RATIONAL CUSPIDAL CURVES IN A MOVING FAMILY OF $\mathbb{P}^2$

RITWIK MUKHERJEE AND RAHUL KUMAR SINGH

Abstract. In this paper we obtain a formula for the number of rational degree $d$ curves in $\mathbb{P}^3$ having a cusp, whose image lies in $\mathbb{P}^2$ and that passes through $r$ lines and $s$ points (where $r + 2s = 3d + 1$). This problem can be viewed as a family version of the classical question of counting rational cuspidal curves in $\mathbb{P}^2$, which has been studied earlier by Z. Ran ([12]), R. Pandharipande ([11]) and A. Zinger ([15]). We obtain this number by computing the Euler class of a relevant bundle and then finding out the corresponding degenerate contribution to the Euler class. The method we use is closely based on the method followed by A. Zinger ([15]) and I. Biswas, S. D’Mello, R. Mukherjee and V. Pingali ([1]). We also verify that our answer for the characteristic numbers of rational cuspidal planar cubics and quartics is consistent with the answer obtained by N. Das and the first author ([2]), where they compute the characteristic number of $\delta$-nodal planar curves in $\mathbb{P}^3$ with one cusp (for $\delta \leq 2$).

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1. Introduction

A classical question in enumerative algebraic geometry is:

Question. What is $N_d$, the number of rational (genus zero) degree $d$ curves in $\mathbb{P}^2$ that pass through $3d - 1$ generic points?

Although the computation of $N_d$ is a classical question, a complete solution to the above problem was unknown until the early 90’s when Ruan–Tian ([13]) and Kontsevich–Manin ([8]) obtained a formula for $N_d$. Generalization of this question to enumerate rational curves with higher singularities (such as cusps, tacnodes and higher order cusps) have been studied by Z. Ran ([12]), R. Pandharipande ([11]) and A. Zinger ([15], [16] and [14]). These results have also been generalized to other surfaces (such as $\mathbb{P}^1 \times \mathbb{P}^1$) by J. Kock ([7]) and more generally for del-Pezzo surfaces by I. Biswas, S. D’Mello, R. Mukherjee and V. Pingali ([1]). The problem

2010 Mathematics Subject Classification. 14N35, 14J45.
of enumerating elliptic cuspidal curves has been solved by Z. Ran ([12]), and more recently a solution to this question in any genus has been obtained by Y. Ganor and E. Shustin ([4]) using methods from Tropical Geometry.

A natural generalization of problems in enumerative geometry (where one studies curves inside some fixed ambient surface such as $\mathbb{P}^2$) is to consider a family version of the same problem. This generalization is considered by S. Kleiman and R. Piene ([1]) and more recently by T. Laarakker ([9]) where they study the enumerative geometry of nodal curves in a moving family of surfaces.

Motivated by this generalization, A. Paul and the authors of this paper studied a family version of computing $N_d$ in ([10]); there the authors find a formula for the characteristic number of rational planar curves in $\mathbb{P}^3$ (i.e. curves in $\mathbb{P}^3$ that lie inside a $\mathbb{P}^2$). In this paper we build up on the results of ([10]) to find the characteristic number of rational planar curves in $\mathbb{P}^3$ having a cusp. The main result of this paper is as follows:

**Main Result.** Let $C_{d}^{\mathbb{P}^3, \text{Planar}}(r, s)$ be the number of genus zero, degree $d$ curves in $\mathbb{P}^3$ having a cusp, whose image lies in a $\mathbb{P}^2$, intersecting $r$ generic lines and $s$ generic points (where $r + 2s = 3d + 1$). We have a recursive formula to compute $C_{d}^{\mathbb{P}^3, \text{Planar}}(r, s)$.

We have written a mathematica program to implement our formula; the program is available on our web page

https://www.sites.google.com/site/ritwik371/home

In section 6 we subject our formula to several low degree checks; in particular, we verify that our numbers are logically consistent with those obtained by N. Das and the first author ([2]).

Let us now give a brief overview of the method we use in this paper; we closely adapt the method applied by A. Zinger ([15]) and I. Biswas, S. D’Mello, R. Mukherjee and V. Pingali ([1]). We express our enumerative number as the number of zeros of a section of an appropriate vector bundle (restricted to an open dense set of an appropriate moduli space). As is usually the case, the Euler class of this vector bundle is our desired enumerative number, plus an extra boundary contribution. In ([15]) and ([1]) the method of “dynamic intersections” (cf. Chapter 11 in [3]) is used to compute the degenerate contribution to the relevant Euler class. We argue in section 5 how the multiplicity computation in ([15]) and ([1]) implies the multiplicity of the degenerate locus that occurs in our case. Finally, computation of the Euler class involves the intersection of tautological classes on the moduli space of planar degree $d$ curves; this in turn involves the characteristic number of rational planar curves in $\mathbb{P}^3$, for which we use the result of our paper [10]. Hence, we can compute both the Euler class and the degenerate contribution, which gives us our desired number $C_{d}^{\mathbb{P}^3, \text{Planar}}(r, s)$.

2. Notation

Let us define a **planar** curve in $\mathbb{P}^3$ to be a curve, whose image lies inside a $\mathbb{P}^2$. We will now develop some notation to describe the space of planar curves of a given degree $d$.

Let us denote the dual of $\mathbb{P}^3$ by $\overline{\mathbb{P}}^3$; this is the space of $\mathbb{P}^2$ inside $\mathbb{P}^3$. An element of $\overline{\mathbb{P}}^3$ can be thought of as a non zero linear functional $\eta : \mathbb{C}^4 \rightarrow \mathbb{C}$ upto
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scaling (i.e., it is the projectivization of the dual of $\mathbb{C}^4$). Given such an $\eta$, we define the projectivization of its zero set as $\mathbb{P}^2_{\eta}$. In other words,

$$\mathbb{P}^2_{\eta} := \mathbb{P}(\eta^{-1}(0)).$$

Note that this $\mathbb{P}^2_{\eta}$ is a subset of $\mathbb{P}^3$. Next, we define the moduli space of planar degree $d$ curves into $\mathbb{P}^3$ as a fibre bundle over $\mathbb{P}^3$. More precisely, we define

$$\pi : \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \mathbb{P}^3$$

to be the fibre bundle, such that

$$\pi^{-1}(\eta) := \mathcal{M}_{0,1}(\mathbb{P}^2_{\eta}, d).$$

Here we are using the standard notation to denote $\mathcal{M}_{0,k}(X, \beta)$ to be the moduli space of genus zero stable maps, representing the class $\beta \in H_2(X, \mathbb{Z})$ and $\mathcal{M}_{0,k}(X, \beta)$ to be its stable map compactification. Since the dimension of a fibre bundle is the dimension of the base, plus the dimension of the fiber, we conclude that the dimension of $\mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ is $3d + 2 + k$.

Next, we note that there is a natural forgetful map

$$\pi_F : \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \mathcal{M}_{0,1}(\mathbb{P}^3, d)$$

where one forgets the plane $\mathbb{P}^2_{\eta}$ and simply thinks about the stable map to $\mathbb{P}^3$. When $d \geq 2$, the map $\pi_F$ is injective when restricted to the open dense subspace of non multiply covered curves (from a smooth domain). This is because every planar degree $d$ degree curve lies in a unique plane, when $d \geq 2$. When $d = 1$, this map is not injective since a line is not contained in a unique plane. Infact we note that the space of lines is 4 dimensional, while the dimension of $\mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, 1)$ is 5.

Let

$$L : \mathcal{M}_{0,1}(\mathbb{P}^3, d) \longrightarrow \mathcal{M}_{0,1}(\mathbb{P}^3, d)$$

denote the universal tangent line bundle over the marked point (the fiber over each point is the tangent space over the given marked point). This line bundle will pullback to a line bundle over $\mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ via the map $\pi_F$; we will denote it by the same symbol $L$ (we will in general avoid writing the pullback symbol $\pi_F^*$ if there is no cause of confusion).

Let us now define a few cycles in $\mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$. Let $\mathcal{H}_L$ and $\mathcal{H}_P$ denote the classes of the cycles in $\mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d)$ that corresponds to the subspace of curves passing through a generic line and a point respectively. We will denote their pullbacks (via the forgetful map that forgets the marked point) to $\mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d)$ by the same symbol $\mathcal{H}_L$ and $\mathcal{H}_P$. We will also denote $H$ and $a$ to be the standard generators of $H^\ast(\mathbb{P}^3; \mathbb{Z})$ and $H^\ast(\mathcal{M}_{0,0}^{\text{Planar}}; \mathbb{Z})$ respectively. As $\pi$ is a projection map from $\mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ to $\mathbb{P}^3$, we denote the pullback $\pi^\ast a$ by the same symbol $a$. Finally, there is an evaluation map from

$$\text{ev} : \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \mathbb{P}^3.$$
We will now define a few numbers by intersecting cycles on $\mathcal{M}_{0,k}^{\text{Planar}}(\mathbb{P}^3, d)$. We will use the convention that
\begin{equation}
\langle \alpha, [M] \rangle = 0 \quad \text{if} \quad \deg(\alpha) \neq \dim(M). \tag{2.1}
\end{equation}

We now define
\begin{align*}
N_{d}^{\text{Planar}}(r, s, \theta) &:= \langle a^\theta, \mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \rangle, \\
\Phi_d(i, j, r, s, \theta) &:= \langle c_1(\mathbb{L}^*)^i(\text{ev}^*H)^j, \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle.
\end{align*}

We note that using the results of our paper ([10]), the numbers $N_{d}^{\text{Planar}}(r, s, \theta)$ are all computable. In section 4, a formula is given to compute the relevant $\Phi_d(i, j, r, s, \theta)$ necessary to obtain the main result of this paper. We note that using the convention introduced in equation (2.1)
\begin{align*}
N_{d}^{\text{Planar}}(r, s, \theta) &= 0 \quad \text{unless} \quad r + 2s + \theta = 3d + 2 \\
\Phi_d(i, j, r, s, \theta) &= 0 \quad \text{unless} \quad r + 2s + \theta + i + j = 3d + 3.
\end{align*}

3. Euler class computation

We will now describe the basic method by which we compute the characteristic number of planar rational cuspidal curves in $\mathbb{P}^3$. We will express this number as the number of zeros of a section of an appropriate bundle restricted to an open dense subspace of the moduli space of planar curves in $\mathbb{P}^3$.

Before we do that, let us make a few abbreviations that we will often use. We denote
\begin{align*}
C_d &:= C_d^{\text{Planar}}(r, s), \\
\mathcal{M} &:= \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s, \\
\overline{\mathcal{M}} &:= \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s.
\end{align*}

Let us now define
\begin{align*}
\mathcal{S} &:= \{([\eta], q) \in \hat{\mathbb{P}}^3 \times \mathbb{P}^3 : \eta(q) = 0\}.
\end{align*}

An element of $\mathcal{S}$ denotes a plane $\mathbb{P}^2_{\eta}$ in $\mathbb{P}^3$ together with a marked point $q$ that lies in the plane. We will now define a rank two bundle $W \to \mathcal{S}$, where the fibre over each point $([\eta], q)$ is $T_q\mathbb{P}^2_{\eta}$. We note that over $\mathcal{S}$, we have the following short exact sequence of vector bundles:
\begin{align*}
0 &\longrightarrow W \longrightarrow T\hat{\mathbb{P}}^3 \longrightarrow \mathbb{L}^* \otimes \mathbb{L}_{\mathbb{P}^3}^* \longrightarrow 0.
\end{align*}

Here the first map is the inclusion map and the second map is $\nabla_{\eta|q}$. Hence,
\begin{align*}
e(W)c(\mathbb{L}^* \otimes \mathbb{L}_{\mathbb{P}^3}^*) &= c(T\hat{\mathbb{P}}^3) \implies c_1(W) = 3H - a \\
c_2(W) &= a^2 - 2aH + 3H^2.
\end{align*}

Next, we note that $C_d$ is the cardinality of the set
\begin{align*}
\{[u, y] \in \mathcal{M} : du|_y = 0\}.
\end{align*}

The process of taking the derivative of $u$ at the marked point induces a section of the rank two vector bundle
\begin{align*}
\mathbb{L}^* \otimes \text{ev}^*W \to \overline{\mathcal{M}}, \quad \text{given by} \quad [u, y] \to du|_y. \tag{3.1}
\end{align*}

This section is transverse to zero, when restricted to $\mathcal{M}$; this is justified in section 5. We now note that there is a degenerate contribution to the Euler class because...
the section vanishes on the boundary $\overline{M} - M$. Let us first describe the boundary component on which it will vanish. The boundary component on which it will vanish is going to be a map from a wedge of three spheres of degree $d_1$, 0 and $d_2$, where the marked point lies on the degree zero component (which is also called a ghost bubble). The cardinality of this set is computed in ([10]) (in the proof of Theorem 3.3); it is given by
\[
B = \frac{1}{2} \sum_{d_1 + d_2 = d, s_1 + s_2 = s, r_1 + r_2 = r} d_1 d_2 B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta) \begin{pmatrix} r \\ r_1 \\ s \\ s_1 \end{pmatrix},
\]
(3.2)
where $B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta)$ as defined in equation (4.7). We claim that this boundary contributes with a multiplicity of one to the Euler class; this is justified in section 5.

It remains to compute the Euler class. By the splitting principle, we note that
\[
e := \langle e(L^* \otimes \text{ev}^* W), \overline{M} \rangle = \langle c_1(L^*)^2 + c_1(L^*)c_1(\text{ev}^* W) + c_2(\text{ev}^* W), \overline{M} \rangle
\]
\[
= \Phi_d(2, 0, r, s, 0) - \Phi_d(1, 0, r, s, 1) + 3\Phi_d(1, 1, r, s, 0)
+ \Phi_d(0, 0, r, s, 2) - 2\Phi_d(0, 1, r, s, 1)
+ 3\Phi_d(0, 2, r, s, 0).
\]
(3.3)
The numbers $\Phi_d(i, j, r, s, \theta)$ that arise in the right hand side of equation (3.3) can be computed using the results of section 4 (namely, Lemmas 4.1, 4.2 and 4.3). Since the boundary contributes with a multiplicity of one, we conclude that
\[
e = C_d + B.
\]
(3.4)
Using equations (3.4), (3.3), the values of $\Phi_d(i, j, r, s, \theta)$ from the results of section 4 equation (3.2) and the values of $N^{p, \text{Planar}}_d(r, s, \theta)$ from the paper ([10]), we obtain the value of $C^{p, \text{Planar}}_d(r, s, \theta)$. This formula has been implemented using mathematica; the program is available on request. In section 6, we present the values of $C_d$ for a few values of $d$.

4. Intersection of Tautological Classes

We will now give a formula for the relevant $\Phi_d(i, j, r, s, \theta)$ that are necessary to compute the Euler class. We will often refer to $\Phi_d(i, j, r, s, \theta)$ as a level $i$ number.

**Lemma 4.1.** The level zero numbers $\Phi_d(0, j, r, s, \theta)$ are given by
\[
\Phi_d(0, j, r, s, \theta) = \begin{cases} 
0 & \text{if } j = 0, \\
dN^{p, \text{Planar}}_d(r, s, \theta) & \text{if } j = 1, \\
N^{p, \text{Planar}}_d(r + 1, s, \theta) & \text{if } j = 2, \\
N^{p, \text{Planar}}_d(r, s + 1, \theta) & \text{if } j = 3, \\
0 & \text{if } j > 3.
\end{cases}
\]
(4.1)
Lemma 4.2. The level one numbers $\Phi_d(1, j, r, s, \theta)$ are given by

$$
\Phi_d(1, j, r, s, \theta) = \begin{cases} 
-2N_d(r, s, \theta) & \text{if } j = 0, \\
\frac{1}{d} \Phi_d(0, 1, r + 1, s, \theta) - \frac{2}{d} \Phi_d(0, 2, r, s, \theta) + \frac{1}{d^2} \sum_{\substack{r_1 + r_2 = r \\ s_1 + s_2 = s \\ d_1 + d_2 = d \\ d_1, d_2 > 0}} d_1^2 d_2^3 (r_1)^{s_1} B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta) & \text{if } j = 1,
\end{cases}
$$

(4.2)

where $B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta)$ is as defined in equation (4.7).

Lemma 4.3. The level two number $\Phi_d(2, 0, r, s, \theta)$ is given by

$$
\Phi_d(2, 0, r, s, \theta) = \frac{1}{d^2} \Phi_d(1, 0, r + 1, s, \theta) - \frac{2}{d} \Phi_d(1, 1, r, s, \theta) + \frac{1}{d^2} (T_1(r, s, \theta) + T_2(r, s, \theta)),
$$

(4.3)

where $B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta)$ is as defined in equation (4.7),

$$
T_1(r, s, \theta) := \sum_{\substack{r_1 + r_2 = r \\ s_1 + s_2 = s \\ d_1 + d_2 = d \\ d_1, d_2 > 0}} \binom{r}{r_1} \binom{s}{s_1} d_1 d_2 B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta),
$$

and

$$
T_2(r, s, \theta) := \sum_{\substack{r_1 + r_2 = r \\ s_1 + s_2 = s \\ d_1 + d_2 = d \\ d_1, d_2 > 0}} \binom{r}{r_1} \binom{s}{s_1} d_1 d_2 B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta),
$$

and

$$
\tilde{B}_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta) := \sum_{i=0}^{3} \Phi_d(1, 0, r_1, s_1, i) \times N_{d_2, \text{Planar}}(r_2, s_2, \theta + 3 - i).
$$

Remark 4.1. The number $\Phi_d(1, j, r, s, \theta)$ for $j > 1$ and $\Phi_d(2, j, r, s, \theta)$ for $j > 0$ can be computed without any further effort; we have not presented the formulas since they are not needed for the Euler class computation.

Before we start proving these Lemmas, let us first recall an important result about $c_1(L^*)$.

Lemma 4.4. On $\overline{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$, the following equality of divisors holds:

$$
c_1(L^*) = \frac{H_L}{d^2} - \frac{2}{d} v^*(H) + \frac{1}{d^2} \sum_{d_1 + d_2 = d, \ d_1, d_2 \neq 0} d_1^2 B_{d_1, d_2},
$$

where $B_{d_1, d_2}$ denotes the boundary stratum corresponding to a bubble map of degree $d_1$ curve and degree $d_2$ curve with the marked point lying on the degree $d_1$ component.

Proof: This lemma is proved in ([5], Lemma 2.3) for $\overline{M}_{0,1}(\mathbb{P}^3, d)$. The corresponding statement for $\overline{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ follows immediately by pulling back the relationship via the natural map $\pi_F$ from $\overline{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ to $\overline{M}_{0,1}(\mathbb{P}^3, d)$ (the map that forgets the plane). $\square$
We are now ready to prove the Lemmas that involve the computation of \( \Phi_d(i, j, r, s, \theta) \).

**Proofs of Lemmas 4.1, 4.2 and 4.3** Let us start by proving Lemma 4.1. We recall that

\[
\Phi_d(0, j, r, s, \theta) = \langle (ev^* H)^j, \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle.
\]

This number is zero unless \( r + 2s + \theta + j = 3d + 3 \). Let us start by considering the case when \( j = 0 \). Let us assume \( r + 2s + \theta = 3d + 3 \) (otherwise the number is zero). In that case, we note that

\[
\Phi_d(0, 0, r, s, \theta) = \langle (ev^* H)^0, \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]

\[
= \langle 1, \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle.
\]

The last equality holds because the intersection is occurring inside \( \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \) and \( r + 2s + \theta = 3d + 3 \).

Next, let us consider the case when \( j = 1 \). Let us assume \( r + 2s + \theta + 1 = 3d + 3 \) (otherwise the number is zero). Let us consider the forgetful map

\[
\delta : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap ev^*(H) \rightarrow \overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d).
\]

We note that the degree of this map is \( d \) (since a degree \( d \) curve intersects a plane at \( d \) points). Hence

\[
\Phi_d(0, 1, r, s, \theta) = \langle (ev^* H), \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]

\[
= \langle 1, \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap ev^*(H) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]

\[
= \deg(\delta)[\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta]
\]

\[
= dN_{d^3}^{\text{Planar}}(r, s, \theta).
\]

Next, let us consider the case when \( j = 2 \). Let us assume \( r + 2s + \theta + 2 = 3d + 3 \) (otherwise the number is zero). Let us consider the forgetful map

\[
\delta : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap ev^*(H^2) \rightarrow \overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L.
\]

We note that the degree of this map is one since in the first case we are considering a curve and a marked point that goes to a line \( (H^2) \), while in the second case we are considering a curve whose image intersects a line. Hence

\[
\Phi_d(0, 2, r, s, \theta) = \langle (ev^* H^2), \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]

\[
= \langle 1, \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap ev^*(H^2) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]

\[
= \deg(\delta)[\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^{r+1} \cap \mathcal{H}_p^s \cap a^\theta]
\]

\[
= N_{d}^{\text{Planar}}(r + 1, s, \theta).
\]

Finally, let us consider the case when \( j = 3 \). Let us assume \( r + 2s + \theta + 3 = 3d + 3 \) (otherwise the number is zero). Let us consider the forgetful map

\[
\delta : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap ev^*(H^3) \rightarrow \overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_p.
\]
We note that the degree of this map is one since in the first case we are considering a curve and a marked point that goes to a point \((H^3)\), while in the second case we are considering a curve whose image passes through a point. Hence
\[
\Phi_d(0, 3, r, s, \theta) = \langle (ev^*H^3), \, \overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]
\[
= \langle 1, \, \overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d) \cap ev^*(H^3) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle
\]
\[
= \deg(\delta) \overline{\mathcal{M}}_{0, 0}^{\text{planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^{s+1} \cap a^\theta
\]
\[
= N_{d, \text{planar}}^\text{planar}(r, s + 1, \theta).
\]
For \(j > 3\), let us assume \(r + 2s + \theta + j = 3d + 3\), since \(ev^*H^3 = 0\) for all \(j > 3\). Therefore, we have \(\Phi_d(0, j, r, s, \theta) = 0\).

Let us now prove Lemma 4.2. We recall that
\[
\Phi_d(1, j, r, s, \theta) = \langle c_1(L^\ast) (ev^*H^3), \, \overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle.
\]
This number is zero unless \(r + 2s + \theta + 1 + j = 3d + 3\). Let us start by considering the case when \(j = 0\). Let us assume \(r + 2s + \theta + 1 = 3d + 3\) (otherwise the number is zero). We note that
\[
\langle \mathcal{H}_L, \, \overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap a^\theta \rangle = 0,
\]
\[
\langle ev^*(H), \, \overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(0, 1, r, s, \theta)
\]
\[
= dN_{d}^{\text{planar}}(r, s, \theta).
\]

Next, let us consider the boundary divisor \(B_{d_1, d_2}\), the boundary stratum of \(\overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d)\) corresponding to a bubble map of degree \(d_1\) curve and degree \(d_2\) curve, with the marked point lying on the degree \(d_1\) component. Let us now compute the degree of the divisor
\[
B_{d_1, d_2}(r, s, \theta) := \deg\left( B_{d_1, d_2} \cap ev^*(H^3) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \right).
\]
As per the convention decided in equation (2.1), we formally declare the degree to be zero unless \(r + 2s + \theta + 1 + j = 3d + 3\). First, let us compute the number of (ordered) two component rational curves of type \((d_1, d_2)\) that lies inside \(\overline{\mathcal{M}}_{0, 0}^{\text{planar}}(\mathbb{P}^3, d) \cap a^\theta\).

This is computed in (110), in the proof of Theorem 3.3 (and in equation (2.2)), given by
\[
B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta) = \sum_{i=0}^{3} N_{d_1}^{\text{planar}}(r_1, s_1, i) \times N_{d_2}^{\text{planar}}(r_2, s_2, \theta + 3 - i).
\]

We note there that each element of \(B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta)\) corresponds to \(d_1 d_2\) bubble maps in \(\overline{\mathcal{M}}_{0, 0}^{\text{planar}}(\mathbb{P}^3, d)\), because there are \(d_1 d_2\) choices for the nodal point of the domain. Furthermore, each such bubble map corresponds to \(d_1\) bubble maps in \(\overline{\mathcal{M}}_{0, 1}^{\text{planar}}(\mathbb{P}^3, d) \cap ev^*(H)\) since each degree \(d_1\) curve intersects a hyperplane in \(d_1\) points. Hence,
\[
B_{d_1, d_2}(r, s, \theta) = \sum_{r_1 + r_2 = r, \, s_1 + s_2 = s} \binom{r}{r_1} \binom{s}{s_1} d_1^2 d_2 B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta).
\]
By equation (4.7) and (4.8), we conclude that $B_{d_1,d_2}(r,s,\theta)$ is zero if $1+r+2s+\theta = 3d+3$. Hence, equations (4.14), (4.5), (4.8) and Lemma 4.4 imply the result of Lemma 4.2 for $j = 0$.

Next, let us prove Lemma 4.2 for the case when $j = 1$. Let us assume $r+2s+\theta+1+1 = 3d+3$ (otherwise the number is zero). We note that

$$\langle \mathcal{H}_L \cdot \text{ev}^*(H), \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(0,1,r+1,s,\theta),$$  \hspace{1cm} (4.9)

$$\langle \text{ev}^*(H^2), \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(0,2,r,s,\theta).$$  \hspace{1cm} (4.10)

We note that equation (4.9) is true irrespective of the value of $r$ and $s$. Hence, using equations (4.9), (4.11), (4.13) and Lemma 4.4, we obtain the result of Lemma 4.2 for the case when $j = 1$.

Finally, let us prove Lemma 4.3. We recall that

$$\Phi_d(2,j,r,s,\theta) = \langle c_1(L^*)^2(\text{ev}^*H), \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle.$$  \hspace{1cm}

This number is zero unless $r+2s+\theta+2+j = 3d+3$. We will only consider the case when $j = 0$. Let us assume $r+2s+\theta+2 = 3d+3$ (otherwise the number is zero). We note that

$$\langle c_1(L^*) \cdot \mathcal{H}_L, \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(1,0,r+1,s,\theta),$$  \hspace{1cm} (4.11)

$$\langle c_1(L^*) \cdot \text{ev}^*(H), \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(1,1,r,s,\theta).$$  \hspace{1cm} (4.12)

Next we will show that

$$\sum_{d_1+d_2=d, \atop d_1,d_2 \neq 0} d_2^2 \langle c_1(L^*) \cdot B_{d_1,d_2}, \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = T_1(r,s,\theta) + T_2(r,s,\theta).$$  \hspace{1cm} (4.13)

We note that equations (4.11), (4.12), (4.13) and Lemma 4.4 implies Lemma 4.3.

Let us now prove equation (4.13). Let us consider the map

$$\pi : \left( \mathcal{B}_{d_1,d_2} \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1} \right) \to \bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d_1) \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1},$$

that maps to the degree $d_1$ component. Let $L_1$ denotes the pullback of the universal tangent bundle over $\bar{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d_1)$ to $B_{d_1,d_2}$ via the map $\pi$. By (5) (equation (2.10), Page 29), we conclude that on $\left( \mathcal{B}_{d_1,d_2} \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1} \right)$, we have the equality of divisors

$$c_1(L^*)|_{B_{d_1,d_2}} = c_1(L_1^*) + G.$$

Now let us compute each of the terms. Let us now consider the space

$$\bar{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3,d_2) \cap \mathcal{H}_L^{r_2} \cap \mathcal{H}_p^{s_2} \cap a^\theta.$$

Hence, we conclude that
\[
\langle c_1(L^a_1), \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap H_L \cap H_p^s \cap a^\theta \rangle
\]
\[
= \sum_{r_1+r_2=r, s_1+s_2=s} \binom{r}{r_1} \binom{s}{s_1} d_1 d_2 \langle c_1(L^a_1 \cdot \Delta_{\overline{\mathbb{P}^3}}, \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap H_L^{r_1} \cap H_p^{s_1} \times \mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap H_L^{r_2} \cap H_p^{s_2} \cap a^\theta \rangle
\]
\[
= \sum_{r_1+r_2=r, s_1+s_2=s} \binom{r}{r_1} \binom{s}{s_1} d_1 d_2 \overline{B}_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta).
\]

Here \(L^a_0\) denotes the universal cotangent bundle over the first marked point (over the moduli space of degree \(d\) curves) and \(\Delta_{\overline{\mathbb{P}^3}}\) denotes the diagonal of \(\overline{\mathbb{P}^3} \times \overline{\mathbb{P}^3}\). We note that the class of the diagonal is given by \(\sum_{i=0}^{3}\pi_i^a a^i a^{3-i}\), where \(\pi_i : \overline{\mathbb{P}^2} \times \overline{\mathbb{P}^2} \to \overline{\mathbb{P}^2}\) is the projection map to the \(i\)th factor.

Similarly, we note that
\[
\langle G, \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap H_L^r \cap H_p^s \cap a^\theta \rangle
\]
\[
= \sum_{r_1+r_2=r, s_1+s_2=s} \binom{r}{r_1} \binom{s}{s_1} d_1 d_2 \langle \Delta_{\overline{\mathbb{P}^3}}, \mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap H_L^{r_1} \cap H_p^{s_1} \times \mathcal{M}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap H_L^{r_2} \cap H_p^{s_2} \cap a^\theta \rangle
\]
\[
= \sum_{r_1+r_2=r, s_1+s_2=s} d_1 d_2 \binom{r}{r_1} \binom{s}{s_1} B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta).
\]

This proves the claim. \(\square\)

5. Transversality and degenerate contribution to the Euler class

Let us start by showing that the section of the bundle, considered in equation (3.1) is transverse to zero (restricted to \(\mathcal{M}\)). This follows from the fact that restricted to each fibre, the section is transverse to zero. Fibre wise transversality is proved in [1] Lemma 8.2, Page 105 and it is also used in [13]. We will now justify the multiplicity.

Let us recapitulate the notation of section 2. We have defined
\[
\pi : \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \to \overline{\mathbb{P}^3}
\]
to be the fiber bundle, such that the fiber over each point is given by
\[
\pi^{-1}([\xi]) := \mathcal{M}_{0,1}(\mathbb{P}^2, d).
\]

Let us abbreviate the Euler class of the rank two bundle considered in section 3 as \(E\), i.e.
\[
E := e(L^a_0 \otimes ev^* W).
\]

Note that this is a (complex) degree two cohomology class (codimension two cycle) in \(\mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)\); it is not a number.

Next, let us denote \(B \subset \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)\) to be the boundary class; i.e. it is
the closure of the space of bubble maps of type \((d_1, d_2)\) such that the marked point
is the nodal point (or more precisely, it is a map from a wedge of three spheres,
where the middle component is constant and the marked point lies on it while the
other two components are of degree \(d_1\) and \(d_2\)).

Finally, let \(C \subset \overline{\mathcal{M}}_{0,1}^{\text{planar}}(\mathbb{P}^3, d)\) denote the class determined by the closure
of the space of cuspidal curves (the cusp being on the marked point).

Let us now focus our attention on a fixed \(\mathbb{P}^2\). Let \((\mathcal{H}_\eta^p)\) denote the cycle in
\(\overline{\mathcal{M}}_{0,1}(\mathbb{P}^2_\eta, d)\) corresponding to the space of curves passing through a generic point
in \(\mathbb{P}^2_\eta\). It is shown by the authors in [1, Section 5 and Section 7], that for a fixed
\([\eta]\), the following equality of numbers

\[
E_{[\eta]} \cdot Z = (C_{[\eta]} + B_{[\eta]}) \cdot Z, \quad (5.1)
\]

hold, where \(Z\) is the cycle \((\mathcal{H}_\eta^p)^{3d-2}\). The above equality is an equality of intersection numbers in \(\overline{\mathcal{M}}_{0,1}(\mathbb{P}^2, d)\). We now note that the proof [1, Section 7] does not
in any way use any specific property of \(Z\) being equal to \((\mathcal{H}_\eta^p)^{3d-2}\); all the authors
used was that \(Z\) was a cycle of the complementary dimension to \(E_{[\eta]}\) (i.e. the
complex codimension of \(Z\) is \(3d - 2\)). Hence, we conclude that equation \((5.1)\) holds
for any cycle \(Z\) of the complementary dimension. Hence, we obtain the following
equality of cycles in \(\overline{\mathcal{M}}_{0,1}(\mathbb{P}^2_\eta, d)\), i.e.

\[
E_{[\eta]} = C_{[\eta]} + B_{[\eta]}, \quad (5.2)
\]

We now note that the cycle \(E\) restricted to each fiber is the left hand side of equation
\((5.2)\), while \((C + B)\) restricted to each fiber is the right hand side of equation \((5.2)\).
Hence, we conclude that

\[
E = C + B. \quad (5.3)
\]

as cycles in \(\overline{\mathcal{M}}_{0,1}^{\text{planar}}(\mathbb{P}^3, d)\). To see why this is so, we just need to justify one simple
fact: if \(\omega\) is a cohomology class (cycle) in \(\overline{\mathcal{M}}_{0,1}^{\text{planar}}(\mathbb{P}^3, d)\), such that its restriction
to each fiber is zero, then \(\omega\) is zero. To see why this is true, we first cover the base
\(\mathbb{P}^3\) with sufficiently small open sets \(U\) such that \(\pi^{-1}(U)\) is trivial and each of the
open sets is contractible (in other words the cohomology of \(U\) is trivial). Let \(\omega_U\)
be the restriction of \(\omega\) to \(\pi^{-1}(U) \approx U \times \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2, d)\). Given a point \(b \in U\), let

\[
i_b : \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2, d) \rightarrow U \times \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2, d)
\]

denote the inclusion map onto the fiber over \(b\). We can now easily see from the
Kunneth formula that this inclusion map induces an injective map on cohomology
(infact a bijective map). Hence, if \(\omega_U\) restricted to each fiber is zero, then \(\omega_U\) is
zero. From this we conclude that \(\omega\) is zero, since the collection of \(U\) we considered
is a covering for \(\mathbb{P}^3\). Equation \((5.3)\) now follows from this observation, by setting
\(\omega := E - (C + B)\) and using equation \((5.2)\).

Let us now consider the cycle in \(\overline{\mathcal{M}}_{0,1}^{\text{planar}}(\mathbb{P}^3, d)\)

\[
Z := \mathcal{H}_L^* \cdot \mathcal{H}_p^* \cdot a^\theta. 
\]

Choose \(r, s\) and \(\theta\) such that dimension of \(Z\) is two (i.e. \(r + 2s + \theta = 3d + 1\)).
Intersecting with equation \((5.3)\), we get the following equality of numbers

\[
E \cdot Z = C \cdot Z + B \cdot Z. 
\]

This is precisely equation \((5.4)\).
6. Low Degree Checks

In this section we subject our formula to certain low degree checks. All these numbers have been computed using our mathematica program. We will abbreviate $C^{\text{Planar}}_d (r, s)$ by $C_d (r, s)$.

First of all our formula gives us the value of zero for $C_d (r, s)$ when $d = 2$. This is as geometrically as expected since there are no conics with a cusp.

Next, in ([2]), N. Das and the first author compute the following numbers: what is $N_d (A^1_1 A^2_2, r, s)$, the number of planar degree $d$ curves in $\mathbb{P}^3$, passing through $r$ lines and $s$ points, that have $\delta$ (ordered) nodes and one cusp, for all $\delta \leq 2$. Note that here $r + 2s = \delta + 2$. For $d = 3$, and $\delta = 0$, this number should be the same as the characteristic number of genus zero planar cubics in $\mathbb{P}^3$ with a cusp, i.e. $C_d (r, s)$. We have verified that is indeed the case. We tabulate the numbers for the readers convenience:

$$C_3 (10, 0) = 17760, \quad C_3 (8, 1) = 2064, \quad C_3 (6, 2) = 240 \quad \text{and} \quad C_3 (4, 3) = 24.$$  

These numbers are the same as $N_d (A^1_1 A^2_2, r, s)$ for $d = 3$ and $\delta = 0$.

Next, we note that when $d = 4$ and $\delta = 2$, the number $\frac{1}{2!} N_d (A^1_1 A^2_2, r, s)$ is same as the characteristic number of genus zero planar quartics in $\mathbb{P}^3$ with a cusp, i.e. $C_d (r, s)$. We have verified that fact. The numbers are

$$C_4 (13, 0) = 10613184, \quad C_4 (11, 1) = 760368, \quad C_4 (9, 2) = 49152 \quad \text{and} \quad C_4 (7, 3) = 2304.$$  

These numbers are the same as $\frac{1}{2!} N_d (A^1_1 A^2_2, r, s)$ for $d = 4$ and $\delta = 2$. We have to divide out by a factor of $\delta!$ because in the definition of $N_d (A^1_1 A^2_2, r, s)$, the nodes are ordered.

7. Acknowledgement

The first author would like to acknowledge the External Grant he has obtained, namely MATRICS (File number: MTR/2017/000439) that has been sanctioned by the Science and Research Board (SERB).

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School of Mathematics, National Institute of Science Education and Research, Bhubaneswar, HBNI, Odisha 752050, India
E-mail address: ritwikm@niser.ac.in

Department of Mathematics, Indian Institute of Technology Patna, Bihta, Patna-801106, India
E-mail address: rahulks@iitp.ac.in