RESTRICTIONS OF ASPHERICAL ARRANGEMENTS

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Abstract. In this note we present examples of $K(\pi, 1)$-arrangements which admit a restriction which fails to be $K(\pi, 1)$. This shows that asphericity is not hereditary among hyperplane arrangements.

1. Introduction

We say that a complex $\ell$-arrangement $\mathcal{A}$ is a $K(\pi, 1)$-arrangement, or that $\mathcal{A}$ is $K(\pi, 1)$ for short, provided the complement $M(\mathcal{A})$ of the union of the hyperplanes in $\mathcal{A}$ in $\mathbb{C}^\ell$ is aspherical, i.e. is a $K(\pi, 1)$-space. That is, the universal covering space of $M(\mathcal{A})$ is contractible and the fundamental group $\pi_1(M(\mathcal{A}))$ of $M(\mathcal{A})$ is isomorphic to the group $\pi$.

By seminal work of Deligne [Del72] (the complexification of) a simplicial real arrangement is $K(\pi, 1)$. Another important class of arrangements which share this topological property are arrangements of fiber type [FR85]. For such an arrangement $\mathcal{A}$ the complement $M(\mathcal{A})$ is described as a tower of successive locally trivial linear fibrations with aspherical fibers and aspherical bases. A repeated application of the long exact sequence in homotopy theory then gives that such $\mathcal{A}$ are $K(\pi, 1)$ (cf. [OT92, Prop. 5.12]). By fundamental work of Terao [Ter86], this property in turn is equivalent to supersolvability of $\mathcal{A}$ (cf. [OT92, Thm. 5.113]).

A particularly important and prominent class of arrangements for which this property is known to hold is the class of reflection arrangements stemming from complex reflection groups. In 1962, Fadell and Neuwirth [FN62] proved that the complexified braid arrangement of the symmetric group is $K(\pi, 1)$. Brieskorn [Br73] extended this result to many finite Coxeter groups and conjectured that this is the case for every finite Coxeter group. As the latter are simplicial, this follows from Deligne’s seminal work [Del72]. Nakamura proved asphericity for the imprimitive complex reflection groups, constructing explicit locally trivial fibrations [Nak83]. Utilizing their approach via Shephard groups, Orlik and Solomon succeeded in showing that the reflection arrangements stemming from the remaining irreducible complex reflection groups admit complements which are $K(\pi, 1)$-spaces with the possible exception of just six cases [OS88]; see also [OT92, §6]. These remaining instances were settled by Bessis in his brilliant proof employing Garside categories [Be15].

Because restrictions of simplicial (resp. supersolvable) arrangements are again simplicial (resp. supersolvable), the $K(\pi, 1)$-property of these kinds of arrangements is inherited by their restrictions.

In contrast, as the restriction of a reflection arrangement need not be a reflection arrangement again, the question whether asphericity passes to restrictions of reflection arrangements is more subtle. Nevertheless, it turns out that restrictions of reflection arrangements

\begin{flushleft}
2010 Mathematics Subject Classification. Primary 52B30, 55P20, 52C35.
Key words and phrases. $K(\pi, 1)$ arrangement, restriction of an arrangement.
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are indeed again $K(\pi, 1)$ with only 11 possible exceptions, see [AR18]. Conjecturally, these remaining instances are also $K(\pi, 1)$.

Nevertheless, the impression of a hereditary behavior of asphericity alluded to by these particular classes is deceptive. In this note we present examples of $K(\pi, 1)$-arrangements which admit non-$K(\pi, 1)$ restrictions, giving the following.

**Theorem 1.1.** The class of $K(\pi, 1)$-arrangements is not closed under taking restrictions.

Specifically, we construct infinite families of $K(\pi, 1)$-subarrangements of Coxeter arrangements of type $D_n$ for any $n \geq 4$ each of which admits a restriction that fails to be $K(\pi, 1)$, see Lemma 3.1 and Example 3.3. Our smallest example of this kind is a rank 4 subarrangement of the Coxeter arrangement of type $D_4$ consisting of just 10 hyperplanes, see Example 3.2. According to our knowledge, this is the first instance of the description of such examples in the literature. Likely, this is not a particularly rare phenomenon.

It is remarkable that this phenomenon appears naturally among canonical subarrangements of reflection arrangements. This shows quite dramatically, while Coxeter arrangements themselves are well understood, their subarrangements still hold some unexpected surprises.

In contrast to the situation with restrictions, any localization of a $K(\pi, 1)$-arrangement is known to be $K(\pi, 1)$ again, by an observation due to Oka, e.g., see [Pa93].

2. Preliminaries

2.1. Hyperplane arrangements. Let $V = \mathbb{C}^\ell$ be an $\ell$-dimensional $\mathbb{C}$-vector space. A hyperplane arrangement $\mathcal{A} = (\mathcal{A}, V)$ in $V$ is a finite collection of hyperplanes in $V$. We also use the term $\ell$-arrangement for $\mathcal{A}$ to indicate the dimension of the ambient space $V$. If the linear equations describing all members of $\mathcal{A}$ are real, then we say that $\mathcal{A}$ is real.

The lattice $L(\mathcal{A})$ of $\mathcal{A}$ is the set of subspaces of $V$ of the form $H_1 \cap \ldots \cap H_i$ where $\{H_1, \ldots, H_i\}$ is a subset of $\mathcal{A}$. The lattice $L(\mathcal{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathcal{A})$.

For $X \in L(\mathcal{A})$, we have two associated arrangements, firstly $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \subseteq \mathcal{A}$, the localization of $\mathcal{A}$ at $X$, and secondly, the restriction of $\mathcal{A}$ to $X$, $(\mathcal{A}^X, X)$, where $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$.

If $0 \in H$ for each $H$ in $\mathcal{A}$, then $\mathcal{A}$ is called central. If $\mathcal{A}$ is central, then the center $\cap_{H \in \mathcal{A}} H$ of $\mathcal{A}$ is the unique maximal element in $L(\mathcal{A})$ with respect to the partial order. We have a rank function on $L(\mathcal{A})$: $r(X) := \text{codim}_V(X)$. The rank $r := r(\mathcal{A})$ of $\mathcal{A}$ is the rank of a maximal element in $L(\mathcal{A})$. Throughout this article, we only consider central arrangements. For $\mathcal{A}$ central, for each $H$ in $\mathcal{A}$ let $\alpha_H$ be a linear form in $V^*$ so that $H = \ker \alpha_H$. Then $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is the defining polynomial of $\mathcal{A}$.

It is easy to see that a central arrangement of rank at most 2 is $K(\pi, 1)$ ([OT92, Prop. 5.6]).

This topological property is not generic among all arrangements. A generic complex $\ell$-arrangement $\mathcal{A}$ is an $\ell$-arrangement with at least $\ell + 1$ hyperplanes and the property that the hyperplanes of every subarrangement $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \ell$ are linearly independent. It follows from work of Hattori [Ha75] that, for $\ell \geq 3$, generic arrangements are never $K(\pi, 1)$ (cf. [OT92, Cor. 5.23]).

**Lemma 2.1.** The real 3-arrangement $\mathcal{A}$ given by $Q(\mathcal{A}) = y(x - y)(x^2 - z^2)(y^2 - z^2)$ is not $K(\pi, 1)$. 

Proof. For \( t \in \mathbb{C} \), let \( \mathcal{A}(t) \) be the 3-arrangement given by \( Q(\mathcal{A}(t)) = xyz(x+y)(x+z)(y+tz) \). Using the transformation \((x, y, z) \mapsto (x - y, y - z, y + z)\), it is easy to see that \( \mathcal{A} \) and \( \mathcal{A}(1) \) are linearly isomorphic, so their complements are diffeomorphic. For \( t \in \mathbb{C} \setminus \{0, -1\} \) the arrangements \( \mathcal{A}(t) \) are all combinatorially isomorphic. Moreover, for real negative \( t \) they satisfy the “simple triangle” condition of Falk and Randell, cf. [FR87, Cor. 3.3, (3.12)], so \( \mathcal{A}(t) \) is not \( K(\pi, 1) \) for \( t < 0 \).

For completeness we show that \( \mathcal{A}(1) \) and \( \mathcal{A}(-2) \) have diffeomorphic complements, which shows that \( \mathcal{A}(1) \cong \mathcal{A} \) is not \( K(\pi, 1) \). For \( t \in \mathbb{C} \) define the 1-parameter family \( \mathcal{B}_t = \mathcal{A}(\frac{3}{2}e^{\pi it} - \frac{1}{2}) \). Then for every \( t \in \mathbb{C} \), \( \mathcal{B}_t \) admits the same lattice, so \( \mathcal{B}_t \) is a lattice isotopy, see [Ra89]. It follows from Randell’s isotopy theorem [Ra89] that \( \mathcal{B}_0 = \mathcal{A}(1) \) and \( \mathcal{B}_1 = \mathcal{A}(-2) \) have diffeomorphic complements. In Figure 1 we show a projective picture of the real arrangement \( \mathcal{A}(-2) \) with the simple triangle shaded in gray.

Note that the arrangement \( X_3 \) considered by Falk and Randell in [FR87, (2.6)] has the same lattice as \( \mathcal{A} \).

\[ \square \]

**Figure 1.** The real arrangement \( \mathcal{A}(-2) \).

### 2.2. Arrangements of ideal type.

Our examples which imply Theorem 1.1 stem from a particular class of subarrangements of Coxeter arrangements which we describe very briefly.

Let \( \Phi \) be an irreducible, reduced root system and let \( \Phi^+ \) be the set of positive roots with respect to some set of simple roots \( \Pi \). An \textit{(upper) order ideal}, or simply \textit{ideal} for short, of \( \Phi^+ \), is a subset \( \mathcal{I} \) of \( \Phi^+ \) satisfying the following condition: if \( \alpha \in \mathcal{I} \) and \( \beta \in \Phi^+ \) so that \( \alpha + \beta \in \Phi^+ \), then \( \alpha + \beta \in \mathcal{I} \).

Recall the standard partial ordering \( \preceq \) on \( \Phi \): \( \alpha \preceq \beta \) provided \( \beta - \alpha \) is a \( \mathbb{Z}_{\geq 0} \)-linear combination of positive roots, or \( \beta = \alpha \). Then \( \mathcal{I} \) is an ideal in \( \Phi^+ \) if and only if whenever \( \alpha \in \mathcal{I} \) and \( \beta \in \Phi^+ \) so that \( \alpha \preceq \beta \), then \( \beta \in \mathcal{I} \). The \textit{generators} of a given ideal \( \mathcal{I} \) are simply the elements in \( \mathcal{I} \) which are minimal with respect to \( \preceq \).

Let \( \mathcal{I} \subseteq \Phi^+ \) be an ideal and let \( \mathcal{I}^c := \Phi^+ \setminus \mathcal{I} \) be its complement in \( \Phi^+ \). Let \( \mathcal{A}(\Phi) \) be the \textit{Weyl arrangement} of \( \Phi \), i.e., \( \mathcal{A}(\Phi) = \{ H_\alpha \mid \alpha \in \Phi^+ \} \), where \( H_\alpha \) is the hyperplane in the...
Euclidean space $V = \mathbb{R} \otimes \mathbb{Z} \Phi$ orthogonal to the root $\alpha$. Following [ST06, §11], we associate with an ideal $\mathcal{I}$ in $\Phi^+$ the arrangement consisting of all hyperplanes with respect to the roots in $\mathcal{I}^c$. The arrangement of ideal type associated with $\mathcal{I}$ is the subarrangement $\mathcal{A}_\mathcal{I}$ of $\mathcal{A}(\Phi)$ defined by

$$\mathcal{A}_\mathcal{I} := \{H_\alpha \mid \alpha \in \mathcal{I}^c\}.$$  

Note that if $\mathcal{I} = \emptyset$, then $\mathcal{A}_\mathcal{I} = \mathcal{A}(\Phi)$ is just the reflection arrangement of $\Phi$ and so $\mathcal{A}_\emptyset$ is $K(\pi, 1)$ by Deligne’s result. It is shown in [AR19, Thm. 1.4] that all arrangements of ideal type $\mathcal{A}_\mathcal{I}$ are $K(\pi, 1)$ provided the underlying root system is classical. This is also known for most $\mathcal{A}_\mathcal{I}$ for root systems of exceptional type, [AR19, Thm. 1.3(iii)] and is conjectured to hold for all $\mathcal{A}_\mathcal{I}$.

### 3. Proof of Theorem 1.1

In this section we label the positive roots in a root system of type $D_n$ as in [Bou68, Planche IV]. In view of [AR19, Thm. 1.4], Theorem 1.1 is a consequence of the following result.

**Lemma 3.1.** Let $\Phi$ be the root system of type $D_n$ for $n \geq 4$. We consider two kinds of ideal arrangements $\mathcal{A}_\mathcal{I}$ by listing the generators of the corresponding ideals $\mathcal{I}$:

(i) $1 \ldots 1^4 = e_1 + e_{n-1};$

(ii) $1 \ldots 1^4 = e_1 + e_{n-1}, 0 \ldots 0 \ldots 12 \ldots 2^4 = e_s + e_t$, where $1 < s < t < n - 1$. Here $s$ is the first position with a coefficient 1 and $t$ is the first position labeled with 2.

Consider

$$Y := \bigcap_{2 \leq i < j \leq n-1} \ker(x_i - x_j)$$

in $L(\mathcal{A}_\mathcal{I})$. Then the rank 3 restriction $\mathcal{A}_\mathcal{Y}$ is not $K(\pi, 1)$.

**Proof.** Let $\mathcal{D}_n$ be the reflection arrangement of $D_n$, i.e.

$$\mathcal{D}_n = \{\ker(x_i \pm x_j) \mid 1 \leq i < j \leq n\}.$$  

If $\mathcal{I}$ is of type (i), then $\mathcal{A}_\mathcal{I}$ is the arrangement

$$\mathcal{A}_\mathcal{I} = \mathcal{D}_n \setminus \{\ker(x_1 + x_i) \mid 2 \leq i \leq n - 1\},$$

and if $\mathcal{I}$ is of type (ii), then we have

$$\mathcal{A}_\mathcal{I} = \mathcal{D}_n \setminus \{\ker(x_1 + x_i), \ker(x_j + x_k) \mid 2 \leq i \leq n - 1, 2 \leq j \leq s < k \leq t\}.$$  

In both cases, restricting to $Y$ we get the rank 3 arrangement $\mathcal{A}_\mathcal{Y}$ with defining polynomial

$$Q(\mathcal{A}_\mathcal{Y}) = (x_1 - x_2)x_2(x_1^2 - x_n^2)(x_2^2 - x_n^2) \in \mathbb{C}[x_1, x_2, x_n],$$

and thus by Lemma 2.1, the restriction is not $K(\pi, 1)$.

The following example illustrates the smallest instance from Lemma 3.1(i).

**Example 3.2.** Let $\mathcal{I}$ be the ideal in the set of positive roots in the root system of type $D_4$ generated by $1^1_1 = e_1 + e_3$, the unique root of height 4 and let $\mathcal{A}_\mathcal{I}$ be the corresponding arrangement. It is obtained from the full Weyl arrangement of type $D_4$ by removing the hyperplanes corresponding to the two highest roots, so that $\mathcal{A}_\mathcal{I}$ has defining polynomial

$$Q(\mathcal{A}_\mathcal{I}) = (x_1 - x_2)(x_1 - x_3)(x_2^2 - x_3^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)(x_3^2 - x_4^2).$$
Consider the restriction of $\mathcal{A}_I$ to the hyperplane $H := \ker(x_2 - x_3)$. Then $\mathcal{A}_I^H$ has defining polynomial $y(x - y)(x^2 - z^2)(y^2 - z^2)$.

Remarkably, this is the smallest instance among the class of arrangements of ideal type which is neither supersolvable nor simplicial. So this phenomenon of a non-$K(\pi, 1)$ restriction appears among the family of arrangements of ideal type already for the smallest instance where this is possible.

The cases from Lemma 3.1 lead to additional instances of $K(\pi, 1)$-arrangements with non-$K(\pi, 1)$ restrictions by means of localizations, as illustrated by our next example.

**Example 3.3.** Let $\Phi$ be of type $D_n$ and let $\mathcal{I}$ be an ideal in $\Phi^+$ which admits the root $0 \ldots 01 \ldots 11_1 = e_r + e_{n-1}$ as a generator, where $1 < r \leq n - 3$. Let $\mathcal{B} := \mathcal{A}_I$ be the corresponding arrangement of ideal type. Let $\Phi_0$ be the standard subsystem of type $D_n$, where $m := n - r + 1$, and let $X := \bigcap_{\gamma \in \Phi_0^+} H_\gamma$. Then $X$ belongs to $L(\mathcal{B})$ and the localization $\mathcal{B}_X$ is isomorphic to one of the arrangements of ideal type in type $D_n$ considered in Lemma 3.1 above. Consequently, choosing the member $Y$ of the lattice of $\mathcal{B}$ corresponding to the one used in the proof of Lemma 3.1, we see that $(\mathcal{B}_X)^Y$ is isomorphic to the non-$K(\pi, 1)$ restriction from Lemma 3.1. Since $(\mathcal{B}_X)^Y \cong (\mathcal{B}_X)^Y$, it follows from Oka’s observation that also $\mathcal{B}^Y$ is not $K(\pi, 1)$. However, $\mathcal{B}$ itself is $K(\pi, 1)$, thanks to [AR19, Thm. 1.4].

The following example illustrates all instances of an arrangement of ideal type $\mathcal{A}_I$ for a root system of type $D_5$ when there is a rank 3 restriction which admits a “simple triangle” and is thus not $K(\pi, 1)$. All of them stem from the constructions outlined in Lemma 3.1 and Example 3.3. Computations for higher rank instances suggest that this is always the case.

**Example 3.4.** Let $\Phi$ be of type $D_5$ and let $\mathcal{I}$ be an ideal in $\Phi^+$. Then $\mathcal{A}_I$ has a rank 3 restriction which admits a “simple triangle” and is thus not $K(\pi, 1)$ if and only if $\mathcal{I}$ is one of the following ideals (we again just list the generators of $\mathcal{I}$):

(i) $111_1^1$ (v) $011_1^1, 110_0^0$
(ii) $111_1^1, 012_1^1$ (vi) $011_1^1, 111_0^0$
(iii) $011_1^1$ (vii) $011_1^1, 111_0^0$
(iv) $011_1^1, 100_0^0$ (viii) $011_1^1, 111_1^1, 111_0^1$

Here the cases (i) and (ii) stem from Lemma 3.1(i) and (ii) and the cases (iii) - (viii) are covered in Example 3.3.

While the arrangement of ideal type in our following example stems from the one considered in Example 3.2 by merely adding a single hyperplane, the topological properties of the restrictions of both arrangements differ sharply.

**Example 3.5.** Let $\mathcal{I}$ be the ideal in the set of positive roots in the root system of type $D_4$ generated by the highest root $121_1^1 = e_1 + e_2$ and let $\mathcal{A}_I$ be the corresponding arrangement, i.e. $\mathcal{A}_I$ is obtained from the full Weyl arrangement of $D_4$ by removing the hyperplane corresponding to the highest root. One checks that $\mathcal{A}_I$ is neither supersolvable nor simplicial. It turns out that every restriction of $\mathcal{A}_I^H$ to a hyperplane $H$ is still factored, see [Ter92]. It thus follows from [Pa95] that each $\mathcal{A}_I^H$ is still $K(\pi, 1)$.

**Remark 3.6.** Note that while all $\mathcal{A}_I$ are free, see [ST06, Th. 11.1] and [ABC+16, Thm. 1.1], the particular restrictions $\mathcal{A}_I^Y$ considered in Lemma 3.1 and in Example 3.3 are not free, see
[FR87, (3.12)]. Consequently, arrangements of ideal type are not hereditarily free. Nevertheless, all $\mathcal{A}$ are actually inductively free, see [Rö17] and [CRS17]. In contrast, the ambient Weyl arrangement $\mathcal{A}(\Phi)$ itself is hereditarily free, see [OT93]. In [Dou99], Douglass gave a uniform proof of this fact using an elegant conceptual Lie theoretic argument.

Acknowledgments: The research of this work was supported by DFG-grant RO 1072/16-1.

References

[ABC+16] T. Abe, M. Barakat, M. Cuntz, T. Hoge, and H. Terao, The freeness of ideal subarrangements of Weyl arrangements, JEMS, 18 (2016), no. 6, 1339–1348.

[AR18] N. Amend and G. Röhrle, The $K(\pi, 1)$-problem for restrictions of complex reflection arrangements, https://arxiv.org/abs/1708.05452, preprint 2017.

[AR19] ______, The topology of arrangements of ideal type, https://arxiv.org/abs/1801.07157, preprint 2018.

[Be15] D. Bessis, Finite complex reflection arrangements are $K(\pi, 1)$, Annals of Math. (2), 181 (2015), no. 3, 809–904.

[Br73] E. Brieskorn, Sur les groupes de tresses [d’après V. I. Arnold]. In Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pages 21–44. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

[Bou68] N. Bourbaki, Éléments de mathématique. Groupes et algèbres de Lie. Chapitre IV-VI, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[CRS17] M. Cuntz, G. Röhrle, and A. Schauenburg, Arrangements of ideal type are inductively free, https://arxiv.org/abs/1711.09760

[Del72] P. Deligne, Les immeubles des groupes de tresses généralisés, Invent. Math. 17 (1972), 273-302.

[Dou99] J. M. Douglass, The adjoint representation of a reductive group and hyperplane arrangements, Represent. Theory 3 (1999), 444–456.

[FN62] E. Fadell and L. Neuwirth, Configuration spaces, Math. Scand. 10 (1962) 111–118.

[FR85] M. Falk and R. Randell, The lower central series of a fiber-type arrangement, Invent. Math. 82 (1985), no. 1, 77–88.

[FR87] ______, On the homotopy theory of arrangements, Complex analytic singularities, 101–124, Adv. Stud. Pure Math., 8, North-Holland, Amsterdam, 1987.

[Ha75] A. Hattori, Topology of $\mathbb{C}^n$ minus a finite number of affine hyperplanes in general position, J. Fac. Sci. Univ. Tokyo 22 (1975) 205–219.

[Nak83] T. Nakamura, A note on the $K(\pi, 1)$-property of the orbit space of the unitary reflection group $G(m, l, n)$, Sci. Papers College Arts Sci. Univ. Tokyo 33 (1983), no. 1, 1–6.

[OS82] P. Orlik and L. Solomon, Arrangements defined by unitary reflection groups, Math. Ann. 261, (1982), 339–357.

[OS88] ______, Discriminants in the invariant theory of reflection groups, Nagoya Math. J. 109 (1988), 23–45.

[OT92] P. Orlik and H. Terao, Arrangements of hyperplanes, Springer-Verlag, 1992.

[OT93] ______, Coxeter arrangements are hereditarily free, Tôhoku Math. J. 45 (1993), 369–383.

[Pa93] L. Paris, The Deligne complex of a real arrangement of hyperplanes, Nagoya Math. J. 131 (1993), 39–65.

[Pa95] ______, Topology of factored arrangements of lines. Proc. Amer. Math. Soc. 123 (1995), no. 1, 7–261.

[Ra89] R. Randell, Lattice-isotopic Arrangements Are Topologically Isomorphic, Proc. Amer. Math. Soc. 107 (1989), no. 2, 555–559.

[Rö17] G. Röhrle, Arrangements of ideal type, J. Algebra, 484, (2017), 126–167.
[ST54] G. C. Shephard and J. A. Todd, Finite unitary reflection groups. Canadian J. Math. 6, (1954), 274–304.

[ST06] E. Sommers and J. Tymoczko, Exponents for B-stable ideals. Trans. Amer. Math. Soc. 358 (2006), no. 8, 3493–3509.

[Ter86] H. Terao, Modular elements of lattices and topological fibrations, Adv. in Math. 62 (1986), no. 2, 135–154.

[Ter92] , Factorizations of the Orlik-Solomon Algebras, Adv. in Math. 92, (1992), 45–53.

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