NEW TWISTED QUANTUM CURRENT ALGEBRAS

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Abstract. We introduce a twisted quantum affine algebra associated to each simply laced finite dimensional simple Lie algebra. This new algebra is a Hopf algebra with a Drinfeld-type comultiplication. We obtain this algebra by considering its vertex representation. The vertex representation quantizes the twisted vertex operators of Lepowsky-Wilson and Frenkel-Lepowsky-Meurman. We also introduce a twisted quantum loop algebra for the Kac-Moody case and give its level one representation.

1. Introduction

Let \( g \) be the complex finite dimensional simple Lie algebra of simply laced type. Let \( \alpha_1, \cdots, \alpha_l \) be the fixed set of simple roots. The associated standard form is then defined by

\[
(\alpha_i | \alpha_j) = a_{ij},
\]

where \( A = (a_{ij}) \) is the Cartan matrix.

Let \( q \) be a generic complex number (i.e. not a root of unity). We define the \( q \)-integer \( [n] \) by

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

Other \( q \)-numbers in this paper are defined accordingly using the same base. Thus the \( q \)-factorial \( [n]! \) is defined by \( [n]! = \prod_{i=1}^{n} [i] \).

In the theory of quantum affine algebras \[ \mathfrak{D}, \mathfrak{F} \] the \( q \)-deformed Heisenberg algebra \( U_q(\hat{h}) \) inside \( U_q(\hat{g}) \) is an associative algebra generated by \( a_i(m) \) (\( m \in \mathbb{Z}^\times \)) and the central element \( \gamma \) subject to the following relations

\[
[a_i(m), a_j(n)] = \delta_{m,-n} \frac{[(\alpha_i | \alpha_j)m]}{m} \frac{\gamma^m - \gamma^{-m}}{q - q^{-1}},
\]

\[
[a_i(m), \gamma] = 0.
\]

In \[ \mathfrak{F} \] we used this \( q \)-deformed Heisenberg algebra to give the first explicit construction of level one irreducible modules of the quantum affine algebras of simply laced types. Later the author \[ \mathfrak{J} \] generalized this construction to the twisted quantum affine algebras using some twisted \( q \)-Heisenberg algebras. Recently in \[ \mathfrak{J} \] we introduced a quantum loop algebra for the quantum Kac-Moody algebra by considering the bilinear form as any symmetric form on an even lattice.

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The purpose of this paper is to study an analogous Heisenberg algebra and obtain a new twisted quantum affine algebra. Following the familiar twisting in conformal field theory \[FLM\] we will restrict the index set to be the set of odd integers in (1.2). With the help of this twisted Heisenberg algebra we will construct a new class of twisted quantum algebras and study their properties. Our construction also follows the more general twisting discussed in \[J4\].

The classical twisted vertex operators played a special role in the construction of the moonshine module \(V^\natural\) [FLM] for the Monster group. It also helped build the twisted boson-fermion correspondence for the double covering group of \(S_n\) [J3]. One can also trace its appearance in various other realizations of affine Lie algebras [LP, DL] and their quantum analogs [J5, DF].

Part of our motivation is to construct a principal bosonization of the quantum affine algebra \(U_q(\widehat{sl}_2)\), which seems quite non-trivial and is still unresolved. The new class of twisted quantum algebras constructed in this paper is different from that of the twisted quantum affine algebras given in [J2]. It is a common phenomenon that one classical object may have different \(q\)-deformations. It is unclear what will be the relation between our new algebras and the untwisted quantum affine algebras, though at least in the case of \(\widehat{sl}_2\) the author conjectures that there exists a homomorphism from \(U_q(\widehat{sl}_2)\) into the endomorphism ring of the Fock space constructed in section 2. We suggest that the verification of this conjecture would provide a principal construction for \(U_q(\widehat{sl}_2)\). The difficulty seems to be the lack of a Drinfeld-Jimbo presentation for our twisted quantum current algebra.

Replacing the underlying root lattice by the root lattice of a Kac-Moody algebra, we also introduce a twisted quantum loop algebra associated to any symmetrizable generalized Cartan matrix. The vertex operator representation of this twisted Kac-Moody loop algebra uses some \(q\)-deformation of the rational function \(\left(1 - q^{-1}\right)\). The commutation relations in this case involve \(q\)-differences of the delta function.

The twisted current algebras first appeared in our work [J1] for the simplest case of \(A_1\). The algebra is similar to certain cases of Ding and Iohara’s algebras [DI], though we have relatively stricter relations. It appears that a certain quotient of our algebra is also a quotient of their algebra. One interesting phenomenon (see [DI]) is that the Serre relations involve with non-constant coefficients. In our case we replace the Serre relations by commutation relations between two operators of same type (see (3.12, 3.13) ) as in the classical case [FLM]. I thank Jintai Ding for helpful discussions and comments on the paper.

2. Twisted vertex representations

Let \(Q\) be the root lattice of the simple finite dimensional Lie algebra \(\mathfrak{g}\) of the simply laced type with the standard bilinear form \(\langle \ , \ \rangle\) such that \(\langle \alpha_i | \alpha_j \rangle = a_{ij}\).

Let the \(q\)-deformed twisted Heisenberg algebra \(U_q(\mathfrak{h})\) be the associative algebra generated by \(a_i(m)\) \((m \in 2\mathbb{Z} + 1)\) and the central element \(\gamma\) subject to the following relations

\[
[a_i(m), a_j(n)] = \delta_{m,-n} \frac{\langle \alpha_i | \alpha_j \rangle m}{2m} \frac{\gamma^m - \gamma^{-m}}{q - q^{-1}},
\]

(2.1)

\[
[a_i(m), \gamma] = [a_i(m), \gamma^{-1}] = 0.
\]

(2.2)

The algebra \(U_q(\mathfrak{h})\) has a canonical representation realized on \(V = S(\mathfrak{h}^-)\), the space of symmetric polynomials in \(a_i(-n), n \in 2\mathbb{Z} + 1, n > 0\). This representation is
actually the induced one from the trivial representation $1_C$ with the central charge
\( \gamma = q \) of the subalgebra $U_q(h^+)$ generated by $\gamma^\pm$ and $a_i(n), n \in 2\mathbb{Z} + 1, n > 0$. The explicit action is given by
\[
\begin{align*}
\gamma^\pm &= \text{the multiplication operator by } q^\pm, \\
a_i(-n) &= \text{the multiplication operator by } a_i(-n), \\
a_i(n) &= \text{the annihilation operator by } a_i(n) \text{ subject to } (2.1),
\end{align*}
\]
for $n \in 2\mathbb{Z} + 1, n > 0$.

We also denote by $a_i(n)$ the operator corresponding to $a_i(n)$, then they satisfy the relation (2.1).

Let $\hat{Q}$ be the central extension of the root lattice $Q$ such that
\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \hat{Q} \longrightarrow Q \longrightarrow 1
\]
with the commutator map
\[
aba^{-1}b^{-1} = (-1)^{(\alpha|\beta)},
\]
where we let $a$ be the preimage of $\alpha$. In the following we will denote the preimage of $\alpha_i$ by $a_i$. Let $T$ be any $\hat{Q}$-module such that as operators on $T$
\[
a_i a_j = (-1)^{(\alpha_i|\alpha_j)} a_j a_i.
\]

Let $V_Q = S(\tilde{h}^-) \otimes T$. We introduce the twisted vertex operators acting on $V_Q$ by the following expressions:
\[
E^\pm(\alpha_i, z) = \exp(\pm \sum_{n=1, odd}^{\infty} \frac{2q^{\mp n/2}}{|n|} a_i(-n)z^n),
\]
\[
E^\pm(\alpha_i, z) = \exp(\mp \sum_{n=1, odd}^{\infty} \frac{2q^{\mp n/2}}{|n|} a_i(n)z^{-n}),
\]
\[
X^\pm_i(z) = E^\pm(\alpha_i, z)E^\pm(\alpha_i, z)q^\mp i = \sum_{n \in \mathbb{Z}} X^\pm_i(n)z^{-n}.
\]

We define the normal ordering as usual by moving the annihilation operators to the right of the creation operators. It is a routine calculation to obtain the operator product expansions for the twisted vertex operators.

**Proposition 2.1.**

\[
X^\pm_i(z)X^\pm_j(w) =: X^\pm_i(z)X^\pm_j(w) = \begin{cases} 1 & \text{if } (\alpha_i|\alpha_j) = 0 \\
\frac{z+w}{z+w} & \text{if } i = j \\
\frac{z+q^{-1}w}{z-q^{-1}w} & \text{if } (\alpha_i|\alpha_j) = -1
\end{cases}
\]

\[
X^\pm_i(z)X^\pm_j(w) =: X^\pm_i(z)X^\pm_j(w) = \begin{cases} 1 & \text{if } (\alpha_i|\alpha_j) = 0 \\
\frac{z+qw}{z-qw} & \text{if } i = j \\
\frac{z+w}{z+w} & \text{if } (\alpha_i|\alpha_j) = -1
\end{cases}
\]

In particular we have the following useful relations.
Lemma 2.1. The operators $\Phi_i(zq^{-1/2}) =: X_i^+(zq^{-1})X_i^-(z)$: and $\Psi_i(zq^{1/2}) =: X_i^+(zq)X_i^-(z)$: are given by

$$
\Phi_i(z) = \exp((q^{-1} - q) \sum_{n > 0, \text{odd}} a_i(-n)z^n) = \sum_{n \geq 0} \Phi_i(-n)z^n,
$$

(2.10)

$$
\Psi_i(z) = \exp((q - q^{-1}) \sum_{n > 0, \text{odd}} a_i(n)z^{-n}) = \sum_{n \geq 0} \Psi_i(n)z^{-n}.
$$

(2.11)

These two exponential operators can generate a complete system of basis for the $q$-Heisenberg algebra $\mathfrak{fj}_q$. Let $G_{ij}(z) = \sum_{n \geq 0} G_{n}z^n$ be the Taylor series of the function

$$
\frac{q^{(\alpha_i|\alpha_j)}z - 1 + zq^{(\alpha_i|\alpha_j)}}{q^{(\alpha_i|\alpha_j)}z + 1 - z}.
$$

(2.12)

at $z = 0$.

Lemma 2.2. The relations (1.5) for the Heisenberg algebras are equivalent to the following formal series one:

$$
\Phi_i(z)\Psi_j(w) = \Psi_j(w)\Phi_i(z)G_{ij}(\frac{z}{w}q^{-1})/G_{ij}(\frac{z}{w}).
$$

Proof. From (1.5) it follows that

$$
\Phi_i(z)\Psi_j(w)
= \Psi_j(w)\Phi_i(z) \exp \left\{ \sum_{n \geq 0, \text{odd}} \frac{2}{n} \frac{q^{(\alpha_i|\alpha_i)n} - q^{-(\alpha_i|\alpha_i)n}}{(q^n - q^{-n})(\frac{z}{w})^n} \right\}
= \Psi_j(w)\Phi_i(z)G_{ij}(z/qw)/G_{ij}(qz/w),
$$

where we use the definition of $G_{ij}(z)$.

It is easy to see that Lemma (2.2) is equivalent to the Heisenberg commutation relations (2.1). We give the complete relations satisfied by the twisted $q$-vertex operators as follows.
Theorem 2.3. The twisted vertex operators obey the relations in Lemma 2.2 and the following relations:

\[(\Phi_i(z), \Phi_j(w)) = (\Psi_i(z), \Psi_j(w)) = 0,\]
\[(\Phi_i(z) X_j^\pm(w)) (\Phi_i(z))^{-1} = X_j^\pm(w) G_{ij}\left(q^{1/2} z/w\right)^{\pm 1},\]
\[(\Psi_i(z) X_j^\pm(w)) (\Psi(z))^{-1} = X_j^\pm(w) G_{ij}\left(q^{1/2} w/z\right)^{\pm 1},\]
\[[X_i^+(z), X_j^-(w)] = [X_i^+(z), X_j^-(w)] = 0, \quad (\alpha_i | \alpha_j) = 0\]
\[[X_i^+(z), X_i^-(w)] = 2z : X_i^+(z) X_i^-(z) : \delta(-\frac{w}{z}), \quad (\alpha_i | \alpha_j) = -1\]
\[[X_i^+(z), X_i^-(w)] = \frac{2(q + q^{-1})}{q - q^{-1}} \left( \Psi_i(q^{-1/2} z) \delta(\frac{w q^{1/2}}{z}) - \Phi_i(q^{-1/2} z) \delta(\frac{w q^{-1/2}}{z}) \right),\]
\[(z - q^{1/2} z)(z + q^{1/2} w) X_i^\pm(z) X_j^\pm(w) = (q^{1/2} z - w)(q^{1/2} z + w) X_i^\pm(w) X_j^\pm(z), \quad (\alpha_i | \alpha_j) = -1,\]
\[(z - q^{1/2} z)(z + q^{1/2} w) X_i^\pm(z) X_i^\pm(w) = (q^{1/2} z - w)(q^{1/2} z + w) X_i^\pm(w) X_i^\pm(z),\]
\[+2(q + q^{-1}) \delta(-\frac{w}{z}) z^2,\]
where \(\delta(z) = \sum_{n \in \mathbb{Z}} z^n\) is the formal series of the \(\delta\)-function.

Proof. Relation (2.13) is a trivial consequence of (2.1). Similar arguments as in the proof of Lemma 2.2 show immediately relations (2.14, 2.13).

To prove the remaining relations we recall the basic property of the \(\delta\)-function: for a formal series \(f(z)\) with \(f(a)\) defined,
\[f(z) \delta(\frac{z}{a}) = f(a) \delta(\frac{z}{a}).\]

From Proposition 2.1 it follows that for \((\alpha_i | \alpha_j) = 2\)
\[[X_i^+(z), X_i^-(w)] = : X_i^+(z) X_i^-(w) : \left( \frac{z + q w z + q^{-1} w}{z - q w z - q^{-1} w} - \frac{w + q z w + q^{-1} z}{w - q w w - q^{-1} z} \right),\]
where the two rational functions in the parenthesis are formal series in \(w/z\) and \(z/w\) respectively.

Observe that
\[\frac{z + q w z + q^{-1} w}{z - q w z - q^{-1} w} = \left( \frac{z + q w}{z - q w} \right) \sum_{n \geq 1} \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} \left( \frac{w}{z} \right)^n \]
as an identity of formal series. Therefore we have that
\[[X_i^+(z), X_i^-(w)] = : X_i^+(z) X_i^-(w) : \left( \frac{z + q w}{z - q w} \right) \delta(\frac{z}{w}) - \delta(\frac{z}{w}) \left( \frac{q^{1/2} z}{w} \delta(\frac{q^{1/2} z}{w}) - \delta(\frac{q^{1/2} z}{w}) \right),\]
where we used Lemma 2.1 and the property of the \(\delta\)-function.

Relation (2.18) follows from similar consideration. For \((\alpha_i | \alpha_j) = -1\) we have
\[(z - q^{1/2} z)(z + q^{1/2} w) X_i^+(z) X_j^\pm(w) = (q^{1/2} z - w)(q^{1/2} z + w) X_j^\pm(w) X_i^+(z) = : X_i^+(z) X_j^\pm(w) : \left( (z + q^{-1} w) - (w + q^{-1} z) \right) = 0.\]
Similarly we proceed to show (2.13):
\[
(z - q^2w)(z + q^2w)x_i^+(z)x_i^+(w) - (q^{1/2}z - w)(q^{1/2}z + w)x_i^+(w)x_i^+(z)
=: X_i^+(z)x_i^+(w) : (z - q^2w)(z - q^{-2}w)\left(\frac{z - w}{z + w} + \frac{w - z}{w + z}\right)
=: X_i^+(z)x_i^+(w) : (z - q^2w)(z - q^{-2}w)(z - w)\delta(\frac{z}{w})
= 2(1 + q^2)(1 + q^{-2})\delta(\frac{z}{w})z^2
\]
which completes the proof.

\[\Box\]

3. Twisted quantum algebras

We now introduce the new quantum affine algebra in this section. Let \(g\) be a complex simple Lie algebra with the standard bilinear form \((\cdot | \cdot)\) on its root lattice \(\Lambda\) as in section 2.

Definition 3.1. The algebra \(U_q(\widehat{g})\) is an associative algebra generated by \(a_{im}, x_{in}, \gamma = q^n \ (n \in \mathbb{Z}, m \in 2\mathbb{Z} + 1)\) with the following defining relations in terms of generating functions:

\[
x_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_{in}^\pm z^{-n}
\]

\[
\phi_i(z) = \exp((q - 1) \sum_{m > 0, \text{odd}} a_{i, -m}z^m) = \sum_{n \geq 0} \phi_{i, -n}z^n,
\]

\[
\psi_i(z) = \exp((q - q^{-1}) \sum_{m > 0, \text{odd}} a_{im}z^{-m}) = \sum_{n \geq 0} \psi_{i, m}z^{-n}.
\]

The defining relation are as follows.

\[
[c, \phi_i(z)] = [c, \psi_i(z)] = [c, x_i^\pm(z)] = 0,
\]

\[
[\phi_i(z), \phi_j(w)] = [\psi_i(z), \psi_j(w)] = 0,
\]

\[
\phi_i(z)\psi_j(w) = \psi_j(w)\phi_i(z)G_{ij}(\frac{z}{w}q^{-c})/G_{ij}(\frac{z}{w}q^c),
\]

\[
\phi_i(z)x_j^\pm(w)\phi_i(z)^{-1} = x_j^\pm(w)G_{ij}(q^{c/2}z/w)^\pm 1,
\]

\[
\psi_i(z)x_j^\pm(w)\psi_i(z)^{-1} = x_j^\pm(w)G_{ij}(q^{c/2}w/z)^\pm 1,
\]

\[
[x_i^\pm(z), x_j^\pm(w)] = [x_i^\pm(z), x_j^\mp(w)] = 0, \quad (\alpha_i|\alpha_j) = 0
\]

\[
(z + w)[x_i^\pm(z), x_j^\pm(w)] = 0, \quad (\alpha_i|\alpha_j) = -1
\]

\[
[x_i^\pm(z), x_{-n}^\pm(w)] = \frac{2(q + q^{-1})}{q - q^{-1}}\left(\psi_i(q^{-c/2}z)\delta(\frac{w}{z}q^c) - \phi_i(zq^{-c/2})\delta(\frac{w}{z}q^{-c})\right),
\]

\[
(z - q^{1/2}w)(z + q^{1/2}w)x_i^\pm(z)x_j^\pm(w) = (q^{1/2}z - w)(q^{1/2}z + w)x_j^\pm(z)x_i^\pm(w), \quad (\alpha_i|\alpha_j) = -1,
\]

\[
(z - q^2z)(z + q^2z)x_i^\pm(z)x_j^\pm(w) = (q^{2}z - w)(q^{2}z + w)x_j^\pm(z)x_i^\pm(w)
\]

where \(G_{ij}\) is defined in (2.11).
In the Drinfeld realization \([\mathfrak{D}f]\) of the usual quantum affine algebras there are several relations similar to our relations, but with a different definition of \(G_{ij}(z)\). Another obvious difference is that we have more complicated commutation relations between operators \(x_i^\pm(z)\) of the same type.

The twisted algebra \(U_q(\hat{\mathfrak{g}})\) has a Hopf algebra structure. Let \(c_1 = c \otimes 1\) and \(c_2 = 1 \otimes c\).

\[
\Delta(x_i^+(z)) = x_i^+(z) \otimes 1 + \phi_i(q^{-i/2}z) \otimes x_i^+(q^i z),
\]
\[
\Delta(x_i^-(z)) = 1 \otimes x_i^+(z) + x_i^-(q^i z) \otimes \phi_i(q^{-i/2}z),
\]
\[
\Delta(\phi_i(z)) = \phi_i(q^{-i/2}z) \otimes \phi_i(q^{i/2}z),
\]
\[
\Delta(\psi_i(z)) = \psi_i(q^{i/2}z) \otimes \psi_i(q^{-i/2}z),
\]
\[
\Delta(c) = c \otimes 1 + 1 \otimes c.
\]

The antipode is given by:

\[
S(x_i^+(z)) = -\phi_i(q^{-c/2}z)x_i^+(q^c z),
\]
\[
S(x_i^-(z)) = -x_i^-(q^{-c}z)\psi_i(q^{-c/2}z)^{-1},
\]
\[
S(\phi_i(z)) = \phi_i(z)^{-1},
\]
\[
S(\psi_i(z)) = \psi_i(z)^{-1},
\]
\[
S(c) = -c.
\]

The counit map is

\[
\epsilon(x_i^+(z)) = 0, \quad \epsilon(\phi_i(z)) = 1, \quad \epsilon(\psi_i(z)) = 1
\]
\[
\epsilon(q^c) = 1.
\]

The relations \((2.14), (2.19)\) resolve the Serre relations for our algebra. We remark that our algebra is different from the algebra of Ding-Iohara \([\mathfrak{D}]\) due to our relations \((2.17), (2.18)\). If we replace the relation \((2.19)\) by the following weaker one:

\[
(z + w)(z - q^{\pm 2}w)(z + q^{\mp 2}w)X_i^+(z)X_i^+(w) = (z + w)(q^{\pm 2}z - w)(q^{\mp 2}z + w)X_i^+(w)X_i^+(z).
\]

Then this modified algebra has a common quotient with the Ding-Iohara algebra in \([\mathfrak{D}]\).

4. A twisted quantum Kac-Moody algebra and its level one representations

We can generalize the algebra to any even lattice. Let \(Q\) be an arbitrary integral lattice generated by \(\alpha_i, i = 1, \cdots, l\), equipped with a symmetric form \((\cdot | \cdot)\) such that

\[
(\alpha_i | \alpha_j) \in \mathbb{Z}, \quad (\alpha_i | \alpha_j) = 2.
\]

In the following we can view this lattice as that of a root lattice of Kac-Moody algebra \([\mathfrak{K}]\). In this sense the following construction parallels to the quantum loop algebra associated to the Kac-Moody algebra studied in \([\mathfrak{K}]\).

Let \(U_q(\hat{\mathfrak{h}}Q)\) be the associative algebra generated by \(\omega_i(m), m \in \mathbb{Z}\) and \(\gamma^{\pm 1}\) with the defining relation \((2.1), (2.2)\). Then we can consider the central extension of \(Q\) by
a finite group $< \kappa >$ similarly (cf. [FLM]). If we use $a_i$ to denote the preimage of $\alpha_i$, then we have

$$(4.2) \quad a_i a_j = (-1)^{\langle \alpha_i | \alpha_j \rangle} a_j a_i.$$

Let $T$ be any $\hat{Q}$-module such that $\kappa = \omega$, a $0(\kappa)$th primitive root of unity. The vertex space is defined accordingly:

$$(4.3) \quad V_Q = S(\tilde{h}_Q) \otimes T.$$

For each $\alpha_i$ we define the vertex operator by the same expression as in (2.7). To compute the operator product expansion we need the following $q$-analogs of series $(z - w)^r$ introduced in [J5, J7]. For $r \in \mathbb{C}$ we define the $q$-analogue to be in base $q^2$:

$$(4.4) \quad (1 - z)^r_{q^2} = \frac{\left( q^{-r+1} z ; q^2 \right)_\infty}{\left( q^r + 1 z ; q^2 \right)_\infty} = \exp(- \sum_{n=1}^{\infty} \frac{|rn|}{n} z^n),$$

where $(b; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n b)$. The twisted $q$-analogue is defined by

$$(4.5) \quad \left( \frac{1 - z}{1 + z} \right)^r_{q^2} = \frac{(1 - z)^r_{q^2}}{(1 + z)^r_{q^2}} = \exp(- \sum_{n \in \mathbb{Z}^+_0} \frac{2|rn|}{n} z^n),$$

The following OPE can be calculated similarly as in section 2.

**Proposition 4.1.**

$$X_i^\pm(z)X_j^\pm(w) =: X_i^\pm(z)X_j^\pm(w) : \left( \frac{1 - q^{\mp 1} w/z}{1 + q^{\mp 1} w/z} \right)^{(\alpha_i | \alpha_j)} q^2,$$

$$X_i^\pm(z)X_j^\mp(w) =: X_i^\pm(z)X_j^\mp(w) : \left( \frac{1 + w/z}{1 - w/z} \right)^{(\alpha_i | \alpha_j)} q^2.$$

Other commutations on the vertex representation space $V_Q$ are given as follows.

**Theorem 4.1.** The twisted vertex operators satisfy the following relations:

$$(4.6) \quad [\Phi_i(z), \Psi_j(w)] = [\Phi_i(z), \Psi_j(w)] = 0,$$

$$(4.7) \quad \Phi_i(z)X_j^\pm(w)\Phi_i(z)^{-1} = X_j^\pm(w)G_{ij}(q^{\mp 1/2} z/w)^{\pm 1},$$

$$(4.8) \quad \Psi_i(z)X_j^\pm(w)\Psi_i(z)^{-1} = X_j^\pm(w)G_{ij}(q^{\mp 1/2} w/z)^{\pm 1},$$

$$(4.9) \quad [X_i^\pm(z), X_j^\pm(w)] = [X_i^\pm(z), X_j^\mp(w)] = 0, \quad (\alpha_i | \alpha_j) = 0,$$

$$(4.10) \quad (z + w)^{-(\alpha_i | \alpha_j)} [X_i^\pm(z), X_j^\mp(w)] = 0, \quad (\alpha_i | \alpha_j) < 0,$$

$$(4.11) \quad (z - w)^{-(\alpha_i | \alpha_j)} (z - q^{\alpha_i | \alpha_j}) (z + q^{\alpha_i | \alpha_j}) X_i^\pm(z) X_i^\pm(w) = (z - w)^{-(\alpha_i | \alpha_j)} (z - w)(q^{-\alpha_i | \alpha_j} z - w)X_i^\pm(w)X_i^\pm(z), \quad (\alpha_i | \alpha_j) < 0,$$

$$(4.12) \quad (z - q^{\mp 2} w)(z + q^{\mp 2} w)X_i^\pm(z)X_i^\pm(w) = (q^{\mp 2} z - w)(q^{\mp 2} z + w)X_i^\pm(w)X_i^\pm(z) + 2(q + q^{-1})^2 \delta\left(-\frac{w}{z}\right).$$


Remark 4.2. Note that we use the factor \((z + w)^{-(\alpha_i|\alpha_j)}\) and \((z - w)^{-(\alpha_i|\alpha_j)}\) to suppress poles in \((4.10)\) and \((4.12)\) respectively. We can cancel these two polynomials and rewrite the commutation relations in terms of delta functions in a representation. It is always possible to refine the commutation relations as in the classical case \(X_2\) and quantum case \([J7]\). In the quantum case one uses the \(q\)-difference of the delta function \(\delta(z - w)\) to expand zero expressions. As in \([J7]\) we have for \(n \geq 0\)

\[
i_{z,w}(z - w)^{-n - 1} - i_{w,z}(z - w)^{-n - 1} = \partial_{q,w}^{(n)} \delta(z - w),
\]

where \(i_{z,w}\) means the expansion in the range \(|z| >> |w|\) and the \(\partial_{q,w}^{(n)}\) denote the \(q\)-divided powers of \(n\)th \(q\)-difference \(\partial_q\):

\[
\partial_q f(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}.
\]

The delta function \(\delta(z - w)\) means \(z\delta(z/w)\). Then we can rewrite for example the relation \((4.11)\)

\[
(z - q^{(\alpha_i|\alpha_j)} w)(z + q^{-(\alpha_i|\alpha_j)} w)X^\pm_j(w)X^\pm_j(z) - (q^{(\alpha_i|\alpha_j)} z - w)(q^{-(\alpha_i|\alpha_j)} z + w)X^\pm_j(w)X^\pm_j(z) = X^\pm_i(z)X^\pm_j(w) : (z + w)^{-(\alpha_i|\alpha_j)} \partial_{q,w}^{(\alpha_i|\alpha_j)} \delta(z - w),
\]

where \((\alpha_i|\alpha_j) < 0\).

Definition 4.3. Let \(\mathfrak{g}\) be a Kac-Moody algebra with the root lattice \(Q = \mathbb{Z} \alpha_1 \oplus \cdots \oplus \mathbb{Z} \alpha_l\) and the symmetric bilinear form \((\cdot | \cdot)\). The algebra \(U_q(\mathfrak{g})\) is an associative algebra with generators \(a_{im}, x^\pm_{im}, \gamma = q^i (n \in \mathbb{Z}, m \in 2\mathbb{Z} + 1)\) and following defining relations:

\[
[c, \Phi_i(z)] = [c, \Phi_j(w)] = [c, X^\pm_i(z)] = 0,
\]

\[
[\Phi_i(z), \Phi_j(w)] = [\Phi_i(z), \Psi_j(w)] = 0,
\]

\[
\Phi_i(z)X^\pm_j(w)\Phi_i(z)^{-1} = X^\pm_j(w)G_{ij}(q^{\pm c/2} z/w)^{\pm 1},\]

\[
\Psi_i(z)X^\pm_j(w)\Psi_1(z)^{-1} = X^\pm_j(z)G_{ij}(q^{\mp c/2} w/z)^{\pm 1},\]

\[
[X^\pm_i(z), X^\pm_j(w)] = [X^+_i(z), X^-_j(w)] = 0, \quad (\alpha_i|\alpha_j) = 0,
\]

\[
(z + w)^{-(\alpha_i|\alpha_j)}[X^+_i(z), X^-_j(w)] = 0, \quad (\alpha_i|\alpha_j) < 0
\]

\[
[X^+_i(z), X^-_j(w)] = \frac{2(q + q^{-1})}{q - q^{-1}} \left( \Psi_i(q^{c/2} z)\delta\left(\frac{w}{z} q\right) - \Phi_i(zq^{c/2})\delta\left(\frac{w}{z} q^{-c}\right) \right),
\]

\[
(z - w)^{-(\alpha_i|\alpha_j)-1}(z - q^{(\alpha_i|\alpha_j)} w)(z + q^{-(\alpha_i|\alpha_j)} w)X^\pm_i(z)X^\pm_j(w) = (z - w)^{-(\alpha_i|\alpha_j)-1}(q^{\alpha_i|\alpha_j}) z - w)(q^{-(\alpha_i|\alpha_j)} z + w)X^\pm_i(w)X^\pm_j(z), \quad (\alpha_i|\alpha_j) < 0,
\]

\[
(z - q^{\pm 2} w)(z + q^{\pm 2} w)X^\pm_i(z)X^\pm_i(w) = (q^{\pm 2} z - w)(q^{\mp 2} z + w)X^\pm_i(w)X^\pm_i(z)
\]

\[
+ 2(q + q^{-1})^2 \delta\left(\frac{w}{z}\right)e.
\]

where the generating functions are defined by \((3.2), (3.3)\).

We close this section with the following statement.
Conjecture 4.1. The quantum affine algebra $U_q(\hat{\mathfrak{sl}_2})$ is isomorphic to the twisted algebra generated by $a_{1m}(m \in 2\mathbb{Z} + 1), \gamma, x^n_1(n \in \mathbb{Z})$ subject to the relations (3.4-3.13) associated to $A_1$.

At $c = 1$ this statement is true by specializing $q \rightarrow 1$ [LW], however we do not know the exact isomorphism at the quantum level. The exact isomorphism will solve the so-called principal vertex construction of $U_q(\hat{\mathfrak{sl}_2})$.

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