Dynamical control of state transfer through noisy quantum channels: optimal tradeoff of speed and fidelity

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We propose a method of optimally controlling state transfer through a noisy quantum channel (spin-chain). This process is treated as qubit state-transfer through a fermionic bath. We show that dynamical modulation of the boundary-qubits levels suffices to ensure fast and high-fidelity state transfer. This is achievable by dynamically optimizing the transmission spectrum of the channel. The resulting optimal control is robust against both static and fluctuating noise.

One dimensional (1D) chains of spin-$\frac{1}{2}$ systems with nearest-neighbor couplings, nicknamed spin chains, constitute a paradigmatic quantum many-body system of the Ising type \[11, 2\], whose treatment is nontrivial yet manageable. As such, spin chains are well suited for studying the transition from quantum to classical transport and from mobility to localization of excitations as a function of disorder and temperature \[3\]. In the context of quantum information (QI), spin chains are envisioned to form reliable quantum channels for QI transmission between nodes (or blocks) of quantum communication or coupling schemes \[4\]. Contenders for the realization of high-fidelity QI transmission are spin chains comprised of superconducting qubits \[5, 6\], cold atoms \[7–10\], nuclear spins in liquid- or solid-state NMR \[11–17\] quantum dots \[18\], ion traps \[19, 20\] and nitrogen-vacancy (NV) centers in diamond \[21–24\].

The distribution of coupling strengths between the spins that form the quantum channel, determines the state transfer-fidelities \[4, 25, 27\]. Perfect state-transfer (PST) channels can be obtained by precisely engineering each of those couplings. Such engineering is however highly challenging at present \[13, 28\]. A much simpler control may involve only the boundary (source and target) qubits that are connected via the channel. Recently, it has been shown that if the boundary qubits are weakly-coupled to a uniform (homogeneous) channel (i.e., one with identical couplings), quantum states can be transmitted with arbitrarily high fidelity at the expense of increasing the transfer time \[29–32\]. Yet such slowdown of the transfer may be detrimental because of omnipresent decoherence. To overcome this problem, we here propose a hitherto unexplored approach for optimizing the tradeoff between fidelity and speed of state-transfer in quantum channels. This approach employs temporal modulation of the couplings between the boundary qubits and the rest of the channel, which is treated as dynamical control of a quantum system coupled to a fermionic bath. The goal of the modulation is to realize an optimal spectral filter \[33–36\] that blocks transfer via the eigenmodes of the channel that are responsible for leakage of the QI \[37\]. We show that under optimal modulation, the fidelity and the speed of transfer can be improved by several orders of magnitude, and the fastest transfer is achievable for a given fidelity.

Quantum channel: Hamiltonian and boundary control.— In keeping with previous studies \[4, 29\], we consider a spin-$\frac{1}{2}$ chain with XX interactions between nearest neighbors. The Hamiltonian is given by

$$H = H_0 + H_{bc}(t),$$

$$H_0 = \frac{1}{2} \sum_{i=1}^{N-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right),$$

$$H_{bc}(t) = \frac{1}{2} \alpha(t) \sum_{i=0}^N \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right),$$

where $H_0$ and $H_{bc}$ stand for the chain and boundary-coupling Hamiltonians, respectively, $\sigma_i^\mu$ are the Pauli matrices, $N$ is the chain length, and $J_\parallel > 0$ is the exchange interaction coupling. The magnetization-conserving $H_0$ can be transformed into a non-interacting fermionic Hamiltonian \[38\], that has the diagonal, particle-conserving form $H_0 = \sum_{k=1}^N \omega_k b_k^\dagger b_k$, where $b_k^\dagger$ populates a fermionic single-particle, eigenstate of energy $\omega_k$.

Under the assumption of mirror symmetry of the couplings $J_i = J_{N-i}$ for odd $N$, there is a single non-degenerate, zero-energy fermionic mode in the quantum channel \[26–28\], corresponding to $k = z = \frac{N+1}{2}$. The two boundary qubits are resonantly coupled to this mode \[24, 29, 31\] with an effective, temporally-modulated coupling strength $J_z \alpha(t)$. This resonant fermionic tunneling is described by the effective Hamiltonian

$$H_S(t) = \tilde{J}_z \alpha(t)(c_0^\dagger b_0 + c_{N+1}^\dagger b_{N+1} + \text{h.c.}).$$

The main idea of our treatment is to consider these three fermionic modes as a system $S$ that interacts with a bath $B$, and thus rewrite the total Hamiltonian as $H = H_S(t) + H_B + H_{SB}(t)$, where $H_B = \sum_{k=1}^N \omega_k b_k^\dagger b_k$ with $k \neq z$, $k = 1 \ldots N$. Upon defining the collective-mode operators $\tilde{b}_{k_{\text{odd(even)}}} = \sum_{k_{\text{odd(even)}}=1}^N \tilde{J}_k b_k$, the system-bath interaction assumes the form

$$H_{SB}(t) = \alpha(t) \left[ (c_0^\dagger + c_{N+1}^\dagger) \tilde{b}_{k_{\text{odd}}} + (c_0^\dagger - c_{N+1}^\dagger) \tilde{b}_{k_{\text{even}}} \right] + \text{h.c.}$$

This form is amenable to optimal dynamical control of the multipartite system \[33–36\] that generalizes single-qubit dynamical control by modulation of the qubit levels \[34–36\].

To this end, we rewrite Eq. (3) in the interaction picture as $H_{SB}^I(t) = \sum_j S_j(t) \otimes \tilde{B}_j(t)$; and decompose $H_{SB}^I(t)$ into symmetric and antisymmetric system operators that are coupled to odd- and even-bath modes (see SI). Upon
representing the system operators $S_j(t)$ via a rotation-matrix $\Omega_{\alpha}(l)$ in a chosen basis of operators $\hat{\nu}_i$, so that $S_j(t) = \sum_i \Omega_{\beta}(t) \hat{\nu}_i$, we can write a time-convolutionless second-order solution for the system density matrix $\rho_S(t)$ in the interaction picture [33]. This solution will be used to calculate and optimize the transfer fidelity in what follows.

Let us consider a generic qubit-state $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ as the source qubit 0, and $|\psi_S\rangle \otimes |0\rangle_B$ with $|\psi_S\rangle = |\psi_0\rangle \otimes \{|0\rangle_0 |0\rangle_{N+1}\}$ as the initial state of $S + B$. We shall be interested in the transfer fidelity of $|\psi_0\rangle$ to the target qubit $N + 1$, averaged over all input states on the Bloch sphere: it is given by [25] $F(T) = \frac{f_{0,N+1}(T)}{f_{0,N+1}(T) + f_{0,N+1}(T) + 1}$, where $f_{0,N+1}(T)$ is simply the transfer fidelity of $|\psi_0\rangle = |\psi_0\rangle$.

This transfer fidelity is expressed in the interaction picture as $f_{0,N+1}(T) = \langle S (\psi | \rho_S(T) | \psi_S) \rangle$, where $T$ is the transfer time. The transfer fidelity remains for any initial state of the bath channel $B$ withing the weak coupling regime [24, 31, 40].

**Optimization method.**— To ensure the best possible state-transfer fidelity, we use modulation as a tool to minimize the infidelity $\zeta(T) = 1 - f_{0,N+1}(T)$ by rendering the overlap between the bath and system spectra as small as possible (see SI) [33]. The infidelity may be written as the convoluted overlap

$$\zeta(T) = \Re \int_0^T dt \int_0^T dt' \sum_{q = \text{even, odd}} \Omega_q(t) \Omega_q(t') \phi_q(t - t')$$

where $\phi_{\text{odd/even}}(\tau) = \sum_{k_{\text{odd/even}}} |\tilde{J}_k|^2 e^{-i\omega_k \tau}$ are the bath-correlation functions, while $\Omega_{\text{odd}}(\tau) = \phi(\tau) \cos(\sqrt{2} \phi(\tau)) / {{\tilde{J}_2}^2}$ and $\Omega_{\text{even}}(\tau) = \phi(\tau) / {{\tilde{J}_2}^2}$ are the dynamical control functions, expressed in terms of the the phase accumulated by the qubit under modulation control $\phi(T) = \tilde{J}_z \int_0^T \alpha(t') dt'$. In the energy domain, Eq. (4) has the form

$$\zeta(T) = \sum_{q = \text{even,odd}} \int \mathcal{G}^q(\omega) \mathcal{F}_q^T(\omega) d\omega,$$

where the Fourier transforms $\mathcal{G}^q(\omega) = \mathcal{F}(\Phi_q(\tau))$ and $\mathcal{F}_q^T(\omega) = \mathcal{F}(\frac{\Omega_q(t)}{\tilde{J}_2})$ are the bath-spectrum and the filter-energy functions, respectively, for even or odd $q$. To determine the optimal modulation control, we minimize the overlap integrals of $\mathcal{G}^q(\omega)$ and $\mathcal{F}_q^T(\omega)$ for a given $T$ by the variational Euler-Lagrange method.

We require the channel to be symmetric with respect to the source and target qubits and the number of eigenvalues to be odd. This requirements allows for a central eigenvalue that is invariant under noise on the couplings, provided a gap exists between this eigenvalue and the adjacent ones, i.e. they are not strongly blurred (mixed) by noise, so as not to make them overlap. The optimized modulations derived here are applicable to any system of this kind. As an example, consider a uniform (homogeneous) spin-chain channel, i.e. $J_i \equiv J$ and energies $\omega_k = 2J \cos(k \pi / N+1)$. In this case, under complete randomization of $\omega_k$, the lineshape is the Wigner semicircle (see Fig. 1a and SI). In the weak-coupling regime $\alpha \ll 1$, the interaction $H_{bc}$ is treated perturbatively, so that $\tilde{J}_z = \sqrt{\frac{2}{N+1}} J$ and $\tilde{J}_k = \tilde{J}_z \sin \left( \frac{k \pi}{N+1} \right)$ are always much smaller than the nearest eigenvalue gap $|\omega_z - \omega_{z\pm1}| \approx \frac{2J}{N+1}$ [29, 31]. This may not happen in the strong coupling regime $\alpha \sim 1$, which requires special consideration (see below).

**Optimal filter design.**— In order to obtain universal solutions for channels with static or fluctuating spin-spin coupling noise, we assume that the discreteness of the quantum channel spectrum is smoothed out by the noise (see SI). On the other hand, the noise-induced broadening is assumed to be lower than the gap around the eigenvalue $\omega_z = 0$ that remain invariant against this noise, whereas higher eigenvalues are affected by it [28]. A filter that is efficient and robust against noise in a system with a central gap must be a narrow band-pass around $\omega_z$. To this end, we look for maximized $F_T(\tau) = \int F_T(\omega) e^{-i\omega \tau} d\omega$ for every $\tau$, in order to ensure that only the lowest frequency components are present in the filter-energy function under the constraint of accumulated phase $\phi(\tau)$ and energy

$$E(T) = \tilde{J}_z^2 \int_0^T |\alpha(t)|^2 dt \geq \frac{\phi(\tau)^2}{\tilde{T}_p}$$

The optimal solutions are found to be (see SI)

$$\alpha_p(t) = \alpha_M \sin^2 \left( \frac{\pi t}{T_p} \right)$$

with $p = 0, 1, 2$, $T_p = c_p \frac{\phi(T)}{\tilde{J}_z}$ and $c_p = \sqrt{\frac{\pi \Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}}$ ($c_0 = 1$, $c_1 = \frac{\pi}{2}$, $c_2 = 2$). Here $p = 0$ means static control, while $p = 1, 2$ stands for dynamical control.
Figure 2. (Color online) Transfer infidelity $1 - F(T)$ for a modulated boundary-controlled coupling $\alpha_p(t) = \alpha_M \sin^p(\frac{\pi t}{T})$ as a function of (a) the transfer time $T$, (inset) the maximum value of the boundary coupling $\alpha_M$ and (b) the perturbation strength $\varepsilon_f$ of the noisy channel, averaged over $10^5$ noise realizations for $\alpha_{\text{opt}} = 0.6$ and $\alpha_{\text{opt}} = 0.7$.

In static noisy channels, the infidelity obtained under static control $p = 0$ (empty circles) is shown to be strongly reduced under dynamical $p = 2$ control (empty squares). A fluctuating noisy channel is less damaging; in the Markovian limit where the correlation time of the noise fluctuations $\tau_c \to 0$ (with $p = 0$, green solid circles), the infidelity converges to its unperturbed value. $N = 2 + 31$ spins and $J = 1$.

$p = 0$, $\alpha_0(t)$ is constant and satisfies the minimal-energy condition, $E_{\text{min}}(T_0) = \frac{\pi^2}{2 T_0}$.

Although the corresponding filter function is a narrow bandpass around 0, it still has many wiggles (Fig. 1b) which overlap with bath-energies that hamper the transfer. Therefore, to improve the fidelity transfer we require a filter that is flatter and lower throughout the bath-energy domain. By allowing $E \gtrsim E_{\text{min}}$, the filter is made lower outside a region around 0 by the modulation $\alpha_1(t)$, with $E_1 = \frac{\alpha}{2} E_{\text{min}}$, or $\alpha_2(t)$, with $E_2 = \frac{3}{2} E_{\text{min}}$ (Fig. 1b).

This modulation control allows the design of optimal filters $F^{\text{even}}(\omega)$ that are sharp around 0 and flat (and low) across the bath-energy range ($F^{\text{odd}}(\omega)$ filters out the same spectral range). The inset in Fig. 1b shows that depending on $T$, different modulations $\alpha_p(t)$ are optimal. They are determined by the overlap between the bath-spectrum, the width of the central peak and the tail of the filter function. The shorter is $T$, the lower is $p$ that yields the highest fidelity, because the central peak that gives the dominant overlap is then the narrowest. However, as $T$ is increased, larger $p$ will give higher fidelity, because now the tails give the dominant contribution to the overlap. As shown in Fig. 1b, the filter for $p = 2$ (similarly for $p = 1$) can improve the transfer fidelity by orders of magnitude in a general noisy gapped-bath.

While the approach based on Eq. (4) strictly holds in the weak coupling-regime ($\alpha_M \ll 1$)[33, 34, 56], the validity of the optimal modulations can also be extended to strong couplings $\alpha_M$, since they become compatible with the weak-coupling regime under the filtering process. This is observed, for example, for a homogeneous channel in Fig. 2b, where the state-transfer infidelity is displayed as a function of $\alpha_M$ and $T$, for $\alpha_p(t) = \alpha_M \sin^p(\frac{\pi t}{T})$ with $p = 0, 2$. In the weak-coupling regime ($\alpha_M \ll 1$) the infidelity decreases with $\alpha_M$ according to a power law, and the transfer time increases as $T \approx c \frac{\pi}{2 \alpha_M}$. Under optimal-filtering in the strong coupling regime [30, 32, 41], there is a minimum infidelity at $\alpha_{\text{opt}}$ that depends of $p$. The corresponding transfer time is $T \approx c \frac{\pi}{2 \alpha_{\text{opt}}}$. Here, the oscillatory behavior of the infidelity reflects the discrete nature of the spectrum. The filter tails are sinc-like functions, so that when a zero of the filter matches a bath-energy eigenvalue, the infidelity exhibits a dip.

While Fig. 2a shows the transfer infidelity at time $T$, it is important to note that the fidelity under optimal modulation, $F(t)$, yields the widest window of time where the fidelity remains high compared with the unmodulated cases. This gives more time for determining the transfered state or using it for further processing, thus increasing the robustness against imperfection in the temporal accuracy of the optimal dynamical control.

The advantages of dynamical control ($p = 1$ or 2) of the boundary-couplings are evident in Fig. 2b. The inset panel shows that by fixing $\max(\alpha(t)) = \alpha_M$, the dynamical modulation increases the transfer fidelity by orders of magnitude only at the expense of slowing down the transfer time at most by a factor of 2. By contrast, without dynamical modulation ($p = 0$), the optimal $\alpha_{\text{opt}}$ value yields faster transfer(main panel), but no significant increase of the fidelity. Namely, the only option for increasing the fidelity is then to reduce $\alpha$, but the transfer time then increases as $\sim \frac{1}{\alpha}$. If the constraint on $\alpha_M$ can be relaxed, i.e. more energy can be used, the great advantages of dynamical control can be appreciated in both respects, i.e. fidelity increase and transfer-time reduction by orders of magnitude. Hence, our main result is that the speed-fidelity tradeoff can be drastically improved under optimal dynamical control. In particular, optimized modulations provide the fastest transfer for a given fidelity.

Robustness against different noises.— We now consider the effects of noise affecting the coupling strengths as follows: $J_i \to J_i + J_i \Delta_i(t)$, $i = 1, ..., N$ with $\Delta_i(t)$ being a uniformly distributed random variable in the interval $[-\varepsilon_j, \varepsilon_j]$ for a given time $t$.

(i) Static-noise. Static control on the boundary-couplings can make the channel robust against static noise [32] but here we show that dynamical boundary-control makes the channel even more robust, because it filters out the bath-energies that damage the transfer. To illustrate this, we compare in Fig. 2b the robustness of modulations $\alpha_p(t)$, $p = 0$ and 2 in the strong-coupling regime for $\alpha_{\text{opt}} = \alpha_{\text{opt}}$. Where the advantage of $p = 2$ compared
with the static control case \( p = 0 \) is evident, at the expense of increasing the transfer time by only a factor of 2. In the weak-coupling regime we may choose \( \alpha_M \), such that the transfer fidelity is similar for \( p = 0 \) and \( p = 2 \), and both cases are similarly robust under disorder, but the modulated case \( p = 2 \) is an order of magnitude faster. This speedup will be important in the presence of other sources of decoherence (see below). We obtain a bound for the fidelity improvement of the state transfer that is intrinsic to the channel: because of Anderson localization [45,47], regardless of how small is \( \alpha_0 \), the fidelity cannot be improved beyond the bound

\[
1 - \tilde{F} \sim \frac{1}{5}N\varepsilon_J^2, \quad (\varepsilon_J \ll 1). \tag{6}
\]

(ii) Markovian noise.— In the limit where the gap-width goes to zero, i.e. for a Markovian noise such that the bath correlation function vanishes at \( t - t' > 0 \), the optimal modulation can be approximated by \( \alpha(t) \approx \alpha_M (a + b \sin^p \left( \frac{\pi t}{\tau} \right)) \), where \( p \approx 3.5, \frac{a}{b} \approx \frac{1}{3} \) and \( \alpha_M = \max \alpha(t) \). However, the infidelity for this optimal modulation almost coincides with the one obtained without modulation. Thus, modulation is not helpful in the Markovian limit. Counterintuitively, arbitrarily high fidelities can be achieved for such a bath by slowing down the transfer time, i.e. by decreasing \( \alpha_M \). This comes about because in a Markovian bath, the very fast coupling fluctuations generate an effective self-decoupling of the disorder, thereby suppressing the Anderson localization effects that hamper the transfer fidelity.

(iii) Non-Markovian noise. We finally consider fluctuating noise \( J_i + J_i \Delta_i(t) \) in a homogeneous channel with constant boundary-couplings. By reducing the noise correlation time \( \tau_c \), we observe a convergence of the transfer fidelity to its value without noise as \( \tau_c \) decrease (Fig. 2b). Consequently the fidelity can be substantially improved by reducing \( \alpha_M \). The effective noise strength scales down as \( \tau_c^{-1/2} \), approaching the Markovian limit when \( \tau_c \rightarrow 0 \). As we saw above, modulation is not helpful in the gapless Markovian limit. By contrast, in the non-Markovian regime that lies between the static and Markovian limits and the bath-spectrum is gapped, dynamical control can strongly reduce the infidelity.

Realizations.— A general procedure applicable to any system which allows control of the boundary spins, consists in modulating the boundary couplings by creating an effective Hamiltonian via Trotter-Susuki decompositions [16,19,20,48]. The corresponding modulation of the boundary-spins energy is only required to set them on-and off-resonance intermittently at suitable times [16] or modulating the boundary couplings by sequences of \( \pi \)-pulses on the boundary spins (see SI). In the weak-coupling regime, efficient transfer through noisy spin chains is realizable by periodically modulating the level distance of the boundary qubits by an off-resonant field, whose effect in this regime is the same as periodically modulating the qubits coupling to the bath [34,36]. The modulation rate must be faster than the inverse transfer time \( 1/T \). On the other hand, decay or leakage of the single excitation shared by the qubits and the bath must be either slower than \( T \) or suppressed by an additional control field [33].

Among the diverse systems that are able to comply with these requirements, we here suggest dipole-dipole (DD) coupled atoms [49], embedded in 1D photonic structures [50,51]. Particular appealing is a chain of atoms trapped just outside an optical fiber whose dipole transition is within the optical bandgap created by a grating in the fiber [51]. If the dipole transition is just below the band edge, the DD coupling is strongly enhanced while the radiative decay is suppressed by the bandgap [51]. The resonance frequency of the boundary atoms can be modulated faster than the DD couplings: Modulation of the boundary-atom frequency shifts at a GHz rate, comparable to the enhanced DD rate, should effectively control the transfer time and fidelity along the chain in the presence of noise caused by sub-Kelvin thermal fluctuations of the atomic positions and/or their random site occupancy.

Conclusions.—We have proposed a general, optimal and robust dynamical-control of the tradeoff between transfer speed and fidelity of qubit state transfer through a quantum channel in the presence of either static or fluctuating noise. The only requirement for this method to apply is for the channel to be symmetric with respect to the source and target qubits and the number of eigenenergies has to be odd. This leads to a central eigenvalue that is invariant against static noise on the couplings, and have a gap separating it from adjacent eigenenergies. Counterintuitively, we have shown that static noise is more detrimental than fluctuating-noise, for a given noise strength on the spin-spin couplings. Dynamical boundary-control has been used to design an optimal spectral-filter that can minimize the leakage to modes of the channel (here considered as a bath), that deteriorate the transfer fidelity. The optimal filter is realizable by universal, simple, modulation shapes that ensure the highest fidelity for a given transfer time in both weak- and strong- coupling regimes, and are robust against static and fluctuating noise on the spin-spin couplings. As a result, the fidelity and/or the transfer time can be improved by orders of magnitude compared with unmodulated transfer, while their robustness against noise on the couplings is maintained or even improved. The principles of this general treatment are extendable to other (non-Ising) quantum channels as well.

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Supplemental information for “Dynamical control of state transfer through noisy quantum channels: optimal tradeoff of speed and fidelity”

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I. INTERACTION PICTURE REPRESENTATION OF THE HAMILTONIAN

The system-bath Hamiltonian (Eq. (3) of the main text) splits into a sum of symmetric and antisymmetric system operators that are coupled to bath odd- and even-modes: $H_{SB}(t) = \sum_{j=1}^{4} S_j \otimes B_j^\dagger$, where $S_{1(3)} = \alpha(t)(c_0 + (-)c_{N+1})$, $S_{2(4)} = S_{1(3)}^\dagger$, $B_{1(3)} = \sum_{k_{\text{odd(even)}}} \tilde{J}_k b_k$ and $B_{2(4)} = B_{1(3)}^\dagger$. In the interaction picture $H_{SB}(t)$ becomes

$$H_{SB}^I(t) = \sum_{j=1}^{4} S_j(t) \otimes B_j^I(t),$$

(1)

where

$$S_{1(3)} = \alpha(t)(c_0 + (-)c_{N+1}), \quad U_S(t) = e^{-i\int_0^t dt' H_S(t')}, \quad B_j(t) = U_B^\dagger(t)B_jU_B(t), \quad U_B(t) = e^{-iH_Bt},$$

(2)

and the evolution operators are

$$U_S(t) = |0\rangle_S\langle 0| + \left(\cos\left(\frac{\sqrt{2}\phi(t)}{2}\right)+1\right)(|0\rangle\langle 0| + |N+1\rangle\langle N+1|) + \left(\cos\left(\frac{\sqrt{2}\phi(t)}{2}\right)-1\right)(|0\rangle\langle N+1| + |N+1\rangle\langle 0|)$$

$$+ \cos(\sqrt{2}\phi(t))|z\rangle\langle z| - i \sin(\sqrt{2}\phi(t))(|0\rangle\langle z| + |N+1\rangle\langle z| + h.c.),$$

$$U_B(t) = \sum_{k=1, k\neq z}^{N} e^{-i\omega_k t}|k\rangle\langle k| + |0\rangle_B\langle 0|,$$

(3)

where the states $|i\rangle = |0.01,0.0\rangle$ denote the one excitation subspace while $|0\rangle_S = |0_0,0_{N+1}\rangle_S$ and $|0\rangle_B = |0_1,0_{N}\rangle_B$ refer to the zero excitation states in the system (S) and bath (B) respectively. Therefore, the bath operators are $B_{1(3)}(t) = \sum_{k_{\text{odd(even)}}} |\tilde{J}_k|^2 e^{-i\omega_k t}|k\rangle_B\langle 0|$, $B_{2(4)}(t) = B_{1(3)}^\dagger(t)$. We define a basis of operators $\hat{\nu}_i$ to describe the rotating system operators $S_j(t)$ via a rotation-matrix $\Omega_{j;i}(t)$, and they are given by

$$\hat{\nu}_1 = |0\rangle_S\langle 0| + |N+1\rangle\langle N+1| \quad \hat{\nu}_2 = \hat{B}_1^\dagger,$$

$$\hat{\nu}_3 = |0\rangle_S\langle z| \quad \hat{\nu}_4 = \hat{B}_3^\dagger,$$

$$\hat{\nu}_5 = |0\rangle_S\langle N+1| \quad \hat{\nu}_6 = \hat{B}_5^\dagger,$$

(4)

such that $S_j(t) = \sum_i \Omega_{j;i}(t)\hat{\nu}_i$. Given that $S_1(t) = \hat{\phi}(t)(\cos(\sqrt{2}\phi(t))\hat{\nu}_2 - i\sqrt{2}\sin(\sqrt{2}\phi(t))\hat{\nu}_3)$, $S_3(t) = \hat{\phi}(t)\hat{\nu}_5$, $S_{2(4)}(t) = S_{1(3)}^\dagger(t)$, the rotation-matrix’s vectors are

$$\Omega_1(t) = \hat{\phi}(t)(\cos(\sqrt{2}\phi(t)), 0, -i\sqrt{2}\sin(\sqrt{2}\phi(t)), 0, 0, 0)$$

$$\Omega_2(t) = \hat{\phi}(t)(0,\cos(\sqrt{2}\phi(t)), 0, i\sqrt{2}\sin(\sqrt{2}\phi(t)), 0, 0)$$

$$\Omega_3(t) = \hat{\phi}(t)(0, 0, 0, 0, 1, 0)$$

$$\Omega_4(t) = \hat{\phi}(t)(0, 0, 0, 0, 0, 1).$$

(5)

II. FIDELITY DERIVATION

From the system-bath interaction, Eq. (1), one can derive the system density matrix $\rho_S(t)$ in the interaction picture for a weak system-bath interaction as [1, 2]

$$\rho_S(t) = \rho_S(0) - t \sum_{i,i'} R_{i,i'}(t)\hat{\nu}_i\hat{\nu}_i\rho_S(0) + h.c.,$$

(6)
where \( R_{i,i'}(t) = \frac{1}{T} \sum_{j,j'} \int_0^T dt' \int_0^t dt'' \Phi_{j,j'}(t''-t') \Omega_{j,i}(t') \Omega_{j',i}^*(t'') \). The correlation between baths \( j \) and \( j' \) is denoted by \( \Phi_{j,j'}(\tau) = \text{Tr}_B \left\{ B_j(\tau) B_{j'}(0) \rho_B(0) \right\} \) and \( R_{i,i'}(t) \) is the average rate of change of the system’s density matrix \( \rho_S \) under the action of \([\hat{\nu}_i, \hat{\nu}_{i'} \rho_S(0)]\) caused by the bath. Using the basis \( \nu_i \), we write the time-independent score matrix \( \Gamma_{i,i'} = \langle \psi | [\hat{\nu}_i, \hat{\nu}_{i'}] | \psi \rangle = \langle \hat{\nu}_i \hat{\nu}_{i'} \rangle - \langle \hat{\nu}_i \rangle \langle \hat{\nu}_{i'} \rangle \) [2], which describes the change of the fidelity with respect to the chosen basis \( \hat{\nu}_i \).

Considering \( |\psi\rangle = |100...0\rangle_{SB} = |100...0\rangle_{0,N+1} S \otimes |0\rangle_B \) as initial state, the score matrix is reduced to

\[
\Gamma_{i,i'} = \delta_{i,2} \delta_{1,i'} + \delta_{i,3} \delta_{5,i'} + \delta_{i,4} \delta_{1,i'} + \delta_{i,5} \delta_{5,i'}.
\]

Then, the correlation functions in terms of the bath operators (described above) are

\[
\Phi_{j,j'}(t-t') = \sum_{k \in \text{odd}} \hat{J}_k|^2 e^{-i \omega_k (t-t')} \delta_{j,2} \delta_{1,j'} + \sum_{k \in \text{even}} \hat{J}_k|^2 e^{-i \omega_k (t-t')} \delta_{j,4} \delta_{3,j'},
\]

and therefore

\[
R_{i,i'}(T) = \frac{1}{T} \int_0^T dt \int_0^t dt' \Phi_{2,1}(t-t') \Omega_{2,i}(t) \Omega_{1,i'}(t') + \Phi_{4,3}(t-t') \Omega_{4,i}(t) \Omega_{3,i'}(t').
\]

In the isolated 3-level system, perfect state transfer of the qubit-state \(|1\rangle\) from the spin 0 (source qubit) to the \( N+1 \) (target qubit) occurs when the accumulated phase due to the modulation control \( \phi(T) = \int_0^T \alpha(t) dt' \) satisfies \( \phi(T) = \frac{\pi}{2} \). In the presence of the bath, the transfer fidelity of this qubit-state is given by \( f_{0,N+1}(T) = |S \langle \psi | \rho_S(T) | \psi \rangle_S| \) in the interaction picture and within the second-order approximation done in Eq. (6). It takes the form

\[
f_{0,N+1}(T) = 1 - \zeta(T), \quad \zeta(T) = T \times \text{Re} \text{Tr} \{ R(T) \Gamma \}.
\]

From Eqs. (7-9)

\[
\zeta(T) = \int_0^T dt \int_0^t dt' \frac{\phi(t) \phi(t')}{J_z^2} (\Phi_{\text{odd}}(t-t') \cos(\sqrt{2} \phi(t')) \cos(\gamma \phi(t)) + \Phi_{\text{even}}(t-t'))
\]

with \( \Phi_{\text{odd(even)}}(\tau) = \sum_{k \in \text{odd(even)}} |\hat{J}_k|^2 e^{-i \omega_k \tau} \).

### III. Euler-Lagrange Optimization

#### A. Optimizing the modulation control \( \alpha(t) \) for general non-Markovian gapped baths

In the energy domain, Eq. (11) has the form

\[
\zeta(T) = \sum_{q=\text{even,odd}} \int G^q(\omega) F_T^q(\omega) d\omega,
\]

where the Fourier transforms \( G^q(\omega) = \mathcal{F}\mathcal{T}(\Phi_q(\tau)) \) and \( F^q_T(\omega) = \mathcal{F}\mathcal{T}(\frac{|\Omega_q(t)|^2}{T}) \) are the bath-spectrum and the filter-energy \( q \) functions, respectively, for even or odd \( q \). To determine the optimal modulation control, we minimize this overlap for a given \( T \) by the variational Euler-Lagrange method. The shape of the bath-spectrum will change from channel to channel, but all of them have a common characteristic: a central gap around \( \omega_z = 0 \). Therefore, to find a general modulation control to minimize Eq. (12), we will assume a bath-spectrum that is continuous in the energy band with the exception of a central gap, and thus, we will maximize the filter function within this gap.

We maximize \( F_T(\tau) = \int F_T(\omega) e^{-i \omega \tau} d\omega \), around \( \omega_z = 0 \), for every \( \tau \) in order to assure the lowest frequency components of the filter-energy function under the accumulated phase \( \phi(T) = J_z \int_0^T \alpha(t) dt \) and energy \( E(T) = J_z^2 \int_0^T |\alpha(t)|^2 dt \geq \frac{\phi(T)^2}{T} \). The Euler-Lagrange equation is then

\[
\frac{\partial F_T(\tau)}{\partial \alpha(t)} = \lambda_E \frac{\partial E(T)}{\partial \alpha(t)} + \lambda \frac{\partial \phi(T)}{\partial \alpha(t)}
\]
Since the desired sharp filter deals with the closest energies to 0, we focus on minimizing the overlap with $G_{\text{even}}(\omega)$. Given that $E_{\text{T}}^{\text{even}}(\tau) = \int_0^T \alpha(t)\alpha(t+\tau)dt$, Eq. (13) becomes $\alpha(t+\tau) + \alpha(t-\tau) = \lambda_E\alpha(t) + \lambda_\phi$. For small $\tau$, it turns to be $\hat{\alpha}(t) = -\hat{\lambda}_E\hat{\alpha}(t) + \hat{\lambda}_\phi$ where $\hat{\lambda}_E = \frac{(\lambda_E - 2)}{\tau^2}$ and $\hat{\lambda}_\phi = \frac{\lambda_\phi}{\tau^2}$ are the rescaled Lagrange multipliers. This differential equation has a general solution $\alpha(t) = A\sin(\omega_t t) + B\cos(\omega_t t) + C$, where the unknown parameters will be optimized according to the required conditions, such as the boundary constraints, the transfer time, energy, etc. Relaxing the constraints and imposing only $\phi(T) = \frac{\pi}{2}$, a condition on the frequency $\omega_{\tau}$ arises from the Fourier transform properties of the convolution between $\alpha(t)$ and the boxcar function on the time interval $[0, T]$.

The total filter will be low and flat outside a small range around 0 only if $\omega_{\tau} = \frac{n\pi}{T}$, $n\in\mathbb{Z}$, since the interference between the FT of the different terms of $\alpha(t)$ that oscillate with $\omega_{\tau}$ interfere destructively. On the other hand only if $n = 0, 1, 2$ the filter has a central and unique peak around 0 reducing the contribution of larger frequencies. For larger values of $n$, the central peak of the filter function is split and peaks at larger frequencies appear.

Therefore, the optimal solutions are found to be

$$\alpha_p(t) = \alpha_M \sin^p \left( \frac{\pi t}{T_p} \right),$$

with $p = 0, 1, 2$, $T_p = c_p \frac{\phi(T)}{\lambda}$ and $c_p = \sqrt{\frac{\pi t}{T_p}} (c_0 = 1, c_1 = \frac{\pi}{2}, c_2 = 2)$.

### B. Optimizing the modulation control $\alpha(t)$ for a specific non-Markovian bath

The minimization of $\zeta(T)$ (11) can also be done for a specific bath-correlation function of a given channel. For example, for a finite homogeneous spin-channel, the exact correlation function of the bath is $\Phi_{\text{odd/even}}(\omega) = \sum_{k_{\text{odd/even}}} \left| \sqrt{2 \pi} \frac{J \sin\left( \frac{\pi k}{N+1} \right)}{N} \right|^2 e^{-ij_2 J \cos\left( \frac{\sqrt{\pi}}{\sqrt{2}} t \right)}$, and has recurrences and time fluctuations due to mesoscopic revivals, while at short times $t$, it behaves as a Bessel function $\Phi(t) = \frac{2(\omega_0 J)^2}{T} J_1(2J t)$. The latter correlation function represents the limiting case of an infinite channel and it gives a continuous bath-spectrum that becomes a semicircle. In the case of a finite channel, $G(\omega)$ will be discrete but modulated by the semicircle with a central gap. If disorder is considered, the position of the spectrum lines fluctuates from channel to channel but they are essentially modulated by the semicircle with a central gap as was considered in the Fig. 1 of the main text, where $G(\omega) = \frac{1}{2} \sqrt{4J^2 - \omega^2(1 - \Theta(\omega - \omega_1)\Theta(\omega + \omega_2))}$, $\omega_{\tau} = \frac{3\omega_{\tau} + 1}{4}$. This is the Wigner-distribution for fully randomized channels [3, 4] with a central gap.

Once the specific channel and $\Phi_{\text{odd/even}}(\tau)$ are given, the Euler-Lagrange method can be implemented as follow. The minimization of $\zeta(T)$ (Eq. 11) can be done under a constraint $\chi(T)$ to avoid unphysical results [1, 2, 5, 6]. The Euler-Lagrange equation turns then

$$\frac{d}{dt} \left( \frac{\partial \zeta}{\partial \phi} - \lambda \frac{\partial \chi}{\partial \phi} - \frac{\partial \zeta}{\partial \phi} \right) = 0,$$

(15)

where $\lambda$ is the Lagrange multiplier factor.

Choosing the energy as a constraint $\chi(T) = E(T) = J_2^T \int_0^T \dot{\phi}^2(t)dt$, the optimal modulation is given by the integro-differential equation

$$\ddot{\phi}(t) = \frac{\sqrt{\pi} Q(t, \phi(t), \dot{\phi}(t))}{J_1 \sqrt{\int_0^T dt' [J_2(t' \dot{\phi}(t'))]^2}},$$

(16)

where

$$Q(t, \phi(t), \dot{\phi}(t)) = \frac{1}{2\tau^2} \int_0^T dt' \Theta(t - t') \dot{\phi}(t') \left( \frac{d \Phi_{\text{odd/even}}(t-t')}{dt} \cos(\sqrt{2} \phi(t)) \cos(\sqrt{2} \dot{\phi}(t')) + \frac{\Phi_{\text{even}}(t-t')}{dt} \right)$$

(17)

+ $\dot{\phi}(t) \left( \Phi_{\text{odd}}(0) \cos^2(\sqrt{2} \phi(t)) + \Phi_{\text{even}}(0) \right)$.

Eq. (16) should satisfy the boundary conditions $\phi(0) = 0$ and $\phi(t = T) = \frac{\pi}{\sqrt{2}}$ to ensure the required state transfer.

### C. Optimizing the modulation control $\alpha(t)$ for a Markovian Bath

For a Markovian bath, the infidelity function (11) to be minimized becomes

$$\zeta(T) = \Re \int_0^T dt \frac{\dot{\phi}^2(t)}{J_2^T} (\Phi_{\text{odd}}(0) \cos^2(\sqrt{2} \phi(t)) + \Phi_{\text{even}}(0)).$$

(18)
Under the Euler-Lagrange method with the energy constraint, the differential equation is obtained
\[
\ddot{\phi}(t) \left( \Phi_{\text{odd}}(0) \cos(\sqrt{2} \phi(t)) + \Phi_{\text{even}}(0) - 2\lambda \tilde{J}_z^2 \right) \\
-\gamma \phi^2(t) \Phi_{\text{odd}}(0) \cos(\sqrt{2} \phi(t)) \sin(\sqrt{2} \phi(t)) = 0.
\]  
This equation has a non-trivial analytical solution and the modulation which minimize $\zeta(T)$ is given by the transcendental equation
\[
T \int_0^\phi(t) \sqrt{\cos(2\gamma \varphi) \Phi_{\text{odd}}(0) k_{\text{odd}} + \Phi_{\text{odd}}(0) + 2\Phi_{\text{even}}(0) - 2\lambda \tilde{J}_z^2} d\varphi \\
-t \int_0^{\phi(T)} \sqrt{2(\Phi_{\text{odd}}(0) \cos^2(\gamma \varphi) + \Phi_{\text{even}}(0) - \lambda \tilde{J}_z^2)} d\varphi = 0.
\]
Without constraint ($\lambda = 0$), the optimal modulation can be approximated by $\alpha(t) \approx \alpha_M(\alpha_M(a + b \sin p(T)))$, where $p \approx 3.5$, $\frac{a}{b} \approx \frac{1}{3}$ and $\alpha_M = \max \alpha(t)$. The infidelity for this optimal modulation almost coincides with the one obtained without modulation $1 - F(T) \approx \frac{\pi^2 N}{6\sqrt{2}J T} (1 - \frac{\pi^2 N}{16\sqrt{2}J T})$ with $T \approx \frac{\pi \sqrt{N}}{2\alpha_M J}$, and they only differ by about 0.1%.

IV. IMPLEMENTATION

The boundary coupling strengths can be engineered as a function of time, using Trotter-Susuki decompositions, by a sequence of $\pi$-pulses applied only at the boundary spins at suitable times; for example, by a series of cycles of duration $\tau_c \ll J$ that contain two $\pi$-pulses separated as $\tau_1^j - \pi - \tau_2^j - \pi - \tau_3^j$, where $\tau_1^j + \tau_2^j + \tau_3^j = \tau_c$ and $j$ represents the cycle number. In this way, the modulation control $\alpha(t)$ (in Eq. (1) from main text) for $t = j\tau_c$ will be given by $\alpha(j\tau_c) = (\tau_1^j - \tau_2^j + \tau_3^j)/\tau_c$. One can modulate $0 \leq \alpha(j\tau_c) = (\tau_1^j - \tau_2^j + \tau_3^j)/\tau_c \leq 1$ as a function of the time $j\tau_c$ as needed, by choosing appropriate values of $\tau_i^j$ at every cycle $j$.

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