Boundedness character of a max-type system of difference equations of second order

Stevo Stević1,2, Mohammed A. Alghamdi2, Abdullah Alotaibi2 and Naseer Shahzad2

1 Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia
2 King Abdulaziz University, Department of Mathematics, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Received 13 April 2014, appeared 20 September 2014
Communicated by Josef Diblík

Abstract. The boundedness character of positive solutions of the next max-type system of difference equations

\[ x_{n+1} = \max \left\{ A, \frac{y^p_n}{x^q_{n-1}} \right\}, \quad y_{n+1} = \max \left\{ A, \frac{x^p_n}{y^q_{n-1}} \right\}, \quad n \in \mathbb{N}_0, \]

with \( \min \{ A, p, q \} > 0 \), is characterized.

Keywords: max-type system of difference equations, positive solutions, bounded solutions, unbounded solutions.

2010 Mathematics Subject Classification: 39A10, 39A20.

1 Introduction

Difference equations and systems which do not stem from the differential ones have attracted some attention in last few decades (see, e.g., [1–47]). Some of the systems that are of interest are symmetric or those obtained from symmetric by modifications of their parameters (see, for example, [5, 9, 13–19, 22, 23, 36, 39–44] and the related references therein). Another subarea, of interest deals with max-type difference equations and systems (see, for example, [1, 7, 10–12, 17, 19, 21, 28–35, 38, 40, 42, 43, 45–47] and the related references therein). However, there are only a few papers which belong to both areas (see [17, 19, 21, 40, 42, 43]). Although majority of the papers in the area treat equations or systems with integer powers of their variables, there are some papers on equations or systems with non-integer powers of their variables (see, for example, [3, 4, 8, 20, 27–33, 35, 47]). Paper [29] is one of the first such papers on max-type difference equations. It studies positive solutions of the difference equation

\[ x_{n+1} = \max \left\{ a, \frac{x^p_n}{x^q_{n-1}} \right\}, \quad n \in \mathbb{N}_0, \]  

(1.1)

Corresponding author. Email: sstevic@ptt.rs
with \( \min\{a, p\} > 0 \).

Motivated by [29], in [43], S. Stević studied the boundedness character and global attractivity of positive solutions of the following symmetric system of max-type difference equations

\[
x_{n+1} = \max\left\{a, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{a, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0,
\]

(1.2)

with \( \min\{a, p\} > 0 \).

For related max-type difference equations see also [28, 30, 33, 35].

Here we continue the line of investigations by studying the boundedness character of positive solutions of the next system of max-type difference equations

\[
x_{n+1} = \max\left\{A, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{A, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0,
\]

(1.3)

where \( \min\{A, p, q\} > 0 \).

Two of our results (Theorem 2.2 and Theorem 2.4) are natural extensions of the results on the boundedness character of positive solutions of system (1.2) appearing in [43]. For the other two results (Theorem 2.1 and Theorem 2.3) we need some other methods, different from the ones used in studying system (1.2). Generally speaking, the paper is also a continuation of studying special cases of the next systems of difference equations

\[
x_{n+1} = \max\left\{A_n, \frac{y_n^p}{x_{n-1}^p}\right\}, \quad y_{n+1} = \max\left\{A_n, \frac{x_n^p}{y_{n-1}^p}\right\}, \quad n \in \mathbb{N}_0,
\]

\[
x_{n+1} = A_n + \frac{y_n^p}{x_{n-1}^p}, \quad y_{n+1} = A_n + \frac{x_n^p}{y_{n-1}^p}, \quad n \in \mathbb{N}_0,
\]

where \( k, l \in \mathbb{N}, \min\{p, q\} > 0 \) and \( (A_n)_{n \in \mathbb{N}_0} \) is a sequence of positive numbers, as well as special cases of their scalar counterparts

\[
x_{n+1} = \max\left\{A_n, \frac{x_n^p}{x_{n-1}^p}\right\}
\]

\[
x_{n+1} = A_n + \frac{x_n^p}{x_{n-1}^p}, \quad n \in \mathbb{N}_0,
\]

where \( k, l \in \mathbb{N}, \min\{p, q\} > 0 \) and \( (A_n)_{n \in \mathbb{N}_0} \) is a sequence of positive numbers.

For some results in the area see, for example, [2, 4, 6, 8, 14, 20, 24, 25, 28–30, 33] and the related references therein.

Solution \((x_n, y_n)_{n \geq -1}\) of system (1.3) is bounded if there is an \( M \geq 0 \) such that

\[
\|(x_n, y_n)\|_2 = \sqrt{x_n^2 + y_n^2} \leq M, \quad n \geq -1.
\]

(1.4)

If

\[
\sup_{n \geq -1} \sqrt{x_n^2 + y_n^2} = +\infty
\]

we say that the solution is unbounded.
2 Boundedness character of positive solutions of system (1.3)

In this section we prove the main results of this paper, which give a complete picture for the boundedness character of positive solutions of system (1.3).

**Theorem 2.1.** Assume that $A > 0$, $2\sqrt{q} \leq p < 1 + q$ and $q \in (0, 1)$. Then all positive solutions of system (1.3) are bounded.

**Proof.** First note that from (1.3) we have
\[
\min\{x_n, y_n\} \geq A, \quad n \in \mathbb{N}. \tag{2.1}
\]

It is not difficult to see that the conditions $2\sqrt{q} \leq p < 1 + q$ and $q \in (0, 1)$ imply that the polynomial $P(t) = t^2 - pt + q$ has zeroes $t_1$ and $t_2$ such that $0 < t_2 < t_1 < 1$.

We have
\[
x_{n+1} = \max\left\{A, \frac{y_n^{t_1 + t_2}}{x_n^{t_1 + t_2}}\right\}, \quad y_{n+1} = \max\left\{A, \frac{x_n^{t_1 + t_2}}{y_n^{t_1 + t_2}}\right\}, \quad n \in \mathbb{N},
\]
which along with (2.1) implies that
\[
\frac{x_{n+1}}{y_n^{t_1 + t_2}} = \max\left\{A, \frac{y_n}{y_n^{t_1 + t_2}, \frac{x_n^{t_1 + t_2}}{x_n^{t_1 + t_2}}}\right\} \leq \max\left\{A^{1-t_1}, \left(\frac{y_n}{x_n^{t_1 + t_2}}\right)^{t_2}\right\} \tag{2.2}
\]
\[
\frac{y_{n+1}}{x_n^{t_1 + t_2}} = \max\left\{A, \frac{x_n}{x_n^{t_1 + t_2}, \frac{y_n^{t_1 + t_2}}{y_n^{t_1 + t_2}}}\right\} \leq \max\left\{A^{1-t_1}, \left(\frac{x_n}{y_n^{t_1 + t_2}}\right)^{t_2}\right\} \tag{2.3}
\]
for every $n \in \mathbb{N}$, and consequently
\[
\max\left\{\frac{x_{n+1}}{y_n^{t_1 + t_2}}, \frac{y_{n+1}}{x_n^{t_1 + t_2}}\right\} \leq \max\left\{A^{1-t_1}, \max\left\{\frac{x_n}{y_n^{t_1 + t_2}}, \frac{y_n}{x_n^{t_1 + t_2}}\right\}\right\} \tag{2.4}
\]

Let
\[
u_n = \max\left\{\frac{x_n}{y_n^{t_1 + t_2}}, \frac{y_n}{x_n^{t_1 + t_2}}\right\}, \quad n \in \mathbb{N},
\]
and
\[
v_{n+1} = \max\left\{A^{1-t_1}, \nu_n^{t_2}\right\}, \quad n \in \mathbb{N}, \tag{2.5}
\]
with
\[
u_1 = u_1.
\]

By induction, we have
\[
u_n \leq v_n, \quad n \in \mathbb{N}. \tag{2.6}
\]

The fact $t_2 \in (0, 1)$ implies that the equation $g(x) = x$, where
\[
g(x) = \max\left\{A^{1-t_1}, x^{t_2}\right\}, \quad x \in (0, \infty), \tag{2.7}
\]
has a unique fixed point \( \bar{x} \geq 1 \) and
\[
(g(x) - x)(x - \bar{x}) < 0, \quad x \in \mathbb{R}_+ \setminus \{\bar{x}\}.
\] (2.8)

Hence, for \( v_1 \in (0, \bar{x}] \), we have
\[
v_n \leq v_{n+1} \leq \bar{x}, \quad n \in \mathbb{N},
\]
and for \( v_1 \geq \bar{x} \), we have
\[
v_n \geq v_{n+1} \geq \bar{x}, \quad n \in \mathbb{N}.
\]

Hence, \((v_n)_{n \in \mathbb{N}}\) is bounded, which along with (2.6) implies that
\[
u_n \leq L_1, \quad n \in \mathbb{N}_0,
\]
for some \( L_1 \geq \bar{x} \geq 1 \).

Therefore
\[
x_{n+1} \leq L_1 y_n, \quad y_{n+1} \leq L_1 x_n, \quad n \in \mathbb{N}_0.
\] (2.9)

From (2.9) we easily get
\[
x_n + y_n \leq 2L_1(x_{n-1} + y_{n-1})^{t_1}, \quad n \in \mathbb{N},
\] (2.10)
from which it easily follows that
\[
x_n + y_n \leq (2L_1)^{\frac{t_1^n}{n+1}} (x_0 + y_0) \leq (2L_1)^{\frac{t_1}{2}} \max \{1, x_0 + y_0\}.
\] (2.11)

From (2.1) and (2.11) the boundedness of sequences \((x_n)_{n \geq -1}\) and \((y_n)_{n \geq -1}\), and consequently the theorem follows.

**Theorem 2.2.** Assume that \( A > 0, \ p > 0\) and \( p^2 < 4q \). Then all positive solutions of system (1.3) are bounded.

**Proof.** Let sequence \((p_n)_{n \in \mathbb{N}_0}\) be defined as follows
\[
p_{k+1} = \frac{q}{p - p_k}, \quad p_0 = 0.
\] (2.12)

Using (1.3) and (2.12) we have
\[
x_{n+1} = \max \left\{ A, \frac{y_n}{x_{n-1}} \right\} = \max \left\{ A, \left( \frac{y_n}{x_{n-1}} \right)^p \right\}
\]
\[
= \max \left\{ A, \max \left\{ \frac{A}{\left( \frac{1}{p^{\frac{1}{p}}/2} \right)^p} \right\} \right\}
\]
\[
= \max \left\{ A, \max \left\{ \frac{A}{\left( \frac{1}{p^{\frac{1}{p}}/2} \right)^p} \right\} \right\}
\]
(2.13)
Max-type system of difference equations

\[ \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p}}, \max \left\{ \frac{A}{y_n^{q/(p-\hat{p})}}, \cdots \max \left\{ \frac{A}{y_n^{q/(p-\hat{p}^d)}}, x_n^{q/(p-\hat{p})} \right\} \right\} \right\} \right\} = \ldots \]

\[ = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p}}, \max \left\{ \frac{A}{y_n^{q/(p-\hat{p})}}, \cdots \max \left\{ \frac{A}{y_n^{q/(p-\hat{p}^d)}}, x_n^{q/(p-\hat{p})} \right\} \right\} \right\} \right\} \]

(2.14)

\[ = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p}}, \max \left\{ \frac{A}{y_n^{q/(p-\hat{p})}}, \cdots \max \left\{ \frac{A}{y_n^{q/(p-\hat{p}^d)}}, x_n^{q/(p-\hat{p})} \right\} \right\} \right\} \right\} \]

(2.15)

If \( p^2 \leq q \), then by using (2.1) in (2.13), for \( n \geq 3 \), we get

\[ x_{n+1} = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p}}, \frac{x_n^{q/p}}{x_n^{q/p}} \right\} \right\} \leq \max \left\{ A, A^{p-q}, \frac{1}{A^q+p-q} \right\}, \]

so \((x_n)_{n \geq 1}\) is bounded, in this case.

The monotonicity of \( g(x) = q/(p-x) \) on the interval \((0, p)\) along with the fact \( 0 = p_0 < p_1 = q/p \) implies that \( p_k \) is increasing as far as \( p_k < p \). If \( p_k < p \) for every \( k \in \mathbb{N}_0 \), then there would exist \( \lim_{k \to \infty} p_k := \hat{p} \) and \((\hat{p})^2 - p\hat{p} + q = 0 \), but the equation does not have real roots because of the condition \( p^2 < 4q \).

Therefore, there is an \( l_0 \in \mathbb{N} \) such that

\[ p_{l_0-1} < p \quad \text{and} \quad p_{l_0} \geq p. \]

If \( l_0 = 2k \), then by using (2.1) in (2.14), we get

\[ x_{n+1} = \max \left\{ A, \max \left\{ \frac{A}{x_n^{q/p}}, \frac{x_n^{q/p}}{x_n^{q/p}} \right\} \right\} \leq \max \left\{ A, A^{p-q}, \frac{1}{A^q+p-q} \right\}, \]

for \( n \geq 2k + 2 \), from which the boundedness of \((x_n)_{n \geq 1}\) follows in this case.

If \( l_0 = 2k + 1 \), then by using (2.1) in (2.15), we get
Theorem 2.3. Assume that $A > 0$, $p = 1 + q$, and $q \in (0, 1)$. Then all positive solutions of system (1.3) are bounded.

Proof. First note that by using the change of variables

$$x_n = A^\frac{q}{p}x_n, \quad y_n = A^\frac{q}{p}y_n, \quad n \in \mathbb{N}_0,$$

system (1.3), in this case, is reduced to the same system with $A = 1$. Hence we may assume that $A = 1$.

Assume that the sequences $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$ are defined by

$$a_0 = q, \quad b_0 = q + 1, \quad a_{2n+1} = (q + 1)b_{2n} - a_{2n}, \quad b_{2n+1} = qb_{2n}, \quad n \in \mathbb{N}_0, \quad a_{2n+2} = (q + 1)a_{2n+1} - b_{2n+1}, \quad a_{2n+2} = qa_{2n+1}, \quad n \in \mathbb{N}_0.

(2.16)$$

From this, by using (1.3) and a simple inductive argument, we have

$$x_{n+1} = \max \left\{ \frac{x_n^{|q+1|}}{x_n^{|q|}}, \ldots, \frac{x_n^{|q|}}{x_n^{|q|}} \right\} = \max \left\{ 1, \frac{b_0}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{(q+1)|a_0|}{y_n^{|q|}} \right\} = \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

$$= \max \left\{ 1, \frac{1}{x_n^{|q|}}, \frac{1}{y_n^{|q|}}, \frac{1}{x_n^{|q|}} \right\}$$

for $n \geq 2k + 3$, from which the boundedness of $(x_n)_{n \geq 1}$ follows in this case.

Since the system (1.3) is symmetric, the boundedness of $(x_n)_{n \geq 1}$ imply the boundedness of $(y_n)_{n \geq 1}$, finishing the proof of the theorem. \qed
we get

for every $k \in \mathbb{N}$.

From (2.16) we have that

$$b_{2n} = \frac{a_{2n+1} + a_{2n}}{q + 1}, \quad n \in \mathbb{N}_0.$$  

Applying this to the following relation

$$b_{2n+2} = (q + 1)a_{2n+1} - qb_{2n}, \quad n \in \mathbb{N}_0,$$

we get

$$a_{2n+3} + a_{2n+2} - (q^2 + q + 1)a_{2n+1} + qa_{2n} = 0, \quad n \in \mathbb{N}_0.$$  

From this and the relation $a_{2n+2} = qa_{2n+1}$, we get

$$a_{2n+3} - (q^2 + 1)a_{2n+1} + q^2a_{2n-1} = 0, \quad n \in \mathbb{N}.$$  

(2.19)

It is easy to see that the general solution of difference equation (2.19) is

$$a_{2n+1} = c_1 + c_2q^{2n}, \quad n \in \mathbb{N}_0.$$  

From this and since

$$a_1 = (q + 1)b_0 - a_0 = q^2 + q + 1, \quad b_1 = qb_0 = q^2 + q,$$

$$a_2 = qa_1 = q^3 + q^2 + q, \quad b_2 = (q + 1)a_1 - b_1 = (q + 1)(q^2 + 1),$$

$$a_3 = (q + 1)b_2 - a_2 = q^4 + q^3 + q^2 + q + 1,$$

we have that

$$c_1 + c_2 = q^2 + q + 1, \quad c_1 + c_2q^2 = q^4 + q^3 + q^2 + q + 1$$

and consequently

$$c_1 = \frac{1}{1 - q}, \quad c_2 = \frac{q^4 + q^3}{q^2 - 1} = \frac{q^3}{q - 1}.$$

Hence

$$a_{2n+1} = \frac{1 - q^{2n+3}}{1 - q}, \quad n \in \mathbb{N}_0.$$  

(2.20)

Letting $n \to +\infty$ we get

$$\lim_{n \to +\infty} a_{2n+1} = \frac{1}{1 - q}.$$  

(2.21)
From this and (2.16) we also have that
\[
\lim_{n \to +\infty} a_{2n} = \lim_{n \to +\infty} b_{2n+1} = \frac{q}{1 - q} = q \lim_{n \to +\infty} b_{2n}.
\] (2.22)

Now note that from (2.17) and (2.18) we have that
\[
x_{2n+1} = \max \left\{ \frac{1}{x_{2n-1}^{a_0}}, \frac{1}{y_{2n-1}^{b_1}}, \ldots, \frac{1}{y_{2n-1}^{b_{2n-2}}}, \frac{1}{x_0^{a_{2n-1}}} \right\}
\]
\[
x_{2n} = \max \left\{ \frac{1}{x_{2n-2}^{a_0}}, \frac{1}{y_{2n-3}^{b_1}}, \ldots, \frac{1}{y_{2n-3}^{b_{2n-2}}}, \frac{1}{x_0^{a_{2n-1}}} \right\},
\]
for \( n \in \mathbb{N} \).

From this, since \( \min \{x_n, y_n\} \geq 1 \) for \( n \in \mathbb{N} \), and by using (2.21) and (2.22) the boundedness of the sequence \((x_n)_{n \geq 1}\) easily follows.

Since system (1.3) is symmetric, the boundedness of \((x_n)_{n \geq 1}\) imply the boundedness of \((y_n)_{n \geq 1}\), finishing the proof of the theorem. \( \square \)

The following theorem shows that positive solutions of system (1.3) are unbounded in the other cases.

**Theorem 2.4.** Assume that \( A > 0 \). If \( p^2 \geq 4q \geq 4 \), or \( p > 1 + q \) and \( q \in (0, 1) \), then system (1.3) has positive unbounded solutions.

**Proof.** Assume that \( p^2 \geq 4q \geq 4 \) and \( p \neq 2 \). From (1.3) we have
\[
x_{n+1} \geq \frac{y_n^p}{x_n^{q}}; \quad y_{n+1} \geq \frac{x_n^p}{y_n^{q}}, \quad n \in \mathbb{N}_0.
\] (2.23)

Let \( a_n = \ln(x_n y_n) \), \( n \geq -1 \). Then from (2.23), it follows that
\[
a_{n+1} - pa_n + qa_{n-1} \geq 0, \quad n \in \mathbb{N}_0.
\] (2.24)

The polynomial \( P(t) = t^2 - pt + q \) has the zeroes \( t_{1,2} = (p \pm \sqrt{p^2 - 4q})/2 \), and \( t_1 > 1 \), and \( t_2 > 0 \).

From (2.24) we get
\[
a_{n+1} + t_1 a_n - t_2 (a_n - t_1 a_{n-1}) \geq 0, \quad n \in \mathbb{N}_0,
\] (2.25)
that is,
\[
\frac{x_{n+1} y_{n+1}}{(x_n y_n)^{t_1}} \geq \left( \frac{x_n y_n}{(x_{n-1} y_{n-1})^{t_1}} \right)^{t_2}, \quad n \in \mathbb{N}_0,
\] (2.26)
which implies that
\[
\frac{x_{n+1} y_{n+1}}{(x_n y_n)^{t_1}} \geq \left( \frac{x_0 y_0}{(x_{-1} y_{-1})^{t_1}} \right)^{t_2^{n+1}}, \quad n \in \mathbb{N}_0.
\] (2.27)

Let \( x_i, y_i, i \in \{-1, 0\} \) be chosen such that
\[
x_0 y_0 > 1 \quad \text{and} \quad x_0 y_0 = (x_{-1} y_{-1})^{t_1}.
\] (2.28)
This, along with (2.27), yields
\[ x_n y_n \geq \left( \frac{x_0 y_0}{(x_{-1} y_{-1})^{t_1}} \right)^{t_2} (x_{n-1} y_{n-1})^{t_1} = (x_{n-1} y_{n-1})^{t_1}, \quad n \in \mathbb{N}_0, \] (2.29)
from which we get
\[ x_n y_n \geq (x_0 y_0)^{t_1}, \quad n \in \mathbb{N}_0. \] (2.30)
Letting \( n \to \infty \) in (2.30), using the first assumption in (2.28) and \( t_1 > 1 \), it follows that
\[ x_n y_n \to +\infty \quad \text{as} \quad n \to \infty, \] (2.31)
which along with the inequality between arithmetic and geometric means implies
\[ \sqrt{x_n^2 + y_n^2} \to +\infty \quad \text{as} \quad n \to \infty, \] (2.32)
from which it follows that \((x_n, y_n)_{n \geq -1}\) is unbounded.

The proof in the case \( p > 1 + q \) and \( q \in (0, 1) \) is similar, since then
\[ t_1 = \frac{p + \sqrt{p^2 - 4q}}{2} > 1. \]
If \( p = q + 1 = 2 \), then \( t_1 = t_2 = 1 \). If we choose \( x_i, y_i, i \in \{-1, 0\} \) such that
\[ x_0 y_0 > x_{-1} y_{-1} > 0, \] (2.33)
then from (2.27) we get
\[ x_n y_n \geq \frac{x_0 y_0}{x_{-1} y_{-1}} x_{n-1} y_{n-1}, \quad n \in \mathbb{N}_0, \]
and consequently
\[ x_n y_n \geq \left( \frac{x_0 y_0}{x_{-1} y_{-1}} \right)^n x_0 y_0, \quad n \in \mathbb{N}_0. \] (2.34)
Letting \( n \to \infty \) in (2.34) we get (2.31) and consequently (2.32), which implies that \((x_n, y_n)_{n \geq -1}\) is unbounded, finishing the proof of the theorem.

\begin{acknowledgements}
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant no. (22-130-1435-HiCi). The authors, therefore, acknowledge technical and financial support of KAU.
\end{acknowledgements}

\begin{references}
[1] K. Berenhaut, J. Foley, S. Stević, Boundedness character of positive solutions of a max difference equation, J. Differ. Equations Appl. 12(2006), 1193–1199. MR2277649; url

[2] K. Berenhaut, J. Foley, S. Stević, The global attractivity of the rational difference equation \( y_n = 1 + (y_{n-k}/y_{n-m}) \), Proc. Amer. Math. Soc. 135(2007), 1133–1140. MR2262916; url
\end{references}
[3] K. Berenhaut, J. Foley, S. Stević, The global attractivity of the rational difference equation $y_n = A + \left(\frac{y_{n-k}}{y_{n-m}}\right)^p$, Proc. Amer. Math. Soc. 136(2008), 103–110. MR2350394; url

[4] K. Berenhaut, S. Stević, The behaviour of the positive solutions of the difference equation $x_n = A + \left(\frac{x_{n-2}}{x_{n-1}}\right)^p$, J. Difference Equ. Appl. 12(2006), 909–918. MR2262329; url

[5] L. Berg, S. Stević, On some systems of difference equations, Appl. Math. Comput. 218(2011), 1713–1718. MR2831394; url

[6] R. DeVault, C. Kent, W. Kosmala, On the recursive sequence $x_{n+1} = p + \left(\frac{x_{n-k}}{x_n}\right)^p$, J. Difference Equ. Appl. 9(2006), 909–918. MR1992905; url

[7] E. A. Grove, G. Ladas, Periodicities in nonlinear difference equations, Chapman & Hall, CRC Press, Boca Raton, 2005. MR2193366

[8] B. Iričanin, The boundedness character of two Stević-type fourth-order difference equations, Appl. Math. Comput. 217(2010), 1857–1862. MR2727930; url

[9] B. Iričanin, S. Stević, Some systems of nonlinear difference equations of higher order with periodic solutions, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 13(2006), 499–508. MR2220850

[10] B. Iričanin, S. Stević, Global attractivity of the max-type difference equation $x_n = \max\{c, \frac{x_n^p}{\prod_{j=2}^x x_{n-j}^p}\}$, Util. Math. 91(2013), 301–304. MR3097907

[11] C. M. Kent, M. A. Radin, On the boundedness nature of positive solutions of the difference equation $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$ with periodic parameters, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms Vol. 2003, suppl., 11–15. MR2015782

[12] G. Papaschinopoulos, V. Hatzifilippidis, On a max difference equation, J. Math. Anal. Appl. 258(2001), 258–268. MR1828104; url

[13] G. Papaschinopoulos, M. Radin, C. J. Schinas, Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form, Appl. Math. Comput. 218(2012), 5310–5318. MR2870051; url

[14] G. Papaschinopoulos, C. J. Schinas, On a system of two nonlinear difference equations, J. Math. Anal. Appl. 219(1998), 415–426. MR1606350; url

[15] G. Papaschinopoulos, C. J. Schinas, On the behavior of the solutions of a system of two nonlinear difference equations, Comm. Appl. Nonlinear Anal. 5(1998), 47–59. MR1621223

[16] G. Papaschinopoulos, C. J. Schinas, Invariants for systems of two nonlinear difference equations, Differential Equations Dynam. Systems 7(1999), 181–196. MR1860787

[17] G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, Nonlinear Anal. 46(2001), 967–978. MR1866733; url

[18] G. Papaschinopoulos, C. J. Schinas, On the dynamics of two exponential type systems of difference equations, Comput. Math. Appl. 64(2012), 2326–2334. MR2966868; url

[19] G. Papaschinopoulos, C. J. Schinas, V. Hatzifilippidis, Global behavior of the solutions of a max-equation and of a system of two max-equations, J. Comput. Anal. Appl. 5(2003), 237–254. MR1980394; url
[20] G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, On the nonautonomous difference equation $x_{n+1} = A_n + (x_{n-1}^{p}/x_n^r)$, Appl. Math. Comput. 217(2011), 5573–5580. MR2770176; url

[21] G. Stefanidou, G. Papaschinopoulos, C. Schinas, On a system of max difference equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 14(2007), 885–903. MR2369919

[22] G. Stefanidou, G. Papaschinopoulos, C. J. Schinas, On a system of two exponential type difference equations, Comm. Appl. Nonlinear Anal. 17(2010), 1–13. MR2669014

[23] S. Stević, A global convergence results with applications to periodic solutions, Indian J. Pure Appl. Math. 33(2002), 45–53. MR1879782

[24] S. Stević, On the recursive sequence $x_{n+1} = a_n + (x_{n-1}/x_n)$ II, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 10(2003), 911–916. MR2008754

[25] S. Stević, On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$, Taiwanese J. Math. 7(2003), 249–259. MR1978014

[26] S. Stević, A short proof of the Cushing–Henson conjecture, Discrete Dyn. Nat. Soc. 2006, Art. ID 37264, 5 pp. MR2272408; url

[27] S. Stević, Asymptotics of some classes of higher order difference equations, Discrete Dyn. Nat. Soc. 2007, Art. ID 56813, 20 pp. MR2324714; url

[28] S. Stević, On the recursive sequence $x_{n+1} = A + (x_n^p/x_{n-1}^r)$, Discrete Dyn. Nat. Soc. 2007, Art. ID 40963, 9 pp. MR2375479

[29] S. Stević, On the recursive sequence $x_{n+1} = \max\{c, x_n^p/x_{n-1}^r\}$, Appl. Math. Lett. 21(2008), 791–796. MR2436166; url

[30] S. Stević, Boundedness character of a class of difference equations, Nonlinear Anal. 70(2009), 839–848. MR2468424; url

[31] S. Stević, Global stability of a difference equation with maximum, Appl. Math. Comput. 210(2009), 525–529. MR2509928; url

[32] S. Stević, Global stability of a max-type difference equation, Appl. Math. Comput. 216(2010), 354–356. MR2596166; url

[33] S. Stević, On a generalized max-type difference equation from automatic control theory, Nonlinear Anal. 72(2010), 1841–1849. MR2577582; url

[34] S. Stević, Periodicity of max difference equations, Util. Math. 83(2010), 69–71. MR2742275

[35] S. Stević, On a nonlinear generalized max-type difference equation, J. Math. Anal. Appl. 376(2011), 317–328. MR2745409; url

[36] S. Stević, On a system of difference equations, Appl. Math. Comput. 218(2011), 3372–3378. MR2851439; url

[37] S. Stević, On the difference equation $x_n = x_{n-2} / (b_n + c_n x_{n-1} x_{n-2})$, Appl. Math. Comput. 218(2011), 4507–4513. MR2862122; url
[38] S. Stević, Periodicity of a class of nonautonomous max-type difference equations, Appl. Math. Comput. 217(2011), 9562–9566. MR2811231; url

[39] S. Stević, On a third-order system of difference equations, Appl. Math. Comput. 218(2012), 7649–7654. MR2892731; url

[40] S. Stević, On some periodic systems of max-type difference equations, Appl. Math. Comput. 218(2012), 11483–11487. MR2943993; url

[41] S. Stević, On some solvable systems of difference equations, Appl. Math. Comput. 218(2012), 5010–5018. MR2870025; url

[42] S. Stević, Solutions of a max-type system of difference equations, Appl. Math. Comput. 218(2012), 9825–9830. MR2916163; url

[43] S. Stević, On a symmetric system of max-type difference equations, Appl. Math. Comput. 219(2013) 8407–8412. MR3037547; url

[44] S. Stević, On the system of difference equations $x_{n+1} = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_{n+1} = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$, Appl. Math. Comput. 219(2013), 4755–4764. MR3001523; url

[45] T. Sun, B. Qin, H. Xi, C. Han, Global behavior of the max-type difference equation $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$, Abstr. Appl. Anal. 2009, Art. ID 152964, 10 pp. MR2506993

[46] H. D. Voulov, On the periodic nature of the solutions of the reciprocal difference equation with maximum, J. Math. Anal. Appl. 296(2004), No. 1, 32–43. MR2070491; url

[47] X. Yang, X. Liao, On a difference equation with maximum, Appl. Math. Comput. 181(2006), 1–5. MR2270463; url