Study of Jordan quasigroups and their construction
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ABSTRACT
Jordan quasigroups are commutative quasigroups satisfying the identity \(x^2(yx) = (x^2y)x\). In this paper we discuss the basic properties of Jordan quasigroups and prove that (i) every commutative idempotent quasigroup is Jordan quasigroup, (ii) if a Jordan quasigroup \(Q\) is distributive then \(Q\) is idempotent, (iii) an idempotent entropic quasigroup satisfies Jordan’s identity, (iv) a unipotent quasigroup \(Q\) satisfying Jordan’s identity satisfies left normal square property, (v) if a quasigroup satisfies JC identity, then alternativity \(\iff\) Jordan’s identity, (vi) for a unipotent Jordan quasigroup \(Q\), \(x^3y = y^3x\ \forall \ x, y \in Q\) and (viii) every Steiner quasigroup is Jordan quasigroup. We also prove some properties of the autotopism of Jordan loops. Moreover, we construct an infinite family of nonassociative Jordan quasigroups whose smallest member is of order 6.

1. Introduction
A magma \((Q, \cdot)\) is a quasigroup if for each \(a, b \in Q\), the equations \(ax = b, ya = b\) have unique solutions \(x, y \in Q\). Standard references on quasigroup are [1, 2]. Jordan quasigroups are commutative quasigroups satisfying \(x^2(yx) = (x^2y)x\). A loop is a quasigroup with identity element.

Jordan loops are among the least studied loops. Powers of elements and order of Jordan loops have been discussed in [3–5]. The identity of Jordan loops consists of two variables \(x\) and \(y\). This is why the properties of Jordan loops are not extensively studied because of the weakness of the identity. In this paper we study Jordan quasigroups.

In Section 1, we construct some basic results regarding the Jordan quasigroups. We also prove different type of quasigroups which are necessarily Jordan quasigroups.

In Section 2, we study some properties of the autotopism of Jordan loops. Also we prove that commutative IP loop of exponent 3 is Jordan loop.

In Section 3, we construct an infinite family of nonassociative Jordan quasigroups whose smallest member is of order 6.

We will need the following definitions.
A quasigroup \(Q\) is,

\begin{itemize}
  \item an idempotent quasigroup if for all \(a \in Q\), we have \(aa = a\).
  \item a medial or entropic quasigroup if for any choice of four elements \(a; b; c; d\), one has \((ab)(cd) = (ac)(bd)\).
  \item a semisymmetric if for all \(a, b \in Q\) we have \(ab = ba\).
  \item a Steiner quasigroup if it is commutative, idempotent and satisfying \((ab)b = a\) for all \(a, b \in Q\).
  \item a left distributive quasigroup if it satisfies \(a(bc) = (ab)(ac)\) for all \(a, b \in Q\).
  \item a right distributive quasigroup if it satisfies \((ab)c = (ab)(ac)\) for all \(a, b \in Q\).
  \item a distributive quasigroup if it is both left and right distributives.
  \item a unipotent quasigroup if \(a^2 = b^2\) for all \(a, b \in Q\).
  \item a left nuclear square quasigroup if \(a^2(bc) = (a^2b)c\) for all \(a, b, c \in Q\).
\end{itemize}

2. Relation of Jordan quasigroups to other quasigroups

\textbf{Theorem 2.1:} A commutative idempotent quasigroup is Jordan quasigroup.

\textbf{Proof:} \(x^2(yx) = x(xy) = (xy)x = (yx)x = (x^2y)x\). \qed

\textbf{Corollary 2.1:} Steiner quasigroup is a Jordan quasigroup.

The converse of Theorem 2.1 is not true as the quasigroup in Example 4.1 is a Jordan quasigroup which is not idempotent. However, the converse holds in the following case.

\textbf{Theorem 2.2:} Every distributive Jordan quasigroup is idempotent.
Proof:

\[ x^2(yn) = (x^2)y \] for all \( x, y \in Q \)

\[ \Rightarrow (x^2)(y^2) = (x^2)(x) \quad \text{by distributivity} \]

\[ \Rightarrow (x^2)(yn) = (x^2)(x^2) \Rightarrow x^3(yn) = (x^2)(x)^3 \]

\[ \Rightarrow (yn)x^3 = (x^2)y \quad \text{by commutativity} \]

\[ \Rightarrow yx = x^2y \quad \text{by right cancellation} \]

\[ \Rightarrow yx = yx^2, \quad \text{by commutativity} \]

\[ \Rightarrow x = x^2 \Rightarrow x^2 = x. \]

\[ \square \]

Theorem 2.3: An idempotent entropic quasigroup \( Q \) satisfies Jordan’s identity.

Proof:

\[ x^2(yn) = (xn)(yn) \] for all \( x, y \in Q \)

\[ = (xy)(yn) \quad \text{by entropic property} \]

\[ = (xy)y \quad \text{by idempotent property} \]

\[ \Rightarrow x^2(yn) = (x^2)y. \]

From Jordan’s identity, it is easy to see that a quasigroup which satisfies the left nuclear square property also satisfies Jordan’s identity. But the converse is not true. However, the converse also holds if the Jordan quasigroup satisfying Jordan’s identity also satisfies unipotency.

Theorem 2.4: A unipotent quasigroup \( L \) satisfying Jordan’s identity satisfies the left nuclear square property.

Proof. By Jordan’s identity \( z^2(yz) = (z^2)y \) since \( L \) is unipotent then \( x^2 = z^2 \forall x, z \in L \) so \( x^2(yz) = (x^2)y \).

Theorem 2.5: For a Jordan quasigroup, if \( x^2 = y^2 \), then \( x^3y = y^3x \).

Proof:

\[ x^3y = (x^2x)y \]

\[ = (y^2x)y \quad \text{since } x^2 = y^2 \]

\[ = y^3(x) \quad \text{by Jordan’s identity} \]

\[ = y^3(yx) \quad \text{by commutativity} \]

\[ = x^3(yx) \quad \text{since } x^2 = y^2 \]

\[ = (x^2y)x \quad \text{by Jordan’s identity} \]

\[ = (y^2y)x \quad \text{since } x^2 = y^2 \]

\[ = y^3x. \]

\[ \square \]

Corollary 2.2: For a unipotent Jordan quasigroup \( J \), \( x^3y = y^3x \) for all \( x, y \in J \).

The following four are left central (LC) identities for loops collected in [7]:

\[ (xy)(yz) = (xxy)z, \]

\[ (xxy)z = (xx)(yz), \]

\[ (xx)(yz) = (xx)(yz), \]

\[ (yxx)z = (yxx)z. \]

These identities are equivalent for loops but not for quasigroups as for example the quasigroup in Example 2.1 satisfies the LC identity \( ((x(xy))z) = (x(x(yz))) \) but does not satisfy the LC identity \( ((xx)(yz)) = (x(xy))z \).

The following theorem establishes a relation among a special LC, Jordan and left alternative identities.

Theorem 2.6: If a quasigroup \( Q \) satisfies LC identity \( (xx)(yz) = (xxy)z \) then \( Q \) is left alternative \( \iff \) \( Q \) satisfies Jordan’s identity.

Proof. By LC Identity

\[ (xx)(yz) = (xxy)z, \]

\[ (xx)(yz) = (xxy)z \quad \text{put } z = x, \]

\[ (xx)(yz) = (xx)(yz) \quad \text{by left alternativity,} \]

\[ x^2(yz) = (x^2y). \]

Conversely by Jordan’s identity

\[ x^2(yz) = (x^2y)z. \]

By LC identity

\[ x^2(yz) = (xxy)z, \]

\[ x^2(yz) = (xxy)z \quad \text{put } z = x. \]

From Equations (1) and (2)

\[ (x^2y)x = (xxy)x, \]

\[ x^3y = (xxy), \]

which is left alternativity.

Remark 2.1: An LC-loop has always left alternative property [8] but this is not necessary for quasigroup as the quasigroup in the following example satisfies the LC identity \( ((xxy)z) = (x(x(yz))) \) but it does not have left alternative property.

Example 2.1: A nonassociative quasigroup of order 4:

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

A commutative quasigroup is trivially entropic but the converse does not hold always. The following theorem shows that the converse also holds if the entropic quasigroup is central square.
Theorem 2.7: A central square entropic quasigroup $Q$ is commutative.

Proof: By definition
\[(xy)(zw) = (xz)(yw) \quad \forall x, y, z, w \in Q,
\]
\[(xy)(x) = (x)(yx) \text{ put } z = x \text{ and } w = x,
\]
\[(xy)x^2 = x^2(xy),
\]
\[(xy)x^2 = (yx)x^2,
\]
\[(xy) = (yx).
\]

Theorem 2.8: An idempotent semisymmetric quasigroup $Q$ satisfies Jordan’s identity.

Proof:
\[x^2(yx) = x(yx) \text{ since } Q \text{ is idempotent}
\]
\[= y \text{ since } Q \text{ is semisymmetric}
\]
\[(x^2y)x = (xy)x \text{ since } Q \text{ is idempotent}
\]
\[= y \text{ since } Q \text{ is semisymmetric}
\]
from (3) and (4) $x^2(yx) = (x^2y)x$.

3. On Jordan loops

Theorem 3.1: For a left alternative loop $L$, if $(L_x, R_x, L_xR_x)$ is a mapping, then $L$ satisfies Jordan’s identity.

Proof. Let $(L_x, R_x, L_x, R_x)$ be an autotopism then
\[(uv)L_xR_x = (uL_x)(vR_x) \quad \forall u, v \in L
\]
\[\Rightarrow (uL_x)(vR_x) = (uv)L_xR_x \text{ by putting } u = x
\]
\[\Rightarrow (xL_x)(vR_x) = ((xy)L_x)x \text{ by alternativity}
\]
\[\Rightarrow (xL_x)(x) = ((xx)y)x.
\]

Theorem 3.2: Commutative IP loop of exponent 3 is Jordan loop.

Proof. Let $L$ be a commutative IP loop, then
\[x^2(yx) = x^2(xy) = x^{-1}(xy) = y
\]
and
\[(x^2y)x = (yx^2)x = (yx^{-1})x = y
\]
hence
\[(x^2y)x = x^2(yx).
\]

Theorem 3.3: For a commutative alternative loop $L$, $x^2y = y^2x \Rightarrow x = y$.

Proof. Let us suppose that $x^2y = y^2x$ then
\[x^2y = y^2x
\]
\[\Rightarrow x(xy) = y^2x \quad \text{by alternativity}
\]
\[\Rightarrow x(xy) = xy^2 \quad \text{by commutativity}
\]
\[\Rightarrow xy = y \quad \text{by left cancellation}
\]
\[\Rightarrow x = y \quad \text{by right cancellation.}
\]

Corollary 3.1: For an alternative Jordan loop $L$, $x^2y = y^2x \Rightarrow x = y$.

Theorem 3.4: For a commutative loop, if $(L_x, L_y, L_z, L_y)$ is an autotopism, then $L$ is a Jordan loop.

Proof:
\[(uv)L_xL_y = (uL_x)(vL_y) \quad \forall u, v \in L
\]
\[\Rightarrow y(x(uv)) = (xu)(yv)
\]
put $u = v = y$
\[\Rightarrow y(xy^2) = (xy)y^2
\]
\[\Rightarrow (y^2x)y = y^2(xy) \quad \text{by commutativity}
\]
interchanging $x$ and $y$ we get
\[(x^2y)x = x^2(yx).
\]

Theorem 3.5: For a Jordan loop, if $(L_x^2, Id_z, L_x)$ is an autotopism, then $L$ is alternative.

Proof:
\[(ul_x^2)(vl_z) = (ul_x)(vl_z) \quad \forall u, v \in L
\]
\[\Rightarrow (x(xu))v = x^2(uv)
\]
\[\Rightarrow (x(xu))v = (x^2u)v \quad \text{by Jordan’s identity}
\]
\[\Rightarrow x(xu) = (x^2u)
\]
\[\Rightarrow x(xu) = (xx)u.
\]
Which is left alternativity and since commutative hence alternative.

4. Construction of Jordan quasigroups

We now construct an infinite family of nonassociative Jordan quasigroups whose smallest member is a quasigroup of order 6. We adopt the same procedure as done for the construction of nonassociative and noncommutative C-loops in [6].

Let $G$ be a multiplicative group with neutral element 1, and $A$ be an abelian group written additively with neutral element 0. Let $\mu : G \times G \rightarrow A$ be a mapping, we can define multiplication on $G \times A$ by
\[(g, a)(h, b) = (gh, a + b + \mu(g, h))
\]
The resulting groupoid is clearly a quasigroup. It will be denoted by $(G, A, \mu)$. Additional properties of $(G, A, \mu)$ can be enforced by additional requirements on $\mu$ [10–12].
Lemma 4.1: Let \( \mu : G \times G \to A \) be a mapping. Then \((G, A, \mu)\) is a Jordan quasigroup if
\[
\mu(h, g) + \mu(g^2, h g) = \mu(g^2, h) + \mu(g^2 h, g) \tag{6}
\]
and
\[
\mu(g, h) = \mu(h, g) \quad \text{for every } g, h \in G \tag{7}
\]

The quasigroup \((G, A, \mu)\) is a Jordan quasigroup if
\[(g, a)(g, a)](h, b)(g, a) = [(g, a)(g, a)](h, b)(g, a)\]
and \((g, a)(h, b) = (h, b)(g, a)\) holds for every \(g, h \in G\) and every \(a, b \in A\). Straightforward calculation with Equation (5) shows that this happens iff Equations (6) and (7) are satisfied.

We now use a particular \((G, A, \mu)\) to construct the above-mentioned family of Jordan quasigroups.

Proposition 4.1: Let \( n > 1 \) be an integer, let \( A \) be an abelian group of order \( n \), and \( \alpha \in A \) be an element of order bigger than 1 let \( G = \{1, x, x^2\} \) be the cyclic group with neutral element 1. Define \( \mu : G \times G \to A \) by
\[
\mu(u, v) = \begin{cases} 
0 & \text{if } (u, v) = (x, x^2), (x^2, x), (x, x), (1, 1) \\
\alpha & \text{otherwise}
\end{cases}
\]

Then \((G, A, \mu)\) is a nonassociative Jordan quasigroup.

Proof. To show that \( J = (G, A, \mu) \) is a Jordan quasigroup, we verify Equation (6) because Equation (7) is obviously satisfied.

Case 1. There is nothing to prove when \( g = 1 \).

Case 2. Assume that \( g = x \) then Equation (6) becomes
\[
\mu(h, x) + \mu(x^2, hx) = \mu(x^2, h) + \mu(x^2 h, x)
\]
If \( h = 1 \Rightarrow \mu(1, x) + \mu(x^2, x) = \mu(x^2, 1) + \mu(x^2, x) \quad \Rightarrow \alpha = \alpha.
\]
If \( h = x \Rightarrow \mu(x, x) + \mu(x^2, x^2) = \mu(x^2, x) + \mu(1, x) \quad \Rightarrow \alpha = \alpha.
\]
If \( h = x^2 \Rightarrow \mu(x^2, x) = \mu(x^2, x) + \mu(x, x) \quad \Rightarrow \alpha = \alpha.
\]

Case 3. Assume that \( g = x^2 \) then Equation (6) becomes
\[
\mu(h, x^2) + \mu(x, hx^2) = \mu(x, h) + \mu(xh, x^2)
\]
If \( h = 1 \Rightarrow \mu(1, x^2) + \mu(x, x^2) = \mu(x, 1) + \mu(x, x^2) \quad \Rightarrow \alpha = \alpha.
\]
If \( h = x \Rightarrow \mu(x, x^2) + \mu(x, 1) = \mu(x, x) + \mu(x^2, x^2) \quad \Rightarrow \alpha = \alpha.
\]
If \( h = x^2 \Rightarrow \mu(x^2, x^2) + \mu(x, x) = \mu(x, x^3) + \mu(1, x^2) \quad \Rightarrow \alpha = \alpha.
\]

Which all are true. Now we show that \( J = (G, A, \mu) \) is nonassociative. For this let \( a \in A \), then we have \((x, a)(1, a)(1, a) = (x, a)(1, 2a) = (x, a)(3a + a)\) and \((x, a)(1, a)(1, a) = (x, 2a + a)(1, a) = (x, a + 2a)\). This shows that \( J \) is not alternative and hence nonassociative.

Example 4.1: The smallest group \( A \) satisfying assumptions of Proposition 4.1 is the 2-element cyclic group \([0, 1]\). Following the construction given in Proposition 4.1 and taking \( \alpha = 1 \), we get the following nonassociative Jordan quasigroup of order 6.

\[
\begin{array}{cccccc}
0 & 1 & 3 & 2 & 5 & 4 \\
1 & 0 & 2 & 3 & 4 & 5 \\
2 & 4 & 5 & 0 & 1 & 3 \\
3 & 2 & 5 & 1 & 0 & 4 \\
4 & 5 & 1 & 0 & 2 & 3
\end{array}
\]

Since there is no command in Loops package of GAP [9] to check Jordan quasigroup. So we write one in the following in GAP codes.

### Returns true if the quasigroup \( <Q> \) is Jordan.

```gap
InstallMethod( IsJordanQuasigroup, [IsQuasigroup], 
function( Q )
local x,y;
if not IsCommutative(Q) then return false; fi;
for x in Q do for y in Q do
if not (x*y)*y=x then return false; fi;
od; od;
return true;
end );
```

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References

[1] Bruck RH. A survey of binary systems. Ergebnisse der Mathematik und ihrer Grenzgebiete. Vol. 20. Berlin: Springer; 1971. New Series.
[2] Vasantha Kandasamy WB. Smarandache loops. Rehoboth: American Research Press; 2002.
[3] Kinyon MK, Pula K, Vojtechovsky P. Admissible orders of Jordan loops. J Combin Des. 2009;17(2):103–118.
[4] Pula K. Power of elements in Jordan loops. Comment Math Univ Carolin. 2008;49(2):291–299.
[5] McCrimmon K. A taste of Jordan algebras. Universitext. New York: Springer; 2004.
[6] Philips JD, Vojtechovsky P. C-loops: an introduction. Publ Math Debrecen. 2006;68(1/2):115–137.
[7] Slaney J, Ali A. Generating loops with the inverse property. In: Sutcliffe G, Colton S, Schulz S, editors. Proceedings of ESARM; 2008, Birmingham. p. 55–66.
[8] Fenyves F. Extra loops II. On loops with identities of Bol-Moufang type. Publ Math Debrecen. 1969;16:187–192.
[9] Nagy GP, Vojtechovsky P. LOOPS: Computing with quasigroups and loops in GAP, version 1.0.0, computational
package for GAP. Available from: http://www.math.du.edu/loops.

[10] Khan A, Shah M, Ali A. Construction of middle nuclear square loops. J Prime Res Math. 2013;9:72–78.

[11] Khan A, Shah M, Ali A. A construction of right alternative loop. Int J Algebra Statist. 2013;2(2):29–32.

[12] Khan A, Shah M, Ali A, et al. On commutative quasigroup. Int J Algebra Statist. 2014;3:42–45.