On the Noise-Information Separation of a Private Principal Component Analysis Scheme

Mario Diaz*, Shahab Asoodeh†, Fady Alajaji‡, Tamás Linder‡, Serban Belinschi§ and James Mingo†

*Arizona State University and Harvard University, mdiaztor@{asu,g.harvard.edu}
†The University of Chicago, shahab@uchicago.edu
‡Queen’s University, {fady,linder,mingo}@queensu.ca
§Institut de Mathématiques de Toulouse, Serban.Belinschi@math.univ-toulouse.fr

Abstract—In a survey disclosure model, we consider an additive noise privacy mechanism and study the trade-off between privacy guarantees and statistical utility. Privacy is approached from two different but complementary viewpoints: information and estimation theoretic. Motivated by the performance of principal component analysis, statistical utility is measured via the spectral gap of a certain covariance matrix. This formulation and its motivation rely on classical results from random matrix theory. We prove some properties of this statistical utility function and discuss a simple numerical method to evaluate it.

I. INTRODUCTION

In the last decades, privacy breaches made clear the necessity of privacy mechanisms with provable guarantees. In this context, additive noise mechanisms are a popular choice among practitioners given their ease of implementation and mathematical tractability [1]. In order to understand the trade-off between the privacy guarantees provided by and the statistical cost of this type of mechanism, it is necessary to precisely quantify privacy and statistical utility. In this paper we consider two common measures of privacy, one based on mutual information and the other one on the minimum mean-squared error (MMSE). In the context of a survey with \( p \) queries and \( n \) respondents, we introduce a measure of statistical utility motivated by the performance of principal component analysis (PCA), a statistical method aimed at finding the least number of variables that explain a given data set [2, Ch. 9]. More specifically, statistical utility is measured by the gap between the eigenvalues of a certain covariance matrix associated with the responses. This formulation and its motivation rely on classical results from random matrix theory. To facilitate mathematical tractability, we focus on a toy model where the eigenvalues of the data covariance matrix are either large or negligible. For this model, we derive a simple numerical method to compute the utility function. A general treatment of spectrum separation can be found in [3, Ch. 6].

Private versions of PCA have been analyzed in the past, specially under the framework of differential privacy, see [4] and references therein. Many of these analyses rely on results stemming from finite dimensional (random) matrix theory, see, e.g., [5]. The approach in the present paper follows a different path, relying on asymptotic random matrix theory considerations. Our main motivation is two-fold: the behavior of the eigenvalues of certain random matrices becomes simpler when the dimensions go to infinity (Thm. [1]) and this asymptotic behavior essentially appears in finite dimension (Thm. [2]).

II. SETTING

In Sec. II we present the setting of our problem. The statistical utility function and some of its properties are then introduced in Sec. III followed by a privacy analysis in Sec. IV. In particular, we study the privacy-utility trade-off in the spirit of [6], which is also related with the privacy-utility trade-offs in [7] and references therein. In Sec. V a simple numerical method for the computation of the utility function is provided. Due to space limitations, all the proofs are deferred to [8].

Notation. Let \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \Re z > 0 \} \) and \( \mathbb{C}^- = -\mathbb{C}^+ \), where \( \mathbb{C} \) is the set of complex numbers and \( \Re z \) is the imaginary part of \( z \in \mathbb{C} \). For a \( p \times p \) complex matrix \( A \in \mathbb{M}_p(\mathbb{C}) \), we let \( A_{ij} \) be its \( i,j \)-entry and \( A^* \) be its conjugate transpose. The indicator function of a set \( E \) is denoted by \( 1_E \). For a probability distribution \( F \), we let \( \text{Supp}(F) \) be its support, i.e., the smallest closed set \( E \) with \( F(E) = 1 \). For \( A \in \mathbb{M}_p(\mathbb{C}) \), we call \( (\lambda_1, \ldots, \lambda_p) \) the eigenvalues of \( A \), \( \lambda_1 \geq \cdots \geq \lambda_p \). The probability distribution defined by \( F_A([a, b]) = \frac{1}{p} \sum k \, 1_{\lambda_k \in [a, b]} \) is called the eigenvalue distribution of \( A \).

Assume that a survey with \( p \) queries is handed to \( n \) respondents. Let \( \hat{X} \) be the \( p \times n \) matrix associated to this survey. We assume that \( \hat{X} \) is a realization of a random matrix

\[
X = \Sigma^{1/2} W,
\]

where \( \Sigma \) is a \( p \times p \) (deterministic) covariance matrix and \( W \) is a \( p \times n \) random matrix whose entries are independent and identically distributed (i.i.d.) real random variables with zero mean and unit variance. Note that the columns of \( \hat{X} \) are independent realizations of a random vector with covariance \( \Sigma \). A popular instance of this model corresponds to the case where the entries of \( W \) are i.i.d. Gaussian random variables; thus the entries of \( X \) are possibly correlated Gaussian random variables. The covariance matrix \( \Sigma \) possesses valuable statistical information about the respondent population. Hence, in many applications the data aggregator is interested in obtaining an estimation of \( \Sigma \). In this setting, the canonical estimator is the sample covariance matrix

\[
\hat{\Sigma} := \frac{1}{n} \hat{X} \hat{X}^*.
\]

Because of privacy concerns, the respondents might not want to disclose their answers, \( \hat{X} \), to the data aggregator. Instead, they might want to use a randomized mechanism to alter their answers, giving them the position of plausible
deniability towards their responses. In this paper we focus on
an additive noise model: instead of providing \( \hat{X} \) to the data
aggregator, the respondents provide
\[
\hat{X}_t := \hat{X} + \sqrt{t} \hat{Z},
\]
where \( t > 0 \) is a design parameter and \( \hat{Z} \) is a realization of
\( Z \), a \( p \times n \) random matrix which is independent of \( X \) and
whose entries are i.i.d. random variables with zero mean and
unit variance. In this case, the sample covariance matrix equals
\[
\hat{\Sigma}_t = \frac{1}{n} \hat{X}_t \hat{X}_t^*,
\]
a realization of \( \Sigma_t = n^{-1} X_t X_t^* \), where \( X_t := X + \sqrt{t} \hat{Z} \).
Note that \( \hat{\Sigma}_0 = \hat{\Sigma} \) and that \( \mathcal{E}(\Sigma_t) = \Sigma + tI_p \), where \( I_p \)
denotes the \( p \times p \) identity matrix. The probability distribution
of the additive noise may change according to the nature of
the data, e.g., discrete or continuous. In particular, both \( t \)
and the distribution of the noise are the design parameters of
the privacy mechanism. Observe that given these parameters,
this additive mechanism can be implemented locally at each
user, making unnecessary the presence of a trustworthy data
aggregator.

If for the application at hand the noise distribution is
fixed, then the trade-off between privacy and statistical utility
becomes evident: when \( t \) increases the respondents’ privacy
improves as their answers get more distorted but, at the same
time, the sample covariance matrix \( \hat{\Sigma}_t \) differs more from \( \Sigma \).
Note that for \( p \) fixed and \( n \) large (\( n \to \infty \)), the latter is not
a problem. Indeed, under some mild assumptions, the law of
large numbers implies
\[
\lim_{n \to \infty} \| \Sigma_t - (\Sigma + tI_p) \|_2^2 \overset{\text{a.s.}}{=} 0,
\]
where a.s. stands for almost surely. Hence, the data aggregator
might use \( \hat{\Sigma}_t - tI_p \) as an estimate of \( \Sigma \) without incurring a big
statistical loss. However, when both \( p \) and \( n \) are large (\( p, n \to \infty \)),
the estimator \( \hat{\Sigma}_t - tI_p \) is known to be a poor estimate of
\( \Sigma \), e.g., the eigenvalues of \( \hat{\Sigma}_t - tI_p \) might be very different
from those of \( \Sigma \), see Thm. 1. Since in many contemporary
applications \( p \) and \( n \) are within the same order of magnitude,
it is necessary to quantify the statistical cost incurred by the
additive noise mechanism in this regime. In the next section
we do so by introducing a utility function connected to the
performance of PCA.

III. STATISTICAL UTILITY FUNCTION

We now introduce a statistical utility function that captures
the performance of PCA applied to \( \hat{\Sigma}_t \). In order to motivate
its definition, let us consider the following example.

To simplify the exposition, in this section we assume that
the entries of \( W \) and \( Z \) are Gaussian. At the end of this section
we comment on the universality of the subsequent analysis.

Example 1. Let \( p = 50 \) and \( n = 2000 \). Assume that \( \Sigma \)
is diagonal with eigenvalues 0, 7, and 10 with multiplicities
35, 10, and 5, respectively. A histogram of the eigenvalues of
an instance of \( \hat{\Sigma} = \hat{\Sigma}_0 \) is given in Fig. 1. Note that this
distribution is a blurred version of the eigenvalue distribution

![Figure 1. Histogram of the eigenvalues of \( \hat{\Sigma} \) for different values of \( t \).](image-url)

As \( t \) increases, the additive noise \( \sqrt{t} \hat{Z} \) becomes stronger,
making the eigenvalue distribution of \( \hat{\Sigma}_t \) more diffuse, as shown in Fig. 1. This behavior has a direct impact on the
benefits of PCA, which provides a dimensionality reduction
inversely proportional to the number of largest eigenvalues.
For example, PCA performed on \( \hat{\Sigma} = \hat{\Sigma}_0 \) would propose the
five largest eigenvalues as the most informative components.
Similarly, PCA performed on \( \hat{\Sigma} \) would suggest the
fifteen largest eigenvalues. Since all the eigenvalues of \( \hat{\Sigma}_0 \)
are merged together, PCA in this latter case might be ineffective.

The forthcoming definition of the statistical utility function
\( U \) relies on the following asymptotic considerations. The
Gaussianity assumed in this section implies
\[
X_t = \Sigma^{1/2} W + \sqrt{t} \hat{Z} \overset{d}{=} (\Sigma + tI_p)^{1/2} W,
\]
where \( \overset{d}{=} \) stands for equality in distribution. In particular,
\[
\Sigma_t \overset{d}{=} \frac{1}{n}(\Sigma + tI_p)^{1/2} WW^*(\Sigma + tI_p)^{1/2}.
\]
The next theorem [9, Thm. 1.1] is a generalization of the
Marchenko-Pastur theorem, a cornerstone of random matrix
theory. For a probability distribution function \( F \), its Cauchy
transform \( G : \mathbb{C}^+ \to \mathbb{C}^- \) is the (analytic) function defined by
\[
G(z) = \int_{\mathbb{R}} \frac{1}{z-x} dF(x).
\]
The Cauchy transform characterizes a distribution function.
Indeed, the Stieltjes inversion formula states that
\[
F([a,b]) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_a^b G(x + i\epsilon) dx,
\]
for all \( a < b \) continuity points of \( F \). When \( F \) is regular enough,
it density equals \( f(x) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} G(x + i\epsilon) \).

Theorem 1 [9]. Assume on a common probability space:

(a) For \( p = 1, 2, \ldots \), \( W_p = (W_p(i,j)) \) is \( p \times n \), \( W_p(i,j) \in \mathbb{C} \)
are identically distributed for all \( p, i, j \), independent across \( i, j \)
for each \( p \), \( \mathbb{E}(|W_1(1,1) - \mathbb{E}(W_1(1,1))|^2) = 1; \)

(b) \( n = n(p) \) with \( p/n \to c \in (0, \infty) \) as \( p \to \infty; \)
Let $T_p^{1/2}$ be the Hermitian nonnegative square root of $T_p$, and let $\Sigma_p = (1/n)T_p^{1/2}W_pW_p^*T_p^{1/2}$. Then, a.s., the eigenvalue distribution of $\Sigma_p$ converges in distribution, as $p \to \infty$, to a non-random probability distribution $F$, whose Cauchy transform $G(z)$ satisfies

$$G(z) = \int_{\mathbb{R}} \frac{1}{x-i(1-c+czG(z))} dH(x), \quad (4)$$

in the sense that, for each $z \in \mathbb{C}^+$, $G(z)$ is the unique solution to (4) in $D_{c,z} = \{G \in \mathbb{C} : (1-c)/z + cG \in \mathbb{C}^-\}$.

Note that if $H$ is discrete, the integral in (4) reduces to a sum. In particular, if $\text{Supp}(H) \cap (0,\infty) = \emptyset$, then $G(z)$ is the only root in $D_{c,z}$ of a polynomial of degree $n+1$. For instance, if $H(x) = 1_{x \geq 1}$, then $G = G(z)$ solves the equation $cz^2 + c(1-c-z)G + 1 = 0$, a quadratic polynomial in $G$. The next example shows the predictive power of Thm. 1.

**Example 2.** In Fig. 2, the histogram of the eigenvalues of a realization of $\Sigma_0$ is depicted for two different values of $p$ and $n$ with $c = 1/40$. In both cases, the eigenvalue distribution of $\Sigma$ is given by (1). The asymptotic density of the eigenvalues provided by Thm. 1 is also depicted. Observe the close agreement between the empirical and asymptotic eigenvalue distributions, even for $p$ as small as 50.

The previous example demonstrates that not only the distribution of the eigenvalues follows closely the corresponding asymptotic density, but also that there are no eigenvalues outside the support of the asymptotic prediction. This observation is formalized in the following theorem (10). Given $c \in (0,\infty)$ and $H$ a probability distribution on $[0,\infty)$, we let $F^{c,H}$ be the limiting distribution determined by (4).

**Theorem 2 (10).** Assume:

(a) $W(i,j) \in \mathbb{C}$, $i,j = 1,2,\ldots$ are i.i.d. random variables with $\mathbb{E}(W(1,1)) = 0$, $\mathbb{E}((W(1,1))^2) = 1$, and $\mathbb{E}(|W(1,1)|^4) < \infty$;

(b) $n = n(p)$ with $c_p = p/n \to c \in (0,\infty)$ as $p \to \infty$;

(c) For each $p$, $T_p$ is $p \times p$ Hermitian nonnegative definite with eigenvalue distribution $H_p$ converging in distribution to a probability distribution $H$;

(d) $\Sigma_p = (1/n)T_p^{1/2}W_pW_p^*T_p^{1/2}$ where $W_p = (W_p(i,j))$ with $i = 1,2,\ldots,p$, $j = 1,2,\ldots,n$ and $T_p^{1/2}$ is any Hermitian square root of $T_p$;

(e) The interval $[a,b]$ with $a > 0$ lies outside the support of $F^{c,H}$ and $F^{c,H}$ for all large $p$.

Then, with probability one, no eigenvalue of $\Sigma_p$ appears in $[a,b]$ for all large $p$.

The previous theorem readily implies that the gaps in the support of $F^{c,H}$ appear in finite dimension. This is of particular interest for this paper, as PCA is more useful when there are few large eigenvalues, i.e., there is a gap between large and small eigenvalues. Now we introduce the promised statistical utility function.

In order to keep the analysis tractable, we consider the following toy model for the situation in which there is a clear distinction between large and small eigenvalues: the covariance matrix $\Sigma$ has only one non-zero eigenvalue, say $s$, with multiplicity $\lfloor rp \rfloor$ for some $r \in (0,1)$. Under this assumption, the eigenvalue distribution of $\Sigma + \delta_0$ equals

$$H(x) = (1-r)1_{x \geq 1} + r1_{x \geq s + r}.$$  

(5)

By (3) and Thm. 1 a.s., the eigenvalue distribution of $\Sigma_t$ converges, as $p \to \infty$, to

$$F_t := F^{c,H}.$$  

(6)

Finally, let $\text{Supp}(F_t)$ be the support of $F_t$ and $N(t)$ be the number of its connected components. Note that, by Lemma 1, $N(t)$ is finite for every $t \in [0,\infty)$.

**Definition 1.** The utility function $U : [0,\infty) \to \mathbb{R}$ is defined as follows. If $N(t) = 1$, we let $U(t) = 0$. If $N(t) = 2$ and $A_t, B_t$ are the connected components of $\text{Supp}(F_t)$,

$$U(t) = \min_{a \in A_t, b \in B_t} |a - b|.$$  

(5)

In words, $U(t)$ approximates the separation between the large and the small eigenvalues of $\Sigma_t$, as long as such separation exists. As exhibited by equations (2) and (3), the large eigenvalues of $\Sigma_t$ correspond mainly to the non-zero eigenvalues of $\Sigma$, while the small ones come from the added noise $\sqrt{\delta}Z$. Note that in order for $U$ to be well defined, it is necessary for the range of $N$ to be a subset of $\{1,2\}$, as established next.

**Theorem 3.** With the assumptions from (5) and (6), we have that $N(t) \in \{1,2\}$ for all $t \in [0,\infty)$.

One way to compute $U(t)$ is finding $F_t$, determining its connected components, and measuring their distance. However, there is a more efficient method based on the discriminant of a cubic equation. To avoid an unnecessary digression, this method is discussed in Sec. V. Using this method, in Fig. 3 we plot the graph of $U$.

Under our standing assumptions, the performance of PCA is heavily compromised for a noise power $t$ such that $U(t) = 0$, as the gap between noise and information disappears. Indeed, for $t$ large enough the gap always disappears, as established by the following proposition.

---

1 More precisely, $\|H_t - F_{2c+1p}\|_\infty \leq 1/p$ which is negligible.
Figure 3. The graph of $U$ for $c = 1/40$, $r = 3/10$, and $s = 10$.

**Proposition 1.** There exists $T = T(c, r, s) \geq 0$ such that $N(t) = 1$ for all $t \geq T$.

Note that in Fig. 3 there exists a $t^*$ such that $U(t) = 0$ if and only if $t \geq t^*$. Thus, in principle, any noise power $t \in [0, t^*)$ does not compromise the performance of PCA. This property makes $t^*$ useful in the design of privacy mechanisms. In view of Thm. 3 and Prop. 1 the existence of such $t^*$ is equivalent to the following.

**Conjecture 1.** $N$ is non-increasing in $t$.

In addition to simulations, there are theoretical reasons to believe in the above conjecture, e.g., similar results are known to be true for other random matrix models [11]. Ultimately, we are interested in the statistical utility function $U$ and not only in the set $\{t \geq 0 : U(t) = 0\} = \{t \geq 0 : N(t) = 1\}$. For this utility function, there is numerical evidence supporting the following stronger conjecture.

**Conjecture 2.** $U$ is non-increasing and convex in $t$.

**Remark.** In this section we assumed that both data and noise are Gaussian. Nonetheless, one can appeal to universality arguments to establish that the conclusions reached in this section hold for a much wider range of random matrix models [11]. Ultimately, we are interested in the statistical utility function $U$ and not only in the set $\{t \geq 0 : U(t) = 0\} = \{t \geq 0 : N(t) = 1\}$. For this utility function, there is numerical evidence supporting the following stronger conjecture.

**Conjecture 1.** $N$ is non-increasing in $t$.

**Conjecture 2.** $U$ is non-increasing and convex in $t$.

In particular, for the toy model of the previous section,

$$P_{IT}^{r, s, p}(t) = \frac{|rp|}{2} \log \left(1 + s \right).$$

In the context of the last remark of the previous section, i.e., when data and/or noise are not necessarily Gaussian, it is relevant to consider the following.

Assume that the noise is Gaussian but the data is drawn from an arbitrary distribution having a density and finite third moment. Let $\theta = \frac{1}{\sqrt{t}}$. With this notation, 

$$P_{IT}(t) = h \left( \theta X^{(1)} + Z^{(1)} \right) - \frac{s}{2} \log 2\pi e,$$

where $h(\cdot)$ denotes differential entropy. In particular, studying $t \mapsto P_{IT}(t)$ amounts to studying $\theta \mapsto h(\theta X^{(1)} + Z^{(1)})$. If $p = 1$, then it follows from [17, Lemma 1] that, as $\theta \to 0$,

$$I(X^{(1)}; \theta X^{(1)} + Z^{(1)}) = \frac{\theta^2}{2} + o(\theta^2),$$

and thus $P_{IT}(t) \sim \frac{1}{2t}$ in the high privacy regime ($t \to \infty$). For $p \geq 1$, the chain rule implies

$$I(X^{(1)}; \theta X^{(1)} + Z^{(1)}) \leq \frac{p \text{Tr}(\Sigma^2)}{2} + o(\theta^2),$$

and hence $P_{IT}(t) \leq \frac{p}{2t} \text{Tr}(\Sigma)$ in the high privacy regime.

Now assume that neither data nor noise is Gaussian. Recall that the non-Gaussianity $D(V)$ of a random vector $V$ is defined as $D(V) := D(V || V G)$, where $D(\cdot || \cdot)$ denotes the Kullback-Leibler divergence, and $V G$ is a Gaussian random vector with the same mean and covariance matrix as $V$. It can be shown that

$$I(X^{(1)}; X^{(1)}_t) = P_{IT}^{G}(t) + D(\sqrt{t} Z^{(1)}) - D(X^{(1)}_t).$$

In this case, regardless of distributions of $X$ and $Z$,

$$P_{IT}(t) \leq P_{IT}^{G}(t) + D(\sqrt{t} Z^{(1)}) = P_{IT}^{G}(t) + p D(\sqrt{t} Z_{11}),$$

where the last equality holds as the entries of $Z$ are i.i.d.

**MMSE.** In [18], see also [16], the authors proposed an estimation-theoretic measure in terms of MMSE. Following this approach, we define

$$P_{ET}(t) := \sum_{i=1}^{p} \text{mse}(X_{i1} | X_{i}) = E \left[ \|X^{(1)} - E[X^{(1)} | X_i] \|^{2} \right],$$

where $\text{mse}(U|V) := E[(U - E[U|V])^2]$. If both data and noise are Gaussian, then we can write

$$P_{ET}^{G}(t) = \text{Tr} \left[ (I_p + t^{-1} \Sigma)^{-1} \Sigma \right].$$

In particular, for the toy model in the previous section

$$P_{ET}^{r, s, p}(t) = \frac{|rp|}{2} \frac{ts}{t + s}.$$
privacy and utility, we define the following privacy-utility function in the spirit of [6], see also [7] and references therein. For \( \epsilon > 0 \), we define

\[
g_t(\epsilon) := \sup_{t : P_T(t) \leq \epsilon} U(t).
\]

In words, \( g_t(\epsilon) \) equals the largest utility \( U(t) \) under the privacy constraint \( P_T(t) \leq \epsilon \). Conditional on the non-increasing behavior of \( U(t) \) (Conj. 2), it is easy to verify that for the model of the previous section

\[
g_{r,s,p}^F(t) = U((P_T^F)^{-1}(\epsilon)),
\]

where \( (P_T^F)^{-1}(\epsilon) = s(e^{2\epsilon/(r p)} - 1)^{-1} \). Since \( U(t) \) can be computed using the tools from the following section, (7) provides a useful way to compute the privacy-utility function \( g_{r,s,p}^F(t) \). Fig. 4 depicts \( g_{r,s,p}^F(t) \) for \( r = 3/10, s = 10 \) and \( p = 50 \). Observe that, conditional on the existence of \( t^* \) (Conj. 1),

\[
g_t(\epsilon) = 0 \text{ if and only if } \epsilon \leq P_T(t^*)
\]

The privacy-utility trade-off for \( P_T \) can be handled similarly by replacing \( P_T(t) \leq \epsilon \) with \( P_T(t^*) \leq \epsilon \), as two highly correlated random variables posses a high mutual information but, at the same time, a small MMSE.

V. NUMERICAL COMPUTATION OF \( U \)

Throughout this section \( c \in (0, \infty) \), \( r \in (0, 1) \), and \( s > 0 \) are fixed. For \( t \geq 0 \), we let \( G_t \) be the Cauchy transform of \( F_t = F^{0,H_t} \), with \( H_t \) defined as in (6). By Thm. 1, for each \( z \in \mathbb{C}^+ \), \( G_t(z) \) is a solution to the equation

\[
G_t(z) = \frac{1 - r}{z - t(1 - c + czG_t)} + \frac{r}{z - (t + s)(1 - c + czG_t)}.
\]

Alternatively, \( G_t(z) \) is a root of the polynomial

\[
P_{t,z}(G) = A_{t,z} G^3 + B_{t,z} G^2 + C_{t,z} G + D_{t,z} \in \mathbb{C}[G],
\]

where \( A_{t,z} := t(t + s)^2 z^2, B_{t,z} := a_{t,z}(t + s)cz + b_{t,z}tcz, C_{t,z} := rtcz + (1 - r)(t + s)cz + a_{t,z}b_{t,z}, \) and \( D_{t,z} := ra_{t,z} + (1 - r)b_{t,z} a_{t,z} = (t(1-c)-z) \) and \( b_{t,z} = (t+s)(1-c)-z \). The following lemma provides a characterization of \( \text{Supp}(F_t) \).

Lemma 1. Let \( \Delta_t : \mathbb{R} \rightarrow \mathbb{R} \) be the (real) polynomial given by

\[
x \mapsto 18 A_{x,t} B_{x,t} C_{x,t} D_{x,t} - 4 B_{x,t}^2 D_{x,t} + B_{x,t}^2 C_{x,t} - 4 A_{x,t} C_{x,t}^2 - 27 A_{x,t}^2 D_{x,t}^2.
\]

Then, \( \text{Supp}(F_t) \) is the closure of \( \{x \in [0, \infty) : \Delta_t(x) < 0\} \).

The above lemma suggests a simple method to compute \( U(t) \): find the positive roots of \( \Delta_t \), identify where \( \Delta_t \) is positive and negative, and subtract the roots delimiting the gap of interest. This process is depicted in Fig. 5 where the support of \( F_t \) is represented by thick blue lines and the value of \( U(t) \) equals the third minus the second positive root of \( \Delta_t \).

REFERENCES

[1] C. Dwork, K. Kenthapadi, F. McSherry, I. Mironov, and M. Naor, Our Data, Ourselves: Privacy Via Distributed Noise Generation. Springer Heidelberg, 2006, pp. 486–503.

[2] R. J. Muirhead, Aspects of multivariate statistical theory. John Wiley & Sons, 2009, vol. 197.

[3] Z. Bai and J. W. Silverstein, Spectral analysis of large dimensional random matrices. Springer, 2010, vol. 20.

[4] K. Chaudhuri, A. D. Sarwate, and K. Sinha, “A near-optimal algorithm for differentially-private principal components,” JMLR, vol. 14, no. 1, pp. 2905–2943, 2013.

[5] L. Wei, A. D. Sarwate, J. Corander, A. Hero, and V. Tarokh, “Analysis of a privacy-preserving PCA algorithm using random matrix theory,” in Signal and Information Processing (GlobalSIP). IEEE, 2016, pp. 1335–1339.

[6] S. Asoodeh, M. Diaz, F. Alajaji, and T. Linder, “Information extraction under privacy constraints,” Information, vol. 7, no. 1, p. 15, 2016.

[7] L. Sankar, S. R. Rajagopalan, and H. V. Poor, “Utility-privacy tradeoffs in databases: An information-theoretic approach,” IEEE Transactions on Information Forensics and Security, vol. 8, no. 6, pp. 838–852, 2013.

[8] M. Diaz, S. Asoodeh, F. Alajaji, T. Linder, S. Belinschi, and J. Mingo, “On the noise-information separation of a private principal component analysis scheme,” To appear.

[9] J. W. Silverstein, “Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices,” Journal of Multivariate Analysis, vol. 55, no. 2, pp. 331–339, 1995.

[10] Z.-D. Bui and J. W. Silverstein, “No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices,” Annals of Probability, pp. 316–345, 1998.

[11] P. Biane, “On the free convolution with a semi-circular distribution,” Indiana University Mathematics Journal, pp. 705–718, 1997.

[12] T. Tao and V. Vu, “Random matrices: Universality of ESDs and the circular law,” The Ann. Probability, vol. 38, no. 5, pp. 2023–2065, 2010, with an appendix by M. Krishnapur.

[13] A. Nica and R. Speicher, Lectures on the combinatorics of free probability. Cambridge University Press, 2006, vol. 13.

[14] J. Mingo and R. Speicher, Free Probability and Random Matrices, ser. Fields Institute Monographs. Springer-Verlag New York, 2017, vol. 35.

[15] F. Benaych-Georges, “Rectangular random matrices, entropy, and Fisher’s information,” Journal of Operator Theory, pp. 371–419, 2009.

[16] F. P. Calmon, A. Makhdoumi, M. Médard, M. Varia, M. Christiansen, and K. R. Duffy, “Principal inertia components and applications,” IEEE Trans. Inf. Theory, vol. 63, no. 8, pp. 5011–5038, Aug 2017.

[17] M. S. Pinsker, V. V. Prelov, and S. Verdú, “Sensitivity of channel capacity,” IEEE Trans. Inf. Theory, vol. 41, no. 6, pp. 1877–1888, Nov 1995.

[18] S. Asoodeh, F. Alajaji, and T. Linder, “Privacy-aware MMSE estimation,” in IEEE International Symposium on Information Theory (ISIT), July 2016, pp. 1989–1993.

[19] D. Guo, S. Shamai, and S. Verdú, “Mutual information and minimum mean-square error in gaussian channels,” IEEE Trans. Inf. Theory, vol. 51, no. 4, pp. 1261–1282, April 2005.