A noncommutative version of the Banach-Stone theorem (II).

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Abstract

In this paper, we extend the Banach-Stone theorem to the non commutative case, i.e, we give a partial answer to the question 2.1 of [13], and we prove that the structure of the postliminal $C^*$-algebras $\mathcal{A}$ determines the topology of its primitive ideals space.

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1 Introduction

Let $X$ be a Banach space and let $C(S, X)$ ($C(S)$) denote the space of $X$-valued (Scalar-valued) continuous functions on a compact hausdorff space $S$ (endowed with the sup-norm). The classical Banach-Stone theorem states that the existence of an isometric isomorphism from $C(S)$ onto $C(S)$ implies that $S$ and $S'$ are homeomorphic. There exists a variety of results in the literature linking the topological structure of a topological space $X$ with algebraic or topological-algebraic structures of $C(X)$, the set of all continuous real functions on $X$. Further results along this line were obtained by Hewitt [1] and Shirota [2]. They proved respectively that, for a realcompact space $X$, the topology of $X$ is determined by the ring structure of $C(X)$ and by the lattice structure of $C(X)$. Moreover, Shirota proved in [2] that the lattices $UC(X)$ and $UB(X)$ determine the topology of a complete metric space $X$, where $UC(X)$ denotes the family of all uniformly continuous real functions on $X$, and $UB(X)$ denotes the subfamily of all bounded functions in
Moreover Behrends [3] proved that if the centralizers (for the definition see also [3]) of \( X \) and \( Y \) are one-dimensional then the existence of an isometric isomorphism between \( C(S, X) \) and \( C(S', X) \) implies that \( S \) and \( S' \) are homeomorphic. Cambern [4] proved that if \( X \) is finite-dimensional Hilbert space and if \( \Psi \) is an isomorphism of \( C(S, X) \) onto \( C(S', X) \) with \( \| \Psi \Psi^{-1} \| < \sqrt{2} \) then \( S \) and \( S' \) are homeomorphic. In [5] Jarosz proved that there is an isometric isomorphism between \( C(S, X) \) and \( C(S', X) \) with a small bound iff \( S \) and \( S' \) are homeomorphic.

In the last few years there has been interest in the connection between the uniformity of a metric space \( X \) and some further structures over \( UC(X) \) and \( UB(X) \). Thus, Araujo and Font in [6], using some results by Lacruz and Llavona [7], proved that the metric linear structure of \( UB(X) \) endowed with the sup-norm determines the uniformity of \( X \), in the case that \( X \) is the unit ball of a Banach space. This result has been extended to any complete metric space \( X \) by Hernández [8]. Garrido and Jaramillo in [9] proved that the uniformity of a complete metric space \( X \) is indeed characterized not only by \( UB(X) \) but also \( UC(X) \). In [13] we proved that the structure of the liminal \( C^* \)-algebra \( A \) determines the topology of its primitive ideal \( \text{Prim}(A) \).

In this note, considering a \( C^* \)-algebra \( A \) and the space of primitive ideals \( \text{Prim}(A) \), we prove that the structure of the postliminal \( C^* \)-algebra \( A \) determines the topology of its primitive ideals.

### 1.1 The hull kernel topology

The topology on \( \text{Prim}(A) \) (The space of all primitive ideals of \( A \)) is given by means of a closure operation. Given any subset \( W \) of \( \text{Prim}(A) \), the closure \( \overline{W} \) of \( W \) is by definition the set of all elements in \( \text{Prim}(A) \) containing \( \cap W = \{ \cap I : I \in W \} \), namely

\[
\overline{W} = \{ I \in \text{Prim}(A) : I \supseteq \cap W \}
\]

It follows that the closure operation defines a topology on \( \text{Prim}(A) \) which called Jacobson topology or hull kernel topology (see [10]).

**Proposition 1.1:** [12] The space \( \text{Prim}(A) \) is a \( T_0 \)-space, i.e. for any two distinct points of the space there is an open neighborhood of one of the points which does not contain the other.

**Proposition 1.2:** [12] If \( A \) is a \( C^* \)-algebra, then \( \text{Prim}(A) \) is locally compact. If \( A \) has a unit, \( \text{Prim}(A) \) is compact.

**Remark 1.1:** The set of \( \mathcal{K}(H) \) of all compact operators on the Hilbert space \( H \) is the largest two sided ideal in the \( C^* \)-algebra \( B(H) \) of all bounded operators.

**Definition 1.1:** A \( C^* \)-algebra \( A \) is said to be liminal if for every irreducible representation \( (\pi, \mathcal{H}) \) of \( A \), one has \( \pi(A) = \mathcal{K}(\mathcal{H}) \).
So, the algebra $\mathcal{A}$ is liminal if it is mapped to the algebra of compact operators under any irreducible representation. Furthermore, if $\mathcal{A}$ is a liminal algebra, then one can prove that each primitive ideal of $\mathcal{A}$ is automatically a maximal closed two-sided ideal. As a consequence, all points of $\text{Prim}(\mathcal{A})$ are closed and $\text{Prim}(\mathcal{A})$ is a $T_1$-space. In particular, every commutative $C^*$-algebra is liminal.

**Definition 1.2:** A $C^*$-algebra $\mathcal{A}$ is said to be postliminal if for every irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ one has $\mathcal{K}(\mathcal{H}) \subset \pi(\mathcal{A})$.

**Remark 1.2:** Every liminal $C^*$-algebra is postliminal but the converse is not true. Postliminal algebras have the remarkable property that their irreducible representations are completely characterized by the kernels: if $\pi_1$ and $\pi_2$ are two irreducible representations with the same kernel, then $\pi_1$ and $\pi_2$ are equivalent, and the space $\mathcal{A}$ and $\text{Prim}\mathcal{A}$ are homeomorphic.

## 2 The main result

In this section, we extend the Banach-Stone Theorem to postliminal $C^*$-algebras. Before we give some lemma.

**Lemma 2.1:** Let $\mathcal{A}$ and $\mathcal{B}$ be postliminal $C^*$-algebras and let $\alpha$ be an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. If $I$ is a primitive ideal of $\mathcal{B}$, then $\alpha^{-1}(I)$ is a primitive ideal of $\mathcal{A}$.

**Proof** It is clear that the kernel of $\pi \circ \alpha$ (representation of $\mathcal{A}$) is $\alpha^{-1}(I_\pi)$, where $I_\pi$ is a primitive ideal of $\mathcal{B}$.

Now, we prove that $\pi \circ \alpha$ is an irreducible representation of $\mathcal{A}$. If, contrary there exists a $\Pi(\mathcal{A})$-invariant subspace $K$ of Hilbert space $H$ ($K \neq 0, K \neq H$), a sample calcule show that $K$ is $\pi(\mathcal{B})$-invariant and $\pi$ is not a irreducible representation of $\mathcal{B}$. This is a contradiction, and we conclude that $\alpha^{-1}(I_\pi)$ is a primitive ideal of $\mathcal{A}$.

**Theorem 2.1:** Let $\mathcal{A}$ and $\mathcal{B}$ be postliminal $C^*$-algebras and let $\alpha$ be an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. If $I$ is a primitive ideal of $\mathcal{B}$, then $\alpha^{-1}(I)$ is a primitive ideal of $\mathcal{A}$. The map $I \rightarrow \alpha^{-1}(I)$ is a homeomorphism of $\text{Prim}(\mathcal{B})$ onto $\text{Prim}(\mathcal{A})$.

**Proof** Let $I_\pi$ be a primitive ideal of $\mathcal{A}$ for some $\pi \in \mathcal{A}$. From Lemma 2.1 $\alpha^{-1}(I_\pi)$ is a primitive ideal, then there is a function $h$:

$$h : \text{Prim}(\mathcal{B}) \rightarrow \text{Prim}(\mathcal{A})$$

such that $\alpha^{-1}(I_\pi) = I_{h(\pi)}$.

Since we can replace $\alpha^{-1}$ by $\alpha$, it follows that $h$ is a bijection. We have induced homomorphisms $\chi_\pi : \mathcal{A}/I_\pi \rightarrow \mathcal{B}(\mathcal{H})$ given by $\chi_\pi(a) =$
\(\pi(a)\) and \(\beta : A/I_\pi \to B/I_{h^{-1}(\pi)}\) given by \(\beta(a + I_\pi) = \alpha(a) + I_{h^{-1}(\pi)}\). Therefore we get a commutative diagram:

\[
\begin{array}{ccc}
A/I_\pi & \xrightarrow{\beta} & B/I_{h^{-1}(\pi)} \\
\downarrow{\chi_\pi} & & \downarrow{\chi_{h^{-1}(\pi)}} \\
B(H) & \xrightarrow{\gamma} & B(H)
\end{array}
\]

and an induced automorphism \(\gamma : B(H) \to B(H)\) defined by

\[
\gamma(\pi(a)) = h^{-1}(\pi)(\alpha(a))
\]

All open set of \(\text{Prim}(A)\) are of the form:

\[
U_I = \{ P \in \text{Prim}(A) : P \not\supseteq I \}
\]

Computation of \(h^{-1}(U_I)\):

\[
\begin{align*}
h^{-1}(U_I) &= \{ \pi : \text{ker}(\pi) \in \text{Prim}(A) \text{ and } \text{ker}(\pi) \not\supseteq I \} \\
&= \{ h^{-1}(\pi) : \text{ker}(\pi) \in \text{Prim}(A) \text{ and } \text{ker}(\pi) \not\supseteq I \} \\
&= \{ \pi' : \text{ker}(\pi') \in \text{Prim}(A) \text{ and } \text{ker}(h(\pi')) \not\supseteq I \} \\
&= \{ \pi' : \text{ker}(\pi') \in \text{Prim}(A) \text{ and } \alpha^{-1}(I_\pi) \not\supseteq I \} \\
&= U_{\alpha(I)}.
\end{align*}
\]

Then \(h^{-1}(U_I)\) is an open set and \(h\) is continuous. Replace \(\alpha\) by \(\alpha^{-1}\), it follows that \(h^{-1}\) is continuous. So, \(h\) is a homeomorphism.

We give now a corollary to our principal result.

**Corollary 2.1:** Let \(A\) be postliminary \(C^*\)-algebra. If \(\alpha\) is an isomorphism of \(A\) onto \(B\), there is an homeomorphism \(h\) from \(\text{Prim}(B)\) to \(\text{Prim}(A)\) and two unitary operators \(U, V \in B(H)\) for some \(H\) such that:

\[
U\pi(a)V = h^{-1}(\pi)(\alpha(a)) \quad \forall a \in A \text{ and } \text{Ker}(\pi) \in \text{Prim}(A)
\]

**Proof** From theorem 2.1, if \(\alpha\) is surjective isometry, then there is an homeomorphism \(h\) from \(\text{Prim}(B)\) to \(\text{Prim}(A)\) and \(\gamma \in \text{Aut}(B(H))\) such that:

\[
\gamma(\pi(a)) = h^{-1}(\pi)(\alpha(a)) \quad \forall a \in A \text{ and } \text{Ker}(\pi) \in \text{Prim}(A)
\]

and from [14, Theorem 4] there are two unitary operators \(U, V \in B(H)\) such that \(\gamma\) is of the form:

\[
\gamma(A) = UAV
\]

then \(U\pi(a)V = h^{-1}(\pi)(\alpha(a)) \quad \forall a \in A \text{ and } \text{Ker}(\pi) \in \text{Prim}(A)\).
References

[1] Hewitt E (1948) Rings of real-valued continuous functions I. Trans. Amer. Math. Soc. 64: 54–99

[2] Shirota, T (1952) A generalization of a theorem of I. Kaplansky. Osaka Math. J. 4: 121–132

[3] Behrends, E (1979) M-Structure and the Banach-Stone theorem. Lect. Notes in Math. 736, Springer-Verlag, Berlin.

[4] Cambern, M (1967) Isomorphisms of space of continuous vector-valued functions. Illinois J. Math. 20: 1–11

[5] Jarosz, K (1982) A generalization of the Banach-Stone theorem. Studia Mathematica, T. LXXIII: 33–39

[6] Araujo J, Font JJ (2000) Linear isometries on subalgebras of uniformly continuous functions. Proc. Edinburgh Math. Soc. 43: 139–147

[7] Lacruz M, Llavona JG (1997) Composition operators between algebras of uniformly continuous functions. Arch. Math. 69: 52–56

[8] Hernández S (1999) Uniformly continuous mappings defined by isometries of spaces of bounded uniformly continuous functions. Topology Atlas # 394

[9] Garrido M I, Jaramillo A J (2000) A Banach-Stone theorem for uniformly continuous functions. Monatshefte Fur Mathematik # 131, 189–192

[10] Dixmier J (1969) Les C*-algèbres et leurs représentations, Gauthier-Villars éditeur, Paris.

[11] Dixmier J (1977) C*-algèbres, North-Holland, New York.

[12] Landi G (1997) An introduction to noncommutative spaces and their geometry, Dipartimento di Scienze Matematiche, Università di Trieste, Italia.

[13] Bouali B (2001) A noncommutative version of the Banach-Stone theorem, preprint.

[14] Molnár L (2000) Some characterisations of the automorphisms of $B(H)$ and $C(X)$, [arXiv:math.FA/0011030] 4 Nov 2000.

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