NOTE ON THE $q$-LOGARITHMIC SOBOLEV AND $p$-TALAGRAND INEQUALITIES ON CARNOT GROUPS

E. BOU DAGHER

Abstract. In the setting of Carnot groups, we prove the $q$–Logarithmic Sobolev inequality for probability measures as a function of the Carnot-Carathéodory distance. As an application, we use the Hamilton-Jacobi equation in the setting of Carnot groups to prove the $p$–Talagrand inequality and hypercontractivity.

1. Introduction

In [18], L. Gross obtained the Logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log \left( \frac{f^2}{\int_{\mathbb{R}^n} f^2 d\mu} \right) d\mu \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu,$$

where $\nabla$ is the standard gradient on $\mathbb{R}^n$ and $d\mu = \frac{e^{-\frac{|x|^2}{4}}}{Z} d\lambda$ is the Gaussian measure. He proved that if $\mathcal{L}$ is the non-positive self-adjoint operator on $L^2(\mu)$ such that $(-\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$, then (1.1) is equivalent to the fact that the semigroup $P_t = e^{t\mathcal{L}}$ generated by $\mathcal{L}$ is hypercontractive: i.e. for $q(t) \leq 1 + (q-1)e^{2t}$ with $q > 1$, we have $\| P_tf \|_q(t) \leq \| f \|_q$ for all $f \in L^q(\mu)$.

In [2], D. Bakry and M. Emery extended the Logarithmic Sobolev inequality for a larger class of probability measures defined on manifolds under an important Curvature-Dimension condition. More generally, if $(\Omega, F, \mu)$ is a probability space, and $\mathcal{L}$ is a non-positive self-adjoint operator acting on $L^2(\mu)$, we say that the measure $\mu$ satisfies a Logarithmic Sobolev inequality if there is a constant $c$ such that, for $f \in D(\mathcal{L})$,

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu \leq c \int f (-\mathcal{L}f) d\mu.$$

F. Otto and C. Villani showed in their celebrated paper [29] that in the setting of manifolds under D. Bakry and M. Emery’s Curvature-Dimension condition, the Logarithmic Sobolev inequality implies the Talagrand transportation cost inequality. Their proof relied on PDE’s methods and

Key words and phrases. Logarithmic Sobolev inequality, Talagrand inequality, hypercontractivity, Hamilton-Jacobi equation, Carnot-Carathéodory distance, Carnot groups.
Otto calculus in Wasserstein space [28]. The Talagrand transportation cost inequality was first introduced in [31] by M. Talagrand:

\begin{equation}
T_w(\mu, \nu) \leq 2 \int \log(f) \, d\mu,
\end{equation}

where \( \mu \) is a measure on \( \mathbb{R}^N \) absolutely continuous with respect to the Gaussian measure \( \nu \), \( f = \frac{d\mu}{d\nu} \) is the relative density, and \( w(x, y) = \sum_{i=1}^{N} (x_i - y_i)^2 \) is the cost of moving a unit mass from \( x \) to \( y \). \( T_w(\mu, \nu) \) is the transportation cost measuring how much effort is required to transport a mass distributed according to \( \mu \) to a mass distributed according to \( \nu \), i.e.

\[
T_w(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^N \times \mathbb{R}^N} w(x, y) \, d\pi(x, y),
\]

where \( \Pi(\mu, \nu) \) is the set of probability measures on \( \mathbb{R}^N \times \mathbb{R}^N \) with \( \mu \) the first marginal and \( \nu \) the second marginal. M. Talagrand showed how the Talagrand transportation cost inequality (1.2) implies the concentration of measure phenomenon, and his approach was motivated by the work of K. Marton [26, 27]. Later on, many works [1, 5, 6, 15, 16, 23, 24, 29] studied links between the Logarithmic Sobolev inequality, the Talagrand inequality, and the concentration of measure phenomenon, which is one of the main tools in probability theory, statistical mechanics, quantum information theory, stochastic dynamics, etc.

In this note, in the setting of Carnot groups, we are first interested in proving the \( q \)-Logarithmic Sobolev inequality (see [7, 8]) for \( q \in (1, \infty) \),

\begin{equation}
\int f^q \log \frac{f^q}{\int f^q \, d\mu} \, d\mu \leq c \int |\nabla f|^q \, d\mu,
\end{equation}

for measures of the form \( d\mu = \frac{e^{-U(d)}}{Z} \, d\lambda \). \( U(d) \) is a function having a suitable growth at infinity, \( \lambda \) is a natural measure like the Lebesgue measure for instance, and \( d \) is a metric related to the sub-gradient \( \nabla = (X_1, \ldots, X_n) \), where \( X_i \)'s are the generators of the group's lie algebra. Since the Laplacian is of Hörmander type and has some degeneracy, D. Bakry and M. Emery’s Curvature-Dimension condition in [2] will no longer hold true. In [19], W. Hebisch and B. Zegarliński developed a method of studying coercive inequalities on general metric spaces that does not require a bound on the curvature of space. Working on a general metric space equipped with non-commuting vector fields \( \{X_1, \ldots, X_n\} \), their method is based on U-bounds, where U stands for the potential, which are inequalities of the form:

\[
\int f^q \eta(d) \, d\mu \leq C \int |\nabla f|^q \, d\mu + D \int f^q \, d\mu,
\]

where \( \eta(d) \) is a function having a suitable growth at infinity. Those methods were applied, for example, in the works [10, 12, 11] in the setting of Carnot groups. In [19], W. Hebisch and B. Zegarliński proved the \( q \)-Logarithmic Sobolev inequality for the measure \( d\mu = \frac{e^{-\alpha d}}{Z} \, d\lambda \), where \( p \) is the finite index conjugate of \( q \) and \( \alpha > 0 \). In this note, we enlarge the class of measures that satisfy the \( q \)-logarithmic Sobolev inequality to include measures of the form \( d\mu = \frac{e^{-U(d)}}{Z} \, d\lambda \), under some growth conditions for \( U(d) \), and get examples such as \( U(d) = (d+1)^p \log(d+1) \) and \( U(d) = \sinh(d) \).

We would like to apply the \( q \)-Logarithmic Sobolev inequality to get hypercontractivity and to
obtain the $p$–Talagrand inequality on $(X, d, \mu)$ with a constant $K$:

\[(1.4) \quad W_p(\mu, \nu)^p \leq K \text{Ent}_\mu \left( \frac{d\nu}{d\mu} \right), \]

with $p$ finite index conjugate of $q$. The $p$–Wasserstein distance between two probability measures on $X$ is defined as $W_p(\mu, \nu)^p = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y)^p d\pi(x, y)$, where $\Pi(\mu, \nu)$ is the set of probability measures on $X \times X$ with $\mu$ the first marginal and $\nu$ the second marginal.

$\text{Ent}_\mu \left( \frac{d\nu}{d\mu} \right) = \int \frac{d\nu}{d\mu} \log \left( \frac{d\nu}{d\mu} \right) d\mu$

is the entropy functional such that $\nu$ is a probability measure absolutely continuous with respect to $\mu$. We note that for $p = 2$, (1.2) is a special case of (1.4).

The $p$–Talagrand inequality was introduced by Z.M. Balogh et al. in [4] who proved that it was implied by the $q$–Logarithmic Sobolev inequality for $p$ the finite index conjugate of $q$; their work generalises the result by J. Lott and C. Villani [25] who proved the implication for the quadratic case $p = q = 2$. J. Lott and C. Villani [25] used the Hamilton-Jacobi infimum convolution operator under the assumption where the space $(X, d, \mu)$ supports local Poincaré inequality and the measure $\mu$ is a doubling measure i.e. the measure of any open ball is positive and finite and there exists a constant $c_d \geq 1$ such that for all $x \in X$ and $r > 0$,

\[(1.5) \quad \mu(B(x, 2r)) \leq c_d \mu(B(x, r)). \]

This idea of using the Hamilton-Jacobi infimum convolution operator was first used by S. Bobkov et al. in [5] who explored the equivalence between the Logarithmic Sobolev inequality with a constant $\rho$ in $\mathbb{R}^n$ and hypercontractivity of the quadratic Hamilton-Jacobi semigroup $Q_t$ i.e.

\[ ||e^{Q_t f}||_{a+\rho t} \leq ||e^f||_a, \]

for every bounded measurable function $f$ on $\mathbb{R}^n$, $t \geq 0$, $a \in \mathbb{R}$, and $||.||_p$ the $L^p$–norm with respect to $\mu$.

In the setting of the Heisenberg group $\mathbb{H}$, Z.M. Balogh et al. in [4] used W. Hebisch and B. Zegarliński’s [19] $q$–Logarithmic Sobolev inequality for the measure $d\mu = \frac{e^{-\alpha d^n}}{Z} d\lambda$, where $d$ is the Carnot-Carathéodory distance, to obtain the $p$–Talagrand inequality. To prove the above implication, Z.M. Balogh et al. observed that by Pansu’s differentiability theorem [17], for Lipschitz continuous functions $f : \mathbb{H} \to \mathbb{R}$, the norm of the sub-Riemannian gradient $|\nabla f(x)| = \left( \sum_{i=1}^N |X_i f|^2 \right)^{\frac{1}{2}}$ used in [19] coincides with the metric subgradient in [4] for $\mu_p$ almost every $x$ for which $|\nabla f(x)| > 0$.

In section 2, we introduce the Carnot group and the Hamilton-Jacobi equation in that setting ([14, 3]). In section 3, we extend W. Hebisch and B. Zegarliński’s results in [19] by enlarging the class of measures that satisfy the $q$–logarithmic Sobolev inequality to include measures of the
form \( d\mu = \frac{e^{-U(d)}}{Z} d\lambda \), under some growth conditions for \( U(d) \), and get examples such as \( U(d) = (d + 1)^p \log(d + 1) \) and \( U(d) = \sinh(d) \). As an application, we adapt Z.M. Balogh et al.’s Theorem 4.1 \([4]\) to get the \( p \)-Talagrand inequality (section 4), and we adapt S. Bobkov et al.’s Theorem 2.1 in \([5]\) to get hypercontractivity (section 5). However, instead, we use the Hamilton-Jacobi equation in the setting of Carnot groups by F. Dragoni \([14]\). The advantage of doing so is that the restriction (1.5) to have \( \mu \) a doubling measure is no longer required!

2. The Carnot Group and the Hamilton-Jacobi Equation

Carnot groups are geodesic metric spaces that appear in many mathematical contexts like harmonic analysis in the study of hypoelliptic differential operators (\([13, 30]\)) and in geometric measure theory (see extensive reference list in the survey paper \([22]\)). The following series of definitions are from \([9]\):

**Definition 1.** We say that a Lie group on \( \mathbb{R}^N \), \( G = (\mathbb{R}^N, \circ) \) is a (homogeneous) Carnot group if the following properties hold:

(C.1) \( \mathbb{R}^N \) can be split as \( \mathbb{R}^N = \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_r} \), and the dilation \( \delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N \)

\[
\delta_\lambda(x) = \delta_\lambda(x^{(1)}, \ldots, x^{(r)}) = (\lambda x^{(1)}, \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)}), \quad x^{(i)} \in \mathbb{R}^{N_i},
\]

is an automorphism of the group \( G \) for every \( \lambda > 0 \). Then \( (\mathbb{R}^N, \circ, \delta_\lambda) \) is a homogeneous Lie group on \( \mathbb{R}^N \). Moreover, the following condition holds:

(C.2) If \( N_1 \) is as above, let \( Z_1, \ldots, Z_{N_1} \) be the left invariant vector fields on \( G \) such that \( Z_j(0) = \partial / \partial x_j |_0 \) for \( j = 1, \ldots, N_1 \). Then

\[
\text{rank}(\text{Lie}\{Z_1, \ldots, Z_{N_1}\}(x)) = N \quad \forall x \in \mathbb{R}^N.
\]

If (C.1) and (C.2) are satisfied, we shall say that the triple \( G = (\mathbb{R}^N, \circ, \delta_\lambda) \) is a homogeneous Carnot group. We also say that \( G \) has step \( r \) and \( N_1 \) generators. The vector fields \( Z_1, \ldots, Z_{N_1} \) will be called the (Jacobian) generators of \( G \).

**Definition 2.** The vector valued operator \( \nabla := (Z_1, Z_2, \ldots, Z_{N_1}) \) is called the sub-gradient on \( G \), and \( \triangle = \sum_{i=1}^{N_1} Z_i^2 \) is called the sub-Laplacian on \( G \).

**Definition 3.** We say that \( \gamma \) is horizontal if there exist measurable functions \( a_1, \ldots, a_{N_1} : [0, 1] \to \mathbb{R} \) such that

\[
\gamma'(t) = \sum_{i=1}^{N_1} a_i(t) Z_i(\gamma(t))
\]
for almost all $t \in [0, 1]$ i.e. $\gamma'(t) \in \text{Span} \{Z_1(\gamma(t)), \ldots, Z_{N_1}(\gamma(t))\}$ almost everywhere. For such a horizontal curve $\gamma$, we define the length of $\gamma$ to be

$$|\gamma| = \int_0^1 \left( \sum_{i=1}^{N_1} a_i^2(t) \right)^{\frac{1}{2}} dt.$$ 

**Definition 4.** The Carnot-Carathéodory distance or the control distance between two points $x$ and $y$ is defined by

$$d(x, y) = \inf \{ t \mid \gamma : [0, t] \rightarrow G, \gamma(0) = x, \gamma(t) = y \mid |\gamma'(s)| \leq 1 \text{ for all } s \in [0, t] \}.$$

where $\gamma : [0, 1] \rightarrow G$ is an absolutely continuous horizontal path on $[0, 1]$, i.e. if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that when a finite number of pairwise disjoint subintervals $[x_k, y_k]$ of $[0, 1]$ satisfy $\sum_k |y_k - x_k| < \delta$, then $\sum_k d(\gamma(y_k), \gamma(x_k)) < \varepsilon$.

It can be shown, proof found in [20], that $d$ is associated to the sub-gradient through:

$$|\nabla f(x)| = \limsup_{d(x, y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We will be using F. Dragoni’s Theorem 4 of [14] to get hypercontractivity and $p$–Talagrand inequality from the $q$–Logarithmic sobolev inequality.

**Theorem 5** (F. Dragoni [14]). Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous, non-decreasing, and convex function with $\phi(0) = 0$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a lower semicontinuous function such that there exists $C > 0$:

$$f(x) \geq -C(1 + d(x)).$$

Then a viscosity solution for the Hamilton-Jacobi problem in the Carnot Group $G$

$$
\begin{cases}
  u_t(x, t) + \phi(|\nabla u(x, t)|) = 0 & (x, t) \in G \times (0, \infty) \\
  u(x, 0) = f(x) & x \in G
\end{cases}
$$

is given by the Hopf-Lax formula

$$Q_t f(x) := \inf_{y \in G} \left\{ t \phi^* \left( \frac{d(x, y)}{t} \right) + f(y) \right\},$$

where $d(x, y)$ is the Carnot-Carathéodory distance as in Definition 4, $\nabla$ is the sub-gradient as in Definition 2, and $\phi^*$ is the Legendre transform of $\phi$.

In this note, we will be considering $f(x)$ to be a continuous and bounded function and we will be choosing the function $\phi(s) = s^q$, where $1 < q \leq 2$.

Thus, the viscosity solution for the Hamilton-Jacobi problem in the Carnot group $G$

$$
\begin{cases}
  u_t(x, t) + \frac{|\nabla u(x, t)|^q}{q} = 0 & (x, t) \in G \times (0, \infty) \\
  u(x, 0) = f(x) & x \in G
\end{cases}
$$

(2.1)

is given by the Hopf-Lax formula
$$Q_tf(x) = \inf_{y \in \mathbb{G}} \left\{ \frac{d(x,y)^p}{t^{p-1}} + f(y) \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3. $q-$Logarithmic Sobolev Inequality

Given the probability measure $d\mu = \frac{e^{-U}}{Z}d\lambda$, where $U$ is an increasing unbounded function and $Z$ is the normalization constant. To obtain the $q-$Logarithmic Sobolev inequality, we will need the following theorems stated in the beginning of this section by W. Hebisch and B. Zegarliński in [19].

Let $\lambda$ be a measure satisfying the $q$-Poincaré inequality for every ball $B_R = \{x : N(x) < R\}$, i.e. there exists a constant $C_R \in (0, \infty)$ such that

$$\frac{1}{|B_R|} \int_{B_R} |f - \frac{1}{|B_R|} \int_{B_R} f|^q d\lambda \leq C_R \frac{1}{|B_R|} \int_{B_R} |\nabla f|^q d\lambda,$$

where $1 \leq q < \infty$.

Note that we have this Poincaré inequality on balls in the setting of Nilpotent lie groups thanks to J. Jerison’s celebrated paper [21].

**Theorem 6** (W. Hebisch, B. Zegarliński [19]). Let $\mu$ be a probability measure on $\mathbb{R}^n$ which is absolutely continuous with respect to the measure $\lambda$ and such that

$$\int f^q \eta d\mu \leq C \int |\nabla f|^q d\mu + D \int f^q d\mu$$

with some non-negative function $\eta$ and some constants $C, D \in (0, \infty)$ independent of a function $f$.

If for any $L \in (0, \infty)$ there is a constant $A_L$ such that $\frac{1}{A_L} \leq \frac{d\mu}{d\lambda} \leq A_L$ on the set $\{\eta < L\}$ and, for some $R \in (0, \infty)$ (depending on $L$), we have $\{\eta < L\} \subset B_R$, then $\mu$ satisfies the $q$-Poincaré inequality

$$\mu |f - \mu f|^q \leq c\mu |\nabla f|^q$$

with some constant $c \in (0, \infty)$ independent of $f$.

**Theorem 7** (W. Hebisch, B. Zegarliński [19]). Suppose the following Sobolev inequality is satisfied

$$\left( \int |f|^{q+\epsilon} d\lambda \right)^{\frac{q}{q+\epsilon}} \leq a \int |\nabla f|^q d\lambda + b \int |f|^q d\lambda,$$

and the following bound is true

$$\mu (|f|^q [\nabla U]^q + U]) \leq C' \mu |\nabla f|^q + D' \mu |f|^q.$$

Then, the following inequality is true

$$\mu \left( f^q \log \frac{f^q}{\mu f^q} \right) \leq C \mu |\nabla f|^q + D \mu |f|^q.$$
Moreover, if \( q \in (1, 2] \) and the following Poincaré inequality holds \( \mu |f - \mu f|^q \leq \frac{1}{\mu |f|^q} |\nabla f|^q \), then one has

\[
\mu \left( f^q \log \frac{f^q}{\mu f^q} \right) \leq c \mu |\nabla f|^q
\]

with some constant \( c \in (0, \infty) \) independent of \( f \).

Under two assumptions on \( d \), outside the unit ball \( B \equiv \{ d(x) < 1 \} \),

\[
\Delta d \leq K + \beta \varepsilon d^{p-1},
\]

and

\[
\frac{1}{\sigma} \leq |\nabla d| \leq 1,
\]

where \( \varepsilon \in \left( 0, \frac{1}{\sigma} \right) \), \( \beta \in (0, \infty) \), and \( \sigma \in [1, \infty) \), W. Hebisch and B. Zegarliński proved that the measure

\[
d\mu_U = e^{-U(d)} Z d\lambda
\]

where \( p > 1 \) satisfies both Poincaré and Logarithmic Sobolev inequalities. To get the Poincaré inequality from Theorem 6, W. Hebisch and B. Zegarliński proved a U-bound inequality (3.1) with \( \eta = d^p \) (Theorem 2.1 of [19]). However, the Logarithmic Sobolev inequality requires an additional U-bound inequality with \( U = d^p \) (Theorem 2.4 of [19]). The additional U-bound inequality is obtained from a more general inequality of the form:

\[
\int d^{q(p-1)} f^q d\mu_p \leq C_q \int |\nabla f|^q d\mu_p + D_q \int |f|^q d\mu_p
\]

which for a given \( p \in (1, \infty) \) is valid for all \( q \in [1, \infty) \) (Theorem 2.3 of [19]). In their proof of the Logarithmic Sobolev inequality, since \( U = d^p \), and \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[
|\nabla U| = |pd^{p-1} \nabla d| \leq pd^{p-1} |\nabla d| \leq pd^{p-1},
\]

which implies that

\[
|\nabla U|^q + U \leq p^q (d^{(p-1)q} + d^p) = p^q d^{pq} + d^p.
\]

Therefore, the second U-bound allows condition (3.2) to be satisfied in Theorem 7. So, they obtain the Logarithmic Sobolev inequality for \( p \) the finite conjugate of \( q \).

In the following, we present a generalised version of Theorems 2.1, 2.3, and 2.4 of [19]: In place of the function \( U(d) = d^p \), we let \( U : [0, \infty) \to [0, \infty) \) be a twice differentiable and increasing function; and let \( d\mu_U = \frac{e^{-U(d)} d\lambda}{Z} \) be a probability measure defined in terms of the function \( U(d) \), where \( Z \) is the normalization constant.

**Theorem 8.** Assume that outside the open unit ball \( B = \{ d(x) < 1 \} \), the metric \( d \) satisfies the following: \( |\nabla d| \) is bounded, say \( |\nabla d| \leq 1 \), and there exist finite positive constants \( K \) and \( c_0 \) such
Using integration by parts and condition (3.4),

(3.4) \[ \Delta d \leq K + U'(d) \left( |\nabla d|^2 - c_0 \right). \]

(i) If \( U'' \leq \beta U' \) for some positive constant \( \beta \), outside \( B \), then for any \( q \in (1, \infty) \), there exist constants \( C_q, D_q \), independent of \( f \), such that

\[ \int |f|^q |U'(d)|^q d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U. \]

(ii) If, in addition, \( U \leq \gamma U'' \) for some positive constant \( \gamma \) and some \( q > 1 \), outside \( B \), then

\[ \int |f|^q U'(d) d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U. \]

In the proof, we adapt, with simplification, the general lines of proof in [6].

Proof. Using integration by parts and condition (3.4),

\[ \int (\nabla d) \cdot \nabla (f e^{-U(d)}) d\lambda = -\int \Delta d (f e^{-U(d)}) d\lambda \]

(3.5) \[ \geq -K \int f e^{-U(d)} d\lambda - \int U'(d) |\nabla d|^2 f e^{-U(d)} d\lambda + c_0 \int U'(d) f e^{-U(d)} d\lambda. \]

Computing

\[ \nabla (f e^{-U(d)}) = \nabla f e^{-U(d)} - U'(d) e^{-U(d)} \nabla d, \]

and taking the dot product with \( \nabla d \),

\[ (\nabla d) \cdot \nabla (f e^{-U(d)}) = (\nabla f) \cdot (\nabla d) e^{-U(d)} - U'(d) f |\nabla d|^2 e^{-U(d)}, \]

and replacing in (3.5) to get

\[ c_0 \int U'(d) f e^{-U(d)} d\lambda \leq K \int f e^{-U(d)} d\lambda + \int |\nabla f||\nabla d|e^{-U(d)} d\lambda. \]

Using the fact that \( |\nabla d| \leq 1 \) (\( |\nabla d| \) bounded is enough), we get

(3.6) \[ \int U'(d) f d\mu_U \leq C \int f d\mu_U + D \int |\nabla f| d\mu_U. \]

To prove (i), we replace \( f \) in (3.6) by \( f^q \) to get

\[ \int f^q (U'(d))^q d\mu_U = \int \left[ f^q U'(d)^q - 1 \right] U'(d) d\mu_U \]

\[ \leq C \int \left[ f^q U'(d)^q - 1 \right] d\mu_U + D \int |\nabla f|^q U'(d)^q d\mu_U \]

\[ = C \int f f^{q-1} U'(d)^{q-1} d\mu_U + D \int q f^{q-1} |\nabla f| U'(d)^{q-1} d\mu_U. \]
Using Young’s inequality with $\alpha$, i.e. $ab \leq \alpha a^p + \frac{b^q}{q (\alpha p)^{q/p}}$ where $p$ and $q$ are conjugate finite indices in each of the three terms on the right-hand side of (3.7), in addition to using the conditions $|\nabla d| \leq 1$ and $U''' < \beta U'$ in the last term, we get

$$
\int f^q U'(d)^q d\mu_U \leq \frac{C}{q (\alpha p)^{q/p}} \int f^q d\mu_U + C \alpha \int f^q U'(d)^q d\mu_U + \frac{D}{\alpha p} \int |\nabla f|^q d\mu_U \\
+ Dq\alpha \int f^q U'(d)^q d\mu_U + \frac{D\beta (q - 1)}{q (\alpha p)^{q/p}} \int f^q d\mu_U + (q - 1) \alpha \int f^q U'(d)^q d\mu_U.
$$

Choosing $\alpha$ small enough so that $C\alpha + Dq\alpha + D (q - 1) \alpha \leq 1$, we get

$$
\int |f|^q U'(d)^q d\mu_U \leq C_q \int |\nabla f|^q d\mu_U + D_q \int |f|^q d\mu_U.
$$

\[\square\]

**Corollary 9.** If the positive twice differentiable and increasing function $U(d)$ satisfies the two inequalities $U''' < \beta U'$ and $U \leq \gamma U^q$, where $\beta$ and $\gamma$ are finite positive constants, then the measure

$$
d\mu_U = \frac{e^{-U(d)} d\lambda}{Z}
$$

satisfies a $q$-Poincaré and a $q$-Logarithmic Sobolev inequality.

**Proof.** To get the Logarithmic Sobolev inequality, we verify inequality (3.2) from Theorem 7

$$
|\nabla U (d)|^q + U (d) = |U' (d) \nabla d|^q + U (d) \leq (1 + \gamma) U'^q (d)
$$

Since $U''' (d) < \beta U' (d)$ by assumption, we can use (i) of Theorem 8, and the result follows. \[\square\]

**Remark 10.** In Corollary 9, it is enough to assume that the inequalities

$$
U''' < \beta U' \text{ and } U \leq \gamma U^q
$$

hold almost everywhere for $\beta$ and $\gamma$ finite positive constants.

**Remark 11.** To get Theorem 2.3 of [19], take $U (d) = d^p$, then

$$
U' (d) = pd^{p - 1} \text{ and } U'' (d) = p(p - 1)d^{p - 2}.
$$

Outside the open unit ball $B = \{d (x) < 1\}$,

$$
\frac{U'' (d)}{U' (d)} = \frac{p - 1}{d} \leq p - 1.
$$

Therefore, (i) from Theorem 8 holds, and the statement of Theorem 2.3 of [19] is recovered.
Suppose $q$ is the finite conjugate index of $p$, then,

$$\frac{U(d)}{U'^q(d)} = \frac{d^p}{p^q d^{(p-1)q}} = \frac{1}{p^q}.$$ 

Therefore, (ii) of Theorem 8 holds, and the statement of Theorem 2.4 of [19] is recovered.

**Example 12.** The $q$-Poincaré and a $q$-Logarithmic Sobolev inequality are satisfied for the measure

$$d\mu_U = \frac{e^{-(d+1)\log(d+1)}}{Z} d\lambda$$

for $q \geq \beta$, where $\beta$ is the finite index conjugate to $p$.

**Proof.** Given $U(d) = (d+1)^p \log(d+1)$, then,

$$U'(d) = (d+1)^{p-1} (p \log(d+1) + 1),$$

and

$$U''(d) = (d+1)^{p-2} (p (p-1) \log(d+1) + 2p - 1).$$

So, outside the open unit ball $B = \{d(x) < 1\}$,

$$\frac{U''(d)}{U'(d)} = \frac{(p (p-1) \log(d+1) + 2p - 1)}{d+1} \leq p (p-1) + \frac{2p - 1}{d+1} \leq p^2 - \frac{1}{2}.$$ 

In addition,

$$\frac{U(d)}{U'^q(d)} \leq \frac{(d+1)^p \log(d+1)}{(d+1)^{(p-1)q} (p \log(d+1) + 1)^q} \leq \frac{(d+1)^p \log(d+1)}{(d+1)^{(p-1)\beta} (p \log(d+1) + 1)^q} \leq 1.$$ 

So, by Corollary 9, the $q$-Poincaré and a $q$-Logarithmic Sobolev inequality are satisfied for the measure $d\mu_U = \frac{e^{-(d+1)\log(d+1)}}{Z} d\lambda$ for $q \geq \beta$, where $\beta$ is the finite index conjugate to $p$. \hfill \Box

**Example 13.** For $U(d) = \sinh(d), U(d) = U''(d) \leq \cosh(d) = U'(d).$

So, by Corollary 9, the $q$-Poincaré and a $q$-Logarithmic Sobolev inequalities hold true for the measure $d\mu_U = \frac{e^{-\sinh(d)}}{Z} d\lambda$ for all $q \geq 1$.

---

4. **$p$–Talagrand Inequality for $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$**

To prove that the $q$–Logarithmic Sobolev inequality implies the $p$–Talagrand inequality, we are inspired by the proof of Theorem 4.1 in [4]. However, instead of using the Hamilton-Jacobi equation as in that paper, we use the following Hamilton-Jacobi equation as in Theorem 4 in the paper by F. Dragoni [14]. In the setting of Carnot groups, the restriction for using a doubling measure $\mu$ is no longer needed.

Let $d$ be the Carnot-Carathéodory distance and $d\mu = \frac{e^{-U(d)}}{Z} d\lambda$ be a measure where $Z$ is a normalization constant, $\lambda$ is the Lebesgue measure, and $U(d)$ satisfies the conditions of Theorem 8.
Theorem 14. Let $1 < q \leq 2$, and $p \geq 2$ be its finite index conjugate, so that \( \frac{1}{p} + \frac{1}{q} = 1 \).

If \((G, d, \mu)\) satisfies the \(q\)-Logarithmic Sobolev inequality with constant \( c = (q-1) \left( \frac{q}{K} \right)^{q-1} \) for some constant \( K > 0 \), then it also satisfies the \(p\)-Talagrand inequality with the same constant \( K \).

For every continuous bounded function \( f \) where \( Qf = Q_{1}f \), we will be proving the dual formulation of the Talagrand inequality

\[
\int_{G} e^{KQf} d\mu \leq e^{K} \int_{G} f d\mu,
\]

which is shown equivalent to the \(p\)-Talagrand inequality (1.4) in [6].

Proof. For some \( n \geq 1 \), define

\[
\phi(t) = \frac{1}{K t^n} \log \left( \int_{G} e^{K(t+s)^n Q_{1+s}f} d\mu \right).
\]

Using the fact that \( f \) is bounded and the dominated convergence theorem, Balogh et al. prove in Theorem 4.1 of [4] that:

\[
\lim_{t \to 0^+} \phi(t) = \int_{G} f d\mu.
\]

The main goal is to prove that \( \phi(t) \) is non-increasing in \( t \), which implies that \( \phi(1) \leq \lim_{t \to 0^+} \phi(t) \). In other words, replacing:

\[
\phi(1) = \frac{1}{K} \log \left( \int_{G} e^{KQf} d\mu \right) \leq \lim_{t \to 0^+} \phi(t) = \int_{G} f d\mu.
\]

Rearranging the terms and exponentiating the last inequality, we get the dual Talagrand inequality (4.1). We now show that \( \phi(t) \) is non-increasing. The two main ingredients to prove that are the \(q\)-Logarithmic Sobolev inequality and the solution to the Hamilton-Jacobi problem (2.1).

Fix \( t \in (0, 1] \). For \( s > 0 \), we have

\[
\lim_{s \to 0^+} \frac{\phi(t+s) - \phi(t)}{s} = \lim_{s \to 0^+} \frac{1}{K (t+s)^n} \left( \frac{1}{K(t+s)^n} - \frac{1}{Kt^n} \right) \log \int_{G} e^{K(t+s)^n Q_{1+s}f} d\mu
\]

\[
+ \lim_{s \to 0^+} \frac{1}{K t^n s} \left( \log \int_{G} e^{K(t+s)^n Q_{1+s}f} d\mu - \log \int_{G} e^{K t^n Q_{1}f} d\mu \right).
\]

As \( s \to 0^+ \), the first term on the right-hand side of (4.2) converges to

\[-\frac{n}{K t^n+1} \log \left( \int_{G} e^{K t^n Q_{1}f} d\mu \right).
\]

The limit of the second term of (4.2) is

\[
\frac{1}{K t^n} \int_{G} e^{K t^n Q_{1}f} d\mu \lim_{s \to 0^+} \left[ \frac{1}{s} \left( \int_{G} e^{K(t+s)^n Q_{1+s}f} d\mu - \int_{G} e^{K t^n Q_{1}f} d\mu \right) \right].
\]
rewriting, we get

\[
\frac{1}{K^t} \int_G e^{K^t Q_t f} d\mu \lim_{s \to 0^+} \left[ \int_G e^{K(t+s)^n Q_{t+s} f} \frac{1}{s} d\mu + \int_G e^{K^t Q_{t+s} f} \frac{1}{s} d\mu \right]
\]

so, using the dominated convergence theorem for the first term, we get

\[
(4.3) = \frac{1}{K^t} \int_G K t^{n-1} Q_t f e^{K^t Q_t f} d\mu + \frac{1}{K^t} \int_G e^{K^t Q_t f} d\mu \lim_{s \to 0^+} \left[ \int_G e^{K(t+s)^n Q_{t+s} f} \frac{1}{s} d\mu \right].
\]

In order to compute the limit in (4.3), we use the Hamilton-Jacobi equation in Theorem 5, which has the solution \(Q_t f\), so

\[
\frac{\partial Q_t f}{\partial t} = -\frac{\nabla Q_t f}{q},
\]

so,

\[
(4.4) \lim_{s \to 0^+} \left( e^{K^t Q_{t+s} f} \frac{1}{s} \right) = K^t e^{K^t Q_t f} \frac{\partial Q_t f}{\partial t} = -K^t e^{K^t Q_t f} \frac{\nabla Q_t f}{q}. \]

Since \(Q_{t+s} f(\cdot)\) is Lipschitz on \(G \times \mathbb{R}_+\), \(Q_{t+s} f = Q_t f + O(s)\) holds uniformly on \(G\). Since \(Q_t f(x)\) is uniformly bounded in \(x\), we can use the dominated convergence theorem and (4.4), to find the limit in (4.3):

\[
\lim_{s \to 0^+} \left[ \int_G e^{K(t+s)^n Q_{t+s} f} \frac{1}{s} d\mu \right] = -K^t \int_G e^{K^t Q_t f} \frac{\nabla Q_t f}{q} d\mu.
\]

In other words, equation (4.3) becomes:

\[
\lim_{s \to 0^+} \frac{1}{K^t} \int_G e^{K^t Q_t f} d\mu - \log \left( \int_G e^{K(t+s)^n Q_{t+s} f} d\mu - \log \left( \int_G e^{K^t Q_t f} d\mu \right) \right)
\]

\[
= \frac{1}{K^{n+1}} \int_G e^{K^t Q_t f} d\mu - \int_G n K^t Q_t f e^{K^t Q_t f} d\mu - \int_G K^{n+1} \frac{\nabla Q_t f}{q} e^{K^t Q_t f} d\mu.
\]

Replacing the last equation in (4.2), we get:

\[
\lim_{s \to 0^+} \frac{\phi(t+s) - \phi(t)}{s} = \frac{1}{K^{n+1}} \int_G e^{K^t Q_t f} d\mu - \int_G n K^t Q_t f e^{K^t Q_t f} d\mu + \int_G K^{n+1} \frac{\nabla Q_t f}{q} e^{K^t Q_t f} d\mu.
\]

Let \(g = e^{K^t Q_t f}\), and \(n = \frac{1}{q-1}\), we obtain:

\[
\lim_{s \to 0^+} \frac{\phi(t+s) - \phi(t)}{s} = \frac{1}{K^{n+1}} \int_G g^q d\mu
\]

\[
\times \left[ -\frac{1}{q-1} \log \left( \int_G g^q d\mu \right) \int_G g^q d\mu + \frac{1}{q-1} \int_G g^q \log(g^q) d\mu \right].
\]
Using the $q$-Logarithmic Sobolev inequality

$$\text{Ent}_\mu(g^q) = \int_G g^q \log (g^q) d\mu - \log \left( \int_G g^q d\mu \right) \int_G g^q d\mu$$

$$\leq (q - 1) \left( \frac{q}{K} \right)^{q-1} \int_G |\nabla g|^q d\mu$$

in (4.5):

$$\lim_{s \to 0^+} \frac{\phi(t + s) - \phi(t)}{s} \leq \frac{1}{Kt^{\frac{1}{q-1}}} \int_G g^q d\mu \times \left[ \left( \frac{q}{K} \right)^{q-1} \int_G |\nabla g|^q d\mu - \int_G Kt^{\frac{1}{q-1}} |\nabla Q_t f|^q g^q d\mu \right] = 0.$$ 

Hence, $\phi(t)$ is non-increasing, and we obtain the dual Talagrand inequality.

5. **Hypercontractivity of Hamilton-Jacobi solutions for $d\mu = e^{-U(d)} Z d\lambda$**

Denote $||.||_p, p \in \mathbb{R}$, the $L^p$-norms with respect to $\mu$, and $||f||_0 = e^{\int \log |f| d\mu}$ whenever $\log|f|$ is $\mu$-integrable. Also, consider the Hamilton-Jacobi problem on a Carnot group $G$ as in Theorem 5.

The following Hypercontractivity theorem is inspired by Theorem 2.1 by Bobkov et al. in [5], but with the measure $d\mu = e^{-U(d)} Z d\lambda$ and the Euclidean gradient replaced by the sub-gradient $\nabla$.

**Theorem 15.** Assume we have the following 2-Logarithmic Sobolev inequality with the measure $d\mu = e^{-U(d)} Z d\lambda$, and in the setting of the Carnot group:

$$\rho \text{Ent}_\mu(f^2) = \rho \int_G f^2 \log (f^2) d\mu - \rho \log \left( \int_G f^2 d\mu \right) \int_G f^2 d\mu \leq 2 \int_G |\nabla f|^2 d\mu.$$ 

Then, for every bounded measurable function $f$ on $G$, every $t \geq 0$, and every $a \in \mathbb{R},$

$$||e^{Q_t f}||_{a + \rho t} \leq ||e^f||_a.$$

The following proof adapts the proof of Theorem 2.1 done by Bobkov et al. in [5] in the Euclidean setting, but uses the Hamilton-Jacobi problem in the Carnot group setting (Theorem 5).

**Proof.** Let $F(t) = ||e^{Q_t f}||_{\lambda(t)}$, with $\lambda(t) = a + \rho t, t > 0$.

For all $t > 0$ and almost every $x$, the partial derivatives $\frac{\partial}{\partial t} Q_t f(x)$ exist, and by the Hamilton-Jacobi problem (Theorem 5),

$$\frac{\partial}{\partial t} Q_t f(x) = - \frac{1}{2} |\nabla Q_t f(x)|^2.$$ 

So, $F$ is differentiable at every point $t > 0$ where $\lambda(t) \neq 0$. 

Computing,

\[(5.2) \quad (F(t)^{\lambda(t)})' = \rho F(t)^{\lambda(t)} \log(F(t)) + \lambda(t) F(t)^{\lambda(t)-1} F'(t).\]

Also,

\[\left( F(t)^{\lambda(t)} \right)' = \frac{\partial}{\partial t} \left( \int_G e^{\lambda(t)Q_t f} d\mu \right) = \int_G \left( \rho Q_t f + \lambda(t) \frac{\partial}{\partial t} Q_t f \right) e^{\lambda(t)Q_t f} d\mu \]

using \((5.1)\),

\[\frac{\rho}{\lambda(t)} \text{Ent}_\mu(e^{\lambda(t) Q_t f}) + \frac{\rho}{\lambda(t)} \log\left( \int_G e^{\lambda(t) Q_t f} d\mu \right) \int_G e^{\lambda(t) Q_t f} d\mu - \frac{\lambda(t)}{2} \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu \]

\[= \frac{\rho}{\lambda(t)} \text{Ent}_\mu(e^{\lambda(t) Q_t f}) + \rho F(t)^{\lambda(t)} \log(F(t)) - \frac{\lambda(t)}{2} \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu.\]

Combining this with \((5.2)\), we get

\[\lambda(t)^2 F(t)^{\lambda(t)-1} F'(t) = \rho \text{Ent}_\mu(e^{\lambda(t) Q_t f})^2 - \frac{\lambda(t)^2}{2} \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu\]

using the 2-Logarithmic Sobolev inequality,

\[\le \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu - \frac{\lambda(t)^2}{2} \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu\]

\[= \frac{\lambda(t)^2}{4} \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu - \frac{\lambda(t)^2}{2} \int_G |\nabla Q_t f(x)|^2 e^{\lambda(t) Q_t f} d\mu\]

\[\le 0,\]

for all \(t > 0\). Thus, \(F'(t) \le 0\) for all \(t > 0\). Since \(F\) is continuous, then it is non-increasing. Hence, \(F(t) \le F(0)\), for all \(t > 0\).

\[\|e^{Q_t f}\|_{\lambda(t)} \le \|e^{Q_0 f}\|_{\lambda(0)} .\]

So,

\[\|e^{Q_t f}\|_{\lambda(t) + \rho t} \le \|e^f\|_{\lambda}.\]

\[\square\]

References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougère, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer. Sur les inégalités de Sobolev logarithmiques. Panoramas et Synthèses, vol. 10, Société Mathématique de France, 2000.

[2] D. Bakry and M. Émery. Diffusions hypercontractives. In Séminaire de Probabilités, XIX, 1983/84, number 1123 in Lecture Notes in Math., pages 177–206. Springer, Berlin, 1985.

[3] Z.M. Balogh, A. Calogero, and R. Pini. The Hopf-Lax formula in Carnot groups: a control theoretic approach. Calculus of Variations and Partial Differential Equations. 49. 10.1007/s00526-013-0627-3.

[4] Z.M. Balogh, A. Engulatov, L. Hunzinker, O.E. Massalo. Functional Inequalities and Hamilton-Jacobi Equations in Geodesic Spaces. Springer Science+Business Media B.V. 2011.
[5] S. Bobkov, I. Gentil, and M. Ledoux. Hypercontractivity of Hamilton–Jacobi equations. J. Math. Pures Appl. 80 (2001) 669–696.

[6] S.G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. J. Funct. Anal. 163 (1999) 1–28.

[7] S. Bobkov and M. Ledoux. From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities. Geom. Funct. Anal., 10(5):1028–1052, 2000.

[8] S. Bobkov and B. Zegarliński. Entropy bounds and isoperimetry. Mem. Amer. Math.Soc., 176(829), 2005.

[9] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. Stratified Lie Groups and Potential Theory for their Sub-Laplacians. Springer Monographs in Mathematics. Springer, 2007.

[10] E. Bou Dagher and B. Zegarliński. Coercive Inequalities and U-Bounds. arXiv:2105.01759 [math.FA].

[11] E. Bou Dagher and B. Zegarliński. Coercive Inequalities in Higher-Dimensional Anisotropic Heisenberg Group. arXiv:2105.02593 [math.FA].

[12] E. Bou Dagher and B. Zegarliński. Coercive Inequalities on Carnot Groups: Taming Singularities. arXiv:2105.03922 [math.FA].

[13] L. Capogna, D. Danielli, S.D. Pauls, and J. Tyson. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem. Progress in Mathematics, 259. Birkhäuser Verlag, Basel, 2007. xvi+223 pp. ISBN: 978-3-7643-8132-5.

[14] F. Dragoni. Metric Hopf–Lax formula with semicontinuous data. Discrete Contin. Dyn. Syst. 17(4), 713–729 (2007).

[15] N. Gozlan. A characterization of dimension free concentration in terms of transportation inequalities. Ann. Probab. 37(6), 2480–2498 (2009)

[16] N. Gozlan, C. Roberto, and P.M. Samson. From concentration to logarithmic Sobolev and Poincaré inequalities. J. Funct. Anal. 260(5), 1491–1522 (2010).

[17] M. Gromov. Carnot–Carathéodory spaces seen from within. In: Sub-Riemannian Geometry, Progr. Math., vol. 144, pp. 79–323. Birkhäuser, Basel (1996).

[18] L. Gross. Logarithmic Sobolev inequalities. Amer. J. Math., 97:1061–1083, 1975.

[19] W. Hebisch and B. Zegarliński. Coercive inequalities on metric measure spaces. J. Funct. Anal., 258:814–851, 2010.

[20] J. Inglis. Coercive Inequalities for Generators of Hörmander Type. Doctor of Philosophy of the University of London and the Diploma of Imperial College, Department of Mathematics Imperial College, 2010.

[21] J. Jerison. The Poincaré inequality for vector fields satisfying Hörmander’s condition. Duke Math. J. 53 (1986), no. 2, 505-523.

[22] E. Le Donne. A Primer on Carnot Groups: Homogeneous Groups, Carnot-Carathéodory Spaces, and Regularity of Their Isometries. Anal. Geom. Metr. Spaces 2017; 5:116–137.

[23] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities. in: Séminaire de Probabilités XXXIII, in: Lecture Notes in Mathematics, vol. 1709, Springer-Verlag, Berlin, 1999, pp. 120–216.

[24] M. Ledoux. The Concentration of Measure Phenomenon. American Mathematical Society, Providence, 2001.

[25] J. Lott and C. Villani. Hamilton–Jacobi semigroup on length spaces and applications. J. Math. Pures Appl. (9) 88(3), 219–229 (2007).

[26] K. Marton. A simple proof of the blowing-up lemma. IEEE Trans. Inform. Theory 32, 445–446 (1986).

[27] K. Marton. Bounding d-distance by informational divergence: a method to prove measure concentration. Ann. Probab. 24, 857–866 (1996).

[28] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. Comm. Partial Differential Equations 26, 101–174 (2001).
[29] F. Otto and C. Villani. *Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality.* J. Funct. Anal. 173, 361–400 (2000).

[30] E. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, New Jersey, 1993.

[31] M. Talagrand. *Transportation cost for Gaussian and other product measures.* Geom. Funct. Anal. 6, 587–600 (1996).

Esther Bou Dagher:
Department of Mathematics
Imperial College London
180 Queen's Gate, London SW7 2AZ
United Kingdom

Email address: esther.bou-dagher17@imperial.ac.uk