Shell models for Hall effect induced magnetic turbulence

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**New Journal of Physics** 9 (2007) 293
Received 3 January 2007
Published 31 August 2007
Online at [http://www.njp.org/](http://www.njp.org/)
doi:10.1088/1367-2630/9/8/293

**Abstract.** The Hall effect occurs in strongly magnetized conductive media and results in non-dissipative currents perpendicular to the electric field. We discuss its influence on the magnetic field dynamics ignoring fluid motion and ambipolar diffusion. The magnetic field evolution can then be basically similar to that of the velocity field in hydrodynamic turbulence resulting in a magnetic turbulence. Shell models for the induction equation with Hall effect are constructed on the basis of the conservation of magnetic energy and helicity in the dissipation-free limit. Numerical simulations of these models indicate that a magnetic energy cascade does occur, but the time behaviour and spatial spectrum of the magnetic field are very different from those of the velocity in shell models of hydrodynamic turbulence.

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1. Introduction

The electric conductivity of strongly magnetized media is anisotropic, depends on the magnetic field strength and gives rise to nonlinearities in the induction equation. These are known as the Hall-drift and the ambipolar diffusion. The two effects are essential in different ranges of the magnetic field strength and can be taken into account separately. Moreover if the conductive matter consists of one sort of ions and electrons and if no neutral particles take part in the transport processes the ambipolar diffusion is absent [1].

In the context of astrophysics, a Hall effect dominating the other transport processes occurs in several classes of objects from weakly ionized accretion discs [2] to white dwarfs [3] and neutron stars ([4] and references therein). Amongst the latter in the form of magnetars with magnetic fields up to \(10^{15} \text{ G}\), objects are to be found in which the dominance of the Hall effect is probably most pronounced.

In general the Hall-term affects the magnetic field evolution as has been discussed by a number of authors, e.g. [5, 6]. Although the Hall-drift itself is non-dissipative it is capable of increasing the decay rate of the field [6] by redistributing magnetic energy from larger to smaller scales [7, 8]. In particular, this transfer can proceed non-locally as a large-scale background field may become unstable to small-scale perturbations [9].

A fundamentally different scenario is provided by the concept of a Hall cascade first advocated in [5]. It has its main support in the facts that the Hall effect in general creates small scales from large ones and that the equation governing the magnetic field under its influence resembles the hydrodynamic vorticity equation remarkably (although not being completely analogous to it). Hence there is good reason to suppose that in analogy to high Reynolds number hydrodynamic turbulence there is for sufficiently strong magnetic fields a Hall-induced magnetic turbulence and that it, as the former, can be described by a cascade. That is, one assumes that if energy is injected at a large scale (say by an e.m.f) it flows down to smaller ones without noticeable losses and is mediated by interactions of neighbouring scales only. This process is thought to continue up to a scale where ohmic dissipation starts to become relevant; thus the cascade stops there at the ohmic or dissipative scale. The range between the injection and dissipative scales corresponds then to the inertial range in hydrodynamic turbulence.
Under the assumption of strong turbulence, in [10] the energy spectral law of the Hall cascade was derived to be \( k^{-7/3} \) in contrast to the Kolmogorov law \( k^{-5/3} \) for strong turbulence in hydrodynamics. From considerations of weakly interacting helicoidal waves in [5] the alternative law \( k^{-2} \) was found. The weakness of the interactions is connected with the fact that they are heavily restricted by the ruling dispersion relation \( \omega \sim k|\vec{k} \cdot \vec{B}_0| \), \( \vec{B}_0 \)—the (largest-scale) background field. As a consequence, energy transfer is not necessarily local in \( \vec{k} \) space. In both models of the Hall cascade most of the energy is contained in the largest scales.

An efficient tool for the modelling of spectral transfer processes in fully developed hydrodynamic turbulence are the so-called shell models, which present a relatively low-order system of ordinary differential equations (ODEs) written for some collective variables, where each of them describes the velocity field in a range of moduli of the wavenumber vector (hence the name shell model).

We discuss possible ways of constructing shell models for the dynamic processes in magnetized media influenced by the Hall effect and introduce two specific types of them. We derive stationary cascade-like solutions in the ideal case and recover them from simulations of the full equations including dissipation. Further, the results are discussed with respect to helicity (being the second conserved quantity in the ideal case) with a focus on its possible cascades in relation to the energy cascade.

2. Statement of the problem

For strongly magnetized media with Hall effect the induction equation reads, in the absence of motions and ambipolar diffusion, in dimensionless form as

\[
\frac{\partial \vec{B}}{\partial t} = R_H^{-1} \triangle \vec{B} - \nabla \times [(\vec{\nabla} \times \vec{B}) \times \vec{B}],
\]

(1)

where \( R_H = B_0/B_e \) is the Hall parameter. Here, \( B_0 \) is the normalization field strength and \( B_e \) corresponds to that level of the magnetic field at which the Hall-drift becomes essential. There is the estimate \( B_e \approx m^*_e c/\tau_e \), valid for a single species of charged particles or vanishing interaction of the charged particles with the fluid of neutrals (see, e.g. [1, 5]), where \( e \) is the elementary charge, \( m^*_e \) the effective mass of an electron and \( \tau_e \) the electron relaxation time. Note that in the crust or the superfluid core of a highly magnetized neutron star (\( B_0 \approx 10^{13} \) G) \( R_H \) may reach some \( 10^3 \) (see [11]).

For the normalization quantities of length and time we chose the dimension of the largest scale \( l_0 \) and

\[
l_0 = \frac{l_0^2 B_e}{\pi c^2 R_0 B_0},
\]

respectively, where \( R_0 \) is the (scalar) Ohmic resistivity, i.e. the resistivity in the limit of vanishing magnetic field. The dimensionless form (1) is especially useful for an analysis based on the theory of hydrodynamic turbulence.

In the limit \( R_H \to \infty \) equation (1) has two integrals of motion, which are conserved quantities. These are the total energy \( E \) and the magnetic helicity \( H \)

\[
E = \frac{1}{2} \int \vec{B}^2 \, dV, \quad H = \int \vec{B} \cdot \vec{A} \, dV,
\]

(2)

where \( \vec{A} \) is the magnetic vector potential and integration is done over all space.

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One of the questions under discussion here concerns the possibility of a non-dissipative (or conservative) energy transfer along the spectrum analogous to the one within the inertial range of hydrodynamic turbulence, provided the Hall parameter $R_H$ is large enough. The corresponding theory for high Reynolds number hydrodynamic turbulence started from Kolmogorov’s famous ‘5/3’ law. Kolmogorov’s scenario implies a large range of scales between the macroscale $l_f$ at which energy is injected and which also contains the major fraction of the total energy, and the microscale $l_ν$, on which viscous dissipation dominates advection. In the range in between $l_f$ and $l_ν$, called the inertial range, the energy is supposed to be transferred mainly due to local nonlinear interactions, but to be almost not dissipated. ‘Local’ (in spectral terms) means that the process is dominated by interactions of vortices of similar scales, and the energy is transported scale by scale. This type of spectral transport is called a cascade. The spectral flux of energy at any scale inside the inertial range is then approximately constant and equal to the energy dissipation rate $ε_U$ or, what is the same, the input power.

Similar arguments for the magnetic energy transfer as described by equation (1), read

$$\frac{dE_l}{dt} \approx \frac{(B_l)^2}{l^2} B_l = \frac{(B_l)^3}{l^2} = ε_B = \text{const.},$$

(3)

where $B_l$ denotes some representative value of the magnetic field constituents with scale $l$ and $E_l$ is the corresponding energy. Hence

$$B_l \approx (ε_B l^2)^{1/3} = ε_B^{1/3} l^{2/3},$$

(4)

and the spectral energy density follows the ‘7/3’ law

$$\hat{E}(k) = C_H ε_B^{2/3} k^{-7/3}.$$  

(5)

Next one can try to define the Hall microscale $l_H$ in analogy to the viscous microscale $l_ν$. However, a superficial view of equation (1) may give the impression that it does not exist, since the estimates of the terms to be compared, that is, the Hall and the dissipation terms, are both inverse quadratic in the scale length. This seems to be in contrast to the hydrodynamic case, where the advection term is reciprocal to and the viscous term is inverse quadratic in the scale length. For the dimensionless microscale $l_ν$ thus

$$l_ν = Re^{-1} u_ν^{-1},$$

(6)

follows where $u_ν$ is a characteristic value of the dimensionless velocity at the scale $l_ν$ and $Re$ is the Reynolds number. However, equation (6) cannot be evaluated based on a priori information alone, but requires knowledge of the cascade law $u_ν(l)$. Analogously, comparing the nonlinear and the dissipation terms in equation (1) one has to define the Hall-microscale $l_H$ by

$$B_H = B_l(l_H) = R_H^{-1}.$$  

(7)

Clearly, there is in principle no difference between the two cases and for Hall-turbulence like for hydrodynamic turbulence the cascade should stop near the microscale. With (4) it follows in dimensionless form as

$$l_H = R_H^{-3/2} ε_B^{-1/2}.$$  

(8)
Like in hydrodynamics we will refer to the spectral range between injection and microscale as the inertial range. Beyond it, dissipation dominates and makes the spectrum decay very quickly (exponentially) to zero.

The spectral law (5) with a direct spectral flux (i.e. a flux towards the small scales) can be expected if the second integral vanishes \((H \approx 0)\). If both conserved quantities (2) are essential in the inertial range of scales (for which the constituents of the magnetic field remain large enough, that is, \(B_1 \gg R_{H}^{-1}\)), the nonlinear interactions lead to a transfer of energy and helicity toward the opposite ends of the spectrum: the dimensional arguments suggest that the energy should follow the direct cascade whereas the helicity is transferred to larger scales. Thus, a stable direct energy cascade can establish itself provided an inverse cascade of magnetic helicity is realized.

3. Shell model I

Shell models were introduced in the seventies as an attempt to describe the energy cascade in a fully developed hydrodynamic turbulence by a dynamical system of relatively few variables. In spite of their simplicity and evident inadequacy for the complicated spatial-temporal structure of real turbulent flows (one says that a shell model is a ‘caricature’ of the turbulence and often the term ‘toy-model of turbulence’ is used), these models describe some very subtle properties of high Reynolds number turbulence and became very popular during the last decade (see, for example [12]–[16]).

Shell models are intended to describe the cascade process in a large range of scales (or wavenumbers) by a series of variables \(U_n(t)\), the square of which characterizes the energy of all velocity constituents with wavenumbers \(k\) ranging from \(k_n = k_0 \lambda^n\) to \(k_{n+1}\). The parameter \(\lambda > 1\) characterizes the ratio between two adjacent scales (i.e. the width of the shell) and is often chosen as \(\lambda = 2\). The model consists of a set of ordinary differential equations for the \(U_n\) reproducing basic properties of the Navier–Stokes equation. Namely, the model has to reflect the square nonlinearity of the latter and to conserve the same integrals of motion (at least the energy) in the dissipation-less limit. One of the widely used shell models for hydrodynamic turbulence is the so-called GOY (Gledzer–Ohkitani–Yamada) model (see, e.g. [12]) which can be represented by the following set of complex ordinary differential equations

\[
(d_t + \nu k_n^2) U_n = i k_n \left( U_{n+2}^* U_{n+1}^* - \frac{\epsilon}{\lambda} U_{n+1}^* U_{n-1}^* - \frac{1 - \epsilon}{\lambda^2} U_{n-1}^* U_{n-2}^* \right) + f_n, \tag{9}
\]

\(n = 0, 1, \ldots\). Here, the asterisk indicates the complex conjugation and \(\epsilon\) is a free parameter. The term \(f_n\) models the external force and is usually specified to be non-zero only for a few shells near \(n = 0\). Equation (9) yields the model of Gledzer [17] for \(\epsilon = 5/4\) and the model of Ohkitani and Yamada [18] for \(\epsilon = 1/2\). The properties of the model (9) for different values of \(\epsilon\) were investigated by Biferale et al [12], and Frick et al [14].

We introduce a shell model in analogy to the GOY model for equation (1)

\[
(d_t + R_{H}^{-1} k_n^2) B_n = k_n^2 (B_{n+2} B_{n+1} + \eta \frac{\lambda}{\lambda^2} B_{n+1} B_{n-1} - \frac{1 + \eta}{\lambda^4} B_{n-1} B_{n-2}) + f_n, \tag{10}
\]

where real variables are used because the real and imaginary parts of the complex variables \(B_n\) would not couple due to the nonlinear interactions as can be seen by writing equation (1) in Fourier space. Here, \(f_n\) refers to an electromotive force.
Equation (10) describes the evolution of the $B_n$ which characterizes the magnetic field in the corresponding shell of wavenumbers $k_n \ldots k_{n+1}$ and guarantee the energy conservation in the dissipation-free limit for any values of parameters $\lambda$ and $\eta$. The latter can be tuned to ensure conservation of a second integral of motion, which we express for our shell model in the general form

$$W = \sum_n z^n B_n^2$$

with

$$z^n = (-1)^p k_n^\alpha, \quad p \in \{1, 2\}.$$  

For $R_H \to \infty$ the quantity $W$ is constant in time if

$$z = \frac{\eta \pm (\eta + 2)}{2(1 + \eta)}.$$  

The first solution $z_1 = 1$ (and hence $p = 2$, $\alpha = 0$) corresponds just to energy conservation independently of the value of $\eta$. Concerning the second solution $z_2 = -1/(1 + \eta)$ two cases have to be distinguished. Firstly, for $\eta < -1$

$$z_2 = \lambda^\alpha, \quad p = 2, \quad \alpha = -\log_\lambda (-\eta - 1),$$

which corresponds to the conservation of the positive definite quantity

$$W^{(1)} = \sum_n k_n^\alpha B_n^2.$$  

Secondly, for $\eta > -1$

$$z_2 = -\lambda^\alpha, \quad p = 1, \quad \alpha = -\log_\lambda (\eta + 1)$$

and the indefinite (non sign-defined) quantity

$$W^{(2)} = \sum_n (-1)^\eta k_n^\alpha B_n^2$$

is conserved.

Some particular cases can be picked out.

1. $\eta = -2$, that is, $\alpha = 0$ or $z = 1$. Again only the energy is conserved ($W^{(1)} = 2E$).

2. $\eta = \lambda - 1$, that is, $\alpha = -1$. Then $W^{(2)} = \sum_n (-1)^\eta k_n^{-1} B_n^2$ and has the dimension of the magnetic helicity $H = \sum_n A_n B_n$. Following the formalism of shell models we assume that $W^{(2)} = H$. Thus this case corresponds to the conservation of both quantities (2), and equation (10) with $\eta = \lambda - 1$ mimics the original equation (1).

3. $\eta = (1 - \lambda)/\lambda$, that is $\alpha = 1$, and $W^{(2)}$ can be related to $\int \vec{B} \cdot \nabla \times \vec{B} \, dV$. This quantity is known as the current helicity and corresponds to the hydrodynamic helicity with the velocity field replaced by the magnetic one; but it is not a conserved quantity in ideal Hall–MHD (magnetohydrodynamics).
4. \( \eta = -1 - \lambda^2 \), that is, \( \alpha = -2 \), and \( W^{(1)} \) can be related to the integral of the squared magnetic potential \( \vec{A} \). This quantity is conserved in ideal two-dimensional (2D) MHD, but meaningless in our case since the Hall effect is essentially a 3D phenomenon.

Particular importance for the dynamics of the inertial range has to be assigned to the spectral flux of the conserved quantity. The spectral flux of energy \( \Pi_n \) from shells with \( k \leq k_n \) to shells with \( k > k_n \) can be defined as the variation of the total energy of all shells with number \( j \leq n \) provided by the nonlinear terms only

\[
\Pi_n = - \sum_{j=0}^{n} d_j^{(n)} E_j = - k_n^2 \left( \frac{1 + \eta}{\lambda^2} B_{n+1} B_n B_{n-1} + B_{n+2} B_{n+1} B_n \right).
\]  

(18)

The spectral flux of the quantity \( W^{(2)} \) (see equation (17)) can be defined similarly to \( \Pi_n \) as

\[
Q_n = - \sum_{j=0}^{n} d_j^{(n)} W_j^{(2)} = (-1)^n k_n^{2+\alpha} \left( \frac{\eta - \lambda^{-\alpha}}{\lambda^2} B_{n+1} B_n B_{n-1} + B_{n+2} B_{n+1} B_n \right).
\]  

(19)

with \( W_j^{(2)} = (-1)^j k_j^{\alpha} B_j^2 \).

In the inertial range, equation (10) has two stationary solutions of the form

\[
B_n = B_0 k_n^{-\gamma}.
\]  

(20)

Being interested in solutions which provide constant flux of conserved quantities, we define them from equations (18) and (19). Presuming the energy flux is independent of the scale (of \( n \)), that is, \( \Pi_n = \text{const.} \), one gets from (18) the solution (20) with \( \gamma = -2/3 \); this corresponds to a pure energy cascade since its spectral helicity flux is zero for any value of \( \eta \). The sign of the energy flux \( \Pi_n = -B_0^3 (2 + \eta) / \lambda^2 \) corresponds to the direction of the energy transfer: the flux is positive if the energy is transferred to high wave numbers (small scales), and negative if the cascade is inverse (the energy goes to large scales). Note that the sign of the flux depends on the sign of \( B_0 \) and on the value of \( \eta \), so the solution (20) allows both directions of the energy flux. An inverse flux means that the inertial range arises at scales larger than the scale of energy injection. Of course, there must then exist a sink of energy at some scales larger than the injection scale.

The same energy flux (up to the sign) arises if the *modules* of the \( B_n \) follow the law

\[
|B_n| = B_0 k_n^{-2/3},
\]  

(21)

but their signs change according to a law \( \cdots + + - + + - + + - \cdots \). The flux is now positive if \( B_0 > 0 \), and then the energy is transferred to small scales and the energy flux is equal to the energy dissipation rate.

The second real solution for \( \gamma \) exists only if \( \eta < -1 \). It reads

\[
\gamma = \frac{-2 + \log_2 |-\eta - 1|}{3},
\]  

(22)

and yields zero spectral energy flux whereas the spectral flux of the second conserved quantity (which cannot be the helicity in this case) does not vanish. But since the sign of \( Q_n \) alternates, the direction of the flux is not defined.
Note that the solutions (20) with $\gamma = -2/3$, and (21) are fully consistent with the ‘$-7/3$’ spectral law (5) by the following consideration: from $B_n \sim k_n^{-2/3}$ it follows that $E_n \sim B_n^2 \sim k_n^{-4/3}$. The relation between this ‘shell energy’ and the spectral energy density $\hat{E}$ is given by

$$E_n = E(k_n \leq k \leq k_{n+1}) = \int_{k_n}^{k_{n+1}} \hat{E}(k) \, dk.$$  \hspace{1cm} (23)

With $\hat{E}(k) \sim k^{-p}$ it follows immediately

$$E_n \sim \frac{\lambda^{-p+1} - 1}{1 - p} k_n^{-p+1}.$$  \hspace{1cm} (24)

Hence, the ‘$-4/3$’ law of $E_n$ corresponds to the ‘$-7/3$’ law of $\hat{E}(k)$. For $\eta > -1$ another stationary solution can be found in the form

$$B_n = B_0 (-1)\eta k_n^\gamma$$  \hspace{1cm} (25)

with $\gamma$ resulting again from (22). In the case $\eta = -1$, that is, conservation of helicity, one obtains $\gamma = -1/3$. Here again, the spectral flux of energy is zero, but the spectral flux of helicity is constant and (for $B_0 > 0$) negative: $Q_n = -B_0^2(\lambda + 1)/\lambda^2$. Hence we have an inverse helicity cascade. Formally, the corresponding law coincides with the Kolmogorov ‘$-5/3$’ solution, but without providing a spectral flux of energy.

In contrast to these stationary solutions, any solutions found in numerical simulations for $\eta = -1$ and high $R_H$ are stochastic and the direction of the temporally averaged flux in the inertial range is determined by the corresponding triple correlations $\langle B_{i-1} B_i B_{i+1} \rangle$ in (18) and (19). (Here $\langle \cdots \rangle$ denotes time averaging.)

In all the numerical simulations discussed below, the model equation (10) was solved using the fourth-order Runge–Kutta integration scheme with adaptive time step. The shell width was always specified by $\lambda = (1 + \sqrt{5})/2$ (the golden section), and the maximum value of $n$ was set to 30–40.

4. Free decay

First, we study how the Hall effect influences the evolution of the energy distribution in a freely decaying system. The initial state ($t = 0$) was chosen such that the energy is concentrated in the two largest scale modes of the magnetic field, that is, in $B_0$ and $B_1$. The magnetic energy evolution for $R_H = 10^5$ and $\eta = \lambda - 1$ obtained from numerical simulations of equation (10), is shown in figure 1 (left) by the solid line. The dashed line in the same panel shows the energy evolution, if only the ohmic dissipation term is kept in (10). It is evident that the Hall effect accelerates the energy dissipation essentially at early stages of the evolution (see figure 1 (right)). Three different stages are observable within the energy evolution: the first one corresponds to the formation of an energy cascade. This stage is very short—a strong spectral flux of energy is already established at $t \approx 1$. Then a well pronounced inertial range with a spectral law close to $\hat{E}(k) \sim k^{-7/3}$ is visible in the spectrum (see figure 2).

The second stage is characterized by a slow decrease of the spectral energy flux and a corresponding slow decrease of the energy dissipation rate. This stage continues up to $t \approx 100$, 

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when the energy cascade really breaks down. During this stage only about a quarter of the initial energy leaves the system by dissipation due to energy transfer to the smallest scales. Note that the helicity does not cascade to smaller scales together with the energy. Instead the former remains almost completely concentrated in the largest scale.

After that, in the last stage of the evolution, all modes decay mainly due to ohmic dissipation that is, independently from each other. Then, the energy spectrum is characterized by a very steep slope. Nevertheless, the nonlinear Hall term affects the dissipation process even at this third stage: for \( t > 200 \) regular oscillations of the dissipation rate arose (figure 1 (right)). Note here, that the oscillating behaviour of the magnetic field during its Hall-influenced decay is a well known effect explained by the so-called ‘helicoidal oscillations’, which arise due to the interaction of poloidal and toroidal field constituents. This effect is here more pronounced at the weakly nonlinear stage of the evolution. It is surprising to find such a feature in a shell model which does not describe the
magnetic field distribution in the physical space and does not possess variables corresponding to poloidal and toroidal constituents.

5. Stationary input of energy

Next, we look for the possibility of a stationary energy cascade in the forced system. First of all, we examine the case in which the largest-scale mode is stationary maintained ($B_0 \equiv 1$) and some small energy is randomly distributed in all shells at the beginning of the simulation. That is, we in fact study the stability of the trivial solution $B_n = \delta_n B_0$, as no direct feeding from shell 0 into other ones occurs. It turns out that equation (10) has a stable fixed point of the type (21), but in the range $-1 < \eta < 0$ only. This range corresponds to conservation of the second integral of motion $W^{(2)}$ with $\alpha > 0$, which cannot be interpreted as the magnetic helicity here. We get $\gamma = -2/3$ and the alternation of the signs of the $B_n$ as described following equation (21) ensures (for $B_0 > 0$) a constant direct flux of energy (see section 3). The spectra of energy and energy flux obtained for $R_H = 3 \times 10^3$ are shown in figure 3. For $\eta > 0$ and $\eta < -1$, the energy flux vanishes and the dissipation range starts just at the largest scale indicating that the trivial solution is a stable fixed point of the system. Thus, especially for our preferred case $\eta = \lambda - 1$, the energy cascade is blocked. The trivial solution is also just the state which the system is approaching during the final stage of the free decay (see above). Secondly, we consider equation (10) with a stationary forcing at the largest scale ($f_n = f_0 \delta_{0n}$). A nontrivial solution exists again only in the range $-1 < \eta < 0$. We did not find any stationary or alternating forcing which for $\eta = \lambda - 1$ excites a permanent cascade.

6. Stochastic forcing

We have seen above that in the free decaying system the Hall effect causes a strong burst of energy flux into small scales. However, this state does not survive very long and the flux vanishes fast. But one can imagine that if the external forcing provides a sequence of alternating (non-correlated) states, the isolated bursts of energy flux can merge into a quasi-stationary spectral energy flux.
To check this idea we use a forcing in two shells with the values $f_0$ and $f_1$ which are piecewise constant in time and changed randomly after constant time intervals $t_c = 0.1$. For such a forcing a well pronounced inertial range with an almost constant spectral energy flux is established (see figure 4). This type of forcing provides a memory loss which is necessary to let the dynamics of the nonlinear system get more pronounced. Note that in the framework of the turbulent dynamo problem it was found that memory effects can drastically change the dynamo process [19].

7. Shell model II (poloidal–toroidal)

Any magnetic field can be split into poloidal and toroidal constituents, but for axisymmetric ones the resulting induction equations (here in dimensionless form) are especially simple

$$\frac{\partial \mathbf{B}^p}{\partial t} = \mathbf{R}^{-1}_H \Delta \mathbf{B}^p - \nabla \times [(\nabla \times \mathbf{B}^t) \times \mathbf{B}^p], \quad (26)$$

$$\frac{\partial \mathbf{B}^t}{\partial t} = \mathbf{R}^{-1}_H \Delta \mathbf{B}^t - \nabla \times [(\nabla \times \mathbf{B}^t) \times \mathbf{B}^t + (\nabla \times \mathbf{B}^p) \times \mathbf{B}^p]. \quad (27)$$

Of course, the Hall effect couples the poloidal and toroidal constituents. For instance, a toroidal field will be generated as a result of the Hall current associated with the term $(\nabla \times \mathbf{B}^p) \times \mathbf{B}^p$ even if the initial magnetic configuration is purely poloidal. In turn, the toroidal field contributes to the generation of a poloidal one. But it is clear that the Hall effect can generate no poloidal field if the original magnetic configuration is purely toroidal. At best, a poloidal field can emerge via an unstable growth of small poloidal perturbations. Employing a linear analysis, an instability of this kind was found for background (initial) fields the profile of which is sufficiently curved [9]. However, up to now the study of the nonlinear regime of this instability by direct numerical methods was not successful.

Now we construct a shell model for the poloidal and toroidal constituents of the magnetic field. In this model the helicity can be described by the product of both field constituents of a
given scale avoiding the artificial definition of helicity used in traditional shell models, which separates the positive and negative helicity contributions from different shells. We apply our model in studying the nonlinear interaction of the constituents and to analyse the cascade process in the essentially nonlinear regime. We use a shell model corresponding to equations (26) and (27), which was first introduced in [20],

\[
d_t B_n^0 + R_H^{-1} k_n^2 B_n^0 = k_n^2 \left( B_{n+2}^0 B_{n+1}^0 - \frac{(\lambda + 1)}{\lambda^2} B_{n+1}^0 B_{n+1}^0 + \frac{1}{\lambda^3} B_{n-1}^0 B_{n-2}^0 \right)
\]

\[+ k_n^2 \left( B_{n+2}^0 B_{n+1}^0 - \frac{(\lambda + 1)}{\lambda^2} B_{n+1}^0 B_{n+1}^0 + \frac{1}{\lambda^3} B_{n-1}^0 B_{n-2}^0 \right),
\]

\[
d_t B_n^t + R_H^{-1} k_n^2 B_n^t = k_n^2 \left( B_{n+2}^t B_{n+1}^t - \frac{(\lambda + 1)}{\lambda^2} B_{n+1}^t B_{n+1}^t + \frac{1}{\lambda^3} B_{n-1}^t B_{n-2}^t \right)
\]

\[+ k_n^2 \left( B_{n+2}^t B_{n+1}^t - \frac{(\lambda + 1)}{\lambda^2} B_{n+1}^t B_{n+1}^t + \frac{1}{\lambda^3} B_{n-1}^t B_{n-2}^t \right). \tag{28}
\]

These equations describe the evolution of the poloidal and toroidal constituents \(B_n^0\) and \(B_n^t\) in the corresponding shell of wavenumbers, defining magnetic energy and helicity as the sums

\[
E = \sum_{n=n_{min}}^{n_{max}} (B_n^{02} + B_n^{t2}), \quad H = \sum_{n=n_{min}}^{n_{max}} k_n^{-1} B_n^0 B_n^t.
\tag{29}
\]

Their conservation is guaranteed for any value of the parameter \(\lambda\). A remarkable advantage of the present shell model is a more natural definition of the magnetic helicity. Now we need not introduce the artificial factor \((-1)^n\) in order to obtain a quadratic quantity with arbitrary sign. The second advantage is the possibility of describing the energy exchange between the poloidal and toroidal constituents of the magnetic field. In particular it was shown in [20] that a poloidal field can develop by an instability from an initially purely toroidal configuration.

We study the inertial range formation at a large value of the Hall parameter. In figure 5 the results of three numerical experiments for \(R_H = 10^4\) are shown. In all cases the initial conditions \(B_0^0 = B_0^t = 1\) and \(B_1^0 = B_1^t = 0.1\) are the same and initially \(H \approx E\). In the first experiment, the range of scales \(0 \leq n \leq 30\) was considered. Here, there is no real inertial range in the spectrum (figure 5(a)). The outflow of energy from the largest scale is blocked by the requirement of helicity conservation. The situation is different if helicity can move to larger scales. Figure 5(b) shows the results of simulations with the same initial conditions, but in an extended range of scales \((-4 \leq n \leq 30\). In this case an inertial range with a spectral slope ‘\(-7/3\)’ is visible up to \(k \approx 100\). Note that the dissipation wavenumber can be evaluated as \(k_D \approx 10^4\) (just at this point the spectral density falls down). So a large range of scales exists between the inertial range and the dissipative one. This ‘intermediate dissipative range’ arises due the fact that the dissipation term and the nonlinear term in equations (28) have the same prefactor \(k_n^2\) (in the Navier–Stokes shell equations the nonlinear terms have the prefactors \(k_n\)). To obtain a well pronounced inertial range we have applied the idea of ‘hyperviscosity’, which was widely used in numerical simulations of 2D turbulence to extend the inertial range of enstrophy transport [21]. Following this idea the dissipation term \(k_n^2 B_n\) is replaced by a term like \(\sim (k_n/k_H)^m B_n\) with a typical value \(m = 4–8\) and \(k_H = 10^4\). The results obtained for \(m = 4\) are shown in figure 5(c) where the inertial range with the spectral law ‘\(-7/3\)’ really extends up to \(k \approx 10^4\).
Figure 5. Temporally averaged energy spectra of free decaying solutions of (28) (model II) for $500 \leq t \leq 600$ and $R_H = 10^4$: (a) normal viscosity and $n_{\text{min}} = 0$; (b) normal viscosity and $n_{\text{min}} = -4$; (c) hyperviscosity and $n_{\text{min}} = -4$; (d) total energy $E$ (thick solid line), helicity $H$ (thick dashed line), poloidal field energy $E_p$ (thin solid line), toroidal field energy $E_t$ (thin dashed line).

Figure 5(d), which corresponds to the first numerical experiment (figure 5(a)), illustrates two facts. Firstly, at high Hall parameter $R_H$ the helicoidal oscillations become more intensive. Secondly, in this case the dissipation of helicity practically vanishes (the corresponding thick dashed line is horizontal).

Analogously to our procedure with model I we performed an analysis of the model dynamics in the case of forced turbulence, adopting the same forcing characteristics. Again, quite similar to the results shown in section 5, steady injection of energy does not enable a cascade. Only a stochastic forcing is capable of creating an inertial range, see left panel of figure 6. Its right panel shows the helicity spectrum and confirms the relation $\hat{H}(k) \sim k^{-1} \hat{E}(k)$.

8. Conclusions

Two types of shell models are introduced for the description of the scale-by-scale or cascade-like energy transfer caused by the Hall effect. Both models are able to ensure conservation of energy and helicity in the dissipation-free limit. The first model describes the helicity in a way, accepted in traditional shell models for hydrodynamic turbulence, where the sign of a shell’s helicity contribution depends on the shell number (positive helicity is attributed to even-numbered, and negative helicity to odd-numbered shells). Our second shell model is based on the separated shell
description of the poloidal and toroidal field constituents of an axisymmetric field and defines the helicity contribution of a given shell as the product of the poloidal and toroidal field variables of this shell thus providing a more natural definition of helicity than that of the common shell models. In contrast to the first model, the second one allows study of the evolution and interaction of two physical constituents of the magnetic field.

Both models reveal a similar general behaviour showing that Hall turbulence turns out to be extremely different from hydrodynamic turbulence at least within the framework of shell models. We have found that any initial magnetic energy concentrated in larger scales triggers the cascade process. However, this effect has temporary character. The cascade actually degrades with energy injection. Only stochastic forcings are able to support stable stationary oscillations which push energy permanently into small scales.

The explanation of this behaviour of the investigated shell models (structurally so similar to those describing hydrodynamic turbulence) is connected with the properties of the integrals describing the conserved quantities. It was shown that the possibility of an energy cascade in Hall turbulence exists for a range of values of the model parameter \( \eta \), which is responsible for what second conservation law exists along with energy conservation. The spectral density of the magnetic helicity decreases with wave number faster than the spectral density of energy \( \hat{H}(k) \sim k^{-10/3} \). Hence, the energy cascade to small scales can be restricted by the magnetic helicity accumulated at the scale of forcing. Numerical simulations confirm that the inertial range with the spectral law \( \hat{E} \sim k^{-7/3} \) is established provided the Hall parameter \( R_H \) is sufficiently large and the outflow of helicity at the left (large scale) end of the inertial range is possible.

Acknowledgments

The study described in this publication was made possible in part by award no. PE-009-0 of the US CRDF and RFBR grant 06-01-00234.
References

[1] Yakovlev D G and Shalybkov D A 1991 Electrical conductivity of neutron star cores in the presence of a magnetic field-I. General solution for a multicomponent Fermi liquid-II. A free particle model of npeSigma/-l-matter Astrophys. Space Sci. 176 171–89
[2] Kunz M W and Balbus S A 2004 Ambipolar diffusion in the magnetorotational instability Mon. Not. R. Astron. Soc. 348 355–60
[3] Muslimov A G, van Horn H M and Wood M A 1995 Magnetic field evolution in white dwarfs: the Hall effect and complexity of the field Astrophys. J. 442 758–67
[4] Cumming A, Arras P and Zweibel E 2004 Magnetic field evolution in neutron star crusts due to the Hall effect and ohmic decay Astrophys. J. 609 999–1017
[5] Goldreich P and Reisenegger A 1992 Magnetic field decay in isolated neutron stars Astrophys. J. 395 250–8
[6] Urpin V and Shalybkov D 1999 Magnetohydrodynamic processes in strongly magnetized young neutron stars Mon. Not. R. Astron. Soc. 304 451–6
[7] Vainshtein S, Chitre S and Olinto A 2000 Rapid dissipation of magnetic fields due to the Hall current Phys. Rev. E 61 4422
[8] Rheinhardt M, Konenkov D and Geppert U 2004 The occurrence of the Hall instability in crusts of isolated neutron stars Astron. Astrophys. 420 631–45
[9] Rheinhardt M and Geppert U 2002 Hall-drift induced magnetic field instability in neutron stars Phys. Rev. Lett. 88 101103
[10] Vainshtein S 1973 Strong plasma turbulence at helicon frequencies Sov. Phys. — JETP 37 73
[11] Geppert U and Rheinhardt M 2002 Non-linear magnetic field decay in neutron stars theory and observations. Astron. Astrophys. 392 1015
[12] Biferale L, Lambart A, Lima R and Paladin G 1995 Transition to chaos in a shell model of turbulence Physica D 80 105
[13] Kadanoff L, Lohse D, Wang J and Benzi R 1995 Scaling and dissipation in the GOY shell model Phys. Fluids 7 617–29
[14] Frick P, Dubrulle B and Babiano A 1995 Scaling properties of a class of shell-models Phys. Rev. E 51 5582–93
[15] Frick P and Sokoloff D 1998 Cascade and dynamo action in a shell model of magnetohydrodynamic turbulence Phys. Rev. E 57 4155–64
[16] Ditlevsen P D 2000 Symmetries, invariant, and cascades in a shell model of turbulence Phys. Rev. E 62 484–9
[17] Gledzer E B 1973 System of hydrodynamic type admitting two quadratic integrals of motion Dokl. Akad. Nauk SSSR 209 1046–8
[18] Yamada M and Ohkitani K 1987 Lyapunov spectrum of a chaotic model of three-dimensional turbulence J. Phys. Soc. Japan 56 4210
[19] Fedotov S, Ivanov A and Zubarev A 2002 Memory effects in a turbulent dynamo: generation and propagation of a large-scale magnetic field Phys. Rev. E 65 036313
[20] Frick P, Stepanov R and Nekrasov V 2003 Shell model of the magnetic field evolution under Hall effect Magnetohydrodynamics 39 327–34
[21] Babiano A, Basdevant C, Legras B and Sadourny R 1987 Vorticity and passive-scalar dynamics in two-dimensional turbulence J. Fluid Mech. 183 379–97