On rainbow trees and cycles

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Abstract
We derive sufficient conditions for the existence of rainbow cycles of all lengths in edge colourings of complete graphs. We also consider rainbow colorings of a certain class of trees.

1 Introduction

Let the edges of the complete graph $K_n$ be coloured so that no colour is used more than max $\{b, 1\}$ times. We refer to this as a $b$-bounded colouring. We say that a subset $S$ of the edges of $K_n$ is rainbow coloured if each edge of $S$ is of a different colour. Various authors have considered the question of how large can $b = b(n)$ be so that any $b$-bounded edge colouring contains a rainbow Hamilton cycle. It was shown by Albert, Frieze and Reed [1] (see Rue [7] for a correction in the claimed constant) that $b$ can be as large as $n/64$. This confirmed a conjecture of Hahn and Thomassen [5]. Our first theorem discusses the existence of rainbow cycles of all sizes. We give a kind of a pancyclic rainbow result.

Theorem 1 There exists an absolute constant $c > 0$ such that if an edge colouring of $K_n$ is $cn$-bounded then there exist rainbow cycles of all sizes $3 \leq k \leq n$.

Having dealt with cycles, we turn our attention to trees.

Theorem 2 Given a real constant $\varepsilon > 0$ and a positive integer $\Delta$, there exists a constant $c = c(\varepsilon, \Delta)$ such that if an edge colouring of $K_n$ is $cn$-bounded, then it contains a rainbow copy of every tree $T$ with at most $(1 - \varepsilon)n$ vertices and maximum degree $\Delta$.

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We conjecture that there is a constant \( c = c(\Delta) \) such that every \( cn \)-bounded edge colouring of \( K_n \) contains a rainbow copy of every spanning tree of \( K_n \) which has maximum degree at most \( \Delta \). We are far from proving this and give a small generalisation of the known case where the tree in question is a Hamilton path. Let \( T^* \) be an arbitrary rooted tree with \( \nu_0 \) nodes. Assume that \( \nu_0 \) divides \( n \) and let \( \nu_1 = n/\nu_0 \). We define \( T(\nu_1) \) as follows: It has a spine which is a path \( P = (x_0, x_1, \ldots, x_{\nu_1 - 1}) \) of length \( \nu_1 - 1 \). We then have \( \nu_1 \) vertex disjoint copies \( T_0, T_1, \ldots, T_{\nu_1 - 1} \) of \( T^* \), where \( T_i \) is rooted at \( x_i \) for \( i = 0, 1, \ldots, \nu_1 - 1 \). \( T(\nu) \) has \( n \) vertices. The edges of \( T(\nu_1) \) are of two types, spine-edges in \( P \) and teeth-edges.

We state our theorem as

\[ \text{Theorem 3} \text{ If an edge colouring of } K_n \text{ is } k \text{-bounded and } \left( \frac{\nu_1 - 2}{\nu_1} \right) > 16kn \text{ then there exists a rainbow copy of every possible } T(\nu_1). \]

2 Proof of Theorem 1

We will not attempt to maximise \( c \) as we will be far from the optimum.

The following lemma is enough to prove the theorem:

\[ \text{Lemma 4} \]

(a) Let \( c_0 = 2^{-7} \) and suppose that \( n \geq 2^{21} \). Then every \( 2c_0n \)-bounded edge colouring of \( K_n \) contains rainbow cycles of length \( k \), \( n/2 \leq k \leq n \).

(b) If \( n \geq e^{1000} \) and \( cn \geq n^{2/3} \) and an edge colouring of \( K_n \) is \( cn \)-bounded, then there exists a set \( S \subseteq [n] \) such that \( |S| = N = n/2 \) and the induced colouring of the edges of \( S \) is \( cN \)-bounded where \( c' = c(1 + 1/(\ln n)^2) \).

We will first show that the lemma implies the theorem. Assume first that \( n \geq e^{1000} \). We let \( N_i = 2^{-i}n \) for \( 0 \leq i \leq r = \lceil \log_2(ne^{-1000}) \rceil \) and note that \( N_i \geq e^{1000} > 2^{21} \) for all \( i \leq r \). Now define a sequence \( c_0, c_1, c_2, \ldots, c_r \) by

\[ c_{i+1} = c_i \left( 1 + \frac{1}{(\ln N_i)^2} \right). \]
Then for $i \geq 1$ we have:
\[
c_i = c_0 \prod_{s=1}^{i} \left(1 + \frac{1}{\ln n - s \ln 2 + 1}\right)
\leq c_0 \exp \left\{ \frac{1}{(\ln n)^2} \sum_{s=1}^{i} \left(1 - \frac{s}{\ln 2}\right)^2 \right\}
= c_0 \exp \left\{ \left(\frac{\log_2 n}{\ln n}\right)^2 \sum_{s=1}^{i} \frac{1}{(\log_2 n - s)^2} \right\}.
\]

Then for all $0 \leq i \leq r$ we have:
\[
c_0 \leq c_i \leq c_0 \exp \left\{ \left(\frac{\log_2 n}{\ln n}\right)^2 \sum_{t=21}^{\infty} \frac{1}{t^2} \right\} \leq c_0 \exp \left\{ 2.1 \int_{t=20}^{\infty} t^{-2} dt \right\} = c_0 \exp \left\{ \frac{2.1}{20} \right\} \leq 2c_0.
\]

Furthermore, for $0 \leq i \leq r$ we have
\[
\frac{c_i}{N_i^{1/3}} \geq \frac{c_0 n^{1/3}}{2i^{1/3}} \geq 1,
\]
which implies that $c_i N_i \geq N_i^{2/3}$.

Assume now we are given a $c_0 n$-bounded coloring of $K_n$ and that $n \geq e^{1000}$. Then by part (a) of the lemma we can find rainbow cycles of length $k$, $n/2 \leq k \leq n$. By part (b) there exists a subset $S$, $|S| = n/2 = N$, such that the induced coloring on $S$ is $c_1 n$-bounded. Now we can apply part (a) of the lemma to the induced subgraph $G[S]$ to find rainbow cycles of length $k$, $n/4 \leq k \leq n/2$. We can continue this halving process for $r$ steps, thus finding rainbow cycles of length $k$, $N_r \leq k \leq n$ where $e^{1000} \leq N_r \leq 2e^{1000}$.

**To summarise:** Assuming the truth of Lemma 4, if $n \geq e^{1000}$ and $c \leq 2^{-7}$ then any $cn$-bounded coloring of $K_n$ contains a rainbow cycle of length $2e^{1000} \leq k \leq n$.

Up to this point, the value of $c$ is quite reasonable. We now choose a very small value of $c$ in order to finish the proof without too much more effort.

Suppose now that $c \leq e^{-3001}$, $n \geq e^{1000}$ and $3 \leq k \leq \min \{2e^{1000}, n\}$. Suppose that $K_n$ is edge colored with $q$ colors and that color $i$ is used $m_i \leq cn$ times. Choose a set $S$ of $k$ vertices. Let $\mathcal{E}$ be the event $S$ contains two edges of the same color. at random. Then,
\[
\Pr(\mathcal{E}) \leq \left(\binom{k}{3}\right)^2 \sum_{i=1}^{q} \left(\frac{m_i}{\binom{n}{2}}\right)^2 + \left(\binom{k}{3}\right) \sum_{i=1}^{q} \left(\frac{m_i}{\binom{2}{2}}\right) \binom{n}{2} \binom{3}{2}
\leq \frac{ck^2}{n-1} + \frac{ck^3}{4} < 1.
\]
The two sums in (1) correspond to having two disjoint edges with the same color and to two edges of the same color sharing a vertex, respectively.

All that is left is the case $n \leq e^{1000}$ but now $c$ is so small that $cn < 1$ and all edges have distinct colors.

### 2.1 Proof of Lemma 4

Part (a) follows immediately from [1] ($n \geq 2^{21}$ is easily large enough for the result there to hold). We can apply the main theorem of that paper to any subset of $[n]$ with at least $n/2$ vertices.

We now prove part (b). Let $S$ be a random $n/2$-subset of $[n]$. Now for each colour $i$ we orient the $i$-coloured edges of $K_n$ so that for each $v \in [n]$,

$$|d_i^+(v) - d_i^-(v)| \leq 1$$

where $d_i^+(v)$ (resp. $d_i^-(v)$) is the out-degree (resp. in-degree) of $v$ in the digraph $D_i = ([n], E_i)$ induced by the edges of colour $i$. Now fix a colour $i$ and let

$$L_i = \{v : d_i^+(v) \geq (\ln n)^6\}.$$

Then with $(v, w)$ denoting an edge oriented from $v$ to $w$ we let

$$A_1 = \{(v, w) \in E_i : v \in L_i\}$$

$$A_2 = \{(v, w) \in E_i : v \notin L_i, w \in L_i \text{ and } \exists \geq (\ln n)^6 \text{ edges of colour } i \text{ from } \overline{L}_i \text{ to } w\}$$

$$A_3 = E_i \setminus (A_1 \cup A_2).$$

Let $|A_j| = \alpha_j n$ where $\alpha_1 + \alpha_2 + \alpha_3 \leq c$.

Let $Z_j, j = 1, 2, 3$, be the number of edges of $A_j$ which are entirely contained in $S$ and let $Z = Z_1 + Z_2 + Z_3$. We write

$$Z_1 = \sum_{v \in L_i} 1_{v \in S} X_{1,v}$$

where $X_{1,v}$ is the number of neighbours of $v$ in $D_i$ that are included in $S$.

Now

$$\Pr(X_{1,v} \geq 1/2d_i^+(v) + 1/4d_i^+(v)^{1/2}\ln n) \leq e^{-(\ln n)^2/24}.$$ 

This follows from the Chernoff bounds (more precisely, using Hoeffding’s lemma [6] about sampling without replacement).

Note that

$$1/2d_i^+(v) + 1/4d_i^+(v)^{1/2}\ln n \leq 1/2d_i^+(v) \left(1 + \frac{1}{2(\ln n)^2}\right).$$

So, on using $n \geq e^{1000}$, we see that with probability at least

$$1 - n e^{-(\ln n)^2/24} = 1 - n^{1-(\ln n)/24} \geq 9/10.$$
we have

\[ Z_1 \leq \frac{1}{2} \alpha_1 n \left( 1 + \frac{1}{2(\ln n)^2} \right). \]

The edges of \( A_2 \) are dealt with in exactly the same manner and we have that with probability at least \( 9/10 \),

\[ Z_2 \leq \frac{1}{2} \alpha_2 n \left( 1 + \frac{1}{2(\ln n)^2} \right). \]

To deal with \( Z_3 \) we observe that if we delete a vertex \( v \) of \( S \) then \( Z_3 \) can change by at most \( 2(\ln n)^6 \). This is because the digraph induced by \( A_3 \) has maximum in-degree and out-degree bounded by \((\ln n)^6 \). Applying a version of Azuma’s inequality that deals with sampling without replacement (see for example Lemma 11 of [4]) we see that for \( t > 0 \),

\[ \Pr \left( Z_3 \geq \frac{1}{4} \alpha_3 n + t \right) \leq \exp \left\{ -\frac{2t^2}{n(\ln n)^{12}} \right\}. \]

So, putting \( t = n^{3/5} \) and using \( n \geq e^{1000} \) and \( cn \geq n^{2/3} \) we see that with probability at least \( 9/10 \),

\[ Z \leq \frac{1}{2}(\alpha_1 + \alpha_2)n \left( 1 + \frac{1}{2(\ln n)^2} \right) + \frac{1}{4} \alpha_3 n + n^{3/5} \leq \frac{1}{2} cn \left( 1 + \frac{1}{(\ln n)^2} \right). \]

So, with probability at least \( 7/10 \) the colouring of the edges of \( S \) is \( c(1 + 1/(\ln n)^2)n/2 \)-bounded and Lemma 4 is proved.

\[ \square \]

### 3 Proof of Theorem 2

We proceed as follows. We choose a large \( d = d(\varepsilon, \Delta) > 0 \) and a small \( c \ll \frac{1}{d^{3/2}} \) and consider a \( cn \)-bounded edge colouring of \( K_n \). We then define \( G_1 = G_{n,p} \), \( p = d/n \). We remove any edge of \( G_1 \) which has the same colour as another edge of \( G_1 \). Call the remaining graph \( G_2 \). The edge set of \( G_2 \) is rainbow coloured. We then remove vertices of low and high degree to obtain a graph \( G_3 \). We then show that \( G_3 \) satisfies the conditions of a theorem of Alon, Krivelevich and Sudakov [2], implying that \( G_3 \) contains a copy of every tree with \( \leq (1 - \varepsilon)n \) vertices and maximum degree \( \leq \Delta \). The theorem we need from [2] is the following:

**Definition:** Given two positive numbers \( a_1 \) and \( a_2 < 1 \), a graph \( G = (V, E) \) is called an \((a_1, a_2)\)-expander if every subset of vertices \( X \subseteq V \) of size \( |X| \leq a_1 |V| \) satisfies \( |N_G(X)| \geq a_2 |X| \). Here \( N_G(X) \) is the set of vertices in \( V(G) \setminus X \) that are neighbours of vertices in \( X \).

**Theorem 5** Let \( \Delta \geq 2 \), \( 0 < \varepsilon < 1/2 \). Let \( H \) be a graph on \( N \) vertices of minimum degree \( \delta_H \) and maximum degree \( \Delta_H \). Suppose that

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\[ T1 \]
\[ N \geq \frac{480\Delta^4 \ln(2/\epsilon)}{\epsilon}. \]

\[ T2 \]
\[ \Delta_H^2 \leq \frac{1}{K} e^{\delta_H/(8K)-1} \text{ where } K = \frac{20\Delta^2 \ln(2/\epsilon)}{\epsilon}. \]

\[ T3 \] Every subgraph \( H_0 \) of \( H \) with minimum degree at least \( \epsilon \delta_H \frac{e^{\delta_H}}{30\Delta^2 \ln(2/\epsilon)} \) is a \( \left( \frac{1}{2\Delta+2}, \Delta+1 \right) \)-expander.

Then \( H \) contains a copy of every tree with \( \leq (1-\epsilon)N \) vertices and maximum degree \( \leq \Delta \).

We now get down to details. In the following we assume that \( cd \ll 1 \ll d \). We will prove that \( \text{whp} \).

\[ P1 \] The number of edges using repeated colours is at most \( d^2cn \).

\[ P2 \] Every set \( X \subseteq [n], |X| \leq n/d^{1/5} \) contains less than \( \alpha d|X| \) edges of \( G_1 \) where, with \( \Delta = 2d \),
\[ \alpha = \frac{\epsilon}{(100\Delta^2(\Delta+2) \ln(2/\epsilon))}. \]

\[ P3 \] \( G_1 \) contains at most \( ne^{-d/10} \) vertices of degree outside \([d/2, 2d]\).

\[ P4 \] Every pair of disjoint sets \( S, T \subseteq [n] \) of size \( n/d^{1/4} \) are joined by at least \( d^{1/2}n/2 \) edges in \( G_1 \).

Before proving that \( P1-P4 \) hold \( \text{whp} \), let us show that they are sufficient for our purposes. Starting with \( G_1 = G_{n,p} \) we remove all edges using repeated colours to obtain \( G_2 \). Then let \( X_0 \) denote the set of vertices of \( G_2 \) whose degree is not in \([d/3, 2d]\). It follows from \( P1, P3 \) that
\[ |X_0| \leq n(e^{-d/10} + 12cd). \] (2)

Note that \( 12cdn \) bounds the number of vertices that lose more than \( d/6 \) edges in going from \( G_1 \) to \( G_2 \).

Now consider a sequence of sets \( X_0, X_1, \ldots \), where \( X_i = X_{i-1} \cup \{x_i\} \) and \( x_i \) has at least \( 2\alpha d \) neighbours in \( X_{i-1} \). We continue this process as long as possible. Let \( G_3 \) be the resulting graph. We claim that the process stops before \( i \) reaches \( |X_0| \). If not, we have a set with \( 2|X_0| \) vertices and at least \( 2\alpha d |X_0| \) edges. For this we need \( 2|X_0| \geq n/d^{1/5} \) (see \( P2 \)) and this contradicts (2) if \( d \) is large and \( c < 1/d^2 \).

Thus \( H = G_3 \) has at least \( n(1-2(e^{-d/10} + 12cd)) \) vertices and this implies that \( T1 \) holds. Also,
\[ d(1/3 - 2\alpha) \leq \delta_H \leq \Delta_H \leq 2d. \]

So if \( d \gg K^2 \), \( T2 \) will also hold.
Now consider a subgraph $\Gamma$ of $H$ which has minimum degree at least $\beta d$ where $\beta = 2(\Delta + 2)\alpha$. Let $\nu = |V(\Gamma)|$. Choose $S \subseteq V(\Gamma)$ where $|S| \leq \frac{n}{\Delta + 2}$ and let $T = N_\Gamma(S)$. Suppose also that $|T| < (|\Delta| + 1)|S|$. 

Suppose first that $|S| \geq n/d^{1/4}$. Then $|S \cup T| \leq \nu(\Delta + 2)/2\Delta + 2$ and so $Y = V(\Gamma) \setminus (S \cup T)$ satisfies $|Y| \geq |S| \geq n/d^{1/4}$. The fact that there are no $S : Y$ edges contradicts $P_1$, $P_4$. 

Now assume that $1 \leq |S| \leq n/d^{1/4}$. Then $|S \cup T| \leq (\Delta + 2)n/d^{1/4} \leq n/d^{1/5}$ and $S \cup T$ contains at least $\beta d|S|/2 \geq \alpha d|S \cup T|$ edges, contradicting $P_2$. 

Thus, $\Gamma$ is $(1/2\Delta + 2, \Delta + 1)$-expander and the minimum degree requirement is $\beta d$ which is weaker than required by $T_3$. 

It only remains to verify $P_1$–$P_4$: 

$P_1$: Let $Z$ denote the number of edges using repeated colours. Let there be $m_i \leq cn$ edges with colour $i$ for $i = 1, 2, \ldots, \ell$. Then

$$E(Z) \leq \sum_{i=1}^{\ell} \binom{m_i}{2} p^2 \leq \frac{(n/2)}{cn} \binom{cn}{2} d^2 \leq \frac{cd^2}{4} n.$$ 

Now whp $G_1$ has at most $dn$ edges and changing one edge can only change $Z$ by at most 2. So, by Azuma’s inequality, we have

$$\Pr(Z \geq E(Z) + t) \leq \exp\left\{-\frac{2t^2}{4dn}\right\},$$

and we get (something stronger than) $P_1$ by taking $t = n^{3/4}$. 

$P_2$: The probability $P_2$ fails is at most

$$\sum_{k=2ad}^{n/d^{1/5}} \binom{n}{k} \left(\frac{k}{2n}\right)^{\alpha d} p^{\alpha d k} \leq \sum_{k=2ad}^{n/d^{1/5}} \left(\frac{k}{2n}\right)^{\alpha d - 1} \left(\frac{e}{\alpha}\right)^{\alpha d} e^k = o(1).$$

$P_3$: If now $Z$ is the number of vertices with degrees outside $[d/2, 2d]$ then the Chernoff bounds imply that

$$E(Z) \leq n(e^{-d/8} + e^{-d/3}),$$

and Azuma’s inequality will complete the proof. 

$P_4$: The probability $P_4$ fails is at most

$$\left(\binom{n}{n/d^{1/4}}\right)^{2d^{1/2}n/2} \sum_{k=0}^{n^2/d^{1/2}} \binom{n^2/d^{1/2}}{k} p^k (1 - p)^{n^2/d^{1/2} - k} \leq 4^e e^{-d^{1/2}n/8} = o(1).$$

4 Proof of Theorem 3

We will use the lop-sided Lovász local lemma as in Erdős and Spencer [3] and in Albert, Frieze and Reed [1]. We state the lemma as
Lemma 6  Let $A_1, A_2, \ldots, A_N$ denote events in some probability space. Suppose that for each $i$ there is a partition of $[N] \setminus \{i\}$ into $X_i$ and $Y_i$. Let $m = \max\{|Y_i| : i \in [N]\}$ and $\beta = \max\{\Pr(A_i | \bigcap_{j \in S} \bar{A}_j) : i \in [N], S \subseteq X_i\}$. If $4m\beta < 1$ then $\Pr(\bigcap_{i=1}^\ell A_i) > 0$.

Suppose now that we have a $k$-bounded colouring of $K_n$ and that $H$ is chosen uniformly from the set of all copies of $T(\nu)$ in $K_n$ where $T$ is an arbitrary rooted tree with $\nu$ vertices. We show that the probability that $H$ is a rainbow copy is strictly positive.

Let $\{e_i, f_i\}, i = 1, 2, \ldots, N$, be an enumeration of all pairs of edges of $K_n$ where $e_i, f_i$ have the same colour (thus $N = \sum e(n_i) \ell$ where $n_i$ is the number of edges of colour $\ell$). Let $A_i$ be the event $H \supset \{e_i, f_i\}$ for $i = 1, 2, \ldots, N$. We apply Lemma 6 with the definition

$$Y_i = \{j \neq i : (e_j \cup f_j) \cap (e_i \cup f_i) \neq \emptyset\}.$$ 

With this definition

$$m \leq 4kn.$$ 

We estimate $\beta$ as follows: Fix $i, S \subseteq X_i$. We show that for each $T \in T_1 = A_i \cap \bigcap_{j \in S} \bar{A}_j$ (this means that $T$ is a copy of $T(\nu_0, \nu_1)$ containing both $e_i, f_i$ and at most one edge from each pair $e_j, f_j$ for $j \in S$) there exists a set $S(T) \subseteq T_2 = A_i \cap \bigcap_{j \in S} \bar{A}_j$ such that (i) $|S(T)| > 4kn$ and (ii) $S(T) \cap S(T') = \emptyset$ for $T \neq T' \in T_1$. This shows that

$$\Pr(A_i | \bigcap_{j \in S} \bar{A}_j) \leq \frac{1}{4m + 1}$$ 

and proves the theorem.

Fix $H \in T_1$. If $e = (x_i, x_{i+1})$ and $f = (x_j, x_{j+1})$ are both spine-edges where $j - i \geq 2$, we define the tree $F_{\text{spine}}(H; e, f)$, which is also a copy of $T(\nu)$, as follows: We delete $e, f$ from $H$ and replace them by $(x_i, x_j)$ and $(x_{i+1}, x_{j+1})$. Suppose now that $e = (a, b) \in T_1 \setminus x_i$ and $f = (c, d) \in T \setminus x_j$ are both teeth-edges and that $\phi(e) = f$ in some isomorphism from $T_i$ to $T_j$. Then we define $F_{\text{teeth}}(H; e, f)$ as follows: We delete $e, f$ from $H$ and replace them by $(a, d)$ and $(b, c)$ to get another copy of $T(\nu)$.

Observe that if $f \neq f_i$ then $H' = F_{\sigma}(H; e_i, f) \in T_2$ for $\sigma \in \{\text{spine, teeth}\}$. This is because $e_i$ is not an edge of $H'$ and the edges that we added are all incident with $e_i$. We cannot therefore have caused the occurrence of $A_j$ for any $j \in X_i$. Similarly, $F_\sigma(H'; f_i, g) \in T_2$ for $g \neq e_i$.

We use $F_{\text{spine}}, F_{\text{teeth}}$ to construct $S(H)$ as follows: We choose an edge $f \neq f_i$ of the same type as $e_i$ and construct $H' = F_{\sigma}(H; e_i, f)$ for the relevant $\sigma$. We then choose $g \neq e_i$ of the same type as $f_i$ and construct $H'' = F_{\sigma'}(H'; f_i, g)$. In this way we construct $S(H) \subseteq T_2$ containing at least $(n_i^2 - 2)$ distinct copies of $T(\nu_1)$.

Notice that knowing $e_i, f_i$ allows us to construct $H'$ from $H''$ and then $H$ from $H'$. This shows that $S(H) \cap S(H') = \emptyset$. After this, all we have to do is choose $k, \nu_1$ so that $(n_i^2 - 2) > 16kn$ in order to finish the proof of Theorem 3.
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