ON ODD RAINBOW CYCLES IN EDGE-COLORED GRAPHS

ANDRZEJ CZYGRINOW, THEODORE MOLLA, BRENDAN NAGLE, AND ROY OURSLER

Abstract. Let $G = (V, E)$ be an $n$-vertex edge-colored graph. In 2013, H. Li proved that if every vertex $v \in V$ is incident to at least $(n + 1)/2$ distinctly colored edges, then $G$ admits a rainbow triangle. We prove that the same condition ensures a rainbow $\ell$-cycle $C_\ell$ whenever $n > 432\ell$. This result is sharp for all odd integers $\ell \geq 3$, and extends earlier work of the authors when $\ell$ is even.

1. Introduction

An edge-colored graph is a pair $(G, c)$, where $G = (V, E)$ is a graph and $c : E \to P$ is a function mapping edges to some palette of colors $P$. A subgraph $H \subseteq G$ is a rainbow subgraph if the edges of $H$ are distinctly colored by $c$. Rainbow subgraph problems are a well-studied area in graph theory (see, e.g., [1]–[9]). Here, we consider degree conditions ensuring the existence of rainbow cycles $C_\ell$ in $(G, c)$ of fixed length $\ell \geq 3$. To that end, a vertex $v \in V$ in an edge-colored graph $(G, c)$ has $c$-degree $\deg^c_G(v)$ given by the number of distinct colors assigned by $c$ to the edges $\{v, w\} \in E$. We set $\delta^c(G) = \min_{v \in V} \deg^c_G(v)$ for the minimum color-degree in $G$. The following result of H. Li [8] motivates our current work.

**Theorem 1.1** (H. Li [8], 2013). Let $(G, c)$ be an $n$-vertex edge-colored graph. If $\delta^c(G) \geq (n + 1)/2$, then $(G, c)$ admits a rainbow 3-cycle $C_3$.

A rainbow $K_{[n/2],[n/2]}$ establishes that Theorem 1.1 is best possible. We prove an analogue of Theorem 1.1 for $\ell$-cycles $C_\ell$ of fixed arbitrary length.

**Theorem 1.2.** For every integer $\ell \geq 3$, every edge-colored graph $(G, c)$ on $n \geq n_0(\ell)$ many vertices satisfying $\delta^c(G) \geq (n + 1)/2$ admits a rainbow $\ell$-cycle $C_\ell$.

Similarly to Theorem 1.1, Theorem 1.2 is best possible for all odd integers $\ell$. When $\ell$ is even, Theorem 1.2 is known in the following stronger form.

**Theorem 1.3** (Czygrinow et al. [4]). For every even integer $\ell \geq 4$, every edge-colored graph $(G, c)$ on $n \geq N_0(\ell)$ many vertices satisfying $\delta^c(G) \geq (n + 5)/3$ admits a rainbow $\ell$-cycle $C_\ell$.

It was shown in [4] that Theorem 1.3 is best possible for every even $\ell \not\equiv 0 \pmod{3}$.

Theorem 1.3 holds non-vacuously when $n \geq 3$, and one may seek to quantify $n_0(\ell)$ and $N_0(\ell)$ in Theorems 1.2 and 1.3. The proof of Theorem 1.3 depends on an application of the Szemerédi Regularity Lemma [10]–[11], and therefore gives very poor bounds on $N_0(\ell)$. Our proof of Theorem 1.2 is elementary, and easily provides $n_0(\ell) = O(\ell^2)$. For the interested Reader, we provide a more detailed analysis in our final section which establishes the order of magnitude for $n_0(\ell)$.

**Theorem 1.4.** The function $n_0(\ell)$ in Theorem 1.2 satisfies $n_0(\ell) \leq 432\ell + 1$.

The remainder of this paper is organized as follows. In Section 2 we sketch Li’s proof [8] of Theorem 1.1 and note the elements there which provide a basis for our approach here. In Section 3 we extend this proof to develop several tools useful for proving Theorems 1.2 and 1.3. In Section 4 we prove Theorem 1.2 and in Section 5 we prove Theorem 1.3. In the entirety of this paper, we employ the following observations.
Remark 1.5. Every edge-colored graph \((G, c)\) admits an edge-minimal spanning subgraph \(H \subseteq G\) satisfying \(\delta^e(G) = \delta^e(H)\). Every rainbow subgraph of \(H\) is also a rainbow subgraph of \(G\), so it suffices to assume in Theorems \([1, 3]\) that \((G, c)\) is edge-minimal. As such, \((G, c)\) admits no three commonly colored edges \(\{u, v\}, \{v, w\}, \{w, x\} \in E(G)\), as removing \(\{v, w\} \in E(G)\) violates edge-minimality. \(\square\)

2. Proof of Theorem \([1]\)

Let \((G, c)\) be an \(n\)-vertex, edge-minimal, edge-colored graph with no rainbow triangle \(C_3\). We show that \(\delta^e(G) \leq n/2\). To that end, for a color \(\alpha \in c(E)\) and a vertex \(v \in V\), we define the \(\alpha\)-neighborhood

\[ N_\alpha(v) = \{ u \in N(v) : c(\{u, v\}) = \alpha \}, \]

where \(N(v) = N_G(v) = \{ u \in V : \{u, v\} \in E \}\) is the usual neighborhood of \(v\) in \(G\), and \(N[v] = \{v\} \cup N(v)\) is the closed neighborhood of \(v\) in \(G\). We define

\[ N_i(v) = \bigcup_{\alpha \in c(E)} \{N_\alpha(v) : |N_\alpha(v)| = 1\} \]

for the set of neighbors \(u \in N(v)\) for which \(c(\{u, v\})\) appears uniquely among \(\{v, w\} \in E\). We define the replication number \(R = R(G, c)\) of \((G, c)\) by

\[ R = R(G, c) = \max_{v \in V} \max_{\alpha \in c(E)} |N_\alpha(v)|. \tag{1} \]

For \(v \in V\) and \(U \subseteq V\), we denote by \(\deg^e_G(v, U)\) the number of colors \(c(\{u, v\})\) among \(u \in N(v) \cap U\).

Fix \((z, \zeta) \in V \times c(E)\) for which \(|N_i(z)| = R\) (cf. \([1]\)). If \(N_i(z) = \emptyset\), then each color incident to \(z\) appears at least twice, so \(\delta^e(G) \leq \deg^e_G(z) \leq (n - 1)/2\) follows. Henceforth, we assume \(N_i(z) \neq \emptyset\), and we define the directed graph \(D = (V_D, \vec{E}_D)\) on vertex set \(V_D = N(z)\) by putting, for each edge \(\{x, y\} \in E(G[N(z)])\), the arc \((x, y) \in \vec{E}_D\) if, and only if, \(y \in N_i(z)\) and \(c(\{x, y\}) = c(\{x, z\})\). Then

\[ |V_D| = \deg_G(z) \geq \deg^e_G(z) + R - 1, \quad \text{and} \quad |\vec{E}_D| = \sum_{y \in N_i(z)} \deg^-_D(y) = \sum_{x \in N(z)} \deg^+_D(x), \tag{2} \]

where \(\deg^+_D(x)\) denotes the out-degree of a vertex \(x \in V_D\) in \(D\), and \(\deg_D(x)\) denotes the corresponding in-degree. We make three observations on \(D:\)

(i) Each \((x, y) \in \vec{E}_D\) places \(x \in N_i(z)\), lest some \(x \neq x' \in N(z) \setminus N_i(z)\) gives \(c(\{x', z\}) = c(\{x, z\}) = c(\{x, y\})\) (cf. Remark \([8]\));

(ii) Each \(x \in N_i(z)\) with \(\alpha = c(\{x, z\})\) satisfies \(\deg^+_D(x) = |N_\alpha(x) \cap N_i(z)| \leq R - 1\) (cf. \([1]\));

(iii) Each \(y \in N_i(z)\) has \(\deg^e_G(y, N[z]) \leq 1 + \deg^-_D(y)\), as \(c(\{y, z\}) = \beta \neq \alpha = c(\{x, z\})\) for \(x \in N(z)\) puts \((x, y) \in \vec{E}_D\), since \(c(\{x, z\}) \neq \alpha\) by \(y \in N_i(z)\) and \(c(\{x, z\}) = \beta\) lest \(\{x, y, z\}\) is rainbow.

Thus,

\[ \sum_{y \in N_i(z)} (\deg^e_G(y, N[z]) - 1) \overset{(iii)}{=} \sum_{y \in N_i(z)} \deg^-_D(y) \overset{(i)}{=} \sum_{x \in N(z)} \deg^+_D(x) \overset{(ii)}{=} \sum_{x \in N_i(z)} \deg^+_D(x) \overset{(i)}{=} |N_i(z)| - |N(z)| + 1 \leq |N_i(z)| (R - 1). \]

Averaging over \(N_i(z)\) guarantees a vertex \(y_0 \in N(z)\) for which

\[ \deg^e_G(y_0, N[z]) \leq R. \tag{3} \]

Since \(\deg^e_G(y_0, V \setminus N[z]) \geq \deg^e_G(y_0) - \deg^e_G(y_0, N[z])\), we conclude

\[ n - 1 - \deg_G(z) = n - |N[z]| \geq \deg^e_G(y_0, V \setminus N[z]) \geq \deg^e_G(y_0) - \deg^e_G(y_0, N[z]) \geq \deg^e_G(y_0) - R \]

\[ \implies n - 1 \geq \deg_G(z) + \deg_G(y_0) - R \geq \deg_G(z) + R - 1 + \deg_G(y_0) - R \]

from which \(2\delta^e(G) \leq \deg_G(z) + \deg_G(y_0) \leq n\) and \(\delta^e(G) \leq n/2\) follow.
3. Tools for proving Theorem 1.2

All tools of this section depend on the following concepts of separation and restriction.

**Definition 3.1** (separates/restricts). Let $(G,c)$ be an edge-colored graph, and fix $v \in V = V(G)$ and $X \subseteq N(v)$. We say a color $\alpha \in c(E)$ will $X$-separate a vertex $y \in V$ from $v$ when some $x \in N(y) \cap X$ satisfies $\alpha = c(\{x,y\}) \neq c(\{v,x\})$. If, additionally, $\alpha \neq c(\{v,y\})$ for all $w \in N(y) \setminus X$, then we say that $(v,X)$ restricts the color $\alpha$ for $y$. We denote by $\sigma_{v,X}(y)$ the number of colors $\alpha \in c(E)$ which $X$-separate $y$ from $v$, and we denote by $\rho_{v,X}(y)$ the number of colors $\alpha \in c(E)$ restricted for $y$ by $(v,X)$.

Every color $\alpha \in c(E)$ restricted for $y$ by $(v,X)$ also $X$-separates $y$ from $v$, and so $\sigma_{v,X}(y) \geq \rho_{v,X}(y)$ holds. The next result formally extends Theorem 1.1 (see Remark 3.3) by averaging these numbers.

**Proposition 3.2.** Let $(G,c)$ be an $n$-vertex, edge-minimal, edge-colored graph with $R = R(G,c)$ from [1], and fix $v \in V$, $X \subseteq N(v)$, and $\emptyset \neq Y \subseteq V \setminus \{v\}$. Then

$$\frac{1}{|Y|}\sum_{y \in Y} \sigma_{v,X}(y) \geq \frac{1}{|Y|}\sum_{y \in Y} \rho_{v,X}(y) \geq \frac{|X| - n - (R - 1)|X \cap N_i(v)|}{|Y|},$$

Proof of Proposition 3.2. Let $(G,c)$, $R$, $v$, $X$ and $Y$ be given as above, where it suffices to prove the rightmost inequality. Define the directed graph $D = (V_D, E_D)$ on vertex set $V_D = X \cup Y$ by putting, for each edge $\{x,y\} \in E$ with $x \in X$ and $y \in Y$, the arc $(x,y) \in E_D$ if and only if, $c(\{x,y\}) = c(\{v,x\})$. Similarly to (i) and (ii) of Section 2 each $(x,y) \in E_D$ gives $x \in N_i(v)$ and $\deg_D(x) \leq R - 1$, so

$$\sum_{y \in Y} \deg_D^-(y) = |E_D| = \sum_{x \in X} \deg_D^+(x) = \sum_{x \in X \cap N_i(v)} \deg_D^+(x) \leq (R - 1)|X \cap N_i(v)|. \quad (4)$$

Similarly to (iii) of Section 2 each $y \in Y$ admits at most $\deg_D^-(y) + \rho_{v,X}(y)$ many colors $\alpha \in c(E)$:

(a) $\alpha = c(\{x,y\})$ for some $x \in N(y) \cap X$;
(b) $\alpha \neq c(\{w,y\})$ for all $w \in N(y) \setminus X$.

Indeed, let $\alpha = c(\{x,y\})$ be such a color. If $\alpha = c(\{x,y\}) = c(\{v,x\})$, then $(x,y) \in E_D$, and otherwise $(v,X)$ restricts $\alpha = c(\{x,y\}) \neq c(\{v,x\})$ for $y$ (cf. Definition 3.1). Consequently,

$$n - |X| \geq \deg_D^-(y, V \setminus X) \geq \deg_D^-(y) - \deg_D^+(y) - \rho_{v,X}(y)$$

$$\implies \deg_D^-(y) \geq \deg_D^-(y) - \rho_{v,X}(y) + |X| - n \geq \delta^c(G) - \rho_{v,X}(y) + |X| - n. \quad (5)$$

Applying (5) to (4) renders the desired result. \hfill $\square$

**Remark 3.3.** Proposition 3.2 implies Theorem 1.1. Let $(G,c)$ be edge-minimal with no rainbow $C_3$, and fix $(z,\zeta)$ with $|N_G(z)| = R$, $x \in N(z) = X$, and (if possible) $y \in N_i(z) = Y$. Then $\rho_{z,X}(y) = 0$ as $c(\{x,y\}) \neq c(\{x,z\})$ gives $c(\{y,z\}) = c(\{x,y\})$ with $z \notin X$, since $(x,y,z)$ is not rainbow and $c(\{y,z\}) = c(\{x,z\})$ violates $y \in N_i(z)$. Now, $\delta^c(G) \leq n - |X| + R - 1 \leq n - \delta^c(G)$ so $\delta^c(G) \leq n/2$. \hfill $\square$

For $(v,X)$ fixed, Proposition 3.2 shows that some vertices $y \in V$ may admit many colors which $X$-separate $y$ from $v$. For relevant $(G,c)$, Proposition 3.4 finds vertices $y \in V$ with few such colors.

**Proposition 3.4.** Fix an integer $\ell \geq 3$, and let $(G,c)$ be an edge-minimal, edge-colored graph with no rainbow $\ell$-cycle $C_\ell$. Fix $v \in V = V(G)$ and $X \subseteq N(v)$, and let $C_{\text{rep}} = C_{\text{rep}}(v,X)$ be the colors $\alpha = c(\{v,x\})$ repeating among $x \in X$. Let $Y = Y(v,C_{\text{rep}})$ be the vertices $y \in V$ which admit an $(\ell - 1)$-vertex, $C_{\text{rep}}$-free, rainbow $\{v,y\}$-path $P_{vy}$ in $(G,c)$. Then every $y \in Y$ satisfies $\sigma_{v,X}(y) \leq 3\ell$.

Proof of Proposition 3.4. Let $(G,c)$, $v$, $X$, $C_{\text{rep}}$, $y \in Y$, and $P_{vy}$ be given as above. For a vertex $x \in N(y) \cap X$, the subgraph $P_{vy} + \{x,y\} + \{v,x\}$ is a rainbow $\ell$-cycle in $(G,c)$ unless:

(A) $x \in V(P_{vy})$; (B) $c(\{x,y\}) \in c(E(P_{vy}))$; (C) $c(\{v,x\}) \in c(E(P_{vy}))$; or (D) $c(\{x,y\}) = c(\{v,x\})$.

At most $|N(y) \cap X \cap V(P_{vy})| \leq \ell - 3$ colors $c(\{x,y\})$ satisfy (A) and at most $|E(P_{vy})| \leq \ell - 2$ colors $c(\{x,y\})$ satisfy (B). At most $|E(P_{vy})| \leq \ell - 2$ colors $c(\{x,y\})$ satisfy (C) as $c(\{v,x\}) \notin C_{\text{rep}}$. All remaining $c(\{x,y\})$ over $x \in N(y) \cap X$ satisfy (D), lest $(G,c)$ admits a rainbow $\ell$-cycle $C_\ell$. \hfill $\square$
3.1. Some corollaries. We now consider several useful corollaries of Propositions 3.2 and 3.4.

**Corollary 3.5.** Fix an integer \( \ell \geq 3 \), and let \((G, c)\) be an \( n \)-vertex, edge-minimal, edge-colored graph with no rainbow \( \ell \)-cycle \( C_\ell \). Let \((z, \zeta) \in V \times c(E)\) satisfy \( |N_\zeta(z)| = R \) (cf. 11), and let \( Y = Y(z, \zeta) \subseteq V\) be the vertices \( y \in V\) which admit an \((\ell - 1)\)-vertex, \( \zeta \)-free, rainbow \( \{y, z\}\)-path \( P_{y,z} \) in \((G, c)\). If \( Y \neq \emptyset \),

\[
\delta^*(G) \leq \frac{n}{2} + \max \left \{ 0, 3\ell + (R - 1) \left( \frac{n + 1}{2|Y|} - 1 \right) \right \}.
\]

**Proof of Corollary 3.5.** Let \((G, c), z, \zeta, R\) and \( Y = Y(z, \zeta) \neq \emptyset \) be given as above, where for sake of argument we assume \( \delta^*(G) \geq n/2\). Let \( X \subseteq N(z) \) satisfy that \( |X| = \lceil n/2 \rceil \), that \( \zeta = c(\{x_0, z\}) \) for some \( x_0 \in X \), and that all \( \{x, z\} \) with \( x \in X \) are colored distinctly. Set \( X^+ = X \cup N_\zeta(z) \), and set \( C_{\text{rep}} = C_{\text{rep}}(z, X^+) \) to be the colors \( \alpha = c(\{x, z\}) \) repeating among \( x \in X^+ \). Then \( C_{\text{rep}} \subseteq \{\zeta\} \), by hypothesis is forbidden on the path \( P_{y,z} \) corresponding to \( y \in Y(z, \zeta) \). Proposition 3.4 guarantees that \( \delta_{\zeta, X^+}(y) \leq 3\ell \) holds for each \( y \in Y \), and Proposition 3.2 then renders

\[
3\ell \geq \frac{1}{|Y|} \sum_{y \in Y} \delta_{\zeta, X^+}(y) \geq \delta^*(G) + |X^+| - n - \frac{(R - 1)|X^+ \cap N_\zeta(z)|}{|Y|},
\]

and using \( |X^+| = |X| + R - 1 \) and \( \lceil n/2 \rceil = |X| \geq |N_\zeta(z) \cap X^+| \) completes the proof. \( \square \)

In practice, the set \( Y = Y(z, \zeta) \) in Corollary 3.5 will be large, and will guarantee the following result.

**Corollary 3.6.** Fix an integer \( \ell \geq 3 \), and let \((G, c)\) be an \( n \)-vertex edge-colored graph with no rainbow \( \ell \)-cycle \( C_\ell \). Then \( \delta^*(G) \leq (n/2 + 3\ell| \)

**Proof of Corollary 3.6.** Let \((G, c)\) be given as above. For sake of argument, we assume \( \delta^*(G) \geq (n/2) + 2\ell - 5 \), and w.l.o.g. we assume \((G, c)\) is edge-minimal. Let \((z, \zeta) \in V \times c(E)\) satisfy that \( |N_\zeta(z)| = R \) (cf. 11). For \( 1 \leq i \leq \ell - 1 \), let \( Y_i = Y_i(z, \zeta) \) be the set of vertices \( y_i \in V\) which admit an \( i \)-vertex, \( \zeta \)-free, rainbow \( \{y_i, z\}\)-path \( P_{y_i,z} \) in \((G, c)\). Inductively, these sets are non-empty, as \( Y_1 = \{z\} \), and for some \( 1 \leq j < \ell - 2 \), a fixed \( y_j \in Y_j \) and corresponding path \( P_{y_j,z} \) provide

\[
|Y_{j+1}| \geq \deg_G(y_j) - 1 - |E(P_{y_j,z})| - (|V(P_{y_j,z})| - 2) \geq \delta^*(G) - 2j + 2 \geq \delta^*(G) - 2\ell + 6 \geq \frac{n + 1}{2}
\]

with \( \delta^*(G) \geq (n/2) + 2\ell - 5 \). Corollary 3.5 now guarantees

\[
\delta^*(G) \leq \frac{n}{2} + 3\ell + (R - 1) \left( \frac{n + 1}{2|Y_{\ell - 1}|} - 1 \right) \leq \frac{n}{2} + 3\ell,
\]

as desired. \( \square \)

The following corollary describes sets similar to \( Y(z, \zeta) \) which are also large.

**Corollary 3.7.** Fix an integer \( \ell \geq 3 \), and let \((G, c)\) be an edge-minimal, edge-colored graph with no rainbow \( \ell \)-cycle \( C_\ell \). Let \( T \) be a triangle in \( G \), let \( v \in V(T) \), and let \( C_T \subseteq c(E)\) satisfy \( C_T \cap c(E(T)) = \emptyset \). Let \( Y = Y(v, C_T) = Y_{\ell - 1}(v, C_T) \) be the vertices \( y \in Y \) which admit an \((\ell - 1)\)-vertex, \( C_T \)-free, rainbow \( \{v, y\}\)-path \( P_{v,y} \) in \((G, c)\). Then \( |Y| \geq (3/2)(\delta^*(G) - |C_T| - 4\ell) \).

**Proof of Corollary 3.7.** Let \((G, c), T, v\) and \( C_T\) be given as above, where for sake of argument we assume \( \delta^*(G) \geq |C_T| + 4\ell + 1 \), and where we set \( \hat{C}_T = C_T \cup c(E(T)) \). Since \( T \) is not monochromatic (cf. Remark 11.9), we label \( V(T) = \{v, x_1, x_2\} \) with \( c(\{v, x_2\}) \neq c(\{x_1, x_2\}) \). For \( 1 \leq i \leq \ell + 1 \), let \( W_i = W_i(x_i, \hat{C}_T) \) be the set of vertices \( w_i \in V \) which admit an \( i \)-vertex, \( \hat{C}_T \)-free, rainbow \( \{w_i, x_i\}\)-path \( P_{w_i,x_i} \) in \((G, c)\) whose vertices meet \( V(T)\) only in \( x_1 \). Inductively, these sets are non-empty, as \( W_1 = \{x_1\} \), and for some \( 1 \leq j < \ell - 2 \), a fixed \( w_j \in W_j \) and corresponding path \( P_{w_j,x_1} \) provide that

\[
|W_{j+1}| \geq \deg_G(w_j) - |\hat{C}_T| - |E(P_{w_j,x_1})| - (|V(P_{w_j,x_1})| - 2) - |\{v, x_2\}| \geq \deg_G(w_j) - 2(j + 1) - |C_T| + \delta^*(G) \geq |C_T| + 4\ell + 1.
\]

It is easy to see that

\[
W_{\ell - 3}(x_1, \hat{C}_T) \cup W_{\ell - 2}(x_1, \hat{C}_T) = W_{\ell - 3} \cup W_{\ell - 2} \subseteq Y = Y_{\ell - 1}(v, C_T),
\]
since, e.g., adding the edges \{v, x_2\} and \{x_1, x_2\} to a path \(P_{w_{\ell-3}x_1}\) corresponding to \(w_{\ell-3} \in W_{\ell-3}\) places \(w_{\ell-3} \in Y\). We bound \(\delta\) as follows. Let \(\Gamma = \Gamma_{w_{\ell-3}}\) be the edge-colored subgraph of \(G\) induced on \(W_{\ell-3}\). Then \(\Gamma\) admits no rainbow \(\ell\)-cycles \(C_{\ell}\), whence Corollary 3.6 guarantees a vertex \(w_{\ell-3} \in W_{\ell-3}\) for which \(\deg^\ell_G(w_{\ell-3}) \leq (1/2)|W_{\ell-3}| + 3\ell\). As such,

\[
|W_{\ell-2} \setminus W_{\ell-3}| \geq \deg^\ell_G(w_{\ell-3}) - 2(\ell - 2) - |C_T| - \deg^\ell_G(w_{\ell-3}) \geq \delta^\ell(G) - \frac{1}{2}|W_{\ell-3}| - 5\ell - |C_T|,
\]

and so

\[
|Y| \geq |W_{\ell-2} \cup W_{\ell-3}| = |W_{\ell-2} \setminus W_{\ell-3}| + |W_{\ell-3}| \geq \delta^\ell(G) + \frac{1}{2}|W_{\ell-3}| - 5\ell - |C_T| \geq \frac{3}{2}\delta^\ell(G) - 6\ell - \frac{3}{2}|C_T|,
\]
as promised. \(\square\)

### 4. Proof of Theorem 1.2

Fix an integer \(\ell \geq 3\). Let \((G, c)\) be an \(n\)-vertex, edge-minimal, edge-colored graph satisfying \(\delta^\ell(G) \geq (n+1)/2\). We assume that \((G, c)\) admits no rainbow \(\ell\)-cycle \(C_{\ell}\), and we bound \(n \leq n(\ell)\) from above in context. Fix \((z, \zeta) \in V \times c(E)\) with \(|N_\zeta(z)| = R\) (cf. (1)). Let \(X \subseteq N(z)\) satisfy that \(|X| = \delta^\ell(G) - 1\) and that all \(c(x, z)\) are distinct and \(\zeta\)-free among \(x \in X\). We distinguish two cases.

**Case 1** (\(\exists e_0 \in E(G[X]) : c(e_0) \neq \zeta\)). By our choice of \(X\), the following hold:

(I) \(\zeta\) does not appear on the triangle \(T = \{z\} \cup e_0\);

(II) \(\zeta\) is the only color possibly repeating among \(c(x, z)\) for \(x \in X^+ = X \cup N_\zeta(z)\).

As such, we set \(C_{\text{rep}} = C_T \subseteq \{\zeta\}\) so that the set \(Y = Y(z, C_{\text{rep}}) = Y(z, \zeta) = Y(z, C_T)\) commonly featured in each of Proposition 3.4 and Corollaries 3.5 and 3.7 has size, by the last of these,

\[
|Y| \geq \frac{3}{2}(\delta^\ell(G) - 1 - 4\ell) \geq \frac{3}{2}(n+1) - 1 - 4\ell \geq \frac{3}{2}(n+1),
\]

where \(*\) holds when \(n \geq 78\ell\), which we assume for sake of argument. Corollary 3.5 then yields

\[
\frac{n + 1}{2} \leq \delta^\ell(G) \leq n + 3\ell + (R - 1) \left(\frac{n + 1}{2|Y|} - 1\right) \leq n + 3\ell \leq 3\ell + (R - 1) \quad \implies \quad R \leq 12\ell.
\]

Now, define the directed graph \(F = (V, \vec{E}_F)\) on vertex set \(V = V(G)\), where

\[
\vec{E}_F = \{(x, y) \in X^+ \times V : \{x, y\} \in E = E(G)\text{ and }c(x, y) \neq c(x, z)\}.
\]

On the one hand, every \(x \in X^+\) clearly satisfies \(\deg^\ell_F(x) \geq \deg^\ell_G(x) - 1\), and so

\[
|\vec{E}_F| = \sum_{x \in X^+} \deg^\ell_F(x) \geq \sum_{x \in X^+} (\deg^\ell_G(x) - 1) \geq |X^+|\delta^\ell(G) - 1.
\]

On the other hand, with \(Y\) defined above,

\[
|\vec{E}_F| = \sum_{y \in V} \deg^\ell_F(y) = \sum_{y \in V \setminus Y} \deg^\ell_F(y) + \sum_{y \in Y} \deg^\ell_F(y) \leq (n - |Y|)|X^+| + \sum_{y \in Y} \deg^\ell_F(y).
\]

For a fixed \(y \in Y\), we bound

\[
\deg^\ell_F(y) = \left|\{x \in N(y) \cap X^+ : c(x, y) \neq c(x, z)\}\right| = \sum_{\alpha \in c(E)} \left|\{x \in N_\alpha(y) \cap X^+ : \alpha \neq c(x, z)\}\right|.
\]

Let \(A = A_y\) be the colors \(\alpha \in c(E)\) where some \(x \in X^+\) satisfies \(\alpha = c(x, y) \neq c(x, z)\). Then

\[
\deg^\ell_F(y) = \sum_{\alpha \in A} \left|\{x \in N_\alpha(y) \cap X^+ : \alpha \neq c(x, z)\}\right| \leq \sum_{\alpha \in A} |N_\alpha(y) \cap X| \leq \sum_{\alpha \in A} |N_\alpha(y)|.
\]

\(^1\)By Theorem 1.3, it suffices to prove Theorem 1.2 for odd integers \(\ell\). However, most of the current argument is independent of parity considerations, so we make no distinction now.
Since \( y \in Y = Y(z, C_{\text{rep}}) \), Proposition 3.4 guarantees that \(|A| \leq 3\ell\), and so
\[
\deg_{F}^c(y) \leq \sum_{\alpha \in A} |N_\alpha(y)| \leq |A| \leq 3\ell.
\]
Applying (14) to (13) yields
\[
|E_F| \leq (n - |Y|)|X^+| + \sum_{y \in Y} \deg_{F}^c(y) \leq (n - |Y|)|X^+| + 36\ell^2 |Y|.
\]
Comparing (12) and (16) yields \(|X^+| (\delta^c(G) - 1) \leq (n - |Y|)|X^+| + 36\ell^2 |Y|\), or equivalently,
\[
N \geq \delta^c(G) - 1 + \left( 1 - \frac{36\ell^2}{|X^+|} \right) |Y|.
\]
Using \(|X^+| = \delta^c(G) - 1 + R \geq \delta^c(G) \geq (n + 1)/2\), we infer
\[
\frac{1}{2} (n + 1) \geq n - \delta^c(G) + 1 \geq \left( 1 - \frac{72\ell^2}{n} \right) |Y| \geq \left( 1 - \frac{72\ell^2}{n} \right) \times \frac{2}{3} (n + 1),
\]
which implies \( n \leq 288\ell^2 \).

Case 2 \((\forall e \in E(G[X]), c(e) = \zeta)\). Set \( Y = V \setminus (\{z\} \cup X) \). We first observe
\[
\delta^c(G) - 2 \leq \deg^c(Y) \leq \delta^c(G) - 1 \quad \text{and} \quad \frac{1}{2} (n - 1) \leq \delta^c(G) - 1 = |X| \leq \frac{1}{2} (n + 1).
\]
Indeed, \(|Y| = n - 1 - |X| = n - \delta^c(G) \leq \delta^c(G) - 1\) holds from \(|X| = \delta^c(G) - 1 \geq (1/2)(n - 1)\).

Now, every edge \( \{x, y\} \in E \) with \( x \in X \) colored neither \( \zeta \) nor \( c(\{x, z\}) \) places \( y \in Y \cap N(x) \), and so \(|Y| \geq |Y \cap N(x)| \geq \deg^c_G(x) - 2 \geq \delta^c(G) - 2\), from which \(|X| \leq (1/2)(n + 1)\) now follows.

We now define two subsets of \( Y \) that we wish to later avoid. For that, let \( H = G[X,Y] \) be the bipartite subgraph of \( G \) induced by the bipartition \( X \cup Y \), and let \( D \) be the subgraph of \( H \) consisting of edges \( \{x, y\} \in E(H) \) with \( c(\{x, y\}) = c(\{x, z\}) \). Let \( Y_H \) be the vertices \( y \in Y \) sending \( \deg^c_H(y) \leq (5/2)\ell \) many distinct colors to \( X \), and let \( Y_D \) be the vertices \( y \in Y \) sending \( \deg^c_D(y) \geq 2 \) many \( D \)-edges to \( X \).

Claim 4.1. \(|Y_H| \leq 11\ell \) and \(|Y_D| \leq |X|/2\).

Proof of Claim 4.1. Let \( \Gamma = G[A] \) be the edge-colored subgraph of \( G \) induced on \( A = Y_H \). Since \( G[A] \) has no rainbow \( \ell \)-cycles \( C_\ell \), Corollary 3.9 guarantees \( a \in A \) with \( \deg^c_G(a) \leq (1/2)|A| + 3\ell \). Since \( a \) sends at most \((5/2)\ell + 1\) distinct colors to \( \{z\} \cup X \) and at most \(|Y| - |A|\) distinct colors to \( Y \setminus A \), we see
\[
\frac{1}{2} |A| + 3\ell \geq \deg^c_G(a) \geq \deg^c_H(a) - \frac{5}{2}\ell - 1 - |Y| + |A| \geq |A| - \frac{5}{2}\ell \quad \implies \quad |Y_H| = |A| \leq 11\ell.
\]
Since each \( x \in X \) sends to \( Y \) precisely \( \deg^c_D(x) \) many \( D \)-edges and \( \deg^c_G(x) - 2 \) many \( \zeta \)-free \( H \setminus D \)-edges,
\[
\deg^c_D(x) \geq \deg^c_G(x) - 2 \leq \deg^c_H(x) \leq |Y| \leq \delta^c(G) - 1 \quad \implies \quad \deg^c_D(x) \leq \deg^c_H(x) \leq \delta^c(G) - 1 \quad \implies \quad \deg^c_D(x) \leq 1
\]
\[
\implies \sum_{y \in Y} \deg_D(y) = |E(D)| = \sum_{x \in X} \deg_D(x) \leq |X|,
\]
and so \(|Y_D| \leq |X|/2\) follows.

Continuing with Case 2, set \( Y_0 = Y \setminus (Y_H \cup Y_D) \), set \( H[X,Y_0] = G[X,Y_0] \) to be the bipartite subgraph of \( H \) induced by the bipartition \( X \cup Y_0 \), and set \( H_0 = H[X,Y_0] \setminus D \). For each \( x \in X \), we already observed (cf. (18)) that \( x \) sends at least \( \deg^c_G(x) - 2 \) many non-\( \zeta \)-non-\( c(\{x, z\}) \) colors into \( Y_0 \), and so
\[
\forall x \in X, \quad \deg^c_{H_0}(x) \geq \deg^c_G(x) - 2 - |Y \setminus Y_0| \geq \delta^c(G) - 2 - 11\ell - \frac{1}{2} |X| \geq \frac{1}{4} (n + 1) - 11\ell - 2. \quad (19)
\]
To \( X \), each \( y \in Y_0 \) sends \( \deg^c_{H_0}(y) \geq (5/2)\ell + 1 \) many colors and \( \deg_D(y) \leq 1 \) many \( D \)-edges, and so
\[
\forall y \in Y_0, \quad \deg^c_{H_0}(y) \geq \frac{5}{2}\ell. \quad (20)
\]
To conclude Case 2, it is convenient to now distinguish between \( \ell \) (mod 2).
Case 2A ($\ell$ is odd). With $(z, \zeta)$ fixed at the start, fix $y_0 \in N_c(z)$ arbitrarily, where necessarily $y_0 \in Y$. The number of non-$\zeta$ colors that $y_0$ sends to $X$ is at least $\deg_G(y_0) - 1 - (|Y| - 1) \geq \delta^c(G) - |Y| \geq 1$ by (13), so fix $x_1 \in X \cap N(y_0)$ to satisfy $c(\{x_1, y_0\}) \neq \zeta$. For an even integer $k \geq 2$, let $Q_{k-1} = (z, y_0, x_1, y_2, \ldots, x_{k-1})$ be a rainbow path, where $x_1, \ldots, x_{k-1} \in X$ and $y_2, \ldots, y_{k-2} \in Y$. Then $Q_{k-1}$ is extended to a rainbow path $Q_k = (z, y_0, \ldots, y_k, x_{k-1}, y_k)$ along
\[
\deg^c_{H_0}(x_{k-1}) - |E(Q_{k-1})| - |\{y_0, \ldots, y_{k-2}\}| = \deg^c_{H_0}(x_{k-1}) - k - \frac{1}{2}(k-2) \geq \frac{1}{2}(n+1) - 11\ell - 1 - \frac{3}{2}k
\]
many $y_k \in Y \setminus \{y_0, \ldots, y_{k-2}\}$, and $Q_k$ is extended to a rainbow path $Q_{k+1} = (z, y_0, \ldots, y_k, x_{k+1})$ along
\[
\deg^c_{H_0}(y_k) - |E(Q_k)| - |\{x_1, \ldots, x_{k-1}\}| = \deg^c_{H_0}(y_k) - (k + 1) - \frac{k-1}{2}
\]
many $x_{k+1} \in X \setminus \{x_1, \ldots, x_{k-1}\}$. More strongly, $x$ was chosen with $c(\{x, z\})$ distinct among $x \in X$, so $Q_k$ is extended to a rainbow path $Q_{k+1} = (z, y_0, \ldots, y_k, x_{k+1})$ along
\[
\deg^c_{H_0}(y_k) - 2(k+1) - \frac{k-1}{2} \geq \frac{3}{2}(\ell - k - 1)
\]
many $x_{k+1} \in X \setminus \{x_1, \ldots, x_{k-1}\}$ where $c(\{x_{k+1}, z\}) \notin c(\{x_1, z\})$. Then $Q_{k+1}$ bears the rainbow $(k+2)$-cycle $(z, x_1, x_2, \ldots, y_k, x_{k+1}, z)$, since $c(\{x_1, z\}) \notin c(\{x_{k+1}, z\})$, and since $c(\{x_{k+1}, y_k\}) \neq c(\{x_{k+1}, z\})$ from $\{x_{k+1}, y_k\} \notin E(D)$. Since $(G, c)$ has no rainbow $\ell$-cycles $C_\ell$, it must be that $k+2 \leq \ell - 1$. Since (22) is positive with $k = \ell - 3$, (21) must be non-positive, whence $n \leq 50\ell$.

Case 2B ($\ell$ is even). The argument above slightly simplifies. Choose $x_1 \in X$ arbitrarily. As before, we extend a rainbow path $Q_{k-1} = (z, x_1, y_2, \ldots, x_{k-1})$ with $x_1, \ldots, x_{k-1} \in X$ and $y_2, \ldots, y_{k-2} \in Y$ to rainbow paths $Q_k = (z, x_1, \ldots, x_{k-1}, y_k)$ and $Q_{k+1} = (z, x_1, \ldots, y_k, x_{k+1})$ where $y_k \in Y \setminus \{y_2, \ldots, y_{k-2}\}$ and $x_{k+1} \in X \setminus \{x_1, \ldots, x_{k-1}\}$, and where $c(\{x_{k+1}, z\}) \notin c(\{x_1, z\})$. The paths $Q_k$ and $Q_{k+1}$ are respectively shorter than $Q_{k-1}$ and $Q_{k+1}$ above, so inequalities analogous to those in (21) and (22) still hold, and with $k + 1 < \ell - 1$, we similarly conclude $n \leq 50\ell$.

5. Proof of Theorem 1.4

Our proof of Theorem 1.4 follows that of Theorem 1.2 where we also use the following corollary of Propositions 3.2 and 3.4 from Section 3.

Corollary 5.1. Fix an integer $\ell \geq 3$, and let $(G, c)$ be an $n$-vertex, edge-minimal, edge-colored graph with no rainbow $\ell$-cycle $C_\ell$ and with $\delta^c(G) \geq 5R+2\ell$ (cf. (11)). Then $\delta^c(G) < n/2$ or $\Delta(G) < \delta^c(G) + 4R + 3\ell$.

Proof of Corollary 5.1. Let $(G, c)$ be given as above. Assume for a contradiction that $\delta^c(G) \geq n/2$ and that some $v \in V = V(G)$ has $\deg_G(v) \geq \delta^c(G) + 4R + 3\ell$. Then $\deg_G(v) \geq (n+1)/2$ whence $v$ is incident to some triangle $T = T_v$ in $G$, where we see $c(E(T)) = \{\alpha, \beta, \gamma\}$. Since $|N_\alpha(v) \cup N_\beta(v) \cup N_\gamma(v)| \leq 3R$, some $X \subseteq N(v)$ has size $|X| = \delta^c(G) + R + 3\ell$, where at least $\delta^c(G)$ of the colors $c(\{v, x\})$ are distinct among $x \in X$, and where $|N_\alpha(v) \cap X|, |N_\beta(v) \cap X|, |N_\gamma(v) \cap X| \leq 1$. Let $C_T = C_{\text{rep}}$ be the $\leq R + 3\ell$ colors $c(\{v, x\})$ repeating among $x \in X$, where $C_T \cap c(E(T)) = \emptyset$. Let $Y = Y(v, C_T) = Y(v, C_T)$ be the set commonly featured in each of Proposition 3.2 and Corollary 3.7. Corollary 3.7 guarantees
\[
|Y| \geq \frac{1}{2}(\delta^c(G) - |C_T| - 4\ell) \geq \frac{1}{2}(\delta^c(G) - R - 7\ell) = \delta^c(G) + \frac{1}{2}R - \frac{3}{2}(R + 7\ell) \geq \delta^c(G) + \frac{1}{2}(5R + 2\ell) - \frac{3}{2}(R + 7\ell) = \delta^c(G) + 3\ell = |X|.
\]
Proposition 3.2 then guarantees
\[
\frac{1}{|Y|} \sum_{y \in Y} \sigma_{v, X}(y) \geq \delta^c(G) + |X| - n - (R - 1) \frac{|X \cap N_i(v)|}{|Y|} \geq \delta^c(G) + |X| - n - (R - 1) = 2\delta^c(G) + R + 3\ell - n - (R - 1) \geq 3\ell + 1,
\]

\(^2\)The colors $\alpha, \beta, \gamma$ are not identical by Remark 1.3 but they need not all be distinct. These considerations, however, play no role in the current context.
which with \( Y = Y(v, C_{\text{rep}}) \) contradicts Proposition 3.4.

5.1. **Proof of Theorem 1.4.** Let \((G, c), (z, \zeta), \) and \(X \subset N(z)\) be given as in Section 4. In Case 2, we already proved that \( n \leq 50\ell \), but in Case 1, we showed only that \( n \leq 288\ell^2 \). The bottleneck of Case 1 arises in (15), where a fixed \( y \in Y \) satisfies \( \deg_F(y) \leq 3\ell R \leq 36\ell^2 \). We claim

\[
\deg_F(y) \leq 4R + 6\ell \leq 54\ell,
\]

which if true updates (17) to say

\[
\frac{1}{2}(n + 1) \geq (1 - \frac{108\ell}{n}) \times \frac{2}{3}(n + 1),
\]

which gives \( n \leq 432\ell \). To see (25), recall from (14) that

\[
\deg_F(y) \leq \sum_{\alpha \in A} |N_\alpha(y)|,
\]

where \( A = A_y \) is the set of colors \( \alpha \in c(E) \) where some \( x \in X^+ \) satisfies \( \alpha = c\{x, y\} \neq c\{x, z\} \). Let \( B = B_y \) consist of all remaining colors incident to \( y \), in which case

\[
\sum_{\beta \in B} |N_\beta(y)| \geq |B| = |\deg_G(y) - |A| \geq \deg_G^c(y) - 3\ell \geq \delta^c(G) - 3\ell.
\]

Then

\[
\Delta(G) \geq \deg_G(y) = \sum_{\alpha \in A} |N_\alpha(y)| + \sum_{\beta \in B} |N_\beta(y)| \geq \deg_F(y) + \sum_{\beta \in B} |N_\beta(y)| \geq \deg_F(y) + \delta^c(G) - 3\ell,
\]

whence

\[
\deg_F(y) \leq \Delta(G) - \delta^c(G) + 3\ell \leq \delta^c(G) + 4R + 3\ell - \delta^c(G) + 3\ell \leq 4R + 6\ell \leq 54\ell.
\]

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