SOME PROPERTIES OF THE TORSION FUNCTION WITH ROBIN BOUNDARY CONDITIONS

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Abstract. In this paper we study some properties of the torsion function with Robin boundary conditions. Here we write the shape derivative of the $L^\infty$ and $L^p$ norms, for $p \geq 1$, of the torsion function, seen as a functional on a bounded simply connected domain $\Omega \in \mathbb{R}^n$, and prove that the balls are a critical shape for these functionals, when the volume is preserved.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2,\alpha}$ and simply connected domain. Let us consider the following torsion problem with Robin boundary conditions

$$\begin{cases}
-\Delta u = 1 & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.1)$$

where $\nu$ stands for the outer unit normal to $\partial \Omega$ and $\beta > 0$ is a positive real number, known as Robin boundary parameter.

A weak solution to (1.1) is a function $u \in H^1(\Omega)$ which satisfies

$$\int_\Omega \nabla u \nabla v \, dx + \beta \int_{\partial \Omega} uv \, d\mathcal{H}^{n-1} = \int_\Omega v \quad \forall v \in H^1(\Omega)$$

When $\beta \to \infty$ we have the well known torsion problem of elasticity or Saint Venant problem, that has been studied by many authors: different estimates and qualitative properties have been studied, for example, by Pólya and Szegő in [PS], or Payne in [P].

It is well known that the solution to problem (1.1) is unique and positive whenever $\partial \Omega$ is sufficiently smooth.

A comparison result à la Talenti has been proved in [ANT]. They consider the symmetrized problem, that is

$$\begin{cases}
-\Delta v = 1 & \text{in } \Omega^t \\
\frac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial \Omega^t,
\end{cases} \quad (1.2)$$
where $\Omega^\sharp$ is the ball centered in the origin having the same measure as $\Omega$. In
dimension 2, they proved that
\[
u^\sharp(x) \leq v(x) \quad x \in \Omega^\sharp,
\]
where $v$ is the solution to the problem (1.2) and $\nu^\sharp$ is the Schwarz symmetrization
of $u$ (for more details see [K]). Moreover they proved, for $n \geq 3$, that
\[
\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega^\sharp)}
\]
and
\[
\|u\|_{L^{p,2}(\Omega)} \leq \|v\|_{L^{p,2}(\Omega^\sharp)},
\]
for all $0 < p \leq \frac{n}{n-2}$. Here $L^{p,q}(\Omega)$ is the Lorentz space (for more detail see [L]). It
follows that in dimensions greater than 3
\[
\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}
\]
when $p = 1, 2$.

In [BG] the authors proved, with different arguments, that $\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)}$, where $u$ is solution to (1.1) and $v$ solution to the symmetrized problem (1.2).

Problem (1.1) is a particular case of the following
\[
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $f \in L^2(\Omega)$ and its symmetrized problem is
\[
\begin{cases}
-\Delta v = f^\sharp & \text{in } \Omega^\sharp \\
\frac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial \Omega^\sharp,
\end{cases}
\]
where $f^\sharp$ is the Schwarz symmetrization of $f$. When in (1.4) and (1.5) we have
Dirichlet boundary conditions, Talenti (see [T]), via the rearrangement of a func-
tion and the Schwarz symmetrization, showed the following
\[
u^\sharp(x) \leq v(x) \quad x \in \Omega^\sharp
\]
Still in [ANT], a Talenti comparison result for the problem (1.4) has been proved: when $n \geq 2$
\[
\|u\|_{L^{p,1}(\Omega)} \leq \|v\|_{L^{p,1}(\Omega^\sharp)} \quad \text{for all } 0 < p \leq \frac{n}{2n}.
\]
and
\[
\|u\|_{L^{p,2}(\Omega)} \leq \|v\|_{L^{p,2}(\Omega^\sharp)} \quad \text{for all } 0 < p \leq \frac{n}{3n-4},
\]
where $v$ is solution to (1.5). So, in dimension 2, we have that
\[
\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}
\]
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when \( p = 1, 2 \). One may ask if (1.7) is still true for larger values of \( p \) in dimension 2 or if it is valid in every dimension and value of \( p \). The authors, though, found counterexamples of the untruthfulness of these questions when \( n = 2 \) and \( p = \infty \), and when \( n = 3 \) and \( p = 2 \). This led to the following open problems:

- \( u^2 \leq v \) in \( \Omega^2 \) for \( n \geq 3 \) and \( f \equiv 1 \);
- \( \|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^2)} \) for \( n \geq 3 \) and \( f \in L^2(\Omega) \).

In this paper we move the first steps in these directions.

In particular we set

\[
M(\Omega) = \|u\|_{L^\infty(\Omega)},
\]

and for every \( p \in [1, +\infty) \) we denote the following functional

\[
F_p(\Omega) = \int_{\Omega} |u(x)|^p \, dx = \int_{\Omega} u^p(x) \, dx = \|u\|_{L^p(\Omega)}^p,
\]

where \( u \) is solution to (1.1). We are interested in computing the shape derivative (see [HP]) of these two functionals and prove that the ball centered at the origin is a critical shape for them.

Let us denote by \( B_R \) the ball centered at the origin in \( \mathbb{R}^n \) with radius \( R > 0 \).

**Main result.** The ball \( B_R \) is a critical shape for the functionals \( M(\Omega) \) and \( F_p(\Omega) \), \( p \geq 1 \), i.e.

\[
M'(B_R, v) = F'_p(B_R, v) = 0,
\]

where \( v \) is a vector field volume preserving of the first order and where \( M'(. \, , v) \) and \( F'_p(\cdot \, , v) \) are the shape derivatives of \( M \) and \( F_p \) respectively.

For the precise definition of vector field volume preserving of the first order see the next section.

Next section will include some preliminary results in order to compute the shape derivative of the two functionals defined before. In section 3 there will be the computation of the \( L^\infty \)-norm of the torsion function with the consequent proof of the main result. In section 4, the same thing for the \( L^p \)-norm will be done.

2. Preliminaries

Throughout this paper we will denote by \( B_R = \{ x \in \mathbb{R}^n : \|x\| < R \} \) the ball centered at the origin with radius \( R > 0 \), where \( \| \cdot \| \) is the classical euclidean distance; by \( \Omega \) a bounded \( C^{2,\alpha} \) and simply connected domain with finite Lebesgue measure, where \( C^{2,\alpha} \) stands for the \( \alpha \)-Hölderian space. We denote by \( \mathcal{H}^{n-1} \) the \((n - 1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \) and by \( | \cdot | \) the Lebesgue measure in \( \mathbb{R}^n \).
2.1. The radial case. It is easy to compute the solution to (1.1) when \( \Omega = B_R \). Indeed, being the solution a superharmonic function in \( B_R \) and possibly radial, it makes sense to look for a solution of this kind

\[
u(x) = u(|x|) = -\frac{|x|^2}{2n} + c,
\]

for some real constant \( c \) that we can compute substituting \( u \) in the boundary condition. On \( \partial B_R \) we have \( |x| = R \), so

\[
0 = \frac{\partial u}{\partial \nu} + \beta u = -\frac{|x|}{n} + \beta \left(-\frac{|x|^2}{2n} + c\right) = -\frac{R}{n} + \beta \left(-\frac{R^2}{2n} + c\right),
\]

hence

\[
u(x) = \frac{R}{\beta n} + \frac{1}{2n}(R^2 - |x|^2).
\] (2.1)

It is worth noticing that \( u \) has a unique maximum point, that is the origin. Its value is

\[
u(0) = \frac{R}{\beta n} + \frac{R^2}{2n}.
\]

2.2. Shape derivative: some definitions and computations. Here we give some preliminary definitions and results that the reader can find in [BW] and [HP]. Let us consider a family of perturbations \( \{\Omega_t\}_t \) of the form

\[
\Omega_t = \{y = x + tv(x) + \frac{t^2}{2}w(x) + o(t^2) : x \in \Omega, t \text{ small enough}\},
\] (2.2)

where \( v, w \) are two \( C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n) \) vector fields.

The Jacobian matrix of \( y := y(t, \Omega) \), up to second order terms, is

\[
I + tD_v + \frac{t^2}{2}D_w,
\]

where \( (D_v)_{ij} = \partial_j v_i \equiv \frac{\partial v_i}{\partial x_j} \). By Jacobi’s formula, for small \( t \), the Jacobian determinant is given by

\[
J(t) = 1 + t \operatorname{div} v + \frac{t^2}{2}((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w) + o(t^2),
\] (2.3)

where \( D_v : D_v := \partial_i v_j \partial_j v_i \). It is clear that for \( t \) small enough, \( J(t) \approx 1 \), so \( y(t, \Omega) \) is a diffeomorphism.

In particular we can write the measure of \( \Omega_t \) in terms of the perturbations defined before

\[
|\Omega_t| = \int_{\Omega} J(t) \, dx = |\Omega| + t \int_{\Omega} \operatorname{div} v \, dx \\
+ \frac{t^2}{2} \int_{\Omega} ((\operatorname{div} v)^2 - D_v : D_v + \operatorname{div} w) \, dx + o(t^2).
\]
Definition 2.1. \( y(t, \Omega) \) is said to be volume preserving of the first order if
\[
\int_\Omega \text{div} \, v \, dx = 0,
\]
and volume preserving of the second order if
\[
\int_\Omega ((\text{div} \, v)^2 - D_v : D_v + \text{div} \, w) \, dx = 0.
\]

The shape derivative of the solution to (1.1) is written by energy arguments. More precisely, let \( \{ \Omega_t \}_t \) be as before, let \( G : \mathbb{R} \rightarrow \mathbb{R} \) a smooth function and \( g \) its derivative \( (G' = g) \). The following functional is called Robin energy
\[
E(\Omega_t, u) = \int_{\Omega_t} |\nabla_y u|^2 \, dx - 2 \int_{\Omega_t} G(u) \, dx + \beta \int_{\partial \Omega_t} u^2 \, dS_t,
\]
(2.4)
where \( dS_t \) is the surface element of \( \Omega_t \). A critical point \( u_t \in H^1(\Omega) \) of (2.4) satisfies the Euler Lagrange equation
\[
\begin{cases}
\Delta_y u_t + g(u_t) = 0 & \text{in } \Omega_t \\
\frac{\partial u_t}{\partial \nu_t} + \beta u_t = 0 & \text{on } \partial \Omega_t.
\end{cases}
\]
(2.5)

By a change of variables, using the inverse function \( x(y) \) of the perturbation (2.2) (which exists for small \( t \)), we have that
\[
E(t) = \int_\Omega \nabla \tilde{u}(t) A \nabla \tilde{u}(t) \, dx - 2 \int_\Omega G(\tilde{u}(t)) J(t) \, dx + \beta \int_{\partial \Omega} \tilde{u}^2(t) m(t) \, dx
\]
where \( \tilde{u}(t) := \tilde{u}(x + tv(x) + \frac{t^2}{2} w(x), t) \), \( A = (A_{ij}(t)) := \frac{\partial y_i}{\partial x_j} \frac{\partial x_i}{\partial y_j} J(t) \) and \( m(t) \) is the index of deformation when passing from \( dS \) to \( dS_t \). The transformed solution \( \tilde{u}(t) \) solves the equation
\[
\begin{cases}
L_A \tilde{u}(t) + g(\tilde{u})(t) = 0 & \text{in } \Omega \\
\frac{\partial \tilde{u}(t)}{\partial \nu_A} + \beta m(t) \tilde{u}(t) = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(2.6)
with \( L_A = \partial_j (A_{ij}(t) \partial_i) \) and \( \frac{\partial}{\partial x_k} = \nu_i A_{ij}(t) \partial_j \).

Note that if we expand
\[
\tilde{u}(t) = \tilde{u}(0) + t \dot{\tilde{u}}(0) + \frac{t^2}{2} \ddot{\tilde{u}}(0) + o(t^2)
\]
then
\[
\dot{\tilde{u}}(0) = u(x)
\]
\[
\ddot{\tilde{u}}(0) = \left[ \frac{d}{dt} \tilde{u}(t) \right]_{t=0}
\]
\[
= \left[ \partial_t \tilde{u}(t) + (v(x) + tw(x)) \cdot \nabla \tilde{u}(t) \right]_{t=0} = u' + v \cdot \nabla u
and \( u' \) is called \textit{shape derivative} of \( \bar{u} \).

If we differentiate (2.6) with respect to \( t \) and evaluate for \( t = 0 \), after lengthy calculations, we will have the equations for \( u' \)

\[
\begin{aligned}
\begin{cases}
\Delta u' + g'(u)u' = 0 & \text{in } \Omega \\
\partial'_\nu u' + \beta u' = -\partial'_\nu (v \cdot \nabla u) + \nabla^\tau u \cdot D_v \nu + \nu \cdot D_v \nabla u - \beta v \cdot \nabla u & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

(2.7)

where \( \nabla^\tau \) is the tangential gradient (the projection of the gradient onto the tangent spaces of \( \partial \Omega \)).

3. \textbf{Shape derivative of the \( L^\infty \)-norm}

We will now examine the case where \( G = I_\mathbb{R} \), that is the identity function on \( \mathbb{R} \), and \( \Omega = B_R \). Clearly \( g \equiv 1 \) and (2.7) becomes

\[
\begin{aligned}
\begin{cases}
\Delta u' = 0 & \text{in } B_R \\
\partial'_\nu u' + \beta u' = \left( \frac{1 + \beta R}{n} \right) (v \cdot \nu) & \text{on } \partial B_R,
\end{cases}
\end{aligned}
\]

(3.1)

where now \( \nu = \frac{x}{R} \).

\textbf{Lemma 3.1.} If \( v \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n) \) is a vector field volume preserving of the first order, then the solution \( u' \) to (3.1) is a function with zero mean value, that is

\[
\int_{B_R} u' \, dx = \int_{\partial B_R} u' \, d\sigma = 0.
\]

\textbf{Proof.} Let us integrate the first equation of (3.1)

\[
0 = \int_{B_R} \Delta u' \, dx = \int_{B_R} \text{div}(\nabla u') \, dx = \int_{\partial B_R} \partial'_\nu u' \, d\sigma.
\]

(3.2)

Since \( v \) is volume preserving of the first order

\[
\int_{B_R} \text{div} v \, dx = \int_{\partial B_R} v \cdot \nu \, d\sigma = 0.
\]

(3.3)

So integrating the second equation on the boundary, by (3.2) and (3.3)

\[
\int_{\partial B_R} u' \, d\sigma = \frac{1}{\beta} \left( \frac{1 + \beta R}{n} \right) \int_{\partial B_R} v \cdot \nu \, d\sigma = 0.
\]

In conclusion, being \( u' \) a harmonic function, by the mean value theorem

\[
\int_{B_R} u' \, dx = \frac{R}{n} \int_{\partial B_R} u' \, d\sigma = 0.
\]
We prove the next result following the proof that can be found in [HLP]. We recall that
\[ M(\Omega) = \|u\|_{L^\infty(\Omega)} \]
where \( u \) is solution to (1.1). In particular we define shape derivative of \( M(\Omega) \) in the direction of the vector field \( v \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n) \), denoted by \( M'(\Omega, v) \), the following
\[ M'(\Omega, v) = \lim_{t \to 0} \frac{M(\Omega_t) - M(\Omega)}{t}, \quad (3.4) \]
Where \( \Omega_t \) is the perturbation of \( \Omega \) defined in the previous section.

**Theorem 3.2.** Let \( B_R \) be a ball centered at the origin with radius \( R > 0 \). Then for every \( C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n) \) vector field \( v \), the shape derivative of \( M \) at \( B_R \) in any direction \( v \) exists and it is given by
\[ M'(B_R, v) = u'(0), \]
where 0 is the maximum point of (2.1) and \( u' \) is the solution to (3.1).

**Proof.** Let \( u_t \) be the the torsion function of the perturbation of the ball \( B_{R,t} \), for \( t > 0 \) and let \( \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) defined as
\[ \psi(t, x) := \nabla u_t(x). \]

Being \( x = 0 \) the unique maximum point of (2.1), \( \psi(0, 0) = 0 \). On the other hand, by the strict concavity of the torsion function \( u \) on \( B_R \), the matrix
\[ D_x \psi(0, 0) = \text{Hess}_u(0) \]
is invertible, since \( \text{Hess}_u \) is negative definite. So for the implicit function theorem, in a neighbourhood of the origin, for \( t \) small enough, there exists a unique \( x_t \) such that \( \nabla u_t(x_t) = 0 \). Moreover the function \( t \to x_t \) is differentiable and \( x_t \) must be a maximum, so \( M(B_{R,t}) = u_t(x_t) \).

We want to prove that
\[ \lim_{t \to 0} \frac{M(B_{R,t}) - M(B_R)}{t} = u'(0), \]
where
\[ \frac{M(B_{R,t}) - M(B_R)}{t} = \frac{u_t(x_t) - u(0)}{t} = \frac{u_t(x_t) - u_t(0)}{t} + \frac{u_t(0) - u(0)}{t}. \]

By the differentiability of the map \( t \to u_t \) and the the fact that \( \nabla u(0) = 0 \), we have
\[ \lim_{t \to 0} \frac{u_t(0) - u(0)}{t} = \frac{d}{dt} [u_t(0)]_{t=0} = u'(0) + v(0) \cdot \nabla u(0) = u'(0). \]

Furthermore, by the differentiability of \( t \to x_t \), by Lagrange theorem on the
segment \([0, x_t]\), the mean value property of \(\nabla u_t\) and the regularity of \(u_t\)
\[
\frac{u_t(x_t) - u_t(0)}{t} = \nabla u_t(\xi_t) \frac{x_t}{t} = \left( \frac{1}{|B_r(\xi_t)|} \int_{B_r(\xi_t)} \nabla u_t(x) \, dx \right) \frac{x_t}{t}
\rightarrow \nabla u(0) \left[ \frac{dx_t}{dt} \right]_{t=0} = 0,
\]
with \(\xi_t\) a suitable point in \([0, x_t]\), and the proof is concluded. \(\square\)

**Corollary 3.3.** The ball is a critical shape for the functional \(M\), for every \(v \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)\) which is volume preserving of the first order, i.e.
\[
M'(B_R, v) = u'(0) = 0.
\]

**Proof.** As a consequence of Lemma 3.1 and Theorem 3.2, applying the mean value theorem, we have that
\[
M'(B_R, v) = u'(0) = \frac{1}{n \omega_n R^{n-1}} \int_{\partial B_R} u'(x) \, d\sigma = \frac{1}{\omega_n R^n} \int_{B_R} u' \, dx = 0.
\]
\(\square\)

### 4. Shape derivative of the \(L^p\)-norm

Let us consider the solution to (1.1). For every \(p \in [1, +\infty)\) we consider the following functional
\[
F_p(\Omega) = \int_{\Omega} |u(x)|^p \, dx = \int_{\Omega} u^p(x) \, dx = \|u\|_{L^p(\Omega)}^p.
\]
As in (3.3) we define the shape derivative of \(F_p(\Omega)\) in the direction of the vector field \(v \in C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)\), denoted by \(F'_p(\Omega, v)\), the following
\[
F'_p(\Omega, v) = \lim_{t \to 0} \frac{F_p(\Omega_t) - F_p(\Omega)}{t},
\]
where \(\Omega_t\) is the perturbation of \(\Omega\).

**Theorem 4.1.** For every \(C^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n)\) vector field \(v\), the shape derivative of \(F_p\) at \(\Omega\) in any direction \(v\) exists and it is given by
\[
F'_p(\Omega, v) = p \int_{\Omega} u^{p-1} u' \, dx + \int_{\partial \Omega} u^p(v \cdot \nu) \, d\sigma,
\]
where \(u'\) is solution to (3.1) and \(\nu\), as before, is the outer normal on \(\partial \Omega\).

**Proof.** Let \(u_t\) be the solution to the perturbed problem
\[
\begin{align*}
-\Delta_y u_t &= 1 & \text{in } \Omega_t \\
\frac{\partial u_t}{\partial \nu_t} + \beta u_t &= 0 & \text{on } \partial \Omega_t,
\end{align*}
\]
where $\Omega_t$ is the perturbed domain defined in (2.2). Then,

$$F_p(\Omega_t) = \int_{\Omega_t} u^p(y) \, dy = \int_{\Omega} \tilde{u}^p(t) J(t) \, dx,$$

with $\tilde{u}(t) = u(x + tv(x) + \frac{\partial}{\partial t}w(x), t)$ and $J(t)$ the Jacobian determinant as in (2.3).

Then it is possible to derive under the sign of integral and

$$\frac{d}{dt} F_p(\Omega_t) = \frac{d}{dt} \int_{\Omega} \tilde{u}^p(t) J(t) \, dx = \int_{\Omega} \frac{d}{dt} \tilde{u}^p(t) J(t) \, dx.$$

Evaluating this derivative for $t = 0$

$$\left[ \frac{d}{dt} F_p(\Omega_t) \right]_{t=0} = p \int_{\Omega} \tilde{u}^p(0) \left[ \frac{d}{dt} \tilde{u}(t) \right]_{t=0} \, J(0) \, dx + \int_{\Omega} \tilde{u}^p(0) \left[ \frac{d}{dt} J(t) \right]_{t=0} \, dx.$$

Remembering that $\tilde{u}(0) = u(x)$, $\left[ \frac{d}{dt} \tilde{u}(t) \right]_{t=0} = u' + v \cdot \nabla u$, $J(0) = 1$ and $J'(0) = \text{div} \, v$, then

$$F'_p(\Omega, v) = p \int_{\Omega} u^{p-1} u' \, dx + \int_{\Omega} u^{p-1} v \cdot \nabla u \, dx + \int_{\Omega} u^p \text{div} \, v \, dx.$$

By previous theorem we know that

$$F'_p(B_R, v) = 0.$$

**Proof.** By previous theorem we know that

$$F'_p(B_R, v) = p \int_{B_R} u^{p-1} u' \, dx + \int_{\partial B_R} u^p (v \cdot \nu) \, d\sigma.$$

Being $u$ constant on the boundary and $v$ a field volume preserving of the first order

$$\int_{\partial B_R} u^p (v \cdot \nu) \, d\sigma = \tilde{u}^p \int_{\partial B_R} (v \cdot \nu) \, d\sigma = \tilde{u}^p \int_{B_R} \text{div} \, v = 0,$$

where $\tilde{u} = u(R) = \frac{R}{\beta_n}$. By corollary 3.3, we know that $u'(0) = 0$ and so by the mean value theorem, we have that

$$\int_{\partial B_R} u'(x) \, dH^{n-1} = u'(0) n \omega_n r^{n-1} = 0,$$
for every $r \in [0; R]$. So, applying the Coarea Formula
\[ p \int_{B_R} u^{p-1} u' \, dx = p \int_{B_R} \left( \frac{R}{\beta n} - \frac{1}{2n} (R^2 - |x|^2) \right)^{p-1} u' \, dx \]
\[ = p \int_0^R \left( \frac{R}{\beta n} - \frac{1}{2n} (R^2 - r^2) \right)^{p-1} \int_{\partial B_r} u'(x) \, d\mathcal{H}^{n-1} \, dr = 0. \]

Hence
\[ F'_p(B_R, v) = 0. \]

\[ \square \]

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