Looking inside a black hole

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Abstract
The cosmic censorship conjecture posits that singularities forming to the future of a regular Cauchy surface are hidden by an event horizon. Consequently any topological structures will ultimately collapse within the horizon of a black hole and so no observer can actively probe them classically. We consider here a quantum analogue of this problem, in which we compare the transition rates of an Unruh–DeWitt detector placed outside the horizon of an eternal BTZ black hole and its associated geon counterpart. We find the transition rates differ, with the latter being time-dependent, implying that we are indeed able to probe the structure of the singularity from outside the horizon.

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(The observed triviality of the spatial topology of our universe (namely its continuous deformability to $\mathbb{R}^3$) stands in stark contrast to the properties of general relativity, which admits all possible topologies. The topological censorship theorem [1, 2] resolves this conundrum insofar as it relegates all isolated topological structures (such as wormholes) inaccessible to observers by any classical experimental means insofar as causal curves beginning near past null infinity passing through the interior of an asymptotically AdS (or flat) spacetime (obeying physically reasonable conditions) detect no topological structure not also present in the boundary-at-infinity [2, 3].

Here we show that observers can probe the global topology of a spacetime via quantum mechanical means. Specifically, we examine two different situations in (2+1)-dimensions: an Unruh–DeWitt detector sitting outside a non-rotating BTZ black hole and the same detector sitting outside the associated $\mathbb{RP}^2$ geon. In both cases the detector travels along the same
trajectory: not rotating and remaining at a fixed distance from the horizon. The difference between the two spacetimes arises in the topological structure of the singularity since the metrics for each are locally identical. The inability to actively probe such topology classically [1, 2] naively suggests identical transition rates, and an investigation of the R P 3 geon relative to its Schwarzschild counterpart led to a conjecture that the transition rates would be the same for a detector that only operates in the asymptotic past/future, as well as for a detector far from the horizon for finite times [4]. However passive observation of topological structures (e.g. via a white hole) is not forbidden, and since quantum-mechanically the detector response is determined by the state we might expect its response rate to be affected by the global properties of the spacetime. Topological identifications in spacetimes with constant curvature have been shown to yield different detector response rates [5–8] and it is therefore reasonable to expect differing responses even though the spacetime geometries near the trajectories in question happen to be identical. We find the detector does indeed measure a difference in the transition rate, yielding a probe of the topological structure of the singularity.

We consider an Unruh–Dewitt detector coupled to a scalar field \( \phi(x) \), treated to first order in perturbation theory [9–11]. The detector is a simplified model of a real particle detector that is linearly coupled to a scalar field. It can be thought of as an idealized atom with two energy levels, denoted by \( |0\rangle_d \) and \( |E\rangle_d \), with respective energy eigenvalues 0 and \( E \). The detector’s interaction with the field is described by the Hamiltonian

\[
H_{\text{int}} = c \chi(\tau) \mu(\tau) \phi(x(\tau)),
\]

where the point-like detector is moving along a timelike trajectory described by \( x^\mu(\tau) \), \( c \) is a small coupling constant, \( \mu(\tau) \) is the monopole moment of the detector, and \( \chi(\tau) \) is a switching function that is positive during the interaction of the detector with the field and vanishing elsewhere.

In general, as the detector moves along the trajectory \( x^\mu(\tau) \), it will transition to \( |E\rangle_d \) and the field will transition to some state \( |\psi\rangle \). For \( c \ll 1 \), the probability of this transition to first order in perturbation theory is [10]

\[
P(E) = c^2 |\langle 0|\mu(0)|E\rangle_d|^2 \mathcal{F}(E),
\]

where \( \mathcal{F}(E) \) is the response function that encodes the information about the detector’s trajectory, the switching function, and the initial state of the field. Explicitly

\[
\mathcal{F}(E) = \text{Re} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) e^{-iEs} G^+(u, u-s),
\]

where \( G^+(x, x') \) is the Wightman Green function associated with the scalar field \( \phi \).

The transition rate in the sharp switching limit is

\[
\dot{\mathcal{F}}(E) = \frac{1}{4} + 2 \text{Im} \int_0^{\Delta\tau} ds e^{-iEs} G^+(\tau, \tau-s),
\]

where \( \Delta\tau = \tau - \tau_0 \) with \( \tau_0 \) being the proper time at which the detector was switched on, \( \tau \) being the read-off time, i.e. the proper time at which the instantaneous transition rate is read off.

We wish to compute the transition rate for a detector in the background of the BTZ black hole [12] and compare this to the associated geon [13, 14]. These solutions to Einstein’s equations can be constructed by making appropriate identifications of three-dimensional anti-de Sitter (AdS 3 ) spacetime. Specifically, AdS 3 can be obtained from the flat space \( \mathbb{R}^{2,1} \) with coordinates \( (X_1, X_2, T_1, T_2) \) and metric

\[
d s^2 = -dT_1^2 -dT_2^2 + dX_1^2 + dX_2^2,
\]
by restriction to the submanifold
\[ X_1^2 - T_1^2 + X_2^2 - T_2^2 = -\ell^2, \]  
where \( \Lambda = -\ell^{-2} \). The resulting metric expressed in both Schwarzschild-like and associated null coordinates is
\[ ds^2 = -f(r) \, dt^2 + dr^2 + r^2 \, d\phi^2 \]
\[ = -\frac{\ell^2}{(1 + UV)^2} [-4 \, dU \, dV + M(1 - UV)^2 \, d\phi^2], \]  
where \( f(r) = -M + r^2/\ell^2 \); the horizon is at \( r_h = \ell \sqrt{M} \). The relationship between each of the coordinates is (in the region where \( T_1 > |X_1| \)):
\[ T_1 = \ell \left( \frac{1 - UV}{1 + UV} \right) \cosh \sqrt{M} \phi = \ell \sqrt{\alpha(r)} \cosh \left( \frac{r_h}{\ell} \phi \right), \]
\[ X_1 = \ell \left( \frac{1 - UV}{1 + UV} \right) \sinh \sqrt{M} \phi = \ell \sqrt{\alpha(r)} \sinh \left( \frac{r_h}{\ell} \phi \right), \]
\[ T_2 = \ell \left( \frac{V + U}{1 + UV} \right) = \ell \sqrt{\alpha(r) - 1} \sinh \left( \frac{r_h}{\ell} t \right), \]
\[ X_2 = \ell \left( \frac{V - U}{1 + UV} \right) = \ell \sqrt{\alpha(r) - 1} \cosh \left( \frac{r_h}{\ell} t \right), \]  
where \( \alpha(r) = r^2/r_h^2, -1 < UV < 1, \) and \( M > 0 \). The BTZ spacetime is obtained by making a periodic identification in the coordinate \( \phi : (U, V, \phi) \sim (U, V, \phi + 2\pi) \).

To obtain the geon quotient of the BTZ spacetime \([14]\) we note from equation (7) that the metric admits the Killing vector \( \xi = \partial_\phi \), which generates the freely-acting involutive isometry \( P = \exp (\pi r_h \xi) \). The map
\[ J : (U, V, \phi) \rightarrow (V, U, P(\phi)), \]  
is then a freely-acting involutive isometry acting on the entire BTZ spacetime \( \mathcal{M}_{\text{BTZ}} \). Together with the identity map it generates the isometry group \( \Gamma := \{ \text{Id}_{\text{BTZ}}, J \} \cong \mathbb{Z}_2 \). The geon spacetime \( \mathcal{M}_{\text{geon}} \) is then the quotient spacetime of the BTZ hole with the group \( \mathbb{Z}_2 \):
\[ \mathcal{M}_{\text{geon}} = \mathcal{M}_{\text{BTZ}}/\mathbb{Z}_2. \]  
It is identical to the BTZ spacetime outside the horizon, but (see figure 1) inside the topology of the singularity changes from that of a point to the real projective space \( \mathbb{RP}^2 \) \([14]\). Likewise, the Hartle–Hawking vacuum, which is the vacuum of mode functions of positive frequency with respect to the affine parameters of the horizon-generating null geodesics, is invariant under the involution \( J \). Hence it induces a unique vacuum on the geon \([4]\).

To obtain the transition rate of a detector in the BTZ spacetime we require the appropriate Wightman function as demanded by equation (4). The Wightman functions of the BTZ spacetime are well-known and obtained via the method of images \([10, 11, 15]\):
\[ G_{\text{BTZ}}^{(\gamma)}(x, x') = \sum_n G_{\gamma}^{(\gamma)}(x, A^n x'), \]  
where \( A x' \) denotes the periodic identification \( (t, r, \phi) \sim (t, r, \phi + 2\pi) \) on the point \( x' \) and the function \( G_{\gamma}^{(\gamma)} \) denotes the AdS\(_{\gamma}\) Wightman function
\[ G_{\gamma}^{(\gamma)}(x, x') = \frac{1}{4\pi} \left( \frac{1}{\sqrt{\Delta X^2(x, x')}} - \frac{\zeta}{\sqrt{\Delta X^2(x, x') + 4\ell^2}} \right), \]  
where \( \Delta X^2(x, x') = \sum_n A^n \Delta X^2(x, A^n x') \).
Figure 1. (a) A conformal diagram of the BTZ hole. Each point in the diagram represents a suppressed $S^1$. (b) A conformal diagram of the geon associated with the BTZ spacetime. The region not on the dashed line is identical to the diagram in (a); however on the dashed line each point in the diagram again represents a suppressed $S^1$ but with half the circumference of the suppressed $S^1$ in diagram (a).

where the parameter $\zeta$ specifies the boundary condition at infinity—either transparent, Dirichlet, or Neumann corresponding to $\zeta$ taking the values $\{0, 1, -1\}$ respectively—and the function $\Delta X^2(x, x')$ is the squared geodesic distance between $x$ and $x'$ in the flat $\mathbb{R}^{2,2}$ space given by

$$\Delta X^2(x, x') = -(T_1 - T_1')^2 - (T_2 - T_2')^2 + (X_1 - X_1')^2 + (X_2 - X_2')^2.$$  (13)

The transition rate of the detector is obtained by substituting equation (11) into equation (4), which yields

$$\dot{\mathcal{F}}_{\text{BTZ}}(E) = \frac{1}{4} + \frac{1}{2\pi\sqrt{2}} \sum_{n=-\infty}^{\infty} \int_0^{\Delta \tau / \ell} d\tilde{s} \text{Re} \left[ e^{-i\Delta \ell \tilde{s}} \left( \frac{1}{\sqrt{\Delta X^2_n}} - \frac{\zeta}{\sqrt{\Delta X^2_n + 2}} \right) \right],$$  (14)

where we have introduced the dimensionless integration variable $\tilde{s} := s/\ell$ and written

$$\Delta X^2_n := \Delta X^2(x(\tau), \Lambda^a x(\tau - \ell \tilde{s})) = -1 + \sqrt{\alpha(r) \alpha(r')} \cosh(r_n/\ell)(\phi - \phi' - 2\pi n)$$

$$- \sqrt{\alpha(r) - 1} \cosh[(r_n/\ell^2)(t - t')],$$  (15)

where the unprimed coordinates are evaluated at $x(\tau)$ and the primed coordinates are evaluated at $x(\tau - \ell \tilde{s})$.

Let us suppose the trajectory of our detector is such that it is sitting at a constant distance from the center of the black hole so that the trajectory is given by

$$r = \text{constant}, \quad t = \frac{\ell}{\sqrt{r^2 - r_n^2}} \tau, \quad \phi = 0,$$  (16)

where $r$ specifies the radial location and $\tau$ is the proper time of the detector. $\Delta X^2_n$ becomes

$$\Delta X^2_n = 2(\alpha - 1)[K_n - \sinh^2(\Xi \tilde{s})],$$  (17)

where we have suppressed the dependence of $\alpha$ on $r$ and have introduced $K_n := (1 - \alpha^{-1})^{-1} \sinh^2(n\pi r_n/\ell)$ and $\Xi := (2\sqrt{\alpha - 1})^{-1}$. The transition rate is now
The transition rate does not depend on the read-off time of the detector. This will not be true in the asymptotic past so that \( \Delta \tau / \ell \) goes to infinity.

This expression can be further simplified to \([10]\):

\[
\dot{J}^{\text{BTZ}}(E) = \frac{e^{-\beta E \ell / 2}}{2\pi} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} dy \cos(y E \ell / \pi) \left[ \frac{1}{\sqrt{K_n + \cosh^2 y}} - \frac{\zeta}{\sqrt{Q_n + \cosh^2 y}} \right],
\]

where \( \beta = \pi \Xi^{-1} = 2\pi \sqrt{\alpha - 1} \) and \( y = \Xi \tilde{s} = \pi \tilde{s} / \beta \). We have dropped the subscript \( \tau \) since the transition rate does not depend on the read-off time of the detector. This will not be true in the case of the geon.

Since the geon spacetime is obtained by acting the isometry \( J \), equation \((9)\), on the BTZ spacetime, we can again make use of the method of images to obtain the geon Wightman functions

\[
G_{\text{geo}}^{+}(x, x') = \sum_{m \in \{0, 1\}} G_{\text{BTZ}}^+(x, J^m x') = \sum_{m \in \{0, 1\}} \sum_{n} G_{m}^{(n)}(x, J^m \Lambda_n x'),
\]

where we only sum over \( m \in \{ 0, 1 \} \) since \( J \) is involutive. The \( m = 0 \) term is identical to equation \((12)\). The \( m = 1 \) term is obtained by applying \( J \) once to the primed coordinates. Solving equations \((8)\) for \((t, r)\) and \((X_2, T_2)\) in terms of \((U, V)\) results in \( t \rightarrow -t \) (and \( X_2 \rightarrow -X_2 \)) upon swapping \( U \) and \( V \). Applying \( P(\phi) \) we have \( \phi' \rightarrow \phi' + \pi \); substituting this into equation \((4)\) yields

\[
\dot{J}_{\text{geo}}(E) = \dot{J}^{\text{BTZ}}(E) + \Delta \dot{J}_{\text{geo}}(E, \tau),
\]

for the transition rate of a detector in the geon spacetime traversing the same trajectory described in equation \((16)\).

The additional contribution due to the geon is

\[
\Delta \dot{J}_{\text{geo}}(E, \tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} dy \cos(E \ell y / \pi) \times \left[ \frac{1}{\sqrt{\tilde{K}_n + \cosh^2(y - 2\tau \Xi)}} - \frac{\zeta}{\sqrt{\tilde{Q}_n + \cosh^2(y - 2\tau \Xi)}} \right]
\]

with

\[
\tilde{K}_n := (1 - \alpha^{-1})^{-1} \sinh^2 \left( (n + \frac{1}{2}) \frac{\pi r_n}{\ell} \right),
\]

\[
\tilde{Q}_n := \tilde{K}_n + (\alpha - 1)^{-1}.
\]

We see that the transition rate of the detector in the geon spacetime now depends on the transition rate read-out time \( \tau \).

Having now derived the transition rate of a detector in the BTZ spacetime, equation \((19)\), and in the associated geon spacetime, equation \((21)\), we are able to evaluate the rates numerically, as shown in figure \( 2 \). The BTZ transition rate depends on four parameters: the energy gap of the detector \( E \), which appears only in the dimensionless ratio \( E \beta \ell \); the horizon
radius $r_h = \ell \sqrt{M}$, which characterizes the mass of the hole; $\alpha(r)$ which encodes the trajectory of the detector; and $\zeta$ which specifies the boundary conditions the AdS$_3$ Wightman function satisfies. The geon transition rate depends on the same four parameters and the additional parameter $\tau$, the read-out time.

The sums appearing in Equations (19) and (21) were evaluated for $-3 \leq n \leq 3$ as larger terms do not contribute significantly to the transition rate. This can be seen by noting that for the $n$th term, the denominator appearing in the integrand goes as $e^n$. The integrals appearing in Equations (19) and (21) were evaluated for $-3 \leq n \leq 3$ as larger terms do not contribute significantly to the transition rate. This can be seen by noting that for the $n$th term, the denominator appearing in the integrand goes as $e^n$. The integrals appearing in Equations (19) and (21) were evaluated for $-3 \leq n \leq 3$ as larger terms do not contribute significantly to the transition rate. This can be seen by noting that for the $n$th term, the denominator appearing in the integrand goes as $e^n$. The integrals appearing...
Figure 3. A plot of the integral of the derivative of the response function (the integral of the curves given in figure 2(b)) as a function of $\tau$. From equation (2) this is proportional to the probability, which must remain positive. A hundred points were uniformly sampled between $-40$ and $40$ and a simple Riemann sum integration method was used.

In the transition rates were evaluated using Gauss–Kronrod quadrature [16] to a precision of ±0.001.

We see from figure 2 that for increasingly negative $\tau$ the BTZ and geon transition rates become virtually identical, whereas for increasingly positive $\tau$ they significantly differ, with the geon transition rate becoming increasingly oscillatory. We also see that for sufficiently large $E$ the transition rate is sometimes negative. The interpretation of this requires some care [6, 7]. While the transition rate is proportional to the derivative of the probability of the transition with respect to the switch off time, this is not the same thing as the response of a detector turned on at time $\tau$ and then turned off at time $\tau + \delta \tau$ in the limit $\delta \tau \to 0$. Rather it corresponds to the comparison of detector responses in two different ensembles of detectors, one switched off at $\tau$ the other switched off at $\tau + \delta \tau$ as $\delta \tau \to 0$. The overall transition probability remains positive, as illustrated in figure 3 for the geon.

In this paper we have shown that a detector placed outside a BTZ hole and the same detector placed outside the associated geon will have different transition rates, even though the local geometry is identical, yielding a probe of the topology of a spacetime via local nonstationary measurements even if the topological structure is hidden behind a Killing horizon. It is reasonable to expect that similar phenomena hold in higher dimensions. Our work suggests that local quantum probes are in general sensitive to quantum states that encode information classically hidden by horizons. The extent to which this has impact for topological censorship—that all causal curves in a spacetime obeying physically reasonable conditions are contractible to a curve on the boundary-at-infinity $I [1, 2]$—remains to be explored. One example would be that of scattering a quantum field off of a black hole and its associated geon to see if topological structures hidden behind the horizon could be detected, a kind of quantum super radiance effect. Another possible means of exploring these ideas would be to investigate higher-dimensional BTZ black holes as compared to their $\mathbb{Z}_2$ quotients [17] so as to ferret out time-dependent effects.
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