Molecular state $N\Xi$ in the coupled-channel formalism

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The relativistic six-quark equations for the molecule $N\Xi$ are found in the dispersion relation technique. The relativistic six-quark amplitudes of the hexaquark including the quarks of three flavors ($u, d, s$) are calculated. The pole of these amplitudes determines the mass of $N\Xi$ state $M = 2252 \text{ MeV}$. The binding energy is equal to $3 \text{ MeV}$.

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I. INTRODUCTION.

The $H$-particle, $N\Omega$-state and $N\Omega$ may be strong interaction stable [1]. Jaffe studied the color-magnetic interaction of the one-gluon-exchange potential in the multiquark system and found the most attractive channel is the flavor singlet with quark content $u^2d^2s^2$. The same symmetry analysis of the chiral boson exchange potential leads to the similar result [2]. Up to now, these three interesting candidates of dibaryons are still not found or confirmed by experiments. It seems that one should go beyond these candidates and should search the possible candidates in a wider region, in terms of a more reliable model.

There were a number of theoretical predictions by using various models: the quark-delocalization model [3, 4], the flavor $SU(3)$ skyrmion model [5], the chiral $SU(3)$ quark model [6], the the quark cluster model [7, 8]. By employing the chiral $SU(3)$ quark model Zhang and Yu studied $\Omega\Omega$ and $\Sigma\Omega$ states [9, 10]. Lomon predicted a deuteron-like dibaryon resonance using R-matrix theory [11].

In our previous paper [12] the relativistic six-quark equations are found in the framework of coupled-channel formalism. The dynamical mixing between the subamplitudes of hexaquark is considered. The six-quark amplitudes of dibaryons are calculated. The poles of these amplitudes determine the masses of dibaryons. We calculated the contribution of six-quark subamplitudes to the hexaquark amplitudes.

II. SIX-QUARK AMPLITUDES OF MOLECULAR STATE $N\Xi$.

We derive the relativistic six-quark equations in the framework of the dispersion relation technique. We use only planar diagrams; the other diagrams due to the rules of $1/N_c$ expansion [13-15] are neglected.

We shall consider the derivation of the relativistic generalization of the Faddeev-Yakubovsky approach [16]. In our case the low-lying dibaryons are considered. We take into account the pairwise interaction of all six quarks in the hexaquark.

For instance, we consider the reduced amplitude $\alpha^{112^*}_{11}$ (Fig. 1). The system of graphical equations in Fig. 1. determines the subamplitudes using the self-consistent method [12].

The coefficients are determined by the permutation of quarks [17, 18]. We should use the coefficient multiplying of the diagrams in the graphical equation Fig. 1.

In Fig. 1 the first coefficient is equal to 4, that the number $4 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 5 and 6); the second coefficient is equal to 4, that the number $4 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4); the third coefficient is equal to 4, that the number $4 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 5 and 6); the fourth coefficient is equal to 8, that the number $8 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 2$ (permutation of particles 5 and 6); the fifth coefficient is equal to 8, that the number $8 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 2$ (permutation of particles 5 and 6); the sixth coefficient is equal to 8, that the number $8 = 2$ (permutation of particles 1 and 2) $\times 2$ (permutation of particles 3 and 4) $\times 2$ (permutation of particles 5 and 6).

The system of equations for the state $N\Xi$ is given in the Appendix A.
III. CALCULATION RESULTS.

The quark masses of model $m_{u,d} = 410 \text{ MeV}$ and $m_s = 557 \text{ MeV}$ coincide with the ordinary baryon ones in our model \cite{19, 20}. The model in question has only three parameters: the cutoff parameter $\Lambda = 11$ (similar to the three quark model) and the gluon coupling constants $g_0$ and $g_1$. These parameters are determined by the $\Lambda\Lambda$ and the di-$\Omega$ masses. We have considered the two type of calculations \cite{12}. In the first case we use the gluon coupling constants $g_0 = 0.653$ (diquark $0^+$) and $g_1 = 0.292$ (diquark $1^+$), which are fitted by the $\Lambda\Lambda$ state with the mass $2173 \text{ MeV}$ and the di-$\Omega$ with the mass $3232 \text{ MeV}$, respectively. In the second case the gluon coupling constants $g_0 = 0.647$ and $g_1 = 0.325$ are determined by the masses of $\Lambda\Lambda$ state with the $M = 2171 \text{ MeV}$ and the di-$\Omega$ state $M = 3093 \text{ MeV}$. The experimental data of these masses are absent, therefore we use the results of paper \cite{4}. In our model the correlation of gluon coupling constants $g_0$ and $g_1$ is similar to the $S$-wave baryon ones. It seems that the first case must prefer that the deuterium state is described more exact.

IV. CONCLUSIONS.

Jaffe considered the most attractive channel of dibaryons (strangeness $S = -2$, isospin $I = 0$, spin-parity $J^P = 0^+$). The molecular state $N\Xi$ ($2252 \text{ MeV}$) with the quantum numbers $SIJ = -2, 0, 0$ possesses the binding energy $B = 3 \text{ MeV}$. The binding energy of deuteron is equal to $B = 2.226 \pm 0.003 \text{ MeV}$. The calculated subamplitudes $A_i$ are given in the Table I. The other interesting states give rise to the following binding energy: $\Lambda\Lambda$ (the binding energy $B = 59 \text{ MeV}$) $SIJ = -2, 0, 0$; $N\Lambda$ (the binding energy $B = 32 \text{ MeV}$) $SIJ = -1, 
\frac{1}{2}, 1$; $N\Omega$ (the binding energy $B = 39 \text{ MeV}$) $SIJ = -3, 
\frac{1}{2}, 2$. The molecule $N\Xi$ will be able to obtain experimentally.

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Appendix A: The system of equations for the molecule $N\Xi$. 

\[ \alpha_1^{uu} = \lambda + 4 \alpha_1^{0,dd} I_1(1^{uu}0^{ud}) + 4 \alpha_1^{0,ss} I_1(1^{uu}0^{us}) + 8 \alpha_2^{0,dd0,us} I_2(1^{uu0,ud0,us}) , \]  

(\text{A1})

\[ \alpha_1^{dd} = \lambda + 4 \alpha_1^{0,dd} I_1(1^{dd}0^{ud}) + 4 \alpha_1^{0,ss} I_1(1^{dd}0^{ds}) + 8 \alpha_2^{0,dd0,ds} I_2(1^{dd}0^{ud0,ds}) , \]  

(\text{A2})

\[ \alpha_1^{ss} = \lambda + 4 \alpha_1^{0,ss} I_1(1^{ss}0^{us}) + 4 \alpha_1^{0,ds} I_1(1^{ss}0^{ds}) , \]  

(\text{A3})

\[ \alpha_1^{0,dd} = \lambda + \alpha_1^{1,ss} I_1(0^{dd}1^{uu}) + \alpha_1^{1,dd} I_1(0^{dd}3^{uu}) + 2 \alpha_1^{0,dd} I_1(0^{dd}0^{ud}) + 2 \alpha_1^{0,ss} I_1(0^{dd}0^{us}) + 2 \alpha_1^{0,ds} I_1(0^{dd}0^{ds}) + 2 \alpha_2^{0,dd0,ss} I_2(0^{dd}0^{ud0,us}) + 2 \alpha_2^{0,dd0,ds} I_2(0^{dd}0^{ud0,ds}) , \]  

(\text{A4})

\[ \alpha_1^{0,ss} = \lambda + \alpha_1^{1,ss} I_1(0^{ss}1^{uu}) + \alpha_1^{1,ss} I_1(0^{ss}3^{uu}) + 2 \alpha_1^{0,ss} I_1(0^{ss}0^{us}) + 2 \alpha_1^{0,ss} I_1(0^{ss}0^{ds}) + \alpha_2^{0,ss0,ss} I_2(0^{ss}0^{us0,us}) + 2 \alpha_2^{0,ss0,ds} I_2(0^{ss}0^{ud0,ds}) , \]  

(\text{A5})

\[ \alpha_1^{0,ds} = \lambda + \alpha_1^{1,dd} I_1(0^{ds}1^{dd}) + \alpha_1^{1,ss} I_1(0^{ds}3^{uu}) + 2 \alpha_1^{0,dd} I_1(0^{ds}0^{ud}) + 2 \alpha_1^{0,ss} I_1(0^{ds}0^{us}) + 2 \alpha_1^{0,ds} I_1(0^{ds}0^{ds}) + 2 \alpha_2^{0,ss0,ds} I_2(0^{ds}0^{us0,ds}) + 2 \alpha_2^{0,dd0,ds} I_2(0^{ds}0^{ud0,ds}) , \]  

(\text{A6})

\[ \alpha_2^{1,ss} = \lambda + 4 \alpha_1^{0,dd} I_4(1^{uu1,ss}0^{ud}) + 4 \alpha_1^{0,ss} I_3(1^{uu1,ss}0^{us}) + 4 \alpha_1^{0,ds} I_4(1^{uu1,ss}0^{ds}) + 8 \alpha_2^{0,dd0,ss} I_7(1^{ss1,uu}0^{us0,us}) + 8 \alpha_2^{0,dd0,ds} I_6(1^{ss1,uu}0^{ud0,us}) + 8 \alpha_3^{0,dd0,0,ds} I_8(1^{uu1,ss}0^{ud0,us0,ds}) , \]  

(\text{A7})

\[ \alpha_2^{1,dd} = \lambda + 4 \alpha_1^{0,dd} I_4(1^{dd1,ss}0^{ud}) + 4 \alpha_1^{0,ss} I_4(1^{dd1,ss}0^{us}) + 4 \alpha_1^{0,ds} I_3(1^{dd1,ss}0^{ds}) + 8 \alpha_2^{0,dd0,ss} I_7(1^{ss1,dd}0^{ud0,us}) + 8 \alpha_2^{0,dd0,ds} I_6(1^{ss1,dd}0^{ud0,ds}) + 8 \alpha_3^{0,dd0,0,ds} I_8(1^{dd1,ss}0^{ud0,us0,ds}) , \]  

(\text{A8})

\[ \alpha_2^{0,ss} = \lambda + \alpha_1^{1,ss} I_4(0^{ds1,uu}1^{ss}) + 2 \alpha_1^{0,dd} I_3(1^{uu0,0,ds}0^{ud}) + 4 \alpha_1^{0,ss} I_4(1^{uu0,0,ds}0^{us}) + 8 \alpha_2^{0,dd0,ss} I_7(1^{uu0,0,ds}0^{ud0,us}) + 8 \alpha_2^{0,dd0,ds} I_6(1^{uu0,0,ds}0^{ud0,ds}) + 8 \alpha_3^{0,dd0,0,ds} I_8(1^{uu0,0,ds}0^{ud0,us0,ds}) , \]  

(\text{A9})

\[ \alpha_2^{1,dd} = \lambda + \alpha_1^{1,ss} I_4(0^{ds1,uu}1^{dd1,ss}) + 2 \alpha_1^{0,dd} I_3(1^{dd0,0,us}0^{ud}) + 4 \alpha_1^{0,ss} I_4(1^{dd0,0,us}0^{us}) + 8 \alpha_2^{0,dd0,ss} I_7(1^{uu0,0,ds}0^{ud0,us}) + 8 \alpha_2^{0,dd0,ds} I_6(1^{uu0,0,ds}0^{ud0,ds}) + 8 \alpha_3^{0,dd0,0,ds} I_8(1^{uu0,0,ds}0^{ud0,us0,ds}) , \]  

(\text{A10})
\[ \begin{align*}
\alpha_2^{ss0d} &= \lambda + \alpha_1^{uu} I_4(0u^d0u^d0u^d0u^d) + \alpha_2^{0d0d} I_5(0u^d0u^d0u^d0u^d) + 2 \alpha_2^{0d0d} I_7(0u^d0u^d0u^d0u^d) \\
&+ 2 \alpha_3^{0d0d0d0d} I_8(0u^d0u^d0u^d0u^d), \quad (A10)
\end{align*} \]

\[ \begin{align*}
\alpha_2^{ss0d} &= \lambda + \alpha_1^{uu} I_4(0u^d0u^d0u^d0u^d) + \alpha_2^{0d0d} I_5(0u^d0u^d0u^d0u^d) + 2 \alpha_2^{0d0d} I_7(0u^d0u^d0u^d0u^d) \\
&+ 4 \alpha_3^{0d0d0d0d} I_8(0u^d0u^d0u^d0u^d), \quad (A11)
\end{align*} \]

\[ \begin{align*}
\alpha_3^{ss0d} &= \lambda + 4 \alpha_1^{uu} I_9(1u^d1u^d1u^d1u^d) + 4 \alpha_2^{0d0d} I_9(1u^d1u^d1u^d1u^d) + 4 \alpha_3^{0d0d} I_9(1u^d1u^d1u^d1u^d) \\
&+ 8 \alpha_3^{0d0d0d0d} I_{10}(1u^d1u^d1u^d1u^d), \quad (A12)
\end{align*} \]

\[ \begin{align*}
\alpha_3^{ss0d} &= \lambda + 4 \alpha_1^{uu} I_9(1u^d1u^d1u^d1u^d) + 4 \alpha_2^{0d0d} I_9(1u^d1u^d1u^d1u^d) + 4 \alpha_3^{0d0d} I_9(1u^d1u^d1u^d1u^d) \\
&+ 8 \alpha_3^{0d0d0d0d} I_{10}(1u^d1u^d1u^d1u^d), \quad (A13)
\end{align*} \]

\[ \begin{align*}
\alpha_3^{ss0d0d} &= \lambda + \alpha_1^{uu} I_9(0u^d0u^d0u^d0u^d) + \alpha_1^{uu} I_9(0u^d0u^d0u^d0u^d) + \alpha_3^{0d0d} I_9(0u^d0u^d0u^d0u^d) + 8 \alpha_3^{0d0d0d0d} I_{10}(0u^d0u^d0u^d0u^d), \quad (A14)
\end{align*} \]
We used the functions $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}$ [12]:

\[
I_1(ij) = \frac{B_j(s_{13}^{(2)})}{B_i(s_{02}^{(2)})} \int_{(m_1 + m_2)^2} \frac{ds_{12} G_1^2(s_{12}^2) \rho_i(s_{12}^{'2})}{\pi} \left[ \frac{1}{s_{12}^{'2} - s_{01}^2} \int_{-1}^{+1} dz_{12} \right]^{1/2} \frac{1}{1 - B_j(s_{13}^{(2)})},
\]

\[
I_2(ikl) = \frac{B_j(s_{13}^{(2)}) B_k(s_{24}^{(2)})}{B_i(s_{02}^{(2)}) B_j(s_{02}^{(2)})} \int_{(m_1 + m_2)^2} \frac{ds_{12} G_1^2(s_{12}^2) \rho_i(s_{12}^{'2})}{\pi} \left[ \frac{1}{s_{12}^{'2} - s_{01}^2} \int_{-1}^{+1} dz_{12} \right]^{1/2} \frac{1}{2} \left[ \int_{-1}^{+1} dz_{29} \right]^{1/2}
\times \frac{z_3^{(2)^{+}}}{z_3^{(2)^{-}}} \int_{z_3^{(2)^{-}}}^{1} \frac{1}{\sqrt{1 - z_3^{(2)^{-}} - z_3^{(2)^{+}}}} \times \frac{1}{1 - B_j(s_{13}^{(2)}) - B_k(s_{24}^{(2)})},
\]

\[
I_3(ijk) = \frac{B_k(s_{23}^{(2)})}{B_i(s_{02}^{(2)}) B_j(s_{02}^{(2)})} \int_{(m_1 + m_2)^2} \frac{ds_{12} G_1^2(s_{12}^2) \rho_i(s_{12}^{'2})}{\pi} \left[ \frac{1}{s_{12}^{'2} - s_{01}^2} \int_{-1}^{+1} dz_{12} \right]^{1/2} \frac{1}{2} \left[ \int_{-1}^{+1} dz_{23} \right]^{1/2}
\times \frac{z_3^{(2)^{+}}}{z_3^{(2)^{-}}} \int_{z_3^{(2)^{-}}}^{1} \frac{1}{\sqrt{1 - z_3^{(2)^{-}} - z_3^{(2)^{+}}}} \times \frac{1}{1 - B_k(s_{23}^{(2)})},
\]

\[
I_4(ijk) = I_1(ik),
\]

\[
I_5(ijk) = I_2(ikl),
\]

\[
I_6(ijk) = I_1(ik) \cdot I_1(1j),
\]

\[
I_7(ijk) = \frac{B_k(s_{02}^{(2)}) B_i(s_{02}^{(2)})}{B_i(s_{02}^{(2)}) B_j(s_{02}^{(2)})} \int_{(m_1 + m_2)^2} \frac{ds_{12} G_1^2(s_{12}^2) \rho_i(s_{12}^{'2})}{\pi} \left[ \frac{1}{s_{12}^{'2} - s_{01}^2} \int_{-1}^{+1} dz_{12} \right]^{1/2} \frac{1}{2} \left[ \int_{-1}^{+1} dz_{23} \right]^{1/2}
\times \frac{z_3^{(2)^{+}}}{z_3^{(2)^{-}}} \int_{z_3^{(2)^{-}}}^{1} \frac{1}{\sqrt{1 - z_3^{(2)^{-}} - z_3^{(2)^{+}}}} \times \frac{1}{1 - B_k(s_{23}^{(2)})},
\]

\[
I_8(ijk) = I_3(ijk) + I_7(ijk).
\]
\[
I_{8}(ijklm) = \frac{B_{l}(s_{0}^{15})B_{m}(s_{0}^{46})}{B_{l}(s_{0}^{12})B_{j}(s_{0}^{45})} \left( \frac{m_{3} + m_{4}}{{\Lambda}_{1}} \right)^{2} \int \frac{ds_{34}^{i} G_{j}^{2}(s_{0}^{34}) \rho_{j}(s_{34}^{i})}{s_{34}^{i} - s_{34}^{i}} \frac{1}{2} \int \frac{dz_{1}(7)}{2} \int \frac{dz_{2}(7)}{2} \int \frac{dz_{3}(7)}{2} \int \frac{dz_{4}(7)}{2} \int \frac{1}{\sqrt{1 - z_{4}^{(7)}(7) - z_{3}^{(7)}(7) - z_{2}^{(7)}(7) + 2z_{1}(7)z_{3}(7)z_{4}(7)}} \\
\times \frac{1}{1 - B_{k}(s_{23}) - B_{l}(s_{45})}.
\]

\[
I_{9}(ijkl) = I_{3}(ijl),
\]

\[
I_{10}(ijklm) = \frac{B_{l}(s_{0}^{23})B_{m}(s_{0}^{45})}{B_{l}(s_{0}^{12})B_{j}(s_{0}^{34})B_{k}(s_{0}^{46})} \left( \frac{m_{3} + m_{4}}{{\Lambda}_{1}} \right)^{2} \int \frac{ds_{ij2}^{i} G_{j}^{2}(s_{ij}^{2}) \rho_{j}(s_{ij}^{2})}{s_{ij}^{2} - s_{ij}^{2}} \frac{1}{2} \int \frac{dz_{1}(10)}{2} \int \frac{dz_{2}(10)}{2} \int \frac{dz_{3}(10)}{2} \int \frac{dz_{4}(10)}{2} \int \frac{dz_{5}(10)}{2} \int \frac{1}{\sqrt{1 - z_{5}^{(10)}(10) - z_{4}^{(10)}(10) - z_{3}^{(10)}(10) + 2z_{1}(10)z_{4}(10)z_{5}(10)}} \\
\times \frac{1}{1 - B_{l}(s_{23}) - B_{m}(s_{45})}.
\]
\[
\begin{align*}
\alpha_2^{1u_10^{u_2}} &= \lambda + 4 \alpha_1^{0u_2} \, I_4(1u_10^{u_2}) + 4 \alpha_1^{0u_2} \, I_3(1u_10^{u_2}) \\
&+ 4 \alpha_1^{0u_2} \, I_4(1s_1u_0^{d_2}) + 8 \alpha_2^{0v_0} \, I_7(1s_1u_0^{d_2}) \\
&+ 8 \alpha_2^{0v_0} \, I_6(1s_1u_0^{d_2}) + 8 \alpha_3^{0v_0} \, I_6(1u_10^{u_2}0^{u_2}0^{d_2})
\end{align*}
\]

Fig. 1. The graphical equations of the reduced amplitude $\alpha_2^{1u_10^{u_2}}$.\cite{A7}.
TABLE I: *N* $\Xi$ (2252*MeV*) (*S1J* = −200). Parameters of model: cutoff $\Lambda = 11.0$, gluon coupling constants $g_0 = 0.653$ and $g_1 = 0.292$. Quark masses $m_{u,d} = 410 MeV$ and $m_s = 557 MeV$.

| Subamplitudes | Contributions, percent |
|---------------|------------------------|
| $A_1^{uu}$    | 1.96                   |
| $A_1^{dd}$    | 1.96                   |
| $A_1^{ss}$    | 2.56                   |
| $A_1^{ud}$    | 4.33                   |
| $A_1^{us}$    | 5.30                   |
| $A_1^{ds}$    | 5.30                   |
| $A_1^{dsu}$   | 7.59                   |
| $A_2^{ud}$    | 7.59                   |
| $A_2^{us}$    | 4.93                   |
| $A_2^{ds}$    | 4.93                   |
| $A_2^{dssu}$  | 12.95                  |
| $A_2^{uu}$    | 12.95                  |
| $A_2^{dd}$    | 11.87                  |
| $A_2^{ss}$    | 11.87                  |
| $A_3^{ud}$    | 4.10                   |
| $A_3^{us}$    | 12.75                  |

$\sum A_1 = 21.41$

$\sum A_2 = 61.73$

$\sum A_3 = 16.86$