ALGEBRA IN THE SUPEREXTENSIONS OF GROUPS, III:
MINIMAL LEFT IDEALS OF $\lambda(\mathbb{Z})$

T.BANAKH, V.GAVRYLKIV

Abstract. We prove that the minimal left ideals of the superextension $\lambda(\mathbb{Z})$ of the discrete group $\mathbb{Z}$ of integers are metrizable topological semigroups, topologically isomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$ of the compact group $\mathbb{Z}_2$ of integer 2-adic numbers.

The superextension $\lambda(X)$ of a discrete group $X$ is the compact Hausdorff right-topological semigroup consisting of maximal linked systems on $X$ and endowed with the semigroup operation $A \ast B = \{ A \subseteq X : \{ x \in X : x^{-1}A \in B \} \in A \}$.

Introduction

After the topological proof (see [HS, p.102], [H2]) of Hindman theorem [H1], topological methods become a standard tool in the modern combinatorics of numbers, see [HS], [P]. The crucial point is that any semigroup operation $\ast$ defined on any discrete space $X$ can be extended to a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of $X$. The extension of the operation from $X$ to $\beta(X)$ can be defined by the simple formula:

\[
\mathcal{U} \ast \mathcal{V} = \left\{ \bigcup_{x \in U} x \ast V_x : U \in \mathcal{U}, \{ V_x \}_{x \in U} \subseteq \mathcal{V} \right\},
\]

where $\mathcal{U}, \mathcal{V}$ are ultrafilters on $X$. Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [HS], [P].

The Stone-Čech compactification $\beta(X)$ of $X$ is the subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$, which is a complete lattice with respect to the operations of union and intersection. In [G2] it was observed that the semigroup operation extends not only to $\beta(X)$ but also to the complete sublattice $G(X)$ of $\mathcal{P}(\mathcal{P}(X))$, generated by $\beta(X)$. This complete sublattice consists of all inclusion hyperspaces over $X$.

By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space $X$ is called an inclusion hyperspace if $\mathcal{F}$ is monotone in the sense that a subset $A \subseteq X$ belongs to $\mathcal{F}$ provided $A$ contains some set $B \in \mathcal{F}$. Besides the operations of union and intersection, the set $G(X)$ possesses an important transversality operation assigning to each inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

\[
\mathcal{F}^\perp = \{ A \subseteq X : \forall F \in \mathcal{F} \, (A \cap F \neq \emptyset) \}.
\]

This operation is involutive in the sense that $(\mathcal{F}^\perp)^\perp = \mathcal{F}$.

1991 Mathematics Subject Classification. 20M12, 54B99, 54H10.
It is known that the family $G(X)$ of inclusion hyperspaces on $X$ is closed in the double power-set $\mathcal{P}(\mathcal{P}(X)) = \{0,1\}^{\mathcal{P}(X)}$ endowed with the natural product topology. The induced topology on $G(X)$ can be described directly: it is generated by the sub-base consisting of the sets

$$U^+ = \{\mathcal{F} \in G(X) : U \in \mathcal{F}\} \quad \text{and} \quad U^- = \{\mathcal{F} \in G(X) : U \in \mathcal{F}^\perp\}$$

where $U$ runs over subsets of $X$. Endowed with this topology, $G(X)$ becomes a Hausdorff supercompact space. The latter means that each cover of $G(X)$ by the sub-basic sets has a 2-element subcover.

The extension of a binary operation $*$ from $X$ to $G(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two inclusion hyperspaces $U, V \in G(X)$. In $[G2]$ it was shown that for an associative binary operation $*$ on $X$ the space $G(X)$ endowed with the extended operation becomes a compact right-topological semigroup. Besides the Stone-Čech extension, the semigroup $G(X)$ contains many important spaces as closed subsemigroups. In particular, the space

$$\lambda(X) = \{\mathcal{F} \in G(X) : \mathcal{F} = \mathcal{F}^\perp\}$$

of maximal linked systems on $X$ is a closed subsemigroup of $G(X)$. The space $\lambda(X)$ is well-known in General and Categorial Topology as the superextension of $X$, see $[BM]$, $[TZ]$. Endowed with the extended binary operation, the superextension $\lambda(X)$ of a semigroup $X$ is a supercompact right-topological semigroup containing $\beta(X)$ as a subsemigroup.

The thorough study of algebraic properties of the superextensions $\lambda(X)$ of groups $X$ was started in $[BGN]$ and continued in $[BGX]$. In this paper we concentrate at describing the minimal (left) ideals of $\lambda(X)$.

Understanding the structure of minimal left ideals of the semigroup $\beta(X)$ had important combinatorial consequences. For example, properties of ultrafilters from a minimal left ideal of $\beta(X)$ were exploited in the topological proof of the classical Van der Waerden Theorem $[HS, 14.3]$ due to Fustenberg and Katznelson $[FK]$. Minimal left ideals of the semigroup $\beta(\mathbb{Z})$ play also an important role in Topological Dynamics, see $[BB]$, $[BF]$, $[HS]$ Ch.19. We believe that studying the structure of minimal (left) ideals of the semigroups $\lambda(X)$ also will have some combinatorial or dynamical consequences.

The main result of this paper is Theorem 5.1 asserting that the minimal left ideals of the semigroup $\lambda(\mathbb{Z})$ are compact metrizable topological semigroups topologically isomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z}_2)$ of the (compact metrizable) group $\mathbb{Z}_2$ of integer 2-adic numbers.

1. Right-topological semigroups

In this section we recall some information from $[HS]$ related to right-topological semigroups. By definition, a right-topological semigroup is a topological space $S$ endowed with a semigroup operation $*: S \times S \to S$ such that for every $a \in S$ the right shift $r_a : S \to S$, $r_a : x \mapsto x \ast a$, is continuous. If the semigroup operation $*: S \times S \to S$ is continuous, then $(S, *)$ is a topological semigroup.

A non-empty subset $I$ of a semigroup $S$ is called a left (resp. right) ideal if $SI \subset I$ (resp. $IS \subset I$). If $I$ is both a left and right ideal in $S$, then $I$ is called an ideal in $S$. Observe that for every $x \in S$ the set $Sx = \{sx : s \in S\}$ (resp. $xS = \{xs : s \in S\}$) is a left (resp. right) ideal in $S$. Such an ideal is called principal. An ideal $I \subset S$
is called \textit{minimal} if any ideal of \( S \) that lies in \( I \) coincides with \( I \). By analogy we define minimal left and right ideals of \( S \). It is easy to see that each minimal left (resp. right) ideal \( I \) is principal. Moreover, \( I = Sx \) (resp. \( I = xS \)) for each \( x \in I \).

If \( S \) is a compact Hausdorff right-topological semigroup, then each minimal left ideal in \( S \), being principal, is closed in \( S \). By [HS 2.6], each left ideal in \( S \) contains a minimal left ideal. The union of all minimal left ideals of \( S \) coincides with the minimal ideal \( K(S) \) of \( S \), [HS 2.8]. By [HS 2.11], all the minimal left ideals of \( S \) are mutually homeomorphic.

An element \( z \) of a semigroup \( S \) is called a \textit{right zero} in \( S \) if \( xz = z \) for all \( x \in S \). It is clear that \( z \in S \) is a right zero in \( S \) if and only if the singleton \( \{z\} \) is a (minimal) left ideal in \( S \).

In the sequel we shall often use the following

\textbf{Lemma 1.1.} Let \( X, Y \) be compact right-topological semigroups. If a semigroup homomorphism \( h : X \to Y \) is injective on some minimal left ideal of \( X \), then \( h \) is injective on each minimal left ideal of \( X \).

\textit{Proof.} Assume that \( h \) is injective on a minimal left ideal \( Xa \) of \( X \) and take any other minimal left ideal \( Xb \) of \( X \). By [HS 2.11], the right shift \( r_a : X \to X \), \( r_a : x \mapsto xa \), is injective on \( Xb \). Next, consider the right shift \( r_{h(a)} : Y \to Y \), \( r_{h(a)} : y \mapsto y \cdot h(a) \). It follows from the equality \( h \circ r_a = r_{h(a)} \circ h \) and the injectivity of the maps \( r_a | Xb \) and \( h | Xa \) that the map \( h | Xb \) is injective. \( \square \)

2. INCLUSION HYPERSPACES AND SUPEREXTENSIONS

A family \( \mathcal{L} \) of subsets of a set \( X \) is called a \textit{linked system on} \( X \) if \( A \cap B \neq \emptyset \) for all \( A, B \in \mathcal{L} \). Such a linked system \( \mathcal{L} \) is \textit{maximal linked} if \( \mathcal{L} \) coincides with any linked system \( \mathcal{L}' \) on \( X \) that contains \( \mathcal{L} \). Each (ultra)filter on \( X \) is a (maximal) linked system. A linked system \( \mathcal{L} \) on \( X \) is maximal linked if and only if for any partition \( X = A \cup B \) either \( A \) or \( B \) belongs to \( \mathcal{L} \).

By \( \lambda(X) \) we denote the family of all maximal linked systems on \( X \). Since each ultrafilter on \( X \) is a maximal linked system, \( \lambda(X) \) contain the Stone-\v{C}ech extension \( \beta(X) \) of \( X \). It is easy to see that each maximal linked system on \( X \) is an inclusion hyperspace on \( X \) and hence \( \lambda(X) \subset G(X) \). Moreover, it can be shown that \( \lambda(X) = \{ A \in G(X) : A = A^\perp \} \). Let also \( N_2(X) = \{ A \in G(X) : A \subset A^\perp \} \) denote the family of all linked inclusion hyperspaces on \( X \). By [Gr] both the subspaces \( \lambda(X) \) and \( N_2(X) \) are closed in the compact Hausdorff space \( G(X) \).

Each function \( f : X \to Y \) between sets \( X, Y \) induces a continuous map \( Gf : G(X) \to G(Y) \) assigning to an inclusion hyperspace \( A \in G(X) \) the inclusion hyperspace

\[ Gf(A) = \{ B \subset Y : f^{-1}(B) \in A \} \in G(Y). \]

The function \( Gf \) maps \( \lambda(X) \) into \( \lambda(Y) \), so we can put \( \lambda f = Gf | \lambda(X) \).

Given any semigroup operation \( * : X \times X \to X \) on a set \( X \) we can extend this operation to \( G(X) \) letting

\[ \mathcal{U} \ast \mathcal{V} = \left\{ \bigcup_{x \in U} x \ast V_x : U \in \mathcal{U}, \{ V_x \}_{x \in U} \subset \mathcal{V} \right\} \]

for inclusion hyperspaces \( \mathcal{U}, \mathcal{V} \in G(X) \). Equivalently, the product \( \mathcal{U} \ast \mathcal{V} \) can be defined as

\[ \mathcal{U} \ast \mathcal{V} = \{ A \subset X : \{ x \in X : x^{-1} A \in \mathcal{V} \} \in \mathcal{U} \} \]
where \( x^{-1}A = \{ z \in X : x \neq z \in A \} \). By \( \mathbb{G}_2 \) the so-extended operation turns \( G(X) \) into a right-topological semigroup. The structure of this semigroup was studied in details in \( \mathbb{G}_2 \). In particular, it was shown that for each group \( X \) the minimal left ideals of \( G(X) \) are singletons containing invariant inclusion hyperspaces.

We call an inclusion hyperspace \( A \subset G(X) \) invariant if \( xA = A \) for all \( x \in X \). More generally, given a subgroup \( H \subset X \) we define \( A \) to be \( H \)-invariant if \( xA = A \) for all \( x \in H \).

It follows from the definition of the topology on \( G(X) \) that the set \( G(X) \) of invariant inclusion hyperspaces is closed in \( G(X) \) and coincides with the minimal ideal \( K(G(X)) \) of the semigroup \( G(X) \). Consequently, \( K(G(X)) \) is a closed rectangular subsemigroup of \( G(X) \). The rectangularity of \( K(G(X)) \) means that \( A \circ B = B \) for all \( A, B \in K(G(X)) \).

### 3. The minimal ideal of \( \lambda(G) \) for odd groups

In this section we characterize groups \( G \) whose superextension \( \lambda(G) \) has one-point minimal left ideals.

Following \( \mathbb{BGN} \), we define a group \( G \) to be odd if the order of each element \( x \) of \( G \) is odd. If \( G \) is a finite odd group, then the maximal linked system

\[
L = \{ A \subset G : |A| > |G|/2 \}
\]

is invariant. In fact, a group \( G \) possesses an invariant maximal linked system if and only if \( G \) is odd, see Theorem 3.2 of \( \mathbb{BGN} \). By Proposition 3.1 of \( \mathbb{BGN} \), a maximal linked system \( Z \in \lambda(G) \) on a group \( G \) is invariant if and only if \( Z \) is a right zero of the semigroup \( \lambda(G) \) if and only if the singleton \( \{ Z \} \) is a minimal left ideal in \( \lambda(G) \). Taking into account that the invariant maximal linked systems form a closed rectangular subsemigroup of \( \lambda(G) \), we obtain the main result of this section.

**Theorem 3.1.** A group \( G \) is odd if and only if all the minimal left ideals of \( \lambda(G) \) are singletons. In this case the minimal ideal \( K(\lambda(G)) \) of \( \lambda(G) \) is a closed rectangular semigroup consisting of invariant maximal linked systems.

Given a subgroup \( H \) of a group \( G \) let \( G/H = \{ xH : x \in G \} \) and \( \pi : G \to G/H \) denote the quotient map. It induces a continuous map \( \lambda\pi : \lambda(G) \to \lambda(G/H) \) between the corresponding superextensions.

**Lemma 3.2.** For any \( H \)-invariant maximal linked system \( A \in \lambda(H) \subset \lambda(G) \) the restriction of \( \lambda \pi : \lambda(G) \to \lambda(G/H) \) to the principal left ideal \( \lambda(G) \ast A \) is injective.

**Proof.** Fix a section \( s : G/H \to G \) of \( \pi \). For every \( L \in \lambda(G) \) let \( \tilde{L} = \lambda\pi(L) \in \lambda(G/H) \) be the projection of \( L \) onto \( G/H \) and \( M = \lambda s(\tilde{L}) \in \lambda(G) \) be the lift of \( \tilde{L} \) by the section \( s \).

We claim that \( L \ast A = M \ast A \). Since \( L \ast A \) and \( M \ast A \) are maximal linked systems, it suffices to check that \( L \ast A \subset M \ast A \). Take any set \( \bigcup_{x \in L} x \ast A_x \in L \ast A \) where \( L \in \mathcal{L} \) and \( \{ A_x \}_{x \in L} \subset A \). Consider the set \( \mathcal{M} = s \circ \pi(L) \in \mathcal{M} \). For every point \( y \in \mathcal{M} \) find a point \( x_y \in L \) with \( y = s \pi(x_y) \) and observe that \( yH = \pi(y) = \pi(x_y) = x_y H \), which implies \( y^{-1}x_y \in H \) and hence \( y^{-1}x_y A_{x_y} \in A \) by the \( H \)-invariantness of \( A \). Since

\[
\mathcal{M} \ast A \ni \bigcup_{y \in \mathcal{M}} y(y^{-1}x_y \ast A_{x_y}) = \bigcup_{y \in \mathcal{M}} x_y \ast A_{x_y} \subset \bigcup_{x \in L} x \ast A_x
\]
we conclude that $\bigcup_{x \in L} x \ast A_x \in \mathcal{M} \ast A$.

Now we are able to prove that $\lambda \pi : \lambda(G) \ast A \to \lambda(G/H)$ is injective. Take any two distinct elements $L_1 \ast A \neq L_2 \ast A$ of $\lambda(G) \ast A$. For every $i \in \{1, 2\}$ consider the maximal linked systems $\tilde{L}_i = \lambda \pi(L_i) = \lambda \pi(L_i \ast A)$ and $M_i = \lambda s(\tilde{L}_i)$. It follows from $M_1 \ast A \neq L_2 \ast A$ that $M_1 \neq M_2$ and hence

$$\lambda \pi(L_1 \ast A) = \tilde{L}_1 \neq \tilde{L}_2 = \lambda \pi(L_2 \ast A).$$

□

**Corollary 3.3.** For a normal odd subgroup $H$ of a group $G$ the map $\lambda \pi : \lambda(G) \to \lambda(G/H)$ is injective on each minimal left ideal of $\lambda(G)$. Consequently, every minimal left ideal of $\lambda(G)$ is topologically isomorphic to a minimal left ideal of $\lambda(G/H)$.

**Proof.** By Lemma 1.1, it suffices to show that $\lambda \pi$ is injective on some minimal left ideal. The group $H$, being odd, admits an $H$-invariant maximal linked system $A \in \lambda(H) \subset \lambda(G)$. By Lemma 3.2 the homomorphism $\lambda \pi$ is injective on the left ideal $\lambda(G) \ast A$ and hence is injective on any minimal left ideal contained in $\lambda(G) \ast A$ (it exists because $\lambda(G)$ is a compact right-topological semigroup). □

4. Maximal invariant linked systems on groups

As we have seen in the preceding section, the property of a maximal system $L \in \lambda(G)$ to be invariant is very strong and forces $L$ to be a right zero of $\lambda(G)$.

Such maximal linked systems exist only on odd groups.

On the other hand, maximal invariant linked systems exist on each group. An invariant linked inclusion hyperspace $L \in \mathcal{N}_2(G)$ is called a **maximal invariant linked system** if $L = L'$ for any invariant linked inclusion hyperspace $L' \in \mathcal{N}_2(G)$ enlarging $L$. By the Zorn Lemma, each invariant linked inclusion hyperspace can be enlarged to a maximal invariant linked system.

**Proposition 4.1.** For any maximal invariant linked system $L_0$ on a group $G$ the set

$$\uparrow L_0 = \{ L \in \lambda(G) : L \supset L_0 \}$$

is a left ideal in $\lambda(G)$.

**Proof.** Let $A, B \in \lambda(X)$ be maximal linked systems with $L_0 \subset B$. Then for every subset $L \in L_0$ we get

$$L = \bigcup_{x \in G} x(x^{-1}L) \in A \ast L_0 \subset A \ast B$$

which means that $L_0 \subset A \ast B$. □

Observe that $L_0 \subset L \subset L_0^+$ for every $L \in \uparrow L_0$. The following theorem shows that the difference $L_0^+ \setminus L_0$ (and consequently, $L \setminus L_0$) is relatively small (for the group $G = \mathbb{Z}$ it is countable!).

**Theorem 4.2.** If $L_0$ is a maximal invariant linked system on an Abelian group $G$, then for any subset $A \in L_0^+ \setminus L_0$ there is a point $x \in G$ such that $xA = G \setminus A$ and consequently, $A = x^2A$. 

Proof. Fix a subset $A \in L_0^+ \setminus L_0$. We claim that
\begin{equation}
(2) \quad aA \cap A = \emptyset
\end{equation}
for some $a \in G$. Assuming the converse, we would conclude that the family \{x \in G : x \in A\} is linked and then the invariant linked system $L_0 \cup \{x \in G : x \in A\}$ is strictly larger than $L_0$, which impossible because of the maximality of $L_0$.

Next, we find $b \in G$ with
\begin{equation}
(3) \quad A \cup bA = G.
\end{equation}
Assuming that no such a point $b$ exist, we conclude that for any $x, y \in G$ the union $xA \cup yA \neq G$. Then $(G \setminus xA) \cap (G \setminus yA) = G \setminus (xA \cup yA) \neq \emptyset$, which means that the family \{G \setminus xA : x \in G\} is linked and invariant. We claim that $G \setminus A \in L_0^+$. Assuming the converse, we would conclude that $G \setminus A$ misses some set $L \in L_0$. Then $L \subset A$ and hence $A \in L_0$ which is not the case. Thus $G \setminus A \in L_0^+$ and hence \{G \setminus xA : x \in G\} because $L_0^+$ is invariant. Since $L_0 \cup \{G \setminus xA : x \in G\}$ is an invariant linked system containing $L_0$, the maximality of $L_0$ guarantees that $G \setminus A \in L_0$ which contradicts $A \in L_0^+$.

Finally we show that $G \setminus A = aA = bA$. Observe that (2) and (3) imply that $aA \subset bA$ and hence $A \subset a^{-1}bA$. On the other hand, (2) and (3) are equivalent to $a^{-1}A \cap A = \emptyset$ and $b^{-1}A \cap A = G$, which implies $a^{-1}A \subset b^{-1}A$ and yields $ba^{-1}A \subset A$. Unifying this inclusion with $A \subset a^{-1}bA = ba^{-1}A$, we conclude that $ba^{-1}A = A$ and hence $bA = aA$. Now looking at (2) and (3) we see that $G \setminus A = aA = bA$. \qed

5. Minimal left ideals of $\lambda(\mathbb{Z})$

In this section we apply the results of the preceding sections to describe the structure of minimal left ideals of the semigroup $\lambda(\mathbb{Z})$. It turns out that they are isomorphic to minimal left ideals of the superextension $\lambda(\mathbb{Z})$ of the compact topological group $\mathbb{Z}$ of integer 2-adic numbers. We recall that $\mathbb{Z} = \lim_{\to} C_{2^k}$ is a totally disconnected compact metrizable Abelian group, which is the limit of the inverse sequence
\[ \cdots \to C_{2^n} \to \cdots \to C_8 \to C_4 \to C_2 \]
of cyclic 2-groups $C_{2^n}$. Let $\pi : \mathbb{Z} \to \mathbb{Z}$ denote the canonic (injective) homomorphism of $\mathbb{Z}$ into $\mathbb{Z}$ (induced by the quotient maps $\pi_{2^k} : \mathbb{Z} \to \mathbb{Z}/2^k\mathbb{Z} = C_{2^k}$, $k \in \mathbb{N}$).

By the continuity of the functor $\lambda$ in the category of compact Hausdorff spaces (see [TZ 2.3.2]), the superextension $\lambda(\mathbb{Z})$ can be identified with the limit of the inverse sequence
\[ \cdots \to \lambda(C_{2^n}) \to \cdots \to \lambda(C_8) \to \lambda(C_4) \to \lambda(C_2) \]
of finite semigroups $\lambda(C_{2^n})$. This implies that $\lambda(\mathbb{Z})$ is a metrizable zero-dimensional compact topological semigroup.

Theorem 5.1. The homomorphism $\lambda \pi : \lambda(\mathbb{Z}) \to \lambda(\mathbb{Z})$ is injective on each minimal left ideal of $\lambda(\mathbb{Z})$. Consequently, the minimal left ideals of the semigroup $\lambda(\mathbb{Z})$ are compact metrizable topological semigroups.

Proof. By Lemma [1.1] it suffices to check that the homomorphism $\lambda \pi$ is injective on some minimal left ideal of $\lambda(\mathbb{Z})$. Fix any maximal invariant linked system $L_0$ on $\mathbb{Z}$ (such a system exists by Zorn Lemma). By Proposition [4.3] the set $\{L \in \lambda(\mathbb{Z}) : L \supset L_0\}$ is a left ideal which necessarily contains a minimal left ideal
$I$ of $\lambda(\mathbb{Z})$. We claim that the homomorphism $\lambda\pi : \lambda(\mathbb{Z}) \to \lambda(\mathbb{Z}_2)$ is injective on $I$. Given two different maximal linked system $A, B \in I$ we need to check that $\lambda\pi(A) \neq \lambda\pi(B)$.

Since the superextension $\lambda(\mathbb{Z}_2)$ is the limit of the inverse sequence

$$
\ldots \to \lambda(C_{2n}) \to \ldots \to \lambda(C_8) \to \lambda(C_4) \to \lambda(C_2),
$$

the inequality $\lambda\pi(A) \neq \lambda\pi(B)$ will follow as soon as we find $k \in \mathbb{N}$ such that $\lambda\pi_{2^k}(A) \neq \lambda\pi_{2^k}(B)$ where $\lambda\pi_{2^k} : \lambda(\mathbb{Z}) \to \lambda(C_{2^k})$ is the homomorphism induced by the quotient homomorphism $\pi_{2^k} : \mathbb{Z} \to C_{2^k}$.

Pick any set $A \in A \setminus B$. Since $A \in L_0 \setminus L_0$, we can apply Theorem 4.2 to conclude that $A = 2n + A$ for some positive number $n \in \mathbb{Z}$. The later equality means that $A = \pi_{2n}^{-1}(\pi_{2n}(A))$ is the complete preimage of the set $\pi_{2n}(A)$ under the quotient homomorphism $\pi_{2n} : \mathbb{Z} \to \mathbb{Z}/2n\mathbb{Z} = C_{2n}$. It follows that $\pi_{2n}(A) \in \lambda\pi_{2n}(A) \setminus \lambda\pi_{2n}(B)$ and hence $\lambda\pi_{2n}(A) \neq \lambda\pi_{2n}(B)$.

Write the number $2n$ as the product $2n = 2^k \cdot m$ for some odd number $m$ and find a (unique) subgroup $H \subseteq C_{2n}$ of order $|H| = m$. It follows that the quotient group $C_{2n}/H$ can be identified with the cyclic 2-group $C_{2^k}$ so that $q \circ \pi_{2n} = \pi_{2^k}$ where $q : C_{2n} \to C_{2n}/H = C_{2^k}$ is the quotient homomorphism. Corollary 3.3 guarantees that the homomorphism $\lambda q : \lambda(C_{2n}) \to \lambda(C_{2^k})$ is injective on each minimal left ideal of $\lambda(C_{2n})$. In particular, it is injective on the minimal left ideal $\lambda\pi_{2n}(I)$. Consequently, $\lambda\pi_{2^k}(A) = \lambda q(A) \neq \lambda q(B) = \lambda\pi_{2^k}(B)$. This completes the proof of the injectivity of $\lambda\pi : \lambda(\mathbb{Z}) \to \lambda(\mathbb{Z}_2)$ on the left ideal $I$ and consequently, on each minimal left ideal $J$ of $\lambda(\mathbb{Z})$.

Since minimal left ideals of $\lambda(\mathbb{Z})$ are compact, the restriction $\lambda\pi|J$ is a topological isomorphism of $J$ onto the minimal left ideal $\lambda\pi(J)$ of $\lambda(\mathbb{Z}_2)$. Since $\lambda(\mathbb{Z}_2)$ is a metrizable topological semigroup, so are the semigroups $\lambda\pi(J)$ and $J$. $\square$

6. Some Open Problems

We saw in Theorem 3.1 that the minimal ideal $K(\lambda(G))$ of the superextension of an odd group $G$ is a compact topological semigroup.

**Problem 6.1.** Characterize groups $G$ such that the minimal ideal $K(\lambda(G))$ is closed in $\lambda(G)$. Is the minimal ideal $K(\lambda(\mathbb{Z}))$ closed in $\lambda(\mathbb{Z})$? Is $K(\lambda(\mathbb{Z}))$ a topological semigroup?

**Problem 6.2.** Characterize groups $G$ such that the minimal left ideals of $\lambda(G)$ are (metrizable) topological semigroups.
References

[BB] B. Balcar, A. Blass, On minimal dynamical systems on Boolean algebras // Commentat. Math. Univ. Carol. 31:1 (1990), 7–11.

[BF] B. Balcar, F. Franek, Structural properties of universal minimal dynamical systems for discrete semigroups // Trans. Am. Math. Soc. 349:5 (1997), 1697–1724.

[BGN] T. Banakh, V. Gavrylkiv, O. Nykyforchyn, Algebra in superextensions of groups, I: zeros and commutativity // preprint (arXiv: 0802.1853).

[BG2] T. Banakh, V. Gavrylkiv, Algebra in superextension of groups, II: cancelativity and centers // preprint (arXiv: 0802.1856).

[FK] H. Furstenberg, Y. Katznelson, Idempotents in compact semigroups and Ramsey Theory // Israel J. Math. 68 (1989), 257–270.

[G1] V. Gavrylkiv, The spaces of inclusion hyperspaces over noncompact spaces // Matem. Studii. 28:1 (2007), 92–110.

[G2] V. Gavrylkiv, Right-topological semigroup operations on inclusion hyperspaces // Matem. Studii (to appear).

[H1] N. Hindman, Finite sums from sequences within cells of partition of \( \mathbb{N} \) // J. Combin. Theory Ser. A 17 (1974), 1–11.

[H2] N. Hindman, Ultrafilters and combinatorial number theory // Lecture Notes in Math. 751 (1979), 49–184.

[HS] N. Hindman, D. Strauss, Algebra in the Stone–Čech compactification, de Gruyter, Berlin, New York, 1998.

[TZ] A. Teleiko, M. Zarichnyi, Categorical Topology of Compact Hausdorff Spaces, VNTL, Lviv, 1999.

[P] I. Protasov, Combinatorics of Numbers, VNTL, Lviv, 1997.

[vM] J. van Mill, Supercompactness and Wallman spaces, Math Centre Tracts. 85. Amsterdam: Math. Centrum., 1977.

[Ve] A. Verbeek, Superextensions of topological spaces. MC Tract 41, Amsterdam, 1972.