CONSISTENT DETECTION AND OPTIMAL LOCALIZATION
OF ALL DETECTABLE CHANGE-POINTS IN PIECEWISE
STATIONARY ARBITRARILY SPARSE NETWORK-SEQUENCES

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Abstract We consider the offline change point detection and localiza-
tion problem in the context of piecewise stationary networks, where the observable is a finite sequence of networks. We develop algorithms involving some suitably modified CUSUM statistics based
on adaptively trimmed adjacency matrices of the observed networks
for both detection and localization of single or multiple change points
present in the input data. We provide rigorous theoretical analysis
and finite sample estimates evaluating the performance of the pro-
posed methods when the input (finite sequence of networks) is gener-
ated from an inhomogeneous random graph model, where the change
points are characterized by the change in the mean adjacency matrix.
We show that the proposed algorithms can detect (resp. localize) all
change points, where the change in the expected adjacency matrix is
above the minimax detectability (resp. localizability) threshold, con-
sistently without any a priori assumption about (a) a lower bound
for the sparsity of the underlying networks, (b) an upper bound for
the number of change points, and (c) a lower bound for the separa-
tion between successive change points, provided either the minimum
separation between successive pairs of change points or the average
degree of the underlying networks goes to infinity arbitrarily slowly.
We also prove that the above condition is necessary to have consist-
tency. Finally, we evaluate the performance and complexity of our
methods empirically using simulated data sets, and demonstrate the
superiority of our algorithms over relevant existing approaches.

1. Introduction. As network data sets have grown in complexity in
the recent decades, so has the prevalence of temporal or time-varying or

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time-series of networks. Time series of networks have emerged in several different fields of study. Examples of temporal or time series of networks in different fields of study include time-series of social networks [30, 52, 42], epidemiological networks [45, 43], animal networks [11, 22], mobile and online communication networks [21, 9, 19], economic networks [37, 56], brain networks [31, 49], genetic networks [41] and ecological networks [4], to name a few. Analysis of temporal networks in terms of modeling, statistical behavior, dynamics, community detection and change point detection has been investigated in several recent works (see [18, 17, 34, 48, 35] for some review of recent works). In this paper, we shall concentrate on the problem of change point detection for time-series of networks.

Change point detection is a classical problem in statistics going all the way back to the early days of statistical quality control [28, 29, 13]. However, the problem of change point detection has gained significant importance and applicability in many fields such as medical diagnostics [54, 51, 5], gene expression [36, 16], online activity [24], speech and image analysis [15, 38], climate science [40], finance [2, 23] and many more. The problem of change point detection started with detection of change in the mean of normal model [28] but since has also been generalized to many different data types such as time-series data [1] and multivariate data [6] as well for detecting change in different parameters of the data distribution such as variance, correlation, density and so on.

The change point detection problem can be broadly classified into two types.

1. **Offline change point detection:** In this type of problem, the whole data sequence is available and the change points are detected within the data sequence. This problem was studied in the beginning by Page (1954) [28] and Girshick and Rubin (1952) [13].

2. **Online change point detection:** In this type of problem, the data is available sequentially and the change points are detected based on the available data. The online version was initially studied by Kolmogorov (1950), Shiyarev (1963) [46], Lorden (1971) [25] and others.

There is a huge literature on the univariate change point problem and possible solutions. An excellent treatment can be found in the book [6]. The multivariate versions of the problem are significantly more complex. Some notable works are [55, 47, 50, 20] in the parametric setting, and [14, 26, 8] in the non-parametric setting.

In this article, we tackle the problem of change point detection in temporal network data, that is one observes a series of networks indexed by time
and wishes the check if there is a time-point (so-called change point) when there is a significant change in the structure of these networks. Potential applications are in, for instance, brain imaging, where one has brain scans of individuals collected over time and is looking for abnormalities, ecological networks observed over time, where one wonders if there is a structural change, and so on. The classical CUSUM statistic [29] for univariate change point problems can be used in the network problem as well, and provides a unified way of constructing estimates of change points. It is also amenable to theoretical analysis because of the averaging structure present. In this paper, we will investigate its theoretical properties in a quite general setup. We stress here that we observe the whole time series ahead of our analysis, this is thus an offline or a posteriori change point problem. We will not discuss the online version of the problem here, which is also quite interesting.

There has been some recent works on the problem of network change points. For example, [33] postulate a hierarchical random graph model and use a Bayesian procedure to detect change points. [32] use local graph statistics for change point and anomaly detection in dynamic networks. For a survey of techniques used in the related problem of anomaly detection in graphs, see [39]. Most two sample graph tests can be used for the change point problem viewed as a multiple testing problem. For example, an eigenvalue based test for the ER vs SBM problem is worked out in [7]. Minimax lower bounds for two sample tests for inhomogeneous Erdős-Rényi graphs have been derived in [12]. As we will see later, such lower bounds are closely related to the lower bounds we derive for the change-point problem we consider. Although much empirical work has been done, not much theory can be found, and most theoretical results focus on particular structures or specialized models. An exception is [44], where the authors [44] model networks as a Markov random field and estimate the change point using a penalized pseudo-likelihood and prove its consistency at a near classical (i.e. fixed dimensional) rate under a restricted strong convexity type assumption on the log-pseudo-likelihood. Although their results are in a high-dimensional setting, and allow more complicated node interaction than random graphs with independent edges, the role of network sparsity in their setup is not clear.

Some recent works [53, 3, 57] propose methods for change point detection in networks generated from block models and graphon models with some theoretical results on the consistency of the detection methods. Graphon estimation based methods [57] only work for dense graphs. On the other hand, [3] only consider block-models. The work most closely related to the current paper is [53]. Although they propose an algorithm which is almost minimax optimal, their algorithm requires two independent copies of the
network time-series, which severely limits its usefulness. In contrast, we provide two efficient algorithms that are minimax optimal and do not require such restrictive assumptions. Our minimax lower bounds are also much more precise than theirs, including provisions for perturbations of specific ranks.

1.1. Problem Description. In this section, we formally represent the inference problem that we consider in this paper. Let us consider that the number of layers in the network sequence is given by $T$ ($T \in \mathbb{N}$), the number of nodes in each network layer is represented by $n$ ($n \in \mathbb{N}$), and number of change points in the network sequence is given by $K$ ($0 \leq K < T$). Let us denote, $[n] := \{1, 2, \ldots, n\}$, and $S_n$ be the set of all $n \times n$ symmetric matrices having entries in $[0, 1]$ and zeros on the diagonal. The set of possible change point locations is given by $U(K, T) := \{\tau = (\tau_0, \tau_1, \ldots, \tau_{K+1}) : 0 = \tau_0 < \tau_1 < \cdots < \tau_K < \tau_{K+1} = T\}$, the set of possible expected adjacency matrices is given by $S_T^n := \{(Q^{(1)}, \ldots, Q^{(T)}) : Q^{(t)} \in S_n, \forall t \in [T]\}$. For $Q \in S_T^n$, let $K(Q) := |\{t \in [T-1] : Q^{(t)} \neq Q^{(t+1)}\}|$ be the number of change points in $Q$, and $T(Q) \in U(K(Q), T)$ be the sequence of all the change points of $Q$. Let the no change point situation is represented by $S_{T,n,0}^T := \{Q \in S_T^n : K(Q) = 0\}$, and at least one change point is presented by $S_{T,n,+}^T := S_T^n \setminus S_{T,n,0}^T$.

Now, we describe the statistical hypothesis testing problem in the context of detecting of change points in a finite sequence of networks. The observable is a sequence $(G^{(1)}_n, \ldots, G^{(T)}_n)$, which is represented by the corresponding sequence of adjacency matrices $\mathcal{A} : (A^{(1)}, \ldots, A^{(T)}) \in S_T^n$, consisting of $T$ simple undirected networks on the same node set $\{v_1, \ldots, v_n\}$ having $n$ nodes. $A^{(t)}_{ij}$ equals 1 (resp. 0) if nodes $v_i$ and $v_j$ are (resp. not) linked in the $t$-th snapshot $G^{(t)}_n$. Let $\mathcal{P} := (P^{(1)}, \ldots, P^{(T)})$, where $P^{(t)} := E A^{(t)}$. So, $\mathcal{P} \in S_T^n$. We define the hypothesis test in terms of the following pairs of competing hypotheses For $X \subset S_{T,n,0}^T$ and $Y \subset S_{T,n,+}^T$, consider the

$$H_0 : \mathcal{P} \in X; H_1 : \mathcal{P} \in Y.$$

In plain English, we want to test whether the sequence of the mean adjacency matrices belongs to a set $X$ having no change point, or to a set $Y$ having at least one change point.

In other words, the question is when can one decide reliably whether there is a sequence of change points in $\mathcal{P}$ or not. The answer depends on the criterion used for judging the performances of the decision rules. There are mainly two paradigms in statistical decision theory, namely the Bayesian
and the minimax approach. We will consider the second approach in this paper. Recall that a nonrandomized test \( \Omega_{n,T} \) is a measurable function of the observable \( \mathcal{A} \) taking values in \( \{0, 1\} \). The minimax risk of such a test \( \Omega_{n,T} \) for the hypotheses \( H_0^X \) and \( H_1^Y \) is

\[
\Pi(\Omega_{n,T}; X, Y) := \sup_{\mathcal{A} \in X} \mathbb{P}_{\mathcal{A}}(\Omega_{n,T} = 1) + \sup_{\mathcal{A} \in Y} \mathbb{P}_{\mathcal{A}}(\Omega_{n,T} = 0).
\]

(1.1)

Here and later \( \mathbb{P}_{\mathcal{A}} \) represents the distribution of \( \mathcal{A} \) when \( E \mathcal{A} = \mathcal{A} \). Recall the following definitions for the asymptotic properties of hypothesis test.

**Definition 1.1.** A test \( \Omega_{n,T} \) is called asymptotically powerful for the hypotheses \( H_0 : \mathcal{P} \in X \) and \( H_1 : \mathcal{P} \in Y \), if \( \limsup_{nT \to \infty} \Pi(\Omega_{n,T}; X, Y) \) equals 0.

A test \( \Omega_{n,T} \) is called asymptotically powerless for the hypotheses \( H_0 : \mathcal{P} \in X \) and \( H_1 : \mathcal{P} \in Y \), if \( \liminf_{nT \to \infty} \Pi(\Omega_{n,T}; X, Y) \) is at least 1.

**Definition 1.2 (Detectability).** For any \( X \in S_{n,0} \) and \( Y \in S_{n,+}^T \), the hypotheses \( H_0 : \mathcal{P} \in X \) and \( H_1 : \mathcal{P} \in Y \) are called

- consistently distinguishable if there is an asymptotically powerful test for them.
- consistently indistinguishable if there is no asymptotically powerful test for them.
- asymptotically indistinguishable if all tests for them are asymptotically powerless.

Our first result (see §3.2.1) in this paper is to obtain the maximum detectable set.

Having considered the problem of detectability, next we consider the problem of localizability. Consider the collection \( \bigcup_{K \in [T-1]} U(K, T) \) of sequences of probability matrices having at least one change point.

**Definition 1.3.** For \( \Lambda \leq T \), a set \( Y \subset \{ \mathcal{A} \in S_n^T : \mathcal{K}(\mathcal{A}) \geq 1 \} \) is called \( \Lambda \)-localizable, if there is an estimator \( \hat{\tau} \) of change points and a constant \( c > 0 \) such that

\[
\liminf_{nT \to \infty} \inf_{\mathcal{A} \in Y} \mathbb{P}_{\mathcal{A}}(\mathcal{K}(\hat{\tau}) = \mathcal{K}(\mathcal{A}) \text{ and } \mathcal{T}_k(\mathcal{A}) \in \hat{\tau}_k \pm cA \text{ for all } k \in [K(\tau)]) = 1.
\]

In our second result (see §3.2.1 and §3.2.2), we have determined which subsets of \( S_n^T \) are \( \Lambda \)-localizable.
1.2. Our results. To determine detectability of a given change point detection problem, one needs to measure the amount of change among the expected observables. In the current case, there are three main quantitative variables that determine the complexity of a change point detection problem, namely (a) the gaps between successive change points, (b) the differences of the probability matrices at the change points, and (c) the level of sparsity of the probability matrices. Researchers have used one of the standard matrix norms for quantifying (b). In this paper, we will use the spectral norm for (b) (denoted by $|| \cdot ||$). For $\mathcal{Q} \in \mathcal{S}_n^T$, define the three quantities as:

(a) Cushion: $g(\mathcal{Q}) := \min_{k \in [K(\mathcal{Q})+1]} (T_k(\mathcal{Q}) - T_{k-1}(\mathcal{Q}))$,

(b) Signal: $S(\mathcal{Q}) := \begin{cases} \min_{k \in [K(\mathcal{Q})]} \|Q^{(\tau_k+1)} - Q^{(\tau_k)}\| & \text{if } K(\mathcal{Q}) \geq 1 \\ 0 & \text{if } K(\mathcal{Q}) = 0 \end{cases}$,

(c) Sparsity: $D(\mathcal{Q}) := \max_{i,j \in [n], t \in [T]} Q_{ij}^{(t)}$.

Also let $\mathcal{S}_n^T$ be the maximum subset of $\mathcal{S}_n^T$ such that

$$\inf_{\mathcal{Q} \in \mathcal{S}_n^T} g(\mathcal{Q})D(\mathcal{Q}) \in \omega(1)$$ as $n$ or $T$ or both goes to $\infty$.

- We have shown (see Algorithm 1 for an asymptotically powerful test procedure and Theorem 3.4 for the precise statement) that if

$$\mathcal{Y} = \left\{ \mathcal{Q} \in \mathcal{S}_n^T : S(\mathcal{Q}) \geq \frac{D(\mathcal{Q})}{g(\mathcal{Q})}, \mathcal{Q} \in \mathcal{S}_n^T \right\},$$

$$\mathcal{V} = \left\{ \mathcal{Q} \in \mathcal{S}_n^T : S(\mathcal{Q}) \leq \frac{D(\mathcal{Q})}{g(\mathcal{Q})} \text{ or } \mathcal{Q} \not\in \mathcal{S}_n^T \right\},$$

$\tilde{\mathcal{V}}$ is the analogue of $\mathcal{V}$ with “$\leq$” replaced by “$\ll$”, and $\mathcal{X} = \mathcal{S}_n^{T,0}$, then $H_0 : \mathcal{P} \in \mathcal{X}$ and $H_1 : \mathcal{P} \in \mathcal{Y}$ are consistently distinguishable,

$H_0 : \mathcal{P} \in \mathcal{X}$ and $H_1 : \mathcal{P} \in \mathcal{V}$ are consistently indistinguishable,

$H_0 : \mathcal{P} \in \mathcal{X}$ and $H_1 : \mathcal{P} \in \tilde{\mathcal{V}}$ are asymptotically indistinguishable

- We have also shown that for any $\Lambda \leq T$,

$$\left\{ \mathcal{Q} \in \mathcal{S}_n^T : S(\mathcal{Q}) \geq \frac{D(\mathcal{Q})}{\Lambda \wedge g(\mathcal{Q})}, \mathcal{Q} \in \mathcal{S}_n^T \right\}$$

is $\Lambda'$-localizable for all $\Lambda' \geq \Lambda$. Thus, if we restrict our consideration to $\{\mathcal{Q} \in \mathcal{S}_n^T : D(\mathcal{Q}) = d_0, g(\mathcal{Q}) = \kappa_0\}$, then the detectability threshold is $\sqrt{d_0/\kappa_0}$ if $d_0\kappa_0 \in \omega(1)$. See Theorem 3.6 for the precise statement.
1.3. Outline of the paper. The remainder of the paper is organized as follows. In §2, we describe the change point detection algorithms proposed in the paper. In §3, we state the theoretical results regarding the performance of the proposed change point detection algorithms and the detectability threshold of change points for a large class of probability distributions.

2. Change Point Detection Methods.

2.1. Notations. Let \([n] := \{1, 2, \ldots, n\}\) for \(n \in \mathbb{N}\), \(\mathcal{M}_{m,n}\) be the set of all \(m \times n\) matrices which have exactly one 1 and \(n - 1\) 0’s in each row. \(\mathbb{R}^{m \times n}\) denotes the set of all \(m \times n\) real matrices. \(\| \cdot \|_2\) is used to denote Euclidean \(\ell_2\)-norm for vectors in \(\mathbb{R}^{m \times 1}\). \(\| \cdot \|_F\) is the Frobenius norm on \(\mathbb{R}^{m \times n}\), namely \(\| M \|_F := \sqrt{\text{trace}(M^TM)}\). \(\mathbf{1}_m \in \mathbb{R}^{m \times 1}\) consists of all 1’s, \(\mathbf{1}_A\) denotes the indicator function of the event \(A\). If \(A \in \mathbb{R}^{m \times n}\), \(I \subseteq [m]\) and \(j \in [n]\), then \(A_{I,j}\) (resp. \(A_{I,*}\)) denotes the submatrix of \(A\) corresponding to row index set \(I\) and column index \(j\) (resp. index set \([n]\)). \(\lambda_i(W)\), \(i \in [n]\), will denote the \(i\)-th largest eigenvalue of \(W \in \mathbb{R}^{n \times n}\). \([X] := X - \mathbb{E}(X)\) for any random variable or random matrix \(X\).

2.2. Network Data and Model. We consider the setup where one observes a sequence of \(T\) networks, \((G_n^{(t)})_{t=1}^T\), with adjacency matrices \((A^{(1)}, A^{(2)}, \ldots, A^{(T)})\) on the same set of nodes \(\{v_1, v_2, \ldots, v_n\}\). For each \(t \in \{1, \ldots, T\}\), the \(t\)-th network \(G_n^{(t)}\) is represented by the corresponding adjacency matrix \(A_n^{(t)}\) whose elements are \(A_n^{(t)} = 1\) if node \(v_i\) is linked to node \(v_j\) at time \(t\), and \(A_n^{(t)} = 0\) otherwise. Thus, the numerical data for the community detection problem consists of \(T \geq 1\) adjacency matrices \((A_{n \times n}^{(1)}, \ldots, A_{n \times n}^{(T)})\). We shall only consider undirected and unweighted graphs in this paper. However, the conclusions of the paper can be extended to positively weighted graphs with non-random weights in a quite straight-forward way by considering weighted adjacency matrices. The theoretical analysis in this paper can be easily extended to positively weighted adjacency matrices.

We consider that the set of adjacency matrices are generated independently from an inhomogeneous random graph model.

**DEFINITION 2.1 (Multilayer Inhomogeneous Random Graph Model (MIR-GraM)).** A sequence of \(T\) (\(T \in \mathbb{N}\)) symmetric adjacency matrices \((A^{(1)}, A^{(2)}, \ldots, A^{(T)})\) follows Multilayer Inhomogeneous Random Graph Model (MIRGraM) with parameters \(\mathcal{P} = (P^{(1)}, \ldots, P^{(T)})\), each of size \(n \times n\),
\( A_{ij}^{(t)} \) iid \( \sim Ber(P_{ij}^{(t)}) \).

for \( i > j \) with \( i, j = 1, \ldots, n \) and \( t = 1, \ldots, T \). Lastly, \( A^{(t)} = (A^{(t)})^T \) and \( \text{diag}(A^{(t)}) = 0 \) for each \( t = 1, \ldots, T \).

We consider the setup where, \( \mathcal{P} \) changes at certain instances, like \( (\tau_1, \ldots, \tau_K) \), where \( 1 = \tau_0 < \tau_1 < \cdots < \tau_K < \tau_{K+1} = T \). Our goal becomes estimating the instances of change-point \( (\tau_1, \ldots, \tau_K) \). We consider the problem of multiple change-point detection and localization.

In this problem we consider that \( K \geq 1 \), so we have to estimate multiple change-points \( (\tau_1, \ldots, \tau_K) \) both in form of a point estimator (\textit{detection}) and interval estimator (\textit{localization}). We propose two different algorithms to address multiple change-point detection problem.

(i) We develop a window-based algorithm (Algorithm 1) to detect and localize the multiple change-points. The formal setup and algorithm is given in §2.3.

(ii) We develop a wild binary segmentation algorithm (Algorithm 2) motivated by [10]. The formal setup and algorithm is given in §2.3.2.

We will now go into the details of algorithms and the setup.

2.3. Multiple change-point detection and localization. The problem of multiple change-point detection also starts with a sequence of networks represented by the symmetric adjacency matrices \( (A^{(1)}_{n \times n}, \ldots, A^{(T)}_{n \times n}) \). The model under a multiple change-point with parameters \( \mathcal{P} = (P^{(1)}, \ldots, P^{(T)}) \) can be framed as -

(i) if \( P^{(t)} = P \), for all \( t \in [T] \), there exists no change-point;

(ii) if there exists a set of change-points \( (\tau_1, \tau_2, \ldots, \tau_K) \) with \( \tau_0 := 0 \leq \tau_1 \leq \cdots \leq \tau_K \leq T =: \tau_{K+1} \) such that

\[ P^{(t)} = Q_k \text{ for } \tau_{k-1} \leq t \leq \tau_k, \ k \in [K] \]

where, \( Q_k \in S_n \) for \( k \in [K] \), then there exists a multiple change-points \( (\tau_1, \tau_2, \ldots, \tau_K) \).

(iii) We additionally assume \( g(\mathcal{P}) \) represents the \textit{minimum cushion} at the boundaries of the network sequence as well as the \textit{minimum cushion} between successive change points, thus satisfying the relationship

\( g(\mathcal{P}) := \min_{0 \leq k \leq K} (\tau_{k+1} - \tau_k) \).
The goal then becomes declaring absence of change-point when case (i) is true and estimating \((\tau_1, \tau_2, \ldots, \tau_K)\) when the case (ii) is true.

Let \(\bar{D} = \frac{1}{nT} \sum_{i,j \in [n],s \in [T]} A_{i,j}^{(s)}\) be the sample average degree of a node over all layers and for some constant \(\mu > 0\), define \(\bar{D}_\mu := \frac{\bar{D}}{1+\sqrt{4\mu}}\) as a normalized version of the average degree, \(\bar{D}\). The degree of each of the vertices average over all layers is given by

\[
D_i = \frac{1}{T} \sum_{j \in [n],s \in [T]} A_{i,j}^{(s)}, \quad \text{for } i \in [n].
\]

Define the function \(\varphi(z) := z \log(z) - z + 1\) \((\varphi : [1, \infty) \mapsto [0, \infty))\). We need to define the following population quantities based on the parameters of MIRGraM, which are given in Definition 2.1,

\[
d := n \left( \max_{i,j \in [n],t \in [T]} \mathbb{E} A_{ij}^{(t)} \right), \quad \varrho := \max_{i \in [n],t \in [T]} \sum_{j \in [n]} \mathbb{E} A_{ij}^{(t)}.
\]

Let \(\mathcal{L}\) be a collection of intervals, each of length \(\Lambda\). Let \(\kappa\) be the cushion at the boundary of each interval in \(\mathcal{L}\). For any \(\ell \in [|\mathcal{L}|]\), let \(T_\ell\) be the set of left end-points of intervals in \(\mathcal{L}\). Then, let us define,

\[
\begin{bmatrix}
\epsilon_\mu \\
\Psi_\mu \\
\eta
\end{bmatrix} := \begin{cases}
\frac{1}{6}, \varphi^{-1} \left( \frac{(2 \log |\mathcal{L}|)^{3/2}}{\frac{1}{4} - \epsilon_\mu (\Lambda \wedge n)^{3/2}} \right), & \text{if } |\mathcal{L}| \not\in (e, e^n), \\
\frac{2n-2}{6n-3}, \varphi^{-1} \left( \frac{(\log |\mathcal{L}|)^{4/3}}{\frac{3}{4} - 2\epsilon_\mu} \right), & \text{if } |\mathcal{L}| \in (e, e^n),
\end{cases}
\]

\[
\Gamma := \begin{cases}
\frac{25n}{4} \exp \left(-\frac{3\epsilon_\mu}{2-6\epsilon_\mu} \left( \log |\mathcal{L}| \vee \frac{3(\Lambda \wedge n)\bar{D}}{1+\sqrt{4\mu}} \right) \right), & \text{if } |\mathcal{L}| \not\in (e, e^n), \\
\frac{25n}{4} \exp \left(-\frac{3\epsilon_\mu}{1-6\epsilon_\mu} \left( 2 \log |\mathcal{L}| \vee \frac{3(\Lambda \wedge n)\bar{D}}{1+\sqrt{4\mu}} \right) \right), & \text{otherwise}.
\end{cases}
\]

A natural statistic is based on the cumulative averages (cusum) of estimates of the \(\mathbf{P}^{(t)}\)'s. Such cusum statistics are widely used in change-point detection problems [6]. A natural estimate of \(\mathbf{P}^{(t)}\) is \(\mathbf{A}^{(t)}\) for any \(t \in [T]\). However for sparse networks, \(\mathbf{A}^{(t)}\) is not a good enough estimate of \(\mathbf{P}^{(t)}\) under operator norm. Hence given \(\Lambda\) and \(\kappa\), for each interval \(I \in \mathcal{L}\), we obtain submatrices \(\tilde{\mathbf{A}}^{(t)}\) of \(\mathbf{A}^{(t)}\) for each \(t \in [T]\) by removing some high degree vertices using the threshold \(\Gamma\), where \(\Gamma\) is defined in (2.5). The thresholding uses the ordered vertices in terms of degree of each vertex, \(D_i\), such that \(D_{(1)} \leq D_{(2)} \leq \cdots \leq D_{(n)}\), and vertices corresponding to \(D_{(j)}\), such that \(j > (n + 1 - \Gamma)\), are pruned. The thresholded matrices \((\tilde{\mathbf{A}}^{(t)})_I^{T-1}\) are better
approximations of \((P(t))_{t=1}^{T}\) in terms of operator norm. We use \((\tilde{A}(t))_{t=1}^{T}\) to construct the cusum statistics, \(G(t)\), for interval \(I_\ell \in \mathcal{L}\),

\[
G(t) := \sqrt{\frac{t}{\Lambda}} \left(1 - \frac{t}{\Lambda}\right) \left(\frac{1}{t} \sum_{i=T_\ell+1}^{t} \tilde{A}^{(i)} - \frac{1}{\Lambda-t} \sum_{i=T_\ell+t+1}^{T_\ell+\Lambda} \tilde{A}^{(i)}\right)
\]

for \(T_\ell + \lfloor \kappa/3 + 1 \rfloor \leq t \leq T_\ell + \Lambda - \lfloor \kappa/3 \rfloor\).

2.3.1. Window-based Algorithm. In order to build the window-based algorithm, Algorithm 1, for multiple change-point detection, we consider the set of intervals -

\[
\mathcal{L} := \{I_\ell\} \text{ where, } I_\ell := (T_\ell, (T_\ell + \Lambda) \wedge T], \text{ and } T_\ell = (\ell - 1)[\Lambda/3] \text{ for } \ell \in \{1, 2, \ldots, [3T/\Lambda] - 2\}.
\]

Along with the set of intervals \(\mathcal{L}\), there are other components of Algorithm 1.

1. Cushion \(\kappa\) and interval length \(\Lambda\) are two tuning parameters of Algorithm 1. For fixed \(\kappa\) and \(\Lambda\), we consider the set of intervals \(\mathcal{L}\) as defined in (2.7). In Algorithm 1, we consider that \(\Lambda \leq \kappa\), so, one can vary \(\Lambda\) from 3 to \(\kappa\) for a given \(\kappa\) to find the intervals of shortest length with a change point.

2. For each \(I_\ell \in \mathcal{L}\), \(\|G(t)\|\) is minimized to get a candidate of change-point estimate, \(\tilde{\tau}\).

3. If \(\|G(t)\|\) is greater than a data-dependent threshold as defined in Algorithm 1, then \(\tilde{\tau}\) is change-point within localized interval \(I_\ell\), otherwise not.
Algorithm 1: Window-based Change Point Detection

Input: Adjacency matrices $A^{(1)}, A^{(2)}, \ldots, A^{(T)}$; cushion $\kappa$.

Output: Change point estimates with localized interval lengths and intervals $(\hat{\tau}, \Lambda, I)$.

For $\Lambda = \kappa, \kappa - 1, \ldots, 3$ do

Obtain $\bar{D} = \frac{1}{nT} \sum_{i,j \in [n], s \in [T]} A^{(s)}_{ij}$.

Obtain $L$ as defined in equation (2.7).

Obtain $\Gamma$ as defined in equation (2.5).

For $\ell = 1, 2, \ldots, \left\lfloor \frac{3T}{\Lambda} \right\rfloor - 2$ do

Define $T_\ell$ and $I_\ell$ as in (2.7).

For $i = 1, 2, \ldots, n$ do

Obtain $D_i$.

Order the values $D_1, \ldots, D_n$ to get $D_1 \leq \cdots \leq D_n$.

Obtain row indices $i_1, \ldots, i_\Gamma$ such that $D_{i_\Gamma} \geq D_{n+1-\Gamma}$.

Obtain $\tilde{A}^{(s)}$ from $A^{(s)}$ for each $s \in I_\ell$ by removing rows and columns with indices $i_1, \ldots, i_\Gamma$.

For $t = T_\ell + \left\lfloor \frac{\Lambda}{3} \right\rfloor + 1, \ldots, T_\ell + \Lambda - \left\lfloor \frac{\Lambda}{3} \right\rfloor$ do

Obtain $G^{(t)}$ as in (2.6).

Obtain $u = \arg \max_{t \in (T_\ell + \left\lfloor \frac{\Lambda}{3} \right\rfloor, T_\ell + \Lambda - \left\lfloor \frac{\Lambda}{3} \right\rfloor)} \|G^{(t)}\|$.

If $\|G^{(u)}\| > \Theta \mu \left( \frac{\bar{D}}{\Lambda} \left( \zeta + 6 + \frac{\log(|L|)}{\log(n)} \right) \right)^{1/2}$, declare $u$ as a change point in interval $I_\ell$ of length $\Lambda$.

Return the detected change-points corresponding to all interval lengths $\Lambda$ and interval $I_\ell$ as $(u, \Lambda, I_\ell)$.

2.3.2. Wild Binary Segmentation. Wild binary segmentation is a randomized algorithm for multiple change-point detection [10]. Let us denote $\mathcal{F}_T^M$ as a set of $M$ random intervals $[s_m, e_m], m = 1, \ldots, M$, whose start and end points have been drawn (independently with replacement) uniformly from the set $\{1, \ldots, T\}$ with the property $e_m > s_m$. Note that, wild binary segmentation algorithm [10] is a recursive one and consequently Algorithm 2 has also been written in a recursive format. The main components of Algorithm 3 are given below.

1. In a specific iteration, Algorithm 2 operates on an interval $(s, e)$ (where, $(s, e) \subseteq [1, T]$). Let $\mathcal{M}_{s,e}$ be the set of indices, $m \in [M]$, of intervals in $\mathcal{F}_T^M$, such that, $(s_m, e_m) \subseteq (s, e)$.

2. Cushion $\kappa$ is the tuning parameter of the algorithm. We consider $\kappa$ is given to the algorithm at any specific iteration.

3. For each $I = (s_m, e_m)$ such that $m \in \mathcal{M}_{s,e}$, say $\|G^{(t)}\|$ is minimized at
$t = u_m$ and if $\|G^{(u_m)}\|$ is greater than a data-dependent threshold as defined in Algorithm 2, then $u_m$ becomes a candidate of change-point estimate.

4. The $\|G^{(u_m)}\|$ is minimized over all $m \in \mathcal{M}_{s,e}$ and say the minimization occurs at $m^*$. Then $u_0 := u_{m^*}$ is declared as a change-point.

5. Algorithm again recursively starts for the two intervals $(s, u_0)$ and $(u_0 + 1, e)$. 
Algorithm 2: Wild Binary Segmentation \((s, e, \kappa)\)

**Input:** Adjacency matrices \(A^{(s)}, A^{(s+1)}, \ldots, A^{(e)}\); cushion \(\kappa\).

**Output:** Change point estimate \(\hat{\tau}\).

1. If \(e - s < \kappa\) then
2. STOP
3. else
4. \(M_{s,e} :=\) set of those indices \(m\) for which \([s_m, e_m] \in \mathcal{F}_T^M\) is such that \([s_m, e_m] \subseteq [s, e]\).
5. Define \(\Lambda = e_m - s_m\).
6. If \(e_m - s_m < \kappa\), then,
7. CONTINUE
8. else
9. For \(i = 1, 2, \ldots, n\) do
10. Obtain degree of each vertex, \(\{D_{i,m}\}_{i=1}^n\).
11. Order the values \(D_{1,m}, \ldots, D_{n,m}\) to get \(D_{(1)}^{(s)} \leq \cdots \leq D_{(n)}^{(s)}\).
12. Obtain row indices \(i_1, \ldots, i_\Gamma\) such that \(D_{i_k,m} \geq D_{(n+1-\Gamma)}^{(s)}\).
13. Obtain \(\tilde{A}^{(s)}\) from \(A^{(s)}\) for each \(s \in (s_m, e_m)\) by removing rows and columns with indices \(i_1, \ldots, i_\Gamma\).
14. For \(t = s_m + \frac{1}{3}\kappa, s_m + \frac{1}{3}\kappa + 1, \ldots, e_m - \frac{1}{3}\kappa\) do
15. Obtain \(c_m^{(t)} := \sqrt{\frac{t}{e_m - s_m}} \left(1 - \frac{t}{e_m - s_m}\right)\).
16. Obtain \(G_m^{(t)} := c_m^{(t)} \times \left[\sum_{s \in (s,t]} \tilde{A}^{(s)} - \sum_{s \in (t,e_m]} \tilde{A}^{(s)}\right]\).
17. Obtain \(u_m = \arg\max_{t \in (s_m + \frac{1}{3}\kappa, e_m - \frac{1}{3}\kappa)} \|G_m^{(t)}\|\).
18. If \(\|G_m^{(u_m)}\| > \Theta \left[\frac{D}{\kappa} \left(\zeta + 6 + \frac{\log(M)}{\log(n)}\right)\right]^{1/2}\), declare \(u_m\) as a candidate change point.
19. end if
20. \(u_0 = \arg\max_{m \in M_{s,e}} \|G_m^{(t)}\|\).
21. Add \(u_0\) to the set of estimated change-points
22. Wild Binary Segmentation \((s,u_0,\kappa)\)
23. Wild Binary Segmentation \((u_0 + 1, e, \kappa)\)
24. else
25. STOP
26. end if
27. end if

3. Theory.
3.1. Lower Bound. Let \( P_\Theta \) denote the distribution of the inhomogeneous random graph having mean adjacency matrix \( \Theta \). For \( \rho, \alpha \in (0, 1), \kappa \in [T] \), and (possibly random) symmetric matrix \( \Gamma \) taking values in \([-1, 1]^{n \times n} \), let

\[
(3.1) \quad \Theta_0 = \rho \langle 1_n, 1_n^T \rangle, \quad P_0^\kappa := P_{\Theta_0} \times \cdots \times P_{\Theta_0}, \quad \Theta(\Gamma, \alpha) = \Theta_0 + \alpha \rho \langle \Gamma \rangle, \\
\quad \quad \quad \quad \quad \quad P_\Gamma^\kappa := P_{\Theta(\Gamma, \alpha)} \times \cdots \times P_{\Theta(\Gamma, \alpha)}, \quad P_{\Gamma, \alpha}^\kappa (\cdot) = E_{\Gamma} P_{\Gamma, \alpha}^\kappa (\cdot),
\]

where \( P_\Gamma \) denotes the distribution of \( \Gamma \).

**Lemma 3.1 (Chi-square divergence bound).** Let \( \bar{P}_\Gamma \) denote the distribution of \( \Gamma \in S_n \), where \( (\Gamma_{ij}, 1 \leq i < j \leq n) \) are i.i.d. with finite second moment. For any \( \mathcal{B} \subset S_n \) satisfying \( \bar{P}_\Gamma (\mathcal{B}) \geq 1/2 \), let \( P_\Gamma (\cdot) = \bar{P}_\Gamma (\cdot | \mathcal{B}) \). For any \( \varepsilon > 0 \), there is a constant \( c(\varepsilon) > 0 \) such that \( \chi^2 (P_0^\kappa, P_\Gamma^\kappa) \leq \varepsilon \) whenever \( \kappa \alpha^2 \rho n \leq c(1 - \rho) \).

**Proof.** Let \( \tilde{\Gamma} \) be an independent copy of \( \Gamma \), \( P_{\Gamma \otimes \tilde{\Gamma}} \) denote their joint distribution, and \( \sigma^2 := E_{\Gamma} (\Gamma_{ij}^2) \). Noting that \( (A_{ij}^{(t)}, 1 \leq i < j \leq n, t \in [\kappa]) \) are i.i.d. \( \text{Ber}(\rho) \) under \( P_0^\kappa \), and using the inequality \( 1 + x \leq e^x \) for any \( x \in \mathbb{R} \),

\[
(3.2) \quad 1 + \chi^2 (P_0^\kappa, P_{\Gamma, \alpha}^\kappa) = E_0^\kappa \left[ \left( \frac{d P_{\Gamma, \alpha}^\kappa}{d P_0^\kappa} \right)^2 \right] = E_0^\kappa \left[ E_{\Gamma \otimes \tilde{\Gamma}} \left( \frac{d P_{\Gamma, \alpha}^\kappa}{d P_0^\kappa} \right) \frac{d P_{\Gamma, \alpha}^\kappa}{d P_0^\kappa} \right] \\
= E_{\Gamma \otimes \tilde{\Gamma}} E_0^\kappa \prod_{1 \leq i < j \leq n} \left( 1 + \alpha \Gamma_{ij} \right) \left( 1 + \alpha \tilde{\Gamma}_{ij} \right) I_{\{A_{ij}^{(t)} = 1\}} + \left( 1 - \frac{\alpha \rho \Gamma_{ij}}{1 - \rho} \right) \left( 1 - \frac{\alpha \rho \tilde{\Gamma}_{ij}}{1 - \rho} \right) \\
I_{\{A_{ij}^{(t)} = 0\}} = E_{\Gamma \otimes \tilde{\Gamma}} \prod_{1 \leq i < j \leq n} \left( 1 + \frac{\alpha^2 \rho \Gamma_{ij} \tilde{\Gamma}_{ij}}{1 - \rho} \right) \leq E_{\Gamma \otimes \tilde{\Gamma}} \exp \left[ \kappa \alpha^2 \rho F(\Gamma, \tilde{\Gamma}) \right],
\]

where \( F(\Gamma, \tilde{\Gamma}) := \sum_{1 \leq i < j \leq n} \Gamma_{ij} \tilde{\Gamma}_{ij} \). Using Bernstein inequality and the fact that \( \bar{P}(\mathcal{B}) \geq 1/2 \),

\[
P_{\Gamma \otimes \tilde{\Gamma}} \left( F(\Gamma, \tilde{\Gamma}) > \ell \sigma^2 n \right) \leq 4 \bar{P}_{\Gamma \otimes \tilde{\Gamma}} \left( F(\Gamma, \tilde{\Gamma}) > \ell \sigma^2 n \right) \\
\leq 4 \exp \left( \frac{-1/2 \ell^2 n^2 \sigma^4}{\sum_{1 \leq i < j \leq n} E_{\Gamma} (\Gamma_{ij}^2) E_{\tilde{\Gamma}} (\tilde{\Gamma}_{ij}^2) + \frac{2}{3} \ell \sigma^2 n} \right) \leq e^{-3 \ell^2 / 7}.
\]
For any $L \geq 1$, we can use the above estimate to have
\[
E_{\Gamma \tilde{\Gamma}} \exp \left[ \frac{\kappa \alpha^2 \rho}{1-\rho} F(\Gamma, \tilde{\Gamma}) \right] \leq E_{\Gamma \tilde{\Gamma}} \left( \exp \left[ \frac{\kappa \alpha^2 \rho}{1-\rho} F(\Gamma, \tilde{\Gamma}) \right] 1_{\{F \leq L \sigma^2 n\}} \right)
\]
\[
+ \sum_{\ell=L}^{\infty} E_{\Gamma \tilde{\Gamma}} \left( \exp \left[ \frac{\kappa \alpha^2 \rho}{1-\rho} F(\Gamma, \tilde{\Gamma}) \right] 1_{\{\ell \sigma^2 n < F \leq (\ell+1) \sigma^2 n\}} \right) \leq \exp \left[ \frac{\kappa \alpha^2 \rho}{1-\rho} L \sigma^2 n \right]
\]
\[
+ \sum_{\ell=L}^{\infty} 4 \exp \left[ \frac{\kappa \alpha^2 \rho}{1-\rho} (\ell + 1) \sigma^2 n - \frac{3}{7} \ell^2 \right].
\]

Given $\varepsilon > 0$, we can choose $L(\varepsilon)$ large enough so that $4 \sum_{\ell=L}^{\infty} e^{(\ell+1) - 3\ell^2/7} \leq \varepsilon/2$. Having chosen $L(\varepsilon)$, we can choose $c(\varepsilon) > 0$ small enough such that $e^{c(\varepsilon)L(\varepsilon) \sigma^2} \leq 1 + \varepsilon/2$. Combining this with (3.2) proves the result. 

For $\Theta_0, \Theta_1 \in S_n$ and $\tau \in [T]$, let $S^{T, \tau}_n := \{ \mathcal{D} \in S^T_n : K(\mathcal{D}) = 1, T_{\mathcal{D}}(\mathcal{D}) = \tau \}$,
\[
\mathcal{D}(\tau, T; \Theta_0, \Theta_1) = \left( \Theta_0, \ldots, \Theta_0, \Theta_1, \ldots, \Theta_1 \right) \in S^{T, \tau}_n.
\]

For any (possibly degenerate) probability distributions $\nu_0, \nu_1$ on $S_n$, let
\[
P_{\mathcal{D}(\tau, T; \nu_0, \nu_1)}(\cdot) := E_{E(\Theta_0, \Theta_1) \sim \nu_0 \otimes \nu_1 P_{\mathcal{D}(\tau, T; \Theta_0, \Theta_1)}(\cdot)}, \quad \text{and} \quad \mathcal{D}(\tau, T; \nu_0, \nu_1)
\]
be the distribution on $S^{T, \tau}_n$ satisfying $P(\mathcal{D}(\tau, T; \nu_0, \nu_1) = \mathcal{D}(\tau, T; \Theta_0, \Theta_1)) = \nu_0 \otimes \nu_1(\Theta_0, \Theta_1)$. For $r \in [n], \kappa \in [T]$, and $\gamma > 0$, let
\[
\mathcal{H}(\gamma, r, \kappa) := \left\{ P_{\mathcal{D}(\tau, T; \nu_0, \nu_1)} : (a) \nu_0 \text{ and } \nu_1 \text{ are distributions on } S_n, \right.
\]
(b) $(\Theta_0, \Theta_1) \sim \nu_0 \otimes \nu_1$ implies $\|\Theta_1 - \Theta_0\|_F^2 \leq \frac{\gamma}{\kappa} n(\|\Theta_0\|_{\max} \vee \|\Theta_1\|_{\max})$,
\[
\text{and } \text{rank}(\Theta_0 - \Theta_1) = r, \quad \text{and } (c) \kappa \leq \tau \leq T - \kappa \}
\]

**Lemma 3.2 (Lower bound for localizability).** For any $\varepsilon > 0$, there is a constant $\gamma(\varepsilon) > 0$ such that for any $r \in [n]$ and $\kappa < T/2$,
\[
\inf_{\mathcal{D}} \sup_{\mathcal{D} \in \mathcal{H}(\gamma, r, \kappa)} E_{\mathcal{D}} |\hat{T} - T_{\mathcal{D}}(\mathcal{D})| \geq (T - 2\kappa)(1 - 2\varepsilon).
\]

**Proof.** Given $\varepsilon > 0$, take $\gamma$ to be equal to $c(\varepsilon)$ (the constant in Lemma 3.1). Let (a) $\alpha, \rho \in (0, 1)$ be such that $\kappa \alpha^2 \rho n \leq \gamma(1 - \rho)$, (b) $\Theta_0 = \langle \rho 1_n, 1_n^T \rangle$, (c) $\Gamma = \sum_{i=1}^{T} 3^{-i} U_i U_i^T$, where $U_1, \ldots, U_r$ are i.i.d. $n \times 1$ vectors with $P_{\Gamma}(U_{ij} = \pm 1) = 1/2$, (d) $\nu_1$ be the distribution of $\Theta_{\alpha, \rho} := \Theta_0 + \langle \alpha \rho \Gamma \rangle$
given the event $B := \{\text{rank}(\Gamma) = r\}$, (e) $\mathcal{D}^{[0]} := \mathcal{D}(\kappa, T; \nu_1, \nu_0)$, (f) $\mathcal{D}^{[1]} := \mathcal{D}(T - \kappa, T; \nu_0, \nu_1)$, and (g) $P_{\kappa}^0, P_{\kappa,\alpha}^0$ are as in (3.1). It is well known that the probability of $B$ goes to 1 as $n \to \infty$. It is easy to see that if $\Theta_1 \sim \nu_1$, then $\|\Theta_1 - \Theta_0\|_F^2 \leq \alpha^2 \rho^2 n^2 \leq \gamma n \rho / \kappa$. Thus $P_{\mathcal{D}^{[i]}} \in \mathcal{H}(\gamma, r, \kappa)$ for both $i = 1, 2$, as $|\Theta_0|_{max} = \rho$. Also, $|T_1(\mathcal{D}^{[0]}) - T_1(\mathcal{D}^{[1]})| = T - 2\kappa$. So, using Le Cann’s lemma $]$, 

$$\inf_{\hat{\tau}} \sup_{P_{\mathcal{D}} \in \mathcal{H}(\gamma, r, \kappa)} E_P |\hat{\tau} - T_1(\mathcal{D})| \geq (T - 2\kappa)(1 - d_{TV}(P_{\mathcal{D}^{[0]}}, P_{\mathcal{D}^{[1]}})),$$

where $d_{TV}(\cdot, \cdot)$ denotes the total variation distance. Using the facts that $d_{TV}(\cdot, \cdot) \leq \chi^2(\cdot, \cdot)$ and $d_{TV}(\mu_1 \otimes \mu_2, \nu_1 \otimes \nu_2) \leq d_{TV}(\mu_1, \nu_1) + d_{TV}(\mu_2, \nu_2)$ for all compatible probability distributions $\mu_i, \nu_i$, we see that

$$d_{TV}(P_{\mathcal{D}^{[0]}}, P_{\mathcal{D}^{[1]}}) \leq 2d_{TV}(P_0^\kappa, P_{1,\alpha}^\kappa) \leq \chi^2(P_0^\kappa, P_{1,\alpha}^\kappa),$$

which is at most $2\varepsilon$ by Lemma 3.1 and the choice of $\alpha$. 

**Lemma 3.3** (Lower bound for detectability). For any $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that if $\mathcal{Y} := \{\mathcal{D} \in S_n^T : \mathcal{K}(\mathcal{D}) \geq 1, \mathcal{O}(\mathcal{D}) \leq c(\varepsilon)\sqrt{\log n_{\gamma, T}}\}$, then $\Pi(\Omega_{n,T}; S_n^T, \mathcal{Y}) \geq 1 - \sqrt{\varepsilon}/2$ for all test function $\Omega_{n,T}$.

**Proof.** Fix any $\rho \in (0, 1)$. Given $\varepsilon > 0$ let $c(\varepsilon)$ be the constant of Lemma 3.1. Choose $\alpha$ so that $\kappa \alpha^2 \rho n \leq c(1 - \rho)$. Let $\Theta_0 = \langle \rho \mathbf{1}_n \mathbf{1}_n^T \rangle, \Theta(\mathbf{U}, \alpha) = \Theta_0 + \alpha \langle \mathbf{U} \mathbf{U}^T \rangle$, where $\mathbf{U} \in \mathbb{R}^{n \times 1}$ has i.i.d. components with $P(U_i = \pm 1) = 1/2$,

$$\mathcal{D}_0 := (\Theta_0, \ldots, \Theta_0), \mathcal{D}(\mathbf{U}, \alpha) = \left(\frac{\kappa}{\Theta(\mathbf{U}, \alpha), \ldots, \Theta(\mathbf{U}, \alpha), \Theta_0, \ldots, \Theta_0} \right) \in S_n^T$$

Let $P_0$ and $P_1$ represent the distributions $P_{\mathcal{D}_0}$ and $E_{\mathbf{U}} P_{\mathcal{D}(\mathbf{U}, \alpha)}$ respectively. Let $\Omega_{n,T}^* := \mathbf{1}_{\left\{d_{P_0} > 1\right\}}$. Then, for any test function $\Omega_{n,T}$,

$$\Pi \left(\Omega_{n,T}; S_n^T, \mathcal{Y}\right) \geq P_0(\Omega_{n,T} = 1) + P_1(\Omega_{n,T} = 0)$$

$$= 1 + \int_{\{\Omega_{n,T} = 1\}} \left[1 - \frac{dP_1}{dP_0}\right] dP_0 \geq 1 + \int_{\{\Omega_{n,T}^* = 1\}} \left[1 - \frac{dP_1}{dP_0}\right] dP_0$$

$$= 1 - \frac{1}{2} E_0 \left|\frac{dP_1}{dP_0} - 1\right| \geq 1 - \frac{1}{2} \sqrt{E_0 \left[\left(\frac{dP_1}{dP_0}\right)^2\right]} \geq 1 - \sqrt{\varepsilon}/2$$

by the choice of $\alpha$ and the result of Lemma 3.1.
3.2. **Multiple Change Points.** Algorithm 1 and 2 were presented in §2.3 to detect and localize multiple change points in network sequences. Recall that the data is given in form of a sequence of networks represented by the symmetric adjacency matrices \( (A_n, \ldots, A_T) \). The generating model (presented in §2.3) with change points \( \tau_1, \tau_2, \ldots, \tau_K \) and parameters \( Q = (Q^{(1)}, \ldots, Q^{(T)}) \), can be recalled as -

(i) if \( Q^{(t)} = Q_k \) for all \( t \in [T] \), there exists no change-point;

(ii) if there exists a set of change-points \( \tau_1, \tau_2, \ldots, \tau_K \) with \( \tau_0 := 0 \leq \tau_1 \leq \cdots \leq \tau_K \leq T =: \tau_{K+1} \) such that

\[
Q^{(t)} = Q_k \text{ for } \tau_{k-1} \leq t \leq \tau_k, \quad k \in [K]
\]

where, \( Q_k \in S_n \) for \( k \in [K] \), then there exists a multiple change-points \( (\tau_1, \tau_2, \ldots, \tau_K) \).

(iii) Additionally \( g(\mathcal{D}) \) represents the minimum cushion as defined in (2.2),

\[
g(\mathcal{D}) := \min_{0 \leq k \leq K} (\tau_{k+1} - \tau_k).
\]

(iv) **Signal strength** is given by

\[
\mathcal{S}(\mathcal{D}) := \begin{cases} 
\min_{k \in [K(\mathcal{D})]} \|Q^{(\tau_k+1)} - Q^{(\tau_k)}\| & \text{if } K(\mathcal{D}) \geq 1 \\
0 & \text{if } K(\mathcal{D}) = 0
\end{cases}
\]

(v) **Sparsity** parameter is given by

\[
D(\mathcal{D}) := \max_{i,j \in [n], t \in [T]} Q^{(t)}_{ij}.
\]

3.2.1. **Window-based Method.** Algorithm 1, which is presented in §2.3.1, outputs \( (\hat{\tau}, \Lambda, I) \), which are the estimates of the change point \( \tau \) along with interval length \( \Lambda \) and interval \( I \) containing the estimated change point. The theoretical results on the performance of the change point estimate \( \hat{\tau} \) is given in Theorems 3.4 and 3.6. We show two different types of theoretical results for the change point detection problem:

1. **Detectability:** The results on detectability focuses on correctly detecting the presence or absence of change point in a network sequence. The loss function for detection is given in terms of minimax loss as defined in (1.1). The theoretical result on detectability for window-based algorithm (Algorithm 1) is given in Theorem 3.4.

2. **\( \Lambda \)-localizability:** The results on \( \Lambda \)-localizability focuses on correctly estimating locations of all the change points in a network sequence and giving an interval estimate of length \( \Lambda \) around the true change points.
The loss function for $\Lambda$-localizability is given in form of the error made in estimating the location of the change point. The theoretical result on $\Lambda$-localizability for window-based algorithm (Algorithm 1) is given in Theorem 3.6 and for wild binary segmentation algorithm (Algorithm 2) is given in Theorem 3.8.

**Theorem 3.4 (Detectability Result).** Let us consider that we have a sequence of networks represented by the symmetric adjacency matrices $\mathcal{A} = \left( A^{(1)}_{n \times n}, \ldots, A^{(T)}_{n \times n} \right)$ generated from the MIRGraM model with parameters $\mathcal{Q} = (Q^{(1)}, \ldots, Q^{(T)}) \in S^n_T$ with the set of change points given by $T(\mathcal{Q}) := (\tau_1, \tau_2, \ldots, \tau_K)$ if $K(\mathcal{Q}) > 0$. Then, we have the following results on detecting change points in the sequence $\mathcal{A}$.

1. (Lower bound) For any $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that if $Y := \{ \mathcal{Q} \in S^n_T : K(\mathcal{Q}) \geq 1, S(\mathcal{Q}) \leq c(\varepsilon) \sqrt{D(\mathcal{Q})} \}$, then

   \[ \Pi(\Omega_{n,T}; S^n_T, 0, Y) \geq 1 - \frac{\varepsilon}{2} \] for all test function $\Omega_{n,T}$.

2. (Upper bound) Consider that Algorithm 1 is applied on $\mathcal{A}$ with cushions of length $\kappa$ ($3 \leq \kappa \leq T$), and intervals of length $\Lambda$ ($3 \leq \Lambda \leq \kappa$). There are constants $C_1, c_1, \zeta_0 > 0$ with $\Psi_\mu$ as in (2.4) for $\mu > 0$, $d$ and $\delta$ as in (2.3) such that if $W$ be the set $\mathcal{Q} \in S^n_T$ satisfying the properties $K(\mathcal{Q}) \geq 1$, and

   \[ \kappa > g(\mathcal{Q}) \text{ and } S(\mathcal{Q}) \geq (C_1 + c_1 \Psi_\mu) \left[ \frac{d}{\Lambda \wedge \kappa} \left( \zeta + \frac{\log(|L|)}{\log(n)} \right) \right]^{1/2} \]

   then,

   \[ \Pi(\Omega_{n,T}; S^n_T, 0, W) \leq 2 \left( \frac{|L|^{9/\log(n)}}{\log(n)} \right) n^{-\zeta} + 3 \exp \left[ -\mu(\Lambda \wedge \kappa) d \right] \text{ for all } \zeta > \zeta_0 \]

**Proof.** 1. The proof follows from Lemma 3.3.

2. The proof follows from the result in Theorem 3.6.

**Remark 3.5.** Note that in Algorithm 1, $\Lambda \leq \kappa$, so $\Lambda \wedge \kappa = \Lambda$ in Theorem 3.4.

**Theorem 3.6 (\Lambda-localizability Result).** Let us consider that we have a sequence of networks represented by the symmetric adjacency matrices $\mathcal{A} = \left( A^{(1)}_{n \times n}, \ldots, A^{(T)}_{n \times n} \right)$ generated from the MIRGraM model with parameters $\mathcal{Q} = (Q^{(1)}, \ldots, Q^{(T)}) \in S^n_T$ with the set of change points given by
\( \mathcal{T}(\mathcal{D}) := (\tau_1, \tau_2, \ldots, \tau_K) \) if \( \mathcal{K}(\mathcal{D}) > 0 \). Then, we have the following results on \( \Lambda \)-localizability of change point estimates in the sequence \( \mathcal{A} \).

1. (Lower bound) For the case of single change point \( \tau \in [T] \), let \( S_n^{T,\tau} := \{ \mathcal{D} \in S_n^T : \mathcal{K}(\mathcal{D}) = 1, \mathcal{T}_I(\mathcal{D}) = \tau \} \), and

\[
\mathcal{D}(\tau, T; \Theta_0, \Theta_1) = \left( \underbrace{\Theta_0, \ldots, \Theta_0}_{\text{\( \tau \) many}}, \underbrace{\Theta_1, \ldots, \Theta_1}_{\text{\( T-\tau \) many}} \right) \in S_n^{T,\tau},
\]

where, \( \Theta_0, \Theta_1 \in S_n \). For any (possibly degenerate) probability distributions \( \nu_0, \nu_1 \) on \( S_n \), let

\[
P_\mathcal{D}(\tau, T; \nu_0, \nu_1)(\cdot) := E_{(\Theta_0, \Theta_1) \sim \nu_0 \otimes \nu_1} P_\mathcal{D}(\tau, T; \Theta_0, \Theta_1)(\cdot), \text{ and } \mathcal{D}(\tau, T; \nu_0, \nu_1)
\]

be the distribution on \( S_n^{T,\tau} \) satisfying \( P(\mathcal{D}(\tau, T; \nu_0, \nu_1) = \mathcal{D}(\tau, T; \Theta_0, \Theta_1)) = \nu_0 \otimes \nu_1(\Theta_0, \Theta_1) \). For \( r \in [n], \kappa \in [T] \), and \( \gamma > 0 \), let

\[
\mathcal{H}(\gamma, r, \kappa) := \{ P_\mathcal{D}(\tau, T; \nu_0, \nu_1) : (a) \nu_0 \text{ and } \nu_1 \text{ are distributions on } S_n, (b) (\Theta_0, \Theta_1) \sim \nu_0 \otimes \nu_1 \text{ implies } \| \Theta_1 - \Theta_0 \|_F \leq \frac{2}{\kappa} \nu(\| \Theta_0 \|_{\text{max}} \vee \| \Theta_1 \|_{\text{max}}), \text{ and } \text{rank}(\Theta_0 - \Theta_1) = r, \text{ and } (c) \kappa \leq |L| \leq T - \kappa \}
\]

Then, for any \( \varepsilon > 0 \), there is a constant \( \gamma(\varepsilon) > 0 \) such that for any \( r \in [n] \) and \( \kappa < T/2 \),

\[
\inf \sup_{\hat{\tau} \in \mathcal{F}(\gamma, r, \kappa)} E_{\mathcal{D}} \left| \hat{\tau} - \mathcal{T}_I(\mathcal{D}) \right| \geq (T - 2\kappa)(1 - 2\varepsilon).
\]

2. (Upper bound) Consider that Algorithm 1 is applied on \( \mathcal{A} \) with cushions of length \( \kappa \) (\( 3 \leq \kappa \leq T \)), and intervals of length \( \Lambda \) (\( 3 \leq \Lambda \leq \kappa \)). There are constants \( C_1, c_1, \zeta_0 > 0 \) with \( \Psi_\mu \) as in (2.4) for \( \mu > 0 \), \( d \) and \( \mathcal{D} \) as in (2.3) such that if \( \mathcal{W} \) be the set \( \mathcal{D} \in S_n^T \) satisfying the properties \( \mathcal{K}(\mathcal{D}) = 1 \), and

\[
(3.6) \quad \kappa > g(\mathcal{D}) \text{ and } \mathcal{G}(\mathcal{D}) \geq (C_1 + c_1 \Psi_\mu) \left[ \frac{d}{\Lambda \wedge \kappa} \left( \zeta + \frac{\log(|L|)}{\log(n)} \right) \right]^{1/2}
\]

then,

\[
P \left( |\hat{\tau}_i - \tau_i| \geq \Lambda \frac{C_1 + c_1 \Psi_\mu}{\mathcal{G}(\mathcal{D})} \left[ \frac{d}{\Lambda \wedge \kappa} \left( \zeta + \frac{\log(|L|)}{\log(n)} \right) \right]^{1/2} \forall i \in [K] \text{ and } \hat{K} = K \right) \leq 2 \left( \frac{|L|^{9/\log(n)}}{\log(n)} \right) n^{-\zeta} + 3 \exp \left[ -\mu(\Lambda \wedge \kappa) \zeta_0 \right] \text{ for all } \zeta > \zeta_0.
\]
Proof. 1. The proof follows from Lemma 3.2. 2. The proof is in the Appendix.

Remark 3.7. Note that in Algorithm 1, \( \Lambda \leq \kappa \), so \( \Lambda \wedge \kappa = \Lambda \) in Theorem 3.6. Also, if \( g(\mathcal{D}) \leq \kappa = O(T) \) and \( \Lambda = O(T) \), then, under the condition weaker condition of \( \mathcal{G}(\mathcal{D}) \geq C \sqrt{\frac{T}{\kappa}} \) (for some constant \( C > 0 \)), \( |\hat{\tau} - \tau| = O(T) \) with high probability. However, if \( g(\mathcal{D}) \leq \kappa = o(T) \) and \( \Lambda = o(T) \), then, under the stronger condition of \( \mathcal{G}(\mathcal{D}) \geq C \sqrt{\frac{T}{\kappa}} \) (for some constant \( C > 0 \)), \( |\hat{\tau} - \tau| = O(\Lambda) \) with high probability. So, as the signal strength \( \mathcal{G}(\mathcal{D}) \) increases, the interval length of change point estimate, \( \Lambda \), decreases, that is, the localization of the change point estimate becomes better.

3.2.2. Wild Binary Segmentation. Algorithm 2, which is presented in §2.3.2, outputs \((\hat{\tau}, \Lambda, I)\), which are the estimates of the change point \( \tau \) along with interval length \( \Lambda \), and interval \( I \) containing the estimated change point. The generating model is presented in §2.3. The theoretical results on the performance of the change point estimate \( \hat{\tau} \) is given in Theorems 3.8.

Theorem 3.8. Consider that Algorithm 2 is applied on \( \mathcal{A} \) with cushions of length \( \kappa \) (\( 3 \leq \kappa \leq T \)). There are constants \( C_1, c_1, \zeta_0 > 0 \) with \( \Psi_\mu \) as in (2.4) for \( \mu > 0 \), \( d \) and \( \psi \) as in (2.3) such that if \( \mathcal{W} \) be the set \( \mathcal{D} \in \mathcal{S}_n^T \) satisfying the properties \( \mathcal{K}(\mathcal{D}) \geq 1 \), and

\[
\kappa > g(\mathcal{D}) \text{ and } \mathcal{G}(\mathcal{D}) \geq (C_1 + c_1 \Psi_\mu) \left[ \frac{d}{\kappa} \left( \frac{\log(M)}{\log(n)} \right) \right]^{1/2}
\]

then, for all \( \zeta > \zeta_0 \),

\[
P \left( |\hat{\tau}_i - \tau_i| \geq \frac{\Lambda_{C_1 + c_1 \Psi_\mu}}{\mathcal{G}(\mathcal{D})} \left[ \frac{d}{\kappa} \left( \frac{\log(M)}{\log(n)} \right) \right]^{1/2} \right) \leq \frac{2(M)^{9/\log(n)}}{\log(n)} n^{-\zeta} + 3 \exp \left[ -\mu \kappa \psi \right] + T \kappa^{-1} \left( 1 - \kappa^2 T^{-2} / 9 \right)^M.
\]

Proof. The proof is in the Appendix.

References.
[1] Aminikhanghahi, S. and Cook, D. J. (2017). A survey of methods for time series change point detection. Knowledge and information systems 51 339–367.
[2] Bai, J. and Perron, P. (1998). Estimating and testing linear models with multiple structural changes. Econometrica 47–78.
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[3] Bhattacharjee, M., Banerjee, M. and Michailidis, G. (2018). Change Point Estimation in a Dynamic Stochastic Block Model. arXiv preprint arXiv:1812.03090.

[4] Blonder, B., Wey, T. W., Dornhaus, A., James, R. and Shi, A. (2012). Temporal dynamics and network analysis. Methods in Ecology and Evolution 3 958–972.

[5] Bosc, M., Heitz, F., Armspach, J.-P., Namer, I., Gounot, D. and Rumbach, L. (2003). Automatic change detection in multimodal serial MRI: application to multiple sclerosis lesion evolution. NeuroImage 20 643–656.

[6] Brodsky, E. and Darkhovsky, B. S. (2013). Nonparametric methods in change point problems 243. Springer Science & Business Media.

[7] Cape, J., Tang, M. and Priebe, C. E. (2017). The Kato–Temple inequality and eigenvalue concentration with applications to graph inference. Electronic Journal of Statistics 11 3954–3978.

[8] Chen, H. and Zhang, N. (2015). Graph-based change-point detection. Ann. Statist. 43 139–176.

[9] Ferraz Costa, A., Yamaguchi, Y., Juci Machado Traina, A., Traina Jr, C. and Faloutsos, C. (2015). Rsc: Mining and modeling temporal activity in social media. In Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining 269–278. ACM.

[10] Fryzlewicz, P. et al. (2014). Wild binary segmentation for multiple change-point detection. The Annals of Statistics 42 2243–2281.

[11] Gates, M. C. and Woolhouse, M. E. (2015). Controlling infectious disease through the targeted manipulation of contact network structure. Epidemics 12 11–19.

[12] Ghoshdastidar, D., Gutzeit, M., Carpentier, A. and von Luxburg, U. (2017). Two-sample hypothesis testing for inhomogeneous random graphs. arXiv preprint arXiv:1707.00833.

[13] Girshick, M. A. and Rubin, H. (1952). A Bayes approach to a quality control model. The Annals of mathematical statistics 114–125.

[14] Harchaoui, Z., Moulines, E. and Bach, F. R. (2009). Kernel change-point analysis. In Advances in neural information processing systems 609–616.

[15] Harchaoui, Z., Vallet, F., Lung-Yut-Fong, A. and Cappé, O. (2009). A regularized kernel-based approach to unsupervised audio segmentation. In 2009 IEEE International Conference on Acoustics, Speech and Signal Processing 1665–1668. IEEE.

[16] Hocking, T. D., Schleiermacher, G., Janoueix-Lerosey, I., Boeva, V., Cappo, J., Delattre, O., Bach, F. and Vert, J.-P. (2013). Learning smoothing models of copy number profiles using breakpoint annotations. BMC bioinformatics 14 164.

[17] Holme, P. (2015). Modern temporal network theory: a colloquium. The European Physical Journal B 88 234.

[18] Holme, P. and Saramäki, J. (2012). Temporal networks. Physics reports 519 97–125.

[19] Jacobs, A. Z., Way, S. F., Ugander, J. and Clauset, A. (2015). Assembling the facebook: Using heterogeneity to understand online social network assembly. In Proceedings of the ACM Web Science Conference 18. ACM.

[20] James, B., James, K. L. and Siegmund, D. (1992). Asymptotic approximations for likelihood ratio tests and confidence regions for a change-point in the mean of a multivariate normal distribution. Statistica Sinica 69–90.

[21] Krings, G., Karsai, M., Bernhardsson, S., Blondel, V. D. and Saramäki, J. (2012). Effects of time window size and placement on the structure of an aggregated communication network. EPJ Data Science 1 4.
[22] Lahiri, M. and Berger-Wolf, T. Y. (2007). Structure prediction in temporal networks using frequent subgraphs. In 2007 IEEE Symposium on Computational Intelligence and Data Mining 35–42. IEEE.

[23] La vianne, M. and Teyssiere, G. (2007). Adaptive detection of multiple change-points in asset price volatility. In Long memory in economics 129–156. Springer.

[24] Lévy-Leduc, C., Roueff, F. et al. (2009). Detection and localization of change-points in high-dimensional network traffic data. The Annals of Applied Statistics 3 637–662.

[25] Lorden, G. et al. (1971). Procedures for reacting to a change in distribution. The Annals of Mathematical Statistics 42 1897–1908.

[26] Lung-Yut-Fong, A., Lévy-Leduc, C. and Cappé, O. (2011). Homogeneity and change-point detection tests for multivariate data using rank statistics. arXiv preprint arXiv:1107.1971.

[27] Mukherjee, S. S. (2018). On Some Inference Problems for Networks, PhD thesis.

[28] Page, E. S. (1954). Continuous inspection schemes. Biometrika 41 100–115.

[29] Page, E. S. (1957). On problems in which a change in a parameter occurs at an unknown point. Biometrika 44 248–252.

[30] Panisson, A., Gauvin, L., Barrat, A. and Cattuto, C. (2013). Fingerprinting temporal networks of close-range human proximity. In 2013 IEEE International Conference on Pervasive Computing and Communications Workshops (PERCOM Workshops) 261–266. IEEE.

[31] Park, H.-J. and Friston, K. (2013). Structural and functional brain networks: from connections to cognition. Science 342 1238411.

[32] Park, Y., Priebe, C. E. and Youssef, A. (2013). Anomaly detection in time series of graphs using fusion of graph invariants. IEEE journal of selected topics in signal processing 7 67–75.

[33] Peel, L. and Clauset, A. (2015). Detecting Change Points in the Large-Scale Structure of Evolving Networks. In AAAI 2914–2920.

[34] Peixoto, T. P. (2015). Inferring the mesoscale structure of layered, edge-valued, and time-varying networks. Physical Review E 92 042807.

[35] Peixoto, T. P. and Gauvin, L. (2018). Change points, memory and epidemic spreading in temporal networks. Scientific reports 8 15511.

[36] Peixoto, T. P., Lavialle, M., Vaissé, C. and Daudin, J.-J. (2005). A statistical approach for array CGH data analysis. BMC bioinformatics 6 27.

[37] Popović, M., Štefančić, H., Sluban, B., Novak, P. K., Grčar, M., Možetič, I., Puliga, M. and Zlatić, V. (2014). Extraction of temporal networks from term co-occurrences in online textual sources. PloS one 9 e99515.

[38] Radke, R. J., Andra, S., Al-Kofahi, O. and Roysam, B. (2005). Image change detection algorithms: a systematic survey. IEEE transactions on image processing 14 294–307.

[39] Ranshous, S., Shen, S., Koutra, D., Harenberg, S., Faloutsos, C., and Samatova, N. F. (2015). Anomaly detection in dynamic networks: a survey. Wiley Interdisciplinary Reviews: Computational Statistics 7 223–247.

[40] Reeves, J., Chen, J., Wang, X. L., Lund, R. and Lu, Q. Q. (2007). A review and comparison of changepoint detection techniques for climate data. Journal of applied meteorology and climatology 46 900–915.

[41] Rigbolt, K. T., Prokhorova, T. A., Akimov, V., Henningsen, J., Johansen, P. T., Kratchmarova, I., Kassem, M., Mann, M., Olsen, J. V. and Blagoev, B. (2011). System-wide temporal characterization of the proteome and phosphoproteome of human embryonic stem cell differentiation. Sci. Signal. 4 rs3–
rs3.

[42] Rocha, L. E., Liljeros, F. and Holme, P. (2010). Information dynamics shape the sexual networks of Internet-mediated prostitution. Proceedings of the National Academy of Sciences 107 5706–5711.

[43] Rocha, L. E., Liljeros, F. and Holme, P. (2011). Simulated epidemics in an empirical spatiotemporal network of 50,185 sexual contacts. PLoS computational biology 7 e1001109.

[44] Roy, S., Atchadé, Y. and Michailidis, G. (2017). Change point estimation in high dimensional Markov random-field models. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 79 1187–1206.

[45] Sala thé, M., Kazandjiev a, M., Lee, J. W., Levis, P., Feldman, M. W. and Jones, J. H. (2010). A high-resolution human contact network for infectious disease transmission. Proceedings of the National Academy of Sciences 107 22020–22025.

[46] Shir y aev, A. N. (1963). On optimum methods in quickest detection problems. Theory of Probability & Its Applications 8 22–46.

[47] Siegmund, D., Yakir, B. and Zhang, N. R. (2011). Detecting simultaneous variant intervals in aligned sequences. The Annals of Applied Statistics 645–668.

[48] Sikdar, S., Ganguly, N. and Mukherjee, A. (2016). Time series analysis of temporal networks. The European Physical Journal B 89 11.

[49] Sporns, O. (2013). Structure and function of complex brain networks. Dialogues in clinical neuroscience 15 247.

[50] Srivastava, M. and Worsley, K. J. (1986). Likelihood ratio tests for a change in the multivariate normal mean. Journal of the American Statistical Association 81 199–204.

[51] Staudacher, M., Telser, S., Amann, A., Hinterhuber, H. and Ritsch-Marte, M. (2005). A new method for change-point detection developed for on-line analysis of the heart beat variability during sleep. Physica A: Statistical Mechanics and its Applications 349 582–596.

[52] Stopczynski, A., Sekara, V., Sapiezynski, P., Cuttone, A., Madsen, M. M., Larsen, J. E. and Lehmann, S. (2014). Measuring large-scale social networks with high resolution. PloS one 9 e95978.

[53] Wang, D., Yu, Y. and Rinaldo, A. (2018). Optimal change point detection and localization in sparse dynamic networks. arXiv preprint arXiv:1809.09602.

[54] Yang, P., Dumont, G. and Ansermino, J. M. (2006). Adaptive change detection in heart rate trend monitoring in anesthetized children. IEEE transactions on biomedical engineering 53 2211–2219.

[55] Zhang, N. R., Siegmund, D. O., Ji, H. and Li, J. Z. (2010). Detecting simultaneous changepoints in multiple sequences. Biometrika 97 631–645.

[56] Zhang, X., Shao, S., Stanley, H. E. and Havlin, S. (2014). Dynamic motifs in socio-economic networks. EPL (Europhysics Letters) 108 58001.

[57] Zhao, Z., Chen, L. and Lin, L. (2019). Change-point detection in dynamic networks via graphon estimation. arXiv preprint arXiv:1908.01823.
