REPRESENTATIONS OF THE $U_q(u_{4,1})$ AND A $q$-POLYNOMIAL THAT DETERMINES BARYON MASS SUM RULES

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Abstract. With quantum groups $U_q(su_n)$ taken as classifying symmetries for hadrons of $n$ flavors, we calculate within irreducible representation $D_{12}^+(p-1, p-3, p-4; p, p-2)$ ($p \in \mathbb{Z}$) of 'dynamical' quantum group $U_q(u_{4,1})$ the masses of baryons $\frac{1}{2}^+$ that belong to 20-plet of $U_q(su_4)$. The obtained $q$-analog of mass relation (MR) for $U_q(su_3)$-octet contains unexpected mass-dependent term multiplied by the factor $\frac{A_q}{B_q}$ where $A_q, B_q$ are certain polynomials (resp. of 7-th and 6-th order) in the variable $q + q^{-1} \equiv [2]_q$. Both values $q = 1$ and $q = e^{\frac{i\pi}{6}}$ turn the polynomial $A_q$ into zero. But, while $q = 1$ results in well-known Gell-Mann–Okubo (GMO) baryon MR, the second root of $A_q$ reduces the $q$-MR to some novel mass sum rule which has irrational coefficients and which holds, for empirical masses, even with better accuracy than GMO mass sum rule.

1. INTRODUCTION

Applications of quantum algebras ($su_q(2)$ first of all) to phenomenological description of rotational spectra of deformed heavy nuclei and diatomic molecules, have appeared a couple of years ago and seem to be encouraging [1-3] (concerning physical applications of quantum groups/algebras in a wider context see [4] and references therein).

Recently, the use of higher rank quantum algebras $su_q(n)$ in order to replace conventional algebras $su(n)$ of unitary groups and their irreducible representations (irreps) in describing global symmetries of hadrons (vector mesons) of $n$ flavours has been proposed [5]. With the help of the corresponding algebras $u_q(n+1)$ of 'dynamical' symmetry, one can realize necessary breaking of flavor symmetries up to exact (for strong interactions alone) isospin symmetry $su_q(2)$, and obtain some $q$-analogs of mass relations (MR’s). It was demonstrated that at every fixed $n$, $n = 3, ..., 6$, all the $q$-dependence in vector meson masses and in coefficients of their $q$-analog MR’s appears, modulo possible additional...
terms of these knot invariants. Corresponding vector meson mass sum rule (MSR).

\[ n \] root of unity \( q \) (equivalently, respective Alexander polynomial of the toroidal \((2n-1)\)-knot). In a sense, the polynomial \([n]_q - [n-1]_q\) through its root \( q(n) \) determines the strength of deformation at every fixed \( n \), and due to this property may be called a defining polynomial for the corresponding vector meson mass sum rule (MSR).

A comparison with empirical data requires appropriate fixation of deformation parameter, and it appears that to every number of flavors \( n \), \( n \geq 3 \), there corresponds a prime root of unity \( q = q(n) = e^{i\pi/(2n-1)} \). The latter turns into zero the polynomial \([n]_q - [n-1]_q\) (equivalently, respective Alexander polynomial of the toroidal \((2n-1)\)-knot). In a sense, the polynomial \([n]_q - [n-1]_q\) through its root \( q(n) \) determines the strength of deformation at every fixed \( n \), and due to this property may be called a defining polynomial for the corresponding vector meson mass sum rule (MSR).

Utilizing the quantum algebras instead of conventional unitary groupies of flavor symmetries, together with 'dynamical' quantum algebras, we get as a result that the collection of torus knots \( 5_1, 7_1, 9_1, 11_1 \) is put into correspondence [6] with vector quarkonia \( ss, cc, bb, \) and \( tt \) respectively. Thus, application of the embedding \( U_q(u_n) \subset U_q(u_{n+1}) \) to vector meson masses provides an appealing possibility of certain topological characterization of flavors, since the number \( n \) just corresponds to \( 2n-1 \) overcrossings of 2-strand braids whose closures give these \((2n-1)\)-torus knots. Equivalently, using \((a,b)\)-presentation of these same knots with \( a = 2n-1, b = 2 \), one is led to the correspondence: \( n \leftrightarrow w \equiv 2n-1, \) where \( w \) (or \( a \)) is nothing but the winding number around the body (tube) of torus (winding number around the hole of torus being equal to 2 for all \( n \geq 3 \)).

Our present goal is to extend the approach of [5,6] to the case of baryons \( \frac{1}{2}^+ \) (including charmed ones) again adopting \( U_q(u_4) \) for the 4-flavor symmetry. However, like in the situation of baryon MSR’s obtained with non-deformed dynamical pseudounitary \( u(4,1) \)-symmetry [7,8], it will be more convenient for us now to exploit representations of 'noncompact' dynamical symmetry, realized by the quantum algebra \( U_q(u_{4,1}) \), in order to effect necessary symmetry breakings.

The paper is organized in the following manner. In Sec.2 and Sec.3, certain amount of information concerning (universal enveloping) quantum algebras \( U_q(u_n), U_q(u_{n+1}) \) as well as their finite- and infinite-dimensional representations necessary for the considered application, is presented.

On the base of calculation of baryon masses (Sec.4) within concrete representation \( D^{+}_{12}(p-1,p-3,p-4;p,p-2) \) (where \( p \) is some fixed integer, precise value of which is unessential since it will not enter final expressions for masses), it is demonstrated in Sec.5 that the resulting \( q \)-analog of baryon octet MR takes somewhat unusual form since it contains an additional mass-dependent structure multiplied by the ratio \( \frac{A_q}{q^2} \) of certain \( q \)-polynomials. The (’rigid’) \( q \)-fixation procedure described in the 6th Section leads to conclusion that the \( q \)-analog of baryon octet MR yields either the usual Gell-Mann–Okubo
(GMO) mass sum rule [9]

$$m_N + m_{\Xi} = \frac{3}{2}m_{\Lambda} + \frac{1}{2}m_{\Sigma}$$

(1)
or a very successful novel MSR with irrational coefficients (see eq.(20) below) if one fixes the deformation parameter as $q = 1$ or $q = e^{i\pi}$ respectively. These values are nothing but two distinct roots of one and the same defining polynomial $A_q$ appearing in the $q$-analog. With the two alternative choices of deformation parameter, consequences for masses of charmed baryons may be obtained and compared to each other.

Another $q$-analog of octet MR (with different defining $q$-polynomial $\tilde{A}_q$) is obtained in Sec.7 by calculations within the other specific representation $\tilde{D}_3^+(p-1, p-3, p-4; p-4, p-2)$ of $U_q(u_{4,1})$. It can be shown, however, that from the $q$-dependent expressions for octet baryon masses obtained by means of this representation $\tilde{D}_3^+(\ldots)$, the same $q$-analog with same $q$-polynomial $A_q$ mentioned in the previous paragraph (modulo minimal modification which consists in the replacement $B_q \rightarrow \tilde{B}_q$ with certain $q$-polynomial $\tilde{B}_q$) can also be derived. Last Section is devoted to some concluding remarks.

2. QUANTUM ALGEBRA $U_q(gl_n)$ AND ITS REAL FORMS

We will use the denotation $[B]_q \equiv [B] \equiv (q^B - q^{-B})/(q - q^{-1})$ where $B$ is either a number or an operator. The elements $1$, $A_{jj+1}$, $A_{j+1j}$, $A_{jj}$, $j = 1, 2, \ldots, n-1$, $A_{nn}$ that generate the $q$-deformed (universal enveloping) algebra $U_q(gl_n)$, satisfy the relations [10]

$$[A_{ii}, A_{jj}] = 0, \quad [A_{ii}, A_{jj+1}] = \delta_{ij}A_{ij+1} - \delta_{ij+1}A_{ji},$$
$$[A_{ii}, A_{j+1j}] = \delta_{ij+1}A_{ij} - \delta_{ij}A_{j+1i},$$
$$[A_{ii+1}, A_{j+1j}] = \delta_{ij}[A_{ii} - A_{i+1i+1}],$$
$$[A_{ii+1}, A_{jj+1}] = [A_{i+1i}, A_{jj+1}] = 0 \quad \text{for} \quad |i - j| \geq 2,$$

(2)

and the trilinear ($q$-Serre) relations

$$(A_{i+1i})^2A_{ii+1} - [2]_qA_{i+1i}A_{ii+1}A_{i+1i} + A_{ii+1}(A_{i+1i})^2 = 0,$$
$$(A_{ii+1})^2A_{i+1i} - [2]_qA_{ii+1}A_{i+1i}A_{ii+1} + A_{i+1i}(A_{ii+1})^2 = 0.$$

(3)

Endowed with comultiplication, counit and antipode (which we do not reproduce here), the $q$-deformed algebra $U_q(gl_n)$ becomes a quantum (Hopf) algebra.

In what follows, we'll need both compact and non-compact real forms of $U_q(gl_n)$.

The 'compact' quantum algebra $U_q(u_n)$ is singled out by means of the $*$-operation

$$(A_{jj})^* = A_{jj}, \quad (A_{j+1j})^* = A_{jj+1}, \quad (A_{jj+1})^* = A_{j+1j}.$$

(4)

The 'noncompact' quantum algebra $U_q(u_{n,1})$ is singled out from $U_q(gl_{n+1})$ by introducing another $*$-operation which includes relations (4) of 'maximal compact' subalgebra $U_q(u_n)$ and, in addition, the relations

$$(A_{n+1n})^* = -A_{nn+1}, \quad (A_{nn+1})^* = -A_{n+1n}, \quad (A_{n+1 n+1})^* = A_{n+1 n+1}.$$

(5)
Finite-dimensional representations of $U_q(u_n)$, similarly to those of the non-deformed algebra $u_n$, are given by sets of ordered integers $\mathbf{m}_n = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and, since standard branching rules survive through $q$-deformation, realized by means of $(q$-analog of) Gel’fand-Tsetlin basis and formulas. Representation formulas for $A_{ii}$ remain unchanged, and $A_{kk+1}, A_{k+1k}$, $k = 1, \ldots, n-1$, act according to formulas given in [10]. Action formulas for the operators which represent nonsimple-root elements must be consistent with $q$-Serre relations (3). We use $A_{ij}$ for $|i - j| = 2$ in the form (see e.g. [5,6])

$$A_{kk+2} = A_{kk+2}(q) \equiv q^{1/2}A_{k+1k}A_{k+2,k} - q^{-1/2}A_{k+1k+2}A_{k+2,k+1},$$

$$A_{k+2,k} = A_{k+2,k}(q) \equiv q^{1/2}A_{k+1k}A_{k+2,k+1} - q^{-1/2}A_{k+2,k+1}A_{k+1k},$$

such that the $q$-Serre relations corresponding to upper signs in (3) follow from (6a) and the commutation rules (CR’s)

$$q^{1/2}A_{kk+2}A_{k+1k+2} - q^{-1/2}A_{kk+2}A_{k+1k+2} = 0,$n

$$q^{1/2}A_{kk+2}A_{kk+1} - q^{-1/2}A_{kk+1}A_{kk+2} = 0,$$

whereas those corresponding to lower signs in (3) follow from (6b) and the CR’s

$$q^{1/2}A_{k+2,k}A_{k+2,k+1} - q^{-1/2}A_{k+2,k}A_{k+2,k+1} = 0,$n

$$q^{1/2}A_{k+2,k}A_{k+1k} - q^{-1/2}A_{k+1k}A_{k+2,k} = 0.$$

Dual definition $\tilde{A}_{kk+2} \equiv -A_{kk+2}(q^{-1})$, $\tilde{A}_{k+2,k} \equiv -A_{k+2,k}(q^{-1})$ is paired with respective dual CR’s. Operators $A_{ij}$ for other nonsimple-root elements ($|i - j| > 2$) are treated analogously.

3. ON THE REPRESENTATIONS OF $U_q(u_{n,1})$

In this section we consider some details concerning irreducible representations (irreps) not just for the particular case of $U_q(u_{4,1})$, but for more general case of quantum algebras $U_q(u_{n,1})$, $2 \leq n < \infty$ (see [11, 12]). First of all, we have to remark that the construction of the ‘principal nonunitary series’ of representations, analysis of their (ir)reducibility as well as the classification of irreps and ‘unitary’ (i.e. infinitesimally unitary) irreps of $U_q(u_{n,1})$ runs in much analogous way to that of the non-deformed (that is, $u(n,1)$) case. Let us refer to [8, 13] and references given therein for the non-deformed case.

Throughout this section, $q$ is considered to be generic (not equal to a root of unity). The representations of the algebra $U_q(u_{n,1})$ are characterized by their signatures $\chi$, that is, by the sets of $n + 1$ numbers: $\chi \equiv (l_1, l_2, \ldots, l_{n-1}; c_1, c_2)$. Here $c_1, c_2$ are complex numbers such that $c_1 + c_2 \in \mathbb{Z}$, and all the $l_i, i = 1, \ldots, n-1$, are integers related with the components $m_1, m_2, \ldots, m_{n-1} \equiv \mathbf{m}$ of the highest weight $\mathbf{m}$ of irrep of the subalgebra $U_q(u_{n-1})$, namely, $l_i = m_i - i - 1$. The condition on the components of highest weight in terms of $l_i$ reads: $l_1 > l_2 > \ldots > l_{n-1}$. Under restriction to the ‘compact’ subalgebra $U_q(u_n)$, the representation $T_\chi$ decomposes into direct sum of all those irreps $T_{\mathbf{l}_n}$ ($\mathbf{l}_n \equiv (l_{1n}, l_{2n}, \ldots, l_{nn})$, $l_{jn} = m_{jn} - j$, $1 \leq j \leq n$),
\( j = 1, \ldots, n \), where \( m_{1n}, m_{2n}, \ldots, m_{nn} \) form the highest weight \( \mathbf{m}_n \) of irrep of \( U_q(u_n) \) for which the condition

\[
l_{1n} > l_1 \geq l_{2n} > l_2 \geq \ldots \geq l_{n-1n} > l_{n-1} \geq l_{nn}
\]

is satisfied. All representations \( T_{\chi} \) which satisfy eq.(8) are contained in \( T_{\chi} \) with unit multiplicity.

Action of the representation \( T_{\chi} \) is defined in the carrier Hilbert space taken as a direct sum of finite-dimensional carrier spaces of irreps of \( U_q(u_n) \). In the carrier space of \( T_{\chi} \), we choose a canonical orthonormal basis formed by the union of canonical (Gel’fand–Tsetlin) bases of \( T_{\chi} \), in accordance with the reduction chain \( U_q(u_{n,1}) \supset U_q(u_n) \supset U_q(u_{n-1}) \supset \ldots \supset U_q(u_2) \). An (orthonormalized) basis vector is completely characterized by the set \( \chi, l_n, l_{n-1}, \ldots, l_1 \) (here \( l_k \equiv (l_{1k}, l_{2k}, \ldots, l_{kk}) \); \( l_{ik} \equiv m_{ik} - i \), \( i = 1, \ldots, k \)) and will be denoted as

\[
|\chi; l_n, l_{n-1}, \ldots, l_1\rangle.
\]

When restricted to subalgebra \( U_q(u_n) \), representation operators act according to formulas of ref. [10]. Operators \( T_{\chi}(A_{nn+1}) \) and \( T_{\chi}(A_{n+1n}) \) that represent ’noncompact’ generators of \( U_q(u_{n,1}) \) act according to formulas

\[
T_{\chi}(A_{nn+1})|\chi; l_n, l_{n-1}, \ldots, l_1\rangle = \\
\sum_{r=1}^{n} \frac{|c_1 - r_n| l_{rn} [l_{rn} - c_2]|l_{rn} - 1| [l_{rn} - l_j]}{|\Pi_{s=1, s \neq r}^{n} [l_{rn} - l_{sn} + 1] [l_{rn} - l_{sn}]|^{1/2}} |\chi; l_n^{1+r}, l_{n-1}, \ldots, l_1\rangle
\]

and

\[
T_{\chi}(A_{n+1n})|\chi; l_n, l_{n-1}, \ldots, l_1\rangle = \\
\sum_{r=1}^{n} \frac{|c_1 - r_n + 1| [l_{rn} - c_2 - 1]|l_{rn} - l_j| [l_{rn} - 1]}{|\Pi_{s=1, s \neq r}^{n} [l_{rn} - l_{sn}] [l_{rn} - l_{sn} - 1]|^{1/2}} |\chi; l_n^{1-r}, l_{n-1}, \ldots, l_1\rangle
\]

where \( l_n^{1+r} \) means that the component \( l_{rn} \) in \( l_n \) is to be replaced respectively by \( l_{rn} \pm 1 \).

To have representation formulas for other ’noncompact’ operators \( T_{\chi}(A_{jn+1}) \) and \( T_{\chi}(A_{n+1j}) \), \( 1 \leq j \leq n - 1 \), one has to utilize relations analogous to eqs. (6).

By means of eqns. (10)-(11) it is not hard to prove the following statement [11].

**Proposition.** The representation \( T_{\chi} \) is irreducible if and only if \( c_1 \) and \( c_2 \) are not integers or \( c_1 \) and \( c_2 \) coincide with some of the numbers \( l_1, l_2, \ldots, l_{n-1} \).

When both \( c_1 \) and \( c_2 \) are integers not coinciding simultaneously with any two of the integers \( l_1, l_2, \ldots, l_{n-1} \), representations from the ’principal nonunitary series’ are no longer irreducible, and the corresponding irreps are extracted from these reducible representations (we call them irreps of integer type).

Two irreps \( T_{\chi} \) and \( T_{\chi'} \) from the Proposition with \( \chi \) and \( \chi' \) differing only by interchange \((c_1, c_2) \leftrightarrow (c_2, c_1)\), are equivalent. Periodicity of the function \( f(w) = \{w\}_q \) implies the following: the representations \( T_{\chi} \) and \( T_{\chi'} \) with \( \chi' = (l_1, \ldots, l_{n-1}; c_1 + \frac{i\pi k}{h}, c_2 - \frac{i\pi k}{h}) \), \( k \in \mathbb{Z} \), are equivalent for \( q = \exp h, \; h \in \mathbb{R} \); the representations \( T_{\chi} \) and \( T_{\chi'} \) with
\[ \chi' = (l_1, ..., l_{n-1}; c_1 + \frac{x_1}{n}, c_2 - \frac{x_2}{n}), \quad k \in \mathbb{Z}, \] are equivalent for \( q = \exp i\hbar, \ h \in \mathbb{R}. \) For this reason, we impose the restriction \( 0 \leq \text{Im} \ c_1 < \frac{\pi}{n} \) (respectively the restriction \( 0 \leq \Re \ c_1 < \frac{\pi}{n} \)) in the case of \( q = \exp h \) (resp. of \( q = \exp i\hbar \) \( h \in \mathbb{R} \) in both cases).

The classification of irreducible representations of the algebra \( U_q(u_{n,1}) \) is completely analogous to that of the non-deformed algebra \( u(n,1) \) (see e.g. [8, 13]).

For our purposes it will be useful to reproduce here the list of different classes of irreps of \( U_q(u_{n,1}) \) which are 'unitary' for \( q = e^{ih}, \ h \in \mathbb{R} \) (the sequence of numbers \( a_1, a_2, ..., a_k \) will be called \textit{contracted} if \( a_{i-1} - a_i = 1 \) for \( i = 2, 3, ..., k \)).

I. Principal continuous series of irreps \( T_\chi \): \( c_1 \) and \( c_2 \) are such that \( c_1 = \bar{c}_2 \).

II. Supplementary continuous series of irreps \( T_\chi \): \( c_1, c_2 \in \mathbb{R} \) and, moreover, there exist such \( l_k \) and \( l_s \) \( (k, s = 1, 2, ..., n - 1) \) that \( |c_1 - l_k| < 1, \ |l_s - c_2| < 1, \) and the sequence \( l_k, l_{k+1}, ..., l_s, \) if \( c_1 > c_2, \) or the sequence \( l_s, l_{s+1}, ..., l_k, \) if \( c_1 < c_2, \) is contracted.

III. Strange series of irreps \( T_\chi \): \( \text{Im} \ c_1 = \text{Im} \ c_2 = \frac{\pi}{n} \).

For these three continuous 'unitary' series, the irreps \( T_\chi \) when restricted to subalgebra \( U_q(u_n) \) contain those irreps \( T_{1n} \) for which the condition (8) is satisfied.

Classes of 'unitary' irreps of integer type \( (c_1, \ c_2 \ \text{not both coincide with some of} \ l_1, ..., l_{n-1}; \ \text{we use the denotation} \ l_0 = \infty, l_n = -\infty). \)

IV. Irreps \( D^{ij}_+(l_1, ..., l_{n-1}; c_1, c_2) \) and \( D^{ij}_-(l_1, ..., l_{n-1}; c_1, c_2) \) where \( l_{i-1} > c_1 > l_i, \ l_{j-1} > c_2 > l_j, \ 1 \leq i \leq j \leq n. \) Moreover, either \( i = j \) holds, or the sequence \( c_1, l_1, l_{i+1}, ..., l_{j-1} \) for \( D^{ij}_+( \) (the sequence \( l_i, l_{i+1}, ..., l_{j-1}, c_2 \) for \( D^{ij}_- \) is contracted. The irrep \( D^{ij}_+(l_1, ..., l_{n-1}; c_1, c_2) \) (resp. irrep \( D^{ij}_-(l_1, ..., l_{n-1}; c_1, c_2) \)) contains with unit multiplicity those and only those irreps of \( U_q(u_n) \) for which the condition (8) and the conditions \( l_{in} > c_1, \ l_{jn} > c_2 \) (resp. \( l_{in} \leq c_1, \ l_{jn} \leq c_2 \)) are satisfied.

V. Irreps \( \tilde{D}^{ij}_+(l_1, ..., l_{n-1}; c_1, c_2) \) and \( \tilde{D}^{ij}_-(l_1, ..., l_{n-1}; c_1, c_2) \) where \( c_1 = l_i, \ 1 \leq i \leq n-1, \) and \( c_2 \) is an integer such that \( l_{j-1} > c_2 > l_j, \ 1 \leq j \leq n. \) For \( \tilde{D}^{ij}_+ \), moreover, either \( i < j \) and the sequence \( l_i, l_{i+1}, ..., l_{j-1}, c_2 \) is contracted, or \( i \geq j \) and the sequence \( l_j, l_{j+1}, ..., l_i \) is contracted. For \( \tilde{D}^{ij}_- \), either \( i < j \) and the sequence \( l_i, l_{i+1}, ..., l_{j-1} \) is contracted, or \( i \geq j \) and the sequence \( c_2, l_j, l_{j+1}, ..., l_i \) is contracted. The \( \tilde{D}^{ij}_+(l_1, ..., l_{n-1}; c_1, c_2) \) (resp. \( \tilde{D}^{ij}_-(l_1, ..., l_{n-1}; c_1, c_2) \)) contains with unit multiplicity those and only those irreps of \( U_q(u_n) \) for which the condition (8) and the condition \( l_{jn} > c_2 \) (resp. \( l_{jn} \leq c_2 \)) are satisfied.

VI. Irreps \( D^{i}_+(l_1, ..., l_{n-1}; c_1, c_2) \) and \( D^{i}_-(l_1, ..., l_{n-1}; c_1, c_2) \) where \( c_1 = c_2 = c \) is an integer such that \( l_{i-1} > c > l_i, \ 1 \leq i \leq n. \) The \( D^{i}_+( \) (resp. \( D^{i}_- \)) contains with unit multiplicity those and only those irreps of \( U_q(u_n) \) for which the condition (8) and the condition \( l_{in} > c \) (resp. \( l_{in} \leq c \)) is satisfied.

There exist additional equivalence relations between irreps from different classes IV–VI completely analogous to the equivalence relations of the non-deformed case (we do not give them here, see e.g. [8]). Remark that two reducible representations \( T_\chi \) and \( T_{\chi'} \) with \( \chi = (l_1, ..., l_{n-1}; c_1, c_2) \) and \( \chi' = (l_1, ..., l_{n-1}; c_2, c_1) \) contain equivalent irreps (from classes IV–VI) of \( U_q(u_{n,1}) \).

At \( q = e^{ih}, \ h \in \mathbb{R}, \) the classes I and III (with minor modification: \( \text{Re} \ c_1 = \text{Re} \ c_2 = \frac{\pi}{n} \) instead of \( \text{Im} \ c_i \)) are the only classes that survive in the classification of 'unitary' irreps.
It is worth to mention the following: the only class from the above presented list of irreps which is absent in the classical limit (disappears at $q \to 1$ or, equivalently, at $h \to 0$) is the class III (strange series) of 'unitary' irreps.

4. EVALUATION OF BARYON MASSES

One has to form state vectors for baryons $\frac{1}{2}^+$ that constitute 20-plet of $U_q(u_4)$ whose decomposition with respect to $U_q(u_3)$ is $20 = 8 + 3 + 3^* + 6$. To this end, we use the embedding $U_q(u_4) \in U_q(u_{4,1})$ and the (orthonormalized) Gel'fand-Tsetlin basis elements in the form (9) constructed in accordance with the aforementioned canonical chain, fixing $n = 4$. For instance, for the isodoublet of nucleons (contained in octet) we have:

$$|N\rangle \leftrightarrow |\chi; l_4, l_3, l_2, l_Q\rangle$$

where $\chi \equiv (l_1, l_2, l_3; c_1, c_2)$ labels some appropriate (that is, such that contains the 20-plet of $U_q(u_4)$) irrep of the 'dynamical' $U_q(u_{4,1})$; $l_4 \equiv (p + 1, p - 1, p - 3, p - 4)$ and $l_3 \equiv (p + 1, p - 1, p - 3)$ label 20-plet of $U_q(u_4)$ and 8-plet of its subalgebra $U_q(u_3)$ respectively; $l_2$ means isodoublet $(p + 1, p - 1)$ of nucleons, and $l_Q$ labels charge states within it. The quantity $l_2$ equal to $(p, p - 2)$ characterizes another ($\Xi$-) isodoublet belonging to octet. State vectors for the rest of members of the 20-plet are constructed analogously.

Mass operator, according to the concept of pseudounitary dynamical group [7-8] extended to present $(q$-deformed) case, is constructed in terms of those "noncompact" generators of $U_q(u_{4,1})$ which respect the isospin-hypercharge symmetry $U_q(u_2)$, but break all higher (flavour) symmetries. We take it in the form

$$\hat{M}_4 = M_o^{(4)} + \gamma A_{45}A_{54} + \delta A_{54}A_{45} + \alpha A_{35}A_{53} + \beta A_{53}A_{35} + \bar{\alpha} \bar{A}_{35}A_{53} + \bar{\beta} A_{53}A_{35}$$

and put $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$ in order to reduce the number of independent parameters. As a result of calculation of the matrix elements $\langle N | \hat{M}_4 | N \rangle \equiv m_N$, $\langle \Xi | \hat{M}_4 | \Xi \rangle \equiv m_\Xi$, etc., within the representation $D_{12}^{p-1,3,4,5} (p - 1, p - 3, p - 4; p, p - 2)$ ($p$ being an arbitrary fixed integer), we obtain the following expressions for masses of baryons $\frac{1}{2}^+$ belonging to 20-plet of $U_q(u_4)$.

(i) Octet baryons:

$$m_N = M_8 + \frac{[2][3]}{[6]} (5) \alpha + \beta,$$

$$m_\Lambda = M_8 + \frac{1}{[6]} (\frac{5[4]^2}{[2]^2} + [2]^2) \alpha + \frac{[2]^2[3]}{[6]} \beta,$$

$$m_\Xi = M_8 + \frac{1}{[6]} (2[5] + [3]) + \frac{[5][3]}{[2]} ([5] - [3] - 2) \alpha + \frac{[2]^2[4]}{[6]} \beta,$$

$$m_\Sigma = M_8 + \frac{[4][5]}{[2]^2[6]} \alpha + \frac{[2][4]}{[6]} \beta.$$
(ii) triplet baryons:

\[ m_{\Xi_{cc}} = M_3 + \frac{[2][3][5]}{6} \alpha + \frac{[2]}{6} \left( ([4] - [2])^2 - 1 \right) \beta, \]

\[ m_{\Omega_{cc}} = M_3 + \frac{[2][3][5]}{6} \alpha + \frac{[2][4]}{6} ([3] - 2) \beta; \]  

(14)

(iii) antitriplet baryons:

\[ m_{\Xi_c'} = M_3^* + \frac{[2][3][5]}{6} \alpha + \frac{[2][3]}{6} ([3] - 2) \beta, \]

\[ m_{\Lambda_c} = M_3^* + \frac{[3]}{6} ([3] - 2)([5] + 2) \alpha + \frac{[3]^2}{6} ([3] - 2) \beta; \]  

(15)

(iv) sextet baryons:

\[ m_{\Sigma_c} = M_6 + \frac{[2][3][5]}{6} \alpha + \frac{[2]}{6} \left( 1 + ([3] - 1)([3] - 2) \right) \beta, \]

\[ m_{\Xi_c} = M_6 + \frac{[3] + 2([3] - 1)[5]}{6} \alpha + \frac{2[5] - [3] + 4}{6} \beta, \]

\[ m_{\Omega_0^c} = M_6 + \frac{[2]}{6} ([3] + ([3] - 2)[5]) \alpha + \frac{[2][4]}{6} \beta. \]  

(16)

In the above expressions for masses, we have used the notations

\[ M_8 = M_0 + \frac{[3]}{6} ([5] \gamma + \delta), \]

\[ M_3 = M_0 + \frac{[2][4]}{6} (\gamma + \delta), \]

\[ M_3^* = M_0 + \frac{[3]}{6} ([4] \gamma + [2] \delta), \]

\[ M_6 = M_0 + \frac{1}{6} ([2][5] \gamma + [4] \delta). \]

As it is seen, the integer \( p \) does not enter the obtained expressions for baryon masses.

In other words, the approach which we follow here is insensitive to the substitution \( U_q(su_{4,1}) \rightarrow U_q(u_{4,1}) \).

Let us check now that in the 'classical' limit \( q \rightarrow 1 \) octet masses satisfy GMO-relation. Indeed, for \( q = 1 \) we have that \( m_N = M_8 + 5 \alpha + \beta, \)

\[ m_\Lambda = M_8 + 4 \alpha + 2 \beta, \]

\[ m_\Xi = M_8 + \frac{8}{3} (\alpha + \beta), \]

\[ m_\Sigma = M_8 + \frac{10}{3} \alpha + \frac{4}{3} \beta \] (here \( M_8 = M_0 + \frac{15}{6} \gamma + \frac{1}{2} \delta \)), and the relation (1) is obviously satisfied.

5. \( q \)-ANALOG OF BARYON OCTET MASS RELATION
We are predominantly interested in an octet MR with \( q \)-dependent coefficients. From eqns. (13), through the differences \([2]m_N - m_\Lambda\), \([2]m_\Sigma - m_\Xi\), and \([2](m_\Sigma - m_N) + m_\Lambda - m_\Xi\), it is straightforward to find the formulas which express independent parameters \( M_8\), \( \alpha \) and \( \beta \) in terms of baryon masses:

\[
\alpha = \frac{[2](m_\Sigma - m_N) + m_\Lambda - m_\Xi}{3 \{3 + [2]^2([4] - [2]) - 3[5]\}},
\]
\[
\beta = \frac{[6]}{[2]^2}(m_\Lambda - m_\Sigma) - \alpha,
\]
\[
M_8 = \frac{[2]m_\Sigma - m_\Xi}{[2] - 1} - \frac{[2]^2}{[2] - 1}\alpha.
\]

Substituting these expressions into the last relation of eq. (13), we arrive at the desired \( q \)-analog MR, namely,

\[
[2]m_N + \frac{[2]}{[2] - 1}m_\Xi = [3]m_\Lambda + \left(\frac{[2]^2}{[2] - 1} - [3]\right)m_\Sigma
+ \frac{A_q}{B_q}([2]m_N + m_\Xi - m_\Lambda - [2]m_\Sigma) \tag{17}
\]

where

\[
A_q = 2^4 + 2^3([5] - [4]) + [2]^2([6] - [5]) - [2]([6] + [4]^2) + [4]^2,
\tag{18}
\]
\[
B_q = \left([2]^3 - [2]^2[4] + 3[5] - [3]\right)([2] - 1). \tag{19}
\]

The relation (17) constitutes our main result. Here we observe nontriviality of the coefficients at \( m_\Xi \) and \( m_\Sigma \) and, as most unexpected thing, the appearance of that additional structure (second line in (17)) with \( A_q/B_q \) as its coefficient.

Strictly speaking, the relation just obtained is not a mass relation. However, at any fixed value of \( q \) it yields some ‘candidate’ MSR. For this reason the \( q \)-analog relation (17) may be viewed as a continuum of candidate MSR’s for baryon masses, only few of which may be considered as realistic mass sum rules.

6. DEFINING \( q \)-POLYNOMIAL AND A NEW BARYON MASS SUM RULE

Now the problem consists in finding the value(s) of deformation parameter at which the relation (17) yields most realistic MSR(s). Clearly, a straightforward way to proceed would be to insert the empirical data for the octet baryon masses into (17) and then solve the equation with respect to \( q \). However, we think of this way as not the best one for two reasons: (i) the equation for \( q \) this way appears to be rather complicated; (ii) so obtained values of deformation parameter would be neccessarily ‘non-rigid’ ones reflecting approximate procedure of solving the equation as well as errors of experimental data and averaging over isomultiplets. Fortunately, there exists another approach which is somewhat analogous to reasonings used in [5,6] for the case of vector mesons and which
leads to 'rigidly fixed' values of $q$. To this end, let us return again to the 'classical' case. As already mentioned, the value $q = 1$ must result in the standard GMO-relation (a kind of the 'correspondence principle'). Indeed, $A_{q=1} = 0$, $(B_{q=1} \neq 0)$, and $m_N + m_\Xi = \frac{3}{2} m_\Lambda + \frac{1}{2} m_\Sigma$ results. But now we observe the point of oversimplification (due to vanishing of $A_q$ at $q = 1$) when reducing the $q$-analog to GMO-relation. Adopting this as a hint of how to search other candidate values of $q$, we proceed by rewrighting the $A_q$ (which is 7-th order polynomial in $q$-deuce $[2]_q$) in its completely factorized form:

$$A_q = ([2] - 2)[2]^3([4] - [2]) = ([2] - 2)[2]^4([2]^2 - 3)$$

(18)

(the recursion $[n]_q = [2]_q[n - 1]_q - [n - 2]_q$ for $q$-numbers is useful in doing this). Since the 'classical' GMO-relation corresponds to vanishing of $A_q$ because of the fact that

(i) $(2 - 2) = 0$,

it is natural to examine the remaining cases when $A_q$ turns into zero:

(ii) $[2] = 0$;

(iii) $[4] - 2 \equiv [2]([3] - 2) \equiv [2][2]^2 - 3 = 0$, $[2] \neq 0$.

The case (ii) leads to the MR $m_\Lambda = m_\Sigma$ which is not very good since, with empirical data, its accuracy is $\approx 6.5\%$. It is interesting, nevertheless, to compare this case with the second nonet Okubo’s formula $m_\rho = m_\omega$ (which was shown to follow from the $q$-analog of vector meson MR, see [5]), just if the same restriction $[2]_q = 0$ has been applied). Remark that in both meson and baryon cases, these MSR’s relate isosinglet and isotriplet masses.

Now let us consider the most interesting case (iii), that is, the values of $q$ that solve $[2]_q = \pm \sqrt{3}$. These are respectively $q_+ = e^{\frac{i \pi}{6}}$ and $q_- = e^{\frac{i 5 \pi}{6}}$. At these values, $[4]_{q_\pm} = \pm \sqrt{3}$ (that is, both $q_+$ and $q_-$ solve the equation $[4]_q - [2]_q = 0$) and also $[3]_{q_\pm} = 2$ ($q_+$ and $q_-$ both solve the equation $[3]_q - 2 = 0$).

Two candidate MSR’s follow from (17) at $q_+$, $q_-$: the relation

$$m_N + \frac{1+\sqrt{3}}{2} m_\Xi = \frac{2}{\sqrt{3}} m_\Lambda + \frac{9-\sqrt{3}}{6} m_\Sigma$$

(20)

and the relation

$$m_N + \frac{1-\sqrt{3}}{2} m_\Xi = -\frac{2}{\sqrt{3}} m_\Lambda + \frac{9+\sqrt{3}}{6} m_\Sigma$$

(20)

respectively. The second candidate MSR (which corresponds to $q_-$) shows bad agreement with data. Fixing $q = q_+$, however, we get a surprizingly good mass relation: with empirical values [14] for octet $1^+$ baryon masses ($m_N = 938.9$ MeV, $m_\Xi = 1318.1$ MeV, $m_\Lambda = 1115.6$ MeV, and $m_\Sigma = 1193.1$ MeV) we have $2739.5$ MeV $\approx 2733.4$ MeV. That is, eq.(20) holds with $0.22\%$ accuracy! For comparison recall that usual GMO-relation (1) is satisfied within $0.57\%$.

Strictly speaking, the case of $q \rightarrow q_+$ essentially differs from the classical ($q = 1$) situation, besides irrationality of the coefficients in (20), yet by the following peculiarity.
All the masses in (13)-(16) become infinite in this case because of pole singularity, since $[6] \equiv [2][3][3] - 2 \rightarrow 0$ when $q$ tends to $q_+$. In order to circumvent this difficulty, one has first to interpret the invariant (background for the 20-plet) mass $M_0$ and all the masses $m_{B_i}$ (with $B_i$ running over the set of baryon symbols in formulas (13)-(16)) as infinite 'bare' masses, then going over to finite 'physical' masses by making use of a multiplicative renormalization through the substitution $M_0 \rightarrow \frac{M_0(\text{phys})}{[6]}$, $m_{B_i} \rightarrow \frac{m_{B_i}(\text{phys})}{[6]}$. Let us remark that such a substitution does not affect the explicit form of the $q$-analog mass relation (17) and the MSR (20).

7. QUANTUM ALGEBRA $U_q(u_{4,1})$ REMOVES 'DEGENERACY'

Calculations analogous to those of section 5 were performed also for another specific representation of dynamical algebra. Namely, with the 20-plet $(p + 1, p - 1, p - 3, p - 4)$ of $U_q(u_{4,1})$ embedded into the (integer type) irrep $\tilde{D}_{32}^p(p - 1, p - 3, p - 4; p - 4, p - 2)$ of $U_q(u_{4,1})$, we have obtained the expressions for baryon masses,

$$m_N = M_8' + \frac{[2][3]}{[6]}(\alpha + [5]\beta),$$
$$m_A = M_8' + \frac{[3]}{[6]}([3]^2 + [3] - 4)\alpha + \frac{[2][3][5]}{[6]}\beta,$$
$$m_\Xi = M_8' + \frac{[2]}{[6]}([3]^3 - 4[3] + 1)\alpha + \frac{[2][4][5]}{[6]}\beta,$$
$$m_\Sigma = M_8' + \frac{([3] - 1)^2}{[6]}\alpha + \frac{[2][4][5]}{[6]}\beta,$$

(here $M_8' \equiv M_0 + \frac{[3]}{[6]}\gamma + \frac{[3][5]}{[6]}\delta$), as well as the following $q$-analog of octet MR

$$m_N - m_A = \frac{[2] - 1}{[2]^2} M_C + \frac{[3][([2] - 1)]}{[2]^2} M_D = \frac{\tilde{A}_q}{B_q} M_C$$

(22)

where

$$M_C = [2](m_N - m_\Sigma) + m_\Xi - m_A,$$
$$M_D = ([2] + 1)m_A - [2]m_N - m_\Xi,$$
$$\tilde{B}_q = [2][3]^2 - [3]^2 - [2][3] - 2[2] + 5,$$

and the defining polynomial is of the form

$$\tilde{A}_q = ([2] - 2)([3]^2 - 5).$$

(23)

Besides the 'classical' root $[2]_q = 2$ (equivalent to $q = 1$) which determines the standard GMO-relation (1), the polynomial $\tilde{A}_q$ has four roots more, corresponding to $[3]_q = \pm\sqrt{5}$. For instance, with $[2]_q = \sqrt{1 + \sqrt{5}}$ (this choice provides $[3]_q = \sqrt{5}$) we have the MSR

$$2(\sqrt{5} - \sqrt{5 + 1})m_N + 2(\sqrt{5 + 1} - 1)m_\Xi =$$
$$\left(2\sqrt{5} - 4 + \sqrt{5 - 1}\right)m_A + \left(2 - \sqrt{5 - 1}\right)m_\Sigma.$$  

(24)
Examination of this MSR with inserted empirical masses shows that it is not very satisfactory (the accuracy is $\approx 2.7\%$). Nevertheless, one could conclude from this second example that distinct dynamical representations may yield essentially different $q$-analogs of octet MSR, with different defining $q$-polynomials. While the presence of factor $[2]_q - 2$ is common feature of all defining polynomials, the difference would lie in sets of extra roots (compare $[2]_q = 0$, $[2]_q = \pm \sqrt{3}$ of the polynomial (31') and the roots $[2]_q^2 = 1 \pm \sqrt{5}$ of $\tilde{A}_q$ in this section). Since when applying classical Lie algebra $u(4,1)$ all the representations yield [7] the GMO-relation and nothing else (a kind of 'degeneracy'), we could say that application of dynamical $q$-algebra $U_q(u_{4,1})$, for octet baryon mass relations, removes this degeneracy.

It turns out, however, that by processing the expressions (21) in some another way we arrive at the $q$-analog of octet MR which coincides with the first $q$-MR, eq.(17), in all the points (coefficients at masses, the combination of masses in curly brackets and, most important, the defining polynomial $A_q$ in front of the $M_C$ in (17) ) except for the change $B_q \to \tilde{B}_q$ of the $q$-polynomial in denominator (it plays no essential role in reducing to MSR's (1) and (20) ). Anyway, we conclude that application of the $q$-algebra $U_q(u_{4,1})$ as dynamical one 'removes degeneracy' at least in the sense that in the framework of the quantum algebra we get, besides eq.(1), mass sum rules of novel type (including such accurate one as eq.(20) ). Those novel MSR's seem to reflect some interesting hadronic dynamics, and this certainly deserves further study.

8. CONCLUSIONS AND OUTLOOK

Extending the approach of dynamical (pseudo)unitary groups to the quantum groups $U_q(su_n)$ ($n$-flavor symmetry) and $U_q(u_{n,1})$ ('dynamical' symmetry) we have obtained at $n = 4$ the $q$-dependent expressions for masses of baryons $^{1\,2}\!\!B$ from which $q$-MR's follow.

The $q$-deformed relation (17) is of interest, first of all, as a 'continuum' of possible MSR’s and also due to the fact that it contains unusual mass-dependent term with $A_q/B_q$ in front of it. It is just this point where the concept of *defining $q$-polynomial* arises: the polynomial $A_q$, by means of its roots, determines concrete octet baryon MSR’s (namely, the classical GMO-relation (1) and this novel, very accurate, relation (20) with $\sqrt{3}$ contained in some of its coefficients, as most successful ones; the MSR (20') and the relation $m_\Lambda = m_\Sigma$, as less successful MSR’s).

It is important to stress once more that, *due to its irrational coefficients*, the relation (20) is of unconventional nature. This property is obviously connected with the root of unity $q_+ = e^{i\pi/6}$ and probably reflects some 'nonperturbative' (topological) information encapsulated in the model under consideration at such value of deformation parameter.

To make last assertion somewhat more transparent, it is useful to present the $q$-MR (17) in the form

$$(1 - \Delta_N)m_N + (1 + \Delta_\Xi)m_\Xi = \left(\frac{3}{2} - \Delta_\Lambda\right)m_\Lambda + \left(\frac{1}{2} + \Delta_\Sigma\right)m_\Sigma,$$

where

$$\Delta_N \equiv \frac{A_q}{B_q}, \quad \Delta_\Xi \equiv -\frac{A_q}{2|B_q|} + \frac{1}{|2| - 1} - 1,$$

$$\Delta_\Lambda \equiv \frac{A_q}{2|B_q|} + \frac{1}{|2| - 1} - 1.$$
\[ \Delta_A \equiv \frac{A_q}{B_q} \left[ 3 - \frac{3}{2} \right], \quad \Delta_\Sigma \equiv -\frac{A_q}{B_q} \left[ 2 - 1 - \frac{3}{2} - \frac{1}{2} \right]. \quad (26) \]

Since \( A_q = 0 \) at \( q = 1 \), we have that all \( \Delta_k \) (here \( k \) takes the ‘values’ \( N, \Xi, \Lambda \) and \( \Sigma \)) equal zero in the classical limit. Therefore, it is natural to consider the quantities \( \Delta_k \) at \( q \neq 1 \) as ‘corrections to the classical coefficients’ due to \( q \)-deformation. Obviously, at values of \( q \) which are very close to unity these corrections do not deviate substantially from zero.

At \( q = q_+ \) again \( A_q \) vanishes. However, now all \( \Delta_k \) other than \( \Delta_N \) are not small (‘perturbative’) quantities, but become of the order of magnitude comparable with the corresponding classical coefficients (for instance, \( \Delta_\Sigma = \frac{\sqrt{3}}{\sqrt{3} - 1} - \frac{2}{\sqrt{3}} - \frac{1}{2} \approx 0.71 \) to be compared with the coefficient \( \frac{1}{2} \) in classical MSR (1)).

Of course, further study is needed in order to clarify, among others, such issues as: (in)dependence of the results, concerning \( q \)-analog MR’s for baryons \( \frac{1}{2}^+ \) and the appearance of new resulting mass sum rules (like the MSR (20)), on the choice of dynamical representation of the quantum algebra \( U_q(u_{4,1}) \); the issue of reducibility of the infinite dimensional representations used within our approach at \( q \) being specific roots of unity, as well as details of dynamics at those values of the deformation parameter; the question about possibility to associate with baryons some topological structures (knot or links) in a manner more or less similar to the treatment in the case of vector mesons [6]. We hope to analyze these problems in subsequent publications.

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