Opening of the Haldane Gap in Anisotropic Two- and Four-Leg Spin Ladders

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We study the opening of the Haldane gap in two-leg and four-leg anisotropic spin ladders using bosonization and renormalization group methods, and we determine the phase diagram as a function of the interchain coupling and relative anisotropy, $J^x/J^xy$. It is found that the opening of the Haldane gap is qualitatively different for the two cases considered. For the two-leg ladder the Haldane gap opens for arbitrarily small interchain coupling, independent of $J^x/J^xy$, and the Haldane phase exists in a large region of parameter space. For the four-leg ladder the opening of the Haldane gap is strongly dependent on both the interchain coupling as well as $J^x/J^xy$, and the Haldane phase exists only in a narrow region about the isotropic antiferromagnet.

I. INTRODUCTION

Recently the properties of systems which can be modeled by spin ladders have been extensively studied both experimentally and theoretically. It is well established that the magnetic properties of these materials depend strongly on the number of legs in the ladder. For even-leg ladders the susceptibility vanishes exponentially at low temperatures, while it shows power-law behavior for odd-leg ladders. This difference is due to the fact that the spectrum of magnetic excitations is gapless for odd-leg ladders and gapped for even-leg ladders.

This resembles the alternation of a gapless and gapped spectrum for isotropic antiferromagnetic spin-$S$ chains, where according to Haldane’s conjecture the spectrum is gapless if $S$ is a half-odd-integer while the spectrum is gapped for integer $S$. The two problems are not unrelated, since a spin-$S$ chain can be described as $2S$ coupled spin-1/2 chains provided the interchain coupling is appropriately chosen.

Although the appearance of the Haldane gap in anisotropic spin-$S$ chains has been studied by several groups, most earlier works on coupled chain or ladder models have considered only the case where the coupling between the spins is isotropic. Some exceptions are the work by Watanabe et al. using bosonization techniques, and by Legeza and Sólyom where the density matrix renormalization group was used to determine the phase diagram of anisotropic two-leg ladders. Since the results of the numerical calculations were not reliable enough to get a definitive answer, in this work we use analytic methods to study the opening of the gap in a ladder model when two or four anisotropic spin-1/2 chains are coupled with anisotropic interchain couplings. Our model is such that when the interchain coupling is equal to the coupling along the chains, the two-leg (four-leg) ladder behaves as a spin-1 (spin-2) chain.

The rest of the paper is organized as follows. In Sec. II the Hamiltonian of the spin ladder model is presented, and its relationship to the composite spin representation of spin-$S$ chains is discussed. The model is bosonized in Sec. III, and the results known for spin-1/2 chains are recapitulated. The two- and four-leg ladders are analyzed in Secs. IV and V, respectively, and the phase diagram as a function of anisotropy and interchain coupling is determined. Finally, in Sec. VI we summarize and present a discussion of our results. The paper also contains two Appendices. The first describes our bosonization conventions, while the second gives a derivation of the renormalization group equations.

II. COUPLED SPIN CHAIN MODELS

Consider a system of $p$ ($p=2$ or 4) coupled anisotropic spin-1/2 chains. Denote the spin operator on chain $\lambda$ ($\lambda=1,...,p$) at site $j$ by $s_{j,\lambda}$. The spins along the chains are coupled by a nearest neighbor Heisenberg exchange; the Hamiltonian for chain $\lambda$ is

$$H_\lambda = J^x \sum_j (s^x_{j,\lambda}s^x_{j+1,\lambda} + s^y_{j,\lambda}s^y_{j+1,\lambda}) + J^z \sum_j s^z_{j,\lambda} s^z_{j+1,\lambda}. \tag{1}$$

The coupling between chains can be taken in various forms. In spin ladder systems it is most natural to assume that the dominant interchain coupling acts between spins on the same rung and therefore has the form

$$H_{\lambda,\lambda'} = J^x_{\perp} \sum_j (s^x_{j,\lambda}s^x_{j,\lambda'} + s^y_{j,\lambda}s^y_{j,\lambda'}) + J^z_{\perp} \sum_j s^z_{j,\lambda}s^z_{j,\lambda'}. \tag{2}$$

This kind of the interchain coupling is shown schematically in Fig. 1.

When the interchain coupling is strong enough and ferromagnetic, the ground state and the low lying excited states of the ladder are those in which the $p$ spins on rung $j$ add up to form a larger spin of length $p/2$. Thus in...
this limit the two-leg ladder is equivalent to a spin-1 chain in some of its physical properties, while the four-leg ladder gives an effective spin-2 chain. Strictly speaking this equivalence is valid only in the limit when the rung coupling becomes very strong. Nevertheless, it was established that in the isotropic two-leg ladder the Hal- dane gap is generated for arbitrarily small interchain coupling between them has the form

\[ H = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + J^z \sum_j S_j^z S_{j+1}^z \]  

and writing the spin operator as a sum of \( p = 2S \) spin-1/2 operators

\[ S_j^\alpha = s_{j,1}^\alpha + s_{j,2}^\alpha + \ldots + s_{j,p}^\alpha. \]  

Inserting this into the equation above, we get

\[ H = \sum_{\lambda=1}^p H_{\lambda} + \sum_{\lambda,\lambda'} H_{\lambda\lambda'}, \]  

where \( H_{\lambda} \) is the same as Eq. (2), so the first part is the Hamiltonian of \( p \) spin-1/2 Heisenberg chains, while the interchain coupling between them has the form

\[ H_{\lambda\lambda'} = J_{\perp}^{xy} \sum_j (s_{j,\lambda}^x s_{j+1,\lambda'}^x + s_{j,\lambda}^y s_{j+1,\lambda'}^y) \]

\[ + J_{\perp}^z \sum_j (s_{j,\lambda}^z s_{j+1,\lambda'}^z + s_{j,\lambda}^x s_{j+1,\lambda'}^x), \]  

with

\[ J_{\perp}^{xy} = J^{xy}, \quad J_{\perp}^z = J^z. \]  

One can view this, as shown schematically in Fig. 2 as a ladder in which the interchain coupling is along diagonals.

In this composite spin model the Hilbert space is larger than in the original spin-S model. For example, when two spin-1/2’s are added, in addition to the symmetric combinations that give the triplet \( S = 1 \) state, the antisymmetric combination can also appear that corresponds to a singlet, \( S = 0 \) state. It has been shown, however, that the ground state and the low lying excited levels of this composite-spin model are identical to that of the true \( S = 1 \) model. The configurations where an \( S = 0 \) appears on at least one site have much higher energies than those we are interested in, since they effectively break the chains. Although it has not been checked numerically, it is reasonable that the same holds for the four-chain case, i.e., the lowest lying levels are those where the model is equivalent to the spin-2 chain; configurations where \( S = 0 \) or \( S = 1 \) appear on at least one site effectively break the homogeneous chain, and therefore have higher energies.

In actual ladder materials, in contrast to the cases discussed above, the interchain coupling seems to be antiferromagnetic and act between nearest neighbor spins on the same rung. However, White showed that the two-leg ladder with (isotropic) antiferromagnetic coupling between nearest neighbor spins is in the same universality class as the spin-1 chain.

White’s arguments were the following. Start with the composite spin representation for the spin-1 chain. Take “chain 2” and slide it by one lattice spacing to the left. Now, gradually turn off the coupling between sites \( \sqrt{5} \) away; denote this coupling by \( J_{\perp}^1 \). White was able to show that the system with \( J_{\perp}^1 = 0 \) evolved continuously from the system with \( J_{\perp}^1 = J_{\perp} \) (i.e., There was no change in symmetry and no disappearance or appearance of gap.) (See Fig. 3.)

White’s arguments can be applied to the four-leg ladder. For now, let us only consider isotropic couplings. (The case of anisotropic couplings will be discussed later.)

FIG. 1. Intra- and interchain couplings in a two-leg ladder

FIG. 2. Intra- and interchain couplings in a composite-spin model

FIG. 3. White’s argument showing that the (isotropic) antiferromagnetic two-leg ladder and the spin-1 chain are in the same universality class. (a) Start with the composite spin representation for a spin-1 chain. (b) Slide “chain-2” by one lattice spacing to the left, and gradually turn off the coupling between sites \( \sqrt{5} \) away. (c) We are left with an antiferromagnetic two-leg ladder.
Start with the composite spin representation for the spin-2 chain. Shift “chain 2” and “chain 4” to the left by one lattice spacing. Then, gradually turn off the interchain couplings between sites which are not nearest neighbors. We are left with an antiferromagnetic ladder with only nearest neighbor couplings. Therefore, as long as we have no change in symmetry and no disappearance or appearance of gaps as we turn off the couplings, the antiferromagnetic four-leg ladder will be in the same universality class as the spin-2 chain. It seems reasonable that the four-leg (isotropic) antiferromagnetic ladder and the spin-2 chain are in the same universality class.

One could ask how the Haldane phase of the ladder arises when the interchain coupling is switched on between anisotropic spin-1/2 chains which are gapless for \(-1 \leq \Delta \leq 1\) and become antiferromagnetic with finite gap for \(\Delta > 1\). It is known that for anisotropic chains with integer spin, the Haldane phase shrinks to narrower and narrower ranges around the anisotropic antiferromagnetic point as the spin length increases. Let \(\Delta = J_z/J_{xy}\). Denote the lower boundary of the Haldane phase for spin-1 by \(\Delta_{c1}(S)\) and the upper boundary by \(\Delta_{c2}(S)\). Then

\[
\Delta_{c1}(S_1) < \Delta_{c1}(S_2) < 1 < \Delta_{c2}(S_2) < \Delta_{c2}(S_1),
\]

for \(S_1 < S_2\). Numerical calculations give \(\Delta_{c1}(S = 1) = \varepsilon (\varepsilon > 0, \varepsilon \ll 1)\) and \(\Delta_{c1}(S = 2) \approx 0.9\). In the continuum model of the spin-1 chain, \(\varepsilon = 0\) is obtained.\[1\]

The phase boundaries of the Haldane phase are shown schematically for the spin-1 and spin-2 chains in Fig. 4. Therefore, the interesting range of anisotropy for the ladder is \(0 < \Delta \leq 1\), where the spin-1/2 chain is gapless, while the spin-1 and spin-2 chains are in the Haldane phase for some range of \(\Delta\).

Starting from the gapless situation and coupling the chains, taking into account that at the isotropic antiferromagnetic point the critical value of the interchain coupling is zero, there are two natural ways

\[
\Delta_{c1}(S=1) \quad \Delta_{c2}(S=1) \quad \Delta
\]

spin-1:

\[
\Delta_{c1}(S=2) \quad \Delta_{c3}(S=2) \quad \Delta
\]

spin-2:

FIG. 4. Lower and upper critical anisotropies for the Haldane phase for \(S = 1\) and \(S = 2\).

![FIG. 5. Possible scenarios for the opening of the Haldane gap. (a) & (b) Scenarios for the two-leg ladder. (c) & (d) Scenarios for the four-leg ladder. The shaded region is the Haldane phase.](image)

for the opening of the Haldane gap as the interchain coupling is switched on, as shown in Fig. 5. Our aim is to decide which scenario occurs. For computational reasons, for the rest of the paper we will work with the composite spin model. In order to study how our system evolves from two (four) decoupled spin-1/2 chains into a ladder that is equivalent to a spin-1 (spin-2) chain, as well the roles of the planar and Ising like parts of the interchain coupling individually, the Hamiltonian in Eq. \(\text{(5)}\) will be generalized by allowing for \(J_{xy}^1 \neq J_{xy}^2\) and \(J_\perp^1 \neq J_\perp^2\) in Eq. \(\text{(4)}\).

## III. BOSONIZING SPIN CHAINS

While the spin-1/2 Heisenberg chain can be solved exactly by the Bethe-Ansatz, models with higher spin values can be treated, even in one-dimension, only approximately. One such approximation is to express the spin model in terms of fermions or bosons and develop a scaling theory for these degrees of freedom. In this section, we outline the key ingredients needed to fermionize and bosonize spin models.

### A. The Spin-1/2 Chain

First consider a single anisotropic spin-1/2 chain described by the Hamiltonian \(\text{(5)}\), where the chain index \(\lambda\) will be dropped. The Jordan-Wigner transformation

\[
\begin{align*}
\tilde{s}_j^+ &= s_j^x + is_j^y = c_j^\dagger \exp \left( i\pi \sum_{l=0}^{j-1} c_l^\dagger c_l \right), \\
\tilde{s}_j^- &= s_j^x - is_j^y = \exp \left( -i\pi \sum_{l=0}^{j-1} c_l^\dagger c_l \right) c_j,
\end{align*}
\]

(9)
where allows us to write this Hamiltonian in terms of spinless fermions

\[ s_j^+ = c_j^\dagger c_j - \frac{1}{2}, \]

or because of the fermionic nature of the \( c_j, c_j^\dagger \) operators the equivalent form

\[ s_j^+ = c_j^\dagger \cos \left( \pi \sum_{i=0}^{j-1} c_i^\dagger c_i \right), \quad s_j^- = \cos \left( \pi \sum_{i=0}^{j-1} c_i^\dagger c_i \right) c_j, \]

allows us to write this Hamiltonian in terms of spinless fermions

\[ H = \frac{1}{2} J^{xy} \sum_j \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + J^z \sum_j \left( c_j^\dagger c_j - \frac{1}{2} \right) \left( c_{j+1}^\dagger c_{j+1} - \frac{1}{2} \right). \]  

(11)

After Fourier transforming to momentum space

\[ H = \frac{1}{2} J^{xy} \sum_k \cos(ka) c_k^\dagger c_k + \frac{J^z}{N} \sum_q \cos(qa) \rho(q) \rho(-q), \]

where \( a \) is the lattice constant, \( N \) the number of sites in the chain, and

\[ \rho(q) = \sum_k c_k^\dagger c_{k+q}. \]

(13)

In the spin liquid phase where there is no net magnetic moment, the band is half filled.

To study the low-energy properties of the model, we can linearize the spectrum around the Fermi points, \( k_F = \pm \pi/2a \). In this way a Luttinger model is obtained, where \( v_F = J^{xy} a \) is the unrenormalized velocity. In what follows we will work with a rescaled Hamiltonian, dividing \( H \) by \( J^{xy} a \), and use a dimensionless coupling \( \Delta = J^z/J^{xy} \) (the relative anisotropy).

Since the low-energy excitations of the Luttinger liquid are density fluctuations (which are bosons), it is convenient to use, in the continuum limit, a further transformation from fermions to bosons. For clarity and completeness, we have given a description of our bosonization conventions in Appendix A.

For \(-1 \leq \Delta \leq 1\) the spin-1/2 Heisenberg chain is known to be critical, the continuum bosonized form of the Hamiltonian is

\[ H = -\frac{u}{2} \int dx \left[ \nabla^2 - \frac{1}{K} (\partial_x \Phi)^2 \right]. \]

(14)

For \(|\Delta| \ll 1\) the parameters \( u \) and \( K \) can be determined perturbatively. However, the general form of \( u \) and \( K \) can be obtained by comparison with the Bethe-Ansatz solution. They are given by

\[ K = \frac{\pi}{2(\pi - \arccos \Delta)}, \quad u = \frac{\pi}{2 \arccos \Delta}. \]

(15)

An important property is that for \( \Delta = 0 \) (i.e., the free fermion point) \( K = 1 \), and for \( \Delta = 1 \) (i.e., the isotropic point) \( K = 1/2 \).

We will also need to know the bosonized form for the spin operators. Using the bosonization rules of Appendix A, in the continuum limit they can be shown to be

\[ s_j^+ (x) = \frac{s_j^+}{a} = \exp \left( -i \pi \frac{x}{a} \Phi \right) \left[ e^{-i\pi(x/a)} + \cos(\sqrt{4\pi} \Phi) \right], \]

\[ s_j^- (x) = \frac{s_j^-}{a} = T \partial_x \Phi + e^{i\pi(x/a)} \sin(\sqrt{4\pi} \Phi) \frac{\pi}{a}. \]

(16)

**B. Coupled Spin-1/2 Chains**

For coupled spin-1/2 chains the same procedure can be used to express the spin operators in terms of boson fields, by simply attaching a \( \lambda \) index to the field. The interchain part of the Hamiltonian will have the same form as Eq. (14). After rescaling by the bare velocity, \( J^{xy} a \), we write the Hamiltonian of the interchain coupling term in the form

\[ H_{\lambda \lambda'} = \frac{1}{a} \sum_j \left( s_{j,\lambda}^x s_{j+1,\lambda'}^x + s_{j,\lambda}^y s_{j+1,\lambda'}^y + s_{j,\lambda}^z s_{j+1,\lambda'}^z \right) \]

\[ + \frac{1}{a} \sum_j \left( s_{j,\lambda}^x s_{j+1,\lambda'}^y + s_{j,\lambda}^y s_{j+1,\lambda'}^x \right), \]

(17)

where we have let

\[ J^{xy}_{\lambda \lambda'} = J^{xy} \] and \( J_{\lambda \lambda'}^z \)

Using the bosonized form for the spin operators, we get

\[ H_{\lambda \lambda'} = \int dx \left[ \frac{g_1}{(2\pi a)^2} \cos(\sqrt{4\pi} (\Phi_\lambda + \Phi_{\lambda'})) \right] \]

\[ + \frac{2J_{\lambda \lambda'}^z}{\pi} \int dx \partial_x \Phi_\lambda \partial_x \Phi_{\lambda'} \]

\[ + g_4 \int dx \left[ \frac{g_4}{(2\pi a)^2} \cos(\sqrt{4\pi} (\Theta_{\lambda} - \Theta_{\lambda'})) \right] \]

\[ \times \cos(\sqrt{4\pi} (\Phi_\lambda + \Phi_{\lambda'})), \]

(18)

where

\[ g_1 = 4J_{\lambda \lambda'}^z, \quad g_2 = -4J_{\lambda \lambda'}^z, \quad g_3 = -4\pi J_{\lambda \lambda'}^{xy}, \quad g_4 = g_5 = 2\pi J_{\lambda \lambda'}^{xy}. \]

(19)

Note that the coupling between the \( x \) and \( y \) components of the spins on different chains is non-local.
IV. THE TWO-LEG LADDER

We first consider how the Haldane gap is generated in the anisotropic two-leg ladder when two gapless spin-1/2 chains are coupled by an interchain coupling of the form given in Eq. (6). The Hamiltonian of the system is

\[ H = H_1 + H_2 + H_{1,2}. \]  

(20)

As mentioned before for \( J_{x}^{xy} = 1 \) and \( J_{z}^{z} = J^{z} \), our Hamiltonian is equivalent, as far as the low-lying states are concerned, to the true spin-1 Heisenberg chain. Therefore, we have a model which interpolates between two uncoupled spin-1/2 chains and a spin-1 chain.

The bosonized form of the Hamiltonian is

\[
H = \frac{u}{2} \int dx \left[ K_{1} \Pi_{1}^{2} + \frac{1}{K_{1}} (\partial_x \phi_1)^2 \right] \\
+ \frac{u}{2} \int dx \left[ K_{2} \Pi_{2}^{2} + \frac{1}{K_{2}} (\partial_x \phi_2)^2 \right] \\
+ \frac{2 J_{\perp}^{z}}{\pi} \int dx \partial_x \phi_1 \partial_x \phi_2 \\
+ \int \frac{dx}{(2\pi a)^2} \left[ g_1 \cos \left( \sqrt{4\pi} (\phi_1 + \phi_2) \right) \\
+ g_2 \cos \left( \sqrt{4\pi} (\phi_1 - \phi_2) \right) \\
+ g_3 \cos \left( \sqrt{2\pi} (\Theta_1 - \Theta_2) \right) \\
\times \cos \left( \sqrt{4\pi} (\phi_1 + \phi_2) \right) \\
+ g_4 \cos \left( \sqrt{2\pi} (\Theta_1 - \Theta_2) \right) \\
\times \cos \left( \sqrt{4\pi} (\phi_1 - \phi_2) \right) \right].
\]

(21)

It is useful to define the fields

\[
\Phi_s = \frac{1}{\sqrt{2}} (\phi_1 + \phi_2), \quad \Phi_a = \frac{1}{\sqrt{2}} (\phi_1 - \phi_2).
\]

(22)

In terms of these fields our Hamiltonian has the form

\[
H = \frac{u_s}{2} \int dx \left[ K_{s} \Pi_{s}^{2} + \frac{1}{K_{s}} (\partial_x \phi_s)^2 \right] \\
+ g_1 \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{4\pi} \Phi_s \right) \\
+ \frac{u_{a}}{2} \int dx \left[ K_{a} \Pi_{a}^{2} + \frac{1}{K_{a}} (\partial_x \phi_a)^2 \right] \\
+ \int \frac{dx}{(2\pi a)^2} \left[ g_2 \cos \left( \sqrt{4\pi} \Phi_a \right) + g_3 \cos \left( \sqrt{2\pi} \Theta_a \right) \right] \\
+ g_4 \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{2\pi} \Theta_a \right) \cos \left( \sqrt{4\pi} \Phi_a \right) \\
+ g_5 \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{2\pi} \Theta_a \right) \cos \left( \sqrt{8\pi} \Phi_a \right),
\]

(23)

where

\[ K_{s} = K \left( 1 + \frac{2K J_{s}^{z}}{u_{s}} \right)^{-1/2}, \quad u_{s} = u \left( 1 + \frac{2K J_{s}^{z}}{u_{s}} \right)^{1/2}, \]

(24)

\[ K_{a} = K \left( 1 - \frac{2K J_{s}^{z}}{u_{s}} \right)^{-1/2}, \quad u_{a} = u \left( 1 - \frac{2K J_{s}^{z}}{u_{s}} \right)^{1/2}. \]

(25)

For \( J_{\perp}^{x} J_{\perp}^{z} \ll 1 \) we have

\[ K_{s} \approx K \left( 1 - \frac{K J_{s}^{z}}{u_{s}} \right), \quad u_{s} \approx u \left( 1 + \frac{K J_{s}^{z}}{u_{s}} \right), \]

\[ K_{a} \approx K \left( 1 + \frac{K J_{s}^{z}}{u_{s}} \right), \quad u_{a} \approx u \left( 1 - \frac{K J_{s}^{z}}{u_{s}} \right). \]

We are interested in whether or not the interchain coupling causes a gap in the excitation spectrum. Therefore, we would like to identify the relevant operators; these operators will “pin” their arguments, thus causing gaps to appear. To do this we consider the scaling dimensions of the operators in the interchain coupling (see Appendix A). The scaling dimension of \( \cos (\sqrt{8\pi} \Phi_a) \) is \( 2K_{a} \); the scaling dimension of \( \cos (\sqrt{8\pi} \Phi_s) \) is \( 2K_{s} \); the scaling dimension of \( \cos (\sqrt{2\pi} \Theta_a) \) is \( 1/(2K_{a}) \); the scaling dimension of \( \cos (\sqrt{2\pi} \Theta_a) \cos (\sqrt{4\pi} \Phi_a) \) is \( 2K_{s}+1/(2K_{a}) \); the scaling dimension of \( \cos (\sqrt{2\pi} \Theta_a) \cos (\sqrt{2\pi} \Theta_a) \cos (\sqrt{4\pi} \Phi_a) \) is \( 2K_{a}+1/(2K_{a}) \); Therefore, \( g_1 \) will grow at large distances for \( K_{a} < 1 \); \( g_2 \) will grow for \( K_{a} < 1 \); \( g_3 \) will grow for \( K_{a} > 1/4 \). The \( g_{1} \) operator is always irrelevant, and therefore will be ignored. However, we will keep the \( g_{4} \) term since, though seemingly irrelevant, it is the most relevant operator generated by the interchain coupling which couples the symmetric and antisymmetric modes. We will see that the \( g_{4} \) term plays a subtle role.

It is interesting to study the behavior of the \( xy \) and \( z \) components of the interchain coupling individually. First we consider the case \( J_{\perp}^{xy} = 0 \) and \( J_{\perp}^{z} \neq 0 \). Therefore, only \( g_3 \) and \( g_2 \) are nonzero. The \( g_{1} \) term is relevant for \( 0 \leq \Delta \leq 1 \). The \( g_{2} \) term is relevant for \( K < 1 - \frac{J_{s}^{z}}{\pi u_{s}} \) (for weak interchain coupling.) Therefore, the symmetric mode is gapped in the entire region \( 0 \leq \Delta \leq 1 \), while the antisymmetric mode remains gapless in a region about \( \Delta = 0 \) and is gapped outside of this region. In the region where \( \Phi_a \) is pinned, the antisymmetric mode is in an ordered phase.

Next we consider the case \( J_{\perp}^{xy} \neq 0 \) and \( J_{\perp}^{z} = 0 \). Therefore, only \( g_3 \) and \( g_4 \) are nonzero. For \( 0 \leq \Delta \leq 1 \), the \( g_{3} \) term is relevant while the \( g_{4} \) term is irrelevant. Therefore, \( g_{3} \) will grow while \( g_{4} \) will initially decrease under the RG. However, once \( g_{3} \) grows to \( O(1) \), we can safely say that \( \Theta_a \) is pinned. At this point, assuming that \( g_{4} \) is renormalized to \( g_{4}^{*} \), the term that couples the symmetric and antisymmetric modes can be written as

\[ g_{4}^{*} \int \frac{dx}{(2\pi a)^2} \left\langle \cos (\sqrt{2\pi} \Theta_a) \cos (\sqrt{8\pi} \Phi_s) \right\rangle \approx g_{4}^{*} \left\langle \cos (\sqrt{2\pi} \Theta_a) \right\rangle \int \frac{dx}{(2\pi a)^2} \cos (\sqrt{8\pi} \Phi_s). \]

(26)
The expectation value is taken with respect to

\[ H_a = \frac{u_a}{2} \int dx \left[ K_a \Pi_s^2 + \frac{1}{K_a} (\partial_x \Phi_a)^2 \right] + g_3 \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{2\pi} \Theta_a \right), \tag{27} \]

where \( K_a \) and \( g_3 \) are the renormalized values of \( K_a \) and \( g_3 \). Then we can define a new coupling constant

\[ g_4 = g_3 \left\langle \cos \left( \sqrt{2\pi} \Theta_a \right) \right\rangle. \tag{28} \]

Therefore, our effective Hamiltonian for the symmetric mode is

\[ H_{\text{eff}} = \frac{u_s}{2} \int dx \left[ K_s \Pi_s^2 + \frac{1}{K_s} (\partial_x \Phi_s)^2 \right] + g_4 \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{8\pi} \Phi_s \right), \tag{29} \]

where \( K_s \) is the renormalized value of \( K_s \). Now \( \cos \left( \sqrt{8\pi} \Phi_s \right) \) is relevant for \( K_s < 1 \). Therefore, the term is relevant. We see that \( J_{\perp}^T \) alone generates a gap in both the symmetric and antisymmetric modes. However, since \( \Theta_a \) was pinned, the antisymmetric mode is in a disordered phase.\[1\]

TABLE I. Results for the two-leg ladder in the region \( 0 \leq \Delta \leq 1 \) (i.e. \( 1 \geq K \geq 1/2 \)).

| \( J_{\perp}^T = 0, J_{\parallel}^T \neq 0 \) | \( J_{\parallel}^T = 0, J_{\perp}^T \neq 0 \) | \( J_{\perp}^T / J_{\parallel}^T = \Delta \) |
|---|---|---|
| \( \Phi_s \) | pinned | pinned |
| \( \Phi_a, \Theta_a \) | pinned for \( K_a < 1 \), ordered | \( \Theta_a \) pinned, disordered |
| | | \( \Theta_a \) pinned, disordered |
| Phase | Gapless for \( K_a < 1 \), Haldane for \( K_a < 1 \) | Haldane phase |

Finally, we consider the case \( J_{\parallel}^T \neq 0 \) and \( J_{\perp}^T \neq 0 \) with \( J_{\parallel}^T / J_{\perp}^T = \Delta \). Therefore, for \( J_{\parallel}^T = 1 \), we recover the results for the anisotropic spin-1 chain. The \( g_1 \) and \( g_3 \) terms are relevant in the entire range \( 0 \leq \Delta \leq 1 \), and \( g_3 \) is always more relevant than \( g_2 \). We see that both the symmetric and antisymmetric modes will get gapped, with the antisymmetric mode being in a disordered phase; the Haldane gap is generated for arbitrarily small interchain coupling.

The results for the two-leg ladder are summarized in Table I.

V. THE FOUR-LEG LADDER

Next we study the opening of the Haldane gap in the four-leg ladder. The Hamiltonian is

\[ H = H_1 + H_2 + H_3 + H_4 + H_{1,2} + H_{1,3} + H_{1,4} + H_{2,3} + H_{2,4} + H_{3,4}. \tag{30} \]

Let us introduce \( \Phi^T = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) \) and \( \Pi^T = (\Pi_1, \Pi_2, \Pi_3, \Pi_4) \). Then in its bosonized form our Hamiltonian is

\[
H = \int dx \left[ \frac{u}{2} \Pi^T \Pi + (\partial_x \Phi)^T M(\partial_x \Phi) \right] + g_1 \int \frac{dx}{(2\pi a)^2} \left[ \cos (\sqrt{4\pi}(\Phi_2 + \Phi_1)) + \cos (\sqrt{4\pi}(\Phi_3 + \Phi_1)) + \cos (\sqrt{4\pi}(\Phi_4 + \Phi_1)) \right. \\
+ \cos \left( \sqrt{4\pi}(\Phi_3 + \Phi_2) \right) + \cos \left( \sqrt{4\pi}(\Phi_4 + \Phi_2) \right) + \cos \left( \sqrt{4\pi}(\Phi_4 + \Phi_3) \right) \\
+ g_2 \int \frac{dx}{(2\pi a)^2} \left[ \cos (\sqrt{4\pi}(\Phi_2 - \Phi_1)) + \cos (\sqrt{4\pi}(\Phi_3 - \Phi_1)) + \cos (\sqrt{4\pi}(\Phi_4 - \Phi_1)) \right. \\
+ \cos \left( \sqrt{4\pi}(\Phi_3 - \Phi_2) \right) + \cos \left( \sqrt{4\pi}(\Phi_4 - \Phi_2) \right) + \cos \left( \sqrt{4\pi}(\Phi_4 - \Phi_3) \right) \\
+ g_3 \int \frac{dx}{(2\pi a)^2} \left[ \cos (\sqrt{4\pi}(\Theta_2 - \Theta_1)) + \cos (\sqrt{4\pi}(\Theta_3 - \Theta_1)) + \cos (\sqrt{4\pi}(\Theta_4 - \Theta_1)) \right. \\
+ \cos \left( \sqrt{4\pi}(\Theta_3 - \Theta_2) \right) + \cos \left( \sqrt{4\pi}(\Theta_4 - \Theta_2) \right) + \cos \left( \sqrt{4\pi}(\Theta_4 - \Theta_3) \right) \\
+ g_4 \int \frac{dx}{(2\pi a)^2} \left. \left[ \cos (\sqrt{4\pi}(\Theta_2 - \Theta_1)) \cos (\sqrt{4\pi}(\Phi_2 + \Phi_1)) + \cos (\sqrt{4\pi}(\Theta_3 - \Theta_1)) \cos (\sqrt{4\pi}(\Phi_3 + \Phi_1)) \right. \\
+ \cos (\sqrt{4\pi}(\Theta_4 - \Theta_1)) \cos (\sqrt{4\pi}(\Phi_4 + \Phi_1)) + \cos (\sqrt{4\pi}(\Theta_3 - \Theta_2)) \cos (\sqrt{4\pi}(\Phi_3 + \Phi_2)) \right].
\]
Define the fields \( \Phi \) and \( \Lambda \). It should be noted that the choice of \( S \) is not unique. Define \( A \) by

\[
\Lambda = S^T MS.
\]

\( M \) is given by

\[
M = \begin{pmatrix}
  a & b & b & b \\
  b & a & b & b \\
  b & b & a & b \\
  b & b & b & a \\
\end{pmatrix}
\] (32)

with \( a = u/(2K) \) and \( b = J_s^+ / \pi \). \( M \) is diagonalized by the orthogonal matrix \( S \)

\[
S = \begin{pmatrix}
  1 & 1 & 1 & 1 \\
  1 & -1 & 1 & -1 \\
  1 & 1 & -1 & -1 \\
  1 & -1 & -1 & 1 \\
\end{pmatrix}
\] (33)

It should be noted that the choice of \( S \) is not unique.

Define \( \Lambda \) by

\[
\Lambda = S^T MS.
\]

\( \Lambda \) is given by

\[
\Lambda = \begin{pmatrix}
  a + 3b & 0 & 0 & 0 \\
  0 & a - b & 0 & 0 \\
  0 & 0 & a - b & 0 \\
  0 & 0 & 0 & a - b \\
\end{pmatrix}
\] (35)

Define the fields \( \Phi_s, \Phi_{a1}, \Phi_{a2}, \) and \( \Phi_{a3} \) by

\[
\begin{pmatrix}
  \Phi_1 \\
  \Phi_2 \\
  \Phi_3 \\
  \Phi_4 \\
\end{pmatrix} = \begin{pmatrix}
  \Phi_s \\
  \Phi_{a1} \\
  \Phi_{a2} \\
  \Phi_{a3} \\
\end{pmatrix}
\] (36)

In terms of the old fields, the new fields are given by

\[
\begin{align*}
\Phi_s & = \frac{1}{2} (\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4) \\
\Phi_{a1} & = \frac{1}{2} (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4) \\
\Phi_{a2} & = \frac{1}{2} (\Phi_1 - \Phi_2 - \Phi_3 + \Phi_4) \\
\Phi_{a3} & = \frac{1}{2} (\Phi_1 - \Phi_2 + \Phi_3 - \Phi_4).
\end{align*}
\] (37)

We see that \( \Phi_s \) is a symmetric mode while \( \Phi_{a1}, \Phi_{a2}, \) and \( \Phi_{a3} \) are antisymmetric modes.

Let us define \( \tilde{\Phi}^T = (\Phi_s, \Phi_{a1}, \Phi_{a2}, \Phi_{a3}) \) and \( \tilde{\Pi}^T = (\Pi_s, \Pi_{a1}, \Pi_{a2}, \Pi_{a3}) \). Then \( \Phi = S \tilde{\Phi} \) and \( \Pi = S \tilde{\Pi} \). In terms of the new fields, the quadratic part of the Hamiltonian is given by

\[
H_0 = \int dx \left[ \frac{u_K}{2} \tilde{\Pi}^T \tilde{\Pi} + (\partial_x \tilde{\Phi})^T \Lambda (\partial_x \tilde{\Phi}) \right].
\] (38)

We can write this as

\[
H_0 = \frac{u_s}{2} \int dx \left[ K_s \Pi_s^2 + \frac{1}{K_s} (\partial_x \Phi_s)^2 \right]
\]

\[
+ \sum_{i=1}^3 \frac{u_a}{2} \int dx \left[ K_{a_i} \Pi_{a_i}^2 + \frac{1}{K_{a_i}} (\partial_x \Phi_{a_i})^2 \right]
\] (39)

with

\[
u_s = u \left( 1 + \frac{6KJ_z^+}{u \pi} \right)^{1/2}, \quad K_s = K \left( 1 + \frac{6KJ_z^+}{u \pi} \right)^{-1/2},
\]

\[
u_a = u \left( 1 - \frac{2KJ_z^+}{u \pi} \right)^{1/2}, \quad K_a = K \left( 1 - \frac{2KJ_z^+}{u \pi} \right)^{-1/2}.
\] (40)

For \( J_{z+}^+, J_{z-}^+ \ll 1 \) we have

\[
u_s \approx u \left( 1 + \frac{3KJ_z^+}{u \pi} \right), \quad K_s \approx K \left( 1 - \frac{3KJ_z^+}{u \pi} \right),
\]

\[
u_a \approx u \left( 1 - \frac{KJ_z^+}{u \pi} \right), \quad K_a \approx K \left( 1 + \frac{KJ_z^+}{u \pi} \right).
\] (41)

In terms of these new fields, the interchain coupling is given by

\[
2g_1 \int \frac{dx}{(2\pi a)^2} \left[ \cos \left( \sqrt{4\pi} \Phi_{a1} \right) \cos \left( \sqrt{4\pi} \Phi_s \right) + \cos \left( \sqrt{4\pi} \Phi_{a2} \right) \cos \left( \sqrt{4\pi} \Phi_s \right) + \cos \left( \sqrt{4\pi} \Phi_{a3} \right) \cos \left( \sqrt{4\pi} \Phi_s \right) \right]
\]

\[
+ 2g_2 \int \frac{dx}{(2\pi a)^2} \left[ \cos \left( \sqrt{4\pi} \Phi_{a1} \right) \cos \left( \sqrt{4\pi} \Phi_{a2} \right) + \cos \left( \sqrt{4\pi} \Phi_{a3} \right) \cos \left( \sqrt{4\pi} \Phi_{a1} \right) + \cos \left( \sqrt{4\pi} \Phi_{a2} \right) \cos \left( \sqrt{4\pi} \Phi_{a3} \right) \right]
\]

\[
+ 2g_3 \int \frac{dx}{(2\pi a)^2} \left[ \cos \left( \sqrt{4\pi} \Phi_{a1} \right) \cos \left( \sqrt{4\pi} \Phi_{a2} \right) + \cos \left( \sqrt{4\pi} \Phi_{a3} \right) \cos \left( \sqrt{4\pi} \Phi_{a1} \right) + \cos \left( \sqrt{4\pi} \Phi_{a2} \right) \cos \left( \sqrt{4\pi} \Phi_{a3} \right) \right]
\]

\[
+ g_4 \int \frac{dx}{(2\pi a)^2} \left[ \cos \left( \sqrt{4\pi} \Phi_{a1} \right) \cos \left( \sqrt{4\pi} \Phi_{a2} \right) + \cos \left( \sqrt{4\pi} \Phi_{a3} \right) \cos \left( \sqrt{4\pi} \Phi_{a1} \right) \right] + \cos \left( \sqrt{4\pi} (\Phi_4 - \Phi_1) \right) \cos \left( \sqrt{4\pi} (\Phi_4 - \Phi_3) \right) \cos \left( \sqrt{4\pi} (\Phi_4 - \Phi_3) \right)
\]
\[ + \cos (\sqrt{\pi} (\Theta_{a2} + \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s + \Phi_{a3})) + \cos (\sqrt{\pi} (\Theta_{a2} - \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s - \Phi_{a3})) \]
\[ + \cos (\sqrt{\pi} (\Theta_{a3} + \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s + \Phi_{a2})) + \cos (\sqrt{\pi} (\Theta_{a3} - \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s - \Phi_{a2})) \]
\[ + g_5 \int \frac{dx}{(2\pi a)^2} \left[ \cos (\sqrt{\pi} (\Theta_{a3} + \Theta_{a2})) \cos (\sqrt{4\pi} (\Phi_s + \Phi_{a3})) + \cos (\sqrt{\pi} (\Theta_{a3} - \Theta_{a2})) \cos (\sqrt{4\pi} (\Phi_s - \Phi_{a3})) \right] \]
\[ + \cos (\sqrt{\pi} (\Theta_{a1} + \Theta_{a3})) \cos (\sqrt{4\pi} (\Phi_s + \Phi_{a1})) + \cos (\sqrt{\pi} (\Theta_{a1} - \Theta_{a3})) \cos (\sqrt{4\pi} (\Phi_s - \Phi_{a1})) \]
\[ + \cos (\sqrt{\pi} (\Theta_{a2} + \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s + \Phi_{a2})) + \cos (\sqrt{\pi} (\Theta_{a2} - \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s - \Phi_{a2})) \]
\[ + \cos (\sqrt{\pi} (\Theta_{a3} + \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s + \Phi_{a3})) + \cos (\sqrt{\pi} (\Theta_{a3} - \Theta_{a1})) \cos (\sqrt{4\pi} (\Phi_s - \Phi_{a3})) \right] . \tag{42} \]

We see that \( g_1 \) and \( g_4 \) couples the symmetric mode to the antisymmetric modes, while \( g_2, g_3, \) and \( g_5 \) only couple the antisymmetric modes among themselves.

Just like for the two-leg ladder, we would like to know the relevance or irrelevance of the operators in the interchain coupling. Therefore, we consider the scaling dimension of the operators in the interchain coupling. The \( g_1 \) terms have scaling dimension \( K_s + K_a \); the \( g_2 \) terms have scaling dimension \( 2K_s \); the \( g_3 \) terms have scaling dimension \( 1/(2K_s) \); the \( g_4 \) terms have scaling dimension \( K_s + K_a + 1/(2K_s) \); the \( g_5 \) terms have scaling dimension \( 2K_a + 1/(2K_a) \). Therefore, the \( g_1 \) terms are relevant for \( K_a + K_s < 2 \); the \( g_2 \) terms are relevant for \( K_a < 1 \); the \( g_3 \) terms are relevant for \( K_a > 1/4 \). The \( g_5 \) terms are always irrelevant and will be dropped. However, we will keep the \( g_4 \) terms. Though they are irrelevant, they are the most relevant terms (arising from the \( xy \) part of the interchain coupling) which couple the symmetric and antisymmetric modes. We will see that the \( g_4 \) terms play a subtle role.

Since the \( g_1, g_2, \) and \( g_3 \) terms are all relevant and all of the modes are coupled together, the physics is determined by the operators which reach strong coupling first under the RG. The RG equations for the parameters are

\[ \frac{dg_1}{dl} = [2 - (K_s + K_a)] g_1 \]
\[ \frac{dg_2}{dl} = (2 - 2K_a) g_2 \]
\[ \frac{dg_3}{dl} = \left[ 2 - \frac{1}{2K_a} \right] g_3 \]
\[ \frac{dg_4}{dl} = \left[ 2 - (K_s + K_a + \frac{1}{2K_a}) \right] g_4 \]

\[ \frac{dK_s}{dl} = -6g_1^2 \left( \frac{K_s}{4\pi u_s} \right)^2 - 3g_1^2 \left( \frac{K_a}{4\pi u_a} \right)^2 \]
\[ \frac{dK_a}{dl} = -2g_2^2 \left( \frac{K_a}{4\pi u_a} \right)^2 - 4g_2^2 \left( \frac{K_a}{4\pi u_a} \right)^2 + g_3^2 \left( \frac{1}{4\pi u_a} \right)^2 \]
\[ - \frac{1}{2} g_4^2 \left( \frac{K_a}{4\pi u_a} \right)^2 + \frac{1}{2} g_4^2 \left( \frac{1}{4\pi u_a} \right)^2 . \tag{43} \]

See Appendix B for a derivation of the RG equations.

Just like for the two-leg ladder, it is interesting to study the behavior of the \( xy \) and \( z \) components of the interchain coupling individually. First we consider the case \( J_{1y}^+ = 0 \) and \( J_{1y}^- \neq 0 \). For this case the initial values of the coupling constants are \( g_1(l = 0) = 4J_{1y}^-, g_2(l = 0) = -4J_{1y}^-, g_3(l = 0) = 0, \) and \( g_4(l = 0) = 0 \). The \( g_1 \) terms are relevant for \( 0 \leq \Delta \leq 1 \). The \( g_2 \) terms are relevant for \( K < 1 - \frac{J_{1y}^-}{\pi} \) (for weak interchain coupling.) However, since all of the modes are coupled (and hence locked together), both the symmetric and antisymmetric modes are gapped in the entire region \( 0 \leq \Delta \leq 1 \). Notice that since the \( \Phi_{ai} \) were pinned, the antisymmetric modes are in an ordered phase.

Next we consider the case \( J_{1y}^+ \neq 0 \) and \( J_{1y}^- = 0 \). For this case the initial values of the coupling constants are \( g_1(l = 0) = 0, g_2(l = 0) = 0, g_3(l = 0) = -4\pi J_{1y}^+, g_4(l = 0) = 2\pi J_{1y}^- / K_s(l = 0) = K, K_a(l = 0) = K, u_s = u, \) and \( u_a = u \). The \( g_3 \) terms are relevant and the \( g_4 \) terms are irrelevant. Therefore, \( g_3 \) will grow and \( g_4 \) will initially decrease under the RG. However, once \( g_3 = \mathcal{O}(1) \) we can safely say that the \( \Theta_{ai} \) are pinned. Then we can write the \( g_4 \) term as

\[ \frac{dK_{1y}^+}{dl} = -6g_1^2 \left( \frac{K_{1y}^+}{4\pi u_s} \right)^2 - 3g_1^2 \left( \frac{K_{1y}^-}{4\pi u_a} \right)^2 \]
\[ \frac{dK_{1y}^-}{dl} = -2g_2^2 \left( \frac{K_{1y}^-}{4\pi u_a} \right)^2 - 4g_2^2 \left( \frac{K_{1y}^+}{4\pi u_a} \right)^2 + g_3^2 \left( \frac{1}{4\pi u_a} \right)^2 \]
\[ - \frac{1}{2} g_4^2 \left( \frac{K_{1y}^+}{4\pi u_a} \right)^2 + \frac{1}{2} g_4^2 \left( \frac{1}{4\pi u_a} \right)^2 . \]
\[ H_a = \sum_{i=1}^{3} \frac{u_g}{2} \int dx \left[ K_a \Pi_{ai}^2 + \frac{1}{K_a} (\partial_x \Phi_{ai})^2 \right] \]
\[ + 2\sqrt{3} \int \frac{dx}{(2\pi a)^2} \left[ \cos (\sqrt{\Theta}_{ai}) \cos (\sqrt{\Theta}_{a1}) + \cos (\sqrt{\Theta}_{a3}) \cos (\sqrt{\Theta}_{a1}) + \cos (\sqrt{\Theta}_{a3}) \cos (\sqrt{\Theta}_{a2}) \right], \]

where \( K_a \) and \( g_1 \) are the renormalized values of \( K_a \) and \( g_1 \). Notice that \( H_a \) is invariant under \( \Theta_{ai} \rightarrow -\Theta_{ai} \); therefore, \( \langle \cos (\sqrt{\Theta}_{ai} - \Theta_{aj}) \rangle = \langle \cos (\sqrt{\Theta}_{ai} + \Theta_{aj}) \rangle \). Now we can define a new effective coupling
\[ g'_4 = \frac{g_4}{4!} \langle \cos (\sqrt{\Theta}_{a1} + \Theta_{a2}) \rangle \]
and write the \( g_4 \) term as
\[ H_1 = 2g'_4 \sum_{i=1}^{3} \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{4\pi} \Phi_s \right) \cos \left( \sqrt{4\pi} \Phi_{ai} \right). \]

Recall that \( [\Phi(x), \Theta(y)] = O(1) \). Since the \( \Theta_{ai} \) are pinned, the \( \Phi_{ai} \) will fluctuate wildly and \( e^{i \beta \Phi_{ai}} \) will have exponentially decaying correlations. Looking at the interchain coupling, one might think that the \( g'_4 \) term is now irrelevant due to the presence of the \( \cos (\sqrt{4\pi} \Phi_{ai}) \) terms. However, this is not entirely correct. To see what can happen we evaluate the (normalized) partition function perturbatively in \( g'_4 \). We write
\[ \frac{Z}{Z_0} = \frac{1}{Z_0} \int D\Phi e^{-S_0} e^{-S_1} = \langle e^{-S_1} \rangle_0 \]
\[ = 1 - \langle S_1 \rangle_0 + \frac{1}{2} \langle S_1^2 \rangle_0 + \cdots \]

where
\[ S_0 = \frac{u_g}{2K_s} \int d^2x \left[ \frac{1}{u_s} (\partial_x \Phi_s)^2 + (\partial_x \Phi_s)^2 \right] + \frac{3}{2\sqrt{3}} \int d^2x \left[ \frac{1}{K_a} (\partial_x \Theta_{ai})^2 + (\partial_x \Theta_{ai})^2 \right] \]
\[ + 2\sqrt{3} \int \frac{d^2x}{(2\pi a)^2} \left[ \cos (\sqrt{\Theta}_{ai}) \cos (\sqrt{\Theta}_{a1}) + \cos (\sqrt{\Theta}_{a3}) \cos (\sqrt{\Theta}_{a1}) + \cos (\sqrt{\Theta}_{a3}) \cos (\sqrt{\Theta}_{a2}) \right] \]

with \( K_s \) the renormalized value of \( K_s \); \( \langle \rangle_0 \) denotes averaging with respect to \( S_0 \), and
\[ S_1 = 2g'_4 \sum_{i=1}^{3} \int \frac{d^2x}{(2\pi a)^2} \cos \left( \sqrt{4\pi} \Phi_s \right) \cos \left( \sqrt{4\pi} \Phi_{ai} \right). \]

The first non-vanishing correction is
\[ \langle S_1^2 \rangle_0 = 3(2g'_4)^2 \int \frac{d^2x_1}{(2\pi a)^2} \frac{d^2x_2}{(2\pi a)^2} \left( \cos \left( \sqrt{4\pi} \Phi_s(x_1) \right) \cos \left( \sqrt{4\pi} \Phi_s(x_2) \right) \cos \left( \sqrt{4\pi} \Phi_{ai}(x_1) \right) \cos \left( \sqrt{4\pi} \Phi_{ai}(x_2) \right) \right) \]
\[ = 1 + 3(g'_4)^2 D \int \frac{d^2x_1}{(2\pi a)^2} \left( \cos \left( 2\sqrt{4\pi} \Phi_s \right) \right) + \cdots \]

Re-exponentiating, we have
\[ \frac{Z}{Z_0} = \langle e^{-S'_1} \rangle_0 \]
\[ S'_1 = g \int \frac{d^2x}{(2\pi a)^2} \cos (\sqrt{16\pi} \Phi_s). \]
Therefore, our effective Hamiltonian for $\Phi_s$ is

$$H_{\text{eff}} = \frac{u_s}{2} \int dx \left[ K_s \Pi^2_s + \frac{1}{K_s} (\partial_x \Phi_s)^2 \right] + g \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{16\pi} \Phi_s \right), \quad (56)$$

where we have introduced a new effective coupling constant, $g$. Now we have an effective Hamiltonian for $\Phi_s$ which is a standard sine-Gordon Hamiltonian. The scaling dimension of $\cos(\sqrt{16\pi} \Phi_s)$ is $4 K_s^2$; it is relevant for $K_s < 1/2$. Therefore, $g$ grows strong for $K_s < 1/2$; $\Phi_s$ gets pinned (and hence becomes massive) for $K_s < 1/2$. However, the $\Theta_{ai}$ are pinned for $0 \leq \Delta \leq 1$. Therefore, the antisymmetric modes are in a disordered phase for $0 \leq \Delta \leq 1$.

Finally, we consider the case $J_{xy}^s \neq 0$ and $J_z^s \neq 0$ with $J_z^s / J_{xy}^s = \Delta$. Therefore, for $J_{xy}^s = 1$, we recover the results for the anisotropic spin-2 chain. For this case, we have the initial values of the coupling constants are $g_1(l=0) = 4 J_z^s$, $g_2(l=0) = -4 J_z^s$, and $g_4(l=0) = 2 \pi J_{xy}^s$. Therefore, the $\Theta_{ai}$ fields will get pinned. Once the $\Theta_{ai}$ fields are pinned, just like the $J_{xy}^s \neq 0$ & $J_z^s = 0$ case, the $g_4$ term becomes

$$2 g_4' \int \frac{d^2 x}{(2\pi a)^2} \cos \left( \sqrt{4\pi} \Phi_s \right) \cos \left( \sqrt{4\pi} \Phi_{ai} \right), \quad (57)$$

where $g_4' = \bar{g}_4 (\cos(\sqrt{\pi}(\Theta_1 + \Theta_2)))$, and $\bar{g}_4$ is the renormalized value of $g_4$. Define $g_4' = 2(\bar{g}_1 + \bar{g}_4)$. ($\bar{g}_1$ is the renormalized value of $g_1$.) Then our interchain coupling is

$$H_{\text{interchain}} = g_1' \sum_{i=1}^3 \int \frac{d^2 x}{(2\pi a)^2} \cos \left( \sqrt{4\pi} \Phi_s \right) \cos \left( \sqrt{4\pi} \Phi_{ai} \right), \quad (58)$$

From this point, the analysis proceeds identically to the case $J_{xy}^s \neq 0$ & $J_z^s = 0$ — derive an effective sine-Gordon Hamiltonian for the symmetric mode and determine the phase boundary from this effective Hamiltonian. Similar to the case $J_{xy}^s \neq 0$ & $J_z^s = 0$, the effective sine-Gordon Hamiltonian is

$$H_{\text{eff}} = \frac{u_s}{2} \int dx \left[ K_s \Pi^2_s + \frac{1}{K_s} (\partial_x \Phi_s)^2 \right] + g \int \frac{dx}{(2\pi a)^2} \cos \left( \sqrt{16\pi} \Phi_s \right), \quad (59)$$

where $K_s$ is the renormalized value of $K_s$, and $g$ is an effective coupling constant. Again, $\Phi_s$ will become massive for $K_s < 1/2$, and the $\Theta_{ai}$ are pinned for $0 \leq \Delta \leq 1$. Therefore, a Haldane gap appears for $K_s < 1/2$ and the antisymmetric modes are in a disordered phase for $0 \leq \Delta \leq 1$.

The above results for the four-leg ladder are summarized in Table II.

| Phase | Haldane for $0 \leq \Delta \leq 1$ | Gapless for $K_s > 1/2$, Gapless for $K_s > 1/2$, | Haldane for $K_s < 1/2$ | Haldane for $K_s < 1/2$ |
|-------|---------------------------------|---------------------------------|-----------------|-----------------|
| $J_{xy}^s = 0, J_z^s \neq 0$ | pinned for $0 \leq \Delta \leq 1$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ |
| $J_{xy}^s \neq 0, J_z^s = 0$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ |
| $J_z^s / J_{xy}^s = \Delta$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ | pinned for $K_s < 1/2$ |

To determine the phase boundary between the gapless and Haldane phases for the case $J_{xy}^s \neq 0$, $J_z^s \neq 0$, with $J_z^s / J_{xy}^s = \Delta$, we numerically integrated up the RG equations. It turned out that $g_3$ grew so much faster than $g_1$ or $g_2$. Our numerical procedure was the following. We integrated up the RG equations to a scale, $\xi$, $\xi$ was defined as the scale where $g_3 = O(1)$ ($g_3 \gg g_1, g_2, g_4$) at that point, we could safely say that the $\Theta_{ai}$ fields are pinned (up to gapped fluctuations). Then we defined

$$K_s = K_s(\xi). \quad (60)$$

From the above arguments, $K_s = 1/2$ defines the phase boundary.

Unfortunately, we were only able to determine the phase boundary qualitatively. This is because $K_s$ is defined when $g_3 = O(1)$. (We need $g_3 = O(1)$ so that we can safely say that the $\Theta_{ai}$ fields are pinned.) However, our RG equations are no longer valid when any of the couplings are of $O(1)$, since they were calculated to lowest order for weak coupling. Therefore, our procedure becomes uncontrolled. However, as long as there is no intermediate fixed point, our results will be qualitatively correct. Although we cannot think of what such a fixed point would physically correspond to, it cannot be ruled out. The phase diagram is shown in Fig. [I].
The perturbative treatment is not valid. The dashed line shows the phase boundary between the gapless and Haldane phases calculated from the RG equations. For the antisymmetric modes, whenever Θai were present, the behavior depended on the number of legs of the ladder. For the two-leg ladder there was a region about ∆ = 0 where Φ⊥ remained gapless; for the four-leg ladder, the Haldane gap is qualitatively different for the two-leg and four-leg ladders.

How do our results apply to spin ladders where the coupling is antiferromagnetic and between nearest neighbor spins on the same rung (as in actual ladder materials)? In Sec. II, we argued that the isotropic composite spin model has the same phase diagram as the isotropic antiferromagnetic ladder. Now what if we include anisotropy and allow for the interchain coupling to be different from the coupling along the chains? It is likely that the phase boundaries (between the massless and Haldane phases) in the $\Delta - J_\perp$ plane, which we found using the composite spin model, will shift. For the two-leg ladder it appears that the Haldane phase still opens from $\Delta = 0, J_\perp = 0$. However, for the four-leg ladder, it is possible that the Haldane phase only exists at the isotropic point or in an extremely narrow sliver about the isotropic point. We leave this (and other possibilities) for future work.

VI. CONCLUDING REMARKS

In this paper we studied the generation of the Haldane gap in two-leg and four-leg anisotropic spin ladders using a particular form for the interchain coupling, namely the case where spins situated diagonally interact. An interchain coupling of this form was chosen so that when $J_\perp = 1$, the two-leg (four-leg) ladder was equivalent to a spin-1 (spin-2) chain (as far as their low lying excitations are concerned.)

After bosonizing our model, we made a (linear) transformation to a new set of fields. One field contained a symmetric combination of the operators on the legs; the others were, in a general sense, antisymmetric combinations. For the antisymmetric modes, whenever $\Theta_a$ or $\Theta_{ai}$ were present, the operators containing $\Theta_a$ or $\Theta_{ai}$ were the most relevant operators; the $\Theta_a$ or $\Theta_{ai}$ were pinned in the entire range $0 \leq \Delta \leq 1$. However, when only $\Phi_a$ or $\Phi_{ai}$ were present, the behavior depended on the number of legs of the ladder. For the two-leg ladder there was a region about $\Delta = 0$ where $\Phi_a$ remained gapless; for the four-leg ladder, the $\Phi_{ai}$ were pinned in the entire range $0 \leq \Delta \leq 1$.

For the symmetric mode, it was always $\Phi_x$ (rather than $\Theta_a$) which came into play. However, its behavior was strongly dependent on the number of legs of the ladder. For the two-leg ladder the symmetric field, $\Phi_x$, was pinned in the entire range $0 \leq \Delta \leq 1$, and the Haldane gap was generated for arbitrarily small interchain coupling. However, for the four-leg ladder, the Haldane gap was strongly dependent on both the interchain coupling and $\Delta$. In order to determine the phase diagram, the RG equations for the parameters were derived and integrated. It was found that the Haldane phase appeared in a narrow range around the isotropic point, as shown in Fig. 6. Therefore, for the two-leg ladder it is scenario (a) of Fig. 3 which occurs, while for the four-leg ladder it is scenario (d) of Fig. 3 which occurs; the opening of the Haldane gap is qualitatively different for the two-leg and four-leg ladders.

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APPENDIX A: BOSONIZATION DICTIONARY

The bosonization procedure is well documented. However, since various conventions exist in the literature, we present our conventions here for clarity and completeness.

To study the low energy properties of our fermion model, we linearize the dispersion about the Fermi points and write

$$\psi(x) = c_j / \sqrt{a} = e^{-ik_F x} \psi_R(x) + e^{ik_F x} \psi_L(x), \quad (A1)$$

where $x = ja$.

Then we have the following bosonization rules:

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a}} \exp(i\sqrt{4\pi} \phi_R),$$

$$\psi_L(x) = \frac{1}{\sqrt{2\pi a}} \exp(-i\sqrt{4\pi} \phi_L),$$

$$\psi_R^\dagger(x) \psi_R(x) + \psi_L^\dagger(x) \psi_L(x) = \rho_0 + \frac{1}{\sqrt{\pi}} \partial_x \phi(x). \quad (A2)$$
where the chiral boson fields are related to the standard Bose-field, $\Phi$, and its dual field, $\Theta$, via

$$
\Phi = \phi_R + \phi_L, \quad \Theta = \phi_R - \phi_L.
$$

(A3)

The fields $\Phi$ and $\Theta$ satisfy the commutation relations

$$
[\Phi(x), \Theta(y)] = i\theta(y - x),
$$

(A4)

where $\theta$ is the step-function. $\Phi$ and $\Theta$ can be thought of as order and disorder fields, respectively, in the usual statistical mechanical sense.

Our fields are normalized such that the free boson action is given by

$$
S_0 = \frac{1}{2} \int d^2 x \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right].
$$

(A5)

The corresponding Green’s function is

$$
G(x, \tau) = \frac{1}{4\pi} \ln \left( \frac{R^2}{|z|} \right),
$$

(A6)

where $R$ is an infrared cutoff ($R \to \infty$). It is the solution to

$$
- (\partial^2_x + \partial^2_\tau) \ G(x, \tau) = \delta(x - x_0) \delta(\tau - \tau_0).
$$

(A7)

Note that in our calculations we use the lattice spacing, $a$, as an ultraviolet regulator.

For a Hamiltonian of the form

$$
H = \frac{u}{2} \int dx \left[ K\Pi^2 + \frac{1}{K} (\partial_x \Phi)^2 \right],
$$

(A8)

we make a canonical transformation

$$
\Pi = \frac{1}{\sqrt{K}} \tilde{\Pi}, \quad \Phi = \sqrt{K} \Phi.
$$

(A9)

In terms of these new variables, our Hamiltonian has the form

$$
H = \frac{u}{2} \int dx \left[ \tilde{\Pi}^2 + (\partial_x \Phi)^2 \right].
$$

(A10)

Using this form, we can pass to a Lagrangian description and obtain an action of the form in Eq. (A5).

In the critical regime of the spin-1/2 Heisenberg chain, the spin-spin correlations functions have power law behavior which can be calculated in the boson representation using the relations

$$
\langle e^{i\alpha \Phi(x)} e^{-i\alpha \Phi(y)} \rangle = \left( \frac{a^2}{|x - y|^2} \right)^{a^2 K}
$$

and

$$
\langle e^{i\beta \Theta(x)} e^{-i\beta \Theta(y)} \rangle = \left( \frac{a^2}{|x - y|^2} \right)^{\beta^2 \pi \pi},
$$

(A11)

where we have first used the transformation in Eq. (A9). From Eq. (A11) we can obtain the scaling dimensions of the operators $e^{i\alpha \Phi(x)}$ and $e^{i\beta \Theta(x)}$; it follows that $e^{i\alpha \Phi(x)}$ has scaling dimension $\frac{a^2 K}{4\pi}$ and $e^{i\beta \Theta(x)}$ has scaling dimension $\frac{\beta^2 \pi \pi}{4\pi}$.

An operator is relevant if its scaling dimension is less than 2; it is irrelevant if its scaling dimension is greater than 2. If an operator is relevant, its coefficient grows strong at large distances; if it is irrelevant, its coefficient becomes smaller at large distances. Therefore, $e^{i\alpha \Phi(x)}$ is relevant for $\frac{a^2 K}{4\pi} < 2$; $e^{i\beta \Theta(x)}$ is relevant for $\frac{\beta^2 \pi \pi}{4\pi} < 2$.

**APPENDIX B: RENORMALIZATION GROUP ANALYSIS**

For weak interchain coupling we can analyze the physics of our model with a perturbative RG analysis. In an RG treatment we coarse grain our system, integrating out short wavelength (high energy) degrees of freedom, and obtain an effective theory for the long wavelength (low energy) degrees of freedom.

We will derive the RG equations for $g_1$, $g_4$ and $K_i$ in detail to illustrate the method. The RG equations for the other parameters can be derived in a similar way.

We consider the correlator $\langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_2)} \rangle$ and evaluate it perturbatively in the interaction. Therefore, we write

$$
\langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_2)} \rangle
$$

$$
= \langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_1)} e^{-S_i} \rangle_0
$$

$$
= \langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_2)} \rangle_0
$$

$$
- \langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_2)} S_i \rangle_0
$$

$$
+ \frac{1}{2} \langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_2)} S_i^2 \rangle_0 + \ldots
$$

(B1)

where $\langle \rangle_0$ denotes averaging with respect to the free boson action. The first nonvanishing correction comes from the $S_i^2$ term.

$$
\langle e^{i/\sqrt{4\pi} \Phi_s(x_1)} e^{-i/\sqrt{4\pi} \Phi_s(x_2)} S_i^2 \rangle_0 =
$$
\begin{align}
\langle e^{i\sqrt{\Phi} a_2(x_3)} e^{-i\sqrt{\Phi} a_2(x_4)} \rangle_0 \langle e^{-i\sqrt{\Phi} a_3(x_3)} e^{i\sqrt{\Phi} a_3(x_4)} \rangle_0.
\end{align}

Evaluate the correlators to get
\begin{align}
&= \frac{3}{4} (2g_1)^2 \int \frac{d^2x_3}{(2\pi a)^2} \frac{d^2x_4}{(2\pi a)^2} \left( \frac{a^2}{|z_{21}|^2} \right)^{K_s} \left( \frac{|z_{31}|^2 |z_{42}|^2}{|z_{32}|^2 |z_{41}|^2} \right)^{K_s} \left( \frac{a^2}{|z_{43}|^2} \right)^{K_s} \left( \frac{a^2}{|z_{43}|^2} \right)^{K_s} \\
&\quad + \frac{3}{2} g_1^2 \int \frac{d^2x_3}{(2\pi a)^2} \frac{d^2x_4}{(2\pi a)^2} \left( \frac{a^2}{|z_{21}|^2} \right)^{K_s} \left( \frac{|z_{31}|^2 |z_{42}|^2}{|z_{32}|^2 |z_{41}|^2} \right)^{K_s} \left( \frac{a^2}{|z_{43}|^2} \right)^{K_s} \left( \frac{a^2}{|z_{43}|^2} \right)^{K_s + \frac{1}{2} K_s}
\end{align}

where \( z_{ij} = u_s(\tau_i - \tau_j) + i(x_i - x_j) \) and \( z_{ij}^0 = u_s(\tau_i - \tau_j) + i(x_i - x_j) \). Define \( r_i = u_s \tau_i \hat{x} + x_i \hat{y} \)

and make the change of variables
\begin{align}
r = r_4 - r_3 \quad ; \quad R = \frac{1}{2} (r_4 + r_3).
\end{align}

Then we have
\begin{align}
&= \frac{3}{4} (2g_1)^2 \frac{1}{u_s^2} \left( \frac{a^2}{|z_{21}|^2} \right)^{K_s} \int \frac{d^2R}{(2\pi a)^2} \frac{d^2r}{(2\pi a)^2} \left[ \left| R - \frac{1}{2} r_4 - r_1 \right|^2 \right]^{K_s} \left[ \left| R - \frac{1}{2} r_4 - r_2 \right|^2 \right]^{K_s} \\
&\quad \times \left( \frac{1}{\sin^2 \theta + \left( \frac{u_s}{u_s} \right)^2 \cos^2 \theta} \right)^{K_s} \\
&\quad + \frac{3}{2} g_1^2 \frac{1}{u_s^2} \left( \frac{a^2}{|z_{21}|^2} \right)^{K_s} \int \frac{d^2R}{(2\pi a)^2} \frac{d^2r}{(2\pi a)^2} \left[ \left| R - \frac{1}{2} r_4 - r_1 \right|^2 \right]^{K_s} \left[ \left| R - \frac{1}{2} r_4 - r_2 \right|^2 \right]^{K_s} \\
&\quad \times \left( \frac{1}{\sin^2 \theta + \left( \frac{u_s}{u_s} \right)^2 \cos^2 \theta} \right)^{K_s + \frac{1}{2} K_s}
\end{align}

where \( u_s(\tau_4 - \tau_3) = r \cos \theta \) and \( x_4 - x_3 = r \sin \theta \). Write
\begin{align}
\left( \frac{1}{\sin^2 \theta + \left( \frac{u_s}{u_s} \right)^2 \cos^2 \theta} \right)^\beta = 1 + \beta \left[ 1 - \left( \frac{u_s}{u_s} \right)^2 \right] \cos^2 \theta + \cdots.
\end{align}

Since \( 1 - \left( \frac{u_s}{u_s} \right)^2 = O(J^2) \), the anisotropy between \( u_s \) and \( u_s \) gives us corrections which are of higher order. Therefore, to this order we ignore the anisotropy.

Using that the most singular part of the integrand is for \( r \) near 0, we expand the integrand about \( r = 0 \). We get
\begin{align}
\left[ \left| R - \frac{1}{2} r_4 - r_1 \right|^2 \right]^{K_s} \left[ \left| R - \frac{1}{2} r_4 - r_2 \right|^2 \right]^{K_s} = -4K_s^2 r^2 \cos^2 \theta \left( \frac{R - r_1}{|R - r_1|^2} \right) \cdot \left( \frac{R - r_2}{|R - r_2|^2} \right) + \text{disconnected pieces},
\end{align}

where we have kept only connected pieces.

Therefore we have
\begin{align}
\langle e^{i\sqrt{\Phi} a_2(x)} e^{-i\sqrt{\Phi} a_2(x)} (g S^2) \rangle_0
\end{align}

\begin{align}
&= \left( \frac{a^2}{|z_{21}|^2} \right)^{K_s} \int \frac{d^2R}{(2\pi a)^2} \nabla_R \ln \left( \frac{|R - r_1|}{a} \right) \cdot \nabla_R \ln \left( \frac{|R - r_2|}{a} \right) \left[ -3g_1^2 \left( \frac{2K_s^2}{u_s^2} \right)^2 \right] \int \frac{d^2r}{(2\pi a)^2} r^2 \cos^2 \theta \left( \frac{a^2}{r^2} \right)^{K_s + K_s} \\
&\quad - \frac{3}{2} g_1^2 \left( \frac{2K_s^2}{u_s^2} \right) \int \frac{d^2r}{(2\pi a)^2} r^2 \cos^2 \theta \left( \frac{a^2}{r^2} \right)^{K_s + K_s + \frac{1}{2} K_s},
\end{align}

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where we have used that
\[
\frac{(\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^2} = \nabla R \ln \left( \frac{|\mathbf{R} - \mathbf{r}|}{a} \right).
\]

Integrate over \( \theta \), integrate by parts over \( \mathbf{R} \) and ignore the surface term, and use that
\[
\nabla^2 R \ln \left( \frac{|\mathbf{R} - \mathbf{r}|}{a} \right) = 2\pi \delta^2(\mathbf{R} - \mathbf{r})
\]
to get
\[
\left\langle e^{i\sqrt{4\pi}\Phi_s(x)} e^{-i\sqrt{4\pi}\Phi_s(y)} S^2 \right\rangle_0 = \left( \frac{a^2}{|z_{21}|^2} \right)^K \left\{ 1 + \ln \left( \frac{a^2}{r_2 - r_1} \right) \left[ -6g_1^2 \left( \frac{K_s}{4\pi u_s} \right)^2 \int \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2(K_s+K_a)} + 3g_2^2 \left( \frac{K_s}{2\pi u_s} \right)^2 \int \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2(K_s+K_a+\frac{1}{2K_a})} \right] \right\}.
\]

Finally, we have
\[
\left\langle e^{i\sqrt{4\pi}\Phi_s(x)} e^{-i\sqrt{4\pi}\Phi_s(y)} e^{-S_1} \right\rangle = \left( \frac{a^2}{|z_{21}|^2} \right)^K \left\{ 1 + \ln \left( \frac{a^2}{r_2 - r_1} \right) \left[ -6g_1^2 \left( \frac{K_s}{4\pi u_s} \right)^2 \int \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2(K_s+K_a)} + 3g_2^2 \left( \frac{K_s}{4\pi u_s} \right)^2 \int \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2(K_s+K_a+\frac{1}{2K_a})} \right] \right\}.
\]

Treat the term in brackets as the first term in a cumulant expansion. Re-exponentiate to get
\[
\left\langle e^{i\sqrt{4\pi}\Phi_s(x)} e^{-i\sqrt{4\pi}\Phi_s(y)} e^{-S_1} \right\rangle = \left( \frac{a^2}{|z_{21}|^2} \right)^{K_s^{\text{eff}}} \left. \left\{ 1 + \ln \left( \frac{a^2}{r_2 - r_1} \right) \left[ -6g_1^2 \left( \frac{K_s}{4\pi u_s} \right)^2 \int \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2(K_s+K_a)} + 3g_2^2 \left( \frac{K_s}{4\pi u_s} \right)^2 \int \frac{dr}{a} \left( \frac{r}{a} \right)^{3-2(K_s+K_a+\frac{1}{2K_a})} \right] \right\} \right\}
\]
where we have introduced
\[
\begin{align*}
\tilde{K}_s &= K_s - 6g_1^2 \left( \frac{K_s}{4\pi u_s} \right)^2 (b - 1) - 3g_2^2 \left( \frac{K_s}{4\pi u_s} \right)^2 (b - 1) \\
\tilde{g}_1 &= g_1 b^{2-(K_s+K_a)} \\
\tilde{g}_4 &= g_4 b^{2-(K_s+K_a+\frac{1}{2K_a})}.
\end{align*}
\]
Since \( \ln b \ll 1 \), we can write \( 1 - b = \ln b \) and \( b^\beta = 1 + \chi \ln b \). Then write \( \tilde{K}_s = K_s + d K_s \), \( \tilde{g}_1 = g_1 + d g_1 \), \( \tilde{g}_4 = g_4 + d g_4 \), and define \( dl = \ln b \). This gives us the RG equations
\[
\begin{align*}
\frac{d K_s}{dl} &= -6g_1^2 \left( \frac{K_s}{4\pi u_s} \right)^2 - 3g_2^2 \left( \frac{K_s}{4\pi u_s} \right)^2, \\
\frac{d g_1}{dl} &= [2 - (K_s + K_a)] g_1, \\
\frac{d g_4}{dl} &= \left[ 2 - (K_s + K_a + \frac{1}{2K_a}) \right] g_4.
\end{align*}
\]

\footnote{For a review see E. Dagotto and T. M. Rice, Science 271, 618 (1996).}
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