Sojourn probabilities in tubes and pathwise irreversibility for Itô processes

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The sojourn probability of an Itô diffusion process, that is its probability to remain in the tubular neighborhood of a smooth path, is a central quantity in the study of path probabilities. For N-dimensional Itô processes with a state-dependent full-rank diffusion tensor, we derive a general expression for the sojourn probability in tubes whose radii are small but finite, and fixed by the metric of the ambient Euclidean space. The central quantity in our study is the exit rate at which trajectories leave the tube for the first time. This has an interpretation as a Lagrangian and can be measured directly in experiment, unlike previously defined sojourn probabilities (involving variable tube radius or shape) which depend on prior knowledge of the state-dependent diffusivity. We find that while in the limit of vanishing tube radius the ratio of sojourn probabilities for a pair of distinct paths is in general divergent, the same for a path and its time-reversal is always convergent and finite. This provides, in turn, a pathwise definition of irreversibility for Itô processes that is agnostic to the state-dependence of the diffusion tensor. For one-dimensional systems we derive an explicit expression for our Lagrangian in terms of the drift and diffusivity, and find that our Lagrangian differs from all previously reported multiplicative-noise generalizations of the Onsager-Machlup Lagrangian. We confirm our result by comparing to numerical simulations for a one-dimensional diffusion process with state-dependent diffusivity, and relate our theory to the classical Stratonovich Lagrangian for multiplicative noise. For one-dimensional systems, we furthermore discuss under which conditions the vanishing-radius limiting ratio of sojourn probabilities for a pair of forward and backward paths recovers the pathwise entropy production found in the literature. Finally, we demonstrate for our one-dimensional example system that the most probable tube for a barrier crossing depends sensitively on the tube radius, and hence on the tolerated amount of fluctuations around the smooth reference path.

I. INTRODUCTION

The overdamped Langevin equation is a fundamental model for diffusive stochastic dynamics with a wide range of applications ranging from chemical physics [1], to ecology [2] and finance [3, 4]. For the special case of additive noise, meaning that the noise term in the Langevin equation does not depend on the current state of the system, relative path probabilities are well-characterized [5–19]. In particular, one experimentally relevant approach to path probabilities is via the sojourn probability, i.e. the probability that a stochastic trajectory always remains within a ball of constant finite radius and with moving center given by a twice continuously differentiable reference path. Since for diffusive dynamics any given individual path is observed with vanishing probability, the sojourn probability becomes zero in the limit of vanishing radius. However, for any given pair of paths the ratio of sojourn probabilities has a finite vanishing-radius limit, so that this limiting ratio can be used to define a ratio of path probabilities. It was Stratonovich [6] who first showed that the limiting ratio of sojourn probabilities is characterized by the Onsager-Machlup (OM) stochastic action functional [6, 9, 10, 14, 16, 20]. An advantage of considering path probabilities as limiting ratios of sojourn probabilities is the connection to physical observables; for any finite tube radius the sojourn probability is positive, and hence an experimental observable. Indeed, the mathematical theory of Stratonovich [6] was recently confirmed experimentally, by extrapolating measured ratios of finite-radius sojourn probabilities to the single-path limit of vanishing radius [21].

While for additive noise the situation is well-understood, the situation is not as simple for multiplicative noise, where the noise strength in the Langevin equation depends on the current state of the system [1]. For state-dependent noise, the limiting ratio of tube probabilities in general does not yield meaningful results, and defining a probability density relative to a quasi translation invariant measure on the space of all paths is not possible [9]. Still, starting with the work of Freidlin and Wentzel [22] (who focused on the weak-noise limit, as discussed more in the conclusions), and Stratonovich [6], there have been several attempts to quantify relative path probabilities also in systems with multiplicative noise [7, 10]. These works can broadly be classified into two approaches. One approach is to define the tubular neighborhood using the metric induced by the diffusion tensor of the stochastic dynamics [22]. The tube is then a moving ellipsoid in $\mathbb{R}^N$, whose principal axes
may vary along the reference path, which is the geometric center of mass of the ellipsoid. Another approach is to locally introduce a new coordinate system, relative to which the diffusivity is state-independent. In this new coordinate system, the established theory for additive noise can be applied. However, since the relation between the two sets of coordinates is nonlinear, in the original coordinates the tube can have an in principle arbitrary geometrical shape. In both approaches mentioned here the tube is therefore not a moving ball in Euclidean space, but an ellipsoid or more general geometrical shape. As we show further below in Sect. II, for one-dimensional systems, these two approaches lead to different small-radius stochastic action Lagrangians. Importantly, in both cases the diffusivity tensor associated with the underlying stochastic dynamics needs to be known to even construct the tube. This severely limits the experimental relevance of these sojourn probabilities, as in an experimental system it is natural to consider a constant-radius tube with respect to the Euclidean metric of the ambient space (and knowledge of the diffusivity profile might not even be available). Indeed, the recent experimental measurement of the OM stochastic action for additive noise employed constant-radius tubes. Hitherto no theory existed for the sojourn probability of a constant-radius scenario for multiplicative noise.

![Diagram](image)

**Figure 1.** Illustration of the tubular neighborhood of a reference path. The black solid line depicts a reference path \( \varphi(t) \) that starts at \( \varphi(t_i) \) and ends at \( \varphi(t_f) \), as indicated by two horizontal gray dashed lines. The gray shaded region around \( \varphi \) depicts a tube with time-dependent radius \( R(t) \) around the reference path, the instantaneous tube radius is indicated by a vertical dashed line. The blue solid line is a realization of one-dimensional Langevin dynamics with multiplicative noise which remains within the tube until the final time, the orange solid line represents a realization that leaves the tube before the final time \( t_f \).

We here fill this gap, by providing a comprehensive theory of tube probabilities for diffusive dynamics with state-dependent diffusivity. We achieve this by establishing an expression for the sojourn probability for tubes with small-but-finite radius, which may vary along the reference path, as illustrated in Fig. 1. The central quantity in our theory is the exit rate at which trajectories first leave the tube, which can be interpreted as a generalization of the stochastic action Lagrangian. We present a series expansion of this exit rate in powers of the time-dependent tube radius. This power series generalizes a previous result for the tubular exit rate for additive noise to which it reduces if the diffusion tensor of the stochastic dynamics is state-independent and isotropic, and if additionally the tube radius is independent of time.

Our theory for finite-radius tubes leads to a physical picture as to why the definition of ratios of path probabilities is not straightforward for state-dependent diffusivity. Namely, because diffusive stochastic dynamics with a full-rank diffusivity tensor is at short length- and time scales dominated by the random noise, as compared to the drift, the sojourn probability is for asymptotically small radius dominated by the diffusivity. For additive noise this dominant term leads to a path-independent factor in the sojourn probability, and so is irrelevant for ratios of path probabilities. However, this is not the case for state-dependent diffusivity, and is the reason why limiting ratios of tube probabilities are in general either zero or divergent. As we discuss further below, finite limiting ratios of sojourn probabilities are only obtained if the tube radius is fine-tuned in such a way that the dominating short-time noise contributions cancel.

While limiting ratios of sojourn probabilities in general do not yield meaningful results, we show that the ratio for a pair of forward and time-reverse reference path is always finite. This provides a pathwise definition of irreversibility for Itô processes that is agnostic to the state-dependence of the diffusion tensor, and which is related to classical measures of irreversibility and entropy production.

For the special case of a one-dimensional system, we derive explicit expressions for the exit rate describing the sojourn probability, and discuss several choices for the time-dependent tube radius. In particular we present an explicit formula for the sojourn probability of a constant-radius tube. We validate our theory by comparing to numerical simulations, and discuss explicitly the relation of the Stratonovich Lagrangian to tubular exit rates. In the context of barrier crossing for a one-dimensional system, we furthermore show that the most probable tube depends sensitively on both the details of the time-dependence of the radius, and the size of the tube.

In our accompanying paper Ref. [26] we compare the theoretical sojourn probabilities derived here also to experimental results, and discuss the radius dependence of the most probable tube based on measured time series. The present work thus provides an experimentally accessible approach to quantifying and generalizing path probabilities for systems with state-dependent diffusivity.

The main part of this paper is structured as follows. In Sect. II we first present our general theory of sojourn
probabilities for $N$-dimensional Langevin dynamics with arbitrary state-dependent full-rank diffusion matrix. In Sect. III we consider one-dimensional systems, $N = 1$, for which we provide explicit expressions for the exit rate in terms of the diffusivity and drift, discuss several possible choices for $R(t)$, and relate the Stratonovich Lagrangian to tubular exit rates. We furthermore discuss how the most probable tube connecting an initial and a final state can depend on the tube radius. We conclude in Sect. IV, where we summarize our findings and discuss their further implications.

II. N-DIMENSIONAL THEORY

We now present our general results for $N$-dimensional diffusive dynamics. For an $N$-dimensional coordinate $X_i(t) \equiv X(t) \equiv (X_1(t), \ldots, X_N(t))$, we consider the Itô-Langevin equation given by \cite{1,27}

$$dX_t = a(X_t, t) \, dt + b(X_t, t) \, dW_t, \quad (1)$$

where $dW_t$ is the increment of the $N$-dimensional Wiener process, $a$ is the drift, and $b$ is the noise matrix, with components $a_i \equiv a_i(x, t)$, $b_{ij} \equiv b_{ij}(x, t)$. While we interpret Eq. (1) in the Itô sense, results for other conventions are obtained from our results by modifying the drift term $a$ appropriately \cite{27}. The stochastic dynamics defined by Eq. (1) can equivalently be described via the Fokker-Planck equation (FPE) \cite{27}

$$\partial_t P = -\nabla_i (a_i P) + \nabla_j (D_{ij} P), \quad (2)$$

where $P \equiv P(x, t | x_i, t_i)$ is the transition probability density for finding a particle at position $x$ and time $t$ after it has started at $x_i$ at time $t_i$, by $\nabla_i \equiv \partial/\partial x_i \equiv \partial_t$ we denote the partial derivative in the $x_i$-direction, and the components of the symmetric diffusion tensor $D$ are given by $D_{ij}(x, t) \equiv b_{ik}(x, t)b_{jk}(x, t)/2$, where we use the Einstein sum convention for repeated indices. We assume that the diffusivity tensor $D$ is a matrix of full rank, which physically means that the noise directly acts on all $N$ degrees of freedom. To denote time derivatives, we use the notation $\partial_t P$ and $P$ interchangeably.

A. The tubular ensemble and sojourn probabilities

We consider the tubular ensemble, which consists of all realizations of the Langevin Eq. (1) that remain within a time-dependent distance $R(t)$ of a continuous reference path $\varphi(t)$ until time $t_f$ \cite{0,7,10,14,16,22,28}.

$$\mathcal{X}_R^\varphi(t_f) \equiv \{ X \mid ||X_s - \varphi(s)|| < R(s) \forall s \in [0, t_f] \}, \quad (3)$$

where in principle any norm $|| \cdot ||$ on $\mathbb{R}^N$ can be used to quantify distances. We use the name tubular ensemble for $\mathcal{X}_R^\varphi$ because a ball with center $\varphi(t)$ and radius $R(t)$ is a tube in spacetime $(x, t)$, as illustrated in Fig. 1.

The sojourn probability

$$P_R^\varphi(t) \equiv P \left( X \in \mathcal{X}_R^\varphi(t) \mid X_0 \sim P_i \right) \quad (4)$$

is the probability that a stochastic trajectory $X$ remains within the tubular neighborhood around the reference path $\varphi$ until time $t$. As indicated in Eq. (4), the sojourn probability depends on the initial distribution $X_0 \sim P_i$ inside the tube; since we are mostly interested in the temporal decay rate of the sojourn probability, which for small tube radius is only affected by the initial condition for a short initial relaxation time $\tau_{rel} \sim R(0)^2$, we suppress the dependence on initial conditions in the following.

The decay of the sojourn probability is described by the instantaneous rate at which stochastic trajectories leave the tubular neighborhood of $\varphi$ for the first time, $\alpha_R^\varphi(t)$, as

$$P_R^\varphi(t) = \exp \left[ -\int_0^t ds \, \alpha_R^\varphi(s) \right]. \quad (5)$$

The sojourn probability is a functional of both the reference path $\varphi(t)$ and the function $R(t)$ which specifies the time-dependence of the radius, and is equivalently described by the functional

$$S[\varphi, R] \equiv S_R[\varphi] \equiv \mathcal{S}_R^\varphi \equiv \int_0^t ds \, \alpha_R^\varphi(s), \quad (6)$$

which we refer to as tubular stochastic action because it describes experimentally observable sojourn probabilities. The exit rate can similarly be interpreted as a finite-radius (tubular) stochastic action Lagrangian $\mathcal{L}_R^\varphi \equiv \alpha_R^\varphi$. As emphasized in Eq. (5), we indicate functional dependencies interchangeably by square brackets and via sub- or superscripts.

In the following we focus on the case where the norm used to define the tube is the standard Euclidean norm, $||x||_2 \equiv \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, and where the reference path $\varphi$ is twice continuously differentiable. To consistently speak of power-series expansions in the time-dependent radius $R(t)$, we furthermore assume that $R(t) = R_0 r(t)$ for a reference radius $R_0$ and a dimensionless differentiable function $r(t)$. We then interpret power series in $R(t)$ as power series in $R_0$, meaning that we scale the radius uniformly (independent of time) along the path.

As we show in more detail in Sect. IIIB below and in App. A the exit rate can be expanded as a perturbation series in $R_0$ for small tube radius, yielding

$$\mathcal{L}_R^\varphi(t) \equiv \alpha_R^\varphi(t) = \frac{\alpha_{\text{free}}^\varphi(t)}{R(t)^2} + \alpha^\varphi(0)(t) + \alpha^\varphi(2) R^2(t) + \mathcal{O}(R_0^4), \quad (7)$$

where

$$\frac{\alpha_{\text{free}}^\varphi(t)}{R(t)^2} = \frac{f(D(\varphi(t), t))}{R(t)^2}. \quad (8)$$
At time \( t \), Eq. \( \text{8} \) is the steady-state free-diffusion exit rate from an \( N \)-dimensional ball of radius \( R(t) \) for Itô-Langevin dynamics Eq. \( \text{1} \) with vanishing drift and a spatially constant diffusion tensor \( D(\varphi(t), t) \). The function \( f(\mathbf{M}) \) is defined for a symmetric full-rank matrix \( \mathbf{M} \) with components \( M_{ij} \) as the smallest negative eigenvalue of the anisotropic Laplace operator \( M_{ij} \nabla_i \nabla_j \), with domain the unit ball and absorbing boundary conditions, see App. \( \text{B} \) for more details. While at time \( t \) the free-diffusion exit rate Eq. \( \text{8} \) scales as \( 1/R(t)^2 \) and only depends on \( D(\varphi(t), t) \), the term \( \alpha^{\varphi,(0)}(t) \) in Eq. \( \text{7} \) is of order \( R(t) \) and depends on \( \alpha(\varphi(t), t), D(\varphi(t), t) \), as well as their spatial and temporal derivatives up to second order, evaluated at \( (\varphi(t), t) \). Note that, as we will see in our explicit one-dimensional example further below, both \( \alpha^{\varphi,(0)} \) and \( \alpha^{\varphi,(2)} \) can depend on \( \dot{r}/r \).

The Itô-Langevin Eq. \( \text{1} \) is on short length- and time scales dominated by the random noise term, which explains that for small radius the exit rate Eq. \( \text{7} \) is dominated by the instantaneous steady-state free-diffusion exit rate Eq. \( \text{8} \) \[20\]. This rate diverges as the radius approaches zero, which via Eq. \( \text{3} \) implies that the probability of any individual path vanishes. On the other hand, the probability Eq. \( \text{7} \) at finite radius is a physical observable, as was noted before in the context of additive noise \[20\]. The expansion Eq. \( \text{7} \) allows to calculate this tube probability for small-but-finite time-dependent radius \( R(t) \), and hence to quantify pathway probabilities for diffusive trajectories in an experimentally measurable way.

### B. Exit rate in terms of FP spectrum

To derive the perturbation series Eq. \( \text{7} \) for the exit rate for a given reference path \( \varphi(t) \) and tube radius \( R(t) \), we consider the equivalent description of the stochastic process Eq. \( \text{1} \) inside the tubular neighborhood of the reference path via the FP Eq. \( \text{2} \). At time \( t \), the spatial domain for Eq. \( \text{2} \) is then

\[
\mathbf{x} \in B_R(t) \equiv \left\{ \mathbf{x} \mid \|\mathbf{x} - \varphi(t)\|_2 < R(t) \right\}, \tag{9}
\]

which is indicated in Fig. \( \text{1} \) as a gray shaded area. The solution \( P_R^\varphi(\mathbf{x}, t) \) to the FPE is subject to absorbing boundary conditions at the tube boundary, \( P_R^\varphi(\mathbf{x}, t) = 0 \) for all \( \mathbf{x} \in \partial B_R(t) \). \( P_R^\varphi(\mathbf{x}, t) \) then describes the spatial distribution of all stochastic trajectories \( \mathbf{X}_t \) that have never left the tubular neighborhood until time \( t \). Note that in Eq. \( \text{6} \) we consider the standard Euclidean norm, so that \( B_R^\varphi \) describes a moving ball in \( \mathbb{R}^N \), with instantaneous radius \( R(t) \) and center \( \varphi(t) \).

To obtain Eq. \( \text{7} \), we use the same strategy as in a recent derivation of finite-radius tubular exit rates for Langevin dynamics with isotropic additive diffusivity \( \mathbf{D} = D_0 \mathbf{1} \), where \( D_0 \) is a positive scalar and \( \mathbf{1} \) is the unit matrix \[20\]. We here shortly summarize the derivation, and refer the reader to App. \( \text{A} \) and Ref. \( \text{20} \) for more details. Our derivation assumes that \( \varphi \) is twice continuously differentiable, and that \( R(t) = R_0 r(t) \) so that \( \dot{R}/R = \dot{r}/r \) is independent of \( R_0 \). To derive Eq. \( \text{7} \) we first introduce dimensionless streaming coordinates that move along the tube center; this removes the time-dependence of the boundary conditions. In the streaming coordinate system we then project the FPE onto the instantaneous eigenbasis of the FP operator. In this eigenbasis, we derive an approximate solution of the FPE as a perturbation series in the small tube radius, using an approach similar to time-dependent perturbation theory in quantum mechanics \[29\]. This perturbative solution \( P_R^\varphi(\mathbf{x}, t) \) is, after an initial relaxation timescale \( \tau_{\text{rel}} \sim R(0)^2 \), dominated by the decay of the slowest-decaying eigenfunction. For our perturbative solution we assume that the diffusivity tensor has full rank, which implies that all degrees of freedom are for small length- and time-scales dominated by random forces.

The sojourn probability up to time \( t \) is simply the survival probability, which we calculate as the spatial integral over the solution \( P_R^\varphi(\mathbf{x}, t) \) of the FPE as \( P_R^\varphi(t) = \int_{B_R(t)}^\mathbb{R}^N d^N \mathbf{x} P_R^\varphi(\mathbf{x}, t) \). From the survival probability we finally evaluate the instantaneous exit rate, defined in Eq. \( \text{5} \) as

\[
\alpha^{\varphi}(t) = -\frac{\dot{P}_R^\varphi}{P_R^\varphi(t)}. \tag{10}
\]

Upon evaluating this expression using the perturbative solution of the FPE, the exit rate Eq. \( \text{7} \) follows. In App. \( \text{A} \) we give resulting expressions for \( f, \alpha^{\varphi,(0)}, \alpha^{\varphi,(2)} \), in terms of the instantaneous FP spectrum inside the tube. In particular we show that the equation that determines \( f \) at time \( t \) only depends on \( D(\varphi(t), t) \) and neither on spatial derivatives of the diffusivity tensor nor on the drift, which justifies the notation \( f(D(\varphi(t), t)) \) in Eq. \( \text{6} \).

### C. Asymptotic ratios of tube probabilities

While the probability to observe any individual path is zero, ratios of probabilities for individual paths can be defined in special cases.

One such case are systems with a constant isotropic diffusion tensor \( \mathbf{D} \equiv D_0 \mathbf{1} \equiv \text{const.} \), where \( D_0 \) is positive and \( \mathbf{1} \) denotes the unit matrix, and considering tubes of time-independent radius, \( R(t) \equiv R_0 = \text{const.} \), with respect to the standard Euclidean norm \[6, 7, 9, 10, 14–16, 20\]. In this scenario the function \( f \) in Eq. \( \text{8} \) is independent of the path \( \varphi \), so that the subleading-order term \( \mathcal{L}_{\text{OM}} \equiv \alpha^{\varphi}(0) \) quantifies relative path probabilities \[20\]. The ratio of path probabilities for two paths \( \varphi, \psi \) is then defined as vanishing-radius limiting ratio of sojourn probabilities, and quantified via a stochastic action \( \mathcal{S}_{\text{OM}} \) as \[6, 7, 9, 10, 14–16, 20\]

\[
e^{-\mathcal{S}_{\text{OM}}(\varphi)} \frac{\dot{P}_R^\varphi(t_f)}{e^{-\mathcal{S}_{\text{OM}}(\psi)}} \equiv \lim_{R_0 \to 0} \frac{P_R^\varphi(t_f)}{P_R^\psi(t_f)}, \tag{11}
\]
where the Onsager-Machlup (OM) action $S_{\text{OM}}[\varphi]$, which is a functional of the twice continuously differentiable path $\varphi$, is found to be

$$S_{\text{OM}}[\varphi] = \int_0^{t_f} dt \; \alpha_{\varphi}(0)(t) \equiv \int_0^{t_f} dt \; L_{\text{OM}}(\varphi(t), \varphi(t), t),$$

(12)

with the OM Lagrangian

$$L_{\text{OM}}(\varphi, \dot{\varphi}) \equiv \alpha_{\varphi}(0) = \frac{1}{4D_0} [\dot{\varphi} - \alpha(\varphi)]^2 + \frac{1}{2} \nabla \cdot \alpha(\varphi).$$

(13)

For additive isotropic noise, ratios of path probabilities can thus be defined as limits of temporally-constant-radius sojourn probabilities, and the OM Lagrangian quantifies such ratios \cite{6,7,9,13,15,20}. In view of Eqs. 6, 7, the OM action $S_{\text{OM}}$ and Lagrangian $L_{\text{OM}}$ are the order-$R_0^0$ contributions of the finite-radius tubular action $S$ and its associated tubular Lagrangian for the special case of additive isotropic noise and a temporally constant tube radius. For a state-dependent diffusion tensor, however, the limit in Eq. 11 in general does not yield meaningful results. In that case, it follows from substituting Eqs. 7, 8, into Eq. 5 that

$$\ln \frac{P_R^\varphi(t_f)}{P_R^\psi(t_f)} = - \int_0^{t_f} ds \left[ \frac{f(D(\varphi(s), s))}{R_\varphi^2(s)} - \frac{f(D(\psi(s), s))}{R_\varphi^2(s)} \right]$$

$$- \int_0^{t_f} ds \left[ \alpha_{\varphi}(0)(s) - \alpha_{\varphi}(s) \right]$$

$$+ O(R_0^2)$$

(14)

where we allow for different choices of $R(t)$ along the paths, which we indicate by the notation $R_{\varphi}(t) = R_0 \varphi(t)$, $R_{\psi}(t) = R_0 \psi(t)$. Since $f$ is a function of the local diffusivity along the path, it follows from Eq. 14 that in general the difference of the free-diffusion exit rates does not vanish, so that in the limit $R_0 \to 0$ the expression Eq. 14 diverges as $1/R_0^2$. The physical origin of this divergence is that in a region with low diffusivity a particle is less likely to diffuse away from a reference path, as compared to a region with large diffusivity \cite{20}.

While in general the log-ratio Eq. 14 diverges in the limit $R_0 \to 0$, we can obtain a finite limit by carefully choosing the path-dependent tube radius $R_{\varphi}$, $R_{\psi}$. To demonstrate this we consider the tube radius

$$R_{\varphi}(t) = R_0 \sqrt{\frac{f(D(\varphi(t), t))}{f_0}},$$

(15)

with constants $R_0$, $f_0$, and also the analogous path-dependent radius for $R_{\psi}$ with the same constants $R_0$, $f_0$. From Eq. 14 it is then evident that the two leading order free-diffusion terms become path-independent, and hence their difference cancels. The limit of vanishing tube radius $R_0 \to 0$ is then finite,

$$\lim_{R_0 \to 0} \ln \frac{P_R^\varphi(t_f)}{P_R^\psi(t_f)} = - \int_0^{t_f} ds \left[ \alpha_{\varphi}(0)(s) - \alpha_{\varphi}(s) \right],$$

(16)

which can be rewritten in a form more reminiscent of Eq 11 as

$$e^{-S_{\text{OM}}[\varphi]} \equiv \lim_{R_0 \to 0} \frac{P_R^\varphi(t_f)}{P_R^\psi(t_f)},$$

(17)

with the action

$$S_{\text{r}}[\varphi] = \int_0^{t_f} ds \; \alpha_{\varphi}(0)(s),$$

(18)

where we choose the subscript r because this action is based on the local rescaling Eq. 15. If the diffusion tensor is constant and isotropic, then according to Eq. 15 the radius is constant and $S_{\text{r}} \equiv S_{\text{OM}}$. Despite the formal similarity between Eqs. 11 and 17, there is an important difference between the two limits. Equation 11 considers the exit rate from a constant-radius tube with respect to the ambient Euclidean metric, which is a natural choice when observing experimental data. On the other hand, in Eq. 17 we consider the path-dependent tube radius Eq. 15, which is designed to scale away the leading-order differences in the small-radius exit rate. To measure the exit rate from a tube with radius Eq. 15 in an experiment, both the diffusion tensor $D$ along the path $\varphi$ and the explicit functional form of $f$ need to be known in order to evaluate $R_{\varphi}$. The ratios of path probabilities Eq. 17 are thus not straightforwardly related to what one would measure in an experiment. Furthermore, the choice Eq. 15 is not the only construction that leads to a finite limiting-ratio of tube probabilities. Another possibility is to define a tube via the metric induced by the diffusion tensor $D$ \cite{22} which corresponds to considering a moving ellipsoid in $\mathbb{R}^N$, whose principal axes vary along the reference path in such a way that the steady-state free-diffusion exit rate is independent of the chosen path. Only for one-dimensional systems, $N = 1$, where ellipsoids and balls are identical and simply given by intervals, do these two constructions lead to the same tube. Yet another construction for obtaining a finite limiting ratio is due to Stratonovich [6], and leads to the standard Lagrangian for multiplicative noise. As we show explicitly in Sect. III for a one-dimensional system, the underlying geometrical idea of the Stratonovich construction is to perform a nonlinear coordinate transformation such that the diffusivity becomes constant, and then to consider a constant-radius tube in this coordinate system. While all three methods (Eq. 15, using the metric induced by the diffusion tensor, the Stratonovich construction) lead to finite limiting ratios of tube probabilities, they in general correspond to different tubes in spacetime; therefore they lead to different finite-radius sojourn probabilities, and consequently different actions. All three methods
also require knowledge of the diffusivity profile, which might not be readily available when working with measured data.

The technical difficulties and ambiguities in extending the vanishing-radius limit of tube probabilities Eq. (11) to systems with multiplicative noise, together with the fact that any individual path has vanishing probability, suggests that instead of considering the limit of vanishing tube radius, focus should be put on the finite-radius sojourn probability Eq. (5) which describes observable events of positive probability. We note that while in general the log-ratio Eq. (14) diverges in the limit of vanishing tube radius, an important exception is the case where \( \psi \), \( R_\psi \) are the time reverse of \( \varphi \), \( R_\varphi \equiv R \), i.e. where \( \psi(t) \equiv \varphi(t) \equiv \varphi(t_f - t) \), \( R_\varphi(t) \equiv R(t_f - t) \), and where we assume that for the reverse path also all explicit time-dependences in \( a, D \) are reversed, as is customary when considering irreversibility in stochastic thermodynamics [23–25]. The limiting ratio of sojourn probabilities for a pair of final positions \( x, x_f \) is obtained by minimizing the action Eq. (6),

\[
\varphi^* \equiv \arg \min_{\varphi} S[\varphi, R_\varphi],
\]

where the minimization is over all twice differentiable paths which fulfill \( \varphi(0) = x_0, \varphi(t_f) = x_f \), and where the time-dependent tube radius \( R_\varphi \) may depend on the path (of course the details of this path-dependence need to be specified before the functional Eq. (20) can be minimized). After \( \varphi^* \) has been obtained, the finite sojourn probability to observe any trajectory that remains within the tube is calculated via Eq. (5). A most probable path (MPP) may be defined from Eq. (20) as most MPT center in the limit of vanishing radius; as we see explicitly in Sect. III D below, the result of course depends on the exact form of \( R_\varphi(t) \).

For additive noise with a constant isotropic diffusion tensor \( D \equiv D_0 \frac{1}{2} \equiv \text{const.} \), and a constant radius, \( R(t) \equiv R_0 = \text{const.} \), the limit \( R_0 \to 0 \) in Eq. (20) is equivalent to finding a path \( \varphi^* \) that minimizes the OM action Eq. (12). For the general Langevin Eq. (1) with multiplicative noise, we see from Eqs. (7), (8), that in the limit of vanishing time-independent tube radius, the MPP in general minimizes the average free-diffusion exit rate along the path. Thus, because diffusive dynamics is on short length- and time scales dominated by the random noise term in Eq. (1), for an Itô-Langevin dynamics with multiplicative noise the drift term is completely irrelevant for the MPP [20]. For a one-dimensional system this is discussed further below and in the accompanying Ref. [26].

### III. ONE-DIMENSIONAL SYSTEMS

#### A. Exit rate

We now consider a one-dimensional system, \( N = 1 \), for which the Itô Eq. (1) becomes

\[
dX_t = a(X_t, t) \, dt + \sqrt{2D(X_t, t)} \, dW_t, \tag{21}
\]

where we use that for one-dimensional systems the noise strength is expressed in terms of the diffusivity as \( b(x, t) = \sqrt{2D(x, t)} \). Similarly, the FP Eq. (2) becomes

\[
\partial_t P = -\partial_x (aP) + \partial_x^2 (DP). \tag{22}
\]

For one-dimensional systems, we calculate the FP spectrum explicitly in terms of the drift and diffusivity, as discussed in more detail in App. C. We use the resulting perturbative spectrum to evaluate the exit rate Eq. (7) to order \( R_0^2 \). The two lowest-order terms in this expansion are

\[
\alpha_{\text{free}}^\varphi(t) \approx \frac{\pi^2}{2} \frac{D(\varphi(t), t)}{R(t)^2}, \tag{23}
\]

and

\[
\mathcal{L}^{\varphi, (0)} = \alpha^{\varphi, (0)} = \frac{1}{4D} \left( \dot{\varphi} - a + \partial_x D \right)^2 + \frac{1}{2} \partial_x a - \frac{1}{16} \left( \frac{\pi^2 - 1}{D} \right) \frac{(\partial_x D)^2}{D} + \frac{1}{4} \left( \frac{\pi^2}{6} - 1 \right) \partial_x^2 D - \frac{1}{2} \frac{\dot{r}}{r}, \tag{24}
\]
where $D$, $a$, and their derivatives are evaluated at $(x,t) \equiv (\varphi(t),t)$, the radius is in general time-dependent, $R(t) \equiv R_0 r(t)$, where $r(t)$ may depend on $\varphi$. The explicit expression for the quadratic term $\alpha \varphi^2(t)$ in Eq. (7) is lengthy, and we provide it in App. C. We furthermore provide a python package which includes all analytical results derived in this paper as symbolic expressions 32.

Our result Eq. (24) is different from the Lagrangians derived for multiplicative noise in the literature [6, 7, 9, 10, 13, 16, 19]. The relevance of our result $\mathcal{L}^{\varphi(t)}$ is that it appears as a term in the perturbative expansion of the exit rate Eq. (7), and hence is a physical observable.

B. Limiting probability ratio for forward-backward path pair

As we discussed in Sect. III C for two paths $\varphi, \psi$ the log-ratio of tube probabilities Eq. (14) in general diverges in the limit of vanishing tube radius. However, if for $\varphi$ we consider the time reverse of $\varphi$, i.e. $\psi(t) \equiv \varphi(t)$, and also reverse all explicit time-dependences in $D$, $a$, $R$, then from Eqs. (19), (24), we obtain

$$\lim_{R_0 \to 0} \ln \frac{P_{R_0}^\varphi(t_f)}{P_{R_0}^\varphi(t_f)} = \int_0^{t_f} dt \frac{a - \partial_x D}{D} \varphi(t) + \ln \frac{r(t_f)}{r(0)}.$$  

To cast the right-hand side of Eq. (25) in a more familiar form, we note that the instantaneous steady-state solution of Eq. (24) subject to instantaneous no-flux boundary conditions at the system boundary (i.e. the full spatial domain of the system without assuming a tube) is given by 27

$$P_{ss}(x,t) = \frac{N(x_0)}{D(x,t)} \exp \left[ \int_{x_0}^x dx' \frac{a(x',t)}{D(x',t)} \right]$$  

where $x_0$ is an arbitrary point in the domain and $N(x_0)$ is a normalization constant which ensures that $P_{ss}$ is a proper probability density. We define the instantaneous potential $U(x,t)$ via Boltzmann inversion of the steady-state solution,

$$U(x,t) = -T \ln [P_{ss}(x,t)L],$$  

where $T$ is the absolute temperature in units of energy and $L$ is an arbitrary length scale to render the argument of the logarithm function dimensionless.

With these definitions, we rewrite the limit Eq. (25) as

$$\lim_{R_0 \to 0} \ln \frac{P_{R_0}^\varphi(t_f)}{P_{R_0}^\varphi(t_f)} = - \frac{\Delta U}{T} + \frac{1}{T} \int_0^{t_f} dt \left[ (\partial_t U)(\varphi(t),t) \right] + \ln \frac{r(t_f)}{r(0)},$$  

where $\Delta U = U(\varphi(t_f),t_f) - U(\varphi(0),0)$.

If the drift and diffusivity are independent of time, then $\partial_t U = 0$ so that the second term on the right-hand side of Eq. (28) vanishes. If furthermore the tube radius at the initial and final time are identical, $r(t_f) = r(0)$, which includes the scenario of a constant tube radius $R(t) \equiv R_0 = const.$, then also the third term on the right-hand side of Eq. (28) is zero. In that case, the remaining potential difference on the right-hand side of Eq. (28) represents the familiar formula for the entropy production along the path $\varphi$, which for multiplicative noise has previously been derived via time-discretization path-integral methods 31.

C. Numerical example: Particle in an asymmetric double well

For our numerical example we fix a length scale $L$ and a diffusivity scale $D_0$, which defines the diffusive time
Figure 3. Tube radius and exit rates for the one-dimensional example system considered in Sect. III. Subplot (a) shows the time-dependent tube radius for constant tube radius (scenario 1; blue horizontal solid line), constant free-diffusion exit rate (scenario 2; orange solid line), and the distance from the reference path to the two interval bounds for the Stratonovich construction (scenario 3; green dashed and dotted lines). For all scenarios, we use $R_0/L = 0.1$. Subplot (b) shows the perturbative exit rate Eq. (7) to order $R^0$ (inclusive) for constant tube radius (scenario 1; blue solid line), constant free-diffusion exit rate (scenario 2; orange solid line), and the Stratonovich construction (scenario 3; green dotted line). Subplot (c) shows the first correction to the steady-state free-diffusion exit rate, $L_1^{φ,(0)} = αφ,(0)$, for the three scenarios considered in subplot (b); these corrections are given by Eq. (33) (scenario 1, blue solid line), Eq. (35) (scenario 2, orange dashed line), and Eq. (36) (scenario 3, green dotted line).

scale $τ_D = L^2/D_0$. We consider a diffusivity profile

$$D(x) = \frac{D_0}{A} \left[ 5 - \cos \left( \frac{\pi x}{L} \right) \right], \quad (29)$$

which features a locally maximal diffusivity $D(x = \pm L) = 3D_0/4$ at $x = \pm L$ and a locally minimal diffusivity $D(x = 0) = D_0$ at $x = 0$. This diffusivity profile is shown in Fig. 2 (a).

We furthermore consider a gradient drift profile $a(x) = -∂xU_a(x)$, with corresponding potential

$$U_a(x) = a_0 \left[ \frac{x}{L} \right]^2 - a_1 \frac{x}{L}, \quad (30)$$

where for our numerical results we use $a_0 = 2L^2/T$ and $a_1 = a_0/20$. Since $a_1 \ll a_0$, Eq. (30) describes a slightly asymmetric double well. We include this asymmetry to avoid any potential issues of degeneracy in determining barrier-crossing MPT centers, as discussed further below.

Note that because the noise is multiplicative, the drift potential Eq. (30) is not proportional to the steady-state potential $U(x)$ defined in Eq. (27). This is most clearly seen from Eq. (26), which relates the drift $a(x) = -∂xU_a$ and the steady state $P_{ss}(x)$. We show the steady-state potential Eq. (27) in Fig. 2 (b), where we observe that the two local minima of $U(x)$ are located at $x_\pm \approx -L$ and $x_+ \approx L$. As Fig. 2 (a), (b) shows, the system we consider here is qualitatively similar to the corrugated channel from the accompanying Ref. 26.

As reference path $φ$ we consider

$$φ(t) = \frac{x_+ + x_-}{2} + \frac{x_+ - x_-}{2} \arctan(\kappa/2) \frac{t - t_I/2}{τ_D}, \quad (31)$$

which is a barrier crossing path that starts at time $t_i = 0$ at $φ(t_i) = x_-$ and ends at time $t_f = τ_D$ at $φ(t_f) = x_+$. For the parameter $κ$, which sets the maximal speed at which $φ$ crosses the barrier, we use $κ = 10$. We show the reference path Eq. (31) in Fig. 2 (c).

We now discuss the tubular exit rate for two particular choices of the time-dependent radius, namely the scenarios of constant radius $R(t) = R_0$, and the scenario of constant free-diffusion exit rate Eq. (15). Subsequently we relate the one-dimensional Stratonovich stochastic action Lagrangian [6] to tubular exit rates.

In this section we only consider the exit rate to order $R^0$ (inclusive). In App. C we compare the theoretical exit rates shown in this section to numerically evaluated exit rates, to demonstrate that for the present system and tube radius considered, the perturbative exit rate Eq. (7) to order $R^0$ approximates the actual exit rate well.

Scenario 1: Constant tube radius. For a tube with constant radius $R(t) \equiv R_0 = \text{const.}$, illustrated as horizontal blue solid line in Fig. 3 (a), we obtain from Eqs. (23), (24) that

$$\frac{α_{φ,(0)}^\text{free}}{R^2} = \frac{π^2}{4} \frac{D}{R_0^2}, \quad (32)$$

$$\mathcal{L}_1^{φ,(0)} = \frac{1}{4D} \left( \dot{φ} - a + 𝜕_xD \right)^2 + \frac{1}{2} 𝜕_x a - \frac{1}{16} \left( 𝜕_x^2 \right)^2 + \frac{1}{4} \left( \frac{π^2}{6} - 1 \right) 𝜕_x^2 D, \quad (33)$$

where as before both $a$, $D$ and their spatial derivatives are evaluated along $φ(t)$; the subscript 1 in Eq. (33) indicates that this is the Lagrangian for the first scenario we consider. In Fig. 3 (b) we show the exit rate Eq. (7) to order $R^0$, evaluated along the path Eq. (31) for a tube radius $R_0/L = 0.1$ and using Eqs. (32), (33). We
observe that the exit rate takes on a minimum at time $t \approx 0.5 \tau_D$, i.e. when the path is close to the barrier top $x \approx 0$. At this point also the diffusivity profile shown in Fig. 2 (b) displays a minimum; this suggests that the radius $R_0/L = 0.1$ is so small that the exit rate Eq. (7) is already dominated by the free-diffusion contribution Eq. (32), which is proportional to $D$. Indeed, when we compare the magnitude of the total exit rate to order $R^0$, shown in Fig. 2 (b), to the typical magnitude of the order-$R^0$ term Eq. (33) in Fig. 3 (c), we conclude that $L^{\phi,(0)}_t$ only contributes less than 10% of the value of the total exit rate in subplot (b).

Scenario 2: Constant free-diffusion exit rate. From Eqs. (14), (22), we see that for constant tube radius the log-ratio of tube probabilities in general diverges as $R_0 \to 0$. As discussed in Sect. 1C, we can obtain a finite limit by choosing a path-dependent tube radius $R(t) = R(\phi(t)) = R_0 \sqrt{\frac{D(\phi(t))}{D_0}}$, for which one-dimensional systems is equivalent to defining the tube with respect to the metric induced by the diffusivity tensor corresponding to the FP Eq. (2) [22]. In this scenario, we obtain from Eqs. (7), (24), that

$$ \frac{d^2}{dx} \alpha^{\phi}_t = \frac{\pi^2}{4} D_0 = \text{const.} \quad (34)$$

$$ L^{\phi,(0)}_2 = \frac{1}{4 D} \left( \phi - a + \frac{1}{2} \partial_x D \right)^2 + \frac{1}{2} \partial_x a - \frac{1}{4} \partial_x D a$$

where the subscript 2 indicates that this is the second scenario we consider. By construction, the free-diffusion exit rate Eq. (34) is now independent of the path and constant as a function of time. In Fig. 3 (a), we show the time-dependent tube radius $R(t)$ for $R_0/L = 0.1$ and the example system Eqs. (29), (30), (31). Figure 3 (b) clearly shows that the total exit rate varies on a much smaller scale as compared to the constant-radius exit rate from scenario 1, which is because the dominant free-diffusion contribution Eq. (34) is now time-independent by design. In Fig. 3 (c) we compare the Lagrangians Eqs. (33), (35) for scenarios 1 and 2. While overall the Lagrangians are rather similar, they deviate from each other for $t/t_D \approx 0.5$, when the path is close to the barrier top. Because the two Lagrangians Eqs. (33), (35) have different functional forms, there is of course no a-priori reason to assume that they should lead to identical curves.

Scenario 3: Stratonovich Lagrangian. Both Lagrangians Eqs. (33), (35) are different from the Lagrangian for multiplicative noise originally derived by Stratonovich [6], which in our notation reads

$$ L^{\phi}_S = \frac{1}{4 D} \left( \phi - a + \frac{1}{2} \partial_x D \right)^2 + \frac{1}{2} \partial_x a - \frac{1}{4} \partial_x D a$$

where the subscript S stands for Stratonovich. We now demonstrate that the Lagrangian Eq. (36) corresponds to an exit rate from a one-dimensional moving ball (i.e. a time-dependent interval) which is not centered at the path $\phi$, and which has a time-dependent radius that is only to leading order identical to the radius from scenario 2. To derive Eq. (36), from the 1D FP Eq. (22), we introduce a new coordinate system $y \equiv \Phi(x)$ by

$$ \frac{d\Phi}{dx} = \frac{1}{\sqrt{D(x)/D_0}}. \quad (37)$$

Transforming Eq. (22) to the $y$-coordinate leads to $\Phi(t) = \alpha(t) P_y - \partial_y \phi P_y + D_0^{\beta_2} P_y$, where $P_y(y,t) = \sqrt{D(y)/D_0} P(x,t)$, $a_y(y) = \sqrt{D(y)/D_0} a(x) - \partial_x D(y)/2$, with $x = \Phi^{-1}(y)$. The coordinate transformation Eq. (37) locally compresses space where the diffusivity is small, and locally stretches space where the diffusivity is large, resulting in a constant diffusivity $D_0$ with respect to the $y$-coordinate, as was remarked by Ito [10]. (As has been emphasized before [31], a coordinate transformation that flattens the diffusivity profile is only guaranteed to exist for one-dimensional multiplicative-noise systems.) The Stratonovich Lagrangian now follows by considering a tube with constant radius $R_0$ in the $y$-coordinate around the path $\phi(t) = \Phi(t)$). Since the diffusivity is constant in this coordinate system, the theory for stochastic dynamics with additive noise is applicable, for which the first correction to freely-diffusive exit from the tube is given by the OM Lagrangian Eq. (13) [22] evaluated using diffusivity $D_0$, drift $a_y$, and the path $\Phi$. Expressing the resulting exit rate back in the original $x$-coordinates yields

$$ \alpha^{\phi}_t = \frac{\pi^2}{4} D_0 + L^{\phi}_S + O(R^2_0) \quad (38)$$

with the Stratonovich Lagrangian Eq. (36). In summary, $L^{\phi}_S$ is obtained by performing a nonlinear coordinate transformation $\Phi$ such that the diffusivity is constant in the new coordinates, then considering a constant-radius tube in the new coordinates, and finally expressing the resulting exit rate in terms of the original coordinates. Importantly, a tube centered around $\phi(t) = \Phi(t)$ in the $y$-coordinate in general does not correspond to a tube centered around $\phi(t)$ in the $x$-coordinate. More explicitly, at time $t$ a tube of radius $R_0$ in the $y$-coordinate is in the $x$-coordinate bounded by the two points $\phi(t) = \Phi^{-1}(\phi(t)) \pm R_0$. The center $\phi_S(t) \equiv (\phi_+(t) + \phi_-(t))/2$ and radius $R_S(t) \equiv (\phi_+(t) - \phi_-(t))/2$ of this one-dimensional ball are given by

$$ \phi_S(t) = \phi(t) + \frac{1}{4} \partial_x D(x=\phi(t) R_0 + O(R^2_0),$$

$$ R_S(t) = \left[ \frac{D}{D_0} R_0 + \frac{1}{12} \partial_x^2 D_0 \right] R_0 \left[ \frac{D}{D_0} R_0 \right] x=\phi(t) + O(R^2_0). \quad (40)$$
Equations (39), (40), are to leading order identical to tube center and radius from scenario 2, but contain additional higher-order terms.

We return to the example system Eqs. (29), (30), (31), and consider a tube of constant radius \( R_0/L = 0.1 \) in the \( y \)-coordinate. In Fig. 3 (a) we show \( R_\pm(t) \equiv \pm(\varphi_\pm(t) - \varphi(t)) \), which is the distance from either of the two tube boundaries to the path \( \varphi \) in the \( x \)-coordinate. The two curves \( R_\pm \) clearly disagree with each other, showing that the tube is in the \( x \)-coordinate not centered at \( \varphi \). Both \( R_\pm \) behave similar to the tube radius from scenario 2, which according to Eq. (40) is their leading order behavior. Indeed, also in Fig. 3 (b), where we show the total exit rate Eq. (38) to order \( R^0 \), the exit rates from scenarios 2 and 3 look very similar. However, subtracting from the exit rate the free-diffusion contributions, which according to Eqs. (34), (38), are equal, we observe in Fig. 3 (c) that the order-\( R^0 \) Lagrangians from scenario 2 and 3, Eqs. (35), (36), clearly deviate from each other, most prominently at the beginning and the end of the path, i.e. for \( t/\tau_D \lesssim 0.3 \) and \( t/\tau_D \gtrsim 0.7 \).

We now consider ratios of tube probabilities for two paths \( \varphi, \psi \) for scenario 3. By construction, the free-diffusion exit rates are equal for any two paths, so that we obtain

\[
\lim_{R_0 \to 0} \frac{P_R^{\varphi}(t_f)}{P_R^{\psi}(t_f)} = -\int_0^{t_f} ds \left[ \mathcal{L}_S^\varphi(s) - \mathcal{L}_S^\psi(s) \right]. \tag{41}
\]

We emphasize that while in both scenarios 2 and 3 the ratio of tube probabilities is well-defined in the limit \( R_0 \to 0 \), the resulting stochastic Lagrangians differ. This highlights that limiting ratios of tube probabilities, and also the Lagrangians associated with them, depend on the detailed nature of the tube. However, if we consider for \( \psi \) the reverse of \( \varphi \), i.e. \( \psi = \varphi_\ast \), then from Eq. (41) we recover Eq. (25) without the boundary term, i.e. we obtain the usual multiplicative-noise pathwise entropy production.

### D. Most probable tube

We now consider the most probable tube for the system Eqs. (29), (30) and the three scenarios discussed in the previous section. For each scenario, we evaluate the most probable tube for a barrier-crossing transition from \( \varphi(0) = x_- \approx -L \) to \( \varphi(t_f) = x_+ \approx L \) in one unit of the diffusive time scale, \( t_f = \tau_D \). We minimize the action functional Eq. (20) to order \( R^0 \) (inclusive) for \( R_0/L = 0.1 \), and in Fig. 4 we compare the resulting MPT centers \( \varphi^\ast \).

For constant tube radius \( R(t) = R_0/L = 0.1 \), the most probable reference path \( \varphi^\ast \) (blue solid line) remains on the barrier top \( x = 0 \) for most of the transition time. This is because the action is, for the small radius considered here, dominated by the free-diffusion exit rate Eq. (32), which is proportional to the diffusivity. Because the diffusivity profile Eq. (29) features a local minimum at \( x/L = 0 \), the free-diffusion exit rate is minimal there. Notably, it follows that for tubes of small constant radius the most probable tube is only weakly influenced by the Lagrangian Eq. (43), and in particular is dominated by the diffusivity profile \( D(x) \) as compared to the drift \( a(x) \). This very effect is also observed in the experimental data analyzed in Ref. [26].

For scenario 2, where \( R(t) = R_0 \sqrt{D(\varphi(t))/D_0} \), the free-diffusion exit rate is constant, so that the extremum of the action is determined by the subleading-order contribution Eq. (35). The resulting MPT shown in Fig. 4 crosses the barrier rather quickly without stopping at the barrier top, and then remains within the target potential well most of the time; this is in sharp contrast to the constant-radius result. Indeed, the most probable tube observed here is qualitatively similar to the most probable barrier-crossing paths for double well systems with constant diffusivity [17] [21], i.e. to the MPPs which follows from the OM Lagrangian Eq. (13).

For the Stratonovich scenario the most probable tube is obtained by minimizing the integrated exit rate Eq. (38) as a functional of \( \varphi \). We show the resulting most probable reference path \( \varphi^\ast \) in Fig. 4 as green dashed line. We observe that this MPT center is almost identical to the result for constant free-diffusion exit rate. This is consistent with the facts that in the Stratonovich scenario the free-diffusion exit rate is also independent of the path, c.f. Eq. (38), and that the \( R^0 \)-order terms for both scenarios 2 and 3 yield similar results also in the previous subsection, see Fig. 3 (c).

To close this section, we consider the radius dependence of the most probable tube Eq. (20); this discussion parallels the corresponding discussion in the accompanying Ref. [26]. We consider a tube of constant radius \( R = R_0 = \text{const.} \) for the example system Eqs. (29), (30). As in Fig. 4, we consider paths that move from \( x = x_- \approx -L \) to \( x = x_+ \approx L \) during a time \( t_f = \tau_D \). We minimize Eq. (20), evaluated using Eq. (7) to order \( R^2 \) (inclusive), for each of the constant-radius tubes \( R_0/L = 0.1, 0.2, 0.3 \). We show the resulting most probable reference paths in Fig. 7 (a). For \( R_0/L = 0.1 \) we obtain the same path \( \varphi^\ast \) as shown in Fig. 4 which shows that the quadratic term in the exit rate is irrelevant for this small tube radius; see App. C for more details. As discussed in the context of Fig. 4 for \( R_0/L = 0.1 \) the MPT center remains on the barrier top for most the transition time because the exit rate is for small radius dominated by the free-diffusion contribution Eq. (32). While for \( R_0/L = 0.2 \) the most probable reference path \( \varphi^\ast \) also rests at the barrier top for most of the transition time, it stays there for a shorter duration as compared to the \( R_0/L = 0.1 \) result; this indicates that for \( R_0/L = 0.2 \) the free-diffusion exit rate is already less dominant. For \( R_0/L = 0.3 \), the path \( \varphi^\ast \) is completely different from its smaller-radius counterparts. Now, the most probable reference path immediately crosses over the potential barrier without stopping, and rests close to the potential
We minimize Eq. (20) using the same algorithm as used for functional minimization in Ref. [21]. More explicitly, we approximate the path in Eq. (20) by a finite number of 40 modes, and solve the resulting finite-dimensional minimization problem for the mode coefficients using a standard algorithm [35]; see Ref. [21] for more details. For the blue and orange solid lines, the minimization is carried out for the temporal integral over the exit rate Eq. (7) to order $R^0$, once with a constant tube radius $R_0/L = 0.1$ (blue solid line) and once with a path-dependent tube radius $R(t) \equiv R_0 \sqrt{D(\varphi(t))}/D_0$ where $R_0/L = 0.1$ (orange solid line); both tubes are indicated as gray shaded area. For the blue dashed line, the temporal integral over the exit rate Eq. (38) is minimized using $R_0/L = 0.1$. Initial and final position of the paths are shown as horizontal dashed lines.

Figure 4. Most probable tubes for the three scenarios considered in Sect. [1] with force and diffusivity given by Eqs. (24), (30). We minimize Eq. (20) using the same algorithm as used for functional minimization in Ref. [21]. More explicitly, we approximate the path in Eq. (20) by a finite number of 40 modes, and solve the resulting finite-dimensional minimization problem for the mode coefficients using a standard algorithm [35]; see Ref. [21] for more details. For the blue and orange solid lines, the minimization is carried out for the temporal integral over the exit rate Eq. (7) to order $R^0$, once with a constant tube radius $R_0/L = 0.1$ (blue solid line) and once with a path-dependent tube radius $R(t) \equiv R_0 \sqrt{D(\varphi(t))}/D_0$ where $R_0/L = 0.1$ (orange solid line); both tubes are indicated as gray shaded area. For the blue dashed line, the temporal integral over the exit rate Eq. (38) is minimized using $R_0/L = 0.1$. Initial and final position of the paths are shown as horizontal dashed lines.

The reason for the crossover that we observe in Fig. 5 (a) is the radius-dependent competition between the terms in the perturbation series Eq. (7). To investigate this competition further we now consider a constant path at a point $x$ with constant radius $R \equiv R_0 \equiv \text{const.}$, and ask for which choice of $x$ the exit rate is minimal, i.e. we consider the function

$$x_{\text{opt}}(R) = \arg\min_{x} \alpha_{R}^{\varphi_x},$$

(42)

where $\varphi_x(t) \equiv x$ is the path resting at $x$. For any value of $R$ the point $x_{\text{opt}}(R)$ is the most stable point in the system, in the sense that the steady-state exit rate from the interval $[x_{\text{opt}} - R, x_{\text{opt}} + R]$ is the minimal exit rate achievable for any interval of width $2R$ in the system. In Fig. 5 (b) we show Eq. (42) as a function of $R$. We observe that for small radius, the optimal resting path is located at $x \approx 0$. This is because for small radius the exit rate Eq. (7) is dominated by the free-diffusion term Eq. (32), which is minimal at $x = 0$. As the radius is increased, the terms at order $R^2$, $R^3$ in Eq. (7) become more important. At the crossover radius $R_c \approx 0.28 L$ the optimal resting point $x_{\text{opt}}$ is discontinuous as a function of $R$, and jumps from $x \approx 0$ to the vicinity of the local minimum $x_+$. Thus, for $R > R_c$ the confinement effects of the potential landscape around $x_+$ outweigh the benefit of the small diffusivity at $x = 0$. This explains the behavior of the MPT centers from Fig. 5 (a), which ac-
concerning to Fig. 5(b) for every radius considered rest at the respective most stable ball in the system. We note that the optimal resting position is never close to the minimum \( x_{-} \); the minimum \( x_{+} \) is preferred because of the linear term in the potential Eq. \( \phi \). In fact, we have included the symmetry-breaking term precisely so that one of the two minima is preferred over the other.

This example shows that the most probable pathway for a transition can depend significantly on the tube radius, i.e. on how much deviation from the reference path one tolerates.

IV. CONCLUSIONS

In this work we present a general theory for the sojourn probability, which is the probability for a diffusive trajectory to remain within a tube of small but finite time-dependent radius \( R(t) \) around a continuous reference path \( \varphi(t) \). We focus on the case of \( N \)-dimensional Langevin dynamics with multiplicative noise and full-rank diffusivity matrix, a tube defined using the standard Euclidean norm, and twice continuously differentiable reference paths. For this scenario we derive an expansion in powers of the tube radius for the instantaneous exit rate at which stochastic trajectories first leave a small-but-finite radius tube. Based on this exit rate, we discuss the vanishing-radius limit for ratios of sojourn probabilities for pairs of reference paths. We show that while in general such limiting ratios are either zero or divergent, for a pair of forward and reverse path they are finite. For the special case of a one-dimensional system, \( N = 1 \), we derive explicit expressions for the exit rate in terms of the drift and diffusivity, consider several choices for the time-dependent tube radius, and illustrate our results with an explicit numerical example.

The Lagrangian Eq. \( \mathcal{L} \) we derive for one-dimensional Langevin dynamics is different from Lagrangians found in the literature \[6, 7, 9, 10, 11, 16, 19\], and has the advantage of being directly related to an observable exit rate. For our one-dimensional example system, we furthermore illustrate how the most probable tube depends on both the choice of time-dependence of the tube radius, as well as the size of the tube. Our results have several important consequences, from both a mathematical and physical point of view.

The exit rate we derive is for small radius dominated by a free-diffusion contribution, which for additive noise is independent of the reference path. In this case limiting ratios of sojourn probabilities for constant-radius tubes probe subleading-order terms of the exit rate, and can be used to define the stochastic action \[6, 7, 9, 10, 11, 16, 20\]. For state-dependent noise, the local free-diffusion exit rate is also state-dependent, and the ratio of sojourn probabilities for two constant-radius tubes is in general either zero or divergent in the limit of vanishing tube radius; this means that one path is typically infinitely more likely than the other. Our theory thus provides an intuitive physical picture as to why classical definitions of stochastic actions for additive-noise systems cannot be simply generalized to systems with multiplicative noise \[9\].

Our work elucidates the geometry behind mathematical attempts to obtain a finite limiting-ratio for pairs of sojourn probabilities \[6, 7, 9\]. These works do not consider the sojourn probability to remain within a moving ball (defined with respect to the standard Euclidean norm) centered at a reference path, but instead the sojourn probability to remain within more complicated geometrical shapes, which are not necessarily centered at the reference path. Since these definitions of the tubular neighborhood use the diffusion tensor of the underlying stochastic dynamics, this diffusion tensor needs to be inferred before such finite-radius sojourn probabilities can be measured. More so, from an experimental point of view it is more natural to simply consider a constant-radius tube with respect to the metric of the ambient Euclidean space. The corresponding sojourn probability is for one-dimensional systems quantified by our explicit results Eqs. \( \text{Eqs. (32), (33)} \), which we compare to experimental measurements in the accompanying Ref. \[26\].

While limiting ratios of sojourn probabilities for arbitrary pairs of paths do not lead to finite results, a pair consisting of forward and reverse path does. This implies that limits of sojourn probabilities can be used to quantify irreversibility along individual paths, and indeed for \( N = 1 \) our results recover established formulas for the path-wise entropy production \[23, 25\].

Besides discussing the technical difficulties and ambiguities arising from trying to define vanishing-radius limits of sojourn probabilities for systems with multiplicative noise, our work focuses on considering finite-radius tubes. Our philosophy here is similar to that of a simplified version of our theory on additive noise \[20\]. Namely, from a mathematical perspective, instead of trying to introduce a probability \textit{density} on the infinite-dimensional space of all continuous paths, we evaluate the probability \textit{measure} induced on that space by Langevin dynamics. Because of this we do not need to consider limiting procedures in our theory, and indeed it has been shown that for Langevin dynamics with multiplicative noise there is no canonical way of defining a probability density on the space of all continuous paths \[9\].

From a physical point of view, considering the finite-radius tubular ensemble is also reasonable: The probability to observe a given individual trajectory vanishes, so that it is not straightforward to quantify it in an experiment. The probability to observe any stochastic trajectory of the finite-radius tubular ensemble is positive, and hence is directly accessible in experiment, simply by counting how many stochastic trajectories that started within the tube remain until a later time \[20\]. Finite-radius tubes can thus be used to probe path-properties in experiment and simulation, and indeed for additive isotropic noise they have been used to infer both ratios of path probabilities \[21\] and the entropy production.
along individual paths \cite{25}. For multiplicative noise we infer finite-radius tube probabilities in the accompanying Ref. \cite{26}.

Our results demonstrate that the most probable tube depends sensitively on both the protocol for the state- or time-dependence of the tube radius, and on the typical size of the tube. Thus, because in practice there is typically a finite amount of deviation from a reference path one is willing to tolerate, considering the single most probable path in general does not yield physically relevant results. The concept of the most probable tube will be useful for understanding in more depth the properties of transition paths \cite{30, 42}, for example by investigating how a small state- or time-dependent tube radius can be chosen so as to capture as many finite-temperature transition paths as possible.

Another interesting direction for future research is to consider the ratio of sojourn probabilities for forward/reverse path pairs also at finite tube radius, and to relate the resulting expression to the path integral of the single-trajectory entropy production over all stochastic trajectories in the corresponding tubular ensemble. This will yield an experimentally relevant generalization of the pathwise entropy production \cite{23, 25, 30, 43} to tubes.

To date, for multiplicative noise and dimension \(N \geq 2\), no explicit representation in terms of \(a, D\) is available for the exit rate Eq. (7) from a tube with small-but-finite constant radius, defined via the standard Euclidean metric. Since this exit rate is arguably the most straightforward experimental observable for quantifying the probability of a given pathway, an important next step will be calculating explicit expressions for the exit rate, in terms of \(a, D\), also for dimensions \(N \geq 2\). From our results for one-dimensional systems, it is expected that the order-\(R^0\) contribution to the resulting exit rate will be different from the Stratonovich Lagrangian \cite{6}.

Similarly, for dimensions \(N \geq 2\), it will be interesting to derive a theory for stochastic dynamics with a diffusivity tensor that is not full rank; this case is not covered by our approach here. An important example system with a diffusivity tensor that is not full rank is given by underdamped Langevin dynamics; there, the degrees of freedom are the position \(x\) and the velocity \(v\) of a massive particle. In this system the noise only acts on the velocity, so that the short-time dynamics of the tuple \((x, v)\) are not as straightforward as in our present theory.

Beyond underdamped Langevin dynamics, the sojourn probability for a tube around a reaction coordinate is an experimentally relevant quantification of observable path probabilities for any kind of stochastic dynamics. It will therefore be interesting to compare tubular exit rates to path integral actions also for e.g. non-Markovian or active stochastic processes \cite{11, 30}.

It will furthermore be interesting to relate our finite-noise theory to the low-noise theory of Freidlin and Wentzell \cite{22}. They consider tubes for asymptotically small noise strength, where the probability distribution on the space of all paths is concentrated around a single most probable tube center (called instanton in this context). At finite noise strength, on the other hand, the probability distribution on the space of paths has a larger support, so that even the probability for the most probable tube at small but finite radius is typically very small (we show this explicitly in Ref. \cite{26}). One should be able to observe a crossover from our theory to that of Freidlin and Wentzell by studying the dependence of the MPT on the typical diffusivity amplitude. More explicitly, upon decreasing the state-dependent diffusivity while keeping the deterministic drift and the small radius constant, the MPT should cross over from being dominated by the free-diffusion behavior we discuss in the present work, to being described by Freidlin-Wentzell theory.

In summary, our present work on sojourn probabilities for diffusive stochastic dynamics provides a comprehensive and physical picture of the rather technical literature on path probabilities for systems with state-dependent noise, relates the concept of path probabilities for such systems to measurement, and in particular for the first time quantifies the probability for a stochastic trajectory to remain within a constant-radius tube around a twice continuously differentiable reference path.

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**Appendix A: Perturbative tubular exit rate for \(N\)-dimensional Itô processes**

We here derive the expression Eq. (7) for the exit rate from a tube with time-dependent radius \(R(t)\) around a path \(\varphi(t)\) for the Itô-Langevin Eq. (1). The present derivation generalizes the calculation in Ref. \cite{20} which considers additive isotropic noise and a time-independent tube radius; we here only highlight the differences to this previous derivation, and refer the reader to the reference for more details. Throughout this appendix, we assume that \(\varphi\) is twice continuously differentiable, and use the standard Euclidean norm \(\|x\|_2 \equiv \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}\) to define the tubular neighborhood of \(\varphi\). To consider power series expansions in the time-dependent radius, we assume that \(R(t)\) is of the form \(R(t) = R_0 + r(t)\) with a differentiable dimensionless function \(r(t)\); any series expansion in powers of \(R(t)\) actually refers to an expansion in powers of \(R_0\).
1. FP equation in dimensionless streaming coordinates.

To solve Eq. (2) for a given reference path φ, we transform to dimensionless coordinates (x̃, t̃) with respect to which the domain of the FPE is time-independent. We define the dimensionless coordinates as

\[ \tilde{t}(t) \equiv \frac{t}{\tau_D}, \quad \tilde{x}(x, t) \equiv \frac{x - \varphi(t)}{R(t)}, \]

(A1)

where we define a diffusive time scale \( \tau_D = L^2/D_0 \), with \( D_0 \) a typical diffusive scale of the system and \( L \) a typical length scale. The definition Eq. (A1) generalizes the coordinates used in Ref. [20], where only a time-independent tube radius was considered. With respect to the coordinates (x̃, t̃), the domain for the FPE is the unit ball, \( \varepsilon \in \hat{B} \equiv \{ ||x||_2 \leq 1 \} \). At time \( t \), the absorbing boundary conditions are then given by \( \tilde{P}^\varphi_R(\tilde{x}, t) = 0 \) for \( ||x||_2 = 1 \), where the dimensionless density is defined as \( \tilde{P}^\varphi_R(\tilde{x}, t) \equiv R(t)^n P^\varphi_R(x, t) \). Casting the FP Eq. (2) inside the tube in dimensionless form yields

\[ \tilde{e}^2 \partial_t \tilde{P}^\varphi_R = \tilde{R}_{\text{app}} \tilde{P}^\varphi_R, \]

(A2)

where we define the dimensionless apparent FP operator

\[ \tilde{R}_{\text{app}} \tilde{P}^\varphi_R = -\tilde{e} \tilde{\nabla}_i \left( \tilde{a}_{\text{app}} \tilde{P}^\varphi_R \right) + \tilde{\nabla}_i \tilde{\nabla}_j \left( \tilde{D}_{ij} \tilde{P}^\varphi_R \right) \]

(A3)

with \( \tilde{a}(\tilde{x}, \tilde{t}) \equiv \tau_D a(x, t) / L \), \( \tilde{D}(\tilde{x}, \tilde{t}) \equiv D(x, t) / D_0 \), \( \tilde{\varphi}(\tilde{t}) \equiv \varphi(t) / L \), \( \tilde{\nabla}_j \equiv \partial / \partial \tilde{x}_j = R \partial / \partial x_j \), \( \tilde{\varepsilon}(\tilde{t}) \equiv \varepsilon(t) / L \), \( \tilde{a}_{\text{app}} = \tilde{a} - \tilde{\varphi} - \tilde{\varepsilon} \tilde{x} \). Throughout this work, a dot on a quantity in physical units indicates a derivative with respect to dimensionless time \( \tilde{t} \), and a dot on a quantity in physical units indicates a derivative with respect to \( t \); with this convention we have for example \( \dot{\varphi} = \varphi \tau_D / L \). For a system with constant isotropic diffusivity, \( D \equiv D_0 = \text{const.} \) and a time-independent tube radius \( R(t) \equiv R_0 = \text{const.} \), the dimensionless apparent FP operator Eq. (A3) reduces to its counterpart in Ref. [20].

2. Perturbative FP propagator.

An approximate propagator (i.e. transition probability density from an initial to a final point) for the dimensionless FP Eq. (A2) can be derived for small tube radius by first projecting the FPE onto its instantaneous eigenbasis, and subsequently solving the resulting projected equation using an approach similar to time-dependent perturbation theory in quantum mechanics [20]. The derivation, which is given in detail in Ref. [20], is also applicable in the present scenario. There are, however, two important differences between the previous derivation and the current scenario. First, in the present work we consider a time-dependent dimensionless tube radius \( \dot{\varepsilon} \), whereas in Ref. [20] the tube radius was assumed constant. Second, because the diffusivity matrix components \( \tilde{D}_{ij}(\tilde{x}) \) in Eq. (A3) are state-dependent, the spectrum of the dimensionless FPE in general depends on \( \tilde{t} \) also to lowest order in \( \dot{\varepsilon} \) (with the exception of the one-dimensional case \( N = 1 \), for which we show in App. C that the spectrum is to lowest order still independent of \( \dot{\varepsilon} \)). By contrast, in Ref. [20], the spectrum was to lowest order independent of \( \dot{\varepsilon} \). Thus, while in this previous work mode-coupling effects were only relevant at order \( \dot{\varepsilon}^3 \), in the present work they can become relevant already at order \( \dot{\varepsilon}^2 \) for dimensions \( N \geq 2 \). Taking these two differences into account, the derivation of a perturbative propagator for Eq. (A2) is carried out as in Ref. [20], resulting in

\[ \tilde{P}^\varphi_R(\tilde{x}, \tilde{t} | \tilde{x}_i, \tilde{t}_i) = \exp \left[ -\int_{\tilde{t}_i}^{\tilde{t}} d\tilde{t}' \frac{\tilde{\lambda}_1(\tilde{t}')}{\tilde{e}^2(\tilde{t}')} \right] \]

(A4)

\[ \times \frac{1}{\rho_{ss}(\tilde{x}_i, \tilde{t}_i) \langle \hat{p}_{\text{app}}(\tilde{x}_i, \tilde{t}_i) \rangle} \]

\[ \times \left[ \tilde{\rho}_1(\tilde{x}, \tilde{t}) = -\sum_{n=2}^{\infty} \frac{\tilde{e}^2(\tilde{t}_i)}{\tilde{\Delta}_{m1}(\tilde{t}_i)} \frac{\langle \tilde{p}_m(\tilde{x}_i, \tilde{t}_i) \rangle |_{\tilde{t}_i} \rho_{m}(\tilde{x}, \tilde{t})} {\tilde{\rho}_m(\tilde{x}_i, \tilde{t}_i)} \right] \]

\[ + O(\dot{\varepsilon}^k), \]

where \( k = 4 \) for \( N \geq 2 \) and \( k = 6 \) for \( N = 1 \), where the instantaneous eigenvalues \( -\tilde{\lambda}_n \) and eigenfunctions \( \tilde{\rho}_n \) of the dimensionless apparent FP operator are defined via

\[ \tilde{R}_{\text{app}}(\tilde{x}, \tilde{t}) \tilde{\rho}_n(\tilde{x}, \tilde{t}) = -\tilde{\lambda}_n(\tilde{t}) \tilde{\rho}_n(\tilde{x}, \tilde{t}) \]

(A5)

with absorbing boundary conditions \( \tilde{\rho}_n(\tilde{x}, \tilde{t}) = 0 \) for \( ||x||_2 = 1 \), and \( \tilde{\Delta}_{m1} = \tilde{\lambda}_n + \tilde{e}^2(\tilde{t}_i) \langle \tilde{p}_{m}(\tilde{x}_i, \tilde{t}_i) \rangle / \tilde{\rho}_{m}(\tilde{x}_i, \tilde{t}_i) \). Taking the time-dependent inner product of two functions \( g, h \) is defined as

\[ \langle g, h \rangle \equiv \int_{\hat{B}} d^N \tilde{x} g(\tilde{x}) h(\tilde{x}) / \rho_{ss}(\tilde{x}, \tilde{t}) \]

(A6)

where \( \rho_{ss} \) is the instantaneous reflecting-boundary steady state corresponding to Eq. (A3) inside the unit ball \( \hat{B} \) Ref. [20]. The solution Eq. (A4) is valid after the transient decay of the initial condition, i.e. for

\[ \tilde{t} - \tilde{t}_i \gg \tilde{\tau}_{\text{rel}} \equiv \frac{\tilde{e}^2(\tilde{t}_i)}{\tilde{\Delta}_{21}(\tilde{t}_i)}. \]

(A7)

Note that for the special case of a one-dimensional system, we in App. C give an expression for the perturbative propagator Eq. (A4) that is also valid for shorter times.

The approximate propagator Eq. (A3) neglects both terms that are exponentially smaller than the expression Eq. (A4), as well as terms that are at least of order \( \dot{\varepsilon}^4 \). In contrast to its counterpart in Ref. [20], the propagator Eq. (A4) i) features a time-dependent \( \dot{\varepsilon} \), and ii) is valid only to order \( \dot{\varepsilon}^2 \) for dimensions \( N \geq 2 \).
3. Perturbative exit rate.

In dimensionless form, the exit rate Eq. (10) is given by

$$\dot{\alpha}_R(\tilde{t}) \equiv \tau_D \alpha_R(t) = -\frac{\hat{P}_R(\tilde{t})}{\hat{P}_R(\tilde{t})}, \quad (A8)$$

where $\hat{P}_R(\tilde{t}) = \int_B d^N \tilde{x} \hat{P}_R(\tilde{x}, \tilde{t})$ is the dimensionless survival probability up to time $\tilde{t}$. Similar to Ref. [20], we evaluate Eq. (A8) using the approximate propagator Eq. (A4). The resulting expression, which is valid for $\tilde{t} - \tilde{t}_1 \gtrsim \tilde{t}_{\text{rel}}$, is independent of the initial condition and given as a power series in $\tilde{c}$ by

$$\dot{\alpha}_R = \dot{\alpha}_{\text{free}} + \dot{\alpha}_{\varphi}(0) + \tilde{c}^2 \dot{\alpha}_{\varphi}(2) + O(\tilde{c}^4), \quad (A9)$$

where the individual terms are expressed in terms of the instantaneous spectrum as

$$\dot{\alpha}_{\text{free}} = \frac{\dot{\lambda}_1^{(0)}}{\tilde{c}^2}, \quad (A10)$$
$$\dot{\alpha}_{\varphi}(0) = \dot{\lambda}_1^{(2)} + \frac{(\hat{p}_1, \hat{p}_1)}{(\hat{p}_1, \hat{p}_1)(0)} - \frac{\tilde{z}_1^{(0)}}{\tilde{z}_1^{(0)}}, \quad (A11)$$
$$\dot{\alpha}_{\varphi}(2) = \dot{\lambda}_1^{(4)} + \frac{(\hat{p}_1, \hat{p}_1)(2)}{(\hat{p}_1, \hat{p}_1)(0)} - \frac{(\hat{p}_1, \hat{p}_1)(0)}{(\hat{p}_1, \hat{p}_1)(0)} - \frac{(\hat{p}_1, \hat{p}_1)(0)}{(\hat{p}_1, \hat{p}_1)(0)}$$
$$- \frac{\tilde{z}_1^{(2)} - \tilde{S}(0)}{\tilde{z}_1^{(0)}} - \frac{\tilde{z}_1^{(2)} - \tilde{S}(0)}{\tilde{z}_1^{(0)}} \left(2 \frac{\dot{\tilde{t}}}{\tilde{t}} - \frac{\dot{\tilde{t}}}{\tilde{t}} \right). \quad (A12)$$

Here, we define

$$\tilde{z}_n(\tilde{t}) \equiv \int_B d^N \tilde{x} \hat{\rho}_n(\tilde{x}, \tilde{t}), \quad (A13)$$
$$\tilde{S}(\tilde{t}) \equiv \sum_{n=2}^{\infty} \frac{1}{\Delta \Lambda_{n1}(\tilde{t})} \left(\hat{p}_n, \hat{p}_1\right) \tilde{z}_n(\tilde{t}), \quad (A14)$$

and by a superscript $(k)$ we denote the $k$-th order term in an expansion in powers of $\tilde{c}$, e.g. $\tilde{z}_n(\tilde{t}) \equiv \int_B d^N \tilde{x} \hat{\rho}_n^{(k)}(\tilde{x}, \tilde{t})$ (the perturbative eigenfunction at order $k$, $\hat{\rho}_n^{(k)}$, is defined in Eq. (B1) below). To obtain the power series expansion Eqs. (A9, A12) we have used the parity properties of the perturbative spectrum; these are derived using the identical strategy as employed for the same purpose in Ref. [20].

With Eqs. (A9, A12), we have an expression of the exit rate in terms of the instantaneous perturbative FP spectrum; the exit rate in physical units is obtained using Eq. (A8) and the definitions of the dimensionless quantities given at the beginning of the present appendix. If the diffusion tensor is isotropic and independent of the state, $D \equiv D_0 \epsilon^i$, and if the tube radius is constant, $\tilde{R} = 0$, then the exit rate given here reduces to the previous result from Ref. [20].

In App. [H] we derive the equations which determine the instantaneous spectrum $(-\lambda_n, \hat{\rho}_n)$ perturbatively as a power series in $\tilde{c}$. In particular, we show that at time $\tilde{t} = t/\tau_D$, Eq. (A10) is the steady-state free-diffusion exit rate from a ball of radius $R(t)$ and with a diffusion tensor given by $D(\varphi(t), \tilde{t})$. Equation (A9) thus shows explicitly that this instantaneous steady-state free-diffusion exit rate dominates the exit rate for small enough tube radius.

Appendix B: Perturbative FP spectrum and reflecting-boundary steady state

1. Perturbative FP spectrum.

To derive a perturbation series for the instantaneous spectrum Eq. (A5), we now generalize the derivation from Ref. [20], which considered diffusive dynamics with additive isotropic noise. For this, we perform a spatial Taylor expansion of both the apparent drift and the diffusion tensor around the reference path $\varphi$ in the eigenvalue Eq. (A5), substitute power series expansions for both the eigenvalue and eigenvector,

$$\hat{\lambda}_n = \sum_{k=0}^{\infty} \tilde{c}^k \hat{\lambda}_n^{(k)}, \quad \hat{\rho}_n = \sum_{k=0}^{\infty} \tilde{c}^k \hat{\rho}_n^{(k)}, \quad (B1)$$

and demand that the resulting equation hold at each power of $\tilde{c}$ separately. This yields a hierarchy of equations which at order $k$ is given by

$$\tilde{D}_{ij}^{(0)} \tilde{\varphi}_i \tilde{\varphi}_j \hat{\rho}_n^{(k)} + \hat{\lambda}_n^{(0)} \hat{\rho}_n^{(k)} = - \sum_{l=1}^{k} \hat{\lambda}_n^{(l)} \hat{\rho}_n^{(k-l)} \quad (B2)$$
$$- \sum_{l=1}^{k} \sum_{m=0}^{l} \tilde{D}^{(l)}_{ij} \tilde{\varphi}_i \tilde{\varphi}_j \left[\hat{\rho}_n^{(m)}\right],$$

where the sums on the right-hand side are zero for $k = 0$, we define

$$\tilde{E}_{i,\alpha}^{(k)}(\tilde{t}) \equiv \tilde{E}_{i,\alpha}^{(k)}(\tilde{t}) \equiv \frac{L_k^{k-1} \delta k!}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_{k-1}} \mid_{\varphi(t), \tilde{t}}} + \delta_k \hat{\varphi}_i(\tilde{t}), \quad (B3)$$

and where we use the Einstein sum convention for the indices $\alpha$. At this point it becomes relevant that we

$$\tilde{D}^{(k)}_{ij,\alpha} \equiv \tilde{D}^{(k)}_{ij,\alpha_1 \cdots \alpha_{k-1}}(\tilde{t}) \equiv \frac{L_k^{k} \delta k!}{D_0 k! \partial x_{\alpha_1} \cdots \partial x_{\alpha_k} \mid_{\varphi(t), \tilde{t}}}, \quad (B5)$$

and where $R(t)$ is the round trip time $\tau_D t/\epsilon$.
The eigenfunction contribution at order $k$ needs to fulfill the absorbing boundary conditions $\hat{\rho}_n^{(k)}(\bar{x}, \bar{t}) = 0$ for $|\bar{x}|_\alpha = 1$. To close the system of equations defined by Eq. (B2) and the absorbing boundary conditions, we introduce the normalization condition $\langle \hat{\rho}_n, \hat{\rho}_n \rangle = 1$, which can be expanded as a power series in $\bar{\epsilon}$ to yield a condition for each $\hat{\rho}_n^{(k)}$ [20]. As we discuss in the remainder of this appendix, the spectrum is then calculated to arbitrary order by solving this system of equations recursively.

At the lowest order, $k = 0$, Eq. (B2) reduces to the eigenvalue equation of the anisotropic Laplace operator,

$$\bar{D}_{ij}^{(0)} \nabla_i \nabla_j \hat{\rho}_n^{(0)} = -\lambda_n^{(0)} \hat{\rho}_n^{(0)}.$$  \tag{B6}

Thus, $\lambda_n^{(0)}$, $\hat{\rho}_n^{(0)}$ is the spectrum of the anisotropic Laplace operator in a unit ball with absorbing boundary conditions. From Eq. (B6) we see that, at time $\bar{t}$, to lowest order the spectrum is that of free diffusion with a diffusion tensor $\bar{D}^{(0)}(\bar{t}) = D_{ij}(\varphi(\bar{t}), \bar{t})/D_0$; in particular, $\lambda_n^{(0)}$ is the corresponding instantaneous steady-state free-diffusion exit rate. Using Eq. (A10), the definition of the function $f$ in Eq. (8) thus follows. Because the diffusion matrix $D_{ij}$ is by definition symmetric, it can be diagonalized via an eigenbasis that is orthonormal with respect to the standard Euclidean inner product. By expressing Eq. (B6) with respect to such an eigenbasis of $D_{ij}$, and subsequently rescaling each axis by the corresponding eigenvalue (which for a full-rank diffusivity tensor is positive), it follows that the equation is equivalent to the eigenvalue equation for the Laplace operator in an $N$-dimensional ellipsoid (with absorbing boundary conditions).

Once the spectrum has been calculated to order $\bar{\epsilon}^{k-1}$, the subsequent order is obtained in two steps [20]. First, we obtain an equation for the eigenvalue contribution $\tilde{\lambda}_n^{(k)}$ by multiplying Eq. (B2) with $\hat{\rho}_n^{(0)}$ and subsequently integrating over $\bar{x}$. This yields

$$\tilde{\lambda}_n^{(k)} = -\frac{1}{k-1} \sum_{l=1}^{k-1} \lambda_n^{(l)} \int_{\partial B} d^N \bar{x} \hat{\rho}_n^{(0)} \hat{\rho}_n^{(k-l)}$$  \tag{B7}

$$- \sum_{l=1}^{k} \int_{\partial B} d^N \bar{x} \hat{\rho}_n^{(0)} \tilde{E}_l, \alpha \nabla_i \left( \tilde{x}_\alpha \rho_n^{(k-l)} \right)$$

$$- \sum_{m \geq 0 \atop l + m = k} \bar{D}_{ij}^{(l)} \int_{\partial B} d^N \bar{x} \left[ \hat{\rho}_n^{(0)} \nabla_i \nabla_j \left( \tilde{x}_\alpha \rho_n^{(m)} \right) \right],$$

which for constant isotropic diffusion reduces to the corresponding result in Ref. [20]. Because on the right-hand side of the equation, only the spectrum up to order $\bar{\epsilon}^{k-1}$ appears, this equation can readily be used to calculate $\tilde{\lambda}_n^{(k)}$. The result is then substituted in Eq. (B2), and $\hat{\rho}_n^{(k)}$ is the solution to the resulting inhomogeneous anisotropic Helmholtz equation with absorbing boundary conditions.

2. Reflecting-boundary steady state.

To evaluate the inner product Eq. (A6) perturbatively, the expansion of $\hat{\rho}_{ss}^{-1} \equiv 1/\hat{\rho}_{ss}$ in powers of $\bar{\epsilon}$ needs to be known. The instantaneous reflecting-boundary steady state of the dimensionless FP operator Eq. (A5) is defined by

$$\nabla_i \tilde{J}_{ss, i} = 0,$$  \tag{B8}

with $\tilde{J}_{ss, i} = \bar{\epsilon} \tilde{\alpha}_{app, i} \hat{\rho}_{ss} - \nabla_j (\bar{D}_{ij} \hat{\rho}_{ss})$. The corresponding reflecting boundary conditions are

$$\left[ \tilde{n}_i \tilde{J}_{ss, i} \right] \big|_{\partial B} = 0,$$  \tag{B9}

with $\tilde{n}_i$ the $i$-th component of the outward-pointing unit normal vector on the unit sphere $\partial B$. Similar to the perturbative calculation of the spectrum, a hierarchy of equations for the coefficients $\hat{\rho}_{ss}^{(k)}$ of the series expansion of $\hat{\rho}_{ss}$ in powers of $\bar{\epsilon}$ is obtained from Eqs. (B8), (B9) by substituting expansions in powers of $\bar{\epsilon}$, and demanding the resulting equation hold at each power of $\bar{\epsilon}$ separately. At order $k$, we obtain

$$\bar{D}_{ij}^{(0)} \nabla_i \nabla_j \hat{\rho}_{ss}^{(k)} = - \sum_{l=1}^{k} \int_{\partial B} d^N \bar{x} \hat{\rho}_n^{(0)} \tilde{E}_l, \alpha \nabla_i \left( \tilde{x}_\alpha \hat{\rho}_{ss}^{(k-l)} \right)$$

$$- \sum_{m \geq 0 \atop l + m = k} \bar{D}_{ij}^{(l)} \tilde{n}_i \tilde{n}_j \left( \tilde{x}_\alpha \hat{\rho}_n^{(m)} \right),$$  \tag{B10}

where the right-hand side is zero for $k = 0$. This equation defines $\tilde{\rho}_{ss}^{(k)}$, the corresponding boundary conditions follow from Eq. (B9) as

$$0 = - \sum_{l=1}^{k} \int_{\partial B} d^N \bar{x} \tilde{n}_i \tilde{n}_j \left( \tilde{x}_\alpha \hat{\rho}_{ss}^{(k-l)} \right) \bigg|_{\tilde{x}}$$

$$- \sum_{m \geq 0 \atop l + m = k} \bar{D}_{ij}^{(l)} \tilde{n}_i \tilde{n}_j \left( \tilde{x}_\alpha \hat{\rho}_n^{(m)} \right) \bigg|_{\tilde{x}},$$  \tag{B11}

where $\tilde{x} \in \partial B$ and where the first sum on the right-hand side of the equation is zero for $k = 0$. Starting from the unnormalized solution $\hat{\rho}_{ss}^{(0)} \equiv 1$ at order $\bar{\epsilon}^0$, this system of equations can be solved recursively. From the resulting perturbation series for $\hat{\rho}_{ss}$, a perturbation series for $\hat{\rho}_{ss}^{-1}$ is then obtained via the definition of the inverse, $\hat{\rho}_{ss}^{-1} \equiv 1/\hat{\rho}_{ss}$. For additive isotropic noise, where $\bar{D}_{ij}^{(k)} \equiv \delta_{ij} \delta_{k,0}$, Eqs. (B10), (B11) simplify to their counterparts in Ref. [20].

Appendix C: Results for one-dimensional systems

In this appendix, we consider the special case of a one-dimensional system, $N = 1$. We derive explicit formulas for the perturbative FP spectrum and the exit rate.
We provide a python module named PyTubular, which contains symbolic implementations of the analytical results from this appendix \[32\]. Beyond the results derived in this appendix, the module PyTubular also contains the normalized version of the perturbative propagator Eq. (A4).

1. Perturbative spectrum of the 1D FP equation.

For a one-dimensional system, \( N = 1 \), we solve Eq. (B2) recursively using the same algorithm as employed in Ref. [20] for the simplified case of additive noise. We now give the resulting lowest order contributions to the both eigenvalues and eigenfunctions; the analytical spectrum up to including order \( \epsilon^5 \) is available in the python module PyTubular [32].

From the parity properties of the spectrum, which are derived as in Ref. [20], it follows that \( \tilde{\lambda}_n^{(k)} = 0 \) for odd \( k \). For even \( k \), the first two nonzero contributions to the eigenvalue are

\[
\tilde{\lambda}_n^{(0)} = \tilde{D}_0 \left( \frac{n\pi}{2} \right)^2 ,
\]

\[
\tilde{D}_0 \tilde{\lambda}_n^{(2)} = \frac{\tilde{E}_1^2}{4} - \tilde{D}_0 \tilde{E}_2 + \frac{1}{16} \left( -\pi^2 n^2 + 3 \right) \tilde{D}_1^2 \]

\[
+ \frac{1}{12} \left( \pi^2 n^2 - 6 \right) \tilde{D}_0 \tilde{D}_2 + \frac{1}{2} \tilde{D}_1 \tilde{E}_1 ,
\]

where according to Eqs. (B4), (B5) we have

\[
\tilde{E}_k (\tilde{t}) = - \frac{L^k}{D_0 k!} \left. \frac{\partial^{k-1} \tilde{\rho}}{\partial x^{k-1}} \right|_{(\tilde{\varphi}(t), \tilde{t})},
\]

\[
\tilde{D}_k (\tilde{t}) = \tilde{D}(\tilde{t}) \frac{D_0^k L^k}{k!} \left. \frac{\partial^k \tilde{\rho}}{\partial x^k} \right|_{(\tilde{\varphi}(t), \tilde{t})}. \tag{C4}
\]

As in the simpler case of additive noise [20], the \( k \)-th order term of \( \tilde{\rho}_n \) is of the form

\[
\tilde{Q}_{n,s}^{(k)} (\tilde{x}, \tilde{t}) = \tilde{Q}_{n,s}^{(k)} (\tilde{x}, \tilde{t}) \sin \left[ \frac{n\pi}{2} (\tilde{x} + 1) \right] ,
\]

where the prefactors \( \tilde{Q}_{n,s}^{(k)} \) are polynomials in \( \tilde{x} \). Up to order \( \tilde{\epsilon}^2 \), they are given by

\[
\tilde{Q}_{n,s}^{(0)} (\tilde{x}) = 1 , \quad \tilde{Q}_{n,c}^{(0)} (\tilde{x}) = 0 ,
\]

\[
\tilde{Q}_{n,s}^{(1)} (\tilde{x}) = - \frac{\tilde{x}^2}{4D_0} (2\tilde{E}_1 + 3\tilde{D}_1 ),
\]

\[
\tilde{Q}_{n,c}^{(1)} (\tilde{x}) = \frac{n\pi \tilde{x}}{8D_0} (1 - \tilde{x}^2) \tilde{D}_1 ,
\]

\[
\tilde{Q}_{n,s}^{(2)} (\tilde{x}) = \frac{\tilde{x}^2}{384D_0^2} \left[ 48\tilde{x}^2 \tilde{E}_1^2 + 240\tilde{x}^2 \tilde{D}_1 \tilde{E}_1 - 192\tilde{x}^2 \tilde{D}_0 \tilde{E}_2 + (32 - 288\tilde{x}^2) \tilde{D}_0 \tilde{D}_2 \right] ,
\]

\[
\tilde{Q}_{n,c}^{(2)} (\tilde{x}) = \frac{\tilde{x}^2 n\pi}{96D_0^2} (1 - \tilde{x}^2) \left[ - 6\tilde{D}_1 \tilde{E}_1 - 15\tilde{D}_1^2 + 8\tilde{D}_0 \tilde{D}_2 \right] ,
\]

with \( \tilde{E}_k \equiv \tilde{E}_k (\tilde{t}) \), \( \tilde{D}_k \equiv \tilde{D}_k (\tilde{t}) \) given by Eqs. (C3), (C4).

If the diffusivity is independent of position and time, \( D(x, t) \equiv D_0 \), then \( \tilde{D}_0 = 1 \) and \( \tilde{D}_k = 0 \) for all \( k \geq 1 \). In that case, Eqs. (C1), (C10) reduce to the corresponding results derived in Ref. [20].

2. Reflecting-boundary steady state.

For a one-dimensional system, Eqs. (B8), (B9) are solved by

\[
\tilde{\rho}_{ss}^{(1)} (\tilde{x}) = \tilde{D}(\tilde{x}) \tilde{\rho}_{ss} (\tilde{x}) \exp \left[ \int_{\tilde{x}} d\tilde{y} \frac{\tilde{E}_{app}}{D} \right] . \tag{C11}
\]

The prefactor in Eq. (C11) can be chosen arbitrarily; we choose it such that \( \tilde{\rho}_{ss} (\tilde{x} = 0) = 1 \), a different choice corresponds to a rescaling of the inner product. Solving Eq. (C11) for \( \tilde{\rho}_{ss}^{(1)} (\tilde{x}) \) and substituting the power series expansions for \( \tilde{a}_{app} \), \( \tilde{D} \), we obtain

\[
\tilde{\rho}_{ss}^{(1)} (\tilde{x}) = \sum_{k=0}^{\infty} \frac{\tilde{E}_k \tilde{x}^k}{D_0} \exp \left[ \int_{\tilde{x}} d\tilde{y} \frac{\sum_{k=0}^{\infty} \frac{k! E_{app}}{k!} \tilde{E}_k \tilde{y}^{k-1}}{\tilde{D}_0} \right] . \tag{C12}
\]

From this, the power-series expansion of \( \tilde{\rho}_{ss}^{(1)} \) is obtained by first expanding the integrand in the exponent to the desired order in \( \tilde{x} \), performing the integral in the exponent, subsequently expanding both the exponential and the prefactor, and finally expanding their product to the desired order in \( \tilde{x} \). To order \( \tilde{x}^2 \), this yields

\[
\tilde{\rho}_{ss}^{(1)} (0) = 1 , \quad \tilde{\rho}_{ss}^{(1)} (1) = \tilde{x} \left( \tilde{E}_1 + \tilde{D}_1 \right) , \tag{C13}
\]

\[
\tilde{\rho}_{ss}^{(1)} (2) = \frac{\tilde{x}^2}{2D_0} \left( \tilde{E}_1 + \tilde{D}_1 \tilde{E}_1 + 2\tilde{D}_0 \tilde{E}_2 + 2\tilde{D}_0 \tilde{D}_2 \right) \tag{C14}
\]

with \( \tilde{E}_k \equiv \tilde{E}_k (\tilde{t}) \), \( \tilde{D}_k \equiv \tilde{D}_k (\tilde{t}) \) given by Eqs. (C3), (C4).

3. Exit rate to order \( R^2 \) for one-dimensional systems.

Employing the perturbative results for the one-dimensional spectrum and instantaneous steady state
discussed just above, the exit rate Eqs. (A9, A12) is readily evaluated. The resulting terms of the perturbation series are

\[
\tilde{\alpha}_{\text{free}} = \frac{\pi^2 \tilde{D}_0}{4e^2},
\]

(C15)

\[
\tilde{\alpha}^{\varphi}_{(0)}(t) = \frac{\tilde{E}^2}{4} - \tilde{D}_0 \tilde{E}_2 + \tilde{D}_1 \tilde{E}_1 - \left( \frac{\pi^2}{16} - \frac{3}{16} \right) \tilde{D}_1^2 + \left( \frac{\pi^2}{12} - \frac{1}{2} \right) \tilde{D}_0 \tilde{D}_2,
\]

(C16)

\[
\tilde{\alpha}^{\varphi}_{(2)}(t) = \left( \frac{1}{8} - \frac{3}{8\pi^2} \right) \tilde{D}_1^2 \tilde{E}_2^2 + \left( \frac{1}{4} - \frac{3}{4\pi^2} \right) \tilde{D}_1^3 \tilde{E}_1 + \left( \frac{3}{32} - \frac{9}{32\pi^2} - \frac{\pi^2}{64} \right) \tilde{D}_1^4 + \left( \frac{1}{12} + \frac{1}{2\pi^2} \right) \tilde{D}_0 \tilde{D}_2 \tilde{E}_1^2
\]

(C17)

\[
\frac{1}{8} - \frac{3}{8\pi^2} \tilde{D}_1^2 \tilde{E}_2^2 + \left( \frac{1}{4} - \frac{3}{4\pi^2} \right) \tilde{D}_1^3 \tilde{E}_1 + \left( \frac{3}{32} - \frac{9}{32\pi^2} - \frac{\pi^2}{64} \right) \tilde{D}_1^4 + \left( \frac{1}{12} + \frac{1}{2\pi^2} \right) \tilde{D}_0 \tilde{D}_2 \tilde{E}_1^2
\]

with \( \tilde{E}_k = \tilde{E}_k(i) \), \( \tilde{D}_k = \tilde{D}_k(i) \) given by Eqs. (C3), (C4), and as before a dot on a dimensionless quantity denotes a derivative with respect to dimensionless time \( i \). By substituting the definitions of \( \tilde{E}_k \), \( \tilde{D}_k \), the exit rate is fully expressed in terms of the drift and diffusion of the 1D FPE. In particular, from Eq. (C15) we obtain the steady-state free diffusion exit rate

\[
\tilde{\alpha}^{\varphi}_{\text{free}} = \frac{1}{\tau_D} \tilde{\alpha}^{\varphi}_{\text{free}} = \frac{\pi^2 D(\varphi(t), t)}{4 R(t)^2},
\]

(C18)

so that the function \( f \) from Eq. (8) is for \( N = 1 \) given by \( f(M) = \pi^2 M/4 \) and Eq. (22) follows. The order-\( R^0 \) contribution Eq. (24) is obtained by substituting the definitions of \( \tilde{E}_k \), \( \tilde{D}_k \) into Eq. (C16) and using the drift and diffusion of the FP Eq. (22).

4. Comparison of small-radius perturbative results to numerical simulations.

In Sect. IIIA we consider the exit rate Eq. (7) to order \( R^0 \) for three different scenarios, namely a constant-radius tube, a constant free-diffusion exit rate tube, and the exit rate related to the Stratonovich construction. We now compare our perturbative analytical results to order \( R^0 \) with exit rates measured in numerical simulations. The results demonstrate that for all three scenarios, and for the radii considered in Sect. IIIA the analytical perturbative exit rate Eq. (7) to order \( R^0 \) describes the actual exit rate very well.

For the numerical results presented here, the dimensionless form of the one-dimensional FP Eq. (22) with absorbing boundary conditions is simulated using the forward Euler algorithm described in Ref. [20]; from the resulting trajectory, the exit rate is obtained via numerical evaluation of Eq. (10).

Scenario 1: Constant radius. We now compare the analytical exit rate Eqs. (7), (22), (33), to numerical results. In Fig. 3(a), we compare the perturbative result Eq. (33) with

\[
\Delta \tilde{\alpha}_R^{\varphi}(t) = \tilde{\alpha}_R^{\varphi}(t) - \tilde{\alpha}_{\text{free}}^{\varphi}(t),
\]

(C19)

where \( \tilde{\alpha}_R^{\varphi} \) is the numerical exit rate and \( \tilde{\alpha}_{\text{free}}^{\varphi} \) is the analytical free-diffusion rate Eq. (32). According to the figure the theoretical expression \( \mathcal{L}_1^{\varphi}(t) \) and the numeri-
we in Fig. 7 compare the results from Fig. 5 with MPT rate and the number of modes. The importance of both the order of the perturbative expansion adequately describes the actual exit rate for the small-radius tube. Thus, also for scenario 3 the perturbative expansion Eq. (7) describes the actual exit rate from the center for constant-radius tubes, based on the theoretical expansion Eq. (37). As for scenarios 1 and 2, we find perfect agreement between perturbative and numerical simulations. For comparison, the exit rate obtained of the constant free-diffusion result Eqs. (34), (35) is shown as dotted line. Subplot (b) considers a time-dependent tube radius such that the free-diffusion exit rate is constant (scenario 2). The solid line depicts the sum of the theoretical results Eqs. (34), (35), the dashed line is obtained by measuring the exit rate in numerical simulations. For comparison, the Stratonovich exit rate Eq. (38) is shown as dotted line. Subplot (c) shows results for the Stratonovich construction (scenario 3). The solid line depicts the Stratonovich exit rate Eq. (38), the dashed line is obtained by measuring the exit rate in numerical simulations. For comparison, the exit rate obtained of the constant free-diffusion result Eqs. (34), (35) is shown as dotted line.

Figure 6. Comparison of theoretical and numerical exit rates for the example system considered in Sect. III.C. All subplots compare the exit rates measured in numerical simulations (solid lines) to the corresponding theoretical prediction $\alpha^{\text{num}}$ (dashed lines). For numerical results, the dimensionless FP Eq. (22) with absorbing boundary conditions is simulated using the algorithm described in Ref. [20]; from the resulting trajectory, the exit rate is obtained via numerical evaluation of Eq. (10). Subplot (a) shows results for constant tube radius (scenario 1). The solid line depicts the theoretical result Eq. (33), the dashed line is obtained by measuring the exit rate in numerical simulations and subtracting Eq. (32). Subplot (b) considers a time-dependent tube radius such that the free-diffusion exit rate is constant (scenario 2). The solid line depicts the sum of the theoretical results Eqs. (34), (35), the dashed line is obtained by measuring the exit rate in numerical simulations. For comparison, the Stratonovich exit rate Eq. (38) is shown as dotted line. Subplot (c) shows results for the Stratonovich construction (scenario 3). The solid line depicts the Stratonovich exit rate Eq. (38), the dashed line is obtained by measuring the exit rate in numerical simulations. For comparison, the exit rate obtained of the constant free-diffusion result Eqs. (34), (35) is shown as dotted line.

5. MPT centers for order $R^0$ exit rate vs. MPT centers for order $R^2$ exit rate.

In Fig. 5 we discuss the radius-dependence of the MPT center for constant-radius tubes, based on the theoretical formula for the exit rate to order $R^2$ and a parametrization of path-space based on $M = 40$ modes. To assess the importance of both the order of the perturbative exit rate and the number of modes $M$ on the MPT center, we in Fig. 7 compare the results from Fig. 5 with MPT centers based on (i) the exit rate to order $R^2$ and using $M = 60$ modes, and (ii) the exit rate to order $R^0$ and using $M = 40$ modes. For radius $R = R_0 = 0.1 L$ and $R = R_0 = 0.2 L$, we observe in Fig. 7 (a), (b) that all three MPT centers agree perfectly with each other, which shows that both $M = 40$ modes and the exit rate to order $R^0$ are sufficient to describe MPT centers at these tube radii. As we see in Fig. 7 (c), for $R = 0.3 L$ the MPT centers based on the exit rate to order $R^2$ agree for both values $M = 40, M = 60$, which demonstrates that $M = 40$ modes are sufficient to describe the most probable tube center. On the other hand, we see that the exit rate to order $R^0$ slightly disagrees with the results based on the order $R^2$ theory; while the differences are not too big, this shows that for this radius the quadratic term Eq. (C17) in the perturbative series Eq. (A9) is already relevant.

6. Transient initial decay of propagator

The propagator Eq. (A1) is dominated by the dynamics of the slowest-decaying eigenmode $\hat{1}$ [20], and as such is only valid after an initial relaxation time defined in Eq. (A7). The transient initial decay of the modes $n \geq 2$ is discussed in the additive-noise derivation of Eq. (A4) in Ref. [20]; since the discussion is based solely on the spectrum of the dimensionless FPE, it carries over directly to the present case of multiplicative noise. With this in mind, since for a one-dimensional system it holds that $\langle \tilde{\rho}_m \rangle = O(\bar{\epsilon})$, the initial transient decay can also be included in the propagator. This leads to
In deriving Eq. (C20), we treat terms proportional to $(\tilde{t} - \tilde{t}_i)^k \exp\left[-(\tilde{t} - \tilde{t}_i)/\tilde{c}_0^2\right]$ (for some constant $\tilde{c}_0 = \mathcal{O}(\tilde{c}_0^0)$) as order $\tilde{c}_0^k$, because such terms are only non-negligible for $\tilde{t} - \tilde{t}_i \lesssim \tilde{c}_0^2/k = \mathcal{O}(\tilde{c}_0^2)$.

Since $\Delta \tilde{\Lambda}_{21} = \min_m \Delta \tilde{\Lambda}_{mn}$ and $\Delta \tilde{\Lambda}_{21} < \tilde{\Lambda}$ for all $t > 1$ and small $\tilde{c}_0$, Eq. (C20) reduces to Eq. (A9) in the limit Eq. (A7).

**Appendix D: Algorithm for functional minimization in path space**

In Sect. III D we consider the most probable tube center as defined in Eq. (20), with the action given by Eq. (6) as the temporal integral over the tubular exit rate. For a one-dimensional system the first three terms in the perturbative tubular exit rate Eq. (A9) are given in terms of the diffusivity and drift by Eqs. (C15), (C16), (C17).

While for given diffusivity, drift, and path it is thus straightforward to evaluate the action, it is not straightforward to perform the functional minimization Eq. (20) which is over the infinite-dimensional space of continuous paths that start at $x_-$ and end at $x_+$ after duration $t_f$. We approximate this infinite-dimensional space by considering paths parametrized via

$$
\varphi(t) = x_- + \frac{t}{t_f} (x_+ - x_-) + \sum_{n=1}^{M} \frac{c_n}{n^2} \sin(n\pi t/f), \quad (D1)
$$

so that a path is represented by a finite-dimensional vector of mode coefficients $(c_1, c_2, ..., c_M) \in \mathbb{R}^M$. Upon substituting this path-space parameterization into any of the analytical formulae for the exit rate, finding the minimum in Eq. (20) for a one-dimensional system becomes a minimization problem in the $M$-dimensional space of coefficients.

To perform this minimization in practice, we use the cma-es algorithm [25]. Unless stated otherwise, we use $M = 40$ modes and start the cma-es algorithm at the initial condition $(c_1, c_2, ..., c_M) = (0, 0, ..., 0)$ with an initial variance $\sigma_0 = 0.5$. We run each minimization five times,
and choose as MPT center the lowest of the five minima.

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