Control of the entanglement of two atoms in an optical cavity via white noise

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Abstract. We employ external optical white-noise fields as the actual driving fields of two atoms inside an optical cavity and investigate how controllable entanglement between two atoms arises in such a situation. Two different action configurations of noise are considered. If two atoms are simultaneously driven by two independent white-noise fields with the same intensity, the entanglement between them is suppressed and eventually completely destroyed by the noise. However, if only one atom is exposed in the white-noise fields, the steady state of the two atoms exhibits entanglement. A stochastic-resonance-like behaviour of steady state entanglement is also revealed. Finally, we examine the Bell violation between atoms and show that the steady state does not violate the Bell inequality even though it is inseparable. The stochastic-resonance-like behaviour cannot be observed in the Bell violation of two atoms during the evolution. The stronger the noise intensity, the more rapidly the Bell violation disappears.
1. Introduction

Quantum entanglement plays a crucial role in quantum information and quantum computation [1]. Entanglement can exhibit the nature of a nonlocal correlation between quantum systems that have no classical interpretation. However, real quantum systems will unavoidably be influenced by surrounding environments. The interaction between the environment and quantum systems of interest can lead to decoherence. It is therefore of great importance to prevent or minimize the influence of environmental noise in the practical realization of quantum information processing. In order to prevent the effect of decoherence, several approaches have been proposed, such as the quantum error-correcting approach [2] and the quantum error-avoiding approach [3]–[5]. Instead of attempting to shield the system from the environmental noise, Plenio and Huelge used white noise to play a constructive role and generate controllable entanglement by incoherent sources. It was found that the noise-assisted entanglement could exhibit the stochastic resonance behaviour [6]. Similar work on this aspect has also been considered by other authors [7]–[11]. A scheme for creating distant entangled atomic states has been proposed, in which two (or more) atoms were driven by a weak laser pulse and the subsequent spontaneous emission was detected [7]. Plenio et al have presented a scheme for preparing the entangled state of two atoms in a lossy cavity by continuously monitoring the leak photon out of the cavity [8], and pointed out that the absence of photon counts was associated with the presence of an atomic entangled state. Beige et al have discussed ways in which entanglement could be established within a dissipative environment [9] and have shown that one could even make use of a strong interaction of the system with its environment to produce entanglement in a controlled way. Schneider and Milburn have also studied the pairwise entanglement in the steady state of the Dicke model and revealed how the steady state of the ion trap with all ions driven simultaneously and coupled collectively to a heat bath could exhibit quantum entanglement [10]. Kim et al have investigated the interaction of the thermal field and a quantum system composed of two qubits and found that such a chaotic field with minimal information could entangle qubits that were prepared initially in a separable state [11]. By repeated measurements of the phase of light transmitted through the cavity, Sørensen and Mølmer proposed a method to prepare entangled states of atoms in optical cavities [12] and showed that this entanglement could be used to implement gates on qubits which are stored in different internal degrees of freedom of the atoms. Another similar way to produce entanglement by measuring the reflection from an optical cavity has also been presented [13], in which the preparation of maximally entangled states of atoms in the cavity was conditional on the
detection of a reflected photon. Recently, Kraus and Cirac have shown how one could entangle distant atoms by using squeezed light [14]. Entanglement was obtained in the steady state, and could be increased by manipulating the atoms locally. For generating multipartite entanglement, Duan and Kimble have proposed an efficient scheme to engineer multi-atom entanglement by detecting cavity decay through single-photon detectors [15].

In this paper, we study the quantum system in which two two-level atoms within a leaky optical cavity are driven by the external optical white-noise fields. We investigate how entanglement between two atoms arises in such a situation. It is shown that white noise exhibits dual aspects, i.e., playing either a destructive or a constructive role in quantum information processing. Recently, Clark and Parkins [16] proposed a scheme to controllably entangle the internal states of two atoms trapped in a high-finesse optical cavity by employing quantum-reservoir engineering. By making use of laser and cavity fields to drive two separate Raman transitions between stable atomic ground states, the two atoms are effectively coupled to a squeezed reservoir. Phase-sensitive reservoir correlations lead to entanglement between the atoms. As distinct from their scheme, we will focus here on the problem of generating entanglement when only incoherent sources are available and show that controllable entanglement can arise in this situation. We show that, if two atoms are simultaneously driven by two independent white-noise fields with the same intensity, the entanglement between them is suppressed and eventually completely destroyed by the noise. However, in another case, in which only one atom is exposed in white-noise field, the steady-state entanglement of the two atoms is a non-monotonic function of both the intensity of noise driving the field and the spontaneous decay rate. A double resonance behaviour emerges. Moreover, the threshold value of the spontaneous decay rate, below which there is no steady-state entanglement, increases with the intensity of the noise field. These results may provide us with a possible way to control the entanglement of two atoms inside the cavity by adjusting the intensity of the external white-noise fields.

This paper is organized as follows: in section 2, we study two atoms trapped in an optical cavity and driven by two independent white-noise fields with the same intensity. We model this system by a master equation and give an explicit analytical solution of the time evolution density matrix. Based on the density matrix, we obtain the analytical expression of the concurrence characterizing the entanglement between two atoms. It is shown that entanglement between them is suppressed and eventually completely destroyed by the noise. In section 3, we consider the situation in which only one of the atoms is driven by the external white-noise field. Both the entanglement during the time evolution and the steady-state entanglement are investigated. A stochastic-resonance-like behaviour of entanglement is revealed. In section 4, we examine the Bell violation of two atoms, and show that the steady state does not violate Bell inequality even though it is inseparable. Moreover, the stochastic-resonance-like behaviour cannot be observed in the Bell violation of two atoms during the evolution, and the stronger the noise intensity, the more rapidly the Bell violation disappears. A conclusion is given in section 5.

2. The master equation describing two atoms trapped in an optical cavity and driven by white-noise fields

The system we consider here is two atoms trapped in a optical cavity as depicted in figure 1. The atoms are driven by two independent thermal fields and separated by a large enough distance that they feel no direct dipole–dipole interaction. The cavity has a field decay $k$ and a frequency
Figure 1. The experimental setup: two atoms are trapped in an optical cavity. The atoms are driven by two independent thermal fields and separated by a large enough distance that they feel no direct dipole–dipole interaction. The cavity has a field decay $k$ and a frequency $\omega$. The two eigenstates of the individual atom ($|0\rangle$, $|1\rangle$) constitute the qubit states.

$\omega$. The two eigenstates of the individual atom ($|0\rangle$, $|1\rangle$) constitute the qubit states. The master equation for the total system density operator is ($\hbar = 1$)

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{L}_{\text{cav}}\rho + \mathcal{L}_{\text{at}}\rho,$$

(1)

where

$$H = \omega a^\dagger a + \frac{\omega_0}{2} \sum_{j=1}^{2} (|1\rangle_{jj}\langle 1| - |0\rangle_{jj}\langle 0|) + g \sum_{j=1}^{2} (a^\dagger |0\rangle_{jj}\langle 1| + a|1\rangle_{jj}\langle 0|),$$

(2)

where $a$ and $a^\dagger$ are the annihilation and creation operators of the cavity field with frequency $\omega$, and $\omega_0$ is the transition frequency of the atoms and $g$ is the atom–cavity coupling constant. The Liouvilleans $\mathcal{L}_{\text{cav}}\rho$ and $\mathcal{L}_{\text{at}}\rho$ are given by

$$\mathcal{L}_{\text{cav}}\rho = \kappa (2a^\dagger a^\dagger a^\dagger - a^\dagger a^\dagger a^\dagger a)$$

(3)

and

$$\mathcal{L}_{\text{at}}\rho = \sum_{j=1}^{2} \left( n_{T}^{(j)} + 1 \right) \Gamma^{(j)}(2|0\rangle_{jj}\langle 1|\rho|1\rangle_{jj}\langle 0|) - |1\rangle_{jj}\langle 1|\rho - \rho|1\rangle_{jj}\langle 1|) + n_{T}^{(j)} \Gamma^{(j)}(2|1\rangle_{jj}\langle 0|\rho|0\rangle_{jj}\langle 1| - |0\rangle_{jj}\langle 0|\rho - \rho|0\rangle_{jj}\langle 0|),$$

(4)

where $\Gamma^{(j)}$ describes the coupling strength of the $j$th atom to the external fields and $n_{T}^{(j)} \Gamma^{(j)}$ is the transition rate due to the thermal field. The spectral width of the thermal field is large compared to the linewidth of the atomic transition so that its effect is that of a white-noise source. Here, $n_{T}^{(j)}$ can be interpreted as an effective photon number and that spontaneous decay of the atom out of the cavities is included in this scenario via the $n_{T}^{(j)} + 1$ term.
In the large detuning limit, i.e., $\Delta = \omega_0 - \omega \gg g\sqrt{n + 1}$ with $n$ being the mean photon number of the cavity field, there is no energy exchange between the atomic system and the cavity. We can obtain the effective Hamiltonian $H_e$ [17, 18]

$$H_e = \frac{g^2}{\Delta} \left[ \sum_{j=1}^{2} (|1\rangle_{ji} \langle 1| aa^\dagger - |0\rangle_{ji} \langle 0|a^\dagger a) + |1\rangle_{11} \langle 0| \otimes |0\rangle_{22} \langle 1| + |0\rangle_{11} \langle 1| \otimes |1\rangle_{22} \langle 0| \right].$$  \hfill (5)

The first and second terms describe the photon-number-dependent Stark shifts, and the third term describes the dipole coupling between the first and second atoms induced by the virtual photon process. When the cavity mode is initially in the vacuum state $|0\rangle$, it will remain in the vacuum state throughout the procedure; the effective Hamiltonian reduces to

$$\tilde{H}_e = \frac{g^2}{\Delta} (|1\rangle_{11} \langle 1| + |1\rangle_{22} \langle 1| + |1\rangle_{11} \langle 0| \otimes |0\rangle_{22} \langle 1| + |0\rangle_{11} \langle 1| \otimes |1\rangle_{22} \langle 0|).$$ \hfill (6)

As the cavity mode will then never be populated, we can disregard it in the following. Now, the master equation (1) can be reduced to

$$\frac{d\rho_s}{dt} = -i\frac{g^2}{\Delta} (|1\rangle_{11} \langle 1| + |1\rangle_{22} \langle 1| + |1\rangle_{11} \langle 0| \otimes |0\rangle_{22} \langle 1| + |0\rangle_{11} \langle 1| \otimes |1\rangle_{22} \langle 0|)\rho_s$$

$$+ i\frac{g^2}{\Delta} \rho_s (|1\rangle_{11} \langle 1| + |1\rangle_{22} \langle 1| + |1\rangle_{11} \langle 0| \otimes |0\rangle_{22} \langle 1| + |0\rangle_{11} \langle 1| \otimes |1\rangle_{22} \langle 0|)$$

$$+ \sum_{j=1}^{2} (n_T^{(j)} + 1) \Gamma^{(j)} (2|0\rangle_{ji} \langle 1|\rho_s |1\rangle_{ji} \langle 0| - |1\rangle_{ji} \langle 1|\rho_s - \rho_s |1\rangle_{ji} \langle 1|)$$

$$+ \sum_{j=1}^{2} n_T^{(j)} \Gamma^{(j)} (2|1\rangle_{ji} \langle 0|\rho_s |0\rangle_{ji} \langle 1| - |0\rangle_{ji} \langle 0|\rho_s - \rho_s |0\rangle_{ji} \langle 0|),$$ \hfill (7)

where $\rho_s$ is the density matrix describing the subsystem containing only two atoms. Firstly, we discuss the case with $n_T^{(1)} = n_T^{(2)} = n_T$ and $\Gamma^{(1)} = \Gamma^{(2)} = \Gamma$, i.e., two atoms are driven by two independent thermal fields with the same intensity. We assume that the atom 1 and atom 2 are initially in the pure product state $|1\rangle_1 \otimes |0\rangle_2$. Then, the explicit analytical solution of the master equation (7) can be obtained as follows:

$$\rho_s(t) = \rho_{11}(t) |1\rangle_{11} \langle 1| \otimes |1\rangle_{22} \langle 1| + \rho_{22}(t) |1\rangle_{11} \langle 1| \otimes |0\rangle_{22} \langle 0| + \rho_{33}(t) |0\rangle_{11} \langle 0| \otimes |1\rangle_{22} \langle 1|$$

$$+ \rho_{44}(t) |0\rangle_{11} \langle 0| \otimes |0\rangle_{22} \langle 0| + \rho_{23}(t) |1\rangle_{11} \langle 0| \otimes |0\rangle_{22} \langle 1| + \rho_{32}(t) |0\rangle_{11} \langle 1| \otimes |1\rangle_{22} \langle 0|$$ \hfill (8)
In order to quantify the degree of entanglement, we adopt the concurrence \( C \) defined by Wooters [19]. The concurrence varies from \( C = 0 \) for an unentangled state to \( C = 1 \) for a maximally entangled state. For two qubits, in the ‘Standard’ eigenbasis: \( |1\rangle \equiv |11\rangle \), \( |2\rangle \equiv |10\rangle \), \( |3\rangle \equiv |01\rangle \), \( |4\rangle \equiv |00\rangle \), the concurrence may be calculated explicitly from the following:

\[
C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0),
\]

where the \( \lambda_i \) \((i = 1, 2, 3, 4)\) are the square roots of the eigenvalues in decreasing order of magnitude of the ‘spin-flipped’ density matrix operator \( R = \rho_s(\sigma^y \otimes \sigma^y)\rho_s^*(\sigma^y \otimes \sigma^y) \), where the asterisk indicates complex conjugation.

It is straightforward to compute analytically the concurrence for the density matrix \( \rho_s(t) \), and the concurrence \( C_s(t) \) related to the density matrix \( \rho_s(t) \) is obtained as follows:

\[
C_s(t) = 2 \max(0, |\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}),
\]

where \( |x| \) gives the absolute value of \( x \). In figure 2, we have plotted the concurrence \( C_s(t) \) as a function of the time \( t \) and the intensity of the thermal field \( n_T \). For \( n_T = 0 \), we may observe that the concurrence exhibits a damped oscillation. The spontaneous emission destroys the entanglement of the two atoms in this case. It is also shown that the entanglement between two atoms decreases with \( n_T \), and there is no entanglement arising between two atoms during the time evolution when \( n_T \) is beyond a threshold value depending on the coupling constant \( \Gamma \) and effective Rabi frequency \( g^2/\Delta \). For a deep understanding of the influence of white noise on the entanglement, we need to clarify the different features of the entanglement in the presence of white noise or only in the presence of spontaneous emission. In figure 3, the concurrence \( C_s(t) \) is plotted as a function of time \( t \) and the coupling constant \( \Gamma \) of the atoms and the external field (note that \( \Gamma \) is equivalent to the spontaneous emission rate if \( n_T = 0 \)) for two different values of the effective photon number \( n_T \) of the thermal field. In the case with \( n_T = 0 \) (see figure 3(a)), the entanglement of two atoms always arises during the time evolution even in the presence of atomic spontaneous emission. However, in the case with \( n_T = 0.3 \) (see figure 3(b)), a threshold value of \( \Gamma \) is found, beyond which there is no entanglement arising during time evolution. The above discussion
Figure 2. The concurrence $C_s(t)$ depicted as a function of the time $t$ and the intensity of the thermal field $n_T$ with $\Gamma = g^2/20\Delta$. In this figure, the unit of the time $t$ is $\Delta/5g^2$.

Figure 3. The concurrence $C_s(t)$ plotted as the function of time $t$ and the coupling constant $\Gamma$ of the atoms and the external field with two different values of effective photon number $n_T$ of the thermal field, (a) $n_T = 0$; and (b) $n_T = 0.3$. In this figure, the unit of the time $t$ and $\Gamma$ is $\Delta/5g^2$ and $5g^2/\Delta$, respectively.
Figure 4. The experimental setup: two atoms are trapped in an optical cavity. Atom 1 is driven by a white-noise field. Atom 2 is not driven by external noise, but its spontaneous emission is involved.

indicates that the two equal intensity-independent white-noise fields suppress the entanglement generation. But that is not the full situation concerning the role played by the white-noise field in the entanglement of two atoms. In the following section, we consider the situation in which only one of the atoms is driven by the external white-noise field. A different aspect of the white-noise field will be found.

3. The steady state entanglement of two atoms

In the above section, we have discussed the case in which the two atoms are simultaneously driven by two independent external white-noise fields with the same intensity; there is no steady state entanglement between the two atoms in that situation. In this section, we consider the situation depicted in figure 4, in which only one of the atoms is driven by the external white-noise field. The master equation is given by

$$\frac{d\rho_s}{dt} = -i\frac{g^2}{\Delta} (|1\rangle_{11}\langle 1| + |1\rangle_{22}\langle 1| + |1\rangle_{11}\langle 0| \otimes |0\rangle_{22}\langle 1| + |0\rangle_{11}\langle 1| \otimes |1\rangle_{22}\langle 0|)\rho_s$$

$$+ i\frac{g^2}{\Delta} \rho_s (|1\rangle_{11}\langle 1| + |1\rangle_{22}\langle 1| + |1\rangle_{11}\langle 0| \otimes |0\rangle_{22}\langle 1| + |0\rangle_{11}\langle 1| \otimes |1\rangle_{22}\langle 0|)$$

$$+ (n_T + 1)\Gamma (2|1\rangle_{11}\langle 1|\rho_s|1\rangle_{11}\langle 0| - |1\rangle_{11}\langle 1|\rho_s - \rho_s|1\rangle_{11}\langle 1|)$$

$$+ n_T\Gamma (2|1\rangle_{11}\langle 0|\rho_s|0\rangle_{11}\langle 1| - |0\rangle_{11}\langle 0|\rho_s - \rho_s|0\rangle_{11}\langle 0|)$$

$$+ \eta (2|0\rangle_{22}\langle 1|\rho_s|1\rangle_{22}\langle 0| - |1\rangle_{22}\langle 1|\rho_s - \rho_s|1\rangle_{22}\langle 1|),$$

(12)

where $\eta$ is the spontaneous emission rate of the atom 2. We assume that the two atoms are initially in the ground state $|0\rangle_1 \otimes |0\rangle_2$. The explicit analytical solution of the steady state of the master equation (12) can be obtained as follows:

$$\rho_{st} = \rho_{11}^s|1\rangle_{11}\langle 1| \otimes |0\rangle_{22}\langle 0| + \rho_{22}^s|1\rangle_{22}\langle 1| \otimes |0\rangle_{11}\langle 0| \otimes |1\rangle_{22}\langle 0|$$

$$+ \rho_{44}^s|0\rangle_{11}\langle 0| \otimes |0\rangle_{22}\langle 0| + \rho_{23}^s|1\rangle_{11}\langle 0| \otimes |0\rangle_{22}\langle 1| + \rho_{32}^s|0\rangle_{11}\langle 1| \otimes |1\rangle_{22}\langle 0|.$$

(13)
where

\[
\rho_{11}^i = \frac{\Omega^2 \Gamma^2 n_T^2}{(\Gamma + \eta + 2n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)},
\]

\[
\rho_{22}^i = \frac{n_T \Gamma[\eta(\Gamma + \eta + 2n_T \Gamma)^2 + \Omega^2(\Gamma + \eta + n_T \Gamma)]}{(\Gamma + \eta + 2n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)},
\]

\[
\rho_{33}^i = \frac{\Omega^2 n_T \Gamma(\Gamma + \eta + n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)}{(\Gamma + \eta + 2n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)},
\]

\[
\rho_{44}^i = \frac{\Gamma \eta(1 + n_T)(\Gamma + \eta + 2n_T \Gamma)^2 + \Omega^2(\Gamma + \eta + n_T \Gamma)^2}{(\Gamma + \eta + 2n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)},
\]

\[
\rho_{23}^i = \frac{-i n_T \Omega \Gamma \eta}{(\Gamma + \eta + 2n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)},
\]

\[
\rho_{32}^i = \frac{-i n_T \Omega \Gamma \eta}{(\Gamma + \eta + 2n_T \Gamma)^2(\Omega^2 + \Gamma \eta + 2n_T \Gamma \eta)},
\]

(14)

where $\Omega = g^2 / \Delta$. The concurrence $C_{st}$ related to the steady state $\rho_{st}$ is obtained as follows:

\[
C_{st} = 2 \max\{0, |\rho_{23}^i| - \sqrt{\rho_{11}^i \rho_{44}^i}\}.
\]

(15)

To date, most works concerning the environment-induced entanglement [9, 11] or noise-assisted entanglement [6, 20] have shown that, in certain specific situations, the environment or the external noise might play a constructive role at the entanglement. Here, we show that not only the external noise driving the atom 1 but also the spontaneous emission of atom 2 may improve the steady-state entanglement. (Note: the two atoms discussed here may have different spontaneous emission rates.) In figure 5, we have plotted the concurrence $C_{st}$ as a function of the spontaneous emission rate $\eta$ and the intensity of the thermal field $n_T$ with $\Gamma = \frac{1}{2} \Omega$. It is shown that the steady-state entanglement exhibits a double stochastic-resonance-like behaviour, which is similar to the results in [6]. If there is no external noise, i.e., $n_T = 0$, the steady state is separable. For weak noise intensity, the steady-state entanglement increases from zero to a maximal value with the increase of spontaneous emission rate $\eta$ and then decreases with $\eta$. Corresponding to every value of the noise intensity, there is a threshold value of the spontaneous emission rate $\eta$ below which the steady state is separable. Similarly, with a fixed moderate value of $\eta$, the steady-state entanglement achieves its maximal value at a moderate value of the external noise intensity. The double stochastic-resonance-like behaviour also emerges in figure 6, in which $C_{st}$ is depicted as a function of the spontaneous emission rate $\eta$ and the coupling constant $\Gamma$ with $n_T = 2$. In figure 7, we plot $C_{st}$ as a function of $\Omega$ and the intensity of the thermal field $n_T$ with $\eta = 5 \Gamma$. We find that, for not too large values of the noise intensity, the steady-state entanglement increases with $\Omega$, and reaches a maximal value at a moderate value of $\Omega$, then decreases with $\Omega$. It can be shown that the threshold value of $n_T$ is strongly dependent on the value of $\Omega$. These results may provide us with a potential way to control the entanglement of two atoms inside the cavity by adjusting the intensity of the external white-noise field. For a deep understanding of the steady-state entanglement, we need to compare the steady state with other mixed entangled states, such as the Werner state, the rank-2 Bell diagonal state, and the maximally entangled mixed state [21]–[23].

New Journal of Physics 7 (2005) 72 (http://www.njp.org/)
Figure 5. The concurrence $C_{st}$ plotted as a function of the spontaneous emission rate $\eta$ and the intensity of the thermal field $n_T$ with $\Gamma = \frac{1}{2} 0.5\Omega$. The unit of $\eta$ is $5\Omega$.

Figure 6. The concurrence $C_{st}$ depicted as a function of the spontaneous emission rate $\eta$ and the coupling constant $\Gamma$ with $n_T = 2$. The units of $\eta$ and $\Gamma$ are both $5\Omega$.

By diagonalizing the steady state in equations (13) and (14), we can see that the steady state does not belong to anyone of the Werner states, the rank-2 Bell diagonal states and maximally entangled mixed states for any possible physical parameters. To illustrate this, the concurrence versus mixedness defined as $M(\rho) = \frac{4}{3}(1 - \text{Tr} \rho^2)$ of the steady state is displayed in figure 8. We can see that, for the same value of mixedness, the concurrence of the steady state is much
Figure 7. The concurrence $C_{st}$ plotted as a function of $\Omega$ and the intensity of the thermal field $n_T$ with $\eta = 5\Gamma$. The unit of $\Omega$ is $10\Gamma$.

smaller than those of the maximally entangled mixed state and the Werner state. It is interesting to point out that the entanglement of the steady state is not a monotonic function of mixedness for fixed values of $\Omega$, $\Gamma$ and $\eta$.

From equation (15), we can find the threshold values of the parameters $\Omega$, $n_T$, $\Gamma$ and $\eta$, beyond which there is no entanglement in the steady state. Some simple inequalities can be derived as follows: $0 < \Omega < \Omega_c$ and $0 < n_T < n_{Tc}$, where $\Omega_c$ and $n_{Tc}$ are given by

$$\Omega_c = \frac{(\Gamma + \eta + 2n_T \Gamma)\sqrt{\eta^2 - \Gamma \eta - n_T \Gamma \eta}}{\Gamma + \eta + n_T \Gamma},$$  \hspace{1cm} (16)$$

$$n_{Tc} = \frac{\eta}{\Gamma} - 1.$$ \hspace{1cm} (17)

The grey area in figure 9 depicts the region where the steady state of the two atoms is entangled in the case with $\eta = 5\Gamma$. In figure 10, the concurrence $C_{st}$ is plotted as a function of the intensity of the thermal field $n_T$ with $\eta = 50\Gamma$ for four different values of $g^2/\Delta$. It clearly tells us that the entanglement increases with the external noise intensity, and achieves a maximal value, then decreases with the noise intensity. The value of $n_T$ corresponding to the maximal value of entanglement slightly depends on $\Omega$. Now, we know some properties of the steady-state entanglement. How will two atoms initially in a pure product state eventually evolve into the entangled steady state? We have calculated the time evolution of the density matrix governed by the master equation (12), and obtained the evolving concurrence of two atoms. In figure 11(b), we show how two atoms initially in various different product states would eventually evolve into the entangled steady state in the presence of the external noise driving one of the atoms. Otherwise, in the absence of a sufficiently intense external noise, two atoms first become entangled due to the

New Journal of Physics 7 (2005) 72 (http://www.njp.org/)
Figure 8. The concurrence versus mixedness of the steady state for selected values of $0 \leq n_T \leq 50$ with: $\Gamma = 0.5\Omega$ and $\eta = 5\Omega$ (······); $\Gamma = 0.5\Omega$ and $\eta = 3\Omega$ (−····−). Solid and dashed lines represent the maximally entangled mixed state and the Werner state, respectively.

dipole coupling induced by the virtual photon process, then they rapidly lose the entanglement, as shown in figure 11(a).

4. Bell violation of two atoms

In this section, we attempt to discuss the nonlocality of two atoms in the steady state. The nonlocal property of two atoms can be characterized by the maximal violation of the Bell inequality. The most commonly discussed Bell inequality is the CHSH inequality [24, 25]. The CHSH operator reads

$$\hat{B} = \vec{a} \cdot \vec{\sigma} \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma} + \vec{a}' \cdot \vec{\sigma} \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma},$$

(18)

where $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$ are unit vectors. In the above notation, the Bell inequality reads

$$|\langle \hat{B} \rangle| \leq 2$$

(19)

The maximal amount of Bell violation of a state $\rho$ is given by [26]

$$\mathcal{B} = 2\sqrt{\lambda + \bar{\lambda}},$$

(20)
Figure 9. This figure depicts the region where the steady state of the two atoms is entangled in the case with $\eta = 5\Gamma$. The unit of $g^2/|\Delta|$ is $10\Gamma$.

Figure 10. The concurrence $C_{st}$ plotted as a function of the intensity of the thermal field $n_T$ with $\eta = 50\Gamma$ for four different values of $g^2/\Delta$ (from top to bottom, $g^2/\Delta = 49\Gamma$, $g^2/\Delta = 50\Gamma$, $g^2/\Delta = 50.5\Gamma$ and $g^2/\Delta = 51\Gamma$).
Figure 11. The concurrence $C$ plotted as a function of the time $t$ with $g^2/\Delta = 2\Gamma$ and $\eta = 5\Gamma$ for two different values $n_T = 10^{-6}$ (a) and $n_T = 2$ (b) of intensity of the thermal field, and for three different initial states: $|0\rangle_1 \otimes |0\rangle_2$; $|1\rangle_1 \otimes |0\rangle_2$; $|0\rangle_1 \otimes |1\rangle_2$. (The values of the solid line in (a) are too small to be seen.) The unit of the time $t$ is $1/10\Gamma$.

Figure 12. The maximal violation $B$ plotted as a function of the time $t$ with $g^2/\Delta = 20\Gamma$ and $\eta = \Gamma$ for three different values of $n_T$, $n_T = 0$ (-----), $n_T = 0.5$ (· · · · · ·), $n_T = 1$ (······). Two atoms are initially prepared in $|1\rangle_1 \otimes |0\rangle_2$. The unit of the time $t$ is $1/100\Gamma$. 

New Journal of Physics 7 (2005) 72 (http://www.njp.org/)
where \( \lambda \) and \( \tilde{\lambda} \) are the two largest eigenvalues of \( T_{\rho}^\dagger T_{\rho} \). The matrix \( T_{\rho} \) is determined completely by the correlation functions being a \( 3 \times 3 \) matrix whose elements are \( (T_{\rho})_{nm} = \text{Tr}(\rho \sigma_n \otimes \sigma_m) \). Here, \( \sigma_1 \equiv \sigma_x \), \( \sigma_2 \equiv \sigma_y \) and \( \sigma_3 \equiv \sigma_z \) denote the usual Pauli matrices. We call the quantity \( B \) the maximal violation measure, which indicates the Bell violation when \( B > 2 \) and the maximal violation when \( B = 2\sqrt{2} \). For the density operator \( \rho_{st} \) in equation (13) characterizing the steady state of two atoms, \( \lambda + \tilde{\lambda} \) can be written as follows:

\[
\lambda + \tilde{\lambda} = 4|\rho_{23}^s|^2 + \max \left[ 4|\rho_{23}^s|^2, (\rho_{11}^s + \rho_{44}^s - \rho_{22}^s - \rho_{33}^s)^2 \right].
\]  

(21)

Recently, Verstraete et al investigated the relations between the violation of the CHSH inequality and the concurrence for systems of two qubits [27, 28]. They showed that the maximal value of \( B \) for a given concurrence \( C \) is \( 2\sqrt{1 + C^2} \), which can be achieved by the pure states and some Bell diagonal states. If the given concurrence \( C \) is larger than \( 1/\sqrt{2} \), the minimal value of \( B \) is \( 2\sqrt{2}C \), which can be achieved by the maximal entangled mixed state. Furthermore, the entangled two-qubit state with the concurrence \( C \leq 1/\sqrt{2} \) may not violate any CHSH inequality, even after all possible local filtering operations, except that their Bell diagonal normal form does violate the CHSH inequalities [27]. So, it is not difficult to understand the following results. Our calculations show that no violation of CHSH inequality will be found in the steady state even though the steady state is entangled. Moreover, the stochastic-resonance-like behaviour cannot be observed in the Bell violation of two atoms during the evolution, and the stronger the noise intensity, the more rapidly the Bell violation disappears, which is shown in figure 12.

5. Conclusion

In this paper, we investigate the problem of generating entanglement when only incoherent sources are available and show that controllable entanglement can arise in this situation. We show that, if two atoms are simultaneously driven by two independent white-noise fields with the same intensity, the entanglement between them is suppressed and eventually completely destroyed by the noise. However, in another case in which only one atom is exposed in the white-noise field, the steady-state entanglement of the two atoms is a non-monotonic function of both the intensity of noise driving the field and the spontaneous decay rate. A double stochastic resonance behaviour emerges. Moreover, the threshold value of the spontaneous decay rate, below which there is no steady-state entanglement, increases with the intensity of the noise field. Finally, we examine the Bell violation of two atoms, and show that the steady state does not violate the Bell inequality even though it is inseparable. Moreover, the stochastic-resonance-like behaviour cannot be observed in the Bell violation of two atoms during the evolution, and the stronger the noise intensity, the more rapid is the disappearance of the Bell violation.

Over the past few years, the system of two atoms trapped in a optical cavity has become a fundamental element realizing various quantum information processes such as quantum computation and teleportation. Entanglement of two atoms plays an essential role in those processes. So our study may be helpful in understanding the role noise plays in certain tasks for quantum information. Our results in this paper can be easily applied to other similar physical systems (e.g. two-coupled quantum dots, two ions in an ion-trap or two spins coupled by exchange interaction).

Recently, much attention has also been paid to entanglement generation in spatially separate cavities [14], [29]–[31]. Browne et al presented a protocol that allows the generation of a
maximally entangled state between individual atoms held in spatially separate cavities [29]. Kraus and Cirac have shown how to scale up their scheme for entangling distant atoms by using squeezed light to build a quantum network [14]. Clark et al. have also proposed an efficient scheme to unconditionally entangle the internal states of atoms trapped in separate high-finesse optical cavities [30]. Mancini and Bose have suggested a way to generate an effective interaction of arbitrary strength between the internal degrees of freedom of two atoms placed in distant cavities connected by an optical fibre [31]. Therefore, it is very interesting to generalize our present study to such a case, in which the atoms are trapped in spatially separated cavities and driven by external white noises. Another possible generalization is to study how the external white noises can affect the multi-partite entanglement of both the evolving state and the steady state in those physical systems containing many particles, such as many atoms in cavity, three or more coupled quantum dots, and many ions in an ion-trap.

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New Journal of Physics 7 (2005) 72 (http://www.njp.org/)
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