ON THE HOMOTOPY TYPE OF THE SPACES OF SPHERICAL KNOTS IN $\mathbb{R}^n$

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Abstract. We study the spaces of embeddings $S^m \hookrightarrow \mathbb{R}^n$ and those of long embeddings $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$, i.e. embeddings of a fixed behavior outside a compact set, assuming the codimension $n - m \geq 3$. More precisely we look at the homotopy fiber of the inclusion of these spaces to the spaces of immersions. We find a natural fiber sequence relating these spaces. We also compare the $L_\infty$-algebras of diagrams that encode their rational homotopy type.

1. Introduction

In this paper we study a relation between the following two spaces:

(1) $\overline{\text{Emb}}(S^m, \mathbb{R}^n) := \text{hofiber} (\text{Emb}(S^m, \mathbb{R}^n) \to \text{Imm}(S^m, \mathbb{R}^n))$;

(2) $\overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) := \text{hofiber} (\text{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \to \text{Imm}_\partial(\mathbb{R}^m, \mathbb{R}^n))$,

where $\text{Emb}(\cdot, \cdot)$ and $\text{Imm}(\cdot, \cdot)$ always refer to spaces of smooth embeddings and immersions, respectively. The homotopy fiber is taken over the inclusions $i_1 : S^m \subset \mathbb{R}^{m+1} \times 0^{n-m-1} \subset \mathbb{R}^n$ and $i_2 : \mathbb{R}^m = \mathbb{R}^m \times 0^{n-m} \subset \mathbb{R}^n$. The subscript $\partial$ means that the embeddings and immersions must coincide with the inclusion $i_2 : \mathbb{R}^m \subset \mathbb{R}^n$ outside a compact subset of $\mathbb{R}^m$. The spaces (1) and (2) are called spaces of embeddings modulo immersions.

The spaces $\overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$, $n - m \geq 3$, have been an object of active study [1–9, 15, 16]. They were shown to be $E_{m+1}$-algebras [4, 6, 15] equivalent to $(m+1)$-loop spaces [3, 8, 15].

To compare their homotopy type to that of $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$ let us begin with a few observations. Given an embedding $\psi \in \overline{\text{Emb}}(S^m, \mathbb{R}^n)$, we can define an inclusion

(3) $\overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \hookrightarrow \overline{\text{Emb}}(S^m, \mathbb{R}^n)$.
The idea of this map is to perturb $\psi$ near some point $p \in S^m$. By standard fibration and transversality arguments it is easy to show that for $n - m \geq 3$, $\pi_0 \overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n) \simeq \pi_0 \overline{\text{Emb}}(S^m, \mathbb{R}^n)$ and $\pi_* \overline{\text{Imm}}(\mathbb{R}^m, \mathbb{R}^n) \simeq \pi_* \overline{\text{Imm}}(S^m, \mathbb{R}^n)$, $* \leq 1$. This implies that the inclusion (3) induces a bijection of the sets of connected components $\pi_0 \overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n) \simeq \pi_0 \overline{\text{Emb}}(S^m, \mathbb{R}^n)$. It is also not hard to show that the inclusion (3) can be enhanced to an $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$-action on $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$. Our main result now states that the homotopy quotient of $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$ by the action of $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$ is the sphere $S^{n-m-1}$, or equivalently that $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$ is homotopy equivalent to a principal $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$-bundle on the sphere.

**Theorem 1.1.** For $n - m \geq 3$, one has an equivalence

$$\overline{\text{Emb}}(S^m, \mathbb{R}^n) \simeq \text{hofiber} \left( S^{n-m-1} \to B \overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n) \right).$$

This in particular implies that all components of $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$ have the same homotopy type and that $\pi_0 \overline{\text{Emb}}(S^m, \mathbb{R}^n) = \pi_0 \overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$.

For an explicit definition of the classifying map $S^{n-m-1} \to B \overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$ appearing in this Theorem we refer to section 2.

As a consequence we will in particular be able to express the rational homotopy types of $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$ and $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$ through each other, see Corollaries 2.5, 2.6 below. It is furthermore well known that the rational homotopy type of $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$, $n - m \geq 3$, may be expressed through hairy graph-complexes. More precisely, in [2] hairy graph-complexes were introduced, denoted by $\text{HGC}_{\tilde{A}, m, n}$ in this paper, which were proved to compute the rational homotopy groups

$$H_*(\text{HGC}_{\tilde{A}, m, n}) \simeq \mathbb{Q} \otimes \pi_* \overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$$

for $n \geq 2m + 2$. The paper [9] determined the rational homotopy type of the $(m + 1)$-st delooping of $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$, $n - m \geq 3$. In particular [9, Theorem 15 and Remark 19] improved the equality (5) to the range $n - m \geq 3$. In that range, the space $\overline{\text{Emb}}(\mathbb{R}^m, \mathbb{R}^n)$ can be disconnected, but since it is an $(m + 1)$-loop space its set of connected components forms an abelian group (in fact finitely generated). The cited theorem proves the isomorphism (5) in degree zero as well. Note, however, that the graph-complex $\text{HGC}_{\tilde{A}, m, n}$ can have non-trivial homology in negative degrees, that has to be ignored.\footnote{In fact the non-positive degree homology $H_{\leq 0}(\text{HGC}_{\tilde{A}, m, n})$, that includes the negative degree and degree zero, is at most one-dimensional.}
Recently, in [10] a more general method has been developed by B. Fresse and the authors to study the rational homotopy type of (connected components of) embeddings modulo immersions spaces $\operatorname{Emb}(L, \mathbb{R}^n)$ and $\operatorname{Emb}_\partial(L, \mathbb{R}^n)$, where $L$ is either a compact submanifold of $\mathbb{R}^{m+1}$ with components of possibly different dimensions, or a closed submanifold whose unbounded connected components coincide with affine subspaces of $\mathbb{R}^{m+1}$ outside a ball of some radius $R$. The main result of [10] provides $L_\infty$-algebras of diagrams that express the rational type of such spaces. In particular, for the first non-trivial case of $L = S^m$ the corresponding $L_\infty$-algebra is a hairy graph-complex denoted by $\text{HGC}_{\text{Am}, n}$.

On the rational homotopy level the comparison of the embedding spaces $\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)$ and $\operatorname{Emb}(S^m, \mathbb{R}^n)$ pursued in this paper hence translates into a comparison of the graph complexes $\text{HGC}_{\text{Am}, n}$ and $\text{HGC}_{\text{Am}, n}$. We shall explain in section 3 how the relation between the spaces of Theorem 1.1 can be seen directly (and independently) on the graph complexes, at least rationally. In fact, this is how we initially discovered our Theorem 1.1. Computations from Section 3 could be useful in further pursuing the graph-complex approach from [10] applying it to other types of manifolds.

**Remark 1.2.** We also want to make the reader aware of related work of Budney and Cohen [7, section 4] that gives a connection between the embedding spaces $\operatorname{Emb}(S^m, \mathbb{R}^n)$ and the long embedding spaces $\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)$.

### 2. Spherical and long embeddings

In this section we describe how the homotopy type of $\overline{\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)}$ is compared to that of $\overline{\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)}$, and in particular prove Theorem 1.1.

#### 2.1. Proof of Theorem 1.1

By the Smale-Hirsch theorem, $\operatorname{Imm}_\partial(R^m, \mathbb{R}^n) \simeq \Omega^m V_m(\mathbb{R}^n)$, where $V_m(\mathbb{R}^n) = \text{SO}(n)/\text{SO}(n-m)$ is the Stiefel manifold of orthogonal $m$-frames in $\mathbb{R}^n$. Thus

$$\overline{\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)} \simeq \text{hofiber}\left(\operatorname{Emb}_\partial(R^m, \mathbb{R}^n) \xrightarrow{D} \Omega^m V_m(\mathbb{R}^n)\right).$$

The space $\overline{\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)}$ is an $(m + 1)$-loop space [3, 8, 15]. We denote by $B\overline{\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)}$ its classifying space and by $g$ the map

$$g : \Omega^m V_m(\mathbb{R}^n) \simeq B\Omega^{m+1} V_m(\mathbb{R}^n) \to B\overline{\operatorname{Emb}_\partial(R^m, \mathbb{R}^n)}$$

obtained by applying the classifying space functor $B$ to the inclusion $\Omega^{m+1} V_m(\mathbb{R}^n) \to \operatorname{Emb}_\partial(R^m, \mathbb{R}^n)$. 

Consider also the map
\[(8) \quad h : S^{n-m-1} \to \Omega^m V_m(\mathbb{R}^n)\]
adjoint to the composition
\[(9) \quad \Sigma^m S^{n-1} \xrightarrow{h_0} \text{SO}(n) \to \text{SO}(n)/\text{SO}(n-m) = V_m(\mathbb{R}^n),\]
where \(h_0\) is the transition map for the tangent bundle of \(S^n = D^m_n \cup S^{n-1} D^m_n\) relating trivializations over the upper and lower discs \(D^m_n\).

To show Theorem 1.1 we will check explicitly that for \(n - m \geq 3\) one has an equivalence
\[(10) \quad \overline{\text{Emb}}(S^m, \mathbb{R}^n) \simeq \text{hofiber} \left( S^{n-m-1} \xrightarrow{g \circ h} B \overline{\text{Emb}}_0(\mathbb{R}^m, \mathbb{R}^n) \right),\]
In other words, the space \(\overline{\text{Emb}}(S^m, \mathbb{R}^n)\) is equivalent to a principal \(\overline{\text{Emb}}_0(\mathbb{R}^m, \mathbb{R}^n)\)-bundle over \(S^{n-m-1}\) with the structure subgroup \(\Omega^m V_m(\mathbb{R}^n) \subset \overline{\text{Emb}}_0(\mathbb{R}^m, \mathbb{R}^n)\).

The equivalence (10) and hence Theorem 1.1 can be shown using the following two propositions.

**Proposition 2.1.** For \(n - m \geq 3\), one has an equivalence
\[
\overline{\text{Emb}}(S^m, \mathbb{R}^n) \simeq \text{hofiber} \left( S^{n-m-1} \xrightarrow{g \circ h} B \overline{\text{Emb}}_0(\mathbb{R}^m, \mathbb{R}^n) \right),
\]
where \(m : \Omega^m V_m(\mathbb{R}^n) \times \Omega^m V_m(\mathbb{R}^n) \to \Omega^m V_m(\mathbb{R}^n)\) is a loop product.

To recall \(D\) and \(h\) denote the maps from (6) and (8).

**Proposition 2.2.** Let \(Y \xrightarrow{f} X\) be a map of pointed spaces, \(Z \xrightarrow{h} \Omega X\) be any map. Let also \(\Omega X \xrightarrow{i} \text{hofiber}(Y \xrightarrow{f} X)\) denote the natural inclusion and \(m : \Omega X \times \Omega X \to \Omega X\) denote the loop product. One has an equivalence
\[(11) \quad \text{hofiber} \left( Z \xrightarrow{ijf} \text{hofiber}(Y \xrightarrow{f} X) \right) \simeq \text{hofiber} \left( \Omega Y \times Z \xrightarrow{m \circ (\Omega f \times h)} \Omega X \right).\]

**Proof of Theorem 1.1** Applying Proposition 2.1 and Proposition 2.2 to the case \(Y \xrightarrow{f} X\) being \(B \overline{\text{Emb}}_0(\mathbb{R}^m, \mathbb{R}^n) \xrightarrow{BD} \Omega^{m-1} V_m(\mathbb{R}^n)\), and \(Z \xrightarrow{h} \Omega X\) being \(S^{n-m-1} \xrightarrow{h} \Omega^m V_m(\mathbb{R}^n)\) immediately yields (10). \(\square\)
Proof of Proposition 2.1. Denote by \(\text{Emb}_*(S^m, \mathbb{R}^n)\) and \(\text{Imm}_*(S^m, \mathbb{R}^n)\) the spaces of embeddings and immersions, respectively, with a fixed behavior near the basepoint \(* \in S^m\). One can easily see that the space

\[
\overline{\text{Emb}}_*(S^m, \mathbb{R}^n) := \text{hofiber} \left( \text{Emb}_*(S^m, \mathbb{R}^n) \xrightarrow{I} \text{Imm}_*(S^m, \mathbb{R}^n) \right)
\]

is weakly equivalent to \(\overline{\text{Emb}}(S^m, \mathbb{R}^n)\). Moreover, we claim that \(\text{Emb}_*(S^m, \mathbb{R}^n) \simeq \text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1}\) and \(\text{Imm}_*(S^m, \mathbb{R}^n) \simeq \Omega^m V_m(\mathbb{R}^n)\) with the map \(I\) of the homotopy type of \(m \circ (D \times h)\).

We decompose \(S^m = D^m_+ \cup_{S^{n-1}} D^m_-\), where \(D^m_+\) is a small closed disc neighborhood of the basepoint \(* \in S^m\), and \(D^m_-\) is its complementary disc. We identify \(\mathbb{R}^n = S^n \setminus \{N\}\) as a sphere without its north pole. Similarly we decompose \(S^n \setminus \{N\} = (D^n_+ \setminus \{N\}) \cup_{S^{n-1}} D^n_-\). One has

\[
\text{Emb}_*(S^m, \mathbb{R}^n) \simeq \text{Emb}_0(D^m_+, D^m_- \setminus \{N\}),
\]

\[
\text{Imm}_*(S^m, \mathbb{R}^n) \simeq \text{Imm}_0(D^m_+, S^n \setminus \{N\}) \simeq \Omega^m V_m(\mathbb{R}^n).
\]

The last equivalence in (13) is by the Smale-Hirsch theorem, as the target manifold \(S^n \setminus \{N\} = \mathbb{R}^n\) is contractible.

Remark 2.3. The transition map between the coordinate framing on \(\mathbb{R}^n = S^n \setminus \{N\}\) and the local coordinates framing near \(N\), when restricted on a small \((n-1)\)-sphere around \(N\), is given by the map \(h_0\) from equation (9).

Consider the space \(\text{Emb}_0(\mathbb{R}^m \sqcup \{\ast\}, \mathbb{R}^n)\), where \(\mathbb{R}^m \sqcup \{\ast\}\) is given the disjoint union topology. Below we define maps

\[
\begin{array}{ccc}
\text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1} & \xrightarrow{A} & \text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n \setminus \{0\}) \\
\downarrow & & \downarrow \\
\text{Emb}_0(\mathbb{R}^m, [\mathbb{R}^n \setminus \{0\}]) & \xrightarrow{B} & \text{Emb}_0([\mathbb{R}^m \sqcup \{\ast\}], \mathbb{R}^n)
\end{array}
\]

Map \(C\) is the inclusion sending \(f \mapsto \tilde{f}\), where \(\tilde{f}(x) = \begin{cases} 
 f(x), & x \in \mathbb{R}^m; \\
 0, & x = \ast. 
\end{cases}\)

By \(S^{n-m-1}\) we understand the unit sphere in \(\mathbb{R}^{n-1}\). Map \(B\) sends a pair \((f, v)\) to \(\tilde{f}\), such that \(\tilde{f}(\ast) = 0^m \times v\) and \(\tilde{f}|_{\mathbb{R}^m}\) is supported in the unit ball with center \(-3 \times 0^{n-1}\) and sending this ball inside the unit ball centered at \(-3 \times 0^{n-1}\). We use here the homeomorphism \(\text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n) \simeq \text{Emb}_0(D^m, D^n)\).
Finally we define $A$. Let $\rho$ be a bump function on $\mathbb{R}^m$ at 0 supported in the unit disc $D^m$. The map $A$ sends $(f, v)$ to $\tilde{f}: \mathbb{R}^m \leftrightarrow \mathbb{R}^n$ supported in the union of two unit discs with centers $-3 \times 0^m$ and $0^m$. Inside the first disc $\tilde{f}$ is the same as $A(f, v)$, while inside the second disc $\tilde{f}(x) = (x, \rho(x)v)$.

**Lemma 2.4.** All three maps $A, B, C$ in (14) are weak homotopy equivalences. Moreover, $B$ is homotopic to $C \circ A$.

**Proof.** Consider two fibrations

\[ \pi_1: \text{Emb}_\partial(\mathbb{R}^m \sqcup \{\ast\}, \mathbb{R}^n) \to \mathbb{R}^n; \]

\[ \pi_2: \text{Emb}_\partial(\mathbb{R}^m \sqcup \{\ast\}, \mathbb{R}^n) \to \text{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \]

obtained by restricting embeddings to one of the components: $\{\ast\}$ or $\mathbb{R}^m$.

Since the target of $\pi_1$ is contractible, the fiber is equivalent to the total space, implying that $C$ is a weak equivalence.

The map $B$ is a morphism of fiber bundles over $\text{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$. By applying the Alexander duality and the fact that both $S^{n-m-1}$ and the fiber of $\pi_2$ (the complement of a knot) are simply connected, we get that $B$ induces an equivalence of fibers, therefore is an equivalence of total spaces.

It is obvious that $B \simeq C \circ A$. By the two out of three property, $A$ is also a weak equivalence. \[ \square \]

To finish the proof of Proposition 2.1 one has to show that the composition

\[ J: \text{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1} \xrightarrow{\sim} \text{Emb}_\partial(\mathbb{R}^m \backslash \{0\}) \xrightarrow{\sim} \text{Emb}_\partial(D^m_+, \mathbb{R}^n) \to \text{Im}\text{m}_\partial(\mathbb{R}^m, S^n \backslash \{N\}) \xrightarrow{\sim} \Omega^m V_m((\mathbb{R}^n)) \]

is homotopic to $m \circ (D \times h)$. It is obvious that $J$ restricted on the first factor $\text{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n)$ is homotopic to $D$. It follows from Remark 2.3 that $J$ restricted on the second factor $S^{n-m-1}$ is homotopic to $h$. Also by construction $J$ is a concatenation of the loop obtained from the first factor with the loop obtained from the second factor, which is exactly what the formula $m \circ (D \times h)$ means. \[ \square \]

**Proof of Proposition 2.2** Recall the standard construction of the homotopy fiber of a map $Y \xrightarrow{f} X$. It is the space of pairs $(y, x)$, where $y \in Y$ and $x: [0, 1] \to X$, such that $x(0) = \ast$ and $x(1) = f(y)$. When this construction is applied, both spaces in (11) are homeomorphic to the space of triples $(z, y, x)$, where $z \in Z$, $y \in \Omega Y$, $x: D^2 \to X$, such that $x|_{\partial D^2}$ is the loop $m(((\Omega f)(h(z))), (\Omega f)(\tilde{y}))$, where $\tilde{y}$ is the inverse of the path $y$, that is $\tilde{y}(t) = y(1 - t)$. \[ \square \]
2.2. Corollaries for the rational homotopy types. The rational homotopy \( \pi_*^Q S^{n-m-1} \) is spanned by the spherical class \( \iota \in \pi_{n-m-1}^Q S^{n-m-1} \) and the Hopf class \([\iota, \iota] \in \pi_{2n-2m-3}^Q S^{n-m-1}\), which is non-zero only if \( n - m \) is odd. The induced map in the rational homotopy \( h_* : \pi_*^Q S^{n-m-1} \to \pi_*^Q \Omega^m V_m(\mathbb{R}^n) \), sends the spherical class \( \iota \) to the SO(\( n \)) Euler class, if \( n \) is even, and sends it to zero if \( n \) is odd. The Hopf class \([\iota, \iota] \) of \( S^{n-m-1} \) is sent to zero, because the rational homotopy of any loop space is an abelian Lie algebra. Recall also that the induced map \( g_* : \pi_*^Q \Omega^m V_m(\mathbb{R}^n) \to \pi_*^Q B \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \), sends the SO(\( n \)) Euler class to the graph-cycle

\[
D = \omega \longrightarrow ^{\bullet}
\]

in HGC, see \([2, 9, 13]\), which is non-zero only if \( n \) is even.

Together with Theorem 1.1, the computations above immediately imply:

**Corollary 2.5.** For \( n - m \geq 3 \), one has \( \text{rk } \pi_i^Q \overline{\text{Emb}}(S^m, \mathbb{R}^n) = \text{rk } \pi_i^Q \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \), except

- for \( n \) even, \( \text{rk } \pi_{n-m-2}^Q \overline{\text{Emb}}(S^m, \mathbb{R}^n) = \text{rk } \pi_{n-m-2}^Q \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) - 1 \);
- for \( n \) odd, \( \text{rk } \pi_{n-m-1}^Q \overline{\text{Emb}}(S^m, \mathbb{R}^n) = \text{rk } \pi_{n-m-1}^Q \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) + 1 \);
- for \( n - m \) odd, \( \text{rk } \pi_{2n-2m-3}^Q \overline{\text{Emb}}(S^m, \mathbb{R}^n) = \text{rk } \pi_{2n-2m-3}^Q \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) + 1 \).

(It follows from Theorem 1.1 that \( \pi_1 \overline{\text{Emb}}(S^m, \mathbb{R}^n) \) is a quotient group of \( \pi_1 \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \) and therefore is abelian.)

Any map from a suspension to an \( H \)-space is rationally coformal (and also formal). For \( n \) odd, the induced map in rational homotopy \( (g \circ h) \), is zero, and for \( n \) even it is non-zero only on the spherical class \( \iota \). This immediately determines the rational homotopy type of \( \overline{\text{Emb}}(S^m, \mathbb{R}^n) \), \( n - m \geq 3 \).

**Corollary 2.6.** For \( n - m \geq 3 \),

- if \( n - m \) or \( n \) is even, each component of \( \overline{\text{Emb}}(S^m, \mathbb{R}^n) \) is rationally equivalent to a product of \( K(\mathbb{Q}, j) \)'s, in other words, it is coformal with an abelian Quillen model;
- if \( n \) is odd, \( \overline{\text{Emb}}(S^m, \mathbb{R}^n) \equiv_\mathbb{Q} \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \times S^{n-m-1} \).

Only in the case \( n \) odd and \( m \) even, the space \( \overline{\text{Emb}}(S^m, \mathbb{R}^n) \) is not rationally abelian. However, the failure of being non-abelian is only in the rational factor \( S^{n-m-1} \).
3. Comparing graph-complexes

As described in the introduction the rational homotopy types of both spaces \(\text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n)\) and \(\text{Emb}(S^m, \mathbb{R}^n)\) have known expressions through graph complexes. The purpose of this section is to illustrate how Theorem 1.1 and in particular Corollaries 2.5 and 2.6 manifest themselves combinatorially on the graph complex level. We shall proceed without using Theorem 1.1 directly, but rather by providing independent arguments, thus essentially re-proving (parts of) the theorem rationally.

3.1. Hairy graph-complexes. In this subsection we describe graph-complexes \(HGC_{\bar{A}_m,n}\), \(HGC_{A_m,n}\) and their \(L_\infty\)-algebra structures that express the rational homotopy type, respectively, of \(\text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n)\) and \(\text{Emb}(S^m, \mathbb{R}^n)\), \(n - m \geq 3\). Here \(\bar{A}_m\) denotes the reduced cohomology algebra \(\tilde{H}^*(S^m, \mathbb{Q})\), and \(A_m\) denotes the cohomology algebra \(H^*(S^m, \mathbb{Q})\). The former is spanned by a single element \(\omega\) of degree \(m\), while the latter is spanned by 1 and \(\omega\). We adapt Whitehead’s grading conventions in which the bracket, higher brackets, and differential of an \(L_\infty\)-algebra are all of degree \(-1\). With these grading conventions

\[
H_\ast(HGC_{\bar{A}_m,n}) = \mathbb{Q} \otimes \pi_\ast \text{Emb}_0(\mathbb{R}^m, \mathbb{R}^n),
\]

\[
H_\ast(HGC_{A_m,n}) = \mathbb{Q} \otimes \pi_\ast \text{Emb}(S^m, \mathbb{R}^n),
\]

and the bracket in graph-complexes corresponds to the Whitehead bracket in the rational homotopy. Note that the latter one is almost always zero according to Corollary 2.6.

The graph-complexes are spanned by connected graphs with two types of vertices: external ones of valence one (called hairs) and internal ones of valence \(\geq 3\). Every external vertex is labeled by \(\omega\) in case of \(HGC_{\bar{A}_m,n}\), and either by \(\omega\) or by 1 in case of \(HGC_{A_m,n}\). Double edges and tadpoles (edges connecting a vertex to itself) are allowed. Such graphs are required to have at least one hair. Let \(E, V, H\) denote, respectively, the sets of edges, internal vertices, and \(\omega\)-hairs of a graph \(\Gamma\). The degree of such graph is

\[
(n - 1)\#E - n\#V - m\#H.
\]

For example the degree of the graph

\[
\begin{array}{c}
\omega \\
\downarrow \\
1 \\
\uparrow \\
\omega
\end{array}
\]
is $4n - 2m - 8$. By orientation of $\Gamma$ we understand orientation of its edges and a linear order of its orientation set $E \cup V \cup H$. Changing orientation of an edge gives the sign $(-1)^v$. Changing the order of the orientation set brings in the Koszul sign of permutation, where edges are assigned degree $n - 1$, internal vertices are assigned degree $-n$, and $\omega$-hairs are assigned degree $-m$.

The differential on $\text{HGC}_{\bar{A}_{m,n}}$ is denoted by $\delta_{\text{split}}$: it acts by splitting the vertices into two:

$$\delta_{\text{split}} \Gamma = \sum_{v \text{ vertex}} \pm \Gamma \text{ split } v$$

The differential on $\text{HGC}_{\bar{A}_{m,n}}$ is $\delta = \delta_{\text{split}} + \delta_{\text{join}}$, where $\delta_{\text{split}}$ is defined by (16), while $\delta_{\text{join}}$ joins a subset of at least two hairs into one hair, multiplying the decorations, schematically:

$$\delta_{\text{join}} \Gamma_{a_1 a_2 \ldots a_k} = \sum_{S \text{ chairs } |S| \geq 2} \pm a_1 \ldots \prod_{j \in S} a_j \Gamma_{\bar{a}_1 \bar{a}_2 \ldots \bar{a}_k}.$$ 

Clearly, a summand in (17) is non-zero only if $S$ contains at most one $\omega$-hair.

The $r$-th $L_\infty$ operation $\ell_r(\Gamma_1, \ldots, \Gamma_r)$, $r \geq 2$, is zero for $\text{HGC}_{\bar{A}_{m,n}}$ and is defined similarly to $\delta_{\text{join}}$ for $\text{HGC}_{A_{m,n}}$. For example, the (homotopy) Lie bracket has the following form:

$$\left[ \Gamma_1, \Gamma_2 \right] = \sum \Gamma_1 \ldots \Gamma_2,$$

where the decorations $\omega$ and 1 on hairs are multiplied whenever hairs are joined. The sum is taken over pairs of non-empty subsets of hairs of $\Gamma_1$ and $\Gamma_2$. More generally, $\ell_r$ is the sum over $r$-tuples of non-empty subsets of hairs of $\Gamma_1, \ldots, \Gamma_r$ with every summand being a new connected hairy graph, where all selected hairs are joined into one.

3.2. Connected components and Maurer-Cartan elements. As it is explained in the introduction, and also stated in Theorem 1.1 one has

$$\pi_0 \text{Emb}(S^m, \mathbb{R}^n) = \pi_0 \text{Emb}_\partial(\mathbb{R}^m, \mathbb{R}^n), \quad n - m \geq 3,$$
are isomorphic as (abelian) groups, and all components of $\overline{\text{Emb}}(S^m, \mathbb{R}^n)$ (as well as all components of $\overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n)$) have the same homotopy type. These groups are almost always finite except two cases:

(a) $m = 2k - 1, n = 4k - 1, k \geq 2$;

(b) $m = 4k - 1, n = 6k, k \geq 1$.

In the latter two cases this group is infinite of rank one [9, Corollary 20]. In case (a), an infinite order generator appears as image, under inclusion

$$\Omega^{2k} V_{2k-1}(\mathbb{R}^{4k-1}) \to \overline{\text{Emb}}_\partial(\mathbb{R}^{2k-1}, \mathbb{R}^{4k-1}),$$

of the SO(2k) Euler class in $\pi_{2k} V_{2k-1}(\mathbb{R}^{4k-1}) = \pi_{2k}(SO(4k-1)/SO(2k))$. In case (b), an infinite order generator corresponds to the Haefliger treffoil $S^{4k-1} \hookrightarrow \mathbb{R}^{6k}$ [11, 12].

By [10, Corollary 1.3], the $L_\infty$-algebras $\text{HGC}_{\mathbb{A}_m,n}$ and $\text{HGC}_{A_m,n}$ do provide some information about the sets (19). Namely, one has naturally defined finite-to-one maps

$$m: \pi_0 \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \to \text{MC}(\text{HGC}_{\mathbb{A}_m,n})/\sim,$$

$$m: \pi_0 \overline{\text{Emb}}(S^m, \mathbb{R}^n) \to \text{MC}(\text{HGC}_{A_m,n})/\sim$$

from the sets of connected components to the sets of Maurer-Cartan elements modulo gauge equivalence. Since the $L_\infty$-algebra $\text{HGC}_{\mathbb{A}_m,n}$ is abelian,

$$\text{MC}(\text{HGC}_{\mathbb{A}_m,n})/\sim = H_0(\text{HGC}_{\mathbb{A}_m,n}).$$

It is not hard to see that $\text{HGC}_{A_m,n}$ in degrees $\leq 0$ can have only trees with all hairs labeled by $\omega$ [10, Proposition 5.1]. Thus, $H_0(\text{HGC}_{A_m,n}) = H_0(\text{HGC}_{\mathbb{A}_m,n})$ and $\text{MC}(\text{HGC}_{A_m,n})/\sim = \text{MC}(\text{HGC}_{\mathbb{A}_m,n})/\sim$, see [10, Corollary 5.2]. By [9, Remark 19],

$$\mathbb{Q} \otimes \pi_0 \overline{\text{Emb}}_\partial(\mathbb{R}^m, \mathbb{R}^n) \simeq H_0(\text{HGC}_{\mathbb{A}_m,n}).$$

The latter group is non-trivial (and is $\mathbb{Q}$) exactly in two cases (a) and (b) above. The Maurer-Cartan elements corresponding to case (a) are multiples of the line graph

$$L_\omega = \omega \longrightarrow \omega.$$

\footnote{This fact can also be easily obtained from Haefliger’s [12, Corollary 6.7 and Remark 6.8].}
For case (b), such elements are multiples of the tripod

\[ T_\omega = \begin{pmatrix} \omega \\ \omega \\ \omega \end{pmatrix} \, . \]

By [10, Corollary 1.3], for an embedding \( \psi \in \overline{\text{Emb}}(\mathbb{R}^n, \mathbb{R}^n) \) (respectively, \( \psi \in \text{Emb}(S^m, \mathbb{R}^n) \)), the rational homotopy type of the component \( \overline{\text{Emb}}(\mathbb{R}^n, \mathbb{R}^n) \) \( \psi \) (respectively, \( \text{Emb}(S^m, \mathbb{R}^n) \) \( \psi \)) is expressed by the positive degree truncation of the \( m(\psi) \) twisted \( L_\infty \)-algebra \( \overline{\text{HGC}}_{m(\psi)} \mathbb{A}_{\mathbb{R}^n} \) (respectively, \( \text{HGC}_{m(\psi)} \mathbb{A}_{\mathbb{R}^n} \)). Since the \( L_\infty \)-algebra \( \overline{\text{HGC}}_{m(n-m)} \mathbb{A}_{\mathbb{R}^n} \) is abelian, such twist makes no effect on it, which corresponds to the fact that all connected components of a loop space have the same homotopy type. In case (a), the twist by \( L_\omega \) does not change neither the differential, nor the bracket of \( \text{HGC}_{m(\psi)} \mathbb{A}_{\mathbb{R}^n} \). This is because for even codimension \( n-m \), any graph with two \( \omega \)-hairs attached to an internal vertex is zero. The twisting by \( T_\omega \) affects the differential and the \( L_\infty \) structure of \( \text{HGC}_{m(\psi)} \mathbb{A}_{\mathbb{R}^n} \). We do not do it here, but one can show that \( \text{HGC}_{T_\omega m(n-m)} \mathbb{A}_{\mathbb{R}^n} \) is \( L_\infty \) isomorphic to the non-deformed one \( \text{HGC}_{m(n-m)} \mathbb{A}_{\mathbb{R}^n} \), which confirms the fact that all components of \( \overline{\text{Emb}}(S^m, \mathbb{R}^n) \), \( n-m \geq 3 \), have the same homotopy type.

3.3. **Computations.** The relation between the long and non-long embedding spaces can be reproduced combinatorially on graph-complexes as follows. There is an obvious inclusion of \( L_\infty \)-algebras \( \overline{\text{HGC}}_{m(n-m)} \mathbb{A}_{\mathbb{R}^n} \rightarrow \text{HGC}_{m(n-m)} \mathbb{A}_{\mathbb{R}^n} \) corresponding to the inclusion \( \tilde{A}_m \rightarrow A_m \).

Let us also consider the following low degree diagrams which are non-zero for certain values of \( m \) and \( n \).

\[
L = 1 \quad \begin{array}{c} \omega \end{array} \\
D = \begin{pmatrix} \omega \\ \omega \end{pmatrix} \quad \begin{array}{c} \omega \end{array} \\
T = 1 \quad \begin{pmatrix} \omega \\ \omega \end{pmatrix} \quad \begin{array}{c} \omega \end{array}
\]

The graph \( L \) is of degree \( n-m-1 \) and is always non-zero; \( D \) is of degree \( n-m-2 \) and is non-zero if \( n \) is even; and \( T \) is of degree \( 2n-2m-3 \) and is non-zero if and only if \( n-m \) is odd. One has that \( dL = D \), i.e., for even \( n \) the corresponding classes cancel in cohomology in \( \text{HGC}_{m(n-m)} \mathbb{A}_{\mathbb{R}^n} \). (But not in \( \text{HGC}_{\tilde{A}_m} \mathbb{R}^n \), since \( L \notin \text{HGC}_{\tilde{A}_m} \mathbb{R}^n \).) Also note that \( dT = 0 \). Using these classes we can completely describe the relation between \( \text{HGC}_{m(n-m)} \mathbb{A}_{\mathbb{R}^n} \) and \( \text{HGC}_{\tilde{A}_m} \mathbb{R}^n \) as follows.

**Theorem 3.1.** The mapping cone \( C \) of the inclusion \( \text{HGC}_{\tilde{A}_m} \mathbb{R}^n \rightarrow \text{HGC}_{A_m} \mathbb{R}^n \) has the following cohomology, depending on \( m \) and \( n \):

- For \( m, n \) even \( H(C) \) is one-dimensional, spanned by a class whose projection to \( \text{HGC}_{\tilde{A}_m} \mathbb{R}^n \) is \( D \).
• For $n$ even and $m$ odd $H(C)$ is two-dimensional, spanned by a class corresponding to $D$ in $HGC_{A_m,n}$ as before and the class of $T \in HGC_{A_m,n}$.

• For $m, n$ odd $H(C)$ is one-dimensional, spanned by the class of $L$ in $HGC_{A_m,n}$.

• For $n$ odd and $m$ even $H(C)$ is two-dimensional, spanned by the class of $L$ and $T$ in $HGC_{A_m,n}$.

**Remark 3.2.** Theorem 3.1 provides a different proof of Corollary 2.5.

The result can alternatively be reformulated as follows.

**Corollary 3.3.** Let $U^I \subset HGC_{A_m,n}$ be the subspace spanned by trees with exactly one $1$-decorated hair. Consider the vector space direct sum $U^I \oplus HGC_{A_m,n} \subset HGC_{A_m,n}$ with the induced (subspace) $L_\infty$-structure. Then the inclusion $U^I \oplus HGC_{A_m,n} \to HGC_{A_m,n}$ is a quasi-isomorphism of $L_\infty$-algebras.

As immediate consequence, the $L_\infty$-algebra $HGC_{A_m,n}$ is homotopy abelian for $n - m$ even. Indeed, for $n - m$ even there can be at most one $\omega$-hair attached to a vertex by symmetry. But then the statement easily follows from Corollary 3.3 since all possible higher $L_\infty$-operations necessarily produce multiple $\omega$-hairs at some vertex. Less trivially, the above arguments can also be extended to show that $HGC_{A_m,n}$ is homotopy abelian for $n$ even and $m$ odd. This gives a different proof of the first statement of Corollary 2.6. Similarly, we can also recover the second statement of Corollary 2.6 which is immediate in case both $m$ and $n$ are odd. In the remaining case $n$ odd and $m$ even, there is a nontrivial bracket, namely

$$[L, L] = T,$$

so that $HGC_{A_m,n}$ is not homotopy abelian. It is possible to upgrade the map $\Phi$ that we construct below, see Lemma 3.6 to an $L_\infty$ map, see the footnote at the end of the proof of Lemma 3.6, that would allow one to split off $L$ and $T$ - the two classes coming from $S^{n-m-1}$, as an $L_\infty$ direct summand.

To prepare for the proof of Theorem 3.1 let us introduce the non-unital dgca

$$A'_m = \mathbb{Q}\epsilon \oplus \mathbb{Q}\omega$$

with $\epsilon$ of degree 0 and $\omega$ of degree $m$, and products $\epsilon^2 = \epsilon$ and $\epsilon\omega = \omega^2 = 0$. We consider the hairy graph-complex $HGC_{A'_m,n}$. Note also that the complexes $HGC_{A_m,n}$ and $HGC_{A'_m,n}$ are isomorphic as vector spaces, identifying $\epsilon$ and 1. In fact, from
now on we shall tacitly identify the decorations $\epsilon$ and 1 on hairs of graphs, keeping in mind however that the differentials on $\text{HGC}_{A_m,n}$ and $\text{HGC}_{A_m,n}'$ are different. Concretely, the differential in $\text{HGC}_{A_m,n}$ has pieces fusing several 1-decorated hairs with one $\omega$-decorated hair, and these terms are absent in the differential on $\text{HGC}_{A_m,n}'$. Note that there is again an inclusion $\text{HGC}_{A_m,n} \to \text{HGC}_{A_m,n}'$.

**Lemma 3.4.** The inclusion map $\mathbb{Q}L \oplus \mathbb{Q}T \oplus \text{HGC}_{A_m,n} \to \text{HGC}_{A_m,n}'$ is a quasi-isomorphism. Here we understand that $\mathbb{Q}T = 0$ in case $n - m$ is even and hence $T = 0$.

**Remark 3.5.** It follows from [10, Corollary 1.3] that the complex $\text{HGC}_{A_m,n}'$ computes the rational homotopy groups of the space $\text{Emb}(\partial(R^m \sqcup \{\ast\}, R^n)$. On the other hand, by Lemma 2.4 $\text{Emb}(\partial(R^m \sqcup \{\ast\}, R^n) \simeq \text{Emb}(\partial(R^m, R^n) \times S^{n-m-1}$. This explains the quasi-isomorphism of Lemma 3.4.

**Proof of Lemma 3.4** There is a splitting of complexes

$$\text{HGC}_{A_m,n} = \text{HGC}_{A_m,n} \oplus U$$

with $U$ being the subcomplex spanned by graphs with at least one hair labeled $\epsilon$. Our goal is to show that $H(U)$ is one- or two-dimensional. To do this we may follow the proof of [14, Theorem 1]. First note that the differential creates one vertex, hence the cohomology is graded by number of vertices. Let $U' \subset U$ be the subcomplex spanned by trees with exactly one $\epsilon$-labelled hair. It is an easy exercise to check that $H(U') = \mathbb{Q}L \oplus \mathbb{Q}T$. We are going to show by induction on the number of internal vertices that the inclusion $U' \subset U$ is a quasi-isomorphism. For zero internal vertices the statement is quickly checked by hand. Suppose we know the statement for less than $k$ internal vertices, and we desire to prove it for $k$ internal vertices. Consider the splitting

$$U = U_1 \oplus U_{>1}$$

where $U_1$ is spanned by diagrams having exactly one $\epsilon$-labelled hair, and $U_{>1}$ being spanned by diagrams having at least two such hairs. One may set up a bounded spectral sequence such that the lowest page differential is the component $f : U_{>1} \to U_1$ that creates one new internal vertex with an $\epsilon$-hair, connecting all $\epsilon$-hairs to it. The map $f$ is injective. The cokernel $V := \text{coker} f$ consists of $L$ and graphs which become disconnected upon removing the vertex at the $\epsilon$-hair. Going further, one may filter $V$ by the number of connected components at that vertex. On the associated graded the complex obtained is just a symmetric power of the complex $U$. Hence, invoking the induction hypothesis, we have shown the desired statement. □
Our next goal is to compare the complexes $\text{HGC}_A^{m,n}$ and $\text{HGC}_{A_{m,n}}$. To this end we will consider the subcomplexes $\text{HGC}_A^{m,n} \subset \text{HGC}_{A_{m,n}}$ and $\text{HGC}_{A_{m,n}} \subset \text{HGC}_{A_{m,n}}$ spanned by diagrams with at least one $\omega$-labeled hair and at least one internal vertex.

Consider now a hairy graph $\Gamma \in \text{HGC}_A^{m,n}$ and let $S$ be some subset of the hairs decorated by $\epsilon$ in $\Gamma$. Denote by $R_S(\Gamma)$ the sum of all graphs obtained by reconnecting the hairs in $S$ to internal vertices of $\Gamma$, not forming tadpoles:\footnote{I.e., a hair cannot be connected to the internal vertex it attaches to.}

\[
\begin{align*}
\Gamma \nabla S & \mapsto \quad R_S(\Gamma) = \sum \Gamma.
\end{align*}
\]

Now consider the map

\[\Phi : \text{HGC}_A^{m,n} \to \text{HGC}_{A_{m,n}}\]

which is defined combinatorially by the formula

\[\Phi(\Gamma) = (-1)^{\#\epsilon} \sum S R_S(\Gamma).\]

where $\Gamma \in \text{HGC}_A^{m,n}$ is a graph with $\#\epsilon$ many $\epsilon$-decorated hairs.

**Lemma 3.6.** The map $\Phi : \text{HGC}_A^{m,n} \to \text{HGC}_{A_{m,n}}$ is an isomorphism of complexes.

**Proof.** It is clear that the map is an isomorphism since $\Phi(\Gamma) = \pm \Gamma + \cdots$, with $\cdots$ representing terms of loop orders higher than that of $\Gamma$. We next show that $\Phi$ commutes with the differentials. Let us first reformulate the problem. We identify $\text{HGC}_A^{m,n}$ and $\text{HGC}_{A_{m,n}}$ as graded vector spaces, and denote the differential of $\text{HGC}_A^{m,n}$ by $d'$ and that of $\text{HGC}_{A_{m,n}}$ by $d$. Let $s : \text{HGC}_A^{m,n} \to \text{HGC}_{A_{m,n}}$ be the map of graded vector spaces that reconnects one hair labelled $\epsilon$ to an internal
vertex.

\[ s(\Gamma) = \sum \Gamma \]

Then we can write \( \Phi = \exp(s) \circ I_e \), where \( I_e(\Gamma) = (-1)^{\#e}\Gamma \). We desire to show that \( \Phi \circ d' = d \circ \Phi \), or equivalently

\[ \exp(\text{ad}_s)(I_e d'I_e) = \sum_{j=0}^{\infty} \text{ad}_s^j d' = d, \]

where \( d' = I_e d'I_e \) and \( \text{ad}_s = [s, -] \) is the commutator as usual.

Furthermore, let us split the differential \( d' \), and similarly \( d' \) in several pieces. To this end it is most convenient to temporarily enlarge our complex \( HGC_{\alpha, n} \) in that we also allow graphs with univalent and bivalent internal vertices. Then we split

\[ d' = d'_1 - B_0 + d'_e + d'_\omega \]

into the following four terms.

- \( d'_1 \) splits a vertex into two vertices, distributing the incoming edges in all possible ways, including such that create uni- or bivalent internal vertices.
- \( B_0 \) attaches a new univalent vertex to the graph. The sign is such that it precisely cancels those terms from \( d'_1 \) that create univalent internal vertices.
- \( d'_e \) creates a new internal vertex with an \( \epsilon \)-decorated hair and attaches a non-empty subset of the \( \epsilon \)-decorated hairs to it,

\[ d'_e \Gamma = \sum_{K \mid |K| \geq 1} A_K(\Gamma), \]

\[ A_K(\Gamma) = \sum K \Gamma \]
• $d'_\omega$ creates a bivalent internal vertex on an $\omega$-decorated hair,

$$d'_\omega = C_\emptyset(\Gamma).$$

$C_\emptyset(\Gamma) = \sum_{\Gamma_\emptyset} \Gamma$.

Note that the $|K| = 1$-term of $d'_\epsilon$ and $d'_\omega$ together cancel all terms in the total differential $d'$ that possibly create a graph with a bivalent internal vertex.

• Finally we note that $I_\epsilon d'_1 I_\epsilon = d'_1$ and $I_\epsilon B_\emptyset I_\epsilon = B_\emptyset$. Denoting

$$\tilde{d}'_\epsilon := (I_\epsilon d'_1 I_\epsilon) \quad \text{and} \quad \tilde{d}'_\omega := (I_\epsilon d'_\omega I_\epsilon)$$

we furthermore have

$$\tilde{d}'_\epsilon \Gamma = \sum_{|K|\geq 1} (-1)^{|K|+1} A_K(\Gamma)$$

and

$$\tilde{d}'_\omega \Gamma = \sum_{|K|\geq 0} (-1)^{|K|} C_K(\Gamma).$$

One quickly checks that $[s, d'_1] = 0$. Furthermore,

$$\frac{1}{j!} (\text{ad}_d^j B_\emptyset)(\Gamma) = \sum_{|J| = j} B_J(\Gamma),$$

where the sum is over subsets $J$ of the set of $\epsilon$-labelled hairs and

$$B_J(\Gamma) = \sum J,$$
Finally, 

$$\frac{1}{j!} (\text{ad}_{j!}^{d_1'})(\Gamma) = \sum_{|J| = j} C_J(\Gamma),$$

where the sum is again over subsets $J$ of the $\epsilon$-decorated hairs, and $C_J(\Gamma)$ is obtained by connecting the hairs in $J$ to one new vertex attached to an $\omega$-decorated hair.

$$C_J(\Gamma) = \sum_{J \setminus \omega} \Gamma$$

Now, putting everything together we get (with the sums being over subsets of the $\epsilon$-decorated hairs)

$$(\Phi d' \Phi^{-1})(\Gamma) = \sum_{j=0}^{\infty} (\text{ad}_{j!}^{d_1'})(\Gamma) =$$

$$d'_1 - \sum_{J \ni K} B_J(\Gamma) + \sum_{J \ni K} (-1)^{|K|+1} A_{J \cup K}(\Gamma) + \sum_{J \ni K} (-1)^{|K|+1} B_{J \cup K}(\Gamma) + \sum_{|J| \geq 0} C_J(\Gamma)$$

$$= d'_1 - \sum_{J \ni K} (-1)^{|K|} B_{J \cup K}(\Gamma) - \sum_{J \ni K} (-1)^{|K|} A_{J \cup K}(\Gamma) + \sum_{|J| \geq 0} C_J(\Gamma).$$

Now we use (twice) that for any function $J \mapsto X_J$ on subsets as above

$$\sum_{J \ni K} (-1)^{|K|} X_{J \cup K} = X_{\emptyset}.$$

This simplifies the above expression to

$$d'_1 - B_0 + \sum_{|J| \geq 1} A_J(\Gamma) + \sum_{|J| \geq 0} C_J(\Gamma) = d.$$

This is precisely $d$, hence the Lemma is proven. \(\square\)

Let us finish the proof of Theorem 3.1.

4The same construction applied to disconnected graphs, interpreted as the Chevalley complex, in fact can be used to construct an $L_\infty$-isomorphism, not just one of complexes.
Proof of Theorem 3.1. For use in the proof let us define the diagrams
\[ L' = 1 \longrightarrow 1 \quad L'' = \omega \longrightarrow \omega \quad D' = 1 \longrightarrow \varnothing \]
We have \( dL' = D' \) and both \( L' \) and \( D' \) vanish if \( n \) is odd. Similarly, \( L'' \) vanishes if \( n - m \) is odd.

Now let \( \text{HGC}_{A_m,n}^{'} \subseteq \text{HGC}_{\bar{A}_m,n} \) be the (codimension one-)subcomplex spanned by diagrams with at least one vertex so that \( \text{HGC}_{\bar{A}_m,n} = \text{HGC}_{A_m,n}^{'} \oplus \mathbb{Q}L'' \). Note that we have a natural inclusion \( \text{HGC}_{A_m,n}^{'} \rightarrow \text{HGC}_{\bar{A}_m,n}^{'} \), fitting into the commutative diagram
\[
\begin{array}{ccc}
\text{HGC}_{A_m,n}^{'} & \longrightarrow & \text{HGC}_{\bar{A}_m,n}^{'} \\
\downarrow & \phi & \\
& \text{HGC}_{\bar{A}_m,n}^{'} &
\end{array}
\]

From Lemma 3.4 and its proof we see that
\[
H(\text{HGC}_{A_m,n}^{'} ) \cong H(\text{HGC}_{A_m,n}^{'} ) \cong H(\text{HGC}_{\bar{A}_m,n}^{'} ) \oplus \mathbb{Q}T \oplus \mathbb{Q}D',
\]
again using the convention that \( \mathbb{Q}T = 0 \) if \( T = 0 \) and similarly for \( \mathbb{Q}D' \).

To show Theorem 3.1 it just remains to compare the cohomology of \( \text{HGC}_{A_m,n}^{'} \) and \( \text{HGC}_{\bar{A}_m,n}^{'} \). We have
\[
\text{HGC}_{A_m,n} = W_0 \oplus W_{>0},
\]
with \( W_0 \) spanned by graphs with zero \( \omega \)-vertices and \( W_{>0} \) the remaining graphs. It is shown in [14, Theorem 1] (see also Lemma 3.4) that \( H(W_0) = 0 \). Now
\[
W_{>0} = \text{HGC}_{A_m,n}^{'} \oplus \mathbb{Q}L,
\]
with nontrivial differential sending \( L \) to \( D \in \text{HGC}_{A_m,n}^{'} \) iff \( D \neq 0 \), i.e., iff \( n \) is even. We find that \( H(\text{HGC}_{A_m,n}^{'} ) \) is either one dimension smaller or bigger than \( H(\text{HGC}_{A_m,n}^{'} ) \) if \( n \) is even or odd. Combining the above observations, depending on the parity of \( n \) and \( n - m \) we arrive at Theorem 3.1. \( \square \)

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