Inflaton and dark matter in a random environment

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Abstract
We consider a Lagrangian of interacting classical fields. We divide the Lagrangian into two parts. The first part is to describe either the dark matter (DM) or the inflaton (IN) depending on the choice of the self-interaction. The second part constitutes an environment of an infinite number of scalar fields interacting linearly with the first part. We approximate the environment by a white noise obtaining a Langevin equation. We show that the resulting Fokker-Planck equation has solutions determining a relation between the diffusion constant, the cosmological constant and the temperature. As a consequence of the Langevin approximation the energy-momentum tensor of the dark matter and the inflaton is not conserved. The compensating energy-momentum tensor is interpreted as the dark energy (DE). We insert the total energy-momentum in Einstein equations. We show that under special initial conditions Einstein equations have a solution with a constant ratio of DM/DE and IN/DE densities.

1 Introduction

The ΛCDM model became the standard cosmological model since the discovery of the universe acceleration [1][2]. It describes very well the large scale structure of the universe. The formation of an early universe is well explained by the inflationary models involving scalar fields [3][4]. However, a model describing universe evolution from its hot early stage till the present day is still missing. Such a model should reveal the nature of the mysterious dark matter (DM), dark energy (DE) and the surprising relation between various parameters in the ΛCDM model (the coincidence problem, the cosmological constant problem). The role of the inflaton field in the high energy physics is also obscure [5]. Some
hints concerning the precise theory connecting the different ingredients of the ΛCDM model may come from the observations on the galactic scale, where the ΛCDM model encounters some difficulties in explaining the dark matter halos. In ref. [6], where the difficulties are reviewed, it has been suggested that DM self-interaction can solve the problems. However, DM self-interaction itself leads to difficulties, e.g., problems with an explanation of spiral halos of DM [7]. In our earlier papers [8][9][10] we introduced a new ingredient in cosmological models: a diffusive interaction between dark matter and dark energy. Our point of view is that the dark energy consists of some unknown particles and fields. They interact in an unknown way with particles of the dark matter and the inflaton. The result of the interaction could be seen in a diffusive behaviour of the dark matter. The diffusion effect does not depend on the details of the interaction but only on its strength and ”short memory” (Markovian approximation). In this paper we follow an approach appearing in many papers (see [11][12][13][14][15] and references quoted there) describing the dark matter and the fields responsible for an inflation (inflatons) by scalar fields. What is new in our model is an introduction of an environment of an infinite set of scalar fields interacting with DM and the inflaton (IN). In a limit of an infinite number of fields the dark energy is described by a random ideal fluid. The model is built in close analogy to the the well-known infinite oscillator model [16][17][18] of Brownian motion. There is some similarity of our model to the warm inflation [19] in the use of the white noise in the description of post-inflationary evolution. The aim of the model is to describe the hot inflationary phase as well as the late evolution with the same set of scalar fields. However, we do not discuss here consequences of the model for inflation. We concentrate in this paper on the late time dynamics. The early inflationary phase could be treated within the quantized version of the model in the Starobinsky-Vilenkin stochastic approximation [20]-[21]. In this preliminary study we show that the time evolution described by the Langevin equation can predict the relation between the diffusion constant, the temperature and the Hubble constant (a version of the fluctuation-dissipation theorem) as well as the relation between DM density, inflaton density and DE density. The plan of the paper is as follows. In sec.2 we define the model in its heuristic form and indicate its relation to the well-known Starobinsky-Vilenkin model [20]-[21]. In sec.3 we introduce a Langevin equation as a limit of an infinite number of the environmental scalar fields. As a consequence of the Fokker-Planck equation resulting from the Langevin equation we derive a relation between the temperature, the diffusion constant and the Hubble constant. In sec.4 we discuss the conservation law of the energy-momentum. In order to preserve the conservation of the total energy-momentum we have to introduce a compensating energy-momentum interpreted as the dark energy. In sec.5 we suggest that an approximate equation of state needed to close Friedmann equations can be obtained by averaging either in time or over configurations. The energy-momentum is inserted in the Einstein equations in sec.6. We obtain a particular solution of these equations which gives a constant ratio of DM/DE density and
IN/DE energy density. The solution is the fixed point of a dynamical system discussed in [9][22].

2 Scalar fields interacting with an environment

The CMB observations show that the universe was once in an equilibrium state. The Hamiltonian dynamics of scalar fields usually discussed in the model of inflation do not equilibrate. We can achieve an equilibration if the scalar field interacts with an environment. We suggest a field theoretic model which is an extension of the well-known oscillator model discussed in [16][17][18]. We consider the Lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) + \sum_a \left( \frac{1}{2} \partial_{\mu} \chi^a \partial^{\mu} \chi^a - \frac{1}{2} m_a^2 \chi^a \chi^a - \lambda_a \phi \chi^a \right). \]  

(1)

Equations of motion read

\[ g^{-\frac{1}{2}} \partial_{\mu} (g^{\frac{1}{2}} \partial^{\mu}) \phi = -V' - \sum_a \lambda_a \chi^a, \]  

(2)

\[ g^{-\frac{1}{2}} \partial_{\mu} (g^{\frac{1}{2}} \partial^{\mu}) \chi^a + m_a^2 \chi^a = -\lambda_a \phi, \]  

(3)

where \( g_{\mu\nu} \) is the metric tensor and \( g = |\det[g_{\mu\nu}]| \). Inserting the solution of eq.(3) in eq.(2) we obtain an equation of the form

\[ g^{-\frac{1}{2}} \partial_{\mu} (g^{\frac{1}{2}} \partial^{\mu}) \phi + m^2 \phi + V'(\phi) = -\int_0^t K(t, t') \phi(t') dt' + W(\chi(0), \partial_t \chi(0)), \]  

(4)

where the kernel \( K \) is an evolution kernel for the linear equation (3) and the noise \( W \) depends linearly on the initial conditions \( (\chi(0), \partial_t \chi(0)) \) for the second order differential equation (3). We could quantize the scalar field equations (2)-(3). Then, the initial values \( W(\chi(0), \partial_t \chi(0)) \) in eq.(4) will be the quantum fields. We could assume that these fields are in the thermal state \( \exp(-\beta H_\chi) \), where \( H_\chi \) is the quantum Hamiltonian of the \( \chi \) fields and \( \frac{1}{\beta} \) is the temperature of the heat bath. Subsequently, we can take the classical limit of the quantum field theory. The first term on the rhs of eq.(4) describes a friction coming from the environment. The second term is the "noise" from the environment. It will have a certain probability distribution in a quantum theory if the initial conditions are quantum fields (or random thermal fields in the classical limit). In the limit of an infinite number of fields with properly chosen masses and couplings we obtain the quantum analogue of the white noise (in complete analogy to the model of refs. [16][17][18]). If we choose the probability distribution as the classical limit of the quantum thermal state \( \exp(-\beta H_\chi) \), then we can obtain the classical noise (discussed in the next section).

We consider in general the metric

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} dt^2 - g^{(3)}_{ij} dx^i dx^j. \]  

(5)
Explicit formulas will be derived for a flat FLWR expanding metric
\[ ds^2 = dt^2 - a^2 dx^2, \]
when \( H = a^{-1} \partial_t a = \text{const (de Sitter space)}. \)

The well-known Starobinsky-Vilenkin stochastic equation [20] [21] does not come from an interaction with an environment but could be obtained from a quantum version of eq.(2) with \( \lambda_a = 0 \). In such a case the field \( \phi \) is cut at high momentum exceeding the de Sitter horizon. We could apply the Starobinsky-Vilenkin approximation by considering two independent noises on the rhs of eq.(4): one describing the quantum fluctuations of the \( \phi \) field and another one coming from an interaction with the \( \chi^a \) fields. The Starobinsky-Vilenkin approximation corresponds to an infinite "friction" \( H \) (or "slow rolling") . Then, neglecting the \( \partial_t^2 \phi \) term in eq.(2) we obtain
\[ 3H \partial_t \phi = -V' + W_{SV}, \]
where the Starobinsky-Vilenkin noise \( W_{SV} \) has the correlations determined by the quantum field (in the de Sitter space) cut at high momenta
\[ \langle W_{SV}(t, x)W_{SV}(t', x) \rangle = \frac{H^3}{4\pi^2} \delta(t - t'). \]

The Starobinsky-Vilenkin equation is supposed to describe fluctuations during inflation. We suggest that eqs.(2)-(3) with a proper choice of the potential \( V \) can describe the time evolution of the inflaton embedded in the environment of dark energy. The potential \( V \) should agree with the observational data [3][23][24][6]. Then, \( V(\phi) \) must fall at large \( \phi \) in order not contradict the observed outcome of the primordial nucleosynthesis [12]. Eq.(4) could also describe the dark matter with another choice of \( V(\phi) \). In particular, Higgs-type potentials are taken into account [11][25]. Models with a single field and a single potential describing dark matter and inflation are also discussed [26].

3 A relation between the diffusion constant, temperature and the Hubble constant

We restrict ourselves to classical field theory and neglect the Starobinsky-Vilenkin noise. We need a formulation of Eq.(4) with an infinite number of fields \( \chi \) in a form which is covariant under the change of coordinates on the manifold. This leads to a direct generalization of the Kramers equation [27] (usually expressed in the phase space)
\[ g^{-\frac{1}{2}} \partial_{\mu}(g^{\mu\nu} g^{\frac{1}{2}} \partial_{\nu}) \phi + V'(\phi) = \gamma W. \]
Here, the noise \( W \) is Gaussian with mean zero and the correlation function
\[ \langle W(x)W(x') \rangle = \delta_{g}(x, x') = g^{-\frac{1}{2}} \delta(x, x'), \]
where the $\delta_g$-function on a manifold is defined by
\[
\int dx \sqrt{g} \delta_g(x, x') f(x) = f(x')
\]
and $\delta$ on the rhs of eq.(17) is understood as the one in local coordinates around $x$. Eqs.(9)-(10) lead to the proper (intrinsic) formulation of the stochastic wave equation on a manifold independent of the choice of coordinates [28]. For a flat expanding metric we have
\[
\partial_t^2 \phi - a^{-2} \Delta \phi + 3H \partial_t \phi + V'(\phi) = \gamma W.
\] (11)

In eq.(11) we have neglected the friction $\Gamma_{en}$ coming from the environment (present in eq.(4), in general $3H \rightarrow 3H + \Gamma_{en}$). We rewrite eq.(11) as a system of first order equations for $\phi$ and $\partial_t \phi$. The Fokker-Planck equation for the stochastic system (11) is (we omit the spatial derivatives in eq.(11))
\[
\partial_t P = \frac{\gamma^2}{2} \int dx g^{-\frac{3}{2}} \delta g_{ij} \Pi(x) P - \int dx V'(\phi(x)) \delta \delta \Pi(x) P - \int dx \delta \Pi(x) \delta \phi(x) P.
\] (12)

We solve eq.(12) with the initial condition
\[
P(0) = \exp \left( - \frac{\beta}{2} \int dx (g^{(3)}(0))^{\frac{3}{2}} \Pi^2 \right),
\]
where $g^{(3)} = \det(g^{(3)}_{ij})$. The solution of eq.(12) for $V = 0$ is
\[
P(t) = L(t) \exp \left( - \frac{1}{2} \alpha(t) \int dx \Pi^2 \right)
\] (13)
with
\[
\alpha(t) = \sqrt{g} a^3 \left( \beta^{-1} \sqrt{g(0)} + \gamma^2 \int_0^t a(s)^3 ds \right)^{-1}
\] (14)
and
\[
L(t) = \exp \left( - \int dx \delta(0) \int_0^t \frac{\gamma^2}{2} (g(s))^{-\frac{3}{2}} a(s) - 3H(s) ds \right).
\]

$L(t)$ contains infinities $\int dx$ and $\delta(0)$ which would cancel if the rhs of eq.(12) was defined with a proper point splitting. However, there is no need to tackle the $L(t)$ factor because it drops out in the calculations of the normalized expectation values with respect to the measure $P$.

For a large time we have
\[
\alpha(t) \simeq \sqrt{g} a^3 \left( \gamma^2 \int_0^t a(s)^3 ds \right)^{-1}
\]
In de Sitter space $a(t) = \exp(\dot{H}t)$. Then, for a large time

$$\alpha(t) \to \sqrt{\frac{gT_{\infty}}{3}}$$

where

$$T_{\infty} = \frac{2}{3H}, \quad (15)$$

In the general metric (5) with $g_{ij} \approx \exp(\dot{H}t)$ we would get $T_{\infty} = T_{st} \sqrt{g_{00}}$ where $T_{st}$ is Tolman’s equilibrium temperature [29] (see also [30]).

In classical field theory in a static background metric $P(0)$ is the canonical Gibbs distribution determined by the maximum of the entropy $S = - \int P \ln P$. $P(t)$ has an interpretation as a Gibbs probability distribution at time $t$. We can define the temperature of the state $P$ by the formula

$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad (16)$$

where the energy $E$ (for $V = 0$) is

$$E = \frac{1}{2} \langle \int dx \sqrt{g^{(3)}} \Pi^{2} \rangle \quad (17)$$

where the expectation value is with respect to $P$. It follows from the formula (13) for $P$ and from eq.(16) that

$$\frac{1}{T} = \alpha(t) a^{-3} \quad (18)$$

We could determine the temperature in another way: as $\frac{1}{T}$ of the mean value of the ”kinetic energy” $\langle \int dx \sqrt{g^{(3)}} \Pi^{2} \Pi(t,x) \Pi(t,x') \rangle$. For the calculation of the temperature we can neglect the interaction and the term $a^{-2} \Delta$ in eq.(11). Then, the solution of eq.(11) is

$$\Pi(t) = \exp( - \int_{0}^{t} H(t') dt') \Pi(0) + \gamma \int_{0}^{t} \exp(-3 \int_{s}^{t} H(t') dt') W(s) ds. \quad (19)$$

We have

$$\langle \Pi(x) \Pi(x') \rangle = a(t)^{-6} a(0)^{6} \Pi(0,x) \Pi(0,x')$$

$$+ \delta(x-x') \gamma^{2} a(t)^{-6} \int_{0}^{t} a(s)^{3} ds \simeq \delta(x-x') \frac{\gamma^{2}}{3H(t)} \quad (20)$$

at large time (for a slowly varying $H$ ). This result coincides with eq.(15). We have got here the same relation (15) between the temperature, diffusion constant and the Hubble constant as for a particle diffusion in [10] (for an exact equality we need $\gamma^{2} = 3\kappa^{2}$, where $\kappa^{2}$ is the diffusion constant for particle’s diffusion). If we assume that the classical fields in eqs.(2)-(3) result as the classical limit of quantum fields, which are defined in the de Sitter space and have a well-defined temperature $T_{\infty}$, then we have in addition the requirement $T_{\infty} = \frac{H}{2\pi}$.
(this relation has been derived in [31][32] as a consequence of the periodicity in the imaginary time). Hence, in the de Sitter stage of the evolution

\[ H^2 = \frac{2\pi\gamma^2}{3}. \]  

(21)

The relation (18) for a general evolution \( a(t) \) coincides with the one discussed in [8][9] if DM equation of state is \( \bar{w} = 1 \). The reason for its applicability to this case \((V = 0)\) will be explained in sec.6. Eqs. (15) and (21) can apply separately at different stages of the universe evolution.

4 The energy-momentum tensor of the interaction with an environment

The total energy-momentum tensor resulting from the Lagrangian (1) is conserved and could be inserted on the rhs of Einstein equations. However, if we replace the infinite set of fields \( \chi^a \) by the noise (as in eq.(9)) then the conservation law for the energy-momentum tensor \( T^{\mu\nu} \) of the field \( \phi \) fails. We have to compensate in the energy-momentum the replacement of the fields \( \chi^a \) by the noise by means of a compensating energy-momentum \( T_\Lambda \) which we interpret as dark energy. Now, the conserved energy-momentum tensor \( T^{\mu\nu}_{tot} \) is

\[ T^{\mu\nu}_{tot} = T^{\mu\nu} + T^{\mu\nu}_\Lambda. \]  

(22)

From the conservation law

\[ (T_\Lambda^{\mu\nu})_{;\mu} = -(T^{\mu\nu})_{;\mu}. \]  

(23)

Many models of dark energy can be described by the energy-momentum of an ideal fluid

\[ T^{\mu\nu}_\Lambda = (\rho_\Lambda + P_\Lambda)u^\mu u^\nu - g^{\mu\nu}P_\Lambda, \]  

(24)

where \( \rho \) is the energy density and \( p \) is the pressure. The velocity \( u^\mu \) satisfies the normalization condition

\[ g_{\mu\nu}u^\mu u^\nu = 1. \]

For the scalar field we have the representation (24) with

\[ u^\mu = \partial^\mu \phi(\partial^\sigma \phi \partial_\sigma \phi)^{-\frac{1}{2}}, \]  

(25)

\[ \rho + p = \partial^\sigma \phi \partial_\sigma \phi, \]  

(26)

\[ p = \frac{1}{2} \partial^\sigma \phi \partial_\sigma \phi - V. \]  

(27)

We have from eq.(9) (with \( T^{\mu\nu} \) defined by eqs.(24)-(25))

\[ (T^{\mu\nu})_{;\mu} = \gamma \partial^\nu \phi W \]  

(28)
The divergence equation (28) in a homogeneous metric in the frame \( u = (1, 0) \) (spatial homogeneity of \( \phi \)) gives

\[
\partial_t \rho + 3(1 + \tilde{w}) H \rho = \gamma W \partial_t \phi, \tag{29}
\]

where

\[
\tilde{w} = \left( \frac{1}{2} \Pi^2 - V \right) \left( \frac{1}{2} \Pi^2 + V \right)^{-1}. \tag{30}
\]

Then, for the compensating energy density we have (from eq.(23))

\[
\partial_t \rho_\Lambda + 3 H (1 + w) \rho_\Lambda = -\gamma W \partial_t \phi, \tag{31}
\]

where \( H = a^{-1} \partial_t a \) and

\[
w = \frac{p_\Lambda}{\rho_\Lambda}. \]

The solution of eq.(31) with a constant \( w \) is

\[
\rho_\Lambda(t) = a^{-3(1+w)} \sigma_\Lambda - \gamma a(t)^{-3(1+w)} \int_{t_0}^t a(s)^{3+3w} W(s) \partial_s \phi(s) ds. \tag{32}
\]

\( \sigma_\Lambda \) is a constant such that

\[
\rho_\Lambda(t_0) = a(t_0)^{-3(1+w)} \sigma_\Lambda.
\]

Various models with a non-zero term on the rhs of eq.(29) (interpreted as a time derivative of the cosmological term) have been discussed [33][34][35][36][37]. In the model (29) the cosmological term is random.

## 5 Averaging over fields

For scalar fields the equation of state \( p = \tilde{w}(\phi) \rho \) depends on the field \( \phi \). In a Hamiltonian system \( \phi \) varies frequently in time without achieving any equilibrium. After adding the noise we may expect a smooth behaviour and an equilibration. Let us calculate (using equations of motion (11) with an omission of spatial derivatives)

\[
\partial_t^2 (a^2 \phi^2) = \left( \frac{3}{2} \partial_t H + \frac{3}{4} H^2 \right) a^2 \phi^2 + 2a^2 \left( (\partial_t \phi)^2 - \phi V' \right) + \gamma \langle \phi W \rangle. \tag{33}
\]

Taking the time average we obtain zero on the lhs. The term \( \langle \phi W \rangle = 0 \) for a causal (“non-anticipating”[38]) solution of eq.(11). Then, the rhs gives the relation between kinetic energy and the potential energy, i.e., the virial theorem.

The environment can make the system ergodic. In an ergodic system the time average is equal to the ensemble average. We can express eq.(33) as

\[
\langle (\frac{3}{2} \partial_t H + \frac{3}{4} H^2) \phi^2 + 2(\partial_t \phi)^2 \rangle = \langle \phi V' \rangle, \tag{34}
\]
where the average is understood either in time or over configurations. We suggest that we can make the approximation for late time evolutions

\[ \tilde{w} \simeq \tilde{w}_{av} = \langle p \rangle \langle \rho \rangle^{-1}. \]  

(35)

Let us calculate \( \tilde{w}_{av} \) in some special cases using for the average over configurations the measure (12) defined by \( P \), corresponding to \( H = 0 \), then

\[ P = \exp \left( - \int dx \left( \frac{1}{2} \Pi^2 + V(\phi) \right) \right). \]

If \( V(\phi) = \mu^2 \phi^2 \) then

\[ \tilde{w}_{av} = 0. \]  

(36)

The result (36) agrees with the usual assumption \( \tilde{w} = 0 \) for massive bodies. We would get the same result from the virial theorem (34) (with \( H = 0 \)). For \( V(\phi) = \mu \phi^4 \) we obtain

\[ \tilde{w}_{av} = \frac{1}{3} \]  

(37)

from the average over configurations as well as from the virial theorem (with \( H = 0 \))(34). The value \( \frac{1}{3} \) is equivalent to the condition \( T^\mu_\mu = 0 \) resulting from conformal invariance of field theory. The \( \phi^4 \) theory could be considered as a version of the Brans-Dicke model (with the cosmological term) which is conformal invariant.

In general, for \( V = \mu |\phi|^k \) by a calculation of elementary integrals we obtain from the average with respect to the measure defined by \( P \)

\[ \tilde{w}_{av} = \frac{k - 2}{k + 2} \]  

(38)

Hence, \( \tilde{w} \to 1 \) for \( k \to \infty \) and \( \tilde{w} \to -1 \) for small \( k \) (eq.(38) breaks down for \( k = 0 \), as if \( V = 0 \) then obviously \( \tilde{w} = 0 \) from eq.(30)). For \( k \geq 2 \) we get from eq.(38) \( 0 \leq \tilde{w}_{av} < 1 \) as a possible value of \( \tilde{w}_{av} \) for the dark matter in the approach of [11]. For the inflaton Peebles and Ratra suggest the behaviour \( V(\phi) \simeq |\phi|^{-\alpha} \) with \( \alpha > 0 \). Hence, \( \tilde{w}_{av} = \frac{2 - \alpha}{4 - \alpha} \) which gives \( \tilde{w}_{av} > 1 \) if \( \alpha > 2 \) and \( \tilde{w}_{av} < 0 \) if \( 0 < \alpha < 2 \).

From the virial theorem (34) we obtain

\[ \hat{w}_{vir} = \left( k + 2 - (\frac{3}{4} \partial_t H + \frac{9}{8} H^2)k \right) \left( k + 2 - (\frac{3}{4} \partial_t H + \frac{9}{8} H^2)k \right)^{-1}. \]  

(39)

The H-correction does not change our conclusion concerning the limits of large and small \( k \). Note that if \( \alpha \simeq t^\sigma \) then \( (\frac{3}{4} \partial_t H + \frac{9}{8} H^2) = \frac{3\sigma}{2\pi}(\sigma - \frac{3}{2}) \sigma \). Hence, it is negative for non-accelerating cosmologies and decreasing in time for a power law expansion.
6 Einstein equations

Einstein equations are written in the form

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}_{\text{tot}}. \]  

(40)

where \( G \) is the Newton constant. The energy-momentum tensor of sec.4 is a random variable. Hence, eq.(40) describes a random evolution. We are not prepared at the moment to discuss the general solutions of eq.(40). We take the mean value on the rhs of eq.(40) treating \( \bar{a}(t) \) as deterministic (this is a conventional approach to the classical approximation of the quantum energy-momentum). In principle, we could solve the wave equation (9) and determine \( T \) and \( T_\Lambda \) as functions of \( a(t) \) from eqs.(29)-(31) (we need to know \( \tilde{w} \) either from the exact solution of the wave equation or by averaging of sec.5). Then, calculation of the expectation value of \( T_{\text{tot}} \) over the noise \( W \) would determine the rhs of Einstein equations (40). It is difficult to do it for non-linear equations. We restrict ourselves here to the simplified case when \( V = 0 \) and the spatial derivatives in eq.(11) are neglected. Then, \( \rho = T^{00} = \frac{1}{2} \Pi^2 \). This energy density and its expectation value \( \rho_{av} \) has been calculated in eq.(20). \( \rho_{av} = \langle T^{00} \rangle \) solves the equation

\[ \partial_t \rho_{av} + 3H(1 + \bar{w})\rho_{av} = \gamma^2 a^{-3} \]  

(41)

with \( \bar{w} = 1 \) as expected from eq.(30) with \( V = 0 \). Then, the compensating energy density \( \rho_{av}^{\Lambda} = \langle T_{\Lambda}^{00} \rangle \) (on the basis of eq.(23)) satisfies the equation

\[ \partial_t \rho_{av}^{\Lambda} + 3H(1 + w)\rho_{av}^{\Lambda} = -\gamma^2 a^{-3} \]  

(42)

The Friedman equation in the FRLW flat metric (6) reads

\[ H^2 = \frac{8\pi G}{3}(\rho_{av} + \rho_{av}^{\Lambda}). \]  

(43)

Inserting the solutions of eqs.(41)-(42) in eq.(43) we obtain

\[ H^2 = \frac{8\pi G}{3} \left( a^{-3(1+w)} \sigma + \gamma^2 a(t)^{-3(1+w)} \int_{t_0}^{t} a(s)^{3\tilde{w}} ds + a^{-3(1+w)} \sigma_{\Lambda} - \gamma^2 a(t)^{-3(1+w)} \int_{t_0}^{t} a(s)^{3\tilde{w}} ds \right). \]  

(44)

We have got the Friedmann equation which coincides with the one for a diffusive matter in [9][10]. This coincidence may be a consequence of the relation between the energy-momentum of quantum fields and the energy-momentum of particles which can be established on the basis of the Wigner function formalism. Although in our simplified model with \( V = 0 \) we have \( \bar{w} = 1 \) (from eq.(30)) we believe that eq.(44) can be also applied to \( V \neq 0 \) by an insertion of a constant \( \tilde{w} \) by means of the method of averaging of sec.5. We have studied eq.(44) numerically in [9] (see also [22]) in order to explore the relation between \( \rho \) and \( \rho_{\Lambda} \) (the coincidence problem). For this purpose we formulated eqs.(41)-(43) as
a closed dynamical system. It comes out that the fixed point of the dynamical system of refs.\cite{9}\cite{22} coincides with the linear evolution of $a$

$$a(t) = a_0 + \lambda t$$

(45)

(a comparison of the general numerical solution of eqs.\cite{41}-\cite{43} with observations is studied in \cite{9}\cite{22}; a consistency of a linear evolution with observations is discussed in \cite{39}\cite{40}).

In fact, we can show directly that (45) is the solution of eq.\cite{44} for some special initial conditions \((a_0, \sigma, \sigma_\Lambda)\). If we set \(t = t_0\) in eq.\cite{44} then we obtain

$$\lambda^2 = \frac{8\pi G}{3 \gamma^2} \left( a_0^{-3(1+\tilde{w})} \sigma + a_0^{-3(1+w)} \sigma_\Lambda \right).$$

(46)

After an integration over \(s\) in eq.\cite{44} (we set \(t_0 = 0\)) with the use of eq.\cite{45} eq.\cite{44} reads

$$3 \frac{8\pi G}{\lambda^2} a(t)^{-2} = \sigma a^{-3(1+\tilde{w})} + \frac{\gamma^2}{\lambda(3w+1)} (a(t)^{-2} - a_0^{-1+3\tilde{w}}) a(t)^{-3(1+\tilde{w})})$$

$$+ \sigma_\Lambda a^{-3(1+w)} - \frac{\gamma^2}{\lambda(3w+1)} a(t)^{-2} + \frac{\gamma^2}{\lambda(3w+1)} a_0^{-3w+1} a(t)^{-3(1+w)}.$$ (47)

If we wish a solution for arbitrary \(w\) and \(\tilde{w}\) then the terms \(a^{-3(1+\tilde{w})}\) and \(a^{-3(1+w)}\) in eq.\cite{47} should cancel. Hence,

$$\sigma_\Lambda = - \frac{\gamma^2}{\lambda} (1 + 3w)^{-1} a_0^{1+3w}$$

(48)

and

$$\sigma = \frac{\gamma^2}{\lambda} (1 + 3\tilde{w})^{-1} a_0^{1+3\tilde{w}}.$$ (49)

From eq.\cite{48} it follows that \(1 + 3w < 0\) if \(\sigma_\Lambda\) is to be non-negative. If the terms at \(a^{-2}\) in eq.\cite{47} are to cancel then

$$\frac{1}{8\pi G} \lambda^3 = \gamma^2 (w - \tilde{w}) (1 + 3w)^{-1} (1 + 3\tilde{w})^{-1}. $$ (50)

Together with eqs.\cite{48}-\cite{49} eq.\cite{50} is the same as eq.\cite{46}. Eq.\cite{50} determines \(\lambda\), then solving eqs.\cite{46},\cite{48}-\cite{49} we determine the initial conditions \(\sigma, \sigma_\Lambda\) and \(a_0\). We obtain \(a_0 = 1\) and \(\sigma = \rho(0) = -(1+3w)(1+3\tilde{w})^{-1} \rho_\Lambda(0) \approx (8\pi G)^{-1} \gamma^2 \). Using the initial conditions \cite{48}-\cite{49} we can calculate the dark energy from eq.\cite{42}

$$\rho_\Lambda(t) = -\frac{\gamma^2}{\lambda} (1 + 3w)^{-1} a(t)^{-2}.$$ (51)

In the units \(8\pi G = 1\) with \(\tilde{w} = 0\) and \(w = -1\) we obtain from eq.\cite{51} the cosmological term at large time \(\Lambda(t) \approx 0.25 t^{-2}\) (this behaviour is tested against observations in \cite{36}). The density of the dark matter follows from eq.\cite{41}

$$\rho(t) = \frac{\gamma^2}{\lambda} (1 + 3\tilde{w})^{-1} a(t)^{-2}.$$ (52)

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Hence,
\[ \rho(t)\rho_\Lambda(t)^{-1} = -(1 + 3\tilde{w})^{-1}(1 + 3w). \] (53)

For the relativistic models of [9][22], when the dark matter density is determined by a phase space distribution we have \(0 < \tilde{w} \leq \frac{1}{3}\). Hence, \(\frac{1}{3} < \rho_{DM}(t)\rho_\Lambda(t)^{-1} \leq 1\) for \(w = -1\). We have a greater range of \(\rho(t)\rho_\Lambda(t)^{-1}\) for scalar field theories. So, for the inflaton of [12] with \(k = -\alpha\) in eq.(38) we can have arbitrarily large \(\tilde{w}\) hence \(\rho_{IN}\rho_\Lambda^{-1}\) can be arbitrarily small whereas for the dark matter with \(0 \leq \tilde{w} < 1\) (when \(k \geq 2\)) \(\rho_{DM}\rho_\Lambda^{-1}\) varies in the interval \(\left(\frac{1}{8}, \frac{1}{2}\right)\) (for \(w = -1\)).

The general solution of eqs.(41)-(43) gives the \(\rho^{-1}\) ratio which varies in time [9][22]. We can show that if the conservation equations (23) are satisfied and we require that \(\rho(t)\rho_\Lambda(t)^{-1} = \text{const}\) then \(a(t)\) satisfying Friedmann equations must be a linear function of \(t\).

7 Discussion and summary

We have drawn some conclusions resulting from the assumption that the relativistic IN/DE or DM/DE interaction has a diffusive character. We have obtained earlier related results in a model of a particle DM/DE interaction resulting from a relativistic diffusion. It seems that some aspects of the diffusive interactions are independent of peculiarities of the models. The diffusive dynamics leads to a fixed ratio between IN/DE and DE/DM densities. The relation between the diffusion constant and the temperature is a version of fluctuation-dissipation theorem well-known in statistical physics, whereas the relation between temperature and the cosmological constant is connected with the definition of temperature in quantum field theory. These relations can supply a new look at the cosmological constant and the coincidence problem. The diffusion constant could be measurable by an observation of the dark matter dynamics through its effect on the luminous matter. The inflaton density can be related to dark energy and some parameters of the CMB spectrum. So, in principle, the formulae (15) and (21) are verifiable. The model is supposed to be applicable in a large range of time. In the same model we could apply the Starobinsky-Vilenkin approximation describing the inflaton fluctuations in a unified way for the inflation era as well as at large time.

The connection between the Hubble constant and the diffusion constant suggests another explanation of the origin of the cosmological term [41]. Its present small value comes from the energy loss of the environment of DE (energy gain of DM). We have compared the diffusive dynamics (resulting from \(\gamma \neq 0\)) with observations in [9]. The relations (15) and (21) could be tested in observations if we could estimate the diffusion constant \(\gamma^2\) and dark matter temperature on the basis of the motion of the luminous matter. From eq.(18) it follows that the temperature of the dark matter may increase during the expansion. Such a conclusion has also been derived from thermodynamics of the dark energy in [42][43]. We have good estimates of the DM density on the
basis of rotational curves [44] and lensing observations [45]. We would need either a time dependence of the DM density or an independent way to determine DM velocities. A dispersion of the velocities could be applied to measure the temperature of the dark matter [46]. Another source of information on the DM phase space distribution could come from computer simulations of the formation of DM halos [47]. It is known that non-interacting DM is unable to describe the galactic DM halos [6]. A dissipative component of the dark matter may be necessary [48]. Some estimates on the diffusion constant might come from the heat transfer inside the halo [49] if the heat diffusion was a result of DM diffusion in an environment of DE. In conclusion, the field theoretic model of the diffusive DM/DE interaction suggested by the relativistic field theory of particle physics can lead to testable consequences on the basis of numerous observational data.

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References

[1] P.A.R. Ade et al, arXiv:1502.01589
[2] K.T. Story et al ApJ, 779, 86(2013)
[3] P.A.R. Ade et al, arXiv:1502.02144
[4] A.H. Guth, D.I. Kaiser and Y. Nomura, Phys.Lett. B733, 112(2014)
[5] F. Bezrukov, D. Gorbunov and M. Shaposhnikov, JCAP 06(2009)029
[6] D.N. Spergel and P.J. Steinhardt, Phys.Rev.Lett. 84, 3760(1999)
[7] J.J. Dalcanton and C.J. Hogan, Ap. J. 561, 35(2001)
[8] Z. Haba, Class.Quant.Grav. 31, 075011(2014)
[9] Z.Haba, A.Stachowski and M.Szydlowski, JCAP 07(2016)024
[10] Z.Haba, Mod.Phys.Lett. A31, 1650146(2016); arXiv:1603.07620[gr-qc]
[11] P.J.E. Peebles and A. Vilenkin, Phys.Rev. D60, 103506(1999)
[12] P.J.E. Peebles and B. Ratra, Ap. J. 325, L17(1988)
[13] Yu.L. Bolotin, A. Kostenko, O.A. Lemets and D.A. Yerokhin, Int. J. Mod. Phys. D24, 1530007 (2014)

[14] M. C. Bento, O. Bertolami and N.C. Santos, Phys. Rev. D65, 067301 (2002)

[15] V. Faraoni, J.B. Dent and E.N. Saridakis, Phys. Rev. D90, 063510 (2014)

[16] G.W. Ford, J.T. Lewis and R.F. O’Connell, Phys. Rev. A37, 4416 (1988)

[17] G.W. Ford and M. Kac, Journ. Stat. Phys. 46, 803 (1987)

[18] H. Kleinert and S.V. Shabanov, Phys. Lett. A200, 171 (1995); arXiv:9503004

[19] A. Berera, I.G. Moss and R.O. Ramos, Rep. Progr. Phys. 72, 026901 (2009)

[20] A. Vilenkin, Phys. Rev. D27, 2848 (1983)

[21] A.A. Starobinsky, in Current Topics in Field Theory, Quantum Gravity and Strings, ed. By H.J. Vega and N. Sanchez, Lecture Notes in Phys. 226, Springer, 1986

[22] A. Stachowski and M. Szydlowski, Phys. Rev. D94, 043521 (2016)

[23] J.E. Lindsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro and M. Abney, Rev. Mod. Phys. 69, 374 (1997)

[24] K. Bamba, S. Nojiri and S.D. Odintsov, Phys. Lett. B737, 374 (2014)

[25] O. Bertolami, C. Cosme and J.G. Rosa, Phys. Lett. B759, 1 (2016); arXiv:1603.06242

[26] A.R. Liddle, C. Pahud and L.A. Urena-Lopez, Phys. Rev. D77, 121301 (2008)

[27] H. Risken, The Fokker-Planck Equation, Springer, 1989

[28] Z. Brzezniak and M. Ondrejat, J. Funct. Anal. 253, 449 (2007)

[29] R.C. Tolman and P. Ehrenfest, Phys. Rev. 36, 1791 (1930)

[30] L. Landau and E.M. Lifshitz, Statistical Physics, Pergamon Press, Oxford, 1980
[31] R. Figari, R. Hoegh-Krohn and C.R. Nappi, 
Commun.Math.Phys. **44**, 265(1975)

[32] G.W.Gibbons and S.W. Hawking, Phys.Rev. **D15**, 2738(1977)

[33] J.M. Overduin and F.I. Cooperstock, Phys.Rev. **D58**, 043506(1998)

[34] Wei Chen and Yong-Shi Wu, Phys. Rev. **D41**, 695(1990)

[35] E.L.D. Perico, J.A.S. Lima, S. Basilakos and J. Sola, 
Phys.Rev. **D88**, 063531(2013)

[36] M. Szydłowski and A. Stachowski, JCAP, **10**(2015)066; arXiv:1507.02114

[37] J.L. Lopez and D.V. Nanopoulos, 
Mod.Phys.Lett. **A11**, 1(1996); arXiv:hep-ph/9501293

[38] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North Holland, 1981

[39] M. Kaplinghat, G. Steigman and T.P. Walker, 
Phys.Rev. **D61**, 1035079(2000)

[40] A. Benoit-Levy and G. Chardin, arXiv:0903.2446

[41] S. Weinberg, Rev.Mod.Phys. **61**, 1(1989)

[42] J.A.S. Lima and J.S. Alcaniz, Phys. Lett. **B600**, 191(2004)

[43] J. P. Mimoso and D. Pavon, arXiv:1610.07788

[44] M. Lisanti, arXiv:1603.0379[hep-ph]

[45] J. Miralda-Escude, Ap.J. **564**, 60(2002); arXiv:0002050

[46] M. Demianski and M. Doroshkevich, MNRAS, **439**, 179(2014)

[47] A. Pontzen and F. Governato, Nature **506**, 171(2014)

[48] J. Fan, A. Katz and J. Shelton, JCAP06(2014)059

[49] O.Y. Gnedin and J.P. Ostriker, Astroph.J. **561**, 61(2001)