THE CRITICAL EXPONENT IS COMPUTABLE FOR
AUTOMATIC SEQUENCES

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The critical exponent of an infinite word is defined to be the supremum of the exponent of each of its factors. For \( k \)-automatic sequences, we show that this critical exponent is always either a rational number or infinite, and its value is computable. Our results also apply to variants of the critical exponent, such as the initial critical exponent of Berthé, Holton, and Zamboni and the Diophantine exponent of Adamczewski and Bugeaud. Our work generalizes or recovers previous results of Krieger and others, and is applicable to other situations; e.g., the computation of the optimal recurrence constant for a linearly recurrent \( k \)-automatic sequence.

1. Introduction

Let \( a = (a(n))_{n \geq 0} \) be an infinite sequence (or infinite word) over a finite alphabet \( \Delta \). We write \( a[i] = a(i) \), and for \( i, n \geq 0 \) we let \( a[i..i+n-1] \) denote the factor of length \( n \) beginning at position \( i \).

If a finite word \( w \) is expressed in the form \( x^n x' \), where \( n \geq 1 \) and \( x' \) is a prefix of \( x \), then we say that \( w \) has period \( x \) and exponent \( |w|/|x| \). The shortest such period is called the period and the largest such exponent is called the exponent and is denoted \( \exp(w) \). For example, the period of \( \text{alfalfa} \) is \( \text{alf} \) and \( \exp(\text{alfalfa}) = 7/3 \). The critical exponent of an infinite word \( a \) is defined to be the supremum, over all nonempty factors \( w \) of \( a \), of the exponent of \( w \); it is denoted by \( c(a) \). It is possible for the critical exponent \( c(a) \) to be rational, irrational, or infinite. If it is rational, it is possible for \( c(a) \) to be attained by some finite factor of \( a \), or not attained by any finite factor.

Critical exponents are an active subject of study. Here are just a few examples.

Example 1. Consider the Thue-Morse sequence

\[
\text{t} = 011010011001011010\ldots,
\]

where \( t[i] \) is the sum, modulo 2, of the digits in the binary expansion of \( i \). Alternatively, \( t \) is the fixed point, starting with 0, of the morphism \( \mu \) defined by 0 \( \rightarrow \) 01 and 1 \( \rightarrow \) 10.

As is well-known, \( t \) contains no overlaps, that is, no factors of the form \( axaxa \), where \( a \in \{0, 1\} \) and \( x \in \{0, 1\}^* \). On the other hand, \( t \) contains square factors such as 00. It follows that the critical exponent of \( t \) is 2, and this exponent is attained by a factor of \( t \).
Example 2. The sequence 0000⋯ clearly has a critical exponent of ∞, as does any ultimately periodic word.

Example 3. The Rudin-Shapiro sequence $r = (r_n)_{n \geq 0} = 0001001000011101\cdots$ counts the number of (possibly overlapping) occurrences of 11, modulo 2, in the base-2 expansion of $n$. Its critical exponent is 4 and it is attained by, for example, the factor 0000 beginning at position 7; see [3].

Example 4. The sequence $c = 2102012101202102012021012102012\cdots$, which counts the number of 1’s between consecutive occurrences of 0 in $t$, is well-known to be squarefree. However, since $t$ contains arbitrarily large squares — for example, the squares $\mu^n(00)$ — it follows that $c$ contains factors of exponent arbitrarily close to 2. Thus its critical exponent is 2, but this is not attained by any finite factor.

Example 5. Consider the Fibonacci word

$$f = 01001001001001001001001001001\cdots,$$

defined to be the fixed point of the morphism $0 \to 01$ and $1 \to 0$. Then Mignosi and Pirillo [17] proved that the critical exponent of $f$ is $(5 + \sqrt{5})/2$, an irrational number.

Example 6. In fact, every real number greater than 1 is the critical exponent of some infinite word [15], and every real number $\geq 2$ is the critical exponent of some infinite binary word [11].

Krieger [12,13,14] showed (among other things) that if an infinite sequence is given as the fixed point of a uniform morphism, then its critical exponent is either infinite or a rational number.

In this paper we generalize this result to the case of $k$-automatic sequences. An infinite sequence $a$ is said to be $k$-automatic for some integer $k \geq 2$ if it is computable by a finite automaton taking as input the base-$k$ representation of $n$, and having $a[n]$ as the output associated with the last state encountered; see, for example, [5,10].

For example, in Figure 1, we see an automaton generating the Thue-Morse sequence $t = t_0t_1t_2\cdots = 011010011001\cdots$. The input is $n$, expressed in base 2, and the output is the number contained in the state last reached.

As is well-known, the class of $k$-automatic sequences is slightly more general than the class of fixed points of uniform morphisms; the former also includes words that can be written as the image, under a coding, of fixed points of uniform morphisms [10]. An example of a word that is 2-automatic but not the fixed point of any uniform morphism is the Rudin-Shapiro sequence $r$, discussed above in Example 3.

(Since this fact does not seem to have been explicitly proved before, we sketch the proof. We know that $r$ is 2-automatic. If $r$ were the fixed point of a $k$-uniform morphism for some $k$ not a power of 2, then by Cobham’s celebrated theorem [9], $r$ would be ultimately periodic, which it is not (since its critical exponent is 4). So it
must be the fixed point of a morphism \( h \) that is \( 2^k \)-uniform for some \( k \geq 1 \). Now \( r \) starts \( 00 \); if \( r = h(r) \) then \( r \) starts with \( h(0)h(0) \). This means \( r[2^k-1] = r[2^{k+1}-1] \).

But clearly the number of occurrences of \( 11 \) in \( 2^k-1 \) is one less than the number of occurrence of \( 11 \) in \( 2^{k+1}-1 \), a contradiction.

Allouche, Rampersad, and Shallit [4] proved that the question

Given a rational number \( r > 1 \), is \( a \) \( r \)-power-free?

is recursively solvable for \( k \)-automatic sequences \( a \). More recently, Charlier, Rampersad, and Shallit [8] showed that

Given \( a \), is its critical exponent infinite?

also has a recursive solution for \( k \)-automatic sequences.

In this paper we show, generalizing some of the results of Krieger mentioned above, that the critical exponent of a \( k \)-automatic sequence is always either rational or infinite. Furthermore, we show that the question

Given \( a \), what is its critical exponent?

is recursively computable for \( k \)-automatic sequences.

There are a number of variants of the critical exponent for infinite words \( a \). For example, instead of taking the supremum of \( \exp(w) \) over all factors \( w \) of \( a \), we could take it over only those factors that occur infinitely often. Or, letting \( x^\beta \) for real \( \beta \geq 1 \) denote the shortest prefix of \( x^x \) of length \( \geq \beta |x| \), we could take the supremum over all real numbers \( \beta \) such that there are arbitrarily large factors of \( a \) of the form \( x^\beta \). We could also restrict our attention to prefixes instead of factors. It turns out that for all of these variants, the resulting critical exponent of automatic sequences is either rational or infinite, and is computable.

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2. Two-dimensional automata

In this paper, we always assume that numbers are encoded in base \( k \) using the digits in \( \Sigma_k = \{0, 1, \ldots, k-1\} \).
The canonical encoding of \( n \) is the one with no leading zeroes, and is denoted \( (n)_k \). Thus, for example, we have \((43)_2 = 101011\). Similarly, if \( w \) is a word over \( \Sigma_k \), then \([w]_k \) denotes the integer represented by \( w \) in base \( k \). Thus \([101011]_2 = 43\).

We will need to encode pairs of integers. We handle these by first padding the representation of the smaller integer with leading zeroes, so it has the same length as the larger one, and then coding the pair as a word over \( \Sigma_k^2 \). This gives the canonical encoding of a pair \((m, n)\), and is denoted \((m, n)_k \). Note that the canonical encoding of a pair does not begin with a symbol that has 0 in both components. For example, the canonical representation of the pair \((20, 13)\) in base 2 is

\[
[1, 0][0, 1][1, 1][0, 0][0, 1],
\]

where the first components spell out \(10100\) and the second components spell out \(01101\).

Given a finite word \( x \in (\Sigma_k^2)^* \), we define the projections \( \pi_i(x) (i = 1, 2) \) onto the \( i \)'th coordinate. Given a finite word \( x \) with \([\pi_2(x)]_k \neq 0\), we define

\[
\text{quo}_k(x) = [\pi_1(x)]_k / [\pi_2(x)]_k
\]

Thus \( \text{quo}_k(x) \) maps words of \((\Sigma_k^2)^*\) to the non-negative rational numbers \(\mathbb{Q}^{\geq 0}\). (We assume, without loss of generality, that no denominator is 0.) For example, \(\text{quo}_2([[1, 0][1, 1][0, 1]]) = 6/3 = 2\). If \( L \subseteq (\Sigma_k^2)^* \), we define \(\text{quo}_k(L) = \{\text{quo}_k(x) : x \in L\} \).

Usually we will assume that the base-\( k \) representation is given with the most significant digit first, but sometimes, as in the following result, it is easier to deal with the reversed representations, where the least significant digit appears first (and shorter representations, if necessary, are padded with trailing zeroes). Since the class of regular languages is (effectively) closed under the map \(L \to L^R\) that sends a regular language to its reversal, this distinction is not crucial to our results, and we will not emphasize it unduly.

**Lemma 7.** Let \( \beta \) be a non-negative real number and define the languages

\[
L_{\leq \beta} = \{x \in (\Sigma_k^2)^* : \text{quo}_k(x) \leq \beta\},
\]

and analogously for the relations \(<, =, \geq, >, \neq\).

(a) If \( \beta \) is a rational number, then the language \( L_{\leq \beta} \) (resp., \( L_{< \beta}, L_{= \beta}, L_{\geq \beta}, L_{> \beta}, L_{\neq \beta} \)) is regular.

(b) If \( L_{\leq \beta} \) (resp., \( L_{< \beta}, L_{\geq \beta}, L_{> \beta} \)) is regular, then \( \beta \) is a rational number.

**Proof.** We handle only the case \( L_{\leq \beta} \), the others being similar.

Suppose \( \beta \) is rational. Then we can write \( \beta = P/Q \) for integers \( P \geq 0, Q \geq 1 \). On input \( x \) representing a pair of integers \( (p, q) \) in the reversed base-\( k \) representation, we need to accept iff \( p/q \leq P/Q \), that is, iff \( pQ \leq qP \). To do so, we simply transduce \( p \) and \( q \) to \( pQ \) and \( qP \) on the fly, respectively, and compare them digit-by-digit. Minor
complications arise if the base-$k$ expansions of $pQ$ and $qP$ have different numbers of digits. To handle this, we accept if some path ending in $[0,0]^i$ leads to an accepting condition. This construction was already given in [4], and the details can be found there.

For the other direction, we use ordinary (most-significant-digit first) representation. Without loss of generality, we can assume $1/k \leq \beta < 1; if not, we can ensure this condition holds by modifying the automaton, shifting one coordinate to the left or right. Take $L \leq k \leq \beta$ and intersect with the (regular) language of words whose second coordinates are of the form $10^*$; then project onto the first coordinate to get $L''$, a regular language over $\Sigma_k$. Now take the lexicographically largest word of each length in $L''$ to get $L''$; by a well-known result (e.g., [20]), this language is also regular. But $L''$ has exactly one word of each length, so by another well-known result (e.g., [16,18,20]), $L''$ must be a finite union of languages of the form $uv^*w$.

3. Computing the sup

In this section, we show that if $L \subseteq (\Sigma_k^2)^*$ is a regular language, then $\alpha := \sup \ quo_k(L)$ is either rational or infinite, and in both cases it is computable.

First, we handle the case where the sup is infinite.

Theorem 8. Let $L \subseteq (\Sigma_k^2)^*$ be a language accepted by a DFA with $n$ states. Assume that no word of $L$ contains leading 0's. Then $\sup \ quo_k(L) = \infty$ if and only if $\quo_k(L) \cap I[k^n, \infty)$ is nonempty.

Proof. If $\sup \ quo_k(L) = \infty$, then clearly $\quo_k(L) \cap I[k^n, \infty)$ is nonempty.

For the other direction, suppose $\quo_k(L) \cap I[k^n, \infty)$ is nonempty. Then $(p,q)_k \in L$ for some integers $p, q$ with $p \geq k^nq$. Writing $x = (p,q)_k$, we have $|x| \geq n$, so we can apply the pumping lemma, writing $x = uvw$ with $|uv| \leq n$ and $|v| \geq 1$, and $uv^iw \in L$ for all $i \geq 0$. Since $p \geq k^nq$, we must have $[\pi_2(uv)]_k = 0$. Since $x$ doesn’t start with $[0,0]$, we have $[\pi_1(uv^iw)]_k \rightarrow \infty$. Hence $\quo_k(uv^iw) \rightarrow \infty$.

Corollary 9. There is an algorithm that, given a DFA $M$ accepting a regular language $L \subseteq (\Sigma_k^2)^*$, decides if $\sup \ quo_k(L(M)) = \infty$.

Proof. We can find a DFA $M'$ accepting $L$ with all leading 0’s removed from words. If $M'$ has $n$ states, then we can compute a DFA $M''$ accepting $L(M') \cap L \geq k^n$, and we can decide if $M''$ accepts anything.

Next, we turn to the case where the sup is finite. We start with two useful lemmas. The first is the classical mediant inequality.

Lemma 10. Let $a, b, c, d$ be non-negative real numbers with $c,d \neq 0$ and $\frac{a}{c} < \frac{b}{d}$. Then $\frac{a}{c} < \frac{a+b}{c+d} < \frac{b}{d}$.
The next is a fundamental inequality for quo.

**Lemma 11.** Let $u, v, w \in (\Sigma^2)^*$ such that $|v| \geq 1$, and such that $[\pi_1(uvw)]_k$ and $[\pi_2(uvw)]_k$ are not both 0. Define

$$\gamma(u, v) := \frac{[\pi_1(uv)]_k - [\pi_1(u)]_k}{[\pi_2(uv)]_k - [\pi_2(u)]_k}$$

and

$$U := \begin{cases} \text{quo}_k(w), & \text{if } [\pi_1(uv)]_k = [\pi_2(uv)]_k = 0; \\ \infty, & \text{if } [\pi_1(uv)]_k > 0 \text{ and } [\pi_2(uv)]_k = 0; \\ \gamma(u, v), & \text{otherwise}. \end{cases}$$

Then exactly one of the following cases occurs:

(i) $\text{quo}_k(uw) < \text{quo}_k(uvw) < \text{quo}_k(uvw^2w) < \cdots < U$;
(ii) $\text{quo}_k(uw) = \text{quo}_k(uvw) = \text{quo}_k(uvw^2w) = \cdots = U$;
(iii) $\text{quo}_k(uw) > \text{quo}_k(uvw) > \text{quo}_k(uvw^2w) > \cdots > U$.

Furthermore, $\lim_{i \to \infty} \text{quo}_k(uv^i w) = U$.

**Proof.** Fix an integer $i \geq 0$. Define

$$A_j := [\pi_j(uv)]_k - [\pi_j(u)]_k$$

and

$$B_j := [\pi_j(uv^i w)]_k$$

for $j = 1, 2$ and define $C := k^{|v|+|w|}$. Then

$$[\pi_j(uv^{i+1} w)]_k = A_j C + B_j$$

for $j = 1, 2$. It follows that

$$\text{quo}_k(uv^{i+1} w) - \text{quo}_k(uv^i w) = \frac{[\pi_1(uv^{i+1} w)]_k - [\pi_1(uv^i w)]_k}{[\pi_2(uv^{i+1} w)]_k - [\pi_2(uv^i w)]_k} = \frac{A_1 C + B_1}{A_2 C + B_2} - \frac{B_1}{B_2}. \quad (3)$$

From the mediant inequality (Lemma [13]) we have

$$\frac{B_1}{B_2} \precsim \frac{A_1}{A_2} \Rightarrow \frac{B_1}{B_2} \precsim \frac{A_1 C + B_1}{A_2 C + B_2} \precsim \frac{A_1}{A_2}$$

where $\precsim$ is any one of the three relations $<, =, >$. In other words,

$$\text{quo}_k(uv^i w) < U \quad \Rightarrow \quad \text{quo}_k(uv^i w) \precsim \text{quo}_k(uv^{i+1} w) \precsim U.$$
The Critical Exponent is Computable for Automatic Sequences

Take $i = 0$ and apply induction to get

$$\exists i \text{ quo}_k (uv^iw) < U \implies \text{ case (i) holds}$$

$$\exists i \text{ quo}_k (uv^iw) = U \implies \text{ case (ii) holds}$$

$$\exists i \text{ quo}_k (uv^iw) > U \implies \text{ case (iii) holds}.$$ 

This proves our first assertion.

We now prove the assertion about the limit. Let $j \in \{1, 2\}$, let $i$ be an integer $\geq 1$, and consider the base-$k$ representation of the rational number

$$\frac{[\pi_j(uv^iw)]_k}{k^{i|v|+|w|}},$$

it looks like

$$\pi_j(u).\pi_j(v)\pi_j(v)\cdots \pi_j(v)\pi_j(w).$$

On the other hand, the base-$k$ representation of

$$[\pi_j(u)]_k + \frac{[\pi_j(v)]_k}{k^{i|v|} - 1}$$

looks like

$$\pi_j(u).\pi_j(v)\pi_j(v)\cdots .$$

Subtracting, we get

$$\left| \frac{[\pi_j(uv^iw)]_k}{k^{i|v|+|w|}} - \left( [\pi_j(u)]_k + \frac{[\pi_j(v)]_k}{k^{i|v|} - 1} \right) \right| < k^{-i|v|}.$$ 

It follows that

$$\lim_{i \to \infty} \frac{[\pi_j(uv^iw)]_k}{k^{i|v|+|w|}} = [\pi_j(u)]_k + \frac{[\pi_j(v)]_k}{k^{i|v|} - 1}.$$ 

Furthermore, this limit is 0 if and only if $[\pi_j(uv)]_k = 0$. Hence, provided $[\pi_2(uv)]_k \neq 0$.
0, we get
\[
\lim_{i \to \infty} \frac{\pi_1(uv^i w)}{\pi_2(uv^i w)} = \lim_{i \to \infty} \left( \frac{\pi_1(uv^i w)}{\pi_2(uv^i w)} \right)^k = \lim_{i \to \infty} \left( \frac{\pi_1(uv^i w)}{\pi_2(uv^i w)} \right)^k \cdot \lim_{i \to \infty} \left( \frac{\pi_1(uv^i w)}{\pi_2(uv^i w)} \right)^k = \lim_{i \to \infty} \left( \frac{\pi_1(uv^i w)}{\pi_2(uv^i w)} \right)^k = \left( \frac{\pi_1(u)}{\pi_2(u)} \right)^k = A_1^k.
\]

Theorem 12. Let \( L \subseteq (\Sigma_k^2)^* \) be a regular language. Then \( \alpha := \sup \frac{\pi_1(uv^i w)}{\pi_2(uv^i w)} \) is either infinite or rational.

Proof. Assume that \( \alpha < \infty \). We will show that \( \alpha \) is rational. In fact, we will show something more: suppose the DFA \( M \) has \( n \) states. Then we claim that \( \alpha \in S \), where

\[
S = S_1 \cup S_2
\]

and

\[
S_1 = \{ \frac{\pi_1(x)}{\pi_2(x)} : |x| < n \text{ and } x \in L \}; \quad (4)
\]

\[
S_2 = \{ \gamma(u, v) : |uv| \leq n, |v| \geq 1, \text{ and there exists } w \text{ such that } uvw \in L \}. \quad (5)
\]

and \( \gamma \) is the function defined in (1).

We will assume, without loss of generality, that no word of \( L \) begins with \([0, 0]\).

There are two cases to consider:

Case 1: \( \alpha = \frac{\pi_1(x)}{\pi_2(x)} \) for some \( x \in L \). Without loss of generality we can assume that \( x \) is a shortest word achieving the sup. We now show \( |x| < n \). If \( |x| \geq n \), then, using the pumping lemma for regular languages, we can write \( x = uvw \) with \( |uv| \leq n \) and \( |v| \geq 1 \), such that \( uv^i w \in L \) for all \( i \geq 0 \). Then by Lemma 11 one of the following two cases must occur:

(a) \( \frac{\pi_1(uvw)}{\pi_2(uvw)} < \frac{\pi_1(uw^2v)}{\pi_2(uw^2v)} < \cdots \);

(b) \( \frac{\pi_1(uvw)}{\pi_2(uvw)} \geq \frac{\pi_1(uvw)}{\pi_2(uvw)} \);

In case (b), we find a shorter word (namely, \( uv \)), for which \( \frac{\pi_1(uv)}{\pi_2(uv)} \geq \alpha \), contradicting our assumption that \( x \) was the shortest word achieving the sup. In case
(a) we find a word (namely, \(uv^2w\)) such that \(\text{quo}_k(uv^2w) > \text{quo}_k(x)\), contradicting the fact that \(\text{quo}_k(x) = \sup \text{quo}_k(L)\). Thus \(|x| < n\) and hence \(\text{quo}_k(x) \in S_1\).

Case 2: The sup is not achieved on \(L\). Then there must be an infinite sequence of distinct words \((x_j)_{j \geq 1}\) with each \(x_j \in L\) and \(\text{quo}_k(x_j)\) converging strictly monotonically to \(\alpha\) from below. Without loss of generality, we can assume that each \(x_j\) is of length \(\geq n\) (the number of states of \(M\)) and further that

\[
\text{quo}_k(x_j) \geq \text{quo}_k(y) \quad \text{for each } y \text{ with } |y| \leq |x_j|.
\]

By the pumping lemma, we can write each \(x_j = u_jv_jw_j\) with \(|u_jv_j| \leq n\) and \(|v_j| \geq 1\) such that \(u_jv_jw_j \in L\) for all \(i \geq 0\).

Since \(|u_jv_j| \leq n\), there are only finitely many choices for \(u_jv_j\). By the infinite pigeonhole principle, there is a single choice of \(u_j, v_j\) (say \(u, v\)) corresponding to infinitely many decompositions of the \(x_j\). Let us restrict ourselves to this particular subsequence, which we write as \((x'_j)_{j \geq 1}\). Thus \(x'_j = uvw_j\) for \(j \geq 1\).

Now, appealing once more to Lemma 11, we see that there are two possibilities:

(a) there exists \(j\) such that \(\text{quo}_k(uvw_j) \geq \text{quo}_k(uvw_j)\);
(b) for all \(j \geq 1\) we have \(\text{quo}_k(uvw_j) < \text{quo}_k(uvw_j) < \text{quo}_k(uvw_j) < \cdots\).

In case (a) we have found a shorter word with a quotient at least as large, contradicting our assumption \(i\). Hence case (b) must occur.

Since \(uv^iw_1 \in L\) for all \(i \geq 1\), we have \(\text{quo}_k(uv^iw_1) \leq \alpha\) for all \(i \geq 1\), and hence \(\sup_{i \geq 1} \text{quo}_k(uv^iw_1) \leq \alpha\). On the other hand, \(\sup_{i \geq 1} \text{quo}_k(uv^iw_1) = \lim_{i \to \infty} \text{quo}_k(uv^iw_1) = \gamma(u, v)\) by Lemma 11. It follows that \(\gamma(u, v) \leq \alpha\).

However, for all \(j \geq 1\) we have \(\text{quo}_k(uvw_j) \leq \lim_{i \to \infty} \text{quo}_k(uvw_j) = \gamma(u, v)\). Hence \(\sup_{j \geq 1} \text{quo}_k(uvw_j) \leq \gamma(u, v)\). But \(\alpha = \lim_{j \geq 1} x'_j = \sup_{j \geq 1} \text{quo}_k(uvw_j) \leq \gamma(u, v)\).

Putting these two results together, we see that \(\gamma(u, v) = \alpha\), and hence \(\alpha \in S_2\).

\[\square\]

**Corollary 13.** There is an algorithm that, given a DFA \(M\) accepting \(L \subseteq (\Sigma_\mathcal{A})^*\), will compute \(\alpha = \sup \text{quo}_k(L)\).

**Proof.** Using Corollary 9 we have an algorithm to decide if \(\alpha\) is infinite.

Otherwise, we know from the proof of Theorem 12 that \(\alpha\) lies in \(S_1 \cup S_2\), where \(S_1\) and \(S_2\) are finite sets that we can compute explicitly from \(M\). Furthermore,

\[\alpha = \min \{ \beta \in S_1 \cup S_2 : L(M) \cap L_{\geq \beta} = \emptyset \}.\]

So it suffices to check, for each \(\beta \in S_1 \cup S_2\), if the language \(L(M) \cap L_{> \beta}\) is empty, which can be done using the usual depth-first search techniques on the automaton for the intersection. \[\square\]
4. Computing the largest special point

Let \( L \subseteq (\Sigma_k^2)^* \). We say an extended real number \( \beta \) is a special point of \( \text{quo}_k(L) \) if there exists an infinite sequence \((x_j)_{j \geq 1}\) of distinct words of \( L \) such that \( \lim_{j \to \infty} \text{quo}_k(x_j) = \beta \). Thus a special point is either an accumulation point of \( \text{quo}_k(L) \), or a rational number with infinitely many distinct representations in \( L \).

Note that every infinite language \( L \) has a special point, and indeed, a largest special point.

**Theorem 14.** Let \( L \) be an infinite regular language accepted by a DFA with \( n \) states. Then the largest special point of \( L \) is either infinite or rational.

Before we begin the proof, we need the following lemma.

**Lemma 15.** Let \( u, v \) be fixed words such that \( \lfloor \pi_2(uv) \rfloor_k \neq 0 \) and let \( i \) be a fixed integer. Let \((w_j)_{j \geq 1}\) be a sequence of words. If \( \lim_{j \to \infty} \text{quo}_k(uv^i w_j) = \lim_{j \to \infty} \text{quo}_k(uv^{i+1} w_j) \), then these limits both equal \( \gamma(u, v) \), where \( \gamma \) is defined in (1).

**Proof.** As in the proof of Lemma 11, define \( A_l = \lfloor \pi_l(uv) \rfloor_k - \lfloor \pi_l(u) \rfloor_k \) for \( l = 1, 2 \).

Also define \( D_{l,j} = \lfloor \pi_l(uv^i w_j) \rfloor_k k^{-(i|v|+|w_j|)} \) for \( l = 1, 2 \) and \( j \geq 1 \). Then, using (3), the hypothesis on the limits can be restated as

\[
\forall \epsilon > 0 \exists N \forall j \geq N \left| \frac{A_1 + D_{1,j}}{A_2 + D_{2,j}} - \frac{D_{1,j}}{D_{2,j}} \right| < \epsilon. \tag{7}
\]

Clearing the denominators and simplifying, we see that (7) implies

\[
\forall \epsilon > 0 \exists N \forall j \geq N |A_1 D_{2,j} - A_2 D_{1,j}| < \epsilon(A_2 + D_{2,j}) D_{2,j}. \tag{8}
\]

Dividing by \( A_2 D_{2,j} \), we see that (8) implies

\[
\forall \epsilon > 0 \exists N \forall j \geq N \left| \frac{A_1}{A_2} - \frac{D_{1,j}}{D_{2,j}} \right| < \epsilon \left( 1 + \frac{D_{2,j}}{A_2} \right). \tag{9}
\]

From the hypothesis on \( u, v \) we have \( A_2 \neq 0 \). But \( \frac{A_1}{A_2} = \gamma(u, v) \) and \( \frac{D_{2,j}}{D_{2,j}} = \text{quo}_k(uv^i w_j) \), so (9) can be restated as \( \lim_{j \to \infty} \text{quo}_k(uv^i w_j) = \gamma(u, v) \).

Now we can return to the proof of Theorem 14.

**Proof.** Let \( \alpha \) be the largest special point in \( \text{quo}_k(L) \). Then there is an infinite sequence \((x_j)_{j \geq 1}\) of distinct words of \( L \) such that \( \lim_{j \to \infty} \text{quo}_k(x_j) = \alpha \). We show that if \( \alpha < \infty \) then \( \alpha \in S_2 \), where \( S_2 \) is the set of rationals defined in (2).

Our proof involves considering more and more refined subsequences of the \((x_j)\); by abuse of notation we refer to each of these subsequences as \((x_j)\).
First, we can assume without loss of generality that
\[ n \leq |x_1| < |x_2| < \cdots , \]
where \( n \) is the number of states in the minimal DFA accepting \( L \). Now apply the pumping lemma to each \( x_j \), obtaining the decompositions \( x_j = u_j v_j w_j \) such that \( |u_j v_j| \leq n \) and \( |v_j| \geq 1 \) and \( u_j v_j^i w_j \in L \) for all \( i \geq 0 \). By the infinite pigeonhole principle, there must be some \( u_j v_j \) that occurs infinitely often, so by replacing the \( (x_j) \) with the appropriate subsequence, we can also assume that the pumping lemma in fact gives the decomposition \( x_j = uv^i w_j \) for each \( j \geq 1 \).

Applying Lemma 11, we see that \( \lim_{i \to \infty} \text{quo}_k(uv^i w_j) = \gamma(u, v) \); and further, for each \( j \geq 1 \) we have either

(a) \( \text{quo}_k(uw_j) < \text{quo}_k(uvw_j) < \text{quo}_k(uv^2 w_j) < \cdots < \gamma(u, v) \); or
(b) \( \text{quo}_k(uw_j) \geq \text{quo}_k(uvw_j) \geq \text{quo}_k(uv^2 w_j) \geq \cdots \geq \gamma(u, v) \).

Again, by the infinite pigeonhole principle, at least one of the two options above must occur for infinitely many \( j \), so by restricting to the appropriate subsequence, we can assume that one of the two sets of inequalities applies for all \( j \). We consider both cases in turn.

Case (a): The sequence \( s = (\text{quo}_k(uv^i w_j))_{j \geq 1} \) cannot be unbounded since \( \alpha < \infty \). From the Bolzano-Weierstrass theorem, we know \( s \) has a convergent subsequence, so we can replace \( (w_j) \) with the appropriate subsequence and define \( \beta := \lim_{j \to \infty} \text{quo}_k(uv^2 w_j) \). Then \( \text{quo}_k(uvw_j) < \text{quo}_k(uv^2 w_j) \), so
\[
\alpha = \lim_{j \to \infty} \text{quo}_k(uvw_j) \leq \lim_{j \to \infty} \text{quo}_k(uv^2 w_j) = \beta .
\]
On the other hand, \( \beta \) is a special point, so \( \beta \leq \alpha \). Therefore \( \alpha = \beta \) and Lemma 16 applies, giving \( \alpha = \beta = \gamma(u, v) \).

Case (b): Just like Case (a), except now we consider the sequence \( s = (\text{quo}_k(uw_j))_{j \geq 1} \) instead.

In both cases, then, we have shown that \( \alpha = \gamma(u, v) \in S_2 \), and so \( \alpha \) is rational.

\[ \square \]

**Corollary 16.** There is an algorithm that, given a DFA \( M \) accepting an infinite language \( L \subseteq (\Sigma_k^2)^* \), will compute the largest special point in \( \text{quo}_k(L) \).

**Proof.** In Theorem 14 we showed that the largest special point is either \( \infty \) or contained in the set
\[ S_2 = \{ \gamma(u, v) : |uw| \leq n, |v| \geq 1, \text{ and there exists } w \text{ such that } uvw \in L \} . \]
The former case occurs iff \( \sup \text{quo}_k(L) = \infty \), which can be checked using Corollary 9.

Otherwise, we (effectively) can list the (finite number of) elements of \( S_2 \). For each \( \beta \in S_2 \), we can check to see if \( \beta \) is an accumulation point using [19], Thm. 24.
We can also check to see if \( \beta \) has infinitely many representations in \( L \) by computing a DFA accepting \( L \cap L_{=\beta} \) and then using the usual method involving cycle detection via depth-first search. Now \( \alpha \) is the largest such \( \beta \) for which one of the two conditions applies.

Every accumulation point is a special point, but the converse is not necessarily true. If, however, our language \( L \) has a certain natural property, then the converse holds.

**Theorem 17.** Let \( L \subseteq (\Sigma_2^*)^* \) be a language such that

- (a) No word of \( L \) has leading \([0,0]'s;
- (b) \( \text{quo}_k(L) \) has infinite cardinality;
- (c) If \( (p,q)_k \in L \), then \( p \geq q \);
- (d) If \( (p,q)_k \in L \) and \( p > q \) then \( (p-1,q)_k \in L \).

Then every special point, except perhaps 1, is an accumulation point. Furthermore, if (a)–(d) hold, the largest accumulation point (that is, \( \limsup \text{quo}_k(L) \)) is rational or infinite, and is computable.

**Proof.** Suppose the conclusion is false. Then there is a special point \( \alpha > 1 \) that is not an accumulation point. Then there are infinitely many representations of \( \alpha \) in \( L \); choose a sequence of these \( (x_i)_{i \geq 1} \) of increasing length. Writing \( x_i = (p_i,q_i)_k \) with \( p_i > q_i \), by hypothesis we get \( (p_i-1,q_i)_k \in L \). But evidently \( \lim_{i \to \infty} (p_i-1)/q_i = \alpha \), so \( \alpha \) is an accumulation point, a contradiction.

The results on rationality and computability now follow from Theorem 14 and Corollary 16.

5. Application to the critical exponent and its variants

We can now apply the results of Sections 3 and 4 to the critical exponent problem.

**Theorem 18.** The critical exponent of a \( k \)-automatic sequence is either rational or infinite, and is effectively computable.

**Proof.** Given a \( k \)-automatic sequence \( a = (a_i)_{i \geq 0} \), we can, using the techniques of [4],[8], (effectively) create a two-dimensional DFA \( M \) accepting

\[
L' = \{(p,q)_k : \exists \text{ a factor of } a \text{ of length } q \text{ with period } p \}
\]

\[
= \{(p,q)_k : \exists i \text{ such that } a[i..i+q-p-1] = a[i+p..i+q-1]\}
\]

\[
= \{(p,q)_k : \exists i \forall j, 0 \leq j < q-p \text{ we have } a[i] = a[i+p]\}.
\]

Then the critical exponent of \( a \) is \( \sup \text{quo}_k(L') \), which, by Theorem 12 is rational or infinite. The infinite case has already been handled in [8] (or we could use Corollary 9). If it is finite, Corollary 13 tells us how to compute it from \( M \).
The Critical Exponent is Computable for Automatic Sequences

The same results also hold for the variant of the critical exponent when the sup is taken over only the factors that occur infinitely often.

**Theorem 19.** If \( a = (a_i)_{i \geq 0} \) is a \( k \)-automatic sequence, the quantity 
\[
c_1(a) := \sup\{\exp(w) : w \text{ is a finite factor of } a \text{ that occurs infinitely often}\}
\]
is either rational or infinite, and is computable.

**Proof.** To see this, it is only necessary to change the appropriate two-dimensional DFA to accept
\[
L'' = \{ (p, q)_k : \exists i \text{ such that } a[i..i + q - p - 1] = a[i + p..i + q - 1] \\
\text{ and for all } j \text{ such that } a[i..i + q - 1] = a[j..j + q - 1] \\
\text{ there exists } \ell > j \text{ such that } a[i..i + q - 1] = a[\ell..\ell + q - 1]\} \quad (11)
\]
The first clause says that the factor of length \( q \) at position \( i \) has period \( p \), and the other two clauses say that if some factor equals this one, then there is another occurrence of that factor further on.

Now apply Theorem 12 and Corollary 13 to \( L'' \).

The same results also hold for the variant where we only consider exponents \( \beta \) that work for arbitrarily large factors. This corresponds precisely to our notion of special point introduced above, so that the largest special point of \( \text{quo}_k(L') \) gives the supremum over all such \( \beta \).

**Theorem 20.** If \( a = (a_i)_{i \geq 0} \) is a \( k \)-automatic sequence, the quantity 
\[
c_2(a) := \sup\{\beta : \forall N \geq 1 \exists w \text{ with } |w| \geq N \text{ and } w^\beta \text{ a factor of } a\}
\]
is either rational or infinite, and is computable.

**Proof.** Consider \( L' \) as defined in the proof of Theorem 18. Then \( c_2(a) \) is the largest special point in \( \text{quo}_k(L') \). Now apply Theorem 14 and Corollary 16.

Yet another variation is the initial critical exponent \( \text{ice}_1(a) \), introduced in [9] (also see [1]), where the supremum of exponents is taken over all prefixes of a given infinite word, as opposed to all factors. There is also the variant \( \text{ice}_2 \), where we consider only the exponents that work for arbitrarily large prefixes. Our results also apply to both these cases.

**Theorem 21.** Let \( \text{ice}_1(a) = \sup\{\exp(w) : w \text{ is a prefix of } a\} \) and let \( \text{ice}_2(a) = \sup\{\beta : \forall N \geq 1 \exists w \text{ with } |w| \geq N \text{ and } w^\beta \text{ a prefix of } a\} \). If \( a \) is a \( k \)-automatic sequence, then both the quantities \( \text{ice}_1(a) \) and \( \text{ice}_2(a) \) are either rational or infinite, and are computable.
Proof. Here the proof is exactly like the proofs of Theorems 18 and 20, with the difference that we replace $L'$ defined in the proof of Theorem 18 with the language $L_i = \{ (p, q) \in \mathbb{N}^2 : a[0..q - p - 1] = a[p..q - 1] \}$.

Then \( \text{ice}_1(a) = \sup \{ \text{quo}_k(L_i) \} \), while \( \text{ice}_2(a) \) is the largest special point in \( \text{quo}_k(L_i) \).

Finally, our results also apply to the so-called Diophantine exponent \( \text{Dio}(a) \), introduced in [2] (also see [1]). It is defined as the supremum of real numbers \( \beta \) for which there exist arbitrarily long prefixes of \( a \) that can be expressed in the form \( uv^\tau \) for some real number \( \tau \) and finite words \( u, v \) such that \( |uv^\tau|/|uv| \geq \beta \). The following results complement those in [7].

Theorem 22. If \( a \) is a \( k \)-automatic sequence, then \( \text{Dio}(a) \) is either rational or infinite, and is computable.

Proof. Here we follow the proof of Theorem 20 once more, except now we work with the language $L_d = \{ ((i+\ell, i+p))_k : \exists i \geq 0, \ell \geq p \geq 1 \text{ such that } a[i..i+\ell-p-1] = a[i+p..i+\ell-1] \}$.

The claims now follow from Theorem 14 and Corollary 16.

6. Other applications

Theorem 12 and Corollary 13 have applications to other problems.

A sequence \( a \) is said to be recurrent if every factor that occurs, occurs infinitely often. It is linearly recurrent if there exists a constant \( C \) such that for all \( \ell \geq 0 \), and all factors \( x \) of length \( \ell \) occurring in \( a \), any two consecutive occurrences of \( x \) are separated by at most \( C\ell \) positions.

Theorem 23. It is decidable if a \( k \)-automatic sequence \( a \) is linearly recurrent. If \( a \) is linearly recurrent, the optimal constant \( C \) is computable.

Proof. First, as in [8], we construct an automaton accepting the language $L = \{ (n, \ell)_k :$

(a) there exists \( i \geq 0 \text{ s. t. } a[i+j] = a[i+n+j] \text{ for all } j, 0 \leq j < \ell, \text{ and} \)

(b) there exists \( t, 0 < t < n \text{ s. t. } a[i+j] = a[i+t+j] \text{ for all } j, 0 \leq j < \ell \} \)

Another way to say this is that \( L \) consists of the base-\( k \) representation of those pairs of integers \( (n, \ell) \) such that (a) there is some factor of length \( \ell \) for which there is another occurrence at distance \( n \) and (b) this occurrence is actually the very next occurrence.

Now from Theorem 12 we know that \( \sup \{ n/\ell : (n, \ell)_k \in L \} \) is either infinite or rational. In the latter case this sup is computable, by Corollary 13 and this gives the optimal constant \( C \) for the linear recurrence of \( a \).
7. Open problems

In this paper we have examined $\sup_{x \in L} \limsup_{k \to \infty} \frac{x^k}{k}$ for $L$ regular. We do not currently know how to prove analogous results for $L$ context-free. Nor do we know how to extend the results on critical exponents to the more general case of morphic sequences.

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In a previous version of this paper that was presented at WORDS 2011, the second author stated that if $L$ is regular, then $\limsup_{x \in L} \limsup_{k \to \infty} \frac{x^k}{k}$ is either rational or infinite, and is computable [21]. Unfortunately the envisioned proof of this more general result contained a subtle flaw that we have not been able to repair. The present paper contains slightly weaker results. We are indebted to Michaël Cadilhac for raising the question.

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