HARMONIC QUASI-ISOMETRIC MAPS INTO GROMOV HYPERBOLIC CAT(0)-SPACES

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ABSTRACT. We show that for every quasi-isometric map from a Hadamard manifold of pinched negative curvature to a locally compact, Gromov hyperbolic, CAT(0)-space there exists an energy minimizing harmonic map at finite distance. This harmonic map is moreover Lipschitz. This generalizes a recent result of Benoist-Hulin.

1. Introduction

The well-known Schoen-Li-Wang conjecture asserts that every quasiconformal self-homeomorphism of the boundary at infinity of a rank one symmetric space $M$ extends to a unique harmonic map from $M$ to itself. This conjecture has recently been settled in the affirmative in a series of break-through papers by Markovic [21], [22], Lemm-Markovic [18], and Benoist-Hulin [2]. Earlier partial results were proved in [23], [27], [11], [20], [5], see also the references in [2]. Benoist-Hulin’s result [2], which goes beyond the Schoen-Li-Wang conjecture, shows that every quasi-isometric map between rank one symmetric spaces $X$ and $Y$ is at finite distance of a unique harmonic map. Even more recently, Benoist-Hulin [3] extended their result in [2] to the case when $X$ and $Y$ are Hadamard manifolds of pinched negative curvature, i.e. simply connected Riemannian manifolds of sectional curvature bounded by $-b^2 \leq K_X, K_Y \leq -a^2$ for some constants $a, b > 0$.

The aim of the present note is to further generalize the existence part of Benoist-Hulin’s result [3] by relaxing the curvature conditions on the target space $Y$. Our methods even work in the context of singular metric spaces $Y$. Recall that Korevaar-Schoen [15] developed a theory of Sobolev and harmonic maps from a Riemannian domain into a complete metric space. We refer to [15] and to Section 3 of the present note for the definition. Our main theorem is:

Theorem 1.1. Let $X$ be a Hadamard manifold of pinched negative curvature and let $Y$ be a locally compact, Gromov hyperbolic, CAT(0)-space. Then for every quasi-isometric map $f: X \to Y$ there exists an energy minimizing harmonic map $u: X \to Y$ which is globally Lipschitz continuous and at bounded distance from $f$.

The precise meaning of being at bounded distance from $f$ is that

$$\sup_{x \in X} d_Y(u(x), f(x)) < \infty.$$ 

It follows in particular that $u$ is also quasi-isometric. Recall that a map $f: X \to Y$ between metric spaces $(X,d_X)$ and $(Y,d_Y)$ is called $(L,c)$-quasi-isometric if

$$L^{-1} \cdot d_X(x,x') - c \leq d_Y(f(x), f(x')) \leq L \cdot d_X(x,x') + c.$$
for all \( x, x' \in X \). The map \( f \) is called quasi-isometric if it is \( (L, c) \)-quasi-isometric for some \( L \geq 1 \) and \( c \geq 0 \). A quasi-isometric map is thus biLipschitz at large scales but no restriction is posed on small scales. In particular, \( f \) need not be continuous. Notice moreover that the image \( f(X) \) need not be quasi-dense in \( Y \).

Recall that a geodesic metric space \( Y \) is called CAT(0) if geodesic triangles in \( Y \) are at least as thin as their Euclidean comparison triangles. Every Hadamard manifold is CAT(0). A geodesic metric space \( Y \) is called Gromov hyperbolic if there exists \( \delta \geq 0 \) such that each side of a geodesic triangle in \( Y \) lies in the \( \delta \)-neighborhood of the other two sides. This is a large scale notion of negative curvature. It poses no restriction on small scales. We refer for example to [6], [10], [9] for comprehensive accounts on CAT(0)-spaces and Gromov hyperbolicity. Since every Hadamard manifold \( Y \) of pinched negative curvature is locally compact, Gromov hyperbolic, and CAT(0) our Theorem 1.1 in particular recovers the existence part of Benoist-Hulin’s result [3, Theorem 1.1].

Unlike in the setting of [6], energy minimizing harmonic maps at finite distance from a fixed quasi-isometric map need not be unique in our more general setting. Indeed, if \( X = \mathbb{H}^2 \) is the hyperbolic plane and \( Y := \mathbb{H}^2 \times [0,1] \) then the maps \( u_t(z) := (z,t) \) for \( t \in [0,1] \) are isometric and energy minimizing harmonic and have finite distance from each other. In the context of singular target spaces uniqueness was shown in [19] for harmonic maps at finite distance from a quasi-isometry between a cocompact Hadamard manifold and a cocompact CAT(0)-space with \( \kappa < 0 \).

The main strategy of proof of our Theorem 1.1 is the same as that in [3], and many of our arguments are in fact similar to those in [3]. On the one hand, existence and (local) Lipschitz regularity of energy minimizing harmonic maps is known in our more general context, see [15]. On the other hand, the smooth structure of the target space \( Y \) and the pinched negative curvature condition on \( Y \) are crucially used at several places in [3]. This is for example essential when establishing bounds on the distance between a quasi-isometric map \( f \) and a harmonic map. One of the principal new ingredients in our proof of similar bounds in our more general context is the use of the Bonk-Schramm embedding theorem [4]. This together with an argument about injective hulls, essentially due to [10], allows us to rough-isometrically embed the (non-geodesic) image \( f(X) \) into the hyperbolic \( k \)-space \( \mathbb{H}^k \) of constant curvature \(-1\) for some \( k \in \mathbb{N} \). The rough-isometric condition, which is much stronger than the quasi-isometric condition, then allows us to prove estimates on the distance between a quasi-isometric map \( f \) and an energy minimizing harmonic map similarly to [3]. A further but more minor difference between our arguments and those in [3] is that we consistently work with the Gromov product in the target space \( Y \) whereas the arguments in [3] rely on an interplay between estimates on the Gromov product and angle estimates. Such estimates on angles are not available in our setting since they require a strictly negative upper curvature bound.

2. Preliminaries

2.1. Basic notation. All metric spaces in our text will be complete. Let \((X,d)\) be a metric space. The open and closed balls in \( X \) centered at \( x \in X \) and of radius \( r > 0 \) are denoted by \( B(x,r) := \{ x' \in X : d(x,x') < r \} \) and \( \overline{B}(x,r) := \{ x' \in X : d(x,x') \leq r \} \), respectively. The distance sphere is \( S(x,r) := \{ x' \in X : d(x,x') = r \} \).

The Hausdorff \( n \)-measure on \( X \) will be denoted by \( \mathcal{H}^n \). The normalization factor
is chosen in such a way that $\mathcal{H}^n$ equals the Lebesgue measure on Euclidean $\mathbb{R}^n$. In particular, if $X$ is a Riemannian manifold of dimension $n$ then $\mathcal{H}^n$ equals the Riemannian volume. The averaged integral will be denoted by

$$
\int_A f \, d\mathcal{H}^n := (\mathcal{H}^n(A))^{-1} \cdot \int_A f \, d\mathcal{H}^n.
$$

2.2. Some Riemannian preliminaries. Let $M$ be a Riemannian manifold. The differential of a smooth function $f: M \to \mathbb{R}$ will be denoted by $Df$. The hessian $D^2f$ of $f$ is the 2-tensor satisfying

$$
D^2f(X, X') = X(X'(f)) - \langle \nabla_X X', f \rangle
$$

for all vector fields $X, X'$ on $M$. The trace of the hessian of $f$ is the Laplace of $f$ and denoted $\Delta f$. The function $f$ is called harmonic if $\Delta f \equiv 0$. If $M$ is an $n$-dimensional Hadamard manifold of sectional curvature $\leq -b^2 \leq K_M \leq -a^2$ for some $a, b > 0$ then the hessian of the distance function $d(x_0) \to x_0 \in M$ satisfies

$$
(1) \ a \coth(ad_{x_0}) \cdot (g - Dd_{x_0} \otimes Dd_{x_0}) \leq D^2d_{x_0} \leq b \coth(bd_{x_0}) \cdot (g - Dd_{x_0} \otimes Dd_{x_0})
$$

on $M \setminus \{x_0\}$, where $g$ denotes the Riemannian metric on $M$. This follows from the hyperbolic law of cosines and comparison estimates, see e.g. [2]. In particular, the laplacian of $d_{x_0}$ satisfies $\Delta d_{x_0} \geq a \cdot (n - 1)$ on $M \setminus \{x_0\}$.

Let $\varphi: M \to N$ be a smooth map into another Riemannian manifolds $N$. We denote by $D\varphi$ the differential of $\varphi$. The second covariant derivative of $\varphi$ is the vector-valued 2-tensor which satisfies

$$
D^2\varphi(X, X') = \nabla_X (D\varphi(X')) - D\varphi(\nabla_X X')
$$

for all vector fields $X, X'$ on $M$, where $\nabla$ denotes the pullback under $\varphi$ of the Riemannian connection on $N$. The trace of $D^2\varphi$ is called the tension field of $\varphi$ and denoted $\tau(\varphi)$. If $\varphi: M \to N$ and $h: N \to \mathbb{R}$ are smooth then one calculates that

$$
(2) \ \Delta (h \circ \varphi) = Dh(\tau(\varphi)) + \sum_{i=1}^n D^2h(D\varphi(e_i), D\varphi(e_i)),
$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis in a tangent space of $M$. The map $\varphi$ is called harmonic if $\tau(\varphi) \equiv 0$.

2.3. Gromov hyperbolicity. Let $(Y, d)$ be a metric space. Recall that the Gromov product of $x, y \in Y$ with respect to a basepoint $w \in Y$ is defined by

$$
(x \mid y)_w := \frac{1}{2} \left[ d(x, w) + d(y, w) - d(x, y) \right].
$$

\textbf{Definition 2.1.} A metric space $Y$ is called $\delta$-hyperbolic, $\delta \geq 0$, if

$$
(x \mid z)_w \geq \min \{ (x \mid y)_w, (y \mid z)_w \} - \delta
$$

for all $x, y, z, w \in Y$. The space is called Gromov hyperbolic if it is $\delta$-hyperbolic for some $\delta \geq 0$.

A geodesic metric space $Y$ is Gromov hyperbolic in the sense of the definition above if and only if there exists $\bar{\delta} \geq 0$ such that every side of a geodesic triangle in $Y$ is contained in the $\bar{\delta}$-neighborhood of the other two sides, see [6] Proposition III.H.1.22.

For a proof of the following lemma see for example [28] Theorem 3.21.
Lemma 2.2. Let $f: X \to Y$ be an $(L, c)$-quasi-isometric map between geodesic $\delta$-hyperbolic metric spaces $X$ and $Y$. Then there exists a constant $c'$ only depending on $L, c,$ and $\delta$ such that for all $x, x', w \in X$ we have

$$L^{-1} \cdot (x | x')_w - c' \leq (f(x) | f(x'))_{f(w)} \leq L \cdot (x | x')_w + c'.$$

The next lemma is also known as exponential divergence of geodesics.

Lemma 2.3. Let $(Y, d)$ be a geodesic $\delta$-hyperbolic metric space. Then there exists $\delta' > 0$ depending only on $\delta$ with the following property. Let $r_1, r_2 \geq 1$ and let $\gamma, \eta: [0, r_1 + r_2] \to Y$ be two geodesics parametrized by arc-length with $\gamma(0) = \eta(0)$. If $d(\gamma(r_1), \eta(r_1)) > 3\delta'$ then any curve connecting $\gamma(r_1 + r_2)$ and $\eta(r_1 + r_2)$ outside the ball $B(\gamma(0), r_1 + r_2)$ has length at least $2^{r_2 - 1} \delta'^{-1}$.

This follows for example from the proof of [6, Proposition III.H.1.25].

2.4. Injective hulls of metric spaces. We will need the following construction of an injective hull due to Isbell [14]. Given a metric space $(Z, d)$, denote by $E(Z)$ the space of all functions $f: Z \to \mathbb{R}$ satisfying

$$f(z) + f(z') \geq d(z, z')$$

for all $z, z' \in Z$ and such that $f$ is extremal in the following sense. If $g: X \to \mathbb{R}$ is another function satisfying (4) and $g \leq f$ then $g = f$. The space $E(Z)$, when equipped with the supremum norm, is called the injective hull of $Z$. It is an injective metric space in the sense that for every subset $A$ of a metric space $B$ and every 1-Lipschitz map $\varphi: A \to E(Z)$ there exists a 1-Lipschitz extension $\tilde{\varphi}: B \to E(Z)$ of $\varphi$. In particular, it follows that $E(Z)$ is a geodesic metric space. The space $Z$ embeds isometrically into $E(Z)$ via the map $z \mapsto d(z, \cdot)$. Moreover, if $Z$ is a subset of another metric space $Z'$ then there exists an isometric embedding $h: E(Z) \to E(Z')$ such that $h(f)|_Z = f$ for all $f \in E(Z)$, see [16, Proposition 3.5]. It was proved in [15, Proposition 1.3] that if $Z$ is Gromov hyperbolic then so is $E(Z)$ and that if $Z$ is moreover geodesic then $E(Z)$ also lies in finite Hausdorff distance of $Z$.

3. Sobolev maps into metric spaces

There are several equivalent definitions of Sobolev maps from a Riemannian domain to a complete metric space, see for example [14] and the approaches described therein. We will use the definition given by Korevaar-Schoen in [15]. As we will only deal with Sobolev maps of exponent $p = 2$ and defined on open balls in a Hadamard manifold, we will restrict to this setting.

Let $X$ be a Hadamard manifold of dimension $n \geq 2$ and let $\Omega \subset X$ be an open, bounded ball. Let $(Y, d_Y)$ be a complete metric space. We denote by $L^2(\Omega, Y)$ the space of all essentially separably valued Borel maps $u: \Omega \to Y$ such that for some and thus every $y_0 \in Y$ we have

$$\int_{\Omega} d_Y^2(y_0, u(x)) \, d\mathcal{H}^n(x) < \infty.$$
whenever \( x \in \Omega \) satisfies \( d(x, \partial \Omega) > \varepsilon \) and \( e_\varepsilon(x) = 0 \) otherwise. The map \( u \) is said to belong to \( W^{1,2}(\Omega, \gamma) \) if its energy, defined by

\[
E(u) := \sup_{f \in C_c(\Omega), 0 \leq f \leq 1} \left( \limsup_{\varepsilon \to 0} \int_\Omega f(x)e_\varepsilon(x) \, d\mathcal{H}^n(x) \right),
\]

is finite. If \( u \in W^{1,2}(\Omega, X) \) then there exists a function \( e_u \in L^1(\Omega) \), called the energy density function of \( u \), such that \( e_\varepsilon \, d\mathcal{H}^n \to e_u \, d\mathcal{H}^n \) as \( \varepsilon \to 0 \) and

\[
E(u) = \int_\Omega e_u(x) \, d\mathcal{H}^n(x),
\]

see [15, Theorems 1.5.1 and 1.10]. In the case that \( Y \) is a Riemannian manifold and \( u \) is smooth the energy defined in (5) coincides with the usual energy as defined for example in [3].

The trace of a Sobolev map \( u \in W^{1,2}(\Omega, \gamma) \) is denoted \( \text{tr}(u) \), see [15, Definition 1.12] for the definition. We mention here that if \( u \) has a continuous representative which has a continuous extension to \( \overline{\Omega} \), again denoted \( u \), then \( \text{tr}(u) = u|_{\partial \Omega} \).

**Definition 3.1.** A map \( u \in W^{1,2}(\Omega, \gamma) \) is said to be energy minimizing harmonic if \( E(u) \leq E(v) \) for all \( v \in W^{1,2}(\Omega, \gamma) \) with \( \text{tr}(v) = \text{tr}(u) \). A map \( u : X \to Y \) is called energy minimizing harmonic if its restriction to every bounded, open ball is energy minimizing harmonic.

It is well-known that if \( Y \) is a Hadamard manifold then a map \( u \in W^{1,2}(\Omega, \gamma) \) is energy minimizing harmonic in the sense above if and only if \( u \) is a harmonic map in the classical sense (vanishing tension field), see [24] and [25].

Now, let \( Y \) be a CAT(0)-space. It follows from [15, Theorem 2.2] that for every Lipschitz map \( f : \partial \Omega \to Y \) there exists a unique energy minimizing harmonic map \( u \in W^{1,2}(\Omega, \gamma) \) with \( \text{tr}(u) = f \). By [15, Theorem 2.4.6] and [26], the map \( u \) is locally Lipschitz continuous in \( \Omega \) and Hölder continuous up to the boundary, in particular \( u|_{\partial \Omega} = f \).

If \( u \in W^{1,2}(\Omega, \gamma) \) is energy minimizing harmonic then, by [7, Lemma 10.2], for every \( y_0 \in Y \) the function \( h : \Omega \to \mathbb{R} \) given by \( h(z) := d_Y(y_0, u(z)) \) is weakly subharmonic in the sense that \( \Delta h \geq 0 \) weakly. Recall that a function \( h \in W^{1,2}(\Omega) \) is said to satisfy \( \Delta h \geq \rho \) weakly for some function \( \rho \in L^1(\Omega) \) if

\[
-\int_\Omega (\nabla h, \nabla \varphi) \, d\mathcal{H}^n \geq \int_\Omega \rho \varphi \, d\mathcal{H}^n
\]

for all non-negative \( \varphi \in C_0^\infty(\Omega) \). By [13, Theorem 1], a continuous and weakly subharmonic function \( h : \overline{\Omega} \to \mathbb{R} \) with \( h|_{\partial \Omega} \leq 0 \) satisfies \( h \leq 0 \) on \( \overline{\Omega} \).

The following result will be used in the proof of Theorem 1.1

**Proposition 3.2.** Let \( X \) be a Hadamard manifold with sectional curvature bounded from below and let \( B = B(x, r) \) be an open ball in \( X \). Let \( u : B \to Y \) be an energy minimizing harmonic map into some CAT(0)-space \( Y \). If the image of \( u \) lies in some ball of radius \( R \) then \( u \) is CR-Lipschitz on the ball \( B(x, r/3) \), where \( C \geq 1 \) only depends on \( r \), the lower bound on sectional curvature of \( X \), and the dimension of \( X \).

**Proof.** Suppose the curvature of \( X \) is bounded by \(-b^2 \leq K_X \leq 0 \) for some \( b > 0 \). By [30, Theorem 1.4], there exists a constant \( C_1 \) depending only on \( r \), the dimension \( n \),
of $X$, and $b$ such that $u$ is $\lambda$-Lipschitz on $B(x, r/3)$ with $\lambda \leq C_1 \cdot E(u|_{B(x,s)})^{\frac{1}{2}}$, where we have set $s := \frac{r}{2}$. It thus suffices to show that $E(u|_{B(x,s)})$ is bounded by $R^2$ times a constant depending only on $r$, $n$, and $b$. Let $y \in Y$ be such that the image of $u$ lies in the ball $B(y, R)$. It is not difficult to show that there exists a smooth function $\eta: X \to \mathbb{R}$ supported in $B(x, r)$ with $0 \leq \eta \leq 1$ everywhere, such that $\eta = 1$ on $B(x, s)$, and $|\Delta \eta| \leq K$ everywhere for some constant $K$ depending only on $r$, $b$, and $n$.

By [26 Equation (6)], we have $\Delta d^2_B(y, u(x')) \geq 2e_u(x')$ weakly, where $e_u$ denotes the energy density of $u$. We thus obtain

$$2E(u|_{B(x,s)}) \leq 2 \int_{B(x,r)} \eta(x')e_u(x') \, d\mathcal{H}^n(x')$$

$$\leq \int_{B(x,r)} \Delta \eta(x') \cdot d^2_B(y, u(x')) \, d\mathcal{H}^n(x')$$

$$\leq K \cdot R^2 \cdot \mathcal{H}^n(B(x, r)).$$

It follows that the Lipschitz constant $\lambda$ of $u$ on $B(x, r/3)$ is bounded by

$$\lambda \leq C_1 \cdot E(u|_{B(x,s)})^{\frac{1}{2}} \leq CR$$

for some constant $C$ depending on $r$, $b$, and $n$. This completes the proof. \hfill \Box

4. Lipschitz quasi-isometric maps

We will need:

**Proposition 4.1.** Let $X$ be a Hadamard manifold with sectional curvature bounded from below and let $Y$ be a CAT(0)-space. Then every quasi-isometric map $f: X \to Y$ is at finite distance from a quasi-isometric map $\tilde{f}: X \to Y$ which is moreover Lipschitz.

We first show the following lemma which will also be used later.

**Lemma 4.2.** Let $X$ be a Hadamard manifold with sectional curvature bounded from below. Then for every $0 < r < R < \infty$ there exists $N \in \mathbb{N}$ such that every ball in $X$ of radius $R$ can be covered by $N$ balls of radius $r$.

**Proof.** Fix $0 < r < R < \infty$ and $x \in X$. Let $A \subset B(x, R)$ be a maximally $r$-separated subset. Thus, distinct points in $A$ have distance at least $r$ and the union of open $r$-balls centered at points in $A$ covers the ball $B(x, R)$. The open balls centered at points in $A$ and with radius $\frac{r}{2}$ are pairwise disjoint and contained in the ball $B(x, R + \frac{r}{2})$. Let $a_1, \ldots, a_k \in A$ be distinct points and denote by $m$ the volume of the ball of radius $\frac{r}{2}$ in Euclidean $\mathbb{R}^n$, where $n$ is the dimension of $X$. Volume comparison with Euclidean space yields $\mathcal{H}^n(B(a_i, \frac{r}{2})) \geq m$ for every $i$, see [8 Theorem 3.101]. By the same theorem, applied to a model space of constant negative curvature, we obtain

$$\mathcal{H}^n \left( B \left( x, R + \frac{r}{2} \right) \right) \leq M$$

for some $M$ depending only on $R$, $n$, and the lower bound on sectional curvature. We conclude that

$$k \cdot m \leq \sum_{i=1}^{k} \mathcal{H}^n \left( B \left( a_i, \frac{r}{2} \right) \right) = \mathcal{H}^n \left( \bigcup_{i=1}^{k} B \left( a_i, \frac{r}{2} \right) \right) \leq \mathcal{H}^n \left( B \left( x, R + \frac{r}{2} \right) \right) \leq M$$

If we set $\tilde{f}(x) := \sum_{i=1}^{k} \frac{1}{k} f(a_i)$, then $\tilde{f}$ is comparable to $f$ and $\tilde{f}$ is Lipschitz with constant $K \cdot \frac{1}{k}$, where $K$ is the Lipschitz constant of $f$. Thus, we have shown that $f$ is at finite distance from a Lipschitz map $\tilde{f}$. We conclude that $f$ is Lipschitz. \hfill \Box

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We first show the following lemma which will also be used later.

**Lemma 4.2.** Let $X$ be a Hadamard manifold with sectional curvature bounded from below. Then for every $0 < r < R < \infty$ there exists $N \in \mathbb{N}$ such that every ball in $X$ of radius $R$ can be covered by $N$ balls of radius $r$.

**Proof.** Fix $0 < r < R < \infty$ and $x \in X$. Let $A \subset B(x, R)$ be a maximally $r$-separated subset. Thus, distinct points in $A$ have distance at least $r$ and the union of open $r$-balls centered at points in $A$ covers the ball $B(x, R)$. The open balls centered at points in $A$ and with radius $\frac{r}{2}$ are pairwise disjoint and contained in the ball $B(x, R + \frac{r}{2})$. Let $a_1, \ldots, a_k \in A$ be distinct points and denote by $m$ the volume of the ball of radius $\frac{r}{2}$ in Euclidean $\mathbb{R}^n$, where $n$ is the dimension of $X$. Volume comparison with Euclidean space yields $\mathcal{H}^n(B(a_i, \frac{r}{2})) \geq m$ for every $i$, see [8 Theorem 3.101]. By the same theorem, applied to a model space of constant negative curvature, we obtain

$$\mathcal{H}^n \left( B \left( x, R + \frac{r}{2} \right) \right) \leq M$$

for some $M$ depending only on $R$, $n$, and the lower bound on sectional curvature. We conclude that

$$k \cdot m \leq \sum_{i=1}^{k} \mathcal{H}^n \left( B \left( a_i, \frac{r}{2} \right) \right) = \mathcal{H}^n \left( \bigcup_{i=1}^{k} B \left( a_i, \frac{r}{2} \right) \right) \leq \mathcal{H}^n \left( B \left( x, R + \frac{r}{2} \right) \right) \leq M$$

If we set $\tilde{f}(x) := \sum_{i=1}^{k} \frac{1}{k} f(a_i)$, then $\tilde{f}$ is comparable to $f$ and $\tilde{f}$ is Lipschitz with constant $K \cdot \frac{1}{k}$, where $K$ is the Lipschitz constant of $f$. Thus, we have shown that $f$ is at finite distance from a Lipschitz map $\tilde{f}$. We conclude that $f$ is Lipschitz. \hfill \Box
and hence that $k \leq \frac{4d}{m}$. This shows that $A$ has at most $\frac{4d}{m}$ points. Since the union of the open $r$-balls centered at points in $A$ covers the ball $B(x, R)$ the proof is complete. □

We now prove Proposition 4.2.

**Proof.** Let $f : X \to Y$ be an $(L, c)$-quasi-isometric map and let $Z \subset X$ be a maximally $1$-separated subset of $X$. It is easy to show that the family of balls given by $\{ B(z, 4) : z \in Z \}$ has bounded multiplicity. Indeed, let $x \in X$ and let $z_1, \ldots, z_k \in Z$ be distinct points such that $d(x, z_i) \leq 4$ for all $i$. By Lemma 4.2, the ball $B(x, 4)$ can be covered by $N$ open balls of radius $\frac{4}{L}$, where $N$ only depends on the lower bound on sectional curvature and the dimension of $X$. Since each of these balls can contain at most one element of $Z$ our claim follows.

Now, the restriction $f|_Z$ of $f$ to $Z$ is $(L + c)$-Lipschitz. Moreover, $Y$ is Lipschitz $k$-connected for every $k \in \mathbb{N}$. Thus, [29, Lemma 5.3] implies that the map $f|_Z$ has a Lipschitz extension $\tilde{f} : X \to Y$ whose Lipschitz constant only depends on $N$. By the triangle inequality, the map $\tilde{f}$ is at bounded distance from $f$ and hence also quasi-isometric. This concludes the proof. □

5. The boundary estimate

Let $(X, d_X)$ be a Hadamard manifold of dimension $n \geq 2$ and of pinched negative curvature $-b^2 \leq K_X \leq -a^2$ for some $a, b > 0$. Let $(Y, d_Y)$ be a CAT(0)-space which is locally compact and Gromov hyperbolic. Suppose $f : X \to Y$ is a quasi-isometric map which is moreover Lipschitz. Thus there exist $L \geq 1$ and $c > 0$ such that

$$L^{-1} \cdot d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq L \cdot d_X(x, x')$$

for all $x, x' \in X$. Let $x_0 \in X$ and set $\bar{B}_R := B(x_0, R)$ whenever $R > 0$. We furthermore set $S_R := S(x_0, R)$. There exists a unique continuous energy minimizing harmonic map $u_R : \bar{B}_R \to Y$ which coincides with $f$ on $S_R$, see Section 3. The main aim of this section is to establish:

**Proposition 5.1.** There exist constants $\alpha, \beta \geq 1$ such that for every $R > 0$ and $x \in \bar{B}_R$ we have

$$d_Y(f(x), u_R(x)) \leq \alpha \cdot d_X(x, S_R) + \beta.$$ 

The proof of the analogous result [3, Proposition 3.7] when $Y$ is a Hadamard manifold with curvature bounded from below heavily depends on the existence, established in [3, Proposition 2.4], of a smooth map at finite distance of $f$ with bounded first and second covariant derivative. In the singular setting we work in, such a result is of course not available. We circumvent this problem by using the following lemma.

**Lemma 5.2.** The set $f(X)$, equipped with the metric from $Y$, admits a rough-isometric map $\psi : f(X) \to \mathbb{H}^k$ for some $k \in \mathbb{N}$.

Recall that a map $\psi : Z \to W$ between metric spaces $(Z, d_Z)$ and $(W, d_W)$ is called $(\lambda, c)$-rough-isometric if

$$\lambda \cdot d_Z(z, z') - c \leq d_W(\psi(z), \psi(z')) \leq \lambda \cdot d_Z(z, z') + c$$

for all $z, z' \in Z$. The idea is to use the well-known Bonk-Schramm embedding theorem [4]. We cannot use their embedding theorem directly since $f(X)$ is not...
Proposition 2.4] constants $A_0$ and $B_0$.

Proof. Denote by $Z$ the set $f(X)$ equipped with the induced metric from $Y$. Denote by $E(Z)$ the injective hull of $Z$ and recall that $E(Z)$ is a geodesic metric space and that $Z$ embeds isometrically into $E(Z)$. Since $Z$ is Gromov hyperbolic (as a subset of $Y$) it follows from [16, Proposition 1.3] that $E(Z)$ is also Gromov hyperbolic.

We claim that $E(Z)$ is in finite Hausdorff distance of $Z$. For this, notice first that the space $E(Z)$ can be viewed as a subset of the injective hull $E(Y)$ of $Y$, see Section 2.2 above, and so the function $\psi$ isometrically into $E(Y)$, see also Section 3.1. By the stability of quasi-geodesics [6, Theorem III.H.1.7], the quasi-geodesic $f(x, x')$ is at distance at most $\delta_2$ from the geodesic $[z, z']$, where $\delta_2$ only depends on the Gromov hyperbolicity constant of $Y$ and the quasi-isometric constants of $f$. Thus $p$ lies at distance at most $\delta_1 + \delta_2$ from a point in $Z$. This proves our claim.

Since $E(Z)$ is at finite distance from $Z$ and $f$ is quasi-isometric it easily follows from Lemma 5.2 that $E(Z)$ has bounded growth at some scale as defined in [4]. That is, there exist $0 < r < R < \infty$ and $N \in \mathbb{N}$ such that every ball of radius $R$ in $E(Z)$ can be covered by at most $N$ balls of radius $r$. Since $E(Z)$ is also geodesic and Gromov hyperbolic it follows from the Bonk-Schramm embedding theorem [4] that $E(Z)$ admits a rough-isometric map $\psi: E(Z) \to \mathbb{H}^k$ for some $k \in \mathbb{N}$. Since $E(Z)$ contains $Z$ isometrically, the proof is complete.

We are now ready for the proof of Proposition 5.1. It uses Lemma 5.2 but is otherwise very similar to that of [3, Proposition 3.7].

Proof. Denote by $Z$ the set $f(X)$ equipped with the induced metric from $Y$. By Lemma 5.2 there exists a $(\lambda, \tilde{c})$-rough-isometric map $\psi: Z \to \mathbb{H}^k$ for some $\lambda, \tilde{c} > 0$ and $k \in \mathbb{N}$. Since the composition $\psi \circ f$ is quasi-isometric there exist by [3, Proposition 2.4] constants $A$ and $M$ and a smooth map $\tilde{f}: X \to \mathbb{H}^k$ such that

$$d_{\mathbb{H}^k}(\psi \circ f(x), \tilde{f}(x)) \leq M$$

for all $x \in X$ and such that the $\|D\tilde{f}\| \leq A$ and $\|\tau(\tilde{f})\| \leq A^2$.

Fix $R > 0$ and $x \in B_R$ and define two continuous functions $\varphi_1, \varphi_2: B_R \to \mathbb{R}$ by

$$\varphi_1(z) := \lambda \cdot d_Y(f(x), u_R(z))$$

and

$$\varphi_2(z) := \frac{2nA^2}{(n-1)} \cdot (d_X(x_0, z) - R).$$

By [7, Lemma 10.2] the function $\varphi_1$ is weakly subharmonic, see also Section 3 above. Furthermore, the function $d_{\mathbb{H}^k}(z) := d_X(x_0, z)$ satisfies $\Delta d_{\mathbb{H}^k} \geq a \cdot (n-1)$ away from $x_0$, see Section 2.2 above, and so the function $\varphi_2$ satisfies $\Delta \varphi_2 \geq 2nA^2$ weakly.

Now, we define a third function $\varphi_3: B_R \to \mathbb{R}$ as follows. Set $y_0 := \psi(f(x))$ and embed $\mathbb{H}^k$ isometrically into $\mathbb{H}^{k+1}$. We pick a point $y_1$ on the geodesic in $\mathbb{H}^{k+1}$ passing through $y_0$ perpendicular to $\mathbb{H}^k$ which is sufficiently far from $y_0$ and define

$$\varphi_3(z) := -d_{\mathbb{H}^{k+1}}(y_1, \tilde{f}(z)) + d_{\mathbb{H}^{k+1}}(y_0, y_1).$$
From (1) and (2) we see that the function \( \varphi_3 \) satisfies
\[
|\Delta \varphi_3| = |\Delta (d_{f_1} \circ \tilde{f})| \leq \|Dd_{f_1}\| \cdot \|\tau(\tilde{f})\| + n \cdot \coth(d_{\mathbb{H}^{k+1}}(y_0, y_1)) \cdot \|D\tilde{f}\|^2
\]
everywhere on \( B_R \), where \( d_{f_1} : \mathbb{H}^{k+1} \to \mathbb{R} \) is given by \( d_{f_1}(w) := d_{\mathbb{H}^{k+1}}(y_1, w) \). If \( y_1 \) is chosen sufficiently far from \( y_0 \) then it follows that \( |\Delta \varphi_3| \leq 2nA^2 \) everywhere on \( B_R \). Consequently, the continuous function \( \varphi : B_R \to \mathbb{R} \) defined by \( \varphi := \varphi_1 + \varphi_2 + \varphi_3 \) is weakly subharmonic.

We now estimate \( \varphi \) on \( S_R \). For this let \( z \in S_R \) and notice that \( \varphi_1(z) = \lambda \cdot d_Y(f(x), f(z)) \) and \( \varphi_2(z) = 0 \). Since \( y_1 \in \mathbb{H}^{k+1} \) is on the geodesic from \( y_0 \) perpendicular to \( \mathbb{H}^k \) it follows from the hyperbolic law of cosines that
\[
d_{\mathbb{H}^{k+1}}(y, y_1) \geq d_{\mathbb{H}^{k+1}}(y, y_0) + d_{\mathbb{H}^{k+1}}(y_0, y_1) - \log(4)
\]
for every \( y \in \mathbb{H}^k \subset \mathbb{H}^{k+1} \). From this we conclude that
\[
\varphi_3(z) = -d_{\mathbb{H}^{k+1}}(y_1, \tilde{f}(z)) + d_{\mathbb{H}^{k+1}}(y_0, y_1)
\leq -d_{\mathbb{H}^{k+1}}(y_0, \tilde{f}(z)) + \log(4)
\leq -d_{\mathbb{H}^{k+1}}(\psi(f(x)), \psi(f(z))) + M + \log(4)
\leq -\lambda d_Y(f(x), f(z)) + M',
\]
where \( M' := \tilde{c} + M + \log(4) \). It follows that \( \varphi(z) \leq M' \) for every \( z \in S_R \). Since \( \varphi \) is weakly subharmonic and continuous we thus obtain from (13) Theorem 1] or from Section [5] above that \( \varphi(z) \leq M' \) for all \( z \in B_R \) and, in particular, also for \( z = x \). Since \( |\varphi_3(x)| \leq M \) we conclude that
\[
d_Y(f(x), u_R(x)) \leq \frac{2nA^2}{\lambda a(n - 1)} \cdot d_X(x, S_R) + \frac{M' + M}{\lambda},
\]
which completes the proof. \( \square \)

6. Distance between harmonic and quasi-isometric maps

The proof of the following proposition is almost identical to that of [3] Proposition 3.5] except that we use the Gromov product instead of angle estimates. The latter are not available in our setting. Let \( (X, d_X), (Y, d_Y), f, x_0, B_R, \) and \( u_R \) be as in Section [5].

**Proposition 6.1.** There exists \( \rho \geq 1 \) such that for every \( R \geq 1 \) we have
\[
d_Y(u_R(x), f(x)) \leq \rho
\]
for all \( x \in B_R \).

We turn to the proof and let \( a, b > 0 \) be such that the sectional curvature of \( X \) satisfies \( -b^2 \leq K_X \leq -a^2 \). Let \( C \geq 1 \) be as in Proposition [3] for the radius \( r = 3 \).

Let \( \delta > 0 \) be such that \( Y \) is \( \delta \)-hyperbolic in the sense of Definition [2.1] and let \( \delta' > 0 \) be the constant from Lemma [2.3]. Denote by \( L \) and \( c \) the constants from [6] and by \( c' \) the constant from Lemma [2.2]. Let \( \alpha, \beta \geq 1 \) be as in Proposition [5.1]. Finally, let \( M \) and \( N \) be the constants appearing in the uniform estimates on the harmonic measure on distance spheres in \( X \) proved in (15) Theorem 1.1].

We choose \( T > 3 \) so large that inequality (12) below holds and that \( \gamma \), as defined in (9) below, satisfies \( \gamma < \frac{\delta}{4} \). We argue by contradiction and assume Proposition [6.1] is false. There then exists a sequence \( R_k \to \infty \) such that
\[
\rho_k := \sup \left\{ x \in B_{R_k} : d_Y(u_{R_k}(x), f(x)) \right\} \to \infty
\]

as \( k \to \infty \). We now abbreviate \( u_k := u_{R_k} \). Let \( k \geq 1 \) be sufficiently large so that
\[
\rho_k > \max \left\{ 2T \alpha + \beta, 2LT + 6 \delta', 4LMT \gamma^{-N} \right\}
\]
and so that \( \rho_k \) satisfies inequality (11) below. Since \( u_k \) and \( f \) are continuous on \( \tilde{B}_{R_k} \), the supremum in (7) is achieved at some point \( x \in \tilde{B}_{R_k} \). By Proposition 5.1 and the choice of \( \rho_k \) we have
\[
d_x(x, S_{R_k}) \geq \frac{\rho_k - \beta}{\alpha} > 2T.
\]
In particular, the ball \( \tilde{B}(x, 2T) \) is contained in \( B_{R_k} \). We first prove:

**Lemma 6.2.** The map \( u_k \) is \( 2C \rho_k \)-Lipschitz on \( \tilde{B}(x, T) \) and satisfies
\[
\frac{\rho_k}{2} \leq d_Y(f(x), u_k(z)) \leq \rho_k + LT
\]
for all \( z \in \tilde{B}(x, T) \).

**Proof.** For every \( z \in X \) with \( d(x, z) \leq 2T \) we have
\[
d_Y(f(x), u_k(z)) \leq d_Y(f(x), f(z)) + d_Y(f(z), u_k(z)) \leq Ld(x, z) + \rho_k,
\]
which implies in particular the second inequality in (8) and that \( u(\tilde{B}(x, 2T)) \subset B(f(x, 2\rho_k)) \) because \( 2LT < \rho_k \). Now, let \( z \in \tilde{B}(x, T) \). Since \( \tilde{B}(z, 3) \subset \tilde{B}(x, 2T) \) it thus follows from Proposition 5.2 applied with \( r = 3 \), that \( u_k \) is \( 2C \rho_k \)-Lipschitz on the ball \( \tilde{B}(z, 1) \) and hence also on the ball \( \tilde{B}(x, T) \) since balls in \( X \) are geodesic.

It remains to verify the first inequality in (8). Suppose it does not hold everywhere. Then there exists \( z_1 \in \tilde{B}(x, T) \) such that
\[
h(z_1) := d_Y(f(x), u_k(z_1)) = \frac{\rho_k}{2}.
\]
Set \( r_1 := d(x, z_1) > 0 \). The Lipschitz continuity just proved implies
\[
h(z) \leq h(z_1) + d_Y(u_k(z_1), u_k(z)) \leq \frac{3\rho_k}{4}
\]
for all \( z \) in the set \( \Sigma := S(x, r_1) \cap \tilde{B}(z_1, \frac{1}{4\rho_k}) \). Using the hyperbolic law of cosines and comparing with the hyperbolic plane of curvature \(-b^2\) we see that \( \Sigma \) contains the intersection of \( S(x, r_1) \) with a geodesic cone \( C_\gamma \) based at \( x \) and with angle
\[
\gamma = \sqrt{\cosh\left(\frac{bT}{\rho_k}\right) - 1} \frac{\sinh(bT)}{\sinh(bT)}.
\]
Let \( \sigma \) denote the harmonic measure on \( S(x, r_1) \). See [1] for the definition. Since \( h \) is continuous and weakly subharmonic the harmonic function \( \xi \) on \( \tilde{B}(x, r_1) \) which equals \( h \) on \( S(x, r_1) \) satisfies
\[
\rho_k = h(x) \leq \xi(x) = \int_{S(x, r_1)} \xi \, d\sigma = \int_{S(x, r_1)} h \, d\sigma
\]
and hence
\[
\int_{S(x, r_1)} (h - \rho_k) \, d\sigma \geq 0.
\]
Since \( h - \rho_k \leq LT \) on \( S(x, r_1) \) and \( h - \rho_k \leq -\frac{\rho_k}{4} \) on \( C_\gamma \cap S(x, r_1) \) it follows that \( \sigma(C_\gamma) \leq \frac{4LT}{\rho_k} \). From the uniform lower bound on the harmonic measure of geodesic cones proved in [1] Theorem 1.1] we thus obtain
\[
\frac{1}{M} \cdot \gamma^N \leq \sigma(C_\gamma) \leq \frac{4LT}{\rho_k},
\]
which contradicts the choice of \( \rho_k \). The proof is complete. \( \square \)

We now define a subset \( U \subset X \) by

\[
U := \left\{ z \in S(x, T) : d_Y(f(x), u_k(z)) \geq \rho_k - \frac{T}{2L} \right\}
\]

and prove:

**Lemma 6.3.** For all \( z_1, z_2 \in U \) we have

\[
(f(z_1) \mid f(z_2))_{f(x)} \geq \frac{T}{4L} - \frac{c}{2} - 2\delta.
\]

**Proof.** Let \( z_1, z_2 \in U \) and notice that for \( i = 1, 2 \), we have

\[
2 \cdot (f(z_i) \mid u_k(z_i))_{f(x)} = d_Y(f(x), f(z_i)) + d_Y(f(x), u_k(z_i)) - d_Y(f(z_i), u_k(z_i))
\]

\[
\geq \frac{T}{L} - c + \rho_k - \frac{T}{2L} - \rho_k
\]

\[
= \frac{T}{2L} - c.
\]

We next claim that

\[
(10) \quad (u_k(z_i) \mid u_k(x))_{f(x)} \geq \frac{\rho_k}{4} - \frac{3\delta'}{2}.
\]

In order to show this, fix \( i \) and set \( y := f(x) \), \( y_1 := u_k(z_i) \), and \( y_2 := u_k(x) \) and recall that \( d_Y(y, y_1) \geq \rho_k - \frac{T}{L} \geq \frac{\rho_k}{2} \) and \( d_Y(y, y_2) = \rho_k \). For \( j = 1, 2 \), let \( y_j' \) be the point on the geodesic from \( y \) to \( y_j \) with \( d_Y(y, y_j') = \frac{T}{2L} \). Let \( \xi \) be the geodesic in \( X \) from \( x \) to \( z_i \). By Lemma 6.2, the curve \( u_k \circ \xi \) stays outside the ball \( B(y, \frac{T}{L}) \) and has length bounded from above by \( 2CT\rho_k \). Since \( \rho_k \) was chosen so large that

\[
(11) \quad \rho_k + LT + 2CT\rho_k < \frac{\rho_k - 4\delta'}{4}
\]

it follows from Lemma 2.3 that \( d_Y(y_1', y_2') \leq 3\delta' \). This is easily seen to imply (10), which proves our claim.

Finally, we use the definition of \( \delta \)-hyperbolicity of \( Y \), the estimates above, and the fact that \( \frac{T}{2L} - \frac{L}{4} \geq \frac{T}{2L} - \frac{7}{12} \) to conclude that

\[
(f(z_1) \mid u_k(x))_{f(x)} \geq \min \left\{ (f(z_1) \mid u_k(z_i))_{f(x)} , (u_k(z_i) \mid u_k(x))_{f(x)} \right\} - \delta \geq \frac{T}{4L} - \frac{c}{2} - \delta
\]

and hence

\[
(f(z_1) \mid f(z_2))_{f(x)} \geq \min \left\{ (f(z_1) \mid u_k(x))_{f(x)} , (f(z_2) \mid u_k(x))_{f(x)} \right\} - \delta \geq \frac{T}{4L} - \frac{c}{2} - 2\delta.
\]

This completes the proof. \( \square \)

The next lemma provides a contradiction to the previous lemma since we had chosen \( T \) so large that

\[
(12) \quad \frac{L}{a} \cdot \log \left( 4M^N(2L^2 + 1)^N \right) + c' < \frac{T}{4L} - \frac{c}{2} - 2\delta.
\]

The lemma will thus finish the proof of Proposition 6.1.

**Lemma 6.4.** There exist \( z_1, z_2 \in U \) such that

\[
(f(z_1) \mid f(z_2))_{f(x)} \leq \frac{L}{a} \cdot \log \left( 4M^N(2L^2 + 1)^N \right) + c'.
\]
Proof. Denote by $\sigma$ the harmonic measure on $S(x,T)$. Let $h$ be the continuous and weakly subharmonic function given by $h(z) := d_Y(f(x), u_k(z))$. Comparing with a harmonic function exactly as in the proof of Lemma 6.2, we obtain that

$$\int_{S(x,T)} (h - \rho_k) \, d\sigma \geq 0.$$  

By the definition of $U$ and by Lemma 6.2, we have $h(z) - \rho_k \leq LT$ for all $z \in S(x,T)$ and $h(z) - \rho_k < -\frac{L}{2}$ whenever $z \in S(x,T) \setminus U$. This together with the above integral inequality yields

$$\sigma(U) \geq \frac{1}{2L^2 + 1}.$$  

The uniform upper bound on the harmonic measure proved in [1, Theorem 1.1] thus shows that there exist $z_1, z_2 \in U$ such that the angle $\gamma'$ between them, as seen from the point $x$, satisfies

$$\gamma' \geq \frac{(\sigma(U))^N}{M^N - (2L^2 + 1)^N}.$$  

From this, [2 Lemmas 2.1 and 2.2] it follows that

$$(f(z_1) \mid f(z_2))(x_1) \leq L(z_1 \mid z_2) \cdot e' \leq \frac{L}{a} \cdot \log \left(4M^N(2L^2 + 1)^N\right) + c',$$  

which concludes the proof.  

7. Completing the proof of the main theorem

We complete the proof of Theorem 1.1. Let $(X, d_X)$ and $(Y, d_Y)$ be spaces as in the statement of the theorem and let $f: X \to Y$ be a quasi-isometric map. By Proposition 4.1, we may assume that $f$ is also $L$-Lipschitz continuous for some $L > 0$. Fix a basepoint $x_0 \in X$ and set $B_R := B(x_0, R)$ and $S_R := S(x_0, R)$ whenever $R > 0$. Let furthermore $u_R: \bar{B}_R \to Y$ be the unique continuous energy minimizing harmonic map which coincides with $f$ on $S_R$, see Section 3. Proposition 6.1 shows that there exists $\rho$ such that

$$(13) \quad d_Y(u_R(x), f(x)) \leq \rho$$  

for all $R \geq 1$ and every $x \in \bar{B}_R$. From this and the Lipschitz continuity of $f$ it follows that for every $x \in B(x_0, R - 4)$ the image of $u_R(B(x, 3))$ is contained in a ball of radius $\rho + 3L$. Proposition 3.2 implies that $u_R$ is $L'$-Lipschitz on $B(x_1, 1)$ for some $L'$ which does not depend on $x$ or $R$. Consequently, $u_R$ is $L'$-Lipschitz on $B(x_0, R - 4)$.

Fix a sequence $R_k \to \infty$ and set $u_k := u_{R_k}$. By Arzela-Ascoli theorem, a diagonal subsequence argument, and by (13) we may thus assume that there exists an $L'$-Lipschitz map $u: X \to Y$ such that $u_k$ converges to $u$ uniformly on compact sets and that $d_Y(u(x), f(x)) \leq \rho$ holds for every $x \in X$.

It remains to show that $u$ is energy minimizing harmonic. Fix $s > 0$. The restriction of $u$ to $B_s$ is in $W^{1,2}(B_s, Y)$ since $u$ is Lipschitz. Now, suppose there exist $\varepsilon > 0$ and $v \in W^{1,2}(B_s, Y)$ such that $\text{tr}(v) = u|_{S_s}$ and $E(v) \leq E(u|_{B_s}) - \varepsilon$. Let $\delta \in (0, 1)$ be sufficiently small, to be determined later. For $k$ sufficiently large (depending on $\delta$) the map $h: S_s \cup S_{s+\delta} \to Y$ defined by $h(u)$ on $S_s$ and $h(u_k)$ on $S_{s+\delta}$ is $2L'$-Lipschitz. Since the ball $B_{s+1}$ is doubling and hence also $A_\delta := B_{s+\delta} \setminus B_s$ and $Y$ is Lipschitz $m$-connected for every $m$ it follows from [17, Theorem 1.5] that $h$ has an $L''$-Lipschitz extension $\hat{h}: A_\delta \to Y$ with Lipschitz constant $L''$ not
depending on $\delta$ or $k$. We now define a map $v_k : \bar{B}_{s+\delta} \to Y$ as follows. For $x \in B_s$ set $v_k(x) := v(x)$ and for $x \in A_\delta$ set $v_k(x) = \bar{h}(x)$. Then $v_k \in W^{1,2}(B_{s+\delta}, Y)$ with $\text{tr}(v_k) = u_k|_{B_{s+\delta}}$, see [15] Theorem 1.12.3]. Since $u_k$ is energy minimizing harmonic we have

$$E((v_k)|_{B_{s+\delta}}) \leq E(v_k) \leq E(v) + n(L'')^2 \cdot \mathcal{H}^n(A_\delta) \leq E(u|_{B_{s+\delta}}) - \epsilon + n(L'')^2 \cdot \mathcal{H}^n(A_\delta).$$

However, the right-hand side is strictly smaller than $E(u|_{B_{s+\delta}}) - \frac{\epsilon}{S}$ whenever $\delta > 0$ is sufficiently small and $k$ is sufficiently large. This contradicts the lower semi-continuity of the energy [15] Theorem 1.6.1]. We conclude that $u$ is indeed energy minimizing harmonic. This completes the proof.

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