Random walk on the incipient infinite cluster on trees

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Abstract. Let $\mathcal{G}$ be the incipient infinite cluster (IIC) for percolation on a homogeneous tree of degree $n_0 + 1$. We obtain estimates for the transition density of the the continuous time simple random walk $Y$ on $\mathcal{G}$; the process satisfies anomalous diffusion and has spectral dimension $\frac{4}{3}$.

2000 MSC. Primary 60K37; Secondary 60J80, 60J35.

Keywords. Percolation, incipient infinite cluster, random walk, branching process, heat kernel.

1. Introduction

We recall the bond percolation model on the lattice $\mathbb{Z}^d$: each bond is open with probability $p \in (0, 1)$, independently of all the others. Let $\mathcal{C}(x)$ be the open cluster containing $x$; then if $\theta(p) = P_p(|\mathcal{C}(x)| = +\infty)$ it is well known (see [Gm]) that there exists $p_c = p_c(d)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$.

If $d = 2$ or $d \geq 19$ (or $d > 6$ for ‘spread out’ models) it is known (see [Gm], [HS]) that $\theta(p_c) = 0$, and it is conjectured that this holds for all $d \geq 2$. At the critical probability $p = p_c$ it is believed that in any box of side $n$ there exist with high probability open clusters of diameter of order $n$ – see [BCKS]. For large $n$ the local properties of these large finite clusters can, in certain circumstances, be captured by regarding them as subsets of an infinite cluster $\mathcal{C}$, called the ‘incipient infinite cluster’ (IIC).

This was constructed when $d = 2$ in [Ke1], by taking the limit as $N \to \infty$ of the cluster $\mathcal{C}(0)$ conditioned to intersect the boundary of a box of side $N$ with center at the origin. See [Ja1], [Ja2] for other constructions of the IIC in two dimensions. For large $d$ a construction of the IIC in $\mathbb{Z}^d$ is given in [HJ], using the lace expansion. It is believed that the results there will hold for any $d > 6$. [HJ] also gives the existence and some properties of the IIC for all $d > 6$ for ‘spread-out’ models: these include the case when there is a bond between $x$ and $y$ with probability $pL^{-d}$ whenever $y$ is in a cube side $L$ with center $x$, and the parameter $L$ is large enough. Rather more is known about the IIC for oriented percolation on $\mathbb{Z}_+ \times \mathbb{Z}^d$ (see [HHS], [HS]), but in this discussion, which mainly concerns what is conjectured rather than what is known, we specialize to the case of $\mathbb{Z}^d$.

We write $\mathcal{C}_d$ for the IIC in $\mathbb{Z}^d$. It is believed that the global properties of $\mathcal{C}_d$ are the same for all $d > d_c$, both for nearest neighbour and spread-out models. In [HJ] it is proved for ‘spread-out’ models that $\mathcal{C}_d$ has one end – that is that any two paths from 0 to infinity intersect infinitely often.

\textsuperscript{1} Research partially supported by a grant from NSERC (Canada).
\textsuperscript{2} Research partially supported by the Grant-in-Aid for Scientific Research for Young Scientists (B) 16740052.
For large $d$, it is believed that the geometry of $\tilde{C}_d$ is also similar to that of the IIC when $d = \infty$ — that is to the IIC on a regular tree; this is supported by the results in [HHS] and [HJ]. For trees the construction of the IIC is much easier than for lattices, and there is a close connection between the IIC and a critical Bienaymé-Galton-Watson branching processes conditioned on non-extinction. In [Ke2] Kesten gave the construction of the IIC $\mathcal{G}$ for critical branching processes. This is an infinite subtree, which contains only one path from the root to infinity. This tree is quite sparse, and has polynomial volume growth: in the case when the offspring distribution has finite variance, a ball $B(x,r)$ in $\mathcal{G}$ has roughly $r^2$ points. (This is when distance in $\mathcal{G}$ is measured using the natural graph distance).

Let $Y = (Y_t, t \geq 0)$ be the simple random walk on $\tilde{C}_d$, and $q_t(x,y)$ be its transition density (see Section 3 for a precise definition). Define the spectral dimension of $\tilde{C}_d$ by

$$d_s(\tilde{C}_d) = -2 \lim_{t \to \infty} \frac{\log q_t(x,x)}{\log t},$$

(1.1)

(if this limit exists). Alexander and Orbach [AO] conjectured that, for any $d \geq 2$, $d_s(\tilde{C}_d) = 4/3$. While it is now thought that this is unlikely to be true for small $d$, the results on the geometry of $\tilde{C}_d$ in [HHS] and [HJ] are consistent with this holding for large $d$. (Or for any $d$ above the critical dimension for spread-out models).

Random walks on supercritical clusters in $\mathbb{Z}^d$ are studied in [B2] (transition density estimates) and [SS] (invariance principle for $d \geq 4$). In these cases the large scale behaviour of the random walk approximates that of the random walk on $\mathbb{Z}^d$, and the unique infinite cluster has spectral dimension $d$.

In what follows, we will specialize to the case of critical percolation on a regular rooted tree with degree $n_0 + 1$, which we denote $\mathbb{B}$. We write 0 for the root of $\mathbb{B}$. We keep $n_0$ fixed, but (in view of possible future applications) wish to obtain estimates which do not depend on $n_0$. For bond percolation with probability $p$ on $\mathbb{B}$, it is easy to see that if $X_n$ is the number of vertices at level $n$ in $\mathcal{C}(0)$, then $X = (X_n)$ is a branching process with $\text{Bin}(n_0,p)$ offspring distribution. Thus $p_c = 1/n_0$. For the construction of the IIC see [Ke2]: we obtain a subtree $\mathcal{G} \subset \mathbb{B}$ with law $\mathbb{P}$, on a probability space $(\Omega_1, \mathcal{F}, \mathbb{P})$. Write $\mathbb{B}_N$ for the $N$-th level of $\mathbb{B}$, and $\mathbb{B} \leq N$ for the union of the first $N$ levels of $\mathbb{B}$. Then the law of $\mathcal{G}$ is characterized by the fact that the law of $\mathcal{G} \cap \mathbb{B} \leq N$ under $\mathbb{P}$ is the same as that of $\mathcal{C}(0)$ under $P_{p_c}$, conditioned on $\mathcal{C}(0)$ reaching level $N$.

Motivated by [AO], in [Ke2] Kesten studied the simple random walk on $\mathcal{G}(\omega)$, and also on $\tilde{C}_2$. Let $X = (X_n, n \geq 0, Q_{x,\omega}^{\epsilon}, x \in \mathcal{G}(\omega))$ be the simple random walk on $\mathcal{G}(\omega)$. We define the annealed law $\mathbb{P}^*$ by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times Q_{x,\omega}^0$, and the rescaled height process $Z^{(n)}$ by

$$Z^{(n)}_t = n^{-1/3}d(0, X_{nt}), \quad t \geq 0,$$

where $d(\cdot, \cdot)$ is the graph distance in $\mathcal{G}(\omega)$.

The following summarizes the main results in of [Ke2] in the tree case.

**Theorem 1.1.** (a) (1.19 in [Ke2].) Let $T_N = \min\{n : d(0, X_n) = N\}$. Then for all $\epsilon > 0$ there exist $\lambda_1, \lambda_2$ such that

$$\mathbb{P}^*(\lambda_1 \leq N^{-3}T_N \leq \lambda_2) \geq 1 - \epsilon, \quad \text{for all } N \geq 1.$$
(b) \((1.16)\) in \([Ke2]\), full proof in \([Ke3]\). Under \(\mathbb{P}^*\) the processes \(Z^{(n)}\) converges weakly in \(C[0, \infty)\) to a process \(Z\) which is not the zero process.

To understand why the \(n^{-1/3}\) scaling arises in (b) it is helpful to consider the behaviour of random walks on regular deterministic graphs with a large scale fractal structure – see for example [Jo], [BB2], [HK], [GT1], [GT2] and [BCK]. Let \(d_f \geq 1\) give the volume growth, so that \(|B(x, r)| \sim r^{d_f}\), and suppose that the effective electrical resistance \(R(x, B(x, r)^c)\) between \(x\) and the exterior of \(B(x, r)\) satisfies \(R(x, B(x, r)^c) \sim r^\zeta\), where \(\zeta > 0\). In this ‘strongly recurrent’ case (see [BCK] for simple recent proofs using ideas that are also used in this paper) one finds that the mean time for \(X\) to escape from \(B(x, r)\) scales as \(r^{d_w}\) where \(d_w = d_f + \zeta\). While the IIC \(G\) is more irregular than the sets considered in these papers, it still has properties similar to regular graphs with \(d_f = 2\). Further, by Proposition 2.10 below, only \(O(1)\) points on \(\partial B(x, r/4)\) are connected to \(B(x, r)^c\) by a path outside \(B(x, r/4)^c\), so one has \(R(x, B(x, r)^c) \sim r\), giving \(\zeta = 1\) and \(d_w = 3\).

In this paper we study the simple random walk on \(G\), and in particular investigate both quenched and annealed properties of its transition densities. For technical convenience we work with the continuous time simple random walk on \(G\), which we denote \(Y = (Y_t, t \in [0, \infty), P_x^\omega, x \in G(\omega))\). Since we consider the law of \(Y\) with general starting points \(x\), we need to consider the measures \(\mathbb{P}_x = \mathbb{P}(:x \in G)\) and \(\mathbb{P}_{x,y} = \mathbb{P}(:|x, y \in G)\).

Unlike [Ke2] we restrict our attention to branching processes with a Binomial offspring distribution. Our main reason for this is to maintain good uniform control of the laws \(\mathbb{P}_x\). It is clear by symmetry that \(\mathbb{P}_x(|B(x, r)| > \lambda)\) is the same for any \(x \in \mathbb{B}_N\), and in fact we have uniform bounds for all \(x \in \mathbb{B}\). (These probabilities are not equal for all \(x\), since a higher level \(x\) is likely to be further from the backbone of the cluster). For a general branching process, the labels of the point \(x\) may give a substantial amount of information about the size of the cluster near \(x\).

**Theorem 1.2.** (a) There exist \(c_0, c_1, c_2, S(x)\) such that for each \(x\),

\[
\mathbb{P}_x(S(x) \geq m) \leq c_0(\log m)^{-1},
\]

and on \(\{\omega : x \in G(\omega)\}\)

\[
c_1 t^{-2/3}(\log \log t)^{-17} \leq q_t^\omega(x, x) \leq c_2 t^{-2/3}(\log \log t)^3 \text{ for all } t \geq S(x).
\]

(b) \(d_s(G) = 4/3 \mathbb{P}-\text{a.s.}\)

The cluster \(G\) contains large scale fluctuations, so that \(q_t(x, x)\) does have oscillations of order \((\log \log t)^\zeta\) as \(t \to \infty\) – see Lemma 5.1.

**Theorem 1.3.** (a) We have

\[
c_1 t^{1/3} \leq \mathbb{E}_x E_x^\omega d(x, Y_t) \leq \mathbb{E}_x E_x^\omega \sup_{0 \leq s \leq t} d(x, Y_s) \leq c_2 t^{1/3}.
\]

(b) There exists \(T(x)\) with \(\mathbb{P}_x(T(x) < \infty) = 1\) such that

\[
c_3 t^{1/3}(\log \log t)^{-12} \leq E_x^\omega[d(x, Y_t)] \leq c_4 t^{1/3} \log t \text{ for all } t \geq T(x).
\]
We also have (annealed) off-diagonal bounds for $q_t^{\omega}(x, y)$. These are of the same form as the bounds
\[ ct^{-d_t/d_w} \exp(-c'(d(x, y)^{d_w}/t)^{1/(d_w-1)}) \]
obtained for regular fractal graphs.

**Theorem 1.4.** (a) Let $x, y \in \mathbb{B}$. Then
\[ E_{x, y} q_t^{\omega}(x, y) \leq c_1 t^{-2/3} \exp \left( - c_2 \left( \frac{d(x, y)^3}{t} \right)^{1/2} \right). \tag{1.6} \]

(b) Let $x, y \in \mathbb{B}$, with $d(x, y) = R$, and $c_3 R \leq t$. Then
\[ E_{x, y} q_t^{\omega}(x, y) \geq c_4 t^{-2/3} \exp(-c_5 (R^3/t)^{1/2}). \tag{1.7} \]

Define the continuous time rescaled height process
\[ \tilde{Z}_t^{(n)} = n^{-1/3} d(0, Y_{nt}), \quad t \geq 0. \]

By Theorem 1.3(a) the processes $(\tilde{Z}^{(n)}, n \geq 1)$ are tight with respect to the annealed law given by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times P_0^\omega$. (This is much easier to prove than the full convergence given in Theorem 1.1(b).) However, the large scale fluctuations in $G$ mean that we do not have quenched tightness.

**Theorem 1.5.** $\mathbb{P}$-a.s., the processes $(\tilde{Z}^{(n)}, n \geq 1)$ are not tight with respect to $P_0^\omega$.

In Section 2 we recall various properties of branching processes, and obtain the geometrical properties of $G$ that we will require. In particular we show that, with high probability, balls $B(x, r) \subset G$ have roughly $r^2$ points, and $O(1)$ disjoint paths between $B(x, r/4)$ and $B(x, r)^c$. Based on this, we define various types of possible ‘good’ behaviour of a ball $B(x, r)$, and the cluster in a neighbourhood of the path between points $x, y \in G$. In Section 3 we review some general properties of random walks on graphs. Our main estimates are given in Section 4, for the random walk on a deterministic subset $G$ of $\mathbb{B}$ for which balls and paths are ‘good’ in the ways given in Section 2. Finally, in Section 5 we tie together the results of Sections 2 and 4, and prove Theorems 1.2–1.5.

Throughout this article, $f_n \sim g_n$ means that $\lim_{n \to \infty} f_n/g_n = 1$. We use $c$, $c'$ and $c''$ to denote strictly positive finite constants whose values are not significant and may change from line to line. We write $c_i$ for positive constants whose values are fixed within each theorem, lemma etc. When we cite a constant $c_1$ in Lemma 2.2, say, we denote it as $c_{2.2.1}$. None of these constants depend on the degree $n_0$ of the tree.

**2. The incipient infinite cluster**

We begin with some estimates for the critical Bienaymé-Galton-Watson branching processes $X_n$, $n \geq 0$, with $X_0 = 1$ and offspring distribution $\text{Bin}(n_0, 1/n_0)$ where $n_0 \geq 2$. These are quite well known, but as we did not find them anywhere in exactly the form we needed, we give the proofs (which are quite short) here.
Let \( f \) be the generator of the offspring distribution, so that
\[
f(s) = E(s^{X_1}) = n_0^{-n_0}(s + n_0 - 1)^{n_0}.
\]

From [Har] p. 21 we have
\[
P(X_n > 0) \sim \frac{2}{nf''(1)} = \frac{2n_0}{(n_0 - 1)n}.
\]

Let
\[
Y_n = \sum_{k=0}^{n} X_k, \quad g_n(s) = E(s^{Y_n}), \quad f_n(s) = E s^{X_n}.
\]

Then conditioning on \( X_1 \) we obtain that \( f_{n+1}(s) = f(f_n(s)) \), and
\[
g_{n+1}(s) = sf(g_n(s)) = \frac{s}{n_0} (g_n(s) + n_0 - 1)^{n_0}.
\]

Set
\[
h_n(\theta) = \log g_n(e^{\theta}), \quad k_n(\theta) = \log f_n(e^{\theta}).
\]

**Lemma 2.1.** (a) Let \( 1 < \alpha \leq 2 \). Then
\[
h_n(\theta) \leq (1 + \alpha n)\theta, \quad \text{provided } 0 \leq \theta \leq \frac{\alpha - 1}{(1 + \alpha n)^2}. \tag{2.3}
\]

(b)
\[
k_n(\theta) \leq \theta + 2n\theta^2, \quad \text{provided } 0 < \theta \leq \frac{1}{6n}. \tag{2.4}
\]

**Proof.** Note that \( h_n \) and \( k_n \) are continuous, strictly increasing and \( h_n(0) = k_n(0) = 0 \).

For (a) we have
\[
h_{n+1}(\theta) = \log \left( \frac{e^{\theta}}{n_0} (e^{h_n(\theta)} + n_0 - 1)^{n_0} \right) = \theta + n_0 \log \frac{1}{n_0} (e^{h_n(\theta)} + n_0 - 1).
\]

Let \( a_n = \min\{\theta : h_n(\theta) = 1\} \). Then since \( e^x \leq 1 + x + x^2 \) on \([0, 1]\), on \([0, a_n]\),
\[
h_{n+1}(\theta) \leq \theta + n_0 \log (1 + \frac{1}{n_0} h_n(\theta)) + \frac{1}{n_0} h_n(\theta)^2 \leq \theta + h_n(\theta) + h_n(\theta)^2. \tag{2.5}
\]

We verify (2.3) by induction. Since \( h_0(\theta) = \theta \), (2.3) holds for \( n = 0 \). Writing \( b_n(\alpha) = (\alpha - 1)/(1 + \alpha n)^2 \), we have \( h_n(\theta) \leq 1 \) for \( \theta \in [0, b_n(\alpha)] \). So, using (2.5) and (2.3) for \( n \)
\[
\]
proving (2.3) for \( n + 1 \).
(b) Similarly, provided \( k_n(\theta) \leq 1 \),

\[
k_{n+1}(\theta) = n_0 \log \left( 1 + \frac{e^{k_n(\theta)} - 1}{n_0} \right) \leq k_n(\theta) + k_n(\theta)^2. \tag{2.6}
\]

Using (2.4) for \( n \) we obtain, since \( \theta + 2n\theta^2 \leq 4\theta/3 \),

\[
k_{n+1}(\theta) \leq (\theta + 2n\theta^2) + (\theta + 2n\theta^2)^2 \leq (\theta + 2n\theta^2) + 16\theta^2/9 \leq (\theta + 2(n + 1)\theta^2),
\]

proving (2.4) for \( n + 1 \).

\[\square\]

**Notation.** Let \( \xi \) be a random variable. We write \( \lambda \xi[n] \) for a r.v. with the distribution of \( \lambda \sum_1^n \xi_i \), where \( \xi_i \) are i.i.d. with \( \xi_i \overset{d}{=} \xi \). We also write \( \text{Ber}(p) \) and \( \text{Bin}(n, p) \) for the Bernoulli and Binomial distributions respectively. Using this notation we have for example \( (\xi[n])[m] = \xi[nm] \), and \( \text{Bin}(n, p) \overset{d}{=} \text{Ber}(p)[n] \). We write \( \succcurlyeq \) for stochastic domination.

**Lemma 2.2.** For any \( \lambda > 0 \)

\[
P(X_n[n] \geq \lambda n) \leq c_1 e^{-\lambda/6}, \tag{2.7}
\]

\[
P(Y_n[n] \geq \lambda n^2) \leq c_2 e^{-\lambda/5}. \tag{2.8}
\]

**Proof.** Let \( \theta = 1/6n \). Using (2.4)

\[
\log P(X_n[n] \geq \lambda n) \leq -\theta \lambda n + nk_n(\theta) \\
\leq -n\theta(\lambda - 2) = -(\lambda - 2)/6,
\]

proving (2.7).

Let If \( \theta \leq b_n(\alpha) \) then

\[
P(Y_n[n] \geq \lambda n^2) = P(e^{\theta Y_n[n]} \geq e^{\theta \lambda n^2}) \leq e^{-\theta \lambda n^2 E e^{\theta Y_n[n]}} \\
= \exp(-\theta \lambda n^2 + nh_n(\theta)) \leq \exp(-\theta \lambda n^2 + (1 + 2n)n\theta).
\]

So taking \( \alpha = 2 \) and \( \theta = b_n(2) = (1 + 2n)^{-2} \)

\[
\log P(Y_n[n] \geq \lambda n^2) \leq -\frac{n^2(\lambda - 2)}{(1 + 2n)^2} + \frac{n}{(1 + 2n)^2} \sim -\frac{1}{5} \lambda + c_3.
\]

\[\square\]

**Lemma 2.3.** (a) There exist \( c_0 > 0, p_0 > 0 \) such that

\[
P(Y_n > c_0 n^2) \geq \frac{p_0}{n}.
\]
Lemma 2.4. \(\eta_n \overset{(d)}{=} \text{Bin}(n, p_0/n)\) then \(Y_n[n] \geq c_0 n^2 \eta_n\).

Proof. (a) This should be in literature, but is also easy to prove directly. Let \(A_n = \{X_{n/2} > 0\}\), and \(a_n = P(A_n)\). Then by (2.2) \(a_n \sim (2n_0/(n_0 - 1))n^{-1}\). We have \(EY_n = n + 1\) and \(EY_n^2 \leq c_1 n^3\), where \(c_1\) does not depend on \(n_0\). On \(A^c\) we have \(Y_{n/2} = Y_n\), so

\[ n + 1 = EY_n = E(Y_n; A_n) + E(Y_n; A_n^c) \leq E(Y_n|A_n)P(A_n) + EY_{n/2}. \]

It follows that

\[ E(Y_n|A_n) \geq \frac{n/2}{a_n} \geq c_2 n^2. \]

Also,

\[ E(Y_n^2|A_n) \leq P(A_n)^{-1}E(Y_n^2; A_n) \leq c_3 n^4. \]

Using the ‘Backwards Chebyshev’ inequality \(P(\xi \geq \frac{1}{2} E\xi) \geq (E\xi^2)/(4E\xi^2)\) with respect to \(P(\cdot|A_n)\) then gives

\[ P(Y_n > \frac{1}{2} c_2 n^2|A_n) \geq P(Y_n > \frac{1}{2} E(Y_n|A_n)|A_n) \geq \frac{c_2 n^4}{4c_3 n^4} = c_4. \]

So \(P(Y_n > \frac{1}{2} c_2 n^2) \geq P(Y_n > c_2 n^2|A_n)P(A_n) \geq c_4 a_n \geq c_5 n^{-1}\), and taking \(c_0 = \frac{1}{2} c_2\), \(p_0 = c_5\), this proves (a).

(b) Let now \(Y_{n(j)}\) be i.i.d. copies of \(Y_n\), and \(F_j = \{Y_{n(j)} > c_0 n^2\}\). Then if \(\xi_j = 1_{F_j}\), by (a) we have \(P(\xi_j = 1) \geq p_0/n\). So,

\[ Y_n[n] = \sum_{j=1}^n Y_{n(j)} \geq \sum_{j=1}^n c_0 n^2 \xi_j \geq c_0 n^2 \eta_n, \]

proving (b). \(\square\)

**Lemma 2.4.** For \(0 < \lambda < 1\),

\[ \exp(-c_1/\lambda) \leq P(Y_n[n] \leq \lambda n^2) \leq \exp(-c_2/\lambda^{1/2}). \]  

(2.9)

Proof. To prove the upper bound let \(c_0 = c_{2.3.0}\), and \(m = (\lambda/c_0)^{1/2} n\). Using Lemma 2.3 we have

\[ Y_n[n] = \sum_{i=1}^n Y_m^{(i)} \geq \sum_{i=1}^n c_0 m^2 \xi_i = \lambda n^2 \sum_{i=1}^n \xi_i; \]

here \(\xi_i\) are i.i.d. \(\text{Ber}(p_0/m)\) r.v. So

\[ P(Y_n[n] < \lambda n^2) \leq P(\sum_{i=1}^n \xi_i < 1) = (1 - p_0/m)^n \leq \exp(-p_0 n/m) = \exp(-c_0^{1/2} p_0 / \lambda^{1/2}). \]
For the lower bound let \( k \geq 1 \) and \( m = n/k \). Let \( G_j = \{ X_m^{(j)} = 0 \} \), and \( G = \bigcap_{1 \leq j \leq n} G_j \). Then \( P(G_j) \geq (1 - c/m)^n \) so

\[
P(Y_n[n] < \lambda n^2) \geq P(Y_n[n] < \lambda n^2 | G) P(G)
\]

\[
\geq (1 - c/m)^n \left( 1 - P(Y_n[n] > \lambda n^2 | G) \right)
\]

\[
\geq c' e^{-c'' k} \left( 1 - P(Y_n[n] > \lambda n^2 | G) \right).
\]

On \( G \) we have \( Y_n[n] = \sum_{j=1}^n Y_m^{(j)} \), so

\[
P(Y_n[n] > \lambda n^2 | G) \leq \frac{E(\sum_{j=1}^n Y_m^{(j)} | G)}{\lambda n^2} = \frac{n E(Y_m^{(1)} | G_1)}{\lambda n^2} \leq \frac{c}{k \lambda}.
\]

Taking \( k \) such that \( c/(k \lambda) = \frac{1}{2} \) completes the proof. \( \Box \)

We will need to consider the following modified branching process. Let \( \tilde{X} = (\tilde{X}_n, n \geq 0) \) be a branching process with \( \tilde{X}_0 = 1 \) and the same \( \text{Bin}(n_0, 1/n_0) \) offspring distribution as \( X \), except that at the first generation we have \( \tilde{X}_1^{(d)} \sim \text{Bin}(n_0 - 1, 1/n_0). \)

**Lemma 2.5.** (a) For any \( \lambda > 0 \)

\[
P(\tilde{X}_n[n] \geq \lambda n) \leq c_1 e^{-c_2 \lambda}, \tag{2.10}
\]

\[
P(\tilde{Y}_n[n] \geq \lambda n^2) \leq c_3 e^{-c_4 \lambda}. \tag{2.11}
\]

(b) For \( 0 < \lambda < 1 \),

\[
\exp(-c_5/\lambda) \leq P(\tilde{Y}_n[n] \leq \lambda n^2) \leq \exp(-c_6/\lambda^{1/2}). \tag{2.12}
\]

(c) There exists \( p_1 > 0 \) such that \( \tilde{Y}_n[n] \geq c_7 n^2 \text{Bin}(n, p_1/n) \).

**Proof.** (a) and the lower bound in (b) are immediate from Lemmas 2.2 and 2.4, since \( \tilde{X}_n \preceq X_n \) and \( \tilde{Y}_n \preceq Y_n \).

For the upper bound in (b), we can write

\[
\tilde{Y}_n[n] = n + \sum_{i=1}^M Y_n^{(i)},
\]

where \( M \sim \text{Bin}(n(n_0 - 1), 1/n_0) \), and \( Y^{(i)} \) are independent copies of \( Y \). Similarly,

\[
Y_m[m] = m + \sum_{i=1}^{M'} Y_m^{(i)},
\]

where \( M' \sim \text{Bin}(m(n_0 - 1), 1/n_0) \), and \( Y^{(i)} \) are independent copies of \( Y \).
where $M'(d) \overset{d}{=} \text{Bin}(nn_0, 1/n_0)$. So if $m = n(n_0 - 1)/n_0$ then

$$\tilde{Y}_n[n] = n + \sum_{i=1}^{M} Y_{n-1}^{(i)} \geq m + \sum_{i=1}^{M} Y_{m-1}^{(i)} = Y_m[m]. \quad \text{(2.13)}$$

(2.12) now follows from Lemma 2.4, since $\frac{1}{2}n \leq m \leq n$.

(c) We have $\text{Ber}(p) \succ \frac{1}{2}\text{Ber}(p/2)[2]$. So, using (2.13), with $m$ as in (b),

$$\tilde{Y}_n[n] \geq Y_m[m] \geq c_0 m^2 \text{Bin}(m, p_0/m)$$

$$\geq \frac{1}{2} c_0 m^2 \text{Bin}(2m, p_0/2m)$$

$$\geq \frac{1}{2} c_0 m^2 \text{Bin}(n, p_0/2m) \geq c_1 n^2 \text{Bin}(n, p_1/n).$$

□

We now define the random graph $G$ we will be working with. We could regard this either as critical percolation on the $n_0$-ary tree $\mathbb{B}$, conditioned on the cluster containing the root 0 being infinite, or as the (critical) Bienaymé-Galton-Watson process with $\text{Bin}(n_0, 1/n_0)$ offspring distribution, conditioned on non-extinction.

Let $\mathbb{B}$ be the $n_0$-ary tree, and let 0 be the root. A point $x$ in the $n$th generation (or level) is written $x = (0, l_1, \cdots, l_n)$, where $l_i \in \{1, 2, \ldots, n_0\}$. Let $\mathbb{B}_n$ be the set of $n_0^n$ points in the $n$th generation, and let $\mathbb{B}_{<n} = \bigcup_{i=0}^{n} \mathbb{B}_i$. If $x \in \mathbb{B}_k$ we write $|x| = k$. If $x = (0, l_1, \cdots, l_n) \in \mathbb{B}_n$, let $a(x, r) = (0, l_1, \cdots, l_{n-r})$ be the ancestor of $x$ at level $|x| - r$.

We regard $\mathbb{B}$ as a graph (in fact a tree) with edge set $E(\mathbb{B}) = \{(x, a(x, 1)), x \in \mathbb{B} - \{0\}\}$. Let $\eta_e, e \in E(\mathbb{B})$, be i.i.d. Bernoulli $1/n_0$ r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$. If $\eta_e = 1$ we say the edge $e$ is open. Let

$$C(0) = \{x \in \mathbb{B} : \text{ there exists an } \eta-\text{open path from 0 to } x\}$$

be the open cluster containing 0. It is clear that $Z_n = |C(0) \cap \mathbb{B}_n|$ is a critical GW process with $\text{Bin}(n_0, 1/n_0)$ offspring distribution. Here and in the following, $|A|$ is a cardinality of the set $A$. As $Z$ has extinction probability 1, the cluster $C(0)$ is $P$–a.s. finite.

We have

**Lemma 2.6. ([Ke2, Lemma 1.14]).** Let $A \subset \mathbb{B}_{\leq k}$. Then

$$\lim_{n \to \infty} P(C(0) \cap \mathbb{B}_{\leq k} = A | Z_n \neq 0) = |A \cap \mathbb{B}_k| P(C(0) \cap \mathbb{B}_{\leq k} = A), \quad \text{(2.14)}$$

and writing $P_0(A) = |A \cap \mathbb{B}_k| P(C_{\leq k} = A)$, $P_0$ has a unique extension to a probability measure $\hat{P}$ on the set of infinite connected subsets of $\mathbb{B}$ containing 0.

Let $G'$ be a rooted labeled tree chosen with the distribution $\hat{P}$: we call this the incipient infinite cluster (IIC) on $\mathbb{B}$. For more information on $G'$ see [Ke2] and [vH] but we remark that $\hat{P}$–a.s. $G'$ has exactly one infinite descending path from 0, which we call the backbone, and denote $H$. 

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It will be useful to give another construction of the IIC, obtained by modifying the cluster $\mathcal{C}(0)$ rather than its law. We can suppose the probability space $(\Omega, \mathcal{F}, P)$ carries i.i.d.r.v. $\xi_i$, $i \geq 1$ uniformly distributed on $\{1, 2, \ldots, n_0\}$, and independent of $(\eta_e)$. For $n \geq 0$ let $\Xi_n = (0, \xi_1, \ldots, \xi_n)$, and let

$$\tilde{\eta}_e = \begin{cases} 1 & \text{if } e = \Xi_n, \Xi_{n+1} \text{ for some } n \geq 0, \\ \eta_e & \text{otherwise.} \end{cases}$$

Then (see [vH]) if

$$\mathcal{G} = \{ x \in \mathbb{B} : \text{there exists a } \tilde{\eta} \text{-open path from } 0 \text{ to } x \},$$

$\mathcal{G}$ has law $P$. It is clear that the backbone of $\mathcal{G}$ is the set $H = \{ \Xi_n, n \geq 0 \}$. For $x, y \in \mathbb{B}$ let

$$P_x(\cdot) = P(\cdot | x \in \mathcal{G}), \quad P_{x,y}(\cdot) = P(\cdot | x, y \in \mathcal{G}),$$

and let $E_x$ and $E_{x,y}$ denote expectation with respect to $P_x$ and $P_{x,y}$ respectively. Given a descending path $b = \{0, b_1, b_2, \ldots\}$, (which we call a possible backbone) let

$$P_{x,b}(\cdot) = P(\cdot | x \in \mathcal{G}, H = b),$$

and define $P_{x,y,b}$ analogously.

For each $x, y \in \mathbb{B}$, let $\gamma(x, y)$ be the unique geodesic path connecting $x$ and $y$. We say that $z$ is a middle point of $\gamma(x, y)$ if $z \in \gamma(x, y)$ and $|d(x, z) - \frac{1}{2}d(x, y)| \leq \frac{1}{2}$. We remark that the construction of $\mathcal{G}$ makes it clear that $P_{x,y,b}(\eta_e = 1) = 1$ if the edge $e$ lies in any of the paths $b, \gamma(0, x)$ and $\gamma(0, y)$, and that under $P_{x,y,b}$ the r.v. $\eta_e, e \notin b \cup \gamma(0, x) \cup \gamma(0, y)$ are i.i.d. with $P_{x,y,b}(\eta_e = 1) = 1/n_0$.

**Notation.** We consider the tree $\mathcal{G} = \mathcal{G}(\omega)$. Let $d(x, y)$ be the graph distance between $x$ and $y$, and

$$B(x, r) = \{ y \in \mathcal{G} : d(x, y) \leq r \}.$$

We write $D(x)$ for the set of descendants of $x$. More precisely, $y \in D(x)$ if and only if $x \in \gamma(0, y)$. Note that $x \in D(x)$. If $y \in D(x)$ we call $x$ an ancestor of $y$ and $y$ a decedent of $x$. We set

$$D_{r}(x) = \{ y \in D(x) : d(x, y) = r \}, \quad D_{\leq r}(x) = \bigcup_{i=0}^{r} D_i(x).$$

We also set

$$D(x; z) = \{ y \in D(x) : \gamma(x, y) \cap \gamma(x, z) = \{ x \} \},$$

and write $D_{r}(x; z) = D_{r}(x) \cap D_{r}(z); D_{\leq r}(x; z) = D_{\leq r}(x) \cap D_{\leq r}(z)$. Thus if $z \in D(x)$ then $y \in D(x; z)$ if and only if the lines of descent from $x$ to $y$ and $z$ are disjoint, except for $x$. (Note that $D(x; x) = D(x)$.) For any $A \subset \mathcal{G}$ we write

$$\partial A = \{ y \in \mathcal{G} - A : y \sim x \text{ for some } x \in A \}.$$
The estimates at the beginning of this Section lead to volume growth estimates for $\mathcal{G}$. For $x \in \mathcal{G}$ let $\mu_x$ be the degree of $x$, and for $A \subset \mathcal{G}$ set $\mu(A) = \sum_{x \in A} \mu_x$. We write

$$V(x, r) = \mu(B(x, r)).$$

Note that as $\mathcal{G}$ is a tree, we have

$$|B(x, r)| \leq V(x, r) \leq 2|B(x, r + 1)|. \tag{2.15}$$

**Proposition 2.7.** (a) Let $\lambda > 0$, $r \geq 1$ and $x, y \in \mathbb{B}$, and $b$ be a possible backbone. Then

$$\mathbb{P}_{x,y,b}(V(x, r) > \lambda r^2) \leq c_0 \exp(-c_1 \lambda), \tag{2.16}$$

and

$$\mathbb{P}_{x,y,b}(V(x, r) < \lambda r^2) \leq c_2 \exp(-c_3/\sqrt{\lambda}). \tag{2.17}$$

(b) The bounds (2.16) and (2.17) also hold for the laws $\mathbb{P}_{x,b}$, $\mathbb{P}_{x,y}$, and $\mathbb{P}_x$.

**Proof.** It is enough to prove (a), since the bounds for $\mathbb{P}_{x,b}$ follow by taking $y = 0$, and those for $\mathbb{P}_{x,y}$ and $\mathbb{P}_x$ then follow on integrating over $b$. Also, using (2.15), it is enough to bound $|B(x, r)|$.

We will assume that $|x| > r$; if not we can use the same arguments with minor modifications. Let $x_i = a(x, i)$ for $0 \leq i \leq r$. If the backbone intersects $B(x, r)$ then let $s$ be the smallest $i$ such that $x_i \in H$, and let $v_0 = x_s$ and $v_i$, $i \geq 1$ be the backbone descending from the point $v_0$. Similarly if $\gamma(0, y)$ intersects $B(x, r)$ then let $t$ be the smallest $j$ such that $y_j \in \gamma(0, y)$, and let $w_0 = y_t$ and $w_i$, $1 \leq i \leq t$ be the path $\gamma(w_0, y)$.

Then we have

$$B(x, r) \subset \left( \bigcup_{i=0}^{r} D_{\leq r}(x_i; x) \right) \cup \left( \bigcup_{i=1}^{r} D_{\leq r}(v_i; v_3r) \right) \cup \left( \bigcup_{i=1}^{r} D_{\leq r}(w_i; y) \right).$$

Under $\mathbb{P}_{x,y,b}$ the r.v. $|D_{\leq r}(\cdot; \cdot)|$ above are i.i.d., with the same law as $\tilde{Y}_r$. Thus $|B(x, r)| \leq \tilde{Y}_r[r][3]$, and by Lemma 2.5(a),

$$\mathbb{P}_{x,y,b}(|B(x, r)| > \lambda r^2) \leq c \exp(-c' \lambda).$$

The proof of (2.17) is very similar. We have $\bigcup_{i=0}^{r/2} D_{\leq r/2}(x_i; x) \subset B(x, r)$, so that $|B(x, r)| \geq \tilde{Y}_{r/2}[r/2]$, and using Lemma 2.5(b) leads to (2.17).

We also wish to show that oscillations in $n^{-2}V(0, n)$ exist. If $W \overset{(d)}{=} \text{Bin}(n, p/n)$ then straightforward calculations give that

$$P(W = k) \geq c_0 e^{-k \log(k/p)}, \quad 0 \leq k \leq n^{1/2}. \tag{2.18}$$

**Proposition 2.8.** (a) For any $\varepsilon > 0$

$$\limsup_{n \to \infty} \frac{V(0, n)}{n^2(\log\log n)^{1-\varepsilon}} = \infty, \quad \mathbb{P} - \text{a.s.}$$
(b) There exists $c_0 < \infty$ such that
\[
\liminf_{n \to \infty} \frac{\log \log n) V(0,n)}{n^2} \leq c_0. \quad \mathbb{P} - \text{a.s.}
\]

**Proof.** It is enough to prove these for the law $\mathbb{P}_b$, for any fixed possible backbone $b = \{0, y_1, y_2, \ldots \}$.

(a) Let
\[
Z_n = |\{x : x \in D(y_i; y_{i+1}), d(x, y_i) \leq 2^{n-2}, 2^{n-1} \leq i \leq 2^{n-1} + 2^{n-2}\}|.
\]
Thus $Z_n$ is the number of descendants off the backbone, to level $2^{n-2}$, of points $y$ on the backbone between levels $2^{n-1}$ and $2^{n-1} + 2^{n-2}$. So $|B(0,2^n)| \geq Z_n$, the r.v. $Z_n$ are independent, and $Z_n \overset{(d)}{=} Y_{2^n-2}[2^{n-2}]$. Using Lemma 2.5(c) we have, if $a_n = (\log n)^{1-\varepsilon}$, and $\eta_n \overset{(d)}{=} \text{Bin}(n, p_1/n)$,
\[
\mathbb{P}_b(|B(0,2^n)| \geq a_n n^4) \geq \mathbb{P}_b(Z_n \geq a_n n^4) \geq P(\tilde{Y}_{2^n-2}[2^{n-2}] \geq a_n n^4) \geq P(\eta_{2^n-2} \geq a_n) \geq c e^{-a_n \log a_n}.
\]
As $Z_n$ are independent, (a) follows by the second Borel-Cantelli Lemma.

(b) Let $n_k = \exp(2k \log k)$, so that $k^2 n_{k-1} \leq n_k$, and let
\[
W_k = \bigcup_{i=0}^{n_k-1} D(y_i; y_{n_k}), \quad V_k = D_{\leq n_k-n_{k-1}}(y_{n_{k-1}}).
\]
Then the r.v. $|V_k|$ are independent and $B(0, n_k) \subset W_{k-1} \cup V_k$.

Fix $0 < \varepsilon < 1/3$ and let
\[
F(i,k) = \{D_{k^{1+\varepsilon} n_{k}}(y_i; y_{i+1}) = \emptyset\}.
\]

Then since $X_n \overset{d}{=} \tilde{X}_n$
\[
\mathbb{P}(F(i,k)) = P(\tilde{X}_{k^{1+\varepsilon} n_{k}} = 0) \geq P(X_{k^{1+\varepsilon} n_{k}} = 0) \geq 1 - \frac{c}{k^{1+\varepsilon} n_{k}}.
\]

Let $G_k = \bigcap_{i=0}^{n_k-1} F(i,k)$; we have
\[
\mathbb{P}(G_c^c) \leq c / k^{1+\varepsilon}.
\]

On the event $G_k$ we have that $|W_k|$ is stochastically dominated by $\sum_{i=1}^{n_k} Y_{k^{1+\varepsilon} n_{k}}^{(i)}$, so
\[
\mathbb{P}(|W_k| \geq k^3 n_k^2) \leq \mathbb{P}(G_c^c) + P(Y_{k^{1+\varepsilon} n_{k}} \geq k^{1+2\varepsilon} (k^{1+\varepsilon} n_{k})^2) \leq c k^{-2} + e^{-c' k^{-1-2\varepsilon}} \leq c'' k^{-2}.
\]
Thus \( |W_k| \leq k^3 n_k^2 \) for all large \( k \). Now \( |V_k| \leq Y_{n_k}[n_k] \), so

\[
\mathbb{P}(|V_k| < c_1 (\log k)^{-1} n_k^2) \geq P(Y_{n_k}[n_k] < c_1 (\log k)^{-1} n_k^2) \geq e^{-c \log k} \geq k^{-1}
\]

if \( c_1 \) is chosen large enough. As the r.v. \( |V_k| \) are independent, we deduce that \( |V_k| < c_1 (\log k)^{-1} n_k^2 \) for all \( k \) in an infinite set \( J \). For all large \( k \in J \),

\[
|B(0, n_k)| \leq |V_k| + (k - 1)^3 n_{k-1}^2 \leq (c_1 (\log k)^{-1} + k^{-1}) n_k^2 \leq \frac{2c_1 n_k^2}{\log \log n_k}.
\]

\[\square\]

**Remark.** Let \( C_\infty \) denote the unique infinite cluster for supercritical bond percolation (i.e. \( p > p_c \)) in \( \mathbb{Z}^d \). Then writing \( Q(x, N) \) for the box side \( N \) and center \( x \)

\[
\frac{|C_\infty \cap Q(x, N)|}{|Q(x, N)|} \to \theta(p).
\]

Propositions 2.7 and 2.8 show that one does not get this kind of convergence for \( \mathcal{G} \), which is a much more irregular set than the clusters considered in \( \text{[B2]} \).

**Definition 2.9.** Let \( x \in \mathcal{G}, r \geq 1 \). Let \( M(x, r) \) be the smallest number \( m \) such that there exists a set \( A = \{z_1, \ldots, z_m\} \) with \( d(x, z_i) \in [r/4, 3r/4] \) for each \( i \), such that any path \( \gamma \) from \( x \) to \( B(x, r)^c \) must pass through the set \( A \). (Since \( \mathcal{G} \) is a tree, the best choice of such a set \( A \) will in fact have the points at a distance \( r/4 \) from \( x \), but we will not need this.)

**Proposition 2.10.** There exist \( c_1, c_2 > 0 \) such that for each \( r \geq 1 \) and each \( x, y \in \mathbb{B} \), and possible backbone \( b \)

\[
\mathbb{P}_{x,y,b}(M(x, r) \geq m) \leq c_1 e^{-c_2 m}.
\]

Similar bounds hold for \( \mathbb{P}_{x,y}, \mathbb{P}_{x,b} \) and \( \mathbb{P}_x \).

**Proof.** We just consider the case \( y = 0 \); the general case is similar but a little more complicated since we would also need to consider offspring on the branch \( \gamma(0,y) \). Let \( w_0 = a(x, r/3) \). If \( w_0 \in b \) then let \( w_1 \) be the point in the backbone at level \( |x| + r/3 \), otherwise let \( w_1 = w_0 \). Let

\[
A_1 = \bigcup_{z \in \gamma(w_0, x)} D_{r/4}(z; x), \quad A_2 = \bigcup_{z \in \gamma(w_0, w_1), z \neq w_1} D_{r/4}(z; w_1).
\]

Let \( N_i = |A_i|; \) we have \( N_1 \leq X_{r/4}[1 + r/4] \) and \( N_2 \leq X_{r/4}[r/2] \). Now let

\[
A_i^* = \{ z \in A_i : D_{r/4}(z) \neq \emptyset \}.
\]

Then any path from \( x \) to \( B(x, r)^c \) must pass through \( A_1^* \cup A_2^* \cup \{w_0, w_1\} \), so \( M = M(x, r) \leq 2 + |A_1^*| + |A_2^*| \).

Let \( p_r = P(z \in A_i^* | z \in A_i) = P(X_{r/4} > 0) \), so that \( p_r \leq c/r \). So, if \( k_i \) are i.i.d. \( \text{Ber}(p_r) \) r.v. independent of \( N_i \), we have

\[
|A_i^*| \overset{(d)}{=} \sum_{j=1}^{N_i} k_j.
\]

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Let
\[ W_n = \sum_{i=1}^{n} (\kappa_i - p_r); \]
then \( W = \{W_n\} \) is a martingale, \( W_n - W_{n-1} \leq 1 \), \( \langle W \rangle_n = np_r(1 - p_r) \), and \( |A_i^*|^{(d)} = W_{N_i} + N_i p_r \). Choose \( r \) large enough so that \( p_r < \frac{1}{2} \). Then
\[ \mathbb{P}_{x,b}(|A_i^*| \geq m) \leq \mathbb{P}_{x,b}(W_{N_i} + N_ip \geq m, N_ip \leq m/2) + \mathbb{P}_{x,b}(N_ip > m/2). \quad (2.19) \]
For the first term in (2.19) we have
\[ \mathbb{P}_{x,b}(W_{N_i} + N_ip \geq m, N_ip \leq m/2) \leq \mathbb{P}_{x,b}(W_{N_i} \geq m/2, \langle W \rangle_{N_i} \leq m(1 - p)/2) \leq \exp\left(-\frac{(m/2)^2}{2((m/2) + m(1 - p)/2)}\right) \leq e^{-cm}, \]
where we used an exponential martingale inequality – see (1.6) in [F]. For the second term, note that \( N_i \leq (X_r/4[r/4])[2] \) and so using Lemma 2.2 we deduce that
\[ \mathbb{P}_{x,b}(N_ip > m/2) \leq ce^{-c3m}. \]
Combining these bounds completes the proof. \( \square \)

**Definition 2.11.** Let \( x \in \mathbb{B}, r \geq 1, \lambda \geq 64 \). We say that \( B(x, r) \) is \( \lambda \)-good if:
(a) \( x \in \mathcal{G} \)
(b) \( r^2 \lambda^{-2} \leq V(x, r) \leq r^2 \lambda \).
(c) \( M(x, r) \leq \frac{1}{60} \lambda \).
(d) \( V(x, r/\lambda) \geq r^2 \lambda^{-4} \).
(e) \( V(x, r/\lambda^2) \geq r^2 \lambda^{-6} \).

**Corollary 2.12.** For \( x \in \mathbb{B} \) and any possible backbone \( b \)
\[ \mathbb{P}_{x,b}(B(x, r) \text{ is not } \lambda\text{-good}) \leq c_1 e^{-c_2 \lambda}. \quad (2.20) \]

**Proof.** By Propositions 2.7 and 2.10 the probability of each of conditions (a)–(d) above failing is bounded by \( \exp(-c\lambda) \). \( \square \)

We now need to introduce some more complicated conditions on the tree \( \mathcal{G} \), and will prove that these hold with high probability. These conditions describe various kinds of ‘good’ behaviour of balls with centers on a path \( \gamma(x, y) \), and will be used when we consider off-diagonal bounds on the transition probabilities of the random walk in Sections 4 and 5.

Fix \( \lambda_1 \geq 64 \) large enough so that the right hand side of (2.20) is less than \( \frac{1}{4} \). For \( x, y \in \mathbb{B} \) and \( k \in \mathbb{N} \), define the event
\[ F_1(x, y, r, k) = \{x, y \in \mathcal{G} \text{ and there exist at least } k \text{ disjoint balls} \}
\[ B(z, r/2) \text{ with } z \in \gamma(x, y) \text{ and which are } \lambda_1\text{-good.}\} \]
For $x, y \in \mathbb{B}$, let $z_0$ be a middle point of $\gamma(x, y)$. Define the events

$$A_*(z, r, N) = \{z \in G \text{ and } B(z, r) \text{ is } N\text{-good.}\},$$

$$F_*(x, y, R, k; r, N) = F_1(x, z_0, R, k/2) \cap F_1(z_0, y, R, k/2)$$
$$\cap A_*(x, r, N) \cap A_*(z_0, r, N) \cap A_*(y, r, N).$$

**Definition 2.13.** The vertex $x \in \mathbb{B}$ satisfies the condition $G_2(N, R)$ if:

(a) $x \in G$,

(b) For every $z \in \partial B(x, NR)$ the event $F_1(x, z, R, 1/8 N)$ holds.

**Proposition 2.14.** Let $x_0, y_0 \in \mathbb{B}$, and $b$ be a possible backbone.

(a) For $R \geq 1$, $N \geq 8$,

$$\mathbb{P}_{x_0, y_0, b}(x_0 \text{ satisfies the condition } G_2(N, R)) \geq 1 - c_1 \exp(-c_2 N).$$

(b) The same bounds as in (a) hold for the laws $\mathbb{P}_{x_0, b}$, $\mathbb{P}_{x_0, y_0}$, and $\mathbb{P}_{x_0}$.

(c) For $x_0, y_0 \in \mathbb{B}$, $8 \leq N < d(x_0, y_0)/8$, $r \geq 1$,

$$\mathbb{P}_{x_0, y_0, b}(F_*(x_0, y_0, d(x_0, y_0)/N, 1/8 N; r, N)) \geq 1 - c_3 \exp(-c_4 N).$$

**Proof.** (a) We prove this for $y_0 = 0$; as in Proposition 2.10 the general case is handled by a similar argument.

Let

$$F_0(y, s) = \{y \in G \text{ and } B(y, s) \text{ is } \lambda_1\text{-good.}\},$$

and write $v_i = a(x, i), R' = RN/4$. We assume that $|x| \geq NR$ and $v_{R'}$ is on the backbone $b$: the other cases can be handled by minor modifications to the arguments below. Let $w_0$ be the highest level point in both $b$ and $\gamma(0, x)$, and $w_i, i \geq 1$ be the backbone $b$ from $w_0$ on.

Under $\mathbb{P}_{x, b}$ the events $F_0(v_{Rj}, R/2), 1 \leq j \leq N$ are independent, and $\mathbb{P}_{x, b}(F_0(v_{Rj}, R/2)^c) \leq \frac{1}{4}$. So standard exponential bounds give

$$\mathbb{P}_{x, b}(F_1(x, v_{R'}, R, N/8)^c) \leq c \exp(-c' N). \quad (2.21)$$

Similarly

$$\mathbb{P}_{x, b}(F_1(w_0, w_{R'}, R, N/8)^c) \leq c \exp(-c' N).$$

Now let $A_1 = \{v_i, 0 \leq i \leq R'\} \cup \{w_i, 0 \leq i \leq R'\}$; note that under $\mathbb{P}_{x, b}$ this set is non-random. Let

$$A_2 = \{y \in \mathbb{B} : a(y, R') \in A_1, \gamma(y, a(y, R')) \cap A_1 = \{a(y, R')\}\}.$$

For $y \in A_2$ let

$$H_1(y) = F_1(a(y, R), a(y, R'), R, N/8)^c,$$

$$H_2(y) = \{y \in G, DR'(y) \neq \emptyset\}.$$
Then
\[ \mathbb{P}_{x,b} \left( \bigcup_{y \in A_2} H_1(y) \cap H_2(y) \right) \leq \sum_{y \in A_2} \mathbb{P}_{x,y,b}(H_1(y) \cap H_2(y)) \mathbb{P}_{x,b}(y \in \mathcal{G}). \]

Under \( \mathbb{P}_{x,y,b} \) the events \( H_1(y) \) and \( H_2(y) \) are independent, and as in (2.21) we obtain \( \mathbb{P}_{x,y,b}(H_1(y)) \leq c \exp(-c'N) \). So,
\[ \mathbb{P}_{x,b} \left( \bigcup_{y \in A_2} H_1(y) \cap H_2(y) \right) \leq c e^{-c'N} \sum_{y \in A_2} \mathbb{P}_{x,y,b}(H_2(y)) \mathbb{P}_{x,b}(y \in \mathcal{G}). \]

The final sum above is bounded by a constant \( c' \) by the same argument as in Proposition 2.10.

Finally, we have
\[ \{ G_2(N, R) \text{ fails for } x \} \subset F_1(x, v_{R'}, R, N/8)^c \cup F_1(w_0, w_{R'}, R, N/8)^c \cup \bigcup_{y \in A_2} (H_1(y) \cap H_2(y)), \]
so combining the bounds above completes the proof. (b) follows on integrating the bounds in (a).

For (c), we first note that, by the argument for (2.21),
\[ \mathbb{P}_{x,y,b}(F_1(x, y, d(x,y), \frac{d(x,y)}{N} \frac{1}{16} N)^c) \leq c' \exp(-cN). \]

So, using Corollary 2.12, we have
\[ \mathbb{P}_{x,y,b}(F_1^c) \leq \mathbb{P}_{x,y,b}(F_1(x, z_0, d(x,y), \frac{d(x,y)}{N} \frac{1}{16} N)^c) + \mathbb{P}_{x,y,b}(F_1(z_0, y, d(x,y), \frac{d(x,y)}{N} \frac{1}{16} N)^c) + \sum_{w = x, z_0} \mathbb{P}_{x,y,b}(A_*(w, r, N)^c) \]
\[ \leq 2c' \exp(-cN) + 3c' \exp(-cN) = 5c' \exp(-cN). \]

\[ \square \]

**Definition 2.15.** Let \( x, y \in \mathbb{B}, m, \theta \in \mathbb{N} \). Define the condition \( G_3(x, y, m, \kappa) \) as follows. Let \( r = d(x,y)/m \), and let \( z_0 = x, z_1, \ldots, z_m = y \) be points on the path \( \gamma(x,y) \) with \( |d(z_{i-1}, z_i) - r| \leq 1 \). (We choose these points in some fixed way – for example so that \( d(z_{i-1}, z_i) \) are non-decreasing.) For each \( i = 1, \ldots, m \) let \( \Theta_i \) be the smallest integer \( \lambda \geq \max(64, 3c_{4,7}^{-1}) \) such that \( B(z_i, \lambda^{20r}) \) is \( \lambda \)-good, and \( |B(z_i, r)| \geq r^2/\lambda^2 \). Then \( G_3(x, y, m, \kappa) \) holds if:
(a) \( x, y \in \mathcal{G} \),
(b) \( \sum_{i=1}^m \Theta_i^{54} \leq \kappa m \).
Proposition 2.16. For each backbone $b$ and $x, y \in \mathcal{B}$

$$\mathbb{P}_{x,y,b}(G_3(x, y, m, \kappa) \text{ holds}) \geq 1 - c_1 \kappa^{-1}.$$  

Proof. By Proposition 2.7 and Corollary 2.12, $\mathbb{P}_{x,y,b}(\Theta_i = k) \leq e^{-ck}$. Thus $\mathbb{E}_{x,y,b} \Theta_i^{54} \leq c'$, and so

$$\mathbb{P}_{x,y,b}(G_3(x, y, m, \kappa) \text{ fails}) = \mathbb{P}_{x,y,b}(\sum_{i=1}^{m} \Theta_i^{54} > \kappa m) \leq c'/\kappa.$$  

$\square$

3. Markov chains on weighted graphs and trees

Let $\Gamma$ be a infinite connected locally finite graph. Assume that the graph $\Gamma$ is endowed by a weight (conductance) $\mu_{xy}$, which is a symmetric nonnegative function on $\Gamma \times \Gamma$ such that $\mu_{xy} > 0$ if and only if $x$ and $y$ are connected by a bond (in which case we write $x \sim y$). We call the pair $(\Gamma, \mu)$ a weighted graph. We can also regard it as an electrical network, in which the bond $\{x, y\}$ has conductance $\mu_{xy}$. We will be mainly concerned with the case when $\mu_{xy} = 1$ if and only if $\{x, y\}$ is an edge: we call these the natural weights on $\Gamma$. Let $\mu_x = \sum_{y \in \Gamma} \mu_{xy}$ for each $x \in \Gamma$, and set $\mu(A) = \sum_{x \in A} \mu_x$ for each $A \subset \Gamma$, so that $\mu$ is then a measure on $\Gamma$.

We next define a quadratic form $\mathcal{E}$ on $\Gamma$ by

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x,y \in \Gamma, x \sim y} (f(x) - f(y))(g(x) - g(y)) \mu_{xy},$$

and set

$$H^2 = H^2(\Gamma, \mu) = \{f \in \mathbb{R}^\Gamma : \mathcal{E}(f, f) < \infty\}.$$  

For $f, g \in H^2$ we define $\mathcal{E}(f, g)$ by polarization. We sometimes abbreviate $\mathcal{E}(f, f)$ as $\mathcal{E}(f)$. Note that if $f = \min_{1 \leq i \leq n} g_i$ then since

$$|f(x) - f(y)|^2 \leq \max_i |g_i(x) - g_i(y)|^2 \leq \sum_i |g_i(x) - g_i(y)|^2,$$

it follows that

$$\mathcal{E}(f, f) \leq \sum_{i=1}^{n} \mathcal{E}(g_i, g_i). \quad (3.1)$$  

Let $Y = \{Y_t\}_{t \geq 0}$ be the continuous time random walk on $\Gamma$ associated with $\mathcal{E}$ and the measure $\mu$. When the natural weights are given on $\Gamma$, $Y$ is called the simple random walk on $\Gamma$. $Y$ is the Markov process with generator

$$\mathcal{L}f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy} (f(y) - f(x)).$$
$Y$ waits at $x$ for an exponential mean 1 random time and then moves to a neighbour $y$ of $x$ with probability proportional to $\mu_{xy}$. We define the transition density (heat kernel density) of $Y$ with respect to $\mu$ by

$$q_t(x, y) = \mathbb{P}^x(Y_t = y)/\mu_y.$$  \hspace{1cm} (3.2)

If $A \subset \Gamma$ we write

$$T_A = \inf\{t \geq 0 : Y_t \in A\}, \quad \tau_A = T_{A^c}.$$

The natural metric on the graph, obtained by counting the number of steps in the shortest path between points, is written $d(x, y)$ for $x, y \in \Gamma$. As before, we write

$$B(x, r) = \{y : d(x, y) \leq r\}, \quad V(x, r) = \mu(B(x, r)).$$

Let $A, B$ be disjoint subsets of $\Gamma$. The effective resistance between $A$ and $B$ is defined by:

$$R(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\}. \hspace{1cm} (3.3)$$

Let $R(x, y) = R(\{x\}, \{y\})$, and $R(x, x) = 0$. In general $R$ is a metric on $\Gamma$ – see [Kig] Section 2.3. If $(\Gamma, \mu)$ has natural weights then $R(x, y) \leq d(x, y)$, and if in addition $\Gamma$ is a tree then $R(x, y) = d(x, y)$.

The following is an easy consequence of (3.3).

**Lemma 3.1.** For all $f \in \mathbb{R}^\Gamma$ and $x, y \in \Gamma$,

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f). \hspace{1cm} (3.4)$$

Further, for each $x, y \in \Gamma$, there exists $f$ so that the equality holds in (3.4).

We recall some basic properties of Green kernels. Let $Y^B_t$ be the continuous time random walk on $(\Gamma, \mu)$ killed outside $B := B_R(x_0, r)$, and $q^B_t(x, y)$ be the transition density of $Y^B_t$. The Green kernel $g_B(x, y)$ of $Y^B_t$ is defined by $g_B(x, y) = \int_0^\infty q^B_t(x, y)dt$. Then $g_B(\cdot, \cdot)$ has the reproducing property that

$$\mathcal{E}(g_B(x, \cdot), f) = f(x)$$

for all $f \in H^2$ such that $f|_{B^c} = 0$.

Using this and the fact that $e_{B, x}(y) := g_B(x, y)/g_B(x, x)$ is the equilibrium potential for $R(x, B^c)$, we have

$$R(x, B^c)^{-1} = \mathcal{E}(e_{B, x}, e_{B, x}) = g_B(x, x)^{-1},$$

so that

$$R(x, B^c) = g_B(x, x) = \int_0^\infty q^B_t(x, x)dt \quad \forall x \in \Gamma, B \subset \Gamma. \hspace{1cm} (3.5)$$
4. Heat kernel estimates on graphs and trees

Recall that for \( x \in \Gamma \) and \( r \geq 0 \), we denote \( V(x, r) = \mu(B(x, r)) \).

**Theorem 4.1.** Let \((\Gamma, \mu)\) be a weighted graph and suppose that the edge weights satisfy \( \mu_{xy} \geq 1 \) for all \( x \) and \( y \). Then

\[
q_{2rV(x, r)}(x, x) \leq \frac{2}{V(x, r)}, \quad x \in \Gamma, \ r > 0.
\]

**Remark.** This is similar to the bound in Proposition 3.2 of [BCK], but has weaker hypotheses: in particular the bound on \( q_t(x, x) \) only uses the volumes of the balls \( V(x, R) \).

**Proof.** Fix \( x_0 \in \Gamma \), write \( B(r) = B(x_0, r) \) and \( V(r) = V(x_0, r) \). Set \( f_t(y) = q_t(x_0, y) \) and

\[
\psi(t) = ||f_t||^2 = q_{2t}(x_0, x_0) = f_{2t}(x_0);
\]

note that \( \psi \) is decreasing. Let \( r > 0 \); since

\[
\sum_{y \in B(r)} f_t(y) \mu_y \leq 1,
\]

there exists \( y = y(t, r) \in B(r) \) with \( f_t(y) \leq V(r)^{-1} \). Note that, since \( \mu_e \geq 1 \) for every edge \( e \), it follows that \( R(x, y) \leq d(x, y) \) for all \( x, y \). Then by (3.4)

\[
\frac{1}{t} f_t(x_0)^2 \leq f_t(y)^2 + |f_t(x_0) - f_t(y)|^2
\]

\[
\leq \frac{1}{V(r)^2} + R(x_0, y)E(f_t, f_t) \leq \frac{1}{V(r)^2} + rE(f_t, f_t).
\]

Hence

\[
\psi'(t) = -2E(f_t, f_t) \leq \frac{2V(r)^{-2} - \psi(t/2)^2}{r}.
\] (4.1)

Since \( -\psi(s/2) \leq -\psi(t) \) for \( t \leq s \leq 2t \), integrating (4.1) from \( t \) to \( 2t \) we obtain

\[
\psi(2t) - \psi(t) \leq 2tr^{-1}V(r)^{-2} - tr^{-1}\psi(t)^2.
\]

So as \( \psi(2t) > 0 \),

\[
tV(r)^2\psi(t)^2 \leq 2t + rV(r)^2\psi(t) \leq (4t) \vee (2rV(r)^2\psi(t)).
\]

Hence

\[
\psi(t) \leq \frac{2}{V(r)} \vee \frac{2r}{t}.
\]

Taking \( r \) such that \( t = rV(r) \) completes the proof. \( \Box \)
Corollary 4.2. Let $V(x, r) \geq r^2/A$, and $t = r^3$. Then

$$q_{2t}(x, x) \leq \frac{2(A \lor 1)}{r^2} = \frac{2(A \lor 1)}{t^{2/3}}.$$

(4.2)

Proof. Let $\lambda = r^{-2}V(x, r)$, so that $\lambda \geq A^{-1}$. Let $t_0 = rV(x, r) = \lambda r^3$. If $\lambda \leq 1$ then $t_0 \leq t$ and so Theorem 4.1 gives

$$q_{2t}(x, x) \leq q_{2t_0}(x, x) \leq \frac{2}{V(x, r)} = \frac{2}{\lambda r^2} = 2\lambda^{-1}t^{-2/3} \leq 2At^{-2/3}.$$

Now suppose that $\lambda \geq 1$. Let $r'$ be such that $t = r'V(x, r')$; as $rV(x, r) = \lambda r^3 = \lambda t$, we have $r' \leq r$. So

$$q_{2t}(x, x) = q_{2r'V(r')}(x, x) \leq \frac{2}{V(x, r')} = \frac{2r'}{t} \leq \frac{2r}{t} = 2t^{-2/3} \leq 2(A \lor 1)t^{-2/3}.$$

□

Lemma 4.3. Let $f_t(y) = q_t(x_0, y)$. Then

$$\left| \frac{f_t(y)}{f_t(x_0)} - 1 \right|^2 \leq \frac{d(x_0, y)}{tf_t(x_0)}.$$  (4.3)

Proof. Let $e(t) = \mathcal{E}(f_t, f_t)$. Then $e$ is decreasing, and

$$|f_t(x_0) - f_t(y)|^2 \leq d(x_0, y)e(t).$$

So as

$$\psi(t) - \psi(t/2) = -2 \int_{t/2}^t e(s)ds,$$

we have

$$2e(t) \cdot t/2 \leq 2 \int_{t/2}^t e(s)ds \leq \psi(t/2).$$

So,

$$|f_t(x_0) - f_t(y)|^2 \leq \frac{d(x_0, y)f_t(x_0)}{t},$$

and dividing by $f_t(x_0)^2$ completes the proof. □

Up to this point we have not needed to use the fact that $\Gamma$ is a tree, but the following lemma relies strongly on this. From now on we take $\Gamma$ to be a subgraph of $\mathbb{B}$, and define $M(x, r)$, and the conditions $\lambda$–good, $G_2(N, R)$ and $G_3(x, y, m, \kappa)$ as in Section 2.
Lemma 4.4. Let $B = B(x_0, r)$, and $x \in B(x_0, r/8)$. Then
\[ \frac{r}{8M(x_0, r)} \leq g_B(x, x) = R(x, B^c) \leq 9r/8. \] (4.4)

Proof. Since $x$ is connected to $B(x, r)^c$ by a path of length $9r/8$, the upper bound is clear.

For the lower bound let $m = M(x_0, r)$ and $A = \{z_1, \ldots, z_m\}$ be the set given in Definition 2.9: note that $d(x, z_i) \geq r/8$ for each $i$. Let $h_i$ be the function on $G$ such that
$h_i(z_i) = 1, h_i(x) = 0$ and $h_i$ is harmonic $G - \{x, z_i\}$. Then $h_i(y) = \mathbb{P}^y(T_{z_i} < T_x)$, and
\[ \mathcal{E}(h_i, h_i) = R(x, z_i)^{-1} = d(x, z_i)^{-1} \leq \frac{8}{r}. \]

If $y \in B(x, r)^c$ then since any path from $y$ to $x$ passes through $A$, we have $h_i(y) = 1$ for at least one $i$. So if $h = \max_i h_i$ then $h(x) = 0$ and $h = 1$ on $B(x, r)^c$. So, using (3.1),
\[ R(x, B^c)^{-1} \leq \mathcal{E}(h, h) \leq \max_i \mathcal{E}(h_i, h_i) \leq \frac{8M(x_0, r)}{r}, \]
proving the lower bound. 

Lemma 4.5. Let $B = B(x_0, r), M = M(x_0, r)$.
(a) \[ E^z \tau_B \leq 2rV(x_0, r), \quad z \in B(x_0, r). \] (4.5)

(b) \[ E^x \tau_B \geq \frac{rV(x_0, r/(32M))}{32M}, \quad \text{for } x \in B(x_0, r/(32M)). \] (4.6)

Proof. For any $z \in B$,
\[ E^z \tau_B = \sum_{y \in B} g_B(z, y)\mu_y. \] (4.7)

The upper bound follows easily from (4.7), since
\[ \sum_{y \in B} g_B(z, y)\mu_y \leq \sum_{y \in B} g_B(z, y)\mu_y = R(z, B^c)V(x, r) \leq 2rV(x, r). \]

For the lower bound, let $x \in B(x_0, r/8)$, and set $p_B^x(y) = g_B(x, y)/g_B(x, x)$. Then
\[ \mathcal{E}(p_B^x, p_B^x) = g_B(x, x)^{-1} \] and so
\[ |1 - p_B^x(y)|^2 \leq d(x, y)R(x, B^c)^{-1} \leq d(x, y)(8M/r). \]

Let $B' = B(x_0, r/(32M))$. Then if $x, y \in B'$, $d(x, y) \leq r/(16M)$ and so $p_B^x(y) \geq 1 - 2^{-1/2} \geq \frac{1}{4}$. So, using Lemma 4.4,
\[ E^x \tau_B \geq \sum_{y \in B'} g_B(x, x)p_B^x(y) \geq \frac{1}{4}\mu(B')R(x, B^c) \geq r\mu(B')/(32M). \]

□
Proposition 4.6. Let \( r \geq 1 \) and \( x_0 \in \Gamma \), and \( B = B(x_0, r) \). Write \( M = M(x_0, r) \), \( V = V(x_0, r) \) and let \( V_1 = V_1(x_0, r) = V(x_0, r/(32M(x_0, r))) \). Then if \( x \in B(x_0, r/(32M)) \),
\[
P^x(\tau_B \leq t) \leq \left( 1 - \frac{V_1}{64MV} \right) + \frac{t}{2rV}.
\]
and
\[
q_2t(x, x) \geq \frac{c_1V_1(x_0, r)^2}{V(x_0, r)3M(x_0, r)^2} \quad \text{for} \quad t \leq \frac{rV_1(x_0, r)}{64M(x_0, r)}.
\]

**Proof.** The proof is standard. By the Markov property,
\[
E^x[\tau_B] \leq t + E^x[1_{\{\tau_B > t\}}E^{Y_t}(\tau_B)],
\]
for all \( t > 0 \). Using this and Lemma 4.5,
\[
\frac{rV_1}{32M} \leq t + P^x(\tau_B > t)2rV,
\]
and rearranging this we have
\[
P^x(Y_t \in B) \geq P^x(\tau_B > t) \geq \frac{(rV_1/32M) - t}{2rV}.
\]
(4.8)

This proves the first assertion.

By (4.8) if \( t \leq rV_1/(64M) \) then
\[
P^x(Y_t \in B) \geq \frac{c_2V_1}{VM}.
\]
By Chapman-Kolmogorov and Cauchy-Schwarz
\[
P^x(Y_t \in B)^2 = \left( \sum \limits_{y \in B} q_t(x, y)\mu_y \right)^2 \leq \mu(B) \sum \limits_{y \in B} q_t(x, y)^2\mu_y \leq q_2t(x, x)V.
\]
So
\[
q_2t(x, x) \geq V^{-1}P^x(Y_t \in B)^2 \geq \frac{c_2V_1^2}{V^3M^2}.
\]
(4.9)

**Theorem 4.7.** Suppose that \( B = B(x_0, r) \) is \( \lambda \)-good for \( \lambda \geq 1 \), and let \( I = I(\lambda, r) = [r^3\lambda^{-6}, r^3\lambda^{-5}] \).

(a) For \( x \in B(x_0, r/\lambda) \),
\[
c_0\frac{r^3}{\lambda^5} \leq E^x\tau_B \leq 2\lambda r^3.
\]
(4.10)

(b) For each \( K \geq 0 \)
\[
q_2t(x_0, y) \leq (1 + \sqrt{K})t^{-2/3}\lambda^3 \quad \text{for} \quad t \in I, \quad y \in B(x_0, Kt^{1/3}).
\]
(4.11)
(c) Let \( x \in B(x_0, r/\lambda) \). Then
\[
q_{2t}(x, y) \geq c_1 t^{-2/3} \lambda^{-17}, \quad \text{if } d(x, y) \leq c_2 \lambda^{-19} r, \quad t \in I. \tag{4.12}
\]

Proof. (a) Let \( B, V, V_1, M \) be as in the previous proof. As \( 32M \leq 64M \leq \lambda, V_1 \geq V(x, r/\lambda) \geq r^4 \lambda^{-4} \), while \( V \leq \lambda r^2 \). Thus (4.10) is immediate from Lemma 4.5.
(b) Let \( t_1 = (r/\lambda^2)^3 \). Then by Corollary 4.2 (taking \( A = \lambda^2 \)), if \( t \in I \),
\[
q_{2t}(x_0, x_0) \leq q_{2t_1}(x_0, x_0) \leq 2\lambda^2 t_1^{-2/3} \leq 2\lambda^{8/3} t_1^{-2/3} \leq \lambda^3 t_1^{-2/3}. \tag{4.13}
\]
Now, for \( t \in I \) and \( y \in B(x_0, KT^{1/3}) \), we have, using Lemma 4.3 and (4.13),
\[
q_{2t}(x_0, y) \leq q_{2t}(x_0, x_0) + |q_{2t}(x_0, y) - q_{2t}(x_0, x_0)| \\
\leq q_{2t}(x_0, x_0) + \sqrt{\frac{K}{2t^{2/3}}} q_{2t}(x_0, x_0) \leq (1 + \sqrt{K}) t^{-2/3} \lambda^3,
\]
proving (4.11).
(c) Let \( x \in B(x_0, r/\lambda) \subset B(x_0, r/(32M)) \). Then \( rV_1/(64M) \geq r^3 \lambda^{-5} \), so for \( t \in I \) by Proposition 4.6,
\[
q_{2t}(x, x) \geq c_2 V_1^2/(V^3 M^2) \geq c_2 r^{-2} \lambda^{-13} \geq c_2 t^{-2/3} \lambda^{-17},
\]
where \( c_2 = c_{4.6.1} \). Hence, by Lemma 4.3, if \( d(x, y) \leq c_2 \lambda^{-19} r, \)
\[
\left| \frac{q_{2t}(x, y)}{q_{2t}(x, x)} - 1 \right|^2 \leq \frac{d(x, y)}{2tq_{2t}(x, x)} \leq \frac{d(x, y)r^2 \lambda^{13}}{2c_2 t} \leq \frac{d(x, y)\lambda^{19}}{2c_2 r} \leq \frac{1}{2},
\]
from which (4.12) follows. \( \square \)

Corollary 4.8. Let \( \lambda \geq 64 \), and \( B(x, r) \) and \( B(x, \lambda^{-5} r) \) be \( \lambda \)-good. Then
\[
E^x d(x, Y_t) \geq c_1 \lambda^{-12} t^{1/3}, \quad \text{for } \frac{r^3}{\lambda^6} \leq t \leq \frac{r^3}{\lambda^5}.
\]

Proof. Let \( I = [r^3 \lambda^{-6}, r^3 \lambda^{-5}] \) and \( B' = B(x, r\lambda^{-5}) \). Let \( t \in I \), and \( y \in B' \). Then since \( r \leq \lambda^{2t^{1/3}} \), \( d(x_0, y) \leq \lambda^{-5} r \leq \lambda^{-3} t^{1/3} \), so by (4.11) (with \( K = 1 \)) we have \( q_{2t}(x_0, y) \leq 2t^{-2/3} \lambda^3 \). Hence since \( B' \) is \( \lambda \)-good,
\[
P^x(Y_{2t} \in B') = \sum_{y \in B'} q_{2t}(x_0, y) \mu_y \leq \mu(B')2t^{-2/3} \lambda^3 \leq 2\lambda^{-2} \leq \frac{1}{2}.
\]
Thus
\[
E^x d(x, Y_{2t}) \geq \lambda^{-5} r P^x(Y_{2t} \not\in B') = \lambda^{-5} r(1 - P^x(Y_{2t} \in B')) \geq \frac{1}{2} r \lambda^{-5}.
\]
\( \square \)
Lemma 4.9. Suppose $x$ satisfies $G_2(N, R)$. Then

$$P^x(\tau_{B(x,NR)} \leq t) \leq e^{-c_1N} \quad \text{provided } N \geq c_2t/R^3. $$

Proof. We use the argument of [BB1]. Let

$$A = \{y \in G : B(y, R/2) \text{ is } \lambda_1\text{-good}\}. $$

Define stopping times $(T_i)$, $(S_i)$ by taking $T_0 = \min\{t : Y_t \in A\}$, and

$$S_n = \min\{t \geq T_{n-1} : Y_t \notin B(Y_{T_{n-1}}, R/2)\},$$

$$T_n = \min\{t \geq S_n : Y_t \in A\}. $$

Since $x$ satisfies $G_2(N, R)$ we have $T_{N/8} \leq \tau_{B(x,NR)}$ $P^x$-a.s. Let $\xi_i = S_{i+1} - T_i$, $i \geq 1$. Then by Proposition 4.6 there exists $p = p(\lambda_1) < 1$ and $c_3 = c_3(\lambda) > 0$ such that

$$P^x(\xi_i \leq s|\sigma(Y_u, 0 \leq u \leq T_i)) \leq p + c_3R^{-3}s. $$

(4.14)

Lemma 1.1 of [BB1] (see also Lemma 3.14 of [B1]) gives that, writing $a = c_3/R^3$, (4.14) implies that

$$\log P^x(\frac{N}{8}) \leq \sum_{i=1}^{N/8} \xi_i \leq t) \leq -\frac{1}{8}N\log(1/p) + 2\left(\frac{aNt}{8p}\right)^{1/2}. $$

Substituting for $a$ we deduce that

$$\log P^x(\tau_{B(x,NR)} \leq t) \leq -N\left(2c_4 - c_5(t/(R^3N))^{1/2}\right) \leq -c_4N,$$

provided $N \geq (c_5/c_4)^2 \cdot (t/R^3). \quad \square$

Theorem 4.10. Let $x, y \in \mathcal{G}$, $t > 0$ be such that $N := \lfloor \sqrt{d(x,y)^3/t} \rfloor \geq 8$ and suppose the event $F_*(x, y, d(x, y)N^{-1}, \frac{1}{8}N; d(x, y)^3t^{-2/3}, N)$ holds. Then

$$q_t(x, y) \leq c_1t^{-2/3}\exp(-c_2N). $$

(4.15)

Proof. Define $T_{z_0} = \inf\{t : Y_t = z_0\}$ and $R = d(x, y)/N$, where $z_0$ is a middle point in $\gamma(x, y)$. Let $G_x$ be the set of points $w$ in $\mathcal{G}$ such that $\gamma(x, w)$ does not contain $z_0$, and let $G_y = \mathcal{G} - G_x$. Then, we have

$$q_t(x, y)|_{\mu_x} = \mu_xP^x(Y_t = y)
= \mu_xP^x(Y_{t/2} \in G_y, Y_t = y) + \mu_xP^x(Y_{t/2} \in G_x, Y_t = y)
= \mu_xP^x(Y_{t/2} \in G_y, Y_t = y) + \mu_yP^y(Y_{t/2} \in G_x, Y_t = x), $$

(4.16)
where in the last line we used the $\mu$-symmetry of $Y$. The two terms in (4.16) are bounded in the same way. For the first,

$$P^x(Y_{t/2} \in G, Y_t = y) \leq P^x(T_{z_0} \leq t/2, Y_t = y)$$

$$= E^x\left(1_{(T_{z_0} \leq t/2)}P^x_0(Y_{t-T_{z_0}} = y)\right)$$

$$\leq P^x(T_{z_0} \leq t/2) \sup_{t/2 \leq s \leq t} q_s(z_0, y)\mu_y,$$

$$\leq \mu_y \sqrt{q_{t/2}(y,y)q_{t/2}(z_0, z_0)P^x(T_{z_0} \leq t/2)}$$

$$\leq \mu_y N^3t^{-2/3}P^x(T_{z_0} \leq t/2),$$

where we used (4.11) with $\lambda = N, r = N^2t^{1/3}$ in the last inequality. Now, $t/R^3 \sim (d(x, y)^3/t)^{1/2} \sim N$, so $N \geq ct/R^3$. Thus, by Lemma 4.9 we have

$$P^x(T_{z_0} \leq t/2) \leq e^{-cN} \quad \text{and} \quad P^y(T_{z_0} \leq t/2) \leq e^{-cN}.$$

Combining these facts

$$q_t(x, y) \leq cN^3t^{-2/3}e^{-cN} \leq ct^{-2/3}e^{-c''N},$$

which completes the proof. \hfill \Box

**Theorem 4.11.** Let $x, y \in \mathcal{G}$, $m \geq 1$, $\kappa \geq 1$ and suppose $G_3(x, y, m, \kappa)$ holds. Then if $T = d(x, y)^3/\kappa/m^2$

$$q_{2T}(x, y) \geq c_1T^{-2/3}e^{-c_2(\kappa+c_3)m}. \quad (4.17)$$

**Proof.** Let $r = d(x, y)/m$, and $(z_i), (\Theta_i)$ be the points and integers given by the condition $G_3(x, y, m, \kappa)$ in Definition 2.15. Let $B_i = B(z_i, \Theta_i^{20}r)$, and $B'_i = B(z_i, r)$. Applying (4.12) to $B_i$ we deduce that if $d(y, y') \leq c_{4.7}T^{19}(\Theta_i^{20}r)$, and

$$\Theta_i^{54}r^3 \leq t_i \leq \Theta_i^{55}r^3, \quad (4.18)$$

then

$$q_{2t_i}(y, y') \geq c_4t_i^{-2/3}\Theta_i^{-17}. \quad (4.19)$$

If $y_i \in B'_i$ then by the choice of $\Theta_i$

$$d(y_{i-1}, y_i) \leq 3r \leq c_{4.7}\Theta_i^{-19}(\Theta_i^{20}r),$$

and so the bound in (4.19) holds for $q_{2t_i}(y, y')$. Therefore for $y_{i-1} \in B'_{i-1}$ and $t_i$ satisfying (4.18),

$$\int_{B'_i} q_{2t_i}(y_{i-1}, y_i)\mu(dy_i) \geq c_4t_i^{-2/3}\Theta_i^{-17}\mu(B'_i) \geq c_4\Theta_i^{-c_5};$$

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we used here the fact that \( \mu(B'_i) \geq \Theta_i^{-2}r^2 \). So if \( t_i \) satisfy (4.18), and \( s = \sum t_i \) then since
\[
\sum \log \Theta_i \leq \sum \Theta_i^{54} \leq m \kappa,
\]
\[
q_{2s}(x, y) \geq \int_{B'_1} \cdots \int_{B'_{m-1}} q_{2t_i}(x, y)q_{2t_1}(x, y_1) \cdots q_{2t_m}(y_{m-1}, x) \mu(dy_1) \cdots \mu(dy_{m-1})
\]
\[
\geq (c\left(\frac{2}{3}\Theta_m^{17}\right) e^{m-1} \prod_{i=1}^{m-1} \Theta_i^{-c_5} \geq s^{-2/3} \exp(-c_6 m - c_5 \sum \log \Theta_i)
\]
\[
\geq s^{-2/3} e^{-(c_5 \kappa + c_6)m}.
\]

As \( G_3(x, y, m, \kappa) \) holds we have \( r^3 \sum \Theta_i^{54} \leq m \kappa r^3 = T \). If \( T \leq r^3 \sum \Theta_i^{55} \) we can choose \( (t_i) \) satisfying (4.18) so that \( s = T \). If not, let \( s' = T - s \), so that \( s' \leq m \kappa r^3 \). Fix a \( j \) such that \( \Theta_j \) is minimal and in the chaining argument above add \( m' \) extra steps (of time length \( t' \) satisfying (4.18) for \( i = j \)) between \( B_{j-1}' \) and \( B_j' \). Since \( c_7^{54} \leq \Theta_j^{54} \leq \kappa \), we have \( c_8 r^3 \leq t' \leq \kappa r^3 \). Then choose \( m', t' \) so that \( m't' + s = T \); we have \( m' \leq cm \). Each extra step gives a factor of \( c_4 \Theta_j^{-c_5} \) in the lower bound in the chaining argument, so the total contribution multiplies the lower bound by a number greater than \( e^{-c(\kappa + c')m} \). Thus (4.17) holds.  

\[\square\]

5. Random walk on the conditioned critical GW-branching precess

In this section, we state and prove our main results on the random walk on the IIC. As in Section 2 we write \( G \) for the IIC on \( B \), and \( \mathbb{P} \) for its law. Let \( Y = \{Y_t\}_{t \geq 0} \) be the simple random walk on \( G(\omega) \) defined in Section 3; we write \( E_\omega \) for its law of \( Y \) started at \( x \). Let \( q^\omega_t(x, y) \) be the transition density of \( Y \).

Proof of Theorem 1.2. Fix \( x \in B \), and let \( c_3 = c_{2.12.2} \). Let \( a = 2/c_3 \) and \( \lambda_n = e + a \log n \), and \( r_n \) satisfy \( r_n^3 \lambda_n^{-6} = e^n \). Let \( F_n \) be the event that \( B(x, r_n) \) is \( \lambda_n \)-good. Then by Corollary 2.12
\[
\mathbb{P}(F_n^{c}) \leq ce^{-c_3 a \log n} = c'n^{-2},
\]
so by Borel-Cantelli \( F_n^{c} \) occurs for only finitely many \( n \), \( \mathbb{P} \)-a.s. Let \( N \) be the largest \( m \) such that \( F_m^{c} \) occurs; then
\[
\mathbb{P}(N > m) \leq \sum_{m+1}^{\infty} \mathbb{P}(F_n^{c}) \leq cm^{-1}.
\]

Set \( S(x) = e^N \). For \( n \geq (\log S(x)) + 1 \) we have, by (4.11) and (4.12),
\[
c't^{-2/3} \lambda_n^{-17} \leq q_n(x, x) \leq c't^{-2/3} \lambda_n^3
\]
for \( e^n \leq t \leq \lambda_n e^n \). Let \( n(t) \) be the unique integer such that \( \log t \in [n(t) - 1, n(t)) \). Hence, if \( t \geq S(x) \), \( n(t) > N \) and so (5.1) holds for \( n = n(t) \). Since
\[
\lambda_{n(t)} = e + a \log n(t) \sim a \log \log t,
\]
we obtain (5.1).  

\[\square\]

While the powers of the terms in \( \log \log t \) given in Theorem 1.2 are not the best possible, we do have oscillations in \( t^{-2/3} q_t^{\omega}(\ldots) \) of that order.

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Lemma 5.1.
\[ \liminf_{t \to \infty} (\log \log t)^{1/6} t^{2/3} q_\omega(0,0) \leq 2, \quad P_\omega^0 - a.s. \quad (5.2) \]

Proof. Define \( a_n \) by \( V(0,2^n) = a_n 2^{2n} \), and let \( t_n = 2^n V(0,2^n) = a_n 2^{3n} \). Then by Theorem 4.1,
\[ q_\omega(0,0) \leq \frac{2}{V(0,2^n)} = \frac{2t_n^{-2/3}}{a_n^{1/3}}. \]

By Proposition 2.8(a), \( a_n > (\log n)^{1/2} \) for infinitely many \( n \), a.s., giving (5.2). \( \square \)

Proof of Theorem 1.3. (a) The lower bound in (1.4) is an immediate consequence of Corollaries 2.12 and 4.8. For the upper bound, let \( Z_t = \sup_{0 \leq s \leq t} d(x,Y_s), R = t^{1/3} \) and \( T_{M} = \tau_{B(x,MR)} \). Let \( K_t(x) = \max n \) such that \( x \) does not satisfy \( G_2(n,R) \).

Then by Proposition 2.14
\[ \mathbb{P}_x(K_t(x) \geq k) \leq \sum_{l=k}^{\infty} \mathbb{P}_x(x \text{ does not satisfy } G_2(l,R)) \leq c e^{-ck}. \quad (5.3) \]

Then \( \{ Z_t \geq nR \} \subset \{ T_n \leq t \} \), and so by Lemma 4.9,
\[ E_x Z_t \leq R \sum_{n=0}^{\infty} P_\omega^x(T_n \leq t) \]
\[ \leq R \left( 1 + K_t(x) + \sum_{n=K_t(x)+1}^{\infty} P_\omega^x(T_n \leq t) \right) \]
\[ \leq R \left( 1 + K_t(x) + \sum_{n=K_t(x)+1}^{\infty} e^{-cn} \right) \leq R(c + K_t(x)). \quad (5.4) \]

Since \( E_x K_t(x) \leq c \) this completes the proof.
(b) Let \( m(t) = \lfloor t \rfloor \); Since
\[ |E_\omega d(x,Y_t) - E_\omega d(x,Y_{m(t)})| \leq E_\omega d(Y_{m(t)},Y_t) \leq c, \]

it is enough to prove (1.5) for integer \( t \). Using (5.3) and Borel-Cantelli there exists \( c' \) such that
\[ \mathbb{P}_x(K_n(x) > c' \log n \text{ i.o.}) = 0. \]
and so by (5.4)
\[ E_\omega d(x,Y_n) \leq c'' n^{1/3} \log n \]
for all sufficiently large \( n \). The lower bound in (1.5) follows from Corollary 4.8 by the same argument as in Theorem 1.2. \( \square \)

Proof of Theorem 1.4. We begin with the on-diagonal case \( x = y \). Let \( \lambda_n = n \) and \( r_n \) be defined by \( 2r_n^3/\lambda_n^6 = t \). Let \( F_n = \{ B(x,r_n) \text{ is } \lambda_n \text{-good } \} \), and \( N(\omega) = \min \{ n : \omega \in F_n \} \).
By Corollary 2.12 $\mathbb{P}_x(N > n) \leq \mathbb{P}_x(F_n^c) \leq e^{-cn}$. On $F_n$ we have, by (4.11), $q_t^\omega(x,x) \leq ct^{-2/3}n^3$, so

$$\mathbb{E}_x[q_t^\omega(x,x)] \leq ct^{-2/3}\mathbb{E}_xN^3 \leq c't^{-2/3}, \quad (5.5)$$

proving the on-diagonal upper bound in (1.6).

For the on-diagonal lower bound choose $m_0$ such that $\mathbb{P}_x(F_{m_0}) \geq \frac{1}{2}$ and then on $F_{m_0}$, by the lower bound in (4.12),

$$q_t^\omega(x,x) \geq ct^{-2/3}m_0^{-17}.$$  

For the off-diagonal bounds, when $d(x,y) \leq 64t^{1/3}$, (1.6) can be proved similarly to (5.5) using Theorem 4.7(b). So we will assume $d(x,y) > 64t^{1/3}$. Now, let $N := [\sqrt{d(x,y)^3/t}] \geq 8$ and define $F_0 = F_1(x,y,d(x,y)N^{-1},\frac{1}{2}N; d(x,y)^3t^{-2/3},N)$. Let $\lambda_0 = N$ and define $\lambda_n = N+n-1$ for $n \geq 1$. For each $n \geq 1$, set $r_n = t^{1/3}\lambda_n^2$ and let $F_n = \{B(x,r_n) \text{ is } \lambda_n \text{-good}\}$. Then, $\mathbb{P}_{x,b}(F_n^c) \leq e^{-c\lambda_n}$. We now apply Theorem 4.7 (b) with $K = \lambda_n^2$ and obtain the following. (Note that we can apply the theorem because $d(x,y)/t^{1/3} \leq cN^{2/3} \leq c\lambda_n^2$.)

$$q_{2t}(x,y) \leq c(1 + \sqrt{\lambda_n^2})t^{-2/3}\lambda_n^3 \leq c't^{-2/3}\lambda_n^4. \quad (5.6)$$

Let $M(\omega) = \min\{n \geq 0 : \omega \in F_n\}$. Then, $\mathbb{P}_x(M = 0) = \mathbb{P}_x(F_0) \geq 1 - 4e^{-N}$ and $\mathbb{P}_x(M > n) \leq \mathbb{P}_x(F_n^c) \leq ce^{-c\lambda_n}$. Thus, using Theorem 4.10 and (5.6), we obtain

$$\mathbb{E}_{x,y}[q_t^\omega(x,y)] = \mathbb{E}_{x,y}[q_t^\omega(x,y) : M = 0] + \mathbb{E}_{x,y}[q_t^\omega(x,y) : M > 0] 
\leq ct^{-2/3}\exp(-c'N) + c''t^{-2/3}\mathbb{E}[\lambda_M^4 : M > 0].$$

Since $\mathbb{E}[\lambda_M^4 : M > 0] \leq c\sum_{k=1}^\infty (N+k-1)^4\exp(-c'(N+k-1)) \leq c\exp(-c''N)$, we obtain (1.6).

We next prove (b). Choose $\kappa = 2c_{2.16.1}$, so that $\mathbb{P}_{x,y}(G_3(x,y,m,\kappa) \text{ holds}) \geq \frac{1}{2}$. Now choose $m = (R^3\kappa/t)^{1/2}$; by Theorem 4.11, for $\omega$ such that $G_3(x,y,m,\kappa)$ holds,

$$q_{2t}^\omega(x,y) \geq ct^{-2/3}\exp(-c'(\kappa + c'')m).$$

Taking expectations gives (1.7). \hfill \Box

Let

$$\bar{Z}_t^{(n)} = n^{-1/3}d(0,Y_{nt}), \quad t \geq 0.$$

By Theorem 1.3(a) the process $\bar{Z}^{(n)}$ is tight with respect to the annealed law given by the semi-direct product $\mathbb{P}^* = \mathbb{P} \times P_0^\omega$. (See Theorem 1.1 for the analogous result for the discrete time simple random walk.)

**Proof of Theorem 1.5.** Let $U_n = \sup_{0 \leq s \leq 1} Z_s^{(n)}$. Then, by (4.5),

$$\mathbb{P}_\omega^0(U_n \leq \lambda) = \mathbb{P}_\omega^0(\sup_{t \leq n} d(0,Y_t) \leq \lambda n^{1/3})
\leq \frac{2\lambda n^{1/3}V(0,\lambda n^{1/3})}{n}.$$  

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So by Proposition 2.8(b), we have, for any $\lambda > 0$, that $\liminf_{n \to \infty} P_0^0(U_n \leq \lambda) = 0$, which shows that the r.v. $U_n$ (and hence the processes $Z^{(n)}$) are not tight. \qed

**Remark.** This result illustrates the difference in the type of results that can arise between the quenched and annealed cases. For the case of supercritical bond percolation in $\mathbb{Z}^d$, while an invariance principle was proved in the annealed case in [DFGW] in 1989, the quenched case (for $d \geq 4$) was only proved recently in [SS].

**Acknowledgment.** The authors thank Antal Járai, Harry Kesten and Gordon Slade for valuable comments.

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Version 1.00, 4 March 2005

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