An Effective Discrete Recursive Method for Stochastic Optimal Control Problems

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Abstract. In this paper, we study the numerical method for stochastic optimal control problems (SOCPs). By reducing the optimal control problem to the discrete case, we derive a discrete stochastic maximum principle (SMP). With the help of this SMP, we propose an effective discrete recursive method for SOCPs with feedback control. We rigorously analyze errors of the proposed method and prove that the cost obtained by our method is of first-order convergence. Numerical experiments are carried out to support our theoretical results.

Keywords. stochastic optimal control, backward stochastic differential equations, maximum principle, recursive method, feedback control

AMS subject classifications. 60H35, 65C20, 93E20

1 Introduction

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P) \) be a complete filtered probability space, on which a \( d \)-dimensional standard Brownian motion \( W_t = (W^1_t, \ldots, W^d_t)^\top \) is given. Consider the following stochastic control system:

\[
\begin{aligned}
   dX_t &= b(t, X_t, u_t) \, dt + \sigma(t, X_t, u_t) \, dW_t, \\
   X_0 &= x_0 \in \mathbb{R}^n,
\end{aligned}
\]

with a cost functional

\[
J(u) = \mathbb{E} \left[ \int_0^T f(t, X_t, u_t) \, dt + h(X_T) \right].
\]

Here, \( u \) is the control variable valued in a convex subset \( U \subset \mathbb{R}^m \), \( X \) is the state process, and \( b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}, f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R} \) and \( h : \mathbb{R}^n \to \mathbb{R} \) are given functions.

An admissible control \( u \) is an \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \)-adapted process with values in \( U \) such that

\[ \mathbb{E} \left[ \int_0^T |u_t|^2 \, dt \right] < \infty. \]

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The set of admissible controls is denoted by $U[0,T]$. Our stochastic optimal control problem (SOCP) is to find a control $u^* \in U[0,T]$ such that

$$J(u^*) = \min_{u \in U[0,T]} J(u).$$

(1.3)

The process $u^*$ is called an optimal control. The state process $X^*$ corresponding to $u^*$ is called an optimal state process, and $(X^*, u^*)$ is called an optimal pair.

In practice, the control usually depends on the historical information of the state process. For example, in option pricing and portfolio optimization, people make current decisions based on the historical stock price information. It is worth noting that the most important control of this type is feedback control, that is, the control is given by the current state. More precisely, there exists a function $\phi$ such that $u_t = \phi(t,X_t)$ (see [26]). By observing the current state information, feedback control can be easily operated. For this reason, we assume that the optimal control is a feedback control in this paper.

However, the SOCP does not directly yield an explicit solution, and thus efficient numerical methods have been widely studied in recent years. Most of the existing numerical algorithms are based on the dynamic programming principle (DPP) and the associated Hamilton-Jacobi-Bellman (HJB) equations (see, e.g., [2, 6, 17, 18, 22–25]). While stochastic maximum principle (SMP) is a popular tool for theoretical studies of stochastic optimal control (see, e.g., [11, 14, 20, 21] and the references therein), it has not been widely used in numerical algorithms. Let us mention some recent works [4, 10], which proposed numerical algorithms for SOCPs based on the SMP, and their discussions are limited to the case where the control $u_t$ is deterministic function of $t$. Furthermore, by introducing the Euler method to solve the adjoint equation, Gong et al. [8] proposed a gradient projection algorithm for SOCPs and first obtained the rate of convergence for the deterministic control case. Recently, in [6], the authors propose a numerical algorithm for SOCPs with feedback control by means of forward backward stochastic differential equations (FBSDEs), but the convergence is not proved theoretically.

Our main results are the following. We first reduce the optimal control problem to the discrete case and obtain a discrete SMP. The discrete SMP coupled with the state and adjoint equations forms a discrete Hamiltonian system. Then we propose a discrete recursive method for SOCPs by approximating the discrete Hamiltonian system. Considering that the goal of the SOCP is to select an appropriate control to achieve the optimal cost, we rigorously analyze errors of the proposed method and proved that the cost obtained by our method is of first-order convergence. We remark that the numerical algorithm of our discrete recursive method is consistent with the algorithm in [6]. Several examples are presented to support the theoretical results.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries. By establishing a discrete SMP, we propose a discrete recursive algorithm for solving SOCPs in Section 3. In Section 4, we prove the main convergence results. In Section 5, various numerical tests are given to demonstrate high accuracy of our method.

## 2 Preliminaries

We recall some basic results about forward and backward stochastic differential equations (SDEs) in this section, which can be found in [12, 13, 19, 26]. We will use the following notations:
Assume the functions $L$ and the Lipschitz constant such that $(F : [0, T] \times \mathbb{R}^n \to \mathbb{R}) = 0$ for $k_1 = k$.

$C_{b,k}$: the set of continuously differentiable functions $\varphi : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with uniformly bounded partial derivatives $\partial^i_x \varphi$ and $\partial^i_x \partial^k_x \varphi$ for $l_1 \leq l$ and $k_1 \leq k$.

$C_{b,k,k}$: the set of continuously differentiable functions $\varphi : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ with uniformly bounded partial derivatives $\partial^i_x \varphi$ and $\partial^i_x \partial^k_x \varphi$ for $l_1 \leq l$ and $k_1 + k_2 \leq k$.

We first recall the following standard estimate of SDE.

**Lemma 2.1** Let $X^i_t$, $i = 1, 2$, be the solution of the following SDE:

$$X^i_t = X^i_0 + \int_0^t b^i(s, X^i_s) \, ds + \int_0^t \sigma^i(s, X^i_s) \, dW_s,$$

where $b^i = b^i(s, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma^i = \sigma^i(s, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are Lipschitz in $x$, $b^i(s, 0) \in L^2_2(0, T; \mathbb{R}^n)$ and $\sigma^i(s, 0) \in L^2_2(0, T; \mathbb{R}^{n \times d})$. Then there exists a constant $C > 0$ depending on $T$ and the Lipschitz constant such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^i_t - X^i_0|^2 \right] \leq C \mathbb{E} \left[ |X^i_0|^2 \right] + C \int_0^T \mathbb{E} \left[ |b^1(s, X^i_s) - b^2(s, X^i_s)|^2 + |\sigma^1(s, X^i_s) - \sigma^2(s, X^i_s)|^2 \right] \, ds.$$

The following lemma is the well-known Feynman-Kac formula, which gives the stochastic representation for the solutions to some parabolic partial differential equations (PDEs).

**Lemma 2.2** Assume the functions $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R}^n$ are uniformly Lipschitz continuous w.r.t. $(x, y, z)$ and continuous w.r.t. $t$, and the matrix-valued function $a = \sigma^T \sigma$ is uniformly elliptic. For any given $(t, x) \in [0, T] \times \mathbb{R}^n$, $(Y^{t,x}, Z^{t,x})$ is the solution of the following FBSDEs:

$$\begin{align*}
\dot{X}^i_s &= b^i(s, X^i_s) \, ds + \sigma^i(s, X^i_s) \, dW_s, \\
\dot{Y}^{t,x}_s &= -F(s, X^i_s, Y^{t,x}_s, Z^{t,x}_s) \, ds + Z^{t,x}_s \, dW_s, \\
\dot{Z}^{t,x}_s &= x, \\
Y^{t,x}_T &= g(X^{t,x}_T).
\end{align*}$$

Then $v(t, x) = Y^{t,x}_t$ is a unique solution of the following PDE:

$$\begin{align*}
\mathcal{L} v(t, x) &= -F(t, x, v(t, x), \sigma(t, x) \partial_x v(t, x)), \\
v(T, x) &= g(x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n,
\end{align*}$$

where $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L} = \partial_t + \sum_{i=1}^n b_i(t, x) \partial_{x_i} + \frac{1}{2} \sum_{i,j=1}^n \left[ \sigma \sigma^T \right]_{i,j}(t, x) \partial_{x_i} \partial_{x_j}.$$

Furthermore, for $k = 0, 1, 2, \ldots$, if $b, \sigma \in C^{1+k,2+2k}_b$, $F \in C^{1+k,2+2k,2+2k}_b$ and $g \in C^{2+2k+\alpha}_b$ for some $\alpha \in (0, 1)$, then $v \in C^{1+k,2+2k}_b$. 

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Remark 2.3 In the case when \( F(t, x, y, z) \equiv 0 \), it is easy to see \( v(t, x) = E \left[ g(X_t^x) \right] \), and (2.2) reduces to

\[
\begin{cases}
\mathcal{L}v(t, x) = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \\
v(T, x) = g(x).
\end{cases}
\] (2.3)

3 The discrete recursive method

In this section, we propose the discrete recursive method for SOCPs. Consider the stochastic control system (1.1) – (1.2). Let \((X^*, u^*)\) be the optimal pair defined in (1.3), and assume that \( u^*_i = \phi^*(t, X_t^i) \), where \( \phi^*: [0, T] \times \mathbb{R}^n \to U \) is a function. For simplicity of presentation, we suppose \( d = 1 \). We need the following assumption.

(A1) For \( \varphi = b, \sigma, f, h \) and \( \phi^*, \varphi_x, \varphi_u \) are continuous in \((t, x, u)\) and \( \varphi_x, \varphi_u \) are bounded.

For the time interval \([0, T]\) and a given positive integer \( N \), we use the following uniform partition:

\[ 0 = t_0 < t_1 < \cdots < t_N = T, \]

with \( \Delta t := t_{i+1} - t_i = T/N \), and denote \( \Delta W_{t_i+1} := W_{t_{i+1}} - W_{t_i} \) for \( 0 \leq i \leq N - 1 \). Define the piecewise admissible control set

\[
\mathcal{U}^N[0, T] = \left\{ u = \sum_{i=0}^{N-1} \phi_i(X_{t_i})I_{[t_i, t_{i+1})}(t) : \phi_i(\cdot) \in C^1_b(\mathbb{R}^n; U) \right\}.
\] (3.1)

This means that it takes the feedback value of the state process at \( t_i \) as the control on \([t_i, t_{i+1})\), and one can check that \( \mathcal{U}^N[0, T] \subset \mathcal{U}[0, T] \) is convex. Now we define the discrete optimal control problem over \( \mathcal{U}^N[0, T] \):

\[
J(\bar{u}) = \min_{u \in \mathcal{U}^N[0, T]} J(u).
\] (3.2)

We call \( \bar{u} \) a discrete optimal control, which has the following expression

\[
\bar{u}_i = \sum_{i=0}^{N-1} \phi_i(\bar{X}_{t_i})I_{[t_i, t_{i+1})}(t), \quad \phi_i(\cdot) \in C^1_b(\mathbb{R}^n; U).
\] (3.3)

The corresponding \( \bar{X} \) and \((\bar{X}, \bar{u})\) are called a discrete optimal state process and discrete optimal pair, respectively. We remark that \((\bar{X}, \bar{u})\) essentially depends on the time partition \( N \). For simplicity, we omit \( N \) without causing confusion. Set

\[
\begin{align*}
\begin{bmatrix} b(\cdot) \cr \sigma(\cdot) \cr b'(t) \cr \sigma'(t) \end{bmatrix} &= \begin{bmatrix} (b_1(\cdot), \ldots, b_n(\cdot))^\top \cr (\sigma_1(\cdot), \ldots, \sigma_n(\cdot))^\top \cr (b(t, \bar{X}_t, \phi_i(\bar{X}_{t_i})), \sigma(t, \bar{X}_t, \phi_i(\bar{X}_{t_i}))) \end{bmatrix}, \\
\begin{bmatrix} b_\ell \cr b_n \end{bmatrix}(\cdot) &= \begin{bmatrix} (b_{1\ell}(\cdot), \ldots, b_{n\ell}(\cdot)) \\
\vdots \\
(b_{n\ell}(\cdot), \ldots, b_{n\ell}(\cdot)) \end{bmatrix},
\end{align*}
\]
and the other derivatives can be similarly defined. Under assumption (A1), the discrete optimal state process \( \hat{X} \) can be uniquely solved by the following piecewise equation:

\[
\begin{align*}
\frac{d \hat{X}_t}{dt} &= b'(t)dt + \sigma'(t)dW_t, \quad t \in [t_i, t_{i+1}], \\
\hat{X}_{t_i} &= \hat{X}_{t_{i-1}}, \quad i = 0, 1, \ldots, N - 1,
\end{align*}
\]

(3.4)

where \( \hat{X}_0 = x_0 \).

### 3.1 Discrete stochastic maximum principle

In this subsection, we derive a discrete SMP, which plays an important role in the proposal of the discrete recursive method for SOCPs. For any fixed integer \( 0 \leq i \leq N - 1 \), take an arbitrary \( \phi_i(\cdot) \in C^1_b(\mathbb{R}^n; U) \). For each \( 0 \leq \varepsilon \leq 1 \), we introduce \( \phi_i^\varepsilon(\cdot) = \bar{\phi}_i(\cdot) + \varepsilon \delta \phi_i(\cdot) \in C^1_b(\mathbb{R}^n; U) \) on \([t_i, t_{i+1}]\) with \( \delta \phi_i(\cdot) = \phi_i(\cdot) - \bar{\phi}_i(\cdot) \).

Denote

\[
u_i^{\varepsilon} = \sum_{j=0}^{i-1} \phi_j(X_{t_i}^\varepsilon)I_{[t_j, t_{j+1}]}(t) + \phi_i(X_{t_i}^\varepsilon)I_{[t_i, t_{i+1}]}(t) + \sum_{j=i+1}^{N-1} \bar{\phi}_j(X_{t_j}^\varepsilon)I_{[t_j, t_{j+1}]}(t).
\]

(3.5)

It is easy to see that \( \nu_i^{\varepsilon} \in \mathcal{U}^N[0, T] \), and the corresponding state process \( X_i^{\varepsilon} \equiv \hat{X}_t \) on \([0, t_i] \),

\[
\begin{align*}
\frac{dX_i^{\varepsilon}}{dt} &= b(t, X_i^{\varepsilon}, \phi_i(X_{t_i}^\varepsilon))dt + \sigma(t, X_i^{\varepsilon}, \phi_i(X_{t_i}^\varepsilon))dW_t, \\
X_{t_i}^{\varepsilon} &= \hat{X}_{t_i}, \quad t \in [t_i, t_{i+1}],
\end{align*}
\]

(3.6)

The variational equation \( \hat{X}_i \) can be given as follows: \( \hat{X}_i \equiv 0 \) on \([0, t_i] \),

\[
\begin{align*}
\frac{d\hat{X}_i}{dt} &= \left[ b_{xz}(t)\hat{X}_i! + b_{z}(t)\delta \phi_i(\hat{X}_{t_i}) \right] dt \\
&\quad + \left[ \sigma_z(t)\hat{X}_i! + \sigma_z(t)\delta \phi_i(\hat{X}_{t_i}) \right] dW_t, \\
\hat{X}_{t_i} &= 0, \quad t \in [t_i, t_{i+1}],
\end{align*}
\]

(3.7)

We also introduce the following adjoint equation:

\[
\begin{align*}
-d\hat{P}_i &= H_x(\hat{X}_{t_i}, \hat{P}_{t_i}, \hat{Q}_i, \bar{\phi}_i(\hat{X}_{t_i}))dt - \hat{Q}_idW_t, \\
\hat{P}_{t_i+1} &= \hat{P}_{t_{i+1}}, \quad t \in [t_i, t_{i+1}], \quad i = 0, 1, \ldots, N - 1,
\end{align*}
\]

(3.8)

with \( \hat{P}_T = h_x(\hat{X}_T) \), where the Hamiltonian \( H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times U \rightarrow \mathbb{R} \) is defined as follows:

\[
H(t, x, p, q, u) = \langle p, b(t, x, u) \rangle + \langle q, \sigma(t, x, u) \rangle + f(t, x, u),
\]

and denote

\[
H_x(\cdot) = (H_{x_1}(\cdot), \ldots, H_{x_n}(\cdot))^T, \quad H_u(\cdot) = (H_{u_1}(\cdot), \ldots, H_{u_m}(\cdot))^T, \quad h_x(\cdot) = (h_{x_1}(\cdot), \ldots, h_{x_n}(\cdot))^T.
\]

Now we establish the following discrete SMP.
Theorem 3.1 Suppose (A1) holds. Let $(\hat{X}, \hat{u})$ be the discrete optimal pair of the problem (3.2), and let $(P, Q)$ be the solution to (3.9). Then for $i = N - 1, \ldots, 1, 0$,

$$
E \left[ \int_{t_i}^{t_{i+1}} H_u(t, \hat{X}_t, \hat{P}_t, Q_t, \hat{\phi}_i(\hat{X}_t)) dt, \phi_i(\hat{X}_t) - \hat{\phi}_i(\hat{X}_t) \right] \geq 0, \quad \forall \phi_i(\cdot) \in C^0_\delta(\mathbb{R}^n; U). \tag{3.10}
$$

Furthermore, if $\phi_i(x)$ is an interior point of $U$, for $x \in \mathbb{R}^n$, then

$$
E \left[ \int_{t_i}^{t_{i+1}} H_u(t, \hat{X}_t, \hat{P}_t, Q_t, \phi_i(x)) dt \bigg| \hat{X}_{t_i} = x \right] = 0, \quad P_{\hat{X}_{t_i}} \text{-a.s.} \tag{3.11}
$$

In order to prove Theorem 3.1 we need the following lemmas.

Lemma 3.2 Suppose (A1) holds. Then for any integer $0 \leq i \leq N - 1$,

$$
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} E \left[ |\hat{X}^{i, \varepsilon}_t|^2 \right] = 0, \tag{3.12}
$$

where $\hat{X}^{i, \varepsilon}_t = \varepsilon^{-1}[X^{i, \varepsilon}_t - \hat{X}_t] - \hat{X}_t.$

Proof. For the proof of lemma, one can refer to Lemma 4.1 of [3].

Lemma 3.3 Suppose that the conditions in Theorem 3.1 hold. The value function

$$
V^i(x) := E \left[ \int_{t_i}^{t_{i+1}} f(t, \hat{X}^{i+1, x}_t, \phi_i(\hat{X}_t)) dt + V^{i+1}(\hat{X}^{i+1, x}_{t_{i+1}}) \right], \quad x \in \mathbb{R}^n, \quad i = N - 1, \ldots, 1, 0,
$$

with $V^N(x) = h(x)$, where $\hat{X}^{i, x}_t$ is the solution of (3.4) starting from $(t_i, x)$, and $(P^{i, x}, Q^{i, x})$ is the solution of (3.9) related to $\hat{X}^{i, x}_t$. Assume $V^{i+1}_{x}(x) = \hat{P}^{i+1, x}_{t_{i+1}}$, for some integer $0 \leq i \leq N - 1$. Then $V^i_x(x) = \hat{P}^{i, x}_t$ if and only if

$$
E \left[ \int_{t_i}^{t_{i+1}} (\bar{\phi}_{i,x}(x))^T H_u(t, \hat{X}^{i+1, x}_t, \hat{P}^{i, x}_t, \hat{Q}^{i, x}_t, \bar{\phi}_i(x)) dt \right] = 0. \tag{3.13}
$$

Proof. For any integer $i$, we define

$$
f^{i}_x(t) = f_x(t, \hat{X}^{i, x}_t, \phi_i(\hat{X}^{i, x}_t)), \quad f^{i}_u(t) = f_u(t, \hat{X}^{i, x}_t, \phi_i(\hat{X}^{i, x}_t)),
$$

where $f^{i}_x(\cdot) = (f^{i}_x(\cdot), \ldots, f^{i}_{i, \lambda}(\cdot))^T$ and $f^{i}_u(\cdot) = (f^{i}_u(\cdot), \ldots, f^{i}_{i, \lambda}(\cdot))^T$. By the classical variational method, one can check that

$$
V^i_x(x) = E \left[ \int_{t_i}^{t_{i+1}} (\bar{X}^{i+1, x}_t)^T f^{i}_x(t) + (\bar{\phi}_{i,x}(x))^T f^{i}_u(t) dt + (\hat{X}^{i+1, x}_{t_{i+1}})^T V^{i+1}(\hat{X}^{i+1, x}_{t_{i+1}}) \right], \quad x \in \mathbb{R}^n, \tag{3.14}
$$

where

$$
\begin{cases}
    d\hat{X}^{i, x}_t = b'_u(t, \hat{X}^{i, x}_t, \phi_i(\hat{X}^{i, x}_t)) dt + \sigma'_u(t, \hat{X}^{i, x}_t, \phi_i(\hat{X}^{i, x}_t)) dW_t, \\
    \hat{X}^{i, x}_{t_{i+1}} = I, \quad t \in [t_i, t_{i+1}].
\end{cases}
$$

Applying Itô’s formula to $(\bar{X}^{i+1, x}_t)^T \hat{P}^{i+1, x}_t$ on $[t_i, t_{i+1}]$, we have

$$
E \left[ (\bar{X}^{i+1, x}_t)^T \hat{P}^{i+1, x}_t - I \hat{P}^{i+1, x}_t \right] = E \left[ \int_{t_i}^{t_{i+1}} ((\bar{\phi}_{i,x}(x))^T (b'_u(t))^T \hat{P}^{i+1, x}_t + (\sigma'_u(t))^T \hat{Q}^{i+1, x}_t) - (\hat{X}^{i+1, x}_t)^T f^i_x(t) dt \right],
$$

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which implies
\[ P_{t_{i+1}} = E \left[ (X_{t_{i+1}} - x)^T \right. \]
\[ \left. + \int_t^{t_{i+1}} \left[ (X_{t_{i+1}} - x)^T \right. \]
\[ \left. \left( \begin{array}{c} f^i_x(t, \tilde{X}_t, \tilde{\phi}_t) \\ f^i_u(t, \tilde{X}_t, \tilde{\phi}_t) \end{array} \right) \right] dt \right]. \quad (3.15) \]

Notice that
\[ V^{i+1}_x(X_{t_{i+1}}) = P_{t_{i+1}} X_{t_{i+1}} = P_{t_{i+1}} x. \quad (3.16) \]

Combining 3.14, 3.15 and 3.16, the proof is complete. \[ \Box \]

**Proof of Theorem 3.1** For simplicity of presentation, in the following of this proof we only consider the case \( n = m = d = 1 \). This method is still applicable to the multi-dimensional case. For convenience, we use the following notations:
\[ V^i_x(t, \lambda) = V^i_x \left( X_t + \lambda (X^i_{t_{i-1}} - X_t) \right), \]
\[ f_x^i(t, \lambda) = f_x \left( t, X_t + \lambda (X^i_{t_{i-1}} - X_t), \tilde{\phi}_t \right) + \lambda (\phi^i_t - \tilde{\phi}_t), \]
\[ f_u^i(t, \lambda) = f_u \left( t, X_t + \lambda (X^i_{t_{i-1}} - X_t), \tilde{\phi}_t \right) + \lambda (\phi^i_t - \tilde{\phi}_t). \]

First, we consider \( u^{N-1, \varepsilon} \) on \([t_{N-1}, T]\). Since \( V^N(x) = h(x), x \in \mathbb{R} \), by Taylor’s expansion, we have
\[ J(u^{N-1, \varepsilon}) - J(\bar{u}) \]
\[ = E \left[ \int_{t_{N-1}}^T f \left( t, X^N_{t_{i-1}}, \phi_{N-1}(X_{t_{i-1}}) \right) \right. \]
\[ \left. \left( \tilde{X}_t, \tilde{\phi}_t \right) \right] dt \]
\[ + E \left[ V^N(\tilde{X}_T) \right] \]
\[ = E \left[ \int_{t_{N-1}}^T \int_0^1 f_x^N(t, \lambda) (X^{N-1, \varepsilon}_t - X_t) d\lambda \right. \]
\[ \left. \int_0^1 f_u^N(t, \lambda) \left( \phi^i_{N-1}(X_{t_{i-1}}) - \phi_{N-1}(X_{t_{i-1}}) \right) d\lambda \right] \]
\[ + E \left[ \int_0^1 V^N(\tilde{X}_T) (X^{N-1, \varepsilon}_t - X_t) d\lambda \right]. \]

Applying Itô’s formula to \( \tilde{X}_t^{N-1} \) on \([t_{N-1}, T]\) and noting that \( \tilde{P}_T = V^N(\tilde{X}_T) \) and \( \tilde{X}_t^{N-1} = 0 \), we have
\[ E \left[ V^N(\tilde{X}_T) \tilde{X}_t^{N-1} \right] = E \left[ (\tilde{P}_T \tilde{X}_t^{N-1} + \tilde{Q}_T \sigma_{\tilde{X}_T}^{N-1}(t)) \right. \]
\[ \left. \left( \phi^i_{N-1}(X_{t_{i-1}}) - f_x^N(t, X_t) \right) \right] dt. \]

Since \( u \) is the discrete optimal control, by Lemma 3.2 and 3.18, for any \( \phi_{N-1} \in C^1_b(\mathbb{R}; U) \), we obtain
\[ 0 \leq \lim_{\varepsilon \to 0} \frac{J(u^{N-1, \varepsilon}) - J(\bar{u})}{\varepsilon} \]
\[ = E \int_{t_{N-1}}^T \left[ f_x^N(t, \tilde{X}_t^{N-1} + f_u^N(t, \phi_{N-1}(X_{t_{i-1}}))) \right. \]
\[ \left. \left( \tilde{X}_t^{N-1}, \phi_{N-1}(X_{t_{i-1}}) \right) \right] dt + V^N(\tilde{X}_T) \tilde{X}_t^{N-1} \]
\[ = E \left[ \int_{t_{N-1}}^T H_u \left( t, \tilde{X}_t, \tilde{P}_t, \tilde{Q}_t, \phi_{N-1}(X_{t_{i-1}}) \right) \phi_{N-1}(X_{t_{i-1}}) dt \right]. \]
Furthermore, if \( \tilde{\phi}_{N-1}(x) \) is an interior point of \( U \), for \( x \in \mathbb{R} \), then \( \delta \phi_{N-1}(x) \) can be positive or negative, which implies
\[
\mathbb{E} \left[ \int_{t_{N-1}}^{T} H_u(t, \tilde{X}_t, \tilde{P}_t, \tilde{Q}_t, \tilde{\phi}_{N-1}(x)) \, dt \middle| \tilde{X}_{t_{N-1}} = x \right] = 0, \quad P_{X_{t_{N-1}}}-\text{a.s.} \quad (3.20)
\]
Second, based on the preceding discussion, we consider \( u^{N-2,\varepsilon} \) on \([t_{N-2}, T]\). By the definition of \( V^{N-1}(\cdot) \) and Taylor’s expansion, we have
\[
J(u^{N-2,\varepsilon}) - J(\bar{u}) = \mathbb{E} \left[ \int_{t_{N-2}}^{t_{N-1}} \left[ f(t, X_t^{N-2,\varepsilon}, \bar{\phi}_{N-2}(X_{t_{N-1}})) - f(t, \bar{X}_t, \bar{\phi}_{N-2}(X_{t_{N-1}})) \right] \, dt \right]
+ \mathbb{E} \left[ V^{N-1}(X_{t_{N-1}}^{N-2,\varepsilon}) - V^{N-1}(\bar{X}_{t_{N-1}}) \right]
= \mathbb{E} \left[ \int_{t_{N-2}}^{t_{N-1}} \int_{0}^{1} f_x^{N-2}(t, \lambda) (X_t^{N-2,\varepsilon} - \bar{X}_t) \, d\lambda \, dt \right]
+ \int_{t_{N-2}}^{t_{N-1}} \int_{0}^{1} f_{u}^{N-2}(t, \lambda) \left[ \bar{\phi}_{N-2}(X_{t_{N-1}}) - \bar{\phi}_{N-2}(\bar{X}_{t_{N-1}}) \right] \, d\lambda \, dt
+ \mathbb{E} \left[ \int_{0}^{1} V_x^{N-1}(t_{N-1}, \lambda) (X_{t_{N-1}}^{N-2,\varepsilon} - \bar{X}_{t_{N-1}}) \, d\lambda \right].
\]
By Lemma \(3.2\) it yields that
\[
0 \leq \lim_{\varepsilon \to 0} \frac{J(u^{N-2,\varepsilon}) - J(\bar{u})}{\varepsilon}
= \mathbb{E} \left[ V_x^{N-1}(\tilde{X}_{t_{N-1}}) \tilde{X}_{t_{N-1}}^{N-2} + \int_{t_{N-2}}^{t_{N-1}} \left[ f_x^{N-2}(t) \tilde{X}_{t}^{N-2} + f_u^{N-2}(t) \delta \phi_{N-2}(\tilde{X}_{t_{N-1}}) \right] \, dt \right]. \quad (3.21)
\]
From (3.20), we know
\[
\mathbb{E} \left[ \int_{t_{N-1}}^{T} H_u(t, X_t^{N-1,x}, P_t^{N-1,x}, Q_t^{N-1,x}, \tilde{\phi}_{N-1}(x)) \, dt \middle| (\phi_{N-1}(x) - \tilde{\phi}_{N-1}(x)) \right] \geq 0,
\]
for any \( \phi_{N-1}(\cdot) \in C^1_b(\mathbb{R}; U) \). If \( \tilde{\phi}_{N-1}(x) \) is an interior point of \( U \),
\[
\mathbb{E} \left[ \int_{t_{N-1}}^{T} H_u(t, X_t^{N-1,x}, P_t^{N-1,x}, Q_t^{N-1,x}, \tilde{\phi}_{N-1}(x)) \, dt \right] = 0,
\]
otherwise if \( \tilde{\phi}_{N-1}(x) \) is a boundary point of \( U \), \( x \) is an extreme point of \( \tilde{\phi}_{N-1} \) and \( \tilde{\phi}_{N-1,x}(x) = 0 \). Thus
\[
\mathbb{E} \left[ \int_{t_{N-1}}^{T} H_u(t, X_t^{N-1,x}, P_t^{N-1,x}, Q_t^{N-1,x}, \tilde{\phi}_{N-1}(x)) \tilde{\phi}_{N-1,x}(x) \, dt \right] = 0. \quad (3.22)
\]
Seeing that \( V_x^{N}(\tilde{X}_T) = \tilde{P}_T \), by Lemma \(3.3\) we derive \( V_x^{N-1}(\tilde{X}_{t_{N-1}}) = \tilde{P}_{t_{N-1}} \). Applying Itô’s formula to \( \tilde{P}_t \tilde{X}_{t}^{N-2} \) on \([t_{N-2}, t_{N-1}]\) and noting that \( \tilde{X}_{t_{N-2}} = 0 \), we have
\[
\mathbb{E} \left[ V_x^{N-1}(\tilde{X}_{t_{N-1}}) \tilde{X}_{t_{N-1}}^{N-2} \right] = \int_{t_{N-2}}^{t_{N-1}} \mathbb{E} \left[ \left( \tilde{P}_t b^{N-2}(t) + \tilde{Q}_t \sigma^{N-2}_a(t) \right) \delta \phi_{N-2}(\tilde{X}_{t_{N-2}}) - f_x^{N-2}(t) \tilde{X}_t^{N-2} \right] \, dt. \quad (3.23)
\]
Combining (3.21) and (3.23), we obtain
\[
E \left[ \int_{t_{N-2}}^{t_{N-1}} H_u (t, \bar{X}_t, \bar{P}_t, \bar{Q}_t, \bar{\phi}_{N-2}(\bar{X}_{t_{N-2}})) \, dt \right] \geq 0, \quad \forall \bar{\phi}_{N-2}(\cdot) \in C^1_b (\mathbb{R}; U).
\]
Furthermore, if \( \bar{\phi}_{N-2}(x) \) is an interior point of \( U \), for \( x \in \mathbb{R} \), then
\[
E \left[ \int_{t_{N-2}}^{t_{N-1}} H_u (t, \bar{X}_t, \bar{P}_t, \bar{Q}_t, \bar{\phi}_{N-2}(x)) \, dt \right] \bigg| \bar{X}_{t_{N-2}} = x = 0, \quad P_{\bar{X}_{t_{N-2}}} \text{-a.s.}
\]
Similarly, consider the above process on \([t_{N-3}, T], [t_{N-4}, T], \ldots, [t_0, T] \), and then our conclusion follows. ■

To sum up, for \( i = N - 1, \ldots, 1, 0 \), the discrete optimal control \( \bar{u}_t = \bar{\phi}_i (\bar{X}_t) \) on \( t \in [t_i, t_{i+1}] \) can be determined by the following discrete Hamiltonian system:
\[
\begin{align*}
\bar{X}_t &= \bar{X}_{t_i} + \int_{t_i}^t b (s, \bar{X}_s, \bar{\phi}_i (\bar{X}_s)) \, ds + \int_{t_i}^t \sigma (s, \bar{X}_s, \bar{\phi}_i (\bar{X}_s)) \, dW_s, \quad (3.24) \\
\bar{P}_t &= P_{t_{i+1}} + \int_{t_i}^t H_x (s, \bar{X}_s, \bar{P}_s, \bar{Q}_s, \bar{\phi}_i (\bar{X}_s)) \, ds - \int_{t_i}^t \bar{Q}_s \, dW_s, \quad (3.25) \\
E \left[ \int_{t_i}^{t_{i+1}} H_u (t, \bar{X}_t, \bar{P}_t, \bar{Q}_t, \bar{\phi}_i (\bar{X}_t)) \, dt, \phi_i (\bar{X}_t) - \bar{\phi}_i (\bar{X}_t) \right] &\geq 0, \quad \forall \phi_i (\cdot) \in C^1_b (\mathbb{R}^n; U), \quad (3.26)
\end{align*}
\]
where \( \bar{X}_0 = x_0 \) and \( \bar{P}_T = \bar{h}_x (\bar{X}_T) \).

### 3.2 Numerical approach for SOCPs

The discrete SMP provides a necessary condition for solving the discrete optimal control. The primary challenge in applying the discrete SMP to solve the discrete optimal control \( \bar{u}_t \) is to obtain \( H_u (t, \bar{X}_t, \bar{P}_t, \bar{Q}_t, \bar{u}_t) \), which is described by the solution \((\bar{X}_t, \bar{P}_t, \bar{Q}_t)\) of the FBSDEs (3.24) – (3.25). Let \( P^N_t (x), Q^N_t (x) \) and \( \phi^N_t (x) \) be the discrete approximations of \( \bar{P} (t, x), \bar{Q} (t, x) \) and \( \bar{\phi}_i (x) \) respectively. Based on the approximation of the discrete SMP (3.26) and the FBSDEs (3.24) – (3.25), we propose the following numerical scheme for the discrete Hamiltonian system (3.24) – (3.26).

**Scheme 3.4** Assume that \( X^N_0 \) and \( P^N_N \) are known. For \( i = N - 1, \ldots, 1, 0 \), solve \( P^N_i (x), Q^N_i (x) \) and \( \phi^N_i (x) \) with \( x \in \mathbb{R}^n \) by
\[
\begin{align*}
X^N_{i+1} &= x + b (t_i, x, \phi^N_i (x)) \Delta t + \sigma (t_i, x, \phi^N_i (x)) \Delta W_{t_{i+1}}, \quad (3.27) \\
Q^N_i (x) &= E^x_{t_i} [P^N_{i+1} \Delta W_{t_{i+1}}] / \Delta t, \quad (3.28) \\
P^N_i (x) &= E^x_{t_i} [P^N_{i+1}] + H_x (t_i, x, P^N_i (x), Q^N_i (x), \phi^N_i (x)) \Delta t, \quad (3.29) \\
\langle H_u (t_i, x, P^N_i (x), Q^N_i (x), \phi^N_i (x)), \phi_i (x) - \phi^N_i (x) \rangle &\geq 0, \quad \forall \phi_i (x) \in U, \quad (3.30)
\end{align*}
\]
where \( P^N_{i+1} \) is the value at space point \( X^N_{i+1} \).

In the case where \( \phi^N_i (x) \) is an interior point of \( U \), Scheme 3.4 becomes Scheme 3.5.
Scheme 3.5 Assume that $X_0^N$ and $P_0^N$ are known. For $i = N - 1, \ldots, 1, 0$, solve $P_i^N(x), Q_i^N(x)$ and $\phi_i^N(x)$ with $x \in \mathbb{R}^n$ by

$$
X_{t+1}^N = x + b(t_i, x, \phi_i^N(x)) \Delta t + \sigma \left(t_i, x, \phi_i^N(x)\right) \Delta W_{t+1},
$$

(3.31)

$$
Q_i^N(x) = \mathbb{E}_{t_i}^x \left[ P_{i+1}^N \Delta W_{t+1} \right] / \Delta t,
$$

(3.32)

$$
P_i^N(x) = \mathbb{E}_{t_i}^x \left[ P_{i+1}^N \right] + H_x \left(t_i, x, P_i^N(x), Q_i^N(x), \phi_i^N(x)\right) \Delta t,
$$

(3.33)

$$
H_u \left(t_i, x, P_i^N(x), Q_i^N(x), \phi_i^N(x)\right) = 0,
$$

(3.34)

where $P_{i+1}^N$ is the value at space point $X_{i+1}^N$.

Moreover, denote

$$
u_i^N = \sum_{i=0}^{N-1} \phi_i^N(X_{t_i}^N) \mathcal{I}_{[t_i,t_{i+1}]}(t),
$$

(3.35)

where the state process

$$
\begin{align*}
\left\{ 
& dX_t^N = b(t, X_t^N, \phi_t^N(X_t^N)) \, dt + \sigma(t, X_t^N, \phi_t^N(X_t^N)) \, dW_t, \\
& X_{t_i}^N = X_0^N, \quad t \in [t_i, t_{i+1}], \quad i = 0, 1, \ldots, N - 1,
\end{align*}
$$

(3.36)

and $X_0^N = x_0$. Then

$$
J(u^N) = \mathbb{E} \left[ \int_0^T f(t, X_t^N, u_t^N) \, dt + h(X_T^N) \right].
$$

Remark 3.6 Numerical methods for FBSDEs have been a hot topic recently (see [1] [2] [3] [4] [5] [27] [28] and the references therein). In this paper, we choose the Euler-type method for solving FBSDEs proposed in [27] and [25]. The conditional expectations $\mathbb{E}_{t_i}^x [P_{i+1}^N] := \mathbb{E}[P_{i+1}^N | X_{t_i}^N = x]$ and $\mathbb{E}_{t_i}^x [P_{i+1}^N \Delta W_{t+1}] := \mathbb{E}[P_{i+1}^N \Delta W_{t+1} | X_{t_i}^N = x]$ in Scheme 3.4 and Scheme 3.5 are functions of Gaussian random variables, which can be approximated by Gauss-Hermite quadrature with high accuracy.

Remark 3.7 For fixed $x$, $H_u \left(t_i, x, P_i^N, Q_i^N, \phi_i^N(x)\right)$ in (3.30) and (3.34) is a deterministic function of the variable $y = \phi_i^N(x)$. Many classical numerical methods can be used to solve (3.30) and (3.34), such as gradient descent method, fixed-point iterative method, Newton’s Method, Bisection method and so on. We assume that $\phi_i^N(x)$ can be solved accurately.

Remark 3.8 Once the control $u^N$ is obtained, by introducing the following equation

$$
Y_t^N = h(X_T^N) + \int_t^T f(s, X_s^N, u_s^N) \, ds - \int_t^T Z_s^N \, dW_s,
$$

(3.37)

then the cost $J(u^N) = Y_0^N$ can be obtained by solving the FBSDEs (3.36) = (3.37).

3.2.1 Summary of the discrete recursive algorithm

To do this, we introduce the following uniform space partition $\mathcal{D}_h = \mathcal{D}_{1,h} \times \mathcal{D}_{2,h} \times \cdots \times \mathcal{D}_{n,h}$, where $\mathcal{D}_{j,h}$ is the partition of the one-dimensional real axis $\mathbb{R}$

$$
\mathcal{D}_{j,h} = \left\{ x_j / x_j = k \right\},
$$

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for $j = 1, 2, \ldots, n$ and $h$ is a suitable spatial step.

In the numerical algorithm, we employ the following iterative algorithm to optimize control

\[ \phi^N_{i+1} (x) = \phi^N_i (x) - \rho^l H_u \left( t_i, x, P^N_i (x), Q^N_i (x), \phi^N_i (x) \right), \quad l = 0, 1, \cdots, \tag{3.38} \]

where $\rho^l$ is the step-size for the iteration. Now we summarize our discrete recursive algorithm.

**Algorithm 1** Framework of the discrete recursive method

1. Set $P^N_i (x) = h_x (x)$, $x \in D_h$ and the error tolerance $\varepsilon$.
2. for $i = N - 1 \rightarrow 0$
   1. for each $x \in D_h$
      1. Choose $\phi^N_0 (x) \in U$ and set $l = 0$.
      2. repeat
         1. Solve $(P^N_i (x), Q^N_i (x))$ by (3.28) − (3.29).
         2. Update $\phi^N_{i+1} (x)$ by (3.38). Set $l = l + 1$.
      3. until $|\phi^N_{i+1} (x) - \phi^N_i (x)| \leq \varepsilon$.
   3. end for
3. end for
4. Compute $u^N_t$ by (3.35).

**Algorithm 1** presents the procedure for our discrete recursive method. We run the algorithm in a backward manner to obtain the values \{ $\phi^N_i (x)$ \}_{i=0}^{N-1}, x \in D_h$, which are the control values in time-space mesh. Then we can compute the control value $u^N_t$ based on grid point interpolation.

## 4 Convergence analysis

We will give the convergence results of the discrete recursive method in this section. In the following, $C$ represents a generic constant which does not depend on the time partition and may be different from line to line. We now give an estimate for the state process $X^N_t$.

**Lemma 4.1** Suppose (A1) holds. We also assume \{ $\phi^N_i (x)$ \}_{i=1}^{N-1} \in C^1_b$, and there exists a positive constant $L$, not depending on $N$, such that $\sup_i |\phi^N_i (0)| \leq L$. Then for $m \geq 2$,

\[ E \left[ \sup_{0 \leq s \leq T} |X^N_s|^{m} \right] \leq C \left( 1 + |x_0|^m \right). \tag{4.1} \]

**Proof.** Rewrite the state equation \[ (3.36) \] as follows:

\[ X^N_s = x_0 + \int_0^s \tilde{b} (r, X^N_r) \, dr + \int_0^s \tilde{\sigma} (r, X^N_r) \, dW_r, \]

where for $r \in [t_i, t_{i+1})$ ($i = 0, \ldots, N - 1$),

\[ \tilde{b} (r, x) = b (r, \phi^N_i (X^N_{t_i})), \quad \tilde{\sigma} (r, x) = \sigma (r, x, \phi^N_i (X^N_{t_i})). \]
For $m \geq 2$, by the standard estimate of SDE, one can derive that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s^N|^m \right] \leq C \left[ |x_0|^m + \mathbb{E} \left[ \int_0^t |\tilde{b}(s,X_s^N)|^m \, ds \right] + \mathbb{E} \left[ \int_0^t |\tilde{\sigma}(s,X_s^N)|^m \, ds \right] \right]
\leq C \left[ |x_0|^m + \int_0^t \mathbb{E} \left[ |\tilde{b}(s,0)|^m + |\tilde{\sigma}(s,0)|^m + |X_s^N|^m \right] \, ds \right].
\]

Notice that for $s \in [t_i,t_{i+1}]$,
\[
|\tilde{b}(s,0)| \leq C (1 + |\phi^N_t(X_s^N)|) \leq C (1 + |X_s^N|),
\]
\[
|\tilde{\sigma}(s,0)| \leq C (1 + |\phi^N_t(X_s^N)|) \leq C (1 + |X_s^N|).
\]

Hence
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s^N|^m \right] \leq C (1 + |x_0|^m) + C \int_0^t \mathbb{E} \left[ \sup_{u \leq s \leq \tau} |X_s^N|^m \right] \, ds,
\]
where $C$ is a positive constant not depending on $N$. By the Gronwall inequality, the required result (4.1) follows.

\[\square\]

**Remark 4.2** The above conclusion also holds for state processes $X_t^*$ and $\bar{X}_1$.

We need the following assumption:

\[(A2)\] There exists a constant $c_0 > 0$, such that for each $(t_i,x) \in [0,T] \times \mathbb{R}^n$,
\[
\langle H_u \left( t_i, x, P_{t_i}^N, Q_{t_i}^N, \phi_i^N(x) \right) - H_u \left( t_i, x, \bar{P}_{t_i}, \bar{Q}_{t_i}, \bar{\phi}_i(x) \right), \phi_i^N(x) - \bar{\phi}_i(x) \rangle \geq c_0 |\phi_i^N(x) - \bar{\phi}_i(x)|^2,
\]
where $(\bar{P}, \bar{Q})$ and $(P^N, Q^N)$ are the adjoint processes with respect to $\bar{u}$ and $u^N$, respectively.

**Remark 4.3** For fixed $(t_i,x) \in [0,T] \times \mathbb{R}^n$, $H_u(t_i, x, P_{t_i}^N, Q_{t_i}^N, \phi_i^N(x))$ is a deterministic function of $\tilde{\phi}_i(x)$. The above assumption means that $\tilde{H}(u_i^N) := H \left( t_i, x, P_{t_i}^N, Q_{t_i}^N, u_i^N \right)$ is uniformly monotone around $u_i^N = \tilde{\phi}_i(x)$, that is, when $\tilde{H}(\phi_i^N(x))$ and $\tilde{H}(\bar{\phi}_i(x))$ are close, $\phi_i^N(x)$ and $\bar{\phi}_i(x)$ are also close. In particular, if $U$ is an open set, the above assumption is also true for $\langle \tilde{H}(\phi_i^N(x)) - \tilde{H}(\bar{\phi}_i(x)), \phi_i^N(x) - \bar{\phi}_i(x) \rangle \leq -c_0 |\phi_i^N(x) - \bar{\phi}_i(x)|^2$.

Now we state our main convergence result.

**Theorem 4.4** Suppose $(A1) - (A2)$ hold. We also assume $b, \sigma \in C_b^{2.5 \cdot 5}$, $f \in C_b^{2.5 \cdot 5}$, $\phi^* \in C_b^{2.4 + \alpha}$, $\{\phi_i\}_{i=1}^{N-1}$, $\{\phi_i\}_{i=1}^{N-1}$, $\{P_i\}_{i=1}^{N-1}$, $\bar{P}_i$ and $\bar{Q}_i$ are close, and there exists a positive constant $L$, not depending on $N$, such that $\sup_i (|\phi_i^N(0)| + |\bar{\phi}_i(0)|) \leq L$. Then for sufficiently small time step $\Delta t$,
\[
|J \left( u^* \right) - J \left( u^N \right) | \leq C \Delta t.
\]

The proof of our convergence theorem will be divided into two parts: discrete approximation $|J \left( u^* \right) - J \left( \bar{u} \right) |$ and recursive approximation $|J \left( \bar{u} \right) - J(u^N)|$. 

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4.1 Discrete approximation

In this subsection, we give the estimate of the discrete approximation error $|J(u^*) - J(\bar{u})|$. 

Theorem 4.5 Suppose (A1) holds. We also assume that $b, \sigma \in C_b^{2+4, 4}, f \in C_b^{2+4+\alpha, 4+\alpha}, \phi^* \in C_b^{2+4+\alpha}$ and $h \in C_b^{4+\alpha}, \alpha > 0$. Then for sufficiently small $\Delta t$,

$$|J(u^*) - J(\bar{u})| \leq C \Delta t. \quad (4.2)$$

Proof. For simplicity of presentation, in the following of this proof we only consider the case $n = 1$. Conclusions still hold for the case $n > 1$. To begin with, we define

$$\bar{u}_t = \sum_{i=0}^{N-1} \phi^*(t, \bar{X}_t) I_{t_i, t_{i+1}}(t), \quad (4.3)$$

where

$$(\{ d\bar{X}_t = b(t, \bar{X}_t, \phi^*(t, \bar{X}_t)) + \sigma(t, \bar{X}_t, \phi^*(t, \bar{X}_t)) dW_t, \quad \bar{X}_t = \bar{X}_{t_i}, \quad t \in [t_i, t_{i+1}], \quad i = 0, 1, \ldots, N - 1, \)$$

and $\bar{X}_0 = x_0$. Since $\bar{u} \in U^N[0, T]$, we have $J(u^*) \leq J(\bar{u}) \leq J(\bar{u})$. So to prove (4.2), it suffices to prove

$$|J(u^*) - J(\bar{u})| \leq C \Delta t. \quad (4.4)$$

Let $J(u^*) - J(\bar{u}) = J_1 + J_2$, where

$$J_1 = \int_0^T \mathbb{E} \left[ f(t, X_t^*, \phi^*(t, X_t^*)) - f(t, \bar{X}_t, \phi^*(t, \bar{X}_t)) \right] dt + \mathbb{E} \left[ h(X_T^*) - h(\bar{X}_T) \right],$$

$$J_2 = \int_0^T \sum_{i=0}^{N-1} \mathbb{E} \left[ f(t, X_t^*, \phi^*(t, X_t^*)) - f(t, \bar{X}_t, \phi^*(t, \bar{X}_t)) \right] I_{t_i, t_{i+1}}(t) dt.$$

Note that

$$X_t^* = x_0 + \int_0^t b^*(s, X_s^*) ds + \int_0^t \sigma^*(s, X_s^*) dW_s, \quad t \in [0, T],$$

where $b^*(s, X_s^*):= b(s, X_s^*, \phi^*(s, X_s^*))$ and $\sigma^*(s, X_s^*):= \sigma(s, X_s^*, \phi^*(s, X_s^*))$. Since $b^*, \sigma^* \in C_b^{2, 4}, f \in C_b^{2+4+\alpha, 4+\alpha}, \phi^* \in C_b^{2+4+\alpha}$ and $h \in C_b^{4+\alpha}, \alpha > 0$, by Remark 2.3, we then have

$$v(s, x; t) = \mathbb{E} \left[ f(t, X_t^{*, s, x}, \phi^*(t, X_t^{*, s, x})) \right], \quad \mu(t, x) = \mathbb{E} \left[ h(X_T^{*, t, x}) \right], \quad (4.5)$$

where $\mu(\cdot, s), v(\cdot, s; t) \in C_b^{4, 4}$ are the solution of (2.3) with the terminal $\mu(T, x) = h(x)$ and $v(t, x; t) = f(t, x, \phi^*(t, x))$, respectively. By applying Itô’s formula to $\mu(T, X_T^*)$ and $v(t, X_t^*; t)$, from (2.3), we obtain

$$\mathbb{E} [v(t, X_t^*; t)] = \mathbb{E} [v(0, x_0; t)] + \int_0^t \mathbb{E} [\mathcal{L} v(s, X_s^*; t)] ds = \mathbb{E} [v(0, x_0; t)], \quad \mathbb{E} [\mu(T, X_T^*]) = \mathbb{E} [\mu(0, x_0)] + \int_0^T \mathbb{E} [\mathcal{L} \mu(s, X_s^*))] ds = \mathbb{E} [\mu(0, x_0)]. \quad (4.6)$$

On the one hand, combining (4.5) - (4.6), we have

$$|J_1| \leq \int_0^T \left[ \mathbb{E} [v(t, X_t^*; t) - v(t, \bar{X}_t; t)] \right] dt + \left[ \mathbb{E} [\mu(T, X_T^*) - \mu(T, \bar{X}_T)] \right]$$

$$\leq \int_0^T \left[ \mathbb{E} [v(t, \bar{X}_t; t) - v(0, x_0; t)] \right] dt + \left[ \mathbb{E} [\mu(T, \bar{X}_T) - \mu(0, x_0)] \right]. \quad (4.7)$$
Set  
\[ b_i(t, \tilde{X}_t) = b(t, \tilde{X}_t, \phi^*(t_i, \tilde{X}_{t_i})), \quad \sigma_i(t, \tilde{X}_t) = \sigma(t, \tilde{X}_t, \phi^*(t_i, \tilde{X}_{t_i})). \]

By Itô’s formula and \((2.3)\), we have
\[
\mathbb{E} \left[ \mu(T, \tilde{X}_T) - \mu(0, X_0) \right] \leq \sum_{i=0}^{N-1} \mathbb{E} \left[ \mu(t_{i+1}, \tilde{X}_{t_{i+1}}) - \mu(t_i, \tilde{X}_t) \right] \tag{4.8}
\]
\[
\leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ \partial_t \mu(s, \tilde{X}_s) + b_i(s, \tilde{X}_s) \partial_x \mu(s, \tilde{X}_s) 
+ \frac{1}{2} \sigma_i^2(s, \tilde{X}_s) \partial_{xx} \mu(s, \tilde{X}_s) - L \mu(t_i, \tilde{X}_t) \right] ds,
\]
which implies
\[
\mathbb{E} \left[ \mu(T, \tilde{X}_T) - \mu(0, X_0) \right] \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ \left( \partial_t \mu(s, \tilde{X}_s) - \partial_t \mu(t_i, \tilde{X}_t) \right) 
+ \left( b_i(s, \tilde{X}_s) - b_i(t_i, \tilde{X}_{t_i}) \right) \partial_x \mu(t_i, \tilde{X}_{t_i}) \right] ds. \tag{4.9}
\]

Using Itô’s formula again, we have
\[
\left| \mathbb{E} \left[ \partial_t \mu(s, \tilde{X}_s) - \partial_t \mu(t_i, \tilde{X}_{t_i}) \right] \right| \leq \int_{t_i}^t \mathbb{E} \left[ \left| \partial_{tt} \mu(r, \tilde{X}_r) + b_i(r, \tilde{X}_r) \partial_{xx} \mu(r, \tilde{X}_r) + \frac{1}{2} \sigma_i^2(r, \tilde{X}_r) \partial_{xxx} \mu(r, \tilde{X}_r) \right| \right] dr \tag{4.10}
\]
\[
\leq C \Delta t + C \int_{t_i}^t \mathbb{E} \left[ 1 + |\tilde{X}_r|^2 \right] dr.
\]

Similarly, we can obtain
\[
\left| \mathbb{E} \left[ b_i(s, \tilde{X}_s) \partial_x \mu(s, \tilde{X}_s) - b_i(t_i, \tilde{X}_{t_i}) \partial_x \mu(t_i, \tilde{X}_{t_i}) \right] \right| \leq C \Delta t + C \int_{t_i}^t \mathbb{E} \left[ 1 + |\tilde{X}_r|^3 \right] dr, \tag{4.11}
\]
\[
\left| \mathbb{E} \left[ \sigma_i^2(s, \tilde{X}_s) \partial_{xx} \mu(s, \tilde{X}_s) - \sigma_i^2(t_i, \tilde{X}_{t_i}) \partial_{xx} \mu(t_i, \tilde{X}_{t_i}) \right] \right| \leq C \Delta t + C \int_{t_i}^t \mathbb{E} \left[ 1 + |\tilde{X}_r|^3 \right] dr. \tag{4.12}
\]

From \((4.9) - (4.12)\), by Lemma \((4.1)\), it follows that
\[
\left| \mathbb{E} \left[ \mu(T, \tilde{X}_T) - \mu(0, X_0) \right] \right| \leq C \Delta t. \tag{4.13}
\]

In the same way, we can estimate \( \int_0^T |\mathbb{E}[v(t, \tilde{X}_t; t) - v(0, X_0; t)]| dt \leq C \Delta t \). Then, by \((4.7)\), it follows that \(|J_1| \leq C \Delta t\). On the other hand, seeing that
\[
|J_2| \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ f(t, \tilde{X}_t, \phi^*(t, \tilde{X}_t)) - f(t, \tilde{X}_t, \phi^*(t_i, \tilde{X}_{t_i})) \right] dt
= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ f(t, \tilde{X}_t, \phi^*(t, \tilde{X}_t)) - f(t_i, \tilde{X}_{t_i}, \phi^*(t_i, \tilde{X}_{t_i})) \right]
- \mathbb{E} \left[ f(t, \tilde{X}_t, \phi^*(t, \tilde{X}_t)) - f(t_i, \tilde{X}_{t_i}, \phi^*(t_i, \tilde{X}_{t_i})) \right] dt.
\]
By Itô’s formula, we have

\[
|J_2| \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left\{ \int_{t_i}^{t} \mathbb{E} \left[ \frac{1}{2} \sigma_i^2(s, \bar{X}_s) \partial^2_{xx} f(s, \bar{X}_s, \phi^*(s, \bar{X}_s)) + \frac{1}{2} \sigma_i^2(s, \bar{X}_s) \partial^2_{zz} f(s, \bar{X}_s, \phi^*(s, \bar{X}_s)) \right] ds \right. \\
+ \int_{t_i}^{t} \mathbb{E} \left[ \frac{1}{2} \sigma_i^2(s, \bar{X}_s) \partial^2_{zz} f(s, \bar{X}_s, \phi^*(s, \bar{X}_s)) \right] ds \right\} dt.
\]

Then, under the conditions of \( b, \sigma, \phi^* \) and \( f \), by Lemma \ref{lemma:4.1} we obtain

\[
|J_2| \leq C \Delta t + C \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \mathbb{E} \left[ 1 + |\bar{X}_s|^2 \right] ds dt \leq C \Delta t.
\]

In conclusion, \( \text{(4.4)} \) holds. The proof is complete. □

### 4.2 Recursive approximation

In this subsection, we estimate the recursive approximation error \( |J(\bar{u}) - J(u^N)| \) generated by the numerical recursive approximation. Consider the following FBSDEs:

\[
\begin{cases}
X_t^N = X_{t_i}^N + \int_{t_i}^{t} b(s, X_s^N, \phi_t^N(X_s^N)) \, ds + \int_{t_i}^{t} \sigma(s, X_s^N, \phi_t^N(X_s^N)) \, dW_s, \\
P_t^N = P_{t_i}^N + \int_{t_i}^{t} \frac{1}{2} \sigma_t^2(s, \bar{X}_s) \partial^2_{zz} f(s, \bar{X}_s, \phi^*(s, \bar{X}_s)) \, ds + \int_{t_i}^{t} Q_s^N \, dW_s,
\end{cases}
\]

for \( t \in [t_i, t_{i+1}], i = 0, 1, \ldots, N - 1 \), with \( X_{t_i}^N = x_0 \) and \( P_{t_i}^N = h_x(X_{t_i}^N) \). For notational simplicity, in the sequel, for \( t \in [t_i, t_{i+1}] \), we let

\[ H_t^N = h_x(t, X_t^N, P_t^N, Q_t^N, \phi_t^N(X_t^N)) \]

Let \( (X_{t_i}^{t_i, \alpha}, P_{t_i}^{t_i, \alpha}, Q_{t_i}^{t_i, \alpha}) \) be the solution of FBSDEs \( \text{(3.24)} - \text{(3.25)} \) with \( \bar{X}_{t_i} = \alpha \), for \( t \in [t_i, T] \). Denote \( b(t, X_{t_i}^{t_i, \alpha}, \phi_t(\alpha)) \) by \( \bar{b}_{t_i}^{N, t_i, \alpha} \), \( \sigma(t, X_{t_i}^{t_i, \alpha}, \phi_t(\alpha)) \) by \( \bar{\sigma}_{t_i}^{N, t_i, \alpha} \) and

\[
\bar{H}_{t_i}^{t_i, \alpha} = h_x(t, X_{t_i}^{t_i, \alpha}, \bar{P}_{t_i}^{t_i, \alpha}, \bar{Q}_{t_i}^{t_i, \alpha}, \bar{\phi}_{t_i}^{t_i, \alpha}).
\]

Then for \( i = 0, 1, \ldots, N - 1 \),

\[
\begin{cases}
X_{t_i+1}^{t_i, N} = X_{t_i}^{t_i, N} + \int_{t_i}^{t_{i+1}} \bar{b}_{t_i}^{N, t_i, \alpha} X_t^{t_i, N} \, dt + \int_{t_i}^{t_{i+1}} \bar{\sigma}_{t_i}^{N, t_i, \alpha} X_t^{t_i, N} \, dW_t, \\
P_{t_i+1}^{t_i, N} = P_{t_i}^{t_i, N} + \int_{t_i}^{t_{i+1}} \bar{H}_{t_i}^{t_i, \alpha} X_t^{t_i, N} \, dt - \int_{t_i}^{t_{i+1}} \bar{Q}_{t_i}^{t_i, N} X_t^{t_i, N} \, dW_t.
\end{cases}
\]

We have the following error estimate.

**Lemma 4.6** Suppose \( (A1) \) holds. Let \( (X_t, P_t, Q_t) \) and \( (X_t^N, P_t^N, Q_t^N), t \in [0, T], \) be the solutions of \( \text{(3.24)} - \text{(3.25)} \) and \( \text{(4.16)} \), respectively. We also assume \( b, \sigma, f \in C_b^{2,5,5} \), \( \{\phi_t\}_{t=1}^{N-1}, \{\bar{\phi}_t\}_{t=1}^{N-1} \in C_b^4 \) and \( h \in C_b^{5+\alpha} \),
α > 0, and there exists a positive constant L, not depending on N, such that supi (|φN i (0)| + |φ̂ i (0)|) \leq L.

Then for i = 0,1,\ldots,N-1,
\[ E \left[ \left| \hat{P}_{t_i}^{t_i,X_i^N} - P_{t_i}^{X_i^N} \right|^2 \right] + \Delta t \sum_{j=i}^{N-1} E \left[ \left| \hat{Q}_{t_j}^{t_j,X_j^N} - Q_{t_j}^{X_j^N} \right|^2 \right] \leq C (\Delta t)^2 + C \Delta t \sum_{j=i}^{N-1} E \left[ \left| \phi_j(X_{t_j}^N) - \phi_{N}^j(X_{t_j}^N) \right|^2 \right]. \]

**Proof.** For simplicity, we introduce the following notation:
\[ \hat{P}_{t_i} = \hat{P}_{t_i}^{t_i,X_i^N} - P_{t_i}^{X_i^N}, \quad \hat{P}_{t_{i+1}} = \hat{P}_{t_{i+1}}^{t_{i+1},X_{i+1}^N} - P_{t_{i+1}}^{X_{i+1}^N}, \]
\[ \hat{Q}_{t_i} = \hat{Q}_{t_i}^{t_i,X_i^N} - Q_{t_i}^{X_i^N}, \quad \hat{Q}_{t_{i+1}} = \hat{Q}_{t_{i+1}}^{t_{i+1},X_{i+1}^N} - Q_{t_{i+1}}^{X_{i+1}^N}. \]

We also denote \( \mathcal{H}_t^{t_i,X_i^N} - \mathcal{H}_t^N \) by \( \hat{\mathcal{H}}_t \). For each integer 0 ≤ i ≤ N - 1, from (4.16) and (4.17), we obtain
\[ \hat{P}_{t_i} = \hat{P}_{t_{i+1}}^{t_i,X_i^N} - \hat{P}_{t_{i+1}}^{t_{i+1},X_{i+1}^N} + \int_{t_{i+1}}^{t_{i+1}} \hat{\mathcal{H}}_t dt - \int_{t_i}^{t_{i+1}} \left( \hat{Q}_{t_i}^{t_i,X_i^N} - Q_{t_i}^{X_i^N} \right) dW_t. \]  

Define the conditional mathematical expectation \( E_t^N [\cdot] := \mathbb{E} [\cdot | \mathcal{X}_t = x] \) and denote \( E_t^N [X_t^N] \) by \( E_t^N [\cdot] \). It is easy to check that the equation above is equivalent to the following equations:
\[ \hat{P}_{t_i} = E_t^N [\hat{P}_{t_{i+1}}] + \hat{\mathcal{H}}_t \Delta t + R_{p,i} + R_{q,i}, \]
\[ \hat{Q}_{t_i} \Delta t = E_t^N [\hat{P}_{t_{i+1}} \Delta W_{t_{i+1}}] + \hat{R}_{q,i} + R_{q,i}, \]

where the error terms
\[ R_{p,i} = \int_{t_i}^{t_{i+1}} E_t^N [\hat{\mathcal{H}} - \hat{\mathcal{H}}_t] dt, \]
\[ R_{q,i} = \int_{t_i}^{t_{i+1}} E_t^N [\hat{\mathcal{H}}_t \Delta W_{t_{i+1}}] dt - \int_{t_i}^{t_{i+1}} E_t^N \left[ \left( \hat{Q}_{t_i}^{t_i,X_i^N} - Q_{t_i}^{X_i^N} \right) - \hat{Q}_t \right] dt, \]
\[ \hat{R}_{p,i} = E_t^N \left[ \hat{P}_{t_{i+1}}^{t_i,X_i^N} - \hat{P}_{t_{i+1}}^{t_{i+1},X_{i+1}^N} \right], \quad \hat{R}_{q,i} = E_t^N \left[ \left( \hat{P}_{t_{i+1}}^{t_i,X_i^N} - \hat{P}_{t_{i+1}}^{t_{i+1},X_{i+1}^N} \right) \Delta W_{t_{i+1}} \right]. \]

By Hölder’s inequality, we have
\[ \left| E_t^N [\hat{P}_{t_{i+1}} \Delta W_{t_{i+1}}] \right|^2 \leq \left( E_t^N \left[ \left| \phi_{t_{i+1}} - \phi_{N}^r \right|^2 \right] \Delta W_{t_{i+1}} \right)^2 \]
\[ \leq \left( E_t^N \left[ \hat{P}_{t_{i+1}}^{t_i,X_i^N} \right] - |E_t^N [\hat{P}_{t_{i+1}}] |^2 \right) \Delta t. \]

Taking square of (4.19) and (4.20) and using the inequalities (4.21) and \((a+b)^2 \leq (1+\gamma \Delta t)a^2 + (1 + \frac{1}{\gamma \Delta t})b^2\), we obtain
\[ |\hat{P}_{t_i}|^2 \leq (1 + \gamma \Delta t) \left| E_t^N [\hat{P}_{t_{i+1}}] \right|^2 + C(1 + \frac{1}{\gamma \Delta t}) \left( |R_{p,i}|^2 + |R_{q,i}|^2 \right) \]
\[ + C(1 + \frac{1}{\gamma \Delta t}) \Delta t \left( |\hat{P}_{t_i}|^2 + |\hat{Q}_{t_i}|^2 + |\phi_{t_i}(X_{t_i}^N) - \phi_N^j(X_{t_i}^N)|^2 \right), \]
\[ |\hat{Q}_{t_i}|^2 \leq C \left( E_t^N \left[ \left| \hat{P}_{t_{i+1}} \right|^2 \right] - |E_t^N [\hat{P}_{t_{i+1}}] |^2 \right) \Delta t + C \left( |R_{p,i}|^2 + |R_{q,i}|^2 \right)^2. \]

By choosing \( \gamma = 2C^2, \Delta t \leq 1 \), and adding up the above inequalities, we obtain
\[ |\hat{P}_{t_i}|^2 + \frac{\Delta t}{2C} |\hat{Q}_{t_i}|^2 \leq (1 + 2C^2 \Delta t) E_t^N \left[ \left| \hat{P}_{t_{i+1}} \right|^2 \right] + \frac{\Delta t}{2C} \left( |\hat{P}_{t_i}|^2 + |\phi_{t_i}(X_{t_i}^N) - \phi_N^j(X_{t_i}^N)|^2 \right) \]
\[ + \frac{1}{2C} \left( |R_{p,i}|^2 + |R_{q,i}|^2 + |R_{q,i}|^2 \right)^2, \]
which yields

\[
|\tilde{P}_t|^2 + C \Delta t |\tilde{Q}_t|^2 \leq (1 + C \Delta t) E_{t_j} \left[ |\tilde{P}_{t_{j+1}}|^2 \right] + C \Delta t \left| \phi_j(X^N_{t_j}) - \phi_j(X^N_{t_j}) \right|^2
\]

Taking mathematical expectation on both sides of (4.22), we have

\[
E \left[ |\tilde{P}_t|^2 \right] + C \Delta t E \left[ |\tilde{Q}_t|^2 \right] \leq (1 + C \Delta t) E \left[ |\tilde{P}_{t_{j+1}}|^2 \right] + C \Delta t E \left[ \left| \phi_j(X^N_{t_j}) - \phi_j(X^N_{t_j}) \right|^2 \right]
\]

which, by induction, leads to the inequality

\[
E \left[ |\tilde{P}_t|^2 \right] \leq C E \left[ |P_{t_{N-1}}|^2 \right] + C \Delta t \sum_{j=1}^{N-1} E \left[ \left| \phi_j(X^N_{t_j}) - \phi_j(X^N_{t_j}) \right|^2 \right]
\]

Based on (4.23), we can deduce

\[
C \Delta t \sum_{j=1}^{N-1} E \left[ |\tilde{Q}_j|^2 \right] \leq C \Delta t \sum_{j=1}^{N-1} E \left[ |\tilde{P}_{t_{j+1}}|^2 \right] + C \Delta t \sum_{j=1}^{N-1} E \left[ \left| \phi_j(X^N_{t_j}) - \phi_j(X^N_{t_j}) \right|^2 \right] \Delta t
\]

Thus

\[
E \left[ |\tilde{P}_t|^2 + \Delta t \sum_{j=1}^{N-1} |\tilde{Q}_j|^2 \right] \leq C E \left[ |P_{t_{N-1}}|^2 \right] + C \sum_{j=1}^{N-1} E \left[ \left| \phi_j(X^N_{t_j}) - \phi_j(X^N_{t_j}) \right|^2 \right] \Delta t
\]

Now we estimate the error terms \( R_{p,j}, R_{q,j}, \tilde{R}_{p,j} \) and \( \tilde{R}_{q,j} \). For \( [t_j, t_{j+1}] \), \( j = N-1, \ldots, i \), under the condition of Lemma 4.6 similar to the variational method in Lemma 3.3 the solutions of (4.17) on \( [t_j, t_{j+1}] \) have the
where \( \bar{\bar{E}} \).

To check that solving the FBSDEs (4.16) is equivalent to finding the solution to the following equations:

\[
\begin{aligned}
|\bar{R}_{p,j}| &= \left| \mathbb{E}^N_{t_j} \left[ m_j \left( t_{j+1}, X^N_{t_{j+1}} \right) - m_j \left( t_j, X^N_{t_j} \right) \right] \right| \\
&= \left| \int_{t_j}^{t_{j+1}} \mathbb{E}^N_{t_j} \left[ \mathcal{L} m_j \left( t_j, X^N_{t_j} \right) + \int_{t_j}^{t} \mathcal{L} \mathcal{L} m_j \left( s, X^N_{s} \right) ds \right] dt \right| \\
&\leq \mathbb{E}^N_{t_j} \left[ |\mathcal{L} m_j \left( t_j, X^N_{t_j} \right) - \mathcal{L} m_j \left( t_j, X^N_{t_j} \right)| \right] \Delta t \\
&\quad + C \int_{t_j}^{t_{j+1}} \int_{t_j}^{t} \left( 1 + \mathbb{E}^N_{t_j} \left[ |X^N_s|^4 + |X^N_{t_j}|^4 \right] \right) ds dt,
\end{aligned}
\]

where

\[
\mathcal{L} = \frac{\partial}{\partial t} + \sum_{k=1}^{n} b_k \left( t, x, \bar{\phi}_j(X^N_{t_j}) \right) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l=1}^{n} [\sigma \sigma^T]_{k,l} \left( t, x, \bar{\phi}_j(X^N_{t_j}) \right) \frac{\partial^2}{\partial x_k \partial x_l},
\]

\[
\bar{\mathcal{L}} = \frac{\partial}{\partial t} + \sum_{k=1}^{n} b_k \left( t, x, \phi_j^N(X^N_{t_j}) \right) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{k,l=1}^{n} [\sigma \sigma^T]_{k,l} \left( t, x, \phi_j^N(X^N_{t_j}) \right) \frac{\partial^2}{\partial x_k \partial x_l}.
\]

Then by Lemma 4.1, we have

\[
\mathbb{E} \left[ |\bar{R}_{p,j}|^2 \right] \leq C (\Delta t)^2 \mathbb{E} \left[ |\bar{\phi}_j(X^N_{t_j}) - \phi_j^N(X^N_{t_j})|^2 \right] + C (\Delta t)^4. \tag{4.27}
\]

Similarly,

\[
\mathbb{E} \left[ |\bar{R}_{q,j}|^2 \right] \leq C (\Delta t)^2 \mathbb{E} \left[ |\bar{\phi}_j(X^N_{t_j}) - \phi_j^N(X^N_{t_j})|^2 \right] + C (\Delta t)^4. \tag{4.28}
\]

In the same way above, we can obtain

\[
\mathbb{E}[|R_{p,j}|^2] \leq C (\Delta t)^4, \quad \mathbb{E}[|R_{q,j}|^2] \leq C (\Delta t)^4. \tag{4.29}
\]

Consequently, the desired conclusion follows from (4.26) - (4.29).

Now we discuss the error produced by the numerical solution of FBSDEs in Schemes 3.4 - 3.5. It is easy to check that solving the FBSDEs (4.16) is equivalent to finding the solution to the following equations:

\[
\begin{aligned}
P^N_{t_i} &= \mathbb{E}_{t_i} \left[ P^N_{t_{i+1}} \right] + \mathcal{H}^N_{t_i} \Delta t + \mathcal{E}_{P,i}, \\
Q^N_{t_i} &= \left( \mathbb{E}_{t_i} \left[ P^N_{t_{i+1}} \Delta W_{t_{i+1}} \right] + \mathcal{E}_{Q,i} \right) / \Delta t,
\end{aligned}
\]

where \( \mathbb{E}_{t_i} \cdot \cdot \cdot = \mathbb{E} \cdot \cdot \cdot \mathbb{F}_{t_i} \) and the truncation errors

\[
\begin{aligned}
\mathcal{E}_{P,i} &= \int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i} \left[ \mathcal{H}^N_{t_i} \right] dt - \mathcal{H}^N_{t_i} \Delta t, \\
\mathcal{E}_{Q,i} &= \int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i} \left[ \mathcal{H}^N_{t_i} \Delta W_{t_{i+1}} \right] dt - \int_{t_i}^{t_{i+1}} \mathbb{E}_{t_i} \left[ Q^N_{t_i} \right] dt + Q^N_{t_i} \Delta t.
\end{aligned}
\]
Lemma 4.7 Suppose (A1) and the conditions in Lemma 4.6 hold. Let \((P_i^{N,i_t,x}, Q_i^{N,i_t,x})\) be the numerical solution at grid point \((t_i, x)\). We assume that \((P_i^{N,i_t,x})_{i=1}^{N-1} \in C^1_0\). Then for \(i = 0, 1, \ldots, N - 1\),

\[
E \left[ P_i^{N,i_t} - P_i^{N,i_t,x} \right]^2 + \Delta t \sum_{j=i}^{N-1} E \left[ Q_i^{N,i_t} - Q_j^{N,i_t,x} \right]^2 \leq C (\Delta t)^2.
\]

Proof. The proof of lemma can be referred to [39].

Based on the above discussion, we have the following lemma.

Lemma 4.8 Suppose (A2) and the conditions in Lemmas 4.6 hold. Then for \(i = 0, 1, \ldots, N - 1\),

\[
\Delta t \sum_{j=i}^{N-1} E \left[ \phi_j (X_i^N) - \phi_j^N (X_i^N) \right]^2 \leq C (\Delta t)^2.
\]

Proof. Provided that \(X_i^N = x\), from (3.10) we know

\[
\left( \phi_i (x) - \frac{\rho}{\Delta t} \int_{t_i}^{t_{i+1}} E [H_u (t, X_i^{t_{i+1}x}, P_i^{t_{i+1}x}, Q_i^{t_{i+1}x}, \phi_i (x))] dt - \phi_i (x), \phi_i (x) - \phi_i (x) \right) \leq 0,
\]

for any \(\phi_i (\cdot) \in C^1_0 (\mathbb{R}^n; U)\) and \(\rho > 0\), which implies

\[
\phi_i (x) = P_U \left( \phi_i (x) - \frac{\rho}{\Delta t} \int_{t_i}^{t_{i+1}} E [H_u (t, X_i^{t_{i+1}x}, P_i^{t_{i+1}x}, Q_i^{t_{i+1}x}, \phi_i (x))] dt \right), \quad (4.32)
\]

where \(P_U\) is the projection operator from \(\mathbb{R}^n\) to \(U\), such that

\[
P_U (v) = \arg \min_{u \in U} |u - v|^2.
\]

Analogously, from (3.30), we have

\[
\phi_i^N (x) = P_U \left( \phi_i^N (x) - \rho H_u \left( t_i, x, P_i^{N,i_t,x}, Q_i^{N,i_t,x}, \phi_i^N (x) \right) \right), \quad \rho > 0. \quad (4.33)
\]

For convenience, we omit the superscript \(^{i_t,x}\) if no ambiguity arises. From (4.32) - (4.33), it is easy to obtain

\[
|\phi_i (x) - \phi_i^N (x)| \leq |\phi_i (x) - \phi_i^N (x)| - \rho [H_u (t_i, x, P_i^{N,i_t,x}, Q_i^{N,i_t,x}, \phi_i^N (x)) - H_u \left( t_i, x, P_i^{N}, Q_i^{N}, \phi_i^N (x) \right)] - \rho R_i^H,
\]

where

\[
R_i^H = \frac{1}{\Delta t} \int_{t_i}^{t_{i+1}} E [H_u (t, X_i^{t_{i+1}x}, Q_i^t, \phi_i (x)) - H_u \left( t_i, x, P_i^{t_{i+1}x}, Q_i^{t_{i+1}x}, \phi_i (x) \right)] dt.
\]

Then, we have

\[
|\phi_i (x) - \phi_i^N (x)|^2 \leq \left| \phi_i (x) - \phi_i^N (x) \right|^2 + \rho^2 |R_i^H|^2 - 2 \rho \langle \phi_i (x) - \phi_i^N (x), R_i^H \rangle + \rho^2 \left[ H_u \left( t_i, x, P_i^{t_{i+1}x}, Q_i^{t_{i+1}x}, \phi_i (x) \right) - H_u \left( t_i, x, P_i^{N,i_t,x}, Q_i^{N,i_t,x}, \phi_i^N (x) \right) \right]^2 + \rho^2 \left[ H_u \left( t_i, x, P_i^{N,i_t,x}, Q_i^{N,i_t,x}, \phi_i^N (x) \right) - H_u \left( t_i, x, P_i^{N,i_t,x}, Q_i^{N,i_t,x}, \phi_i^N (x) \right) \right]^2
\]

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\[-2\rho \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), \phi_i(x) - \phi^N_i(x) \rangle \]
\[-2\rho \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), \phi_i(x) - \phi^N_i(x) \rangle \]
\[+ 2\rho^2 \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), \phi_i(x) - \phi^N_i(x) \rangle \]
\[H_u (t_i, x, P_t, Q_t, \phi^N_i(x)) - H_u (t_i, x, P_t, Q_t, \phi^N_i(x)) \]
\[+ 2\rho^2 \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), R^H \rangle \]
\[+ 2\rho^2 \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), R^H \rangle \].

Notice that
\[-2\rho \langle H_u (t_i, x, P_t, Q_t, \phi^N_i(x)) - H_u (t_i, x, P_t, Q_t, \phi^N_i(x)), \phi_i(x) - \phi^N_i(x) \rangle \leq \rho^2 |H_u (t_i, x, P_t, Q_t, \phi^N_i(x)) - H_u (t_i, x, P_t, Q_t, \phi^N_i(x))|^2 \]
\[+ |\phi_i(x) - \phi^N_i(x)|^2 \].

Similarly,
\[2\rho^2 \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), R^H \rangle \]
\[\leq \rho^2 |H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x))|^2 + \rho^2 |R^H|^2 \],
\[2\rho^2 \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), R^H \rangle \]
\[\leq \rho^2 |H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x))|^2 + \rho^2 |R^H|^2 \],
\[2\rho^2 \langle H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x)), R^H \rangle \]
\[\leq \rho^2 |H_u (t_i, x, P_t, Q_t, \phi_i(x)) - H_u (t_i, x, P_t, Q_t, \phi_i(x))|^2 + \rho^2 |R^H|^2 \],
\[\text{and} \]
\[-2\rho \langle \phi_i(x) - \phi^N_i(x), R^H \rangle \leq \rho^2 |\phi_i(x) - \phi^N_i(x)|^2 + |R^H|^2 \].

From (4.35) \(- (4.40)\) and assumption (A2), we have
\[|\phi_i(x) - \phi^N_i(x)|^2 \leq (1 - 2c_{0\rho} + 8C\rho^2) |\phi_i(x) - \phi^N_i(x)|^2 \]
\[+ 2C (1 + 4\rho^2) \left(|P_t - P^N_t|^2 + |Q_t - Q^N_t|^2 \right) \]
\[+ 9C\rho^2 \left(|P_t - P^N_t|^2 + |Q_t - Q^N_t|^2 \right) \]
\[+ (1 + 3\rho^2) |R^H|^2 \].

Choosing sufficiently small \(\rho\) in (4.41), such that \(2c_{0\rho} - 8C\rho^2 \geq c_{0\rho}/2\), we obtain
\[|\phi_i(x) - \phi^N_i(x)|^2 \leq \frac{C\rho}{c_{0\rho}} \left(|P_t - P^N_t|^2 + |Q_t - Q^N_t|^2 \right) \]
\[+ \frac{C}{c_{0\rho}} \left(|P_t - P^N_t|^2 + |Q_t - Q^N_t|^2 \right) \]
\[+ \frac{C}{c_{0\rho}} |R^H|^2 \].
Then we can deduce
\[
\Delta t \sum_{j=1}^{N-1} \mathbb{E} \left[ \left| \phi_j(X_i^N) - \phi_j^N(X_i^N) \right|^2 \right] \leq \frac{C}{C_0 \rho} \Delta t \sum_{j=1}^{N-1} \mathbb{E} \left[ \left| P_j^N - P_j^{N,t_j}X_i^N \right|^2 + \left| Q_j^N - Q_j^{N,t_j}X_i^N \right|^2 \right] \\
+ \frac{C \rho}{C_0} \Delta t \sum_{j=1}^{N-1} \mathbb{E} \left[ \left| P_j^{t_j}X_i^N - P_j^{t_j} \right|^2 \right] + \frac{C}{C_0 \rho} \Delta t \sum_{j=1}^{N-1} \mathbb{E} \left[ |R_H|^2 \right].
\]

By Itô’s formula, it is easy to check \( \mathbb{E} \left[ |R_H|^2 \right] \leq C (\Delta t)^2 \). Then, by Lemmas 4.6 and 4.7 we have
\[
\Delta t \sum_{j=1}^{N-1} \mathbb{E} \left[ \left| \phi_j(X_i^N) - \phi_j^N(X_i^N) \right|^2 \right] \leq \frac{C \rho}{C_0} \Delta t \sum_{j=1}^{N-1} \mathbb{E} \left[ \left| \phi_j(X_i^N) - \phi_j^N(X_i^N) \right|^2 \right] + \frac{C}{C_0 \rho} (1 + \rho^2) (\Delta t)^2.
\]

Further choosing the constant \( \rho \), such that \( C \rho / C_0 \leq 1/2 \), the desired result follows. \( \blacksquare \)

**Theorem 4.9** Suppose (A2) and the conditions in Lemmas 4.6, 4.7 hold. Then
\[
|J (\bar{u}) - J (u^N)| \leq C \Delta t.
\]

**Proof.** First, we rewrite the state equations (3.4) and (3.36) as follows:
\[
X_t = x_0 + \int_0^t \bar{b} (s, X_s) \, ds + \int_0^t \bar{\sigma} (s, X_s) \, dW_s,
\]
\[
X_i^N = x_0 + \int_0^t \bar{b} (s, X_i^N) \, ds + \int_0^t \bar{\sigma} (s, X_i^N) \, dW_s,
\]
where for \( s \in [t_i, t_{i+1}] \) (\( i = 0, \ldots, N - 1 \)),
\[
\bar{b} (s, x) = b (s, \phi_i (X_t)), \quad \bar{\sigma} (s, x) = \sigma (s, \phi_i (X_t)),
\]
\[
\tilde{b} (s, x) = b (s, \phi_i^N (X_i)), \quad \tilde{\sigma} (s, x) = \sigma (s, \phi_i^N (X_i)).
\]

By Lemmas 2.4 and 4.8 for \( 0 \leq r \leq T \), we have
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s - X_r^N|^2 \right] \leq C \int_0^t \mathbb{E} \left[ \left( \tilde{b} (s, X_s) - \bar{b} (s, X_s) \right)^2 + \left( \tilde{\sigma} (s, X_s) - \bar{\sigma} (s, X_s) \right)^2 \right] \, ds
\]
\[
\leq C \int_0^t \sum_{i=0}^{\tau} \mathbb{E} \left[ \left( \phi_i (X_t) - \phi_i^N (X_i) \right)^2 \right] I_{[t_i, t_{i+1}]}(s) \, ds
\]
\[
\leq C \sum_{i=0}^{\tau} \left\{ \mathbb{E} \left[ \left( \phi_i (X_t) - \phi_i^N (X_i) \right)^2 \right] + \mathbb{E} \left[ \left( X_t - X_i^N \right)^2 \right] \right\} \Delta t
\]
\[
\leq C \int_0^t \left\{ \sup_{0 \leq r \leq s} |X_r - X_r^N|^2 \right\} ds + C (\Delta t)^2,
\]
where \( \tau \) is an integer, satisfying \( t_{\tau} < t \leq t_{\tau+1} \). Then, by Gronwall’s inequality, we obtain
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s - X_r^N|^2 \right] \leq C (\Delta t)^2,
\]
(4.44)
for $0 \leq t \leq T$. Notice that

\[ |J(\bar{u}) - J(u^N)| \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \mathbb{E} \left[ |f_t(X_t, \phi_X(X_t)) - f_t(X^N_t, \phi_X^N(X^N_t))| \right] dt + \mathbb{E} \left[ |h(X_T) - h(X^N_T)| \right]. \]

Since the continuity of $f$ and $h$, by Hölder’s inequality, we have

\[ |J(\bar{u}) - J(u^N)| \leq C \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left( \mathbb{E} \left[ |X_t - X^N_t|^2 \right] \right)^{1/2} + \mathbb{E} \left[ |X_T - X^N_T|^2 \right]^{1/2} \]

\[ + C \sum_{i=0}^{N-1} \mathbb{E} \left[ |\phi_X(X^N_t) - \phi_X^N(X^N_t)| \right] \Delta t + C \left( \mathbb{E} \left[ |X_T - X^N_T|^2 \right] \right)^{1/2}. \]

In addition, by Hölder’s inequality, we have

\[ \Delta t \sum_{i=0}^{N-1} \mathbb{E} \left[ |\phi_X(X^N_t) - \phi_X^N(X^N_t)| \right] \leq \sqrt{T} \left( \sum_{i=0}^{N-1} \mathbb{E} \left[ |\phi_X(X^N_t) - \phi_X^N(X^N_t)|^2 \right] \right)^{1/2}. \]

Combining (4.44), (4.46) – (4.47) and Lemma 4.8, we complete the proof. ■

4.3 Proof of the main results

Proof of Theorem 4.4. Notice that

\[ |J(u^*) - J(u^N)| = |J(u^*) - J(\bar{u}) + J(\bar{u}) - J(u^N)| \leq |J(u^*) - J(\bar{u})| + |J(\bar{u}) - J(u^N)|. \]

By Theorems 4.3 and 4.9 we complete our proof. ■

5 Numerical experiments

In this section, some numerical experiments have been presented to illustrate the high accuracy of our algorithm for solving SOCPs. The first two examples are deterministic control, and the latter two examples
are feedback control. In our tests, we use Gauss-Hermite quadrature rule to approximate the conditional mathematical expectation and use cubic spline interpolation to compute spatial non-grid points. To make sure the first-order convergence of our method, the Euler method is also adopted to solve the related FBSDEs when calculating the cost. In the following tables, CR stands for the convergence rate.

Example 5.1 We first consider the control problem of the Black-Scholes type in \[4\]

\[
\begin{cases}
    dX_t = u_t X_t dt + \sigma X_t dW_t, \\
    X_0 = x_0,
\end{cases}
\]

with the cost functional

\[
J(u) = \frac{1}{2} \int_0^T E[(X_t - \eta^*_t)^2] dt + \frac{1}{2} \int_0^T u_t^2 dt.
\]

The function $\eta^*$ and the corresponding optimal control $u^*$ can be expressed as

\[
\eta^*_t = \frac{e^{\sigma^2 t} - (T - t)^2}{x_0 - T + \frac{\sigma^2 t}{2}} + 1, \quad u^*_t = \frac{T - t}{x_0 - T + \frac{\sigma^2 t}{2}}.
\]

We set $x_0 = 1, T = 1$ and $\sigma = 0.1$ and the reference optimal cost is $J(u^*) = 0.514898066090988$. Numerical results by using our discrete recursive method are listed in Table 1.

Table 1: Errors and convergence rates for Example 5.1 a.

| N   | 8    | 16   | 32   | 64   | 128  | CR  |
|-----|------|------|------|------|------|-----|
| $|J(u^*) - J(u_N)|$ | 1.393E-01 | 1.364E-01 | 8.512E-02 | 3.243E-02 | 9.068E-03 | 0.996 |

Next, we choose a different $\eta^*$ and $u^*$, which is

\[
\eta^*_t = \frac{e^{\sigma^2 t} - (e^{-T} - e^{-t})^2 - e^{-t}}{x_0 + 1 - e^{-t} - te^{-T}}, \quad u^*_t = \frac{e^{-T} - e^{-t}}{x_0 + 1 - e^{-t} - te^{-T}}.
\]

Set $x_0 = 1, T = 1$ and $\sigma = 0.1$. The reference optimal cost is $J(u^*) = 0.345819897539892$. Numerical results in Table 2 demonstrate that our method is stable and admits a first order rate of convergence.

Table 2: Errors and convergence rates for Example 5.1 b.

| N   | 8    | 16   | 32   | 64   | 128  | CR  |
|-----|------|------|------|------|------|-----|
| $|J(u^*) - J(u_N)|$ | 5.931E-02 | 2.826E-02 | 1.369E-02 | 6.554E-03 | 3.056E-03 | 1.067 |

Example 5.2 The second example is the inventory control problem in \[4\]. The inventory level satisfies the following equation

\[
\begin{cases}
    dX_t = (u_t - r_t) dt + \sigma dW_t, \\
    X_0 = x_0,
\end{cases}
\]

with the total cost

\[
J(u) = \frac{1}{2} \int_0^T E[(X_t - \eta_t)^2] dt + \frac{1}{2} \int_0^T u_t^2 dt.
\]
The demand rates \( r_t = (T - t)/2 \) and \( \eta_t = 0.5Tt - 0.25t^2 + 1 \). Then the optimal production \( u^*_t \) and the optimal cost can be expressed as

\[
u^*_t = T - t, \quad J(u^*) = \frac{1}{6}T^3 + \frac{\sigma^2}{4}T^2 + T.
\]

We set \( x_0 = 0 \) and \( T = 1 \). Table 3 shows the numerical results of the Example 5.2 with \( \sigma = 0.0, 0.1 \) and \( 0.3 \), respectively. It clearly shows that the cost obtained by our numerical method admits a first order rate of convergence.

Example 5.3 The third example is a LQ problem in \[26\]

\[
\begin{cases}
    dX_t = u_t dt + \delta u_t dW_t, \\
    X_0 = x_0,
\end{cases}
\]

with the cost functional

\[
J(u) = \frac{1}{2} \int_0^T \mathbb{E} [X_t^2] dt.
\]

The optimal control and the corresponding optimal cost are given by

\[
u^*_t = -\frac{X_t}{\delta^2}, \quad J(u^*) = \frac{1}{2} \delta^2 \left( 1 - e^{-T/\delta^2} \right).
\]

Set \( x_0 = 1, T = 1 \) and \( \delta = 2 \). Numerical results are listed in Table 4. It is clearly shown that our method is stable and admits a first order rate of convergence.

Example 5.4 In last example we consider a portfolio problem

\[
\begin{cases}
    d\tilde{X}_t = (\alpha \tilde{u}_t + \gamma) \tilde{X}_t dt + \beta \tilde{u}_t \tilde{X}_t d\tilde{W}_t, \quad t \in (0, 1], \\
    \tilde{X}_0 = \tilde{x}_0,
\end{cases}
\]

with the cost functional

\[
\tilde{J}(u^*) = \min_{u \in K} \frac{1}{2} \mathbb{E} \left[ (\tilde{X}_1 - \kappa)^2 \right],
\]

Table 3: Errors and convergence rates for Example 5.2

| \(N\)  | 8    | 16    | 32    | 64    | 128    | CR  |
|-------|------|-------|-------|-------|--------|-----|
| \(\sigma = 0.0\) | 5.654E-02 | 2.746E-02 | 1.380E-02 | 6.870E-03 | 3.532E-03 | 1.000 |
| \(\sigma = 0.1\) | 7.888E-02 | 4.408E-02 | 2.286E-02 | 1.108E-02 | 4.908E-03 | 1.000 |
| \(\sigma = 0.3\) | 1.321E-01 | 8.608E-02 | 5.204E-02 | 2.548E-02 | 7.563E-03 | 1.001 |

Table 4: Errors and convergence rates for Example 5.3

| \(N\)  | 8    | 16    | 32    | 64    | 128    | CR  |
|-------|------|-------|-------|-------|--------|-----|
| \(J(u^*) - J(u^N)\) | 9.611E-03 | 4.653E-03 | 2.338E-03 | 1.193E-03 | 6.114E-04 | 0.991 |

Example 5.4 In last example we consider a portfolio problem

\[
\begin{cases}
    d\tilde{X}_t = (\alpha \tilde{u}_t + \gamma) \tilde{X}_t dt + \beta \tilde{u}_t \tilde{X}_t d\tilde{W}_t, \quad t \in (0, 1], \\
    \tilde{X}_0 = \tilde{x}_0,
\end{cases}
\]

with the cost functional

\[
\tilde{J}(u^*) = \min_{u \in K} \frac{1}{2} \mathbb{E} \left[ (\tilde{X}_1 - \kappa)^2 \right],
\]
and

\[ K = \{ \tilde{u} \in U(0,1) : -1 \leq \tilde{u} \leq 1, \text{ a.e. a.s.} \}. \]

Set \( \tilde{x}_0 = 6, \kappa = 20, \alpha = 0.25, \gamma = 1 \) and \( \beta = \sqrt{2}/2. \) The reference optimal cost with a fine mesh is \( \tilde{J}(u^*) = 6.00909101172000. \) Since the control set \( U = [-1,1], \) we need to project \( \tilde{u}_t \) into \([-1,1]\) by \( \tilde{u}_t = \max(\min(\tilde{u}_t,1),-1). \) We remark that the Bisection method can also be used to solve this example. Numerical results are listed in Table 5. It is clearly shown that our method admits a first order rate of convergence.

Table 5: Errors and convergence rates for Example 5.4

| N   | 8      | 16     | 32     | 64     | 128    | CR   |
|-----|--------|--------|--------|--------|--------|------|
| \(|J(u^*) - \tilde{J}(u^N)|\) | 3.592E+00 | 1.797E+00 | 9.761E-01 | 4.622E-01 | 2.205E-01 | 1.001 |

Remark 5.5 The control problem above is obtained from Example 4 in [8] through the following transformation

\[ \tilde{X}_t = \frac{1}{T} X_{Tt}, \quad \tilde{u}_t = u_{Tt}, \quad \tilde{J}(\tilde{u}) = \frac{1}{T^2} J(u), \]

and the process \( \tilde{W}_t = \frac{1}{\sqrt{T}} W_{Tt} \) with the \( \sigma \)-field \( \mathcal{F}_{Tt}^{\tilde{W}} = \mathcal{F}_{Tt}^W. \)

6 Conclusion

In this work, we reduce the optimal control problem to the discrete case and derive a discrete SMP. By means of this discrete SMP, we propose an effective discrete recursive method for solving SOCPs. The Euler scheme is used to approximate the discrete Hamilton system that is given by the discrete SMP condition and the state and adjoint equations. We conducted a rigorous error analysis and prove that our method admits a first order rate of convergence. Several numerical examples powerful support the theoretical results.

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