**W_∞ AND w_∞ GAUGE THEORIES AND CONTRACTION**

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**Abstract**

We present a general method of constructing $W_∞$ and $w_∞$ gauge theories in terms of $d+2$ dimensional local fields. In this formulation the $W_∞$ gauge theory Lagrangians involve non-local interactions, but the $w_∞$ theories are entirely local. We discuss the so-called classical contraction procedure by which we derive the Lagrangian of $w_∞$ gauge theory from that of the corresponding $W_∞$ gauge theory. In order to discuss the relationship between quantum $W_∞$ and quantum $w_∞$ gauge theory we solve $d = 1$ gauge theory models of a Higgs field exactly by using the collective field method. Based on this we conclude that the $W_∞$ gauge theory can be regarded as the large $N$ limit of the corresponding $SU(N)$ gauge theory once an appropriate coupling constant renormalization is made, while the $w_∞$ gauge theory cannot be.

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1. Introduction and Summary

$W_\infty$ algebra and its so-called classically contracted $w_\infty$ algebra appeared recently in various problems in physics, in particular in the study of $c = 1$ string theory [1] and quantum Hall system [2, 3]. The gauge theory based on these algebras also appeared in these studies [4, 5]. In fact the same algebras and the gauge theories based on them had been proposed previously as the theories [6, 7, 8] relevant to the large $N$ limit of $SU(N)$ gauge theories [9]. But it seems to us that not enough studies have been done to differentiate the formal and dynamic aspect of $W_\infty$ and $w_\infty$ theories. In view of the recent developments we study in this paper this subject as systematically as possible by using the technique developed in the study of quantum Hall system [2, 5].

The $W_\infty$ algebra is a commutator algebra of Hermitian operators of one harmonic oscillator [10]. It is an infinite-dimensional Lie algebra. If we choose a set of linearly independent real function of $z$ and $\bar{z}$ as the parameters of $W_\infty$ group, the structure constants of the algebra are expressed in terms of Moyal bracket [11]. Replacing the Moyal bracket by a Poisson bracket we define the $w_\infty$ algebra. It is an algebra of area-preserving diffeomorphism. As an introduction we discuss this issue in section 2 together with the so-called classical contraction procedure by which the $W_\infty$ algebra is transformed to the $w_\infty$ algebra.

$W_\infty$ gauge theory is a gauge field theory of $W_\infty$ as an internal symmetry algebra. The $W_\infty$ gauge potential is a space-time dependent Hermitian operator of harmonic oscillator. In the coherent state representation it is a function of $z$, $\bar{z}$, which we call the color space coordinates, and $x^\mu$ ($\mu = 1, 2, \ldots d$), the space time coordinates. Thus, we can express the $W_\infty$ gauge theories in terms of $d + 2$ dimensional local fields. The interactions of the fields are necessarily non-local in the color space in $W_\infty$ theories, but they are local in $w_\infty$ theories. In section 3 we define $W_\infty$ theories as $d + 2$ dimensional field theories with non-local interactions and $w_\infty$ gauge theories as $d + 2$ dimensional local field theories. Since the $W_\infty$ algebra is closely related to the $w_\infty$ algebra, the gauge theories based on
these algebras may also be closely related. In order to see the relationship at classical Lagrangian level, we introduce the $l \to 0$ contraction procedure by which we derive the $w_\infty$ gauge theories from the corresponding $W_\infty$ gauge theories. The procedure consists of the introduction of a length scale $l$ in the color space, an appropriate scale transformation of the fields, and the $l \to 0$ limit. In this section we also introduce matter fields analogous to the quark fields and the Higgs fields.

The $W_\infty$ algebra can be considered as the $N \to \infty$ limit of the $SU(N)$ algebra [10]. Therefore, the $W_\infty$ gauge theory or its variation $w_\infty$ gauge theory [7, 12] might be used for the large $N$ gauge theory. Since the $w_\infty$ gauge theory is a local theory and much easier to be handled, it is important to determine whether this theory can serve for the large $N$ gauge theory or not. For this purpose we solve $d = 1$ gauge theory of Higgs field exactly in section 4, which reveals also the quantum mechanical relationship between the $W_\infty$ theory and the $w_\infty$ theory. In $d = 1$ there exists only the time component of the gauge potential and the pure gauge theory is trivial but it constrains the states of Higgs field to be gauge invariant. The $W_\infty$ theory becomes essentially $N = \infty$ limit of $d = 1$ gauged matrix model. On the other hand the $w_\infty$ model becomes an infinite number of non-interacting quantum mechanical systems. We use the collective field method [13] to solve these theories. The spectrum of these two theories are in general different and coincides with the $N = \infty$ limit of $SU(N)$ theory only for $W_\infty$ model. Based on this result we conclude that $W_\infty$ gauge theory can but $w_\infty$ gauge theory cannot serve for the purpose of large $N$ gauge theory.

2. $W_\infty$ and $w_\infty$ Algebra.

We define the $W_\infty$ algebra as a commutator algebra of all Hermitian operators $\xi(\hat{a}, \hat{a}^\dagger)$ in the Hilbert space of a harmonic oscillator [2]. A convenient parametrization for these operators is achieved by using a real function $\xi(z, \bar{z})$ as

$$\xi(\hat{a}, \hat{a}^\dagger) = \frac{i}{2}\xi(z, \bar{z})|_{z=\hat{a}^\dagger, \bar{z}=\hat{a}} \frac{\xi(z, \bar{z})}{z} = \int d^2z e^{-|z|^2}|z\rangle \xi(z, \bar{z}) \langle z|, \tag{2.1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are standard creation and annihilation operators and $\frac{\xi(z, \bar{z})}{z}$ stands for
the anti-normal-order symbol, i.e., all the creation operators stand to the right of the annihilation operators. The last expression (2.1) is in the coherent state representation (see A1). Obviously the product of two ξ’s is not anti-normally ordered and we bring it to the anti-normal-order form by using the commutation relation [^a, ^a†] = 1. We obtain (see A.1)

$$\xi_1(\hat{a}, \hat{a}†)\xi_2(\hat{a}, \hat{a}†) = \frac{i}{4} \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \partial^n_z \xi_1(z, \bar{z}) \partial^n_{\bar{z}} \xi_2(z, \bar{z}) \bigg|_{z=\hat{a}, \bar{z}=\hat{a}†},$$

(2.2)

from which the following commutation relation follows

$$[\xi_1(\hat{a}, \hat{a}†), \xi_2(\hat{a}, \hat{a}†)] = i\{\xi_1, \xi_2\}(\hat{a}, \hat{a}†),$$

(2.3)

where \{\xi_1, \xi_2\} is a Moyal bracket [11] defined by

$$\{\xi_1, \xi_2\}(z, \bar{z}) \equiv \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \left( \partial^n_z \xi_1(z, \bar{z}) \partial^n_{\bar{z}} \xi_2(z, \bar{z}) - \partial^n_{\bar{z}} \xi_1(z, \bar{z}) \partial^n_z \xi_2(z, \bar{z}) \right).$$

(2.4)

The commutation relation (2.3) is that of the $W_\infty$ Lie algebra in the fundamental representation. The $W_\infty$ is an infinite-dimensional Lie group with parameters being a set of linearly independent real functions $\xi(z, \bar{z})$. The generators of $W_\infty$ are the linear functionals of $\xi(z, \bar{z})$. Thus we write for arbitrary representation:

$$[\rho[\xi_1], \rho[\xi_2]] = i\rho[\{\xi_1, \xi_2\}],$$

(2.5)

where $\rho$ is the generator of $W_\infty$ group.

The Lie algebra of $w_\infty$, the area-preserving diffeomorphisms, is defined by the commutation relation

$$[\rho[\xi_1], \rho[\xi_2]] = i\rho[\{\xi_1, \xi_2\}],$$

(2.6)

where \{ , \} is the Poisson bracket symbol.

It is well known [2] that one can obtain the $w_\infty$ algebra from the $W_\infty$ by a contraction. To explain it let us introduce a length scale $l$ in the $z, \bar{z}$ space, which we call the color space, and set

$$z = \frac{1}{\sqrt{2l}}(\sigma_x + i\sigma_y), \quad \bar{z} = \frac{1}{\sqrt{2l}}(\sigma_x - i\sigma_y).$$

(2.7)
The Poisson bracket is the leading surviving term of the Moyal bracket in the \( l \to 0 \) limit.

To be more specific we set \( \xi(z, \bar{z}) = l^{-2} \xi(\bar{\sigma}) \) and obtain

\[
\lim_{l \to 0} l^2 \{ \xi_1, \xi_2 \}(z, \bar{z}) = \partial_x \xi_1(\bar{\sigma}) \partial_y \xi_2(\bar{\sigma}) - \partial_y \xi_1(\bar{\sigma}) \partial_x \xi_2(\bar{\sigma}) \equiv \epsilon^{ij} \partial_i \xi_1(\bar{\sigma}) \partial_j \xi_2(\bar{\sigma}) \equiv \{ \xi_1, \xi_2 \}(\bar{\sigma}).
\]

(2.8)

In this paper we call this procedure as the \( l \to 0 \) contraction.

3. \( W_\infty \) and \( w_\infty \) Gauge Invariant Lagrangians

The \( W_\infty \) gauge theory is a gauge field theory of \( W_\infty \) as an internal symmetry algebra.

Let us discuss first the pure Yang-Mills theory. We introduce a gauge potential \( \hat{A}_\mu \) which is a Hermitian operator in the harmonic oscillator Hilbert space as well as a function of space time:

\[
\hat{A}_\mu(x) \equiv A_\mu(x, \hat{a}, \hat{a}^\dagger) = \int d^2 z e^{-|z|^2} |z \rangle A_\mu(x, z, \bar{z}) \langle z|.
\]

(3.1)

The action is given by

\[
S_{YM} = -\frac{1}{4g^2} \int d^4 x \text{tr} (\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}),
\]

(3.2)

where \( \hat{F}_{\mu\nu} \) is the field strength defined by

\[
\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu].
\]

(3.3)

We then rewrite these by using the coherent state representation:

\[
S_{YM} = -\frac{1}{4g^2} \int d^4 x d^2 z \sum_{n=0}^\infty \frac{(-)^n}{n!} \partial_z^n A_{\mu\nu}(x, z, \bar{z}) \partial_{\bar{z}}^n \bar{F}^{\mu\nu}(x, z, \bar{z}),
\]

(3.4)

where

\[
A_{\mu\nu}(x, z, \bar{z}) = \partial_\mu A_\nu(x, z, \bar{z}) - \partial_\nu A_\mu(x, z, \bar{z}) + \{ A_\mu, A_\nu \}(x, z, \bar{z}).
\]

(3.5)

This action is invariant under the \( W_\infty \) gauge transformation:

\[
\delta \hat{A}_\mu(x) = \partial_\mu \hat{\xi}(x) + i[\hat{\xi}(x), \hat{A}_\mu(x)], \quad \delta A_\mu(x, z, \bar{z}) = \partial_\mu \xi(x, z, \bar{z}) - \{ \xi, A_\mu \}(x, z, \bar{z}).
\]

(3.6)
In the coherent state representation the gauge fields are formally \(d + 2\) dimensional local fields, \(d\) for space time and \(2\) for color space. However, the interactions are non-local since the action involves derivatives of infinite order.

In the action (3.4) we no longer have damping factor \(e^{-|z|^2}\) because it is a trace expression. Therefore we have to restrict the field configurations so that we can integrate by parts in the color space. In this paper we define our \(W_\infty\) gauge theories as \(d + 2\) dimensional field theories such that all the fields and their derivatives vanish at \(z = \infty\).

Next, let us introduce a fermion field, which is a fundamental representation of \(W_\infty\), namely a field which transforms as a bra or ket vector in the Hilbert space of harmonic oscillator:

\[
|\psi(x)\rangle = \int |z\rangle d^2z e^{-|z|^2} \langle z|\psi(x)\rangle \equiv \int |z\rangle d^2z e^{-|z|^2} \psi(x, \bar{z}). \tag{3.7}
\]

We write the action as

\[
S_F = \int d^n x \langle \psi(x) | \gamma^\mu (i \partial_\mu - \hat{A}_\mu (x)) | \psi(x) \rangle = \int \int d^n x d^2z e^{-|z|^2} \bar{\psi}(x, \bar{z}) \gamma^\mu (i \partial_\mu - A_\mu (x, \bar{z})) \psi(x, \bar{z}), \tag{3.8}
\]

which is invariant under the \(W_\infty\) gauge transformation (3.6) and

\[
\delta |\psi(x)\rangle = -i \hat{\xi}(x)|\psi(x)\rangle, \quad \delta \psi(x, \bar{z}) = -i \hat{\xi}(\partial_{\bar{z}}, \bar{z}) \hat{\xi} \psi(x, \bar{z}), \tag{3.9}
\]

where \(\hat{\xi} \quad \dagger\) indicates that the derivatives are placed on the left of \(\bar{z}\).

As a last example of \(W_\infty\) gauge theory let us consider next a scalar field which is in an adjoint representation. In this paper we call this field simply a Higgs field.

\[
\hat{M}(x) \equiv M(x, \hat{a}, \hat{a}^\dagger) = \int d^2z e^{-|z|^2} \langle \bar{z} | M(x, z, \bar{z}) | z \rangle. \tag{3.10}
\]
The action is given by:

\[
S_H = \int d^d x \text{tr} \left[ \frac{1}{2} (\partial_\mu \hat{M}(x) - [\hat{A}_\mu, \hat{M}](x)) (\partial^\mu \hat{M}(x) - [\hat{A}^\mu, \hat{M}](x)) - v(\hat{M}) \right]
\]

\[
= \int d^d x \left[ \int d^2 z \frac{1}{2} \sum_{m=0}^\infty \frac{(-)^m}{m!} \partial_z ^m (\partial_\mu M(x, z, \bar{z}) - \{A_\mu, M\}(x, z, \bar{z})) \times \partial_{\bar{z}} ^m (\partial^\mu M(x, z, \bar{z}) - \{A^\mu, M\}(x, z, \bar{z})) \right] - \text{tr} v(\hat{M}),
\]

\[v(\hat{M}) = \sum_n g_n \hat{M}^n, \quad \text{dim}(g_n) = d \left( \frac{n}{2} - 1 \right) - n.\]  

Here again we require that the fields and their \(z, \bar{z}\) derivatives should fall off to zero at \(z = \infty\). We can check that this action is invariant under the \(W_\infty\) gauge transformation (3.6) and

\[
\delta M(x, z, \bar{z}) = \{\xi, M\}(x, z, \bar{z}), \quad \delta \hat{M}(x) = -i [\hat{\xi}(x), \hat{M}(x)]. \quad (3.12)
\]

Notice that here again the interactions are non-local in the color space.

The quantization of the theory is done by the standard canonical quantization. Although the interactions are non-local in the color space, they are local in the ordinary space. Accordingly there is no problem for the quantization. We shall see it explicitly in an example in the next section.

Let us discuss next the contraction procedure which will allow us to obtain the \(w_\infty\) gauge invariant actions from \(W_\infty\) ones. For the fields in the adjoint representation such as \(A_\mu\) and \(M\) this procedure is straightforward. As we mentioned earlier, by introducing two real coordinates \(\sigma_x\) and \(\sigma_y\) as in (2.6) and by taking the \(l \to 0\) limit we reduce Moyal bracket to Poisson bracket. If we simultaneously rescale the fields and the coupling constants as

\[
A_\mu(x, z, \bar{z}) = l^{-2} A_\mu(x, \bar{\sigma})
\]

\[
M(x, z, \bar{z}) = (2\pi)^{\frac{d}{2}} l M(x, \bar{\sigma})
\]

\[
g^2 = \tilde{g}^2 l^{-6}, \quad g_n = \tilde{g}_n (2\pi)^\frac{d}{2} l^{2-n},
\]

we obtain from (3.4) and (3.11) the following \(w_\infty\) gauge invariant \(d + 2\) dimensional local
field theory:

\[ S_{YM} = -\frac{1}{4\tilde{g}^2} \int d^d x d^2 \sigma F_{\mu\nu}(x, \sigma) F^{\mu\nu}(x, \sigma), \]

\[ F_{\mu\nu}(x, \sigma) = \partial_{\mu} A_{\nu}(x, \sigma) - \partial_{\nu} A_{\mu}(x, \sigma) + \epsilon^{ij} \partial_i A_{\mu}(x, \sigma) \partial_j A_{\nu}(x, \sigma); \]

\[ S_H = \int d^d x d^2 \sigma \left[ \frac{1}{2} (\partial_{\mu} M(x, \sigma) - \epsilon^{ij} \partial_i M(x, \sigma) \partial_j M(x, \sigma)) \times \right. \]

\[ \times \left( \partial^{\mu} M(x, \sigma) - \epsilon^{ij} \partial_{\mu} A^{\mu}(x, \sigma) \partial_j M(x, \sigma) \right) - \tilde{v}(M) \right], \]

\[ \tilde{v}(M) = \sum_n \tilde{g}_n M^n(x, \sigma). \]

Here again we require that the fields vanish at \( \sigma = \infty \).

Setting \( \xi(x, z, \bar{z}) = l^{-2} \xi(x, \bar{\sigma}) \) we obtain the \( w_\infty \) gauge transformation:

\[ \delta A^\mu(x, \bar{\sigma}) = \partial^\mu \xi(x, \bar{\sigma}) - \epsilon^{ij} \partial_i \xi(x, \bar{\sigma}) \partial_j A^\mu(x, \bar{\sigma}) \]

\[ \delta M(x, \bar{\sigma}) = \epsilon^{ij} \partial_i \xi(x, \bar{\sigma}) \partial_j M(x, \bar{\sigma}). \]

Even though the transformations (3.15) are obtained from (3.6) and (3.12) by the \( l \to 0 \) limit, we can independently check that the actions (3.14) is really invariant by the \( w_\infty \) gauge transformation (3.15). We remark that the second equation of (3.15) can be written as

\[ \delta M(x, \bar{\sigma}) = M(x, \bar{\sigma} + \delta \sigma(x, \bar{\sigma})) - M(x, \bar{\sigma}), \]

\[ \delta \sigma^i(x, \bar{\sigma}) = -\epsilon^{ij} \partial_j \xi(x, \bar{\sigma}), \]

which is a local area-preserving coordinate transformation.

As we mentioned earlier the damping factor \( e^{-|z|^2} \) cancels out in Lagrangians for the fields in the adjoint representation such as \( A_\mu \) and \( M \), due to the property of the trace in coherent state representation. But it remains there for the fields in fundamental representation, such as Fermi field (see (3.8)). Therefore, a naive \( l \to 0 \) limit leads to the trivial result (\( S_F \equiv 0 \)). This may imply difficulty in introducing a Fermi field of fundamental representation in \( w_\infty \) theory.

We should mention that the YM lagrangian (3.14) had already been written down in the literature [7, 8].

4. One Dimensional Higgs Model: One Dimensional \( W_\infty \) and \( w_\infty \) Matrix model
In the previous section we presented a general method for constructing $W_\infty$ gauge theory and then we obtained the $w_\infty$ gauge theories by using the $l \to 0$ contraction procedure from $W_\infty$ theories.

Several questions arise. Since $W_\infty$ group can be considered as an $N = \infty$ limit of $SU(N)$ group, can one use the $W_\infty$ gauge theory for the large $N$ limit of $SU(N)$ gauge theory, especially for the large $N$ QCD [9]? In the large $N$ QCD one takes the $N \to \infty$ limit keeping $g^2 N$ finite. Since in $W_\infty$ theories $N$ is already at infinity, how can one implement the large $N$ QCD condition? A simple Feynman diagramatic calculation of $w_\infty$ theory shows that the coupling constants are multiplicatively renormalized to absorb the infinite volume of color space. In a similar way in $W_\infty$ gauge theory the question arises whether or not one can implement the QCD condition as a multiplicative renormalization of coupling constants? As shown in the previous section it is possible to obtain $w_\infty$ theory from $W_\infty$ theory by contraction. This shows the relationship between these theories as classical theories. How about in quantum theory? Is there any physical region where one can use the $w_\infty$ gauge theory for large $N$ QCD? In this section we address these questions by solving the simplest one-dimensional case exactly.

Since in $d = 1$ there exists only time component of gauge field and the pure gauge field model becomes trivial, we consider the gauge invariant Higgs model. This model may be thought of as a gauged one dimensional $W_\infty$ (or $w_\infty$ ) matrix model [14]. We solve it by using the collective field method [13]. Since we can carry out the discussions entirely in parallel for both $W_\infty$ and $w_\infty$ models, we present the corresponding expressions simultaneously and put label $a$ for $W_\infty$ model and label $b$ for $w_\infty$ model.

One dimensional $W_\infty$ and $w_\infty$ matrix model Lagrangians are given by (compare with (3.11) and (3.14) respectively):

$$L = \text{tr} \left[ \frac{1}{2} (\partial_t \hat{M} - [\hat{A}_0, \hat{M}])^2 - v(\hat{M}) \right]$$

$$= \frac{1}{2} \int d^2 z (\partial_t M(t, z, \bar{z}) - \{ A_0, M \}(t, z, \bar{z})) e^{\partial_t \partial_z (\partial_t M(t, z, \bar{z}) - \{ A_0, M \}(t, z, \bar{z}))} - \text{tr} v(\hat{M}),$$

(4.1a)
\[ L = \int d\bar{\sigma} \left[ \frac{1}{2} (\partial_i M(t, \bar{\sigma}) - \epsilon^{ij} \partial_i A_0(t, \bar{\sigma}) \partial_j M(t, \bar{\sigma}))^2 - v(M) \right]. \quad (4.1b) \]

The canonical quantization leads to the following Hamiltonians:

\[ H = \int P(z, \bar{z}) \partial_t M(z, \bar{z}; t) d^2 z - L = \int d^2 z \sum_{n=0}^{\infty} \frac{1}{n!} e^{n} P(z, \bar{z}) \partial_z^n P(z, \bar{z}) + \text{tr} \langle \hat{M} \rangle, \quad (4.2a) \]

\[ H = \int P(\bar{x}) \partial_t M(\bar{x}; t) d^2 x - L = \int d\bar{\sigma} \left( \frac{1}{2} P(\bar{\sigma})^2 + v(M(\bar{\sigma})) \right), \quad (4.2b) \]

where \( P \)'s are the canonical momentum operators conjugated to \( M \)'s, with the following commutation relations:

\[ [\hat{M}(z, \bar{z}), \hat{P}(z', \bar{z}')] = i\delta^{(2)}(z - z'), \quad (4.3a) \]

\[ [\hat{M}(\bar{\sigma}), \hat{P}(\bar{\sigma}')] = i\delta(\bar{\sigma} - \bar{\sigma}'). \quad (4.3b) \]

The \( W_\infty \) (and \( w_\infty \)) gauge invariance of the actions (4.1) leads to the following constraints, which we impose on the state vector \( |\Psi\rangle \):

\[ \hat{\Pi}_\xi |\Psi\rangle \equiv \int d^2 z \{\xi, \hat{M}\} \hat{P}(z, \bar{z}) |\Psi\rangle = 0, \quad (4.4a) \]

\[ \hat{\Pi}_\xi |\Psi\rangle \equiv \int d\bar{\sigma} e^{ij} \partial_i \xi(\bar{\sigma}) \partial_j \hat{M}(\bar{\sigma}) \hat{P}(\bar{\sigma}) |\Psi\rangle = 0, \quad (4.4b) \]

which simply state that the wave function (or the state vector) is gauge invariant and depends only on gauge invariant singlet variables. Therefore in order to solve the problem we choose the following \( W_\infty \) (and \( w_\infty \)) invariant collective field as dynamical variables:

\[ \phi(x) = \text{tr} \delta(x - M(\hat{a}, \hat{a}^\dagger)), \quad (4.5a) \]

\[ \phi(x) = \int d\bar{\sigma} \delta(x - M(\bar{\sigma})). \quad (4.5b) \]

We then change variables from \( P(z, \bar{z}), M(z, \bar{z}) \) to \( \pi(x), \phi(x) \), where \( \pi(x) \) is a canonical momentum conjugate to \( \phi(x) \):

\[ [\pi(x), \phi(x)] = -i\delta(x - x') + \text{const.} \quad (4.6) \]
The standard procedure (see A2) to do this is to compute \( \Omega(x, x') \) and \( \omega(x) \). We describe this calculation in Appendix A3. The definitions and the results are

\[
\Omega(x, x') \equiv - \int d^2z \sum_{n=0}^{\infty} \frac{1}{n!} [\partial^n_z P(z, \bar{z}), \phi(x)][\partial^n_{\bar{z}} P(z, \bar{z}), \phi(x')] = \partial_x \partial_{x'} [\delta(x - x') \phi(x)],
\]

(4.7a)

\[
\Omega(x, x') \equiv - \int d\bar{\sigma} [P(\bar{\sigma}), \phi(x)][P(\bar{\sigma}), \phi(x')] = \partial_x \partial_{x'} [\delta(x - x') \phi(x)],
\]

(4.7b)

and

\[
\omega(x) \equiv \int d^2z \sum_{n=0}^{\infty} \frac{1}{n!} [\partial^n_z P(z, \bar{z}), [\partial^n_{\bar{z}} P(z, \bar{z}), \phi(x)]] = 2\partial_x [\phi(x) G(x; \phi)],
\]

(4.8a)

\[
\omega(x) \equiv \int d\bar{\sigma} [P(\bar{\sigma}), [P(\bar{\sigma}), \phi(x)]] = -\kappa^2 \partial^2_x \phi(x),
\]

(4.8b)

where \( \kappa^2 = \delta^2(0) \), and \( G(x; \phi) = P \int \frac{\phi(x')}{\phi(x)} \). Notice that we obtained the same expression for \( \Omega \) for both theories but quite different expressions for \( \omega \).

In the collective field theory the hermiticity requirement of the Hamiltonian leads to the following equation for the Jacobian \( J \) of change of variables:

\[
\omega(x) + 2 \int dx' \Omega(x, x') C(x') = 0, \quad C(x) = \frac{1}{2} \frac{\delta}{\delta \phi(x)} J,
\]

(4.9)

Using (4.7) and (4.8) and assuming \( \partial_x \phi(-\infty) = \phi(-\infty) \partial_x C(-\infty) = 0 \) we obtain

\[
\partial_x C(x) = G(x; \phi),
\]

(4.10a)

\[
\partial_x C(x) = -\frac{1}{2} \kappa^2 \frac{\partial_x \phi(x)}{\phi(x)} = -\frac{1}{2} \kappa^2 \partial_x \ln \phi(x).
\]

(4.10b)

Since the kinetic energy part of the hermitian Hamiltonian in the collective field theory is given by

\[
K = \frac{1}{2} \int \int dx dx' [\pi(x) \Omega(x, x') \pi(x') + C(x) \Omega(x, x') C(x')],
\]

(4.11)

using (4.7) and (4.10) we obtain the following Hamiltonians:

\[
H = \int dx \left( \frac{1}{2} (\partial_x \pi(x))^2 \phi(x) + \frac{\pi^2}{6} \phi(x)^3 + v(x) \phi(x) \right) - e \left( \int dx \phi(x) - N \right)
\]

(4.12a)
\[
H = \int dx \left( \frac{1}{2} \partial_x \pi(x) \phi(x) \partial_x \pi(x) + \frac{1}{8} \kappa^4 \left( \frac{\partial_x \phi(x)}{\phi(x)} \right)^2 + \tilde{v}(x) \phi(x) \right) - e \left( \int dx \phi(x) - L^2 \right) \tag{4.12b}
\]

where \(e\) is a Lagrange multiplier to insure
\[
\int dx \phi(x) = \text{tr} 1 \equiv N \quad \text{(for } W_\infty), \tag{4.13a}
\]
\[
\int dx \phi(x) = \int d\sigma \equiv L^2 \quad \text{(for } \omega_\infty) \tag{4.13b}
\]

which follows from the definition of collective field (4.5). Notice that eventually we have to take \(N \to \infty, \kappa \to \infty, L \to \infty\). For this purpose we first make the following scale transformations:
\[
x \to N^{1/2} x, \quad \phi(x) \to N^{1/2} \phi(x), \quad \pi(x) \to N^{-1} \pi(x), \quad e \to Ne \tag{4.14a}
\]
\[
x \to \kappa x, \quad \phi(x) \to L^2 \kappa \phi(x), \quad \pi(x) \to L^{-2} \pi(x), \quad e \to \kappa^2 e \tag{4.14b}
\]

which preserve the canonical commutation relation (4.6). We obtain
\[
H = \int dx \left[ \frac{1}{2N^2} \left( \partial_x \pi(x) \right)^2 \phi(x) + N^2 \left( \frac{\pi^2}{6} \phi(x)^3 + u(x) \phi(x) - e \phi(x) \right) \right] + N^2 e \tag{4.15a}
\]
\[
H = \int dx \left[ \frac{1}{2N^2} \left( \partial_x \pi(x) \right)^2 \phi(x) + N^2 \left( \frac{1}{8} \left( \frac{\partial_x \phi(x)}{\phi(x)} \right)^2 + \tilde{u}(x) \phi(x) - e \phi(x) \right) \right] + N^2 e \tag{4.15b}
\]

where for \(w_\infty\) we set \(N \equiv L \kappa\) and
\[
u(x) = \sum_n N \left( \frac{\pi}{2} - 1 \right) g_n x^n \equiv \sum_n \alpha_n x^n, \tag{4.16a}
\]
\[
\tilde{\nu}(x) = \sum_n \kappa^{-2} \tilde{g}_n x^n = \sum_n (\kappa l)^{-2} g_n x^n \equiv \sum_n \tilde{\alpha}_n x^n. \tag{4.16b}
\]

We know from the previous study [13, 15] that the \(1/N\) expansion of Hamiltonians (4.15) is a standard semi-classical expansion and in the \(N \to \infty\) limit the excitation spectrum is finite provided that \(u(x)\) is finite and given by
\[
H = \frac{1}{2} \sum_{n=0}^{\infty} \left( p_n^2 + \omega_n^2 q_n^2 \right), \quad [q_n, p_m] = i \delta_{nm}, \tag{4.17}
\]
where

\[ \omega_n = n\pi/T, \quad T = \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{dx'}{\sqrt{2(\tilde{c}_0 - u(x'))}}, \quad \frac{1}{\pi} \int_{\tilde{x}_1}^{\tilde{x}_2} dx' \sqrt{2(\tilde{c}_0 - u(x'))} = 1, \quad (4.18a) \]

\[ \omega_n = E_n - E_0, \quad \left(-\frac{1}{2} \partial_x^2 + \tilde{u}(x)\right)\chi_n(x) = E_n\chi_n(x) \quad (4.18b) \]

The derivation of these results will be given in detail in Appendix A4 and A5.

The result for \( W_\infty \) model exactly coincides with the \( N \to \infty \) limit of the matrix model discussed in [13, 15]. Of course it is expected already from the equations (4.7a) and (4.8a) since these equations coincide with the ones obtained in [13]. The infinite volume of color space (i.e. \( N \)) is absorbed by the multiplicative renormalization into the coupling constants (see (4.16)).

The \( W_\infty \) effective Hamiltonian (4.15a) is actually an element of the \( w_\infty \) spectrum generating algebra [1]. However this \( w_\infty \) symmetry is not directly related to the original \( W_\infty \) gauge symmetry of the Lagrangian, since the dynamical \( w_\infty \) transformations are realized non-trivially in the physical Hilbert space while the original \( W_\infty \) gauge symmetry acts trivially in the physical Hilbert space.

As we see in (4.18) the energy spectrum of \( W_\infty \) model is that of bosons with equally spaced frequencies irrespective of its interactions. For the \( w_\infty \) model the equally spaced spectrum occurs only when \( \tilde{\alpha}_n = 0 \) for \( n \neq 2 \), namely the free \( w_\infty \) theory. It appears that in quantum theory the \( l \to \infty \) contraction of \( W_\infty \) theory is a free \( w_\infty \) theory. Therefore, it is not possible to learn \( W_\infty \) theory by studying \( w_\infty \) theory.

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Appendix A1. Coherent state representation

We list the definitions and basic properties of the coherent state representation which we use throughout the paper:

\[ |z\rangle = e^{\hat{a}^\dagger z}|0\rangle, \quad \langle z| = \langle 0|e^{\hat{a}z}, \quad \langle z'|z\rangle = e^{z'z} \]

\[ \hat{a}|z\rangle = z|z\rangle, \quad \langle z|\hat{a}^\dagger = \langle z|\bar{z}, \quad \int d^2z e^{-|z|^2}|z\langle z| = 1, \quad d^2z \equiv \frac{dRez \, dImz}{\pi} \] (A1.1)

For a given real function \( \xi(z, \bar{z}) = \sum_{m,n} \xi_{mn} z^n \bar{z}^m \) we obtain

\[ \xi(\hat{a}, \hat{a}^\dagger) = \sum_{m,n} \xi_{mn} \hat{a}^n (\hat{a}^\dagger)^m = \int d^2z e^{-|z|^2} \sum_{m,n} \xi_{mn} \hat{a}^n |z\langle z|(\hat{a}^\dagger)^m = \int d^2z e^{-|z|^2}|z\rangle \xi(z, \bar{z}) \langle z| \] (A1.2)

A proof of (2.2) goes as follows.

\[ \xi_1(\hat{a}, \hat{a}^\dagger) \xi_2(\hat{a}, \hat{a}^\dagger) = \int d^2z e^{-|z|^2} \int d^2z' e^{-|z'|^2}|z\rangle \xi_1(z, \bar{z}) e^{\bar{z}'z'} \xi_2(z', \bar{z}') \langle z'| 
\]

\[ = \int d^2z e^{-|z|^2} \int d^2z' e^{-|z'|^2}|z\rangle \xi_1(z, \bar{z}) \xi_2(z - \frac{\bar{z}}{\bar{z}'}, \bar{z}') e^{\bar{z}'z'} \langle z'| 
\]

\[ = \int d^2z |z\rangle \xi_1(z, \bar{z}) \xi_2(z - \frac{\bar{z}}{\bar{z}'}, \bar{z}') e^{-|z|^2} \langle z| 
\]

\[ = \int d^2z e^{-|z|^2}|z\rangle \sum_{n=1}^\infty \left( \frac{-1}{n!} \partial^n_z \xi_1(z, \bar{z}) \partial^n_z \xi_2(z, \bar{z}) \right) \big|_{z=\bar{z}', \bar{z}'=\bar{z}} \] (2.2)

A2. Change of variables in quantum mechanics [13]

We start with a standard form for the Hamiltonian and Schrödinger equation which is given by

\[ \hat{H}\psi = \left( -\frac{1}{2} \sum_{a=1}^N \frac{\partial^2}{\partial q^a \partial q^a} + V(q) \right) \psi(q) = E\psi(q) \] (A2.1)

We consider a transformation given by

\[ q^a \rightarrow Q^a = f^a(q), \quad q^a = F^a(Q) \] (A2.2)
We use the chain rule of differentiation to convert the derivatives with respect to \( q \)'s into derivatives with respect to \( Q \)'s.

\[
-\frac{1}{2} \sum_a \frac{\partial^2}{\partial q^a} \frac{\partial}{\partial \psi(q)} \left( \sum_{ab} \frac{\partial f^b}{\partial q^a} \frac{\partial}{\partial iQ^b} + \sum_{abc} \frac{\partial f^c}{\partial q^a} \frac{\partial}{\partial iQ^b} \frac{\partial}{\partial iQ^c} \right) \psi(F(Q))
\]  

(A2.3)

We define

\[
\omega^a(Q) \equiv -\sum_b \frac{\partial^2 Q^a}{\partial q^b} = -\sum_b \frac{\partial^2 f^a}{\partial q^b}, \quad \Omega^{ab}(Q) \equiv \sum_c \frac{\partial Q^a}{\partial q^c} \frac{\partial Q^b}{\partial q^c} = \sum_c \frac{\partial f^a}{\partial q^c} \frac{\partial f^b}{\partial q^c}.
\]  

(A2.4)

Then we obtain

\[
\hat{H} \psi = \left[ i \sum_a \omega^a(Q) P_a + \sum_{ab} \Omega^{ab}(Q) P_a P_b \right] \psi(F(Q)),
\]  

(A2.5)

where we set \( \frac{1}{i} \frac{\partial}{\partial Q^a} = P_a, \quad \tilde{V}(Q) \equiv V(F(Q)). \)

The Hamiltonian after the change of variables appears to be non-Hermitian if we take the naive Hermitian conjugate: \( P_a^\dagger = P_a, Q^a \dagger = Q^a \). This is because \( H \) is Hermitian in the original \( q \)-space. But after the change of variables, the \( Q \)-space should be defined by multiplying the wave function by the square root of the Jacobian in order to satisfy the naive Hermitian conjugation prescription:

\[
\int dq \psi^*_1(q) \psi_2(q) = \int J(Q) dQ \psi^*_1(F(Q)) \psi_2(F(Q)) = \int dQ \Psi^*_1(Q) \Psi_2(Q),
\]  

(A2.6)

where \( \Psi(Q) = J^{1/2}(Q) \psi(F(Q)) \). The Hamiltonian in \( Q \)-space is then obtained by a similarity transformation

\[
H_{\text{eff}} = J^{1/2}HJ^{-1/2},
\]  

(A2.7)

which should be Hermitian.

In practice the Jacobian is difficult to calculate while \( \omega \) and \( \Omega \) defined by (A2.4) are relatively easy to compute. So, it would be nice if \( H_{\text{eff}} \) is expressed in terms of \( \omega \) and \( \Omega \). Notice first \( J^\dagger(Q) = J(Q^\dagger) = J(Q) \), accordingly \( J^{1/2}P_a J^{-1/2} = P_a + iC_a(Q) \), where \( C_a(Q) = \frac{1}{2} \frac{\partial}{\partial Q^a} \ln J(Q) \) and \( C_a^\dagger = C_a \). We obtain

\[
H_{\text{eff}} = \frac{1}{2} \left[ i \sum_a \omega^a(Q)(P_a + iC_a) + \sum_{ab} \Omega^{ab}(P_a + iC_a)(P_b + iC_b) \right] + \tilde{V}(Q).
\]  

(A2.8)
Since $H_{\text{eff}}$ should be Hermitian, $H_{\text{eff}} - H_{\text{eff}}^\dagger = i \sum_a \{ (\omega^a + 2 \sum_b \Omega^{ab} C_b + \sum_b \Omega^{ab}) \}_a = 0$, and by taking a commutator bracket with $Q_a$ we obtain

$$\omega^a + 2 \sum_b \Omega^{ab} C_b + \sum_b \Omega^{ab} = 0. \quad (A2.9)$$

This is the equation that determines $C_a$. $H_{\text{eff}}$ is then computed as

$$H_{\text{eff}} = \frac{1}{2} \sum_{ab} [P_a \Omega^{ab} P_b + C_a \Omega^{ab} C_b + (C_a \Omega^{ab})_b] + \tilde{V} \quad (A2.10)$$

(A2.9) and (A2.10) are the main results.

**A3. Calculation of $\Omega$ and $\omega$**

In order to calculate $\Omega(x, x')$ and $\omega(x)$ for $W_\infty$ theory one needs the following equations:

$$[P(z, \bar{z}), e^{-ikM(\hat{a}, \hat{a}^\dagger)}] = -e^{-|z|^2} k \int_0^1 d\tau e^{-ik\tau M(\hat{a}, \hat{a}^\dagger)} \langle z | e^{-ik(1-\tau)M(\hat{a}, \hat{a}^\dagger)} | z \rangle,$$

$$[P(z, \bar{z}), \phi(k)] = -ke^{-|z|^2} \langle z | e^{-ikM(\hat{a}, \hat{a}^\dagger)} | z \rangle,$$

$$[P(z, \bar{z}), [P(w, \bar{w}), \phi(k)]] = k^2 e^{-|z|^2 - |w|^2} \int_0^1 d\tau \langle w | e^{-ik\tau M(\hat{a}, \hat{a}^\dagger)} | z \rangle \langle z | e^{-ik(1-\tau)M(\hat{a}, \hat{a}^\dagger)} | w \rangle. \quad (A3.1)$$

The proof is straightforward once we realize the following identity:

$$e^{-|z|^2} \sum_n \frac{1}{n!} ((\partial_z - \bar{z})^n | z \rangle) \langle ((\partial_z - z)^n \rangle | z \rangle) = 1. \quad (A3.2)$$

The (A3.2) expresses the completeness property of the generators of $W_\infty$ fundamental representation. It can be considered as a generalization of the completeness property of $SU(N)$ generators to the case of $N = \infty$. One can prove this identity by multiplying both sides by arbitrary ket vector $|z'\rangle$. Then the right hand side is $|z'\rangle$ and the left hand side is
equal to:

\[
e^{-|z|^2} \sum_n \frac{1}{n!} (\hat{a}^\dagger - \bar{z})^n |z\rangle (\partial_\bar{z} - z)^n e^{\bar{z}z'}
\]

\[
= \sum_n \frac{1}{n!} ((\hat{a}^\dagger - \bar{z})(z' - z))^n |z\rangle e^{-|z|^2 + \bar{z}z'}
\]

\[
= \epsilon^{(\hat{a}^\dagger - \bar{z})(z' - z)} |z\rangle e^{-|z|^2 + \bar{z}z'}
\]

\[
= \epsilon^{\hat{a}^\dagger z'} e^{-\hat{a}^\dagger z} |z\rangle = |z'\rangle
\]

(A3.3)

A4. Solution of $W_\infty$ matrix model

Although we can solve this model starting from (4.15a) in a straightforward fashion [15] as we shall discuss it in A5 for $w_\infty$ model, in this Appendix we solve it in a slightly different way which illuminates the dynamical group structure of the theory.

We start with the Hamiltonian (4.12a) with the constraint (4.13a):

\[
H = \int dx \left( \frac{1}{2} (\partial_x \pi(x))^2 \phi(x) + \frac{\pi^2}{6} \phi(x)^3 + \nu(x) \phi(x) \right), \quad \int dx \phi(x) = \text{tr}1 = N. \quad (A4.1)
\]

Using $y_\pm(x) = \pm \pi \phi(x) - \partial_x \pi(x)$ one [16] writes the Hamiltonian and the constraint as

\[
H = \frac{1}{2\pi} \int dx \int_{y_+(x)}^{y_-(x)} dy \left( \frac{1}{2} y^2 + \nu(x) \right), \quad \frac{1}{2\pi} \int dx \int_{y_-(x)}^{y_+(x)} dy = N, \quad (A4.2)
\]

where $y_\pm(x)$'s satisfy the commutation relation

\[
[y_\pm(x), y_\pm(x')] = \mp 2\pi i \delta'(x - x'). \quad (A4.3)
\]

We diagonalize this Hamiltonian by a canonical transformation. In the integral of (A4.2) we change variables from $y, x$ to $e, \xi$ such that $e = \frac{1}{2} y^2 + \nu(x)$. The action integral is the generator of transformation $S(x, e) = \int^x y dy = \pm \int^x \sqrt{2(e - \nu(x'))} dx'$, accordingly

\[
\xi = \frac{\partial S}{\partial e} = \pm \int^x \frac{dx'}{\sqrt{2(e - \nu(x'))}}, \quad \text{where} \pm \text{is for positive and negative} \ \xi \ \text{respectively. The boundary of the phase space is transformed from} \ y_\pm(x) \ \text{to} \ e(\xi)
\]

\[
e(\xi) = \frac{1}{2} y_\pm^2(x) + \nu(x) \quad (A4.4)
\]
and the Hamiltonian and the constraint are given by

\[
H = \frac{1}{2\pi} \oint d\xi \int_{e(\xi)}^{e(\xi)} e de, \quad \frac{1}{2\pi} \oint d\xi \int_{e(\xi)}^{e(\xi)} de = N. \tag{A4.5}
\]

We expand \(e(\xi)\) around the minimum configuration \(e_0\) of \(H\): \(e(\xi) = e_0 + \delta e(\xi)\), where \(e_0\) is given by a solution of

\[
\frac{1}{\pi} \int_{x_1}^{x_2} \sqrt{2(e_0 - v(x))} dx = N, \tag{A4.6}
\]

where \(x_1\) and \(x_2\) are the turning points.

Since we see from (A4.6) that \(e_0\) is of order \(N\), we may assume \(e_0 \gg \delta e(\xi)\) and we may approximate \(\xi\) on the boundary by \(\xi(x) = \pm \int_{x_1}^{x} \frac{dx'}{\sqrt{2(e_0 - v(x'))}}\) and the half period by

\[
T = \int_{x_1}^{x_2} \frac{dx'}{\sqrt{2(e_0 - v(x'))}} = \int_{\tilde{x}_1}^{\tilde{x}_2} \frac{dx'}{\sqrt{2(\tilde{e}_0 - u(x'))}}, \tag{A4.7}
\]

where \(\tilde{e}_0 = N^{-1}e_0\), \(\tilde{x}_1 = N^{-\frac{1}{2}}x_1\) and \(u(x) = N^{-1}v(N^{-\frac{1}{2}}x)\) (see (4.16a)).

Since the commutation rules of \(e(\xi)\)'s are given by

\[
[e(\xi), e(\xi')] = \left[\frac{1}{2} y_\pm^2(x) + v(x), \frac{1}{2} y_\pm^2(x') + v(x')\right] = \mp 2\pi i y_\pm(x) y_\pm(x') \delta'(x - x')
\]

\[
= -2\pi i \delta'(\xi - \xi'),
\]

if we define \(\theta\) and \(r(\theta)\) by \(\theta = \omega \xi\), \(r(\theta) = \omega^{-1} \delta e(\xi)\), \(\omega = \frac{\pi}{T}\), we obtain

\[
[r(\theta), r(\theta')] = -2\pi i \delta'(\theta - \theta') \tag{A4.8}
\]

The normal mode expansion of \(r(\theta)\) is given by \(r(\theta) = \sqrt{\frac{2}{\omega}} \sum_{n>0} (\sin(n\theta) p_n + n \omega \cos(n\theta) q_n)\), and leads to the following Hamiltonian:

\[
H = E_0 + H_{\text{coll}},
\]

\[
H_{\text{coll}} = \oint \frac{d\xi}{2\pi} \int_0^{\delta e(\xi)} e de = \omega \oint \frac{d\theta}{2\pi} \frac{1}{2} (r(\theta))^2 = \frac{1}{2} \sum_n (p_n^2 + \omega_n^2 q_n^2), \tag{A4.9}
\]

\[
\omega_n = n \omega = n \frac{\pi}{T}.
\]
From (A4.6) and (A4.7) it is obvious that $T^{-1}$ is finite in the large $N$ limit provided $u(x)$ is finite. In the double scaling limit one of the turning points goes to infinity so that $T \to \infty$ and we obtain the continuous spectrum (chiral Boson).

The reason why we could solve the $W_\infty$ model by a canonical transformation is that there exists a dynamical $w_\infty$ algebra in the physical Hilbert space and the Hamiltonian is a generator of the algebra. We simply quote a result of [2], namely $\rho[\xi]$’s defined by

$$\rho[\xi] = \oint \frac{d\theta}{2\pi} \int_{r(\theta)}^r dr \xi(r, \theta),$$

(A4.10)

satisfy the $w_\infty$ commutation relation (2.6).

A5. $w_\infty$ Matrix Model

We first obtain the field configuration at which the potential energy of (4.15b) is minimum. The equations are

$$-\frac{1}{4} \frac{\partial}{\partial \phi} \left( \frac{\partial \phi}{\partial \phi} \right) - \frac{1}{8} \left( \frac{\partial \phi}{\partial \phi} \right)^2 + \tilde{u}(x) = e$$

and

$$\int dx \phi(x) = 1.$$  

For the variable $\varphi(x) = \sqrt{\phi(x)}$ the first equation is the Schrödinger equation

$$\left( -\frac{1}{2} \frac{\partial^2}{\partial \varphi^2} + \tilde{u}(x) \right) \varphi(x) = e \varphi(x)$$

and the second equation is the normalization of the wave function. Therefore we consider an orthonormal set of eigenfunction $\chi_n$ with eigenvalue $E_n$. We set $\phi(x) = \chi_0^2(x) + \frac{1}{\sqrt{N}} \eta(x), \quad \pi(x) = \sqrt{N} \zeta(x)$ and expand the Hamiltonian. We obtain

$$H = \frac{1}{2} \int dx \left[ \chi_0^2(x) \left( \varphi'(x) \right)^2 + \frac{1}{4} \left( \frac{(\partial \eta(x))^2}{\chi_0^2(x)} + (2 \varphi^2 \ln \chi_0) \frac{\eta^2(x)}{\chi_0^2(x)} \right) \right]$$

(A5.1)

$\eta(x)$ and $\zeta(x)$ satisfy the canonical commutation relation:

$$[\varphi'(x), \eta(x')] = -i \delta'(x - x')$$

(A5.2)

and the constraint $\int dx \eta(x') = 0$. The normal mode expansion is now straightforward:

$$\eta(x) = \chi_0(x) \sum_{n=1}^\infty \sqrt{2 \omega_n} \chi_n(x) q_n, \quad \zeta(x) = \chi_0^{-1}(x) \sum_{n=1}^\infty \frac{1}{\sqrt{2 \omega_n}} \chi_n(x) p_n,$$

(A5.3)

and

$$H = \frac{1}{2} \sum_{n=0}^\infty \left( p_n^2 + \omega_n^2 q_n^2 \right), \quad [q_n, p_m] = i \delta_{n,m}, \quad \omega_n = E_n - E_0.$$  

(A5.4)
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