The Feynman propagator for a free bosonic scalar field on the discrete spacetime of a causal set is presented. The formalism includes scalar field operators and a vacuum state which define a scalar quantum field theory on a causal set. This work can be viewed as a novel regularisation of quantum field theory based on a Lorentz invariant discretisation of spacetime.

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I. FREE SCALAR QUANTUM FIELD THEORY IN THE CONTINUUM

We briefly review the quantum field theory for a free real scalar field on $\mathbb{M}^4$ (following the sign conventions of [5]). The defining equation for a Green’s function $G^{(d)}$ for the Klein-Gordon equation is

$$\Box G^{(d)}(x) = \delta^d(x).$$

Here $x = (x^0, \vec{x}) = (x^0, x^i)$ (for $i = 1, \ldots, d-1$), $m$ is the mass of the field, $\delta^d$ is the $d$-dimensional delta function and we choose units with $\hbar = c = 1$. The d’Alambertian is $\Box := \partial^2_{x^0} - \partial^2_{\vec{x}}$.

Three important Green’s functions (or propagators) are the retarded $G_R$, advanced $G_A$ and Feynman $G_F$. In $\mathbb{M}^4$ we have [7, 9, p23]:

$$G^{(2)}_R(x) := \theta(x^0)\theta(s^2)\frac{1}{2}J_0(ms),$$

$$G^{(2)}_A(x) := G^{(2)}_R(-x),$$

$$G^{(2)}_F(x) := \frac{1}{4}H^{(2)}_0(ms),$$

where $s = \sqrt{(x^0)^2 - (x^1)^2}$ for $(x^0)^2 \geq (x^1)^2$ and $s = -i\sqrt{(x^1)^2 - (x^0)^2}$ for $(x^0)^2 < (x^1)^2$. $J_0$ is a Bessel function of the first kind, $H^{(2)}_0$ is a Hankel function of the
second kind and

\[ \theta(\alpha) = \begin{cases} 1 & \text{if } \alpha \geq 0 \\ 0 & \text{if } \alpha < 0. \end{cases} \]  

In \( \mathbb{M}^4 \) we have \[ \text{Appendix 1} \]:

\[ G_{R}^{(4)}(x) := \theta(x^0) \theta(s^2) \left( \frac{\delta(s^2)}{2\pi} - \frac{m}{4\pi s} J_1(ms) \right), \]  
\[ G_{A}^{(4)}(x) := G_{R}^{(4)}(-x), \]  
\[ G_{F}^{(4)}(x) := \frac{1}{4\pi} \delta(s^2) - \frac{m}{8\pi s} H_1^{(2)}(ms), \]

where \( s = \sqrt{(x^0)^2 - x^2} \) for \( (x^0)^2 \geq x^2 \) and \( s = -i \sqrt{x^2 - (x^0)^2} \) for \( (x^0)^2 < x^2 \). \( J_1 \) is a Bessel function of the first kind and \( H_1^{(2)} \) is a Hankel function of the second kind and \( \delta(s^2) \) is the delta function of \( s^2 \).

The Pauli-Jordan function is defined as \[ \Delta(x) := G_R^{(4)}(x) - G_A^{(4)}(x). \]  

We drop the dimension-superscripts whenever we refer to arbitrary dimension.

A. Fields in the continuum

A free real bosonic scalar field is represented by an algebra of field operators \( \hat{\phi}(x) \) (which act on a Fock space \( F \)) satisfying the following conditions:

1. \( (\Box + m^2) \hat{\phi}(x) = 0, \]
2. \( \hat{\phi}(x) = \hat{\phi}(x)^\dagger, \]
3. \[ [\hat{\phi}(x), \hat{\phi}(y)] = i \Delta(y - x), \]

In addition there exists a Poincaré invariant vacuum state \( |0\rangle \) in \( F \). With the fields so defined the Feynman propagator is given by the vacuum expectation value of the time-ordered product of two field operators:

\[ G_F(y - x) = i\langle 0 | T \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle. \]

The time-ordering has time increasing from right to left.

Applying \( (\Box + m^2) \) to the commutator in \[ \text{(13)} \] gives

\[ [(\Box + m^2) \hat{\phi}(x), \hat{\phi}(y)] = (\Box + m^2)i \Delta(y - x) = 0. \]

That is, even if only \[ \text{(12)} \] and \[ \text{(13)} \] hold, we have that \( (\Box + m^2) \hat{\phi}(x) \) commutes with all the \( \hat{\phi}(y) \) \[ \text{(11)} \].

\[ \theta(\alpha) = \begin{cases} 1 & \text{if } \alpha \geq 0 \\ 0 & \text{if } \alpha < 0. \end{cases} \]

(II. FREE SCALAR QUANTUM FIELD THEORY ON A CAUSAL SET)

Let \( (\mathcal{C}, \preceq) \) be a finite causal set with \( p \) elements \( v_1, \ldots, v_p \), generated by sprinkling (with density \( \rho \)) into a finite causal interval in \( \mathbb{M}^d \).

For sprinklings into \( \mathbb{M}^2 \) the \( p \times p \) matrix

\[ K_{R}^{(2)} := aC(I - abC)^{-1}, \quad a = \frac{1}{2}, \quad b = -\frac{m^2}{\rho}, \]

where \( I \) is the identity matrix, \( C \) the causal matrix for the causal set and \( m \) is the mass of the field) is the appropriate definition of the retarded propagator \[ \text{[7]} \].

For sprinklings into \( \mathbb{M}^4 \) the \( p \times p \) matrix

\[ K_{R}^{(4)} := aL(I - abL)^{-1}, \quad a = \frac{\sqrt{b}}{2\pi \sqrt{6}}, \quad b = -\frac{m^2}{p}, \]

(\( L \) is the link matrix for the causal set) is the appropriate definition of the retarded propagator. Both these matrices were defined and studied in detail in \[ \text{[7]} \].

Our goal now is to define a \( p \times p \) matrix \( K_F \) to serve as the Feynman propagator on a causal set. We drop the dimension-superscripts to be able to refer to sprinklings in either \( \mathbb{M}^2 \) or \( \mathbb{M}^4 \).

If \( K_R \) is the retarded propagator it follows that its transpose \( K_A := K_R^T \) is the advanced propagator and the real matrix defined by

\[ \Delta := K_R - K_A, \]

is the causal set analogue of the Pauli-Jordan function.

The matrix \( i\Delta \) is skew-symmetric and Hermitian. These two properties ensure its rank is even \[ \text{[10]} \] and its non-zero eigenvalues appear in real positive and negative pairs. In particular (if its rank is non-zero) there exist non-zero eigenvalues and normalised eigenvectors such that:

\[ i\Delta u_i = \lambda_i u_i, \quad i\Delta v_i = -\lambda_i v_i, \]

(\( \lambda_i > 0 \) for \( i = 1, \ldots, s \) where \( 2s \) is the rank of \( i\Delta \)).

These eigenvectors are each defined up to a phase factor and can be chosen such that \( u_i = v_i^* \), \( u_i^* v_j = 0 \) for all \( i, j = 1, \ldots, s \) (where \( z^* \) denotes complex conjugate of \( z \) and \( u_i := (i)^T \) is the Hermitian conjugate of a column vector \( u \)).

It’s useful to define the Hermitian \( p \times p \) matrix

\[ Q := \sum_{i=1}^{s} \lambda_i u_i u_i^*, \]

such that \( i\Delta = Q - Q^* = Q - Q^T \).

A. Fields on a causal set

We now define an algebra of field operators \( \hat{\phi}_x \) (acting on some Hilbert space \( H \)) to represent a free real bosonic scalar field on the causal set. For each causal set element \( v_x \) (\( x = 1, \ldots, p \)), we suppose there exists a field operator \( \hat{\phi}_x \) such that

1. \[ \hat{\phi}_x = \hat{\phi}_x^\dagger, \]
2. \[ [\hat{\phi}_x, \hat{\phi}_y] = i\Delta_{xy}, \]
3. \[ i\Delta \omega = 0 \Rightarrow \sum_{x' = 1}^{p} \omega_{x'} \hat{\phi}_{x'} = 0, \]
for \( x, y = 1, \ldots, p \). The first two conditions are natural generalisations of the continuum case. The last condition ensures that any linear combination of field operators that commutes with all the field operators must be zero. By [15] this is the analogue of imposing the Klein-Gordon equation on the field operators.

From these field operators we can define new operators

\[
\hat{a}_i := \sum_{x=1}^{p} (v_i)_x \hat{\phi}_x, \quad \hat{a}_i^\dagger := \sum_{x=1}^{p} (u_i)_x \hat{\phi}_x.
\]  

(24)

for \( i = 1, \ldots, s \). They satisfy \((\hat{a}_i^\dagger)^i = \hat{a}_i\) and

\[
[\hat{a}_i, \hat{a}_j] = v_i^T i \Delta v_j = -\lambda_j u_i^\dagger v_j = 0,
\]  

(25)

\[
[\hat{a}_i, \hat{a}_j^\dagger] = u_i^T i \Delta u_j = \lambda_j v_i^\dagger u_j = 0,
\]  

(26)

\[
[\hat{a}_i, \hat{a}_j^\dagger] = v_i^T i \Delta u_j = \lambda_j u_i^\dagger u_j = \lambda_j \delta_{ij},
\]  

(27)

and can be interpreted as (unnormalised) creation and annihilation operators.

The transformation can be inverted to give

\[
\hat{\phi}_x = \sum_{i=1}^{s} (u_i)_x \hat{a}_i + (v_i)_x \hat{a}_i^\dagger,
\]  

(28)

(here it is important we imposed [23]).

We now define a vacuum state vector \(|0\rangle \in H\) by the conditions that \(\hat{a}_i|0\rangle = 0\) for all \(i = 1, \ldots, s\) and \(|0\rangle = 1\). This allows us to recognise that \(H\) is the Fock space spanned by basis vectors \((\hat{a}_i^\dagger)^{n_1}(\hat{a}_j^\dagger)^{n_2} \cdots (\hat{a}_i^\dagger)^{n_s}|0\rangle\) for integers \(n_i \geq 0, i = 1, \ldots, s\).

The two-point function can be evaluated as

\[
\langle 0| \hat{\phi}_x \hat{\phi}_y |0\rangle = \sum_{i=1}^{s} \sum_{j=1}^{s} (u_i)_x (v_j)_y \langle 0| \hat{a}_i \hat{a}_j^\dagger |0\rangle
\]  

(29)

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} (u_i)_x (v_j)_y \lambda_j \delta_{ij} = Q_{xy}.
\]

We can now define the Feynman propagator by analogy with [14]. With time increasing from right to left we have

\[
(K_F)_{xy} := i \langle 0| T \hat{\phi}_x \hat{\phi}_y |0\rangle := i (\bar{A}_{xy} Q_{yx} + \bar{A}_{yx} Q_{xy} + \delta_{xy} Q_{xy}),
\]  

(30)

where \(\bar{A}\) is the causal matrix for a linear extension of the causal set and \(\delta_{xy}\) is the Kronecker delta. In general there are multiple different linear extensions which assign an arbitrary order to pairs of unrelated elements. This arbitrariness does not affect \(K_F\) because the field operators for unrelated elements commute.

Observing that \(\bar{A}_{xy}(i\Delta_{xy}) = (iK_R)_{xy}\) we have

\[
\bar{A}_{xy} Q_{yx} = \bar{A}_{xy} (Q_{yx} - i\Delta_{xy}) = \bar{A}_{xy} Q_{xy} - i(K_R)_{xy}.
\]  

(31)

Substituting this into (30) gives an alternative form

\[
K_F = K_R + iQ,
\]  

(32)

since \(\bar{A}_{xy} + \bar{A}_{yx} + \delta_{xy} = 1\) for all \(x, y = 1, \ldots, p\).

Since \(i\Delta = Q - Q^*\) the imaginary part of \(Q\) is \(\Im(Q) = \Delta/2\). Combining this with (32) and looking at the real and imaginary parts of \(K_F\) gives

\[
\Re(K_F) = K_R - \frac{\Delta}{2} = \frac{K_R + K_A}{2},
\]  

(33)

\[
\Im(K_F) = \Re(Q).
\]  

(34)

III. COMPARISON WITH THE CONTINUUM

The causal set propagators depend on the particular random causal set that is sprinkled. By calculating their average value for different sprinklings we can compare the causal set and continuum propagators. To do this, first fix a finite causal interval \(\mathbb{A} \subset \mathbb{M}^d\). Pick two points \(X, Y \in \mathbb{A}\). Sprinkle a finite causal set into \(\mathbb{A}\) with density \(\rho\). Almost surely this will not contain \(X\) and \(Y\) so add \(X\) and \(Y\) to it to obtain a finite causal set \((\mathbb{C}, \preceq)\).

For definiteness label the causal set element \(X\) as \(v_1\) and \(Y\) as \(v_2\).

We now calculate \(K_R\) and \(K_F\) for \((\mathbb{C}, \preceq)\) and look at \((K_R)_{12}\) and \((K_F)_{12}\), i.e. the propagator values for the pair \((X, Y)\). Let \(\mathbb{E}(K_R^{(d)}|X, Y, \mathbb{M}^d, \rho)\) denote the expected value of \((K_R^{(d)})_{12}\) and \(\mathbb{E}(K_F^{(d)}|X, Y, \mathbb{M}^d, \rho)\) the expected value of \((K_F^{(d)})_{12}\) averaged over all causal sets sprinkled into \(\mathbb{A} \subset \mathbb{M}^d\) with \(X\) and \(Y\) added in the manner described and for a fixed density \(\rho\). It was shown in [7] that

\[
\mathbb{E}(K_R^{(d)}|X, Y, \mathbb{M}^d, \rho) = G_R^{(2)}(Y - X),
\]  

(35)

\[
\lim_{\rho \to \infty} \mathbb{E}(K_R^{(d)}|X, Y, \mathbb{M}^d, \rho) = G_R^{(4)}(Y - X).
\]  

(36)

Using these and (33) we have

\[
\Re(\mathbb{E}(K_F|X, Y, \mathbb{M}^d, \rho)) = \Re(G_F^{(2)}(Y - X)),
\]  

(37)

\[
\lim_{\rho \to \infty} \Re(\mathbb{E}(K_F|X, Y, \mathbb{M}^d, \rho)) = \Re(G_F^{(4)}(Y - X)).
\]  

(38)

That is, the real part of the expected value of \(K_F\) is correct for \(\mathbb{M}^2\) and correct in the infinite density limit for \(\mathbb{M}^4\) [12]. We can compare the imaginary parts of \(K_F\) and \(G_F\) through numerical simulations.

By using a computer to perform sprinkling into finite regions of 1+1 and 3+1 Minkowski spacetime we can compute \(K_R, K_A, i\Delta, Q\) and \(K_F\) for a range of sprinkling densities and field masses. Plotting the real and imaginary parts of \(K_F\) against the absolute value of the proper time for causally related and spacelike separated pairs of sprinkled points allows us to see if \(K_F\) agrees with \(G_F\).

Simulations using standard linear algebra software on a desktop computer give clear results for the 1+1 dimensional case for causal sets with as few as 600 elements (see Fig. [1]). The agreement is very good provided \(0 \ll m \ll \sqrt{\rho}\). There is disagreement between the imaginary parts of \(K_F\) and \(G_F^{(2)}\) as we take the field mass to zero but this seems to be related to the lack of a massless limit of \(\Im(G_F^{(2)})\). There are also small deviations due to the calculation being sprinkled into a finite region of
FIG. 1: Plots of $\Im((K_F)_{xy})$ for pairs of sprinkled points $v_x$ and $v_y$ in a single 600 element causal set generated by sprinkling into a unit-volume causal interval in $M^2$ and $\Im(G^2_F(X))$ for $X \in M^2$.

Minkowski spacetime. These “edge effects” get smaller if the results are only plotted for pairs of points away from the edges of the region.

In 3+1 dimensions the simulations are less clear. We expect good agreement with the continuum for large $\rho$ (since the infinite density limit is needed in (36)) but larger sprinklings are needed to investigate this.

IV. DISCUSSION

The formalism provokes interesting questions about the eigenvectors $u_i, v_i$. They seem to be related to linear combinations of positive and negative frequency plane wave solutions of the Klein-Gordon equation. Plotting the eigenvector values as a function of position for causal sets sprinkled into $M^d$ shows relatively smooth oscillatory behaviour for the eigenvectors associated with large eigenvalues. They show some dependence on the shape of the region into which the points are placed but this dependence is lost when $\lambda_i u_i u_i^\dagger$ is summed in the $Q$ and $K_F$ matrices (details to appear elsewhere).

Extending $K_F$ to infinite causal sets as well as further investigating the behaviour of $K_F$ for sprinklings into $M^4$ as well as other conformally flat or curved spacetimes would be of significant interest. Investigating the variance of $K_F$ for different sprinkling densities is also important. The propagator $K_R$ (and therefore $K_F$) can be extended to non-sprinkled causal sets [7 Sec 4.2]. This ensures both $K_R$ and $K_F$ are relevant to a fundamental theory based only on causal sets with no reference to Lorentzian manifolds or sprinklings.

The work presented here opens the door to interacting scalar field theories, scalar field phenomenology, the evaluation of Feynman diagrams, S-matrices and scattering amplitudes on a causal set. Extending the quantum field theory to include interacting spinor and vector fields remains to be done—advances here could lead to physical predictions for matter on a causal set.

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\[\Re(G_F(x)) = (G_R(x) + G_A(x))/2.\]