Abstract

We show that first approximations to the bulk viscosity \( \eta \) are expressible in terms of factors that depend on the sound speed \( v_s \), the enthalpy, and the interaction (elastic and inelastic) cross section. The explicit dependence of \( \eta \) on the factor \( \left( \frac{1}{3} - v_s^2 \right) \) is demonstrated in the Chapman-Enskog approximation as well as the variational and relaxation time approaches. The interesting feature of bulk viscosity is that the dominant contributions at a given temperature arise from particles which are neither extremely nonrelativistic nor extremely relativistic. Numerical results for a model binary mixture are reported.

1. Introduction

Recent interest in the bulk viscosity \( \eta \) of strongly interacting matter stems from the observation that in the phase transition from hadrons to strongly interacting quarks and gluons \( \eta \) exhibits a drastic change \[1\]. The purpose of this work is to establish the dependence of \( \eta \) on the sound speed \( v_s \), the thermodynamic properties of the system (particularly, the enthalpy) and the interaction (elastic and inelastic) cross sections between the constituents of a hadronic system.

2. Bulk viscosity and the speed of sound

2.1. Chapman-Enskog approximation (Single component gas)

In this approach, the first approximation to bulk viscosity can be written as \[2, 3\]

\[
\eta = kT \left( \frac{\alpha_2^2}{2w_0^{(2)}} \right) ,
\]

where

\[
\alpha_2 = \frac{3}{2} \left( \hat{h} \left( \gamma - \frac{5}{3} \right) + \gamma \right) , \quad z = \frac{mc^2}{kT} , \quad \hat{h} = \frac{K_3(z)}{K_2(z)} \quad \text{and} \quad \gamma = \frac{c_p}{c_v} .
\]

Above, \( \hat{h} = h/c^2 \) (given in terms of modified Bessel functions for a Boltzmann gas) is the reduced enthalpy and \( \gamma \) is the ratio of specific heats at constant pressure and volume, respectively. The quantity \( w_0^{(2)} \) is the so-called omega integral which contains information about the cross section of the scattering particles. Explicitly (and for illustration, for elastic scattering),

\[
w_i^{(s)} = \frac{2\pi^3 c}{K_2(z)^2} \int_0^\infty d\psi \sinh^2 \psi \cosh^2 \psi K_i(2z \cosh \psi) \int_0^\pi d\Theta \sin \Theta \sigma(\psi, \Theta) \left( 1 - \cos^2 \Theta \right) ,
\]

Preprint submitted to Nuclear Physics A

September 16, 2009
where \(\sigma(\psi, \Theta)\) is the differential cross section and \(j = \frac{5}{3} + \frac{1}{2} (-1)^3\); the others symbols are:

\[
g = \frac{1}{2}(p_1 - p_2) \quad \text{and} \quad P = (-p_\alpha p^\alpha)^{1/2}
\]

\[
\sinh \psi = \frac{g}{mc} \quad \text{and} \quad \cosh \psi = \frac{P}{2mc}.
\]

The adiabatic speed of sound \(v_s = \sqrt{\frac{\partial P}{\partial \varepsilon}}\) and the ratio of specific heats \(\gamma = c_p/c_v\) can be related using the relation

\[
\gamma = 1 + \frac{\partial P}{\partial \varepsilon} = 1 + v_s^2.
\]

Then, \(\alpha_2\) in Eq. (1) can be rewritten in terms of the speed of sound as

\[
\alpha_2 = \frac{3}{2} \left\{ -(\hat{\varepsilon} + 1) \left( \frac{1}{3} - v_s^2 \right) - \frac{1}{3} \hat{\varepsilon} + \frac{4}{3} \right\}.
\]

Thus, the bulk viscosity takes the form

\[
\eta = kT \frac{a^2 \left( \frac{1}{3} - v_s^2 \right)^2 + 2a \left( \frac{1}{3} - v_s^2 \right) + b^2}{2w_0^{(2)}},
\]

where \(a = -\frac{3}{2} (\hat{\varepsilon} + 1)\) and \(b = -\frac{1}{2} (\hat{\varepsilon} - 4)\).

**Limiting Situations**

It is instructive to consider the limiting cases of ultrarelativistic and nonrelativistic situations. For nearly massless particles or very high temperatures \(T, z = m/T \ll 1\). In this case, \(\hat{\varepsilon} \rightarrow 4\) as \(z \rightarrow 0\). Consequently, \(a \rightarrow -\frac{15}{2}z\) and \(b \rightarrow 0\). Utilizing these values, the bulk viscosity can be written as

\[
\eta \rightarrow \frac{225}{8} \frac{kT}{2w_0^{(2)}} \left( \frac{1}{3} - v_s^2 \right)^2.
\]

Notice that for weakly interacting massless particles \(v_s^2 \rightarrow \frac{1}{3}\), so that \(\eta \rightarrow 0\). Note also the quadratic dependence of \(\eta\) on \(v_s^2\).

On the other hand, for masses such that \(z = m/T \gg 1\), the coefficient \(a \rightarrow -\frac{3}{2}z\) and \(b \rightarrow -\frac{1}{2}z\). These limiting forms render the bulk viscosity as

\[
\eta = \frac{kT}{2w_0^{(2)}} \frac{z^2}{4} \left[ 9 \left( \frac{1}{3} - v_s^2 \right)^2 + 6 \left( \frac{1}{3} - v_s^2 \right) + 1 \right].
\]

Weakly interacting, nonrelativistic particles are characterized by \(v_s^2 \rightarrow \frac{1}{2}\) so that, again \(\eta \rightarrow 0\).

From the above analysis, we learn the important lesson that for a given temperature and for weakly interacting particles, intermediate mass particles contribute significantly to the bulk viscosity. It would be interesting to investigate the extent to which this conclusion is modified by strong interactions.
2.2. Variational and relaxation time approximations

Here one starts from the general definition of the stress energy tensor \[4\]

\[
T^{ij} = T_0^{ij} + \int d\Gamma P^{ij} \delta f_p, \quad \delta f_p = -\tau \left( \frac{\partial f_p}{\partial \Gamma} + v_p \cdot \nabla f_p \right) \quad \text{and} \quad d\Gamma = \frac{d^3 p}{(2\pi)^3 \rho^0}, \tag{11}
\]

where \(f_p^0\) is the equilibrium distribution function, \(v_p\) is the velocity of a particle with momentum \(p\) and energy \(\epsilon_p = p^0\). In terms of the fluid velocity field \(u\), the dissipative part of the energy stress tensor can be written as

\[
T^{ij}_{\text{diss}} = -\eta \left( \frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right) - \left( \eta_v - \frac{2}{3} \right) \nabla \cdot u \delta^{ij}. \tag{12}
\]

Inserting the deviation function \(\delta f_p\) into the second part of the stress energy tensor and comparing Eqs. (11) and (12), one gets \[4\]

\[
\eta_v = \frac{\tau}{9T} \int d\Gamma f^0 \left[ \left( 1 - \frac{3h}{c_i T} \right) \rho^2 - m^2 \frac{n^2}{\epsilon_p^2} \right] = \frac{\tau}{9T} \int d\Gamma f^0 \left[ \left( 1 - 3v_i^2 \right) \epsilon_p - \frac{m^2}{\epsilon_p^2} \right]^2, \tag{13}
\]

where \(\tau\) is a momentum-independent relaxation time. In writing the rightmost equality, the identity \(v_i^2 = h/(c_i T)\) and other integral identities have been used. Equation (13) lends itself to straightforward manipulations in the two limiting situations studied in the previous section. For \(m \to 0\) and \(v_i^2 \to \frac{1}{3}\), one obtains the result that \(\eta_v \to 0\). In the case of massive particles, for which \(\epsilon_p \approx m\) and \(v_i^2 \to \frac{1}{3}\), the result \(\eta_v \approx \frac{m^2}{\epsilon_p^2} \to 0\) is obtained. It is gratifying that the variational approach yields results similar to those obtained using the Chapman-Enskog approximation.

It is easy to verify that similar conclusions emerge in the case that the expression for bulk viscosity features an energy dependent relaxation time \(\tau_a(\epsilon_a)\) [1]:

\[
\eta_v = \frac{1}{9T} \sum_a \int d\Gamma \frac{\tau_a(\epsilon_a)}{\epsilon_a^2} \left[ \left( 1 - 3v_i^2 \right) \epsilon_a^2 - \frac{m_a^2}{\epsilon_a^2} \right]^2 f_a^0, \tag{14}
\]

where the subscript \(a\) denotes the particle species. In conclusion, we learn that intermediate mass particles contribute the most to the bulk viscosity.

3. Bulk viscosity of a binary mixture

In the Chapman-Enskog approach, the first order approximation to the bulk viscosity of a binary mixture has the same formal expression as that for a one component gas \[3, 5\]. Explicitly,

\[
[\eta_{\text{vis}}] = kT \frac{a_2^2}{a_{22}^2}, \quad \alpha_2 = \frac{x_i (\gamma_{(1)} - \gamma)}{\gamma_{(1)} - 1}, \tag{15}
\]

where \(x_i = n_i / n\), with \(n_i\) denoting the number densities and \(\gamma_{(i)}\) denoting the ratio of specific heats of species \(i\). The coefficient \(a_{22}\) differs from that in a one-component gas as it receives contributions from two sources: \(a_{22} \equiv a_{22}^o + a_{22}^v\). The quantity \(a_{22}^o\) accounts for collisions between the same type of particles, and is calculated from the expressions given earlier for a one-component gas. The quantity \(a_{22}^v\) accounts for collisions between different types of particles. Explicitly,

\[
a_{22}^o = \frac{16 \rho_1 \rho_2}{M^2 n^2} \frac{\omega^{(1)}_{1200}}{3} \Omega_{(12)}, \quad \tag{16}
\]
where $M = m_1 + m_2$, $\rho_i = n_i m_i$ and the appropriate omega integral is

$$w^{(1)}_{1200}(\sigma_{12}) = \frac{\pi \mu c^3}{4 k T K_2(z_1)K_2(z_2)} \int_0^\infty d\psi_{12} \sinh^3 \psi_{12} \left( \frac{g_{12}^2}{2 \mu k} \right) \left( \frac{M c P_{12}}{P_{12}} \right)^2 K_2 \left( \frac{c P_{12}}{k T} \right)$$

$$* \int_0^\infty d\Theta \sin \Theta_{12} \sigma_{12}(\psi_{12}, \Theta_{12})(1 - \cos \Theta_{12}), \quad (17)$$

where $P_{12}^2 = m_1^2 c^2 + m_2^2 c^2 + 2 m_1 m_2 c^2 \cosh \psi_{12}$ and $g_{12} = m_1 c \sinh \psi_{12} = m_2 c \sinh \psi_{12}$.

Figure 1 shows illustrative results of bulk viscosity versus temperature for different mass configurations with constant cross sections. From the results shown, we learn that intermediate mass configurations contribute the most to the bulk viscosity.

Figure 1: Bulk viscosity $\eta_v$ versus temperature $T$ in a binary mixture of dissimilar mass particles. The mass $m_1$ of particle 1 is fixed at 140 MeV, whereas the mass $m_2$ of particle 2 is varied as indicated alongside the different curves. Results are for a constant energy independent cross section of 1 fm$^2$ for all collisions.

Work is in progress to calculate the bulk viscosity of a hadronic mixture comprising of many hadronic resonances whose masses extend up to 2 GeV.

Acknowledgments

This research was supported by the Department of Energy under the grant DE-FG02-93ER40756.

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