Stable Non–Perturbative Minimal Models Coupled to 2D Quantum Gravity

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Abstract

A generalisation of the non–perturbatively stable solutions of string equations which respect the KdV flows, obtained recently for the \((2m - 1, 2)\) conformal minimal models coupled to two–dimensional quantum gravity, is presented for the \((p, q)\) models. These string equations are the most general string equations compatible with the \(q\)-th generalised KdV flows. They exhibit a close relationship with the bi–hamiltonian structure in these hierarchies. The Ising model is studied as a particular example, for which a real non-singular numerical solution to the string susceptibility is presented.

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1. Introduction and Conclusions.

Pure two dimensional quantum gravity is known to have a KdV flow symmetry to all orders in genus perturbation theory. In a series of papers\cite{1}–\cite{4} it was established that a complete formulation of non-perturbative pure two dimensional quantum gravity can be developed from the single principle that this symmetry is respected non-perturbatively i.e. that the KdV flows are exact. In particular the principle leads to a unique string equation which has as special cases the non-perturbatively sick solutions of hermitian matrix models, but also has a unique $\lambda_1$ real non-singular solution. The above discoveries are briefly reviewed in section 2, together with the matrix model reasons for expecting these successes. The primary reason for the present paper is to show that these successes generalise to 2D quantum gravity coupled to a general conformal minimal model. For $(p, q)$ matter the single principle is that the $q$-th generalised KdV flows (known to exist perturbatively\cite{7}) are preserved non-perturbatively. We will demonstrate that this leads to unique string equations for each system as conjectured in ref.\cite{2} (and indeed for massive theories interpolating between all $(p, q)$ critical points for given $q$) and display a unique $\lambda_2$ real singularity-free solution for the Ising model coupled to gravity.

The success of the present formulation demands an analysis of its principle: non-perturbative preservation of flows. Let us first note that if this principle is discarded then other formulations are possible\cite{11}; Clearly some input is needed to define 2D gravity (a.k.a. 1D string theory) beyond the genus expansion. One might hope that unitarity ensures a unique non-perturbative extension. For 2D string theory (the lowest dimension where the concept of an $S$-matrix makes sense)

\footnote{We believe. In ref.\cite{2} it was proven that there is at most a discrete number of such solutions with real asymptotics, and a numerical study uncovered only one.}

\footnote{Subject to similar caveats.}
this appears to be insufficient[5]. In this case it is a natural conjecture that the perturbation expansion has a form of KP flow symmetry and it would certainly be interesting to trace this out and determine whether or not an exact KP flow symmetry picks out a unique unitary non-perturbative extension. Of course some other general constraint (causality?) conceivably might provide the missing information but this is not really the point: If one succeeds in providing an ‘unprincipled’ extension (in the sense that the perturbation theory must be separately determined) then non-perturbative string theory remains logically incomplete.

Having argued for principles, what form should they take? A common attempt in the past has been to try to formulate non-perturbative string theory from some symmetry principle on the world-sheet i.e. a symmetry of 2D quantum gravity considered on a single (possibly pinched) genus. In view of the work of the last two years on low dimensional string theory, and of the simplicity in this case of the world-sheet theory when appropriately formulated, this now surely seems as unlikely as expecting, say, non-perturbative QCD to arise from some symmetry of the world-line. Indeed in a second quantized theory one expects the symmetries to be best manifested on the second quantized fields. Recalling that their expectation values – the background fields – are nothing but the couplings $t_r$ in string theory, we expect the symmetry to be manifested as active transformations on the $t_r$. Thus a non-perturbative string theory symmetry principle should be a symmetry of “theory space”, the space of all world-sheet couplings, and not of a single world-sheet theory. This is precisely what the KP (generalized KdV) flows are. Thus it seems highly probable that this symmetry principle is a hint of a much larger symmetry determining the non-perturbative form of 2D quantum gravity coupled to general conformal matter.

In this paper our primary purpose, as already mentioned, is to develop the formulation for gravity coupled to $(p, q)$ matter. The first steps towards this were already taken in ref.[2] where, motivated by a matrix model whose eigen-value space was $\mathbb{R}_+$, use was made of $[\hat{P}, Q] = Q$: the appropriate generalisation of Douglas’ $[P, Q] = 1$ formalism[7]. By using the Lax pair formalism for (generalized) KdV flows we will see that in this case the Douglas’ formalism is nothing but a trivial expression of scaling: i.e. it follows directly from the fact that KP flows have a
grading. Many other intimate connections with the reductions of the KP hierarchy are uncovered. For example the classical $W\lambda(q)$-algebra of the second Hamiltonian structure is seen to play a central rôle. The integrability of the hierarchy implies the existence of a first integral of the scaling equation: our string equation. The existence of a natural ‘gauge’ parameter $\sigma$ – the boundary of $\mathbb{R}_+$ and physically a world-sheet boundary coupling – is seen to be a consequence of coordination of the bi-Hamiltonian structure. Indeed another reason for the present paper is to provide a complete discussion of this parameter, partial results having been reported earlier[3][11][4].

In section 5 we recall[2] that the $L_{-1}$ symmetry of KP flows is not a symmetry of the vacuum, leading to an analogy with spontaneous symmetry breaking in which $\sigma$ is identified with the Goldstone boson. Since, as we will show, all the $W\lambda(k)_n$ generators with negative index $n$ are ‘spontaneously broken’ there are further generalizations of the $\sigma$ parameter for the $(p, q)$ models with $q \geq 3$. For pure gravity $\sigma$ is the boundary cosmological constant. For the Ising model it is the boundary magnetic field. There are two other parameters for the Ising model associated with $W_{-1}$ and $W_{-2}$. The latter we tentatively associate with the boundary cosmological constant – a parameter missing in previous formulations[19].

The structure of the paper is as follows: Section 2 is a short review of previous work followed by a review of our construction for the $(2m - 1, 2)$ models. We emphasise the Lax pair formulation and bring together the earlier results on $\sigma$.

Section 3 deals with the Ising model coupled to 2D quantum gravity. In particular we review the $[P, Q] = 1$ equations and combine the reasons for expecting the solutions to have similar non-perturbative sicknesses to the $[P, Q] = 1$ solutions of pure 2D quantum gravity. Next we derive the most general string equation compatible with an exact Boussinesque flow symmetry (assuming no new dimensionful parameters appear at the non-perturbative level). We study the solution for vanishing magnetic field in particular. The cosmological constant $z \to +\infty$ limit is fixed by the established genus expansion. In the $z \to -\infty$ limit we assume that all singularity-free solutions have an asymptotic expansion in which case there are two possibilities. One of these leads to the problematic $[P, Q] = 1$ solution, while the other has $\rho \to 0$ as previously[2]. There are at most a discrete number of solutions
with the latter asymptotic, and a numerical study reveals only one. It is real and free of singularities. Since our Lee-Yang (5,2) solution has the same leading asymptotics we display it too, for comparison.

In section 4 we construct the most general string equations compatible with the qth KdV flows. As mentioned above we generalise the introduction of the σ parameter, utilise the close relationship to the generalized hierarchies, and in sect. 5 discuss the modifications of the Dyson-Schwinger W-algebra constraints.

2. Review

In this section we review the \([\tilde{P}, Q] = Q\) formulation of the \((2m − 1, 2)\) models which was developed in [1][2][3], including a complete discussion of the rôle played by the non–perturbative parameter σ, the boundary cosmological constant. Central to the discussion is the requirement of scaling in the models and the principle that the KdV flows are preserved. We derive the most general string equations compatible with these requirements, using the Lax pair representation of the KdV flows. This representation readily makes contact with Douglas’ differential operator formulation of one–matrix models, and prepares the way for the generalisations presented in later sections. These equations have been shown to have real, pole–free solutions [1][2][18].

2.1 Matrix Models

The original one–hermitian matrix models [6] provided an exact solution to the \((2m − 1, 2)\) models via the string equations for the string susceptibility, \(\rho\), together with the KdV flows. The solutions for \(\rho\) obtained from these string equations, although well defined in perturbation theory, produce problematic non–perturbative solutions for the \(m\)–even models: The only relevant real solutions to the string equations possess poles. The physical interpretation of these poles is unclear, and their presence violates the Dyson–Schwinger equations of the models. The relevant pole–free solutions to the \(m\)–even equations are the triply truncated solution of Boutroux and its generalisations[8][14] which are complex and therefore physically unacceptable.
The problems of the definition may be traced to an instability of the one–hermitian matrix model at its $m$–even critical points, as a careful study of the associated scaled eigenvalue problem reveals[10]. A complementary study of the asymptotic behaviour of the $m$–even string equations reveals the presence of real ‘instanton’ solutions in the single–well eigenvalue problem. The presence of the instantons is not by itself a signature of instability. However, the local topology of the eigenvalue space and the form of the effective potential for the scaled eigenvalues demonstrates that the definition of the $m$–even critical points is unstable to eigenvalues tunneling into a different configuration.

Later, by studying one–complex matrix models[17][1] an alternative exact solution to the $(2m−1,2)$ models was constructed. A different set of string equations which have the same perturbation theory as the previous definition was found. These equations possess real pole–free solutions however, thereby providing a more satisfactory non–perturbative definition. The KdV flows of the earlier definition assume a central rôle in these models, since it turns out that the equations are the most general consistent with this structure.

The stability of these models may be traced back to the local topology of the scaled eigenvalues: The one–complex matrix models studied were formulated in terms of the combination $M\lambda \dagger M$ and the new solutions had positive definite scaled eigenvalues. The local topology of the critical theory is thus $\mathbb{R}_+$ in contrast to the $\mathbb{R}$ of the original one–hermitian matrix models. The resulting ‘infinite wall’ at the boundary has the effect of removing the eigenvalue tunneling problem. The same effect may be obtained simply by imposing an $\mathbb{R}_+$ topology on scaled eigenvalue space.

The differential of the string equations of the original one–hermitian matrix model definition of the $(2m−1,2)$ models may be written as the canonical commutation relation between the operator $Q$ representing position $\lambda_s$ and the operator $P$ representing momentum $d/d\lambda_s$: $[P,Q]=1$. The operators $P$ and $Q$ are differential operators[7] in the scaled parameter $z$.\footnote{For unitary models, $z$ is the cosmological constant and hence couples to the puncture operator $P$.} In particular, $Q = d\lambda_s + u_2$ where $u_2 = -\rho$ and $d \equiv \partial / \partial z$.  

For a one–matrix model defined on a half–line (i.e. the scaled eigenvalue space has topology $\mathbb{R}_+$) we may write the differentiated string equations as the canonical commutation relation between $Q$ and the relevant conjugate momentum, $\tilde{P}$, which is scale transformations $\lambda_s \frac{d}{d\lambda_s}$ about the wall at $\lambda_s = 0$: $[\tilde{P}, Q] = Q$. This represents a sort of gauge fixed version. For full generality we should introduce another parameter into the theory: the scaled position, $\sigma$, of the wall. Thus the canonical momentum is $(\lambda_s - \sigma) \frac{d}{d\lambda_s} \equiv \tilde{P} - \sigma P$. We thus have:

$$[\tilde{P} - \sigma P, Q] = Q - \sigma$$

(2.1)

which represents the differentiated string equations of the stable definition of the $(2m - 1, 2)$ models.

2.2 The $[\tilde{P}, Q] = Q$ Definition of the $(2m - 1, 2)$ Models

In Douglas’ differential operator prescription for the $(2m - 1, 2)$ series, local operators in the theory are constructed via fractional powers of $Q$, the coordinate operator $\lambda 4$, giving the infinite set $O_k \sim Q\lambda^k + \frac{1}{2} + \ldots$ and their dimensionful couplings $t_k$. Operator insertions are structured according to the KdV flows:

$$\frac{\partial Q}{\partial t_k} = \kappa [Q, Q\lambda^k + \frac{1}{2}, Q]$$

(2.2)

Here $t_0$ and the cosmological constant $z$ are seen to be related by the non–universal normalisation $\kappa$ by setting $k$ to zero in the above: $\kappa t_0 = z$. Thus in the unitary model ($(3, 2) \equiv$ pure gravity) where $z$ is the cosmological constant, we have $Q\lambda^{1/2} \sim P$ the puncture operator. In what follows, we shall normalise the KdV flows with $\kappa = -1$.

The string equations realising the commutation relations (2.1) may be derived using the principle that the KdV flows are preserved even beyond perturbation theory.

The scale transformation operator $\tilde{P}$ is simply

$$\tilde{P} = \sum_{k=0}^{\infty} \lambda^k (k + \frac{1}{2}) t_k \frac{\partial}{\partial t_k}$$

(2.3)

$\lambda 4$ See section 4 for a brief review of the fractional powers of differential operators.
where, setting the scaling dimension of $Q$ to 1, the dimensions of the $t_k$ may be
derived from (2.2). Using (2.3) and (2.2), the translation operator follows as

$$P = \sum_{k=1}^{\infty} \lambda \infty (k + \frac{1}{2}) t_k \frac{\partial}{\partial t_{k-1}}$$

Using these definitions and (2.1) the differentiated string equation for the $(2m-1, 2)$
models is:

$$[\sum_{k=1}^{\infty} \lambda \infty (k + \frac{1}{2}) t_k Q_+ \lambda_+ + \frac{1}{2} - \sigma \sum_{k=1}^{\infty} \lambda \infty (k + \frac{1}{2}) t_k Q_+ \lambda_+ - \frac{1}{2} - \frac{z}{2} d, Q] = Q - \sigma$$

(2.4)

where $Q = d \lambda 2 + u_2$. Using the fact that $[Q_+ \lambda + \frac{1}{2}, Q] = \mathcal{R} \lambda' [-u_2]_{k+1}$ equation (2.4) is a differential equation for $u_2$:

$$\frac{1}{4} \mathcal{R} \lambda'' + (u_2 - \sigma) \mathcal{R} \lambda' + \frac{1}{2} u \lambda' 2 \mathcal{R} = 0$$

(2.5)

where

$$\mathcal{R} = \sum_{k=1}^{\infty} \lambda \infty (k + \frac{1}{2}) t_k R_k - z$$

The $R_k$'s are the Gel’fand–Dikii differential polynomials $\lambda 5$ in $-u_2$. In the above, we
have used the recursion relation $D_1 R_{k+1} = D_2 R_k$ where $D_1 \equiv d$ and $D_2 \equiv \frac{1}{4} d \lambda 3 + u_2 d + \frac{1}{2} u \lambda' 2$. The requirement for them to vanish at $u_2 = 0$ fixes them uniquely up
to the normalisation $\mathcal{R}_0$, which we set to 2. When multiplied by $\mathcal{R}$ (2.5) may be
once integrated to give:

$$(u_2 - \sigma) \mathcal{R} \lambda 2 + \frac{1}{2} \mathcal{R} \mathcal{R} \lambda'' - \frac{1}{4} (\mathcal{R} \lambda') \lambda 2 = 0$$

(2.6)

where the matrix model tells us to fix the constant of integration to zero by the
requirement that in the $z \to +\infty$ limit we must have the asymptotic expansion of
$\mathcal{R} = 0$ coinciding with the hermitian matrix model perturbative physics. It is easily
verified that the string equation (2.6) has the generalised Galilean transformations

$$u_2 \to u_2 + \epsilon$$
$$\sigma \to \sigma + \epsilon$$
$$z \to z + \frac{3}{2} t_1$$
$$t_k \to t_k - \epsilon (k + \frac{3}{2}) t_{k+1} \quad k \geq 0$$

(2.7)

$\lambda 5$ In the original work [21] the differential polynomials $R_k[u]$ are normalised such
that $4R_k = \mathcal{R}_k$. 7
as a symmetry. Using this symmetry we may perform a redefinition of the $t_k$’s in order to set $\sigma$ to zero. This corresponds to putting the potential wall in the matrix model at the origin of eigenvalue space. The resulting string equation was discussed in refs.[1] and [2] where it was argued that it possesses real, pole–free solutions for the $m$th model with the asymptotics $u_2 \rightarrow z\lambda 1/m$ (0) in the $z \rightarrow +\infty$ ($-\infty$) limits. A numerical solution for pure gravity ($m = 2$) is presented in [2]. Further analytical and numerical study has demonstrated the consistency of KdV differential flows between all the $m$–critical models, and the $m = 1$ and $m = 3$ solutions were displayed [18]. The symmetry (2.7) may be rewritten as a “flow” for $u_2$ under the parameter $\sigma$:

$$\frac{\partial u_2}{\partial \sigma} = 1 + \sum_{k=0} \lambda_\infty(k + \frac{3}{2})t_{k+1} \frac{\partial u_2}{\partial t_k} = -\mathcal{R}\lambda'$$

where we have used the KdV flows for $u_2$: $\partial_{t_k} u_2 = -\mathcal{R}\lambda'_{k+1}$ in the last step. Equation (2.8) is consistent with the interpretation of the differentiated string equation (2.4) or (2.5) as a scale–invariance equation for $u_2$:

$$\sum_{k=1} \lambda_\infty(k + \frac{1}{2})t_k \frac{\partial u_2}{\partial t_k} + z \frac{\partial u_2}{\partial z} + \sigma \frac{\partial u_2}{\partial \sigma} + u_2 = 0$$

Equations (2.8) and (2.9) may be written as constraints on the partition function of the theory, which plays the rôle of a tau function, $\tau$, of the KdV hierarchy. They are then seen to be the familiar $L_{-1}$ and $L_0$ Virasoro constraints, but with modifications in the presence of arbitrary $\sigma$:

$$L_{-1} \tau \equiv \sum_{k=1} \lambda_\infty(k + \frac{1}{2})t_k \frac{\partial \tau}{\partial t_{k-1}} + \frac{1}{4} t_\lambda 2_0 \tau = \frac{\partial \tau}{\partial \sigma}$$

$$L_0 \tau \equiv \sum_{k=0} \lambda_\infty(k + \frac{1}{2})t_k \frac{\partial \tau}{\partial t_k} + \frac{1}{16} \tau = \sigma \frac{\partial \tau}{\partial \sigma}$$

The rest of the Virasoro constraints are similarly modified:

$$L_n \tau \equiv \sum_{k=0} \lambda_\infty(k + \frac{1}{2})t_k \frac{\partial \tau}{\partial t_{k+n}} + \frac{1}{4} \sum_{k=1} \lambda_n \frac{\partial \lambda 2\tau}{\partial t_{k-1} t_{n-k}} = \sigma \lambda n + 1 \frac{\partial \tau}{\partial \sigma} \quad n \geq 1$$

$\lambda^6 \tau$ and $u_2$ are related by $u_2 = 2d\lambda^2 \ln \tau$
That $\sigma$ produces a term of the form $\sigma \lambda n + 1 \partial / \partial \sigma$ in the $L_n$ is consistent with the fact that the Virasoro constraints represent diffeomorphisms in the space of eigenvalues of the one–matrix model. The terms in $\sigma$ then arise naturally as boundary terms from infinitesimal variations of the position of the “wall” at $\lambda_s = \sigma$.

It was also noted in ref.[3] that $\sigma$ plays the role of a boundary cosmological constant in the theory. It takes on the mantle of the combination $t_{m-1}/(m + \frac{1}{2})t_m$ which was identified as such for the $m$–critical $[P, Q] = 1$ theory to first order in $t_k, k < m[19]$. Here the identification is not perturbative. It is also cleaner and more natural since $\sigma$ couples directly to $L_{-1}$, which through (2.7) is seen to lead to an $e^{\lambda - \sigma \ell}$ dependence for the macroscopic loop $\omega(\ell) \sim< e^{\lambda \ell (u_2 + d \lambda 2)}>$, identifying $\sigma$ as the boundary cosmological constant. Thus $L_{-1}$ is the conjugate i.e. boundary length operator—as is already obvious in the one–hermitian matrix model from the corresponding Ward identity[19]. We are taking consistently the one–hermitian matrix model definition for a macroscopic loop; these issues and the full loop equations are discussed in depth in ref.[4]. The above observations lead naturally to generalisations when we consider the general $(p, q)$ model. These are discussed, together with a derivation of (2.11) and (2.10) through the Dyson–Schwinger equations, in section 5.

3. The Ising model

In preparation for the generalisation of the $[\tilde{P}, Q] = Q$ formulation to all of the $(p, q)$ models, we study the Ising model (4,3). We derive the most general string equation compatible with the Boussinesque flows, but postpone the now subtle issue of boundary parameters until sections 4 and 5. We begin with a brief review of the $[P, Q] = 1$ formulation of the Ising model[20].

3.1 $[P, Q] = 1$ Ising Model

Consider the following two–hermitian matrix partition function:

$$ Z(H, g, c) = \int \mathcal{D}M_+ \mathcal{D}M_- \exp \left( - \frac{N}{\gamma} S(M_+, M_-, H, g, c) \right) \quad (3.1) $$
where

\[ S(M_+, M_-, H, g, c) = \text{Tr}(M_+ \lambda^2 + M_- \lambda^2 - 2cM_+M_- - ge\lambda HM_+\lambda^4 - ge\lambda - H M_-\lambda^4) \] (3.2)

This defines the Ising model on a random surface where \( M_\pm \) represent the two Ising spin states at the vertices of the diagrams in the usual large-\( N \) expansion of (3.1). \( H \) is the magnetic field.

Following refs.[15] and [16] the partition function may be expressed in terms of the norms \( h_n \) of certain polynomials \( P_n \lambda \pm \) which are orthogonal with respect to a measure weighted by the potential (3.2):

\[ \int d\lambda d\mu e^{\lambda - \frac{N}{\gamma} S(\lambda, \mu)} P_n \lambda + (\lambda) P_m \lambda - (\mu) = h_n \delta_{nm} \] (3.3)

The \( P_n \lambda \pm \) satisfy a recursion relation:

\[ \lambda P_n \lambda \pm (\lambda) = P_{n+1} \lambda \pm (\lambda) + R_n \lambda \pm P_{n-1} \lambda \pm (\lambda) + S_n \lambda \pm P_{n-3} \lambda \pm (\lambda) \] (3.4)

The string equations arise as the double scaling limit [20] of identities derived using (3.4) and (3.1). The scaling functions \( u_2 \) and \( u_3 \) (related to the string susceptibility and the magnetisation), together with the variables \( \nu, \mu \) and \( B \) (the physical string coupling, cosmological constant and magnetic field) arise as the scaling parts of the quantities in the recursion relation. They are related to the free energy \( \Gamma \) as follows: \( u_2 = -3/2\nu \lambda^2 \theta_2 \lambda \mu \Gamma \) and \( u_3 = \nu \lambda^2 \partial_\mu \theta_2 \mu \Gamma \). Henceforth we shall absorb \( \nu \) into the quantities \( \mu \) and \( B \) defining \( z = \mu/\nu \) and \( B = B/\nu \).

Another approach to derive the string equations is to study the double scaled limits, \( P \) and \( Q \), of operators \( P_{mn} \) and \( Q_{mn} \) defined by:

\[ Q_{mn} \equiv \int d\lambda d\mu e^{\lambda - \frac{N}{\gamma} S(\lambda, \mu)} \frac{P_m \lambda + (\lambda)}{\sqrt{h_m}} \frac{P_n \lambda - (\mu)}{\sqrt{h_n}} \]

and

\[ P_{mn} \equiv \int d\lambda d\mu e^{\lambda - \frac{N}{\gamma} S(\lambda, \mu)} \frac{P_m \lambda + (\lambda)}{\sqrt{h_m}} \frac{d}{d\mu} \frac{P_n \lambda - (\mu)}{\sqrt{h_n}} \]

The string equations then arise from the requirement that \( P \) and \( Q \), which are differential operators in \( z \), satisfy \([P, Q] = 1\). This approach was first proposed in
[7] and carried out explicitly for the two–matrix model in [22] to yield the string equations for the Ising model.\[7

In the double scaled limit, the operator $Q$ becomes a third order differential operator in $z$:

$$Q = d\lambda 3 + \frac{3}{4}\{u_2, d\} + u_3$$

Here, $d$ denotes $d/dz$. The critical point for the Ising model is realised when $P$ is a fourth order differential operator. The requirement that $P$ and $Q$ satisfy $[P, Q] = 1$ fixes $P$ to be $Q_+\lambda 4/3$ where the ‘+’ denotes the differential operator part of the pseudo–differential\[8 operator $Q\lambda 4/3$.

A simple calculation yields the following string equations:

$$-\frac{1}{12}(u\lambda''''2 + 9u_2u\lambda''2 + \frac{9}{2}(u_2\lambda')\lambda 2 + 6u_2\lambda 3 - 8u_3\lambda 2) = z$$

$$\frac{2}{3}(u\lambda'''3 + 3u_2u_3) = B$$

(3.5)

where we have integrated once with respect to $z$. One integration constant has been absorbed into $z$, and the other is identified with the magnetic field $B$ [7]. (To match some of the conventions used in refs.[20] we must redefine $z \rightarrow z/2$, $\nu \rightarrow \nu/\sqrt{6}$ and $u_3 \rightarrow -u_3$ and recall that the string susceptibility $\rho$ is $-u_2$.) We may study some of the tree level physics obtained from these equations by taking the leading behaviour for large cosmological constant $z$. This amounts to neglecting all derivatives:

$$-\frac{1}{2}u\lambda 3_2 + \frac{2}{3}u\lambda 2_3 = z$$

$$2u_2u_3 = B$$

(3.6)

Eliminating $u_3$ from the resulting equations, and adopting the above conventions, we have the following equation for the string susceptibility at the sphere level:

$$\rho\lambda 3 + \frac{1}{3}\frac{B\lambda 2}{\rho\lambda 2} = z$$

(3.7)

\[7 \text{The authors in ref.[22] also studied potentials of higher order than quartic to discover that the two–matrix model can be tuned to critical points other than the (\ast, 3) models. This issue will not concern us at present.}

\[8 \text{These objects are briefly reviewed in section 4. The reader is referred to one of many fine works on the calculus of pseudo–differential operators such as ref.[23][24] for a comprehensive treatment of the subject.}
From this we may derive expressions for tree level correlators of the puncture operator $\mathcal{P}$ and the spin operator $\mathcal{S}$.

The Ising model (3.1) with $H \neq 0$ is not symmetric under the interchange of $M_+ \text{ and } M_-$. Such an interchange translates into $d \rightarrow -d$ in the differential operator formalism[20] and this is the origin of the $\mathbb{Z}_2$–odd transformation properties of $\mathcal{B}$ and $u_3$. Studying the $\mathbb{Z}_2$ symmetric sector by setting $\mathcal{B}$ and $u_3$ to zero, we have the string equation for the string susceptibility:

$$\frac{2}{27}\rho \lambda''' - \frac{1}{2}(\rho \lambda')\lambda'2 - \rho \rho \lambda'' + \rho \rho 3 = z$$

(3.8)

Large $z$ expansion supplies the genus perturbation theory which is uniquely determined from the the spherical contribution $\rho(z) = z\lambda1/3$.

Equation (3.8) is very similar to the string equation for the Lee–Yang singularity (the (5, 2) model [25]) where instead the coefficient of $\rho \lambda'''$ is $1/10$. A numerical solution to that equation with the asymptotics $\rho \rightarrow \pm z\lambda1/3$ for $z \rightarrow \pm \infty$ was presented in [26]. The family of $(2m - 1, 2)$ string equations was studied in ref.[14] in order to construct the solutions analogous to the ‘triply truncated’ solutions found by Boutrourx for Painlevé'. In that work, the authors demonstrated that the generalised truncated solutions were real for the $m$–odd models and complex for the $m$–even models. The Lee–Yang string equation falls into the former category and the Painlevé' equation into the latter.

It is a simple matter to repeat the analysis for the Ising equation (3.8). The reality of the truncated solutions is determined by the nature of the solutions to the associated polynomial $\lambda^9 s\lambda 2 - As + 3A = 0$, where $1/A = 2/27$ for the Ising model and $1/10$ for Lee–Yang. If $s$ is complex then the truncated solutions will be real, and for real $s$ they are complex. We see that $s$ is complex for $A > 1/12$ and so the Lee–Yang solution is real ($s = 5 \pm i\sqrt{5}$) and the Ising model solution ($s = 27, 9/2$) is complex.

The alert reader will note that the same polynomial determines the reality of the ‘instantons’ in the asymptotic expansion of (3.8) when addressing the question of Borel resummability [12][13]. These ‘instanton’ solutions are merely the leading exponential corrections to the perturbation expansion as a representation.

\lambda9 We refer the reader to ref.[14] for details.
of the full non–perturbative solution. The real instantons and corresponding non–
resummability for the Ising model is purely a consequence of the unitarity of the
theory (whereby all terms in the genus expansion contribute with positive sign) and
the typical \((2n)!\) growth of the perturbative series.

Later we will find that the \([\hat P, Q] = Q\) formulation supplies the same genus
expansion as for (3.8) and hence the same resummation properties for positive
cosmological constant. Nevertheless, we shall explicitly have a real non–perturbative
solution. This situation was already discussed for the \((2m - 1, 2)\) models in [2].

3.2 \([\hat P, Q] = Q\) Ising Model.

We must first recall the underlying structure which exists perturbatively for the
\([P, Q] = 1\) definition of the \((*, 3)\) models, the Boussinesque hierarchy, which defines
the flows of \(Q = d\lambda 3 + (3/4)\{u_2, d\} + u_3:\)

\[
\frac{\partial Q}{\partial t_{l,k}} = \kappa [Q\lambda \frac{3k + l}{3}, Q] \quad l = 1, 2; \quad k = 0, 1 \ldots \infty \quad (3.9)
\]

The \(t_{l,k}\) are an infinite set of parameters which parametrize the hamiltonian flows
of \(Q\). \(\kappa\) is a non–universal normalisation parameter. We may write the equation
(3.9) as a pair of equations for \(u_2\) and \(u_3:\)

\[
\alpha^{(i)} \frac{\partial u_i}{\partial t_{l,k}} = \kappa D\lambda^{ij} R\lambda^{ji}_{l,k+1} \equiv \kappa D\lambda^{ij} R\lambda^{ji}_{l,k} \quad i, j = 2, 3 \quad (3.10)
\]

The \(R\lambda_{ji,k}\) are differential polynomials in \(u_2\) and \(u_3\), and \(\alpha^{(2)} = 3/2, \alpha^{(3)} = 1\). The
subscript bracketed \((i)\) indicates “no sum on \(i\)”. The \(R\lambda_{ji,k}\) are the generalisation of
the Gel’fand–Dikii differential polynomials for the KdV hierarchy. The equivalence
of the two hamiltonian structures defined in (3.10) implies a recurrence relation
between them. Requiring them to vanish at \(u_2 = u_3 = 0\) fixes them completely up
to the normalisations \(R\lambda_{i0}\). The first few are:

\[
\begin{align*}
R\lambda_{21,0} &= 3; & R\lambda_{31,0} &= 0; \\
R\lambda_{22,0} &= 0; & R\lambda_{32,0} &= 3; \\
R\lambda_{21,1} &= 2u_3; & R\lambda_{31,1} &= \frac{3}{2} u_2; \\
R\lambda_{22,1} &= -\frac{1}{4} (u\lambda''_2 + 3u_2 \lambda_2); & R\lambda_{32,1} &= 2u_3; \\
R\lambda_{21,2} &= -\frac{1}{12} u\lambda'''_2 - \frac{3}{4} u_2 u\lambda''_2 - \frac{3}{2} (u_2 \lambda')\lambda_2 - \frac{1}{2} u\lambda_3 + \frac{4}{3} u\lambda_2; & R\lambda_{31,2} &= \frac{2}{3} (u\lambda''_3 + 3u_3 u_2);
\end{align*}
\]
The objects $\mathcal{D}\lambda ij_1$ and $\mathcal{D}\lambda ij_2$ define the first and second Hamiltonian structures of the Boussinesq hierarchy. They are a shorthand notation for the fundamental structures defining the Poisson bracket for functionals of $u_2$ and $u_3$. The explicit expressions for them are:

\[
\begin{align*}
\mathcal{D}\lambda 22_2 &= \frac{2}{3} d\lambda 3 + \frac{1}{2} u_2 \lambda' + u_2 d \\
\mathcal{D}\lambda 23_2 &= u_3 d + \frac{2}{3} u_3 \lambda' \\
\mathcal{D}\lambda 32_2 &= u_3 d + \frac{1}{3} u_3 \lambda' \\
\mathcal{D}\lambda 33_2 &= -\frac{1}{18} d\lambda 5 - \frac{5}{12} u_2 d\lambda 3 - \frac{5}{8} u_2 \lambda' d\lambda 2 + (-\frac{1}{2} u_2 \lambda 2 - \frac{3}{8} u_2 \lambda'') d + (-\frac{1}{2} u_2 u_2 \lambda' - \frac{1}{12} u_2 \lambda''')
\end{align*}
\]

and $\mathcal{D}\lambda 21_1 = \mathcal{D}\lambda 31_1 = 0$; $\mathcal{D}\lambda 23_1 = \mathcal{D}\lambda 32_1 = d$. From the equation

\[\alpha(i) \frac{\partial u_i}{\partial t_{1,0}} = \kappa \mathcal{D}\lambda ij_1 R\lambda j_{1,1} \equiv \kappa \mathcal{D}\lambda ij_2 R\lambda j_{1,0} = \alpha(i) \kappa u_i \lambda'_i\]

we make the identification $z = \kappa t_{1,0}$. In what follows we shall set $\kappa = -1$. The scaling dimension of $Q$ supplies a natural length scale in the theory. Fixing its dimension fixes the scaling of the $t_{l,k}$. If we assign the scaling dimension $2$ to $u_2$, we obtain $[Q] = 3$, $[t_{l,k}] = -(3k + l)$, and $[u_3] = 3$.

The central assumption which leads uniquely to the string equation is that the Boussinesq flows (3.9) hold at the non-perturbative level. We construct $\tilde{P}$, the generator of scale transformations in the theory, out of the parameters in the Ising model with magnetic field: $t_{1,2}$ (which defines the $(4,3)$ model), $t_{1,0} = -z$, and $t_{2,0} \propto B$.

\[
\tilde{P} = -7t_{1,2} \frac{\partial}{\partial t_{1,2}} - z \frac{\partial}{\partial z} - 2B \frac{\partial}{\partial B}
\]

Using $[\partial_{t_{1,2}}, Q] = -[Q + \lambda 7/3, Q]$, we have for our differentiated string equation:

\[
\left[-7t_{1,2}Q + \lambda 7/3 - z \frac{\partial}{\partial z} - 2B \frac{\partial}{\partial B}, Q\right] - 3Q = 0
\]

which is a pair of scaling equations for $u_2$ and $u_3$:

\[
\alpha(i) \left(-7t_{1,2} \frac{\partial u_i}{\partial t_{1,2}} + z \frac{\partial u_i}{\partial z} + 2B \frac{\partial u_i}{\partial B} + iu_i\right) = 0
\]
Identifying $B = -2t_{2,0}$, these equations may be succinctly written as:

$$\mathcal{D}\lambda_{ij}^{2}\mathcal{R}\lambda_{j} = 0 \quad (3.11)$$

where $\mathcal{R}\lambda_{2} = \mathcal{R}\lambda_{2,1} - z$ and $\mathcal{R}\lambda_{3} = \mathcal{R}\lambda_{3,1} - B$ and we have set $t_{1,2} = 1/7$. It should be noted here that we may write the derivative of the $[P, Q] = 1$ string equations (3.5) as

$$\mathcal{D}\lambda_{ij}^{3}\mathcal{R}\lambda_{j} = 0 \quad (3.12)$$

Finally, to obtain the string equation, we multiply equation (3.11) on the left by $\mathcal{R}\lambda_{i}$ giving us a total differential and integrate once with respect to $z$, to give:

$$\frac{1}{2}u_{2}\mathcal{R}_{2}\lambda_{2} + \frac{2}{3}\mathcal{R}_{2}\mathcal{R}\lambda''_{2} - \frac{1}{3}(\mathcal{R}\lambda'_{2})\lambda_{2} + u_{3}\mathcal{R}_{2}\mathcal{R}_{3} - \frac{1}{18}\left(\mathcal{R}_{3}\mathcal{R}\lambda(4)_{3} - \mathcal{R}\lambda'_{3}\mathcal{R}\lambda''_{3} - \frac{1}{2}(\mathcal{R}\lambda''_{3})\lambda_{2}\right)$$

$$- \frac{5}{12}\left(u_{2}\mathcal{R}_{3}\mathcal{R}\lambda''_{3} - \frac{1}{2}u_{2}(\mathcal{R}\lambda'_{3})\lambda_{2} + \frac{1}{2}u_{2}\lambda'_{3}\mathcal{R}\lambda'_{3}\right) - \frac{1}{12}(3u_{2}\lambda_{2} + u_{2}\lambda'_{3})\mathcal{R}\lambda_{2} = 0 \quad (3.13)$$

(For convenience of notation we have exchanged the superscripts on the $\mathcal{R}\lambda_{i}$'s for subscripts.) We have set the constant of integration to zero by requiring that our perturbative physics obtained in the $z \to +\infty$ limit is the same as that obtained from the matrix model via equations (3.12). Indeed, with the constant in place, the tree level string equation is:

$$(-\frac{1}{2}u_{2}\lambda_{3} + \frac{2}{3}u\lambda_{2} - z)\lambda_{2} - u_{2}(2u_{2}\lambda_{3} - B) = \text{constant}$$

from which we may obtain the equations (3.6) by setting each bracket and the constant to zero. We can then obtain the same tree level physics as the $[P, Q] = 1$ definition. With the constant set to zero we must always follow this procedure at any level of perturbation theory to match the physics of the $[P, Q] = 1$ equations. This is a direct consequence of the fact that the structure of the string equation (3.13) always admits a $\mathcal{R}_{2} = \mathcal{R}_{3} = 0$ solution.

### 3.3 The Solution for the String Susceptibility.

We now study the physics of equation (3.13) in the absence of the $Z_{2}$ breaking quantities $u_{3}$ and $B$. Setting them to zero and adopting the conventions of (3.8) we have the following string equation for the string susceptibility, $\rho$:

$$\frac{9}{16}\rho\mathcal{R}\lambda_{2}I - \frac{1}{2}\mathcal{R}_{I}\mathcal{R}\lambda''_{I} + \frac{1}{4}(\mathcal{R}\lambda'_{I})\lambda_{2} = 0 \quad (3.14)$$
where
\[ R_I \equiv \frac{2}{27} \rho \lambda''' - \frac{1}{2} (\rho \lambda') \lambda 2 - \rho \rho \lambda'' + \rho \lambda 3 - z \]

We now have an equation for \( \rho \) which is structurally identical to that obtained for the string susceptibility of the Lee–Yang model \((5, 2)\): The coefficient of the first term in (3.14) would instead be 1, and the expression for \( R_{LY} \) would have \( 1/10 \) as the coefficient of the first term. The two equations are similarly analysed. Using dimensional arguments the asymptotic expansion for \( z \to +\infty \) may be shown to be of the form
\[ \rho = z \lambda \frac{1}{3} \sum_{i=0}^{\infty} \lambda A_i \left( \frac{1}{z \lambda^3} \right) \lambda^i \]
where the \( A_i \)'s are dimensionless constants. By substitution these can be seen to be determined uniquely once \( A_0 \) is known. Since the same is true of the \([P, Q] = 1\) string equation (3.8) it follows that the resulting perturbative expansion is identical in both cases once we have set \( A_0 = 1 \). As in ref.[2] we assume that all real solutions without an asymptotic expansion in the \( z \to -\infty \) limit have poles. Requiring an asymptotic expansion in this limit we find that the sphere solution is either \( \rho = -|z| \lambda 1/3 \) or \( \rho = 0 \). It is clear from the above discussion that the first choice leads to the same perturbative physics as the \([P, Q] = 1\) definition and probably the same problematic non–perturbative solution. (Certainly there are at most a discrete number of solutions in this case with the latter being one of them \( \lambda 10 \).) This expectation is reinforced by the matrix model understanding of the \((2m - 1, 2)\) models; in particular the Lee–Yang model’s \([\tilde{P}, Q] = Q\) definition has the spherical physics fixed in the one–matrix model [1]: The end–point of the eigenvalue density of the model corresponds to the string susceptibility \( \rho \) in the spherical limit. In the \( z \to +\infty \) limit the endpoint of the density pulls away from the wall where we recover locally the \( \mathbb{R} \) topology and hence the physics is identical to that of the \([P, Q] = 1\) models; \( \rho_0 = z \lambda 1/3 \). In the \( z \to -\infty \) limit, the endpoint pushes up against the wall, and \( \rho_0 \) vanishes.

The appropriateness of the choice \( \rho = 0 \) in the \( z \to -\infty \) limit for the string equation (3.14) is made manifest by a detailed analytic study of the family of

\[ \lambda 10 \] A numerical study of the string equation, using the techniques described later, failed to find a pole–free solution with these asymptotics.
equations of this form and their solutions[18]. The KdV and Boussinesque flow structure of the \((2m - 1, 2)\) and \((*, 3)\) models respectively, may be shown to preserve the monodromy data of the linear problem associated to the string equations if and only if the \(\rho_0 = 0\) asymptotic is preserved. This asymptote is fixed in pure gravity[2]. Note that \((2, 3) = (3, 2)\), a fact which is trivially verified using the expression for \(\mathcal{R}\lambda 2_{2,1}\).

Using the same procedure as for the \(z \to +\infty\) limit, in the \(z \to -\infty\) limit with \(\rho_0 = 0\) we have the asymptotic series

\[
\rho = \frac{1}{z\lambda^2} \sum_{j=0}^{\infty} \lambda^j B_j \left( \frac{1}{z\lambda^7} \right) \lambda^j
\]

The \(B_j\)'s are again fixed uniquely by the initial \(B_0 = -4/9\).

We continue by studying linear perturbations \(\epsilon\) about the leading behaviour of the string susceptibility in the large \(z\) regions. In the \(z \to +\infty\) limit we have \(\rho_0 = z\lambda 1/3\) and so we try to find a solution for \(\epsilon(z)\) by substituting \(\rho = z\lambda 1/3 + \epsilon(z)\) into equation (3.14). Following the WKB prescription for large \(z\) we expect the exponential behaviour \(\epsilon \sim e^{\lambda - f(z)}\) with \(|f| >> |f'\lambda| >> |f''\lambda| \ldots\) so that \(\epsilon \lambda(n)/\epsilon \approx (-f\lambda')\lambda n\). Using this and keeping only leading order we find that

\[
\mathcal{R}_I(\rho_0 + \epsilon) \approx \epsilon \left( \frac{2}{27} (f\lambda')\lambda 4 - (f\lambda')\lambda 2z\lambda^1 \frac{1}{3} + 3z\lambda^2 \frac{2}{3} \right)
\]

and \(\mathcal{R}_I(\rho_0 + \epsilon) \approx (-f\lambda')\lambda n \mathcal{R}_I(\rho_0 + \epsilon)\). (Derivatives of \(\rho_0\) do not survive in this limit.) The string equation (3.14) becomes

\[
\left( \frac{2}{27} (f\lambda')\lambda 4 - (f\lambda')\lambda 2z\lambda^1 \frac{1}{3} + 3z\lambda^2 \frac{2}{3} \right) \lambda^2 \left( \frac{9}{4} z\lambda^1 \frac{1}{3} - (f\lambda')\lambda 2 \right) = 0
\]

From this we find \(\epsilon(z) = Ae^{\lambda - 6/7\alpha z\lambda 7/6}\) with \(\alpha \lambda 2 = 9/2\), \(9/4\) or \(9/4\). As three of these solutions are exponentially growing perturbations we must set their coefficients to zero.

In the \(z \to -\infty\) limit we study perturbations around the leading non-vanishing behaviour for the string susceptibility, the torus term \(\rho_0 = -4/9z\lambda 2\). This time, as \(\rho_0\) and it derivatives are subleading, we have \(\mathcal{R}_I(\rho_0 + \epsilon) \approx \epsilon 2/27(f\lambda')\lambda 4 - z\). The string equation then becomes, to leading order \(9/16z\lambda 2 + z/27(f\lambda')\lambda 6 = 0\) which gives \((f\lambda')\lambda 6 = -243/16|z|\). This yields the following solution for the exponential
corrections in the $z \to -\infty$ limit: $\epsilon(z) = B\lambda - 6/7\beta z\lambda 7/6$ with $\beta \lambda 6 = -243/16$. Again, three of these solutions have positive real part and their coefficients must be set to zero to match the chosen asymptotes. Six integration constants have now been determined locally in a sixth order differential equation and so we expect at most a discrete number of solutions with the above asymptotics.

Further progress was made with numerical techniques identical to those used in [2] to find the solution for the $[\tilde{P}, Q] = Q$ pure gravity model ($m = 2$) and in [18] for the $m = 1$ and $m = 3$ models. The solution of the differential equation was treated as a two-point boundary value problem. We employed a NAG FORTRAN library routine (D02RAF) to solve the problem by relaxation. The program uses Newton iteration with deferred correction, and allows user specification of the initial mesh and approximate solution. The absolute error tolerance was set at $\sim 10^{-5}$.

We chose to solve the string equation differentiated once, removing a factor of $R_I$. This allows for a more numerically well-behaved highest derivative, since otherwise the expression for the highest derivative obtained from the string equation contains factors of $1/R_I$. It was ensured that the correct solution was found by including more terms in the asymptotic series to calculate the boundary conditions. The stability of the solution was tested by performing the integration with a number of different values for the boundary, e.g. $z = \pm 100$ and $z = \pm 10$.

The non-perturbative solution for the string susceptibility of the Ising model is displayed in figure 1(a), where the integration was carried out on a mesh of 1600 points in the range $\pm 200$. For contrast, in figure 1(b), we display the non-perturbative solution for the Lee–Yang model.

4. The $(p, q)$ String Equations and the $q$th KdV Hierarchy.

We present here the generalisation of the $[\tilde{P}, Q] = Q$ formulation to the $(p, q)$ minimal models, deriving the unique string equation consistent with the requirement of preservation of the $q$th KdV flows. A parameter analogous to $\sigma$ is included in the discussion, which in the Ising model is seen to be the boundary magnetic field.
4.1 The qth KdV Hierarchy.

We begin by reviewing the basic tools of the formalism, the KP hierarchy and its $q$–reductions, referred to as the $q$th KdV hierarchies. The KP hierarchy may be formulated in terms of the pseudo differential operator $
abla = d + \sum_{i=1}^{\infty} \lambda_i f_i(t) d\lambda - i$, where $d\lambda - 1$ is defined by $d\lambda - 1 f = \sum_{j=0}^{\infty} \lambda(-1)^j f\lambda(j) d\lambda - j - 1$. Integer powers of $\nabla$ generate a basis for the complete set of objects which commute with it. Taking the differential operator part of these (denoted by a ‘+’ subscript) generates a set of evolution equations for $\nabla$:

$$\frac{\partial \nabla}{\partial t_r} = \kappa [\nabla + \lambda r, \nabla]$$ (4.1)

The $t_r$ parametrise the infinite set of flows thus defined. Equation (4.1) defines the KP hierarchy.

The $q$th reduction may be constructed in terms of the object $Q = \nabla \lambda^q$ and then requiring that $Q_{-} = 0$:

$$\frac{\partial Q}{\partial t_r} = \kappa [Q + \lambda^q r, Q]$$ (4.2)

When $r = 0 \mod q$ the flows of equation (4.2) are trivial, so we modify our notation explicitly to highlight the values of $r$ which are mutually prime with $q$:

$$\frac{\partial Q}{\partial t_{l,k}} = \kappa [Q + \lambda^q \frac{kq + l}{q}, Q] \quad l = 1, 2, \ldots q - 1; \quad k = 0, 1, \ldots \infty$$ (4.3)

The indices $l$ and $k$ now span the set of non–trivial flows $r = qk + l$. Equation (4.3) defines the $q$th KdV hierarchy. The differential operator $Q$ may be written in the form:

$$Q = d\lambda q + \sum_{i=2}^{\infty} \lambda q \alpha_i \{ u_i, d\lambda q - i \}$$ (4.4)

and (4.2) defines a pair of Hamiltonian equations for the $\{ u_i \}$:

$$\frac{\partial u_i}{\partial t_{l,k}} = \kappa \{ H_{l,k+1}, u_i \}_1 \equiv \kappa \{ H_{l,k}, u_i \}_2$$ (4.5)

The hamiltonians are constructed from fractional powers of $Q$ in the following way:

$$H_{l,k} = \frac{q}{kq + l} \int \text{Res} Q_\lambda \frac{kq + l}{q} dz = \frac{q}{kq + l} \text{Tr} Q_\lambda \frac{kq + l}{q}$$
where the residue of a pseudo–differential operator is simply the coefficient of the $d\lambda - 1$ term. We note here that the $\mathcal{H}_{l,k}$ are not well defined for the relevant solutions $u_i$, which grow as a power of $z$ as $z \to +\infty$. Their explicit appearance here uncovers the structure of the $q$–KdV system. They themselves are never used with these solutions, both here and later.

These $q$–KdV systems are ‘bi–hamiltonian’: They possess two Poisson brackets between functionals $W$, $V$ of the $\{u_i\}$:

$$\{W[u], V[u]\}_{1,2} = \int dx dy \frac{\delta W}{\delta u_i(x)} \{u_i(x), u_j(y)\}_{1,2} \frac{\delta V}{\delta u_j(y)}$$

The fundamental Poisson brackets $\{u_i(x), u_j(y)\}_{1,2}$ may be written

$$\{u_i(x), u_j(y)\}_{1,2} = \mathcal{D}_{1,2} \lambda ij(x) \delta(x - y)$$

The objects $\mathcal{D} \lambda ij_1$ and $\mathcal{D} \lambda ij_2$ are a set of differential operators. Using them we may develop (4.5) further

$$\frac{\partial u_i}{\partial h_{l,k}} = \kappa \{\mathcal{H}_{l,k+1}, u_i\}_1 = \kappa \{\mathcal{H}_{l,k}, u_i\}_2$$

$$= \kappa \mathcal{D}_1 \lambda ij \frac{\delta \mathcal{H}_{l,k+1}}{\delta u_j} = \kappa \mathcal{D}_2 \lambda ij \frac{\delta \mathcal{H}_{l,k}}{\delta u_j}$$

$$\Rightarrow \mathcal{D}_1 \lambda ij \mathcal{R} \lambda ji_{k+1} = \mathcal{D}_2 \lambda ij \mathcal{R} \lambda ji_k$$

where the $\mathcal{R} \lambda ji_{k}$ are differential polynomials in the $\{u_i\}$. They are the generalisation of the Gel’fand–Dikii differential polynomials encountered in the $q = 2$ case.

The last line in (4.8) is a recursion relation among them. Requiring them to vanish at $\{u_i\} = 0$ fixes them uniquely, up to the normalisations $\mathcal{R} \lambda i_{l,0} = q \delta \lambda i - 1_i$. In what follows we set these at $\mathcal{R} \lambda i_{l,0} = q \delta \lambda i - 1_i$, and we set the overall normalisation $\kappa$ to -1. The second Poisson bracket in (4.7) is in fact the $W \lambda(q) \equiv \mathcal{W} A_{q-1}$–algebra (at a particular value of the central charge) where the $\{u_i\}$ are the $q - 2$ currents. In particular, $u_2$ corresponds to the energy–momentum tensor. For example in the $q = 2$ case we have:

$$\{u_2(z), u_2(y)\}_2 = \left( \frac{1}{4} d\lambda 3 + u_2 d + \frac{1}{2} u_2 \lambda' \right) \delta(z - y)$$
which is the $W\lambda(2)$ or Virasoro algebra, and for the case $q = 3$ we have a $W\lambda(3)$–algebra:

\[
\{u_2(z), u_2(y)\}_2 = \left(\frac{2}{3}d\lambda^3 + \frac{1}{2}u_2\lambda' + u_2d\right) \delta(z - y)
\]

\[
\{u_2(z), u_3(y)\}_2 = \left(u_3d + \frac{2}{3}u_3\lambda'\right) \delta(z - y)
\]

\[
\{u_3(z), u_2(y)\}_2 = \left(u_3d + \frac{1}{3}u_3\lambda'\right) \delta(z - y)
\]

\[
\{u_3(z), u_3(y)\}_2 = \left(\frac{1}{18}d\lambda^5 - \frac{5}{12}u_2d\lambda^3 - \frac{5}{8}u_2\lambda'd\lambda^2 + \left(-\frac{1}{2}u_2\lambda^2 - \frac{3}{8}u_2\lambda''\right)d + \left(-\frac{1}{2}u_2\lambda' - \frac{1}{12}u_2\lambda''\right)\right) \delta(z - y)
\]

which forms the second hamiltonian structure for the Boussinesq hierarchy[27].

4.2 The $(p, q)$ String Equations.

As discussed before, the matrix model formalism motivates us to work with the operator $Q$ which plays the rôle of the continuum limit of a position operator in the orthogonal polynomial basis. We construct a realisation of the equation $[\tilde{P} - \sigma P, Q] = Q - \sigma$ in terms of the parameters $t_{l,k}$. Heat kernels of $Q$ generate the correlators of the observables in the theory, which are the macroscopic loops. $Q$ therefore supplies a natural length scale in the theory. From the definition of $Q$ in (4.4) and the $q$–KdV flows in (4.2) the scaling of the $\{u_i\}$ and $\{t_{l,k}\}$ may be deduced. If we assign the scaling dimension 2 to $u_2$ we obtain the following dimensions:

$[u_i] = i; [t_{l,k}] = -(qk + l); [Q] = q$. Using these, we construct the generator of scale transformations, $\tilde{P}$:

\[
\tilde{P} = \sum_{l=1}^{\infty} \lambda q - 1 \sum_{k=0}^{\infty} \lambda \infty (qk + l)t_{l,k} \frac{\partial}{\partial t_{l,k}}
\]

and the generator of translations, $P$:

\[
P = \sum_{l=1}^{\infty} \lambda q - 1 \sum_{k=1}^{\infty} \lambda \infty (qk + l)t_{l,k} \frac{\partial}{\partial t_{l,k-1}}
\]

By then adopting the principle that the $q$th KdV hierarchy holds we have

\[
[\sum_{l=1}^{\infty} \lambda q - 1 \sum_{k=0}^{\infty} \lambda \infty \kappa (qk + l)t_{l,k} \left(Q\lambda k + \frac{l}{q} + \sigma Q\lambda k + \frac{l}{q} - 1\right), Q] = q(Q - \sigma)
\]

(4.9)
We may rewrite this as a set of scaling equations for the \( \{ u_i \} \):

\[
\alpha(i) \left( \sum_{l=1}^{q-1} \lambda q - 1 \sum_{k=1}^{q-1} \lambda \infty (qk + l) t_{l,k} \frac{\partial u_i}{\partial t_{l,k}} + \sum_{m=1}^{q-1} \lambda q - 1 m t_{m,0} \frac{\partial u_i}{\partial t_{m,0}} + \sigma \frac{\partial u_i}{\partial \sigma} + i u_i \right) = 0
\]  

(4.10)

The \( q - 1 \) objects \( t_{m,0} \) are proportional to the parameters coupling to the relevant operators in the theory, the \( \mathcal{O}_{m,0} \). In particular we have \( t_{1,0} = -z, 2t_{2,0} = -\mathcal{B} \), etc (after setting \( \kappa = -1 \)), and so in rewriting (4.9) we must use the identity

\[
[z \frac{\partial}{\partial z}, Q] = \sum_{q=2}^{\infty} \lambda q - 1 \sum_{k=0}^{\infty} \lambda \infty (k + \frac{l}{q}) t_{l,k} R \lambda_i l,k
\]

and in order to interpret it as a scaling equation we have made the following identification:

\[
\alpha(i) \frac{\partial u_i}{\partial \sigma} = -q \mathcal{D} \lambda i j_1 R \lambda j
\]

(4.11)

The scaling equations (4.10) may be written succinctly as

\[
(\mathcal{D} \lambda i j_2 - \sigma \mathcal{D} \lambda i j_1) R \lambda j = 0
\]

(4.12)

where the objects \( R \lambda i \) in the above equation are

\[
R \lambda i \equiv \sum_{l=1}^{q-1} \lambda q - 1 \sum_{k=0}^{q-1} \lambda \infty (k + \frac{l}{q}) t_{l,k} R \lambda i l,k
\]

\[
= \sum_{l=1}^{q-1} \lambda q - 1 \sum_{k=0}^{q-1} \lambda \infty (k + \frac{l}{q}) t_{l,k} R \lambda i l,k + (i - 1) t_{i-1,0}
\]

In the above, we have used that \( R l_0 \lambda j = q \delta \lambda j - 1 \). Equation (4.12) is the differentiated string equation. Its structure is an explicit realisation of the \( W \lambda(q) \)-algebra structure inherent in the second Hamiltonian structure of the \( q \)-KdV hierarchy. In analogy with the case explicitly worked out from the matrix model, the string equation is obtained by multiplying on the left by \( R \lambda i \) giving a total derivative, and integrating once with respect to \( z \). (The integration constant is then set to zero using perturbation theory. See section 3.3)

It should be noted here that the \( \sigma \)-deformed differentiated string equation for the \( (2m - 1, 2) \) models may also be written in the form of (4.12): \( (\mathcal{D}_2 - \sigma \mathcal{D}_1) R = 0 \) 

It is now apparent that the process of introducing \( \sigma \) into the formalism may be regarded as forming a linear combination of the second hamiltonian structure \( \mathcal{D}_2 \)
and the first, $D_1$. This is always possible as the two structures are ‘coordinated’, in the sense of ref.[24]. In this picture (4.12) is indeed the natural generalisation of the equations first found for $[\hat{P}, Q] = Q$ definition of the $(2m - 1, 2)$ models.

That $\mathcal{R}_i(\mathcal{D}\lambda_i j_2 - \sigma \mathcal{D}\lambda_i j_1) \mathcal{R}_j$ is always a total derivative must be demonstrated. The integral $\int dz \mathcal{R}_i \mathcal{D}\lambda_i j_{1,2} \mathcal{R}_j$ is

$$\sum_{l,m=1}^{\lambda q - 1} \lambda \infty (k + \frac{l}{q})(n + \frac{m}{q}) t_{l,k} t_{m,n} \int \mathcal{R}_i \mathcal{D}\lambda_i j_{1,2} \mathcal{R}_j m_{n}$$

We then have using equations (4.6) and (4.7):

$$\int \mathcal{R}_i \mathcal{D}\lambda_j m_{n} dz = \int dz dy \mathcal{R}_i \mathcal{D}\lambda_j m_{n} \delta(z - y) \mathcal{R}_j m_{n} = \{\mathcal{H}_{l,k}, \mathcal{H}_{m,n}\} = 0$$

where the last line is simply the statement that all the Hamiltonians in the $q$–KdV hierarchy are in involution. However, the above equation holds for all functions $u_i$ satisfying the boundary conditions (which are those necessary for the Hamiltonians $\mathcal{H}_{l,k}$ to be well-defined). This can only be true if the integrand on the left hand side of the equation is a total $z$–derivative. Thus we conclude that we may always integrate (4.12) to obtain the string equation.

As an example, we have the string equation for the $(\ast, 3)$ models:

$$\int \frac{1}{2} u_2 R_2 \lambda 2 + \frac{2}{3} R_2 R_2' - \frac{1}{3}(R\lambda' 2) \lambda 2 + (u_3 - \sigma) R_2 R_3 - \frac{1}{18} \left( R_3 R_\lambda 4 - R_\lambda' 3 R_\lambda'' 3 - \frac{1}{2}(R\lambda'' 3) \lambda 2 \right)$$

$$- \frac{5}{12} \left( u_2 R_3 R_\lambda' 3 - \frac{1}{2} u_2 (R\lambda' 3) \lambda 2 + \frac{1}{2} u_2 \lambda' R_3 R_\lambda' 3 \right) - \frac{1}{12} (3 u_2 \lambda 2 + u_2 \lambda'') R_\lambda 2 = 0$$

where

$$R_2 = \sum_{l=1,2} \sum_{k=1}^{\lambda q - 1} (k + \frac{l}{3}) t_{l,k} R_\lambda 2_{l,k}$$

and

$$R_3 = \sum_{l=1,2} \sum_{k=1}^{\lambda q - 1} (k + \frac{l}{3}) t_{l,k} R_\lambda 3_{l,k} - B$$

(For convenience of notation we have exchanged the superscripts on the $\mathcal{R}_i$'s for subscripts.) From this, the Ising model is obtained by setting $t_{l,k} = 3/7 \delta l_1 \delta k_2$. 23
5. The W–algebra Constraints.

In this section the constraints for the $[\tilde{P}, Q] = Q$ formulation of the $(p, q)$ models are derived. We complete the discussion of the parameter $\sigma$ in the $(2m - 1, 2)$ models in terms of the algebra of constraints, which provides the appropriate method for generalisation to the $(p, q)$ models. We discuss the possible significance of the analogous $q - 2$ extra parameters.

5.1 The $(p, q)$–Model W–constraints

In the $[P, Q] = 1$ formalism, the string equation leads to an infinite set of constraints on the partition function of the theory, which plays the rôle of a $\tau$–function of the associated $q$th reduction of the KP hierarchy. The $[P, Q] = 1$ string equation (3.12) is equivalent to the $L_{-1}$ constraint on the $\tau$–function:

$$D\lambda i j_1 R\lambda j \equiv d\lambda i \left(\frac{L_{-1} \cdot \tau}{\tau}\right) = 0$$  (5.1)

(For what follows we shall work with all the $t_r$’s which parameterise all of the flows, including the trivial ones[28]. The $R\lambda i$’s from the previous section are now defined as $\sum_{r=1}^{\infty} \frac{1}{q} t_r R\lambda i_r$, and in equation (5.1) we have $d\lambda i = \partial / \partial t_{i-1}$. We also have $u_i = (i/2) d\lambda i (\ln \tau)$.)

In ref.[28] it was shown that the constraints derived from the string equation may be written as:

$$W \lambda(k)_n \cdot \tau = 0, \quad n \geq -k + 1, \quad k = 2, 3, \cdots, q$$  (5.2)

where the $W \lambda(k)_n$ are the $n$th Fourier modes of the $W \lambda(q)$–algebra generator with spin ‘$k$’. For example $W \lambda(2)$ is the stress tensor and its Fourier modes (usually denoted $L_n$) satisfy the Virasoro algebra. The constraints $L_n . \tau = 0$ for $n \geq -1$ then form a consistent set in the sense that no further constraints upon $\tau$ are generated using the commutation relations of the modes.

In this section we will show that equations (3.11), obtained from the $[\tilde{P}, Q] = Q$ string equation by differentiation, imply the following constraints:

$$W \lambda(k)_n \cdot \tau = 0, \quad n \geq 0, \quad k = 2, 3, \cdots, q$$  (5.3)
which also form a consistent set. These constraints follow from the differentiated string equation, which is equivalent to the $L_0$ constraint:

$$\mathcal{D} \lambda_{ij2} \mathcal{R} \lambda_j \equiv \partial \partial_t (L_0 \tau) = 0$$

where

$$L_0 = \sum_{i=1}^{\lambda} \lambda_{\infty} \partial_i + \text{const.}$$

Using the techniques and notation $\lambda_{11}$ of ref.[28], the constraint $L_0 \tau = 0$ may be written:

$$\text{res}_\lambda (\lambda \partial_\mu \lambda_{2X(t, \lambda, \mu)} |_{\mu=\lambda}) \cdot \tau = 0$$

From equation (2.8) of ref.[28],

$$\text{res}_\lambda (\lambda \omega \lambda^* (t', \lambda) \partial_\lambda \omega (t, \lambda)) = \text{res}_\nu \left( X(t, \nu) \text{res}_\lambda (\lambda \partial_\mu \lambda_{2X(t, \lambda, \mu)} |_{\mu=\lambda}) \tau(t) \frac{X \lambda^* (t'(t', \nu) \tau(t')} {\tau(t')} \right)$$

and thus equation (5.4) implies that the left hand side of (5.5) vanishes. However, following ref.[28] one has

$$\text{res}_\lambda (\lambda \omega \lambda^* (t', \lambda) \partial_\lambda \omega (t, \lambda)) = (MQ \lambda_{1/q}(t)) \cdot \delta(x - x')$$

We have shown that the $[\tilde{P}, Q] = Q$ string equation implies the condition

$$(MQ \lambda_{1/q}) \cdot = 0.$$ \hspace{1cm} (5.6)

Taking powers of $T \equiv MQ \lambda_{1/q}$, it is then straightforward to show that equation (5.6) implies the further conditions

$$(M \lambda^n Q \lambda^n/q) \cdot = 0$$

for $n = 1, 2, \ldots, q - 1$. Following the discussion of ref.[28], these lead to the constraints (5.3).

The above equations correspond to the case where the scaled eigenvalue space is taken to be $\mathbb{R}_+$ with the boundary, or ‘wall’, being at $\lambda_s = 0$. In fact there

$\lambda_{11}$ Beware of the interchange of the names of the operators $Q$ and $L$ in ref.[28]
is no reason for this restriction on the coordinates and in general one may take the boundary position to be $\lambda_s = \sigma$. The effect of this on the string equation and the Virasoro constraints of the one-matrix model has already been described in refs.[3][11][4]. approach most suited to There it is shown that the eigenvalue boundary plays the rôle of the boundary cosmological constant on the worldsheet.

In sections 4.2 and 3.2, by identifying the eigenvalue space with the position operator $Q$ in Douglas’ $PQ$ formalism, we derived the effect of $\sigma$ on the string equation in the general $(p, q)$ case, and on the Ising model in particular. In the latter case the $\sigma$ parameter will be seen, by precisely parallel arguments to refs.[19][4], to be the $(Z_2$ odd) boundary magnetic field. This method however takes into account only one of the flavours of eigenvalue in a multi-matrix model, each of which may have their own boundary.

Rather than working directly in the continuum limit the clearest picture might be expected to emerge from working explicitly with a $q - 1$-matrix model. (For the one-matrix model this was done in ref.[4] where it was used primarily to derive the effect on macroscopic loops.) Integrating over the angular modes leaves an integral over $q - 1$ coupled eigenvalues, and the obvious generalisation is to introduce $q - 1$ boundary parameters giving the position of the $q - 1$ ‘walls’ in scaled eigenvalue space. eigenvalues restricted to being greater than their In the Ising model, a critical point in the two-matrix model, linear combinations of the two flavours of loop give the $Z_2$ even boundary length and $Z_2$ odd boundary magnetization[19]. Given the results of the one-matrix model, it is then natural to conjecture that here the wall parameters will in a similar way provide the conjugate parameters: the boundary magnetic field and the boundary cosmological constant. Note that the latter is not apparent in the $[P, Q] = 1$ KP description[19]. It is therefore an important question to determine whether such a parameter exists in our formulation. We will below identify a $Z_2$ even parameter, conjugate to a redundant $Z_2$ even operator (namely $W_{-2}$) which we therefore suggest is (perhaps non-linearly) related to the boundary cosmological constant. Curiously, we will also uncover a further redundant $Z_2$ odd parameter (conjugate to $W_{-1}$) for which we have as yet no physical interpretation.

The reason we cannot be more definite in our identifications is because we

\[ \lambda_{12} \] Strictly speaking it is a linear combination[19].
were unable to carry through a direct analysis of the two-matrix model, due to certain technical difficulties in the orthogonal polynomial approach (outlined below). Other technical difficulties nullify the standard steepest desents approach as an alternative method for any multi-matrix model.

Let us now turn briefly to the two-matrix model inserting eigenvalue boundaries at $\sigma_1 = \sigma_2 = 0$. The orthogonal polynomials are now normalised as

$$\int_0^\infty \lambda e^{-\frac{N}{\gamma} S(\lambda, \mu)} P_n \lambda^+ + (\lambda) P \lambda - (\mu) = h_n \delta_{nm}$$

The problem with the method arises when we consider the generalisation of eqn.(3.4). The reason the recursion relation (3.4) finishes at $P_{n-3}$ is because the potential $S$ is of order 4, as can be seen by noting that $\lambda P \lambda + n (\lambda) P \lambda - (\mu) h_{n-k}$ vanishes for any $k > 3$ on integration with the measure implied in eqn.(3.3). The proof follows by converting $\lambda$ into $\frac{\gamma}{2 N c} \frac{\partial}{\partial \mu} \exp(-\frac{N}{\gamma} S) + \text{corrections}$, integrating by parts and then using (3.3) together with the definition $P \lambda - (\mu) = \mu \lambda m + \text{lower powers}$. Unfortunately in our case we pick up a boundary term

$$h_n \frac{\gamma}{2 N c} P \lambda - (\mu) \int_0^\infty \lambda e^{-\frac{N}{\gamma} S(\lambda, 0)} P_n \lambda + (\lambda)$$

for any odd $k > 0$ which is therefore equal to the coefficient $P \lambda + n - k$ in an infinite order generalisation of (3.4). The coefficients (in particular the $h_n$) can now be determined, in principle, from an infinite set of simultaneous recurrence relations generalising the usual construction. We conclude that the orthogonal polynomial approach is at best inappropriate for analysing the continuum limit. Of course these arguments are unaffected by choosing general positions $\sigma_1, \sigma_2$ for the walls or using a non-even potential.

We now turn to an approach based on the continuum Dyson-Schwinger equations. We first review the results of the one-matrix model using this approach and then generalise to general $(p, q)$. Our starting point in the one-matrix model is the algebra of constraints\cite{2}:

$$L_n \tau = 0 \quad n \geq 0 \quad (5.7)$$

This differs from the hermitian matrix model in that the $L_{-1}$ constraint is missing. The KdV flows on the other hand, upon which our formulation is based, are invariant\footnote{Recall that we are interested in $n \approx N \rightarrow \infty$ for the continuum limit.}
under transformations generated by the full set $L_n : n \geq -1$. Thus we have a situation reminiscent of spontaneous symmetry breaking: the ‘dynamics’ i.e. the KdV flows are invariant under the full group (generated by the $L_n : n \geq -1$) whereas the ‘vacuum’ $\tau$ is invariant only under the little group generated by $L_n : n \geq 0$. The ‘Goldstone boson’ is $\sigma$ which in (5.7) has been gauge fixed to zero. Indeed there are now an infinite number of vacua $\tau(\sigma)$ connected by the broken generator:

$$
\tau(\sigma) = e^{\lambda \sigma L_{-1}} \tau
$$

(5.8)

where we identify $\tau(0)$ with the $\tau$ function in (5.7). The constraints on $\tau(\sigma)$ are an inner automorphism of those of (5.7) plus a constraint arising from $\sigma$ independence of $\tau$:

$$
L'_n \tau(\sigma) = 0 \quad n \geq 0
$$

$$
\partial'_\sigma \tau(\sigma) = 0
$$

(5.9)

where

$$
L'_n = e^{\lambda \sigma L_{-1}} L_n e^{\lambda \sigma L_{-1}} \tau(\sigma)
$$

and

$$
\partial'_\sigma = e^{\lambda \sigma L_{-1}} \frac{\partial}{\partial \sigma} e^{\lambda \sigma L_{-1}} \tau(\sigma)
$$

Taking linear combinations of these constraints:

$$
L\lambda\sigma_{-1} = -\partial'_\sigma
$$

$$
L\lambda\sigma_0 = L'_0 + \sigma L\lambda\sigma_{-1}
$$

$$
L\lambda\sigma_1 = L'_1 + 2\sigma L'_0 + \sigma 2L\lambda\sigma_{-1}
$$

$$
\vdots = \vdots
$$

(5.10)

we get $L\lambda\sigma_n = L_n - \sigma \lambda n + 1 \frac{\partial}{\partial \sigma}$ for $n \geq -1$. These corrections can be computed most simply as follows:

$$
\frac{\partial}{\partial \sigma} \tau(\sigma) = L_{-1} \tau(\sigma)
$$

(5.11)

$$
L_n \tau(\sigma) = e^{\lambda \sigma L_{-1}} (-\sigma \text{ad} L_{-1}) \lambda n + 1 L_n \tau(0)
$$

$$
= \sigma \lambda n + 1 L_{-1} e^{\lambda \sigma L_{-1}} \tau(0)
$$

$$
= \sigma \lambda n + 1 \frac{\partial}{\partial \sigma} \tau(\sigma)
$$

(5.12)

where use is made of (5.8), the notation $(\text{ad}X)Y = [X, Y]$ (for two operators), and the identity $e^{\lambda - X Y} e^{\lambda X} = e^{\lambda - \text{ad}X Y}$.
Our string equation (2.6) follows from integrating the modified $L_0$ constraint. It is clear that the system is now invariant under translations generated by $L_{-1}$. Indeed, using (5.11), the explicit formula $L_{-1} = \sum_{k=1}^{\infty} \lambda \infty (k + \frac{1}{2}) t_k \partial_{t_{k-1}} + z \lambda 2/4$, and differentiating, one obtains

$$\frac{\partial u_2}{\partial \sigma} - \sum_{k=1}^{\infty} \lambda \infty (k + \frac{1}{2}) t_k \frac{\partial u_2}{\partial t_{k-1}} = 1$$

which, on multiplying by an infinitesimal $\epsilon$, may be interpreted as the invariance (2.7). Clearly under a finite translation by $-\sigma$ we return to our original equations. Thus the boundary length $L_{-1}$ and boundary cosmological constant $\sigma$ are redundant just as was the case in ref.[19], however the symmetries and physical significance of our formulation are far more transparent with $\sigma \neq 0$.

Now consider 2D gravity coupled to a general $(p, q)$ minimal model. As shown at the beginning of this section, the $W\lambda(k)_{\alpha}$ constraints, where $\alpha$ is restricted to $-k + 1 \leq \alpha < 0$ (and $2 \leq k \leq q$), are missing as compared to the hermitian matrix model formulation. These, together with our constraints (5.2), generate the full set of symmetries of the generalised KdV hierarchy. Thus we now have spontaneous breaking of $\sum_{k=2}^{q} \lambda q (k - 1) = \frac{1}{2} q (q - 1)$ symmetries to which we may associate $\frac{1}{2} q (q - 1)$ new parameters: $\sigma \lambda(k)_{\alpha}$. These parameters (and associated operators) will be redundant since there will be analogous symmetries to (2.7) which will ‘gauge’ them away, however we again expect that they will have physical significance. Generalising (5.8) we have

$$\tau (\sigma \lambda(k)_{\alpha}) = S (\sigma \lambda(k)_{\alpha}; W \lambda(k)_{\alpha}) \tau, \quad \text{with} \quad S (0; W \lambda(k)_{\alpha}) = 1.$$  \hspace{1cm} (5.13)

Since the $W$-algebra of constraints is no longer a Lie algebra something more general for $S$ than $\exp \{ \sum_{k,\alpha} \sigma \lambda(k)_{\alpha} W \lambda(k)_{\alpha} \}$ may be more appropriate. The transformation (5.13) induces a similarity transformation on the constraints $C' = SCS\lambda - 1$ generalising (5.9), where $C$ runs over the $W \lambda(k)_{n}$ with $n \geq 0$ and the new constraints $\partial/\partial \sigma \lambda(k)_{\alpha}$ (which are trivially zero on $\tau$). The latter constraints give rise to the generalised symmetries by similar derivations to (2.7); Except for $L_{-1}$ these are non-local. Since the similarity transformation preserves the $W$-algebra we see that the $W\lambda(k)_{n}$ still form a $W$-algebra while each $\partial'_{\sigma \lambda(k)_{\alpha}}$ commutes with all the other constraints. It can be shown that, by taking linear combinations of these constraints
(generalising (5.10)), one can form a new simpler set of constraints of the form 
\[(W\lambda\sigma)\lambda(k)_n = W\lambda(k)_n + \text{corrections}\] where now \(n \geq 1 - k\) and the corrections do not involve any \(W\lambda(k)_m\) with \(m \geq 0\). It may also be shown that these corrections can be written as a finite sum of the form \(\sum_{k,\alpha} c\lambda(k)_\alpha W\lambda(k)_\alpha\) where each \(c\lambda(k)_\alpha\) is a power series expansion in all the \(\sigma\lambda(k)_\alpha\) with operator valued coefficients. This follows because the original \(W\)-algebra with \(n \geq 1 - k\) forms a closed algebra of constraints under commutation. By construction the commutator of two of these \(W\lambda\sigma\) constraints also closes, however, as we will show by example below, these constraints do not form a \(W\) algebra.

In particular, consider the Ising model (or more generally coprime \(p > q = 3\)). In this case we have broken ‘symmetries’ \(L_{-1}\), \(W_{-1}\) and \(W_{-2}\). The most general case generalising (5.8) will involve some combination of these three generators together with three parameters. The corresponding formulæ to (5.11) will be more complicated involving mixed combinations of parameters and operators, because these operators do not commute. However, we can classify the operators under the \(Z_2\) symmetry which flips spin (exchanges \(M_+\) with \(M_-\)). Since this corresponds to \(d \rightarrow -d\) (cf. comments below equation (3.5)), we have \(Q\lambda 1/3 \rightarrow -Q\lambda 1/3\) and using (4.1), it follows that \(t_r\) is \(Z_2\)-odd(even) if \(r\) is odd(even). Using the explicit formulæ for \(W_n\) and \(L_n\) (see e.g. ref.[28]) we find that these are \(Z_2\)-odd(even) if \(n\) is odd(even). Thus we propose that \(W_{-2}\) is (perhaps non-linearly) related to the boundary length. We do not know what rôle is played by the \(Z_2\) odd \(W_{-1}\). As mentioned above, \(L_{-1}\) is the boundary magnetization. We will now confirm this.

Avoiding the complications of mixing let us first consider \(L_{-1}\) in isolation. In this case eqn.(5.8) again applies. A direct calculation of the modified constraints \(\lambda 14\), via the method of (5.12), gives:

\[W\lambda\sigma_n = W_n - (n + 2)\sigma\lambda n + 1W_{-1} + (n + 1)\sigma\lambda n + 2W_{-2} \quad n \geq 0 \quad (5.14)\]

with the \(L\lambda\sigma_n, n \geq -1\), as before. It is clear (by counting) from the discussion below (5.13) that there are no \(W\lambda\sigma_{-1}\) or \(W\lambda\sigma_{-2}\) constraints; Indeed it is amusing to note that this is already incorporated in (5.14) if we take \(n \geq -2\). This is preserved by the modified algebra of constraints which by explicit computation is found to be:

\(\lambda 14\) We use the conventions of ref.[28].
\[ [L\lambda\sigma_n, L\lambda\sigma_m] = (n - m)L\lambda\sigma_{n+m} \]
\[ [L\lambda\sigma_n, W\lambda\sigma_m] = (2n - m)W\lambda\sigma_{n+m} - (2n + 1)(m + 2)\sigma\lambda m + 1W\lambda\sigma_{n-1} + 2(n + 1)(m + 1)\sigma\lambda m + 2W\lambda\sigma_{n-2} \]
\[ [W\lambda\sigma_n, W\lambda\sigma_m] = -\frac{1}{3}(n - m)\{(n\lambda 2 + 4nm + m\lambda 2) + 9(n + m) + 14\}L\lambda\sigma_{n+m} + 2(n - m)(U\lambda\sigma_{n+m} + \sigma\lambda n + m + 3U\lambda\sigma_{-3}) + \frac{1}{3}(n + 1)(m + 2)(m + 1)m\sigma\lambda n + 2L\lambda\sigma_{m-2} - \frac{1}{3}(n + 2)(m + 3)(m + 2)(m + 1)\sigma\lambda n + 1L\lambda\sigma_{m-1} + 2(n + 2)(m + 1)\sigma\lambda n + 1U\lambda\sigma_{m-1} - 2(m + 2)(n + 1)\sigma\lambda n + 2U\lambda\sigma_{m-2} -(n \leftrightarrow m) \]

where \( U\lambda\sigma_n \) may be taken to be \( \sum_{k \leq -2} L\lambda\sigma_k L\lambda\sigma_{n-k} + \sum_{k \geq -1} L\lambda\sigma_{n-k} L\lambda\sigma_k \). The corresponding (galilean) symmetry follows, cf. (2.7), from equation (4.11):

\[
\begin{align*}
  u_3 & \to u_3 + \epsilon \\
  u_2 & \to u_2 \\
  \sigma & \to \sigma + \epsilon \\
  z & \to z + \frac{4}{3}t_{1,1} \\
  B & \to B + \frac{5}{3}t_{2,1} \\
  t_{l,k} & \to t_{l,k} - \epsilon(k + 1 + \frac{l}{3})t_{l,k+1} \quad k \geq 0; l = 1, 2
\end{align*}
\]

and the string equation, from the \( L'_0 \) constraint, is the one in (4.13). Since it is \( u_3 \) that shifts by a constant we identify \( \sigma \), by arguments similar to those below (2.11), as the boundary magnetic field and \( L_{-1} \) as the boundary magnetization[19].

Finally consider \( W_{-2} \). If we take \( \tau(\theta) = \exp(\theta W_{-2})\tau \) then the corrections to
the constraints (5.2) can be computed perturbatively in \( \theta \). We find:

\[
\begin{align*}
L_\lambda \theta_0 &= L_0 - 2\theta \frac{\partial}{\partial \theta} \\
L_\lambda \theta_1 &= L_1 - 4\theta W_{-1} + 8\theta \lambda^2 L_{-2} L_{-1} + \cdots \\
L_\lambda \theta_2 &= L_2 - 12\theta \lambda^2 L_{-1} \lambda^2 + \cdots \\
L_\lambda \theta_3 &= L_3 - 16\theta \lambda^2 L_{-1} + \cdots \\
W_\lambda \theta_{-2} &= W_{-2} - \frac{\partial}{\partial \theta} \\
W_\lambda \theta_0 &= W_0 - 4\theta L_{-1} \lambda^2 - 8\theta \lambda^2 (L_{-2} W_{-2} + 2L_{-3} W_{-1}) + \cdots \\
W_\lambda \theta_1 &= W_1 - 4\theta L_{-1} - 6\theta \lambda^2 (L_{-1} W_{-2} + 2L_{-2} W_{-1}) + \cdots \\
W_\lambda \theta_2 &= W_2 - 8\theta \lambda^2 (3W_{-2} + 2L_{-1} W_{-1}) + \cdots 
\end{align*}
\]

The remaining (positive \( n \)) constraints receiving no corrections to order \( \theta \lambda^2 \). The commutation algebra is evidently even less illuminating. The corresponding symmetry follows from the explicit formula for \( W_{-2} \) and is highly non-local.

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Figure Captions.

Figure 1(a): The numerical solution to the Ising model $[\tilde{P}, Q] = Q$ string equation (3.14) for the string susceptibility $\rho(z)$, with asymptotics $\rho(\infty) = z\lambda^{1/3}$ and $\rho(-\infty) = 0$. The integration was performed on a mesh of 1600 points over a range $z = \pm 200$, with a maximum error of $10\lambda-5$. Here, $z$ is the cosmological constant.

Figure 1(b): The numerical solution to the Lee–Yang model $[\tilde{P}, Q] = Q$ string equation (equation (2.6) with $t_k = 16/35\delta k^3$) for $\rho(z)$, with asymptotics $\rho(\infty) = z\lambda^{1/3}$ and $\rho(-\infty) = 0$. An identical integration procedure to that used for the Ising model string equation was employed.