MULTIDIMENSIONAL Λ-WRIGHT-FISHER PROCESSES WITH GENERAL FREQUENCY-DEPENDENT SELECTION

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Abstract. We construct a constant size population model allowing for general selective interactions and extreme reproductive events. It generalizes the idea of Krone and Neuhauser [37] who represented the selection by allowing individuals to sample potential parents in the previous generation before choosing the ‘strongest’ one, by allowing individuals to use any rule to choose their real parent. Via a large population limit, we obtain a generalisation of Λ-Fleming Viot processes allowing for non transitive interactions between types. We provide fixation properties, and give conditions for these processes to be realised as solutions of stochastic differential equations.

1. Introduction

Modelling selection falls within the most delicate problems in ecology and evolution. A variety of hypotheses have been proposed to describe how competing allelic types jostle against each other in trying to propagate successfully in the next generation [24, 27, 34, 35]. Despite the complexity of the debate on the concept of selection itself, in population genetics there is agreement on the idea that an appropriate measure of the fitness strength of a given allelic type is related to the probability of its eventual fixation in the population. This agreement relies on the fact that generally, in population genetics models, fitnesses are considered as transitive in the sense that if in some conditions an allele 1 has a higher fitness than an allele 0 and an allele 2 has a higher fitness than an allele 1, then the allele 2 will have a higher fitness than the allele 0. This assumption impedes the modelling of non transitive interactions, as for instance the well known Rock-Paper-Scissors (RPS) interaction, where any of the three alleles involved may rise in frequency depending on the frequencies of the two other alleles present in the population [17].

Eco-evolutionary models aim at taking into account feedbacks between ecological and evolutionary processes. In this setting, interactions are defined at the individual scale, which allows to model a much larger variety of interactions than in population genetics models (as RPS interactions for instance [9]). The drawback however, is that the variation of population size makes more complex the study. For instance, even when studying processes faster than the time scale of population size evolution, we have to call upon technical results, as large deviations results, to control the population size dynamics (see [13] for a classical example).

The aim of this paper is to construct a constant size population model allowing to take into account any interaction we could think of, as well as extreme reproductive events. This is achieved by a generalisation of a construction recently introduced in [28], based on the concept of ancestral selection graph introduced by Krone and Neuhauser [37]. The main idea of this construction is that individuals first sample a random number of potential parents from the previous generation, and then choose their parent in the previous generation, with a rule which depends on the number and types of sampled potential parents. Notice that the term ‘potential parents’ may be misleading as in the current work we allow the real parent not to be one of the potential parents, to model mutations. However, to make the link with [28] clear we kept their notations. Following [28], we also consider high-fecundity extreme reproductive events (Λ-events), where one individual may give birth to a number of offsprings.
of order $N$, the size of the population. After a proper rescaling of time, we derive a large population limit of our population model described by a generator, and which can be realised as the strong solution of a stochastic differential equation (SDE) under suitable conditions. This SDE generalizes classical Wright-Fisher diffusion with selection and $\Lambda$-Wright-Fisher processes to a multidimensional case with general frequency-dependent selection and jumps given by a $\Lambda$-measure. We prove general properties on fixation and extinction for this class of models, and apply them to classical ecological interactions, as RPS interactions or negative frequency-dependent selection for instance.

Notice that in a work conducted in parallel, Cordero and coauthors [18] also consider a generalisation of [28] to a general selection case. Their focus differs substantially from ours, as they consider the 1-dimensional setting, and concentrate on genealogies, duality properties of the frequency process with the ancestral process. Furthermore, they seek for genealogies that minimise the number of potential ancestors. We encourage interested readers to also have a look at [18].

The paper is structured as follows. In Section 2 we describe the discrete model. Section 3 is dedicated to the derivation of the large population limit as well as the statement of general results on the processes under consideration. In Section 4 we present applications of our construction to model a set of ecological interactions, as well as properties on the long time behaviour of these specific processes. Finally, the proofs are given in Section 5.

In the sequel, $\mathbb{N}^* := \{1, 2, ..., \}$ will denote the set of positive integers, $\mathbb{N} := \mathbb{N}^* \cup \{0\}$, $\mathbb{Z} := -\mathbb{N}^* \cup \mathbb{N}$, and for $N \in \mathbb{N}^*$, $|N| := \{1, 2, ..., N\}$. Finally, $|D|$ will denote the cardinal of a discrete set $D$, and for $K \in \mathbb{N}^*$ and $x \in [0, 1]^K$, $||x|| := x_1 + ... + x_K$.

2. The discrete model

This section is dedicated to the description of our class of Wright-Fisher type models, which are constant size population models, with size $N \in \mathbb{N}^*$, and with discrete non-overlapping generations. They are a generalisation of the ones introduced in [28]. The main novelty of our approach is that it includes the following features:

- We are in a multidimensional setting. We denote by $E = [K]$ the allelic type space, with $K \in \mathbb{N}^*$.
- To choose a real parent, knowing the set of potential parents, we will introduce a colouring rule, which may depend on $N$ and on all the characteristics of the set of potential parents (number and frequencies of the different types), and may be random.
- The real parent may not be one of the potential parents. This allows us to take into account mutations for instance.

Multitype models are widespread in the literature, both in population genetics (see for example [7, 21, 45, 46]) and in eco-evolutionary models (for instance [15, 17, 11, 9, 10]). There are also interesting instances of colouring (2-types) models in which individuals observe several potential parents and are coloured according to some rule depending on the observed sample [8, 18, 28] and questions regarding the existence and (lack of) uniqueness of a colouring rule that leads to a prescribed stochastic differential equation are addressed in [18]. The literature that accounts for mutations is vast [23, 22, 3]. However, as far as we are aware of, the family of models introduced in this paper constitutes the first class that integrates all the above points, and luckily the flexibility of this family does not compromise its simplicity.

We begin in Section 2.1 with the description of births in generations without extreme reproductive events, and describe in Section 2.2 extreme reproductive events. In Section 2.3
we introduce the multidimensional frequency process, which will be the process of interest in the rest of the paper.

2.1. Births without extreme reproductive events. In the sequel, \( g \in \mathbb{Z} \) will always denote a generation. For \( N \in \mathbb{N}^* \), we parametrize the strength of the selection via a probability distribution \( Q_N \) on \( \mathbb{N}^* \). It will correspond to a number of 'potential parents' sampled by an individual. In order to describe the dynamics of our discrete model, we now introduce a graph and the concept of colouring rule:

**Definition 2.1.** Let \( N \in \mathbb{N}^* \) and \( V := \mathbb{Z} \times [N] \). Consider a family of independent uniformly on \([N]\) distributed random variables

\[
(U_{v,k}, v \in V, k \in \mathbb{N}^*)
\]

and a family of independent \( Q_N \) distributed random variables

\[
(K_v, v \in V)
\]

with values in \( \mathbb{N}^* \). Let \( E \) be the set of directed edges

\[
E = \{ (v, (g - 1, U_{v,k})) \}, \text{ for all } v = (g, l) \in V \text{ such that } 1 \leq l \leq N \text{ and } k \leq K_v \}.
\]

The genealogical random di-graph with parameters \( N \) and \( Q_N \) is the random di-graph \((V, E)\).

The idea of this graph is that we assign types to all the vertices in some generation, and we want to see types propagating in the subsequent generations. To do this we need to specify how a vertex is coloured (receive its type), given the types of the vertices in the previous generation which are connected to it.

**Definition 2.2 (Colouring rule).** Let

\[
C = \{ (z_1, z_2, ..., z_n), n \in \mathbb{N}^*, z_i \in E, \forall i \in [n] \}
\]

be the possible sets of potential parents with their type that can be sampled by an individual. A \((N, E)\)-colouring rule is a family of probability distributions over \( E \)

\[
C_N = \{ c^N_z(\cdot) \}_{z \in C}
\]

such that for any \( z \in E \),

\[
c^N_z = \delta_z.
\]

Notice that the last condition will ensure that if an individual samples only one potential parent, it will be its real parent. We could not do this assumption but the notations would become cumbersome (however, see Remark \( \Box \)). Notice also that we allow the colouring rule to depend on \( N \), which leaves more freedom.

**Definition 2.3 (Graph colouring without extreme event).** Fix \( N \in \mathbb{N}^* \), \( Q_N \) a probability distribution on \( \mathbb{N}^* \), the space of types \( E \) and a \((N, E)\)-colouring rule \( C_N \). A colouring of the graph \((V, E)\) is a function \( f : V \rightarrow E \). Let \( V_g \) be the vertices in the \( g \)-th generation. We construct a colouring \( f \) recursively by first arbitrarily defining \( f \) in \( V_0 \) and then extending the colouring to the subsequent generations using the colouring rule. Given that the colouring has been constructed in generation \( g - 1 \), for any \( v \in V_g \), let

\[
z_v = (f(u_1), ..., f(u_{K_v}))
\]

where \( u_k \in V_{g-1} \) is such that \((v, u_k) = (v, U_{v,k}) \) for \( k \leq K_v \) (in other words, \( U_k \) is a potential parent of \( v \)). For every type \( i \in E \),

\[
\mathbb{P}(f(v) = i) = c^N_{z_v}(i).
\]
In the sequel, we will use the terms type and colour interchangeably.

Following the three previous definitions, the reproduction mechanism thus works as follows when there is no extreme reproductive event:

1. (Choice of potential parents). The individual \( v \) of generation \( g \) samples a number \( K_v \) distributed as \( Q_N \) of potential parents independently and uniformly distributed on \( (g - 1) \times [N], ((g - 1, U_{(v,1)}), \ldots, (g - 1, U_{(v,K_v)}) \). The choice is with replacement.

2. (Choice of type). The type of \( v \) is chosen according to the colouring rule, as described in Definition 2.3.

2.2. Births with extreme reproductive events. We now want to include the possibility of extreme reproductive events, that is to say events when one individual may generate a non-negligible fraction of the next generation. The motivation to consider such reproductive events comes from the observation that some species, in particular marine species \(^2 \)\(^{30} \) and most viruses, have offspring distributions with very large variance, and that the Kingman coalescent is not a good model to describe their genealogy \(^{31} \). It is instead necessary to include the possibility of merging more than two individuals simultaneously. Such kinds of models have seen a growing interest from the discovery of \( \Lambda \)-coalescent trees by Sagitov \(^{44} \) and their full description by Pitman \(^{43} \).

To this purpose, we introduce a random background formed by a sequence of i.i.d. Bernoulli trials

\[
H = \{H_g : g \in \mathbb{Z}\} \in \{0,1\}^\infty,
\]

with a probability of success \( \gamma_N \in [0,1] \) and a sequence

\[
Z = \{Z_g : g \in \mathbb{Z}\}
\]

of i.i.d. \([0,1]\)-valued random elements with common distribution \( \Lambda \). \( H \) and \( Z \) are assumed to be independent, and we also assume that \( \Lambda([0,1]) < \infty \). For every \( g \in \mathbb{Z}, H_g = 1 \) (resp., \( H_g = 0 \)) indicates that at generation \( g \) extreme reproduction does (resp., does not) occur. \( Z \) will give the expected sizes of extreme reproductive events, when they occur. Let

\[
Y^* = \{Y^*_g : g \in \mathbb{Z}\},
\]

be a sequence of i.i.d. random variables, with \( Y^*_{g-1} \) uniformly distributed among individuals in generation \( g - 1 \) and \( Y^* \) independent of \( (H,Z) \). Conditionally on \( Y^*_{g-1} \), we thus assume that when \( H_g = 1 \) individuals in generation \( v \) choose one parent independently with the law:

\[
\eta_g := B(Z_g)dy_{g-1} + (1 - B(Z_g))U_g,
\]

where \( B(Z_g) \) is distributed as a Bernoulli random variable with parameter \( Z_g \) and \( U_g \) is uniformly distributed on \([N]\). \( B(Z_g) \) and \( U_g \) are independent, and are drawn independently for each individual in generation \( g \).

**Definition 2.4** (Graph colouring with extreme event). Fix \( N \in \mathbb{N}^* \), \( Q_N \) a probability distribution on \( \mathbb{N}^* \), the space of types \( \mathcal{E} \) and a finite measure \( \Lambda \) on \([0,1]\). Given that the colouring has been constructed until generation \( g - 1 \), that there is an extreme reproductive event at generation \( g \), and denoting for \( i \in \mathcal{E} \) by

\[
|i/(g - 1)| := \text{Card}\{u \in V_{g-1}, f(u) = i\}
\]

the number of individuals with colour \( i \) in generation \( g - 1 \), we have the following colouring for every type \( i \in \mathcal{E} \) and \( v \in V_{g} \):

\[
P(f(v) = i|Z_g, Y^*_{g-1}) = Z_g1_{\{i,f(y^*_{g-1})\}} + (1 - Z_g)\frac{|i/(g - 1)|N}{N}.
\]

Following the previous definitions, the reproduction mechanism thus works as follows when there is an extreme reproductive event:
(1) (Choice of the individual with a large progeny) An individual \( Y_{g-1}^* \) is drawn uniformly among individuals of the generation \( g - 1 \). A variable \( Z_g \) is also drawn according to the distribution \( \Lambda \). It will give the size of the extreme reproductive event.

(2) (Choice to be or not in the extreme event). The individual \( v \) of generation \( g \) draws a Bernoulli variable \( B_v(Z_g) \) with parameter \( Z_g \).

(3) If \( B_v(Z_g) = 1 \), the parent of \( v \) is \( Y_{g-1}^* \).

(4) If \( B_v(Z_g) = 0 \), the individual \( v \) chooses its parent uniformly in generation \( g - 1 \).

2.3. Frequency processes. We have described in the two previous sections how individuals choose their parent in the previous generation. It allows us to construct a forward in time process, describing the dynamics of the composition of the population in terms of types. Fix \( N \in \mathbb{N}^* \), \( Q_N \) a probability distribution on \( \mathbb{N}^* \), \( \Lambda \) a finite measure on \([0, 1] \), \( \gamma_N \in [0, 1] \), and a colouring rule \( C_N \). Colour arbitrary the graph at generation 0. Then colour generations \( g \geq 1 \) as follows:

- Colour the graph following Definition 2.3 with colouring rule \( C_N \) and distribution of the number of potential parents \( Q_N \) when there is no extreme reproductive event \((H_g = 0 \text{ with probability } 1 - \gamma_N)\).
- Colour the graph following Definition 2.4 with \( Z \) \( \Lambda \)-distributed when there is an extreme reproductive event \((H_g = 1 \text{ with probability } \gamma_N)\).

Then the frequency process with parameters \((\Lambda, \gamma_N, Q_N, C_N)\) is by definition \( \bar{X}_g^N = (X_{g,i}^N, i \in E) \) such that

\[
X_{g,i}^N = \frac{1}{N} |\{v \in V_g : f(v) = i\}|.
\]

This process describes the genetic composition of the population at any generation. The rest of the paper is devoted to the study of this process.

3. General results

The models and subsequent frequency processes we have just introduced allow us to take into account any interaction we could think of. Indeed we have a total freedom on the colouring rule, and may even choose a parent which does not belong to the potential parents sampled by an individual. In this section, we will first present a rescaling of the population process, which generalizes the class of Wright-Fisher processes considered until now. We will then describe the class of selection functions that we can obtain with our colouring rules, and we will finally prove general results on allele extinction and fixation of our class of models.

3.1. Large population limit. We will now prove that by rescaling time and taking the large population limit \((N \to \infty)\), we may obtain a time and space continuous process. We focus on the case

\[
1 - Q_N(\{1\}) =: \rho_N \to 0, \quad (N \to \infty),
\]

and to get a proper limit process, we will rescale the time by a factor of order \( \rho_N \). Before stating this result we need to introduce the following notations, for any finite measure \( \Lambda \) on \([0, 1] \) and \( \alpha \in (0, 1/2)\):

\[
\hat{\Lambda} := \frac{\Lambda}{\Lambda([0, 1])}, \quad \Lambda_\alpha(z \in .) := \frac{\Lambda(z \in .)}{z^2} \mathbf{1}_{\{z \geq N^{-\alpha}\}} \quad \text{and} \quad \hat{\Lambda}_\alpha^N := \frac{\Lambda_\alpha^N}{\Lambda_N^\alpha([0, 1])}.
\]

Let us also denote for \( K \in \mathbb{N}^* \) the \( K \)-th simplex

\[
\Delta_K := \{x \in [0, 1]^K, \|x\| \leq 1\}.
\]

Moreover, for \( N \in \mathbb{N}^* \), given the colouring rule \( C_N \) (recall Definition 2.2), consider the function \( p^N : \Delta_K \mapsto \mathbb{R} \), \( p^N = (p_1^N, ..., p_K^N) \) such that

\[
p_i^N(x) = \mathbb{P}(f(v) = i | v \in V_1, X_0^N = x), \quad \forall x \in \Delta_K, i \in E
\]
and \( \mu^N : \Delta_K \to \mathbb{R} \), \( \mu^N = (\mu^N_1, ..., \mu^N_K) \) such that
\[
(3.3) \quad \mu^N_i(x) = p^N_i(x) - x_i, \quad \forall \ x \in \Delta_K, i \in E.
\]

Then we have the following convergence result:

**Proposition 3.1.** Let \( \Lambda \) be a finite measure on \([0, 1]\), \((C_N, N \in \mathbb{N})\) a sequence of colouring rules and \( \alpha \) in \((0, 1/2)\). Consider for \( N \in \mathbb{N}^* \) \((\tilde{X}_N^t, g \in \mathbb{Z})\) the frequency process with parameters \((\tilde{\Lambda}_N, \gamma_N, Q_N, C_N)\). Assume that there exist \( \kappa > 0 \) and \( \sigma \geq 0 \) such that

(i) \( \lim_{N \to \infty} \rho_N^{-1} \mu^N_i(x) = \mu_i(x) \in \mathbb{R}, \forall i \in E, x \in \Delta_K \),

(ii) \( \gamma_N = \Lambda_N^{\alpha}([0, 1]) \rho_N / \kappa + o(\Lambda_N^{\alpha}([0, 1]) \rho_N) \),

(iii) \( \lim_{N \to \infty} 1/(N\rho_N) = \sigma/\kappa < \infty \) and \( \rho_N N^{2\alpha} \to 0 \),

(iv) \( \lim_{N \to \infty} Q_N(K_{(0, 1)} = k|K_{(0, 1)} > 1) = \pi_{k-1} \) for every \( k \geq 2 \),

(v) \( \beta := \lim_{N \to \infty} \mathbb{E}[K_{(0, 1)} - 1|K_{(0, 1)} > 1] = \sum_{k=1}^{\infty} k \pi_k < \infty \).

Then, the sequence \((X^N_{\kappa t/\rho_N})^t \geq 0 \) is tight. Further, if we denote by \( \mathcal{A}^N \) the discrete generator of \( X^N_{\kappa t/\rho_N} \), the sequence of generators \( \mathcal{A}^N \), applied on \( f \in \mathcal{C}_2(\Delta_K) \to \mathbb{R} \) and \( x \in \Delta_K \), converges to \( \mathcal{A} \) given by
\[
(3.4) \quad \mathcal{A} f(x) = \sum_{i=1}^{K} \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \sigma \sum_{i,j=1}^{K} \sigma_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{K} x_i \int_{0}^{1} [f((1-z)x + z e_i) - f(x)] \frac{\Lambda(dz)}{z^2},
\]

where \( \sigma_{ij}(x) = (1_{j=i} - x_j) x_i \) and \( (e_i, \ 1 \leq i \leq K) \) is the canonical basis of \( E \).

The first term of the generator describes a general frequency-dependent selective function, the second term is the classical Wright-Fisher diffusion, and the last part corresponds to the generator of a \( K \)-dimensional \( \Lambda \)-Fleming-Viot process. Such a general multidimensional process including these three elements has not been neither introduced nor studied until now to the best of our knowledge.

**Remark 1.** Notice that if we do not assume that when only one potential parent is sampled it is necessarily the real parent (see the end of Definition 2.2), we can obtain a different diffusion term. More precisely, if we introduce for \( i \in E \) and \( x \in \Delta_K \),
\[
g_i(x) := \sum_{j=1}^{K} x_j g^N_j(i),
\]

and if \( G(x) \) is a random variable with values in \( \Delta_K \), and law given by
\[
\mathbb{P}(G(x) = Y) = \begin{pmatrix} N \\ Y_1, ..., Y_K \end{pmatrix} g^N_1(x)...g^K_N(x),
\]

then for \( (i, j) \in E^2 \) and if the limit exists,
\[
\sigma \sigma_{ij}(x) = \lim_{N \to \infty} \frac{\kappa}{\rho_N} \mathbb{E} \left[ \left( \frac{G_i(x)}{N} - x_i \right) \left( \frac{G_j(x)}{N} - x_j \right) \right].
\]

Under some assumptions on the parameters (see Corollary 3.1 for examples) we can link the infinitesimal generator \( \mathcal{A} \) to the solutions to a stochastic differential equation (SDE) generalizing the one dimensional case derived in [28].

**Proposition 3.2.** Assume that the following SDE is well posed
\[
dX(t) = \mu(X(t))dt + \sqrt{\sigma^2(X(t))}dB_t
\]
\[
(3.5) \quad + \sum_{i=1}^{K} e_i \int_{0}^{1} \int_{0}^{1} z \left( I_{\{\sum_{j=1}^{i-1} X_j(t^-) \leq u < \sum_{j=1}^{i} X_j(t^-)\}} - X_i(t^-) \right) \tilde{N}(dt, du, dz),
\]
where $B$ is a $K$-dimensional Brownian motion, $(e_i, i \leq K)$ is the canonical basis of $E$, $\tilde{N}$ is a compensated Poisson random measure on $\mathbb{R}_+ \times [0, 1] \times [0, 1]$ with intensity $dtdu\Lambda(dz)/z^2$, and $\zeta$ has the following expression for $x \in \Delta^K$:

\begin{align*}
\zeta_{ij}(x) &= 0 & \text{if } i < j, \\
\zeta_{ii}(x) &= \sqrt{x_i(1-x_1-\ldots-x_i)} \\
\zeta_{ij}(x) &= -x_i \sqrt{x_j(1-x_1-\ldots-x_j)(1-x_1-\ldots-x_{j-1}-x_j)} & \text{if } i > j.
\end{align*}

Then every solution of (3.5) admits the generator in Equation (3.4).

Theorem 1. Let us assume that hypotheses of Proposition 3.1 hold and there exists a unique (in law) solution to (3.5) that we denote $(X(t), t \geq 0)$. Then the frequency process $(X^N_{\lfloor \kappa t/\rho N \rfloor}, t \geq 0)$ converges in law to the space and time continuous process $(X(t), t \geq 0)$, and $(X(t), t \geq 0)$ admits $A$ defined in (3.4) as infinitesimal generator.

To our knowledge, similar results have only been derived in the one dimensional case in [28] and recently in [18]. In this special case (where there are only two alleles) we can ensure the existence of a unique strong solution for the SDE under consideration. We can also provide this property in the multidimensional setting when specific assumptions are fulfilled. In general, the applicability of Theorem 1 depends on the existence and uniqueness of a solution to the SDE under consideration. There had been a lot of recent developments in proving existence and uniqueness of multidimensional SDEs (see for example [5, 38, 39, 40, 49]). The next corollary presents some sufficient conditions.

Corollary 3.1. If the hypotheses of Proposition 3.1 hold, and at least one of the following three statements is true

(1) $K = 2$
(2) $\sigma \equiv 0$ and $\int_0^\infty \Lambda(dy)/y < \infty$
(3) Selection is transitive (See Section 4.1 below)

Then, using the notation of Theorem 1

$$(X^N_{\lfloor \kappa t/\rho N \rfloor}, t \geq 0) \Rightarrow (X(t), t \geq 0),$$

where $(X(t), t \geq 0)$ is the pathwise unique strong solution to the SDE (3.5) and admits $A$, defined in (3.4) as its infinitesimal generator.

The assumption $\sigma \equiv 0$ in point (2) of the previous result comes from the fact that the diffusion coefficient of (3.5) is not Lipschitz on $\Delta_K$ if $\sigma > 0$. However, as long as the solution to (3.5) is not too close to the boundaries of $\Delta_K$ this problem does not arise. Hence, if we introduce the stopping times, for any $0 < \varepsilon < 1/2$: $T(\varepsilon, X) := \inf \left\{ t \geq 0, \min_{1 \leq i \leq K} X_i(t) \leq \varepsilon \right\}$,

we can prove the following result:

Lemma 3.1. Let $0 < \varepsilon < 1/2$ and assume that

$$\int_0^\infty \Lambda(dy)/y < \infty.$$

Then the SDE (3.5) admits a pathwise unique strong solution $X$ on the time interval $[0, T(\varepsilon, X))$. 
3.2. Selection functions. This section generalizes the ideas of [18] to the multidimensional setting. As we have a large degree of freedom in the choice of the colouring rule, the class of selection functions we can obtain with this construction is very general. It is the content of the next lemma:

**Lemma 3.2.** Let $N \in \mathbb{N}^*$ and $\mu : \Delta_K \to \mathbb{R}^K$ be a continuous function such that there exists $\lambda > 0$, with the property that

$$\frac{\mu(x)}{\lambda} + x \in \Delta_K, \quad \forall x \in \Delta_K.$$ 

Assume also that there is a unique (in law) solution to (3.5). Then there exists a sequence of couples of distribution and colouring rule $(Q_N, C_N)$ such that the sequence of frequency processes $X^N_{\lfloor \kappa t/\rho_N \rfloor}$ converges to $(X(t), t \geq 0)$ which has infinitesimal generator $A$ given by (3.4).

The previous Lemma has a weakness: it does not provide a colouring rule independent of $N$ leading to $\mu$ as a limit selection function. This is done in the next lemma, but with a less general class of functions:

**Lemma 3.3.** Let $\mu : \Delta_K \to \mathbb{R}^K$ be a continuous function such that there exists $\lambda > 0$, with the property that

$$g(x) := \frac{\mu(x)}{\lambda} + x \in \Delta_K, \quad \forall x \in \Delta_K,$$ 

and $g$ is a polynomial function of degree $n$. Let for each $i \in E$

$$g_i(x) = \sum_z \binom{n}{z} \alpha_i^z \prod_{r=1}^k x_r^z$$

be the representation of $g_i$ in the Bernstein basis. Then the choices

$$Q_N = (1 - \rho_N) \delta_1 + \rho_N \delta_n$$

and

$$c_i^N = \alpha_i^1$$

for every $N$-dimensional vector $z$ and $c_i^n = \delta_z$ for every one dimensional vector $z$ lead to the same conclusions as in the previous lemma.

3.3. On the alleles extinction and fixation. The last part of Section 3 concerns the properties of extinction and fixation of the frequency process $(X(t), t \geq 0)$. The first result provides a sufficient condition for one of the alleles to reach fixation in a finite time. Before stating it, we introduce a parameter which depends on the mean number of potential parents $\beta$ (defined in Proposition 3.1) and on the measure $\Lambda$:

$$(3.7) \quad \kappa^* := \frac{1}{\beta} \int_0^1 \log(1 - y) \Lambda(dy) \frac{\Lambda(dy)}{y^2}.$$ 

**Proposition 3.3.** Assume that there are no mutations (the real parent is one of the potential parents) and the infinitesimal generator of the frequency process $(X(t), t \geq 0)$ is given by (3.4). Then there is almost sure fixation of one of the alleles in finite time if at least one of the two following conditions is satisfied:

(i) $\sigma > 0$

(ii) $\kappa < \kappa^* < \infty$

**Remark 2.** Following Definitions 2.1 and 2.3, the assumption 'there are no mutations' can be written rigorously:

$$\mathbb{P}(f(v) = i) = 0 \quad \text{if} \quad f(U_{v,k}) \neq i \quad \forall \ (v, g, k) \quad \text{such that} \quad \{v, (g - 1, U_{v,k})\} \in \mathcal{E}.$$
We end this section with a result on the way alleles get extinct in the case of a Wright-Fisher diffusion (without jumps). This is an extension of a recent result by Coron and coauthors who proved this result in the neutral case \cite{10}.

**Proposition 3.4.** Assume that the SDE (3.5) is well posed and note \((X(t), t \geq 0)\) a solution. Suppose that \(\Lambda \equiv 0\), \(\sigma > 0\) and that \(\mu\) satisfies

\[
\mu_i(x) = x_i (1 - x_i) s_i(x), \quad \text{for all } x \in \Delta_K
\]

with \(s\) continuous on \(\Delta_K\). Then one of the alleles ultimately fixes, and before its fixation, the population experiences successive (and non simultaneous) allele extinctions.

**Remark 3.** Notice that as soon as we impose that the real parent is among the potential parents (no mutations), \(\mu_i(x)\) will be of the form \(\mu_i(x) = x_i \tilde{s}_i(x)\), with \(\tilde{s}\) bounded. Indeed, by definition, for \(i \in E, N \in \mathbb{N}^*\) and \(x \in \Delta_K\),

\[
\mu_i^N(x) = (Q_N(1) - 1) x_i + \sum_{k=2}^{\infty} Q_N(k) \mathbb{P}(f(v) = i|v \in V_1, X_0^N = x, K_v = k).
\]

But, as a type \(i\) individual has to be sampled among the potential parents for \(v\) to be of type \(i\), we get

\[
\mathbb{P}(f(v) = i|v \in V_1, X_0^N = x, K_v = k) \leq 1 - (1 - x_i)^k = x_i \left(1 + (1 - x_i) + \ldots + (1 - x_i)^{k-1}\right).
\]

Hence in the absence of mutation, to check the conditions of Proposition 3.4 on \(\mu\), we will just have to prove that \(\mu_i(x)/(1 - x_i)\) is bounded.

The next section is dedicated to examples of selection functions we can consider with our models. All of them satisfy the assumption of Proposition 3.4.

**Lemma 3.4.** If \(\mu\) is of the form (4.2), (4.4), (4.5), (4.6) or (4.7), \(\Lambda \equiv 0\) and the SDE (3.5) is well posed, then the assumptions of Proposition 3.4 are satisfied.

4. Applications

In this section, we illustrate the generality of our construction by applying it to model various ecological interactions. We also provide, when the calculations are not too cumbersome, the selective function obtained as a limit.

4.1. Transitive ordering. It is the classical assumption in population genetics models, consisting in ordering transitively the fitnesses of the different alleles. In this scheme, 1 has a smaller fitness than \(\{2, \ldots, K\}\), 2 has a larger fitness than 1, but a smaller fitness than \(\{3, \ldots, K\}\), and so on. Thus the colouring rule consists in taking one parent among the potential parents with the highest type, \(c_z(.) = \delta_{\text{sup}_z}\), where \(\text{sup}_z\) is the supreme norm of the vector \(z \in E\).

If we consider only two alleles and that the number of potential parents is one or two, we find as a limit the classical Wright-Fisher process with selection (plus jumps), where the frequency \((Y_t, t \geq 0)\) of type 1 individuals is the unique strong solution to:

\[
dY_t = -\kappa Y_t (1 - Y_t) dt + \sqrt{\sigma Y_t (1 - Y_t)} dW_t + \int_0^1 f \left(1_{\{u < Y_t\}} - Y_t\right) \tilde{M}(dt, du, dz)
\]

where \(\tilde{M}\) is a compensated Poisson measure with intensity \(dtdud\Lambda(dz)/z^2\).

If we allow the number of potential parents to exceed two, but still with only two alleles, it amounts to the model described and studied in \cite{28}, where the selective function is

\[
\mu_1(x) = x_1 x_1, x_1, 1 - x_1 = \sum_{k=1}^{\infty} \pi_k \left(x_1^{k+1} - x_1\right),
\]
where we recall that \( \pi \) has been introduced in Proposition 3.1.

Finally, if we consider a finite number of alleles larger than two, we obtain the following selective function for \( i \in E \):

\[
\mu_i(x) = \kappa \sum_{k=1}^{\infty} \pi_k \left( (x_0 + \ldots + x_i)^{k+1} - (x_0 + \ldots + x_{i-1})^{k+1} - x_i \right).
\]

This can be deduced from (4.1). Indeed for any \( i \in E \), if we divide the alleles in two groups: 1 to \( i \) and \( i+1 \) to \( K \), the relative frequencies of the two groups evolve as if there were only two types in the population. Doing the same with the groups 1 to \( i+1 \) and \( i+2 \) to \( K \), and so on, allows us to conclude.

By extending results in [28] we can provide the probability of fixation of the different alleles. In [28] the authors prove moment duality between the frequency process of the weakest allele and a process \((D_t, t \geq 0)\) named the ancestral process and corresponding to the limit of the number of potential ancestors of a sample taken at a given generation. The ancestral process has the following generator:

\[
L_f(n) = \kappa \sum_{i=0}^{\infty} \pi_i [f(n+i-1) - f(n)] + \frac{\sigma}{2} [f(n-1) - f(n)] + \sum_{k=2}^{n} \binom{n}{k} \lambda_{nk} [f(n-k+1) - f(n)],
\]

for every \( n \in \mathbb{N} \) and \( f : \mathbb{N} \rightarrow \mathbb{R} \) twice continuously differentiable, where we recall that \( \pi \) has been defined in Proposition 3.1, and for \( 2 \leq k \leq n \), \( \lambda_{nk} \) is given by:

\[
\lambda_{nk} = \int_0^1 y^k (1-y)^{n-k} \Lambda(dy) y^2.
\]

It appears that the behaviour of \((D_t, t \geq 0)\) provides the long time equilibrium of the frequency process \((X(t), t \geq 0)\). Recall the definition of \( \kappa^* \) in (3.7). Then we have the two following possible long time behaviours:

**Lemma 4.1.** Let us assume that the hypotheses of Proposition 3.1 are satisfied and that \( \mu \) is given by (4.2). Then

(i) If \( \kappa < \kappa^* < \infty \), \((D_t, t \geq 0)\) has a unique stationary distribution \( \nu \) and for \( i \in E \) and \( x \in \Delta_K \),

\[
P_x \left( \lim_{t \to \infty} X_i(t) = 1 \right) = \phi_\nu(x_0 + \ldots + x_i) - \phi_\nu(x_0 + \ldots + x_{i-1}),
\]

where \( \phi_\nu \) is the probability generating function of \( \nu \).

(ii) If \( \kappa \geq \kappa^* \), and if we denote by \( S \) the maximal label of alleles present at the beginning:

\[ S := \sup \{ i \in E, x_i > 0 \}, \]

we get

\[
P_x \left( \lim_{t \to \infty} X_S(t) = 1 \right) = 1.
\]

This result says that if the selection function is not strong enough (\( \kappa \) small) or if the variance generated by the choice of parents in the previous generation (\( \sigma > 0 \)) or extreme reproductive events is large enough, the latter can override the selection, and a deleterious allele may fixate.

**Remark 4.** In the case of transitive fitnesses (\( \mu \) given by (4.2)), we could consider much more general extreme reproductive events (more precisely \( \Xi \) reproductive events, see [28]) and we would still obtain the point (3) of Corollary 3.1 and Lemma 4.1. We do not provide details here for the sake of readability.
4.2. **Transitive ordering with mutations.** We can still consider that the highest type is the best when choosing among potential parents, but add mutations with some probability. As a consequence, the offspring can carry a type different from the types of all its potential parents. Notice that this colouring rule is not deterministic. The form of the mutation probability may depend on what we want to model (Muller’s ratchet, fixation of a beneficial mutant, ...).

4.3. **Logistic competition.** We want to model competitive interactions, which can depend on the types of the competing individuals. These kinds of competitive interactions have been widely studied by ecologists and in the setting of varying size stochastic population models, as they allow to model for instance the preference for different types of resources or space depending on the individual’s type, and lead to non-monotonic dynamics (see [41, 15] for instance). To model these interactions, we assume that $K_v$ is only supported on $\{1, 2\}$, and that the assumptions of Proposition 3.1 are satisfied. When $K_v = 1$ the real parent is the potential parent sampled uniformly at random in the previous generation by $v$. When $K_v = 2$ and the two potential parents have the same type, the real parent is chosen among them with probability $1/2$ for each. Now assume that they have different types that we denote by $i$ and $j$. Then the potential parent of type $i$ transmits its type with probability $p_{ij}$, and the potential parent of type $j$ transmits its type with probability $p_{ji}$, with $p_{ij} + p_{ji} = 1$. We thus get

$$
\mu_i(x) = \kappa(x_i^2 + 2x_i \sum_{j \neq i} p_{ij} x_j - x_i) = \kappa x_i + 2 \sum_{j \neq i} (1 - p_{ji}) x_j - 1
$$

(4.4)

$$
= \kappa x_i [1 - x_i + 2 \sum_{j \neq i} p_{ji} x_j - 1] = \kappa x_i [1 - x_i - 2 \sum_{j \neq i} p_{ji} x_j].
$$

We thus obtain a competitive Lotka-Volterra like function, whose competition coefficients are given by the probabilities $p_{ij}$. Notice that the intracompetition coefficient (one in the present case) is constrained by the fact that the frequency of type $i$ cannot exceed 1, and equals 1 when the population is monomorphic.

4.4. **Rock-Paper-Scissors.** RPS is a children’s game where rock beats scissors, which beat paper, which in turn beats rock. Such competitive interactions between morphs or species in nature can lead to cyclical dynamics, and have been documented in various ecological systems [11, 48, 47, 33, 36, 12, 42]. Let us describe two examples of such cycles. The first one [47] is concerned with pattern of sexual selection in some male lizards. Males are associated to their throat colours, which have three morphs. Type 1 individuals (orange throat) are polygamous and very aggressive. They control a large territory. Type 2 individuals (dark-blue throat) are monogamous. They control a smaller territory. Finally type 3 individuals (prominent yellow stripes on the throat, similar to receptive females) do not engage in female-guarding behaviour but roam around in search of sneaky matings. As a consequence of these different strategies, the type 1 outcompetes the type 2 (because males are more aggressive), which outcompetes the type 3 (as males of type 2 are able to control their small territory and guard their female), which in turn outcompetes the type 1 (as males of type 3 are not very efficient in defending their territory, having to split their efforts on several females). The second example [36] is concerned with the interactions between three strains of Escherichia coli bacteria. Type 1 individuals release toxic colicin and produce an immunity protein. Type 2 individuals produce the immunity protein only. Type 3 individuals produce neither toxin nor immunity. Then type 1 is defeated by type 2 (because of the cost of toxic colicin production), which is defeated by type 3, (because of the cost of immunity protein production), which in turn is defeated by type 1 (not protected against toxic colicin).
To model such interactions in our setting, we assume as in the previous example, that $K_v$ is supported on $\{1,2\}$, and we introduce the partial order $1 < 2$, $2 < 3$, $3 < 1$. When two potential parents with different types are sampled as potential parents, they compete and the ‘strongest’ transmits its type. We thus define the colouring rule $c^N_2(.) = \delta_{\text{max}z_1,z_2}$, and obtain as limit selective function, for $i \in \mathcal{E}$ and $x \in \Delta_K$,

$$
\mu_i(x) = \kappa x_i (x_{\text{mod}_3(i-1)} - x_{\text{mod}_3(i+1)}),
$$

where $\text{mod}_3(i)$ is the rest of the division of $i$ by 3.

The next result tells that either there is fixation in the RPS selection case, or at least the product of allele frequencies is closer and closer to 0 when time increases.

**Lemma 4.2.** Let us consider the process $(X(t),t \geq 0)$ with infinitesimal generator $A$ defined in (3.4) and $\mu$ given by (4.5), that is to say the RPS selection. Then

$$E[\ln (X_1(t)X_2(t)X_3(t))] \sim -\infty, \quad t \to \infty$$

if and only if $\sigma \neq 0$ or $\Lambda \neq 0$.

**Remark 5.** Notice that if $\sigma$ and $\kappa$ satisfy assumptions of Proposition 3.3, we know that one of the three alleles gets fixed in finite time.

4.5. Food web. The previous example can be generalized to any number of types and any partial order. In particular, it is well suited to represent food webs. A possibility could be that $i < j$ indicates that $i$ is eaten by $j$, and in case $i$ and $j$ do not eat each other $i = j$ would indicate that we take $p_{ij} = p_{ji} = 1/2$ in the example of logistic competition.

4.6. Negative frequency-dependent selection. Negative frequency-dependent selection is a common form of selection in nature. It refers to the fact that it is more advantageous for an individual to be of a type in minority in the population. In the case of the orchid Dactylorhiza sambucina for instance, there is a negative frequency-dependent selection on colours due to behavioural responses of pollinators to lack of reward availability [26]. In other populations, such a selection can be due to the fact that some predators concentrate on common varieties of prey and overlook rare ones [1].

To represent this form of selection, we sample $K_v$ potential parents as usually, and choose the real parent uniformly at random among the potential parents with the less frequent type. If there are several ‘less frequent types’ we choose one of them uniformly at random. Notice that to represent such a form of selection, it is necessary to sample more than 2 potential parents with a positive probability. Otherwise it would result in no selection ($\mu \equiv 0$).

For the sake of simplicity, to compute $\mu$, we will assume that $K_v$ is supported on $\{1,3\}$, but we could consider very general situations. In this case, if there are 3 potential parents, for a type $i$ parent to be chosen there are three possibilities. Either, all the potential parents are of type $i$, or only one parent is of type $i$ and the two other parents of type $j$ different from $i$, or the three parents are of different types, $i$, $j$ and $k$. Summing the probabilities of these three events, we get

$$
\mu_i(x) = \kappa \left[ x_i^3 + 3x_i \left( \sum_{j \neq i} x_j^2 + \frac{1}{3} \sum_{j \neq k,j,k \neq i} x_j x_k \right) - x_i \right]
$$

$$
(4.6) \quad = \kappa x_i \left[ x_i^2 - 1 + 3 \left( \frac{2}{3} \sum_{j \neq i} x_j^2 + \frac{1}{3} (1 - x_i)^2 \right) \right] = 2\kappa x_i \left[ \sum_{j \neq i} x_j^2 - x_i (1 - x_i) \right].
$$
4.7. Positive frequency-dependent selection. Positive frequency-dependent selection refers to the fact that it is more advantageous for an individual to be of a type in majority in the population. This is for instance the fact for warning signal in butterflies, indicating that one individual is poisonous for predators [16].

The modelling of this type of selection is similar in spirit to the previous one. Among the potential parents, we choose uniformly a parent with the most frequent type. If there are several ‘most frequent types’ we choose one of them uniformly at random.

For an easy computation, we make the same assumptions on \( K \) as in the previous example.

In this case, if there are 3 potential parents, for a type \( i \) parent to be chosen there are three possibilities. Either, all the potential parents are of type \( i \), or two parents are of type \( i \) and the other one of type \( j \) different from \( i \), or the three parents are of different types, \( i \), \( j \) and \( k \). Summing the probabilities of these three events, we get

\[
\mu_i(x) = \kappa x_i \left[ x_i^2 - 1 + 3x_i(1 - x_i) + \sum_{j \neq k, j, k \neq i} x_j x_k \right]
\]

(4.7)

The remainder of the paper is dedicated to the proofs.

5. Proofs

5.1. Proofs of Section 3. Before focusing on the convergence of the frequency process, we will first prove Proposition 3.2 which states that the infinitesimal generator \( A \) defined in (3.4) caracterizes the solution to (3.5) when this equation is well posed.

Proof of Proposition 3.2 We will show that \( A \) is the infinitesimal generator of the process \((X(t), t \geq 0)\). The drift part (with the selection function \( \mu \)) is straightforward, and the last part is the classical generator of a Λ-Fleming Viot process (see [29] for instance). Let us check that the diffusive part has the good form. For \( i < j \in E^2 \), from (3.5) and Itô formula, we have:

\[
\frac{d\langle X_i, X_i \rangle}{dt} = \sigma \sum_{k=1}^{i} \zeta_{ik}(X) = \sigma \zeta_{ii}^2(X) + \sigma \sum_{k=1}^{i-1} \zeta_{ik}^2(X)
\]

\[
= \sigma \frac{X_i(1 - X_1 - ... - X_i)}{1 - X_1 - ... - X_i - 1} + \sigma X_i^2 \sum_{k=1}^{i-1} \frac{X_k}{(1 - X_1 - ... - X_{k-1})(1 - X_1 - ... - X_k)}
\]

\[
= \sigma \frac{X_i(1 - X_1 - ... - X_i)}{1 - X_1 - ... - X_i - 1} + \sigma X_i^2 \sum_{k=1}^{i-1} \left( \frac{1}{1 - X_1 - ... - X_k} - \frac{1}{1 - X_1 - ... - X_{k-1}} \right)
\]

\[
= \sigma \frac{X_i(1 - X_1 - ... - X_i)}{1 - X_1 - ... - X_i - 1} + \sigma X_i^2 \left( \frac{1}{1 - X_1 - ... - X_i} - 1 \right)
\]

\[
= \sigma \frac{X_i}{1 - X_1 - ... - X_i} \{(1 - X_1 - ... - X_i) + X_i \{1 - X_1 - ... - X_{i-1}\}\}
\]

\[
= \sigma X_i(1 - X_i),
\]

The remainder of the paper is dedicated to the proofs.
Under the assumption of existence and uniqueness of the solution of which was the claim. This ends the proof.

Equation (3.5), it is enough to prove the convergence of the sequence of generators of (computation, we will adopt the following representation for the law of the notations of Sections 2.1 and 2.2, as well as (3.2) and (3.3). In order to simplify the and

\[ x_{\lfloor Nt \rfloor} = \sum_{i \leq K} Y_i N_{i,k} \]

where the laws of \( M^{(1, N)}_z \) and \( M^{(2, N)}_z \) are defined as follows:

\[ \mathbb{P} \left( M^{(1, N)}_z = Y \right) = \left( \frac{N}{Y_1, \ldots, Y_K} \right) \left( p_1^N(x) \right)^{Y_1} \cdots \left( p_K^N(x) \right)^{Y_K}, \]

\[ M^{(2, N)}_z | \{ Z_g = z \} = \sum_{k=1}^N \{ (1 - B_k) e_{C_k} + B_k e_B \}, \]

where \( (Y_i, i \leq K) \in \mathbb{N}^K \) such that \( \sum_{1 \leq i \leq K} Y_i = N \), \( (B_k, 1 \leq k \leq N) \) are Bernoulli random variables with parameter \( z \), for \( i \in E, 1 \leq k \leq N, \mathbb{P}(B = i) = \mathbb{P}(C_k = i) = x_i \), all these variables are independent and we recall that \((e_i, 1 \leq i \leq K)\) is the canonical basis of \( E \). Then the discrete generator \( \mathcal{A}^N \) of \( (X^N_{\lfloor Nt \rfloor})_t \) applied to any function \( f \in C^2(\Delta_K) \) in a point \( x \) of \( \Delta_K \) satisfies:

\[ \mathcal{A}^N f(x) := \lim_{t \to 0} \frac{\mathbb{E} \left[ f \left( X^N_{\lfloor Nt \rfloor} \right) \right] - f(x)}{t} \]

\[ = \lambda_N^N([0, 1]) \rho_N K^{-1} \mathbb{E} \left[ \frac{f \left( \frac{M_z^{(1, N)}}{N} \right)}{\rho_N K^{-1}} \right] - f(x) \]

\[ + \left( 1 - \lambda_N^N([0, 1]) \rho_N K^{-1} \right) \mathbb{E} \left[ \frac{f \left( \frac{M_z^{(2, N)}}{N} \right)}{\rho_N K^{-1}} \right] - f(x). \]
Using Taylor expansion and the representation of $M^{(2,N)}_x$ gives:

\[
A^N f(x) = \Lambda^N_x([0,1]) \int_0^1 \mathbb{E} \left[ f \left( \sum_{k=1}^N \{(1-B_k)e_{C_k} + B_k e_B \} / N \right) \right] \Lambda^N_x(dz)
\]

\[
+ \frac{1 - \Lambda^N_x([0,1]) \rho N^{-1}}{\rho N^{-1}} \sum_{i=1}^K \mathbb{E} \left[ \left( M^{(1,N)}_x \right)_i / N - x \right] \frac{\partial f}{\partial i}(x)
\]

\[
+ \frac{1 - \Lambda^N_x([0,1]) \rho N^{-1}}{2 \rho N^{-1}} \sum_{i,j=1}^K \mathbb{E} \left[ \left( M^{(1,N)}_x \right)_i / N - x \right] \left( \left( M^{(1,N)}_x \right)_j / N - x \right) \frac{\partial^2 f}{\partial i \partial j}(x) + o(1)
\]

\[
=: A_N + B_N + C_N + o(1).
\]

First, from assumption (i) of Proposition 3.1, we get that

\[
\lim_{N \to \infty} B_N = \kappa \sum_{i=1}^K \mu_i(x) \frac{\partial f}{\partial i}(x).
\]

Second, as $1 - Q_N(\{1\}) = o(1)$ when $N$ goes to infinity it is enough to study $C_N$ to consider the case when individuals sample only one potential parent. We get, using assumption (iii) in Proposition 3.1

\[
\lim_{N \to \infty} C_N = \frac{\sigma}{2} \sum_{i,j=1}^K x_i (1_{i=j} - x_j) \frac{\partial^2 f}{\partial i \partial j}(x).
\]

The limit of $A_N$ is more involved to obtain. First notice that $M^{(2,N)}_x$ can be rewritten

\[
M^{(2,N)}_x \{ Z = z \} = \sum_{k=1}^N \{(1-z)e_{C_k} + ze_B \} + \sum_{k=1}^N (z - B_k)(e_{C_k} - e_B),
\]

and that the expectation of the $i$th coordinate ($i \in E$) of the second sum is

\[
\mathbb{E} \left[ \sum_{k=1}^N (z - B_k)(1_{C_k=i} - 1_{B=i}) \right] = \sum_{k=1}^N \mathbb{E}[z - B_k](x_i - x_i) = 0.
\]

Hence

\[
\mathbb{E} \left[ f \left( M^{(2,N)}_x / N \right) \right] = \mathbb{E} \left[ f \left( \sum_{k=1}^N \{(1-z)e_{C_k} + ze_B \} / N \right) \right]
\]

\[
+ \frac{1}{2N^2} \sum_{i,j=1}^K \mathbb{E} \left[ H_i^N H_j^N \frac{\partial^2 f}{\partial i \partial j}(\Xi^N) \right],
\]

where $\Xi^N(z)$ belongs to the segment $[\sum_{k=1}^N \{(1-z)e_{C_k} + ze_B \} / N, M^{(2,N)}_x / N]$ and

\[
H_i^N(z) := \sum_{k=1}^N (z - B_k)(1_{C_k=i} - 1_{B=i}).
\]

But by Cauchy-Schwarz inequality, we have

\[
\mathbb{E} \left[ \left( H_i^N(z)H_j^N(z) \frac{\partial^2 f}{\partial i \partial j}(\Xi^N(z)) \right) \right] \leq \sqrt{\mathbb{E} \left[ (H_i^N(z))^2 \right] \mathbb{E} \left[ (H_j^N(z))^2 \right]} \left\| \frac{\partial^2 f}{\partial i \partial j} \right\|_{\infty}
\]

\[
\leq C \sqrt{\mathbb{E} \left[ (H_i^N(z))^2 \right] \mathbb{E} \left[ (H_j^N(z))^2 \right]},
\]

where $C$ is a constant.
for a finite \( C \) where for a function \( g \) on \( \Delta_K \), \( \|g\| := \sup_{x \in \Delta_K} |g(x)| \), and we have used that a continuous function is bounded on a compact set.

We thus need to bound the term in the square root, which is a simple calculation, as we know explicitly the laws of \( B \), \((B_k, 1 \leq k \leq N)\) and \((C_k, 1 \leq k \leq N)\).

\[
E \left[ (H_i^N(z))^2 \right] = E \left[ \left( \sum_{k=1}^{N} (z - B_k)(1_{C_k = i} - 1_{B_k = i}) \right)^2 \right] = \\
E \left[ \sum_{k=1}^{N} (z - B_k)^2 (1_{C_k = i} - 1_{B_k = i})^2 \right] + E \left[ \sum_{k \neq l}^{N} (z - B_k)(z - B_l)(1_{C_k = i} - 1_{B_k = i})(1_{C_l = i} - 1_{B_l = i}) \right].
\]

As the \((B_k, 1 \leq k \leq N)\) are mutually independent, with mean \( z \), and independent of \( B \) and \((C_k, 1 \leq k \leq N)\), the last term is null in the previous equality, which implies

\[
\left| E \left[ H_i^N(z)H_j^N(z) \frac{\partial^2 f}{\partial i \partial j}(\Xi^N(z)) \right] \right| \leq CN.
\]

In particular this implies that

\[
\frac{1}{N^2} \int_0^1 \left| E \left[ H_i^N(z)H_j^N(z) \frac{\partial^2 f}{\partial i \partial j}(\Xi^N(z)) \right] \Lambda_N^\alpha(dz) \right| \leq \frac{C}{N} \Lambda_N^\alpha([0, 1]) \leq CN^{2n-1} \Lambda([0, 1]),
\]

by definition of \( \Lambda_N^\alpha \) in (3.1). We thus have shown that for large \( N \),

\[
A_n = \int_0^1 \left[ E \left[ f \left( \sum_{k=1}^{N} \{(1-z)e_{C_k} + ze_B\}/N \right) \right] - f(x) \right] \Lambda_N^\alpha(dz) + o(1).
\]

The next step consists in proving that

\[
\int_0^1 \left[ E \left[ f \left( \sum_{k=1}^{N} \{(1-z)e_{C_k} + ze_B\}/N \right) \right] - f \left( \sum_{k=1}^{N} \{(1-z)x + ze_B\}/N \right) \right] \Lambda_N^\alpha(dz) = o(1).
\]

The result is obtained as before by doing a Taylor expansion around \( \sum_{1 \leq k \leq N} \{(1-z)e_{C_k} + ze_B\}/N \), and by introducing

\[
H_i^N(z) := (1-z) \sum_{k=1}^{N} (1_{C_k = i} - x_i).
\]

As the calculations are very similar to the previous ones we do not provide details. This yields that for large \( N \),

\[
A_N = \int_0^1 (E[f((1-z)x + ze_B)] - f(x)) \Lambda_N^\alpha(dz) + o(1).
\]

To end the proof we need to show that \( \Lambda_N^\alpha(dz) \) can be replaced by \( \Lambda(dz)/z^2 \). For this step we will again make use of Taylor expansion. Indeed we have

\[
E[f((1-z)x + ze_B) - f(x)] = \sum_{i=1}^{K} zE[1_{B_i = i} - x_i] \frac{\partial f}{\partial i}(x) + \frac{1}{2} \sum_{i,j=1}^{K} z^2E \left[ (1_{B_i = i} - x_i)(1_{B_j = j} - x_j) \frac{\partial^2 f}{\partial i \partial j}(\Phi^N(z)) \right] = \frac{1}{2} \sum_{i,j=1}^{K} z^2E \left[ (1_{B_i = i} - x_i)(1_{B_j = j} - x_j) \frac{\partial^2 f}{\partial i \partial j}(\Phi^N(z)) \right],
\]

\[
\sum_{i=1}^{K} zE[1_{B_i = i} - x_i] \frac{\partial f}{\partial i}(x) = \sum_{i=1}^{K} \frac{z}{N} \sum_{l=1}^{N} \{ 1_{C_l = i} - x_i \} \frac{\partial f}{\partial i}(x) = \frac{1}{N} \sum_{l=1}^{N} \sum_{i=1}^{K} \{ 1_{C_l = i} - x_i \} \frac{\partial f}{\partial i}(x).
\]

Each summand has the form

\[
\sum_{i=1}^{K} \{ 1_{C_l = i} - x_i \} \frac{\partial f}{\partial i}(x) = \sum_{i=1}^{K} \frac{1}{N} \sum_{l=1}^{N} 1_{C_l = i} \frac{\partial f}{\partial i}(x) - \sum_{i=1}^{K} x_i \frac{\partial f}{\partial i}(x) = \frac{1}{N} \sum_{l=1}^{N} \{ 1_{C_l = i} - x_i \} \frac{\partial f}{\partial i}(x) - \sum_{i=1}^{K} x_i \frac{\partial f}{\partial i}(x) = O(N^{-\frac{1}{2}}).
\]

Thus

\[
E[f((1-z)x + ze_B) - f(x)] = O(N^{-\frac{1}{2}}).
\]

This completes the proof.
where \( \Phi_N(z) \) is on the segment \( [x, x + z(e_B - x)] \). Thus there exists a finite constant \( C \) such that
\[
\left| \mathbb{E} [f ((1 - z)x + ze_B) - f(x)] \right| \leq Cz^2.
\]
In particular,
\[
\int_0^1 \left( \mathbb{E} [f ((1 - z)x + ze_B)] - f(x) \right) \left( \Lambda_N^0(dz) - \frac{\Lambda(dz)}{z^2} \right) \leq C \int_0^1 1_{z \leq N^{-a}} \Lambda(dz) \to 0.
\]
This ends the proof.

**Proof of Proposition 3.1.** If we don’t assume that Equation (3.5) has a unique strong solution, the convergence of the generators is no longer sufficient to claim the weak convergence of the processes. However, we can show that the sequence of processes is tight using the robust theory introduced in [4]. More precisely, we will apply their Theorem 2.3. To this aim, we need to check that their hypotheses (H0), (H1) and (H1') are satisfied.

Hypothesis (H0) is trivially true because we are working in a compact space. To prove (H1) it is enough to extend the functions and functional spaces in Section 4 of [4] to more dimensions. Following the notation of [4] we introduce the function
\[
h(u_1, u_2, \ldots, u_K) = (1 - e^{-u_1}, \ldots, 1 - e^{-u_K})
\]
and the functional space
\[
\mathcal{H} = \{(u_1, \ldots, u_k) \in \Delta_K \to H_{\bar{z}}, \bar{z} \in \mathbb{R}^K \} \text{ with } H_{\bar{z}}(\bar{u}) = 1 - e^{-\sum_{i=1}^K u_i}.
\]
Then as in [4], (H1.1) is trivial by construction, (H1.2) follows an application of the Local Stone-Weierstrass Theorem (See Appendix 6.4 of [4] for details). Hypothesis (H1') follows from the fact that \( \mathcal{H} \subset C_2 \) and thus we can apply the uniform convergence of the generators that we verified in the proof of Theorem 4.

**Proof of Corollary 3.1.** To prove that the SDE (3.5) has a unique strong solution, under hypothesis (1) we will apply Theorem 5.1 of [40], under hypothesis (2) we will apply Corollary 2.9 in [18], and for hypothesis (3) we will use Lemma 3.6 of [25] and induction.

We will first work in the direction of proving the statement under hypothesis (2) and (1) will be obtained after an additional computation.

Consider any colouring rule, as defined in Definition 2.2. Let \( i \leq K \). For any configuration of potential parents with \( k_i \) parents of type \( i \) (\( 1 \leq i \leq K \)), there is a probability \( p_i^N(k_1, \ldots, k_K) \) for the offspring to be of type \( i \). Notice that we may have \( p_i^N > 0 \) even if \( k_i = 0 \) when we take mutations into account. Moreover, knowing that there are \( k_1 + \ldots + k_K = K \) potential parents, such a configuration has a probability
\[
\mu_i(x) = \lim_{N \to \infty} \frac{1}{\rho_N} \sum_{k=2}^{\infty} \prod_{k_i} \left( \sum_{k_1+\ldots+k_K=k} \left( \begin{array}{c} k \\ k_1, \ldots, k_K \end{array} \right) x_1^{k_1} \ldots x_K^{k_K} p_i(k_1, \ldots, k_K) - x_i \right).
\]
But notice that for any \( K \)-tuple \( (k_1, \ldots, k_K) \) of integers, and \( (z, x) \in \Delta_K^2 \),
\[
\left| x_1^{k_1} \ldots x_K^{k_K} - z_1^{k_1} \ldots z_K^{k_K} \right| \leq \left| x_1 - z_1 \right| + \left| x_2 - z_2 \right| \ldots + \left| x_K - z_K \right| \leq \sum_{j=1}^{K} \left| x_j - z_j \right|.
\]
Hence
\[ |\mu_i(x) - \mu_i(z)| \leq \lim_{N \to \infty} \frac{1}{\rho_N} \sum_{k=2}^{\infty} \pi_k \frac{1}{k_{1+\ldots+k_N}} \left( k_{1+\ldots+k_N} \right) \rho_i^N(k_{1+\ldots+k_N}) |x_{1+\ldots+k_N} - z_{1+\ldots+k_N}| \]
\[ = |x_i - z_i| + \lim_{N \to \infty} \frac{1}{\rho_N} \left( \sum_{k=2}^{\infty} \pi_k \left( \sum_{k_{1+\ldots+k_N}=k} k_{1+\ldots+k_N} \right) \rho_i^N(k_{1+\ldots+k_N}) |x_{1+\ldots+k_N} - z_{1+\ldots+k_N}| \right) \]
\[ \leq C_i \sum_{j=1}^{K} |x_j - z_j|, \]
where \( C_i \) is a finite constant. As a consequence,
\[ |(\mu(x) - \mu(z), x - z)| \leq \sup_{1 \leq i \leq K} C_i \sum_{i,j=1}^{K} |x_i - z_i||x_j - z_j| \]
\[ \leq \sup_{1 \leq i \leq K} C_i \sum_{i,j=1}^{K} (|x_i - z_i|^2 + |x_j - z_j|^2) \leq \sup_{1 \leq i \leq K} 2C_i K |x - z|^2. \]

Now, take \( \zeta \equiv 1 \) in Assumption 2.1 and \( \rho = C \) in Assumption 2.3 of [49]. We have to check that the following inequalities hold for \((x, z) \in \Delta_K^2\):
\[
\langle \mu(x), x \rangle + |\zeta(x)|^2 + \int_0^1 \int_0^1 |c(x, u, y)|^2 \frac{\Lambda(dy)}{y^2} \leq C(|x|^2 + 1), \tag{5.1}
\]
\[
\langle \mu(x) - \mu(z), x - z \rangle + |\zeta(x) - \zeta(z)|^2 \leq C|x - z|^2, \tag{5.2}
\]
and
\[
\int_0^1 \int_0^1 |c(x, u, y) - c(z, u, y)| du \frac{\Lambda(dy)}{y^2} \leq C|x - z|, \tag{5.3}
\]
where \( c(x, u, y) = y(1_{0 \leq u - (x_1 + \ldots + x_{i-1}) < x_i - x_i}) \) and \( C \) is a positive constant, in order to apply Corollary 2.9 in [49]. The function \( \sigma \) and \( \mu \) are bounded. Moreover,
\[ \int_0^1 \int_0^1 |c(x, u, y)|^2 du \frac{\Lambda(dy)}{y^2} \leq \int_0^1 \int_0^1 du \Lambda(dy) = \int_0^1 \Lambda(dy), \]
which is finite by assumption. Hence (5.1) holds.

Let us now prove that (5.3) holds. We have
\[ \int_0^1 \int_0^1 |c_i(x, u, y) - c_i(z, u, y)| du \frac{\Lambda(dy)}{y^2} \]
\[ = \int_0^1 \int_0^1 \left| (1_{0 \leq u - (x_1 + \ldots + x_{i-1}) < x_i} - x_i) - (1_{0 \leq u - (z_1 + \ldots + z_{i-1}) < z_i} - x_i) \right| du \frac{\Lambda(dy)}{y} \]
\[ \leq (2|x_i - z_i| + |x_i - z_i|) \int_0^1 \frac{\Lambda(dy)}{y}, \]
with the convention \( x_{-1} = z_{-1} = 0 \). Hence (5.3) holds with \( \kappa = 3 \int_0^1 \Lambda(dy)/dy \). This proves that (3.3) admits a unique strong solution under assumption (2).
To obtain a general result in multiple dimensions we would need to verify that assumption (5.2) holds. Let us focus on \( \zeta \) and see what is the problem. We have for \( i \in \mathbb{E} \)

\[
|\zeta_{ii}(x) - \zeta_{ii}(z)|^2 = \left| \frac{x_i(1 - x_1 \ldots - x_i)}{1 - x_1 \ldots - x_{i-1}} - \frac{z_i(1 - z_1 \ldots - z_i)}{1 - z_1 \ldots - z_{i-1}} \right|^2 \\
\leq \left| \frac{x_i(1 - x_1 \ldots - x_i)}{1 - x_1 \ldots - x_{i-1}} - \frac{z_i(1 - z_1 \ldots - z_i)}{1 - z_1 \ldots - z_{i-1}} \right| < |x - z|,
\]

for \( j > i \in \mathbb{E}^2 \),

\[
|\zeta_{ij}(x) - \zeta_{ij}(z)|^2 = 0,
\]

and for \( j < i \in \mathbb{E}^2 \),

\[
|\zeta_{ij}(x) - \zeta_{ij}(z)|^2 = \left| \frac{x_j}{(1 - x_1 \ldots - x_{j-1})(1 - x_1 \ldots - x_j)} - \frac{z_j}{(1 - z_1 \ldots - z_{j-1})(1 - z_1 \ldots - z_j)} \right|^2 \\
\leq \left| \frac{x_j}{(1 - x_1 \ldots - x_{j-1})(1 - x_1 \ldots - x_j)} - \frac{z_j}{(1 - z_1 \ldots - z_{j-1})(1 - z_1 \ldots - z_j)} \right| < 3|x - z|.
\]

Unfortunately, it is not enough to apply Corollary 2.9 in [49]; we would need \( |x - y|^2 \).

However, it is exactly what one needs to apply Theorem 5.1 of [40], which works only in the case \( K = 2 \). To be more precise, to check that the result is true under condition (1), we need to check conditions (2.a), (2.b), (5.a), (5.b) and (5.c) in [40]. The calculations are either already done in the previous lines, or very similar. We thus do not give the details. Notice that similarly to the case of [28], the assumption \( \int_0^\infty \Lambda(dy)/y \) is not needed in this case. The finitness of \( \Lambda([0, 1]) \) is enough. We conclude that the result follows also under hypothesis (1).

To prove (3) we will make a change of variable and apply Lemma 3.6 in [28]. Let us choose \( i_0 \in [K - 1] \) and consider the process

\[
(Y_{i_0}(t), t \geq 0) := (X_1 + \ldots + X_{i_0})(t), t \geq 0).
\]

Adding the \( i_0 \) equations, and recalling that (1.2) implies that for \( x \in \Delta_K \)

\[
\sum_{i=1}^{i_0} \mu_i(x) = \kappa \sum_{k=1}^{\infty} \pi_k \left( (x_1 + \ldots + x_{i_0})^{k+1} - (x_1 + \ldots + x_{i_0}) \right),
\]

\( Y_{i_0} \) should be solution to

\[
dY_{i_0}(t) = \kappa \sum_{k=1}^{\infty} \pi_k \left( Y_{i_0}^{k+1}(t) - Y_{i_0}(t) \right) dt + \sqrt{\sigma} \sum_{i=1}^{i_0} \sum_{j=1}^{K} \zeta_{ij}(X(t)) dB^{(j)}_t \\
+ \sum_{i=1}^{i_0} \int_0^1 \int_0^1 z \left( 1_{\{t \leq u < t \leq t \}} - X_i(t^{-}) \right) \tilde{N}(dt, du, dz).
\]

Notice first that the jump term may be reduced to

\[
\int_0^1 \int_0^1 z \left( 1_{u < Y_{i_0}(t^{-})} - Y_{i_0}(t^{-}) \right) \tilde{N}(dt, du, dz).
\]
Let us now focus on the diffusion term. By definition of $Y_i$, we get using the calculations derived in the proof of Proposition 3.2:

$$
\frac{d\langle Y_{i_0}, Y_{i_0} \rangle}{dt} = \frac{d\langle X_1 + \ldots + X_{i_0}, X_1 + \ldots + X_{i_0} \rangle}{dt}
$$

$$
= \sum_{i=1}^{i_0} d\langle X_i, X_i \rangle + \sum_{1 \leq i, j \leq i_0, i \neq j} d\langle X_i, X_j \rangle = \sigma \sum_{i=1}^{i_0} X_i(1 - X_i) - \sigma \sum_{1 \leq i, j \leq i_0, i \neq j} X_iX_j
$$

$$
= \sigma \sum_{i=1}^{i_0} X_i \left(1 - X_i - \sum_{1 \leq j \leq i_0, i \neq j} X_j\right) = \sigma (X_1 + \ldots + X_{i_0})(1 - X_1 - \ldots - X_{i_0})
$$

$$
= \sigma Y_{i_0} (1 - Y_{i_0}).
$$

In particular it implies that there exists a Brownian motion $W^{(i_0)}$ such that (see Theorem (4.4) in [20] for instance):

$$
\sqrt{\sigma} \sum_{i=1}^{i_0} \sum_{j=1}^{K} \zeta_{i,j}(X(t))dB_{i}^{(j)} = \sqrt{\sigma} Y_{i_0}(t) (1 - Y_{i_0}(t))dW_t^{(i_0)},
$$

and thus $Y_{i_0}$ should be solution to

$$
dY_{i_0}(t) = \kappa \sum_{k=1}^{\infty} \pi_k \left(Y_{i_0}^{k+1}(t) - Y_{i_0}(t)\right) dt + \sqrt{\sigma} Y_{i_0}(t) (1 - Y_{i_0}(t))dW_t^{(i_0)}
$$

$$
+ \int_{0}^{1} \int_{0}^{1} z \left( 1_{\{u < Y_{i_0}(t^-)\}} - Y_{i_0}(t^-) \right) \tilde{N}(dt, du, dz).
$$

But from Lemma 3.6 in [28] we know that this equation has a unique strong solution. We can make the same reasoning to prove that $Y_{i_0-1}$ (if $i_0 > 1$) is the unique strong solution to

$$
dY_{i_0-1}(t) = \kappa \sum_{k=1}^{\infty} \pi_k \left(Y_{i_0-1}^{k+1}(t) - Y_{i_0-1}(t)\right) dt + \sqrt{\sigma} Y_{i_0-1}(t) (1 - Y_{i_0-1}(t))dW_t^{(i_0-1)}
$$

$$
+ \int_{0}^{1} \int_{0}^{1} z \left( 1_{\{u < Y_{i_0-1}(t^-)\}} - Y_{i_0-1}(t^-) \right) \tilde{N}(dt, du, dz),
$$

where $W^{(i_0-1)}$ is uniquely determined from the Brownian motions $(B^{(j)}, 1 \leq j \leq K)$. As $X_{i_0} = Y_{i_0} - Y_{i_0-1}$, this concludes the proof. 

\[\square\]

**Proof of Lemma 3.7** Assume that hypotheses of Lemma 3.1 hold. In the proof of Corollary 3.1 we saw that the reason for which we could not ensure the existence of a strong solution was that all the $\zeta_{ij}$’s had not bounded continuous partial derivatives on $\Delta_K$. But for any $0 < \varepsilon < 1/2$, there exists a finite constant $C(\varepsilon)$ such that for $x$ in $(\varepsilon, 1-\varepsilon)^K$ and $(i,j,k) \in E^2$, 

$$
\left|\frac{\partial \zeta_{ij}(x)}{\partial_k}\right| \leq C(\varepsilon).
$$

As a consequence, applying Taylor Formula we get, for $x, z$ in $(\varepsilon, 1-\varepsilon)^K$,

$$
|\zeta_{ij}(x) - \zeta_{ij}(z)|^2 \leq KC(\varepsilon)|x - z|^2.
$$

We thus can apply Corollary 2.9 in [49] and conclude. 

\[\square\]
5.2. Proofs of general results on the selective functions.

Proof of Lemma 5.2. For each $N \in \mathbb{N}^*$ consider the random graph with parameters

$$Q_N = (1 - \rho_N)\delta_1 + \rho_N\delta_N$$

and colouring rule $c_N^z = g(z/N)$ for every $N$-dimensional vector $z$ and $c_N^z = \delta_z$ for every one dimensional vector $z$. Note that this is indeed a colouring rule because we assumed that $g : \Delta_K \mapsto \Delta_K$. For every $i \in E$, note that

$$p_i^N(x) = \sum_{z \in E^N} \mathbb{P}(z_v = z)c_N^z(i).$$

Observe that $z_v$ is a multinomial random variable with parameter $x$ and 1 with probability $1 - \rho_N$ and is a multinomial random variable with parameter $x$ and $N$ with probability $\rho_N$, so

$$p_i^N(x) = \mathbb{E}[c_N^z] = (1 - \rho_N)x_i + \rho_N\mathbb{E}\left[\frac{z_v}{N}\right] = x_i + \rho_N(g_i(x) - x_i) + o(\rho_N), \quad (N \to \infty)$$

where in the right hand side we used the Law of Large Numbers. The rest of the proof consists in applying Theorem 11.

Proof of Corollary 5.3. The proof is similar to the proof of Corollary 3.2 and follows the observation that by construction

$$\mu_i(x) = g_i(x) - x_i$$

for all $i \in E$ and $x \in \Delta_K$. \hfill \Box

5.3. Proofs of general results on alleles extinction and fixation.

Proof of Proposition 5.3. Recall the definition of the ancestral process $(D_t, t \geq 0)$ in (4.3). Then if either of the conditions of Proposition 5.3 are satisfied, the process $(D_t, t \geq 0)$ positive recurrent (see Theorem 3 of [29] and Theorem 1.1 of [24]). In particular, it will reach one in finite time almost surely. This means that all the individuals in the population at a given time will have the same ancestor. As we did not allow for mutations in this Proposition, it implies that all the descendants of this ancestor (that is to say all the individuals alive at subsequent generations and related to this ancestor by a sequence of edges) will have the same type. \hfill \Box

The proof of Proposition 5.4 is based on the two following lemmas:

**Lemma 5.1.** Assume that the process $Z = (Z_t, t \geq 0)$ satisfies an equation of the form:

$$dZ_t = (1 - Z_t)S(Z_t,t)dt + \sqrt{Z_t(1 - Z_t)}dB_t, \quad Z_0 \in (0,1)$$

where $B$ is a Brownian motion, and for all $t \geq 0$ and $z \in [0,1]$, $|S(z,t)| \leq C < \infty$ and $S(0,t) = 0$. Then $Z_t \in [0,1]$ for all $t \geq 0$, and

$$\int_0^{T_a} \frac{1}{1 - Z_s} ds = \infty \quad a.s.,$$

where for $a \in [0,1], T_a := \inf\{t \geq 0, Z_t = a\}$.

**Lemma 5.2.** Let $n \geq 3$ be in $\mathbb{N}^*$ and $V = (V_1(t),...,V_n(t), t \geq 0)$ a process with (possibly inhomogeneous) infinitesimal generator acting on $f$ at $v \in \Delta_n$ of the form

$$\mathcal{A}_tf(v) = \sum_{i=1}^n \mu_i(v,t)\frac{\partial f}{\partial v_i}(v) + \sigma \sum_{i,j=1}^n \sigma_{ij}(v)\frac{\partial^2 f}{\partial v_i v_j}(v),$$
Lemma 5.1, we have
\[ \int_t \]
and define the time change \( \tau \) on \([0, \infty)\) by
\[ \int_0^{\tau(t)} \frac{1}{1 - V_n(s)} \, ds = t, \quad \forall \ t \geq 0. \]

Next let us introduce the process
\[ Y = (Y_1(t), ..., Y_{n-2}(t), 1 - Y_1(t) - ... - Y_{n-1}(t), t \geq 0) \]
\[ := \left( \frac{V_1}{1 - V_n}(\tau(t)), ..., \frac{V_{n-2}}{1 - V_n}(\tau(t)), \frac{V_{n-1}}{1 - V_n}(\tau(t)), t \geq 0 \right). \]

Then the stochastic process \( Y \) has a (possibly inhomogenous) infinitesimal generator acting on \( f \) at \( y \in \Delta_{n-1} \) of the form
\[ \tilde{A}_t f(y) = \sum_{i=1}^{n} \tilde{\mu}_i(y,t) \frac{\partial f}{\partial y_i}(v) + \sigma \sum_{i,j=1}^{n} \sigma_{ij}(y) \frac{\partial^2 f}{\partial y_i y_j}(y), \]
where
\[ \tilde{\mu}(y,t) = (y_i(1 - y_i) \tilde{s}_i(y,t), 1 \leq i \leq n - 1), \]
\[ |\tilde{s}| \leq C' \text{ for a finite } C', \] and
\[ \sigma(y) = ((1_{j=i} - y_j)y_i, 1 \leq i, j \leq n - 1). \]

Before proving these two Lemmas, we prove Proposition 3.4.

**Proof of Proposition 3.4.** The fact that one of the alleles ultimately fixates is a consequence of Proposition 3.3. If \( n = 2 \), the result is immediate. Hence we assume that \( n \geq 3 \). From Lemma 5.1, we have
\[ \int_0^{T_{1}^V_n} \frac{1}{1 - V_n(s)} \, ds = \infty, \]
where \( T_{1}^V_n \) is the hitting time of 1 by the process \( V_n \). Indeed,
\[ \left| - \sum_{i=1}^{n-1} \mu_i(V(t), t) \right| \leq \sum_{i=1}^{n-1} V_i(t) (1 - V_i(t)) |s_i(V(t), t)| \leq C \sum_{i=1}^{n-1} V_i(t) = C \left( 1 - V_n(t) \right). \]

We thus may introduce the time change \( \tau \) on \([0, \infty)\) such that
\[ \int_0^{\tau(t)} \frac{1}{1 - V_n(s)} \, ds = t, \quad \forall \ t \geq 0. \]
As for any \( t < \infty, V_n(\tau(t)) < 1 \), we may consider the process
\[ Y(t) = (Y_1(t), ..., Y_{n-2}(t)) := \left( \frac{V_1(\tau(t))}{1 - V_n(\tau(t))}, ..., \frac{V_{n-2}(\tau(t))}{1 - V_n(\tau(t))} \right). \]
Thanks to Lemma 5.2, we know that the stochastic process \( (Y_1(t), ..., Y_{n-2}(t))_{t \geq 0} \) has a (possibly inhomogenous) infinitesimal generator acting on \( f \) at \( y \in \Delta_{n-1} \) of the form
\[ \tilde{A}_t f(y) = \sum_{i=1}^{n} \tilde{\mu}_i(y,t) \frac{\partial f}{\partial y_i}(v) + \sigma \sum_{i,j=1}^{n} \sigma_{ij}(y) \frac{\partial^2 f}{\partial y_i y_j}(y), \]
By applying Itô’s Formula to the function $f_{\tilde{s}}$ or $\tilde{s}$ for a finite $C'$, and

$$|\tilde{s}| \leq C'$$

for a finite $C'$, and

$$\sigma(y) = ((1_{j=i} - y_j)y_i, 1 \leq i, j \leq n - 1).$$

We end the proof following the proof of Theorem 2.1 in [19]. By the induction hypothesis, the process $Y$ undergoes $n - 2$ successive extinctions at times

$$E_1^Y < \ldots < E_{n-2}^Y < \infty.$$ 

Hence,

$$\tau (E_1^Y) < \ldots < \tau (E_{n-2}^Y) < \tau (\infty) = T_1^{V_n}.$$ 

if $\{T_1^{V_n} < \infty\}$, the times $\tau (E_1^Y), \ldots, \tau (E_{n-2}^Y), T_1^{V_n}$ correspond to the $n - 1$ extinction times of alleles $\{1, 2, \ldots, n - 1\}$. This concludes the proof, as from the first part of the theorem we know that

$$P(\bigcup_{n=1}^{\infty} \{T_1^{V_n} < \infty\}) = 1.$$

□

To prove Lemma 5.1, we cannot use general results of [19] because of the selection term. We will instead apply Itô formula to an auxiliary function of the process $Z$.

Proof of Lemma 5.1. First notice that $Z$ is continuous and that the states $0$ and $1$ are absorbing. This implies that $Z$ stays in the interval $[0, 1]$. Let $\varepsilon, Z_0 > 0$ be such that $Z_0 < 1 - \varepsilon$. By applying Itô’s Formula to the function $f(x) := -\ln(1 - x)$ we get for any positive $t$:

$$-\ln(1 - Z_{t \wedge T_1 - \varepsilon}) + \ln(1 - Z_0)$$

$$= \int_0^{t \wedge T_1 - \varepsilon} S(Z_s, s)ds + \int_0^{t \wedge T_1 - \varepsilon} \sqrt{Z_s \over 1 - Z_s} dB_s + \frac{1}{2} \int_0^{t \wedge T_1 - \varepsilon} Z_s \over 1 - Z_s ds.$$

We have

$$P \left( \lim_{\varepsilon \to 0^+} \left\{ -\ln(1 - Z_{t \wedge T_1 - \varepsilon}) + \ln(1 - Z_0) \right\} = \infty, T_1 < \infty \right) = P(T_1 < \infty),$$

and

$$P \left( \limsup_{\varepsilon \to 0^+} \left\| \int_0^{T_1 - \varepsilon} S(Z_s, s)ds \right\| < \infty, T_1 < \infty \right) = P(T_1 < \infty),$$

as $S$ is bounded by assumption. We deduce

$$P \left( \lim_{\varepsilon \to 0^+} \left\{ \int_0^{T_1 - \varepsilon} \sqrt{Z_s \over 1 - Z_s} dB_s + \int_0^{T_1 - \varepsilon} Z_s \over 1 - Z_s ds \right\} = \infty, T_1 < \infty \right) = P(T_1 < \infty).$$

We will now prove that (5.5) implies the following property:

$$P \left( \int_0^{T_1} \frac{Z_s}{1 - Z_s} ds = \infty, T_1 < \infty \right) = P(T_1 < \infty).$$

The random variable

$$\int_0^{T_1 - \varepsilon} \frac{Z_s}{1 - Z_s} ds$$

is non negative and non increasing with $\varepsilon$. As a consequence, it has a nonnegative limit when $\varepsilon$ goes to 0, which can be finite or infinite. Let us consider a measurable event $A$ such that

$$\lim_{\varepsilon \to 0^+} \left\{ \int_0^{T_1 - \varepsilon} \frac{Z_s}{1 - Z_s} ds \right\} = \int_0^{T_1} \frac{Z_s}{1 - Z_s} ds < \infty \quad \text{a.s. on } A \cap \{T_1 < \infty\}. $$
Then from (5.3), we get that
\begin{equation}
\lim_{\epsilon \to 0^+} \left\{ \int_0^{T_1-\epsilon} \sqrt{\frac{Z_s}{1 - Z_s}} dB_s \right\} = \infty \quad \text{a.s. on } A \cap \{T_1 < \infty\}.
\end{equation}

Let us introduce the process
\[ M^{(\epsilon)}_t := \int_0^{t \wedge T_1-\epsilon} \sqrt{\frac{Z_s}{1 - Z_s}} dB_s. \]

$M^{(\epsilon)}$ is a continuous martingale. In particular, it is a time change of a Brownian motion, and there exists a Brownian motion $W$ such that (see Theorem (4.4) in [20] for instance):
\[ M^{(\epsilon)}_t = W_{\int_0^{t \wedge T_1-\epsilon} \frac{Z_s}{1 - Z_s} ds}. \]

This implies:
\[
\mathbb{E} \left[ 1_{A \cap \{T_1 < \infty\}} e^{-\int_0^{T_1-\epsilon} \frac{Z_s}{1 - Z_s} dB_s} \right] = \mathbb{E} \left[ 1_{A \cap \{T_1 < \infty\}} e^{-\int_0^{T_1-\epsilon} \frac{Z_s}{1 - Z_s} ds} \right] \geq \mathbb{E} \left[ 1_{A \cap \{T_1 < \infty\}} e^{-\sup \{W_{u, a} \leq \int_0^{T_1-\epsilon} \frac{Z_s}{1 - Z_s} ds\} \right].
\]

As $\{T_1 < \infty\}$ implies $\{T_1-\epsilon < \infty\}$, we may let $t$ go to infinity, and obtain
\begin{equation}
\mathbb{E} \left[ 1_{A \cap \{T_1 < \infty\}} e^{-\int_0^{T_1-\epsilon} \frac{Z_s}{1 - Z_s} dB_s} \right] \geq \mathbb{E} \left[ 1_{A \cap \{T_1 < \infty\}} e^{-\sup \{W_{u, a} \leq \int_0^{T_1-\epsilon} \frac{Z_s}{1 - Z_s} ds\} \right].
\end{equation}

But from (5.8), we get that
\[
\lim_{\epsilon \to 0^+} \mathbb{E} \left[ 1_{A \cap \{T_1 < \infty\}} e^{-\int_0^{T_1-\epsilon} \frac{Z_s}{1 - Z_s} dB_s} \right] = 0,
\]
and (5.7) implies that the right hand side of (5.8) is positive if and only if the event $A \cap \{T_1 < \infty\}$ has a positive probability. We thus deduce that
\[ \mathbb{P} \left( A \cap \{T_1 < \infty\} \right) = 0, \]
which implies (5.6). We conclude the proof of Lemma 5.1 by noticing that
\[ \mathbb{P} \left( \int_0^{T_1} \frac{1}{1 - Z_s} ds = \infty, T_1 = \infty \right) = \mathbb{P}(T_1 = \infty). \]

\[ \square \]

\textbf{Proof of Lemma 5.2.} Let us denote by $\tilde{\mathcal{L}}$ the infinitesimal generator of the process
\[ \left( \frac{V_1}{1 - V_n(t)}, \ldots, \frac{V_n-2}{1 - V_n(t)}, V_n(t) \right)_{t \geq 0}. \]

For any real valued function $f$ defined on $\{(y_1, \ldots, y_{n-2}, 1 - x_n) \in \Delta_{n-2} \times \Delta_1\}$, twice differentiable, we can write for $x_n \neq 1$,
\[
\tilde{\mathcal{L}} f(y_1, \ldots, y_{n-2}, 1 - x_n) = \mathcal{L}(f \circ g)(x_1, \ldots, x_{n-1}),
\]
where by definition, for $(x_1, \ldots, x_{n-1}) \in [0,1]^{n-1}$ such that $0 < x_1 + \ldots + x_{n-1} \leq 1$,
\[
y = (y_1, \ldots, y_{n-2}, 1 - x_n) = g(x_1, \ldots, x_{n-1}) = \left( \frac{x_1}{x_1 + \ldots + x_{n-1}}, \ldots, \frac{x_{n-2}}{x_1 + \ldots + x_{n-1}}, \frac{x_1 + \ldots + x_{n-1}}{x_1 + \ldots + x_{n-1}} \right).
\]
Recall that

\[ \mathcal{L} f(x) = \mathcal{L} f(x_1, \ldots, x_{n-1}) := \sum_{i=1}^{n-1} \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \sum_{i=1}^{n-1} x_i(1-x_i) \frac{\partial^2 f}{\partial x_i^2}(x) - \sum_{i \neq j=1}^{n-1} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(x). \]

The calculations for the diffusion part of \( \hat{\mathcal{L}} \) have been done in [19]. We thus only need to compute the term of drift.

\[ A(x) := \sum_{i=1}^{n-1} \mu_i(x) \frac{\partial}{\partial x_i} (f \circ g(x)) \]

\[ = \sum_{i=1}^{n-2} \mu_i(x) \left[ \sum_{j=1}^{n-2} \frac{\partial y_j}{\partial x_i} \frac{\partial f}{\partial y_j}(y) + \frac{\partial (1-x_n)}{\partial x_i} \frac{\partial f}{\partial (1-x_n)}(y) \right] + \mu_{n-1}(x) \left[ \sum_{j=1, j \neq i}^{n-2} \frac{1-x_n}{(1-x_n)^2} \frac{\partial f}{\partial y_j}(y) + \frac{1-x_n}{(1-x_n)^2} \frac{\partial f}{\partial (1-x_n)}(y) \right] \]

Rearranging the terms, we get:

\[ A(x) = \frac{1}{1-x_n} \left( \sum_{i=1}^{n-2} \left( \mu_i(x) - y_i \sum_{j=1}^{n-1} \mu_j(x) \right) \frac{\partial f}{\partial y_i}(y) \right) + \sum_{j=1}^{n-1} \mu_j(x) \frac{\partial f}{\partial (1-x_n)}(y). \]

If we introduce the notation

\[ \tilde{\mu}_i(y, x) = \mu_i(x) - y_i \sum_{j=1}^{n-1} \mu_j(x), \]

we get, adding the diffusive part:

\[ \tilde{\mathcal{L}}(y) = \frac{1}{1-x_n} \sum_{i=1}^{n-2} \left( \tilde{\mu}_i(y, x) \frac{\partial f}{\partial y_i}(y) + \sigma y_i (1-y_i) \frac{\partial^2 f}{\partial y_i^2}(y) - \sum_{j=1, j \neq i}^{n-2} y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j}(y) \right) \]

\[ + \sum_{i=1}^{n-1} \mu_i(x) \frac{\partial f}{\partial (1-x_n)}(y) + \sigma x_n (1-x_n) \frac{\partial^2 f}{\partial (1-x_n)^2}(y). \]

This proves that the diffusive part of the process \( Y \) is the same as the diffusive part of an \((n-2)\)-dimensional Wright-Fisher process. We still have to prove that \( \tilde{\mu} \) satisfies the assumptions we want. \( \tilde{\mu} \) can be rewritten in two different ways. First we have

\[ \tilde{\mu}_i(y, x) = x_i(1-x_i)s_i(x) - y_i \sum_{j=1}^{n-1} \mu_j(x) = y_i \left( (1-x_n)(1-x_i)s_i(x) - \sum_{j=1}^{n-1} \mu_j(x) \right). \]
Second,
\begin{equation}
\tilde{\mu}_i(y, x) = (1 - y_i)\mu_i(x) - y_i \sum_{j=1, j\neq i}^{n-1} \mu_j(x) = (1 - y_i)\mu_i(x) - y_i \sum_{j=1, j\neq i}^{n-1} x_j(1 - x_j)s_j(x).
\end{equation}

Let us focus on the last term. By assumption, we know that there exists a finite $C$ such that for any $x \in \Delta_{n-1}$, and for $1 \leq j \leq n-1$,
\[ |y_i(1 - x_j)s_j(x)| \leq C. \]

Hence, we get
\begin{equation}
\sum_{j=1, j\neq i}^{n-1} x_j(1 - x_j)s_j(x) \leq C \sum_{j=1, j\neq i}^{n-1} x_j = C(1 - x_i - x_n) = C(1 - x_n)(1 - y_i).
\end{equation}

Thus, from (5.10), we deduce that there exists a finite $C'$ such that
\[
\limsup_{y \in [0,1]^{n-1}, x \in \Delta_{n-1}, y_i \to 0^+} \frac{|\tilde{\mu}_i(y, x)|}{y_i(1 - y_i)} < C',
\]
and from (5.11) and (5.12) we deduce that
\[
\limsup_{y \in [0,1]^{n-1}, x \in \Delta_{n-1}, y_i \to 1^-} \frac{|\tilde{\mu}_i(y, x)|}{y_i(1 - y_i)} < C'.
\]

As we are working on compact sets this concludes the proof. \hfill \Box

5.4. Proofs of results on specific examples.

Proof of Lemma 3.4. From Remark 3 we know that it is enough to check that in the examples under consideration, $\mu_i(x)/(1 - x_i)$ is bounded, for $i \in E$ and $x \in \Delta_K$.

**Transitive ordering case:** Recall that according to (1.2),
\[ \mu_i(x) = \kappa \sum_{j=1}^{\infty} \pi_j \left( (x_0 + \ldots + x_j)^{j+1} - (x_0 + \ldots + x_{i-1})^{j+1} - x_i \right). \]

We can rewrite $\mu_i$ as the sum of two functions $\alpha_i$ and $\beta_i$ as follows
\[
\mu_i(x) = \kappa \sum_{j=1}^{\infty} \pi_j \left( x_i \sum_{k=0}^{j} (x_0 + \ldots + x_{i-1})^{j-k} - x_i \right)
\]
\[ = \kappa x_i \sum_{j=1}^{\infty} \pi_j \left( \sum_{k=0}^{j} (x_0 + \ldots + x_{i-1})^{j-k} - 1 \right)
\]
\[ = \kappa x_i \sum_{j=1}^{\infty} \pi_j \left( (x_0 + \ldots + x_i)^{j} - 1 + (x_0 + \ldots + x_{i-1}) \sum_{k=0}^{j-1} (x_0 + \ldots + x_i)^{k} (x_0 + \ldots + x_{i-1})^{j-k-1} \right)
\]
\[ =: \alpha_i + \beta_i. \]
First notice that
\[ |a_i| = \kappa x_i \sum_{j=1}^{\infty} \pi_j \left( 1 - (x_0 + \ldots + x_i)^j \right) \]
\[ = \kappa x_i (1 - (x_0 + \ldots + x_i)) \sum_{j=1}^{\infty} \pi_j \left( \sum_{k=0}^{j-1} (x_0 + \ldots + x_i)^k \right) \]
\[ \leq \kappa x_i (1 - x_i) \sum_{j=1}^{\infty} j \pi_j = x_i (1 - x_i) \beta, \]
where we recall that \( \beta \) has been defined in point (v) of Proposition 3.1. Second, we have,
\[ |\beta_i| = \kappa x_i (x_0 + \ldots + x_{i-1}) \sum_{j=1}^{\infty} \pi_j \left( \sum_{k=0}^{j-1} (x_0 + \ldots + x_i)^k (x_0 + \ldots + x_{i-1})^{j-k-1} \right) \]
\[ \leq \kappa x_i (1 - x_i) \sum_{j=1}^{\infty} j \pi_j = x_i (1 - x_i) \beta. \]
As a consequence,
\[ |\mu_i(x)| \leq 2\kappa \beta x_i (1 - x_i). \]

**RPS or food web case:** For \( i \in E \),
\[ |\mu_i(x)| = \kappa x_i \left| \sum_{j \neq i, j < i} x_j - \sum_{j \neq i, j < j} x_j \right| \leq \kappa x_i \left( \sum_{j \neq i, j < i} x_j + \sum_{j \neq i, j < j} x_j \right) \leq 2\kappa x_i (1 - x_i). \]

**Negative frequency-dependent selection:** Recall that when the distribution of \( K_v \) is concentrated on \( \{1, 3\} \), we get
\[ \mu_i(x) = 2\kappa s x_i \left[ \sum_{j \neq i} x_j^2 - x_i (1 - x_i) \right]. \]
As
\[ \sum_{j \neq i} x_j^2 \leq \sum_{j \neq i} x_j = 1 - x_i, \]
the assumptions of Proposition 3.1 are satisfied. If \( K_v = p \notin \{1, 3\} \), for a parent of type \( i \) to be chosen, a potential parent of type \( i \) has to be present. This ensures that \( \mu_i(x) \) can be written \( x_i \tilde{s}_i(x) \) with \( \tilde{s} \) bounded. Moreover, if we exclude the case when there are only parents of type \( i \) (which contributes with a term \( \rho_N x_i (x_i^{p-1} - 1) \) in \( \mu_i \)), we get terms of the form (for \( u < p \))
\[ \kappa x_i^u \sum_{B_k} A_k \sum_{j_1, \ldots, j_{p-u} \in B_k} x_{j_1 \ldots j_{p-u}}, \]
where \( A_k \in \mathbb{Q} \) and \( B_k \) are sets which do not contain \( i \). And
\[ \kappa \sum_{j_1, \ldots, j_{p-u} \in B_k} x_{j_1 \ldots j_{p-u}} \leq \kappa |B_k| \sup_{j \neq i} x_j \leq \kappa |B_k| (1 - x_i). \]
This ensures that the negative frequency-dependent selection rule satisfies the assumptions of Proposition 3.1.
**Positive frequency-dependent selection:** In this case, the calculations are very similar to the previous case. We thus do not give details.

**Logistic competition:** The result is straightforward in this case.

Proof of Lemma 4.1. Lemma 4.1 is a consequence of Lemma 4.7 in [28]. Let us first assume condition (i) of Lemma 4.1. Then from Lemma 4.7 in [28], \((D_1, t \geq 0)\) has a unique stationary distribution. Let us choose \(i \in E\) and divide \(E\) into two subsets, \(E_1 := \{0, ..., i\}\) and \(E_2 := \{i + 1, ..., K\}\). Treating the types of \(E_1\) as the weak type 0 in [28], and the types of \(E_1\) as the selected type 1 in [28], we also get applying this Lemma that

\[
\mathbb{P}_x \left( \lim_{t \to \infty} (X_0(t) + ... + X_i(t)) = 1 \right) = \phi_o(x_0 + ... + x_i)
\]

and

\[
\mathbb{P}_x \left( \lim_{t \to \infty} (X_0(t) + ... + X_i(t)) = 0 \right) = 1 - \phi_o(x_0 + ... + x_i).
\]

Applying the same trick to \(E_1 := \{0, ..., i - 1\}\) and \(E_2 := \{i, ..., K\}\) allows to conclude the proof of point (i).

Let us now assume (ii). Then the process \(X_{\theta}\) has the same properties that the process \(X_1\) in [28], for which there is almost sure fixation in finite time. This ends the proof.

Proof of Lemma 4.4. The proof will be based on studying the action of the generator \(A\) of \(X\) over the Logarithm. In other words, we will use the Logarithm as a Lyapunov function to study the long term behaviour of \((X_t, t \geq 0)\).

First, applying Theorem 1 of [29], we obtain the existence of random variables \(V\) and \(W\) in [0,1] with continuous densities such that the infinitesimal generator \(A\) of \((X(t), t \geq 0)\) applied to a function \(g\) at \(x\) on \(\Delta_3\) can be rewritten

\[
A g(x) = \sum_{i=1}^{3} \mu_i(x) \frac{\partial g}{\partial x_i}(x) + \sum_{i,j=1}^{3} \sigma_{ij}^2(x) \left( \sigma \frac{\partial^2 g}{\partial x_i x_j}(x) + \frac{1}{2} \mathbb{E} \left[ \frac{\partial^2 g}{\partial x_i x_j}(x(1-W) + VW e_i) \right] \right)
\]

\[
= \sum_{i=1}^{3} \mu_i(x) \frac{\partial g}{\partial x_i}(x) + \sum_{i=1}^{3} \left( \sum_{j=1}^{3} \sigma_{ij}^2(x) \left( \sigma \frac{\partial^2 g}{\partial x_i x_j}(x) + \frac{1}{2} \mathbb{E} \left[ \frac{\partial^2 g}{\partial x_i x_j}(x(1-W) + VW e_i) \right] \right) \right).
\]

Let

\[
f_i(X(t)) = \ln(X_i(t)) = \ln \left( 1 - X_{\text{mod}3(i-1)}(t) + X_{\text{mod}3(i+1)}(t) \right).
\]

Then, a direct calculation leads to

\[
\sum_{i=1}^{3} \mu_i(x) \frac{\partial f_1}{\partial x_i}(x) = 2\kappa(x_3 - x_2).
\]

Moreover, for any \(y \in \Delta_3\),

\[
\sum_{j=1}^{3} \sigma_{ij}^2(x) \frac{\partial^2 f_1}{\partial x_i x_j}(y) = x_1(1 - x_1) \left( \frac{1}{y_1^2} \right) - x_1 x_2 \left( \frac{1}{y_1^2} \right) - x_1 x_3 \left( \frac{1}{y_1^2} \right) + \frac{2x_1(x_2 + x_3)}{y_1^2},
\]

\[
\sum_{j=1}^{3} \sigma_{ij}^2(x) \frac{\partial^2 f_1}{\partial x_i x_j}(y) = -x_1 x_2 \left( \frac{1}{y_1^2} \right) + x_2(1 - x_2) \left( \frac{1}{y_1^2} \right) - x_2 x_3 \left( \frac{1}{y_1^2} \right) + \frac{2x_1 x_2}{y_1^2},
\]

and

\[
\sum_{j=1}^{3} \sigma_{ij}^2(x) \frac{\partial^2 f_1}{\partial x_i x_j}(y) = -2 \frac{x_1 x_3}{y_1^2},
\]
by a similar computation. Adding all the terms yields for the function $f_1$:

$$A f_1(x) = 2\kappa(x_3 - x_2) - (1 - x_1) \left( \frac{4\sigma}{x_1} + \mathbb{E} \left[ \frac{x_1}{x_1(1 - W) + VW} + \frac{1}{1 - W} \right] \right),$$

and adding over the three functions yields

$$A(f_1 + f_2 + f_3)(x) = -\sum_{i=1}^{3} (1 - x_i) \left( \frac{4\sigma}{x_i} + \mathbb{E} \left[ \frac{x_i}{x_i(1 - W) + VW} + \frac{1}{1 - W} \right] \right) \leq -C,$$

where $C$ is a positive constant if and only if $\sigma \neq 0$ or $\Lambda \neq 0$. We can now conclude using the generator equation that

$$\lim_{t \to \infty} \mathbb{E} \left[ \ln(X_1(t) X_2(t) X_3(t)) \right] - \ln(x_1 x_2 x_3) = \lim_{t \to \infty} \int_0^t \mathbb{E} [Af_1(X(s)) + Af_2(X(s)) + Af_3(X(s))] ds \leq \lim_{t \to \infty} -\int_0^t C ds = -\infty.$$

\[\square\]

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