CLASSIFICATION OF INTEGRABLE EVOLUTION EQUATIONS
OF THE FORM \( u_t = u_{xxx} + f(t, x, u, u_x, u_{xx}) \)

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Abstract.

We obtain the classification of integrable equations of the form \( u_t = u_{xxx} + f(t, x, u, u_x, u_{xx}) \) using the formal symmetry method of Mikhailov et al [A.V.Mikhailov, A.B.Shabat and V.V.Sokolov, in What is Integrability edited by V.E. Zakharov (Springer-Verlag, Berlin 1991)]. We show that all such equations can be transformed to an integrable equation of the form \( v_t = v_{xxx} + f(v, v_x, v_{xx}) \) using transformations \( \Phi(x, t, u, v, u_x, v_x) = 0 \), and the \( u_{xx} \) dependence can be eliminated except for two equations.

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1. Introduction.

The existence of “formal symmetries” has been proposed as an integrability test by Mikhailov et al [1]. A formal symmetry is a truncated expansion of the recursion operator in inverse powers of \( D = \frac{\partial}{\partial x} \), hence the equations admitting a formal symmetry are candidates for those equations admitting a recursion operator.

The classifications of evolution equations \( u_t = F(x, u, u_x, u_{xx}, u_{xxx}) \) (except for a class essentially nonlinear in \( u_{xxx} \)), and \( u_t = h(x,t)u_3 + f(x,t,u,u_x) \) are given respectively in [1] and [2]. In this paper we obtain the classification of evolution equations \( u_t = u_{xxx} + f(t, x, u, u_x, u_{xx}) \) using the existence of canonical conserved densities \( \rho_i, i = 0, \ldots, 3 \). We show that all such equations can be transformed to an equation without explicit \( x \) and \( t \) dependence using first order differential substitutions. The list of integrable equations given in [1] is further simplified by the availability of time dependent transformations. Furthermore, the \( u_{xx} \) dependence is eliminated, but two of the transformed equations are nonlocal. Thus, up to first order differential substitutions, there are only three equations in the class considered.

2. Formal symmetries, recursion operators and differential substitutions.

Let \( u_t = F[u] \) be an evolution equation where \( F \) is a differential polynomial. A symmetry \( \sigma \) of this equation is a differential polynomial which satisfies the linearized equation \( \sigma_t = F'\sigma \) where \( F' \) is the Frechet derivative of \( F \). The recursion operator \( R \) is a linear operator that sends symmetries to symmetries, i.e. \( R\sigma \) is a symmetry whenever \( \sigma \) is a symmetry [3]. In general the recursion operator \( R \) is an integro-differential operator, and satisfies \( (R_t + [R,F'])\sigma = 0 \). If the symmetries are dense in the solutions of this equation, then \( R \) can be defined as the solution of the operator equation

\[
R_t + [R,F'] = 0. \tag{2.1}
\]

In 1+1 dimensions the integral terms in the recursion operator can be expanded in inverse powers of \( D = \frac{\partial}{\partial x} \), and Eq.(2.1) can be solved recursively, leading to linear first order differential equations for the coefficients of \( R \). The solvability of these equations in the class of local functions gives certain conserved density conditions, \( (\rho_i)_t = D\eta_i \). These conserved density conditions in turn give nonlinear partial differential equations for \( F \), whose solutions lead to the classification.

We consider the classification of evolution equations

\[
u_t = u_3 + f(x, t, u, u_1, u_2). \tag{2.2}\]
In the following, subscripts denote differentiation with respect to \( x \). The classification is obtained using the existence of conserved densities \( \rho_0, \ldots, \rho_3 \), and we have chosen the coordinate transformations so as to put the equations in the form given in [1].

We shall use the following general result for the elimination of \( u_2 \). Consider the evolution equation 
\[ u_t = u_m + f(u_{m-1}, \ldots, u, x, t). \]
If this equation is integrable then \( \rho_0 = \partial f / \partial u_{m-1} \) is a conserved density, hence \( \int \rho_t \) is a local function. The dynamical variable \( s \) defined by \( s = u_{m-1}e^{\frac{1}{m-1}\int \rho_0} \), satisfies an equation of the form \( s_t = s_m + g(s_{m-2}, s, u, \ldots, u_{m-1}) \) i.e. \( s_{m-1} \) is eliminated. If \( \rho_0 \) is itself a total derivative, then \( s \) is a local function. Otherwise, one has to introduce a new dynamical variable \( v = \int \rho_0 \), and apply the transformation to the new equation. This transformation of course introduces non-locality, and the crucial point is to ensure that the transformed equation is local. We shall also use the coordinate transformation \( x \rightarrow x + h(t)t \) to eliminate the term \( h'(t)u_1 \). Other differential substitutions will be described as needed.

3. Classification results.

For equations of the form (2.2), the first four nontrivial conserved density conditions are given below. We shall obtain a classification based on the existence of these conserved densities.

\[
\begin{align*}
\rho_0 & = \frac{\partial f}{u_2}, \\
\rho_1 & = \frac{1}{3} \left( \frac{\partial f}{u_2} \right)^2 - \frac{\partial f}{\partial u_1}, \\
\rho_2 & = \frac{\partial f}{\partial u} + \frac{2}{27} \left( \frac{\partial f}{u_2} \right)^3 - \frac{1}{3} \frac{\partial f}{\partial u_2} \frac{\partial f}{\partial u_1} + \frac{1}{3} \eta_0, \\
\rho_3 & = \eta_1,
\end{align*}
\]

where \( \rho_i = D \eta_i, i = 0, 1. \)

The method is to compute the time derivative of the \( \rho_i \)’s and integrate by parts until a term that involves the highest derivative non-linearly appears. Then the coefficient of this highest derivative has to be set to zero, and we obtain partial differential equations for \( f \).

As a first step, it can easily be seen that \( f \) has to be a third order polynomial in \( u_2 \), hence, we obtain the \( u_2 \) dependence of \( f \), and the evolution equation has the form

\[ u_t = u_3 + A(x, t, u, u_1)u_2^2 + B(x, t, u, u_1)u_2 + C(x, t, u, u_1). \]

Then, we obtain the following nonlinear differential equation for the \( u_1 \) dependence of \( A \),

\[
\frac{\partial^2 A}{\partial u_1^2} - 4 \frac{\partial A}{\partial u_1} A + \frac{16}{9} A^3 = 0.
\]
For $A \neq 0$, the solution is given by

$$A = -\frac{3}{4} \frac{\partial Z/\partial u_1}{Z}, \quad \frac{\partial^3 Z}{\partial u_1^3} = 0. \quad (3.4)$$

Hence

$$Z = a(x, t, u) u_1^2 + b(x, t, u) u_1 + c(x, t, u). \quad (3.5)$$

In Sections 4, 5 and 6 we shall study respectively the cases where $A = B = 0$, $A = 0$, $B \neq 0$ and $A \neq 0$.

4. Classification for the case $A = B = 0$.

We will show that the equations in this class consist of the KdV, mKdV, their potential forms and the Callegero-Degasperis-Fokas equation. Actually, the classification of integrable equations of the form $u_t = h(x, t) u_{xxx} + f(t, x, u, u_x)$ has been obtained in [2], hence we will only give an outline of the derivations for completeness. The use of the coordinate transformations (4.2) given below is crucial in obtaining the classification. Using the conserved density conditions we first obtain

$$u_t = u_3 - \frac{1}{2} k(t)^2 u_1^3 + a_2(x, t, u) u_1^2 + a_1(x, t, u) u_1 + a_0(x, t, u). \quad (4.1)$$

The first branching is given by $k(\partial a_2/\partial u) = 0$. The coordinate transformations leaving the form of (4.1) invariant are

$$\tilde{u}_t = \tilde{u}_3 - \frac{1}{2} \frac{k^2}{\alpha^2} \tilde{u}_1^3 + \tilde{u}_1^2 \left[ \frac{3 \beta}{2} k^2 + \frac{1}{\alpha} a_2 \right] + \tilde{u}_1 \left[ -\frac{3 \beta^2}{2 \alpha^2} k^2 - 2 \frac{\beta}{\alpha} a_2 + a_1 \right]$$

$$- \frac{\alpha}{\alpha} (\tilde{u} - \beta) + \beta_t - \beta_{xxx} + \frac{1}{2} \frac{\beta^2}{\alpha^2} k^2 + \frac{\beta^2}{\alpha} a_2 - \beta x a_1 + a_0 \alpha. \quad (4.2)$$

For $k \neq 0$, we set $k = 1$, $a_2 = 0$, and using the conserved density conditions we obtain the Callegero-Degasperis-Fokas equation

$$u_t = u_3 - \frac{1}{2} u_1^3 + u_1 [c_1 e^{2u} + c_2 e^{-2u}]. \quad (4.3)$$

For $k = 0$, we first obtain $u_t = u_3 + a_2(x, t, u) u_1^2 + a_1(x, t, u) u_1 + a_0(x, t, u)$, where $a_2 = b_0(x, t)$, $a_1 = c_2(x, t) u_2 + c_1(x, t) u + c_0(x, t)$, $a_0 = d_3(x, t) u^3 + d_2(x, t) u^2 + d_1(x, t) u + d_0(x, t)$. The branching is given by the condition $b_0 c_2 = 0$. For $b_0 \neq 0$ we have the potential KdV equation,

$$u_t = u_3 + u_1^2, \quad (4.4)$$

For $b_0 = 0$, using the conserved density conditions and appropriate coordinate transformations we obtain the KdV

$$u_t = u_3 + uu_1 \quad (4.5)$$
and mKdV equations

\[ u_t = u_3 + u^2 u_1. \] (4.6)

It is well known that all the equations in this class are related via Miura transformations, hence up to first order differential substitutions there is a single equation in the class \( u_t = u_3 + f(x, t, u, u_x) \).

5. Classification for the case \( A = 0, B \neq 0 \).

We will show that all equations in this class are linearizable. The conserved density conditions imply that the evolution equation has to be in the form

\[ u_t = u_3 + b_1(x, t, u) u_1 u_2 + b_0(x, t, u) u_2 + c_3(x, t, u) u_1^3 + c_2(x, t, u) u_2^2 + c_1(x, t, u) u_1 + c_0(x, t, u). \] (5.1)

Then the transformation \( u \rightarrow \phi(u) \) gives \( b_1 \rightarrow \frac{b_1}{\phi'} - 2 \phi'' \phi' \) and we can set \( b_1 = 0 \). Then the allowable transformations are of the form \( u \rightarrow \alpha(x, t) u + \beta(x, t) \). The conserved density conditions give \( \partial c_3 / \partial u = 0 \), and we set \( c_3 \) constant by choosing \( \alpha \). Then we have the branching \( b_0 c_3 = 0 \). If \( b_0 = 0 \), (5.1) is independent of \( u_2 \), hence we set \( c_3 = 0 \). Then we have

\begin{align*}
    b_0 &= b_{02}(x, t) u_2^2 + b_{01}(x, t) u + b_{00}(x, t), \\
    c_2 &= c_{21}(x, t) u + c_{20}(x, t), \\
    c_1 &= \frac{1}{3} b_{02}^2 u_4^2 + \frac{2}{3} b_{01} b_{02} u_3^2 + c_{12}(x, t) u_2^2 + c_{11}(x, t) u_1 + c_{10}(x, t). \quad (5.2)
\end{align*}

We have two subcases depending on whether \( b_{02} = 0 \) or \( b_{02} \neq 0 \).

**The case** \( b_{02} \neq 0 \): In this case we use coordinate transformations \( u \rightarrow \alpha(x, t) u + \beta(x, t) \) to set \( b_{02} = 3 \) and \( b_{01} = 0 \). Then we obtain the following equation.

\[ u_t = u_3 + 3 u_2 u_2 + 9 u u_2^2 + 3 u_4 u_1 + c_{10}(x, t) u_1 + \frac{1}{2} (c_{10}) u. \] (5.3)

This equation can be linearized by the following sequence of Miura and coordinate transformations. Recall that the transformation (2.5) eliminates \( u_2 \) but in this case \( \int \rho_0 \) is not a local function. Thus we first use the potentiation \( u_1 \rightarrow u^2 \) to obtain

\[ u_t = u_3 - \frac{3}{4} u_2^2 + 3 u_2 u_1 + u_1^3 + u_1 c_{10}(x, t) + d(t). \]

Then using the transformation \( u \rightarrow e^{2u} \) we obtain

\[ u_t = u_3 - \frac{3}{4} u_2^2 + c_{10}(x, t) u_1 + 2d(t) u. \]
Finally the transformation \( u \to 2\sqrt{u} \) we obtain the linear equation

\[
\frac{\partial u}{\partial t} = u_3 + c_{10}(x, t)u_1 + \frac{1}{2}u((c_{10})_x + 2d).
\]  

\( (5.4) \)

**The case** \( b_{02} = 0 \): In this case we use point transformations to set \( b_{01} = 3 \) and \( b_{00} = 0 \). Then the conserved density conditions give

\[
\frac{\partial u}{\partial t} = u_3 + 3u_1u_2 + \frac{3}{2}u_1^2 + u_1c_{10}(x, t) + (c_{10})_xu + c_{00}(x, t).
\]  

\( (5.5) \)

The potentiation \( u_1 \to u \) gives

\[
\frac{\partial u}{\partial t} = u_3 + 3u_1u_2 + u_1^2 + u_1c_{10}(x, t) + \int c_{00}(x, t) \, dx.
\]

Then the point transformation \( u \to e^u \) gives the linear equation

\[
\frac{\partial u}{\partial t} = u_3 + u_1c_{10} + \int c_{00} \, dx.
\]  

\( (5.6) \)

For completeness, we give the interacting soliton equation of the KdV equation that we would obtain if we had taken \( c_2 = 0 \) instead of \( b_1 = 0 \) in (5.1)

\[
\frac{\partial u}{\partial t} = u_3 - \frac{3}{2} \frac{u_2u_1}{u} + \frac{3}{2} \frac{u_1^3}{u^2} + \frac{3}{2}u u_1.
\]  

\( (5.7) \)

**6. Classification for the case** \( A \neq 0 \).

When \( A \neq 0 \), the coordinate transformation \( u \to \phi(x, t, u) \) results in

\[
A \to (2au_1 + \phi_b b - 2\phi_x a) \left[ au_1^2 + u_1(\phi_b b - 2\phi_x a) + (\phi_u^2 c - \phi_u \phi_x b + \phi_x^2 a) \right]^{-1}.
\]  

\( (6.1) \)

Thus for \( a = 0 \), these coordinate transformations can be used to take \( b = 1, c = 0 \), similarly for \( a \neq 0 \), we can take \( a = 1 \) and \( b = 0 \). The cases \( c = 0 \) and \( c \neq 0 \) has to be considered separately in the last subclass. As a result \( A \) has the forms studied in Sections 6.1-3 below. In all cases the allowable coordinate transformations should satisfy \( \phi_x = 0 \). We will use the transformation

\[
v = \int e^{\frac{1}{2}\int \rho_0 u_2}
\]  

\( (6.2) \)

to eliminate \( u_2 \) term.

**6.1 Classification for the case** \( A = -\frac{4}{u_1} u_1^{-1} \).

We will show that all equations in this class are either linearizable, or transformable to the mKdV equation. We first obtain

\[
B = a_{11}(x, t, u)u_1^{1/2} + a_{12}(x, t, u)u_1 + a_{13}(x, t, u).
\]
Then $C$ can be obtained by solving a fourth order o.d.e., and if $a_{13}$ is nonzero, it has $u_1^3 ln(u_1)$ and $u_1 ln(u_1)$ terms. The first branching is given by the condition $a_{11}a_{13} = 0$. If we assume $a_{11} = 0$, $a_{13} \neq 0$, the conserved density conditions give $a_{13} = 0$. Hence we assume $a_{11} \neq 0$.

The case $a_{11} \neq 0$ (linearizable equation): We will show that the equation with $a_{11} \neq 0$ is linearizable. We have

$$C = a_{01}(x,t,u)u_1^{3/2} + a_{02}(x,t,u)u_1^2 + a_{03}(x,t,u)u_1 + a_{04}(x,t,u) + \left[\frac{2}{3}(a_{12})_u + \frac{1}{3}a_{12}^2\right]u_1^3 + \frac{2}{3}a_{11}a_{12}u_1^{5/2}.$$ \(\text{(6.3)}\)

The conserved density conditions give $(a_{04})_x = 0$, and we set $a_{04} = 0$ by a coordinate transformation. As a result for $a_{11} \neq 0$, we obtain the following equations.

$$\begin{align*}
(a_{11})_t &= (a_{11})_{xxx} + (a_{11})_x a_{03} + \frac{1}{2}(a_{03})_x a_{11}, \\
(a_{12})_t &= -\frac{2}{3}(a_{11})_{xxx} a_{11} + \frac{2}{3}(a_{11})_x^2 - \frac{1}{3}a_{11}^2 a_{03}, \\
(a_{11})_u &= \frac{1}{3}a_{11}a_{12}, \\
(a_{12})_x &= -\frac{1}{3}a_{11}^2, \\
(a_{03})_u &= \frac{2}{3}(a_{11})_x a_{11}, \\
a_{01} &= 2(a_{11})_x, \\
a_{02} &= 0, \\
a_{04} &= 0. 
\end{align*}$$ \(\text{(6.4)}\)

We make the differential substitution $s = \psi(x,t,u,u_1)$ which gives an equation where $s_2^2$ term is not present. Elimination of the coefficient of $s_2s_1$ requires that

$$\psi = \psi_1(x,t,u)\sqrt{u_1} + \psi_0(x,t,u).$$

Then $s_2s_1$ and $s_2$ are eliminated by choosing $\psi_1$ and $\psi_0$ as solutions of

$$\begin{align*}
3(\psi_1)_u - a_{12}\psi_1 &= 0, \\
3(\psi_1)_x + a_{11}\psi_0 &= 0. 
\end{align*}$$ \(\text{(6.4)}\)

The resulting equation for $s$ is independent of $u$ by virtue of the equations (6.3) and (6.4) and it can be transformed to equation in the previous section by a linear coordinate transformation, and then it is linearized.

The case $a_{11} = 0$ (equations linearizable or transformable to mKdV): When $a_{11} = 0$, we have $(a_{12})_x = 0$, and we can make $a_{12} = 0$ by a point transformation, hence $B = 0$. Then we are allowed to make transformations of the form $u \rightarrow \alpha(t)u + \beta(t)$. The conserved density conditions give two equations in this class, the first one being a constant coefficient equation.

$$\begin{align*}
(6.5a) 
\quad u_t &= u_3 - \frac{3}{4}u_1^{-1}u_2^2 + a_{01}u_1^{3/2} + a_{02}u_1^2 + a_{03}u_1 + a_{04}, \\
(6.5b) 
\quad u_t &= u_3 - \frac{3}{4}u_1^{-1}u_2^2 + a_{03}(x,t)u_1.
\end{align*}$$
We will show that both equations are transformable to an equation independent of $u_2$. The conserved density $\rho_0 = -\frac{3}{4} \frac{u_2}{u_1}$ is trivial, the transformation (6.2) reduces to $v = 2\sqrt{u_1}$, and we have

$$u_t - u_3 + \frac{3}{4} \frac{u_2^2}{u_1} \rightarrow v_t - v_3.$$

Thus we obtain respectively the following equations

$$v_t = v_3 + v_1 \left( \frac{1}{2} a_{02} u_1^2 + \frac{3}{4} a_{01} v + a_{03} \right), \quad (6.6a)$$

$$v_t = v_3 + a_{03} v_1 + \frac{1}{2} (a_{03})_v. \quad (6.6b)$$

The second equation is linear and the first one is transformable to the mKdV equation. Therefore, all integrable equations with $A = -\frac{3}{4} u_1^{-1}$ are either linearizable, or transformable to the mKdV equation.

**6.2 Classification for the case $A = -\frac{3}{2} u_1^{-1}$.

In this case, using the conserved density conditions we obtain

$$u_t = u_3 - \frac{3}{2} \frac{u_2^2}{u_1} + a_{01}(t, u) \frac{1}{u_1} + a_{02}(t, u) u_1^4 + a_{03} u_1 + a_{04}(t, u), \quad (6.7)$$

where the coefficients have to satisfy the following equations.

$$\begin{align*}
(a_{01})_t &= - (a_{01})_u a_{04} + 2(a_{04})_u a_{01}, \quad (6.8a) \\
(a_{02})_t &= - (a_{02})_u a_{04} - (a_{04})_{uuu} - 2(a_{04})_u a_{02}, \quad (6.8b) \\
0 &= (a_{01})_{5u} + 10(a_{01})_{uuu} a_{02} + 15(a_{01})_{uu} (a_{02})_u \\
&\quad + a_{01}(u [6(a_{02})_{uuu} + 16 a_{02}^2] + 2a_{01} [(a_{02})_{uuu} + 8(a_{02})_u a_{02}]. \quad (6.8c)
\end{align*}$$

The equations (6.8a-c) form a consistent system, hence (6.7) is an integrable equation. The conserved density $\rho_0 = -\frac{3}{2} \frac{u_2}{u_1}$ is trivial and the transformation $v = \ln u_1$ gives

$$u_t - u_3 + \frac{3}{2} \frac{u_2^2}{u_1} \rightarrow v_t - v_3 + \frac{1}{3} v_1^3. \quad (6.9)$$

We will show that, generically (6.7) cannot be transformed to a local equation independent of $u_2$. The coordinate transformation $u \rightarrow \phi(u, t)$ gives

$$\begin{align*}
a_{01} &\rightarrow \frac{1}{\phi_3^2} a_{01}, \\
a_{02} &\rightarrow \frac{1}{\phi_3^2} \left[ -\phi_{uuu} \phi_u + \frac{3}{2} \phi_{uu} + \phi_u^2 a_{02} \right], \\
a_{03} &\rightarrow a_{03}, \\
a_{04} &\rightarrow \phi_t + \phi_u a_{04}. \quad (6.10)
\end{align*}$$
Using these coordinate transformations we take \( a_{04} = 0 \), which implies that \( a_{01} \) and \( a_{02} \) are independent of \( t \), hence the evolution equation in independent of \( x \) and \( t \). If we set \( (a_{01})_u = 0 \), then

\[
(a_{02})_{uuu} + 8(a_{02})_u a_{02} = 0,
\]

(6.11)

hence the integrable equation in this class is

\[
u_t = u_3 - \frac{3}{2} u_1^2 + a_{01} \frac{1}{u_1} + a_{02}(u) u_1^3
\]

(6.12)

where \( a_{02} \) satisfies (6.11). It can be seen that when \( a_{02} \) is independent of \( u \), it is transformed to the Callegero-Degasperis-Fokas equation,

\[
u_t = v_3 - \frac{1}{2} v_1^3 + v_1[-a_{01}e^{-2v} + 3a_{02}e^{2v}].
\]

On the other hand, if \( a_{02} \) is not constant it can not be transformed to a local equation.

6.3 Classification for the case \( A = -\frac{3}{2}(u_1^2 + c(x,t,u))^{-1}u_1 \).

In this case, using the conserved density conditions we first solve \( B \), as a function depending on the derivatives of \( c \). Then the compatibility of the differential equations for \( C \) implies that \( c_x = 0 \), and we can set \( c = 1 \) by choosing \( \phi_u \). It then follows that \( B = 0 \). As a result we obtain

\[
C = a_{01}(t)(u_1^2 + 1)^{3/2} + a_{02}(u,t)u_1(u_1^2 + 1) + a_{03}(t)u_1 + a_{04}(t).
\]

We can set \( a_{04} = 0 \) by choosing \( \phi_t \), and \( a_{03} = 0 \) by a linear change in \( x \). Then we obtain the following equation

\[
u_t = u_3 - \frac{3}{2} u_1(u_1^2 + 1)^{-1} u_2^2 + a_{01}(u_1^2 + 1)^{3/2} + a_{02}(u) u_1(u_1^2 + 1),
\]

(6.13)

where

\[
(a_{02})_{uuu} + 8(a_{02})_u a_{02} = 0, \quad (a_{02})_u a_{01} = 0.
\]

(6.14)

Thus for \( a_{01} \neq 0 \) we have an equation independent of \( u \).

The conserved density \( \rho_0 = -3(1 + u_1^2)^{-1} u_1 u_2 \) is trivial and using the transformation \( v = \int (1 + u_1^2)^{-1/2} u_2 = \ln | u_1 + \sqrt{1 + u_1^2} | \), we obtain

\[
u_t - u_3 + \frac{3}{2} \frac{u_1}{1 + u_1^2} u_2^2 \rightarrow v_t - v_3 + \frac{1}{3} v_1^3.
\]

Then if \( a_{02} \) is constant (6.13) is transformed to

\[
v_t = v_3 - \frac{1}{2} v_1^3 + \frac{3}{4} v_1 [(a_{01} + a_{02}) e^{2v} + (a_{02} - a_{01}) e^{-2v} - 2a_{02}].
\]

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However for $a_{01} = 0$, $a_{02}$ need not be constant and (6.13) is not transformable to a local equation.

7. Conclusion.

We have shown that all evolution equations of the form (2.2) can be transformed to an equation independent of $x$ and $t$. The surprising fact is that the classification of equations with explicit $t$ dependence is considerably simplified by the availability of time dependent transformations.

It is known that integrable equations of the form $u_t = u_3 + f(x, t, u, u_x)$ are all related to the KdV equation via Miura transformations. The equations with $u_{xx}$ dependence that are not transformable to one of these equations are

$$u_t = u_3 - \frac{3}{2} u_1^2 + a_{01} \frac{1}{u_1} + a_{02}(u)u_1^3$$

$$u_t = u_3 - \frac{3}{2} u_1(u_1^2 + 1)^{-1} u_2 + a_{02}(u)u_1(u_1^2 + 1),$$

where

$$(a_{02})_{uuu} + 8(a_{02})_u a_{02} = 0.$$

An equivalent form of the first equation is

$$u_t = u_3 - \frac{3}{2} \frac{u_2}{u_1} + \frac{1}{u_1}[4u_3 + k_1 u + k_2].$$

It has been checked that this equation do not have fifth and seventh order local symmetries. This result suggest the following possibilities. (i) The formal symmetry condition will not hold at a later stage, but the formal symmetry condition has been checked at later stages and no inconsistency has been observed. (ii) The formal symmetry condition will hold at all orders, but it will not be possible to write the recursion operator in closed form. It has been shown that such a situation occurs for a system of evolution equations [4]. Hence these two equations need further investigation.

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