ON THE VANISHING OF THE LOCAL RAYLEIGH CONDITION OF THE HYDROSTATIC EULER EQUATION AND THE FINITE TIME BLOW UP OF THE SEMI-LAGRANGIAN EQUATIONS

VICTOR CAÑULEF-AGUILAR

Abstract. This paper is devoted to the large time behaviour of the smooth solutions of the hydrostatic Euler equation satisfying the local Rayleigh condition. It is proven under certain assumptions that some quantities will blow up in finite time implying the vanishing of the local Rayleigh condition. Additionally, we find some conditions satisfied by the global smooth solutions of the hydrostatic Euler equation satisfying the local Rayleigh condition, which added to the compatibility condition found in [Bre99] suggest that the above global solutions must be the stationary ones. Additionally, the finite time blow up of smooth solutions of the Semi-Lagrangian equations introduced by Brenier in [Bre99] is proved under certain conditions.

1. Introduction

The hydrostatic Euler equation is a limit equation obtained after passing to the limit in the solutions of the rescaled 2D Euler equation. More precisely, starting from the 2D Euler equation in a narrow periodic domain, that is:

\begin{align*}
\begin{cases}
  u_t + uu_x + vu_y + p_x = 0 & \text{in } T \times (0, \varepsilon), \\
  v_t + uv_x + vv_y + p_y = 0 \\
  u_x + v_y = 0 \\
  v|_{y=0,1} = 0 \\
  (u, v)|_{t=0} = (u_0, v_0)
\end{cases}
\end{align*}

(1.1)

After the following rescale $\varepsilon\tilde{y} = y$, $\tilde{u}(x, \tilde{y}, t) = u(x, \varepsilon\tilde{y}, t)$, $\varepsilon\tilde{v}(x, \tilde{y}, t) = v(x, \varepsilon\tilde{y}, t)$ and $\tilde{p}(x, \tilde{y}, t) = p(x, \varepsilon\tilde{y}, t)$, we get the rescaled 2D Euler equation:

\begin{align*}
\begin{cases}
  u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + v^\varepsilon u_y^\varepsilon + p_x^\varepsilon = 0 & \text{in } T \times (0, 1), \\
  \varepsilon^2 (u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + v^\varepsilon u_y^\varepsilon) + p_y^\varepsilon = 0 \\
  u_x^\varepsilon + v_y^\varepsilon = 0 \\
  v^\varepsilon|_{y=0,1} = 0 \\
  u^\varepsilon|_{t=0} = u_0^\varepsilon
\end{cases}
\end{align*}

(1.2)

Passing to the limit, we get the Hydrostatic Euler Equation:

\textit{Date:} December 27, 2021.
\[
\begin{align*}
\begin{cases}
    u_t + uu_x + vu_y + p_x = 0 & \text{in } T \times (0,1), \\
    u_x + v_y = 0 \\
    p_y = 0 \\
    v|_{y=0,1} = 0 \\
    u|_{t=0} = u_0
\end{cases}
\end{align*}
\] (1.3)

Whose vorticity (namely \(\omega = u_y\)) formulation (following the notation in [MW12]) is given by:

\[
\begin{align*}
\begin{cases}
    \omega_t + u\omega_x + v\omega_y = 0 & \text{in } T \times (0,1), \\
    (u, v) = \nabla^\perp A\omega \\
    \omega|_{t=0} = \omega_0
\end{cases}
\end{align*}
\] (1.4)

Where the stream function satisfies:

\[
\begin{align*}
\begin{cases}
    -\partial^2_y A(\omega) = \omega & \text{in } T \times (0,1), \\
    A(\omega)|_{y=0,1} = 0
\end{cases}
\end{align*}
\] (1.5)

So, the stream function is equal to:

\[
A(\omega) = (1 - y) \int_0^y z\omega(x, z, t)dz + y \int_y^1 (1 - z)\omega(x, z, t)dz
\]

\[
= -\frac{1}{2} \int_0^1 \{|y - z| - y - z + 2yz\}\omega(x, z, t)dz.
\]

The local existence of solutions to the above problem satisfying the local Rayleigh condition was proven in [Bre99] in the \(C\) class and in [MW12] in Sobolev spaces, as well as the convergence of the solutions of the rescaled Euler Equation to the Hydrostatic Euler Equation when considering convex initial profiles (see for instance [Bre03], [Gre99] and [MW12]). In spite of the local existence results in the analytic setting (for instance [KTVZ10] and [KTVZ11]), It is known that for initial profiles with inflexion points, the convergence may not hold, which was proved in [Gre00] and stated in [Gre99]. Additionally, the ill posedness of the hydrostatic Euler equation (in the sense of Hadamard) was proved in [Ren09] for the linearized equation and in [HKN16] near an analytic shear flow. Moreover, It is well known that the solutions of the hydrostatic Euler Equation may develop singularities in finite time when the initial profile does not satisfy the convexity condition (more precisely, when \(u(x_0, y, 0) = 0\) for a fixed \(x_0\) and \(\forall y \in [0,1]\)) as was shown in [KTV12] and in [CINT12] by using similar techniques to the one showed in [EE97]. Nevertheless, It is important to mention that It is not clear if there exist solutions for such initial degenerate profiles and that the solutions studied here are not like those mentioned above. In the last section can be found a couple of results concerning the relation between the possible formation of singularities and the validity of the convexity condition.

The aim of this article is to study the long time behaviour of the smooth solutions of the hydrostatic Euler equation satisfying the local Rayleigh condition, more
precisely, we want to know how long the local Rayleigh condition will remain valid and if there exist global smooth solutions that will satisfy the above condition for every time. Finally, let us state the main result:

**Theorem 1.** Let \( \omega_0 \) be a smooth function in \( \mathbb{T} \times (0, 1) \) satisfying the local Rayleigh condition (i.e. \( \partial_y \omega > 0 \)) and whose \( x \) derivative is not identically 0. Let \( \omega(x, y, t) \) be the smooth solution to the problem \( \frac{\partial}{\partial t} \omega = -\lambda \omega \) set

\[
E_1(t) = \int_{\mathbb{T} \times (0, 1)} \frac{\omega_x}{\omega_y} dxdy = \int_{\mathbb{T} \times (0, 1)} \left( \frac{\omega_x}{\omega_y} - u_x \right) dxdy
\]

and

\[
E_2(t) = \int_{\mathbb{T} \times (0, 1)} u^2 \frac{\omega_x}{\omega_y} dxdy = \int_{\mathbb{T} \times (0, 1)} u^2 \left( \frac{\omega_x}{\omega_y} - u_x \right) - u P_x dxdy.
\]

Then if \( E_1(t = 0) \geq 0 \) or \( E_2(t = 0) \geq 0 \) there exist a time \( T^* \) such that the local Rayleigh condition will be vanished, that is \( \omega \big|_{t = T^*} \) will be vanished in some points. Moreover, if \( E_1(t = 0) > 0 \), then \( T^* \leq E_1^{-1}(t = 0) \) (or \( T^* \leq E_2^{-1}(t = 0) \)) if \( E_2(t = 0) > 0 \).

**Remark 1.** If \( \omega_0 \) is \( x \) independent, smooth and \( \partial_y \omega > 0 \), then the solution to \( \frac{\partial}{\partial t} \omega = -\lambda \omega \) is stationary, and the smooth stationary solutions satisfying \( \partial_y \omega_0 > 0 \) are \( x \) independent (see Corollary 7).

**Remark 2.** In the particular case when \( \omega_0(x, y) = \omega_0(-x, y) \), we will have that \( E_1(t = 0) = 0 \). Additionally, if \( \omega(x, y, t) \) solves the problem \( \frac{\partial}{\partial t} \omega = -\lambda \omega \), then \( \omega(x, y, t) = \omega(-x, y, -t) \) since \( \omega(-x, y, -t) \) satisfies the problem \( \frac{\partial}{\partial t} \omega = -\lambda \omega \) with the same initial condition. So in that case, the local Rayleigh condition will be vanished in finite time going forward and backward in time.

**Remark 3.** To see that the set of initial conditions satisfying \( \partial_y \omega_0 > 0 \) and \( E_1(t = 0) > 0 \) is non empty, we can choose \( \omega_0(x, y) = 2y - \sin(2\pi x - y) \). Then: \( \partial_x \omega_0 = -2\pi \cos(2\pi x - y), \partial_y \omega_0 = 2 + \cos(2\pi x - y) \) and so:

\[
E_1(t = 0) = \int_{\mathbb{T} \times (0, 1)} \frac{(2y - \sin(2\pi x - y)) \cdot -2\pi \cos(2\pi x - y)}{2 + \cos(2\pi x - y)} dxdy
\]

\[
= -2\pi \int_{\mathbb{T} \times (0, 1)} (2y - \sin(2\pi x - y)) dxdy + 4\pi \int_{\mathbb{T} \times (0, 1)} \frac{(2y - \sin(2\pi x - y))}{2 + \cos(2\pi x - y)} dxdy
\]

\[
= -2\pi + \int_{\mathbb{T} \times (0, 1)} \frac{4\pi \cdot 2y}{2 + \cos(2\pi x - y)} dxdy = -2\pi + \frac{4\pi}{\sqrt{3}} \int_0^1 \frac{2ydy}{2} = 2\pi \left( \frac{2}{\sqrt{3}} - 1 \right) > 0.
\]

**Remark 4.** Note that if for an initial condition \( \omega_0(x, y) \) we have \( E_1(t = 0) < 0 \), then, by changing the initial condition to \( \omega_0(-x, y) \), we will have that \( E_1(t = 0) > 0 \).

1.1. Additional results.

**Theorem 2.** Let \( \omega_0 \) be a smooth function in \( \mathbb{T} \times (0, 1) \) satisfying the local Rayleigh condition (i.e. \( \omega_y > 0 \)), let \( \omega(x, y, t) \) be the smooth solution to the problem \( \frac{\partial}{\partial t} \omega = -\lambda \omega \). Set \( E_1(t) \) and \( E_2(t) \) as above, then if \( \omega \) is a global smooth solution satisfying the local Rayleigh condition, the following properties are satisfied:
In that case, it was not clear if the derivatives of $u$ could develop singularities (note that if the derivatives of $h$ blow up, does not imply the blow up of the derivatives of $u$ since the vanishing of $\omega_y$ produce the blow up of $h_a$) or they were just losing the convexity of $u$ (that is, $h_a$ could get unbounded). The above explains why Theorem 1 gives a further description of the solutions and moreover, it gives a bound for the validity time of the local Rayleigh condition.

Remark 5. Note that in the semilagrangian formulation introduced in [Bre99] (that is, $\omega(x,0,t) = k$, $\omega(x,1,t) = k + 1$, $\omega(x,h(x,t,a),t) = k + a$, where $k$ is a constant and $0 < a < 1$), the above energies are equal to:

$$E_1(T) = -\int_{\mathbb{T} \times (0,1)} v_x h_a dx da = E_1(t = 0) + \int_0^T \int_{\mathbb{T} \times (0,1)} v_x^2 h_a \, dx \, da \, dt,$$

$$E_2(T) = -\int_{\mathbb{T} \times (0,1)} v^2 v_x h_a dx da = E_2(t = 0) + \int_0^T \int_{\mathbb{T} \times (0,1)} v^2 v_x h_a \, dx \, da \, dt,$$

where $v = u(x,h(x,t,a),t)$ (and so $v_x(x,t,a) = u_x - \frac{\omega_x}{\omega_y}(x,h(x,t,a),t)$) and $v_t(x,t,a) = -P_x - u_x - \frac{\omega_x}{\omega_y}(x,h(x,t,a),t)$ and $h_a(x,t,a) = \frac{1}{\omega_y}(x,h(x,t,a),t)$.

In that case $v$ and $h_a$ satisfy the following conservation laws:

$$\begin{cases}
  v_t + \partial_x \left( \frac{v^2}{2} + P(x,t) \right) = 0 & \text{in } \mathbb{T} \times (0,1), \\
  \partial_t h_a + \partial_x (vh_a) = 0 \\
  \partial_a P = 0 \\
  \int_0^1 h_a \, da = 1 \\
  v|_{t=0} = v_0
\end{cases}$$ (1.6)

In [Bre99] was found a compatibility condition for the global smooth solutions in the $C^1$ class, which is given by:

$$\int_{\mathbb{T} \times (0,1)} v^2 h_a \, dx \, da = \int_0^1 \left( \int_{\mathbb{T}} v \, dx \right)^2 \left( \int_{\mathbb{T}} h_a \, dx \right) \, da,$$

which seems to be the only previous result concerning the long time behavior of the solutions of the hydrostatic Euler equation satisfying the local Rayleigh condition.

Remark 6. It is worth mentioning that the above compatibility condition does not give additional information about why the solutions are not global, more specifically, it was not clear if the derivatives of $u$ could develop singularities (note that if the derivatives of $h$ blow up, does not imply the blow up of the derivatives of $u$ since the vanishing of $\omega_y$ produce the blow up of $h_a$) or they were just losing the convexity of $u$ (that is, $h_a$ could get unbounded). The above explains why Theorem 1 gives a further description of the solutions and moreover, it gives a bound for the validity time of the local Rayleigh condition.
1.2. **Semilagrangian equations in higher dimensions.** Despite we do not get a transport equation for the vertical derivative of $u$ (that is, the "hydrostatic" vorticity) for the hydrostatic approximation by adding extra horizontal periodic variables (and so, at a first sight we are not able to get a semilagrangian formulation), equations (1.6) have a natural analogous in higher dimensions:

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla P &= 0, \text{where } (x,a) \in \mathbb{T}^d \times (0,1), \\
\partial_t h_a + \nabla \cdot (vh_a) &= 0 \\
\partial_a P &= 0 \\
\int_0^1 h_a da &= 1
\end{aligned}
\]

(1.7)

Note that since $\partial_i v_j = \partial_j v_i$, the first equation is equivalent to:

\[
\partial_t v + \nabla \left( \frac{|v|^2}{2} + P \right) = 0.
\]

Now, for getting an expression for the pressure, note that:

\[
\partial_t \int_0^1 vh_a da = -\int_0^1 \left( \nabla \left( \frac{|v|^2}{2} + P \right) h_a + v \nabla \cdot (vh_a) \right) da = -\nabla \cdot \int_0^1 (v \otimes v) h_a da,
\]

where we used that $\partial_i v_j = \partial_j v_i$. Now, applying divergence on both sides we get:

\[
0 = -\Delta P - \nabla \cdot \int_0^1 (v \otimes v) h_a da,
\]

from which we obtain:

\[
P(x,t) = (-\Delta)^{-1} \left( \nabla \cdot \int_0^1 (v \otimes v) h_a db \right).
\]

**Remark 7.** In particular, smooth solutions to (1.7) satisfy:

\[
\begin{aligned}
\partial_t \int_0^1 vh_a db + \nabla \cdot \int_0^1 v \otimes vh_a db + \nabla P &= 0, \text{where } x \in \mathbb{T}^d, \\
\nabla \cdot \int_0^1 vh_a db &= 0 \\
\partial_i v_j - \partial_j v_i &= 0 \text{ for } i, j \leq d,
\end{aligned}
\]

(1.8)

**Remark 8.** The compatibility condition mentioned above was also shown in [Bre99] in higher dimension, that is, global smooth solutions to (1.7) satisfy:

\[
\int_0^1 \int_{\mathbb{T}^d} |v|^2 h_a dx da = \int_0^1 \left| \int_{\mathbb{T}^d} v dx \right|^2 \left( \int_{\mathbb{T}^d} h_a dx \right) da.
\]

Note that both sides are still time independent.

Now, let us state the second main result:
Remark 9. Recently in [KV21] was shown a relation between generalized solutions to incompressible Euler equation in 1D and permutation processes.

Theorem 3. Let \((v, h_a)\) be a smooth solution to \(1.4\) with smooth initial condition. Set
\[
E_1(t) = -\int_{\T^d \times (0,1)} (\nabla \cdot v) \, h_a \, dx \, da = \partial_t \int_{\T^d \times (0,1)} h_a \log(h_a) \, dx \, da
\]
and
\[
E_2(t) = \int_{\T^d \times (0,1)} \partial_t \left( \frac{|v|^2}{2} \right) h_a \, dx \, da = -\int_{\T^d \times (0,1)} \left( v \cdot \nabla \left( \frac{|v|^2}{2} + P \right) \right) h_a \, dx \, da.
\]
Suppose that \(E_1(t) = 0\) (or \(E_2(t) = 0\)). Then, there exists a time \(T^* < \infty\) such that \(E_1(t) \to \infty\) (respectively \(E_2(t) \to \infty\)) as \(t \to T^*\). Moreover, \(T^* \leq E_1^{-1}(t = 0)\) (respectively \(T^* \leq E_2^{-1}(t = 0)\)) \(\cdot \int_{\T^d \times (0,1)} |v|^2 h_a \, dx \, da\).

Theorem 4. Let \((v, h_a)\) be a smooth solution to \(1.7\) with smooth initial condition. Set \(E_1\) as in Theorem 3. Suppose that \(E_1(t = 0) > 0\), then:
\[
\exp \left( \int_{\T^d \times (0,1)} h_a \log(h_a) \, dx \, da(t = 0) \right) \frac{1}{1 - TE_1(t = 0)} \leq \exp \left( \int_{\T^d \times (0,1)} h_a \log(h_a) \, dx \, da\right) (t = T) \leq \left( \int_{\T^d \times (0,1)} h_a^{1+p} \, dx \, da \right)^{1/p} (t = T).
\]
For every \(p \in (0, \infty)\).

Remark 9. Recently in [KV21] was shown a relation between generalized solutions to incompressible Euler equation in 1D and permutation processes.

Remark 10. Note that Theorem 4 gives a rate for the blow up of any \(L^p(\T^d \times (0, 1))\) norm of \(h_a\) with \(p > 1\) and seems to be sharp in the sense that the \(L^1\) norm of \(h_a\) is finite (recall that \(\int_{\T^d \times (0,1)} h_a \, dx \, da = 1\)).

2. Monotonicity and Blow up of the Energies

In order to prove the main results, we need to show some properties of \(E_1\) and \(E_2\). More specifically, the blow up of \(E_1\) will be used for proving the vanishing of the local Rayleigh condition. On the other hand, for proving the bounds for the integral of \(P^2\), the following monotonicity results will be used:

Proposition 1. Let \(\omega\) be a smooth solution to \(1.4\) satisfying the local Rayleigh Condition. Set \(E_1(t) = \int_{\T^d \times (0,1)} \frac{\omega_x \omega_y}{\omega} \, dx \, dy\) and \(E_2(t) = \int_{\T^d \times (0,1)} u \frac{\omega_x \omega_y}{\omega} \, dx \, dy\), then the following holds
\[
\partial_t E_1 = \int_{\T^d \times (0,1)} \left| u_x - \frac{\omega_x}{\omega} \right|^2 \, dx \, dy,
\]
\[
\partial_t E_2 = \int_{\T^d \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega_x}{\omega} \right) \right|^2 \, dx \, dy.
\]

Proof: The first equality follows by computing the material derivative of \(\frac{\omega_x \omega_y}{\omega}\):

First note that since \(D_t(\omega) = 0\), \(D_t(\omega_x) = -u_x \omega_x - v_x \omega_y\) and \(D_t(\omega_y) = u_x \omega_y - v_x \omega_x\), we have that:
Lemma 1. Let \( u \) which implies that

\[
D_t \left( \frac{\omega \omega_x}{\omega_y} \right) = -u_x \omega + \frac{\omega^2 \omega_x^2}{\omega_y^2} - 2 \frac{u_x \omega \omega_x}{\omega_y} = \left| \frac{\omega \omega_x}{\omega_y} - u_x \right|^2 - \omega v_x - u_x^2.
\]

Now, since \( D_t(u_x) = -u_x^2 - \omega v_x - P_{xx} \), we get:

\[
D_t \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) = \left| \frac{\omega \omega_x}{\omega_y} - u_x \right|^2 + P_{xx}.
\]

from which we obtain:

\[
\partial_t E_1 = \partial_t \int_{T \times (0,1)} \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) dxdy = \int_{T \times (0,1)} \left( \frac{\omega \omega_x}{\omega_y} - u_x \right)^2 + P_{xx}.
\]

Analogously:

\[
D_t \left( u^2 \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) - uP_x \right) = \left| u \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) - P_x \right|^2 - uP_{xt},
\]

integrating the above we get:

\[
\partial_t E_2 = \partial_t \int_{T \times (0,1)} u^2 \left( \frac{\omega \omega_x}{\omega_y} - u_x \right) - uP_x dxdy
\]

\[
= \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right|^2 dxdy,
\]

where we have used that \( \int_0^1 u(x, z, t)dz = 0 \).

Lemma 1. Let \( \omega \) be a smooth solution to 1.4 satisfying the local Rayleigh Condition. If \( \int_{T \times (0,1)} \left| u_x - \frac{\omega \omega_x}{\omega_y} \right|^2 dxdy = 0 \) (at any time) then \( \omega \) is stationary and independent of \( x \).

Proof. It follows from the fact that

\[
\left| u_x - \frac{\omega \omega_x}{\omega_y} \right|^2 = \left| \frac{\omega^2}{\omega_y} \cdot \partial_y \left( \frac{u_x}{\omega} \right) \right|^2 = 0,
\]

which implies that \( \frac{u_x}{\omega} \) is constant in \( y \). Since \( \int_0^1 u_x(x, z, t)dz = 0 \), \( u_x \) must be equal to 0 (if \( \omega \) has a change of sign, the argument is analogous).

Lemma 2. Let \( \omega \) be a smooth solution to 1.4 satisfying the local Rayleigh Condition. If \( \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right|^2 dxdy = 0 \) (at any time) then \( \omega \) is stationary and independent of \( x \).

Proof. Analogously to the previous proof, we have that:

\[
\left| P_x + u \left( u_x - \frac{\omega \omega_x}{\omega_y} \right) \right|^2 = \left| \frac{\omega^2}{\omega_y} \cdot \partial_y \left( \frac{u_x}{\omega} \right) \right|^2 = 0,
\]

which implies that \( \frac{u_x}{\omega} \) is constant in \( y \). Since \( \int_0^1 u_t(x, z, t)dz = 0 \), \( u_t \) must be equal to 0. Now, by uniqueness (which It was shown in [MW12] u must be stationary
and consequently independent of \(x\) (if \(\omega\) has a change of sign, the argument is analogous).

\[
\square
\]

**Corollary 1.** The stationary solutions to (1.4) satisfying the local Rayleigh condition are independent of \(x\) (and the solutions that are independent of \(x\) are evidently stationary).

**Proposition 2.** Let \(\omega\) be a non stationary smooth solution to (1.4) satisfying the local Rayleigh condition and \(E_1(t = 0) \geq 0\), then \(E_1(t)\) blows up at finite time.

**Proof.** It follows directly by differentiating \(E_1(t)\):

\[
\partial_t E_1 = \int_{T \times (0,1)} \left| u_x - \frac{\omega \omega_y}{\omega_y} \right|^2 \, dx \, dy \geq \left( \int_{T \times (0,1)} \left( u_x - \frac{\omega \omega_y}{\omega_y} \right) \, dx \, dy \right)^2 = E_1^2(t).
\]

Note that since \(\omega\) is non stationary, \(\partial_t E_1(t = 0) > 0\), so there exists a time \(t' > 0\) such that the local Rayleigh condition holds and \(E_1(t = t') > 0\), which yields the finite time blow up of \(E_1(t)\).

\[
\square
\]

**Proposition 3.** Let \(\omega\) be a non stationary smooth solution to (1.4) satisfying the local Rayleigh condition and \(E_2(t = 0) \geq 0\), then \(E_2(t)\) blows up at finite time.

**Proof.** Following the same procedure as in the previous proof, note that:

\[
\|u\|_{L^2}^2 \cdot \partial_t E_2(t) = \left( \int_{T \times (0,1)} u^2 \, dx \, dy \right) \cdot \int_{T \times (0,1)} \left| P_x + u \left( u_x - \frac{\omega \omega_y}{\omega_y} \right) \right|^2 \, dx \, dy \geq \left( \int_{T \times (0,1)} u \left( P_x + u \left( u_x - \frac{\omega \omega_y}{\omega_y} \right) \right) \right)^2 = \left( \int_{T \times (0,1)} u^2 \frac{\omega \omega_x}{\omega_y} \right)^2 = E_2^2(t).
\]

Finally, we obtain: \(\partial_t E_2 \geq \left( \|u\|_{L^2}^2 \right)^{-1} E_2^2\) (note that \(\|u\|_{L^2}^2\) is time independent).

\[
\square
\]

3. **Proof of Theorem 1**: Vanishing of the local Rayleigh condition

**Proof.** First note that by Jensen’s inequality we know that:

\[
\exp \left( \int_{T \times (0,1)} \frac{1}{\omega_y} \log \left( \frac{1}{\omega_y} \right) \, dx \, dy \right) \leq \int_{T \times (0,1)} \frac{1}{\omega_y} \, dx \, dy.
\]

On the other hand:

\[
\partial_t \int_{T \times (0,1)} \frac{1}{\omega_y} \log \left( \frac{1}{\omega_y} \right) \, dx \, dy = \int_{T \times (0,1)} \frac{\omega \omega_x}{\omega_y} \, dx \, dy.
\]

Now, if \(E_1(t = 0) > 0\), we have:

\[
E_1(T) \geq \frac{E_1(t = 0)}{1 - TE_1(t = 0)},
\]

which implies:
\[
\int_{(0,1)} \log \left( \frac{1}{\omega_y} \right) dx dy \bigg|_{t=0}^{t=T} = \int_0^T E_1(t) dt \geq \log \left( \frac{1}{1 - T E_1(t = 0)} \right).
\]

The above yields:

\[
\exp \left( \int_{(0,1)} \log \left( \frac{1}{\omega_y} \right) dx dy(t = 0) \right) \frac{1}{1 - T E_1(t = 0)} \leq \int_{(0,1)} \frac{1}{\omega_y} dx dy(t = T).
\]

Note that actually the first term is less or equal to

\[
\left( \int_{(0,1)} \frac{1}{\omega_y} dx dy \right)^{1/p} (t = T)
\]

for every \( p \in (0, \infty) \).

\[\square\]

**Remark 11.** Note that if \( E_2(t = 0) > 0 \), then, \( E_2(t) \geq (E_2^{-1}(t = 0) - t\|u\|_2^{-2})^{-1} \), which yields:

\[
\|u\|_2^2 \int_0^T \frac{dt}{\left( \frac{\|u\|_2^2}{E_2(t)} - t \right)^2} \leq \int_0^T E_2(t) dt \leq \int_0^T \partial_t E_1(t) \|u\|_2^2 dt \leq \|u\|_{\infty}^2 E_1(t) \bigg|_{t=0},
\]

which produce the vanishing of the local Rayleigh condition.

4. **Proof of Theorem** Properties of the global smooth solutions

**Proof.** Global smooth solutions satisfying the local Rayleigh condition must satisfy: \( E_1(t), E_2(t) < 0, \forall t > 0 \), which yields (recall that \( E_1 \) and \( E_2 \) are strictly increasing at any time so they must converge to 0 as \( t \to \infty \) due to the boundedness of \( E_1 \) and \( E_2 \)):

\[
\int_0^\infty \int_{(0,1)} \left| P_x + u \left( u_x - \frac{\omega_y}{\omega_x} \right) \right|^2 dx dy dt = -E_2(t = 0).
\]

\[
\int_0^\infty \int_{(0,1)} \left| u_x - \frac{\omega_y}{\omega_x} \right|^2 dx dy dt = -E_1(t = 0).
\]

From the above we get the following bounds for the integral of \( P_x^2 \):

\[
\int_0^\infty \int_T P_x^2 dx dt = \int_0^\infty \int_{(0,1)} \left| P_x + u \left( u_x - \frac{\omega_y}{\omega_x} \right) - u \left( u_x - \frac{\omega_y}{\omega_x} \right) \right|^2 dx dy dt \\
\leq \int_0^\infty \int_{(0,1)} 2 \left| P_x + u \left( u_x - \frac{\omega_y}{\omega_x} \right) \right|^2 + 2 \left| u \left( u_x - \frac{\omega_y}{\omega_x} \right) \right|^2 dx dy dt \\
\leq -2E_2(t = 0) - 2\|u\|_{\infty}^2 E_1(t = 0).
\]

(Note that \( u \) is evidently bounded). On the other hand we have:

\[
\int_0^T |E_1(t)| dt = -\int_0^T E_1(t) dt = \int_{(0,1)} \log(\omega_y) dx dy \bigg|_{t=0}^{t=T},
\]
\[ \leq \int_{T \times (0,1)} \log \left( \frac{1}{\omega_y} \right) dxdy \bigg|_{t=0} + \log \left( \int_{T \times (0,1)} \omega_y dxdy \bigg|_{t=T} \right) \]
\[ \leq \int_{T \times (0,1)} \log \left( \frac{1}{\omega_y} \right) dxdy \bigg|_{t=0} + \log \left( 2\|\omega\|_{\infty} \right), \]
that is uniformly bounded, from which we conclude the result.

5. Singularity formation and the Rayleigh regime

Despite It is not known if the smooth solutions of the hydrostatic Euler equations satisfying the local Rayleigh condition develop singularities before or during the vanishing of the local Rayleigh condition, here are two results supporting the idea that there is no formation of singularities.

**Proposition 4.** Let \( \omega_0 \) be a smooth function in \( T \times (0,1) \) satisfying the local Rayleigh condition (i.e. \( \omega_y > 0 \)), let \( \omega(x, y, t) \) be the smooth solution to the problem \( \ref{eq:Rayleigh} \) Then, we have:

\[ \int_0^T\int_T P^2 dx dt \leq 2E_2 \bigg|_{t=0} + 2\|u\|_E1 \bigg|_{t=T}. \]

**Proof.** It follows by applying the same procedure as in the proof of Theorem \( \ref{th:existence} \).

**Proposition 5.** Let \( \omega_0 \) be a smooth function in \( T \times (0,1) \) satisfying the local Rayleigh condition (i.e. \( \omega_y > 0 \)), let \( \omega(x, y, t) \) be the smooth solution to the problem \( \ref{eq:Rayleigh} \). Suppose that \( |\omega| > 0 \) in \( T \times (0,1) \). Then, we have:

\[ |u_x|_{\infty} \leq C(\omega_0) \frac{\omega_x}{\omega_y} - u_x \bigg|_{\infty}. \]

\[ \int_{T \times (0,1)} \frac{\omega_x^2}{\omega_y} dxdy \bigg|_{t=T} \leq \int_{T \times (0,1)} \frac{\omega_y^2}{\omega_x} dxdy \bigg|_{t=0} \exp \left( \int_0^T \frac{\omega_x}{\omega_y} - u_x \bigg|_{\infty} + 2|u_x|_{\infty} dt \right). \]

**Proof.** Since \( \int_0^1 u_x(x, y, t)dy = 0 \), at every \( x \) there exists \( y_0(x) \) such that \( u_x(x, y_0(x), t) \) will be vanished. Now, using that \( \omega \) does not vanishes, we get the following identity:

\[ \frac{u_x}{\omega}(x, \tilde{y}, t) = \int_{y_0(x)}^{\tilde{y}} \frac{\omega \omega_x - u_x \omega_y}{\omega^2} dy = \int_{y_0(x)}^{\tilde{y}} \frac{\omega \omega_x - u_x \omega_y}{\omega_y} \omega_y dy, \]

from which we get:

\[ |u_x|(x, \tilde{y}, t) \leq |\omega|(x, \tilde{y}, t) \frac{\omega \omega_x - u_x \omega_y}{\omega_y} \int_{y_0(x)}^{\tilde{y}} \omega_y dy \]

\[ = \frac{\omega \omega_x - u_x \omega_y}{\omega_y} \bigg|_{\omega(y, y_0(x), t) - 1}. \]

On the other hand we have:

\[ \partial_t \int_{T \times (0,1)} \frac{\omega^2}{\omega_y} dxdy = \int_{T \times (0,1)} \frac{\omega^2}{\omega_y} \left( \frac{\omega \omega_x}{\omega_y} - 3u_x \right) dxdy \]
\begin{equation}
\leq \left( \frac{\omega_y u_x - u_x \omega_y}{\omega_y} \right)_\infty^2 + 2 |u_x|_\infty \int_{T \times (0,1)} \frac{\omega_y^2}{\omega_x} dx dy,
\end{equation}
which ends the proof after applying the Grönwall inequality.
\hfill \Box

\section{Proof of Theorem \ref{thm:3} and Theorem \ref{thm:4}}

For proving Theorem \ref{thm:3} we can follow the same procedure as the one shown in the previous sections. First let us state the following proposition:

\begin{proposition}
Let \((v, h_a)\) be a smooth solution to \eqref{eq:1.7} with smooth initial condition. Set \(E_1\) and \(E_2\) as in Theorem \ref{thm:3}. Then, the time derivative of \(E_1\) and \(E_2\) are given by:
\begin{align*}
\partial_t E_1 &= \int_{T^2 \times (0,1)} |\nabla v|^2 h_a dx da \\
\partial_t E_2 &= \int_{T^2 \times (0,1)} |\partial_t v|^2 h_a dx da = \int_{T^2 \times (0,1)} \left| \nabla \left( P + \frac{|v|^2}{2} \right) \right|^2 h_a dx da.
\end{align*}
\end{proposition}

\begin{proof}
The time derivative of \(E_1\) is equal to:
\begin{align*}
\partial_t E_1(t) &= \int_{T^2 \times (0,1)} \nabla \cdot \nabla \left( |v|^2/2 + P \right) h_a + \nabla \cdot v \nabla \cdot (vh_a) dx da \\
&= \int_{T^2 \times (0,1)} (v \cdot \Delta v) h_a + (\nabla v \cdot \nabla v) h_a + \nabla \cdot v \nabla \cdot (vh_a) dx da + \int_{T^4} \Delta P \int_0^1 h_a dtdx \\
&= \int_{T^2 \times (0,1)} |\nabla v|^2 h_a + \nabla \cdot (\nabla v \cdot (vh_a)) dx da = \int_{T^2 \times (0,1)} |\nabla v|^2 h_a dx da
\end{align*}
On the other hand, the time derivative of \(E_2\) is given by:
\begin{align*}
\partial_t E_2(t) &= \int_{T^2 \times (0,1)} v_t \cdot v h_a + v \cdot v_t h_a - v \cdot v_t \nabla \cdot (vh_a) dx da.
\end{align*}
Now, the second term is equal to:
\begin{align*}
\int_{T^2 \times (0,1)} v \cdot v_t h_a dx da &= - \int_{T^2 \times (0,1)} v \cdot \nabla (v \cdot v_t + P) h_a dx da \\
&= \int_{T^2} P_t \nabla \cdot \int_0^1 v h_a dtdx + \int_{T^2 \times (0,1)} v \cdot v_t \nabla \cdot (vh_a) dx da, \quad \text{which yields (note that } \nabla \cdot \int_0^1 v h_a dtdx = 0): \]
\begin{align*}
\partial_t E_2(t) &= \int_{T^2 \times (0,1)} |\partial_t v|^2 h_a dx da = \int_{T^2 \times (0,1)} \left| \nabla \left( P + \frac{|v|^2}{2} \right) \right|^2 h_a dx da.
\end{align*}
\hfill \Box
\end{proof}

Now, let us prove Theorem \ref{thm:3}.
Proof. By using the previous proposition, we know that: \( \partial_t E_1 \geq E_1^2 \) and

\[
\int_{T^d \times (0,1)} |v|^2 h_a dx da \cdot \partial_t E_2 \geq \left( \int_{T^d \times (0,1)} |v| |v| h_a dx da \right)^2.
\]

\[
\geq \left( \int_{T^d \times (0,1)} v \cdot v h_a dx da \right)^2 = E_2^2.
\]

Which yields the blow up of the two quantities (note that \( \int_{T^d \times (0,1)} |v|^2 h_a dx da \) is time independent).

The proof of Theorem \( \Box \) is analogous to the one showed in Section 3.

Proposition 7. Let \( (v, h_a) \) be a smooth solution to \( T.7 \) with smooth initial condition. Then, the following estimate holds:

\[
\int_{T^d \times (0,1)} |\log(h_b)|^2 h_b dx db|_{t=0} \leq C(T) \left( \int_{T^d \times (0,1)} \log(h_b) h_b dx db|_{t=0} \right),
\]

where

\[
C(T) = \int_{T^d \times (0,1)} |\log(h_b)|^2 h_b dx db|_{t=0} + \log(E_1) \int_{T^d \times (0,1)} \log(h_b) h_b dx db|_{t=0} + \int_0^T (\partial_t E_1) dt.
\]

Proof. First note that:

\[
\int_{T^d \times (0,1)} |\log(h_b)|^2 h_b dx db|_{t=0} = -2 \int_0^T \int_{T^d \times (0,1)} \log(h_b) \nabla \cdot v h_b dx db dt.
\]

\[
\leq \int_0^T \varepsilon(t) \int_{T^d \times (0,1)} |\log(h_b)|^2 h_b dx db dt + \int_0^T \varepsilon^{-1}(t) \partial_t E_1(t) dt.
\]

Now, by choosing \( \varepsilon(t) = \frac{E_1(t)}{\int_{T^d \times (0,1)} \log(h_b) h_b dx db} \), the last term will be equal to:

\[
\int_0^T \varepsilon(t) \int_{T^d \times (0,1)} |\log(h_b)|^2 h_b dx db dt + \log(E_1) \int_{T^d \times (0,1)} \log(h_b) h_b dx db|_{t=0} + \int_0^T \varepsilon^{-1}(t) \partial_t E_1(t) dt.
\]

from which, the result follows by applying the integral Grönwall inequality.

\( \Box \)

Acknowledgements

The author is indebted to Yann Brenier for useful discussions, for mentioning the possibility to extend the blow up result to higher dimensions and for his hospitality during the author’s stay at Laboratoire de Mathématiques d’Orsay.

This work is part of the grant SEV-2015-0554-17-4 funded by: MCIN/AEI/10.13039/501100011033.
ON THE VANISHING OF THE LOCAL RAYLEIGH CONDITION

References

[Bre99] Yann Brenier. Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity*, 12:495, 01 1999.

[Bre03] Yann Brenier. Remarks on the derivation of the hydrostatic euler equations. *Bulletin Des Sciences Mathematiques - BULL SCI MATH*, 127:585–595, 09 2003.

[CINT12] Chongsheng Cao, Slim Ibrahim, Kenji Nakanishi, and Edriss Titi. Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics. *Communications in Mathematical Physics*, 337, 10 2012.

[EE97] Weinan E and Bjorn Engquist. Blowup of solutions of the unsteady prandtl’s equation. *Communications on Pure and Applied Mathematics*, 50(12):1287–1293, 1997.

[Gre99] Emmanuel Grenier. On the derivation of homogeneous hydrostatic equations. *Mathematical Modelling and Numerical Analysis*, 33:965–970, 09 1999.

[Gre00] Emmanuel Grenier. On the nonlinear instability of euler and prandtl equations. *Communications on Pure and Applied Mathematics*, 53:1067 – 1091, 09 2000.

[HKN16] Daniel Han-Kwan and Toan Nguyen. Ill-posedness of the hydrostatic euler and singular vlasov equations. *Archive for Rational Mechanics and Analysis*, 221, 09 2016.

[KTVZ10] Igor Kukavica, Roger Temam, Vlad Vicol, and Mohammed Ziane. Existence and uniqueness of solutions for the hydrostatic euler equations on a bounded domain with analytic data. *Comptes Rendus Mathematique - C R MATH*, 348:639–645, 06 2010.

[KTVZ11] Igor Kukavica, Roger Temam, Vlad Vicol, and Mohammed Ziane. Local existence and uniqueness for the hydrostatic euler equations on a bounded domain. *Journal of Differential Equations*, 250:1719–1746, 02 2011.

[KV21] Michał Kotowski and Bálint Virág. Large deviations for the interchange process on the interval and incompressible flows, 2021.

[KW12] Tak Kwong Wong. Blowup of solutions of the hydrostatic euler equations. *Proc. Amer. Math. Soc.*, 143, 11 2012.

[MW12] Nader Masmoudi and Tak Kwong Wong. On the theory of hydrostatic euler equations. *Archive for Rational Mechanics and Analysis*, 204(1):231–271, Apr 2012.

[Ren09] Michael Renardy. Ill-posedness of the hydrostatic euler and navier–stokes equations. *Arch. Ration. Mech. Anal.*, 194:877–886, 12 2009.

Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, Nicolás Cabrera, 13-15, Madrid, 28049, Spain

Email address: vacanulef@uc.cl