} e_{\tau} \left(\frac{T_{\nu}}{T_{\tau}} \right)^{u-1} \tag{2.13}
$$

and

$$
\Gamma_{(2)}^{(2)}_{m,a,b,n}[X,q,t] = -\chi(m=1) \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left(\begin{array}{c} \frac{1}{M} \\ r \end{array}\right) \frac{h_{s}}{\frac{1}{M}} (-1)^{n-1-r-s} (-1) \times
$$

$$
\sum_{\nu \to \tau + s} \frac{e_{\nu} \left(B_{\nu} \right)}{w_{\nu}} e_{n-1} \left[ \frac{X D_{a}}{M} \right]. \tag{2.14}
$$

Now recall that Lemma 2.1 gave

$$
\nabla h_{a} e_{b} e_{n-a-b} = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left(\begin{array}{c} \frac{1}{M} \\ r \end{array}\right) \frac{h_{s}}{\frac{1}{M}} (-1)^{n-r-s} \sum_{\nu \to \tau + s} \frac{e_{\nu} \left(B_{\nu} \right)}{w_{\nu}} e_{n} \left[ \frac{X D_{a}}{M} \right].
$$

Thus, 2.14 is none other than

$$
\Gamma_{(2)}^{(2)}_{m,a,b,n}[X,q,t] = \chi(m=1) \nabla h_{a} e_{b} e_{n-1-a-b} \tag{2.15}
$$

and we see that 2.12, 2.13 and 2.14 complete our proof.

Our next task is to obtain the desired expression for the polynomial in 2.11. That is,

$$
\Gamma_{(1)}^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left(\begin{array}{c} \frac{1}{M} \\ r \end{array}\right) \frac{h_{s}}{\frac{1}{M}} (-1)^{n-r-s} \times
$$

$$
\sum_{\nu \to \tau + s} \frac{e_{\nu} \left(B_{\nu} \right)}{w_{\nu}} \sum_{u=m}^{n} e_{n-u} \left[ \frac{X D_{a}}{M} \right] e_{u-m} \left[ \frac{X}{(1-t)} \right] t^{u-1} M \sum_{\tau \to \nu} e_{\tau} \left(\frac{T_{\nu}}{T_{\tau}} \right)^{u-1}. \tag{2.16}
$$

This will be obtained by a sequence of transformations.
To begin, rearranging the order of summations we get

\[
\Gamma^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r}\left[\frac{1}{\tau}\right]h_{b-s}\left[\frac{1}{\tau}\right](-1)^{n-r-s} \sum_{u=m}^{n} e_{u-m}\left[\frac{X}{(1-t)}\right]t^{u-1} \times \\
\sum_{\tau^{+}r+s-1} \frac{1}{u_{r}} \sum_{\nu^{+}\tau} M_{u_{r}}^{w_{r}} c_{\nu^{+}\tau}^{(}\frac{T_{e}}{T_{r}}\rceil)^{u-1} e_{\tau^{+}} \left[\frac{X_{D_{e}}}{M_{e}}\right] \\
\]

(by 1.16)

\[
\times \sum_{\tau^{+}r+s-1} \frac{1}{u_{r}} \sum_{\nu^{+}\tau} d_{\nu^{+}\tau}^{(}\frac{T_{e}}{T_{r}}\rceil)^{u-1} (e_{\tau^{+}} \left[\frac{X_{B_{r}}}{B_{r}}\right] + e_{\tau^{+}-1} \left[\frac{X_{B_{r}}}{B_{r}}\right] \sum_{k=0}^{n-u} e_{n-u-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] e_{k} \left[\frac{X}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{k} \\
\]

and this is best rewritten as

\[
\Gamma^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r}\left[\frac{1}{\tau}\right]h_{b-s}\left[\frac{1}{\tau}\right](-1)^{n-r-s} \sum_{u=m}^{n} e_{u-m}\left[\frac{X}{(1-t)}\right]t^{u-1} \times \\
\sum_{\tau^{+}r+s-1} \frac{1}{u_{r}} \sum_{\nu^{+}\tau} \sum_{k=0}^{n-u} e_{k} \left[\frac{X}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{u+k-1} (e_{\tau^{+}} \left[\frac{X_{B_{r}}}{B_{r}}\right] + e_{\tau^{+}-1} \left[\frac{X_{B_{r}}}{B_{r}}\right] \sum_{k=0}^{n-u} e_{n-u-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] e_{k} \left[\frac{X}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{k} . \\
\]

Notice next that the following summation interchanges

\[
\sum_{u=m}^{n} \sum_{k=0}^{n-u} \chi(k \leq n-u) = \sum_{k=0}^{n} \sum_{u=m}^{n} \chi(u+k \leq n) \\
\]

give

\[
\Gamma^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r}\left[\frac{1}{\tau}\right]h_{b-s}\left[\frac{1}{\tau}\right](-1)^{n-r-s} \sum_{k=0}^{n} \sum_{u=m}^{n} \chi(u+k \leq n) e_{u-m} \left[\frac{X}{(1-t)}\right] t^{u-m} e_{k} \left[\frac{X}{(1-t)}\right] \\
\times \sum_{\tau^{+}r+s-1} \frac{1}{u_{r}} e_{n-u-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] \sum_{\nu^{+}\tau} \sum_{k=0}^{n-u} e_{n-u-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] e_{k} \left[\frac{X}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{u+k-1} (e_{\tau^{+}} \left[\frac{X_{B_{r}}}{B_{r}}\right] + e_{\tau^{+}-1} \left[\frac{X_{B_{r}}}{B_{r}}\right] \sum_{k=0}^{n-u} e_{n-u-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] e_{k} \left[\frac{X}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{k} . \\
\]

Making the change of variables \(u \rightarrow v = u + k\) we get

\[
\Gamma^{(1)}_{m,a,b,n}[X,q,t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r}\left[\frac{1}{\tau}\right]h_{b-s}\left[\frac{1}{\tau}\right](-1)^{n-r-s} \sum_{k=0}^{n} \sum_{v=m+k}^{n} e_{v-m-k} \left[\frac{tX}{(1-t)}\right] e_{k} \left[\frac{tX}{(1-t)}\right] \\
\times \sum_{\tau^{+}r+s-1} \frac{1}{u_{r}} e_{n-v} \left[\frac{X_{D_{e}}}{M_{e}}\right] \sum_{\nu^{+}\tau} \sum_{k=0}^{n-v} e_{n-v-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] e_{k} \left[\frac{tX}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{v+k-1} (e_{\tau^{+}} \left[\frac{X_{B_{r}}}{B_{r}}\right] + e_{\tau^{+}-1} \left[\frac{X_{B_{r}}}{B_{r}}\right] \sum_{k=0}^{n-v} e_{n-v-k} \left[\frac{X_{D_{e}}}{M_{e}}\right] e_{k} \left[\frac{tX}{(1-t)}\right] \left(\frac{T_{e}}{T_{r}}\right)^{k} . \\
\]

Or better yet, since

\[
\sum_{k=0}^{n} \sum_{v=m+k}^{n} e_{v-m-k} \left[\frac{tX}{(1-t)}\right] e_{k} \left[\frac{tX}{(1-t)}\right] = \sum_{v=m}^{n} \sum_{k=0}^{n} \chi(v \leq n-m) e_{v-m-k} \left[\frac{tX}{(1-t)}\right] e_{k} \left[\frac{tX}{(1-t)}\right] = \sum_{v=m}^{n} e_{v-m} \left[\frac{tX}{(1-t)}\right] \\
\]

now we have

\[
\sum_{k=0}^{n} \sum_{v=m+k}^{n} e_{v-m-k} \left[\frac{tX}{(1-t)}\right] e_{k} \left[\frac{tX}{(1-t)}\right] = \sum_{v=m}^{n} e_{v-m} \left[\frac{tX}{(1-t)}\right] \\
\]

is the desired result.
we have
\[
\Gamma^{(1)}_{m,a,b,n}[X,q,t] = (-t)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left[ \frac{1}{M} \right] b_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{v=m}^{n} e_{v-m} \left[ \frac{X}{(1-t)} \right] 
\times \sum_{\tau \vdash r+s-1} \frac{1}{w_{\tau}} e_{n-v} \left[ \frac{X D_{\tau}}{M} \right] \sum_{\nu \vdash \tau} d_{\nu \tau} (\frac{T_\tau}{T_\tau})^{v-1} (e_{r} [B_{\tau}] + e_{r-1} [B_{\tau}] \frac{D_{\tau}}{T_\tau}).
\]

Now note that since \( \tau \vdash r+s-1 \) we must have \( r+s-1 \geq r \) for \( e_{r} [B_{\tau}] \neq 0 \) and \( r \geq 1 \) for \( e_{r-1} [B_{\tau}] \neq 0 \). We are thus brought to split \( \Gamma^{(1)}_{m,a,b,n}[X,q,t] \) into two terms obtained by the corresponding variable changes \( s \mapsto s+1 \) and \( r \mapsto r+1 \), obtaining
\[
\Gamma^{(1)}_{m,a,b,n}[X,q,t] = \Gamma^{(1b)}_{m,a,b,n}[X,q,t] + \Gamma^{(1a)}_{m,a,b,n}[X,q,t]
\]
with
\[
\Gamma^{(1b)}_{m,a,b,n}[X,q,t] = (-t)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left[ \frac{1}{M} \right] b_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{v=m}^{n} e_{v-m} \left[ \frac{X}{(1-t)} \right] 
\times \sum_{\tau \vdash r+s} \frac{e_{r} [B_{\tau}]}{w_{\tau}} e_{n-v} \left[ \frac{X D_{\tau}}{M} \right] \sum_{\nu \vdash \tau} d_{\nu \tau} (\frac{T_\tau}{T_\tau})^{v-1}
\]
and
\[
\Gamma^{(1a)}_{m,a,b,n}[X,q,t] = (-t)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-1-r} \left[ \frac{1}{M} \right] b_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{v=m}^{n} e_{v-m} \left[ \frac{X}{(1-t)} \right] 
\times \sum_{\tau \vdash r+s} \frac{e_{r} [B_{\tau}]}{w_{\tau}} e_{n-v} \left[ \frac{X D_{\tau}}{M} \right] \sum_{\nu \vdash \tau} d_{\nu \tau} (\frac{T_\tau}{T_\tau})^{v}.
\]

Now 1.18 gives
\[
\sum_{\nu \vdash \tau} d_{\nu \tau} (\frac{T_\tau}{T_\tau})^{v} = (-1)^{v-1} e_{v-1} [D_{\tau}] + \chi(v = 0),
\]
thus
\[
\sum_{\nu \vdash \tau} d_{\nu \tau} (\frac{T_\tau}{T_\tau})^{v-1} = (-1)^{v-2} e_{v-2} [D_{\tau}] + \chi(v = 1) \quad \text{and} \quad \sum_{\nu \vdash \tau} d_{\nu \tau} (\frac{T_\tau}{T_\tau})^{v} = (-1)^{v-1} e_{v-1} [D_{\tau}] + \chi(v = 0).
\]

Using these identities we get (since \( v \geq m \geq 1 \))
\[
\Gamma^{(1a)}_{m,a,b,n}[X,q,t] = (-t)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-1-r} \left[ \frac{1}{M} \right] b_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{v=m}^{n} e_{v-m} \left[ \frac{X}{(1-t)} \right] 
\times \sum_{\tau \vdash r+s} \frac{e_{r} [B_{\tau}]}{w_{\tau}} e_{n-v} \left[ \frac{X D_{\tau}}{M} \right] (-1)^{v-1} e_{v-1} [D_{\tau}].
\]

On the other hand, in the previous case we have two terms
\[
\Gamma^{(1b)}_{m,a,b,n}[X,q,t] = (-t)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left[ \frac{1}{M} \right] b_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{v=m}^{n} e_{v-m} \left[ \frac{X}{(1-t)} \right] 
\times \sum_{\tau \vdash r+s} \frac{e_{r} [B_{\tau}]}{w_{\tau}} e_{n-v} \left[ \frac{X D_{\tau}}{M} \right] (-1)^{v} e_{v-2} [D_{\tau}] +
\]
\[
+ (-t)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} \left[ \frac{1}{M} \right] b_{b-s} \left[ \frac{1}{M} \right] (-1)^{n-r-s} \sum_{v=m}^{n} e_{v-m} \left[ \frac{X}{(1-t)} \right] 
\times \sum_{\tau \vdash r+s} \frac{e_{r} [B_{\tau}]}{w_{\tau}} e_{n-v} \left[ \frac{X D_{\tau}}{M} \right] \chi(v = 1)
\]
and since \( v = 1 \) forces \( m = 1 \) we see that the second term reduces to

\[
(-t)^{0} \chi(m = 1) \sum_{r=0}^{a} \sum_{s=0}^{b-1} e_{a-r} M h_{b-s} \left[ (-1)^{n-1-r-s} \sum_{\tau^{r+s}} e_{r} B_{\tau} \right] \frac{e_{n-1} \left[ XD_{M} \right]}{w_{\tau}} .
\]

Recalling from Lemma 2.1 that for \( a + b + c = n \)

\[
\nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} M h_{b-s} \left[ (-1)^{n-1-r-s} \sum_{\tau^{r+s}} e_{r} B_{\tau} \right] \frac{e_{n-1} \left[ XD_{M} \right]}{w_{\tau}} ,
\]

we immediately recognize that the second term is none other than \( \nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = e_{n-1} \).

Thus putting together 2.12, 2.15, 2.16, 2.17 and 2.18, our manipulations carried out in the last three pages have yielded the following final expression for the polynomial \( \Gamma_{m,a,b,n}[X, q, t] = C_{m} \nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = e_{n-1} \).

**Theorem 2.1**

For all integers \( a, b \geq 0, m \geq 1 \) and \( n > a + b \) we have

\[
C_{m} \nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = t^{m-1} \Phi_{m,a,b,n}^{(1)}[X, q, t] + t^{m-1} \Phi_{m,a,b,n}^{(2)}[X, q, t] + \chi(m = 1) \left( \nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = e_{n-1} \right)
\]

with

\[
\Phi_{m,a,b,n}^{(1)}[X, q, t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b-1} e_{a-r} M h_{b-s} \left[ (-1)^{n-1-r-s} \sum_{v=m}^{n} e_{v-m} \left[ X \right] \right]
\]

\[
\times \sum_{\tau^{r+s}} e_{r} B_{\tau} \frac{e_{n-1} \left[ XD_{M} \right]}{w_{\tau}} (-1)^{v-1} e_{v-1} \left[ D_{\tau} \right] .
\]

and

\[
\Phi_{m,a,b,n}^{(2)}[X, q, t] = (-1)^{m-1} \sum_{r=0}^{a} \sum_{s=0}^{b-1} e_{a-r} M h_{b-s} \left[ (-1)^{n-1-r-s} \sum_{v=m}^{n} e_{v-m} \left[ X \right] \right]
\]

\[
\times \sum_{\tau^{r+s}} e_{r} B_{\tau} \frac{e_{n-1} \left[ XD_{M} \right]}{w_{\tau}} (-1)^{v} e_{v-2} \left[ D_{\tau} \right] .
\]

Our next task is to identify the sums occurring on the righthand sides of 2.19 and 2.20 as simply given by the following

**Theorem 2.2**

a) \( \Phi_{m,n,a,b}^{(1)} = B_{m-1} \nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = e_{n-1} \)  

b) \( \Phi_{m,n,a,b}^{(2)} = B_{m-2} \nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = e_{n-1} \).

**Proof**

By Lemma 2.1,

\[
\nabla h_{a}^{*} e_{b}^{*} e_{c}^{*} = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r} M h_{b-s} \left[ (-1)^{n-1-r-s} \sum_{\tau^{r+s}} e_{r} B_{\tau} \right] \frac{e_{n-1} \left[ XD_{M} \right]}{w_{\tau}} .
\]

Recall that we have

\[
B_{a}^{*} P[X] = P[X + \frac{M}{r}] \Omega \left[ \frac{X}{r} \right]_{z-a} .
\]
So to compute $B_{m-1}^{*} \nabla h_{a-1}^{*} e_{b}^{*} e_{n-a-b}^{*}$ we need

$$B_{m-1}^{*} e_{n-1} \left[ \frac{XD_{a}}{M} \right] = e_{n-1} \left[ \frac{X+M D_{a}}{M} \right] \Omega \left[ \frac{X}{1-M} \right] \bigg|_{z^{-(m-1)}} = \frac{e_{n-1} \left[ \frac{XD_{a}}{M} + \frac{D_{a}}{z} \right] \Omega \left[ \frac{X}{1-M} \right] \bigg|_{z^{-(m-1)}}}{z^{-(m-1)}}$$

$$= \sum_{u=0}^{n-1} e_{n-1-u} \left[ \frac{XD_{a}}{M} \right] e_{u} [D_{\gamma}] \Omega \left[ \frac{X}{1-M} \right] \bigg|_{z^{-(m-1)}}$$

(by 2.3) $= \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_{a}}{M} \right] e_{u-1} [D_{\gamma}] (-1)^{u-m} e_{u-m} \left[ \frac{X}{1-M} \right].$

Thus

$$B_{m-1}^{*} \nabla h_{a-1}^{*} e_{b}^{*} e_{n-a-b}^{*} = (-1)^{n-1} \sum_{r=0}^{a-1} \sum_{s=0}^{b} e_{a-1-r} \left[ \frac{1}{M} \right] h_{b-1-s} \left[ \frac{1}{M} \right] (-1)^{n-1-r-s}$$

$$\times \sum_{\gamma^{r+s}} e_{r} [B_{\gamma}] \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_{a}}{M} \right] e_{u-1} [D_{\gamma}] (-1)^{u-m} e_{u-m} \left[ \frac{X}{1-M} \right].$$

This proves 2.21 a).

To prove 2.21 b) we use again Lemma 2.1, to write

$$\nabla h_{a}^{*} e_{b-1}^{*} e_{n-1-a-b}^{*} = \sum_{a} \sum_{b} \left[ \frac{1}{M} \right] h_{b-1-s} \left[ \frac{1}{M} \right] (-1)^{n-2-r-s} \sum_{\gamma^{r+s}} e_{r} [B_{\gamma}] \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_{a}}{M} \right] e_{u-1} [D_{\gamma}] (-1)^{u-m} e_{u-m} \left[ \frac{X}{1-M} \right].$$

Consequently we need $B_{m-2}^{*} e_{n-2} \left[ \frac{XD_{a}}{M} \right]$, which, using 2.22 again, can be rewritten as

$$B_{m-2}^{*} e_{n-2} \left[ \frac{XD_{a}}{M} \right] = e_{n-2} \left[ \frac{XD_{a}}{M} + \frac{D_{a}}{z} \right] \Omega \left[ \frac{X}{1-M} \right] \bigg|_{z^{-(m-2)}} = \sum_{u=0}^{n-2} e_{n-2-u} \left[ \frac{XD_{a}}{M} \right] e_{u} [D_{\gamma}] \Omega \left[ \frac{X}{1-M} \right] \bigg|_{z^{-(m-2)}}$$

$$= \sum_{u=m-2}^{n-2} e_{n-u} \left[ \frac{XD_{a}}{M} \right] e_{u-1} [D_{\gamma}] (-1)^{u-m} e_{u-m} \left[ \frac{X}{1-M} \right].$$

This gives

$$B_{m-2}^{*} \nabla h_{a}^{*} e_{b-1}^{*} e_{n-1-a-b}^{*} = (-1)^{m-1} \sum_{r=0}^{a-1} \sum_{s=0}^{b} e_{a-1-r} \left[ \frac{1}{M} \right] h_{b-1-s} \left[ \frac{1}{M} \right] (-1)^{n-1-r-s}$$

$$\sum_{\gamma^{r+s}} e_{r} [B_{\gamma}] \sum_{u=m}^{n} e_{n-u} \left[ \frac{XD_{a}}{M} \right] e_{u-2} [D_{\gamma}] (-1)^{u} e_{u-m} \left[ \frac{X}{1-M} \right].$$

This proves 2.21 b) and completes our argument.

Since Theorems 2.1 and 2.2 combined give Theorem I.2, this section is completed as well.
3. The combinatorial side

For \( \gamma \models n \) and \( a+b+c = n \), let \( \Gamma_{\gamma}^{a,b,c} \) denote the family of parking functions with diagonal composition \( \gamma \) whose diagonal word is in the collection of shuffles \( \downarrow A \cup B \cup C \) with \( A, B, C \) successive segments of the integers \( 1, 2, \ldots, n \) of respective lengths \( a, b, c \). For instance in the display below we have an element of \( \Gamma_{\gamma}^{3,5,4} \).

Notice that reading the cars by decreasing diagonal numbers we obtain \( \sigma(PF) = 459103116712281 \), which is easily seen to be a shuffle of the three words

\[
\downarrow A = 3 \ 2 \ 1 \ , \ \ B = 4 \ 5 \ 6 \ 7 \ 8 \ and \ \ C = 9 \ 10 \ 11 \ 12 .
\]

Here and after it will be convenient to call the elements of \( A \) small cars, the elements of \( B \) middle cars and the elements of \( C \) big cars. At times, referring to these elements, we will use the abbreviations an \( S \), an \( M \) or a \( B \) respectively.

The goal of this section is to show that our family \( \Gamma_{\gamma}^{a,b,c} \) satisfies the same recursion satisfied by polynomials \( \langle \nabla S, e_q h_{b}, h_{c} \rangle \) from Theorem I.2. More precisely our goal is to show that, when \( a+b+c = n \) and \( \alpha \models n-m \), the polynomials

\[
\Pi^{a,b,c}_{m,\alpha}(q,t) = \sum_{PF \in \mathcal{PF}_n, p(PF) = (m,\alpha)} q^{\text{area}(PF)} t^{\text{dinv}(PF)} \chi(\sigma(PF) \in \downarrow A \cup B \cup C) \quad 3.1
\]

satisfy the recursion

**For** \( m > 1 \):

\[
\Pi^{a,b,c}_{m,\alpha}(q,t) = t^{m-1} q^{\ell(\alpha)} \left( \sum_{\beta \models m-1} \Pi^{a,b,c}_{\alpha,\beta}(q,t) + \sum_{\beta \models m-2} \Pi^{a,b-1,c-1}_{\alpha,\beta}(q,t) \right) . \quad 3.2
\]

**For** \( m = 1 \):

\[
\Pi^{a,b,c}_{1,\alpha}(q,t) = q^{\ell(\alpha)} \Pi^{a-1,b,c}_{\alpha}(q,t) + \Pi^{a,b-1,c}_{\alpha-1}(q,t) + \Pi^{a,b,c-1}_{\alpha}(q,t) + (q-1) \sum_{\iota,\delta=1} q^{\ell(\alpha+\delta)} \Pi^{a,b-1,c-1}_{\alpha+\delta}(q,t) . \quad 3.3
\]

Note that this is what Theorem I.2 forces us to write if Theorem I.1 is to be true. Our task is to establish these identities using only combinatorial properties of our families \( \Gamma_{\gamma}^{a,b,c} \).

Before we proceed to carry this out, we need to point out a few properties of the parking functions in \( \Gamma_{\gamma}^{a,b,c} \). To begin, note that since middle as well as big cars are to occur in increasing order in \( \sigma(PF) \) we will never run into an \( M \) on top of an \( M \) nor a \( B \) on top of a \( B \). Thus along any given column of the Dyck path there is at most one middle car and at most one big car. Clearly, on any column, a middle car can only lie above a small car and big car can only be on top of a column. By contrast small cars can be on top of small cars.

Note that since in the general case

\[
\downarrow A = a \cdots 321 , \ \ B = a + 1 \cdots a + b , \ C = a + b + 1 \cdots a + b + c = n
\]

Then if the car in cell \((1,1)\) is an \( S \) it has to be a 1 while if in that cell we have an \( M \) or a \( B \) they must respectively be an \( a + b \) or an \( a + b + c \). For the same reason a small car in any diagonal makes a primary dinv with any car in the same diagonal and to its right. While a middle car can make a primary dinv only with a big car to its right. Likewise a big car in any diagonal makes a secondary dinv only with a middle or a small car to its right and in the adjacent lower diagonal.
Keeping all this in mind we will start by proving the identity in 3.2. We do this by constructing a bijection $\phi$ between the family below and to the left and the union of the families below and to the right

$$ \Gamma_{a,b,c}^{m,\alpha} \iff \sum_{\beta=\alpha}^{m-1} \Gamma_{a,\beta}^{a-1,b,c} + \sum_{\alpha=\beta+1}^{m-2} \Gamma_{a,\beta}^{a,b,c-1}. $$

3.4

Here sums denote disjoint unions. Notice that each $PF \in \Gamma_{m,a,b,c}^{m,\alpha}$ necessarily starts with $m$ cars, the first of which is in cell $(1,1)$ and the next $m-1$ cars are in higher diagonals. We will refer to this car configuration as the first section of $PF$ and the following car configuration as the rest of $PF$. Recall that if the first car is small then it must be a 1. Next note that if the first car is not small then it must be $a+b$, for the first car can be an $a+b+c$ only when $m = 1$. Moreover, on top of $a+b$, also be a big car for otherwise we are again back to the $m = 1$ case. Also notice that, unless $m = 2$, then the third car in the first section is in diagonal 1 next to the big car atop $a+b$. To construct $\phi(PF)$ we proceed as follows

(i) If the first car is 1: Remove the main diagonal cells from the first section and cycle it to the end, as we do in the display below for $PF \in \Gamma_{5,2,5}^{3,5,4}$

(ii) If the first car is $a+b$: Remove the entire first column and the remaining main diagonal cells from the first section and cycle it to the end, as we do in the display below for another $PF \in \Gamma_{5,2,5}^{3,5,4}$

(iii) If the first car is $a+b$ and $m = 2$: Remove the entire first section.

To complete the construction of $\phi(PF)$, in case (i) we need only decrease by one all the remaining cars numbers. In cases (i) and (ii), if the big car in the removed column is a $u$, we must decrease by one all the middle numbers and the big numbers smaller than $u$ and finally decrease by 2 all the big cars numbers larger than $u$. This given, notice that in case (i) $\phi(PF)$ lands in the first sum on the right hand side of 3.4 while in cases (ii) and (iii) $\phi(PF)$ lands in the second sum. Moreover since the families in 3.4 are disjoint, $\phi$
has a well defined inverse, and therefore it is a bijection. Note further that since all the diagonal numbers of surviving cars of the first section have decreased by one as they have been cycled, not only the desired shuffle condition for \(\phi(PF)\) is achieved but all the primary dinvs between pairs of elements in different sections have become secondary dinvs, and all the secondary ones have become primary. For pairs of elements in the same section there is clearly no dinv changes since their relative order has been preserved. Remarkably in all cases, we have the same dinv loss of \(l(\alpha)\) units, in case (i) caused by the loss of primary dinvs created by the removed car 1 and in cases (ii) and (iii) by the loss of a primary dinv created by a big car to the right of \(a + b\) plus the loss of secondary dinv created by a small or middle car with the removed big car. Finally we note that the lowering by one of all the diagonal numbers of the first section causes a loss of area of \(m - 1\) in each of the three cases. These observations, combined, yield us a proof of the equality in 3.2.

Our next task is to establish 3.3. Here we have three distinct possibilities for the first car. The polynomial on the left of 3.3 will be split into three parts

\[
\Pi_{\alpha,a,s}^{a,b,c}(q,t) = \Pi_{\alpha,a,m}^{a,b,c}(q,t) + \Pi_{\alpha,a,M}^{a,b,c}(q,t) + \Pi_{\alpha,a,B}^{a,b,c}(q,t)
\]

according as the first car is a \(S\), an \(M\) or a \(B\). We will show that

\[\begin{align*}
a) & \quad \Pi_{\alpha,a,S}^{a,b,c}(q,t) = q^{l(\alpha)}\Pi_{\alpha,a}^{a,b,c-1}(q,t) \\
b) & \quad \Pi_{\alpha,a,M}^{a,b,c}(q,t) = \Pi_{\alpha,a}^{a,b,c-1}(q,t) + (q-1) \sum_{i:a_i=1} q^{i-1}\Pi_{\alpha,a}^{a,b,c-1}(q,t) \\
c) & \quad \Pi_{\alpha,a,B}^{a,b,c}(q,t) = \Pi_{\alpha,a}^{a,b,c-1}(q,t)
\end{align*}\]

Note, that in cases a), b) and c) the first cars must be 1, \(a + b\) and \(n\) respectively. Cases a) and c) are easily dealt with. If \(PF \in \Gamma_{\alpha,a,M}^{a,b,c}\) and the first car is a 1 we simply let \(\phi(PF)\) be parking function in \(\Gamma_{\alpha,a}^{a,b,c-1}\) obtained by removing the 1 and decreasing by one all remaining car numbers. Note that since the removed 1 made a primary dinv with every car in the 0-diagonal the dinv loss is precisely \(l(\alpha)\) in this case. This proves 3.6 a). If the first car is \(n\) we only need to remove the first car to obtain \(\phi(PF) \in \Gamma_{\alpha,a,b,c-1}^{a,b,c-1}\). The identity in 3.6 c) then follows immediately since the removed \(n\) was causing no dinv with any cars in its diagonal.

We are left with 3.6 b). This will require a more elaborate argument. Note that removing a first car \(a + b\) causes a loss of dinv equal to the number of big cars in its diagonal. The problem is that we have to figure out a way of telling what that number is. What is remarkable, is that our work on the symmetric function side led us to predict 3.6 b), which, as we shall see, solves this problem in a truly surprising manner.

To proceed we need some definitions. Firstly, recall that \(\Gamma_{1,a}^{a,b,c}\) denotes the family of \(PF \in \Gamma_{1,a}^{a,b,c}\) whose first car is \(a + b\). It will also be convenient to view the parking functions in \(\Gamma_{\alpha,a}^{a,b,c-1}\), for \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)\), as made up of \(k\) sections, the \(i^{th}\) one consisting of the configuration of cars whose indices are in the interval

\[
[\alpha_1 + \cdots + \alpha_{i-1}, \alpha_1 + \cdots + \alpha_i].
\]

Note further that if a big car is in the diagonal it must consist, of a single section, since no car can be placed on top of it. Thus, \(a\) being fixed, any given \(PF' \in \Gamma_{\alpha}^{a,b,c-1}\) can have big main diagonal cars only in the sections with indices in the set

\[
S(\alpha) = \{i : \alpha_i = 1\}.
\]

This given, it will convenient to denote by \(S(PF')\) the subset of \(S(\alpha)\) giving the indices of the sections containing the big cars in the main diagonal of \(PF'\).
Keeping this in mind, for $PF' \in \Gamma^a_{i_0,M}$ let $PF'' \in \Gamma^a_{i,b,c}$ be the parking function obtained by removing $a + b$ and lowering by one all big car numbers of $PF$. If $r = |S(PF')|$ is the number of big cars in the 0-diagonal of $PF''$ then, since the removed $a + b$ gave no contribution to the area of $PF$, it follows that

$$\text{tarea}(PF) q^{\text{dinv}(PF)} = \text{tarea}(PF') q^{\text{dinv}(PF')} q^r.$$  \hspace{1cm} 3.7

Guided by 3.6 b) we start by writing

$$q^r = 1 + (q - 1) + (q - 1)q + (q - 1)q^2 + \cdots + (q - 1)q^{r-1},$$

so that 3.7 may be rewritten as

$$\text{tarea}(PF) q^{\text{dinv}(PF)} = \text{tarea}(PF') q^{\text{dinv}(PF')} + (q - 1) \sum_{s=1}^{r} \text{tarea}(PF') q^{\text{dinv}(PF') + s - 1}. \hspace{1cm} 3.8$$

Next let

$$S(PF') = \{1 \leq i_1 < i_2 < \cdots < i_r \leq k\}.$$  \hspace{1cm} 3.9

This given, let $PF^{(i_s)}$ be the parking function obtained by removing from $PF'$ the big car in section $i_s$, and lowering by one all the big car numbers higher the removed one. Using our notational convention about $\overset{i_s}{\alpha}^{(i_s)}$, we may write

$$PF^{(i_s)} \in \Gamma^a_{\overset{i_s}{\alpha}^{(i_s)} - c}. \hspace{1cm} 3.10$$

Now we have

$$\text{dinv}(PF) = \text{dinv}(PF^{(i_s)}) + i_s - 1 + 1 + r - s.$$  \hspace{1cm} 3.11

To see this recall that we obtained $PF^{(i_s)}$ from $PF$ by removing car $a + b$ and the big diagonal car of section $i_s$. Now every diagonal car between these two removed cars was contributing a dinv for $PF$: the big ones with $a + b$ and and the rest of them with the removed big car. That accounts for $i_s - 1$ dinvs (one for each section of $PF'$ preceding section $i_s$). An additional dinv was caused by the removed pair and finally we must not forget that the remaining big diagonal cars past section $i_s$, $r - s$ in total, created a dinv with $a + b$. Thus 3.8 holds true precisely as stated. Since $\text{dinv}(PF) = \text{dinv}(PF') + r$ we may rewrite 3.10 as

$$\text{dinv}(PF') + s - 1 = \text{dinv}(PF^{(i_s)}) + i_s - 1$$

and thus 3.8 itself becomes

$$\text{tarea}(PF) q^{\text{dinv}(PF)} = \text{tarea}(PF') q^{\text{dinv}(PF')} + (q - 1) \sum_{s=1}^{r} \text{tarea}(PF') q^{\text{dinv}(PF^{(i_s)}) + i_s - 1}.\hspace{1cm}$$

Since removing diagonal cars does not change areas, this may be rewritten as

$$\text{tarea}(PF) q^{\text{dinv}(PF)} = \text{tarea}(PF') q^{\text{dinv}(PF')} + (q - 1) \sum_{s=1}^{r} \text{tarea}(PF^{(i_s)}) q^{\text{dinv}(PF^{(i_s)}) + i_s - 1},$$

or even better, using 3.9

$$\text{tarea}(PF) q^{\text{dinv}(PF)} = \text{tarea}(PF') q^{\text{dinv}(PF')} + (q - 1) \sum_{i \in S(PF')} \text{tarea}(PF^{(i)}) q^{\text{dinv}(PF^{(i)}) + i - 1}.$$  \hspace{1cm} 3.12
Since the pairing $PF \leftrightarrow PF'$ is clearly a bijection of $\Gamma_{a,b,c}^{\alpha}$ onto $\Gamma_{a,b-1,c}^{\alpha}$, to prove 3.6 b) we need only show that summing the right hand side of 3.12 over $PF' \in \Gamma_{a,b-1,c}^{\alpha}$ gives the right hand side of 3.6 b). Since by definition
\[ \sum_{PF' \in \Gamma_{a,b,c}^{\alpha}} q^{area(PF')} \prod \alpha_{a,b}^{b-1,c}(q,t) \]
and
\[ \sum_{PF' \in \Gamma_{a,b-1,c}^{\alpha}} q^{area(PF')} \prod \alpha_{a,b-1,c}^{b-1}(q,t) \]
we are reduced to showing that
\[ \sum_{PF' \in \Gamma_{a,b-1,c}^{\alpha}} \sum_{i \in S(PF')} q^{i} \prod \alpha_{a,b-1,c}^{b-1}(q,t) = \sum_{i \in S(\alpha)} q^{i} \prod \alpha_{a,b-1,c}^{b-1}(q,t) \]

It develops that this identity is due to a beautiful combinatorial mechanism that can best be understood by a specific example. Suppose that $\alpha$ is a composition with only three parts equal to one. In the following figure and on the left we display a sectionalization of the family $\Gamma_{a,b-1,c}^{\alpha}$ according to which of the three sections of length one of $\alpha$ contains a big car and which does not. We depicted a singleton section with a big car by a boxed $B$ and a singleton section with a small or middle car by a boxed cross. On the right of the display we have three columns depicting the analogous sectionalization of the the three families $\Gamma_{a,b-1,c-1}^{\alpha(1)}, \Gamma_{a,b-1,c-1}^{\alpha(2)}$ and $\Gamma_{a,b-1,c-1}^{\alpha(3)}$. Here the deleted section is depicted as a darkened box. The arrows connecting the two sides indicate how the summands on the left hand side of 3.13 are to be arranged to give the summands on the right hand side.

Notice that only the terms with a non empty $S(PF')$ do contribute to the left hand side of 3.13. Accordingly, on the left hand side of this diagram we have depicted the seven terms with non empty $S(PF')$. Notice also that the number of summands on the right hand side that correspond to a summand on the left hand side is precisely given by the size of the subset $S(PF')$. Finally as $PF'$ varies among the elements of $\Gamma_{a,b-1,c}^{\alpha}$ with a non empty $S(PF')$ the corresponding terms on the right hand side pick up all the sections of $\Gamma_{a,b-1,c-1}^{\alpha(1)}, \Gamma_{a,b-1,c-1}^{\alpha(2)}$ and $\Gamma_{a,b-1,c-1}^{\alpha(3)}$ each exactly once an only once.

**Remark 3.1**

We find this last argument truly remarkable. Indeed we can only wonder how the $C$ and $B$ operators could be so savvy about the combinatorics of parking functions. For we see here that the ultimate consequences of the innocent looking commutativity property in 1.38 b) as expressed by I.18 provide exactly what is needed to establish the equality in 3.13.
We terminate our writing with a few words concerning the equality of the base cases. To begin we notice from 3.2 and 3.3 that at each iteration we loose at least one of the cars. At the moment all the small cars are gone, the identity reduces to the $hh$ result established in [10]. At the moment all middle or big cars are gone the identity reduces to the $eh$ result established in [9]. If we are left with only middle cars or only big cars the identity reduces to the equality

$$\langle \nabla C_1^n 1, h_n \rangle = \sum_{PF \in PF_n, p(PF) = (1^n)} t^\text{area}(PF) q^\text{dinv}(PF).$$

Indeed, since middle (or big) cars can’t be placed on top of each other all columns of north steps of the supporting Dyck path must have length exactly 1. This forces the cars to be all in the main diagonal. Thus $p(PF) = 1^n$ and the area vanishes. Since the cars must increase as we go down the diagonal the dinv vanishes as well and the right hand side reduces to a single term equal to 1. Now, using the definition in I.9, it was shown in [10] (Proposition 1.3) that

$$C_1 C_1 \cdots C_1 1 = q^{-\binom{n}{2}} \tilde{H}_n[X; q, t],$$

and the definition of $\nabla$ gives

$$\nabla C_1 C_1 \cdots C_1 1 = \tilde{H}_n[X; q, t].$$

Thus 3.13 reduces to

$$\langle \tilde{H}_n[X; q, t], h_n \rangle = 1$$

which is a well known equality (see [6]).

In the case that we are left with only small cars our equality is much deeper since we are essentially back to a non-trivial special case of the $eh$ result.

However, we do not need to use any of our previous results, since in at most $n-1$ applications of our recursion we should be left with a single car, where there is only one parking function with no area and no dinv, forcing the right hand side of I.11 to be equal to 1, while the left hand side reduces to $\langle \nabla C_1 1, e_1 \rangle$ or $\langle \nabla C_1 1, h_1 \rangle$ as the case may be. Both of these scalar products are trivially equal to 1, since $C_1 1 = h_1$ and $\nabla h_1 = h_1$.

This terminates our treatment of the combinatorial side.
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