COMPLEX MANIFOLDS OF SOBOLEV MAPPINGS AND A HARTOGS-TYPE THEOREM IN LOOP SPACES

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Abstract. We recall the complex structure on the generalised loop spaces \( W^k,2(S, X) \), where \( S \) is a compact real manifold with boundary and \( X \) is a complex manifold, and prove a Hartogs-type extension theorem for holomorphic maps from certain domains in generalized loop spaces.

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Introduction

All manifolds in this paper are supposed to be Hausdorff and second countable. In the first section we recall the notion of the Sobolev class \( W^{k,2}(\Omega) \) for a domain \( \Omega \subset \mathbb{R}^n \) as well as its basic properties. We explain that a composition of a smooth map with a map of class \( W^{k,2} \) is still in the class \( W^{k,2} \). Namely we prove the following theorem which goes back to [M].

Theorem 1. Let \( k > \frac{n}{2} \), then for every \( u \in W^{k,2}_{loc}(\Omega, \mathbb{R}^m) \) and every \( f \in C^k(\mathbb{R}^m) \) the function \( f \circ u \) is in \( W^{k,2}_{loc}(\Omega) \).

This permits us to define correctly the space of Sobolev maps \( W^{k,2}(S, X) \) between real manifolds \( S \) and \( X \) and provide the natural structure of a Hilbert manifold on this space.

In the second section following Lempert [L] we discuss the complex Hilbert structure of the Sobolev manifold \( W^{k,2}(S, X) \), where \( X \) is now a complex manifold. \( S \) everywhere is a compact real manifold with boundary.

For positive integers \( q \geq 1, n \geq 1 \) and real \( r \in ]0,1[ \) the \( q \)-concave Hartogs figure in \( \mathbb{C}^{q+n} \) is defined as

\[
H^q_n(r) := (\Delta^q \times \Delta^n(r)) \cup (A^q_{1-r,1} \times \Delta^n) .
\]

where \( A^q_{1-r,1} = \Delta^q \setminus \Delta^q_{1-r} \). Here \( \Delta^q \) stands for the polydisk in \( \mathbb{C}^q \) centered at zero of radius \( r \). The envelope of holomorphy of \( H^q_n(r) \) is \( \Delta^{q+n} \). We say that a complex manifold \( X \) is \( q \)-Hartogs if every holomorphic mapping \( f : H^q_n(r) \to X \) extends to a holomorphic mapping \( \tilde{f} : \Delta^{q+1} \to X \). If the same holds for a complex Hilbert manifold \( \mathcal{X} \) we say that \( \mathcal{X} \) is \( q \)-Hilbert-Hartogs. We proved in [A-Z] that if \( \mathcal{X} \) is \( q \)-Hilbert-Hartogs then every holomorphic mapping \( f : H^q_n(r) \to \mathcal{X} \) extends to a holomorphic mapping \( \tilde{f} : \Delta^{q+n} \to \mathcal{X} \). For finite dimensional \( X \) this was proved in [I]. In the last section of this paper we prove the following Hartogs-type extension theorem.

Theorem 2. Let \( \mathcal{X} \) be a \( q \)-Hilbert-Hartogs manifold. Then every holomorphic map \( F : W^{k,2}(S, H^q_n(r)) \to \mathcal{X} \) extends to a holomorphic map \( \tilde{F} : W^{k,2}(S, \Delta^q \times \Delta^n) \to \mathcal{X} \).

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This statement gives us an example of pairs of open sets $U \subseteq \hat{U}$ in a complex Hilbert manifold such that holomorphic mappings with values in $q$-Hilbert-Hartogs manifolds extend from $U$ to $\hat{U}$. It shows that $\mathcal{U} := W^{k,2}(S, \Delta^q \times \Delta^n)$ is, in some sense, the "envelope of holomorphy" of $\mathcal{U} := W^{k,2}(S, H^n_q(r))$. For compact $S$ without boundary this theorem was proved in [A-I].

1. **Sobolev maps between smooth manifolds**

Our goal in this section is to define the space of Sobolev $W^{k,2}$-maps from a manifold $S$ to a manifold $X$ and explain that this space possesses a natural topology. Let $\Omega \subset \mathbb{R}^n$ be a domain.

**Definition 1.1.** The Sobolev class $W^{k,2}(\Omega)$ is defined as follows.

$$W^{k,2}(\Omega) := \{ u \in L^2(\Omega) \mid \forall \alpha \in \mathbb{N}^n, \ |\alpha| \leq k, \ D^\alpha u \in L^2(\Omega) \},$$

where $L^2(\Omega)$ is the Hilbert space of equivalence classes of real valued square integrable functions with respect to the Lebesgue measure in $\mathbb{R}^n$.

The space $W^{k,2}(\Omega)$ endowed with the scalar product

$$< u, v >_{W^{k,2}(\Omega)} = \sum_{|\alpha| \leq k} < D^\alpha u, D^\alpha v >_{L^2(\Omega)}$$

is a Hilbert space. Recall that the Fourier transform of a function from the Schwarz space $u \in \mathcal{S}(\mathbb{R}^n)$ is defined as

$$\hat{u}(\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} u(x)e^{-ix\xi}d\mu(x),$$

where $\mu$ is the Lebesgue measure in $\mathbb{R}^n$. By the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and using the Placherel identity we can extend the Fourier transform to the isometry of $L^2(\mathbb{R}^n)$ onto itself, i.e. for $f \in L^2(\mathbb{R}^n)$ one has that $||f||_{L^2(\mathbb{R}^n)} = ||\hat{f}||_{L^2(\mathbb{R}^n)}$. One can give the characterisation of Sobolev maps by the Fourier transform.

$$u \in W^{k,2}(\mathbb{R}^n) \iff \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k, \ D^\alpha u \in L^2(\mathbb{R}^n)$$

$$\iff \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k, \ \xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$$

$$\iff (1 + |\xi|^2)^\frac{k}{2} \hat{u} \in L^2(\mathbb{R}^n)$$

Moreover the corresponding norms are equivalent. More precisely we have the following.

**Proposition 1.1.** For $u \in W^{k,2}(\mathbb{R}^n)$ we have the inequalities:

i) $||u||_{W^{k,2}(\mathbb{R}^n)} \leq ||(1 + |\xi|^2)^\frac{k}{2} \hat{u}||_{L^2(\mathbb{R}^n)}$;

ii) there exists $a \in \mathbb{R}^+$ wich depends on $k$ and $n$ only such that

$$(2) \quad ||(1 + |\xi|^2)^\frac{k}{2} \hat{u}||_{L^2(\mathbb{R}^n)} \leq a ||u||_{W^{k,2}(\mathbb{R}^n)}.$$

**Proof.** First we compute the following:

$$||u||_{W^{k,2}}^2 = \sum_{|\alpha| \leq k} ||D^\alpha u||_{L^2}^2 = \sum_{|\alpha| \leq k} ||\hat{D^\alpha u}||_{L^2}^2 = \sum_{|\alpha| \leq k} ||\xi^{\alpha_1} \ldots \xi^{\alpha_n} \hat{u}||_{L^2}^2$$

$$= \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \xi^{2\alpha_1} \ldots \xi^{2\alpha_n} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi.$$

So we can establish the following equality

$$||u||_{W^{k,2}} = \sqrt{\int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi}.$$
For $(i)$ we have $(1 + \xi^2)^k = (1 + \xi_1^2 + \ldots + \xi_n^2)^k = \sum_{|\alpha| \leq k} a_\alpha \xi^{2\alpha} \geq \sum_{|\alpha| \leq k} \xi^{2\alpha}$ since $a_\alpha \geq 1.$ And therefore

$$\|u\|_{W^{k,2}} = \left( \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \leq \left( \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \leq \left( \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} 2^{2\alpha} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \leq a \|u\|_{W^{k,2}}.$$ 

As for $(ii),$ since $(1 + |\xi|^2)^k = \sum_{|\alpha| \leq k} a_\alpha \xi^{2\alpha} \leq a^2 \sum_{|\alpha| \leq k} \xi^{2\alpha}$ with $a = \max_{|\alpha| \leq k} \sqrt{a_\alpha},$ we obtain

$$\left\| (1 + |\cdot|^2)^{\frac{k}{2}} \hat{u} \right\|_{L^2} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \leq a \left( \int_{\mathbb{R}^n} \sum_{|\alpha| \leq k} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} \leq a \|u\|_{W^{k,2}}.$$ 

Proposition is proved. \(\square\)

Denote by $W^{k,2}_{\text{loc}}(\Omega)$ the space of distributions $u \in \mathcal{D}'(\Omega)$ which are locally in $W^{k,2},$ i.e. for every relatively compact $D \Subset \Omega$ the restriction $u|_D \in W^{k,2}(D).$ It is clear that for $u \in W^{k,2}_{\text{loc}}(\Omega)$ and a mapping $f : B \to \Omega$ from a domain $B \subset \mathbb{R}^m$ to $\Omega$ of class $C^k$ the composition $u \circ f \in W^{k,2}_{\text{loc}}(B).$

Recall the Sobolev imbedding theorem: if $f \in W^{k,2}(\Omega)$ and $k > \frac{n}{2} + m,$ then there exists an $m$-times continuously differentiable function on $\Omega$ that is equal to $f$ almost everywhere. Therefore the condition $k > \frac{n}{2}$ will be always assumed along this text in order to insure that all $u \in W^{k,2}$ are at least continuous.

**Theorem 1.1.** Let $k > \frac{n}{2},$ then for every $u \in W^{k,2}_{\text{loc}}(\Omega)$ and $f \in C^k(\mathbb{R})$ the function $f \circ u$ is in $W^{k,2}_{\text{loc}}(\Omega).$

**Proof.** The proof will be achieved in three steps.

Step 1: We shall state it in the form of a lemma.

**Lemma 1.1.** For $\xi, \eta \in \mathbb{R}^n$ and $t \geq 0$ we have the following inequality:

$$(3) \quad (1 + |\xi|^2)^t \leq 4^t ((1 + |\xi - \eta|^2)^t + (1 + |\eta|^2)^t).$$

**Proof.** Since $|\xi - \eta + \eta|^2 \leq \left( |\xi - \eta| + |\eta| \right)^2 \leq 2 \left( |\xi - \eta|^2 + |\eta|^2 \right)$

$$(1 + |\xi|^2)^t = (1 + |\xi - \eta + \eta|^2)^t \leq (1 + 2|\xi - \eta|^2 + 2|\eta|^2)^t \leq (4 + 2|\xi - \eta|^2 + 2|\eta|^2)^t \leq 2^t (1 + |\xi - \eta|^2 + 1 + |\eta|^2)^t \leq 2^t (\max(1 + |\xi - \eta|^2, 1 + |\eta|^2))^t \leq 4^t (\max(1 + |\xi - \eta|^2, 1 + |\eta|^2))^t \leq 4^t (\max(1 + |\xi - \eta|^2, 1 + |\eta|^2))^t + 4^t (\min(1 + |\xi - \eta|^2, 1 + |\eta|^2))^t = 4^t (1 + |\xi - \eta|^2)^t + 4^t (1 + |\eta|^2)^t.$$ 

Lemma is proved. \(\square\)

Step 2: Let us state it again as a lemma.

**Lemma 1.2.** If $k > n/2$ then the product of two Sobolev maps $u, v \in W^{k,2}(\mathbb{R}^n)$ is in $W^{k,2}(\mathbb{R}^n)$ with the estimate

$$\|uv\|_{W^{k,2}(\mathbb{R}^n)} \leq C \|u\|_{W^{k,2}(\mathbb{R}^n)} \|v\|_{W^{k,2}(\mathbb{R}^n)}.$$
Proof. Since \( \hat{uv} = \hat{u} \ast \hat{v} \) we have
\[
\|uv\|_{L^{k,2}} \leq \left\| \hat{uv}(1 + |\cdot|^2)^{\frac{k}{2}} \right\|_{L^2} = \sqrt{\int_{\mathbb{R}^n} |(\hat{u} \ast \hat{v})(\xi)|^2(1 + |\xi|^2) \, d\xi} = \\
= \sqrt{\int_{\xi \in \mathbb{R}^n} \left( \int_{\eta \in \mathbb{R}^n} \hat{u}(\xi - \eta) \hat{v}(\eta) (1 + |\eta|^2)^{\frac{k}{2}} \, d\eta \right)^2 \, d\xi} \leq \\
\leq \left\| \int_{\xi \in \mathbb{R}^n} \left( \int_{\eta \in \mathbb{R}^n} |\hat{u}(\xi - \eta) \hat{v}(\eta)| (1 + |\eta|^2)^{\frac{k}{2}} \, d\eta \right)^2 \, d\xi + \left\| \int_{\xi \in \mathbb{R}^n} 4^k |\hat{u}(\xi - \eta) \hat{v}(\eta)| (1 + |\eta|^2)^{\frac{k}{2}} \, d\eta \right\|_{L^2} \right. \\
\leq \left. \|u\|_{L^p} \|v\|_{L^q} \right\|_{L^p} \leq \int_X \|F(x,y)\|_{L^p} \, dx.
\]

Therefore by Minkowski integral inequality \([4]\) we obtain
\[
\|uv\|_{L^{k,2}} \leq \\
\leq 4^k \left\| \int_{\eta \in \mathbb{R}^n} |\hat{u}(\xi - \eta) \hat{v}(\eta)| (1 + |\eta|^2)^{\frac{k}{2}} \, d\eta \right\|_{L^2} + 4^k \left\| \int_{s \in \mathbb{R}^n} |\hat{u}(s) \hat{v}(\xi - s)| (1 + |s|^2)^{\frac{k}{2}} \, ds \right\|_{L^2} \\
\leq 4^k \left\| \int_{\eta \in \mathbb{R}^n} |\hat{u}(\xi - \eta) \hat{v}(\eta)| (1 + |\eta|^2)^{\frac{k}{2}} \, d\eta \right\|_{L^2} + 4^k \left\| \int_{s \in \mathbb{R}^n} |\hat{u}(s) \hat{v}(\xi - s)| (1 + |s|^2)^{\frac{k}{2}} \, ds \right\|_{L^2} \\
\leq 4^k \left\| \hat{v}(\eta)| \hat{u}(\xi - \eta) \hat{v}(\eta)| (1 + |\eta|^2)^{\frac{k}{2}} \, d\eta \right\|_{L^2} + 4^k \left\| \hat{v}(\eta)| \hat{u}(s) \hat{v}(\xi - s)| (1 + |s|^2)^{\frac{k}{2}} \, ds \right\|_{L^2} \\
\leq 4^k \left\| \hat{v}(\eta)| a \right\|_{W^{k,2}} d\eta + 4^k \left\| \hat{v}(\eta)| a \right\|_{W^{k,2}} ds.
\]
In the first integral we transform as follows.

\[ 4^k \int_{\eta \in \mathbb{R}^n} |\hat{\nu}(\eta)| |a| \|u\|_{W^{k,2}} d\eta \leq 4^k a \|u\|_{W^{k,2}} \int_{\eta \in \mathbb{R}^n} |\hat{\nu}(\eta)| (1 + |\eta|^2)^{\frac{k}{2}} (1 + |\eta|^2)^{-\frac{k}{2}} d\eta \leq 4^k a^2 \|u\|_{W^{k,2}} \|v\|_{W^{k,2}} \left\| (1 + |\xi|^2)^{-\frac{k}{2}} \right\|_{L^2}. \]  

We make the same with the second integral and obtain

\[ \|uv\|_{W^{k,2}} \leq 4^k a^2 \|u\|_{W^{k,2}} \|v\|_{W^{k,2}} \left\| (1 + |\xi|^2)^{-\frac{k}{2}} \right\|_{L^2} + 4^k a^2 \|u\|_{W^{k,2}} \|v\|_{W^{k,2}} \left\| (1 + |\xi|^2)^{-\frac{k}{2}} \right\|_{L^2} \]

\[ \leq \left( 2^{2k+1} a^2 \left\| (1 + |\xi|^2)^{-\frac{k}{2}} \right\|_{L^2} \right) \|u\|_{W^{k,2}} \|v\|_{W^{k,2}}. \]

Lemma is proved. \[\square\]

This step implies that the product of two \( W^{k,2}_{\text{loc}}(\Omega) \) maps are still in \( W^{k,2}_{\text{loc}}(\Omega) \) and consequently this implies that for any polynomial \( P \) and for any \( u \in W^{k,2}_{\text{loc}}(\Omega) \) the function \( P(u) \in W^{k,2}_{\text{loc}}(\Omega) \).

Step 3 : For \( I \) a closed interval one defines the norm on the space \( C^k(I) \) by

\[ \|f\|_{C^k(I)} = \sum_{j=0}^{k} \sup_{x \in I} |f^{(j)}(x)|. \]

Remark that:

- The norm \( \|f\|_{C^0(I)} = \sup_{x \in I} |f(x)| \) correspond to the usual sup-norm.
- The space \( C^k(I) \) endowed with the norm \( \| \cdot \|_{C^k(I)} \) is a Banach space.

Recall the approximation theorem of Weierstrass:

Let \( f \in C^0(I) \). Then for every \( \epsilon > 0 \), there exists a polynomial \( P \) such that for all \( x \in [a, b] \), we have \( |f(x) - P(x)| < \epsilon \), i.e. \( \|f - P\|_{C^0(I)} < \epsilon \).

From that statement, one can deduce an approximation with respect to the norm \( C^k \):

**Proposition 1.2.** Any function \( f : I \to \mathbb{R} \) of class \( C^k \) defined on a closed interval \( I \) can be uniformly approximated by polynomials with respect to the norm \( C^k \), i.e.

\[ \forall \epsilon > 0, \ \exists P \in \mathbb{R}[X], \text{ such that } \|f - P\|_{C^k(I)} < \epsilon. \]

**Proof.** Let \( I = [a, b] \) with \( a, b \in \mathbb{R} \). Since \( f \) is \( C^k \), use the approximation theorem on \( f^{(k)} \). It exists \( P_k \in \mathbb{R}[X] \) such that \( \|P_k - f^{(k)}\|_{C^0(I)} < \epsilon \). Then we define

\[ \forall l \in [1, k], \ P_{k-l}(x) = \int_a^x P_{k-l+1}(t) dt + f^{(k-l)}(a). \]

We can prove by induction on \( l \) that

\[ \forall l \in [0, k], \forall x \in [a, b], \ |P_{k-l}(x) - f^{(k-l)}(x)| \leq \epsilon \frac{(x-a)^l}{l!}. \]  

Indeed for \( l = 0 \) the assertion \( (7) \) is true and we suppose that \( (7) \) is true for \( l - 1 \). Then we can write the following.

\[ |P_{k-l}(x) - f^{(k-l)}(x)| = \left| \int_a^x P_{k-l+1}(t) dt + f^{(k-l)}(a) - f^{(k-l)}(x) \right| = \]

\[ = \left| \int_a^x \left( P_{k-l+1}(t) - f^{(k-l+1)}(t) \right) dt \right| \leq \int_a^x \left| P_{k-l+1}(t) - f^{(k-l+1)}(t) \right| dt \]

\[ \leq \int_a^x \epsilon \frac{(t-a)^{l-1}}{(l-1)!} dt \leq \epsilon \frac{(x-a)^l}{l!}. \]
For all \( l \in [0, k] \), we obtain that \( \| P_0^{k-l} - f^{(k-l)} \|_{C^0(I)} \leq \epsilon \frac{(b-a)^l}{l!} \) and then \( \| P_0 - f \|_{C^k(I)} \leq \epsilon \sum_{l=0}^{k} \frac{(b-a)^l}{l!} \).

Proposition is proved. \( \square \)

Now we can finish the proof of the theorem. Fix a relatively compact subdomain \( D \Subset \Omega \). Since \( u \) is continuous there exists a closed interval \( I \supset u(D) \). Let \( (P_s)_{s \in \mathbb{N}} \) be a sequence of polynomials that converges to \( f \) in \( C^k(I) \). Then \( (P_s)_{s \in \mathbb{N}} \) is a Cauchy sequence: for every \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for any \( l, s > N \) we have \( \| P_l - P_s \|_{C^k(I)} < \epsilon \). By Step 2 for any \( s \in \mathbb{N} \) the function \( P_s(u) \) is in \( W^{l,2}(D) \). Let us prove that \( (P_s(u))_{s \in \mathbb{N}} \) is a Cauchy sequence in the Hilbert space \( W^{k,2}(D) \). Note \( P_{sl} = P_s - P_l \).

For \( \alpha \) with \( |\alpha| \leq k \) and any polynomial \( P \) we have

\[
D^\alpha P(u) = \sum_{r=1}^{s} \left( \sum_{\alpha_1 + \ldots + \alpha_r = \alpha \atop \alpha_1, \ldots, \alpha_r \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_r} u \right) P^{(r)}(u). \tag{11}
\]

For \( P(X) = X^p \) relation (11) becomes

\[
\sum_{\alpha_1 + \ldots + \alpha_p = \alpha \atop \alpha_1, \ldots, \alpha_p \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_p} u = \frac{1}{p!} \left( D^\alpha (u^p) - \sum_{r=1}^{p-1} \left( \sum_{\alpha_1 + \ldots + \alpha_r = \alpha \atop \alpha_1, \ldots, \alpha_r \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_r} u \right) \frac{p!}{(p-r)!} u^{p-r} \right).
\]

Let us prove by induction on \( p \leq |\alpha| \) that

\[
\sum_{\alpha_1 + \ldots + \alpha_p = \alpha \atop \alpha_1, \ldots, \alpha_p \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_p} u \in L^2(D).
\]

For \( p = 1 \) it is true because \( \frac{\partial |\alpha|}{\partial x_j} u \in L^2(D) \) for every \( j = 1, \ldots, n \) since \( u \in W^{k,2}(D) \). Let suppose that is true for all \( s \leq p - 1 \). Then for \( p \leq |\alpha| \) using the previous expression we obtain

\[
\left\| \sum_{\alpha_1 + \ldots + \alpha_p = \alpha \atop \alpha_1, \ldots, \alpha_p \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_p} u \right\|_{L^2} \leq \frac{1}{p!} \left\| D^\alpha (u^p) \right\|_{L^2} + \sum_{r=1}^{p-1} \left\| \sum_{\alpha_1 + \ldots + \alpha_r = \alpha \atop \alpha_1, \ldots, \alpha_r \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_r} u \right\|_{L^2}.
\]

By Lemma 1.2 the map \( u^p \in W^{k,2}(D) \), therefore \( D^\alpha (u^p) \in L^2(D) \) for \( |\alpha| \leq k \). For the second term since \( u \) is continuous we have that \( u \) is bounded on \( D \) by \( M \). Then

\[
\left\| \sum_{\alpha_1 + \ldots + \alpha_r = \alpha \atop \alpha_1, \ldots, \alpha_r \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_r} u \right\|_{L^2} \leq \frac{M^{p-r}}{(p-r)!} \left\| \sum_{\alpha_1 + \ldots + \alpha_r = \alpha \atop \alpha_1, \ldots, \alpha_r \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_r} u \right\|_{L^2},
\]

and by induction hypothesis

\[
\left\| \sum_{\alpha_1 + \ldots + \alpha_r = \alpha \atop \alpha_1, \ldots, \alpha_r \neq \emptyset} D^{\alpha_1} u \cdots D^{\alpha_r} u \right\|_{L^2} < \infty.
\]
We obtain from (11) that
\[ \|P_s(u) - P_\alpha(u)\|_{W^{k,2}}^2 = \|P_s(u)\|_{W^{k,2}}^2 = \sum_{|\alpha| \leq k} \|D^\alpha P_s(u)\|_{L^2}^2 \leq \]
\[ \sum_{|\alpha| \leq k} \sum_{r=1}^{|\alpha|} \left( \sum_{a_1 + \cdots + a_r = \alpha} D^{a_1} u \cdots D^{a_r} u \right)^2 \|P_s(u)^r\|_{L^2}^2 \]
\[ \leq \sum_{|\alpha| \leq k} \sum_{r=1}^{|\alpha|} \left( \sum_{a_1 + \cdots + a_r = \alpha} D^{a_1} u \cdots D^{a_r} u \right)^2 \|P_s(u)^r\|_{L^2}^2 \]

Then we can conclude that \((P_s(u))_{s \in \mathbb{N}}\) is a Cauchy sequence on the Hilbert space \(W^{k,2}(D)\). Therefore the sequence converge to \(f(u) \in W^{k,2}(D)\). Theorem is proved. \(\square\)

**Definition 1.2.** Let \(\Omega \subset \mathbb{R}^n\) be a domain. The Sobolev space \(W^{k,2}(\Omega, \mathbb{R}^m)\) is defined as
\[ W^{k,2}(\Omega, \mathbb{R}^m) = \{ u \in L^2(\Omega, \mathbb{R}^m) \mid \forall \alpha \in \mathbb{N}^n, |\alpha| \leq k, D^\alpha u \in L^2(\Omega, \mathbb{R}^m) \}. \]

As in the one dimensional case one can prove an analogous of the Theorem 1.1. Namely

**Theorem 1.2.** Let \(k > \frac{n}{2},\) then for every \(u \in W^{k,2}_{\text{loc}}(\Omega, \mathbb{R}^m)\) and \(f \in C^k(\mathbb{R}^m)\) the function \(f \circ u\) is in \(W^{k,2}_{\text{loc}}(\Omega)\).

**Proof.** The idea of the proof follows the one dimensional case. Fix a relatively compact subdomain \(D \Subset \Omega\).

For any polynomial \(P(T_1, \ldots, T_n)\) and \(|\alpha| \leq k\) one has
\[ D^\alpha(P(u)) = \sum_{r=1}^{|\alpha|} \sum_{a_1 + \cdots + a_r = \alpha} (d^r P)_{a_1, \ldots, a_r} D^{a_1} u \cdots D^{a_r} u \]
\[ = \sum_{r=1}^{|\alpha|} \sum_{a_1 + \cdots + a_r = \alpha} \left( \sum_{i_1, \ldots, i_r = 1}^m \frac{\partial^r P}{\partial t_{i_1} \cdots \partial t_{i_r}}(u) D^{a_1} u_{i_1} \cdots D^{a_r} u_{i_r} \right) \]
\[ = \sum_{r=1}^{|\alpha|} \sum_{i_1, \ldots, i_r = 1}^m \frac{\partial^r P}{\partial t_{i_1} \cdots \partial t_{i_r}}(u) \left( \sum_{a_1 + \cdots + a_r = \alpha} D^{a_1} u_{i_1} \cdots D^{a_r} u_{i_r} \right) \]

Fix \(\alpha \in \mathbb{N}^n\) with \(|\alpha| \leq k\). We prove by induction on \(p\) with \(1 \leq p \leq |\alpha|\) that
\[ \forall I = (j_1, \ldots, j_p) \in [1, m]^p, \sum_{a_1 + \cdots + a_p = \alpha} D^{a_1} u_{j_1} \cdots D^{a_p} u_{j_p} \in L^2(D). \]

For \(p = 1\) the property is true because \(D^\alpha u_j \in L^2(D)\) for \(j = 1, \ldots, m\) since \(D^\alpha u \in L^2(D, \mathbb{R}^m)\). Let suppose that is true for all \(s \leq p - 1\). For \(p \leq |\alpha|\) one can define for every \(J = (j_1, \ldots, j_p) \in [1, m]^p \beta = e_{j_1} + \cdots + e_{j_p}\) with \((e_j)_{i=1, \ldots, m}\) the standard basis in \(\mathbb{R}^m\). Then consider \(P(T) = T^\beta = T_{i_1}^{\beta_1} \cdots T_{i_m}^{\beta_m}\) in relation (11)
\[ D^\alpha(u^\beta) = \sum_{r=1}^{|\alpha|} \sum_{i_1, \ldots, i_r = 1}^m \frac{\partial^r P}{\partial t_{i_1} \cdots \partial t_{i_r}}(u) \left( \sum_{a_1 + \cdots + a_r = \alpha} D^{a_1} u_{i_1} \cdots D^{a_r} u_{i_r} \right) \]
\[ = \sum_{r=1}^p \sum_{i_1, \ldots, i_r = 1}^m \frac{\partial^r T^\beta}{\partial t_{i_1} \cdots \partial t_{i_r}}(u) \left( \sum_{a_1 + \cdots + a_r = \alpha} D^{a_1} u_{i_1} \cdots D^{a_r} u_{i_r} \right). \]
Note that \( \frac{\partial^p T^\beta}{\partial t_{i_1} \cdots \partial t_{i_p}} = \beta ! \) and the number of indices \( J \) such that \( \beta = e_{j_1} + \cdots + e_{j_p} \) is equal again to \( \beta ! \). The other term are equal to zero. Then we obtain

\[
(\beta!)^2 \sum_{\substack{\alpha_1 + \cdots + \alpha_p = \alpha \\
\alpha_1, \ldots, \alpha_p \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_p} u_{i_p} = \\
= D^\alpha(u^\beta) - \sum_{r=1}^{m-1} \sum_{i_1, \ldots, i_r = 1}^m \frac{\partial^r T^\beta}{\partial t_{i_1} \cdots \partial t_{i_r}}(u) \left( \sum_{\substack{\alpha_1 + \cdots + \alpha_r = \alpha \\
\alpha_1, \ldots, \alpha_r \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_r} u_{i_r} \right),
\]

where \( \beta! = \beta_1! \cdots \beta_m! \).

The term \( D^\alpha(u^\beta) = D^\alpha(u^\beta_1 \cdots u^\beta_m) \) is in \( L^2(D) \) since \( u_1, \ldots, u_m \) are in \( W^{k,2}(D) \) and Lemma 1.2. For the other term one can remark first that \( \frac{\partial^r X^\beta}{\partial x_{i_1} \cdots \partial x_{i_r}}(u) \) is a polynomial evaluate on \( u \) and since \( u \) is bounded on \( D \) this polynomial is bounded by \( M \in \mathbb{R} \). Therefore for every \( (i_1, \ldots, i_r) \in [1, m]^r \) one has

\[
\left\| \frac{\partial^r X^\beta}{\partial x_{i_1} \cdots \partial x_{i_r}}(u) \left( \sum_{\substack{\alpha_1 + \cdots + \alpha_r = \alpha \\
\alpha_1, \ldots, \alpha_r \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_r} u_{i_r} \right) \right\|_{L^2(D)} \leq M \left\| \sum_{\substack{\alpha_1 + \cdots + \alpha_r = \alpha \\
\alpha_1, \ldots, \alpha_r \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_r} u_{i_r} \right\|_{L^2(D)}
\]

and by induction hypothesis this is in \( L^2(D) \). Then for every \( |\alpha| \leq k \)

\[
\forall p \leq |\alpha|, \forall I = (j_1, \ldots, j_p) \in [1, m]^p, \sum_{\substack{\alpha_1 + \cdots + \alpha_p = \alpha \\
\alpha_1, \ldots, \alpha_p \neq 0}} D^{\alpha_1} u_{j_1} \cdots D^{\alpha_p} u_{j_p} \in L^2(D).
\]

Now, since \( u \) is continuous there exists a compact set \( K \supset u(D) \). Let \( (P_s)_{s \in \mathbb{N}} \) be a sequence of polynomials that converges to \( f \) in \( C^k(K) \). Then \( (P_s)_{s \in \mathbb{N}} \) is a Cauchy sequence : for every \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for any \( l, s > N \) we have \( \|P_l - P_s\|_{C^k(K)} < \epsilon \). By Lemma 1.2 for any \( s \in \mathbb{N} \) the function \( P_s(u) \) is in \( W^{k,2}(D) \). Let us prove that \( (P_s(u))_{s \in \mathbb{N}} \) is a Cauchy sequence in the Hilbert space \( W^{k,2}(D) \). Note \( P_d = P_s - P_l \).

\[
(15) \|D^\alpha(P_d(u))\|_{L^2(D)}^2 \leq \sum_{r=1}^{m-1} \sum_{i_1, \ldots, i_r = 1}^m \left\| \frac{\partial^r P_d}{\partial t_{i_1} \cdots \partial t_{i_r}}(u) \right\|_{L^2(D)}^2 \sum_{\substack{\alpha_1 + \cdots + \alpha_r = \alpha \\
\alpha_1, \ldots, \alpha_r \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_r} u_{i_r} \right\|_{L^2(D)}^2
\]

\[
\leq \sum_{r=1}^{m-1} \sum_{i_1, \ldots, i_r = 1}^m \epsilon^2 \left\| \sum_{\substack{\alpha_1 + \cdots + \alpha_r = \alpha \\
\alpha_1, \ldots, \alpha_r \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_r} u_{i_r} \right\|_{L^2(D)}^2
\]

\[
(17) \leq \epsilon^2 \sum_{r=1}^{m-1} \sum_{i_1, \ldots, i_r = 1}^m \left\| \sum_{\substack{\alpha_1 + \cdots + \alpha_r = \alpha \\
\alpha_1, \ldots, \alpha_r \neq 0}} D^{\alpha_1} u_{i_1} \cdots D^{\alpha_r} u_{i_r} \right\|_{L^2(D)}^2
\]

Therefore \( (P_s(u))_s \) is a Cauchy sequence in the Hilbert space \( W^{k,2}(D) \) so the sequence converge to \( f(u) \in W^{k,2}(D) \). Theorem is proved.

From that one can define Sobolev spaces of maps between manifolds.

**Definition 1.3.** Let \( S \) and \( X \) be real manifolds of class \( C^k \). A map \( u : S \to X \) is said to be in \( W^{k,2}(S,X) \) if for every \( s \in S \), every coordinate chart \((V, \psi)\) which contains \( s \) and every coordinate chart \((U, \phi)\) which contains \( u(s) \) one has that \( \phi \circ u \circ \psi^{-1} \in W^{k,2} \).
Theorem 1.2 insures that this definition is correct. The space of $W^{k,2}$-maps from $S$ to $X$ we denote as $W^{k,2}(S, X)$. Notice that $W^{k,2}(S, X)$ inherits the natural topology from $W^{k,2}_{loc}$. We shall call this topology the Sobolev topology.

2. COMPLEX STRUCTURE ON THE SPACE OF SOBOLEV MAPS

From now on let $S$ be a compact connected $n$-dimensional real manifold with boundary. Let $X$ be a finite dimensional complex manifold. Our goal in this section is to equip the Sobolev space $W^{k,2}(S, X)$ with a natural structure of a complex Hilbert manifold.

For a set $U \subset S \times X$ and $s \in S$ we write

$$U^s := \{ x \in X \mid (s, x) \in U \} \quad \text{and} \quad \epsilon^s : U^s \to U \quad x \mapsto (s, x).$$

**Lemma 2.1.** Given $g \in W^{k,2}(S, X)$ with $k > \frac{n}{2}$, there exists a $W^{k,2}$-diffeomorphism $G$ between a neighborhood $U \subset S \times X$ of the graph $\{(s, g(s)) \mid s \in S\}$ of $g$ and a neighborhood of the zero section in $g^*TX$ such that

i) $\{G(s, g(s)) \mid s \in S\}$ is the zero section of $g^*TX$;

ii) $G^* = G \circ \epsilon^s$ maps $U^s$ biholomorphically on a neighborhood of $0 \in T_{g(s)}X$;

iii) $dG^s_{g(s)}$ is the identity map.

**Proof.** i) We recall the argument from [L] pointing out the smoothness of $G$. Let $(\Omega_j, \phi_j)$ be an atlas of the complex manifold $X$. Then the sets $S_j = g^{-1}(\Omega_j) \subset S$ form an open covering of $S$. Consider $U_j \subset S_j \times \Omega_j$ a neighborhood of the graph of $g|_{S_j}$. We can construct locally the diffeomorphism $G_j$ by

$$G_j : U_j \to g^*TX \quad (s, x) \mapsto \left( s, (d\phi_j^{-1})_{g(s)}(\phi_j(x) - \phi_j(g(s))) \right).$$

Notice that $G_j$ is of class $W^{k,2}$ because such is $g$. It remains to glue all the $G_j$. Take $\{\eta_j\}$ a $C^k$-partition of unity subordinated to the covering $\{S_j\}$ and define $G(s, x) = \sum_j \eta_j(s)G_j(s, x)$. Choose then the restriction of $G$ to a suitable $U \subset \bigcup_j U_j$. Remark that $G(s, g(s)) = (s, 0)$.

ii) We have that $G_j^* = G_j \circ \epsilon^s(x) = (d\phi_j^{-1})_{g(s)}(\phi_j(x) - \phi_j(g(s)))$. The map $G_j^* : U_j^s \to g^*TX$ is holomorphic for every $s \in S_j$ since $x \mapsto (d\phi_j^{-1})_{g(s)}(\phi_j(x) - \phi_j(g(s)))$ is holomorphic. Here $U_j^s = \{ x \in X \mid (s, x) \in U_j \}$. Holomorphicity of $G_j^* = \sum_j \eta_j(s)G_j^*(s)$ follows.

iii) We compute the differential of $G^*$ and obtain

$$dG^s_{g(s)} = (d\phi_j^{-1})_{g(s)}(d\phi_j)_{g(s)} = \text{Id} \quad \square$$

For $g \in W^{k,2}(S, X)$ choose $U_g$ and $G$ as in the previous lemma. Those $h \in W^{k,2}(S, X)$ whose graph $\Gamma_h := \{(s, h(s)) \mid s \in S\}$ is contained in $U_g$ form a neighborhood $U_g \subset W^{k,2}(S, X)$ of $g$:

$$U_g = \{ h \in W^{k,2}(S, X) \mid \Gamma_h \subset U_g \}.$$

For $h \in U_g$ define the section $\psi_g(h) = G(\cdot, h(\cdot)) \in W^{k,2}(S, g^*TX)$. Thus $\psi_g$ is a homeomorphism between $U_g$ and an open neighborhood of zero section in $W^{k,2}(S, g^*TX)$. We can define the chart $(U_g, \psi_g)$ where local coordinates are in a complex Hilbert space $W^{k,2}(S, g^*TX)$. Now we need to verify that transition maps are holomorphic.

Let $h \in W^{k,2}(S, X)$ be such that $\Gamma_h \subset U_g \cap U_{g'}$, i.e. $h \in U_g \cap U_{g'}$. For $s \in S$ we have

$$\psi_{g'}(h)(s) = (\psi_{g'} \circ \psi_{g}^{-1})(\psi_{g}(h)(s)) = \left[ G' \circ G_s^{-1} \right](\psi_{g}(h)(s)) = \left[ G'(s, \cdot) \circ G(s, \cdot)^{-1} \right](\psi_{g}(h)(s)).$$

Due to item iii) of the Lemma just proved we have that $G' \circ G_s^{-1}$ is a biholomorphism between an appropriate open subsets of $T_{g(s)}X$ and $T_{g'(s)}X$. Therefore the value $\psi_{g'}(h)(s)$ depends holomorphically on $\psi_{g}(h)(s)$.

In more details we have a $W^{k,2}$-regular maps $P = G' \circ G^{-1}$ between open sets $V \subset g^*TX$ and $V' \subset g'^*TX$ such that for every $s \in S \ P(s, v)$ holomorphically depends on $v \in V \cap T_{g(s)}X$. 
where $P(s,v) = (s,P^s(v))$. Let $V$ be the open set of sections of $g^*TX$ which are contained in $V$. The same for $V'$. We obtain a mapping $\mathcal{P} : V \rightarrow V'$ defined as $\mathcal{P}(h)(s) = P(s,P^h(s))$, where $h$ is a $W^{k,2}$-section of $g^*TX$ contained in $V$. Then $\mathcal{P}$ is holomorphic. This easily follows from Gâteaux differentiability and continuity of $\mathcal{P}$. Thus $W^{k,2}(S, X)$ has the structure of a complex Hilbert manifold.

**Lemma 2.2.** Let $D$ and $X$ be finite dimensional complex manifolds and let $S$ be an $n$-dimensional compact real manifold with boundary. A mapping $F : D \times S \rightarrow X$ represents a holomorphic map $F_* : D \rightarrow W^{k,2}(S, X)$ if and only if the following holds:

i) for every $s \in S$ the map $F(\cdot, s) : D \rightarrow X$ is holomorphic;

ii) for every $z \in D$ one has $F(z, \cdot) \in W^{k,2}(S, X)$ and the correspondence $D \ni z \mapsto F(z, \cdot) \in W^{k,2}(S, X)$ is continuous with respect to the Sobolev topology on $W^{k,2}(S, X)$ and the standard topology on $D$.

**Proof.** $\Rightarrow$ Given $F_* : D \rightarrow W^{k,2}(S, X)$ we construct $F : D \times S \rightarrow X$ as follows. For $s \in S$ and $z \in D$ we define $F(z, s) = F_*(z)(s)$. By the assumption about $F_*$ for any $z \in D$ the map $F(z, \cdot)$ is in $W^{k,2}(S, X)$ and the map $z \mapsto F(z, \cdot) = F_*(z)$ is continuous (in fact it is holomorphic). To prove the holomorphicity of $F(z, s)$ for a fixed take any $s \in S$ in a neighborhood of some $z_0$ take any chart $(U, \psi_g)$ which contains the graph of $F_*(z_0)$. We have $\psi_g(F(z, s))(s) = G(s, F_*(z)(s)) = G(s, F(z, s))$ for $z$ close to $z_0$ by definition of $\psi_g$. This is holomorphic by the definition of the complex structure of $W^{k,2}(S, X)$. Therefore $F : D \times S \rightarrow X$ is holomorphic in $z$.

Conversely given $F : D \times S \rightarrow X$ satisfying i) and ii) we can construct $F_*$ as follows. For $z \in D$ we define $F_*(z) = F(z, \cdot)$. Mapping $F_*$ is well defined since $F(z, \cdot) \in W^{k,2}(S, X)$ and is continuous by ii). For the holomorphicity again we take a chart $(U, \psi_g)$ and we consider the map $\psi_g \circ F_*$ defined for $z \in D$ by $\psi_g \circ F_*(z) = G(\cdot, F(z, \cdot)(\cdot))$. By i) for every $s \in S$ the map $s \mapsto G(s, F(z, s)(s))$ is holomorphic by composition of holomorphic maps $F(\cdot, s)$ and $G^*$. Then $\psi_g \circ F_*$ is holomorphic. $\square$

**Remark.** If $X$ is a smooth real manifold one can repeat the same construction to ensure that $W^{k,2}(S, X)$ has a structure of a smooth Hilbert manifold. In fact the holomorphy of item ii) in lemma 2.1 should be replaced by smoothness since coordinate charts $\psi_g$ are smooth.

### 3. A Hartogs-type theorem

In [1] it was proved the following result.

**Theorem 3.1.** If a complex manifold $X$ is $q$-Hartogs then for any $(p, n)$ with $p \geq q, n \geq 1$ the map $f : H^p_\omega(r) \rightarrow X$ extends holomorphically to $\Delta^{q+n}$.

For Hilbert $\mathcal{X}$ it was proved in [A-Z]. The proof of this result for Hilbert $\mathcal{X}$ lies on the following two statements proved in [A-Z], and we shall need them here too. Recall first the definition of a 1-complete neighborhood.

**Definition 3.1.** A 1-complete neighborhood of a compact $\mathcal{K} \subset \mathcal{X}$ is an open set $U \supset \mathcal{K}$ such that

i) $U$ is contained in a finite union of open coordinate balls centered at points of $\mathcal{K}$, i.e.

$$U \subset \bigcup_{\alpha = 1}^n B_\alpha$$

where $B_\alpha = \phi_\alpha^{-1}(B^\infty)$ and $\phi_\alpha^{-1}(0) = k_\alpha \in \mathcal{K}$.

ii) $U$ possesses a strictly plurisubharmonic exhaustion function $\psi : U \rightarrow [0, t_0)$, i.e.

- for every $t < t_0$ one has that $\psi^{-1}([0, t)) \subset U$.

Here by a strictly plurisubharmonic function we mean the following.

**Definition 3.2.** Let $U$ be an open subset of $\mathcal{X}$. A function $f \in \mathcal{C}^2(U, \mathbb{R})$ is said to be strictly plurisubharmonic on $U$ if the Levi form $\mathcal{L}_{f,a}$ is positive definite for every $a \in U$, i.e.

$$\mathcal{L}_{f,a}(v) > 0 \text{ for each } a \in U \text{ and } v \in T_a \mathcal{X} \setminus \{0\}.$$
In [A-Z] we give a strongest definition of a strictly plurisubharmonic function, see definition 2.1 there. In fact we states that the Levi form satisfies
\[ L_{f,a}(v) \geq c(a)||v||^2 \quad \text{for each} \quad a \in U \quad \text{and} \quad v \in T_a \mathcal{X}\backslash \{0\}, \]
with \( c \) a positive function in \( \mathcal{C}^0(U, \mathbb{R}) \). These definition are not equivalent as it shows the counter-example given by Lempert:
Take for example the function \( f \) defined by \( f(z) = \sum_{j=1}^{\infty} \frac{|z_j|^2}{j} \) for \( z \in \mathbb{C}^2 \). The Levi form on a point \( a \in \mathbb{C}^2 \) is
\[ \forall v \in \mathbb{C}^2, \quad L_{f,a}(v) = \sum_{j=1}^{\infty} \frac{1}{j} |v_j|^2, \]
and this cannot be bound from below by \( c(\|v\|^2) \). Nevertheless plurisubharmonic functions we need in this paper satisfies this strongest definition.

**Theorem 3.2.** Let \( \phi : \bar{D} \to \mathcal{X} \) be an imbedded analytic \( q \)-disk in a complex Hilbert manifold \( \mathcal{X} \). Then \( \phi(\bar{D}) \) has a fundamental system of \( 1 \)-complete neighborhoods.

Here an analytic \( q \)-disk in a complex Hilbert manifold \( \mathcal{X} \) is a holomorphic imbedding \( \phi \) of a neighborhood of a closure of a relatively compact strongly pseudoconvex domain \( D \subset \mathbb{C}^q \) into \( \mathcal{X} \). We shall also need the following lemma from [A-Z], see Lemma 3.1 there.

**Lemma 3.1.** Let \( \phi_n : \bar{D} \to \mathcal{X} \) be a sequence of analytic \( q \)-disks in a complex Hilbert manifold \( \mathcal{X} \) and let \( \Phi_n \) be their graphs. Suppose that there exists an analytic disk \( \phi_0 : \bar{D} \to \mathcal{X} \) with the graph \( \Phi_0 \) such that for any neighborhood \( \mathcal{V} \supset \Phi_0 \) one has \( \Phi_n \subset \mathcal{V} \) for \( n \gg 1 \). Then \( \phi_n \) converges uniformly on \( \bar{D} \to \phi_0 \).

Now we return to the proof of the theorem 2 from the introduction.

**Theorem 3.3.** Let \( \mathcal{X} \) be a \( q \)-Hilbert-Hartogs manifold. Then every holomorphic map \( F : W^{k,2}(S, H_q^n(r)) \to \mathcal{X} \) extends to a holomorphic map \( \tilde{F} : W^{k,2}(S, \Delta^g \times \Delta^n) \to \mathcal{X} \).

**Proof.** The proof will be achieved in a number of steps. First we shall construct some “natural extension” of \( F \). After that we shall prove that the extension is continuous and finally that it is holomorphic.

**Step 1. Natural extension.** Let \( f = (f^q, f^n) : S \to \Delta^g \times \Delta^n \) be an element of \( W^{k,2}(S, \Delta^g \times \Delta^n) \). Here \( f^q \) and \( f^n \) are components of \( f \). We want to extend \( F \) to \( f \). This will be done along an appropriate analytic disc which passes through \( f \). Consider the mapping \( \phi_f : \Delta^q \times S \to \Delta^g \times \Delta^n \) defined as

\[ \phi_f : (z, s) \to \left( h_{f^q(s)}(z), f^n(s) \right). \]

Here \( h_a \) is the following automorphism of \( \Delta^q \) interchanging \( a = (a_1, ..., a_q) \) and 0:

\[ h_a(z) = \left( \frac{a_1 - z_1}{1 - a_1 z_1}, ..., \frac{a_q - z_q}{1 - a_q z_q} \right). \]

Notice that due to the compactness of \( f^q(S) \subset \Delta^g \) automorphisms \( h_{f^q(s)} \) are defined for \( z \) in a fixed (independent on \( s \)) neighborhood of \( \Delta^g \), and therefore such is \( \phi_f \). Denote by \( \phi_{f^q} : \Delta^q \to W^{k,2}(S, \Delta^{q+n}) \) the analytic \( q \)-disk in \( W^{k,2}(S, \Delta^g \times \Delta^n) \) represented by \( \phi_f \), i.e \( \phi_{f^q}(z) \in W^{k,2}(S, \Delta^{q+n}) \) acts as follows

\[ \phi_{f^q}(z) : s \to \left( \frac{f^q(s) - z_1}{1 - f^q(s) z_1}, ..., \frac{f^q(s) - z_q}{1 - f^q(s) z_q}, f^n(s) \right). \]

Here \( f^q(s) = (f^q_1(s), ..., f^q_q(s)) \) is the \( q \)-component of \( f \). Denote by \( \Phi_f = \phi_{f^q}(\Delta^q) \) the image of \( \phi_{f^q} \). As in lemma 3.1 we mark here and everywhere with * analytic disks \( D \to W^{k,2}(S, \mathcal{X}) \) represented by maps \( D \times S \to \mathcal{X} \) and by capital letters we mark the images of these discs. Our \( \phi_f \) possesses the following properties:

i) \( \phi_{f^q}(0) = f \), i.e. is our loop \( f \).
ii) For \( z \in \partial \Delta^q \) one has that \( \phi_{f_\ast}(z)(S) \subset A^q_{r-1} \times \Delta^n \), therefore
\[
\partial \Phi_f := \phi_{f_\ast}(\partial \Delta^q) \subset W^{k,2}(S, H^n_q(r)) .
\]
Indeed, for \( z \) close to \( \partial \Delta^q \) some \( z_j \) is close to \( \partial \Delta \) and then the \( j \)-component of \( h_{f_\ast}(z) \) also is close to \( \partial \Delta \) for all \( s \in S \) as required.

Remark that due to the second item above our map \( F \) is defined and holomorphic near the boundary of the analytic \( q \)-disc \( \Phi_f \). In order to assign to \( f \) some “natural” value \( \tilde{F}(f) \), which we shall call a “natural extension” of \( F \), we shall extend \( F \) holomorphically to the \( q \)-disc \( \Phi_f \ni f \).

Consider the analytic \((q + 1)\)-disk \( \psi_* \) in \( W^{k,2}(S, \Delta^q \times \Delta^n) \) represented by
\[
(20) \quad \psi_\ast(z, s) := h_{f_\ast}(z, t f^n(s)) , \quad z \in \Delta^q , \quad |t| \leq 1 + \delta , \quad s \in S
\]
for an appropriate \( \delta > 0 \) small enough. Here we use the compactness of \( f^n(S) \). Remark that
\begin{itemize}
  \item[iii)] \( \psi^0(\cdot, \cdot) \) takes its values in \( \Delta^q \times \{0\} \subset H^n_q(r) \) and therefore for \( |t| \) small \( \psi^t(\cdot, \cdot) \) takes its values in \( H^n_q(r) \) by continuity.
  \item[iv)] \( \psi^t = \tilde{\phi}_f \).
  \item[v)] For all \( t \in \Delta_{1+\delta}^q \) one has that \( \partial \Psi^t = \Psi^t_\ast(\partial \Delta^q) \subset W^{k,2}(S, H^n_q(r)) \) by the second item as before.
\end{itemize}

Here \( \Psi \) and \( \Psi^t \) stand for the images of \( \psi_* \) and \( \psi^t_* \) respectively, i.e. for every \( t \) we see \( \Psi^t_\ast \) as an analytic \( q \)-disk, while \( \Psi_* \) as an analytic \((q + 1)\)-disk. Therefore for \( \delta > 0 \) small enough the Hartogs figure
\[
(21) \quad H^n_q(\delta) := \{ (z, t) : ||z|| < 1 + \delta , |t| < \delta \text{ or } 1 - \delta < ||z|| < 1 + \delta , |t| < 1 + \delta \}
\]
is mapped by \( \psi_* \) to \( W^{k,2}(S, H^n_q(r)) \) and consequently the composition \( F \circ \psi_* : H^n_q(\delta) \to \mathcal{X} \) is well defined and holomorphic. Due to the assumed \( q \)-Hartogsness of \( \mathcal{X} \) this composition \( F \circ \psi_* \) holomorphically extends to the associated polydisc \( \Delta_{1+\delta}^q \). In particular it extends to \( \Delta^q \times \{1\} \), i.e. \( F \) holomorphically extends onto the \( q \)-disc \( \Phi_f \ni f \).

(22) Denote by \( \tilde{F} \) the extension \( F \circ \psi_* \) of \( F \) and set \( \tilde{F}(f) := (F \circ \psi_*)(0, 1) \).

Remark that \( \tilde{F} \) is indeed an extension of \( F \), i.e. that \( \tilde{F}(f) = F(f) \) for \( f \in W^{k,2}(S, H^n_q(r)) \) because \( \tilde{F} \) is a holomorphic extension of \( F \) from \( \Psi \cap W^{k,2}(S, H^n_q(r)) \). We call \( \tilde{F} \) the “natural extension” of \( F \).

**Step 2.** The natural extension is continuous. Let \( f' \in W^{k,2}(S, \Delta^q \times \Delta^n) \) be close to \( f \). Construct \( q \)-disc \( \tilde{\phi}_{f'_*} \) and \((q + 1)\)-disc \( \psi'_* \) for \( f' \) as we did for \( f \). Denote by \( \Gamma_\Psi \) and \( \Gamma_{\Psi'} \) the graphs of \( F \circ \psi_* \) and \( F \circ \psi'_* \) in \( \Delta_{1+\delta}^q \times \mathcal{X} \) correspondingly. Due to Lemma 3.1 all we need to prove is that \( \Gamma_{\Psi'} \) enters to a given neighborhood of \( \Gamma_\Psi \) provided \( f' \) is sufficiently close to \( f \). Due to Theorem 3.2 one can choose a \( 1 \)-complete neighborhood \( \mathcal{V} \) of \( \Gamma_\Psi \). Moreover, for an appropriate \( \delta > 0 \) the graph \( \Gamma_{\Psi'} \) over the Hartogs figure \( H^n_q(\delta) \) as in (21) enters to the neighborhood \( \mathcal{V} \) of the graph \( \Gamma_\Psi \) because \( \psi' \) is close to \( \psi \) and takes it values in \( W^{k,2}(S, H^n_q(r)) \) when restricted to \( H^n_q(\delta) \). A priori there is no reason that all the graph enters in \( \mathcal{V} \). But the maximum principle applied to the plurisubharmonic exhausting function of \( \mathcal{V} \) all the graph \( \Gamma_{\Psi'} \) enters in the neighborhood \( \mathcal{V} \).

The lemma 3.1 permits us to obtain the continuity of the map and the step is proved.

**Step 3.** Holomorphicity. All what is left to prove is that our natural extension \( \tilde{F} \) is Gâteaux holomorphic. Fix \( f \in W^{k,2}(S, \Delta^q \times \Delta^n) \), \( g \in W^{k,2}(S, C^{q+n}) \) and consider the complex affine line \( L := \{ f + \lambda g : \lambda \in \mathbb{C} \} \subset W^{k,2}(S, C^{q+n}) \). We need to prove that \( \tilde{F}|_L \) is holomorphic in a neighborhood of zero. Fix an \( \epsilon > 0 \) sufficiently small and consider an analytic \((q + 2)\)-disc \( \theta_* : \Delta^q \times \Delta^2 \to W^{k,2}(S, C^{q+n}) \) represented by the mapping:
\[
(23) \quad \theta : (z, \lambda, \mu, s) \to \left( \frac{f^q_1(s) + \lambda g^q_1(s) - z_1}{1 - (f^q_1(s) + \mu g^q_1(s))z_1} , \ldots , \frac{f^q_n(s) + \lambda g^q_n(s) - z_q}{1 - (f^q_n(s) + \mu g^q_n(s))z_q} , f^n(s) + \lambda g^n(s) \right) .
\]
Denote by $\Theta = \text{Im}\theta$, its image. Notice that for $\varepsilon > 0$ small enough the $q$-component of \text{(23)} defines for every fixed $|\lambda|,|\mu| \leq \varepsilon$ and $s \in S$ a holomorphic imbedding of $\overline{\Delta}^q$ to a neighborhood of $\overline{\Delta}^q$ which is uniformly close to the automorphism

$$z \rightarrow \left( \frac{f_1^q(s) - z_1}{1 - f_1^q(s)z_1}, \ldots, \frac{f_q^q(s) - z_q}{1 - f_q^q(s)z_q} \right)$$

of $\Delta^q$. Therefore for every $|\lambda|,|\mu| \leq \varepsilon$ the disk $\theta_s$ has values in $W^{k,2}(S, \Delta^q_{1+r} \times \Delta^n)$ and the arguments of Step 1 can be repeated to $\theta(\cdot, \lambda, \mu, \cdot)$ on the place of $\phi_f$.

Consider an analytic $(q + 3)$-disk $\psi_s$ in $W^{k,2}(S, C^{q+n})$ represented by \text{(24)}

$$\psi^i(z, \lambda, \mu, s) := \psi(z, \lambda, \mu, t, s) := \left( \theta^q(z, \lambda, \mu, s), t^m(z, \lambda, \mu, s) \right), \quad z \in \overline{\Delta}^q, \quad |t| \leq 1 + \delta, \quad s \in S$$

for an appropriate $\delta > 0$ small enough. Here $\theta^q$ and $\theta^m$ are components of $\theta = (\theta^q, \theta^m)$. Remark that

- $\psi^0$ takes its values in $\Delta^q \times \{0\} \subset H^n_q(r)$ and therefore for $|t|$ small $\psi^t$ takes its values in $H^n_q(r)$ by continuity.
- $\psi^1 = \theta$.
- For all $t \in \overline{\Delta}^q_{1+\delta}$ one has that $\partial \Psi^t \subset W^{k,2}(S, H^n_q(r))$.

This gives a holomorphic extension of $F \circ \theta_s$ from $\partial \Delta^q \times \Delta^2$ to $\overline{\Delta}^q \times \Delta^2$. We denote this extension as $\tilde{F}$. Notice now the following properties of $\theta$ and $F$:

1) $\theta(\cdot, \lambda, \cdot, \cdot) = \phi_{f + \lambda g}(\cdot, \cdot)$;
2) $\theta(0, \lambda, \mu, \cdot) = f(\cdot) + \lambda g(\cdot)$, in particular, this doesn’t depend on $\mu$.

Properties iv) and v) imply that

$$\tilde{F}(f + \lambda g) = \left( \text{extF}_{|\theta_{f,\lambda g}} \right) |z=0 = \left( \text{extF}_{|\theta(\cdot, \lambda, \cdot, \cdot)} \right) |z=0 = \left( \text{extF}_{|\theta(\cdot, \lambda, \mu, \cdot)} \right) |\mu=0, z=0 =$$

$$= \tilde{F} |\mu=0, z=0 = \tilde{F}(0, \lambda, 0).$$

Here by $\text{extF}_{|\theta_{f,\lambda g}}$ we denote the extension of $F$ along the $q$-disc $\phi_{f + \lambda g}$ (and then taking the value of this extension at $z = 0$). This was the definition of the natural extension. Therefore the first equality is justified. As for second it is justified by (i) . The third equality follows from (i) since at $z = 0$ nothing depends on $\mu$ and $\mu = \lambda$ can be replaced by $\mu = 0$. But this is the extension $\tilde{F}$ evaluated at $z = 0, \lambda, \mu = 0$ and the latter holomorphically depends on $\lambda$. Therefore such is the left hand side $\tilde{F}(f + \lambda g)$. The holomorphicity of $\tilde{F}$ is proved.

**Remark.** The theorem just proved was stated in [A-I], see Theorem 3.1 there, for the case of compact $S$ without boundary. The step 1 of the proof was presented there as well. As for step 2 and 3 the details were missing. □

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