On Hausdorff integrations of Lie algebroids

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Abstract

In this note we present Hausdorff versions for Lie Integration Theorems 1 and 2, and apply them to study Hausdorff symplectic groupoids arising from Poisson manifolds. To prepare for these results we include a discussion on Lie equivalences and propose an algebraic approach to holonomy. We also include subsidiary results such as a generalization of the integration of subalgebroids to the non-wide case, and explore in detail the case of foliation groupoids.

1 Introduction

Lie groupoids and Lie algebroids play a central role in Higher Differential Geometry, serving as models for classic geometries such as actions, foliations and bundles, and with applications to Symplectic Geometry, Noncommutative Algebra and Mathematical Physics, among others [8, 19]. Every Lie groupoid yields a Lie algebroid through differentiation, setting a rich interplay between global and infinitesimal data, which is ruled by the so-called Lie Theorems. Lie 1 constructs a maximal (source-simply connected) Lie groupoid integrating the algebroid of a given groupoid, and Lie 2 shows that a Lie algebroid morphism can be integrated to a Lie groupoid morphism under a certain hypothesis [16, 19]. The hardest one, Lie 3, provides computable obstructions to the integrability of a Lie algebroid [5].

When working with Lie groupoids one usually allows the manifold of arrows to be non-Hausdorff. One reason for that is to include the monodromy and holonomy groupoids arising from foliations. Other reason is that the maximal groupoid given by Lie 1 may be non-Hausdorff even when the original groupoid is. A Poisson manifold yields a Lie algebroid on its cotangent bundle, which is integrable if and only if the Poisson manifold has a complete symplectic realization [6]. The canonical symplectic structure on the cotangent bundle integrates to a symplectic structure on the source-simply connected groupoid [4], which may be non-Hausdorff, and smaller integrations may be Hausdorff but not symplectic. This note is motivated by the problem of understanding Hausdorff symplectic groupoids arising from Poisson manifolds.

Our first main result is a Hausdorff version for Lie 1, Theorem 4.2, showing that every Hausdorff groupoid yields a maximal Hausdorff integration. We illustrate with examples that this may or may not agree with the source-simply connected integration or with the original one. In order to prove this theorem we pay special attention to Lie equivalences, namely Lie groupoid morphisms which are isomorphism at the infinitesimal level, and their characterization by their kernels, see Proposition 2.5. This characterization appears for instance in [13] and [15]. We review it to set notations and to serve for quick reference. Our first result also relies on the construction of monodromy and holonomy groupoids, for which we include here an original algebraic approach, see Section 3, that the reader may find interesting on its own.

In analogy with the classic case, one might a priori expect that a Lie algebroid morphism can be integrated to a morphism between Hausdorff groupoids if the first one is maximal. We
show with the Example 6.2 that this is not the case. Our second main result is a Hausdorff version of Lie 2 that includes an holonomy hypothesis, Theorem 6.3. In order to prove this we first develop a generalized version of the integration of Lie subalgebroids [20] that works in the non-wide case, in Proposition 5.1, and which also allow us to complete a neat conceptual proof for classic Lie 2. We include a second Hausdorff version of Lie 2, Theorem 6.5, in the context of Lie groupoids arising from foliations, where the holonomy hypothesis becomes automatic, and which implies Corollary 6.6, the uniqueness of the maximal Hausdorff integration.

Finally, building over the theory of VB-groupoids and VB-algebroids [2], we apply our results to show Theorem 7.6, stating that if the algebroid induced by a Poisson manifold is integrable by a Hausdorff groupoid, then the maximal Hausdorff integration is symplectic. We achieve this by looking at the canonical symplectic form on the cotangent bundle as a VB-algebroid morphism, and then integrate it to a VB-groupoid morphism. This should be compared with [16], where Lie 2 first appeared, as a device to integrate a bialgebroid to a Poisson groupoid in a similar fashion. Our application shows that the holonomy hypothesis can indeed be computable in concrete situations. Corollary 7.7 concludes that if a Poisson manifold is integrable by a Hausdorff groupoid then it admits a Hausdorff complete symplectic realization.

A Hausdorff version for Lie 3, namely the problem of deciding whether a Lie algebroid admits a Hausdorff integration at all, is left to be addressed elsewhere. Besides the usual obstructions to integrability, we know of examples of integrable algebroids which does not admit a Hausdorff integration, such as in the foliation described in Example 7.6, and in the Lie algebra bundle constructed in [11 VI.5]. A similar question, whether the canonical (source-simply connected) integration is Hausdorff, is studied in [17] for Poisson manifolds and in [1] for foliations. In this direction, we show in Corollary 4.3 that if there is a Hausdorff integration then the maximal integration is Hausdorff if and only if the foliation by source-fibers has no vanishing cycles.

Organization. In section 2 we provide a systematic study of Lie equivalences, featuring the characterization by their kernels, and in section 3 we give the algebraic approach to monodromy and holonomy. These two sections do not contain original results and their originality, if any, lies in the developed approach. Section 4 reviews classic Lie 1, reformulated in a categorical way, and prove its Hausdorff version, our first main result, illustrated with several examples. In section 5 we generalize some of the results regarding integration of Lie subalgebroids to the non-wide case, and we apply them in section 6, where we provide two versions for Hausdorff Lie 2. Finally, in section 7, we prove our last main result, a Hausdorff version of the integration of Poisson manifolds by Hausdorff symplectic groupoids.

Notations and conventions. We assume certain familiarity with the basic theory of Lie groupoids and Lie algebroids, and refer to [8, 9, 19] for further details. We denote a Lie groupoid either by $G \rightrightarrows M$ or simply $G$, as the ambiguity between the whole groupoid and its manifold of arrows should be solved by the context. Given $x \in M$ we write $G(-, x)$ for the source fiber and $G_x$ for the isotropy group. We write $A_G \Rightarrow M$ or simply $A_G$ for the corresponding algebroid, and use the convention that $A_G = \ker ds|_M$. If $\phi : G' \rightarrow G$ is a Lie groupoid morphisms, we write $A_\phi$ for the map induced among the algebroids. The objects $M$ are always Hausdorff, though $G$ need not to be. The last section uses notions of VB-groupoids and VB-algebroids that the reader may consult in [2].

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2 Basics on Lie equivalences

We review basic facts about morphisms inducing an isomorphism at the infinitesimal level. The results here are elementary and are scattered in the literature. We collect them to set notations and for quick reference throughout the paper.

We say that a Lie groupoid morphism \( \phi : G' \to G \) is a Lie equivalence if it induces an isomorphism \( A_\phi : A_{G'} \to A_G \) between the corresponding Lie algebroids. Given a Lie groupoid, its source-connected component \( G_0 \subset G \) is an open subgroupoid with the same objects and the same Lie algebroid, and therefore, \( \phi : G' \to G \) is a Lie equivalence if and only if \( \phi : G'_0 \to G_0 \) is so. In particular, the inclusion \( G_0 \to G \) is a Lie equivalence. We will focus our attention on Lie equivalences \( \phi \) between source-connected Lie groupoids. Without loss of generality, we will assume that \( M' = M \) and that \( \phi|_M \) is just the identity. Lie equivalences relate the several integrations of a given Lie algebroid.

Example 2.1. Lie groups are the same as Lie groupoids with a single object. A Lie equivalence between Lie groups \( \phi : G' \to G \) is a homomorphism inducing isomorphism on the Lie algebras. If \( G \) and \( G' \) are connected then \( \phi \) is surjective, its kernel \( K \) is discrete and lies in the center, and it follows that \( \phi : G' \to G'/K \cong G \) is a topological covering (see e.g. [14, I.11]).

Example 2.2. If \( M \) is connected, the projection \( \phi : \pi_1(M) \to P(M) \) from its fundamental groupoid to its pair groupoid is a Lie equivalence, as both integrate \( TM \Rightarrow M \). There is a correspondence between (pointed) transitive groupoids and principal bundles (see eg. [9, 3.5.3]), under which the Lie equivalence \( \phi \) is associated with the universal covering map \( \tilde{M} \to M \).

Next example shows the relevance of the source-connected hypothesis.

Example 2.3. Let \( G \rightrightarrows M \) be an étale Lie groupoid, namely one on which \( G \) and \( M \) have the same dimension. Then the Lie algebroid \( A_G \Rightarrow M \) is the zero vector bundle, and the inclusion from the unit groupoid \( U(M) \to G \) is a Lie equivalence. This means, for instance, that if \( G \rightrightarrows M \) is a discrete group acting on a manifold, then the dynamics is not seen at the infinitesimal level.

As shown in previous examples, there are strong ties between Lie equivalences and coverings maps. We organize this and other basic properties in the following proposition.

Proposition 2.4. Let \( \phi : G' \to G \) be a Lie equivalence between source-connected groupoids. Then (i) the map between the arrows \( \phi_1 : G'_1 \to G_1 \) is étale, namely a local diffeomorphism, (ii) the orbits of \( G' \) and \( G \) agree, and (iii) the map between the isotropies \( \phi_x : G'_x \to G_x \) and between the source-fibers \( \phi|_{G'(-,x)} : G'(-,x) \to G(-,x) \) are covering maps.

Proof. Given \( y \overset{\phi}{\to} x \) an arrow in \( G \), the right multiplication \( R_y : G(-,y) \to G(-,x) \) and the source map yield a natural short exact sequence relating the fiber of the Lie algebroid \( A_y = \ker u(y) ds = T_y G(-,y) \), the tangent to the arrows and the tangent to the objects:

\[
0 \to A_y \xrightarrow{dR_y} T_y G \xrightarrow{ds} T_x M \to 0
\]

If \( \phi \) is a Lie equivalence, then it yields isomorphisms on \( A_y \) and \( T_x M \), and by the five lemma, it should also induces another on \( T_y G \). This proves the first claim.
Given $G \rightrightarrows M$ a Lie groupoid and $x \in M$, the anchor map $\rho_x : A_x \to T_x M$ is natural, it has kernel the Lie algebra of the isotropy $G_x$ and cokernel the tangent space to the orbit $T_x O_x$. Then, a Lie equivalence $\phi$ should preserve kernel and cokernel of the anchor. Since in a source-connected groupoid the orbits are connected, (ii) follows, and it also follows that the morphisms between the isotropy groups are Lie equivalences.

For $x \in M$, we see $\phi|_{G'(-,x)} : G'(-,x) \to G(-,x)$ as a map over the orbit $O_x \subset M$ using $t'|_{G'(-,x)}$ and $t|_{G(-,x)}$, which are principal bundles with groups $G'_x$ and $G_x$ via right multiplication. Thus we can locally split $\phi|_{G'(-,x)}$ as $\phi_x \times \id : G'_x \times U \to G_x \times U$, from where both its image and its complement are opens. Since $G$ is source-connected, $\phi|_{G'(-,x)}$ is surjective, and so are the morphisms $\phi_x$. Claim (iii) easily follows from here. 

Previous proposition implies the well-known fact that source-simply-connected Lie groupoids are maximal integrations among the source-connected ones. In fact, if $G$ is source-simply connected, the covering map $\phi|_{G'(-,x)} : G'(-,x) \to G(-,x)$ is also injective, hence invertible.

Lie equivalences can be characterized by their kernels. The kernel $K \subset G'$ of a groupoid morphism $\phi : G' \to G$ is the subgroupoid of arrows that are mapped into an identity. It is wide, namely it has the same objects as $G'$, and it is normal, namely if $x \xrightarrow{k} x$ is in $K$ then $y \xrightarrow{\phi^{-1}k} y$ is also in $K$ for every $x \xrightarrow{g} y$. If $\phi$ is injective on objects then $K$ is intransitive, namely its source and target maps agree. The kernel may fail to be smooth in general. We say that a subgroupoid $K \subset G$ is swind if it is smooth, wide, intransitive, normal and have discrete isotropy.

**Proposition 2.5.** Let $G'$ be a source-connected Lie groupoid, possibly non-Hausdorff. There is a 1-1 correspondence between Lie equivalences $\phi : G' \to G$ with $G$ source-connected, and swind subgroupoids $K \subset G'$. Moreover, $G$ is Hausdorff if and only if $K$ is closed. In particular, $G'$ is Hausdorff if and only if $M'$ is closed.

**Proof.** Given a Lie equivalence $\phi : G' \to G$ with both $G', G$ source-connected, $\phi$ is a submersion between the arrows and between the isotropies, as shown in Proposition 2.2, so the kernel $K = \phi^{-1}(M) \subset G'$ is smooth, intransitive, and the isotropies $K_x \subset G'_x$ are discrete.

Conversely, given a swind subgroupoid $K \subset G'$, then right multiplication defines a Lie groupoid action of $K$ over $G'$, this action is free and proper, so the orbit space is a manifold $G = G'/K$ and the projection $G' \to G$ a surjective submersion [9, 3.6.2]. Since $K$ is normal, the quotient $G$ inherits a groupoid structure over $M$, becoming a Lie groupoid. To see that the quotient map $G' \to G$ is a Lie equivalence, note that it yields a fiberwise epimorphism $A_{G'} \to A_G$ between the Lie algebroids, and that both algebroids have the same rank.

Finally, since a Lie equivalence $\phi : G' \to G$ is a quotient map, $G$ is Hausdorff if and only if $M$ is closed, see Lemma 2.6 and this is the case if and only if $K = \phi^{-1}(M) \subset G'$ is closed. 

Previous proposition can be found in [13]. It can also be seen as an instance of a more general result, characterizing fibrations by their kernel systems (cf. [15] §1.2.4]). A Lie equivalence $\phi : G' \to G$ is an example of a fibration, and its kernel system is simply the kernel $K \subset G'$. For the sake of completeness, we include the following.

**Lemma 2.6.** A Lie groupoid $G \rightrightarrows M$ is Hausdorff if and only if $M \subset G$ is closed.

**Proof.** If the sequence $g_n$ has two different limits $g, g' \in G$ then $g, g'$ must have the same source and target, for $M$ is Hausdorff, and then $g^{-1} q = \lim(g_n)^{-1} q_n$ is in the closure of the units, so $M$ is not closed. Conversely, if $M$ is not closed then there is a sequence $u(x_n) \to g$ with $g$ not a unit, then $u(x_n) =usu(x_n)$ has two limits $g, u(s(g))$ and $G$ cannot be Hausdorff. 


3 An algebraic approach to holonomy

We propose an algebraic approach to the monodromy and holonomy groupoid, which is equivalent to that in the literature, provides a clean definition of holonomy, and makes it explicit some of the fundamental properties of them.

Given $M$ a manifold and $F$ a foliation, by a foliated chart $(U, \phi)$ we mean a chart $\phi = (\phi_1, \phi_2) : U \xrightarrow{\sim} \mathbb{R}^p \times \mathbb{R}^q$ that is a foliated diffeomorphism between $F|_U$ and the foliation by the second projection. Given a foliated chart $(U, \phi)$, the local monodromy groupoid $\text{Mon}(F|_U)$ is the Lie groupoid arising from the submersion $\phi_2 : U \to \mathbb{R}^q$. Its objects are $U$, and it has one arrow $y \xleftarrow{\gamma} x$ if $\phi_2(x) = \phi_2(y)$, there is no isotropy and the orbits are the plaques. An inclusion of foliated charts $U \subset V$ yields an inclusion $\text{Mon}(F|_U) \to \text{Mon}(F|_V)$. The monodromy groupoid of $F$ can be defined as the colimit of the local monodromy groupoids and inclusions:

$$\text{Mon}(F) = \text{colim}_{(U,\phi)} \text{Mon}(F|_U)$$

$\text{Mon}(F)$ is well-defined, at least set-theoretically, for the category of groupoids is cocomplete, namely every colimit exists [12] p. 4. We will later show that this naturally inherits a smooth structure, but first, let us relate our approach with the one in the literature (eg. [3, 19]).

**Lemma 3.1.** Set-theoretically, $\text{Mon}(F)$ is the disjoint union of the fundamental groupoids of the leaves, namely it has objects the points of $M$, and an arrow $y \xleftarrow{\gamma} x$ in $\text{Mon}(F)$ identifies with the homotopy class of a path $y \xleftarrow{\gamma} x$ within a leaf.

**Proof.** Every groupoid has an underlying graph, consisting of its objects, arrows, source and target. The 3-steps construction of a groupoid colimit $\text{colim}_a G^a$ goes as follows [12] p. 4: (i) compute the graph colimit $G^\infty$ levelwise, namely $G^\infty_0 = \text{colim}_a G^a_0$, and $G^\infty_1 = \text{colim}_a G^a_1$, and $s^\infty, t^\infty$ are the map induced by the $s^a, t^a$; (ii) build the path category $P(G^\infty)$, with the same objects as $G^\infty$ and arrows the chains of arrows in $G^\infty$, and (iii) mod out $P(G^\infty)/\sim$ by all the relations spanned by the commutative triangles on each $G^a$.

From this, it is rather clear that the objects of $\text{Mon}(F)$ are the points of $M$, and that we can regard an arrow $y \xleftarrow{\gamma} x$ in $\text{Mon}(F)$ as the class of a discrete path $(g_k, \ldots, g_1)$ where $g_i = (y_i \xleftarrow{\gamma_i} x_i) \in \text{Mon}(F|_{U_i})$, $x_i = y_{i-1}$, $x_1 = x$ and $y_k = y$, under the equivalence relation generated by:

1. **(i)** replacing some $g_i$ by $\iota(g_i)$ if $\iota : \text{Mon}(F|_U) \to \text{Mon}(F|_V)$ is a chart inclusion;

2. **(ii)** replacing $g_i, g_{i-1}$ by the product $g_i g_{i-1}$ if they belong to the same chart; and

3. **(iii)** insert $h = id_{x_i} \in \text{Mon}(F|_{U_i})$ between $g_i$ and $g_{i-1}$.

To each arrow $(g_k, \ldots, g_1)$ we can associate the juxtaposition of the segment paths within each chart, hence defining a groupoid map $\text{Mon}(F) \to \coprod_L \pi_1(L)$. The proof that this is in fact a groupoid isomorphism can be done leafwise, and it is a basepoint-free version of Van Kampen theorem, similar to that in [18, 1.7], subdividing continuous paths and homotopies into small enough pieces, each of them included in some foliated chart.

Given $(U, \phi), (U', \phi')$ foliated charts and given $x \in U \cap U'$, the transverse transition map at $x$ is the (germ of a) diffeomorphism $\gamma_{U|U'} : (\mathbb{R}^q, \phi_2(x)) \to (\mathbb{R}^q, \phi'_2(x))$ given by $\gamma_{U|U'}(y) = (\phi'(\phi)_2^{-1}(\phi_1(x), y)$. More generally, given $y \xleftarrow{\gamma} x$ in $\text{Mon}(F)$, and given $(U, \phi), (U', \phi')$ foliated
charts around $x$ and $y$, the holonomy $\gamma^g_{U} : (\mathbb{R}^q, \phi_2(x)) \to (\mathbb{R}^q, \phi_2(y))$ of $g$ with respect to $U, U'$ is defined by representing $g$ as a discrete path $(g_k, \ldots, g_1)$, $g_i \in \text{Mon}(F|U_i)$, $U = U_1, U' = U_k$, as the composition of the transverse transition maps $\gamma^g_{U_{i+1}U_i}$. This is well-defined for the above composition is invariant under the three elementary moves described in Lemma \[2.1\].

**Proposition 3.2.** The monodromy groupoid $\text{Mon}(F)$ naturally inherits the structure of a Lie groupoid, possibly non-Hausdorff. The inclusions $\text{Mon}(F|U) \to \text{Mon}(F)$ are smooth, and the colimit is valid within the category of Lie groupoids.

**Proof.** Given $y \xleftarrow{g} x$ in $\text{Mon}(F)$, we show now how to construct a foliated chart around $g$. Regard $g$ as the class of a discrete path $(g_k, \ldots, g_1)$ with $g_i = (y_i \leftarrow x_i) \in \text{Mon}(F|U_i)$, and realize the germ $\gamma^g_{U_kU_1}$ as a diffeomorphism $B \to \gamma^g_{U_kU_1}(B)$ with $(\phi_1)_2(x) \in B \subset \mathbb{R}^3$ an open ball. Then we can set $(\tilde{U}, \tilde{\phi})$ a chart for $\text{Mon}(F)$ around $g$ by

$$\tilde{U} = \{(g_k', \ldots, g_1') : x_i' \xleftarrow{\tilde{\phi}} x_{i-1}' \in \text{Mon}(F|U_i), \phi_2(x_1') \in B\}$$

and by $\tilde{\phi} : \tilde{U} \to (\mathbb{R}^p \times \gamma^g_{U_kU_1}(B)) \times_B (\mathbb{R}^p \times B)$, $g' \mapsto (\phi_k(x_k'), \phi_1(x_1'))$, where the fiber product over $\pi_2(\gamma^g_{U_kU_1})^{-1}$ and $\pi_2$ is an Euclidean open of dimension $2p + q$. Given two such charts and fixing an arrow $g$ on that intersection, it is straightforward to check that the three elementary moves lead to a smooth transition map, well-defined around $g$, from where the result easily follows.

From this definition, it is immediate that the inclusions $\text{Mon}(F|U) \to \text{Mon}(F)$ are not only smooth but open embeddings at the level of arrows. The colimit is valid within the category of Lie groupoids because the images of these inclusions cover a neighborhood of the identities, and a groupoid map $\text{Mon}(F)$ is smooth if and only if it is so in such a neighborhood. \hfill \Box

If $x \xleftarrow{g} x$ is an arrow in $\text{Mon}(F)$ and $(U, \phi)$ is a foliated chart around $x$, then we can build a chart $(\tilde{U}, \tilde{\phi})$ around $g$ as above, by using $(U, \phi)$ as the initial and final foliated chart, and then

$$\tilde{\phi}(\tilde{U} \cap I(\text{Mon}(F))) = \{(x, \gamma^g_{UU}(y), x, y) \cap \{(x, y, x, y) \subset (\mathbb{R}^p \times \gamma^g_{U_kU_1}(B)) \times_B (\mathbb{R}^p \times B)\},$$

where $I(\text{Mon}(F))$ denotes the isotropy of $\text{Mon}(F)$. It follows that $s : I(\text{Mon}(F)) \to M$ is always locally injective, and it is locally surjective at $g$ if and only if $\gamma^g_{UU}$ is trivial. Based on this, we say that $g$ has trivial holonomy if $s : I(\text{Mon}(F)) \to M$ is locally bijective at $g$. The arrows with trivial holonomy $K^h \subset \text{Hol}(F)$ define a subgoupoid, so the quotient of $\text{Mon}(F)$ by $K^h$ is a well-defined holonomy groupoid $\text{Hol}(F) \rightrightarrows M$, and the projection $\text{Mon}(F) \to \text{Hol}(F)$ is a Lie equivalence, see Proposition \[2.3\]. While $\text{Mon}(F)$ is source-simply connected, $\text{Hol}(F)$ is just source-connected and its isotropy groups are the holonomy groups $\text{Hol}_{\text{loc}}(F)$. Two paths $g, g'$ with the same initial and final points have the same holonomy if they induce the same diffeomorphism on small transversals.

**Proposition 3.3** (cf. [7] Prop. 1). If $G$ is a source-connected Lie groupoid integrating $F$, then the canonical projection $\text{Mon}(F) \to \text{Hol}(F)$ factors through $G$.

**Proof.** Given $(U, \phi)$ a foliated chart, the composition $\psi : (G_U)^0 \to G_U \to \text{Mon}(F|U)$ is a Lie equivalence, and since $\text{Mon}(F|U)$ is source-simply connected, $\psi$ is an isomorphism. Then the local inclusions $\text{Mon}(F|U) \cong (G_U)^0 \to G_U \to G$ induce a morphism from the colimit $\phi : \text{Mon}(F) \to G$ preserving the underlying foliation, hence being a Lie equivalence. In light of Proposition \[2.3\], we need to show that $K = \ker \phi \subset K^h$, namely that the arrows in $K$ have no holonomy. Since $K \subset I(\text{Mon}(F))$ and $s : K \to M$ is a surjective submersion, then $s|I(\text{Mon}(F))$ is locally surjective at every $g \in K$, proving $K \subset K^h$. \hfill \Box
As we have seen, the monodromy and the holonomy groupoid are always Lie groupoids, though the manifold of arrows may be non-Hausdorff. Let us illustrate with some simple examples.

**Example 3.4.** Let \( F \) be the foliation on \( M = \mathbb{R}^3 \setminus 0 \) by horizontal planes. \( F \) is simple, its leaves are the fibers of \( (x,y,z) \mapsto z \), there is no holonomy, and \( \text{Hol}(F) \) is a submersion groupoid, in particular it is Hausdorff. On the other hand, \( \text{Mon}(F) \) is non-Hausdorff, for \( u(1,0,1/n) \in \text{Mon}(F) \) converges to any element of the isotropy at \((1,0,0)\), that is isomorphic to \( \mathbb{Z} \).

**Example 3.5.** Write \( f : \mathbb{R} \to \mathbb{R} \) for the smooth map given by \( f(t) = e^{-1/t} \) if \( t > 0 \) and \( f(t) = 0 \) otherwise. Let \( M = \mathbb{R} \times S^1 \) be the cylinder with coordinates \( t,r \) and \( F \) the 1-dimensional foliation spanned by the vector field \( X(t,r) = \frac{∂}{∂r} + f(t) \frac{∂}{∂t} \). Then \( \text{Mon}(F) \) is non-Hausdorff, as the non-trivial loops at \( z = 0 \) are in the closure of the units. And since these are all the non-trivial loops and they have non-trivial holonomy, \( K^h = M \) and \( \text{Hol}(F) = \text{Mon}(F) \).

It follows from Proposition 3.3 that \( F \) does not admit any Hausdorff integration.

We close with an example of a foliation which does not admit a Hausdorff integration.

**Example 3.6.** Let \( M = \mathbb{R}^3 \setminus \{(0,0,z) : z \geq 0\} \), and let \( F \) be the foliation given by the 1-form \( \frac{f(z) y}{x^2+y^2} dx - \frac{f(z) x}{x^2+y^2} dy + dz \). The leaves of \( F \) on \( z < 0 \) are the horizontal planes, and on \( z > 0 \) are spirals spanned by the vector fields \( x \frac{∂}{∂x} + y \frac{∂}{∂y} \) and \( -y \frac{∂}{∂x} + x \frac{∂}{∂y} + f(z) \frac{∂}{∂z} \). Then \( \text{Mon}(F) \) is non-Hausdorff, as the non-trivial loops at \( z = 0 \) are in the closure of the units. And since these are all the non-trivial loops and they have non-trivial holonomy, \( K^h = M \) and \( \text{Hol}(F) = \text{Mon}(F) \).

We present here our first main contribution, which is a Hausdorff version of Lie 1. We start reviewing the classic version, establish the new result, characterize the associated kernel via a maximal Hausdorff integration








\begin{align*}
\text{Proposition 4.1} \text{ (Lie 1).} & \text{ Given } G \text{ a Lie groupoid, there exists a Lie groupoid } \tilde{G} \text{ and a universal Lie equivalence } \tilde{\phi} : \tilde{G} \to G, \text{ in the sense that for any other Lie equivalence } \phi' : G' \to G \text{ there exists a unique factorization } \tilde{\phi} = \phi' \phi.
\end{align*}

\begin{center}
\begin{tikzcd}
\tilde{G} \ar[r, swap, \exists! \phi] \ar[d, \tilde{\phi}] & G' \ar[d, \phi'] \\
G
\end{tikzcd}
\end{center}

**Proof.** Without loss of generality we can assume \( G \) source-connected. Let \( F^s \) be the foliation on \( G \) by the source-fibers, which is invariant under the free and proper \( G \)-action by right multiplication. Then \( \text{Mon}(F^s) \) and \( \text{Hol}(F^s) \) also inherit free and proper \( G \)-actions on their objects and arrows, and the quotients are well-defined Lie groupoids. \( F^s \) has no holonomy, \( \text{Hol}(F^s) \) is the submersion groupoid induced by \( s : G \to M \). Let \( \tilde{G} \) be the quotient \( \text{Mon}(F^s)/G \). The projection \( \text{Mon}(F^s) \to \text{Hol}(F^s) \) induces a Lie equivalence \( \tilde{\phi} : \tilde{G} = \text{Mon}(F^s)/G \to \text{Hol}(F^s)/G = G \). To see that it is universal, let \( \phi' : G' \to G \) be a Lie equivalence, and consider the Lie groupoid theoretic
fibered product $G' \times_G \Hol(F^s)$ (cf. 2 Appendix). Since $\phi'$ is a Lie equivalence, the same holds for the base-change morphism $G' \times_G \Hol(F^s) \to \Hol(F^s)$, so $G' \times_G \Hol(F^s)$ integrates $F^s$. Then Proposition \ref{prop:holonomies} gives a uniquely defined Lie equivalence $\Mon(F^s) \to G' \times_G \Hol(F^s)$, which modding out by the free proper $G$-action gives the desired map $\phi : \hat{G} \to G'$.

Note that since $\hat{\phi}$ and $\phi'$ are Lie equivalences, the same holds for $\phi$. The source-fibers of $\hat{G}$ identify with those of $\Mon(F^s)$, from where it is clear that $\hat{G}$ is source-simply connected. Our proof exploits that every Lie groupoid is the quotient of a holonomy groupoid. The infinitesimal analog to this statement will play a key role in next section.

**Theorem 4.2** (Hausdorff Lie 1). Given $G$ a Hausdorff Lie groupoid, there exists a Hausdorff Lie groupoid $\hat{G}$ and a universal Lie equivalence $\phi : \hat{G} \to G$, in the sense that for any other Lie equivalence $\phi' : G' \to G$ with $G'$ Hausdorff, there exists a unique factorization $\phi = \phi' \phi$.

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{\exists \phi} & G' \\
\phi \downarrow & & \downarrow \forall \phi' \\
G & & \\
\end{array}
\]

**Proof.** We can assume $G$ source-connected. Let $\hat{\phi} : \hat{G} \to G$ be the universal Lie equivalence given by Lie 1, and let $K \subset \hat{G}$ its kernel, which is closed by Proposition \ref{prop:holonomies}.

Given a Lie equivalence $\phi' : G' \to G$ from a Hausdorff groupoid $G'$ we denote by $K' \subset \hat{G}$ the kernel of the factorization $\phi : \hat{G} \to G'$, which is closed and is included in $K$. Then we can identify $G' = \hat{G}/K'$, see Proposition \ref{prop:holonomies}. In particular, if $\phi : \hat{G} \to G$ is a universal Lie equivalence, it follows from $\hat{\phi} = \phi' \phi$ that the kernel $\hat{K}$ of $\hat{G} \to \hat{G}$ is included in $K'$, for every $K'$. Thus, in order to construct $\hat{G}$, we need a minimal closed swind subgroupoid $\hat{K}$ inside $\hat{K}$.

If $M \subset K \subset \hat{K}$ is a subgroupoid then $K$ is automatically wide, intransitive and with discrete isotropy. Define $\hat{K}$ as the intersection of all the closed swind subgroupoid $K$ of $\hat{K}$. The intersection is non-trivial, for at least $K = \hat{K}$ is closed swind. It is clear that $\hat{K}$ is closed and normal. In order to show that $\hat{K}$ is swind, we only need to prove that $\hat{K}$ is smooth, or equivalently open, as $M, \hat{K}$ are manifolds of the same dimension. Given $K \subset \hat{K}$ one of the groupoids we are intersecting, if $g \in \hat{K}$, and if $g \in U \subset \hat{K}$ is a connected open neighborhood, then $U \subset \hat{K}$ for every $K$, as both $K$ and $\hat{K} \setminus K$ are open. It follows that $U \subset \hat{K}$, so $\hat{K}$ is open, as claimed.

Finally, since $\hat{K}$ is closed swind, the quotient $\hat{G} = \hat{G}/K$ is a Hausdorff Lie groupoid, again by Proposition \ref{prop:holonomies} and the map $\hat{G} \to G$ is a Lie equivalence, as well as $\hat{G} \to G$. By construction, the latter is universal among the Lie equivalences from a Hausdorff groupoid. 

$\hat{K} \subset \hat{K}$ is both open and closed, and therefore it has to be a union of connected components. In particular, $\hat{K}$ must contain every component of $\hat{K}$ intersecting $M$. We can think of arrows in $\hat{K}$ as $G$-classes of (homotopy types of) loops within a leaf of the foliation $F^s$. A non-trivial loop $\alpha_0$ in a leaf $L_0$ is a vanishing cycle if it can be extended to a continuous family $\alpha_t$, $0 \leq t \leq 1$, such that $\alpha_t$ is a trivial loop on some leaf $L_t$ for all $t > 0$. If $F^s$ has vanishing cycles then they must belong to $\hat{K}$, and if $F^s$ has no vanishing cycles then $M$ is closed in $\hat{G}$.

**Corollary 4.3.** Given $G$ a Hausdorff groupoid, the maximal integration $\hat{G}$ is Hausdorff, namely $\hat{G} = \hat{G}$, if and only if the foliation $F^s$ on $G$ has no vanishing cycles.

Next examples show that $\hat{G}$ can be either $\hat{G}$ or $G$ or something in between.
Example 4.4. Given $G$ a Hausdorff groupoid, if the source map $s : G \to M$ is trivial, as in the case of groupoids arising from Lie group actions $K \curvearrowright M$, and more generally, if $s$ is locally trivial, as in the case of strict linearizable groupoids (cf. [10]), then there are no vanishing cycles on $F^s$, and therefore, the universal groupoid $\hat{G}$ is Hausdorff and equal to $\check{G}$.

Example 4.5. Let $G$ be the holonomy groupoid of the foliation $F$ by horizontal planes on $\mathbb{R}^3 \setminus 0$. In Example 3.4 we see that $G$ is Hausdorff and that $\check{G} = \text{Mon}(F)$ is not. The only non-identity arrows in the kernel $\check{K}$ are the non-trivial loops in the leaf $z = 0$, and they are all vanishing cycles, so $\check{K} = \hat{K}$ and the universal Hausdorff groupoid $\hat{G}$ is in this case equal to $G$.

Example 4.6. We can modify previous example, by considering the foliation $F$ by horizontal planes on $\mathbb{R}^3 \setminus (\{0\} \cup L)$, where $L$ is a vertical line other than the $z$-axis, and setting $G = \text{Hol}(F)$. Then $G$ is Hausdorff, $\hat{G} = \text{Mon}(F)$ is non-Hausdorff, and $\hat{G}$ does not agree with $G$ nor $\check{G}$. The kernel $\check{K}$ has the non-trivial loops coming from the missing point, but it does not contain the non-trivial loops corresponding to the missing line.

5 Integrating Lie subalgebroids

We generalize now the results on integration of Lie subalgebroids from [20] to the non-wide case, namely when the subalgebroid is defined over a proper submanifold. This extension will be used later to give a nice conceptual proof of Lie 2 that is adaptable to the Hausdorff case.

Given $G$ a Lie groupoid, the foliation $F^s$ on $G$ by $s$-fibers can be seen as the pullback vector bundle of its algebroid $A_G$ along the target map, or alternatively, we can see $A_G$ as the quotient of $F^s$ under the action of $G \curvearrowright G \to M$ by right multiplication, which keeps this foliation invariant. This way every Lie algebroid is the quotient of a foliation.

\[
\begin{array}{c}
F^s \\
\downarrow \\
G_1 \\
\downarrow \\
M
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
A_G
\end{array}
\]

Proposition 5.1. Given $G \to M$ a Lie groupoid and $S \subset M$, there is a 1-1 correspondence between Lie subalgebroids $B \Rightarrow S$ of $A_G \Rightarrow M$ and $G$-invariant foliations $F$ on $t^{-1}(S) \subset G$.

Proof. Given $B \Rightarrow S$ a Lie subalgebroid, we can compute the Lie-algebroid-theoretic fibered product between the projection $F^s \to A_G$ and the inclusion $B \to A_G$ (cf. [2]), which turns out to be a subalgebroid $t^*B = F$ of $F^s$. Since a subalgebroid of a foliation must be a foliation, $F$ is a foliation itself, and since $F^s$ is $G$-invariant for the right multiplication, so does $F$.

\[
\begin{array}{c}
t^{-1}(S) \\
\downarrow \\
G \\
\downarrow \\
M
\end{array}
\quad \begin{array}{c}
t^*B = F \\
\rightarrow \\
B = F/G \\
\rightarrow \\
A_G
\end{array}
\]
$S = t^{-1}(S)/G$. The inclusion $F \subset F^s$ induces a Lie algebroid inclusion in the quotient $B \subset A_G$. It is straightforward to check that these two constructions are mutually inverses.

Let $G \Rightarrow M$ be a Lie groupoid, $B \Rightarrow S$ a non-necessarily wide subalgebroid of $A_G \Rightarrow M$, and $F^B$ the corresponding $G$-invariant foliation on $t^{-1}(S)$. Since $G$ acts freely and properly over $F^B$, then $\text{Mon}(F^B)$ and $\text{Hol}(F^B)$ also inherit free proper actions of $G$, and we get two new Lie groupoids as the quotients of these actions, which in the notations of [20], are $H_{max}$ and $H_{min}$.

$$H_{max} = \text{Mon}(F^B)/G \quad H_{min} = \text{Hol}(F^B)/G$$

Next we extend Proposition 3.3 and generalize some results from [20] to the non-wide case.

**Proposition 5.2.** The Lie groupoids $H_{max}$ and $H_{min}$ are integrations of $B \Rightarrow S$. The canonical map $H_{max} \to H_{min}$ is a Lie equivalence. The inclusion $B \subset A_G$ integrates to immersive morphisms $H_{max} \to \tilde{G}$ and $H_{min} \to G$ fitting in the following commutative diagram:

$$\begin{array}{ccc}
H_{max} & \to & \tilde{G} \\
\downarrow & & \downarrow \\
H_{min} & \to & G
\end{array}$$

**Proof.** Since $\text{Mon}(F^B)$ and $\text{Hol}(F^B)$ are integrations of $F^B$, the quotients $H_{max}$ and $H_{min}$ are integrations of $F^B/G = B$, and since $\text{Mon}(F^B) \to \text{Hol}(F^B)$ is the identity at the infinitesimal level, the same holds for quotient morphism $H_{max} \to H_{min}$.

The inclusion $F^B \subset F^s$ gives rise to a commutative diagram of monodromy and holonomy groupoids, for even though the holonomy groupoids is not functorial, it does yield a map in this particular situation, as the foliation $F^s$ has no holonomy.

$$\begin{array}{ccc}
\text{Mon}(F^B) & \to & \text{Mon}(F^s) \\
\downarrow & & \downarrow \\
\text{Hol}(F^B) & \to & \text{Hol}(F^s)
\end{array}$$

Modding out by the $G$-action we get the desired commutative square. That $H_{max} \to \tilde{G}$ and $H_{min} \to G$ are immersive follows from the natural sequence $0 \to A_y \xrightarrow{dR_y} T_y G \xrightarrow{d_s} T_x M \to 0$. \[\square\]

Previous proposition shows that a subalgebroid of an integrable algebroid is integrable, for instance by $H_{min}$, but even though there is a canonical map $H_{min} \to G$ induced by the inclusion, this map is not injective in general. Think for instance in a pair groupoid $G = P(M)$ and a foliation $B = F \subset TM = A_G$ with non-trivial holonomy.

**Corollary 5.3.** $H_{min} \to G$ is injective if and only if the foliation $F^B$ has trivial holonomy.

**Proof.** In the commutative diagram below one of the horizontal morphisms is injective if and only if the other is so, as the vertical maps are principal $G$-bundles.

$$\begin{array}{ccc}
\text{Hol}(F^B) & \to & \text{Hol}(F^s) \\
\downarrow & & \downarrow \\
H_{min} \cong \text{Hol}(F^B)/G & \to & G \cong \text{Hol}(F^s)/G
\end{array}$$

10
And regarding \((\text{Hol}(F^B) \Rightarrow t^{-1}(N)) \to (\text{Hol}(F^s) \Rightarrow G)\), since it is injective at the level of objects, it is going to be injective at the level of arrows if and only if it is so in the isotropies. Since \(F^s\) has no isotropy, this holds if and only if \(F^B\) has no holonomy, and the result follows. \(\square\)

6 Hausdorff versions for Lie’s second Theorem

Using the results from previous section we give here a nice proof of classic Lie 2, and derive a Hausdorff version, where a subtle hypothesis must be included, as we show in an example. We work out a second Hausdorff version valid for foliation groupoids.

Let us review Lie 2, with a proof similar to the original one [16], improved by our results from previous section, which allow us to integrate the graph of an algebroid morphism.

**Proposition 6.1** (Lie 2). Let \(G\) and \(H\) be Lie groupoids and \(\varphi : A_G \to A_H\) a Lie algebroid morphism. If \(G = \tilde{G}\) then \(\varphi\) integrates to a groupoid morphism \(\phi : G \to H\), which is unique.

**Proof.** Let \((B \Rightarrow S) \subset (A_G \times A_H \Rightarrow M \times N)\) be the graph of \(\varphi\), seen as a non-wide Lie subalgebroid of the cartesian product. Formally, we can construct \(B \Rightarrow S\) as the fibered product between the identity of \(A_G\) and \(\varphi\), as in [2, Appendix]. Since \(G \times H\) integrates \(A_G \times A_H\), Proposition 5.2 yields a Lie groupoid \(H_{min}\) integrating \(B\) and a morphism \(\alpha\) as follows:

\[
\begin{array}{ccc}
H_{min} & \xrightarrow{\alpha} & G \times H \\
\downarrow{\beta} & & \downarrow{pr_1} \\
G & \leftarrow & H
\end{array}
\]

The composition \(pr_1\alpha : H_{min} \to G\) is a Lie equivalence, and since \(G = \tilde{G}\), by Lie 1, there must exists a section \(\beta\) for it. We can finally integrate \(\varphi\) to the composition \(\phi = pr_2\alpha\beta\). \(\square\)

Given \(G, H\) Hausdorff Lie groupoids and \(\varphi : A_G \to A_H\) a Lie algebroid map, it is natural to wonder whether the integration \(\phi : \tilde{G} \to H\) descends to the maximal Hausdorff quotient \(\tilde{G}\). This is not always true, as next example shows.

**Example 6.2.** Let \(M = \mathbb{R}^3 \setminus \{(0,0,z) : z \geq 0\}\), and let \(F\) be the simple foliation given by the third projection. Let \(G = \text{Hol}(F)\), which is a submersion groupoid, hence Hausdorff. The kernel \(\tilde{K}\) of the projection \(\text{Mon}(F) \to \text{Hol}(F)\) is connected, as every non-trivial loop \(\gamma\) in \(F\) can be deformed into a trivial one just by decreasing \(z\). It follows that \(\tilde{K} = \tilde{K}\) and \(G = \tilde{G}\). Let \(\phi : (\text{Mon}(F) \rightrightarrows M) \to (\mathbb{R} \rightrightarrows *)\) be the morphism given by \(\phi([\gamma]) = f(z)\omega(\gamma)\), where \(f : \mathbb{R} \to \mathbb{R}\) is the smooth map in Example 5.3 and \(\omega(\gamma) = \frac{1}{2\pi} \int_{\gamma} \frac{zdy - ydz}{x^2 + y^2}\) is the winding number of \(\gamma\) at 0. \(\ker \phi\) consists of the loops at \(z = 0\), so it does not include \(\tilde{K}\). Thus, even though \(\mathbb{R} \rightrightarrows *\) is Hausdorff, \(\phi\) does not descend to a morphism \(G \to \mathbb{R}\).

Let us present now our first Hausdorff version for Lie 2. Given \(G, H\) Lie groupoids and \(\varphi : A_G \to A_H\) a Lie algebroid morphism, denote by \(B \Rightarrow S\) the graph of \(\varphi\), and by \(F^e\) to the \((G \times H)\)-invariant foliation on \((t \times t)^{-1}(S) \subset G \times H\), given in Proposition 5.1. The key hypothesis in next theorem should be compared with that of Corollary 2.5 in [20].

**Theorem 6.3** (Hausdorff Lie 2, v1). Let \(G\) and \(H\) be Hausdorff Lie groupoids and \(\varphi : A_G \to A_H\) a Lie algebroid morphism. If \(G = \tilde{G}\) and the foliation \(F^e\) has trivial holonomy then \(\varphi\) integrates to a groupoid morphism \(\phi : G \to H\), which is unique.
Proof. As in Proposition 6.1, the inclusion of the graph $S \subset A_G \times A_H$ of $\varphi$ integrates to a Lie groupoid immersion $\alpha : H_{\text{min}} \to G \times H$. Since $F^\varphi$ has no holonomy, the morphism $\alpha$ is injective, as seen in Corollary 5.3, and since $G \times H$ is Hausdorff we conclude that $H_{\text{min}}$ is Hausdorff as well. The composition $pr_1 \alpha : H_{\text{min}} \to G$ is a Lie equivalence, and since $G = \hat{G}$, by Theorem 1.2 there must exist $\beta : G \to H_{\text{min}}$ a section for $pr_1 \alpha$. The composition $\phi = pr_2 \alpha \beta$ is the desired integration of $\varphi$. \hfill \Box

Let us focus now in the case of foliation groupoids, namely those on which the isotropy groups are discrete [7]. Foliation groupoids can be characterized as those whose algebroid has injective anchor map, or equivalently, those integrating a foliation. They can also be characterized as those groupoids which are Morita equivalent to an étale one.

**Lemma 6.4.** If $H \rightrightarrows N$ is a foliation groupoid then the units $N$ form an open set within the isotropy $I(H) = \bigcup_{x \in N} H_x$ of $H$.

**Proof.** The characteristic foliation $F$ of $H$ is a regular foliation on $N$. Let $x \in N$ and $(U, \phi)$ be a foliated chart around it, so $U = V \times W$ where $V \cong \mathbb{R}^p$, $W \cong \mathbb{R}^q$ and $\dim(M) = p + q$. As shown in the proof of Proposition 6.3, the source-connected component of the restriction $H_U^\varphi$ is the local monodromy groupoid $\text{Mon}(F|_U)$, which is isomorphic to the product $P(V) \times U(W)$ of the pair groupoid of $V$ times the unit groupoid of $W$. Since $H_U^\varphi \subset H_U \subset H$ are open inclusions, we conclude that $I(H) \cap H_U^\varphi = I(H_U^\varphi) = U \subset N$ is an open neighborhood of $x$ within $I(H)$. \hfill \Box

Our second Hausdorff version for Lie 2 disregards the holonomy hypothesis when assuming that $H$ is a foliation groupoid.

**Theorem 6.5** (Hausdorff Lie 2, v2). Let $G$ and $H$ be Hausdorff Lie groupoids and $\varphi : A_G \to A_H$ a Lie algebroid morphism. If $G = \hat{G}$ and $H$ is a foliation groupoid then $\varphi$ integrates to a groupoid morphism $\phi : G \to H$, which is unique.

**Proof.** We know that $\varphi$ can be integrated to a morphism $\tilde{\phi} : \hat{G} \to H$ by Lie 1. This descends to a morphism $\tilde{G} \to H$ if and only if the kernel $\hat{K}$ of the projection $\hat{G} \to \hat{G}$ is included in $\ker \phi$.

We know that $N$ is open in $I(H)$ by Lemma 6.4 and it is also closed by Lemma 2.6. It follows that $\tilde{\phi}^{-1}(N) \cap \hat{K}$ is a closed wind subgroupoid, and by the proof of Theorem 1.2 it contains $\hat{K}$, or in other words, $\hat{K}$ is included in the kernel of $\tilde{\phi}$, which completes the proof. \hfill \Box

Given a Lie algebroid $A$ and a Hausdorff integration $G$, unlike the case of $\hat{G}$, the maximal Hausdorff integration $\hat{G}$ from Theorem 1.2 strongly depends on $G$. More precisely, if $H$ is another Hausdorff integration of $A$, then $\hat{H}$ and $\hat{G}$ may a priori be different. In light of Proposition 2.5 this is because the intersection of the two closed wind subgroupoids $\hat{K}_G, \hat{K}_H \subset \hat{G}$ may fail to be smooth. Nevertheless, when $A$ is a foliation, the situation is simpler, as described below.

**Corollary 6.6.** If $F$ is a foliation over $M$ admitting a Hausdorff integration, then there is a maximal Hausdorff integration $\hat{G}_F$ that covers any other Hausdorff integration of $F$.

**Proof.** Let $G_1$ and $G_2$ be Hausdorff Lie groupoids integrating $F$. Consider $\tilde{G}_1$ and $\tilde{G}_2$ the Hausdorff covering groupoids associated to $G_1$ and $G_2$, as constructed in Theorem 1.2. Since $G_1$ and $G_2$ are foliation groupoids, we can apply Theorem 6.3 to integrate the identity of $F$ to Lie groupoid morphisms $G_1 \to \tilde{G}_2$ and $G_2 \to \tilde{G}_1$, which should be mutually inverse by the uniqueness of the integration. It follows that $\tilde{G}_1$ and $\tilde{G}_2$ are isomorphic, hence defining $\hat{G}_F$. \hfill \Box
7 Application to symplectic geometry

As an application of our results, we will show now that if the Lie algebroid induced by a Poisson manifold has a Hausdorff integration, then it can be integrated by a Hausdorff symplectic groupoid, and the Poisson manifold admits a Hausdorff complete symplectic realization \[6\]. We assume here familiarity with the concepts of VB-groupoids and VB-algebroids \[2\].

A Poisson manifold \((M, \pi)\) gives rise to an induced Lie algebroid \(A \Rightarrow M\) with \(A = T^*M\) and bracket and anchor are given by \([df, dg] = d\{f, g\}\) and \(\rho(df) = X_f\). The canonical symplectic form \(\omega\) can is compatible with the algebroid structure in \(A = T^*M\), in the sense that the induced map \(\omega^b: TA \rightarrow T^*A\) is VB-algebroid isomorphism with respect to the tangent and cotangent structures \(TA \Rightarrow TM\) and \(T^*A \Rightarrow A^*\) \[2\]. It turns out that any Lie algebroid \(A\) with a compatible symplectic structure \(\omega \in \Omega^2(A)\) turns out to be a Poisson manifold, as the isomorphism \(\omega^b\)

\[
\begin{array}{ccc}
TA & \xrightarrow{\omega^b} & T^*A \\
\downarrow & & \downarrow \\
TM & \xrightarrow{} & A^*
\end{array}
\]

allow us to identify \(A \cong T^*M\), \(\omega\) with \(\omega_{can}\), and the core-anchor map of \(TA \Rightarrow TM\) with a Poisson bivector \(\pi^\#: T^*M \rightarrow TM\). Further details can be found in \[21\].

This way symplectic groupoids arise naturally as the global counterpart of Poisson manifolds.

A symplectic groupoid \((G \Rightarrow M, \omega)\) is a Lie groupoid with a symplectic structure on \(G\) such that the induced map \(\omega^b: (TG \Rightarrow M) \rightarrow (T^*G \Rightarrow A^*)\) is a VB-groupoid isomorphism. The compatibility can also be described by requiring \(\omega\) to be multiplicative, or its graph to be Lagrangian within \(G \times G \times G\) \[4\], \[15\]. A symplectic groupoid \(G\) induces a Poisson structure on the units such that the source map \(s: G \rightarrow M\) is Poisson, and therefore a complete symplectic realization. It has been proven in \[9\] that a Poisson manifold admits a complete symplectic realization if and only if the associated algebroid is integrable.

Given \((M, \pi)\) a Poisson manifold, \(A \Rightarrow M\) the induced Lie algebroid, and \(G \Rightarrow M\) an integration of it, the canonical form \(\omega_{can}\) on \(A\) may not be integrable to a symplectic form \(\omega\) on \(G\), so \(G \Rightarrow M\) may not be a symplectic groupoid. We illustrate it with a very simple example.

**Example 7.1.** Let \(M = S^3\) and \(\pi = 0\). The induced Lie algebroid is \(T^*M \Rightarrow M\), with bracket and anchor map equal to 0. An integration of \(A\) is the cotangent bundle \(G = T^*M \cong M\), with fiberwise addition as multiplication. Any other integration is obtained from \(G\) by modding out by a wide discrete group bundle \(K\) that is Lagrangian. When \(K\) has rank 3 the quotient \(G/K \cong S^3 \times T^3\) is compact, \(\alpha^2 = 0\) for every \(\alpha \in H^2(G/K)\), and \(G/K\) is not symplectic.

The first proof of Lie 2 for groupoids and algebroids appeared in the appendix of \[16\], with the intention to integrate the compatible Poisson bivector on a Lie bialgebroid, viewed as a Lie algebroid map, to a Poisson groupoid. This idea, in the context of symplectic groupoids, gives the following well-known result, that we recall here before developing a Hausdorff version.

**Proposition 7.2.** Given \((M, \pi)\) a Poisson manifold, if the induced Lie algebroid \(A \Rightarrow M\) is integrable by a Lie groupoid \(G \Rightarrow M\), then the source-simply connected integration \(\tilde{G} \Rightarrow M\) inherits the structure of a (possible non-Hausdorff) symplectic groupoid.
**Sketch of proof.** The Lie equivalence \( \tilde{G} \rightarrow G \) yields another one \( T\tilde{G} \rightarrow TG \), and since the source-fibers of \( TG \) are affine bundles over those of \( \tilde{G} \), we have that \( T\tilde{G} \) is the source-simply connected integration of \( TA \). Then the canonical symplectic structure, viewed as a Lie algebroid map \( \omega^b_{can} : TA \rightarrow T^*A \), integrates by Lie 2 (Proposition 6.1) to a VB-groupoid isomorphism \( TG \rightarrow T^*\tilde{G} \), which turns out to be a multiplicative symplectic structure on \( \tilde{G} \).

In order to establish the Hausdorff version, our first step is to show that the tangent of the maximal Hausdorff integration is the maximal Hausdorff integration of the tangent.

**Lemma 7.3.** Given \( G \) a Hausdorff groupoid with algebroid \( A \), the Lie groupoid \( T\tilde{G} \) is the maximal Hausdorff integration of \( TA \) over \( TG \).

**Proof.** Let \( \hat{K} \) be the kernel of \( \tilde{G} \rightarrow \tilde{G} \), which is the intersection of all the swind subgroupoids \( M \subset K \subset \hat{K} \). The groupoid \( TG \) is Hausdorff and it projects into \( TG \) via a Lie equivalence. It remains to show that \( T\tilde{G} \) is maximal in the sense of Theorem 4.2 or equivalently, that the intersection of all the closed swind subgroupoids \( M \subset K' \subset \hat{K} \) is exactly \( T\hat{K} \). Given such a \( K' \), it is open and closed in \( T\hat{K} \) and therefore a union of connected components, hence equal to \( T\hat{K} \) for some subgroupoid \( K \) of \( \hat{K} \). It is easy to check that this \( K \) must be closed, smooth, wide, intransitive, normal and with discrete isotropy, so \( \tilde{K} \subset K \) and the result follows.

The second step is a linear version of Proposition 5.1 showing that VB-subalgebroids correspond to invariant linear foliations. Given \( E \rightarrow M \) a vector bundle, we say that \( F \) is a **linear foliation** on \( E \) if it is invariant under the multiplication by scalars. This is equivalent to say that \( F \Rightarrow E \) is a VB-algebroid over \( F_0 \Rightarrow M \), the foliation restricted to the zero section. Note that \( \text{Mon}(F) \) is then canonically a vector bundle over \( \text{Mon}(F_0) \).

**Lemma 7.4.** Given \( \Gamma \Rightarrow E \) a VB-groupoid and \( S \subset E \) a vector subbundle, there is a 1-1 correspondence between VB-subalgebroids \( B \Rightarrow S \) of \( A_{\Gamma} \Rightarrow E \) and \( \Gamma \)-invariant linear foliations \( F \) on \( t^{-1}(S) \subset \Gamma \).

**Proof.** It follows by combining Proposition 5.1 with the characterization of VB-groupoids and VB-algebroids as Lie groupoids and Lie algebroids endowed with a regular compatible action of the multiplicative monoid \((\mathbb{R}, \cdot)\) [2].

A peculiarity about linear foliations is that the holonomy at the zero section somehow controls the holonomy on the total space. Intuitively, if there is a loop \( \gamma \) with non-trivial holonomy, then it has a transverse dislocation \( \gamma' \) that ceases to be a loop, and therefore, the paths \( \epsilon \gamma' \) are transverse dislocations of \( 0 \gamma \) which are not loops, proving that \( 0 \gamma \) has also holonomy. We give a concise prove using our algebraic approach to holonomy.

**Lemma 7.5.** Let \( E \rightarrow M \) be a vector bundle and \( F \subset TE \) a linear foliation. If \( \text{Hol}_x(F) = 0 \) for some \( x \in M \) then \( \text{Hol}_x(F) = 0 \) for every \( e \in E_x \).

**Proof.** Since \( \text{Hol}_x(F) = 0 \), we know that \( s : I(\text{Mon}(F)) \rightarrow M \) is locally bijective at \( x = \text{id}_x \). Let \( e \overset{g}{\sim} e \) be a loop at \( e \in E_x \), and let \( x \overset{0g}{\rightarrow} x \) its projection on the zero section. Since \( 0g \) has no holonomy, we know that \( s : I(\text{Mon}(F)) \rightarrow E \) is locally bijective at \( 0g \). But this is a vector bundle map over \( s : I(\text{Mon}(F_0)) \rightarrow M \), and therefore, it has to be a linear isomorphism between the fibers \( I(\text{Mon}(F))_{0g} \rightarrow E_x \) and locally bijective at \( 0g \) in the base. This proves that the vector bundle map is also locally bijective around \( g \) and at any other point over \( 0g \).
We are ready to present the main result of the section, roughly saying that if a Poisson manifold is integrable by a Hausdorff groupoid, then it is also integrable by a Hausdorff symplectic one.

**Theorem 7.6.** Given \((M, \pi)\) a Poisson manifold, if the induced Lie algebroid \(A \Rightarrow M\) is integrable by a Hausdorff groupoid \(G \rightrightarrows M\), then \(G \rightrightarrows M\) is a Hausdorff symplectic groupoid.

**Proof.** We may suppose \(G = \hat{G}\). We want to integrate the isomorphism of Lie algebroids \(\varphi = \omega^{\text{can}} : (TA \Rightarrow TM) \rightarrow (T^*A \Rightarrow A^*)\) to an isomorphism of Lie groupoids \(TG \rightarrow T^*G\) defining a multiplicative symplectic form on \(G\). Writing \(B \Rightarrow S\) for the graph of \(\varphi\), which is a VB-subalgebroid of \(TA \times T^*A \Rightarrow TM \times A^*\), by Theorem 6.3 and Lemma 7.3 we just need to show that the foliation \(F^\varphi\) on \((t \times t)^{-1}(S) \subset TG \times T^*G\) has no holonomy.

Since \(B \rightarrow A_{TG} \times A_{T^*G} \rightarrow A_G\) is an isomorphism, the map \(\pi : (t \times t)^{-1}(S) \subset TG \times T^*G \rightarrow TG\) is étale when restricted to each leaf of \(F^\varphi\), so the fibers of \(\pi\) are transversal to \(F^\varphi\), and it defines an Ehresmann connection. To see that \(F^\varphi\) has no holonomy it is enough to show that if \(\gamma\) is a horizontal loop for that connection, then every horizontal lift of \(\pi \gamma\) is also a loop. The foliation \(F^\varphi\) is linear by Lemma 7.3 so we can suppose that \(\gamma\) is in the zero section by Lemma 7.3.

It is convenient to recall the exact sequence \(0 \rightarrow A_y \xrightarrow{dR_y} T_y G \xrightarrow{d_s} T_x M \rightarrow 0\) from where the formulas for the source and target maps of \(TG\) and \(T^*G\) can be derived [2]. Given \(y \xleftarrow{h} x\) and \(y \leftrightarrow z\) in \(G\), and given \(v \in T_y G\) and \(\alpha \in T^*_y G\), it follows that

\[(v, \alpha) \in (t \times t)^{-1}(S) \iff \varphi(dt(v)) = (dR_i)^*(\alpha|_{G_{(-,z)}}) \in A^*_y.

Then a curve \(\gamma(r) = (g_r, v_r, h_r, \alpha_r) \in (t \times t)^{-1}(S)\) is in a leaf of \(F^\varphi\), or in other words, it is horizontal for the Ehresmann connection, if and only if for every \(r_0\) we have

\[
\varphi \left[ \frac{d}{dr} \right]_{r=r_0} (g_r, v_r)(g_{r_0}, v_{r_0})^{-1} = \frac{d}{dr} \big|_{r=r_0} (h_r, \alpha_r)(h_{r_0}, \alpha_{r_0})^{-1}
\]

where the multiplications are in \(TG\) and \(T^*G\), respectively.

So let \(\gamma\) be a horizontal loop such that \(\gamma(r) = (g_r, 0, h_r, 0)\), namely the loop sits within the zero section. If \(\gamma'\) is another horizontal lift for \(\pi \gamma\), then \(\gamma'(0) = (g_r, 0, h_0', \alpha_0')\), and the uniqueness of solutions in the above differential equation readily implies

\[h'_r = g_r g_0^{-1} h_0' \quad \alpha'_r = (0_{h'_r}(h'_r, \alpha'_r))_0 \alpha_0'.
\]

Note that the product \(0_{h'_r} \alpha'_0\) in \(T^*G\) makes sense because \(t(\alpha'_0) = (dR_{h'})(\alpha'_0|_{G_{(-,z)}}) = \varphi(dt(v_0))\) and \(v_0 = 0\). Finally, since \(\gamma(1) = \gamma(0)\) we have that \(g_1 = g_0\), and this applied to the explicit formulas for \(\gamma'\) show that \(\gamma'(1) = \gamma'(0)\), namely that \(\gamma'\) is also a loop, so \(F^\varphi\) has no holonomy.

It is straightforward to check that the integrated morphism \(\phi : TG \rightarrow T^*G\) is indeed \(\omega^h\) where \(\omega\) is a multiplicative symplectic form on \(G\).

**Corollary 7.7.** A Poisson manifold with Hausdorff integration admits a Hausdorff complete symplectic realization.

**Proof.** Let \((M, \pi)\) be the Poisson manifold, \(A \Rightarrow M\) its induced algebroid, and \(G \rightrightarrows M\) a Hausdorff integration. Then \(\hat{G} \rightrightarrows M\) is a Hausdorff symplectic groupoid, and therefore, \(s : \hat{G} \rightarrow M\) is a Hausdorff complete symplectic realization [6].
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