OPTIMAL CONDITIONS FOR \( L^\infty \)-REGULARITY AND A PRIORI ESTIMATES FOR ELLIPTIC SYSTEMS, I: TWO COMPONENTS

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Abstract. In this paper we present a new bootstrap procedure for elliptic systems with two unknown functions. Combining with the \( L^p \)-\( L^q \)-estimates, it yields the optimal \( L^\infty \)-regularity conditions for the three well-known types of weak solutions: \( H^1_0 \)-solutions, \( L^1 \)-solutions and \( L^1_\delta \)-solutions. Thanks to the linear theory in \( L^p_\delta(\Omega) \), it also yields the optimal conditions for a priori estimates for \( L^1_\delta \)-solutions. Based on the a priori estimates, we improve known existence theorems for some classes of elliptic systems.

1. Introduction

The aim of this paper is to present a new alternate-bootstrap procedure to obtain \( L^\infty \)-regularity and a priori estimates for solutions of semilinear elliptic systems. This method enables us to obtain the optimal \( L^\infty \)-regularity conditions for the three well-known types of weak solutions: \( H^1_0 \)-solutions, \( L^1 \)-solutions and \( L^1_\delta \)-solutions of elliptic systems (for their definitions, see Section 2). Combining with the linear theory in \( L^p_\delta \)-spaces, our method also enables us to obtain a priori estimates for \( L^1_\delta \)-solutions, therefore to improve existence theorems for various classes of elliptic systems.

Let us consider the Dirichlet system of the form

\[
\begin{align*}
-\Delta u &= f(x, u, v), \quad \text{in } \Omega, \\
-\Delta v &= g(x, u, v), \quad \text{in } \Omega, \\
u &= v = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a smoothly bounded domain and \( f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) are Carathéodory functions. A typical case is

\[
\begin{align*}
-\Delta u &= u^r v^p, \quad \text{in } \Omega, \\
-\Delta v &= u^s v^q, \quad \text{in } \Omega, \\
u &= v = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( r, s \geq 0, \ p, q > 0 \).

As a motivation, let us mention that in an important recent article [QS], Quittner & Souplet developed an alternate-bootstrap method in the scale of weighted Lebesgue spaces \( L^p_\delta(\Omega) \). Their bootstrap procedure works well for system (1.1) with

\[
\begin{align*}
-h_1(x) \leq f &\leq C_1(|v|^p + |u|^q) + h_2(x), \quad u, v \in \mathbb{R}, \ x \in \Omega, \\
-h_1(x) \leq g &\leq C_1(|u|^q + |v|^p) + h_2(x), \quad u, v \in \mathbb{R}, \ x \in \Omega,
\end{align*}
\]

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where \( p, q > 0, \ pq > 1, \ \gamma, \sigma \geq 1, \ C_1 > 0, \ h_1 \in L^1_\theta(\Omega), \ h_2 \in L^\theta \) with \( \theta > n/2 \). They obtained the optimal conditions for \( L^\infty \)-regularity and a priori estimates for \( L^1_\delta \)-solutions, see [QS Theorem 2.1]. The optimality was shown by Souplet [S, Theorem 3.3]. Using this method, they obtained new existence theorems for various classes of elliptic systems.

Our bootstrap procedure works for system (1.1) with \( f, g \) satisfying more general assumptions

\[
|f| \leq C_1(|u|^p + |u|\gamma) + h(x), \\
|g| \leq C_1(|u|^\sigma + |v|^\delta) + h(x), \quad u, v \in \mathbb{R}, \ x \in \Omega, \tag{1.4}
\]

where \( r, s, \gamma, \sigma \geq 0, \ p, q > 0, \ C_1 > 0 \) and the regularity of \( h \) will be specified later. The bootstrap procedure is only based on the \( L^m-L^k \)-estimates in the linear theories of weak solutions. So we are able to obtain the optimal \( L^\infty \)-regularity conditions for the three well-known types of weak solutions: \( H^1_0 \)-solutions, \( L^1 \)-solutions and \( L^1_\delta \)-solutions of elliptic systems. Under some additional appropriate conditions on \( f, g \), this method also enables us to obtain a priori estimates for \( L^1_\delta \)-solutions.

1.1. Optimal conditions for \( L^\infty \)-regularity. First we consider the case where \( pq > (1-r)(1-s) \). Set

\[
\alpha = \frac{p+1-s}{pq - (1-r)(1-s)}, \quad \beta = \frac{q+1-r}{pq - (1-r)(1-s)}. \tag{1.5}
\]

Note that \( (\alpha, \beta) \) is the solution of

\[
\begin{bmatrix}
  r-1 \\
  q \\
  s-1
\end{bmatrix}
\begin{bmatrix}
  \alpha \\
  \beta
\end{bmatrix}
= \begin{bmatrix}
  1 \\
  1
\end{bmatrix}.
\]

Throughout this paper, we assume that \( \alpha, \beta > 0 \), which is obvious if \( r, s \leq 1 \). The numbers \( \alpha, \beta \) are related to its scaling properties of system (1.2) (see for instance [CFMT]). For the parabolic counterpart of (1.2), these numbers appear for instance in [DE, Wang, Zh] in the study of blow-up.

For the \( L^\infty \)-regularity, we obtain the following theorems.

**Theorem 1.1.** (Optimal \( L^\infty \)-regularity for \( H^1_0 \)-solutions)

Assume that \( f, g \) satisfy (1.4) with \( pq > (1-r)(1-s) \).

(i) If

\[
\max\{\alpha, \beta\} > \frac{n-2}{4}, \quad r, s, \gamma, \sigma < \frac{n+2}{n-2}, \tag{1.6}
\]

then any \( H^1_0 \)-solution of system (1.1) belongs to \( L^\infty(\Omega) \);

(ii) If \( n \geq 3 \) and

\[
\max\{\alpha, \beta\} < \frac{n-2}{4}, \tag{1.7}
\]

system (1.1) in \( B_1 \), the unit ball in \( \mathbb{R}^n \), with \( f = (u+c_1)^r(v+c_2)^p \) and \( g = (u+c_1)^\sigma(v+c_2)^\delta \) for some \( c_1, c_2 > 0 \) admits a positive \( H^1_0 \)-solution \((u, v)\) such that \( u \notin L^\infty(B_1) \) and \( v \notin L^\infty(B_1) \).

**Theorem 1.2.** (Optimal \( L^\infty \)-regularity for \( L^1 \)-solutions)

Assume that \( f, g \) satisfy (1.4) with \( pq > (1-r)(1-s) \).
(i) If
\[ \max\{\alpha, \beta\} > \frac{n-2}{2}, \quad r, s, \gamma, \sigma < \frac{n}{n-2}, \]
\[ \min\{p + r, q + s\} < \frac{n}{n-2}, \quad h \in L^\theta(\Omega), \quad \theta > \frac{n}{2}, \]
then any \( L^1 \)-solution of system (1.1) belongs to \( L^\infty(\Omega) \);

(ii) If \( n \geq 3 \) and
\[ \max\{\alpha, \beta\} < \frac{n-2}{2}, \]
system (1.1) in \( B_1 \), the unit ball in \( \mathbb{R}^n \), with \( f = (u+c_1)^r(v+c_2)^p \) and \( g = (u+c_1)^q(v+c_2)^s \) for some \( c_1, c_2 > 0 \) admits a positive \( L^1 \)-solution \((u, v)\) such that \( u \notin L^\infty(B_1) \) and \( v \notin L^\infty(\Omega) \).

Theorem 1.3. (Optimal \( L^\infty \)-regularity for \( L^1_\delta \)-solutions)
Assume that \( f, g \) satisfy (1.4) with \( pq > (1-r)(1-s) \).

(i) If
\[ \max\{\alpha, \beta\} > \frac{n-1}{2}, \quad r, s, \gamma, \sigma < \frac{n+1}{n-1}, \]
\[ \min\{p + r, q + s\} < \frac{n+1}{n-1}, \quad h \in L^\theta(\Omega), \quad \theta > \frac{n+1}{2}, \]
then any \( L^1_\delta \)-solution of system (1.1) belongs to \( L^\infty(\Omega) \);

(ii) If \( n \geq 2 \) and
\[ \max\{\alpha, \beta\} < \frac{n-1}{2}, \]
there exist functions \( a, b \in L^\infty(\Omega) \), \( a, b > 0 \) such that system (1.1) with \( f = a(x)u^rv^p \) and \( g = b(x)u^qv^s \) admits a positive \( L^1_\delta \)-solution \((u, v)\) such that \( u \notin L^\infty(\Omega) \) and \( v \notin L^\infty(\Omega) \).

Our theorems are closely related to the three critical exponents:

\[ p_S := \begin{cases} \infty & \text{if } n \leq 2, \\ (n+2)/(n-2) & \text{if } n \geq 3, \end{cases} \]

\[ p_{sg} := \begin{cases} \infty & \text{if } n \leq 2, \\ n/(n-2) & \text{if } n \geq 3, \end{cases} \]

\[ p_{BT} := \begin{cases} \infty & \text{if } n \leq 1, \\ (n+1)/(n-1) & \text{if } n \geq 2. \end{cases} \]

\( p_S \) is the Sobolev exponent. \( p_{sg} \) and \( p_{BT} \) appear in study of \( L^1 \)-solutions and \( L^1_\delta \)-solutions of scalar elliptic equations respectively. Note that

\[ \frac{n-2}{4} = \frac{1}{p_S - 1}, \quad \frac{n-2}{2} = \frac{1}{p_{sg} - 1}, \quad \frac{n-1}{2} = \frac{1}{p_{BT} - 1}. \]

So if we write each critical exponent as \( p_c \), the optimal conditions for \( L^\infty \)-regularity of the above three types of weak solutions have a consistent form \( \max\{\alpha, \beta\} > 1/(p_c - 1) \) and \( r, s, \gamma, \sigma, \min\{p + r, q + s\} < p_c \).
Remark 1.1. If \( r, s \leq 1 \), \( \min\{p + r, q + s\} < p_c \) in Theorem 1.1-1.3 (i) is superfluous, see Remark 2.2.

For \( pq \leq (1 - r)(1 - s) \), we have the following theorem.

**Theorem 1.4.** Assume that \( f, g \) satisfy (1.4) with \( pq \leq (1 - r)(1 - s) \). Then Theorem 1.1-1.3 (i) also hold if \( \max\{\alpha, \beta\} > 1/(p_c - 1) \) is replaced by \( pq - (1 - r)(1 - s) < (p_c - 1) \max\{p + 1 - s, q + 1 - r\} \).

In order to justify the above theorems, let us recall the optimal \( L^\infty \)-regularity for the scalar equation

\[
-\Delta u = f(x, u), \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial \Omega,
\]

(1.12)

where \(|f| \leq C(1 + |u|^p)\) with \( p \geq 1 \). It is well-known that the Sobolev exponent \( p_S \) plays an important role in the optimal \( L^\infty \)-regularity and a priori estimates of the \( H^1_0 \)-solutions, see [FLN, GS, JL, ZZ] and the references therein. Any \( H^1_0 \)-solution of (1.12) belongs to \( L^\infty (\Omega) \) if and only if \( p \leq p_S \), see for instance [BK, SI]. For the \( L^1 \)-solutions, the critical exponent is \( p_{sg} \). Any \( L^1 \)-solution of (1.12) belongs to \( L^\infty (\Omega) \) if and only if \( p < p_{sg} \), see for instance [A, NS, P].

The critical exponent \( p_{BT} \) first appeared in the work of Brézis & Turner in [BT]. They obtained a priori estimates for all positive \( H^1_0 \)-solutions of (1.12) for \( p < p_{BT} \) using the method of Hardy-Sobolev inequalities. However the meaning of \( p_{BT} \) was clarified only recently. It was shown by Souplet [S, Theorem 3.1] that \( p_{BT} \) is the critical exponent for the \( L^\infty \)-regularity of \( L^1 \)-solutions of (1.12) by constructing an unbounded solution with \( f = a(x)u^p \) for some \( a \in L^\infty (\Omega), a \geq 0 \) if \( p > p_{BT} \). The critical case \( p = p_{BT} \) was recently shown to belong to the singular case for \( f = u^p \), see [DMP], also [MR] for related results. Moreover, the results of [S] was extended to the case \( f = u^p \) when \( p > p_{BT} \) is close to \( p_{BT} \).

If we set \( \alpha = 1/(p - 1) \), i.e., the solution of \((p - 1)\alpha = 1\), the optimal conditions for \( L^\infty \)-regularity of the above three types of weak solutions also have a consistent form \( \alpha > 1/(p_c - 1) \). For more detailed discussions, we refer to the book [QS2, Chapter I].

Using the bootstrap procedure they developed based on linear theory in \( L^1_0(\Omega) \), Quittner \& Souplet [QS, Theorem 2.1] obtained similar \( L^\infty \)-regularity condition as Theorem 1.3 (i) assuming that \( f, g \) satisfy (1.3). In [S, Theorem 3.3], Souplet proved a similar result as in Theorem 1.3 (ii) in the case \( f = a(x)v^p \) and \( g = b(x)w^q \) for some functions \( a, b \in L^\infty (\Omega), a, b \geq 0 \).

**Remark 1.2.** Using the method of moving planes and Pohozaev-type identities, in the case \( f = v^p \) and \( g = w^q \), \( p, q > 1 \), it is proved if \( \Omega \) is convex and bounded, and \( \alpha + \beta > (n - 2)/2 \), then there exists a positive classical solution of (1.1); If \( n \geq 3 \), \( \Omega \) is starshaped and bounded, and \( \alpha + \beta \leq (n - 2)/2 \), then (1.1) has no positive solution, see [CFM, M2]. Note that the optimal \( L^\infty \)-regularity condition in Theorem 1.1 is weaker than the existence condition, i.e., the so-called Sobolev hyperbola.

**Remark 1.3.** We shall use a bootstrap procedure to prove the above theorems. Based on another bootstrap procedure, using the method of Rellich-Pohozaev identities and moving planes, [CFM, Lemma 2.2] obtained a priori estimates for \( H^1_0 \)-solutions of (1.1) with \( f, g \) satisfying some conditions similar to (1.3).
1.2. Optimal conditions for a priori estimates and existence theorems. Combining with the linear theory in $L^p_\delta$-spaces, developed in [FSW], see also [BV], our bootstrap procedure enables us to obtain a priori estimates for system (1.1) with $f,g$ satisfying (1.4) and

$$f + g \geq -C_2(u + v) - h_1(x), \quad u, v \in \mathbb{R}, \quad x \in \Omega,$$

where $C_2 > 0$, $h_1 \in L^1_\delta(\Omega)$. By an a priori estimate, we mean an estimate of the form

$$\|u\|_{L^\infty} \leq C, \quad \|v\|_{L^\infty} \leq C$$

for all possible nonnegative solutions of (1.1) (in a given set of functions), with some constant $C$ independent of $(u, v)$. Our main result of the a priori estimates is the following theorem.

**Theorem 1.5.** Let $f,g$ satisfy (1.4) and (1.13) with $pq > (1 - r)(1 - s)$ and (1.10). Then there exists $C > 0$ such that for any nonnegative solution $(u, v)$ of (1.1) satisfying

$$\|u\|_{L^1_\delta} + \|v\|_{L^1_\delta} \leq M,$$

it follows that $u, v \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty} + \|v\|_{L^\infty} \leq C.$$

The constant $C$ depends only on $M, \Omega, p, q, r, s, \gamma, \sigma, C_1, C_2$.

(1.10) is optimal for the a priori estimates for the $L^1_\delta$-solutions of the system (1.1) under the assumptions (1.4) and (1.13), see Theorem 1.3 (ii).

There are several methods for the derivation of a priori estimates: The method of Rellich-Pohozaev identities and moving planes, see [CFM, FLN]; The scaling or blow-up methods, which proceeds by contradiction with some known Liouville-type theorems, see [BM, CFMT, FY, GS, Lou, So, Zou] and references therein, for the related Liouville-type results, see [BM, BuM, CMM, FF, M, PQS, RZ, Sa, SZ, SZ2] and the references therein; The method of Hardy-Sobolev inequalities, see [BT, CFM2, C, CFS, GW]. For the detailed comments of the above methods and the advantages of the bootstrap methods, we refer to [QS], see also a survey paper [S2].

A similar theorem for system (1.1) with $f,g$ satisfying (1.3) was proved by Quittner & Souplet [QS, Theorem 2.1]. Based on their a priori estimates, they obtained new existence theorems for various classes of elliptic systems.

**Theorem 1.6.** Assume that $f,g$ satisfy (1.4) and (1.10) with $pq > (1 - r)(1 - s)$ and (1.10). Then

(a) any nonnegative $L^1_\delta$-solution $(u, v)$ of (1.1) belongs to $L^\infty(\Omega)$ and satisfies the a priori estimate (1.14);

(b) system (1.1) admits a positive $L^1_\delta$-solution $(u, v)$ if in addition $f, g$ satisfy

$$f + g = o(u + v), \quad \text{as } u, v \to 0^+,$$

uniformly in $x \in \Omega$. 
Remark 1.4. If \( pq > (1 - r)(1 - s) \) and \( \max\{\alpha, \beta\} > \frac{n-1}{2} \) are replaced by \( pq \leq (1 - r)(1 - s) \) and \( pq - (1 - r)(1 - s) < \frac{2}{n-1} \max\{p + 1 - s, q + 1 - r\} \) respectively, then the conclusions of Theorem 1.5 and 1.6 also hold.

Remark 1.5. Consider system (1.1) with boundary conditions of the form \( u_\nu = au, v_\nu = bv \), where \( a, b \in \mathbb{R} \) and \( u_\nu \) denotes the derivative of \( u \) with respect to the outer unit normal on \( \partial \Omega \). If, for example, \( f, g \) satisfy
\[
 f + g \geq C_1 (\lambda_1(a)u + \lambda_1(b)v) - C_2, \quad u, v \geq 0, \; x \in \Omega,
\]
where \( C_1 > 1, C_2 \geq 0 \) and \( \lambda_1(a) \) denotes the first eigenvalue of \(-\Delta\) with boundary conditions \( u_\nu = au \), then it is easy to deduce that
\[
 \|u\|_{L^1} + \|v\|_{L^1} \leq M,
\]
with \( M \) independent of \( u, v \). The proof of Theorem 2.4 (in Section 2) implies (1.14). Using this a priori estimate, we also have a similar existence theorem of \( L^1 \)-solutions of system (1.1) with Neumann conditions as Theorem 1.6.

If \( r = s = 0 \), under assumptions (1.3), (1.16), the system (1.1) was studied by several authors. Using another bootstrap method, similar results as the above theorem was obtained in [QS, Theorem 1.1], see also [CFM2, F, FY, Zou] for more related results.

The second existence theorem is about the system
\[
\begin{align*}
-\Delta u &= a(x)u^r v^p - c(x)u, \quad \text{in } \Omega, \\
-\Delta v &= b(x)u^q v^s - d(x)v, \quad \text{in } \Omega, \\
u &= v = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( r, s \leq 1, pq > (1 - r)(1 - s), a, b, c, d \in L^\infty(\Omega), a, b \geq 0, \int_\Omega a, \int_\Omega b > 0, \inf\{\text{spec}(-\Delta + c)\} > 0, \inf\{\text{spec}(-\Delta + d)\} > 0.\)

**Theorem 1.7.** Assume that
\[
\max\{\alpha, \beta\} > \frac{n-1}{2}. \tag{1.19}
\]
Then
(a) any nonnegative \( L^1_c \)-solution \((u, v)\) of (1.18) belongs to \( L^\infty(\Omega) \) and satisfies the a priori estimate (1.14);  
(b) system (1.18) admits a positive \( L^1_c \)-solution \((u, v)\).

From the above theorem, we obtain the existence theorem for system (1.2).

**Corollary 1.8.** Assume that \( r, s \leq 1, pq > (1 - r)(1 - s) \) and (1.19) holds. Then system (1.2) admits a positive classical solution \((u, v)\).

A similar existence result was proved in [QS, Theorem 1.4] but under more stronger assumptions. Set
\[
\begin{align*}
\hat{p} &= \frac{(n+1)p}{n+1-(n-1)r}, \quad \hat{q} = \frac{(n+1)q}{n+1-(n-1)s}, \\
\hat{\alpha} &= \frac{\hat{p}+1}{(\hat{p}\hat{q}-1)_+}, \quad \hat{\beta} = \frac{\hat{q}+1}{(\hat{p}\hat{q}-1)_+}.
\end{align*}
\]
Instead of (1.19), they required that \( \max\{\hat{\alpha}, \hat{\beta}\} > (n - 1)/2 \). The a priori estimates and existence of positive solutions for (1.2) was studied in [CFMT] in the case when \( \Omega = B_R(0) \) and the parameters satisfy \( 0 \leq r, s \leq 1 \), \( pq \geq (1 - r)(1 - s) \), plus some additional conditions. Note that the results there also cover the case when the Laplace operators are replaced by \( \Delta_m u, \Delta_n u \), \( m, n > 1 \). We refer to [M, RZ, TV, Zh, B] for existence/nonexistence results for (1.2) and to [DE, Li, Wang, Zh] and the references therein for related results on the associated parabolic systems.

Remark 1.6. It was shown in [RZ] that system (1.2) has no positive solutions if \( p, q, r, s \geq 1 \), \( \min\{p + r, q + s\} \geq (n + 2)/(n - 2)_+ \) and \( \Omega \) is star-shaped. It was also proved in [Zou2] that system (1.2) has a positive solution if \( r, s \geq 1 \), \( pq > (r - 1)(s - 1) \) and

\[
\max\{p + r, q + s\} \leq (n + 2)/(n - 2)_+, \tag{1.20}
\]

see also [Zou]. Our result is that system (1.2) has a positive solution if \( 0 \leq r, s \leq 1 \), \( pq > (1 - r)(1 - s) \) and (1.19) holds. If \( r = s = 0 \), for the existence of positive solutions of the system (1.2), we have the optimal condition \( \alpha + \beta > (n - 2)/2 \), see Remark 1.2. We would like to point out that

(i) \( \max\{p + r, q + s\} \leq (n + 1)/(n - 1) \) implies (1.19), but (1.20) does not;
(ii) (1.19) is much more general than (1.20). (1.19) allows very large \( p \) or \( q \);
(iii) If \( r = s = 0 \), (1.19) is stronger than \( \alpha + \beta > (n - 2)/2 \).

So it is still a widely open question what should be the optimal conditions on \( p, q, r, s, n \) for existence of positive solutions to system (1.2).

A special case of (1.18) is the following model of a nuclear reactor

\[
\begin{align*}
-\Delta u &= uv - au, & \text{in } \Omega, \\
-\Delta v &= bu, & \text{in } \Omega, \\
u &= v &= 0, & \text{on } \partial \Omega,
\end{align*}
\tag{1.21}
\]

where \( u, v \) present the neutron flux and the temperature, respectively. This system and the corresponding parabolic system were studied in [Ch, GW, GW2, Q, QS, QS2]. In [GW2], the existence and a priori estimate were obtained under the assumption \( n \leq 3 \), or \( \Omega \) convex and \( n \leq 5 \). In [QS Theorem 1.2] and [QS2 Theorem 31.17], the existence and a priori estimate were obtained under weaker assumption \( n \leq 4 \) without assuming \( \Omega \) convex. Our theorem recover their result since \( \max\{\alpha, \beta\} = 2 > (n - 1)/2 \) implies \( n < 5 \).

In next section, we present our bootstrap procedure. In Section 3, we prove Theorem 1.1-1.3. In Section 4, we prove Theorem 1.5-1.7

2. The Bootstrap Procedure

In what follows we give the definitions of three types of weak solutions of system (1.1), see [QS2 Chapter I].

Definition 2.1. (i) By an \( H^1_0 \)-solution of system (1.1), we mean a couple \((u, v)\) with

\[
u, v \in H^1_0(\Omega), \quad f, g \in H^{-1}(\Omega),
\]
satisfying
\[ \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \int_{\Omega} \nabla v \cdot \nabla \varphi = \int_{\Omega} g \varphi, \]
for all \( \varphi \in H^1_0(\Omega) \).

(ii) By an \( L^1 \)-solution of system (1.1), we mean a couple \((u, v)\) with
\( u, v \in L^1(\Omega), \quad f, g \in L^1(\Omega) \),
satisfying
\[ -\int_{\Omega} u \Delta \varphi = \int_{\Omega} f \varphi, \quad -\int_{\Omega} v \Delta \varphi = \int_{\Omega} g \varphi, \]
for all \( \varphi \in C^2(\Omega), \quad \varphi \vert_{\partial \Omega} = 0 \).

(iii) Set \( \delta(x) := \text{dist}(x, \partial \Omega) \) and \( L^1_\delta(\Omega) := L^1(\Omega; \delta(x)dx) \). By an \( L^1_\delta \)-solution of system (1.1), we mean a couple \((u, v)\) with
\( u, v \in L^1(\Omega), \quad f, g \in L^1_\delta(\Omega) \),
satisfying (2.1). The three types of weak solutions of the scalar equation (1.1 2) and the linear equation
\[ -\Delta u = \phi, \quad \text{in } \Omega; \quad u = 0, \quad \text{on } \partial \Omega, \]
are defined similarly. According to [BCMR, Lemma 1], if \( \phi \in L^1_\delta(\Omega) \), (2.2) admits a unique \( L^1_\delta \)-solution \( u \in L^1(\Omega) \). Moreover, \( \|u\|_{L^1_{\delta}} \leq C(\Omega, m, k) \|\phi\|_{L^m} \). But \( \phi \in L^m(B_1) \) and \( U \notin L^k(B_1) \), see also [QS2, Chapter I].

Obviously, Proposition 2.1 holds for the \( H^1_0 \)-solution of (2.2). But it is not convenient to derive the optimal condition for \( L^\infty \)-regularity of the \( H^1_0 \)-solutions of system (1.1). For our purpose, we develop an \( L^m-L^k \)-estimate for the \( H^1_0 \)-solution of (2.2). It is an invariant of Proposition 2.1. Let \( n \geq 3 \), set \( 2_* := 2n/(n + 2) \). It is the conjugate number of the Sobolev imbedding exponent, \( 2n/(n - 2) \).

Proposition 2.2. Let \( 1 \leq m \leq k \leq \infty \) satisfy
\[ \frac{1}{m} - \frac{1}{k} < \frac{4}{n + 2}. \]

Let \( u \in H^1_0(\Omega) \) be the unique \( H^1_0 \)-solution of (2.2). If \( \phi \in L^{2_*}(\Omega) \), then \( u \in L^{2_*}(\Omega) \) and satisfies the estimate \( \|u\|_{L^{2_*}} \leq C(\Omega, m, k) \|\phi\|_{L^{2_*}} \).
The above proposition in hand, the $L^\infty$-regularity of the $H^1_0$-solutions of (1.12) with $|f| \leq C(1 + |u|^p)$ with $1 \leq p < p_S$ follows immediately from a simple bootstrap argument. It is much simpler than the usual proof, see [BK, St, QS].

For all $1 \leq k \leq \infty$, define the spaces $L^k_\delta(\Omega) = L^k(\Omega; \delta(x)dx)$. For $1 \leq k < \infty$, $L^k_\delta(\Omega)$ is endowed with the norm
\[ \|u\|_{L^k_\delta} = \left( \int_{\Omega} |u(x)|^k \delta(x)dx \right)^{1/k}. \]

Note that $L^\infty_\delta(\Omega) = L^\infty(\Omega; dx)$, with the same norm $\|u\|_\infty$. For the $L^1_\delta$-solutions, we have the following regularity result.

**Proposition 2.3.** (see [FSW], also [QS, QS]) Let $1 \leq m \leq k \leq \infty$ satisfy
\[ \frac{1}{m} - \frac{1}{k} < \frac{2}{n + 1}. \] (2.5)
Let $u \in L^1(\Omega)$ be the unique $L^1_\delta$-solution of (2.2). If $\phi \in L^m_\delta(\Omega)$, then $u \in L^k_\delta(\Omega)$ and satisfies the estimate $\|u\|_{L^k_\delta} \leq C(\Omega, m, k)\|\phi\|_{L^m_\delta}$.

The condition (2.5) is optimal, since for $1 \leq m < k \leq \infty$ and $1/m - 1/k > 2/(n + 1)$, there exists $\phi \in L^m_\delta(\Omega)$ such that $u \notin L^k_\delta(\Omega)$, where $u$ is the unique $L^1_\delta$-solution of (2.2), see [S, Theorem 2.1].

**Remark 2.1.** According to Proposition 2.1-2.3, the assumptions of $h$ in Theorem 1.1-1.3 are natural.

In order to give a uniform proof of Theorem 1.1-1.3 (i), we write the three critical exponents $p_S$, $p_{sg}$, $p_{BT}$ as $p_c$. Denote $B^k$ the spaces $L^{2+k}(\Omega)$, $L^k(\Omega)$, $L^k_\delta(\Omega)$, and $\|\cdot\|_{B^k}$ in $B^k$ the norms $\|\cdot\|_{L^{2+k}}$, $\|\cdot\|_{L^k}$, $\|\cdot\|_{L^k_\delta}$. Note that (2.3)-(2.5) can be written in one form
\[ \frac{1}{m} - \frac{1}{k} < \frac{1}{p'_c}, \] (2.6)
where $1/p'_c + 1/p_c = 1$. The optimal conditions of $L^\infty$-regularity in Theorem 1.1-1.3 (i) can also be written in one form
\[ \max\{\alpha, \beta\} > \frac{1}{p_c - 1}, \quad r, s, \gamma, \sigma < p_c, \] \[ \min\{p + r, q + s\} < p_c, \quad h \in B^\theta, \quad \theta > p'_c. \] (2.7)

We shall prove the following theorem.

**Theorem 2.4.** Assume that $f, g$ satisfy (1.4) with (2.4). Then there exists $C > 0$ such that for any $(H^1_0, \ L^1, \ L^1_\delta)$-solution $(u, v)$ of (1.1) satisfying
\[ \|u\|_{B^k} + \|v\|_{B^k} \leq M_1(k), \quad \text{for all } 1 \leq k < p_c, \] (2.8)
it follows that $u, v \in L^\infty(\Omega)$ and
\[ \|u\|_{L^\infty} + \|v\|_{L^\infty} \leq C. \]

The constant $C$ depends only on $M_1(k), \Omega, p, q, r, s, \gamma, \sigma, C_1$. 

Without loss of generality, we assume that \( q + s \geq p + r \). Then \( \beta \geq \alpha \). From (2.7), we have
\[
\beta > \frac{1}{p_c - 1},
\] (2.9)
and
\[
p + r < p_c.
\] (2.10)

**Remark 2.2.** If \( r \leq 1 \), (2.10) can be deduced by (2.9). In fact, we have
\[
p + r - 1 \leq \frac{pq - (1 - r)(1 - s)}{q + 1 - r} = \frac{1}{\beta} < p_c - 1.
\]

We first prove two lemmas, which assert that by bootstrap only on the first equation of system (1.1), the integrability of \( u \) can be improved to such an extent that the bootstrap on the second equation is possible. In the following, \( C = C(M_1, r, s, p, q, \gamma, \sigma, \Omega, C_1) \) is different from line to line, but it is independent of \( (u, v) \) satisfying (2.8). For simplicity, we denote by \( | \cdot |_k \) the norm \( \| \cdot \|_{B^k} \).

**Lemma 2.5.** Let \( f, g \) satisfy (1.4) with (2.7). If
\[
p < p_c/p'_c,
\] (2.11)
then \( |u|_\infty \leq C \).

**Proof.** We shall carry out the bootstrap only on the first equation of system (1.1) to prove \( |u|_\infty \leq C \).

**Case I.** \( r < 1 \).

Thanks to (2.7), (2.10) and (2.11) there exists \( k \) such that
\[
(p + r) \wedge \gamma < k < p_c, \quad \frac{p}{k} < \frac{1}{p'_c}.
\] (2.12)

For such \( k \) fixed, there exists \( \varepsilon > 0 \) small enough to satisfy
\[
\frac{\gamma}{k + m\varepsilon} - \frac{1}{k + (m + 1)\varepsilon} < \frac{1}{p'_c}, \quad \text{for any integer } m \geq 0,
\] (2.13)
and
\[
r < \frac{k}{k + \varepsilon},
\] (2.14)
since \( r < 1 \). From (2.12) and (2.14), we have
\[
\frac{r}{k + m\varepsilon} + \frac{p}{k} - \frac{1}{k + (m + 1)\varepsilon} < \frac{1}{p'_c}, \quad \text{for any integer } m \geq 0.
\] (2.15)

For \( m \geq 0 \), set
\[
\rho_m = \frac{r}{k + m\varepsilon} + \frac{p}{k} < 1, \quad \frac{\gamma}{k + m\varepsilon} < 1.
\]

From (2.12), when \( m \) is large enough, we have \( \rho_m \wedge \varepsilon_m > p'_c \). Denote \( m_0 = \min\{m : \rho_m \wedge \varepsilon_m > p'_c\} \). We claim that after \( m_0 \)-th bootstrap on the first equation, we arrive at the desired result \( |u|_\infty \leq C \).
According to (2.8), we have \(|u|_k \leq C, |v|_k \leq C\). If \(m_0 = 0\), we can take \(k\) such that \(p'_c < \rho_0 \cup \theta_0 = k/(p + r + \gamma) \leq \theta\) and (2.12) holds. Then applying Proposition 2.1-2.3 using the first equation of system (1.1), we obtain

\[
|u|_\infty \leq C|f|_{\rho_0 \cup \theta_0} \\
\leq C(||u||^p|_{\rho_0 \cup \theta_0} + ||u||^\gamma|_{\rho_0 \cup \theta_0}) + |h|_{\rho_0 \cup \theta_0} \\
\leq C(||u||^p|_{\rho_0} + ||u||^\gamma|_{\theta_0} + 1) \\
\leq C(|u|_m^p + |u|_m^\gamma + 1) \\
\leq C. \tag{2.16}
\]

Now we consider \(m_0 > 0\). If we have got the estimate \(|u|_{k+m_0} \leq C\) for some \(0 \leq m < m_0\), then applying Proposition 2.1-2.3 using (2.13), (2.15) and the first equation of system (1.1), we obtain

\[
|u|_{k+m_0} \leq C|f|_{\rho_m \cup \theta_m} \\
\leq C(||u||^p|_{\rho_m \cup \theta_m} + ||u||^\gamma|_{\rho_m \cup \theta_m}) + |h|_{\rho_m \cup \theta_m} \\
\leq C(||u||^p|_{\rho_m} + ||u||^\gamma|_{\theta_m} + 1) \\
\leq C(|u|_{k+m_0}^p + |u|_{k+m_0}^\gamma + 1) \\
\leq C. \tag{2.17}
\]

So we have \(|u|_{k+m_0} \leq C\). We can take \(m : m_0 - 1 < m \leq m_0\) such that \(p'_c < \rho_m \cup \theta_m \leq \theta\). A similar argument to (2.16) yields \(|u|_\infty \leq C\).

**Case II.** \(r \geq 1\).

Since \((p + r) \cup \gamma < p_c\), there exist

\[
k : (p + r) \cup \gamma < k < p_c, \\
\eta : \eta > 1, \text{ close enough to } 1,
\]

such that

\[
\frac{r}{k} + \frac{p}{k} - \frac{1}{\eta k} < \frac{1}{p'_c}, \\
\frac{\gamma}{k} - \frac{1}{\eta k} < \frac{1}{p'_c}
\]

from which we obtain

\[
\frac{r}{\eta^m k} + \frac{p}{k} - \frac{1}{\eta^{m+1} k} < \frac{2}{n + 1}, \\
\frac{\gamma}{\eta^m k} - \frac{1}{\eta^{m+1} k} < \frac{2}{n + 1}
\]

for any integer \(m \geq 0\). Similarly to the arguments of Case I, we also have \(|u|_\infty \leq C\).

The proof of the lemma is complete. \(\square\)

**Lemma 2.6.** Let \(f, g\) satisfy (1.4) with (2.7). If

\[
p \geq p_c/p'_c. \tag{2.18}
\]

Let \(k^* : p_c < k^* \leq \infty\) be the solution of

\[
\frac{r}{k^*} + \frac{p}{p_c} - \frac{1}{k^*} = \frac{1}{p'_c}. \tag{2.19}
\]
Then for any $1 \leq k_1 < k^*$, we have $|u|_{k_1} \leq C$.

**Proof.** According to (2.10) and (2.18), we necessarily have $r < 1$. We shall also carry out the bootstrap only on the first equation of system (1.1) to prove $|u|_{k_1} \leq C$. We first consider the case where $p > p_c/p'_c$. So $p_c < k^* < \infty$. For any $\varepsilon : 0 < \varepsilon \ll 1$, set $k_\varepsilon = k^* - \varepsilon$. Thanks to (2.10) and (2.19), we shall also carry out the bootstrap only on the first equation of system (1.1) to prove $|u|_{k_1} \leq C$. We shall also carry out the bootstrap only on the first equation of system (1.1) to prove $|u|_{k_1} \leq C$. We first consider the case where $p > p_c/p'_c$. So $p_c < k^* < \infty$. For any $\varepsilon : 0 < \varepsilon \ll 1$, set $k_\varepsilon = k^* - \varepsilon$. Thanks to (2.10) and (2.19), since $r < 1$, there exist

$$
k : (p + r) \lor \gamma < k < p_c,
$$

such that

$$
\frac{r}{k_\varepsilon} + \frac{p}{k} - \frac{1}{k_\varepsilon} < \frac{1}{p'_c},
$$

(2.20)

$$
\frac{r k_\varepsilon^\gamma}{k_\varepsilon} < \tau k,
$$

(2.21)

$$
\frac{\gamma}{k} - \frac{1}{k_\varepsilon} < \frac{1}{p'_c},
$$

(2.22)

where $k_\varepsilon^m = k_\varepsilon - \tau^m (k_\varepsilon - k)$ for $m \geq 0$. In fact, (2.20) is a small perturbation of (2.19) with respect to $k^*$ and, (2.21) is a small perturbation of itself with $\tau = 1$. A careful computation yields that

$$
\frac{r}{k_\varepsilon^m} - \frac{1}{k_\varepsilon^m+1} < \frac{r}{k_\varepsilon} - \frac{1}{k_\varepsilon}, \quad \text{for any integer } m \geq 0, \quad \text{(using (2.21))}
$$

$$
\frac{\gamma}{k_\varepsilon^m} - \frac{1}{k_\varepsilon^m+1} < \frac{\gamma}{k_\varepsilon^m-1} - \frac{1}{k_\varepsilon^m}, \quad \text{for any integer } m \geq 1. \quad \text{(using } \gamma \geq 1)\n$$

So, according to (2.20) and (2.22), we have

$$
\frac{r}{k_\varepsilon^m} + \frac{p}{k} - \frac{1}{k_\varepsilon^m+1} < \frac{1}{p'_c},
$$

(2.23)

$$
\frac{\gamma}{k_\varepsilon^m} - \frac{1}{k_\varepsilon^m+1} < \frac{1}{p'_c},
$$

(2.24)

for any integer $m \geq 0$.

Set

$$
\frac{1}{\rho_m} = \frac{r}{k_\varepsilon^m} + \frac{p}{k} < 1, \quad \frac{1}{\varrho_m} = \frac{\gamma}{k_\varepsilon^m} < 1.
$$

Note that

$$
\frac{1}{\varrho} = \frac{r}{k_\varepsilon} + \frac{p}{k} > \frac{r}{k^*} + \frac{p}{p_c} \geq \frac{1}{p'_c}.
$$

So $\rho_m \land \varrho_m < \varrho < p'_c$. Then $|h|_{\rho_m \land \varrho_m} \leq C |h|_\varrho \leq C$ for all $m \geq 0$.

We already have $|u|_k \leq C$, $|v|_k \leq C$ from (2.8). If we have got $|u|_{k_\varepsilon^m} \leq C$ for some $m \geq 0$, applying Proposition 2.1-2.3, using (2.23), (2.24) and the first equation of system (1.1), similarly to (2.17), we obtain $|u|_{k_\varepsilon^{m+1}} \leq C$. So, for any integer $m \geq 0$, there holds $|u|_{k_\varepsilon^m} \leq C$. Noting that $k_\varepsilon^m \to k_\varepsilon$ as $m \to \infty$, we prove the lemma for $p > p_c/p'_c$.

If $p = p_c/p'_c$, we have $k^* = \infty$. The above proof is also valid when $k_\varepsilon$ is replaced by any arbitrary large number. The proof is complete. □
Lemma 2.5 and 2.6 in hand, we can prove Theorem 2.4.

**Proof of Theorem 2.4**

**Case I.** $p < p_c/p_c'$.  
According to Lemma 2.5, $|u|_\infty \leq C$. Since $s, \sigma < p_c$, a simple bootstrap argument on the second equation yields that $|v|_\infty \leq C$.

**Case II.** $p = p_c/p_c'$.  
According to Lemma 2.6, $|u|_{k_1} \leq C$ for any $k_1 \geq 1$. Take $k_1$ large enough and $k : s \lor \sigma < k < p_c$ such that

$$\frac{q}{k_1} < \frac{1}{p_c}, \quad \frac{q}{k_1} + \frac{s}{k} < 1.$$  

Similarly to the proof of Lemma 2.5 we have $|v|_\infty \leq C$. So we also have $|u|_\infty \leq C$ since $r, \gamma < p_c$.

**Case III.** $p > p_c/p_c'$. In this case we necessarily have $r < 1$.

According to (2.9) and (2.19), there exist

- $k_1 : p_c < k_1 < k^*$, close enough to $k^*$,
- $k : (p + r) \lor \gamma \lor \sigma < k < p_c$, close enough to $p_c$,
- $\eta : \eta > 1$ close enough to 1,

such that

$$\frac{q}{k_1} + \frac{s}{k} < 1, \quad \frac{q}{k_1} + \frac{s}{k} - \frac{1}{\eta k} < \frac{1}{p_c'}, \quad \frac{q}{k_1} + \frac{s}{k} - \frac{1}{\eta k} + \frac{1}{\eta k_1} < \frac{1}{p_c'}.$$  

(2.25)  

In fact, (2.27) is equivalent to (2.25). (2.25) with $k_1 = k^*$ and $k = p_{BT}$ is exactly (2.9). So, (2.25) and (2.27) are just small perturbations of (2.9). (2.26) is a small perturbation of (2.19). Multiplying the LHS of (2.26)-(2.29) by $1/\eta^m$, we have

$$\frac{r}{\eta^m k_1} + \frac{p}{\eta^{m+1} k_1} - \frac{1}{\eta^{m+1} k_1} - \frac{1}{\eta^{m+1} k_1} < \frac{1}{p_c'}, \quad \frac{\gamma}{\eta^m k_1} - \frac{1}{\eta^{m+1} k_1} - \frac{1}{\eta^{m+1} k_1} < \frac{1}{p_c'}, \quad \frac{\sigma}{\eta^m k_1} - \frac{1}{\eta^{m+1} k_1} - \frac{1}{\eta^{m+1} k_1} < \frac{1}{p_c'}, \quad \frac{\rho}{\eta^m k_1} + \frac{s}{\eta^{m+1} k_1} < 1, \quad \frac{\rho}{\eta^m k_1} + \frac{s}{\eta^{m+1} k_1} < 1.$$  

(2.30)  

(2.31)  

for any integer $m \geq 0$.

Set

$$\begin{align*}
\frac{1}{\mu_m} &= \frac{r}{\eta^m k_1} + \frac{p}{\eta^{m+1} k_1} < 1, \quad \frac{1}{\nu_m} = \frac{\gamma}{\eta^m k_1} < 1, \\
\frac{1}{\rho_m} &= \frac{q}{\eta^m k_1} + \frac{s}{\eta^{m+1} k_1} < 1, \quad \frac{1}{\varphi_m} = \frac{\sigma}{\eta^m k_1} < 1.
\end{align*}$$  

(2.30)  

(2.31)
Since \( \eta > 1 \), for \( m \) large enough, we have \( \rho_m \land q_m > p'_c \) and \( \mu_m \land \nu_m > p'_c \). Denote \( m_0 = \min\{m : (\rho_m \land q_m) \lor (\mu_m \land \nu_m) > p'_c\} \). We may assume that \( \rho_{m_0} \land q_{m_0} > p'_c \). We claim that after \( m_0 \)-th alternate bootstrap on system (1.1), we shall arrive at the desired result \( |v|_\infty \leq C \).

We already have \( |u|_{k_1} \leq C \) (from Lemma 2.6) and \( |v|_k \leq C \) (from (2.8)). If \( m_0 = 0 \), we can take \( k, k_1 \) such that \( p'_c < \rho_0 \land q_0 \leq \theta \). Then applying Proposition 2.1-2.3 using the second equation of system (1.1), a similar argument to (2.17) yields that \( |v|_\infty \leq C \). So we also have \( |u|_\infty \leq C \) since \( r, \gamma < p_c \).

Now we consider \( m_0 > 0 \). If we have got the estimate \( |u|_{\eta^m k_1} + |v|_{\eta^m k} \leq C \) for some \( 0 \leq m < m_0 \), then applying Proposition 2.1-2.3 using (2.31) and the second equation of system (1.1), a similar argument to (2.17) yields that \( |v|_{\eta^{m+1} k} \leq C \). Then using (2.30) and the first equation of system (1.1), we obtain \( |u|_{\eta^{m+1} k} \leq C \). So we have \( |u|_{\eta^m k} + |v|_{\eta^m k} \leq C \). We can take \( m : m_0 - 1 < m \leq m_0 \) such that \( p'_c < \rho_m \land q_m \leq \theta \). A similar argument to (2.16) yields \( |v|_\infty \leq C \). So we also have \( |u|_\infty \leq C \) since \( r, \gamma < p_c \). The proof is complete.

\[ \square \]

Theorem 2.7 also holds if \( pq \leq (1 - r)(1 - s) \) in (1.4), we have the following theorem.

**Theorem 2.7.** Assume that \( f, g \) satisfy (1.4) with \( pq \leq (1 - r)(1 - s) \) and (2.7) where \( \max\{\alpha, \beta\} > 1/(p_c - 1) \) is replaced by \( pq - (1 - r)(1 - s) < (p_c - 1) \max\{p + 1 - s, q + 1 - r\} \). Then the conclusion of Theorem 2.4 holds.

**Proof.** Assume that \( q + s \geq q + r \). Note that

\[ \frac{q}{k^s} + \frac{s}{pc} < 1 \]

is equivalent to \( pq - (1 - r)(1 - s) < (p_c - 1) \max\{p + 1 - s, q + 1 - r\} \). So the proof is essentially word by word the same as the proof of Theorem 2.4 \( \square \)

### 3. \( L^\infty \)-regularity

In this section, we prove Theorem 1.1-1.3.

**Proof of Theorem 1.1.**

(i) If \( n = 1, 2 \), the \( L^\infty \)-regularity of \( H^1_0 \)-solutions follows directly from the Sobolev imbedding theorem and Proposition 2.1. If \( n \geq 3 \), since \( u, v \in H^1_0(\Omega) \), we have (2.8) from the Sobolev imbedding theorem. Then the \( L^\infty \)-regularity follows from Theorem 2.4 with \( p_c = (n + 2)/(n - 2) \) and \( B^1 = L^2(\Omega) \) according to (1.6).

(ii) Let \( (u, v) = (c_1|x|^{-2a} - c_1, c_2|x|^{-2\beta} - c_2) \), where \( c_1, c_2 \) are determined by \( c_1^{r-1}c_1^p = 2\alpha(n - 2 - 2\alpha) \), \( c_1^{s-1}c_2^p = 2\beta(n - 2 - 2\beta) \). Since \( \alpha, \beta < (n - 2)/4 < (n - 2)/2 \), we have \( c_1, c_2 > 0 \). Obviously,

\[ -\Delta u = 2c_1\alpha(n - 2 - 2\alpha)|x|^{-2a-2} = c_1^{r-1}|x|^{-2\alpha r-2\beta p}(u + c_1)^r(v + c_2)^p, \]

\[ -\Delta v = 2c_2\beta(n - 2 - 2\beta)|x|^{-2\beta-2} = c_2^{s-1}|x|^{-2\alpha q-2\beta s}(u + c_1)^q(v + c_2)^s. \]

It is easy to verify that \( (u, v) \) is an \( H^1_0 \)-solution of system (1.1) in \( B_1 \) with \( f = (u + c_1)^r(v + c_2)^p, g = (u + c_1)^q(v + c_2)^s. \) \( \square \)

**Proof of Theorem 1.2.**

(i) If \( n = 1, 2 \), the \( L^\infty \)-regularity of \( L^1 \)-solutions follows directly from Proposition 2.1. If \( n \geq 3 \), since \( f(\cdot, u, v), g(\cdot, u, v) \in L^1(\Omega) \), we have (2.8) from Proposition 2.1. Then the
$L^\infty$-regularity follows from Theorem 2.3 with $p_c = n/(n-2)$ and $B^1 = L^1(\Omega)$ according to (1.8). (ii) Since $\alpha, \beta < (n-2)/2$, $(u,v)$ constructed in the proof of Theorem 1.1 (ii) is also a $L^1$-solution of system (1.1) in $B_1$ with $f = (u + c_1)^p(v + c_2)^p, g = (u + c_1)^q(v + c_2)^q$. □

Proof of Theorem 1.3.
(i) If $n = 1$, the $L^\infty$-regularity of $L^1_\delta$-solutions follows directly from Proposition 2.3. If $n \geq 2$, we have (2.8) since $f(\cdot, u, v), g(\cdot, u, v) \in L^1_\delta(\Omega)$ from Proposition 2.3. Then the $L^\infty$-regularity follows from Theorem 2.4 with $p = n/(n-1)$ and according to [S, Lemma 5.1], the solution $L^1$-solutions follows directly from Proposition 2.3. If $\phi = |x|^{-2(\alpha+1)}1_\Sigma$ and $\psi = |x|^{-2(\beta+1)}1_\Sigma$, and $u, v > 0$ be the corresponding solutions of (2.2).

Proof of Theorem 1.4.
The proof is word by word the same as the proof of Theorem 1.1, 1.3 (i). □

4. A priori estimates of $L^1_\delta$-solutions and existence theorems

In order to prove Theorem 1.5, we recall a special property of the $L^1_\delta$-solutions, which is a consequence of Proposition 2.3, see [QS, Proposition 2.2, 2.3].

Proposition 4.1. Let $(u, v)$ be the $L^1_\delta$-solution of system (1.1) with $f, g$ satisfying (1.13) and let $1 \leq k < p_{BT}$. Then, $u, v \in L^k_\delta(\Omega)$ and satisfies the estimate \[ \|u\|_{L^k_\delta} + \|v\|_{L^k_\delta} \leq C(\Omega, k, C_2)(\|u\|_{L^k_\delta} + \|v\|_{L^k_\delta} + \|h_1\|_{L^k_\delta}). \]

Proof. The proof is similar to that of [QS, Proposition 2.2]. Let $\varphi_1(x)$ be the first eigenfunction of $-\Delta$ in $H^1_\delta(\Omega)$. Recall that \[ c_1\delta(x) \leq \varphi_1(x) \leq c_2\delta(x), \quad x \in \Omega, \]
for some $c_1, c_2 > 0$. We have

\[
\int_\Omega (|f| + |g|)\varphi_1 = \int_\Omega (|\Delta u| + |\Delta v|)\varphi_1 = 2\int_\Omega ((\Delta u)_+ + (\Delta v)_+)\varphi_1 - \int_\Omega \varphi_1(\Delta u + \Delta v) \\
\leq 2\int_\Omega (C_2(u_+ + v_+ + h_+))\varphi_1 + \lambda_1\int_\Omega (u + v)\varphi_1 \\
\leq C(\Omega, C_2)(\|u_+\|_{L^k_\delta} + \|v_+\|_{L^k_\delta} + \|h_+\|_{L^k_\delta}) \\
\leq C(\Omega, C_2)(\|u\|_{L^k_\delta} + \|v\|_{L^k_\delta} + \|h\|_{L^k_\delta}).
\]
Applying Proposition 2.3 with $m = 1$, we have
\[
\|u\|_{L_k^1} + \|v\|_{L_k^1} \leq C(\Omega, k, C_2)(\|u\|_{L_\delta^1} + \|v\|_{L_\delta^1} + \|h_1\|_{L_\delta^1}).
\]

**Proof of Theorem 1.5.**
Since $f, g$ satisfy (1.13), from Proposition 4.1, (2.8) can be deduced by (1.15). So this theorem follows immediately from Theorem 2.4 with $p_c = (n + 1)/(n - 1)$ and $B^1 = L_\delta^1(\Omega)$. 

From Theorem 1.5, in order to obtain the a priori estimate (1.14), we only have to obtain, for all $L_\delta^1$-solutions $(u, v)$ of system (1.1),
\[
\|u\|_{L_1^\delta} + \|v\|_{L_1^\delta} \leq M
\]
for some $M$ independent of $u, v$. In the following we give some propositions which assert the a priori estimate (1.14).

**Proposition 4.2.** [QS, Proposition 3.1] If $f, g$ satisfy (1.16) with $\lambda > \lambda_1$, then any nonnegative $L_\delta^1$-solution of system (1.1) satisfies (1.15) with $M$ independent of $u, v$.

**Proposition 4.3.** [QS, Proposition 3.2] If $f, g$ satisfy
\[
\begin{align*}
f &\geq C_1 u^r v^q - C_2 u, \\
g &\geq C_1 u^q v^s - C_2 v,
\end{align*}
\]
where $r, s < 1$, $pq > (1 - r)(1 - s)$. Then any nonnegative $L_\delta^1$-solution of system (1.1) in $H^1_0 \cap L^\infty$ satisfies (1.15) with $M$ independent of $u, v$.

Proposition 4.3 can be extended to some case where $r, s \geq 1$, see [QS, Proposition 3.5], see also [QS, Theorem 1.4 (ii), (iii)] for the precise assumptions.

The following proposition gives the uniform $L_\delta^1$-estimates of the $L_\delta^1$-solutions of system (1.18) where $r, s \leq 1$.

**Proposition 4.4.** Any nonnegative $L_\delta^1$-solution $(u, v)$ of system (1.18) satisfies (1.15) with $M$ independent of $u, v$.

**Proof.** We use the idea of [S, Proposition 4.1]. Denote $G(x, y), V(x, y)$ the Green functions in $\Omega$ for $-\Delta$ and $-\Delta + q(x)$. If $\inf\{\text{spec}(-\Delta + q)\} > 0$, by [Zhao, Theorem 8], there exists a positive constant $C = C(\Omega, q)$ such that
\[
\frac{1}{C}G(x, y) \leq V(x, y) \leq CG(x, y).
\]
By [BC, Lemma 3.2], we know that
\[
G(x, y) \geq C\delta(x)\delta(y) \quad \text{for } x, y \in \overline{\Omega}.
\]
So we also have
\[
V(x, y) \geq C\delta(x)\delta(y) \quad \text{for } x, y \in \overline{\Omega},
\]
for some constant $C > 0$. Denote $\varphi_q(x)$ the first eigenfunction of $-\Delta + q(x)$ in $H^1_0(\Omega)$ and $\lambda_q$ the first eigenvalue. Recall that
\[
c_1 \delta(x) \leq \varphi_q(x) \leq c_2 \delta(x), \quad x \in \Omega,
\]
for some $c_1, c_2 > 0$. Let $w$ be the solution of the linear equation
\[
-\Delta w + q(x)w = \phi(x), \quad x \in \Omega; \quad w = 0, \quad x \in \partial \Omega.
\]
If \( \phi \in L^1_\delta \) is nonnegative, then we have
\[
w = \int_\Omega V(x, y)\phi(x) \geq C(\int_\Omega \phi\delta)\delta \geq C(\int_\Omega \phi \varphi_q)\varphi_q
\]
with \( C \) depending only on \( \Omega, q(x) \). Let \( (u, v) \) be a nonnegative \( L^1_\delta \)-solution of (1.18). Set
\[
A = \int_\Omega a(x)u^r v^s \varphi_c, \quad B = \int_\Omega b(x)u^q v^s \varphi_d.
\]
Then we have
\[
u \geq CA \varphi_c, \quad v \geq CB \varphi_d.
\]
Therefore we obtain
\[
A \geq C \int_\Omega a \varphi_c^{r+1} \varphi_d^q A^r B^p \geq CA^r B^p, \quad (4.2)
\]
\[
B \geq C \int_\Omega b \varphi_c^{q+1} \varphi_d^q A^q B^s \geq CA^q B^s. \quad (4.3)
\]
If \( r = 1 \) or \( s = 1 \), \( A, B \leq C \) obviously. We consider \( r < 1 \). From (4.2), we have \( A^{1-r} \geq CB^p \). So combining with (4.3), we obtain \( B \geq CB^p/(1-r)+s \). Since \( pq > (1-r)(1+s) \), we have \( B \leq C \). From (4.3), we also have \( A \leq C \). Using \( \varphi_c \) as a testing function in the first equation of (1.18) and \( \varphi_d \) in the second equation, this yields that
\[
\int_\Omega u \varphi_c = \int_\Omega a(x)u^r v^s \varphi_c = A \leq C, \\
\int_\Omega v \varphi_d = \int_\Omega b(x)u^q v^s \varphi_d = B \leq C.
\]
The proof is complete. \( \square \)

Now we can prove our existence theorems. The proof is standard, see [QS]. For the readers’ convenience, we give the details.

**Proof of Theorem 1.6.**

(a) This is a direct consequence of Theorem 1.5 and Proposition 4.2.

(b) Let \( K \) be the positive cone in \( X := L^\infty(\Omega) \times L^\infty(\Omega) \) and let \( S : X \to X : (\phi, \psi) \mapsto (u, v) \) be the solution operator of the linear problem
\[
-\Delta u = \phi, \quad -\Delta v = \psi, \quad \text{in } \Omega, \\
u = v = 0, \quad \text{on } \partial \Omega.
\]
Since any nonnegative \( L^1_\delta \)-solution of (1.1) is in \( L^\infty \) by part (a), the system (1.1) is equivalent to the equation \( (u, v) = T(u, v) \), where \( T : X \to X \) is a compact operator defined by \( T(u, v) = S(f(\cdot, u, v), g(\cdot, u, v)) \). Let \( W \subset K \) be relatively open, \( Tz \neq z \) for \( z \in W \setminus W \), and let \( i_K(T, W) \) be the fixed point index of \( T \) with respect to \( W \) and \( K \) (see [AF] the definition and basic properties of this index).

If \( W_\varepsilon = \{(u, v) \in K : \|u, v\|_X < \varepsilon \} \) and \( \varepsilon > 0 \) is small enough, then (1.17) guarantees \( H_1(\mu, u, v) \neq (u, v) \) for any \( \mu \in [0, 1] \) and \( (u, v) \in W_\varepsilon \setminus W_\varepsilon \), where
\[
H_1(\mu, u, v) = \mu T(u, v) = S(\mu f(\cdot, u, v), \mu g(\cdot, u, v)).
\]
Consequently,
\[
i_K(T, W_\varepsilon) = i_K(H_1(1, \cdot, \cdot), W_\varepsilon) = i_K(H_1(0, \cdot, \cdot), W_\varepsilon) = i_K(0, W_\varepsilon) = 1.
\]
On the other hand, if $R > 0$ is large, then our a priori estimates guarantee $H_2(\mu, u, v) \neq (u, v)$ for any $\mu \in [0, C_1 + 1]$ and $(u, v) \in \overline{W_R} \setminus W_R$, where

$$H_2(\mu, u, v) = S(f(\cdot, u, v) + \mu, g(\cdot, u, v)).$$

Using $\varphi_1$ as a testing function we easily see that $H_2(C_1 + 1, u, v) = (u, v)$ does not possess nonnegative solutions, hence

$$i_K(T, W_R) = i_K(H_2(C_1 + 1, \cdot, \cdot), W_R) = 0.$$ 

Consequently, $i_K(T, W_R \setminus \overline{W_\varepsilon}) = -1$, which implies existence of a positive solution of (1.1). The proof is complete. □

**Proof of Theorem 1.7.**

(a) This is a direct consequence of Theorem 1.5 and Proposition 4.4.

(b) Let $K, X, W_\varepsilon$ be the same as in the proof of Theorem 1.6 (b), let $S$ be the solution operator of the linear problem

$$-\Delta u + c(x)u = \phi, \quad \text{in } \Omega,$$

$$-\Delta v + d(x)v = \psi, \quad \text{in } \Omega,$$

$$u = v = 0, \quad \text{on } \partial \Omega.$$ Let us show that $H_1(\mu, u, v) \neq (u, v)$ for any $\mu \in (0, 1]$ and $(u, v) \in \overline{W_\varepsilon} \setminus W_\varepsilon$ for $\varepsilon$ small. Assume by contrary $(u, v) \in \overline{W_\varepsilon} \setminus W_\varepsilon$, $H_1(\mu, u, v) = (u, v)$. Then $u \neq 0$, $v \neq 0$ and the standard $L^2$-estimates (with $z > n/2$) guarantee

$$\|u\|_\infty \leq C\|u\|_\infty^r \|v\|_\infty^p, \quad \|v\|_\infty \leq C\|u\|_\infty^q \|v\|_\infty^s.$$ Hence

$$\|u\|_{\infty}^{(1-r)(1-s)} \leq C\|u\|_{\infty}^{pq},$$

which contradicts $pq > (1 - r)(1 - s)$ if $\varepsilon$ is small enough.

On the other hand, if $R > 0$ is large, then our a priori estimates guarantee $H_2(\mu, u, v) \neq (u, v)$ for any $\mu \in [0, \lambda_c]$ and $(u, v) \in \overline{W_R} \setminus W_R$, where

$$H_2(\mu, u, v) = S(f(\cdot, u, v) + \mu(u + 1), g(\cdot, u, v)).$$

and $\lambda_c$ is the first eigenvalue of $-\Delta + c(x)$ in $H_0^1(\Omega)$. Using $\varphi_\varepsilon$ as a testing function we easily see that $H_2(\lambda_c, u, v) = (u, v)$ does not possess nonnegative solutions, hence

$$i_K(T, W_R) = i_K(H_2(\lambda_c, \cdot, \cdot), W_R) = 0.$$ Consequently, $i_K(T, W_R \setminus \overline{W_\varepsilon}) = -1$, which implies existence of a positive solution of (1.18). The proof is complete. □

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