ON THE FACTORIZATION OF THREE-DIMENSIONAL TERMINAL FLOPS

HSIN-KU CHEN

Abstract. We factorize three-dimensional terminal flops into a composition of divisorial contractions to points and blowing-up smooth curves.

1. Introduction

Flops are one of the typical birational maps which occurs naturally in the minimal model program. It is proved by Kollár [14] (for three-dimensional case) and Kawamata [13] (in general) that minimal models of terminal varieties are connected by flops. As a consequence, we know that Calabi-Yau threefolds which belong to the same birational class can be connected by flops. On the other hands, J. A. Chen and Hacon proved that every step of three-dimensional terminal MMP can be factorize into a combination of (inverses of) divisorial contractions to points, blowing-up smooth curves and flops, and the former two kinds of birational maps are well-understood nowadays. Thus understanding flops becomes an important issue in three-dimensional birational geometry.

The simplest flops were constructed by Atiyah and Reid [22, Part II]. The construction is as follows: they consider a smooth threefold $X$ which contains a $K$-trivial smooth rational curve $C$ with normal bundle $O(-1)\oplus O(-1)$ or $O\oplus O(-2)$ (the threefold can be taken as a small resolution of a deformation of an $A$-type Du Val singularity). After suitably blowing-up and blowing-down smooth rational curves, one can get another smooth threefold $X'$, such that $X-C$ and $X'-C'$ are isomorphic for some $K$-trivial smooth rational curve $C'$ on $X'$. These flops are known as Atiyah/pagoda flops.

In the late twentieth century rich theories about three-dimensional flops were developed. Pinkham [21] and Katz-Morrison [10] classified three-dimensional simple smooth flops (that is, smooth flops with only one flopping curve). Kollár [14] gives an explicit local description of three-dimensional terminal flops. Nevertheless, it is still unclear that how to construct a meaningful factorization, as in the case of Atiyah flops or pagoda flops. In [21] Pinkham described an example, which is a factorization of a simple smooth flop with normal bundle $O(1)\oplus O(-3)$. But he had not developed a general theory, and his technique (computing the normal bundle sequence) can not be applied to non-smooth flops.

In this paper we construct a factorization for three-dimensional terminal flops using the minimal model program. We will prove the following.

Theorem 1.1. Let $X \rightarrow X'$ be a three-dimensional terminal $\mathbb{Q}$-factorial flop. Then $X \rightarrow X'$ can be factorize into a composition of divisorial contractions to points, blowing-up smooth curves, and inverses of the above maps.

Combining [6, Theorem 1.1] and [3, Theorem 1.2], we know that

Corollary 1.2. Each step of MMP beginning with terminal threefolds can be factorized into a composition of divisorial contractions to points, blowing-up smooth curves, and inverses of the above maps.

To prove Theorem 1.1 we first factorize three-dimensional terminal flops into a combination of (inverses of) divisorial contractions and simple smooth flops.
Theorem 1.3. Let $X \to X'$ be three-dimensional terminal $\mathbb{Q}$-factorial flop. Then $X \to X'$ can be factorize into a composition of divisorial contractions to points, blowing-up smooth curves, simple smooth flops, and inverses of the above maps.

The basic idea of the proof of Theorem 1.3 is the follows: assume that $X \to X'$ is a flop over $W$. We first construct a divisorial contraction $W_1 \to W$ such that $W_1$ has better singularities than $W$. Let $Y$ be a $\mathbb{Q}$-factorization of $W_1$ and one can run the $K_Y$-MMP over $W$. The minimal model is a $\mathbb{Q}$-factorization of $W$, and it may be $X$ or $X'$. Assume that the minimal model of $Y$ over $W$ is $X$, then the birational map $Y \to X$ is a sequence of flips followed by a divisorial contraction. By the result of J. A. Chen and Hacon we can factorize this birational map into (inverses of) divisorial contractions and flops.

On the other hand, one can show that there exists $Y'$ such that either $Y' \to Y''$ is either a flop or an isomorphism, and $X'$ is a minimal model of $Y'$ over $W$. One can also factorize $Y'' \to X'$ into (inverses of) divisorial contractions and flops. In this way we get a factorization of $X \to X'$. If $X \to X'$ is not a simple smooth flop, then one can show that every flop appear in the factorization has better singularities compared with $X \to X'$. Thus we can factorize $X \to X'$ into a composition of (inverses of) divisorial contractions and simple smooth flops by induction on the singularity.

Now assume that $X \to X'$ is a simple smooth flop over $W$. One can first verify that the singularities of $W$ have only finitely many possibilities. We can construct the factorization of $X \to X'$ as before, and every step of the factorization can be written down explicitly. By direct computation, one can figure out that every flop appear in the factorization is better than $X \to X'$ (in fact, most of them are Atiyah flops). One can prove the following theorem. Here a $w$-morphism is a divisorial contraction to a point with minimal discrepancy. Please see Section 2.4 for the precise definition.

Theorem 1.4. Let $X \to X'$ be a three-dimensional simple smooth flop over $W$. Then $X \to X'$ can be factorize into a composition of $w$-morphisms, blowing-up smooth curves contained in smooth loci, and inverses of the above maps.

Moreover,

(1) Assume that $W$ has $cA$ singularities, then $X \to X'$ has a factorization of type $(A^{(k)})$ for some $k \geq 1$.
(2) Assume that $W$ has $cD$ singularities, then $X \to X'$ has a factorization of type $(D^{(k)})$ for some $k \geq 0$.
(3) Assume that $W$ has $cE_n$ singularities, then $X \to X'$ has a factorization of type $(E_n)$.

In the following notation, the label $w$ means a $w$-morphism, and the label $c$ means a blowing-up a smooth curve.

\[
\begin{align*}
(A^{(0)}) : & X \cong X' ; & (A^{(1)}) : & \xymatrix{ Y \ar[dr]^{c} \ar[rr]^{c} & & X' \ar[dl]_{c} } ; & (A^{(k)}) : & \xymatrix{ Y \ar[r]^{(A^{(k-1)})} & \ar[d]_{c} X' } ; & (A^{(2)}) : & \xymatrix{ Y_{(0,1)} \ar[d]_{c} \ar[r]_{(A^{(1)})} & Y_{(0,0)} \ar[d]_{w} } & (A^{(1)}) : & \xymatrix{ Y_{(0,0)} \ar[r]_{(A^{(1)})} & \ar[d]_{w} Y_{(0,1)} } \\
(D^{(0)}) : & \xymatrix{ Y_{(1)} \ar[d]_{c} \ar[r]_{(A^{(1)})} & Y_{(0)} \ar[r]_{(A^{(1)})} & \ar[d]_{c} Y_{(0,1)} } & \xymatrix{ Y_{(0)} \ar[r]_{(A^{(1)})} & \ar[d]_{w} Y_{(0,0)} } & \xymatrix{ Y_{(0,0)} \ar[r]_{(A^{(1)})} & \ar[d]_{w} Y_{(0,1)} } & \xymatrix{ Y'_{(0,1)} \ar[d]_{c} } & \xymatrix{ Y'_{(0)} \ar[r]_{(A^{(1)})} & \ar[d]_{w} Y'_{(0,0)} } & \xymatrix{ Y'_{(0,0)} \ar[r]_{(A^{(1)})} & \ar[d]_{w} Y'_{(0,1)} } & \xymatrix{ Y'_{(0,1)} \ar[d]_{c} } \end{align*}
\]
ON THE FACTORIZATION OF THREE-DIMENSIONAL TERMINAL FLOPS

\[(D^{(k)}) : \]

\[
Y_{(0,1)} \leftarrow \leftarrow (A^{(1)}) \quad Y_{(0,0)} \quad Y'_{(0,0)} \rightarrow \rightarrow (A^{(1)}) \quad Y'_{(0,1)} \\
\downarrow \quad \downarrow \quad \downarrow \\
Y_{(1)} \quad Y_{(0)} \quad Y'_{(0)} \\
\downarrow \quad \downarrow \quad \downarrow \\
X \quad X' \\
\]

\[
X \leftarrow \leftarrow c \quad Y_{1} \leftarrow \leftarrow c \quad Y_{(0,2)} \leftarrow \leftarrow c \quad Y_{(0,1,1)} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
Y_{(0,1)} \leftarrow \leftarrow (A^{(1)}) \quad Y_{(0,0)} \quad Y'_{(0,0)} \rightarrow \rightarrow (A^{(1)}) \\
\downarrow \quad \downarrow \quad \downarrow \\
Z_{(0,0)} \\
\downarrow \quad \downarrow \quad \downarrow \\
Z \\
\downarrow \quad \downarrow \quad \downarrow \\
Z' \\
\downarrow \quad \downarrow \quad \downarrow \\
\]

\[
(E_{6}) : \\
Y'_{(0,0)} \leftarrow \leftarrow (A^{(1)}) \quad Y'_{(0,2)} \leftarrow \leftarrow (A^{(1)}) \quad Y'_{(0,1,1)} \\
\downarrow \quad \downarrow \quad \downarrow \\
Y'_{(0,1)} \leftarrow \leftarrow (A^{(1)}) \quad Y'_{(0,1,0)} \\
\downarrow \quad \downarrow \quad \downarrow \\
X' \leftarrow \leftarrow c \quad Y'_{1} \leftarrow \leftarrow c \quad Y'_{(0,2)} \leftarrow \leftarrow c \quad Y'_{(0,1,1)}
\]
\[
X \xrightarrow{c} Y \xrightarrow{c} \bar{Y} \xrightarrow{c} Y^{(0,2,1)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
Y^{(0,2)} \xrightarrow{w} Y^{(0,2,0)}
\]
\[
Y^{(0,1)} \xrightarrow{w} Y^{(0,1,0)}
\]
\[
\bar{Z}^{(1)} \xrightarrow{w} \bar{Z}^{(0,2)} \xrightarrow{c} \bar{Z}^{(0,1,1)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}^{(0,1)} \xrightarrow{w} \bar{Z}^{(0,1,0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}^{(0)} = \bar{Z}^{(0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}^{(0,1)} \xrightarrow{w} \bar{Z}^{(0,1,0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
Y \xrightarrow{w} Y^{(0,0)} \xrightarrow{w} Z^{(1)} \xrightarrow{w} Z^{(0,2)} \xrightarrow{c} Z^{(0,1,1)} \quad \text{\scriptsize \{(D)\}}
\]
\[
Y'' \xrightarrow{w} Y''^{(0,0)} \xrightarrow{w} Z''^{(1)} \xrightarrow{w} Z''^{(0,2)} \xrightarrow{c} Z''^{(0,1,1)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}''^{(0,1)} \xrightarrow{w} \bar{Z}''^{(0,1,0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}''^{(0)} = \bar{Z}''^{(0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}''^{(0,1)} \xrightarrow{w} \bar{Z}''^{(0,1,0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}''^{(1)} \xrightarrow{w} \bar{Z}''^{(0,2)} \xrightarrow{c} \bar{Z}''^{(0,1,1)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}''^{(0,1)} \xrightarrow{w} \bar{Z}''^{(0,1,0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Z}''^{(0,2)} \xrightarrow{w} \bar{Z}''^{(0,2,0)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Y}'' \xrightarrow{c} \bar{Y}'' \xrightarrow{c} \bar{Y}'' \xrightarrow{c} Y''^{(0,2,1)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]
\[
\bar{Y}'' \xrightarrow{c} \bar{Y}'' \xrightarrow{c} \bar{Y}'' \xrightarrow{c} Y''^{(0,2,1)} \quad \text{\scriptsize \{(A^{(1)})\}}
\]

\[
\begin{align*}
X & \overset{c}{\longleftarrow} Y_1 \overset{c}{\longleftarrow} Y_{(0,2)} \overset{c}{\longleftarrow} Y_{(0,1,1)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
Y_{(0,1)} & \overset{w}{\longleftarrow} Y_{(0,1,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
Y_{(0,0)} & \overset{w}{\longleftarrow} Z_{(0,4)} \overset{c}{\longleftarrow} Z_{(0,3,1)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
Z_{(0,3)} & \overset{w}{\longleftarrow} Z_{(0,3,0)} \\
& \overset{c}{\downarrow} \\
\overset{\lambda}{\downarrow} (A^{(1)}) \\
Z_{(0,2,1)} & \overset{c}{\longleftarrow} Z_{(0,1,1,1)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
Z_{(0,2,0)} & \overset{w}{\longleftarrow} Z_{(0,1,1,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
Z_{(0,1,1)} & \overset{w}{\longleftarrow} Z_{(0,1,1,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
Z_{(0,0)} & \overset{w}{\longleftarrow} \tilde{Z}_{(0,2)} \overset{w}{\longleftarrow} \tilde{Z}_{(0,1,2)} \overset{c}{\longleftarrow} \tilde{Z}_{(0,1,1,1)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
\tilde{Z}_{(0,1,1)} & \overset{w}{\longleftarrow} \tilde{Z}_{(0,1,1,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
\tilde{Z}_{(0,1)} & \overset{w}{\longleftarrow} \tilde{Z}_{(0,1,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
\tilde{Z}_{(0,0)} & \overset{w}{\longleftarrow} \tilde{Z}_{(0,2)} \overset{w}{\longleftarrow} \tilde{Z}_{(0,1,2)} \overset{c}{\longleftarrow} \tilde{Z}_{(0,1,1,1)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
\tilde{Z}_{(0,1,1)} & \overset{w}{\longleftarrow} \tilde{Z}_{(0,1,1,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
\tilde{Z}_{(0,1)} & \overset{w}{\longleftarrow} \tilde{Z}_{(0,0)} \\
& \overset{\lambda}{\downarrow} (A^{(1)}) \\
\tilde{Z}_{(0)} & \overset{A}{\downarrow} \overset{OR \ (D)}{=} \\
& \overset{\lambda}{\downarrow} \tilde{Z}_{(0)}
\end{align*}
\]
and the diagram from $\tilde{Z}'(0)$ to $X'$ is symmetric.

\[(E_{8}^{(2)}) \]:
and the diagram from $\tilde{Z}'_0$ to $X'$ is symmetric.

We only need to prove Theorem 1.3 and Theorem 1.4. In Section 2 we will recall some known result about three-dimensional simple smooth flops as well as singularities and factorizations of terminal threefolds. The proof of Theorem 1.3 is given in Section 3. In Section 4 we describe the factorization of flips with very simple singularities, which is useful in the rest of the article. We construct the factorization of simple smooth flops over $cD$ points as well as smooth flops over $cA_2$ points in Section 5. and construct the factorization of simple smooth flops over $cE$ points in Section 6. This finishes the proof of Theorem 1.4.

I want to thank Jungkai Alfred Chen and Paolo Cascini for their helpful comments. Part of this work was done while the author was visiting the University of Cambridge DPMMS. The author want to thank the University of Cambridge DPMMS for its hospitality.

2. Preliminaries

2.1. Conventions. In this paper, every variety is assumed to be terminal threefolds which are projective and over the complex number.

A flop is diagram of small birational morphisms

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
W & \xrightarrow{f'} & W
\end{array}
$$

such that $W$ is $\mathbb{Q}$-Gorenstein, and $\rho(X/W) = \rho(X'/W) = 1$. Notice that we have

$$K_X = f^*K_W$$

and

$$K_{X'} = f'^*K_W.$$ 

Let $X \dasharrow X'$ be a three-dimensional flop having terminal singularities. By [14] we know that $X$ and $X'$ have the same singularities and $exc(X/W) \cong exc(X'/W)$. We say that $X \dasharrow X'$ is simple if $exc(X/W)$ is irreducible. If $X$ is smooth (resp. Gorenstein, non-Gorenstein), then we say that $X \dasharrow X'$ is a smooth (resp. Gorenstein, non-Gorenstein) flop. If $X \dasharrow X'$ is Gorenstein flop, we say that the flop is of type $A$, $D$ or $E$ if $W$ has $cA$, $cD$ or $cE$ singularities, respectively.

A divisorial contraction is a birational morphism $Y \to X$ such that the exceptional locus is irreducible and $K_Y$ is anti-ample over $X$.

Assume that $\phi : Z \dasharrow Y$ is a birational map and $H$ is a divisor on $Z$. If $\phi$ is an isomorphism on the generic point of $H$, then we denote the proper transform of $H$ on $Y$ by $H_Y$.

2.2. Simple smooth flops. We briefly introduce the classification of three-dimensional simple smooth flops.

**Definition.** Let $X \dasharrow X'$ be a three-dimensional simple smooth flop. Let $X_0 = X$ and $C_0$ be the flopping curve. Define $X_{i+1} = Bl_{C_i}X_i$. The exceptional divisor $E_{i+1}$ of $X_{i+1} \to X_i$ has a ruled surface structure over $C_i$. If $E_{i+1}$ do not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then let $C_{i+1}$ be the unique negative section.

The normal bundle sequence is a sequence $\{(a_i, b_i)\}_{i=0}^N$ such that $N_{C_i/X_i} = \mathcal{O}_{C_i}(a_i) \oplus \mathcal{O}_{C_i}(b_i)$ with $a_i \geq b_i$, and $a_N = b_N$. 


By [21] and [10] there are seven types of simple smooth flops:

| No. | Singularity of W | normal bundle sequence |
|-----|------------------|------------------------|
| 1   | cA₁              | (−1, −1)               |
| 2   | cA₁              | (0, −2), ..., (0, −2), (−1, −1) |
| 3   | cD₄              | (1, −3), (−1, −2), (−1, −1) |
| 4   | cE₆              | (1, −3), (0, −3), (−1, −2), (−1, −1) |
| 5   | cE₇              | (1, −3), (0, −3), (−1, −2), (−1, −1) |
| 6   | cE₈              | (1, −3), (0, −3), (−1, −2), (−1, −1) |
| 7   | cE₈              | (1, −3), (0, −3), (−1, −2), (−1, −1) |

Case No.1 is called an *Atiyah flop* and Case No.2 is called a *pagoda flop*. Using the notation in Theorem [1.3] it is well-known that (cf. [22, Part II]) the Atiyah flop has a factorization of type \(A^{(1)}\) and a pagoda flop with a length \(n\) normal bundle sequence has a factorization of type \(A^{(n)}\).

**Lemma 2.1.** Let \(X \rightarrow X'\) be an Atiyah flop. Let \(C \subset X\) be the flopping curve and \(C' \subset X'\) be the flopped curve. Let \(H\) be a divisor on \(X\). Then the following holds.

1. Assume that \(H\) intersects \(C\) transversally at \(m\) points, then \(\text{mult}_{C'}H_{X'} = m\).
2. Assume that \(H.C = m\), then \(H_{X'}.C' = −m\).

**Proof.** Let \(\phi : Y = \text{Bl}_C X \rightarrow X\) and \(\phi' : Y = \text{Bl}_{C'} X' \rightarrow X'\) be the common resolution and let \(E = \text{exc}(\phi) = \text{exc}(\phi')\). Then \(H_Y\) intersects \(E\) transversally at \(m\) curves \(l_1, ..., l_m\), such that \(l_i\) maps bijectively to \(C'\). Hence \(\text{mult}_{C'}H_{X'} = m\).

Now we prove (2). First assume that \(m > 0\) and \(H\) intersects \(C\) transversally at \(m\) points. Let \(l\) be a curve on \(Y\) which maps bijectively to \(C'\). Then we have

\[
H_{X'}.C' = H_Y.l + (\text{mult}_{C'}H_{X'})E.l = (\phi'H).l + (\text{mult}_{C'}H_{X'})E.l = 0 - m = −m.
\]

In general we can write \(H = A - B\) where \(A\) and \(B\) are ample divisors such that \(B\) and some multiple of \(A\) intersects \(C\) transversally. One can easily see that \(H\) has the desired property. \(\square\)

We have the following observation for the underlying space of simple smooth flops.

**Lemma 2.2.** Assume that \((P \in W)\) is a germ of Gorenstein terminal threefold. Then there exists a simple smooth flop over \(W\) if and only if \(W\) is not \(\mathbb{Q}\)-factorial and there are exactly one exceptional divisor which has discrepancy one over \(P\).

**Proof.** Assume that \(X \rightarrow W\) is a flopping contraction such that \(X\) is smooth and \(\text{exc}(X/W)\) is irreducible. Then there is only one exceptional divisor \(E\) over \(W\) with \(a(E, X) = 1\), namely the divisor obtained by blowing-up the flopping curve on \(X\).

Conversely, if \(X \rightarrow W\) is a flopping contraction with \(n\) flopping curves for \(n > 1\), then blowing-up those curves produces \(n\) different exceptional divisors which have discrepancy one over \(W\). Likewise, if \(X \rightarrow W\) is a flopping contraction such that \(X\) has a Gorenstein singular point \(P\). Then there exists an exceptional divisor \(F\) over \(P\) such that \(a(F, X) = a(F, W) = 1\) (cf. [1]). Since blowing-up a flopping curve always induces an exceptional divisor with discrepancy one, there are at least two discrepancy one exceptional divisors over \(W\). \(\square\)
Remark 2.3. Given a flop \( X \rightarrow X' \) over \( W \) such that \( X \) as well as \( X' \) are \( \mathbb{Q} \)-factorial. Then \( W \) has exactly two \( \mathbb{Q} \)-factorizations, namely \( X \) and \( X' \). Indeed, if \( X_1 \rightarrow W \) is a small contraction such that \( X_1 \) is \( \mathbb{Q} \)-factorial. Let \( A_X \) be an ample divisor on \( X_1 \). Then either \( A_X \) or \( A_{X'} \) is ample because of the condition \( \rho(X/W) = 1 \). Thus \( X_1 \) is either isomorphic to \( X \) or \( X' \).

2.3. Weighted blow-ups. Let \( G = \langle \tau \mid \tau^r = id \rangle \) be a cyclic group of order \( r \). For any \( \mathbb{Z} \)-valued \( n \)-tuple \((a_1, ..., a_n)\) one can define a \( G \)-action on \( \mathbb{A}^n_{(x_1, ..., x_n)} \) by \( \tau(x_i) = \xi^{a_i} x_i \), where \( \xi = e^{2\pi i/n} \). We will denote the quotient space \( \mathbb{A}^n/\mathbb{Z} \) by \( \mathbb{A}^n_{(x_1, ..., x_n)}/\frac{1}{r}(a_1, ..., a_n) \).

Let \( W \cong \mathbb{A}^n_{(x_1, ..., x_n)}/\frac{1}{r}(a_1, ..., a_n) \) be a cyclic-quotient singularity. There is an elementary way to construct a birational morphism \( Y \rightarrow W \), so called the weighted blow-up, defined as follows.

We write everything using the language of toric varieties. Let \( N \) be the lattice \( \langle e_1, ..., e_n \rangle \mathbb{Z} \), where \( \{e_1, ..., e_n\} \) is the standard basis of \( \mathbb{R}^n \) and \( v = \frac{1}{r}(a_1, ..., a_n) \). Let \( \sigma = \langle e_1, ..., e_n \rangle_{\mathbb{R}_{\geq 0}} \).

We have \( W \cong \text{Spec } \mathbb{C}[N^\vee \cap \sigma^\vee] \).

Let \( w = \frac{1}{r}(b_1, ..., b_n) \) be a vector such that \( b_i = \lambda a_i + k_i r \) for \( \lambda \in \mathbb{N} \) and \( k_i \in \mathbb{Z} \). The weighted blow-up of \( W \) with weight \( w \) is the toric variety defined by the fan consists of those cones

\[ \sigma_i = \langle e_1, ..., e_{i-1}, w, e_{i+1}, ..., e_n \rangle. \]

Let \( U_i \) be the toric variety defined by the cone \( \sigma_i \) and lattice \( N \).

Lemma 2.4. Let

\[ v' = \frac{1}{b_i}(-b_1, ..., -b_{i-1}, r, -b_{i+1}, ..., -b_n) \]

and

\[ w' = \frac{1}{r b_i}(a_1 b_i - a_i b_1, ..., a_{i-1} b_i - a_i b_{i-1}, r a_i, a_{i+1} b_i - a_i b_{i+1}, ..., a_n b_i - a_i b_n). \]

Assume that \( u = \frac{1}{r}(a'_1, ..., a'_n) \) is a vector such that

\[ \langle e_1, ..., e_n, v', w' \rangle_{\mathbb{Z}} = \langle e_1, ..., e_n, u \rangle_{\mathbb{Z}}, \]

then

\[ U_i \cong \mathbb{A}^n_{\frac{1}{r'}(a'_1, ..., a'_n)}. \]

In particular, if \( \lambda = 1 \), then \( U_i \cong \frac{1}{b_i}(-b_1, ..., -b_{i-1}, r, -b_{i+1}, ..., -b_n) \).

Corollary 2.5. Assume that the hypothesis of Lemma 2.4 is satisfied. Let \( x_1, ..., x_n \) be local coordinates of \( X \) and \( y_1, ..., y_n \) be local coordinates of \( U_i \) defined by the cyclic quotient structure. The change of coordinates of \( U_i \rightarrow X \) is given by \( x_i = \frac{b_i}{b_j} y_i \) and \( x_j = \frac{b_j}{b_i} y_i \) for \( j \neq i \).

Corollary 2.6. Assume that

\[ S = (f_1(x_1, ..., x_n) = ... = f_k(x_1, ..., x_n) = 0) \subset W \]

is a complete intersection subvariety and \( S' \) is the proper transform of \( S \) on \( Y \). Assume that the exceptional locus \( E \) of \( S' \rightarrow S \) is irreducible and reduced. Then

\[ a(S, E) = \frac{b_1 + ... + b_n}{r} - \sum_{i=1}^{k} wt_i f_i(x_1, ..., x_n) - 1. \]
Please see [2, Section 2.2] for the proofs of the above lemma and corollaries.

In this paper we consider terminal threefolds which are embedding into cyclic quotient of $\mathbb{A}^4$ or $\mathbb{A}^5$

\[ X \hookrightarrow \mathbb{A}_{(x,y,z,u)}^4 \left/ \left( a, b, c, d \right) \right. \] or \[ X \hookrightarrow \mathbb{A}_{(x,y,z,u,t)}^5 \left/ \left( a, b, c, d, e \right) \right. \]

We say that $Y \to X$ is a weighted blow-up with weight $w$ if $Y$ is the proper transform of $X$ inside the weighted blow-up of $\mathbb{A}_{(x,y,z,u)}^4 \left/ \left( a, b, c, d \right) \right.$ or $\mathbb{A}_{(x,y,z,u,t)}^5 \left/ \left( a, b, c, d, e \right) \right.$ with weight $w$. 

**Convention 2.7.** Assume that $X$ is of the above form and let $Y \to X$ be a weighted blow-up. The notation $U_x, U_y, U_z, U_u$ and $U_t$ will stand for $U_1, ..., U_5$ in Lemma 2.4.

2.4. Terminal threefolds.

2.4.1. Singularities of terminal threefolds.

**Definition.** Let $Y \to X$ be a divisorial contraction contracts a divisor $E$ to a point $P$.

We say that $Y \to X$ is a $w$-morphism if $a(X, E) = 1/r_P$, where $r_P$ is the Cartier index of $K_X$ near $P$.

**Definition.** The depth of a terminal singularity $P \in X$, $\text{dep}(P \in X)$, is the minimal length of the sequence

\[ X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X, \]

such that $X_m$ is Gorenstein and $X_i \to X_{i-1}$ is a $w$-morphism for all $1 \leq i \leq m$.

The generalize depth of a terminal singularity $P \in X$, $\text{gdep}(P \in X)$, is the minimal length of the sequence

\[ X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X, \]

such that $X_n$ is smooth and $X_i \to X_{i-1}$ is a $w$-morphism for all $1 \leq i \leq n$. The variety $X_n$ is called a feasible resolution of $P \in X$.

The Gorenstein depth of a terminal singularity $P \in X$, $\text{dep}_{\text{Gor}}(P \in X)$, is defined by $\text{gdep}(P \in X) - \text{dep}(P \in X)$.

For a terminal threefold we can define

\[ \text{dep}(X) = \sum_P \text{dep}(P \in X), \]

\[ \text{gdep}(X) = \sum_P \text{gdep}(P \in X) \]

and

\[ \text{dep}_{\text{Gor}}(X) = \sum_P \text{dep}_{\text{Gor}}(P \in X). \]

**Remark 2.8.** In the above definition, the existence of a sequence

\[ X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X, \]

such that $X_m$ is Gorenstein follows from [9, Theorem 1.2]. The existence of a sequence

\[ X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X, \]

such that $X_n$ is smooth follows from [4, Theorem 2].

**Definition.** Assume that $Y \to X$ is a $w$-morphism such that $\text{gdep}(Y) = \text{gdep}(X) - 1$. Then we say that $Y \to X$ is a strict $w$-morphism.

**Remark 2.9.** It has been proved in [3, Section 5], that if $Y \to X$ is a strict $w$-morphism, then $\text{dep}_{\text{Gor}}(Y) = \text{dep}_{\text{Gor}}(X)$. 
 Remark 2.10. Assume that $X$ is terminal threefold with $\text{dep}_{\text{Gor}}(X) = 0$ and $P \in X$ is a singular point. By [19, Corollary 3.4] we know that $g\text{dep}(P \in X) = 1$ if and only if $P$ is a $\frac{1}{2}(1, 1, 1)$ point (since $\text{dep}_{\text{Gor}}(X) = 0$ implies that $P$ is not a Gorenstein singular point). Also assume that $g\text{dep}(P \in X) = 2$, then $P$ is a $\frac{1}{3}(1, 2, 1)$ point or a $cA/2$ point defined by

$$(xy + z^2 + u^2 = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{2}(1, 1, 1, 0).$$

Indeed, in this case there exists a $w$-morphism $X_1 \to X$ such that $X_1$ contains exactly one singular point which is a $\frac{1}{2}(1, 1, 1)$ point. Now $w$-morphisms between terminal threefolds are listed in [8] and [9]. One can verify that the preceding two cases are the only possibility.

We will need the following properties of the depth:

Proposition 2.11 ([3], Porposition 5.1).

(1) Assume that $Y \to X$ is a divisorial contraction between terminal and $\mathbb{Q}$-factorial threefolds.

(1-1) If $Y \to X$ is a divisorial contraction to a point, then $g\text{dep}(X) \leq g\text{dep}(Y) + 1$ and $\text{dep}(X) \leq \text{dep}(Y) + 1$.

If $Y \to X$ is a divisorial contraction to a curve, then $g\text{dep}(X) \leq g\text{dep}(Y)$ and $\text{dep}(X) \leq \text{dep}(Y)$.

(1-2) $\text{dep}_{\text{Gor}}(X) \geq \text{dep}_{\text{Gor}}(Y)$ and the inequality is strict if the non-isomorphic locus on $X$ contains a Gorenstein singular point.

(2) Assume that $X \dashrightarrow X'$ is a flip between terminal and $\mathbb{Q}$-factorial threefolds.

(2-1) $g\text{dep}(X) > g\text{dep}(X')$ and $\text{dep}(X) > \text{dep}(X')$.

(2-2) $\text{dep}_{\text{Gor}}(X) \leq \text{dep}_{\text{Gor}}(X')$ and the inequality is strict if the non-isomorphic locus on $X'$ contains a Gorenstein singular point.

As an easy corollary one has that:

Corollary 2.12. Assume that

$$X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k$$

is a process of the minimal model program. Then $\text{dep}_{\text{Gor}}(X_k) \geq \text{dep}_{\text{Gor}}(X_0)$. The inequality is strict if the non-isomorphic locus of $X_0 \dashrightarrow X_k$ on $X_k$ contains a Gorenstein singular point.

Remark 2.13. Assume that $P \in X$ is a three-dimensional cyclic quotient point, then there is only one terminal divisorial contraction $Y \to X$ over $P$, which is a weighted blow-up in [12, Theorem 5]. An easy computation shows that $Y$ has only cyclic quotient singularities.

Likewise, assume that $P \in X$ is a three-dimensional $cA/r$ singularity with $r > 1$, then every terminal divisorial contraction $Y \to X$ is a weighted blow-up in [11, Theorem 1.2 (i)]. One can easily compute that $Y$ has only $cA/r$ singularities.

2.4.2. Chen-Hacon factorization.

Theorem 2.14 ([10] Theorem 3.3). Assume that either $Y \dashrightarrow Y_1$ be a flip over $U$, or $Y \to U$ is a divisorial contraction to a curve and $Y$ is not Gorenstein over $U$. Then there
exists a diagram

\[
\begin{array}{ccc}
Z_0 \rightarrow \ldots \rightarrow Z_i & & Z_{i+1} \rightarrow \ldots \rightarrow Z_k \\
\downarrow & & \downarrow \\
Y & & V_i \\
\downarrow & & \downarrow \\
U & & Y_1
\end{array}
\]

such that \(Z_0 \rightarrow Y\) is a strict \(w\)-morphism (\cite[Corollary 5.8]{[6]}), \(Z_k \rightarrow Y_1\) is a divisorial contraction, \(Z_0 \rightarrow Z_1\) is a flip or a flop over \(V_0\) and \(Z_i \rightarrow Z_{i+1}\) is a flip over \(V_i\) for \(i > 0\). \(Y_1 \rightarrow U\) is a divisorial contraction to a point if \(Y \rightarrow U\) is divisorial.

**Remark 2.15.** Notation as in the above theorem. Assume that \(Y \rightarrow Y_1\) is a flip, \(C_{Z_0}\) is the flipping/flopping curve of \(Z_0 \rightarrow Z_1\) and \(C_Y\) is the image of \(C_{Z_0}\) on \(Y\). Then \(C_Y\) is a flipping curve of \(Y \rightarrow Y_1\). This fact follows from the construction of the diagram.

### 3. First factorization

#### 3.1. General elephants

Let \(X\) be a germ of a three-dimensional terminal singularity. The general hyperplane section of \(X\) (so called the general elephant of \(X\)) has only Du Val singularities, please see \([23, (6.4)]\) for more details. The singularity of the general elephant can be used to measure the singularities of \(X\). In this subsection we will discuss how to estimate the general elephants of varieties appearing in the Chen-Hacon factorization.

**Lemma 3.1.** Let \(W\) be a terminal threefold. Assume that \(H\) is a reduced and irreducible effective Weil divisor on \(W\) such that \(K_W + H\) is \(\mathbb{Q}\)-Cartier and \(H\) has only Du Val singularities. Then \((W,H)\) is canonical.

**Proof.** Let \(W_m \rightarrow W_{m-1} \rightarrow \ldots \rightarrow W_1 \rightarrow W_0 = W\) be a sequence of \(w\)-morphisms resolving the singularities of \(\text{Sing}(W) \cap H\). Let \(W_k \rightarrow \ldots \rightarrow W_m\) be a sequence of smooth blow-ups, such that \((W_k,H_{W_k})\) is log smooth. Let \(E_i = \text{exc}(W_i \rightarrow W_{i-1})\). By \([6, \text{Lemma } 2.7 (2)]\) we know that \(a(E_i,W_{i-1},H) = 0\) for all \(1 \leq i \leq m\) and it is easy to compute that \(a(E_i,W_{i-1},H) \leq 0\) for \(m+1 \leq i \leq k\). It follows that \(a(E_i,W,H) \leq 0\) for all \(i\). This implies that \(a(E_i,W,H) = 0\) for all \(i\), since otherwise \(H\) can not have canonical singularities. Hence \((W,H)\) is canonical. \(\Box\)

**Corollary 3.2.** Assume that \(f : X \rightarrow W\) is a birational morphism between terminal threefolds. Let \(H_W\) be a reduced and irreducible Weil divisor on \(W\) such that \(K_W + H\) is \(\mathbb{Q}\)-Cartier and \(H\) has only Du Val singularities. If \(K_X + H_X = f^*(K_W + H)\), then \(H_X\) has only Du Val singularities and \(H_X \rightarrow H\) is a partial resolution.

**Proof.** We have \((W,H)\) is canonical, hence \((X,H_X)\) is canonical. \([18, \text{Proposition } 5.51]\) implies that \(H_X\) is normal. Hence \(H_X\) has also Du Val singularities and \(H_X \rightarrow H\) is a partial resolution. \(\Box\)

**Corollary 3.3.** Assume that \(Y \rightarrow Y_1\) is a flip over \(U\) and \(Z_k \rightarrow Y_1\) is the last step in the factorization of Theorem 2.14. Assume that there exists a Du Val section \(H_U \in |-K_U|\) and \(Z_k \rightarrow Y_1\) is a divisorial contraction to a curve \(C_1\), then \(C_1\) is smooth.

**Proof.** Note that \(C_1\) is a flipped curve since \(C_1 = \text{Center}_{Y_1} F\) and \(\text{Center}_Y F\) is a point, where \(F = \text{exc}(Z_k \rightarrow Y_1)\). We know that \(H_{Y_1} \rightarrow H_U\) is a partial resolution by Corollary 3.2. Since \(C_1\) is a flipped curve, \(H_{Y_1} C_1 < 0\), so \(C_1 \subset H_{Y_1}\) and it is an exceptional curve of \(H_{Y_1} \rightarrow H_U\). Hence \(C_1\) should be smooth. \(\Box\)
Assume that either $Y \to Y_1$ be a flip over $U$, or $Y \to U$ is a divisorial contraction to a curve and $Y$ is not Gorenstein over $U$. We have the following diagram

$$
\begin{array}{c}
Z_0 \longrightarrow \cdots \longrightarrow Z_i \\
\downarrow h \\
Y \\
\downarrow \tau_i \\
V_i \\
\downarrow \phi_i \\
U \\
\end{array} \quad \begin{array}{c}
Z_{i+1} \longrightarrow \cdots \longrightarrow Z_k \\
\downarrow g \\
Y_1 \\
\end{array}
$$

as in Theorem 2.14.

**Lemma 3.4.** Notation as above. Assume that $H_U \in |-K_U|$ is a reduced and irreducible effective Weil divisor such that $K_U + H_U$ is $\mathbb{Q}$-Cartier, $H_U$ has only Du Val singularities and contains the non-isomorphic locus of $Y \to U$. Then $H_{V_i} \in |-K_{V_i}|$ and $H_{V_i} \to H_U$ is a partial resolution. Moreover, we have $H_{V_0} \not\cong H_U$.

**Proof.** If $Y \to U$ is small, then $K_Y + H_Y = g^*(K_U + H_U)$. Assume that it is a divisorial contraction to a curve $C$. Since $H_U$ is normal, it is smooth at the generic point of $C$. Thus we also have $K_Y + H_Y = g^*(K_U + H_U)$. This implies that $H_Y \in |-K_Y|$, hence $H_Y$ passes through every non-Gorenstein singular point of $Y$. Corollary 3.2 implies that $H_Y$ has also Du Val singularities. Thus we have

$$K_{Z_0} + H_{Z_0} = h^*(K_Y + H_Y) = (g \circ h)^*(K_U + H_U) = (\phi_0 \circ \tau_0)^*(K_U + H_U)$$

by [3] Lemma 2.7 (2)]. Since all $Z_i$ are isomorphic in codimension one, we have $K_{Z_i} + H_{Z_i} = (\phi_i \circ \tau_i)^*(K_U + H_U)$ for all $i$. Hence $K_{V_i} + H_{V_i} = \phi^*(K_U + H_U)$ for all $i$. This implies that $(V_i, H_{V_i})$ is canonical and so $H_{V_i}$ is normal by [13] Proposition 5.51] for all $i$. Hence $H_{V_i}$ has also Du Val singularities and $H_{V_i} \to H$ is a partial resolution. The fact that $H_{V_0} \not\cong H_U$ follows from [3] Lemma 2.7 (3)]. \qed

Now let $W$ be a germ of three-dimensional non-$\mathbb{Q}$-factorial terminal singularity. Let $W_1 \to W$ be a $w$-morphism and let $Y \to W_1$ be a $\mathbb{Q}$-factorization. Let

$$
\begin{array}{c}
Y = Y_0 \to \cdots \to Y_i \\
Y_{i+1} \to \cdots \to Y_k \\
\downarrow Y_0 \to \cdots \to Y_i \\
W_1 \\
\downarrow W_1 \\
W \\
\end{array} \quad \begin{array}{c}
\downarrow U_i \\
\downarrow X \\
\end{array}
$$

be a sequence of $K_Y$-MMP over $W$.

**Lemma 3.5.** Assume that $H \in |-K_W|$ is a Du Val section. Then $H_{U_i} \to H$ is a partial resolution.

**Proof.** By [3] Lemma 2.7 (2)] and the fact that $Y \to W_1$ is isomorphic in codimension one, we know that $K_Y + H_Y = \phi^*(K_W + H)$, where $\phi$ is the morphism $Y \to W$. It follows that $K_{U_i} + H_{U_i} = \phi_i^*(K_W + H)$ where $\psi_i$ is the morphism $U_i \to W$, since $Y \to U_i$ is isomorphic in codimension one. Thus $H_{U_i}$ has only Du Val singularities and $H_{U_i} \to H$ is a partial resolution by Corollary 3.2. \qed
3.2. **Factorizing flops.** Let $X \rightarrow X'$ be a three-dimensional terminal flop over $W$. Let $W_1 \rightarrow W$ be a strict $w$-morphism and let $Y$ be a $Q$-factorization of $W_1$.

**Lemma 3.6.** $\rho(Y/W) = 2$.

*Proof.* Let $X''$ be the relative minimal model of $Y \rightarrow W$. Let $E = exc(W_1 \rightarrow W)$, then $E_Y$ is covered by $K_Y$-negative curves, so the MMP will contract $E_Y$. Thus $X'' \rightarrow W$ is isomorphic in codimension one. This means that $X''$ is a $Q$-factorization of $W$. By Remark 2.3 we know that $X''$ is isomorphic to either $X$ or $X'$, hence $\rho(X''/W) = 1$. On the other hand, since there are only one exceptional divisor on $Y$, there are only one divisorial contraction in the MMP. Hence $\rho(Y/X'') = 1$ and so $\rho(Y/W) = 2$. \hfill \Box

If $W_1$ is $Q$-factorial, then there are two $K_{W_1}$-negative extremal rays over $W$. One can run two different MMP and one may assume that $X$ is one of the minimal model. We denote the other minimal model by $X'$. If $W$ is not $Q$-factorial, then $Y \rightarrow W$ is a flopping contraction. One can construct a flop $Y \rightarrow Y'$ over $W_1$ as in [14]. We may assume that $X$ is a minimal model of $Y$ over $W$. Let $X_1$ be a minimal model of $Y' \rightarrow W$.

**Lemma 3.7.** Notation as above. One has $X_1 = X'$.

*Proof.* Assume not, then $X_1 \neq X$. We will show that this is impossible.

If $W_1$ is not $Q$-factorial, we denote

$$Y = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_k \rightarrow X$$

be the $K_Y$-MMP over $W$, where $Y_i \rightarrow Y_{i+1}$ is a flip and $Y_k \rightarrow X$ is a divisorial contraction. Similarly we denote

$$Y' = Y'_0 \rightarrow Y'_1 \rightarrow \cdots \rightarrow Y'_k \rightarrow X_1$$

be the $K_{Y'}$-MMP over $W$. If $W_1$ is $Q$-factorial, we denote

$$X \leftrightarrow Y_k \leftrightarrow \cdots \leftrightarrow Y_1 \leftrightarrow Y_0 = W_1 = Y'_0 \rightarrow Y'_1 \rightarrow \cdots \rightarrow Y'_k \rightarrow X_1$$

be the two different $K_{W_1}$-MMP over $W$. Assume that $X_1 = X$, then $Y_k = Y'_k$ because $Y_k \rightarrow X$ and $Y'_k \rightarrow X_1 = X$ extracts the same exceptional divisor. Interchanging $Y_j$ and $Y'_j$ if necessary we may assume that $k \geq k'$. Now $NE(Y_k/W)$ is generated by a $K_{Y_k}$-negative extremal ray and a $K_{Y'_k}$-positive extremal ray. $Y_{k-1}$ and $Y'_{k-1}$ are both the anti-flip along the $K_{Y_k}$-positive extremal ray, hence we have $Y_{k-1} \cong Y'_{k-1}$. By induction we may assume that $Y_{k-i} \cong Y'_{k-i}$ for all $i > 0$, and so $Y_i \cong Y'_i$ for some $i$. Since $K_{Y'}$ is anti-nef over $W$ but $K_{Y_k}$ is not unless $i = 0$, we know that $Y \cong Y'$ and both $Y \rightarrow Y_i$ and $Y' \rightarrow Y'_i$ contract the same extremal ray, this contradicts to our construction. \hfill \Box

In conclusion, we have

**Corollary 3.8.** Let $X \rightarrow X'$ be a three-dimensional terminal flop over $W$, then there exists a factorization

$$X \leftrightarrow Y_k \leftrightarrow \cdots \leftrightarrow Y_1 \leftrightarrow Y_0 = Y \rightarrow Y' = Y'_0 \rightarrow Y'_1 \rightarrow \cdots \rightarrow Y'_{k} \rightarrow X_1$$

such that $Y_i \rightarrow Y_{i+1}$ and $Y'_i \rightarrow Y'_{i+1}$ are flips and either $Y = Y'$ or $Y \rightarrow Y'$ is a flop.

The following lemma will be used in Section 6.

**Lemma 3.9.** Let $X \rightarrow X'$ be a terminal flop over $W$. Then $\text{dep}(X) \leq \text{dep}(W)$. The equality holds if and only if $X \rightarrow X'$ is a Gorenstein flop.

*Proof.* If $X \rightarrow X'$ is a Gorenstein flop, then $\text{dep}(X) = \text{dep}(W)$. So we may assume that $X \rightarrow X'$ is a non-Gorenstein flop and we want to prove that $\text{dep}(X) < \text{dep}(W)$.

We will prove the statement by induction on $\text{gdep}(W)$. We have a factorization as in Corollary 3.8. If $\text{gdep}(W) = 1$ then $W_1 = Y = Y'$ is smooth. In this case $k = k' = 0$ and
we define a divisorial contraction to a curve such that \( Y \) to be the factorization of the flip \( Y \) by Proposition 2.11 and so \( \text{dep}(X) < \text{dep}(Y) + 1 \leq \text{dep}(Y) \). If \( k = 0 \) then \( Y_k \to X \) is a divisorial contraction to the flopping curve. One also has \( \text{dep}(X) \leq \text{dep}(Y) \) by Proposition 2.11. Thus
\[
\text{dep}(X) \leq \text{dep}(Y) \leq \text{dep}(W_1) = \text{dep}(W) - 1 < \text{dep}(W).
\]

\[\Box\]

**Convention 3.10.** Let \( Y_{(i)}^{(0)} = Y_i, U_{(i)}^{(0)} = U_i \). For any \( n \)-tuple \( I = (a_1, ..., a_n) \), we denote \( I_n + 1 \) by the tuple \((a_1, ..., a_n + 1)\). Define

\[
\begin{align*}
Y_{(I,0)}^{(j)} \to \ldots \to Y_{(I,i)}^{(j)} & \quad Y_{(I,i+1)}^{(j)} \to \ldots \to Y_{(I,k_j)}^{(j)} \\
Y_I^{(j)} & \quad U_{(I,i)}^{(j)} & \quad Y_{I+1}^{(j)}
\end{align*}
\]

to be the factorization of the flip \( Y_I^{(j)} \to Y_{I+1}^{(j)} \) as in Theorem 2.14. Also, if \( Y_{(I,k_j)}^{(j)} \to Y_{I+1}^{(j)} \) is a divisorial contraction to a curve such that \( Y_{(I,k_j)}^{(j)} \) is not Gorenstein over \( Y_{I+1}^{(j)} \), then we define \( Y_I^{(j+1)} = Y_{(I,k_j)}^{(j)} \), \( U_I^{(j+1)} = Y_{I+1}^{(j)} \) and

\[
\begin{align*}
Y_{(I,0)}^{(j+1)} \to \ldots \to Y_{(I,i)}^{(j+1)} & \quad Y_{(I,i+1)}^{(j+1)} \to \ldots \to Y_{(I,k_j)}^{(j+1)} \\
Y_I^{(j+1)} & \quad U_{(I,i)}^{(j+1)} & \quad Y_{I+1}^{(j+1)}
\end{align*}
\]

to be the factorization of \( Y_{(I,k_j)}^{(j+1)} \to Y_{I+1}^{(j)} \).

We will keep this convention in the rest of this article. Nevertheless, usually every variety we are studying lies on the same level, namely the superscript \((j)\) are all the same. In those situations we will omit the superscript and write \( Y_I^{(j)} \) as \( Y_I \). Moreover, if \( H \) is a divisor on some variety \( Z \) which is birational to \( Y_I^{(j)} \) and \( H \) intersects the isomorphism locus of \( Z \to Y_I^{(j)} \) non-trivially. Then we will denote the proper transform of \( H \) on \( Y_I^{(j)} \) by \( H_I^{(j)} \) or \( H_I \) (in the case that the superscript can be omit) instead of \( H_{Y_I^{(j)}} \).

**Lemma 3.11.** *If \( Y_{(I,0)} \to Y_{(I,1)} \) is a flop, then it is simple.*

**Proof.** The flipping curve of \( Y_I \to Y_{I+1} \) is a tree of \( \mathbb{P}^1 \)'s. Let \( C_1, ..., C_k \) be those flipping curves which passing though the non-isomorphic locus of \( Y_{(I,0)} \to Y_I \). Then the proper transform of \( C_i \) on \( Y_{(I,0)} \) are all disjoint. Since flopping curves should be connected, the flop \( Y_{(I,0)} \to Y_{(I,1)} \) is simple. \(\Box\)
Combining Lemma 3.4 and Lemma 3.5 one has

**Lemma 3.12.** If $H \in |-K_W|$ is a Du Val section, then $H_{U^i} \in |-K_{U^i}|$ and $H_{U^i} \to H$ is a partial resolution for all possible $i$.

**Lemma 3.13.** We have $\text{dep}_{\text{Gor}}(X) \geq \text{dep}_{\text{Gor}}(Y^{(j)})$ for all possible $j$, $I$. Moreover, if the exceptional locus of $X$ over $W$ contains a Gorenstein singular point, then the inequality is strict.

**Proof.** By Corollary 2.12 we know that $\text{dep}_{\text{Gor}}(X) \geq \text{dep}_{\text{Gor}}(Y_i) = \text{dep}_{\text{Gor}}(Y^{(0)})$ for all $i$, and the inequality is strict if the exceptional locus of $X$ over $W$ contains a Gorenstein singular point. Now for any possible $I$ and $j$ one has that $\text{dep}_{\text{Gor}}(Y^{(j)}_{I,l}) \leq \text{dep}_{\text{Gor}}(Y^{(j)}_{I+1})$ for all $0 \leq l \leq k^{(j)}_I$ and also for $j > 0$ one has $\text{dep}_{\text{Gor}}(Y^{(j)}_{I+1}) \leq \text{dep}_{\text{Gor}}(U^{(j)}_I) = \text{dep}_{\text{Gor}}(Y^{(j-1)})$. This finishes the proof. □

**Proposition 3.14.** Any three-dimensional terminal flop can be factorize into a composition of divisorial contractions to points, blow-up smooth curves, smooth flops, and inverses of above maps.

**Proof.** In this proof we say that a birational map has a factorization if it can be factorize into a composition of (inverses of) divisorial contractions to points, blow-up smooth curves and smooth flops.

Assume that $X \to Y'$ is a singular flop over $W$. By Corollary 3.8 we have a decomposition $X \dashrightarrow Y \dashrightarrow Y' \to X'$. We need to show that $Y' \to X$ has a factorization and $Y \to Y'$ has a factorization. Then by the symmetry we can also assume that $Y' \to X'$ has a factorization, so $X \to X'$ has a factorization.

As in Convention 3.10 we can a factorize the birational map $Y \to X$ to the following maps:

(i) $Y^{(j)}_{I,l} \to Y^{(j)}_{I,0}$ which is an inverse of $w$-morphism.

(ii) $Y^{(j)}_{I,0} \to Y^{(j)}_{I,1}$ which is a flop.

(iii) $Y^{(j)}_{I,k}$ is either a divisorial contraction to a point or a divisorial contraction to a curve such that $Y^{(j)}_{I,k}$ do not have non-Gorenstein point over $Y^{(j)}_{I+1}$.

Notice that if $Y^{(j)}_{I,k} \to Y^{(j)}_{I+1}$ is a divisorial contraction to a curve, then the curve is smooth by Lemma 3.12 and Corollary 3.5. In this case by 7 Theorem 4 we know that $Y^{(j)}_{I,k} \to Y^{(j)}_{I+1}$ is a blowing-up a smooth curve. Thus $Y' \to X$ has a factorization if whenever $Y^{(j)}_{I,0} \to Y^{(j)}_{I,1}$ is a singular flop, it has a factorization. Let $S_0$ be the set consists the flops $Y \to Y'$ and $Y^{(j)}_{I,0} \to Y^{(j)}_{I,1}$ for all $j$ and $I$, and let $S = \{ Z \dashrightarrow Z' \text{ is a singular flop} \} \subset S_0$. It is enough to show that $Z \dashrightarrow Z'$ has a factorization if $Z \dashrightarrow Z' \in S$.

Let $P \in W$ be the image of flopping curves of $X$ on $W$ and let $GE(P) = A_i, D_j$ or $E_k$ denotes the type of a general elephant near $P$. We will prove the statement by induction on the pair $(\text{dep}_{\text{Gor}}(X), GE(P))$, where the relation between general elephants are given by

$$A_i < A_{i'} < D_j < D_{j'} < E_6 < E_7 < E_8$$

if $i < i'$, $j < j'$. Notice that by Lemma 3.13 one has $\text{dep}_{\text{Gor}}(Y^{(j)}) \leq \text{dep}_{\text{Gor}}(X)$ for all possible $j$ and $I$. This means that for all $Z \dashrightarrow Z' \in S$ one has that $\text{dep}_{\text{Gor}}(Z) \leq \text{dep}_{\text{Gor}}(X)$ Moreover, if $X \dashrightarrow X'$ is a Gorenstein flop, then the inequality is strict. In this case by the induction hypothesis we know that the flop $Z \dashrightarrow Z'$ has a factorization for all $Z \dashrightarrow Z' \in S$. 
Now assume that $X \dashrightarrow X'$ is a non-Gorenstein flop. Given $Z \dashrightarrow Z' \in \mathcal{S}$. If $Z \dashrightarrow Z'$ is a Gorenstein flop, then we already know that $Z \dashrightarrow Z'$ has a factorization. Assume that $Z \dashrightarrow Z'$ is a non-Gorenstein flop over $V$ and $Q \in V$ is the image of flopping curves, then $Q$ is a non-Gorenstein point. By Lemma 3.12 we know that $H_V \in \mid - K_V \mid$ and $H_V \rightarrow H$ is a partial resolution, for any Du Val section $H \in \mid - K_W \mid$. This means that $GE(Q) < GE(P)$, hence $Z \dashrightarrow Z'$ also has a factorization.

Proof of Theorem 3.3. Assume that $X \dashrightarrow X'$ is a flop over $W$. By Proposition 3.11 we already know that $X \dashrightarrow X'$ can be factorize into a composition of (inverse of) divisorial contractions and smooth flops. We only need to show that we may further assume that those smooth flops are all simple. Notice that those flops $Y_{(i)} \dashrightarrow Y_{(j)}$ are always simple by Lemma 3.11 so we only need to consider the flop $Y \dashrightarrow Y'$.

If $X \dashrightarrow X'$ is a smooth flop and $gdep(W) = 1$, then $Y = W_1 = Y'$ is smooth and so $Y_k = Y$. In this case the flop is simple (in fact it is an Atiyah flop). Now $Y \dashrightarrow Y'$ is a flop over $W_1$ and $gdep(W_1) = gdep(W) - 1$, hence one can induction on $gdep(W)$ and assume that every smooth flops in the factorization of $Y \dashrightarrow Y'$ is simple. Hence every smooth flops in the factorization of $X \dashrightarrow X'$ is simple.

4. Factorization of flips with simple singularities

We begin with the following important property of three-dimensional flips:

**Lemma 4.1** (II, Theorem 0). Assume that $X \dashrightarrow X'$ is a three-dimensional canonical flip and $C$ is a flipping curve, then $-1 < K_X.C < 0$.

**Lemma 4.2.** Assume that $Y \dashrightarrow Y'$ is an Atiyah flop. Let $C_Y \subset Y$ be the flopping curve, $C_{Y'} \subset Y'$ be the flopped curve and $S \subset Y$ be a surface. Assume that either $S$ is smooth or $Sing(S)$ is pure of dimension one and $C_Y \not\subset Sing(S)$. Then either $S_{Y'}$ is smooth or $Sing(S_{Y'})$ is pure of dimension one.

**Proof.** Let $\pi : \tilde{Y} = Bl_{C_Y}Y \rightarrow Y$ and $\pi' : \tilde{Y} = Bl_{C_{Y'}}Y' \rightarrow Y'$ be a common resolution of $Y \dashrightarrow Y'$. Let $E = exc(\pi) = exc(\pi')$. First assume that $S$ is smooth.

(i) Assume that $C_Y \not\subset S$. If $S.C_Y > 1$ then $S_{Y'}$ is singular along $C_{Y'}$ by Lemma 2.4 and we have done. Assume that $S.C_Y = 1$. We have that $S_{Y'} = Bl_{S \cap C_{Y'}}S$ and $S_{Y'} \cong S_Y$. Hence $S_{Y'}$ is smooth.

(ii) Assume that $C_Y \subset S$ and $C_{Y'} \not\subset S_{Y'}$. In this case we have $S_{Y'.C_{Y'}} = 1$ since otherwise $S$ won’t be smooth. Now

$$S \cong S_{Y'} = Bl_{S_{Y'} \cap C_{Y'}}S_{Y'}.$$

Since $S.C_Y = -S_{Y'.C_{Y'}} = -1$, $C_Y$ is a $-1$-curve and $S_{Y'}$ is obtained by contracting this curve, so $S_{Y'}$ is also smooth.

(iii) Assume that $C_Y \subset S$ and $C_{Y'} \subset S_{Y'}$. We have $\pi^*S = S_{Y'} + E$. Assume that $S_{Y'}$ is not singular along $C_{Y'}$, then we also have $\pi'^*S_{Y'} = S_{Y'} + E = \pi^*S$. This implies that $S.C_Y = S_{Y'.C_{Y'}} = 0$. Let $\phi : Y \rightarrow U$ and $\phi' : Y' \rightarrow U$ be the flopping contractions. Since $S$ is smooth, $S_U$ has only a $A_1$ singularity. Thus $S_{Y'}$ is smooth because $S_{Y'} \not\subset S_U$ and there are no intermediate varieties between a $A_1$ singularity and its minimal resolution.

Now assume that $S$ is not smooth, $Sing(S)$ is pure of dimension one and $C_Y \not\subset Sing(S)$. Let $Sing(S)_{Y'}$ be the proper transform of $Sing(S)$ on $Y'$. We want to say that $Sing(S_{Y'})$ is either $Sing(S)_{Y'}$ or $C_{Y'} \cup Sing(S)_{Y'}$. For any $P \in Sing(S) \cap C_Y$, we have $mult_P S \geq 2$. Let $l_P = \pi^*P$. Since $C_Y \not\subset Sing(S)$, $\pi^*S = S_{Y'} + \lambda E$ for $\lambda = 0$ or 1. Since $mult_P E = 1$,
we have $I_P \subset S_Y$. We may write
\[ S_Y \cap E = \sum_{P \in \text{Sing}(S) \cap C_Y} m_P I_P + \lambda \Gamma \]
for some curve $\Gamma$ which maps bijectively to $C_Y$ via $\pi$. If $\sum_P m_P > 1$ or $\pi'(\Gamma) = C_{Y'}$ then $S_Y$ is singular along $C_{Y'}$, so $\text{Sing}(S_{Y'}) = C_{Y'} \cup \text{Sing}(S)_{Y'}$. Now assume that $\sum_P m_P = 1$ and $\pi'(\Gamma)$ is a point. Notice that in this case $\lambda = 1$ or $P$ can not be a singular point of $S$. We have $\text{Sing}(S_{Y'}) \cap C_{Y'} = \pi'(\Gamma)$. It follows that $\text{Sing}(S_{Y'}) \cap C_{Y'} = \pi'(\Gamma)$ and so $\text{Sing}(S_{Y'}) = \text{Sing}(S)_{Y'}$. \hfill \Box

Conjecture 4.3. Let $Y_I \rightarrow Y_{I+1}$ be a flip. We say that the flip has a factorization (*) if we have the following diagram
\[
\begin{array}{ccc}
Y_I & - - \rightarrow & Y_{I+1} \\
\vdots & & \vdots \\
Y_{(I,0)} & - - \rightarrow & Y_{(I,1)} \\
& \downarrow & \downarrow \\
& Y_I & \rightarrow \cdots Y_{(I,2)} \rightarrow Y_{(2)} \rightarrow Y_{I+1}
\end{array}
\]
with $I = (I, 1, \ldots, 1, 0)$, such that
\[ Y_{I+1} \rightarrow \cdots Y_{(I,1,2)} \rightarrow Y_{(2)} \rightarrow Y_{I+1} \]
is a sequence of $w$-morphisms or blowing-up smooth curves.

Lemma 4.4. Let $Y_I \rightarrow Y_{I+1}$ be a flip. Assume that $Y_I \rightarrow Y_{I+1}$ has a factorization (*) as in Conjecture 4.3 and every flop in the factorization is an Atiyah flops. If $S$ is a surface on $Y_{I+1}$ such that $S_I$ is smooth in the smooth locus of $Y_I$ for $I' = (I, 1, \ldots, 1, 2)$ or $\bar{I}$, then either $S_I$ has one-dimensional singularities, or $S_I$ is smooth in the smooth locus of $Y_I$.

Proof. We know that $Y_I \rightarrow Y_I$ can be factorized into a composition of Atiyah flops and $w$-morphism. Notice that the singular locus of $S_{(I,1,\ldots,1,1)}$ do not contain the flopped curve of $Y_{(I,1,\ldots,1,0,1)} \rightarrow Y_{(I,1,\ldots,1,1,1)}$ since the flopped curve appears on the smooth locus of $Y_{(I,1,\ldots,1,2)}$. Lemma 4.2 implies that either $S_{(I,1,\ldots,1,0)}$ has one-dimensional singularities or $S_{(I,1,\ldots,1,1)}$ is smooth in the smooth locus of $Y_{(I,1,\ldots,1,0)}$. One can see that $S_Y$ satisfies the desired property. \hfill \Box

Lemma 4.5. Let $Y_I \rightarrow Y_{I+1}$ be a flip such that $Y_{(I,0)} \rightarrow Y_{(I,1)}$ is a simple smooth flop. Let $F_{(I,0)}$ be the exceptional divisor of $Y_{(I,0)} \rightarrow Y_I$. If $F_{(I,1)}$ is smooth along the flopped curve of $Y_{(I,0)} \rightarrow Y_{(I,1)}$, then $Y_{(I,0)} \rightarrow Y_{(I,1)}$ is an Atiyah flop.

Proof. Let $C_{(I,1)}$ be the flopped curve on $Y_{(I,1)}$. We have $(C_{(I,1)})^2 F_{(I,1)} < 0$ by the Hodge index theorem [18, Lemma 3.40]. Since $F_{(I,0)}, C_{(I,0)} > 0$ where $C_{(I,0)}$ is the flopping curve, we have $F_{(I,1)}, C_{(I,1)} < 0$. So
\[ 0 \geq F_{(I,1)}, C_{(I,1)} = K_{F_{(I,1)}, C_{(I,1)}} = -2 - (C_{(I,1)})^2 F_{(I,1)} > -2, \]
which implies that $F_{(I,1)}, C_{(I,1)} = -1$. Hence the normal bundle of $Y_{(I,1)}$ along $C_{(I,1)}$ is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (cf. [22, Remark 5.2]) and so $Y_{(I,0)} \rightarrow Y_{(I,1)}$ is an Atiyah flop. \hfill \Box
Lemma 4.6. Let $Y_I \to Y_{I+1}$ be a flip such that $Y_{(I,0)} \to Y_{(I,1)}$ is a smooth flop. Assume that both $Y_I$ and $Y_{I+1}$ has only cA/r singularities of the form
\[(xy + z^r + f(z,u) = 0) \subset A^4_{(x,y,z,u)}/(a - a, 1, r),\]

$Y_I \to Y_{I+1}$ has a factorization $(\ast)$ as in Convention 4.2, and $Y_{(I',0)} \to Y_{(I',1)}$ is either an isomorphism or an Atiyah flop for all possible non-empty $I' = (1, \ldots, 1)$. Then $Y_{(I,0)} \to Y_{(I,1)}$ is an Atiyah flop.

Proof. Let $F_{(I,2)}$ be the exceptional divisor of $Y_{(I,2)} \to Y_{I+1}$. By the assumption that $Y_{I+1}$ has only simple cA/r singularities one can verify that $F_{(I,2)} \subset Y_{(I,2)}$ satisfies the condition in Lemma 4.4 (applied to the flip $Y_{(I,1)} \to Y_{(I,2)})$. Hence either $F_{(I,1)}$ has one-dimensional singularities, or it is smooth in the smooth locus of $Y_{(I,1)}$. However it can not have one-dimensional singularities since every curve on $Y_{(I,1)}$ appears either on $Y_{(I,0)}$ or $Y_{(I,2)}$, and $F_{(I,1)}$ is smooth in the smooth locus of $Y_{(I,0)}$ for $i = 0, 2$ by the assumption about the singularities of $Y_{I}$ and $Y_{I+1}$. Hence $F_{(I,1)}$ is smooth in the smooth locus of $Y_{(I,1)}$ and now $Y_{(I,0)} \to Y_{(I,1)}$ is an Atiyah flop by Lemma 4.5.

Lemma 4.7. Assume that $\text{dep}(Y_I) = \text{dep}(Y_{I+1}) + 1$. Then $Y_{(I,0)} \to Y_{(I,1)}$ is a flop. Either $Y_{(I,1)} \to Y_{I+1}$ is a divisorial contraction to a curve which do not change the depth, or $Y_{(I,1)} \to Y_{(I,2)}$ is a flip with $\text{dep}(Y_{I+1}) = \text{dep}(Y_{(I,2)}) + 1$. Then $Y_{(I,2)} \to Y_{I+1}$ is a $w$-morphism.

Proof. Let $C_{(I,0)}$ be the flipping/flopping curve on $Y_{(I,0)}$ and let $F_{(I,0)}$ be the exceptional divisor of $Y_{(I,0)} \to Y_I$. Then we have $F_{(I,0)} \subset Y_{(I,0)} > 0$, hence $F_{(I,1)} \subset Y_{(I,1)}$, where $C_{(I,1)}$ denotes the flipped/flopped curve on $Y_{(I,1)}$. This implies that $C_{(I,1)} \subset F_{(I,1)}$.

Assume that $Y_{(I,0)} \to Y_{(I,1)}$ is a flip. Then by Proposition 2.11 we have
\[\text{dep}(Y_{I+1}) \leq \text{dep}(Y_{(I,0)}) - 1 = \text{dep}(Y_I) - 2 = \text{dep}(Y_{I+1}) - 1.\]

This implies that $k_I = 1$ and $Y_{(I,1)} \to Y_{I+1}$ is a $w$-morphism by Proposition 2.11. Since $F_{(I,1)}$ is contracted to a point on $Y_{I+1}$, one can see that $K_{Y_{I+1}} \subset C_{(I,1)} < 0$, this leads to a contradiction. Hence $Y_{(I,0)} \to Y_{(I,1)}$ is a flop.

If $k_I = 1$, then $Y_{(I,1)} \to Y_{(I+1)}$ is a divisorial contraction to a curve because we have $\text{exp}(Y_{(I,1)} \to U_I) = F_{(I,1)}$. One can see that $\text{dep}(Y_{(I,1)}) = \text{deg}(Y_I) - 1 = \text{deg}(Y_{I+1})$. If $k_I > 1$, then $k_I$ should be 2 and we have $\text{deg}(Y_{I+1}) - 1 = \text{deg}(Y_{I+1}) - 1$. This proves the last statement.

Lemma 4.8. Assume that $Y_I \to Y_{I+1}$ is a flip such that the flipping curve passes through only one singular point. If $Y_{(I,0)} \to Y_{(I,1)}$ is a flop, then it is a Gorenstein flop.

Proof. Assume that then the flipping curve $C_{(I,0)}$ passes through a non-Gorenstein point. Let $H_I \in |- K_{Y_I}|$ be a general elephant of the flipping contraction $Y_I \to U_I$, then $H_{(I,0)} \in |- K_{Y_{I+1}}|$ by [6, Lemma 2.7 (2)]. Since $C_{(I,0)}$ contains a non-Gorenstein point, $H_{(I,0)}$ intersects $C_{(I,0)}$ non-trivially. Since $H_{(I,0)} \subset C_{(I,0)} = 0$, $H_{(I,0)}$ contains $C_{(I,0)}$. This implies that $H_I$ contains $C_I$ where $C_I$ is the image of $C_{(I,0)}$ on $Y_I$. Notice that $C_I$ is a flipping curve of $Y_I \to Y_{I+1}$ by Remark 2.15. Since $C_I$ passes through only one singular point, the flip $Y_I \to Y_{I+1}$ is of type $IC$ in Kollár-Mori’s classification [17, Theorem 2.2]. It follows that $Y_I$ has only a cyclic quotient singular point over $U_I$, and $Y_{(I,0)}$ also has only a cyclic quotient singular point over $U_{(I,0)}$ by Remark 2.13. According to [17, Theorem 2.2.2], the Dynkin diagram of the $H_U$ is of type $D_m$

\[
\begin{array}{c}
\circ \\
\circ \cdot \cdot \cdot \circ \circ \circ \circ \circ
\end{array}
\]
Let $P_{(I,0)}$ be the non-$\mathbb{Q}$-factorial point of $U_{(I,0)}$. It follows that near $P_{(I,0)}$ we have $H_{U_{(I,0)}}$ is of type $A_i$ for $i = 3$ or $m - 1$, or $D_j$ for some $4 \leq j \leq m - 1$. We will prove that any of those cases can not happen.

(i) $H_{U_{(I,0)}}$ is of type $A_i$ near $P_{(I,0)}$. Then the flopping curve of $Y_{(I,0)} \rightarrow Y_{(I,1)}$ contains a cyclic quotient point of index $i$, so $P_{(I,0)}$ is of index $i$. One can easily see that this can not happen because $P_{(I,0)}$ can not be a cyclic quotient point, and the general elephant of a non-cyclic-quotient index $i$ point has at least $A_{2i+1}$ singularities ([23 (6.4)]).

(ii) $H_{U_{(I,0)}}$ is of type $D_j$ near $P_{(I,0)}$. In this case the flopping curve of $Y_{(I,0)} \rightarrow Y_{(I,1)}$ contains a cyclic quotient point of index $j$ and hence $P_{(I,0)}$ is also an index $j$ point. This means that $j = 4$ and $P_{(I,0)}$ is a $cAx/4$ point. However, the general elephant of a $cAx/4$ point has at least $D_3$ singularities ([23 (6.4)]). Hence this case won’t happen.

\[\square\]

**Lemma 4.9.** Assume that $Y_I \rightarrow Y_{I+1}$ is a flip such that the flipping curve passes through only one singular point $P$ with $\text{dep}_{\text{Gor}}(P \in X) = 0$ and $\text{dep}(P \in X) = 1$. Then $Y_{(I,0)} \rightarrow Y_{(I,1)}$ is an Atiyah flop and $Y_{(I,1)} \rightarrow Y_{I+1}$ is a blowing-up a smooth curve.

Moreover, we have $K_{Y_{I+1}}C_{I+1} = 1$ if $C_{I+1}$ is the flipped curve.

**Proof.** Note that $g\text{dep}(P \in X) = 1$, hence $Y_{(I,0)}$ is smooth. Since there are no flipping contraction begin with a smooth variety, we have that $Y_{(I,0)} \rightarrow Y_{(I,1)}$ is a flop and $Y_{(I,1)} \rightarrow Y_{I+1}$ is a blowing-up a smooth curve by [6] Remark 3.4]. This flop is an Atiyah flop by Lemma 4.9.

Now we know that $F_{(I,1)}C_{(I,1)} = -1$ by the computation in Lemma 4.5, hence $K_{Y_{I+1}}C_{I+1} = 1$.

\[\square\]

**Lemma 4.10.** Assume that $Y_I \rightarrow Y_{I+1}$ is a flip such that the flipping curve passes through only one singular point $P$ with $\text{dep}_{\text{Gor}}(P \in X) = 0$ and $\text{dep}(P \in X) = 2$. Then we have one of the following factorization.

(1) $Y_{(I,1,0)} \rightarrow Y_{(I,1,1)}$

\[\begin{array}{c}
Y_{(I,0)} \rightarrow Y_{(I,1)} & \rightarrow Y_{(I,2)} \\
\downarrow w & \downarrow w \\
Y_I & Y_{I+1}
\end{array}\]

$Y_{I+1}$ has a $\frac{1}{2}(1,1,1)$ singularity.

(2) $Y_{(I,0,0)} \rightarrow Y_{(I,0,1)}$

\[\begin{array}{c}
Y_{(I,0)} \rightarrow Y_{(I,1)} \\
\downarrow w & \downarrow c \\
Y_I & Y_{I+1}
\end{array}\]

$Y_{I+1}$ is smooth and $a(\text{exc}(Y_{(I,0,1)} \rightarrow Y_{(I,1)}), Y_{I+1}) = 2.$
flipping curve of $Y$ is negatively, hence it contains the flipped curve. This implies that singular point on $Y$.

Moreover, if the singular point on $Y$ is of type $cA/2$, then we are in the case (1) or (3).

Proof. Notice that $\text{dep}_G(Y_{(1,0)}) = 0$ by Remark 2.7. Assume first that $Y_{(1,0)} \rightarrow Y_{(1,1)}$ is a flip. We have $\text{dep}(Y_{(1,1)}) = 0$ and so $k_I = 1$. The factorization of $Y_{(1,0)} \rightarrow Y_{(1,1)}$ is given by Lemma 4.9. Notice that $\text{exc}(Y_{(1,1)} \rightarrow Y_{(1,2)})$ intersects the flipped curve of $Y_{(1,0)} \rightarrow Y_{(1,1)}$ negatively, hence it contains the flipped curve. This implies that $Y_{(1,1)} \rightarrow Y_{I+1}$ should be a divisorial contraction to a curve. It is easy to see that $a(\text{exc}(Y_{(1,0,1)} \rightarrow Y_{(1,1)}), Y_{I+1}) = 2$ and we are in the case (2).

Now assume that $Y_{(1,0)} \rightarrow Y_{(1,1)}$ is a flop. The flop is a smooth flop by Lemma 4.8 and the fact that $\text{dep}_G(Y_{(1,0)}) = 0$. If $k_I = 1$ then $Y_{(1,1)} \rightarrow Y_{I+1}$ is a divisorial contraction to a curve (cf. [6] Remark 3.4). However we know that $\text{dep}_G(Y_{(1,1)}) = \text{dep}_G(Y_{(1,0)}) = 0$ and $\text{dep}(Y_{(1,1)}) = \text{dep}(Y_{(1,0)}) = 1$. By Remark 2.10 we know that $Y_{(1,1)}$ has a $\frac{1}{2}(1, 1, 1)$ singularity, so $Y_{I+1}$ cannot be smooth. Since $\text{dep}(Y_{I+1}) < 2$, we know that $\text{dep}(Y_{I+1}) = 1$ and $Y_{I+1}$ has a $\frac{1}{2}(1, 1, 1)$ singularity. However, there is no divisorial contraction to a curve which contains a cyclic quotient singularity [12, Theorem 5]. Hence $k_I > 1$ and we have a flip $Y_{(1,1)} \rightarrow Y_{(1,2)}$. Now $\text{dep}(Y_{(1,2)}) = 0$ so $k_I = 2$ and we have a divisorial contraction $Y_{(1,2)} \rightarrow Y_{I+1}$. The factorization of $Y_{(1,1)} \rightarrow Y_{(1,2)}$ is given by Lemma 4.9. If $Y_{(1,2)} \rightarrow Y_{I+1}$ is a $w$-morphism, then $\text{dep}(Y_{I+1}) = 1$ and we are in the case (1). Otherwise $Y_{I+1}$ is smooth and we are in the case (3).

Now every flop appear in the factorization is an Atiyah flop by Lemma 4.6. If the singular point on $Y$ is of type $cA/2$, then $K_{Y_{(1,0)}}C_{Y_{(1,0)}} = -\frac{1}{2}$ by Lemma 4.1, where $C_{Y_{(1,0)}}$ is the flipping curve of $Y_{(1,0)} \rightarrow Y_{I+1}$. An easy computation shows that $K_{Y_{(1,0)}}C_{Y_{(1,0)}} > -\frac{1}{2}$. On the other hand, we know that $Y_{(1,0)}$ has only a $\frac{1}{2}(1, 1, 1)$ singularity, so $K_{Y_{(1,0)}}C_{Y_{(1,0)}} \in \frac{1}{2}\mathbb{Z}_{\leq 0}$. This means that $K_{Y_{(1,0)}}C_{Y_{(1,0)}} = 0$ and so $Y_{(1,0)} \rightarrow Y_{(1,1)}$ is a flop. Hence we are in the case (1) or (3).

5. Factorizing $D$-type simple smooth flops

The goal of this section is to construct the factorization of a $D$-type simple smooth flop.

Lemma 5.1. Assume that

$$W = (x^2 + y^2z + z^n + u^g(x, y, z, u) = 0) \subset \mathbb{A}^4$$

is an isolated cD singularity. After a suitable coordinate change we may write the defining equation of $W$ as $x^2 + y^2z + \lambda yu^k + g(z, u)$. We say that this equation is a normal form of the singularity of $W$.

Proof. After several steps of coordinate change one may assume that $W$ is defined by $f(x, y, z, u) = x^2 + y^2z + \lambda yu^k + \mu u^n + y^2u^k(y, u) + g(z, u)$. We use the notation that $k = \infty$ if $\lambda = 0$ and $n = \infty$ if $\mu = 0$. Let $n(f) = \min\{n, k\}$. Then $n(f)$ is finite or $W$ do
We may assume that for all monomial \(y^iu^j \in h(y, u)\) we have \(j + 1 < n(f)\). We call such kind of equation a sub-normal form.

In the sub-normal form we may assume that \(h(y, u)\) contains only finitely many monomials. If \(h(y, u) = \sum\limits_{i,j} a_{ij}y^iu^j\), we define \(h(y, u)_{\text{deg } y > 0} = \sum\limits_{i,j} a_{ij}y^iu^j\). Let

\[
\delta(f) = \begin{cases} 
\min \{ j \mid y^iu^j \in h(y, u), i \neq 0 \} & \text{if } h(y, u)_{\text{deg } y > 0} \neq 0, \\
n(f) - 1 & \text{otherwise}.
\end{cases}
\]

If \(\delta(f) = n(f) - 1\), then after replace \(z\) by \(z - uh(u)\) the equation becomes a normal form and we have done. Now assume that \(\delta(f) < n(f) - 1\). Let \(z' = z + uh(y, u)\) and \(f'(x, y, z', u)\) be a sub-normal form of \(f(x, y, z, u)\). One may write \(f'(x, y, z', u) = x^2 + y^2z' + \lambda yu^k + \mu u^n + y^2uh'(y, u) + g'(z', u)\), where

\[
y^2uh'(y, u) = \sum\limits_{i \neq 0,j} b_{ij}(-uh(y, u)_{\text{deg } y > 0})^iw^j - \left( \sum\limits_{i \neq 0,j} b_{ij}(-uh(y, u)_{\text{deg } y > 0})^iw^j \right)_{\text{deg } y \leq 1; \text{deg } u \geq n(f')}
\]

if \(g(z, u) = \sum\limits_{i,j} b_{ij}z^iw^j\). It follows that \(n(f') \leq n(f)\) and \(\delta(f') \geq \delta(f)\) if \(\delta(f') \neq n(f') - 1\). Thus by induction on the integer \(n(f) - \delta(f)\) one can show that every isolated \(cD\) singularity admits a normal form.

\[\Box\]

**Lemma 5.2.** Assume that \(W\) has a \(cD_4\) singularity such that there is only one exceptional divisor of discrepancy one over \(W\). Let \(W_1\) be the unique \(w\)-morphism over \(W\). Then \(W_1\) has a non-Gorenstein singularity of type \(\frac{1}{2}(1, 1, 1)\). The other singularities of \(W_1\) may be \(cD_1\), \(cA_1\) or \(cA_2\). Furthermore, assume that there exists a non-\(\mathbb{Q}\)-factorial Gorenstein point \(P\) of type \(cD_4\) on \(W_1\), then there is only one exceptional divisor of discrepancy one over \(P\).

**Proof.** We may assume that \(W\) is defined by \(f(x, y, z, u) = x^2 + y^2z + \lambda yu^k + g(z, u)\). Let \(f_3(y, z, u)\) be the homogeneous part of degree 3 of \(f\). As discussed in [4, Proposition 15], we have that a \(w\)-morphism over \(W\) can be obtained by the weighted blow-up with following weights:

(i) \(f_3(y, z, u)\) is irreducible. One can define \(W_1\) to be the weighted blow-up with weight \((2, 1, 1, 1)\). The only non-Gorenstein singularity of \(W_1\) is the origin of the chart \(U_x\), which is a \(\frac{1}{2}(1, 1, 1)\) singularity.

(ii) \(k > 2\) and \(f_3(y, z, u)\) is reducible. After a suitable change of coordinates we may assume that \(z^3 \in g(z, u)\). One can weighted blow-up \((2, 1, 2, 1)\). In this case there is a \(cA_2\) singularity on the origin of the chart \(U_z\), which is locally defined by

\[
(x_1^2 + y_1^2 + z_1^2 + u_1^{2k}) = 0 \subset A_4^{x_1,y_1,z_1,u_1}/\frac{1}{2}(0, 1, 1, 1),
\]

for some \(k > 0\). Let \(F\) be the exceptional divisor which is obtained by weighted blowing-up \(W_1\) with weight \(w\) such that \(w(y_1 + z_1, y_1 - z_1, u_1, x_1) = \frac{1}{2}(1, k - 1, k, 2)\). Then \(F\) is also an exceptional divisor of discrepancy one over \(W\), which contradicts to our assumption. Thus this case won’t happen.

(iii) \(k = 2\) and \(f_3(y, z, u)\) is reducible. Let \(W_1\) be the weighted-blow-up with weight \((2, 2, 1, 1)\). Then there is a non-Gorenstein singularity on \(W_1\) which is locally defined by

\[
(x_2^2 + y_2z_2 + u_2^2) = 0 \subset A_4^{x_2,y_2,z_2,u_2}/\frac{1}{2}(2, 1, 1, 1).
\]

Let \(F\) be the exceptional divisor obtained by weighted blow-up \(\frac{1}{2}(2, 1, 1, 1)\). One can compute that \(a(W, F) = 1\), hence this case won’t happen.
From now on we assume that \( f_3(y, z, u) \) is irreducible and \( W_1 \) is obtained by weighted blowing-up \((2, 1, 1, 1)\). One can check that every Gorenstein singular point on \( W_1 \) is contained in \( U_z \). Fix a singular point \( P \in W_1 \). We may assume that \( P \) is the origin of the chart \( U_z \) and the local defining equation is \( y^2 + x^2z + g'(z, u) \) with mult \( g'(z, u) \leq 3 \). Hence \( P \) is a \( cA_1 \), a \( cA_2 \) or a \( cD_4 \) point.

Assume that \( P \) is a \( cD_4 \) point. Note that in this case we have \( u^3 \in f_3(y, z, u) \) and hence the degree 3 part of \( y^2 + x^2z + g'(z, u) \) is irreducible. Thus there are only one discrepancy one exceptional divisor over \( P \) by the discussion above. \( \square \)

Because \( W_1 \) may have \( cA_2 \) singularities, smooth flops over \( cA_2 \) points may appear in the factorization of a type \( D \) simple smooth flop. Luckily, the factorization of a smooth flop over a \( cA_2 \) point is similar to the factorization of a \( D \)-type simple smooth flop. We will construct the factorization of this two kinds of flops at the same time.

**Proposition 5.3.** Notation as in Convention \[3.16\]. Assume that \( X \dashrightarrow X' \) is a smooth flop over \( W \) such that either

(i) \( X \dashrightarrow X' \) is simple and \( W \) has a \( cD \) singularity, or

(ii) \( W \) has a \( cA_2 \) singularity.

Then \( Y_{(0,1)} \rightarrow Y_{(1)} \rightarrow X \) is a sequence of blowing-up smooth curves, \( Y_{(0,0)} \rightarrow Y_{(0,1)} \) is an Atiyah flop, and \( Y_{(0,0)} \rightarrow Y_{(0)} \) is a blowing-up a \( \frac{1}{2}(1, 1, 1) \) point.

On the other hand, if \( Y \not\cong Y' \), then \( Y \dashrightarrow Y' \) is a flop which is either an \( A \)-type simple smooth flop or satisfies one of the condition (i), (ii) above.

**Proof.** By Lemma \[5.2\] in case (i) or by direct computation in case (ii), we know that \( Y_{(0)} \) has only one non-Gorenstein singular point, which is of type \( \frac{1}{2}(1, 1, 1) \). Notice that this point is the only singular point of \( Y_{(0)} \) because \( \text{deg}_{Gor}(X) = 0 \) implies that \( \text{deg}_{Gor}(Y_{(0)}) = 0 \) by Corollary \[2.12\]. The factorization now follows from Lemma \[4.9\]. The last statement follows from Lemma \[5.2\] (2). \( \square \)

6. **Factorizing \( E \)-type simple smooth flop**

6.1. **Resolving \( cE \) singularities.** Let \( W = (x^2 + y^3 + yg_{\geq 3}(z, u) + h_{\geq 4}(z, u) = 0) \subset \mathbb{A}^4 \) be a \( cE \) singularity. We have

(1) \( h_4(z, u) \neq 0 \) if \( X \) has a \( cE_6 \) singularity.

(2) \( g_3(z, u) \neq 0 \) and \( h_4(z, u) = 0 \) if \( W \) has a \( cE_7 \) singularity.

(3) \( g_3(z, u) = h_4(z, u) = 0 \) and \( h_5(z, u) \neq 0 \) if \( W \) has a \( cE_8 \) singularities.

We may assume that \( u^4 \not\in h(z, u) \) (resp. \( u^5 \not\in h(z, u) \)) if \( W \) is \( cE_6 \) (resp. \( cE_8 \)).

**Lemma 6.1.** Let \( u_k(x, y, z, u) = (a, b, c, 1) \) be a weight such that after weighted blowing-up \( W \) with weight \( u_k \) we get a \( w \)-morphism. Here \( k = a + b + c - 1 \) equals to the weight of the defining equation of \( X \). Assume that there is only one exceptional divisor with discrepancy one over \( X \), then

(1) If \( 2a = 3b = k \), then \( (a, b, k) = (3, 2, 6) \).

(2) If \( k > 3 \), then neither \( 2a - 1 = 3b = k \) nor \( 3b - 1 = 2a = k \).

**Proof.** Let \( W' \rightarrow W \) be the weighted blow-up with weight \( w \) and let \( E \) be the exceptional divisor. We have \( a(E, W) = 1 \). We want to say that if (1) or (2) is not true, then there exists an exceptional divisor \( F \) over \( W \) such that \( a(F, W) = 1 \), which contradicts to our assumption.

Assume first that (1) is not true, so that \( 2a = 3b = k \) but \( k > 3 \). In this case we have \( a = 3d \) and \( b = 2d \) for some \( d > 1 \). Consider the chart \( U_g' \subset W' \) is defined by
Let \( W'' \to W' \) be the weighted blow-up \( \frac{1}{a}(d-1, 1, 1) \). One can take \( F = \text{exc}(W'' \to W) \).

Now assume that (2) is not true. If \( 3b-1 = 2a = k \), then the origin of the chart \( U'_y \) on \( W' \) is a cyclic quotient point of type \( \frac{1}{a}(a, c, 1) \). Note that \( a+c = k+1-b = 3b-b = 2b \) and \( a > b \) since \( 3b-1 = 2a \). Let \( W'' \to W' \) be the weighted blow-up with weight \( \frac{1}{a}(a-b, c, 1) \) and we can take \( F = \text{exc}(W'' \to W) \).

Finally assume that \( 2a-1 = 3b = k \). The origin of the chart \( U'_x \) on \( W' \) is a cyclic quotient point of type \( \frac{1}{a}(b, c, 1) \). Notice that we have \( b+c = k+1-a = a \) and \( 2b = \frac{1}{a}(2a-1) > a \) because \( k > 3 \) implies \( a > 2 \). Let \( W'' \to W' \) be the weighted blow-up with weight \( \frac{1}{a}(2b-a, 2c, 2) \) and let \( F \) be the exceptional divisor. One can compute that

\[
a(F, W) \leq \frac{1}{a}(3(2b-a)+2) = \frac{1}{a}(2(2a-1)-3a+2) = 1,
\]

hence \( a(W, F) = 1 \).

Assume that \( W \) is a \( cE_n \) singularity for \( n = 6, 7 \) or \( 8 \). We assume that \( W \) is defined by \( x^2 + y^3 + yg_{\geq 3}(z, u) + h_{\geq 4}(z, u) \). By [4], Theorem 34, 36, 37] we know that there exists a weight \( w_k \) in the following set

\[
\begin{align*}
w_4 &= (2, 2, 1, 1)_{n=6}, \quad w_5 = (3, 2, 1, 1)_{n=6,7}, \quad w_6 = (3, 2, 2, 1), \quad w_8 = (4, 3, 2, 1), \\
w_9 &= (5, 3, 2, 1)_{n=7,8}, \quad w_{12} = (6, 4, 3, 1), \quad w_{14} = (7, 5, 3, 1)_{n=7,8}, \\
w_{18} &= (9, 6, 4, 1)_{n=7,8}, \quad w_{24} = (12, 8, 5, 1)_{n=8}, \quad w_{30} = (15, 10, 6, 1)_{n=8}
\end{align*}
\]

such that the weighted blow-up with weight \( w_k \) gives a \( w \)-morphism \( W' \to W \).

**Lemma 6.2.** Notation as above. Assume that \( W \) has only one exceptional divisor with has discrepancy one, then \( W' \to W \) is obtained by weighted blowing-up \( w_{n-2} \). Moreover, if \( n = 6 \) then \( h_4(z, u) \) is not a perfect square. If \( n = 7 \), then either \( u^3 \in g(z, u) \) or \( u^5 \in h(z, u) \).

**Proof.** By Lemma [6,1] we know that if \( W' \to W \) is obtained by weighted blowing-up the weight \( w_k \), then \( k \leq 6 \). This proves the case when \( n = 8 \).

Let \( E = \text{exc}(W' \to W) \). Assume that \( n = 7 \). We want to show that if \( W' \to W \) is obtained by weighted blowing-up \( w_6 \), then there exists an exceptional divisor \( F \neq E \) such that \( a(F, W) = 1 \). We know that \( W \) is defined by \( x^2 + y^3 + yz^3 + yg_{\geq 3}(z, u) + h_{\geq 5}(z, u) \). The chart \( U'_x \subset W' \) is defined by

\[
(x^2 + y^3 + yz^3 + yg_{\geq 2}(z, u) + h_{\geq 4}(z, u) = 0) \subset \mathbb{A}^4_{(x, y, z, u)}/\frac{1}{2}(1, 0, 1, 1),
\]

which is a \( cD/2 \) point.

(i) \( \text{mult } h'(z, u) \geq 6 \). Let \( W'' \to W' \) be the weighted blow-up with weight \( \frac{1}{2}(3, 4, 1, 1) \) and let \( F = \text{exc}(W'' \to W) \).

(ii) \( \text{mult } h'(z, u) = 4 \) and \( u^4 \notin h'(z, u) \). Note that after suitable change of coordinate we may assume that \( u^2 \notin g'(z, u) \). Let \( W_1 \to W' \) be the weighted blow-up with weight \( \frac{1}{2}(3, 2, 3, 1) \). The chart \( U_{1, z} \subset W_1 \) is defined by

\[
x^2 + y^3 + yz + g_1(y, z, u) = 0 \subset \mathbb{A}^4_{(x, y, z, u)}/\frac{1}{3}(3, 2, 1, 1).
\]
Let $W_2 \to W_1$ be the weighted blow-up $\frac{1}{2}(3,2,1,1)$ (resp. $\frac{1}{3}(3,5,1,1)$) if $u^3 \in g_1(y,z,u)$ (resp. $u^3 \not\in g_1(y,z,u)$). One can see that either $F_1 = \text{exc}(W_1 \to W')$ or $F_2 = \text{exc}(W_2 \to W_1)$ has discrepancy one over $W$. Let $F$ be this exceptional divisor.

(iii) $u^3 \in h'(z,u)$. Let $W'' \to W'$ be the weighted blow-up with weight $\frac{1}{2}(3,2,1,1)$ and let $F = \text{exc}(W'' \to W)$.

Since there always exists an exceptional divisor $F$ over $W$ such that $a(F,W) = 1$, we get a contradiction. Thus when $n = 7$, $W' \to W$ should be obtained by weighted blowing-up $w_5$. It is easy to see that $w_5$ defines a $w$-morphism implies either $u^3 \in g(z,u)$ or $u^5 \in h(z,u)$.

Finally assume that $n = 6$ and we are going to show that $W' \to W$ is obtained by weighted blowing-up $w_4$. Equivalently, we want to say that if $W' \to W$ is obtained by weighted blowing-up $w_6$ or $w_5$, then there exists a exceptional divisor $F$ over $W$ such that $a(F,W) = 1$.

(a) $W' \to W$ is obtained by weighted blowing-up $w_6$. We have that $W$ is defined by $x^2 + y^3 + z^4 + g(y,z,u)$. The chart $U'_x \subset W'$ is defined by

$$\{x^2 + y^3 + z^2 + g'(y,z,u) = 0\} \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{2}(1,0,1,1),$$

which is a $CA/2$ point. For a suitable change of coordinates we may write the defining equation as $(x + iz)(x - iz) + g_1(y,u)$. Define a weight $w'(x + iz, x - iz, u, y) = (1, 2m - 1, 1, 2)$, where $m = \text{wt}_{w'}(g_1(y,u))$. Let $W'' \to W'$ be the weighted blow-up with weight $w'$, then $F = \text{exc}(W'' \to W')$ satisfies that $a(F,W) = 1$.

(b) $W' \to W$ is obtained by weighted blowing-up $w_5$. In this case $W$ is defined by

$$x^2 + xq(z,u) + y^3 + g(y,z,u)$$

and the weight of the defining equation of $W$ is 5. Note that we have $q(z,u)$ is a non-zero homogeneous polynomial of degree 2. The chart $U'_x \subset W'$ is a cyclic quotient point of type $\frac{1}{2}(2,1,1)$. Let $W'' \to W'$ be the weighted blow-up with weight $\frac{1}{3}(2,1,1)$, then $F = \text{exc}(W'' \to W')$ satisfies $a(F,W) = 1$.

Hence $W' \to W$ is defined by $w_4$. One can easily see that $w_4$ is a $w$-morphism only when $h_4(z,u)$ is not a perfect square. □

6.2. Factorizing $cE_6$ flops. In this subsection we assume that $X \dashrightarrow X'$ is a simple smooth flop over $W$, such that $W$ has a $cE_6$ singularity. As before we let $W_1 \to W$ be a $w$-morphism, let $Y \to W_1$ be a $\mathbb{Q}$-factorization of $W_1$ and construct a diagram as in Convention [6.10]

Lemma 6.3.

(1) $W_1$ has only one singular point, which is a depth three $CAx/2$ point.

(2) $Y \dashrightarrow Y'$ is a simple flop over the $CAx/2$ point. $Y$ has a $CA$ point which is of depth 2. There are no other singular point on $Y$.

Proof. By Lemma 6.2 we know that $W_1 \to W$ is obtained by weighted blowing-up the weight $(2,2,1,1)$. One can check that there is only one non-Gorenstein point $P \in W_1$, which is a $CAx/2$ point defined by

$$\{x^2 + y^2 + h'(y,z,u) = 0\} \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{2}(0,1,1,1)$$

such that mult $h'(y,z,u) = 4$ and $h'(0,z,u)$ is not a perfect square (the latter statement follows from Lemma 6.2). The only $w$-morphism $W_2 \to W_1$ over this point is given by weighted blowing-up the weight $\frac{1}{4}(2,3,1,1)$ ([8 Section 8]). An easy computation shows that the only non-Gorenstein point on $W_2$ is a $\frac{1}{4}(2,1,1)$ point, hence $\text{dep}(W_2) = 2$ and
Lemma 6.11. It follows that the birational map $Z \dashrightarrow W$ such that $D$ has at worst a type flop. We know that $W$ is smooth, $Y$ is a flip over $W$. Note that $Y$ has no Gorenstein singular points since $0 \leq \text{dep}_\text{Gor}(Y) \leq \text{dep}_\text{Gor}(X) = 0$. Hence $W_1$ does not have any Gorenstein singular point other than $P$.

Claim. $\text{dep}(Y) > 1$.

To prove the claim, assume that $\text{dep}(Y) = 1$. We have that $Y \dashrightarrow Y(1)$ can be factorize as in Lemma 6.9. Hence there exists a sequence of blowing-up smooth curves $Y_{(0,1)} \rightarrow Y(1) \rightarrow X$, so that there exists an Atiyah flop $Y_{(0,1)} \rightarrow Y_{(0,0)}$. However since $W$ has a $cE_6$ singularity, the normal bundle sequence is of length 4, this leads a contradiction.

Since $\text{dep}(Y) < \text{dep}(W_1) = 3$ by Lemma 3.9, we have $\text{dep}(Y) = 2$. Since $Y \rightarrow W_1$ is a flop over a $cA_2$ point, the singular point of $Y$ is a $cA_2$ point described in Remark 2.10.

Lemma 6.4. The flop $Y \dashrightarrow Y'$ in Lemma 6.3 can be factorize into

\[ Y \leftarrow Z(1) \leftarrow Z(0) \rightarrow Z'(0) \rightarrow Z'(1) \rightarrow Y', \]

such that $Z(0) \rightarrow Z(1)$ and $Z'(0) \rightarrow Z(1)$ are flips in Lemma 4.10 (1), $Z(0) \rightarrow Z'(0)$ is at worst a $D$-type smooth flop, and other arrows are $w$-morphisms.

Proof. Let $W_2 \rightarrow W_1$ be a $w$-morphism over the $cA_2$ point. Form the computation in Lemma 6.11 we know that the non-Gorenstein point on $W_2$ is a $\frac{1}{3}(2,1,1)$ point and $W_2$ has at worst $cD$ Gorenstein singularities. Hence $\text{dep}(Z(0)) = 2$ and $Z(0) \rightarrow Z'(0)$ is at worst a $D$-type flop. We know that $\text{dep}(Y) = 2$ and $Y \dashrightarrow Y'$ is a non-Gorenstein flop by Lemma 6.11. It follows that the birational map $Z(0) \dashrightarrow Y$ is a flop $Z(0) \dashrightarrow Z(1)$ followed by a $w$-morphism $Z(1) \rightarrow Y$. One has that $\text{dep}(Z(1)) = 1$ and $Z(0) \rightarrow Z(1)$ is given by Lemma 4.10 (1). This proves the lemma.

Proposition 6.5. Assume that $X \dashrightarrow X'$ is a simple smooth flop over $W$ such that $W$ has a $cE_6$ singularity. Then we have a factorization

\[ X \leftarrow Y(1) \leftarrow Y(0) = Y \dashrightarrow Y' = Y(0) \dashrightarrow Y'(1) \rightarrow X', \]

such that $Y(0) \rightarrow Y(1)$ is given by Lemma 4.10 (3), $Y \dashrightarrow Y'$ is given by Lemma 6.4, and all other maps are blowing-up smooth curves. The diagram of $Y' \dashrightarrow X'$ is symmetric.

Proof. By Lemma 6.3 we know that $\text{dep}(Y) = 2$. Since $Y$ has $cA_2$ singularities and $Y(1)$ is smooth, $Y(0) \rightarrow Y(1)$ is given by Lemma 4.10 (3).

6.3. Factorizing $cE_7$ flops. In this subsection we assume that $X \dashrightarrow X'$ is a simple smooth flop over $W$, such that $W$ has a $cE_7$ singularity.

Lemma 6.6. $Y \dashrightarrow Y'$ is a smooth flop over a $cA_2$ point. There are two singular points on $Y$. One of them is a $\frac{1}{3}(2,1,1)$ point and the other one is a $\frac{1}{2}(1,1,1)$ point. Moreover, the flipping curve of $Y = Y(0) \dashrightarrow Y(1)$ passes through the both singular points.

Proof. Locally $W$ is defined by $x^2 + y^3 + z^3 + g(y,z,u)$. By Lemma 6.2 we know that $W_1 \rightarrow W$ is obtained by weighted blowing-up the weight $(3,2,1,1)$. It is easy to see that there are two non-Gorenstein points, one of them is a $\frac{1}{3}(2,1,1)$ point and the other one is a $\frac{1}{2}(1,1,1)$ point. Note that the chart $U_y \subset W_1$ is defined by

\[ (y(x^2 + 1) + z^3 + g'(y,z,u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{2}(1,0,1,1). \]
One can see that the point \( P = (\pm 1, 0, 0, 0) \) is a \( cA_2 \) point. Since \( \text{dep}_{\text{Gor}}(Y) \leq \text{dep}_{\text{Gor}}(X) = 0 \), \( P \) is not \( \mathbb{Q} \)-factorial and \( Y \rightarrow Y' \) should be a smooth flop over \( P \).

Let \( H_W = (u = 0) \subset W \) be a Du Val section. Then \( H_{W_1} \) passes through the non-\( \mathbb{Q} \)-factorial point of \( W_1 \), hence \( H_Y \) contains flopping curves of \( Y \rightarrow Y' \). Let \( C_Y \) be the flipping curve of \( Y = Y(0) \rightarrow Y(1) \). Then \( C_Y \) intersects a flopping curve non-trivially, hence \( C_Y \) intersects \( H_Y \) non-trivially at a smooth point. Now there is an index \( I = (0, ..., 0) \) and a sequence of \( w \)-morphisms

\[
Y_I \rightarrow \cdots \rightarrow Y(0,0) \rightarrow Y(0) = Y
\]

such that \( K_{Y_I}.C_{Y_I} = 0 \), hence \( H_{Y_I}.C_{Y_I} = 0 \). We also know that \( H_{Y_I} \) intersects \( C_{Y_I} \) non-trivially, hence \( C_{Y_I} \subset H_{Y_I} \), which means that \( C_Y \subset H_Y \). Now \( C_{W_1} \subset H_{W_1}.E \) and \( H_{W_1}.E = (z = u = 0) \subset \mathbb{P}(3,2,1,1) \) is irreducible where \( E = \text{exc}(W_1 \rightarrow W) \). Hence \( C_{W_1} = H_{W_1}.E \) and one can see that \( C_{W_1} \) contains the two non-Gorenstein points. Hence \( C_Y \) contains the two non-Gorenstein points.

\[\square\]

**Lemma 6.7.** Let \( Y \) be a terminal threefold contains two \( \frac{1}{2}(1,1,1) \) points. Assume that \( Y \rightarrow U \) is a birational map contracting \( C \cong \mathbb{P}^1 \) to a point such that \(-K_Y \) is nef over \( U \) and the two \( \frac{1}{2}(1,1,1) \) points are contained in \( C \). Then \(-K_Y.C = 0 \). Moreover, assume that there exists a flop \( Y \rightarrow Y' \) over \( U \) such that the general elephant \( H_U \in \left| -K_U \right| \) has \( A \)-type singularities, then we have the following factorization

\[
Y \leftarrow Z(1) \leftarrow Z(0) = Z'(0) \rightarrow Z'(1) \rightarrow Y',
\]

where \( Z(1) \rightarrow Y \) as well as \( Z'(1) \rightarrow Y' \) are \( w \)-morphisms and \( Z(0) \rightarrow Z(1) \) as well as \( Z'(0) \rightarrow Z'(1) \) has a factorization given in Lemma 4.10 (1).

**Proof.** Let \( P_1 \) and \( P_2 \) be the two cyclic quotient points. We have that \( Y \rightarrow U \) can not be a flipping contraction because \( \sum_{i=1,2} w_{P_i}(0) \geq 1 \) (please see [20 (2.3), Theorem 4.9]). Hence it is a flopping contraction.

Now assume that we have a flop \( Y \rightarrow Y' \) over \( Q \subset U \) and assume that a general elephant near \( Q \) has \( A \)-type singularities. Then \( Q \) is a \( cA/2 \) point. There are exactly two discrepancy \( \frac{1}{2} \) exceptional divisors and one discrepancy 1 exceptional divisor over \( Q \subset U \).

By [2 Proposition 3.4] we may assume that \( Q \) is defined by

\[
(xy + z^4 + u^2 + g(z,u) = 0) \subset A^4_{(x,y,z,u)}\left(\frac{1}{2}(1,1,1,0)\right).
\]

Let \( Z = Z' \) be the weighted blow-up with the weight \( \frac{1}{2}(3,1,1,2) \), then the only singular point on \( Z \) is a \( \frac{1}{2}(1,1,2) \) point. Hence \( \text{dep}(Z) = 2 \).

On the other hand, \( Z(\kappa) \rightarrow Y \) is a \( w \)-morphism, so \( \text{dep}(Z(\kappa)) = \text{dep}(Y) - 1 = 1 \). This implies that \( k = 1 \). The factorization of \( Z(0) \rightarrow Z(1) \) is given by Lemma 4.10 (1). \[\square\]

**Proposition 6.8.** Assume that \( X \rightarrow X' \) is a simple smooth flop over \( W \) such that \( W \) has a \( cE_7 \) singularity. Then there exists a factorization

\[
X \leftarrow \bar{Y} \leftarrow \tilde{Y} \leftarrow Y(0,2) \leftarrow Y(0,1) \leftarrow Y(0,0) \rightarrow Y(0) = Y \rightarrow Y' = Y'(0) \leftarrow Y'(0,0)
\]

\[
\rightarrow Y'(0,1) \rightarrow Y'(0,2) \rightarrow \tilde{Y}' \rightarrow Y' \rightarrow X'
\]

such that \( \tilde{Y} \rightarrow \bar{Y} \rightarrow X \) is a sequence of blowing-up smooth curves, \( Y(0,1) \rightarrow Y(0,2) \) and \( Y(0,2) \rightarrow \tilde{Y} \) are given in Lemma 4.9, \( Y(0,0) \rightarrow Y(0,1) \) is a flop given in Lemma 6.7 and \( Y \rightarrow Y' \) is a smooth flop over a \( cA_2 \) point. The diagram of \( Y' \rightarrow X' \) is symmetric.

**Proof.** By Lemma 6.6 there are two cyclic quotient points on \( Y(0) \) which have of indices 2 and 3 respectively and the flopping curve \( C_Y \) of \( Y = Y(0) \rightarrow Y(1) \) passes through the both singular point. By the classification [17 Theorem 2.2] we know that we are in the case semistable \( IA + IA \), hence a general elephant \( H_W \in \left| -K_W \right| \) has \( A \)-type singularities.
Now $Y_{(0,0)}$ contains two $\frac{1}{2}(1,1,1)$ point. Since $H_W$ has $A$-type singularities, $H_{U_{(0,0)}}$ has $A$-type singularities. Thus by Lemma 6.7 we know that $Y_{(0,0)} \to Y_{(0,1)}$ is a flop and its factorization is given by Lemma 6.7.

**Claim.** We have $k_{(0)} \geq 2$. There exists a flip $Y_{(0,2)} \to \hat{Y}$ and a sequence of blowing-up smooth curves $\hat{Y} \to Y \to X$.

To prove the claim, assume first that $k_{(0)} = 1$. In this case we have a divisorial contraction to a curve $Y_{(0,1)} \to Y_{(1)}$ (Remark 3.4). Let $\Gamma \subset Y_{(1)}$ be this curve. Then there are two singular fibers over $\Gamma$ because there are two singular points on $Y_{(0,1)}$ which are connected by the flopped curve of the flop $Y_{(0,0)} \to Y_{(0,1)}$, and the flopped curve maps bijectively to $\Gamma$. On the other hand, we have $\text{dep}(Y_{(1)}) < \text{dep}(Y_{(0)}) = 3$, hence $\text{dep}(Y_{(1)}) \leq 2$. This implies that $Y_{(1)}$ contains two depth 1 points, which are $\frac{1}{2}(1,1,1)$ points. However there is no divisorial contraction to a curve which contains cyclic quotient points by Theorem 5. This proves that $k_{(0)} \geq 2$.

Now assume that $k_{(0)} = 2$. There is a flip followed by a divisorial contraction $Y_{(0,1)} \to Y_{(0,2)} \to Y_{(1)}$. Notice that if $\text{dep}(Y_{(1)}) = 1$ then the $K_Y$-MMP over $W$ is given by

$$Y = Y_{(0)} \to Y_{(1)} \to Y_{(2)} \to X$$

such that $Y_{(1)} \to Y_{(2)}$ is given by Lemma 4.9 and $Y_{(2)} \to X$ is a blowing-up a smooth curve on $X$. As shown in the proof of Lemma 6.3 in this case the normal bundle sequence of the flop $X \to X'$ has only length three. This leads to a contradiction. Hence $\text{dep}(Y_{(1)}) \neq 1$ and we have $\text{dep}(Y_{(1)})$ is either 0 or 2. Assume that $\text{dep}(Y_{(1)}) = 0$, then $Y_{(1)}$ is smooth. Hence $Y_{(0,2)}$ is also smooth and $Y_{(0,2)} \to Y_{(1)}$ is divisorial contraction to a curve because the exceptional divisor of this divisorial contraction has discrepancy two over $X$, hence discrepancy one over $Y_{(1)}$. This implies that the two singular points of $Y_{(0,1)}$ are both contained in the flipping curve of $Y_{(0,1)} \to Y_{(0,2)}$. However it is impossible since the both two singular points are $\frac{1}{2}(1,1,1)$ points and there are no flip along two $\frac{1}{2}(1,1,1)$ points by Lemma 6.7. So $\text{dep}(Y_{(1)})$ should be 2.

In this case we have $\text{dep}(Y_{(0,2)}) = 1$ and $Y_{(0,2)} \to Y_{(1)}$ is a $w$-morphism. In other words, we have $Y_{(0,2)} = Y_{(1,0)}$. Let $\hat{Y} = Y_{(1,1)}$ and $\tilde{Y} = Y_{(2)}$. We know that $Y_{(2)}$ can not has depth one or the length of the normal bundle sequence should be 3. Thus $Y_{(2)}$ is smooth and so $\hat{Y} = Y_{(2)} \to X$ is a blowing-up a smooth curve. Since $\text{dep}(Y_{(0,2)}) = \text{dep}(Y_{(1,0)}) = 1$, $Y_{(1,1)}$ is also smooth and $Y_{(1,1)} \to Y_{(2)}$ is also a blowing-up a smooth curve.

Finally if $k_{(0)} > 2$ then it is easy to see that $k = 3$ and $\text{dep}(Y_{(i)}) = 3 - i$ for $i = 2, 3$. As we saw before, $\text{dep}(Y_{(1)})$ can not be one. Hence $\text{dep}(Y_{(1)}) = 0$. This implies that there exists a divisorial contraction $Y_{(1)} \to X$ which is a blowing-up a smooth curve. Let $F = \text{exc}(Y_{(0,3)} \to Y_{(1)})$. One can check that $a(F, X) = 2$, hence $a(F, Y_{(1)}) = 1$ and so $Y_{(0,3)} \to Y_{(1)}$ is also a blowing-up a smooth curve. We let $\hat{Y} = Y_{(0,3)}$ and $\tilde{Y} = Y_{(1)}$ and the claim is proved.

Notice that $Y_{(0,i)}$ has only $\frac{1}{2}(1,1,1)$ singularities for $i = 1, 2$. Hence the factorization of the two flops $Y_{(0,1)} \to Y_{(0,2)}$ and $Y_{(0,2)} \to \hat{Y}$ are given by Lemma 4.9. $Y \to Y'$ is a smooth flop over a $cA_2$ point by Lemma 6.6. 

**6.4. Factorizing $cE_8$ flops.** In this subsection we assume that $X \to X'$ is a simple smooth flop over $W$, such that $W$ has a $cE_8$ singularity.

**Lemma 6.9.** There exists a sequence of $w$-morphisms $W_8 \to \cdots W_1 \to W_0 = W$ such that $W_8$ is smooth. $W_1$ has a non-$\mathbb{Q}$-factorial $cE/2$ singular point, $W_2$ has a non-$\mathbb{Q}$-factorial $cD/3$ singular point, $W_3$ has a $cAx/4$ singular point, $W_4$ has a $\frac{1}{4}(3, 2, 1)$ point, and $W_5$ has a $\frac{1}{3}(1, 2, 1)$ and a $\frac{1}{2}(1, 1, 1)$ point. $W_6$ and $W_7$ is obtained by resolving the $\frac{1}{3}(1, 3, 1)$
singularity and \( W_8 \) is the blowing-up the \( \frac{1}{2}(1, 1, 1) \) point. Moreover, let \( E_i = \text{exc}(W_i \to W_{i-1}) \), then one can compute \( a(E_j, W_i) \) as in the following table:

|   | \( W_1 \) | \( W_2 \) | \( W_3 \) | \( W_4 \) | \( W_5 \) | \( W_6 \) | \( W_7 \) |
|---|---|---|---|---|---|---|---|
| \( E_1 \) | 1 | - | - | - | - | - | - |
| \( E_2 \) | 2 | \( \frac{1}{2} \) | - | - | - | - | - |
| \( E_3 \) | 3 | 1 | \( \frac{1}{3} \) | - | - | - | - |
| \( E_4 \) | 4 | \( \frac{3}{2} \) | \( \frac{2}{3} \) | \( \frac{1}{4} \) | - | - | - |
| \( E_5 \) | 5 | 2 | 1 | \( \frac{1}{2} \) | \( \frac{1}{3} \) | - | - |
| \( E_6 \) | 2 | 1 | \( \frac{2}{3} \) | \( \frac{1}{2} \) | \( \frac{2}{5} \) | \( \frac{1}{3} \) | - |
| \( E_7 \) | 4 | 2 | \( \frac{3}{2} \) | 1 | \( \frac{4}{5} \) | \( \frac{2}{3} \) | \( \frac{1}{2} \) |
| \( E_8 \) | 3 | \( \frac{3}{2} \) | 1 | \( \frac{3}{4} \) | \( \frac{3}{5} \) | \( \frac{1}{2} \) | \( \frac{1}{2} \) |

**Proof.** We may assume that \( W \) is defined by \( x^2 + y^3 + z^5 + g(y, z, u) \) such that \( \frac{\partial^2}{\partial y \partial u} g(y, z, u) = 0 \) and \( u^5 \not\in g(y, z, u) \). By Lemma 6.2 we know that \( W_1 \to W \) is obtained by weighted blowing-up \( (3, 2, 2, 1) \). The only non-Gorenstein singular point on \( W_1 \) is a \( cE/2 \) point. If this point is \( \mathbb{Q} \)-factorial, then the only non-Gorenstein point on \( Y \) is this \( cE/2 \) point, so the flipping curve of \( Y = Y_0 \to Y_1 \) should pass through this point. However it is impossible because of the classification \([17], \text{Theorem } 2.2\). So the \( cE/2 \) point is not \( \mathbb{Q} \)-factorial. In this case \( W_1 \) do not have any Gorenstein singular point since \( \text{deg}_{\text{Gor}}(Y) \leq \text{deg}_{\text{Gor}}(X) = 0 \). In particular, we have that at least one of the following monomials

\[ yu^4, zu^4, u^6, u^7 \]

appears in \( g(y, z, u) \).

Now the chart \( U_{z,1} \subset W_1 \) is defined by

\[ (x^2 + y^3 + z^4 + g(y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{2}(1, 0, 1, 1) \]

and at least one of the following following monomials

\[ yu^4, u^6, zu^7 \]

appears in \( g'(y, z, u) \). Assume that \( u^4 \in g'(y, z, u) \), we choose a suitable change of coordinates \( z \to z + \lambda u \) so that \( u^4 \not\in g'(y, z + \lambda u, u) \), and construct the weighted blow-up with weight \( \frac{1}{2}(3, 2, 3, 1) \). Under this construction we get a \( w \)-morphism \( W' \to W_1 \) and one can see that the exceptional divisor of \( W' \to W_1 \) has discrepancy one over \( W \). This leads a contradiction since there should be only one discrepancy one exceptional divisor over \( W \). Hence \( u^4 \not\in g'(y, z, u) \).

Now let \( W_2 \to W_1 \) be the weighted blow-up with weight \( \frac{1}{2}(3, 2, 3, 1) \). The only non-Gorenstein singular point on \( W_2 \) is the origin of the chart \( U_{z,2} \subset W_2 \) which is defined by

\[ (x^2 + y^3 + z^3 + g''(y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{3}(0, 2, 1, 1), \]

such that \( yu^4, u^6 \) or \( z^2 u^7 \) \( \in g''(y, z, u) \). Let \( w(y, z, u) = \frac{1}{3}(2, 4, 1) \) be a weight. The monomials with \( w \)-weights less than twelve are the following:

(a) \( w \)-weight equals to 6: \( yu^4, u^6 \) and \( zu^2 \).
(b) \( w \)-weight equals to 9: \( yz^3, yu^7, z^4 u, zu^5 \) and \( u^9 \). Notice that \( yu^7 \) and \( u^9 \) cannot appear in \( g''(y, z, u) \) because \( g''(y, z, u) = g'(yz, z^2, z^2 u)/z^3 \).
One can see that

(i) If $\text{wt}_u g''(y, z, u) < 12$, then the weighted blow-up with weight $\frac{1}{4}(3, 2, 4, 1)$ defines a $w$-morphism $W_3 \to W_2$.

(ii) If $\text{wt}_u g''(y, z, u) \geq 12$, then the weighted blow-up with weight $\frac{1}{4}(6, 5, 4, 1)$ defines a $w$-morphism $W_3 \to W_2$.

Notice that there are exactly one exceptional divisor which has discrepancy $\frac{1}{3}$ over $W_1$ because any exceptional divisor which do not appear on $W_2$ has discrepancy greater than $\frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$ over $W_1$. Hence the $Y$ has only one non-Gorenstein point.

**Claim.** The $cD/3$ point on $W_2$ is not $\mathbb{Q}$-factorial.

To prove the claim, consider the following three possibilities:

**Case (i-1):** $zu^2 \in g''(y, z, u)$. We will show that this case won’t happen. In this case the chart $U_{z,3} \subset W_3$ is defined by

$$\{x^2 + y^3 + z^2 + g''(z, u) = 0\} \subset \mathbb{A}^4_{(x, y, z, u)}/\frac{1}{4}(3, 2, 1, 1),$$

with $u^2 \in g''(y, z, u)$. After a suitable change of coordinates $z \mapsto z + \lambda u$ we may assume that $U_{z,3}$ is defined by $x^2 + y^3 + z u$. Let $W_4 \to W_3$ be the weighted blow-up with weight $\frac{1}{4}(3, 2, 5, 1)$. One can verify that the exceptional divisor of $W_4 \to W_3$ has discrepancy one over $W$. This leads to a contradiction.

**Case (i-2):** $zu^2 \notin g''(y, z, u)$. The origin of the chart $U_{z,3}$ is a $cA_4/4$ point and weighted blow-up this point with weight $\frac{1}{4}(3, 2, 5, 1)$ defines a $w$-morphism $W_4 \to W_3$. $W_4$ has a $\frac{1}{3}(3, 2, 1)$ point. Let $F$ be the exceptional divisor which has discrepancy $\frac{1}{2}$ over $W_4$. Then one can compute that $a(F, W_2) = 1$ and $a(F, W_1) = \frac{5}{2}$. Lemma [6.10] now implies the claim.

**Case (ii):** The only non-Gorenstein singular point on $W_3$ is a cyclic quotient point of type $\frac{1}{3}(1, 4, 1)$. Let $F$ be the exceptional divisor which has discrepancy $\frac{1}{5}$ over $W_3$. We have $a(F, W_2) = 1$ and $a(F, W_1) = \frac{5}{2}$. The statement follows from Lemma [6.10].

Now the claim implies that $W_2$ do not have any Gorenstein singularity because the $\mathbb{Q}$-factorization of $W_2$ has zero Gorenstein depth. This implies that $zu^7$ do not appear in $g''(y, z, u)$. Hence either $uy^4$ or $u^4$ appear in $g''(y, z, u)$ and weighted blowing-up the $cD/3$ point on $W_2$ with the weight $\frac{1}{3}(3, 2, 4, 1)$ defines a $w$-morphism $W_3 \to W_2$.

The only non-Gorenstein point on $W_3$ is the origin of the chart $U_{z,3}$ defined by

$$\{x^2 + y^3 + z^2 + g''(z, u) = 0\} \subset \mathbb{A}^4_{(x, y, z, u)}/\frac{1}{4}(3, 2, 1, 1)$$

and the resolution of this point is described in Case (i-2) of the proof of the claim. $W_4$ has a $\frac{1}{3}(3, 2, 1)$ point. $W_5 \to \cdots \to W_4$ is the economic resolution.

Finally the discrepancies of exceptional divisors follows from direct computation. □

**Lemma 6.10.** Assume that $V \to W$ is a $w$-morphism. Let $X$ be a $\mathbb{Q}$-factorization of $W$ and let $Y$ be a $\mathbb{Q}$-factorization of $V$. Assume that $\text{exc}(V \to W)$ contains only one non-Gorenstein point $P$ which is of index $r + 1$ and either $r = 1$, $X$ is smooth and $\text{exc}(X \to W)$ is irreducible, or $r > 1$ and $X$ contains only one non-Gorenstein point. Let $F$ be an exceptional divisor over $P$ such that $a(F, V) = 1$. If $a(F, W) = 1 + \frac{1}{r}$, then $P$ is not $\mathbb{Q}$-factorial and Center$_FY$ is a curve.

**Proof.** We run $K_Y$-MMP over $W$ and we may assume that $X$ is the minimal model. Then $Y \to X$ factorize into

$$Y \to Y_1 \to \cdots \to Y_k \to X,$$
such that $Y \dashrightarrow Y_k$ is a sequence of flips and $Y_k \to X$ is a divisorial contraction. Note that $Y_k \to X$ is a divisorial contraction to a curve if $r = 1$, and a $w$-morphism if $r > 1$.

Assume that $P$ is $\mathbb{Q}$-factorial, then $P$ appears on $Y$ and the flipping curve of $Y \dashrightarrow Y_1$ passes through $P$. By the negativity lemma, we know that $a(Y_k, F) > a(Y, F) = 1$. On the other hand, every singular point on $Y_k$ has index less than $r + 1$ since otherwise there exists an exceptional divisor $G$ such that $a(G, Y) < a(G, Y_k) \leq \frac{1}{r+1}$, which is impossible. Hence $a(F, Y_k) \geq 1 + \frac{1}{r} = a(F, W) = a(F, X)$.

This leads to a contradiction since $\text{Center}_{Y_k} F$ is contained in $\text{exc}(Y_k \to X)$. 

\[ \text{Lemma 6.11.} \text{ Notation as in Lemma 6.9. The cAx/4 point on } W_3 \text{ is not } \mathbb{Q}\text{-factorial. Let } Z \to W_3 \text{ be a } \mathbb{Q}\text{-factorization and let } Z \dashrightarrow Z' \text{ be the corresponding flop. Then } Z \text{ has two singular points. One of them is a } \frac{1}{3}(1, 3, 1) \text{ point and the other one is a } \frac{1}{2}(1, 1, 1) \text{ point. We have } \text{Center}_2 E_7 \text{ is the flopping curve, Center}_2 E_6 \text{ is the } \frac{1}{2}(1, 1, 1) \text{ point and Center}_2 E_4 \text{ is the } \frac{1}{2}(1, 1, 1) \text{ point for } i = 4, 5 \text{ and } 8. \\

\text{Moreover, let } Z \text{ be a } \mathbb{Q}\text{-factorization of } W_4. \text{ If } Z \neq W_4, \text{ then the corresponding flop } Z \dashrightarrow Z' \text{ is at worst a smooth flop over a cAx point. Assume that } Z \text{ is the minimal model of } Z \text{ over } W_3. \text{ The birational map } Z_0 = Z \dashrightarrow Z \text{ can be factorize into } \\

\begin{align*}
\tilde{Z}_{(0,1,1,1)} &\dashrightarrow \tilde{Z}_{(0,1,1,0)} \\
\tilde{Z}_{(0,1,2)} &\dashrightarrow \tilde{Z}_{(0,1,1)} \dashrightarrow \tilde{Z}_{(0,1,0)} \\
\tilde{Z}_{(0,2)} &\dashrightarrow \tilde{Z}_{(0,1)} \dashrightarrow \tilde{Z}_{(0,0)} \\
\tilde{Z}_1 &\dashrightarrow \tilde{Z}_0 \\
\tilde{Z} &\dashrightarrow Z \\
\end{align*}

\text{such that every dash arrow is an Atiyah flop.} \\

\text{Proof.} \text{ The cAx/4 point is not } \mathbb{Q}\text{-factorial and Center}_2 E_7 \text{ is the flopping curve by Lemma 6.10. By the table in Lemma 6.9 we know that } Z \text{ contains a singular point of index 4, but there is no discrepancy one exceptional divisor over this point. It follows that the index 4 point should be a cyclic quotient point. Since there are two exceptional divisor of discrepancy } \frac{1}{2} \text{ over } Z, Z \text{ contains another } \frac{1}{2}(1, 1, 1) \text{ point.} \\

\text{We are going to factorize the flop } Z \dashrightarrow Z'. \text{ By direct computation, one can see that the Gorenstein singularities on } W_4 \text{ are at worst cAx singularities. Let } Z \to W_4 \text{ be a } \mathbb{Q}\text{-factorization. If } W_4 \text{ is not } \mathbb{Q}\text{-factorial, let } Z \dashrightarrow Z' \text{ be the corresponding flop. Otherwise let } Z' = Z. \text{ We know that } Z \dashrightarrow Z' \text{ is at worst a smooth flop over a cAx point.} \\

\text{Now } \tilde{Z}_{(0)} = Z \text{ has only a } \frac{1}{3}(3, 2, 1) \text{ singularity and } \tilde{Z}_{(k)} \text{ has a } \frac{1}{3}(1, 2, 1) \text{ and } \frac{1}{2}(1, 1, 1) \text{ point. Hence } \text{dep}(\tilde{Z}) = \text{dep}(\tilde{Z}_{(k)}) + 1, \text{ so } k = 1. \text{ We know that } \text{exc}(\tilde{Z}_{(0,0)} \to \tilde{Z}_{(0)}) = E_5 \text{ and } a(E_5, \tilde{Z}) = \frac{1}{2}. \text{ Hence } \tilde{Z}_{(0,0)} \dashrightarrow \tilde{Z}_{(0,1)} \text{ is a flop and } \tilde{Z}_{(0,2)} \to \tilde{Z}_{(1)} \text{ is a } w\text{-morphism by Lemma 1.7.} \\

\text{Since Center}_{\tilde{Z}_{(0,0)}} E_7 \text{ is the } \frac{1}{3}(1, 2, 1) \text{ point, this point lies on the flipping curve of } \tilde{Z}_{(0,1)} \dashrightarrow \tilde{Z}_{(0,2)}. \text{ We have that } \tilde{Z}_{(0,1,0)} \dashrightarrow \tilde{Z}_{(0,1,1)} \text{ is a flop and } \tilde{Z}_{(0,1,2)} \to \tilde{Z}_{(0,2)} \text{ is a } w\text{-morphism by Lemma 1.7.} \text{ } \tilde{Z}_{(0,1,1)} \text{ has only two } \frac{1}{2}(1, 1, 1) \text{ point. One can check that}
\[
a(E_8, \tilde{Z}_{(0,1,2)}) = a(E_8, \tilde{Z}_{(0,1,1)}) = \frac{1}{2}.\]
This implies that Center\(\tilde{Z}_{(0,1,2)}\) \(E_8\) is a \(\frac{1}{2}(1,1,1)\) point which is not contained in the flipped curve. Thus the flipping curve of \(\tilde{Z}_{(0,1,1)} -\rightarrow \tilde{Z}_{(0,1,2)}\) contains only a \(\frac{1}{2}(1,1,1)\) point and the factorization of this flip is given by Lemma 4.9.

Finally every flop appears in the factorization is an Atiyah flop by Lemma 4.8 and Lemma 4.10.

**Lemma 6.12.** Notation as in Lemma 6.9 and Lemma 6.11. Let \(Z\) be a \(\mathbb{Q}\)-factorization of \(W_2\) and assume that \(Z\) is a minimal model of \(\tilde{Z}\) over \(W_2\). Then \(Z\) has a \(cA/3\) singular point defined by

\[
(xy + z^3 + u^2 = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{3}(1,2,1,0).
\]

There is a unique \(w\)-morphism \(\tilde{Z}_1 \rightarrow Z\) such that \(\tilde{Z}_1\) has a \(\frac{1}{2}(1,1,1)\) point and a \(\frac{1}{3}(2,1,1)\) point. The \(\frac{1}{2}(1,1,1)\) point is the center of \(E_6\) and the \(\frac{1}{3}(2,1,1)\) point is the center of \(E_4\) and \(E_5\). The center of \(E_8\) on \(Z\) is a curve.

We have the following factorization

\[
\begin{array}{cccccc}
\tilde{Z}_{(0,1,1,1)} & \xrightarrow{c} & E_8 & \xrightarrow{w} & \tilde{Z}_{(0,1,1,0)} \\
\tilde{Z}_{(0,1,2)} & \xrightarrow{w} & E_5 & \xrightarrow{w} & \tilde{Z}_{(0,1,0)} \\
\tilde{Z}_{(0,2)} & \xrightarrow{w} & E_4 & \xrightarrow{w} & \tilde{Z}_{(0,0)} \\
\tilde{Z}_1 & \xrightarrow{w} & E_3 & \xrightarrow{w} & \tilde{Z}_0 \\
\end{array}
\]

such that every dash arrow is an Atiyah flop.

**Proof.** Notice that a general elephant of a singular point on \(Z\) have at most 5 components, hence \(Z\) do not have \(cD/3\) singularity. Thus \(Z\) has only a \(cA/3\) singular point. Because there are exactly one exceptional divisor of discrepancy \(\frac{1}{3}\), two exceptional divisor of discrepancy \(\frac{2}{3}\) and one exceptional divisor of discrepancy \(1\) over this singular point, we may assume that this point is of the following form

\[
(xy + z^3 + u^2 = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{3}(1,2,1,0)
\]

by [2] Proposition 3.4. We know that \(\tilde{Z}_k \rightarrow Z\) is a \(w\)-morphism and there is only one \(w\)-morphism over this \(cA/3\) point. One can verify that \(\tilde{Z}_k\) has a \(\frac{1}{2}(1,1,1)\) point and a \(\frac{1}{3}(2,1,1)\) point. Thus \(\text{dep}(\tilde{Z}_k) = 3 = \text{dep}(\tilde{Z}) - 1\) and so \(k = 1\). Notice that \(\tilde{Z}\) contains two singular points, one is a \(\frac{1}{4}(1,3,1)\) point and the other one is a \(\frac{1}{2}(1,1,1)\) point. The flipping curve of \(\tilde{Z}_0 -\rightarrow \tilde{Z}_1\) passes through only one of these two points since the flipped curve on \(\tilde{Z}_1\) (which is the proper transform of the flopping curve of \(Z -\rightarrow Z'\) on \(Z_1\)) passes through only one singular point. Lemma 6.9 implies that Center\(Z E_8\) is a curve, hence the flipping curve should passes through the \(\frac{1}{4}(1,3,1)\) point and \(a(E_6, \tilde{Z}_1) = a(E_6, \tilde{Z}_0) = \frac{1}{2}\). Hence the \(\frac{1}{2}(1,1,1)\) point on \(\tilde{Z}_1\) is the center of \(E_6\). Now the factorization of \(\tilde{Z}_0 -\rightarrow \tilde{Z}_1\) can be contracted using the same method in the proof of Lemma 6.11. \qed
Lemma 6.13. Notation as in Lemma 6.9 and Lemma 6.12. We have that $Y$ has a $cA/2$ singularity with depth equals to 2 or 3 and $Z(1)$ has also a $cA/2$ singularity if $Z(1) \rightarrow Y$ is a $w$-morphism, in the case that $\text{dep}(Y) = 3$. We have Center$_{Y}E_{6}$ is a curve on $Y$. $E_{2}$, $E_{3}$ (resp. $E_{2}$, $E_{3}$ and $E_{4}$) appears on the feasible resolution of $Y$ when $\text{dep}(Y) = 2$ (resp. $\text{dep}(Y) = 3$), Center of $E_{5}$ on the feasible resolution is a curve contained in $E_{3}$ and Center of $E_{4}$ is a curve contained in $E_{2}$ and not contained in $E_{3}$ if $\text{dep}(Y) = 2$.

Moreover, we have the following factorization

$$Y \leftarrow Z(1) \leftarrow Z_{(k_{(0)},)} \leftarrow \cdots \leftarrow Z_{(0,1)} \leftarrow Z_{(0,0)} \rightarrow Z(0) = Z \rightarrow Z'$$

such that $Z_{(k_{(0)},)} \rightarrow Z_{(1)} \rightarrow Y$ and $Z_{(0,0)} \rightarrow Z(0)$ are $w$-morphisms, $Z_{(0,0)} \rightarrow Z_{(0,1)}$ is an Atiyah flop, $k_{(0),} = 3$ if $\text{dep}(Y) = 3$ and $k_{(0),} = 4$ if $\text{dep}(Y) = 2$, $Z_{(0,1)} \rightarrow Z_{(0,2)}$ is given by Lemma 4.10 (1) and other flips have the factorization in Lemma 4.10.

Proof. Obviously $Y$ can not have $cE/2$ singularities because a general elephant of a singular point on $Y$ has at most 6 components. Notice that there is only one exceptional divisor of discrepancy $\frac{1}{2}$ over the singular point of $Y$. Furthermore, let $Z_{k} \rightarrow Y$ be the $w$-morphism, then every non-Gorenstein point on $Z_{k}$ should have index 2 since $Z$ has only a $cA/3$ singular point. By the classification in [8] and by a direct computation one can check that $Y$ can not have $cAx/2$ or $cD/2$ singularities. Thus the singular point of $Y$ is a $cA/2$ point and the general elephant of this point has at worst $A_{5}$ singularities (23 (6.4)). Moreover, there exists one discrepancy one exceptional divisor over this point. This implies that the singular point on $Y$ is of the form

$$(xy + z^{2} + u^{n} = 0) \subset A^{4}_{(x,y,z,u)}/\frac{1}{2}(1,1,1,0),$$

with $n = 2, 3$. One can see that $\text{dep}(Y) = n = 2$ or 3.

By Lemma 6.10 we know that Center$_{Y}E_{6}$ is a curve. $E_{2}$ and $E_{3}$ appear in the feasible resolution of $Y$ because they are only two exceptional divisors with discrepancy less than or equal to one. The $w$-morphism $Z_{k} \rightarrow Y$ extracts $E_{2}$. Since Center$_{Z}E_{8}$ is a curve and $a(E_{8}, Z_{k}) = 1$, Center$_{Z}E_{8}$ is a curve. We have $a(E_{4}, Z_{k}) = 1$. If $\text{dep}(Y) = 2$ then $Z_{k}$ has only a $\frac{1}{4}(1,1,1)$ singular point, so Center$_{Z}E_{4}$ is a curve. If $\text{dep}(Y) = 3$, then $Z_{k}$ has a non-cyclic quotient $cA/2$ point and the center of $E_{4}$ should be this point. Finally since $a(E_{5}, Y) = 2$, $E_{5}$ should be a curve contained in $E_{3}$.

Let $CZ_{(0),}$ be the flipping curve of $Z_{(0),} \rightarrow Z_{(1)}$ and let $CZ_{(0,0),}$ be the proper transform of $CZ_{(0),}$ on $Z_{(0,0),}$. Note that $Z_{(0,0),}$ has two singular points. One of them is a $\frac{1}{4}(1,2,1)$ point and the other one is a $\frac{1}{4}(1,1,1)$ point. The explicit equation of the singular point on $Z_{(0),}$ is given by Lemma 6.12. According to Mori’s local classification [20] Page 243], we know that the $CZ_{(0),}$ won’t pass through the $\frac{1}{4}(1,2,1)$ point. We claim that $CZ_{(0,0),}$ do not pass through any singular point and so $Z_{(0,0),} \rightarrow Z_{(0,1)}$ is a smooth flop.

Indeed, assume that $CZ_{(0,0),}$ passes through the $\frac{1}{4}(1,1,1)$ point. If $Z_{(0,0),} \rightarrow Z_{(0,1)}$ is a flop, then the non-$Q$-factorial point of $U_{(0,0),}$ has exactly one exceptional divisor of discrepancy $\frac{1}{2}$ and one exceptional divisor of discrepancy 1. This implies that this non-$Q$-factorial point in defined by

$$(xy + z^{2} + u^{2} = 0) \subset A^{4}_{(x,y,z,u)}/\frac{1}{2}(1,1,1,0).$$

However this singularity is $Q$-factorial by [15], Proposition 2.2.7, which leads to a contradiction. Hence $Z_{(0,0),} \rightarrow Z_{(0,1)}$ is a flip. The flipped curve is the center of $E_{6}$ and is contained in $\text{exc}(Z_{(0,k_{(0),})} \rightarrow Z_{(1)}) = E_{3}$. In this case we have $a(E_{6}, Z_{(0,k_{(0),})}) = a(E_{6}, Z_{(0,1)}) = 1$, hence this curve appears on $Z_{(0,k_{(0),})}$ and is contracted by $Z_{(0,k_{(0),})} \rightarrow Z_{(1)}$. This is impossible since Center$_{Y}E_{6}$ is a curve.
So we know that \( Z_{(0,0)} \to Z_{(0,1)} \) is a smooth flop. We need to say that the flipping curve of \( Z_{(0,1)} \to Z_{(0,2)} \) do not connect the two singular points on \( Z_{(0,1)} \). If it is true, then the flip is of the type semistable \( IA + IA \) [17] Theorem 2.2. Let \( H_{Z_{(0)}} \) be a general elephant of the flip \( Z_{(0)} \to Z_{(1)} \). Then \( H_{Z_{(0)}} \) contains the flipping curve of \( Z_{(0,1)} \to Z_{(0,2)} \). On the other hand, the flipping curve intersects the flopped curve of \( Z_{(0,0)} \to Z_{(0,1)} \) non-trivially, hence \( H_{Z_{(0,1)}} \) intersects the flopped curve non-trivially. Since \( H_{Z_{(0,1)}} \) intersects trivially to the flopped curve, it contains the flopped curve. Thus \( H_{Z_{(0,0)}} \) contains \( C_{Z_{(0,0)}} \) and \( H_{Z_{(0)}} \) contains \( C_{Z_{(0)}} \). However it is impossible by the classification [17] Theorem 2.2 and the fact that there are only one \( caA/3 \) singular point on \( Z_{(0)} \).

Hence the flipping curve of \( Z_{(0,1)} \to Z_{(0,2)} \) passes through only one singular point. Assume first that \( dep(Y) = 3 \). In this case \( Center_{Z_{(0,k(0))}} E_4 \) is a \( \frac{1}{2}(1,1,1) \) point. It follows that \( k(0) = 3 \) and \( Z_{(0,1)} \to Z_{(1)} \) can be factorized into two flips followed by a \( w \)-morphism

\[ Z_{(0,1)} \to Z_{(0,2)} \to Z_{(0,3)} \to Z_{(1)} \]

The two flipped curves correspond to \( E_5 \) and \( E_6 \). Since \( Center_{Z_{(0,3)}} E_5 \) is a curve contained in \( E_3 \) and \( Z_{(0,3)} \to Z_{(1)} \) contracts \( E_3 \) to a point, the flipped curve of \( Z_{(0,2)} \to Z_{(0,3)} \) corresponds to \( E_6 \). Thus \( Z_{(0,1)} \to Z_{(0,2)} \) is a flip around the \( \frac{1}{3}(1,2,1) \) point and its factorization is given by Lemma 4.10 (1) since the flipped curve on \( Z_{(0,2)} \) contains \( Center_{Z_{(0,2)}} E_4 \), which is a \( \frac{1}{2}(1,1,1) \) point. Now \( Z_{(0,2)} \to Z_{(0,3)} \) is a flip around a \( \frac{1}{2}(1,1,1) \) point, and its factorization is given by Lemma 4.9.

Now assume that \( dep(Y) = 2 \). One has that every possible flip \( Z_{(0,i)} \to Z_{(0,i+1)} \) is a flip around only one singular point. Let \( Z_{(0,j-1)} \to Z_{(0,j)} \) be the flip around a \( \frac{1}{3}(1,1,1) \) point such that the flipped curve corresponds to \( E_6 \). We want to say that \( j = k(0) \). Indeed, let \( L_1 = Center_{Z_{(0,i)}} E_6 \), then we have \( K_{Z_{(0,i)}} L_j = 1 \) by Lemma 4.9. If \( j \neq k(0) \) then \( K_{Z_{(0,k(0))}} L_{k(0)} < K_{Z_{(0,i)}} L_j = 1 \), hence \( K_{Z_{(0,k(0))}} L_{k(0)} \leq 0 \) since \( Z_{(0,k(0))} \) is smooth. However it is impossible because \( L_{k(0)} \) intersects \( exc(Z_{(0,k(0))} \to Y) \) non-trivially and the image of \( L_{k(0)} \) on \( Y \) is \( Center_{E_6} \), which is a \( K_Y \)-trivial curve. Hence the last flip \( Z_{(0,k(0)-1)} \to Z_{(0,k(0))} \) is a flip around a \( \frac{1}{2}(1,1,1) \) point.

The flip \( Z_{(0,1)} \to Z_{(0,2)} \) is a flip around a \( \frac{1}{3}(1,2,1) \) point. We claim that \( dep(Z_{(0,2)}) = dep(Z_{(0,1)}) - 1 \). Hence \( k(0) = 4 \), the factorization \( Z_{(0,1)} \to Z_{(0,2)} \) is given by Lemma 4.10 (1) and \( Z_{(0,2)} \to Z_{(0,3)} \) is also a flip around a \( \frac{1}{2}(1,1,1) \) point.

Assume that \( dep(Z_{(0,2)}) = dep(Z_{(0,1)}) - 2 \), then \( k(0) = 3 \). Let \( \Gamma_i = Center_{Z_{(0,i)}} E_5 \). Then \( \Gamma_3 \) is a flipped curve so \( K_{Z_{(0,2)}} \Gamma_2 > 0 \). We have seen before that \( K_{Z_{(0,3)}} L_3 = 1 \). One has \( \phi^* K_Y = K_{Z_{(0,3)}} - \frac{1}{2} E_2 - E_3 \) and \( \phi^* K_Y L_3 = 0 \) where \( \phi \) denotes the morphism \( Z_{(0,3)} \to Y \). Hence \( E_3 L_3 \leq 1 \). Since \( \Gamma_3 \) is smooth and \( L_3 \) meets \( E_3, \Gamma_3 \) meets \( L_3 \) at most one point transversally. This will imply that \( K_{Z_{(0,2)}} \Gamma_2 - K_{Z_{(0,3)}} \Gamma_3 \leq 1 \), hence \( K_{Z_{(0,3)}} \Gamma_3 \geq 0 \).

However it is impossible since \( \Gamma_3 \subset L_3 \) is contracted by \( Z_{(0,3)} \to Z_{(1)} \).

Finally we need to check that the flop \( Z_{(0,0)} \to Z_{(0,1)} \) is an Atiyah flop. First assume that \( dep(Y) = 3 \) and we have \( k(0) = 3 \). Let \( \Xi_1 \) be the flipping curve of \( Z_{(0,1)} \to Z_{(0,1+1)} \). One can see that the proper transform of \( \Xi_2 \) on \( Z_{(0,1)} \) is not the flopped curve of \( Z_{(0,0)} \to Z_{(0,1)} \) because the flopped curve do not pass through the \( \frac{1}{2}(1,1,1) \) point. This implies that \( \Xi_1 \) and \( \Xi_2 \) both appear on \( Z_{(0,0)} \). Let \( F_{Z_{(0,0)}} = E_3 \) be the exceptional divisor of \( Z_{(0,0)} \to Z_{(0)} \). Then we have \( F_{Z_{(0,0)}} \) is generically smooth along \( \Xi_1 \). This implies that \( F_{Z_{(0,0)}} \) is smooth in the smooth locus of \( Z_{(0,1)} \) by applying Lemma 4.4 twice. Hence \( Z_{(0,0)} \to Z_{(0,1)} \) is an Atiyah flop by Lemma 4.3.

Now assume that \( dep(Y) = 2 \) and we have \( k(0) = 4 \). As before let \( \Xi_i \) be the flipping curve of \( Z_{(0,i+1)} \). Then \( \Xi_i \) appears on \( Z_{(0,0)} \) for \( i = 1, 3 \) by the same reason as in the previous case. We are going to show that \( \Xi_2 \) also appears on \( Z_{(0,0)} \), or equivalently,
the proper transform of $\Xi_2$ on $Z_{(0,1)}$ is not the flopped curve of $Z_{(0,0)} \rightarrow Z_{(0,1)}$. Let $C_{Z_{(0,1)}}$ be the flopped curve of $Z_{(0,0)} \rightarrow Z_{(0,1)}$ and let $C_i$ be the proper transform of $C_{Z_{(0,1)}}$ on $Z_i$, for all possible $i$. We are going to show that $C_{Z_{(0,2)}} \neq \Xi_2$. Assume that $C_{Z_{(0,2)}} = \Xi_2$. In this case we have $C_{Z_{(0,1)}}$ intersects $\Xi_1$ non-trivially. We know that $K_{Z_i} C_i = 0$ for $i = (0, 1), (0, 1, 0), (0, 1, 1), (0, 1, 1, 0)$ and $(0, 1, 1, 1)$. However, $K_{Z_{(0,1)}} C_{Z_{(0,1)}} < 0$ since $C_{Z_{(0,1)}}$ intersects $\Xi_1$ non-trivially. On the other hand, we have $Z_{(0,2)} = Z_{(0,2,0)}$ and the proper transform of $\Xi_2$ on $Z_{(0,2,0)}$ is $K_{Z_{(0,2,0)}}$-trivial, since $Z_{(0,2,0)} \rightarrow Z_{(0,2,1)}$ is a flop. Thus $C_{Z_{(0,2)}} \neq \Xi_2$ and $F_{Z_{(0,1)}}$ is smooth in the smooth locus of $Z_{(0,1)}$, which implies that $Z_{(0,0)} \rightarrow Z_{(0,1)}$ is an Atiyah flop.

Proposition 6.14. Assume that $X \rightarrow X'$ is a simple smooth flop over $W$ such that $W$ has a $cE_8$ singularity. We have the factorization

$$X \leftarrow Y_{(1)} \leftarrow Y_{(2)} \leftarrow Y_{(1)} \leftarrow Y_{(0)} \rightarrow Y \rightarrow Y'$$

such that $Y_{(0)} \rightarrow Y_{(1)} \rightarrow X$ is a sequence of blowing-up smooth curves, $Y_{(0,0)} \rightarrow Y_{(0,1)}$ is an Atiyah flop. $Y_{(0,1)} \rightarrow Y_{(0,2)}$ is given by Lemma 4.9 (reps. Lemma 4.10 (1)) if $\text{dep}(Y) = 2$ (resp. $\text{dep}(Y) = 3$). $Y \rightarrow Y'$ is a flop given in Lemma 6.13 and the diagram $Y' \rightarrow X'$ is symmetric. \[ □ \]

Proof. By Lemma 6.13 we know that $Y$ contains only a $cA/2$ point with depth 2 or 3. We use the notation as in Convention 3.10. Notice that there are only one exceptional divisor with discrepancy less than one over $Y = Y_{(0)}$, hence $k_{(0)} = 1$. Thus $Y_{(1)} \rightarrow X$ is a blowing-up a smooth curve. If $\text{dep}(Y) = 2$, then $Y_{(0)} \rightarrow Y_{(1)}$ is given by Lemma 4.10 (3).

Now assume that $\text{dep}(Y) = 3$. Since $Y$ has only a $cA/2$ singular point, by the same argument as in the last paragraph of the proof of Lemma 4.10 we can show that $Y_{(0,0)} \rightarrow Y_{(0,1)}$ is a flop. It is a smooth flop by Lemma 4.8. Now $\text{dep}(Y_{(0,1)}) = 2$ and $Y_{(0,1)}$ contains only a $cA/2$ point. Since $\text{exc}(Y_{(0,0)} \rightarrow Y_{(0)}) = E_2$ and $\text{Center}_{(1)} E_2$ is a curve, $Y_{(0,2)} \rightarrow Y_{(1)}$ is a blowing-up a smooth curve. Hence $Y_{(0,2)}$ is smooth and $Y_{(0,1)} \rightarrow Y_{(0,2)}$ is given by Lemma 4.10 (3). Finally $Y_{(0,0)} \rightarrow Y_{(0,1)}$ is an Atiyah flop by Lemma 4.6. \[ □ \]

Proof of Theorem 1.4. The statement (1) is well-known. (2) follows from Proposition 5.8 and (3) follows from Proposition 6.8 and Proposition 6.13. \[ □ \]

References

[1] X. Benveniste, *Sur le cone des 1-cycles effectifs en dimension 3*, Math. Ann. 272 (1985), 257-265.
[2] H-K Chen, *On the Nash problem for terminal threefolds of type cA/r*, preprint, arXiv:1907.06326.
[3] H-K Chen, *Minimal resolutions of threefolds*, preprint, arXiv:2304.08760.
[4] J. A. Chen, *Explicit resolution of three dimensional terminal singularities*, Minimal Models and Extremal Rays (Kyoto, 2011), Adv. Stud. Pure Math. 70 (2016), 323-360.
[5] J. A. Chen, *Birational maps of 3-folds*, Taiwanese J. Math. 19 (2015), 1619-1642.
[6] J. A. Chen and C. D. Hacon, *Factoring 3-Fold Flips and Divisorial Contractions to Curves*, J. Reine Angew. Math., 657 (2011), 173-197.
[7] S. Cutkosky, *Elementary contractions of Gorenstein threefolds*, Math. Ann. 280 (1988), no. 3, 521-525.
[8] T. Hayakawa, *Blowing Ups of 3-dimensional Terminal Singularities*, Publ. Res. Inst. Math. Sci. 35 (1999), 515-570.
[9] T. Hayakawa, *Blowing Ups of 3-dimensional Terminal Singularities, II*, Publ. Res. Inst. Math. Sci. 36 (2000), 423-456.
[10] S. Katz, D. R. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups*, J. Algebraic Geom. 1 (1992), 449-530.
[11] M. Kawakita, *Three-fold divisorial contractions to singularities of higher indices*, Duke Math. J. 130 (2005), 57-126.
[12] Y. Kawamata, *Divisorial contractions to 3-dimensional terminal quotient singularities*, Higher-dimensional complex varieties (Trento, 1994), 241–246, de Gruyter, Berlin, 1996.

[13] Y. Kawamata, *Flops Connect Minimal Models*, Publ. Res. Inst. Math. Sci. **44** (2008), 419-423.

[14] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989), 15-36.

[15] J. Kollár, *Flips, flops, minimal models, etc*, Surveys in Differential Geometry, **1** (1991), 113-199.

[16] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Astérisque 211, Société Mathématique de France, Paris, 1992.

[17] J. Kollár, S. Mori, *Classification of three-dimensional flips*, J. Amer. Math. Soc. **5** (1992), 533-703.

[18] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics 134, Cambridge Univ. Press, 1998.

[19] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. **116** (1982), 113-176.

[20] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988), 117-253.

[21] H. Pinkham, *Factorization of birational maps in dimension 3*, Proc. Symp. Pure Math. **40** (1983), 343-371.

[22] M. Reid, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math. **1** (1983), 131-180.

[23] M. Reid, *Young person’s guide to canonical singularities*, Proc. Symp. Pure Math. **46** (1987), 345-414.

School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea

Email address: hkchen@kias.re.kr