A note on minor antichains of uncountable graphs

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Abstract
A simplified construction is presented for Komjáth’s result that for every uncountable cardinal $\kappa$, there are $2^\kappa$ graphs of size $\kappa$ none of them being a minor of another.

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antichain, minor, stationary sets, well-quasi ordering

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1 | INTRODUCTION

The famous Robertson–Seymour Theorem asserts that the class of finite graphs is well-quasi-ordered under the minor relation $\leq$: For every sequence $G_1, G_2, \ldots$ of finite graphs there are indices $i < j$ such that $G_i \leq G_j$.

This is no longer true for arbitrary infinite graphs. Thomas [7] has constructed a sequence $G_1, G_2, \ldots$ of binary trees with tops of size continuum, such that $G_i \not\leq G_j$ whenever $i < j$.

Here, binary tree with tops describes the class of graphs where one selects in the rooted infinite binary tree $T_2$ a collection $\mathcal{R}$ of rays all starting at the root, adds for each $R \in \mathcal{R}$ a new vertex $v_R$, and makes $v_R$ adjacent to all vertices on $R$. Let us write $G(\mathcal{R})$ for the resulting graph. In his proof, Thomas carefully selects continuum-sized collections of rays $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots$ such that $G_i = G(\mathcal{R}_i)$ form the desired bad sequence.

1Recall that a graph $H$ is a minor of another graph $G$, written $H \leq G$, if to every vertex $x \in H$ we can assign a (possibly infinite) connected set $V_x \subset V(G)$, called the branch set of $x$, so that these sets $V_x$ are pairwise disjoint and $G$ contains a $V_x - V_y$ edge whenever $xy$ is an edge of $H$. This is an open access article under the terms of the Creative Commons Attribution-NonCommercial-NoDerives License, which permits use and distribution in any medium, provided the original work is properly cited, the use is non-commercial and no modifications or adaptations are made. © 2022 The Authors. Journal of Graph Theory published by Wiley Periodicals LLC.
Thomas’s result raises the question whether infinite graphs smaller than size continuum are well-quasi ordered. While this question for countable graphs is arguably the most important open problem in infinite graph theory, Komjáth [2] has established that for all other (uncountable) cardinals $\kappa$, there are in fact $2^\kappa$ pairwise minor-incomparable graphs of size $\kappa$.

The purpose of this note is to give an alternative construction for Komjáth’s result which is simpler than the original, and also more integrated with other problems in the area:

First, our construction reinstates a pleasant similarity to Thomas’s original strategy: The desired minor-incomparable graphs can already be found amongst the $\kappa$-regular trees with $\kappa$ many tops. Second, our construction bears a surprising similarity to a family of rays considered in the 60’s by A.H. Stone in his work on Borel isomorphisms [5]. And finally, a very similar family of graphs had recent applications for results about normal spanning trees in infinite graphs [4].

2 | TREES WITH TOPS AND STONE’S EXAMPLE

Consider the order tree $(T, \preceq)$ where the nodes of $T$ are all sequences of elements of $\kappa$ of length $\leq \omega$ including the empty sequence, and let $t \preceq t'$ if $t$ is a proper initial segment of $t'$. The graph on $T$ where any two comparable vertices are connected by an edge was considered by Kriz and Thomas in [3] where they showed that any tree-decomposition of this graph must have a part of size $\kappa$, despite not containing a subdivision of an uncountable clique.

For our purposes, however, it suffices to consider a graph $G$ on $T$ such that any node represented by finite sequences of length $n$ is connected to all its successors of length $n + 1$ in the tree order $\preceq$, and any node represented by an $\omega$-sequence is connected to all elements below in the tree order $\preceq$. Clearly, $G$ is connected. We later use the simple fact that

(i) every connected subgraph $H \subset G$ has a unique minimal node $t_H$ in $(T, \preceq)$.

Now given a set $S \subset \kappa$ consisting just of cofinality $\omega$ ordinals, choose for each $s \in S$ a cofinal sequence $f_s : \omega \to s$, and let $F = F(S) := \{f_s : s \in S\}$ be the corresponding collection of sequences in $\kappa$. Let $T^S$ denote the subtree of $T$ given by all finite sequences in $T$ together with $F(S)$, and let $G(S)$ denote the corresponding induced subgraph of $G$. We will refer to $G(S)$ as a ‘$\kappa$-regular tree with tops’, where the elements of $F(S)$ are of course the ‘tops’.

To the author's best knowledge, such a collection of tree branches $F(S) = \{f_s : s \in S\}$ for $S$ the set of all cofinality $\omega$ ordinals was first considered by Stone in [5, §5] for the case $\kappa = \omega_1$ and in [6, §3.5] for the general case of uncountable regular $\kappa$.

We consider below graphs $G(S)$ where $S \subset \kappa$ is stationary. Recall that a subset $A \subset \kappa$ is stationarv if $\sup A = \kappa$, and closed if $\sup(A \cap \ell) = \ell$ implies $\ell \in A$ for all limits $\ell < \kappa$. The set $A$ is a club in $\kappa$ if it is both closed and unbounded. A subset $S \subset \kappa$ is stationary (in $\kappa$) if $S$ meets every club of $\kappa$. Below, we use the following two elementary properties of stationary sets of regular uncountable cardinals $\kappa$ (for details see e.g. [[1], §8]):

• If $S \subset \kappa$ is stationary and $S = \bigcup\{S_n : n \in \mathbb{N}\}$, then some $S_n$ is stationary.
• Fodor’s lemma: If $S \subset \kappa$ is stationary and $f : S \to \kappa$ is such that $f(s) < s$ for all $s \in S$, then there is $i < \kappa$ such that $f^{-1}(i)$ is stationary.
3 CONSTRUCTING FAMILIES OF MINOR-INCOMPARABLE GRAPHS

At the heart of Komjáth’s proof [2] lies the construction, for regular uncountable \( \kappa \), of \( \kappa \) pairwise minor-incomparable connected graphs of cardinality \( \kappa \). From this, the singular case follows, and by considering disjoint unions of these graphs, one may obtain an antichain of size \( 2^\kappa \). Here, we will prove directly the maximum bound in the regular case.

**Theorem 1.** For regular uncountable \( \kappa \), the class of \( \kappa \)-regular trees with \( \kappa \) many tops contains a minor-antichain of size \( 2^\kappa \).

**Proof.** As the cofinality \( \omega \) ordinals of a regular uncountable \( \kappa \) split into \( \kappa \) many disjoint stationary subsets [1, Lemma 8.8], it is routine and well-known that there is a family \( \Sigma \) of \( 2^\kappa \) stationary subsets consisting of cofinality \( \omega \) ordinals such that for any \( \Sigma \neq \emptyset \), the differences \( \Delta S \) and \( \Delta R \) are still stationary, cf. [8, Proposition 1.1. We claim that the family \( \Sigma \subset GS \) is the desired antichain. Towards to aim, it clearly suffices to show: If \( \Sigma \) and \( \Sigma \) are disjoint stationary subsets consisting of cofinality \( \omega \) ordinals, then \( \Sigma \prec \Sigma \).

Suppose for a contradiction that \( \Sigma \prec \Sigma \). For ease of notation, we identify \( s \) with \( f(s) \) for all \( s \in S \), and similarly for \( R \). For \( v \in \Sigma \) write \( \Sigma(v) \) for the by (i) unique minimal node of the branch set of \( v \) in \( \Sigma \). Note that if \( v, w \) are adjacent in \( \Sigma \), then \( v \) and \( w \) are comparable in \( \Sigma \). Since \( \Sigma \) has countable height, by (ii) there is a stationary subset \( S' \subset S \) such that all \( t_s \) for \( s \in S' \) belong to the same level of \( \Sigma \). Suppose for a contradiction this level has finite height \( n \). By applying Fodor’s lemma (iii) iteratively \( n + 1 \) times, we obtain a stationary subset \( S'' \subset S' \) such that all \( f_s \) for \( s \in S'' \) agree on \( f_s(i) \) for \( i < n \). So distinct \( t_s \) for \( s \in S'' \) have at least \( n + 1 \) common neighbours below them in \( \Sigma \), a contradiction.

Thus, we may assume that \( t_s \in \Sigma \) for all \( s \in S \), giving rise an injective function \( f : S \to R, s \to t_s \). Since \( f \) is injective, we cannot have \( f(x) < x \) on a stationary subset of \( S \) by Fodor’s lemma (iii). Hence, we may further assume that \( f(x) \geq x \) for all \( x \in S \).

For \( \kappa < \ell \) let \( \Sigma(\ell) \) be the subtree of \( \Sigma \) of all elements whose coordinates are strictly less than \( \ell \), and consider the function \( g : \kappa \to \kappa, i \mapsto \min \{ j < \kappa : t_v \in \Sigma(\ell) \text{ for all } v \in \Sigma(\ell) \} \). Since \( \kappa \) is regular, the function \( g \) is well-defined. And clearly, \( g \) is increasing. The function \( g \) is also continuous. Indeed, for a limit \( \ell < \kappa \) consider any \( v \in \Sigma(\ell) \setminus \bigcup_{i < \ell} \Sigma(i) \). Clearly, \( v \) is a top, and so all its neighbours belong to \( \bigcup_{i < \ell} \Sigma(i) \). Hence, \( t_v \) must be comparable to infinitely many nodes in \( \bigcup_{i < \ell} \Sigma(\ell) \), implying that \( t_v \in \bigcup_{i < \ell} \Sigma(\ell) \), too.

Hence, the set of fixed points \( C \) of \( g \) forms a club in \( \kappa \), see [1, Exercise 8.1]. But any \( s \in S \cap C \) satisfies \( \leq f(s) \leq g(s) = s \), showing that \( s = f(s) \in S \cap R \), a contradiction. \( \square \)

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Data sharing is not applicable to this article as no new data were created or analysed in this study.
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