QUANTIZATION OF CONIC LAGRANGIAN SUBMANIFOLDS OF COTANGENT BUNDLES

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Abstract. Let $M$ be a manifold and $\Lambda$ a compact exact connected Lagrangian submanifold of $T^*M$ (with Maslov class 0). We can associate with $\Lambda$ a conic Lagrangian submanifold $\Lambda'$ of $T^*(M \times \mathbb{R})$. We prove that there exists a canonical sheaf $F$ on $M \times \mathbb{R}$ whose microsupport is $\Lambda'$ outside the zero section. We deduce the already known result that the projection from $\Lambda$ to $M$ induces isomorphisms between the homotopy groups.

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1. Introduction

Let $M$ be a $C^\infty$ manifold and $\Lambda$ a closed conic Lagrangian submanifold of $T^*M$, the cotangent bundle of $M$ with the zero section removed. Motivated by the paper [11] where D. Tamarkin used the microlocal theory of sheaves of M. Kashiwara and P. Schapira to obtain results in symplectic geometry (see [11] and the survey [4]), we consider the problem of constructing a sheaf on $M$ whose microsupport coincides with $\Lambda$ outside the zero section. We call such a sheaf a “quantization” of $\Lambda$. As explained to me by C. Viterbo it is possible to define a quantization of $\Lambda$ by means of Floer homology, under some topological assumptions (see [13]), and, conversely, to recover some aspects of Floer homology from a quantization (for example, analogs of the spectral invariants are introduced in [12] using a quantization).

The main purpose of this paper is to construct a quantization using only the microlocal theory of sheaves. We also deduce from the existence of the quantization known results of Fukaya-Seidel-Smith [2] and Abouzaid [1] on the topology of $\Lambda$. However we assume, as in [2] and [1], that $\Lambda$ has Maslov class 0. This is proved by Kragh [8] but we do not know for the moment how to recover this result with our methods. We also point out that the link between the microlocal theory of sheaves and the symplectic geometry is studied in another way in [10] [9].

In order to understand topological obstructions for this problem (the Maslov class of $\Lambda$ and some relative Stiefel-Whitney class) we give a description of a stack over $\Lambda$ that we call the Kashiwara-Schapira stack. In the first part we study the Kashiwara-Schapira stack of $\Lambda$ when $\Lambda$ is a locally closed conic Lagrangian submanifold of $T^*M$. In the second and third parts we construct a quantization of $\Lambda$ when $\Lambda$ is a closed conic Lagrangian submanifold of $T^*(M \times \mathbb{R})$ obtained by adding a variable from a closed compact exact Lagrangian submanifold of $T^*M$ with Maslov class zero and relative Stiefel-Whitney class zero. Using this quantization we recover the results of [2] and [1] mentioned above.

Let us first recall one of the main ingredients of the microlocal theory of sheaves (see [5] [6] [7]), namely, the microsupport of sheaves. Let $k$ be a commutative unital ring of finite global dimension. We denote
by $\mathcal{D}^b(k_M)$ the bounded derived category of sheaves of $k$-modules on $M$. In loc. cit. the authors attach to an object $F$ of $\mathcal{D}^b(k_M)$ its singular support, or microsupport, $SS(F)$, a closed subset of $T^*M$. We recall its definition in Section 2. The microsupport is conic for the action of $(\mathbb{R}^+, \times)$ on $T^*M$ and is co-isotropic. It gives some information on how the cohomology groups $H^i(U; F)$ vary when the open subset $U \subset M$ moves. The microsupport was introduced as a tool for the study of sheaves. In [11] Tamarkin uses it in the other direction: if a given conic Lagrangian submanifold $\Lambda$ of $\dot{T}^*M$ admits a quantization $F \in \mathcal{D}^b(k_M)$, then we may use the cohomology of $F$ or its extension groups to obtain results on $\Lambda$. Tamarkin constructs quantizations of Lagrangian submanifolds of $T^*SU(N)$ associated with subsets of the complex projective space, in particular the real projective space and the Clifford torus. He deduces non-displaceability results for these subsets. In [3], building on Tamarkin’s ideas, the authors consider a Hamiltonian isotopy, say $\Phi: \dot{T}^*M \times \mathbb{I} \to \dot{T}^*M$, homogeneous for the $\mathbb{R}^+$-action on $T^*M$. We can see its graph as a conic Lagrangian submanifold of $\dot{T}^*(M \times M \times \mathbb{I})$. The authors prove that this graph admits a quantization and they deduce a new proof of a non-displaceability conjecture of Arnold and results on non-negative isotopies.

1.1. Sheaf of microlocal germs. Let us explain the results of the first part of this paper. Let $S$ be a subset of $T^*M$. Following [7] we denote by $\mathcal{D}^b_S(k_M)$ the full triangulated subcategory of $\mathcal{D}^b(k_M)$ formed by the $F$ such that $SS(F) \subset S$. We let $\mathcal{D}^b(k_M; S)$ be the quotient $\mathcal{D}^b(k_M)/\mathcal{D}^b_{T^*M \setminus S}(k_M)$. We also denote by $\mathcal{D}^b_S(k_M)$ the full triangulated subcategory of $\mathcal{D}^b(k_M)$ formed by the $F$ such that $SS(F) \cap \Omega \subset S$, for some neighborhood $\Omega$ of $S$. Let $\Lambda$ be a locally closed conic submanifold of $\dot{T}^*M$. We define the Kashiwara-Schapira stack of $\Lambda$, denoted $\mathcal{G}(k_\Lambda)$, as the stack associated with the prestack $\mathcal{G}^0_\Lambda$ given as follows. For $\Lambda_0$ open in $\Lambda$ the objects of $\mathcal{G}^0_\Lambda(\Lambda_0)$ are those of $\mathcal{D}^b_{\Lambda_0}(k_M)$. The morphisms between two objects $F, G$ are

$$\text{Hom}_{\mathcal{G}^0_\Lambda(\Lambda_0)}(F, G) := \text{Hom}_{\mathcal{D}^b_{(\Lambda_0)}}(F, G).$$

Our main result in the first part is that $\mathcal{G}(k_\Lambda)$ is equivalent to a stack of twisted local systems on $\Lambda$. We prove it in the three main steps described below. For $p \in \Lambda$ we have the Lagrangian subspaces of $T_pT^*M$ given by $\lambda_\Lambda(p) = T_p\Lambda$ and $\lambda_0(p) = T_p\pi^{-1}(p)$, where $\pi: T^*M \to M$ is the projection. Let $\sigma: \mathcal{L}_M \to T^*M$ be the Lagrangian Grassmannian of $T^*M$ and let $U_\Lambda$ be the open subset of $\mathcal{L}_M|_\Lambda$ formed by the $l$ which are transversal to $\lambda_\Lambda(p)$ and $\lambda_0(p)$, where $p = \sigma(l)$. 
Our first step is the definition of a functor
\[ m_\Lambda : \text{D}^b_{(-)}(k_M) \to \text{D}^b(k_{U_\Lambda}). \]
We call \( m_\Lambda(F) \) the sheaf of microlocal germs of \( F \). To explain the definition of \( m_\Lambda(F) \) we first recall that \( \text{SS}(F) \) is the closure of the set of \( p = (x; \xi) \) satisfying: there exists \( \varphi : M \to \mathbb{R} \) with \( d\varphi_x = \xi \) such that \( (R\Gamma_{\varphi \geq 0}F)_x \neq 0 \). We assume that \( F \in \text{D}^b_{(-)}(k_M) \) and \( p \in \Lambda \). We set \( l = T_p\Lambda_\varphi \), where \( \Lambda_\varphi = \{ (x; d\varphi_x) \} \). It is proved in [7] that, if \( l \in U_\Lambda \), then \( (R\Gamma_{\varphi \geq 0}F)_x \) only depends on \( l \). We set \( m_{\Lambda,l}(F) = (R\Gamma_{\varphi \geq 0}F)_x \). It is also proved in [7] that all \( m_{\Lambda,l}(F) \), for \( l \in U_\Lambda \), are isomorphic, up to a shift given by the Maslov index. If \( m_{\Lambda,l}(F) \) is concentrated in one degree, then \( F \) is said pure along \( \Lambda \). If \( m_{\Lambda,l}(F) \simeq k[d] \) for some \( d \in \mathbb{Z} \), then \( F \) is said simple along \( \Lambda \). We prove that it is possible to define \( m_\Lambda(F) \in \text{D}^b(k_{U_\Lambda}) \) with stalks \( m_{\Lambda,l}(F) \) and with locally constant cohomology sheaves. We let \( \text{DL}(k_{U_\Lambda}) \) be the stack associated with the subprestack of \( \text{D}^b(k_{U_\Lambda}) \) formed by the complexes with locally constant cohomology sheaves. Then we obtain a functor \( m_\Lambda : \mathcal{G}(k_\Lambda) \to \text{DL}(k_{U_\Lambda}). \)

The second step is to understand the link between the \( m_\Lambda(F)|_U \) for the different connected components \( U \) of \( U_\Lambda \). We let \( U_{\Lambda}^n \) be the fiber product of \( U_\Lambda \) over \( \Lambda \), \( n \) times. If \( I \subset [1,n] \) is a set of indices, we let \( q_I : U_{\Lambda}^n \to U_{\Lambda}^{[I]} \) be the projection to the corresponding factors. We introduce the Maslov sheaf of \( \Lambda \), \( \mathcal{M}_\Lambda \in \text{DL}(k_{U_\Lambda^2}) \), obtained as \( \mathcal{M}_\Lambda = m_U(\mathcal{K}_\Lambda) \), where \( U \) is a neighborhood of the diagonal of \( \Lambda \times \Lambda \) and \( \mathcal{K}_\Lambda \in \mathcal{G}(k_U) \) is a canonical object defined on \( U \). For \( (l,l') \in U_{\Lambda}^2 \) we can see that \( (\mathcal{M}_\Lambda)(l,l') \simeq k[d] \), where \( d \in \mathbb{Z} \) is given by some Maslov index associated with \( (l,l') \). Moreover we have isomorphisms
\[ q_{12}^{-1}\mathcal{M}_\Lambda \otimes q_{23}^{-1}\mathcal{M}_\Lambda \sim q_{13}^{-1}\mathcal{M}_\Lambda \quad \text{and} \quad \mathcal{M}_\Lambda \otimes q_2^{-1}m_\Lambda(F) \sim q_1^{-1}m_\Lambda(F), \]
for any \( F \in \mathcal{G}(k_\Lambda) \), which satisfy natural commutative diagrams. In particular, for any \( p \in \Lambda \) and any connected components \( U, V \subset U_\Lambda \cap \sigma^{-1}(p) \) the restriction \( m_\Lambda(F)|_U \) determines \( m_\Lambda(F)|_V \). We introduce the stack \( \mathcal{S}_{mg}(k_\Lambda) \) of pairs \((L,u)\), where \( L \in \text{DL}(k_{U_\Lambda}) \) and \( u \) is an isomorphism \( \mathcal{M}_\Lambda \otimes q_2^{-1}L \sim q_1^{-1}L \) satisfying the same diagram as \( m_\Lambda(F) \). Then we prove that \( m_\Lambda \) induces an equivalence \( m'_\Lambda : \mathcal{G}(k_\Lambda) \sim \mathcal{S}_{mg}(k_\Lambda) \).

The third step is a description of the monodromy of \( m_\Lambda(F) \). Let \( l \in U_\Lambda \) and set \( p = \sigma(l) \). Since \( l \) is transversal to \( \lambda_0(p) \) and \( \lambda_\Lambda(p) \) we have a decomposition \( T_p\Lambda \approx \lambda_0(p) \oplus \lambda_\Lambda(p) \). It gives a projection \( T_p\Lambda \to \lambda_\Lambda(p) \) whose restriction to \( \lambda_0(p) \) is an isomorphism. We obtain in this way an isomorphism \( \nu_p(l) : \lambda_0(p) \sim \lambda_\Lambda(p) \). For a vector space \( E \) of dimension \( n \) we set \( E^+ = E \oplus \Lambda^n E \). Then \( E^+ \) is canonically oriented. Let \( T_\Lambda \) be the fiber bundle over \( \Lambda \) with fiber the space of orientation preserving
isomorphisms $\text{Iso}^+(\lambda_1^n(p), \lambda_2^n(p))$. Then $l \mapsto u_p(l) \oplus \Lambda^n u_p(l)$, where $n$ is the dimension of $M$, gives an embedding

$$i_{U_A} : U_A \hookrightarrow \mathcal{I}_A.$$ 

The fiber of $\mathcal{I}_A$ is isomorphic to $GL^+_{n+1}(\mathbb{R})$. It is well-known that the fundamental group of $GL^+_{n+1}(\mathbb{R})$ is $\mathbb{Z}$ for $n = 1$ and $\mathbb{Z}/2\mathbb{Z}$ for $n > 1$. Hence it has a canonical morphism, say $\varepsilon$, to $\mathbb{Z}/2\mathbb{Z}$. We also have an obvious morphism $\mathbb{Z}/2\mathbb{Z} \to k^\times$, the multiplicative group of $k$. Composing these morphisms we define $\varepsilon' : \pi_1(U_A \cap \sigma^{-1}(p)) \to k^\times$, for any given $p \in \Lambda$. We prove that the monodromy of $\mathfrak{m}_A(F)$ along a loop $\gamma$ is the multiplication by $\varepsilon'(\gamma)$. Since the stalks $\mathfrak{m}_A(F)_l$ are independent of $l$, up to shift, we deduce the existence of $L_p \in DL(\mathcal{I}_A,p)$ such that, for any connected component $U$ of $U_A \cap \sigma^{-1}(p)$ there exist $d_U \in \mathbb{Z}$ and an isomorphism $\mathfrak{m}_A(F)|_U \simeq L_p|_{\mathfrak{m}_A(U)}[d_U]$. We also prove the following global version of this result. If the Maslov class of $\Lambda$ is zero and $F$ is simple, then there exist $L \in DL(\mathcal{I}_A)$ such that, for any connected component $U$ of $U_A$ there exist $d_U \in \mathbb{Z}$ and an isomorphism $\mathfrak{m}_A(F)|_U \simeq L|_{\mathfrak{m}_A(U)}[d_U]$.

In the case of the Maslov sheaf we deduce that there exists a local system $\mathcal{L}_A$ on $\mathcal{I}_A \times_A \mathcal{I}_A$, with stalks $k$ and monodromy given by $\varepsilon$, such that

$$\mathcal{M}_A|_U \sim \sim i^{-1}_{U_A} \mathcal{L}_A|_U[\tau_U]$$

for all connected component $U$ of $U^2_A$, where $\tau_U$ is given by the Maslov index. Moreover the isomorphism $q_{12}^{-1} \mathcal{M}_A \otimes q_{23}^{-1} \mathcal{M}_A \sim \sim q_{13}^{-1} \mathcal{M}_A$ extends to a similar isomorphism for $\mathcal{L}_A$. Using this we can make the link between $\mathfrak{m}_A(F)$ and $L$ canonical. It remains the problem of choosing the shifts $d_U$, but this is solved by pull-back to a covering of $\Lambda$. Our final result is the following. Let $r : \Lambda \to \Lambda$ be the covering of $\Lambda$ corresponding to the Maslov class and let $p : r^*\mathcal{I}_A \to \Lambda$ be the projection. Then $r^{-1}(\mathcal{S}(k\Lambda))$ is equivalent to the substack of $p_*DL(k_{r^*\mathcal{I}_A})$ of objects with monodromy $\varepsilon$. In particular if the Maslov class of $\Lambda$ is 0 and the image of $rw_2(\Lambda) \in H^2(\Lambda; (\mathbb{Z}/2\mathbb{Z})\Lambda)$ (see Definition $\S\S 3.3$) in $H^2(\Lambda; k\Lambda^\times)$ is 0, then $\mathcal{S}(k\Lambda)$ admits a global simple object.

1.2. $\mu$-sheaves. Now we describe the second part of this paper, where we construct a prestack of categories on the base with morphisms given by sections of $\mu\text{hom}$ outside the zero-section, where $\mu\text{hom}$ is Sato’s microlocalization functor (recalled in section $\S$). We assume that our manifold is of the form $M \times \mathbb{R}$. We let $(t; \tau)$ be the coordinates on $T^*\mathbb{R}$ and, for an open subset $U \subset M \times \mathbb{R}$, we set $T^r_{\tau>0}U = (T^rM \times \{(t; \tau) ; \tau > 0\}) \cap T^*U$. For $F \in D^b(k_U)$ we set $SS(F) = SS(F) \cap T^*U$. We denote
by \(D_{r>0}^b(k_U)\) the full subcategory of \(D^b(k_U)\) formed by the \(F\) such that \(SS(F) \subset T_{r>0}U\). We will define a prestack of triangulated categories on \(M \times \mathbb{R}\) denoted \(U \mapsto D^b(k_U^\mu)\), together with a functor of prestacks \(\Psi_U: D_{r>0}^b(k_U) \to D^b(k_U^\mu)\) such that for all \(F, G \in D_{r>0}^b(k_U)\) we have 
\[
\text{Hom}_{D^b(k_U^\mu)}(\Psi_U(F), \Psi_U(G)) \simeq H^0(\tilde{T}^*U; \mu\text{hom}(F, G)).
\]
Moreover any \(F \in D^b(k_U^\mu)\) is locally isomorphic to some \(\Psi_V(G)\). The category \(D^b(k_U^\mu)\) is intermediate between \(D^b(k_U; T_{r>0}^*U)\), which is used to define \(\mathcal{G}(k_A)\), and \(D^b(k_U)\), which is the category where we want to obtain our quantization of \(\Lambda\).

For any \(F \in D^b(k_U^\mu)\) and any relatively compact open subset \(V \subset U\) there exist \(\varepsilon_0 > 0\) and “restrictions” \(F_\varepsilon \in D^b(k_V)\), for \(\varepsilon \in \mathbb{R}\). We have \(SS((\Psi(U)(F)_\varepsilon) \subset SS(F) \cup T^*_\varepsilon(SS(F))\), where \(T^*_\varepsilon\) is the translation \(T^*_\varepsilon(x, t; \xi, \tau) = (x, t + \varepsilon; \xi, \tau)\). We have a gluing property in \(D^b(k_U^\mu)\): let \(U = \bigcup_{i \in I} U_i\) be an open covering, let \(F_i \in D^b(k_{U_i})\) and let \(F_i|_{U_{ij}} \simeq F_j|_{U_{ij}}\) be isomorphisms, for all \(i, j \in I\), satisfying the cocyle condition. If \(\text{Hom}(F_i, F_i[d]) = 0\) for all \(i \in I\) and all \(d < 0\), then we can glue the \(F_i\)’s into \(F \in D^b(k_U^\mu)\).

Now we give some details on the functor \(\Psi_U\) and the category \(D^b(k_U^\mu)\). One problem when we want to glue simple sheaves is that two non isomorphic simple sheaves in \(D^b(k_{M \times \mathbb{R}})\) may represent the same object in \(\mathcal{G}(k_A)\). For example we consider \(\Lambda = \{(x, 0; 0, \tau); \tau > 0\}\), \(F = k_{M \times [0, +\infty)}\) and \(G = k_{M \times (-\infty, 0]}[1]\). The microsupports of \(F\) and \(G\) are contained \(T^*_{M \times \mathbb{R}}(M \times \mathbb{R}) \cup \Lambda\), \(F\) and \(G\) are isomorphic in \(\mathcal{G}(k_A)\), but not in \(D^b(k_{M \times \mathbb{R}})\). However their convolutions with \(k_{[0, \varepsilon]}\) are isomorphic: \(F \star k_{[0, \varepsilon]} \simeq G \star k_{[0, \varepsilon]} \simeq k_{M \times [0, \varepsilon]}\). The convolution is defined for \(F \in D^b(k_{M \times \mathbb{R}})\) and \(F' \in D^b(k_{\mathbb{R}})\) by \(F \star F' := \text{RS}_s(F \boxtimes F')\), where \(s: M \times \mathbb{R}^2 \to M \times \mathbb{R}\) is the sum \((x, t, t') \mapsto (x, t + t')\). This is a general fact: for \(F, G \in D^b(k_{M \times \mathbb{R}})\), if \(F\) and \(G\) are isomorphic in \(D^b(k_{M \times \mathbb{R}}; \tilde{T}^*(M \times \mathbb{R}))\), then \(F \star k_{[0, \varepsilon]} \simeq G \star k_{[0, \varepsilon]}\). The idea is to introduce the category of sheaves which are of the form \(F \star k_{[0, \varepsilon]}\). If \(F \in D^b(k_U)\) for \(U \subset M \times \mathbb{R}\), then \(F \star k_{[0, \varepsilon]}\) is only defined over \(U \cap T_{\varepsilon}(U)\). Hence we have to consider all \(\varepsilon > 0\) at once. We define a functor \(\Psi_U: D^b(k_U) \to D^b(k_{U \times [0, +\infty)})\) by \(\Psi_U(F) = F \star k_\gamma\), where \(\gamma = \{(t, u) \in \mathbb{R} \times [0, +\infty]; 0 \leq t < u\}\). Then \(D^b(k_U^\mu)\) is the category of “boundary values” of sheaves on \(U \times \mathbb{R}\) which are locally of the form \(\Psi_V(F)\), for \(V\) open in \(U\) and \(F \in D^b_{r>0}(k_V)\).

1.3. Quantization. In the last part of the paper we built a quantization of a closed conic Lagrangian \(\Lambda \subset T^*_r(M \times \mathbb{R})\) satisfying some topological assumptions and we deduce that the projection \(\Lambda \to M\) is
a homotopy equivalence. We assume that $\Lambda/\mathbb{R}_{>0}$ is compact and that the map $T^*_\tau(\mathbb{M} \times \mathbb{R}) \to T^*\mathbb{M}$, $(x, t; \xi, \tau) \mapsto (x; \xi/\tau)$ gives an immersion of $\Lambda/\mathbb{R}_{>0}$ in $T^*\mathbb{M}$ (then $\Lambda$ can be recovered up to translation from its image in $T^*\mathbb{M}$ which is a compact exact Lagrangian submanifold).

Then, for a given pure global object $G \in \mathcal{S}(k_\Lambda)$, we prove that there exists $F \in \mathcal{D}^b(\mathcal{K}_{M \times \mathbb{R}})$ such that $SS(F) = \Lambda$ outside the zero-section and $F$ is mapped to $G$ by the functor $\mathcal{D}^b_0(k_{M \times \mathbb{R}})$ to $\mathcal{S}(k_\Lambda)$. We call this $F$ a quantization of $G$. We recall that the existence of a global object $G$ is treated in the first part of the paper.

We first prove the existence of $F' \in \mathcal{D}^b(\mathcal{K}_{M \times \mathbb{R}})$ with microsupport bounded by $\Lambda$ and which represents $G$. We proceed as follows. By the main result of [3] we can move $\Lambda$ by a Hamiltonian isotopy and assume that the projection $\Lambda/\mathbb{R}_{>0} \to \mathbb{M} \times \mathbb{R}$ is finite. We can find a finite open covering $\Lambda = \bigcup_{i \in I} \Lambda_i$ and open subsets $U_i \subset \mathbb{M} \times \mathbb{R}$ such that $\Lambda_i$ is a connected component of $\Lambda \cap T^*U_i$ and there exists $F^0_i \in \mathcal{D}^b_{(\Lambda_i)}(\mathcal{K}_{U_i})$ which is simple along $\Lambda_i$. Since $\Lambda/\mathbb{R}_{>0} \to \mathbb{M} \times \mathbb{R}$ is finite, we can in fact assume that $SS(F^0_i) \subset T^*_{\mathbb{M} \times \mathbb{R}}(\mathbb{M} \times \mathbb{R}) \cup \Lambda_i$. Then $G|_{\Lambda_i}$ is represented by $F_i = F^0_i \otimes L_i$, where $L_i$ is some local system on $U_i$, up to some shift. Since $G$ is pure we have $\mu_{\text{hom}}(F_i, F_j)|_{T^*U_{ij}} \simeq (\text{Hom}_{\mathcal{S}(k_\Lambda)}(G, G))_{\Lambda_{ij}}$, for all $i, j \in I$. Denoting by $u_{ji} \in \mu_{\text{hom}}(F_i, F_j)$ the image of $\text{id}_G$, we have $u_{kj} \circ u_{ji} = u_{ki}$. Then we can glue the $F'_i := \Psi_{U_i}(F_i)$ and we obtain $F' \in \mathcal{D}^b_{\tau > 0}(\mathcal{K}'_{U})$, where $U = \bigcup_{i \in I} U_i$.

Then we derive from $F'$ a representative of $G$ in $\mathcal{D}^b(\mathcal{K}_{M \times \mathbb{R}})$, as follows. The restriction of $F'$ along $\mathbb{M} \times \mathbb{R} \times \{\varepsilon\}$, for $\varepsilon$ small enough, gives $F'_\varepsilon$ which can be extended by $0$ as an object of $\mathcal{D}^b(\mathcal{K}_{M \times \mathbb{R}})$ and satisfies $SS(F'_\varepsilon) = \Lambda \cup T^*_\varepsilon(\Lambda)$ outside the zero section. This is almost our quantization but the microsupport contains an extra part $T^*_\varepsilon(\Lambda)$. We can move this part away as follows. We can find a Hamiltonian isotopy $\phi: T^*(\mathbb{M} \times \mathbb{R}) \times T^*(\mathbb{M} \times \mathbb{R})$ such that, for all $s \in [0, +\infty[$, we have $\phi_s(\Lambda) = \Lambda$ and $\phi_s(T^*_\varepsilon(\Lambda)) = T^*_\varepsilon+\varepsilon(\Lambda)$. By the main result of [3] we can quantize $\phi$ and compose the resulting kernel with $F'_\varepsilon$. We obtain $F_s \in \mathcal{D}^b(\mathcal{K}_{M \times \mathbb{R}})$ such that $SS(F_s) = \Lambda \cup T^*_\varepsilon+\varepsilon(\Lambda)$. For $s$ big enough there exists $a \in \mathbb{R}$ such that $\Lambda \subset T^*(\mathbb{M} \times ]-\infty, a])$ and $T^*_\varepsilon+\varepsilon(\Lambda) \subset T^*(\mathbb{M} \times ]a, +\infty[)$. Then $F_s \otimes \mathcal{K}_{M \times ]-\infty, a[}$ can be extended by a locally constant sheaf over $\mathbb{M} \times ]a, +\infty[$ and gives the representative $F$ of $G$ that we were looking for.

Now, using standard properties of the microsupport, we can prove the following isomorphism. Let $G, G' \in \mathcal{S}(k_\Lambda)$ be pure objects and let $F, F' \in \mathcal{D}^b(\mathcal{K}_{M \times \mathbb{R}})$ be the quantizations of $G, G'$ obtained by the above procedure. By construction $F'|_{\mathbb{M} \times \{t\}} \simeq 0$ for $t \ll 0$. We can also see that, for $t \gg 0$, $F|_{\mathbb{M} \times \{t\}}$ is independent of $t$ and is a local system on
M. We set $L = F|_{M \times \{t\}}$, $L' = F'|_{M \times \{t\}}$, for $t \gg 0$. Then we have

$$R\text{Hom}(L, L') \simto R\text{Hom}(F, F') \simto R\Gamma(\Lambda; \mathcal{H}om_{\mathcal{G}(k_\Lambda)}(G, G'))$$

The $\mathcal{H}om$ sheaf in $\mathcal{G}(k_\Lambda)$ is in fact a local system on $\Lambda$. Hence we obtain an isomorphism between the cohomology of some local systems on $M$ and corresponding local systems on $\Lambda$. We can use it to prove that $M$ and $\Lambda$ have the same cohomology. We can also see that there exists a unique $F \in D^b_k(M \times \mathbb{R})$ such that $SS(F) = \Lambda$ outside the zero section, $F|_{M \times \{t\}} \cong k_M$ for $t \ll 0$ and $F|_{M \times \{t\}} \cong k_M$ for $t \gg 0$. Using this $F$ we can show that the quantization procedure induces an equivalence between local systems on $M$ and local systems on $\Lambda$, proving that $M$ and $\Lambda$ have the same homotopy groups.

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2. Microlocal theory of sheaves

In this section, we recall some definitions and results from [7], following its notations with the exception of slight modifications. We consider a real manifold $M$ of class $C^\infty$.

Some geometrical notions ([7, §4.2, §6.2]). For a locally closed subset $A$ of $M$, we denote by $\text{Int}(A)$ its interior and by $\overline{A}$ its closure. We denote by $\Delta_M$ or simply $\Delta$ the diagonal of $M \times M$.

We denote by $\pi_M : T^*M \to M$ the cotangent bundle of $M$. If $N \subset M$ is a submanifold, we denote by $T^*_N M$ its conormal bundle, which is naturally a fiber bundle over $N$ and a submanifold of $T^* M$. We identify $M$ with $T^*_M M$, the zero-section of $T^* M$. We set $\tilde{T}^* M = T^* M \setminus T^*_M M$ and we denote by $\tilde{\pi}_M : \tilde{T}^* M \to M$ the projection. For any subset $A$ of $T^* M$ we define its antipodal $A^\circ = \{(x; \xi) \in T^* M; (x; -\xi) \in A\}$.

Let $f : M \to N$ be a morphism of real manifolds. It induces morphisms on the cotangent bundles:

$$T^* M \xleftarrow{f^*} M \times_N T^* N \xrightarrow{f^*} T^* N.$$ 

Let $N \subset M$ be a submanifold and $A \subset M$ any subset. We denote by $C_N(A) \subset T_N M$ the cone of $A$ along $N$. If $M$ is a vector space and $x_0 \in N$, then $C_N(A) \cap T_{N, x_0} M$ is the image in $T_{N, x_0} M$ of the union of all
lines which can be obtained as a limit of lines \((x_0, x_n)\), where \(\{x_n\}_{n \in \mathbb{N}}\) is a sequence in \(A \setminus \{x_0\}\) with \(x_n \to x_0\). If \(A, B\) are two subsets of \(M\), we set \(C(A, B) = C_{\Delta M}(A \times B)\). Identifying \(T_{\Delta M}(M \times M)\) with \(TM\) through the first projection, we consider \(C(A, B)\) as a subset of \(TM\).

The cotangent bundle \(T^*M\) carries an exact symplectic structure. We denote the Liouville 1-form by \(\alpha_M\). It is given in local coordinates \((x; \xi)\) by \(\alpha_M = \sum_i \xi_i dx_i\). We denote by \(H: T^*T^*M \xrightarrow{\sim} TT^*M\) the Hamiltonian isomorphism. We have \(H(dx_i) = -\frac{\partial}{\partial \xi_i}\) and \(H(d\xi_i) = \frac{\partial}{\partial x_i}\). Following [7] we usually identify \(T^*T^*M\) and \(TT^*M\) by \(-H\).

**Microsupport.** We consider a commutative unital ring \(k\) of finite global dimension (e.g. \(k = \mathbb{Z}\)). We denote by \(\text{Mod}(k)\) the category of \(k\)-modules and by \(\text{Mod}(k_M)\) the category of sheaves of \(k\)-modules on \(M\). We denote by \(\mathcal{D}(k_M)\) (resp. \(\mathcal{D}^b(k_M)\)) the derived category (resp. bounded derived category) of \(\text{Mod}(k_M)\).

We recall the definition of the microsupport (or singular support) \(\text{SS}(F)\) of a sheaf \(F\), introduced by M. Kashiwara and P. Schapira in [7] and [6].

**Definition 2.1.** (see [7] Def. 5.1.2) Let \(F \in \mathcal{D}^b(k_M)\) and let \(p \in T^*M\). We say that \(p \notin \text{SS}(F)\) if there exists an open neighborhood \(U\) of \(p\) such that, for any \(x_0 \in M\) and any real \(C^1\)-function \(\phi\) on \(M\) satisfying \(d\phi(x_0) \in U\) and \(\phi(x_0) = 0\), we have \((R\Gamma_{\{x;\phi(x)\geq 0\}}(F))_{x_0} \simeq 0\).

We set \(\ddot{\text{SS}}(F) = \text{SS}(F) \cap T^*M\).

In other words, \(p \notin \text{SS}(F)\) if the sheaf \(F\) has no cohomology supported by “half-spaces” whose conormals are contained in a neighborhood of \(p\). The following properties are easy consequences of the definition.

- The microsupport is closed and \(\mathbb{R}^+\)-conic, that is, invariant by the action of \((\mathbb{R}^+, \times)\) on \(T^*M\).
- \(\text{SS}(F) \cap T_{\pi_M}^*M = \pi_M(\text{SS}(F)) = \text{supp}(F)\).
- The microsupport satisfies the triangular inequality: if \(F_1 \to F_2 \to F_3 \xrightarrow{+1} \) is a distinguished triangle in \(\mathcal{D}^b(k_M)\), then \(\text{SS}(F_i) \subset \text{SS}(F_j) \cup \text{SS}(F_k)\) for all \(i, j, k \in \{1, 2, 3\}\) with \(j \neq k\).

**Example 2.2.** (i) If \(F\) is a non-zero local system on a connected manifold \(M\), then \(\text{SS}(F) = T_M^*M\), the zero-section. Conversely, if \(\text{SS}(F) \subset T_M^*M\), then the cohomology sheaves \(H^i(F)\) are local systems, for all \(i \in \mathbb{Z}\).

(ii) If \(N\) is a smooth closed submanifold of \(M\) and \(F = k_N\), then \(\text{SS}(F) = T_N^*M\).
(iii) Let $\phi$ be $C^1$-function with $d\phi(x) \neq 0$ when $\phi(x) = 0$. Let $U = \{x \in M; \phi(x) > 0\}$ and let $Z = \{x \in M; \phi(x) \geq 0\}$. Then

$$SS(k_U) = U \times_M T^*_M M \cup \{(x; \lambda d\phi(x)); \ \phi(x) = 0, \ \lambda \leq 0\},$$

$$SS(k_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\phi(x)); \ \phi(x) = 0, \ \lambda \geq 0\}.$$ 

(iv) Let $\lambda$ be a closed convex cone with vertex at 0 in $E = \mathbb{R}^n$. Then $SS(k_\lambda) \cap T^*_0 \mathbb{R}^n$ is the polar cone of $\lambda$, that is, $\lambda^o = \{\xi \in E^*; \langle v, \xi \rangle \geq 0 \ 	ext{for all} \ v \in E\}$.

**Notation 2.3.** For a subset $S$ of $T^*M$ we denote by $\mathcal{D}^b_S(k_M)$ the full triangulated subcategory of $\mathcal{D}^b(k_M)$ of the $F$ such that $SS(F) \subset S$. We denote by $\mathcal{D}^b(k_M; S)$ the quotient of $\mathcal{D}^b(k_M)$ by $\mathcal{D}^b_{T^*M \setminus S}(k_M)$. If $p \in T^*M$, we write $\mathcal{D}^b(k_M; p)$ for $\mathcal{D}^b(k_M; \{p\})$. We denote by $\mathcal{D}^b_S(k_M)$ the full triangulated subcategory of $\mathcal{D}^b(k_M)$ of the $F$ for which there exists a neighborhood $\Omega$ of $S$ in $T^*M$ such that $SS(F) \cap \Omega \subset S$.

**Functorial operations.** Let $M$ and $N$ be two real manifolds. We denote by $q_i$ ($i = 1, 2$) the $i$-th projection defined on $M \times N$ and by $p_i$ ($i = 1, 2$) the $i$-th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$.

**Definition 2.4.** Let $f : M \to N$ be a morphism of manifolds and let $\Lambda \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. We say that $f$ is non-characteristic for $\Lambda$ if $f^{-1}_\pi(\Lambda) \cap T^*_M N \subset M \times_N T^*_N N$.

A morphism $f : M \to N$ is non-characteristic for a closed $\mathbb{R}^+$-conic subset $\Lambda$ of $T^*N$ if and only if $f_d : M \times_N T^*N \to T^*M$ is proper on $f^{-1}_\pi(\Lambda)$ and in this case $f_d f^{-1}_\pi(\Lambda)$ is closed and $\mathbb{R}^+$-conic in $T^*M$.

We denote by $\omega_M$ the dualizing complex on $M$. Recall that $\omega_M$ is isomorphic to the orientation sheaf shifted by the dimension. We also use the notation $\omega_{M/N}$ for the relative dualizing complex $\omega_M \otimes f^{-1}_* \omega_{N}^{\otimes -1}$.

We have the duality functors

$$D_M(\ast) = R\mathbb{H}om(\ast, \omega_M), \quad D'_M(\ast) = R\mathbb{H}om(\ast, k_M).$$

**Theorem 2.5.** (See [7 §5.4].) Let $f : M \to N$ be a morphism of manifolds, $F \in \mathcal{D}^b(k_M)$ and $G \in \mathcal{D}^b(k_N)$. Let $q_1 : M \times N \to M$ and $q_2 : M \times N \to N$ be the projections.

(i) We have

$$SS(F \boxtimes G) \subset SS(F) \times SS(G),$$

$$SS(R\mathbb{H}om(q_1^{-1}F, q_2^{-1}G)) \subset SS(F)^a \times SS(G).$$

(ii) We assume that $f$ is proper on $\text{supp}(F)$. Then $SS(Rf_!F) \subset f_!f^{-1}_d SS(F)$. 

---

**References:**

[7] Section 5.4
(iii) We assume that \( f \) is non-characteristic with respect to \( SS(G) \). Then the natural morphism \( f^{-1}G \otimes \omega_{M/N} \to f^!(G) \) is an isomorphism. Moreover \( SS(f^{-1}G) \cup SS(f^!G) \subset f_{d}f_{\pi}^{-1}SS(G) \).

(iv) We assume that \( f \) is a submersion. Then \( SS(F) \subset M \times_{N} T^*N \) if and only if, for any \( j \in \mathbb{Z} \), the sheaves \( H^j(F) \) are locally constant on the fibers of \( f \).

For the definition of cohomologically constructible we refer to \cite{[7]} §3.4.

**Corollary 2.6.** Let \( F, G \in D^b(k_M) \).

(i) We assume that \( SS(F) \cap SS(G)^a \subset T^*_M M \). Then \( SS(F \otimes G) \subset SS(F) + SS(G) \).

(ii) We assume that \( SS(F) \cap SS(G) \subset T^*_M M \). Then \( SS(R\mathbb{H}om(F,G)) \subset SS(F)^a + SS(G) \). Moreover, assuming that \( F \) is cohomologically constructible, the natural morphism \( D^F \otimes G \to R\mathbb{H}om(F,G) \) is an isomorphism.

The next result follows immediately from Theorem 2.5 (ii) and Example 2.2 (i). It is a particular case of the microlocal Morse lemma (see \cite{[7]} Cor. 5.4.19), the classical theory corresponding to the constant sheaf \( F = k_M \).

**Corollary 2.7.** Let \( F \in D^b(k_M) \), let \( \phi: M \to \mathbb{R} \) be a function of class \( C^1 \) and assume that \( \phi \) is proper on \( \text{supp}(F) \). Let \( a < b \) in \( \mathbb{R} \) and assume that \( d\phi(x) \notin SS(F) \) for \( a \leq \phi(x) < b \). Then the natural morphisms \( R\Gamma(\phi^{-1}([-\infty, b]; F) \to R\Gamma(\phi^{-1}([-\infty, a]; F) \) and \( R\Gamma(\phi^{-1}([a, +\infty]); M; F) \to R\Gamma(\phi^{-1}([a, +\infty]); M; F) \) are isomorphisms.

Here we only explained the proper and non-characteristic cases but more general results can be found in \cite{[7]}. In particular we will use the following generalization of Corollary 2.6 (ii). For two closed conic subsets \( A, B \subset T^*M \) we set

\[
A \hat{\oplus} B = \hat{\pi}_{M,M,d}(\pi_{A,B}^d(-H^{-1}(C(A, B^a)))
\]

\[
A \hat{\oplus} B = (A \oplus B) \cup (A \hat{\oplus} B).
\]

**Theorem 2.8.** (See \cite{[7]} Cor. 6.4.5.) Let \( F, G \in D^b(k_M) \). Then

\( SS(R\mathbb{H}om(F,G)) \subset SS(F)^a \hat{\oplus} SS(G) \).

**Microlocalization.** Let \( N \) be a submanifold of \( M \). Sato’s microlocalization is a functor \( \mu_N: D^b(k_M) \to D^b(k_{T^*_N M}) \). We refer to \cite{[7]} for the definition and the main properties. When \( N \) is closed in \( M \) we write \( \mu_N \) for \( Rj_!\mu_N \), where \( j \) is the embedding of \( T^*_N M \) in \( T^*M \). We let
\(i_M: M \to T^*M\) be the inclusion of the zero section. Then, for any \(F \in D^b(k_M)\) we have

\[
\begin{align*}
R\pi_{M*}\mu_N(F) & \simeq i_M^{-1}\mu_N(F) \simeq R\Gamma_N(F), \\
R\pi_{M!}\mu_N(F) & \simeq i_M^!\mu_N(F) \simeq F \otimes \omega_{N|M},
\end{align*}
\]

and we deduce Sato’s distinguished triangle (triangle (4.3.1) in [7]):

\[
\begin{align*}
F \otimes \omega_{N|M} & \to R\Gamma_N(F) \to R\pi_{M*}(\mu_N(F)|_{\dot{T}^*M}) \xrightarrow{+1}.
\end{align*}
\]

We recall the definition of the bifunctor \(\mu_{hom}\), which is a variant of the microlocalization introduced in [7]. Let \(\Delta M\) the diagonal of \(M \times M\). Let \(q_1, q_2: M \times M \to M\) be the projections. We identify \(T^*_{\Delta M}(M \times M)\) with \(T^*M\) through the first projection. For \(F, G \in D^b(k_M)\) we have

\[
\mu_{hom}(F, G) = \mu_{\Delta M}(R\mathcal{H}om(q_2^{-1}F, q_1^{-1}G)) \in D^b(k_{T^*M}).
\]

For a submanifold \(N\) of \(M\) we have \(\mu_N(G) \simeq \mu_{hom}(k_N, G)\), for any \(G \in D^b(k_M)\). The formulas (2.2) and (2.3) give

\[
\begin{align*}
R\pi_{M*}\mu_{hom}(F, G) & \simeq R\mathcal{H}om(F, G), \\
R\pi_{M!}\mu_{hom}(F, G) & \simeq \delta_M^{-1}R\mathcal{H}om(q_2^{-1}F, q_1^{-1}G),
\end{align*}
\]

where \(\delta_M: M \to M \times M\) is the diagonal embedding. If \(F\) is cohomologically constructible, then \(\delta_M^{-1}R\mathcal{H}om(q_2^{-1}F, q_1^{-1}G) \simeq D'(F) \otimes G\) and Sato’s distinguished triangle gives

\[
D'(F) \otimes G \to R\mathcal{H}om(F, G) \to R\pi_{M*}(\mu_{hom}(F, G)|_{\dot{T}^*M}) \xrightarrow{+1}.
\]

**Proposition 2.9.** (Cor. 6.4.3 of [7]) Let \(F, G \in D^b(k_M)\). Then

\[
\begin{align*}
\text{supp } \mu_{hom}(F, G) & \subset \text{SS}(F) \cap \text{SS}(G),
\text{SS}(\mu_{hom}(F, G)) & \subset -H^{-1}(C(\text{SS}(G), \text{SS}(F))).
\end{align*}
\]

### 3. The Kashiwara-Schapira stack

We follow the notations of [7 §7.5]. Let \(M\) be a manifold of dimension \(n\). We introduce the Kashiwara-Schapira stack of a general locally closed subset \(\Lambda\) of \(T^*M\). In the first part of the paper we only consider the case where \(\Lambda\) is a locally closed conic Lagrangian submanifold of \(\dot{T}^*M\). However Theorem [15.2] makes sense also when \(M\) is an analytic manifold and \(\Lambda\) is a subanalytic locally closed conic Lagrangian subset of \(\dot{T}^*M\) (perverse sheaves are examples of objects satisfying hypothesis (ii) of the theorem).
3.1. **Definition of the Kashiwara-Schapira stack.** We will use the categories associated with a subset of $T^*M$ introduced in Notation 2.3.

**Definition 3.1.** Let $\Lambda \subset T^*M$ be a locally closed conic subset. We define a prestack $S^0_\Lambda$ on $\Lambda$ as follows. Over an open subset $\Lambda_0$ of $\Lambda$ the objects of $S^0_\Lambda(\Lambda_0)$ are those of $D^b(\Lambda_0)$. For $F, G \in S^0_\Lambda(\Lambda_0)$ we set

$$\text{Hom}_{S^0_\Lambda}(F, G) := \text{Hom}_{D^b(\Lambda_0)}(F, G).$$

We define the Kashiwara-Schapira stack of $\Lambda$ as the stack associated with $S^0_\Lambda$. We denote it by $S(k_\Lambda)$ and, for $\Lambda_0 \subset \Lambda$, we write $S^0(k_{\Lambda_0})$ instead of $S(k_{\Lambda_0})(\Lambda_0)$.

We denote by $s_\Lambda: D^b(\Lambda) \rightarrow S(k_\Lambda)$ the obvious functor. However, for $F \in D^b(\Lambda)$, we often write $F$ instead of $s_\Lambda(F)$ if there is no risk of ambiguity.

Several results in Part I make a link between $S(k_\Lambda)$ and stacks of the following type.

**Definition 3.2.** Let $X$ be a topological space. We let $D^0_L(k_X)$ be the subprestack of $U \mapsto D^b(k_U)$, $U$ open in $X$, formed by the $F \in D^b(k_U)$ with locally constant cohomologically sheaves. We let $D_L(k_X)$ be the stack associated with $D^0_L(k_X)$. We denote by $\text{Loc}(k_X)$ the substack of $\text{Mod}(k_X)$ formed by the locally constant sheaves.

We remark that $D_L(k_X)$ is only a stack of additive categories (the triangulated structure is of course lost in the “stackification”). However the cohomological functors $H^i: D^b(k_U) \rightarrow \text{Mod}(k_U)$ induce functors of stacks $H^i: D_L(k_X) \rightarrow \text{Loc}(k_X)$ and the natural embedding $\text{Mod}(k_U) \hookrightarrow D^b(k_U)$ induces $\text{Loc}(k_X) \hookrightarrow D_L(k_X)$.

When $\Lambda$ is a locally closed conic Lagrangian submanifold of $T^*M$, our main result on $S(k_\Lambda)$ is Theorem 8.3 which says that it is equivalent to a stack of twisted local systems.

3.2. **Link with $\mu hom$.** For $\Omega \subset T^*M$, we recall the link between the morphisms in $D^b(k_M; \Omega)$ and the sections of $\mu hom$. We recall that a morphism $u: F \rightarrow G$ in $D^b(k_M; \Omega)$ is represented by a triple $(F', s, u')$ with $F' \in D^b(k_M)$ and

$$F \leftarrow F' \xrightarrow{u'} G$$

such that the $L$ defined (up to isomorphism) by the distinguished triangle $F' \rightarrow F \rightarrow L \xrightarrow{+1} \Omega$ satisfies $\Omega \cap \text{SS}(L) = \emptyset$. By (2.9) we see that $s$ induces an isomorphism $\mu hom(F, G)|_\Omega \simeq \mu hom(F', G)|_\Omega$. On the other hand (2.3) gives a morphism $\text{Hom}(F', G) \rightarrow H^0(\Omega; \mu hom(F', G))$. 
Hence \( u' \) induces an element in \( H^0(\Omega; \mu hom(F, G)) \). Finally we obtain a well-defined morphism
\[
\text{(3.1)} \quad \text{Hom}_{\mathcal{D}^b(k_M;\Omega)}(F, G) \to H^0(\Omega; \mu hom(F, G)|_\Omega).
\]

**Theorem 3.3.** (Theorem 6.1.2 of [7]) If \( \Omega = \{p\} \) for some \( p \in T^*M \), then (3.1) is an isomorphism.

For \( F, G, H \in \mathcal{D}^b(k_M) \) we have a composition morphism (see [7 Cor. 4.4.10])
\[
\text{(3.2)} \quad \mu hom(F, G) \otimes \mu hom(G, H) \to \mu hom(F, H).
\]

It is compatible with the composition morphism for \( R\text{Hom} \) through the isomorphism \( R\text{Hom}(F, G) \simeq R\pi_M*\mu hom(F, G) \). Hence for a given open subset \( \Omega \) of \( T^*M \) the morphism induced by (3.2) on the sections over \( \Omega \) is compatible with the composition in \( \mathcal{D}^b(k_M;\Omega) \) through (3.1).

It follows from Theorem (3.3) that the sheaf associated with \( \Omega \mapsto \text{Hom}^b_{\mathcal{D}^b(k_M;\Omega)}(F, G) \) is \( H^0(\mu hom(F, G)) \), for given \( F, G \in \mathcal{D}^b(k_M) \). We obtain an alternative definition of \( \mathcal{S}(k_M) \):

**Corollary 3.4.** Let \( \Lambda \subset T^*M \) be as in Definition 3.1. We define a prestack \( \mathcal{S}^1_\Lambda \) on \( \Lambda \) as follows. Over an open subset \( \Lambda_0 \) of \( \Lambda \) the objects of \( \mathcal{S}^1_\Lambda(\Lambda_0) \) are those of \( \mathcal{D}^b_{(\Lambda_0)}(k_M) \). For \( F, G \in \mathcal{S}^1_\Lambda(\Lambda_0) \) we set \( \text{Hom}_{\mathcal{S}^1_\Lambda(\Lambda_0)}(F, G) := \text{Hom}^0(\Lambda_0; \mu hom(F, G)|_{\Lambda_0}) \). The composition is induced by (3.2). Then, the natural functor of prestacks \( \mathcal{S}^0_\Lambda \to \mathcal{S}^1_\Lambda \) induces an isomorphism on the associated stacks.

**Notation 3.5.** For \( F, G \in \mathcal{D}^b(k_M) \) and sections \( a \) of \( \mu hom(F, G) \) and \( b \) of \( \mu hom(G, H) \), we denote by \( b \circ a \) the image of \( a \otimes b \) by (3.2).

### 3.3. Pure and simple sheaves.

For the remaining part of this section we assume that \( \Lambda \) is a locally closed Lagrangian submanifold of \( T^*M \). For a function \( \varphi : M \to \mathbb{R} \) of class \( C^\infty \) we define
\[
\text{(3.3)} \quad \Lambda_\varphi = \{(x; d\varphi(x)); \ x \in M\}.
\]

We notice that \( \Lambda_\varphi \) is a closed Lagrangian submanifold of \( T^*M \). For a given point \( p = (x; \xi) \in \Lambda \cap \Lambda_\varphi \) we have the following Lagrangian subspaces of \( T_p(T^*M) \)
\[
\text{(3.4)} \quad \lambda_0(p) = T_p(T_x^*M), \quad \lambda_\Lambda(p) = T_p\Lambda, \quad \lambda_\varphi(p) = T_p\Lambda_\varphi.
\]

We recall the definition of the inertia index (see for example §A.3 in [7]). Let \( (E, \sigma) \) be a symplectic vector space and let \( \lambda_1, \lambda_2, \lambda_3 \) be three Lagrangian subspaces of \( E \). We define a quadratic form \( q \) on \( \lambda_1 \oplus \lambda_2 \oplus \lambda_3 \) by \( q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1) \). Then \( \tau_E(\lambda_1, \lambda_2, \lambda_3) \)
is defined as the signature of \(q\), that is, \(p_+ - p_-\), where \(p_\pm\) is the number of \(\pm 1\) in a diagonal form of \(q\). We set
\[
\tau_\varphi = \tau_{p,\varphi} = \tau_{T_pM}(\lambda_0(p), \lambda_\Lambda(p), \lambda_\varphi(p)).
\]

**Proposition 3.6.** (Proposition 7.5.3 of [7]) Let \(\varphi_0, \varphi_1: M \to \mathbb{R}\) be functions of class \(C^\infty\), let \(p = (x; \xi) \in \Lambda\) and let \(F \in D^b_{(\Lambda)}(k_M)\). We assume that \(\Lambda\) and \(\Lambda_\varphi\) intersect transversally at \(p\), for \(i = 0, 1\). Then \((R\Gamma_{\{\varphi_0\geq 0\}}(F))_x\) is isomorphic to \((R\Gamma_{\{\varphi_0\geq 0\}}(F))_x[\frac{1}{2}(\tau_{\varphi_0} - \tau_{\varphi_1})]\).

**Definition 3.7.** (Definition 7.5.4 of [7]) In the situation of Proposition 3.6 we say that \(F\) is pure at \(p\) if \((R\Gamma_{\{\varphi_0\geq 0\}}(F))_x\) is concentrated in a single degree, that is, \((R\Gamma_{\{\varphi_0\geq 0\}}(F))_x \simeq L[d]\), for some \(L \in \text{Mod}(k)\) and \(d \in \mathbb{Z}\). If moreover \(L \simeq k\), we say that \(F\) is simple at \(p\).

If \(F\) is pure (resp. simple) at all points of \(\Lambda\) we say that it is pure (resp. simple) along \(\Lambda\).

We know from [7] that, if \(\Lambda\) is connected and \(F \in D^b_{(\Lambda)}(k_M)\) is pure at some \(p \in \Lambda\), then \(F\) is in fact pure along \(\Lambda\). Moreover the \(L \in \text{Mod}(k)\) in the above definition is the same at every point. We will make this more precise in Part 1.

**Example 3.8.** The generic situation is easy. We consider the hypersurface \(X = \mathbb{R}^{n-1} \times \{0\}\) in \(M = \mathbb{R}^n\). We let \(\Lambda = \{(x; 0; 0, \xi_n); \xi_n > 0\}\) be the “positive” half part of \(T^*_XM\). We set \(Z = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}\). Let \(F \in D^b_{\Lambda, T^*_XM}(k_M)\). Then, there exists \(L \in D^b(k)\) such that the image of \(F\) in the quotient category \(D^b(k_M; T^*_M)\) is isomorphic to \(L_Z\).

For any \(p \in \Lambda\) we can find an integral transform that sends a neighborhood of \(p\) in \(\Lambda\) to the conormal bundle of a smooth hypersurface. Then, Theorem 7.2.1 of [7] reduces the general case to Example 3.8 and we can deduce:

**Lemma 3.9.** Let \(p = (x; \xi)\) be a given point of \(\Lambda\). Then there exist a neighborhood \(\Lambda_0\) of \(p\) in \(\Lambda\) such that
(i) there exists \(F \in D^b_{(\Lambda_0)}(k_M)\) which is simple along \(\Lambda_0\),
(ii) for any \(G \in D^b_{(\Lambda_0)}(k_M)\) there exist a neighborhood \(\Omega\) of \(\Lambda_0\) in \(T^*_M\) and an isomorphism \(F \otimes L_\Lambda \cong G\) in \(D^b(k_M; \Omega)\), where \(L \in D^b(k)\) is given by \(L = \mu\text{hom}(F, G)_p\).

**Definition 3.10.** Let \(\Lambda \subset T^*_M\) be a locally closed conic Lagrangian submanifold. We let \(\mathcal{S}^p(k_\Lambda)\) (resp. \(\mathcal{S}^s(k_\Lambda)\)) be the substack of \(\mathcal{S}(k_\Lambda)\) formed by the pure (resp. simple) sheaves along \(\Lambda\).
Let $F, G \in D^b_{(\Lambda)}(k_M)$. By \cite[(2.10)]{2.10} we know that $\mu_{hom}(F, G)$ has locally constant cohomology sheaves on $\Lambda$. Moreover, for a given $p = (x; \xi) \in \Lambda$ we have
\[(3.6) \quad \mu_{hom}(F, G)_p \simeq R\text{Hom}((R\Gamma_{\{\varphi_0 \geq 0\}}(F))_x, (R\Gamma_{\{\varphi_0 \geq 0\}}(G))_x),\]
where $\varphi_0$ is such that $\Lambda$ and $\Lambda_{\varphi_i}$ intersect transversally at $p$ (see Proposition \cite{3.6}). Hence, if $F$ and $G$ are simple along $\Lambda$ (or $F$ and $G$ are pure along $\Lambda$ and $k$ is a field), then $\mu_{hom}(F, G)$ is concentrated in one degree.

3.4. The canonical object on the diagonal. Let $\Lambda$ be a locally closed Lagrangian submanifold of $\tilde{T}^*M$. One question considered in the first part of this paper is to give conditions so that $\mathcal{G}(k_\Lambda)$ admits a global object. We prove in this paragraph that $\mathcal{G}(k_{\Lambda \times \Lambda})$ admits a canonical object defined on some neighborhood of the diagonal
\[(3.7) \quad \Delta_\Lambda = \{(p, p^a); p \in \Lambda\}.\]

Let $X$ be a manifold and $Y, Z$ two submanifolds of $X$. We recall that $Y$ and $Z$ have a clean intersection if $W = X \cap Y$ is a submanifold of $X$ and $TW = TY \cap TZ$. This means that we can find local coordinates $(x, y, z, w)$ such that $Y = \{z = 0\}$ and $Z = \{x = 0\}$. Using these coordinates the following lemma is easy:

**Lemma 3.11.** Let $X$ be a manifold and $Y, Z$ two submanifolds of $X$ which have a clean intersection. We set $W = X \cap Y$. Then $C(X, Y) = W \times_X TY + W \times_X TZ$.

**Lemma 3.12.** Let $X$ be a manifold and $\Lambda_1, \Lambda_2$ be two Lagrangian submanifolds of $\tilde{T}^*X$. Let $F_1 \in D^b_{(\Lambda_1)}(k_X)$ and $F_2 \in D^b_{(\Lambda_2)}(k_X)$. We assume that $\Lambda_1$ and $\Lambda_2$ have a clean intersection and we set $\Xi = \Lambda_1 \cap \Lambda_2$. Then there exists a neighborhood $U$ of $\Xi$ in $T^*X$ such that $SS(\mu_{hom}(F_1, F_2)|_U) \subset T^*_\Xi \tilde{T}^*X$, that is, $\mu_{hom}(F_1, F_2)|_U$ has locally constant cohomology sheaves on $\Xi$.

**Proof.** We have $SS(\mu_{hom}(F_1, F_2)) \subset -H^{-1}(C(SS(F_2), SS(F_1)))$ by the bound \cite[(2.10)]{2.10}. Let $U_i$ be a neighborhood of $\Lambda_i$ such that $SS(F_i) \cap U_i \subset \Lambda_i$, $i = 1, 2$. Then $U = U_1 \cap U_2$ is a neighborhood of $\Xi$ and we have $-H^{-1}(C(SS(F_2), SS(F_1))) \cap T^*U \subset -H^{-1}(C(\Lambda_2, \Lambda_1))$.

Since $\Lambda_i$ is Lagrangian we have $-H^{-1}(T\Lambda_i) = T^*\Lambda_i T^*X$, for $i = 1, 2$. In particular $-H^{-1}(\Xi \times_{T^*X} T\Lambda_i) \subset T^*_\Xi \tilde{T}^*X$ and the result follows from Lemma \cite{3.11}. \hfill \Box

Let $\Lambda_0$ be an open subset of $\Lambda$. Let $\Delta_M \subset M \times M$ be the diagonal. Let $F \in D^b_{(\Lambda_0)}(k_M)$. Theorem \cite[(2.5)]{2.5} (i) gives $D^F \in D^b_{(\Lambda_0)}(k_M)$. We have
\[ \Delta_{\Lambda_0} = T^*_M (M \times M) \cap (\Lambda_0 \times \Lambda_0^0) \] and we deduce a morphism
\[
\text{Hom}(F, F) \simeq \text{Hom}(\omega_{\Delta_M|\mathcal{M} \times \mathcal{M}}, F \boxtimes D'F)
\]
\[
\rightarrow H^0(\Delta_{\Lambda_0}; \mu_{\text{hom}}(\omega_{\Delta_M|\mathcal{M} \times \mathcal{M}}, F \boxtimes D'F)).
\]
We denote by \( \delta_F \in H^0(\Delta_{\Lambda_0}; \mu_{\text{hom}}(\omega_{\Delta_M|\mathcal{M} \times \mathcal{M}}, F \boxtimes D'F)) \) the image of \( \text{id}_F \) by (3.8).

**Proposition 3.13.** Let \( \Lambda \) be a locally closed Lagrangian submanifold of \( T^*M \). Let \( \Lambda_0 \) be an open subset of \( \Lambda \). Let \( F, G, H \in D^b_{(\Lambda_0)}(k_M) \). We assume that \( F \) is simple along \( \Lambda_0 \). Then there exists a unique
\[
\delta_{F,G} \in H^0(\Delta_{\Lambda_0}; \mu_{\text{hom}}(\omega_{\Delta_M|\mathcal{M} \times \mathcal{M}}, F \boxtimes D'F)),
\]
such that \( \delta_{F,G} \circ \delta_F = \delta_G \) (where \( \circ \) is defined in Notation 3.3). If \( G \) also is simple, then we have \( \delta_{G,H} \circ \delta_{F,G} = \delta_{F,H} \).

**Proof.** We set for short \( A_F := \mu_{\text{hom}}(\omega_{\Delta_M|\mathcal{M} \times \mathcal{M}}, F \boxtimes D'F) \) and \( B := \mu_{\text{hom}}(F \boxtimes D'F, G \boxtimes D'G) \).

The intersection \( \Delta_{\Lambda_0} = T^*_M (M \times M) \cap (\Lambda_0 \times \Lambda_0^0) \) is clean. Hence by Lemma 3.12, \( A_F \) and \( A_G \) are locally constant on \( \Delta_{\Lambda_0} \). Since \( F \) is simple we deduce by (3.0) that \( A_F \simeq k_{\Delta_{\Lambda_0}} \). The same argument shows that \( B \) is locally constant on \( \Lambda_0 \times \Lambda_0^0 \). By (3.0) again the stalks of \( A_G \) and \( B \) are isomorphic. It follows that \( \cdot \circ \delta_F \) gives an isomorphism \( A_G \simeq B \) and \( \delta_{F,G} \) is uniquely determined by \( \delta_G \).

The last formula follows from the unicity of \( \delta_{F,G} \). \( \square \)

**Corollary 3.14.** There exists a neighborhood \( U \) of \( \Delta_{\Lambda} \) in \( \Lambda \times \Lambda^a \) and \( K_{\Delta_M} \) in \( \mathfrak{S}^*(k_U) \) such that, for any open subset \( \Lambda_0 \subset \Lambda \),

(i) for any \( F \in D^b_{(\Lambda_0)}(k_M) \), there exist a neighborhood \( V \) of \( \Delta_{\Lambda_0} \) in \( \Lambda \times \Lambda^a \) and a canonical morphism in \( \mathfrak{S}^*(k_V) \):
\[
\gamma_F: K_{\Delta_M}|_V \rightarrow (\mathfrak{s}_{\Lambda_0 \times \Lambda_0^0}(F \boxtimes D'F))|_V,
\]
which is an isomorphism as soon as \( F \) is simple,

(ii) for \( F, G \in D^b_{(\Lambda_0)}(k_M) \) with \( F \) simple along \( \Lambda_0 \), there exists a neighborhood \( W \) of \( \Delta_{\Lambda_0} \) in \( \Lambda \times \Lambda^a \) such that \( \delta_{F,G}|_W \circ \gamma_F|_W = \gamma_G|_W \).

Moreover, for other \( (U', K'_{\Delta}, \gamma'_{F}) \) satisfying (i) and (ii) there exist a neighborhood \( U_1 \) of \( \Delta_{\Lambda} \) in \( \Lambda \times \Lambda^a \) and a unique isomorphism \( \gamma: K_{\Delta} \rightarrow K'_{\Delta} \) in \( \mathfrak{S}^*(k_{U_1}) \) such that \( \gamma'_{F}|_{U_1} = \gamma \circ \gamma_{F}|_{U_1} \), for all \( F \) as in (i).

**Proof.** We can find a locally finite open covering \( \Lambda = \bigcup_{i \in I} \Lambda_i \) and \( F_i \in D^b_{(\Lambda_i)}(k_M) \) which is simple along \( \Lambda_i \), for all \( i \in I \). We set \( G_i = \mathfrak{s}_{\Lambda_i \times \Lambda_i^0}(F_i \boxtimes D'F_i) \in \mathfrak{S}^*(k_{\Lambda_i \times \Lambda_i^0}) \). By Proposition 3.13 and Corollary 3.3, for any \( i, j \in I \), there exist a neighborhood \( U_{ij}^2 \) of \( \Delta_{\Lambda_{ij}} \) in \( \Lambda \times \Lambda^a \) and an
isomorphism \( \delta_{ij} : G_i \xrightarrow{\sim} G_j \) in \( \mathfrak{S}^*(\mathfrak{k}_{u_{ij}^3}) \). Moreover, for \( i, j, k \in I \), there exists a neighborhood \( U_{ijk}^3 \) of \( \Delta_{\Lambda_{ijk}} \) in \( \Lambda \times \Lambda^a \) such that \( \delta_{ik} = \delta_{jk} \circ \delta_{ij} \) in \( \mathfrak{S}^*(\mathfrak{k}_{u_{ijk}^3}) \).

Since the covering is locally finite we can find a neighborhood \( U_i \) of \( \Delta_{\Lambda_i} \) in \( \Lambda \times \Lambda^a \), for each \( i \in I \), such that \( U_i \cap U_j \subset U_{ij}^2 \) and \( U_i \cap U_j \cap U_k \subset U_{ijk}^3 \), for all \( i, j, k \in I \). Then, the \( G_i \) glue into an object \( \mathcal{K}_{\Delta_{\Lambda}} \in \mathfrak{S}^*(\cup_{i \in I} U_i) \).

Then (i) and (ii) follow from Proposition 3.13. The unicity follows easily from (i) and (ii).

}\qed

\section*{Part 1. Sheaf of microlocal germs}

\subsection*{4. Definition of microlocal germs}

We use the notations of [7, §7.5], in particular the notations (3.4) and (3.5). Let \( M \) be a manifold of dimension \( n \) and \( \Lambda \) a locally closed conic Lagrangian submanifold of \( T^*M \). We let

\begin{equation}
\sigma_{T^*M} : \mathcal{L}_M \to T^*M
\end{equation}

be the fiber bundle of Lagrangian Grassmannian of \( T^*M \). By definition the fiber of \( \mathcal{L}_M \) over \( p \in T^*M \) is the Grassmannian manifold of Lagrangian subspaces of \( T_p T^*M \). We let

\begin{equation}
\sigma^0_{T^*M} : \mathcal{L}^0_M \to T^*M
\end{equation}

be the subbundle of \( \mathcal{L}_M \) whose fiber over \( p \in T^*M \) is the set of Lagrangian subspaces of \( T_p T^*M \) which are transversal to \( \lambda_0(p) \). Then \( \mathcal{L}^0_M \) is an open subset of \( \mathcal{L}_M \). For a given \( p \in T^*M \) we set \( V = T_{\pi_M(p)}M \) and we identify \( T_p T^*M \) with \( V \times V^* \). We use coordinates \( (\nu; \eta) \) on \( T_p T^*M \). Then we can see that any \( l \in (\mathcal{L}^0_M)_p \) is of the form

\begin{equation}
l = \{ (\nu; \eta) \in T_p T^*M; \eta = A \cdot \nu \},
\end{equation}

where \( A : V \to V^* \) is a symmetric matrix. This identifies the fiber \( (\mathcal{L}^0_M)_p \) with the space of \( n \times n \)-symmetric matrices.

For a function \( \varphi \) defined on a product \( X \times Y \) and for a given \( x \in X \) we use the general notation \( \varphi_x = \varphi|_{\{x\} \times Y} \).

\begin{lemma}
There exists a function \( \varphi : \mathcal{L}^0_M \times M \to \mathbb{R} \) of class \( C^\infty \) such that, for any \( l \in \mathcal{L}^0_M \) with \( \sigma_{T^*M}(l) = (x; \xi) \),

\[ \varphi(l)(x) = 0, \quad d_x \varphi(l)(x) = \xi, \quad \lambda_x(\sigma_{T^*M}(l)) = l. \]
\end{lemma}

\begin{proof}
(i) We first assume that \( M \) is the vector space \( V = \mathbb{R}^n \). We identify \( T^*M \) and \( M \times V^* \). For \( p = (x; \xi) \in M \times V^* \) the fiber \( (\mathcal{L}^0_M)_p \) is identified with the space of quadratic forms on \( V \) through (4.3). For
l ∈ (L^0_M)_p we let q_l be the corresponding quadratic form. Now we define φ_0 by

φ_0(l, y) = ⟨y − x; ξ⟩ + \frac{1}{2} q_l(y − x), \quad \text{where } (x; ξ) = σ^0_{T^*M}(l).

We can check that φ_0 satisfies the conclusion of the lemma.

(ii) In general we choose an embedding i: M ↪ X := \mathbb{R}^N. For a given \( p' = (x; ξ') \in M \times_X T^*X \) the subspace \( T_{p'}(M \times_X T^*X) \) of \( T_{p'} T^*X \) is co-isotropic. The symplectic reduction of \( T_{p'} T^*X \) by \( T_{p'}(M \times_X T^*X) \) is canonically identified with \( T_p T^*M \), where \( p = i_d(p') \). The symplectic reduction sends Lagrangian subspaces to Lagrangian subspaces and we deduce a map, say \( r_{p'}: L_{X,p'} \to L_{M,p} \). The restriction of \( r_{p'} \) to the set of Lagrangian subspaces which are transversal to \( T_{p'}(M \times_X T^*X) \) is an actual morphism of manifolds. In particular it induces a morphism \( r^0_{p'}: L^0_{X,p'} \to L^0_{M,p} \). We can see that \( r^0_{p'} \) is onto and is a submersion. When \( p' \) runs over \( M \times_X T^*X \) we obtain a surjective morphism of bundles, say \( r \):

\[
\begin{array}{ccc}
L^0_X |_{M \times_X T^*X} & \xrightarrow{r} & L^0_M \\
\downarrow & & \downarrow \\
M \times_X T^*X & \xrightarrow{i_d} & T^*M.
\end{array}
\]

We can see that \( r \) is a fiber bundle, with fiber an affine space. Hence we can find a section, say \( j: L^0_M \to L^0_X \). For \( (l, x) \in L^0_M \times M \) we set \( \varphi(l, x) = \varphi_0(j(l), i(x)) \), where \( \varphi_0 \) is defined in (i). Then \( \varphi \) satisfies the conclusion of the lemma.

We come back to the Lagrangian submanifold Λ of \( \dot{T}^*M \). We let

\[
U_Λ \subset L^0_M|_Λ
\]

be the subset of \( L^0_M|_Λ \) consisting of Lagrangian subspaces of \( T_p T^*M \) which are transversal to \( \lambda_Λ(p) \). We define \( σ_Λ = σ_{T^*M}|_{U_Λ} \) and \( τ_M = π_M|_Λ \circ σ_Λ \):

\[
\begin{array}{ccc}
U_Λ & \xrightarrow{σ_Λ} & Λ \\
\quad & \searrow & \swarrow \\
\quad & π_M|_Λ & \quad
\end{array}
\]

We note that \( U_Λ \) is not a fiber bundle over \( Λ \) but only an open subset of \( L^0_M|_Λ \). However, for a given \( p ∈ Λ \), we will use the notation

\[
U_{Λ,p} = σ_Λ^{-1}(p).
\]
We also introduce a notation for the graph of \(\tau_M\) and the natural “half-line bundle” over it:

\[
I_\Lambda \subset U_\Lambda \times M, \quad I_\Lambda = \{(l, \tau_M(l)); \ l \in U_\Lambda\},
\]

\[
J_\Lambda \subset \hat{T}^*(U_\Lambda \times M), \quad J_\Lambda = \{(l, x; 0, \lambda \xi); \ (x; \xi) = \sigma_\Lambda(l), \lambda > 0\}.
\]

**Definition 4.2.** We let \(\mathcal{T}_\Lambda\) be the space of functions \(\varphi: U_\Lambda \times M \to \mathbb{R}\) of class \(C^\infty\) such that, for any \(l \in U_\Lambda\) with \(\sigma_\Lambda(l) = (x; \xi)\),

\[
\varphi(l, x) = 0, \quad d\varphi(l)(x) = \xi, \quad \lambda = \varphi(l) = l.
\]

**Lemma 4.3.** For any \(\varphi \in \mathcal{T}_\Lambda\) and any \(l_0 \in U_\Lambda\) we have

\[
\frac{\partial \varphi}{\partial l}(l_0, \tau_M(l_0)) = 0.
\]

**Proof.** For a given \(l_0\) and \((x_0; \xi_0) = \sigma_\Lambda(l_0)\), we have the transpose derivatives of \(\tau_M, \sigma_\Lambda, \pi_M\):

\[
\tau_{M,d}: T_{x_0}^* M \to T_{l_0}^* (U_\Lambda), \quad \sigma_{\Lambda,d}: T_{(x_0; \xi_0)} \Lambda \to T_{l_0}^* (U_\Lambda),
\]

\[
\pi_{M,d}: T_{x_0} M \to T_{(x_0; \xi_0)} \Lambda, \quad (\pi_M\mid_\Lambda)_d: T_x M \to T_{(x_0; \xi_0)} \Lambda.
\]

By definition we have \(\varphi(l, \tau_M(l)) = 0\) for all \(l \in U_\Lambda\). By differentiation we obtain

\[
-\frac{\partial \varphi}{\partial l}(l_0, x_0) = \tau_{M,d}(\varphi(l_0, x_0)) = \tau_{M,d}(x_0; \xi_0) = \sigma_{\Lambda,d}(\pi_M\mid_\Lambda)_d(x_0; \xi_0).
\]

We remark that \(\pi_{M,d}(x_0; \xi_0)\) is the Liouville 1-form at \((x_0; \xi_0)\). Since \(\Lambda\) is conic Lagrangian it vanishes on \(\Lambda\) and \((\pi_M\mid_\Lambda)_d(x_0; \xi_0) = 0\). \(\square\)

**Lemma 4.4.** Let \(X\) be a manifold and \(p: E \to X\) a fiber bundle over \(X\). Let \(Y_1, Y_2 \subset E\) be two submanifolds of \(E\) and set \(Y_3 = Y_1 \cap Y_2\). We assume that \(Y_3\) is a submanifold, that \(p|_{Y_3}: Y_3 \to X\) is a submersion and that, for any \(x \in X\), the submanifolds \(Y_1 \cap p^{-1}(x)\) and \(Y_2 \cap p^{-1}(x)\) have a clean intersection. Then \(Y_1\) and \(Y_2\) have a clean intersection.

**Proof.** We have to check that \(T_y Y_3 = T_y Y_1 \cap T_y Y_2\), for all \(y \in Y_3\). Since this is a local problem we can write \(E = X \times F\) and \(y = (x, z)\). We set \(F_i = Y_i \cap p^{-1}(x) \subset F\). By hypothesis the projection \(T_y Y_3 \to T_x X\) is onto, hence a fortiori \(T_y Y_i \to T_x X\), \(i = 1, 2\). We deduce \(T_z F_i = T_y Y_i \cap \{(0) \times T_z F\}\). Let \(v = (v_x, v_z) \in T_z X \times T_z F\) be in \(T_y Y_1 \cap T_y Y_2\). Since \(p|_{Y_3}\) is a submersion we can find \(w_z \in T_z F_3\) such that \(w = (v_x, w_z) \in T_y Y_3\). Then \(v - w = (0, v_z - w_z) \in T_z F_1 \cap T_z F_2\). By hypothesis \(v_z - w_z \in T_z F_3\) and it follows that \(v \in T_y Y_3\). This proves the lemma. \(\square\)

For a function \(\varphi: M \to \mathbb{R}\) of class \(C^\infty\) we have introduced the Lagrangian submanifold \(\Lambda_\varphi\) in \(\text{[3.3]}\). We also define

\[
N_\varphi = \{(x; \lambda \cdot d\varphi(x)); \ x \in M, \lambda > 0, \varphi(x) = 0\}.
\]
Since $\varphi^{-1}(0)$ is smooth at the points where $d\varphi$ is not zero, $\Lambda_\varphi \cap \dot{T}^*M$ is a locally closed conic Lagrangian submanifold of $\dot{T}^*M$.

**Proposition 4.5.** For any $\varphi \in \mathcal{T}_\Lambda$ there exists a neighborhood $V$ of $I_\Lambda$ (defined in (4.6)) in $U_\Lambda \times M$ such that

(i) $SS(k_{\varphi^{-1}([0, +\infty[)} \cap \dot{T}^*V = \Lambda_\varphi \cap \dot{T}^*V$ is a submanifold of $\dot{T}^*V$,

(ii) $(T^*_U \Lambda \times \Lambda) \cap \Lambda_\varphi \cap \dot{T}^*V = J_\Lambda$, with $J_\Lambda$ given in (4.7),

(iii) $(T^*_U \Lambda \times \Lambda) \cap \dot{T}^*V$ and $\Lambda_\varphi \cap \dot{T}^*V$ have a clean intersection.

**Proof.** (a) For a given $l \in U_\Lambda$, the manifolds $\Lambda_\varphi$ and $\Lambda$ have a transverse intersection at $\sigma_\Lambda(l) = (x_l; \xi_l)$. Hence we can find a neighborhood $V_l$ of $x_l$ in $M$ such that:

(a-i) $SS(k_{\varphi^{-1}([0, +\infty[)} \cap \dot{T}^*V_l = \Lambda_\varphi \cap \dot{T}^*V_l$ is a submanifold of $\dot{T}^*V_l$,

(a-ii) $\Lambda \cap \Lambda_\varphi \cap \dot{T}^*V_l = \mathbb{R}_{>0} \cdot \sigma_\Lambda(l)$,

(a-iii) $\Lambda \cap \dot{T}^*V_l$ and $\Lambda_\varphi \cap \dot{T}^*V_l$ have a clean intersection.

The assertion (a-i) follows from Example 2.2 (iii), the assertions (a-ii) and (a-iii) from the transversality of $\Lambda_\varphi$ and $\Lambda$. We can also assume that $V := \sqcup_{l \in U_\Lambda} \{l\} \times V_l$ is a neighborhood of $I_\Lambda$ in $U_\Lambda \times M$.

(b) Let $((l; 0), (x; \xi)) \in (T^*_U \Lambda \times \Lambda) \cap \dot{T}^*V$. If $((l; 0), (x; \xi)) \in \Lambda_\varphi$, then $(x; \xi) \in \Lambda \cap \Lambda_\varphi \cap \dot{T}^*V_l$. Hence, by (a-ii), $(l; 0), (x; \xi) \in J_\Lambda$. Conversely, Lemma 4.3 implies $J_\Lambda \subset T^*_U \Lambda \times \Lambda$ and we deduce (ii).

Now (iii) follows from (a-iii) and Lemma 4.4 applied to $E = T^*(U_\Lambda \times M)$, $X = U_\Lambda, Y_1 = (T^*_U \Lambda \times \Lambda)$ and $Y_2 = \Lambda_\varphi \cap \dot{T}^*V$.

**Theorem 4.6.** Let $\varphi \in \mathcal{T}_\Lambda$ and let $F \in D^b_{(\Lambda)}(k_M)$. Let $q_1: U_\Lambda \times M \to U_\Lambda$ and $q_2: U_\Lambda \times M \to M$ be the projections. We set

$$\mathcal{M}_{\varphi, F} = \mu_{hom}(k_{\varphi^{-1}([0, +\infty[)}(q_2^{-1}F)) \in D^b(k_{T^*(U_\Lambda \times M)}),$$

$$\mathcal{N}_{\varphi, F} = (R\Gamma_{\varphi^{-1}([0, +\infty[)](q_2^{-1}F)))_{I_\Lambda} \in D^b(k_{U_\Lambda \times M}).$$

Then there exists a neighborhood $V$ of $I_\Lambda$ in $U_\Lambda \times M$ such that

(i) $\dot{T}^*V \cap \text{supp}(\mathcal{M}_{\varphi, F}) \subset J_\Lambda$ and $SS(\mathcal{M}_{\varphi, F}|_{\dot{T}^*V}) \subset T^*_U \Lambda \times \text{supp}(\mathcal{M}_{\varphi, F}|_{\dot{T}^*V})$,

(ii) $R\tilde{\pi}_{\Lambda, *}(\mathcal{M}_{\varphi, F}|_{\dot{T}^*V}) \simeq \mathcal{N}_{\varphi, F}$,

(iii) $SS(Rq_{1*}(\mathcal{N}_{\varphi, F}) \subset T^*_U \Lambda$,

(iv) for any $l \in U_\Lambda$ we have $(Rq_{1*}(\mathcal{N}_{\varphi, F}))_l \simeq (R\Gamma_{\varphi^{-1}([0, +\infty[)}(F))_x$, where $x = \tau_M(l)$.

**Proof.** (i) We take the neighborhood $V$ given by Proposition 4.5. Then the result follows from Proposition 4.5 and Lemma 3.12

(ii) Sato’s triangle (2.8) gives

$$D'(k_{\varphi^{-1}([0, +\infty[}) \otimes q_2^{-1}F \otimes k_{I_\Lambda} \to \mathcal{N}_{\varphi, F} \to R\tilde{\pi}_{U_\Lambda \times M, *}(\mathcal{M}_{\varphi, F})_{I_\Lambda} \to 1.$$
By definition $d\varphi$ does not vanish in a neighborhood of $I_A$. Hence $\varphi^{-1}(0)$ is a smooth hypersurface near $I_A$ and $\text{D}^j(k_{\varphi^{-1}(0)}) \simeq k_{\varphi^{-1}(0)}$. Since $I_A \subset \varphi^{-1}(0)$, the first term of the above triangle is zero. By (i) the support of $R\check{\pi}_V_* \left( \mathcal{M}_{\varphi,F} \right)_{T_V}$ is already contained in $I_A$. So we can forget the subscript $I_A$ in the third term and we obtain (ii).

(iii) By (i) the cohomology sheaves of $M$ from (ii) that $A$. Theorem 2.8. We set for short

Since microsupports are closed subsets it is enough to prove

for $R\check{\pi}_M$.

By definition

By Theorem 4.6 the cohomology sheaves of $M$ and we deduce (iii).

(iv) We first prove that $i_l: \{l\} \times M \hookrightarrow U_A \times M$ is non-characteristic for $R\check{\pi}_M^{-1}(\varphi^{-1}([0, +\infty[: q^{-1}])$ in a neighborhood of $x$. We use the bound in Theorem 2.3. We set for short $A = T\check{\pi}_A U_A \times SS(F)$ and $B = T\check{\pi}_{U_A} U_A \times \Lambda$. Since microsupports are closed subsets it is enough to prove

(a) $(A + \Lambda') \cap (I_l U_A \times T^* M) \subset \{(l, x; 0, 0)\}$,

(b) $\check{\pi}_l \check{\pi}_d^{-1}(-H^{-1}C(A, \Lambda')) \cap (I_l U_A \times T^* M) \subset \{(l, x; 0, 0)\}$.

Since $A \subset T\check{\pi}_A U_A \times T^* M$ and $\Lambda' \cap T^* (U_A \times M) \subset T\check{\pi}_{U_A} U_A \times T^* M$, by Lemma 4.3 the statement (a) is clear. Since $A \subset B$ in some neighborhood of $A \Lambda'$, we may replace $A$ by $B$ in (b). We have seen that $-H^{-1}C(B, \Lambda') \subset T\check{\pi}_B T^* (U_A \times \Lambda)$. Hence $\check{\pi}_l \check{\pi}_d^{-1}(-H^{-1}C(B, \Lambda')) \subset T\check{\pi}_{U_A} U_A \times \Lambda$ and this gives (b).

Now the non-characteristicity implies

$$i_l^{-1} \mathcal{N}_{\varphi,F} \simeq (i_l R\check{\pi}_M^{-1}(\varphi^{-1}([0, +\infty[: q^{-1}])) \otimes \omega_{\{l\}}^{\otimes -1} \varphi^{-1}(\varphi^{-1}([0, +\infty[: q^{-1}]))_{U_A} \simeq (R\check{\pi}_M^{-1}(\varphi^{-1}([0, +\infty[: q^{-1}]))_{U_A}$$

and we deduce (iv). □

For $\varphi \in T_A$ we define a functor, using the notations of Theorem 4.6

$$m_{\varphi}: \text{D}^b(\mathcal{A}) \rightarrow \text{D}^b(\mathcal{A})$$

By Theorem 4.6 the cohomology sheaves of $m_{\varphi}(F)$ are locally constant sheaves on $U_A$ and we have $(m_{\varphi}(F))_l \simeq (R\check{\pi}_l^{-1}(\varphi^{-1}([0, +\infty[: q^{-1}]))_{U_A}$, for any $l \in U_A$ and $x = \tau_M(l)$.

Let $\varphi_0, \varphi_1 \in T_A$. We define $\varphi: U_A \times M \times \mathbb{R} \rightarrow \mathbb{R}$ by $\varphi(l, x, t) = t\varphi_0(l, x) + (1 - t)\varphi_1(l, x)$. We let $q_{13}: U_A \times M \times \mathbb{R} \rightarrow U_A \times \mathbb{R}$ and $q_{2}: U_A \times M \times \mathbb{R} \rightarrow M$ be the projections and we define a functor

$$m_{\varphi_0, \varphi_1}: \text{D}^b(\mathcal{A}) \rightarrow \text{D}^b(\mathcal{A})$$

Theorem 4.6 works as well with the parameter $t$ and we obtain:
Lemma 4.7. Let \( \varphi_0, \varphi_1 \in \mathcal{T}_\Lambda \). For \( t \in \mathbb{R} \) we let \( i_t : U_\Lambda \times \{ t \} \hookrightarrow U_\Lambda \times \mathbb{R} \) be the inclusion. Then, for any \( F \in D^b_{(\Lambda)}(k_M) \), the cohomology sheaves of \( m_\Lambda^{\varphi_0, \varphi_1}(F) \) are locally constant sheaves on \( U_\Lambda \times \mathbb{R} \) and we have natural isomorphisms \( i_t^* m_\Lambda^{\varphi_0, \varphi_1}(F) \simeq m_\Lambda^t(F) \), for all \( t \in \mathbb{R} \). In particular we have a canonical isomorphism \( m_\Lambda^0(F) \simeq m_\Lambda^{\varphi_1}(F) \).

By this lemma the following definition is meaningful.

Definition 4.8. Let \( \Lambda \) be a locally closed conic Lagrangian submanifold of \( T^* M \). We let \( m_\Lambda : D^b_{(\Lambda)}(k_M) \rightarrow D^b(k_{U_\Lambda}) \) be the functor \( m_\Lambda^\varphi \) for an arbitrary \( \varphi \in \mathcal{T}_\Lambda \). For a given \( l \in U_\Lambda \) and \( F \in D^b_{(\Lambda)}(k_M) \) we set \( m_{\Lambda,l}(F) = (m_\Lambda(F))_l \) and call it the microlocal germ of \( F \) at \( l \).

Proposition 4.9. Let \( \Lambda \) be a locally closed conic Lagrangian submanifold of \( T^* M \). Then the functors \( m_{\Lambda_0} : D^b_{(\Lambda_0)}(k_M) \rightarrow D^b(k_{U_{\Lambda_0}}) \), where \( \Lambda_0 \) runs over the open subsets of \( \Lambda \), induce a functor of stacks
\[
m_\Lambda : \mathcal{S}(k_\Lambda) \rightarrow \sigma_{\Lambda^*}(DL(k_{U_\Lambda})).
\]
In particular, for \( F, G \in D^b_{(\Lambda)}(k_M) \), we have a canonical morphism
\[
\sigma_\Lambda^{-1} H^0 \mu hom(F, G) \xrightarrow{\sim} H^0 R\mu hom(m_\Lambda(F), m_\Lambda(G)),
\]
which is actually an isomorphism.

Proof. (i) Let \( \Lambda_0 \) be an open subset of \( \Lambda \). Let \( F \in D^b_{(\Lambda_0)}(k_M) \) be such that \( SS(F) \cap \Lambda_0 = \emptyset \). Then, by (iv) of Theorem 4.6 we have \( m_{\Lambda_0}(F) = 0 \). Hence the functor \( m_{\Lambda_0} \) factorizes through \( D^b(k_{U_{\Lambda_0}} ; \Lambda_0) \). On the other hand, by (iii) of Theorem 4.6 this functor takes value in the subcategory of \( D^b(k_{U_{\Lambda_0}}) \) of objects with locally constant cohomology sheaves. Hence we obtain a functor \( m_0^\Lambda : \mathcal{S}_0(\Lambda_0) \rightarrow \sigma_{\Lambda^*} DL^0(k_{U_\Lambda}) \) between the prestacks of Definitions 3.1 and 3.2. We deduce \( m_\Lambda \) as the composition of \( (m_0^\Lambda)_{\Lambda^*} \) with the natural functor \( (\sigma_{\Lambda^*} DL^0(k_{U_\Lambda}))_{\Lambda^*} \rightarrow \sigma_{\Lambda^*} DL(k_{U_\Lambda}) \), where \( (\cdot)_{\Lambda^*} \) denotes the associated stack.

(ii) By Corollary 3.4 the \( Hom \) sheaf in the stack \( \mathcal{S}(k_\Lambda) \) is \( H^0 \mu hom(\cdot, \cdot) \). It follows from Definition 3.2 that the \( Hom \) sheaf in the stack \( DL(k_X) \) is \( H^0 R\mu hom(\cdot, \cdot) \). This gives the morphism (4.9). It is an isomorphism by (3.6).

Now let \( M' \) be another manifold and \( \Lambda' \) be a locally closed conic Lagrangian submanifold of \( T^* M' \). We have the obvious embedding \( i_{\Lambda, \Lambda'} : U_\Lambda \times U_{\Lambda'} \hookrightarrow U_{\Lambda \times \Lambda'}, (l, l') \mapsto l \oplus l' \). Proposition 7.5.10 of [7] gives

Proposition 4.10. There exists an isomorphism of functors
\[
i_{\Lambda, \Lambda'}^{-1} \circ m_{\Lambda \times \Lambda'} \simeq m_\Lambda \boxtimes m_{\Lambda'}.\]
5. The Maslov sheaf

5.1. Definition of the Maslov sheaf. Let $\Lambda$ be a locally closed conic Lagrangian submanifold of $T^*M$. By Corollary 3.14, we have a canonical object $\mathcal{K}_{\Delta,\Lambda}$ in $\mathcal{G}^*(\mathcal{K}_U)$, where $U$ is a neighborhood of $\Delta, \Lambda$ in $\Lambda \times \Lambda^\circ$.

We set $\tilde{U}_\Lambda = \Delta \times_{\Lambda \times \Lambda^\circ} U_{\Lambda \times \Lambda^\circ}$. The antipodal map $\cdot^a: T^*M \to T^*M$ induces a morphism on the Grassmannian of Lagrangian subspaces of $T_pT^*M$ and $T_p^aT^*M$. We also denote this morphism by $(\cdot)^a: \mathcal{L}_{M,p} \to \mathcal{L}_{M,p^a}, l \mapsto l^a$. We define an embedding $\upsilon_\Lambda: U_{\Lambda \times \Lambda} \to \tilde{U}_\Lambda$, $(l_1, l_2) \mapsto l_1 \oplus l_2^a$.

**Definition 5.1.** We define the Maslov sheaf $\widetilde{\mathcal{M}}_\Lambda = \mathcal{K}_{\Delta,\Lambda}|_{\tilde{U}_\Lambda}$. We also set $\mathcal{M}_\Lambda = \upsilon_{\Lambda}^{-1}(\widetilde{\mathcal{M}}_\Lambda) \in \mathcal{D}(k_{U_{\Lambda \times \Lambda}})$.

By Corollary 3.14 for any open subset $\Lambda_0 \subset \Lambda$ and $F \in \mathcal{D}_\Lambda(k_M)$, we have a canonical morphism in $\mathcal{D}(k_{U_{\Lambda}})$

\begin{equation}
\gamma_F = m_{\Lambda \times \Lambda^*}(\gamma_F): \widetilde{\mathcal{M}}_\Lambda|_{\tilde{U}_{\Lambda_0}} \to m_{\Lambda \times \Lambda^*}(F \boxtimes \mathcal{D}(F)|_{\tilde{U}_{\Lambda_0}}),
\end{equation}

which is an isomorphism as soon as $F$ is simple along $\Lambda_0$. The following results are well-known and can be deduced for example from [7, Appendix].

**Proposition 5.2.** Let $U \subset \tilde{U}_\Lambda$ be a connected component of $\tilde{U}_\Lambda$. Let $l \in U$ and let $(p, p^a) \in \Delta_\Lambda$ be the projection of $l$ to $\Delta_\Lambda$ (so that $l$ is a Lagrangian subspace of $T_{(p, p^a)}T^*(M \times M)$). Then the Maslov index

\begin{equation}
\tau_\Lambda(l) := \tau_{\Lambda \times \Lambda^*}(\lambda_0(p) \times \lambda_0(p^a),, \lambda_\Lambda(p) \times \lambda_\Lambda^*(p^a), l)
\end{equation}

is independent of $l \in U$. It is an even number. Moreover, for any $p \in \Lambda, l_1 \in U_{\Lambda,p}$ and $l_2 \in U_{\Lambda^*,p^a}$, we have

\begin{equation}
\tau_\Lambda(l_1, l_2) = \tau_\Lambda(\lambda_0(p), \lambda_\Lambda(p), l_1) - \tau_\Lambda(\lambda_0(p^a), \lambda_\Lambda^*(p^a), l_2).
\end{equation}

**Notation 5.3.** For a connected component $U$ of $\tilde{U}_\Lambda$ we set $\tau(U) = \tau_\Lambda(l) \in 2 \mathbb{Z}$, for any $l \in U$. If $U_1$ and $U_2$ are connected components of $U_{\Lambda}$ and $U_{\Lambda^*}$ such that $\sigma_\Lambda(U_1) \cap \sigma_{\Lambda^*}(U_2)$ is non empty and connected, then $U_1 \times_{\Lambda} U_2$ is contained in a connected component, say $U$, of $\tilde{U}_\Lambda$. In this case we set $\tau(U_1, U_2) = \tau(U)$. For $k \in 2 \mathbb{Z}$ we define

\begin{equation}
\tilde{U}_\Lambda^k = \{ l \in \tilde{U}_\Lambda; \tau_\Lambda(l) = k \}.
\end{equation}

**Proposition 5.4.** Let $\tilde{\sigma}_\Lambda: \tilde{U}_\Lambda \to \Lambda$ be the projection to the base. Then we have, for any $k \in 2 \mathbb{Z}$,

(i) for any $p \in \Lambda$, the fiber $\tilde{U}_\Lambda^k \cap \tilde{\sigma}_\Lambda^{-1}(p)$ is connected,
(ii) the image of the restriction \( \tilde{\sigma}_U|_{\tilde{U}_k^k} : \tilde{U}_k^k \to \Lambda \) is
\[
\{ p \in \Lambda ; \ \text{codim}_{\lambda_0(p)}(\lambda_0(p) \cap \lambda_\Lambda(p)) \geq |k|/2 \}.
\]

In particular, the restriction \( \tilde{\sigma}_U|_{\tilde{U}_k^k} : \tilde{U}_0^k \to \Lambda \) is onto.

**Proposition 5.5.** Let \( U \subset \tilde{U}_\Lambda \) be a connected component of \( \tilde{U}_\Lambda \). Then \( \tilde{\mathcal{M}}_{\Lambda|U} \in \mathcal{D}(k_U) \) is concentrated in degree \( \frac{1}{2} \tau(U) \).

5.2. **Composition with the Maslov sheaf.** All Grothendieck operations do not induce functors on the categories \( \mathcal{D}(k_* \Lambda) \). But this works for the tensor product and the inverse image. Let \( X \) be a topological space. The functor \( \tilde{\otimes}^L \) on \( \mathcal{D}^0(k_X) \) clearly induces a bifunctor on the subprestack \( \mathcal{D}^0(k_X) \). Taking the associated stack we obtain a bifunctor on \( \mathcal{D}(k_X) \). We denote it also by \( \otimes^L \) since it commutes with the natural functor \( \mathcal{D}^0(k_X) \to \mathcal{D}(k_X) \).

Let \( f : X \to Y \) be a continuous map between topological spaces. Then \( f^{-1} \) induces a functor of prestack \( \mathcal{D}^0(k_Y) \to f_* \mathcal{D}^0(k_X) \). We note that we have a functor of stacks \( (f_* \mathcal{D}^0(k_X))^a \to f_* \mathcal{D}(k_X) \), where \((.)^a\) denotes the associated stack. Hence \( f^{-1} \) induces a functor of stacks \( \mathcal{D}(k_Y) \to f_* \mathcal{D}(k_X) \), which commutes with the functor \( \mathcal{D}^0(k_* \Lambda) \to \mathcal{D}(k_* \Lambda) \).

We denote by \( U^n_i \Lambda \) the fiber product of \( n \) factors \( U_i \Lambda \) over \( \Lambda \). In Propositions 5.6 and 5.8 below we denote by \( q_{ij} : U^3_i \Lambda \to U^3_j \Lambda \) the projection to the factors \( i \) and \( j \). We use similar notations \( q_0 : U^2_i \Lambda \to U^2_i \Lambda \), \( q_{ij} : U^4_i \Lambda \to U^4_i \Lambda \), and \( q_{ijk} : U^4_i \Lambda \to U^4_k \Lambda \).

**Proposition 5.6.** There exists a canonical isomorphism in \( \mathcal{D}(k_{U^2_i}^4) \)
\[
u : q_{12}^{-1} \mathcal{M}_\Lambda \otimes q_{23}^{-1} \mathcal{M}_\Lambda \simeq q_{13}^{-1} \mathcal{M}_\Lambda
\]
such that the following diagram commutes in \( \mathcal{D}(k_{U^2_i}^4) \)

\[
\begin{array}{ccc}
q_{12}^{-1} \mathcal{M}_\Lambda \otimes q_{23}^{-1} \mathcal{M}_\Lambda & \xrightarrow{id \otimes q_{23}^{-1} u} & q_{12}^{-1} \mathcal{M}_\Lambda \otimes q_{24}^{-1} \mathcal{M}_\Lambda \\
q_{13}^{-1} \mathcal{M}_\Lambda \otimes q_{34}^{-1} \mathcal{M}_\Lambda & \xrightarrow{q_{13}^{-1} u} & q_{14}^{-1} \mathcal{M}_\Lambda.
\end{array}
\]

If we had a good notion of direct image, we could define a composition of kernels in \( \mathcal{D}(k_* \Lambda) \), as in [7 §3.6]. We could restate the result as \( \mathcal{M}_\Lambda \circ \mathcal{M}_\Lambda \simeq \mathcal{M}_\Lambda \) (working on connected components of \( U_\Lambda \)) which justifies the title of this paragraph.

**Proof.** By Proposition 5.5, \( \mathcal{M}_\Lambda \) is concentrated in a single degree over each connected component of \( U^2_i \Lambda \). Hence it is enough to define \( \nu \) locally.
(i) We consider an open subset $\Lambda_0$ of $\Lambda$ such that there exists $F \in \mathcal{D}^b_{(\Lambda_0)}(k_M)$ which is simple along $\Lambda_0$. Then we have the isomorphism $\gamma_F : \mathcal{M}_\Lambda|\tilde{\mathcal{V}}_{\Lambda_0} \simeq m_{\Lambda_0 \times \Lambda_0}^*(F \boxtimes \mathcal{D}'F)|\tilde{\mathcal{U}}_{\Lambda_0}$ given in (5.1). Let us set for short $V = U_{\Lambda_0}$ and $L = m_{\Lambda_0}(F)$. Then Proposition 4.10 gives $v_{\Lambda_0}^{-1}\mathcal{M}_\Lambda \simeq \mathcal{L} \boxtimes_{\Lambda_0} \mathcal{D}'L$. On each connected component $V_i$ of $V$ we have $L|_{V_i} \simeq L_i[d_i]$ where $L_i$ is a local system of rank 1 and $d_i \in \mathbb{Z}$. Hence we have a canonical isomorphism $\mathcal{D}'L \otimes L \simeq k_V$ and we deduce the sequence of isomorphisms:

$$q_{12}^{-1}\mathcal{M}_\Lambda \otimes q_{23}^{-1}\mathcal{M}_\Lambda$$

$$\simeq (L \boxtimes_{\Lambda_0} \mathcal{D}'L \boxtimes_{\Lambda_0} k_V) \otimes (k_V \boxtimes_{\Lambda_0} L \boxtimes_{\Lambda_0} \mathcal{D}'L)$$

$$\simeq L \boxtimes_{\Lambda_0} (\mathcal{D}'L \otimes L) \boxtimes_{\Lambda_0} \mathcal{D}'L$$

$$\simeq L \boxtimes_{\Lambda_0} k_V \boxtimes_{\Lambda_0} \mathcal{D}'L$$

$$\simeq q_{13}^{-1}\mathcal{M}_\Lambda.$$

We denote by $u(F)$ the composition in (5.6).

(ii) Let us check that $u(F)$ is independent of $F$. Let $F' \in \mathcal{D}^b_{(\Lambda_0)}(k_M)$ be another simple sheaf along $\Lambda_0$. Up to shrinking $\Lambda_0$ we have $F \simeq F'[i]$ in $\mathcal{D}^b(k_M; \Omega)$, for some neighborhood $\Omega$ of $\Lambda_0$ and some $i \in \mathbb{Z}$. Since morphisms in $\mathcal{D}^b(k_M; \Omega)$ are compositions of morphisms in $\mathcal{D}^b(k_M)$ and their inverses, we may even assume that we have $v : F \to F'[i]$ in $\mathcal{D}^b(k_M)$ which induces an isomorphism in $\mathcal{D}^b(k_M; \Omega)$, hence in $\mathfrak{S}(k_{\Lambda_0})$. Then $w = m_{\Lambda_0}(v) : L = m_{\Lambda_0}(F) \simeq L' = m_{\Lambda_0}(F'[i])$ is an isomorphism and we have $\gamma_{F'} \circ (\gamma_F)^{-1} = w \boxtimes \mathcal{D}'(w^{-1})$. Using $w$ it is easy to draw a commutative square between any two consecutive lines of the definitions of $u(F)$ and $u(F')$ in (5.6). Then we obtain $u(F) = u(F')$ as required. In particular taking a covering of $\Lambda$ by open subsets like $\Lambda_0$ we can glue the local definitions $u(F)$ and obtain our morphism $u$.

(iii) The commutativity of the diagram (5.5) is also a local question. It is a straightforward consequence of the defining sequence (5.6). □

**Definition 5.7.** We define a stack $\mathfrak{S}_{mg}(k_\Lambda)$ as follows (the subscript "mg" stands for microlocal germs). For an open subset $\Lambda_0$ of $\Lambda$, the objects of $\mathfrak{S}_{mg}(k_{\Lambda_0})$ are the pairs $(L, u_L)$, where $L \in \mathcal{D}L(k_{U_{\Lambda_0}})$ and $u_L$ is an isomorphism in $\mathcal{D}L(k_{U_{\Lambda_0}^2})$

$$u_L : \mathcal{M}_\Lambda \otimes q_2^{-1}L \simeq q_1^{-1}L$$
such that the following diagram commutes in $\text{DL}(k_U^3)$

\[
\begin{array}{ccc}
q^{-1}_{12}L & \to & q^{-1}_{12}L \\
\downarrow & & \downarrow \\
q^{-1}_{13}L & \to & q^{-1}_{13}L
\end{array}
\]

(5.8)

We define $\text{Hom}_{S_{mg}}((L,u_L),(L',u_{L'}))$, for two objects $(L,u_L)$ and $(L',u_{L'})$ of $S_{mg}(k_U^3)$, as the set of $v \in \text{Hom}_{DL}(k_U^3)$ such that

\[
\text{Hom}_{S_{mg}}((L,u_L),(L',u_{L'}))
\]

is commutative.

**Proposition 5.8.** Let $\Lambda_0$ be an open subset of $\Lambda$ and $F \in \mathcal{S}(k_{\Lambda_0})$. Then we have a canonical isomorphism in $\text{DL}(k_U^3)$

\[
u_F : \mathcal{M}_\Lambda \otimes q^{-1}_L \simto q^{-1}_L
\]

such that $(\mathcal{M}_\Lambda \otimes q^{-1}_L, \nu_F) \in \mathcal{S}_{mg}(k_{\Lambda_0})$. In other words $\mathcal{M}_\Lambda$ induces a functor

\[
\mathcal{M}_\Lambda' : \mathcal{S}(k_\Lambda) \to \mathcal{S}_{mg}(k_{\Lambda_0}).
\]

(5.10)

**Proof.** By Lemma 3.9 we can write locally $F = G \otimes L_M$, where $G$ is simple along $\Lambda$ and $L \in D^b(k)$. Then we have locally the canonical isomorphisms $m_\Lambda(F) \simto q^{-1}_L \mathcal{M}_\Lambda$ and $\widetilde{\mathcal{M}} \simto m_{\Lambda \times \Lambda^*}(G \otimes D'G)$. Now the proof goes on like the proof of Proposition 5.8.

**Theorem 5.9.** Let $\Lambda$ be a locally closed conic Lagrangian submanifold of $T^*M$. Then the functor $\mathcal{M}_\Lambda'$ of Proposition 5.8 is an equivalence of stacks.

**Proof.** We prove that $\mathcal{M}_\Lambda'$ is faithful in (A-i), full in (A-ii) and essentially surjective in (B). The proofs of (A-ii) and (B) follow the same idea. In both cases we are given a morphism or an object, say $\alpha$, on a neighborhood of a fiber $U_{\Lambda,p}$ and we want to see that $\alpha = m_\Lambda(\beta)$ for some $\beta$ in a neighborhood of $p$. We pick up $l_0 \in U_{\Lambda,p}$ and easily define $\beta$ so that $\alpha = m_\Lambda(\beta)$ in the connected component of $U_{\Lambda}$ containing $l_0$. To check that this isomorphism holds everywhere we use the isomorphisms (5.1), (5.9) and Proposition 5.8.
(A) We first show that \( m'_\Lambda \) is fully faithful. Let \( \Lambda_0 \) be an open subset of \( \Lambda \) and \( F, G \in \mathcal{S}(k_{\Lambda_0}) \).

(A-i) Let \( v : F \to G \) be such that \( m'_\Lambda(v) = 0 \). We have to show that \( v = 0 \). Since we deal with stacks this is a local problem and we may as well assume, by Corollary 3.4, that \( F \) and \( G \) are objects of \( D^b_{(\Lambda_0)}(k_M) \) and that \( v \in H^0(\Lambda_0; \mu hom(F, G)) \). For any \( l \in U_{\Lambda_0, p} \) the formula (3.6) implies \( v_p = (m'_\Lambda(v))_l = 0 \). Since this holds for all \( p \in \Lambda_0 \) we conclude that \( v = 0 \). Hence \( m'_\Lambda \) is faithful.

(A-ii) Let \( v' \in Hom_{\mathcal{S}_{mg}(k_{\Lambda_0})}(m'_\Lambda(F), m'_\Lambda(G)) \). We want to prove that \( v' = m'_\Lambda(v) \) for some \( v : F \to G \). Again, since we deal with stacks and since we already know that \( m'_\Lambda \) is faithful, it is enough to find such a \( v \) in a neighborhood of each \( p \in \Lambda_0 \). So we can assume that \( F \) and \( G \) are objects of \( D^b_{(\Lambda_0)}(k_M) \). We choose \( l_0 \in U_{\Lambda_0, p} \). By (3.6) there exists a unique \( v \in \mu hom(F, G)_p \) such that \( v_p = (m_{\Lambda}(v))_{l_0} = v'_{l_0} \). Since \( m_{\Lambda}(F) \) and \( m_{\Lambda}(G) \) are locally constant, we can choose a neighborhood \( \Lambda_1 \) of \( p \) in \( \Lambda_0 \) such that

- (a) the connected component of \( U_{\Lambda_1} \) containing \( l_0 \), say \( U_{\Lambda_1}^0 \), satisfies \( \sigma_{\Lambda}(U_{\Lambda_1}^0) = \Lambda_1 \),
- (b) \( v \) is defined on \( \Lambda_1 \),
- (c) \( m_{\Lambda}(v))_l = v'_l \) for all \( l \in U_{\Lambda_1}^0 \).

We will prove that

\[
(m_{\Lambda}(v))_{l'} = v'_{l'} \quad \text{for all } l' \in U_{\Lambda_1}.
\]

This means that \( (m_{\Lambda}(v))_{|U_{\Lambda_1}} = v'_{|U_{\Lambda_1}} \) and thus achieves the proof that \( m'_\Lambda \) is fully faithful. So let us prove (5.11). By definition, \( v' \) satisfies the diagram (5.9), where \( L = m_{\Lambda}(F), L' = m_{\Lambda}(G), u_L = u_F \) and \( u_{L'} = u_G \) (with \( u_F, u_G \) given by Proposition 5.8). On the other hand, by Proposition 5.8, \( m_{\Lambda}(v) \) satisfies the same diagram. Since \( \sigma_{\Lambda}(U_{\Lambda_1}^0) = \Lambda_1 \), we can choose \( l \in U_{\Lambda_1}^0 \) such that \( \sigma_{\Lambda}(l') = \sigma_{\Lambda}(l) \). Now we obtain (5.11) by taking the stalks of the diagram (5.9) at \( (l', l) \) and using the following properties: \( (\mathcal{M}_\Lambda)_{(l, l)} \) is free of rank 1, \( (m_{\Lambda}(v))_l = v'_l, u_L \) and \( u_{L'} \) are isomorphisms.

(B) Now we prove that \( m'_\Lambda \) is essentially surjective. Let \( \Lambda_0 \) be an open subset of \( \Lambda \) and \( (L, u) \in \mathcal{S}_{mg}(k_{\Lambda_0}) \). Since \( m'_\Lambda \) is fully faithful, it is enough to prove that, for any \( p \in \Lambda_0 \), there exists a neighborhood \( \Lambda_1 \) of \( p \) in \( \Lambda_0 \) and \( F \in \mathcal{S}(k_{\Lambda_1}) \) such that \( m'_{\Lambda_1}(F) \cong (L, u)_{|U_{\Lambda_1}} \).

We choose \( l_0 \in U_{\Lambda_0, p} \) and a neighborhood \( \Lambda_1 \) of \( p \) in \( \Lambda_0 \) such that

- (a) the connected component of \( U_{\Lambda_1} \) containing \( l_0 \), say \( U_{\Lambda_1}^0 \), satisfies \( \sigma_{\Lambda}(U_{\Lambda_1}^0) = \Lambda_1 \),
- (b) \( v \) is defined on \( \Lambda_1 \),
- (c) \( (m_{\Lambda}(v))_l = v'_l \) for all \( l \in U_{\Lambda_1}^0 \).

This means that \( (m_{\Lambda}(v))_{|U_{\Lambda_1}} = (L, u)_{|U_{\Lambda_1}} \) and thus achieves the proof that \( m'_\Lambda \) is essentially surjective. So let us prove (5.11). By definition, \( v' \) satisfies the diagram (5.9), where \( L = m_{\Lambda}(F), L' = m_{\Lambda}(G), u_L = u_F \) and \( u_{L'} = u_G \) (with \( u_F, u_G \) given by Proposition 5.8). On the other hand, by Proposition 5.8, \( m_{\Lambda}(v) \) satisfies the same diagram. Since \( \sigma_{\Lambda}(U_{\Lambda_1}^0) = \Lambda_1 \), we can choose \( l \in U_{\Lambda_1}^0 \) such that \( \sigma_{\Lambda}(l') = \sigma_{\Lambda}(l) \). Now we obtain (5.11) by taking the stalks of the diagram (5.9) at \( (l', l) \) and using the following properties: \( (\mathcal{M}_\Lambda)_{(l, l)} \) is free of rank 1, \( (m_{\Lambda}(v))_l = v'_l, u_L \) and \( u_{L'} \) are isomorphisms.
Then a morphism making the diagram commutative. Since \( w \) fibers we have \( \mathcal{M} \). The restriction of this diagram to \( U \) where \( (id, q) \) consider the projection \( M \leq \Lambda 1 \) product with \( U \Lambda 1 \). Hence we have obtained \( m_{\Lambda 1}(F) \). We first recall well-known results on locally constant sheaves and introduce some notations. Let \( X \in D_L^0(\mathbb{k}_X) \) such that \( SS(L) \subset T_X^*X \), that is, \( L \) has locally constant cohomology sheaves.

(b) the map \( \sigma_{\Lambda 1} : U^{0}_{\Lambda 1} \to \Lambda 1 \) has connected fibers and admits a section, say \( i_1 : \Lambda 1 \to U^{0}_{\Lambda 1} \).

(c) there exists \( F_1 \in D^b(\mathbb{k}_U) \) which is simple along \( \Lambda 1 \).

Then \( m_{\Lambda 1}(F_1) \) is a local system free of rank 1 on \( U^{0}_{\Lambda 1} \), with a shift. We define \( L_1 = D'(m_{\Lambda 1}(F_1)) \otimes L \) and \( F = F_1 \otimes i_1^{-1}(L_1) \). We have a canonical isomorphism \( v_1 : m_{\Lambda 1}(F)|_{U^{0}_{\Lambda 1}} \cong L|_{U^{0}_{\Lambda 1}} \). Now we prove that it extends naturally to an isomorphism on \( U_{\Lambda 1} \). We consider the diagram in \( DL(\mathbb{k}_{U_{\Lambda 1} \times \Lambda 1} U^{0}_{\Lambda 1}) \)

\[
\begin{array}{ccc}
\mathcal{M}_\Lambda \otimes q_2^{-1}m_{\Lambda 1}(F) & \xrightarrow{u_F} & q_1^{-1}m_{\Lambda 1}(F) \\
\text{id}_{\mathcal{M}_\Lambda} \otimes q_2^{-1}(v_1) & \downarrow & \downarrow i_1 \circ w \\
\mathcal{M}_\Lambda \otimes q_2^{-1}L & \xrightarrow{u} & q_1^{-1}L, \\
\end{array}
\]

where \( u_F \) is given by Proposition 5.8. We define \( w \) as the unique morphism making the diagram commutative. Since \( U^{0}_{\Lambda 1} \) has connected fibers we have \( w = q_1^{-1}(v) \), where \( v : m_{\Lambda 1}(F) \cong L \) is defined by \( v = (id_{U_{\Lambda 1}}, i_1)^{-1}(w) \).

Hence we have obtained \( m_{\Lambda 1}(F) \cong L \). To see that \( (m_{\Lambda 1}(F), u_F) \cong (L, u) \) we need the commutativity of the diagram (5.12) not only on \( U_{\Lambda 1} \times \Lambda 1 U^{0}_{\Lambda 1} \) but on \( U_{\Lambda 1} \times \Lambda 1 U^{0}_{\Lambda 1} \) (with \( v_1 \) replaced by \( v \)). For this we consider the projection \( q_{23} : U_{\Lambda 1} \times \Lambda 1 U_{\Lambda 1} \times \Lambda 1 U^{0}_{\Lambda 1} \to U_{\Lambda 1} \times \Lambda 1 U^{0}_{\Lambda 1} \), we take the pull-back of the diagram (5.12) by \( q_{23} \) and we take the tensor product with \( \mathcal{M}_\Lambda \boxtimes \mathbb{k}_{U^{0}_{\Lambda 1}} \). Using the diagram (5.8), which is valid for \( (L, u) \) and \( (m_{\Lambda 1}(F), u_F) \), we obtain the commutative diagram in \( DL(\mathbb{k}_{U_{\Lambda 1} \times \Lambda 1 U_{\Lambda 1} \times \Lambda 1 U^{0}_{\Lambda 1}}) \)

\[
\begin{array}{ccc}
q_{23}^{-1}\mathcal{M}_\Lambda \otimes q_2^{-1}m_{\Lambda 1}(F) & \xrightarrow{q_{23}^{-1}u_F} & q_1^{-1}m_{\Lambda 1}(F) \\
\downarrow & & \downarrow \\
q_{23}^{-1}\mathcal{M}_\Lambda \otimes q_2^{-1}L & \xrightarrow{q_{23}^{-1}u} & q_1^{-1}L. \\
\end{array}
\]

The restriction of this diagram to \( U_{\Lambda 1} \times \Lambda 1 U_{\Lambda 1} \times \Lambda 1 i_1(\Lambda 1) \) is the diagram (5.12) extended to \( U_{\Lambda 1} \times \Lambda 1 U_{\Lambda 1} \), as required. \( \square \)

6. Monodromy Morphism

We first recall well-known results on locally constant sheaves and introduce some notations. Let \( X \) be a manifold and \( L \in D^b(\mathbb{k}_X) \) such that \( SS(L) \subset T_X^*X \), that is, \( L \) has locally constant cohomology sheaves.
Then any path $\gamma : [0, 1] \to X$ induces an isomorphism

$$M_\gamma (L) : L_{\gamma(0)} \cong L_{\gamma(1)}.$$  

Moreover, $M_\gamma (L)$ only depends on the homotopy class of $\gamma$ with fixed ends. We will use the notation $M_{[\gamma]} (L) := M_\gamma (L)$, where $[\gamma]$ is the class of $\gamma$. For another path $\gamma' : [0, 1] \to X$ such that $\gamma'(0) = \gamma(1)$, we have $M_{\gamma' \cdot \gamma} (L) = M_{\gamma'} (L) \circ M_\gamma (L)$. In particular, if we fix a base point $x_0 \in X$, we obtain the monodromy morphism

$$M(L) : \pi_1(X; x_0) \to \text{Iso}(L_{x_0})$$

$$\gamma \mapsto M_\gamma (L),$$

where $\pi_1(X; x_0)$ is the fundamental group of $(X, x_0)$ and $\text{Iso}(L_{x_0})$ is the group of isomorphisms of $L_{x_0}$ in $D^b(k)$. For any $k$-module $M$ we have the sign morphism $s_M : \mathbb{Z}/2\mathbb{Z} \to \text{Iso}(M)$, which sends $1 \in \mathbb{Z}/2\mathbb{Z}$ to the multiplication by $-1 \in k$. When we have a morphism $\varepsilon : \pi_1(X; x_0) \to \mathbb{Z}/2\mathbb{Z}$, we will say “$L$ has monodromy $\varepsilon$” if $M(L) = s_{L_{x_0}} \circ \varepsilon$. We remark that a morphism $\pi_1(X; x_0) \to \mathbb{Z}/2\mathbb{Z}$ is necessarily invariant by conjugation in $\pi_1(X; x_0)$. Hence we do not have to choose a base point and we will write abusively $\varepsilon : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$.

Now we go back to the situation of section 4. In particular $M$ is a manifold of dimension $n$ and $\Lambda$ is a locally closed conic Lagrangian submanifold of $\dot{T}^* M$. For $F \in D^b_{(\Lambda)}(k_M)$ we have defined $m_{\Lambda}(F) \in D^b(k_{U_{\Lambda}})$, which has locally constant cohomology sheaves. For a given $p \in \Lambda$ we will describe $M(m_{\Lambda}(F) |_{U_{\Lambda,p}})$. We first define an embedding of $U_{\Lambda,p}$ into a connected manifold.

By definition, an element $l \in U_{\Lambda,p}$ is a Lagrangian subspace of $T_p \Lambda$ which is transversal to $\lambda_0(p)$ and $\lambda_{\Lambda}(p)$. The decomposition $T_p \Lambda \cong l \oplus \lambda_{\Lambda}(p)$ gives a projection $T_p \Lambda \to \lambda_{\Lambda}(p)$ and its restriction to $\lambda_0(p)$ is an isomorphism that we denote by

$$u_p(l) : \lambda_0(p) \cong \lambda_{\Lambda}(p).$$

For two vector spaces $V, W$ of dimension $n$ we denote by $\text{Iso}(V, W)$ the space of isomorphisms from $V$ to $W$. If $V$ and $W$ are oriented, we let $\text{Iso}^+(V, W)$ be the connected component of orientation preserving isomorphisms. We remark that $V \oplus \Lambda^n V$ has a canonical orientation and we set

$$\overset{\sim}{\text{Iso}}(V, W) = \text{Iso}^+(V \oplus \Lambda^n V, W \oplus \Lambda^n W).$$

Now we have a natural embedding

$$i_{U_{\Lambda,p}} : U_{\Lambda,p} \to \overset{\sim}{\text{Iso}}(\lambda_0(p), \lambda_{\Lambda}(p))$$

$$l \mapsto u_p(l) \oplus \Lambda^n u_p(l).$$
The topological space $\widehat{\text{Iso}}(\lambda_0(p), \lambda_\Lambda(p))$ is isomorphic to $GL_{n+1}^+(\mathbb{R})$. The fundamental group $\pi_1(GL_n^+(\mathbb{R}))$ is isomorphic to $\mathbb{Z}$ for $n = 2$ and to $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. In any case we have a canonical morphism $\pi_1(GL_n^+(\mathbb{R})) \to \mathbb{Z}/2\mathbb{Z}$ which does not depend on the choice of a base point. So we obtain canonical morphisms, for any connected component $U_{\Lambda,p}^0$ of $U_{\Lambda,p}$

$$
\pi_1(U_{\Lambda,p}^0) \xrightarrow{\pi_1(\iota_{U_{\Lambda,p}})} \pi_1(\widehat{\text{Iso}}(\lambda_0(p), \lambda_\Lambda(p)))
$$

(6.6)

**Theorem 6.1.** Let $F \in D^b_{(\Lambda)}(k_M)$ and $p \in \Lambda$. Then, for any connected component $U_{\Lambda,p}^0$, the monodromy of $m_\Lambda(F)|_{U_{\Lambda,p}^0}$ is $\epsilon'_p$.

**Proof.** (i) We let $U_{\Lambda,p}^0$ be the connected component of $U_{\Lambda}$ which contains $U_{\Lambda,p}^0$. Since $U_{\Lambda,p}^0$ is open in $L_{\Lambda,p}^0$, we can deform any loop $\gamma$ in $U_{\Lambda,p}^0$ into a loop $\gamma'$ in a nearby fiber $U_{\Lambda,q}^0$. This does not change the monodromy. We also have $\epsilon'_p(\gamma) = \epsilon'_q(\gamma')$. Hence we may as well assume that $p$ is a generic point of $\Lambda$, that is, in a neighborhood of $p$ we have $\Lambda = T_{\Lambda,q}^\ast M$, for a submanifold $N \subset M$. Then $F$ is isomorphic to $L_N$ in $\mathfrak{G}(k_{\Lambda})$, for some $L \in D^b(k)$, and we have $m_\Lambda(F) \simeq m_\Lambda(Z_N) \otimes_{\mathbb{Z}} L$. Hence we can also assume that $k = \mathbb{Z}$ and $F = Z_N$.

(ii) We take coordinates $(x_1, \ldots, x_n)$ so that $N = \{x_1 = \cdots = x_k = 0\}$ and $p = (0; 1, 0)$. We identify $(L_{\Lambda,p}^0)_p$ with a space of matrices as in [4.13]. Then $U_{\Lambda,p}$ is the space of symmetric matrices $A$ such that $\det(A_k) \neq 0$, where $A_k$ is the matrix obtained from $A$ by deleting the $k$ first lines and columns. Choosing a connected component $U_{\Lambda,p}^0$ means prescribing the signature of $A_k$. We can choose a base point $B \in U_{\Lambda,p}^0$ represented by a diagonal matrix $B = \text{diag}(0, \ldots, 0, 1, \ldots, 1, -1, \ldots, -1)$ with $k$ zeroes and $l$ 1’s. We choose indices $i < j$ such that $B_{ii} = -1$ and $B_{jj} = 1$. For $\theta \in [0, 2\pi]$, we define the matrix $B(\theta)$ which is equal to $B$ except

$$
\begin{pmatrix}
B_{ii}(\theta) & B_{ij}(\theta) \\
B_{ji}(\theta) & B_{jj}(\theta)
\end{pmatrix} = 
\begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
\sin(\theta) & -\cos(\theta)
\end{pmatrix}
$$

Then $\gamma: \theta \mapsto B(\theta)$ defines a loop in $U_{\Lambda,p}^0$ and $\pi_1(U_{\Lambda,p}^0)$ is generated by loops of this form, where $i, j$ run over the possible indices. We have $\epsilon'_p(\gamma) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Hence it remains to prove that the monodromy of $m_\Lambda(Z_N)$ around $\gamma$ is $-1$. Since $m_\Lambda(Z_N)$ has stalk $\mathbb{Z}$ up to some shift, the monodromy can only be $1$ or $-1$. So we only have to check that the monodromy of $m_\Lambda(Z_N)$ around $\gamma$ is not trivial.
(iii) We define \( \varphi: [0, 2\pi] \times M \to \mathbb{R} \) by \( \varphi(\theta, x) = x + \varphi B(\theta) \cdot x \). Then \{\varphi_\theta \geq 0\} \cap N is a quadratic cone which is homotopically equivalent to the subspace \( V_\theta = \langle e_\theta, e_p; p = l+1, \ldots, l+k, p \neq i \rangle \), where \( e_p = (0, 1, 0) \) and \( e_\theta = (0, \cos(\theta), 0, \sin(\theta), 0) \). The stalk of \( m_\Lambda(Z_N) \) at \( B(\theta) \in U_{\Lambda, p}^0 \) is

\[
(6.7) \quad m_\Lambda(Z_N)_{B(\theta)} \simeq R\Gamma_{\{\varphi_\theta \geq 0\}}(Z_N) \simeq R\Gamma_{V_\theta}(Z_N) \simeq \mathbb{Z}[n - k - l]
\]

and the choice of the isomorphism \( (6.7) \) is equivalent to the choice of an orientation of \( V_\theta \). Since we can not choose compatible orientations of all \( V_\theta, \theta \in [0, 2\pi] \), the monodromy of \( m_\Lambda(Z_N) \) is not 1, as required. \( \square \)

Let \( L_p \) be the locally constant sheaf on \( \widehat{\text{Iso}}(\lambda_0(p), \lambda_\Lambda(p)) \) with stalk \( k \) and monodromy \( \varepsilon_p \). Let \( F \in D^b(\Lambda)(k_M) \) be simple along \( \Lambda \). Then, for any connected component \( U_{\Lambda, p}^0 \) of \( U_{\Lambda, p} \), \( m_\Lambda(F)|_{U_{\Lambda, p}^0} \) is concentrated in a single degree. Proposition \( 6.1 \) says that

\[
(6.8) \quad m_\Lambda(F)|_{U_{\Lambda, p}^0} \simeq L_p|_{U_{\Lambda, p}^0} [d_0],
\]

for some integer \( d_0 \).

7. Extension of microlocal germs

We have seen in \( (6.8) \) that the sheaf of microlocal germs \( m_\Lambda(F) \) of a simple sheaf \( F \) extends from \( U_{\Lambda, p} \) to \( \widehat{\text{Iso}}(\lambda_0(p), \lambda_\Lambda(p)) \), via the natural embedding \( i_{U_{\Lambda, p}} \), as a local system on \( \widehat{\text{Iso}}(\lambda_0(p), \lambda_\Lambda(p)) \). In this section we prove that such an extension exists not only over \( p \) but globally over \( \Lambda \) if the Maslov class of \( \Lambda \) is zero.

We also prove that in the case of the Maslov sheaf we can also extend the structural morphism of Propositional \( 5.6 \). We deduce a description of a twisted version of the Kashiwara-Schapira stack in Corollary \( 7.9 \).

7.1. Global extension. We will use a notion of twisted local systems. We consider a fiber bundle \( p: E \to X \) over a connected manifold \( X \). We let \( E_x \) be the fiber over \( x \in X \). We assume that \( E_x \) is path connected. We let \( H_1(E) \in \text{Mod}(\mathbb{Z}_X) \) be the local system with stalk \( H_1(E_x; \mathbb{Z}) \) (we have \( H_1(E_x; \mathbb{Z}) \simeq (\pi_1(E_x; b_x))^{ab} \), for any base point \( b_x \in E_x \)).

Definition 7.1. We assume to be given a morphism of local systems \( \varepsilon: H_1(E) \to (\mathbb{Z}/2\mathbb{Z})_X \). We consider an open subset \( U \subset E \) such that, for all \( x \in X \), the “fiber” \( U_x := U \cap E_x \) is non-empty and connected. We let \( DL(\varepsilon(k_U|X)) \) be the substack of \( p_*(DL(k_U)) \) formed by the \( F \) such that, for all \( x \in X \), \( F|_{U_x} \) has monodromy \( \varepsilon \). Similarly we let \( \text{Loc}^\varepsilon(k_U|X) \) be the substack of \( p_*(\text{Loc}(k_U)) \) formed by the local systems \( F \) such that, for all \( x \in X \), \( F|_{U_x} \) has monodromy \( \varepsilon \).
Lemma 7.2. In the situation of Definition 7.1 the restriction map from $E$ to $U$ induces an equivalence of stacks $\text{Loc}^\varepsilon(k_{E|X}) \cong \text{Loc}^\varepsilon(k_{U|X})$.

Proof. (i) Let $\varepsilon_0 : H_1(E) \to (\mathbb{Z}/2\mathbb{Z})_X$ be the zero morphism. Both stacks $\text{Loc}^{\varepsilon_0}(k_{E|X})$ and $\text{Loc}^{\varepsilon_0}(k_{U|X})$ are equivalent to $\text{Loc}(k_X)$ through the inverse image by $p$. Hence the result is true for $\varepsilon = \varepsilon_0$.

(ii) In general, let $W \subset X$ be a contractible open subset and set $U_W = U \cap p^{-1}(W)$. Then $\text{Loc}^{\varepsilon}(k_{E|W})$ contains a unique object with stalk $k$, say $L_W$. Taking the tensor product with $L_W$ induces an equivalence $\text{Loc}^{\varepsilon}(k_{E|W}) \cong \text{Loc}^{\varepsilon_0}(k_{E|W})$. The same holds for $U_W$ and $L_W|_{U_W}$ and we have the commutative diagram

$$\begin{array}{ccc}
\text{Loc}^{\varepsilon}(k_{E|W}) & \xrightarrow{r_W} & \text{Loc}^{\varepsilon}(k_{U_W|W}) \\
\circlearrowleft_{L_W} & \downarrow \cong & \downarrow \circlearrowleft_{L_W|_{U_W}} \\
\text{Loc}^{\varepsilon_0}(k_{E|W}) & \cong & \text{Loc}^{\varepsilon_0}(k_{U_W|W}),
\end{array}$$

where the bottom arrow is an equivalence by (i). Hence the restriction map $r_W$ is an equivalence. Since this holds for all contractible open subsets $W$ and we deal with stacks, the restriction map is an equivalence over $X$. \hfill \Box

We let $\mathcal{I}_\Lambda$ be the fiber bundle over $\Lambda$ whose fiber over a point $p$ is $\tilde{\text{Iso}}(\lambda_0(p), \lambda_\Lambda(p))$. The inclusions $i_{U_{\Lambda,p}}$ in (6.5) give

$$(7.1) \quad i_{U_{\Lambda}} : U_{\Lambda} \hookrightarrow \mathcal{I}_{\Lambda}.$$

For an integer $N > 0$ we define $\Xi_N = \{(\xi, 0; 0, \xi_N) ; \xi_N > 0\} \subset \mathcal{T}_{\mathbb{R}^{N-1}} \simeq \mathbb{R}^N$ and $p_N = (0; 0, 1) \in \Xi_N$. We have the inclusions over $\Lambda \times \Xi_N$

$$\begin{array}{ccc}
U_{\Lambda} \times U_{\Xi_N} & \xrightarrow{i_{U_{\Lambda} \times \Xi_N}} & U_{\Lambda \times \Xi_N} \\
\downarrow i_{U_{\Lambda} \times \Xi_N} & & \downarrow i_{U_{\Lambda \times \Xi_N}} \\
\mathcal{I}_{\Lambda} \times \mathcal{I}_{\Xi_N} & \xrightarrow{i_{\mathcal{I}_{\Lambda} \times \Xi_N}} & \mathcal{I}_{\Lambda \times \Xi_N}.
\end{array}$$

We remark that $\Xi_N \simeq \mathbb{R}^N$ and that the objects in this diagram are products of their restrictions over $\Lambda \times \{p_N\}$ by $\Xi_N$. The Maslov index $\tau(\lambda_0, \lambda_{\Xi_N}, l)$ is constant for $l$ in a given connected component of $U_{\Xi_N}$ and takes distinct values for distinct components. It can take the values $-N + 1, -N + 3, \ldots, N - 1$. So $U_{\Xi_N}$ has $N$ connected components that we label by the Maslov index $U_{\Xi_N}^{-N+1}, \ldots, U_{\Xi_N}^{N-1}$.

Proposition 7.3. Let $\Lambda$ be a locally closed conic Lagrangian submanifold of $\mathcal{T}^*M$. We assume that the Maslov class of $\Lambda$ is zero and that $U_{\Lambda}$ has a finite number of connected components, say $U_i, i \in I$. Then there exist $N$ and a family of connected components $V_i \subset U_{\Xi_N}, i \in I$,
We obtain finally
\[ L \]
composes in a unique way \[ U \] such that all the products \[ W \subset U_{\Lambda \times \Xi_N} \]. Moreover the projection \[ W \to \Lambda \times \Xi_N \] is onto.

\textbf{Proof.} (i) For \( l \in \Lambda \) with \( \sigma_\Lambda(l) = p \in \Lambda \) we set for short \( \tau_\Lambda(l) = \tau(\lambda_0(p), \lambda_\Lambda(p), l) \). For \( l \in U_{\Xi_N} \) we define \( \tau_{U_{\Xi_N}}(l) \) in the same way.

Let \( p \in \Lambda \) and \( q \in \Xi_N \) be two given points and let \( U_1, U_2 \subset U_\Lambda \) and \( V_1, V_2 \subset U_{\Xi_N} \) be connected components such that \( U_i \cap U_{\Lambda, p} \) and \( V_i \cap U_{\Xi_N, q}, i = 1, 2, \) are non empty. We choose \( l_i \in U_i \cap U_{\Lambda, p} \) and \( l_i' \in V_i \cap U_{\Xi_N, q}, i = 1, 2, \). Since the connected components of \( U_{\Lambda \times \Xi_N, (p, q)} \) are distinguished by the Maslov index, we see that \( U_1 \times V_1 \) and \( U_2 \times V_2 \) are in the same connected component of \( U_{\Lambda \times \Xi_N} \) if \( \tau_\Lambda(l_1) - \tau_{U_{\Xi_N}}(l_1') = \tau_\Lambda(l_2) - \tau_{U_{\Xi_N}}(l_2') \).

(ii) With the notations in (i) we have \( \tau_\Lambda(l_1) - \tau_\Lambda(l_1') = \tau(U_1, U_2) \) (recall Notations 5.3). Setting \( U_i' = \sigma_\Lambda(U_i) \) we obtain a Čech cocycle on the covering \( \Lambda = \bigcup_{i \in I} U_i' \) defined by \( c_{ij} = \frac{1}{\lambda} \tau(U_i, U_j), i, j \in I \). Its class in \( H^1(\Lambda; \mathbb{Z}_0) \) is the Maslov class of \( \Lambda \). By hypothesis it is zero, so we can find a family of integers \( n_i, i \in I, \) such that \( n_i - n_j = \frac{1}{\lambda} \tau(U_i, U_j) \). We choose \( N \) odd such that \( N > \sup \{2|n_i|; i \in I \} \). By (i) the products \( U_i \times U_{\Xi_N}^{2n_i} \) all are in the same connected component of \( U_{\Lambda \times \Xi_N} \). It is clear that \( \bigcup_{i \in I} U_i \times U_{\Xi_N}^{2n_i} \) maps surjectively onto \( \Lambda \times \Xi_N \) and this implies the last assertion. \( \Box \)

\textbf{Corollary 7.4.} With the hypothesis of Proposition 7.3 let \( F \in D^b_{(\Lambda)}(k_M) \) be simple along \( \Lambda \). Then there exists \( L \in \text{Loc}^c(k_{\mathcal{I}_\Lambda}(\Lambda)) \) such that, for any connected component, \( U \) of \( U_\Lambda \) there exists an isomorphism \( m_\Lambda(F)|_U \simeq L|_U[d_U], \) for some \( d_U \in \mathbb{Z} \).

\textbf{Proof.} We take \( N \) and \( W \) given by Proposition 7.3. We set \( K_N = k_{\mathbb{R}^{N-1}} \in D^b(k_{\mathbb{R}^N}) \) and \( L^0 = m_{\Lambda \times \Xi_N}(F \boxtimes K_N)|_W[d] \in \text{Loc}^c(k_{W_{\Lambda \times \Xi_N}}) \), where the shift \( d \) is chosen so that \( L^0 \) is in degree 0. By Lemma 7.2 we have an equivalence of stacks \( \text{Loc}^c(k_{\mathcal{I}_{\Lambda \times \Xi_N}}(\Lambda \times \Xi_N)) \sim \text{Loc}^c(k_{W_{\Lambda \times \Xi_N}}) \). Hence there exists \( L^1 \in \text{Loc}^c(k_{\mathcal{I}_{\Lambda \times \Xi_N}}(\Lambda \times \Xi_N)) \) such that \( L^1|_W \simeq L^0 \). We remark that the inclusion of the fiber \( \mathcal{I}_{\Xi_N, pN} \subset \mathcal{I}_{\Xi_N} \) is a homotopy equivalence. Hence \( \text{Loc}^c(k_{\mathcal{I}_{\Xi_N}}) \) contains a unique object, say \( L_{\Xi_N} \), up to isomorphism, with stalks isomorphic to \( k \). Then \( L^1|_{\mathcal{I}_{\Lambda \times \Xi_N}} \) decomposes in a unique way \( L^1|_{\mathcal{I}_{\Lambda \times \Xi_N}} \simeq L \boxtimes L_{\Xi_N} \), with \( L \in \text{Loc}^c(k_{\mathcal{I}_{\Lambda}}) \).

We obtain finally
\[ m_\Lambda(F) \boxtimes m_{\Xi_N}(K_N) \simeq m_{\Lambda \times \Xi_N}(F \boxtimes K_N)|_{U_\Lambda \times U_{\Xi_N}} \simeq (L \boxtimes L_{\Xi_N})|_{U_\Lambda \times U_{\Xi_N}}. \]

We set \( i_k = \lfloor \frac{1}{2} k \rfloor \), where \( \lfloor \cdot \rfloor \) is the integer part. Then \( m_{\Xi_N}(K_N)|_{U_{\Xi_N}^{2k}} \simeq L_{\Xi_N}|_{U_{\Xi_N}^{2k}}[i_k] \) and we deduce that \( L \) satisfies the conclusion of the corollary. \( \Box \)
7.2. Extension of the Maslov sheaf. We first remark that the isomorphisms in Corollary 7.4 can be made canonical by requiring compatibility conditions between them.

**Lemma 7.5.** Let \( F \in D^b_{(\Lambda)}(k_M) \) be simple along \( \Lambda \). We assume that there exists a connected component \( U_0 \) of \( U_\Lambda \), such that \( \sigma|U_0 : U_0 \to \Lambda \) is onto. Then there exist \( L \in \text{Loc}^\epsilon(\kappa_{\tau_{\Lambda}|\Lambda}), K \in \text{Loc}^\epsilon(\kappa_{\tau_{\Lambda}}|\Lambda) \) and isomorphisms

\[
\begin{align*}
(i) \quad & \alpha_U : m_\Lambda(F)|_U \sim \sim L|_U[d_U], \text{ for any connected component } U \text{ of } U_\Lambda, \text{ where } d_U \text{ is some integer}, \\
(ii) \quad & \beta : m_{\Lambda \times \Lambda^0}(F \boxtimes D')|_{\tilde{U}_\Lambda^2} \sim \sim K|_{\tilde{U}_\Lambda^2}[-1], \\
(iii) \quad & \gamma : L \boxtimes \Lambda D'L \sim \sim K|_{\tilde{U}_\Lambda^2},
\end{align*}
\]

such that, for all connected components, \( U, V \), of \( U_\Lambda \), with \( \tau(U, V) = 2 \), we have \( \gamma|_{U \times V} \circ \alpha_U \boxtimes D'(\alpha_U^{-1}) = \beta|_{U \times V} \).

Moreover, for other \( (U', K', \alpha_{U'}, \beta', \gamma') \) as in (i)-(iii) satisfying the same conclusion, there exists a unique isomorphism \( u : L \sim \sim L' \) in \( \text{Loc}^\epsilon(\kappa_{\tau_{\Lambda}|\Lambda}) \) such that \( \alpha_{U'} = u|_U \circ \alpha_U \), for all connected components \( U \) of \( U_\Lambda \).

**Proof.** (a) The hypothesis on \( U_0 \) implies that the Maslov class of \( \Lambda \) is zero. Hence, by Corollary 7.4, we can find \( L \in \text{Loc}^\epsilon(\kappa_{\tau_{\Lambda}|\Lambda}), K \in \text{Loc}^\epsilon(\kappa_{\tau_{\Lambda}}|\Lambda) \), isomorphisms \( \alpha_U : m_\Lambda(F)|_U \sim \sim L|_U[d_U] \) as in (i) and \( \beta \) as in (ii) (the shift \(-1\)) in (ii) is \(-\frac{1}{2} \tau(\tilde{U}_\Lambda^2)\)). Then \( \alpha_U' \) and \( \beta \) give an isomorphism between \( L \boxtimes \Lambda D'L|_{U_0 \times \Lambda U_0} \) and \( K|_{U_0 \times \Lambda U_0} \). By Lemma 7.2 it follows that \( L \boxtimes \Lambda D'L \) and \( K|_{\tilde{U}_\Lambda^2} \) are isomorphic and we can choose \( \gamma \) as in (iii).

(b) For connected components, \( U, V \), of \( U_\Lambda \), with \( \tau(U, V) = 2 \), we set \( a(U, V) = \gamma|_{U \times V} \circ (\alpha_U' \boxtimes D'(\alpha_U'^{-1})) \). Then \( a(U, V) \) and \( \beta|_{U \times V} \) are two isomorphisms between two local systems with stalk \( \mathbb{Z} \). Hence they are equal up to sign. Now we adjust the signs inductively as follows so that \( a(U, V) = \beta|_{U \times V} \).

For \( k \in \mathbb{Z} \) we let \( I_k \) be the set of connected components \( U \) of \( U_\Lambda \) such that \( \tau(U_0, U) = 2k \) (so \( I_0 = \{U_0\} \)). For any \( p \in \Lambda \) the possible Maslov index \( \tau(\lambda_0(p), \lambda_\Lambda(p), l) \) run over \( \{-N, -N+2, \ldots, N\} \), for some integer \( N \). Hence there exists integers \( d_p \leq \epsilon_p \) such that \( I_k \) has exactly one component meeting \( U_{\Lambda, p} \) if \( k \in [d_k, e_k] \) and none if \( k \notin [d_k, e_k] \). We deduce, for \( 0 \leq k \leq l \) and \( U_1 \in I_k, U_2 \in I_l \), that we have either \( \sigma_\Lambda(U_2) \cap \sigma_\Lambda(U_1) = \emptyset \) or \( \sigma_\Lambda(U_2) \subset \sigma_\Lambda(U_1) \): if not we would have a contradiction by considering \( p \in \sigma_\Lambda(U_2) \cap \partial \sigma_\Lambda(U_1) \). Now it follows that, for any \( 0 \leq k \) and any \( U \in I_k \), we have a unique sequence \( U_0, U_1, \ldots, U_k = U \) with \( U_l \in I_l \) and \( \sigma_\Lambda(U_{l+1}) \subset \sigma_\Lambda(U_l) \), for \( l = 0, \ldots, k - 1 \).
Assuming $\alpha_V \in \sigma_V$ and $(\alpha')_V = \pm \alpha'_V$ so that $\gamma|_{W \times V} \cdot (\alpha \boxtimes D'(\alpha^{-1})) = \beta|_{W \times V}$. We use the same argument to define $\alpha_V$ for $\tau(U, V) \leq 0$.

The unicity of the $\alpha_U$’s follows by the same inductive argument. □

Recall the fiber bundle $\mathcal{I}_\Lambda \to \Lambda$ and the inclusion $i_{U_\Lambda} : U_\Lambda \hookrightarrow \mathcal{I}_\Lambda$ defined in (5.1). We also set $\tilde{\mathcal{I}}_\Lambda = \mathcal{I}_\Lambda \times \Lambda_\sigma$ and we denote by $\tilde{i}_{U_\Lambda} : \bar{U}_\Lambda \hookrightarrow \tilde{\mathcal{I}}_\Lambda$ the natural inclusion. Similarly we set $\cal{L}^3 = \mathcal{I}_\Lambda \times \Lambda_\sigma$ and we have the inclusions $i_{U^3_\Lambda} : U^3_\Lambda \hookrightarrow \mathcal{L}^3_\Lambda$, $i_{U^3_\Lambda} : U^3_\Lambda \hookrightarrow \mathcal{L}^3_\Lambda$.

By Proposition 5.5 we can shift $\mathcal{M}_\Lambda$ to obtain an object of $\text{Loc}(k_{\mathcal{L}^3_\Lambda})$ as follows. We define $\mathcal{M}'_\Lambda \in \text{Loc}(k_{\mathcal{L}^3_\Lambda})$ by $\mathcal{M}'_\Lambda|_U : = \mathcal{M}_\Lambda|_U [\frac{1}{2}\tau(U)]$, for any connected component $U$ of $U^3_\Lambda$. Proposition 5.6 gives the isomorphism:

$$u' : q_{12}^{-1}(\mathcal{M}') \otimes q_{23}^{-1}(\mathcal{M}_\Lambda) \simeq q_{13}^{-1}(\mathcal{M}_\Lambda').$$

Theorem 7.6. There exist $\mathcal{L}_\Lambda \in \text{Loc}^c(k_{\mathcal{L}^3_\Lambda})$ together with two isomorphisms $\alpha : \mathcal{M}_\Lambda' \simeq i^{-1}_{U^3_\Lambda} \mathcal{L}_\Lambda$ and

$$v : q_{12}^{-1}(\mathcal{L}_\Lambda) \otimes q_{23}^{-1}(\mathcal{L}_\Lambda) \simeq q_{13}^{-1}(\mathcal{L}_\Lambda),$$

such that the following diagram is commutative

$$\begin{array}{ccc}
q_{12}^{-1}(\mathcal{M}_\Lambda') \otimes q_{23}^{-1}(\mathcal{M}_\Lambda) & \xrightarrow{u'} & q_{13}^{-1}(\mathcal{M}_\Lambda') \\
\downarrow & & \downarrow \\
q_{12}^{-1}(\mathcal{L}_\Lambda) \otimes q_{23}^{-1}(\mathcal{L}_\Lambda) & \xrightarrow{v'} & q_{13}^{-1}(\mathcal{L}_\Lambda),
\end{array}$$

where $u'$ is (7.2), $v' = i^{-1}_{U^3_\Lambda}(v)$ and the vertical arrows are induced by $\alpha$. Moreover $(\mathcal{L}_\Lambda, v)$ satisfies a commutative diagram similar to (5.5).

Proof. (i) We first define $\mathcal{L}_\Lambda$ locally. We choose an open subset $\Lambda_0$ of $\Lambda$ such that

(a) there exists a connected component $U_0$ of $U_{\Lambda_0}$, such that $\sigma_{U_0} : U_0 \to \Lambda_0$ is onto,

(b) there exists $F_0 \in \mathcal{D}^\ell_{\Lambda_0}(k_M)$ which is simple along $\Lambda_0$.

We choose $L_0 \in \text{Loc}^c(k_{\mathcal{L}_{\Lambda_0}})$ and $\alpha_U : m_{\Lambda}(F_0)|_U \simeq L_0[d_U]$, for $U$ any connected component of $U_{\Lambda_0}$, given by Lemma 7.5. We set $\mathcal{L}_0 = L_0 \boxtimes_{\Lambda_0} D'L_0$. The contraction $L_0 \otimes D'L_0 \to k_{U_{\Lambda_0}}$ induces $v_0 : q_{12}^{-1}(\mathcal{L}_0) \otimes
\( q_{23}^{-1}(\mathcal{L}_0) \sim \sim q_{13}^{-1}(\mathcal{L}_0) \) and the \( \alpha_U \)'s induce \( \alpha_0 : m_{\Lambda_0}(F_0) \boxtimes m_{\Lambda_0}(D'F_0) \sim \sim i_{U,\Lambda}^{-1}\mathcal{L}_0 \).

(ii) Let \( F'_0 \in D^b(\Lambda_0)_M \) be another simple sheaf and \( a : F_0 \rightarrow F'_0 \) a morphism such that \( m_{\Lambda}(a) \) is an isomorphism. We choose \( L'_0 \in \text{Loc}^c(\kappa_{\Lambda_0}|_{\Lambda_0}) \) and \( \alpha_U' : m_{\Lambda}(F_0)|_{U} \sim \sim L'_0|_{U}d_U \) as in part (i) of the proof. Then there exists an isomorphism \( b : L_0 \sim \sim L'_0 \) such that \( \alpha_U' \circ m_{\Lambda}(a)|_{U} = b|_{U} \circ \alpha_U \), for all connected components \( U \) of \( U_{\Lambda_0} \).

Defining \( \mathcal{L}'_0, v'_0 \) and \( \alpha'_0 \) the same way as \( \mathcal{L}_0, v_0 \) and \( \alpha_0 \), we see that \( b \) induces \( \beta : L_0 \sim \sim L'_0 \) such that the following diagrams commutes

\[
\begin{array}{ccc}
m_{\Lambda_0}(F_0) \boxtimes m_{\Lambda_0}(D'F_0) & \xrightarrow{\alpha_0} & i_{U,\Lambda}^{-1}\mathcal{L}_0 \\
m_{\Lambda_0}(F'_0) \boxtimes m_{\Lambda_0}(D'F'_0) & \xrightarrow{\alpha'_0} & i_{U,\Lambda}'^{-1}\mathcal{L}'_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
m_{\Lambda_0}(F_0) \boxtimes m_{\Lambda_0}(D'F_0) & \xrightarrow{\alpha_0} & q_{12}^{-1}(\mathcal{L}_0) \boxtimes q_{23}^{-1}(\mathcal{L}_0) \xrightarrow{v_0} q_{13}^{-1}(\mathcal{L}_0) \\
m_{\Lambda_0}(F'_0) \boxtimes m_{\Lambda_0}(D'F'_0) & \xrightarrow{\alpha'_0} & q_{12}'^{-1}(\mathcal{L}'_0) \boxtimes q_{23}'^{-1}(\mathcal{L}'_0) \xrightarrow{v'_0} q_{13}'^{-1}(\mathcal{L}'_0),
\end{array}
\]

where the vertical arrows are induced by \( a \) or \( \beta \).

(iii) We cover \( \Lambda \) by open subsets \( \Lambda_i, i \in I \), satisfying (a) and (b) in part (i) of the proof. We choose simple sheaves \( F_i \in D^b(\Lambda_0)_M \) and we construct \( \mathcal{L}_i, v_i \) and \( \alpha_i \) the same way as \( \mathcal{L}_0, v_0 \) and \( \alpha_0 \). By part (ii) we can glue them in \( \mathcal{L}, v \) and \( \alpha \). The commutativity of the diagram \((7.3)\), as well as the last assertion, are local statements and are clear by the construction of \( v_0 \) and \( \alpha_0 \).

**Definition 7.7.** Replacing \( \mathcal{M}_\Lambda \) by \( \mathcal{M}'_\Lambda \) in Definition \([5.7]\), we define a similar stack \( \mathcal{S}'_{mg}(\kappa_\Lambda) \) similar to \( \mathcal{S}_{mg}(\kappa_\Lambda) \): for an open subset \( \Lambda_0 \) of \( \Lambda \), the objects of \( \mathcal{S}'_{mg}(\kappa_{\Lambda_0}) \) are the pairs \((L, u_L)\), where \( L \in DL(\kappa_{U_{\Lambda_0}}) \) and \( u_L \) is an isomorphism in \( DL(\kappa_{U_{\Lambda_0}}) \)

\[
(7.4) \quad u_L : \mathcal{M}'_\Lambda \otimes q_2^{-1}L \sim \sim q_1^{-1}L
\]

such that the diagram obtained from \((5.8)\) by replacing \( \mathcal{M}_\Lambda \) by \( \mathcal{M}'_\Lambda \) is commutative. The morphisms in \( \mathcal{S}'_{mg}(\kappa_\Lambda) \) are defined as in the case of \( \mathcal{S}_{mg}(\kappa_\Lambda) \), replacing \( \mathcal{M}_\Lambda \) by \( \mathcal{M}'_\Lambda \) in the diagram \((5.9)\).

In the next lemma we use the same notations \( q_1, \ldots \) for the projections from \( \mathcal{T}'_\Lambda \) to \( \mathcal{T}'_\Lambda \) as in the case of \( U_{\Lambda} \).
Lemma 7.8. Let $L \in \mathcal{DL}^e(k_{\Lambda|\Lambda})$. Then there exists a unique isomorphism $v_L : L \otimes p_2^{-1}L \sim p_1^{-1}L$ such that

\[
\begin{array}{ccc}
q_{12}^{-1}L \otimes q_{23}^{-1}L \otimes q_{34}^{-1}L & \xrightarrow{id \otimes q_{23}^{-1}v_L} & q_{12}^{-1}L \otimes q_2^{-1}L \\
\downarrow v \otimes q_3^{-1}id & & \downarrow (q_1^{-1}v_L) \\
q_{13}^{-1}L \otimes q_3^{-1}L & \xrightarrow{q_1^{-1}v_L} & q_1^{-1}L
\end{array}
\]

commutes.

Proof. Since the statement contains the unicity of $v_L$ it is enough to prove the result locally on $\Lambda$. So we can assume that $\Lambda$ is contractible. Hence $\text{Loc}^e(k_{\Lambda|\Lambda})$ contains a unique object with stalk $\mathbb{Z}$, say $L_0$. Then $L \simeq L_0 \otimes \sigma_\Lambda^{-1}(L')$ for a unique $L' \in \mathcal{DL}(k_{\Lambda})$. We also have $L_\Lambda \simeq L_0 \boxtimes_{\Lambda_0} D''L_0$ and the result is obvious. $\square$

Theorem 7.6 and Lemma 7.8 imply the following result.

Corollary 7.9. The restriction functor $\mathcal{DL}^e(k_{\Lambda|\Lambda}) \to \mathcal{DL}^e(k_{U|\Lambda})$ induces an equivalence of stacks $\mathcal{DL}^e(k_{\Lambda|\Lambda}) \sim \mathcal{S}'_{mg}(k_{\Lambda})$.

8. The Maslov covering

The stacks $\mathcal{S}_{mg}(k_{\Lambda})$ and $\mathcal{S}'_{mg}(k_{\Lambda})$ of Definitions 5.7 and 7.7 are locally equivalent. We will see that their inverse images on the covering $\hat{\Lambda} \to \Lambda$ given by the Maslov class of $\Lambda$ are equivalent.

Let us first assume that the Maslov class of $\Lambda$ vanishes. Then we can find integers $d_U$, for all connected components $U$ of $U_{\Lambda}$, such that

\[
(8.1) \quad d_U - d_V = \frac{1}{2} \tau(U, V), \quad \text{for all components } U, V \subset U_{\Lambda}.
\]

We choose such integers $d_U$ and we define a functor $S : \mathcal{S}_{mg}(k_{\Lambda}) \to \mathcal{S}'_{mg}(k_{\Lambda})$ as follows. For an open subset $A_0$ of $\Lambda$ and $(L, u) \in \mathcal{S}_{mg}(k_{A_0})$, we define $S(L, u) = (L_1, u_1)$ by $L_1|_{U} := L|_{U}[d_U]$, for all connected components $U$ of $U_{\Lambda}$, and $u_1$ is induced by $u$. Then the next lemma follows from the definitions of $\mathcal{S}_{mg}(k_{\Lambda})$ and $\mathcal{S}'_{mg}(k_{\Lambda})$.

Lemma 8.1. Let $\Lambda$ be a locally closed conic Lagrangian submanifold of $T^*M$. We assume that the Maslov class of $\Lambda$ vanishes. Then the functor $S : \mathcal{S}_{mg}(k_{\Lambda}) \to \mathcal{S}'_{mg}(k_{\Lambda})$ is an equivalence of stacks.

To obtain a global version of Lemma 8.1 we consider the covering $\hat{\Lambda} \to \Lambda$ given by the Maslov class of $\Lambda$. Let us give a construction adapted to our previous definitions. Recall the map $\sigma_{\Lambda} : U_{\Lambda} \to \Lambda$. For $l \in U_{\Lambda}$ we set for short $\tau(l) = \tau(\lambda_0(p), \lambda_\Lambda(p), l)$, where $p = \sigma_{\Lambda}(l)$. We
define the topological space \( \hat{\Lambda}^0 = U_\Lambda \times \mathbb{Z} / \sim \), where \( \sim \) is the equivalence relation given by:

\[
(l, a) \sim (l', a') \iff \sigma_\Lambda(l) = \sigma_\Lambda(l') \text{ and } a - \frac{1}{2} \tau(l) = a' - \frac{1}{2} \tau(l').
\]

Then we have a well-defined map \( r: \hat{\Lambda}^0 \to \Lambda \), \( (l, a) \mapsto \sigma_\Lambda(l) \), which turns \( \hat{\Lambda}^0 \) into a covering of \( \Lambda \) with fiber \( \mathbb{Z} \). We choose one connected component of \( \hat{\Lambda}^0 \) which we denote by \( \hat{\Lambda} \). We still denote by \( r: \hat{\Lambda} \to \Lambda \) the restriction of \( r \) to \( \hat{\Lambda} \). We let \( q: U_\Lambda \times \mathbb{Z} \to \hat{\Lambda}^0 \) be the quotient map

and we define \( U_\hat{\Lambda} = q^{-1}(\hat{\Lambda}) \). We can identify \( U_\hat{\Lambda} \) with the pull-back of \( U_\Lambda \to \Lambda \) by \( r \). We denote by \( d: U_\hat{\Lambda} \to \mathbb{Z} \) the restriction of the projection \( U_\Lambda \times \mathbb{Z} \to \mathbb{Z} \) to \( U_\hat{\Lambda} \). Then \( d \) is constant on the connected components of \( U_\hat{\Lambda} \) and we denote by \( d_U \) its value on a component \( U \).

Now we define a functor \( \hat{S}: r^{-1} \mathcal{G}_{mg}(k_\Lambda) \to r^{-1} \mathcal{G}'_{mg}(k_\Lambda) \). It is enough to define \( \hat{S} \) locally. So we consider an open subset \( \Lambda_0 \subset \hat{\Lambda} \), with image \( \Lambda_1 := r(\Lambda_0) \) by \( r \), such that \( r|_{\Lambda_0}: \Lambda_0 \sim \Lambda_1 \). Then \( r^{-1} \mathcal{G}_{mg}(k_\Lambda)(\Lambda_0) \simeq \mathcal{G}_{mg}(k_{\Lambda_1}) \) and the same holds for \( \mathcal{G}'_{mg} \). For \( (L, u) \in \mathcal{G}_{mg}(k_{\Lambda_1}) \), we define \( \hat{S}(L, u) = (L_1, u_1) \) by \( L_{1|U} := L|_U[d_U] \), for all connected components \( U \) of \( U_{\Lambda_1} \), and \( u_1 \) is induced by \( u \). Now Lemma 8.1 becomes in this general setting:

**Lemma 8.2.** The functor \( \hat{S}: r^{-1} \mathcal{G}_{mg}(k_\Lambda) \to r^{-1} \mathcal{G}'_{mg}(k_\Lambda) \) is an equivalence of stacks.

Putting together Theorem 5.9, Lemma 8.2, and Corollary 7.9, we obtain a description of the Kashiwara-Schapira stack. We let \( \mathcal{I}_\Lambda = r^* \mathcal{I}_\Lambda \) be the pull-back of \( \mathcal{I}_\Lambda \) by \( r \).

**Theorem 8.3.** Let \( \Lambda \) be a locally closed conic Lagrangian submanifold of \( T^* M \). Let \( r: \hat{\Lambda} \to \Lambda \) be the covering defined by the Maslov class of \( \Lambda \). Then we have a equivalence of stacks \( r^{-1} \mathcal{G}(k_\Lambda) \simeq DL^\varepsilon(k_{\mathcal{I}_\Lambda}^\varepsilon) \).

**Topological obstructions.** We quickly recall the definition of the second Stiefel-Whitney class. We first recall some facts on topological obstructions. We consider a connected manifold \( X \) and a fiber bundle \( p: E \to X \) with path connected fibers. We assume to be given a morphism of local systems \( \varepsilon: H_\varepsilon(E) \to (\mathbb{Z}/2\mathbb{Z})_X \). The second obstruction class of \( E \) and \( \varepsilon \) is the obstruction for the stack \( \text{Loc}^\varepsilon(\mathbb{Z}_X) \) to have a global object, locally free of rank 1. It is defined as follows. We first remark that, if \( U \) is contractible, \( \text{Loc}^\varepsilon(\mathbb{Z}_U) \) has only one rank 1 locally free object, up to isomorphism. Clearly the automorphism group of this object is \( \mathbb{Z}/2\mathbb{Z} = \{ \pm 1 \} \). We consider a covering \( X = \bigcup_{i \in I} U_i \) such that, for all \( i, j, k \in I \), the open sets \( U_i, U_{ij} \) and \( U_{ijk} \) are contractible. We choose \( L_i \in \text{Loc}^\varepsilon(\mathbb{Z}_{U_i}) \) and isomorphisms \( u_{ij}: L_i|_{U_{ij}} \sim L_j|_{U_{ij}} \), for
Definition 8.4. For a bundle $E \to X$ and a morphism $\varepsilon : H_1(E) \to (\mathbb{Z}/2\mathbb{Z})_X$, we let $o_2(E, \varepsilon) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ be the class defined by the cocyle $\{c_{ijk}\}_{i,j,k \in I}$.

By construction $\mathrm{Loc}^\varepsilon(\mathbb{Z}_X)$ has a global object locally free of rank 1 if and only if $o_2(E, \varepsilon) = 0$.

We will use this obstruction class in the following case. Let $F_1, F_2 \to X$ be two real vector bundles over $X$ of the same rank, say $r$. Let $\mathcal{I}_{F_1, F_2}$ be the fiber bundle with fiber $\tilde{\mathrm{Iso}}(F_1(x), F_2(x))$ at $x \in X$ (see (6.3)). It comes with a morphism $\varepsilon : H_1(\mathcal{I}_{F_1, F_2}) \to (\mathbb{Z}/2\mathbb{Z})_X$.

Definition 8.5. We define the relative second Stiefel-Whitney class of $\Lambda$ of the covering, the objects $\{c_{ijk}\}_{i,j,k \in I}$ of $\mathrm{DL}(\mathbb{Z})$ are equivalent to the direct sum $\mathcal{S}^{\langle i \rangle}$.

Let us assume moreover that the image of $\rho_2(\lambda_0, \lambda_\Lambda)$ in $H^2(X; \mathbb{k}^\times)$ is zero. Then the stack of simple sheaves $\mathcal{S}(\mathbb{k}_\Lambda)$ admits a global object.

We remark that the equivalence of Theorem 8.3 (and the equivalence $\mathcal{S}(\mathbb{k}_\Lambda) \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Loc}^\varepsilon(\mathbb{k}_\Lambda)[i]$) is not canonical since $\hat{\Lambda}$ is defined as “one” connected component of $\hat{\Lambda}^0$. If the Maslov class is zero, we have $\hat{\Lambda}^0 = \Lambda \times \mathbb{Z}$, but this decomposition is not canonical.

Part 2. $\mu$-sheaves

We set for short $\mathbb{R}_{>0} = ]0, +\infty[\setminus \mathbb{Z}$ and $\mathbb{R}_{\geq 0} = [0, +\infty[\setminus \mathbb{Z}$. We usually endow $\mathbb{R}$ with the coordinate $t$ and $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ with the coordinate $u$.

We set $T^*_\tau \mathbb{R} = \{ (t, \tau) \in T^*\mathbb{R} ; \tau \geq 0 \}$ and we define $T^*_\tau \mathbb{R}$ similarly.

For a manifold $M$ and an open subset $U \subset M \times \mathbb{R}$ we define

$$
T^*_\tau \mathbb{R} \cap T^*U = \begin{cases} 
T^*M \times T^*_\tau \mathbb{R} & \text{if } \tau \geq 0, \\
T^*_\tau \mathbb{R} \cap T^*U & \text{if } \tau < 0.
\end{cases}
$$

(8.2)
Definition 8.7. Let $U$ be an open subset of $M \times \mathbb{R}$. We let $\mathbf{D}^b_{\tau \geq 0}(k_U)$ (resp. $\mathbf{D}^b_{\tau > 0}(k_U)$) be the full subcategory of $\mathbf{D}^b(k_U)$ of sheaves $F$ satisfying $\mathcal{SS}(F) \subseteq T^*_{\tau \leq 0}U$ (resp. $\mathcal{SS}(F) \subseteq T^*_{\tau > 0}U$).

9. Convolution

The convolution product is a variant of the “composition of kernels” considered in [11] (denoted by $\circ$). It is used in [11] to study the localization of $\mathbf{D}^b(k_{M \times \mathbb{R}})$ by the objects with microsupport in $T^*_{\tau \leq 0}(M \times \mathbb{R})$, in a framework similar to the present one. Namely, Tamarkin proves that the functor $F \mapsto k_{M \times [0, +\infty[} \ast F$ is a projector from $\mathbf{D}^b(k_{M \times \mathbb{R}})$ to the left orthogonal of the subcategory $\mathbf{D}^b_{T^*_{\tau \leq 0}(M \times \mathbb{R})}(k_{M \times \mathbb{R}})$ of objects with microsupport in $T^*_{\tau \leq 0}(M \times \mathbb{R})$ (see [4] for a survey). For a manifold $N$ we consider the maps:

$$s, q_1, q_2 : N \times \mathbb{R}^2 \rightarrow N \times \mathbb{R}$$

defined by $s(x, t_1, t_2) = (x, t_1 + t_2), q_1(x, t_1, t_2) = (x, t_1)$ and $q_2(x, t_1, t_2) = (x, t_2)$.

Definition 9.1. For $F, G \in \mathbf{D}^b(k_{N \times \mathbb{R}})$ we set

$$(9.1) \quad F \ast G := R s_! (q_1^{-1}F \otimes q_2^{-1}G) \in \mathbf{D}^b(k_{N \times \mathbb{R}}).$$

In fact we will mainly consider the case where $N$ is of the type $N = M \times \mathbb{R}_{>0}$ for some manifold $M$ and we will use the product with the kernels $k_{\lambda}, k_{-\lambda} \in \mathbf{D}^b(k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$, where $\lambda$ is one of

$$\begin{align*}
\gamma &= M \times \{(t, u); 0 \leq t < u\}, \\
\gamma' &= \text{Int}(\gamma), \\
\gamma &= \text{In}(\gamma), \\
\lambda_0 &= M \times \{0\} \times \mathbb{R}_{\geq 0}, \\
\lambda_1 &= M \times \{(t, u) \in \mathbb{R} \times \mathbb{R}_{>0}; t = u\},
\end{align*}$$

and $-\lambda$ is given in Definition 9.2. For $F \in \mathbf{D}^b(k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$ we thus have $k_{\gamma} \ast F \in \mathbf{D}^b(k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$. As we will see now the product with $k_{\gamma}$ also makes sense for $F \in \mathbf{D}^b(k_U)$ where $U$ is an open subset of $M \times \mathbb{R} \times \mathbb{R}_{>0}$.

Definition 9.2. For two subsets $A, B$ of $N \times \mathbb{R}$ we define

$$\begin{align*}
-A &= \{(x, t) \in N \times \mathbb{R}; (x, -t) \in A\}, \\
A \ast B &= s q_1^{-1}(A \cap q_2^{-1}B) \\
&= \{(x, t) \in N \times \mathbb{R}; \exists t_1 \in \mathbb{R}, (x, t_1) \in A, (x, t - t_1) \in B\}, \\
B \ast A &= \{(x, t) \in N \times \mathbb{R}; (-A) \ast \{(x, t)\} \subseteq B\}.
\end{align*}$$
We remark that $B_{sA}$ is the largest subset $V$ of $N \times \mathbb{R}$ such that $(-A) \star V \subset B$. When $N = M \times \mathbb{R}_{\geq 0}$ we will mostly use

$$B_{sT} = \{(x, t, u) \in M \times \mathbb{R} \times \mathbb{R}_{>0}; \{x\} \times [t - u, t] \times \{u\} \subset B\}.$$  

**Lemma 9.3.** Let $N$ be a manifold and $U \subset N \times \mathbb{R}$ an open subset. Let $F, G \in D^b(k_{N \times \mathbb{R}})$. We set $A = \text{supp} \, G$. Then the natural morphism $F_U \to F$ induces an isomorphism

$$(G \star F) |_{U_* A} \sim (G \star F) |_{U_* A}.$$  

**Proof.** For $(x, t) \in U_* A$ we have by definition

$$s^{-1}(x, t) \cap q_2^{-1}(A) = s^{-1}(x, t) \cap q_1^{-1}(U) \cap q_2^{-1}(A).$$  

It follows that $(G \star F_U)(x, t) \sim (G \star F)(x, t)$ and this proves the isomorphism.  

**Definition 9.4.** Let $N$ be a manifold. Let $j: U \hookrightarrow N \times \mathbb{R}$ be the inclusion of an open subset and let $A$ be a closed subset of $N \times \mathbb{R}$. Let $G \in D^b(k_{N \times \mathbb{R}})$ be such that $\text{supp} \, G \subset A$. Then, for $F \in D^b(k_U)$, we define $G \star F \in D^b(k_{U_* A})$ by

$$G \star F = (G \star j_! F) |_{U_* A}.$$  

By Lemma 9.3 we also have $G \star F = (G \star R j_* F) |_{U_* A}$. The base change formula gives the following result.

**Lemma 9.5.** In the situation of Definition 9.4 we let $N'$ be a submanifold of $N$ and we set $U' = U \cap (N' \times \mathbb{R})$ and $A' = A \cap (N' \times \mathbb{R})$. Then $U_* A \cap (N' \times \mathbb{R}) = U'_* A'$ and $(G \star F) |_{U'_* A'} \simeq G |_{N' \times \mathbb{R}} \star F |_{U'}$.

**Lemma 9.6.** Let $G, G' \in D^b(k_{N \times \mathbb{R}})$. Let $U$ be an open subset of $N \times \mathbb{R}$ and $F \in D^b(k_U)$. We set $A = \text{supp} \, G$, $A' = \text{supp} \, G'$ and $V = U_* (A_* A')$. Then $V = (U_* A)_* A' = (U_* A'_*)_A$ and

$$G \star G' \simeq G' \star G \quad \text{in } D^b(k_{N \times \mathbb{R}}),$$  

$$G \star (G' \star F) \simeq (G \star G') \star F \quad \text{in } D^b(k_V).$$  

**Proof.** The commutativity is clear from the definition of $\star$. To prove the associativity we use the base change formula to show that both terms are isomorphic to $R s_3! (q_1^{-1} G \otimes q_2^{-1} G' \otimes q_3^{-1} j_! F)$, where $j: U \hookrightarrow N \times \mathbb{R}$ is the inclusion and $s_3, q_i: M \times \mathbb{R}^3 \to M \times \mathbb{R}$ are respectively the sum and the projections.

Now we consider a manifold $M$ and $N = M \times \mathbb{R}_{>0}$. We use the notations 9.2 and we define the “translation”

$$T: M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R} \times \mathbb{R}_{>0}, \quad (x, t, u) \mapsto (x, t + u, u).$$
For any $F \in D^b(k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$ we have $k_{\lambda_0} \ast F \simeq F$ and $k_{\lambda_1} \ast F \simeq T_*(F)$. Hence the distinguished triangles

(9.3) \[ k_{\gamma} \to k_\gamma \to k_{\lambda_0} \xrightarrow{+1} , \quad k_{\lambda_1} \to k_\gamma[1] \xrightarrow{+1} \]

yield the distinguished triangles, for $F \in D^b(k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$,

(9.4) \[ k_\gamma \ast F \to k_\gamma \ast F \to F \xrightarrow{+1} , \quad k_{\lambda_1} \ast F \to k_\gamma[1] \ast F \xrightarrow{+1} . \]

An easy computation gives the isomorphisms

(9.5) \[ k_{\lambda_1} \ast (k_\gamma[1]) \simeq k_{\lambda_1}, \quad k_{\lambda_1} \ast (k_{\gamma}[1]) \simeq k_{\lambda_0}, \quad k_{\lambda_1} \ast k_\gamma \simeq k_\gamma. \]

Using Lemma 9.6 we deduce the following result.

**Lemma 9.7.** Let $U \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ be an open subset. We set $U' = U_{\ast}(\pi_1(-\gamma))$. Then, for any $F \in D^b(k_U)$, we have natural isomorphisms in $D^b(k_{U'})$

\[ k_{\gamma} \ast (k_{\gamma}[1]) \ast (k_\lambda \ast F) \simeq k_{\lambda} \ast (k_{\gamma}[1] \ast F) \simeq F \vert_{U'}. \]

**Lemma 9.8.** Let $a < b \in \mathbb{R}$ and let $F \in D^b_{\tau_2 \geq 0}(k_{\mathbb{R}})$. Then $k_{[a,b]} \ast F \simeq 0$.

**Proof.** We may as well assume $a = 0$, $b = 1$. We have $k_{[0,1]} \ast F = R\mathcal{S}_{1}(k_{[0,1]} \boxtimes F)$ and $\text{SS}(k_{[0,1]} \boxtimes F) \subset \{(t_1, t_2; \tau_1, \tau_2); \tau_1 \leq 0, \tau_2 \geq 0\}$. It follows from Theorem 2.5 (ii) and (iv)) that $F' = k_{[0,1]} \ast F$ has constant cohomology sheaves. Hence $k_{[0,1]} \ast F' = 0$. We can check that $k_{[0,1]} \ast k_{[0,1]} \simeq k_{[0,1]}[-1] \oplus k_{[1,2]}$ and Lemma 9.6 gives $k_{[0,1]}[-1] \ast F \oplus k_{[1,2]} \ast F \simeq 0$. In particular $k_{[0,1]} \ast F \simeq 0$. \hfill \Box

We define the projections

(9.6) \[ q = q_M: M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R}, \quad (x, t, u) \mapsto (x, t), \]

\[ r = r_M: M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R}, \quad (x, t, u) \mapsto (x, t - u). \]

**Lemma 9.9.** Let $F \in D^b_{\tau \geq 0}(k_{\mathbb{R}})$. Then we have $k_\gamma \ast q^{-1}F \simeq q^{-1}F$ and $r^{-1}F \simeq k_\gamma \ast q^{-1}F[1]$. In particular both triangles in (9.4), applied to $q^{-1}F$, reduce to the same one

(9.7) \[ r^{-1}(F)[-1] \to k_\gamma \ast q^{-1}(F) \to q^{-1}(F) \xrightarrow{+1} . \]

**Proof.** We set $\gamma'' = \gamma \setminus \lambda_0$. Recall that $k_{\lambda_0} \ast q^{-1}F \simeq q^{-1}F$. Hence the excision exact sequence gives the distinguished triangle $k_{\gamma''} \ast q^{-1}F \to k_{\gamma} \ast q^{-1}F \to q^{-1}F \xrightarrow{+1}$. For a given $(x, u) \in M \times \mathbb{R}_{>0}$, the base change by $i_{(x,u)}: \{x\} \times \mathbb{R} \times \{u\} \hookrightarrow M \times \mathbb{R} \times \mathbb{R}_{>0}$, together with Lemma 9.8 shows that $i_{(x,u)}^{-1}(k_{\gamma''} \ast q^{-1}F) \simeq 0$. Since this holds for any $(x, u)$ we have $k_{\gamma''} \ast q^{-1}F \simeq 0$ and then $k_{\gamma} \ast q^{-1}F \simeq q^{-1}F$. 


we have the identity morphisms. We check it for \( (\text{id}_\gamma \circ a) \circ (a \circ \text{id}) \), the other proofs being similar. We have to prove that the composition of morphisms induced by (9.10), \( k_\gamma \to (k_\gamma \star (k_\gamma'[1])) \star k_\gamma \) and \( k_\gamma \star (k_\gamma'[1]) \star k_\gamma \to k_\gamma \), is the identity morphism of \( k_\gamma \).

By Lemma 9.5, it is enough to fix a point \((x,u)\) in \( M \times ]0,\varepsilon[\) and work in the fiber \( \mathbb{R} \) over this point. We set \( \Gamma_2 = \gamma \times \gamma' \subset \mathbb{R}^2 \) and \( D = s^{-1}(0) \cap \Gamma_2 \). Then \( (k_\gamma'[1]) \star k_\gamma \simeq R_{s_3!}(k_{\Gamma_2[1]}) \). We see that the first morphism in (9.10) is the direct image by \( s \) of \( k_{\Gamma_2[1]} \to k_{\Gamma_2[1]} \) and the second one is the direct image of \( k_{\Gamma_2[1]} \to k_{D[1]} \).

We set \( \Gamma_3 = \gamma \times \gamma' \times \gamma \subset \mathbb{R}^3 \), \( D_1 = \overline{D} \times \gamma \) and \( D_2 = \gamma \times D \). We define \( s_3 : \mathbb{R}^3 \to \mathbb{R} \), \((t_1,t_2,t_3) \mapsto t_1 + t_2 + t_3 \). Then \( k_\gamma \star (k_\gamma'[1]) \star k_\gamma \simeq R_{s_3}!(k_{\Gamma_3[1]}) \). By the previous discussion the morphism \( k_\gamma \to (k_\gamma \star (k_\gamma'[1])) \star k_\gamma \), is induced by \( k_{D_1} \to k_{\Gamma_3[1]} \) and the morphism \( k_\gamma \star (k_\gamma'[1]) \star k_\gamma \), is induced by \( k_{\Gamma_3[1]} \to k_{D_2[1]} \). Their composition is the natural morphism \( k_{D_1} \to k_{D_2[1]} \). When we take its image by \( R_{s_3!} \) we find the identity morphism of \( k_\gamma \) as required. \( \square \)
10. Microlocalization and convolution

The main result of this section is Proposition 10.5 which gives the global sections of $\mu\text{hom}$ outside the zero-section. The formula proved here will be given a more symmetric form in Theorem 11.5.

10.1. Third term of Sato’s triangle. Let $M$ be a manifold and $N$ a submanifold of $M$. We assume for simplicity that $N$ is connected. We suppose to be given a submanifold $N'$ of $M$, defined in a neighborhood of $N$ in $M$, such that $N$ is a hypersurface of $N'$ and $N' \setminus N$ is the union of two connected components, say $N'^+$ and $N'^-$. We set $N^\pm = N \cup N'^\pm$. We let $T^- N \subset N \times_N TN'$ be the subbundle in half spaces of $N \times_N TN'$ determined by $N^-$ and we let

$$T^-_N M = \{ (x; \xi) \in T^* M; \ x \in N, \langle \xi, \nu \rangle \geq 0 \ \text{for all} \ \nu \in T^- N \}$$

be its polar subset.

**Proposition 10.1.** Let $U$ be a neighborhood of $N$ in $M$. Let $F \in D^b(k_U)$ such that $SS(F) \cap T^-_N M = \emptyset$. Then

$$R\pi_U_* (\mu_N(F)|_{T^* U}) \simeq (R\Gamma_{U \cap N^+}(F))|_N. \tag{10.1}$$

**Proof.** (i) Let us first prove

$$R\Gamma_{U \cap (N^+ \setminus N)}(F)|_N[-1] \xrightarrow{\sim} F \otimes \omega_{N|M}. \tag{10.2}$$

By Example 2.2 (iii) and Theorem 2.5 (ii) we have $SS(k_{N^+ \setminus N}) \cap (N \times_M T^* M) \subset T^- N M$. Since microsupports are closed subsets of $T^* M$, we deduce that, up to shrinking $U$, we have $SS(k_{N^+ \setminus N}) \cap SS(F) \cap T^*U \subset T^*_U U$. Then Corollary 2.6 (i) gives $R\Gamma_{N^+ \setminus N}(F) \simeq D'_M(k_{N^+ \setminus N}) \otimes F$.

Applying $D'_M$ to the triangle $k_{N^+ \setminus N} \to k_{N'} \to k_N \xrightarrow{\Delta} 1$, we can check that $D'_M(k_{N^+ \setminus N})|_N \simeq \omega_{N|M}[1]$, which implies (10.2).

(ii) The distinguished triangle $R\Gamma_{U \cap (N^+ \setminus N)}(F)[-1] \to R\Gamma_N(F) \to R\Gamma_{U \cap N^+}(F) \xrightarrow{\Delta} 1$ and (10.2) give the distinguished triangle

$$F \otimes \omega_{N|M} \to R\Gamma_N(F) \to R\Gamma_{U \cap N^+}(F)|_N \xrightarrow{\Delta} 1.$$ 

Comparing with Sato’s triangle (2.3), we deduce the proposition. □

10.2. The case of $\mu\text{hom}$. Here we apply Proposition 10.1 to $\mu\text{hom}$. For an open subset $U \subset M \times \mathbb{R}$ we denote by $\Delta_U$ the diagonal. We also set

$$\Delta'_U = \{(x, t_1, t_2) \in (M \times \mathbb{R})^2; \ (x, t_1) \in U, \ (x, t_2) \in U \},$$

$$\Delta''_U = \{(x, t_1, t_2) \in \Delta'_U; \ t_1 \geq t_2 \}.$$

We denote by $q'_i$ the restriction of $q_i$ to $\Delta'_{M \times \mathbb{R}}$ or $\Delta''_U$, $i = 1, 2$. 

Corollary 10.2. Let $M$ be a manifold and $U$ an open subset of $M \times \mathbb{R}$. Let $F, G \in D^b_{\tau>0}(\mathcal{K}_U)$ (see Definition 8.7). Then
\[ R\tilde{\nu}_*(\mu\text{hom}(F,G)|_{\tilde{\tau}-U}) \simeq R\mathcal{H}\text{om}((q_2^{-1}F)|_{\Delta_{U}'}, q_1^*G)|_{\Delta_{U}}. \]

Proof. We apply Proposition 10.1 to the case $M = U^2$, $N = \Delta_{U}$, $N' = \Delta_{U}'$, $N^+ = \Delta_{U}^+$. We have $T^*_N M = \{(x, t, x, t; \xi, -\xi, -\tau); \tau \leq 0\}$. Since $SS(R\mathcal{H}\text{om}(q_2^{-1}F, q_1^*G))$ is contained in $\{(x_1, t_1, x_2, t_2; \xi_1, \tau_1, \xi_2, \tau_2); \tau_1 > 0, \tau_2 < 0\}$, the hypothesis of Proposition 10.1 are satisfied and we obtain
\[ R\tilde{\nu}_*(\mu\text{hom}(F,G)|_{\tilde{\tau}-U}) \simeq (R\Gamma_{\Delta_{U}'}R\mathcal{H}\text{om}(q_2^{-1}F, q_1^*G))|_{\Delta_{U}}. \]

We enter $\Delta_{U}^+$ inside $R\mathcal{H}\text{om}$ and the result follows from $(q_2^{-1}F)|_{\Delta_{U}'} \simeq (q_2^{-1}F)|_{\Delta_{U}^+}$ and $R\Gamma_{\Delta_{U}'}(q_1^*G) \simeq q_1^*G$. \hfill \Box

10.3. Microlocalization and convolution. In (10.7) below, we express $\mu\text{hom}$ using the convolution product of section 9. This is the main step in the proof of (11.7). We consider an open subset $U \subset M \times \mathbb{R}$. We will use the maps
\begin{align*}
\iota & = \iota_U : U \to U \times \mathbb{R}_{\geq 0}, \quad (x, t) \mapsto (x, t, 0), \\
\iota & = j_U : U \times \mathbb{R}_{> 0} \to U \times \mathbb{R}_{\geq 0}, \quad (x, t, u) \mapsto (x, t, u).
\end{align*}

We also use the notations of (10.2) and $q = q_M$ defined in (9.3). We define the projections $q'', q_1'', q_2''$ from $\Delta_{U}' \times \mathbb{R}_{> 0}$ so that we have the commutative diagram
\begin{equation}
\begin{array}{ccc}
U \times \mathbb{R}_{> 0} & \xrightarrow{q'' = q_1'' \times \text{id}} & \Delta_{U}' \times \mathbb{R}_{> 0} \\
\downarrow q & & \downarrow q_2'' = q_2' \times \text{id} \\
U & \xrightarrow{q_1''} & \Delta_{U}' \\
\end{array}
\end{equation}

We define $\Gamma \subset M \times \mathbb{R}^2 \times \mathbb{R}_{> 0}$ by $\Gamma = \{(x, t_1, t_2, u); |t_1 - t_2| < u\}$.

Lemma 10.3. For any $F \in D^b(\mathcal{K}_{\Delta_{U}'})$ we have
\begin{equation}
F|_{\Delta_{U}} \simeq i^{-1}Rj_*Rq_1'' \circ R\Gamma_{\Gamma}(q''^{-1}(F)).
\end{equation}

Proof. We let $\overline{q_1''}, \overline{q''} : \Delta_{U}' \times \mathbb{R} \to U \times \mathbb{R}$ be the projections extending $q_1''$ and $q''$, that is, $\overline{q_1''}(x, t) = (x, t_1, t_3)$, $\overline{q''}(x, t) = (x, t_1, t_2)$. We let $j' : \Delta_{U}' \times \mathbb{R}_{> 0} \to \Delta_{U}' \times \mathbb{R}$ be the inclusion and $\Gamma$ the closure of $\Gamma$ in $M \times \mathbb{R}^3$. Then $j' \circ q_1'' = \overline{q_1''} \circ j'$ and
\[ Rj'_*R\Gamma_{\Gamma}(q''^{-1}(F)) \simeq R\Gamma_{\Gamma}(\overline{q''}^{-1}(F)). \]

Using Example 2.2 (iv) and Theorem 2.5 (iii) we see that $SS(\mathcal{K}_{\Gamma}) \cap SS(q''^{-1}(F))$ is contained in the zero section. Hence $R\Gamma_{\Gamma}(\overline{q''}^{-1}(F)) \simeq
We define the functor $\Psi$ of Corollary 2.6 (ii) and the right hand side of (10.5) is isomorphic to $i^{-1}\text{R}q''_*((q''^{-1}F)|_{\Gamma \cap q''^{-1}\Delta U})$. We conclude with the isomorphism $(q''^{-1}F)|_{\Delta U \times \{0\}} \simeq (F)|_{\Delta U}$ and the base change formula. \hfill $\Box$

**Lemma 10.4.** For any $G \in \mathbf{D}^b(k_{U \times \mathbb{R}_>0})$ we have

\[(10.6) \quad \text{R}q''_1((q''^{-1}G)|_{\Gamma \cap q''^{-1}\Delta U})|(U \times \mathbb{R}_>) \simeq k_\gamma \ast G.\]

**Proof.** We define $\phi : M \times \mathbb{R}_2 \times \mathbb{R}_{>0} \xrightarrow{\sim} M \times \mathbb{R}_2 \times \mathbb{R}_{>0}, (x, t_1, t_2, u) \mapsto (x, t_1 - t_2, t_2, u)$. We have $q''_1 = s \circ \phi$, where $s(x, t_1, t_2, u) = (x, t_1 + t_2, u)$. $q''_2 \circ \phi^{-1} = q''_2$ and $\phi(\Gamma \cap q''^{-1}\Delta U) = q''_1^{-1}(\gamma)$. Replacing $q''_1$ and $q''_2$ by these expressions we see that the left hand side of (10.6) is isomorphic to $\text{R}s_!(q''_2^{-1}G \otimes q''_1^{-1}k_\gamma)$, which is the definition of $k_\gamma \ast G$. \hfill $\square$

**Proposition 10.5.** Let $M$ be a manifold and $U$ an open subset of $M \times \mathbb{R}$. Let $F, G \in \mathbf{D}^b_{\mathbb{Q}}(k_U)$. Then

\[(10.7) \quad \text{R}s_!(\mu\text{hom}(F, G)|_{\Gamma \cap q''^{-1}\Delta U}) \simeq i^{-1}\text{R}j_!(\text{R}\text{Hom}(k_\gamma \ast q^{-1}F, q^{-1}G)).\]

**Proof.** By Corollary 10.2 the formula (10.5) and the commutative diagram (10.4), the left hand side of (10.7) is isomorphic to

\[
i^{-1}\text{R}j_*\text{R}q''_*\text{R}\text{Hom}((q''^{-1}F)|_{\Delta U'}, q''_1^{-1}(G))
\simeq i^{-1}\text{R}j_*\text{R}q''_*\text{R}\text{Hom}((q''^{-1}q^{-1}F)|_{\Gamma \cap q''^{-1}\Delta U}, q''_1^{-1}q^{-1}G))
\simeq i^{-1}\text{R}j_*\text{R}\text{Hom}(\text{R}q''_!(q''^{-1}q^{-1}F)|_{\Gamma \cap q''^{-1}\Delta U}, q^{-1}G)).\]

We conclude with (10.6). \hfill $\square$

11. The functor $\Psi_U$

Motivated by the formula (10.7) we define the functor $\Psi_U$ below. The main result of this section is the formula (11.7). Together with (11.3) it says that the functor $\Psi_U$ induces an isomorphism between sections of $\mu\text{hom}$ outside the zero-section and sections of a usual $\text{R}\text{Hom}$.

We consider a manifold $M$ and we use the notations $i, j, q, r$ of (9.6) and (10.3). We recall that $s : M \times \mathbb{R}_2 \times \mathbb{R}_{>0} \to M \times \mathbb{R} \times \mathbb{R}_{>0}$ is the sum $s(x, t_1, t_2, u) = (x, t_1 + t_2, u)$. We also consider an open subset $U \subset M \times \mathbb{R}$ and we set for short

\[(11.1) \quad U_\gamma := (q^{-1}U)|_{\Gamma} = \{(x, t, u) \in M \times \mathbb{R} \times \mathbb{R}_{>0}; \{x\} \times [t - u, t] \subset U\}.

**Definition 11.1.** We define the functor $\Psi_U : \mathbf{D}^b(k_U) \to \mathbf{D}^b(k_{U_\gamma})$ by $\Psi_U(F) = k_\gamma \ast q^{-1}F \simeq \text{R}s_!(F \boxtimes k_{\{(t_1, u); 0 \leq t_1 < u\}})$. 
We recall that $D^b_{\tau>0}(k_U)$ is introduced in Definition 8.7. For $F, G \in D^b_{\tau>0}(k_U)$ the distinguished triangle (9.7) and the formula (10.7) are rewritten,

\[(11.2) \quad r^{-1}(F)[-1] \xrightarrow{\beta} \Psi_U(F) \xrightarrow{\alpha} q^{-1}(F) \xrightarrow{\delta},\]

\[(11.3) \quad R\pi_U(\mu\text{hom}(F,G)|_{{\mathcal{T}}\cdot U}) \simeq i^{-1}Rj_*\langle R\text{Hom}(\Psi_U(F),q^{-1}(G))\rangle.\]

**Definition 11.2.** If $V$ is an open subset of $M \times \mathbb{R} \times \mathbb{R}_{>0}$, we set

$$b(V) = \text{Int}_{M \times \mathbb{R} \times \mathbb{R}_{>0}}([M \times \mathbb{R} \times \mathbb{R}_{>0}] \cap (M \times \mathbb{R} \times \{0\}),$$

where the closure and the interior are taken in $M \times \mathbb{R} \times \mathbb{R}_{>0}$.

In other words a point $(x,t) \in M \times \mathbb{R}$ belongs to $b(V)$ if there exists a neighborhood $W$ of $(x,t,0)$ in $M \times \mathbb{R} \times \mathbb{R}_{>0}$ such that $W \cap (M \times \mathbb{R} \times \mathbb{R}_{>0}) \subset V$. In particular $b(V)$ is an open subset of $M \times \mathbb{R}$.

11.1. **Link with microlocalization.** In this paragraph we prove Theorem 11.5 which gives a symmetric version of formula (11.3) with respect to $F$ and $G$. This is the starting point of the definition of $\mu$-sheaves in Section 13.

Let $V$ be an open subset of $M \times \mathbb{R} \times \mathbb{R}_{>0}$. Then, for $F, G \in D^b(k_U)$, $k \in \mathbb{Z}$ and $(x,t) \in b(V)$ we have

\[(11.4) \quad H^k(i^{-1}Rj_*(F))_{(x,t)} \simeq \lim_{W} H^k(W;F),\]

\[(11.5) \quad H^k(i^{-1}Rj_*(\mathcal{R}\text{Hom}(F,G)))_{(x,t)} \simeq \lim_{W} \text{Hom}(F|_W,G|_W[k]),\]

where $W$ runs over the open subsets of $M \times \mathbb{R} \times \mathbb{R}_{>0}$ such that $(x,t) \in b(W)$.

**Lemma 11.3.** Let $U \subset M \times \mathbb{R}$ be an open subset and $F \in D^b(k_U)$. Let $U' \subset U$ be a relatively compact open subset and let $\varepsilon > 0$ be small enough so that $U_\varepsilon := r^{-1}(U') \cap (M \times \varepsilon \times \varepsilon)$ is contained in $U_{r_\gamma}$. We let $r_\varepsilon: U_\varepsilon \to U'$ be the restriction of $r$. Then for any $F \in D^b(k_U)$ we have $\mathcal{R}r_\varepsilon!(\Psi_U(F)|_{U_\varepsilon}) \simeq 0$.

**Proof.** We define $r': M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R}^2$, $(x,t_1,t_2,u) \mapsto (x,t_1,t_2-u)$ and $s': M \times \mathbb{R}^2 \to M \times \mathbb{R}$, $(x,t_1,t_2) \mapsto (x,t_1+t_2)$. We have the commutative diagram

\[
\begin{array}{ccc}
M \times \mathbb{R}^2 \times \mathbb{R}_{>0} & \xrightarrow{r'} & M \times \mathbb{R}^2 \\
\downarrow s & & \downarrow s' \\
M \times \mathbb{R} \times \mathbb{R}_{>0} & \xrightarrow{r} & M \times \mathbb{R}.
\end{array}
\]
Proof.

We set \( \gamma_{\varepsilon} = \{(t, u) \in \mathbb{R} \times \mathbb{R}_{>0}; 0 \leq t < u < \varepsilon\}\). Then we have

\[
Rr_{\varepsilon!}((k_{\gamma} \ast q^{-1}F)|_{U_{\varepsilon}'}) \simeq R(r \circ s)_!(F \boxtimes k_{\gamma_{\varepsilon}})
\]

\[
\simeq R(s' \circ r')!(r'^{-1}(F \boxtimes k_{\mathbb{R}}) \otimes k_{M \times \mathbb{R} \times \gamma_{\varepsilon}})
\]

\[
\simeq Rs'!((F \boxtimes k_{\mathbb{R}}) \otimes Rr'!(k_{M \times \mathbb{R} \times \gamma_{\varepsilon}}))
\]

and it is enough to prove that \( Rr'!(k_{M \times \mathbb{R} \times \gamma_{\varepsilon}}) \simeq 0\). For any \((x, t_1, t_2) \in M \times \mathbb{R}^2\) the intersection \(r'^{-1}(x, t_1, t_2) \cap (M \times \mathbb{R} \times \gamma_{\varepsilon})\) is empty or a half closed interval. This implies \( Rr'!(k_{M \times \mathbb{R} \times \gamma_{\varepsilon}}) \simeq 0\).

\[\square\]

**Proposition 11.4.** Let \( U \) be an open subset of \( M \times \mathbb{R} \). For any \( F, G \in D^b(k_U) \), we have

\[
i^{-1}Rj_*R\mathcal{H}om(\Psi_U(F), r^{-1}(G)) \simeq 0,
\]

\[
i^{-1}Rj_*R\mathcal{H}om(\Psi_U(F), \Psi_U(G)) \simeq 0.
\]

Proof. (i) Let us prove (11.6). By (11.5) it is sufficient to prove, for any \( k \in \mathbb{Z}\),

\[
\text{Hom}(\Psi_U(F)|_{U'}, (r^{-1}(G)[k])|_{U'}) \simeq 0,
\]

where \( U' \) is any relatively compact open subset of \( U \), \( \varepsilon > 0 \) and \( U'_{\varepsilon} \) is defined in Lemma 11.3. Since \( r \) is a submersion we have \( r^{-1} \simeq r'[-1] \) and the result follows from Lemma 11.3 and the adjunction formula between \( Rr_1 \) and \( r' \).

(ii) The mapping cone of (11.7) is \( i^{-1}Rj_*R\mathcal{H}om(\Psi_U(F), k_{\varepsilon} \ast q^{-1}(G)) \).

By Lemma 9.9 we have \( \kappa_{\varepsilon} \ast q^{-1}(G) \simeq r^{-1}(G)[-1] \) and the result follows from (11.6). \[\square\]

The formulas (11.3) and (11.7) give

**Theorem 11.5.** Let \( U \) be an open subset of \( M \times \mathbb{R} \) and \( F, G \in D^b_{r>0}(k_U) \). Then

\[
R\hat{\pi}_U!(\mu\text{hom}(F, G)|_{T^*_U}) \simeq i^{-1}Rj_*R\mathcal{H}om(\Psi_U(F), \Psi_U(G))).
\]

11.2. **Microsupport of \( \Psi_U(F) \).** Let \( U \) be an open subset of \( M \times \mathbb{R} \).

**Lemma 11.6.** Let \( L \in D^b_{r>0}(k_U) \) and let \( V \subset U \) be an open subset. We assume that \( SS(L|_V) \subset T^*_V \). Then \( \Psi_U(L)|_{(q^{-1}(V))_{qq}} \simeq 0 \). In particular \( \text{supp}(\Psi_U(L)) \subset (T^*_V \ast \pi(U)) \cap U'_{\gamma} \).

Proof. It is enough to prove \( \Psi_U(L)|_{(q^{-1}(V))_{qq}} \simeq 0 \), for any \( x \in M \). We set \( V_x = V \cap (\{x\} \times \mathbb{R}) \). By Lemma 11.5 we have \( \Psi_U(L)|_{(q^{-1}(V))_{qq}} \simeq 0 \). But \( V_x \) is a disjoint union of open intervals of \( \mathbb{R} \) and \( L|_V \) is constant on each of these intervals. A direct computation gives \( \Psi_U(L|_V) \simeq 0 \) and we obtain the result. \[\square\]
Proposition 11.7. Let $F, G \in D^b_{>0}(k_U)$. Then

\begin{equation}
\hat{\text{SS}}(\Psi_U(F)) = (q_dq_\pi^{-1}(\text{SS}(F)) \cup r_dr_\pi^{-1}(\text{SS}(F))) \cap \hat{T}^*U_\gamma,
\end{equation}

\begin{equation}
\muhom(\Psi_U(F), \Psi_U(G))|_{\hat{T}^*U_\gamma} \simeq (q_dq_\pi^{-1}\muhom(F, G) \oplus r_dr_\pi^{-1}\muhom(F, G))|_{\hat{T}^*U_\gamma}.
\end{equation}

Proof. (i) The first equality follows from the triangle (11.2), the triangular inequality for the microsupport and the fact that $\hat{\text{SS}}(q^{-1}F)$ and $\text{SS}(r^{-1}F)$ are disjoint.

(ii) We set for short \(Z_1 = q_dq_\pi^{-1}(\hat{T}^*(M \times \mathbb{R})) \cap \hat{T}^*U_\gamma, Z_2 = r_lr_\pi^{-1}(\hat{T}^*(M \times \mathbb{R})) \cap \hat{T}^*U_\gamma\) and $A = \muhom(\Psi_U(F), \Psi_U(G))|_{\hat{T}^*U_\gamma}$. By (11.8) we have $\text{supp} A \cap \hat{T}^*U_\gamma \subset Z_1 \cup Z_2$. Since $Z_1$ and $Z_2$ are disjoint closed subsets of $T^*U$, we deduce $A \simeq A_{Z_1} \oplus A_{Z_2}$.

Since $\text{SS}(r^{-1}F) \cap Z_1 = \emptyset$ we have $\muhom(r^{-1}F, \Psi_U(G))|_{Z_1} \simeq 0$. By the distinguished triangle (11.2) we deduce $A|_{Z_1} \simeq \muhom(q^{-1}F, \Psi_U(G))|_{Z_1}$. Using (11.2) for $G$ the same argument gives

$$A|_{Z_1} \simeq \muhom(q^{-1}F, q^{-1}G)|_{Z_1}.$$  

By [7, Prop. 4.4.7] we have $\muhom(q^{-1}F, q^{-1}G) \simeq q_dq_\pi^{-1}\muhom(F, G)$. In particular its support is contained in $Z_1$. A similar result holds for $A|_{Z_2}$, with $q$ replaced by $r$, and we deduce (11.9).

Remark 11.8. We can use the decomposition (11.9) to recover the morphism in Theorem 11.5. In the situation of Proposition 11.7 we obtain the projection

$$\muhom(\Psi_U(F), \Psi_U(G))|_{\hat{T}^*U_\gamma} \rightarrow q_dq_\pi^{-1}(\muhom(F, G))|_{\hat{T}^*U_\gamma}.$$  

For $A \in D^b(k_{\hat{T}^*U})$ we have $R\hat{\pi}_{U_\gamma*}((q_dq_\pi^{-1}(A))|_{\hat{T}^*U_\gamma}) \simeq q^{-1}R\hat{\pi}_{U_\gamma*}(A|_{\hat{T}^*U})$ and we deduce the morphism

$$R\hat{\text{Hom}}(\Psi_U(F), \Psi_U(G)) \rightarrow R\hat{\pi}_{U_\gamma*}(\muhom(\Psi_U(F), \Psi_U(G))|_{\hat{T}^*U_\gamma})$$

$$\rightarrow q^{-1}R\hat{\pi}_{U_\gamma*}(\muhom(F, G)|_{\hat{T}^*U}).$$

For any $F \in D^b(k_U)$ we have $i^{-1}Rj_*q^{-1}F \simeq F$ and we obtain a morphism

$$i^{-1}Rj_*R\hat{\text{Hom}}(\Psi_U(F), \Psi_U(G)) \rightarrow R\hat{\pi}_{U*}(\muhom(F, G)|_{\hat{T}^*U}).$$

By Theorem 11.5 this is an isomorphism.
12. Adjunction properties

Let $U$ be an open subset of $M \times \mathbb{R}$ and let $F, G \in \mathcal{D}^b_{\tau > 0}(k_U)$. We consider a morphism $u: \Psi_U(F) \to \Psi_U(G)$ in $\mathcal{D}^b(k_{U_\gamma})$. The main result of this section is Theorem [12.3] which says roughly that, locally near the boundary $U \times \{0\}$ of $U_\gamma$, $u$ is of the form $\Psi_U(v)$. This will be useful to show that the category of $\mu$-sheaves introduced in Section 13 is triangulated.

For a given $\varepsilon > 0$ we denote by $q_\varepsilon: M \times \mathbb{R} \times [0, \varepsilon[ \to M \times \mathbb{R}$ the projection. We recall the functors $\Phi_\varepsilon$, $\Phi'_\varepsilon: \mathcal{D}^b(k_{M \times \mathbb{R} \times [0,\varepsilon[}) \to \mathcal{D}^b(k_{M \times \mathbb{R} \times [0,\varepsilon[})$, introduced in (9.8), and we define

$$
\Psi_\varepsilon: \mathcal{D}^b(k_{M \times \mathbb{R}}) \to \mathcal{D}^b(k_{M \times \mathbb{R} \times [0,\varepsilon[}), \quad F \mapsto \Phi_\varepsilon(q_\varepsilon^{-1}(F)),
$$

$$
\Psi'_\varepsilon: \mathcal{D}^b(k_{M \times \mathbb{R} \times [0,\varepsilon[}) \to \mathcal{D}^b(k_{M \times \mathbb{R}}), \quad F \mapsto Rq_{\varepsilon!}(\Phi'_\varepsilon(F)[1]),
$$

By Proposition 9.10 the functor $\Psi'_\varepsilon$ is left adjoint to $\Psi_\varepsilon$. For an open subset $U \subset M \times \mathbb{R}$ we set $U_\varepsilon = U_\gamma \cap (M \times \mathbb{R} \times [0,\varepsilon[)$. Lemma 9.3 implies $\Psi_\varepsilon(F)|_{U_\varepsilon} \simeq \Psi_U(F|_U)|_{U_\varepsilon}$.

**Lemma 12.1.** For any $F \in \mathcal{D}^b(k_{M \times \mathbb{R}})$ and any $\varepsilon > 0$ we have

$$
\Psi'_\varepsilon \circ \Psi_\varepsilon(F) \simeq k_{[0,\varepsilon[} \star F \simeq \Psi_M(F)|_{M \times \mathbb{R} \times [\varepsilon]}.
$$

**Proof.** By (9.9) we have $\Psi'_\varepsilon \circ \Psi_\varepsilon(F) \simeq Rq_{\varepsilon!}((k_{\varepsilon'})[2] \oplus k_\gamma[1]) \star q_\varepsilon^{-1}(F))$. By the projection formula we deduce $\Psi'_\varepsilon \circ \Psi_\varepsilon(F) \simeq K \star F$, where $K = Rq_{\varepsilon!}((k_{\varepsilon'})[2] \oplus k_\gamma[1])$. We have $Rq_{\varepsilon!}(k_{\varepsilon'}) \simeq 0$ and $Rq_{\varepsilon!}(k_\gamma) \simeq k_{[0,\varepsilon[}[−1]$. This gives the first isomorphism. The second one follows by base change.

**Lemma 12.2.** Let $F \in \mathcal{D}^b(k_{M \times \mathbb{R}})$, $t, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Let $U$ be an open subset of $M \times \mathbb{R}$. We assume that $\text{supp}(F) \subset M \times [t, +\infty[$, that $U \subset M \times ]t, +\infty[$ and that $F|_U \in \mathcal{D}^b_{\tau > 0}(k_U)$. We set $F' = \Psi'_\varepsilon \circ \Psi_\varepsilon(F)$ and we consider a distinguished triangle

$$
F' \xrightarrow{a} F \to L \xrightarrow{+1},
$$

where $a$ is the adjunction morphism. Then $\text{SS}(L|_U)$ is contained in the zero section of $T^*U$. In particular $F'|_U \in \mathcal{D}^b_{\tau > 0}(k_U)$ and $a$ induces an isomorphism $\Psi_U(F'|_U) \to \Psi_U(F|_U)$ in $\mathcal{D}^b(k_{U_\gamma})$.

**Proof.** By Lemma 12.1 we have $L \simeq k_{[0,\varepsilon[} \star F \simeq Rs_!(F \boxtimes k_{[0,\varepsilon[})$. The map $s: M \times \mathbb{R}^2 \to M \times \mathbb{R}$ is proper on $S := \text{supp}(F \boxtimes k_{[0,\varepsilon[})$ and we can bound $\text{SS}(L)$ by Theorem 2.5 (ii) as follows.

For a given $(x, t_1, t_2) \in s^{-1}(U) \cap S$, we have $0 \leq t_2 \leq \varepsilon$, $t < t_1 + t_2 < t + \varepsilon$ by the hypothesis on $U$ and $t \leq t_1$ by the hypothesis on $\text{supp}(F)$.
Then we can bound $SS(F \boxtimes k_{[0, \varepsilon]})$ at $(x, t_1, t_2)$, in the first case by $(T_{r>0}^*(M \times \mathbb{R}) \cup T_{M \times \mathbb{R}}^*(M \times \mathbb{R})) \times T_{r<0}^* \mathbb{R}$ and in the second case by $T^*(M \times \mathbb{R}) \times T_{\mathbb{R}}^* \mathbb{R}$. In both cases we obtain

$$s_d^{-1}(SS(F \boxtimes k_{[0, \varepsilon]}) \cap T_{(x, t_1, t_2)}^*(M \times \mathbb{R}^2)) \subset s_n^{-1}(T_U^* U)$$

and we deduce $SS(L|_U) \subset T_U^* U$. By the triangular inequality for the microsupport it follows that $F''|_U \in D_{r>0}(k_U)$ and the last assertion follows from Lemma [11.6]

**Theorem 12.3.** Let $U$ be an open subset of $M \times \mathbb{R}$ and let $F, G \in D_{r>0}^b(k_U)$. We consider a morphism in $D^b(k_{U'})$

$$u: \Psi_U(F) \rightarrow \Psi_U(G).$$

and $(x_0, t_0) \in U$. Then there exist an open neighborhood, $V$, of $(x_0, t_0)$, $F', G' \in D_{r>0}^b(k_V)$ and morphisms

$$a: F' \rightarrow F|_V, \quad b: G' \rightarrow G|_V, \quad v: F' \rightarrow G'$$

such that the diagram in $D^b(k_{V'})$

$$\begin{array}{ccc}
\Psi_V(F') & \xrightarrow{\Psi_V(v)} & \Psi_V(G') \\
\Psi_V(a) \downarrow & & \downarrow \Psi_V(b) \\
\Psi_V(F) & \xrightarrow{u|_{V'}} & \Psi_V(G) \\
\end{array}$$

commutes and $\Psi_V(a), \Psi_V(b)$ are isomorphisms.

**Proof.** The idea of the proof is to define $F' = \Psi'_\varepsilon \circ \Psi'_\varepsilon(F)$, $G' = \Psi'_\varepsilon \circ \Psi'_\varepsilon(G)$ and $v = \Psi'_\varepsilon(u)$. Then the diagram of the theorem is nothing but the right hand square in the adjunction diagram (12.1) below. To make this idea work we use Lemma [12.2] and we begin with cutting $F$ and $G$ by suitable open subsets so that the hypothesis of the lemma are satisfied.

(i) Let $i_U$ be the embedding of $U$ in $M \times \mathbb{R}$. We define $F_1 = Ri_{U*}(F)$, $G_1 = Ri_{U*}(G)$. By Lemma [9.3] we have $\Psi_\varepsilon(F_1)|_{U_\varepsilon} \simeq \Psi_U(F)|_{U_\varepsilon}$ and $\Psi_\varepsilon(G_1)|_{U_\varepsilon} \simeq \Psi_U(G)|_{U_\varepsilon}$.

(ii) For any $H \in D^b(k_{M \times \mathbb{R}})$ and $\varepsilon > 0$ we have

$$\text{supp}(\Psi_\varepsilon(H)) \subset (\gamma * q^{-1}(\text{supp}(H))) \cap (M \times \mathbb{R} \times ]0, \varepsilon[).$$
Hence we can choose an open neighborhood \( V_1 \) of \((x_0, t_0)\) and \( \varepsilon > 0 \) such that \( \text{supp}(\Psi_\varepsilon((F_1)_V)) \subset U_\varepsilon^x \). We may also assume \((F_1)_V \simeq F_{V_1}\). This implies

\[
\text{Hom}_{D^b(k_M \times \mathbb{R} \times [a_\varepsilon])}(\Psi_\varepsilon((F_1)_V), \Psi_\varepsilon(G_1)) \\
\simeq \text{Hom}_{D^b(k_{U_\varepsilon^x})}(\Psi_U((F_1)_V)|_{U_\varepsilon^x}, \Psi_U(G)|_{U_\varepsilon^x})
\]

Hence \( u \) and the morphism \( F_{V_1} \to F \) induce \( u_1: \Psi_\varepsilon((F_1)_V) \to \Psi_\varepsilon(G_1) \) and we see that \( u_1|_{(V_1)_\varepsilon} = u|_{(V_1)_{\varepsilon}} \).

(iii) We choose \( t \) such that \( t_0 \in \]t, t+\varepsilon[\). We define \( V = V_1 \cap (M \times [t, t+\varepsilon]) \) and we set \( F_2 = (F_1)_V \), \( G_2 = (G_1)_V \) and we let \( u_2: \Psi_\varepsilon(F_2) \to \Psi_\varepsilon(G_2) \) be the morphism induced by \( u_1 \) and the natural morphisms \( F_2 \to F_1 \) and \( G_1 \to G_2 \). Then \( \Psi_\varepsilon(F_2)|_{V_\varepsilon} \simeq \Psi_U(F)|_{V_\varepsilon}, \Psi_\varepsilon(G_2)|_{V_\varepsilon} \simeq \Psi_U(G)|_{V_\varepsilon} \) and \( u_2|_{V_\varepsilon} = u|_{V_\varepsilon} \) (we note that \( V_\varepsilon \subset M \times \mathbb{R} \times [0, \varepsilon] \)). We remark that \( F_2|_V \simeq F|_V \) and \( G_2|_V \simeq G|_V \).

(iv) The morphisms of functors \( a: \text{id} \to \Psi_\varepsilon \circ \Psi_\varepsilon' \) and \( b: \Psi_\varepsilon' \circ \Psi_\varepsilon \to \text{id} \) give the diagram

\[
\begin{array}{c}
\Psi_\varepsilon(F_2) \xrightarrow{a_2 := a(\Psi_\varepsilon(F_2))} \Psi_\varepsilon \circ \Psi_\varepsilon'(F_2) \xrightarrow{b_2 := \Psi_\varepsilon(b(F_2))} \Psi_\varepsilon(G_2) \\
\Psi_\varepsilon(G_2) \xrightarrow{u_2} \Psi_\varepsilon \circ \Psi_\varepsilon'(G_2) \xrightarrow{u_2} \Psi_\varepsilon(G_2)
\end{array}
\]

(12.1)

where the left hand square commutes. We have \( b_2 \circ a_2 = \text{id} \) and \( b_2|_{V_\varepsilon} \) is an isomorphism by Lemma 12.2. Hence \( a_2|_{V_\varepsilon} \) is the inverse isomorphism. The same results hold for the bottom line. Since the big square commutes we deduce that the restriction of the right hand square to \( V_\varepsilon \) also commutes. We conclude by setting \( F' = (\Psi_\varepsilon \circ \Psi_\varepsilon(F_2))|_V, G' = (\Psi_\varepsilon \circ \Psi_\varepsilon(G_2))|_V \) and \( v = \Psi_\varepsilon|_{(u_2)} \).

13. \( \mu \)-sheaves

Let \( M \) be a manifold. We introduce a category of sheaves on \( M \times \mathbb{R} \times \mathbb{R}_{>0} \) defined in a neighborhood of the boundary \( M \times \mathbb{R} \times \{0\} \) which are locally of the type \( \Psi_U(F) \).

13.1. Definition and first properties.

**Definition 13.1.** For an open set \( U \subset M \times \mathbb{R} \) we define the category of \( \mu \)-sheaves on \( U \), denoted \( D^b(k_U^\mu) \), as follows.

(i) An object of \( D^b(k_U^\mu) \) is a pair \((V, F)\), where \( V \) is an open subset of \( M \times \mathbb{R} \times \mathbb{R}_{>0} \) and \( F \in D^b(k_V) \), such that

(a) \( U \subset b(V) \), where \( b(V) \) is given in Definition 11.2.
(b) there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $V \subseteq \bigcup_{i \in I} (U_i)_\gamma$,

(c) for each $i \in I$ there exist $F_i \in D^b_{r>0}(\mathbb{k}_{U_i})$ and an isomorphism $F|_{V \cap (U_i)_\gamma} \simeq \Psi_U(F_i)|_{V \cap (U_i)_\gamma}$.

(ii) For two sheaves $(V_1, F_1)$ and $(V_2, F_2)$ we let $j_{12}$ be the embedding of $V_1 \cap V_2$ in $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ and we set

$$\text{Hom}((V_1, F_1), (V_2, F_2)) = H^0(U; i_{12}^{-1} R\text{Hom}(F_1, F_2)).$$

The composition law is induced by the natural morphism $R\text{Hom}(F_1, F_2) \otimes R\text{Hom}(F_2, F_3) \to R\text{Hom}(F_1, F_3)$.

In other words a morphism $u$ from $(V_1, F_1)$ to $(V_2, F_2)$ is represented by a morphism $\bar{u}: F_1 \to F_2$ in $D^b(\mathbb{k}_V)$ defined on some open subset $V$ of $V_1 \cap V_2$ such that $U \subseteq b(V)$. Moreover, $u = 0$ if and only if there exists another open subset $V' \subseteq V$ such that $U \subseteq b(V')$ and $\bar{u}|_{V'} = 0$.

In particular for a sheaf $(V, F)$ and another open subset $V' \subseteq V$ with $U \subseteq b(V')$, the sheaves $(V, F)$ and $(V', F|_{V'})$ are isomorphic. Conversely, if $(V_1, F_1)$ and $(V_2, F_2)$ are isomorphic, then there exists an open subset $V$ of $V_1 \cap V_2$ such that $U \subseteq b(V)$ and $F_1|_V \simeq F_2|_V$. Hence we often write $F$ instead of $(V, F)$.

For an open subset $U' \subseteq U$ we have a restriction morphism

$$D^b(\mathbb{k}^\mu_{U'}) \to D^b(\mathbb{k}^\mu_U), \quad (V, F) \mapsto (V, F)|_{U'} := (V', F|_{V'}),$$

where $V' = V \cap \bigcup_{i \in I} (U'_i \cap U_i)_\gamma$ for some covering $U = \bigcup_{i \in I} U_i$ satisfying (i-b) in Definition [13.1]

**Definition 13.2.** For $(V_1, F_1)$ and $(V_2, F_2)$ in $D^b(\mathbb{k}^\mu_U)$ we set

$$\text{Hom}^\mu(F_1, F_2) = i_{12}^{-1} Rj_{12} R\text{Hom}(F_1, F_2) \in D^b(\mathbb{k}_U),$$

where $j_{12}$ is the embedding of $V_1 \cap V_2$ in $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$. Then $\text{Hom}^\mu$ is a bifunctor from $D^b(\mathbb{k}^\mu_U)$ to $D^b(\mathbb{k}_U)$.

For any open subset $U' \subseteq U$ we obtain

$$\text{Hom}_{D^b(\mathbb{k}_{U'})}(F|_{U'}, G|_{U'}) \simeq H^0(U'; \text{Hom}^\mu(F, G)).$$

By definition the functor $\Psi_U: D^b(\mathbb{k}_U) \to D^b(\mathbb{k}_{U'})$ induces a functor denoted in the same way $\Psi_U: D^b_{r>0}(\mathbb{k}_U) \to D^b(\mathbb{k}^\mu_{U'})$.

Let $F \xrightarrow{u} G \xrightarrow{v} H \xrightarrow{w} F[1]$ be a triangle in $D^b(\mathbb{k}^\mu_U)$. We represent $u, v, w$ by morphisms $\tilde{u}, \tilde{v}, \tilde{w}$ in $D^b(\mathbb{k}_V)$, for some open subset $V$ of $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ such that $U \subseteq b(V)$. We say that our triangle is distinguished if $F|_{V'} \xrightarrow{\tilde{u}} G|_{V'} \xrightarrow{\tilde{v}} H|_{V'} \xrightarrow{\tilde{w}} F|_{V'}[1]$ is distinguished in $D^b(\mathbb{k}_V)$, where $V' \subseteq V$ is another open subset such that $U \subseteq b(V')$. 
Proposition 13.3. The assignment $U \mapsto D^b(k_U^\mu)$ is a prestack of triangulated categories on $M \times \mathbb{R}$.

Proof. The axioms of triangulated categories are easily verified except the fact that any morphism $F \xrightarrow{\mu} G$ may be embedded in a distinguished triangle. We represent $\mu$ by $\overline{\mu}$ and consider a distinguished triangle $F \xrightarrow{\overline{\mu}} G \xrightarrow{1} H \xrightarrow{+1}$ in $D^b(k_V)$, for some $V$ such that $U \subset b(V)$. For a given $(x,t) \in U$ we can find a neighborhood $W$ of $(x,t)$ and $F_1, G_1 \in D^b(k_W)$ such that $F \simeq \Psi_W(F_1)$ and $G \simeq \Psi_W(G_1)$. By Theorem [12.3] up to shrinking $W$, we can find $F', G' \in D^b(k_W)$, $a : F' \to F_1$, $b : G' \to G_1$ and $u : F' \to G'$ so that we have a commutative diagram in $D^b(k_W)$

$$
\begin{array}{ccc}
\Psi_W(F') & \xrightarrow{\Psi_W(u')} & \Psi_W(G') \\
\Psi_W(a) & \downarrow & \Psi_W(b) \\
\Psi_W(F_1) & \xrightarrow{u} & \Psi_W(G_1)
\end{array}
$$

and the vertical arrows are isomorphisms. Hence $H|_{W_y}$ is isomorphic to $\Psi_W(H')$, where $H'$ is the mapping cone of $u'$. Since this holds for any $(x,t) \in U$ we obtain that $H \in D^b(k_U^\mu)$.

13.2. Microsupport.

Lemma 13.4. Let $U \subset M \times \mathbb{R}$ be an open subset and let $(V,F) \in D^b(k_U^\mu)$. Then there exists a subset $\Lambda \subset T_{>0}^*U$ such that (recall that $q,r$ are given in (9.6))

$$
(13.1) \quad SS(F) \cap \hat{T}^*V = (q_dq_r^{-1}(\Lambda) \cup r_d^{-1}r_r^{-1}(\Lambda)) \cap \hat{T}^*V,
$$

$$
(13.2) \quad \text{supp}(F) \subset (\gamma^* (\pi_{M \times \mathbb{R}}(\Lambda))) \cap V.
$$

Moreover $\Lambda$ only depends on the isomorphism class of $(V,F)$ in $D^b(k_U^\mu)$.

Proof. (i) We choose a covering $U = \bigcup_{i \in I} U_i$ and $F_i \in D^b_{>0}(k_U)$ as in Definition [13.1]. We set $\Lambda = \bigcup_{i \in I} SS(F_i)$. Then the first equality follows from (11.8). We see that $\Lambda$ is independent of the choices with the following remark. Let $U' \subset M \times \mathbb{R}$ and $V' \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ be open subsets such that $U' \subset b(V')$. Let $\Lambda_1, \Lambda_2 \subset \hat{T}^*U'$ such that $(q_dq_r^{-1}(\Lambda_1) \cup r_d^{-1}r_r^{-1}(\Lambda_1)) \cap \hat{T}^*V' = (q_dq_r^{-1}(\Lambda_2) \cup r_d^{-1}r_r^{-1}(\Lambda_2)) \cap \hat{T}^*V'$. Then $\Lambda_1 = \Lambda_2$.

(ii) To bound $\text{supp}(F)$ we apply Lemma [11.6] to each $F_i$.

Definition 13.5. In the situation of Lemma [13.4] we denote by $SS^\mu(F)$ the subset $\Lambda \subset T_{>0}^*U$ satisfying [13.1].
13.3. **Morphisms in** $\mathcal{D}^b(k_U^\mu)$. Extending the maps $q$ and $r$ to $M \times \mathbb{R}^2$, we define, for an open subset $V$ of $M \times \mathbb{R}^2$,

$$
\check{T}^*V = T^*V \cap q_dq^{-1}_{\pi}(M \times \mathbb{R}), \quad \check{T}^*V = T^*V \cap r_d\pi^{-1}T^*(M \times \mathbb{R}).
$$

Let $(V, F), (W, G) \in \mathcal{D}^b(k_U^\mu)$. By Lemma 13.4, we have, setting $V' = V \cap W$,

$$
(13.3) \quad \mu_{\text{hom}}(F, G)|_{\check{T}^*V'} \simeq (\mu_{\text{hom}}(F, G))_{\check{T}^*V'} \oplus (\mu_{\text{hom}}(F, G))_{\check{T}^*V'}. \quad \text{(13.3)}
$$

For an open subset $U \subset M \times \mathbb{R}$ and $V$ such that $U \subset b(V)$ we have $T^*b(U \times \mathbb{R}) = T^*U \times \mathbb{R}$ and we have natural embeddings $i_U^*: T^*U \to T^*b(U \times \mathbb{R})$, $j_U^*: T^*b(U \times \mathbb{R}) \to T^*b(U \times \mathbb{R})$.

**Definition 13.6.** For $(V, F), (W, G) \in \mathcal{D}^b(k_U^\mu)$ we define

$$
\mu_{\text{hom}}^\mu(F, G) = j_{V\cap W}^{-1}(\mu_{\text{hom}}(F, G)|_{T^*b(V \cap W)}) \in \mathcal{D}^b(k_{T^*b}.U).
$$

We can see that $\mu_{\text{hom}}^\mu$ induces a bifunctor from $\mathcal{D}^b(k_U^\mu)$ to $\mathcal{D}^b(k_{T^*b.U})$. The projection to the first term in (13.3) induces a morphism

$$
R\mathcal{H}om(F, G) \to R\check{\pi}_{V'}(\mu_{\text{hom}}(F, G)) \to R\check{\pi}_{V'}(\mu_{\text{hom}}(F, G))_{\check{T}^*b.V'}.
$$

where $V' = V \cap W$. Applying the functor $j^{-1}Rf_*$ we deduce

$$
(13.4) \quad \mathcal{H}om^\mu(F, G) \to R\check{\pi}_*((\mu_{\text{hom}}^\mu)(F, G)). \quad \text{(13.4)}
$$

**Proposition 13.7.** We consider an open subset $U$ of $M \times \mathbb{R}$ and $F, G \in \mathcal{D}^b(k_U^\mu)$. Then the morphism (13.4) is an isomorphism. In particular

$$
\text{Hom}_{\mathcal{D}^b(k_U^\mu)}(F, G) \simeq \lim_{V, U \subset U} H^0(\check{T}^*b.V; \mu_{\text{hom}}(F, G)|_{\check{T}^*b.V}).
$$

**Proof.** Since the statement is local on $U$ we may assume, up to shrinking $U$, that $F = \Psi_U(F_0), G = \Psi_U(G_0)$ for some $F_0, G_0 \in \mathcal{D}^b(k_U)$. Then the result follows from Theorem 11.5 and Remark 11.8. \qed

13.4. **Inverse image.** Let $N$ be another manifold and let $f: M \to N$ be a morphism of manifolds. We set $\tilde{f} = f \times \text{id}_\mathbb{R}: M \times \mathbb{R} \to N \times \mathbb{R}$. Let $V$ be an open subset of $N \times \mathbb{R}$ and let $G \in \mathcal{D}^b_{r>0}(k_V)$. We see that $\tilde{f}$ is non-characteristic for $G$ and that $\tilde{f}^{-1}(G) \in \mathcal{D}^b_{T>0}(k_{\tilde{f}^{-1}(V)})$. The base change formula also gives $(\tilde{f} \times \text{id}_{\mathbb{R}>0})^{-1}\Psi_V(G) \simeq \Psi_{\tilde{f}^{-1}(V)}(\tilde{f}^{-1}(G))$. We leave the following result to the reader.

**Lemma 13.8.** The inverse image by $\tilde{f} \times \text{id}_{\mathbb{R}>0}$ induces a functor $\tilde{f}^{-1}: \mathcal{D}^b(k_V^\mu) \to \mathcal{D}^b(k_{\tilde{f}^{-1}(V)}^\mu)$. Moreover, for any $G \in \mathcal{D}^b(k_V^\mu)$ we have the bound $SS^\mu(\tilde{f}^{-1}G) \subset \tilde{f}_d\tilde{f}^{-1}(SS^\mu(G))$. 

14. Gluing $\mu$-sheaves

In this section we see how it is possible to glue $\mu$-sheaves. For two subsets $U_i, U_j$ of a given set we use the notation $U_{ij} := U_i \cap U_j$.

**Lemma 14.1.** Let $U_1, U_2 \subset M \times \mathbb{R}$ and $V, V_1, V_2 \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ be open subsets. We assume that $V \subset V_{12}, U_i \subset b(V_i), i = 1, 2,$ and $U_{12} \subset b(V)$. Then there exist open subsets $V'_i \subset V_i, i = 1, 2,$ such that $U_i \subset b(V'_i)$ and $V'_1 \cap V'_2 = V$.

**Proof.** By shrinking $V_i, i = 1, 2,$ we can assume that $b(V_i) = U_i$. We choose a distance $d$ on $M \times \mathbb{R}^2$ and define

$$V'_1 = \{ y \in V_1 ; d(y, V_1 \setminus V_2) < d(y, V_2 \setminus V_1) \} \cup V.$$ We define $V'_2$ by exchanging the roles of $V_1$ and $V_2$. Then the required properties are easily checked. \qed

**Lemma 14.2.** Let $U_1, U_2 \subset M \times \mathbb{R}$ be open subsets. We consider $F_i \in D^b(k_{U_i}^\mu), i = 1, 2,$ and an isomorphism $\varphi : F_1|_{U_{12}} \sim F_2|_{U_{12}}$. We assume that $\text{Hom}^b(F_i, F_i) \in D^b(k_{U_i}^\mu)$ is concentrated in degrees $\geq 0$, for $i = 1, 2$. Then there exists a unique $\mu$-sheaf $F \in D^b(k_{U_{12}}^\mu)$ with isomorphisms $\varphi_i : F_i \sim F|_{U_i}, i = 1, 2$ such that $\varphi_1|_{U_{12}} = \varphi_2|_{U_{12}} \circ \varphi|_{U_{12}}$.

Moreover for any open subset $U' \subset U$ and $F' \in D^b(k_{U'}^\mu)$ such that $\text{Hom}^b(F_i|_{U' \cap U_i}, F'|_{U' \cap U_i}) \in D^b(k_{U'}^\mu)$ is concentrated in degrees $\geq 0$, for $i = 1, 2,$ we have the exact sequence

$$0 \to \text{Hom}_{D^b(k_{U_i}^\mu)}(F, F') \to \text{Hom}_{D^b(k_{U_{12}}^\mu)}(F_1, F') \oplus \text{Hom}_{D^b(k_{U_{12}}^\mu)}(F_2, F') \to \text{Hom}_{D^b(k_{U_{12}}^\mu)}(F_1, F'),$$

where $\text{Hom}_{D^b(k_{U_i}^\mu)}(F'', F')$ means $\text{Hom}_{D^b(k_{U_i}^\mu)}(F''|_W, F'|_W)$.

**Proof.** (i) We first define $F$. We choose representatives $(V_i, G_i)$ of $F_i$. Then there exist $V \subset V_{12}$ such that $U_{12} \subset b(V)$ and an isomorphism $\tilde{\varphi} : G_1|_V \to G_2|_V$ which represents $\varphi$. By Lemma 14.1 we may assume that $V = V_{12}$. Let $j_i : V_i \to V_1 \cup V_2$ be the embedding. We define $G \in D^b(k_{V_1 \cup V_2}^\mu)$ by the distinguished triangle

$$(14.2) \quad j_{1!}(G_1)|_{V_1} \xrightarrow{a_1 + a_2} j_{1!}(G_1) \oplus j_{2!}(G_2) \xrightarrow{b} G \xrightarrow{+1}$$

where $a_1$ is induced by $(G_1)|_{V_1} \to G_1$ and $a_2$ is induced by $\tilde{\varphi}$ and $(G_2)|_{V_2} \to G_2$. Then $b$ induces $b_i : j_{i!}(G_i) \to G, i = 1, 2,$ and we have $b_i|_{V_i} : G_i \sim G|_{V_i}$. Since $b \circ (a_1 + a_2) = 0$ the restriction to $V_{12} = V$ gives $b_1|_V = b_2|_V \circ \tilde{\varphi}$. Since any point $(x, t) \in U$ has a neighborhood $W$ in $M \times \mathbb{R} \times \mathbb{R}_{>0}$ such that $W \subset U_1 \cup V_1$ or $W \subset U_2 \cup V_2$, the pair $F := (V_1 \cup V_2, G)$ is a $\mu$-sheaf. Then $b_i$ induces $\varphi_i : F_i \sim F|_{U_i}, i = 1, 2$ such that $\varphi_1|_{U_{12}} = \varphi_2|_{U_{12}} \circ \varphi|_{U_{12}}$.\[\]
(ii) Let us prove the second part of the lemma. The morphism \( \varphi_i \) induces \( \text{Hom}^\mu(F|_{U_i}, F')|_{U_i} \cong \text{Hom}^\mu(F_i|_{U_i} \cap U_i, F'|_{U_i} \cap U_i) \). This proves that \( \text{Hom}^\mu(F|_{U_i}, F') \) is concentrated in degrees \( \geq 0 \) and also
\[
H^0(U_i \cap U_i; \text{Hom}^\mu(F|_{U_i}, F')) \cong \text{Hom}_{D^b(k_U)}(F_i, F'),
\]
\[
H^0(U_i \cap U; \text{Hom}^\mu(F|_{U_i}, F')) \cong \text{Hom}_{D^b(k_U \cap U)}(F_i, F').
\]
Now (14.1) follows from the Mayer-Vietoris sequence associated with the covering \( U = U_1 \cup U_2 \) and the object \( \text{Hom}^\mu(F|_{U}, F') \). The unicity of \( F \) follows from (14.1).

\[\square\]

**Lemma 14.3.** Let \( U \subset M \times \mathbb{R} \) be an open subset and \( U = \bigcup_{i=1}^{n} U_i \subset \) a finite covering of \( U \). We consider \( \mu \)-sheaves \( F_i \in D^b(k_U^\mu) \), \( i = 1, \ldots, n \), and isomorphisms \( \varphi_{ij} : F_i|_{U_{ij}} \cong F_j|_{U_{ij}} \), for all \( i < j \) such that, for all \( i < j < k \), we have \( \varphi_{ij}|_{U_{ijk}} \circ \varphi_{ji}|_{U_{ijk}} = \varphi_{ki}|_{U_{ijk}} \). We assume that \( \text{Hom}^\mu(F_i, F_i) \in D^b(k_U^\mu) \) is concentrated in degrees \( \geq 0 \), for any \( i \). Then there exists a unique \( \mu \)-sheaf \( F \in D^b(k_U^\mu) \) with isomorphisms \( \varphi_i : F_i \cong F|_{U_i}, i = 1, \ldots, n \) such that \( \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}} \circ \varphi_{ji} \), for all \( i < j \).

**Proof.** We prove the result by induction on \( n \). The case \( n = 2 \) is given by Lemma 14.2.

Let us assume the result is true for \( n - 1 \). We set \( U = U_1 \cup U_2 \) and consider \( F_0 \in D^b(k_U^\mu) \) given by Lemma 14.2, with the isomorphisms \( \varphi_i : F_i \cong F_0|_{U_i}, i = 1, 2 \). The exact sequence of Lemma 14.2 gives morphisms \( \psi_j : F_0|_{U \cap U_j} \to F_j|_{U \cap U_j}, \) for all \( j = 3, \ldots, n \) such that \( \psi_j|_{U_i \cap U_j} \circ \varphi_{ij}|_{U_i \cap U_j} = \varphi_{jij}|_{U_i \cap U_j} \), for all \( i, j, k \). In particular we can see that the \( \psi_j \) are isomorphisms.

Now we apply the case \( n - 1 \) to the family \( F_0, F_3, \ldots, F_n \) and the morphisms \( \psi_i, j = 3, \ldots, n, \varphi_{ji}, 3 \leq i < j \leq n \). We obtain a \( \mu \)-sheaf \( F \in D^b(k_U^\mu) \) and we can see that \( F \) also gives a gluing of the original family \( F_1, F_2, \ldots, F_n \).

\[\square\]

**Proposition 14.4.** Let \( U \subset M \times \mathbb{R} \) be an open subset and \( U = \bigcup_{i \in \mathbb{N}} U_i \) a covering of \( U \). We consider \( \mu \)-sheaves \( F_i \in D^b(k_U^\mu) \), \( i \in \mathbb{N} \), and isomorphisms \( \varphi_{ij} : F_i|_{U_{ij}} \cong F_j|_{U_{ij}} \), for all \( i < j \), such that, for all \( i < j < k \), we have \( \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi_{ki}|_{U_{ijk}} \). We assume that \( \text{Hom}^\mu(F_i, F_i) \in D^b(k_U^\mu) \) is concentrated in degrees \( \geq 0 \), for any \( i \). Then there exists a unique \( \mu \)-sheaf \( F \in D^b(k_U^\mu) \) with isomorphisms \( \varphi_i : F_i \cong F|_{U_i}, i \in \mathbb{N} \) such that \( \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}} \circ \varphi_{ji} \) for all \( i < j \).

**Proof.** For \( n \in \mathbb{N} \) we set \( U'_n = \bigcup_{i=0}^{n-1} U_i \). Lemma 14.3 gives a unique \( \mu \)-sheaf \( F'_n \in D^b(k_{U'_n}^\mu) \) which glue the \( F_i, i = 0, \ldots, n \), together.

We set \( U''_n = U'_n \setminus U'_{n-2} \) and \( F''_n = F'_n|_{U''_n} \). We define \( V_0 = \bigcup_{n \in \mathbb{N}} U_{2n} \) and \( V_1 = \bigcup_{n \in \mathbb{N}} U_{2n+1} \). Since \( V_0 \) is the disjoint union of the \( U_{2n} \) we can...
glue the $F''_{2n}$ into $G_0 \in \mathcal{D}^b(\mathbf{k}_{V_0}^\mu)$ (it is enough to give representatives $(W_{2n}, H_{2n})$ of the $F''_{2n}$ such that the $W_{2n}$ are disjoint). In the same way we can glue the $F''_{2n+1}$ into $G_1 \in \mathcal{D}^b(\mathbf{k}_{V_1}^\mu)$. We conclude by Lemma 14.3 applied to $G_0$ and $G_1$. \hfill \Box

\section*{Part 3. Quantization}

\section*{15. Quantization as a $\mu$-sheaf}

Let $\Lambda \subset T^*_{\tau > 0}(M \times \mathbb{R})$ be a locally closed conic subset and let $F \in \mathcal{G}(\mathbf{k}_\Lambda)$ be a global object of the Kashiwara-Schapira stack. Assuming that $F$ locally admits a representative in $\mathcal{D}^b_{\Lambda,T^*_{\tau > 0}M}(\mathbf{k}_{M \times \mathbb{R}})$ we prove that it admits a global representative $G \in \mathcal{D}^b(\mathbf{k}_V^\mu)$, where $U$ is an open subset containing $\pi_{M \times \mathbb{R}}(\Lambda)$, such that $\mathcal{S}\mathcal{S}^\mu(G) \subset \Lambda$. We also check that this hypothesis on $F$ is fulfilled as soon as $\Lambda$ is a locally closed conic connected Lagrangian submanifold of $T^*_{\tau > 0}(M \times \mathbb{R})$ in generic position.

\begin{definition}
Let $U \subset M \times \mathbb{R}$ be an open subset and let $\Lambda \subset T^*_{\tau > 0}U$ be a locally closed conic subset. We let $\mathcal{D}^b(\mathbf{k}_{U}^\mu)$ be the full subcategory of $\mathcal{D}^b(\mathbf{k}_V^\mu)$ formed by the $F$ such that $\mathcal{S}\mathcal{S}^\mu(F) \subset \Lambda$ (recall that $\mathcal{S}\mathcal{S}^\mu(F) \subset T^*_{\tau > 0}U$ is introduced in Definition 13.5).

Let $U \subset M \times \mathbb{R}$ and $V \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ be open subsets such that $b(V) = U$. We assume that $q : V \to M \times \mathbb{R}$ has connected fibers. Let $\Lambda \subset T^*_{\tau > 0}U$ be a locally closed conic subset. Then $q^{-1}$ induces an equivalence of stacks

\begin{equation}
\mathcal{G}(\mathbf{k}_\Lambda) \xrightarrow{\cong} q^* q^{-1}_d \mathcal{G}(\mathbf{k}_{q_d q^{-1}_d(\Lambda)}).
\end{equation}

Hence the functor $\mathbf{s}_{q_d q^{-1}_d(\Lambda)} : \mathcal{D}^b_{(q_d q^{-1}_d(\Lambda))}(\mathbf{k}_V) \to \mathcal{G}(\mathbf{k}_{q_d q^{-1}_d(\Lambda)})$ (see Definition 3.1) induces a functor

\begin{equation}
\mathbf{s}_\Lambda^\mu : \mathcal{D}^b(\mathbf{k}_U^\mu) \to \mathcal{G}(\mathbf{k}_\Lambda).
\end{equation}

\begin{theorem}
Let $\Lambda \subset T^*_{\tau > 0}(M \times \mathbb{R})$ be a locally closed conic subset. Let $F \in \mathcal{G}(\mathbf{k}_\Lambda)$ be a global object in the Kashiwara-Schapira stack. We assume

\begin{itemize}
\item[(i)] for any $p = (x; \xi) \in \Lambda$ there exists a neighborhood $\Lambda_0$ of $p$ in $\Lambda$ and a neighborhood $V$ of $x$ in $M \times \mathbb{R}$ such that $F|_{\Lambda_0}$ admits a representative in $\mathcal{D}^b_{V \cup \Lambda_0}(\mathbf{k}_V)$,
\item[(ii)] for any $k < 0$, we have $\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{G}(\mathbf{k}_\Lambda))(F,F[k]) = 0$.
\end{itemize}

Then there exist an open subset $U \subset M \times \mathbb{R}$ and $G \in \mathcal{D}^b(\mathbf{k}_U^\mu)$ such that $\mathbf{s}_\Lambda^\mu(G) \simeq F$ in $\mathcal{G}(\mathbf{k}_\Lambda)$, where $\mathbf{s}_\Lambda^\mu$ is defined in (15.2).
\end{theorem}
Proof. By the hypothesis (i) we can find a countable covering \( \Lambda = \bigcup_{i \in I} \Lambda_i \), open subsets \( U_i \subset M \times \mathbb{R} \) and \( F_i \in \mathcal{D}^b(k_{U_i}) \), for each \( i \in I \), such that \( SS(F_i) \subset \Lambda_i \) and the image of \( F_i \) in \( \mathcal{S}(k_{\Lambda_i}) \) is \( F|_{\Lambda_i} \).

For \( i \in I \) we set \( G_i = \Psi_{U_i}(F_i) \in \mathcal{D}^b(k^\mu_{U_i}) \). Setting \( U_{ij} = U_i \cap U_j \) and \( \Lambda_{ij} = \Lambda_i \cap \Lambda_j \), we find, by Theorem 11.5 and Definition 13.1, \n\begin{equation}
(15.3) \quad \text{Hom}_{\mathcal{D}^b(k^\mu_{ij})}(G_i|_{U_{ij}}, G_j|_{U_{ij}}) \simeq H^k(\Lambda_{ij}; \mu\text{hom}(F_i|_{U_{ij}}, F_j|_{U_{ij}})),
\end{equation}
for any \( i, j \in I \) and any \( k \in \mathbb{Z} \). By Corollary 3.4 we also have \( H^l\mu\text{hom}(F_i, F_i) \simeq \text{Hom}_{\mathcal{S}(k_{\Lambda_i})}(F, F[l])|_{\Lambda_i} \), for any \( i \in I \) and any \( l \in \mathbb{Z} \). By hypothesis (ii) it follows that \( \mu\text{hom}(F_i, F_i) \) is concentrated in degrees \( \geq 0 \). Hence (15.3) (with \( i = j \)) gives
\[ \text{Hom}_{\mathcal{D}^b(k^\mu_{ij})}(G_i, G_i[k]) \simeq 0 \quad \text{for } k \leq 0. \]

By Corollary 3.4 again, we obtain from (15.3)
\[ \text{Hom}_{\mathcal{D}^b(k^\mu_{ij})}(G_i|_{U_{ij}}, G_j|_{U_{ij}}) \simeq H^b(\Lambda_{ij}; \text{Hom}_{\mathcal{S}(k_{\Lambda_i})}(F, F)), \]
for any \( i, j \in I \). We define \( u_{ji} \in \text{Hom}_{\mathcal{D}^b(k^\mu_{ij})}(G_i|_{U_{ij}}, G_j|_{U_{ij}}) \), for \( i, j \in I \), as the image of \( id_F \) by this isomorphism. We have \( u_{kj}|_{V_{ijk}} \circ u_{ji}|_{V_{ijk}} = u_{ki}|_{V_{ijk}} \), for all \( i, j, k \in I \). By Proposition 14.4 we obtain \( G \in \mathcal{D}^b(k^\mu_U) \), where \( U = \bigcup_{i \in I} U_i \), together with isomorphisms \( v_i : G_i \sim \rightarrow G|_{U_i} \), for all \( i \in I \), such that \( u_{ji} = v_j|_{U_{ij}} \circ v_i^{-1}|_{U_{ij}} \), for all \( i, j \in I \). Then \( U \) and \( G \) satisfy the conclusions of the theorem. \[ \square \]

Now we check that hypothesis (i) and (ii) of Theorem 15.2 are satisfied when \( \Lambda \) is a locally closed conic connected Lagrangian submanifold such that the projection \( \Lambda/\mathbb{R}_{>0} \to M \times \mathbb{R} \) is finite. We will use the “refined microlocal cut-off lemma”:

**Proposition 15.3.** [7 Prop. 6.1.4] Let \( N \) be a manifold and \( x \in N \). Let \( V \subset T^*_xN \) be an open cone such that \( \nabla \) is proper. Let \( F \in \mathcal{D}^b(k_N) \) and let \( W \subset T^*_xN \) be a conic neighborhood of \( \nabla \cap SS(F) \). Then there exist \( F' \in \mathcal{D}^b(k_N) \) and a distinguished triangle \( F' \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{\mu} \) such that \( SS(G) \cap V = \emptyset \) and \( T^*_xN \cap SS(F') \subset W \).

**Lemma 15.4.** Let \( N \) be a manifold and let \( \Lambda \) be a locally closed conic Lagrangian submanifold of \( \tilde{T}^*_xN \) such that the projection \( \Lambda/\mathbb{R}_{>0} \to N \) is finite. Let \( p = (x, \xi) \in \Lambda \). Then there exist a neighborhood \( U \) of \( x \) and \( F \in \mathcal{D}^b(k_U) \) such that \( SS(F) \subset \Lambda_0 \), where \( \Lambda_0 \) is the connected component of \( \Lambda \cap T^*_xU \) containing \( p \), and \( F \) is simple along \( \Lambda_0 \).

**Proof.** By Lemma 3.9 there exists a neighborhood \( \Omega \) of \( p \) in \( T^*_xN \) and \( F \in \mathcal{D}^b(k_N) \) such that \( SS(F) \cap \Omega \subset \Lambda \) and \( F \) is simple along \( \Lambda \) at \( p \). Up to shrinking \( \Omega \) we can assume that \( T^*_xN \cap \Omega \cap \Lambda = \mathbb{R}_{>0}\xi \).
We choose an open convex cone $V \subset T^*_xN$ such that $\xi \in V$, $\overline{V}$ is proper and $\overline{V} \subset T^*_xN \cap \Omega$. Hence $\overline{V} \cap \text{SS}(F) = \mathbb{R}_{>0}\xi$. In particular $W := V$ is a neighborhood of $\overline{V} \cap \text{SS}(F)$. By Proposition 15.3 there exist $F' \in \mathcal{D}^b(\mathcal{k}_N)$ and a distinguished triangle $F' \xrightarrow{u} F \to G \xrightarrow{+1} \text{ such that SS}(G) \cap V = \emptyset$ and $T^*_xN \cap \text{SS}(F') \subset W$.

Since microsupports are closed conic subsets there exists a conic neighborhood $V_1$ of $V$ in $T^*_xN$ such that $\text{SS}(G) \cap V_1 = \emptyset$. Hence $\text{SS}(F') \cap V = \text{SS}(F') \cap V_1 \subset \Lambda \cap V_1$. Since $T^*_xN \cap \text{SS}(F') \subset W = V$, it follows that $T^*_xN \cap \text{SS}(F') \subset \Lambda \cap V = \mathbb{R}_{>0}\xi$. Using once again that microsupports are closed and conic we deduce that, for any conic neighborhood $V \cap \text{SS}(G) \cap V_1$ containing $p$. Shrinking $U$ we may assume that $T^*_xU \cap \Lambda \cap V_2$ is connected and this gives the lemma. 

**Corollary 15.5.** Let $\Lambda \subset T^*_x(M \times \mathbb{R})$ be a locally closed conic connected Lagrangian submanifold such that the projection $\Lambda/\mathbb{R}_{>0} \to M \times \mathbb{R}$ is finite. Let $F \in \mathcal{S}^p(\mathcal{k}_\Lambda)$ be a pure object in the Kashiwara-Schapira stack. Let $L \in \text{Mod}(\mathcal{k})$ be such that $\text{m}_{\Lambda,I}(F) \simeq L[d]$ for some $l \in U_\Lambda$ and $d \in \mathbb{Z}$. We assume that $\text{Ext}^i(L, L) = 0$ for all $i > 0$. Then there exist an open subset $U \subset M \times \mathbb{R}$ and $G \in \mathcal{D}^b(\mathcal{k}_U^\mu)$ such that $\mathcal{s}^\mu_{\Lambda}(G) \simeq F$ in $\mathcal{S}(\mathcal{k}_\Lambda)$.

**Proof.** By Lemma 15.4 we can find a countable covering $\Lambda = \bigcup_{i \in I} \Lambda_i$, open subsets $U_i \subset M \times \mathbb{R}$ and $F_i \in \mathcal{D}^b(\mathcal{k}_{U_i})$, for each $i \in I$, such that $\Lambda_i$ is a connected component of $\Lambda \cap T^*_xU_i$, $\text{SS}(F_i) \subset \Lambda_i$ and $F_i$ is simple along $\Lambda_i$. We can also assume that $\Lambda_i$ is contractible. Then, up to shifting $F_i$ by some integer, there exists an isomorphism in $\mathcal{S}(\mathcal{k}_\Lambda)$, for each $i \in I$, $u_i : F \mid_{\Lambda_i} \xrightarrow{\sim} \mathcal{s}_{\Lambda_i}(F_i \otimes L)$. Now the result follows from Theorem 15.2.

16. **From $\mu$-sheaves to sheaves**

In Theorem 15.2 we have given conditions so that a global object $F \in \mathcal{S}(\mathcal{k}_\Lambda)$ admits a representative $G \in \mathcal{D}^b(\mathcal{k}_U^\mu)$. Now we want to build a representative in $\mathcal{D}^b(\mathcal{k}_{U,M \times \mathbb{R}})(\mathcal{k}_{M \times \mathbb{R}})$. Lemma 16.2 below is a first step between $\mathcal{D}^b(\mathcal{k}_U^\mu)$ and $\mathcal{D}^b(\mathcal{k}_{U,M \times \mathbb{R}})(\mathcal{k}_{M \times \mathbb{R}})$.

16.1. **Restriction near the boundary.** For $u \in \mathbb{R}$ we define the translations:

\[
T_u : M \times \mathbb{R} \to M \times \mathbb{R} \quad T_u' : T^*(M \times \mathbb{R}) \to T^*(M \times \mathbb{R})
\]

\[\begin{align*}
(x, t) & \mapsto (x, t + u) \\
(x, t; \xi, \tau) & \mapsto (x, t + u; \xi, \tau).
\end{align*}\]
We define the map $\rho: T^*_\mathbb{R}(M \times \mathbb{R}) \to T^*M$, $(x, t; \xi, \tau) \mapsto (x; \xi/\tau)$. We will often consider the following condition on the subset $\Lambda$ of $T^*(M \times \mathbb{R})$:

\begin{equation}
\begin{cases}
\Lambda \text{ is a closed conic subset of } T^*_\mathbb{R}(M \times \mathbb{R}), \\
\Lambda/\mathbb{R}_{>0} \text{ is compact,} \\
\rho \text{ induces an injective map } \Lambda/\mathbb{R}_{>0} \hookrightarrow T^*M.
\end{cases}
\end{equation}

**Lemma 16.1.** Let $\Lambda \subset T^*_\mathbb{R}(M \times \mathbb{R})$ be a conic subset such that $\rho$ induces an injective map $\Lambda/\mathbb{R}_{>0} \hookrightarrow T^*M$. Then the translated sets $T'_u(\Lambda), u \in \mathbb{R}$, are mutually disjoint.

**Lemma 16.2.** Let $\Lambda \subset T^*_\mathbb{R}(M \times \mathbb{R})$ be a closed conic subset satisfying (16.2). Let $U$ be an open subset of $M \times \mathbb{R}$ such that $\Lambda \subset U$ and let $F \in \mathcal{D}_\Lambda^b(k_M^*)$. Then there exist $\varepsilon > 0$ and $G \in \mathcal{D}^b(k_{M \times \mathbb{R}})$ such that

(i) $\text{SS}(G) \subset \Lambda \sqcup T'_\varepsilon(\Lambda) \sqcup T^*_\mathbb{R}(M \times \mathbb{R})$, 
(ii) $\text{supp } G \subset \bigcup_{u \in [0, \varepsilon]} T_u(\pi_M \times \mathbb{R}(\Lambda))$, 
(iii) $s_A(G) \simeq s_A^b(F)$ in $\mathcal{G}(k_A)$ (where $s_A$ and $s_A^b$ are defined in Definition 3.1 and (15.2)).

**Proof.** Let $V \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ be an open subset such that $b(V) = U$ and $F$ admits a representative in $\mathcal{D}^b(k_V)$. Let us set $S = \pi_M \times \mathbb{R}(\Lambda)$. This is a compact subset of $U$. By Lemma 13.4 we have supp$(F) \subset (\pi \times S) \cap V$.

For $\varepsilon > 0$ we set $S_\varepsilon = \bigcup_{u \in [0, \varepsilon]} T_u(S)$. When $\varepsilon$ is small enough this is still a compact subset of $U$ and we have

\begin{equation}
\text{supp}(F|_{V \cap (U \times [0, \varepsilon])}) \subset V \cap (S_\varepsilon \times [0, \varepsilon]).
\end{equation}

Since $b(V) = U$ we can choose $\varepsilon$ so that $S_\varepsilon \times [0, \varepsilon] \subset V$. Let us set $V_\varepsilon = V \cap (M \times \mathbb{R} \times \{\varepsilon\})$. Then $F_\varepsilon := F|_{V_\varepsilon} \in \mathcal{D}^b(k_{V_\varepsilon})$ has a compact support. We let $G \in \mathcal{D}^b(k_{M \times \mathbb{R}})$ be its unique extension by 0 to $M \times \mathbb{R}$.

Then the assertion (i) follows from Lemmas 13.4 and 16.1. The point (ii) is given by (16.3). Since $\Lambda$ and $T'_\varepsilon(\Lambda)$ are disjoint, $G$ belongs to $\mathcal{D}^b(\Lambda)(k_{M \times \mathbb{R}})$, and we can consider $s_A(G)$. Then (iii) follows from the equivalence (15.1). □

16.2. **Hamiltonian isotopy.** The object $G$ given by Lemma 16.2 is not a quantization of $\Lambda$ because its microsupport contains the term $T'_\varepsilon(\Lambda)$. The next step is to use a Hamiltonian isotopy which sends $\Lambda \cup T'_\varepsilon(\Lambda)$ to $\Lambda \cup T'_{u+\varepsilon}(\Lambda)$ for $u$ arbitrarily large. For this we will need the main result of [3] that we quickly recall now.

Let us first recall the definition of the composition of kernels considered in [7]. Let $M_i$, $i = 1, 2, 3$, be manifolds. We write for short $M_{ij} := M_i \times M_j$, $1 \leq i, j \leq 3$, and $M_{123} = M_1 \times M_2 \times M_3$. We denote by $q_{ij}$ the projection $M_{123} \to M_{ij}$. Similarly, we denote by $p_{ij}$ the projection $T^*M_{123} \to T^*M_{ij}$. We also define the map $p_{123}$, the composition
of $p_{12}$ and the antipodal map on $T^*M_2$. The composition of kernels is defined by

$$\circ : D^b(k_{M_{12}}) \times D^b(k_{M_{23}}) \to D^b(k_{M_3})$$

where $K_1, K_2 \mapsto K_1 \circ K_2 := Rq_{13}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2)$.

For subsets $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$, we define in the same way

$$\Lambda_1 \circ \Lambda_2 := p_{13}(p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2).$$

Let $\Lambda_i = SS(K_i) \subset T^*M_{i,i+1}$ and assume that $p_{13}$ is proper on $p_{12}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2$. Then it follows from Theorem [2.5] that:

$$SS(K_1 \circ K_2) \subset \Lambda_1 \circ \Lambda_2.$$  

When $M_1 = M_2 = M_3 = M$, we have a natural candidate for an inverse of a kernel. We define $v : M \times M \to M \times M$, $(x,y) \mapsto (y,x)$ and we set, for $K \in D^b(k_{M \times M})$,

$$K^{-1} = v^{-1}R\mathcal{H}om(K, \omega_M \boxtimes k_M).$$

Then there exists a natural morphism $K^{-1} \circ K \to k_{\Delta M}$.

Let $M$ be a manifold and let $I$ be an open interval of $\mathbb{R}$ containing $0$. We consider a homogeneous Hamiltonian isotopy $\phi : \dot{T}^*M \times I \to \dot{T}^*M$ of class $C^\infty$, that is, $\phi$ is a $C^\infty$-map and, denoting by $\phi_t : \dot{T}^*M \times \{t\} \to \dot{T}^*M$ the restriction at time $t$, we have

(i) $\phi_t$ is a homogeneous symplectic isomorphism for each $t \in I$,

(ii) $\phi_0 = \text{id}_{\dot{T}^*M}$.

The graph of $\phi_t$, twisted by the antipodal map, is the Lagrangian submanifold of $\dot{T}^*(M \times M)$ given by $\Gamma_{\phi_t} := \{(x_1,x_2;\xi_1,\xi_2); (x_1;\xi_1) = \phi_t(x_2;\xi_2)\}$. We let $\Gamma_\phi \subset \dot{T}^*(M \times M) \times I$ be the union of the $\Gamma_{\phi_t}$, $t \in I$.

It is not difficult to check that there exists a unique conic Lagrangian submanifold $\Lambda_\phi \subset \dot{T}^*(M \times M \times I)$ which is identified with $\Gamma_\phi$ through the projection induced by $T^*I \to I$:

$$\begin{array}{ccc}
\Lambda_\phi & \subset & \dot{T}^*(M \times M \times I) \\
\Gamma_\phi & \subset & \dot{T}^*(M \times M) \times I.
\end{array}$$

**Theorem 16.3.** (Theorem 3.7 of [3]) We consider a homogeneous Hamiltonian isotopy $\phi : \dot{T}^*M \times I \to \dot{T}^*M$ of class $C^\infty$. Then there exists a unique $K \in D(k_{M \times M \times I})$ satisfying

(i) for any compact subset $C \subset M \times M \times I$ we have $K|_C \in D^b(k_C)$,

(ii) $SS(K) = \Lambda_\phi$,

(iii) $K_0 \simeq k_\Delta$, where $K_t := K|_{M \times M \times \{t\}}$ for any $t \in I$. 


Moreover, both projections $\text{supp}(K) \Rightarrow M \times I$ are proper, and we have $K_t \circ K_t^{-1} \simeq K_t^{-1} \circ K_t \simeq \kappa$ for all $t \in I$.

We can then check that $K$ is non-characteristic for the embeddings $i_t: M \times M \times \{t\} \hookrightarrow M \times M \times I$ and that we have $\text{SS}(K_t) = \Gamma_{\phi_t}$. For any $F \in D^b(k_M)$ we also obtain $\text{SS}(K_t \circ F) = \phi_t(\text{SS}(F))$.

In the next paragraph we will apply Theorem 16.3 to the following Hamiltonian isotopy. In the situation of Lemma 16.2 we can find a Hamiltonian isotopy which keeps $\Lambda$ fixed and translates $T^*_\varepsilon(\Lambda)$.

**Lemma 16.4.** Let $\Lambda \subset T^*_{\varepsilon>0}(M \times \mathbb{R})$ be a closed conic subset satisfying (16.2). Let $\varepsilon > 0$ and $A > 0$ be given. Then there exist a neighborhood $\Omega$ of $\Lambda$ in $\dot{T}^*(M \times \mathbb{R})$ and a homogeneous Hamiltonian isotopy $\phi: \dot{T}^*(M \times \mathbb{R}) \times \mathbb{R} \rightarrow \dot{T}^*(M \times \mathbb{R})$ of class $C^\infty$ such that, for all $s \in [0, A]$, we have $\phi_s|_{\Omega} = \text{id}_{\Omega}$ and $\phi_s(T^*_{\varepsilon}(\Lambda)) = T^*_{\varepsilon+s}(\Lambda)$.

**Proof.** We set $\Lambda' = \bigcup_{s \in [0, A]} T^*_{\varepsilon+s}(\Lambda)$. Then $\Lambda'/\mathbb{R}_{>0}$ is compact. By lemma 16.1 the sets $\Lambda$ and $\Lambda'$ are disjoint. Let $\Omega, \Omega' \subset \dot{T}^* M$ be disjoint conic neighborhoods of $\Lambda$ and $\Lambda'$ such that $\Omega/\mathbb{R}_{>0}$ and $\Omega'/\mathbb{R}_{>0}$ are compact. We choose a $C^\infty$-function $h: \dot{T}^*(M \times \mathbb{R}) \rightarrow \mathbb{R}$ such that $h$ is homogeneous of degree 1, $h|_{\Omega} = 0$, $h|_{\Omega'} = -\varepsilon$ and $\text{supp}(h)/\mathbb{R}_{>0}$ is compact. Hence the Hamiltonian flow of $h$, say $\phi$, is defined on $\mathbb{R}$. Since the Hamiltonian vector field of the function $-\varepsilon$ is $H_{-\varepsilon} = \partial/\partial t$ we have $\phi_s(x, t; \xi, \tau) = (x, t + s; \xi, \tau)$, for all $(x, t; \xi, \tau) \in \Omega'$, and the lemma follows easily. \hfill \Box

### 16.3. Quantization as a sheaf.

Now we sum up the results in paragraphs 16.1 and 16.2 to deduce the existence of a quantization.

**Theorem 16.5.** Let $\Lambda \subset T^*_{\varepsilon>0}(M \times \mathbb{R})$ be a closed conic subset satisfying (16.2). Let $F \in \mathcal{S}(k_\Lambda)$ be a global object in the Kashiwara-Schapira stack satisfying hypothesis (i) and (ii) of Theorem 15.2. Then there exists $G \in D^b(k_{M \times \mathbb{R}})$ such that

(i) $\text{SS}(G) \subset \Lambda \sqcup T^*_{M \times \mathbb{R}}(M \times \mathbb{R})$,

(ii) $G|_{M \times \{t\}} \simeq 0$ for $t \ll 0$,

(iii) $s_\Lambda(G) \simeq F$ (where $s_\Lambda$ is given in Definition 7.3).

**Proof.** (a) By Theorem 15.2 there exist an open subset $U \subset M \times \mathbb{R}$ and $G_1 \in D^b(k_U)$ such that $s_\Lambda(G_1) \simeq F$ in $\mathcal{S}(k_\Lambda)$. By Lemma 16.2 we deduce $G_2 \in D^b(k_{M \times \mathbb{R}})$, with compact support, such that $\text{SS}(G_2) \subset \Lambda \sqcup T^*_\Lambda(\Lambda) \sqcup T^*_{M \times \mathbb{R}}(M \times \mathbb{R})$ and $s_\Lambda(G_2) \simeq F$.

(b) Since $\Lambda/\mathbb{R}_{>0}$ is compact we can find $a < b \in \mathbb{R}$ such that $\Lambda \subset T^*(M \times [a, b])$. By Lemma 16.4 there exists a neighborhood $\Omega$ of $\Lambda$ in $\dot{T}^*(M \times \mathbb{R})$ and a homogeneous Hamiltonian isotopy $\phi: \dot{T}^*(M \times \mathbb{R}) \times
$\mathbb{R} \to \hat{T}^*(M \times \mathbb{R})$ of class $C^\infty$ such that, for all $s \in [0, b-a]$, we have $\phi_s|_{\Omega} = \text{id}_\Omega$ and $\phi_s(T'_s(\Lambda)) = T'_{s+\varepsilon}(\Lambda)$.

(c) Let $K \in D(k_{(M\times \mathbb{R})^2})$ be the sheaf associated with $\phi$ by Theorem [16.3]. We set $G' = K \circ G_2 \in D(k_{M\times \mathbb{R} \times \mathbb{R}})$. We recall that $K$ satisfies condition (i) in Theorem [16.3] and that the projections supp$(K) \Rightarrow (M \times \mathbb{R}) \times \mathbb{R}$ are proper. Since $G_2$ has a compact support, it follows that, for any compact interval $I$ of $\mathbb{R}$, the restriction $G'|_{M\times \mathbb{R} \times I}$ also has a compact support and belongs to $D^b(k_{M\times \mathbb{R} \times I})$. For $s \in \mathbb{R}$ we set $G'_s = G'|_{M \times \mathbb{R} \times \{s\}} \simeq K_s \circ G_2 \in D^b(k_{M \times \mathbb{R}})$. Then, for any $s \in [0, b-a]$, we have

$$\text{SS}(G'_s) = \phi_s(\text{SS}(G_2)) \subset \Lambda \cup T'_{s+\varepsilon}(\Lambda).$$

In particular $\text{SS}(G'_{b-a}|_{M \times [b, b+\varepsilon[}) \subset T^*_M (M \times \mathbb{R})$ and $G'_{b-a}|_{M \times [b, b+\varepsilon[}$ has locally constant cohomology sheaves.

(d) We set $L = G'_{b-a}|_{M \times [b, b+\varepsilon[}$. Then $L \in D^b(k_{M})$ has locally constant cohomology sheaves and we define $G$ by gluing $G'_{b-a}|_{M \times [b, b+\varepsilon[}$ and $L_{M \times [b, +\infty[}$, that is, we define $G$ by the distinguished triangle

$$L_M|_{0, b, b+\varepsilon[} \to (G'_{b-a})_{M \times [b, b+\varepsilon[} \oplus L_M|_{b, +\infty[} \to G \overset{\text{id}}{\to} .$$

Then $G$ clearly satisfies (i). Since $G'_{b-a}$ has a compact support, $G$ also satisfies (ii). Since $G \simeq G_2$ in $D^b(k_{M \times \mathbb{R}}; \Omega)$ we obtain (iii). \hfill $\square$

**Corollary 16.6.** Let $\Lambda \subset T_{\tau>0}^*(M \times \mathbb{R})$ be a closed conic connected Lagrangian submanifold such that $\Lambda/\mathbb{R}_{>0}$ is compact and the map $\Lambda/\mathbb{R}_{>0} \to T^*M$, $(x, t; \xi, \tau) \mapsto (x; \xi/\tau)$, is an embedding. Let $F \in \mathfrak{S}^p(k_\Lambda)$ be a pure object in the Kashiwara-Schapira stack. Let $L \in \text{Mod}(k)$ be such that $m_{A, t}(F) \simeq L[d]$ for some $l \in U_\Lambda$ and $d \in \mathbb{Z}$. We assume that $\text{Ext}^i(L, L) = 0$ for all $i > 0$. Then there exists $G \in D^b(k_{M \times \mathbb{R}})$ such that

1. $\text{SS}(G) \subset \Lambda \cup T^*_M (M \times \mathbb{R})$,
2. $G|_{M \times \{t\}} \simeq 0$ for $t \ll 0$,
3. $\mathfrak{g}_\Lambda(G) \simeq F$.

**Proof.** By Theorem [16.3] we may move $\Lambda$ by a Hamiltonian isotopy. Hence we can assume that $\Lambda$ satisfies (16.2) and the result follows from Theorem [16.5] and Corollary [15.3]. \hfill $\square$

**Remark 16.7.** The assumption on $\Lambda$ in Corollary [16.6] is equivalent to

$$\Lambda = \{(x, t; \xi, \tau); \tau > 0, (x; \xi/\tau) \in \tilde{\Lambda}, t = -f(x; \xi/\tau)\},$$

where $\tilde{\Lambda} \subset T^*M$ is a compact exact Lagrangian submanifold and $f: \tilde{\Lambda} \to \mathbb{R}$ satisfies $df = \alpha_M|_{\tilde{\Lambda}}$. By Theorem [8.3] the existence of a
simple global object in $\mathcal{G}^p(k_A)$ is equivalent to the vanishing of the Maslov and relative Stiefel-Whitney classes of $\tilde{\Lambda}$. This vanishing is proved by Kragh in [8].

17. INVARIANCE BY TRANSLATION

In this section we study cohomological properties of the sheaves obtained in Theorem 16.5. The main result is Theorem 17.10. We will later need a pull-back to the universal cover of $M$. Hence we consider conditions on $\Lambda \subset \tilde{T}^*(M \times \mathbb{R})$ which are slightly more general than (16.2):

$$\begin{aligned}
\{ & \text{there exist a locally trivial covering } \alpha: M \to N \text{ and } \\
& \Lambda_N \subset T^*_{\tau>0}(N \times \mathbb{R}) \text{ satisfying (16.2) such that } \\
& \Lambda \subset \beta_d\beta^{-1}_\tau(\Lambda_N), \text{ where } \beta := \alpha \times \text{id}_\mathbb{R}: M \times \mathbb{R} \to N \times \mathbb{R}. \}
\end{aligned}$$

17.1. The morphism to the translated sheaf. We have defined $T_u$, $u \in \mathbb{R}$, in (16.1). Recall also the category $D^b_{\tau \geq 0}(k_{M \times \mathbb{R}})$ introduced in Definition 8.7. The direct image by $T_u$ induces a functor from $D^b_{\tau \geq 0}(k_{M \times \mathbb{R}})$ to itself. We have the following lemma from [11], where

$$q_M: M \times \mathbb{R} \to M$$
denotes the projection.

**Lemma 17.1.** ([11, §2.2.2], see also [4, §5]) For any $u \geq 0$ there exists a morphism of functors $\tau_u: \text{id}_{D^b_{\tau \geq 0}(k_{M \times \mathbb{R}})} \to T_u^*$, such that $\tau_u$ induces isomorphisms, for any $F \in D^b_{\tau \geq 0}(k_{M \times \mathbb{R}})$,

$$\begin{aligned}
Rq_{M*}(\tau_U(F)) : Rq_{M*}(F) & \xrightarrow{\sim} Rq_{M*T_u(F)}, \\
Rq_{M!}(\tau_U(F)) : Rq_{M!}(F) & \xrightarrow{\sim} Rq_{M!T_u(F)},
\end{aligned}$$

which coincide with the natural isomorphisms obtained from $R(q_M \circ T_u)_* \simeq Rq_{M*T_u*}$ and $R(q_M \circ T_u)_! \simeq Rq_{M!T_u*}$.

We will also use the following result.

**Lemma 17.2.** Let $F \in D^b(k_{M \times \mathbb{R}})$. We assume that there exists $A > 0$ such that $\text{supp}(F) \subset M \times [-A, A]$. We also assume either $SS(F) \subset T^*_{\tau \geq 0}(M \times \mathbb{R})$ or $SS(F) \subset T^*_{\tau \leq 0}(M \times \mathbb{R})$. Let $p_M: M \times \mathbb{R} \to M$ be the projection. Then $Rp_{M!}(F) \simeq 0$.

**Proof.** By base change we may assume that $M$ is a point. Then the result follows from the “Morse lemma” Corollary 2.7. \[\square\]
17.2. Bounds for the microsupports. Lemma 17.3 below will be used to bound some microsupport in Proposition 17.9. Lemma 17.5 gives a bound for the microsupport of the direct image by a covering. We define the maps

\[ q: M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}, \quad (x, t, u) \mapsto (x, t), \]

\[ r: M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}, \quad (x, t, u) \mapsto (x, t - u), \]

\[ p: M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, t, u) \mapsto u. \]

We consider a closed conic subset \( \Lambda \subset T^*_{r>0}(M \times \mathbb{R}) \). We set \( \Lambda_0 = \Lambda \cup T^*_{M \times \mathbb{R}}(M \times \mathbb{R}) \) and

\[ \Lambda' = q_d q^{-1}_\pi(A_0^\delta) + r_d r^{-1}_\pi(A_0). \]

We use coordinates \((x, t; u; \xi, \eta, v)\) on \( T^*(M \times \mathbb{R} \times \mathbb{R}) \).

**Lemma 17.3.** We assume that the map \( \rho, (x, t; \xi, \eta) \mapsto (x; \xi/\eta) \), induces an injective map \( \Lambda/\mathbb{R}_{>0} \mapsto T^* M \). Then we have

\[ q_d q^{-1}_\pi(A_0^\delta) \cap r_d r^{-1}_\pi(A_0) \subset \hat{T}^*_{M \times \mathbb{R} \times \mathbb{R}}(M \times \mathbb{R} \times \mathbb{R}), \]

\[ \Lambda' \subset \{(x, t, u; \xi, \tau, v) \in T^*(M \times \mathbb{R} \times \mathbb{R}) ; \ v \leq 0 \}, \]

\[ \Lambda' \cap (\hat{T}^*_{M \times \mathbb{R}}(M \times \mathbb{R}) \times \hat{T}^*(\mathbb{R} \setminus \{0\})) = \emptyset. \]

In particular \( p_\pi p_d^{-1}(\Lambda') \subset T^*_R \mathbb{R} \cup \{(0; v); \ v \leq 0 \} \).

**Proof.** (i) Let us assume that \((x, t, u; \xi, \tau, v)\) is a point in the intersection (17.5). Hence it belongs to \( q_d q^{-1}_\pi(T^*(M \times \mathbb{R})) \cap r_d r^{-1}_\pi(T^*(M \times \mathbb{R})) \). Hence \( v = 0 \) and \( \tau + v = 0 \). Then \( \tau = 0 \) and we find \((x, t; \xi, 0) \in \Lambda_0 \). Since \( \Lambda \subset T^*_{r>0}(M \times \mathbb{R}) \) we obtain \( \xi = 0 \).

(ii) We have \( q_d q^{-1}_\pi(A_0^\delta) \subset \{v = 0\} \) and \( r_d r^{-1}_\pi(A_0) \subset \{\tau \geq 0, \ \tau + v = 0\} \subset \{v \leq 0\} \). Then (17.6) follows.

(iii) We assume that \( \Lambda' \) contains a point \((x, t, u; 0, 0, v)\) with \( v \neq 0 \). Let us prove that this implies \( u = 0 \), which will give (17.7).

By definition of \( \Lambda' \) there exists \((\xi_i, \tau_i, v_i) \in T^*_{(x, t, u)}(M \times \mathbb{R} \times \mathbb{R}), \ i = 1, 2, \) such that \( \xi_1 + \xi_2 = 0, \ \tau_1 + \tau_2 = 0, \ v_1 + v_2 = v, \ (x, t, u; \xi_1, \tau_1, v_1) \in q_d q^{-1}_\pi(A_0^\delta) \) and \((x, t, u; \xi_2, \tau_2, v_2) \in r_d r^{-1}_\pi(A_0) \). The last two conditions give \( v_1 = 0, \ (x, t; -\xi_1, -\tau_1) \in \Lambda_0, \ \tau_2 + v_2 = 0 \) and \((x, t - u; \xi_2, \tau_2) \in \Lambda_0 \). Summing up we find \( \xi_2 = -\xi_1, \ \tau_2 = -\tau_1 = -v, \ (x, t; \xi_2, \tau_2) \in \Lambda_0 \) and \((x, t - u; \xi_2, \tau_2) \in \Lambda_0 \).

Since \( \tau_2 = -v \neq 0 \) we even obtain \((x, t; \xi_2, \tau_2) \in \Lambda \) and \((x, t - u; \xi_2, \tau_2) \in \Lambda \). Both these points of \( \Lambda \) have the same image by \( \rho \). By the hypothesis on \( \Lambda \) they also have the same image in \( \Lambda/\mathbb{R}_{>0} \), which implies \((x, t) = (x, t - u) \). Hence \( u = 0 \), as required.
Lemma 17.4. Let $X, Z$ be manifolds. Let $q_X: X \times Z \to X$ be the projection and $\delta = \delta_X = \text{id}_Z: X \times Z \to X \times X \times Z$ the diagonal embedding. Let $K \in D^b(k_{X \times X})$ be a kernel such that both projections $T^* (X \times X) \to (T^* X) \times X$ and $T^* (X \times X) \to X \times T^* X$ are proper on $\text{SS}(K)$. We set $L = \delta q_X^{-1} K$. Then, for any $F \in D^b(k_{X \times Z})$, we have $Rq_{X*}(L^{-1} \circ F) \simeq K^{-1} \circ Rq_{X*} F$, where $(\cdot)^{-1}$ is defined in (16.5).

Proof. The hypothesis on $\text{SS}(K)$ gives both a properness property and a non-characteristicity property. It implies, for any $G \in D^b(k_X)$,

$$K^{-1} \circ G \simeq Rq_1(v^{-1} R\text{Hom}(K, \omega_M \otimes k_M) \otimes q_2^{-1} G) \simeq Rq_1 R\text{Hom}(v^{-1} K, \omega_M \otimes q_2^{-1} G),$$

where $q_i: X \times X \to X$ is the projection to the $i$th-factor. A similar result holds for $L$ and $F$. Then the lemma follows from the adjunction formula $Rq_{X*} R\text{Hom}(q_1^{-1} \cdot, \cdot) \simeq R\text{Hom}(\cdot, Rq_{X*} \cdot)$.

Lemma 17.5. Let $X, Y$ be manifolds and let $\alpha: X \to Y$ be a locally trivial covering. Let $S \subset T^* Y$ be a closed conic subset and let $F \in D^b(k_X)$ such that $\text{SS}(F) \subset \alpha_\pi \alpha^{-1}(S)$. Then $\text{SS}(\alpha_!(F)) \cup \text{SS}(R\alpha_*(F)) \subset S$.

We assume moreover that $S \cap \hat{T}^* Y$ is a smooth Lagrangian manifold. Then, for any Cartesian square,

$$\begin{array}{ccc}
X' & \xrightarrow{i'} & X \\
\alpha' \downarrow & & \alpha \downarrow \\
Y' & \xrightarrow{i} & Y,
\end{array}$$

where $Y'$ is a submanifold of $Y$, we have $i^{-1} R\alpha_* F \simeq R\alpha'_* i'^{-1} F$ and $i' \alpha' F \simeq \alpha_3 i^* F$.

Proof. (i) The first part of the lemma is Lemma 1.15 of [4].

(ii) Let us prove the isomorphisms of the second part. The second isomorphism follows from the first by adjunction. To check that $i^{-1} R\alpha_* F \to R\alpha'_* i'^{-1} F$ is an isomorphism is a local problem on $Y$, so we may assume that $Y = \mathbb{R}^n$ and $X = \mathbb{R}^n \times Z$, for some discrete set $Z$. By Theorem 16.3 and Lemma 17.4, we may move $S \cap \hat{T}^* \mathbb{R}^n$ by any Hamiltonian isotopy of $\hat{T}^* \mathbb{R}^n$. Hence we can assume that $S$ is a subanalytic manifold.

By Lemma 8.4.7 of [7] it follows that, for any $x \in \mathbb{R}^n$, there exists an open ball $B$ containing $x$ such that $R\Gamma(B \times \{z\}; F|_{\mathbb{R}^n \times \{z\}}) \to F_{(x,z)}$ is an isomorphism. We notice indeed that the ball given by loc. cit. only depends on $\text{SS}(F)$ and not on $F$. Hence the same ball works for all
Now the lemma follows from $\text{R} \Gamma(B; \text{R} \alpha_* F) \cong (\text{R} \alpha_* F)_x$. 

17.3. **Invariance of global sections.** In this section we will use Sato’s distinguished triangle for $\mu \text{hom}$ (see (2.4) and (2.8)) in a case where $F$ is not cohomologically constructible. Hence we cannot replace the expression (2.7) by $D(F') \otimes G$ and we introduce the following notation.

**Definition 17.6.** Let $X$ be a manifold. Let $q_{X,1}, q_{X,2}: X \times X \to X$ be the projections and $\delta_X: X \to X \times X$ the diagonal embedding. Let $F, F' \in D^b(k_X)$. We set

$$\mathcal{H}om'(F, F') := \delta_X^{-1} \mathcal{R} \text{Hom}(q_{X,2}^{-1} F, q_{X,1}^{-1} f') \quad (17.8)$$

Then (2.2)–(2.7) give a distinguished triangle, for any $F, F' \in D^b(k_X)$,

$$\mathcal{H}om'(F, F') \rightarrow \mathcal{R} \text{Hom}(F, F')$$

$$\rightarrow \mathcal{R} \hat{\pi}_M(\mu \text{hom}(F, F')|_{I^*_xM}) \rightarrow + \quad (17.9)$$

**Lemma 17.7.** (i) Let $f: X \to Y$ be a morphism of manifolds. Let $F, F' \in D^b(k_Y)$ such that $f$ is non-characteristic for $SS(F)$ and $SS(F')$. Then

$$f^{-1} \mathcal{H}om'(F, F') \cong \mathcal{H}om'(f^{-1} F, f^{-1} F').$$

(ii) Let $F, F' \in D^b(k_Y)$ such that $SS(F) \cap SS(F') \subset T^*_x X$. Then $SS(\mathcal{H}om'(F, F')) \subset SS(F)^a + SS(F')$.

**Proof.** (i) We use the notations of Definition 17.6. We set $G = \mathcal{R} \text{Hom}(q_{Y,1}^{-1} F, q_{Y,1}^{-1} F')$. Then $SS(G) \subset SS(F)^a \times SS(F')$ is non-characteristic for $f \times f$. Hence $(f \times f)^{-1} G \cong (f \times f)^{-1} G \otimes \omega_{X \times X|Y \times Y}^{-1}$ by Theorem 2.5(iii). We also have $(f \times f)^{-1} q_{Y,1}^{-1} F' \cong (f \times f)^{-1} q_{Y,1}^{-1} F' \otimes \omega_{X \times X|Y \times Y}$ and we deduce the isomorphisms:

$$f^{-1} \mathcal{H}om'(F, F') \cong f^{-1} \delta_Y^{-1} \mathcal{R} \text{Hom}(q_{Y,2}^{-1} F, q_{Y,1}^{-1} F')$$

$$\cong \delta_X^{-1} (f \times f)^{-1} \mathcal{R} \text{Hom}(q_{Y,2}^{-1} F, q_{Y,1}^{-1} F')$$

$$\cong \delta_X^{-1} \mathcal{R} \text{Hom}((f \times f)^{-1} q_{Y,2}^{-1} F, (f \times f)^{-1} q_{Y,1}^{-1} F')$$

$$\cong \delta_X^{-1} \mathcal{R} \text{Hom}(q_{Y,2}^{-1} F, q_{Y,1}^{-1} F').$$

(ii) follows from Theorem 2.5(i) and (iii). Θ

**Lemma 17.8.** Let $I$ be an open interval of $\mathbb{R}$ containing 0. Let $G \in D^b(k_I)$ such that $SS(G) \subset T^*_I I \cup \{(0, v); \ v \leq 0\}$. Then, for any
Let \( u \in I \cap \mathbb{R}_{\leq 0} \) and \( v \in I \cap \mathbb{R}_{\geq 0} \) we have canonical isomorphisms

\[
(17.10) \quad G_0 \cong R\Gamma(I; G) \cong G_u,
\]

\[
(17.11) \quad R\Gamma_{\{0\}} G \cong R\Gamma_c(I; G) \cong R\Gamma_{\{v\}} G.
\]

**Proof.** Corollary 2.7 implies, for any \( a \leq b \leq c \leq d \) in \( I \),

\[
(17.12) \quad R\Gamma([a, d]; G) \cong R\Gamma([b, c]; G) \quad \text{if } b < c \text{ and } b < 0,
\]

\[
(17.13) \quad R\Gamma_{[a, d]}(I; G) \cong R\Gamma_{[a, d]}(I; G) \quad \text{if } 0 \leq c.
\]

For a given \( u \in I \cap \mathbb{R}_{\leq 0} \) we choose \( b < u < c \) and we let \( b, c \to u \) in (17.12). We obtain \( R\Gamma([a, d]; G) \cong G_u \). Letting \( [a, d] \) grow to \( I \) we obtain (17.10). The isomorphisms (17.11) follow in the same way from (17.13). \( \square \)

**Proposition 17.9.** Let \( \alpha: M \to N \) be a locally trivial covering and let \( \Lambda \subset T^* (M \times \mathbb{R}) \), \( \Lambda_N \subset T^*_{\geq 0}(N \times \mathbb{R}) \) be closed conic Lagrangian submanifolds such that (17.1) is satisfied. Let \( F, F' \in D^b(k_{M\times \mathbb{R}}) \) such that

1. SS(\( F \)) \& SS(\( F' \)) \( \subset T^*_{\mathbb{R}}(M \times \mathbb{R}) \cup \Lambda \),
2. \( F|_{M \times (t)} \simeq F'|_{M \times (t)} \simeq 0 \) for \( t \ll 0 \).

Then, for any \( u \geq 0 \), the morphism \( \tau_u(F') \) of Lemma 17.7 induces isomorphisms

\[
R\Gamma(M \times \mathbb{R}; R\mathcal{H}om(F, F')) \cong R\Gamma(M \times \mathbb{R}; R\mathcal{H}om(F, T_{us} F')),
\]

\[
R\Gamma(M \times \mathbb{R}; \mathcal{H}om'(F, F')) \cong R\Gamma(M \times \mathbb{R}; \mathcal{H}om'(F, T_{us} F')).
\]

**Proof.** (i) Since \( \Lambda_N / \mathbb{R}_{>0} \) is compact we can choose \( A > 0 \) such that \( \Lambda \subset T^* (M \times ] - A, A[) \). We choose \( B > A \). We have the distinguished triangle

\[
(17.14) \quad F_{M \times ] - \infty, B[} \to F \to F_{M \times [B, +\infty[} \xrightarrow{+1}.
\]

We will prove the isomorphisms of the proposition with \( F \) replaced by \( F_{M \times ] - \infty, B[} \) (in parts (ii) and (iii) below) or by \( F_{M \times [B, +\infty[} \) (in part (iv)). We set, with the notations \( q, r \) of (17.3),

\[
K = R\mathcal{H}om(q^{-1}(F_{M \times ] - \infty, B[}), r^{-1}F'),
\]

\[
L = \mathcal{H}om'(q^{-1}(F_{M \times ] - \infty, B[}), r^{-1}F').
\]

For \( u \in \mathbb{R} \) we define the embedding \( i_u: M \times \mathbb{R} \to M \times \mathbb{R} \times \mathbb{R}, \)

\[
(x, t) \mapsto (x, t, u).
\]

Then (17.15) below follows from standard adjunction formulas. By Theorem 2.5(iii), objects of the type \( q^{-1}(\cdot) \) and \( r^{-1}(\cdot) \) are non-characteristic for \( i_u \). Hence Lemma 17.7 gives (17.16):

\[
(17.15) \quad i_u^* K \simeq R\mathcal{H}om(F_{M \times ] - \infty, B[}; T_{us} F')[-1],
\]

\[
(17.16) \quad i_u^{-1} L \simeq \mathcal{H}om'(F_{M \times ] - \infty, B[}; T_{us} F').
\]
(ii) We set $I_B = \{ -\infty, B - A \}$ and $\Lambda_B = T^*_M M \times \{(B; \tau); \tau \geq 0\}$. Recall $\Lambda_0 = \Lambda \cup T^*_M M \times \mathbb{R}$ and $\Lambda$ defined in \text{(17.14)}. Then $SS(F_{M \times [-\infty, B]}) \subset \Lambda_0 \cup \Lambda_B$. Since the intersection $\pi_{M \times \mathbb{R}}(\Lambda') \cap (M \times \mathbb{R} \times IB) \cap (M \times \{B\} \times \mathbb{R})$ is empty, we deduce from Corollary 2.6 Lemma 17.7 and \text{(17.5)} that

\begin{equation}
(17.17)
SS(K|_{M \times \mathbb{R} \times IB}) \cup SS(L|_{M \times \mathbb{R} \times IB}) \subset \Lambda_B \cap T^*(M \times \mathbb{R} \times IB),
\end{equation}

where $\Lambda_B = \Lambda' \cup (\Lambda^a_B \times T^*_\mathbb{R} \mathbb{R})$.

Let $p': N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(y, t, u) \mapsto u$ be the projection. We have $p = p' \circ (\alpha \times id_{\mathbb{R}^2})$. By Lemma 17.3 and by the definition of $\Lambda_B$ we have $p_\pi p^{-1}_d(\Lambda'_B) \subset T^*_\mathbb{R} \mathbb{R} \cup \{(0; v); v \leq 0\}$.

Let $C \subset N$ be a compact subset such that $\pi_N(\Lambda_N) \subset C \times \{ -A, A\}$. Then, by the hypothesis (i) and (ii), we have $F|_{\alpha^{-1}(C) \times \mathbb{R}} \simeq 0$. It follows that $p'$ is proper on $(\alpha \times id_{\mathbb{R}^2})(supp K)$ and $(\alpha \times id_{\mathbb{R}^2})(supp L)$.

Using $p = p' \circ (\alpha \times id_{\mathbb{R}^2})$, we deduce from \text{(17.17)}, Lemma 17.5 and Theorem 2.5

\begin{equation}
(17.18)
SS(Rp_*K) \cup SS(Rp_*L) \subset T^*_\mathbb{R} \mathbb{R} \cup \{(0; v); v \leq 0\}.
\end{equation}

(iii) Lemma 17.3 and \text{(17.18)} give

\begin{align}
(17.19) & \quad R\Gamma_{\{0\}}(Rp_*K) \simeq R\Gamma_{\{u\}}(Rp_*K) \quad & \text{if } u \in [0, B - A[,

(17.20) & \quad (Rp_*L)_0 \simeq (Rp_*L)_v \quad & \text{if } v \leq 0.
\end{align}

Then the base change $R\Gamma_{\{u\}}(Rp_*K) \simeq R\Gamma(M \times \mathbb{R}; i_u^1 K)$ and \text{(17.19)} give \text{(17.22)} below. Let $i_u^1: N \times \mathbb{R} \times \{v\} \to N \times \mathbb{R} \times \mathbb{R}$ be the inclusion. We have

\begin{equation}
(17.21)
(Rp_*L)_v \simeq (Rp'_R(\alpha \times id_{\mathbb{R}^2})_L)_v \\
\simeq R\Gamma(N \times \mathbb{R}; i_v^{-1}R(\alpha \times id_{\mathbb{R}^2})_L) \\
\simeq R\Gamma(N \times \mathbb{R}; R(\alpha \times id_{\mathbb{R}^2})_v^{-1}L) \\
\simeq R\Gamma(M \times \mathbb{R}; i_v^{-1}L),
\end{equation}

where the first isomorphism follows from the fact that $p'$ is proper on $(\alpha \times id_{\mathbb{R}^2})(supp L)$, the second one from the usual base change and the third one from Lemma 17.5. Then \text{(17.20)} and \text{(17.21)} give \text{(17.23)}:

\begin{align}
(17.22) & \quad R\Gamma(M \times \mathbb{R}; i_0^1 K) \hookrightarrow R\Gamma(M \times \mathbb{R}; i_u^1 K) \quad & \text{if } u \in [0, B - A[,

(17.23) & \quad R\Gamma(M \times \mathbb{R}; i_v^{-1}L) \hookrightarrow R\Gamma(M \times \mathbb{R}; i_0^{-1}L) \quad & \text{if } v \leq 0.
\end{align}

(iv) For a given $u \geq 0$ we have $F'|_{A + u, +\infty[} \xrightarrow{\simeq} (T_{u,F'})|_{A + u, +\infty[}$. Hence for any $B > A + u$, we obtain

\[ R\mathcal{H}om(F_{M \times [B, +\infty]}, F') \xrightarrow{\simeq} R\mathcal{H}om(F_{M \times [B, +\infty]}, T_{u,F'}). \]
Together with (17.22), (17.15) and the distinguished triangle (17.14),
this gives the first isomorphism of the proposition. The second one
follows in the same way from (17.23) and (17.16). □

In the situation of Proposition 17.9 we choose $A > 0$ such that
$\Lambda \subset T^*(M \times [A, A])$. We set
$$L = F|_{M \times \{A+1\}}, \quad L' = F'|_{M \times \{A+1\}}.$$ (17.24)
Since $\text{SS}(F) \cap T^*(M \times (\mathbb{R} \setminus [-A, A]))$ is contained in the zero section,
$L$ has locally constant cohomology sheaves outside $M \times (\mathbb{R} \setminus [-A, A])$
and we obtain
$$F|_{M \times \{A+\infty\}} \simeq L \otimes k_{[A, +\infty[}, \quad F|_{M \times \{-\infty, -A\}} \simeq 0.$$ (17.25)
The same result holds for $F'$ and $L'$.

**Theorem 17.10.** In the situation of Proposition 17.9 we define $L, L' \in D^b(k_M)$ by (17.24). Then

$$\text{RG}(M \times \mathbb{R}; \mathcal{R}\text{Hom}(F, F')) \sim \text{RG}(M; \mathcal{R}\text{Hom}(L, L')).$$ (17.26)
$$\text{RG}(M \times \mathbb{R}; \text{Hom}'(F, F')) \simeq 0,$$ (17.27)
$$\text{RG}(M \times \mathbb{R}; \text{Hom}(F, F')) \sim \text{RG}(\Lambda; \mu\text{hom}(F, F')).$$ (17.28)
In particular we have a canonical isomorphism
$$\text{RHom}(L, L') \simeq \text{RG}(\Lambda; \mu\text{hom}(F, F')).$$ (17.29)

*Proof. (i) Let us prove (17.26). Let $p_M: M \times \mathbb{R} \to M$ be the projection.
Let us choose $A > 0$ so that (17.25) holds for $F$ and $F'$ and let $u > 2A$
Hence $\text{supp}(T_{u*}F') \subset M \times \{A, +\infty[ \}$ and we obtain

$$\text{RPM}_*\mathcal{R}\text{Hom}(F, T_{u*}F')) \simeq \text{RPM}_*\mathcal{R}\text{Hom}(p_M^{-1}(L), T_{u*}F'))$$

$$\simeq \mathcal{R}\text{Hom}(L, \text{RPM}_* T_{u*}F').$$ (17.30)
Let us set $G = (T_{u*}F') \otimes k_{[A, +\infty]}$. By (17.25) we know that $T_{u*}F'$
has locally constant cohomology sheaves in a neighborhood of $M \times \{A+u\}$. We deduce that $SS(G) \subset T^*_{\geq 0}(M \times \mathbb{R})$. By Lemma 17.2 we obtain

$$\text{RPM}_*(G) \simeq 0.$$ (17.28)
Again we have the distinguished triangle

$$G \to T_{u*}F' \to L' \otimes k_{[A+u, +\infty]} \xrightarrow{+1}.$$ (17.29)
Hence we obtain $\text{RPM}_* T_{u*}F' \simeq \text{RPM}_*(L' \otimes k_{[A+u, +\infty]}) \simeq L'$. By (17.30) we have $\text{RPM}_*\mathcal{R}\text{Hom}(F, T_{u*}F'))$

$$\simeq \mathcal{R}\text{Hom}(L, L')$$ and we conclude with Proposition 17.9.

(ii) Let us prove (17.27). We choose $A$ and $u$ as in (i). By (17.25) we have $F|_{M \times \{-\infty, -A\}} \simeq 0$ and $T_{-u*}F'|_{M \times \{-A, +\infty[} \simeq L \otimes k_{[-A, +\infty[}$. We deduce the distinguished triangle

$$\text{Hom}'(F, L' \otimes k_{[A, +\infty[}) \to \text{Hom}'(F, T_{-u*}F'))$$

$$\to \text{Hom}'(F, L' \otimes k_{[-A, A]} \xrightarrow{+1}).$$ (17.31)
We have \( \text{SS}(\mathbb{H}om'(F, L' \boxtimes k_{(-A,A)})) \subset T^*_\tau(M \times \mathbb{R}) \). By Lemma 17.2 we deduce \( R^pM_* (\mathbb{H}om'(F, L' \boxtimes k_{(-A,A)})) \simeq 0 \). Since \( F \) is locally constant on \( M \times [A, +\infty[ \), we also have \( R^pM_* (\mathbb{H}om'(F, L' \boxtimes k_{[A, +\infty[}) \simeq 0 \). We conclude with the triangle (17.31) and Proposition 17.9 as in (i).

(iii) The isomorphism (17.28) follows from (17.27) and the distinguished triangle (17.9).

\[ \square \]

Remark 17.11. We have recalled in (3.2) that \( \mu_{\text{hom}} \) admits a composition morphism (denoted by \( \circ \) in Notation 3.5) compatible with the composition morphism for \( R\mathbb{H}om \). In particular the isomorphism (17.28) is compatible with the composition morphisms \( \circ \) and \( \circ' \). Since (17.26) is clearly compatible with \( \circ \), we deduce that (17.29) also is compatible with \( \circ \) and \( \circ' \).

18. Behaviour at infinity

In this section we restrict to the case where the subset \( \Lambda \) of \( T^*_\tau(M \times \mathbb{R}) \) is a smooth Lagrangian submanifold satisfying (16.2). We deduce from Theorem 17.10 some information on the restriction of a quantization of \( \Lambda \) along \( M \times \{ t \} \), for \( t \gg 0 \).

We first give a general lemma on the stalks of a simple sheaf in the following situation. Let \( N \) be a connected manifold and let \( \Lambda \subset \dot{T}^*N \) be a smooth closed conic Lagrangian submanifold. We set

\[ Z_\Lambda = \{ x \in \dot{\pi}_N(\Lambda); \text{ there exist a neighborhood } W \text{ of } x \text{ and a smooth hypersurface } S \subset W \text{ such that } \Lambda \cap T^*W \subset T^*_S W \}. \]

Lemma 18.1. Let \( N \) be a connected manifold and let \( \Lambda \subset \dot{T}^*N \) be a smooth closed conic Lagrangian submanifold. Let \( x, y \in N \setminus \dot{\pi}_N(\Lambda) \). Let \( I \) be an open interval containing 0 and 1. Then there exists a \( C^\infty \) embedding \( c: I \to N \) such that \( c(0) = x, c(1) = y \) and \( c([0,1]) \) only meets \( \dot{\pi}_N(\Lambda) \) at points of \( Z_\Lambda \), with a transverse intersection.

Proof. (i) Let \( n \) be the dimension of \( N \) and let \( B \subset \mathbb{R}^{n-1} \) be the open ball of radius 1 and center 0. We choose a \( C^\infty \) embedding \( \gamma: B \times I \to N \) such that \( \gamma_0(0) = x \) and \( \gamma_0(1) = y \), where \( \gamma_s := \gamma|_{\{s\} \times I} \) for \( s \in B \). For example we can define \( \gamma \) by integrating a vector field which admits an integral curve from \( x \) to \( y \). We may also assume that \( \gamma(B \times \{0\}) \) and \( \gamma(B \times \{1\}) \) do not meet \( \dot{\pi}_N(\Lambda) \).

We choose \( \gamma': B \times I \times \mathbb{R}^n \to T^*N \) so that \( \gamma'_s := \gamma|_{\{s\} \times I \times \mathbb{R}^n} \) is a trivialization of \( \gamma_s(I) \times N \). Then \( \gamma' \) is an open embedding. In particular \( \gamma' \) is transversal to \( \Lambda \).
By the transversality theorem there exists $s \in B$ such that $\gamma_s'$ is transversal to $\Lambda$. In particular $\Lambda \cap \gamma_s'(0) \times \mathbb{R}^n$ consists of finitely many lines, say $\mathbb{R}_{>0} \cdot p_i, i = 1, \ldots, N$. The transversality also implies that $V_i := T_{p_i} \Lambda \cap T_{p_i} \gamma_s'(I \times \mathbb{R}^n)$ is of dimension 1. We write $p_i = \gamma_s'(t_i, v_i)$ and $x_i = \gamma_s(t_i)$.

Since $T_{p_i} \Lambda \cap T_{p_i} T_{x_i} N$ is contained in $V_i$, it is also of dimension 1. This means that $\hat{\pi}_N|_{\Lambda} : \Lambda \to N$ is of maximal rank $n - 1$ at $p_i$. Hence there exists a neighborhood $\Omega_i$ of $p_i$ and a smooth hypersurface $S_i$ around $x_i$ such that $\Lambda \cap \Omega_i = T^*_S T_{x_i} N \cap \Omega_i$. If $x_i = x_j$ for $i \neq j$, then $S_i$ and $S_j$ meet transversally at $x_i$ and $S_i \cap S_j$ is a submanifold of codimension 2 in a neighborhood of $x_i$. Hence, by deforming $\gamma_s$, we can assume moreover that $\gamma_s([0, 1])$ avoids $S_i \cap S_j$. Then all $x_i$ are distinct and belong to $Z_\Lambda$. We also see that the intersection of $S_i$ and $\gamma_s(I)$ is transversal. By joining $x$ to $\gamma_s(0)$ and $y$ to $\gamma_s(1)$, we obtain the embedding $c$ of the lemma.

**Lemma 18.2.** Let $N$ be a connected manifold and let $\Lambda \subset T^* N$ be a smooth closed conic Lagrangian submanifold. Let $F \in D^b(k_N)$ be such that $SS(F) \subset \Lambda$ and $F$ is simple along $\Lambda$. We set $U = N \setminus \hat{\pi}_N(\Lambda)$. We assume that there exists $x_0 \in U$ such that $H^i F x_0$ is free over $k$, for all $i \in \mathbb{Z}$. Then $H^i F x$ is free over $k$, for all $x \in U$ and all $i \in \mathbb{Z}$.

**Proof.** (i) Let $x \in U$ and let $I$ be an open interval containing 0 and 1. By Lemma 18.1 we can choose a $C^\infty$ path $\gamma : I \to N$ such that $\gamma(0) = x_0, \gamma(1) = x$ and $\gamma([0, 1])$ meets $\hat{\pi}_N(\Lambda)$ at finitely many points, all contained in $Z_\Lambda$ and with a transversal intersection. We denote these points by $\gamma(t_i)$, where $0 < t_1 < \cdots < t_k < 1$.

(ii) The stalk $F_{\gamma(t)}$ is constant for $t \in ]t_i, t_{i+1}[$. By Example 3.8 for $t_{i-1} < t < t_i < u < t_{i+1}$, the stalks $F_{\gamma(t)}$ and $F_{\gamma(u)}$ differ by $k[d_i]$, for some degree $d_i \in \mathbb{Z}$. Hence $H^i F_{\gamma(t)}$ is free over $k$, for all $i \in \mathbb{Z}$, if and only if the same holds for $F_{\gamma(u)}$. The lemma follows.

**Theorem 18.3.** Let $\Lambda \subset T^*_{\tau > 0} (M \times \mathbb{R})$ be a smooth connected Lagrangian submanifold satisfying 16.2. Let $F \in D^b(k_{M \times \mathbb{R}})$ be such that

1. $SS(F) \subset T^*_{M \times \mathbb{R}} (M \times \mathbb{R}) \cup \Lambda$;
2. $F$ is simple along $\Lambda$;
3. $F|_{M \times \{t\}} \simeq 0$ for $t \ll 0$.

Let $A \in \mathbb{R}$ be such that $\Lambda \subset T^* (M \times \mathbb{R}) \sim -\infty, A]$ and set $L = F|_{M \times \{A\}}$. Then $L \simeq L'[d]$, where $d \in \mathbb{Z}$ and $L'$ is a local system on $M$ whose stalk is free of rank 1 over $k$.

Moreover the projection $\pi_{M \times \mathbb{R}}|_\Lambda : \Lambda \to M \times \mathbb{R}$ induces an isomorphism $R\Gamma(M; k_M) \rightarrow R\Gamma(\Lambda; k_\Lambda)$.
Proof. (i) Let $\alpha: \tilde{M} \to M$ be the universal covering of $M$. We have $T^*\tilde{M} = \tilde{M} \times_M T^*M$. We set $\tilde{F} = (\alpha \times \text{id}_\mathbb{R})^{-1}F$, $\tilde{L} = \alpha^{-1}L$ and $\tilde{\Lambda} = (\tilde{M} \times \mathbb{R}) \times_{(\tilde{M} \times \mathbb{R})} \Lambda \subset T^*(\tilde{M} \times \mathbb{R})$. Then $\tilde{\Lambda}$ is a smooth Lagrangian submanifold of $T^*\tilde{M}$ (maybe non connected) which satisfies (17.1). We also have: $\text{SS}(\tilde{F}) \subset \tilde{\Lambda}$, $\tilde{\alpha}$ is simple along $\tilde{\Lambda}$, $\tilde{\alpha}^{-1}\tilde{F}$ is concentrated in non-negative degrees we deduce

$SS(\tilde{F}|_{\tilde{M} \times \{t\}}) \simeq 0$ for $t \ll 0$ and $\tilde{\alpha}^{-1}\tilde{F}|_{\tilde{M} \times \{t\}} \simeq 0$ for $t \ll 0$.

The first statement of the theorem is now equivalent to $\tilde{L} \simeq k_{\tilde{M}}[d]$. Let $d_0 \leq d_1$ be integers such that $H^{d_0}(\tilde{L}) \neq 0$, $H^{d_1}(\tilde{L}) \neq 0$ and $H^{k}(\tilde{L}) \neq 0$, for all $k \not\in [d_0, d_1]$.

By Lemma 18.2 $H^iF_x$ is free over $k$, for all $x \not\in \pi_{\tilde{M} \times \mathbb{R}}(\Lambda)$ and all $i \in \mathbb{Z}$. In particular the local systems $H^iL$ are locally free over $k$, for all $i \in \mathbb{Z}$. Since $\tilde{M}$ is simply connected, we obtain $H^{d_0}(\tilde{L}) \simeq k_{\tilde{M}}^{n_0}$ and $H^{d_1}(\tilde{L}) \simeq k_{\tilde{M}}^{n_1}$, for some $n_0, n_1 \in \mathbb{N}$.

(ii) We first prove that $\tilde{L}$ is concentrated in one degree, that is, $d_0 = d_1$. Since $\tilde{F}$ is simple along $\tilde{\Lambda}$ we have $\mu_{\text{hom}}(\tilde{F}, \tilde{F})|_{T^*(\tilde{M} \times \mathbb{R})} \simeq k_{\tilde{\Lambda}}$. Hence Theorem 17.10 applied with $F' = F = \tilde{F}$, gives

(18.1) $\text{RHom}(\tilde{L}, \tilde{L}) \simeq \text{R}\Gamma(\tilde{\Lambda}; k_{\tilde{\Lambda}})$.

We set for short $G = \text{R}\tens{\text{Hom}}(\tilde{L}, \tilde{L})$. Then $H^kG \simeq 0$ for all $k < d_0 - d_1$ and $H^{d_0 - d_1}G \simeq \text{Hom}(k_{\tilde{M}}^{n_1}, k_{\tilde{M}}^{n_0})$. Hence

$H^{d_0 - d_1}\text{RHom}(\tilde{L}, \tilde{L}) \simeq H^{d_0 - d_1}(\tilde{M}; G) \simeq \text{Hom}(k_{\tilde{M}}^{n_1}, k_{\tilde{M}}^{n_0}) \neq 0$.

Since $\text{R}\Gamma(\tilde{\Lambda}; k_{\tilde{\Lambda}})$ is concentrated in non-negative degrees we deduce $d_0 = d_1$.

(iii) Now we have $\tilde{L} \simeq k_{\tilde{M}}^{n_0}[-d_0]$. Hence, taking $H^0$ in (18.1) gives

(18.2) $\text{Hom}(k_{\tilde{M}}^{n_0}, k_{\tilde{M}}^{n_0}) \simeq H^0(\tilde{\Lambda}; k_{\tilde{\Lambda}}) \simeq H^0(\tilde{\Lambda}; \mu_{\text{hom}}(\tilde{F}, \tilde{F})|_{T^*(\tilde{M} \times \mathbb{R})})$.

It follows that $\tilde{\Lambda}$ has $n_0^2$ connected components, say $\tilde{\Lambda}_i$, $i = 1, \ldots, n_0^2$. Let $e_i \in H^0(\tilde{\Lambda}; k_{\tilde{\Lambda}})$ be the section with support $\tilde{\Lambda}_i$ and equal to 1 on $\tilde{\Lambda}_i$. Let $\text{id}_F \in H^0(\tilde{\Lambda}; k_{\tilde{\Lambda}})$ be the section induced by $\text{id}_F \in \text{Hom}(\tilde{F}, \tilde{F})$. Then we have $\text{id}_F = \sum_{i=1}^{n_0} e_i$, $e_i \circ e_i = e_i$ and $e_i \circ e_j = 0$ for all $i \neq j$ in $\{1, \ldots, n_0^2\}$.

By Remark 17.11 the isomorphism (18.2) is compatible with the compositions $\circ$ of $\text{Hom}(k_{\tilde{M}}^{n_0}, k_{\tilde{M}}^{n_0})$ and $\circ$ of $\mu_{\text{hom}}(\tilde{F}, \tilde{F})$. Let $p_i \in \text{Hom}(k_{\tilde{M}}^{n_0}, k_{\tilde{M}}^{n_0})$ be the image of $e_i$ by (18.2). Then we have $p_i^2 = p_i$ and $p_i p_j = 0$ for all $i \neq j$ in $\{1, \ldots, n_0^2\}$. Hence we have $n_0^2$ orthogonal projectors in $\text{Hom}(k_{\tilde{M}}^{n_0}, k_{\tilde{M}}^{n_0})$. This implies $n_0^2 \leq n_0$ and we deduce
\[ n_0 = 1, \text{ that is, } \tilde{L} \cong k_M[-d_0], \text{ which proves the first statement of the theorem.} \]

(iv) We have \( \mu_{\text{hom}}(F, F) \cong k_\Lambda \) and \( R\text{Hom}(L, L) \cong k_M \) by the first part of the theorem. Hence Theorem 17.10 applied to \( F' = F \), gives
\[
R\Gamma(M; k_M) \cong R\hom(L, L) \cong R\Gamma(\Lambda; k_\Lambda),
\]
which is the last part of the theorem. We remark that this isomorphism is induced by \( R\text{Hom}(F, F) \cong R(\pi_{M \times \mathbb{R}})_* \mu_{\text{hom}}(F, F) \), hence by the inverse image by \( \pi_{M \times \mathbb{R}} \), as claimed. \( \Box \)

By Theorem 8.3 and Corollary 16.6 (see also Remark 16.7), Theorem 18.3 gives the following result of [2]. We recall that Kragh [8] obtained a better result, namely that the Maslov class of \( \tilde{\Lambda} \) vanishes.

**Corollary 18.4.** Let \( \tilde{\Lambda} \subset T^*M \) be a compact exact Lagrangian submanifold. We assume that the Maslov class of \( \tilde{\Lambda} \) vanishes and that the image of the relative Stiefel-Whitney class \( \text{rw}_2(\lambda_0, \lambda_{\tilde{\Lambda}}) \) in \( H^2(X; k^x) \) is zero (see Definition 8.3 and Corollary 8.6). Then the projection \( \pi_M|_{\tilde{\Lambda}}: \tilde{\Lambda} \to M \) induces an isomorphism \( R\Gamma(M; k_M) \cong R\Gamma(\tilde{\Lambda}; k_{\tilde{\Lambda}}) \).

**19. Homotopy equivalence**

In [1] Abouzaid gives a result more precise than Corollary 18.4 in the situation of the corollary the projection \( \pi_M \) induces an isomorphism of the Poincaré groups. Since we already have an isomorphism between the cohomology groups it is enough to see that \( \pi_1(\tilde{\Lambda}) \to \pi_1(M) \) is an isomorphism. It is equivalent to show that the inverse image by \( \pi_M \) induces an equivalence of categories \( \text{Loc}(k_M) \cong \text{Loc}(k_{\tilde{\Lambda}}) \), for some field \( k \), which we prove in this section.

Let \( \Lambda \subset T^*_{\tau>0}(M \times \mathbb{R}) \) be a closed conic connected Lagrangian submanifold such that \( \Lambda/\mathbb{R}_{>0} \) is compact and the map \( \Lambda/\mathbb{R}_{>0} \to T^*M \), \( (x, t; \xi, \tau) \mapsto (x; \xi/\tau) \), is an embedding. It is equivalent to say that \( \Lambda \) is associated with a compact exact Lagrangian submanifold \( \tilde{\Lambda} \subset T^*M \) as in Remark 16.7. We assume for simplicity that \( k \) is a field. We make the same hypothesis as in Corollary 18.4 on the vanishing of the Maslov and Stiefel-Whitney classes of \( \tilde{\Lambda} \) (or \( \Lambda \)). By Theorem 8.3 there exists a simple global object in \( \mathcal{G}^*(k_\Lambda) \). By Corollary 16.6 there exists \( F_0 \in D^b(k_{M \times \mathbb{R}}) \) satisfying (a), (b), (c) below:

\[
(19.1) \left\{ \begin{array}{l}
(a) \quad \text{SS}(F_0) \cap \hat{T}^*(M \times \mathbb{R}) \subset \Lambda, \\
(b) \quad F_0|_{M \times \{t\}} \simeq 0 \text{ for } t \ll 0, \\
(c) \quad F_0 \text{ is simple along } \Lambda, \\
(d) \quad F_0|_{M \times \{t\}} \simeq k_M \text{ for } t \geq A.
\end{array} \right.
\]
Theorem 18.3 implies that there exists $A \in \mathbb{R}$, $d \in \mathbb{Z}$ and a local system $L$ on $M$ with stalk $k$ such that $F_0|_{M \times \{t\}} \simeq L[d]$, for $t \geq A$. Hence, tensorizing $F_0$ by $q^{-1}_M(D'L[-d])$ we may assume that $F_0$ also satisfies (d) in (19.1).

Recall that $\mathcal{G}(k)$ is the substack of $\mathcal{G}(k_A)$ formed by the objects whose microlocal germs are concentrated in one degree. Since $k$ is a field, the functor $\mathcal{H}om_{\mathcal{G}(k)}(F_0, \cdot)$ induces an equivalence of stacks

$$h_F : \mathcal{G}(k_A) \simeq \bigoplus_{i \in \mathbb{Z}} \text{Loc}(k_A)[i].$$

The inverse image by $q_M$ and $\pi_{M \times \mathbb{R}}|_{\Lambda} : \Lambda \to M \times \mathbb{R}$ induce

$$\pi_M^* : \text{Loc}(k_M) \xrightarrow{q_M^{-1}} \text{Loc}(k_{M \times \mathbb{R}}) \xrightarrow{\pi_{M \times \mathbb{R}}|_{\Lambda}^{-1}} \text{Loc}(k_A).$$

The tensorization by $F_0$ composed with $s_{\Lambda} : D^{b}_{(\Lambda)}(k_{M \times \mathbb{R}}) \to \mathcal{G}(k_A)$ gives

$$s_{F_0} : \bigoplus_{i \in \mathbb{Z}} \text{Loc}(k_M)[i] \to \mathcal{G}(k_A), \quad L \mapsto s_{\Lambda}(F_0 \otimes q^{-1}_M L),$$

where $q_M : M \times \mathbb{R} \to M$ is the projection.

**Lemma 19.1.** Let $F_0 \in D^{b}_{(\Lambda)}(k_{M \times \mathbb{R}})$ be a quantization of $\Lambda$ satisfying (19.1). Then the functors (19.2)–(19.4) give the commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{i \in \mathbb{Z}} \text{Loc}(k_M)[i] & \xrightarrow{\bigoplus_{i \in \mathbb{Z}} \pi_M^*[i]} & \bigoplus_{i \in \mathbb{Z}} \text{Loc}(k_A)[i] \\
F_0 \otimes q^{-1}_M(\cdot) & \downarrow & \downarrow h_{F_0} \\
D^{b}_{\Lambda \cup T^{*}_M \times \mathbb{R}}(k_{M \times \mathbb{R}}) & \xrightarrow{s_{\Lambda}} & \mathcal{G}(k_A).
\end{array}$$

**Proof.** Recall that $\mathcal{H}om_{\mathcal{G}(k)}(\cdot, \cdot) \simeq H^0\mu\text{hom}(\cdot, \cdot)$ by Corollary 3.4. Hence, for $L \in \text{Loc}(k_M)$ we have $h_{F_0}(s_{F_0}(L)) \simeq \mu\text{hom}(F_0, F_0 \otimes q^{-1}_M L)$.

By [7 Prop. 4.4.8] we have a general morphism, for a manifold $X$ and $F_1, F_2, G_1, G_2 \in D^{b}(k_X)$,

$$R \mu\text{hom}(F_1, F_2) \boxtimes_X \mu\text{hom}(G_1, G_2) \to \mu\text{hom}(F_1 \otimes G_1, F_2 \otimes G_2),$$

where $s : T^*X \times X T^*X \to T^*X$ is the fiberwise sum. In our case, for $F_1 = F_2 = F_0$ and $G_1 = k_{M \times \mathbb{R}}$, $G_2 = L$, we obtain a natural morphism

$$\mu\text{hom}(F_0, F_0) \otimes \pi^{-1}_M q^{-1}_M L \to \mu\text{hom}(F_0, F_0 \otimes q^{-1}_M L).$$

Since $L$ is locally constant this is an isomorphism and we have $\mu\text{hom}(F_0, F_0 \otimes q^{-1}_M L) \simeq k_{A} \otimes \pi^{-1}_M q^{-1}_M L$, which proves the commutativity of the diagram. □
Theorem 19.2. Let $M$ be a manifold and and let $\tilde{\Lambda} \subset T^*M$ be a compact exact Lagrangian submanifold. We assume that the Maslov class and the relative Stiefel-Whitney class of $\tilde{\Lambda}$ vanish. Then the projection $\pi_M|_{\tilde{\Lambda}} : \tilde{\Lambda} \to M$ is a homotopy equivalence.

Proof. (i) We let $\Lambda \subset T^*_{t>0}(M \times \mathbb{R})$ be a closed conic connected Lagrangian submanifold associated with $\tilde{\Lambda}$ as in Remark 16.7. We choose $F_0 \in D^b(k_{M \times \mathbb{R}})$ a quantization of $\Lambda$ satisfying (19.1). The functor $h_{F_0}$ is an equivalence of categories. Hence, by Lemma 19.1 it is enough to prove that $\mathfrak{s}_{F_0}$ also is an equivalence. We prove the fully faithfulness in (ii) and the essential surjectivity in (iii) below.

(ii) Let us prove that $\mathfrak{s}_{F_0}$ is fully faithful. Let $L, L' \in \text{Loc}(k_M)$ and $i, i' \in \mathbb{Z}$. We have to prove that

$$\text{Hom}(L[-i], L'[-i']) \simeq \text{Hom}_{\mathfrak{g}^p(k_{\Lambda})}(\mathfrak{s}_{F_0}(L[-i]), \mathfrak{s}_{F_0}(L'[-i'])).$$

If $i \neq i'$ both terms are zero by definition. So we assume $i = i' = 0$. Let $A \in \mathbb{R}$ be such that $\Lambda \subset T^*(M \times [0, A])$. Then

$$\text{Hom}_{\mathfrak{g}^p(k_{\Lambda})}(\mathfrak{s}_{F_0}(L), \mathfrak{s}_{F_0}(L'))$$

$$\simeq H^0(\Lambda; H^0 \mu_{\hom}(F_0 \otimes q_M^{-1}L, F_0 \otimes q_M^{-1}L'))$$

$$\simeq H^0(\Lambda; \mu_{\hom}(F_0 \otimes q_M^{-1}L, F_0 \otimes q_M^{-1}L'))$$

$$\simeq \text{Hom}((F_0 \otimes q_M^{-1}L)|_{M \times (A+1)}, (F_0 \otimes q_M^{-1}L')|_{M \times (A+1)})$$

$$\simeq \text{Hom}(L, L'),$$

where the first isomorphism follows from Corollary 3.1, the second one from the fact that $\mu_{\hom}(F_0, F_0)$ is in degree 0, the third one from Theorem 17.10 and the last one from (d) in (19.1).

(iii) Let us prove that $\mathfrak{s}_{F_0}$ is essentially surjective. Let $F \in \mathfrak{g}^p(k_{\Lambda})$. By Theorem 16.9 there exists $G \in D^b(k_{M \times \mathbb{R}})$ such that

$$\begin{cases} 
(a) \text{SS}(G) \subset \Lambda \cup T^*_{M \times \mathbb{R}}(M \times \mathbb{R}), \\
(b) G|_{M \times \{t\}} \simeq 0 \text{ for } t \ll 0, \\
(c) \mathfrak{s}_{A}(G) \simeq F.
\end{cases}$$

(iii-a) We set $L = G|_{M \times \{A+1\}}$. Then the $H^iL$ are local systems on $M$. Let us prove that $L$ is concentrated in one degree. Let $a \leq b$ be such that $H^aL \neq 0$, $H^bL \neq 0$ and $H^iL = 0$ for $i \notin [a, b]$. We have canonical non zero morphisms $u : H^aL[-a] \to L$ and $v : L \to H^bL[-b]$. By Theorem 17.10 we have

$$u \neq 0 \in \text{Hom}(H^aL, L[a]) \simeq H^a(\Lambda; \mu_{\hom}(F_0 \otimes q_M^{-1}H^aL, G)),$$

$$v \neq 0 \in \text{Hom}(L[b], H^bL) \simeq H^{-b}(\Lambda; \mu_{\hom}(G, F_0 \otimes q_M^{-1}H^bL))$$
Since $F_0$ and $G$ are pure, $\mu_{\text{hom}}(F_0, G)$ is concentrated in one degree, say $d$. Then $\mu_{\text{hom}}(F_0 \otimes q_M^{-1}H^aL, G)$ also is concentrated in degree $d$ and $\mu_{\text{hom}}(G, F_0 \otimes q_M^{-1}H^aL)$ is concentrated in degree $-d$. Hence (19.6) implies $a \geq d$ and (19.7) implies $-b \geq -d$. Since $b \geq a$ we deduce $a = d = b$.

(iii-b) By (iii-a) $L = G|_{M \times \{A+1\}}$ is concentrated in degree $d$. We set $G_0 = F_0 \otimes q_M^{-1}L [-d]$. Hence $\mu_{\text{hom}}(G_0, G)$ is concentrated in degree 0 and we have $H^0(\Lambda; \mu_{\text{hom}}(G_0, G)) \simeq \text{Hom}_{\mathcal{D}^b(\Lambda)}(s_\Lambda(G_0), s_\Lambda(G))$. The same holds with $G_0$ and $G$ exchanged. Then (19.6) and (19.7) translate into

$$(19.8) \quad \text{Hom}(L, L) \simeq \text{Hom}_{\mathcal{D}^b(\Lambda)}(s_\Lambda(G_0), F) \simeq \text{Hom}_{\mathcal{D}^b(\Lambda)}(F, s_\Lambda(G_0)).$$

Moreover Remark 17.11 implies that (19.8) is compatible with the composition. Letting $a : s_\Lambda(G_0) \to F$ and $b : F \to s_\Lambda(G_0)$ be the images of $\text{id}_L$ by (19.8) we deduce that $a$ and $b$ are mutually inverse isomorphisms. Hence $F \simeq s_\Lambda(G_0) \simeq s_{F_0}(L[-d])$ and $s_{F_0}$ is essentially surjective. 

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