Correlated spontaneous symmetry breaking induced by zero-point fluctuations in a quantum mixture

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We propose a form of spontaneous symmetry breaking driven by zero-point quantum fluctuations. To be specific, we consider the low-energy dynamics of a mixture of two species of spin-1 Bose gases. It is demonstrated that the quantum fluctuations lift a degeneracy regarding the relative orientations of the spin directors of the two species, and result in correlation or locking between these macroscopic variables. This locking persists in the presence of the trapping potential and weak magnetic fields, allowing, in principle, an experimental probe of this correlated spontaneous symmetry breaking, as a macroscopic manifestation of zero-point quantum fluctuations.

PACS numbers: 03.75.Mn, 03.75.Kk, 67.85.Fg

I. INTRODUCTION

Spontaneous symmetry breaking is a central concept in all of physics [1]. The quantum collective phenomena induced by zero-point motion of dynamical variables are also ubiquitous [1,2] and the dynamics governed by the quantum fluctuations of the collective degrees of freedom are extremely important for our understanding of macroscopic emergent phenomena [3]. In this paper we demonstrate a different type of spontaneous symmetry breaking, namely, one in which the macroscopic variables of two interacting many-body systems are locked together through microscopic zero-point quantum fluctuations. We frame our theory in the context of spinor Bose gases [7–10], where quantum fluctuations can be dominating in energetics or dynamics and are highly controllable [11–15]. Specifically, we consider a mixture of two spinor Bose gases with interspecies spin exchanges [16–19] and show that in the symmetry-breaking ground state, the spin directors of the two gases are locked together by the zero-point quantum fluctuations. The spin director is the spin direction modulo $Z_2$ symmetry due to the compensation of the inversion of the spin direction by a $\pi$ phase transformation [20]. Our results give more motivation to study interspecies spin exchanges in mixtures of different spinor Bose gases [16–19], which are under experimental investigation [21].

This article is organized as follows. In Sec. II we define the many-body Hamiltonian, and discuss the mean-field theory. In Sec. III we consider quantum fluctuations and find the energies in the Bogoliubov theory. In Sec. IV we investigate the fluctuation-induced locking between the directors of the two species. The fluctuation-induced spin dynamics is discussed in Sec. V. In Sec. VI we discuss how the locking effect survives a trapping potential or an external magnetic field. Finally, we summarize our investigation in Sec. VII.

II. HAMILTONIAN AND MEAN FIELD THEORY

The many-body Hamiltonian we apply to study this problem is

$$\mathcal{H} = \sum_{\alpha=a,b} \mathcal{H}_\alpha + \mathcal{H}_{ab},$$ (1)

$$\mathcal{H}_\alpha = \int d\mathbf{r} \psi_\alpha^\dagger h_\alpha(\mathbf{r})_{\mu\nu} \psi_\alpha + \frac{i}{2} \int d\mathbf{r} \psi_\alpha^\dagger \psi_\alpha (c_{0}^2\delta_{\mu\nu}\delta_{\rho\sigma} + c_{2}^2 \mathbf{F}_{\alpha\mu\nu} \cdot \mathbf{F}_{\alpha\rho\sigma} ) \psi_\alpha \psi_\alpha$$

is the Hamiltonian of species $\alpha$. Here $h_\alpha = -\frac{\hbar^2}{2m_\alpha} \nabla^2 + V_\alpha(\mathbf{r})$ is the spin-independent single-particle Hamiltonian of each atom of species $\alpha$, $c_{0}^2$ is the intraspecies density-density interaction strength of species $\alpha$, $c_{2}^2$ is the intraspecies spin-exchange interaction strength of species $\alpha$, and $\psi_\alpha$ represents the field operator for the $\mu$ component of species $\alpha$, with $\mu = a, b$ and $\mu = -1, 0, 1$ or $x, y, z$, depending on the basis used. For the case of $\mu = x, y, z$, the $\eta$ component $F_{\mu\nu}^\eta$ of $\mathbf{F}_{\mu\nu}$ is $-i\epsilon_{\mu\nu\eta}$, where $\epsilon_{\mu\nu\eta}$ is the Levi-Civita antisymmetric tensor [22]. In addition,

$$\mathcal{H}_{ab} = \int d\mathbf{r} \psi_a^\dagger \psi_b^\dagger (c_{0b}^a\delta_{\mu\nu}\delta_{\rho\sigma} + c_{2b}^a \mathbf{F}_{a\mu\nu} \cdot \mathbf{F}_{b\rho\sigma} ) \psi_b \psi_a$$

is the interaction between the two species, with $c_{0b}^a$ the interspecies density-density interaction strength and $c_{2b}^a$ the interspecies spin-exchange interaction strength. Repeated indices are summed over. We focus on the regime of $c_{0b}^a > 0$, $c_{2b}^a > 0$ and $c_{2b}^a > 0$ and assume that $c_{2b}^a > (c_{0b}^a)^2$.

The three-component vector field $\psi_\alpha$ of the species $\alpha$ can be written as

$$\psi_\alpha = \Phi_\alpha \mathbf{n}_\alpha,$$ (2)
where \( \mathbf{n}_a \) is the spin director, which is also a three-component vector, and

\[
\Phi_\alpha \equiv \sqrt{\rho_\alpha} e^{i \chi_\alpha},
\]

where

\[
\rho_\alpha = \psi^\dagger_{\alpha \mu} \psi_{\alpha \mu}
\]
is the number-density operator. In terms of \( \rho_\alpha \) and the spin-density operator

\[
L_\alpha = \psi^\dagger_{\alpha \mu} F_{\alpha \mu \nu} \psi_{\alpha \nu},
\]

the Hamiltonian can be rewritten as

\[
\mathcal{H} = \mathcal{H}_p + \mathcal{H}_s,
\]

where

\[
\mathcal{H}_p = \sum_{\alpha=a,b} \frac{1}{2} \int d^3 \mathbf{r} \left[ \frac{1}{m_\alpha} |\nabla \Phi_\alpha (\mathbf{r})|^2 + V_\alpha (\mathbf{r}) \rho_\alpha (\mathbf{r}) + \frac{1}{2} \epsilon_0^2 \rho_\alpha (\mathbf{r})^2 \right]
\]
is the phase part,

\[
\mathcal{H}_s = \sum_{\alpha=a,b} \frac{1}{2} \int d^3 \mathbf{r} \left[ \frac{\epsilon_0^2}{m_\alpha} \nabla \Phi_\alpha (\mathbf{r})^2 + c_0^2 L_\alpha^2 (\mathbf{r}) \right]
\]
is the spin part, and a spin-phase coupling part is negligible in the long-wavelength limit or when \( \mathbf{L}_\alpha = 0 \). Hence the phase and spin degrees of freedom are decoupled and are described by the collective variables \( \{ \rho_\alpha (\mathbf{r}), \chi_\alpha (\mathbf{r}) \} \) and \( \{ \mathbf{n}_\alpha (\mathbf{r}), \mathbf{L}_\alpha (\mathbf{r}) \} \), respectively. Here \( \mathcal{H}_p \) simply describes a mixture of two scalar Bose gases, and is independent of the relative orientation of \( \mathbf{n}_a \) and \( \mathbf{n}_b \). Henceforth we focus on the spin part \( \mathcal{H}_s \).

First consider the uniform case \( V_\alpha = V_b = 0 \). In the ground state, \( \rho_\alpha \) and \( \rho_b \) are both constants and the total energy

\[
E = \mathcal{V} \left[ \frac{1}{4} c_0^2 L_a^2 + \frac{1}{4} c_0^2 L_b^2 + \frac{1}{2} \epsilon_0^2 \rho_a^2 \right. + \frac{1}{2} \epsilon_0^2 \rho_b^2 + \left. \frac{1}{2} \epsilon_0^2 \rho_\alpha \rho_b \right],
\]

where \( \mathcal{V} \) is the volume of the system. Minimization of \( E \) implies that in the ground state, \( \Phi_a \), \( \Phi_b \), \( \mathbf{n}_a \) and \( \mathbf{n}_b \) are all position independent, while \( \mathbf{L}_a = \mathbf{L}_b = 0 \), implying that the total spin is also 0. The mean-field ground state is \( |N_a, n_a \rangle \otimes |N_b, n_b \rangle \), where each species is in its own spin nematic state uncorrelated with the other species. Here \( |N_a, n_a \rangle \) denotes the state in which the spin director of each atom of species \( \alpha \) is aligned along the direction of \( \mathbf{n}_\alpha \). There are no constraints on \( \mathbf{n}_a \) and \( \mathbf{n}_b \), hence the ground state manifold becomes \( S^2 \otimes S^2 \), which possesses a huge degeneracy.

III. QUANTUM FLUCTUATIONS

In the following we show that this degeneracy is lifted by zero-point quantum fluctuations. Denoted by \( n_a^0 \) and \( n_b^0 \), the spin directors of the two species in a mean-field symmetry-breaking ground state satisfy the relation \( n_a^0 \cdot n_b^0 = \cos \theta \). Let us arbitrarily set \( n_a^0 = \mathbf{e}_x \), \( n_b^0 = \mathbf{e}_{x'} \) and \( \mathbf{e}_z \cdot \mathbf{e}_{z'} = \cos \theta \), as depicted in Fig. 1. The degeneracy implies that \( \theta \) is arbitrary. The range of \( \theta \) is limited to \(-\pi/2 \leq \theta < \pi/2 \) because of \( Z_2 \) symmetry.

Consider

\[
\mathbf{n}_\alpha = \mathbf{n}_\alpha^0 + \mathbf{u}_\alpha,
\]

for \( \alpha = a, b \), where quantum fluctuations are

\[
\mathbf{u}_a \approx \frac{\psi_{a x}}{\sqrt{\rho_a}} \mathbf{e}_x + \frac{\psi_{a y}}{\sqrt{\rho_a}} \mathbf{e}_y,
\]

\[
\mathbf{u}_b \approx \frac{\psi_{b x}}{\sqrt{\rho_b}} \mathbf{e}_{x'} + \frac{\psi_{b y}}{\sqrt{\rho_b}} \mathbf{e}_{y'},
\]

with unit vectors \( \mathbf{e}_x \), \( \mathbf{e}_y \), and \( \mathbf{n}_a^0 \) forming a Cartesian coordinate system with \( \mathbf{e}_{x'} \), \( \mathbf{e}_{y'} \), and \( \mathbf{n}_b^0 \) forming another one, as shown in Fig. 1. We find that

\[
\mathcal{H}_s = \sum_{i=x,y} \mathcal{H}_{si},
\]

where

\[
\mathcal{H}_{si} = \sum_{\alpha, k} \frac{\mu_\alpha^2}{2 m_\alpha} \psi_{\alpha \text{i}, k}^\dagger \psi_{\alpha \text{i}, k} + \frac{1}{2} \epsilon_0^2 \rho_\alpha \sum_{\alpha, k} \left[ |\psi_{\alpha \text{i}, k}^\dagger|^2 - (\psi_{\alpha \text{i}, k}^\dagger \psi_{\alpha \text{i}, -k} + H.c.) \right] + \epsilon_2 \sqrt{\rho_\alpha \rho_b} \zeta_{i} (\theta)
\]

\[
\sum_k \left( \psi_{\alpha \text{i}, k}^\dagger \psi_{\beta \text{i}, k} - \psi_{\alpha \text{i}, k}^\dagger \psi_{\beta \text{i}, -k} + H.c. \right),
\]

written in terms of momentum \( \mathbf{k} \), with \( k \equiv |\mathbf{k}| \), \( \zeta_{i}(\theta) = \cos \theta \), and \( \zeta_{i y}(\theta) = 1 \). Here \( \mathcal{H}_{si} \) depends only on \( \psi_{\alpha \text{i}} \), \( \mathcal{H}_{xx} \) depends on \( \theta \), and \( \mathcal{H}_{xy} \) is independent of \( \theta \).
By performing a Bogoliubov transformation, one obtains
\[ H_{si} = \sum_{\lambda=\pm, k} \omega_{\lambda, k} (A_{\lambda, k}^\dagger A_{\lambda, k} + \frac{1}{2}), \]
where \( A_{\lambda, k} \) is some Bosonic operator and
\[ \omega_{\lambda, \pm, k} = \frac{1}{2} \left( c_{\lambda ak}^2 + c_{\lambda bk}^2 \pm \sqrt{(c_{\lambda ak}^2 + c_{\lambda bk}^2)^2 - 4E_{ak}E_{bk}(4g_ag_b - 4g_{ab}^2\cos^2 \theta - 2g_aE_{bk} + 2g_bE_{ak} + E_{ak}E_{bk})^{1/2}} \right), \]
with \( E_{ak} \equiv \frac{k^2}{2m_a}, c_{\lambda ak}^2 = E_{\lambda ak}^2 + 2g_aE_{bk} + 2g_bE_{ak} + E_{ak}E_{bk}^{1/2} \),
where \( \omega_{\lambda, \pm, k} \) depends on both \( g_{ab} \) and \( \theta \), while \( \omega_{\theta, \pm, k} \) depends on \( g_{ab} \) but is independent of \( \theta \).

IV. FLUCTUATION-INDUCED LOCKING

According to the spectra of \( H_{sx} \) and \( H_{sy} \) obtained above, we know that the quantum fluctuations lead to a \( \theta \)-dependent zero-point energy
\[ E_0(\theta) = E_0(\theta) + E_0(\theta = 0), \]
where on the righthand side, the first term \( E_0(\theta) \) is the zero-point energy of \( H_{sx} \), with
\[ E_0(\theta) = \frac{1}{2} \sum_k \Omega_k(\theta), \]
where
\[ \Omega_k(\theta) \equiv \omega_{\theta, +, k} + \omega_{\theta, -, k} \]
\[ = \left[ c_{\theta ak}^2 + c_{\theta bk}^2 + 2E_{\theta ak}^{1/2}E_{\theta bk}^{1/2} \left( 4g_ag_b - 4g_{ab}^2\cos^2 \theta + 2g_aE_{bk} + 2g_bE_{ak} + E_{ak}E_{bk} \right)^{1/2} \right]^{1/2}, \]
which reaches its minimum at \( \theta = 0 \). The second term \( E_0(\theta = 0) \) on the righthand side of (10), which is \( \theta \)-independent, is the zero-point energy of \( H_{sy} \).

Hence the total zero-point energy \( E_{0}(\theta) \) also reaches its minimum at \( \theta = 0 \). Therefore, in the ground state, \( \Omega_{\theta}^{a} \) and \( \Omega_{\theta}^{b} \) are actually locked in the low-energy limit, that is, they tend to align in the same direction. This is in contrast to what was suggested by the mean-field analysis.

For large \( k \), the \( \theta \)-dependent part of \( \Omega_k \) behaves as
\[ -\frac{g_{ab}^2 m_am_b}{k^2(m_a + m_b)} \cos^2 \theta + O\left( \frac{k^4}{k^6} \right), \]
thus the \( \theta \)-dependent part of the energy has ultraviolet divergence. This divergence originates from the use of contact interaction, which fails at short range or large momentum. It can be removed by introducing a momentum cutoff or by the renormalization of interaction strengths, that is, by evaluating the ground-state energy in terms of the renormalized quantities \( c_{0, r}^{ab} \) and \( c_{2, r}^{ab} \), which are directly related to the experimentally observed scattering lengths and correspond to the bare quantities \( c_{0, 0}^{ab} \) and \( c_{2, 0}^{ab} \) respectively.

It is known that \( c_{0, 0}^{ab} = 2U_{2, r} + U_0 \) and \( c_{2, 0}^{ab} = U_{2, r} - U_0 \), and \( U_0 \) and \( U_2 \) are the interaction strengths for the total spin \( F = 0 \) and \( 2 \) channels, respectively [17]. Accordingly \( c_{0, r}^{ab} = (2U_{2, r} + U_{0, r})/3 \) and \( c_{2, r}^{ab} = (U_{2, r} - U_{0, r})/3 \), with
\[ U_{F, r} = \lim_{k \to 0} \langle k', F|\hat{T}|k, F \rangle, \]
with \( k = |k| = |k'| \), given by the zero-energy \( \hat{T} \)-matrix element for two-body scattering [22]. The Lippmann-Schwinger equation reads
\[ \hat{T} = \hat{U}_F + \hat{U}_F G_0 \hat{T}, \]
with \( \hat{U}_F = U_F \delta (r) \) is the two-body potential and
\[ G_0(k) = \frac{-2M}{k^2}, \]
is the zero-energy Green’s function for the relative motion of atoms \( a \) and \( b \). Consequently, \( \frac{1}{U_{F, r}} = \frac{1}{U_F} + \int d^3k \frac{2M}{k^2} \).

Since the divergence occurs in the second order of interaction strengths, we expand the above formula to second order and obtain
\[ U_F = U_{F, r} + U_{F, r}^2 \int d^3k \frac{2M}{k^2}. \]
Therefore,
\[ c_{0, r}^{ab} = \frac{2U_{2, r} + U_0}{3} = \frac{2U_{2, r} + U_{0, r}}{3} + \frac{2U_{2, r}^2 + U_{0, r}^2}{3} \int d^3k \frac{2M}{k^2} = c_{0, r}^{ab} + [(c_{0, r}^{ab})^2 + 2(c_{2, r}^{ab})^2] \int d^3k \frac{2M}{k^2}, \]
which is essential for the cancellation of the divergence.

Renormalization effect should be considered for all the terms of \( c_{0, r}^{ab} \) and \( c_{2, r}^{ab} \), including those in the mean field energy, in fluctuations of \( H_p \) and \( H_s \). In the mean field energy, by substituting (13) for \( c_{0, r}^{ab} \rho_a \rho_b \) in the mean field energy, it can be seen that it becomes \( c_{0, r}^{ab} \rho_a \rho_b \) after the \((c_{0, r}^{ab})^2 \) term cancels the divergent term in the Bogoliubov ground state energy of \( H_p \), while \( (c_{2, r}^{ab})^2 \) term cancels the divergent \((c_{2, r}^{ab})^2 \) terms in \( E_0(\theta) \), with exact cancellation at \( \theta = 0 \).

Therefore, the zero-point energy \( E_0(\theta) \) should be regularized by using
\[ E_0(\theta) = \frac{\nu}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \Omega_k(\theta) - \frac{\partial \Omega_k(\theta)}{g_{ab}^2} g_{ab}^2 \right], \]

where
where the summation has been replaced with an integral. It can be shown that after the subtraction, \( \frac{\partial E_0}{\partial \theta} \big|_{\theta=0} = 0 \) and \( \frac{\partial^2 E_0}{\partial \theta^2} \big|_{\theta=0} < 0 \) still hold, thus \( E_0(\theta) \) remains minimal at \( \theta = 0 \).

Without a loss of the generality, we focus on the case \( g_a = g_b = g \). By introducing \( k = k_0x \), where \( k_0 \equiv \sqrt{2gM} \) is a characteristic momentum, we rewrite \( E_0(\theta) = \sqrt{k_0^2gI} \frac{\alpha}{g} \cos^2 \theta \), where

\[
I \left( \frac{\alpha}{g}, \cos^2 \theta \right) \equiv \int \frac{d^3x}{(2\pi)^3} f \left( \frac{\alpha}{g}, \cos^2 \theta, x \right)
\]

is dimensionless and \( f \) is also dimensionless and depends on \( \alpha/g, \cos^2 \theta \), and the dimensionless quantity \( x \). Hence

\[
E_0(\theta) = g \sqrt{2M^2N_\alpha \sqrt{\rho_\alpha(e^\alpha)}^3} I \left( \frac{\alpha}{g}, \cos^2 \theta \right),
\]

where \( \sqrt{\rho_\alpha(e^\alpha)}^3 \) is analogous to the Lee-Huang-Yang parameter in dilute gas theory [24].

The behavior of \( E_0(\theta) \) for various values of \( \alpha/g \) was numerically investigated and is shown in Fig. 2. The result indicates that \( E_0(\theta) \) increases with \( \alpha/g \) because of the enhancement of spin fluctuations. The zero-point energy plays the role of an effective potential and can strongly influence the coherent spin dynamics. Although the mean-field spin dynamics has already been studied in quite a few laboratories [10] [24] [28], the macroscopic quantum spin dynamics driven by microscopic quantum fluctuations so far has not yet been explored in experiments and remains to be observed. The phenomenon studied here provides motivation and a different venue in which to understand fluctuation-induced dynamics.

### V. FLUCTUATION-INDUCED SPIN DYNAMICS

In the present case, the effective Hamiltonian that controls the fluctuations of the spin directors is

\[
\mathcal{H}_e^{\text{eff}} = \sum_\alpha \frac{c^\alpha_2}{2\Omega} l^2_\alpha + \frac{c^\alpha_{ab}}{\Omega} n\cdot l + E_0(\theta) + E_0(0).
\]

where \( n = \sum_\alpha \frac{c^\alpha_{ab}}{c^\alpha_2 - c^\alpha_{ab}} n_\alpha + \frac{(c^\alpha_2 - c^\alpha_{ab})n_\alpha}{c^\alpha_2 - c^\alpha_{ab}} + \frac{c^\alpha_{ab}}{c^\alpha_2 - c^\alpha_{ab}} - 2c^\alpha_{ab} l_\alpha + E_0(\theta) + E_0(0)
\]

and the relative quantities

\[
l = \frac{(c^\alpha_2 - c^\alpha_{ab})l_\alpha + (c^\alpha_2 - c^\alpha_{ab})l_\alpha}{c^\alpha_2 + c^\alpha_{ab} - 2c^\alpha_{ab}},
\]

and \( n = n_\alpha - n_\beta \), we can rewrite \( \mathcal{H}_e^{\text{eff}} \) as \( \mathcal{H}_e^{\text{eff}} = \mathcal{H}_c + \mathcal{H}_r \), where

\[
\mathcal{H}_c = \frac{1}{2\Omega} \left( \frac{c^\alpha_2}{c^\alpha_2} + \frac{c^\alpha_{ab}}{c^\alpha_2 - 2c^\alpha_{ab}} \right) l^2 + E_0(0)
\]

is the center-of-mass part describing a free rotor and \( \mathcal{H}_r = \frac{1}{2\Omega} \left( \frac{c^\alpha_2}{c^\alpha_2} + \frac{c^\alpha_{ab}}{c^\alpha_2 - 2c^\alpha_{ab}} \right) l^2 + E_0(\theta) \) is the relative part. Note that \( \cos \theta = 1 - n^2/2 \) and \( \mathcal{H}_c \) and \( \mathcal{H}_r \) are decoupled.

Let us focus on the relative motion and consider small oscillations around the minimum \( \theta = 0 \). To the lowest order, we can write \( n_i = q_i e_x + q_i e_y \) and \( l = l_{xy} e_x + l_{xy} e_y \), with \( |q_i, l_{xy}| = i\epsilon_{ij} (i, j = x, y) \). Then up to a constant,

\[
\mathcal{H}_r = \sum_{i=x,y} \left[ \frac{c^\alpha_2}{c^\alpha_2} l^2 + \frac{c^\alpha_{ab}}{2\Omega} l^2 + \frac{K l^2}{2q_i^2} \right]
\]

where \( K = -\frac{\partial E_0}{\partial \theta} \big|_{\theta=0} \). Hence \( \mathcal{H}_r \) describes two independent and identical harmonic oscillators, both with frequency

\[
\omega_0 = \sqrt{\frac{c^\alpha_2 + c^\alpha_{ab}}{2\Omega}} = \left( \frac{g_\alpha}{N_\alpha} + \frac{g_b}{N_b} - \frac{2g_{ab}}{\sqrt{N_aN_b}} \right) K.
\]

For typical values \( g_a \sim g_b \sim 100 \text{ Hz} \) (in the unit of \( h \)), \( N_a \sim N_b \sim 10^4 \), and \( g_{ab}/g = 0.6 \), \( K \) can be numerically estimated as \( \sim 6g_a \). The corresponding frequency of the oscillation about the locked position is about 2Hz. It can be substantially enhanced, even by a few orders of magnitude, when an optical lattice is applied and the amplitude of fluctuation is tuned [13]. More investigations are needed to address this circumstance.
The oscillation of the spin directors results in the oscillation of occupation numbers in the Zeeman sublevels. As an example, let us consider the case \( l_1 = l_2 = 0 \) so that \( n \) is independent of time. Then
\[
n_a = n + \frac{c_a^b - c_a^b}{c_a^2 + c_b^2 - 2c_a^b} n_r
\]
and
\[
n_b = n - \frac{c_b^a - c_b^a}{c_a^2 + c_b^2 - 2c_a^b} n_r.
\]
Thus the spin states of species \( a \) and \( b \) in the Zeeman basis state of \( m = \pm 1 \) are
\[
\xi_{a \pm 1} = \frac{c_a^2 - c_b^2}{\sqrt{2(c_a^2 + c_b^2 - 2c_a^b)}} (i q_y \mp q_x)
\]
and
\[
\xi_{b \pm 1} = -\frac{c_b^2 - c_a^2}{\sqrt{2(c_a^2 + c_b^2 - 2c_a^b)}} (i q_y \mp q_x),
\]
while the spin states of species \( a \) and \( b \) in the Zeeman basis state of \( m = 0 \) are both \( n \). Since \( q_x \) and \( q_y \) both oscillate with frequency \( \omega_0 \), the occupation number \( N_\alpha|\xi_{a \pm 1}|^2 \) oscillates with frequency \( 2\omega_0 \). The occupation numbers may be probed by using, say, an optical cavity \[14\].

VI. LOCKING IN A TRAP

Now we examine how the locking effect survives a trapping potential, which is an experimental necessity. Supposing that the potential has the harmonic form and the clouds of the two species have the same size \( R \), then we have
\[
\rho_\alpha = A_\alpha (R^2 - r^2),
\]
where \( A_\alpha \) is a positive constant.

The spin dynamics is determined by the Heisenberg equations
\[
\begin{align*}
  i\partial_t n_\alpha (r) &= \{n_\alpha (r), \mathcal{H}_s\}, \\
  i\partial_t L_\alpha (r) &= \{L_\alpha (r), \mathcal{H}_s\}.
\end{align*}
\]
For small fluctuations, we can impose the commutation relations
\[
\begin{align*}
  \{L_\alpha^i (r), n_\beta^j (r')\} &= i\delta_{\alpha\beta} \epsilon^{ijk} n_\beta^k (r) \delta (r - r'), \\
  \{L_\alpha^i (r), L_\beta^j (r')\} &= i\delta_{\alpha\beta} \epsilon^{ijk} L_\beta^k (r) \delta (r - r'),
\end{align*}
\]
where \( i, j, k = x, y, z \). One obtains
\[
\begin{align*}
  \partial_t n_\alpha &= c_a^b (n_a \times L_\alpha) + c_b^a (n_a \times L_\beta), \\
  \partial_t L_\alpha &= \frac{1}{m} n_a \times \nabla \cdot (\rho_\alpha \nabla n_\alpha) + c_a^b L_\beta \times L_\alpha,
\end{align*}
\]
where \( \alpha \neq \beta \) and \( \nabla^2 \sqrt{\mathcal{F}} \) terms are neglected for large clouds. In the Thomas-Fermi approximation, the spin structure for the ground state remains unaffected, thus the fluctuating directors can be generally written as
\[
\begin{align*}
  n_a &= u_{ax} e_x + u_{ay} e_y + \sqrt{1 - u_{az}^2} n_a^0, \\
  n_b &= u_{bx} e_x' + u_{by} e_y' + \sqrt{1 - u_{bz}^2} n_b^0.
\end{align*}
\]
To first order of \( u_{ai} \) and \( L_{ai} \),
\[
\begin{align*}
  \partial_t u_{ay} &= c_a^2 L_{ax} + c_b^2 L_{bx} \cos \theta, \\
  \partial_t L_{ax} &= \frac{1}{m_a} \nabla \cdot (\rho_\alpha \nabla u_{ay}).
\end{align*}
\]
The two eigenfrequencies are given by
\[
\omega^2 = \frac{f_{nl}}{2} [\nu_a^2 + \nu_b^2 \mp (\nu_a^2 - \nu_b^2)^2 + 4\nu_a^2 \nu_b^2 \cos^2 \theta]^{1/2},
\]
where \( f_{nl} \equiv 2n^2 + 2nl + 3n + l \), \( l \) is the angular quantum number, \( 2n \) is the order of a polynomial of even powers describing the radial wave function \[20\], \( \nu_a^2 = \frac{\alpha_c^2}{m_a} \), \( \nu_b^2 = \frac{\alpha_c^b}{m_b} \), and \( \nu_b^2 = \frac{\alpha_c^b}{m_a} \).

Therefore \( \omega_+ = \omega_- = (f_{nl})^{1/2} [\nu_a^2 + \nu_b^2 + 2(\nu_a^2 - \nu_b^2)^2 \cos^2 \theta]^{1/2} \), which reaches its minimum at \( \theta = 0 \). Hence the locking also occurs in a trap.

Now let us address the effect of an external magnetic field along the \( z \) direction, the presence of which breaks the full rotational symmetry down to \( S^1 \) symmetry, leading to a nonzero mean-field value of \( L_{oz} \). It can be shown that in the mean-field ground state \[30\], each species \( \alpha \) undergoes Bose-Einstein condensation with the Zeeman basis wave function \( \langle \psi_{\alpha,1}, 0, \psi_{\alpha,-1} \rangle \), hence the ground state only possesses spin rotation symmetry \( S^1 \times S^1 \), as the two spins can rotate around the \( z \) axis independently without changing the mean-field energy. Due to the Zeeman barrier, the low energy dynamics is dominated by the phase fluctuations of \( \psi_{\alpha,1} \) and \( \psi_{\alpha,-1} \), while \( \psi_{\alpha,0} \) remains zero, thus the effective Hamiltonian describes a mixture of two pseudospin-1/2 Bose gases \[16\]. For low enough magnetic fields, the spin fluctuation can overcome the Zeeman barrier and restore \( S^2 \) symmetry \[20\], consequently the locking persists.

VII. SUMMARY

Note that the spin directors are macroscopic collective variables of the spinor Bose gases, due to Bose-Einstein condensation. We have shown that in our system, microscopic quantum fluctuations dramatically change the nature of spontaneous symmetry breaking. Two interacting macroscopic systems undergo spontaneous symmetry breaking in a correlated way, and consequently the two macroscopic collective variables are locked. The symmetry-breaking states of the system are
$|N_a, n⟩ \otimes |N_b, n⟩$, where the spin directors of the two species are locked to be $n$ along an arbitrary direction. They are in contrast to states $|N_a, n_a⟩ \otimes |N_b, n_b⟩$, as suggested by the simple mean-field analysis, where $n_a$ and $n_b$ are arbitrary and independent of each other.

To summarize, by considering a mixture of two distinct species of spin-1 atoms with interspecies spin exchange, we have shown that the zero-point quantum fluctuations lift the ground state degeneracy suggested by the mean field theory and lead to the locking between the spin directors of the two species under the experimentally realistic conditions. This is a type of quantum phenomenon in which the microscopic quantum fluctuations fundamentally control the macroscopic collective phenomenon, by changing the very nature of symmetry breaking.

**ACKNOWLEDGMENTS**

This work was supported by the National Science Foundation of China (Grant No. 11374060) and NSERC (Canada). F.Z. was also supported by Canadian Institute for Advanced Research.

[1] P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Benjamin-Cummings, Menlo Park, 1984).
[2] W. E. Lamb and R. C. Retherford, Phys. Rev. **72**, 241 (1947).
[3] E. F. Shender, Sov. Phys. JETP **56**, 178 (1982).
[4] C. L. Henley, Phys. Rev. Lett. **62**, 2056 (1989).
[5] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[6] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
[7] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998).
[8] M. Koashi and M. Ueda, Phy. Rev. Lett. **84**, 1066 (2000).
[9] T. L. Ho and S. K. Yip, Phy. Rev. Lett. **84**, 4031 (2000).
[10] D. M. Stamper-Kurn and M. Ueda, Rev. Mod. Phys. **85**, 1191 (2013).
[11] J. L. Song, G. W. Semenoff and F. Zhou, Phys. Rev. Lett. **98**, 160408 (2007).
[12] A. M. Turner, R. Barnett, E. Demler, and A. Vishwanath, Phys. Rev. Lett. **98**, 190404 (2007).
[13] J. L. Song and F. Zhou, Phys. Rev. A **77**, 033628 (2008).
[14] X. L. Cui, Y. P. Wang and F. Zhou, Phy. Rev. A **78**, 050701(R) (2008).
[15] R. Barnett, S. Powell, T. Graß, M. Lewenstein and S. Das Sarma, Phys. Rev. A **85**, 023615 (2012).
[16] Y. Shi and Q. Niu, Phy. Rev. Lett. **96**, 140401 (2006).
[17] M. Luo, Z. B. Li, C. G. Bao, Phy. Rev. A **75**, 043609 (2007).
[18] Z. Xu, Y. Zhang and L. You, Phys. Rev. A **79**, 023613 (2009).
[19] Y. Shi, Phy. Rev. A **82**, 023603 (2010); Y. Shi and L. Ge, *ibid.* **83**, 013616 (2011); Y. Shi and L. Ge, Int. J. Mod. Phys. B. **26**, 1250002 (2012).
[20] F. Zhou, Phys. Rev. Lett. **87**, 080401 (2001); F. Zhou, Int. J. Mod. Phys. B. **17**, 2643 (2003).
[21] D. Xiong, X. Li, F. Wang and D. Wang. [arXiv:1305.7091](https://arxiv.org/abs/1305.7091).
[22] M. Snoek and F. Zhou, Phys. Rev. B **69**, 094410 (2004).
[23] S. Uchino, M. Kobayashi, and M. Ueda, Phys. Rev. A **81**, 063632 (2010).
[24] T. D. Lee, K. Huang and C. N. Yang, Phys. Rev. **106**, 1135 (1957).
[25] H. Schmaljohann, M. Erhard, J. Kronjager, M. Kottke, S. van Staa, L. Cacciapuoti, J. J. Arlt, K. Bongs, and K. Sengstock, Phys. Rev. Lett. **92**, 040402 (2004).
[26] M. S. Chang, Q. S. Qin, M. X. Zhang, L. You, and M. S. Chapman, Nature Phys. **1**, 111 (2005).
[27] J. M. Higbie, L. E. Sadler, S. Inouye, A. P. Chikkatur, S. R. Leslie, K. L. Moore, V. Savalli, and D. M. Stamper-Kurn, Phys. Rev. Lett. **95**, 050401 (2005).
[28] A. Widera, F. Gerbier, S. Folling, T. Gericke, O. Mandel, and I. Bloch, Phys. Rev. Lett. **95**, 100405 (2005).
[29] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[30] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur and W. Ketterle, Nature (London) **396**, 345 (1998).