On the non-3-colourability of random graphs.

Olivier Dubois ∗ Jacques Mandler†

Abstract. We show that for $c \geq 2.4682$, a random graph on $n$ vertices with $cn(1 + o(1))$ edges almost surely has no 3-colouring. This improves on the current best upper bound of 2.4947.

1 Introduction

An old problem on random graphs remaining open to this day is that of the existence and determination of a $k$-colourability threshold, already posed in the paper [10] which launched the whole subject. Using the uniformly distributed model $G(n, m)$ of graphs with $m$ edges on $n$ vertices, it reads: does there exist a constant $c_k$ such that if $m \sim (c_k - \varepsilon)n$ for some $\varepsilon > 0$ as $n \to \infty$, then almost all graphs in $G(n, m)$ are $k$-colourable, while if $m \sim (c_k + \varepsilon)n$ almost none is? The conjectured positive answer has been mostly pursued in the case $k = 3$, the smallest value for which $k$-colourability is an NP-complete problem. It is now supported by computer experiments which put $c_3$ at about 2.3, and a non-uniform version is known to hold [1], stating that at least an $n$-dependent $c_k(n)$ exists with the required property. Whether $c_k(n)$ converges remains open, but its behaviour in the large $n$ limit is constrained by proven upper and lower bounds which are getting progressively tighter.

The first lower bounds were by-products of studies on the existence of a $k$-core, which is a necessary condition for non-$k$-colourability, but understandably perhaps, even the exact threshold for the $k$-core [20] yields a mediocre bound for colourability, e.g. 1.675 for $k = 3$. More recently, better bounds were achieved by analyzing simple colouring algorithms, using the powerful differential-equation techniques of [21]. This gave $c_3 > 1.923$ [2], and to go beyond that, more complex extensions of Wormald’s techniques had to be sought [1], leading to the current best lower bound of 2.015 for $c_3$.

Upper bounds are generally based on the ubiquitous first-moment method, starting with the ‘naive’ bound $c_k < k \log k$ as obtained by Devroye see [3]. For $k = 3$, this is 2.71. More recent bounds have used the expected number of colourings with a local minimality property similar to that introduced for $k$-SAT [5, 18]. In [6], however, the property was weaker than it might have been, giving nevertheless $c_3 < 2.60$. This was corrected by [8], introducing the more restrictive ‘rigid colourings’ but using an untight bound for a probability appearing in the expectation. This gave $c_3 < 2.522$. Finally, this was improved to 2.495 by [13], correcting an error in [17], and independently by [11]. They used a rather sophisticated occupancy result of [16] to evaluate exactly the probability just mentioned. We seem to be reaching a point where the sheer complexity of the calculations needed to extend the method becomes a hindrance to further progress.

This paper lowers the upper bound on $c_3$ to 2.468155, i.e. by a similar amount as between the current best and previous best. We still use rigid colourings, but now the graphs themselves are restricted to a subspace sufficiently ‘dense’ for the first-moment (Markov) estimate still to apply in the limit, but that also leaves out many graphs that contribute to raise the first moment. Perhaps the crucial point is that the calculations are actually fairly straightforward if well taken. On the other hand, some side issues may be a bit tedious to check explicitly. In this extended abstract, we will concentrate on the main flow of the calculation. We first introduce the subspace of graphs we are

∗LIP6, Box 169, CNRS-Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, France.
†LIP6, Box 169, CNRS-Université Paris 6, 4 place Jussieu, 75252 Paris Cedex 05, France.
restricting our attention to, and show how calculating the first moment of rigid colourings in this space yields an upper bound for $c_3$. We then perform the actual calculation, leading to a pair of nonlinear equations in two unknowns. Existence and uniqueness of a solution is then discussed, justifying a simple iterative procedure, the result of which is plugged into the expectation. For the above-quoted value of $c$, this gives an expectation just below 1.

2 A large subspace of random graphs.

Throughout the paper, $m = cn(1 + o(1))$, and $2.4 < c < 2.5$. (Non-colourability is known to hold a.s. above 2.5, and what happens below 2.4 is not our topic here.) We also denote the probability space by $G(n, m)$, and, setting $\lambda = 2c$, by $p(x, \lambda)$ or $p_x$ the Poisson probability function of mean $\lambda$, i.e. $e^{-\lambda \lambda^x / x!}$. To avoid irrelevancies in our enumerations, we will consider directed graphs given by an ordered list of edges; multiple edges and single-vertex loops are allowed (although statistically insignificant). This changes nothing as to existence and value of the threshold. With these conventions, $|G(n, m)| = n^{\lambda_n}$.

First, let the random variable $\theta_x$ denote the number of vertices of a random graph having degree $x$. We show that $\theta_x$ is concentrated around its mean which is $p_x$.

**Lemma 2.1** There is an absolute constant $C > 0$ such that for any $\varepsilon > 0$,

$$\Pr (|\theta_x - p_x| > \varepsilon) \leq C \sqrt{\lambda_n} e^{-c(\varepsilon, p_x)n},$$  

where $c(\xi, \eta) = \min \left( (\xi + \eta) \log (1 + \xi/\eta) - \xi, \frac{\xi^2}{2\eta} \right)$. In particular, $\lim_{n \to \infty} \Pr (|\theta_x - p_x| > \varepsilon) = 0$.

**Proof** If $K_i$ is the degree of vertex $i$, the random vector $(K_i)_{1 \leq i \leq n}$ follows a multinomial distribution, which we can view as describing $\lambda n$ indistinguishable balls (the extremities of the $cn$ edges) being thrown into $n$ bins (the vertices). A simple Poissonization argument (considering the situation where a Poisson number $M$ of balls with mean $\lambda n$ are thrown) shows that there are independent r.v.’s $L_i$, with mean $\lambda$, such that the $K_i$’s are distributed as the $L_i$’s, conditional on $M = \lambda n$. The sum $W_x = \sum^n_{i=1} 1_{\{L_i = x\}}$ obeys a binomial large-deviation inequality:

$$\Pr \left( \left| \frac{W_x}{\lambda n} - p_x \right| > \varepsilon \right) \leq 2 e^{-c(\varepsilon, p_x)n}$$

where $c(\varepsilon, p_x) = c_\varepsilon/p_x p_x$, and $c_\varepsilon$ is as in [5], Corollary A.14. Decomposing w.r.t. the values of $M$ and using a standard inequality for Poisson r.v.’s yields [6].

It is interesting to note that in the sequel, we do not need the full strength of Lemma 2.1, but only the weak form of concentration stated at the end.

Now, to $\varepsilon > 0$ and $x_{\text{max}} \in \mathbb{N}$, we associate the set $G(\varepsilon, x_{\text{max}}, n, m)$ of graphs such that for $0 \leq x \leq x_{\text{max}}$, the number of vertices with degree $x$ lies between $(p_x - \varepsilon) n$ and $(p_x + \varepsilon) n$. The idea is that in view of Lemma 2.1, for $x_{\text{max}}$ large enough and $\varepsilon$ small enough, $G(\varepsilon, x_{\text{max}}, n, c)$ contains ‘most well-behaved’ members of $G(n, m)$, and therefore that the expected number of (rigid) 3-colourings of graphs drawn uniformly from $G(\varepsilon, x_{\text{max}}, n, m)$ provides an upper bound for $\Pr (3-\text{Col})$, the probability of 3-colourability. But, the discrepancy between the first-moment (say, rigid-colouring) bound and $c_3$ is due to a ‘small’ (yet exponential) number of ‘rogue’ graphs having a huge number of colourings, many of which are left out from $G(\varepsilon, x_{\text{max}}, n, m)$. Consequently, the expectation just mentioned actually gives a better bound than the rigid colourings by themselves. Of course, in dealing with $G(\varepsilon, x_{\text{max}}, n, m)$ we have to control approximations. As for the calculations, performing them in a constrained subset of $G(n, m)$ is actually in some ways beneficial, e.g. we do not need the balls-and-bins occupancy results of [11].

The following proposition makes precise the general idea just explained while staying at the level of the original probability space $G(n, m)$, which is technically simpler than working in the subspace $G(\varepsilon, x_{\text{max}}, n, m)$. The r.v. $R(G)$ is defined as the number of rigid colourings of $G \in G(n, m)$.
Proposition 2.2 Let the r.v. \( X_{\varepsilon, x_{\max}, n, m} \) on \( G(n, m) \) be defined by

\[
X_{\varepsilon, x_{\max}, n, m}(G) = \begin{cases} 
R(G) & \text{if } G \in \mathcal{G}(\varepsilon, x_{\max}, n, m) \\
0 & \text{otherwise.}
\end{cases}
\]

If, for some integer \( x_{\max} \) and some \( \varepsilon > 0 \), \( \mathbb{E}[X_{\varepsilon, x_{\max}, n, m}] \) tends to 0 as \( n \to \infty \), then so does \( \Pr(3\text{-Col}) \).

Proof Let \( 3\text{-col}(n, m) = \{G \in G(n, m) : G \text{ is 3-colourable}\} \). Then, by Lemma 2.1 and since \( R(G) \geq 1 \) for \( G \in 3\text{-col}(n, m) \):

\[
\Pr(3\text{-Col}) = \frac{|3\text{-col}(n, m)|}{|G(n, m)|} = \frac{|\mathcal{G}(\varepsilon, x_{\max}, n, m) \cap 3\text{-col}(n, m)|}{|G(n, m)|} + \frac{|\{G \in G(n, m) : \exists x \ (0 \leq x \leq x_{\max} \text{ and } |\theta_x(G) - p_x| \geq \varepsilon)\}|}{|G(n, m)|} \\
\leq \frac{1}{|G(n, m)|} \sum_{G \in \mathcal{G}(\varepsilon, x_{\max}, n, m) \cap 3\text{-col}(n, m)} 1 + (x_{\max} + 1) o(1) \\
\leq \frac{1}{|G(n, m)|} \sum_{G \in \mathcal{G}(\varepsilon, x_{\max}, n, m) \cap 3\text{-col}(n, m)} R(G) + o(1) \\
= \mathbb{E}[X_{\varepsilon, x_{\max}, n, m}] + o(1).
\]

3 Combinatorial analysis of the expectation.

First, let \( \Theta_{\varepsilon, x_{\max}, n, m} \) be the set of vectors \( \theta = (\theta_x)_{0 \leq x \leq x_{\max}} \) in \( I_{x_{\max}+1}^n = \{0, 1/n, 2/n, ..., 1\}^{x_{\max}+1} \) with \( \sum_{x=0}^{x_{\max}} \theta_x \leq 1, \sum_{x=0}^{x_{\max}} x \theta_x \leq \lambda \), and all \( |\theta_x - p_x| < \varepsilon \). Since this set is of polynomial size \( \leq (2\varepsilon n)^{x_{\max}+1} \), and polynomial factors are irrelevant in our study, counting graphs in \( \mathcal{G}(\varepsilon, x_{\max}, n, m) \) with some property \( \mathcal{P} \) really boils down to counting, for fixed \( \theta \in \Theta_{\varepsilon, x_{\max}, n, m} \), graphs with \( \mathcal{P} \) in the set \( \mathcal{G}(\theta) \) of \( G \in \mathcal{G}(\varepsilon, x_{\max}, n, m) \) with \( (\theta_x(G))_{0 \leq x \leq x_{\max}} = \theta \).

To say that \( G \in \mathcal{G}(\theta) \) is 3-colourable means that there is a partition of the vertices into vertices of types 0, 1 and 2 (‘blue’, ‘red’, ‘green’) such that there are only three types of edges: types 0 (joining a blue and a red vertex), 1 (red and green), and 2 (green and blue).

Recall also that a rigid coloring is one in which every vertex of type 0 has edges joining it to at least a vertex of type 1 and at least a vertex of type 2, while every vertex of type 1 is joined to at least a vertex of type 2.

Now, given such a partition and rationals \( \beta_0, \beta_1, \beta_2 \in I_n \), as well as \( \mu^0_{x, j}, \mu^1_{x, j}, \mu^2_{x, j} \) \( (0 \leq x \leq x_{\max}, 0 \leq j \leq x) \), we count the graphs in \( \mathcal{G}(\theta) \) with:

- \( \beta_i m \) edges of type \( i \), and:
- the vertices of degree \( x \) being distributed as follows: for \( i = 0, 1, 2 \) and \( 0 \leq j \leq x \), there are \( \mu^i_{x, j} \beta_x n \) vertices of degree \( x \) and type \( i \), each of which has \( j \) (type-\( i \)) edges joining it to vertices of type \( i + 1 \) (mod 3),

which are rigidly coloured. Let \( \mathcal{Z}(\theta, \beta, \mu, n, m) \) be the number of such graphs.

Note that the colouring being rigid says exactly that \( \mu^0_{x, 0} = \mu^0_{x, x} = 0 \) and \( \mu^1_{x, 0} = 0 \). Also, any vertex of degree 0 must be of type 2, and any vertex of degree 1 must be of type 1 or 2. Accordingly,
we will be considering only $\mu_{x,j}^0$ for $2 \leq x \leq x_{\text{max}}, 1 \leq j \leq x - 1$, and $\mu_{x,j}^1$ for $1 \leq x \leq x_{\text{max}}, 1 \leq j \leq x$. For $\mu_{x,j}^2$, no restriction applies.

We first choose, among $m$ empty templates representing the edges, those corresponding to each type of edge, and within each edge template, the colour of each vertex (recall that in our model the edges are directed). This can be done in $A_n(\beta, m)$ ways, where

$$A_n(\beta, m) = \frac{m!}{(\beta_0m)! (\beta_1m)! (\beta_2m)!} 2^m.$$ 

Second, we attribute each vertex a type. Within each group of $\mu_{x,j}^i \theta_x n$ vertices of degree $x \leq x_{\text{max}}$, we comply with the above-stated requirements. The remaining $\tau n$ vertices, with $\tau = 1 - \sum_{x=0}^{x_{\text{max}}} \theta_x$, will be those of degree $> x_{\text{max}}$. The number of ways this can be done is:

$$B_n(\theta, \mu, n) = \frac{n!}{(\theta_0n)! (\theta_1n)! ... (\theta_{x_{\text{max}}n})! (\tau n)!} \times \prod_{x=0}^{x_{\text{max}}} \left( \mu_{x,1}^0 \theta_x n \right)! ... \left( \mu_{x,x-1}^1 \theta_x n \right)! \left( \mu_{x,x}^1 \theta_x n \right)! \left( \mu_{x,0}^2 \theta_x n \right)! ... \left( \mu_{x,x}^2 \theta_x n \right)!.$$ 

Finally, we effectively fill the template locations with the vertices of various types. Let $M_n(\theta, \beta, \mu, n, m)$ be the number of possibilities here. To begin with, consider the vertices of high degree ($> x_{\text{max}}$). They are to occupy $\sigma n$ places, with $\sigma = (\lambda - \sum_{x=0}^{x_{\text{max}}} x \theta_x) (1 + o(1))$ and $\sigma \to 0$ as $x_{\text{max}} \to +\infty$ and $\varepsilon \to 0$, uniformly in $\theta$. The ways to assign them are certainly less than

$$\eta(\theta, n, m) = \frac{\lambda n}{\sigma n} (\tau n)^{\sigma n}.$$ 

The $\beta_0m$ type-0 edge templates contain $2 \beta_1 m$ vertices, $\beta_1 m$ of which are already known to be blue, and $\beta_1 m$ red. Let us say that among the blue ones, $\sigma_0 m$ are of high degree and already assigned, and $\sigma^0 m$ among the red ones. The number of ways to fill the still-free places in the type-0 templates is, then:

$$M_0 = \frac{[(\beta_0 - \sigma_0^0) m]!}{\prod_{x=2}^{x_{\text{max}}} x \prod_{j=1}^{x_{\text{max}}} J(x-j) \mu_{x,j}^0 \theta_x n} \times \frac{[(\beta_0 - \sigma_1^0) m]!}{\prod_{x=2}^{x_{\text{max}}} x \prod_{j=1}^{x_{\text{max}}} J(x-j) \mu_{x,j}^0 \theta_x n}.$$ 

We fill similarly the type-1 and type-2 templates, with

$$M_1 = \frac{[(\beta_1 - \sigma_1^0) m]!}{\prod_{x=1}^{x_{\text{max}}} x \prod_{j=1}^{x_{\text{max}}} J(x-j) \mu_{x,j}^1 \theta_x n} \times \frac{[(\beta_1 - \sigma_1^0) m]!}{\prod_{x=2}^{x_{\text{max}}} x \prod_{j=1}^{x_{\text{max}}} J(x-j) \mu_{x,j}^1 \theta_x n}$$

and

$$M_2 = \frac{[(\beta_2 - \sigma_0^2) m]!}{\prod_{x=0}^{x_{\text{max}}} x \prod_{j=0}^{x_{\text{max}}} J(x-j) \mu_{x,j}^2 \theta_x n} \times \frac{[(\beta_2 - \sigma_0^2) m]!}{\prod_{x=2}^{x_{\text{max}}} x \prod_{j=1}^{x_{\text{max}}} J(x-j) \mu_{x,j}^2 \theta_x n}$$

possibilities, respectively. By construction $\sigma_0^0 + \sigma_1^0 + \sigma_1^1 + \sigma_2^1 + \sigma_2^2 + \sigma_0^2 = \sigma$. All in all,

$$M_n(\theta, \beta, \mu, n, m) \leq M_0 M_1 M_2 \eta(\theta, n, m),$$

$$Z(\theta, \beta, \mu, n, m) = A_n(\beta, m) B_n(\theta, \mu, n) M_n(\theta, \beta, \mu, n, m) \leq A_n(\beta, m) B_n(\theta, \mu, n) M_0 M_1 M_2 \eta(\theta, n, m).$$

and, for the expectation referred to in Proposition 2.2:

$$\mathbb{E}[X_{\varepsilon, x_{\text{max}}, n, m}] \leq \frac{1}{n^{\beta \mu}} (2^{zn})^{x_{\text{max}}+1} \sum_{\beta, \mu} A_n(\beta, m) B_n(\theta, \mu, n) M_n(\theta, \beta, \mu, n, m),$$

where $\beta$ and $\mu$ in the sum are constrained by a relationship which we shall examine later.
4 Asymptotics.

Using an inequality version of Stirling’s formula and/or upper bounds on multinomial coefficients derived from it, it is seen that (with the convention that an empty product is equal to 1):

\[ A_n (\beta, m) \frac{1}{n} \leq (1 + o (1)) \left( \frac{2e}{\beta_0^2 \beta_1^2 \beta_2^2} \right)^n, \]

\[ B_n (\theta, \mu, n) \frac{1}{n} \leq \frac{1}{\tau^e} \prod_{x=0}^{x_{\text{max}}} \left( \theta_x \prod_{j=1}^{x-1} \mu_{x,j} \prod_{j=1}^{x} \mu_{x,j} \prod_{j=0}^{x} \mu_{x,j}^2 \right) \theta_x, \]

\[ \mathcal{M}_0 \frac{1}{n} \leq (1 + o (1)) \left( \beta_0 cn/e \right)^{2(\beta_0 - \sigma^0_0 - \sigma^0_1)} \prod_{x=0}^{x_{\text{max}}} \left( \prod_{j=1}^{x-1} j \mu_{x,j+1} \prod_{j=1}^{x} \mu_{x,j} \right)^{\theta_x}, \]

\[ \mathcal{M}_1 \frac{1}{n} \leq (1 + o (1)) \left( \beta_1 cn/e \right)^{2(\beta_1 - \sigma^1_0 - \sigma^1_1)} \prod_{x=0}^{x_{\text{max}}} \left( \prod_{j=1}^{x-1} j \mu_{x,j+1} \prod_{j=1}^{x} \mu_{x,j} \right)^{\theta_x}, \]

\[ \mathcal{M}_2 \frac{1}{n} \leq (1 + o (1)) \left( \beta_2 cn/e \right)^{2(\beta_2 - \sigma^2_0 - \sigma^2_1)} \prod_{x=0}^{x_{\text{max}}} \left( \prod_{j=1}^{x-1} j \mu_{x,j+1} \prod_{j=1}^{x} \mu_{x,j} \right)^{\theta_x}, \]

so that, using \( j!(x-j) = x!/(x-j) \):

\[ (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2) \frac{1}{n} \leq \zeta_0 (\varepsilon, x_{\text{max}}) \left( \frac{\beta_0 cn/e}{\beta_0^2 \beta_1^2 \beta_2^2} \right)^{2c} \prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x-1} \left( x! \mu_{x,j} + \mu_{x,j+1} + \mu_{x,j}^2 \right) \theta_x, \]

where \( \lim_{\varepsilon \to 0, x_{\text{max}} \to +\infty} \zeta_0 (\varepsilon, x_{\text{max}}) = 1 \), uniformly in \( \theta \), and that

\[ (A_n (\beta, m) B_n (\theta, \mu, n) \eta (\theta, n, m) \mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2) \frac{1}{n} \leq \frac{\zeta'_0 (\varepsilon, x_{\text{max}})}{\prod_{x=0}^{x_{\text{max}}} (x! \theta_x)^{\theta_x}} \times \left( \prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x-1} \left[ \mu_{x,j+1}^0 \mu_{x,j+1}^1 \mu_{x,j+1}^2 \right] \right) \left( \prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x-1} \left[ \mu_{x,j+1}^0 \mu_{x,j+1}^1 \mu_{x,j+1}^2 \right] \right) \theta_x, \]

where \( \zeta'_0 \) has the same property (and by convention \( 0^0 = 1 \)).

Further, we can get \( \theta \)-free estimates where \( \theta_x \) is replaced throughout by \( p_x \), at the price of additional factors which all tend to 1 as \( \varepsilon \to 0 \) and \( x_{\text{max}} \to +\infty \). In the end result, the sum in (2) being of a polynomial number of exponentially-behaved terms,

\[ \lim_{n \to \infty} E \left[ X_{\varepsilon, x_{\text{max}}, n, m} \right] \frac{1}{n} \leq \zeta_1 (\varepsilon, x_{\text{max}}) \frac{\lambda/e}{2^c} \prod_{x=0}^{x_{\text{max}}} \left( x! p_x \right)^{p_x} \times \left( \beta_0 \beta_1 \beta_2 \right)^{c} \prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x-1} \left[ \mu_{x,j+1}^0 \mu_{x,j+1}^1 \mu_{x,j+1}^2 \right] \right) \theta_x, \]

where \( \lim_{\varepsilon \to 0, x_{\text{max}} \to +\infty} \zeta_1 (\varepsilon, x_{\text{max}}) = 1 \), and the max is under constraints to which we now come.
5 The optimization.

We set:

\[
\begin{align*}
\sum_{j=1}^{x-1} \mu_{x,j}^0 &= \alpha_x^0, & \sum_{j=1}^{x} \mu_{x,j}^1 &= \alpha_x^1, & \sum_{j=0}^{x} \mu_{x,j}^0 &= \alpha_x^2,
\end{align*}
\]

and note that for \(0 \leq x \leq x_{\text{max}}\),

\[
\alpha_x^0 + \alpha_x^1 + \alpha_x^2 = 1,
\]

and that \(\alpha_x^0 = \mu_{0,0}^0 = 1, \alpha_x^1 = \mu_{0,0}^1 = \alpha_x^0 = \mu_{0,0}^0 = \alpha_x^0 = \mu_{1,0}^0 = \mu_{1,1}^0 = 0\); also, \(\mu_{1,1}^1 = \alpha_x^1\).

Introduce the reduced blue, red and green spreads \(\varphi_0, \varphi_1\) and \(\varphi_2\) of the coloured graph. These are the quotients by \(\lambda n\) of the numbers of places of the corresponding colours in our filled graph template (excluding the places occupied by vertices of degree \(> x_{\text{max}}\)); namely,

\[
\varphi_i = \lambda^{-1} \sum_{x=0}^{x_{\text{max}}} xp_x \alpha_x^i = \lambda^{-1} \sum_{x=0}^{x_{\text{max}}} xp_x \sum_{j=0}^{x} \mu_{x,j}^i.
\]

Note that \(\beta_0 + \beta_2 = 2\varphi_0 + \eta_0 (\varepsilon, x_{\text{max}})\), where \(\lim_{\varepsilon \to 0} x_{\text{max}} \to +\infty \eta_0 (\varepsilon, x_{\text{max}}) = 0\). Similarly, \(\beta_0 + \beta_1 = 2\varphi_1 + \eta_1, \beta_1 + \beta_2 = 2\varphi_2 + \eta_2,\) and \(\varphi_0 + \varphi_1 + \varphi_2 = 1 + \eta_3\), (actually \(\eta_3\) depends only on \(x_{\text{max}}\), see below). Thus, \(\beta_0 = 1 - 2\varphi_2 + \eta_4,\) and similarly for \(\beta_1\) and \(\beta_2\). From this we see that (3) still holds with \(1 - 2\varphi_1\) replacing \(\beta_{i+1(\text{mod}3)}\), and the maximization on \(\mu\) alone (subject to \(0 \leq \varphi_i \leq 1/2\)), under the penalty of a slightly larger \(\zeta_1 (\varepsilon, x_{\text{max}})\) which still tends to 1 in the limit of \(\varepsilon \to 0, x_{\text{max}} \to +\infty\).

Also, the max may be extended to \(\mu\) being a vector of reals in \([0, 1]\). We are therefore looking to solve the problem: minimize the function

\[
f(\mu) = \sum_{x=0}^{x_{\text{max}}} p_x \sum_{j=0}^{x} \left[ \mu_{x,j}^0 \log \left( \frac{\mu_{x,j}^0}{j} \right) + \mu_{x,j}^1 \log \left( \frac{\mu_{x,j}^1}{j} \right) + \mu_{x,j}^2 \log \left( \frac{\mu_{x,j}^2}{j} \right) \right]
\]

\[+ c(1 - 2\varphi_0) \log(1 - 2\varphi_0) + c(1 - 2\varphi_1) \log(1 - 2\varphi_1) + c(1 - 2\varphi_2) \log(1 - 2\varphi_2)\]

(\text{where by convention } 0 \log 0 = 0), subject to the constraints \(0 = C_x = \alpha_x^0 + \alpha_x^1 + \alpha_x^2 - 1\) for \(0 \leq x \leq x_{\text{max}}, \mu \geq 0, 0 \leq \varphi_0, \varphi_1, \varphi_2 \leq 1/2,\) and \(\mu_{0,0} = \mu_{x,x} = \mu_{x,0} = 0\) (so that these are not really variables, and we view \(\mu\) as a vector in \(\mathbb{R}^{3(x_{\text{max}} + 1)/2}\) and not \(\mathbb{R}^{3(x_{\text{max}} + 1)(x_{\text{max}} + 2)/2}\)). Setting \(\varphi_{\text{min}} = 0.26\) and \(\varphi_{\text{max}} = 0.4\), simple calculations indicate that within our chosen \(\varepsilon\) domain, the expected number of (unrestricted) 3-colourings such that \(\varphi_0 \leq \varphi_{\text{min}}\) or \(\varphi_0 \geq \varphi_{\text{max}}\) tends to zero anyway, and similarly for \(\varphi_1\) and \(\varphi_2\). This means that we can restrict \(\mu\) to the set \(U = \mathbb{R}_{+}^{3(x_{\text{max}} + 1)/2} \cap \{ \mu : \varphi_{\text{min}} < \varphi_i < \varphi_{\text{max}}, i = 0, 1, 2 \}\). \(U\) is not open, but it can be seen directly that a vector \(\mu\) with a null coordinate (recall that we have excluded the coordinates \(\text{required to be null}\)) cannot be a local minimum. So we can replace \(\mathbb{R}_{+}\) with \([0, +\infty]\) and minimize on the resulting open set \(D\), where differential techniques can be used.

Since the constraints above are linear, the classical method of Lagrange multipliers \([\text{13}]\) applies without having to check for some constraint qualification such as linear independence of gradients (which is true, though). Associating a Lagrange multiplier \(\Lambda_x\) to the constraint \(C_x = 0\), a necessary condition for a local minimum is

\[
0 = \nabla f + \sum_{x=0}^{x_{\text{max}}} \Lambda_x \nabla C_x.
\]

This gives, for \(i = 0, 1, 2\), with \(j \notin \{0, x\}\) if \(i = 0\), and \(j \neq 0\) if \(i = 1\):

\[
\mu_{x,j}^i = \frac{(x)}{\exp[\Lambda_x/p_x + 1 - x]} = \frac{(x)}{\exp[\Lambda_x/p_x + 1 - x]} = \frac{(x)}{\exp[\Lambda_x/p_x + 1 - x]} = \frac{(x)}{\exp[\Lambda_x/p_x + 1 - x]},
\]

(6)
which we plug back into $C_x = 0$ to find that the denominator is

$$B(x, \varphi) = \max \left( 0, 2^x - 2 \right) \left( 1 - 2\varphi_0 \right)^x + \left( 2^x - 1 \right) \left( 1 - 2\varphi_1 \right)^x + 2^x \left( 1 - 2\varphi_2 \right)^x.$$  

With the $\varphi_i$ defined by (5), our necessary condition (6) is a rather hopeless system of $3x_{\max} \left( x_{\max} + 1 \right)/2$ nonlinear equations in as many unknowns. However, its peculiar form means that if we view (6, 5) as a system of equations in $\mu$ and $\varphi$, there is an 'easier' way to solve it, namely eliminating the $\mu$’s by plugging the r.h.s’s $r_{x,j}^i(\varphi)$ of (6) into (5). Noting further that the constraints $C_x = 0$ imply $\varphi_2 = U(x_{\max}) - \varphi_0 - \varphi_1$, with $U(x_{\max}) = \lambda^{-1} \sum_{x=0}^{x_{\max}} x p_x$, we obtain a much nicer necessary condition, namely two equations in the two unknowns $\varphi_0$ and $\varphi_1$:

$$0 = \lambda \varphi_0 - \sum_{x=0}^{x_{\max}} x p_x \sum_{j=0}^{x} r_{x,j}^0(\varphi_0, \varphi_1, U(x_{\max}) - \varphi_0 - \varphi_1),$$  

$$0 = \lambda \varphi_1 - \sum_{x=0}^{x_{\max}} x p_x \sum_{j=0}^{x} r_{x,j}^1(\varphi_0, \varphi_1, U(x_{\max}) - \varphi_0 - \varphi_1).$$  

(In practice, we take $x_{\max}$ sufficiently large so that $U(x_{\max})$ can be replaced by 1.) Having solved (7, 8), we recover $\mu$ from (5).

Of course, even a system of two nonlinear equations can be unmanageable, but in this case a change of variables $\varphi_0 = y_0 + y_1, \varphi_1 = y_0 - y_1$ turns (7, 8) into

$$K_0(y_0, y_1) = 0,$$

$$K_1(y_0, y_1) = 0,$$

where $K_0$ and $K_1$ are functions that, within our restricted range $\varphi_{\min} < \varphi_i < \varphi_{\max}$, are found to be monotone in each variable separately, with partial derivatives

$$\frac{\partial K_0}{\partial y_0} > 0, \quad \frac{\partial K_0}{\partial y_1} > 0, \quad \frac{\partial K_1}{\partial y_0} > 0, \quad \frac{\partial K_1}{\partial y_1} < 0.$$

Since along $K_i = 0$, we have $\partial y_1/\partial y_0 = - (\partial K_i/\partial y_0) / (\partial K_i/\partial y_1)$, it follows that $y_1$ decreases in $y_0$ along $K_0 = 0$, while it increases along $K_1 = 0$ (see Fig. 1). Further, for the smallest attainable value of $y_0$ in our range, the solution in $y_1$ of $K_0 = 0$ is larger than that of $K_1 = 0$; while the reverse holds for the largest attainable $y_0$. Thus, the existence and uniqueness of the solution to (9), and therefore also to (7, 8), simply follow from the intermediate value theorem. Since a minimum of $f(\mu)$ in $U$ must exist (corresponding, in the limit, to the maximum term of $\mathbb{E}[X_{\varepsilon,x_{\max},n,m}]$), and since there is no local minimum on the boundary of $U$, it must be a point of null gradient in $D$, so this is it.

Figure 1: The implicit functions defined by (10). The first equation defines the decreasing function, the second the increasing one.
6 The numerical calculations.

We now evaluate our modified estimate (3) with the \( \varphi_j \) and \( \mu_{x,j}^i \) derived from (7), 8) and (3) together with \( \mu_{x,0}^i = \mu_{x,x}^i = \mu_{0,0}^1 = 0 \). We have, for \( i = 0, 1, 2 \) (since the factors corresponding to \( \mu_{x,j}^i = 0 \) evaluate to 1 on both sides of the first equality):

\[
\prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x} \left( \begin{array}{c}
\mu_{x,j}^0 \\
\mu_{x,j}^1 \\
\mu_{x,j}^2
\end{array} \right) = \prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x} \left( \begin{array}{c}
(1-2\varphi_0)\varphi_j \lambda \\
B(x, \varphi) \n(1-2\varphi_1)\varphi_j \lambda \\
B(x, \varphi) \n(1-2\varphi_2)\varphi_j \lambda
\end{array} \right),
\]

so, taking account of (3), i.e., \( C_x = 0 \),

\[
\prod_{x=0}^{x_{\text{max}}} \prod_{j=0}^{x} \left( \begin{array}{c}
\mu_{x,j}^0 \\
\mu_{x,j}^1 \\
\mu_{x,j}^2
\end{array} \right) = \prod_{x=0}^{x_{\text{max}}} \frac{(1-2\varphi_0)^{\mu_{x,0}}^1 (1-2\varphi_1)^{\mu_{x,1}}^1 (1-2\varphi_2)^{\mu_{x,2}}^1}{B(x, \varphi)^{\mu_{x,0}}^1 B(x, \varphi)^{\mu_{x,1}}^1 B(x, \varphi)^{\mu_{x,2}}^1}.
\]

Finally, since \( \lim_{x_{\text{max}} \to +\infty} \prod_{x=0}^{x_{\text{max}}} (x!p_x)^{p_x} = (\lambda/e)^{\lambda} \),

\[
\lim_{n \to \infty} E[X_{\varepsilon, x_{\text{max}}, n, m}]^{1/n} \leq \zeta_2(\varepsilon, x_{\text{max}}) \prod_{x=0}^{x_{\text{max}}} \frac{B(x, \varphi)^{p_x}}{2^c}
\]

\[
(1-2\varphi_0)^{\mu_{x,0}}^1 (1-2\varphi_1)^{\mu_{x,1}}^1 (1-2\varphi_2)^{\mu_{x,2}}^1
\]

\[
\leq \zeta_2(\varepsilon, x_{\text{max}}) \prod_{x=0}^{x_{\text{max}}} \frac{B(x, \varphi)^{p_x}}{2^c}
\]

\[
(1-2\varphi_0)^{\mu_{x,0}}^1 (1-2\varphi_1)^{\mu_{x,1}}^1 (1-2\varphi_2)^{\mu_{x,2}}^1
\]

where \( \lim_{\varepsilon \to 0, x_{\text{max}} \to +\infty} \zeta_2(\varepsilon, x_{\text{max}}) = 1 \), and an explicit \( \zeta_2(\varepsilon, x_{\text{max}}) \) can be obtained by tracking down the successive approximations made. Recall that \( \varphi_2 \) is \( U(x_{\text{max}}) - \varphi_0 - \varphi_1 \) with \( \lim_{x_{\text{max}} \to +\infty} U(x_{\text{max}}) = 1 \).

The simple monotonic behaviour described above makes it possible provably to solve the system (7), 8) using a very basic iterative procedure that starts from an angle of the admissible rectangle in \( \varphi_0, \varphi_1 \) and spirals towards the solution. Doing so for \( c = 2.468155 \), a sufficiently large \( x_{\text{max}} \) and small \( \varepsilon \), one finds \( \lim_{n \to \infty} E[X_{\varepsilon, x_{\text{max}}, n, m}]^{1/n} < 0.9999995 \). By monotonicity, this value of \( c \), then, is an upper bound for \( c_3 \).

References

[1] D. Achlioptas, E. Friedgut, A Sharp Threshold for k-Colorability, Random Structures & Algorithms, 14 (1), (1999), p.63-70.

[2] D. Achlioptas, M. Molloy, The Analysis of a List-Coloring Algorithm on a Random Graph. FOCS 1997: 204-212.

[3] D. Achlioptas and M. Molloy, Almost All Graphs with 2.522 n Edges are not 3-Colourable. Electronic J. Comb. (6), R29 (1999).

[4] D. Achlioptas and C. Moore, Almost all graphs with average degree 4 are 3-colorable, presented at the Workshop on Computational Complexity and Statistical Physics, Santa Fe, N.M. USA, 2001. Also STOC02.

[5] N. Alon and J. H. Spencer. The Probabilistic Method, Wiley-Interscience, New York, NY, 1992.
[6] B. Bollobás, *Random Graphs*, Academic Press, New York, 1985.

[7] V. Chvátal, *Almost all graphs with $1.44n$ edges are 3-colorable*, Random Structures Algorithms 2 (1991) 11-28.

[8] O. Dubois, Y. Boufkhad, *A General Upper Bound for the Satisfiability Threshold of Random r-SAT Formulae*. J. Algorithms 24(2): 395-420 (1997)

[9] P.E. Dunne and M. Zito, *An improved upper bound on the non-3colourability threshold*. Information Processing Letters, 65:17–23, 1998.

[10] P. Erdős, A. Rényi, *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutat’o Int. Közl 5 (1960), 17-61.

[11] N. Fountoulakis and C. McDiarmid, unpublished.

[12] A. M. Frieze, and N. Wormald, *k-SAT: a tight threshold for meoderatly growing k*, 5th Int. Symp. on the Th. and Appl. of Satisfiability testing, Cincinnati, Ohio, May 2002.

[13] A. Frieze, M.E. Dyer, *Randomly colouring graphs with lower bounds on girth and maximum degree*, Proceedings of FOCS 2001, 579-587.

[14] A. Frieze, C. McDiarmid, *Algorithmic theory of random graphs*, Random Structures and Algorithms 10, John Wiley and Sons, 5-42.

[15] I. Giotis, A. Kaporis, L. Kirousis, *Corrigendum to ”A Note on the Non-Colorability Threshold of a Random Graph*, Electron. J. Combin., 2002.

[16] A. Kamath, R. Motwani, K. Palem, P. Spirakis *Tail Bounds for Occupancy and the Satisfiability Threshold Conjecture*, FOCS 1994: 592-603.

[17] A. Kaporis, L. Kirousis, Y. Stamatiou, *A note on the non-colorability threshold of a random graph*, Electron. J. Combin., 7:R29, 2000.

[18] L. Kirousis, E. Kranakis, D. Krizanc, Y. Stamatiou, *Approximating the unsatisfiability threshold of random formulas*. Random Structures and Algorithms 12(3): 253-269 (1998).

[19] DG Luenberger, *Linear and Nonlinear Programming*, AddisonWesley, Reading, Mass., 2nd edition, (1984).

[20] B. Pittel, J. Spencer, and NC Wormald, *Sudden emergence of a giant k-core in a random graph*, J. Combinatorial Theory, Series B 67 (1996), 111–151.

[21] NC Wormald, *Differential equations for random processes and random graphs*, Annals of Applied Probability 5 (1995), 1217–1235.