Topological Coherent Modes in Trapped Bose Gas

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Abstract

The report reviews the problem of topological coherent modes, which are nonlinear collective states of Bose-condensed atoms. Such modes can be generated by means of alternating external fields, whose frequencies are in resonance with the transition frequencies between the related modes. The Bose gas with generated topological coherent modes is a collective nonlinear analog of a resonant atom. Such systems exhibit a variety of nontrivial effects, e.g.: interference fringes, interference current, mode locking, dynamic transitions, critical phenomena, chaotic motion, harmonic generation, parametric conversion, atomic squeezing, and entanglement production.

1 Introduction

Trapped Bose gas, being dilute and cooled down to temperatures below the Bose-Einstein condensation temperature, forms a coherent system well described by the Gross-Pitaevskii equation \[1–5\]. Stationary solutions to the latter are called the topological coherent modes \[6\]. The ground state, corresponding to the lowest energy level of the Gross-Pitaevskii stationary equation, represents the standard Bose-Einstein condensate. The states, associated with the higher energy levels, describe nonground-state condensates. The higher topological coherent modes can be generated by means
of alternating fields, with the frequencies that are in resonance with the chosen transition frequencies [6].

The name topological coherent modes comes from the following. Different stationary solutions to the Gross-Pitaevskii equation, related to distinct energy levels, possess essentially dissimilar spatial shapes, with different number of zeros. The modes, displaying distinct spatial topology, can be named topological. These modes can be characterized as coherent due to the fact that the Gross-Pitaevskii equation can be interpreted as an exact equation for coherent states [7].

The topological coherent modes, described by the nonlinear Gross-Pitaevskii equation, should not be confused with elementary collective excitations, defined by the linear Bogolubov - de Gennes equations. The elementary excitations describe small oscillations around a given nonlinear topological mode and do not change the topology of the latter [1].

The general notion of the nonlinear coherent modes and the possibility of their generation by means of resonant fields was advanced in Ref. [6]. A particular case of vortex creation was considered in Refs. [8,9]. Various properties of topological coherent modes were also studied in theoretical papers [10–27]. A dipole topological mode was generated in experiment [28]. The feasibility of resonant formation of multimode condensates, consisting of several topological coherent modes, was investigated [29,30].

Bose gas of trapped atoms with resonantly generated topological coherent modes exhibits a variety of unusual features. The aim of this report is to present a general picture of such a resonant system (Section 2) and to give a survey of its most interesting properties (Section 3).

2 Topological Coherent Modes

The topological coherent modes are defined [6] as the stationary solutions to the eigenvalue problem

$$\hat{H}[\varphi_n] \varphi_n(r) = E_n \varphi_n(r) ,$$  

which is represented by the stationary Gross-Pitaevskii equation with the nonlinear Hamiltonian

$$\hat{H}[\varphi] \equiv -\frac{\hbar^2}{2m} \nabla^2 + U(r) + N A_s |\varphi|^2 ,$$

containing a trapping potential $U(r)$ and the interaction intensity $A_s \equiv 4\pi\hbar^2 a_s/m$, with $m$ being atomic mass; $a_s$ scattering length; and $N$ the total number of atoms.

In what follows, we shall use the notation for the scalar product $(\varphi_m, \varphi_n) \equiv \int \varphi_m(r) \varphi_n(r) dr$. The eigenfunctions $\varphi_n(r)$, because of the nonlinearity of problem (1), are not necessarily orthogonal, so that $(\varphi_m, \varphi_n)$ is not compulsory the Kroneker delta $\delta_{mn}$. But the functions $\varphi_n(r)$ can always be normalized, with $(\varphi_n, \varphi_n) \equiv ||\varphi_n||^2 = 1$.

If the trapped Bose gas is initially in one of the coherent modes $n$, then to generate another mode requires applying an alternating field

$$V(r, t) = V_1(r) \cos \omega t + V_2(r) \sin \omega t ,$$

with a frequency $\omega$ being close to one of the transition frequencies

$$\omega_{mn} \equiv \frac{1}{\hbar} (E_m - E_n) .$$
Say the modes with the energies $E_1$ and $E_2$ are connected, such that $E_1 < E_2$, with the transition frequency being $\omega_{21}$. The related resonance condition reads as

$$\left| \frac{\Delta \omega}{\omega} \right| \ll 1, \quad \Delta \omega \equiv \omega - \omega_{21}.$$  

(4)

With an applied time-dependent field, we have the temporal Gross-Pitaevskii equation

$$i \hbar \frac{\partial}{\partial t} \varphi(r, t) = \left( \hat{H}[\varphi] + \hat{V} \right) \varphi(r, t),$$

(5)
in which $\hat{V} = \hat{V}(r, t)$. The alternating field can be represented as

$$V(r, t) = \frac{1}{2} B(r) e^{i\omega t} + \frac{1}{2} B^*(r) e^{-i\omega t},$$

(6)

where $B(r) \equiv V_1(r) - iV_2(r)$. Experimentally, the alternating field can be realized as the modulation of the trapping potential. Another way could be by alternating the scattering length $a_s(t)$ by modulating an external magnetic field close to a Feshbach resonance [31–33].

One possibility of studying the resonant generation of coherent modes would be by a direct numerical solution of the temporal equation (5), which we have done in our works [29,30]. However to get a deep physical insight into the problem, it is necessary to develop an analytical theory. Such a general theory, based on the averaging technique [34], was developed in Refs. [6,12,15,20–24,29,30]. In order to show that the basic equations of the analytical approach can be accurately derived and all consideration is well mathematically grounded, we describe, first of all, the main steps of this derivation.

We can look for the solution of Eq. (5) in the form of the mode expansion

$$\varphi(r, t) = \sum_n c_n(t) \varphi_n(r) \exp \left( -\frac{i}{\hbar} E_n t \right),$$

(7)

with the coefficients $c_n(t)$ being slow functions of time, such that

$$\frac{\hbar}{E_n} \left| \frac{d c_n}{dt} \right| \ll 1.$$  

(8)

Then the functions $c_n(t)$ can be treated as temporal quasi-invariants with respect to the fast exponentials $\exp(-iE_n t/\hbar)$. Substituting expansion (7) into Eq. (5), we multiply the latter by the mentioned exponential and average over time according to the rule

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(c_n, t) \, dt,$$

where the quasi-invariants $c_n$ are kept fixed. Averaging in this way the equality $||\varphi||^2 = 1$, we have the normalization condition

$$\sum_n |c_n(t)|^2 = 1.$$  

(9)

For what follows, we need the notation for the transition amplitudes: one of them being due to atomic interactions,

$$\alpha_{mn} \equiv A_s \frac{N}{\hbar} \left( |\varphi_m|^2, 2|\varphi_n|^2 - |\varphi_m|^2 \right),$$

(10)
and another related to the external field (6),

\[ \beta_{mn} \equiv \frac{1}{\hbar} \left( \varphi_m, \vec{B} \varphi_n \right), \tag{11} \]

where \( \vec{B} \equiv B(\mathbf{r}) \). Note that from normalization (9) one has \( |c_n|^2 = 1 - \sum_{m(\neq n)} |c_m|^2 \).

Using this, the nonlinear part, resulting from Eq. (5), can be represented as

\[ A_s \frac{N}{\hbar} \sum_m (2 - \delta_{mn})(|\varphi_n|^2, |\varphi_m|^2) = \sum_{m(\neq n)} \alpha_{nm} |c_m|^2 c_n + \alpha_{nn} c_n. \]

Thus we come to the equation

\[ i \frac{dc_n}{dt} = \sum_{m(\neq n)} \alpha_{nm} |c_m|^2 c_n + \alpha_{nn} c_n + \frac{1}{2} \delta_{n1} \beta_{12} c_2 e^{i\Delta\omega t} + \frac{1}{2} \delta_{n2} \beta_{12}^* c_1 e^{-i\Delta\omega t}. \tag{15} \]

What is of physical interest is the behaviour of the fractional mode populations

\[ w_n(t) \equiv |c_n(t)|^2. \tag{13} \]

These do not depend on the phase of \( c_n \). Therefore, we may employ the gauge transformation

\[ c_n \rightarrow c_n \exp(-i\alpha_{nn} t). \tag{14} \]

Then Eq. (15) reduces to

\[ i \frac{dc_n}{dt} = \sum_{m(\neq n)} \alpha_{nm} |c_m|^2 c_n + \frac{1}{2} \delta_{n1} \beta_{12} c_2 e^{i\Delta t} + \frac{1}{2} \delta_{n2} \beta_{12}^* c_1 e^{-i\Delta t}, \tag{15} \]

where

\[ \Delta \equiv \Delta\omega + \alpha_{11} - \alpha_{22}. \tag{16} \]

For neighboring modes, one has \( \alpha_{11} \approx \alpha_{22} \) and \( \Delta \approx \Delta\omega \). Otherwise, it is always possible to choose such a detuning \( \Delta\omega \) that \( \Delta \) be small, which is assumed in what follows, \( |\Delta| \ll \omega \).

To preserve well-defined resonance, it is necessary, in analogy with the case of resonant atoms [35], that the transition amplitudes, involved in the process, be small. In the considered problem, such amplitudes are given by Eqs. (10) and (11). So, it is necessary that the transition amplitudes, due to atomic interactions, be small,

\[ \left| \frac{\alpha_{12}}{\omega_{21}} \right| \ll 1, \quad \left| \frac{\alpha_{21}}{\omega_{21}} \right| \ll 1, \tag{17} \]

as well as the amplitude determined by the modulating field,

\[ \left| \frac{\beta_{12}}{\omega_{21}} \right| \ll 1. \tag{18} \]

From Eq. (15) it stems that if at the initial time \( c_n(0) = 0 \) for \( n \neq 1,2 \), then always

\[ c_n(t) = 0, \quad (n \neq 1,2) \tag{19} \]

for all \( t \geq 0 \). Hence Eq. (15) reduces to the system of two complex-valued equations

\[ i \frac{dc_1}{dt} = \alpha_{12} |c_2|^2 c_1 + \frac{1}{2} \beta_{12} c_2 e^{i\Delta t}, \quad i \frac{dc_2}{dt} = \alpha_{21} |c_1|^2 c_2 + \frac{1}{2} \beta_{12}^* c_1 e^{-i\Delta t}. \tag{20} \]
Such equations, though resembling the case of two coupled electromagnetic modes [36],
differ from that by the presence of the nonlinearity caused by atomic interactions.

The complex-valued system (20) is equivalent to four real-valued equations. However,
because of the global gauge symmetry and due to the normalization condition
\(|c_1|^2 + |c_2|^2 = 1\), the related dynamical system is, actually, two-dimensional. To show
this, it is convenient to define \(c_j = |c_j| \exp(i \pi_j t)\) and \(\beta_{12} \equiv \beta e^{i \gamma}\), where \(\beta \equiv |\beta_{12}|\). Let
us also introduce the notation
\[
\alpha \equiv \frac{1}{2} (\alpha_{12} + \alpha_{21}) , \quad \delta \equiv \Delta + \frac{1}{2} (\alpha_{12} - \alpha_{21}) .
\]
Then for the population difference
\[
s \equiv |c_2|^2 - |c_1|^2 \tag{21}
\]
and effective phase difference
\[
x \equiv \pi_2 - \pi_1 + \gamma + \Delta , \tag{22}
\]
we find the system of two equations
\[
\frac{ds}{dt} = -\beta \sqrt{1 - s^2} \sin x , \quad \frac{dx}{dt} = \alpha s + \frac{\beta s}{\sqrt{1 - s^2}} \cos x + \delta . \tag{23}
\]

The consideration can be generalized to the case of the multiple generation of
topological coherent modes [29,30]. This requires, instead of one modulating field (6),
the action of several oscillating fields
\[
V(r, t) = \frac{1}{2} \sum_j \left[ B_j(r) e^{i \omega_j t} + B_j^*(r) e^{-i \omega_j t} \right] , \tag{24}
\]
whose frequencies are tuned to the resonance with the chosen transition frequencies.
For instance, in the case of three coexisting modes, we obtain
\[
i \frac{dc_1}{dt} = (a_{12} |c_2|^2 + a_{13} |c_3|^2) c_1 + f_1 ,
\]
\[
i \frac{dc_2}{dt} = (a_{21} |c_1|^2 + a_{23} |c_3|^2) c_2 + f_2 ,
\]
\[
i \frac{dc_3}{dt} = (a_{31} |c_1|^2 + a_{32} |c_2|^2) c_3 + f_3 , \tag{25}
\]
where the forces \(f_j\), related to the modulating fields, depend on the type of the
mode generation, whether this is the cascade generation, \(V\)-type, or \(\Lambda\)-type generation
[29,30]. The effective detuning \(\Delta_{mn} \equiv \Delta \omega_{mn} + \alpha_{nn} - \alpha_{mm}\), where \(\Delta \omega_{mn} \equiv \omega_j - \omega_m\),
is again assumed to be small, \(|\Delta_{mn}| \ll \omega_{mn}\).

In the same manner, one can derive the evolution equations for the amplitudes \(c_n(t)\)
for an arbitrary number of generated coherent modes, whose populations are given
by \(|c_n|^2\). The resulting equations, such as (20), (23), or (25), are nonlinear because
of the binary atomic interactions. One could also include three-body interactions,
which would yield the fifth-order nonlinearity with respect to \(|c_n|\). Such three-body
interactions can play an important role in describing the dissipation caused by three-
body recombinations [1,37].

One could also take into account nonadiabatic corrections to the atomic evolution
equations. However nonadiabatic description is of vital importance only for noncon-
fined motion of atoms [38], while for trapped atoms nonadiabatic corrections amount
to at most a few percent [39].
3 Dynamic Resonant Effects

Trapped condensed Bose gas, with resonantly generated topological coherent modes, resembles a resonant atom, hence, such a resonant condensate should display the features typical of resonant finite-level atoms. But this resonant condensate is, in addition, a collective nonlinear system, because of which it can possess many other unusual properties, not existing in finite-level atoms.

(1) Interference Fringes. The total density \( \rho(r, t) = |\varphi(r, t)|^2 \) of trapped atoms, with generated topological modes, is not simply the sum of the partial mode densities \( \rho_n(r, t) = |c_n(t)\varphi_n(r)|^2 \), but the interference fringes arise, described by the interference density
\[
\rho_{\text{int}}(r, t) \equiv \rho(r, t) - \sum_n \rho_n(r, t),
\]
with fast oscillation in time [20,23].

(2) Interference Current. Similarly, there exists a fastly oscillating interference current
\[
j_{\text{int}}(r, t) \equiv j(r, t) - \sum_n j_n(r, t),
\]
sometimes called the internal Josephson current [20,23].

(3) Mode Locking. Under this effect, the fractional mode populations \( w_n \) are locked in the vicinity of their initial values, so that either
\[
0 \leq w_n(t) \leq \frac{1}{2}
\]
for all \( t \geq 0 \), or
\[
\frac{1}{2} \leq w_n(t) \leq 1,
\]
ever crossing the line \( w_n = 1/2 \), but being either below it or above it, depending on initial conditions [6,23,24].

(4) Dynamic Transition. Varying the system parameters, the dynamics of the mode populations can be qualitatively changed from the mode locked regime to the mode unlocked regime, when the mode populations fluctuate in the whole region
\[
0 \leq w_n(t) \leq 1,
\]
individually of their initial values [15,23,24].

(5) Critical Phenomena. On the manifold of the system parameters, there exists a critical surface in the vicinity of which the dynamics of the mode populations becomes unstable. Crossing the critical surface, when varying some parameters, changes the dynamics between the mode locked and mode unlocked regimes. Close to the surface, a tiny variation of some of the system parameters, say of the pumping amplitude or of the detuning, provokes drastic changes in the dynamics of the mode populations [15,16,20,23,24]. The location of the critical surface also depends on the initial setup. Thus, for the case of two coexisting topological modes, we have the critical surface described by the relation
\[
\frac{1}{2} \alpha s_0^2 - \beta \sqrt{1 - s_0^2} \cos x_0 + \delta s_0 = \beta \text{ sgn } \alpha,
\]
in which \( s_0 \equiv s(0) \) and \( x_0 \equiv x(0) \). For each given initial conditions, this is a surface in the three-dimensional space of the parameters \( \alpha, \beta, \) and \( \delta \). For \( s_0 = \mp 1 \), the critical surface reads as

\[
\beta \text{ sgn } \alpha \pm \delta = \frac{1}{2} \alpha .
\]

Fixing \( s_0 = -1 \), we can reduce the above relation to two critical lines on the manifold of the parameters

\[
b \equiv \beta |\alpha|, \quad \varepsilon \equiv \delta |\alpha| .
\]

These lines are

\[
b + \varepsilon = \frac{1}{2} \quad (\alpha > 0),
\]

\[
b - \varepsilon = \frac{1}{2} \quad (\alpha < 0).
\]

A time-averaged system displays on the critical surface critical phenomena typical of statistical systems with the second-order phase transitions. For the averaged system it is possible to define an effective capacity and susceptibility, which diverge on the critical surface \([15,16,20]\).

(6) Chaotic Motion. Fractional mode populations for a two-mode condensate are always periodic functions of time. But if the number of modes in the condensate is three or larger, then, depending on the system parameters, one has either quasiperiodic or chaotic motion. For instance, in the case of the three-mode condensate, for which \( \alpha_{ij} = \alpha, \beta_{ij} = \beta, \) and \( \Delta_{ij} = 0 \), chaotic motion appears when

\[
|\frac{\beta}{\alpha}| \geq 0.639448,
\]

that is, under a sufficiently strong pumping \([29,30]\).

(7) Harmonic Generation. The generation of topological coherent modes may occur not solely under the direct resonance condition \( \omega = \omega_{21} \), but also under the condition of harmonic generation

\[
n \omega = \omega_{21} \quad (n = 1, 2, \ldots),
\]

when just one modulating field, with a frequency \( \omega \), is applied \([29,30]\).

(8) Parametric Conversion. Another possibility of generating the topological modes is under the condition of parametric conversion

\[
\sum_j (\pm \omega_j) = \omega_{21},
\]

when several alternating fields, with frequencies \( \omega_j \), are involved \([29,30]\).

(9) Atomic Squeezing. Squeezing is a quantum effect that does not exist for classical quantities. So, for treating it, one has to quantize the modes by considering the mode amplitudes \( c_n \) as Bose operators. Then employing the pseudospin representation, one can define the squeezing factor

\[
Q \equiv \frac{2 \Delta^2 (S_z)}{\langle S_\pm \rangle},
\]
where $\Delta^2(S_z)$ is the dispersion of the $z$-component of the total spin and $S_\pm$ are rising and lowering operators, respectively. The defined squeezing factor describes the relation between the dispersion, associated with the mode populations, and the dispersion of a relative phase. When $Q < 1$, one says that the atomic squeezing occurs. Then the mode populations can be measured with a higher accuracy as compared to the measurement of current [23].

(10) Entanglement Production. In the condensate with topological coherent modes, entanglement can be produced. A general measure of entanglement production, valid for arbitrary systems, was introduced in Ref. [40]. The measure of entanglement, generated by an operator $\hat{A}$, is defined as

$$\varepsilon(\hat{A}) \equiv \log \frac{||\hat{A}||_D}{||\hat{A} \otimes||_D},$$

where $|| \cdot ||_D$ implies the restricted norm over a set $D$ of disentangled functions, and $\hat{A} \otimes$ is a nonentangling counterpart of $\hat{A}$. Considering a $p$-particle reduced density operator $\hat{\rho}_p$ for a multimode condensate, we find

$$\varepsilon(\hat{\rho}_p) = (1 - p) \log \sup_n w_n,$$

where $w_n = w_n(t) = |c_n(t)|^2$. Hence the above measure $\varepsilon(\hat{\rho}_p) \equiv \varepsilon_p(t)$ is a function of time, characterizing the entanglement evolution [40].

Summarizing, we have shown that by applying resonant modulating fields to a Bose-condensed gas of trapped atoms, one can create topological coherent modes. We have developed an analytical theory describing the condensate with such coherent modes and also made numerical calculations by directly solving the Gross-Pitaevskii equation [29,30]. All results of the analytical theory are in very good agreement with numerical solutions. The condensate with topological coherent modes is a novel system possessing a rich variety of nontrivial properties which could be employed in many applications.

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