On the Green-Functions of the classical offshell electrodynamics under the manifestly covariant relativistic dynamics of Stueckelberg

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Abstract

In previous papers derivations of the Green function have been given for 5D off-shell electrodynamics in the framework of the manifestly covariant relativistic dynamics of Stueckelberg (with invariant evolution parameter \( \tau \)). In this paper, we reconcile these derivations resulting in different explicit forms, and relate our results to the conventional fundamental solutions of linear 5D wave equations published in the mathematical literature. We give physical arguments for the choice of the Green function retarded in the fifth variable \( \tau \).

1 Introduction

Classical 5D electrodynamics arises as a \( U(1) \) gauge of the relativistic quantum mechanical Schrödinger equation [20, 8, 13, 14, 18], similar to the construction of Maxwell fields from the \( U(1) \) gauge of the classical Schrödinger equation.

We have studied the configuration of such fields associated with a uniformly moving source [1] as well as from a uniformly accelerating one [2]. The action of the resulting generalized Lorentz force on the source (radiation reaction) is under study; the results, very different in nature from the Abraham-Lorentz-Dirac analysis (e.g., [5, 17]), will be reported elsewhere [3].

By requiring local gauge invariance of

\[ i \frac{\partial}{\partial \tau} \psi_{\tau}(x) = \frac{1}{2M} p^\mu p_\mu \psi_{\tau}(x), \]  

(1)

where \( p^\mu \) is represented by \(-i\partial/\partial x^\mu\), five compensation fields are introduced [20, 8, 13, 14].

1
Under the 5D generalized Lorentz gauge, these fields obey a 5D wave equation of the form \( \eta_{\mu\nu} \partial^\mu \partial^\nu + \sigma_{55} \partial^2 \equiv \partial_\alpha \partial^\alpha \),

\[
\left( \eta_{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} + \sigma_{55} \frac{\partial^2}{\partial \tau^2} \right) a^\alpha(x, \tau) \equiv \partial_\beta \partial^\beta a^\alpha(x, \tau) = j^\alpha(x, \tau) \tag{2}
\]

where \( x = (x^\mu) = (t, x^i) \) is a 4D spacetime coordinate and \( \alpha, \beta \in \{0, 1, 2, 3, 5\} \) run over the entire 5D coordinates. Here, \( x^5 \equiv \tau \), whereas \( \mu, \nu \in \{0, 1, 2, 3\} \) run over the 4D spacetime coordinates; \( \sigma_{55} = \pm 1 \) is the signature of \( \tau \) coordinate in the wave equation, denoting either \( O(4, 1) \) or \( O(3, 2) \) symmetry of the homogeneous wave equation.

We shall use \( \sigma_{55} = +1 \) (corresponding to \( O(4, 1) \)) here, although most of the results can easily be extended to the \( \sigma_{55} = -1 \) case as well.

The Green function (GF) associated with (2) obeys the equation

\[
\partial_\alpha \partial^\alpha g(x, \tau) = \delta^4(x) \delta(\tau) \tag{3}
\]

There are numerous ways to solve (3) without referring directly to the Fourier transform; most of these involve using the \( O(4, 1) \) symmetry of the equation.

Nevertheless, in the works of Land and Horwitz \cite{13} and Oron et al. \cite{16} mentioned above, the Fourier method was used, for which \( g(x, \tau) \) is represented by

\[
g(x, \tau) = \frac{1}{(2\pi)^5} \int_{\mathbb{R}^5} d^4k \, dk_5 \, \frac{e^{i(k_\mu x^\mu + k_5 \tau)}}{k_\mu k^\mu + k_5^2} = \frac{1}{(2\pi)^5} \int_{\mathbb{R}^5} d^5k \, \frac{e^{i(k_\alpha x^\alpha)}}{k_\alpha k^\alpha} \tag{4}
\]

Solutions of (4) were obtained in

- Land and Horwitz \cite{13}, using an integral over Schwinger’s result \cite{19} obtaining

\[
g_P(x, \tau) = -\frac{1}{4\pi} \delta(x^2) \delta(\tau) - \frac{1}{2\pi^2} \frac{\partial}{\partial x^2} \begin{cases}
\theta(x^2 - \tau^2) & O(3, 2) \\
\theta(-x^2 - \tau^2) & O(4, 1)
\end{cases} \tag{5}
\]

where \( g_P \) refers to the Principal Part solution, and \( x^2 = x_\mu x^\mu = r^2 - t^2 \).

- Oron and Horwitz \cite{16}, integrating first using \( k_5 \), in which the result obtained is (for \( O(4, 1) \)):

\[
g(x, \tau) = \frac{2\theta(\tau)}{(2\pi)^3} \times \begin{cases}
\frac{1}{[r^2 - \tau^2]^2} \tan^{-1} \left( \frac{1}{2} \frac{\sqrt{-x^2 - \tau^2}}{\sqrt{r^2 + \tau^2}} \right) - \frac{\tau}{x^2(r^2 + \tau^2)} & O(3, 2) \\
\frac{1}{2} \frac{1}{[r^2 + \tau^2]^2} \ln \frac{\tau^2 \sqrt{r^2 + \tau^2}}{r^2 + \tau^2} - \frac{\tau}{x^2(r^2 + \tau^2)} & O(4, 1)
\end{cases} \tag{6}
\]

- Aharonovich and Horwitz \cite{1} resulting in equation (7) (below). Using a different method, a \( \tau \)-retarded form of (7) was obtained by these authors in \cite{2}.

- in the physics (e.g. \cite{4, 6, 10, 11}) and mathematics (e.g. \cite{7, 12}) literature, for fundamental solutions to the linear \( N \)-dimensional wave equation, many of which are a \( t \)-retarded form of (7), whereas the others are without specific retardation \cite{2}, and have a form closely related to (7).
Even though the previous methods have obtained different results, as displayed above, in this paper we shall show that all of these methods result in essentially the following form

\[
g(x, \tau) = \frac{1}{2\pi^2} \lim_{a \to 0} \frac{\partial}{\partial a} \frac{\theta(-x_\mu x^\mu - \tau^2 + a)}{\sqrt{-x_\mu x^\mu - \tau^2 + a}} \bigg|_{a=0},
\]

(7)

consistently with the solutions in the general mathematical literature [7, 12] (the form of (7) implies a well defined regularization [7]).

To the best of our knowledge, however, explicit \(\tau\)-retarded solutions could only be reproduced by methods of the type developed by Nozaki [15], as was used in [2]. For applications to physical problems such as that of self-interaction, we favor the \(\tau\)-retarded form. This paper is primarily devoted to discussion of this form. We discuss this point in the last section.

The remainder of the paper is organized as follows:

1. In section 2 we examine the method employed in ref. [13]. We shall term this method as the Klein-Gordon method, since it essentially reproduces the Klein-Gordon propagator in the first 4D spacetime coordinates, and then integrates over \(k_5\), essentially the Klein-Gordon mass term.

2. In section 3 we examine the method of ref. [16], which integrates over \(k_5\) first, and then over the spacetime \(k^\mu\) coordinates.

## 2 Klein-Gordon method

In the following, we discuss the analysis of Land and Horwitz [13]. Starting from (4), we shall work with 3D spherical coordinates \((k, \theta, \phi)\). After integrating over the spherical angles \((\theta, \phi)\), one integrates over \(k_0\). As the denominator has poles at \(k_0 = \pm \sqrt{k^2 + k_5^2}\), the Principal Part solution is taken, using the contour given in figure [1].

One can see that (4) could be seen as an inverse Fourier transform in \(m\) for the Klein-Gordon propagator:

\[
g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{im\tau} G_{KG}(x, m)
\]

where \(G_{KG}\) is the Principal-Part Klein-Gordon GF with the well known form [19]:

\[
G_{KG}(x, m) = -\frac{\delta(x_\mu x^\mu)}{4\pi} + \frac{m\theta(-x_\mu x^\mu)}{4\pi} J_1(m\sqrt{-x_\mu x^\mu})
\]

(8)

We shall refer to [6] in due course.

Going back to (4), we find (after the integration over \(\theta\) and \(\phi\)):

\[
g(x, \tau) = \frac{1}{(2\pi)^3 r} \int_{-\infty}^{+\infty} dk_5 \int_0^{+\infty} d k \int_{-\infty}^{+\infty} dk_0 \sin(kr) \frac{e^{i(k_5 \tau - k_0 t)}}{k^2 - k_0^2 + k_5^2} = -\frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \int_{-\infty}^{+\infty} dk_5 \int_0^{+\infty} d k \int_{-\infty}^{+\infty} dk_0 \cos(kr) \frac{e^{i(k_5 \tau - k_0 t)}}{k^2 - k_0^2 + k_5^2}
\]
where $k = |k|$.  

Now, the principal-part solution of the $k_0$ integral is  

$$  \int_{-\infty}^{+\infty} \frac{e^{-ik_0 t}}{k^2 + k_5^2 - k_0^2} dk_0 = i\pi \epsilon(t) (a_{-1}(-) + a_{-1}(+)) $$

where  

$$ a_{-1}(-) = \left[ \left( k_0 + \sqrt{k^2 + k_5^2} \right) \times \frac{e^{-ik_0 t}}{k^2 + k_5^2 - k_0^2} \right]_{k_0 = -\sqrt{k^2 + k_5^2}} = \frac{e^{i\sqrt{k^2 + k_5^2} t}}{2\sqrt{k^2 + k_5^2}} $$

$$ a_{-1}(+) = \left[ \left( k_0 - \sqrt{k^2 + k_5^2} \right) \times \frac{e^{-ik_0 t}}{k^2 + k_5^2 - k_0^2} \right]_{k_0 = +\sqrt{k^2 + k_5^2}} = -\frac{e^{-i\sqrt{k^2 + k_5^2} t}}{2\sqrt{k^2 + k_5^2}} $$

Thus:

$$ g(x, \tau) = -\frac{i^2 \pi}{(2\pi)^4} \frac{2\epsilon(-t)}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{0}^{\infty} dk \cos(kr) \sin \left( \frac{t\sqrt{k^2 + k_5^2}}{\sqrt{k^2 + k_5^2}} \right) $$

$$ = -\frac{1}{(2\pi)^4} \frac{\epsilon(t)}{r} \frac{1}{2} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{0}^{\infty} dk \cos(kr) \sin \left( \frac{t\sqrt{k^2 + k_5^2}}{\sqrt{k^2 + k_5^2}} \right) $$

where we have extended the $k$ integration to the negative real axis as well, since the integrand is even in $k$. Further progress is made by substituting...
\[ k(\beta) = |k_5| \sinh(\beta) \]

\[ g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{\epsilon(t)}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{-\infty}^{\infty} (|k_5| \cosh(\beta) \, d\beta) \times \cos(r|k_5| \sinh(\beta)) \sin(t|k_5| \cosh(\beta)) \frac{\sin(t|k_5| \cosh(\beta))}{|k_5| \cosh(\beta)} \]

\[ = -\frac{1}{(2\pi)^3} \frac{\epsilon(t)}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{-\infty}^{\infty} d\beta \times \cos(r|k_5| \sinh(\beta)) \sin(t|k_5| \cosh(\beta)) \]

\[ = -\frac{1}{(2\pi)^3} \frac{\epsilon(t)}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{-\infty}^{\infty} d\beta \]

\[ \times \frac{1}{2} [\sin(|k_5|(r \sinh(\beta) + t \cosh(\beta))) - \sin(|k_5|(r \sinh(\beta) - t \cosh(\beta)))] \]

If \(|t| < r\), we can substitute:

\[ t = \rho \sinh(\alpha) \quad r = \rho \cosh(\alpha) \quad \rho^2 = r^2 - t^2 \]

and thus:

\[ g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{2\epsilon(t)}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{0}^{\infty} d\beta \]

\[ \times \frac{\theta(r^2 - t^2)}{2} [\sin(|k_5|\rho \sinh(\beta + \alpha)) - \sin(|k_5|\rho \sinh(-\beta + \alpha))] \]

\[ = 0 \]

The result for \(r^2 > t^2\) has the integrand \(\sin(|k_5|\rho \sinh(\alpha \pm \beta))\) which is odd around the center \(\beta = \alpha = 0\), and since the bounds are even at \(\pm \infty\), we obtain the null result.

On the other hand, when \(|t| > r\) we find:

\[ t = \epsilon(t)\rho \cosh(\alpha) \quad r = \rho \sinh(\alpha) \quad \rho^2 = t^2 - r^2 \]

And thus:

\[ t \cosh(\beta) + r \sinh(\beta) = \epsilon(t)\rho \cosh(\alpha) \cosh(\beta) + \rho \sinh(\alpha) \sinh(\beta) \]

\[ = \epsilon(t)\rho \cosh(\alpha + \epsilon(t)\beta) \]

\[ r \sinh(\beta) - t \cosh(\beta) = -(t \cosh(\beta) - r \sinh(\beta)) \]

\[ = \epsilon(-t)\rho \cosh(\alpha + \epsilon(-t)\beta) \]

Substituting back in \(g(x, \tau)\) we find:

\[ g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{2\epsilon(t)}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{-\infty}^{\infty} d\beta \]

\[ \times \frac{\theta(t^2 - r^2)}{2} [\sin(\epsilon(t)|k_5|\rho \cosh(\alpha + \epsilon(t)\beta)) - \sin(\epsilon(-t)|k_5|\rho \cosh(\alpha + \epsilon(-t)\beta))] \]

\[ = -\frac{1}{(2\pi)^3} \frac{1}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \int_{-\infty}^{\infty} d\beta \times \theta(t^2 - r^2) \times \sin(|k_5|\rho \cosh(\beta)) \]

Substituting \(u = \cosh(\beta)\) we find:

\[ g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{1}{r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \times 2 \times \int_{1}^{\infty} \frac{du}{\sqrt{u^2 - 1}} \times \theta(t^2 - r^2) \times \sin(|k_5|\rho u) \]

\[ = -\frac{1}{(2\pi)^3} \frac{2}{r} \frac{\partial}{\partial r} \times \theta(t^2 - r^2) \int_{1}^{\infty} \frac{du}{\sqrt{u^2 - 1}} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} \sin(|k_5|\rho u) \]

\[(9)\]

\(^1|k_5|\) ensures the bounds on \(\beta\) are invariant under the sign of \(k_5\).
As the $k_5$ integration picks up only the even part, we can rewrite it as follows:

\[
g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{2}{r} \frac{\partial}{\partial r} \times \theta(t^2 - r^2) \int_1^\infty \frac{du}{\sqrt{u^2 - 1}} \times 2 \times \int_0^\infty dk_5 \cos(k_5 \tau) \sin(k_5 \rho u) \]

\[
= -\frac{1}{(2\pi)^3} \frac{2}{r} \frac{\partial}{\partial r} \times \theta(t^2 - r^2) \int_1^\infty \frac{du}{\sqrt{u^2 - 1}}
\times 2 \times \int_0^\infty dk_5 \frac{1}{4i} \left[ e^{ik_5(\tau + \rho u)} - e^{ik_5(\tau - \rho u)} + e^{ik_5(-\tau + \rho u) - e^{-ik_5(\tau + \rho u)}} \right]
\]

Since\footnote{E.g., in \cite{[citation]}.}

\[
\int_0^\infty dk_5 e^{ik_5(\tau + \rho u)} = +\pi \delta(\tau + \rho u) - P \frac{1}{i(\tau + \rho u)}
\]

we find:

\[
g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{2}{r} \frac{\partial}{\partial r} \times \theta(t^2 - r^2) \int_1^\infty \frac{du}{\sqrt{u^2 - 1}}
\times \frac{1}{2i} \left[ +\pi (\delta(\tau + \rho u) - \delta(\tau - \rho u)) + \delta(-\tau + \rho u) - \delta(-\tau - \rho u) \right]
\]

\[
= -\frac{1}{(2\pi)^3} \frac{2}{r} \frac{\partial}{\partial r} \times \theta(t^2 - r^2) \int_1^\infty \frac{du}{\sqrt{u^2 - 1}} \times P \left[ \frac{1}{\rho u + \tau} + \frac{1}{\rho u - \tau} \right]
\]

Let us inspect an integral of the form:

\[
I(a, b) = \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \frac{1}{ax + b}
\]

Substituting $x(\alpha) = \cosh(\alpha), dx(\alpha) = \sinh(\alpha) d\alpha$ we find:

\[
I(a, b) = \int_0^\infty \frac{\sinh(\alpha) d\alpha}{\sinh \alpha} \frac{1}{a \cosh(\alpha) + b} = \int_0^\infty \frac{d\alpha}{a \cosh(\alpha) + b} = \frac{1}{2} \times \int_{-\infty}^{\infty} \frac{d\alpha}{a \cosh(\alpha) + b}
\]

where we have utilized the evenness of $\cosh(\alpha)$ around $\alpha = 0$.

After a further substitution of $u(\alpha) = e^\alpha, d\alpha(u) = du/u$, we find:

\[
I(a, b) = \frac{1}{2} \int_0^\infty \frac{du/u}{a^{1/2}(u + 1/a) + b} = \frac{1}{a} \int_0^\infty \frac{du}{u^2 + 2ub/a + 1}
\]

The roots of the denominator are:

\[
u_{1,2} = -\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - 1} = \frac{1}{a} \left[ -b \pm \sqrt{b^2 - a^2} \right]
\]

If $b^2 < a^2$, then we can rewrite the denominator as:

\[
u^2 + 2u \frac{b}{a} + 1 = \left( u + \frac{b}{a} \right)^2 + 1 - \frac{b^2}{a^2}
\]
Therefore, if \( a^2 < b^2 \), we have:

\[
I(a, b) = \frac{1}{a} \int_{a}^{\infty} \frac{du}{(u - u_1)(u - u_2)} = \frac{1}{a} \int_{0}^{\infty} \frac{du}{u - u_1} \left[ \frac{1}{u - u_1} - \frac{1}{u - u_2} \right] \\
= \frac{1}{a(u_1 - u_2)} \ln \left| \frac{u - u_1}{u - u_2} \right| \bigg|_{0}^{\infty} = \frac{1}{a(u_1 - u_2)} \ln \left| 0 - \frac{u_1}{u_2} \right| \\
= -\frac{1}{2a\sqrt{b^2 - a^2}} \ln \left| \frac{b + \sqrt{b^2 - a^2}}{b - \sqrt{b^2 - a^2}} \right|
\]

where we have taken the principal part of the integration, as in (11).

However, in our case, eq. (11), we have (with principal part) \( I(a, b) + I(a, -b) \), but \( I(a, b) = -I(a, -b) \). Therefore, when \( b^2 > a^2 \), which corresponds to \( \tau^2 > \rho^2 \), \( g(x, \tau) \) vanishes.

In the other case where \( b^2 < a^2 \), we find:

\[
I(a, b) = \frac{1}{a} \int_{0}^{\infty} \frac{du}{u^2 + 2ab/a + 1} = \frac{1}{a} \int_{0}^{\infty} \frac{du}{(u + b/a)^2 + D^2}
\]

where we have put \( D^2 = 1 - b^2/a^2 \).

Substituting \( v = (u + b/a)/D \), we find:

\[
I(a, b) = \frac{1}{a} \int_{b/aD}^{\infty} \frac{D \, dv}{v^2D^2 + D^2} = \frac{1}{aD} \tan^{-1}(v)\bigg|_{b/aD}^{\infty} = \frac{1}{aD} \left( \frac{\pi}{2} - \tan^{-1}\left( \frac{b}{aD} \right) \right) \\
= \frac{1}{a\sqrt{1 - b^2/a^2}} \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{b}{a\sqrt{1 - b^2/a^2}} \right) \right] = \frac{1}{\sqrt{a^2 - b^2}} \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{b}{\sqrt{a^2 - b^2}} \right) \right]
\]

Now:

\[
I(a, b) = \frac{\pi}{\sqrt{a^2 - b^2}} \left[ \frac{\pi}{2} - \tan^{-1}\left( \frac{b}{\sqrt{a^2 - b^2}} \right) \right] \\
I(a, -b) = \frac{\pi}{\sqrt{a^2 - b^2}} \left[ \frac{\pi}{2} + \tan^{-1}\left( \frac{b}{\sqrt{a^2 - b^2}} \right) \right]
\]

And thus:

\[
I(a, b) + I(a, -b) = \frac{\pi}{\sqrt{a^2 - b^2}}
\]

Thus, we are left with the solution

\[
g(x, \tau) = -\frac{1}{(2\pi)^3} \frac{2}{r} \frac{\partial}{\partial r} \times \theta(t^2 - r^2) \times \frac{\pi \theta(\rho^2 - \tau^2)}{\sqrt{\rho^2 - \tau^2}} \\
= \frac{1}{4\pi^2} \frac{\partial}{\partial r} \theta(t^2 - r^2) \times \theta(t^2 - r^2) \times \frac{\pi \theta(\rho^2 - \tau^2)}{\sqrt{\rho^2 - \tau^2}}
\]

Now, clearly, \( \theta(t^2 - r^2) \times \theta(t^2 - r^2 - \tau^2) = \theta(t^2 - r^2 - \tau^2) \). Moreover, writing

\[
\frac{1}{r} \frac{\partial}{\partial r} = \frac{1}{r} \frac{\partial r^2}{\partial r} \frac{\partial}{\partial r} \frac{2r}{r} \frac{\partial}{\partial r^2} = \frac{2}{r^2} \frac{\partial}{\partial r^2}
\]
we find:
\[
g(x, \tau) = \frac{1}{2\pi^2} \frac{\partial}{\partial r^2} \frac{\theta(t^2 - r^2 - \tau^2)}{\sqrt{t^2 - r^2 - \tau^2}}
\]
and this is the GF expected, which differs from the one found by Land and Horwitz [13], i.e.,
\[
G_P(x, \tau) = -\frac{1}{4\pi} \delta(t^2 - r^2) \delta(\tau) - \frac{1}{4\pi^2} \frac{\partial}{\partial r^2} \frac{\theta(t^2 - r^2 - \tau^2)}{\sqrt{t^2 - r^2 - \tau^2}} \tag{12}
\]
The reason is due to carrying out the derivative of \(\theta(t^2 - r^2)\) in \([9]\), before performing the integration. The integration results in a \(\theta(t^2 - r^2 - \tau^2)\) factor that subsumes the \(\theta(t^2 - r^2)\) prefactor. And therefore, the first term in \([12]\) should not appear. Taking this derivative prematurely, results in the following construction, which leads directly to \([12]\):
\[
g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_5 \cos(k_5 \tau) \int_{1}^{\infty} \frac{du}{\sqrt{u^2 - 1}} \sin(k_5 \rho u)
\]
We can rewrite it as:
\[
g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_5 e^{ik_5 \tau} G_{KG}(x, k_5)
\]
where \(G_{KG}(x, k_5)\) is given in \([8]\).
Moving on with the integration, we immediately find:
\[
g(x, \tau) = \frac{1}{8\pi^2} \delta(t^2 - r^2) \int_{0}^{\infty} dk_5 \cos(k_5 \tau) J_0(k_5 \rho) + \frac{\theta(t^2 - r^2)}{2\pi^2} \frac{\partial}{\partial r^2} \frac{\theta(t^2 - r^2 - \tau^2)}{\sqrt{t^2 - r^2 - \tau^2}}
\]
\[
= \frac{1}{4\pi^2} \delta(t^2 - r^2) \frac{1}{2} \int_{0}^{\infty} dk_5 \cos(k_5 \tau) \times 1 + \frac{\theta(t^2 - r^2)}{2\pi^2} \frac{\partial}{\partial r^2} \frac{\theta(t^2 - r^2 - \tau^2)}{\sqrt{t^2 - r^2 - \tau^2}}
\]
\[
= \frac{1}{4\pi^2} \delta(t^2 - r^2) \delta(\tau) + \frac{\theta(t^2 - r^2)}{2\pi^2} \frac{\partial}{\partial r^2} \frac{\theta(t^2 - r^2 - \tau^2)}{\sqrt{t^2 - r^2 - \tau^2}}
\]
We see that extra term \(\delta(t^2 - r^2) \delta(\tau)\) arises from the condition on the Klein-Gordon Green-function \([8]\) which results in a contribution on the 4D light-cone \(x_\mu x^\mu\). However, we note that integrating \(G_{KG}(x, m)\) directly with respect to \(m\) does not reproduce the 5D GF as in \([7]\). To see this we first show that it is indeed expected that such an integration would produce \([7]\), as \(G_{KG}(x, m)\) is essentially the GF of the Klein-Gordon equation
\[
(\partial_\mu \partial^\mu - m^2) G_{KG}(x, m) = \delta^4(x) \tag{13}
\]
and therefore, integrating over \(m\)
\[
\int_{-\infty}^{\infty} e^{im\tau}(\partial_\mu \partial^\mu - m^2) G_{KG}(x, m) \, dm = \delta^4(x) \int_{-\infty}^{\infty} e^{im\tau} \, dm \tag{14}
\]
\[
\int_{-\infty}^{\infty} e^{im\tau}(\partial_\mu \partial^\mu + \partial_\tau) G_{KG}(x, m) \, dm = 2\pi \delta^4(x) \delta(\tau) \tag{15}
\]
and therefore
\[ g(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{im\tau} G_{KG}(x, m) \, dm \]  \quad (16)

as expected. Nevertheless, taking \( G_{KG}(x, m) \) directly as in (8) would not produce (7). However, one can rewrite (8) in such a way that results in (16) correctly, as follows:

\[ G_{KG}(x, m) = \frac{\partial}{\partial r^2} \left[ \frac{m\theta(-x, x^\mu)}{4\pi} J_0(m \sqrt{-x, x^\mu}) \right] \]  \quad (17)

Therefore, under the integral in (16), the \( \theta(-x, x^\mu) \) in eq. (17) would be subsumed by \( \theta(-x, x^\mu - \tau^2) \), in accordance with (7), and therefore, would eliminate the \( \delta(x, x^\mu)\delta(\tau) \).

## 3 Integration over \( k_5 \) first

In this method, Oron and Horwitz [16] split (4) into 2 regions in \((k, k_0)\) space, the timelike region \( k_5 k_0^\mu < 0 \) and the spacelike region \( k_5 k_0^\mu > 0 \). We shall reexamine this calculation here and find that correcting a sign error in (16), the result (7) emerges from this method as well.

Thus, one finds

\[ g(x, \tau) = g_1(x, \tau) + g_2(x, \tau) \]

\[ g_1(x, \tau) = \frac{1}{(2\pi)^3 r} \int_{-\infty}^{+\infty} dk_5 \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk_0 \theta(k^2 - k_0^2) \sin(kr) e^{i(k_5 \tau - k_0 t)} \]

\[ g_2(x, \tau) = \frac{1}{(2\pi)^3 r} \int_{-\infty}^{+\infty} dk_5 \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk_0 \theta(-k^2 + k_0^2) \sin(kr) e^{i(k_5 \tau + k_0 t)} \]

Then each of the functions can be contour integrated over \( k_5 \). Clearly, in \( g_1(x, \tau) \) the integral is well defined as the poles are in the complex plane \( k_5 = \pm i \sqrt{k^2 - k_0^2} \), whereas in \( g_2(x, \tau) \), the Principal Part is taken over \( k_5 = \pm \sqrt{k_0^2 - k^2} \).

The result is

\[ g_1(x, \tau) = \frac{1}{(2\pi)^3 r} \int_{0}^{\infty} dl \int_{-\infty}^{+\infty} d\alpha \ l \ \cosh(\alpha) \sin(l \sinh(\alpha)) \cos(l \cosh(\alpha)) e^{-l|\tau|} \]

\[ g_2(x, \tau) = -\frac{1}{(2\pi)^3 r} \int_{0}^{\infty} dl \int_{-\infty}^{+\infty} d\alpha \ l \ \sinh(\alpha) \sin(l \sinh(\alpha)) \cos(l \cosh(\alpha)) \sin(l |\tau|) \]

where \( l = \sqrt{\pm(k^2 - k_0^2)} \) and \( \alpha \) is the corresponding hyperbolic angle.

In both cases we can simplify by absorbing \( l \cosh(\alpha) \) or \( l \sinh(\alpha) \) as follows:

\[ g_1(x, \tau) = -\frac{1}{(2\pi)^3 r} \int_{0}^{\infty} dl \int_{-\infty}^{+\infty} d\alpha \ \cos(l \sinh(\alpha)) \cos(l \cosh(\alpha)) e^{-l|\tau|} \]

\[ g_2(x, \tau) = \frac{1}{(2\pi)^3 r} \int_{0}^{\infty} dl \int_{-\infty}^{+\infty} d\alpha \ \cos(l \sinh(\alpha)) \cos(l \cosh(\alpha)) \sin(l |\tau|) \]
Expanding the \( \cos(\ldots) \) terms:

\[
g_1(x, \tau) = -\frac{1}{2(2\pi)^3 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\alpha \times \left( \cos(l(r \cosh(\alpha) + t \sinh(\alpha))) + \cos(l(r \cosh(\alpha) - t \sinh(\alpha)) \right) e^{-l|\tau|}.
\]

For \( r > |t| \) we can write \( r = \rho \cosh(\beta) \) and \( t = \rho \sinh(\beta) \) to find:

\[
g_1(x, \tau) = -\frac{1}{(2\pi)^3} \frac{\partial}{\partial \rho} \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\alpha \times \left( \cos(l\rho \cosh(\alpha + \beta)) + \cos(l\rho \cosh(\alpha - \beta)) \right) e^{-l|\tau|}.
\]

Performing the \( l \) integration first, one obtains:

\[
g_1(x, \tau) = -\frac{1}{(2\pi)^3} \frac{\partial}{\partial \rho} \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} d\alpha \times \cos(l\rho \cosh(\alpha)) e^{-l|\tau|}.
\]

Integrating \( g_1(x, \tau) \) over \( \alpha \) results in

\[
g_1(x, \tau) = \frac{1}{2i(2\pi)^3} \frac{\partial}{\partial \rho} \frac{2}{\sqrt{(i|\tau|)^2 - \rho^2}} \ln \left( \frac{i|\tau| + \sqrt{(i|\tau|)^2 - \rho^2}}{-i|\tau| + \sqrt{(-i|\tau|)^2 - \rho^2}} \right).
\]
and since $\rho^2 = r^2 - t^2 > 0$, we have
\[
\sqrt{-t^2 - \rho^2} = \sqrt{(-1)(\rho^2 + \tau^2)} = i\sqrt{\rho^2 + \tau^2}
\]
and thus:
\[
g_1(x, \tau) = -\frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \frac{1}{\sqrt{\tau^2 + \rho^2}} \ln \left( \frac{|\tau| + \sqrt{\tau^2 + \rho^2}}{-|\tau| + \sqrt{\tau^2 + \rho^2}} \right)
\]
Similarly for $g_2(x, \tau)$:
\[
g_2(x, \tau) = \frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \frac{1}{\sqrt{\tau^2 + \rho^2}} \ln \left( \frac{|\tau| + \sqrt{\tau^2 + \rho^2}}{-|\tau| + \sqrt{\tau^2 + \rho^2}} \right)
\]
Clearly, $g_1(x, \tau) = -g_2(x, \tau)$, and thus, for the case of $\rho^2 = r^2 - t^2 > 0$, we find $g(x, \tau) = 0$.
Returning back to (18) with $0 \leq r < |l|$, we write $r = \rho \sinh(\beta)$ and $t = \epsilon(t) \rho \cosh(\beta)$ where $\rho^2 = t^2 - r^2$, and thus:
\[
g_1(x, \tau) = -\frac{1}{2(2\pi)^3 r} \frac{\partial}{\partial r} \int_0^\infty dl \int_{-\infty}^{+\infty} d\alpha \left( \cos (l\rho \sinh(\beta + \epsilon(t)\alpha)) + \cos (l\rho \sinh(\beta - \epsilon(t)\alpha)) \right) e^{-l|\tau|} \times \cos (l(\rho \sinh(\alpha)) \sin(l|\tau|))
\]
\[
g_2(x, \tau) = \frac{1}{2(2\pi)^3 r} \frac{\partial}{\partial r} \int_0^\infty dl \int_{-\infty}^{+\infty} d\alpha \left( \cos (l\rho \cosh(\alpha + \epsilon(t)\beta)) + \cos (l(\rho \cosh(\alpha + \epsilon(t)\beta)) \sin(l|\tau|)) \right. \times \cos (l(\rho \cosh(\alpha)) \sin(l|\tau|))
\]
Realigning the integration bounds, one finds:
\[
g_1(x, \tau) = -\frac{1}{2(2\pi)^3 r} \frac{\partial}{\partial r} \int_0^\infty dl \int_{-\infty}^{+\infty} d\alpha \left( \frac{1}{|\tau| + i\rho \sinh(\alpha)} \right. \times \left. \frac{1}{|\tau| + i\rho \sinh(\alpha)} \right)
\]
\[
g_2(x, \tau) = \frac{1}{2(2\pi)^3 r} \frac{\partial}{\partial r} \int_0^\infty dl \int_{-\infty}^{+\infty} d\alpha \left( \frac{1}{\rho \cosh(\alpha) + |\tau|} \right. \times \left. \frac{1}{\rho \cosh(\alpha) + |\tau|} \right)
\]
Once again, after integrating first over $l$ we find:
\[
g_1(x, \tau) = -\frac{1}{2} \frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \int_{-\infty}^{+\infty} d\alpha \left| \frac{1}{|\tau| - i\rho \sinh(\alpha)} \right| \times \left| \frac{1}{|\tau| + i\rho \sinh(\alpha)} \right|
\]
\[
g_2(x, \tau) = \frac{1}{2} \frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \int_{-\infty}^{+\infty} d\alpha \left| \frac{1}{\rho \cosh(\alpha) + |\tau|} \right| \times \left| \frac{1}{\rho \cosh(\alpha) + |\tau|} \right|
\]
We can now do the $\alpha$ integration:
\[
g_1(x, \tau) = -\frac{1}{2} \frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \times 2 \times \frac{-1}{\sqrt{(|\tau|)^2 + \rho^2}} \ln \left( \frac{i|\tau| + \sqrt{(|\tau|)^2 + \rho^2}}{-i|\tau| + \sqrt{(|\tau|)^2 + \rho^2}} \right)
\]
\[
g_2(x, \tau) = \frac{1}{2} \frac{1}{(2\pi)^3 r} \frac{\partial}{\partial r} \times \frac{1}{\sqrt{(|\tau|)^2 - \rho^2}} \ln \left( \frac{|\tau| + \sqrt{(|\tau|)^2 - \rho^2}}{-|\tau| + \sqrt{(|\tau|)^2 - \rho^2}} \right)
\]
Now, we have 2 cases:

• \( \rho^2 - r^2 = t^2 - r^2 - \tau^2 > 0 \), i.e., the 5D timelike region.

• \( t^2 - r^2 - \tau^2 < 0 \), which is part of the 5D spacelike region, since we still have \( t^2 - r^2 > 0 \).

Let us first consider the second case, namely, \( t^2 - r^2 - \tau^2 < 0 \). We then find:

\[
|i| + \sqrt{(i|\tau|)^2 + \rho^2} = |i| + \sqrt{-\tau^2 + \rho^2} = |i| + \sqrt{(-1)(\tau^2 - \rho^2)} = |i|(|i| + \sqrt{\tau^2 - \rho^2})
\]

In which case, once again, one finds

\[
g_1(x, \tau) = -\frac{1}{(2\pi)^3 r^2} \ln \left( \frac{|i| + \sqrt{\tau^2 + r^2 - t^2}}{|i| + \sqrt{\tau^2 + r^2 - t^2}} \right)
\]

\[
g_2(x, \tau) = \frac{1}{(2\pi)^3 r^2} \ln \left( \frac{|i| + \sqrt{\tau^2 + r^2 - t^2}}{|i| + \sqrt{\tau^2 + r^2 - t^2}} \right)
\]

in which case, once again, \( g_1(x, \tau) = -g_2(x, \tau) \).

On the other hand, in the 5D timelike region, we have \( t^2 - r^2 - \tau^2 > 0 \), in which case, the numerator and denominator of the \( \ln \) argument in both \( g_1 \) and \( g_2 \) are complex conjugates.

Let us use a shortened notation, in which \( a = |i| \) and \( b = \sqrt{t^2 - r^2 - \tau^2} \).

For \( g_1 \), we find the argument of the \( \ln \) to be:

\[
\frac{ia + b}{-ia + b} = e^{i\tan^{-1}(a/b) - i\tan^{-1}(-a/b)} = e^{2i\tan^{-1}(a/b)}
\]

Similarly, for \( g_2 \):

\[
\frac{a + ib}{-a + ib} = \frac{b - ia}{b + ia} = e^{i\tan^{-1}(-a/b) - i\tan^{-1}(a/b)} = e^{-2i\tan^{-1}(a/b)} = e^{2i\pi - 2i\tan^{-1}(a/b)}
\]

Thus, in the shortened notation, and recalling that \( \sqrt{\tau^2 + r^2 - t^2} = i\sqrt{t^2 - r^2 - \tau^2} = ib \) we find:

\[
g_1 = -\frac{1}{(2\pi)^3 r^2} \frac{\partial}{\partial r} \frac{-1}{ib} 2i\tan^{-1} \left( \frac{a}{b} \right)
\]

\[
g_2 = \frac{1}{(2\pi)^3 r^2} \frac{\partial}{\partial r} \frac{1}{ib} \left[ 2i\pi - 2i\tan^{-1} \left( \frac{a}{b} \right) \right]
\]

Clearly, when summing \( g = g_1 + g_2 \), the \( \tan^{-1}(\ldots) \) terms cancel, and we find:

\[
g(x, \tau) = \frac{1}{2\pi^2 \sqrt{\tau^2 + r^2 - t^2}} \frac{\partial}{\partial r} \theta(t^2 - r^2 - \tau^2)
\]

which is, once again, the conventional solution.

Thus, even for this method of integrating \( k_5 \) first, we have obtained \( 7 \) (a different result was obtained in \[10\] due to an error in sign).
4 Conclusions and discussion

We have shown that the 5D Green Function is reproduced with the same methods used in [16] and [13], showing that the different methods used lead to equivalent results. In this, we believe that the apparent form of the Green function discrepancy has been removed, and one can utilize the $\tau$-retarded conventional Green function for computing the fields. In [2], we have used the method of Nozaki [15], who derived generalized fundamental solutions for the $O(p, q)$ wave equation, to obtain an explicit $\tau$-retarded Green function. The form of the Green function has direct implications on the form of the fields produced by charges, and in particular, on the problem of radiation reaction.

We have chosen the $\tau$-retarded Green-function in this work, and for future applications, in accordance with what appears to be the dynamical structure of the 5D fields. In contrast to the solutions of the standard Maxwell theory, where the $t$ variable is subject to the action of the Lorentz transformation as a fundamental symmetry, the $\tau$ variable does not necessarily participate in a higher symmetry. Although the equations for the fields have $O(4, 1)$ symmetry, in this case, the manifestly covariant dynamics of the sources evolve by the $O(3, 1)$ invariant $\tau$ serving as a universal evolution parameter.

Non-relativistic mechanics is conventionally considered to evolve according to a Galilean invariant parameter $t$ (which may be identified with the parameter $\tau$ of the relativistic theory), corresponding to Newton’s basic postulate of a universal time. The relativistic dynamics of Stueckelberg [21] [22] considers the events $x^\mu(\tau)$ to be the basic dynamical elements of the relativistic theory of matter. Jackson’s construction [9] for a covariant current is an example for this interpretation (see also [14] [18]). The quantization of the 5D fields that arise as the $U(1)$ gauge fields from the Stueckelberg-Schrödinger equation [20] (see eq. (1)) indicate that these fields evolve in $\tau$, unlike the Maxwell fields which have no distinguished evolution parameter. One may therefore impose a causal structure on the fields governed by this distinguished parameter $\tau$, through the use of the $\tau$-retarded propagator.

Application of the explicit $\tau$-retarded solution to the radiation reaction problem will be reported in a succeeding publication.
References

[1] I. Aharonovich and L. P. Horwitz, *Green functions for wave propagation on a five-dimensional manifold and the associated gauge fields generated by a uniformly moving point source*, Journal of Mathematical Physics 47 (2006), no. 12, 122902.

[2] I. Aharonovich and L. P. Horwitz, *Radiation fields of a uniformly accelerating point source in the framework of Stueckelberg’s manifestly covariant relativistic dynamics*, Journal of Mathematical Physics 51 (2010), no. 5, 052903.

[3] I. Aharonovich and L. P. Horwitz, *Radiation reaction in the classical off-shell electrodynamics under the manifestly covariant relativistic dynamics of stueckelberg: (i) above shell case.*, To be published (2011).

[4] R. Courant and D. Hilbert, *Methods of mathematical physics*, 1 ed., vol. 2, Wiley-Interscience, 1989.

[5] P. A. M. Dirac, *Classical theory of radiating electrons*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1934-1990) 167 (1938), no. 929, 148–169.

[6] Gal’tsov and V. Dmitri, *Radiation reaction in various dimensions*, Phys. Rev. D 66 (2002), no. 2, 025016.

[7] I. M. Gel’fand and G. E. Shilov, *Generalized functions, properties and operations*, Generalized Functions, vol. 1, Academic Press, 1964, Translated from Russian.

[8] L. P. Horwitz and N. Shnerb, *Second quantization of the stueckelberg relativistic quantum theory and associated gauge fields*, Foundations of Physics 28 (1998), no. 10, 1509–1519.

[9] John David Jackson, *Classical electrodynamics*, 3 ed., Wiley, August 1998.

[10] P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, *Radiation reaction and renormalization in classical electrodynamics of a point particle in any dimension*, Phys. Rev. D 66 (2002), no. 2, 025017.

[11] Boris Kosyakov, *Introduction to the classical theory of particles and fields*, Springer, 2007.

[12] Prem K. Kythe, *Fundamental solutions for differential operators and applications*, Birkhäuser, July 1996.

[13] M. C. Land and L. P. Horwitz, *Green’s functions for off-shell electromagnetism and spacelike correlations*, Foundations of Physics 21 (1991), no. 3, 299–310.

[14] M. C. Land, N. Shnerb, and L. P. Horwitz, *On Feynman’s approach to the foundations of gauge theory*, Journal of Mathematical Physics 36 (1995), no. 7, 3263–3288.
[15] Yasuo Nozaki, *On Riemann-Liouville integral of ultra-hyperbolic type*, Kodai Math. Sem. Rep. 16 (1964), no. 2, 69–87.

[16] O. Oron and L. P. Horwitz, *Classical radiation reaction off-shell corrections to the covariant lorentz force*, Physics Letters A 280 (2001), no. 5-6, 265 – 270.

[17] Fritz Rohrlich, *Classical charged particles*, John Wiley & Sons, December 1990.

[18] D. Saad, L.P. Horwitz, and R.I. Arshansky, *Off-shell electromagnetism in manifestly covariant relativistic quantum mechanics*, Foundations of Physics 19 (1989), no. 10, 1125–1149.

[19] Julian Schwinger, *On gauge invariance and vacuum polarization*, Phys. Rev. 82 (1951), no. 5, 664–679.

[20] N. Shnerb and L. P. Horwitz, *Canonical quantization of four- and five-dimensional U(1) gauge theories*, Phys. Rev. A 48 (1993), no. 6, 4068–4074.

[21] E. C. G. Stueckelberg, *Remarks on the creation of pairs of particles in the theory of relativity*, Helv. Phys. Acta 14 (1941), 588–594.

[22] ———, *La Mecanique du point materiel en theorie de relativite et en theorie des quanta*, Helv. Phys. Acta 15 (1942), 23–37.