Mass spectra of singly heavy baryons in the self-consistent chiral quark-soliton model

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We investigate the mass spectra of the lowest-lying singly heavy baryons, based on the self-consistent chiral quark-soliton model. We take into account the rotational $1/N_c$ and strange current quark mass ($m_s$) corrections. Regarding $m_s$ as a small perturbation, we expand it to the second order. The mass spectra of heavy baryons are computed and compared with the experimental data. Treating $m_s$ as a parameter to fix, we find that the value of the strange current quark mass tends to decrease as the mass scale of the baryon system increases. It is determined to be $m_s \simeq 174$ MeV for charmed baryons and $m_s \simeq 166$ MeV for bottom baryons. Both values are quite smaller than that in the light-baryon sector, where $m_s$ is fixed to be around 200 MeV. Fitting the classical masses of the heavy baryon to the center mass of each representation, we determine the masses of all the lowest-lying singly heavy baryons. We predict the mass of the $\Omega_b^*$ baryon to be 6066.7 MeV, when the second-order $m_s$ corrections are included.

Keywords: Heavy baryons, pion mean fields, chiral quark-soliton model, flavor SU(3) symmetry breaking

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I. INTRODUCTION

Interest in heavy baryons is renewed as a series of new experimental data on them was reported [110]. A conventional heavy baryon is composed of a heavy quark and two light quarks. Since the mass of the heavy quark is very large in comparison with that of the light quarks, we can take the limit of the infinitely heavy mass of the heavy quark, i.e. $m_Q \to \infty$. It leads to the conservation of the heavy-quark spin $J_Q$, which results also in the conservation of the total spin of light quarks: $J = J' - J_Q$, where $J'$ is the spin of the heavy baryon [111][113]. This is called heavy-quark spin symmetry that makes $J$ a good quantum number. In this limit, the heavy quark can be regarded merely as a static color source and dynamics inside a heavy baryon is mostly governed by the light quarks consisting of it. Thus the two light quarks determine which flavor SU(3) representation a heavy baryon belongs to. There are two different representations: $3 \otimes 3 = 3 \oplus 6$. The anti-triplet ($\bar{3}$) has $J = 0$ and total $J' = 1/2$ whereas the sextet ($6$) has $J = 1$. Thus, the spin of a heavy baryon is determined by the spin alignment of the light-quark pair together with a heavy quark. It becomes either $J' = 1/2$ or $J' = 3/2$. So, there are 15 different lowest-lying heavy baryons classified as shown in Fig. 1 in the case of charmed baryons.

![Diagram of baryons](image)

FIG. 1. The anti-triplet ($\bar{3}$) and sextet ($6$) representations of the lowest-lying heavy baryons. The left panel draws the weight diagram for the anti-triplet with the total spin $1/2$. The centered panel corresponds to that for the sextet with the total spin 1/2 and the right panel depicts that for the sextet with the total spin 3/2.

Very recently, Ref. [14] put forward a mean-field approach to describe the masses of singly heavy baryons, being motivated by Ref. [15]. The main idea of this mean-field approach is rooted in Refs. [16][17], where Witten has suggested that in the limit of the large number of colors ($N_c$) the nucleon can be viewed as a bound state of $N_c$ valence quarks in a pion mean field with a hedgehog symmetry [18][19], as the quantum fluctuation around the saddle point of the pion field is $1/N_c$ suppressed. In this large $N_c$ limit, the presence of $N_c$ valence quarks that constitute the lowest-lying baryons brings about the vacuum polarization, which produces the pion mean field. The $N_c$ valence quarks are also self-consistently influenced by this pion mean field. Because of the hedgehog symmetry, an SU(2) soliton is embedded into the isospin subgroup of the flavor $SU(3)_f$ [17], which was also employed by various chiral soliton models [20][22]. The collective quantization of the chiral soliton yields the collective Hamiltonian with effects of flavor $SU(3)_f$ symmetry breaking. This mean-field approach is called the chiral-quark soliton model (χQSM) [23][25]. One salient feature of the χQSM is that the right hypercharge is constrained to be $Y' = N_c/3$ imposed by the $N_c$ valence quarks. This right hypercharge selects allowed representations of light baryons such as the baryon octet ($8$), the decuplet ($10$), etc. The χQSM described successfully the properties of the lowest-lying light baryons such as the mass splittings [26], the form factors [27][30], parton distributions [31].

In the present work, we investigate the mass spectra of singly heavy baryons in the ground states within the framework of the χQSM. Since a singly heavy baryon contains $N_c - 1$ light valence quarks, the imposed constraint $Y'$ should be modified as $Y' = (N_c - 1)/3$. This allows the lowest-lying representations: the baryon anti-triplet ($\bar{3}$) and the baryon sextet ($6$). While in Ref. [13] all dynamical parameters were fixed by using the experimental data, we will compute them here explicitly in a self-consistent way. This explicit calculation has a certain advantage over the previous model-independent analysis: Since we calculate the valence and sea contributions separately, we can correctly consider the pion mean fields that is produced only by the $N_c - 1$ valence quarks whereas the vacuum polarization is kept to be the same as in the case of light baryons. On the other hand, the model calculation suffers from a deficiency: the classical soliton mass turns out to be rather large in the model, which is a usual problem in any chiral soliton models. It means that the predicted values of baryon masses from the model tend systematically to be overestimated. Thus, we will first concentrate on the mass splittings of the lowest-lying heavy baryons in the present work, focussing on the effects of SU(3) symmetry breaking.
Regarding the mass of the strange current quark as a small perturbation, we first consider its linear-order corrections to the masses of heavy baryons. Then, we include the second-order corrections of the strange current quark mass. However, a caveat on the second-order corrections should be mentioned. In principle the effective chiral action may include a term that is proportional to the square of the current quark masses. However, so far we do not know any rigorous theoretical method to do it. Thus, the second-order corrections in the present work imply only the contributions arising from the second-order perturbation theory. Bearing this warning in mind, we will show that the results are very intriguing: The value of the strange current quark mass \( m_s \) tends to be smaller than that in the case of the light-baryon sector. While we cannot find a quantitative explanation for the reason, we can infer from the results that the value of the strange current quark mass may be influenced by the presence of heavy quarks. Once the value of \( m_s \) is determined, we take a practical point of view, so the center masses in each representation are fixed by using the experimental data as in Ref. [14]. Then we are able to produce all the values of the lowest-lying singly heavy baryons. We also predict the mass of the \( \Omega_b^* \) baryon, of which the value is experimentally yet unknown.

The structure of the present work is sketched as follows: In Section II, we briefly review the QSM for singly heavy baryons. In Section III, we examine numerically the effects of SU(3) breaking. We first consider the dependence of the mass splittings on strange quark mass as perturbation to the first order and then the second-order corrections to the mass splittings. We also present the prediction of the heavy baryon masses, fixing the center masses in each representation by the data. The last Section is devoted to the summary and conclusions of the present work. In Appendices, we have compiled all necessary formulae explicitly.

II. GENERAL FORMALISM

A heavy quark inside a heavy baryon can be regarded as a static color source in the limit of the infinite heavy-quark mass \( m_Q \to \infty \). In this case, the heavy quark is only required to make the heavy baryon a color singlet. So, the heavy baryon can be described as the correlation function of the \( N_c-1 \) light-quark field operators in Euclidean space, defined by

\[
\Pi_B(0, T) = \langle J_B(0, T/2), J_B^\dagger(0, -T/2) \rangle_0 = \frac{1}{Z} \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} \langle 0 | \mathcal{O}_B(0, T/2) | 0 \rangle e^{i \int d^4x \psi^\dagger \gamma^a (i\partial_a + iMU^\gamma_5 + i\hat{m})\psi},
\]

where \( J_B \) denotes the light-quark current consisting of \( N_c - 1 \) light quarks for a heavy baryon \( B \)

\[
J_B(x, t) = \frac{1}{(N_c - 1)!} \varepsilon^{\beta_1 \cdots \beta_{N_c - 1}} \Gamma_{j'_1 j'_3}^{f_1 \cdots f_{N_c-1}} \Psi_{\beta_1 f_1}(x, t) \cdots \Psi_{\beta_{N_c-1} f_{N_c-1}}(x, t).
\]

\( \beta_i \) represent color indices and \( \Gamma_{j'_1 j'_3}^{f_1 \cdots f_{N_c-1}} \) is a matrix with both flavor and spin indices. \( J' \) and \( T \) are the spin and isospin of the heavy baryon, respectively. \( J'_3 \) and \( T_3 \) are their third components, respectively. The notation \( \langle \cdots \rangle_0 \) in Eq. (1) stands for the vacuum expectation value. \( M \) denotes the dynamical quark mass and \( U^\gamma_5 \) is defined as

\[
U^\gamma_5 = U^1 \frac{1 + \gamma_5}{2} + U^\dagger \frac{1 - \gamma_5}{2}
\]

with

\[
U = \exp(i\pi^a \lambda^a).
\]

\( \pi^a \) represents the pseudo-Goldstone field. \( \hat{m} \) is the flavor matrix of the current quarks, written as \( \hat{m} = \text{diag}(m_u, m_d, m_s) \).

We assume in the present work isospin symmetry, i.e. \( m_u = m_d \). The strange current quark mass will be treated perturbatively.

Integrating over the quark fields, we obtain the correlation function as

\[
\Pi_B(0, T) = \frac{1}{Z} \Gamma_{j'_1 j'_3}^{f_1 \cdots f_{N_c-1}} \Gamma_{j'_1 j'_3}^{(g)^*} \int \mathcal{D}U \prod_{i=1}^{N_c-1} \left\langle 0, T/2 \left| \frac{1}{D(U)} \right| 0, -T/2 \right\rangle e^{-S_{\text{eff}}(U)},
\]

where \( D(U) \) is defined as

\[
D(U) = i\gamma_4 \partial_4 + i\gamma_k \partial_k + iMU^\gamma_5 + i\hat{m}
\]

and \( S_{\text{eff}} \) represents the effective chiral action written as

\[
S_{\text{eff}} = -N_c \text{Tr} \log D(U).
\]
The correlation function at large separation of the Euclidean time $\tau$ picks up the ground-state energies [23, 24]

$$\lim_{\tau \to \infty} \Pi_B(\tau) \sim \exp[-(N_c - 1)E_{\text{val}} + E_{\text{sea}}\tau],$$

where $E_{\text{val}}$ and $E_{\text{sea}}$ the valence and sea quark energies. The soliton mass is then derived by minimizing self-consistently the energies around the saddle point of the chiral field $U$

$$\frac{\delta}{\delta U}[(N_c - 1)E_{\text{val}} + E_{\text{sea}}]\bigg|_{U_c} = 0,$$

which yields the soliton mass

$$M_{\text{sol}} = (N_c - 1)E_{\text{val}}(U_c) + E_{\text{sea}}(U_c).$$

Since a singly heavy baryon contains the heavy quark, its classical mass of a heavy baryon should be expressed as

$$M_{\text{cl}} = M_{\text{sol}} + m_Q,$$

where $m_Q$ is the effective heavy quark mass that is different from that discussed in QCD and will be absorbed in the center mass of each representation, which will be discussed later.

As in the light-baryon sector, we expect that the low-lying heavy baryons will arise from rotational excitations of the light-quark soliton whereas the heavy quark is kept to be static. Keeping in mind that the SU(2) soliton $U_c(r)$ has hedgehog symmetry, we embed it into SU(3) [17]

$$U(r) = \begin{pmatrix} U_c(r) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To find the $1/N_c$ quantum fluctuations, we need to integrate the meson fields over small oscillations of the $U(r)$ field around the saddle point. However, we will not carry out this procedure and this is often called the mean-field approximation. On the other hand, we have to consider explicitly the rotational zero modes that are not small and cannot be neglected. Thus, we restrict ourselves to take into account these zero modes only. Considering a slowly rotating hedgehog field $U(r)$ in Eq. (12)

$$U(r, t) = A(t)U(r)A^\dagger(t),$$

where $A(t)$ is an element of flavor SU(3) matrix, we can find the collective Hamiltonian to describe heavy baryons. For a detailed formalism of the semiclassical quantization, we refer to Ref. [24]. Regarding the angular velocity of the soliton and the current strange quark mass as small parameters, we can expand the quark propagator in Eq. (5) with respect to them.

Having quantized the chiral soliton, we arrive at the collective Hamiltonian for singly heavy baryons

$$H = H_{\text{sym}} + H_{\text{sb}}^{(1)} + H_{\text{sb}}^{(2)},$$

where $H_{\text{sym}}$ represents the flavor SU(3) symmetric part, $H_{\text{sb}}^{(1)}$ and $H_{\text{sb}}^{(2)}$ the SU(3) symmetry-breaking parts respectively to the first and second orders, which will be discussed later. $H_{\text{sym}}$ is expressed as

$$H_{\text{sym}} = M_{\text{cl}} + \frac{1}{2I_1} \sum_{i=1}^{3} \hat{J}_i^2 + \frac{1}{2I_2} \sum_{a=4}^{7} \hat{J}_a^2,$$

where $I_1$ and $I_2$ denote the moments of inertia of the soliton. The explicit expressions for $I_1, I_2$ are given in Eq. (A6). The operators $\hat{J}_i$ are the SU(3) generators. In the $(p, q)$ representation of the SU(3) group, we find the eigenvalue of the quadratic Casimir operator $\sum_{i=1}^{8} J_i^2$, given as

$$C_2(p, q) = \frac{1}{3} [p^2 + q^2 + pq + 3(p+q)],$$

so the eigenvalues of $H_{\text{sym}}$ is obtained as

$$E_{\text{sym}}(p, q) = M_{\text{cl}} + \frac{1}{2I_1} J(J+1) + \frac{1}{2I_2} [C_2(p, q) - J(J+1)] - \frac{3}{8I_2} \hat{Y}^2.$$
The right hypercharge $\mathcal{Y}$ is constrained to be $(N_c - 1)/3$, which is imposed by the $N_c - 1$ valence quarks inside a singly heavy baryon. The wave functions of the singly heavy baryon is derived as

$$ \psi_{B}^{(R)}(J', J'_q, J; A) = \sum_{m_3=\pm 1/2} C_{J_Q}^{J'J'_q} \chi_{m_3} \sqrt{\text{dim}(p,q)} (-1)^{\frac{\mathcal{Y} + J_3}{2}} D_{(Y,T,T_3)}^{(R)*}(\mathcal{Y},J,J_3)(A), \quad (18) $$

where

$$ \text{dim}(p,q) = (p+1)(q+1) \left(1 + \frac{p+q}{2}\right). \quad (19) $$

$J'$ and $J'_q$ denote the spin angular momentum and its third component of the heavy baryon, respectively. Note that a similar expression can be found in Ref. [38], though its formalism is rather different from the present one. $J$ and $J_Q$ stand for the soliton spin and heavy-quark spin, respectively. $J_3$ and $m_3$ represent the corresponding third components, respectively. Since the spin operator for the heavy baryon is given by the addition of the soliton and heavy-quark spin operators

$$ J' = J_Q + J, \quad (20) $$

the relevant Clebsch-Gordan coefficients appear in Eq. (18). The SU(3) Wigner $D$ function in Eq. (18) is just the wave-function for the quantized soliton consisting of the $N_c - 1$ valence quarks, whereas $\chi_{m_3}$ is the Pauli spinor for the heavy quark. $R$ stands for a SU(3) irreducible representation corresponding to $(p,q)$.

Since a singly heavy baryon consists of $N_c - 1$ valence quarks, we have two irreducible representations when $N_c = 3$: $3 \otimes 3 = 3 \oplus 6$. Thus, we have the following representations for the lowest-lying singly heavy baryons

$$ \begin{align*}
[3_0] &= D(0,1) : \text{anti-triplet with } J = 0 \\
[6_1] &= D(2,0) : \text{sextet with } J = 1.
\end{align*} \quad (21) $$

The soliton being coupled to the heavy quark, we finally get three different representations which have been illustrated already in Fig. 1. Since the soliton in the sextet ($J = 1$) is coupled to the heavy quark ($J_Q = 1/2$), we have two sextet representations with spin 1/2 and 3/2 respectively, which are degenerate. The hyperfine spin-spin interaction will lift this degeneracy.

Since a singly heavy baryon consists of $N_c - 1$ valence quarks, the pion mean fields should be changed. In Refs. [14, 32], a scale factor was introduced to explain the modification of the mean field, of which the value was taken to be in the range of $1 - 0.66$. Because all dynamical variables being proportional to the color factor were fixed by the experimental data in Refs. [14, 32], it was impossible to decompose the valence and sea parts. On the other hand, we can treat separately the valence and sea quark contributions in the present work. So, we will replace $N_c$ factor with $N_c - 1$ only in front of the valence part of the dynamical parameters, while we keep the sea part intact.

In order to describe the mass splittings of SU(3) baryons in a specific representation, we have to consider the effects of flavor SU(3) symmetry breaking, dealing with the mass of the strange current quark, $m_s$, as a small perturbation. First, we consider the first-order corrections that are proportional to the linear $m_s$, and then we proceed to take into account the second-order corrections. In this case, the baryon wave functions are no more in pure states but are mixed with higher representations. Thus, there are two different contributions: one from the collective Hamiltonian and the other from the baryon wave functions. Both corrections will be considered as the second-order contributions.

### A. Mass splittings to the linear order

The symmetry-breaking part of the collective Hamiltonian is given as [24, 26]

$$ H_{sb}^{(1)} = \frac{\sum_i m_i}{m_0} \frac{m_s}{3} + \alpha D_{8s}^{(8)} + \beta \chi + \frac{\gamma}{\sqrt{3}} \sum_{i=1}^{3} D_{8i}^{(8)} J_i, \quad (22) $$

where

$$ \alpha = \left( -\frac{\sum_i m_i}{3m_0} + \frac{K_2}{I_2} \mathcal{Y} \right) m_s, \quad \beta = -\frac{K_2}{I_2} m_s, \quad \gamma = 2 \left( \frac{K_1}{I_1} - \frac{K_2}{I_2} \right) m_s. \quad (23) $$

The first term in Eq. (22) can be absorbed into the symmetric part of the Hamiltonian, since it does not contribute to the mass splittings of heavy baryons in a given representation. The $m_0$ represents the averaged mass of the
up and down quarks. The three parameters $\alpha$, $\beta$, and $\gamma$ are expressed in terms of the moments of inertia $I_{1,2}$ and $K_{1,2}$, of which the valence parts are different from those in the light baryon sector by the color factor $N_c - 1$ in place of $N_c$. The valence part of $\Sigma_{\pi N}$ are related to the $\pi N$ sigma term: \[ \Sigma_{\pi N} = (N_c - 1)N_c^{-1}\Sigma_{\pi N}, \]
where $\Sigma_{\pi N} = (m_u + m_d)|N|\bar{u}u + d\bar{d}|N|$ and $\Sigma_{\pi N} = (m_u + m_d)\sigma$. The explicit expressions for the moments of inertia and the $\pi N$ sigma term can be found in Appendix A. Note that their sea parts are the same as in the light baryon sector.

Taking into account the $m_s$ corrections to the first order, we can write the masses of the singly heavy baryons in representation $\mathcal{R}$ as

\[ M_{B,\mathcal{R}} = M_{\mathcal{R}}^Q + M_{B,\mathcal{R}}^{(1)}, \quad (24) \]

where

\[ M_{\mathcal{R}}^Q = m_Q + E_{sym}(p, q). \quad (25) \]

$M_{\mathcal{R}}^Q$ is called the center mass of a heavy baryon in representation $\mathcal{R}$. $E_{sym}(p, q)$ is defined in Eq. (17). Note that the lower index $B$ denotes a certain baryon belonging to a specific representation $\mathcal{R}$. The upper index $Q$ stands for either the charm sector ($Q = c$) or the bottom sector ($Q = b$). The center masses for the anti-triplet and sextet representations can be explicitly written as

\[ M_{\mathcal{R}}^{Q_3} = M_{\mathcal{R}}^{Q_3} + \frac{1}{2}I_2, \quad M_{\mathcal{R}}^{Q_6} = M_{\mathcal{R}}^{Q_6} + \frac{1}{1}I_1, \quad (26) \]

where $M_{cl}$ was defined in Eq. (11). The second term in Eq. (24) denotes the linear-order $m_s$ corrections to the heavy baryon mass

\[ M_{B,\mathcal{R}}^{(1)} = \langle B, \mathcal{R}|H_{sb}^{(1)}|B, \mathcal{R} \rangle = Y \delta_\mathcal{R}, \quad (27) \]

where

\[ \delta_\mathcal{R} = \frac{3}{8}\alpha + \beta, \quad \delta_6 = \frac{3}{20}\alpha + \beta - \frac{3}{10}\gamma. \quad (28) \]

The values of the matrix elements for the relevant SU(3) Wigner $D$ functions are tabulated in Appendix B. Thus, we obtain the masses of the lowest-lying singly heavy baryons

\[ M_{B,\mathcal{R}}^{Q_3} = M_{\mathcal{R}}^{Q_3} + Y \delta_\mathcal{R}, \quad M_{B,\mathcal{R}}^{Q_6} = M_{\mathcal{R}}^{Q_6} + Y \delta_6, \quad (29) \]

with the linear-order $m_s$ corrections taken into account.

**B. Mass splittings to the second order**

We now consider the second-order $m_s$ corrections. When we include the second-order corrections, the collective wave function of baryons is no more in a pure state but is mixed with those in higher representations. Using the standard method of perturbation theory, we can derive the second-order $m_s$ corrections to the baryon mass, which arise from the baryon wave functions \[ M_{B}^{(2)(wf)} = \sum_{\mathcal{R} \neq \mathcal{R}'} \left| \frac{\langle B, \mathcal{R}'|H_{sb}^{(1)}|B, \mathcal{R} \rangle}{M_{\mathcal{R}}^Q - M_{\mathcal{R}'}^Q} \right|^2, \quad (30) \]

where $\mathcal{R}'$ denote higher representations that are different from $\mathcal{R}$. These representations are determined by the irreducible decomposition of the following products $3 \otimes 8 = 3 \oplus 6 \oplus 15$ and $6 \otimes 8 = 3 \oplus 6 \oplus 15 \oplus 24$. The corresponding baryon wave function is then expressed as a mixed state with those in higher representations

\[ |B^{(\mathcal{R})} \rangle = |\mathcal{R}, B \rangle - \sum_{\mathcal{R} \neq \mathcal{R}'} \left| \frac{\langle\mathcal{R}', B|H_{sb}^{(1)}|\mathcal{R}, B \rangle}{M_{\mathcal{R}}^Q - M_{\mathcal{R}'}^Q} \right| |\mathcal{R}', B \rangle. \quad (31) \]
Explicit calculation yields the collective wave functions of the baryon anti-triplet and sextet, respectively, as

\[ |B_{30}\rangle = |3_0, B\rangle + p_{30}^B |\bf{1}S_0, B\rangle, \]
\[ |B_{61}\rangle = |6_1, B\rangle + q_{61}^B |\bf{1}S_1, B\rangle + q_{61}^B |\bf{2}S_1, B\rangle, \]

with the mixing coefficients

\[ p_{30}^B = p_{\bf{3}0} \left[ \frac{2}{\sqrt{3}} \right], \quad q_{61}^B = q_{\bf{1}S_1} \left[ \frac{2\sqrt{3}}{\sqrt{3}} \right], \quad q_{2S_1} = q_{\bf{2}S_1} \left[ \frac{1}{\sqrt{3/2}} \right], \]

where

\[ p_{\bf{3}0} = -\frac{3}{16\sqrt{3}} \alpha I_2, \quad q_{\bf{1}S_1} = -\frac{1}{4\sqrt{3}} (\alpha + \frac{2}{3} \gamma) I_2, \quad q_{\bf{2}S_1} = -\frac{2}{25} (\alpha - \frac{1}{3} \gamma) I_2, \]

in the bases of \([\Lambda_Q, \Xi_Q]\) and \([\Sigma_Q, \Xi_Q, \Omega_Q]\), respectively. Then, we obtain the second-order corrections to the masses of the singly heavy baryons from the baryon wave functions as

\[ M_{\Lambda_Q}^{(2)(\text{wf})} = -I_2 \frac{9}{160} \alpha^2, \]
\[ M_{\Xi_Q}^{(2)(\text{wf})} = -I_2 \frac{27}{640} \alpha^2, \]
\[ M_{\Sigma_Q}^{(2)(\text{wf})} = -I_2 \frac{1}{90} (3\alpha + 2\gamma)^2 - I_2 \frac{2}{1125} (3\alpha - \gamma)^2, \]
\[ M_{\Xi_Q}^{(2)(\text{wf})} = -I_2 \frac{1}{240} (3\alpha + 2\gamma)^2 - I_2 \frac{1}{375} (3\alpha - \gamma)^2, \]
\[ M_{\Omega_Q}^{(2)(\text{wf})} = -I_2 \frac{1}{375} (3\alpha - \gamma)^2. \]

There are yet another second-order \(m_s\) corrections that come from the collective Hamiltonian \[21, 31]:

\[ H_{sb}^{(2)} = \text{m}_s^2 \left[ \frac{2}{3} K_1^2 \sum_{i=1}^{3} D_{s_i}(A)D_{s_i}(A) + \frac{2}{3} K_2^2 \sum_{a=4}^{7} D_{s_a}(A)D_{s_a}(A) \right. \]
\[ - \left. \frac{2}{3} N_1 \sum_{i=1}^{3} D_{s_i}(A)D_{s_i}(A) - \frac{2}{3} N_2 \sum_{a=4}^{7} D_{s_a}(A)D_{s_a}(A) - \frac{2}{9} N_0 \left( 1 - D_{s_8}(A) \right)^2 \right], \]

where \(N_0, N_1,\) and \(N_2\) are defined in Appendix A. Computing the matrix elements of Eq. (36), we obtain the second-order \(m_s\) corrections to the masses of the singly heavy baryons, which arise from the collective Hamiltonian

\[ M_{\Lambda_Q}^{(2)(\text{op})} = \text{m}_s^2 \left( \frac{3}{20} K_1^2 + \frac{1}{5} K_2^2 \right) \left( \frac{13}{180} N_0 - \frac{3}{20} N_1 - \frac{2}{5} N_2 \right), \]
\[ M_{\Xi_Q}^{(2)(\text{op})} = \text{m}_s^2 \left( \frac{3}{10} K_1^2 + \frac{3}{10} K_2^2 \right) \left( - \frac{7}{90} N_0 - \frac{3}{10} N_1 - \frac{3}{10} N_2 \right), \]
\[ M_{\Sigma_Q}^{(2)(\text{op})} = \text{m}_s^2 \left( \frac{19}{90} K_1^2 + \frac{16}{45} K_2^2 \right) \left( \frac{1}{90} N_0 - \frac{19}{45} N_1 - \frac{16}{45} N_2 \right), \]
\[ M_{\Xi_Q}^{(2)(\text{op})} = \text{m}_s^2 \left( \frac{4}{15} K_1^2 + \frac{4}{15} K_2^2 \right) \left( - \frac{2}{45} N_0 - \frac{4}{15} N_1 - \frac{1}{3} N_2 \right), \]
\[ M_{\Omega_Q}^{(2)(\text{op})} = \text{m}_s^2 \left( \frac{1}{3} K_1^2 + \frac{1}{3} K_2^2 \right) \left( - \frac{1}{9} N_0 - \frac{1}{3} N_1 - \frac{4}{15} N_2 \right). \]

We will call them as the second-order \(m_s\) corrections from the operator, so that we distinguish them from those coming from the wave-function corrections. Considering these second-order \(m_s\) corrections, we can extend Eq. (24) to

\[ M_{B, R}^Q = M_{B, R}^Q + M_{B, R}^{(1)} + M_{B, R}^{(2)}, \]

where \(M_{B, R}^{(2)}\) denote the second-order corrections to a baryon in representation \(R\).
III. RESULTS AND DISCUSSION

We are now in a position to compute the mass splittings of the lowest-lying singly heavy baryons. Reference [26] showed in detail how model parameters such as the cutoff masses and the current quark masses can be fixed in the vacuum sector. We follow it except for the mass of the strange current quark, which was taken to be $m_s = 150$ MeV as a canonical value in Ref. [26]. As will be shown in the present work, that value of the strange quark mass does not yield good results of both the light and heavy baryon masses in comparison with the experimental data. Thus, instead of fixing the value of $m_s$, we will regard it as a free parameter and will examine the dependence of the baryon mass splittings on it. We will soon see that this approach provides a new feature of the strange current quark mass. In the present work, we choose the constituent quark mass $M = 420$ MeV, which provided the best prediction of baryon observables [24]. We also follow Ref. [26] for the numerical methods of diagonalizing the Dirac equation in the presence of the pion field and deriving the self-consistent solutions of equations of motion. However, we use a much larger size of the box in solving the Dirac equation such that we are able to reduce a numerical instability and uncertainties [1]. Detailed numerical techniques and relevant references are also given in Ref. [26].

The definition of the deviation function $\Delta M$ specific single heavy quark (either a charm quark or a bottom quark) and that of the experimental data on the $\pi N$ sigma term, and the classical mass of the soliton. Note that the valence part of the moments of inertia for singly heavy baryons have the $N_c - 1$ factor, whereas $N_c$ for light baryons.

TABLE I. Numerical results of the moments of inertia, the $\Sigma_{\pi N}$ term, and the classical soliton mass $M_{\text{sol}}$. As discussed in the previous Section II, the expressions for the valence parts of all relevant quantities should be modified. The prefactor $N_c$ in those expressions for light baryons, which counts the number of quarks, should be replaced by the factor $N_c - 1$, since a singly heavy baryon consists of $N_c - 1$ light valence quarks. So, the difference between the left panel of Table I and the right one arises from the different prefactor of each valence part.

The definition of $\Sigma_{\pi N}$ is just the same as $\Sigma_{\pi N}$ except for the valence contribution as shown in Eq. (11).

It is already well known that any chiral soliton approaches cannot reproduce the absolute values of the SU(3) baryon masses, because the classical nucleon mass turns out to be large. So, in Ref. [26] the theoretical values were shifted in such a way that the mass of the $\Sigma^*$ was matched to the corresponding experimental data. Then the masses relative to $m_N$, were drawn as a function of $m_s$ to examine the $m_s$ dependence of the baryon octet and decuplet masses. However, since we have to compute the masses of the heavy baryon anti-triplet and sextet together with those of the baryon octet and decuplet, we will not follow Ref. [26] but define a more general mass deviation function $\Delta M_B$

$$\Delta M_B(m_s) = M^{Q}_{B,R} - (\overline{M}^{Q}_{B,R} - M^{Q,\text{exp}}_{B,R}), \quad (39)$$

where $\overline{M}^{Q}_{B,R}$ and $M^{Q,\text{exp}}_{B,R}$ represent the average of theoretical predictions of all the heavy baryon masses with a specific single heavy quark (either a charm quark or a bottom quark) and that of the experimental data on the corresponding masses. Thus, the deviation function $\Delta M_B(m_s)$ shows how much the theoretical prediction of the mass splitting is deviated from the experimental data. Since the mass of the strange current quark will be considered as a free parameter in the present work, we seek the minimum value of $\Delta M_B(m_s)$ with $m_s$ varied. In this way, we are able to determine the value of $m_s$, which provides the best predictions of the baryon masses in comparison with the experimental data. Since we want to investigate the effects of flavor SU(3) symmetry breaking in the present work, we first examine the leading-order corrections.

Combining Eq. (24) with Eq. (39), we can see how the first-order $m_s$ corrections can describe the experimental data on the mass splittings of baryons. In Fig. 2, we draw the results of the deviation $\Delta M_B$ as a function of $m_s$, taking into account the linear-order $m_s$ corrections. In fact, the results illustrated in Fig. 2 were already discussed in Fig. 4.

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1 10 fm is taken for the box size in the present work whereas 5 fm was used in Ref. [26].
FIG. 2. The mass deviations $\Delta M_B$ of the SU(3) baryon octet and decuplet as a function of $m_s$ with the linear-order $m_s$ corrections considered, reproducing those given in Ref. [26]. The dashed lines draw the mass deviations of the baryon octet whereas the solid ones depict those of the baryon decuplet. The minimal deviation is found at $m_s \approx 210$ MeV.

of Ref. [26], where the constituent quark mass $M = 391$ MeV was employed. Since the theoretical results are overall overestimated, Blotz et al. shifted the theoretical values so that the experimental mass of $\Sigma^*$ can be matched by the model calculation. They obtained an overall deviation of less than 2% in comparison with the data, when $m_s = 200$ MeV was used. Examining carefully the results depicted in Fig. 2, we find that $m_s = 210$ MeV yields the minimum deviation from the experimental data.

FIG. 3. The mass deviations $\Delta M_B$ of the SU(3) baryon anti-triplet and sextet with the linear-order $m_s$ corrections considered, given as a function of $m_s$. The left panel shows those of the charmed baryons, whereas the right panel draws those of the bottom baryons. The dashed lines draw the mass deviations of the baryon anti-triplet, the solid ones depict those of the baryon sextet with spin $J = 1/2$, and the dot-dashed ones represent those of the baryon sextet with spin $J = 3/2$. The minimal deviations are found at around $m_s = 177$ MeV for the charmed baryons, and $m_s = 167$ MeV for bottom baryons, when the linear-order $m_s$ corrections are considered.

Figure 3 shows the results of the linear-order mass deviation for the singly heavy baryons with $m_s$ varied from 0 to 250 MeV. In the left panel of Fig. 3 $\Delta M_B$ for the charm baryons are depicted whereas in the right panel those
for the bottom baryons are drawn. The value of $m_s$ can be also determined, which yields the minimum of the mass deviation. We obtain $m_s = 177$ MeV and $m_s = 167$ MeV, which provide the minimum values of the mass deviation for the charmed and bottom baryon mass splittings, respectively. Interestingly, the value of $m_s$ tends to decrease as the mass scale of the baryon system increases.

We now proceed to the second-order $m_s$ corrections to the baryon masses by putting Eq. (38) into Eq. (39). Figure 4 illustrates the dependence of the mass deviation on $m_s$. Adding the second-order $m_s$ corrections, we see that the results of the baryon masses are in better agreement with the experimental data. As shown in Fig. 4, the mass deviation reaches the lowest value at $m_s = 202$ MeV that is approximately 8 MeV smaller than that with the first-order $m_s$ corrections only included.

In Fig. 5, the results of the mass deviations are drawn for the charmed baryons and the bottom baryons in the left and right panels, respectively. As in the case of the light baryons presented in Fig. 4, the results of the mass splittings with the second-order $m_s$ corrections are much better than those with the linear $m_s$ corrections only. Moreover, the values of $m_s$ are determined to be smaller, i.e. $m_s = 174$ MeV for the charmed baryons and $m_s = 166$ MeV for the bottom baryons, compared to those with the first-order corrections considered only.

We observe two important features of the $m_s$ corrections within the SU(3) chiral quark-soliton model. Firstly, the results of the mass splittings of both the light and heavy baryons exhibit definitely better convergence to the experimental data when one includes the second-order $m_s$ corrections. Secondly, if one determines the value of the strange current quark mass based on the results of the baryon mass splittings, then the value of $m_s$ tends to become smaller as the mass scale of the baryon system increases. That is, the value of $m_s$ for the bottom baryons turns out be approximately 20 % smaller than that for the light ones. So far we cannot provide firm theoretical grounds on the reason why the value of the strange current quark mass decreases for a better description of the mass splittings of the heavy baryons. However, we want to stress that the strange current quark mass is still an effective one in the present framework. In general, light quarks may undergo changes differently in the presence of a heavy quark. This will be an interesting subject to study in future.

Though we are not able to determine the masses of singly heavy baryons, because the center mass (26) in each representation seems overestimated, compared with the experimental data. In addition, we must know the hyperfine interaction which will lift the degeneracy of different spin states in the sextet representation. Thus, we will fix each center mass and parameters for the hyperfine splitting, using the experimental data, so that we can obtain the values of the lowest-lying singly heavy baryons. We will follow the method proposed by Ref. [14] in which the spin-spin
FIG. 5. The mass deviations $\Delta M_B$ of the SU(3) baryon anti-triplet and decuplet with the second-order $m_s$ corrections considered, given as a function of $m_s$. The left panel shows those of the charmed baryons, whereas the right panel draws those of the bottom baryons. The dashed lines draw the mass deviations of the baryon anti-triplet, the solid ones depict those of the baryon sextet with spin $J = 1/2$, and the dot-dashed ones represent those of the baryon sextet with spin $J = 3/2$. The minimal deviations are found at around $m_s = 174$ MeV for the charmed baryons, and $m_s = 166$ MeV for bottom baryons, when the second-order $m_s$ corrections are considered.

The interaction Hamiltonian is given as

$$H_{\text{sol}Q} = \frac{2}{3} \kappa m_Q \langle J \cdot J_Q \rangle = \frac{2}{3} \kappa \langle J \cdot J_Q \rangle,$$

where $\kappa$ represents the flavor-independent hyperfine coupling constant. Note that the baryon anti-triplet does not acquire any contribution from the hyperfine interaction, since the corresponding soliton has spin $J = 0$. On the other hand, the baryon sextet has $J = 1$. Being coupled to the heavy quark spin, it produces two different multiplets $J' = 1/2$ and $J' = 3/2$, of which the masses are expressed respectively as

$$M_{B,6}^{Q,1/2} = M_{B,6}^{Q} - \frac{2}{3} \kappa \langle J \cdot J_Q \rangle, \quad M_{B,6}^{Q,3/2} = M_{B,6}^{Q} + \frac{1}{3} \kappa \langle J \cdot J_Q \rangle.$$  (41)

Thus, we find the hyperfine mass splitting as

$$M_{B,6}^{Q,3/2} - M_{B,6}^{Q,1/2} = \frac{\kappa}{m_Q},$$  (42)

of which the value can be determined by using the center value of the sextet masses. In the charm and bottom sectors, we obtain the corresponding numerical values respectively

$$\frac{\kappa}{m_c} = 67.1 \text{ MeV}, \quad \frac{\kappa}{m_b} = 20.3 \text{ MeV}.$$  (43)

Combining Eq. (41) with Eq. (38), we can derive the final masses of the lowest-lying singly heavy baryons

$$M_{B,3}^{Q} = M_{3}^{Q} + M_{B,3}^{(1)} + M_{B,3}^{(2)},$$

$$M_{B,6}^{Q,1/2} = M_{6}^{Q} + M_{B,6}^{(1)} + M_{B,3}^{(2)} - \frac{2}{3} \kappa \langle J \cdot J_Q \rangle,$$

$$M_{B,6}^{Q,3/2} = M_{6}^{Q} + M_{B,6}^{(1)} + M_{B,6}^{(2)} + \frac{1}{3} \kappa \langle J \cdot J_Q \rangle.$$  (44)

The numerical results of the charmed baryons and the bottom ones are listed in Table II and Table IV respectively. Note that we have used the mass of the strange current quark, which were defined in Figs. 3 and 5. As expected, the masses of the heavy baryons are in better agreement with the experimental data, when the second-order $m_s$ corrections are included. The mass of the $\Omega_b^*$ is predicted to be 6067 MeV, whereas the model-independent approach of Ref. [14] predicts $M_{\Omega_b^*} = (6095 \pm 4.4)$ MeV. The difference is less than 1%.
TABLE II. Results of the masses of the charmed baryon masses in unit of MeV. In the third and fourth columns those with the first-order and second-order $m_s$ corrections are listed. The last column represents the experimental data.

| $\mathcal{R}_Q^0$ | $B_c$ | 1st order | 2nd order | Experiment |
|------------------|------|-----------|-----------|------------|
| $\frac{3}{2}$   | $\Lambda_c$ | 2276.1 | 2283.7 | 2286.5±0.1 |
| $\frac{1}{2}$   | $\Xi_c$     | 2479.8 | 2472.2 | 2469.4±0.3 |
| $\frac{5}{2}$   | $\Sigma_c$  | 2457.7 | 2453.0 | 2453.5±0.1 |
| $\frac{3}{2}$   | $\Xi_{c}'$  | 2575.2 | 2576.7 | 2576.8±2.1 |
| $\frac{1}{2}$   | $\Omega_c$  | 2692.6 | 2695.8 | 2695.2±1.7 |
| $\frac{5}{2}$   | $\Sigma_{c}'$| 2525.9 | 2521.1 | 2518.1±0.8 |
| $\frac{3}{2}$   | $\Xi_{c}'$  | 2643.3 | 2644.9 | 2645.9±0.4 |
| $\frac{1}{2}$   | $\Omega_{c}'$| 2760.4 | 2763.9 | 2765.9±2.0 |

TABLE III. Results of the masses of the bottom baryon masses in unit of MeV. In the third and fourth columns those with the first-order and second-order $m_s$ corrections are listed. The last column represents the experimental data.

| $\mathcal{R}_Q^0$ | $B_b$ | 1st order | 2nd order | Experiment |
|------------------|------|-----------|-----------|------------|
| $\frac{3}{2}$   | $\Lambda_b$ | 5610.2  | 5616.2  | 5619.5±0.2 |
| $\frac{1}{2}$   | $\Xi_b$     | 5802.4 | 5796.5 | 5793.1±0.7 |
| $\frac{5}{2}$   | $\Sigma_b$  | 5821.3 | 5815.9 | 5813.4±1.3 |
| $\frac{3}{2}$   | $\Xi_{b}'$  | 5932.1 | 5933.6 | 5935.0±0.05 |
| $\frac{1}{2}$   | $\Omega_b$  | 6043.0 | 6047.0 | 6048.0±1.9 |
| $\frac{5}{2}$   | $\Sigma_{b}'$| 5839.0 | 5835.6 | 5833.6±1.3 |
| $\frac{3}{2}$   | $\Xi_{b}'$  | 5949.9 | 5953.3 | 5955.3±0.1 |
| $\frac{1}{2}$   | $\Omega_{b}'$| 6060.7 | 6066.7 | -          |

IV. SUMMAY AND CONCLUSIONS

In the present work we investigate the mass spectra of the lowest-lying singly heavy baryons within the framework of the self-consistent chiral-quark soliton model. In the model, the $N_c - 1$ light valence quarks polarize the Dirac sea, so that we obtained the soliton energy consisting of the $N_c - 1$ valence- and sea-quark energies. Minimizing the soliton energy around the saddle point of the classical pion field self-consistently, we derived the soliton mass. Because of the hedgehog symmetry, we embedded the SU(2) soliton into the flavor SU(3). Since we employ the mean-field approximation which indicates that the $1/N_c$ quantum fluctuations are ignored. However the rotational zero modes or rotational $1/N_c$ corrections were taken into account, a rigid rotation of the soliton being assumed. All the moments of inertia were computed in the present work explicitly.

We consider the effects of flavor SU(3) symmetry breaking to the second-order in perturbation. As expected, the inclusion of the second-order $m_s$ corrections leads to the better results of the mass splittings of heavy baryons than those with the linear $m_s$ corrections, in comparison with the experimental data. In the course of the calculation, we obtained the value of the mass of the strange current quark in such a way that it produces the minimum of the mass deviation for each heavy quark sector (c and b), that is, $m_s = 174$ MeV for the charmed baryons and $m_s = 166$ MeV for the bottom baryons. Both values are smaller than that determined in the light baryon sector ($m_s = 202$ MeV).

Having fixed the center mass in each representation, we were able to obtain the numerical values of all the lowest-lying singly heavy baryons both in the charm and bottom sectors. With the second-order $m_s$ corrections included, the present results are in very good agreement with the experimental data. The mass of the $\Omega_b$ baryon is predicted to be 6066.7 MeV in the present work.

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Appendix A: Moments of inertia

In this Appendix, we compile all relevant formulae for the modified \( \pi N \) sigma term, the moments of inertia \( I_{1,2}, K_{1,2}, \) and \( N_{1,2} \). All terms consist of the vacuum and sea parts. The modified \( \pi N \) sigma term is written as :

\[
\Sigma_{\pi N} = \Sigma_{\pi N}^{\text{val}} + \Sigma_{\pi N}^{\text{sea}},
\]

where the valence and sea parts are expressed respectively as

\[
\Sigma_{\pi N}^{\text{val}} = m_0 (N_e - 1) \langle \text{val} | \gamma_4 | \text{val} \rangle, \quad \Sigma_{\pi N}^{\text{sea}} = \frac{m_0}{2} N_c \sum_n \langle n | \gamma_4 | n \rangle \text{sign}(E_n) R_{\Sigma}(E_n),
\]

where \( \gamma_4 \) is the Dirac \( \gamma \) matrix in Euclidean space represented as

\[
\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

The function \( R_{\Sigma}(E_n) \) denotes a regularization function written as

\[
R_{\Sigma}(E_n) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{du}{\sqrt{u}} e^{-u \phi(u/E_n^2)},
\]

where \( \phi(u) \) is a cutoff function defined by

\[
\phi(u) = c \theta(u - 1/\Lambda_1^2) + (1 - c) \theta(u - 1/\Lambda_2^2).
\]

The free parameters \( \Lambda_1, \Lambda_2, \) and \( c \) are determined in the mesonic sector by reproducing the pion decay constant \( f_\pi = 93 \text{ MeV} \) and the pion mass \( m_\pi = 139 \text{ MeV} \). Their numerical values are explicitly given as \( \Lambda_1 = 381.15 \text{ MeV}, \Lambda_2 = 1428.00 \text{ MeV}, \) and \( c = 0.7276 \).

The moment of inertia tensor \( I_{ab} \) is given as

\[
I_{ab} = I_{ab}^{\text{val}} + I_{ab}^{\text{sea}},
\]

where

\[
I_{ab}^{\text{val}} = \frac{(N_e - 1)}{2} \sum_{\text{val}, n \neq \text{val}} \frac{\langle n | \lambda_a | \text{val} \rangle \langle \text{val} | \lambda_b | n \rangle}{E_n - E_{\text{val}}},
\]

\[
I_{ab}^{\text{sea}} = \frac{N_c}{4} \sum_{m, n} \langle n | \lambda_a | m \rangle \langle m | \lambda_b | n \rangle R_I(E_n, E_m),
\]

with the different regularization function \( R_I(E_n, E_m) \)

\[
R_I(E_n, E_m) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{du}{\sqrt{u}} \phi(u) \left[ e^{-uE_n^2} - e^{-uE_m^2} \right] \frac{E_ne^{-uE_n^2} + E_me^{-uE_m^2}}{E_n + E_m}.
\]

\( \lambda_a \) in Eq. [A7] denote the Gell-Mann matrices for flavor SU(3) group, satisfying \( \text{tr}(\lambda_a \lambda_b) = 2\delta_{ab} \) and \( [\lambda_a, \lambda_b] = 2if_{abc} \lambda_c, \ a, b, c = 1, \cdots, 8 \). The moments of inertia \( I_1 \) and \( I_2 \) are defined by

\[
I_{ab} = \begin{cases} 
I_1 \delta_{ab} & a, b = 1, 2, 3 \\
I_2 \delta_{ab} & a, b = 4, 5, 6, 7 \\
0 & a, b = 8
\end{cases}
\]

Similarly, the anomalous moments of inertia tensor is expressed as

\[
K_{ab} = K_{ab}^{\text{val}} + K_{ab}^{\text{sea}},
\]

where

\[
K_{ab}^{\text{val}} = \frac{(N_e - 1)}{2} \sum_{\text{val}, n \neq \text{val}} \frac{\langle n | \lambda_a | \text{val} \rangle \langle \text{val} | \gamma_4 | n \rangle}{E_n - E_{\text{val}}},
\]

\[
K_{ab}^{\text{sea}} = \frac{N_c}{8} \sum_{m, n} \langle n | \lambda_a | m \rangle \langle m | \gamma_4 \lambda_b | n \rangle \frac{\text{sign}(E_n) - \text{sign}(E_m)}{E_n - E_m}.
\]
The anomalous moments of inertia \( K_1 \) and \( K_2 \) are defined by

\[
K_{ab} = \begin{cases} 
K_1 \delta_{ab} & a, b = 1, 2, 3 \\
K_2 \delta_{ab} & a, b = 4, 5, 6, 7 \\
0 & a, b = 8 
\end{cases} \quad (A12)
\]

Finally we express the third moments of inertia tensor, which appears only when the second-order \( m_s \) corrections are considered.

\[
N_{ab} = N^\text{val}_{ab} + N^\text{sea}_{ab}, \quad (A13)
\]

then the moment of inertia

\[
N^\text{val}_{ab} = \frac{(N_c - 1)}{2} \sum_{\text{val}, n \neq \text{val}} \frac{\langle n|\lambda_a \gamma_4|\text{val}\rangle\langle\text{val}|\lambda_b \gamma_4|n\rangle}{E_n - E_{\text{val}}},
\]

\[
N^\text{sea}_{ab} = \frac{N_c}{4} \sum_{m,n} \langle n|\lambda_a \gamma_4|m\rangle\langle m|\lambda_b \gamma_4|n\rangle \mathcal{R}_N(E_n, E_m), \quad (A14)
\]

with the regularization function

\[
\mathcal{R}_N(E_n, E_m) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{du}{\sqrt{u}} \frac{E_n e^{-uE_n^2} - E_m e^{-uE_m^2}}{E_n - E_m}. \quad (A15)
\]

\( N_0, N_1, \) and \( N_2 \) are defined by

\[
N_{ab} = \begin{cases} 
N_1 \delta_{ab} & a, b = 1, 2, 3 \\
N_2 \delta_{ab} & a, b = 4, 5, 6, 7 \\
\frac{1}{3}N_0 & a, b = 8 
\end{cases} \quad (A16)
\]

### Appendix B: Matrix elements of the SU(3) Wigner \( D \) functions

In Appendix B we tabulate all relevant matrix elements of the SU(3) Wigner \( D \) functions in each representation.

| \( \mathcal{R} \) | \( T \) | \( Y \) | \( \langle RYT J|D^{(8)}_{ij}\rangle\) | \( \langle RYT J|D^{(8)}_{si}\rangle\) | \( \langle RYT J|D^{(8)}_{si}|J_i\rangle\) | \( \langle RYT J\rangle\) |
|-----------------|-------|-------|----------------|----------------|----------------|----------------|
| \( \Lambda_c \)  | \( 3 \) | 0      | 2/3            | 1/4            | 0              | 0              |
| \( \Xi_c \)     | 1/2   | -1/3  | -1/8           | -1/8           | 0              | 0              |
| \( \Sigma_c \)  | 1     | 2/3   | 1/10           | -\sqrt{3}/5   | 0              | 0              |
| \( \Xi_c \)     | 6     | 1/2   | -1/3           | -1/20          | \sqrt{3}/10    | 0              |
| \( \Omega_c \)  | 0     | -4/3  | -1/5           | 2\sqrt{3}/5   | 0              | 0              |

| \( \mathcal{R} \) | \( T \) | \( Y \) | \( \langle RYT J|D^{(8)}_{ss}\rangle\) | \( \langle RYT J|D^{(8)}_{si}\rangle\) | \( \langle RYT J|D^{(8)}_{si}|J_i\rangle\) | \( \langle RYT J|D^{(8)}_{sp}\rangle\) | \( \langle RYT J|D^{(8)}_{sp}|J_i\rangle\) | \( \langle RYT J\rangle\) |
|-----------------|-------|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( \Lambda_c \)  | \( 3 \) | 0      | 2/3            | 7/40           | 9/40           | 3/5            | 3/5            | 3/5            |
| \( \Xi_c \)     | 1/2   | -1/3  | 1/10           | 9/20           | 9/20           | 3/5            | 3/5            | 3/5            |
| \( \Sigma_c \)  | 1     | 2/3   | 3/20           | 19/60          | 8/15           | 3/5            | 3/5            | 3/5            |
| \( \Xi_c \)     | 6     | 1/2   | -1/3           | 2/5            | 1/2            | 2/5            | 2/5            | 2/5            |
| \( \Omega_c \)  | 0     | -4/3  | 1/10           | 1/2            | 2/5            | 2/5            | 2/5            | 2/5            |

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TABLE VI. Transition matrix elements of the SU(3) Wigner $D$ functions $D_{36}^{(8)}$ and $D_{81}^{(8)} J_i$, which appear from the second-order perturbation.

| $\mathcal{R}' \mathcal{R} \ Y$ | $\langle \mathcal{R}' Y T J | D_{36}^{(8)} | \mathcal{R} Y T J \rangle$ | $\langle \mathcal{R}' Y T J | D_{81}^{(8)} J_i | \mathcal{R} Y T J \rangle$ |
|----------------|--------------------------------|--------------------------------|
| $\Lambda_c$ | $\bar{15} 3$ | 0 | 3$\sqrt{5}$/20 | 0 |
| $\Xi_c$ | $1/2$ | 1/3 | 3$\sqrt{15}$/40 | 0 |
| $\Sigma_c$ | $1$ | 2/3 | $\sqrt{19}$/10 | $\sqrt{5}$/10 |
| $\Xi_c$ | $\bar{15} 6$ | 1/2 | $\sqrt{5}$/20 | $\sqrt{5}$/10 |
| $\Omega_c$ | 0 | $-4$/3 | 0 | 0 |
| $\Xi_c$ | 1 | 2/3 | 1/5 | $-\sqrt{5}$/15 |
| $\Xi_c$ | $\Xi_c$ | 1/2 | $-\sqrt{6}$/10 | $-\sqrt{2}$/10 |
| $\Omega_c$ | 0 | $-4$/3 | $\sqrt{6}$/10 | $-\sqrt{2}$/10 |

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