CONTRACTIBLE FLOW OF STABILITY CONDITIONS VIA GLOBAL DIMENSION FUNCTION

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Abstract. We introduce an analytic method that uses the global dimension function $\text{gldim}$ to produce contractible flows on the space $\text{Stab} \mathcal{D}$ of stability conditions on a triangulated category $\mathcal{D}$. In the case when $\mathcal{D} = \mathcal{D}(S^\lambda)$ is the topological Fukaya category of a graded surface $S^\lambda$, we show that $\text{gldim}^{-1}(0,y)$ contracts to $\text{gldim}^{-1}(0,x)$ for any $1 \leq x \leq y$, provided $(x,y)$ does not contain ‘critical’ values $\{1 + w_\partial/m_\partial | w_\partial \geq 0, \partial \in \partial S^\lambda\}$, where the pair $(m_\partial, w_\partial)$ consists of the number $m_\partial$ of marked points and the winding number $w_\partial$ associated to a boundary component $\partial$ of $S^\lambda$. One consequence is that the global dimension of $\mathcal{D}(S^\lambda)$ must be one of these critical values.

Besides, we remove the assumptions in Kikuta-Ouchi-Takahashi’s classification result on triangulated categories with global dimension less than 1.

Key words: global dimension function, stability conditions, contractible flow, topological Fukaya categories

1. Introduction

1.1. Deformation of stability conditions. The space of stability conditions on a triangulated category, introduced by Bridgeland [B1], is an interesting homological invariant, which relates representation theory of algebras and algebraic/symplectic geometry. Original motivation comes from the study of D-brands in string theory, mirror symmetry, Donaldson-Thomas theory, etc. One of the breakthroughs in this direction is the correspondence between this type of spaces and the moduli spaces of (framed) quadratic differentials, shown by Bridgeland-Smith [BS] for the Calabi-Yau-3 surface case (cf. [KQ]) and Haiden-Katzarkov-Kontsevich [HKK] for the Calabi-Yau-∞ surface case. Aiming to make a precise link between these two works, we introduce $q$-deformation of categories, stability conditions and quadratic differentials in the prequels [IQ1, IQ2]. Namely, for a Calabi-Yau-$X$ category $\mathcal{D}_X$, whose Grothendieck group is the $q$-deformation of a rank $n$ lattice, and any complex number $s$, we identify a subspace $\text{QStab}_s \mathcal{D}_X$ (of complex dimension $n$) in the space $\text{Stab} \mathcal{D}_X$ of stability conditions of $\mathcal{D}_X$.

We show that one can glue these subspaces $\text{QStab}_s \mathcal{D}_X$ under certain conditions into a complex manifold of dimension $n + 1$. Moreover, $\text{QStab}_s \mathcal{D}_X$ can be embedded into the usual spaces of stability conditions on the corresponding Calabi-Yau-$N$ categories, when $s = N$ is a positive integer. So the next question is how $\text{QStab}_s \mathcal{D}_X$ deforms when the ‘Calabi-Yau dimension’ $s$ varies, which will lead to deformation of spaces of stability conditions along $s$-direction.

From our construction of $\text{QStab}_s \mathcal{D}_X$ in [IQ1, IQ2], the question is closely related to the study of stability condition on Calabi-Yau-∞ categories (e.g. usual bounded derived
categories of algebras or of coherent sheaves on Fano varieties). One of the key tools here is the global dimension function $\text{gldim}$ (see [Q1, IQ1]). Our philosophy is that such a function is piecewise Morse and could shed light on deformation of stability conditions as well as contractibility of spaces of stability conditions (cf. [FLLQ] for the case of coherent sheaves on the projective plane).

1.2. Global dimension of triangulated categories. Global dimension is a classical homological invariant of algebras $[A]$, or equivalently of their abelian categories. From 90’s, triangulated-derived categories become more popular than abelian categories as they carry more symmetries and are ‘better’ in certain sense. It is natural to explore the corresponding invariant for triangulated categories as global dimension for abelian categories. In [Q1], we proposed the infimum $Gd_D$ of the global dimension function $\text{gldim}$ on the space of stability conditions of $D$ to be a nice candidate as the global dimension for a triangulated category $D$ (cf. [Q3]).

In [Q1], we have shown that $Gd_{D_{\infty}}(Q) = 1$ if $Q$ is a non-Dynkin acyclic quiver and $Gd_{D_{\infty}}(Q) = 1 - 2/h_Q$ if $Q$ is a Dynkin quiver, where $D_{\infty}(Q) = D^b(\text{mod } kQ)$, $kQ$ is the path algebra of $Q$ and $h_Q$ the Coxeter number of $Q$ (when $Q$ is a Dynkin quiver). In [KOT], Kikuta-Ouchi-Takahashi (KOT) showed that in fact, under some minor condition, any triangulated category $D$ with $Gd_D < 1$ is equivalent to $D_{\infty}(Q)$ for some Dynkin quiver $Q$. This is a classification theorem of finite type triangulated categories via our global dimension $Gd$, comparing to the classical version for abelian categories—Gabriel’s famous theorem $[G]$: 

- The module category $\text{mod } kQ$ of a quiver $Q$ is of finite type if and only if $Q$ is a Dynkin quiver.

In Section 3, we refine KOT’s classification theorem by removing their assumptions, where the statement becomes (Theorem 3.2):

- any triangulated category $D$ with $Gd_D < 1$ must be of the form $D_{\infty}(Q)/\iota$ for some Dynkin quiver $Q$ and a graph automorphism $\iota$ of $Q$.

This is the analogue of Dlab-Ringel’s refinement ([DR]) of Gabriel’s result.

In Section 6, we calculate global dimensions of graded affine type A quivers (Theorem 6.4) as a first example of non-integer global dimensions of (non-Calabi-Yau) triangulated categories.

Note that we will actually use this classification result in the later part of the paper.

1.3. Test field: topological Fukaya categories. We mainly focus on topological Fukaya category $D_{\infty}(S^3)$ of a graded marked surface $S^3$ in this paper, which can be also realized as the bounded derived category of a graded gentle algebra. There have been a lot of works on this categories, namely,

- the classification of objects in $D_{\infty}(S^3)$ in [HKK];
- the description of stability conditions on $D_{\infty}(S^3)$ via quadratic differentials in [HKK], cf. [T].
the study of triangle equivalence between different $D_\infty(S^\lambda)$ in [LP];
- the description on morphisms in $D_\infty(S^\lambda)$ in [IQZ] (as a simplified case).

Based on these works, we prove the following.

**Theorem 1.** Let $S^\lambda$ be a graded marked surface as in Section 4.1.

- Any stability condition is gldim-reachable (Corollary 5.11).
- If $\text{gldim} \sigma \geq 1$, then it equals the maximal angle of the core of the corresponding quadratic differentials (Proposition 5.10 and 5.14).
- If $1 \leq \text{gldim} \sigma / \in V(S^\lambda)$, then there is a real submanifold of $\text{Stab} D_\infty(S^\lambda)$, open in its closure and, restricted to which, gldim is differentiable with no critical point (Theorem 5.15).
- If $1 \leq x < y$ such that $(x, y) \cap V(S^\lambda) = \emptyset$, then $\text{Stab} \leq y D_\infty(S^\lambda)$ contracts to $\text{Stab} \leq x D_\infty(S^\lambda)$ (Corollary 5.16).
- $\text{Gd} D_\infty(S^\lambda)$ is in $V(S^\lambda)$ (Corollary 5.18).

Here $V(S^\lambda) = \{1 + w_\partial/m_\partial \mid \partial \subset \partial S, w_\partial \geq 0\}$ is the set of critical values, where the pair $(m_\partial, w_\partial)$ consist of the number $m_\partial$ of marked points and the winding number $w_\partial$ associated to a boundary component $\partial$ of $S^\lambda$.

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2. Preliminaries

2.1. Global dimension function of stability conditions. Following Bridgeland [B1], we recall the notion of stability conditions on triangulated categories.

Throughout the paper, $\mathcal{D}$ is a triangulated category with Grothendieck group $K(\mathcal{D}) \cong \mathbb{Z}^n$ for some integer $n$. Denote by $\text{Ind} \mathcal{D}$ the set of (isomorphism classes of) indecomposable objects in $\mathcal{D}$. Let $k$ be an algebraically closed field.

**Definition 2.1.** A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z: K(\mathcal{D}) \to \mathbb{C}$, called the central charge, and a family of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for $\phi \in \mathbb{R}$, called the slicing, satisfying the following conditions:

(a) if $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E)e^{i\pi \phi}$ for some $m(E) \in \mathbb{R}_{>0}$,
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
(c) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i) (i = 1, 2)$, then $\text{Hom}(A_1, A_2) = 0$,
(d) for $0 \neq E \in \mathcal{D}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_l$$

(2.1)
and a collection of exact triangles (known as the HN-filtration)

\[ 0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \ldots \rightarrow E_{l-1} \rightarrow E_l = E \]

with \( A_i \in P(\phi_i) \) for all \( i \).

Nonzero objects in \( P(\phi) \) are called semistable of phase \( \phi \) and simple objects in \( P(\phi) \) are called stable of phase \( \phi \). For semistable object \( E \in P(\phi) \), denote by \( \phi_\sigma(E) = \phi \) its phase. For any object \( E \), define its upper/lower phases

\[ \phi^+_\sigma(E) = \phi_1, \quad \phi^-_\sigma(E) = \phi_l \]

via the HN-filtration, respectively.

In this paper, we will always assume that stability condition satisfies the technical condition, known as the support property, see e.g. [IQ1] for more details. There is a natural \( \mathbb{C} \)-action on the set \( \text{Stab}(\mathcal{D}) \) of all stability conditions on \( \mathcal{D} \), namely:

\[ s \cdot (Z, P) = (Z \cdot e^{-ixs}, P_{\text{Re}(s)}) \]

where \( P_x(\phi) = P(\phi + x) \). Any auto-equivalence \( \Phi \in \text{Aut}(\mathcal{D}) \) also acts naturally on \( \text{Stab}(\mathcal{D}) \) as

\[ \Phi(Z, P) = (Z \circ \Phi^{-1}, \Phi(P)) \]

Recall Bridgeland’s key result [B1], that \( \text{Stab} \mathcal{D} \) is a complex manifold with local homeomorphism

\[ Z : \text{Stab} \mathcal{D} \rightarrow \text{Hom}_2(K(\mathcal{D}), \mathbb{C}), \quad (Z, P) \mapsto Z. \quad (2.2) \]

**Definition 2.2.** Given a slicing \( P \) on a triangulated category \( \mathcal{D} \). Define the global dimension of \( P \) by

\[ \text{gldim } P = \sup \{ \phi_2 - \phi_1 | \text{Hom}(P(\phi_1), P(\phi_2)) \neq 0 \} \in \mathbb{R}_{\geq 0} \cup \{ +\infty \}. \quad (2.3) \]

The global dimension of a stability condition \( \sigma = (Z, P) \) is defined to be \( \text{gldim } P \). The global dimension \( \text{Gd} \mathcal{D} \) of \( \mathcal{D} \) is defined as

\[ \text{Gd} \mathcal{D} : = \inf \text{gldim } \text{Stab} \mathcal{D}. \]

We say \( P \) (or \( \sigma \)) is gldim-reachable if there exist \( \phi_1 \) and \( \phi_2 \) such that

\[ \text{Hom}(P(\phi_1), P(\phi_2)) \neq 0 \quad \text{and} \quad \text{gldim } P = \phi_2 - \phi_1. \]

We say \( \mathcal{D} \) is gldim-reachable if there exists \( \sigma \) such that \( \text{gldim } \sigma = \text{Gd } \mathcal{D} \). Note that it is possible that \( \text{Stab} \mathcal{D} = \emptyset \) and then \( \text{Gd } \mathcal{D} \) is not defined.

**Example 2.3.** By [Q1, KOT], we have the following:

- If \( \mathcal{D} = \mathcal{D}_\infty(Q) \) is the bounded derived category of the path algebra of an acyclic quiver \( Q \), then \( \mathcal{D} \) is gldim-reachable.
- If \( \mathcal{D} = \mathcal{D}^{b}(\text{coh } X) \) is the bounded derived category of the coherent sheaves on a smooth projective curve \( X \) of genus \( g \) (over \( \mathbb{C} \)), then \( \text{Gd } \mathcal{D} = 1 \) and
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- \( \mathcal{D} \) is gldim-reachable if \( g = 0, 1 \);
- \( \mathcal{D} \) is not gldim-reachable if \( g > 1 \).

In [IQ1], we have shown that gldim is a continuous function on Stab \( \mathcal{D} \), which is invariant under the \( \mathbb{C} \)-action and Aut \( \mathcal{D} \).

**Notations 2.4.** Let
\[
\text{Stab}_{\mathcal{D}} G := \text{Stab} \mathcal{D} \cap \text{gldim}^{-1}(I)
\]
for any \( I \subset \mathbb{R} \).

A stability condition \( \sigma \) on \( \mathcal{D} \) is *totally (semi)stable* if every indecomposable object is (semi)stable with respect to \( \sigma \). Note that \( \text{Stab}_{<1} \mathcal{D} \) consists of precisely all totally semistable stability conditions, and \( \text{Stab}_{<1} \mathcal{D} \) consists of all totally stable stability conditions which are gldim-reachable. ([Q1, Prop. 3.5]).

**2.2. Type A example.** Let us describe all totally stable stability conditions for type A quiver and give explicit formula of global dimension function in such a case. Denote by Poly\((n + 1)\) the moduli space of convex \((n + 1)\)-gon \( \mathbf{P} \subset \mathbb{C} \), where the vertices \( \{V_i \in \mathbb{C} \mid 0 \leq i \leq n\} \) of the polygons are labelled in anticlockwise order with \( V_0 = 0 \) and \( V_1 = 1 \). The local coordinate of a polygon \( \mathbf{P} \) in Poly\((n + 1)\) is given by its vertices \( V_i \in \mathbb{C} \) for \( 2 \leq i \leq n \).

Consider the \( A_n \) quiver with straight orientation
\[
Q = A_n : 1 \leftarrow 2 \leftarrow \cdots \leftarrow n.
\] (2.4)

Denote by \( \mathcal{D}_\infty(A_n) \) its bounded derived category. By abuse of notation, let \( P_j \) be the projective \( kQ \) module at \( j \). Denote by \( M_{ij} = \text{Cone}(P_{i-1} \rightarrow P_j) \) for \( 1 \leq i \leq j \leq n \) (where we set \( P_0 = 0 \)).

**Proposition 2.5.** ([Q1, Prop. 3.6]) There is a natural bijection \( \mathfrak{Z} : \text{Stab}_{<1} \mathcal{D}_\infty(A_n) / \mathbb{C} \to \text{Poly}(n + 1) \), sending a stability condition \( \sigma \) to a \((n + 1)\)-gon \( \mathbf{P}_\sigma \) such that the oriented diagonals \( \overrightarrow{V_{i-1}V_j} \) of \( \mathbf{P}_\sigma \) gives the central charges \( Z(M_{ij}) \) of indecomposable objects in \( \mathcal{D}_\infty(A_n) \).

More precisely, let \( \mathbb{C} \cdot \sigma \in \text{Stab}_{<1} \mathcal{D}(A_n) / \mathbb{C} \) with representative \( \sigma \) such that \( Z(P_1) = 1 \). Let \( \mathbf{P}_\sigma = \mathfrak{Z}((\mathbb{C} \cdot \sigma) \sigma) \) be the corresponding \((n + 1)\)-gon so that \( V_i = Z(P_i) \) for \( 1 \leq i \leq n \) and then
\[
\text{gldim} \sigma = \frac{1}{\pi} \max \{\arg \overrightarrow{V_jV_i} - \arg \overrightarrow{V_{i+1}V_{j+1}} \mid 0 \leq i < j \leq n\},
\] (2.5)

where \( V_{n+1} = V_0 \) (cf. Figure 1).

### 3. Classification of finite type categories after KOT

For an acyclic quiver \( Q \), denote by \( \mathcal{D}_\infty(Q) \) the bounded derived category of the path algebra \( kQ \). Similarly when \( Q \) is a specie, cf. [CQ] for details. Note that any Dynkin specie can be folded from a Dykin quiver.
Let $h_Q$ be the Coxeter number associated to a Dynkin diagram $Q$. Recall the following, which is a combination of [Q1, Thm. 4.7] for the quiver case and [CQ, Cor. 6.5] for the specie case.

**Theorem 3.1.** $\text{Gd} D_\infty(Q) = 1 - 2/h_Q$ for a Dynkin quiver or specie $Q$, where the minimal value of $\text{gldim}$ on $\text{Stab} D_\infty(Q)$ is given by the solution of the Gepner equation $\tau \cdot \sigma = (-2/h_Q) \cdot \sigma$. Moreover, the solution of $\tau \cdot \sigma = (-2/h_Q) \cdot \sigma$ is unique up to $\mathbb{C}$-action.

If $\text{Gd} D < 1$, we have the classification theorem for $D$ (Theorem 3.2 below). This is essentially due to Kikuta-Ouchi-Takahashi [KOT, Theorem 5.12], where we are going to remove the assumption there:

- the category $D$ is the perfect derived category per $A$ of some smooth proper differential graded (dg) $\mathbb{C}$-algebra $A$.

Recall the following notions.

- An object $E$ in $D$ is **exceptional** if $\text{Hom}^\bullet(E, E) = k$.
- An **exceptional sequence** $(E_1, \ldots, E_m)$ in $D$ is a collection of exceptional objects such that $\text{Hom}^\bullet(E_i, E_j) = 0$ for any $i > j$.
- An exceptional sequence is **strong** if in addition that $\text{Hom}^k(E_i, E_j) = 0$ for any $i, j$ and $k \neq 0$.
- An exceptional sequence is **full** if the smallest full triangulated subcategory of $D$ containing $\{E_i\}$ coincides with $D$.

**Theorem 3.2.** Let $D$ be a connected triangulated category. Then $\text{Gd} D < 1$ if and only if $D = D_\infty(Q)/\iota$ for some Dynkin quiver $Q$ and some $\iota \in \text{Aut} D_\infty(Q)$ induced from some graph automorphism of $Q$.

**Proof.** By Theorem 3.1, we only need to show that when $D$ admits a stability condition $\sigma = (Z, P)$ with $\text{gldim} \sigma < 1$, then $D$ must be of Dynkin type as stated.
First we remove the condition that the category $\mathcal{D}$ is over $\mathbb{C}$ but still assuming it is the perfect derived category per $A$ of some smooth proper differential graded (dg) $k'$-algebra $A$ over some field $k'$ (which is not necessarily algebraically closed). Then applying the argument in [KOT, § 5.1], we deduce that $\mathcal{D}$ is locally finite. By [XZ], the Auslander Reiten quiver of such a locally finite triangulated category $\mathcal{D}$ is isomorphic to the orbit $\mathbb{Z}Q/\iota$, where $\mathbb{Z}Q$ is the translation quiver of some Dynkin quiver $Q$ and $\iota$ is an automorphism of $\mathbb{Z}Q$. Note that $\text{Aut} \mathbb{Z}Q$ is generated by $[1], \tau$ and graph automorphisms (if exists) of $Q$. If $\iota^r = [N]$ for some $N \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}$, then $P(\geq 0) = P(\geq 0)[N] = P(\geq N) \subset P(\geq 1) \subset P(\geq 0)$.

Thus $P(\geq 0) = P(\geq 1)$ or $P = P[1]$, which is a contradiction. Therefore $\mathbb{Z}Q \cap \mathbb{Z}[1] = \emptyset$. Noticing that $\tau_Q^Q = [-2]$, we deduce that $\iota$ can only be an automorphism of/induced by $Q$. So $\mathcal{D}$ must be of the form $\mathcal{D}_\infty(Q)/\iota$ as required (cf. [CQ, Example 1.1]).

Next, let us remove all constraints, only assuming that $\mathcal{D}$ is some triangulated category. We still follow [KOT, § 5.1]. Let

$S(I) = \{ \phi \in I \mid P(\phi) \neq 0 \}$.

If $S(0, 1]$ is an infinite set, then we can take a monotone increasing (similar for decreasing) sequence

$\phi - \epsilon < \phi_1 < \phi_2 < \cdots < \phi_m < \cdots < \phi$

such that $\lim_{m \to \infty} \phi_k = \phi$ and $0 < \epsilon < 1 - \text{gldim} \sigma$. Let $E_k \in P(\phi_k)$. For any integer $i, j, m \geq 1$, we have

\[
\begin{cases}
\phi_i + m > \phi_j, \\
\text{gldim} \sigma < 1 - \epsilon < (\phi_j + m) - \phi_i,
\end{cases}
\]

which implies

\[
\begin{align*}
\text{Hom}(E_i[m], E_j) &= 0, \\
\text{Hom}(E_i, E_j[m]) &= 0,
\end{align*}
\]

i.e. $\text{Hom}^*(E_i, E_j) = \text{Hom}(E_i, E_j)$. If in addition $i > j$, we also have $\text{Hom}(E_i, E_j) = 0$. So $(E_1, \ldots, E_m)$ is a full strongly exceptional sequence in the full thick subcategory $\mathcal{D}^{(m)} \subset \mathcal{D}$ they generated. This subcategory falls into the case above, i.e. $\mathcal{D}^{(m)}$ is of the form $\mathcal{D}_\infty(Q^{(m)})/\iota$, where $m$ is the rank/number of vertices of some Dynkin quiver $Q^{(m)}$. Restricted $\sigma$ to $\mathcal{D}^{(m)}$, we have ([KOT, Prop. 5.2])

$\text{gldim} \sigma \geq \text{gldim} \sigma|_{\mathcal{D}^{(m)}} \geq 1 - 2/h_{Q^{(m)}}$.

But

$\lim_{m \to \infty} 1 - 2/h_{Q^{(m)}} = 1$,

which contradicts to $\text{gldim} \sigma < 1$. Thus $S(0, 1]$ is a finite set.

Then we deduce $\mathcal{D}$ is locally finite as in [KOT, § 5.1] and finish the proof as the previous case. \qed
4. Topological Fukaya categories

4.1. Graded marked surface. In this subsection, we partially follow [IQZ, § 2], cf. [HKK, LP]. A graded marked surface $S^\lambda = (S, Y, \lambda)$ consists:

- a smooth oriented surface $S$;
- a set of closed marked points $Y$ in $\partial S$ (cf. [IQZ, § 6.2]), such that $Y \cap \partial_i \neq \emptyset$ for each boundary component $\partial_i$ of $\partial S$.
- a grading/foliation $\lambda$ on $S$, that is, a section of the projectivized tangent bundle $\mathbb{P}T S$.

Let $b = |\partial S|$ and $\aleph = |Y|$. Then $\partial S$ is divided into $\aleph$ many boundary arcs. The rank of $S$ is

$$n = 2g + b + \aleph - 2.$$  (4.1)

We will require $n \geq 2$ to exclude the trivial case. Denote by $S^\circ := S \setminus \partial S$ its interior.

Note that the projection $\mathbb{P}T S \to S$ with $\mathbb{R}^1 \simeq S^1$-fiber leads to a short exact sequence (cf. [IQZ, § 2.1])

$$0 \to H^1(S) \to H^1(\mathbb{P}TS) \xrightarrow{\text{proj.}} H^1(S^1) = \mathbb{Z} \to 0.$$  

In fact, [LP, Lem. 1.2] shows that $\lambda$ is determined by a class in $H^1(\mathbb{P}T S)$, denoted by $[\lambda]$. Moreover, such a data $\lambda$ is equivalent to a $\mathbb{Z}$-covering $\text{cov}: \mathbb{R}T S^\lambda \to \mathbb{P}T S$, known as the Maslov covering, where $\mathbb{R}T S^\lambda$ is the $\mathbb{R}$-bundle of $S$ that can be constructed via gluing $\mathbb{Z}$ copies of $\mathbb{P}T S$ cut by $\lambda$.

A morphism $(f, \tilde{df}): S^\lambda \to S_1^\lambda$ between two graded marked surfaces is a map $f: S \to S_1$ such that it preserves the marked points and $[\lambda] = f^* [\lambda_1]$, regarding $[\lambda] \in H^1(\mathbb{P}T S)$, together with a map $\tilde{df}: \mathbb{R}T S^\lambda \to \mathbb{R}T S_1^\lambda$ that fits into the commutative diagram

$$\begin{array}{ccc}
\mathbb{R}T S^\lambda & \xrightarrow{\tilde{df}} & \mathbb{R}T S_1^\lambda \\
\text{cov} & & \downarrow \text{cov}_1 \\
\mathbb{P}T S & \xrightarrow{df} & \mathbb{P}T S_1.
\end{array}$$

There is a natural automorphism [1], known as the grading shift on $S^\lambda$, given by the deck transformation of $\mathbb{R}T S^\lambda \to \mathbb{P}T S$, or equivalently, by rotating $\lambda: S \to \mathbb{P}^1$ by $\pi$ clockwise.

For a curve $c: [0, 1] \to S$, we always assume $c(t) \in S^\circ$ for any $t \in (0, 1)$. A graded curve $\tilde{c}$ is a lift of the tangent $\dot{c}$ of $c$ in $\mathbb{R}T S$, of an usual curve $c$ on $S$. There are exactly $\mathbb{Z}$ lifts of $\dot{c}$ on $\mathbb{R}T S^\lambda$, related by the grading shift [1] (i.e. the deck transformation of $\mathbb{R}T S^\lambda$).

This definition of graded curves is taken from [IQZ], see [HKK] for alternative/original version. For any graded curves $\tilde{c}_1$ and $\tilde{c}_2$, let $p = c_1(t_1) = c_2(t_2) \in S \setminus (\partial S \cup \Delta)$ be an intersection of $c_1$ and $c_2$. Note that we always require that any curves intersect
transversally. The intersection index \( i = i_p(\tilde{c}_1, \tilde{c}_2) \) from \( \tilde{c}_1 \) to \( \tilde{c}_2 \) at \( p \) is the shift \([i]\) such that the lift \( \tilde{c}_2[i] |_p \) of \( \tilde{c}_2[i] \) at \( p \) is in the interval 

\[
(\tilde{c}_1 |_p, \tilde{c}_1[1] |_p) \subset \mathbb{R} T_p S \cong \mathbb{R}.
\]

Note that when the index 0 intersection \( p \) from \( \tilde{c}_1 \) to \( \tilde{c}_2 \) can be viewed as a sharp angle from the tangent direction of \( \tilde{c}_1 \) at \( p \) to the one of \( \tilde{c}_2 \), where sharp means that it is less than \( \pi \) in \( \mathbb{P} T_p S \). Further details see [IQZ, § 2.4] and cf. Figure 2.

**Figure 2.** Intersection index (as angle)

Let \( \text{Int}^\rho(\tilde{c}_1, \tilde{c}_2) \) be the number of intersections between \( \tilde{c}_1 \) and \( \tilde{c}_1 \) with index \( \rho \) in \( S \). Denote by

\[
\text{Int}^q(\tilde{c}_1, \tilde{c}_2) = \sum_{\rho \in \mathbb{Z}} q^\rho \cdot \text{Int}^\rho((\tilde{c}_1, \tilde{c}_2))
\]

the number of \( q \)-intersections between \( \tilde{c}_1 \) and \( \tilde{c}_1 \). Note that \( \text{Int}^q \) becomes the usual geometric intersection number when specializing \( q = 1 \).

Let \( D_\infty(S^\lambda) \) be the topological Fukaya category associated to \( S^\lambda \) and

\[
\dim_q \text{Hom}^\bullet(X, Y) = \sum_{d \in \mathbb{Z}} q^d \cdot \text{Hom}^d(X, Y).
\]

Recall the following result about \( D_\infty(S^\lambda) \), where the first part is due to [HKK] and the second part (on morphisms) is due to [IQZ].

**Theorem 4.1.** [HKK, IQZ] There is a bijection \( X \) between the set of isotopy classes of graded curves \( \{\tilde{\eta}\} \) on \( S^\lambda \) with local system and the set of isomorphism classes of indecomposable objects \( \{X_\tilde{\eta}\} \) in \( D_\infty(S^\lambda) \). Furthermore, let \( \tilde{\alpha}, \tilde{\beta} \) be two graded curves which are not closed curves (and hence no local system is needed). Then each index \( \rho \) intersection between them induces a (non-trivial) morphism in \( \text{Hom}^\rho(X_\tilde{\alpha}, X_\tilde{\beta}) \). Moreover, these morphisms form a basis for the \( \text{Hom}^\bullet \) space so that we have

\[
\dim_q \text{Hom}^\bullet(X_\tilde{\alpha}, X_\tilde{\beta}) = \text{Int}^q(\tilde{\alpha}, \tilde{\beta}).
\]

**4.2. Quadratic differentials.** In this section, we quickly review the theory of stability conditions as quadratic differentials in the topological Fukaya category setting.

Let \( X \) be a compact Riemann surface and \( \xi \) a non-zero meromorphic quadratic differential on \( X \), that is, a meromorphic section of the square of the cotangent bundle. The
set of singularities of $X$ is denoted by $\text{Sing}(\xi)$. Usually, the singularities considered are zeroes or poles of order $k \geq 1$, i.e. local coordinate can be chosen to be $z^{\pm k} d^{\otimes 2}$.

The (horizontal) foliation $\lambda(\xi)$ of $\xi$ gives a line field (see Section 4) on $X$. In fact, these are certain geodesics on $X$, where the metric is induced from $\xi$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Foliations/line fields near zeroes/poles}
\end{figure}

For instance, near a zero of order $1/2/3$, the foliation $\lambda(\xi)$ is shown in the upper pictures of Figure 3; near a pole of order $3/4/5$, the foliation $\lambda(\xi)$ is shown in the lower pictures of Figure 3. When performing real blow-up at a (higher order) pole $p$ of order $k \geq 3$, one gets a boundary component $\partial_p$ with $k - 2$ marked points, where points on $\partial_p$ correspond to tangent direction at $p$ and marked points are distinguished directions, as shown (the black lines) in Figure 3. In the pictures we presented, our convention is the following: red circles are zeros and blue bullets are poles. For details, see [BS, KQ].

However, in our case, the singularities are of exponential type, in the sense of [HKK], cf. [IQ2]. Namely, the local coordinate around a given singularity $p$ is of the form (up to scaling a holomorphic function)

$$z^{-l} e^{-\xi} d z^{\otimes 2},$$

(4.3)

where the numerical data here is $(k, l) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ and $l$ can be calculated as two minus the winding number of the line field around $p$ (cf. [IQ2]). When performing real blow-up at a (higher order) pole $p$ of type $(k, l)$, one gets a boundary component $\partial_p$ with $2k$ distinguished points:

- $k$ of which are in the metric completions (call closed marked points) that behave as infinity order zeroes;
- $k$ of which (called open marked points) behave as infinity order poles.

The closed and open marked points are in alternative order on $\partial_p$. 
The neighbourhood of such an infinity order zero/pole is as the neighbourhood of zero in the Riemann surface for $\log z$, cf. the left picture (taken from [Wiki]) of Figure 4 (and thus they should sometimes be considered as marked/unmarked boundary arcs, cf. [HKK]). Also, the foliation $\lambda(\xi)$ on a real blow-up of a Riemann sphere with a single singularity of type $(3,4)$ is shown in the right picture (taken from [IQ2]) of Figure 4. Our convention is that red circles are closed marked points and we do not use points to represent open ones in this paper.

Denote by $X_\xi^{\text{cd}}$ the graded marked surface (of closed type), which is the real blow-up of $X$ with respect to $\xi$ equipped with closed marked points $Y(\xi)$ and foliation $\lambda(\xi)$ as its grading.

The foliations induce the horizontal strip decomposition of $S$ (cf. [HKK, § 2.4]), where the surface is divided into regions/strips consisting of horizontal foliations. Each strip is either isomorphic to the upper half plane $\mathbb{H}$ (with finite height) or a strip $\{z \in \mathbb{C} \mid 0 \leq \Im(z) \leq \Im(z_0)\}$ for some $z_0 \in \mathbb{C}$ with $\Im(z_0) > 0$ (with finite height).

An infinite height strip is shown in right picture of Figure 5. A finite height strip is shown in the right picture of Figure 5. In this case, there is exactly one closed marked point on each boundary of a strip mentioned above, namely 0 and $z_0$ respectively. So there is an unique geodesic connecting these two points, which is known as the saddle connection of this strip, whose angle is $\arg z_0$. In fact, up to a small rotation of the quadratic differential, we can assume that there is no horizontal saddle trajectories (known as saddle-free), so that any finite height strip is as the case mentioned above.

In general, a saddle connection is a maximal geodesic connecting zeroes (or points in the metric completion). Thus, one can integral the (square root of) the quadratic differential along saddle connections.

4.3. Stability conditions as quadratic differentials.

**Definition 4.2.** A $S^\lambda$-framed quadratic differential $\Xi = (X, \xi, \psi)$ consists of a Riemann surface $X$, a meromorphic quadratic differential $\xi$ with only exponential type
singualrities and a diffeomorphism $\psi: \mathbb{S}^\lambda \to (\mathbb{X}_{cd}, \lambda(\xi))$ preserving marked points. Two $\mathbb{S}^\lambda$-framed quadratic differentials $(\mathbb{X}_1, \xi_1, \psi_1)$ and $(\mathbb{X}_2, \xi_2, \psi_2)$ are equivalent, if there exists a biholomorphism $f: \mathbb{X}_1 \to \mathbb{X}_2$ such that $f^*(\phi_2) = \phi_1$ and $\psi_2^{-1} \circ f \circ \psi_1$ is a homeomorphism of $\mathbb{S}^\lambda$ that is isotopic to identity. Denote by $\text{FQuad}_{\infty}(\mathbb{S}^\lambda)$ the moduli space of $\mathbb{S}^\lambda$-framed quadratic differentials on $\mathbb{S}^\lambda$.

The main result of [HKK] is the following, where the surjectivity part is improved by [T].

**Theorem 4.3.** [HKK, T] There is an isomorphism of complex manifolds

$$\iota = \iota(\mathbb{S}^\lambda): \text{FQuad}_{\infty}(\mathbb{S}^\lambda) \xrightarrow{\cong} \text{Stab} \mathcal{D}_{\infty}(\mathbb{S}^\lambda),$$

(4.4)

Moreover, the graded saddle connections of $\Xi$ correspond (semi)stable objects of $\sigma$ under the bijection $X$ in Theorem 4.1 and, up to $2\pi \mathbb{Z}$, the angles of a saddle connection equals $\pi$ times the phase of the corresponding semistable object.

4.4. **Winding numbers.** Denote by $\mathfrak{N}(\mathbb{S}^\lambda) = (k, w)$ the partial numerical data of $\mathbb{S}^\lambda$, for $k = (k_1, \ldots, k_b)$ and $w = (w_1, \ldots, w_b)$, where $k_i$ is the number of closed marked points on a boundary component $\partial_i$ and $w_i$ the (clockwise) winding number around $\partial_i$. Note that comparing with (4.3), we have $w_i = 2 - l_i$ and they satisfy

$$\sum_{i=1}^{b} l_i = 4 - 4g \iff \sum_{i=1}^{b} w_i = 4g - 4 + 2b$$

(see [LP, IQ2] for details).

We are interested in a particular class of arcs on $\mathbb{S}$, i.e. the minimal arcs. A *minimal arc* on $\mathbb{S}$ is an arc connecting two adjacent closed marked points on some boundary component, such that it is isotopic to a boundary segment. For instance, the arcs $\eta_j$ in Figure 6 are minimal arcs.

**Example 4.4.** Consider the case that $\mathbb{S}$ is an annulus with boundaries $\partial_m$ and $\partial_r$. Then $\mathcal{D}_{\infty}(\mathbb{S}^\lambda)$ is triangle equivalent to the bounded derived category $\mathcal{D}_{\infty}(\text{A}_{m,r})$ of a graded $\text{A}_{m,r}$ quiver (with $m + r$ vertices whose arrows form a non-oriented cycle, $m$ of
which are clockwise and the other \( r \) are anticlockwise).

\[ \tilde{A}_{m,r}: \]

Note that the sum of the winding numbers is zero in this case. Then norm form of the numerical data can be chosen to be \( \mathcal{R}(S^3) = ((m, r), (w, -w)) \) for \( m, r \in \mathbb{Z}_+ \) and \( w \in \mathbb{Z}_{\geq 0} \).

**Lemma 4.5.** \( \text{Gd} \mathcal{D}_\infty(\tilde{A}_{m,r}) \leq 1 + w/m \).

**Proof.** When \( w = 0 \), we have \( \text{Gd} \mathcal{D}_\infty(\tilde{A}_{m,r}) = 1 \), which was calculated in [Q1]. Now assume that \( w > 0 \). Let \( \eta_1, \ldots, \eta_m \) be the minimal arcs on \( \partial_m \) in clockwise order, as shown (red arcs) in Figure 6. By convention, the subscript will be in \( \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} \).

![Figure 6. A full formal arc system containing certain minimal arcs in the annulus case](image)

For any graded lifts \( \tilde{\eta}_j \) of \( \eta_j \), we have

\[
\sum_{j=1}^{m} i_{p_j}(\tilde{\eta}_j, \tilde{\eta}_{j+1}) = w + m, \tag{4.5}
\]

where \( i_{p_j} \) is the intersection index of \( \tilde{\eta}_j \) and \( \tilde{\eta}_{j+1} \) at \( p_j \), cf. Figure 6. Therefore, we can choose certain graded lifts of \( \eta_j \) such that

\[
i_{p_j}(\tilde{\eta}_j, \tilde{\eta}_{j+1}) = \lfloor j(m + w)/m \rfloor - \lfloor (j - 1)(m + w)/m \rfloor, \quad \forall j.
\]

In particular, \( i_{p_1}(\tilde{\eta}_1, \tilde{\eta}_2) \geq 2 \) and

\[
1 + w/m \leq i_{p_j}(\tilde{\eta}_j, \tilde{\eta}_{j+1}) \leq [1 + w/m] + 1.
\]

Then we can complete \( \{ \tilde{\eta}_j \mid j \in \mathbb{Z}_m \} \) to a full formal arc system \( A \) (cf. dashed arcs in Figure 6) such that
• there is exactly one arc \( \alpha \) that is incident \( \partial_m \) at \( p_1 \) and connects two boundaries;
• for any two graded arcs \( \eta, \eta' \) in \( A - \{ \tilde{\eta}_j \mid j \in \mathbb{Z}_m \} \), there is at most one intersection between them and, if they intersect, the intersection index is 1.

Here, a full formal arc system is a collection of (graded) arcs that divide \( S \) into polygons, such that each polygon contains exactly one boundary segment. The objects corresponding to a full formal arc system is a set of generators for \( \mathcal{D}_\infty(\widetilde{A}_{m,r}) \). The condition on intersection index can be translated to

\[
\text{Hom}^{\leq 0}(X_{\tilde{\eta}}, X_{\tilde{\eta}'}), \quad \forall \eta, \eta' \in A.
\] (4.6)

Thus \( \{ X_{\tilde{\eta}} \mid \eta \in A \} \) form a so-called simple minded collection. Equivalently, they are the set of simple objects of a heart \( \mathcal{H}_A \) that they generate \( \mathcal{D}_\infty(\widetilde{A}_{m,r}) \). Furthermore (cf. \[B1, Lem 5.2\]), to give a stability condition \( \sigma \) in \( \text{Stab} \mathcal{D}_\infty(S^1) \) with heart \( \mathcal{H}_A \) is equivalent to make a choice of central charges for simples

\[
\{ Z(X_{\tilde{\eta}}) \in \mathcal{H} \mid \eta \in A \},
\]

where

\[
\mathcal{H} = \{ z = re^{i\pi\theta} \mid r \in \mathbb{R}_{>0}, \theta \in [0, 1) \} \subset \mathbb{C}
\]
is the upper half plane. We can find a stability condition \( \sigma \) such that

\[
\begin{align*}
\phi_\sigma(X_{\tilde{\eta}_{j+1}}) &= j(m + w)/m - [j(m + w)/m], \quad \forall j \\
\phi_\sigma(X_{\tilde{\eta}'}) &= \phi_\sigma(X_{\tilde{\eta}}), \quad \text{for any } \eta, \eta' \notin A - \{ \eta_j \mid j \in \mathbb{Z}_m \}, \\
\phi_\sigma(X_{\tilde{\alpha}}) &= \phi_\sigma(X_{\tilde{\eta}_1}).
\end{align*}
\] (4.7)

Then we have

\[
\phi_\sigma(X_{\tilde{\eta}_{j+1}}) - \phi_\sigma(X_{\tilde{\eta}_j}) = 1 + \frac{w}{m},
\]

for any \( j \). One can check that \( \text{gldim} \sigma = 1 + \frac{w}{m} \) (or use Proposition 5.14), which completes the lemma. \( \square \)

5. Contractible flow

5.1. General strategy. In this section, we develop a strategy to attack the contractibility conjecture of spaces of stability conditions. The idea is to use the function \( \text{gldim} \) to induce a contractible flow. Of cause, this strategy should only apply to the ‘Calabi-Yau-\( \infty \)’ case, as \( \text{gldim} \) is constant on Calabi-Yau-\( N \) categories (for \( N \in \mathbb{Z}_+ \)).

**Definition 5.1.** Given a stability condition \( \sigma \), define a set \( \mathcal{P}(\sigma) \)

\[
\{ (M_1, M_2) \mid M_i \in \text{Sim} \mathcal{P}(\phi_i), M_1[Z] \neq M_2[Z], \text{Hom}(M_1, M_2) \neq 0, \phi_2 - \phi_1 = \text{gldim} \sigma \}
\] (5.1)

which consists of pairs of stable indecomposable objects whose phase difference achieves the value \( \text{gldim} \sigma \) and has non-zero morphisms in \( \mathcal{P} \). Note that we also require that the objects in such a pair are not in the same shift orbit to exclude the case of nontrivial higher self-extension of an object.
Define a subspace

\[ \text{Stab}_{\mathfrak{P}(\sigma)} D := \{ \sigma' \in \text{Stab} D \mid \mathfrak{P}(\sigma') = \mathfrak{P}(\sigma) \}. \tag{5.2} \]

We will prove that (5.2) is determined by a collection of equations and the following conjecture, which allow us to use the differential of gldim to contract (certain part of) the space of stability conditions piecewise.

**Conjecture 5.2.** \( \text{Stab}_{\mathfrak{P}(\sigma)} D \) is a real submanifold of \( \text{Stab} D \) where the function gldim is differentiable without critical points restricted to its interior.

Moreover, we expect the following for many cases, which holds for the case of coherent sheaves on the projective plane (cf. [Q1, FLLQ]).

**Conjecture 5.3.** The differential of gldim provide a flow such that \( \text{Stab}^y D \) contracts to \( \text{Stab}^y D \) for any \( y \in \text{gldim} D \)-reachable, then \( \text{Stab}^y D \) contracts to \( \text{Stab}^y D \) for \( y = \text{gldim} D \) and any \( y < x \).

### 5.2. Max angle as gldim and reachability.

Recall that the rank \( n \) of \( S \) in (4.1) is required to be at least 2. We apply the general strategy above to the topological Fukaya categories \( D_{\infty}(S^\lambda) \).

Take \( \sigma \in \text{Stab} D_{\infty}(S^\lambda) \) with \( \mathfrak{P}(\sigma) \) as in Definition 5.1. Let \( \Xi = (X, \xi, \psi) = \iota^{-1}(\sigma) \) be the \( S^\lambda \)-framed quadratic differential as in Theorem 4.3. We will identify \( (S, Y, \lambda) \) with \( (X^{\xi}, Y(\xi), \lambda(\xi)) \) via \( \psi \) when there is no confusion.

**Lemma 5.4.** Let \( M_1, M_2 \) be two \( \sigma \)-semistable indecomposable objects with corresponding graded curves \( \tilde{\gamma}_i \) on \( S^\lambda \), respectively. If \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) intersect in the interior of \( S \) of index 0, then

\[ \phi_{\sigma}(M_1) < \phi_{\sigma}(M_2) < \phi_{\sigma}(M_1) + 1 \]  \hspace{1cm} (5.3)

**Proof.** Since \( \tilde{\gamma}_1 \) intersects \( \tilde{\gamma}_2 \) at a point \( p \) with index 0, \( \tilde{\gamma}_2 \) intersects \( \tilde{\gamma}_1[1] \) at \( p \) with index 0. By (4.2) in Theorem 4.1, we have

\[ \text{Hom}(M_1, M_2) \neq 0 \neq \text{Hom}(M_2, M_1[1]). \]

As \( M_1 \) and \( M_2 \) are both \( \sigma \)-semistable, we have \( \phi_{\sigma}(M_1) \leq \phi_{\sigma}(M_2) \leq \phi_{\sigma}(M_1) + 1 \). To get (5.3), we need to rule out the possible equality. Suppose that \( \phi_{\sigma}(M_1[k]) = \phi_{\sigma}(M_2) \) for \( k \in \{0, 1\} \). After rotating an angle of \( -\phi_{\sigma}(M_2) \cdot \pi \), the underlying curves \( \gamma_i \) become horizontal foliations. But such foliations can not intersect in the interior of \( S \), which is a contradiction. \( \square \)

An immediate consequence is the following.

**Corollary 5.5.** Let \( M_1, M_2 \) be two \( \sigma \)-semistable indecomposable objects with corresponding graded curves \( \tilde{\gamma}_i \) on \( S^\lambda \), respectively. If \( \phi_{\sigma}(M_2) - \phi_{\sigma}(M_1) \geq 1 \) with \( \text{Hom}(M_1, M_2) \neq 0 \), then

- either \( \gamma_1 \) and \( \gamma_2 \) intersect (and only intersect) at marked points in \( Y \),
- or \( \gamma_1 = \gamma_2 \) that corresponds the same simple closed curve.
In the latter case, $M_2$ is some shift of $M_1$ and $\gamma_i$ corresponds to a ring domain in the foliation of $\Xi$ with angle $\phi_i(M_i) \cdot \pi$.

**Proof.** The lemma above shows that there is no intersection between $\gamma_i$ in the interior of $S$. If $\gamma_1 \neq \gamma_2$ and they do not intersect, then Theorem 4.1 implies that there is no Hom between $M_i$, which is a contradiction. Thus, the only cases left is the ones listed in the corollary. Note that in the latter case, $\gamma_i$ can not have self intersection since it is a foliation of a fixed angle. □

**Definition 5.6.** Consider the set $T$ of all saddle connections of $\Xi$, which corresponds to the set $\text{Ind}^\sigma$ of all $\sigma$-semistable indecomposable objects, where $\sigma = (\mathbb{Z}, \mathcal{P})$ is the stability condition that corresponds to $\Xi$. Denote by $\text{Core}(\xi)$ the core of $\xi$, which is the convex hull of $T$.

At each marked point $p \in Y(\Xi)$, denote by $(\eta^p_-, \ldots, \eta^p_+)$ the set of all ungraded saddle connections in clockwise order (with respect to $p$).

![Figure 7. Max angle](image)

**Proposition 5.7.** The core $\text{Core}(\xi)$ and all $\eta^p_\pm$ are well defined.

**Proof.** By [HKK, Prop. 2.2], $\text{Core}(\xi)$ is the union of finite saddle connections and triangles cut out by $A$, for any maximal geodesic arc system $A$. Moreover, it is well-defined and independent of the choice of $A$. Consider the boundaries of these triangles together with the saddle connections (for any chosen $A$), we see that $\eta^p_\pm$ must be among them. □

**Remark 5.8.** It is possible that there are infinite many saddle trajectories coming out of a marked point $p \in Y(\Xi)$, e.g. in the annulus case of § 6.3, cf. Figure 8. However, there are still leftmost/rightmost saddle trajectories bounding all of them. For instance, the orange loop in Figure 8, which corresponds to (some shifts of) a skyscaper sheaf in the usual Kronecker case. See Example 6.2 for more details in the Kronecker case.

Choose a grading $\widetilde{\eta}^p_\pm$ for both of them so that the intersection indexes are zero, i.e. $i_p(\widetilde{\eta}^-_\pm, \widetilde{\eta}^+_\pm) = 0$. Note that $\eta^p_\pm$ may be the two endpoints of the same arc, and in such a case, their graded version $\eta^p_\pm$ may still differ by shifts. Let $M^p_\pm$ be the $\sigma$-semistable
object corresponding to $\tilde{\eta}_\pm^p$ with proper shifts, such that the intersection of $\eta_\pm^p$ at $p$ induces a non-zero homomorphism in $\text{Hom}(M^p_+,M^p_-)$. Denote by

$$\angle_p \text{Core}(\sigma) = \phi_\sigma(M^p_+) - \phi_\sigma(M^p_-).$$

Note that $M^p_\pm$ are only well-defined up to some shifts simultaneously but $\angle_p \text{Core}(\sigma)$ is independent of such shifts.

Now we can describe a formula for $\text{gldim}$ under certain conditions.

**Proposition 5.9.** If $\text{gldim} \sigma > 1$, then $\sigma$ is $\text{gldim}$-reachable and

$$\text{gldim} \sigma = \max \angle \text{Core}(\xi) = \max_{p \in Y} \{\angle_p \text{Core}(\sigma)\}. \quad (5.4)$$

Moreover, if $\text{gldim} \sigma \notin \mathbb{Z}$, then any pair $(M_1,M_2)$ in (5.1) corresponds to an angle of $\text{Core}(\xi)$.

**Proof.** Firstly, consider the case when $\text{gldim} > 1$. Let $x = \text{gldim} \sigma$. For any $0 < \epsilon \ll 1$ such that

$$(x - \epsilon, x) \cap \mathbb{Z} = \emptyset, \quad (5.5)$$

take any $y \in (x - \epsilon, x)$ which is achieved by

$$y = \phi_\sigma(M_2) - \phi_\sigma(M_1)$$

for some indecomposable objects $M_1,M_2$. Then $M_1$ is not the shift of $M_2$ as $y \notin \mathbb{Z}$. By Corollary 5.5, we deduce that $M_1,M_2$ correspond to the graded curves $\tilde{\eta}_1,\tilde{\eta}_2$ which connect marked points and only intersect at marked points/endpoints. This implies that $y \leq \angle_p \text{Core}(\sigma)$ for some $p \in \eta_1 \cap \eta_2 \subset Y$. Thus, we have

$$\text{gldim} \sigma \leq \max \angle \text{Core}(\xi)$$

and clearly the $\max \angle \text{Core}(\xi)$ is reachable.
Finally, the condition $M_1[\mathbb{Z}] \neq M_2[\mathbb{Z}]$ in (5.1) says that $M_1$ is not a shift of $M_2$ and the deduction above also implies that when $x = \phi_\sigma(M_2) - \phi_\sigma(M_1)$, the corresponding curves $\eta_k$ intersect at a point in $\mathcal{Y}$. Thus this pair corresponds to an angle of Core($\xi$).

\textbf{Proposition 5.10.} Suppose that gldim $\sigma \leq 1$. Then $\sigma$ is gldim-reachable and (5.4) still holds.

\textit{Proof.} If gldim $\sigma < 1$, Theorem 3.2 says that this happens if and only if $D_\infty(\mathcal{S}^\lambda)$ is of the form $D_\infty(Q)/\iota$ for a Dynkin quiver $Q$. This will force $\mathcal{S}$ being a disk and $Q$ being an $A_n$ quiver (with $\iota = \text{id}$). Then $\sigma$ is gldim-reachable due to finiteness of the category.

Next, consider the case gldim $\sigma = 1$ and we can exclude the disk case as above. By [Q1, Prop. 3.5], $\sigma$ is totally semistable, i.e. any indecomposable object $M$ is $\sigma$-semistable. Take any boundary component $\partial_0$ with winding number $w$ and let $p \in \partial_0 \cap \mathcal{Y}(\xi)$. So there is a loop $\gamma_k$ (non-trivial since $\mathcal{S}^\lambda$ is not a disk), for any $k \geq 1$, based at $p$ and go around $\partial_0$ for $k$ times, with a self-intersection of index $1 + kw$. See the orange loop for $\gamma_1$ in Figure 8, where $\partial_0$ is the outer boundary. Let $M_k$ be the indecomposable object corresponding to some graded version of $\gamma_k$. So by (4.2) we have $\text{Hom}(M_k, M_1[1 + kw]) \neq 0$. As $M_k$ is semistable, we have $1 + kw \geq 0$ and hence $w \geq 0$. But gldim = 1 forces $1 + kw \leq 1$, i.e. $w \leq 0$. Thus, $w = 0$ and gldim is achieved by $M_1$ and $M_1[1]$ corresponding to an angle at $p$. Note that in this case gldim is also achieved by a $k\mathbb{P}^1$-family of objects. \hfill $\square$

Combing the propositions above, we know that any stability condition on $D_\infty(\mathcal{S}^\lambda)$ is gldim-reachable.

\textbf{Corollary 5.11.} Any $\sigma \in \text{Stab} D_\infty(\mathcal{S}^\lambda)$ is gldim-reachable.

5.3. Cycles of saddle connections and critical values.

\textbf{Proposition 5.12.} Suppose that gldim $\sigma$ is reached at $p_1$ and $p_2$, i.e.

\[ \text{gldim} \sigma = \angle_{p_1} \text{Core}(\sigma) = \phi_\sigma(M_{p_1}^+) - \phi_\sigma(M_{p_1}^-), \quad j = 1, 2. \]

Let $\eta_{p_1}^{p_2}$ be the arcs corresponding to $M_{p_1}^{p_2}$. If $\eta_{p_1}^{p_1} = \eta_{p_2}^{p_2}$, then $\eta_{p_1}^{p_1}$ is a minimal arc (cf. Section 6.3).

\textit{Proof.} Consider the arc $\eta = \eta_{p_1}^{p_1} = \eta_{p_2}^{p_2}$. Any geodesics starting from $p_1$ that is on the right hand side (clockwise side with respect to $p_1$) can not end at a closed marked point (infinity order zero) since $\eta_{p_1}^{p_1}$ is the rightmost saddle connection. Therefore, they can only end at the boundary where $p_1$ lives. Similarly, any geodesics starting from $p_2$ that is on the left hand side (anticlockwise side with respect to $p_2$) can not end at a closed marked point. Hence, they can only end at the boundary where $p_2$ lives.

Take all horizontal strips that intersect $\eta$. They must have finite height and the saddle connections of these strips form a broken geodesic connecting $p_1$ and $p_2$ (dashed line segment in Figure 9). They will be on the right hand side of $\eta$ when walking from $p_2$ to $p_1$. Therefore on the other/left hand side, the infinities of these strips tend to an infinity order pole/open marked point on some boundary of $\mathcal{S}$. By the discussion above, such an open marked point is the boundary where both $p_1$ and $p_2$ live. See Figure 9. Thus
we have shown that $p_1$ and $p_2$ are in the same boundary $\partial$ of $S$ and they are adjacent closed marked points.

\[ \eta_{p_1} \pm = \eta_{p_2} \pm = \eta_{p_3} \pm = \eta_{p_4} \pm = \eta_{p_5} \pm. \]

\[ \eta_{p_2} = \eta_{p_3} \pm = \eta_{p_4} \pm = \eta_{p_5} \pm. \]

**Figure 10.** A cycle of saddle connections

**Corollary 5.13.** Suppose that $\text{gldim} \sigma$ is reached at $p_1, p_2, \ldots, p_m$, i.e.

$\text{gldim} \sigma = \angle_{p_j} \text{Core}(\sigma) = \phi_\sigma(M_{p_j}^+) - \phi_\sigma(M_{p_j}^-), \quad j = 1, \ldots, m.$

Let $\eta_{p_j}^\pm$ be the arcs corresponding to $M_{p_j}^\pm$. If $\eta_{p_j}^\pm = \eta_{p_{j+1}}^\pm$ for $1 \leq j \leq m$ and $\eta_{p_m}^\pm = \eta_{p_1}^\pm$, cf. the left picture in Figure 10, then the arcs $\eta_{p_j}^\pm$ are precisely all the minimal arcs (cf. Section 6.3) at some boundary $\partial$ of $S$ in clockwise order (cf. the right picture in Figure 10). Note that in such a case we will have

\[ \text{gldim} \sigma = 1 + w_\partial/m_\partial, \quad (5.6) \]

for $m = m_\partial$ the number of marked point on $\partial$ and $w_\partial$ the winding number of $\partial$. 
Proof. By repeatedly using Proposition 5.12 above we end up as the right picture of Figure 10.

For the final calculation, we only need to notice that, by properly shifting $M^p_j$, we can arrange that

$$M^p_j = M^{p,j+1}, \quad 1 \leq j \leq m - 1$$

and then $M^{p,m} = M^p_1[t]$, where $t$ can be calculated as in (4.5), that equals $w + m$. So we have

$$m \cdot \text{gldim} \sigma = \sum_{j=1}^{m} \phi_\sigma(M^p_j) - \phi_\sigma(M^p_{j-1}) = w + m$$

as claimed. □

Note that in the situation of the proposition above, we have $w \geq 0$ unless $S$ is a disk. This follows from the fact that $\text{gldim} \sigma \geq 1$ unless $S$ is a disk (of type A).

Denote by

$$V(S^\lambda) = \{ 1 + w_\partial/m_\partial \mid \partial \subset \partial S, w_\partial \geq 0 \}$$

be the set of critical values of $\text{gldim}$.

We can upgrade the second statement of Proposition 5.9 a little bit.

**Proposition 5.14.** If $\text{gldim} \sigma > 1$, then $\sigma$ is $\text{gldim}$-reachable exactly by the pair $(M_1, M_2)$ of objects corresponding to two edges of an angle of $\text{Core}(\xi)$. Moreover, $M_1[Z] = M_2[Z]$ can only happen if $\text{gldim} \sigma = 1 + w_\partial/m_\partial$ for some $\partial$ with $m_\partial = 1$. If this does not happen, then (5.1) consists of precisely all such pairs $(M_1, M_2)$.

**Proof.** This follows from the fact that different arcs corresponds to different objects. So $M_1[Z] = M_2[Z]$ implies that two edges of an angle of $\text{Core}(\xi)$ coincide. And Corollary 5.13 implies that they bound a boundary with exactly one closed marked point. □

5.4. **Main result.** Recall that for $\sigma = (Z, P)$, we define a set $\mathcal{P}(\sigma)$ of pairs of objects in (5.1), whose phase difference of each pair reaches $\text{gldim} \sigma$. Let $\mathcal{P}(\sigma) = [\mathcal{P}(\sigma)]/[1]$ be the set of shift orbits of such pairs.

**Theorem 5.15.** If $1 \leq \text{gldim} \sigma \in V(S^\lambda)$, then $\text{Stab}_{\mathcal{P}(\sigma)} D_\infty(S^\lambda)$ consists of an real submanifold $\text{Stab}_{\mathcal{P}(\sigma)} D_\infty(S^\lambda)$ of $\text{Stab} D_\infty(S^\lambda)$ with

$$3 \leq \dim_{\mathbb{R}} \text{Stab}_{\mathcal{P}(\sigma)} D_\infty(S^\lambda) = 2n + 1 - s,$$

for $s = \#\mathcal{P}(\sigma)$. Moreover, $\text{Stab}_{\mathcal{P}(\sigma)} D_\infty(S^\lambda)$ is open in its closure and, restricted to which, $\text{gldim}$ is differentiable without critical point.

**Proof.** By Proposition 5.9 and 5.10, we know that $\text{gldim} \sigma$ will be only reached at certain closed marked points $p_1, p_2, \ldots, p_s$, in the sense that

$$\text{gldim} \sigma = \angle_{p_j} \text{Core}(\sigma) = \phi_\sigma(M^p_j) - \phi_\sigma(M^p_{j-1}), \quad j = 1, \ldots, s. \quad (5.8)$$

Thus, we have

$$\mathcal{P}(\sigma) = \{ (M^p_j[m]) \mid m \in \mathbb{Z}, 1 \leq j \leq s \}.$$
Up to the $\mathbb{C}$-action, we can assume that the heart $\mathcal{H}_\sigma$ of $\sigma$ is finite/algebraic (i.e. a length category with finitely many simples). Then (cf. [B1, Lem. 5.2] and [QW]) $\sigma$ is the half-open-half-closed cube $U(\mathcal{H}) \cong H^n \subset \text{Stab} \mathcal{D}_\infty(S^\lambda)$ (recall that $H$ is the upper half plane), where the coordinates are given by the central charges $Z_\sigma = \{Z(S_i)\}$ of simples $S_i$ in $\mathcal{H}_\sigma$.

Let $m_\pm \in \mathbb{Z}$ such that $M^\rho_j[m_\pm]$ is in $\mathcal{H}_\sigma$. Then $Z_\sigma^\pm = Z(M^\rho_j[m_\pm])$ will be the linear combinations of central charges of simples in $Z_\sigma$. Let $G^\sigma$ be the directed graph whose vertices are $\{M^\rho_j[m_\pm] | 1 \leq j \leq s\}$ and whose arrows are $\{M^\rho_i[m_-] \rightarrow M^\rho_j[m_+] | 1 \leq j \leq s\}$.

As $\text{gldim} \sigma \notin \mathbb{V}(S^\lambda)$, Corollary 5.13 implies that there is no cycle in $G^\sigma$. In fact, any connected component of $G^\sigma$ has the following form

Moreover, Proposition 5.12 can be translated to: if some $M^\rho_j[m_\pm]$ is neither a source nor a sink in $G^\sigma$, then it corresponds to a minimal arc. A consequence is that the ungraded arcs $\{\eta^\rho_j | 1 \leq j \leq s\}$ can be completed to a full formal arc system. Therefore, in the Grotendieck group

$$K \mathcal{D}_\infty(S^\lambda) = \langle [S_i] | \text{simple } S_i \text{ in } \mathcal{H}_\sigma \rangle,$$

the classes of $M^\rho_j[m_\pm]$ form a partial basis. Since the central charge $Z$ is a group homomorphism, $\{Z^\pm_j | 1 \leq j \leq s\}$ are linear independent in the coordinate $Z_\sigma$. Furthermore, the no-cycle condition in $G^\sigma$ implies that the differences $\{Z^+_j - Z^-_j | 1 \leq j \leq s\}$ are also linear independent in the coordinate $Z_\sigma$. Thus, by change of coordinates, we can choose $\{Z^+_j - Z^-_j | 1 \leq j \leq s\}$, together with some $Z(S_i)$ (or their linear combinations), to be the coordinates in the neighbourhood of $U(\sigma)$ of $\sigma$, where we use polar coordinate system $z = m \cdot e^{i\pi\theta}$ for complexes ($m \in \mathbb{R}_+ , \theta \in \mathbb{R}$) regarding Stab as a real manifold.

Next, we claim that there is a neighbourhood $U(\sigma)$ of $\sigma$ in $\text{Stab} \mathcal{D}_\infty(S^\lambda)$, so that

$$\text{gldim} \varsigma = \max \{\angle_{p_j} \text{Core}(\varsigma) | 1 \leq j \leq s\}$$

for any $\varsigma \in U(\sigma)$. To see this, let

$$\epsilon = \text{gldim} \sigma - \max \{\angle_p \text{Core}(\sigma) | p \neq p_j, 1 \leq j \leq s\},$$

so that for any other pair of $\sigma$-semistable indecomposable objects $(M'_1, M'_2) \notin \mathfrak{P}(\sigma)$ with $\text{Hom}(M'_1, M'_2) \neq 0$ and $M'_1[Z] \neq M'_2[Z]$, we have

$$\phi_\sigma(M'_2) - \phi_\sigma(M'_1) \leq \text{gldim} - \epsilon.$$
Take $U(\sigma)$ be the open ball with center $\sigma$ and radius $\epsilon/4$ and recall that the distance on Stab is defined by

$$d(\sigma, \varsigma) := \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma}(E) - \phi^-_{\varsigma}(E)|, |\phi^+_{\sigma}(E) - \phi^+_{\varsigma}(E)|, \left| \log \frac{m_{\sigma}(E)}{m_{\varsigma}(E)} \right| \right\}. \quad (5.10)$$

Then for any $\varsigma = (W, Q) \in U(\sigma)$, we have

$$(W, Q) \subset (\mathcal{P}(\varphi - \epsilon/4, \varphi + \epsilon/4), \forall \varphi \in \mathbb{R},$$

where $\sigma = (Z, \mathcal{P})$. Then we deduce that for any pair $(M_1^+, M_2^+) \notin \mathcal{P}(\sigma)$ as above, we will have

$$\phi_{\varsigma}(M_1^+) - \phi_{\varsigma}(M_2^+) \leq \phi_{\sigma}(M_1^+) - \phi_{\sigma}(M_2^+) + \epsilon/2.$$

Similarly, $\text{gldim} \varsigma > \text{gldim} \sigma - \epsilon/4$ (cf. [IQ1]) which implies the claim.

In a neighbourhood of $\sigma$ in $\text{Stab} \mathcal{D}_\infty(\mathcal{S}^\lambda)$, consider a submanifold $\mathcal{V}$ defined by the equations (5.8), or equivalently

$$\theta_1 = \theta_2 = \cdots = \theta_s$$

for $Z^+_j - Z^-_j = m_j \cdot e^{i\pi \theta_j}$. It is a real submanifold in $\text{Stab} \mathcal{D}_\infty(\mathcal{S}^\lambda)$ with dimension $2n - (s - 1)$ and we have

$$U_0(\sigma) \subset \left( \mathcal{V} \cap \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda) \right)$$

for any small neighbourhood $U_0(\sigma)$ of $\sigma$ in $\text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda)$. Let $\varsigma \in \mathcal{V} \cap \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda)$.

Since the objects $M^{P_j}_\pm$ that appears in $\mathcal{P}(\varphi)$ is stable, there is a neighbourhood $U_0(\varsigma)$ of $\varsigma$ in $\text{Stab} \mathcal{D}_\infty(\mathcal{S}^\lambda)$, such that $M^{P_j}_\pm$ remains stable (before it gets destabilized to be semistable). Then as (5.9) holds and $\mathcal{P}(\varsigma)$ remains unchange, we have

$$\left( U_0(\varsigma) \cap \mathcal{V} \right) \subset \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda),$$

that implies $\text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda)$ is open locally (in $\mathcal{V}$) and hence open in its closure as claimed.

Furthermore, when restricted to this submanifold, $\text{gldim}$ is in fact given by a single coordinate. Thus, it is differentiable without critical point.

Finally, we estimate the real dimension of $\text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda)$. Semi-stable objects and their phase differences are invariant under the $\mathcal{C}$-action. Thus, $\text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda)$ is closed under the $\mathcal{C}$-action, which implies that its real dimension is at least two. Moreover, $\text{gldim}$ is invariant under the $\mathcal{C}$-action. Together with the fact that $\text{gldim}$ has no critical point as we showed above, we deduce that $\text{dim}_{\mathbb{R}} \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(\mathcal{S}^\lambda) \geq 3$. Another way to see this is via a direct calculation. Namely, we have (from (4.1))

$$2n + 1 - s = 4g + 2b + 2N - 3 - s.$$

If $g \geq 2$ or $b \geq 2$, we have $8 \geq b \geq 1$ and $8 \geq s$, which implies $2n + 1 - s \geq 3$. If $g = 0, b = 1$, then $n \geq 2$ implies $8 \geq 3$. We claim that $s \leq 8 - 1$. Otherwise there is a cycle of saddle connections as in Figure 10, such that they correspond to a collection of semistable objects whose phase difference is gldim. But the winding number $w$ is $-2$, which implies (5.6) for $m = 8$. This contradicts to gldim $\geq 1$. So we always have $2n + 1 - s \geq 3$. $\square$
We proceed to analyze what happens on the boundary of subspaces \( \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(S^\lambda) \).

**Corollary 5.16.** If \( 1 \leq y < x \) such that \( (y,x) \cap \mathcal{V}(S^\lambda) = \emptyset \), then \( \text{Stab}_{\leq y} \mathcal{D}_\infty(S^\lambda) \) contracts to \( \text{Stab}_{\leq y} \mathcal{D}_\infty(S^\lambda) \) via a flow induced by the differential of \( \text{gldim} \).

**Proof.** By Theorem 5.15, if \( \sigma \) is in \( \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(S^\lambda) \), then \( d \text{gldim} \) gives a contractible flow. When the flow hits the boundary of \( \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(S^\lambda) \), it necessarily enters another \( \text{Stab}_{\mathcal{P}(\varsigma)} \mathcal{D}_\infty(S^\lambda) \). Then the corollary follows. \( \square \)

**Remark 5.17.** When following the contractible flow in \( \text{Stab}_{\mathcal{P}(\sigma)} \mathcal{D}_\infty(S^\lambda) \) and hits its boundary and then enters another \( \text{Stab}_{\mathcal{P}(\varsigma)} \mathcal{D}_\infty(S^\lambda) \) at \( \varsigma \), typical scenarios are

- \( \mathcal{P}(\sigma) \) gets bigger that \( \text{gldim} \) is achieved at another angle of the core or;
- some of the \( M^\pm \) get destabilized to be semistable and the corresponding pairs in \( \mathcal{P}(\sigma) \) are replaced.

In the next section, we will examine type A and (graded) affine type A cases in more details to show such phenomenons.

Another immediate consequence is the following.

**Corollary 5.18.** If \( S^\lambda \) is not a disk, then \( \text{Gd} \mathcal{D}_\infty(S^\lambda) \) is in \( \mathcal{V}(S^\lambda) \).

6. **Examples**

Denote by \( \mathbb{P}\text{Stab}(-) = \text{Stab}(-)/\mathbb{C} \) the spaces of projective stability conditions, where \( \text{gldim} \) is well-defined.

6.1. **Rank 2 cases and deformation.**

**Example 6.1.** Consider the case when \( S \) is a disk with three marked points, where

\[
\mathcal{D}_\infty(S^\lambda) \cong \mathcal{D}^b(kA_2)
\]

is the bounded derived category of an \( A_2 \) quiver \( 1 \to 2 \). Let \( Z_1 = Z(S_1) = Z(P_1) \), \( Z_2 = Z(S_2) \) and \( Z_3 = Z(P_2[1]) \). Then \( \mathbb{P}\text{Stab} \mathcal{D}_\infty(S^\lambda) \) decomposes into:

- three 1-dim \( \mathbb{R} \) subspaces (blue lines in Figure 11), which correspond to equations \( |Z_i| = |Z_j|, \) \( \{i,j\} \in \{1,2,3\} \);
- three 2-dim \( \mathbb{R} \) subspaces green areas in Figure 11 that are bounded by the 1 subspaces above;
- one critical point \( \mathbb{C} \cdot \sigma_G \) with \( |Z_1| = |Z_2| = |Z_3| \), which is the solution in Theorem 3.1.

The contractible flow is shown in Figure 11:

**Example 6.2.** Consider the case when \( S \) is an annulus with one marked point on each boundary, where

\[
\mathcal{D}_\infty(S^\lambda) \cong \mathcal{D}_\infty(kK_2) \cong \mathcal{D}^b(\text{coh } \mathbb{P}^1)
\]
is the bounded derived category of the Kronecker quiver $K_2: 1 \rightarrow 2$ or the one of coherent sheaves on $\text{coh} \mathbb{P}^1$. Similarly to the $A_2$ case, $\text{PStab} \mathcal{D}_\infty(S^\lambda)$ decomposes into (cf. [O] and [Q2, § 7.5.2]):

- a core 2-dim$_\mathbb{R}$ subspaces $\text{PStab}_{=1} \mathcal{D}^b(\text{coh} \mathbb{P}^1)$;
- $\mathbb{Z}$ many copies of 1-dim$_\mathbb{R}$ subspaces (dashed blue lines in Figure 12) in $\text{PStab}_{=1} \mathcal{D}^b(\text{coh} \mathbb{P}^1)$, which correspond to equations

$$|Z(O(j-1)[1])| = |Z(O(j))|, \quad j \in \mathbb{Z}.$$  

They are related by $- \otimes O(1) \in \text{Aut} \mathcal{D}^b(\text{coh} \mathbb{P}^1)$;
- $\mathbb{Z}$ many copies of 2-dim$_\mathbb{R}$ subspaces green areas in Figure 11, each of which is a connected component of

$$\text{PStab}_{>1} \mathcal{D}^b(\text{coh} \mathbb{P}^1).$$ (6.1)
They are also related by \( -\otimes \mathcal{O}(1) \in \text{Aut} \mathcal{D}^b(\text{coh} \mathbb{P}^1) \).

The contractible flow is shown in Figure 12 (exists in (6.1)):

6.2. Disk case revisit. Let \( S \) be a disk with \( n + 1 \) marked points, i.e. \( g = 0, b = 1, \delta = n + 1 \) and \( w = -2 \). Then \( \mathcal{D}_\infty(S^\lambda) \cong \mathcal{D}_\infty(A_n) \) for an \( A_n \) quiver. Recall that \( \text{Stab}_{<1} \mathcal{D}_\infty(A_n) \), consists of all totally stable stability conditions in this case. Its projective version is isomorphic to the space of convex \((n + 1)\)-gon (Proposition 2.5).

In this case, \( \mathcal{V}(S^\lambda) = \{1, (n - 1)/(n + 1)\} \), where \((n - 1)/(n + 1)\) is in fact \( \text{Gd} \mathcal{D}_\infty(A_n) \).

Then Theorem 5.15 can be rephrased as following corollary.

**Corollary 6.3.** \( \text{Stab}_{<y} \mathcal{D}_\infty(A_n) \) contracts to \( \text{Stab}_{<x} \mathcal{D}_\infty(A_n) \) for any \( 1 \leq x \leq y \). In particular, \( \text{Stab} \mathcal{D}_\infty(A_n) \) contracts to \( \text{Stab}_{<1} \mathcal{D}_\infty(A_n) \).

**Proof.** We only need to show the second statement. On one hand, we have

\[
\text{Stab} \mathcal{D}_\infty(A_n) = \lim_{y \to \infty} \text{Stab}_{<y} \mathcal{D}_\infty(A_n).
\]

On the other hand, any \( \sigma \in \text{Stab} \mathcal{D}_\infty(A_n) \) with \( \text{gldim} \sigma = 1 \) is in some (open) real submanifold \( \text{Stab}_{\eta(\sigma)} \mathcal{D}_\infty(A_n) \) and thus can be further contracted. Thus, the statement follows. \( \square \)

6.3. Annulus case revisit. We keep the notation in Example 4.4, i.e. we have \( \mathcal{D}_\infty(S^\lambda) \cong \mathcal{D}_\infty(A_{m,r}) \) and \( \mathcal{N}(S^\lambda) = ((m,r),(w,-w)) \) for \( m, r \in \mathbb{Z}_+ \) and \( w \in \mathbb{Z}_{\geq 0} \).

Then \( \mathcal{V}(S^\lambda) = \{1, 1 + w/m\} \).

**Theorem 6.4.** \( \text{Stab}_{<y} \mathcal{D}_\infty(A_{m,r}) \) contracts to \( \text{Stab}_{<x} \mathcal{D}_\infty(A_{m,r}) \) for any \( 1 + w/m \leq x \leq y \). Moreover, \( \text{Gd} \mathcal{D}_\infty(A_{m,r}) = 1 + w/m \) and in particular \( \text{Stab} \mathcal{D}_\infty(A_{m,r}) \) contracts to \( \text{Stab}_{1+w/m} \mathcal{D}_\infty(A_{m,r}) \).

**Proof.** By Theorem 5.15, we only need to prove \( \text{Gd} \mathcal{D}_\infty(A_{m,r}) = 1 + w/m \). The \( w = 0 \) case is contained in [Q1, Thm. 5.2].
Consider the case when \( w > 0 \) and suppose that \( \text{Gd} \, D_\infty(\tilde{A}_{m,r}) < 1 + \frac{w}{m} \). We have \( \text{Stab}_{<1+w/m} \, D_\infty(\tilde{A}_{m,r}) \) contracts to \( \text{Stab}_{<1} \, D_\infty(\tilde{A}_{m,r}) \) by Theorem 5.15, which implies that \( \text{Gd} \, D_\infty(\tilde{A}_{m,r}) \leq 1 \). By Corollary 5.11, this can only happen for \( w = 0 \), which is a contradiction. Thus, \( \text{Gd} \, D_\infty(\tilde{A}_{m,r}) \geq 1 + \frac{w}{m} \), which forces \( \text{Gd} \, D_\infty(\tilde{A}_{m,r}) = 1 + \frac{w}{m} \) by Lemma 4.5.

\[ \square \]

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