Euler Coefficients and Restricted Dyck Paths
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We consider the problem of enumerating Dyck paths staying weakly above the \(x\)-axis with a limit to the number of consecutive \(\uparrow\)'s, or a limit to the number of consecutive \(\downarrow\)'s. We use Finite Operator Calculus to obtain formulas for the number of all such paths reaching a given point in the first quadrant. All our results are based on the Eulerian coefficients.

1 Introduction

One of the most recent papers on patterns occurring \(k\) times in Dyck paths was written by A. Sapounakis, I. Tasoulas, P. Tsikouras, Counting strings in Dyck paths, 2007, to appear in *Discrete Mathematics* [5]. The authors find generating functions for all 16 patterns generated by combinations of four up (\(\uparrow\)) and down (\(\downarrow\)) steps. A Dyck path starts at \((0,0)\), takes only up and down steps, and ends at \((2n,0)\), staying weakly above the \(x\)-axis. Returning to the \(x\)-axis at the end of the path has the advantage that every path containing the pattern \(uduu\), say, \(k\) times, will contain the reversed pattern \(ddud\) also \(k\) times when read backwards. This reduces significantly the number of patterns under consideration. Dyck paths containing \(k\) strings of length 3 were discussed by E. Deutsch in [4].

In this paper we consider only the patterns \(u^r\) and \(d^r\), for all integers \(r > 2\), and we will investigate only the case \(k = 0\), which means pattern avoidance. It has been shown in [5] that the generating function \(f(t)\) for avoiding \(u^r\) (or \(d^r\)) satisfies the equation

\[
\frac{1}{1-t^r} f(t) = 1 + \sum_{i=1}^{r-1} t^i f(t)^i.
\]

However, we will allow the Dyck paths to end at \((n,m)\), \(m \geq 0\), which removes the above mentioned symmetry, as shown in the following two tables.

| \(m\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| 7    |   |   |   |   |   | 1 |   |   |   | 10 |    |    |    |    |
| 6    |   |   |   |   | 3 |   |   |   |   |   | 19 |    |    |    |    |
| 5    |   | 1 | 6 | 28 | 112|    |    |    |    |    |    |    |    |    |
| 4    | 2 | 9 | 33 | 116|    |    |    |    |    |    |    |    |    |    |
| 3    | 1 | 3 | 10 | 32 | 101| 321|    |    |    |    |    |    |    |    |
| 2    | 1 | 3 | 8  | 23 | 68 | 205|    |    |    |    |    |    |    |    |
| 1    | 1 | 2 | 5  | 13 | 36 | 104| 309|    |    |    |    |    |    |    |
| 0    | 1 | 1 | 2  | 5  | 13 | 36 | 104|    |    |    |    |    |    |    |

\(n:\) 0 1 2 3 4 5 6 7 8 9 10 11 12 13

The number of Dyck paths avoiding \(uuuu\)
The number of Dyck paths avoiding dddd

The two tables indicate the differences between the two problems, both starting out from equal counts on the x-axis \((m = 0)\). Because only points \((n, m)\) with \(n + m = 0 \mod 2\) can be reached by a Dyck path, we consider the lattice points \((2n + b, 2m + b)\), for \(b = 0, 1\). We first show that the number of Dyck paths to \((2n + b, 2m + b)\) avoiding \(d^r\) equals

\[
D \text{yck}(2n + b, 2m + b; d^r) = \frac{2m + b + 1}{n + m + b + 1} \binom{n + m + b + 1}{n - m},
\]

where the Euler coefficient \([\text{2}]\) is denoted by

\[
\binom{n + m + b + 1}{n - m} = \sum_{i=0}^{\lfloor (n-m)/r \rfloor} (-1)^i \binom{n + m + b + 1}{i} (2n + b - ri)^{n + m + b - ri}
\]

(see Definition \([\text{3}]\) and expansion \([\text{12}]\)). More about Euler coefficients can be found in Section \([\text{4}]\). For given \(m\), the number of Dyck paths \(D \text{yck}(n + m, m - n; d^r)\) to \((n + m, m - n)\) avoiding \(d^r\) has the generating function (over \(n\)) \((1 - t^r)^m (1 - t)^{-m-2} (rt^r (1 - t) - (1 - t^r) (2t - 1)),\) as shown in \([\text{11}]\). Note that the coefficient of \(t^m\) in this generating function equals the original \(D \text{yck}(2m, 0; d^r)\).

Next we show that the number of Dyck paths to \((2n + b, 2m + b)\) avoiding \(u^r\) equals

\[
D \left(2m + b, 2m + b; u^r\right) = \sum_{i=0}^{2m+b-1} \frac{1}{n + m + b + 1 - i} \binom{i - 2m - b}{i} \binom{n + m + b + 1 - i}{n + m + b - i},
\]

except for the original Dyck path counts to \((2n, 0)\), which either must be gotten from those to \((2n - 1, 1)\), or from the Dyck paths to \((2n, 0)\) avoiding
The case $r = 4$ seems to be very special. We conjecture in Section 3.1 that in this case the generating function for the Dyck paths equals

\[
\sum_{n \geq 0} \text{Dyck} (4m - n - 1, 2m - n + 1; u^4) = \left(3 + t - \sqrt{(1 + t)^2 + 4t^3} \right) \left(\frac{1 - t^4}{1 - t}\right)^m / 2,
\]

hence $\text{Dyck} (2m, 0; u^4) = \left[t^{2m}\right] \left(3 + t - \sqrt{(1 + t)^2 + 4t^3} \right) \left(\frac{1 - t^4}{1 - t}\right)^m / 2$.

Throughout the following sections we will discuss ballot paths (weakly above $y = x$), with steps $\uparrow$ and $\rightarrow$, instead of Dyck paths. The transformations $D(n, m) = \text{Dyck}(n + m, m - n)$ and $\text{Dyck}(2n + b, 2m + b) = D(n - m, m + n + b)$, with $D(n, m)$ counting ballot path to $(n, m)$, go back and forth between the two equivalent setups. Of course, the pattern $u^r$ becomes the pattern $\uparrow^r$, or $N^r$, and $d^r$ becomes $\rightarrow^r$, or $E^r$.

2 Ballot paths without the pattern $\rightarrow^r$

Definition 1. $s_n(m; r) = s_n(m)$ is the number of $\{\uparrow, \rightarrow\}$ paths staying weakly above the diagonal $y = x$ from $(0, 0)$ to $(n, m) \in \mathbb{Z}^2$ avoiding a sequence of $r > 0$ consecutive $\rightarrow$ steps. We get $s_0(m) = 1$ for all $m \geq 0$. We set $s_n(m) = 0$ if $n < 0$ or if $m + 1 = n > 0$.

Lemma 2. The following recurrence holds for all $m \geq n > 0$:

\[
s_n(m) = s_{n-1}(m) + s_n(m - 1) - s_{n-r}(m - 1). \tag{1}
\]

Proof: The number of paths reaching $(n, m)$ is obtained by adding the number of paths reaching $(n - 1, m)$ and $(n, m - 1)$, but subtracting paths that would have exactly $r \rightarrow$ steps. Those forbidden steps occur necessarily at the end of the path, so they are preceded by an up step, and must come from $(n - r, m - 1)$.

We now extend $s_n(m)$ to all integers $m$ by first setting $s_0(m) = 1$ and using (1) to define the remaining $s_n(m)$ for $m < n - 1$.

Lemma 3. $(s_n)$ is a polynomial sequence with $\deg s_n = n$.

Proof: We proceed by induction on $n$. Clearly, $\deg (s_0) = 0$. Suppose $s_k(m)$ is a polynomial of degree $k$ for $0 \leq k \leq l$. Then $s_{l+1}(m) - s_{l+1}(m - 1) = s_l(m) - s_{l-r+1}(m - 1)$, which implies the first difference of $s_{l+1}(m)$ is a polynomial of degree $l$. Thus, $s_{l+1}(m)$ is a polynomial of degree $l + 1$.

By interpolation we can define $(s_n)$ on all real numbers.

3
Definition 4 The Eulerian Coefficient is defined as

\[
\left(\begin{array}{c}
x \\ n \\
\end{array}\right)_r = [t^n](1 + t + \cdots + t^{r-1})^x \\
= \sum_{i=0}^{\lfloor n/r \rfloor} (-1)^i \binom{x}{i} \binom{x + n - ri - 1}{n - ri} \\
\]

(see (12)). Note that for \( r = 2 \) the Euler coefficient equals the binomial coefficient \( \binom{x}{n} \).

The following table shows the polynomial extension of \( s_n(m) \). The number of \( \{\uparrow, \rightarrow\} \) paths to \((n, m)\) avoiding a sequence of 4 \( \rightarrow \) steps appear above the \( y = x \) diagonal. The numbers on the diagonal \((n, n)\), 1, 1, 2, 5, 13, 36, \ldots, are the number of Dyck paths to \((2n, 0)\). Of course, \( s_n(n) \leq C_n \), the \( n \)-th Catalan number.

\begin{table}[h]
\centering
\begin{tabular}{l|cccccccc}
\hline
\( m \) & 1 & 7 & 27 & 75 & 161 & 273 & 357 & 309 & 0 \\
\hline
6 & 1 & 6 & 20 & 48 & 87 & 104 & 0 & -22 & -222 \\
5 & 1 & 5 & 14 & 28 & 40 & 36 & 0 & -76 & -182 \\
4 & 1 & 4 & 9 & 14 & 13 & 0 & -27 & -62 & -93 \\
3 & 1 & 3 & 5 & 5 & 0 & -10 & -22 & -30 & -31 \\
2 & 1 & 2 & 2 & 0 & -4 & -8 & -10 & -8 & -5 \\
1 & 1 & 1 & 0 & -2 & -3 & -3 & -2 & 0 & 0 \\
0 & 1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & 3 \\
\hline
\( n \) : & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{tabular}
\caption{Path counts \( s_n(m) \) and their polynomial extension \( (r = 4) \).}
\end{table}

Theorem 5

\[
s_n(x) = \frac{x - n + 1}{x + 1} \binom{x + 1}{n} = \frac{x - n + 1}{x + 1} \sum_{i=0}^{\lfloor n/r \rfloor} (-1)^i \binom{x + 1}{i} \binom{x + n - ri}{n - ri} \\
\]

Proof: We saw that \( (s_n(x)) \) is a basis for the vector space of polynomials. Using operators on polynomials, we can write the recurrence relation as

\[
1 - E^{-1} = B - B^r E^{-1} \\
\]

(2)

where \( B \) and \( E^a \) are defined by linear extension of \( B s_n(x) = s_{n-1}(x) \) and \( E^a s_n(x) = s_n(x + a) \), the shift by \( a \). The operators \( \nabla = 1 - E^{-1} \) and \( E^{-1} \) both have power series expansions in \( D \), the derivative operator. Hence
$B$ must have such an expansion too, and therefore commutes with $\nabla$ and $E^a$. The power series for $B$ must be of order 1, because $B$ reduces degrees by 1. Such linear operators are called \textit{delta operators}. The basic sequence $(b_n(x))_{n \geq 0}$ of a delta operator $B$ is a sequence of polynomials such that $\deg b_n = n$, $Bb_n(x) = b_{n-1}(x)$ (like the Sheffer sequence $s_n(x)$ for $B$), and initial conditions $b_n(0) = \delta_{0,n}$ for all $n \in \mathbb{N}_0$. In our special case, the basic sequence is easily determined. Solving for $E_1$ in (2) shows that

$$E_1 = \sum_{i=0}^{r-1} B^i.$$ 

Finite Operator Calculus tells us that if $E_1 = 1 + \sigma(B)$, where $\sigma(t)$ is a power series of order 1 \cite[(2.5)]{3}, then the basic sequence $b_n(x)$ of $B$ has the generating function

$$\sum_{n \geq 0} b_n(x)t^n = (1 + \sigma(t))^x.$$ 

Thus, in our case $b_n(x) = [t^n](1 + t + t^2 + \cdots + t^{r-1})^x = \binom{x}{n}$. Since the Sheffer sequence $(s_n)$ has initial values $s_n(n-1) = \delta_{n,0}$, using Abelization \cite{3} gives us

$$s_n(x) = \frac{x - n + 1}{x + 1} b_n(x + 1) = \frac{x - n + 1}{x + 1} \binom{x + 1}{n} = \frac{x - n + 1}{x + 1} \binom{x + 1}{n}.$$ 

\hfill \blacksquare

\textbf{Corollary 6} \textit{The number of Dyck paths to $(2n,0)$ avoiding $r$ down steps is}

$$s_n(n) = \frac{1}{n+1} \binom{n+1}{n}.$$ 

\textbf{3 Ballot paths without the pattern $\uparrow^r$}

\textbf{Definition 7} $t_n(m;r) = t_n(m)$ is the number of \{\uparrow, \rightarrow\} paths staying weakly above the line $y = x$ from $(0,0)$ to $(n,m)$ avoiding a sequence of $r > 0$ consecutive $\uparrow$ steps. We set $t_n(m) = 0$ if $n < 0$ or $m + 1 = n > 0$.

This time we do not immediately have a polynomial sequence, as the table below shows. The path $N^{r-1} (EN^{r-1})^k$ to $(k, (r-1)(k+1))$ is the only admissible path reaching the point $(k, (r-1)(k+1))$ (all others would have $r$ or more $N$-steps). Hence $t_{n-1}((r-1)n) = 1$ for all $n \geq 1$, and $t_{n-1}(m) = 0$ for $m > (r-1)n$. The only other 1’s in the table occur in column 0, $t_0(m) = 1$ for $m = 0, \ldots, r-1$, and 0 for all other values of $m$. 

5
The table contains a strip weakly above the diagonal $y = x$ where

$$t_n (m) = t_n (m-1) + t_{n-1} (m) \quad (5)$$

This happens for $0 < n \leq m < n + r$ because paths in this strip cannot have $r$ consecutive vertical steps. All paths that reach a point $(n, m)$ for $m \geq n + r$ and violate the condition of not containing $N^r$ must have this pattern exactly at the end of the path, which means that they end in the pattern $EN^r$. Hence for $m \geq n + r$ we get the recurrence

$$t_n (m) = t_n (m-1) + t_{n-1} (m) - t_{n-1} (m-r) \quad (6)$$

We assume that $t_n (m) = 0$ for all $m < n$ (also for $n = 0$).

We can find a recursion that holds for all $m \geq n$ as follows: For $n \geq 1$ we always have $t_n (n) = t_{n-1} (n)$, because $t_n (n-1) = 0$. From (5) follows by induction (inside the exceptional strip) that $t_n (m) = \sum_{i=n}^{m} t_{n-1} (i)$ for all $n \leq m < n + r$. For the values of $m$ on the boundary of the strip we have

$$t_n (n+r) = \sum_{i=n}^{n+r} t_{n-1} (i) - t_{n-1} (n) = \sum_{i=n+1}^{n+r} t_{n-1} (i)$$

from (5) and (6), and after that by induction using (6), $t_n (m) = \sum_{i=m+1-r}^{m} t_{n-1} (i)$ for all $m \geq n+r$. We can write both recursions together as

$$t_n (m) = \sum_{i=\max\{n,m+1-r\}}^{m} t_{n-1} (i) \quad (7)$$

for all $m \geq n$. We can avoid the difficulty with the lower bound in the summation by setting $t_n (m) = 0$ for all $m \leq n$. Call the modified numbers $t'_n (m)$. The new table follows the recursion

$$t'_n (m) = \sum_{i=m+1-r}^{m} t'_{n-1} (i)$$

for all $m > n > 0$. The ‘lost’ value $t_n (n)$ can be easily recovered, because $t_n (n) = t_{n-1} (n) = t'_{n-1} (n)$. 

6
The restricted ballot path counts $t_n(m)$ ($r = 4$).

In order to show the polynomial structure in the above table, we transform it into the table below by a 90° counterclockwise turn, and shifting the top 1’s flush against the $y$-axis. In formulas, we define $p_n(m) = t'_{m-1}((r-1)m-n)$ for $m(r-1)\geq n \geq 0$ (or $t'_n(m) = p_{(r-1)(n+1) - m}(n+1)$). The recursion $t'_n(m) = \sum_{i=m+1-r}^{m} t'_{i-1}(i)$ ‘along the previous column’ becomes now a recursion $p_n(m) = \sum_{i=0}^{r-1} p_{n-j}(m-1)$ ‘along the previous row’. More precisely, for $(r-1)m-n > m-1 \geq 1$, i.e., $m \geq 2$ and $n \leq (r-2)m$ holds

$$p_n(m) = t'_{m-1}((r-1)m-n) = \sum_{i=(r-1)(m-1)-n}^{(r-1)m-n} t'_{m-2}(i)$$

$$= \sum_{i=(r-1)(m-1)-n}^{(r-1)m-n} p_{(r-1)(m-1)-i}(m-1) = \sum_{i=0}^{r-1} p_{n-j}(m-1) \quad (8)$$

The numbers $p_n(m)$ for $0 \leq n \leq (r-2)m$ are exactly the cases where $t'_n(m)$ is positive, and the only additional numbers needed in the recursion (8) are the numbers

$$p_{(r-2)m-j}(m-1) = t'_{m-2}((r-1)(m-1) - (r-2)m + j) = t'_{m-2}(m+j-r+1) = 0 \text{ for } j = 0, \ldots, r-3.$$  

We also add a row $p_0(0) = \delta_{n,0}$ for $n = 0, \ldots, r-2$ to the table, so that the recursion (8) holds for $m = 1$. This part of the $p$-table, $p_n(m)$ for $0 \leq m$ and $0 \leq n \leq (r-2)(m+1)$, is shown below for $r = 4$. Note that $p_0(m) = t_{m-1}((r-1)m) = 1$ for all $m \geq 1$, and also $p_0(0) = 1$.
The rotated and shifted table $p_n(m)$ and its polynomial extension below the staircase, for $r = 4$. The bold numbers occur also on the second subdiagonal in the table below.

We obtained the recursion (7) as a discrete integral from (6) and (5). We can now take differences in recursion (8) and get

$$p_n(m) - p_n(m - 1) = p_{n-1}(m) - p_{n-r}(m - 1),$$

for all $m \geq 1$ and $0 \leq n \leq (r - 2)(m + 1)$. The column $p_0(m)$ can be extended as a column of ones to all integers $m$; hence $p_0(m)$ can be extended to the constant polynomial $1$. The recursion shows by induction that the $n$-th column can be extended to a polynomial of degree $n$, and by interpolation we can assume that we have polynomials in a real variable. The extension of $p_n(m)$ is again denoted by $p_n(m)$. The above table shows some values of the polynomial expansion in cursive. The expansion follows the same recursion, hence

$$p_n(x) - p_{n-1}(x - 1) = p_{n-1}(x) - p_{n-r}(x - 1)$$

(9)

with initial values $p_{(r-2)m+j}(m) = 0$ for $j = 1, \ldots, r - 2$ and $m \geq 0$. These conditions, together with $p_0(0) = 1$, determine the solution uniquely.

Recursion (9) shows that $(p_n(x))$ is a Sheffer sequence for the same operator $B$ as the sequences $(s_n(x))$ in recursion (2). Hence $p_n(x)$ can be written in terms of the same basis, the Eulerian coefficients, as $s_n(x)$. However, the initial values (zeroes) for $(p_n(x))$ are more difficult, because they are not on a line with positive slope. We introduce know a Sheffer sequence $(q_n(x; \alpha))_{n \geq 0}$ for the delta operator $B$ that has roots on the parallel to the diagonal shifted by $\alpha + 1$, $q_n(n - \alpha - 1; \alpha) = 0$, and agrees with $(p_n)$ at one position left of the roots, for each $n$. 

| $m$ | 1 8 36 119 315 699 1338 2246 3344 |
|-----|-----------------------------------|
| 7   | 1 7 28 197 391 667 991 1295       |
| 6   | 1 6 21 55 115 200 297 379         |
| 5   | 1 5 15 34 61 90 112 116 101      |
| 4   | 1 4 10 19 28 32 23 13            |
| 3   | 1 3 6 9 10 8 5 0                 |
| 2   | 1 2 3 2 0 0 -2 2                |
| 1   | 1 1 1 0 0 -1 1 -2 4             |
| 0   | 1 0 0 -1 1 -1 2 -4 7            |

$n$: 0 1 2 3 4 5 6 7 8
Lemma 8 For the Sheffer sequence \( (q_n(x; \alpha)) \) for \( B \) with initial values
\[ q_0(m; \alpha) = 1, \quad q_n(0) = \delta_{n,0} \text{ for } 0 \leq n \leq \alpha \text{ and } q_n(n - \alpha - 1; \alpha) = 0 \text{ for } n > \alpha, \]
holds
\[ q_{n+\alpha}(n; \alpha) = p_{(r-2)m-\alpha}(n) \]
for all \( n \geq \lfloor \alpha/(r-2) \rfloor \).

We will proof this Lemma in Subsection 3.2.

| \( m \) | 1 | 6 | 21 | 56 | 120 | 214 | 320 | 386 | 321 |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 5 | 15 | 35 | 65 | 99 | 121 | 101 | 0 |
| 4 | 1 | 4 | 10 | 20 | 31 | 38 | 0 | -70 |
| 3 | 1 | 3 | 6 | 10 | 12 | 10 | 0 | -22 | -58 |
| 2 | 1 | 2 | 3 | 4 | 3 | 0 | -7 | -18 | -33 |
| 1 | 1 | 1 | 1 | 0 | -2 | -6 | -10 | -15 |
| 0 | 1 | 0 | 0 | 0 | -2 | -4 | -5 |

The polynomials \( q_n(m, 2) \) for \( r = 4 \)

The sequence \( (q_n) \) agrees with the Euler coefficients \( b_n(x) = \binom{x}{n} \) for the first degrees \( n = 0, \ldots, \alpha \). It follows from the Binomial Theorem for Sheffer sequences that
\[ q_n(x; \alpha) = \sum_{i=0}^{\alpha} \binom{i - \alpha - 1}{i} \frac{x + \alpha + 1 - n}{x + \alpha + 1 - i} \binom{x + \alpha + 1 - i}{n - i}. \quad (10) \]

Corollary 9 The number of ballot paths avoiding \( r \uparrow \)-steps equals
\[ t_n(m; r) = \sum_{i=0}^{m-n-1} \frac{1}{m+1-i} \binom{i + m + n}{i} \binom{m + 1 - i}{m - i}, \]
for \( m > n \geq 0 \). Furthermore, \( t_n(n) = t_{n-1}(n) = \frac{1}{n+1} \binom{n+1}{n} \) for all \( n > 0 \).

Proof: \( t_n(m; r) = p_{(r-1)(n+1)-m}(n+1) = q_m(n+1; m-1-n) \).

The Corollary shows that the number of Dyck paths to \( (2n, 0) \) avoiding \( r \) up steps, \( \frac{1}{n+1} \binom{n+1}{n} \), equals the number of Dyck paths to \( (2n, 0) \) avoiding \( r \) down steps (see formula (4)).
3.1 A Conjecture for the Case \( r = 4 \).

A Motzkin path can take horizontal unit steps in addition to the up and down steps of a Dyck path. Suppose a Motzkin path is “peakless”, i.e., the pattern \( uu \) and \( ud \) does not occur in the path. Denote the number of peakless Motzkin paths to \((n, 0)\) by \( M'(n) \). Starting at \( n = 0 \) we get the following sequence, \( 1, 1, 1, 2, 4, 7, 13, 26, 52, 104, 212, 438, 910, \ldots \) for \( M'(n) \) (see [http://www.research.att.com/~njas/sequences/A023431](http://www.research.att.com/~njas/sequences/A023431)). It is easy to show that
\[
M'(n) = \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n-i}{2i} \binom{2i}{i} \binom{i+1}{i+1}.
\]

For the case \( r = 4 \) we conjecture that
\[
p_n(0) = (-1)^n M'(n-3) \quad \text{for all} \quad n \geq 3.
\]
That would imply
\[
\sum_{n \geq 0} p_n(0) t^n = \frac{3 + t - \sqrt{(1+t)^2 + 4t^3}}{2} / 2,
\]
and therefore
\[
\sum_{n \geq 0} p_n(x) t^n = \frac{3 + t - \sqrt{(1+t)^2 + 4t^3}}{2} \left( \frac{1-t^4}{1-t} \right)^x.
\]

For example, the coefficient of \( t^7 \) in this power series equals
\[
(x-3) \left( x^6 + 24x^5 + 247x^4 + 426x^3 - 38x^2 - 2340x + 6720 \right) / 7!,
\]
which in turn equals \( p_7(x) \), as can be checked using the table for \( p_r(m) \).

3.2 Proof of Lemma 8

Because of recursion (8) we obtain the operator identity
\[
I = E^{-1} \left( B^0 + B^1 + \cdots + B^{r-1} \right)
\]
which holds for \((p_n(x))\) and \((q_n(x; \alpha))\), and shows that both polynomials enumerate lattice paths with steps \((0, 1), (1, 1), (2, 1), \ldots, (r-1, 1)\) (above the respective boundaries). The number of such paths reaching \((n, m)\) can also be seen as compositions of \(m\) into \(n\) terms taken from \([0, 1, \ldots, r-1]\). In the case of \(p_n(m)\) the terms \(a_1, \ldots, a_n\) also have to respect the boundary,

which means that \( \sum_{i=1}^{k} a_i \leq (r-2) k \) for all \( k = 1, \ldots n \). For \(q_n(m; \alpha)\) we get

for the same reason that \( \sum_{i=1}^{k} b_i \leq k + \alpha - 1 \). Such restricted compositions have the following nice property.

**Lemma 10** Let \( c \in \mathbb{N}_1, \alpha \in \mathbb{N}_0, \) and \( n \) be a natural number such that \( n - \alpha \geq 0 \). Let \( P_n^\alpha \) be the number of compositions of \( cn - \alpha \) into \( n \) parts from \([0, c+1]\) such that \( a_1 + a_2 + \ldots + a_n = cn - \alpha \), and \( k \sum_{i=1}^{k} a_i \leq ck \) for all \( k = 1, \ldots, n-1 \). Let \( Q_n^\alpha \) be the number of compositions of \( n + \alpha \) into \( n \) parts from \([0, \alpha+1]\) such that \( b_1 + b_2 + \ldots + b_n = n + \alpha \), and \( k \sum_{i=1}^{k} b_i \leq \alpha k \) for all \( k = 1, \ldots, n-1 \). Then
\[
\frac{Q_n^\alpha}{P_n^\alpha} = \frac{\binom{n+\alpha}{\alpha}}{\binom{n}{\alpha}}.
\]
parts from $[0, c+1]$ such that $b_1 + b_2 + \ldots + b_n = n + \alpha$, and $\sum_{i=1}^{k} b_i \leq k + \alpha$
for all $k = 1, \ldots, n-1$. Then $P_n^\alpha = Q_n^\alpha$.

**Proof:** Suppose, $b_1 + b_2 + \ldots + b_n = n + \alpha$, and $\sum_{i=1}^{k} b_i \leq k + \alpha$. Define $a_i = c+1-b_{n+1-i}$ for $i = 1, \ldots, n$. Note that $a_i \in [0, c+1]$, and

$$k \sum_{i=1}^{k} b_i \leq k \sum_{i=1}^{k} b_{n+1-i}.$$ Hence

$$\sum_{i=1}^{k} a_i = (c+1)k - \sum_{i=1}^{k} b_{n+1-i} \leq (c+1)k - n + (n - k) = ck$$

and

$$\sum_{i=1}^{n} a_i = (c+1)n - \sum_{i=1}^{n} b_{n+1-i} = cn - \alpha.$$ We apply this Lemma with $c = r-2$ to obtain $p_{(r-2)n-\alpha} (n) = q_{n+\alpha} (n)$.

### 3.3 Abelization

Let $(b_n)$ be the basic sequence for some arbitrary delta operator $B$, i.e., $Bb_n = b_{n-1}$ and $b_n(0) = \delta_{0,n}$. Every basic sequence is also a sequence of binomial type, which means that $\sum_{n \geq 0} b_n (x) t^n = e^{x\beta(t)}$, where $\beta(t) = t + a_2 t^2 + \ldots$ is a formal power series. The compositional inverse of $\beta(t)$ is the power series that represents $B$,

$$B = \beta^{-1} (D) = D + b_2 D^2 + \ldots$$

where $D = \partial/\partial x$ is the $x$-derivative. The Abelization of $(b_n)$ (by $a \in \mathbb{R}$) is the basic sequence $\left( \frac{x}{x+an} b_n (x+an) \right)_{n \geq 0}$ for the delta operator $E^{-a}B$ (see [3]). Note that with any Sheffer sequence $(s_n)$ for $B$ the sequence $(s_n (x + c - an))$ is a Sheffer sequence for $E^a B$. Hence

$$\left( \frac{x+c-an}{x+c} b_n (x+c) \right)_{n \geq 0}$$

is a Sheffer sequence for $E^a E^{-a}B = B$ again. Choosing $c = a = 1$ shows [3].

Sheffer sequences and the basic sequence for the same delta operator are connected by the Binomial Theorem for Sheffer sequences,

$$s_n (y + x) = \sum_{i=0}^{n} s_i (y) b_{n-i} (x).$$

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Applying this Theorem to \( \left( \frac{x}{x + an} b_n(x + an) \right) \) and \( (b_n(x + c + an)) \) shows that

\[
b_n(y + x + c + an) = \sum_{i=0}^{n} b_i(y + c + ai) \frac{x}{x + a(n - i)} b_{n-i}(x + a(n - i)).
\]

Choosing \( x \) as \( x + \alpha + 1 - an, a = 1, c = 0, \) and \( y = -\alpha - 1 \) gives \( b_n(x) = \sum_{i=0}^{n} b_i(i - \alpha - 1) \frac{x}{x + \alpha + 1 - i} b_{n-i}(x + \alpha + 1 - i). \) This is not quite what we have in (10); there the summation stops at \( \alpha. \) This effect in (10) is due to the ‘initial values’ \( b_i(i - \alpha - 1) \) which are 0 for \( i > \alpha. \)

The generating function of a Sheffer sequence \( (s_n) \) for \( B \) is of the form \( \phi(t) e^{x\beta(t)}, \) where \( \phi(t) = \sum_{n \geq 0} s_n(0) t^n. \) If \( s_n(x) = \frac{x + c}{x + c} b_n(x + c) = b_n(x + c) - \frac{an}{x + c} b_n(x + c) \) then

\[
\sum_{n \geq 1} \frac{n}{x + c} b_n(x + c) t^n = \frac{t}{x + c} \frac{\partial}{\partial t} e^{(x+c)\beta(t)} = t\beta'(t) e^{(x+c)\beta(t)}
\]

and

\[
\sum_{n \geq 0} s_n(x) t^n = e^{(x+c)\beta(t)} (1 - at\beta'(t))
\]

If \( c = a = 1 \) and \( e^{\beta(t)} = (1 + t + \cdots + t^{r-1}) \), then \( \beta'(t) = \frac{1 - (r - t + 1) t^{r-1}}{(1-t^r)(1-t)}, \) and therefore

\[
\sum_{n \geq 0} s_n(m) t^n = \frac{(1 - t^r)^m}{(1-t)^{m+2}} (rt^r (1-t) + (1-t^r) (1-2t)), \quad (11)
\]

the generating function of the number of ballot paths to \( (n, m) \), avoiding \( \rightarrow_r. \)

### 4 Euler Coefficients

The coefficients of the polynomial

\[
(1 + t + t^2 + \cdots + t^{r-1})^n
\]

were considered by Euler in [2], where he gives the following recurrence:

\[
\binom{n}{k}_{r+1} = \sum_{i=0}^{k/2} \binom{n}{k-i}\binom{k-i}{i}_{r}
\]

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To calculate the Euler coefficients in terms of only binomial coefficients, we rewrite the polynomial as follows:

\[
(1 + t + t^2 + \cdots + t^{r-1})^n = \left(\frac{1 - t^r}{1 - t}\right)^n = \sum_{i=0}^{n} \binom{n}{i} t^i \sum_{j \geq 0} \binom{n + j - 1}{j} t^j.
\]

Thus we have proven

\[
\binom{n}{k}_r = \sum_{i=0}^{\lfloor k/r \rfloor} (-1)^i \binom{n}{i} \binom{n + k - ri - 1}{k - ri}.
\] (12)

Note that this identity implies \(\lim_{r \to \infty} \binom{n}{k}_r = \binom{n + k - 1}{k}\). Combinatorially, these are all \(\{\uparrow, \to\}\) paths avoiding \(\to^r\). This problem occurs in Wilf’s *generatingfunctionology* [6], Section 4.12. Note that identity (12) implies \(\lim_{r \to \infty} \binom{n}{k}_r = \binom{n + k - 1}{k}\).

| \(x\) | 1 8 36 120 322 728 1428 2472 3823 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 7 | 1 7 28 84 203 413 728 1128 1554 |
| 6 | 1 6 21 56 120 216 336 456 546 |
| 5 | 1 5 15 35 65 101 135 155 155 |
| 4 | 1 4 10 20 31 40 31 20 10 |
| 3 | 1 3 6 10 12 12 10 6 3 |
| 2 | 1 2 3 4 3 2 1 0 0 |
| 1 | 1 1 1 0 0 0 0 0 0 |
| 0 | 1 0 0 0 0 0 0 0 0 |

A table of Euler coefficients \(\binom{x}{n}_4\) for \(r = 4\)

We now show some properties about Euler Coefficients similar to the basic properties of binomial coefficients.

1. For binomial coefficients, this property is usually called Pascal’s Identity:

\[
\binom{n}{k}_r = \sum_{i=0}^{r-1} \binom{n-1}{k-i} \binom{r-1}{i}.
\]
Proof

\[(1 + t + \cdots + t^{r-1})^n = (1 + t + \cdots + t^{r-1})^{n-1}(1 + t + \cdots + t^{r-1})\]

\[= \sum_{i=0}^{r-1} t^i(1 + t + \cdots + t^{r-1})^{n-1}\]

so

\[\binom{n}{k}_r = \sum_{i=0}^{r-1} t^{k-i}(1 + t + \cdots + t^{r-1})^{n-1} = \sum_{i=0}^{r-1} \binom{n-i}{k-i}_r\]

2. The table of Euler Coefficients is symmetric similar to Pascal’s Triangle:

\[\binom{n}{k}_r = \binom{n}{n(r-1)-k}_r\]

Proof We proceed by induction on \(n\), fixing \(r \geq 2\). For \(n = 2\) we have the well known symmetry for binomial coefficients. Suppose true for some \(l > 2\). From the above recurrence we have

\[\binom{l+1}{k}_r = \sum_{i=0}^{r-1} \binom{l}{k-i}_r = \sum_{i=0}^{r-1} \binom{l(r-1) - (k-i)}{r}\]

\[= \sum_{i=0}^{r-1} \binom{l(r+1)(r-1) - k - i}{r} = \binom{l+1}{(l+1)(r-1) - k}_r\]

and the induction follows. \(\blacksquare\)

3. This property is similar to Vandermondt Convolution for binomial coefficients:

\[\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}_r \binom{m}{k-i}_r\]

It follows because the Euler coefficients are of binomial type [4].

4. Here we have an identity that is trivial for binomial coefficients, i.e. \(r = 2\), and gives and identity for the Catalan numbers as \(r \to \infty\).

\[\frac{1}{n+1} \binom{n+1}{n}_r = \binom{n}{n}_r - \sum_{i=1}^{n-2} \binom{n}{n-i-1}_r\]

Proof Let \(s_n(x) = \frac{x - n + 1}{x + 1} \binom{x + 1}{n}_r\), as in (3). The binomial theorem for Sheffer sequences states that \(s_n(x+y) = \sum_{i=0}^{n} s_i(y) \tilde{h}_{n-i}(x)\).
Let \( x = n, y = 0 \) and noting that \( s_n(0) = (1 - n){1\choose n}_r = 1 - n \) for \( 0 < n < r \) and 0 otherwise, we have \( s_n(n) = \sum_{i=0}^{n} s_i(0)b_{n-i}(n) \), hence

\[
\frac{1}{n+1}{n+1\choose n}_r = \sum_{i=0}^{r-1} (1-i){n\choose n-i}_r
= {n\choose n}_r - \sum_{i=1}^{r-2} i{n\choose n-i-1}_r.
\]

We have already noted that \( {n\choose k}_r \rightarrow {n+k-1\choose k} \) as \( r \rightarrow \infty \), so

\[
\lim_{r \to \infty} \frac{1}{n+1}{n+1\choose n}_r = (2n-1){2n-1\choose n} - \sum_{i=1}^{n-1} i{2n-i-2\choose n-1} = C_n.
\]

\[\blacksquare\]

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