Harmonic oscillations and their switching in elliptical optical waveguide arrays

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Abstract
We have studied harmonic oscillations in an elliptical optical waveguide array in which the coupling between neighboring waveguides is varied in accord with a Kac matrix so that the propagation constant eigenvalues can take equally spaced values. As a result, long-living Bloch oscillations (BO) and dipole oscillations (DO) are obtained when a linear gradient in the propagation constant is applied. Moreover, we achieve a switching from DO to BO or vice versa by ramping up the gradient profile. The various optical oscillations as well as their switching are investigated by field-evolution analysis and confirmed by Hamiltonian optics. The equally spaced eigenvalues in the propagation constant allow viable applications in transmitting images, switching and routing of optical signals.

Keywords: harmonic oscillations, elliptical optical waveguide arrays, Bloch oscillation, dipole oscillation

(Some figures in this article are in colour only in the electronic version)

1. Introduction
Optical oscillations in optical waveguide arrays (OWA) are harmonic when the propagation constants of the transverse modes are equally spaced [1–3]. These oscillator waveguide arrays are of particular interest not only due to their applications in transmitting images [4], focusing and steering [5], tunneling [6–8], and switching and routing optical signals [9, 10], but they are also good candidates for realizing the optical analogies of the dynamics in quantum systems [11]. Among these, Bloch oscillations (BO) and dipole oscillations (DO) are two fundamental and important types of oscillations [12, 13]. BO is known as the oscillatory motion of a particle in a periodic potential under a constant force [14], whose optical equivalent in an OWA is a linear gradient of the onsite propagation constants [15, 16]. DO usually occurs at the bottom of a parabolic band [17]. In the reciprocal space, DO is distinguished from BO in that the BO momentum is accelerated monotonically by the force while the DO momentum oscillates due to the confinement of a band [18, 19]. The steering between BO and DO has been realized in a parabolic optical waveguide array [19]. However, both BO and DO decay fairly quickly due to the unequally spaced propagation constants in a parabolic band. In order to obtain long-living BO and DO, we need to construct OWAs so that the eigenvalues of the propagation constants are equally spaced.

In the literature, several models have been proposed to achieve this goal, for example, the Wannier–Stark waveguide arrays, which require an infinitely varying transverse refractive-index distribution [15]. In real experiments [9, 10], we need to truncate the system, which unavoidably disrupt the equally spaced propagation constants. As a result, the oscillations can decay rapidly, especially when the light beam is incident at the boundaries of the system. Besides, there also exist dissipation [20], defects [21] and other kinds of damping effects in real systems. Recently, a finite waveguide array with graded coupling constants described by the elements of a Kac matrix was proposed [22, 23]. This is to achieve the required equally spaced propagation constants [3]. From now on, we refer these graded couplings $\kappa$ to circular couplings as $\kappa$ varies circularly as a function of waveguide index. In the finite
waveguide arrays with a constant onsite potential, harmonic oscillations other than BO have been observed. The harmonic oscillation is similar to DO, as it also oscillates in the reciprocal space.

In this work, we apply a linear gradient to the onsite propagation constants, which leads to the occurrence of BO. Strictly speaking, BO is approximate, because the band structure is elliptical caused by the combination of linear graded onsite potential and the circular couplings. Due to the elliptical profile of the dispersion relation, we call the array the elliptical optical waveguide array (EOWA). We propose to realize a switching between DO and BO or vice versa by the elliptical potential profiles (solid line) and a linearly graded onsite potential constant (dashed line).

2. Model and formalism

The EOWA consists of $N = 100$ waveguides as shown schematically in figure 1. This array is divided into two zones (zone 1: $0 \leq z \leq z_1$ and zone 2: $z_1 < z \leq z_2$) along the longitudinal $z$-axis direction, where there are two elliptical potential profiles $H_1$ and $H_2$, respectively. As shown in figure 2, each elliptical potential profile is formed by the circular couplings between adjacent waveguides ($\kappa$, solid line) and a linearly graded onsite potential constant ($\beta_0$, dashed line). The switching between BO and DO can be achieved by ramping up the gradient of the propagation constant from zone 1 to zone 2. The circular couplings are realized by carefully designing the structure of EOWA. The linear gradient in propagation constants are obtained by taking advantage of the electro-optical effect or thermo-optical effect [15, 9], and the gradient strength is adjusted by the external voltage or temperature difference. The size of each waveguide is in the micrometer scale. However, the real physical parameters should be calibrated through experiments. As shown in figure 1, the intensity of the discrete input beam has a Gaussian distribution and its wavefront is not a plane wave. The finite phase difference of excitations between adjacent waveguides is realized by putting a dielectric block just in front of the waveguide array. The light beam propagates along the axis of the waveguide array, that is, the $z$ direction. The waveguide array is labeled by $n$ ($n = 1, 2, \ldots, N$) in the transverse direction.

As mentioned in the literature [24], the waveguide arrays are analyzed by the coupled-mode theory [15]. The evolutionary equation of modal amplitude $a_n$ in the $n$th waveguide is written as

$$\frac{d}{dz} a_n(z) + V_n a_n(z) + \kappa_{n,n-1} a_{n-1}(z) + \kappa_{n,n+1} a_{n+1}(z) = 0, \quad (1)$$

where $V_n = \alpha n/N + V_0$ is the onsite propagation constant, $\alpha$ is the gradient of the linear potential, $V_0$ is the onset propagation constant, which is set to be 4 in this study, and $\kappa_{n,n-1}$ and $\kappa_{n,n+1}$ are the coupling constants between nearest-neighbor waveguides. In order to obtain harmonic BO and DO, the values of the coupling constants should be designed appropriately, as described in the following.

Substituting the solution $a_n^m(z) = u_n^m \exp(i\beta_n^m z)$ into (1), we have

$$\beta_n^m u_n^m = V_n u_n^m + \kappa_{n,n-1} u_{n-1}^m + \kappa_{n,n+1} u_{n+1}^m, \quad (2)$$

where $\beta_n^m$ means the wavenumber of the supermode $m$ and the transverse mode profile is given by a superposition of the mode amplitudes $u_n^m$ of the individual waveguides. Equation (2) is rewritten in the matrix form

$$\beta |u\rangle = H |u\rangle, \quad (3)$$

where the Hamiltonian matrix $H$ is defined as $H_{n,m} = V_n = \alpha n/N + 4$, $H_{n,n-1} = \kappa_{n,n-1}$ and $H_{n,n+1} = \kappa_{n,n+1}$. In order to achieve the harmonic BO and DO, the eigenvalues of $H$ should be equally spaced. It was shown that the $N + 1$ by
which is a function of waveguide index \( n \) [22, 23]. The Kac matrix is a tridiagonal matrix with elements \( K_{n,n+1} = n, K_{n,n+1} = N - n \) and \( K_{n,m} = 0 \) (otherwise), and is asymmetric (non-Hermitian). The Kac matrix is symmetrized [23] and the super- and sub-boundaries of the phase diagram for EOWA, as shown in figure 3(b). Separated by \( \beta_{c1} \) and \( \beta_{c2} \), there are also three different oscillations: lower dipole oscillation, Bloch oscillation and upper dipole oscillation. (b) The contour plots of mode patterns of the eigenvectors in the real space. The shape of the contour plots is similar to the elliptical band. The corresponding mode patterns for the three different oscillations are also separated by the two critical lines \( \beta_{c1} \) and \( \beta_{c2} \).

Figure 3. (a) Phase diagram for the elliptical optical waveguide array with \( N = 100 \) waveguides. The pseudo-dispersion relation lines \( \beta(n, 0) \) and \( \beta(n, \pi) \) serve as the upper and lower boundaries of the band, whose shape is elliptical. Separated by the two critical lines \( \beta_{c1} = \beta(1, \pi) \) and \( \beta_{c2} = \beta(n, 0) \), three regions are formed, which indicate three different kinds of oscillations: lower dipole oscillation, Bloch oscillation and upper dipole oscillation. (b) The contour plots of mode patterns of the eigenvectors in the real space. The shape of the contour plots is similar to the elliptical band. The corresponding mode patterns for the three different oscillations are also separated by the two critical lines \( \beta_{c1} \) and \( \beta_{c2} \).

3. Results

3.1. Various oscillations and normal modes in EOWA

To analyze the various oscillations in EOWA, we resort to an effective diagrammatic approach with the aid of a phase diagram [18]. From the pseudo-dispersion relation (7), two curves \( \beta(n, 0) \) and \( \beta(n, \pi) \) serve as the upper and lower boundaries of the phase diagram for EOWA, as shown in figure 3(a). Separated by two critical lines \( \beta_{c1} = \beta(1, \pi) \) and \( \beta_{c2} = \beta(N, 0) \), there are three different regions: the lower DO region, BO region and upper DO region. In these three regions, DO at the smaller \( n \) side, BO at the middle and DO at the larger \( n \) side take place, respectively. As stated in the previous research [18], there exist correspondences between various optical oscillations and the localization of different gradon modes in OWA [18]. The mode patterns of all the eigenmodes are shown by the contour plots of square moduli of eigenmodes as a function of eigenvalue \( \beta \) and waveguide index \( n \), as shown in figure 3(b). Separated by \( \beta_{c1} \) and \( \beta_{c2} \), there are also three different regions, which indicate three different gradon modes.
corresponding to the three kinds of optical oscillations. If we construct the light beam using components of a specific kind of gradon modes in a particular region, the light beam will undergo a certain type of oscillation, as shown by the three input Gaussian beams in different oscillation regions in figure 3(a). For a certain input beam, is it possible to undergo DO under certain conditions and BO under other conditions? The answer is yes, and we will present it in detail in the following sections.

3.2. DO–BO transition

The switching between DO and BO is realized by ramping up the gradient of the propagation constant. Let us first sketch an example of the DO–BO transition as shown in figure 4(a). In the range 0 ≤ z ≤ z1, the original onsite propagation constant has a linear gradient α = 1.0 and the Hamiltonian is \( H_1 = \beta(n, k, 1.0) \) (solid lines). This linear gradient together with the circular couplings leads to the long-living DO for an input Gaussian beam \( (n_0 = 40, k_0 = 0, \sigma = 5) \). In the original potential profile, the input light beam undergoes DO between points A and B. After two periods, the gradient of the propagation constant is changed to be \( \alpha = 4.0 \) when \( z_1 \leq z \leq z_2 \). As a consequence, the elliptical potential profile is changed to a new one (as the dashed lines showed): the light beam is lifted from point A to point C and undergoes BO between points C and D in the new potential profile. In other words, the DO–BO transition has been realized by varying the gradient strength of the propagation constant from \( \alpha = 1.0 \) to 4.0. As shown in figure 4(b), the DO–BO transition process A → B → C → D is also marked on the phase-space orbit. The solid lines are for the original potential profile when \( \alpha = 1.0 \) and \( \beta = 5.38 \). Since these curves are closed and their wavenumbers are confined to a certain range, these features indicate the occurrence of DO. The dashed lines are for the new potential profile when \( \alpha = 4.0 \) and \( \beta = 6.58 \). The value of the wavenumber \( kd \) has no limitation along these dashed lines, which indicates the occurrence of BO. The transition takes place between points A and C, which coincide on the phase-space orbit.

3.3. Field-evolution analysis

The process of DO–BO transition is investigated through the field-evolution analysis. The analysis is performed with an input wavefunction at \( z = 0 \):

\[
\psi(0) = \frac{1}{(2\pi \sigma^2)^{1/4}} e^{-\frac{1}{2} \sigma^2} e^{-i k_0(n_0-z_0)},
\]

where \( k_0 \) is the input transverse wavenumber. The incoming field at \( z (z < 0) \) is \( \psi(z) = \psi(0) \exp(i \beta_0 z) \), where \( \beta_0 \) is the propagation constant of an individual homogeneous channel. The intensity profile \( |\psi(0)|^2 \) has a discrete Gaussian distribution centered at the \( n_0 \)th waveguide and spatial width \( \sigma \). This input beam is a discrete Gaussian beam, whose intensity distribution is schematically shown as the dashed line in figure 1. The exponential factor \( \exp \{-i k_0(n - n_0)\} \) denotes the phase differences between input beams excited on the \( n \)th and the \( n_0 \)th waveguides. Other than the Gaussian distribution, the input beam can take other functional forms.

We expand the input wavefunction in terms of the supermodes \( |u_m\rangle \):

\[
|\psi(0)\rangle = \sum_m A_m |u_m\rangle,
\]

where \( A_m = \langle u_m | \psi(0) \rangle \) is the constituent component of the input Gaussian beam. The subsequent wavefunction at propagation distance \( z \) is given by

\[
|\psi(z)\rangle = \sum_m A_m e^{i \beta_m z} |u_m\rangle.
\]

At a certain propagation distance \( z \), the wavefunction in the reciprocal space can be obtained by taking the following Fourier transform:

\[
|\phi(k, z)\rangle = \mathcal{F}(|\psi(n, z)\rangle).
\]
By using these wavefunctions, the mean values of $n$ and $k$ are obtained as

$$
\langle n \rangle = \frac{\langle \psi | n | \psi \rangle}{\langle \psi | \psi \rangle}, \quad \langle k \rangle = \frac{\langle \phi | k | \phi \rangle}{\langle \phi | \phi \rangle}.
$$

The field evolution of the DO–BO transition is demonstrated by the contour plots of $|\psi(n, z)|^2$ in the real space and $|\phi(k, z)|^2$ in the reciprocal space, as shown in figures 5(a) and (b), respectively. When $0 \leq z \leq z_1$, oscillatory motions take place in both real space and reciprocal space (in a confined range), which indicates the occurrence of DO in this range. When $z_1 \leq z \leq z_2$, there is still oscillatory motion in the real space (with a smaller amplitude), while in the reciprocal space, the wavenumber $kd$ is accelerated and can cover the whole range $[-\pi, \pi]$ in the reduced Brillouin zone. These features clearly show the occurrence of the DO–BO transition. The period of these harmonic oscillations $Z$ can be obtained from $\psi(z + Z) = \psi(z)$ and (10), where the condition $\exp(i \Delta \beta Z) = 1$ should be satisfied. Thus the period is derived as $Z = 2\pi/\Delta \beta = 2\pi N/\sqrt{4 + \alpha^2}$. We need to emphasize that there is almost no leakage of field from the main orbit of field evolution. Both DO and BO can persist for a long propagation distance. This property is important in practical applications of propagating light in optical waveguide arrays.

### 3.4. Hamiltonian optics

The results of the DO–BO transition obtained through the field-evolution analysis are now confirmed by the Hamiltonian optics approach. From the pseudo-dispersion relation (7), the evolution of the mean position and the mean wavenumber of the beam can be solved by using the equation of motion

$$
\frac{dx}{dz} = \frac{\partial \beta(x, k, \alpha)}{\partial k}, \quad \frac{dk}{dz} = -\frac{\partial \beta(x, k, \alpha)}{\partial x}.
$$

where the auxiliary variable $x = 2n/N - 1$ ($-1 < x \leq 1$) is used to replace the discrete waveguide index $n$ and $\beta(x, k, \alpha) = 4 + \alpha(1 + x)/2 + \sqrt{1 - x^2}\cos k$. The numerical Hamiltonian optics results of the mean position $\langle x \rangle$ and the mean wavenumber $\langle kd \rangle$ are shown by solid lines in figures 5(c) and (d), respectively. The dashed lines and dots in figures 5(c) and (d) are the mean values of $\langle x \rangle$ and $\langle kd \rangle$ calculated from the field-evolution analysis results. The mean values obtained from two methods match very well with each other.
4. Discussion and conclusion

As suggested in [3], the experimental realization of the EOWA is similar to the experimental structures GaAs/AlGaAs [10]. To obtain the required $\kappa$ values, the spacing between the waveguides was carefully designed based on the evanescent codirectional couplings [25]. In the above investigation, we considered the nearest-neighbor couplings between waveguides. This consideration is valid when the field overlapping is significant between neighboring waveguides only. We can also extend this study to more general OWAs with higher-order couplings. As is shown in [26], the second-order coupling in OWA leads to nontrivial BO. It is instructive to consider higher-order couplings and the effects of defects in EOWA. Isolated defects can always disrupt the equally spaced eigenvalues and lead to damping, but Longhi designed localized defects carefully by supersymmetric quantum mechanics and obtained undamped optical oscillations [21]. From the basic physics point of view, an intrinsic harmonic system can naturally help us to access the competing damping mechanisms, like defects and disorder or even nonlinearity. Thus in future work, we will investigate the nature of the harmonic oscillation against these effects.

In summary, we have studied the dipole oscillation and Bloch oscillation in the elliptical optical waveguide arrays, which possess a linear gradient in the onsite propagation constants and circular couplings between neighboring waveguides. The couplings between neighboring waveguides are designed according to the tridiagonal elements of a Kac matrix, which is to achieve equally spaced eigenvalues. As a consequence, both DO and BO are long-living, which is important to optical steering applications. We also proposed to realize the DO–BO transition or vice versa by ramping up the gradient of propagation constants. The spatial evolution of the DO–BO transition is demonstrated through the field-evolution analysis and confirmed by the Hamiltonian optics approach. The long-living DO and BO and their switching have viable applications in transmitting images, switching and routing of optical signals.

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