Motion and gravitational radiation of a binary system consisting of an oscillating and rotating coplanar dusty disk and a point-like object

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A binary system composed of an oscillating and rotating coplanar dusty disk and a point mass is considered. The conservative dynamics is treated on the Newtonian level. The effects of gravitational radiation reaction and wave emission are studied to leading quadrupole order. The related waveforms are given. The dynamical evolution of the system is determined semi-analytically exploiting the Hamiltonian equations of motion which comprise the effects both of the Newtonian tidal interaction and the radiation reaction on the motion of the binary system in elliptic orbits. Tidal resonance effects between orbital and oscillatory motions are considered in the presence of radiation damping.

1. Introduction

Compact binaries are among the most promising sources for the gravitational waves that should be detected by gravitational wave observatories such as GEO600, LIGO, TAMA, and VIRGO on Earth, or LISA in space. On the other hand, it is well known that oscillating compact objects also emit gravitational waves that should be detectable by gravitational wave observatories. Though the ring-down phases of merged binaries or collapsed objects are the strongest sources for gravitational waves from oscillating objects, oscillations of the components of binary systems generated by tidal interaction during their inspiralling phase should induce measurable characteristic modifications in the gravitational waveforms compared with those from binaries with point-like components. Particularly in the leading resonance cases where the orbital frequency is a factor of two or three smaller than the oscillation frequencies, the internal motion of the binaries is expected to be strongly excited.

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The treatment of the motion of compact binaries within the full Einstein theory of gravity is a demanding challenge. Much effort has been invested in this challenge on the numerical simulation side because an exact analytic solution cannot be expected to be achievable. However, a fully numerically simulation is still far ahead.

The aim of the present paper is the development of a quite simple approximate model to the full Einstein theory which shows typical features of inspiralling compact binaries and which is fully analytic apart from the numerical integration of a system of ordinary differential equations. This model of a binary system is given by a point-like object and an oscillating, rotating dusty disk, where the orbital plane and the plane of the disk coincide. As there is no pressure involved, an extreme simplification occurs for the field equations as well as for the equations of motion. The dynamics is treated at the level of Newtonian gravity augmented by the gravitational radiation reaction dynamics, which occurs in the full Einstein theory to leading approximation at the order \(1/c^5\) \((c\) denotes the speed of light\) which is, counted in powers of \(1/c^2\), 2.5 orders beyond the Newtonian order of \(1/c^0\). Within this model we further assume that the most general motion of the dusty disk, apart from its orbital motion, is rotation and oscillation, i.e. we do not allow the disk to leave this configuration. This means that the tidal force is considered as a perturbation only. The same is true for the radiation reaction force which is of 2.5pN (post-Newtonian) order.

The Newtonian oscillating rotating disk of dust is interesting for it’s own reasons. While stellar oscillations are usually treated as small perturbations, the oscillations of the disk can be arbitrarily large, they are bulk oscillations. Moreover, the equation of motion for the disk, which can be derived from the equations of hydrodynamics, exhibits a surprising similarity to the well-known Keplerian problem. Exploiting this analogy, it is possible to write down a parametrized solution for the disk’s oscillations \[1\].

Having fully specified our model, let us shortly summarize the present knowledge of the analytical motion of binary systems in the Einstein theory of gravity now. Regarding point-like binaries without proper spins the conservative motion is known fully analytically up to the 3pN order, where the 2.5pN order is neglected \[2\]; the motion of a single oscillating and rotating dusty disk in it’s rest frame is known fully analytically to 1pN order \[3\]. If the spins are included into the binary motion, in particular, if the spin orientations are parallel or anti-parallel to the orbital angular momentum as in our model, the analytical solution is known up to 2pN order \[4\]. Taking into account Newtonian tidal interaction in close binary systems, several semi-analytical investigations to the evolution of binaries have been undertaken \[5\], \[6\], \[7\]. In \[8\] and \[9\] the semi-analytical solutions of close Newtonian binaries systems have been obtained, including 2.5pN gravitational damping effects.

The paper is organized as follows: In section II we derive the Hamiltonian formalism for the Newtonian disk, starting from the equations of hydrodynamics. We will not consider quadrupole radiation damping as a secular effect, but include radiation reaction terms in the Hamiltonian equations of motion. To that end we derive the leading order radiation reaction Hamiltonian. Then we calculate the leading order gravitational waveform for the isolated Newtonian disk.

In section III we give a short review of orbital Hamiltonian formalism and calculate the contributions to the leading order gravitational waveform of the orbital motion.

In section IV we focus on the binary system. The tidal interaction Hamiltonian is calculated explicitly and given in form of a series expansion. In a next step we derive
the radiation reaction Hamiltonian for the coupled binary system. The leading order Hamiltonian equations of motion, governing the dynamics of the system, are solved numerically for different initial conditions. Since the contribution of the interaction of the disk quadrupole with the orbital quadrupole is much smaller than all the other contributions it can be neglected in most applications. On the other hand, the influence of the tidal interaction potential on the dynamics of the binary is not negligible because the exchange of orbital energy with oscillation energy can be quite strongly.

Taking into account leading order gravitational damping, orbital energy is dragged from the orbit via tidal interaction. In other words, tidal interaction speeds up the inspiral process of our binary system.

The energy transfer is strongest in tidal resonance, where the oscillation frequency is \( n \) times the orbital frequency (\( n \) is a small integer number). We study the effect of tidal interaction on the orbit and on the leading order gravitational waveform. In section V we shall discuss our results and give an outlook on future projects.

2. Oscillations of a rotating disk of dust – Hamiltonian formulation

One of the simplest axisymmetric systems that exhibit the essential new features of general relativity is a thin disk of dust. During their evolution these disks emit gravitational radiation and may eventually form a black hole. No wonder that general relativistic disks of dust have been subjects of interest for many years. Bardeen and Wagoner \[9\] were the first to study relativistic stationary rotating disks. They succeeded to find a solution for a rigidly rotating disk in form of a series expansion. Unfortunately this solution can be obtained only numerically beyond 2pN approximation. In terms of ultraelliptic functions, Neugebauer and Meinel \[10\] found an exact solution for a rigidly rotating disk in full general relativity using inverse scattering methods. However, these solutions are restricted to rigidly rotating stationary disks. Nonstationary, rigidly rotating disks of dust were first studied by Hunter \[1\]. These disks are particularly interesting, as they are sources of gravitational waves. In the present paper we will focus on oscillating Maclaurin disks. These are infinitely thin, rotating disks of dust which are obtained as a limiting case of Maclaurin spheroids. For these configurations Kley and Schäfer \[3\] succeeded to derive a fully analytically solution up to first post-Newtonian approximation in a parametrized form. That solution exhibits a surprising similarity to the 1pN parametrized solution of the Keplerian problem.

However, the study of a weakly relativistic disk as a component of a binary system is beyond the scope of this paper. We shall restrict ourselves on leading order results, thus considering only the Newtonian disk.

Let us start by briefly reviewing the known analytic solution for the oscillating disk, obtained from the equation of hydrodynamics for a pressureless system. Since we are interested in the leading order gravitational waveform emitted by the oscillating rotating disk we shall also discuss the related 2.5pN radiation reaction. Note that we do not treat quadrupole radiation damping as a secular effect. The corresponding radiation reaction terms will be included into the Hamiltonian equations of motion. We shall solve the Hamiltonian equations of motion for the isolated Newtonian disk numerically and calculate the leading order gravitational waveform in terms of canonical conjugate variables.
2.1. The Newtonian disk revisited

The Newtonian dynamics of MacLaurin disks has been treated by several authors [1], [11], [3], starting with the equations of hydrodynamics in Eulerian form. Let us briefly review this standard approach before we consider the disk in Hamiltonian formalism. For an axisymmetric, pressureless configuration confined to the $z = 0$ plane the equations of hydrodynamics and mass conservation in cylindrical coordinates $(r, \phi)$ read

$$\frac{\partial \Sigma}{\partial t} + v_r \frac{\partial \Sigma}{\partial r} + \frac{\Sigma}{r} \frac{\partial}{\partial r}(rv_r) = 0, \tag{1}$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{\partial U}{\partial r} - \frac{v_r^2}{r} = 0,$$

$$\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} = 0.$$

Here $\Sigma(r, t)$ denotes the surface density and $v_r$ and $v_\phi$ are radial and azimuthal velocities, respectively. The gravitational potential $U$ is the solution to Poisson’s equation

$$\Delta U = -4\pi G \Sigma(r, t) \delta(z). \tag{2}$$

For $\Sigma, v_r, v_\phi$ we shall choose an ansatz where the surface density is that of a MacLaurin disk,

$$\Sigma(r, t) = \sigma(t) \sqrt{1 - \frac{r^2}{r_d(t)^2}}, \quad v_r(r, t) = -rf(t), \quad v_\phi(r, t) = r\Omega(t), \tag{3}$$

where $\sigma(t)$ is the surface density at the center and $r_d(t)$ denotes the disk radius at a given time. The disk is rigidly rotating with angular velocity $\Omega(t)$. Using this ansatz, it is possible to solve (1) analytically. To that end we note that the Poisson equation (2) can be solved explicitly inside the disk. Inserting (3) we find

$$U(r, t) = G\sigma(t)r_d(t) \frac{\pi^2}{4} \left[ 2 - \left( \frac{r}{r_d(t)} \right)^2 \right], \quad (r \leq r_d, z = 0). \tag{4}$$

Substituting equations (3) and (4) into (1) we can derive a system of four differential equations for the unknown variables $r_d, \sigma, f$ and $\Omega$:

$$\dot{\sigma} - 2f\sigma = 0,$$

$$\dot{r}_d + fr_d = 0,$$

$$\dot{\Omega} - 2f\Omega = 0,$$

$$-\dot{f} + f^2 + \frac{G\pi^2\sigma}{2r_d} - \Omega^2 = 0. \tag{5}$$

This system is integrated with respect to the initial conditions $f(0) = 0$ and $r_d(0) = R_d$. Combining the first two equations, one finds

$$\sigma(t)r_d^2(t) = \frac{3}{2\pi}M_d, \tag{6}$$

where $M_d$ is the total Newtonian mass of the disk. The third equation in (5) gives

$$\Omega(t)r_d^2(t) = \Omega_0 R_d^2, \quad \Omega_0 = \Omega(0) \tag{7}$$

where $\Omega_0$ is the initial angular velocity.
which is constant at the Newtonian level. The remaining equation containing $\ddot{r}_d$ reads

$$\ddot{r}_d + \frac{2C}{r_d^2} - \frac{h^2}{r_d^3} = 0,$$

where

$$C := G\sigma(t)r_d(t)^2\pi^2 = \frac{3\pi}{8}GM_d$$

and

$$h^2 := 2C\xi^2R_d = \Omega_0^2R_d^4. \quad (10)$$

$\xi^2$ is defined by $\Omega_0$ as follows,

$$\Omega_0^2 = \frac{2C\xi^2}{R_d^4}. \quad (11)$$

Equation (8) exhibits a surprising similarity to the radial equation in the Keplerian problem. Exploiting this similarity, the solution to equation (8) can be given in parametrized form,

$$r_d = a_d(1 - \epsilon \cos \varphi), \quad \frac{2\pi}{P}t = u - \epsilon \sin u - \pi, \quad (12)$$

where the definitions

$$a_d = \frac{R_d}{1 + \epsilon}$$

and $\epsilon = 1 - \xi^2$ hold. For not being confused with the orbital eccentricity $e$, we shall speak of $\epsilon$ as the *ellipticity* of the disk’s motion.

### 2.2. Hamiltonian formulation - Newtonian level

After this short review we shall now derive the Hamiltonian formulation of the disk’s dynamics. On the Newtonian level this can be easily achieved by exploiting the analogy of (8) to the radial equation of the Keplerian problem. The Lagrangian, leading to Eq. (8), reads

$$L = \alpha \left[ \frac{\dot{r}_d^2}{2} + \frac{\beta}{r_d} + \frac{\gamma}{r_d^2} \right],$$

$\alpha, \beta$ and $\gamma$ being coefficients which have to be determined. The Euler-Lagrangian equation corresponding to this ansatz reads

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_d} - \frac{\partial L}{\partial r_d} = 0 \quad \Rightarrow \quad \ddot{r}_d + \frac{\beta}{r_d^2} + \frac{2\gamma}{r_d^3} = 0.$$

Comparing this with Eq. (8), we have to identify

$$\beta = 2C, \quad \gamma = -\frac{h^2}{2} = -CR_d\xi^2.$$
Substituting the canonical conjugate momentum to $r_d$,  

$$p_r = \frac{\partial L}{\partial \dot{r}_d} = \alpha \dot{r}_d,$$  

the Newtonian Hamiltonian reads

$$H_d^{(N)} = \frac{p_r^2}{2\alpha} - \alpha \left( \frac{\beta}{r_d} \frac{\gamma}{r_d} \right) = \frac{1}{2\alpha} \left( \frac{p_r^2}{r_d^2} + \frac{p_\phi^2}{r_d^2} \right) - \frac{2C\alpha}{r_d},$$  

where we have identified $p_\phi^2 \equiv 2\alpha^2 C R_d \xi^2$. To fix the value of $\alpha$ we require that $H_d^{(N)}$ should be the conserved energy of the disk. The later is given by

$$E_d = \int d^3x \frac{1}{2} \rho (v^2 - U) = \frac{3\pi GM_d^2}{20 R_d} (\xi^2 - 2) = \frac{2}{5} \frac{C}{R_d} M_d (\xi^2 - 2).$$

In particular, this must hold at $t = 0$, where by virtue of the initial condition $r_d(0) = R_d$ and $\dot{r}_d(0) = 0$, the momentum conjugate has to fulfill $p_r(0) = 0$. Inserting this into Eq. (13), we end up with

$$H_d^{(N)} = \frac{1}{2\alpha} \left( \frac{p_r^2}{r_d^2} + \frac{p_\phi^2}{r_d^2} \right) - \frac{2C\alpha}{r_d}, \quad \alpha = \frac{2}{5} M_d.$$  

### 2.3. Radiation reaction part

It is well known that the leading order part of the gravitational radiation is of the quadrupolar type. In the case of continuous matter the related radiation reaction Hamiltonian is given by [12]

$$H_{\text{reac}}(t) = \frac{2G}{5c^3} \ddot{Q}_{ij}(t) \int d^3x \rho (\pi_i \pi_j) + \frac{1}{4\pi G} \partial_i U \partial_j U),$$

where

$$Q_{ij} := \int d^3x \rho (x^i x^j - \frac{1}{3} \delta^{ij} r^2)$$

is the Newtonian mass-quadrupole tensor of the system. At Newtonian order we may identify $\rho_* = \rho, \pi_i = v^i \rho,$ and $U_* = U$. Using the continuity equation for the mass and the equations of motion for the matter, the following equation, relating an integral of non-compact support to an integral of compact support, can be shown to hold,

$$\int d^3x (\rho v^i v^j + \frac{1}{4\pi G} \partial_i U \partial_j U) = \frac{1}{2} \ddot{Q}_{ij}.$$  

Thus we are allowed to write the reaction Hamiltonian of our disk as

$$H_{\text{reac}}^{(d)}(t) = \frac{G}{5c^3} \ddot{Q}_{ij}^{(d)}(t) \tilde{Q}_{ij}^{(d)}(p, q),$$  

where $\tilde{Q}_{ij}^{(d)}(p, q)$ denotes that $\tilde{Q}_{ij}^{(d)}$ has to be treated as a function of position and momentum variables, i.e. making use of the equations of motion the second time derivatives have to be eliminated. The mass quadrupole tensor of the disk is diagonal, $Q^{(d)}_{ij} = (Q^{(d)}_{11}, Q^{(d)}_{11}, -2Q^{(d)}_{11})$. This simplifies the Eq. (18) enormously. In the appendix
the disk’s quadrupole tensor and it’s time derivatives are given explicitly. In particular, we get

\[ \dddot{Q}_{11}(p, q) = \frac{1}{3} \left[ \frac{1}{\alpha} (p_r^2 + p_\phi^2) - \frac{2C\alpha}{r_d} \right]. \quad (19) \]

Inserting this into Eq. (18) we finally arrive at

\[ H^{(d)}_{\text{reac}}(t) = \frac{2G}{5c^5} \dot{\mathcal{V}}^{(d)}_{xx}(t) \left[ \frac{1}{\alpha} (p_r^2 + p_\phi^2) - \frac{2C\alpha}{r_d} \right]. \quad (20) \]

### 2.4. Equations of motion

The total Newtonian plus leading order radiation reaction (non-autonomous) Hamiltonian of the oscillating rotating disk reads

\[ H^{(d)}_{\text{total}} = H^{(N)}_{d} + H^{(d)}_{\text{reac}}(t), \quad (21) \]

where \( H^{(N)}_{d} \) and \( H^{(d)}_{\text{reac}} \) are given by (14) and (20), respectively. The Hamiltonian equations of motion are obtained in the usual way as

\[ \dot{p}_i = -\frac{\partial H^{(d)}_{\text{total}}}{\partial q^i}, \quad \dot{q}_i = \frac{\partial H^{(d)}_{\text{total}}}{\partial p_i}. \]

However, some care is needed when calculating the reaction part of the Hamiltonian equations of motion. It is obtained by taking the derivatives of \( H^{(d)}_{\text{reac}} \) with respect to the canonical conjugate variables, but one must take \( \dot{Q}^{(d)}_{ij}(t) \) as a function of \( t \). Only after differentiation we shall express the third time derivative of the quadrupole tensor as a function of \( p_r \) and \( r_d \). Inserting Eq. (19) the reaction part of the Hamiltonian equations of motion reads

\[ (\dot{r})_{\text{reac}} = \frac{\partial H^{(d)}_{\text{reac}}}{\partial p_r} = -8 \frac{GC}{15} \frac{p_r^2}{c^5 \alpha r_d^2}, \]

\[ (\dot{\phi})_{\text{reac}} = \frac{8 \alpha GC}{15} \frac{p_r p_\phi}{c^5 r_d^4}, \]

\[ (\dot{p}_r)_{\text{reac}} = -8 \frac{GC}{15} \frac{p_r}{c^5 r_d^2} \left( \frac{p_\phi^2}{\alpha r_d^3} - \frac{C\alpha}{r_d^2} \right), \]

\[ (\dot{p}_\phi)_{\text{reac}} = 0. \]

The Hamiltonian equations for the oscillating disk, including leading order gravitational radiation reaction, read

\[ \dot{r}_d = \frac{p_r}{\alpha} \left[ 1 - \frac{8 \alpha GC}{15} \frac{p_r}{c^5 r_d^2} \right], \quad (22) \]

\[ \dot{\phi} = \frac{p_\phi}{\alpha r_d^2} \left[ 1 - \frac{8 \alpha GC}{15} \frac{p_r}{c^5 r_d^2} \right], \quad (23) \]

\[ \dot{p}_r = \frac{p_\phi^2}{\alpha r_d^3} - \frac{2C}{r_d^2} - 8 \frac{GC}{15} \frac{p_r}{c^5} \left( \frac{p_\phi^2}{\alpha r_d^3} - \frac{C\alpha}{r_d^2} \right), \quad (24) \]

\[ \dot{p}_\phi = 0. \quad (25) \]
Figure 1: In leading order approximation $I_{\text{disk}}^{20}$ is the only nonvanishing contribution to the gravitational radiation field of the rotating, oscillating disk. In the figure $I_{\text{disk}}^{20}$ is plotted for $\xi^2 = 0.9, R_d(0) = 10$ (in units of $G\alpha/c^2$).

Note that we do not treat gravitational damping as a secular effect. The leading order radiation reaction terms enter directly into the equations of motion. It is remarkable that, although the total energy of the disk is not conserved, the angular momentum is a conserved quantity at 2.5pN as indicated by Eq. (25). This is due to the rotational symmetry of the disk and well expected. For a binary point-mass system, however, 2.5pN radiation reaction destroys orbital momentum conservation, as we shall see in the next section.

2.5. Gravitational waves from oscillating rotating disks

The 2.5pN radiation reaction terms in Eqs. (22)-(25) are the leading order dissipative terms. Energy is dragged from the oscillating disk by emission of gravitational waves. Asymptotically, the gravitational radiation field can be represented by

$$h_{ij}^{\text{rad}} = \frac{G}{c^4 D} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left[ \frac{1}{c} \right]^{l-2} (l) I_{lm} (t - D/c) T_{ij}^{E2,lm}(\theta, \phi)$$

$$+ \left[ \frac{1}{c} \right]^{l-1} (l) S_{lm} (t - D/c) T_{ij}^{B2,lm}(\theta, \phi),$$

where $D$ denotes the source-observer distance, the indices $i, j$ refer to Cartesian coordinates in the asymptotic space, $T_{ij}^{E2,lm}$ and $T_{ij}^{B2,lm}$ are the pure-spin tensor-spherical harmonics of magnetic and electric type and $I_{lm}$ and $S_{lm}$ are the spherical radiative mass and current multipole moments, respectively. The upper pre-index $(l)$ denotes the number of time derivatives. In leading order approximation only $l = 2$ terms contribute to the radiation field, i.e.

$$h_{ij}^{\text{rad}} = \frac{G}{c^4 D} \sum_{m=-2}^{2} I_{2m}^{2m} T_{ij}^{E2,2m}(\theta, \phi).$$

This is the famous quadrupole radiation.

In many cases it is more convenient to work with STF mass multipole moments $Q_{A_l}$. 
They are related to the spherically radiative mass multipole $I^{ij}$ according to, e.g. \[ I^{lm}(t) = \frac{16\pi}{(2l + 1)!!} \left[ \frac{2l + 1}{4\pi(l - m)!(l + m)!} \right]^{1/2} \delta_{i_1}^1 \delta_{i_2}^1 \cdots (\delta_{i_m}^1 + i\delta_{i_m}^2)(\delta_{i_{m+1}}^3 + \cdots \delta_{i_l}^3). \] (28)

where

$Y_{lm}^{lm} := (-1)^m(2l - 1)!! \left[ \frac{2l + 1}{4\pi(l - m)!(l + m)!} \right]^{1/2} (\delta_{i_1}^1 + i\delta_{i_1}^2) \cdots (\delta_{i_m}^1 + i\delta_{i_m}^2)(\delta_{i_{m+1}}^3 + \cdots \delta_{i_l}^3). \] (29)

The brackets denote the symmetric part. In particular, we get

$Y_{1,1}^{20} = 3\sqrt{\frac{5}{16\pi}} \delta_{i_1}^3 \delta_{i_2}^3, \] (30)

$Y_{1,1}^{21} = -\sqrt{\frac{15}{8\pi}} (\delta_{i_1}^1 \delta_{i_2}^3 + i\delta_{i_1}^2 \delta_{i_2}^3), \] (31)

$Y_{1,1}^{22} = \sqrt{\frac{15}{32\pi}} (\delta_{i_1}^1 \delta_{i_2}^2 - \delta_{i_1}^2 \delta_{i_2}^2 + 2i\delta_{i_1}^3 \delta_{i_2}^3). \] (32)

Inserting these expressions, we readily find

$I^{20} = \frac{16\pi}{15} \sqrt{3} Y_{1,1}^{20} Q_{1,1} = 4 \sqrt{\frac{3\pi}{5}} Q_{33}, \] (33)

$I^{21} = \sqrt{3} \frac{16\pi}{15} Y_{1,1}^{21} Q_{1,1} = -4 \sqrt{\frac{2\pi}{5}} (Q_{13} - iQ_{23}), \] (34)

$I^{22} = \sqrt{3} \frac{16\pi}{15} \sqrt{\frac{15}{32\pi}} (Q_{11} - Q_{22} - 2iQ_{12}) = 2 \sqrt{\frac{2\pi}{5}} (Q_{11} - Q_{22} - 2iQ_{12}). \] (35)

The quadrupole tensor of the disk is diagonal and, moreover, $Q_{11}^{(d)} = Q_{22}^{(d)}$. Hence only $I_{\text{disk}}^{20}$ contributes to the leading order gravitational radiation field of the disk. In terms of $r_d, p_r, \varphi, p_{\varphi}$ the second time derivative of $Q_{33}^{(d)}$ reads

$\ddot{Q}_{33}^{(d)} = -2 \frac{1}{3} \left[ \frac{1}{\alpha} (p_r^2 + \frac{p_{\varphi}^2}{r_d^2}) - \frac{2C\alpha}{r_d} \right]. \] \] (36)

Inserting this into Eq. (33), the leading order time dependence of the gravitational waveform of the oscillating rotating disk of dust is given by

$I_{\text{disk}}^{20} = -8 \sqrt{\frac{\pi}{15}} \left[ \frac{1}{\alpha} (p_r^2 + \frac{p_{\varphi}^2}{r_d^2}) - \frac{2C\alpha}{r_d} \right]. \] \] (36)

In figure the $I_{\text{disk}}^{20}$ component of the leading order gravitational radiation field is given for a particular example.

Exploiting the analogy of the analytic solution Eq. (12) we may express $I_{\text{disk}}^{20}$ in terms of $u$. To that end we start with Eq. (19), where $\ddot{Q}_{11}^{(d)}$ is given as a function of the Newtonian energy $E_d$ and $r_d$. Inserting

$E_d = -\frac{\alpha C}{R_d} (1 + \epsilon), \quad r_d = a_d (1 - \epsilon \cos u) = \frac{R_d}{1 + \epsilon} (1 - \epsilon \cos u)$


into (19) we readily obtain

\[
\ddot{Q}_{11}^{(d)} = \frac{2\alpha}{3} \left( \frac{E_d}{\alpha} + \frac{C}{r_d} \right) = \frac{2}{3} \frac{E_d}{\alpha} \left( 1 - \frac{1}{1 - \epsilon \cos u} \right),
\]

and hence

\[
\ddot{I}_{20}^{20} = 4 \sqrt{\frac{3\pi}{5}} \dot{Q}_{33}^{(d)} = -16 \sqrt{\frac{\pi}{15}} E_d \left( 1 - \frac{1}{1 - \epsilon \cos u} \right).
\]

This expression is quite similar to the result given for \( \ddot{I}_{20}^{20} \) (see Eq. (55)).

### 3. The orbital motion reviewed

#### 3.1. Hamiltonian formalism

The dynamics of a binary point-mass system, including leading order gravitational radiation reaction, is well known, and we shall only briefly review the basic points. The orbital Hamiltonian reads

\[
H_{\text{orb}} = H_{\text{orb}}^{(N)} + H_{\text{reac}}^{(o)}(t),
\]

Calculating the leading order reaction Hamiltonian according to Eq. (15) we arrive at [15]

\[
H_{\text{reac}}^{(o)} = 2 \frac{G^2}{c^5} \dot{Q}_{ij}^{(o)} \left[ \frac{P_i P_j}{\mu} - \frac{GM\mu}{R^3} \frac{R_i R_j}{R^3} \right],
\]

and the Newtonian Hamiltonian is given by

\[
H_{\text{orb}}^{(N)} = \frac{1}{2\mu} \left( P_R^2 + \frac{P_\Phi^2}{R^2} \right) - \frac{GM\mu}{R},
\]

where \( \mathcal{M} \) is the total and \( \mu \) the reduced mass, respectively. The time evolution of the system is governed by the Hamiltonian equations of motion,

\[
\dot{P}_i = -\frac{\partial H_{\text{orb}}}{\partial R^i}, \quad \dot{R}^i = \frac{\partial H_{\text{orb}}}{\partial P_i}.
\]

Note that \( \dot{Q}_{ij}^{(o)}(t) \) has to be taken as a function of time. Only after the differentiation it shall be expressed as a function of \( P_R, P_\Phi, R \) and \( \Phi \). In polar coordinates, the Hamiltonian equations of the orbital motion read

\[
\dot{R} = \frac{P_R}{\mu} - \frac{8}{15} \frac{G^2}{R^2 c^5 \nu R^4} \left[ 2P_R^2 + 6 \frac{P_\Phi^2}{R^2} \right],
\]

\[
\dot{\Phi} = \frac{P_\Phi}{\mu R^2} - \frac{8}{3} \frac{G^2}{c^5 \nu R^4} P_R P_\Phi,
\]

\[
\dot{P}_R = -\frac{8}{5} \frac{G^2 P_\Phi}{c^5 \nu R^5} \left[ \frac{2GM^3 \nu^2}{R} + \frac{2P_R^2}{R^2} - P_R^2 \right],
\]

\[
\dot{P}_\Phi = -\frac{8}{3} \frac{G^2 P_R}{c^5 \nu R^4} \left[ \frac{GM^3 \nu^2}{R} + \frac{8G^2 P_\Phi}{3} - \frac{P_\Phi^2}{5} - \frac{P_R^2}{\nu R} \right],
\]

(41)
where \( \nu = \mu / M \).

For numerical calculations we shall apply rescaled variables. Defining \( X, P_R', P_\Phi \) and \( \tau \) by

\[
R = \frac{G M}{c^2} X, \quad P_R = \mu c P'_R, \quad P_\Phi = \frac{G M \mu}{c} P'_\Phi, \quad t = \frac{G M}{c^3} \tau
\]

we arrive at

\[
\frac{dX}{d\tau} = P'_R - \frac{8}{15} \frac{\nu}{X^2} \left[ 2P'^2_R + 6 \frac{P'^2_\Phi}{X^2} \right]
\]

\[
\frac{d\Phi}{d\tau} = \frac{P'_\Phi}{X^2} - \frac{8}{3} \frac{\nu}{X^3} P'_R P'_\Phi,
\]

\[
\frac{dP'_R}{d\tau} = -\frac{8 \nu P'_R}{5} \left[ \frac{2}{X} + \frac{2P'^2_\Phi}{X^2} - P'^2_R \right],
\]

\[
\frac{dP'_\Phi}{d\tau} = \frac{1}{X^2} + \frac{8 \nu}{3} \frac{P'_R}{X^4} \left[ \frac{1}{5} - \frac{P'^2_\Phi}{X} \right].
\]

### 3.2. Leading order gravitational waveforms

The leading order gravitational radiation field is given in Eq. (27). If we identify the orbital plane with the \( \Theta = \pi / 2 \) plane, \( Q^{(1)}_{13} \) and \( Q^{(1)}_{23} \) vanish and thus only \( I_{20}^{\text{orb}} \) and \( I_{22}^{\text{orb}} \) are nonzero. The time derivatives of \( Q^{(1)}_{ij} \) as functions of the canonical variables are given in the Appendix B. Inserting this in (35) and (33), respectively, we find

\[
\ddot{I}_{22}^{\text{orb}} = 4 \sqrt{\frac{2\pi}{5}} c^{-2i\Phi} \left[ \frac{P_R'^2}{\mu} - \frac{P'^2_\Phi}{\mu R^2} - \frac{G M \mu}{R} - 2i \frac{P_R P_\Phi}{\mu R} \right]
\]

\[
= 4 \sqrt{\frac{2\pi}{5}} c^{-2i\Phi} \mu c^2 \left[ \frac{P'^2_R}{X^2} - \frac{P'^2_\Phi}{X^2} - \frac{1}{X} - 2i \frac{P'_R P'_\Phi}{X} \right]
\]

or, if split into real and imaginary parts,

\[
\Re(\ddot{I}_{22}^{\text{orb}}) = 4 \sqrt{\frac{2\pi}{5}} \mu c^2 \left[ \cos(2\Phi) \left\{ P'^2_R - \frac{P'^2_\Phi}{X^2} - \frac{1}{X} \right\} - 2 \sin(2\Phi) \frac{P'_R P'_\Phi}{X} \right]
\]

\[
\Im(\ddot{I}_{22}^{\text{orb}}) = -4 \sqrt{\frac{2\pi}{5}} \mu c^2 \left[ \sin(2\Phi) \left\{ P'^2_R - \frac{P'^2_\Phi}{X^2} - \frac{1}{X} \right\} + 2 \cos(2\Phi) \frac{P'_R P'_\Phi}{X} \right],
\]

and

\[
\ddot{I}_{20}^{\text{orb}} = -8 \sqrt{\frac{\pi}{15}} \mu c^2 \left[ P'^2_R + \frac{P'^2_\Phi}{X^2} - \frac{1}{X} \right].
\]

### 3.3. Gravitational waveforms – analytical results

The contributions to the gravitational waveform in Eqs. (47) and (50) are given in terms of \( P'_R, P'_\Phi, \Phi \) and \( X \). These expressions are applied when solving the dynamics fully numerically. It is instructive, however, to compare those results with semi-analytical ones where the conservative dynamics is solved fully analytically and where the gravitational radiation damping is treated as a secular effect. If the two components are well separated, the curves turn out to be in good agreement. At distances of around \( 10 \frac{G M}{c^2} \), \( \ddot{I}_{22}^{\text{orb}} \) as calculated by the secular approach is in a good agreement to the one calculated by full
numerical simulation over a few periods only.
The parametrized solution to the Newtonian problem reads
\[ X = a'(1 - e \cos u), \quad \frac{2\pi}{P} t \equiv n \cdot t = u - e \cos u. \]
Here \( u \) is the mean anomaly, the period \( P \) is given by
\[ n = \frac{2\pi}{P} = \sqrt{\frac{GM}{a^3}}, \quad (51) \]
and the azimuthal angle \( \Phi \) is related to \( u \) by
\[ \Phi = 2 \arctan \left[ \sqrt{1 + e^2} \tan \frac{u}{2} \right]. \quad (52) \]
From Eq. (52) we readily obtain
\[ \dot{\Phi} = \frac{\sqrt{1 - e^2}}{1 - e \cos u} \frac{\dot{u}}{\sqrt{(1 - e \cos u)^2}}. \quad (53) \]
The Newtonian energy of the orbit is \( E = \mu c^2 E' \) with \( E' = -\frac{1}{2\alpha} \). Using this relation, it is now easy to derive from the eqs. (33) and (35) the leading order contributions
\[ \ddot{I}^{22} = 8 \frac{2\pi}{5} e^{-2i\Phi} \mu c^2 E' \left[ 1 - \frac{1}{A(u)} + \frac{2(1 - e^2)}{A(u)^2} + 2i e \sqrt{1 - e^2} \sin u \right], \quad (54) \]
\[ \ddot{I}^{20} = -16 \frac{\pi}{15} \mu c^2 E' \left[ 1 - \frac{1}{A(u)} \right], \quad (55) \]
where \( A(u) = 1 - e \cos u \). In particular, \( \ddot{I}^{20} \) vanishes identically for circular orbits. The luminosity of the gravitational wave emission in leading order approximation is given by
\[ \mathcal{L} = \frac{G}{5c^5} \frac{\partial^3 Q_{ij}}{dt^3} \frac{\partial^3 Q_{ij}}{dt^3}. \]
Averaging \( \mathcal{L} \) over the orbital period gives the leading order energy loss
\[ \left\langle \frac{dE}{dt} \right\rangle = -\frac{1}{P} \int_0^P \mathcal{L} \, dt = -32 \frac{G^4 M^3 \mu^2}{5c^5 a^5 (1 - e^2)^{7/2}} \left[ 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right]. \]
while the decay of the orbital eccentricity is given by
\[ \left\langle \frac{de}{dt} \right\rangle = -\frac{1}{15} \frac{\nu}{GMc^5} \left( \frac{GM}{a} \right)^4 \frac{e}{(1 - e^2)^{5/2}} (304 + 121 e^2). \]

4. The binary system

In previous chapters we have considered the Newtonian oscillating rotating disk and the orbital motion separately. In the following we shall focus on a binary system composed of a point-like object and a dusty disk. Due to tidal interaction neither the internal dynamics of the disk nor the orbital motion are completely independent. In fact, the influence of tidal coupling on the form of the orbit and on the inspiral process can be rather strong, as we will see later on. For simplicity we assume the orbital plane to be the \( z = 0 \) plane, i.e. the plane of the disk.
4.1. Tidal coupling on the Newtonian level

The Newtonian interaction potential for a general matter distribution reads

\[ U = -\frac{G}{2} \int \int \frac{\rho(r) \rho(\hat{r})}{|r - \hat{r}|} d^3r d^3\hat{r} \]

\[ = -\frac{G}{2} \int \int \frac{(\rho_p(r) + \rho_d(r))(\rho_p(\hat{r}) + \rho_d(\hat{r}))}{|r - \hat{r}|} d^3r d^3\hat{r}. \]  

(56)

Not taking into account self-interaction terms, we are left with

\[ U_{\text{int}} = U_{\text{int}}^{(1,2)} + U_{\text{int}}^{(2,1)} = -G \int \frac{\rho_d(r) \rho_p(\hat{r})}{|r - \hat{r}|} d^3r d^3\hat{r}. \]  

(57)

One of the two volume integrals is trivial since the point mass density is given by \( \rho_p(\hat{r}) = M_p \delta(\hat{r} - R) \). The remaining integral reads

\[ U_{\text{int}} = -GM_p \int \frac{\rho_d(r) \rho_p(\hat{r})}{|r - \hat{r}|} d^3r. \]

(58)

Inserting the density of the disk \( \rho_d = \delta(z)\Sigma(r,t) \), where the surface density \( \Sigma \) is defined by Eq. (3) and where the origin of the coordinate system has been shifted to the center of the disk, we end up with

\[ U_{\text{int}} = -GM_p \sigma(t) \int_0^{2\pi} d\phi \int_0^{r_d} dr r \frac{\sqrt{1 - \frac{r^2}{r_d^2}}}{\sqrt{r^2 + R^2 - 2rR \cos \phi}} \int_0^{r_d} \frac{\sqrt{1 - \frac{r^2}{r_d^2}}}{\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right) \cos \phi}} \int_0^{r_d} \frac{\sqrt{1 - \frac{r^2}{r_d^2}}}{\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right) \cos \phi}} dx. \]

(61)

Unfortunately, this integral seems not to be solvable analytically for \( R > r_d \) (in case of \( R < r_d \), see e.g. Kley and Schäfer [3]). Expanding the integrand in terms of Legendre polynomials, we obtain

\[ J = \int_0^{2\pi} d\phi \int_0^{r_d} dr r \frac{\sqrt{1 - \frac{r^2}{r_d^2}}}{\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right) \cos \phi}} \int_0^{r_d} \frac{\sqrt{1 - \frac{r^2}{r_d^2}}}{\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right) \cos \phi}} dx. \]

(60)

Defining

\[ Q_l := \int_0^1 x^{l+1} \sqrt{1 - x^2} dx = \frac{\sqrt{\pi} \Gamma\left(1 + \frac{l}{2}\right)}{4 \Gamma\left(\frac{3}{2} + \frac{l}{2}\right)}, \]

we end up with

\[ J = r_d^2 \sum_{l=0}^{\infty} \left(\frac{r_d}{R}\right)^l Q_l \int_0^{2\pi} P_l(\cos \phi') d\phi'. \]
The integral \( \int_0^{2\pi} P_l(\cos \varphi) d\varphi \) is only nontrivial for even numbers of \( l \). With the substitutions \( y = \cos \varphi, \ d\varphi = -\frac{dy}{\sqrt{1-y^2}} \) the case \( l = 2n \) can be rewritten as

\[
\int_0^{2\pi} P_{2n}(\cos \varphi) d\varphi = 2 \int_{-1}^{1} \frac{P_{2n}(y)}{\sqrt{1-y^2}} dy.
\]

This is a special case of the more general relation \[16\]

\[
\int_{-1}^{1} (1-y^2)^{\lambda-1} P_\mu(y) dy = \frac{2^{\mu} \pi \Gamma(\lambda + \frac{\mu}{2}) \Gamma(\lambda - \frac{\mu}{2})}{\Gamma(\lambda + \frac{\mu+1}{2}) \Gamma(\lambda - \frac{\mu}{2}) \Gamma(1 + \frac{\mu}{2}) \Gamma(\frac{1-\mu}{2})},
\]

which holds for \( \Re \lambda > \frac{1}{2} |\Re \mu| \). Taking \( \lambda = \frac{1}{2}, \ \mu = 0, \ \nu = 2n \) we arrive at

\[
\int_{-1}^{1} \frac{P_{2n}(y)}{\sqrt{1-y^2}} dy = \frac{\pi \Gamma(\frac{1}{2})^2}{\Gamma(1+n)^2 \Gamma(\frac{1}{2} - n)^2} = \left( \frac{\pi}{n! \Gamma(\frac{1}{2} - n)} \right)^2,
\]

and thus

\[
\int_0^{2\pi} P_{2n}(\cos \varphi) d\varphi = 2 \left( \frac{\pi}{n! \Gamma(\frac{1}{2} - n)} \right)^2.
\]

Inserting this into Eq. \[61\] the tidal potential Eq. \[59\] reads

\[
U_{int} = -\frac{GM_p}{2R} \sigma(t) r_d^2 \pi \sum_{l=0}^{\infty} \left( \frac{r_d}{R} \right)^{2l} W_l = -\frac{3 GM_p M_d}{4} \sum_{l=0}^{\infty} \left( \frac{r_d}{R} \right)^l W_l
\]

\[
= U_{orb} + U_{tidal},
\]

where

\[
W_l := \left( \frac{\pi^{3/2}}{(l)! \Gamma(\frac{1-l}{2})^2 \Gamma(\frac{5+l}{2})} \right), \quad l \text{ even,}
\]

and \( W_l = 0 \) for odd values of \( l \). It is easy to see that the \( l = 0 \) term in Eq. \[62\] gives just the point-mass interaction,

\[
U_0 = -\frac{3 GM_p M_d}{4} \frac{\pi^{3/2}}{\Gamma(\frac{1}{2})^2 \Gamma(\frac{5}{2})} = -\frac{GM_p M_d}{R} \equiv U_{orb},
\]

where we used \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \) and \( \Gamma(\frac{5}{2}) = \frac{3}{4} \sqrt{\pi} \). The tidal interaction potential can be immediately read off from Eq. \[62\],

\[
U_{tidal} = -\frac{3 GM_p M_d}{4} \sum_{l=2}^{\infty} \left( \frac{r_d}{R} \right)^l W_l.
\]

As expected, the first nonvanishing contribution comes from the quadrupole term \[17\]. Correspondingly, we expect the tidal interaction to introduce a periastron shift into the Newtonian orbit (see e.g. \[18\]).
4.2. Radiation reaction Hamiltonian and Hamiltonian equations of motion

Though it is tempting to take the leading order radiation reaction Hamiltonian of the binary system as the superposition of the radiation reaction Hamiltonians of the isolated disk and pure orbital motion, respectively, a careful analysis shows that coupling between orbital and disk quadrupole moment already enters at leading order. This can be easily seen from the general 2.5 pN reaction Hamiltonian of a fluid configuration \[12\],

\[
H_{\text{reac}} = \frac{2G}{5c^5} \tilde{Q}_{ij} (t) \int d^3r \left( \rho v_i v_j + \frac{1}{4\pi G} \partial_i U \partial_j U \right) = \frac{G}{5c^5} \left( \tilde{Q}^{(o)}_{ij} (t) + \tilde{Q}^{(d)}_{ij} (t) \right) \left( \tilde{Q}^{(o)}_{ij} (R^i, P_i) + \tilde{Q}^{(d)}_{ij} (x^i, p_i) \right) \quad (66)
\]

Here \( \tilde{Q}^{(d)}_{ij} \) and \( \tilde{Q}^{(o)}_{ij} \) are the STF mass quadrupole moments of the disk and the orbit, respectively, \( R^i, P_i \) denote the canonical conjugate variables of the orbital motion and \( x^i, p_i \) are the canonical conjugate variables of the disk relative to the disk’s center. With \( H_{\text{reac}}^{(d)} \) and \( H_{\text{reac}}^{(N)} \) given by the eqs. (20) and (38) the leading order reaction Hamiltonian of the binary system is

\[
H_{\text{reac}} (t) = H_{\text{reac}}^{(d)} (t) + H_{\text{reac}}^{(o)} (t) + \frac{G}{5c^5} \left[ \tilde{Q}^{(d)}_{ij} (t) \tilde{Q}^{(o)}_{ij} (R^i, P_i) + \tilde{Q}^{(o)}_{ij} (t) \tilde{Q}^{(d)}_{ij} (x^i, p_i) \right] \quad (67)
\]

The last two terms describe the coupling between disk and orbital quadrupole moments. Finally, using the diagonal structure of the disk’s mass quadrupole tensor we arrive at

\[
H_{\text{reac}} (t) = H_{\text{reac}}^{(d)} (t) + H_{\text{reac}}^{(o)} (t) + \frac{G}{5c^5} \left[ \tilde{Q}^{(d)}_{11} (R^i, P_i) + \tilde{Q}^{(o)}_{22} (R^i, P_i) - 2 \tilde{Q}^{(o)}_{33} (R^i, P_i) \right] \tilde{Q}^{(d)}_{11} (x^i, p_i) + \frac{G}{5c^5} \left[ \tilde{Q}^{(o)}_{11} (t) \tilde{Q}^{(d)}_{11} (x^i, p_i) + \tilde{Q}^{(d)}_{11} (t) \tilde{Q}^{(o)}_{33} (R^i, P_i) \right] \quad (68)
\]

As before, the radiation reaction parts of the Hamiltonian equations of motion are obtained by differentiating \( H_{\text{reac}} \) with respect to the generalized coordinates and momenta. Thus the reaction part of the equations of motion reads,

\[
(r_d)_{\text{reac}} = -\frac{8}{15} \frac{Gc^2 p_r^2}{\alpha c^2 r_d^2} - \frac{4}{15} \frac{G^2 M p_r P_R}{R^2} \quad (69)
\]

\[
(\dot{r})_{\text{reac}} = -\frac{8}{15} \frac{Gc^2 p_r p_{\dot{r}}}{\alpha c^2 r_d^2} - \frac{4}{15} \frac{G^2 M p_r P_R}{R^2} \quad (70)
\]

\[
(\dot{p}_r)_{\text{reac}} = \frac{8}{15} \frac{Gc^2 p_r}{\alpha c^2 r_d^2} \left[ -\frac{p_r^2}{\alpha r_d} + C\alpha \right] + \frac{4}{15} \frac{G^2 M P_R}{c^5 R^2} \left[ \frac{C\alpha}{r_d^2} - \frac{p_r^2}{\alpha r_d^2} \right] \quad (71)
\]

\[
(\dot{p}_{\dot{r}})_{\text{reac}} = 0 \quad (72)
\]

\[
(\ddot{R})_{\text{reac}} = -\frac{16}{5} \frac{G^2 M}{\mu c^5 R^2} \left[ \frac{P_R^2}{R^2} + \frac{1}{3} \frac{P_R}{R^2} \right] - \frac{8}{15} \frac{Gc^2 p_r P_R}{\alpha c^2 r_d^2} \quad (73)
\]

\[
(\ddot{\Phi})_{\text{reac}} = -\frac{8}{3} \frac{G^2 M P_R P_\Phi}{R^4} - \frac{8}{15} \frac{Gc^2 p_r P_\Phi}{\alpha c^2 R^2 r_d^2} \quad (74)
\]

\[
(\dot{P}_R)_{\text{reac}} = -\frac{8}{3} \frac{G^2 M P_R}{c^5 R^4} \left[ \frac{P_R^2}{\mu R} + \frac{1}{5} \frac{G M \mu}{R} \right] + \frac{4}{15} \frac{Gc^2 p_r}{c^5 R^2} \left[ GM \mu - \frac{2 P_R^2}{\mu R} \right] \quad (75)
\]

\[
(\dot{P}_\Phi)_{\text{reac}} = \frac{8}{5} \frac{G^2 M P_\Phi}{c^5 R^3} \left[ \frac{P_R^2}{\mu R} - \frac{2 G M \mu}{R} - 2 \frac{P_R^2}{\mu R} \right] \quad (76)
\]
Note that the coupling of disk and orbital mass quadrupole moments is already present at leading order. However, numerically these terms turn out to be very small even in close orbits and thus can be neglected in most scenarios.

4.3. Hamiltonian equations of motion

The total Hamiltonian of the binary takes the form

\[ H_{\text{tot}} = H_{\text{orb}}^{(N)} + H_{d}^{(N)} + H_{\text{tidal}} + H_{\text{reac}}, \]  

(77)

where

\[ H_{\text{orb}}^{(N)} = \frac{1}{2\mu} \left( \frac{P_R^2}{R^2} + \frac{P_\Phi^2}{R^2} \right) - \frac{GM\mu}{R}, \]  

(78)

\[ H_{d}^{(N)} = \frac{1}{2\alpha} \left( \frac{p_r^2}{r_d^2} + \frac{p_\varphi^2}{r_d^2} \right) - \frac{2C\alpha}{r_d}, \]  

(79)

\[ H_{\text{tidal}} = -\frac{3GM\mu}{4R} \sum_{l=2}^{\infty} \left( \frac{r_d}{R} \right)^l W_l \]  

(80)

and where \( H_{\text{reac}} \) is given by Eq. (68). The radiation reactive part of the equations of motion has been calculated in the previous section and is given by the Eqs. (69) - (76).

The dynamics of the binary, including tidal coupling and leading order gravitational radiation reaction effects, is described by the following set of first order differential equations:

\[ \dot{P_R} = \frac{P_\Phi^2}{\mu R^3} - \frac{GM\mu}{R^2} - \frac{3GM\mu}{4R^2} \sum_{l=2}^{\infty} (l+1) \left( \frac{r_d}{R} \right)^l W_l - \frac{8G^2M}{3} \frac{P_R P_\Phi^2}{\mu c^5} \left[ \frac{p_R}{\mu R} \right] \]  

(81)

\[ \dot{P_\Phi} = \frac{8G^2M P_\Phi}{5} \frac{\left[ \frac{P_R^2}{\mu^2} + \frac{2GM\mu}{R} - \frac{2P_\Phi^2}{\mu R^2} \right]}{c^5 R^3} \]  

(82)

\[ \dot{R} = \frac{P_R}{\mu} - \frac{1}{5} \frac{G^2M}{\mu c^5} \left[ \frac{P_\Phi^2}{R^4} + \frac{1}{3} \frac{P_R^2}{R^2} \right] - \frac{8GC p_r P_R}{15 \mu c^5} \left( \frac{r_d}{R} \right)^2 \]  

(83)

\[ \dot{\Phi} = \frac{P_\Phi}{\mu R^2} - \frac{8G^2M}{3} \frac{P_R P_\Phi}{R^4} - \frac{8GC p_r P_\Phi}{15 \mu c^5} \left( \frac{r_d^2}{R^2} \right) \]  

(84)

\[ \dot{r_d} = \frac{p_r}{\alpha} - \frac{8GC p_r^2}{15 \alpha c^5} \frac{r_d^2}{R^2} - \frac{4G^2M}{15 \alpha c^5} \frac{p_r P_R}{R^2} \]  

(85)

\[ \dot{\varphi} = \frac{p_\varphi}{\alpha r_d^2} - \frac{8GC p_r p_\varphi}{15 \alpha c^5} \frac{r_d^2}{R^2} - \frac{4G^2M}{15 \alpha c^5} \frac{p_r P_R}{r_d^2 R^2} \]  

(86)

\[ \dot{p_r} = \frac{p_r^2}{\alpha r_d^2} - \frac{2C\alpha}{r_d^2} + \frac{3GM\mu}{4R^2} \sum_{l=2}^{\infty} \left( \frac{r_d}{R} \right)^l W_l + \frac{8GC p_r}{15 \mu c^5} \left[ C\alpha - \frac{2p_\varphi^2}{\alpha r_d} \right] \]  

(87)

\[ \dot{p_\varphi} = 0. \]  

(88)
In order to solve these equations numerically we shall introduce scaled variables. We choose the same scaling as in Eq. (42) for the disk variables

\[ p_r = \mu c p_r', \quad p_\varphi = \frac{G M \mu}{c} p_\varphi', \quad r_d = \frac{G M}{c^2 x_d}. \tag{89} \]

Substituting this into the Eqs. (81 - 88) and defining

\[ A := \frac{C}{G M} = \frac{3 \pi M_d}{8 \mathcal{M}}, \quad B := \frac{\alpha}{\mu} = \frac{2 \mathcal{M}}{5 M_p} \]

we finally arrive at

\[
\begin{align*}
\frac{dX}{d\tau} &= P_R' - \frac{16}{5} \nu X \left( \frac{P_\varphi'^2}{X^2} + \frac{1}{3} P_R'^2 \right) - \frac{8}{15} \frac{A \nu}{x_d^2} P_\varphi' P_R' \\
\frac{d\Phi}{d\tau} &= \frac{P_\varphi'}{X^2} - \frac{8}{3} \nu X \left[ P_R'^2 + \frac{A p_\varphi'}{x_d^2} \right] \\
\frac{dP_\varphi'}{d\tau} &= \frac{8}{5} \nu X^3 \left[ P_R'^2 - \frac{2}{X} - 2 \frac{P_\varphi'^2}{X^2} \right] \\
\frac{dP_R'}{d\tau} &= \frac{P_\varphi'^2}{X^2} - \frac{1}{X^2} - \frac{3}{4 X^2} \sum_{l=2}^{\infty} \left( l + 1 \right) \left( \frac{x_d}{X} \right)^l W_l - \frac{8}{3} \nu X P_R' P_\varphi' + \frac{8}{15} \frac{\nu}{X^4} P_R' \\
&\quad + \frac{4}{15} \frac{A \nu}{X^2 x_d^2} \left[ 1 - 2 \frac{P_\varphi'^2}{X^2} \right] p_r' \\
\frac{dp_r'}{d\tau} &= \frac{1}{B} \frac{p_\varphi'^2}{x_d^3} - 2B \frac{A}{x_d^2} + \frac{3}{4} \frac{1}{X^2} \sum_{l=2}^{\infty} \left( \frac{x_d}{X} \right)^{l-1} W_l + \frac{8}{15} \frac{\nu}{x_d^4} \left[ BA - \frac{1}{B} \frac{p_\varphi'^2}{x_d} \right] p_r' \\
&\quad + \frac{4}{15} \frac{\nu P_R'}{X^2 x_d^2} \left[ AB - \frac{1}{B} \frac{P_\varphi'^2}{x_d} \right] \\
\frac{d\varphi}{d\tau} &= \frac{1}{B} \left[ \frac{p_\varphi'}{x_d} - \frac{8}{15} \frac{A \nu}{x_d^2} p_\varphi' p_\varphi' - \frac{4}{15} \frac{\nu}{X^2 x_d^2} p_\varphi' P_R' \right] \\
\frac{dx_d}{d\tau} &= \frac{1}{B} \left[ p_r' - \frac{8}{15} A \nu \frac{p_\varphi'^2}{x_d^2} - \frac{4}{15} \frac{\nu}{X^2 x_d^2} p_\varphi' P_R' \right] \\
\frac{dp_\varphi'}{d\tau} &= 0. \tag{97} \end{align*}
\]

4.3.1. Initial conditions

Both, the dynamics of the binary as well as the leading order gravitational waveform depend on the choice of initial conditions. Various initial conditions correspond to various initial phase differences between orbital and disk angular variables. Let \( t = 0 \) be the time of the periastron passage where by convention \( \Phi(0) = 0 \). The initial values for the orbital variables shall read

\[ X(0) = a'(1 - e), \quad \Phi(0) = 0, \quad P_\varphi'(0) = \sqrt{a'(1 - e^2)}, \quad P_R' = 0. \tag{98} \]

To fix the initial conditions for the disk variables, we require

\[
H_d'(0) = \frac{1}{2B} \left( p_r'(0)^2 + \frac{p_\varphi'(0)^2}{x_d(0)^2} \right) - \frac{2AB}{x_d(0)} \equiv E_d'(0) = - \frac{AB}{X_d(1 + \epsilon)}, \tag{99} \]
where \( \epsilon \) is the ellipticity of the disk at \( t = 0 \) and \( X_d \) is the maximal disk radius. Note that if \( x_d(0) = X_d \) the momentum conjugate \( p'_r(0) \) vanishes. The momentum conjugate to \( \varphi \) is a constant as can be seen from the equations of motion. It is given by

\[
\begin{align*}
\dot{p}_\varphi^2 &= 2 \alpha^2 C R_d \xi^2 = \frac{G^2 M^2 \mu^2}{c^2} p_\varphi^2 \quad \Longrightarrow \quad p'_\varphi(0) = B \sqrt{2AX_d(1 - \epsilon)}.
\end{align*}
\]

The initial value of \( p'_r \) for a given \( x_d(0) \) can be easily derived from Eq. (99),

\[
\begin{align*}
&x_d(0) = x_{d,0}, \quad \varphi(0) = 0, \quad p'_\varphi(0) = B \sqrt{2AX_d(1 - \epsilon)}, \\
p'_r(0) &= 2B^2 A \left[ -\frac{1 + \epsilon}{X_d} + \frac{(\epsilon - 1) X_d}{x_{d,0}^2} + 2 \frac{x_{d,0}^2}{x_{d,0}} \right].
\end{align*}
\]

5. Discussion

In the last decade several authors investigated tidal and tidal-resonant effects in binary systems. Close binary neutron stars were studied by e.g., Schäfer and Kokkotas [15] and Lai and Ho [8]. Recently, resonant excitations of white dwarf oscillations in a compact binary were investigated by Rathore et al. [19]. In these papers the stellar oscillations were treated numerically. In our model there is only one oscillation mode, but it is known analytically. Although our model is much simpler than the binary neutron stars of [15], [8] or the white dwarf-compact object binary of [19], it exhibits all essential relativistic features of a compact binary system to leading order approximation. Moreover, we were able to give the Hamiltonian equations of motion in analytic form.

In their recent paper Rathore et al. investigated a white dwarf-compact object system with non-rotating white dwarf. Gravitational damping was taken into account as a secular effect and the change of the orbital motion due to tidal interaction was discussed only qualitatively. The symmetry of our disk-compact object binary enabled us to study the effects, in particular the influence of tidal interaction on the actual orbit, in great detail (see Fig. 2). Note that while the white dwarf of Rathore et al. was nonrotating, rotation is essential in our case to stabilize the dusty disk. The single eigenmode of the oscillating disk corresponds to the f-mode of Rathore et al.

In our paper we studied the oscillations of an isolated rotating Newtonian disk of dust in Hamiltonian formalism. Thin oscillating disks of dust with a diameter of a few hundred kilometers could eventually form after the merging of two white dwarfs. The similarity of the disk’s equation of motion to the well-known Keplerian problem is remarkable. Exploiting this analogy one can easily write down the analytic solution for the disk’s oscillations. This allows to study the effects of tidal interaction and its influence on the actual gravitational waveform in great detail (see Figs. 3, 4, 5, 6, 7, 8). For a suitable choice of parameters (i.e. the initial major semi-axis not being too small) the evolution of the system can be followed over hundreds or even thousands of orbital periods with considerable accuracy (see e.g. Fig. 5). Our model might thus be a good test to run data analysis codes that intend to extract physical parameters of the binary system from a given gravitational waveform.

A particular feature of the tidally coupled binary system is the so-called tidal resonance (see Fig. 8). In Newtonian theory the tidal coupling between the internal dynamics of a star and the orbital dynamics becomes particularly strong if one of the star’s eigenfrequencies is \( n \) times the orbital frequency. Tidal interaction and in particular tidal resonances may speed up the inspiral process by a considerable amount (see Figs. 2 and 8). In fact, for a realistic choice of parameters (i.e. of \( M_d \) and \( M_p \)) the 2:1 resonance
leads to an immediate destruction of the disk-compact object binary. At this point, the analysis breaks down. Higher order resonances, starting with the 3:1 one, can be treated without problems for a suitable choice of parameters.

Some remarks should be made on the validity of our model. It relies on the assumption that the disk is not getting disrupted and that the change in the disk’s density does result from the oscillation of the disk only. In particular we do not take into account mass accretion on the compact companion. Thus, even for tidal resonance, the maximal radius of the disk has to be located far inside it’s Roche lobe. In other words, the maximal disk radius must be considerably smaller than the periastron distance between the two objects.

In a forthcoming paper we will extend our analysis to a system of two oscillating disks. In this case there will occur quadrupole-quadrupole coupling between the disk’s mass quadrupole moments. These terms are of course small but they might become important when extending the analysis to include first order post-Newtonian corrections. By investigating post-Newtonian corrections we hope to get a better intuition for a semi-analytical treatment of the much more complicated analysis of inspiraling NS-NS binaries.

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**A. The quadrupole tensor of the disk and it’s derivatives**

To calculate the quadrupole tensor of the disk, we start with the general definition

\[
Q_{ij} = \int d^3x \rho(x^ix^j - \frac{1}{3}\delta^{ij}r^2).
\]

Since \(\rho_d\), the density of the disk, is confined to the \(z = 0\) plane and depends on \(r\), the radial coordinate, only, the nondiagonal elements of \(Q^{(d)}_{ij}\) vanish. It is now easy to see that

\[
Q^{(d)}_{11} = \int r dr d\varphi \rho(r) r^2 (2\cos^2 \varphi - \sin^2 \varphi) = \frac{M_d}{15} r_d^2 = \frac{\alpha}{6} r_d^2.
\]  

Due to the symmetry it is clear that the relations

\[
Q^{(d)}_{11} = Q^{(d)}_{22}, \quad Q^{(d)}_{33} = -2Q^{(d)}_{11}
\]

hold. The second and third time derivative of \(Q^{(d)}_{11}\) can be calculated using the equation of motion Eq. (8). With

\[
\dot{r}_d^2 = \frac{2E_d}{\alpha} + \frac{4C}{r_d} - \frac{\dot{h}^2}{r_d^2},
\]

\[
\ddot{r}_d = \frac{\dot{h}^2}{r_d^3} - \frac{2C}{r_d^2} = \frac{\dot{p}_\varphi^2}{\alpha^2 r_d^3} - \frac{2C}{r_d^2}.
\]
we obtain
\[ \ddot{Q}_{11}^{(d)} = \frac{\alpha}{3} (r_d^2 + r_d^2) = \frac{2\alpha}{3} \left( \frac{E_d}{\alpha} + \frac{C}{r_d} \right) \]
\[ \ddot{Q}_{11} = \frac{\alpha}{3} \left( \frac{p_r^2}{\alpha^2} + \frac{p_\phi^2}{\alpha^2 r_d^2} - \frac{2C}{r_d} \right) = \frac{1}{3} \left( \frac{1}{\alpha} \left( \frac{p_r^2 + p_\phi^2}{\alpha^2 r_d^2} - \frac{2C}{r_d} \right) \right) \]
and
\[ \ddot{Q}_{11}^{(d)} = \frac{2}{3} \frac{C p_r}{r_d^2}. \] (106)

B. Time derivatives of the orbital quadrupole tensor

By definition, the \( z = 0 \) plane is the orbital plane. The \( Q_{13}^{orb} \) and \( Q_{23}^{orb} \) components of the orbital quadrupole tensor thus vanish and we are left with
\[ Q_{11}^{(o)} = \frac{\mu R^2}{6} [1 + 3 \cos(2\Phi)], \] (107)
\[ Q_{22}^{(o)} = \frac{\mu R^2}{6} [1 - 3 \cos(2\Phi)], \] (108)
\[ Q_{33}^{(o)} = -\frac{\mu R^2}{3}, \quad Q_{12}^{(o)} = \frac{\mu R^2}{2} \sin(2\Phi). \] (109)

The time derivatives are calculated using the (Newtonian) equations of motion,
\[ \dot{R}^2 = \frac{2E_{orb}}{\mu} + \frac{2GM}{R} - R^2 \dot{\Phi}^2, \quad \ddot{R} = \dot{R} \dot{\Phi} - \frac{GM}{R^2}, \quad \ddot{\Phi} = -\frac{2}{R} \dot{R} \dot{\Phi}. \]

One finds
\[ \ddot{Q}_{11}^{(o)} = \frac{1}{3} \left[ 2E^{(o)} + \frac{GM \mu}{R} \right] + \cos(2\Phi) \left[ 2E^{(o)} + \frac{GM \mu}{R} - 2\mu R^2 \dot{\Phi}^2 \right] - 2\mu R \dot{R} \dot{\Phi} \sin(2\Phi) \]
\[ = \frac{1}{3} \left[ \frac{p_r^2}{\mu} + \frac{p_\phi^2}{\mu R^2} - \frac{GM \mu}{R} \right] + \cos(2\Phi) \left[ \frac{p_r^2}{\mu} - \frac{p_\phi^2}{\mu R^2} - \frac{GM \mu}{R} \right] - \frac{2P_r P_\phi}{\mu R} \sin(2\Phi), \]
\[ \ddot{Q}_{22}^{(o)} = \frac{1}{3} \left[ 2E^{(o)} + \frac{GM \mu}{R} \right] - \cos(2\Phi) \left[ 2E^{(o)} + \frac{GM \mu}{R} - 2\mu R^2 \dot{\Phi}^2 \right] + 2\mu R \dot{R} \dot{\Phi} \sin(2\Phi) \]
\[ = \frac{1}{3} \left[ \frac{p_r^2}{\mu} + \frac{p_\phi^2}{\mu R^2} - \frac{GM \mu}{R} \right] - \cos(2\Phi) \left[ \frac{p_r^2}{\mu} - \frac{p_\phi^2}{\mu R^2} - \frac{GM \mu}{R} \right] + \frac{2P_r P_\phi}{\mu R} \sin(2\Phi), \]
\[ \ddot{Q}_{12}^{(o)} = \left[ 2E^{(o)} + \frac{GM \mu}{R} - 2\mu R^2 \dot{\Phi}^2 \right] \sin(2\Phi) + 2\mu R \dot{R} \dot{\Phi} \cos(2\Phi) \]
\[ = \left[ \frac{p_r^2}{\mu} - \frac{p_\phi^2}{\mu R^2} - \frac{GM \mu}{R} \right] \sin(2\Phi) + \frac{2P_r P_\phi}{\mu R} \cos(2\Phi) \]
\[ \ddot{Q}_{33}^{(o)} = -\frac{2}{3} \left[ 2E^{(o)} + \frac{GM \mu}{R} \right] \]
\[ = -\frac{2}{3} \left[ \frac{p_r^2}{\mu} + \frac{p_\phi^2}{\mu R^2} - \frac{GM \mu}{R} \right]. \]
\[ \ddot{Q}_{11}^{(o)} = -\frac{GMP_R}{3R^2} \left[ 1 + 3\cos(2\Phi) \right] + \frac{4GMP_\Phi}{R^3} \sin(2\Phi), \]
\[ \ddot{Q}_{22}^{(o)} = -\frac{GMP_R}{3R^2} \left[ 1 - 3\cos(2\Phi) \right] - \frac{4GMP_\Phi}{R^3} \sin(2\Phi), \]
\[ \ddot{Q}_{12}^{(o)} = -\frac{4GMP_\Phi}{R^3} \cos(2\Phi) - \frac{GMP_R}{R^2} \sin(2\Phi), \]
\[ \ddot{Q}_{33}^{(o)} = \frac{2GMP_P}{3R^2}. \]

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Figure 2: Tidal interaction and gravitational radiation effects are shown for a slightly elliptic ($e(0) = 3/10$) binary with $a'(0) = 19, x_d(0) = 6, M_d = M_p = 1/2, \epsilon = 1/10$. At $t = 0$ the disk to orbital frequency relation is $\omega_d/\omega_o = 7.05$. The Newtonian point-particle orbit is given in the upper left figure. Including tidal interaction leads to a periastron shift (upper right). The figures on the bottom are calculated including leading order gravitational radiation effects in the equations of motion. Tidal interaction speeds up the inspiralling process as can be seen by comparing the orbit including (right) and excluding (left) tidal interaction.
Figure 3: The influence of the tidal interaction on the $I_{22}^{\text{total}}$ component of the binary’s gravitational radiation field is shown. Real and imaginary parts of $I_{22}^{\text{total}}$ are given without (left) and including (right) tidal interaction. In order to make the influence of tidal interaction on the gravitational waveform more visible, the orbital separation is taken to be small ($M_d = M_p = 1/2, e(0) = 0, a'(0) = 10, x_d(0) = X_d = 3, \epsilon = 1/10$). The time is measured in units of $T_0$, the initial orbital period.

Figure 4: $I_{20}^{d}$ (solid line) and $I_{20}^{\text{orb}}$ (dotted line) for the tidally coupled system (left), and the $I_{20}^{\text{total}}$ component of the binary’s leading order gravitational waveform (right). The parameters are the same as in Fig. 3. Although the unperturbed orbit is assumed to be circular, $I_{20}^{\text{orb}}$ is not zero due to tidal interaction. $I_{20}^{\text{total}}$ is dominated by the disk’s contribution. The time is measured in units of $T_0$, the initial orbital period.
Figure 5: The \( \dddot{I}^{20} \) component of the leading order gravitational radiation field of the binary for \( a'(0) = 15 \), all other parameters are the same as in figures (3), (4). On the left, the contributions of the disk is given, the right figure shows \( \dddot{I}^{20}_{\text{total}} \), which is dominated by the disk’s contribution. The time is measured in units of the initial orbital period \( T_0 \).

Figure 6: Real (solid line) and imaginary (dotted line) part of \( \dddot{I}^{22}_{\text{total}} \) for an elliptic orbit with \( e(0) = 3/10, a'(0) = 25, x_d(0) = X_d = 6, \epsilon = 1/10, M_d = M_p = 1/2 \). Time is measured in fraction of the initial orbital period \( T_0 \).
Figure 7: $\tilde{I}^{20}$ for an elliptic orbit ($e(0) = 0.3$). In the left graph orbital (dotted line) and disk (solid line) contributions are given separately, the right graph shows the $\tilde{I}^{20}_{\text{total}}$ part of the binary’s leading order gravitational waveform. The time is given in fraction of the initial orbital period $T_0$. 
Figure 8: The $\ddot{I}_{20}^\text{disk}$ contribution to the leading order gravitational wave radiation is shown for a binary with $a(0) = 100, e(0) = 0.3, \epsilon(0) = 0.1, x_d(0) = X_d = 40, M_p = M_d = 1/2$. The system starts at $\omega_d/\omega_0 = 4.95$. The upper left figure shows $\ddot{I}_{20}^\text{disk}$, while $\ddot{I}_{20}^\text{orb}$ is given in the upper right figure. The total $\ddot{I}_{20}^\text{total}$ contribution is shown in the last figure. The time is measured in units of the initial orbital period $T_0$. 