The general classical solution of the superparticle

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The theory of vectors and spinors in 9+1 dimensional spacetime is introduced in a completely octonionic formalism based on an octonionic representation of the Clifford algebra $Cl(9,1)$. The general solution of the classical equations of motion of the CBS superparticle is given to all orders of the Grassmann hierarchy. A spinor and a vector are combined into a $3 \times 3$ Grassmann, octonionic, Jordan matrix in order to construct a superspace variable to describe the superparticle. The combined Lorentz and supersymmetry transformations of the fermionic and bosonic variables are expressed in terms of Jordan products.

I. INTRODUCTION

The relationship between the division algebras and the existence of supersymmetric theories has been observed before [1–3], especially in the context of string theory. In particular, the division algebras have been used to solve the classical equations of motion for the superparticle and the superstring [4]. However, because of the non-associativity of the octonions, there have been difficulties in the the case of this division algebra of highest dimension. For example, the Lorentz invariance of the formalism was unclear. Since the CBS superparticle [5] is an ideal testing ground to introduce techniques using division algebras and explore supersymmetry, it has attracted some attention; see I. Oda et al. [6] and H. Tachibana & K. Imaeda [7]. The connection of supersymmetric theories to the division algebras can also be made in terms of Jordan algebras as in [8] and especially [9].

This article carries on these previous attempts to cast the description of the superparticle in a form that clearly displays its symmetries; however, a more transparent and powerful octonionic formalism is used. We go beyond a mere rewriting of vector and spinor variables in terms of octonionic expressions, with supersymmetry and Lorentz transformations acting differently on these variables. We succeed in introducing a unified superspace variable as a Jordan matrix, which includes both fermionic and bosonic variables. Both the supersymmetry transformations and the general solution are expressed in terms of Jordan matrices involving both kinds of variables in this unified way.

There are many other approaches, not involving division algebras, which investigate supersymmetry in the superparticle, see for example [11].

This article is organized as follows. Section II introduces the octonionic formalism for vectors and spinors and their Lorentz transformations in 9+1 dimensions. (We deal exclusively with the 9+1-dimensional case. The analogues in 5+1, 3+1, and 2+1 dimensions can be found easily.) A subsection using the octonionic analogue of the Fierz-matrix [10] derives what we call the 3-$\Psi$’s rule, an identity that is needed for the Green-Schwarz superstring to be supersymmetric [12]. A note on the notion of octonionic dotted and undotted spinors concludes this introductory section. Section III derives the general classical solution of the equations of motion for the CBS superparticle. Section IV develops the Jordan matrix formalism combining bosonic and fermionic variables into one object. Lorentz and supersymmetry transformations and the superparticle action are expressed in this way.

II. OCTONIONIC SPINORS AND THE 3-$\Psi$’S RULE

A. Octonionic spinors

Octonionic spinors are based on an octonionic representation of a Clifford algebra. The non-associativity of the octonions raises obstacles which can be removed with care. A rigorous treatment and resolution of this issue can be found in [13], which also contains an introduction to octonions. Only general properties of octonions independent of a specific multiplication table will be used here. However, because we make frequent use of a variety of octonionic identities, the reader may find more information on octonions helpful; see [14,15].
The full Clifford algebra $\mathcal{C}l(9,1)$ in 9+1 dimensions has a real, faithful, irreducible, Weyl representation in terms of 32×32-matrices. (As a reference for the general topic of Clifford algebras see [16,13].) An octonionic Majorana-Weyl representation is given in terms of 4×4-matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & \Gamma_\mu \\ \tilde{\Gamma}_\mu & 0 \end{pmatrix}, \quad (1)$$

where

$$\begin{align*}
\Gamma_0 &= -\tilde{\Gamma}_0 = 1 = \begin{pmatrix} 10 \\ 01 \end{pmatrix}, \\
\Gamma_j &= \tilde{\Gamma}_j = \begin{pmatrix} 0 & e_j \\ e_j^* & 0 \end{pmatrix} \quad (1 \leq j \leq 8), \\
\Gamma_9 &= \tilde{\Gamma}_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma_{11} &= \gamma_0\gamma_1\ldots\gamma_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)
\end{align*}$$

(In our conventions an octonion $x$ has real components $x^j \ (1 \leq j \leq 8)$, i.e., $x = x^j e_j$, where $e_j \ (1 \leq j \leq 8)$ are the octonionic units and $e_j^*$ their octonionic conjugates. The signature of the metric is $- + \ldots +$.) This representation is understood to act on a column of four octonions, a spinor, by left multiplication. This notion is necessary in order to remove the ambiguity that arises from the fact that octonionic multiplication is not associative. The fundamental property $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ remains valid under this interpretation. (For a rigorous treatment see [13].)

A vector with components $x^\mu \ (0 \leq \mu \leq 9)$ is embedded in the Clifford algebra via

$$x^\mu = x^\mu\gamma_\mu = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \quad (3)$$

where

$$X = x^\mu\Gamma_\mu \quad \text{and} \quad \tilde{X} = x^\mu\tilde{\Gamma}_\mu. \quad (4)$$

(Boldface capitals always denote the 2×2 hermitian matrix associated as in (4) with the vector denoted by the same lowercase letter.) The inverse of this relationship is

$$x^\mu = \frac{1}{4} \text{Re tr} (\#^\mu) = \frac{1}{4} \text{Re tr} \left( X\tilde{\Gamma}_\mu + \tilde{X}\Gamma_\mu \right) = \frac{1}{2} \text{Re tr} \left( X\Gamma_\mu \right) = \frac{1}{2} \text{Re tr} \left( \tilde{X}\tilde{\Gamma}_\mu \right), \quad (5)$$

where indices are raised with the metric tensor $g$. Also note that

$$\Gamma_\mu = \tilde{\Gamma}_\mu \quad (6)$$

and

$$\tilde{X} = X - (\text{tr} (X)) 1, \quad (7)$$

which implies

$$XX = \tilde{X}X = X^2 - (\text{tr} (X)) X = -\det (X) 1, \quad (8)$$

since the characteristic polynomial for a hermitian 2×2-matrix $A$ is $p_A(z) = z^2 - \text{tr} (A) z + \det (A)$. The combination $\tilde{X}$ appears in the matrix product

$$\#\# = x_\mu x^\mu 1_{4\times4}, \quad (9)$$

so that we have

$$x_\mu x^\mu 1 = XX = -\det (X) 1$$

or its polarized form,
\[ 2x_\mu y^\mu 1 = X\tilde{Y} + Y\tilde{X} = \tilde{X}Y + \tilde{Y}X. \]

Now our convention for the numbering of the components of an octonion allows us to simply write
\[ X = \begin{pmatrix} x^+ & x^- \\ x^+ & -x^- \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} -x^+ & x^- \\ x^+ & x^- \end{pmatrix}, \quad \text{where} \quad x^\pm = x^0 \pm x^9. \]  

A full spinor \( \Psi \) is given by a column of four arbitrary octonions. It can be decomposed into its positive and negative chiral projections,
\[ \Psi_\pm := P_\pm \Psi, \]
via the projection operators
\[ P_\pm = \frac{1}{2}(1 \pm \gamma_{11}). \]

For the chiral projections either the top or the bottom two components vanish. Depending on the context we will often regard a chiral spinor as just the column of the two non-vanishing components. We also define the adjoint spinor:
\[ \overline{\Psi} := \Psi^\dagger A. \]

\( A \) is the matrix that intertwines the given representation with the hermitian conjugate representation:
\[ \gamma_\mu^\dagger A = A \gamma_\mu. \]

Then \( A \) is given up to a constant by
\[ A = \gamma_0 \gamma_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

\( ^\dagger \) denotes matrix transposition composed with octonionic conjugation.) The construction of a vector \( y \) out of two spinors \( \Phi \) and \( \Psi \) is done in the usual way:
\[ y_\mu := \text{Re} \left[ \Phi \gamma_\mu \Psi \right] = \text{Re} \left[ \Phi_+ \dagger \Gamma_\mu \Psi_+ \right] + \text{Re} \left[ \Phi_- \dagger \Gamma_\mu \Psi_- \right]. \]

So far we have built everything out of real octonions, i.e., the components \( x^j \) of an octonion \( x = x^j e_j \) were real numbers. However, in order to consider anticommuting spinors we need to introduce elements of a Grassmann algebra. This notion can be incorporated into the octonionic formalism by letting the octonionic components take values in a real Grassmann algebra of arbitrary, possibly infinite, dimension. (Formally, we extend the octonionic algebra by taking the tensor product with a Grassmann algebra over a real vector space, e.g., octonionic conjugation only affects the octonionic part.) Then the components of the octonions that make up an anticommuting spinor are odd Grassmannian. For the previous relation \( 13 \) we now obtain, in addition:
\[ y_\mu = \text{Re} \left[ \Phi \gamma_\mu \Psi \right] = -\text{Re} \left[ \overline{\Psi} \gamma_\mu \Phi \right] = -\text{Re} \left[ \Psi_+ \dagger \Gamma_\mu \Phi_+ \right] - \text{Re} \left[ \Psi_- \dagger \Gamma_\mu \Phi_- \right]. \]

The cyclic properties of the trace and the vanishing of the real parts of graded commutators and associators imply the following identities:
\[ y_\mu = -\text{Re tr} (\Psi \bar{\Phi}^\dagger_\mu) \]
\[ = \text{Re tr} (\Phi \bar{\Phi}^\dagger_\mu) \]
\[ = -\text{Re tr} (\Psi_+ \Phi_+^\dagger \bar{\Gamma}_\mu) - \text{Re tr} (\Psi_- \phi_-^\dagger \bar{\Gamma}_\mu) \]
\[ = \text{Re tr} (\Phi_+ \Psi_+^\dagger \bar{\Gamma}_\mu) + \text{Re tr} (\Phi_- \Psi_-^\dagger \bar{\Gamma}_\mu) \]
\[ = \frac{1}{2} \text{Re tr} \left( (\Phi_+ \Psi_+^\dagger - \Psi_+ \Phi_+^\dagger) \bar{\Gamma}_\mu \right) + \frac{1}{2} \text{Re tr} \left( (\Phi_- \Psi_-^\dagger - \Psi_- \Phi_-^\dagger) \bar{\Gamma}_\mu \right). \]   

(20)

The full power of the octonionic formalism becomes evident when we write \( y \) in terms of its Clifford representation \( Y \) and \( \bar{Y} \) without the use of the Dirac matrices, as follows:

\[ Y = (\Phi_+ \Psi_+^\dagger - \Psi_+ \Phi_+^\dagger) + (\Phi_- \Psi_-^\dagger - \Psi_- \Phi_-^\dagger), \]
\[ \bar{Y} = (\Phi_+ \Psi_+^\dagger - \Psi_+ \Phi_+^\dagger) + (\Phi_- \Psi_-^\dagger - \Psi_- \Phi_-^\dagger). \]

(21)

This form of writing \( Y \) and \( \bar{Y} \) involves the hermitian matrix product of two component spinors for which we have the following identity:

\[ (\Phi_\sigma \Psi_\sigma^\dagger - \Psi_\sigma \Phi_\sigma^\dagger) = (\Phi_\sigma \Psi_\sigma^\dagger - \Psi_\sigma \Phi_\sigma^\dagger) - \left( \text{tr} \left( \Phi_\sigma \Psi_\sigma^\dagger - \Psi_\sigma \Phi_\sigma^\dagger \right) \right) \]
\[ = (\Phi_\sigma \Psi_\sigma^\dagger - \Psi_\sigma \Phi_\sigma^\dagger) + (\Psi_\sigma^\dagger \Phi_\sigma - \Phi_\sigma^\dagger \Psi_\sigma) \mathbf{1}, \]

(22)

where \( \sigma \in \{+,-\} \). This relationship allows us to rewrite (21):

\[ Y = (\Phi_+ \Psi_+^\dagger - \Psi_+ \Phi_+^\dagger) + (\Phi_- \Psi_-^\dagger - \Psi_- \Phi_-^\dagger) + (\Psi_-^\dagger \Phi_- - \Phi_-^\dagger \Psi_-) \mathbf{1}, \]
\[ \bar{Y} = (\Phi_+ \Psi_+^\dagger - \Psi_+ \Phi_+^\dagger) + (\Psi_+^\dagger \Phi_+ - \Phi_+^\dagger \Psi_+) \mathbf{1} + (\Phi_- \Psi_-^\dagger - \Psi_- \Phi_-^\dagger). \]

(23)

These identities are plausible because of equation (20). To prove them we need to use the fact that the \( \Gamma_\mu \) are a basis for the space of the hermitian matrices. (Note that the matrices are grouped so that the combinations in parentheses are hermitian. In particular, their traces are real, which means that we may commute octonionic products and/or take octonionic conjugates.)

The distinct advantage of the octonionic formalism is that we can deal with spinors without having to use Dirac matrices. In fact, in the rest of the article Dirac matrices only appear to relate our results to the usual notation, but they are not used in any of the derivations.

### B. Lorentz transformations

In Clifford language the orthogonal group is the Clifford group which is generated by unit vectors. A unit vector \( p \) induces a reflection at a line both on vectors and on spinors via the transformations

\[ \hat{\Psi} \to \hat{\Psi}' = \hat{p} \hat{\Psi} \hat{p}^{-1}, \]
\[ \Psi \to \hat{p} \Psi. \]

(24)

Components of vectors parallel to \( p \) remain fixed, whereas those perpendicular to \( p \) are inverted. A pair of unit vectors \( p, q \) induces a rotation in the plane spanned by them, which means that the even part of the Clifford group corresponds to the simple orthogonal group. Specifically the 9+1-dimensional proper orthochronous Lorentz transformations are generated by

\[ X \to P(QXQ)P, \]
\[ \Psi_+ \to P(Q\Psi_+), \]
\[ \Psi_- \to \bar{P}(Q\Psi_-), \]

(25)

where \( p_\mu p^\mu = q_\mu q^\mu = 1 \). More details specifically about the effects of the non-associativity of the octonions are given in [13][14].
C. The 3-Ψ’s rule

The previous relationships (24), which represent part of the octonionic analogue of the Fierz identities, allow us to deduce the 3-Ψ’s rule for anticommuting 9+1-D Majorana-Weyl spinors: (Given \( \sigma \in \{+,-\} \), we take \( \Psi_k = P_\sigma \psi_k \forall k \in \{1,2,3\} \).)

\[
\gamma^\mu \Psi_1 \overrightarrow{\gamma}_3 \Psi_3 = (\Psi_2 \Psi_3^\dagger - \Psi_3 \Psi_2^\dagger) \Psi_1 = (\Psi_2 \Psi_3^\dagger - \Psi_3 \Psi_2^\dagger) \Psi_1 - \text{Re} \text{ tr} \left( \Psi_2 \Psi_3^\dagger - \Psi_3 \Psi_2^\dagger \right) \Psi_1 = (\Psi_2 \Psi_3^\dagger) \Psi_1 - (\Psi_3 \Psi_2^\dagger) \Psi_1 + \Psi_1 (\Psi_3^\dagger \Psi_2) - \Psi_1 (\Psi_2^\dagger \Psi_3).
\]

(26)

When we add the terms generated by cyclic permutations of the spinors, we may express the result in terms of associators of octonions. We may even treat both spinor components simultaneously by defining an associator for \( \Psi_1 \), \( \Psi_2 \), and \( \Psi_3 \).

\[
\left[ \Psi_1, \Psi_2^\dagger, \Psi_3 \right] := \Psi_1 (\Psi_2^\dagger \Psi_3) - (\Psi_1 \Psi_2^\dagger) \Psi_3
\]

(27)

This spinor associator is just a shorthand for the following expression involving associators of the components:

\[
\left[ \Psi_1, \Psi_2^\dagger, \Psi_3 \right] = \left( \begin{array}{c}
\left[ \Psi_{11}, \Psi_{21}^*, \Psi_{31} \right] + \left[ \Psi_{11}, \Psi_{22}^*, \Psi_{32} \right] \\
\left[ \Psi_{12}, \Psi_{21}^*, \Psi_{31} \right] + \left[ \Psi_{12}, \Psi_{22}^*, \Psi_{32} \right]
\end{array} \right),
\]

(28)

where \( \Psi_\alpha = \left( \begin{array}{c}
\psi_{\alpha,1}^\dagger \psi_{\alpha,2}^\dagger \\
\psi_{\alpha,2} \psi_{\alpha,1}^\dagger
\end{array} \right) \quad (\alpha = 1,2,3) \). The expression (28) is symmetric in the last two anticommuting spinors \( \Psi_2 \) and \( \Psi_3 \):

\[
\left[ \Psi_1, \Psi_2^\dagger, \Psi_3 \right] = \left[ \Psi_1, \Psi_3^\dagger, \Psi_2 \right],
\]

(29)

(The derivation is a little tricky. It uses the facts that octonionic conjugation of one of the arguments of an associator merely changes its sign, that the associator for (non-Grassmann) octonions is an antisymmetric function of its three arguments, and that the components of \( \Psi_2 \) and \( \Psi_3 \) appear symmetrically in (28).) Therefore we see that

\[
\gamma^\mu \Psi_1 \overrightarrow{\gamma}_3 \Psi_3 + \text{ cyclic} = -\left[ \Psi_2, \Psi_3^\dagger, \Psi_1 \right] + \left[ \Psi_3, \Psi_2^\dagger, \Psi_1 \right] + \left[ \Psi_1, \Psi_3^\dagger, \Psi_2 \right] - \left[ \Psi_1, \Psi_2^\dagger, \Psi_3 \right] + \left[ \Psi_2, \Psi_1^\dagger, \Psi_3 \right] - \left[ \Psi_3, \Psi_1^\dagger, \Psi_2 \right] = 0.
\]

(30)

This identity is required for the Green-Schwarz superstring to exhibit global fermionic supersymmetry [12]. This derivation shows that the 3-Ψ’s rule is a direct consequence of the alternativity of the octonionic algebra, i.e., the relevant part of the Fierz identities are naturally built into the algebraic structure of the octonions.

D. A note on dotted and undotted spinors

In 3+1 dimensions, the usual notion of dotted and undotted spinors for a complex representation of \( C\ell(3,1) \) arises from two facts: complex conjugation of the Dirac matrices induces another faithful irreducible representation of the Clifford algebra \( C\ell(3,1) \), and matrix transposition induces a faithful representation of the opposite Clifford algebra \( C\ell_{\text{opp}}(3,1) \). \( C\ell_{\text{opp}}(3,1) \) is the algebra obtained by defining \( a \cdot_{\text{opp}} b = b \cdot a \), where \( \cdot \) (resp. \( \cdot_{\text{opp}} \)) denotes multiplication in the abstract algebra (resp. its opposite). Therefore, the two irreducible representations \( \Gamma \) and \( \tilde{\Gamma} \) of the even subalgebra \( C\ell_0(3,1) \), are essentially just complex conjugates of each other, more precisely they are related by charge conjugation:

\[
\tilde{X}^{\tilde{B} B} = c^{BA} (X_{AA})^* e^{AB}
\]

(31)

(Note that for \( x \in \mathbb{C} \), (1) and (2) define a representation of \( C\ell(3,1) \).) This relationship (31) still holds in the octonionic case, although octonionic conjugation does not result in another representation, nor does matrix transposition give a representation for the opposite Clifford algebra, because octonionic multiplication is not commutative.
\[(\hat{p}\hat{q})^* \neq \hat{p}^* \hat{q}^*, \quad (\hat{p}\hat{q})^T \neq \hat{q}^T \hat{p}^T. \quad (32)\]

As a consequence \(\Phi\), defined by
\[
\Phi^\beta := \epsilon^\beta \bar{A} (\Psi_A)^* = \epsilon^\beta \bar{A} \Psi_A
\]
\[
\iff \Phi := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Psi_+^*, \quad (34)
\]
does not transform like a negative chirality spinor according to (25).

Therefore the naive expectation that there is no essential difference between the analogous octonionic and complex representations of \(\text{Cl}(9, 1)\) and \(\text{Cl}(3, 1)\) turns out to be wrong. For this reason we prefer to use the original relationship (7) as a definition for \(\tilde{X}\) rather than (31). Remarkably, we never have to use (31) in any derivation, which confirms that this relationship is not of primary importance.

Hermitian conjugation is still an antiautomorphism:
\[
(\hat{p}\hat{q})^\dagger = q^\dagger p^\dagger. \quad (35)
\]

We already used this fact to obtain a Dirac hermitian form which defines the spinor adjoint, see (15). So only two pairs of the four spinor spaces with lowered/raised, undotted/dotted indices are in close correspondence. This difference is related to the fact that spinors can be both Majorana and Weyl in 9+1 dimensions.

Actually, it is still possible to restore relations (32) and (33). Namely, one has to switch to the opposite octonionic algebra. For example, the octonionic conjugate of an octonionic representation is another representation when the original octonionic product is replaced by its opposite. This idea of utilizing various octonionic multiplication rules, for example, the rule for the opposite octonionic algebra, will be pursued further in [13].

### III. THE SUPERPARTICLE ACTION, THE EQUATIONS OF MOTION AND THEIR SOLUTION

The action in the Lagrangian form, or second order action, for the CBS-superparticle [5] is given by
\[
S = \int d\tau L(\tau), \quad (36)
\]
where
\[
L = \frac{1}{e} \pi_\mu \pi^\mu \\
= \frac{1}{e} \text{tr} \left( \Pi \tilde{\Pi} \right),
\]
\[
\pi_\mu = e^{-1} [\dot{x}_\mu + \sum_{A=1}^N \text{Re} \bar{\theta}^A \gamma_\mu \theta^A] = e^{-1} [\dot{x}_\mu + \sum_{A=1}^N \text{Re} \bar{\theta}^A \gamma_\mu \theta^A],
\]
\[
\Pi = e^{-1} [\dot{X} + \sum_{A=1}^N (\dot{\theta}^A \theta^A\dagger - \theta^A \dot{\theta}^A\dagger)],
\]
and the variables describing the superparticle are its spacetime position \(x_\mu\), a set of \(N\) Majorana-Weyl spinors \(\theta^A\), and \(e\) is the einbein or induced metric on the worldline. The following equations of motion are obtained from varying the action,

- with respect to \(e\):
  \[
  \pi_\mu \pi^\mu = 0
  \iff \text{tr} \left( \Pi \tilde{\Pi} \right) = 0; \quad (38)
  \]

- with respect to \(x\):
  \[
  \dot{\pi}_\mu = 0
  \iff \dot{\Pi} = 0; \quad (39)
  \]
• and with respect to $\theta^A$:

$$\mathbb{D}_\mu \dot{\tilde{\theta}}^A = \pi_\mu \tilde{\Gamma}_\mu \dot{\tilde{\theta}}^A = 0$$

$$\Longleftrightarrow \tilde{\Pi} \dot{\tilde{\theta}}^A = 0. \quad (40)$$

We solve the algebraic equations for $\tilde{\Pi}$ and $\dot{\tilde{\theta}}^A$. Equations (33) and (38) imply that $\pi$ is a constant lightlike vector. Such vectors can be parametrized by 9 even Grassmann parameters $\{\pi_1, \ldots, \pi_9\}$. This parametrization is unique for the future or past light cone in the regular case, i.e., if at least one of these components is invertible and therefore has non-zero body. In this case $\sum_{i=1}^9 \pi_i^0$ is invertible and has up to a sign a unique square root $\pi_0$, whence $\pi^+$ or $\pi^-$ is invertible.

Otherwise, in the singular case when all components of $\pi$ have zero body, there may not exist any $\pi_0$ to make $\pi_\mu$ lightlike, or there may be multiple possibilities. (For example, if the spatial components are all zero, then $\pi_0$ may be any even Grassmann number which squares to zero. These difficulties arise, because $x \mapsto x^2$ is not injective in the neighborhood of zero.) We do not have a parametrization of this variation of the trivial solution.

We parametrize $\Pi$ by two real even Grassmann numbers $|a|, |b|$ and an even Grassmann unit octonion $\tilde{r}$, where $\tilde{r}\tilde{r}^* = 1$.

$$\Pi = \begin{pmatrix} |a|^2 & |a||b|\tilde{r}^* \\ |a||b|\tilde{r}^* & |b|^2 \end{pmatrix}. \quad (41)$$

This parametrization does not cover the case where one of $\pi^+$ or $\pi^-$ is neither invertible nor a square. If $|a| = 0$ or $|b| = 0$, then $\tilde{r}$ is undetermined. We can solve (40) for $\Pi$, even in the pathological cases, by letting

$$\dot{\tilde{\theta}}^A = \Pi \xi^A = \left( \begin{array}{c} \pi^+ \\ \pi^- \end{array} \right) \xi_1^A + \left( \begin{array}{c} \pi^+ \\ -\pi^- \end{array} \right) \xi_2^A, \quad (42)$$

where $\xi^A$ is an odd Grassmann spinor. This solution relies on the weak form of associativity, the so-called alternativity, of the octonions, which makes products which involve not more then two full octonions and their octonionic conjugates associative. If $\pi^+$ (resp. $\pi^-$) is invertible, we may redefine $\xi_1^A \rightarrow \xi_1^A - \pi^\pm \xi_2^A$ (resp. $\xi_2^A \rightarrow \xi_2^A - \pi^\pm \xi_2^A$) to see that our solution only depends on one arbitrary odd Grassmann octonion function.

If we can write $\Pi$ as in (41), then

$$\dot{\tilde{\theta}}^A = \left( \begin{array}{c} |a| \\ |b|\tilde{r}^* \end{array} \right) (|a|\xi_1^A + |b|\tilde{r}\xi_2^A) = \Psi_0 \xi_0^A, \quad (43)$$

where $\Psi_0 = (\begin{array}{c} |a| \\ |b|\tilde{r}^* \end{array})$ is a commuting spinor and $\xi_0$ is an arbitrary odd Grassmann octonion function. So we have given the general classical solution for the CBS superparticle (except for a parametrization of the lightlike vector in the pathological cases). This parametrization depends on two real even parameters $|a|$ and $|b|$ and one even unit octonion $\tilde{r}$, and a set of $N$ odd octonion functions of $\pi$, $\xi_0^A$.

Our solution, involving the asymmetric $\Psi_0 = (\begin{array}{c} |a| \\ |b|\tilde{r}^* \end{array})$, raises questions about the Lorentz invariance of the parametrization. In the work of others [5], $\Pi$ is parametrized in terms of the most general possible commuting spinor $\Psi = (\begin{array}{c} a \\ b \end{array})$:

$$\Pi = \Psi \Psi^\dagger = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}. \quad (44)$$

However, this parametrization introduces a redundancy of 7 extra parameters that correspond to an octonionic unit sphere $S^7$, since only the combination $\tilde{r} = \frac{ab^*}{|a||b|}$ enters into the off diagonal elements of $\Pi$. For this general $\Psi = (\begin{array}{c} a \\ b \end{array})$, $\dot{\tilde{\theta}}^A = \Psi \xi^A$, with $\xi^A$ an arbitrary odd Grassmann octonion, is not necessarily a solution of (40) because of the non-associativity of the octonions. By choosing one specific $\Psi_0 = (\begin{array}{c} |a| \\ |b|\tilde{r}^* \end{array})$ we removed the redundancy and obtained the general solution $\dot{\tilde{\theta}}^A = \Psi_0 \xi_0^A$. In this case all products are associative because they involve only two independent octonionic directions, namely $\xi_0^A$ and $\tilde{r}$.

A recent article by Cederwall & Preitschopf [17] proposes to modify the octonionic product in a way that keeps track of non-associativity. We can apply their ideas to obtain an alternate form of the solution which avoids the specification of $\Psi_0$. 

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\[ \dot{\theta}^A = \Psi \circ \zeta^A \text{ or } \dot{\theta}^A = \Psi \circ \zeta_b^A, \]  

(45)

where \( x \circ y := |a|^{-2} (xa^*)(ay) \) and similar for \( \circ_b \). The new Grassmann functions \( \zeta^A_a \) and \( \zeta^A_b \) are related to \( \zeta^A_0 \) via

\[ \frac{a}{|a|} \zeta^A_0 = \zeta^A_a \quad \text{and} \quad \frac{b}{|b|} \zeta^A_0 = \zeta^A_b. \]

In line with [17] the proper interpretation of \( \Pi = \Psi \Psi^\dagger \) is to view it as an element of \( \mathbb{R} \times \mathbb{O} P^1 \), \( \mathbb{O} P^1 \) being the octonionic projective line. The sixteen parameters of \( \Psi \) are collapsed, using the even Grassmannian generalization of the Hopf [18] map: \( \mathbb{R} \times S^{15} \approx \mathbb{R} \times S^8 \times S^7 \). The singularities for \( |a| \) or \( |b| \) not invertible are caused by the fact that the particular coordinate maps cannot be extended to include both of these “points”.

Lorentz invariance is broken by specifying a certain multiplication rule for the octonionic product. Using a modified product adapted to the spinor components can be shown to restore the Lorentz invariance.

From any of the forms for \( \dot{\theta}^A \) we get \( \theta^A \) by simply integrating the arbitrary odd Grassmann octonion function, using the form of (42), for example,

\[ \theta^A = \Pi Z^A + \theta^A_0. \]  

(46)

So given \( \Pi, \theta^A \) is parametrized by an arbitrary odd Grassmann octonion function \( Z^A \) and a constant anticommuting spinor \( \theta^A_0 \).

The choice of \( Z^A \) corresponds to the freedom of the local fermionic supersymmetry discussed in the next section. In fact, this supersymmetry can be used to eliminate \( Z^A \).

**IV. THE JORDAN MATRIX FORMALISM**

This section carries on the attempts of Foot & Joshi [9] and Gürsey [2] to understand the symmetries of the superparticle. These include global supersymmetry, a local bosonic symmetry, usually called the \( \lambda \)-transformation, and a local fermionic supersymmetry, the \( \kappa \)-transformation. We combine a fermionic spinor variable \( \beta \) and a bosonic vector \( B \) and scalar \( b \) into one superspace object, namely a 3 × 3 Jordan matrix \( \mathcal{B} \):

\[ \mathcal{B} = \begin{pmatrix} B & \beta \\ \beta^\dagger & b \end{pmatrix}. \]  

(47)

(\( \beta \) corresponds to a positive chirality spinor.) The Jordan product for Jordan matrices with Grassmannian entries is taken to be defined as in [9], which is equivalent to taking the hermitian part of the matrix product:

\[ A \circ B := \frac{1}{2} (AB + (AB)^\dagger). \]  

(48)

The results of section II can be applied to obtain a generating set for all Lorentz transformations for a Jordan matrix:

\[ \mathcal{A} \rightarrow \mathcal{M} \mathcal{A} \mathcal{M}^\dagger, \text{ where } \mathcal{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \text{ } M = \tilde{\Gamma}_1, \text{ and } p_{\mu} p^\mu = 1. \]  

(49)

(\( \tilde{Q} \) in (25) has been replaced by the constant \( \tilde{\Gamma}_1 \), which is purely real and allows us to move the parentheses. This subset of transformations, of course, still generates all of the Lorentz transformations.)

For the superparticle we consider

\[ \mathcal{X} = \begin{pmatrix} X & e^{\frac{1}{2}\theta} \\ e^{\frac{1}{2}\theta} & e \end{pmatrix} \]  

(50)

as the fundamental superspace matrix. (It is clear from the solution in the previous section that the various fermionic variables decouple, which reflects a symmetry of the Lagrangian. In this section, we consider a single fermionic variable, i.e., \( N = 1 \).) The global supersymmetry transformation may then be written as

\[ (1 + \delta_\epsilon) \mathcal{X} = Z_\epsilon \circ \mathcal{X} \]

\[ = \begin{pmatrix} 12e^{-\frac{1}{2}\epsilon} & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} X & e^{\frac{1}{2}\theta} \\ e^{\frac{1}{2}\theta} & e \end{pmatrix}. \]  

(51)
Note that we used the non-hermitian matrix $Z$ for this transformation, which avoids the extension to larger matrices in $\mathbb{C}$.

The $\lambda$-transformation has a simple structure as well:

$$(1 + \delta_\lambda)X = X \circ Z_{\lambda^\dagger}$$

$$= \left(\begin{array}{cc} e^{\frac{i}{2} \theta} & \frac{1}{2} \theta \hat{\theta}^\dagger 1 \\ e^{\frac{i}{2} \hat{\theta}} & e \end{array}\right) \circ \left(\begin{array}{cc} 1 & 0 \\ \frac{1}{2} \lambda \hat{\lambda}^\dagger 1 \\ \end{array}\right)$$

$$= \left(\begin{array}{cc} X + \lambda (\theta \hat{\theta}^\dagger - \hat{\theta} \theta^\dagger) e^{\frac{i}{2} \theta (\theta^\dagger + \lambda \hat{\lambda}^\dagger)} \\ e^\frac{i}{2} (\theta^\dagger + \lambda \hat{\lambda}^\dagger) \end{array}\right).$$

We can also construct a superspace variable that contains the conjugate momentum $\Pi$ of $X$:

$$\mathcal{P}' := e^{-1} \hat{X} \circ \hat{Z}^\dagger$$

$$= e^{-1} \left(\begin{array}{cc} X (e^{\frac{i}{2} \theta}) & 1 \\ \frac{1}{2} \theta \hat{\theta}^\dagger 1 \\ \end{array}\right) \circ \left(\begin{array}{cc} \Pi & e^\frac{i}{2} \hat{\theta} \\ e^{-\frac{i}{2} \hat{\theta}} & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc} e^{-\frac{i}{2} \hat{\theta}} + \frac{1}{2} \hat{\theta} e^{-\frac{i}{2} \hat{\theta}} \\ e^{-\frac{i}{2} \hat{\theta}}^{\dagger} & 0 \end{array}\right).$$

However, we prefer to work better to postulate another superspace variable as the “conjugate” to $X$:

$$\mathcal{P} := \left(\begin{array}{cc} \Pi & e^{-\frac{i}{2} \hat{\theta}} \\ e^{-\frac{i}{2} \hat{\theta}}^{\dagger} & 0 \end{array}\right).$$

It can be used to give a pretty form for the solution of the equations of motion:

$$\mathcal{P} = (\phi_a \phi_a^\dagger)_a \text{ resp. } \mathcal{P} = (\phi_b \phi_b^\dagger)_b,$$

where

$$\phi_a = \left(\begin{array}{cc} a \\ b \\ e^{-\frac{i}{2} \zeta_a^*} \end{array}\right) \text{ resp. } \phi_b = \left(\begin{array}{cc} a \\ b \\ e^{-\frac{i}{2} \zeta_b^*} \end{array}\right),$$

products are evaluated using the modified product, taking the hermitian part is implied. This causes the $(3,3)$ component to vanish. This form exactly reproduces (35). It can be interpreted to be a Grassmannian extension of the octonionic projective line, which can also be defined as the matrices which are idempotent up to scale:

$$(\mathcal{P} \circ \mathcal{P})_a = (\text{tr } (\mathcal{P}))_a = (\mathcal{P} \circ \mathcal{P})_b.$$

The $\kappa$-transformation can also be obtained using $\mathcal{P}$:

$$\delta_\kappa \mathcal{X} = 4 \left(\begin{array}{cc} 0 & \theta \\ 0 & \hat{\theta} \end{array}\right) \circ \left(\begin{array}{cc} 0 & \kappa \\ \kappa^\dagger & 0 \end{array}\right).$$

Taking a closer look at this local fermionic symmetry, we realize that

$$\delta_\kappa \theta = \Pi \kappa, \quad \delta_\kappa X = \hat{\theta} \delta_\kappa \theta^\dagger - \delta_\kappa \theta \hat{\theta}^\dagger, \quad \delta_\kappa e = 2 (\hat{\theta}^\dagger \kappa - \kappa^\dagger \hat{\theta})$$

$$\Rightarrow \delta_\kappa \Pi = 2 [\kappa (\hat{\Pi} \hat{\theta})^\dagger - (\hat{\Pi} \hat{\theta}) \kappa^\dagger],$$

i.e., on shell $\delta_\kappa \Pi = 0$ and $\delta_\kappa \theta$ has the form of the general solution for $\hat{\theta}$. (The form of the transformation simplifies due to our choice to include the scale in the definition of $\Pi$.) We see that the $\kappa$-supersymmetry can be used to absorb
the arbitrary odd Grassmann octonion function in the solution for $\theta$, so that just a constant spinor remains. Therefore all solutions are generated by acting with $\kappa$-transformations on solutions of the form

$$\begin{align*}
\theta &= \theta_0, \\
\dot{X} &= e^{\pi} \\
\Rightarrow X &= E \Pi + X_0,
\end{align*}$$

(60)

where $\theta_0$ is a constant anticommuting spinor, $X_0$ is a constant vector, $\Pi$ is a constant lightlike vector, and $E = \int e(\tau) \, d\tau$ is the arclength along the worldline.

The Freudenthal product for Jordan matrices is defined by

$$X \ast Y := X \circ Y - \frac{1}{2} X(\text{tr}(Y)) - \frac{1}{2} (\text{tr}(X))Y + \frac{1}{2} [(\text{tr}(X))(\text{tr}(Y)) - \text{tr}(X \circ Y)] I.$$ 

(61)

This notion can be extended to Grassmannian Jordan matrices. The Lagrangian $L$ for the superparticle is then given by the following form which has the same appearance as the $E_6$ invariant trilinear form on the non-Grassmannian Jordan algebra:

$$L = -\text{tr} \left((P \ast X) \circ P\right).$$

(62)

Due to the antisymmetry in (62) with respect to the spinor variables only the $(3, 3)$ component of $X$ contributes, i.e., $X$ could be replaced by $\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$:

$$L = -\text{tr} \left(\left(P \ast \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}\right) \circ P\right).$$

(63)

Also only the upper $2 \times 2$ matrix $\Pi$ in $P$ contributes to the Lagrangian, so that $P$ can be replaced by $P'$ in either form. Alternate versions of this formalism, where the einbein $e$ is substituted by 1 in the superspace variable $X$, are possible, since the trilinear form only contains certain combinations of variables, i.e., products of two vectors and a scalar or of two spinors and a vector, both of which have units of length squared.

V. CONCLUSION

We have demonstrated the usefulness of the octonionic formalism in several ways in this article. We have solved the classical equations of motion for the CBS superparticle. The question of Lorentz covariance of the solution was answered using a modified octonionic product. The local fermionic transformation was seen to relate solutions and to absorb the arbitrary odd Grassmann octonion function in the solution for the fermionic variable. We have been able to express Lorentz and all known supersymmetry transformations in terms of Jordan products involving Jordan matrices with Grassmannian entries. However, the exact form of the objects that should be used in these expressions was unclear because of the cancellations due to the anticommuting variables. We believe that an extension to the Green-Schwarz superstring will fix the form of the expressions, if such an extension is possible. Another interesting avenue is to explore further the symmetries of the theory in terms of the Jordan matrices. In group manifolds are generalized to $S^7$ transformations. By taking a varying octonionic product into account, it may be possible to generalize (super) Lie groups in a similar way. An extension of the octonionic formalism off shell is needed to lead to a quantization of the theory in this formalism, but it may be the key to unlock the mysteries of the superstring.

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soli deo gloria
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