A note on the Nielsen realization problem for connected sums of $S^2 \times S^1$

Bruno P. Zimmermann  
Università degli Studi di Trieste  
Dipartimento di Matematica e Geoscienze  
34127 Trieste, Italy

Abstract. We consider finite group-actions on 3-manifolds $\mathcal{H}_g$ obtained as the connected sum of $g$ copies of $S^2 \times S^1$, with free fundamental group $F_g$ of rank $g$. We prove that, for $g > 1$, a finite group of diffeomorphisms of $\mathcal{H}_g$ inducing a trivial action on homology is cyclic and embeds into an $S^1$-action on $\mathcal{H}_g$. As a consequence, no non-trivial element of the twist subgroup of the mapping class group of $\mathcal{H}_g$ (generated by Dehn twists along embedded 2-spheres) can be realized by a periodic diffeomorphism of $\mathcal{H}_g$ (in the sense of the Nielsen realization problem). We also discuss when a finite subgroup of the outer automorphism group $\text{Out}(F_g)$ of the fundamental group of $\mathcal{H}_g$ can be realized by a group of diffeomorphisms of $\mathcal{H}_g$.

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1. Introduction

All finite group-actions in the present paper will be faithful, smooth and orientation-preserving, all manifolds orientable. We are interested in finite group-actions on connected sums $\mathcal{H}_g = \#_g (S^2 \times S^1)$ of $g$ copies of $S^2 \times S^1$; we will call $\mathcal{H}_g$ a closed handle of genus $g$ in the following. The fundamental group of $\mathcal{H}_g$ is the free group $F_g$ of rank $g$. Considering induced actions on the fundamental group and on the first homology $H_1(\mathcal{H}_g) \cong \mathbb{Z}^g$, there are canonical maps

$$\text{Diff}(\mathcal{H}_g) \to \text{Out}(F_g) \to \text{GL}(g, \mathbb{Z})$$

where $\text{Diff}(\mathcal{H}_g)$ denotes the orientation-preserving diffeomorphism group of $\mathcal{H}_g$ and $\text{Out}(F_g) = \text{Aut}(F_g)/\text{Inn}(F_g)$ the outer automorphism group of its fundamental group.

**Theorem 1.** Let $G$ be a finite group acting on a closed handle $\mathcal{H}_g$ of genus $g > 1$ such that the induced action on the first homology of $\mathcal{H}_g$ is trivial. Then $G$ is cyclic and a subgroup of an $S^1$-action on $\mathcal{H}_g$; in particular, all elements of $G$ are isotopic to the identity.
For a description and classification of circle-actions on 3-manifolds and closed handles, see [14].

Denoting by $\text{Mod}(\mathcal{H}_g)$ the mapping class group of isotopy classes of orientation-preserving diffeomorphisms of $\mathcal{H}_g$, there are induced maps

$$\text{Mod}(\mathcal{H}_g) \to \text{Out}(F_g) \to \text{GL}(g, \mathbb{Z}).$$

Let $\text{Twist}(\mathcal{H}_g)$ denote the subgroup of $\text{Mod}(\mathcal{H}_g)$ generated by all Dehn twists along embedded 2-spheres in $\mathcal{H}_g$ (i.e., by cutting along a 2-sphere and regluing after twisting by one full turn around an axis; since such a twist represents a generator of $\pi_1(SO(3)) \cong \mathbb{Z}_2$, its square is isotopic to the identity). By classical results of Laudenbach [6],[7] there is a short exact sequence

$$1 \to \text{Twist}(\mathcal{H}_g) \hookrightarrow \text{Mod}(\mathcal{H}_g) \to \text{Out}(F_g) \to 1;$$

moreover $\text{Twist}(\mathcal{H}_g) \cong (\mathbb{Z}_2)^g$ is generated by the sphere twists around the core spheres $S^2 \times \ast$ of the $g$ different $S^2 \times S^1$ summands of $\mathcal{H}_g$ (twists around separating 2-spheres instead are isotopic to the identity). It is proved in [1] that $\text{Mod}(\mathcal{H}_g)$ is isomorphic to a semidirect product $\text{Twist}(\mathcal{H}_g) \rtimes \text{Out}(F_g)$. Theorem 1 has the following consequence (in the sense of the Nielsen realization problem).

**Corollary 1.** No nontrivial element of the twist group $\text{Twist}(\mathcal{H}_g)$ can be realized (represented) by a periodic diffeomorphism of $\mathcal{H}_g$.

For $g > 1$ this follows from Theorem 1 but the methods apply also to the case $g = 1$ of $\mathcal{H}_1 = S^2 \times S^1$, using the fact that $S^2 \times S^1$ is a geometric 3-manifold belonging to the $(S^2 \times \mathbb{R})$-geometry (one of Thurston’s eight 3-dimensional geometries, see [15]), and that finite group-actions on $S^2 \times S^1$ are geometric ([10, Theorem 8.4]).

For a solution of the Nielsen realization problem for aspherical and Haken 3-manifolds, see [21] (here finite groups of mapping classes can always be realized, except for a purely algebraic obstruction in the case of Seifert fiber spaces where, however, a finite inflation of the group can always be realized).

By [6], homotopic diffeomorphisms of $\mathcal{H}_g$ are isotopic but this does not remain true for arbitrary connected sums of 3-manifolds. By [4], twists around separating 2-spheres in a 3-manifold may or may not be homotopic to the identity, moreover by [3] there are sphere-twists which are homotopic but not isotopic to the identity (see also the discussion in the introduction of [1]). As an example, considering a connected sum $M = M_1 \# M_2$ of two closed hyperbolic 3-manifolds $M_1$ and $M_2$, the sphere-twist around the connecting 2-sphere is not homotopic to the identity; also, it cannot be realized by a periodic map (e.g., if $M_1$ or $M_2$ does not admit a nontrivial periodic map then also the connected sum $M = M_1 \# M_2$ has no periodic maps).
There arises naturally the question of which finite subgroups of Out($F_g$) can be realized by a finite group of diffeomorphisms of $H_g$. Finite groups $G$ of diffeomorphisms of $H_g$ which act faithfully on the fundamental group (i.e., inject into Out($F_g$)) are considered in [17] where, for $g \geq 15$, the quadratic upper bound $|G| \leq 24g(g-1)$ for their orders is obtained. Since Out($F_g$) has finite subgroups of larger orders, these subgroups cannot be realized by finite groups of diffeomorphisms (by [16] the maximal order of a finite subgroup of Out($F_g$) is $2^g g!$, for $g > 2$). A precise result is as follows (we refer to [17, section 2] for definitions and the proof).

**Theorem 2.** Let $G$ be a finite subgroup of Out($F_g$) and $1 \to F_g \to E \to G \to 1$ the corresponding group extension associated to $G$. Then $G$ can be realized by an isomorphic group of diffeomorphisms of $H_g$ if and only if $E$ is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of a finite graph of finite groups $(\Gamma, \mathcal{G})$ in normal form associated to a closed handle-orbifold (in particular, the vertex groups $(\Gamma, \mathcal{G})$ have to be isomorphic to finite subgroups of $SO(4)$ and the edge groups to finite subgroups of $SO(3)$).

We note that, for a finite group $G$ acting on a closed handle $H_g$, the quotient $H_g/G$ has the structure of a closed handle-orbifold (see [17]). Analogous results on finite group-actions on 3-dimensional handlebodies are obtained in [8] and [12] (and in [9] for finite group-actions on handlebodies in arbitrary dimensions).

The case $g = 2$ is special. By well-known results,

$$\text{Out}(F_2) \cong \text{Aut}(\mathbb{Z}^2) \cong \text{GL}(2, \mathbb{Z}) \cong D_6 * D_2 D_4,$$

so up to conjugation the maximal finite subgroups of $\text{Out}(F_2)$ are the dihedral groups $D_6$ and $D_4$ of orders 12 and 8, and both can be realized by diffeomorphisms of the torus with one boundary component (hence, if the realizations of the amalgamated subgroups $D_2$ coincide, one obtains a realization of the whole group $\text{Out}(F_2) \cong D_6 * D_2 D_4$). Considering the product with a closed interval, one obtains realizations on the handlebody $V_2$ of genus 2 and also on its double $H_2$ along the boundary.

Concerning the case $g = 3$, by [18] there are exactly five maximal finite subgroups of $\text{Out}(F_3)$ up to conjugation; by an easy application of Theorem 2, all of these maximal finite subgroups can be realized by diffeomorphisms of the closed handle $H_3$ of genus 3 (but not of a handlebody $V_3$ of genus 3).

**2. Proof of Theorem 1**

Let $G$ be a finite group acting faithfully and orientation-preservingly on a closed handle $H_g = \sharp_g(S^2 \times S^1)$ of genus $g$. By the equivariant sphere theorem (see [10] for an approach by minimal surface techniques, [2] and [5] for topological-combinatorial proofs), there exists an embedded, homotopically nontrivial 2-sphere $S^2$ in $H_g$ such that $x(S^2) = S^2$.
or \( x(S^2) \cap S^2 = \emptyset \) for all \( x \in G \). We cut \( \mathcal{H}_g \) along the system of disjoint 2-spheres \( G(S^2) \), by removing the interiors of \( G \)-equivariant regular neighbourhoods \( S^2 \times [-1,1] \) of these 2-spheres, and call each of these regular neighbourhoods \( S^2 \times [-1,1] \) a 1-handle. The result is a collection of 3-manifolds with 2-sphere boundaries, with an induced action of \( G \). We close each of the 2-sphere boundaries by a 3-ball and extend the action of \( G \) by taking the cone over the center of each of these 3-balls, so \( G \) permutes these 3-balls and their centers. The result is a finite collection of closed handles of lower genus on which \( G \) acts (cf. [17]). Applying inductively the procedure of cutting along 2-spheres, we finally end up with a finite collection of 3-spheres or 0-handles (closed handles of genus 0). Note that the construction gives a finite graph \( \Gamma \) on which \( G \) acts whose vertices correspond to the 0-handles and whose edges to the 1-handles. Note that \( \Gamma \) has no free edges, i.e. edges with one vertex of valence 1.

On each 3-sphere (0-handle) there are finitely many points which are the centers of the attached 3-balls (their boundaries are the 2-spheres along which the 1-handles are attached). For each of these 3-spheres, let \( G_v \) denote its stabilizer in \( G \) (by the geometrization of finite group-actions on 3-manifolds, one may assume that the action of a stabilizer \( G_v \) on the corresponding 3-sphere is orthogonal but this is not needed for the following). Denoting by \( G_e \) the stabilizer in \( G \) of a 1-handles \( S^2 \times [-1,1] \), we can assume that each stabilizer \( G_e \) preserves the product structure of \( S^2 \times [-1,1] \) of the corresponding 1-handle (by choosing small equivariant regular neighbourhoods of the 2-spheres). If some element of a stabilizer \( G_e \) acts as a reflection on \([-1,1] \), we split the 1-handle into two 1-handles by introducing a new 0-handle obtained from a small regular neighbourhood \( S^2 \times [-\epsilon,\epsilon] \) of \( S^2 \times \{0\} \) by closing up with two 3-balls. Hence we can assume that each stabilizer \( G_e \) of a 1-handle \( S^2 \times [-1,1] \) does not interchange its two boundary 2-spheres; that is, \( G \) acts without inversions on the graph \( \Gamma \).

Suppose now that \( g > 1 \) and that the induced action of \( G \) on the first homology of \( \mathcal{H}_g \) and hence also of \( \Gamma \) is trivial. As before, \( G \) acts without inversions on \( \Gamma \) and \( \Gamma \) has no free edges. We will prove in next Proposition that under these hypotheses the action of \( G \) on \( \Gamma \) is trivial, that is each element of \( G \) acts as the identity on \( \Gamma \). Hence \( G \) fixes each vertex and each edge of \( \Gamma \).

Since \( G \) fixes each 1-handle \( S^2 \times [-1,1] \), it maps each 2-sphere \( S^2 \times \{0\} \) to itself. By construction, \( G \) does not interchange the two sides of such a 2-sphere and acts faithfully on it (otherwise some element of \( G \) would act trivially on an invariant regular neighbourhood of such a 2-sphere and then act trivially also on all of \( \mathcal{H}_g \) (well-known in particular for smooth actions)). It follows that \( G \) is isomorphic to a finite subgroup of the orthogonal group \( \text{SO}(3) \), i.e. cyclic \( \mathbb{Z}_n \), dihedral \( \mathbb{D}_{2n} \), tetrahedral \( A_4 \), octahedral \( S_4 \) or dodecahedral \( A_5 \). It is easy to see that an orientation-preserving action of \( \mathbb{D}_{2n} \), \( A_4 \), \( S_4 \) or \( A_5 \) on \( S^3 \) has at most two global fixed points around which a 1-handle can be attached; but then the graph \( \Gamma \) would be a segment or a circle, that is \( g \leq 1 \). Since \( g > 1 \), \( G \) is a cyclic group which acts by rotations around an axis \( S^1 \) in each 0-handle.
$S^3$. By the positive solution of the Smith-conjecture [13], each of these axes is a trivial knot in $S^3$, and hence the action of the cyclic group $G$ embeds into an $S^1$-action on each 0-handle. Since these $S^1$-actions on the 0-handles extend to the connecting 1-handles $S^2 \times [-1, 1]$, the cyclic $G$-action on $H_g$ embeds into an $S^1$-action.

To complete the proof of Theorem 1, it remains to prove the following Proposition (which may be considered as an analogue of Theorem 1 for finite graphs).

**Proposition.** Let $G$ be a finite group acting faithfully on a finite connected graph $\Gamma$ without free edges and of genus $g > 1$ (or cycle rank, or rank of its free fundamental group). Then also the induced action of $G$ on the first homology $H_1(\Gamma) \cong \mathbb{Z}^g$ of $\Gamma$ is faithful.

**Proof.** By subdividing edges, we can assume that $G$ acts without inversion of edges on $\Gamma$. Suppose that an element $x \in G$ acts trivially on the first homology of $\Gamma$. Then its Lefschetz number is $1 - g$ which, by the Hopf trace formula, is equal to the Euler characteristic of the fixed point set of $x$ which is a subgraph $\Gamma'$ of $\Gamma$ (since $G$ acts without inversions of edges). The graph $\Gamma$ of genus $g$ has Euler characteristic $1 - g$; passing from $\Gamma'$ to $\Gamma$ by adding successively the missing edges, the Euler characteristic remains unchanged (when adding a free edge) or decreases. Since $\Gamma$ has no free edges, this implies $\Gamma' = \Gamma$, and hence $x$ acts trivially on $\Gamma$. This completes the proof of the Proposition.

By [19, proof of Satz 3.1], each finite subgroup of $\text{Out}(F_g)$ can be realized by an action of the group on a finite graph $\Gamma$ without free edges (this is a version of the Nielsen realization problem for finite graphs which several years later was "rediscovered" by various authors); the Proposition implies then the following well-known result.

**Corollary 2.** The canonical projection $\text{Out}(F_g) \rightarrow \text{GL}(g, \mathbb{Z})$ is injective on finite subgroups of $\text{Out}(F_g)$.

We note that not all finite subgroups of $\text{GL}(g, \mathbb{Z})$ are induced in this way by finite subgroups of $\text{Out}(F_g)$; in fact, for $g = 2, 4, 6, 7, 8, 9$ and 10 there are finite subgroups of $\text{GL}(g, \mathbb{Z})$ of orders larger than $2^g g!$ (which, by [16], is the maximal order of a finite subgroup of $\text{Out}(F_g)$). On the other hand, there are also small cyclic subgroups of $\text{GL}(g, \mathbb{Z})$ which cannot be realized in this way, see the discussion in [20, section 5].
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Erratum to:

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Bruno P. Zimmermann

Department of Mathematics, University of Trieste, Italy

In the recent paper [1], we considered the Nielsen realization problem for finite subgroups of the mapping class groups $\text{Mod}(\mathcal{H}_g)$ of the 3-manifolds $\mathcal{H}_g = \#_g(S^2 \times S^1)$ obtained as the connected sums of $g$ copies of $S^2 \times S^1$ (called closed handles of genus $g$ in [1]).

Denoting by $\text{Twist}(\mathcal{H}_g)$ the subgroup of $\text{Mod}(\mathcal{H}_g)$ generated by all Dehn twists along embedded 2-spheres in $\mathcal{H}_g$, there is a short exact sequence

$$1 \to \text{Twist}(\mathcal{H}_g) \hookrightarrow \text{Mod}(\mathcal{H}_g) \to \text{Out}(\pi_1(\mathcal{H}_g)) \to 1,$$

and $\text{Twist}(\mathcal{H}_g) \cong (\mathbb{Z}_2)^g$ is generated by the twists around non-separating 2-spheres in $\mathcal{H}_g$. As a consequence of the main theorem, it is claimed in [1] that no non-trivial element of the twist group $\text{Twist}(\mathcal{H}_g)$ can be realized (represented) by a periodic diffeomorphism of $\mathcal{H}_g$ (in the sense of the Nielsen realization problem); the proof of the main theorem in [1] is incomplete and the theorem has to be modified as follows.

**Theorem.** Let $G$ be a finite group of orientation-preserving diffeomorphisms of a closed handle $\mathcal{H}_g$ of genus $g > 1$. If $G$ induces the trivial action on the first homology of $\mathcal{H}_g$ then $G$ is a cyclic group.

It is further claimed in [1] that $G$ is a subgroup of an $S^1$-action on $\mathcal{H}_g$ but the proof is incomplete since, in general, the various $S^1$-actions on the building blocks (0-handles) of $\mathcal{H}_g$ don’t fit together along the 1-handles to define an $S^1$-action on all of $\mathcal{H}_g$.

**Corollary.** A non-cyclic subgroup of the twist group $\text{Twist}(\mathcal{H}_g)$ cannot be realized by diffeomorphisms of $\mathcal{H}_g$.

It is a consequence of the main theorem in a recent preprint by Lei Chen and Bena Tshishiku [2] that cyclic subgroups of $\text{Twist}(\mathcal{H}_g)$ can be realized by diffeomorphisms; more generally, the authors show that, for every closed oriented 3-manifold $M$ which is a connected sum, a subgroup $G$ of the twist group can be realized if and only if $G$ is cyclic and $M$ is a connected sum of lens spaces (including $S^2 \times S^1$).
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