On The Exact and JWKB Solution of 1D Quantum Harmonic Oscillator by Mathematica

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Abstract. Although being the fundamental semiclassical approximation method mainly used in quantum mechanics and optical waveguides, JWKB method along with the application of the associated JWKB asymptotic matching rules is known to give exact solutions for the Quantum Harmonic Oscillator (QHO). Asymptotically matched JWKB solutions are typically accurate or exact in the entire domain except for a narrow domain around the classical turning points where potential energy equals the total energy of the related quantum mechanical system. So, one has to cope with this diverging behavior at the classical turning points since it prohibits us from using continuity relations at the related boundaries to determine the required JWKB coefficients. Here, a computational diagram and related mathematica codes to surmount the problem by applying parity matching for even and odd JWKB solutions rather than boundary continuities are being presented. In effect, JWKB coefficients as well as the conversion factor for the dimensionless form of the Schrödinger’s equation, which is common to both exact and JWKB solutions, is being successfully obtained.

1. Introduction

(JWKB)$^1$ method$^{[1]}$ is conventionally known to be a strong and effective semiclassical approximation method enabling accurate analytical solutions in quantum mechanical systems, described by the Time Independent Schroedinger’s Equation (TISE), given by:

$$\varphi''(x) + f(x)\varphi(x) = 0; f(x) = k^2(x) = \frac{2m}{\hbar^2} [E - U(x)]$$  \hspace{1cm} (1)

where these terms are in usual meanings: $m$ represents mass, $\hbar$ represents Planck’s constant divided by $2\pi$, $E$ represents total energy, and $U(x)$ represents function of potential energy, namely, i.e.,$^{[1,5,7]}$. Although being an approximation method, JWKB method is known to give exact solutions for the Shape Invariant Potentials (SIP) involving the QHO where the potential in $^{[1]}$ is given by $^{[1,3]}$:

$$U(x) = \frac{1}{2} mw^2 x^2$$  \hspace{1cm} (2)

Although given exact solutions for the QHO, the $(JWKB)_1$ method has a typical property that both complementary $(JWKB)_1$ solutions (and hence the general $(JWKB)_1$ solution) diverge at a small region around the classical turning points where $f(x) = 0 \Rightarrow E = V(x)$ in eqn. $(1)$.

$^1$ We simply refer to "nth order JWKB (or WKB)" by a simple abbreviation: $(JWKB)_n$ here.
i.e., \[1, 2, 4, 5\]. So, it does not enable us to apply the continuity relations for the wavefunctions at these turning points to obtain the necessary constant JWKB coefficients. Moreover, although the general (JWKB) wavefunction is accurate for a given potential (and exact for the QHO) in the Classically Accessible Region (CAR), it needs asymptotic matching in the Classically Inaccessible Region (CIR) for accurate (or exact for the QHO) (JWKB) solutions \[1, 2, 2, 4\].

CAR is the region where the particle can classically exist since its potential energy is smaller than its total energy: \( f(x) > 0 \Rightarrow E > V(x) \), and CIR is the region where it can not classically exist since its potential energy is greater than its total energy: \( f(x) < 0 \Rightarrow E < V(x) \).

In this work, we study the JWKB solutions of the QHO in 1D by mathematica and present the results of the applications of parity conditions of wavefunctions rather than their continuity at the turning points to surmount the problem similar to \[2,4\]. By following this procedure, we also find the conversion factor for the dimensionless form TISE.

2. Exact solution

Exact solution of the QHO by different conventional approaches (such as analytical solution, or algebraic methods, etc.) is given in any fundamental textbooks, i.e., \[4–6\]. Results of the solutions can be summarized as follows:

TISE for the QHO in 1D (by using (2) in (1)):

\[
\varphi''(x) + \frac{2m}{\hbar^2}(E_n - \frac{1}{2}mw^2x^2)\varphi(x) = 0
\]  

(3)

Change of variable:

\[
y(x) = \beta x = \sqrt{\frac{mw}{\hbar}} x
\]  

(4a)

TISE for the QHO in 1D in dimensionless form:

\[
\varphi''(y) + k^2[\lambda(E_n), y]\varphi(y) = 0 \Leftrightarrow \left\{ k^2[\lambda(E_n), y] = \lambda^2 - y^2, \lambda^2 = \frac{2E_n}{hw} \right\}
\]  

(4b)

Exact eigenenergies:

\[
\Lambda_{EX} := \lambda = \lambda^2 = 2n + 1 \Rightarrow E_n = (n + \frac{1}{2})hw, n = 0, 1, 2, ...
\]  

(4c)

Exact eigenfunctions (wavefunctions) in \(y\):

\[
\varphi_n(y) = \sqrt{\frac{1}{\sqrt{\pi}2^n n!}}H_n(y)e^{-\frac{y^2}{2}}, n = 0, 1, 2, ...
\]  

(4d)

And by using (4a), we have the wavefunctions in \(x\):

\[
\varphi_n(x) = \sqrt{\frac{\beta}{\sqrt{\pi}2^n n!}}H_n(\beta x)e^{-\frac{(\beta x)^2}{2}}, n = 0, 1, 2, ...
\]  

(4e)

3. Conventional 1st order JWKB approximation ((JWKB) \(n=1\)) method

(i) JWKB Eigenenergies of the dimensionless form in (4b):

They can be found by applying the Bohr-Sommerfeld quantization formula \[1,6\]:

\[
\int_{y_1}^{y_2} k[\lambda(E_n), y]dy = (n + \frac{1}{2})\pi
\]  

(5)

where \(\tilde{\lambda}\) represents JWKB eigenvalues and \(y_1\&y_2\) are the classical turning points of the TISE for the QHO in dimensionless form given in (4b), which are solutions of \(k[\lambda(E), y] = 0\) for \(y\).
(ii) JWKB Eigenfunctions (wavefunctions) of the dimensionless form in \((4b)\):

Conventional \(n = 1\)st order JWKB (= \((JWKB)_{1}\)) eigenfunctions involve only the first two terms \((n = 0\) and up to \(N = 1\) only) of the JWKB expansion terms, \(S_{ij}(i = 0, 1, 2, \ldots; j = 1, 2)\), to give the famous JWKB wavefunction formula as follows:

\[
(\varphi_{JWKB})_{n}(y) =: \tilde{\varphi}(\lambda, y) = \exp[\frac{1}{4}S_0(\lambda, y) + S_1(\lambda, y)], \quad \delta = \hbar/i \rightarrow 0
\]

\[
\Rightarrow \quad \tilde{\varphi}_1(\lambda, y) = \exp[\frac{S_{11}(\lambda, y)}{\delta} + S_{11}(\lambda, y)], \quad \delta = \hbar/i \rightarrow 0
\]

\[
\Rightarrow \quad \tilde{\varphi}_2(\lambda, y) = \exp[\frac{S_{12}(\lambda, y)}{\delta} + S_{12}(\lambda, y)], \quad \delta = \hbar/i \rightarrow 0
\]

\[
\Rightarrow \quad \tilde{\varphi}(\lambda, y) = c_1 \tilde{\varphi}_1(\lambda, y) + c_2 \tilde{\varphi}_2(\lambda, y)
\]

where \(\gamma_i\) is either of the classical turning points (either \(y_{11}\) or \(y_{22}\)) and \(c_1\) and \(c_2\) are arbitrary constants \(1,2,7\). One of the advantages of the JWKB method is that once the solution of one region has been obtained, the other adjacent region, is not required to be calculated, they can directly be written via the following conventional \((JWKB)_{1}\) connection formulae as in \(1,2,4,8\):

\[
\sin \left[ \int_{y_i}^{y_f} k(\lambda, y) dy + \frac{\pi}{4} \right] \ll - \quad \frac{1}{\sqrt{\kappa(\lambda, y)}} \exp \left[ - \int_{y_i}^{y_f} \kappa(\lambda, y) dy \right]
\]

where \(\kappa^2(\lambda, y) = -k^2(\lambda, y)\) as usual. The integrals here are the definite integrals whose upper and lower values include the turning point \(y_i\) and the variable \(y\) whose sequence is chosen in such a way that the connection spans a continuous integration through the \(y\) axis. JWKB connections are made in the double arrow direction, but the connections in the reverse direction are normally not allowed. To make a connection in the reverse direction, one has to interfere a small phase term as in \(2,4\). For example, the reverse connection of the first one of \((7)\) can be achieved as follows:

\[
\frac{2}{\sqrt{k(\lambda, y)}} \sin \left[ \int_{y_i}^{y_f} k(\lambda, y) dy + \frac{\pi}{4} + \varepsilon \right] = \frac{2}{\sqrt{k(\lambda, y)}} \sin \left[ \mu(\lambda, y) + \varepsilon \right]
\]

\[
\frac{\cos \varepsilon}{\sqrt{\kappa(\lambda, y)}} \exp \left[ - \int_{y_i}^{y_f} \kappa(\lambda, y) dx \right] + \frac{2 \sin \varepsilon}{\sqrt{\kappa(\lambda, y)}} \exp \left[ \int_{y_i}^{y_f} \kappa(\lambda, y) dy \right]
\]

In this case, the solution of the classically allowed region should have taken to have a solution of a \(\text{sin}e\) function with an extra small phase, \(\varepsilon\), added to the present phase of \(\pi/4\) to have a total phase of \(\pi/4 + \varepsilon\) to give the right-hand-side of \((8)\) in order to make a successful JWKB connection in the reverse direction of the double arrow in \(7,2,4\).

(iii) Asymptotic Matching in the CIR: Conventional \((JWKB)_{1}\) asymptotic matching rules require either of the complementary solutions in \((6)\) to be cancelled in the CIR as follows \(1,2,4,7\):

\[
\tilde{\varphi}^m(y) = \begin{cases} \tilde{\varphi}(y) \text{ for CAR:} & f(y) > 0 \\ \text{either } k_1 \tilde{\varphi}_1(y) \text{ or } k_2 \tilde{\varphi}_2(y) \text{ for CIR:} & f(y) < 0 \end{cases}
\]

so that the CIR should meet the following requirement regarding the asymptotical matching:

\[
\lim_{y \to -\infty} \tilde{\varphi}(y) = 0 \text{ if CIR lies on the LHS}\]

\[
\lim_{y \to -\infty} \tilde{\varphi}(y) = 0 \text{ if CIR lies on the RHS}
\]
where $\tilde{\psi}^m(y)$ represents the asymptotically matched general (JWKB) solution. In other words, asymptotically diverging term for $y \to \pm \infty$ in the CIR should be cancelled in the general solution so that (9b) can hold. Alternative JWKB asymptotic matching rule regarding the JWKB expansion terms is also available in [2,3].

4. JWKB solution

4.1. Dealing with the discontinuities at the classical turning points

Since JWKB method is useless at the classical turning points, they diverge and hence, we have discontinuities at these points binding two adjacent regions via the unidirectional JWKB connection formulae given in [7]. Although we can find the structure of the wavefunctions in two adjacent regions via these connection rules, we can not use continuity requirements of quantum mechanical wavefunctions to determine the energy dependent coefficients for these wavefunctions ($A(\lambda)\& B(\lambda)$ in Sect. in [4,3]). Let us follow the parity conditions for even potentials as in [2, 4] but by considering (4d) and (4e) in multivariables as follows:

$$\varphi_n[\beta, \lambda(n), x] = \sqrt{\frac{\beta}{\sqrt{2}2^nn!}} H_n(\beta x)e^{-\frac{q^2}{4}} = \sqrt{\beta} \psi_n[\lambda(n), y] \bigg|_{y \to \beta x}, n = 0, 1, 2, \ldots \quad (10a)$$

$$\psi_n[\lambda(n), y] = \sqrt{\frac{1}{\sqrt{2}2^nn!}} H_n(y)e^{-\frac{q^2}{4}} = \left\{ \varphi_n[\beta, \lambda(n), x] \bigg|_{x \to y/\beta} \right\}_{\beta \to 1} = \varphi_n[1, \lambda(n), y], n = 0, 1, 2, \ldots \quad (10b)$$

By using the parity conditions (E.P.: even parity wavefunctions, O.P.: odd parity wavefunctions) we get:

**E.P.:** $\varphi_n(\beta, \lambda, -x) = \varphi_n(\beta, \lambda, x), n = 0, 2, 4, \ldots$ \implies \begin{align*}
 i) & \varphi_n(\beta, \lambda, 0) = [\sqrt{\beta} \varphi_n(1, \lambda, x)]_{x=0} = \sqrt{\beta} \psi_n(\lambda, y)|_{y=0} = \pm p \Rightarrow \psi_n(\lambda, 0) = \pm \frac{p}{\sqrt{\beta}} \\
 ii) & \frac{\partial \varphi_n(\beta, \lambda, x)}{\partial x} \bigg|_{x=0} = \frac{\partial}{\partial x} [\sqrt{\beta} \psi_n(\lambda, y)] \bigg|_{x=0} = 0 \Rightarrow \frac{\partial \psi_n(\lambda, y)}{\partial y} \bigg|_{y=0} = 0
\end{align*} \quad (11a)

**O.P.:** $\varphi_n(\beta, \lambda, -x) = -\varphi_n(\beta, \lambda, x), n = 1, 3, 5, \ldots$ \implies \begin{align*}
 i) & \varphi_n(\beta, \lambda, 0) = [\sqrt{\beta} \varphi_n(1, \lambda, x)]_{x=0} = \sqrt{\beta} \psi_n(\lambda, y)|_{y=0} = 0 \Rightarrow \psi_n(\lambda, 0) = 0 \\
 ii) & \frac{\partial \varphi_n(\beta, \lambda, x)}{\partial x} \bigg|_{x=0} = \frac{\partial}{\partial x} [\sqrt{\beta} \psi_n(\lambda, y)] \bigg|_{x=0} = \sqrt{\beta} \frac{\partial \psi_n(\lambda, y)}{\partial y} \bigg|_{x=0} = \pm q \Rightarrow \frac{\partial \psi_n(\lambda, y)}{\partial y} \bigg|_{y=0} = \pm \frac{q}{\sqrt{\beta}}
\end{align*} \quad (11b)

from which we can also find a relation with the conversion factor for O.P. and E.P. wavefunctions as follows:

$$\beta_{E.P.} = \left[ \frac{\pm p}{\varphi_n(1, \lambda, y) \bigg|_{y=0}} \right]^2 \equiv \tilde{\beta}_{E.P.} = \left[ \frac{\pm p}{\tilde{\varphi}_n(1, \lambda, y) \bigg|_{y=0}} \right]^2, n = 0, 2, 4, \ldots \quad (12a)$$

$$\beta_{O.P.} = \left[ \frac{\pm q}{\frac{\partial \varphi_n(1, \lambda, y)}{\partial y} \bigg|_{y=0}} \right]^{2/3} \equiv \tilde{\beta}_{O.P.} = \left[ \frac{\pm q}{\frac{\partial \tilde{\varphi}_n(1, \lambda, y)}{\partial y} \bigg|_{y=0}} \right]^{2/3}, n = 1, 3, 5, \ldots \quad (12b)$$
where \( p \)\&\( q \) are positive real constants regarding the EP and OP initial values of the physical
system. We already have from (10a)-(10b):
\[
\tilde{\psi}(\lambda, y) = \frac{1}{\sqrt{\beta}} \tilde{\varphi}(1, \lambda, y/\beta) \equiv \psi(\lambda, y) = \frac{1}{\sqrt{\beta}} \varphi(1, \lambda, y/\beta), y \neq y_t
\]
(12c)
where symbols with tilde represent JWKB terms. Although we derived these results by using
exact wavefunctions (\( \varphi \)), it should also be valid for the JWKB wavefunctions (\( \tilde{\varphi} \)) since the parity
conditions require the same initial values for exact and JWKB wavefunctions at
\( y = 0 \), namely:
\[
\begin{align*}
|\varphi_n(1, \lambda, x)|_{x=0} &= |\tilde{\varphi}_n(1, \lambda, x)|_{x=0} = \begin{cases} 0 & \text{for O.P., } n=1,3,5,\ldots \\ \pm p & \text{for E.P., } n=0,2,4,\ldots \end{cases} \\
\frac{\partial \varphi_n(1, \lambda, x)}{\partial x} |_{x=0} &= \frac{\partial \tilde{\varphi}_n(1, \lambda, x)}{\partial x} |_{x=0} = \begin{cases} \pm q & \text{for O.P., } n=1,3,5,\ldots \\ 0 & \text{for E.P., } n=0,2,4,\ldots \end{cases}
\end{align*}
\]
(13)
For the JWKB solutions we assume that the asymptotic matching in the CIR (and also the
parity matching in the entire domain) has been achieved, however it is not necessary for
\( x = 0 \) since it is not in the CIR. Mathematica codes for the determination of
\( A(\lambda) \& B(\lambda) \) are given in
[Fig. 1](#).

### 4.2. JWKB Eigenenergies

JWKB quantization rule given in (5) for the QHO gives [1,2,4,5]:
\[
\int_{y_1}^{y_2} k[\tilde{\lambda}(E_n), y] dy = \int_{y_1}^{y_2=\lambda} \sqrt{\tilde{\lambda}^2 - y^2} dy = (n + \frac{1}{2})\pi
\]
which gives exactly the exact eigenenergies given in (4c):
\[
\lambda_{JWKB}^2 = \tilde{\lambda}^2 = 2n + 1 \equiv \lambda_{EX}^2 = \lambda^2
\]
\[
\Rightarrow E_n(\text{JWKB}) =: \tilde{E}_n = (n + \frac{1}{2})\hbar \omega = E_n := E_{\text{JWKB}}, (n = 0, 1, 2,\ldots)
\]
(15)
Simple mathematica codes to determine JWKB eigenenergies is given in Fig. [1](#).

### 4.3. JWKB Eigenfunctions

One can find the E.P. and O.P. JWKB wavefunctions by working only on the first quadrant
of the \( f - y \) graph using the connection formulae given in (7)-(8) and then by applying the
symmetry relations to extend it to the second quadrant, i.e., in [1,2,4,5,7]. We also apply both
asymptotic matching (modification) and parity matchings given in [1, 2, 4]. We study here the
specific cases for \( p = q = 1 \):
\[
p = q = 1 \Rightarrow \psi(\lambda, y) =: \psi^{(1)}(\lambda, y)
\]
(16a)
from which the other cases with \( p \neq 1 \& q \neq 1 \) can be determined as follows:
\[
p \neq 1 \& q \neq 1 \Rightarrow \psi(\lambda, y) =: \psi^{(2)}(\lambda, y) = \psi^{(1)}(\lambda, y) \times \begin{cases} p & \text{for EP} \\ q & \text{for OP} \end{cases}
\]
(16b)
4.3.1. Even Parity (EP) Wavefunctions

For the EP wavefunctions, if we start with a cosine-like wavefunction in region II:

\[ \tilde{\varphi}_{II}(\tilde{\lambda}_n, y) = A(\lambda_n) \sqrt{k(\lambda_n, y)} \cos \left( \int_0^y k(\lambda_n, y) dy \right), \text{ for } 0 < y < \lambda_n \]  

and by applying the JWKB connection formulae, we get:

\[ \tilde{\varphi}_{III}(\tilde{\lambda}_n = \lambda_n, y) = A(\lambda_n) \sqrt{\kappa(\lambda_n, y)} \left\{ \cos [\alpha(\lambda_n)] \exp[\zeta(\lambda_n, y)] + \frac{1}{2} \sin [\alpha(\lambda_n)] \exp[-\zeta(\lambda_n, y)] \right\}, \text{ for } \lambda_n < y < \infty \]  

However, we have to use the asymptotically matched (modified) wavefunction in region III via (9a)-(9b) of [1,2,4]:

\[ \tilde{\varphi}_{III}^{(m)}(\tilde{\lambda}_n = \lambda_n, y) = \frac{A(\lambda_n)}{2\sqrt{\kappa(\lambda_n, y)}} \sin [\alpha(\lambda_n)] \exp[-\zeta(\lambda_n, y)], \text{ for } \lambda_n < y < \infty \]  

So far, we have obtained the general structure of the JWKB wavefunctions in the first quadrant \((0 < x < \infty)\) and once the coefficients are determined, they can be extended to the second quadrant \((-\infty < x < 0)\) by using the parity conditions of the wavefunctions. However, we cannot apply the continuity of wavefunctions between region II and III since the JWKB solutions diverge at the classical turning points coinciding this interface here. So, we apply the parity conditions in (11b)-(11a) to determine the coefficients residing in the wavefunction formulae obtained above. For the EP wavefunctions, we have:

\[ \tilde{\varphi}_{(E.P.)}^{(m)}(\lambda_n, y) = P_{(E.P.)}(\lambda_n) \times \begin{cases} \tilde{\varphi}_{II}^{(m)}(\lambda_n, -y) & \text{for } -\infty < x \leq -\lambda \\ \tilde{\varphi}_{II}^{(m)}(\lambda_n, -y) & \text{for } -\lambda < x \leq 0 \\ \tilde{\varphi}_{III}^{(m)}(\lambda_n, y) & \text{for } 0 < x \leq \lambda \\ \tilde{\varphi}_{III}^{(m)}(\lambda_n, y) & \text{for } \lambda \leq x < \infty \end{cases} \]  

where \(P_{(E.P.)}(\lambda_n)\) is for parity matching and defined by:

\[ P_{(E.P.)}(\lambda_n) = (-1)^{(\lambda_n^2 - 1)/4} \]
Abbreviations we use for EP JWKB solutions here (and also for the OP solutions in the next subsection) are as follows:

\[
\begin{align*}
\alpha(\lambda_n) &= \int_0^{\lambda_n} k(\lambda_n, y) dy + \frac{\pi}{4} = \eta(\lambda_n, 0) + \frac{\pi}{4} \\
\eta(\lambda_n, y) &= \int_y^{\lambda_n} k(\lambda_n, y) dy \to = \frac{\lambda^2_n - \lambda^2_{n+1}}{4} - \frac{\lambda^2_n}{2} \sin^{-1} \left( \frac{y}{\lambda_n} \right) - \frac{\pi}{4} \sqrt{\lambda^2_n - y^2} \\
\zeta(\lambda_n, y) &= \int_y^{\lambda_n} \kappa(\lambda_n, y) dy \to = \frac{\pi}{4} \sqrt{y^2 - \lambda^2_n} - \frac{1}{2} \lambda^2_n \ln \left| \frac{y + \sqrt{y^2 - \lambda^2_n}}{\lambda_n} \right|
\end{align*}
\]  

(17f)

However, since the determination of constant \(A(\lambda_n)\) requires the values of \(\beta\) in (11a) and it is not convenient to apply (11a) for all \(\beta\) values repeatedly (we might also not have found them yet), we have a little trick here in our algorithm as follows:

\[
\begin{align*}
\text{our IVP: } \psi(\lambda, y) &= \frac{\pm p}{\sqrt{\beta}} \to A(\lambda_n) = ? \\
\text{modified IVP: } \psi(\lambda, y) &= \pm p \bigg|_{p=1} \to A(\lambda_n) = ?
\end{align*}
\]  

(18a)

Once we find \(A(\lambda_n)\) from the modified IVP, then by using (11a) and (12a) we see that we should replace it by \(p\beta^{-1/2}A(\lambda_n)\) but the \(\beta\) value does not change, namely:

\[
A(\lambda_n) =: p\beta^{-1/2}A(\lambda_n) \text{, } \beta =: \beta
\]  

(18b)

Simple mathematica codes to determine the conversion factor \(\beta\) for the EP wavefunctions according to the algorithm in (11a), (12a) with \(p = 1\) is given in Fig. 2, the related mathematica codes to find \(A(\lambda_n)\) via this algorithm are given in Fig. 3 and the graphs of some exact and JWKB EP wavefunctions calculated according to the prescription discussed here are given for \(p = 1\) in Fig. 4.

4.3.2. Odd Parity (OP) Wavefunctions

For the OP wavefunctions, if we start with a sine-like wavefunction in region II:

\[
\tilde{\psi}_{II}(\tilde{\lambda}_n = \lambda_n, y) = \frac{A(\lambda_n)}{\sqrt{k(\lambda_n, y)}} \sin \left[ \int_0^y k(\lambda_n, y) dy \right], \text{ for } 0 < y < \lambda_n
\]  

(19a)

and by applying the JWKB connection formulae along with the asymptotic matching in region III, we similarly get:

\[
\tilde{\psi}_{III}(\tilde{\lambda}_n = \lambda_n, y) = \frac{B(\lambda_n)}{\sqrt{\kappa(\lambda_n, y)}} \left\{ \sin \left[ \alpha(\lambda_n) \right] \exp \left[ \zeta(\lambda_n, y) \right] - \frac{1}{2} \cos \left[ \alpha(\lambda_n) \right] \exp \left[ -\zeta(\lambda_n, y) \right] \right\}, \text{ for } \lambda_n < y < \infty
\]  

(19b)

\[
\tilde{\psi}_{III}(m, \lambda_n, y) = -\frac{B(\lambda_n)}{2\sqrt{\kappa(\lambda_n, y)}} \cos \left[ \alpha(\lambda_n) \right] \exp \left[ -\zeta(\lambda_n, y) \right], \text{ for } \lambda_n < y < \infty
\]  

(19c)

We have the following results in the entire domain for the OP wavefunctions:

\[
\tilde{\psi}_{(O,P)}^{(m, \lambda)}(\lambda_n, y) = P_{(O,P)}(\lambda_n) \times \begin{cases} 
-\tilde{\psi}_{III}(\lambda_n, y) & \text{for } -\infty < x \leq -\lambda \\\n-\tilde{\psi}_{III}(\lambda_n, y) & \text{for } -\lambda < x \leq 0 \\
\tilde{\psi}_{II}(\lambda_n, y) & \text{for } 0 < x \leq \lambda \\
\tilde{\psi}_{III}(\lambda_n, y) & \text{for } \lambda \leq x < \infty
\end{cases}
\]  

(19d)
Figure 2. Mathematica codes to determine conversion factor $\beta$ for EP wavefunctions.

```mathematica
C[p_] := \[Lambda] = 1.5; Clear[\[Beta], \[Psi]exa];
\[Psi]exa[\[Beta]_, n_, x_] = \[Beta]/(\[Sqrt][2]*2*\[Pi]*\[SquareRoot][-1]*\[Beta]*x)*HermiteH[n, \[Beta]*x]*Exp[-0.5*(\[Beta]*x)^2];
Table[\[Beta] = \[Sqrt][2]*n + 1; Print["\[Beta]", \[Beta]],
Print["\[Beta]", \[Beta]]];
```

| n  | $\lambda = \sqrt{2n+1}$ | $\beta$     |
|----|---------------------------|--------------|
| 0  | 1.000000                  | 1.772457     |
| 2  | 2.236068                  | 3.544917     |
| 4  | 3.000000                  | 4.726547     |
| 6  | 3.605551                  | 5.671857     |
| 8  | 4.123106                  | 6.462127     |
| 10 | 4.582576                  | 7.202357     |
| 12 | 5.000000                  | 7.857117     |
| 14 | 5.385165                  | 8.46157      |
| 16 | 5.744563                  | 9.02567      |
| 18 | 6.082763                  | 9.556527     |
| 20 | 6.403124                  | 10.05957     |
| 22 | 6.708204                  | 10.53957     |
| 24 | 7.000000                  | 10.99677     |
| 26 | 7.280110                  | 11.43667     |
| 28 | 7.549834                  | 11.86027     |
| 30 | 7.810250                  | 12.26917     |

Figure 3. Simple Mathematica codes to determine JWKB coefficient $A$ for EP wavefunctions.

```mathematica
Print[
'Determination of $A(\lambda)$ for EP. wf. in Region II by even parity condition at $x=0$:'] A = ; \[Lambda] = ;
\[Psi]II[\[lambda]_, y_] = Assuming[(\[lambda] , y) \[Re] & 0 < y < \[lambda], \frac{A}{\[Sqrt]([\[lambda] , y]} \times \text{Cos}[\text{int}[\[lambda] , y]]]
A = A /. A -> Quiet[A = A /. Solve[\[Psi]II[\[lambda] , 0] = 1, \[lambda]]] Determination of $A(\lambda)$ for EP. wf. in Region II by even parity condition at $x=0$: 
\{(\lambda^2)^{1/4}\}
```
where \( P_{(O.P.)}(\lambda_n) \) is for parity matching and defined by:

\[
P_{(O.P.)}(\lambda_n) = (-1)^{\left\lfloor \frac{\lambda_n^2 - 1}{2} \right\rfloor}
\]  

As to the determination of the conversion factor, similar to our EP calculations, but we rather use (11b)-(12b) here for the OP wavefunctions as given in Fig. 5. We find the desired coefficient (=\( B(\lambda_n) \)) regarding the OP wavefunctions by using the initial values in (13) according to (11b). However, since the determination of constant \( B \) requires the values of \( \beta \) in (11b) and it is not convenient to apply (11b) for all \( \beta \) values repeatedly (we might also not have found them yet), we similarly apply the followings:

our IVP: \[
\quad \frac{\partial \psi(\lambda, y)}{\partial y} \bigg|_{y=0} = \pm q \sqrt{\beta} \Rightarrow B(\lambda_n) =?
\]  

modified IVP: \[
\quad \frac{\partial \psi(\lambda, y)}{\partial y} \bigg|_{y=0} = \pm q \bigg|_{q=1} \Rightarrow B(\lambda_n) =?
\]  

Once we find \( B(\lambda_n) \) from the modified IVP, then by using (11b) and (12b) we see that we should we replace it by \( q\beta^{-3/2}B(\lambda_n) \) but the \( \beta \) value does not change, namely:

\[
B(\lambda_n) =: q\beta^{-3/2}B(\lambda_n) , \quad \beta =: \beta
\]  

Related mathematica codes to find \( B \) via this algorithm are given in Fig. 6 and the graphs of some exact and JWKB OP wavefunctions calculated according to the prescription discussed here are given for \( q = 1 \) in Fig. 7.

5. Conclusion

Although JWKB method fails at the classical turning points and hence we can not apply continuity relations to determine the JWKB wavefunction coefficients in the adjacent domains.
Figure 5. Mathematica codes to determine conversion factor $\beta$ for OP wavefunctions.

Figure 6. Mathematica codes to determine $B$ for OP wavefunctions.
around these classical turning points, we find the JWKB solutions of the QHO successfully in the entire domain by applying the parity conditions of the wavefunctions. We have both exact JWKB eigenenergies and exact JWKB eigenfunctions (wavefunctions) in the entire domain except for a very narrow region around the classical turning points as typical to the JWKB method. The algorithm and the related mathematica codes presented here are very simple and can be used to find JWKB solutions of the QHO along with the conversion factor for the dimensionless form. Moreover, it is possible to adopt them to mathematica cdf (computable document file) player for interactive simulation and animation applications since EP and OP calculations here are simultaneous but in distinct variable and $\beta$-matrix representations.

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