INCENTIVE-COMPATIBLE ELICITATION OF QUANTILES

NICHOLAS M. KIEFER

ABSTRACT. Incorporation of expert information in inference or decision settings is often important, especially in cases where data are unavailable, costly or unreliable. One approach is to elicit prior quantiles from an expert and then to fit these to a statistical distribution and proceed according to Bayes rule. Quantiles are often thought to be easier to elicit than moments. An incentive-compatible elicitation method using an external randomization is available. Such a mechanism will encourage the expert to exert the care necessary to report accurate information. A second application might be called posterior elicitation. Here an analysis has been done and the results must be reported to a decision maker. For a variety of reasons (possibly including the reward system in the corporate hierarchy) the modeler might need the right incentive system to report results accurately. Again, eliciting posterior quantiles can be done with an incentive compatible mechanism.

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1. INTRODUCTION

Incorporation of prior information is important in any decision or inference setting, whether it is done formally or informally. The formal Bayesian approach encourages transparency in assumptions, clear thinking and coherence. One example is risk management in financial institutions in which prudent management requires understanding default probabilities for groups of homogeneous assets. In the case of new types of assets there may not be enough data information to support a practical conventional estimator, for example the frequency estimator in the case of binomial defaults. Zero is not an estimated default probability that is acceptable to regulators. This issue has attracted regulatory and industry as well as academic attention, see Kiefer (2009), and references given there. Kiefer (2010) proposes eliciting prior quantiles for an expert’s prior on the value of the default probability for a particular group of assets. For example, the median can be assessed by asking the expert at what value of a default rate θ would he be equally surprised to see a realization above or below θ. These quantiles (perhaps after feedback and revision) are assembled into a distribution, either by fitting a specific functional form or as proposed by Kiefer (2010) fit to a smoothed maximum-entropy distribution. The idea is to impose as little information as possible beyond that elicited from the expert. This distribution is then used to process data information through Bayes rule.

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Cornell University, Departments of Economics and Statistical Science, 490 Uris Hall, Ithaca, NY 14853-7601, US, email:nicholas.kiefer@cornell.edu, and CREATES, funded by the Danish Science Foundation, University of Aarhus, Denmark.
and the likelihood function. The latter is itself typically a representation of a large number of probabilities in terms of a few parameters, so the statistical treatment and the approximations involved are the same in the prior and the likelihood.

Other examples of elicitation of quantiles and their use to form a prior distribution are cited in O’Hagan, Buck, Daneshkhah, Eiser, Garthwaite, Jenkinson, Oakley, and Rakow (2006) and include applications to drug testing, sales of engines, the effect of nuclear waste on temperature, and future earnings. A discussion of the statistical and psychological issues involved in elicitation is Garthwaite, Kadane, and O’Hagan (2005). These issues are not reviewed here. The incentive compatibility question does not seem to have been stressed.

A second application is eliciting accurate assessments from a modeling group within an organization. Here, the risks and rewards to proper reporting incentives are clear. In banks, risk modelers are required by regulators to report to senior management, not to business line management. In other institutions the separation between incentives for modelers and those acting on the results might be less clear. Senior management may choose to put in place incentive compatible reporting mechanisms. Regulators may question the compensation plan for modelers.

Finally, reporting of financial forecasts and results to counterparties and regulators should be as accurate as possible. Here too, incentive-compatible mechanisms might play a role.

The difficult part is the elicitation of the quantiles, which requires thought and therefore some effort from the expert in our first example (the expert must have the incentive to provide this effort) and simply rewards for accuracy in the second. Since the quantiles can never be observed, and therefore the assessment checked, there is an issue of providing an incentive for the expert to provide the required thought. The problem of eliciting probabilities for given sets is well-studied and a widely-used approach is scoring. The scoring method does not naturally extend to the quantile assessment problem, as shown in Section 2. Savage (1971) reviews techniques for assessing probabilities and notes an interesting interpretation of probabilities as prices. He notes that the device of outside randomization, used by Marschak (1964) to compel a true valuation for a bid or asked price applies also to probability assessment. An ingenious recent method due to Karni (2009) introduces a second outside source of randomness to eliminate possible effects of risk aversion. This method does extend naturally to eliciting quantiles as shown in Section 2 with a different development than that of Marschak or Karni. Section 3 concludes.

2. Eliciting Probabilities and Quantiles

Assume at the outset that the expert’s information about the unknown quantity \( \theta \) is coherent, that is that it can be described by a probability distribution. Classical discussions of the necessity of describing uncertainty in terms of probability are Savage (1954), De Finetti (1974), and Lindley (1982). We do not review these well-known demonstrations and simply assume that the expert’s information is described in a probability distribution with \( cdf F(\theta) \) and \( pdf f(\theta) \). We wish to elicit the probability \( \alpha \) quantile \( q_\alpha \) with \( F(q_\alpha) = \alpha \). Assume that \( f(q_\alpha) > 0 \) so that the
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Quantile is well-defined. Assume for convenience that $\text{Supp}(f) \subseteq [0, 1]$. This is natural when the uncertain quantity $\theta$ is a default probability; in other cases a parameter transformation may be appropriate. The method of scoring for eliciting the probability $\beta$ associated with a given set, say $[0, q]$, rewards the expert with

$$r(\beta|q) = (I(\theta \in [0, q])g_1(\beta) + (1 - I(\theta \in [0, q]))g_0(\beta)$$

where $I()$ is the indicator function $g_1(\beta)$ is a nondecreasing function, $g_0(\beta)$ is a nonincreasing function and $\beta$ is the elicited probability, after the single realization of $\theta$ is seen. The scoring rule is proper if the expectation $Er(\beta|q) = F(q)g_1(\beta) + (1 - F(q))g_0(\beta)$ is maximized at $\beta = q_\alpha$ (some applications minimize instead). The optimal choice of $\beta$ for a risk-neutral expert rewarded by a proper scoring rule is $F(q)$. A classical example of a proper scoring rule has $g_1(\beta) = (1 - \beta)^2$ and $g_0(\beta) = -\beta^2$ in which case the first-order condition for maximizing $Er(\beta|q)$ implies $F(q)/(1 - F(q)) = \beta/(1 - \beta)$.

Scoring rules for probabilities have been widely studied. A classical application is the assessment of the quality of weather forecasts. Most of the literature does not consider scoring for probability assessment, rather for estimation when the score is an objective function or for measuring the accuracy of assessments already arrived at, or for evaluating the fit of statistical models. Thus, most of the literature does not consider risk aversion and its effects on probability assessment, although the issue is well known, see Savage (1971) or for a general treatment that also considers the effects of state preference (a very difficult problem) Kadane and Winkler (1988). Schervish (1989) provides a characterization of the class of proper scoring rules for probabilities. It is clear from (11) that scoring rules need not be symmetric. Winkler (1994) argues that in forecasting a short-horizon weather event the maximum score should not occur at probability 0.5 as with symmetric scores but at the long-run event probability, reflecting maximum uncertainty. This argument may also be relevant for assessing financial risks. Scoring the assessment of the probability of a binary event can be readily extended to scoring the assessment of a full probability distribution. A simple device is to calculate the score for a distribution assessment by choosing an interval randomly and scoring the probability implied by the assessed distribution using a scoring rule for a binary event. Matheson and Winkler (1976) propose a score that integrates over the random interval with a weighting function emphasizing more important parts of the distribution being assessed. Scoring rules for probabilities can be related to information measures and utilities, see Jose, Nau, and Winkler (2008). Proper scoring rules typically do not lead to accurate assessments in the presence of risk aversion.

With $u(x)$ the utility of a payoff $x$ the expected utility associated with the scoring rule (11) is

$$Eu(r(\beta|q)) = F(q)u(g_1(\beta)) + (1 - F(q))u(g_0(\beta))$$

and the expected-utility maximizing choice of $\beta$ is not typically $F(q)$ when $u$ is nonlinear. In the quadratic case for example the first-order condition implies $F(q)/(1 - F(q)) = u'(-\beta^2)\beta/(u'(1 - \beta)^2(1 - \beta))$. Karni (2009) provides a method for eliciting probabilities that works in the presence of risk aversion. That method extends to quantile elicitation.
There is far less work on quantile scoring. Fixing $\beta$ and using (11) to elicit the quantile $q$ does not work even without risk aversion as the optimal choice of $q$ is 0 for $g_0(\beta) \geq g_1(\beta)$ and 1 for $g_0(\beta) \leq g_1(\beta)$. Gneiting and Raftery (2007) note that $w(q; \beta) = \beta s(q) + (s(\theta) - s(q))1\{\theta \leq q\} + h(\theta)$ for $s$ nondecreasing and $h$ arbitrary gives a proper scoring rule for quantiles. They note that the characterization of the full class of proper scoring rules for quantiles remains open. We give a simple proof for differentiable and increasing $s$.

**Theorem 1.** $w(q|\beta) = \beta s(q) + (s(\theta) - s(q))1\{\theta \leq q\} + h(\theta)$ with $s()$ increasing and $h()$ arbitrary is a proper scoring rule for the $\beta$th quantile $q$.

Proof. $E_q w(q|\beta) = \beta s(q) + \int_0^q (s(\theta) - s(q))dF + \int_0^1 h(\theta)dF$. The first order condition is $\beta s'(q) - F(q)s'(q) = 0$, hence the optimal $q$ satisfies $F(q) = \beta$.  

A widely used rule in econometrics (for estimation and goodness of fit assessment, not for elicitation) is $w(q|\beta) = (\theta - q)(1\{\theta \leq q\} - \beta)$. This scoring rule, with $s(q) = q$, is behind most work in quantile estimation, see Koenker and Bassett (1978) and Koenker and Machado (1999). With risk aversion, proper scoring rules for quantiles need not elicit the true quantiles.

3. A NEW METHOD FOR QUANTILE ELICITATION

The new method based on outside randomness works as follows. Suppose the elicitor wishes $q_\alpha$. The elicitor has access to a genie which generates a random variable $\xi$ from the uniform distribution on $[0, 1]$ and independently a random variable $d \in \{0, 1\}$ from the Bernoulli with probability $\alpha$. The expert supplies a value $q$ for the $\alpha$th quantile. Nature supplies one realization of $\theta$. The expert receives a reward equal to $rI(\theta \in [0, \xi])$ if $q < \xi$ and $rd$ if $q \geq \xi$. The expert’s utility of a payoff $x$ is $u(x)$.

It is the random variable $\xi$ which leads to true revelation in the risk neutral case and the Bernoulli $d$ which allows for risk aversion. To develop intuition, consider the method with $d$ replaced by its expectation $\alpha$. Then the reward from 4) upon observing $\theta$ is

$$v(q|\xi, \theta) = u(r)I(\theta \in [0, \xi])I(q < \xi) + u(r\alpha)I(q \geq \xi)$$

(normalizing $u(0) = 0$). Marginalizing wrt $\theta$ and then $\xi$ leads to

$$v(q) = u(r) \int_{-\infty}^{\xi} F(t)dt + u(r\alpha)q$$

and the FOC $u(r)(-F(q)) + u(r\alpha)$ leads to $q = q_\alpha$ when utility is linear but not otherwise.

Now suppose the rv $d$ is supplied along with $\xi$. Then

$$v(q|\xi, d) = u(r)F(\xi)I(q < \xi) + u(rd)I(q \geq \xi)$$

and marginalizing wrt $d$ gives

$$v(q|\xi) = u(r)F(\xi)I(q < \xi) + u(r)I(q \geq \xi)\alpha.$$
The FOC implies \( F(q) = \alpha \), so revelation is optimal with risk aversion. The random variable \( d \) supplied by the genie essentially moves the expectation through the utility function. In short:

**Theorem 2.** With the reward described above the optimal policy for the expert is to report the true quantile.

**Proof.** Consider the expected utility to the expert of supplying \( q \). First, suppose the genie supplies \( \xi \) to the expert. Marginalizing with respect to \( \theta \) and \( d \) gives the expected utility

\[
v(q|\xi) = u(r)(F(\xi)I(q < \xi) + F(\alpha)I(q \geq \xi))
\]

piecewise constant with a break at \( \xi \). Marginalizing with respect to the uniform random variable \( \xi \) yields the unconditional expected utility function

\[
v(q) = u(r)\left(\int_{q}^{1} F(t)dt + F(\alpha)q\right)
\]

The first-order condition is

\[
v'(q) = u(r)(-F(q) + F(\alpha)) = 0
\]

and the function is concave, so the optimal policy for the expert is to report the true quantile. \( \square \)

An alternative proof from a decision-theoretic point of view and using lotteries can be given. This proof uses preferences over lotteries but does not require the full expected utility framework. Let \((x, p) \in R \times [0, 1]\) denote the lottery that pays \$\(x\) with probability \(p\) and \$0 with probability \((1-p)\). The expert payoff is \((r, F(\xi))\) if \(\xi > q\) and \((r, \alpha)\) if \(\xi \leq q\). Consider the report \(q > q_\alpha\). If \(\xi > q\) then the expert’s payoff is \((r, F(\xi))\) whether he reports \(q\) or \(q_\alpha\). If \(\xi \leq q_\alpha\) then the expert’s payoff is \((r, \alpha)\) whether he reports \(q\) or \(q_\alpha\). However, had he reported \(q_\alpha\) instead, his reward would have been \((r, F(\xi))\). But \(F(\xi) > \alpha\) hence \((r, F(\xi))\) first-order stochastically dominates \((r, \alpha)\) so the expert cannot win and may lose as a result of reporting \(q > q_\alpha\). Similarly, reporting \(q < q_\alpha\) is dominated.

4. Conclusion

The classical elicitation problem concerns eliciting probabilities for given events. This paper studies the complementary problem of eliciting events for given probabilities. This is the problem involved in obtaining prior quantiles. Although the reward \(r\) does not affect the optimality condition, it is clear that the actual effort expended by the expert will depend on the value of the reward (precisely, on its utility). Perhaps some part of a bonus could be tied into the probability assessment. Interesting open questions include: Which quantiles and how many should be assessed? How much accuracy can be expected in a quantile assessment? Can experts be trained to improve their assessments? How can prior quantiles be assessed from a group of experts?

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