PERMUTATION-EQUIVARIANT QUANTUM K-THEORY VIII.
EXPLICIT RECONSTRUCTION

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Abstract. In Part VII, we proved that the range \( \mathcal{L}_X \) of the big \( J \)-function in permutation-equivariant genus-0 quantum K-theory is an overruled cone, and gave its adelic characterization. Here we show that the ruling spaces are \( D_q \)-modules in Novikov’s variables, and moreover, that the whole cone \( \mathcal{L}_X \) is invariant under a large group of symmetries defined in terms of \( q \)-difference operators. We employ this for the explicit reconstruction of \( \mathcal{L}_X \) from one point on it, and apply the result to toric \( X \), when such a point is given by the \( q \)-hypergeometric function.

Adelic characterization

We begin where we left in Part VII: at a description of the range \( \mathcal{L} \subset \mathcal{K} \) in the space \( \mathcal{K} \) of \( K^0(X) \otimes \Lambda \)-value rational functions of \( q \) of the \( J \)-function of permutation-equivariant quantum K-theory of a given Kähler target space \( X \):

\[
\mathcal{J} := 1 - q + t(q) + \sum_{\alpha} \phi_{\alpha} \sum_{n,d} Q^d \left( \frac{\phi_{\alpha}^L}{1 - qL}; t(L), \ldots, t(L) \right) S_n^{s_{0,1+n,d}}.
\]

We proved that \( \mathcal{L} \) is an overruled cone, i.e. it is swept by a family of certain \( \Lambda[q,q^{-1}] \)-modules, called ruling spaces:

\[
\mathcal{L} = \bigcup_{t \in \Lambda^+} (1 - q) S(q)^{-1} \mathcal{K} + t,
\]

where \( S_t \) is a certain family of “matrix” functions rational in \( q \), whose construction we are not going to remind here. Let us recall the adelic characterization of \( \mathcal{L} \), which will be our main technical tool.

It is given in terms of the overruled cone \( \mathcal{L}^{fake} \subset \hat{\mathcal{K}} \) in the space of vector-valued Laurent series in \( q - 1 \), describing the range of the

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J-function in fake quantum K-theory of $X$:

$$\mathcal{L}^{\text{fake}} = \bigcup_{t \in \Lambda} (1 - q)T_t, \quad T_t = S^{\text{fake}}_t(q)^{-1}\hat{\mathcal{K}}_+.$$  

Here $S^{\text{fake}}_t$ is some “matrix” Laurent series, $T_t$ is a tangent space to $\mathcal{L}^{\text{fake}}$, containing $(1 - q)T_t$, and tangent to $\mathcal{L}^{\text{fake}}$ at all points of the ruling space $(1 - q)T_t$.

According to the last section of Part VII, a rational function $f \in \mathcal{K}$ lies in $\mathcal{L}$ if and only if its Laurent series expansions $f_\zeta$ near $q = 1/\zeta$ satisfy the following three conditions:

(i) $f_\zeta(1) \in \mathcal{L}^{\text{fake}}$;

(ii) when $\zeta \neq 0, 1, \infty$ is a primitive $m$th root of unity,

$$f_\zeta(q^{1/m}/\zeta) \in \mathcal{L}_t^{(c)}$$

a certain subspace in $\hat{\mathcal{K}}$, determined by the tangent space $T_t$ to $\mathcal{L}^{\text{fake}}$ at the point $f_\zeta(1)$;

(iii) when $\zeta \neq 0, \infty$ is not a root of unity, $f_\zeta$ is a power series in $q - 1/\zeta$, i.e. $f$ has no pole at $q = 1/\zeta$.

The subspace $\mathcal{L}_t^{(c)}$ is described as $\nabla_\zeta\Psi^m(T_t) \otimes \Psi^m(\Lambda) \Lambda$, where the Adams operation $\Psi^m$ acts by $\Psi^m(q) = q^m$ and naturally on the $\lambda$-algebra $K^0(X) \otimes \Lambda$, and $\nabla_\zeta$ is the operator of multiplication by

$$e^{\sum_{k>0} \left( \frac{\Psi^k(T^*_X)}{k(1-\zeta^{-k}q^{k/m})} - \frac{\Psi^m(T^*_X)}{k(1-q^{km})} \right)}.$$  

In its turn, the cone $\mathcal{L}^{\text{fake}} \subset \hat{\mathcal{K}}$ (and hence its tangent spaces $T_t$) can be expressed in terms of the cone $\mathcal{L}^H \subset \mathcal{H}$, describing the range of cohomological J-function in the space $\mathcal{H}$ of Laurent series in one indeterminate $z$ with coefficients in $H^{\text{even}}(X) \otimes \Lambda$. Namely, according to the Hirzebruch–Riemann–Roch theorem [2] in fake quantum K-theory,

$$\text{qch}(\mathcal{L}^{\text{fake}}) = \triangle \mathcal{L}^H,$$

where the quantum Chern character $\text{qch} : \hat{\mathcal{K}} \to \mathcal{H}$ acts by $\text{qch} q = e^z$, and by the natural Chern character $\text{ch} : K^0(X) \otimes \Lambda \to H^{\text{even}}(X) \otimes \Lambda$ on the vector coefficients, while $\triangle$ acts as the multiplication in the classical cohomology of $X$ by the Euler–Maclaurin asymptotics (see [3, 2, 6]) of the infinite product:

$$\triangle \sim \prod_{r=1}^{\infty} \text{td}(T_X \otimes q^{-r}).$$

Using all these descriptions, we are going on explore how the string and divisor equations of quantum cohomology theory manifest in the genus-0 permutation-equivariant quantum K-theory.
**Divisor equations and \( D_\text{q} \)-modules**

Let \( p_1, \ldots, p_K \) be a basis in \( H^2(X, \mathbb{R}) \) consisting if integer numerically effective classes, and let \( Q^d = Q_1^{d_1} \cdots Q_K^{d_K} \), where \( d_i = p_i(d) \), represent degree-\( d \) holomorphic curves in the Novikov ring. We remind that the Novikov variables are included into the ground \( \lambda \)-algebra \( \Lambda \).

The loop space \( \mathcal{H} \) of Laurent series in \( z \) with vector coefficients in \( H^{\text{even}}(X) \otimes \Lambda \) is equipped with the structure of a module over the algebra \( \mathcal{D} \) of differential operators in the Novikov variables, so that \( Q_i \) acts as multiplication by \( Q_i \), and \( Q_i \partial_{Q_i} \) acts as \( zQ_i \partial_{Q_i} - p_i \). The divisor equations in quantum cohomology theory imply (see e.g. [4]), that

linear vector fields \( f \mapsto (Q_i \partial_{Q_i} - p_i/z)f \) in \( \mathcal{H} \) are tangent to \( \mathcal{L}^H \subset \mathcal{H} \).

In follows that the ruling spaces (as well as tangent spaces) of \( \mathcal{L}^H \) are \( \mathcal{D} \)-modules, i.e. are invariant with respect to each differential operator \( D(Q_i zQ_i \partial_{Q_i} - p, z) \), and moreover the flow \( \epsilon \mapsto e^{\epsilon D/z} \) of the vector field \( f \mapsto Df/z \) preserves \( \mathcal{L}^H \).

Indeed, for \( f \in \mathcal{L}^H \), the vector \( (Q_i \partial_{Q_i} - p_i)f \) lies in \( T_f \mathcal{L}^H \), and hence \( (zQ_i \partial_{Q_i} - p_i)f \) lies in the same ruling space \( zT_f \mathcal{L}^H \) as \( f \) does. Therefore so does \( Df \), and hence \( Df/z \in T_f \mathcal{L}^H \), i.e. the vector field \( f \mapsto Df/z \) is tangent to \( \mathcal{L}^H \).

Note that the operator \( \triangle \) relating \( \mathcal{L}^H \) and \( \mathcal{L}^\text{fake} \) involves multiplication in the commutative classical cohomology algebra \( H^{\text{even}}(X) \), but does not involve Novikov’s variables. Consequently, the tangent and ruling spaces of \( q\text{ch}(\mathcal{L}^\text{fake}) \) are \( \mathcal{D} \)-modules too, and moreover, the flows \( \epsilon \mapsto e^{\epsilon D/z} \) preserve \( q\text{ch}(\mathcal{L}^\text{fake}) \).

We equip the space \( \mathcal{K} \) of vector-valued rational functions of \( q \) with the structure of a module over the algebra \( \mathcal{D}_q \) of finite difference operators. It is generated (over the algebra of Laurent polynomials in \( q \)) by multiplication operators, acting as multiplications by \( Q_i \), and translation operators, acting as \( P_i q_i Q_i \partial_i \), where \( P_i \) is the multiplication in \( K^0(X) \) by the line bundle with the Chern character \( \text{ch} P_i = e^{-p_i} \).

**Proposition** (cf. [6, 4]). The ruling spaces of the overruled cone \( \mathcal{L} \subset \mathcal{K} \) of permutation-equivariant quantum \( K \)-theory is are \( \mathcal{D}_q \)-modules.

**Proof.** If \( f \in \mathcal{L} \), it passes the tests (i),(ii),(iii) of adelic characterization. We need to show that \( g := P_i q_i Q_i \partial_i f \), which obviously lies in \( \mathcal{K} \), also passes the tests (and with the same \( t \in K^0(X) \otimes \Lambda_+ \)). This is obvious for test (iii), and is true about test (i) because of the above \( \mathcal{D} \)-module (and hence \( \mathcal{D}_q \)-module) property of the ruling spaces \( (1 - q)T_i \) of \( \mathcal{L}^\text{fake} \). To verify test (ii), we write:

\[
g(q^{1/m}/\zeta) = P_i (q^{1/m}) Q_i \partial_i \zeta^{-Q_i \partial_i} f(q^{1/m}/\zeta).
\]
First, note that the operator $\nabla_\zeta$ relating $L^\zeta$ with $\Psi^m(T_t)$ does not involve Novikov’s variables and commutes with $D_q$. Next, let us elucidate the notation $\Psi^m(T_t) \otimes \Psi^m(\Lambda) \Lambda$. In fact the space so indicated consists of linear combinations $\sum a_\lambda \Psi^m(Q, q, \lambda)$, where $f_a \in T_t$, and $\lambda \in \Lambda[[q^{-1}]]$. We have the following commutation relations:

$$
P_i \Psi^m = \Psi^m p_i^{m \lambda},$$

$$(q^{1/m})Q_i \partial_{Q_i} \Psi^m = q^{m \lambda} Q_i \partial_{Q_i} \Psi^m = \Psi^m (q^{1/m})Q_i \partial_{Q_i},$$

$$\zeta Q_i \partial_{Q_i} \Psi^m = \zeta^{m \lambda} Q_i \partial_{Q_i} \Psi^m = \Psi^m.$$

Therefore

$$
P_i (q^{1/m})Q_i \partial_{Q_i} \zeta Q_i \partial_{Q_i} \left( \sum a_\lambda \Psi^m(f_a) \right) =$$

$$
\sum a_\lambda (q^{1/m} / \zeta^{Q_i \partial_{Q_i}}} \Psi^m(p_i^{m \lambda} Q_i \partial_{Q_i}, f_a),
$$

which lies in $\Psi^m(T_t) \otimes \Psi^m(\Lambda) \Lambda$ since $T_t$ is invariant under the operator $p_i^{m \lambda} / (q^{1/m})Q_i \partial_{Q_i} = e^{(zQ_i \partial_{Q_i} - p_i)/m}$.

Let $D(Pq^{Q\partial Q}, q)$ be a constant coefficient finite difference operator, by which we mean a Laurent polynomial expression in translation operators $P_i Q_i \partial_{Q_i}$, and maybe $q$, with coefficients from $\Lambda$ independent of $Q$. We assume below that $\epsilon \in \Lambda$ to assure $\epsilon$-adic convergence of infinite sums.

**Theorem 1.** The operator

$$
e \sum_{k>0} \Psi^k(\epsilon D(Pq^{kQ\partial Q}, q))/k(1-q^k)
$$

preserves $L \subset K$.

**Proof.** We show that if $(1-q)f$ passes tests (i), (ii), (iii) of the adelic characterization of $L$, then $(1-q)g$, where

$$g := e \sum_{k>0} \Psi^k(\epsilon D(Pq^{kQ\partial Q}, q))/k(1-q^k) f,$$

also does.

(i) Suppose $(1-q)f(1)$ lies in the ruling space $(1-q)T_t \subset L^{fake}$. Note that the exponent $\sum_{k>0} \Psi^k(\epsilon D(Pq^{kQ\partial Q}, q))/k(1-q^k)$ has first order pole at $q = 1$. According to the discussion above the flow defined by such an operator on $\hat{K}$ preserves $L^{fake}$, and therefore maps its tangent spaces to tangent spaces, and ruling spaces to ruling spaces, and moreover, the operators regular at $q = 1$ preserve each ruling and tangent space. It follows that $(1-q)g(1) \in (1-q)T_t' \subset L^{fake}$, where

$$T_t' := e \sum_{k>0} \Psi^k(\epsilon D(Pq^{kQ\partial Q}, 1))/k^2(1-q)T_t.$$

(ii) We have
\[ \Psi^m(T'_t) = e \sum_{k>0} \Psi^{mk}(\epsilon D(P q^{kQ\partial_Q}, 1))/k^2(1 - q^m) \Psi^m(T_t). \]

On the other hand, for a primitive \( m \)th root of unity \( \zeta \),
\[ g(\zeta)(q^{1/m}/\zeta) = e \sum_{k>0} \Psi^k(\epsilon D(Q, P q^{kQ\partial_Q}, 1))/k(1 - q^k) \Psi^{1/m}(\zeta), \]
where \( A \) is some operator regular at \( q = 1 \). It comes out of refactoring \( e^A + B/(1 - q) \), where \( A \) and \( B \) are regular at \( q = 1 \), as \( e^A e^B/(1 - q) \). We use here the fact that the operators \( A \) and \( B \) have constant coefficients, and hence commute.

Note that the exponents \( \sum_{k>0} \Psi^{mk}(\epsilon D(Q, P q^{kQ\partial_Q}, 1))/k^2(1 - q^m) \) and \( \sum_{l>0} \Psi^{ml}(\epsilon D(Q, P q^{lQ\partial_Q}, 1))/ml(1 - q^l) \) agree modulo terms regular at \( q = 1 \) (which, again, commute with the singular terms). Since we are given that \( f(\zeta)(q^{1/m}/\zeta) \in \nabla_{\zeta} \Psi^m(T_t) \otimes \Psi^m(\Lambda) \), and since \( \nabla_{\zeta} \) commutes with \( D_q \), we conclude (using the refactoring again), that
\[ g(\zeta)(q^{1/m}/\zeta) \in \nabla_{\zeta} \Psi^m(T'_t) \otimes \Psi^m(\Lambda). \]

Note that the exponent in \( e^A \) involves translations \( P_i q_i Q_i \partial_Q \) as well as \( \zeta^{-Q_i\partial_Q} \), and so it is important, that (as we’ve checked in the proof of above Proposition), such operators preserve the space \( \Psi^m(T'_t) \otimes \Psi^m(\Lambda) \).

(iii) If \( f \) is regular at \( q = 1/\zeta \), where \( \zeta \neq 0, \infty \) is not a root of unity, \( g \) is obviously regular there too. \( \square \)

**Corollary (the \( q \)-string equation).** The range \( L \subset K \) of permutation-equivariant \( J \)-function is invariant under the multiplication operators:
\[ f \mapsto e \sum_{k>0} \Psi^k(\epsilon)/k(1 - q^k) f, \quad \epsilon \in \Lambda_+. \]

**Proof:** Use Theorem 1 with \( D = 1 \).

**Examples**

**Example 1:** \( d = 0 \). In degree 0, i.e., modulo Novikov’s variables, the cone \( L \subset K \) coincides with the cone \( L_{pt} \) over the \( \lambda \)-algebra \( K^0(X) \otimes \Lambda \). Theorem 1 and Proposition allow one to recover the part of \( L_{pt} \) over the \( \lambda \)-algebra \( \Lambda' = K^0(X)_{pt} \otimes \Lambda \), where by \( K^0(X)_{pt} \) (the primitive part) we denote the part of the ring \( K^0(X) \) generated by line bundles.
Let monomials \( P^n := P_1^{a_1} \cdots P_K^{a_K} \) run a basis of \( K^0(X)_{pr} \). Applying the above theorem to the finite difference operator

\[
D = \sum a \epsilon_a P^a q^a \partial Q := \sum a \epsilon_a \prod_{i=1}^{K} P_i^{a_i} q_i^{a_i} \partial Q_i, \quad \epsilon_a \in \Lambda_+,
\]

and acting on the point \( J \equiv 1 - q \) modulo Novikov’s variables, we recover over \( \Lambda' \) the small J-function of the point:

\[
(1 - q) e \sum a \sum_{k > 0} \Psi^k(\epsilon_a) P^k a / k(1 - q^k) \equiv 1 - q + \sum a \epsilon_a P^a \mod \mathcal{K}_-.
\]

Furthermore, applying linear combinations

\[
\sum a c_a(q) P^a q^a \partial Q
\]

with coefficients \( c_a \in \Lambda[q, q^{-1}] \) which are arbitrary Laurent polynomials in \( q \), we get, according to Proposition, points in the same ruling space of the cone \( L \). Modulo Novikov’s variables this effectively results in multiplying by arbitrary elements \( \sum a c_a(q) P^a \) from \( \Lambda'[q, q^{-1}] \), and therefore yields the entire cone \( L_{pt} \) over \( \Lambda' \).

**Example 2:** \( X = \mathbb{C}P^1 \). We know\(^1\) one point on \( L = L_{\mathbb{C}P^1} \), the small J-function:

\[
J(0) = (1 - q) \sum_{d \geq 0} \frac{Q^d}{(1 - Pq)^2(1 - Pq^2)^2 \cdots (1 - Pq^d)^2}.
\]

Here \( P = \mathcal{O}(-1) \) is the generator of \( K^0(\mathbb{C}P^1) \). It satisfies the relation \((1 - P)^2 = 0 \). The K-theoretic Poincaré pairing is determined by

\[
\chi(\mathbb{C}P^1, \phi(P)) = \text{Res}_{P=1} \frac{\phi(P)}{(1 - P)^2} \frac{dP}{P}.
\]

We use Theorem 1 with the operator \( D = \lambda + \epsilon Pq Q \partial Q \), \( \lambda, \epsilon \in \Lambda_+ \), and obtain a 2-parametric family of points on \( L_{\mathbb{C}P^1} \):

\[
(1 - q) e \sum_{k > 0} \Psi^k(\lambda) + \Psi^k(\epsilon) P^k q^k \partial Q / k(1 - q^k) J(0) =
\]

\[
(1 - q) e \sum_{k > 0} \Psi^k(\lambda) / k(1 - q^k) \sum_{d \geq 0} \frac{Q^d e \sum_{k > 0} \Psi^k(\epsilon) P^k q^kd / k(1 - q^k)}{(1 - Pq)^2(1 - Pq^2)^2 \cdots (1 - Pq^d)^2}.
\]

Examine now two specializations.

\(^1\)From various sources: Part IV (by localization), or [6] (by adelic characterization), or [5] (by toric compactifications).
Thus, the correlator sum indeed coincides with the degree-1 part of \( F_0 \). Modulo \( Q^2 \), we are left with

\[
\mathcal{J} \equiv e^{\sum_{k>0} \Psi^k(\lambda)/k(1-q^k)} \left( 1 - q + \frac{(1-q)Q}{(1-Pq)^2} e^{\sum_{k>0} \Psi^k(\epsilon) P^k q^k/k(1-q^k)} \right).
\]

Modulo \( K_- \) (and \( Q^2 \)), we have: \([\mathcal{J}]_+ \equiv 1 - q + \lambda + \epsilon P\). According to Part VII, Corollary 3,

\[
\mathcal{F}_0(t) = -\frac{1}{2} \Omega([\mathcal{J}]_+, \mathcal{J}(t)) - \frac{1}{2} (\Psi^2(t(1)), 1).
\]

For degree \( d = 1 \) part \( \mathcal{J}_1 \) of \( \mathcal{J} \), we have

\[
-\Omega([\mathcal{J}]_+, \mathcal{J}_1) = \text{Res}_{q=0, \infty} \left( 1 - \frac{1}{q} + \lambda + \epsilon P, \frac{(1-q)(1-Pq)^2 e^{A(q)}}{(1-Pq)^2} \right) \frac{dq}{q},
\]

where \( A(q) = \sum_{k>0} (\Psi^k(\lambda) + \Psi^k(\epsilon) P^k q^k)/k(1-q^k) \). The 1-form has no pole at \( q = \infty \). Since \((1-q)/(1-Pq)^2\) \( t = 2P - 1 \), and \( A'(0) = \lambda + \epsilon P \), the residue at \( q = 0 \) is calculated as

\[
\left( 1 + \lambda + \epsilon P, e^{A(0)} \right) - \left( 1, (2P - 1) e^{A(0)} + (\lambda + \epsilon P) e^{A(0)} \right) = \text{Res}_{P=1} \frac{2(1-P) e^{A(0)} dP}{(1-P)^2} = 2 e^{A(0)} = 2 e^{\sum_{k>0} \Psi^k(\lambda)/k}.
\]

Let us check this rather trivial result “by hands”. The degree \( d = 1 \) part of \( \mathcal{F}_0(t) \) at \( t = \lambda + \epsilon P \) is defined as \( \sum_{n \geq 0} (\lambda + \epsilon P, \ldots, \lambda + \epsilon P)_{n,0,1} S_n \). Since there is only one rational curve of degree 1 in \( \mathbb{CP}^1 \), the moduli space \( X_{0,1} = \overline{\mathcal{M}}_{0,0,n}(\mathbb{CP}^1, 1) \) is obtained from \( (\mathbb{CP}^1)^n \) by some blow-ups along the diagonals. The evaluation maps \( \text{ev}_i : X_{0,0,1} \to \mathbb{CP}^1 \) factor through \( (\mathbb{CP}^1)^n \) as the projections \( (\mathbb{CP}^1)^n \to \mathbb{CP}^1 \). Therefore the correlator sum can be evaluated as

\[
\sum_{n \geq 0} \left( H^\ast \left( \mathbb{CP}^1; \lambda + \epsilon P \right)^\otimes_n \right) S_n = \sum_{n \geq 0} (\lambda^\otimes_n) S_n
\]

because for \( P = \mathcal{O}(-1) \) we have \( H^\ast(\mathbb{CP}^1; P) = 0 \). Let us remind from Part I that for elements of a \( \lambda \)-algebra,

\[
(\lambda^\otimes_n) S_n := \frac{1}{n!} \sum_{h \in S_n} \prod_{k>0} \Psi^k(h)(\lambda),
\]

where \( l_k(h) \) is the number of cycles of length \( k \) in the permutation \( h \). Thus, the correlator sum indeed coincides with \( e^{\sum_{k>0} \Psi^k(\lambda)/k} \).
Secondly, let us return to our 2-parametric family of points on $\mathcal{L}_{\mathbb{C}P^1}$, and specialize it to the symmetrized theory, where only the $S_n$-invariant part of sheaf cohomology is taken into account. For this, we specialize the $\lambda$-algebra to $\Lambda = \mathbb{Q}[\lambda, \epsilon, Q]$ with $\Psi^k(\lambda) = \lambda^k, \Psi^k(\epsilon) = \epsilon^k$ (and $\Psi^k(Q) = Q^k$ as before). Some simplifications ensue. Since

$$q^{kd} = 1 - (1 - q^k)(1 + q^k + \cdots + q^{k(d-1)}),$$

we have

$$e^{\sum_{k>0} \epsilon^k P^k q^{kd}/k(1 - q^k)} = e^{\sum_{k>0} \epsilon^k P^k/k(1 - q^k) \prod_{r=0}^{d-1} \left(1 - \epsilon P q^r\right)}. $$

Thus, we obtain the following 2-parametric family of points on $\mathcal{L}_{\mathbb{C}P^1}^{sym}$:

$$\mathcal{J}_{\mathbb{C}P^1}^{sym} = (1 - q) e^{\sum_{k>0} (\lambda^k + \epsilon^k P^k)/k(1 - q^k) \sum_{d \geq 0} Q^d \prod_{r=0}^{d-1} (1 - \epsilon P q^r) / \prod_{r=1}^d (1 - P q^r)^2}.$$ 

Note that the projection of this series to $K_+$ along $K_-$ picks contributions only from the terms with $d = 0$ and $k = 1$:

$$[\mathcal{J}_{\mathbb{C}P^1}^{sym}]_+ = 1 - q + \lambda + \epsilon P.$$

Therefore the series represents the small $J$-function of the symmetrized quantum $K$-theory of $\mathbb{C}P^1$. The exponential factor is actually equal to

$$\exp_q(\lambda/(1 - q)) \exp_q(\epsilon P/(1 - q)).$$

Thus, we obtain:

$$\mathcal{J}_{\mathbb{C}P^1}^{sym}(\lambda + \epsilon P) = \text{mod } (1 - P)^2

(1 - q) \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{d=0}^{\infty} \frac{\lambda^m \epsilon^l P^l Q^d \prod_{r=0}^{d-1} (1 - \epsilon P q^r) \prod_{s=1}^d (1 - q^s) \prod_{r=0}^d (1 - P q^r)}{\prod_{i=1}^m (1 - q^i) \prod_{j=1}^l (1 - q^j)}.$$ 

Reconstruction theorems

As in Example 1, assume that $p_1, \ldots, p_K$ is a numerically effective integer basis in $H^2(X, \mathbb{Q})$, that Novikov’s monomials $Q^d = Q_1^d \cdots Q_K^d$ represent degree $d$ holomorphic curves in $X$ in coordinates $d_i = p_i(d)$ on $H^2(X)$, that $P_i$ are line bundles with $\text{ch} P_i = e^{-P_i}$, and that monomials $P^a = P_1^{a_1} \cdots P_K^{a_K}$ run a basis in $K^0(X)_{pr}$, the primitive part of the $K$-ring. We also write $a.d$ for the value $\sum a_i d_i$ of $-c_1(P^a)$ on $d$. 

Theorem 2 (explicit reconstruction). Let $I = \sum_d I_d Q^d$ be a point in the range $L \subset K$ of the $J$-function of permutation-equivariant quantum $K$-theory on $X$, written as a vector-valued series in Novikov’s variables. Then the following family also lies in $L$:

$$\sum_d I_d Q^d e^{\sum_{k>0} \sum_a \Psi^k(\epsilon_a) P^{ka} q^{k(d)}/k(1-q^k), \quad \epsilon_a \in \Lambda_+}.$$

Moreover, for arbitrary Laurent polynomials $c_a \in \Lambda[q, q^{-1}]$, the following series also lies in $L$:

$$\sum_d I_d Q^d e^{\sum_{k>0} \sum_a \Psi^k(\epsilon_a) P^{ka} q^{k(d)}/k(1-q^k) \sum_a c_a(q) P^a q^{a(d)}}.$$

Furthermore, when $K^0(X) = K^0(X)_{pr}$, the whole cone $L \subset K$ is parameterized this way.

Proof. We first work over $\Lambda'$ freely generated as $\lambda$-algebra by the “time” variables $\epsilon_a$, and use Theorem 1 with the $Q$-independent finite difference operator $D = \sum_a \epsilon_a P^a q^{aQ\partial_Q}$. We conclude that the family

$$e^{\sum_{k>0} \sum_a \Psi^k(\epsilon_a) P^{ka} q^{kaQ\partial_Q}/k(1-q^k) I} = \sum_d I_d Q^d e^{\sum_{k>0} \sum_a \Psi^k(\epsilon_a) P^{ka} q^{k(d)}/k(1-q^k)}$$

lies in the cone $L$, defined over $\Lambda'$. To obtain the second statement, we apply Proposition, using finite difference operators $\sum_a c_a(q) P^a q^{aQ\partial_Q}$. Afterwards we specialize the “times” $\epsilon_a$ to any values $\epsilon_a \in \Lambda$ (which at this point may become dependent on $Q$). Finally, when $K^0(X) = K^0(X)_{pr}$, we use the formal Implicit Function Theorem to conclude that the whole cone $L$ is parameterized, because this is true modulo Novikov’s variables, as Example 1 shows.

Example: $X = \mathbb{CP}^N$. According to Theorem 2, the entire cone $L$ is parameterized as follows:

$$J = (1-q) \sum_{d \geq 0} Q^d \frac{e^{\sum_{k>0} \sum_{a=0}^N \Psi^k(\epsilon_a) P^{ka} q^{kad}/k(1-q^k) \sum_{a=0}^N c_a(q) P^a q^{a(d)}}}{(1-Pq)^{N+1}(1-Pq^2)^N \cdots (1-Pq^d)^N+1}.$$

Of course, this is obtained by applying Theorem 2 to the small $J$-function $J(0)$ from [5] (also [6], or Parts II–IV in the non-equivariant limit). Here $\epsilon_a \in \Lambda$, $c_a(q)$ are arbitrary Laurent polynomials in $q$ with coefficients in $\Lambda$, and $P^a$, $a = 0, \ldots, N$, $P = O(-1)$, are used for a basis in $K^0(X)$. Perhaps, the basis $(1-P)^a$, $a = 0, \ldots, N$, is more useful (cf. [4]), and we get yet another parameterization of $L$:

$$(1-q) \sum_{d \geq 0} Q^d \frac{e^{\sum_{k>0} \sum_{a=0}^N \Psi^k(\epsilon_a) (1-P^k q^{d})^a/k(1-q^k) \sum_{a=0}^N c_a(q) (1-Pq^d)^a}}{(1-Pq)^{N+1}(1-Pq^2)^N \cdots (1-Pq^d)^N+1}.$$
We return now to the context of Part VII, where we studied the mixed J-function $J(x, t)$, involving two types of inputs: permutable $t$ and non-permutable $x$, both taken from $K_+$. The cone $L \subset K$ represents the range of $t \mapsto J(0, t)$. Recall that according to the general theory, it is the union of ruling spaces $(1 - q)T_t$, where $t = T(t)$ is given by a certain non-linear map

$$T : K^0(X) \otimes \Lambda_+ \oplus (1 - q)K_+ \to K^0(X) \otimes \Lambda_+.$$ 

At the same time, for a fixed value of $t$, the range of the ordinary J-function $x \mapsto J(x, t)$ is an overruled Lagrangian cone $L_t \subset K$, which shares with $L$ one ruling space, $T_t$, corresponding to $t = T(t)$. Each tangent space of each cone $L_t$ is tangent to $L_t$ along one of the ruling spaces (e.g. $T_t$ is tangent along $(1 - q)T_t$), and is related with this ruling space by the multiplication by $1 - q$. As a consequence, not only each ruling (and tangent) space of each $L_t$ is a $D_q$-module (which is proved on the basis of adelic characterization as in Proposition above), but also each cone $L_t$ is invariant under the flow

$$f \mapsto e^{\epsilon D(Q, P_q Q^0 Q, q)/(1-q)} f,$$

where $D \in D_q$. We use this to reconstruct the family $L_t$.

**Theorem 3.** Let $I = \sum I_d Q^d$ (as in Theorem 2). Then

$$I(\epsilon) = \sum_d I_d(\epsilon) Q^d := \sum_d I_d(\epsilon) Q^d e^{\sum_{k>0} \sum_a \Psi(k(\epsilon_a) P_{k a} q^{k(a,d)}/(1-q^k)}, \quad \epsilon_a \in \Lambda_+$$

represent a family of points on the cones $L_{t(\epsilon)}$ (one point on each cone), and the following family of points, parameterized by $\tau_a \in \Lambda$ and by $c_a \in \Lambda[q, q^{-1}]$, lies on $L_{t(\epsilon)}$:

$$\sum_d I_d(\epsilon) Q^d e^{\sum_{a} \tau_a P_{a} q^{a,d} / (1-q)} \sum_a c_a(q) P_{a} q^{a,d}.$$ 

Moreover, if $K^0(X) = K^0(X)_{pr}$, for each $t \in K^0(X) \otimes \Lambda_+$ the whole cone $L_t$ is thus parameterized.

**Proof.** It is clear from computation modulo Novikov’s variables that the family $I(\epsilon)$ has no tangency with the ruling spaces, hence represents at most one point from each $L_t$ (and does represent one, when $K^0(X) = K^0(X)_{pr}$). Given one point, $I(\epsilon)$, on $L_{t(\epsilon)}$, we generate more points by machinery discussed above: applying the commuting flows

$$e^{\sum_{a} \tau_a P_{a} q^{\partial Q, q} / (1-q)} I(\epsilon) = \sum_d Q^d I_d(\epsilon) e^{\sum_{a} \tau_a P_{a} q^{a,d} / (1-q)}.$$
followed by the application of the operators \( \sum_a c_a(q) P^a q^{aQ \partial Q} \), where \( \tau_a \) and the coefficients of \( c_a \) are independent variables. Afterwards they can be specialized to some values in \( \Lambda \) (in particular, depending on \( Q \)). In the case when \( K^0(X) = K^0(X)_{pr} \), it follows from the Implicit Function Theorem and Example 1 about the limit to \( d = 0 \), that the entire cone \( \mathcal{L}_t \) for each \( t \) is thus obtained.

**Example:** \( X = \mathbb{C}P^N \). It follows that for fixed values of \( \epsilon_a \), the corresponding cone \( \mathcal{L}_{t(\epsilon)} \) is parameterized as

\[
(1 - q) \sum_{d \geq 0} Q^d \frac{\sum_{a=0}^N \left( \tau_a P^{aq \partial a} + \sum_{k>0} \Psi^k(\epsilon_a) \frac{P^{ka \partial a}}{k(1-q^k)} \right) \sum_{a=0}^N c_a(q) P^a q^{ad}}{(1 - Pq)^{N+1}(1 - Pq^2)^{N+1} \cdots (1 - Pq^d)^{N+1}},
\]

and all \( \mathcal{L}_t \) are so obtained.

**Remarks.** Reconstruction theorems in quantum cohomology and K-theory go back to Kontsevich–Manin [9] and Lee–Pandharipande [10] respectively. Theorem 3 is a slight generalization (from the case \( t = 0 \)) of the “explicit reconstruction” result [4] in the ordinary (non-permutation-equivariant) quantum K-theory, which in its turn mimics the results of quantum cohomology theory already found in [1, 7], and shares the methods based on finite difference operators with the K-theoretic results of [8].

Theorems of this section show that when \( K^0(X) \) is generated by line bundles, the entire range \( \mathcal{L} \) of the J-function in the permutation-equivariant genus-0 quantum K-theory of \( X \), as well as the entire family \( \mathcal{L}_t \) of the overruled Lagrangian cones representing the “ordinary” J-functions, depending on the permutable parameter, \( t \), can be explicitly represented in a parametric form, \textit{given one point on any of these cones}. In essence, all genus-0 K-theoretic GW-invariants of \( X \), permutation-equivariant, ordinary, or mixed, are thereby reconstructed from any one point: a \( K^0(X) \)-valued series \( \sum_d I_d Q^d \) in Novikov’s variables.

In the case of a toric \( X \), the results of Part V exhibit such a point in the form of the \textit{q-hypergeometric series} mirror-symmetric to \( X \). Needless to say, the same applies to toric bundles spaces, or super-bundles (a.k.a. toric complete intersections), as well as to the torus-equivariant versions of K-theoretic GW-invariants. Thus “all” (torus-equivariant or not; permutation-equivariant, ordinary, or mixed) K-theoretic genus-0 GW-invariants of toric manifolds, toric bundles, or toric complete intersections are computed in a geometrically explicit form, illustrated by the above example.
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