Random trees and moduli of curves

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Abstract

This is an expository account of the proof of Kontsevich’s combinatorial formula for intersections on moduli spaces of curves following the paper [14]. It is based on the lectures I gave on the subject in St. Petersburg in July of 2001.

These are notes from the lectures I gave in St. Petersburg in July of 2001. Our goal here is to give an informal introduction to intersection theory on the moduli spaces of curves and its relation to random matrices and combinatorics. More specifically, we want to explain the proof of Kontsevich’s formula given in [14] and how it is connected to other topics discussed at this summer school such as, for example, the combinatorics of increasing subsequences in a random permutations.

These lectures were intended for an audience of mostly analysts and combinatorialists interested in asymptotic representation theory and random matrices. This is very much reflected in both the selection of the material and its presentation. Since absolutely no background in geometry was assumed, there is a long and very basic discussion of what moduli spaces of curves and intersection theory on them are about. We hope that a reader trained in analysis or combinatorics will get some feeling for moduli of curves (without worrying too much about the finer points of the theory, all but a few of which were swept under the rug).

Conversely, in the second, asymptotic, part of the text, I allowed myself to operate more freely because the majority of the audience was experienced in asymptotic analysis. Also, since many fundamental ideas such as e.g. the KdV equations for the double scaling limit of the Hermitian 1-matrix model

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were at length discussed in other lectures of the school, their discussion here is much more brief that it would have been in a completely self-contained course. A much more detailed treatment of both geometry and asymptotics can be found in the paper [14], on which my lectures were based.

It is needless to say that, since this is an expository text based on my joint work with Rahul Pandharipande, all the credit should be divided while any blame is solely my responsibility. Many people contributed to the success of the St. Petersburg summer school, but I want to especially thank A. Vershik for organizing the school and for the invitation to participate in it. I am grateful to NSF (grant DMS-0096246), Sloan foundation, and Packard foundation for partial financial support.

1 Introduction to moduli of curves

1.1
Let me begin with an analogy. In the ideal world, the moduli spaces of curves would be quite similar to the Grassmann varieties

$$Gr_{k,n} = \{ L \subset \mathbb{C}^n, \dim L = k \}$$

of $k$-dimensional linear subspaces $L$ of an $n$-dimensional space. While any such subspace $L$ is geometrically just a $k$-dimensional vector space $\mathbb{C}^k$, nontrivial things can happen if $L$ is allowed to vary in families, and this nontriviality is captured by the geometry of $Gr_{k,n}$.

A convenient formalization of the notion of a family of linear spaces parameterized by points of some base space $B$ is a (locally trivial) vector bundle over $B$. There is a natural tautological vector bundle over the Grassmannian $Gr_{k,n}$ itself, namely the space

$$\mathcal{L} = \{(L,v), v \in L \subset \mathbb{C}^n\}$$

formed by pairs $(L,v)$, where $L$ is a $k$-dimensional subspace of $\mathbb{C}^n$ and $v$ is a vector in $L$. Forgetting the vector $v$ gives a map $\mathcal{L} \rightarrow Gr_{k,n}$ whose fiber over $L \in Gr_{k,n}$ is naturally identified with $L$ itself.

Given any space $B$ and a map

$$\phi : B \rightarrow Gr_{k,n}$$
we can form the pull-back of $\mathcal{L}$

$$\phi^*\mathcal{L} = \{(b, v), b \in B, v \in C^n, v \in \phi(b)\}$$

which is a rank $k$ vector bundle over $B$. For a compact base $B$, in the $n \to \infty$ limit this becomes a bijection between (homotopy classes) of maps $B \to Gr_{k,\infty}$ and (isomorphism classes) of rank $k$ vector bundles on $B$.

In particular, one can associate to a vector bundle $\phi^*\mathcal{L}$ its characteristic cohomology classes obtained by pulling back the elements of $H^*(Gr_{k,n})$ via the map $\phi$. Intersections of these classes describe the enumerative geometry of the bundle $\phi^*\mathcal{L}$. It is thus especially important to understand the intersection theory on the space $Gr_{k,n}$ itself — and this leads to a very beautiful classical combinatorics, in particular, Schur functions play a central role (see for example [5], Chapter 14).

1.2

One would like to have a similar theory with families of linear spaces replaced by families of curves of some genus $g$. That is, given a family $F$ of, say, smooth genus $g$ algebraic curves parameterized by some base $B$ we want to have a natural map $\phi : B \to M_g$ that captures the essential information about the family $F$. Here $M_g$ is the moduli space of smooth curves of genus $g$, that is, the space of isomorphism classes of smooth genus $g$ curves. At this point it may be useful to be a little naive about what we mean by a family of curves etc., in this way we should be able to understand the basic issues without too many technicalities getting in our way. Basically, a family $F$ of curves with base $B$ is a “nice” morphism

$$\pi : F \to B$$

of algebraic varieties whose fibers $\pi^{-1}(b), b \in B$, are smooth complete genus $g$ curves. We want the moduli space $M_g$ and the induced map $\phi : B \to M_g$ to be also algebraic.

We will see that the first difficulty with the above program is that, in general, the family $F$ will not be a pull-back of any universal family over $M_g$. To get a sense of why this is the case we can cheat a little and consider the (normally forbidden) case $g = 0$. Up to isomorphism, there is only one curve of genus 0, namely the projective line $\mathbb{P}^1$. Hence the map $\phi$ in this case can only be the trivial map to a point. There exist, however, highly nontrivial
families with fibers isomorphic to $\mathbb{P}^1$ or even $\mathbb{C}^1$ as we, in fact, already saw above in the example of the tautological rank 1 bundle over $Gr_{1,n} \cong \mathbb{P}^{n-1}$.

The reason why there exist locally trivial yet globally nontrivial families with fiber $\mathbb{P}^1$ is that $\mathbb{P}^1$ has a large automorphism group which one can use to glue trivial pieces in a nontrivial way. Basically, the automorphisms are the principal issue behind the nonexistence of a universal family of curves over $\mathcal{M}_g$. The situation becomes manageable, if not entirely perfect, once one can get the automorphism group to be finite (which is automatic for smooth curves of genus $g > 1$). A standard way to achieve this is to consider curves with sufficiently many marked points on them ($\geq 3$ marked points for $g = 0$ and $\geq 1$ for $g = 1$). Since curves with marked points arise very naturally in many other geometric situations, the moduli spaces $\mathcal{M}_{g,n}$ of smooth genus $g$ curves with $n$ distinct marked points should be considered on equal footing with the moduli spaces $\mathcal{M}_g$ of plain curves.

1.3

As the first example, let us consider the space $\mathcal{M}_{1,1}$ of genus $g = 1$ curves $C$ with one parked point $p \in C$. By Riemann-Roch the space of $H^0(C, \mathcal{O}(2p))$ of meromorphic functions on $C$ with at most double pole at $p$ has dimension 2. Hence, in addition to constants, there exists a (unique up to linear combinations with constants) nonconstant meromorphic function

$$f : C \to \mathbb{P}^1,$$

with a double pole at $p$ and no other poles (this is essentially the Weierstrass function $\wp$). Thus, $f$ defines at a 2-fold branched covering of $\mathbb{P}^1$ doubly ramified over $\infty \in \mathbb{P}^1$. For topological reasons, it has three additional ramification points which, after normalization, we can take to be 0, 1 and some $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Now it is easy to show that such a curve $C$ must be isomorphic to the curve

$$C \cong \{y^2 = x(x-1)(x-\lambda)\} \in \mathbb{P}^2$$  \hspace{1cm} (1)

in such a way that the point $p$ becomes the unique point at infinity and the function $f$ becomes the coordinate function $x$. It follows that every smooth pointed $g = 1$ curves occurs in the following family of curves

$$F = \{(x, y, \lambda), y^2 = x(x-1)(x-\lambda)\}$$  \hspace{1cm} (2)
with the base

\[ B = \{ \lambda \} = \mathbb{P}^1 \setminus \{ 0, 1, \infty \} , \]

where the marked point is the point \( p \) at infinity. For example, the curve (1) corresponding to \( \lambda = \frac{3}{2} \) is plotted in Figure 1.

![Figure 1: The \( \lambda = \frac{3}{2} \) member of the family (2).](image)

However, a given curve \( C \) occurs in the family (2) more than once. Indeed, we made arbitrary choices when we normalized the 3 critical values of \( f \) to be 0, 1, and \( \lambda \), respectively. At this stage, we can choose any of \( 6 = 3! \) possible assignments which makes the symmetric group \( S(3) \) act on the base \( B \) preserving the isomorphism class of the fiber. Concretely, this group is generated by involutions

\[ \lambda \mapsto 1 - \lambda , \]

which interchanges the roles of 0 and 1, and by

\[ \lambda \mapsto 1 / \lambda , \]

exchanging the roles of 1 and \( \lambda \). It can be shown that two members of the family \( F \) are isomorphic if and only if they belong to the same \( S(3) \) orbit. Thus, the structure map \( \phi : B \to \mathcal{M}_{1,1} \) should be just the quotient map

\[ \phi : B \to B / S(3) = \text{Spec} \mathbb{C}[B]^{S(3)}. \]
Here $\mathbb{C}[B]^{S(3)}$ is the algebra of $S(3)$-invariant regular functions on $B$. This algebra is a polynomial algebra with one generator, the traditional choice for which is the following

$$j(\lambda) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$ 

Thus, $\mathcal{M}_{1,1}$ is simply a line

$$\mathcal{M}_{1,1} = \text{Spec } \mathbb{C}[j] \cong \mathbb{C}.$$ 

It is now time to point out that the family $F$ is not a pull-back $\phi^*$ of some universal family over $\mathcal{M}_{1,1}$. The simplest way to see this is to observe that the group $S(3)$ fails to act on $F$. Indeed, let us try to lift the involution $\lambda \to 1 - \lambda$ from $B$ to $F$. There are two ways to do this, namely

$$(x, y, \lambda) \mapsto (1 - x, \pm iy, 1 - \lambda),$$

neither of which is satisfactory because the square of either map

$$(x, y, \lambda) \mapsto (x, -y, \lambda)$$

yields, instead of identity, a nontrivial automorphism of every curve in the family $F$. One should also observe that both choices act by a nontrivial automorphism on the fiber over the fixed point $\lambda = \frac{1}{2} \in B$. In fact, the fibers of $F$ over fixed points of a transposition and a 3-cycle in $S(3)$, respectively, (with $j(\lambda) = 1728$ and $j(\lambda) = 0$, resp.) are precisely the curves with extra large automorphism groups (of order 4 and 6, resp.).

The existence of a nontrivial automorphism of every pointed genus 1 curve leads to the somewhat unpleasant necessity to consider every point of $\mathcal{M}_{1,1}$ as a “half-point” in some suitable sense in order to get correct enumerative predictions. Again, automorphisms make the real world not quite ideal.

While it is important to be aware of these automorphism issues (for example, to understand how intersection numbers on moduli spaces can be rational numbers), there is no need to be pessimistic about them. In fact, by allowing spaces more general than algebraic varieties (called stacks) one can live a life in which $\mathcal{M}_{g,n}$ is smooth and with a universal family over it. This is, however, quite technical and will remain completely outside the scope of these lectures.
1.4

Clearly, the space $\mathcal{M}_{1,1} \cong \mathbb{C}$ is not compact. The $j$-invariant of the curve (1) goes to $\infty$ as the parameter $\lambda$ approaches the three excluded points $\{0, 1, \infty\}$. As $\lambda$ approaches 0 or 1, the curve $C$ acquires a nodal singularity; for example, for $\lambda = 1$ we get the curve

$$y^2 = x(x - 1)^2$$

plotted in Figure 2. It is natural to complete the family (2) by adding the corresponding nodal cubics for $\lambda \in \{0, 1\}$. All plane cubic with a node being isomorphic, the function $j$ extends to a map

$$j : \mathbb{C} \to \mathbb{P}^1/S(3) = \overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$$

to the moduli space of $\overline{\mathcal{M}}_{1,1}$ of curves of arithmetic genus 1 with at most one node and a smooth marked point.\(^1\)

In general, it is very desirable to have a nice compactification for the noncompact spaces $\mathcal{M}_{g,n}$. First of all, interesting families of curves over a

\(^1\)For future reference we point out that something rather different happens in the family (2) as $\lambda \to \infty$. Indeed, the equation

$$\lambda^{-1} y^2 = x(x - 1)(\lambda^{-1} x - 1)$$

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complete base $B$ are typically forced to have singular fibers over some points in the base (as in the example above). Fortunately, as we will see below, it often happens that precisely these special fibers contain key information about the geometry of the family. Also, since eventually we will be interested in intersection theory on the moduli spaces of curves, having a complete space can be a significant advantage.

A particularly remarkable compactification $\overline{M}_{g,n}$ of $M_{g,n}$ was constructed by Deligne and Mumford. The space $\overline{M}_{g,n}$ is the moduli space of stable curves $C$ of arithmetic genus $g$ with $n$ distinct marked points. Stability, by definition, means that the curve $C$ is complete and connected with at worst nodal singularities, all marked points are smooth, and that $C$, together with marked points, admits only finitely many automorphisms. In practice, the last condition means that every rational component of the normalization of $C$ should have at least 3 special (that is, lying over marked or singular points of $C$) points. Observe that, in particular, the curve $C$ is allowed to be reducible. A typical stable curve can be seen in Figure 3.

Figure 3: A boundary element of $\overline{M}_{1,5}$

\[ x(x-1) = 0 \text{ in the } \lambda \to \infty \text{ limit, which means that we get three lines (one of which is the line at infinity), all three of them intersecting in the marked point at infinity. In other words, the fiber of the family at } \lambda = \infty \text{ is very much not the kind of curve by which we want to compactify } M_{1,1}. \] This problem can be cured, but in a not completely trivial way, see below.
Those who have not seen this before are probably left wondering how it is possible for $\overline{M}_{g,n}$ to be compact. What if, for example, one marked point $p_1$ on some fixed curve $C$ approaches another marked point $p_2 \in C$? We should be able to assign some meaningful stable limit to such a 1-parametric family of curves, but it is somewhat nontrivial to guess what it should be.

A family of curves with a 1-dimensional base $B$ is a surface $F$ together with a map $\pi : F \to B$ whose fibers are the curves of the family. Marking $n$ points on the curves means giving $n$ sections of the map $\pi$, that is, $n$ maps

$$p_1, \ldots, p_n : B \to F,$$

such that

$$\pi(p_k(b)) = b, \quad b \in B, \quad k = 1, \ldots, n.$$

We will denote by

$$S_i = p_i(B)$$

the trajectories of the marked points on $F$; they are curves on the surface $F$.

Now suppose we have a 1-dimensional family of 2-pointed curves such that at some bad point $b_0$ of the base $B$ we have $p_1(b_0) = p_2(b_0)$, that is, over this point two marked points hit each other, see Figure 4, and therefore the fiber $\pi^{-1}(b_0)$ is not a stable 2-pointed curve. It is quite easy to repair this family: just blow up the offending (but smooth) point $P = p_1(b_0) = p_2(b_0)$ on the surface $F$. Let

$$\sigma : \widetilde{F} \to F$$
be the blow-up at $P$. Then

$$\tilde{\pi} = \pi \circ \sigma : \tilde{F} \to B$$

is a new family of curves with base $B$. Outside $b_0$ this the same family as before, whereas the fiber $\tilde{\pi}^{-1}(b_0)$ is the old fiber $\pi^{-1}(b_0)$ plus the exceptional divisor $E = \sigma^{-1}(P) \cong \mathbb{P}^1$ of the blow-up, see Figure 5. Assuming the sections $S_1$ and $S_2$ met each other and the fiber $\pi^{-1}(b_0)$ transversally at $P$, as in Figure 4 the marked points on $\tilde{\pi}^{-1}(b_0)$ are two distinct points on the exceptional divisor $E$. Therefore, $\tilde{\pi}^{-1}(b_0)$ is a stable 2-pointed curve which is the stable limit of the curves $\pi^{-1}(b)$ as $b \to b_0$.

To summarize, if one marked point on a curve $C$ approaches another then $C$ bubbles off a projective line $\mathbb{P}^1$ with these two points on it as in Figure 6.

More generally, if $F$ is any family of curves with a smooth 1-dimensional base $B$ that are stable except over one offending point $b_0 \in B$ then after a sequence of blow-ups and blow-downs and, possibly, after passing to a branched covering of $B$, one can always arrive at a new family with all fibers stable. Moreover, the fiber over $b_0$ in this family is determined uniquely. This process is called the stable reduction and how it works is explained, for example, in [8], Chapter 3C.
In particular, there exists a stable reduction of the family (2) which, as we saw in Section 1.3, fails to have a stable fiber over the point $\lambda = \infty$ in the base. This is an example where only blow-ups and blow-downs will not suffice, that is, a base change is necessary.

1.6

The topic of this lectures is intersection theory on the Deligne-Mumford spaces $\overline{M}_{g,n}$ and, specifically, intersections of certain divisors $\psi_i$ which will be defined presently. It was conjectured by Witten [19] that a suitable generating function for these intersections is a $\tau$-function for the Korteweg–de Vries hierarchy of differential equations. This conjecture was motivated by an analogy with matrix models of quantum gravity, where the same KdV hierarchy appears (this was already discussed in other lectures at this school). The KdV equations were deduced by Kontsevich in [9] from an explicit combinatorial formula for the intersections of the $\psi$-classes (see also, for example, [2] for more information about the connection to the KdV equations). The main goal of these lectures is to explain a proof of this combinatorial formula of Kontsevich following the paper [14].

The definition of the divisors $\psi_i$ is the following. A point in $\overline{M}_{g,n}$ is a stable curve $C$ with marked points $p_1, \ldots, p_n$. By definition, all points $p_i$ are smooth points of $C$, hence the tangent space $T_{p_i}C$ to $C$ at $p_i$ is a line. Similarly, we have the cotangent lines $T^*_{p_i}C$, $i = 1, \ldots, n$. As the point $(C, p_1, \ldots, p_n) \in \overline{M}_{g,n}$ varies, these cotangent lines $T^*_{p_i}C$ form $n$ line bundles over $\overline{M}_{g,n}$. By definition, $\psi_i$ is the first Chern class of the line bundle $T^*_{p_i}C$. In other words, it is the divisor of any nonzero section of the line bundle $T^*_{p_i}C$.

1.7

To get a better feeling for these classes let us intersect $\psi_i$ with a curve in $\overline{M}_{g,n}$. The answer to this question should be a number. Let $B$ be a curve. A map $B \to \overline{M}_{g,n}$ is morally equivalent to a 1-dimensional family of curves with base $B$ (in reality we may have to pass to a suitable branched covering of $B$ to get an honest family. \footnote{We already saw an example of this in Section 1.3. Indeed, $\overline{M}_{1,1}$ is itself a line $\mathbb{P}^1$. However, in order to get an actual family over it we have to go to a branched covering.} So, let us consider a family $\pi : F \to B$ of
stable pointed curves with base $B$ and the induced map $\phi : B \to \overline{M}_{g,n}$. As usual, the marked points $p_1, \ldots, p_n$ are sections of $\pi$ and

$$S_i = p_i(B), \quad i = 1, \ldots, n,$$

are disjoint curves on the surface $F$. A section $s$ of $\phi^*(T_{p_i})$ is a vector field on the curve $S_i$ which is tangent to fibers of $\pi$ and, hence, $s$ is a section of the normal bundle to $S_i \subset F$, see Figure 7. The degree of this normal bundle is the self-intersection of the curve $S_i$ on the surface $F$, that is,

$$\deg (s) = (S_i, S_i)_F,$$

where $(s)$ is the divisor of $s$. In other words,

$$\int_B c_1 (\phi^*(T_{p_i})) = (S_i, S_i)_F,$$

where $c_1$ denotes the 1st Chern class. Dually, we have

$$\int_{\phi(B)} \psi_i = -(S_i, S_i)_F.$$

We will now use this formula to compute the intersections of $\psi_i$ with $\overline{M}_{g,n}$ in the cases when the space $\overline{M}_{g,n}$ is itself 1-dimensional. Since

$$\dim \overline{M}_{g,n} = 3g - 3 + n,$$

this happens for $\overline{M}_{0,4}$ and $\overline{M}_{1,1}$.

As always, the automorphisms are to blame.
The space $\overline{M}_{0,4}$ is easy to understand. After all, there is only one smooth curve of genus 0, namely $\mathbb{P}^1$. Moreover, any 3 distinct points of $\mathbb{P}^1$ can be taken to the points $\{0,1,\infty\}$ by an automorphism of $\mathbb{P}^1$ (in particular, this means that $\overline{M}_{0,3}$ is a point). After we identified the first three marked points with $\{0,1,\infty\}$, we can take any point $x \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ as the fourth marked point. Thus the locus of smooth curves in $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Singular curves are obtained as we let $x$ approach the 3 excluded points $\{0,1,\infty\}$, which, by the process described in Section 1.5, bubbles off a new $\mathbb{P}^1$ with two marked points on it. This completes the description of $\overline{M}_{0,4} \cong \mathbb{P}^1$.

In addition, this gives a description of the universal family over $\overline{M}_{0,4}$ (oh yes, in genus 0 it does exist!). Take $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(x,y)$. The map $(x,y) \mapsto x$ with 4 sections

$$p_1(x) = (x,0), \quad p_2(x) = (x,1), \quad p_3(x) = (x,\infty), \quad p_4(x) = (x,x),$$

defines a family of 4-pointed smooth genus 0 curves for $x \in \mathbb{P}^1 \setminus \{0,1,\infty\}$. The section $p_4$ collides with the other three at the points $(0,0)$, $(1,1)$, and $(\infty, \infty)$, see Figure 8. To extend this family over all of $\mathbb{P}^1$, we blow up these collision points as in Section 1.5 and get the surface $F$ shown in Figure 9. The closures of the curves $p_1(x), \ldots, p_4(x)$ give 4 sections which are now disjoint everywhere.

Incidentally, this surface $F$ can be naturally identified with $\overline{M}_{0,5}$ and, more generally, for any $n$ there exists a natural map $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ giving the universal family over $\overline{M}_{0,n}$. This map forgets the $(n+1)$st marked point and, if the curve becomes unstable after that, blows down all unstable components.
Now let us compute $\int_{\overline{M}_{0,4}} \psi_1$ using the recipe given in Section 1.7. Recall that $S_1 \subset F$ denotes the closure of the curve \{(x,0), x \neq 0\} in $F$ (a.k.a. the proper transform of the corresponding curve in $\mathbb{P}^1 \times \mathbb{P}^1$). Let $E$ denote the preimage of $(0,0)$ under the blow-up, that is, let $E$ be the exceptional divisor. The self-intersection of $S_1$ with any curve $\{y = c\}, c \neq 0$, on $\mathbb{P}^1 \times \mathbb{P}^1$ is clearly zero. Letting $c \to 0$ we get

$$(S_1, E + S_1) = 0.$$ 

Since, obviously, $(S_1, E) = 1$ we conclude that

$$\int_{\overline{M}_{0,4}} \psi_1 = -(S_1, S_1) = -(-1) = 1.$$ 

1.9

Now let us analyze the integral $\int_{\overline{M}_{1,1}} \psi_1$. In the absence of a universal family, we have to look for another suitable family to compute this integral. A particularly convenient family can be obtained in the following way. Consider the projective plane $\mathbb{P}^2$ with affine coordinates $(x, y)$. Pick two generic cubic polynomials $f(x, y)$ and $g(x, y)$ and consider the family of cubic curves

$$F = \{(x, y, t), f(x, y) - t g(x, y) = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1,$$ 

with base $B = \mathbb{P}^1$ parameterized by $t$. The cubic curves $f(x, y) = 0$ and $g(x, y) = 0$ intersect in 9 points $p_1, \ldots, p_9$ and we can choose any of those points as the marked point in our family. An example of such family of plane cubics is plotted in Figure 10.
Our first observation is that the surface $F$ is the blow-up of $\mathbb{P}^2$ at the points $p_1, \ldots, p_9$ (which are distinct for generic $f$ and $g$). Indeed, we have a rational map

$$\mathbb{P}^2 \ni (x, y) \to \left(x, y, \frac{g}{f}\right) \in F,$$

which is regular away from $p_1, \ldots, p_9$. Each of the points $p_i$ is a transverse intersection of $f = 0$ and $g = 0$, which is another way of saying that at those points the differentials $df$ and $dg$ are linearly independent. Thus, this map identifies $F$ with the blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_9$. The graph of the function $\frac{g(x, y)}{f(x, y)}$ is shown in Figure 11. This graph goes vertically over the points $p_1, \ldots, p_9$ that are blown up.

Since the section $S_1$ is exactly one of the exceptional divisors of this blow-up, arguing as in Section 1.8 above we find that

$$(S_1, S_1) = -1.$$ 

It does not mean, however, that we are done with the computation of the integral, because the induced map $\phi : B \rightarrow \overline{M}_{1,1}$ is very far from being one-to-one. In fact, set-theoretically, the degree of the map $\phi$ is 12 as we shall now see.

To compute the degree of $\phi$ we need to know how many times a fixed generic elliptic curve appears in the family $F$. This is a classical computation. First, one can show that the singular cubic is generic enough. Then we claim that, as $t$ varies, there will be precisely 12 values of $t$ that produce a singular
Figure 11: A fragment of the surface

There are various ways to see this. For example, the singularity of the curve is detected by vanishing of the discriminant. The discriminant of a cubic polynomial is a polynomial of degree 12 in its coefficients, hence a polynomial of degree 12 in \( t \).

A alternative way to obtain this number 12 is to compute the Euler characteristic of the surface \( F \) in two different ways. On the one hand, viewing \( F \) as a blow-up, we get

\[
\chi(F) = \chi(\mathbb{P}^2) + 9(\chi(\mathbb{P}^1) - 1) = 12.
\]

On the other hand, \( F \) is fibered over \( B \) and the generic fiber is a smooth elliptic curve whose Euler characteristic is 0. The special fibers are the nodal elliptic curves with Euler characteristic equal to 1. Hence, there are 12 special fibers.

However, as remarked in Section [1.3] each point of \( \overline{M}_{1,1} \) is really a half–point because of automorphism of order 2 of any pointed genus 1 curve. Therefore, the \( 24 = 2 \cdot 12 \) is the true degree of the map \( \phi \). By the push-pull formula we thus obtain

\[
\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{\deg \phi} \int_B \phi^* \psi_1 = \frac{1}{\deg \phi} (-(S_1, S_1)) = \frac{1}{24}.
\]
An interesting corollary of this computation is that if $F \to B$ is a smooth family of 1-pointed genus 1 stable curves over a smooth complete curve $B$ then the set-theoretic degree of the induced map $B \to \overline{M}_{1,1}$ has to be divisible by 12.

1.10

It is difficult to imagine being able to compute many intersections of the $\psi$-classes in the above manner. To begin with, it is essentially impossible to write down a sufficiently explicit family of general high genus curves, see the discussion in Chapter 6F of [8]. It is therefore amazing that there exist several complete and beautiful descriptions of the all possible intersection numbers of the form

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle \overset{\text{def}}{=} \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}, \quad k_1 + \cdots + k_n = 3g - 3 + n.$$  \hspace{1cm} (5)

The most striking description was conjectured by Witten [19] and says the exponential of the following generating function for the numbers (5)

$$F(t_1, t_2, \ldots) = \sum_{n} \frac{1}{n!} \sum_{k_1, \ldots, k_n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle \ t_{k_1} \cdots t_{k_n}$$  \hspace{1cm} (6)

is a $\tau$-function for the KdV hierarchy of differential equations. This conjecture was motivated by the (physical) analogy with the random matrix models of quantum gravity and, in fact, the $\tau$-function thus obtained is the same as the one that arises in the double scaling of the 1-matrix model (and discussed in other lectures of this school). The KdV equation and the string equation satisfied by the $\tau$-function uniquely determine all numbers (5). Alternatively, the numbers (5) are uniquely determined by the associated Virasoro constraints. Further discussion can be found, for example, in [2].

1.11

Kontsevich in [9] obtained the KdV equations for (5) from a combinatorial formula for the following (somewhat nonstandard) generating function

$$K_{g,n}(z_1, \ldots, z_n) = \sum_{k_1 + \cdots + k_n = 3g - 3 + n} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle \prod \frac{(2k_i - 1)!!}{z_i^{2k_i + 1}},$$  \hspace{1cm} (7)
for the numbers \[^5\] with fixed \(g\) and \(n\).

The main ingredient in Kontsevich’s combinatorial formula is a 3-valent graph \(G\) embedded in the a topological surface \(\Sigma_g\). A further condition on this graph \(G\) is that the complement \(\Sigma_g \setminus G\) is a union of \(n\) topological disks (in particular, this forces \(G\) to be connected). These disks, called cells, have to (bijectively) numbered by \(1, \ldots, n\). Two such graphs \(G\) and \(G'\) are identified if there exist an orientation preserving homeomorphism of \(\Sigma_g\) that takes \(G\) to \(G'\) and preserves the labels of the cells. In particular, every graph \(G\) has an automorphism group \(\text{Aut } G\), which is finite and only seldom nontrivial. Let \(\mathcal{G}^3_{g,n}\) denote the set of distinct such graphs \(G\) with given values of \(g\) and \(n\); this is a finite set. An example of an element of \(\mathcal{G}^3_{2,3}\) is shown in Figure 12.

![Figure 12: A trivalent map on a genus 2 surface](image)

Another name for a graph \(G \subset \Sigma_g\) such that \(\Sigma_g \setminus G\) is a union of cells is a map on \(\Sigma_g\). One can imagine that the cells are the countries in which the graph \(G\) divides the surface \(\Sigma_g\).

Kontsevich’s combinatorial formula for the function \[^7\] is the following:

\[
K_{g,n}(z_1, \ldots, z_n) = 2^{2g-2+n} \sum_{G \in \mathcal{G}^3_{g,n}} \frac{1}{|\text{Aut } G|} \prod_{\text{edges } e \text{ of } G} \frac{1}{z_{\text{one side of } e} + z_{\text{other side of } e}}, \tag{8}
\]

where the meaning of the term

\[z_{\text{one side of } e} + z_{\text{other side of } e}\]

is the following. Each edge \(e\) of \(G\) separates two cells (which may be identical). These cell carry some labels, say, \(i\) and \(j\). Then \((z_i + z_j)^{-1}\) is the factor in \[^8\] corresponding to the edge \(e\).
To get a better feeling for how this works let us look at the cases \((g, n) = (0, 3), (1, 1)\) that we understand well. The space \(\overline{M}_{0,3}\) is a point and the only nontrivial integral over it is
\[
\langle \tau_0 \tau_0 \tau_0 \rangle = \int_{\overline{M}_{0,3}} 1 = 1.
\]
Thus,
\[
K_{0,3} = \frac{1}{z_1 z_2 z_3}.
\]
The combinatorial side of Kontsevich’s formula, however, is not quite trivial. The set \(G^3_{0,3}\) consists of 4 elements. Two of them are shown in Figure 13, the other two are obtained by permuting the cell labels of the graph of the left. All these graphs have only trivial automorphisms. Hence, we get
\[
K_{0,3} = 2 \left( \frac{1}{2z_1(z_1 + z_2)(z_1 + z_3)} + \text{permutations} \right) + \frac{1}{(z_1 + z_2)(z_1 + z_3)(z_2 + z_3)},
\]
and, indeed, this simplifies to \((z_1 z_2 z_3)^{-1}\). What is apparent in this example is that it is rather mysterious how \(\mathbb{S}\), which a priori is only a rational function of the \(z_i\)’s, turns out to be a polynomial in the variables \(z_i^{-1}\).

Perhaps this example created a somewhat wrong impression because in this case \(\mathbb{S}\) was much more complicated than \(\mathbb{L}\). So, let us consider the case \((g, n) = (1, 1)\), where the computation of the unique integral
\[
\langle \tau_1 \rangle = \frac{1}{24}
\]
already does require some work. The unique element of $G^3_{1,1}$ is shown in Figure 14. This graph can be obtained by gluing the opposite sides of a hexagon, which also explains why the automorphism group of this graph is the cyclic group of order 6 (acting by rotations of the hexagon). Thus, (8) specializes in this case to

$$2 \frac{1}{6} \frac{1}{(z_1 + \bar{z}_1)^3} = \frac{1}{24} \frac{1}{z_1^3},$$

as it should.

1.12

Kontsevich was led to the formula (8) by considering a cellular decomposition of $M_{g,n}$ coming from Strebel differentials. In these lectures we shall explain, following [14], different approach to the formula (8) via the asymptotics in the Hurwitz problem of enumerating branched covering of $\mathbb{P}^1$. This approach is based on the relation between the Hurwitz problem and intersection theory on $\overline{M}_{g,n}$ discovered in [3, 4] and on the asymptotic analysis developed in [10]. It has several advantages over the approach based on Strebel differentials.

2 Hurwitz problem

2.1

Intersection theory on $\overline{M}_{g,n}$ is about enumerative geometry of families of stable $n$-pointed curves of genus $g$. The significance of the space $\overline{M}_{g,n}$ is that its geometry captures some essential information about all possible families of curves. Through the space $\overline{M}_{g,n}$, one can learn something about curves.
in general from any specific enumerative problem. If the specific enumerative problem is sufficiently rich, one can gather a lot of information about intersection theory on $\overline{M}_{g,n}$ from it. Potentially, one can get a complete understanding of the whole intersection theory, which then can be applied to any other enumerative problem.

Our strategy will be to study such a particular yet representative enumerative problem. This specific problem will be the Hurwitz problem about branched covering of $\mathbb{P}^1$. That there exists a direct connection between Hurwitz problem and the intersection theory on $\overline{M}_{g,n}$ was first realized in [3, 4]. The beautiful formula of [3] for the Hurwitz numbers will be the basis for our computations.

In fact, we will see that the (exact) knowledge of the numbers (5) is equivalent to the asymptotics in the Hurwitz problem. This is, in some sense, very fortunate because asymptotic enumeration problems often tend to be more structured and accessible than exact enumeration.

2.2

It is a century-old theme in combinatorics to enumerate branched coverings of a Riemann surface by another Riemann surface (an example of which is shown schematically in Figure 16). Given degree $d$, positions of ramification points downstairs, and their types (that is, given the conjugacy class in $S(d)$ of the monodromy around each one of them), there exist only finitely many possible coverings and the natural question is: how many? This very basic enumerative problem arises all over mathematics, from complex analysis to ergodic theory. These numbers of branched coverings are directly connected to other fundamental objects in combinatorics, namely to the class algebra of the symmetric group and — via the representation theory of finite groups — to the characters of symmetric groups.

We also mention that there is a general, and explicit, correspondence between enumeration of branched covering of a curve and the the Gromov-Witten theory of the same curve, see [15]. From this point of view, the computation of the numbers (5), that is, the Gromov-Witten theory of a point, arises as a limit in the Gromov-Witten theory of $\mathbb{P}^1$ as the degree goes to infinity. This is parallel to how the free energy (6) equation arises as the limit in the 1-matrix model.
The particular branched covering enumeration problem that we will be concerned with can be stated as follows. The data in the problem are a partition $\mu$ and genus $g$. Let

$$f : C \rightarrow \mathbb{P}^1$$

be a map of degree

$$d = |\mu| = \sum \mu_i,$$

where $C$ is smooth connected complex curve of genus $g$. We require that $\infty \in \mathbb{P}^1$ is a critical value of the map $f$ and the corresponding monodromy has cycle type $\mu$. Equivalently, this can be phrased as the requirement that divisor $f^{-1}(\infty)$ has the form

$$f^{-1}(\infty) = \sum_{i=1}^{n} \mu_i [p_i],$$

where $n = \ell(\mu)$ is the length of the partition $\mu$ and $p_1, \ldots, p_n \in C$ are the points lying over $\infty \in \mathbb{P}^1$. We further require that all other critical values of $f$ are distinct and nondegenerate. In other words, the map $f^{-1}$ has only square-root branch points in $\mathbb{P}^1 \setminus \{\infty\}$. The number $r$ of such square-root branch points is given by the Riemann–Hurwitz formula

$$r = 2g - 2 + |\mu| + \ell(\mu). \tag{9}$$

An example of such a covering can be seen in Figure 15 where $\mu = (3)$ and $r = 2$, hence $d = 3$, $n = 1$, and $g = 0$.

Figure 15: A Hurwitz covering with $\mu = (3)$
We will call a covering satisfying the above conditions a \textit{Hurwitz covering}. Once the positions of the $r$ simple branchings are fixed, there are only finitely many Hurwitz coverings provided we identify two coverings

$$f : C \to \mathbb{P}^1, \quad f' : C' \to \mathbb{P}^1$$

for which there exists an isomorphism $h : C \to C'$ such that $f = f' \circ h$.

Similarly, we define automorphisms of $f$ as an automorphisms $h : C \to C$ such that $f = f \circ h$. We will see that, with a very rare exception, Hurwitz coverings have only trivial automorphisms.

By definition, the \textit{Hurwitz number} \( \text{Hur}_g(\mu) \) is the number of isomorphism classes of Hurwitz coverings with given positions of branch points. In the special case when such a covering has a nontrivial automorphism, it should be counted with multiplicity $\frac{1}{2}$.

\section*{2.4}

The Hurwitz problem can be restated as a problem about factoring permutations into transpositions. This goes as follows.

Let us pick a point \( x \in \mathbb{P}^1 \) which is not a ramification point. Then, by basic topology, all information about the covering is encoded in the homomorphism

$$\pi_1(\mathbb{P}^1 \setminus \{\text{ramification points}\}, x) \to \text{Aut} f^{-1}(x) \cong S(d).$$

The identification of \( \text{Aut} f^{-1}(x) \) with \( S(d) \) here is not canonical, but it is convenient to pick any one of the \( d! \) possible identifications. Then, by construction, the loop around \( \infty \) goes to a permutation \( s \in S(d) \) with cycle type \( \mu \) and loops around finite ramification points correspond to some transpositions \( t_1, \ldots, t_r \) in \( S(d) \).

The unique relation between those loops in \( \pi_1 \) becomes the equation

$$t_1 \cdots t_r = s.$$  \hfill (10)

This establishes the equivalence of the Hurwitz problem with the problem of factoring general permutations into transpositions (up to conjugation, since we picked an arbitrary identification of \( \text{Aut} f^{-1}(x) \) with \( S(d) \)). More precisely, the Hurwitz number \( \text{Hur}_g(\mu) \) is the number (up to conjugacy, and possibly with an automorphism factor) of factorization of the form [10] that
correspond to a connected branched covering. A branched covering is connected when we can get from any point of $f^{-1}(x)$ to any other point by the action of the monodromy group. Thus, the transpositions $t_1, \ldots, t_n$ have to generate a transitive subgroup of $S(d)$, which is then automatically forced to be the whole of $S(d)$.

The fact that $t_1, \ldots, t_n$ generate $S(d)$ greatly constraints the possible automorphisms of $f$. Indeed, the action of any nontrivial automorphism on $f^{-1}(x)$ has to commute with $t_1, \ldots, t_n$, and hence with $S(d)$, which is only possible if $d = 2$.

By the usual inclusion–exclusion principle, it is clear that one can go back and forth between enumeration of connected and possibly disconnected coverings. Thus, the Hurwitz problem is essentially equivalent to decomposing the powers of one single element of the class algebra of the symmetric group, namely of the conjugacy class of a transposition

$$\sum_{1 \leq i < j \leq d} (ij)$$

in the standard conjugacy class basis. There is a classical formula, going back to Frobenius, for all such expansion coefficients in terms of irreducible characters. The character sums that one thus obtains can be viewed as finite analogs of Hermitian matrix integrals, with the dimension of a representation $\lambda$ playing the role of the Vandermonde determinant and the central character of $\prod$ in the representation $\lambda$ playing the role of the Gaussian density, see, for example [11, 12] for a further discussion of properties of such sums.

2.5

For us, the crucial property of the Hurwitz problem is its connection with the intersection theory on the Deligne-Mumford spaces $\overline{M}_{g,n}$. This connection was discovered, independently, in [4] and [3], the latter paper containing the following general formula

$$\text{Hur}_g(\mu) = \frac{r!}{|\text{Aut} \mu|} \prod_{i=1}^{n} \frac{\mu_i!}{\mu_i^{\mu_i}} \int_{\overline{M}_{g,n}} \frac{1 - \lambda_1 + \cdots + \lambda_g}{\prod (1 - \mu_i \psi_i)},$$

where $r$ is number of branch points given by (9), $n = \ell(\mu)$ is the length of the partition $\mu$, $\text{Aut} \mu$ is the stabilizer of the vector $\mu$ in $S(n)$,

$$\lambda_i \in H^{2i} (\overline{M}_{g,n}), \quad i = 1, \ldots, g,$$
are the Chern classes of the Hodge bundle over $\overline{M}_{g,n}$ (it is not important for what follows to know what this is), and finally, the denominators are supposed to be expanded into a geometric series

$$\frac{1}{1 - \mu_i \psi_i} = 1 + \mu_i \psi_i + \mu_i^2 \psi_i^2 + \ldots,$$

which terminates because $\psi_i \in H^2(\overline{M}_{g,n})$ is nilpotent.

In particular, the integral in the ELSV formula (12) is a polynomial in the $\mu_i$'s. The monomials in this polynomial are obtained by picking a term in the expansion (13) for each $i = 1, \ldots, n$ and then adding a suitable $\lambda$-class to bring the total degree to the dimension of $\overline{M}_{g,n}$. It is, therefore, clear that the top degree term of this polynomial involves only intersections of the $\psi$-classes and no $\lambda$-classes. That is,

$$\int_{\overline{M}_{g,n}} = \sum_{k_1 + \cdots + k_n = 3g - 3 + n} \prod \mu_i^{k_i} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle + \text{lower degree}. \quad (14)$$

These top degree terms are precisely the numbers (5) that we want to understand.

2.6

A natural way to infer something about the top degree part of a polynomial is to let its arguments go to infinity. The behavior of the prefactors in (12) is given by the Stirling formula

$$\frac{m^m}{m!} \sim \frac{e^m}{\sqrt{2\pi m}}, \quad m \to \infty.$$

Let $N$ be a large parameter and let $\mu_i$ depend on $N$ in such a way that

$$\frac{\mu_i}{N} \to x_i, \quad i = 1, \ldots, n, \quad N \to \infty,$$

where $x_1, \ldots, x_n$ are finite. We will also additionally assume that all $\mu_i$'s are distinct and hence $|\text{Aut } \mu| = 1$. Then by (14) and the Stirling formula, we have the following asymptotics of the Hurwitz numbers:

$$\frac{1}{N^{3g - 3 + n/2}} \frac{\text{Hur}_g(\mu)}{e^{\mu_i r_i!}} \to \frac{1}{(2\pi)^{n/2}} \sum_{k_1 + \cdots + k_n = 3g - 3 + n} \prod \mu_i^{k_i - \frac{1}{2}} \langle \tau_{k_1} \cdots \tau_{k_n} \rangle =: H_g(x). \quad (15)$$
It is convenient to Laplace transform the asymptotics \( H_g(x) \). Since
\[
\int_0^\infty e^{-sx} x^{k-1/2} \, dx = \frac{\Gamma(k + 1/2)}{s^{k+1/2}} = \sqrt{\pi} \frac{(2k - 1)!!}{2^k s^{k+1/2}},
\]
we get
\[
\int_{\mathbb{R}^n_{>0}} e^{-s \cdot x} H_g(x) \, dx = \sum_{k_1+\ldots+k_n=3g-3+n} \langle \tau_{k_1} \ldots \tau_{k_n} \rangle \prod_{i=1}^n \frac{(2k_i - 1)!!}{(2s_i)^{k_i+1/2}}, \tag{16}
\]
which up to the following change of variables
\[
z_i = \sqrt{2s_i}, \quad i = 1, \ldots, n,
\]
is precisely the Kontsevich generating function \( \langle \tau \rangle \) for the numbers \( \langle \tau \rangle \).

Thus, we find ourselves in situation which looks rather comfortable: the generating function that we seek to compute is not only related to a specific enumerative problem but, in fact, it is the Laplace transform of the asymptotics in that enumerative problem. People who do enumeration know that asymptotics tends to be simpler than exact enumeration and, usually, the Laplace (or Fourier) transform of the asymptotics is the most natural thing to compute.

This general philosophy is, of course, only good if we can find a handle on the Hurwitz problem. In the following subsection, we will discuss a re-statement of the Hurwitz problem in terms of enumeration of certain graphs on genus \( g \) surfaces that we call branching graphs. This description will turn out to be particularly suitable for our purposes (which may not be a huge surprise because, after all, Kontsevich’s formula \( \langle \tau \rangle \) is stated in terms of graphs on surfaces).

2.7

A very classical way to study branched coverings is to cut the base into simply-connected pieces. Over each of the resulting regions the covering becomes trivial, that is, consisting of \( d \) disjoint copies of the region downstairs, where \( d \) is the degree of the covering. The structure of the covering is then encoded in the information on how those pieces are patched together upstairs. Typically, this gluing data is presented in the form of a graph, usually with some additional labels etc.
There is, obviously, a considerable flexibility in this approach and some choices may lead to much more convenient graph enumeration problems than the others. For the Hurwitz problem, we will follow the strategy from [1], which goes as follows.

Let

\[ f : C \to \mathbb{P}^1 \]

be a Hurwitz covering with partition \( \mu \) and genus \( g \). In particular, the number \( r \) of finite ramification points of \( f \) is given by the formula (9). Without loss of generality, we can assume these ramification points to be \( r \)th roots of unity in \( \mathbb{C} \). Let us cut the base \( \mathbb{P}^1 \) along the unit circle \( S = \{ |z| = 1 \} \), that is, let us write

\[ \mathbb{P}^1 = D_- \cup S \cup D_+ \]

where \( D_\pm = \{ |z| \lesssim 1 \} \) are the Southern and Northern hemisphere in Figure 16 respectively.

Since the map \( f \) is unramified over \( D_- \), its preimage \( f^{-1}(D_-) \) consists of \( d \) disjoint disks. Their closures, however, are not disjoint: they come together precisely at the critical points of \( f \). By construction, critical points of \( f \) are in bijection with its critical values, that is, with the \( r \)th roots of unity in \( \mathbb{P}^1 \). Thus, the the set \( f^{-1}(D_-) \subset C \) looks like the structure in Figure 17. This structure is, in fact, a graph \( \Gamma \) embedded in a genus \( g \) surface. Its vertices are the components of \( f^{-1}(D_-) \) and its edges are the critical points of \( f \) that join those components together. In addition, the edges of \( \Gamma \) (there are \( r \) of them) are labeled by the roots of unity.

This edge-labeled graph \( \Gamma \subset \Sigma_g \) is subject to some additional constraints. First, the cyclic order of labels around any vertex should be in agreement
with the cyclic order of roots of unity. Next, the complement of $\Gamma$ consists of $n$ topological disks, where $n$ is the length of the partition $\mu$. Indeed, the complement of $\Gamma$ corresponds to $f^{-1}(D_+)$ and $z = \infty$ is the only ramification point in $D_+$. The connected components of $f^{-1}(D_+)$ thus correspond to parts of $\mu$.

The partition $\mu$ can be reconstructed from the edge labels of $\Gamma$ as follows. Pick a cell $U_i$ in $f^{-1}(D_+)$. The length of the corresponding part $\mu_i$ of $\mu$ is precisely the number of times the map $f$ wraps the boundary $\partial U_i$ around the circle $S$. As we follow the boundary $\partial U_i$, we see the edge labels appear in a certain sequence. As we complete a full circle around $\partial U_i$, the edge labels will make exactly $\mu_i$ turns around $S$. It is natural to call this number $\mu_i$ the perimeter of the cell $U_i$. This perimeter is $(2\pi)^{-1}$ times the sum of angles between pairs of the adjacent edges on $\partial U_i$, where the angle is the usual angle in $(0, 2\pi)$ between the corresponding roots of unity, see Figure 17.

We call an edge-labeled embedded graph $\Gamma$ as above a branching graph. By the above correspondence, the number $\text{Hur}_g(\mu)$ is the number of genus $g$ branching graphs with $n$ cells of perimeter $\mu_1, \ldots, \mu_n$. As usual, in the trivial $d = 2$ case, those graphs have to be counted with automorphism factors.

It is this definition of Hurwitz numbers that we will use for the asymptotic analysis in the next lecture.
2.8

It may be instructive to consider an example of how this correspondence between coverings and graphs works. Consider the covering corresponding to factorization
\[(12) (13) (24) (14) (13) = (1243)\]
of the form \([10]\). The degree of this covering is \(d = 4\), it has \(r = 5\) ramification points, and the monodromy \(\mu = (4)\) around infinity. It follows that its genus is \(g = 1\). Let us denote the five finite ramification points by
\[\{a, b, c, d, e\} = \{1, e^{2\pi i / 5}, \ldots, e^{8\pi i / 5}\} .\]

The preimage of \(D_-\) on the torus \(\Sigma_1\) consists of 4 disks and the monodromies tell us which disk is connected to which at which critical point: for example, at the critical point lying over \(a\), the 1st disk is connected to the 2nd disk. This is illustrated in Figure 18 where, among the 3 preimages of any critical value, the one which is a critical point is typeset in boldface. Clearly, any disk in \(f^{-1}(D_-)\) has the alphabet \(\{a, b, c, d, e\}\) going counterclockwise around its boundary and, in particular, the cyclic order of the critical values on its boundary is in agreement with the orientation on \(\Sigma_1\).

Observe that the preimage \(f^{-1}(D_+)\) is one cell whose boundary is a 4-fold covering of the equator. In particular, the alphabet \(\{a, b, c, d, e\}\) is repeated 4 times around the boundary of \(f^{-1}(D_+)\). Finally, Figure 19 shows the branching graph translation of Figure 18.

![Figure 18: Preimage of \(f^{-1}(D_-)\) on the torus \(\Sigma_1\)](image)
2.9

Finally, a few remarks about how one can prove a formula like (12). This will necessarily be a very sketchy account; the actual details of the proof can be found in [7, 14], as well as in the original paper [3].

As mentioned before, the numbers like Hur$_g$($\mu$) a special case of in the integrals in the Gromov-Witten theory of $\mathbb{P}^1$, that is, certain intersections on the Kontsevich moduli space $\bar{M}_{g,d,n}(\mathbb{P}^1)$ of stable degree $d$ maps

$$f : C \to \mathbb{P}^1$$

from a varying $n$-pointed genus $g$ domain curve $C$ to the fixed target curve $\mathbb{P}^1$.

Since such a map can be composed with any automorphism of $\mathbb{P}^1$, we have a $\mathbb{C}^\times$-action on $\bar{M}_{g,d,n}(\mathbb{P}^1)$. A theory due to Graber and Pandharipande [12] explains how to localize the integrals in Gromov-Witten theory to the fixed points of the action of the torus $\mathbb{C}^\times$. These fixed point loci in $\bar{M}_{g,d,n}(\mathbb{P}^1)$ are, essentially, products of Deligne-Mumford spaces $\bar{M}_{g_i,n_i}$ for some $g_i$'s and $n_i$'s. Indeed, only very few maps are fixed by the action of the torus. Namely, for the standard $\mathbb{C}^\times$-action on $\mathbb{P}^1$ and an irreducible domain curve $C$ the only choices are the degree 0 constant maps to $\{0, \infty\} = (\mathbb{P}^1)^{\mathbb{C}^\times}$ or the degree $d$ map

$$\mathbb{P}^1 \ni z \mapsto z^d \in \mathbb{P}^1.$$

In general, the domain curve is allowed to be reducible, but still any torus-invariant map has to be of the above form on each component $C_i$ of $C$. Once all discrete invariants of the curve $C$ are fixed (that is, the combinatorics of its irreducible components, their genera and numbers of marked points on them) the remaining moduli parameters are only a choice of a bunch of
curves to collapse plus a choice of where to attach the non-collapsed \( \mathbb{P}^1 \)'s to them. That is, the torus-fixed loci are products of Deligne-Mumford spaces, modulo possible automorphisms of the combinatorial structure.

In this way integrals in the Gromov-Witten theory of \( \mathbb{P}^1 \) can be reduced, at least in principle, to computing intersections on \( \overline{M}_{g,n} \). An elegant localization analysis leading to the ELSV formula is presented in [7], see also [14].

3 Asymptotics in Hurwitz problem

3.1

Our goal now is to see how the Laplace transform (16) of the asymptotics (15) in the Hurwitz problem turns into Kontsevich’s combinatorial formula (8). The formulation of the Hurwitz problem in terms of branching graphs, see Section 2.7, looks promising. Indeed, a branching graph \( \Gamma \) is by definition embedded in a topological genus \( g \) surface \( \Sigma_g \) and it cuts \( \Sigma_g \) into \( n \) cells. Here the numbers \( g \) and \( n \) are the same as the indices in \( \overline{M}_{g,n} \), on the intersection theory on which we are trying to understand. Similarly, in Kontsevich’s formula we have a graph \( G \) embedded into \( \Sigma_g \) and cutting it into \( n \) cells. This graph \( G \), however, is a more modest object: it does not have any edge labels and it is allowed to have only 3-valent vertices.

Recall that we denote by \( G^3_{g,n} \) the set of all possible 3-valent graphs as in Kontsevich’s formula (8). Let us introduce two larger sets

\[
G^3_{g,n} \subset G^{\geq 3}_{g,n} \subset G_{g,n},
\]

on which, by definition, the 3-valence condition is weakened to allow vertices of valence 3 or more, and dropped altogether, respectively. The elements of \( G^{\geq 3}_{g,n} \) can be obtained from elements of \( G^3_{g,n} \) by contracting some edges. In particular, the set \( G^{\geq 3}_{g,n} \) is still a finite set. Similarly, denote by \( H_{g,\mu} \) the set of all branching graphs with given genus \( g \) and perimeter partition \( \mu \). Our first order of business is to construct a map

\[
H_{g,\mu} \to G^{\geq 3}_{g,n},
\]

which we call the *homotopy type* map. This map is the composition of the map

\[
H_{g,\mu} \to G_{g,n}
\]
which simply forgets the edge labels with the map
\[ G_{g,n} \to G_{g,n}^{\geq 3}, \]
which does the following. First, we remove all univalent vertices together with the incident edge. After that, we remove all the remaining 2-valent vertices joining their two incident edges. What is left, by construction, has only vertices of valence 3 and higher and still cuts \( \Sigma_g \) into \( n \) cells.

We remark that in the two exceptional cases \( (g, n) = (0, 1), (0, 2) \), which correspond to unstable moduli spaces, what we get in the end (a point and a circle, respectively) is not really an element of \( G_{g,n}^{\geq 3} \). In all other cases, however, we do get an honest element of \( G_{g,n}^{\geq 3} \). Figure 20 illustrates this procedure applied to the branching graph from Figure 17.

![Figure 20: The homotopy type of the branching graph from Figure 17](image)

3.2

Now let us make the following simple but important observation. Since the set \( G_{g,n}^{\geq 3} \) is finite and we are interested in the asymptotics of \( \text{Hur}_g(\mu) \) as \( \mu \to \infty \) while keeping \( g \) and \( n \) fixed, we can just do the asymptotics separately for each homotopy type and then sum over all possible homotopy types. The Laplace transform (16) will then be also expressed as a sum over all corresponding homotopy types \( G \) in \( G_{g,n}^{\geq 3} \).

We now claim that not only Kontsevich’s combinatorial formula (8) is the Laplace transformed asymptotics (16) but, in fact, the summation over
\( G \in G^3_{g,n} \) in Kontsevich’s formula corresponds precisely to summation over possible homotopy types. Since there are non-trivalent homotopy types, implicit in this claim is the statement that non-trivalent homotopy types do not contribute to asymptotics.

3.3

What do we need to do to get the asymptotics of the number of branching graphs of a given homotopy type \( G \)? What would suffice is to have a simple way to enumerate all such branching graphs. To enumerate all branching graphs with given homotopy type \( G \), we need to retrace the steps of the homotopy type map. Imagine that the homotopy type graph \( G \) is a fossil from which we want to reconstruct some prehistoric branching graph \( \Gamma \). What are the all possible ways to do it?

The answer to all these rhetoric questions is quite simple. It is easy to see that the preimage of any edge in \( G \) is some subtree in the original branching graph \( \Gamma \). In addition, all these trees carry edge-labels which were erased by the homotopy type map. Thus, for any edge \( e \) of \( G \), we need to take a tree \( T_e \) whose edges are labeled by roots of unity. In particular, there is a canonical way to make this tree planar, that is, embed it in the plane in such a way that the cyclic order of edges around each vertex agrees with the order of their labels. In particular, each such tree is a branching tree, that is, it satisfies the \((g,n) = (0,1)\) case of our definition of a branching graph\(^3\).

Next, these trees are to be glued into the graph \( \Gamma \) by identifying some of their vertices, as in Figure 21. This means that each of these branching trees carries two special vertices, which we call its root and top. These special vertices of \( T_e \) mark the places where \( T_e \) is attached to the other trees in \( \Gamma \). We will call such a branching tree with two marked vertices an edge tree.

3.4

Now we have a procedure which from a homotopy type \( G \) and a collection of edge trees \( \{T_e\} \) with distinct labels assembles a branching graph \( \Gamma \). This procedure, which we will call assembly, does have some imperfections. Those imperfections will be discussed momentarily, but first we want to make the following important observation.

\(^3\)A small and inessential detail is that the labels \( T_e \) are taken from a larger set of roots of unity.
Since the homotopy type graph $G$ is something fixed and finite, the whole asymptotics of the branching graphs lies in the edge trees. For a large random branching graph $\Gamma$, those edge trees will be large random trees. This is how the theory of random trees enters the scene. Fortunately, a large random tree is a very well studied and a very nicely behaved object, see for example [16] for a particularly enjoyable introduction. It turns out that all the information we need about random trees is either already classical or can be easily deduced from known results.

In fact, all required knowledge about random trees can be quite easily deduced (as was done in [14]) from the first principles, which in this case, is the following formula going back to Cayley [17]. Consider all possible trees $T$ with the vertex set $\{1, \ldots, m\}$. For any such tree $T$, we have a function $\text{val}_T(i)$ which takes the vertex $i = 1, \ldots, m$ to its valence in $T$. The information about all vertex valences in all possible trees $T$ is encoded in the following generating function

$$\sum_T z_1^{\text{val}_T(1)} \cdots z_m^{\text{val}_T(m)} = z_1 \cdots z_m (z_1 + \cdots + z_m)^{m-2}. \quad (17)$$

A probabilistic restatement of this result is the following. The valence $\text{val}_T(i)$ is the number of edges of $T$ incident to the vertex $i$. Let us cut all edges in half; since there were $m - 1$ edges of $T$, we get $2m - 2$ half edges. The formula (17) says that the same distribution of half-edges can be obtained as follows: give every vertex a half-edge and the remaining $m - 2$ edges just
throw at the vertices randomly like darts.

What is then the valence of a given vertex in a random tree $T$? It is 1 for the half edge allowance that it always gets plus its share in the random distribution of $m - 2$ darts among $m$ targets. As $m \to \infty$, this share goes to a Poisson random variable with mean 1. In other words, as $m \to \infty$ we have

$$\text{Prob}\{\text{val}_T(i) = v\} \to \frac{e^{-1}}{(v-1)!}, \quad v = 1, 2, \ldots . \quad (18)$$

For different vertices, their valences become independent in the $m \to \infty$ limit.

Also, setting all variables in (17) to 1 we find that the total number of trees with vertex set $\{1, \ldots, m\}$ is $m^{m-2}$.

### 3.5

Now it is time to talk about how the assembly map differs from being one-to-one (it clear that it is onto).

First, it may happen that the cyclic order of edge labels is violated at one of the vertices of $G$ where we patch together different edge trees. If this is the case, we simply declare the assembly to be a failure and do nothing. The probability of such an assembly failure in the large graph limit can be computed as follows. Suppose that we need to glue together three vertices with valences $v_1$, $v_2$, and $v_3$. From (18), the chance of seeing these particular valences is

$$e^{-3} \frac{1}{(v_1 - 1)!(v_2 - 1)!(v_3 - 1)!}.$$ 

On the other hand, the conditional probability that the edge labels in the resulting graph are cyclically ordered, given that they were cyclically ordered before gluing is easily seen to be

$$\frac{(v_1 - 1)!(v_2 - 1)!(v_3 - 1)!}{(v_1 + v_2 + v_3 - 1)!}.$$

Hence the success rate of the assembly at a particular trivalent vertex is

$$e^{-3} \sum_{v_1, v_2, v_3 \geq 1} \frac{1}{(v_1 + v_2 + v_3 - 1)!} = \frac{e^{-2}}{2}.$$
Assembly failures at distinct vertices being asymptotically independent events, this goes into an overall factor and, eventually in the prefactor in \((8)\).

At this point it should be clear that there in no need to consider nontrivalent vertices. Indeed, a homotopy type graph with a vertex of valence \(\geq 4\) can be obtained from a trivalent graph by contracting some edges, hence corresponds to the case when some of the edge graphs are trivial. It is obvious that the chances that a large random tree came out empty are negligible. Hence, nontrivalent graphs make indeed no contribution to the asymptotics and can safely be ignored.

3.6

The second (minor) issue with the assembly map is that we can get the same, that is, isomorphic branching graphs starting from different collections of the edge trees. This happens if the homotopy type graph \(G\) has nontrivial automorphisms. It is clear that the group \(\text{Aut}(G)\) acts on edges of \(G\) and, hence, acts by permutations on collections of edge trees preserving the isomorphism class of the assembly output. It is also clear that the chance for a large edge tree to be isomorphic to another edge tree (or to itself with root and top permuted) is, asymptotically, zero. Hence almost surely this \(\text{Aut}(G)\) action is free and hence there is an overcounting of branching graphs by exactly a factor of \(|\text{Aut}(G)|\). This explains the division by \(|\text{Aut}(G)|\) in \((8)\).

3.7

Now, after explaining the summation over trivalent graphs and the automorphism factor in \((8)\), we get to the heart of Kontsevich’s formula — the product over the edges.

It is at this point that the convenience (promised in Section 2.6) of working with the Laplace transform \((16)\) rather then the asymptotics \((15)\) itself can be appreciated. We will see shortly that, asymptotically, the cell perimeters of a branching graph \(\Gamma\) assembled from a 3-valent graph \(G\) and bunch of random edge trees \(\{T_e\}\) is a sum of independent contributions from each edge of \(G\). This makes the Laplace transform \((16)\) factor over the edges of \(G\) as in \((8)\). To justify the above claim, we need to take a closer look at a large typical edge tree.
Let $T$ be an edge tree. It has two marked vertices, root and top; let us call
the path joining them the *trunk* of $T$. The tree $T$ naturally splits into 3
parts: the root component, the top component, and the trunk component,
according to their closest trunk point. This is illustrated in Figure 22.

Figure 22: The components of an edge tree

Figure 22 may give a wrong idea of the relative size of these components
for a typical large edge tree. Let $M \to \infty$ be the size (e.g. the number
of vertices) in $T$. It is known and can be without difficulty deduced from
(17) (see for example [14]) that the size $M$ distributes itself among the three
components of $T$ in the $M \to \infty$ limit as follows.

First, the size of the root and the top component stays finite in the
$M \to \infty$ limit. In fact, it goes to the *Borel distribution*, given by the following
formula

$$\text{Prob}(k) = \frac{k^{k-1} e^{-k}}{k!}, \quad k = 1, 2, \ldots.$$  

Second, the typical size of the trunk is of order $\sqrt{M}$. More precisely, scaled
by $\sqrt{M}$, the trunk size distribution goes to the Rayleigh distribution with
density

$$x e^{-x^2/2} \, dx, \quad x \in (0, \infty).$$

For our purposes, however, it only matters that the size of all these parts is
$o(M)$ as $M \to \infty$.
The overwhelming majority of vertices lie, therefore, somewhere in the branches of the trunk component of $T$. What is very important is that, after assembly, any such vertex will find itself completely surrounded by a unique cell. As a result, it will contribute exactly 1 to that cell’s perimeter. What this analysis shows it that, asymptotically, the cell perimeters are determined simply by the number of such interior trunk vertices ending up in a given cell, all other contributions to perimeters being $o(M)$. It should be clear that such contributions of distinct edges of $G$ are indeed independent, leading to the factorization in (8).

3.9

What remains is to determine is what the edge factors are, that is, to determine the actual contribution of an edge tree $T$ to the perimeters of the adjacent cells.

All we need to know for this is to know how vertices in the trunk component distribute themselves between the two sides of the trunk, as in Figure 22. One shows, see [14] and below, that the fraction of the vertices that land on a given side of the trunk is, asymptotically, uniformly distributed on $[0,1]$. This reduces the computation of the edge factor to computing one single integral. That computation will be presented in a moment, after we review the knowledge that we have accumulated so far.

3.10

Let $G$ be a 3-valent map with $n$ cells. It has

$$|E(G)| = 6g - 6 + 3n$$

edges and

$$|V(G)| = 4g - 4 + 2n$$

vertices, which follows from the Euler characteristic equation

$$|V(G)| - |E(G)| + n = 2 - 2g$$

combined with the 3-valence condition $3|V(G)| = 2|E(G)|$.

Let $e \in E(G)$ be an edge of $G$ and let $T_e$ be the corresponding edge tree. Let $d_e$ be the number of vertices of $T_e$. Ignoring the few vertices on the trunk
itself, the vertices of $T_e$ distribute themselves between the two sides of the 
trunk of $T_e$. Let’s say that $p_e$ vertices are on the one side and define the 
number $q_e$ by

$$p_e + q_e = d_e. \quad (19)$$

It is clear that $q_e$ is the approximate number of vertices on the other side of 
the trunk. We call the numbers $p_e$ and $q_e$ the semiperimeters of the tree $T_e$.

The basic question, which we now can answer in the large graph limit, 
is how many branching graphs $\Gamma$ have given semiperimeters \{(pe, qe)\}e∈E(G).
This distribution can be computed asymptotically as follows.

3.11

First, there are some overall factors that come from automorphisms of $G$ 
and the assembly success rate. Recall that in Section 3.5 we saw that the 
assembly success rate is $e^{-2|V(G)||2^{-|V(G)|}}$.

Second, for every edge $e \in E(G)$ we need to pick an edge tree $T_e$ with $d_e$ 
vertices. As we already learned from (17), the number of vertex-labeled trees 
with $d_e$ vertices is $d_e^{d_e - 2}$. Vertex labels can be traded for edge labels at the 
expense of the factor $d_e!/(d_e - 1)! = d_e$, hence there are $d_e^{d_e - 3}$ edge labeled 
trees with $d_e$ vertices. The choice of the root vertex brings in additional factor 
of $d_e$ choices. Once the root is fixed, the condition (19) dictates the pos 
tion of the top, so there is no additional freedom in choosing it. To summarize, 
there are $\sim d_e^{d_e - 2}$ edge trees with given semiperimeters $p_e$ and $q_e$.

Third, the edge labels of $\Gamma$ is a shuffle of edge labels of the trees $T_e$. Let

$$r = \sum_e (d_e - 1)$$

be the total number of edges in $\Gamma$ (and, hence, also the total number of simple 
branch points in the Hurwitz covering corresponding to $\Gamma$). Obviously, there are

$$r! \prod_{e \in E(G)} (d_e - 1)!$$

ways to shuffle edge labels of \{T_e\} into edge labels of $\Gamma$.

Putting it all together, we obtain the following approximate expression for 
the number of branching graphs with given semiperimeters \{(pe, qe)\}e∈E(G)

$$\frac{r! e^{-2|V(G)||2^{-|V(G)|}}}{|\text{Aut}(G)|} \prod_{e \in E(G)} \frac{d_e^{d_e - 2}}{(d_e - 1)!} \sim \frac{r! e^d 2^{-|V(G)|}}{|\text{Aut}(G)|} \prod_{e \in E(G)} \frac{1}{\sqrt{2\pi d_e^{d_e/2}}}, \quad (21)$$
where
\[ d = |V(\Gamma)| = \sum_e d_e - 2|V(G)| \]
is the degree of the corresponding Hurwitz covering and the RHS of (21) is obtained from the LHS by the Stirling formula.

Note that the factor \( r!e^d \) precisely cancels with prefactor in (15).

3.12

Since the cell perimeters of \( \Gamma \) are the sums of edge tree semiperimeters along the boundaries of the cells, the computation of the Laplace transform (16) indeed boils down to the computation of a single edge factor

\[ \frac{1}{\sqrt{2\pi}} \int_{p,q>0} \frac{e^{-ps_1-qs_2}}{(p+q)^{3/2}} dp dq = \frac{1}{\sqrt{2\pi}} \frac{1}{s_1-s_2} \int_{x>0} (e^{-s_1x} - e^{-s_2x}) \frac{dx}{x^{3/2}} \]
\[ = \frac{1}{\sqrt{2\pi}} \Gamma \left( -\frac{1}{2} \right) \frac{\sqrt{s_1-s_2}}{s_1-s_2} \]
\[ = \frac{\sqrt{2}}{\sqrt{s_1} + \sqrt{s_2}}, \]

where we set \( x = p + q \).

Recall that the relation between the Laplace transform variables \( s_i \) in (16) and the variables \( z_i \) in Kontsevich’s generation function (7) is
\[ z_i = \sqrt{2s_i}, \quad i = 1, \ldots, n. \]
Thus we get indeed the LHS of (8), including the correct exponent of 2, which is
\[ |E(G)| - |V(G)| = 2g - 2 + n. \]
This completes the proof of Kontsevich’s formula (8).

4 Remarks

4.1

Since random matrices are the common thread of many talks at this school, let us point out various connections between moduli of curves and random
matrices. As we already discussed, the original KdV conjecture of Witten was based on physical parallelism between intersection theory on $\overline{M}_{g,n}$ and the double scaling limit of the Hermitian 1-matrix model. Despite many spectacular achievements by physicists as well as mathematicians, this double scaling seems to remain a source of serious mathematical challenges, in particular, it appears that no direct mathematical connection between it and moduli of curves is known. On the other hand, there is a very direct connection between what we did and another, much simpler, matrix model, namely, the edge scaling of the standard GUE model. This connection goes as follows.

Recall that by Wick formula the coefficients of the $1/N$ expansion of the following $N \times N$ Hermitian matrix integral

$$\int e^{-\text{tr} H^2} \prod_1^m \text{tr} H^{k_i} dH$$

are the numbers of ways to glue a surface of given topology from $m$ polygons with perimeters $k_1, \ldots, k_n$. The double scaling limit of the 1-matrix model is concerned with gluing a given surface out of a very large number of small pieces. An opposite asymptotic regime is when the number $m$ of pieces stays fixed while their sizes $k_i$ go to infinity. Since for large $k$ the trace $\text{tr} H^k$ picks out the maximal eigenvalues of $H$, this asymptotic regime is about largest eigenvalues of a Hermitian random matrix. In the large $N$ limit, the distribution of largest eigenvalues of $H$ is well known to be the Airy ensemble. This edge scaling random matrix ensemble is very rich, yet susceptible to a very detailed mathematical analysis. In particular, the individual distributions of eigenvalues were found by Tracy and Widom in [18]. They are given in terms of certain solutions of the Painlevé II equation.

The connection between GUE edge scaling and what we were doing is the following. If one takes a branching graph as in Figure 17 and strips it off its edge labels, one gets a map on genus $\Sigma_g$ with a few cells of very large perimeter, that is, an object of precisely the kind enumerated by (22) in the edge scaling regime. We argued that almost all vertices of a large branching graph are completely surrounded by a unique cell, hence contribute exactly 1 to that cell’s perimeter regardless of the edge labels. This shows that edge labels play no essential role in the asymptotics, thus establishing a direct connection between Hurwitz numbers asymptotics and GUE edge scaling. A similar direct connection can be established in other situations, for example, between GUE edge scaling and distribution of long increasing subsequences.
in a random permutation, see [10]. Since a great deal is known about GUE edge scaling, one can profit very easily from having a direct connection to it. In particular, one can give closed error-function-type formulas for a natural generating functions (known as \( n \)-point functions) for the numbers [13], see [13].

There exists another matrix model, namely the Kontsevich’s matrix model [9], specifically designed to reproduce the graph summation in (8) as its diagrammatic expansion. Once the combinatorial formula (8) is established, this Kontsevich’s model can be used to analyze it, in particular, to prove the KdV equations, see [9] and also [2].

Alternatively, the KdV equations can be pulled back from the GUE edge scaling (where they have been studied in depth by Adler, Shiota, and van Moerbeke) via the above described connection, see the exposition in [13].

4.2

In our approach, the intersections (5), the combinatorial formula (8), the KdV equations etc. appear through the asymptotic analysis of the Hurwitz problem. The ELSV formula (12), which is the bridge between enumeration of branched coverings and the intersection theory of \( \overline{M}_{g,n} \) is, on the hand, an exact formula. It is, therefore, natural to ask for more exact bridges between intersection theory, combinatorics, and integrable systems.

After the moduli space \( \overline{M}_{g,n} \) of stable curves, a natural next step is the Gromov-Witten theory of \( \mathbb{P}^1 \), that is, the intersection theory on the moduli space \( \overline{M}_{g,n}(\mathbb{P}^1,d) \) of stable degree \( d \) maps

\[
C \rightarrow \mathbb{P}^1
\]

from an \( n \)-pointed genus \( g \) curve \( C \) to the projective line \( \mathbb{P}^1 \). More generally, one can replace \( \mathbb{P}^1 \) by some higher genus target curve \( X \). It turns out, see [15], that there is a simple dictionary, which we call the Gromov-Witten-Hurwitz correspondence, between enumeration of branched coverings of \( \mathbb{P}^1 \) and Gromov-Witten theory of \( \mathbb{P}^1 \). This correspondence naturally connects with some very beautiful combinatorics and integrable systems, the role of random matrices now being played by random partitions. The connection with the integrable systems is seen best in the equivariant Gromov-Witten theory of \( \mathbb{P}^1 \), where the 2-Toda lattice hierarchy of Ueno and Takasaki plays the role that KdV played for \( \overline{M}_{g,n} \).
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