Non-invariant solutions of the three–dimensional semi–empirical model of the far turbulent wake

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Abstract

A semi-empirical three-dimensional model of turbulence in the approximation of the far turbulent wake behind a body of revolution in a passive stratified medium is considered. The sought quantities are the kinetic turbulent energy, kinetic energy dissipation rate, averaged density defect and density fluctuation variance. The full group of transformations admitted by this model is found. The model is reduced to the system of the ordinary differential equations due to similarity presentations obtained and B–determining equations method. System of ordinary differential equations satisfying natural boundary conditions was solved numerically. The solutions obtained agree with experimental data.

1 Introduction

The turbulence play an important role in the formation of the ocean structure [1,2]. For example, the role of turbulence on the evolution of the spatial structure of a thin phytoplankton layer was examined in [3].

Semi–empirical models of turbulence are now widely used in methods of calculation turbulent flows. However practically there are few analytical approaches to research of this models.

One of the examples of a three–dimensional free turbulent flow is a turbulent wake behind a body of revolution in a stratified medium. Sufficiently complete experimental data on the dynamics of a turbulent wake behind a body of revolution in a linearly stratified medium were obtained by Lin and Pao and presented in [4].

The turbulent wake behind an axisymmetrical body in a linearly stratified medium was numerically simulated in [5]. Based on hierarchy of semi–empirical turbulence models of second order, the numerical simulation of the dynamics of a turbulent wake in a stable stratified medium was carried out by Chernykh et al. [6]. A satisfactory agreement with experimental data [4] was obtained in [5,6].

A series of papers [11,12,16] was devoted to construction of similarity solutions of semi–empirical turbulence models. The present paper is a continuation of our
investigations. In this paper we consider three–dimensional semi–empirical model of the far turbulent wake behind an axisymmetric self–propelled body in a passive stratified medium [6–8]. Considering problem is equivalent to the problem of the development of a turbulent mixing zone in a passive stratified medium [7,8].

In section 3 we have to define the admissible differential operators of the point groups of transformations [13,14] for considering model, which will allow us to pass to the system of the degenerate elliptic equations. In section 4 we focus on the solutions of the second order to the B–determining equation [15] for degenerate elliptic equations. This gives the corresponding differential constraints and allow us to pass to the system of ordinary differential equations. In section 4 we will present the calculation results.

2 Model

To calculate the characteristics of the far turbulent wake behind an axisymmetric self–propelled body in a passive stratified medium we use the three–dimensional semi–empirical turbulence model [6–8]

\[
\frac{u_0}{\partial x} \epsilon = \frac{\partial}{\partial y} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} + \frac{\partial}{\partial z} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} - e, \\
\frac{u_0}{\partial x} \epsilon = \frac{\partial}{\partial y} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} + \frac{\partial}{\partial z} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} - C_{e^2} \frac{e^2}{e}, \\
\frac{u_0}{\partial x} \langle \rho_1 \rangle = \frac{\partial}{\partial y} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} + \frac{\partial}{\partial z} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} - C_{\rho^2} \frac{e^2}{e}, \\
\frac{u_0}{\partial x} \langle \rho^2 \rangle = \frac{\partial}{\partial y} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} + \frac{\partial}{\partial z} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} + 2C_{\rho^2} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} + 2C_{\rho^2} \frac{e^2}{\epsilon} \left( \frac{\partial}{\partial z} \frac{e^2}{\epsilon} \frac{e^2}{\epsilon} - 1 \right)^2 - C_T \frac{\langle \rho^2 \rangle}{e}.
\]

In this equations \( u_0 \) is the velocity of an incoming undisturbed flow, \( e(x, y, z) \) is the turbulent kinetic energy, \( \epsilon(x, y, z) \) is the kinetic energy dissipation rate, \( \langle \rho_1 \rangle(x, y, z) \) is the averaged density defect, and \( \langle \rho^2 \rangle(x, y, z) \) is the density fluctuation variance. The quantities \( C_e = 0.136, C_\epsilon = 0.105, C_{\rho^2} = 1.92, C_{\rho} = 0.208, C_{1\rho} = 0.087, C_T = 1.25 \) are generally accepted empirical constants [9,10].

In what follows, we assume that the velocity of an incoming undisturbed flow equals unity. The marching variable \( x \) in equations (1)–(4) acts as the time.

By analogy with [11,12], for the model (1)–(4) we have to define the admissible differential operators of the point groups of transformations.

3 Similarity solutions

Group analysis of the system (1)–(4) performed by a standard scheme [13,14]. The infinitesimal symmetry group of the model (1)–(4) is spanned by eight vector
Changing to polar coordinates

Note that the presentations (10) are satisfied all the reduced equations (6)–(9).

Next consider the linear combination of the scaling vector fields $X_6$ and $X_7$

$$Z = x \frac{\partial}{\partial x} + ay \frac{\partial}{\partial y} + az \frac{\partial}{\partial z} + 2(\alpha - 1) \frac{\partial}{\partial \xi} + (2 \alpha - 3) \frac{\partial}{\partial \eta} + \alpha \langle \rho_1 \rangle \frac{\partial}{\partial \rho} + 2 \alpha \langle \rho^2 \rangle \frac{\partial}{\partial \rho^2}.$$  

The solution of the model (1)–(4) invariant with respect to operator $Z$ has the form

$$e = x^{2 \alpha - 2} E(\xi, \eta), \quad \epsilon = x^{2 \alpha - 3} G(\xi, \eta), \quad \langle \rho_1 \rangle = x^\alpha H(\xi, \eta), \quad \langle \rho^2 \rangle = x^{2 \alpha} R(\xi, \eta),$$

where $\xi = y/x^\alpha$, $\eta = z/x^\alpha$ is the similarity variables. Substituting presentation (5) into (1)–(4), we obtain the reduced system

$$C_\epsilon E^2 \left( \frac{\partial^2 E}{\partial \xi^2} + \frac{\partial^2 E}{\partial \eta^2} \right) - C_\epsilon G^2 \left( \frac{\partial E \partial G}{\partial \xi \partial \xi} + \frac{\partial E \partial G}{\partial \eta \partial \eta} \right) + 2 C_\epsilon E \left( \frac{\partial^2 E}{\partial \xi^2} + \frac{\partial^2 E}{\partial \eta^2} \right) +$$

$$\alpha \left( \xi \frac{\partial E}{\partial \xi} + \eta \frac{\partial E}{\partial \eta} \right) + 2(1 - \alpha) E - G = 0, \quad (6)$$

$$C_\epsilon E^2 \left( \frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2} \right) + 2 C_\epsilon E \left( \frac{\partial E \partial G}{\partial \xi \partial \xi} + \frac{\partial E \partial G}{\partial \eta \partial \eta} \right) - C_\epsilon G^2 \left( \frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2} \right) +$$

$$\alpha \left( \xi \frac{\partial G}{\partial \xi} + \eta \frac{\partial G}{\partial \eta} \right) + (3 - 2 \alpha) G - C_{\epsilon 2} \frac{G^2}{E} = 0, \quad (7)$$

$$C_\rho E^2 \left( \frac{\partial^2 H}{\partial \xi^2} + \frac{\partial^2 H}{\partial \eta^2} \right) + 2 C_\rho E \left( \frac{\partial H \partial E}{\partial \xi \partial \xi} + \frac{\partial H \partial E}{\partial \eta \partial \eta} \right) + \alpha \left( \xi \frac{\partial H}{\partial \xi} + \eta \frac{\partial H}{\partial \eta} \right) -$$

$$C_\rho G^2 \left( \frac{\partial H \partial G}{\partial \xi \partial \xi} + \frac{\partial H \partial G}{\partial \eta \partial \eta} \right) - 2 C_\rho E \frac{G}{\partial \eta} + C_\rho E \frac{G}{\partial \eta} - \alpha H = 0, \quad (8)$$

$$C_{1\rho} E^2 \left( \frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} \right) + 2 C_{1\rho} E \left( \frac{\partial R \partial E}{\partial \xi \partial \xi} + \frac{\partial R \partial E}{\partial \eta \partial \eta} \right) + \alpha \left( \xi \frac{\partial R}{\partial \xi} + \eta \frac{\partial R}{\partial \eta} \right) -$$

$$C_{1\rho} G^2 \left( \frac{\partial R \partial G}{\partial \xi \partial \xi} + \frac{\partial R \partial G}{\partial \eta \partial \eta} \right) + 2 C_{1\rho} E \frac{G}{\partial \eta} + 2 C_{1\rho} E \frac{H^2}{\partial \eta} -$$

$$2 C_{1\rho} E \frac{G}{\partial \eta} - C_T \frac{G \rho}{E} - 2 \alpha R = 0. \quad (9)$$

Numerical analysis of degeneration of the far turbulent wake in a passive stratified medium show that the functions $E$ and $G$ must be presented in the form

$$E(\xi, \eta) = E(\sqrt{\xi^2 + \eta^2}), \quad G(\xi, \eta) = G(\sqrt{\xi^2 + \eta^2}). \quad (10)$$

Note that the presentations are satisfied all the reduced equations (6)–(9).

Changing to polar coordinates $\xi = r \cos(\phi)$, $\eta = r \sin(\phi)$, and by virtue of (10) the
reduced system become

\[ C_{\epsilon} \frac{E}{G} \left( EE'' + 2E'^2 - \frac{E}{G} E'G' + \frac{E}{r} E' \right) + \alpha r E' + 2(1 - \alpha) E - G = 0, \]  

(11)

\[ C_{\epsilon} \frac{E}{G} \left( EG'' - \frac{E}{G} G'^2 + 2E'G' + \frac{E}{r} G' \right) + \alpha r G' + (3 - 2\alpha) G - C_{\epsilon 2} \frac{G^2}{E} = 0, \]  

(12)

\[ C \rho \frac{E^2}{G} \left( H_{rr} + \frac{1}{r^2} H_{\phi\phi} \right) + \left( C \rho \frac{E}{G} \left( 2E' - \frac{E}{G} G' + \frac{E}{r} \right) + \alpha r \right) H_r - \alpha H + \]  

\[ C \rho \frac{E}{G} \left( \frac{E}{G} G' - 2E' \right) \sin(\phi) = 0, \]  

(13)

where \( E = E(r), \ G = G(r), \ H = H(r, \phi), \ R = R(r, \phi). \) Here and elsewhere, subscripts denote derivatives, so \( H_r = \partial H / \partial r, \) etc. We now apply the BDE method [15] to reduce the equations (13), (14) to some ordinary differential equations.

4 BDE method

Consider more general equation than (13)

\[ H_{\phi\phi} + r^2 H_{rr} + A(r) H_r + B(r) H + C(r) \sin(\phi) = 0, \]  

(15)

where \( A(r), B(r), C(r) \) are arbitrary functions. We take the B–determining equation corresponding to (15) of the form

\[ D^2_{\phi} h + r^2 D^2_{r} h + b_1(r, \phi) D_r h + b_2(r, \phi) h = 0. \]  

(16)

Here and throughout \( D_{\phi}, D_r \) are the operators of total differentiation with respect to \( \phi \) and \( r. \) The functions \( b_1(r, \phi) \) and \( b_2(r, \phi) \) are to be determined together with the function \( h. \) Note that for classical determining equations [13,14] holds

\[ b_1(r, \phi) = A(r), \quad b_2(r, \phi) = B(r). \]

We seek second order solution of (16) of the form

\[ h = H_{\phi\phi} + h_1(\phi, H, H_{\phi}). \]  

(17)

Substituting (17) into BDE (16) leads to an equation which includes derivatives of the fourth order. We can express the derivatives \( H_{\phi\phi\phi\phi}, H_{\phi\phi\phi}, H_{r\phi\phi}, H_{\phi\phi\phi}, H_{\phi\phi} \) by means of (15). Setting the coefficient of \( H_{rrr} \) equal to zero we obtain \( b_1(r, \phi) = A(r). \)
The left-hand side of (16) is a polynomial with respect to $H_{rr}$ and $H_{\phi\phi}$. This polynomial must identically vanish. Collecting similar terms we obtain the equations

$$
\begin{align*}
  h_{1,H_{r\phi}} &= 0, \quad h_{1,H_{\phi\phi}} = 0, \\
  2(A(r)H_r + B(r)H + C(r)\sin(\phi))h_{1,H_{\phi\phi}} - 2H_{\phi}h_{1,H_{r\phi}} - 2h_{1,H_{r\phi}} + B(r) - b_2(r, \phi) &= 0.
\end{align*}
$$

(18)

It is easy to show that the general solution of the equations (18) is

$$
\begin{align*}
  h_1(\phi, H, H_{\phi}) &= h_2(\phi)H_{\phi} + h_3(\phi, H), \\
  b_2(r, \phi) &= B(r) - 2h_2'(\phi).
\end{align*}
$$

Substituting the functions $b_1$, $b_2$, and $h_1$ into BDE (16) we obtain that the left-hand side of (16) is a polynomial with respect to $H_r$ and $H_{\phi\phi}$. This polynomial must identically vanish. Collecting similar terms leads to the following equations

$$
\begin{align*}
  h_{3,H_{rr}} &= 0, \\
  2h_{3,H_{r\phi}} + h_2''(\phi) - 2h_2'(\phi)h_2(\phi) &= 0, \\
  (B(r)H + C(r)\sin(\phi))h_{3,H_{\phi\phi}} - h_{3,H_{r\phi}} + (2h_2'(\phi) - B(r))h_3 + C(r)(\cos(\phi)h_2(\phi) - \sin(\phi)) &= 0.
\end{align*}
$$

(19)

The equations (19) imply

$$
\begin{align*}
  h_3(\phi, H) &= \left(\frac{1}{2}h_2(\phi)^2 - \frac{1}{2}h_2'(\phi) + h_4\right)H, \\
  h_2'(\phi) - h_2''(\phi) - 2\cot(\phi)h_2(\phi) + 2(1 - h_4) &= 0.
\end{align*}
$$

(20)

Here $h_4$ is arbitrary constant.

Clearly, that the Riccati equation (20) has the partial solution

$$
h_2(\phi) = \tan(\phi)
$$

(21)

for $h_4 = 1/2$.

Thus we find the second order solution of the BDE (16)

$$
h = H_{\phi\phi} + \tan(\phi)H_{\phi}.
$$

The corresponding differential constraint $h = 0$ has the general solution

$$
H = H_1(r)\sin(\phi) + H_2(r),
$$

(22)

where $H_1$ and $H_2$ are arbitrary functions.

Substitution (22) into equation (14) gives

$$
\begin{align*}
  C_1\frac{E^2}{G} \left(R_{rr} + \frac{1}{r^2}R_{r\phi\phi}\right) + \left(C_1\frac{E}{G} \left(\frac{E}{r} + 2E' - \frac{E'}{G'} + ar\right)\right)R_r - \\
  \left(Cr\frac{E^2}{G} + 2a\right)R + 2C_1E^2\left((rH_1' - H_1)(rH_1' + H_1 - 2r)\sin^2(\phi) + \\
  2r^2(H_1' - 1)H_2\sin(\phi) + r^2H_2'^2 + (H_1 - r)^2\right) &= 0.
\end{align*}
$$

(23)

By analogy with the case of the equation (13), consider more general equation than (23)

$$
R_{\phi\phi} + r^2R_{rr} + K(r)R_r + L(r)R + M(r)\sin^2(\phi) + N(r)\sin(\phi) + P(r) = 0,
$$

(24)
where \( K(r), L(r), M(r), N(r) \) and \( P(r) \) are arbitrary functions. The BDE method applied to equation (24) gives rise to the following results:

\[
\begin{align*}
    b_1(r, \phi) &= K(r), \\
    b_2(r, \phi) &= L(r) - \frac{8}{\sin^2(2\phi)}, \\
    h &= R_{\phi\phi} - 2\cot(2\phi)R_{\phi}, \\
    N(r) &= 0.
\end{align*}
\]

(25)

(26)

The formula (26) for the equation (23) takes the form

\[
(H'_1 - 1)H'_2 = 0.
\]

Clearly, that we must explain the case

\[
H'_2 = 0.
\]

(27)

Integrating differential constraint \( h = 0 \) corresponding to the BDE solution (25), we find

\[
R = R_1(r)\sin^2(\phi) + R_2(r),
\]

(28)

where \( R_1(r) \) and \( R_2(r) \) are arbitrary functions.

Thus in the similarity variables \( \xi \) and \( \eta \) from (22), (27), (28) we have

\[
H(\xi, \eta) = H_3(\sqrt{\xi^2 + \eta^2})\eta + H_2, \quad R(\xi, \eta) = R_3(\sqrt{\xi^2 + \eta^2})\eta^2 + R_2(\sqrt{\xi^2 + \eta^2}),
\]

(29)

where \( H_3 = H_1/\sqrt{\xi^2 + \eta^2} \) and \( R_3 = R_1/(\xi^2 + \eta^2) \).

This allow us to reduce the model (1)–(4) to the system of ordinary differential equations. Substituting presentations (10), (29) into the reduced system (6)–(9) we obtain

\[
\begin{align*}
    H_2 &= 0, \\
    E'' &= E' \left( \frac{G'}{G} - 2\frac{E'}{E} - \frac{1}{\tau} \right) + \frac{G}{C_\tau E} \left( 2(\alpha - 1) + \frac{G}{E} - \alpha \tau \frac{E'}{E} \right), \\
    G'' &= G' \left( \frac{G'}{G} - 2\frac{E'}{E} - \frac{1}{\tau} \right) + \frac{G}{C_\tau E^2} \left( 2(\alpha - 3)G + \frac{C_\tau G^2}{E} - \alpha \tau G' \right), \\
    H_3'' &= H_3 \left( \frac{G'}{G} - 2\frac{E'}{E} - \frac{5}{\tau} - \frac{\alpha \tau G}{C_\tau E^2} \right) + \frac{H_3 - 1}{\tau} \left( \frac{G'}{G} - 2\frac{E'}{E} \right), \\
    R_3'' &= R_3 \left( \frac{G'}{G} - 2\frac{E'}{E} - \frac{1}{\tau} - \frac{\alpha \tau G}{C_\tau E^2} \right) + \frac{2R_3}{\tau} \left( \frac{G'}{G} - 2\frac{E'}{E} + \frac{C_\tau G^2}{2C_1 E^3} \right) - \frac{2C_\rho H_3'}{C_{1\rho}} \left( \frac{2(H_3 - 1)}{\tau} + H_3' \right), \\
    R_2'' &= R_2 \left( \frac{G'}{G} - 2\frac{E'}{E} - \frac{1}{\tau} - \frac{\alpha \tau G}{C_\tau E^2} \right) + \frac{R_2G}{C_1 E^2} \left( \frac{C_\tau G}{E} + 2\alpha \right) - 2R_3 \frac{2C_\rho}{C_{1\rho}} (H_3 - 1),
\end{align*}
\]

where \( \tau = \sqrt{\xi^2 + \eta^2} \).
5 Calculation results

System (30)–(34) has to satisfy the conditions

\[ E' = G' = H'_1 = R'_1 = R'_2 = 0, \tau = 0, \]
\[ E = G = H_1 = R_1 = R_2 = 0, \tau \to \infty. \]  

(35)  

(36)

Conditions (35) takes into account flow symmetry with respect to the OX axis. The boundary conditions (36) imply that all functions take zero values outside the turbulent wake.

The system (30)–(34) of ordinary differential equations satisfying boundary condition (35), (36) was solved numerically. Additional difficulties are caused by the

![Figure 1: Calculated profiles as $\xi = 0$: (a) normed profile of $E$, (b) normed profile of $G$, (c) profile of $H$, (d) normed profile of $R$.](image)

Figure 1: Calculated profiles as $\xi = 0$: (a) normed profile of $E$, (b) normed profile of $G$, (c) profile of $H$, (d) normed profile of $R$.  

The system (30)–(34) of ordinary differential equations satisfying boundary condition (35), (36) was solved numerically. Additional difficulties are caused by the
fact that the coefficients of ordinary differential equations have singularities. The problem was solved by a modified shooting method and asymptotical expansion of the solution in the vicinity of the singular point \[16\].

Figure 2: Calculated functions: (a) function $E/E_0$, (b) function $G/G_0$, (c) function $H$, (d) function $R/R_0$.

Value of $\alpha$ a taken to be equal to 0.23. The results for the problem solution are illustrated in Figs. 1, 2. Figure 1 shows the profiles of the functions $E/E_0, G/G_0, H$ and $R/R_0$ as $\xi = 0$, where subscript 0 denote axial value. The functions $E/E_0, G/G_0, H$ and $R/R_0$ are plotted in Fig. 2.

The function $H(0, \eta)$ characterizing the degree of fluid mixing in the turbulent wake a given in Fig. 1c. As can be seen, the maximum value of this function slightly differ from 0.25, which is consistent with the present notions of incomplete fluid mixing in the wakes \[17\].
In Fig. 3 the axial values of the turbulent energy are compared with Lin and Pao's experimental data [4], Hassid's computational results [5] and results of numerical computations [6,7]. We have borrowed this figure from work [6] and have put the values. We can see satisfactory agreement with Lin and Pao's experimental data here as well.

Conclusion

The main results of the paper are as follows. The three-dimensional semi–empirical turbulence model of the far turbulent wake behind an axisymmetric self–propelled body in a passive stratified medium was reduced to the system of ordinary differential equations due to similarity presentations obtained and B–determining equations method. The system of ordinary differential equations satisfying natural boundary conditions was solved numerically. The solutions constructed agree with experimental data.

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References

[1] J.S. Turner, Buoyancy Effects in Fluids, 1973, Cambridge University Press, Cambridge.
[2] A.S. Monin, R.V. Ozmidov, Ocean turbulence, 1981, Gidrometeoizdat, Leningrad, (in russian).

[3] Zh. Wang, L. Goodman, Evolution of the spatial structure of a thin phytoplankton layer into a turbulent field, Mar. Ecol. Prog. Ser. 374 (2009) 57–74.

[4] J.T. Lin, Y.H. Pao, Wakes in stratified fluids, Ann. Rev. Fluid Mech. 11 (1979) 317–338.

[5] S. Hassid, Collapse of turbulent wakes in stable stratified media, Journal of Hydronautics 14 (1980) 25–32.

[6] G.G. Chernykh, A.V. Fomina, N.P. Moshkin, Numerical models of turbulent wake dynamics behind a towed body in a linearly stratified medium, Russ. J. Numer. Anal. Numer. Math. Model. 21 (2006) 395–424.

[7] Yu.D. Chashechkin, G.G. Chernykh, O.F. Voropaeva, The propagation of a passive admixture from a local instantaneous source in a turbulent mixing zone, Int. J. Comp. Fluid Dyn. 619 (2005) 517–529.

[8] O.F. Voropaeva, G.G. Chernykh, On numerical simulation of the dynamics of the turbulentized fluid regions in stratified medium, Vychisl. Tekhnol. 11 (1992) 93–104, (in russian).

[9] M.M. Gibson, B.E. Launder, On the calculation of horizontal, turbulent, free shear flows under gravitational influence, Trans. ASME, Ser. C, J. Heat Transfer 98C (1976) 81–87.

[10] W. Rodi, Examples of calculation methods for flow and mixing in stratified fluids, J. Geophys. Res. 92 (1987) 5305–5328.

[11] O.V. Kaptsov, I.A. Efremov, Invariant properties of the far turbulent wake model, Vychisl. Tekhnol. 610 (2005) 45–51, (in russian).

[12] O.V. Kaptsov, I.A. Efremov, A.V. Schmidt, Self-similar solutions of the second-order model of the far turbulent wake, J. Appl. Mech. Tech. Phys. 249 (2008) 74–78, (in russian).

[13] L.V. Ovsyannikov, Group analysis of differential equations, 1982, Academic Press, New York.

[14] N.H. Ibragimov, Transformation groups applied to mathematical physics, 1985, Reidel, Boston.

[15] V.K. Andreev, O.V. Kaptsov, V.V. Pukhnachov, A.A. Rodionov, Applications of group-theoretical methods in hydrodynamics, 1998, Kluwer, Dordrecht.

[16] O.V. Kaptsov, Yu.V. Shan’ko, Family of self–similar solutions of one model of the far turbulent wake, Computational and Information Technologies in Sciences, Engineering, and Education, Proc. Int. Conf. (Pavlodar, Kazakhstan, September 20–22, 2006), Pavlodar (2006) 576–579.
[17] O.F. Vasiliev, B.G. Kuznetsov, Yu.M. Lytkin, G.G. Cherhykh, Development of the turbulized fluid region in a stratified medium, *Izv. USSR Acad. Sci., Mech. Zhidk. Gaza* 3 (1974) 45–52.