LOW PERTURBATIONS FOR A CLASS OF NONUNIFORMLY ELLIPTIC PROBLEMS

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ABSTRACT. We introduce and study a new functional which was motivated by our paper on the Caffarelli-Kohn-Nirenberg inequality with variable exponent (Bahrouni, Rădulescu & Repovš, Nonlinearity 31 (2018), 1518-1534). We also study the eigenvalue problem for equations involving this new functional.

1. Introduction

The Caffarelli-Kohn-Nirenberg inequality plays an important role in studying various problems of mathematical physics, spectral theory, analysis of linear and nonlinear PDEs, harmonic analysis, and stochastic analysis. We refer to Chaudhuri & Ramaswamy [2], Baroni, Colombo & Mingione [4], Colasuonno & Pucci [7], and Colombo & Mingione [8] for relevant applications of the Caffarelli-Kohn-Nirenberg inequality.

Let \( Ω \subset \mathbb{R}^N \) (\( N \geq 2 \)) be a bounded domain with smooth boundary. The following Caffarelli-Kohn-Nirenberg inequality (see Caffarelli, Kohn & Nirenberg [5]) establishes that given \( p \in (1, N) \) and real numbers \( a, b, q \) such that

\[-∞ < a < \frac{N - p}{p}, \quad a \leq b \leq a + 1, \quad q = \frac{Np}{N - p(1 + a - b)},\]

there is a positive constant \( C_{a,b} \) such that for every \( u \in C^1_c(Ω) \),

\[ \left( \int_Ω |x|^{-bq} |u|^q \, dx \right)^{p/q} \leq C_{a,b} \int_Ω |x|^{-ap} |\nabla u|^p \, dx. \]

This inequality has been extensively studied (see, e.g. Abdellaoui & Peral [1], Chaudhuri & Ramaswamy [2], Bahrouni, Rădulescu & Repovš [3], Catrina & Wang [6], and Mihăilescu, Rădulescu & Stancu [11], and the references therein).

In particular, Bahrouni, Rădulescu & Repovš [3] gives a new version of the Caffarelli-Kohn-Nirenberg inequality with variable exponent. The next theorem is proved under the following assumptions: let \( Ω \subset \mathbb{R}^N \) (\( N \geq 2 \)) be a bounded domain with smooth boundary and suppose that the following hypotheses are satisfied

(A) \( a : \overline{Ω} \to \mathbb{R} \) is a function of class \( C^1 \) and there exist \( x_0 \in Ω \), \( r > 0 \), and \( s \in (1, +\infty) \) such that:

1. \( |a(x)| \neq 0 \), for every \( x \in \overline{Ω} \setminus \{x_0\} \);
2. \( |a(x)| \geq |x - x_0|^s \), for every \( x \in B(x_0, r) \);

(P) \( p : \overline{Ω} \to \mathbb{R} \) is a function of class \( C^1 \) and \( 2 < p(x) < N \) for every \( x \in Ω \).

Theorem 1.1. (see Bahrouni, Rădulescu & Repovš [3]) Suppose that hypotheses (A) and (P) are satisfied. Let \( Ω \subset \mathbb{R}^N \) (\( N \geq 2 \)) be a bounded domain with smooth boundary. Then there exists a
positive constant \( \beta \) such that
\[
\int_{\Omega} |a(x)|^{p(x)}|u(x)|^{p(x)}dx \leq \beta \int_{\Omega} |a(x)|^{p(x)-1}||\nabla a(x)||u(x)|^{p(x)}dx
+ \beta \left( \int_{\Omega} |a(x)|^{p(x)}|\nabla u(x)|^{p(x)}dx - \int_{\Omega} |a(x)|^{p(x)}|\nabla p(x)||u(x)|^{p(x)+1}dx \right)
+ \beta \int_{\Omega} |a(x)|^{p(x)-1}|\nabla p(x)||u(x)|^{p(x)-1}dx.
\]
for every \( u \in C_0^1(\Omega) \).

Motivated by Bahrouni, Rădulescu & Repovš [3], we introduce and study in the present paper a new functional \( T : E_1 \to \mathbb{R} \) via the Caffarelli-Kohn-Nirenberg inequality, in the framework of variable exponents. More precisely, we study the eigenvalue problem in which functional \( T \) is present. Our main result is Theorem 4.2 and we prove it in Section 5.

2. Function spaces with variable exponent

We recall some necessary properties of variable exponent spaces. We refer to Hajek, Santalucia, Vanderwerff & Zizler [10], Musielak [12], Papageorgiou, Rădulescu & Repovš [13], Rădulescu [15], Rădulescu [16], and Rădulescu & Repovš [17], and the references therein.

Consider the set
\[
C_+ (\Omega) = \{ p \in C(\Omega) \mid p(x) > 1 \text{ for all } x \in \Omega \}.
\]

For any \( p \in C_+ (\Omega) \), let
\[
p^+ = \sup_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \inf_{x \in \Omega} p(x),
\]
and define the variable exponent Lebesgue space as follows
\[
L^{p(x)}(\Omega) = \left\{ u \mid u \text{ is measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},
\]
with the Luxemburg norm
\[
|u|_{p(x)} = \inf \left\{ \mu > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.
\]

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces if and only if \( 1 < p^- \leq p^+ < \infty \), and continuous functions with compact support are dense in \( L^{p(x)}(\Omega) \) if \( p^+ < \infty \).

Let \( L^{q(x)}(\Omega) \) denote the conjugate space of \( L^{p(x)}(\Omega) \), where \( 1/p(x) + 1/q(x) = 1 \). If \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{q(x)}(\Omega) \) then the following Hölder-type inequality holds:
\[
\left| \int_{\Omega} uv \ dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}.
\]

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the \( p(.) \)-modular of the \( L^{p(x)}(\Omega) \) space, which is the mapping \( \rho : L^{p(x)}(\Omega) \to \mathbb{R} \) defined by
\[
\rho(u) = \int_{\Omega} |u|^{p(x)} dx.
\]

**Proposition 2.1.** (see Rădulescu & Repovš [17]) The following properties hold
\( i \) \( |u|_{p(x)} < 1 \) \( \text{resp.} = 1; > 1 \) \( \Leftrightarrow \rho(u) < 1 \) \( \text{resp.} = 1; > 1 \);
\( ii \) \( |u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-} \); and
\( iii \) \( |u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-} \).
Proposition 2.2. (see Rădulescu & Repovš [17]) If $u, u_n \in L^{p(x)}(\Omega)$ and $n \in \mathbb{N}$, then the following statements are equivalent

1. $\lim_{n \to +\infty} |u_n - u|_{p(x)} = 0$.
2. $\lim_{n \to +\infty} \rho(u_n - u) = 0$.
3. $u_n \to u$ in measure in $\Omega$ and $\lim_{n \to +\infty} \rho(u_n) = \rho(u)$.

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \}.$$ 

On $W^{1,p(x)}(\Omega)$ we consider the following norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$ 

Then $W^{1,p(x)}(\Omega)$ is a reflexive separable Banach space.

3. Functional $T$

We shall introduce a new functional $T : E_1 \to \mathbb{R}$ motivated by the Caffarelli-Kohn-Nirenberg inequality obtained in Bahrouni, Rădulescu & Repovš [3].

We denote by $E_1$ the closure of $C^1_c(\Omega)$ under the norm

$$\|u\| = \|B(x)||^{\frac{1}{p(x)}} |\nabla u(x)|_{p(x)} + \|A(x)|^{\frac{1}{p(x)}} u(x)|_{p(x)} +$$

$$\|D(x)|^{\frac{1}{p(x)}} u(x)|_{p(x)+1} + \|C(x)|^{\frac{1}{p(x)}} u(x)|_{p(x)-1},$$

where the potentials $A, B, C,$ and $D$ are defined by

$$\begin{cases}
A(x) = |a(x)|^{p(x)-1} |\nabla a(x)| \\
B(x) = |a(x)|^{p(x)} \\
C(x) = |a(x)|^{p(x)-1} |\nabla p(x)| \\
D(x) = B(x)|\nabla p(x)|.
\end{cases}$$

We now define $T : E_1 \to \mathbb{R}$ as follows

$$T(u) = \int_{\Omega} \frac{B(x)}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_{\Omega} \frac{A(x)}{p(x)} |u(x)|^{p(x)} dx$$

$$+ \int_{\Omega} \frac{D(x)}{p(x)+1} |u(x)|^{p(x)+1} dx + \int_{\Omega} \frac{C(x)}{p(x)-1} |u(x)|^{p(x)-1} dx.$$ 

The following properties of $T$ will be useful in the sequel.

Lemma 3.1. Suppose that hypotheses (A) and (P) are satisfied. Then the functional $T$ is well-defined on $E_1$. Moreover, $T \in C^1(E_1, \mathbb{R})$ with the derivative given by

$$\langle L(u), v \rangle = \langle T'(u), v \rangle = \int_{\Omega} B(x) |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx + \int_{\Omega} A(x) |u(x)|^{p(x)-2} u(x) v(x) dx$$

$$+ \int_{\Omega} D(x) |u(x)|^{p(x)-1} u(x) v(x) dx + \int_{\Omega} C(x) |u(x)|^{p(x)-3} u(x) v(x) dx,$$

for every $u, v \in E_1$.

Proof. The proof is standard, see Rădulescu & Repovš [17].

Lemma 3.2. Suppose that hypotheses (A) and (P) are satisfied. Then the following properties hold

(i) $L : E_1 \to E_1^*$ is a continuous, bounded and strictly monotone operator;

(ii) $L$ is a mapping of type $(S_+)$, i.e. if $u_n \to u$ in $E_1$ and

$$\limsup_{n \to +\infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0,$$

then $u_n \to u$ in $E_1$. 


Proof. (i) Evidently, $L$ is a bounded operator. Recall the following Simon inequalities (see Simon [18]):

\[
\begin{cases}
|x - y|^p \leq c_p \left( \frac{x^{p-2} + y^{p-2}}{x} \right) (|x| - |y|) \quad \text{for } p \geq 2 \\
|x - y|^p \leq C_p \left[ \left( \frac{|x|^{p-2} + |y|^{p-2}}{x} \right) (|x| + |y|) \right]^{\frac{2}{p-2}} \quad \text{for } 1 < p < 2,
\end{cases}
\]

for every $x, y \in \mathbb{R}^N$, where $c_p = \left( \frac{1}{2} \right)^{-p}$ and $C_p = \frac{1}{p-1}$.

Using inequalities (4) and recalling that $2 < p^*$, we can prove that $L$ is a strictly monotone operator.

(ii) The proof is identical to the proof of Theorem 3.1 in Fan & Zhang [9].

\[\square\]

4. Main theorem

We recall our Compactness Lemma:

**Lemma 4.1.** (see Bahrouni, Rădulescu & Repovš [3]) Suppose that hypotheses (A) and (P) are satisfied and that $p^* > 1 + s$. Then $E_1$ is compactly embeddable in $L^q(\Omega)$ for each $q \in (1, \frac{Np^-}{N + sp^+})$. Moreover, the same conclusion holds if we replace $L^q(\Omega)$ by $L^{q(x)}(\Omega)$, provided that $q^+ < \frac{Np^-}{N + sp^+}$.

We are concerned with the following nonhomogeneous problem

\[
\begin{aligned}
-\text{div} (B(x)|\nabla u|^{p(x)-2} \nabla u) + (A(x)|u|^{p(x)-2} + C(x)|u|^{p(x)-3})u &= 0 \\
|u|^{q(x)} - D(x)u^{p(x)-1}u &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $\lambda > 0$ is a real number and $q$ is continuous on $\overline{\Omega}$. We assume that $q$ satisfies the following basic inequalities

\[Q \quad 1 < \min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} (p(x) - 1) < \max_{x \in \overline{\Omega}} q(x) < \frac{Np^-}{N + sp^+}.\]

We can now state the main result of this paper.

**Theorem 4.2.** Suppose that all hypotheses of Lemma 4.1 are satisfied and that inequalities (Q) hold. Then there exists $\lambda_0 > 0$ such that every $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (5).

In order to prove Theorem 4.2 (this will be done in the Section 5), we shall need some preliminary results. We begin by defining the functional $I_\lambda : E_1 \rightarrow \mathbb{R}$,

\[
I_\lambda(u) = \int_{\Omega} \frac{B(x)}{p(x)} |\nabla u(x)|^{p(x)} + \frac{A(x)}{p(x)} |u(x)|^{p(x)} + \frac{C(x)}{p(x)-1} |u(x)|^{p(x)-1} dx \\
+ \lambda \int_{\Omega} |u(x)|^{q(x)} dx - \lambda \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} dx.
\]

Standard argument shows that $I_\lambda \in C^1(E_1, \mathbb{R})$ and

\[
(I_\lambda'(u), v) = \int_{\Omega} B(x) |\nabla u(x)|^{p(x)} \nabla u(x) \nabla v(x) + \int_{\Omega} A(x) |u(x)|^{p(x)-2} u(x) v(x) dx \\
+ \int_{\Omega} D(x) |u(x)|^{p(x)-1} u(x) v(x) dx + \int_{\Omega} C(x) |u(x)|^{p(x)-3} u(x) v(x) dx \\
- \lambda \int_{\Omega} |u(x)|^{q(x)-2} u(x) v(x),
\]

for every $u, v \in E_1$.

Thus the weak solutions of problem (5) coincide with the critical points of $I_\lambda$. 
Lemma 4.3. Suppose that all hypotheses of Theorem 4.2 are satisfied. Then there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \) there exist \( \rho, \alpha > 0 \) such that

\[
I_\lambda(u) \geq \alpha \quad \text{for any} \quad u \in E_1 \quad \text{with} \quad \|u\| = \rho.
\]

Proof. By Lemma 4.1, there exists \( \beta > 0 \) such that

\[
|u|_{r(x)} \leq \beta \|u\|, \quad \text{for every} \quad u \in E_1 \quad \text{and} \quad r^+ \in \left(1, \frac{Np^-}{N+sp^+}\right).
\]

We fix \( \rho \in (0, \min(1, \frac{1}{\beta})) \). Invoking Proposition 2.1, for every \( u \in E_1 \) with \( \|u\| = \rho \), we can get

\[
|u|_{q(x)} < 1.
\]

Combining the above relations and Proposition 2.1, for any \( u \in E_1 \) with \( \|u\| = \rho \), we can then deduce that

\[
I_\lambda(u) \geq \frac{1}{p^+} \left( \int_{\Omega} B(x)|\nabla u(x)|^{p(x)} \, dx + \int_{\Omega} A(x)|u(x)|^{p(x)} \, dx \right)
+ \frac{1}{p^+ + 1} \int_{\Omega} D(x)|u(x)|^{p(x)+1} \, dx
+ \frac{1}{p^+ - 1} \int_{\Omega} C(x)|u(x)|^{p(x)-1} \, dx - \lambda \frac{q^-}{q} \int_{\Omega} |u(x)|^{q(x)} \, dx
\geq \frac{1}{4p^+(p^+ + 1)} \|u\|^{p^+ - 1} - \lambda \frac{q^-}{q} \|u\|^{q^-}
\geq \frac{1}{4p^+(p^+ + 1)} \rho^{p^+ - 1} - \lambda \frac{q^-}{q} \rho^{q^-}
= \rho^{q^-} \left( \frac{1}{4p^+(p^+ + 1)} \rho^{p^+ - 1} - \lambda \frac{q^-}{q} \right).
\]

Put \( \lambda_0 = \frac{\rho^{p^+ + 1} - q^-}{4p^+ (2p^+ + 2)} \frac{q^-}{\beta^q} \). It now follows from (6) that for any \( \lambda \in (0, \lambda_0) \),

\[
I_\lambda(u) \geq \alpha \quad \text{with} \quad \|u\| = \rho,
\]

and \( \alpha = \frac{\rho^{p^+ + 1}}{4p^+ (2p^+ + 2)} > 0 \). This completes the proof of Lemma 4.3. \( \square \)

Lemma 4.4. Suppose that all hypotheses of Theorem 4.2 are satisfied. Then there exists \( \varphi \in E_1 \) such that \( \varphi > 0 \) and \( I_\lambda(t\varphi) < 0 \), for small enough \( t \).

Proof. By virtue of hypotheses \((P)\) and \((Q)\), there exist \( \epsilon_0 > 0 \) and \( \Omega_0 \subset \Omega \) such that

\[
q(x) < q^- + \epsilon_0 < p^- - 1, \quad \text{for every} \quad x \in \Omega_0.
\]

Let \( \varphi \in C_0^\infty(\Omega) \) such that \( \overline{\Omega_0} \subset \text{supp}(\varphi) \), \( \varphi = 1 \) for every \( x \in \overline{\Omega_0} \) and \( 0 \leq \varphi \leq 1 \) in \( \Omega \). It then follows that for \( t \in (0, 1) \),

\[
I_\lambda(t\varphi) = \int_{\Omega} \frac{t^{p(x)}B(x)}{p(x)} |\nabla \varphi(x)|^{p(x)} \, dx + \int_{\Omega} \frac{t^{p(x)}A(x)}{p(x)} |\varphi(x)|^{p(x)} \, dx + \int_{\Omega} \frac{t^{p(x)-1}C(x)}{p(x) - 1} |\varphi(x)|^{p(x)-1} \, dx
+ \int_{\Omega} \frac{t^{p(x)+1}D(x)}{p(x) + 1} |\varphi(x)|^{p(x)+1} \, dx - \lambda t^{q^-} \int_{\Omega} \frac{t^{q(x)}|\varphi(x)|^{q(x)}}{q(x)} \, dx
\leq \frac{t^{p^- - 1}}{p^- - 1} \left( \int_{\Omega} \frac{B(x)}{p(x)} |\nabla \varphi(x)|^{p(x)} \, dx + \int_{\Omega} \frac{A(x)}{p(x)} |\varphi(x)|^{p(x)} \, dx + \int_{\Omega} \frac{C(x)}{p(x) - 1} |\varphi(x)|^{p(x)-1} \, dx \right)
+ \int_{\Omega} \frac{D(x)}{p(x) + 1} |\varphi(x)|^{p(x)+1} \, dx - \lambda t^{q^-} \int_{\Omega} \frac{\varphi(x)|^{q(x)}}{q(x)} \, dx.
\]

Combining (7) and (8), we finally arrive at the desired conclusion.

This completes the proof of Lemma 4.4. \( \square \)
5. Proof of Theorem 4.2

In the last section we shall prove the main theorem of this paper. Let $\lambda_0$ be defined as in Lemma 4.3 and choose any $\lambda \in (0, \lambda_0)$.

Again, invoking Lemma 4.3, we can deduce that
\[
\inf_{u \in \partial B(0, \rho)} I_\lambda(u) > 0.
\]

On the other hand, by Lemma 4.4, there exists $\varphi \in E_1$ such that $I_\lambda(t \varphi) < 0$ for every small enough $t > 0$.

Moreover, by Proposition 2.1, when $\|u\| < \rho$, we have
\[
I_\lambda(u) \geq \frac{1}{4p^+(p^++1)}\|u\|^{p^++1} - c\|u\|^{q-},
\]
where $c$ is a positive constant. It follows that
\[
-\infty < m = \inf_{u \in B(0, \rho)} I_\lambda(u) < 0.
\]

Applying Ekeland's variational principle to the functional $I_\lambda : B(0, \rho) \to \mathbb{R}$, we can find a (PS) sequence $(u_n) \in B(0, \rho)$, that is,
\[
I_\lambda(u_n) \to m \quad \text{and} \quad I'_\lambda(u_n) \to 0.
\]

It is clear that $(u_n)$ is bounded in $E_1$. Thus there exists $u \in E_1$ such that, up to a subsequence, $(u_n) \rightharpoonup u$ in $E_1$.

Using Theorem 4.1, we see that $(u_n)$ strongly converges to $u$ in $L^{q(x)}(\Omega)$.

So, by the H"older inequality and Proposition 2.2, we can obtain the following
\[
\lim_{n \to +\infty} \int_\Omega |u_n|^{q(x)-2}u_n(u_n - u)dx = \lim_{n \to +\infty} \int_\Omega |u|^{q(x)-2}u(u - u)dx = 0.
\]

On the other hand, since $(u_n)$ is a (PS) sequence, we can also infer that
\[
\lim_{n \to +\infty} \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u \rangle = 0.
\]

Combining the above pieces of information with Lemma 3.2, we can now conclude that
\[
(u_n) \to u \quad \text{in} \quad E_1.
\]

Therefore
\[
I_\lambda(u) = m < 0 \quad \text{and} \quad I'_\lambda(u) = 0.
\]

We have thus shown that $u$ is a nontrivial weak solution for problem (5) and that every $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (5).

This completes the proof of Theorem 4.2. \qed

Acknowledgements

The second author was supported by the Slovenian Research Agency grants P1-0292, J1-7025, J1-8131, N1-0064, N1-0083, and N1-0114. We thank the referee for comments and suggestions.
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