The Existence of a Weak Solution to Volume Preserving Mean Curvature Flow in Higher Dimensions

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Abstract

In this paper, we construct a family of integral varifolds, which is a global weak solution to the volume preserving mean curvature flow in the sense of $L^2$-flow. This flow is also a distributional BV-solution for a short time, when the perimeter of the initial data is sufficiently close to that of a ball with the same volume. To construct the flow, we use the Allen–Cahn equation with a non-local term motivated by studies of Mugnai, Seis, and Spadaro, and Kim and Kwon. We prove the convergence of the solution for the Allen–Cahn equation to the family of integral varifolds with only natural assumptions for the initial data.

1. Introduction

Let $d \geq 2$ be an integer and $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. Assume that $T > 0$ and that $U_t \subset \Omega$ is an open set with the smooth boundary $M_t := \partial U_t$ for any $t \in [0, T)$. The family of the hypersurfaces $(M_t)_{t \in [0, T)}$ is called the volume preserving mean curvature flow if the normal velocity vector $\vec{v}$ satisfies

$$\vec{v} = \vec{h} - \left( \frac{1}{\mathcal{H}^{d-1}(M_t)} \int_{M_t} \vec{h} \cdot \vec{v} \; d\mathcal{H}^{d-1} \right) \vec{v}, \quad \text{on } M_t, \; t \in (0, T). \quad (1)$$

Here, $\mathcal{H}^{d-1}$ is the $(d - 1)$-dimensional Hausdorff measure, and $\vec{h}$ and $\vec{v}$ are the mean curvature vector and the inner unit normal vector of $M_t$, respectively. Note that the solution $(M_t)_{t \in [0, T)}$ to (1) satisfies

$$\frac{d}{dt} \mathcal{H}^{d-1}(M_t) \leq 0 \quad \text{and}$$

$$\frac{d}{dt} \mathcal{L}^d(U_t) = -\int_{M_t} \vec{v} \cdot \vec{v} \; d\mathcal{H}^{d-1} = 0, \quad t \in (0, T), \quad (2)$$
where $\mathcal{L}^d$ is the $d$-dimensional Lebesgue measure. From (2), $\{M_t\}_{t \in [0, T)}$ has the volume preserving property, that is, $\mathcal{L}^d(U_t)$ is constant with respect to $t$.

For when $U_0$ is convex, Gage [15] and Huisken [19] proved that there exists a solution to (1) and it converges to a sphere as $t \to \infty$. Escher and Simonett [10] showed the short time existence of the solution to (1) for smooth initial data $M_0$ and they also proved that if $M_0$ is sufficiently close to a sphere in the sense of the little Hölder norm $h^{1+\alpha}$, then there exists a global solution and it converges to some sphere as $t \to \infty$ (see also [3,4,29] for related results). Mugnai, Seis, and Spadaro [35] studied the minimizing movement for (1) and they proved the global existence of the flat flow, that is, there exist $C = C(d, U_0) > 0$ and a family of Caccioppoli sets $\{U_t\}_{t \in [0, \infty)}$ such that $\mathcal{L}^d(U_s \triangle U_t) \leq C \sqrt{s - t}$ for any $0 \leq t < s$, $\mathcal{H}^{d-1}(\partial_s U_t)$ is monotone decreasing, and $\mathcal{L}^d(U_t)$ is constant. Here, $\partial^s U_t$ is the reduced boundary of $U_t$. In addition, for $d \leq 7$, they proved the global existence of the weak solution to (1) in the sense of the distribution, under the reasonable assumption for the convergence, that is,

$$
\lim_{k \to \infty} \int_0^T \mathcal{H}^{d-1}(\partial_s^k U^k_t) \, dt = \int_0^T \mathcal{H}^{d-1}(\partial_s^s U_t) \, dt,
$$

where $\{U^k_t\}_{t \in [0, T)}$ is the time-discretized approximate solution to (1). This kind of condition was introduced in [31] (see also [1,26]). Laux and Swartz [28] also proved similar results in the case of the phase field method. On the other hand, the author [43] proved the existence of the weak solution (family of integral varifolds) to (1) in the sense of $L^2$-flow for $2 \leq d \leq 3$ without any such convergence assumption, via the phase field method studied by Golovaty [17]. Recently, Kim and Kwon [24] proved the existence of the viscosity solution to (1) for the case where $U_0$ satisfies a geometric condition called $\rho$-reflection. Moreover, they also proved that the viscosity solution converges to some sphere uniformly as $t \to \infty$.

Let $\{\delta_i\}_{i=1}^\infty$ be a positive sequence with $\delta_i \to 0$ as $i \to \infty$ and we denote $\delta_i$ as $\delta$ for simplicity. Suppose that $U^\delta_t$ is an open set with smooth boundary $M^\delta_t$ for any $t \in [0, T)$. The approximate solutions studied in [24,35] correspond to the following mean curvature flow $\{M^\delta_t\}_{t \in [0, T)}$ with non-local term:

$$
\bar{v} = \bar{h} - \lambda^\delta \bar{\nu}, \quad \text{on } M^\delta_t, \quad t \in (0, T),
$$

where

$$
\lambda^\delta(t) = \frac{1}{\delta}(\mathcal{L}^d(U^\delta_0) - \mathcal{L}^d(U^\delta_t)).
$$

One can check that (4) is a $L^2$-gradient flow of

$$
E^\delta(t) = \mathcal{H}^{d-1}(M^\delta_t) + \frac{1}{2\delta}(\mathcal{L}^d(U^\delta_0) - \mathcal{L}^d(U^\delta_t))^2,
$$

that is,

$$
\frac{d}{dt}E^\delta(t) = -\int_{M^\delta_t} |\bar{v}|^2 \, d\mathcal{H}^{d-1} \leq 0 \quad \text{for any } t \in (0, T).
$$
Hence \( \{M^\delta_t\}_{t \in [0,T)} \) satisfies a relaxed volume preserving property, namely,
\[
(\mathcal{L}^d(U^\delta_0) - \mathcal{L}^d(U^\delta_t))^2 \leq 2\delta E^\delta(t) \leq 2\delta E^\delta(0) = 2\delta \mathcal{H}^{d-1}(M^\delta_0).
\]
Therefore \( \{M^\delta_t\}_{t \in [0,T)} \) converges to the solution \( \{M_t\}_{t \in [0,T)} \) to (1) as \( \delta \to 0 \) formally. Note that we cannot directly obtain the monotonicity of \( \mathcal{H}^{d-1}(M_t) \) by the energy estimates above. However, if we have a natural energy estimate
\[
\liminf_{i \to \infty} \frac{1}{2\delta_i} (\mathcal{L}^d(U^\delta_0) - \mathcal{L}^d(U^\delta_i))^2 = \liminf_{i \to \infty} \frac{\delta_i}{2} |\lambda^{\delta_i}(t)|^2 = 0 \quad \text{for a.e. } t \in [0, T),
\]
by Fatou’s lemma (see Proposition 10). The reason why the \( L^2 \)-estimate is natural is because the non-local term of the solution to (1) satisfies it (see Proposition 19). Mugnai, Seis, and Spadaro [35] used a minimizing movement scheme corresponding to (4), and Kim and Kwon [24] used (4) to prove the existence of the viscosity solution to (1). Based on these results, in this paper we show the global existence of the weak solution to (1), via the phase field method corresponding to (4).

We denote \( W(a) := \frac{(1-a^2)^2}{2} \) and \( k(s) = \int_0^s \sqrt{2W(a)} \, da = s - \frac{1}{3}s^3 \). Let \( \varepsilon \in (0, 1) \), \( T > 0 \), and \( \alpha \in (0, 1) \). With reference to [35] and [24], in this paper we consider the following Allen–Cahn equation with non-local term:
\[
\begin{aligned}
\varepsilon \partial_t \varphi^\varepsilon &= \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} + \lambda^\varepsilon \sqrt{2W(\varphi^\varepsilon)}, \quad (x, t) \in \Omega \times (0, \infty), \\
\varphi^\varepsilon(x, 0) &= \varphi^\varepsilon_0(x), \quad x \in \Omega,
\end{aligned}
\]
where \( \lambda^\varepsilon \) is given by
\[
\lambda^\varepsilon(t) = \frac{1}{\varepsilon^\alpha} \left( \int_\Omega k(\varphi^\varepsilon_0(x)) \, dx - \int_\Omega k(\varphi^\varepsilon(x, t)) \, dx \right).
\]
Note that if \( \varphi^\varepsilon_0 \) satisfies suitable assumptions, the standard PDE theories imply the global existence and uniqueness of the solution to (5) (see Remark 8). Set
\[
E^\varepsilon(t) = \int_\Omega \left( \frac{\varepsilon |\nabla \varphi^\varepsilon(x, t)|^2}{2} + \frac{W(\varphi^\varepsilon(x, t))}{\varepsilon} \right) \, dx
\]
\[
\quad + \frac{1}{2\varepsilon^\alpha} \left( \int_\Omega k(\varphi^\varepsilon_0(x)) \, dx - \int_\Omega k(\varphi^\varepsilon(x, t)) \, dx \right)^2
\]
\[
=: E^\varepsilon_S(t) + E^\varepsilon_P(t).
\]
As above, one can check that the solution \( \varphi^\varepsilon \) to (5) satisfies
\[
\frac{d}{dt} E^\varepsilon(t) = - \int_\Omega \varepsilon (\partial_t \varphi^\varepsilon)^2 \, dx \leq 0 \quad \text{for any } t \in (0, \infty),
\]
\[
E^\varepsilon(T) + \int_0^T \int_\Omega \varepsilon (\partial_t \varphi^\varepsilon)^2 \, dx \, dt = E^\varepsilon(0) = E^\varepsilon_S(0) \quad \text{for any } T \geq 0,
\]
and
\[
\left( \int_\Omega k(\varphi^\varepsilon_0(x)) \, dx - \int_\Omega k(\varphi^\varepsilon(x,t)) \, dx \right)^2 = 2\varepsilon^\alpha E_p^\varepsilon(t) \leq 2\varepsilon^\alpha E^\varepsilon_S(0) \tag{10}
\]
for any \( t \in ]0, \infty[ \). Assume \( \sup_{p \in (0,1)} E^\varepsilon_S(0) < \infty \) (this assumption corresponds to \( \mathcal{H}^{d-1}(M_0) < \infty \) for (1)). Then, we can expect that \( \varphi^\varepsilon(x,t) \approx 1 \) or \(-1\) when \( x \) is outside the neighborhood of the zero level set \( M^\varepsilon_t = \{ x \in \Omega \mid \varphi^\varepsilon(x,t) = 0 \} \) for sufficiently small \( \varepsilon \). Then we have \( \int_\Omega k(\varphi^\varepsilon) \, dx \approx \frac{2}{3} \int_\Omega \varphi^\varepsilon \, dx \) and thus we can regard (10) as a relaxed volume preserving property. The function \( \sqrt{2W(\varphi^\varepsilon)} \) expresses that the non-local term is almost zero when \( x \) is outside the neighborhood of \( M^\varepsilon_t \). In addition, \( \sqrt{2W(\varphi^\varepsilon)} \) plays important roles in \( L^\infty \)-estimates and energy estimates (see Proposition 6 and Theorem 12).

The first main result of this paper is that there exists a global-in-time weak solution to (1) for any \( d \geq 2 \) in the sense of \( L^2 \)-flow, under the assumptions on the regularity of \( M_0 \) (see Theorem 3). We employ (5) to construct the solution. Note that we do not require assumptions such as (3). The second main result is that, when \( M_0 \) is \( C^1 \) and the value \( \mathcal{H}^{d-1}(M_0)/(\mathcal{L}^d(U_0))^{d-1} \) is sufficiently close to that of a ball, there exists \( T_1 > 0 \) such that the flow has a unit density for a.e. \( t \in [0,T_1) \) and is also a distributional BV-solution up to \( t = T_1 \) (see Theorem 4). To obtain the main results, we need to prove that the varifold \( V^\varepsilon_t \) defined by the Modica–Mortola functional [33] converges to an integral varifold for a.e. \( t \geq 0 \) (roughly speaking, the condition (3) corresponds to this convergence). For the standard Allen–Cahn equation without non-local term, this convergence was shown by Ilmanen [23] and Tonegawa [47]. Therefore we can expect the convergence for (5) if \( \lambda^\varepsilon \) has suitable properties. In fact, \( \lambda^\varepsilon \) can be regarded as an error term when we consider the parabolic rescaled equation of (5). We explain this more precisely. Define \( \tilde{\varphi}^\varepsilon(\tilde{x}, \tilde{t}) = \varphi^\varepsilon(\varepsilon \tilde{x}, \varepsilon^2 \tilde{t}) \). Then \( \tilde{\varphi}^\varepsilon \) satisfies
\[
\partial_{\tilde{t}} \tilde{\varphi}^\varepsilon = \Delta_{\tilde{x}} \tilde{\varphi}^\varepsilon - W'(\tilde{\varphi}^\varepsilon) + \varepsilon \lambda^\varepsilon(\varepsilon^2 \tilde{t}) \sqrt{2W(\tilde{\varphi}^\varepsilon)},
\tag{11}
\]
where \( \Delta_{\tilde{x}} \) is a Laplacian with respect to \( \tilde{x} \). Assume \( \sup_{x} |\varphi^\varepsilon_0(x)| < 1 \). Then Proposition 5 below yields \( \sup_{x,t} |\varphi^\varepsilon(x,t)| < 1 \). Thus we have
\[
\sup_{\tilde{t} \geq 0} |\varepsilon \lambda^\varepsilon(\varepsilon^2 \tilde{t}) \sqrt{2W(\tilde{\varphi}^\varepsilon)}| \leq \sup_{\tilde{t} \geq 0} |\varepsilon \lambda^\varepsilon(\varepsilon^2 \tilde{t})| \leq \frac{4}{3} \mathcal{L}^d(\Omega) \varepsilon^{1-\alpha} = \frac{4}{3} \varepsilon^{1-\alpha}, \tag{12}
\]
where we used \( \max_{s \in [-1,1]} |k(s)| = \frac{2}{3} \). Therefore, broadly speaking, the non-local term \( \varepsilon \lambda^\varepsilon(\varepsilon^2 \tilde{t}) \sqrt{2W(\tilde{\varphi}^\varepsilon)} \) is a perturbation (to the best of our knowledge, for (1), no phase field model with such a property has been known). Hence we can show the rectifiability and the integrality of the varifold \( V_t \) with arguments similar to that in [23,47] (see also [45]). However, the proofs are not exactly the same as those, because the monotonicity formula for (5) is different from the standard one (see Proposition 8). Therefore we give the proofs in Section 4. In addition, as another good property of \( \lambda^\varepsilon \), the \( L^2 \)-norm can be controlled (see Lemma 1). This property is useful when proving the monotonicity formula and the rectifiability of \( V_t \).
In [23], to construct the weak solution to the mean curvature flow (Brakke flow), the simplest Allen–Cahn equation was considered. As generalizations of the result, the equations with external forces (see [30,34,44,45]) and with Laplace-Beltrami operators (see [36,37]) have been studied. The most well-known phase field model for (1) studied by Rubinstein and Sternberg [39] is the equation
\begin{equation}
\begin{aligned}
\varepsilon \partial_t \varphi^\varepsilon &= \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} + \Lambda^\varepsilon, \quad (x, t) \in \Omega \times (0, T), \\
\varphi^\varepsilon (x, 0) &= \varphi^\varepsilon_0 (x), \quad x \in \Omega,
\end{aligned}
\end{equation}
where \(\Lambda^\varepsilon(t) = \frac{1}{|\mathcal{L}^d(\Omega)|} \int_{\Omega} W' (\varphi^\varepsilon(x,t)) \varepsilon \, dx\). As above, the solution to (13) has the volume preserving property \(\frac{d}{dt} \int_{\Omega} \varphi^\varepsilon \, dx = 0\). Chen, Hilhorst, and Logak [8] proved that for the smooth solution \(\{M_t\}_{t \in [0,T]}\) to (1), there exists a family of functions \(\{\varphi_{0i}\}_{i=1}^\infty\) with \(\varepsilon_i \to 0\) such that the level set \(M_t^{\varepsilon_i} = \{x \in \Omega \ | \ \varphi_{^{\varepsilon_i}}(x,t) = 0\}\) converges to \(M_t\), where \(\varphi^{\varepsilon}\) is a solution to (13) with initial data \(\varphi^\varepsilon_0\). In addition, as mentioned above, Laux and Simon [27] proved the convergence of the vector-valued version of (13) to the weak volume preserving multiphase mean curvature flow under an assumption that corresponds to (3). However, it is an open problem to show its convergence for (13) without such assumptions. One of the difficulties is that the boundedness of \(\sup_{\varepsilon>0} \int_0^T \int_{\Omega} \varepsilon \left(\Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}\right)^2 \, dx \, dt < \infty\).

Note that (14) corresponds to \(\int_0^T \int_{M_t} |\overrightarrow{h}|^2 \, d\mathcal{H}^{d-1} \, dt < \infty\) of the solution to (1) and is important to show the rectifiability of the varifold (see Theorem 12 and Theorem 14). As another phase field method for (1), the study of Brassel and Bretin [6] is known (see [43, Section 1] for a comparison of these equations).

Recently, the weak-strong uniqueness of the mean curvature flow and the relationship between weak solutions have been well studied. In [12], they proved the weak-strong uniqueness of the BV solution to the multiphase mean curvature flow. More precisely, if both BV solution and strong solution with the same initial data exist, then the BV solution agrees with the strong solution while both exist. As a result strongly related to our study, Laux [25] obtained the weak-strong uniqueness for the volume preserving mean curvature flow (see Remark 5). Hensel and Laux [18] proposed a new weak solution to the mean curvature flow via varifolds and the De Giorgi type inequality. Moreover, they proved the existence of the varifold solution and the weak-strong uniqueness. In [41], Stuvard and Tonegawa showed that the multiphase Brakke flow which they construct is a \(L^2\)-flow. In addition, they proved that the flow is also a BV solution for a short time, under the suitable assumptions for initial data (therefore, the results of [12] can be used in this case).

The organization of the rest of this paper is as follows: in Section 2, we set our notations and state the main results. In Section 3, to obtain the existence theorem we prove the energy estimates and \(L^\infty\)-estimates for the solution to (5). In addition, for \(d = 2\) or 3, we give a short proof for the integrality of the limit measure \(\mu_t\).
constructed as a weak solution to (1). In Section 4, we show the integrality of $\mu_t$ for any $d \geq 2$. In Section 5, we prove the main results. In Section 6, we give some supplements for this paper.

2. Preliminaries and Main Results

2.1. Notations and definitions

For $r > 0, d \in \mathbb{N}$, and $x \in \mathbb{R}^d$, we denote $B^d_r(x) := \{ y \in \mathbb{R}^d \mid |x - y| < r \}$ (we often write this as $B_r(x)$ for simplicity). We define $\omega_d := \mathcal{L}^d(B^d_1(0))$. For $d \times d$ matrix $A = (a_{ij})$ and $B = (b_{ij})$, we denote the usual matrix multiplication by $A \circ B$ and $A \cdot B := \sum_{i,j} a_{ij} b_{ij}$. For $a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$, we define a $d \times d$ matrix $a \otimes a$ by $a \otimes a := (a_i a_j)$. Next we recall notations and definitions from geometric measure theory and refer to [11,16,40,47] for more details. For a Caccioppoli set $E \subset \mathbb{R}^d$, we denote the reduced boundary of $E$ by $\partial^* E$. For the characteristic function $\chi_E$, we denote the total variation measure of the distributional derivative $\nabla \chi_E$ by $\|\nabla \chi_E\|$. Let $U \subset \mathbb{R}^d$ be an open set. We write the space of bounded variation functions on $U$ as $BV(U)$. For any Radon measure $\mu$ on $U$ and $\phi \in C_c(U)$, we often write $\int \phi \ d\mu$ as $\mu(\phi)$. For $p \geq 1$, we write $f \in L^p(\mu)$ if $f$ is $\mu$-measurable and $\int |f|^p \ d\mu < \infty$. For $d, k \in \mathbb{N}$ with $k < d$, let $G(d,k)$ be the space of $k$-dimensional subspace of $\mathbb{R}^d$. For an open set $U \subset \mathbb{R}^d$, let $G_k(U) := U \times G(d,k)$. We say $V$ is a general $k$-varifold on $U$ if $V$ is a Radon measure on $G_k(U)$. We denote the set of all general $k$-varifolds on $U$ by $\mathbb{V}_k(U)$. For a general varifold $V \in \mathbb{V}_k(U)$, we define the weight measure $\|V\|$ by

$$\|V\|(\phi) := \int_{G_k(U)} \phi(x) \ dV(x, S) \quad \text{for any } \phi \in C_c(G_k(U)).$$

We call $V \in \mathbb{V}_k(U)$ is rectifiable if there exist a $\mathcal{H}^k$-measurable $k$-countably rectifiable set $M \subset U$ and $\theta \in L^1_{loc}(\mathcal{H}^k|_M)$ such that $\theta > 0 \mathcal{H}^k$-a.e. and

$$V(\phi) = \int_M \phi(x, T_x M) \theta(x) \ d\mathcal{H}^k(x) \quad \text{for any } \phi \in C_c(G_k(U)),$$

where $T_x M$ is the approximate tangent space of $M$ at $x$. Note that such $x$ exists for $\mathcal{H}^k$-a.e. on $M$. If $\theta \in N \mathcal{H}^k$-a.e. on $M$, we call $V$ is integral. In addition, if $\theta = 1 \mathcal{H}^k$-a.e. on $M$, we say $V$ has unit density.

For $V \in \mathbb{V}_k(U)$ and a smooth diffeomorphism $f : U \to U$, we define the push-forward of $V$ by

$$f_# V(\phi) := \int_{G_k(U)} \phi(x, \nabla f(x) \circ S) |A_k \nabla f(x) \circ S| \ dV(x, S) \quad (15)$$

for any $\phi \in C_c(G_k(U))$, where $|A_k \nabla f(x) \circ S|$ is the Jacobian of the map.

For $V \in \mathbb{V}_k(U)$, we define the first variation $\delta V$ by

$$\delta V(\tilde{\phi}) := \int_{G_k(U)} \nabla \tilde{\phi}(x) \cdot S \ dV(x, S) \quad \text{for any } \tilde{\phi} \in C^1_c(U; \mathbb{R}^d).$$
Here, we identify $S \in \mathcal{G}(d, k)$ with the corresponding orthogonal projection of $\mathbb{R}^d$ onto $S$. When the total variation $\|\delta V\|$ of $\delta V$ is locally bounded and absolutely continuous with respect to $\|V\|$, there exists a measurable vector field $\vec{h}$ such that

$$
\delta V(\vec{\phi}) = -\int_U \vec{\phi}(x) \cdot \vec{h}(x) d\|V\|(x) \quad \text{for any } \vec{\phi} \in C_c^1(U; \mathbb{R}^d).
$$

The vector valued function $\vec{h}$ is called the generalized mean curvature vector of $V$. In addition, a Radon measure $\mu$ is called $k$-rectifiable if there exists a $k$-rectifiable varifold such that $\mu$ is represented by $\mu = \|V\|$. Note that this $V$ is uniquely determined, so the first variation and the generalized mean curvature vector of $\mu$ is naturally determined by $V$. The definition of an integral Radon measure is determined in the same way.

The formulation of the following is similar to that of the Brakke flow [5,48]:

**Definition 1.** (*$L^2$-flow* [34]) Let $T > 0$, $U \subset \mathbb{R}^d$ be an open set, and $\{\mu_t\}_{t \in [0, T)}$ be a family of Radon measures on $U$. Set $d\mu := d\mu_t dt$. We call $\{\mu_t\}_{t \in [0, T)}$ an $L^2$-flow with a generalized velocity vector $\vec{v}$ if the following hold:

1. For a.e. $t \in (0, T)$, $\mu_t$ is $(d-1)$-integral, and also has a generalized mean curvature vector $\vec{h} \in L^2(\mu_t; \mathbb{R}^d)$.
2. The vector field $\vec{v}$ belongs to $L^2(0, T; (L^2(\mu_t))^d)$ and

$$
\vec{v}(x, t) \perp T_x \mu_t \quad \text{for } \mu\text{-a.e. } (x, t) \in U \times (0, T),
$$

where $T_x \mu_t \in \mathcal{G}(d, d-1)$ is the approximate tangent space of $\mu_t$ at $x$.
3. There exists $C_T > 0$ such that

$$
\left| \int_0^T \int_U (\partial_t \eta + \nabla \eta \cdot \vec{v}) d\mu_t dt \right| \leq C_T \|\eta\|_{C^0(\overline{U} \times (0, T))} \tag{16}
$$

for any $\eta \in C_c^1(U \times (0, T))$.

**Remark 1.** If there exists a family of smooth hypersurfaces $\{M_t\}_{t \in (0, T)}$ with the normal velocity vector $\vec{w}$, then (16) holds with $\vec{v} = \vec{w}$ and $\mu_t = \mathcal{H}^{d-1}\lfloor M_t$. In addition, if $\vec{v}$ satisfies (16) with $\mu_t = \mathcal{H}^{d-1}\lfloor M_t$, then $\vec{v} = \vec{w}$. This proof is almost identical to the proof in [48, Proposition 2.1].

The $L^2$-flow has the following property:

**Proposition 1.** (See Proposition 3.3 of [34]) Assume that $\{\mu_t\}_{t \in (0, T)}$ is an $L^2$-flow with the generalized velocity vector $\vec{v}$ and set $d\mu := d\mu_t dt$. Then

$$
(\vec{v}(x_0, t_0), 1) \in T_{(x_0, t_0)} \mu
$$

at $\mu$-a.e. $(x_0, t_0) \in \Sigma(\mu)$, where $T_{(x_0, t_0)} \mu \in \mathcal{G}(d+1, d)$ is the approximate tangent space of $\mu$ at $(x_0, t_0)$ and $\Sigma(\mu) = \{(x, t) \mid T_{(x, t)} \mu \text{ exists at } (x, t)\}$.
2.2. Assumptions for initial data

Let $U_0 \subset \subset (0, 1)^d$ be a bounded open set with the following properties:

1. There exists $D_0 > 0$ such that

$$\sup_{x \in (0,1)^d, 0 < R < 1} \frac{\mathcal{H}^{d-1}(M_0 \cap B_r(x))}{\omega_{d-1}r^{d-1}} \leq D_0,$$

where $M_0 = \partial U_0$.

2. There exists a family of open sets $\{U_i^0\}_{i=1}^\infty$ such that $U_i^0$ has a $C^3$ boundary $M_i^0 = \partial U_i^0$ for any $i$ and it holds that

$$\lim_{i \to \infty} \mathcal{L}^d(U_0 \triangle U_i^0) = 0$$

and

$$\lim_{i \to \infty} \| \nabla \chi_{U_i^0} \| = \| \nabla \chi_{U_0} \|$$

as Radon measures.

Note that the second assumption is satisfied when $U_0$ is a Caccioppoli set, and both conditions are fulfilled when $M_0$ is $C^1$ (see [16]).

We denote $q^\varepsilon(r) := \tanh(r/\varepsilon)$ for $r \in \mathbb{R}$. Then denoting derivatives by subscript $r$, $q^\varepsilon$ satisfies

$$\frac{\varepsilon(q^\varepsilon(r))^2}{2} = \frac{W(q^\varepsilon(r))}{\varepsilon}$$

for any $r \in \mathbb{R}$

and

$$q^\varepsilon_{rr}(r) = \frac{W'(q^\varepsilon(r))}{\varepsilon^2}$$

for any $r \in \mathbb{R}$.

In addition, (19) yields

$$\int_{\mathbb{R}} \left( \frac{\varepsilon(q^\varepsilon(r))^2}{2} + \frac{W(q^\varepsilon(r))}{\varepsilon} \right) dr = \int_{\mathbb{R}} \sqrt{2W(q^\varepsilon)}q^\varepsilon_r dr = \int_{-1}^{1} \sqrt{2W(q)} dq =: \sigma.$$

This means that the Radon measure $\mu_i^\varepsilon$ defined below needs to be normalized by $\sigma$.

Next we extend $U_i^0$ and $M_i^0$ periodically to $\mathbb{R}^d$ with period $\Omega = \mathbb{T}^d$ and define

$$r_i(x) = \begin{cases} \text{dist} (x, M_i^0), & \text{if } x \in U_i^0, \\ -\text{dist} (x, M_i^0), & \text{if } x \notin U_i^0. \end{cases}$$

Then $|\nabla r_i(x)| \leq 1$ for a.e. $x \in \mathbb{R}^d$ and there exists $b_i > 0$ such that $r_i$ is $C^3$ on $N_{b_i} := \{x \mid \text{dist} (x, M_i^0) < b_i\}$ (see [9]). Let $d_i$ be a smooth monotone non-decreasing function such that

$$d_i(r) = \begin{cases} r, & \text{if } |r| < \frac{1}{4}b_i, \\ \frac{2}{3}b_i, & \text{if } r > \frac{3}{4}b_i, \\ -\frac{2}{3}b_i, & \text{if } r < -\frac{3}{4}b_i, \end{cases}$$

for $r \in \mathbb{R}$.
and $|\frac{d}{dt}d_i| \leq 1$. Set $\overline{r}_i := d_i(r_i)$. Then $\overline{r}_i \in C^3(\Omega), \overline{r}_i = r_i$ on $N_{b_i/4}$, and $|\nabla \overline{r}_i(x)| \leq 1$ for any $x \in \mathbb{R}^d$. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a positive sequence with $\varepsilon_i \to 0$ and $\frac{\varepsilon_i}{b_i^2} \to 0$ as $i \to \infty$, and

$$\sup_{x \in \mathbb{R}^d} |\nabla^{j+1}\overline{r}_i(x)| \leq \varepsilon_i^{-j} \quad \text{for any } i \in \mathbb{N}, \ j = 1, 2. \quad (21)$$

Note that (21) corresponds to the condition (29) below. We define a periodic function $\varphi_0^{\varepsilon_i} \in C^3(\Omega)$ by

$$\varphi_0^{\varepsilon_i}(x) := \varphi^{\varepsilon_i}(\overline{r}_i(x)) = \tanh \left( \frac{d_i(r_i(x))}{\varepsilon_i} \right) \quad \text{for any } i \in \mathbb{N}. \quad (22)$$

We define a Radon measure $\mu_i^{\varepsilon_i}$ by

$$\mu_i^{\varepsilon_i}(\phi) := \frac{1}{\sigma} \int_{\Omega} \phi \left( \frac{\varepsilon_i |\nabla \varphi_i^{\varepsilon_i}(x, t)|^2}{2} + \frac{W(\varphi_i^{\varepsilon_i}(x, t))}{\varepsilon_i} \right) \, dx, \quad \phi \in C_c(\Omega), \quad (23)$$

where $\varphi_i^{\varepsilon_i}$ is the solution to (5) with initial data $\varphi_0^{\varepsilon_i}$ defined by (22) and $\sigma = \int_{-1}^{1} \sqrt{2W(s)} \, ds$.

**Remark 2.** In this paper we choose a typical function $W(a) = (1-a^2)^2$ as a double-well potential for simplicity. Since more general potentials have been considered for the convergence of the standard Allen–Cahn equation (see [23,45,47]), generalizations of our results regarding $W$ can be made as well.

For $\varphi_0^{\varepsilon_i}$ and $\mu_0^{\varepsilon_i}$, we have the following properties (see [23, p.423] and [30, Section 5]):

**Proposition 2.** There exists a subsequence $\{\varepsilon_i\}_{i=1}^{\infty}$ (denoted by the same index and the subsequence is taken only for $\{\varepsilon_i\}_{i=1}^{\infty}$, not for $\{M_i\}_{i=1}^{\infty}$) such that the following hold:

1. For any $i \in \mathbb{N}$ and $x \in \Omega$, we have

$$\frac{\varepsilon_i |\nabla \varphi_i^{\varepsilon_i}(x)|^2}{2} \leq \frac{W(\varphi_0^{\varepsilon_i}(x))}{\varepsilon_i}.$$

2. There exists $D_1 = D_1(D_0) > 0$ such that

$$\max \left\{ \sup_{i \in \mathbb{N}} \mu_i^{\varepsilon_i}(\Omega), \sup_{i \in \mathbb{N}, \ x \in \Omega} \mu_i^{\varepsilon_i}(B_r(x)) \right\} \leq D_1. \quad (24)$$

3. $\mu_0^{\varepsilon_i} \rightharpoonup \mathcal{H}^{d-1}[-M_0]$ as Radon measures, that is,

$$\int_{\Omega} \phi \, d\mu_0^{\varepsilon_i} \to \int_{M_0} \phi \, d\mathcal{H}^{d-1} \quad \text{for any } \phi \in C_c(\Omega).$$

4. For $\psi_i^{\varepsilon_i} := \frac{1}{2}(\varphi_i^{\varepsilon_i} + 1)$, $\lim_{i \to \infty} \psi_i^{\varepsilon_i} = \chi_{U_0}$ in $L^1$ and $\lim_{i \to \infty} \|\nabla \psi_i^{\varepsilon_i}\| = \|\nabla \chi_{U_0}\|$ as Radon measures.

**Remark 3.** The first property (1) is obtained from $|\nabla \overline{r}_i| \leq 1$ (see the proof of Proposition 6). The assumption $\frac{\varepsilon_i}{b_i^2} \to 0$ is used to show $\int_{\Omega \setminus N_{b_i/4}} \left( \frac{\varepsilon_i |\nabla \varphi_i^{\varepsilon_i}|^2}{2} + \frac{W(\varphi_0^{\varepsilon_i})}{\varepsilon_i} \right) \, dx \to 0$. 
2.3. Main results

We denote the approximate velocity vector \( \tilde{v}^{\varepsilon_i} \) by

\[
\tilde{v}^{\varepsilon_i} = \begin{cases} 
- \partial_t \varphi^{\varepsilon_i} \frac{\nabla \varphi^{\varepsilon_i}}{|\nabla \varphi^{\varepsilon_i}|}, & \text{if } |\nabla \varphi^{\varepsilon_i}| \neq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

The first main result of this paper is

**Theorem 3.** Suppose that \( d \geq 2 \) and \( U_0 \) satisfies (17) and (18). For any \( i \in \mathbb{N} \), let \( \varphi_0^{\varepsilon_i} \) be defined so that all the claims of Proposition 2 are satisfied and \( \varphi^{\varepsilon_i} \) be a solution to (5) with initial data \( \varphi_0^{\varepsilon_i} \). Then there exists a subsequence \( \{\varepsilon_i\}_{i=1}^{\infty} \) (denoted by the same index) such that the following hold:

(a) There exist a countable subset \( B \subset [0, \infty) \) and a family of \( (d-1) \)-integral Radon measures \( \{\mu_t\}_{t \in [0, \infty)} \) on \( \Omega \) such that

\[
\mu_0 = \mathcal{H}^{d-1}|_M, \quad \mu^{\varepsilon_i}_t \rightharpoonup \mu_t \quad \text{as Radon measures for any } t \geq 0,
\]

and

\[
\mu_s(\Omega) \leq \mu_t(\Omega) \quad \text{for any } s, t \in [0, \infty) \setminus B \text{ with } 0 \leq t < s < \infty.
\]

(b) There exists \( \psi \in BV_{loc}(\Omega \times [0, \infty)) \cap C_{loc}^1([0, \infty); L^1(\Omega)) \) such that the following hold:

(b1) \( \psi^{\varepsilon_i} \rightharpoonup \psi \text{ in } L^1_{loc}(\Omega \times [0, \infty)) \) and a.e. pointwise, where \( \psi^{\varepsilon_i} = \frac{1}{2}(\varepsilon_i + 1). \)

(b2) \( \psi|_{t=0} = \chi_{U_0} \text{ a.e. on } \Omega. \)

(b3) For any \( t \in [0, \infty) \), \( \psi(x, t) = 1 \) or \( 0 \) for \( \mathcal{L}^d \)-a.e. \( x \in \Omega \) and \( \psi \) satisfies the volume preserving property, that is,

\[
\int_\Omega \psi(x, t) \, dx = \mathcal{L}^d(U_0) \quad \text{for all } t \in [0, \infty).
\]

(b4) For any \( t \in [0, \infty) \) and for any \( \phi \in C_c(\Omega; [0, \infty)) \), we have \( \|\nabla \psi(\cdot, t)\| (\phi) \leq \mu_t(\phi). \)

(c) For \( \lambda^{\varepsilon_i} \) given by (6), we have

\[
\sup_{i \in \mathbb{N}} \int_0^T |\lambda^{\varepsilon_i}|^2 \, dt < \infty \quad \text{for any } T > 0
\]

and there exists \( \lambda \in L^2_{loc}(0, \infty) \) such that \( \lambda^{\varepsilon_i} \rightharpoonup \lambda \) weakly in \( L^2(0, T) \) for any \( T > 0. \)

(d) There exists \( \tilde{f} \in L^2_{loc}([0, \infty); (L^2(\mu_t))^d) \) such that

\[
\lim_{i \to \infty} \frac{1}{\sigma} \int_0^\infty \int_\Omega -\lambda^{\varepsilon_i} \sqrt{2W(\varphi^{\varepsilon_i})} \nabla \varphi^{\varepsilon_i} \cdot \tilde{f} \, dx \, dt = \int_\Omega \int_0^\infty \tilde{f} \cdot \tilde{\phi} \, d\mu_t \, dt \quad \text{for any } \tilde{\phi} \in C_c(\Omega \times [0, \infty); \mathbb{R}^d), \text{ where } \tilde{v} \text{ is the inner unit normal vector of } \{\psi(\cdot, t) = 1\} \text{ on } \text{spt} \|\nabla \psi(\cdot, t)\|.}
\]
(e) The family of Radon measures \( \{ \mu_t \}_{t \in (0, \infty)} \) is an \( L^2 \)-flow with a generalized velocity vector \( \tilde{v} = \tilde{h} + \tilde{f} \), where \( \tilde{h} \in L^2_{loc}((0, \infty); (L^2(\mu_t))^d) \) is the generalized mean curvature vector of \( \mu_t \). Moreover, for any \( \tilde{\phi} \in C_c(\Omega \times [0, \infty); \mathbb{R}^d) \),

\[
\lim_{t \to \infty} \int_0^\infty \int_\Omega \tilde{v} \cdot \partial_t \mu_t dt = \int_0^\infty \int_\Omega \tilde{v} \cdot \tilde{\phi} d\mu_t dt.
\]

Remark 4. From (b4), (d), and (e), we have

\[
\text{Remark 4. For any } \phi \in C_c(\Omega \times (0, \infty); \mathbb{R}^d), \text{ where } \frac{d\|\nabla \psi(\cdot, t)\|}{d\mu_t} \text{ is the Radon–Nikodym derivative. Hence we have } \tilde{v} = \tilde{h} - \lambda \tilde{v} \text{ in the sense of } L^2 \text{-flow if } \mu_t = \|\nabla \psi(\cdot, t)\| \text{ for a.e. } t \text{ (from Theorem 4 below, this is correct for a short time if the initial data is close to a ball).}
\]

We also show that if \( U_0 \approx B \), then there exists \( T_1 > 0 \) such that \( \{ \partial^* U_t \}_{t \in [0, T_1)} \) is a distributional solution to (1) in the framework of BV functions.

Theorem 4. For any \( r \in (0, \frac{1}{4}) \), there exists \( \delta_1 > 0 \) depending only on \( d \) and \( r \) with the following properties. Assume that \( U_0 \subset (\frac{1}{4}, \frac{3}{4})^d \) satisfies \( \mathcal{L}^d(U_0) = \mathcal{L}^d(B_r(0)) \) and has a \( C^1 \) boundary \( M_0 \) with \( \mathcal{H}^{d-1}(M_0) \leq 2\mathcal{H}^{d-1}(\partial B_r(0)) \) and

\[
\mathcal{H}^{d-1}(M_0) - d\omega_d^\frac{1}{d}(\mathcal{L}^d(U_0)) \frac{d-1}{d} \leq \delta_1.
\]

Then there exists \( T_1 = T_1(d, r, M_0) > 0 \) such that the following hold:

(a) For a.e. \( t \in [0, T_1) \), \( \mu_t = \|\nabla \psi(\cdot, t)\| = \mathcal{H}^{d-1}|_{\partial^* U_t} \), where \( \{ \mu_t \}_{t \in (0, \infty)} \) is the \( L^2 \)-flow with initial data \( \mu_0 = \mathcal{H}^{d-1}|_{M_0} \), given by Theorem 3.
(b) Let \( \tilde{v}, \tilde{h}, \tilde{v}, \) and \( \lambda \) be functions given by Theorem 3. Then \( \{ \partial^* U_t \}_{t \in [0, T_1)} \) is a distributional solution to (1) with initial data \( \partial U_0 = M_0 \) in the following sense.

(b1) For any \( t \in [0, T_1) \), \( \mathcal{L}^d(U_t) = \mathcal{L}^d(U_0) \).
(b2) For a.e. \( t \in [0, T_1) \), \( \tilde{h} \) is also a generalized mean curvature vector of \( \mathcal{H}^{d-1}|_{\partial^* U_t} \).

(b3) For any \( \phi \in C_c(\Omega \times [0, T_1); \mathbb{R}^d) \), we have

\[
\int_0^{T_1} \int_{\partial^* U_t} (\tilde{v} - \tilde{h} + \lambda \tilde{\nu}) \cdot \tilde{\phi} d\mathcal{H}^{d-1} dt = 0.
\]

(b4) For any \( \phi \in C^1_c(\Omega \times (0, T_1)) \), we have

\[
\int_0^{T_1} \int_{U_t} \partial_t \phi dx dt = \int_0^{T_1} \int_{\partial^* U_t} \tilde{v} \cdot \tilde{\nu} \phi d\mathcal{H}^{d-1} dt.
\]
(b5) (Additional volume preserving property). For a.e. \( t \in [0, T_1] \), we have
\[
\int_{\partial^* U_t} \tilde{v} \cdot \nu \, dH^{d-1} = \int_{\Omega} \tilde{v} \cdot \nu \| \nabla \psi (\cdot, t) \| = 0.
\]

(b6) For a.e. \( t \in [0, T_1] \), we have
\[
\lambda(t) = \frac{1}{H^{d-1}(\partial^* U_t)} \int_{\partial^* U_t} h \cdot \nu \, dH^{d-1}.
\]

Remark 5. Recently, Laux [25] showed that if both strong solution and BV solution to the volume preserving mean curvature flow with same initial data exist, then the strong solution matches the BV solution while both exist (see also [12, 18]).

Remark 6. The isoperimetric inequality tells us that \( d\omega_{\frac{1}{d}}(\mathcal{L}^d(U)) \frac{d-1}{d} \leq H^{d-1} (\partial^* U) \) for any Caccioppoli set \( U \subset \mathbb{R}^d \) with \( \mathcal{L}^d(U) < \infty \) and the equality holds if and only if there exists a ball \( B \subset \mathbb{R}^d \) such that \( \mathcal{L}^d(U \triangle B) = 0 \) (see [13, 46] and references therein). Moreover, by the quantitative isoperimetric inequality (see [14, Theorem 1.1]) and the assumption (27), we have
\[
\min \left\{ \frac{\mathcal{L}^d(U_0 \triangle (B_r(x)))}{r^d} \left| x \in \mathbb{R}^d \right. \right\} \leq \frac{C \sqrt{\delta_1}}{\sqrt{\mathcal{H}^{d-1}(B_r(0))}},
\]
where \( r > 0 \) is a constant given by Theorem 4 and \( C > 0 \) depends only on \( d \) (thus, \( U_0 \) needs to be close to a sphere in the above sense). On the other hand, \( M_0 \) does not have to be close to a sphere in \( C^0 \) (for example, \( U_0 \) does not have to be connected).

Remark 7. The property (b4) claims that \( \tilde{v} \) is a normal velocity vector in a weak sense, since
\[
\frac{d}{dt} \int_{U_t} \phi \, dx = \int_{U_t} \partial_t \phi \, dx - \int_{\partial U_t} \tilde{v} \cdot \nu \phi \, dH^{d-1}
\]
holds for any \( \phi \in C^1_c(\Omega \times (0, T_1)) \) and \( t \in (0, T_1) \), where \( \{U_t\}_{t \in [0, T_1]} \) is a family of open sets and the smooth boundary \( \partial U_t \) moves by the normal velocity vector \( \tilde{v} \). By (2), we can regard (b5) as a volume preserving property in a weak sense.

### 3. Energy and Pointwise Estimates

In this section we show standard estimates for (5) such as the uniform \( L^2 \)-estimate for \( \lambda^\varepsilon \) and the monotonicity formula.
3.1. Assumptions

Let \( \{\varepsilon_i\}_{i=1}^{\infty} \) be a positive sequence with \( \varepsilon_i \to 0 \) as \( i \to \infty \). In this section, we assume that there exist \( D_1 > 0 \) and \( \omega > 0 \) such that (24) and

\[
\frac{2}{3} - \left| \int_{\Omega} k(\varphi_{0}^{\varepsilon_i}(x)) \, dx \right| > \omega > 0 \tag{28}
\]

hold for any \( i \in \mathbb{N} \). The set \( \{x \in \Omega \mid \varphi_{0}^{\varepsilon_i}(x) = 0\} \) corresponds to the initial data \( M_0 \) of (1), and (28) yields that \( \mathcal{L}^d(\{x \in \Omega \mid \varphi_{0}^{\varepsilon_i} \approx 1\}) > 0 \) formally, since \( \int_{\Omega} k(\pm 1) \, dx = \pm \frac{2}{3} \mathcal{L}^d(\Omega) = \pm \frac{2}{3} \). For some \( C_1 > 0 \), we also assume that the initial data \( \varphi_{0}^{\varepsilon_i} \) of the solution to (5) satisfies

\[
\varphi_{0}^{\varepsilon_i} \in C^3(\Omega), \quad \sup_{x \in \Omega} |\varphi_{0}^{\varepsilon_i}(x)| < 1, \quad \text{and} \quad \varepsilon_i^{j} \sup_{x \in \Omega} |\nabla^j \varphi_{0}^{\varepsilon_i}(x)| \leq C_1 \tag{29}
\]

for any \( i \in \mathbb{N} \) and \( j = 1, 2, 3 \). In addition, to control the discrepancy measure \( \xi_{i}^{\varepsilon} \) defined below, we assume

\[
\frac{\varepsilon_i |\nabla \varphi_{0}^{\varepsilon_i}(x)|^2}{2} \leq \frac{W(\varphi_{0}^{\varepsilon_i}(x))}{\varepsilon_i} \quad \text{for any } x \in \Omega \text{ and } i \in \mathbb{N}. \tag{30}
\]

Note that the function \( \varphi_{0}^{\varepsilon_i} \) defined by (22) satisfies all the assumptions above, for sufficiently large \( i \). Throughout this paper, we often write \( \varepsilon \) as \( \varepsilon_i \) for simplicity.

3.2. Pointwise estimates

The comparison principle implies

**Proposition 5.** The solution \( \varphi^{\varepsilon} \) to (5) with (29) satisfies

\[
|\varphi^{\varepsilon}(x, t)| < 1, \quad x \in \Omega, \quad t \geq 0. \tag{31}
\]

**Remark 8.** The estimate (31) implies \( \sqrt{2W(\varphi^{\varepsilon})} = 1 - (\varphi^{\varepsilon})^2 \). By a priori estimates including Proposition 6 below, standard PDE theories imply the global existence and uniqueness of the classical solution to (5) with initial data \( \varphi_{0}^{\varepsilon} \) satisfying (29).

**Proof.** Suppose that \( t_0 := \inf\{t \in [0, \infty) \mid \sup_{x \in \Omega} \varphi^{\varepsilon}(x, t) \geq 1\} < \infty \). Then \( t_0 > 0 \) since \( \sup_{x \in \Omega} \varphi_{0}^{\varepsilon}(x) < 1 \). We may assume that there exists \( t_1 \in (t_0, \infty) \) such that \( \sup_{x \in \Omega} \varphi^{\varepsilon}(x, t) \leq 2 \) for any \( t < t_1 \). Let \( \varphi_{+}^{\varepsilon} \) be a solution to

\[
\varepsilon \partial_t(\varphi_{+}^{\varepsilon}) = \varepsilon \Delta \varphi_{+}^{\varepsilon} - \frac{W'(\varphi_{+}^{\varepsilon})}{\varepsilon} + L^\varepsilon \sqrt{2W(\varphi_{+}^{\varepsilon})}, \quad (x, t) \in \Omega \times (0, t_1) \tag{32}
\]

with initial data \( \varphi_{+}^{\varepsilon}(x, 0) = \sup_{x \in \Omega} \varphi_{0}^{\varepsilon}(x) \), where \( L^\varepsilon := 2\varepsilon^{-\alpha} \max_{|s| \leq 2} |k(s)| \). Note that \( \sup_{t \in (0, t_1)} |\lambda^{\varepsilon}(t)| \leq L^\varepsilon \), where \( \lambda^{\varepsilon} \) is given by the solution \( \varphi^{\varepsilon} \) to (5), and this implies that \( \varphi_{+}^{\varepsilon} \) is a supersolution to (5) if we regard \( \lambda^{\varepsilon} \) as a given function. Since the initial data is constant and \( W'(s), \sqrt{2W(s)} \to 0 \) as \( s \to 1 \), one can easily check that the solution \( \varphi_{+}^{\varepsilon} \) to (32) depends only on \( t \) and satisfies \( \varphi_{+}^{\varepsilon}(t) < 1 \) for any \( t \in (0, t_1) \). Therefore the comparison principle implies that \( \varphi^{\varepsilon}(x, t) \leq \varphi_{+}^{\varepsilon}(t) < 1 \) for any \( (x, t) \in \Omega \times (0, t_1) \). This yields a contradiction. Hence \( \varphi^{\varepsilon}(x, t) < 1 \) for any \( (x, t) \in \Omega \times [0, \infty) \) and the remaining inequality \( \varphi^{\varepsilon}(x, t) > -1 \) can be proved similarly. \( \square \)
In addition, by Proposition 5 and the maximum principle we have the following proposition (see [23]):

**Proposition 6.** If the solution \( \varphi^\varepsilon \) to (5) satisfies (29) and (30), then we have

\[
\frac{\varepsilon |\nabla \varphi^\varepsilon(x, t)|^2}{2} \leq \frac{W(\varphi^\varepsilon(x, t))}{\varepsilon}, \quad x \in \Omega, \ t \geq 0.
\]

**(Proof.** By (31), we can define a function \( r^\varepsilon \) by

\[
r^\varepsilon(x, t) = (q^\varepsilon)^{-1}(\varphi^\varepsilon(x, t)), \quad x \in \Omega, \ t \geq 0,
\]

since \( q^\varepsilon : \mathbb{R} \to (-1, 1) \) is one to one and surjective. Denoting the derivatives of \( q^\varepsilon \) by subscript \( r \), we compute that

\[
\varepsilon q^\varepsilon r \frac{\partial t r^\varepsilon}{\partial t} = \varepsilon q^\varepsilon r \frac{\Delta r^\varepsilon}{\Delta 1} + \varepsilon q^\varepsilon r \frac{\nabla r^\varepsilon}{\nabla 2} - \frac{W'(q^\varepsilon)}{\varepsilon} + \lambda^\varepsilon \sqrt{2W(q^\varepsilon)}
\]

where we used (19) and (20). Then we obtain

\[
\partial_t r^\varepsilon = \Delta r^\varepsilon - \frac{2q^\varepsilon(r^\varepsilon)}{\varepsilon}(|\nabla r^\varepsilon|^2 - 1) + \lambda^\varepsilon,
\]

(34)

where we used \( W'(q^\varepsilon)/\sqrt{2W(q^\varepsilon)} = -2q^\varepsilon \). We compute

\[
\frac{1}{2} \partial_t |\nabla r^\varepsilon|^2 = \frac{1}{2} \Delta |\nabla r^\varepsilon|^2 - |\nabla r^\varepsilon|^2 - \nabla r^\varepsilon \cdot \nabla \left( \frac{2q^\varepsilon(r^\varepsilon)}{\varepsilon}(|\nabla r^\varepsilon|^2 - 1) \right),
\]

where we used \( \nabla \lambda^\varepsilon = 0 \). Set \( w^\varepsilon = |\nabla r^\varepsilon|^2 - 1 \). Then \( w^\varepsilon \) satisfies

\[
\partial_t w^\varepsilon \leq \Delta w^\varepsilon - \frac{4q^\varepsilon(r^\varepsilon)}{\varepsilon} \nabla r^\varepsilon \cdot \nabla w^\varepsilon - 2 \left( \nabla r^\varepsilon \cdot \nabla \frac{2q^\varepsilon(r^\varepsilon)}{\varepsilon} \right) w^\varepsilon.
\]

In addition, we have \( w^\varepsilon(x, 0) \leq 0 \), because

\[
\frac{\varepsilon |\nabla \varphi^\varepsilon_0(x)|^2}{2} - \frac{W(\varphi^\varepsilon_0(x))}{\varepsilon} = \frac{W(q^\varepsilon(r^\varepsilon(x, 0)))}{\varepsilon}(|\nabla r^\varepsilon(x, 0)|^2 - 1) \leq 0
\]

by (30). Hence the maximum principle implies \( w^\varepsilon(x, t) \leq 0 \) for any \( x \in \Omega \) and \( t \in [0, \infty) \), and we obtain (33) by \( \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} - \frac{W(\varphi^\varepsilon)}{\varepsilon} = \frac{W(q^\varepsilon)}{\varepsilon} w^\varepsilon \leq 0. \)

\( \square \)
3.3. Energy estimates

By \( E^\varepsilon(t) = \sigma \mu^\varepsilon_t(\Omega) \), (8), (9), and (24), we can easily obtain the following estimates:

**Proposition 7.** For any \( \varepsilon > 0 \) and \( T > 0 \), we have

\[
\mu^\varepsilon_T(\Omega) + \frac{1}{\sigma} \int_0^T \int_\Omega \varepsilon (\partial_t \varphi^\varepsilon)^2 \, dx \, dt \leq \mu^\varepsilon_0(\Omega) \tag{35}
\]

and

\[
\sup_{\varepsilon \in (0,1)} \mu^\varepsilon_T(\Omega) \leq \sup_{\varepsilon \in (0,1)} \mu^\varepsilon_0(\Omega) \leq D_1. \tag{36}
\]

**Remark 9.** Generally, \( "\mu^\varepsilon_s(\Omega) \leq \mu^\varepsilon_t(\Omega)" \) for any \( 0 \leq t < s < \infty \) can not be shown from the energy estimates above. However, there exists a countable set \( B \) such that \( \mu^\varepsilon_s(\Omega) \leq \mu^\varepsilon_t(\Omega) \) holds for any \( t, s \in [0, \infty) \) \( \setminus B \) with \( t < s \), where \( \mu^\varepsilon_t(\Omega) = \lim_{\varepsilon \to 0} \mu^\varepsilon_t(\Omega) \) (see Proposition 10).

Set \( D'_1 := \sup_{\varepsilon \in (0,1)} \mu^\varepsilon_0(\Omega) \). Note that \( D'_1 \leq D_1 \). By an argument similar to that in [7], we have the following lemma:

**Lemma 1.** There exist constants \( C_2 = C_2(\omega, d, D'_1) > 0 \), \( C_3 = C_3(\omega, d, D'_1) > 0 \), and \( \varepsilon_1 = \varepsilon_1(\omega, d, D'_1, \alpha) > 0 \) such that

\[
\int_0^T |\lambda^\varepsilon(t)|^2 \, dt \leq C_2(\mu^\varepsilon_0(\Omega) - \mu^\varepsilon_T(\Omega) + T) \quad \text{for any } \varepsilon \in (0, \varepsilon_1) \text{ and } T > 0, \tag{37}
\]

and

\[
\sup_{\varepsilon \in (0, \varepsilon_1)} \int_{t_1}^{t_2} |\lambda^\varepsilon(t)|^2 \, dt \leq C_3(1 + t_2 - t_1) \quad \text{for any } 0 \leq t_1 < t_2 < \infty. \tag{38}
\]

**Proof.** Let \( \tilde{\xi} = (\xi^1, \xi^2, \ldots, \xi^d) : \Omega \times [0, \infty) \to \mathbb{R}^d \) be a smooth periodic test function. By integration by parts, we have

\[
\int_\Omega \sqrt{2W(\varphi^\varepsilon)} \nabla \varphi^\varepsilon \cdot \tilde{\xi} \, dx = \int_\Omega \nabla k(\varphi^\varepsilon) \cdot \tilde{\xi} \, dx = -\int_\Omega k(\varphi^\varepsilon) \text{div} \tilde{\xi} \, dx \tag{39}
\]

and

\[
\int_\Omega \Delta \varphi^\varepsilon \nabla \varphi^\varepsilon \cdot \tilde{\xi} \, dx = -\sum_{i,j=1}^d \int_\Omega \partial_i \varphi^\varepsilon \partial_i \varphi^\varepsilon \partial_j \xi^j \, dx + \int_\Omega \frac{|
abla \varphi^\varepsilon|^2}{2} \text{div} \tilde{\xi} \, dx. \tag{40}
\]
Multiply (5) by $\nabla \varphi^\varepsilon \cdot \tilde{\zeta}$ and integrate over $\Omega$. Then, using (39) and (40), we have

$$
\int_\Omega \varepsilon \partial_t \varphi^\varepsilon \nabla \varphi^\varepsilon \cdot \tilde{\zeta} \, dx + \sum_{i,j=1}^d \int_\Omega \varepsilon \partial_{x_i} \varphi^\varepsilon \partial_{x_j} \varphi^\varepsilon \partial_{x_i} \xi^j \, dx \\
- \int_\Omega \left( \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon} \right) \text{div } \tilde{\zeta} \, dx = -\lambda^\varepsilon \int_\Omega k(\varphi^\varepsilon) \text{div } \tilde{\zeta} \, dx. \tag{41}
$$

The Cauchy–Schwarz inequality, (35), and (36) imply

$$
\left| \int_\Omega \varepsilon \partial_t \varphi^\varepsilon \nabla \varphi^\varepsilon \cdot \tilde{\zeta} \, dx + \sum_{i,j=1}^d \int_\Omega \varepsilon \partial_{x_i} \varphi^\varepsilon \partial_{x_j} \varphi^\varepsilon \partial_{x_i} \xi^j \, dx \\
- \int_\Omega \left( \frac{\varepsilon |\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon} \right) \text{div } \tilde{\zeta} \, dx \right| \leq C_4 \left\| \tilde{\zeta} (\cdot, t) \right\|_{C^1(\Omega)} \left( (D'_1)^{\frac{1}{2}} \left( \int_\Omega \varepsilon (\partial_t \varphi^\varepsilon)^2 \, dx \right)^{\frac{1}{2}} + D'_1 \right), \tag{42}
$$

where $C_4 > 0$ depends only on $d$. Let $\eta \in C^\infty_c (B_1(0))$ be a smooth non-negative function with $\int_{B_1(0)} \eta \, dx = 1$ and define the standard mollifier $\eta_\delta$ by $\eta_\delta(x) = \delta^{-d} \eta(x/\delta)$ for $\delta > 0$. Let $u = u(x, t)$ be the periodic solution to

$$
\begin{cases}
-\Delta u = k(\varphi^\varepsilon) * \eta_\delta - \int_\Omega (k(\varphi^\varepsilon) * \eta_\delta) \quad \text{in } \Omega, \\
\int_\Omega u \, dx = 0.
\end{cases}
$$

Note that

$$
\int_\Omega \left\{ k(\varphi^\varepsilon) * \eta_\delta - \int_\Omega (k(\varphi^\varepsilon) * \eta_\delta) \right\} = 0
$$

and there exists $C > 0$ depending only on $\mathcal{L}^d(\Omega)$ such that

$$
\left\| k(\varphi^\varepsilon (\cdot, t)) * \eta_\delta - \int_\Omega (k(\varphi^\varepsilon (\cdot, t)) * \eta_\delta) \right\|_{C^1(\Omega)} \leq C(1 + \delta^{-1}), \quad t \geq 0,
$$

where we used $\| \varphi^\varepsilon \|_{L^\infty} \leq 1$. Therefore the standard PDE arguments imply the existence and uniqueness of the solution $u$ and

$$
\| u(\cdot, t) \|_{C^2, \beta(\Omega)} \leq C_5, \quad t \geq 0,
$$

where $\beta \in (0, 1)$ and $C_5 > 0$ depends only on $\beta, d,$ and $\delta$. Set $\tilde{\zeta}(x, t) = \nabla u(x, t)$. Then, by (41) and (42), we have

$$
|\lambda^\varepsilon| \int_\Omega k(\varphi^\varepsilon) \text{div } \tilde{\zeta} \, dx \leq C_4 C_5 \left( (D'_1)^{\frac{1}{2}} \left( \int_\Omega \varepsilon (\partial_t \varphi^\varepsilon)^2 \, dx \right)^{\frac{1}{2}} + D'_1 \right). \tag{43}
$$
We compute
\[- \int_{\Omega} k(\varphi^\varepsilon) \text{div} \vec{\zeta} \, dx = \int_{\Omega} k(\varphi^\varepsilon)(-\Delta u) \, dx \]
\[= \int_{\Omega} k(\varphi^\varepsilon) \left\{ k(\varphi^\varepsilon) * \eta_{\delta} - \int_{\Omega} k(\varphi^\varepsilon) * \eta_{\delta} \right\} \, dx \]
\[= \frac{4}{9} \mathcal{L}^d(\Omega) + \int_{\Omega} (k(\varphi^\varepsilon))^2 - \frac{4}{9} \, dx + \int_{\Omega} k(\varphi^\varepsilon)(k(\varphi^\varepsilon) * \eta_{\delta} - k(\varphi^\varepsilon)) \, dx - \frac{1}{\mathcal{L}^d(\Omega)} \left( \int_{\Omega} k(\varphi^\varepsilon) \, dx \right)^2 + \frac{1}{\mathcal{L}^d(\Omega)} \]
\[\times \int_{\Omega} k(\varphi^\varepsilon) \, dx \left( \int_{\Omega} k(\varphi^\varepsilon) \, dx - \int_{\Omega} k(\varphi^\varepsilon) * \eta_{\delta} \, dx \right). \quad (44)\]

By \((k(s))^2 - \frac{4}{9} \geq -W(s)\) for any \(s \in [-1, 1]\), we have
\[\int_{\Omega} (k(\varphi^\varepsilon))^2 - \frac{4}{9} \, dx \geq -\varepsilon \sigma \mu^\varepsilon(\Omega) - \varepsilon \sigma \sigma D'_1. \quad (45)\]

By using
\[\int_{\Omega} |\nabla(k(\varphi^\varepsilon))| \, dx = \int_{\Omega} \sqrt{2W(\varphi^\varepsilon)}|\nabla \varphi^\varepsilon| \, dx \leq \sigma \mu^\varepsilon(\Omega) \leq \sigma D'_1,\]
\[\|\varphi^\varepsilon\|_{L^\infty} \leq 1,\] and Proposition 18, we have
\[\left| \int_{\Omega} k(\varphi^\varepsilon)(k(\varphi^\varepsilon) * \eta_{\delta} - k(\varphi^\varepsilon)) \, dx \right| \leq C_6 \delta, \quad (46)\]
and
\[\left| \frac{1}{\mathcal{L}^d(\Omega)} \int_{\Omega} k(\varphi^\varepsilon) \, dx \left( \int_{\Omega} k(\varphi^\varepsilon) \, dx - \int_{\Omega} k(\varphi^\varepsilon) * \eta_{\delta} \, dx \right) \right| \leq C_6 \delta, \quad (47)\]
where \(C_6 > 0\) depends only on \(D'_1\) and \(\mathcal{L}^d(\Omega)\). Set \(\delta = \frac{\omega^2}{4C_6 \mathcal{L}^d(\Omega)}\). By (10), (28), (44), (45), (46), and (47), there exists \(\varepsilon_1 > 0\) depending only on \(\alpha, D'_1, \mathcal{L}^d(\Omega),\) and \(\omega\) such that
\[- \int_{\Omega} k(\varphi^\varepsilon) \text{div} \vec{\zeta} \, dx \]
\[\geq \frac{4}{9} \mathcal{L}^d(\Omega) - \frac{1}{\mathcal{L}^d(\Omega)} \left( \int_{\Omega} k(\varphi^\varepsilon) \, dx \right)^2 - \varepsilon \sigma D'_1 - 2C_6 \delta \]
\[\geq \frac{1}{\mathcal{L}^d(\Omega)} \left( \omega^2 - \frac{4\sqrt{2}}{3} \varepsilon \frac{\omega^2}{3} \left( D'_1 \right)^{\frac{1}{2}} \right) - \varepsilon \sigma D'_1 - 2C_6 \delta \]
\[\geq \frac{1}{4 \mathcal{L}^d(\Omega)} \omega^2 \]
holds for any \(\varepsilon \in (0, \varepsilon_1)\), where we used \((\int_{\Omega} k(\varphi^\varepsilon_0) \, dx)^2 - (\int_{\Omega} k(\varphi^\varepsilon) \, dx)^2 \leq \frac{4\sqrt{2}}{3} \varepsilon \frac{\omega^2}{3} \left( D'_1 \right)^{\frac{1}{2}}\) by (10). From (35), (36), (43), and (48), we obtain (37) and (38). □
Remark 10. For the classical solution to the volume preserving mean curvature flow, we can obtain a similar estimate for the non-local term (see Proposition 19).

We define the discrepancy measure \( \xi_t^\varepsilon \) on \( \Omega \) by

\[
\xi_t^\varepsilon(\phi) := \frac{1}{\sigma} \int_{\Omega} \phi(x)\xi_t^\varepsilon(x, t) \, dx, \quad \phi \in C_c(\Omega),
\]

where

\[
\xi_t^\varepsilon(x, t) = \varepsilon |\nabla \varphi_t^\varepsilon(x, t)|^2 - \frac{W(\varphi_t^\varepsilon(x, t))}{\varepsilon}.
\]

Proposition 6 implies the following lemma.

Lemma 2. Assume (30). Then \( \xi_t^\varepsilon(x, t) \leq 0 \) for any \( (x, t) \in \Omega \times [0, \infty) \). In addition, \( \xi_t^\varepsilon \) is a non-positive measure for any \( t \geq 0 \).

We denote the backward heat kernel \( \rho = \rho(y,s)(x,t) \) by

\[
\rho(y,s)(x,t) = \frac{1}{(4\pi(s-t))^{d/2}} e^{-\frac{|x-y|^2}{4(s-t)}}, \quad x, y \in \mathbb{R}^d, \ 0 \leq t < s.
\]

With exactly the same proof as in [43, p. 2028], we obtain the following estimates similar to the monotonicity formula obtained by Huisken [20] and Ilmanen [23] (for convenience, we call thus the monotonicity formula)

Proposition 8. (See [43]) Let \( \xi_t^\varepsilon(x, 0) \leq 0 \) for any \( x \in \Omega \). Assume (30). Then

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\mu_t^\varepsilon(x)
\]

\[
\leq \frac{1}{2(s-t)} \int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\xi_t^\varepsilon(x) + \frac{1}{2}(\lambda_t^\varepsilon)^2 \int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\mu_t^\varepsilon(x)
\]

holds for any \( 0 \leq t < s < \infty \) and for any \( y \in \mathbb{R}^d \). Here, \( \mu_t^\varepsilon \) and \( \xi_t^\varepsilon \) are extended periodically to \( \mathbb{R}^d \). In addition, we have

\[
\int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\mu_t^\varepsilon(x) \bigg|_{t=t_1}^{t=t_2} \leq \left( \int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\mu_t^\varepsilon(x) \bigg|_{t=t_1}^{t=t_2} \right) e^{\frac{1}{2} \int_{t_1}^{t_2} |\lambda_t^\varepsilon|^2 \, dt}
\]

\[
\leq \left( \int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\mu_t^\varepsilon(x) \bigg|_{t=t_1}^{t=t_2} \right) e^{C_3(t_2-t_1+1)}
\]

for any \( y \in \mathbb{R}^d, 0 \leq t_1 < t_2 < \infty, \) and \( \varepsilon \in (0, \varepsilon_1) \).

Remark 11. Ilmanen [23] proved the monotone decreasing of \( \int_{\mathbb{R}^d} \rho(y,s)(x,t) \, d\mu_t^\varepsilon(x) \) with respect to \( t \), for the solution to the Allen–Cahn equation without the non-local term under suitable assumptions. In general, one can show that the Brakke flow with smooth initial data has unit density for a short time by using the monotonicity formula see [45]). However, in order to show a similar conclusion for our problem, it is necessary that \( \mu_0^\varepsilon(\Omega) - \mu_t^\varepsilon(\Omega) \) is small enough, due to (37) (see Lemma 12 below).
As a corollary of the monotonicity formula, we can obtain the following upper bounds of the densities of $\mu_t^\varepsilon$:

**Corollary 1.** (See [23,42]) There exists $0 < D_2 < \infty$ depending only on $d$, $C_3$, $D_1$, and $T$ such that

$$
\mu_t^\varepsilon(B_R(y)) \leq D_2 R^{d-1}
$$

for all $y \in \mathbb{R}^d$, $R \in (0,1)$, $\varepsilon \in (0, \varepsilon_1)$, and $t \in [0,T]$.

**Proof.** Using $\int_0^1 \left( \log \frac{1}{k} \right)^{d-1} \frac{1}{k} dk = \Gamma \left( \frac{d-1}{2} + 1 \right) = \pi^{\frac{d-1}{2}}/\omega_d$, and the same calculation as (155) below, we have

$$
\int_{\mathbb{R}^d} \rho_{(y,s)}(x,0) \, d\mu_0^\varepsilon \leq D_1 \frac{d_1}{d^{\frac{d-1}{2}}} \int_0^1 \left( \log \frac{1}{k} \right)^{\frac{d-1}{2}} \, dk = D_1
$$

for any $s > 0$ and $y \in \mathbb{R}^d$. By (51) and (53),

$$
\int_{\mathbb{R}^d} \rho_{(y,t)}(x,t) \, d\mu_t^\varepsilon \leq D_1 e^{C_3(t+1)}
$$

for any $t \in [0,T)$ with $0 < t < s$ and $y \in \mathbb{R}^d$. Set $R = 2\sqrt{s-t}$. We compute

$$
\int_{\mathbb{R}^d} \rho_{(y,t)}(x,t) \, d\mu_t^\varepsilon = \frac{1}{\pi^{d/2}} \frac{d_1}{R^{d-1}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2R^2}} \, d\mu_t^\varepsilon \geq \frac{1}{\pi^{d/2}} \frac{d_1}{R^{d-1}} \int_{B_R(y)} e^{-1} \, d\mu_t^\varepsilon.
$$

Therefore we have (52) by (54) and (55).

By integration by parts, we have the following estimate:

**Lemma 3.** For any non-negative test function $\phi \in C^2_c(\Omega)$, there exists $C_7 > 0$ depending only on $D_1$, $\|\phi\|_{C^2(\Omega)}$, $\omega$, and $d$ such that

$$
\int_0^T \left| \frac{d}{dt} \mu_t^\varepsilon(\phi) \right| \, dt \leq C_7 (1+T) \quad \text{for any } \varepsilon \in (0, \varepsilon_1) \text{ and } T > 0.
$$

**Proof.** By integration by parts, we have

$$
\frac{d}{dt} \int_\Omega \phi \left( \frac{\varepsilon |\nabla \phi^\varepsilon|^2}{2} + \frac{W(\phi^\varepsilon)}{\varepsilon} \right) \, dx
$$

$$
= - \int_\Omega \varepsilon \phi (\partial_t \phi^\varepsilon)^2 \, dx + \lambda^\varepsilon \int_\Omega \phi \sqrt{2W(\phi^\varepsilon)} \partial_t \phi^\varepsilon \, dx - \int_\Omega \varepsilon (\nabla \phi \cdot \nabla \phi^\varepsilon) \partial_t \phi^\varepsilon \, dx.
$$

By Cauchy’s mean-value theorem, there exists $c = c(d) > 0$ such that $\sup_{\phi > 0} \frac{\|\nabla \phi\|_{C^0(\Omega)}^2}{\phi} \leq c(d) \|\nabla^2 \phi\|_{C^0(\Omega)}$. Hence,

$$
\frac{d}{dt} \int_\Omega \phi \left( \frac{\varepsilon |\nabla \phi^\varepsilon|^2}{2} + \frac{W(\phi^\varepsilon)}{\varepsilon} \right) \, dx \leq \lambda^\varepsilon \int_\Omega \phi \frac{2W(\phi^\varepsilon)}{\varepsilon} \, dx + \int_\Omega \frac{|\nabla \phi^\varepsilon|^2}{\phi} \varepsilon |\nabla \phi^\varepsilon|^2 \, dx + 2 \int_\Omega \varepsilon (\partial_t \phi^\varepsilon)^2 \, dx
$$

$$
\leq C \sigma D_1 \left( 1 + |\lambda^\varepsilon|^2 + \int_\Omega \varepsilon (\partial_t \phi^\varepsilon)^2 \, dx \right),
$$
where $C > 0$ depends only on $\|\phi\|_{C^2(\Omega)}$ and $d$. Thus (35), (38), and (58) imply (56).

With an argument similar to that of [23], we can show the following proposition by using (56):

**Proposition 9.** There exist a subsequence $\{\varepsilon_i\}_{i=1}^\infty$ and a family of Radon measures $\{\mu_t\}_{t \geq 0}$ such that

$$
\mu_t^{\varepsilon_i} \rightharpoonup \mu_t \quad \text{as Radon measures on } \Omega
$$

(59)

for any $t \in [0, \infty)$ and for any $d \geq 2$. In addition, there exists a countable set $B \subset [0, \infty)$ such that $\mu_t(\Omega)$ is continuous on $[0, \infty) \setminus B$.

**Proof.** Let $\{\phi_k\}_{k=1}^\infty \subset C_c(\Omega)$ be a dense subset with $\phi_k \in C^2_c(\Omega)$ for any $k$, and for any $x \in \Omega \cap \mathbb{Q}^d$ and $r \in (0, 1) \cap \mathbb{Q}$, there exists $k \in \mathbb{N}$ such that $\phi_k \in C^2_c(B_r(x))$. Let $f_k^i(t) = \mu_0^{\varepsilon_i}(\phi_k) + \int_0^t \left( \frac{d}{ds} \mu_s^{\varepsilon_i}(\phi_k) \right)_+ ds$ and $g_k^i(t) = \int_0^t \left( \frac{d}{ds} \mu_s^{\varepsilon_i}(\phi_k) \right)_- ds$. Then

$$
\mu_t^{\varepsilon_i}(\phi_k) = f_k^i(t) - g_k^i(t),
$$

and $f_k^i(t)$ and $g_k^i(t)$ are non-decreasing functions with

$$
0 \leq f_k^i(t) \leq C_k \quad \text{and} \quad 0 \leq g_k^i(t) \leq C_k
$$

for any $i$ and $t \in [0, T)$, where $C_k > 0$ depends only on $D_1, \|\phi_k\|_{C^2(\Omega)}, \omega$, and $d$, by (56). Then Helly’s selection theorem implies that there exist a subsequence $\varepsilon_i \rightarrow 0$ (denoted by the same index), $f_k, g_k : [0, T) \rightarrow [0, \infty)$ such that $\lim_{i \rightarrow \infty} f_k^i(t) = f_k(t)$ and $\lim_{i \rightarrow \infty} g_k^i(t) = g_k(t)$ for any $t \in [0, T)$. Therefore we have

$$
\lim_{i \rightarrow \infty} \mu_t^{\varepsilon_i}(\phi_k) = f_k(t) - g_k(t)
$$

(60)

for any $t \in [0, T)$. By this and the diagonal argument, we can choose a subsequence such that (60) holds for any $t \in [0, T)$ and $k \in \mathbb{N}$. On the other hand, for any $t \in [0, T)$, the compactness of Radon measures yields that there exist $\mu_t$ and a subsequence $\varepsilon_i \rightarrow 0$ (depending on $t$) such that $\mu_t^{\varepsilon_i} \rightharpoonup \mu_t$ as Radon measures. However, $\mu_t$ is uniquely determined by (60). Hence we obtain (59) for any $t \in [0, T)$. By the diagonal argument with $T \rightarrow \infty$, we have (59) for any $t \in [0, \infty)$. From a similar argument as to that above, there exists monotone increasing functions $f$ and $g$ such that $\mu_t(\Omega) = f(t) - g(t)$. By the monotonicity, there exists a countable set $B$ such that $f$ and $g$ are continuous on $[0, \infty) \setminus B$. This concludes the proof. \qed

**Proposition 10.** Let $B$ be the countable set given by Proposition 9. For any $t, s \in [0, \infty) \setminus B$ with $t < s$, we have $\mu_s(\Omega) \leq \mu_t(\Omega)$.\qed
Theorem 12. Assume that $d_i > 0$ for any $i$. We recall that $E^i_S$ and $E^i_P$ are energies defined by (7). By (8) and $E^i_S(0) \leq D_1$, Helly’s selection theorem yields that there exist a subsequence $i \to 0$ (denoted by the same index) and a monotone decreasing function $E(t)$ such that $E^i_S(t) \to E(t)$ for any $t \in [0, \infty)$. For any $T > 0$, the estimate (38) and Fatou’s lemma imply

$$\lim_{i \to \infty} \int_0^T E^i_P(t) \, dt \leq \liminf_{i \to \infty} \int_0^T E^i_P(t) \, dt = \liminf_{i \to \infty} \int_0^T \frac{\varepsilon_i}{2} |\lambda^i| \, dt = 0.$$ 

Therefore $\liminf_{i \to \infty} E^i_P(t) = 0$ a.e. $t \geq 0$ and hence $E(t) = \sigma \mu_t(\Omega)$ for a.e. $t \geq 0$. By this, the monotonicity of $E(t)$, and the continuity of $\mu_t(\Omega)$ on $[0, \infty) \setminus B$, we obtain the claim. \hfill \Box

Proof. From Proposition 9, we may assume that $\mu^\varepsilon_i(\Omega) \to \mu_t(\Omega)$ for any $t \in [0, \infty)$. We recall that $E^i_S$ and $E^i_P$ are energies defined by (7). By (8) and $E^i_S(0) \leq D_1$, Helly’s selection theorem yields that there exist a subsequence $i \to 0$ (denoted by the same index) and a monotone decreasing function $E(t)$ such that $E^i_S(t) \to E(t)$ for any $t \in [0, \infty)$. From Proposition 9, we recall that $\mu^\varepsilon_i(\Omega)$ is a countable set $\tilde{B}$ such that $\mu^\varepsilon_i(\Omega) = \sigma \mu_t(\Omega)$ for a.e. $t \geq 0$. By this, the monotonicity of $E(t)$, and the continuity of $\mu_t(\Omega)$ on $[0, \infty) \setminus B$, we obtain the claim. \hfill \Box

We define a Radon measure $\mu$ on $\Omega \times [0, \infty)$ by $d \mu := d \mu_t dt$. By the boundedness of $\sup_i \mu^\varepsilon_i(\Omega)$, the dominated convergence theorem implies

$$\lim_{i \to \infty} \int_0^T \int_{\Omega} \phi \, d\mu^\varepsilon_i \, dt = \int_{\Omega \times [0, T)} \phi \, d\mu \quad \text{for any } \phi \in C_c(\Omega \times [0, T)).$$

For our measures $\mu$ and $\mu_t$, we have the following property:

Proposition 11. There exists a countable set $\tilde{B} \subset [0, \infty)$ such that

$$\text{spt} \, \mu_t \subset \{ x \in \Omega \mid (x, t) \in \text{spt} \, \mu \} \quad (61)$$

for any $t \in (0, \infty) \setminus \tilde{B}$.

Proof. Let $f_k$ and $g_k$ be monotone increase functions given by Proposition 9. Then there exists a countable set $\tilde{B}$ such that $f_k$ and $g_k$ are continuous on $[0, \infty) \setminus \tilde{B}$ for any $k$. Suppose that there exists $t_0 \in [0, \infty) \setminus \tilde{B}$ such that $x \in \text{spt} \, \mu_{t_0}$ and $(x, t_0) \notin \text{spt} \, \mu$. Then we may assume that there exists $k$ such that $x \in \text{spt} \, \phi_k$ and $\mu(\phi_k \times (t_0 - \delta, t_0 + \delta)) = 0$ for sufficiently small $\delta > 0$, where $\phi_k$ is a function given by Proposition 9. From $x \in \text{spt} \, \mu_{t_0}$, $\mu_{t_0}(\phi_k) > 0$ and there exists $\delta' > 0$ such that $\mu_t(\phi_k) > 0$ for any $t \in (t_0 - \delta', t_0 + \delta')$ by the continuity of $f_k$ and $g_k$. However, this contradicts $\mu(\phi_k \times (t_0 - \delta, t_0 + \delta)) = 0$. Therefore we obtain (61) for $t \in [0, \infty) \setminus \tilde{B}$. \hfill \Box

3.4. Integrality of $\mu_t$ for $d \leq 3$

In the case of $d \leq 3$, we can use the results of [38]. For $d \geq 4$, we employ the arguments of [23,32,45] in Section 4 below.

Theorem 12. Assume that $d = 2$ or $3$ and (59). Then $\mu_t$ is integral for a.e. $t \geq 0$. 

Proof. The estimates (35), (36), and (38) imply
\begin{align}
& \int_{0}^{T} \int_{\Omega} \epsilon \left( \Delta \phi^{\epsilon} - \frac{W'(\phi^{\epsilon})}{\epsilon^{2}} \right)^{2} \, dx \, dt \\
& \leq \int_{0}^{T} \int_{\Omega} \epsilon (\partial_{t} \phi^{\epsilon})^{2} \, dx \, dt + \int_{0}^{T} \int_{\Omega} |\lambda^{\epsilon}|^{2} \frac{2W'(\phi^{\epsilon})}{\epsilon} \, dx \, dt \\
& \leq \sigma \mu^{0}_{0}(\Omega) + 2D_{1}C_{3}(1 + T) \leq \sigma D_{1} + 2D_{1}C_{3}(1 + T)
\end{align}
for any $T > 0$. Then Fatou’s lemma yields
\begin{align}
& \int_{0}^{T} \lim \inf_{i \to \infty} \int_{\Omega} \epsilon_{i} \left( \Delta \phi^{\epsilon_{i}} - \frac{W'(\phi^{\epsilon_{i}})}{\epsilon_{i}^{2}} \right)^{2} \, dx \, dt \\
& \leq \lim \inf_{i \to \infty} \int_{0}^{T} \int_{\Omega} \epsilon_{i} \left( \Delta \phi^{\epsilon_{i}} - \frac{W'(\phi^{\epsilon_{i}})}{\epsilon_{i}^{2}} \right)^{2} \, dx \, dt < \infty.
\end{align}
Therefore
\begin{align}
\lim \inf_{i \to \infty} \int_{\Omega} \epsilon_{i} \left( \Delta \phi^{\epsilon_{i}} - \frac{W'(\phi^{\epsilon_{i}})}{\epsilon_{i}^{2}} \right)^{2} \, dx < \infty \quad \text{for a.e. } t \geq 0.
\end{align}
By this, $2 \leq d \leq 3$, and (36), $\mu_{t}$ is integral for a.e. $t \geq 0$ (see [38, Theorem 5.1]).
\[\Box\]

4. Rectifiability and Integrality of $\mu_{t}$

We already proved the rectifiability and integrality of $\mu_{t}$ with $d \leq 3$ in Theorem 12. Next we consider the case of $d \geq 2$ and basically follow [23,32,45].

4.1. Assumptions

We assume (24) and (28–30) again in this section. Let $\{\epsilon_{i}\}_{i=1}^{\infty}$ be a positive sequence such that $\epsilon_{i} \to 0$ as $i \to \infty$. By the weak compactness of the Radon measures and Proposition 9, we may assume that there exist Radon measures $\mu$, $|\xi|$ and a family of Radon measures $\{\mu_{t}\}_{t \in [0,T)}$ such that
\begin{align}
\mu(\phi) &= \lim_{i \to \infty} \int_{0}^{T} \mu_{t}^{\epsilon_{i}}(\phi) \, dt, \quad |\xi|(\phi) = \lim_{i \to \infty} \int_{0}^{T} \int_{\Omega} \sigma^{-1} |\xi^{\epsilon_{i}}| \phi \, dx \, dt \\
\mu_{t}(\phi) &= \mu_{t}^{\epsilon_{i}}(\phi), \quad \phi \in C_{c}(\Omega), \ t \in [0,T).
\end{align}
Remark 12. In the discussion above, we proved that there exists $\mu_t = \lim_{\varepsilon \to 0} \mu^\varepsilon_t$ for any $t \geq 0$, however such a property does not necessarily hold for $\xi^\varepsilon_t$, because we do not know anything about the monotonicity of $\xi^\varepsilon_t$ (which was the key to the argument for $\mu^\varepsilon_t$).

By the standard PDE theories and the rescaling arguments, we obtain the following lemma. The proof is almost the same as [45, Lemma 4.1]. So, we skip this.

Lemma 4. There exists $C_8 > 0$ depending only on $d$ and $C_1$ such that

$$\sup_{\Omega \times [0,T]} \varepsilon |\nabla \varphi^\varepsilon| + \sup_{x,y \in \Omega, t \in [0,T]} \frac{\varepsilon^3 |\nabla \varphi^\varepsilon(x, t) - \nabla \varphi^\varepsilon(y, t)|}{|x - y|^{\frac{3}{2}}} \leq C_8$$

for any $\varepsilon \in (0, 1)$.

4.2. Vanishing of $\xi$

First we show $|\xi| = 0$ for any $d \geq 2$.

Lemma 5. Assume $(x', t') \in \text{spt} \mu$ and $\alpha_1 \in (0, 1)$. Then there exist a sequence $\{(x_j, t_j)\}_{j=1}^\infty$ and a subsequence $\{\varepsilon_{ij}\}_{j=1}^\infty$ such that $|(x_j, t_j) - (x', t')| < \frac{1}{j}$ and $|\varphi^\varepsilon_{ij}(x_j, t_j)| < \alpha_1$ for all $j$.

Proof. Define $Q_r = B_r(x') \times (t' - r, t' + r)$ for $r > 0$. If the claim is not true, then there are $r > 0$ and $N > 1$ such that $\inf_{Q_r} |\varphi^\varepsilon_{ij}| \geq \alpha_1$ for any $i > N$. Without loss of generality, we may assume that $\inf_{Q_r} \varphi^\varepsilon_{ij} \geq \alpha_1$ for any $i > N$. For $s \in [\alpha_1, 1)$, we have $W(s) = \frac{1}{4s} W'(s)(s^2 - 1) \leq \frac{1}{4\alpha_1} W'(s)(s^2 - 1)$. Note that $W'(s)(s^2 - 1) \geq 0$ for $s \in [\alpha_1, 1)$. Assume that $\phi \in C^\infty_c(B_{r/2}(x'))$ satisfies $0 \leq \phi \leq 1$ and $\phi = 1$ on $B_{r/2}(x')$. We compute

$$\int_{Q_r} \phi^2 W'(\varphi^\varepsilon) \mathbf{d}x \mathbf{d}t \leq \frac{1}{4\alpha_1} \int_{Q_r} \phi^2 W'(\varphi^\varepsilon) \left( (\varphi^\varepsilon)^2 - 1 \right) \mathbf{d}x \mathbf{d}t = \frac{1}{4\alpha_1} \int_{Q_r} \phi^2 \left( -\partial_t \varphi^\varepsilon + \Delta \varphi^\varepsilon + \lambda^\varepsilon \sqrt{2W(\varphi^\varepsilon)} \right) \times ((\varphi^\varepsilon)^2 - 1) \mathbf{d}x \mathbf{d}t.$$

Now we estimate the three terms on the right hand side above. We compute

$$\left| \int_{Q_r} \phi^2 \partial_t \varphi^\varepsilon ((\varphi^\varepsilon)^2 - 1) \mathbf{d}x \mathbf{d}t \right| = \left| \int_{t' - r}^{t' + r} \frac{d}{dr} \int_{B_r(x')} \phi^2 \left( \frac{1}{3} ((\varphi^\varepsilon)^3 - \varphi^\varepsilon) \right) \mathbf{d}x \mathbf{d}t \right| \leq C,$$
where $C > 0$ depends only on $r$. Here we used $\| \varphi^\varepsilon \|_{L^\infty} \leq 1$ and $0 \leq \phi \leq 1$. By $\inf_Q \varphi^\varepsilon_i \geq \alpha_1$, integration by parts, and Young’s inequality,

$$\int_{Q_r} \phi^2 \Delta \varphi^\varepsilon ((\varphi^\varepsilon)^2 - 1) \, dx \, dt$$

$$= \int_{Q_r} -2\phi(\nabla \phi \cdot \nabla \varphi^\varepsilon)((\varphi^\varepsilon)^2 - 1) - 2\phi^2 |\nabla \varphi^\varepsilon|^2 \varphi^\varepsilon \, dx \, dt$$

$$\leq \int_{Q_r} \alpha_1 \phi^2 |\nabla \varphi^\varepsilon|^2 + \frac{1}{\alpha_1} |\nabla \phi|^2 ((\varphi^\varepsilon)^2 - 1)^2 - 2\phi^2 |\nabla \varphi^\varepsilon|^2 \alpha_1 \, dx \, dt$$

$$\leq \frac{1}{\alpha_1} \int_{Q_r} |\nabla \phi|^2 \, dx \, dt \leq C,$$

where $C > 0$ depends only on $\alpha_1$, $r$, and $\| \nabla \phi \|_{L^\infty}$. Here we used $0 \leq \phi \leq 1$. By (36), (38), and $\sqrt{2W(\varphi^\varepsilon)}((\varphi^\varepsilon)^2 - 1) = -2W(\varphi^\varepsilon)$,

$$\left| \int_{Q_r} \phi^2 \sqrt{\frac{2W(\varphi^\varepsilon)}{\varepsilon}} ((\varphi^\varepsilon)^2 - 1) \, dx \, dt \right| \leq \int_{Q_r} \phi^2 |\lambda|^2 \sqrt{\frac{2W(\varphi^\varepsilon)}{\varepsilon}} \, dx \, dt$$

$$\leq \int_{t'-r}^{t'+r} |\lambda|^2 \int_{B_r(x')} \frac{2W(\varphi^\varepsilon)}{\varepsilon} \, dx \, dt$$

$$\leq 2\sigma D_1 \sqrt{2r} \left( \int_{t'-r}^{t'+r} |\lambda|^2 \, dx \right)^{\frac{1}{2}} \leq 2\sigma D_1 \sqrt{2r} C_3^\frac{1}{2} (1 + r)^{\frac{1}{2}}.$$

Therefore there exists $C > 0$ depending only on $\alpha_1$, $r$, $C_3$, $\| \nabla \phi \|_{L^\infty}$, and $D_1$ such that

$$\int_{Q_r} \phi^2 \frac{W(\varphi^\varepsilon)}{\varepsilon^2} \, dx \, dt \leq C.$$

By (33), $\mu^\varepsilon(B_{r/2}(x')) \leq 2\sigma^{-1} \int_{B_{r/2}(x')} \frac{W}{\varepsilon} \, dx$. Thus

$$\int_{t'-r}^{t'+r} \mu^\varepsilon(B_{r/2}(x')) \, dt \leq 2\sigma^{-1} \int_{Q_r} \phi^2 \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx \, dt \leq 2\sigma^{-1} \varepsilon C,$$

where $C > 0$ depends only on $\alpha_1$, $r$, $C_3$, $\| \nabla \phi \|_{L^\infty}$, and $D_1$. However, this implies $(x', t') \notin \text{spt } \mu$. This is a contradiction. \hfill $\square$

Set

$$\rho^r_s(x) := \frac{1}{(\sqrt{2\pi r})^{d-1}} e^{-\frac{|x-y|^2}{2r^2}}, \quad r > 0, \quad x, y \in \mathbb{R}^d. \quad (64)$$

Note that $\rho_{(y,s)}(x, t) = \rho^r_s(x) = \rho^r_s(y)$ for $r = \sqrt{2(s-t)}$.

**Lemma 6.** There exist $\gamma_1$, $\eta_1$, $\eta_2 \in (0, 1)$ depending only on $d$, $W$, $T$, $D_2$, and $C_3$ such that the following hold. For $t, s \in [0, T/2)$ with $0 < s - t \leq \eta_1$, we denote $r = \sqrt{2(s-t)}$ and $t' = s + r^2/2$. If $x \in \Omega$ satisfies

$$\int_{\mathbb{R}^d} \rho^r_s(y) \, d\mu_s(y) = \int_{\mathbb{R}^d} \rho_{(x,t')}(y, s) \, d\mu_s(y) < \eta_2,$$

then $(B_{\gamma_1 r}(x) \times \{t'\}) \cap \text{spt } \mu = \emptyset$. \hfill (65)
Proof. First we remark that \(0 \leq t < s < t' < T\), \(s = \frac{t + t'}{2}\), and \(r = \sqrt{2(t' - t)} = \sqrt{2(t'' - s)}\). Assume that \(x \in \Omega\) satisfies (65), \((x', t') \in \text{spt } \mu\), and \(x' \in B_{\gamma t}(x)\). We choose \(\gamma_1, \eta_1, \text{ and } \eta_2\) later. Let \(\alpha_1 \in (0, 1)\) be a constant. By Lemma 5, there exist a sequence \(\{(x_j, t_j)\}_{j=1}^{\infty}\) and a subsequence \(\varepsilon_j \to 0\) such that \(\lim_{j \to \infty}(x_j, t_j) = (x', t')\) and \(|\varphi^{\varepsilon_j}(x_j, t_j)| < \alpha_1\) for all \(j\). Then we may assume that for \(\alpha' = (\alpha_1 + 1)/2 > \alpha_1\), there exists \(\gamma_2 = \gamma_2(W, \alpha_1) > 0\) such that \(\frac{W(\varphi^{\varepsilon_j}(y, t_j))}{\varepsilon_j} \geq \frac{W(\alpha')}{\varepsilon_j}\) for any \(j\) and for any \(y \in B_{R\varepsilon_j}(x_j)\), because \(W(\alpha_1) > W(\alpha')\) and
\[
|\varphi^{\varepsilon_j}(y, t_j) - \varphi^{\varepsilon_j}(x_j, t_j)| \leq \sup_{z \in \Omega} \|\nabla \varphi^{\varepsilon_j}(z, t_j)\| |y - x_j| 
\leq \varepsilon_j^{-1} W(0)|y - x_j| \leq W(0)\gamma_2
\]
for any \(y \in B_{R\varepsilon_j}(x_j)\), where we used (33). Thus, there exists \(\eta_3 = \eta_3(d, \gamma_2) > 0\) such that
\[
\eta_3 \leq \int_{B_{R\varepsilon_j}(x_j)} \frac{W(\alpha')}{\varepsilon_j} \rho(x_j, t_j + \varepsilon_j^2)(y, t_j) \, dy
\leq \int_{B_{R\varepsilon_j}(x_j)} \frac{W(\varphi^{\varepsilon_j}(y, t_j))}{\varepsilon_j} \rho(x_j, t_j + \varepsilon_j^2)(y, t_j) \, dy.
\]
Here we used
\[
\inf_{y \in B_{R\varepsilon_j}(x_j)} \rho(x_j, t_j + \varepsilon_j^2)(y, t_j) > C_9 \varepsilon_j^{-d} > 0,
\]
where \(C_9 > 0\) depends only on \(d\) and \(\gamma_2\). By the monotonicity formula, we have
\[
\eta_3 \leq \int_{\mathbb{R}^d} \rho(x_j, t_j + \varepsilon_j^2)(y, t_j) \, d\mu_{t_j}^{\varepsilon_j}(y) \leq e^{C_3(T + 1)} \int_{\mathbb{R}^d} \rho(x_j, t_j + \varepsilon_j^2)(y, s) \, d\mu_{s}^{\varepsilon_j}(y).
\]
Choose \(\eta_2 = \eta_2(d, \gamma_2, T, C_3) > 0\) such that
\[
2\eta_2 \leq \int_{\mathbb{R}^d} \rho(x_j, t_j + \varepsilon_j^2)(y, s) \, d\mu_{s}^{\varepsilon_j}(y)
\]
and letting \(j \to \infty\), we have
\[
2\eta_2 \leq \int_{\mathbb{R}^d} \rho(x', t')(y, s) \, d\mu_{s}(y). \tag{66}
\]
Changing the center of the backward heat kernel by using (153), we have
\[
\eta_2 \leq \int_{\mathbb{R}^d} \rho(x', t')(y, s) \, d\mu_{s}(y)
\]
when \(|x - x'| \leq \gamma_1 r\). Here \(\gamma_1\) depends only on \(\eta_2\) and \(D_2\). This is a contradiction to (65). Therefore \((x', t') \notin \text{spt } \mu\). \(\square\)

We can also show the following using the estimate (66):
Lemma 7. There exists $C_{10} \leq 0$ depending only on $d$, $T$, $C_3$, and $D_2$ such that
\[ \mathcal{H}^{d-1}(\text{spt } \mu_t \cap U) \leq C_{10} \liminf_{r \downarrow 0} \mu_{t-r^2}(U) \quad (67) \]
for any $t \in (0, T) \setminus \tilde{B}$ and for any open set $U \subset \Omega$, where $\tilde{B}$ is the countable set given by Proposition 11.

Proof. We only need to prove (67) for any compact set $K \subset U$. Let $X_t := \{ x \in K \mid (x, t) \in \text{spt } \mu \}$ with $t \in (0, T) \setminus \tilde{B}$. For any $x \in X_t$, by (66), we have
\[ 2\eta_2 \leq \int_{\mathbb{R}^d} \rho(x, t)(y, t - r^2) \, d\mu_{t-r^2}(y) \]
for sufficiently small $r > 0$. By (154), we deduce that
\begin{align*}
\int_{\mathbb{R}^d} \rho(x, t)(y, t - r^2) \, d\mu_{t-r^2}(y) \\
\leq \int_{B_{Lr}(x)} \rho(x, t)(y, t - r^2) \, d\mu_{t-r^2}(y) + 2^{d-1}e^{-\frac{3L^2}{2r^2}}D_2
\end{align*}
for any $L > 0$. Therefore for sufficiently large $L > 0$ depending only on $d$, $\gamma_2$, $T$, $C_3$, and $D_2$, we have
\[ \eta_2 \leq \int_{B_{Lr}(x)} \rho(x, t)(y, t - r^2) \, d\mu_{t-r^2}(y) \leq (4\pi)^{-\frac{d-1}{2}}r^{1-d} \mu_{t-r^2}(B_{Lr}(x)), \]
where we used $\rho(x, t)(y, t - r^2) \leq (4\pi)^{-\frac{d-1}{2}}r^{1-d}$. Hence there exists $C_{11} > 0$ depending only on $d$, $\gamma_2$, $T$, $C_3$, and $D_2$ such that
\[ \omega_{d-1}r^{d-1} \leq C_{11}\mu_{t-r^2}(B_{Lr}(x)) \quad (68) \]
holds for any sufficiently small $r > 0$. Set $B := \{ B_{Lr}(x) \subset U \mid x \in X_t \}$. By the Besicovitch covering theorem, there exists a finite sub-collection $B_1, B_2, \ldots, B_{N(d)}$ such that each $B_i$ is a family of the disjoint closed balls and
\[ X_t \subset \bigcup_{i=1}^{N(d)} \bigcup_{x \in B_{Lr}(x)} B_{Lr}(x). \quad (69) \]
Let $\mathcal{H}^{d-1}_0$ be defined in [11, Chapter 2]. Note that $\mathcal{H}^{d-1} = \lim_{\delta \downarrow 0} \mathcal{H}^{d-1}_\delta$. By (68) and (69), we compute
\begin{align*}
\mathcal{H}^{d-1}_{2Lr}(X_t) \leq \sum_{i=1}^{N(d)} \sum_{x \in B_{Lr}(x) \in B_i} \omega_{d-1}(Lr)^{d-1} \\
\leq \sum_{i=1}^{N(d)} L^{d-1}C_{11}\mu_{t-r^2}(B_{Lr}(x)) \\
\leq \sum_{i=1}^{N(d)} L^{d-1}C_{11}\mu_{t-r^2}(U) = N(d)L^{d-1}C_{11}\liminf_{r \downarrow 0} \mu_{t-r^2}(U).
\end{align*}
Letting $r \downarrow 0$, we have $\mathcal{H}^{d-1}(X_t) \leq N(d)L^{d-1}C_{11}\liminf_{r \downarrow 0} \mu_{t-r^2}(U)$. By this and (61), we obtain (67). \qed
By Lemma 6, we obtain

**Lemma 8.** (see [23, 32]) For $T \in [1, \infty)$, let $\eta_2$ be a constant as in Lemma 6. Set

$$
Z_T := \left\{(x, t) \in \text{spt } \mu \mid 0 \leq t \leq T/2, \limsup_{s \downarrow t} \int_{\mathbb{R}^d} \rho(y, s)(x, t) \, d\mu_s(y) \leq \eta_2/2 \right\}.
$$

Then $\mu(Z_T) = 0$ holds.

**Proof.** Let $\eta_1, \eta_2,$ and $\gamma_1$ be constants as in Lemma 6. For $\tau \in (0, \eta_1)$, we denote

$$
Z^{\tau} := \left\{(x, t) \in \text{spt } \mu \mid 0 \leq t \leq T/2, \int_{\mathbb{R}^d} \rho(y, s)(x, t) \, d\mu_s(y) < \eta_2, \forall s \in (t, t + \tau) \right\}.
$$

Let $\{\tau_m\}_{m=1}^{\infty}$ be a positive sequence with $\tau_m \to 0$ as $m \to \infty$. Then $Z_T \subset \bigcup_{m=1}^{\infty} Z^{\tau_m}$. Therefore we need only show $\mu(Z^{\tau}) = 0$ for any $\tau \in (0, \eta_1)$. Set

$$
P_{\tau}(x, t) := \{(x', t') \mid \tau > |t - t'| > \gamma_1^{-2}|x - x'|^2\}, \quad x \in \Omega, \ t \in [0, T/2).
$$

We now show that if $(x, t) \in Z^{\tau}$, then

$$
P_{\tau}(x, t) \cap Z^{\tau} = \emptyset. \tag{70}
$$

Assume that $(x', t') \in P_{\tau}(x, t) \cap Z^{\tau}$ for a contradiction. First we consider the case of $t' > t$. Set $s = \frac{t + t'}{2}$ and $r = \sqrt{t' - t} = \sqrt{2(s - t)}$. Since $(x, t) \in Z^{\tau}$,

$$
\int_{\mathbb{R}^d} \rho_s^r(y) \, d\mu_s(y) = \int_{\mathbb{R}^d} \rho(y, s)(x, t) \, d\mu_s(y) < \eta_2.
$$

Therefore Lemma 6 yields $(x', t') \notin \text{spt } \mu$, because $x' \in B_{\gamma_1}^{\tau}(x)$ by the definition of $P_{\tau}(x, t)$. This yields a contradiction. In the case of $t' < t$, we can show $(x, t) \notin \text{spt } \mu$ similarly. This is a contradiction. Therefore (70) holds.

For $(x_0, t_0) \in \Omega \times [\tau/2, T/2]$, we denote

$$
Z^{\tau, x_0, t_0} := Z^{\tau} \cap \left( B_{\frac{\tau}{2}}^{\sqrt{\tau}}(x_0) \times (t_0 - \tau/2, t_0 + \tau/2) \right).
$$

We can choose a countable set $\{(x_j, t_j)\}_{j=1}^{\infty}$ such that $Z^{\tau} \subset \bigcup_{j=1}^{\infty} Z^{\tau, x_j, t_j}$. Thus we only need to prove $\mu(Z^{\tau, x_0, t_0}) = 0$. Let $P : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be a projection such that $P(x, t) = x$. For $\rho \in (0, 1)$ and $r \leq \rho$, let $\{(\overline{B}_{r/\tau}(x_\lambda))_{\lambda \in \Lambda}\}$ be a covering of $P(Z^{\tau, x_0, t_0}) \subset B_{\frac{\tau}{2\sqrt{\tau}}}(x_0)$. Then, we may choose a countable covering $\mathcal{F} = \{\overline{B}_{r}(x_i)\}_{i=1}^{\infty}$ of $P(Z^{\tau, x_0, t_0})$ with $(x_i, t_i) \in Z^{\tau, x_0, t_0}$ for some $t_i$, by Vitali’s covering theorem. Let $A$ be a set of centers of all balls in $\{\overline{B}_{r}(x_i)\}_{i=1}^{\infty}$. Then, by Besicovitch’s covering theorem, there exist $N(d)$ and subcollections $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{N(d)} \subset \mathcal{F}$ of disjoint balls such that

$$
A \subset \bigcup_{k=1}^{N(d)} \bigcup_{B_{k,i} \in \mathcal{F}_k} B_{k,i} \tag{71}
$$

Note that $\mathcal{F}_k$ is finite ($\mathcal{F}_k = \{B_{k,1}, \ldots, B_{k,n_k}\}$) and

$$
\mathcal{L}^d \left( \bigcup_{i=1}^{n_k} B_{k,i} \right) = \sum_{i=1}^{n_k} \mathcal{L}^d (B_{k,i}) \leq \mathcal{L}^d \left( \overline{B}_{\frac{\tau}{2\sqrt{\tau+\rho}}}(x_0) \right).
$$
since each balls in $\mathcal{F}_k$ are disjoint and $B_{k,i} \subset \overline{B}_{\frac{\gamma_1}{2}\sqrt{\tau + \rho}}(x_0)$. Therefore

$$
\sum_{k=1}^{N(d)} \sum_{i=1}^{n_k} \omega_d r^d = \sum_{k=1}^{N(d)} \sum_{B_{k,i} \in \mathcal{F}_k} \mathcal{L}^d(B_{k,i}) \leq N(d) \mathcal{L}^d(\overline{B}_{\frac{\gamma_1}{2}\sqrt{\tau + \rho}}(x_0)) =: N'.
$$

(72)

If $(x, t) \in Z^{\tau, x_0, t_0}$, then there exists $B_{k,i} = \overline{B}_r(x_{k,i}) \in \mathcal{F}_k$ for some $k$ and $i$ such that $x \in \overline{B}_{2r}(x_{k,i})$ and $|t_{k,i} - t| \leq \gamma_1^{-1} |x_{k,i} - x|^2 \leq 4\gamma_1^{-1} r^2$ by (70) and (71) (note that we should change the radius because $A$ is not a covering of $Z^{\tau, x_0, t_0}$). Hence, we have

$$
Z^{\tau, x_0, t_0} \subset \bigcup_{k=1}^{N(d)} \bigcup_{i=1}^{n_k} \overline{B}_{2r}(x_{k,i}) \times (t_{k,i} - 4r^2\gamma_1^{-2}, t_{k,i} + 4r^2\gamma_1^{-2})
$$

By this, (52), and (72) we obtain

$$
\mu(Z^{\tau, x_0, t_0}) \leq \sum_{k=1}^{N(d)} \sum_{i=1}^{n_k} \mu(\overline{B}_{2r}(x_{k,i}) \times (t_i - 4r^2\gamma_1^{-2}, t_i + 4r^2\gamma_1^{-2}))
$$

$$
\leq \sum_{k=1}^{N(d)} \sum_{i=1}^{n_k} D_2(2r)^{d-1} \times 8\gamma_1^{-2}r^2 \leq 2^{d+2}\gamma_1^{-2} \omega_d^{-1} N'D_2.
$$

Letting $\rho \to 0$, we have $\mu(Z^{\tau, x_0, t_0}) = 0$. Thus $\mu(Z_T) = 0$ holds. \hfill \Box

**Theorem 13.** (see [23]) We see that $|\xi| = 0$ and $\lim_{i \to \infty} |\xi_{\hat{t}_i}|(\Omega) = 0$ for a.e. $t \in (0, T)$.

**Proof.** First we show that

$$
\int_{\Omega \times (0, s)} \frac{\rho(y,s)(x,t)}{s-t} \frac{d|\xi|}{d\mu_0}(x, t) \leq C
$$

(73)

for some $C > 0$. By (33) and (51), integrating (50) on $(0, s - \delta)$ with $\delta > 0$, we obtain

$$
\int_0^{s-\delta} \frac{1}{2\sigma(s-t)} \int_{\mathbb{R}^d} \rho(y,s)(x,t) \left| \frac{W(\varphi^\varepsilon)}{\varepsilon} - \frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} \right| dxdt
$$

$$
\leq \left(1 + e^{C_3(s+1)} \frac{1}{2} \int_0^{s-\delta} |\lambda^\varepsilon|^2 dt \right) \int_{\mathbb{R}^d} \rho(y,s)(x,0) d\mu_0^\varepsilon.
$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, we obtain (73). Next, integrating (73) on $\Omega \times (0, T)$ by $d\mu_sds$ we have

$$
\left( \int_{\Omega \times (0, T)} \rho(y,s)(x,t) d\mu_s(y)ds \right) \int_{\Omega \times (0, T)} \frac{d|\xi|}{d\mu_0}(x, t) \leq CD_1 T,
$$

where we used Fubini’s theorem. Then this boundedness implies

$$
\int_{\Omega \times (0, T)} \frac{\rho(y,s)(x,t)}{s-t} d\mu_s(y)ds < \infty \quad \text{for $|\xi|$-a.e. $(x, t) \in \Omega \times (0, T)$}.
$$

(74)
Next we claim
\[ a(x, t) := \limsup_{\varepsilon \downarrow 0} \int_{\Omega} \rho_{(y, s)}(x, t) \, d\mu_s(y) = 0 \quad \text{for } |\xi|\text{-a.e. } (x, t) \in \Omega \times (0, T). \]  
(75)

Define \( \beta := \log(s - t) \) and
\[ h(s) := \int_{\Omega} \rho_{(y, s)}(x, t) \, d\mu_s(y). \]

Assume that \((x, t)\) satisfies (74). Then
\[ \int_{-\infty}^{\log(T-t)} h(t + e^\beta) \, d\beta < \infty. \]  
(76)

Let \( \theta \in (0, 1] \) and \( \{\beta_i\}_{i=1}^\infty \) be a negative monotone decreasing sequence such that
\[ \beta_i \downarrow -\infty, \quad 0 < \beta_i - \beta_{i+1} \leq \theta, \quad \text{and} \quad h(t + e^{\beta_i}) \leq \theta. \]

For any \( \beta \in (-\infty, \beta_1) \), choose \( i \) such that \( \beta \in [\beta_i, \beta_{i-1}) \) holds. One can check that
\[ \sup_{y \in B_M(x)} \frac{\rho_{(y, t+2e^\beta-e^{\beta_i})}(x, t)}{\rho_{(y, t+e^{\beta_i})}(x, t)} \leq e^{M^2(1-e^\beta-\beta_i)} \leq e^{M^2(1-e^\theta)} \]  
(77)

for \( M > 0 \), where \( r = \sqrt{2(2e^\beta - e^{\beta_i})} \). We compute
\[
\begin{align*}
    h(t + e^\beta) &= \int_{\Omega} \rho_{(y, t+e^\beta)}(x, t) \, d\mu_{t+e^\beta}(y) = \int_{\Omega} \rho_{(y, t+2e^\beta)}(x, t + e^\beta) \, d\mu_{t+e^\beta}(y) \\
    &\leq e^{C_3(\beta - \beta_i + 1)} \int_{\Omega} \rho_{(y, t+2e^\beta-e^{\beta_i})}(x, t) \, d\mu_{t+e^{\beta_i}}(y) \\
    &\leq e^{2C_3} \int_{\Omega} \rho_{(y, t+2e^\beta-e^{\beta_i})}(x, t) \, d\mu_{t+e^{\beta_i}}(y) \\
    &\leq e^{2C_3} \int_{B_M(x)} \rho_{(y, t+2e^\beta-e^{\beta_i})}(x, t) \, d\mu_{t+e^{\beta_i}}(y) + e^{2C_3} 2^{d-1}e^{-\frac{3M^2}{2}} D_2 \\
    &\leq e^{2C_3} e^{M^2(1-e^\theta)} \int_{B_M(x)} \rho_{(y, t+e^{\beta_i})}(x, t) \, d\mu_{t+e^{\beta_i}}(y) + e^{2C_3} 2^{d-1}e^{-\frac{3M^2}{2}} D_2 \\
    &\leq e^{2C_3} e^{M^2(1-e^\theta)} \theta + e^{2C_3} 2^{d-1}e^{-\frac{3M^2}{2}} D_2,
\end{align*}
\]  
(78)

where we used (51), (154), and
\[
\int_{\Omega} \rho_{(y, t+e^{\beta_i})}(x, t) \, d\mu_{t+e^{\beta_i}}(y) = h(t + e^{\beta_i}) \leq \theta.
\]

Thus, for any \( \delta > 0 \), we can choose \( \theta \in (0, 1] \) and \( M > 0 \) such that \( h(t + e^\beta) \leq \delta \) for any \( \beta < \beta_1 \). This proves (75). Set
\[ A := \{(x, t) \in \Omega \times (0, T) \mid a(x, t) = 0\} \quad \text{and} \quad B := \{(x, t) \in \Omega \times (0, T) \mid a(x, t) > 0\}.\]
Then $\Omega \times (0, T) = A \cup B$ and $|\xi|(B) = 0$ by (75). Moreover, Lemma 8 and (154) imply $\mu(A) = 0$ and thus $|\xi|(A) = 0$, because $|\xi|$ is absolute continuous with respect to $\mu$. Therefore $|\xi|(\Omega \times (0, T)) = 0$. The rest of the claim can be shown from the dominated convergence theorem. 

4.3. Rectifiability

Next we show the rectifiability of $\mu_t$.

**Definition 2.** For $\phi \in C_c(G_{d-1}(\Omega))$, we define $V_t^\varepsilon \in V_{d-1}(\Omega)$ by

\[
V_t^\varepsilon(\phi) := \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(x,t)| \neq 0\}} \phi \left( x, I - \frac{\nabla \varphi^\varepsilon(x,t)}{|\nabla \varphi^\varepsilon(x,t)|} \right) \otimes \frac{\nabla \varphi^\varepsilon(x,t)}{|\nabla \varphi^\varepsilon(x,t)|} \, d\mu_t^\varepsilon(x).
\]

(79)

Here, $\varphi^\varepsilon$ is a solution to (5).

Note that the first variation of $V_t^\varepsilon$ is given by

\[
\delta V_t^\varepsilon(\vec{\phi}) = \int_{G_{d-1}(\Omega)} \nabla \vec{\phi}(x) \cdot S \, dV_t^\varepsilon(x, S)
\]

\[
= \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(x,t)| \neq 0\}} \nabla \vec{\phi}(x) \cdot \left( I - \frac{\nabla \varphi^\varepsilon(x,t)}{|\nabla \varphi^\varepsilon(x,t)|} \right) \otimes \frac{\nabla \varphi^\varepsilon(x,t)}{|\nabla \varphi^\varepsilon(x,t)|} \, d\mu_t^\varepsilon(x)
\]

for $\vec{\phi} \in C^1_c(\Omega; \mathbb{R}^d)$. By integration by parts, we have

\[
\delta V_t^\varepsilon(\vec{\phi}) = \int_{\Omega} (\vec{\phi} \cdot \nabla \varphi^\varepsilon) \left( \varepsilon \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon} \right) \, dx - \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(x,t)| \neq 0\}} \frac{W(\varphi^\varepsilon)}{\varepsilon} \, div \vec{\phi} \, dx
\]

\[
+ \int_{\Omega \cap \{|\nabla \varphi^\varepsilon(x,t)| \neq 0\}} \nabla \vec{\phi} \cdot \left( \frac{\nabla \varphi^\varepsilon(x,t)}{|\nabla \varphi^\varepsilon(x,t)|} \right) \otimes \frac{\nabla \varphi^\varepsilon(x,t)}{|\nabla \varphi^\varepsilon(x,t)|} \, d\mu_t^\varepsilon(x)
\]

(80)

Note that the second and third terms of the right hand side converges to 0 for a.e. $t \in [0, T)$ by Theorem 13. By (35) and (38), we have

\[
\sup_{i \in \mathbb{N}} \int_0^T \int_{\Omega} (\Delta \varphi^\varepsilon_i - \frac{W'(\varphi^\varepsilon_i)}{\varepsilon_i^2}) \, dx \, dt \leq C
\]

for some $C > 0$ (see the proof of Theorem 12). Thus Fatou’s lemma implies

\[
\liminf_{i \to \infty} \int_{\Omega} (\Delta \varphi^\varepsilon_i - \frac{W'(\varphi^\varepsilon_i)}{\varepsilon_i^2}) \, dx < \infty
\]

(81)
for a.e. \( t \in [0, T) \). Hence, by (80), (81), and the Cauchy–Schwarz inequality, we have
\[
\liminf_{i \to \infty} |\delta V_{t_i}^\varepsilon(\tilde{\phi})| \\
\leq \liminf_{i \to \infty} \left( \int_{\Omega} \varepsilon_i |\nabla \phi_{i \varepsilon}^\varepsilon|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \varepsilon_i \left( \Delta \phi_{i \varepsilon}^\varepsilon - \frac{W'(\phi_{i \varepsilon}^\varepsilon)}{\varepsilon_i^2} \right)^2 \, dx \right)^{\frac{1}{2}} \quad (82)
\]
\[
\leq D_1 \liminf_{i \to \infty} \left( \int_{\Omega} \varepsilon_i \left( \Delta \phi_{i \varepsilon}^\varepsilon - \frac{W'(\phi_{i \varepsilon}^\varepsilon)}{\varepsilon_i^2} \right)^2 \, dx \right)^{\frac{1}{2}} < \infty
\]
for a.e. \( t \in [0, T) \) and for any \( \tilde{\phi} \in C^1_c(\Omega; \mathbb{R}^d) \) with \( \sup |\tilde{\phi}| \leq 1 \). Let \( t \in [0, T) \) and \( \tilde{B} \) satisfy (82), where \( \tilde{B} \) is given by Proposition 11. Taking a subsequence \( i_j \to \infty \) (note that the subsequence depends on \( t \)), there exists a varifold \( V_t \) such that \( V_{t_i}^\varepsilon \rightharpoonup V_t \) as Radon measures and \( \delta V_t \) is a Radon measure by (82). In addition, Proposition 11, Lemma 7, and the standard measure theoretic argument imply
\[
V_t = V_t_{\{x \in \Omega \mid \limsup_{r \to 0} r^{1-d} \|V_t\|(B_r(x)) > 0 \} \times G(d,d-1)}.
\]
Therefore Allard’s rectifiability theorem yields the following theorem.

**Theorem 14.** For a.e. \( t \geq 0 \), \( \mu_t \) is rectifiable. In addition, for a.e. \( t \geq 0 \), \( \mu_t \) has a generalized mean curvature vector \( \tilde{h}(\cdot,t) \) with
\[
\delta V_t(\tilde{\phi}) = -\int_\Omega \tilde{\phi} \cdot \tilde{h}(\cdot,t) \, d\mu_t = \lim_{i \to \infty} \int_\Omega (\tilde{\phi} \cdot \nabla \phi_{i \varepsilon}^\varepsilon) \left( \varepsilon_i \Delta \phi_{i \varepsilon}^\varepsilon - \frac{W'(\phi_{i \varepsilon}^\varepsilon)}{\varepsilon_i} \right) \, dx
\]
and
\[
\int_\Omega \phi^2 \, d\mu_t \leq \frac{1}{\sigma} \liminf_{i \to \infty} \int_\Omega \varepsilon_i \phi \left( \Delta \phi_{i \varepsilon}^\varepsilon - \frac{W'(\phi_{i \varepsilon}^\varepsilon)}{\varepsilon_i^2} \right)^2 \, dx < \infty
\]
for any \( \phi \in C_c(\Omega; [0, \infty)) \) and \( \tilde{\phi} \in C_c(\Omega; \mathbb{R}^d) \).

Detailed proof of this is in [23,45], so we omit it (however, the essential part has already been discussed above). Note that (59) and \( \mu_t = \|V_t\| \) imply that \( V_t \) does not depend on the choice of subsequence \( \{V_{t_i}^\varepsilon\}_{i=1}^\infty \) above.

**4.4. Integrality**

To prove the integrality, we mainly follow [22,45,47]. The propositions that are directly applicable to our problem are in Appendix for readers’ convenience. Let \( \{r_i\}_{i=1}^\infty \) be a positive sequence with \( r_i \to 0 \) and \( \frac{\tilde{e}}{r_i} \to 0 \) as \( i \to \infty \). Set \( u^\varepsilon(\tilde{x}, \tilde{t}) = \phi^\varepsilon(x, t) \) and \( g^\varepsilon(\tilde{t}) = r \lambda^\varepsilon(t) \) for \( \tilde{x} = \frac{x}{\varepsilon}, \tilde{t} = \frac{t}{\varepsilon}, \) and \( \tilde{e} = \frac{e}{\varepsilon} \). Then, \( u^\varepsilon \) is a solution to
\[
\tilde{e} \partial_t u^\varepsilon = \tilde{\varepsilon} \Delta \tilde{z} u^\varepsilon - \frac{W'(u^\varepsilon)}{\tilde{\varepsilon}} + g^\varepsilon \sqrt{2W(u^\varepsilon)}. \quad (83)
\]
We remark that the monotonicity formula (51) and the upper bound of the density (52) hold for \( d\tilde{\mu}^\varepsilon_1(\tilde{x}) = \sigma^{-1} \left( \frac{\epsilon |\nabla \tilde{x} u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\epsilon} \right) d\tilde{x} \), because the value
\[
\int_{\mathbb{R}^d} \rho(y, s)(x, t) \, d\mu^\varepsilon_1(x)
\]
is invariant under this rescaling, and for any \( s > 0 \) we have
\[
\frac{1}{s^{d-1}} \int_{B_s(0)} \left( \frac{\epsilon |\nabla \tilde{x} u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\epsilon} \right) \, d\tilde{x} = \frac{1}{(sr)^{d-1}} \int_{B_{sr}(0)} \left( \frac{\epsilon |\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\epsilon} \right) \, dx \leq \sigma D_2
\]
by (52). We subsequently drop \( \tilde{\cdot} \) for simplicity. First we consider the energy estimate on \( \{ x \in B_1(0) \mid |u^\varepsilon(x, t)| \geq 1 - b \} \).

**Proposition 15.** (See [47]) For any \( s > 0 \) and \( a \in (0, T) \), there exist positive constants \( b \) and \( \varepsilon_2 \) depending only on \( D_1, D_2, C_3, a, \alpha, \) and \( s \) such that
\[
\int_{\{ x \in B_1(0) \mid |u^\varepsilon(x, t)| \geq 1 - b \}} \frac{W(u^\varepsilon(x, t))}{\varepsilon} \, dx \leq s
\]
for all \( t \in (a, T) \) whenever \( \varepsilon \in (0, \varepsilon_2) \).

To prove Proposition 15, we prepare following two lemmas:

**Lemma 9.** (See [47]) For any \( \delta \in (0, T) \), there exist positive constants \( C_{12} \) and \( \varepsilon_3 \) depending only on \( d, \delta, \alpha, \) and \( C_1 \) with the following property. Assume that there exist \( (x_0, t_0) \in B_1(0) \times (\delta, T) \) and \( \gamma \in (0, \frac{\delta}{2}] \) such that

\[
u^\varepsilon(x_0, t_0) < 1 - \varepsilon^\gamma \quad \text{(or } u^\varepsilon(x_0, t_0) > -1 + \varepsilon^\gamma) \quad \text{(84)}
\]

and

\[
1 \leq \tilde{r} := C_{12} \gamma |\log \varepsilon| \leq \varepsilon^{-1} \min \left\{ \sqrt{\frac{\delta}{2}}, \frac{1}{2} \right\}. \quad \text{(85)}
\]

Then
\[
\inf_{B_{\tilde{r}}(x_0) \times (t_0 - \varepsilon^2 \tilde{r}^2, t_0)} u^\varepsilon < \frac{1}{2} \quad \text{(resp. } \sup_{B_{\tilde{r}}(x_0) \times (t_0 - \varepsilon^2 \tilde{r}^2, t_0)} u^\varepsilon > -\frac{1}{2})
\]
for any \( \varepsilon \in (0, \varepsilon_3) \).

**Proof.** We may assume that \( B_{\tilde{r}}(x_0) \times (t_0 - \varepsilon^2 \tilde{r}^2, t_0) \subset B_2(0) \times (0, T) \) by (85). We consider the rescaling of (83) by \( \tilde{x} = \frac{x-x_0}{\varepsilon} \) and \( \tilde{t} = \frac{t-t_0}{\varepsilon^2} \). Then we obtain
\[
\partial_t \tilde{u}^\varepsilon = \Delta \tilde{x} \tilde{u}^\varepsilon - W'(\tilde{u}^\varepsilon) + \varepsilon \tilde{g}^\varepsilon \sqrt{2W(\tilde{u}^\varepsilon)}, \quad (x, t) \in B_{\tilde{r}}(0) \times (-\tilde{r}^2, 0), \quad \text{(86)}
\]
where $\tilde{u}^\varepsilon(\tilde{x}, \tilde{t}) = u^\varepsilon(x, t)$ and $\tilde{g}^\varepsilon(\tilde{t}) = g^\varepsilon(t)$. Note that (12) and $L^d(\Omega) = 1$ yield
\[ \|\varepsilon \tilde{g}^\varepsilon\|_{L^\infty} \leq \frac{4}{3} \varepsilon^{1-\alpha} \] (87)
for $\alpha \in (0, 1)$. Let $\psi$ be a function with
\[ \begin{cases} 
\partial_{\tilde{t}} \psi \geq & \Delta_{\tilde{x}} \psi - \frac{1}{10} \psi \quad \text{on } \mathbb{R}^d \times (-\infty, 0), \\
\psi(\tilde{x}, \tilde{t}) \geq e^{\frac{\tilde{t}}{C_{13}}} \quad & \text{on } (\mathbb{R}^d \times (-\infty, 0)) \setminus B_1(0,0), \\
\psi(0,0) = & 1,
\end{cases} \] (88)
for some constant $C_{13} > 0$. For example, $\psi = e^{\frac{-\tilde{t}}{100} - \frac{1}{10}} e^{\frac{\tilde{x}}{C_{13}}} \sqrt{1+|\tilde{x}|^2}$ satisfies (88). Set $\tilde{r} := C_{13} \sqrt{\log \varepsilon}$. We may assume that $\tilde{r} \geq 1$ for sufficiently small $\varepsilon$. Note that
\[ 1 - \varepsilon \psi^\tilde{t} e^{\frac{\tilde{t}}{C_{13}}} = 0. \] (89)
The assumption (84) is equivalent to
\[ \tilde{u}^\varepsilon(0,0) < 1 - \varepsilon \psi. \] (90)
For a contradiction, we assume that
\[ \inf_{B_{\tilde{r}}(0) \times (-\tilde{r}^2,0)} \tilde{u}^\varepsilon \geq \frac{1}{2}. \] (91)
Set $\phi^\varepsilon := 1 - \varepsilon \psi$. Then (88) and (90) imply
\[ \partial_{\tilde{t}} \phi^\varepsilon \leq \Delta_{\tilde{x}} \phi^\varepsilon + \frac{1}{10} (1 - \phi^\varepsilon) \quad \text{on } \mathbb{R}^d \times (-\infty, 0) \]
and
\[ \phi^\varepsilon(0,0) = 1 - \varepsilon \psi(0,0) = 1 - \varepsilon \psi > \tilde{u}^\varepsilon(0,0). \] (92)
Moreover, by $\tilde{r} \geq 1$,
\[ \psi \geq e^{\frac{|\tilde{x}|}{C_{13}}} \geq e^{\frac{\tilde{r}}{C_{13}}} \quad \text{on } \partial(B_{\tilde{r}}(0) \times (-\tilde{r}^2,0)). \]
Therefore
\[ \phi^\varepsilon = 1 - \varepsilon \psi \leq 1 - \varepsilon \psi e^{\frac{\tilde{r}}{C_{13}}} = 0 < \frac{1}{2} \leq \tilde{u}^\varepsilon \quad \text{on } \partial(B_{\tilde{r}}(0) \times (-\tilde{r}^2,0)) \] (93)
by (89) and (91). We consider a function $w = \phi^\varepsilon - \tilde{u}^\varepsilon$ on $B_{\tilde{r}}(0) \times (-\tilde{r}^2,0)$. By (92) and (93), $w$ attains its positive maximum at an interior point $(x', t') \in \mathbb{R}^d \times (-\tilde{r}^2,0)$.
depending only on \( \delta \), and hence \( \partial_t w - \Delta \tilde{\tau} w \geq 0 \) and \( w > 0 \) at \( (x', t') \). At \( (x', t') \), we compute that

\[
0 \leq \partial_t w - \Delta \tilde{\tau} w \leq \frac{1}{10} (1 - \phi^\varepsilon) + W' (\tilde{u}^\varepsilon) - \varepsilon \tilde{g}^\varepsilon \sqrt{2W (\tilde{u}^\varepsilon)}
\]

\[
= \frac{1}{10} (1 - \phi^\varepsilon) - 2 \tilde{u}^\varepsilon (1 - (\tilde{u}^\varepsilon)^2) - \varepsilon \tilde{g}^\varepsilon (1 - (\tilde{u}^\varepsilon)^2)
\]

\[
\leq \frac{1}{10} (1 - \phi^\varepsilon) + (-1 + \frac{8}{3} \varepsilon^{1-\alpha}) (1 - (\tilde{u}^\varepsilon)^2)
\]

\[
\leq \frac{1}{10} (1 - \phi^\varepsilon) + \frac{3}{2} (-1 + \frac{8}{3} \varepsilon^{1-\alpha}) (1 - \phi^\varepsilon) < 0
\]

for sufficiently small \( \varepsilon \), where we used (87) and \( 1 > \phi^\varepsilon > \tilde{u}^\varepsilon \geq \frac{1}{2} \) at \( (x', t') \). This is a contradiction. The other case can be proved similarly. \( \square \)

**Lemma 10.** (See [47]) For any \( \delta \in (0, T) \), there exist positive constants \( C_{14} \) and \( \varepsilon_4 \) depending only on \( \delta, \alpha, d, C_3, \) and \( D_2 \) such that the following holds. For \( t \in (\delta, T) \) and \( r \in (0, \frac{1}{2}) \), set

\[
Z_{r,t_0} := \left\{ x_0 \in B_1(0) \mid \inf_{B_r(x_0) \times (t_0 - r^2, t_0)} |u^\varepsilon| < \frac{1}{2} \right\}.
\]

Then for any \( \varepsilon \in (0, \varepsilon_4) \), we have

\[
\mathcal{L}^d (Z_{r,t_0}) \leq C_{14} r, \quad \varepsilon \leq r < \frac{1}{2}. \tag{94}
\]

**Proof.** First we claim that there exist some constants \( \varepsilon_4, C_{15}, \) and \( C_{16} \) such that if \( x_0 \in Z_{r,t_0} \) and \( \varepsilon \in (0, \varepsilon_4) \) then

\[
\sigma \mu_{t_0 - 2r^2}^\varepsilon (B_{C_{15} r}(x_0)) = \int_{B_{C_{15} r}(x_0)} \frac{\varepsilon |
abla \frac{u^\varepsilon}{\varepsilon} |^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \mathrm{d}x \bigg|_{t = t_0 - 2r^2} \geq C_{16} r^{d-1}
\]

\tag{95}

holds for any \( r \in [\varepsilon, \frac{1}{2}] \). We may assume that \( (x_1, t_1) \in B_r(x_0) \times (t_0 - r^2, t_0) \) with \( |u^\varepsilon(x_1, t_1)| < \frac{1}{2} \). By the monotonicity formula (51), for any \( \varepsilon \in (0, \varepsilon_1) \) we have

\[
\int_{\mathbb{R}^d} \rho_{(x, t_1 + \varepsilon^2)} (x, t) \, \mathrm{d} \mu_{t}^\varepsilon (x) \bigg|_{t = t_1}
\]

\[
\leq \left( \int_{\mathbb{R}^d} \rho_{(x_1, t_1 + \varepsilon^2)} (x, t) \, \mathrm{d} \mu_{t}^\varepsilon (x) \bigg|_{t = t_0 - 2r^2} \right) \, e^{C_3 (3r^2 + 1)}. \tag{96}
\]

By \( |u^\varepsilon(x_1, t_1)| < \frac{1}{2} \), repeating the proof of Lemma 6, there exists \( \eta = \eta(\alpha, d) > 0 \) such that

\[
\eta \leq \int_{\mathbb{R}^d} \rho_{(x_1, t_1 + \varepsilon^2)} (x, t) \, \mathrm{d} \mu_{t}^\varepsilon (x) \bigg|_{t = t_1}. \tag{97}
\]
Then (96), (97), and (154) imply
\[
\eta' \leq \int_{B_R(x_1)} \rho(x_1, t_1 + \varepsilon^2) (x, t) \, d\mu^e_t (x) \bigg|_{t=t_0 - 2r^2} + 2^{d-1} e^{-\frac{3R^2}{16(t_0 + \varepsilon^2 - r^2 + 2r^2)}} D_2,
\]
where \(\eta' = \eta' (\alpha, d, C_3) > 0\). By \(|t_1 - t_0| < r^2\) and \(\varepsilon \leq r\), we have \(e^{-\frac{3R^2}{16(t_1 + \varepsilon^2 - r^2 + 2r^2)}} \leq e^{-\frac{3R^2}{64r^2}}\). Thus there exists \(\gamma > 0\) depending only on \(\alpha, d, C_3\), and \(D_2\) such that
\[
\eta' \leq \int_{B_r(x_1)} \rho(x_1, t_1 + \varepsilon^2) (x, t_0 - 2r^2) \, d\mu^e_{t_0 - 2r^2} (x).
\]
Note that since \(t_1 + \varepsilon^2 - (t_0 - 2r^2) \geq 2r^2\) there exists \(C_17 > 0\) depending only on \(d\) such that
\[
\rho(x_1, t_1 + \varepsilon^2) (x, t_0 - 2r^2) \leq \frac{C}{r^{d-1}}.
\]
Hence we obtain (95) for some \(C_{15}\) and \(C_{16}\).

Finally we prove (94). The inequality (95) yields that there exists \(C_17 > 0\) depending only on \(\alpha, d, C_3\), and \(D_2\) such that
\[
\mathcal{L}^d (\overline{B}_{C_{15}r} (x_0)) \leq r C_17 \mu^e_{t_0 - 2r^2} (\overline{B}_{C_{15}r} (x_0))
\]  
(98)
for any \(x_0 \in Z_{r, t_0}\) and \(r \in [\varepsilon, \frac{1}{2}]\). Set \(\tilde{r} := C_{15}r\). By an argument similar to that in the proof of Lemma 8, there exist \(\mathcal{F}_1, \ldots, \mathcal{F}_{N(d)}\) such that \(N(d)\) depends only on \(d\), \(\mathcal{F}_k = \{\overline{B}_{\tilde{r}} (x_{k, 1}), \ldots, \overline{B}_{\tilde{r}} (x_{k, n_k})\}\) is a family of disjoint closed balls for any \(k\), and
\[
Z_{r, t_0} \subset \bigcup_{k=1}^{N(d)} \bigcup_{i=1}^{n_k} \overline{B}_{2\tilde{r}} (x_{k, i}), \quad x_{k, i} \in Z_{r, t_0} \quad \text{for any } k \text{ and } i.
\]
Therefore
\[
\mathcal{L}^d (Z_{r, t_0}) \leq \sum_{k=1}^{N(d)} \sum_{i=1}^{n_k} \mathcal{L}^d (\overline{B}_{2\tilde{r}} (x_{k, i})) = 2^d \sum_{k=1}^{N(d)} \sum_{i=1}^{n_k} \mathcal{L}^d (\overline{B}_{\tilde{r}} (x_{k, i}))
\]
\[
\leq 2^d \sum_{k=1}^{N(d)} \sum_{i=1}^{n_k} r C_17 \mu^e_{t_0 - 2r^2} (\overline{B}_{\tilde{r}} (x_{k, i}))
\]
\[
= 2^d r C_17 \sum_{k=1}^{N(d)} \mu^e_{t_0 - 2r^2} (\bigcup_{i=1}^{n_k} \overline{B}_{\tilde{r}} (x_{k, i}))
\]
\[
\leq 2^d r C_17 \sum_{k=1}^{N(d)} \mu^e_{t_0 - 2r^2} (B_{1 + C_{15}/2} (0)) \leq 2^d r C_17 N(d) D_2,
\]
where we used (52), (98), and the property that \(\mathcal{F}_k\) is a family of disjoint balls. Hence, we obtain (94). \qed
Proof of Proposition 15. First, we restrict \( b \in (0, 1) \) to be small enough so that
\[
1 - \sqrt{b} > \frac{1}{2}, \quad \log \sqrt{b} \leq -1, \quad C_{12} |\log b| \geq 1. \tag{99}
\]
and restrict \( \varepsilon \) to be small enough to use Lemmas 9 and 10. We choose a positive integer \( J \) such that
\[
\varepsilon^{\frac{1}{2^{J+1}}} \in (b, \sqrt{b}]. \tag{100}
\]
Then, (85), (99), and (100) imply
\[
1 \leq C_{12} |\log b| \leq \frac{1}{2} C_{12} |\log \varepsilon|. \tag{101}
\]
Set \( t_0 \in (\delta, T) \) and
\[
A_j := \left\{ x \in B_1(0) \mid 1 - \varepsilon^{\frac{1}{2^{J+1}}} \leq |u^\varepsilon(x, t_0)| \leq 1 - \varepsilon^{\frac{1}{2^J}} \right\}, \quad j = 1, \ldots, J.
\]
For \( x_0 \in A_j \), we use Lemma 9 with \( \gamma = \frac{1}{2^J} \). Note that (85) holds with \( \tilde{r} = \frac{1}{2^J} C_{12} |\log \varepsilon| \) by (101). Then we obtain
\[
\inf_{B_{\varepsilon \tilde{r}}(x_0) \times (t_0 - \varepsilon \tilde{r}^2, t_0)} |u^\varepsilon| < \frac{1}{2}
\]
and hence
\[
A_j \subset Z_{\varepsilon \tilde{r}, t_0}. \tag{102}
\]
By (85), we have \( \varepsilon \leq \varepsilon \tilde{r} < \frac{1}{2} \) for sufficiently small \( \varepsilon \). Therefore (94) and (102) yield
\[
\mathcal{L}^d(A_j) \leq \mathcal{L}^d(Z_{\varepsilon \tilde{r}, t_0}) \leq \frac{1}{2^J} C_{12} C_{14} \varepsilon |\log \varepsilon| \tag{103}
\]
for any \( j = 1, \ldots, J \). On the other hand, since \( |u^\varepsilon(x, t_0)| \geq 1 - \varepsilon^{\frac{1}{2^{J+1}}} \) for any \( x \in A_j \), we obtain
\[
\frac{W(u^\varepsilon(x, t_0))}{\varepsilon} \leq \frac{W(1 - \varepsilon^{\frac{1}{2^{J+1}}})}{\varepsilon} \leq C_{18} \varepsilon^{\frac{1}{2^J} - 1} \tag{104}
\]
for some constant \( C_{18} \) depending only on \( W \). We define \( Y := \{ x \in B_1(0) \mid 1 - b \leq |u^\varepsilon(x, t_0)| \leq 1 - \sqrt{\varepsilon} \} \). Note that
\[
Y \subset \bigcup_{j=1}^J A_j \tag{105}
\]
by (100). Set \( p(t) = 2^{-t} \varepsilon^{2^{-t}} \). Then \( p \) satisfies
\[
p'(t) = -(\log 2) 2^{-t} \varepsilon^{2^{-t}} (1 + 2^{-t} \log \varepsilon) > 0 \quad \text{for any } t \in [1, J + 1], \tag{106}
\]
because $2^{-J-1} \log \varepsilon \leq \log \sqrt{b} \leq -1$ by (99) and (100). Set $C_{19} = C_{12} C_{14} C_{18}$. Then from (100), (103), (104), (105), and (106) we have

$$
\int_Y \frac{W(u^\varepsilon(x, t_0))}{\varepsilon} \, dx \leq \sum_{j=1}^J \int_{A_j} \frac{W(u^\varepsilon(x, t_0))}{\varepsilon} \, dx \leq C_{19} |\log \varepsilon| \sum_{j=1}^J 2^{-j} e^{-2^j} \leq C_{19} |\log \varepsilon| \int_1^{J+1} 2^{-t} e^{-2^t} \, dt = C_{19} \frac{e^{-2^{J+1}} - \sqrt{e}}{\log 2} \leq C_{19} \frac{\sqrt{b} \log 2}{\log 2}. 
$$

(107)

Using the same argument above, we can show that

$$
\int_{\{x \in B_1(0) \mid 1 - \sqrt{\varepsilon} \leq |u^\varepsilon(x, t_0)| \leq 1 - \varepsilon^{\frac{2}{3}}\}} \frac{W(u^\varepsilon)}{\varepsilon} \, dx \leq C_{18} \mathcal{L}^d(\{x \in B_1(0) \mid 1 - \sqrt{\varepsilon} \leq |u^\varepsilon(x, t_0)| \leq 1 - \varepsilon^{\frac{2}{3}}\}) \leq 2C_{19} |\log \varepsilon|,
$$

where we used Lemma 9 with $\gamma = \frac{2}{3}$. Since $|u^\varepsilon| \leq 1$, we have

$$
\int_{\{x \in B_1(0) \mid 1 - \varepsilon^{\frac{2}{3}} \leq |u^\varepsilon(x, t_0)|\}} \frac{W(u^\varepsilon)}{\varepsilon} \, dx \leq \frac{W(1 - \varepsilon^{\frac{2}{3}})}{\varepsilon} \mathcal{L}^d(\{x \in B_1(0) \mid 1 - \varepsilon^{\frac{2}{3}} \leq |u^\varepsilon(x, t_0)|\}) \leq \varepsilon^{\frac{1}{2}} \mathcal{L}^d(B_1(0)).
$$

(109)

By (107), (108), and (109), Proposition 15 holds for sufficiently small $b$ and $\varepsilon$. □

Now we prove the integrality of $\mu_t$.

**Theorem 16.** For a.e. $t > 0$, there exist a countably $(d-1)$-rectifiable set $M_t$ and $\mathcal{H}^{d-1}$-measurable function $\theta_t : M_t \to \mathbb{N}$ with $\theta_t \in L^1_{\text{loc}}(\mathcal{H}^{d-1} \setminus M_t)$ such that $\mu_t = \theta_t \mathcal{H}^{d-1} \setminus M_t$ holds.

**Proof.** Set $H^\varepsilon := \Delta \varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon^2}$. Then for a.e. $t_0 > 0$, we can choose a subsequence $\{V_{t_0j}\}_{j=1}^\infty$ such that $V_{t_0j} \rightharpoonup V_{t_0}$,

$$
\lim_{j \to \infty} \int_{\Omega} |\xi_{t_0j}(x, t_0)| \, dx = 0,
$$

and

$$
c_H(t_0) := \sup_{j \in \mathbb{N}} \int_{\Omega} \varepsilon_{t_0j} |H^{\varepsilon_{t_0j}} \nabla \varphi_{t_0j} |(x, t_0) \, dx < \infty
$$

hold by Theorem 13 and (81). Note that $V_{t_0}$ is a countably $(d-1)$-rectifiable varifold and determined by $\mu_{t_0}$ uniquely from Theorem 14. We fix such $t_0 > 0$ and show the
claim for \( \mu_{t_0} \). In this proof, even if we take a subsequence \( \varepsilon_{i,j} \), we always abbreviate \( \varepsilon_{i,j} \) by \( \varepsilon_i \) for simplicity. Set

\[
A_{i,m} := \left\{ x \in \Omega \mid \int_{B_r(x)} \varepsilon_i |H^{\varepsilon_i} \nabla \psi^{\varepsilon_i}|(x, t_0) \, dx \leq m \mu_{t_0}^{\varepsilon_i}(B_r(x)) \text{ for any } r \in \left(0, \frac{1}{2}\right) \right\}
\]

and

\[
A_m := \{ x \in \Omega \mid \text{there exists } x_i \in A_{i,m} \text{ for any } i \in \mathbb{N} \text{ such that } x_i \to x \}
\]

for any \( m \in \mathbb{N} \). Then the Besicovitch covering theorem implies

\[
\mu_{t_0}^{\varepsilon_i}(\Omega \setminus A_{i,m}) \leq \frac{c(d)c_H(t_0)}{m}, \quad (112)
\]

where \( c(d) > 0 \) is a constant depending only on \( d \). Set \( A := \bigcup_{m=1}^{\infty} A_m \). Next we prove

\[
\mu_{t_0}(\Omega \setminus A) = 0. \quad (113)
\]

If (113) is not true, there exists a compact set \( K \subset\Omega \setminus A \) with \( \mu_{t_0}(K) > \frac{1}{2} \mu_{t_0}(\Omega \setminus A) > 0 \). Since \( A_1 \subset A_2 \subset A_3 \subset \cdots \), we have \( K \subset \Omega \setminus A_m \) for any \( m \in \mathbb{N} \). For any \( x \in K \), there is a neighborhood \( B_r(x) \) such that \( B_r(x) \cap A_{i,m} = \emptyset \) for sufficiently large \( i \), by the definition of \( A_m \). This and the compactness of \( K \) imply that there exist an open set \( O_m \) and \( i_0 \in \mathbb{N} \) such that \( K \subset O_m \) and \( O_m \cap A_{i,m} = \emptyset \) for any \( i \geq i_0 \). Let \( \phi_m \in C_c(O_m) \) be a non-negative test function such that \( 0 \leq \phi_m \leq 1 \) and \( \phi_m = 1 \) on \( K \). We compute

\[
\mu_{t_0}(K) \leq \int_\Omega \phi_m \, d\mu_{t_0} = \lim_{i \to \infty} \int_\Omega \phi_m \, d\mu_{t_0}^{\varepsilon_i} = \lim_{i \to \infty} \int_{\Omega \setminus A_{k,m}} \phi_m \, d\mu_{t_0}^{\varepsilon_i} \\
\leq \liminf_{i \to \infty} \mu_{t_0}^{\varepsilon_i}(\Omega \setminus A_{k,m})
\]

(114)

for any \( k \geq i_0 \). Combining (112) and (114), we obtain \( \mu_{t_0}(K) = 0 \). Therefore we have proved (113).

By the rectifiability of \( \mu_{t_0} \) and (113), for \( \mu_{t_0} \) a.e. \( x \in \text{spt} \mu_{t_0} \), it has an approximate tangent space \( P \) and \( x \in A_m \) for some \( m \). Fix such \( x \). We may assume that \( x = 0 \) and \( P = \{ x \in \mathbb{R}^d \mid x_d = 0 \} \) by a parallel translation and a rotation. Set

\[ \theta := \lim_{0 \to 1-} \frac{\mu_{t_0}(B_{r_0}(0))}{C_d \cdot r_0^d - 1}. \]

We only need to prove \( \theta \in \mathbb{N} \). Let \( \Phi_r(x) = \frac{x}{r} \) for \( r > 0 \) and \( (\Phi_r)_{\#} V_{t_0} \) be the push-forward of the varifold (see (15)). Then for any positive sequence \( r_i \to 0 \), we have \( \lim_{i \to \infty} (\Phi_{r_i})_{\#} V_{t_0} = \theta |P| \), where \( |P| \) is the unit density varifold generated by \( P \). By the assumption \( 0 \in A_m \), there exists \( \{ x_i \}_{i=1}^{\infty} \) such that \( x_i \in A_{i,m} \) and \( x_i \to 0 \) as \( i \to \infty \). Passing to a subsequence if necessary, we may assume that

\[
\lim_{i \to \infty} \frac{x_i}{r_i} = 0, \quad \lim_{i \to \infty} \frac{\varepsilon_i}{r_i} = 0, \quad (115)
\]

and

\[
\lim_{i \to \infty} (\Phi_{r_i})_{\#} V_{t_0}^{\varepsilon_i} = \theta |P|. \quad (116)
\]
Set $u^{\tilde{\varepsilon}_i}(\tilde{x}, \tilde{t}) = \varphi^{\varepsilon_i}(x, t)$ and $g^{\tilde{\varepsilon}_i}(\tilde{t}) = r_i \lambda^{\varepsilon_i}(t)$ for $\tilde{x} = \frac{x}{r_i}$, $\tilde{t} = \frac{t-r_0}{r_i}$, and $\tilde{\varepsilon}_i = \frac{\varepsilon_i}{r_i}$ (another functions $\tilde{\xi}_i$ and $\tilde{H}^{\varepsilon_i}$ are defined in the same way). Note that $\tilde{x}_i := \frac{x}{r_i} \to 0$ and $\tilde{\varepsilon}_i \to 0$ by (115) and $u^{\tilde{\varepsilon}_i}$ is a solution to (83) with $\tilde{\varepsilon}_i$ instead of $\varepsilon$. We compute

$$
\int_{B_3(0)} |\tilde{\xi}^{\varepsilon_i}(\tilde{x}, 0)| \, d\tilde{x} = \frac{1}{r_i^{d-1}} \int_{B_{3r_i}(0)} |\xi^{\varepsilon_i}(x, 0)| \, dx.
$$

Thus, by (110) we may assume that

$$
\lim_{i \to \infty} \int_{B_3(0)} |\tilde{\xi}^{\varepsilon_i}(\tilde{x}, 0)| \, d\tilde{x} = 0,
$$

passing to a subsequence if necessary. We compute

$$
\frac{\varepsilon_i}{r_i^{d-2}} \int_{B_{3r_i}(0)} |H^{\varepsilon_i} \nabla_{\tilde{z}} u^{\varepsilon_i}(\tilde{x}, 0)| \, d\tilde{x} \leq \frac{m}{r_i^{d-2}} \mu_{t_0}(B_{4r_i}(x_i))
$$

$$
\leq 4^{d-1} \nu_0 d_{d-1} D_{2r_i} \to 0 \quad \text{as } i \to \infty,
$$

where we used (52), (115), and $x_i \in A_{i,m}$. Let $\tilde{V}^{\varepsilon_i}$ be a varifold defined by (79) with $u^{\tilde{\varepsilon}_i}$ instead of $\varphi^{\varepsilon_i}$. Then $\tilde{V}^{\varepsilon_i}_0 = (\Phi_{r_i})_# V^{\varepsilon_i}_0$. Next we show

$$
\lim_{i \to \infty} \int_{B_3(0)} (1 - (v^{i}_d)^2) \tilde{\varepsilon}_i |\nabla_{\tilde{z}} u^{\varepsilon_i}|^2 \, d\tilde{x}\bigg|_{\tilde{r} = 0} = 0,
$$

where $v^{i} = (v^{i}_1, v^{i}_2, \ldots, v^{i}_d) = \frac{\nabla_{\tilde{z}} u^{\varepsilon_i}}{|\nabla_{\tilde{z}} u^{\varepsilon_i}|}$. For $S \in \mathbb{G}(d, d-1)$, set $\psi(S) := 1 - v^{2}_d$, where $v \in \mathbb{S}^{d-1}$ be one of the unit normal vectors to $S$. Then $\psi : \mathbb{G}(d, d-1) \to \mathbb{R}$ is well-defined, continuous, and $\psi(P) = 0$. Hence, for any $\phi \in C_c(\mathbb{R}^d)$, $\phi \psi \in C_c(G_{d-1}(\mathbb{R}^d))$ and

$$
\lim_{i \to \infty} \tilde{V}^{\varepsilon_i}_0(\phi \psi) = \int \phi(\tilde{x})(1 - (v^{i}_d)^2) \, d\|\tilde{V}^{\varepsilon_i}_0\|_{(\tilde{x})} = \lim_{i \to \infty} (\Phi_{r_i})_# V^{\varepsilon_i}_0(\phi \psi) = \theta |P|(\phi \psi) = \theta \int \phi(\tilde{x}) \psi(P) \, dH_{d-1}(\tilde{x}) = 0,
$$

where we used (116) and $\psi(P) = 0$. Thus (120) proves (119). We subsequently fix the subsequence and drop $\tilde{\varepsilon}$ and time variable (for example, we write $u^{\tilde{\varepsilon}_i}(\tilde{x}, 0)$ as $u^{\varepsilon_i}$) for simplicity. We assume that $N \in \mathbb{N}$ is the smallest positive integer greater than $\theta$, namely,

$$
\theta \in [N - 1, N).
$$
Let \( s > 0 \) be an arbitrary number. Then Proposition 15 and (33) imply that there exists \( b > 0 \) such that
\[
\int_{\{x \in B_3(0) \mid |u^\varepsilon_i(x)| \geq 1-b\}} \varepsilon_i |\nabla u^\varepsilon_i|^2 \, dx \leq s
\]  
for sufficiently large \( i \). Note that we may use Proposition 15 with \( t = 0 \) since \( \tilde{t} = \frac{t-t_0}{r_i} \) in this proof. For these \( s > 0, b > 0, \) and \( c > 0 \) given by Lemma 4, we choose \( \varrho \) and \( L \) given by Propositions 20 and 21 in the Appendix with \( R = 2 \) (we may restrict \( \varrho \) to be small if necessary). We choose \( a = L \varepsilon_i \) as a constant in Proposition 20. Set
\[
G_i := B_2(0) \cap \{|u^\varepsilon_i| \leq 1-b\}
\]
and
\[
\{ x \mid \int_{B_r(x)} \varepsilon_i |H^\ell \nabla u^\varepsilon_i| + |\xi_{\ell i}| + (1-\nu_d^2)\varepsilon_i |\nabla u^\varepsilon_i|^2 \, dx \leq \varrho \mu_0^\varepsilon_i (B_\varepsilon_i(x)) \}
\]
for sufficiently large \( i \). The Besicovitch covering theorem, (117), (118), and (119) yield
\[
\mu_0^\varepsilon_i ((B_2 \cap \{|u^\varepsilon_i| \leq 1-b\}) \setminus G_i) \leq \frac{c(d)}{\varrho} \int_{B_3(0)} \varepsilon_i |H^\ell \nabla u^\varepsilon_i| + |\xi_{\ell i}| + (1-\nu_d^2)\varepsilon_i |\nabla u^\varepsilon_i|^2 \, dx \to 0 \quad \text{as } i \to \infty.
\]  
Next we show that for sufficiently large \( i \)
\[
\frac{\mu_0^\varepsilon_i (B_r(x))}{\omega d_{d-1} r^{d-1}} \geq 1 - 2s, \quad \text{for any } x \in G_i \text{ and } r \in [L \varepsilon_i, 1].
\]  
Note that all the assumptions in Proposition 21 are satisfied by Lemma 4, (33), and (123). Thus we have (125) with \( r = L \varepsilon_i \). By integration by parts, we have
\[
\frac{d}{dr} \left\{ \frac{1}{d-1} \int_{B_r(x)} e^{\ell i} \, dy \right\} + \frac{1}{d} \int_{B_r(x)} (\xi_{\ell i} + \varepsilon_i H^\ell_i (y \cdot \nabla u^\varepsilon_i)) \, dy - \frac{\varepsilon_i}{d+1} \int_{\partial B_r(x)} (y \cdot \nabla u^\varepsilon_i)^2 \, dH^{d-1}(y) = 0.
\]
Thus we can compute
\[
\frac{1}{\sigma d-1} \int_{B_r(x)} e^{\ell i} \, dy \bigg|_{\tau=L \varepsilon_i} \geq - \int_{L \varepsilon_i}^{r} \frac{1}{\sigma d} \int_{B_r(x)} \varepsilon_i H^\ell_i (y \cdot \nabla u^\varepsilon_i) \, dy \, d\tau \geq - \int_{L \varepsilon_i}^{r} \frac{1}{\sigma d} \int_{B_r(x)} \varepsilon_i \tau |H^\ell_i \nabla u^\varepsilon_i| \, dy \, d\tau \geq \frac{\varrho D_2}{\sigma},
\]
where we used (52), (123), and $\xi_{\varepsilon_i} \leq 0$. Therefore we obtain (125) for sufficiently large $i$ by restricting $\phi$ to be small. Let $\delta > 0$ and $\phi \in C_c(B_3(0))$ be a non-negative test function such that $\phi = 1$ on $B_2(0) \cap \{|x_d| > \delta\}$. Then there exists $i_0 \geq 1$ such that

$$\mu_0^{\varepsilon_i}(\phi) \leq (1 - 2s)\omega_{d-1} \frac{\delta^{d-1}}{2}, \quad \text{for any } i \geq i_0,$$

(126)
since $\mu_0^{\varepsilon_i} = \|V_0^{\varepsilon_i}\| \to \theta \mathcal{H}^{d-1}|_P$. Assume that $x \in G_i \cap \{|x_d| > 2\delta\}$ for $i \geq i_0$. Then (125) and (126) imply

$$(1 - 2s)\omega_{d-1} \delta^{d-1} \leq \mu_0^{\varepsilon_i}(B_{\delta_1}(x)) \leq \mu_0^{\varepsilon_i}(\phi) \leq (1 - 2s)\omega_{d-1} \frac{\delta^{d-1}}{2},$$

for any $i \geq i_0$.

This is a contradiction. Thus

$$\text{dist} (P, G_i) \to 0 \quad \text{as } i \to \infty.$$  

(127)

Set $Y := P^{-1}(x) \cap G_i \cap \{|x| = l\}$ for $x \in P \cap B_1(0)$. Next we show that for sufficiently large $i$

$$\#Y \leq N - 1, \quad \text{for any } x \in P \cap B_1(0) \text{ and } |l| \leq 1 - b.$$  

(128)

For a contradiction, assume that $\#Y \geq N$ and choose $y_j \in Y$ for $j = 1, 2, \ldots, N$. We use Proposition 20 with $R = 1$, $a = L\varepsilon_i$ and $Y' = \{y_j\}_{j=1}^N$ instead of $Y$. Note that the smallness of diam $Y'$ is true from (127) and $|y_j - y_k| > 3L\varepsilon_i$ for any $1 \leq j < k \leq N$ holds by (159). Then (158) yields

$$\sum_{j=1}^N \frac{1}{(L\varepsilon_i)^{d-1}} \mu_0^{\varepsilon_i}(B_{L\varepsilon_i}(y_j)) \leq s + (1 + s)\mu_0^{\varepsilon_i}(\{|z| \text{ dist } (z, Y') < 1\})$$  

(129)

for sufficiently large $i$. By (127) and $\mu_0^{\varepsilon_i} = \|V_0^{\varepsilon_i}\| \to \theta \mathcal{H}^{d-1}|_P$,

$$\limsup_{i \to \infty} \mu_0^{\varepsilon_i}(\{|z| \text{ dist } (z, Y') < 1\}) \leq \theta \mathcal{H}^{d-1}|_P(B_1(0)) = \theta \omega_{d-1}.$$  

By this, $\#Y' = N$, (125), and (129) we have

$$N \omega_{d-1}(1 - 2s) \leq s + (1 + s)\theta \omega_{d-1}.$$  

However, this contradicts (121) by restricting $s$ to be small. Thus (128) holds for sufficiently large $i$.

Finally, we complete the proof. Set $\hat{V}_0^{\varepsilon_i} := V_0^{\varepsilon_i} \mathcal{L}_{|\{x_d| \leq 1|} \times \mathcal{G}(d,d-1)$. We regard $P$ as a diagonal matrix $(p_{jk})$ with $p_{kk} = 1$ for $1 \leq k \leq d - 1$ and $p_{dd} = 0$. Then the push-forward of $\hat{V}_0^{\varepsilon_i}$ by $P$ is given by

$$P_# \hat{V}_0^{\varepsilon_i}(\phi) = \int_{\{|x_d| \leq 1\}} \phi(Px, \nabla Px \circ (I - v^i \otimes v^i))|\Lambda_{d-1}\nabla Px \circ (I - v^i \otimes v^i)| \, d\mu_0^{\varepsilon_i}$$

$$= \int_{\{|x_d| \leq 1\}} \phi(Px, P \circ (I - v^i \otimes v^i))|v^i_d| \, d\mu_0^{\varepsilon_i}$$
for any \( \phi \in C_0(P \cap B_2(0) \times \mathbb{R}^d) \). Here \( \Lambda_{d-1} \nabla P \circ (I - v^i \otimes v^i) \) is the Jacobian and \( v^i_d = \frac{\partial v^i}{\partial u^i} \). Due to (116), \( P^i \to \hat{V}_0 \) are satisfied. Then one can check that all the assumptions in Sections 3 and 4 are fulfilled. Therefore (a) holds by Propositions 2, 9, and 10. By (117),

\[
\lim_{i \to \infty} \int_{B_1(0)} \left| \frac{\varepsilon_i}{2} \frac{\nabla u^{\varepsilon_i}}{\varepsilon_i} + \frac{W(u^{\varepsilon_i})}{\varepsilon_i} - \sqrt{2W(u^{\varepsilon_i})} \nabla u^{\varepsilon_i} \right| \, dx = 0
\]

(130) holds (see (133) below). We compute

\[
\omega_{d-1} \theta = \mathcal{H}^{d-1} |_{P(B_1(0))} \leq \lim \inf_{i \to \infty} \| P^i \hat{V}_0 \| (B_1(0)) = \lim \inf_{i \to \infty} \int_{B_1(0)} |v_d^i| \, d\mu^\varepsilon_0
\]

\[
\leq \lim \inf_{i \to \infty} \int_{B_1(0) \cap G_i} |v_d^i| \, d\mu^\varepsilon_0 + 2s
\]

\[
\leq \lim \inf_{i \to \infty} \frac{1}{\sigma} \int_{B_1(0) \cap G_i} |v_d^i| \sqrt{2W(u^{\varepsilon_i})} \nabla u^{\varepsilon_i} \, dx + 2s,
\]

where we used (122), (124), and (130). By the co-area formula and the area formula, we have

\[
\int_{B_1(0) \cap G_i} |v_d^i| \sqrt{2W(u^{\varepsilon_i})} \nabla u^{\varepsilon_i} \, dx
\]

\[
= \int_{-1+b}^{1+b} \sqrt{2W(\tau)} \int_{B_1(0) \cap G_i \cap \{u^i = \tau\}} |\Lambda_{d-1} \nabla P \circ (I - v^i \otimes v^i)| \, d\mathcal{H}^{d-1} \, d\tau
\]

\[
= \int_{-1+b}^{1+b} \sqrt{2W(\tau)} \int_{B_1(0) \cap \{u^i = \tau\}} \mathcal{H}^0 \left( \{u^i = \tau\} \cap G_i \cap P^{-1}(x) \right) \, d\mathcal{H}^{d-1} \, d\tau
\]

\[
\leq \int_{-1+b}^{1+b} \sqrt{2W(\tau)} \int_{B_1(0) \cap \{x_d = 0\}} (N - 1) \, d\mathcal{H}^{d-1} \, d\tau
\]

\[
\leq \sigma (N - 1) \omega_{d-1},
\]

(132)

where we used (128) and \( \sigma = \int_{-1}^{1} \sqrt{2W(\tau)} \, d\tau \). Hence \( \theta \leq N - 1 \) due to (131) and (132) and the arbitrariness of \( s \). By this and (121), \( \theta = N - 1 \).

\[\square\]

5. Proofs of Main Theorems

In this section we prove Theorem 3 and Theorem 4 on the existence of the weak solution in the sense of \( L^2 \)-flow and distributional \( BV \)-solution.

**Proof of Theorem 3.** Let \( \{\varphi_0^{\varepsilon_i}\}_{i=1}^{\infty} \) be a family of functions such that all the claims of Proposition 2 are satisfied. Then one can check that all the assumptions in Sections 3 and 4 are fulfilled. Therefore (a) holds by Propositions 2, 9, and 10. By taking a subsequence \( \varepsilon_i \to 0 \), we obtain (b) (the proof is standard and is exactly the same as that in [43], so we omit it). By Lemma 1 and the weak compactness of \( L^2(0, T) \), we may take a subsequence \( \varepsilon_i \to 0 \) such that (c) holds (for the weak convergence for all \( T > 0 \), we only need to use the diagonal argument).
Next we show (d). We compute

$$\frac{\varepsilon|\nabla \varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon} - \sqrt{2W(\varphi^\varepsilon)}|\nabla \varphi^\varepsilon| = \left(\frac{\sqrt{\varepsilon}|\nabla \varphi^\varepsilon|}{\sqrt{2}} - \frac{\sqrt{W(\varphi^\varepsilon)}}{\sqrt{\varepsilon}}\right)^2 \leq |\xi_\varepsilon|.$$ 

(133)

Set \(d\hat{\mu}^\varepsilon := \frac{1}{\sigma} \sqrt{2W(\varphi^\varepsilon)}|\nabla \varphi^\varepsilon| \, dx \, dt\). By (133), Proposition 9, and Theorem 13, we have

$$\hat{\mu}^\varepsilon \rightharpoonup \mu \quad \text{as Radon measures},$$

(134)

where \(d\mu := d\mu_t \, dt\). By (35), (38), (133), and (134), we obtain

$$\sup_{i \in \mathbb{N}} \int_{\Omega \times (0,T)} |\lambda_{\varepsilon i}|^2 \, d\hat{\mu}_{\varepsilon i} \leq \sup_{i \in \mathbb{N}} \int_{\Omega \times (0,T)} |\lambda_{\varepsilon i}|^2 \, d\mu_i \, dt \leq D_1 C_3 (1 + T).$$

Then there exist \(\tilde{\varepsilon} \in (L^2_{loc}(\mu))^d\) and the subsequence \(\varepsilon_i \to 0\) such that

$$\frac{1}{\sigma} \int_{\Omega \times (0,T)} -\lambda_{\varepsilon i} \sqrt{2W(\varphi_{\varepsilon i})} \nabla \varphi_{\varepsilon i} \cdot \tilde{\phi} \, dx \, dt = \int_{\Omega \times (0,T) \cap \{||\nabla \varphi_{\varepsilon i}|| \neq 0\}} -\lambda_{\varepsilon i} \frac{\nabla \varphi_{\varepsilon i}}{|\nabla \varphi_{\varepsilon i}|} \cdot \tilde{\phi} \, d\hat{\mu}_{\varepsilon i} \to \int_{\Omega \times (0,T)} \tilde{\mu} \cdot \tilde{\phi} \, d\mu_t \, dt$$

for any \(\tilde{\phi} \in C_c(\Omega \times [0, T); \mathbb{R}^d)\) (see [21, Theorem 4.4.2]). Moreover, if \(\tilde{\phi}\) is smooth, we have

$$\int_{\Omega \times (0,T)} \tilde{\phi} \, d\mu = \lim_{i \to \infty} \frac{1}{\sigma} \int_{\Omega \times (0,T)} -\lambda_{\varepsilon i} \sqrt{2W(\varphi_{\varepsilon i})} \nabla \varphi_{\varepsilon i} \cdot \tilde{\phi} \, dx \, dt$$

$$= \lim_{i \to \infty} \frac{1}{\sigma} \int_{\Omega \times (0,T)} \lambda_{\varepsilon i} k(\varphi_{\varepsilon i}) \nabla \tilde{\phi} \, dx \, dt = \int_0^T \int_{\Omega} \lambda \tilde{\phi} \, dx \, dt$$

$$= \left(- \int_0^T \lambda \int_{\Omega} \tilde{\phi} \, dx \right) \cdot \nabla \psi (-, t) \|dt,$$

where we used (c), \(k'(s) = \sqrt{2W(s)}\), \(\lim_{i \to \infty} k(\varphi_{\varepsilon i}) = \sigma (\psi - \frac{1}{2})\) for a.e. \(x, t\), and the dominated convergence theorem. Hence we have (25).

Now we prove (e). By replacing \(d\hat{\mu}^\varepsilon\) with \(d\hat{\mu}^\varepsilon := \frac{\varepsilon}{\sigma} |\nabla \varphi_{\varepsilon i}|^2 \, dx \, dt\), the convergence (26) is obtained in the same way as (d). In addition, for any \(\tilde{\phi} \in C_c(\Omega \times [0, T); \mathbb{R}^d)\), we compute

$$\int_0^T \int_{\Omega} \tilde{\phi} \, d\mu_t \, dt = \lim_{i \to \infty} \int_0^T \int_{\Omega} \tilde{\phi} \, d\mu_{\varepsilon i} \, dt = \lim_{i \to \infty} \int_0^T \int_{\Omega} \tilde{\phi} \, d\hat{\mu}_{\varepsilon i}$$

$$= \lim_{i \to \infty} \int_0^T \int_{\Omega} \tilde{\phi} \, d\mu_t \, dt = \int_0^T \int_{\Omega} \tilde{\phi} \, d\mu_t \, dt,$$

where we used Theorem 14 and (d). Thus \(\tilde{\phi} = \tilde{h} + \tilde{f}\). One can check that \(\{\mu_t\}_{t \in [0, \infty)}\) is an \(L^2\)-flow with the generalized velocity vector \(\tilde{v}\) (see [43, Proposition 4.3] for the inequality (16) and [34, Lemma 6.3] for the perpendicularity).
To prove Theorem 4, we use the next proposition and lemmas. In the original proof of the proposition, $2 \leq d \leq 3$ is assumed to use the results of [38, Proposition 4.9, Theorem 5.1]. however, we already know that $|\xi| = 0$ and $\mu_t$ is integral for a.e. $t$, so we can show the claim in the same way.

**Proposition 17.** (See Proposition 4.5 of [34]) Let $\psi$, $\tilde{v}$, and $\tilde{v}$ are given by Theorem 3. Then $\int_0^T \int_\Omega \phi \tilde{v} \cdot \tilde{v} d\|\nabla \psi(\cdot, t)\| dt < \infty$ and

$$\int_0^T \int_\Omega \phi \tilde{v} \cdot \tilde{v} d\|\nabla \psi(\cdot, t)\| dt = \int_0^T \int_\Omega \partial_t \phi \psi \, dx \, dt \quad (135)$$

for any $\phi \in C^1_c(\Omega \times (0, T))$ and for any $T > 0$.

**Proof.** Set $\nabla_{x,t} = (\nabla, \partial_t)$ in the sense of BV. One can check that $\|\nabla_{x,t} \psi\| \lesssim \mu, \mu|\partial|^\phi(\psi=1)$ is rectifiable, $\int_0^T \|\nabla \psi(\cdot, t)\| dt < \infty$, and the approximate tangent space coincides with that of $\|\nabla_{x,t} \psi\|$ for $\mu$-a.e. and $\|\nabla_{x,t} \psi\|$-a.e. (see [34, Proposition 8.1–8.3] and [2, Proposition 2.85]). By this and Proposition 1, we have

$$0 = \int_0^T \int_\Omega \phi(\tilde{v}, 1) \cdot \tilde{v}_{x,t} \, d\|\nabla_{x,t} \psi\| = \int_0^T \int_\Omega \phi \tilde{v} \cdot \tilde{v} \, d\|\nabla \psi(\cdot, t)\| dt + \int_0^T \int_\Omega \phi \partial_t \psi \, dx \, dt$$

for any $\phi \in C^1_c(\Omega \times (0, T))$, where $\tilde{v}_{x,t}$ is the inner unit normal vector of $\{(x, t) \mid \psi(x, t) = 1\}$. Therefore we obtain (135). \qed

**Lemma 11.** Let $\gamma$ and $\delta$ be positive constants with $\delta < \gamma$. Under the same assumptions of Theorem 4, there exist $T_2 \in (0, 1)$ and $\varepsilon_S \in (0, 1)$ depending only on $\gamma$, $\delta$, and $C_3(\omega, d, D_1)$ with the following property. Let $g : \mathbb{R} \to [0, \infty)$ be a smooth even function such that $g(0) = 0$, $0 \leq g''(s) \leq 2$ for any $s \in \mathbb{R}$, and $g(s) = |s| - \frac{1}{2} \delta \varepsilon$ if $|s| \geq 1$. Set $g^\delta(s) := \varepsilon g(s/\delta)$ and define $\tilde{r}^\varepsilon, \delta \in C^\infty(\mathbb{R}^d \times [0, \infty))$ by

$$\tilde{r}^\varepsilon, \delta(x, t) := g^\delta(x_1) + \int_0^t \lambda^\varepsilon(\tau) \, d\tau + 2\delta^{-1}t - \gamma,$$

where $\lambda^\varepsilon$ is given by (6). Set $\tilde{\phi}^\varepsilon, \delta := q^\varepsilon(\tilde{r}^\varepsilon, \delta)$ and assume that $\tilde{\phi}^\varepsilon(x, 0) \geq \phi^\varepsilon_0(x)$ for any $x \in \mathbb{R}^d$. Then

$$\tilde{\phi}^\varepsilon, \delta \geq \phi^\varepsilon \quad \text{in} \, \mathbb{R}^d \times [0, T_2) \quad (136)$$

for any $\varepsilon \in (0, \varepsilon_S)$.

**Proof.** We denote $\tilde{r} = \tilde{r}^\varepsilon, \delta$ for simplicity. By (34) and the comparison principle, we only need to prove

$$\partial_t \tilde{r} \geq \Delta \tilde{r} - \frac{2q^\varepsilon(\tilde{r})}{\varepsilon} (|\nabla \tilde{r}|^2 - 1) + \lambda^\varepsilon \quad \text{in} \, \mathbb{R}^d \times (0, T_2) \quad (137)$$

for sufficiently small $T_2 > 0$ and $\varepsilon > 0$, since $\tilde{\phi}^\varepsilon, \delta \geq \phi^\varepsilon$ if and only if $\tilde{r} \geq r^\varepsilon$. In the case of $|x_1| \geq \delta$, (137) holds by $\partial_t \tilde{r} = 2\delta^{-1} + \lambda^\varepsilon, |\nabla \tilde{r}| = 1$, and $\Delta \tilde{r} = 0$. Next we
consider the case of \(|x_1| \leq \delta\). Set \(O_\delta := \{ x \in \mathbb{R}^d \mid |x_1| \leq \delta \}\). By \(\tilde{r}(x, 0) \leq -\gamma + \frac{\delta}{2}\) on \(O_\delta\) and
\[
|\tilde{r}(x, t) - \tilde{r}(x, 0)| \leq \int_0^t |\lambda^e(\tau)|\,d\tau + 2\delta^{-1}t \leq \sqrt{C_3(1+t)}\sqrt{r} + 2\delta^{-1}t,
\]
there exists \(T_2 > 0\) such that
\[
\tilde{r}(x, t) \leq -\frac{\gamma}{4} < 0 \quad \text{for any } (x, t) \in O_\delta \times [0, T_2). \tag{138}
\]
By (138) and \(|\nabla \tilde{r}| \leq 1\),
\[
\frac{2q^e(\tilde{r})}{\epsilon} (|\nabla \tilde{r}|^2 - 1) \geq 0.
\]
By using this, for any \((x, t) \in O_\delta \times [0, T_2),\)
\[
\partial_t \tilde{r} - \Delta \tilde{r} + \frac{2q^e(\tilde{r})}{\epsilon} (|\nabla \tilde{r}|^2 - 1) - \lambda^e \geq 2\delta^{-1} - \Delta \tilde{r} \geq 0, \tag{139}
\]
where we used \(\Delta \tilde{r} \leq \delta^{-1} g''(x_1/\delta) \leq 2\delta^{-1}\). Therefore we obtain (137).

**Lemma 12.** Let \(r \in (0, \frac{1}{4})\). Then there exists \(T_3 > 0\) depending only on \(d\) and \(r\) with the following property. Let \(U_0 \subset \subset (\frac{1}{4}, \frac{3}{4})^d\) satisfies \(\mathcal{L}^d(U_0) = \mathcal{L}^d(B_r(0))\) and has a \(C^1\) boundary \(M_0\) with (27) for \(\delta_1 > 0\). In addition, we assume \(\mathcal{H}^{d-1}(M_0) \leq 2\mathcal{H}^{d-1}(\partial B_r(0))\). Then we have
\[
0 \leq \mu_0(\Omega) - \mu_1(\Omega) \leq \delta_1 \quad \text{for any } t \in [0, T_3), \tag{140}
\]
where \(\mu_t\) is a weak solution to (1) with initial data \(M_0\).

**Proof.** First we claim that there exists \(T_3 > 0\) depending only on \(d\) and \(r\) such that
\[
U_t = \{ x \in (0, 1)^d \mid \psi(x, t) = 1 \} \subset \left(\frac{1}{10}, \frac{9}{10}\right)^d \tag{141}
\]
for any \(t \in [0, T_3)\), where \(\psi = \lim_{i \to \infty} \psi^{\epsilon_i} = \lim_{i \to \infty} \frac{1}{2}(\phi^{\epsilon_i} + 1)\). Let \(\tilde{\phi}^{\epsilon, \delta}\) be a function given by Lemma 11 with \(\gamma = \frac{1}{10}\) and \(\delta = \frac{1}{20}\). By (136) and (138), one can check that there exists \(T_3 = T_3(C_3(\omega, d, D'_1)) > 0\) such that \(\lim_{i \to \infty} \phi^{\epsilon_i}(x, t) = -1\) on \(\{ x \in \mathbb{R}^d \mid |x_1| \leq \frac{1}{10}\} \) for any \(t \in [0, T_3)\). Note that \(\omega\) and \(D'_1\) depend only on \(r\) by \(\mathcal{L}^d(U_0) = \mathcal{L}^d(B_r(0))\) and \(\mathcal{H}^{d-1}(M_0) \leq 2\mathcal{H}^{d-1}(\partial B_r(0))\). Hence \(T_3\) depends only on \(d\) and \(r\). Therefore \(U_t \cap \{ x \in \mathbb{R}^d \mid |x_1| \leq \frac{1}{10}\} = \emptyset\) for any \(t \in [0, T_3)\). Similarly we have (141). Thus \(\partial^*(U_t \cap (0, 1)^d) = \partial^*U_t\) for any \(t \in [0, T_3)\). Hence, by using the isoperimetric inequality for Caccioppoli sets (see [13,46]), and (b3) and (b4) of Theorem 3, we have
\[
d\omega_d^{\frac{1}{d}}(\mathcal{L}^d(U_0)) \leq d\omega_d^{\frac{1}{d}}(\mathcal{L}^d(U_t)) \leq \mathcal{H}^{d-1}(\partial^* U_t) \leq \mu_t(\Omega), \tag{142}
\]
for any \(t \in [0, T_3)\).

By (27) and (142), we obtain (140).

Finally we prove Theorem 4.
Proof of Theorem 4. First we show (a). From (37),
\[ \int_0^T |\lambda^e(t)|^2 \, dt \leq C_2(\mu^0(\Omega) - \mu^e_T(\Omega) + T) \]
for any \( T > 0 \) and for any \( \varepsilon \in (0, \varepsilon_1) \). By this and (140), we can choose \( \delta_1 = \delta_1(C_2(\omega, d, D^1_1)) > 0 \) so that
\[ \limsup_{i \to \infty} e^{\frac{1}{2} \int_0^{T_4} |\lambda^e(t)|^2 \, dt} \leq \limsup_{i \to \infty} e^{\frac{1}{2} C_2 \delta_1} e^{\frac{1}{2} C_2 T_4} \leq \frac{10}{9}, \quad (143) \]
where \( T_4 = T_4(d, r) = \min\{T_3, \frac{2}{C^2} \log \frac{100}{\eta T_3} \} > 0 \) and \( \delta_1 \) also depends only on \( d \) and \( r \) since \( \mathcal{L}^d(U_0) = \mathcal{L}^d(B_r(0)) \) and \( \mathcal{H}^{d-1}(M_0) \leq 2 \mathcal{H}^{d-1}(\partial B_r(0)) \). Then (51) and (143) imply
\[ \int_{\mathbb{R}^d} \rho_{(y,s)}(x, t) \, d\mu_t(x) \leq \frac{10}{9} \int_{\mathbb{R}^d} \rho_{(y,s)}(x, 0) \, d\mu_0(x) \quad (144) \]
for any \( y \in \mathbb{R}^d \), \( t \in [0, T_4) \), and \( s > 0 \) with \( 0 \leq t < s \). Recall that \( \rho_{(y,s)}(x, 0) \) converges to \((d - 1)\)-dimensional delta function at \( y \) as \( s \downarrow 0 \). Therefore, since \( M_0 \) is \( C^1 \), we may assume that there exists \( s_0 > 0 \) depending only on \( M_0 \) such that
\[ \int_{\mathbb{R}^d} \rho_{(y,s)}(x, 0) \, d\mu_0(x) \leq \frac{3}{2} \quad (145) \]
for any \( y \in \mathbb{R}^d \) and \( s \in (0, s_0) \). Set \( T_1 = T_1(d, r, M_0) := \min\{T_4, s_0\} \). Let \( t_0 \in (0, T_1) \) be a number such that \( \mu_{t_0} \) is integral. Then there exist a countably \((d - 1)\)-rectifiable set \( M_{t_0} = \mathcal{L}^d(B_{r_0}) \) such that \( \mu_{t_0} = \theta_{t_0} \mathcal{H}^{d-1}[M_{t_0}] \). Assume that there exist \( x_0 \in M_{t_0} \) and \( N \geq 2 \) such that \( M_{t_0} \) has an approximate tangent space at \( x_0 \) and
\[ \lim_{r \to 0} \frac{\omega_{d-1} r^{d-1}}{\omega_{d-1} r^{d-1}} = \theta_{t_0}(x_0) = N. \]
Set \( r = \sqrt{2(s-t_0)} \) for \( t_0 < s < T_1 \). Using the same calculation as (155), for any \( \delta \in (0, 1) \), we obtain
\[ \int_{\mathbb{R}^d} \rho_{(y,s)}(x, t) \, d\mu_{t_0} \geq \frac{1}{(\sqrt{2\pi r})^{d-1}} \int_{\delta}^1 \mu_{t_0}(B \sqrt{2r \log \frac{1}{k}}(x_0)) \, dk \]
\[ \to \frac{N \omega_{d-1}}{\pi^{d-1}} \int_{\delta}^1 \left( \log \frac{1}{k} \right)^{\frac{d-1}{2}} \, dk \quad \text{as } r \to 0 (s \downarrow t_0). \]
By this and \( \int_0^1 \left( \log \frac{1}{k} \right)^{\frac{d-1}{2}} \, dk = \Gamma\left(\frac{d-1}{2} + 1\right) = \pi^{\frac{d-1}{2}} / \omega_{d-1} \), we have
\[ \lim_{s \downarrow t_0} \int_{\mathbb{R}^d} \rho_{(y,s)}(x, t) \, d\mu_{t_0} = N. \]
Then we have a contradiction by (144) and (145). Therefore \( \theta_{t_0}(x) = 1 \mathcal{H}^{d-1} \)-a.e. on \( M_{t_0} \). By an argument similar to that in [45, Theorem 2.2 (2d)], we have \( \mathcal{H}^{d-1}(\partial^* U_{t_0} \setminus M_{t_0}) = 0 \) and \( \mathcal{H}^{d-1}(M_{t_0} \setminus \partial^* U_{t_0}) = 0 \) because \( \theta_{t_0}(x) \) is an even integer for \( \mathcal{H}^{d-1} \)-a.e. \( x \in M_{t_0} \setminus \partial^* U_{t_0} \). Hence we obtain (a).

The claim (b1) and (b2) are clear and (b3) is also obvious by Remark 4 and \( \mu_t = \| \nabla \psi(\cdot, t) \| \) for a.e. \( t \in (0, T_1) \). By (133), we have (b4).
Next we prove (b5). By (135), for any $\phi \in C^1_c((0, T))$, we compute
\[
\int_0^{T_1} \phi \int_\Omega \tilde{v} \cdot \nabla \psi (\cdot, t) \, dt = \int_0^{T_1} \partial_t \phi \int_\Omega \psi \, dx \, dt = 0,
\]
where we used (b3) of Theorem 3. Thus $\int_\Omega \tilde{v} \cdot \nabla \psi (\cdot, t) \, \|dx = 0$ for a.e. $t \in (0, T_1)$.

Now we prove (b6). Set $d\nu := d\mathcal{H}^{d-1} \lfloor_{\partial^* U_t} dt$. Since the space $C_c(\Omega)$ is dense in $L^2(\nu)$, for any $\eta \in C_c((0, T_1))$ we have
\[
0 = \int_0^{T_1} \int_{\partial^* U_t} \{\tilde{v} - \bar{h} + \lambda \tilde{v}\} \cdot \tilde{v} \eta \, d\mathcal{H}^{d-1} \, dt
= \int_0^{T_1} \left( - \int_{\partial^* U_t} \bar{h} \cdot \tilde{v} \, d\mathcal{H}^{d-1} + \lambda \mathcal{H}^{d-1}(\partial^* U_t) \right) \eta \, dt,
\]
where we used (b3) and (b5). Hence we obtain (b6).

\[\square\]

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6. Appendix

Proposition 18. (See Lemma 3.24 in [2]) Let $U \subset \mathbb{R}^d$ be an open set. Assume that $u \in BV(U)$ and $K \subset U$ is a compact set. Then
\[
\int_K |u \ast \eta_\delta - u| \, dx \leq \delta \|\nabla u\|(U)
\]
for all $\delta \in (0, \text{dist}(K, \partial U))$, where $\eta_\delta$ is the standard mollifier defined in Section 3.
As in Lemma 1, we can obtain the following estimate for the classical solution to the volume preserving mean curvature flow (see also [7,24,27,28,35] for the weak solutions).

**Proposition 19.** Let $\Omega = \mathbb{R}^d$ and $U_t \subset \Omega$ be a bounded open set with smooth boundary $M_t$ for any $t \in [0, T)$ and $0 < \mathcal{L}^d (U_0) < \mathcal{L}^d (\Omega)$. Assume that \{\{M_t\}_{t \in [0,T)}\} is the volume preserving mean curvature flow. Then there exists $C_\lambda > 0$ depending only on $d$, $\mathcal{L}^d (U_0)$, and $H^{d-1} (M_0)$ such that

$$\int_0^s |\lambda(t)|^2 \, dt \leq C_\lambda (1 + s), \quad s \in (0, T).$$

(146)

where $\lambda(t) = \frac{1}{H^{d-1} (M_t)} \int_{M_t} \vec{h} \cdot \nu \, d H^{d-1}$.

**Proof.** Let $\vec{\zeta} : \Omega \times [0, \infty) \to \mathbb{R}^d$ be a smooth periodic function. By (1), (2), the divergence theorem, and the property of the mean curvature, we have

$$\frac{d}{dt} H^{d-1} (M_t) = - \int_{M_t} \vec{h} \cdot \vec{v} \, d H^{d-1} = - \int_{M_t} |\vec{v}|^2 \, d H^{d-1} \leq 0$$

(147)

and

$$\int_{M_t} \vec{v} \cdot \vec{\zeta} \, d H^{d-1} = \int_{M_t} \vec{h} \cdot \vec{\zeta} \, d H^{d-1} - \lambda \int_{M_t} \vec{v} \cdot \vec{\zeta} \, d H^{d-1}$$

$$= - \int_{M_t} \text{div} M_t \vec{\zeta} \, d H^{d-1} + \lambda \int_{U_t} \text{div} \vec{\zeta} \, d x.$$  

(148)

By (147) and (148), we obtain

$$|\lambda| \left| \int_{U_t} \text{div} \vec{\zeta} \, d x \right| \leq \| \vec{\zeta} (\cdot, t) \|_{C^1} \left( H^{d-1} (M_t) + \int_{M_t} |\vec{v}| \, d H^{d-1} \right)$$

$$\leq \| \vec{\zeta} (\cdot, t) \|_{C^1} \left( H^{d-1} (M_0) + \int_{M_t} |\vec{v}| \, d H^{d-1} \right).$$

(149)

Let $\alpha, \delta \in (0, 1)$ and $u = u(x, t)$ be a periodic solution to

$$\begin{cases}
-\Delta u = \chi U_t \ast \eta_\delta - \frac{1}{\mathcal{L}^d (\Omega)} \int_{\Omega} (\chi U_t \ast \eta_\delta) & \text{in } \Omega, \\
\int_{\Omega} u \, d x = 0.
\end{cases}$$

Then the standard PDE arguments imply the existence and uniqueness of the solution $u$ and

$$\|u(\cdot, t)\|_{C^{2,\alpha} (\Omega)} \leq C_\delta, \quad t \in [0, T),$$
where \( C_\delta > 0 \) depends only on \( d \) and \( \delta \). Set \( \zeta(x, t) = \nabla u(x, t) \). We compute that

\[
- \int_{U_t} \text{div} \zeta \, dx = \int_{U_t} (-\Delta u) \, dx
\]

\[
= \int_{U_t} \left( \chi U_t * \eta \delta - \int_{\Omega} (\chi U_t * \eta \delta) \right) \, dx
\]

\[
= \int_{\Omega} \chi U_t (\chi U_t * \eta \delta) \, dx - \frac{\mathcal{L}^d(U_t)}{\mathcal{L}^d(\Omega)} \int_{\Omega} (\chi U_t * \eta \delta) \, dx
\]

\[
= \int_{\Omega} \chi U_t (\chi U_t * \eta \delta - \chi U_t) \, dx + \int_{\Omega} \chi U_t^2 \, dx
\]

\[
- \frac{\mathcal{L}^d(U_t)}{\mathcal{L}^d(\Omega)} \int_{\Omega} (\chi U_t * \eta \delta - \chi U_t) \, dx - \frac{\mathcal{L}^d(U_t)}{\mathcal{L}^d(\Omega)} \int_{\Omega} \chi U_t \, dx
\]

\[
\geq - C\delta \mathcal{H}^{d-1}(M_t) + \mathcal{L}^d(U_t) \left( 1 - \frac{\mathcal{L}^d(U_t)}{\mathcal{L}^d(\Omega)} \right)
\]

\[
\geq - C\delta \mathcal{H}^{d-1}(M_0) + \mathcal{L}^d(U_0) \left( 1 - \frac{\mathcal{L}^d(U_0)}{\mathcal{L}^d(\Omega)} \right)
\]

where we used Proposition 18, \( \| \nabla \chi U_t \|_{\mathcal{H}^{d-1}(\Omega)} = \mathcal{H}^{d-1}(M_t), (147) \), and the volume preserving property. We choose \( \delta > 0 \) such that

\[
- \int_{U_t} \text{div} \zeta \, dx \geq \frac{1}{2} \mathcal{L}^d(U_0) \left( 1 - \frac{\mathcal{L}^d(U_0)}{\mathcal{L}^d(\Omega)} \right) > 0.
\]

By this and (149),

\[
|\lambda| \leq \frac{C\delta}{\omega'} \left( \mathcal{H}^{d-1}(M_0) + \int_{M_t} |\tilde{v}| \, d\mathcal{H}^{d-1} \right),
\]

(150)

where \( \omega' = \frac{1}{2} \mathcal{L}^d(U_0) \left( 1 - \frac{\mathcal{L}^d(U_0)}{\mathcal{L}^d(\Omega)} \right) \). The equality (147) implies

\[
\int_{0}^{s} \int_{M_t} |\tilde{v}|^2 \, d\mathcal{H}^{d-1} \, dt \leq \mathcal{H}^{d-1}(M_0), \quad s \in [0, T).
\]

(151)

Therefore we obtain (146) by (150) and (151).

Next we show some properties of the backward heat kernel.

**Lemma 13.** (See [23]) Let \( D > 0 \) and \( \nu \) be a Radon measure on \( \mathbb{R}^d \) satisfying

\[
\sup_{R>0, x \in \mathbb{R}^d} \frac{\nu(B_R(x))}{\omega_{d-1} R^{d-1}} \leq D.
\]

(152)

Then the following hold:

1. For any \( a > 0 \) there exists \( \gamma_1 = \gamma_1(a) > 0 \) such that for any \( r > 0 \) and for any \( x, x_1 \in \mathbb{R}^d \) with \( |x - x_1| \leq \gamma_1 r \), we have the estimate

\[
\int_{\mathbb{R}^d} \rho_{x, x_1}^r(y) \, d\nu(y) \leq \int_{\mathbb{R}^d} \rho_{x}^r \, d\nu(y) + aD,
\]

(153)

where \( \rho_{x}^r \) is given by (64).
2. For any $r$, $R > 0$ and for any $x \in \mathbb{R}^d$, we have
\[
\int_{\mathbb{R}^d \setminus B_R(x)} \rho_x^r(y) \, d\nu(y) \leq 2^{d-1} e^{-3R^2/8r^2} D.
\] (154)

**Proof.** We only show (153) here (the estimate (154) can be shown more easily). For $\beta \in (0, 1)$, we have
\[
\int_{\mathbb{R}^d} \rho_x^r(y) \, d\nu(y) = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} e^{-|x-y|^2/2r} \, d\nu(y)
\]
\[
= \frac{1}{(2\pi)^{d-1}} \int_0^1 \nu(y \mid e^{-|x-y|^2/2r} > k) \, dk = \frac{1}{(2\pi)^{d-1}} \int_0^1 \nu(B_{r \sqrt{2 \log \frac{1}{k}}(x_1)) \, dk
\]
\[
\leq \omega^{-1} D \int_0^\beta \left( \log \frac{1}{k} \right)^{d-1} \, dk + \frac{1}{(2\pi)^{d-1}} \int_0^1 \nu(B_{r \sqrt{2 \log \frac{1}{k}}(x_1)) \, dk,
\]
where we used (152). By \( \int_0^1 \left( \log \frac{1}{k} \right)^{d-1} \, dk = \Gamma(d-1+1) = \pi \frac{d-1}{2} / \omega_{d-1}, \) we have
\[
\int_0^\beta \left( \log \frac{1}{k} \right)^{d-1} \, dk \to 0 \quad \text{as} \quad \beta \to 0.
\] (156)

We choose $\gamma_1 > 0$ depending only on $\beta$ such that
\[
\sqrt{2 \log \frac{1}{k} + \gamma_1} \leq \sqrt{2 \log \frac{1}{k - \beta}} \quad \text{for any} \quad k \in (\beta, 1].
\]

For any $x \in \mathbb{R}^d$ with $|x - x_1| \leq \gamma_1 r$, we have $B_{r \sqrt{2 \log \frac{1}{k_1}}(x_1)} \subset B_{r \sqrt{2 \log \frac{1}{k_1+\gamma_1}}(x)} \subset B_{r \sqrt{2 \log \frac{1}{k+\gamma_1}}(x)}$ for $k \in (\beta, 1]$. Therefore
\[
\frac{1}{(2\pi)^{d-1}} \int_\beta^1 \nu(B_{r \sqrt{2 \log \frac{1}{k_1}}(x_1)) \, dk \leq \frac{1}{(2\pi)^{d-1}} \int_\beta^1 \nu(B_{r \sqrt{2 \log \frac{1}{k+\gamma_1}}(x)) \, dk
\]
\[
\leq \frac{1}{(2\pi)^{d-1}} \int_0^1 \nu(B_{r \sqrt{2 \log \frac{1}{k}}}(x)) \, dk' = \int_{\mathbb{R}^d} \rho_x^r(y) \, d\nu(y).
\] (157)

Hence (155), (156), and (157) imply (153).

Let $u^\varepsilon = u^\varepsilon(x)$ be a smooth function and define
\[
e_\varepsilon(x) = \frac{\varepsilon |\nabla u^\varepsilon(x)|^2}{2} + \frac{W(u^\varepsilon(x))}{\varepsilon}, \quad \xi_\varepsilon(x) = \frac{\varepsilon |\nabla u^\varepsilon(x)|^2}{2} - \frac{W(u^\varepsilon(x))}{\varepsilon}.
\]

The following propositions are used in the proof of the integrality of $\mu_t$.

**Proposition 20.** (See [22,45,47]) For any $R \in (0, \infty)$, $E_0 \in (0, \infty)$, $s \in (0, 1)$, and $N \in \mathbb{N}$, there exists $Q \in (0, 1)$ with the following property: Assume that a set $Y \subset \mathbb{R}^d$ has no more than $N + 1$ elements and $Y \subset \{(0, \ldots, 0, x_d) \in \mathbb{R}^d \mid x_d \in \mathbb{R}\}$, diam $Y \leq Q R$, and there exists $x \in (0, R)$ such that $|y - z| > 3a$ holds for any $y, z \in Y$ with $y \neq z$. Moreover, we assume the following:

1. $u^\varepsilon \in C^2(\{y \in \mathbb{R}^d | \text{dist}(y, Y) < R\}).$
2. For any \( x \in Y \) and \( r \in [a, R] \),
\[
\int_{B_r(x)} |\xi_\varepsilon| + (1 - (v_d)^2)\varepsilon|\nabla u^\varepsilon| \leq \varrho d^{d-1}.
\]
Here \( v = (v_1, \ldots, v_d) = \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|} \).

3. For any \( x \in Y \),
\[
\int_R^d d\tau \int_{B_\tau(x)} (\xi_\varepsilon)_+ dy \leq \varrho.
\]

4. For any \( x \in Y \) and \( r \in [a, R] \),
\[
\int_{B_r(x)} \varepsilon|\nabla u^\varepsilon|^2 dy \leq E_0 d^{d-1}.
\]

Then, we have
\[
\sum_{x \in Y} \frac{1}{a^{d-1}} \int_{B_a(x)} e^\varepsilon \leq s + \frac{1+s}{R^{d-1}} \int_{[x|\text{dist}(x,Y) < R]} e^\varepsilon.
\]

Proposition 21. (See [22,45,47]) For any \( s, b, \beta \in (0, 1) \), and \( c \in (1, \infty) \), there exist \( \varrho, \varepsilon \in (0, 1) \) and \( L \in (1, \infty) \) with the following property: Assume that \( \varepsilon \in (0, \varepsilon) \), \( u^\varepsilon \in C^2(B_{4\varepsilon L}(0)) \) and
\[
\sup_{B_{4\varepsilon L}(0)} \varepsilon|\nabla u^\varepsilon| \leq c, \quad \sup_{x,y \in B_{4\varepsilon L}(0), x \neq y} \frac{1}{|x - y|^{1/2}} \left| \frac{|\nabla u^\varepsilon(x) - \nabla u^\varepsilon(y)|}{|x - y|^{1/2}} \right| \leq c, \quad |u^\varepsilon(0)| < 1 - b,
\]
\[
\int_{B_{4\varepsilon L}(0)} (|\xi_\varepsilon| + (1 - (v_d)^2)\varepsilon|\nabla u^\varepsilon|) dx \leq \varrho (4\varepsilon L)^{d-1},
\]
and
\[
\sup_{B_{4\varepsilon L}(0)} (\xi_\varepsilon)_+ \leq \varepsilon^{-\beta}.
\]

Then we have
\[
[-1 + b, 1 - b] \subset u^\varepsilon(J) \quad \text{and} \quad \inf_{x \in J} \partial_{x_d} u^\varepsilon(x) > 0 \quad \text{or} \quad \sup_{x \in J} \partial_{x_d} u^\varepsilon(x) < 0,
\]
where \( J = B_{3\varepsilon L}(0) \cap \{(0, \ldots, 0, x_d) \in \mathbb{R}^d | x_d \in \mathbb{R}\} \). In addition, we have
\[
\left| \frac{1}{\omega_{d-1}(L \varepsilon)^{d-1}} \int_{B_{\varepsilon L}(0)} e^\varepsilon \right| \leq s.
\]

Remark 13. Note that the assumptions for \((\xi_\varepsilon)_+\) in the propositions above hold for the solution to (5) with suitable initial data (see 33).

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