Summation of Power Series by Self-Similar Factor Approximants

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Abstract

A novel method of summation for power series is developed. The method is based on the self-similar approximation theory. The trick employed is in transforming, first, a series expansion into a product expansion and in applying the self-similar renormalization to the latter rather than the former. This results in self-similar factor approximants extrapolating the sought functions from the region of asymptotically small variables to their whole domains. The method of constructing crossover formulas, interpolating between small and large values of variables is also analysed. The techniques are illustrated on different series which are typical of problems in statistical mechanics, condensed-matter physics, and, generally, in many-body theory.

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1 Introduction

It is the standard situation for practically all realistic problems in physics and applied mathematics that their solutions are obtained by some kind of perturbation theory or iterative procedure resulting in asymptotic series in powers of a parameter or a variable. Such series are usually only asymptotic and diverge for finite values of the expansion variable, while what is of physical interest are exactly the finite values of the latter. That is why the problem of defining an effective sum of divergent asymptotic series is of paramount importance. The most often employed summation techniques are Padé and Borel summations [1,2]. These techniques have a number of limitations, because of which in many cases they are either of bad accuracy or often are not applicable at all.

An alternative approach to the problem of defining effective sums of asymptotic series or effective limits of iterative sequences has been developed, being based on the ideas of optimal control theory, renormalization group theory, and general dynamical theory. The principal concepts of this approach are as follows.

The first pivotal step is the introduction of \textit{control functions} whose role is to govern an optimal convergence of approximation sequences [3]. This results in the \textit{optimized perturbation theory} which has been widely employed for a variety of applications (see surveys in [4,5]).

The second idea is to consider the passage from one successive approximation to another as the motion on the manifold of approximants, where the approximant order plays the role of discrete time. The recurrent relations, representing this motion, are formalized by means of \textit{group self-similarity} [6–11].

A dynamical system in discrete time, whose trajectory is bijective to an approximation sequence, is called the \textit{approximation cascade}. Embedding the latter into an approximation flow makes it possible to derive differential and integral equations of motion, whose fixed points represent \textit{self-similar approximants} to the sought function [6–11]. The stability of the calculational procedure is characterized by \textit{local multipliers} [8–11].

Another key point of the approach is the introduction of control functions with the help of \textit{fractal transforms} [12–14]. This allows one to analyse asymptotic series by transforming them into their fractal counterparts and then invoking all the machinery of the self-similar approximation theory for the transformed series. In this way, the approximants, possessing a nice self-similar structure, were obtained, such as the \textit{self-similar exponential approximants} [13–15] and \textit{self-similar root approximants} [16–18].

In the present paper, we suggest and give a detailed analysis of one more type of approximants possessing a nice self-similar structure, which, because of their form, we name the \textit{self-similar factor approximants}. In Section 2, we explain the origin of these approximants and discuss some variants of their usage. Then, in Section 3, we illustrate the method on several functions containing exponentials. Section 4 contains examples of more complicated and more realistic physical problems. Quartic anharmonic models are treated in section 5. Calculations of critical points and critical indices for some critical phenomena are presented in Section 6. The results are summarized in Section 7, where further possible developments of the method are discussed.
2 Method of self-similar factor approximants

Suppose we are looking for a physical quantity described by a real function \( f(x) : \mathbb{R} \to \mathbb{R} \). It is the common case for realistic physical problems that the sought function is defined through so complicated equations that the problem can be solved only approximately, by invoking a kind of perturbation theory. Assume that the latter yields perturbative approximants

\[
f(x) \approx f_k(x) \quad (x \to 0),
\]

of order \( k = 0, 1, 2, \ldots \), in the asymptotic vicinity of \( x = 0 \). For a while, the concrete meaning of the function \( f(x) \) and of the variable \( x \) is not important. In particular applications, \( x \) may be a spatial variable, a coupling parameter, or any other physical parameter. This may also be a time variable in expansions having to do with differential evolution equations [19–21].

We may assume, without loss of generality, that the sought function \( f(x) \) has the property \( f(0) = 1 \), so that \( f_k(0) = 1 \). As is evident, this can be achieved for any function by the appropriate scaling and normalization. Then an asymptotic series in powers of \( x \) takes the form

\[
f_k(x) = 1 + \sum_{n=1}^{k} a_n x^n.
\]

If we apply the technique of the self-similar approximation theory to the fractal transform of series (2), we would come, depending on the choice of control functions, to the self-similar exponential or root approximants [13–18]. But here, we follow another route.

We may notice [22] that each series (2) can always be equivalently presented as a product

\[
f_k(x) = \prod_{i=1}^{k} (1 + b_i x),
\]

with the coefficients \( b_i \) defined by the accuracy-through-order relations with respect to series (2). Now, we apply the procedure of self-similar renormalization not to the whole series (2) but separately to each factor of the product (3). This results in the transformation

\[
1 + b_i x \to (1 + A_i x)^{n_i},
\]

in which \( A_i \) and \( n_i \) are control parameters (see details in [12–18]). In this way, starting with product (3), we come to the self-similar factor approximants

\[
f_k^*(x) = \prod_{i=1}^{k} (1 + A_i x)^{n_i}.
\]

Control parameters \( A_i \) and \( n_i \) can be found by means of the re-expansion procedure [23], expanding expression (5) in powers of \( x \) and equating the like terms from series (2). It is also possible to define some of the controllers by matching the behaviour of the factor approximant (5) with the large-\( x \) asymptotic behaviour of \( f(x) \), provided this is available. For instance, if the asymptotic form

\[
f(x) \simeq f_\infty x^\beta \quad (x \to \infty)
\]
is known, then we may require the validity of the crossover condition

$$\prod_{i=1}^{k} A_i^{n_i} = f_\infty, \quad \sum_{i=1}^{k} n_i = \beta. \quad (7)$$

The resulting crossover formula (5) would sew together two asymptotic expressions, at small $x \to 0$ and large $x \to \infty$.

As a zero-order approximant, we may set

$$f_0^*(x) \equiv 1 + a_1 x. \quad (8)$$

The first-order approximant (5) is

$$f_1^*(x) = (1 + A_1 x)^{n_1}, \quad (9)$$

and for the second-order, we have

$$f_2^*(x) = (1 + A_1)^{n_1} (1 + A_2 x)^{n_2}. \quad (10)$$

Thus, we construct approximants of any order. For each approximant $f_k^*(x)$, the accuracy-through-order relation defines its own controllers $A_i = A_{ik}$ and $n_i = n_{ik}$. However, to avoid cumbersome notation, we shall not employ the double indexation, keeping in mind that the controllers $A_i$ and $n_i$ are a sort of variables that are different for different approximation orders $k = 1, 2, \ldots$.

Each self-similar factor approximant (5) of order $k$ contains $2k$ controllers $A_1, \ldots, A_k$ and $n_1, n_2, \ldots, n_k$. To define these through the re-expansion procedure, one needs to have $2k$ nontrivial terms of series (2). Thus, an even-order perturbative series (2) uniquely defines the corresponding factor approximant (5). The possibility of using odd-order series will be discussed in Section 7.

It may happen that, calculating the controllers $A_i$ and $n_i$, we shall obtain for some of them complex values. If the sought function is real, then the complex-valued controllers should come in complex conjugate pairs, so that approximant (5) be real. This means that the latter may contain the factors of the type $|z^\alpha|^2$, with both $z$ and $\alpha$ being complex valued. Since

$$|z^\alpha| = |z|^{\text{Re} \alpha} \exp \{-(\text{Im} \alpha) \arg z\}$$

is a nonalgebraic function of $z$, hence expression (5) may, in general, be a nonalgebraic function. According to definition [24], an analytical function that is not algebraic is termed transcendental.

In this way, formula (5) can represent the following functions. When all powers $n_i$ are real integers, positive or negative, then $f_k^*(x)$ is a rational function. In the case when all powers are equal to $\pm 1$, expression (5) coincides with the form of Padé approximants, which are, therefore, just a narrow particular case of the self-similar factor approximants. When some of $n_i$ are real but not integer, then $f_k^*(x)$ is an irrational function. And if some of $A_i$ and $n_i$ are complex valued, then $f_k^*(x)$ represents a transcendental function. As far as any sought function, having the same form (5), is exactly reproducible by $f_k^*(x)$, whose controllers are
defined by the re-expansion procedure [23], this means that the class of functions, that can be exactly reconstructed by the self-similar factor approximants, is rather wide, including rational, irrational, and some transcendental functions. This class is essentially wider than that related to Padé approximants and consisting solely of rational functions. Consequently, the factor approximants (5) can provide much better accuracy than Padé approximants for the majority of functions and also can be employed when the latter are not applicable at all.

Approximants (5) can well approximate such entire transcendental functions as exponentials. This can be easily understood noticing that if a factor \((1 + Ax)^n\) is such that \(A = a/n\), where \(a\) is a constant and \(n \to \infty\), then such a factor gives

\[
(1 + \frac{a}{n} x)^n \to e^{ax} \quad (n \to \infty),
\]

that is, it is equivalent to an exponential. Therefore, if in some particular calculations we find a very small \(A\) but a very large \(n\), this hints that the corresponding factor \((1 + Ax)^n\) is close to the exponential \(\exp(nAx)\). This is why the factor approximants can provide a reasonable approximation for exponential functions.

### 3 Reconstruction of functions including exponentials

Exponential functions are ubiquitous in natural sciences representing various physical laws. At the same time, this is a simple example of an entire transcendental function. The best way of approximating these functions would be by means of the self-similar exponentials [13–15], which often reconstruct exponential-type functions exactly [25]. But the factor approximants, as is explained in the previous section, should also mimic well exponential behaviour. This is illustrated in the following examples.

#### 3.1 Nonmonotonic exponential function with maximum

We choose a nonmonotonic function in order to stress that such a nonmonotonic behaviour presents no problem for the method. Let us consider the function

\[
f(x) = \exp \left\{ \frac{x}{(1 + x)^{5/2}} \right\}.
\]

Its asymptotic expansion at \(x \to 0\) yields the finite sums (2) with the coefficients

\[
\begin{align*}
a_1 &= 1, & a_2 &= -2, & a_3 &= 2.042, & a_4 &= -0.271, \\
a_5 &= -3.572, & a_6 &= 8.636, & a_7 &= -12.661, & a_8 &= 12.183,
\end{align*}
\]

and so on. To approximate function (11), we construct the self-similar factor approximants (5). The first-order approximant (9) is yet too simple to catch well the peculiarities of the function variation. So, here and in what follows, we shall start the analysis with the second-order approximant (10).
Here for $f^*_2(x)$, we have

\[ A_1 = 1.658 + 0.840 \, i, \quad A_2 = A_1^*; \]

\[ n_1 = -0.244 - 1.077 \, i, \quad n_2 = n_1^*. \]

But, as $x \to \infty$, the function $f^*_2(x)$ decreases to zero, since $n_1 + n_2 = -0.488$, while function (11) tends to unity.

For $f^*_3(x)$, we get

\[ A_1 = 1.3540, \quad A_2 = 1.0015 + 0.4034 \, i, \quad A_3^* = A_2^*; \]

\[ n_1 = 11.0051, \quad n_2 = -5.4336 + 3.7402 \, i, \quad n_3 = n_2^*. \]

Note that the enumeration of the controllers $A_i$ and $n_i$ can be arbitrary, as far as the order of factors in Eq. (5) is interchangeable. As a rule, we shall enumerate them according to the descending order of the values $|\text{Re} \, A_i|$. Now, $n_1 + n_2 + n_3 = 0.138$, and the approximation is not bad till $x \sim 150$.

For $f^*_4(x)$, we find

\[ A_1 = 1.1455, \quad A_2 = 1.0076 + 0.1703 \, i, \quad A_3 = A_2^*, \quad A_4 = 0.5156; \]

\[ n_1 = 84.0147, \quad n_2 = -40.1038 + 36.6118 \, i, \quad n_3 = n_2^*, \quad n_4 = -3.7834. \]

The sum of the powers decreases, $n_1 + n_2 + n_3 + n_4 = 0.024$. And the accuracy is very good up to $x$ of order of hundred, with the error being less than 5%.

### 3.2 Nonmonotonic exponential function with minimum

Let us now turn to the function

\[ f(x) = \exp \left\{- \frac{x}{(1 + x)^{5/2}} \right\} \]  

having a minimum. The corresponding coefficients in series (2) are

\[ a_1 = -1, \quad a_2 = 3, \quad a_3 = -7.042, \quad a_4 = 15.354, \]

\[ a_5 = -32.261, \quad a_6 = 65.951, \quad a_7 = -131.714, \quad a_8 = 257.734. \]

This function also tends to 1 at $x \to \infty$.

For $f^*_2(x)$, we have

\[ A_1 = 1.65790 + 0.83984 \, i, \quad A_2 = A_1^*; \]

\[ n_1 = 0.24381 + 1.07665 \, i, \quad n_2 = n_1^*. \]

This approximant works well till $x \sim 10$, but at $x \to \infty$ its behaviour is not correct since $n_1 + n_2 = 0.488$. 

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The controllers of $f_3^*(x)$ are

\[ A_1 = 1.35398, \quad A_2 = 1.00149 + 0.40337i, \quad A_3 = A_2^*; \]

\[ n_1 = -11.00514, \quad n_2 = 5.43355 - 3.74022i, \quad n_3 = n_2^*. \]

This function decreases at $x \to \infty$, since $n_1 + n_2 + n_3 = -0.133$. Reasonable accuracy is provided till $x \sim 10$.

The next approximant $f_4^*(x)$ has the parameters

\[ A_1 = 1.14550, \quad A_2 = 1.00759 + 0.17032i, \quad A_3 = A_2^*, \quad A_4 = 0.51563; \]

\[ n_1 = -84.01466, \quad n_2 = 40.10379 - 36.61184i, \quad n_3 = n_2^*, \quad n_4 = 3.78340. \]

The function decreases much slower than the previous approximations, since $n_1 + n_2 + n_3 + n_4 = -0.024$. The accuracy is good up to $x \sim 100$. For instance, the percentage error at $x = 40$ equals $-2\%$.

### 3.3 Spectral density of black-body radiation

This is given by the Planck formula $\rho(x) = Cx^3/(e^{x} - 1)$. This is also a nonmonotonic function with a maximum. To get the presentation (2), we define $f(x) \equiv \rho(x)/Cx^2$, which results in

\[ f(x) = \frac{x}{e^x - 1}, \quad (13) \]

where $x \equiv \hbar \omega/k_B T$. The coefficients of series (2) are

\[ a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{12}, \quad a_3 = 0, \quad a_4 = -\frac{1}{720}, \]

\[ a_5 = 0, \quad a_6 = \frac{1}{30240}, \quad a_7 = 0, \quad a_8 = -\frac{1}{1209600}, \]

\[ a_9 = 0, \quad a_{10} = \frac{1}{47900160}, \quad a_{11} = 0, \quad a_{12} = -\frac{691}{1307674368000}. \]

For $f_2^*(x)$, we get

\[ A_1 = -0.05 + 0.119024i, \quad A_2 = A_1^*; \]

\[ n_1 = -1 + 2.520504i, \quad n_2 = n_1^*. \]

The function decreases at $x \to \infty$, since $n_1 + n_2 = -2$, but not so fast as Eq. (13).

For $f_3^*(x)$, one has

\[ A_1 = -0.052201, \quad A_2 = -9.613987 \times 10^{-3} + 0.150710i, \quad A_3 = A_2^*; \]

\[ n_1 = 7.654768, \quad n_2 = -1.327382 + 0.417823i, \quad n_3 = n_2^*. \]

The function increases at $x \to \infty$ because $n_1 + n_2 + n_3 = 5$, which is not correct.

The controllers of $f_4^*(x)$ are

\[ A_1 = -0.026326 + 0.044214i, \quad A_2 = A_1^*, \quad A_3 = -1.452250 \times 10^{-3} + 0.158034i, \quad A_4 = A_3^*; \]
The sum $n_1 + n_2 + n_3 + n_4 = -4$ shows that the function decreases as $x \to \infty$, in agreement with the behaviour of Eq. (13).

To analyse the behaviour of higher-order approximants, we have also calculated $f^*_5(x)$, for which $\sum_{k=1}^5 n_k = 0.367295$; because of this $f^*_5(x)$ increases at infinity, though not as fast as $f^*_3(x)$. And for $f^*_6(x)$, we find $\sum_{k=1}^6 n_k = -3.300722$; hence this approximant decreases as $x \to \infty$. With increasing order $k$, the interval of $x$, where $f^*_k(x)$ provides a very good approximation to $f(x)$, becomes larger. Figure 1 presents the difference $f^*_4(x) - f(x)$.

### 3.4 Specific heat of diatomic gas

The oscillational part of the specific heat for a diatomic gas has the form [26]

$$C(x) = \frac{x^2 e^x}{(e^x - 1)^2} ,$$

with $x \equiv \hbar \omega/k_B T$. The expansion coefficients of series (2) are

$$a_1 = -\frac{1}{12} , \quad a_2 = \frac{1}{240} , \quad a_3 = -\frac{1}{6048} , \quad a_4 = -\frac{1}{172800} ,$$

$$a_5 = -\frac{1}{5322240} , \quad a_6 = \frac{691}{118879488000} ,$$

and so on. Expression (14) tends to zero as $x \to \infty$.

For the second approximant $C^*_2(x)$, we find

$$A_1 = 0.0251 , \quad A_2 = 0.0026 ;$$

$$n_1 = -2.0656 , \quad n_2 = -11.9344 .$$

The approach to zero, as $x \to \infty$, is sufficiently fast, since $n_1 + n_2 = -14$.

For $C^*_3(x)$, one gets

$$A_1 = 0.025148 , \quad A_2 = 198.278139 , \quad A_3 = 2.629939 \times 10^{-3} ;$$

$$n_1 = -2.065650 , \quad n_2 = 0 , \quad n_3 = -11.934396 .$$

Here, $n_1 + n_2 + n_3 = -14$. Both $C^*_k(x)$ are of good accuracy, with the difference $C^*_k(x) - C(x)$ less than $10^{-4}$. Note that small $A_3$ with large $n_3$ hint at the exponential dependence. The comparison of $C^*_3(x)$ with $C(x)$, given in Eq. (14), shows that this factor approximant is essentially more accurate than the best Padé approximants $P_{[4/2]}(x)$ and $P_{[5/1]}(x)$, as is illustrated in Fig. 2.
3.5 Regular solution from Debye-Huckel theory

Consider the function

\[ f(x) = \frac{2}{x} - \frac{2}{x^2} \left(1 - e^{-x}\right) \]  

(15)

appearing in the Debye-Huckel theory [26]. This function contains an exponential, but the large-\(x\) asymptotic behaviour is of power law,

\[ f(x) \approx 2 \left(\frac{1}{x} - \frac{1}{x^2}\right) \quad (x \to \infty). \]

The coefficients of series (2) are

\[ a_1 = -\frac{1}{3}, \quad a_2 = \frac{1}{12}, \quad a_3 = -\frac{1}{60}, \quad a_4 = \frac{1}{360}, \]

\[ a_5 = -\frac{1}{2520}, \quad a_6 = \frac{1}{20160}, \quad a_7 = -\frac{1}{181440}, \quad a_8 = \frac{1}{1814400}. \]

Constructing the factor approximants (5), we shall invoke the crossover condition (7).

For \( f_2^*(x) \), we have

\[ A_1 = 0.177337 + 0.176694 i, \quad A_2 = A_1^*; \]

\[ n_1 = -0.5 + 0.441432 i, \quad n_2 = n_1^*. \]

The maximal error of this approximant is about 2.5\% at \( x \approx 15 \).

The controllers of \( f_3^*(x) \) are

\[ A_1 = 0.153026, \quad A_2 = 0.076787 + 0.156796 i, \quad A_3 = A_2^*; \]

\[ n_1 = -1.146332, \quad n_2 = 0.073166 + 0.539398 i, \quad n_3 = n_2^*. \]

The maximal error now is around −1\% at \( x \approx 20 \). Percentage errors of \( f_3^*(x) \) and the two-point Padé approximant \( P_{3/2}(x) \) are shown in Fig. 3.

For \( f_4^*(x) \), we get

\[ A_1 = 0.095338 + 0.056647 i, \quad A_2 = A_1^*, \quad A_3 = 0.043027 + 0.139489 i, \quad A_4 = A_3^*; \]

\[ n_1 = -0.966783 + 0.967772 i, \quad n_2 = n_1^*, \quad n_3 = 0.466783 + 0.285025 i, \quad n_4 = n_3^*. \]

Here the maximal, with respect to all \( x \in [0, \infty) \), error is about 0.5\% at \( x \approx 30 \). The best two-point Padé approximant [1] involving the same number of terms has the maximal error −6\% at \( x \approx 20 \), which is an order larger than the maximal error of \( f_4^*(x) \). The accuracy of the latter may be improved even more, if to take into account two terms from the large-\(x\) expansion. Then the maximal error can be reduced to 0.15\%.

4 Structure factors and radiation intensity

The quantities such as structure factors and radiation intensity are directly observable in experiments. That is why much attention is paid to their theoretical description. Below we show how one can derive rather simple and quite accurate expressions for such quantities, by means of the self-similar factor approximants.
4.1 Structure factor of branched polymers

The structure factor $S(x)$ is usually expressed in scaled units as a function of $x \equiv k^2 R^2/d$, where $k$ is a wave vector, $R$ is gyration radius, and $d$ is spatial dimensionality [27,28]. For $d = 3$, the structure factor of branched polymers is given [28] by a confluent hypergeometric function $S(x) = F_1 (1; 3/2; 3x/2)$. The behaviour at large $x$ is

$$S(x) \simeq \frac{1}{3} x^{-1} \quad (x \to \infty).$$

At small $x$, the coefficients of series (2) are

$$a_1 = -1, \quad a_2 = 0.6, \quad a_3 = -0.257.$$

We shall use the crossover condition (7). Then for $S_2^*(x)$, we find

$$A_1 = 0.247387 + 0.483295 i, \quad A_2 = A_1^*;$$
$$n_1 = -0.5 - 0.778627 i, \quad n_2 = n_1^*.$$

The approximant $S_2^*(x)$ agrees well with the exact $S(x)$, as is shown in Fig. 4.

4.2 Structure factor of ring polymers

The structure factor of three-dimensional dilute ring polymers $S(x)$ can be considered as a function of $x \equiv (kR)^2$, where $k$ is wave vector and $R$ is gyration radius [29]. In series (2), we have

$$a_1 = - \frac{1}{3}, \quad a_2 = 0.0610, \quad a_3 = -0.0073.$$

Knowing the large-$x$ asymptote

$$S(x) \simeq 0.63 x^{-1/2\nu} \quad (\nu = 0.588),$$

we shall employ the crossover condition (7). Then we get for $S_2^*(x)$

$$A_1 = 0.065483 + 0.185055 i, \quad A_2 = A_1^*;$$
$$n_1 = -0.425170 + 0.750184 i, \quad n_2 = n_1^*.$$

This gives a rather good accuracy, as compared to numerical results [29].

4.3 Luminescent intensity of donor-acceptor recombination

The expression for the temporal dependence of the luminescence intensity, arising from radiative recombination of donor-acceptor pairs, has the form [30,31]

$$I(z) = - \exp\{aK(z)\} \frac{d}{dz} K(z),$$

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in which $z \equiv wt$, $w$ is a transition probability per unit time, $t$ is time, $a$ is a constant, and

$$K(z) \equiv \int_0^\infty x^2 \left[ \exp \left( -ze^{-x} \right) - 1 \right] dx .$$

We shall consider the reduced quantity

$$f(z) \equiv \frac{I(z)}{I(0)} , \quad I(0) = 2 ,$$

with the typical parameter $a = 6.514 \times 10^{-7}$. Then the coefficients of short-time series (2) are

$$a_1 = -0.125 , \quad a_2 = 0.019 , \quad a_3 = -2.604 \times 10^{-3} , \quad a_4 = 3.334 \times 10^{-4} , \quad a_5 = -3.858 \times 10^{-5} , \quad a_6 = 4.050 \times 10^{-6} .$$

In the long-time limit, one has

$$f(z) \simeq \frac{(\ln z)^2}{4z} \exp \left\{ -\frac{a}{3} (\ln z)^3 \right\} \quad (z \to \infty) ,$$

which tends to zero.

For $f^*_2(z)$ we find

$$A_1 = 0.132653 + 0.072661 i , \quad A_2 = A^*_1 ;$$

$$n_1 = -0.256845 + 0.391265 i , \quad n_2 = n^*_1 .$$

This approximant tends to zero, as $z \to \infty$, since $n_1 + n_2 = -0.514$.

For $f^*_3(z)$, we obtain

$$A_1 = 0.098681 , \quad A_2 = 0.087209 + 0.090440 i , \quad A_3 = A^*_2 ;$$

$$n_1 = -1.292027 , \quad n_2 = 0.293510 + 0.269217 i , \quad n_3 = n^*_2 .$$

This also tends to zero, as $z \to \infty$, with the power law $n_1 + n_2 + n_3 = -0.705$. In the range $0 \leq z \leq 10^3$, the maximal error is about $2\%$ at $z \approx 200$. The best Padé approximant $P_{3/3}(z)$ is much less accurate, as is seen in Fig. 5.

5 Models of quartic anharmonic oscillators

Anharmonic oscillator models, especially low-dimensional ones, are usually considered as touchstones for checking the validity of any approximation technique. This is because these models mimic the main mathematical features of more realistic, but much more complicated, problems in many-body theory and field theory. Below, we illustrate the applicability of the method of self-similar factor approximants to two low-dimensional anharmonic models.
5.1 Partition function of zero-dimensional model

The partition function is modelled by the integral
\[
Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp \left( -\varphi^2 - g\varphi^4 \right) \, d\varphi ,
\]
(16)
in which \( g \) imitates a coupling parameter. Coefficients of series (2) are given by the formula
\[
a_n = \frac{(-1)^n}{\sqrt{\pi} \, n! \, \Gamma \left( 2n + \frac{1}{2} \right)} .
\]

In the strong-coupling limit, one has
\[
Z(g) \simeq 1.022765 \, g^{-1/4} \quad (g \to \infty) .
\]
(17)

Following the technique of Section 2, we find for \( Z_2^*(g) \) the controllers
\[
A_1 = 19.141 , \quad A_2 = 4.859 ;
\]
\[
n_1 = -0.00862 , \quad n_2 = -0.120 .
\]
At large \( g \), this yields
\[
Z_2^*(g) \simeq 0.806 \, g^{-0.129} \quad (g \to \infty) .
\]

The approximant gives a reasonable accuracy at finite \( g \) and is qualitatively correct at large \( g \), though the decrease of the partition function (16) is not as fast as in Eq. (17). The amplitude of the decay is predicted with an error \(-21\%\) and the power, with an error 48%.

For \( Z_3^*(g) \), we get
\[
A_1 = 31.220 , \quad A_2 = 13.317 , \quad A_3 = 3.464 ;
\]
\[
n_1 = -0.000526 , \quad n_2 = -0.022 , \quad n_3 = -0.125 .
\]
In the strong-coupling limit,
\[
Z_3^*(g) \simeq 0.807 \, g^{-0.148} \quad (g \to \infty) .
\]

Here the prediction errors are \(-21\%\) for the amplitude and 41% for the power.

The approximant \( Z_4^*(g) \) contains
\[
A_1 = 43.965 , \quad A_2 = 23.064 , \quad A_3 = 10.294 , \quad A_4 = 2.677 ;
\]
\[
n_1 = -0.0000526 , \quad n_2 = -0.00259 , \quad n_3 = -0.035 , \quad n_4 = -0.124 .
\]

The strong-coupling limit gives
\[
Z_4^*(g) \simeq 0.810 \, g^{-0.161} \quad (g \to \infty) .
\]

The errors are \(-21\%\) for the amplitude and 36% for the power index.
For $Z_5^*(g)$, we find

$$A_1 = 57.165211 , \quad A_2 = 33.720143 , \quad A_3 = 18.543443 ,$$
$$A_4 = 8.396565 , \quad A_5 = 2.174639 ;$$
$$n_1 = -1.025651 \times 10^{-6}, \quad n_2 = -2.150967 \times 10^{-4}, \quad n_3 = -6.028095 \times 10^{-3},$$
$$n_4 = -0.044164 , \quad n_5 = -0.119596 .$$

The strong-coupling limit is

$$Z_5^*(g) \simeq 0.81445 g^{-0.170} \quad (g \to \infty).$$

Here, the errors of the amplitude is $-20\%$ and that of the power index is $32\%$.

And for $Z_6^*(g)$, we obtain

$$A_1 = 66.087061 , \quad A_2 = 41.445220 , \quad A_3 = 25.096672 ,$$
$$A_4 = 13.752049 , \quad A_5 = 6.067701 , \quad A_6 = 1.511098 ;$$
$$n_1 = -1.153412 \times 10^{-7}, \quad n_2 = -3.352991 \times 10^{-5}, \quad n_3 = -1.330855 \times 10^{-3},$$
$$n_4 = -0.014434 , \quad n_5 = -0.057625 , \quad n_6 = -0.110557 .$$

In the strong-coupling limit, this leads to

$$Z_6^*(g) \simeq 0.82548 g^{-0.184} \quad (g \to \infty).$$

The errors are $-19\%$ for the amplitude and $26\%$ for the index. These results demonstrate monotonic numerical convergence.

The accuracy of approximants can be essentially improved by invoking the crossover condition (7). Then for $Z_2^*(g)$, we have

$$A_1 = 8.57593 , \quad A_2 = 0.30987 ; \quad n_1 = -0.08136 , \quad n_2 = -0.16864 ;$$
$$A_1^{n_1} A_2^{n_2} = 1.023 , \quad n_1 + n_2 = -0.25 .$$

This approximant provides a good accuracy in the whole region of $g \in [0, \infty)$, with the maximal error of $6\%$ at $g \approx 5$.

For $Z_3^*(g)$, we find

$$A_1 = 19.47265 , \quad A_2 = 5.21916 , \quad A_3 = 0.18018 ;$$
$$n_1 = -7.94804 \times 10^{-3}, \quad n_2 = -0.10947 , \quad n_3 = -0.13258 ;$$
$$A_1^{n_1} A_2^{n_2} A_3^{n_3} = 1.023 , \quad n_1 + n_2 + n_3 = -0.25 .$$

The maximal error is about $4\%$ at $g \approx 10$.

The next approximant $Z_4^*(g)$ contains

$$A_1 = 31.45590 , \quad A_2 = 13.55695 , \quad A_3 = 3.72340 , \quad A_4 = 0.12535 ;$$

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\[ n_1 = -4.96209 \times 10^{-4}, \quad n_2 = -0.02119, \quad n_3 = -0.11631, \quad n_4 = -0.11200; \]
\[ A_1^{n_1} A_2^{n_2} A_3^{n_3} A_4^{n_4} = 1.023, \quad n_1 + n_2 + n_3 + n_4 = -0.25. \]

The maximal error is around 3.5\% at \( g \approx 10 \).

And for \( Z_5^g(g) \), we obtain
\[ A_1 = 44.148724, \quad A_2 = 23.247563, \quad A_3 = 10.482932, \]
\[ A_4 = 2.878801, \quad A_5 = 0.095481; \]
\[ n_1 = -2.395229 \times 10^{-5}, \quad n_2 = -2.477078 \times 10^{-3}, \quad n_3 = -0.033250, \]
\[ n_4 = -0.115813, \quad n_5 = -0.098436. \]

Now the maximal error is close to 3\% at \( g \approx 20 \).

Finally, for \( Z_6^g(g) \), we have
\[ A_1 = 57.315168, \quad A_2 = 33.870044, \quad A_3 = 18.694023, \]
\[ A_4 = 8.552264, \quad A_5 = 2.338677, \quad A_6 = 0.076825; \]
\[ n_1 = -9.879942 \times 10^{-7}, \quad n_2 = -2.072431 \times 10^{-4}, \quad n_3 = -5.811092 \times 10^{-3}, \]
\[ n_4 = -0.042533, \quad n_5 = -0.112765, \quad n_6 = -0.088682. \]

For this approximant, the maximal error is about 2.5\% at \( g \approx 30 \).

### 5.2 Ground-state energy of the anharmonic oscillator

We consider the one-dimensional anharmonic quartic oscillator with the Hamiltonian
\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + gx^4, \tag{18}
\]
where \( x \in (-\infty, +\infty) \). Perturbation theory gives [32] the ground-state energy \( E(g) \) as the series
\[
E(g) = \sum_{n=0}^{\infty} a_n g^n, \tag{19}
\]
where the first eleven coefficients are
\[
a_0 = 0.5, \quad a_1 = 0.75, \quad a_2 = -2.625, \quad a_3 = 20.8125, \quad a_4 = -241.2890625, \]
\[ a_5 = 3580.98046875, \quad a_6 = -63982.8134766, \quad a_7 = 1329733.72705, \]
\[ a_8 = -31448214.6928, \quad a_9 = 833541603.263, \quad a_{10} = -24478940702.8. \]

Since here \( a_0 = 1/2 \), the reduced function \( f(g) \), for which \( f(0) = 1 \), has to be defined as
\[
f(g) \equiv \frac{E(g)}{E(0)} = 2E(g),
\]

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with the corresponding change of the expansion coefficients in series (2). The strong-coupling limit of the ground state energy is

$$ E(g) \simeq 0.667986 \, g^{1/3} \quad (g \to \infty). \quad (20) $$

For the self-similar factor approximant $E^*_2(g)$, we have

$$ A_1 = 17.5973, \quad A_2 = 5.3122; $$
$$ n_1 = 0.0221, \quad n_2 = 0.2091. $$

The large-$g$ limit is

$$ E^*_2(g) \simeq 0.7554 \, g^{0.2312} \quad (g \to \infty), $$

Comparing the latter with the asymptotic form (20), we see that the error of the amplitude here is 13% and that of predicting the power is −31%.

In the case of $E^*_3(g)$, we get

$$ A_1 = 26.7402, \quad A_2 = 12.4688, \quad A_3 = 3.8380; $$
$$ n_1 = 1.8017 \times 10^{-3}, \quad n_2 = 0.0547, \quad n_3 = 0.2005. $$

The strong-coupling limit gives

$$ E^*_3(g) \simeq 0.7562 \, g^{0.2570} \quad (g \to \infty). $$

Here the error of the amplitude is again 13%, while that of the power is −23%. If we compare $E^*_3(g)$ with the Padé approximant $P_{3/3}(g)$, providing the best accuracy at finite $g$, then we see that the latter is wrong at large $g$, tending to a constant, as $g \to \infty$, instead of increasing.

For $E^*_4(g)$, we find

$$ A_1 = 36.27784, \quad A_2 = 20.11620, \quad A_3 = 9.69400, \quad A_4 = 2.98137; $$
$$ n_1 = 1.05387 \times 10^{-4}, \quad n_2 = 8.41555 \times 10^{-3}, \quad n_3 = 0.08068, \quad n_4 = 0.18273. $$

In the strong-coupling limit, we have

$$ E^*_4(g) \simeq 0.75229 \, g^{0.27193} \quad (g \to \infty). $$

The errors are around 13% for the amplitude and −18% for the power.

The approximant $E^*_5(g)$ contains

$$ A_1 = 46.160836, \quad A_2 = 28.182719, \quad A_3 = 16.332980, $$
$$ A_4 = 7.924575, \quad A_5 = 2.422310; $$
$$ n_1 = 4.992768 \times 10^{-6}, \quad n_2 = 8.787692 \times 10^{-4}, \quad n_3 = 0.018549, $$
$$ n_4 = 0.097564, \quad n_5 = 0.164673. $$
The strong-coupling limit yields
\[ E_5^*(g) \approx 0.747711 \, g^{0.28167} \quad (g \to \infty) . \]
Now the error of the amplitude is 12% and that of the power is −15%. There is a monotonic numerical convergence to exact results.

The accuracy of the approximants \( E_k^*(g) \) can be essentially improved with the help of the crossover condition (7). Invoking the latter, we get for \( E_2^*(g) \)
\[
A_1 = 9.457715 , \quad A_2 = 0.869296 ; \\
n_1 = 0.140915 , \quad n_2 = -0.192419 ; \\
\frac{1}{2} A_1^{n_1} A_2^{n_2} = 0.667986 , \quad n_1 + n_2 = \frac{1}{3} .
\]
For \( E_3^*(g) \), we find
\[
A_1 = 18.112324 , \quad A_2 = 5.896948 ; \quad A_3 = 0.483681 ; \\
n_1 = 0.019178 , \quad n_2 = 0.184859 , \quad n_3 = 0.129296 ; \\
\frac{1}{2} A_1^{n_1} A_2^{n_2} A_3^{n_3} = 0.667986 , \quad n_1 + n_2 + n_3 = \frac{1}{3} .
\]
In the approximant \( E_4^*(g) \), the controllers are
\[
A_1 = 27.098877 , \quad A_2 = 12.848678 ; \quad A_3 = 4.253915 , \quad A_4 = 0.322173 ; \\
n_1 = 1.616412 \times 10^{-3} , \quad n_2 = 0.049729 , \quad n_3 = 0.184752 , \quad n_4 = 0.097236 ,
\]
with condition (7) being valid.
And for \( E_5^*(g) \), we obtain
\[
A_1 = 36.555891 , \quad A_2 = 20.393831 , \quad A_3 = 9.992627 , \\
A_4 = 3.302859 , \quad A_5 = 0.236809 ; \\
n_1 = 9.656933 \times 10^{-5} , \quad n_2 = 7.776204 \times 10^{-3} , \quad n_3 = 0.075166 , \\
n_4 = 0.172047 , \quad n_5 = 0.078248 ,
\]
again with condition (7) exactly satisfied.

6 Critical points and critical indices

The self-similar factor approximants (5) look very appropriate for describing critical phenomena, where various thermodynamic functions usually possess power-law behaviour. The direct application of Padé approximants [1] or their variants [33–35] does not allow for a correct description of critical indices. But the factor forms, analogous to approximants (5), should provide an accurate account of typical thermodynamic characteristics in the vicinity of critical points [36]. Note that critical phenomena exist not only in the models of statistical mechanics and condensed matter physics but also in the models of rupture and materials failure [37–40], earthquakes [41,42], and stock market crashes [43–45]; all such critical phenomena having many common features [46,47]. Below we show that the self-similar factor approximants present, probably, the best possibility for characterizing different critical phenomena.
6.1 Logarithmic divergence in elliptic integral

This example illustrates that even when the divergence in a critical point is not of power law but logarithmic, it can, anyway, be reasonably well described by approximants (5). For this purpose, we consider the elliptic integral

\[ f(t) \equiv \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - t \sin^2 x}}, \tag{21} \]

which diverges logarithmically at \( t_c = 1 \). The logarithmic divergence implies that if one defines the critical index as

\[ \alpha \equiv \lim_{t \to t_c^-} \left( t - t_c \right) \frac{d}{dt} \ln f(t), \tag{22} \]

then it has to be close to zero. Expanding integral (21) in powers of \( t \), one gets the asymptotic series of type (2), with the coefficients

\[
\begin{align*}
a_1 & = \frac{1}{4}, & a_2 & = \frac{9}{64}, & a_3 & = \frac{25}{256}, & a_4 & = \frac{1225}{16384}, \\
a_5 & = \frac{3969}{65536}, & a_6 & = \frac{53361}{1048576}, & a_7 & = \frac{184041}{4194304}, \\
a_8 & = \frac{41409225}{1073741824}, & a_9 & = \frac{147744025}{4294967296}, & a_{10} & = \frac{21334223721}{68719476736}.
\end{align*}
\]

For the approximant \( f_2^*(t) \), we have

\[
A_1 = -0.9640, \quad A_2 = -0.3485; \\
n_1 = -0.2218, \quad n_2 = -0.1038.
\]

Divergence happens at a critical point \( t_c \), with a critical index, defined in Eq. (22), which are

\[ t_c = \frac{1}{|A_1|} = 1.0373, \quad \alpha = |n_1| = 0.222. \]

The critical point is defined with an error 3.7%.

For \( f_3^*(t) \), we find

\[
A_1 = -0.9839, \quad A_2 = -0.6311; \quad A_3 = -0.1767; \\
n_1 = -0.1938, \quad n_2 = -0.0721, \quad n_3 = -0.0783.
\]

Divergence is characterized by

\[ t_c = \frac{1}{|A_1|} = 1.0163, \quad \alpha = |n_1| = 0.194. \]

Here, the error of \( t_c \) is 1.6%.
The controllers of the approximants $f_4^*(t)$ are

$$A_1 = -0.99112, \quad A_2 = -0.77048, \quad A_3 = -0.41456, \quad A_4 = -0.10519;$$

$$n_1 = -0.17705, \quad n_2 = -0.05864, \quad n_3 = -0.05405, \quad n_4 = -0.06587.$$

The critical point and index are

$$t_c = \frac{1}{|A_1|} = 1.0090, \quad \alpha = |n_1| = 0.177.$$

The error of $t_c$ is $0.9\%$.

For $f_5^*(t)$, we obtain

$$A_1 = -0.9944, \quad A_2 = -0.8449, \quad A_3 = -0.5782, \quad A_4 = -0.2867, \quad A_5 = -0.0692;$$

$$n_1 = -0.1657, \quad n_2 = -0.0508, \quad n_3 = -0.0438, \quad n_4 = -0.0452, \quad n_5 = -0.0582.$$

The critical characteristics are

$$t_c = \frac{1}{|A_1|} = 1.0056, \quad \alpha = |n_1| = 0.166.$$

Now, the error of $t_c$ is just $0.6\%$. Both the critical point and index show good numerical convergence to exact results. The best Padé approximant, based on the same number of perturbative terms, is much less accurate giving $t_c = 1.117$ and $\alpha = 1$; the error of $t_c$ being $12\%$.

### 6.2 Critical indices from Wilson expansion

The scalar quantum field theory with $\varphi^4$-interaction is an ideal ground for testing new methods in perturbation theory [2,48]. Much efforts has been invested into studying critical phenomena by means of the $O(n)$-symmetric $\varphi^4$-theory. A very popular problem consists in calculating critical exponents by field theoretic renormalization group techniques in $d = 4 - \varepsilon$ dimensions, resulting in the Wilson $\varepsilon$-expansions [2,49–51]. To illustrate a good accuracy of self-similar factor approximants in summing such $\varepsilon$-expansions, we consider here the critical index $\nu$ for the correlation length $\xi(T)$, defined as

$$\nu \equiv -\lim_{T \to T_c+0} (T - T_c) \frac{d}{dT} \ln \xi(T),$$

where $T$ is temperature. At the critical temperature $T_c$, the correlation length diverges as $\xi \sim (T - T_c)^{-\nu}$. We shall use the expansion [49] for the function

$$f(\varepsilon) \equiv \frac{1}{2\nu(\varepsilon)} \quad (\varepsilon \equiv 4 - d).$$
From here, one can get the critical index in three dimensions

$$\nu = \frac{1}{2f(1)} \quad (d = 3)$$

(25)

or in two dimensions

$$\nu = \frac{1}{2f(2)} \quad (d = 2).$$

(26)

Function (24), presented as expansion (2) in powers of $\varepsilon$, in the single-component case possesses the coefficients

$$a_1 = -0.1665, \quad a_2 = -0.05865, \quad a_3 = 0.06225, \quad a_4 = -0.1535, \quad a_5 = 0.4755.$$

The straightforward usage of such expansions does not give reliable results, since the $\varepsilon$-series are asymptotic and divergent. For instance, the critical index

$$\nu_k \equiv \frac{1}{2f_k(\varepsilon)} \quad (\varepsilon \equiv 4 - d),$$

(27)

defined through a $k$-order perturbative polynomial $f_k(\varepsilon)$, as in Eq. (2), for $d = 3$ becomes

$$\nu_1 = 0.6, \quad \nu_2 = 0.645, \quad \nu_3 = 0.597, \quad \nu_4 = 0.731, \quad \nu_5 = 0.431 \quad (d = 3).$$

The sequence $\{\nu_k\}$, evidently, does not converge.

For the factor approximant $f_2^*(\varepsilon)$, we obtain

$$A_1 = 3.018517, \quad A_2 = -0.439949;$$

$$n_1 = 6.874957 \times 10^{-3}, \quad n_2 = 0.425622.$$

The corresponding index

$$\nu_2^* \equiv \frac{1}{2f_2^*(\varepsilon)} \quad (\varepsilon \equiv 4 - d)$$

(28)

in $\nu_2^* = 0.634$ for $d = 3$ and $\nu_2^* = 1.216$ for $d = 2$.

We know that for the two-dimensional Ising model, when $\varepsilon = 2$, the index $\nu = 1$. Imposing this restriction, we find for $d = 3$ the index $\nu_2^* = 0.629$. This is to be compared with the Borel-transform based calculations for the $\varepsilon$-expansion, yielding $\nu = 0.628 \pm 0.001$ [52] and $\nu = 0.629 \pm 0.003$ [53] and also with the lattice numerical results $\nu = 0.631 \pm 0.002$ [2].

### 6.3 Critical characteristics from high-temperature expansions

High-temperature series expansions are often used in studying critical phenomena. Here, we shall consider the so-called $(2 + 1)$-dimensional Ising model, whose Hamiltonian reads

$$H = \sum_i (1 - \sigma_i^z) - g \sum_{<ij>} \sigma_i^x \sigma_j^x - h \sum_i \sigma_i^x,$$

(29)
in which $\sigma_i^a$ are Pauli matrices; $g$, coupling parameter; $h$, magnetic field; summation is over a 2-dimensional spatial lattice; $<ij>$ denotes nearest-neighbour pairs of sites. This model is in the same universality class as the three-dimensional Ising model. High-temperature expansions for this model are equivalent to the weak-coupling expansions in powers of $g$.

Consider, first, the mass gap

$$F(g) \equiv E_1(g) - E_0(g) ,$$

being the energy difference between the first excited and ground state levels, at zero magnetic field. For the reduced quantity

$$f(g) \equiv \frac{F(g)}{F(0)} , \quad F(0) = 2 ,$$

the series expansions in powers of $g$ are of type (2).

The coefficients in series (2) for a square lattice are [54]

$$a_1 = -2 , \quad a_2 = -1 , \quad a_3 = -\frac{3}{2} , \quad a_4 = -\frac{9}{4} , \quad a_5 = -\frac{11}{2} ,$$

$$a_6 = -10.253906 , \quad a_7 = -28.849609 , \quad a_8 = -57.418152 ,$$

$$a_9 = -175.053360 , \quad a_{10} = -365.267989 .$$

When $g$ approaches the critical value $g_c$, the mass gap tends to zero, as $F \sim (g_c - g)^\nu$, with the critical index

$$\nu \equiv - \lim_{g \to g_c - 0} (g_c - g) \frac{d}{dg} \ln f(g) .$$

Note that the coefficients $a_i$ in series (2), for the present case, are of the same sign, which is known to be a difficult situation for summation. The absolute values of these coefficients are quickly increasing, which manifests the series divergence.

For the self-similar factor approximant $f_2^*(g)$, we find

$$A_1 = 4.03553 , \quad A_2 = -3.03553 ;$$

$$n_1 = -2.4905 \times 10^{-3} , \quad n_2 = 0.65555 .$$

The critical coupling and critical index are

$$g_c = \frac{1}{|A_2|} = 0.32943 , \quad \nu = n_2 = 0.656 .$$

For the factor approximant $f_3^*(g)$, we have

$$A_1 = -3.03055 , \quad A_2 = 2.34432 + 1.17108i ; \quad A_3 = A_2^* ;$$

$$n_1 = 0.65826 , \quad n_2 = (1.57037 + 5.33043i) \times 10^{-3} , \quad n_3 = n_2^* ,$$

which gives the critical characteristics

$$g_c = \frac{1}{|A_1|} = 0.32997 , \quad \nu = n_1 = 0.658 .$$
Approximant $f_4^*(g)$ possesses

\[
A_1 = -3.03764, \quad A_2 = 2.80311, \quad A_3 = -0.85714 + 1.50878 i, \quad A_4 = A_3^*; \quad n_1 = 0.6514, \quad n_2 = -7.37667 \times 10^{-3}, \quad n_3 = (-7.68487 + 4.56213 i) \times 10^{-3}, \quad n_4 = n_3^*.
\]

This results in

\[
g_c = \frac{1}{|A_1|} = 0.32920, \quad \nu = n_1 = 0.651.
\]

Finally, for $f_5^*(g)$, we obtain

\[
A_1 = -3.04301, \quad A_2 = 2.85769, \quad A_3 = -2.08221, \quad A_4 = 0.05019 + 1.48535 i, \quad A_5 = A_4^*; \quad n_1 = 0.64147, \quad n_2 = -6.64215 \times 10^{-3}, \quad n_3 = 0.02454, \quad n_4 = (-1.27747 + 7.47775 i) \times 10^{-3}, \quad n_5 = n_4^*.
\]

The corresponding critical characteristics are

\[
g_c = \frac{1}{|A_1|} = 0.32862, \quad \nu = n_1 = 0.641.
\]

The found results can be compared with other calculations, summarized in Ref. [54], where the critical coupling is in the interval $0.328 \leq g_c \leq 0.329$, while the critical index lies in the range $0.629 \leq \nu \leq 0.646$. This is in good agreement with our estimates.

Let us also consider the susceptibility

\[
\chi(g) \equiv -\frac{1}{N} \frac{\partial^2 E_0}{\partial h^2} \bigg|_{h=0}, \quad (33)
\]

which behaves as $\chi(g) \sim (g_c - g)^{-\gamma}$ in the vicinity of the critical coupling $g_c$, with the critical index

\[
\gamma \equiv \lim_{g \to g_c} (g_c - g) \frac{d}{dg} \ln \chi(g). \quad (34)
\]

Expanding the susceptibility (33) in powers of $g$, one gets the series of type (2). For the triangular lattice the coefficients are

\[
a_1 = 6, \quad a_2 = 32.95, \quad a_3 = 166.5, \quad a_4 = 843.046875, \quad a_5 = 4218.416666, \quad a_6 = 20941.023004, \quad a_7 = 103361.512587, \quad a_8 = 507986.371687, \quad a_9 = 2488222.50870, \quad a_{10} = 12155136.2137.
\]

Again, all coefficients are of the same sign and fastly increasing in their values. Such a series is divergent for any finite $g$.

For the factor approximant $\chi_2^*(g)$, we have

\[
A_1 = 6.22771, \quad A_2 = -4.58786;
\]
\[ n_1 = -0.03523 , \quad n_2 = -1.35562 . \]

The critical coupling and index are

\[ g_c = \frac{1}{|A_2|} = 0.21797 , \quad \gamma = |n_2| = 1.356 . \]

In the next order, we get \( \chi^*_3(g) \), with the controllers

\[ A_1 = -4.81702 , \quad A_2 = 1.6542 + 2.20379 i ; \quad A_3 = A_2^* ; \]
\[ n_1 = -1.18552 , \quad n_2 = 0.22051 + 0.09988 i , \quad n_3 = n_2^* . \]

From here,

\[ g_c = \frac{1}{|A_1|} = 0.20760 , \quad \gamma = |n_1| = 1.185 . \]

For \( \chi^*_4(g) \), we find

\[ A_1 = -4.76379 , \quad A_2 = -0.89137 , \quad A_3 = 0.79806 + 1.56616 i , \quad A_4 = A_3^* ; \]
\[ n_1 = -1.25533 , \quad n_2 = 2.32460 , \quad n_3 = 1.06769 - 0.12380 i , \quad n_4 = n_3^* , \]

which yields

\[ g_c = \frac{1}{|A_1|} = 0.20992 , \quad \gamma = |n_1| = 1.255 . \]

And the approximant \( \chi^*_5(g) \) contains

\[ A_1 = -4.76723 , \quad A_2 = -2.66216 , \quad A_3 = -1.30481 , \]
\[ A_4 = 0.84329 + 1.55612 i , \quad A_5 = A_4^* ; \]
\[ n_1 = -1.24715 , \quad n_2 = -0.10924 , \quad n_3 = 1.54693 , \]
\[ n_4 = 1.02499 - 0.01717 i , \quad n_5 = n_4^* . \]

The corresponding critical characteristics are

\[ g_c = \frac{1}{|A_1|} = 0.20977 , \quad \gamma = |n_1| = 1.247 . \quad (35) \]

These results are in good agreement with numerical calculations, summarized in Ref. [54], according to which the critical coupling for the triangular lattice is in the interval \( 0.20972 \leq g_c \leq 0.20980 \) and the critical index is in the range \( 1.236 \leq \gamma \leq 1.250 \).
7 General discussion and possible variants

The self-similar factor approximants, as is illustrated above by a variety of examples, provide a powerful tool for defining effective sums of divergent power series. The latter in the majority of applications are asymptotic. However, the presented method is not limited to only asymptotic series. The sole thing that is required is that the perturbative form (2) be a finite power series. If all coefficients $a_n$ in Eq. (2) do not depend on the approximation order $k = 1, 2, \ldots$, then the difference

$$\Delta f_k(x) \equiv f_k(x) - f_{k-1}(x)$$

composes as asymptotic sequence $\{\Delta f_k(x)\}$, where $\Delta f_k(x) = a_k x^k$. But we also may consider the case [5], when an iterative procedure results in the approximations

$$f_k(x) = 1 + \sum_{n=1}^{k} a_{nk} x^n ,$$

with different coefficients $a_{nk} \neq a_{nk-1}$. In such a situation, the difference $\Delta f_k(x)$ does not constitute an asymptotic sequence $\{\Delta f_k(x)\}$. Nevertheless, the method of self-similar factor approximants is applicable to this case as well.

The mathematical structure of formula (5) resembles, in some sense, such geometric structures as fractals [55,56]. With regard to functions, one usually says that a function possesses fractal properties if it manifests a kind of scaling. In particular, it can be an asymptotic scaling [56,57], arising in a limit of the function variable. In that sense, formula (5) displays the property of asymptotic scaling

$$\lim_{x \to \infty} \frac{f_\ast^k(\lambda x)}{f_\ast^k(x)} = \lambda^{\nu_k} , \quad \nu_k = \sum_{i=1}^{k} n_i ,$$

(36)

where $\lambda$ is a scaling parameter. We also may notice another scaling property of the factor form (5). To this end, introducing functions

$$\varphi_i(x) \equiv (1 + A_i x)^{n_i} ,$$

(37)

we can present Eq. (5) as

$$f_\ast^k(x) = \prod_{i=1}^{k} \varphi_i(x) .$$

(38)

Then, the property of functional scaling

$$f_\ast^k(x) \to \left( \prod_{i=1}^{k} \lambda_i \right) f_\ast^k(x) , \quad \varphi_i(x) \to \lambda_i \varphi_i(x) ,$$

(39)

is valid.

It is worth emphasizing that the factor approximant (5) is principally different from the continued-function representation mentioned by Bender and Orszag [58]. According to their scheme, approximants $\beta_k(x)$ for a function $f(x)$ could be constructed in the following
way. Let us take an arbitrary function $\beta(c, x)$, where $c$ is a parameter. Then a $k$-order approximation to the sought function $f(x)$ is given by

$$
\beta_k(x) \equiv \beta(c_1, x\beta(c_2, \ldots, x\beta(c_k, x)) \ldots).
$$

However, such a manner of acting has several essential drawbacks. First of all, this scheme is too much ambiguous, since there exists an infinity of functions that could be accepted as $\beta(c, x)$. Contrary to this, the form of the factor approximants (5) is not arbitrary postulated but is derived from the given perturbative series by means of the self-similar approximation theory. Second, Eq. (5) does not have the form of a continued function $\beta_k(x)$. Really, even if we suppose that $\beta(c, x) = (1 + cx)^n$, then

$$
\beta_k(x) = (1 + c_1(1 + c_2(1 + \ldots + c_k x)^n)^n \ldots)^n,
$$

which has nothing to do with formula (5). The latter is not a function of the same function but a product (38) of different functions (37). This product is not only different from the above $\beta_k(x)$ but also is more general. Last but not least, the sequence $\{\beta_k(x)\}$ of the continued-function representation is, generally, speaking, divergent. This can be easily demonstrated for the above mentioned case of $\beta(c, x)$ resulting in the power-law representation $\beta_k(x)$. The latter, at large $x$, are of the type

$$
\beta_k(x) \simeq \left( \prod_{i=1}^{k} c_i \right)^n x^{nk}, \quad (x \to \infty),
$$

which shows that the subsequent approximations $\beta_k(x)$ differ significantly from each other. Thus, $\beta_1(x) \sim x^n$, $\beta_2(x) \sim x^{2n}$, and so on, each $\beta_k(x) \sim x^{nk}$ having principally different dependence on $x$. It is evident that such a sequence cannot converge.

The self-similar factor approximant (5) of order $k$ requires the knowledge of $2k$ nontrivial (except the first term equal to one) perturbative terms of series (2). Thus, for constructing $f^*_k(x)$, we need an even number of terms. A natural question would be: how could we use an odd number of such terms? There are two straightforward ways of doing this, which can be called additive and multiplicative.

The additive way is as follows. The perturbative form (2) can always be rewritten as

$$
f_k(x) = 1 + a_1 x \prod_{i=1}^{k-1} (1 + c_i x).
$$

By applying the self-similar renormalization (4) to the factors in the product of Eq. (40), we come to the approximant

$$
f^*_k(x) \equiv 1 + a_1 x \prod_{i=1}^{k} (1 + B_i x)^{m_i},
$$

containing the additive unitary term. The approximant (41) requires the availability of an odd number of terms in series (2). If the large-$x$ behaviour (6) is known, it can be employed for approximant (41) as the crossover condition

$$
a_1 \prod_{i=1}^{k} B_i^{m_i} = f_{\infty}, \quad \sum_{i=1}^{k} m_i + 1 = \beta.
$$

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We have analysed the usage of approximants (41) to the examples considered above. Sometimes these approximants \( f_{k+}^*(x) \) are not bad, being of comparable accuracy with \( f_k^*(x) \) or even better. This, e.g., is the case for defining the critical index (23) from the Wilson \( \varepsilon \)-expansion, for which we find \( \nu_{2+}^* = 0.628 \) for \( d = 3 \) and \( \nu_{2+}^* = 0.914 \) for \( d = 2 \). However, for the majority of cases, the accuracy of \( f_{k+}^*(x) \) is essentially worse than that of \( f_k^*(x) \). On occasion, this accuracy can be improved by considering the inverse function \( f^{-1}_k(x) \), instead of \( f_k(x) \). This, for instance, gives good results for the structure factors of Section 4. The crossover condition (42), if available, also may certainly cure the behaviour of \( f_{k+}^*(x) \). For example, this yields good approximations for the ground-state energy of the anharmonic oscillator of Section 5.2. But, in general, the approximants \( f_{k+}^*(x) \) are inferior to \( f_k^*(x) \).

As an alternative, we could try the multiplicative way of extracting information from odd numbers of perturbative terms. For this purpose, we may present Eq. (2) as

\[
f_k(x) = (1 + Ax) \prod_{i=1}^{k-1} (1 + b_i x) .
\]

(43)

Accomplishing the self-similar renormalization of the right-hand-side factors in Eq. (43), except the first factor \( 1 + Ax \), we define the self-similar approximant

\[
f_{k+1/2}^*(x) \equiv (1 + Ax) \prod_{i=1}^{k} (1 + A_i x)^{n_i}
\]

(44)

In the starting trivial case,

\[
f_{1/2}^* = f_0^* = 1 + a_1 x.
\]

(45)

The following nontrivial approximants are

\[
f_{3/2}^* = (1 + Ax)(1 + A_1 x)^{n_1} ,
\]

\[
f_{5/2}^* = (1 + Ax)(1 + A_1 x)^{n_1}(1 + A_2 x)^{n_2} ,
\]

(46)

and so on. Each \( f_{k+1/2}^*(x) \) requires for its definition \( 2k + 1 \) terms of series (2).

The main difference in employing approximants \( f_{k+1/2}^*(x) \), as compared to \( f_k^*(x) \), is in the following. When determining the related control parameters by means of the order-through-accuracy matching, for the former approximants we may get nonunique solutions, while for the latter the solutions are unique. Different solutions for the control parameters yield several possible variants of \( f_{k+1/2}^*(x) \). As we have checked, all of these several variants, generally, have reasonable accuracy, being close to each other. However, dealing with several approximants of the same order complicates the calculational procedure. In this way, the self-similar factor approximants \( f_k^*(x) \) look to be more preferable, compared to both \( f_{k+}^*(x) \) and \( f_{k+1/2}^*(x) \).

In all cases, we have considered, the accuracy of \( f_k^*(x) \) improves with the increasing order \( k \). This implies the existence of the so-called numerical convergence for the sequence \( \{ f_k^*(x) \} \). It is, of course, impossible to guarantee that the latter sequence always converges to the exact function \( f(x) \). This kind of uncertainty, is actually, common for practically any sequence of numerical approximants. To render the procedure uniquely defined requires imposing on the calculational algorithm additional conditions making the latter single-valued [59].
More information of the self-similar factor approximants \( f_k^*(x) \) can be extracted by invoking the probabilistic analysis [60]. To this end, we introduce the multiplier

\[
M_k(x) \equiv \frac{\delta f_k^*(x)}{\delta f_0^*(x)}.
\]

Taking account Eq. (48), we get

\[
M_k(x) = \frac{1}{a_1} \frac{d}{dx} f_k^*(x).
\]

The probability of \( f_k^*(x) \) is defined as

\[
p_k(x) = \frac{|M_k^{-1}(x)|}{\sum_{i=1}^{N} |M_i^{-1}(x)|},
\]

where \( N \) is the total number of available approximants \( f_k^*(x) \), with \( k = 1, 2, \ldots, N \). Having defined these quantities, we now may find the most probable approximants, corresponding to the maximal probability (49). Also, we can calculate a weighted approximant

\[
\overline{f}_k(x) \equiv \sum_{j=1}^{k} p_j(x) f_j^*(x).
\]

This averaging is especially useful when the subsequent approximants \( f_j^*(x) \) and \( f_{j+1}^*(x) \) display large variations. The latter become smoothed by the averaging procedure (50). For example, for the critical index of the correlation length, discussed in section 6.2, defined by this averaging procedure, we find \( \nu = 0.630 \).

Additional possibilities arise, when, for the same problem, there exist expansions for several different functions. Say, for simplicity, there are two functions \( f(x) \) and \( g(x) \), for which the expansions \( f_k(x) \) and \( g_k(x) \) are available at \( x \to 0 \). Then, in addition to the factor approximants \( f_k^*(x) \) and \( g_k^*(x) \), we may, considering some combinations \( F_k(f_k(x), g_k(x)) \) of the series \( f_k(x) \) and \( g_k(x) \), construct the approximants \( F_k^*(x) \). For instance, we can consider the sums \( f_k(x) + g_k(x) \), products \( f_k(x)g_k(x) \), or the ratios \( f_k(x)/g_k(x) \), or other combinations of the given series. A more detailed consideration of such possibilities will be done in future.

Finally, we would like to emphasize that the accuracy of the self-similar factor approximants (5), for the examples considered in the present paper, is always essentially better than that of the best Padé approximants, constructed with the same number of perturbative terms of series (2). This fact is easy to understand, since the factor approximants (5) have a much more general form as compared to Padé approximants, including the latter just as a narrow particular class. As is explained in Section 2, the self-similar factor approximants can represent rational, irrational, and transcendental functions.

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Figure Captions

Fig. 1. Difference of the self-similar factor approximant $f_4^*(x)$ and function (13).

Fig. 2. Specific heat (14) (solid line), the approximant $C^*_3(x)$ (dashed line), and the best Padé approximants $P_{4/2}(x)$ (dashed-dotted line) and $P_{5/1}(x)$ (dotted line) versus $1/x$.

Fig. 3. Percentage errors of the approximant $f_3^*(x)$ (solid line) and the two-point Padé approximant $P_{3/2}(x)$ (dashed line), as compared to the exact function (15).

Fig. 4. Structure factor of branched polymer $S(x)$ (solid line) and its factor approximant $S_2^*(x)$ (dashed line).

Fig. 5. Luminiscent intensity $f(z)$ (solid line), approximant $f_3^*(z)$ (dashed line) and the best Padé approximant $P_{3/3}(x)$ (dashed-dotted line).
Fig. 2

Approximants

1/x
Fig. 5

Approximants

Z