THE LAX INTEGRABLE DIFFERENTIAL-DIFFERENCE DYNAMICAL SYSTEMS ON EXTENDED PHASE SPACES

The Lax type flows in the forms [1, 2]

\[ L_{tp} = [(L^p)_+, L], \quad p \in \mathbb{N}, \]  

on the dual space \( G^* \) to the Lie algebra of linear operators

\[ L := \sum_{j<\infty} u_j(n)T^j, \]

where \( u_j \in C^\infty(\mathbb{Z}/q\mathbb{Z}; \mathbb{R}), \quad j \in \mathbb{Z}, \quad q \in \mathbb{N}, \) and \( T \) is the shift operator, satisfying the following rule

\[ T^j u = (T^j u)^T \]

and the lower index \( + \) signs a projection of the corresponding operator on the Lie subalgebra \( G_+ \subset G \), which consists of the elements \( \sum_{0 \leq j<\infty} u_j(n)T^j \), with respect to the scalar product

\[ (A, B) := \sum_{n \in \mathbb{Z}/q\mathbb{Z}} \sum_{j \in \mathbb{Z}} a_j(n)b_j(n), \quad A, B \in G, \]

\[ A := \sum_{j<\infty} a_j(n)T^j, \quad B := \sum_{i<\infty} T^{-i}b_i(n), \]

are considered. The corresponding evolutions for eigenfunctions \( f_k \in W := L_\infty(\mathbb{Z}/q\mathbb{Z}; \mathbb{R}) \) and adjoint eigenfunctions \( f^*_k \in W \), \( k = 1, N \), of the associated with (1) isospectral problem take the forms

\[ f_{k, tp} = ((L^p)_+ f_k), \quad f^*_{k, tp} = -((L^p)^*_+ f^*_k), \]

where functions \( f_k, f^*_k \in W \) are related to \( N \in \mathbb{N} \) different eigenvalues.

The existence of Hamiltonian representation for the hierarchy of dynamical systems (1)-(2) on an extended phase space \( G^* \times W^{2N} \) is investigated by use of the invariant Casimir functionals’ property under the Lie-Backlund transformation on the space \( G^* \)

\[ L_{>0} \mapsto L = L_{>0} + \sum_{k=1}^{N} f_k T(T - 1)^{-1} f^*_k, \]

where \( L_{>0} := \sum_{0<j<\infty} u_j(n)T^j \). The corresponding hierarchies of Lax type additional symmetries [3] are stated to be Hamiltonian too. It is established that the additional
symmetry is generated by the Poisson structure, being equal to the tensor product of the 
$R$-deformed canonical Lie-Poisson bracket [1] on $G^*$ and the standard Poisson bracket 
on the space $W^{2N}$, and some power of a suitable spectral eigenvalue is its Hamiltonian function.

The similar problems for the central extension [3, 4] of the Lie algebra $G$ are studied also.

References

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