ON THE GRIFFITHS-YUKAWA COUPLING LENGTH OF SOME CALABI-YAU FAMILIES

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Abstract. We determine the Griffiths-Yukawa coupling length of the Calabi-Yau universal families coming from hyperplane arrangements.

1. Introduction

In the study of moduli spaces of Calabi-Yau (CY) manifolds, an effective method is to investigate the associated variation of Hodge structure. To every variation of Hodge structure $\mathcal{V}$, we can associate an interesting numerical invariant $\varsigma(\mathcal{V})$, called the Griffiths-Yukawa coupling length, which is introduced in [10]. The connection of Griffiths-Yukawa coupling length with Shararevichs conjecture for CY manifolds has been intensively studied (see e.g., [4, 12]). It has been shown that, for example, a CY family with maximal Griffiths-Yukawa coupling length is rigid.

On the other hand, among the various Calabi-Yau moduli spaces, the ones coming from hyperplane arrangements are particular interesting, due to their analogue of elliptic curves and their relations to Gross’ geometric realization problem (see e.g., [2, 3, 8]).

The main purpose of this note is the determination of the Griffiths-Yukawa coupling length for the CY families coming from hyperplane arrangements. More precisely, let $m, n, r$ be positive integers satisfying the condition:

\[(1.0.1) \quad r|m, \quad n = m - \frac{m}{r} - 1.\]

Let $\mathcal{M}_{AR}$ be the coarse moduli space of ordered $m$ hyperplane arrangements in $\mathbb{P}^n$ in general position and $\mathcal{X}_{AR} \xrightarrow{\tilde{f}} \mathcal{M}_{AR}$ be the family of CY $n$-folds which is obtained by a resolution of $r$-fold covers of $\mathbb{P}^n$ branched along $m$ hyperplanes in general position. This family gives a weight $n$ complex variation of Hodge structure ($\mathbb{C}$-VHS) $\tilde{\mathcal{V}}_{pr} := (R^n\tilde{f}_*\mathcal{C})_{pr}$ over $\mathcal{M}_{AR}$. Our main result is:

**Theorem 1.1.** The Griffiths-Yukawa coupling length of $\tilde{\mathcal{V}}_{pr}$ is $\varsigma(\tilde{\mathcal{V}}_{pr}) = \frac{m}{r} - 1$. 
Remark 1.2. By the theorem above, among the Calabi-Yau families coming from hyperplane arrangements, the ones with minimal Griffiths-Yukawa coupling length (i.e. $\zeta(\tilde{V}_{pr}) = 1$) are those with $(m, n, r) = (n + 3, n, \frac{n+3}{2})$, and these are exactly the families considered in [7]. In particular, we get a positive answer to one of the questions listed at the end of [7]. Besides, the ones with maximal Griffiths-Yukawa coupling length (i.e. $\zeta(\tilde{V}_{pr}) = n$) are those with $(m, n, r) = (2n + 2, n, 2)$, and these are exactly the Calabi-Yau families coming from double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position.

In section 2 we recall the notion of $\mathbb{C}$-VHS, which is more general than the usual notion of $\mathbb{Q}$-VHS or $\mathbb{R}$-VHS. Then we recall the definition of the Griffiths-Yukawa coupling length in the context of $\mathbb{C}$-VHS. In section 3 we introduce the various $\mathbb{C}$-VHS coming from hyperplane arrangements. The key observation leading to the main result is a special locus $\mathcal{M}_C$ in $\mathcal{M}_{AR}$ which parameterizes distinct points on $\mathbb{P}^1$. We use the weight one $\mathbb{C}$-VHS associated to $\mathcal{M}_C$ to compute some of the Hodge numbers of $\tilde{V}_{pr}$, hence obtain an upper bound of $\zeta(\tilde{V}_{pr})$. In section 4 we use the tool of Jacobian rings to compute the Higgs map associated to the weight one $\mathbb{C}$-VHS on $\mathcal{M}_C$. In this way, we obtain the lower bound of $\zeta(\tilde{V}_{pr})$ and finish the proof of the main result.

2. Definitions

Throughout this section, we let $M$ be a complex manifold and $\mathcal{O}_M$ be the sheaf of holomorphic functions on $M$. We identify holomorphic vector bundles of finite rank on $M$ and sheaves of locally free $\mathcal{O}_M$-modules of finite rank on $M$ by the well known way. We first review the definition of complex variation of Hodge structure, which seem to vary slightly according to the source.

Definition 2.1. (c.f. [1]) A complex variation of Hodge structure ($\mathbb{C}$–VHS) of weight $n$ on $M$ is a local system $\mathcal{V}$ of finite dimensional $\mathbb{C}$-vector spaces on $M$ together with a filtration of holomorphic vector bundles:

$$F^n\mathcal{V} \subset F^{n-1}\mathcal{V} \subset \cdots \subset F^1\mathcal{V} \subset F^0\mathcal{V} = \mathcal{V} := \mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_M$$

satisfying the following Griffiths transversality condition:

$$DF^p\mathcal{V} \subset F^{p-1}\mathcal{V} \otimes_{\mathcal{O}_M} \Omega_M, \quad \forall \ 1 \leq p \leq n,$$
where $D : \mathcal{V} \to \mathcal{V} \otimes \mathcal{O}_M \Omega_M$ is the Gauss-Manin connection induced by the local system $\mathcal{V}$. The filtration $F^p \mathcal{V}$ is called the Hodge filtration and the integers $h^{p,n-p}(\mathcal{V}) := \text{rank}_{F^p \mathcal{V}} \mathcal{V} (0 \leq p \leq n)$ are called the Hodge numbers. Here by convention, $F^{n+1} \mathcal{V} = 0$.

**Remark 2.2.** If $\mathcal{V}$ is a weight $n$ rational variation of Hodge structure ($\mathbb{Q}$-VHS), then it is easy to see $\mathcal{V} \otimes_{\mathbb{Q}} \mathbb{C}$ is admits a structure of $\mathbb{C}$-VHS of weight $n$.

Obviously, a morphism $\phi : \mathcal{W} \to \mathcal{V}$ between two $\mathbb{C}$-VHS of weight $n$ on $M$ is a morphism of local systems preserving the Hodge filtration.

If $(\mathcal{V}, F^n \mathcal{V} \subset F^{n-1} \mathcal{V} \subset \cdots \subset F^1 \mathcal{V} \subset F^0 \mathcal{V} = \mathcal{V} := \mathcal{V} \otimes \mathbb{C} \mathcal{O}_M)$ is a $\mathbb{C}$-VHS of weight $n$ on $M$, then by the Griffiths transversality, $\forall 0 \leq p \leq n$, the Gauss-Manin connection $D$ induces a $\mathcal{O}_M$-linear homomorphism:

$$\theta^{p,n-p} : E^{p,n-p} \to E^{p-1,n-p+1} \otimes \mathcal{O}_M \Omega_M$$

where $E^{p,n-p} := \frac{F^p \mathcal{V}}{F^{p+1} \mathcal{V}}$. For $x \in M$ and $v \in T_x M$, we denote the contraction of $\theta^{p,n-p}$ with $v$ by

$$\theta_x^{p,n-p}(v) : E^{p,n-p}_x \to E^{p-1,n-p+1}_x,$$

which is a $\mathbb{C}$-linear map between the fibers $E^{p,n-p}_x$ and $E^{p-1,n-p+1}_x$. We call $(E := \bigoplus_{p=0}^n E^{p,n-p}, \theta := \bigoplus_{p=0}^n \theta^{p,n-p})$ the Higgs bundle associated to the $\mathbb{C}$-VHS $\mathcal{V}$. A morphism between two $\mathbb{C}$-VHS of weight $n$ induces a morphism between the associated Higgs bundles in an obviously way.

Let $\mathcal{V}$ be a $\mathbb{C}$-VHS of weight $n$ over $M$ and $(E, \theta)$ the associated Higgs bundle. For every $q$ with $1 \leq q \leq n$, the $q$th iterated Higgs field

$$E^{n,0} \xrightarrow{\theta^{n,0}} E^{n-1,1} \otimes \Omega_M \xrightarrow{\theta^{n-1,1}} \cdots \xrightarrow{\theta^{n-q+1,q-1}} E^{n-q,q} \otimes S^q \Omega_M$$

defines a morphism

$$\theta^q : \text{Sym}^q(TM) \to \text{Hom}(E^{n,0}, E^{n-q,q})$$

where $TM$ is the holomorphic tangent bundle of $M$. The Griffiths-Yukawa coupling length $\varsigma(\mathcal{V})$ of $\mathcal{V}$ is defined by

$$\varsigma(\mathcal{V}) = \min\{q \geq 1 \mid \theta^q = 0 \} - 1.$$

**Lemma 2.3.** Let $(\mathcal{V}, F^\cdot \mathcal{V})$ and $(\tilde{\mathcal{V}}, F^{\cdot} \tilde{\mathcal{V}})$ be $\mathbb{C}$-VHS of weight $n$ on $M$ with Hodge numbers $h^{n,0}(\mathcal{V}) = h^{n,0}(\tilde{\mathcal{V}}) = 1$. Suppose $\phi : \mathcal{V} \to \tilde{\mathcal{V}}$ is a morphism of $\mathbb{C}$-VHS satisfying: $\forall 0 \leq p \leq n, \forall x \in M$, the induced linear map between the fibers of the associated Higgs bundles $\phi_x^{p,n-p} : E^{p,n-p}_x \to \tilde{E}^{p,n-p}_x$ is injective. Then $\varsigma(\mathcal{V}) = \varsigma(\tilde{\mathcal{V}})$. 
Similarly, if \( \varphi \) where
\[
\begin{align*}
\text{Proof.} \quad & \text{The assumptions give us the following commutative diagram, } \forall \ x \in M, \forall \ 1 \leq q \leq n : \\
E_x^{n,0} & \xrightarrow{\phi_x^{n,0}} E_x^{n-1,1} \otimes \Omega_{M,x} \xrightarrow{\phi_x^{n-1,1}} \cdots \xrightarrow{\phi_x^{n-q+1,q-1}} E_x^{n-q,q} \otimes S^q\Omega_{M,x} \\
E_x^{n,0} & \xrightarrow{\tilde{\phi}_x^{n,0}} \tilde{E}_x^{n-1,1} \otimes \bar{\Omega}_{M,x} \xrightarrow{\tilde{\phi}_x^{n-1,1}} \cdots \xrightarrow{\tilde{\phi}_x^{n-q+1,q-1}} \tilde{E}_x^{n-q,q} \otimes S^q\Omega_{M,x}
\end{align*}
\]
where \( \phi_x^{n,0} \) is an isomorphism, and \( \phi_x^{n-q,q} \) is injective, \( \forall \ 1 \leq q \leq n \). It is easy to see that \( \zeta(V) = \zeta(\bar{V}) \) follows from this commutative diagram and the definitions of \( \zeta(V) \) and \( \zeta(\bar{V}) \).
\[ \square \]

If \( V_1, V_2 \) are linear subspaces of a \( \mathbb{C} \)-linear space \( V \), for \( 0 \leq p \leq n \), we let \( \wedge^p V_1 \wedge^{n-p} V_2 \) denote the linear subspace of \( \wedge^n V \) spanned by elements in the set
\[
\{ e_1 \wedge \cdots \wedge e_p \wedge e_{p+1} \wedge \cdots \wedge e_n | e_1, \ldots, e_p \in V_1, e_{p+1}, \ldots, e_n \in V_2 \}.
\]
Similarly, if \( W_i \) (\( 1 \leq i \leq n \)) are linear subspaces of a \( \mathbb{C} \)-linear space \( W \), we let \( W_1 \wedge \cdots \wedge W_n \) denote the linear subspace of \( \wedge^n V \) spanned by elements in the set
\[
\{ e_1 \wedge \cdots \wedge e_n | e_i \in W_i, 1 \leq i \leq n \}.
\]

Now suppose \((V, F)\) is a \( \mathbb{C} \)-VHS of weight \( n \) on \( M \), for any \( m \geq 1 \), we endow the local system \( \wedge^m V \) a Hodge filtration \( F^\cdot \wedge^m V \) such that \( \wedge^m V \) becomes a \( \mathbb{C} \)-VHS of weight \( mn \) with this Hodge filtration. The Hodge filtration is defined as follows:
\[
\forall \ 0 \leq p \leq mn, \forall \ x \in M, \text{ the fiber of the holomorphic bundle } F^p \wedge^m V \text{ at } x \text{ is }
\sum_{i_1 + \cdots + i_n = p} F^{i_1}V_x \wedge F^{i_2}V_x \wedge \cdots \wedge F^{i_n}V_x \subset \wedge^n V_x.
\]

With this definition of Hodge filtration, the Griffiths transversality is easy to verify.

We give the following lemma for the purpose of later use:

**Lemma 2.4.** Let \( \mathcal{W} \) be a weight one \( \mathbb{C} \)-VHS on \( M \) with associated Higgs bundle \( (F = F^{1,0} \oplus F^{0,1}, \eta : F^{1,0} \to F^{0,1} \otimes \Omega_M) \). The Hodge numbers are \( h^{1,0}(\mathcal{W}) = n, h^{0,1}(\mathcal{W}) = k - 1 \leq n \). Suppose there exist \( x \in M \) and \( v \in T_xM \) satisfying that the Higgs map \( \eta_x(v) : F_x^{1,0} \to F_x^{0,1} \) is surjective. Then the Griffiths-Yukawa coupling length of \( \wedge^n \mathcal{W} \) is \( \zeta(\wedge^n \mathcal{W}) = k - 1 \).

**Proof.** Let \( \mathcal{V} = \wedge^n \mathcal{W} \) be the \( \mathbb{C} \)-VHS of weight \( n \) on \( M \). Denote the Higgs bundle associated to \( \mathcal{V} \) by \( (E = \oplus E^{p,q}, \theta = \oplus \theta^{p,q}) \). A direct computation shows that \( h^{n-q,q}(\mathcal{V}) = 0 \),
∀ q ≥ k. From this we get \( \zeta(\mathcal{V}) \leq k - 1 \). In order to prove \( \zeta(\mathcal{V}) \geq k - 1 \), it suffices to show that the iterated Higgs map

\[
\theta^{k-1}_x(v^{k-1}) : E^{n,0}_x \to E^{n-k+1,k-1}_x
\]

is nonzero.

By definitions, for any 0 ≤ q ≤ n, we can identify \( E^{n-q,q}_x \) with \( \bigwedge^n F^1_{x,0} \otimes \bigwedge^q F^0_{x,1} \) \( \mathbb{C} \). With these identifications, we have the following commutative diagram:

\[
\begin{array}{ccc}
E^{n,0}_x & \xrightarrow{\theta^{k-1}_x(v^{k-1})} & E^{n-k+1,k-1}_x \\
\downarrow & & \downarrow \\
\bigwedge^n F^{1,0}_x & \xrightarrow{\bigwedge^n \eta_x(v)} & \bigwedge^{n-k+1} F^{1,0}_x \otimes k-1 F^{0,1}_x
\end{array}
\]

where for \( e_1 \wedge \cdots e_n \in \bigwedge^n F^{1,0}_x \),

\[
\bigwedge^n \eta_x(v)(e_1 \wedge \cdots e_n) := (k-1)! \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq n} e_1 \wedge \cdots \wedge \eta_x(v)e_{i_1} \wedge \cdots \wedge \eta_x(v)e_{i_{k-1}} \wedge \cdots \wedge e_n.
\]

From this diagram and the explicit expression of \( \bigwedge^n \eta_x(v) \), it is easy to deduce the non-vanishing of the map \( \theta^{k-1}_x(v^{k-1}) \) from the surjectivity of the map \( \eta_x(v) : F^{1,0}_x \to F^{0,1}_x \).

\[\square\]

3. \( \mathbb{C} \)-VHS from hyperplane arrangements

Now the meaning of letters in the tuple \((m, n, r, \zeta)\) will be fixed to the end of the paper: \( m, n, r \) are positive integers satisfying the condition (1.0.1), and \( \zeta \) is a fixed primitive \( r \)-th root of unity. If the cyclic group \( \mathbb{Z}/r\mathbb{Z} = < \sigma > \) acts on a \( \mathbb{C} \)-VHS \( \mathbb{V} \), we denote \( \mathbb{V}_{(i)} \) as the \( i \)-th eigen-sub \( \mathbb{C} \)-VHS of \( \mathbb{V} \), i.e. the sections of \( \mathbb{V}_{(i)} \) consist sections \( s \) of \( \mathbb{V} \) satisfying \( \sigma \cdot s = \zeta^is \).

An ordered arrangement \( \mathfrak{A} = (H_1, \cdots, H_m) \) of \( m \) hyperplanes in \( \mathbb{P}^n \) is in general position if no \( n+1 \) of the hyperplanes intersect in a point. Let \( \mathcal{M}_{AR} \) denote the coarse moduli space of ordered arrangements of \( m \) hyperplanes in \( \mathbb{P}^n \) in general position. As shown in [8], \( \mathcal{M}_{AR} \) can be realized as an open subvariety of the affine space \( \mathbb{C}^m(\frac{m}{r}-1) \) and it admits a natural family \( f : \mathcal{X}_{AR} \to \mathcal{M}_{AR} \), where each fiber \( f^{-1}(\mathfrak{A}) \) is the \( r \)-fold cyclic cover of \( \mathbb{P}^n \) branched along the hyperplane arrangement \( \mathfrak{A} \). It is easy to see the crepant resolution process in [7] gives a simultaneous crepant resolution \( \pi : \tilde{\mathcal{X}}_{AR} \to \mathcal{X}_{AR} \) for the family \( f \). We denote this smooth projective family of CY manifolds by \( \tilde{f} : \tilde{\mathcal{X}}_{AR} \to \mathcal{M}_{AR} \).
Let $\mathcal{M}_C$ be the moduli space of ordered distinct $m$ points on $\mathbb{P}^1$ and $g : C \rightarrow \mathcal{M}_C$ be the universal family of $r$-fold cyclic covers of $\mathbb{P}^1$ branched at $m$ distinct points. There is a natural embedding $\mathcal{M}_C \hookrightarrow \mathcal{M}_{AR}$ (for details, see [8], section 2.3).

We consider the various $C$-VHS attached to the three families $f, \tilde{f}, g$:

$$\mathcal{V} := R^n f_* \mathcal{C}, \quad \tilde{\mathcal{V}} := R^n \tilde{f}_* \mathcal{C}, \quad \tilde{\mathcal{V}}_{pr} := (R^n \tilde{f}_* \mathcal{C})_{pr}, \quad \mathcal{W} := R^1 g_* \mathcal{C},$$

where $(R^n \tilde{f}_* \mathcal{C})_{pr}$ means the local system on $\mathcal{M}_{AR}$ whose fiber over $A \in \mathcal{M}_{AR}$ is the primitive $n$-th cohomology of $\tilde{f}^{-1}(A)$. Note that $\mathcal{V}$ is indeed a $C$-VHS, although the family $f$ is not smooth (see [8], section 6). Note also the weights of $\mathcal{V}, \tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}_{pr}$ are $n$, while the weight of $\mathcal{W}$ is one.

Since $\mathbb{Z}/r\mathbb{Z}$ acts naturally on the families $f : X_{AR} \rightarrow \mathcal{M}_{AR}$ and $g : C \rightarrow \mathcal{M}_C$, we have a decomposition of the $C$-VHS into eigen-sub $C$-VHS:

$$\mathcal{V} = \oplus_{i=0}^{r-1} \mathcal{V}(i), \quad \mathcal{W} = \oplus_{i=0}^{r-1} \mathcal{W}(i).$$

**Proposition 3.1.** Notations as above, then

1. as $C$-VHS of weight $n$ on $\mathcal{M}_C$, we have $\mathcal{V}(1)|_{\mathcal{M}_C} \simeq \wedge^n \mathcal{W}(1)$;
2. the Hodge numbers of $\mathcal{W}(1)$ are: $h^{1,0}(\mathcal{W}(1)) = n, \quad h^{0,1}(\mathcal{W}(1)) = \frac{m}{r} - 1$;
3. the Hodge numbers of $\mathcal{V}(1)$ are:

   $$h^{n-q,q} = \begin{cases} 
   \binom{n}{q} \left(\frac{m-1}{q}\right), & 0 \leq q \leq \frac{m}{r} - 1; \\
   0, & \frac{m}{r} \leq q \leq n. 
   \end{cases}$$

4. $\varsigma(\tilde{\mathcal{V}}_{pr}) = \varsigma(\mathcal{V}(1))$;
5. $\varsigma(\tilde{\mathcal{V}}_{pr}) \leq \frac{m}{r} - 1$.

**Proof.** For (1), one can see [8], Proposition 6.3.

(2) follows from a standard computation of the Hodge numbers of cyclic covers of $\mathbb{P}^1$. One can see for example [5], (2.7).

(3) follows from (1) and (2).

(4) follows from Lemma 2.3. Indeed, one can verify directly that the embeddings $\mathcal{V}(1) \hookrightarrow \mathcal{V}$ and $\tilde{\mathcal{V}}_{pr} \hookrightarrow \tilde{\mathcal{V}}$ satisfy the hypothesis of Lemma 2.3 so we get $\varsigma(\mathcal{V}(1)) = \varsigma(\mathcal{V})$, $\varsigma(\tilde{\mathcal{V}}_{pr}) = \varsigma(\tilde{\mathcal{V}})$. One can use Theorem 5.41 in [6] to show the natural morphism $\mathcal{V} \rightarrow \tilde{\mathcal{V}}$ satisfies the hypothesis of Lemma 2.3 which gives $\varsigma(\mathcal{V}) = \varsigma(\tilde{\mathcal{V}})$. Combining these equalities, we get $\varsigma(\tilde{\mathcal{V}}_{pr}) = \varsigma(\mathcal{V}(1))$.

(5) follows from (3) and (4).
4. Computations in Jacobian ring

In this section, we keep the notations in section 3. We want to analyse the Higgs maps associated to the universal family $g : C \rightarrow \mathcal{M}_C$ in some detail. Recall $\mathcal{M}_C$ is the coarse moduli space of ordered pairwise distinct $m$ points in $\mathbb{P}^1$. It is well known that $\mathcal{M}_C$ can be identified with a Zariski open subset $U$ of $\mathbb{C}^{m-3}$ via the map:

$$U \sim \mathcal{M}_C \quad (a_1, \ldots, a_{m-3}) \mapsto ([1 : 0], [1 ; 1], [0 : 1], [1 : a_1], \ldots, [1 : a_{m-3}])$$

where $[z_0 : z_1]$ are the homogeneous coordinates on $\mathbb{P}^1$. We fix this identification and view $\mathcal{M}_C$ as a Zariski open subset of $\mathbb{C}^{m-3}$.

For $a = (a_1, \ldots, a_{m-3}) \in \mathcal{M}_C \subset \mathbb{C}^{m-3}$, let $C = g^{-1}(a)$ be the fiber over $a$ of the universal family $C$, then $C$ is the $r$-fold cyclic cover of $\mathbb{P}^1$ branched at the $m$ points $[1 : 0], [1 ; 1], [0 : 1], [1 : a_1], \ldots, [1 : a_{m-3}]$. We let $Y$ denote the smooth curve which is the complete intersection of the $m-2$ hypersurfaces in $\mathbb{P}^{m-1}$ defined by the equations:

$$y_2^r - (y_0^r + y_1^r) = 0;$$

$$y_{2+i}^r - (y_0^r + a_i y_1^r) = 0, \quad 1 \leq i \leq m - 3.$$ 

Here $[y_0 : \cdots : y_{m-1}]$ are the homogeneous coordinates on $\mathbb{P}^{m-1}$. $Y$ is called the Kummer cover of $C$, and when $a$ varies in $\mathcal{M}_C$, the Kummer covers of $g^{-1}(a)$ form a family of curves over $\mathcal{M}_C$.

Let $N = \oplus_{j=0}^{m-1} \mathbb{Z}/r\mathbb{Z}$. Consider the following group

$$N_1 := Ker(N \rightarrow \mathbb{Z}/r\mathbb{Z})$$

$$(a_j) \mapsto \sum_{j=0}^{m-1} a_j$$

We define a natural action of $N$ on $Y$. $\forall \alpha = (a_0, \ldots, a_{m-1}) \in N$, the action of $\alpha$ on $Y$ is induced by

$$\alpha \cdot y_j := \zeta^{a_j} y_j, \quad \forall 0 \leq j \leq m - 1.$$ 

Recall $\zeta$ is a fixed $r$-th primitive root of unity.

**Proposition 4.1.** The following statements hold:

1. The map $\pi_1 : Y \rightarrow \mathbb{P}^1$, $[y_0 : \cdots : y_{m-1}] \mapsto [y_0^r : y_1^r]$ defines a cover of degree $r^{m-1}$. 

(2) \( C \simeq Y/N_1 \).

(3) There exists a natural isomorphism of rational Hodge structures \( H^1(C, \mathbb{Q}) \simeq H^1(Y, \mathbb{Q})^{N_1} \), where \( H^1(Y, \mathbb{Q})^{N_1} \) denotes the subspace of invariants under \( N_1 \).

Proof. (1) can be verified directly.

(2): By (1), one can verify \( Y/N_1 \) is a \( r \)-fold cyclic cover of \( \mathbb{P}^1 \) branched at the \( m \) points \([1 : 0], [1; 1], [0 : 1], [1 : a_1], \cdots, [1 : a_m-3]\). Then \( C \simeq Y/N_1 \) follows from the uniqueness of this kind of covers.

(3) follows from (2) directly. \( \square \)

Recall \( \mathcal{W} = R^1g_*\mathcal{C} \) is the weight one \( \mathbb{C} \)-VHS coming from the universal family \( g : \mathcal{C} \to \mathfrak{M}_C \), and under the natural \( \mathbb{Z}/r\mathbb{Z} \)-action, \( \mathcal{W}(1) \) is the first eigen-sub \( \mathbb{C} \)-VHS of \( \mathcal{W} \). Let \((F = F^{1,0} \oplus F^{0,1}, \eta : F^{1,0} \to F^{0,1} \otimes \Omega^1_{\mathcal{M}_C}) \) be the Higgs bundle associated to \( \mathcal{W}(1) \). At the point \( a = (a_1, \cdots, a_m-3) \in \mathfrak{M}_C \subset \mathbb{C}^{m-3} \), we can identify the fiber \( F^1_a \) with \( H^1(C, \mathbb{C})(1) \), where \( H^1(C, \mathbb{C})(1) := \{ \alpha \in H^1(C, \mathbb{C}) | \sigma^* \alpha = \zeta \alpha \} \) is the first eigen subspace of \( H^1(C, \mathbb{C}) \) under the natural \( \mathbb{Z}/r\mathbb{Z} = \langle \sigma \rangle \)-action. Similarly, we can identify \( F^0_a \) with \( H^0(C, \mathbb{C})(1) \). Under these identifications, it is a standard fact that we have the following commutative diagram (see e.g., [11], Theorem 10.21):

\[
\begin{array}{ccc}
F^1_a \otimes T_a \mathfrak{M}_C & \xrightarrow{\eta_a} & F^0_a \\
\downarrow \iota & & \downarrow \iota \\
H^1(C, \mathbb{C})(1) \otimes T_a \mathfrak{M}_C & \xrightarrow{\psi_a} & H^0(C, \mathbb{C})(1)
\end{array}
\]

(4.1.1)

where \( \psi_a \) means the composition of the Kodaira-Spencer map \( T_a \mathfrak{M}_C \to H^1(C, TC) \) and the cup product \( H^1(C, \mathbb{C})(1) \otimes H^1(C, TC) \to H^0(C, \mathbb{C})(1) \).

Since the isomorphism in Proposition [11] (3) is equivariant with respect to the \( \mathbb{Z}/r\mathbb{Z} \)-action (note \( \mathbb{Z}/r\mathbb{Z} = N/N_1 \) acts naturally on \( H^1(Y, \mathbb{Q})^{N_1} \)), we can identify the corresponding eigen subspaces: \( H^1(C, \mathbb{C})(1) = H^1(C, \mathbb{C})^{N_1}(1), H^0(C, \mathbb{C})(1) = H^0(C, \mathbb{C})^{N_1}(1) \).

Under these identifications, we have the commutative diagram:

\[
\begin{array}{ccc}
H^1(C, \mathbb{C})(1) \otimes T_a \mathfrak{M}_C & \xrightarrow{\tilde{\psi}_a} & H^0(C, \mathbb{C})(1) \\
\downarrow \iota & & \downarrow \iota \\
H^1(Y, \mathbb{C})^{N_1}(1) \otimes T_a \mathfrak{M}_C & \xrightarrow{\tilde{\psi}_a} & H^0(Y, \mathbb{C})^{N_1}(1)
\end{array}
\]

(4.1.2)

where \( \tilde{\psi}_a \) is defined similarly as \( \psi_a \).

In order to represent \( \tilde{\psi}_a \) more explicitly, we use the tool of Jacobian ring. It is constructed as follows. In the polynomial ring of \( 2m-2 \) variables \( \mathbb{C}[\mu_0, \cdots, \mu_{m-3}, y_0, \cdots, y_{m-1}] \),
consider the polynomial
\[ F = \mu_0 F_0 + \cdots + \mu_{m-3} F_{m-3} \]
where
\[ F_0 := y_0^2 - (y_0^r + y_1^r), \]
\[ F_i := y_{i+2}^r - (y_0^r + \alpha_i y_1^r), \quad 1 \leq i \leq m-3. \]

Let \( J = \langle \frac{\partial F}{\partial \mu_1}, \frac{\partial F}{\partial y_j} \mid 0 \leq i \leq m-3, 0 \leq j \leq m-1 \rangle \) be the ideal of \( \mathbb{C}[\mu_0, \cdots, \mu_{m-3}, y_0, \cdots, y_{m-1}] \)
generated by the partial derivatives of \( F \). Define the Jacobian ring to be
\[ R := \mathbb{C}[\mu_0, \cdots, \mu_{m-3}, y_0, \cdots, y_{m-1}]/J \]

There is a natural bigrading on the polynomial ring \( \mathbb{C}[\mu_0, \cdots, \mu_{m-3}, y_0, \cdots, y_{m-1}] \), that is: the \((p, q)\)-part \( \mathbb{C}[\mu_0, \cdots, \mu_{m-3}, y_0, \cdots, y_{m-1}]_{(p, q)} \) is linearly spanned by the monomials \( \Pi_{i=0}^{m-3} \mu_i^{\alpha_i} \Pi_{j=0}^{m-1} y_j^{\beta_j} \) with \( \sum_{i=0}^{m-3} \alpha_i = p, \sum_{j=0}^{m-1} \beta_j = q \). Since the ideal \( J \) is a homogeneous ideal, there is a naturally induced bigrading on \( R \), written as \( R = \bigoplus_{p, q \geq 0} R_{(p, q)}. \)

The group \( N = \bigoplus_{j=0}^{m-1} \mathbb{Z}/r\mathbb{Z} \) acts on \( R \) through \( y_0, \cdots, y_{m-1} \). Explicitly, recall \( \zeta \) is a fixed primitive \( r \)-th root of unit, \( \forall \alpha = (\alpha_j) \in N \), we define the action of \( \alpha \) on \( R \) by
\[ \alpha \cdot y_j := \zeta^{\alpha_j} y_j, \quad \forall 0 \leq j \leq m - 1. \]
\[ \alpha \cdot \mu_i := \mu_i, \quad \forall 0 \leq i \leq m - 3. \]

It is obviously that the action of \( N \) on \( R \) preserves the bigrading. Let \( R^N_{(p, q)} \) be the \( N \)-invariant part of \( R_{(p, q)} \), then we have the decomposition of the \( N \)-invariant subring: \( R^N = \bigoplus_{p, q \geq 0} R^N_{(p, q)} \). Recall \( N_1 = Ker((\bigoplus_{j=0}^{m-1} \mathbb{Z}/r\mathbb{Z} \xrightarrow{r} \mathbb{Z}/r\mathbb{Z}) \) is the kernel subgroup of \( N = \bigoplus_{j=0}^{m-1} \mathbb{Z}/r\mathbb{Z} \) under the summation homomorphism.

**Proposition 4.2.** The following statements hold:

1. There are isomorphisms
   \[ H^{1,0}(Y, \mathbb{C})_{(1)}^N \cong R^N_{(0, mr-m-2r)}, \quad H^{0,1}(Y, \mathbb{C})_{(1)}^N \cong R^N_{(1, mr-m-r)}. \]

2. Let \((x_1, \cdots, x_{m-3})\) be the coordinates on \( \mathcal{M}_C \subset \mathbb{C}^{m-3} \), define a map
   \[ T_a \mathcal{M}_C \xrightarrow{\phi_i} R_{(1, r)} \]
   \[ \frac{\partial}{\partial x_i} \mapsto -\mu_i y_i^r \]
   Under this map and the isomorphisms in (1), we have a commutative diagram.
\[ H^1,0(Y, \mathcal{C})^N_1 \otimes T_a \mathcal{M}_C \xrightarrow{\tilde{\psi}_a} H^0,1(Y, \mathcal{C})^N_1 \]

\[ \downarrow id \otimes \phi \quad \downarrow \cong \]

\[ R^N_{(0, mr - m - 2r)} \otimes R_{(1, r)} \rightarrow R^N_{(1, mr - m - r)} \]

where the lower horizontal map is the natural multiplication in \( R \).

**Proof.** (1) follows from [9], Corollary 2.5 and its proof. \( \square \)

(2) follows from (1) and [9], Proposition 2.6.

For \( \beta = \sum_{i=1}^{m-3} \lambda_i \mu_i y_i^r \in R_{(1, r)} \), let \( \psi_\beta \) be the following multiplication homomorphism

\[ R^N_{(0, mr - m - 2r)} \xrightarrow{\psi_\beta} R^N_{(1, mr - m - r)} \]  
\[ \alpha \mapsto \alpha \cdot \beta \]

Now our key computations are included in the following

**Proposition 4.3.** For a generic \( \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^{m-3} \), the homomorphism \( \psi_\beta \) is surjective.

**Proof.** We first analyse the relations in \( R \), in order to obtain bases of the \( \mathbb{C} \)-linear spaces \( R^N_{(0, mr - m - 2r)} \) and \( R^N_{(1, mr - m - r)} \).

By the definition, the following relations hold in \( R \):

\[ \frac{\partial F}{\partial \mu_0} = y_2^r - (y_0^r + y_1^r) = 0; \]

\[ \frac{\partial F}{\partial \mu_i} = y_{i+2}^r - (y_0^r + a_i y_1^r) = 0, \quad 1 \leq i \leq m - 3; \]

\[ -\frac{\partial F}{r \partial y_0} = y_0^{r-1}(\mu_0 + \mu_1 + \cdots + \mu_{m-3}) = 0; \]

\[ -\frac{\partial F}{r \partial y_1} = y_1^{r-1}(\mu_0 + a_1 \mu_1 + \cdots + a_{m-3} \mu_{m-3}) = 0; \]

\[ -\frac{\partial F}{r \partial y_{i+2}} = \mu_i y_{i+2}^{r-1} = 0, \quad 0 \leq i \leq m - 3; \]

From these relations, it is easy to see the \( \mathbb{C} \)-linear space \( R^N_{(0, mr - m - 2r)} \) is linearly spanned by elements in the set

\[ \{ y_0^b y_1^c | b, c \in \mathbb{Z}_{\geq 0}, \quad b + c = m - \frac{m}{r} - 2 \}. \]

By Proposition 3.1 (2), Proposition 4.1 (2) and Proposition 4.2 (1), we can deduce

\[ \dim R^N_{(0, mr - m - 2r)} = \dim H^{1,0}(Y, \mathcal{C})^N_1 = \dim H^{1,0}(\mathcal{C}, \mathcal{C})_1 = h^{1,0}(\mathcal{M}_C) = n = m - \frac{m}{r} - 1. \]
This implies that \( \{ y_0^b y_1^c | b, c \in \mathbb{Z}_{\geq 0}, b + c = m - \frac{m}{r} - 2 \} \) is a \( \mathbb{C} \)-basis of \( R^N_{(0,mr-m-2)} \).

From the relations (4.3.1), we get the following relations in \( R \):

\[
\begin{align*}
\sum_{i=0}^{m-3} \mu_i y_0^r &= 0; \\
\mu_0 y_1^r + \sum_{i=1}^{m-3} a_i \mu_i y_1^r &= 0; \\
\mu_0 y_0^r + \mu_0 y_1^r &= 0; \\
\mu_i y_0^r + a_i \mu_i y_1^r &= 0, \quad 1 \leq i \leq m - 3.
\end{align*}
\]

(4.3.2)

The relations above imply directly that the \( \mathbb{C} \)-linear space \( R^N_{(1,mr-m-r)} \) is linearly spanned by elements in the set \( \{ \mu_i y_1^{mr-m-r} | i = 1, 2, \ldots, m - 3 \} \). Next we want to get a \( \mathbb{C} \)-basis of \( R^N_{(1,mr-m-r)} \) from this set.

By the relations (4.3.2), we know in \( R \),

\[
\begin{align*}
\mu_0 y_0^r + \mu_0 y_1^r &= 0; \\
\mu_i y_0^r + a_i \mu_i y_1^r &= 0, \quad 1 \leq i \leq m - 3.
\end{align*}
\]

Then we get \( \forall e \geq 0 \),

\[
\begin{align*}
\mu_0 y_0^{er} &= (-1)^e \mu_0 y_1^{er}; \\
\mu_i y_0^{er} &= (-1)^e a_i \mu_i y_1^{er}, \quad 1 \leq i \leq m - 3.
\end{align*}
\]

(4.3.3)

From the relations (4.3.2) again, we get \( \forall e \geq 0 \),

\[
\begin{align*}
\mu_0 y_0^{er} + \mu_1 y_0^{er} + \cdots + \mu_{m-3} y_0^{er} &= 0, \\
\mu_0 y_1^{er} + a_1 \mu_1 y_1^{er} + \cdots + a_{m-3} \mu_{m-3} y_1^{er} &= 0.
\end{align*}
\]

(4.3.4)

From the identities (4.3.3) and (4.3.4), we get \( \forall e \geq 0 \),

\[
(a_1^e - a_1) \mu_1 y_1^{er} + \cdots + (a_{m-3}^e - a_{m-3}) \mu_{m-3} y_1^{er} = 0.
\]

From this we get the following identity

\[
\begin{pmatrix}
 a_1^2 - a_1 & a_2^2 - a_2 & \cdots & a_{m-3}^2 - a_{m-3} \\
 a_1^3 - a_1 & a_2^3 - a_2 & \cdots & a_{m-3}^3 - a_{m-3} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_1^{m-\frac{m}{r}-1} - a_1 & a_2^{m-\frac{m}{r}-1} - a_2 & \cdots & a_{m-3}^{m-\frac{m}{r}-1} - a_{m-3}
\end{pmatrix}
\begin{pmatrix}
 \mu_1 y_1^{mr-m-r} \\
 \mu_2 y_1^{mr-m-r} \\
 \vdots \\
 \mu_{m-3} y_1^{mr-m-r}
\end{pmatrix} = 0.
\]

Note that \( a \in \mathfrak{M}_C \) implies that \( \forall 1 \leq i \leq m - 3, a_i \neq 0, 1 \) and \( \forall i \neq j, a_i \neq a_j \). So that from the matrix equality above, we know that any \( m - 3 - (m - \frac{m}{r} - 2) = \frac{m}{r} - 1 \).
distinct elements in \( \{ \mu_i y_1^{mr-m-r} \mid i = 1, 2, \ldots, m-3 \} \) form a \( \mathbb{C} \)-basis of \( R_{(1, mr-m-r)}^N \).

We can represent the map \( \psi_\beta \) as follows:

\[
\psi_\beta \left( \begin{array}{c}
 y_1^{mr-m-2r} \\
 y_1^{mr-m-3r} y_0^r \\
 \vdots \\
 y_0^{mr-m-2r}
\end{array} \right) = \left( \begin{array}{cccccc}
 \lambda_1 & \lambda_2 & \cdots & \lambda_{m-3} \\
 -a_1 \lambda_1 & -a_2 \lambda_2 & \cdots & -a_{m-3} \lambda_{m-3} \\
 \vdots & \vdots & \vdots & \vdots \\
 (-a_1)^{m-r-2} \lambda_1 & (-a_2)^{m-r-2} \lambda_2 & \cdots & (-a_{m-3})^{m-r-2} \lambda_{m-3}
\end{array} \right) \left( \begin{array}{c}
 \mu_1 y_1^{mr-m-r} \\
 \mu_2 y_1^{mr-m-r} \\
 \vdots \\
 \mu_{m-3} y_1^{mr-m-r}
\end{array} \right)
\]

From this matrix representation, we can easily see that for a generic \( \lambda = (\lambda_1, \cdots, \lambda_{m-3}) \in \mathbb{C}^{m-3} \), the homomorphism \( \psi_\beta \) is surjective. \( \square \)

Now we can prove our main theorem. Recall the notations from section 3. We have:

\textbf{Theorem 4.4.} \( \varsigma(\tilde{V}_{pr}) = \frac{m}{r} - 1 \).

\textit{Proof.} In Proposition 3.1 (5), we have established the inequality \( \varsigma(\tilde{V}_{pr}) \leq \frac{m}{r} - 1 \). So it suffices to prove \( \varsigma(\tilde{V}_{pr}) \geq \frac{m}{r} - 1 \).

Combining the commutative diagrams (4.1.1), (4.1.2), (4.2.1) and Proposition 4.3 together shows that for any \( a \in \mathcal{M}_C \), for a generic tangent vector \( v \in T_a \mathcal{M}_C \), the Higgs map \( \eta_a(v) : F_a^{1,0} \to F_a^{0,1} \) associated to the \( \mathbb{C} \)-VHS \( \mathcal{W}_{(1)} \) is surjective. This \( \) and Proposition 3.1 (2) imply \( \mathcal{W}_{(1)} \) satisfies the hypothesis of Lemma 2.4. Applying this lemma to \( \mathcal{W}_{(1)} \), we get \( \varsigma(\wedge^n \mathcal{W}_{(1)}) = \frac{m}{r} - 1 \). This gives \( \varsigma(\mathcal{V}_{(1)}|_{\mathcal{M}_C}) = \frac{m}{r} - 1 \) by Proposition 3.1 (1). By the definition of the length of Griffith-Yukawa coupling, \( \varsigma(\mathcal{V}_{(1)}) \geq \varsigma(\mathcal{V}_{(1)}|_{\mathcal{M}_C}) \). So finally we get the desired inequality \( \varsigma(\tilde{V}_{pr}) \geq \frac{m}{r} - 1 \) by combining the (in)equalities above and Proposition 3.1 (4). \( \square \)

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ON THE GRIFFITHS-YUKAWA COUPLING LENGTH

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