Parameter estimation of a second order system via nonlinear identification algorithm

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Abstract. In this paper, a parameter estimation for a class of second order nonlinear system is presented. The considered system, can be represented in such way that it is linear respect its parameters. Since the system output is given, only, for the second integrator a state estimation of unmeasured state variables is reconstructed via nonlinear observer based on the terminal sliding mode observer. Therefore, the main contribution of this paper deals with a full order nonlinear observer algorithm design to enhance parameter identification of a nonlinear second order system. Finally, to illustrate the theoretical performance of the proposed identification algorithm, an experimental result of a mechanical system is presented.

1. Introduction

In system dynamics theory, study of trajectories performance is essential to job to analyze its performance. However, in deterministic systems, not all information is available to be measured or it is an estimate value of a real one. This is the case of mechanical, electric and electronic devices [1, 2, 3]. For this reason, successful parameters identification algorithms and/or trajectory estimation approaches of unmeasured state space variables, are very common problems in data processing and control theory [4, 5, 6, 7]. Today, several parameter identification algorithms were designed where the considered system fulfills the linearity respect its parameters [4, 5, 7, 8]. Actually, the Least-Square Method (LSM), attracts a lot attention of engineers by the easiness of implementation of this algorithm in real time processes and its applications on adaptive-control and neural-network systems [4, 5, 7, 9, 10, 11]. In fact, parameter identification approaches use the regressor-vector scheme into a recursive algorithm which gives a numerical solution of a differential equation to update the current parameter estimation [9, 12, 13, 14, 15]. However, almost all algorithms to identify system parameters, algorithms like instrumental variables (IV), LSM, autoregressive moving average model with exogenous input model, IV with forgetting factor, among others need all variable states [9, 11, 13, 16].

In this work, it is assumed that only one of the state variables is available to be measured. Since, the vast majority of algorithms to identify parameter require an accurately measured state
variable. For this reason, the reconstruction of high quality state signals is key piece before to identify the system parameters. In this context, the state estimation may be given by different procedures as: signal processing with the use of advanced filters and state estimation via full order or reduced type observers \cite{17,18,19,12}. Due that, the LSM needs accurately measured state variables to work efficiently, it is necessary to have a workable state estimation. This problem can be attached by using Sliding Mode Observer (SMO), this one uses the concept of finite time convergence to guarantee correct state estimation \cite{20,21}. Actually, the concept of terminal sliding mode control guarantees the trajectory convergence of second order systems \cite{22,23}. Thus, if the finite time convergence of the estimated states to real one is guaranteed, then the parameter identification via LSM could be suitable. In this way, a state estimation based on terminal sliding mode observer (TSMO) was successfully implemented \cite{24}. In fact, the designed observer function supplies of relevant information to the LSM for the parameter identification.

The outline of this paper is as follows. The problem statement, system description and preliminaries are presented in section 2. Next, Section 3 provides the main paper contribution, parameter identification where the no available state variable is obtained via TSMO. Subsequently, in Section 4 the identification-state estimation algorithm is tested in numerical example. Finally, concluding remarks are displayed in Section 5.

2. Problem statement and preliminaries

Consider the system

\begin{align}
\dot{x}_1 &= x_2, & x(0) &= x_0, \\
\dot{x}_2 &= f(x,t) + g(x_1)u, & y &= x_1, \quad \text{a. e. } t \in \mathbb{R}^+, \\
\end{align}

where \( x \in \mathbb{R}^2 \) is the state space, \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a nonlinear vector function which fulfills

\[ \| f(x,t) \| \leq \delta_0 < \infty, \quad 0 < \| g(x_1) \| \leq \delta_1 < \infty \text{ and } \| x_2 \| \leq \delta_2 < \infty. \]  

(2)

The control input is defined as \( u \in \mathbb{R}, g \in \mathbb{R} \) can be a linear or nonlinear function. Since, \( x_2 \) is not available to be measured and system dynamics contain unknown parameters. The problem here, design a full order observer in order to obtain the estimate \( \hat{x} \), which in finite time converges to the real one. Furthermore, in same time that the state estimation algorithm works, identify the system parameters. In this way, the state observer is proposed in next format

\begin{align}
\dot{\hat{x}}_1 &= \hat{x}_2, \\
\dot{\hat{x}}_2 &= \varphi(\sigma), \quad \hat{x}(0) = \hat{x}_0,
\end{align}

(3)

where the estimation error is defined as \( e = x - \hat{x}, \varphi \in \mathbb{R} \) is the observer function, \( \sigma \in \mathbb{R} \) is the sliding manifold, defined below. Moreover, nonlinear functions \( \hat{f} \) and \( \hat{g} \) has known dynamics with unknown but estimate parameters. Thus, the dynamic error function follows as

\begin{align}
\dot{\hat{e}}_1 &= x_2 - \hat{x}_2, \\
\dot{\hat{e}}_2 &= f(x,t) + g(x_1)u - \varphi(\sigma), \quad \hat{e}_1(0) = \hat{e}_2(0) = 0
\end{align}

(4)

Notice that, systems (1)-(3) can be written as

\begin{align}
\hat{x}_1 &= x_2, \quad \hat{x}_2 = \Theta \psi(x,u),
\end{align}

(5)

where \( \Theta \in \mathbb{R}^{1 \times p} \) is a matrix which contains "p" parameters of system in linear form, \( \psi(x,u) \in \mathbb{R}^p \) is the system regressor vector and it contains all state functions.

Summarizing, the problem covered by this work, is design a full order nonlinear observer such that the state estimate error function converges to zero solution in finite time; at the same time joins the designed observer function in such way that it be used in the LSM to identify the system parameters. According to the above, the following definition is introduced.
\textbf{Definition 1 (Least-Square estimation)} For a given joint uncertainty $\Delta(\hat{x}, u)$, the function
\[
\hat{\Theta} = \arg\min_{\Theta} \int_{t_0}^{t} \| \Delta(\hat{x}, u) \|^2 d\tau, \tag{6}
\]
denotes the Least-Square estimate $\hat{\Theta}$, obtained from the measurement of variables $(\hat{x}, u)$.

3. State estimation and parameter identification algorithms

Here, the estimate dynamic (3), is based on Sliding Mode Observer function $\varphi(\sigma)$. The sliding manifold is established, in such way, that the estimation error $e_1$ converges to zero in finite time. To this end, consider next quadratic storage function
\[
V(\sigma) = \frac{1}{2} \sigma^2, \quad \sigma = e_2 + Re_1|e_1|^{-\frac{1}{2}}, \quad 0 < R \in \mathbb{R},
\]
where $\dot{e}_1 = e_2$, and $\sigma \in \mathbb{R}$ is the sliding manifold. Notice that, its time derivative along the trajectories (4) fulfills next relation
\[
\frac{d}{dt} V(\sigma) \big|_1 = \sigma \left\{ \dot{e}_2 + R\dot{e}_1|e_1|^{-\frac{1}{2}} + Re_1 \left( -\frac{1}{3} |e_1|^{-\frac{2}{3}} \frac{d}{dt} |e_1| \right) \right\}. \tag{7}
\]
Further, time derivative of $|e_1|$ yields to $\frac{d}{dt} |e_1| = \text{Sign}(e_1)\dot{e}_1$. Thus, (7) yields to
\[
\frac{d}{dt} V(\sigma) \big|_1 = \sigma \left\{ f(x, t) + g(x_1)u - \varphi(\sigma) + \frac{3}{2} R\dot{e}_1|e_1|^{-\frac{1}{2}} \right\}.
\]

By introducing the nonlinear observer function $\varphi(\sigma) = \frac{3}{2} R\dot{e}_1|e_1|^{-\frac{1}{2}} + \phi(\sigma)$, previous equation yields to
\[
\frac{d}{dt} V(\sigma) \big|_1 = \sigma \left\{ f(x, t) + g(x_1)u - \phi(\sigma) \right\},
\]
without loss of generality, $\dot{V}(\sigma) \leq |\sigma||f(x, t)| + |\sigma||g(x_1)||u| - \sigma \phi(\sigma)$ holds. Using assumption (2), and form previous inequality, $\dot{V}(\sigma) \leq \delta_0 |\sigma| + \delta_1 |\sigma||u| - \sigma \phi(\sigma)$ is satisfied. In this way, if the function $\phi(\sigma)$ is given by
\[
\phi(\sigma) = (\rho_1 + \rho_2|u|)\text{Sign}(\sigma), \tag{8}
\]
the storage function time derivative yields to $\dot{V}(\sigma) \leq -\alpha_1 |\sigma| - \alpha_2 |u||\sigma| \leq -\alpha_1 |\sigma|$, then $V_1(\sigma) \leq -\alpha_1 |\sigma| = -\alpha_1 \sqrt{2} V_1^{\frac{1}{2}}(\sigma)$. Previous differential inequality holds next solution
\[
V_1^{\frac{1}{2}}(\sigma(t)) \leq V_1^{\frac{1}{2}}(\sigma(t_0)) - \frac{\alpha_1}{\sqrt{2}}(t - t_0),
\]
which means that the sliding variable converges to the origin in finite time
\[
t_r = \frac{\sqrt{2}}{\alpha_2} V_1^{\frac{1}{2}}(\sigma(t_0)) + t_0, \tag{9}
\]
where $\alpha_2 = \rho_2 - \delta_1 > 0$. After this time, the sliding motion $\sigma = 0$ for all $t \geq t_r$. On the other hand, under assumption that $\dot{e}_1 = e_2$, so where $\sigma = 0$ is fulfilled, then
\[
\dot{e}_1 = -Rx_1|e_1|^{-\frac{1}{2}}, \quad e_1(0) = e_1^0,
\]
solving, for all $\tau \in [t_r, t)$, next relation is obtained
\[
|e_1(t)|^{\frac{1}{2}} = |e_1(t_r)|^{\frac{1}{2}} - \frac{R}{3}(t - t_r).
\]
Thus, trajectory $e_1(t)$ converges to zero in finite time $T = \frac{2}{\pi} |x_1(0)|^{\frac{1}{2}} + t_r$. Once fixed the sliding manifold, it is evident that, after time $T$, the estimate state $\hat{x}_1 = x_1$, so $\hat{x}_2 = x_2$ and $\psi(x, u) = \psi(\hat{x}, u)$. In this sense

$$0 = \Theta \psi(\hat{x}, u) - \varphi(\sigma) \quad \forall \ t \geq T,$$

However, matrix $\Theta$ is unknown, and the aim of this paper, is to identify the parameters of matrix $\hat{\Theta}$. Additionally, $\hat{\Theta} = \Theta - \hat{\Theta} \in \mathbb{R}^{1 \times p}$, defines the estimate error matrix, and it can be estimated from previous statement. By adding and subtracting $\Theta$, $\hat{\Theta} \psi(\hat{x}, u) = \Theta \psi(\hat{x}, u) - \varphi(\sigma)$ is fulfilled. Thereby, it can be defined the adjoint uncertainty $\Delta(\hat{x}, u)$ as

$$\Delta(\hat{x}, u) = \hat{\Theta} \psi(\hat{x}, u) = \left( \Theta \psi(\hat{x}, u) - \hat{\Theta} \psi(\hat{x}, u) \right). \quad (10)$$

From this statement, the joint uncertainty allows to apply the identification of considered parameters $\hat{\Theta} \in \mathbb{R}^{1 \times p}$. Using Definition 1 and selecting the following performance index

$$\mathcal{J}(\Theta) = \int_{0}^{t} \|\Delta(\hat{x}, u)\|^2 d\tau = \int_{0}^{t} \left\| (\hat{\Theta} \psi(\hat{x}, u) - \hat{\Theta} \psi(\hat{x}, u)) \right\|^2 d\tau$$

$$= \operatorname{tr}\left\{ \hat{\Theta} \left( \int_{0}^{t} \psi(\hat{x}, u) \psi^T(\hat{x}, u) d\tau \right) \hat{\Theta}^T \right\} + 2 \operatorname{tr}\left\{ \left( \int_{0}^{t} \hat{x}_2 \psi^T(\hat{x}, u) d\tau \right) \hat{\Theta}^T \right\}. \quad (11)$$

The derivative respect to the parameters $\Theta$ yields to

$$\frac{\partial}{\partial \Theta} \mathcal{J}(\Theta) = \frac{\partial}{\partial \Theta} \int_{0}^{t} \hat{x}_2 \psi^T(\hat{x}, u) d\tau + 2 \hat{\Theta} \int_{0}^{t} \psi^T(\hat{x}, u) \psi(\hat{x}, u) d\tau.$$

The minimization of the estimate $\hat{\Theta}$ can be achieved as $\frac{\partial}{\partial \Theta} \mathcal{J}(\Theta) = 0$, then

$$\hat{\Theta} = Z(t) \Gamma, \quad Z(t) = \int_{0}^{t} \hat{x}_2 \psi^T(\hat{x}, u) d\tau, \quad \Gamma = \left[ \int_{0}^{t} \psi(\hat{x}, u) \psi^T(\hat{x}, u) d\tau \right]^{-1}. \quad (12)$$

Notice that, for $t \geq T$, the solution $\hat{x} = x$ is fulfilled, consequently $\dot{\hat{x}} = \dot{x} = (\rho_1 + \rho_2 |u|) \operatorname{Sign}(\sigma)$. Introducing a small enough $0 < \varepsilon \in \mathbb{R}$, such that

$$\Gamma^{-1}(t) := \int_{0}^{t} \psi(\hat{x}, u) \psi^T(\hat{x}, u) d\tau + \varepsilon I_n,$$

the matrix $\Gamma^{-1}$ is a non singular matrix. Hence, it is evident that $\frac{d}{dt} \left( \Gamma^{-1}(t) \right) = \psi(\hat{x}, u) \psi^T(\hat{x}, u)$. Accordingly with $\Gamma^{-1} \Gamma = I_n$, the time derivative of this expression yields to $\dot{\hat{\Theta}}(t) = -\Gamma(t) \psi(\hat{x}, u) \psi^T(\hat{x}, u) \Gamma(t)$, thus, $\left(12\right)$ is rewritten as

$$\frac{d}{dt} \hat{\Theta}(t) = -\varphi(\sigma(t)) \psi^T(\hat{x}, u) \Gamma(t),$$

which defines the identification algorithm based on estate estimation $\left(3\right)-\left(8\right)$.

4. Illustrative example

The mathematical model of the pendulum is given by the next ordinary differential equation

$$\ddot{q} = -\frac{g}{l} \sin(q) - \frac{b}{Ml} \dot{q} + \frac{1}{Ml}, \quad (13)$$
| Description          | Notation | value | Units   |
|----------------------|----------|-------|---------|
| Mass 1               | \( M \)  | 0.37  | kg      |
| Pole length          | \( l \)  | 0.25  | m       |
| Gravity constant     | \( g \)  | 9.81  | \( m/s^2 \) |
| Damping coefficient  | \( b \)  | unknown | N \( s/m \) |

Table 1. Parameters of the pendulum system.

where the measured output is the angular position as \( y = q \). Introducing the change of variable \( x_1 = q, x_2 = \dot{q} \) and \( u = \tau \), next nonlinear system control affine, in state space representation, is reached as

\[
\dot{x} = f(x) + g(x)u, \quad y = Cx, \quad C = [1, 0]
\]

\[
f(x) = \begin{bmatrix} -L \sin(x_1) - \frac{M}{m}x_2 \\ \frac{1}{m} \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

Previous system has the format as (1). System parameters are given in Table 1. The observer algorithm is implemented in Matlab-Simulink R2016a. The numerical solver is the fixed step 4th-order Runge Kutta with step size \( h = 0.001 \). The experimental results were obtained on the PendCon system, see Fig. 1. A National Instruments Multifunction DAQ NI USB-6343 is used to acquire signals and to transmit the control input to a 24 volt DC motor with 3600 Cnt/Rev optical encoder from Maxon Inc. Appropriately, a 24VDC, 2.1AMP power supply is used.

![Diagram of pendulum system](image)

Figure 1. Realtime pendulum system.

The initial conditions of the real and estimate system are fixed as \( x_0^T = [1.3, 0] \) and \( \hat{x}_0^T = [-0.5, 0.5] \) respectively. In a sake to clarify the results of this algorithm, the estimation of system (13) is addressed for dynamics free of control, \( u = 0 \). Initial time is selected as \( t_0 = 0 \), final simulation time is \( t_f = 20 \) seconds, and parameters of observer are

\[
R = 17, \; \delta_1 = 20, \; \delta_2 = 10.5, \; \rho_1 = 31, \; \rho_2 = 83, \; \alpha_2 = 72.5.
\]

Thus, sliding manifold convergence time is \( t_r = 1.1166 \) and error convergence time \( T = 6.6198 \) seconds. For identification estimation algorithm, its parameters are selected as

\[
\hat{\Theta}_0 = [-0.1, 0.34, 0.25], \; \varkappa = 0.0001, \; L = 5.
\]

Real parameters of considered system are \( \Theta = [39.2400, 0.1330, 10.8108] \). The experimental results are depicted on figures 2, 3, and 4.
Figure 2. Numerical result for state estimation and real state, where pendulum velocity is not available.

Figure 2 depicts system estimate and real position and velocity. Figure 3 shows the observer function $\varphi$ and the sliding manifold $\sigma$. Finally, Figure 4 presents the estimation of system parameters. In this figure, solid line refers to the estimated parameter and dashed one deals with the real parameter.

Figure 3. Experimental result for state estimation and real state, where pendulum velocity is not available.

5. Concluding remarks
In this paper, a full order nonlinear observer algorithm is combined with Least square method to parameter identification of a nonlinear second order system. Further, by using terminal sliding mode observer concept, the full order observer guarantees the finite time convergence of the
estimated state of real one. Finally, to validate the effectiveness of this method, the proposed algorithm is tested in a real-time process.

Figure 4. System parameter estimate and real one.

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