Multiplication on self-similar sets with overlaps

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Abstract

Let $A, B \subset \mathbb{R}$. Define $A \cdot B = \{x \cdot y : x \in A, y \in B\}$.

In this paper, we consider the following class of self-similar sets with overlaps. Let $K$ be the attractor of the IFS $\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\}$, where $f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of $K$. The main result of this paper is $K \cdot K = [0, 1]$ if and only if $(1 - \lambda)^2 \leq c$. Equivalently, we give a necessary and sufficient condition such that for any $u \in [0, 1], u = x \cdot y$, where $x, y \in K$.

1 Introduction

Given $A, B \subset \mathbb{R}$. Define $A * B = \{x * y : x \in A, y \in B\}$, where * is $+, -, \times$ or $\div$ (when $* = \div, y \neq 0$). The arithmetic sum of two Cantor sets was studied by many scholars. There are many results concerning with this topic, see [1, 9, 5, 6, 11] and references therein. It is an important problem in homoclinic bifurcations [17]. The sum of two fractal sets is similar to the projection of the product of these two sets through some angle [7]. Therefore, one can consider the sum of two fractal sets from the projection perspective [9, 18, 14]. For the multiplication on two fractal sets, however, to the best of our knowledge, few papers analyzed this topic. From the physical point of view, this problem arises naturally in the study of the spectrum of the Labyrinth model [21]. In [20], Athreya, Reznick, and Tyson considered the multiplication and division on the middle-third Cantor sets. They proved that $17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9$, where $\mathcal{L}$ denotes the Lebesgue measure and $C$ is the middle-third Cantor set. There are still many open questions. For instance, if the middle-third Cantor set is replaced by the overlapping self-similar sets [10], then how can we obtain the sharp result, i.e. giving a necessary and sufficient condition such that the multiplication of two overlapping self-similar sets is exactly some interval. This is one of the main motivations of this paper. Another motivation of analyzing the multiplication on self-similar sets is that we want to give a new representation for any number in the unit interval, namely, given any $u \in [0, 1]$, then how can we find $x, y$ in the same self-similar set such that $u = x \cdot y$.

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In this paper, we consider the following class of overlapping self-similar sets \[10\]. Let \( K \) be the self-similar set of the IFS

\[
\{ f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda, 0 < \lambda < 1 \}.
\]

We assume that \( f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset \), where \( I = [0,1] \) is the convex hull of \( K \). This class of self-similar set, which is indeed a classical example allowing overlaps \[10\], was investigated by many people. The celebrated conjecture posed by Furstenberg states that the self-similar set \( \Lambda = \Lambda_3 \cup \Lambda_3 + \gamma \cup \Lambda_3 + 2 \) has Hausdorff dimension 1 for any irrational \( \gamma \). Hochman \[9\] proved this conjecture is correct. Keyon \[15\], Rao and Wen \[19\] studied the Hausdorff dimension of \( \Lambda \) if \( \gamma \) is rational. Ngai and Wang \[16\] came up with the finite type condition, and gave an algorithm which can calculate the Hausdorff dimension of \( \Lambda \) when \( \Lambda \) is of finite type. In \[2,3,12\], Dajani et al. analyzed the points in \( \Lambda \) with multiple codings, and obtained that when the overlaps are the exact overlaps, then the set of points with exactly \( k \) codings has the same Hausdorff dimension as the univoque set. In \[8\], Guo et al. considered the bi-Lipschitz equivalence of overlapping self-similar sets when \( \gamma \) differs. In \[13\], Jiang, Wang and Xi considered when the self-similar set \( \Lambda \) is bi-Lipschitz equivalent to another self-similar set with the strong separation condition. All these results analyzed the overlapping self-similar sets from different aspects.

In this paper, we consider the multiplication on \( K \). The assumptions on \( K \) allow very complicated overlaps. We, however, have the following result.

**Theorem 1.1.** Let \( K \) be the self-similar set defined above. Then

\[ K \cdot K = [0,1] \text{ if and only if } (1 - \lambda)^2 \leq c. \]

**Remark 1.2.** The necessary condition is due to the following observation:

\[ K \subset [0,c] \cup [1-\lambda,1], \text{ which implies } K \cdot K \subset [0,c] \cup [(1-\lambda)^2,1]. \]

This paper is arranged as follows. In section 2, we give a proof of Theorem 1.1. In section 3, we give two examples. Finally, we give some remarks.

## 2 Proof of Theorem 1.1

In this section, we first prove two useful lemmas.

### 2.1 Preliminaries

Let \( I = [0,1] \). For any \( (i_1 \cdots i_n) \in \{1,2,3\}^n \), we call \( f_{i_1 \cdots i_n}(I) \) a basic interval with length \( \lambda^n \). Denote by \( E_n \) the collection of all the basic intervals with length \( \lambda^n \). Let \( J \in E_n \).
Denote \( \tilde{J} = \bigcup_{i=1}^{3} I_{n+1,i} \), where \( I_{n+1,i} \in E_{n+1}, I_{n+1,i} \subset J, i = 1, 2, 3 \). Let \( [A, B] \subset [0, 1] \), where \( A \) and \( B \) are the left and right endpoints of some basic intervals in \( E_k \) for some \( k \geq 1 \), respectively. \( A \) and \( B \) may not in the same basic interval. In the following lemma, we choose \( A \) and \( B \) in this way. Let \( F_k \) be the collection of all the basic intervals in \([A, B]\) with length \( \lambda^k, k \geq k_0 \) for some \( k_0 \in \mathbb{N}^+ \), i.e. the union of all the elements of \( F_k \) is denoted by \( G_k = \bigcup_{i=1}^{k} I_{k,i} \), where \( t_k \in \mathbb{N}^+, I_{k,i} \in E_k \) and \( I_{k,i} \subset [A, B] \). Clearly, by the definition of \( G_n \), it follows that \( G_{n+1} \subset G_n \) for any \( n \geq k_0 \).

**Lemma 2.1.** Assume \( F : \mathbb{R}^2 \to \mathbb{R} \) is a continuous function. Suppose \( A \) and \( B \) are the left and right endpoints of some basic intervals in \( E_{k_0} \) for some \( k_0 \geq 1 \), respectively. Then \( K \cap [A, B] = \cap_{n=k_0}^{\infty} G_n \). Moreover, if for any \( n \geq k_0 \) and any basic intervals \( I_1, I_2 \subset G_n \),

\[
F(I_1, I_2) = F(\tilde{I}_1, \tilde{I}_2),
\]

then \( F(K \cap [A, B], K \cap [A, B]) = F(G_{k_0}, G_{k_0}) \).

**Proof.** Let \( G_n = \bigcup_{i=1}^{t_n} I_{n,i} \) for some \( t_n \in \mathbb{N}^+ \), where \( I_{n,i} \in E_n \) and \( I_{n,i} \subset [A, B] \). Then by the construction of \( G_n \), i.e. \( G_{n+1} \subset G_n \) for any \( n \geq k_0 \), it follows that

\[
K \cap [A, B] = \cap_{n=k_0}^{\infty} G_n.
\]

By the continuity of \( F \), we conclude that

\[
F(K \cap [A, B], K \cap [A, B]) = \cap_{n=k_0}^{\infty} F(G_n, G_n).
\]

By virtue of the relation \( G_{n+1} = \tilde{G}_n \) and the condition in the lemma, we have

\[
F(G_n, G_n) = \bigcup_{1 \leq i,j \leq t_n} F(I_{n,i}, I_{n,j}) = F(\bigcup_{1 \leq i,j \leq t_n} \tilde{I}_{n,i}, \tilde{I}_{n,j}) = F(G_{n+1}, G_{n+1}).
\]

Therefore, \( F(K \cap [A, B], K \cap [A, B]) = F(G_{k_0}, G_{k_0}) \) follows immediately from identity (1) and \( F(G_n, G_n) = F(G_{n+1}, G_{n+1}) \) for any \( n \geq k_0 \). 

**Lemma 2.2.** Let \( I_1 = [a, a + t], I_2 = [b, b + t] \) be two basic intervals. If \( a \geq b \geq 1 - c - \lambda \) and \( c \) satisfies the following inequalities

\[
\begin{align*}
(1 - \lambda)^2 &\leq c \\
1 - 2c &\leq \lambda \\
\frac{1 - c}{2} &\leq \lambda
\end{align*}
\]

then \( f(I_1, I_2) = f(\tilde{I}_1, \tilde{I}_2) \), where \( f(x, y) = xy \).

**Proof.** Since \( I_1 = [a, a + t], I_2 = [b, b + t] \), it follows that

\[
\tilde{I}_1 = [a, a + ct] \cup [a + t - \lambda t, a + t], \tilde{I}_2 = [b, b + ct] \cup [b + t - \lambda t, b + t].
\]

Clearly, \( f(I_1, I_2) = [ab, (a + t)(b + t)] \). Now we calculate \( f(\tilde{I}_1, \tilde{I}_2) \). Without loss of generality, we may assume \( a \geq b \). After simple calculation,

\[
f(\tilde{I}_1, \tilde{I}_2) = J_1 \cup J_2 \cup J_2 \cup J_4,
\]
where

\[ J_1 = [ab, (a + ct)(b + ct)] = [e_1, h_1] \]
\[ J_2 = [b(a + dt), (a + t)(b + ct)] = [e_2, h_2] \]
\[ J_3 = [a(b + dt), (a + ct)(b + t)] = [e_3, h_3] \]
\[ J_4 = [(a + dt)(b + dt), (a + t)(b + t)] = [e_4, h_4], \]

and \( d = 1 - \lambda \). Note that \( f(\tilde{I}_1, \tilde{I}_2) = f(I_1, I_2) = [ab, (a + t)(b + t)] \) if and only if

\[
\begin{cases}
  h_1 - e_2 \geq 0 \\
  h_2 - e_3 \geq 0 \\
  h_3 - e_4 \geq 0
\end{cases}
\]

Therefore, it suffices to prove the above inequalities.

(I) If \( a \geq b \), then

\[
\begin{align*}
  h_1 - e_2 &= (a + ct)(b + ct) - (a + dt)b \\
          &= t\left(c^2t + ac + bc - bd\right) \\
          &\geq t\left(c^2t + bc + bc - bd\right) \\
          &= t\left(c^2t + b(2c - d)\right) \geq 0.
\end{align*}
\]

Therefore, if \( 2c - d \geq 0 (\Leftrightarrow 2c \geq 1 - \lambda) \)

then

\[ (a + ct)(b + ct) - (a + dt)b \geq 0. \]

(II) We need to show

\[
\begin{align*}
  h_2 - e_3 &= (a + t)(b + ct) - a(b + dt) = t(b + ac - ad + ct) = ct^2 + t(b + ac - ad) \geq 0.
\end{align*}
\]

In fact, the following inequality is sufficient,

\[
\sup_{a \in [b, 1]} a(d - c) \leq b
\]

i.e.,

\[ 1 - \lambda - c = (d - c) \leq b, \]

which is the assumption in lemma.

(III) If \( a \geq b \), then

\[
\begin{align*}
  h_3 - e_4 &= (b + t)(a + ct) - (a + dt)(b + dt) \\
          &= t\left(a - d^2t - ad + bc - bd + ct\right).
\end{align*}
\]

It suffices to prove that \( a - d^2t - ad + bc - bd + ct \geq 0 \) if \( a \geq b \). When \( a \geq b \), we obtain that \( a - ad \geq b - bd \) as \( a - ad - (b - bd) = (1 - d)(a - b) \geq 0 \). As such

\[
\begin{align*}
  a - d^2t - ad + bc - bd + ct \\
  \geq t\left(c - d^2\right) + b - bd + bc - bd \\
  \geq t\left(c - d^2\right) + b(1 - 2d + c).
\end{align*}
\]
If \( c - d^2 \geq 0 \) and 
\[
1 - 2d + c \geq 0 (\Leftrightarrow c \geq 1 - 2\lambda)
\]
then
\[
a - d^2 t - ad + bc - bd + ct \geq 0
\]
Under the condition
\[
a \geq b \geq d - c = 1 - \lambda - c,
\]
if \( c \) and \( d \) satisfy the following inequalities
\[
1 - 2d + c \geq 0, \quad 2c \geq d \geq c \text{ and } c \geq d^2,
\]
then \( f(I_1, I_2) = f(\tilde{I}_1, \tilde{I}_2) \). \( \square \)

### 2.2 Proofs of some lemmas

We first give an outline of the proof of Theorem 1.1. First, by the conditions for \( c \) and \( \lambda \), see Lemma 2.3 and Remark 1.2, we have
\[
\lambda \leq c \leq 2\lambda, \quad (1 - \lambda)^2 \leq c < 1 - \lambda.
\]
Therefore, if \( K \cdot K = [0, 1] \), then \((\lambda, c)\) should be in the purple region (the first picture of Figure 1). Conversely, we shall prove that for any \((\lambda, c)\) in the purple region, \( K \cdot K = [0, 1] \). We partition the purple region into five subregions, see the last picture of Figure 1. More precisely, in Lemma 2.5, we prove that for the brown region in the last picture, \( K \cdot K = [0, 1] \). In Lemma 2.6, we prove that for the gray region (the second picture), \( K \cdot K = [0, 1] \). In Lemma 2.7, we show that if \((\lambda, c)\) in the orange region (the third picture), we prove \( K \cdot K = [0, 1] \). Note that the union of the regions generated from Lemma 2.5, Lemma 2.6, and Lemma 2.7 is precisely the purple region in the first picture.

Before, we prove Lemmas 2.5, 2.6, 2.7, and 2.9. We give the following lemmas which are useful to our analysis.

**Lemma 2.3.** Let \( K \) be the self-similar set of the following IFS
\[
\{ f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda, 0 < \lambda < 1 \}\.
\]
If \( f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset \), then \( \lambda \leq c \leq 2\lambda, 0 < \lambda < 1 - c \). If \( K \cdot K = [0, 1] \), then \( \lambda \geq 2 - \sqrt{3} \).

**Proof.** The first statement is trivial. We only prove the second one. Since \( \lambda \leq c \leq 2\lambda \), it follows that \( K \cdot K \subset [0, 2\lambda] \cup [(1 - \lambda)^2, 1] \). If \( 0 < \lambda < 2 - \sqrt{3} \), then \( 2\lambda < (1 - \lambda)^2 \), which contradicts with \( K \cdot K = [0, 1] \). \( \square \)

**Lemma 2.4.** If \((\lambda, c)\) satisfies the following conditions
\[
\begin{cases}
(1 - \lambda)^2 \leq c < 1 - \lambda \\
\lambda \leq c \leq 2\lambda
\end{cases}
\]
then
\[
\begin{cases}
1 - 2c \leq \lambda \\
1 - c \leq \frac{1}{2} \leq \lambda.
\end{cases}
\]
Figure 1: Lemma 2.5 (brown region in the last picture), Lemma 2.6 (gray region), Lemma 2.7 (orange region), Lemma 2.9 (blue region)

Proof. The proof is due to the first picture of Figure 1. 

In terms of this Lemma 2.8 in Lemma 2.2, the condition

\[
\begin{align*}
(1 - \lambda)^2 &\leq c \\
1 - 2c &\leq \lambda \\
\frac{1 - c}{2} &\leq \lambda
\end{align*}
\]

can be replaced by

\[
\begin{align*}
(1 - \lambda)^2 &\leq c < 1 - \lambda \\
\lambda &\leq c \leq 2\lambda.
\end{align*}
\]

Lemma 2.5. If \((\lambda, c)\) satisfies the following conditions

\[
\begin{align*}
(1 - \lambda)^2 &\leq c < 1 - \lambda \\
\lambda &\leq c \leq 2\lambda \\
\frac{3 - \sqrt{5}}{2} &\leq \lambda < 1
\end{align*}
\]

then \(K \cdot K = [0, 1]\).

Proof. Since \(1 - \lambda \geq 1 - c - \lambda\), in terms of Lemmas 2.4 and 2.5 we let \([A, B] = [1 - \lambda, 1]\). It is easy to check that

\[f(K \cap [A, B], K \cap [A, B]) = [(1 - \lambda)^2, 1].\]
Therefore,
$$[0, 1] \supset K \cdot K \supset \bigcup_{n=0}^{\infty} \lambda^n [(1 - \lambda)^2, 1] \cup \{0\} = [0, 1].$$

\[\square\]

**Lemma 2.6.** For any $0 < \lambda < 1$, the following inequality holds
$$c(1 - \lambda + c\lambda) > (c - \lambda^2)(1 - \lambda^2).$$

Suppose $c$ and $\lambda$ satisfy the following inequalities,
\[
\begin{cases}
2 - \sqrt{3} \leq \lambda < \frac{3 - \sqrt{5}}{2} \\
(1 - \lambda)^2 \leq c < 1 - \lambda \\
\lambda \leq c \leq 2\lambda \\
c \leq \lambda^2 + \frac{\lambda}{1 - \lambda}
\end{cases}
\]
then $K \cdot K = [0, 1]$.

**Proof.** First, we prove that if $0 < \lambda < 1$, then $c(1 - \lambda + c\lambda) > (c - \lambda^2)(1 - \lambda^2)$.

It is easy to check that
$$x_1 = \frac{-(\lambda - 1) - \sqrt{(\lambda - 1)^2 - 4(\lambda - \lambda^3)}}{2}, x_2 = \frac{-(\lambda - 1) + \sqrt{(\lambda - 1)^2 - 4(\lambda - \lambda^3)}}{2}$$
are the roots of $c^2 + (\lambda - 1)c + \lambda - \lambda^3 = 0$. Since $0 < \lambda < 1$, it follows that
$$(\lambda - 1)^2 - 4(\lambda - \lambda^3) < 0.$$ In other words, $x_1$ and $x_2$ are complex numbers rather than the reals. Therefore,
$$c^2 + (\lambda - 1)c + \lambda - \lambda^3 \geq 0.$$ With a similar discussion of Lemma 2.5 it follows that
$$K \cdot K \supset \bigcup_{n=0}^{\infty} \lambda^n [(1 - \lambda)^2, 1]).$$

By the assumptions
$$2 - \sqrt{3} \leq \lambda < \frac{3 - \sqrt{5}}{2}, (1 - \lambda)^2 \leq c,$$ we conclude that
$$2c \geq 2(1 - \lambda)^2 \geq 1 - \lambda + \lambda^2.$$ In other words, $c - \lambda^2 \geq 1 - c - \lambda$. Therefore, we may make use of Lemma 2.2 for $\bar{K} = K \cap [c - \lambda^2, 1] = ([c - \lambda^2, c] \cup [1 - \lambda, 1 - \lambda + c\lambda] \cup [1 - \lambda^2, 1]) \cap K$. Simple calculation yields that
$$K \cdot K \supset f(\bar{K}, \bar{K}) = [(c - \lambda^2)^2, c^2] \cup [(c - \lambda^2)(1 - \lambda), c(1 - \lambda + c\lambda)]$$
$$\cup [(c - \lambda^2)(1 - \lambda^2), c] \cup [(1 - \lambda^2)(1 - \lambda), 1 - \lambda + c\lambda] \cup [(1 - \lambda^2)^2, 1].$$
By the condition \((1 - \lambda)^2 \leq c\) and the consequence
\[
K \cdot K \supset \bigcup_{n=0}^{\infty} \lambda^n\left([\lambda^2, 1]\right),
\]
we obtain that \(K \cdot K \supset \bigcup_{n=0}^{\infty} \lambda^n[(c - \lambda^2)(1 - \lambda), 1]\). Since \(c \leq \lambda^2 + \frac{\lambda}{1 - \lambda}\), it follows that \(\lambda \geq (c - \lambda^2)(1 - \lambda)\). Therefore,
\[
[0, 1] \supset K \cdot K \supset \bigcup_{n=0}^{\infty} \lambda^n[(c - \lambda^2)(1 - \lambda), 1] \cup \{0\} = [0, 1],
\]
as required. \(\Box\)

**Lemma 2.7.** Suppose \(c\) and \(\lambda\) satisfy the following inequalities,
\[
\begin{cases}
2 - \sqrt{3} \leq \lambda \leq \frac{3 - \sqrt{5}}{2} & \\
(1 - \lambda)^2 \leq c < 1 - \lambda & \\
\lambda \leq c \leq 2\lambda & \\
(c - \lambda^2)(1 - \lambda) \leq c^2 &
\end{cases}
\]
Then \(K \cdot K = [0, 1]\).

**Proof.** If \((c - \lambda^2)(1 - \lambda) \leq c^2\), then with a similar discussion as the proof of Lemma 2.6, we have the following inclusion
\[
[0, 1] \supset K \cdot K \supset \bigcup_{n=0}^{\infty} \lambda^n[(c - \lambda^2)(1 - \lambda), 1] \cup \{0\} = [0, 1].
\]
Here we need to assume \(\lambda \geq (c - \lambda^2)^2\), that is, \(c \leq \sqrt{\lambda + \lambda^2}\). Since \(c \leq 2\lambda\), it suffices to prove \(2\lambda \leq \sqrt{\lambda + \lambda^2}\), which is a direct consequence of \(2 - \sqrt{3} \leq \lambda \leq \frac{3 - \sqrt{5}}{2}\). \(\Box\)

**Lemma 2.8.** Suppose \(c\) and \(\lambda\) satisfy the following inequalities,
\[
\begin{cases}
2 - \sqrt{3} \leq \lambda \leq \frac{3 - \sqrt{5}}{2} & \\
(1 - \lambda)^2 \leq c \leq 2\lambda & \\
(c - \lambda^2)(1 - \lambda) \geq c^2 & \\
c \geq \lambda^2 + \frac{\lambda}{1 - \lambda} &
\end{cases}
\]
Then \(c \geq \frac{1}{2}\), i.e. \(c - \lambda \geq 1 - c - \lambda\).

**Proof.** The proof is due to the help of computer, see the fourth picture in Figure 1. We plot the blue region which satisfies the conditions in lemma, and find that \(c \geq \frac{1}{2}\). \(\Box\)

**Lemma 2.9.** Suppose \(c\) and \(\lambda\) satisfy the following inequalities,
\[
\begin{cases}
2 - \sqrt{3} \leq \lambda \leq \frac{3 - \sqrt{5}}{2} & \\
(1 - \lambda)^2 \leq c \leq 2\lambda & \\
(c - \lambda^2)(1 - \lambda) \geq c^2 & \\
c \geq \lambda^2 + \frac{\lambda}{1 - \lambda} &
\end{cases}
\]
Then \(K \cdot K = [0, 1]\).
Proof. By Lemma 2.8, it follows that $c - \lambda \geq 1 - c - \lambda$. Therefore, we can utilize Lemmas 2.2 and 2.1 by taking $\tilde{K} = ([c - \lambda, c] \cup [1 - \lambda, 1]) \cap K$. Therefore,

$$f(\tilde{K}, \tilde{K}) = f([c - \lambda, c] \cup [1 - \lambda, 1]), [c - \lambda, c] \cup [1 - \lambda, 1])$$

$$= ((c - \lambda)^2, c^2] \cup [(c - \lambda)(1 - \lambda), c] \cup [(1 - \lambda)^2, 1]$$

The equation $c^2 = (c - \lambda)(1 - \lambda)$ has two roots, i.e.

$$c_1 = \frac{-(\lambda - 1) - \sqrt{(\lambda - 1)^2 - 4(\lambda - \lambda^2)}}{2}, c_2 = \frac{-(\lambda - 1) + \sqrt{(\lambda - 1)^2 - 4(\lambda - \lambda^2)}}{2}.$$ 

Since $2 - \sqrt{3} \leq \lambda < \frac{3 - \sqrt{5}}{2}$, it follows that

$$(\lambda - 1)^2 - 4(\lambda - \lambda^2) < 0.$$ 

Therefore, $c^2 > (c - \lambda)(1 - \lambda)$. Subsequently,

$$f(\tilde{K}, \tilde{K}) = [(c - \lambda)^2, 1]$$

Next, we prove that $\lambda \geq (c - \lambda)^2$ which is equivalent to $\sqrt{\lambda + \lambda} \geq c$. This is trivial as $\sqrt{\lambda + \lambda} \geq 2\lambda \geq c$. Therefore,

$$[0, 1] \supset K \cdot K \supset \cup_{n=0}^{\infty} \lambda^n [(c - \lambda)^2, 1] \cup \{0\} = [0, 1].$$

Proof of Theorem 1.1. By Remark 1.2, we only need to prove the sufficiency of Theorem 1.1. If $c \geq (1 - \lambda)^2$ and $\frac{3 - \sqrt{5}}{2} \leq \lambda \leq 1$, then by Lemma 2.5 $K \cdot K = [0, 1]$. If $2 - \sqrt{3} \leq \lambda < \frac{3 - \sqrt{5}}{2}$ and $c \geq (1 - \lambda)^2$, then by Lemmas 2.6, 2.7 and 2.9 $K \cdot K = [0, 1]$. Note that the union of the associated regions of $(\lambda, c)$ satisfying the conditions in Lemmas 2.5, 2.6, 2.7 and 2.9 is the brown, gray, orange, blue regions in the pictures of Figure 1, which is exactly the purple region of the first picture of Figure 1.

3 Examples

Example 3.1. Let $K$ be the attractor of the following IFS,

$$\left\{ f_1(x) = \frac{x}{3}, f_2(x) = \frac{x}{3} + c - \frac{1}{3}, f_3(x) = \frac{x + 2}{3} \right\}, \frac{1}{3} \leq c < \frac{2}{3}.$$ 

If $c \in \left[\frac{4}{9}, \frac{2}{3}\right]$, then $K \cdot K = [0, 1]$. Moreover, $c = \frac{4}{9}$ is sharp, i.e. for any $\frac{1}{3} \leq c < \frac{4}{9}$,

$$K \cdot K \subseteq [0, 1].$$ 

In this example $2 - \sqrt{3} \leq \lambda = 1/3 < 1$. Hence for any $c \geq (1 - \lambda)^2 = \frac{4}{9}$, i.e. $c \in \left[\frac{4}{9}, \frac{2}{3}\right]$, then $K \cdot K = [0, 1]$. Moreover, $c = \frac{4}{9}$ is sharp.
Example 3.2. Let $K$ be the self-similar set of the following IFS,
\[ \{ f_1(x) = \lambda x, f_2(x) = \lambda x + \lambda - \lambda^n, f_3(x) = \lambda x + 1 - \lambda \}, \]
where $0 < \lambda < \beta$, $n \geq 2$, and $\beta \in (0, 1)$ is the smallest real root of $x^n - 3x + 1 = 0$. Then $K \cdot K = [0, 1]$ if and only if $\alpha \leq \lambda < \beta$, where $\alpha \in (0, 1)$ is the smallest real root of $x^n + x^2 - 4x + 1 = 0$.

First, we prove the following lemma.

Proposition 3.3. Let $\alpha$ and $\beta$ be the smallest real roots of $x^n + x^2 - 4x + 1 = 0$ and $x^n - 3x + 1 = 0$, respectively. Then $\alpha < \beta$.

Proof. By the Rouché theorem and the intermediate value theorem, it follows that $\alpha$ and $\beta$ are the unique real roots of $x^n + x^2 - 4x + 1 = 0$ and $x^n - 3x + 1 = 0$ in $(0, 1)$, respectively. Now, we prove $\alpha < \beta$. If $\beta \leq \alpha$. Let $H(x) = x^n - 3x + 1$. Then
\[ H(\alpha) = \alpha^n - 3\alpha + 1 = \alpha^n + \alpha^2 - 4\alpha + 1 + \alpha - \alpha^2 = \alpha - \alpha^2 \geq 0. \]
$H(1) < 0$. Therefore, by the intermediate value theorem, we conclude that there is another root of $H(x)$ in $(\alpha, 1)$, which contradicts to the uniqueness of the root in $(0, 1)$.

By Theorem 1.1, $K \cdot K = [0, 1]$ if and only if $c \geq (1 - \lambda)^2$, where $c = 2\lambda - \lambda^n$. Therefore, $\lambda^n + \lambda^2 - 4\lambda + 1 \leq 0$. The condition $\lambda^n - 3\lambda + 1 > 0$ is equivalent to $c < 1 - \lambda$. Therefore, $K \cdot K = [0, 1]$ if and only if $\alpha \leq \lambda < \beta$.

4 Final remarks

In this paper, we only consider the multiplication on the self-similar sets. It is natural to consider the division on the overlapping self-similar sets. Moreover, we can prove the following result.

Theorem 4.1. Let $K$ be the attractor defined in Theorem 1.1. Given $u \in [0, 1]$, then there exist some $\lambda, c$ and some $x_1, x_2, x_3, x_4, x_5, x_6 \in K$ such that
\[ u = x_1 \cdot x_2 = x_3^2 + x_4^2 = \frac{x_5}{x_6}. \]

We will publish these results elsewhere.

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