Abstract In this paper, we address the problem of constructing a (canonical) uniform probability measure on $\mathbb{N}$, which is not possible within the bounds of the Kolmogorov axioms. We argue that if we want to construct such a uniform probability measure, we should define a probability measure as a finitely additive measure assigning 1 to the whole space, with a domain that is closed under complements and finite disjoint unions. We introduce a specific condition involving a thinning operation and motivate why this is a necessary condition for a probability measure to be uniform. Then a probability measure that has this property is constructed and we show that, on its domain, it is the only probability measure that has this property, making it canonical. We use this result to derive canonical uniform probability measures on other metric spaces, including $\mathbb{R}^n$.

Keywords Uniform probability · Foundations of probability · Kolmogorov axioms · Finite additivity

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1 Introduction and main results

Within the bounds of the Kolmogorov axioms [7], a probability measure on $\mathbb{N} = \{1, 2, 3, \ldots\}$ cannot assign the same probability to every singleton and therefore, a uniform probability measure on $\mathbb{N}$ does not exist. Despite this, we have some intuition about what a uniform probability measure on $\mathbb{N}$ should look like. According to this intuition, for example, we would assign probability $1/2$ to the subset of all odd numbers. If we want to capture this intuition in a mathematical framework, we have to violate at least one of the axioms of Kolmogorov.

A suggestion by De Finetti [5] is to relax countable additivity of the measure to finite additivity. To see why this suggestion is reasonable, we must first understand why it is possible, within the axioms of Kolmogorov, to set up a uniform probability measure on $[0, 1]$, namely Lebesgue measure [8]. The type of additivity we demand plays a crucial role here. In the standard theory one always demands countable additivity. If every singleton has the

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same probability, in an infinite space, every singleton must have probability zero. With countable additivity this means that every countable set must have probability zero. This is no problem if we are working on the uncountable \([0,1]\), since we still have freedom to assign different probabilities to different uncountable subsets of \([0,1]\). The interval \([0,1/2]\), for example, has Lebesgue measure 1/2, while it is equipotent with \([0,1]\), which has Lebesgue measure 1. This works because the cardinality of the set over which we sum is smaller than the cardinality of the space itself.

On \(\mathbb{N}\) the problem of countable additivity is immediately clear: since every subset of \(\mathbb{N}\) is countable, every subset should have probability zero, which is of course impossible. In analogy with Lebesgue measure, we only want finite subsets to have probability zero and we want to assign all kinds of different probabilities to countable subsets. To do this, we should change the type of additivity to finite additivity. In short: since the cardinality of the space changes from uncountable to countable, the additivity should change from countable to finite.

Lebesgue measure does not measure every subset of \([0,1]\). Similarly, we will not ask that every element of \(\mathbb{N}\) is measurable. On what kind of collection do we want to define a finitely additive probability measure? Since we want to take finite disjoint unions, the domain should at least be closed under such finite disjoint unions. Further, we insist that the domain should also be closed under taking complements, corresponding to the intuition that if we know that a certain set has probability \(p\), the complement must have probability \(1 - p\). This motivates Definition 1.1.

**Definition 1.1** Let \(X\) be space. Write \(\mathcal{P}(X)\) for the power set of \(X\) and let \(\mathcal{H} \subseteq \mathcal{P}(X)\). An \(f\)-system of \((X, \mathcal{H})\) is a nonempty collection \(\mathcal{F} \subseteq \mathcal{H}\) such that

1. \(A, B \in \mathcal{F}\) with \(A \cap B = \emptyset\) implies that \(A \cup B \in \mathcal{F}\),
2. \(A \in \mathcal{F}\) implies that \(A^c \in \mathcal{F}\).

A probability measure on \((X, \mathcal{H})\) is a map \(\mu : \mathcal{F} \to [0,1]\) such that

1. \(\mathcal{F}\) is an \(f\)-system of \((X, \mathcal{H})\),
2. \(A, B \in \mathcal{F}\) with \(A \cap B = \emptyset\) implies that \(\mu(A \cup B) = \mu(A) + \mu(B)\),
3. \(\mu(X) = 1\).

A probability pair on \((X, \mathcal{H})\) is a pair \((\mathcal{F}, \mu)\) such that \(\mathcal{F}\) is an \(f\)-system of \((X, \mathcal{H})\) and \(\mu \in [0,1]^\mathcal{F}\) is a probability measure on \((X, \mathcal{H})\).

**Remark 1.2** Schurz and Leitgeb [10, p. 261] call an \(f\)-system a pre-Dynkin system, since in case of closure under countable unions of mutually disjoint sets, such a collection would be called a Dynkin system [11, p. 19].

We define

\[
\mathcal{M} := \left\{ \bigcup_{i=1}^{\infty} [a_{2i-1}, a_{2i}) : 0 \leq a_1 < a_2 < a_3 < \cdots \right\} \subseteq \mathcal{P}([0,\infty)). \tag{1.1}
\]
From a probability pair \((F, \mu)\) on \(([0, \infty), \mathcal{M})\) we can immediately derive a corresponding probability pair \((F', \mu')\) on \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\) given by

\[
F' := \left\{ A \subseteq \mathbb{N} : \bigcup_{n \in A} [n-1, n) \in F \right\},
\]

\[
\mu'(A) := \mu \left( \bigcup_{n \in A} [n-1, n) \right).
\]

It turns out that by working on \(([0, \infty), \mathcal{M})\) instead of \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\), we can formulate and prove our claims much more elegantly. We should emphasize, however, that conceptually there is no difference between \(([0, \infty), \mathcal{M})\) and \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\) and that the work we do in Sections 2 and 3 can be done in the same way for \((\mathbb{N}, \mathcal{P}(\mathbb{N}))\).

For \(A \in \mathcal{M}\) we define \(S_A : [0, \infty) \to [0, \infty)\) by

\[
S_A(x) := m(A \cap [0, x)),
\]

where \(m\) is the Lebesgue measure on \(\mathbb{R}\). We set \(\rho_A : [0, \infty) \to [0, 1]\) by \(\rho_A(0) := 0\) and

\[
\rho_A(x) := \frac{S_A(x)}{x},
\]

for \(x > 0\). A set \(A \in \mathcal{M}\) is said to have natural density \(\lambda\) \([11, p. 270]\) if \(\rho_A(x) \to \lambda\) as \(x \to \infty\). We denote the collection of sets in \(\mathcal{M}\) with a natural density by \(\mathcal{C}\) and set \(\lambda : \mathcal{C} \to [0, 1]\) by

\[
\lambda(A) := \lim_{x \to \infty} \rho_A(x),
\]

which is also known as the Cesàro limit.

Schurz and Leitgeb \([10]\) propose the following uniform probability measure on \(([0, \infty), \mathcal{M})\). With the Hahn-Banach Theorem \([4, p. 78]\) we can extend the limit operator on the subspace of convergent functions in \(L^\infty([0, \infty))\), to a linear operator HB-lim that is bounded by the limsup operator and is defined on \(L^\infty([0, \infty))\). Then

\[
\lambda^*(A) := \text{HB-lim}(\rho_A),
\]

is a probability measure defined on \(\mathcal{M}\). Also a hyperreal variant of this probability measure, that is not only finitely additive but hypercountably additive, has been proposed \([13]\).

The probability measure \(\lambda^*\) is defined for every element of \(\mathcal{M}\), but is not unique: for every \(A \in \mathcal{M} \setminus \mathcal{C}\) there are infinitely many different Hahn-Banach extensions of the limit operator that give different values for \(\lambda^*(A)\). This can be seen as follows. Let \(\mathcal{W}\) be the linear subspace of \(L^\infty([0, \infty))\) consisting of convergent functions and let \(A \in \mathcal{M}\). Choose any \(V \in \mathbb{R}\) such that \(\lim \inf(\rho_A) \leq V \leq \lim \sup(\rho_A)\). Then consider the linear operator \(L : \{ f + c\rho_A : f \in \mathcal{L}, c \in \mathbb{R} \} \to \mathbb{R}\) given by \(L(f + c\rho_A) := \lim(f) + cV\). Note that \(L\) is dominated by the limsup operator. Apply the Hahn-Banach
Theorem to extend $L$ to a linear operator $L'$ on $L^\infty([0, \infty))$. Then $L'$ extends the limit operator and $L'(\rho_A) = V$.

In this paper, we avoid arbitrariness like that in $\lambda^*$ by constructing a canonical uniform probability measure on $([0, \infty), \mathcal{M})$, which we then generalize to canonical uniform probability measures on other spaces (see Section 4). How do we make such a canonical choice? Let us again look at what happens with Lebesgue measure on $[0, 1]$. Lebesgue measure is unique in the sense that every member $A$ of the Lebesgue $\sigma$-algebra on $[0, 1]$ is assigned the same probability by every probability measure (in the Kolmogorov sense) on $[0, 1]$ that is translation invariant and at least measures all the intervals in $[0, 1]$ [2]. On $([0, \infty), \mathcal{M})$ we want to do something similar. We want to define 'uniformity' by some property $P$ of probability pairs on $([0, \infty), \mathcal{M})$. Once we have this property, we want that our probability pair does not only have property $P$, but also is unique with respect to $P$ in the sense of Definition 1.3.

**Definition 1.3** Let $P$ be some property of probability pairs on $(X, \mathcal{H})$. The probability pair $(\mathcal{F}, \mu)$ on $(X, \mathcal{H})$ has unique values with respect to $P$ if it has property $P$ and for every $A \in \mathcal{F}$ and probability pair $(\mathcal{F}', \mu')$ on $(X, \mathcal{H})$ with property $P$ and $A \in \mathcal{F}'$ it is true that $\mu(A) = \mu'(A)$.

Notice that having unique values with respect to $P$ is stronger than uniqueness in the sense that there is no other measure pair with the same $f$-system that has property $P$. The latter says only something about other measure pairs with the same $f$-system, while having unique values says something about other measure pairs with an $f$-system that intersect the $f$-system. We want to have unique values because we want to avoid the situation that we have some set $A$ in our $f$-system, for which there is another measure pair with property $P$ assigning a different probability to $A$. In that scenario, we would not have any ground to decide what the unique probability of $A$ is.

The structure of this paper is as follows. In Section 2, we discuss the property of being a 'weakly thinnable pair' (WTP) and motivate why this is a natural choice for $P$. In Section 3, we explicitly construct a measure pair $(\mathcal{A}_{uni}, \alpha)$ such that

\[ C \subset \mathcal{A}_{uni} \subset \left\{ A \in \mathcal{M} : \frac{1}{\log(D)} \int_1^D \frac{1_A(y)}{y} \, dy \text{ converges} \right\} \tag{1.7} \]

and

\[ \alpha(A) = \lim_{D \to \infty} \frac{1}{\log(D)} \int_1^D \frac{1_A(y)}{y} \, dy. \tag{1.8} \]

The expression in (1.8) is called the logarithmic density of $A$ [11, p. 272]. We give the precise definition of $\mathcal{A}_{uni}$ in Section 4. We also present the following theorem about $(\mathcal{A}_{uni}, \alpha)$, which is the main result of our paper.

**Theorem 1.4** The pair $(\mathcal{A}_{uni}, \alpha)$ is a WTP that has unique values with respect to being a WTP. In addition, any WTP $(\mathcal{F}, \mu)$ has unique values with respect to being a WTP if and only if $\mathcal{F} \subseteq \mathcal{A}_{uni}$. 
On the basis of Theorem 1.4, we propose $\alpha$ as our canonical choice for a uniform probability measure on $([0, \infty), \mathcal{M})$. In Section 4, we derive from $\alpha$ canonical uniform probability measures on certain metric spaces including Euclidean space (the collection of subsets we work on in these spaces is specified in Section 4). The proofs of the results in Sections 2-4 are given in Section 5.

There are some small notational remarks. We write $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For real-valued sequences $x, y$ or real-valued functions $x, y$ on $[0, \infty)$ we write $x \sim y$ or $x_i \sim y_i$ if $\lim_{i \to \infty} x_i - y_i = 0$. Because we work only on $([0, \infty), \mathcal{M})$ in Sections 2 and 3, every time we speak of an $f$-system, probability pair or probability measure we omit specifying that this is on $([0, \infty), \mathcal{M})$.

2 Defining uniformity on $([0, \infty), \mathcal{M})$

From a purely mathematical standpoint, we can choose any $P$ to define uniformity, but we want to make a natural choice that somehow corresponds to our intuition about uniformity. Furthermore, we want to choose $P$ in such a way that we can find a measure pair that has unique values with respect to $P$ and has a ‘big’ $f$-system.

Let us look at natural options for $P$. If we have $A \in \mathcal{M}$, the average $\rho_A(x)$ represents the uniform probability of $A \cap [0, x)$ on $[0, x)$. We would like to view the uniform probability measure on $([0, \infty), \mathcal{M})$ as the ‘limit’ of these uniform measures for $x \to \infty$. In other words, a uniform probability measure should assign to elements of $\mathcal{M}$ with a natural density their natural density. So, it seems reasonable to let $P$ be the property that for a probability pair $(\mathcal{F}, \mu)$ we have that $C \subseteq \mathcal{F}$ and $\mu$ extends the Cesàro limit. In that case, $(\mathcal{C}, \lambda)$ is a probability pair that has unique values with respect to $P$. In fact, no measure pair with an $f$-system containing anything outside $C$, has unique values with respect to $P$.

Notice that in our approach, it suffices that the domain of a measure is an $f$-system. Consequently, there is no need to extend $\lambda$ to $\mathcal{M}$ as Schurz and Leitgeb do, since $C$ is already an $f$-system. The probability measure $\lambda$ would be the canonical choice for a uniform probability measure on $([0, \infty), \mathcal{M})$, if we choose $P$ as above. But we can do better. We introduce a new property $P$ that allows for a probability pair that has unique values with respect to $P$, and has an $f$-system which is bigger than $C$.

For Lebesgue measurable $Y \subseteq \mathbb{R}$ with $0 < m(Y) < \infty$ the uniform probability measure on $Y$ is given by

$$\mu_Y(X) := \frac{m(X)}{m(Y)},$$

where $X \subseteq Y$ is Lebesgue measurable. Let $A \subseteq B \subseteq C$ be all Lebesgue measurable with $m(B) > 0$ and $m(C) < \infty$. Observe that

$$\mu_C(A) = \mu_C(B)\mu_B(A).$$
We want to generalize this property to a property of probability pairs on 
\(([0, \infty), \mathcal{M})\). Write
\[
\mathcal{M}^* := \{ A \in \mathcal{M} : m(A) = \infty \}.
\] (2.3)

Consider for \(A \in \mathcal{M}^*\) the map \(f_A : A \to [0, \infty)\) given by \(f_A(x) := S_A(x)\). The map \(f_A\) gives a one-to-one correspondence between \(A\) and \([0, \infty)\). If \(A \in \mathcal{M}^*\) and \(B \in \mathcal{M}\), we want to introduce notation for the set
\[
\{ f_A^{-1}(b) : b \in B \},
\] (2.4)
that gives the subset of \(A\) that corresponds to \(B\) under \(f_A\). Inspired by van Douwen [6], we introduce the following operation.

**Definition 2.1** For \(A, B \in \mathcal{M}\), define
\[
A \circ B := \{ x \in [0, \infty) : x \in A \land S_A(x) \in B \}.
\] (2.5)

Note that if \(A, B \in \mathcal{M}\), then \(A \circ B \in \mathcal{M}\) and that for \(A \in \mathcal{M}^*\) we have
\[
A \circ B = \{ f_A^{-1}(b) : b \in B \}.
\] (2.6)

We can view this operation as thinning \(A\) by \(B\) because we create a subset of \(A\), where \(B\) is ‘deciding’ which parts of \(A\) are removed. We also can view the operation \(A \circ B\) as thinning out \(B\) over \(A\), since we ‘spread out’ the set \(B\) over \(A\).

Let \((\mathcal{F}, \mu)\) be a probability pair and let \(A \in \mathcal{F} \cap \mathcal{M}^*\). If \(B \in \mathcal{M}\), the set \(A \circ B\) is the subset of \(A\) corresponding to \(B\). We can use this to transform \(\mu\) into a measure on \(A\) as follows. We set \(\mathcal{F}_A := \{ A \circ B : B \in \mathcal{F} \}\) and then define \(\mu_A : \mathcal{F}_A \to [0, 1]\) by
\[
\mu_A(A \circ B) := \mu(B).
\] (2.7)

Given \(B \in \mathcal{F}\) such that \(A \circ B \in \mathcal{F}\), analogous to (2.2), we insist that
\[
\mu(A \circ B) = \mu(A)\mu_A(A \circ B)
\] (2.8)
is a necessary condition for \(\mu\) to be uniform. Using (2.7), this translates into
\[
\mu(A \circ B) = \mu(A)\mu(B).
\] (2.9)

We now have the restriction that \(A \in \mathcal{F} \cap \mathcal{M}^*\). However, if \(A \in \mathcal{F} \setminus \mathcal{M}^*\), then \(\mu(A \circ B) \leq \mu(A)\). Clearly, we want that a uniform probability measure assigns probability zero to a set of finite Lebesgue measure. Hence, \(\mu(A) = 0\) if \(\mu\) is uniform and (2.9) still holds. We call the property that (2.9) holds for every \(A, B \in \mathcal{F}\) with \(A \circ B \in \mathcal{F}\) the thinnability of \(\mu\).

In Appendix A, we show that thinnability can only be achieved on probability pairs with relatively small \(f\)-systems. Therefore, we choose to use a weakened version of thinnability in this paper.
Definition 2.2 Let \((\mathcal{F}, \mu)\) be a probability pair. The probability measure \(\mu\) is weakly thinnable if
\[
\mu(C \circ A) = \mu(C)\mu(A) \quad (2.10)
\]
for every \(A \in \mathcal{F}\) and \(C \in \mathcal{C}\) such that \(C \circ A \in \mathcal{F}\).

If \(A \in \mathcal{F}\) and \(C \in \mathcal{C}\), we want to use \(2.10\) to determine the probability of \(C \circ A\). In other words: we want that \(C \circ A \in \mathcal{F}\) if \(C \in \mathcal{C}\) and \(A \in \mathcal{F}\). We say that \(\mathcal{F}\) is closed under weak thinning if this holds. In particular, closedness of \(\mathcal{F}\) under weak thinning implies that \(\mathcal{C} \subseteq \mathcal{F}\), since \([0, \infty) \in \mathcal{F}\) and \(C \circ [0, \infty) = C\) for every \(C \in \mathcal{C}\).

Besides weak thinnability, there is another property that we want to include in the new approach of ‘uniformity’. Let \((\mathcal{F}, \mu)\) be a probability pair. Let \(A, B \in \mathcal{F}\) and suppose it is true for every \(x \in [0, \infty)\) that
\[
S_A(x) \geq S_B(x) \quad (2.11)
\]
Since this inequality is true for every \(x\), it is clear that \(B\) is ‘sparser’ than \(A\). Therefore, we insist that \(\mu(A) \geq \mu(B)\) is a necessary condition for \(\mu\) to be uniform. We call this property ‘preserving ordering by \(\rho\)’ since \(2.11\) can trivially be rewritten as \(\rho_A(x) \geq \rho_B(x)\).

Since we have \(\mathcal{C} \subseteq \mathcal{F}\), it seems natural to also ask \(\mu|_C = \lambda_c\), but it turns out to be sufficient to ask the weaker property that \(\mu([c, \infty)) = 1\) for every \(c \in [0, \infty)\). So, to reduce redundancy we require the latter and then prove that \(\mu|_C = \lambda\). Putting everything together, we obtain the following definition.

Definition 2.3 A probability pair \((\mathcal{F}, \mu)\) is a weakly thinnable pair (WTP) if it satisfies the following conditions:

F \(\mathcal{F}\) is closed under weak thinning,
M1 \(\mu([c, \infty)) = 1\) for every \(c \in [0, \infty)\),
M2 \(\mu\) is weakly thinnable,
M3 \(\mu\) preserves ordering by \(\rho\).

We start to state that the conditions of Definition 2.3 are sufficient to obtain extension of the Cesàro limit. In fact, we get the following slightly stronger result.

Proposition 2.4 Let \((\mathcal{F}, \mu)\) be a WTP. Then for \(A \in \mathcal{F}\) we have
\[
\liminf_{x \to \infty} \rho_A(x) \leq \mu(A) \leq \limsup_{x \to \infty} \rho_A(x) \quad (2.12)
\]
As motivated, we choose being a WTP to be the property \(P\) that defines uniformity. In the next section, we introduce an important WTP.
3 The pair \((\mathcal{A}_{\text{uni}}, \alpha)\)

For \(A \in \mathcal{M}\) set \(\sigma_A : (0, \infty)^2 \to [0, 1]\) given by
\[
\sigma_A(D, x) := \frac{1}{D} \int_x^{x+D} 1_A(y) dy,
\]
which is the average of \(1_A\) over the interval \([x, x+D]\). Then set for any \(A \in \mathcal{M}\)
\[
U(A) := \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \sigma_A(D, x) \tag{3.2}
\]
and
\[
L(A) := \liminf_{D \to \infty} \inf_{x \in (0, \infty)} \sigma_A(D, x). \tag{3.3}
\]
Define
\[
\mathcal{W}_{\text{uni}} := \{A \in \mathcal{M} : L(A) = U(A)\} \tag{3.4}
\]
and \(\kappa : \mathcal{W}_{\text{uni}} \to [0, 1]\) given by
\[
\kappa(A) := L(A) = U(A). \tag{3.5}
\]
It is easy to check that \((\mathcal{W}_{\text{uni}}, \kappa)\) is a probability pair. For any \(A \in \mathcal{M}\), we set
\[
\log(A) := \{\log(a) : a \in A \cap [1, \infty)\}. \tag{3.6}
\]

Definition 3.1 We define
\[
\mathcal{A}_{\text{uni}} := \{A \in \mathcal{M} : \log(A) \in \mathcal{W}_{\text{uni}}\} \tag{3.7}
\]
and \(\alpha : \mathcal{A}_{\text{uni}} \to [0, 1]\) given by
\[
\alpha(A) := \kappa(\log(A)). \tag{3.8}
\]
A typical example of a set in \(\mathcal{A}_{\text{uni}}\) that is not in \(\mathcal{C}\), is
\[
A = \bigcup_{n=0}^{\infty} [e^{2n}, e^{2n+1}). \tag{3.9}
\]
It is easy to check that \(A \notin \mathcal{C}\), but
\[
\log(A) = \bigcup_{n=0}^{\infty} [2n, 2n+1), \tag{3.10}
\]
so \(\log(A) \in \mathcal{W}_{\text{uni}}\) with \(\kappa(A) = 1/2\). Hence \(A \in \mathcal{A}_{\text{uni}}\) with \(\alpha(A) = 1/2\).

That \((\mathcal{A}_{\text{uni}}, \alpha)\) is a probability pair follows directly from the fact that \((\mathcal{W}_{\text{uni}}, \kappa)\) is a probability pair. The pair \((\mathcal{A}_{\text{uni}}, \alpha)\) is also a WTP.

Theorem 3.2 The pair \((\mathcal{A}_{\text{uni}}, \alpha)\) is a WTP.

We do not only want that a uniform probability pair is a WTP, but that it has unique values with respect to being a WTP.
Theorem 3.3 The pair \((A^{\text{uni}}, \alpha)\) has unique values with respect to being a WTP.

Theorem 3.3 justifies our choice of \(\alpha\) as a canonical uniform probability measure on \(([0, \infty), \mathcal{M})\). Moreover, the \(f\)-system \(A^{\text{uni}}\) is maximal in the following sense.

Theorem 3.4 Let \((\mathcal{F}, \mu)\) be a probability pair that has unique values with respect to being a WTP. Then \(\mathcal{F} \subseteq A^{\text{uni}}\).

4 Generalization to metric spaces

In this section we use \((A^{\text{uni}}, \alpha)\) to derive canonical probability measures on a class of metric spaces. Of course one could also try to construct such a probability measure by working more directly on these metric spaces, instead of constructing a derivative of \((A^{\text{uni}}, \alpha)\). Since probability pairs on \(([0, \infty), \mathcal{M})\), motivated from the problem of a uniform probability measure on \(\mathbb{N}\), is the priority of this paper, we do not make such an effort here.

Let us first sketch the idea of the generalization. Let \(A \in \mathcal{M}\). Whether \(A\) is in \(A^{\text{uni}}\) depends completely on the asymptotic behavior of \(\rho_A\) (Lemma 5.2). If \(A \in A^{\text{uni}}\), then also \(\alpha(A)\) only depends on the asymptotic behavior of \(\rho_A\) (Lemma 5.2). Now suppose that on a space \(X\), we have a subset \(B \subseteq X\) and we can somehow define a function \(\tilde{\rho}_B : [0, \infty) \to [0, 1]\) that represents the ‘density’ of \(B\). Then, intuitively, if \(\tilde{\rho}_B\) is asymptotically equivalent with \(\rho_A\), then the probability of \(B\) should be \(\alpha(A)\). The goal of this section is to make this idea precise.

Let \((X, d)\) be a metric space. For \(x \in X\) and \(r \geq 0\), write
\[
B(x, r) := \{y \in X : d(y, x) < r\}. 
\]
(4.1)

Write \(B(X)\) for the Borel \(\sigma\)-algebra of \(X\). We need a ‘uniform’ measure on this space to measure density of subsets in open balls. It is clear that the measure of an open ball should at least be independent of where in the space we look, i.e. it should only depend on the radius of the ball. This leads to the following definition.

Definition 4.1 We say that a Borel measure \(\nu\) on \(X\) is uniform if for all \(r > 0\) and \(x, y \in X\) we have
\[
0 < \nu(B(x, r)) = \nu(B(y, r)) < \infty. 
\]
(4.2)

On \(\mathbb{R}^n\) with Euclidean metric, the standard Borel measure as obtained by assigning to products of intervals the product of the length of those intervals, is a uniform measure. In general, on normed locally compact vector spaces, the invariant measure with respect to vector addition, as given by the Haar measure, is a uniform measure.

A result by Christensen [3] tells us that uniform measures that are Radon measures are unique up to multiplicative constants on locally compact metric
spaces. This, however, does not cover all cases. The set of irrationals numbers, for example, is not locally compact, but the Lebesgue measure restricted to Borel sets of irrational numbers is a uniform measure and unique up to a multiplicative constant. We give a slightly more general version of the result of Christensen.

**Proposition 4.2** If $\nu_1$ and $\nu_2$ are two uniform measures on $X$, then there exists some $c > 0$ such that $\nu_1 = c \nu_2$.

Proposition 4.2 gives us uniqueness, but not existence. To see that there are metric spaces without a uniform measure, consider the following example. Let $X$ be the set of vertices in a connected graph that is not regular. Let $d$ be the graph distance on $X$. If we suppose that $\nu$ is a uniform measure on $X$, from (4.2) with $r < 1$ it follows that for some $C > 0$ we have $\nu(B(x, 2)) = C(1 + deg(x))$ for every $x \in V$, which implies (4.2) cannot hold for $r = 2$ since the graph is not regular. A characterization of metric spaces on which a uniform measure exist, does not seem to be present in the literature.

We now assume $X$ has a uniform measure $\nu$ and that $\nu(X) = \infty$. In addition to that, we write $h(r) := \nu(B(x, r))$ for $r \geq 0$ and assume that

$$\forall C > 0 \lim_{r \to \infty} \frac{h(r + C)}{h(r)} = 1, \quad (4.3)$$

which is equivalent with amenability in case $(X, d)$ is a normed locally compact vector space. For the importance of this assumption, see Remark 4.4 below.

Set $r^-(u) := \sup \{r \in [0, \infty) : h(r) \leq u\}$, $r^+(u) := r^- + 1$ (4.4)

for $u \in [0, \infty)$. Note that $h(r^-(u)) \leq u$ and $h(r^+(u)) \geq u$. Write $(X, \mathcal{L}(X), \bar{\nu})$ for the (Lebesgue) completion of $(X, \mathcal{B}(X), \nu)$. Fix some $o \in X$. For $A \in \mathcal{L}(X)$ define the map $\bar{\rho}_A : [0, \infty) \to [0, \infty)$ given by $\bar{\rho}_A(0) := 0$ and

$$\bar{\rho}_A(u) := \frac{\bar{\nu}(B(o, r^-(u)) \cap A)}{h(r^-(u))} \quad (4.5)$$

for $r > 0$. The value $\bar{\rho}_A(u)$ is the density of $A$ in the biggest open ball around $o$ of at most measure $u$. Notice that $\bar{\rho}_A$ is independent of the choice of $\nu$ as a result of Proposition 4.2. The function $\bar{\rho}_A$ does depend on the choice of $o$, but in Proposition 4.3 we show that the asymptotic behavior of $\bar{\rho}_A$ does not depend on the choice of $o$. We also show in Proposition 4.3 that the asymptotic behavior of $\bar{\rho}_A$ is not affected if we replace $r^-(u)$ by $r^+(u)$ in (4.5).

**Proposition 4.3** Fix $x, y \in X$ and $A \in \mathcal{L}(X)$. Then

$$\frac{\bar{\nu}(B(x, r^-) \cap A)}{h(r^-(u))} \sim \frac{\bar{\nu}(B(y, r^+) \cap A)}{h(r^+(u))} \quad (4.6)$$
Remark 4.4 Proposition 4.3 is not necessarily true if we do not assume (4.3), as illustrated by the following example. Suppose $X$ is the set of vertices of a 3-regular tree graph and $d$ is the graph distance. Let $\nu$ be the counting measure, which is a uniform measure on this metric space. Then clearly (4.3) is not satisfied. Now pick any $x \in X$ and let $y$ be a neighbor of $x$. Let $A \subseteq \mathcal{P}(X)$ be the connected component containing $y$ in the graph where the edge between $x$ and $y$ is removed. Then
\[
\lim_{r \to \infty} \frac{\nu(B(x, r) \cap A)}{h(r)} = \frac{1}{3} \text{ and } \lim_{r \to \infty} \frac{\nu(B(y, r) \cap A)}{h(r)} = \frac{2}{3}.
\] (4.7)

Let $(\mathcal{F}, \mu)$ be a probability pair on $(X, \mathcal{L}(X))$. Suppose that $A \in \mathcal{L}(X)$ and that there is a $B \in \mathcal{A}^{\text{uni}}$ such that $\rho_B(u) \sim \hat{\rho}_A(u)$. This means that the density of $A$ in $X$ asymptotically behaves the same as the density of $B$ in $[0, \infty)$. Since we already have a canonical uniform probability measure that assigns a probability to $B$, we insist that $A \in \mathcal{F}$ with $\mu(A) = \alpha(B)$ is a necessary condition for $\mu$ to be uniform.

Definition 4.5 A probability pair $(\mathcal{F}, \mu)$ on $(X, \mathcal{L}(X))$ is an extension pair (EP) if for every $A \in \mathcal{L}(X)$ and $B \in \mathcal{A}^{\text{uni}}$ such that $\rho_B(x) \sim \hat{\rho}_A(x)$, we have $A \in \mathcal{F}$ and $\mu(A) = \alpha(B)$.

As motivated, we choose being an EP as the property that defines ‘uniformity’ for probability pairs on $(X, \mathcal{L}(X))$.

Define for $A \in \mathcal{L}(X)$ the map $\xi_A : (1, \infty)^2 \to [0, 1]$ given by
\[
\xi_A(D, x) := \frac{1}{\log(D)} \int_x^D \frac{\rho_A(y)}{y} \, dy.
\] (4.8)

We set
\[
\mathcal{A}^{\text{uni}}(X) := \left\{ A \in \mathcal{L}(X) : \limsup_{D \to \infty} \sup_{x > 1} \xi_A(D, x) = \liminf_{D \to \infty} \inf_{x > 1} \xi_A(D, x) \right\}
\] (4.9)
and $\alpha^X : \mathcal{A}^{\text{uni}}(X) \to [0, 1]$ by
\[
\alpha^X(A) := \limsup_{D \to \infty} \sup_{x \in (1, \infty)} \xi_A(D, x) = \liminf_{D \to \infty} \inf_{x \in (1, \infty)} \xi_A(D, x).
\] (4.10)

We now give the analogue of Theorem 1.4.

Theorem 4.6 The pair $(\mathcal{A}^{\text{uni}}(X), \alpha^X)$ is an EP that has unique values with respect to being an EP. In addition, if an EP $(\mathcal{F}, \mu)$ has unique values with respect to being an EP, then $\mathcal{F} \subseteq \mathcal{A}^{\text{uni}}(X)$.

So $\alpha^X$ gives us a canonical uniform probability measure on $(X, \mathcal{L}(X))$. In case of Euclidean space, we have the following expression for $(\mathcal{A}^{\text{uni}}(X), \alpha^X)$.
Proposition 4.7 Suppose $X = \mathbb{R}^n$ and $d$ is Euclidean distance. Let $\sigma$ be the surface measure on the unit sphere in $\mathbb{R}^n$. Then for $A \in \mathcal{L}(\mathbb{R}^n)$ we can replace $\bar{\xi}_A(D, x)$ in (4.9) and (4.10) by

$$\frac{1}{\log(D)} \int_D \frac{K_A(y)}{y} dy,$$

(4.11)

where $K_A : [0, \infty) \to [0, 1]$ is given by

$$K_A(r) := \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{S^{n-1}} 1_A(ru) \sigma(du).$$

(4.12)

5 Proofs

First we show that every $f$-system of a WTP is closed under translation and that every probability measure of a WTP is invariant under translation.

Lemma 5.1 Let $(\mathcal{F}, \mu)$ be a WTP. Let $A \in \mathcal{F}$ and $c \in [0, \infty)$. Then

$$A' := \{c + a : a \in A\} \in \mathcal{F}$$

(5.1)

and $\mu(A) = \mu(A')$.

Proof Let $(\mathcal{F}, \mu)$ be a WTP. Let $A \in \mathcal{C}$ and $c \in [0, \infty)$. Set $B := [c, \infty)$. We have $B \in \mathcal{C}$ and by M1 we find that $\mu(B) = 1$. Therefore, $A' = B \circ A \in \mathcal{F}$ by F1 and

$$\mu(A') = \mu(B) \mu(A) = \mu(A)$$

(5.2)

by M2.

We give the proof of Proposition 4.4.

Proof of Proposition 4.4 Let $(\mathcal{F}, \mu)$ be a WTP and $A \in \mathcal{F}$. Set $u := \lim \sup_{x \to \infty} \rho_A(x)$. If $u = 1$ there is nothing to prove, so assume $u < 1$. Let $\epsilon > 0$ be given. Let $u' \in [0, 1] \cap \mathbb{Q}$ such that $u' > u$ and $u' - u < \epsilon$. The idea is to construct a $Y \in \mathcal{M}$ such that we can easily see that $\mu(Y) = u'$ and $\rho_A(x) \leq \rho_Y(x)$ for all $x$, so that with M3 we get $\mu(A) \leq u'$.

First we observe that there is a $K > 0$ such that for all $x \geq K$ we have $\rho_A(x) \leq u'$. We can write $u'$ as $u' = \frac{p}{q}$ for some $p, q \in \mathbb{N}_0$ with $p \leq q$. Now we introduce the set $Y$ given by

$$Y := [0, K) \cup \bigcup_{i=0}^{\infty} [iq, iq + p).$$

Note that $Y \in \mathcal{C} \subseteq \mathcal{F}$. Lemma 4.1 and the fact that $\mu$ is a measure, gives us that $\mu(Y) = u'$. Further, observe that for each $x \in [0, \infty)$ we have $\rho_A(x) \leq \rho_Y(x)$, so with M3 we get

$$\mu(A) \leq \mu(Y) = u' < u + \epsilon.$$
Letting $\epsilon \downarrow 0$ we find
\[ \mu(A) \leq u = \limsup_{x \to \infty} \rho_A(x). \]
By applying this to $A^c$ we find
\[ \mu(A) = 1 - \mu(A^c) \geq 1 - \limsup_{x \to \infty} \rho_{A^c}(x) = \liminf_{x \to \infty} \rho_A(x). \]

Before we prove Theorem 3.2, we present the following alternative representation of $(A^{\text{uni}}, \alpha)$. We define for $A \in \mathcal{M}$ the map $\xi_A : (1, \infty)^2 \to [0, 1]$ given by
\[ \xi_A(D, x) := \frac{1}{\log(D)} \int_{D}^{x} \frac{\rho_A(y)}{y} dy. \]
(5.3)

Set
\[ S := \{ s \in (1, \infty)^\mathbb{N} : \lim_{n \to \infty} s_n = \infty \}. \]
(5.4)

If $s \in S$ and $f \in (1, \infty)^\mathbb{N}$, then we can interpret the pair $(s, f)$ as the sequence $(s_1, f_1), (s_2, f_2), \ldots$ in $(1, \infty)^2$. Write
\[ \mathcal{P} := \{ (s, f) : s \in S, f \in (1, \infty)^\mathbb{N} \} \]
(5.5)
for the collection of all such sequences.

For every $(s, f) \in \mathcal{P}$ we set
\[ A^{s, f} := \{ A \in \mathcal{M} : \lim_{n \to \infty} \xi_A(s_n, f_n) \text{ exists} \} \]
(5.6)
and
\[ \alpha^{s, f}(A) := \lim_{n \to \infty} \xi_A(s_n, f_n). \]
(5.7)

**Lemma 5.2 (Alternate Representation)** We have
\[ A^{\text{uni}} = \bigcap_{(s, f) \in \mathcal{P}} A^{s, f} \]
(5.8)
with for any $(s, f) \in \mathcal{P}$ and $A \in A^{\text{uni}}$
\[ \alpha(A) = \alpha^{s, f}(A). \]
(5.9)

**Proof** Let $A \in \mathcal{M}$. We start to relate $\log_\mu(A)$ and $\xi_A$. If $D, x \in (1, \infty)$, then
\[
\sigma_{\log_\mu(A)}(\log(D), \log(x)) = \frac{1}{\log(D)} \int_{\log(x)}^{\log(Dx)} 1_A(e^y)dy = \frac{1}{\log(D)} \int_{x}^{Dx} \frac{1_A(u)}{u} du = \frac{1}{\log(D)} \int_{x}^{Dx} S_A(u) u^{-1} du = \frac{1}{\log(D)} \left( \frac{S_A(u)}{u} \right)_{u=x}^{u=Dx} + \int_{x}^{Dx} \frac{S_A(u)}{u^2} du = \frac{\rho_A(Dx) - \rho_A(x)}{\log(D)} + \xi_A(D, x).
\]
(5.10)
This implies that for \((s, f) \in \mathcal{P}\) we have

\[
\mathcal{A}^{s,f} = \left\{ A \in \mathcal{M} : \lim_{n \to \infty} \sigma_{\log(A)}(\log(s_n), \log(f_n)) \text{ exists} \right\} \quad (5.11)
\]

with for \(A \in \mathcal{A}^{s,f}\)

\[
\alpha^{s,f}(A) = \lim_{n \to \infty} \sigma_{\log(A)}(\log(s_n), \log(f_n)). \quad (5.12)
\]

Since for any \(A \in \mathcal{M}\) and \((s, f) \in \mathcal{P}\)

\[
L(\log(A)) \leq \liminf_{n \to \infty} \sigma_{\log(A)}(\log(s_n), \log(f_n)) \quad (5.13)
\]

and

\[
\limsup_{n \to \infty} \sigma_{\log(A)}(\log(s_n), \log(f_n)) \leq U(\log(A)), \quad (5.14)
\]

we find that if \(\log(A) \in W_{\text{uni}}\), then \(A \in \mathcal{A}^{s,f}\) with \(\alpha^{s,f}(A) = \alpha(A)\).

The only thing left to show is that

\[
\bigcap_{(s, f) \in \mathcal{P}} \mathcal{A}^{s,f} \subseteq \mathcal{A}^{\text{uni}}. \quad (5.15)
\]

So assume \(A \in \bigcap_{(s, f) \in \mathcal{P}} \mathcal{A}^{s,f}\). Suppose we have \((s, f) \in \mathcal{P}\) such that \(\alpha^{s,f}(A) = L(\log(A))\) and \((s', f') \in \mathcal{P}\) such that \(\alpha^{s',f'}(A) = U(\log(A))\). Then we can create a new sequence given by

\[
s'' := (s_1, s'_1, s_2, s'_2, ...) \quad \text{and} \quad f'' := (f_1, f'_1, f_2, f'_2, ...). \quad (5.16)
\]

Because by assumption \(A \in \mathcal{A}^{s'',f''}\), we then have \(\alpha^{s,f}(A) = \alpha^{s',f'}(A)\). Hence \(A \in \mathcal{A}^{\text{uni}}\). So it is sufficient to show that we can choose \((s, f)\) and \((s', f')\) in the desired way.

Choose \(s \in \mathcal{S}\) such that

\[
\liminf_{n \to \infty} \sigma_{\log(A)}(\log(s_n), \log(x)) = \liminf_{D \to \infty} \inf_{x \in (1, \infty)} \sigma_{\log(A)}(\log(D), \log(x)). \quad (5.17)
\]

Choose \(f \in (1, \infty)^\mathbb{N}\) such that

\[
\left| \inf_{x \in (1, \infty)} \sigma_{\log(A)}(\log(s_n), \log(x)) - \sigma_{\log(A)}(\log(s_n), \log(f_n)) \right| < \frac{1}{n} \quad (5.18)
\]

for every \(n \in \mathbb{N}\). Then \((s, f) \in \mathcal{P}\) with

\[
\alpha^{s,f}(A) = \liminf_{D \to \infty} \inf_{x \in (1, \infty)} \sigma_{\log(A)}(\log(D), \log(x)) = L(\log(A)). \quad (5.19)
\]

In the same way choose \((s', f') \in \mathcal{P}\) such that

\[
\alpha^{s',f'}(A) = \limsup_{D \to \infty} \sup_{x \in (1, \infty)} \sigma_{\log(A)}(\log(D), \log(x)) = U(\log(A)). \quad (5.20)
\]

\[\square\]
We are ready to give the proof of Theorem \[ \text{3.2} \]

**Proof of Theorem \[ \text{3.2} \]** Notice that any intersection of \( f \)-systems satisfying \( F \) is again an \( f \)-system satisfying \( F \). Therefore, if we show that \(( \mathcal{A}^{s,f}, \alpha^{s,f} )\) is a WTP for every \((s, f) \in \mathcal{P}\), it follows from Lemma \[ \text{5.2} \] that \(( \mathcal{A}^{\text{uni}}, \alpha )\) is a WTP.

Let \((s, f) \in \mathcal{P}\). It immediately follows that \(( \mathcal{A}^{s,f}, \alpha^{s,f} )\) is a probability pair and that \(M_1\) and \(M_3\) hold, so we have to verify \(F\) and \(M_2\). Note that for every \(A, B \in \mathcal{M}\) and \(x > 0\) we have

\[
\rho_{A \circ B}(x) = \frac{1}{x} \int_0^x 1_{A \circ B}(y) \, dy = \frac{1}{x} \int_0^x 1_A(y) 1_B(S_A(y)) \, dy = \frac{1}{x} \int_0^{S_A(x)} 1_B(u) \, du = \frac{S_A(x)}{x} \frac{1}{S_A(x)} \int_0^{S_A(x)} 1_B(u) \, du = \rho_A(x) \rho_B(S_A(x)) = \rho_A(x) \rho_B(x \rho_A(x)) \tag{5.21}
\]

Let \( A \in \mathcal{C} \) and \( B \in \mathcal{A}^{s,f} \). Then

\[
\xi_{A \circ B}(s_n, f_n) = \frac{1}{\log(s_n)} \int_{f_n}^{s_n f_n} \frac{\rho_A(y) \rho_B(y \rho_A(y))}{y} \, dy \sim \lambda(A) \frac{1}{\log(s_n)} \int_{f_n}^{s_n f_n} \frac{\rho_B(\lambda(A) y)}{y} \, dy \tag{5.22}
\]

If \( \lambda(A) = 0 \) it is clear that \( A \circ B \in \mathcal{A}^{s,f} \) with \( \alpha^{s,f}(A \circ B) = 0 = \lambda(A) \alpha^{s,f}(B) \).

If \( \lambda(A) > 0 \), then we see that

\[
\int_{f_n}^{s_n f_n} \frac{\rho_B(\lambda(A) y)}{y} \, dy = \int_{\lambda(A) f_n}^{\lambda(A) s_n f_n} \frac{\rho_B(u)}{u} \, du \tag{5.23}
\]

Since

\[
\left| \int_{\lambda(A) f_n}^{\lambda(A) s_n f_n} \frac{\rho_B(u)}{u} \, du - \int_{f_n}^{s_n f_n} \frac{\rho_B(u)}{u} \, du \right| \leq \int_{\lambda(A) s_n f_n}^{s_n f_n} \frac{1}{u} \, du + \int_{\lambda(A) f_n}^{f_n} \frac{1}{u} \, du = 2 \log \left( \frac{1}{\lambda(A)} \right), \tag{5.24}
\]

we have

\[
\frac{1}{\log(s_n)} \int_{f_n}^{s_n f_n} \frac{\rho_B(\lambda(A) y)}{y} \, dy \sim \frac{1}{\log(s_n)} \int_{f_n}^{s_n f_n} \frac{\rho_B(u)}{u} \, du \sim \alpha^{s,f}(B) \tag{5.25}
\]

Thus \( A \circ B \in \mathcal{A}^{s,f} \) with

\[
\alpha^{s,f}(A \circ B) = \lambda(A) \alpha^{s,f}(B) = \alpha^{s,f}(A) \alpha^{s,f}(B) \tag{5.26}
\]

\( \square \)
For our proof of Theorem 3.3, we need an alternate expression for $U(\log(A))$. For $A \in \mathcal{M}$ set $\tau_A : (1, \infty) \times \mathbb{N} \to [0, 1]$ given by

$$
\tau_A(C, j) := \sigma_A(C^{j-1}(C-1), C^j-1) \quad (5.27)
$$

$$
= \frac{1}{C^j-1} \int_{C^j-1}^{C^j} 1_A(y)dy. \quad (5.28)
$$

Also set for $C > 1$ and $A \in \mathcal{M}$

$$
U^*(C, A) := \limsup_{n \to \infty} \sup_{k \in \mathbb{N}} \frac{1}{n} \sum_{j=k}^{k+n-1} \tau_A(C, j). \quad (5.29)
$$

**Lemma 5.3** For every $A \in \mathcal{M}$ we have

$$
\lim_{C \downarrow 1} U^*(C, A) = U(\log(A)). \quad (5.30)
$$

**Proof** Let $A \in \mathcal{M}$ and fix $C > 1$.

**Step 1** We show that

$$
U(\log(A)) = \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \frac{Q(D, x)}{D} \sum_{j=P(x)+1}^{P(D, x)} \frac{1}{C^j} \int_{C^j-1}^{C^j} 1_A(u)du, \quad (5.31)
$$

where

$$
P(x) := \left\lfloor \frac{x}{\log(C)} \right\rfloor \text{ and } Q(D, x) := \left\lfloor \frac{D + x}{\log(C)} \right\rfloor \quad (5.32)
$$

for $D, x \in (0, \infty)$.

Define

$$
E(D, x) := \sigma_{\log(A)}(D, x) - \frac{1}{D} \int_{C^{P(x)}}^{C^{Q(D, x)}} \frac{1}{u} 1_A(u)du. \quad (5.33)
$$

Since

$$
\sigma_{\log(A)}(D, x) = \frac{1}{D} \int_x^{x+D} 1_A(e^y)dy = \frac{1}{D} \int_{e^x}^{e^{x+D}} \frac{1}{u} 1_A(u)du, \quad (5.34)
$$

we have

$$
|E(D, x)| \leq \frac{1}{D} \int_{C^{P(x)+1}}^{C^{Q(D, x)+1}} \frac{1}{u} du + \frac{1}{D} \int_{C^{Q(D, x)}}^{C^{Q(D, x)+1}} \frac{1}{u} du = \frac{2}{D} \log(C). \quad (5.35)
$$

This implies

$$
U(\log(A)) = \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \sigma_{\log(A)}(D, x)
$$

$$
= \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \frac{1}{D} \int_{C^{P(x)+1}}^{C^{Q(D, x)+1}} \frac{1}{u} 1_A(u)du
$$

$$
= \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \frac{1}{D} \sum_{j=P(x)+1}^{Q(D, x)} \int_{C^j-1}^{C^j} \frac{1}{u} 1_A(u)du. \quad (5.36)
$$
**Step 2** We give an upper and lower bound for
\[ \int_{C^{j-1}}^{C^j} \frac{1_A(u)}{u} \, du \] (5.37)
in terms of \( \tau_A(C, j) \).
If we set for \( j \in \mathbb{N} \)
\[ \zeta(j) := \int_{C^{j-1}}^{C^j} 1_A(y) \, dy = \tau_A(C, j)(C - 1)C^{j-1}, \] (5.38)
then
\[ \int_{C^j - \zeta(j)}^{C^j} \frac{1}{u} \, du \leq \int_{C^{j-1}}^{C^j} \frac{1}{u} \, du \leq \int_{C^{j-1} + \zeta(j)}^{C^j} \frac{1}{u} \, du. \] (5.39)
We now observe that
\[ \int_{C^{j-1} + \zeta(j)}^{C^j} \frac{1}{u} \, du = \log \left( \frac{C^{j-1} + \zeta(j)}{C^j} \right) = \log(1 + (C - 1)\tau_A(C, j)) \] (5.40)
and
\[ \int_{C^j - \zeta(j)}^{C^{j-1}} \frac{1}{u} \, du = \log \left( \frac{C^j}{C^j - \zeta(j)} \right) = \log(C) - \log(1 + (C - 1)(1 - \tau_A(C, j))). \] (5.41)
The fact that \( \log(1 + y) \leq y \) for every \( y \geq 0 \), combined with (5.39), (5.40) and (5.41) gives
\[ \log(C) - (C - 1)(1 - \tau_A(C, j)) \leq \int_{C^{j-1}}^{C^j} \frac{1}{u} \, du \leq (C - 1)\tau_A(C, j). \] (5.42)

**Step 3** We combine Step 1 and Step 2 to finish the proof.
Set
\[ \gamma(D, x) := \frac{1}{Q(D, x) - P(x)} \sum_{j=P(x)+1}^{Q(D, x)} \tau_A(C, j) \] (5.43)
and observe that
\[ \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \gamma(D, x) = \limsup_{n \to \infty} \sup_{k \in \mathbb{N}} \frac{1}{n} \sum_{j=k}^{k+n-1} \tau_A(C, j) = U^*(C, A). \] (5.44)
We use (5.42) and (5.44) to find an upper bound for the expression in (5.36), giving us

\[ U(\log(A)) = \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \frac{1}{D} \sum_{j = \frac{Q(D,x)}{P(x)}}^{\frac{C-1}{D}} \int_{C^{j-1}}^{C^j} \frac{1}{u} \, du \]

\[ \leq \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \frac{C - 1}{D} (Q(D,x) - P(x)) \gamma(D, x) \]

\[ = \frac{C - 1}{\log(C)} \limsup_{D \to \infty} \sup_{x \in (0, \infty)} \gamma(D, x) \]

\[ = \frac{C - 1}{\log(C)} U^*(C, A). \]  

(5.45)

Analogously, we find that

\[ U(\log(A)) \geq 1 - \frac{C - 1}{\log(C)} (1 - U^*(C, A)). \]  

(5.46)

Combining (5.45) and (5.46) we obtain

\[ \frac{\log(C)}{C - 1} U(\log(A)) \leq U^*(C, A) \leq 1 - \frac{\log(C)}{C - 1} (1 - U(\log(A))), \]

(5.47)

which implies

\[ \lim_{C \downarrow 1} U^*(C, A) = U(\log(A)). \]  

(5.48)

We also need the following lemma.

**Lemma 5.4** Let \((\mathcal{F}, \mu)\) be a WTP. Then for any \(A \in \mathcal{F}\) and \(C > 1\)

\[ \mu(A) \leq C \sup_{j \in \mathbb{N}} \tau_A(C, j). \]  

(5.49)

**Proof** Let \((\mathcal{F}, \mu)\) be a WTP with \(A \in \mathcal{F}\). Fix \(C > 1\) and write

\[ S := \sup_{j \in \mathbb{N}} \tau_A(C, j). \]  

(5.50)

The idea is to introduce a set \(B \in \mathcal{M}\) for which we have \(\limsup_{x \to \infty} \rho_B(x) \leq CS\) and \(\rho_A(x) \leq \rho_B(x)\) for all \(x\). Set

\[ B := \bigcup_{j=1}^{\infty} [C^{j-1}, C^{j-1} + SC^{j-1}(C - 1)]. \]  

(5.51)
By construction of $B$ we have $\rho_A(x) \leq \rho_B(x)$ for every $x \in (0, \infty)$. So

$$\limsup_{x \to \infty} \rho_A(x) \leq \limsup_{x \to \infty} \rho_B(x)$$

$$= \limsup_{n \to \infty} \rho_B(C^n + SC^n(C - 1))$$

$$= \limsup_{n \to \infty} \frac{SC^{n+1}(C - 1)}{C^n + SC^n(C - 1)}$$

$$= \limsup_{n \to \infty} \frac{S(C^n+1 - 1)}{C^n + SC^n(C - 1)}$$

$$= \limsup_{n \to \infty} \frac{CS}{C^n + SC^n(C - 1)}$$

$$= \frac{CS}{1 + S(C - 1)} \leq CS.$$ (5.52)

By Proposition 2.4 we then find

$$\mu(A) \leq \limsup_{x \to \infty} \rho_A(x) \leq CS.$$ (5.53)

We are ready to give the proof of Theorem 3.3.

Proof of Theorem 3.3 Let $(F, \mu)$ be a WTP and $A \in F$. It is sufficient to show that

$$L(\log(A)) \leq \mu(A) \leq U(\log(A)).$$ (5.54)

We give the following example to give an idea of the proof that follows. Set

$$Z_1 := [2, 3) \cup [4, 5) \cup [6, 7) \cup \ldots,$$

$$Z_2 := [5, 6) \cup [9, 10) \cup [11, 12) \cup \ldots,$$

$$Z_3 := [7, 8) \cup [11, 12) \cup [15, 16) \cup \ldots.$$

Note that $Z_1, Z_2, Z_3 \in C$ are pairwise disjoint. Now, we set

$$A' := Z_1 \circ A + Z_2 \circ A + Z_3 \circ A.$$ (5.55)

Observe that for $j \geq 3$

$$\tau_{A'}(2, j) = \frac{1}{2} \left( \tau_A(2, j - 1) + \tau_A(2, j - 2) \right).$$ (5.56)

So we constructed a set $A'$ that on each interval $[2^{j-1}, 2^j)$ with $j \geq 3$ has an average that equals the average of the averages of $A$ on two consecutive intervals. By weak thinnability we find that $\mu(A') = \frac{1}{2} \mu(A) + \frac{1}{4} \mu(A) + \frac{1}{4} \mu(A) = \mu(A)$. If $\tau_A(2, j)$ is convergent or only oscillates a little, we can give a good upper bound of $\mu(A)$ using Lemma 5.4. Applying this strategy not only for $C = 2$ but for any $C > 1$ and averages of not only two but arbitrarily many averages on consecutive intervals, is what happens in the proof.
**Step 1** We construct a $\hat{A} \in F$.

Fix $C > 1$ and $n \in \mathbb{N}$. We split up $[C^{j-1}, C^j)$ into intervals of length 1 plus a remainder interval for every $j$. Set for $j \in \mathbb{N}$

$$N_j := \lfloor C^{j-1}(C - 1) \rfloor$$

(5.57)

and for $j \in \mathbb{N}$ and $l \in \{1, ..., N_j\}$

$$I(j, l) := [C^{j-1} + l - 1, C^{j-1} + l).$$

(5.58)

Write

$$R(j) := [C^{j-1} + N_j, C^j)$$

(5.59)

for the remainder interval, so that for every $j \in \mathbb{N}$ we have

$$[C^{j-1}, C^j) = R(j) \cup \bigcup_{l=1}^{N_j} I(j, l).$$

(5.60)

Set for $p \in \{0, ..., n\}$

$$a_p := \lfloor C^p \rfloor \text{ and } a := \sum_{p=0}^{n} a_p.$$

(5.61)

Now choose $u \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ we have

$$N_{u+j} \geq a(N_{n+j} + 1),$$

(5.62)

which can be done since $N_j$ is asymptotically equivalent with $C^{j-1}(C - 1)$. For $p \in \{0, ..., n\}$ and $k \in \{1, ..., a_p\}$ set

$$\zeta(p, k) := \sum_{i=0}^{p-1} a_i + k$$

(5.63)

and then define for $j \in \mathbb{N}$ and

$$I^{p,k}(j) := \bigcup_{l \in \mathbb{N}} I(j, al + \zeta(p, k)).$$

(5.64)

Let $p \in \{0, .., n\}$ and $k \in \{1, ..., a_p\}$. Note that

$$\bigcup_{j=1}^{\infty} I^{p,k}(u + j) \in \mathcal{C}$$

(5.65)

and that (5.62) guarantees that

$$m(I^{p,k}(u + j)) \geq C^{n+j-1}(C - 1) \geq C^{n-p+j-1}(C - 1)$$

(5.66)
for every $j \in \mathbb{N}$. Now choose
\[
  Z(p, k) \subseteq \bigcup_{j=1}^{\infty} I^{p,k}(u + j)
\]  
(5.67)
such that $Z(p, k) \in C$ and
\[
m(Z(p, k) \cap I^{p,k}(u + j)) = C^{n-p+j-1}(C - 1).
\]  
(5.68)
for every $j \in \mathbb{N}$. From this it directly follows that
\[
  \lambda(Z(p, k)) = \frac{C^n}{C^{p+u}}.
\]  
(5.69)

We now introduce
\[
  \hat{A} := \sum_{p=0}^{n} \sum_{k=1}^{a_p} Z(p, k) \circ A.
\]  
(5.70)
Observe that all the $Z(p, k)$ are disjoint. So the closedness of $F$ under weak thinning and disjoint unions implies that $\hat{A} \in F$.

**Step 2** We give an upperbound for $\mu(A)$ by first giving an upperbound for $\mu(\hat{A})$ and then relating $\mu(A)$ and $\mu(\hat{A})$.

A crucial property of $\hat{A}$ is that for $j \in \mathbb{N}$
\[
m([C^{u+j-1}, C^{u+j}] \cap \hat{A}) = \sum_{p=0}^{n} a_p m([C^{j+n-p-1}, C^{j+n-p}] \cap A).
\]  
(5.71)
Hence
\[
  \tau_{\hat{A}}(C, u + j) = C^{n-u} \sum_{p=0}^{n} a_p C^{-p} \tau_A(C, j + n - p)
\]  
\[
  \leq C^{n-u} \sum_{p=0}^{n} \tau_A(C, j + n - p)
\]  
(5.72)
\[
  \leq C^{n-u} \sup_{k \in \mathbb{N}} \sum_{j=k}^{k+n} \tau_A(C, j).
\]
We apply Lemma 5.4 for $\hat{A}$ and find with (5.72) that
\[
  \mu(\hat{A}) \leq C^{n-u+1} \sup_{k \in \mathbb{N}} \sum_{j=k}^{k+n} \tau_A(C, j).
\]  
(5.73)
The weak thinnability of $\mu$ gives that
\[
  \mu(\hat{A}) = \sum_{p=0}^{n} \sum_{k=1}^{a_p} \mu(Z(p, k)) \mu(A) = \mu(A) C^{n-u} \sum_{p=0}^{n} a_p C^{-p}.
\]  
(5.74)
Combining (5.73) and (5.74) gives
\[ \mu(A) = \frac{C_{u-n}}{\sum_{p=0}^{n} \alpha_p C_{-p}} \mu(\hat{A}) \]
\[ \leq \frac{C_{u-n}}{\sum_{p=0}^{n}(C_{p} - 1)C_{-p}} \mu(\hat{A}) \]
\[ \leq \frac{C_{u-n}}{n + 1 - \frac{1}{1+C_{u-n}}} \mu(\hat{A}) \]
\[ \leq C \frac{n + 1}{n + 1 - \frac{1}{1+C_{u-n}}} \sup_{k \in \mathbb{N}} \frac{1}{n + 1} \sum_{j=k}^{k+n} \tau_{A}(C, j). \quad (5.75) \]

**Step 3** We take limits in (5.75).
Unfix \( n \) and \( C \). We first take the limit superior for \( n \to \infty \) in (5.75), giving
\[ \mu(A) \leq C \limsup_{n \to \infty} \sup_{k \in \mathbb{N}} \frac{1}{n + 1} \sum_{j=k}^{k+n} \tau_{A}(C, j) = C^U(C, A). \quad (5.76) \]

Then we take the limit superior for \( C \downarrow 1 \) and find by Lemma 5.3 that
\[ \mu(A) \leq \limsup_{C \downarrow 1} U^*(C, A) = U(\log(A)). \quad (5.77) \]

The lower bound we can now easily obtain by applying our upper bound for the complement of \( A \). Doing this, we see that
\[ 1 - \mu(A) = \mu(A^c) \leq U(\log(A^c)) \leq 1 - L(\log(A)), \quad (5.78) \]
giving that \( \mu(A) \geq L(\log(A)) \). \( \Box \)

Using the alternate representation of \((A_\text{uni}, \alpha)\) (Lemma 5.2), we can prove Theorem 3.4.

**Proof of Theorem 3.4** We prove the contrapositive. Let \((\mathcal{F}, \mu)\) be a WTP with \( \mathcal{F} \setminus A_\text{uni} \neq \emptyset \). Let \( A \in \mathcal{F} \setminus A_\text{uni} \). By Lemma 5.2 this means that there is a \((s, f) \in \mathcal{P}\) such that
\[ I := \liminf_{n \to \infty} \xi_A(s_n, f_n) \neq \limsup_{n \to \infty} \xi_A(s_n, f_n) =: S. \]

Clearly, we can find \( m, l \in \mathbb{N}^\infty \) such that \( \xi_A(s_m, f_m) \) tends to \( I \) and \( \xi_A(s_l, f_l) \) tends to \( S \). Now set \( s'_n := s_m + n, f'_n := f_m, s'_n := s_l + n, f''_n := f_l + n \). Then we see that \( A \in A^{s', f'} \) and \( A \in A^{s'', f''} \) with
\[ \alpha^{s', f'}(A) = I \quad \text{and} \quad \alpha^{s'', f''}(A) = S. \]

In the proof of Theorem 3.2 we showed that \((A^{s', f'}, \alpha^{s', f'})\) and \((A^{s'', f''}, \alpha^{s'', f''})\) are both a WTP. Thus \((\mathcal{F}, \mu)\) does not have unique values. \( \Box \)
Without any further additional notation or lemma, the proofs of results in Section 4 are given below.

**Proof of Proposition 4.2** We give a proof along the lines of Mattila [9, p. 45], with small adaptations for completeness and more generality.

Let $(X, d)$ be a metric space and $\nu_1, \nu_2$ uniform measures on $X$. Write $h_1(r) := \nu_1(B(x, r))$ and $h_2(r) := \nu_2(B(x, r))$ for $r > 0$, which are well defined since $\nu_1$ and $\nu_2$ are uniform. We show that $\nu_1 = c \nu_2$ for some $c > 0$. It is sufficient to show that $\nu_1 = \nu_2$ on all open sets.

First let $A$ be an open set of $(X, d)$ with $\nu_1(A) < \infty$ and $\nu_2(A) < \infty$. Suppose that $r > 0$ is such that $h_2$ is continuous in $r$. Then

$$|\nu_2(A \cap B(x, r)) - \nu_2(A \cap B(y, r))| \leq \nu_2(B(x, r) \triangle B(y, r)) \leq \nu_2(B(x, r + d(x, y)) \setminus B(x, r)) \quad (5.79)$$

$$= h_2(r + d(x, y)) - h_2(r).$$

Hence $x \mapsto \nu_2(A \cap B(x, r))$ is a continuous mapping from $X$ to $[0, \infty)$. Since $h_2$ is nondecreasing, it can have at most countable many discontinuities. So we can choose $r_1, r_2, r_3, \ldots$ such that $\lim_{n \to \infty} r_n = 0$ and $h_2$ is continuous in every $r_n$.

For $n \in \mathbb{N}$ let $f_n : X \to [0, 1]$ be given by

$$f_n(x) := 1_A(x) \frac{\nu_2(A \cap B(x, r_n))}{h_2(r_n)}. \quad (5.80)$$

Notice that by our previous observation $f_n$ is continuous on $A$, hence $f_n$ is measurable. Because $A$ is open, we have $\lim_{n \to \infty} f_n(x) = 1$ for every $x \in A$. With Fatou’s Lemma we find

$$\nu_1(A) = \int_A \lim_{n \to \infty} f_n(x) \nu_1(dx) \leq \liminf_{n \to \infty} \frac{1}{h_2(r_n)} \int_A \nu_2(A \cap B(x, r_n)) \nu_1(dx) \quad (5.81)$$

$$\leq \liminf_{n \to \infty} \frac{1}{h_2(r_n)} \int_X \int_A 1_{B(x, r_n)}(y) \nu_2(dy) \nu_1(dx).$$

Note that any uniform measure is $\sigma$-finite. Applying Fubini’s theorem we obtain

$$\nu_1(A) \leq \liminf_{n \to \infty} \frac{1}{h_2(r_n)} \int_A \int_X 1_{B(x, r_n)}(y) \nu_1(dx) \nu_2(dy) \quad (5.82)$$

$$= \liminf_{n \to \infty} \frac{1}{h_2(r_n)} \int_A \nu_1(B(y, r_n)) \nu_2(dy)$$

$$= \liminf_{n \to \infty} \frac{h_1(r_n)}{h_2(r_n)} \nu_2(A).$$

By interchanging $\nu_1$ and $\nu_2$ we get

$$\nu_2(A) \leq \liminf_{n \to \infty} \frac{h_2(r_n)}{h_1(r_n)} \nu_1(A). \quad (5.83)$$
Note that \( \liminf_{n \to \infty} \frac{h_2(r_n)}{h_1(r_n)} > 0 \) since (5.83) would otherwise imply that all open balls are null sets. So we may rewrite (5.83) as

\[
\nu_1(A) \geq \frac{1}{\liminf_{n \to \infty} \frac{h_2(r_n)}{h_1(r_n)}} \nu_2(A)
\]

\[
= \limsup_{n \to \infty} \frac{h_1(r_n)}{h_2(r_n)} \nu_2(A)
\]

\[
\geq \liminf_{n \to \infty} \frac{h_1(r_n)}{h_2(r_n)} \nu_2(A).
\]

Hence \( \nu_1(A) = cv_2(A) \) with

\[
c := \liminf_{n \to \infty} \frac{h_1(r_n)}{h_2(r_n)} > 0.
\]  

(5.85)

Now let \( A \) be any open set of \((X,d)\). Let \( x \in X \) and set \( A_n := A \cap B(x,n) \) for \( n \in \mathbb{N} \). Note that \( A_n \) is open with \( \nu_1(A_n) \leq \nu_1(B(x,n)) < \infty \) and \( \nu_2(A_n) \leq \nu_2(B(x,n)) < \infty \). Hence, by the first part of the proof, we find \( \nu_1(A_n) = cv_2(A_n) \). But then

\[
\nu_1(A) = \lim_{n \to \infty} \nu_1(A_n) = \lim_{n \to \infty} cv_2(A_n) = cv_2(A).
\]  

(5.86)

\[\square\]

**Proof of Proposition 4.3** Fix \( A \in \mathcal{L}(X) \) and \( x,y \in X \). By (4.3) we have

\[
\lim_{u \to \infty} \frac{h(r^+(u))}{h(r^-(u))} = \lim_{u \to \infty} \frac{h(r^-(u)) + 1}{h(r^+(u))} = \lim_{r \to \infty} \frac{h(r+1)}{h(r)} = 1.
\]  

(5.87)

Hence

\[
\frac{\bar{\nu}(B(x,r^-) \cap A)}{h(r^-)} \sim \frac{\bar{\nu}(B(x,r^+)) \cap A)}{h(r^+)}.
\]  

(5.88)

Observe that for any \( r \in [0, \infty) \) we have

\[
\left| \frac{\bar{\nu}(B(x,r) \cap A)}{h(r)} - \frac{\bar{\nu}(B(y,r) \cap A)}{h(r)} \right| = \frac{1}{h(r)} |\bar{\nu}(A \cap B(x,r)) - \bar{\nu}(A \cap B(y,r))| \leq \frac{1}{h(r)} \nu(B(x,r) \Delta B(y,r)) \leq \frac{1}{h(r)} \nu(B(x,r + d(x,y)) \setminus B(y,r)) \leq \frac{h(r + d(x,y)) - h(r)}{h(r)}.
\]  

(5.89)

By (4.3), it follows that

\[
\frac{\bar{\nu}(B(x,r^-) \cap A)}{h(r^-)} \sim \frac{\bar{\nu}(B(y,r^-) \cap A)}{h(r^-)}.
\]  

(5.90)

Combining (5.88) and (5.90) gives the desired result. \[\square\]
By construction we have \( \rho \). Proof of Proposition 4.7

Suppose \( \nu(\alpha(A \cap B(o,u_n))) \). Then set \( w_1 := v_1 \) and \( w_n := v_n - v_{n-1} \) for \( n \geq 2 \). Define

\[
B := \bigcup_{n=1}^{\infty} [w_n, n].
\]

By construction we have \( \rho_B(n) = \bar{\rho}_A(n) \) for every \( n \in \mathbb{N} \). From (1.3) it follows that

\[
\lim_{n \to \infty} \sup_{x, y \in [n, n+1]} |\rho_A(x) - \rho_A(y)| = 0.
\]

This means \( \rho_B(u) \sim \bar{\rho}_A(u) \). So \( B \in \mathcal{A}^{uni} \) with \( \alpha(B) = \alpha^X(A) \). By Definition 4.5 we have that \( A \in \mathcal{F} \) and \( \mu(A) = \alpha^X(A) \). Hence \( \mathcal{A}^{uni}(X), \alpha^X \) has unique values with respect to being an EP.

Now let \( B \in \mathcal{L}(X) \setminus \mathcal{A}^{uni}(X) \). That means there are \( (s,f), (s', f') \in \mathcal{P} \) such that

\[
\lim_{n \to \infty} \tilde{\xi}_B(s_n, f_n) \neq \lim_{n \to \infty} \tilde{\xi}_B(s'_n, f'_n).
\]

Then \( (\mathcal{F}, \mu) \) and \( (\mathcal{F}', \mu') \) given by

\[
\mathcal{F} := \{ A \in \mathcal{L}(X) : \tilde{\xi}_A(s_n, f_n) \text{ converges} \},
\]

\[
\mu(A) := \lim_{n \to \infty} \tilde{\xi}_A(s_n, f_n),
\]

\[
\mathcal{F}' := \{ A \in \mathcal{L}(X) : \tilde{\xi}_A(s'_n, f'_n) \text{ converges} \},
\]

\[
\mu'(A) := \lim_{n \to \infty} \tilde{\xi}_A(s'_n, f'_n),
\]

are both an EP by Lemma 4.3 but \( \mu(B) \neq \mu'(B) \).

**Proof of Proposition 4.7** Suppose \( X = \mathbb{R}^n \) with \( d \) Euclidean distance. Set

\[
\delta_n := \frac{2\pi^{n/2}}{\Gamma(n/2)}.
\]

Let \( \nu \) be the Borel measure on \( \mathbb{R}^n \). Note that \( h(r) = n^{-1} \delta_n r^n \). If we set \( u = \sqrt{md_{n+1}} y \), then

\[
\int_{\mathbb{R}^d} \frac{\bar{\rho}_A(y)}{y} dy = \int_x^{x+D} \frac{\delta_n}{y^2} \int_0^{\sqrt{n\delta_n^{-1} y}} r^{n-1} K_A(r) dr dy
\]

\[
= \frac{\sqrt{n\delta_n^{-1} xD}}{n^{n+1}} \int_0^{u} r^{n-1} K_A(r) dr du.
\]

Now observe that by partial integration

\[
\int \frac{n^2}{u^{n+1}} \int_0^u r^{n-1} K_A(r) dr du = -u \int_0^u r^{n-1} K_A(r) dr + \int \frac{K_A(u)}{u} du.
\]

(5.97)
If we set for $D, x \in (1, \infty)\)

$$
\zeta_A(D, x) := \frac{1}{\log(D)u^n} \int_0^u r^{n-1}K_A(r)dr \bigg|_{u=x},
$$

then

$$
\bar{\zeta}_A(D^n, n^{-1}\delta_n x^n) = \zeta_A(D, x) + \frac{1}{\log(D)} \int_x^D \frac{K_A(u)}{u}du.
$$

Since $|\zeta_A(D, x)| \leq \frac{1}{\log(D)}$, the desired result follows.

6 Discussion

The natural analogue of an $\sigma$-algebra in finite additive probability theory is an algebra. Both Schurz and Leitgeb [10] and Wenmackers and Horsten [13] remark that the restriction of $M$ to $C$ is problematic since $C$ is not an algebra. However, any collection extending $C$ that is not $M$ itself, is not an algebra since $a(C) = M$. This can be seen as follows. Let $A \in M$ and set

\begin{align*}
A_+ &:= \{ a + 1 : a \in A \} \\
A_- &:= \{ a - 1 : a \in A \setminus [0, 1) \} \\
M_1 &:= \cup_{i=0}^{\infty} [2i, 2i+1), \\
M_2 &:= M_1^c, \\
X &:= (A \cap M_1) \cup (A_+^c \cap M_2), \\
Y &:= (A \cap M_2) \cup (A_-^c \cap M_1).
\end{align*}

Then $M_1, M_2, X, Y \in C$ with $\lambda(M_1) = \lambda(M_2) = \lambda(X) = \lambda(Y) = 1/2$ and $A = (M_1 \cap X) \cup (M_2 \cap Y)$. Hence $A \in a(C)$ and since $A \in M$ was arbitrary, we have $a(C) = M$.

This observation bring us to the conclusion that the requirement of an algebra, despite the fact that an algebra is the natural analogue of a $\sigma$-algebra, is too restrictive. Furthermore, finite additivity only dictates how a probability measure behaves when taking disjoint unions, and thus only suggests closedness under disjoint unions. Therefore, we think the requirement of an $f$-system rather than an algebra in Definition [1.1] is justified.

It should be noted that even if one prefers an algebra as the domain of a probability measure, hence $M$, our argument from Section 2 that being a WTP defines uniformity for a probability pair, remains to have implications. The probability measure $\lambda^*$ from [1.0], for example, is in general not a WTP. This is because the probability of $A \in A^{uni} \setminus C$ is uniquely determined by Theorem [3.3] whereas $\lambda^*(A)$ may be any value between $\liminf(\rho_A)$ and $\limsup(\rho_A)$. So $\lambda^*$ is not uniform for every Hahn-Banach extension of the limit operator. If HB-$\lim$ is a multiplicative Hahn-Banach extension (dominated by the limsup...
operator) of the limit operator on $L^\infty([0,\infty))$, the probability pair $(\mathcal{M}, \mu)$ given by
\[ \mu(A) := \lim_{D \to \infty} \frac{1}{\log(D)} \int_1^D \frac{1_A(y)}{y} dy, \]
(6.2)
is a WTP (this can be proven completely analogous to the proof of Theorem 3.2).

It might strike as a serious problem that in Section 2 we motivated thinnability as a natural property of a uniform probability measure, while the proposed probability measure $\alpha$ is only weakly thinnable, but not thinnable (appendix A). A thinnable probability measure on $\mathcal{A}^{\text{uni}}$, however, does not exist (appendix A). Since we are not looking for a strong $P$, but for a $P$ that leads to a canonical probability pair with a ‘big’ $f$-system, we prefer weak thinnability. It is for the same reason we, for example, do not require a probability measure to be countably additive: although this is in combination with M1 an a priori justifiable $P$, a measure pair with property $P$ does not even exist.

There may, of course, be another $P$ for which there is a probability pair which has unique values with respect to $P$ and has a bigger $f$-system than $\mathcal{A}^{\text{uni}}$. Choosing such a $P$, however, requires a justification for using $P$ as the definition of ‘uniformity’. At this point, we can not see any convincing motivation for such a property.

A typical example of a set in $\mathcal{M}$ that does not have natural density, but is assigned a probability by $\alpha$, is
\[ A := \bigcup_{n=0}^{\infty} [e^{2n}, e^{2n+1}), \]
(6.3)
for which we have $\alpha(A) = 1/2$. It is, however, unclear how ‘many’ of such sets there are, i.e. how much ‘bigger’ the $f$-system $\mathcal{A}^{\text{uni}}$ is than $\mathcal{C}$ and how much ‘smaller’ it is than $\mathcal{M}$. If we could construct a uniform probability measure on $\mathcal{M}$ by the method of Section 4 we could determine the probability of $\mathcal{A}^{\text{uni}}$ if $\mathcal{A}^{\text{uni}} \in \mathcal{A}^{\text{uni}}(\mathcal{M})$. To construct such a probability measure, we need to equip $\mathcal{M}$ with a metric $d$ such that $(\mathcal{M}, d)$ has a uniform measure. It is, however, not at all clear how we should choose $d$. So at this point, it is not clear if there is a useful way of measuring the collections $\mathcal{C}$ and $\mathcal{A}^{\text{uni}}$.

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Consider the set
\[ A = \bigcup_{n=0}^{\infty} [2^{2n}, 2^{2n+1}) \]  
(A.1)

We have \( A, A^C \in A^{uni} \) with \( \alpha(A) = \alpha(A^C) = 1/2 \). But also, we have \( A \circ A^C \in A^{uni} \) with
\[
\alpha(A \circ A^C) = \alpha \left( \bigcup_{n=0}^{\infty} \left[ 2^{2n} + \frac{1}{6} 2^{2n}, 2^{2n} + \frac{2}{3} 2^{2n} \right] \right)
\]
\[
= \alpha \left( \bigcup_{n=0}^{\infty} [2n \log(2) + \log(1 + 1/6), 2n \log(2) + \log(1 + 1/3)) \right) \]  
(A.2)
\[
= \frac{\log(1 + 2/3) - \log(1 + 1/6)}{2 \log(2)} \neq \frac{1}{4} = \alpha(A) \alpha(A^C). \]

So \( \alpha \) is not thinnable.

If \((\mathcal{F}, \mu)\) is a WTP with \( A, A^C \in \mathcal{F} \), then by Theorem 3.3 we also find that \( \mu \) is not thinnable.