Neural Status Registers

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Abstract. Neural networks excel at approximating functions and finding patterns in complex and challenging domains. Yet, they fail to learn simple but precise computation. Recent work addressed the ability to add, subtract, and multiply numbers but is lacking a component to drive control flow. True computer intelligence should also be able to decide when to perform what operation. In this paper, we introduce the Neural Status Register (NSR), inspired by physical Status Registers. At the heart of the NSR are arithmetic comparisons between inputs. With theoretically principled changes to physical Status Registers, the NSR allows end-to-end differentiation and learns such comparisons reliably. But the NSR also extrapolates: it generalizes to unseen data distributions. For example, the NSR trains on single digits and correctly predicts numbers that are up to 14 orders of magnitude larger. This suggests that the NSR captures the true underlying arithmetic. In follow-up experiments, we use the NSR to control the computation of a downstream arithmetic unit to learn piecewise functions. We can also learn more challenging tasks through redundancy. Finally, we use the NSR to learn an upstream convolutional neural network to compare images of MNIST digits to decide which image contains the larger digit.

1 Introduction

Neural networks are powerhouses to achieve state-of-the-art results in many interesting domains. A prime example of their strength is computer vision, where they set new records in classifying objects in images \cite{17, 30} or generating natural-looking pictures \cite{11}. Another area of success lies in natural language understanding and translation \cite{5, 32}. Plugged into a reinforcement learning setting, neural networks learn to capture the rules of various challenging games (such as chess, shogi, and go) and convincingly beat humans and other programs alike \cite{27}. With neural networks, computers start to catch up with humans in pattern recognition tasks, with which they historically struggled. Conversely, neural networks still share the human struggle with tasks that traditional computers do well, in particular with exact calculations.

Research has investigated the domain of mathematics with neural networks and achieved remarkable results in several domains \cite{19, 26}, however, with a problematic limitation: These math neural networks are often not able to extrapolate to larger, unseen numbers after training \cite{18, 22, 29}. We argue that this ability to extrapolation is essential to testify that a model truly learned the underlying logic, or whether it simply exploited some patterns in the training...
data. One could even say that extrapolation is a second-level overfitting test. Regular overfitting occurs when a model memorizes the training set so well that it cannot solve unseen instances from a test set. Missing extrapolation occurs when a model memorizes its training data distribution so well that it cannot solve instances from an unseen distribution. In particular in mathematics, it is easy to show how this behavior can happen and why this behavior is undesirable. For instance, we can train a model that learns to add two-digit numbers and generalize to unseen two-digit numbers. However, this model is unlikely able to robustly add numbers with three or more digits. Yet, as humans we know that it is possible to understand addition of numbers on a level that allows correctly handling numbers of any size. Thus, we want an arithmetic model that can do so as well. While the above early work achieved remarkable results, the authors observed struggles with extrapolation. For example, Saxton et al. [26] report that their models perform much worse on dedicated extrapolation intervals; Lample and Charton [19] see their models perform much worse for expressions that are generated differently than the training set.

It is for this reason that recent research addressed architectures that emphasize extrapolation. A seminal work in this field is the Neural GPU by Kaiser and Sutskever [15]. Neural GPUs handle binary numbers as bit sequences to compute addition and multiplication. Remarkably, some of their models handle numbers that are 100 times larger than those seen during training. Trask et al. [31] devise a component, that is later improved upon by Madsen and Johansen [23], to perform addition, subtraction, and multiplication. Both components show much stronger extrapolation than other state-of-the-art neural architectures. These three examples tackle the problem of how to compute with neural networks.

To the best of our knowledge, we are the first to tackle what to compute with neural networks. In other words, we want to learn the control logic to complement and control computation units. We take inspiration from physical computer circuits, that realize control logic by branching control flow into two paths. The core principle underlying branching are comparisons between numbers, whose results decide which branch to take. The commonly used comparisons are \( >, <, \geq, \leq, =, \neq \) which are evaluated through a Status Register. More precisely, the Status Register uses a sign and a zero bit to track if the last arithmetic operation resulted in a negative number or zero, respectively. For example, to conditionally execute a program branch if two numbers are equal, a CPU subtracts these two numbers and then checks the zero bit in the Status Register: the CPU takes one branch for a set bit and the other on for an unset bit.

In this work, we present a Neural Status Register (NSR) that allows such comparison-based control logic. With theoretically principled changes to physical Status Registers, the NSR is differentiable end-to-end. Therefore, the NSR can be learned from just input-output training examples using gradient descent. In particular, the NSR learns the right operands and the needed comparison to control downstream layers. We experimentally show that our NSR construction can learn all commonly-needed comparisons quickly, and from little data. We can use the NSR to control downstream layers such as Neural Arithmetic Units [23].
Through computational redundancy, the NSR can reliably learn more complex functions. We can also learn upstream layers, exemplified by learning to compare MNIST digits in a convolutional architecture. Moreover, our NSR architecture shows strong extrapolation capabilities, handling numbers up to 14 orders of magnitude larger than those seen during training.

2 Neural Status Registers

![High-Level NSR architecture](image)

Fig. 1: High-Level NSR architecture. $V_1$ and $V_2$ each learn one operand $o_1$ and $o_2$, respectively. The subtraction of these operands is fed through `sign` and `zero` bits. Weights $W^\pm$, $W^0$ for the bit activations and a bias term $b$ learn the comparison. Resulting logits $z_1$ and $z_2$ produce the results after a softmax.

Figure 1 shows the high-level idea of the NSR. The NSR is a neural layer that takes a vector $x$ of numbers as an input. The NSR transforms $x$ into a two-neuron output. One of those output neurons encodes the likelihood of a comparison being true and the other of the same comparison being false. The NSR learns what numbers of the input vector to compare and how to compare these numbers. First, parameters $V_1$ and $V_2$ learn the operands as weighted sums of the elements of $x$. The `sign` and `zero` bits of the operand difference are computed. Second, weights $W^\pm$, $W^0$ for the bits and a bias $b$ learn the type of comparison. A softmax activation transforms this weighted sum into final likelihoods.

We now look at the derivatives for the learnable parameters using backpropagation. Here, we assume learning with the Mean Absolute Error (MAE), and that the MAE is directly applied to the NSR’s output neurons $y_0$ and $y_1$. We further assume that the true label is $y_{true} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If the target label were $y_{true} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the derivatives would only swap signs. We allow slight slack in the notation and
allow derivatives of vectors. Appendix C contains the complete calculations. The
derivatives for the bit-weighing parameters $b$, $W^\pm$, $W^0$ are:

$$\frac{\partial L}{\partial b} = -2y_0y_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{\partial L}{\partial W^\pm} = -2y_0y_1 \text{sign} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ (1)

$$\frac{\partial L}{\partial W^0} = -2y_0y_1 \text{zero} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ (2)

The derivatives for the two operand selection parameters $V_1$ and $V_2$ are:

$$\frac{\partial L}{\partial V_1} = 2y_0y_1 (\text{sign}'(W^1_1 - W^1_0) + \text{zero}'(W^0_1 - W^0_0)) \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$ (3)

$$\frac{\partial L}{\partial V_2} = -2y_0y_1 (\text{sign}'(W^1_1 - W^1_0) + \text{zero}'(W^0_1 - W^0_0)) \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$ (4)

These equations require derivatives for the $\text{sign}$ and $\text{zero}$ bits. The physical bits
are discrete and non-differentiable, so we require a continuous approximation.

2.1 Differentiability through Approximation

To allow end-to-end differentiability, we need to relax the sign and zero functions
to a continuous space. We need to adapt both the output space and the input
space since most CPUs work with integers only and have a dedicated circuit for
floating-point numbers.

Moreover, Equations (1) and (2) suggest to not use the original value range
of $[0; 1]$. In these equations, bit values appear as factors in the derivatives for
$W^\pm$ and $W^0$. Therefore, having a bit output 0 would cause vanishing gradients.
For uniformly distributed inputs, $\text{sign}$ would be 0 around half the time, $\text{zero}$
would almost always be 0, having no gradient in all those cases is not acceptable.

Therefore, we shift the unset value of the $\text{sign}$ and $\text{zero}$ bit from 0 to $-1$
instead to provide gradient signals over all data points. Figures 2a and 2b show
the discrete bit values as dots, plus proposed relaxations as solid blue lines.
We suggest to approximate $\text{sign}$ through $\tanh$ function and $\text{zero}$ through the
(rescaled) derivative of $\tanh$ function.

$$\hat{\text{sign}}(x) = \tanh x$$

$$\hat{\text{zero}}(x) = 1 - 2(\tanh x)^2$$

2.2 Reliability through Redundancy

In preliminary experiments, we observed that the NSR does not always learn, for
example, the comparison $\neq$. We hypothesize that the NSR finding the solution is
dependant on having a good parameter initialization. The same phenomenon has
been observed for the related XOR function \cite{1,9}. Frankle and Carbin \cite{9} show that having neural networks with redundant hidden units increases the chances of learning XOR. Not because more than two hidden neurons are needed, but because the chance of having two correctly-initialized neurons increases. These two neurons can then learn the function. Kaiser and Sutskever \cite{15} observe a similar phenomenon. They create redundant parameter sets which are gradually forced to converge onto one set of values. We propose a similar redundancy scheme for the NSR. We extract multiple independent operands and compute independently weighted bits for each of them. Next, we sum these bits into one joint signal. Eventually all but one sign and zero bit — the ones with right initialization — are pruned. We show in a later experiment how this approach enables the NSR to learn difficult problems reliably.

2.3 Extrapolation through Regularization

To promote extrapolation, we employ a three-way regularization loss $L_{\text{Reg}}$. The loss ensures each operand consists of exactly one element of the input vector, uses only one of the redundant bit structures, and the output is either 0 or 1.

$$L_{\text{reg}} = L_{\text{op}}(V_1) + L_{\text{op}}(V_2) + L_{\text{prune}} + L_{\text{out}}$$

Inspired from the sparsity loss in Madsen and Johansen \cite{23}, the loss $L_{\text{op}}$ biases both $V_1$ and $V_2$ to become one-hot vectors, so that they choose each exactly one element form the input. Regularization moves the largest element towards 1 and every other element towards 0. The following equation shows the loss in the simple case without redundancy.

$$L_{\text{op}}(V) = \frac{1}{|V|} \sum_{j \neq x} (V_{ij}) + (1 - V_{ix})$$

with $x = \arg \max_i (V_i)$
Bit redundancy increases the probability of having correctly initialized weights (a winning ticket). Frankle and Carbin [9] argue when having a winning ticket, weights not included in the winning ticket can be pruned without loss of performance. The loss $L_{\text{prune}}$ does this pruning by biasing all weights in $W^\pm$ and $W^0$ to become 0 apart from the largest absolute value per column. This weight has the strongest influence on the final prediction, indicative of the overall most helpful signal. The loss is computed as:

$$L_{\text{prune}} = \frac{1}{|W^\pm| + |W^0|} - 4\left(\sum_{j=1}^{2} \sum_{i \neq x_j} |W^\pm_{ij}| + \sum_{k=1}^{2} \sum_{i \neq x_k} |W^0_{ik}|\right)$$

with $x_j = \arg\max_n |W^\pm_{nj}|$, $x_k = \arg\max_n |W^0_{nk}|$

Ultimately, the NSR offers branched execution of downstream layers. To this end, the NSR should output a clear decision in the form of a one-hot encoded output. In particular, all real occurring bit combinations of the sign and zero bit should have a one-hot output. These combinations are $(-1, -1)$, $(1, -1)$, or $(0, 1)$, meaning the first operand is smaller, greater, or equal to the second. The loss $L_{\text{out}}$ biases the larger output towards 1 and the smaller output towards 0. The following equation shows $L_{\text{out}}$ for the simple architecture without redundancy:

$$\text{out}(s, z) = \text{softmax}(\text{bias} + s \cdot W^\pm + z \cdot W^0)$$

$$L_{\text{out}} = \frac{1}{6} \sum_{s, z \in \text{bits}} \sum_{i=1}^{2} \min(\text{out}(s, z)_i, 1 - \text{out}(s, z)_i)$$

with $\text{bits} = \{(-1, -1), (1, -1), (0, 1)\}$

2.4 A Simple Example

We conclude the NSR presentation with a brief example on how the NSR operates. In this example, we assume to learn comparisons between the two entries $x_1$ and $x_2$ of a two-element input vector $x$. Thus $V_1$ and $V_2$ also each have two elements, where $V_{ij}$ denotes the weight of using $x_i$ in operand $o_j$. Even in the case without redundancy, the NSR can easily learn all six comparisons $>$, $<$, $\geq$, $\leq$, $=$, $\neq$, with weights as shown in Table 3. For example, when expressing $x_1 > x_2$, the NSR uses $x_1$ as $o_1$ and $x_2$ as $o_2$. Then, the NSR defaults to the comparison being false ($b_0 = -50$) unless the sign bit says otherwise ($W^\pm_0 = 100$). Similarly, we can learn the other five comparisons. Note that the last three negations are negations of the first three. Some ways to negate a comparison are: negating every entry in $W^\pm$ and $W^0$; swapping the weights for $o_1$ and $o_2$; or swapping the weights for $y_0$ and $y_1$ (which is done in Table 3).
Table 3: Values for the learnable weights of the NSR in Figure 1 to solve different comparisons. Inputs are vectors with two elements $x_1$ and $x_2$.

| Comparison $\square$ | $V_{11}$ | $V_{12}$ | $V_{21}$ | $V_{22}$ | $W_0^\pm$ | $W_1^\pm$ | $W_0^0$ | $W_1^0$ | $b_0$ | $b_1$ |
|----------------------|---------|---------|---------|---------|----------|----------|--------|--------|------|-----|
| $x_1 > x_2$          | 1       | 0       | 0       | 1       | 100      | 0        | 0      | 0      | -50  | 0   |
| $x_1 < x_2$          | 1       | 0       | 0       | 1       | -100     | 0        | -100   | 0      | 50   | 0   |
| $x_1 \neq x_2$       | 1       | 0       | 0       | 1       | 0        | 0        | -100   | 0      | 50   | 0   |
| $x_1 \leq x_2$       | 1       | 0       | 0       | 1       | 0        | 100      | 0      | 0      | 0    | -50 |
| $x_1 \geq x_2$       | 1       | 0       | 0       | 1       | 0        | -100     | 0      | -100   | 0    | 50  |
| $x_1 = x_2$          | 1       | 0       | 0       | 1       | 0        | 0        | 0      | -100   | 0    | 50  |

3 Experiments and Results

We seek to answer the following questions in our experimental evaluation:\footnote{1} Can the NSR learn all commonly-used comparisons on numbers? Can we use those comparisons to the control execution of downstream layers? What is the impact of having redundant bit structures? Can we also learn upstream layers?

For all our experiments we initially supervise using the Mean Absolute Error (MAE). After this error becomes small ($\ll 0.1$), we start regularizing. This approach of first learning good weights and then regularizing them to an extrapolating solution is taken from Madsen and Johansen\cite{23}. We start with a learning rate of 0.1, which we decay with a factor of 0.7 every 1000 epochs, training for $5 \cdot 10^4$ epochs total. Weights are updated with the Adam algorithm\cite{16}.

We use a vanilla two-layer feedforward network, which we refer to as 2NN, as a reference architecture. The hidden units are activated with sigmoid. Theoretically, 2NN can model any comparison with already two hidden units. We attach the architecture of 2NN plus weights to solve all comparisons in Appendix A. Unless noted otherwise, we use 10 redundant bits for the NSR. For fairness, we increase the hidden dimension of 2NN to 20, matching the number of parameters.

3.1 Learning Binary Comparisons

In a first experiment, we expect NSR and 2NN to learn comparisons and extrapolate to larger numbers. For every type comparison $\square \in \{>, <, \geq, \leq, =, \neq\}$, training examples are all the tuples $(x_1, x_2, y^\square)$, with $x_1, x_2 \in [-9 : 9]^2 \subset \mathbb{Z}^2$ and $y^\square$ is 1 if $x_1 \square x_2$ is true. Otherwise $y^\square$ is 0. Out of those examples, $\neq$ and $=$ are the most challenging\footnote{1} and also come with an unbalanced training set — having 95%/5% label split. Note, that there is no test set, we test their model purely on their extrapolation performance on unseen examples from unseen distributions.

\footnote{1} Code is supplemented as archive nsr.zip and can be accessed online at https://anonymous.4open.science/r/003add9b-9684-44a8-b93c-1ac5f3db42ba/
After training, we test the learned model if it can perfectly predict all training cases, counting a prediction as correct if it is at most 0.1 off from the target value. Next, we test extrapolation using the series \( a_n = \begin{cases} 10^{\lceil \frac{n}{2} \rceil} \cdot 10 & \text{if } n \text{ odd} \\ 10^{\lceil \frac{n}{2} \rceil} \cdot 10 & \text{if } n \text{ even} \end{cases} \). Every sequence element corresponds to one extrapolation test. For one \( a_n \), we create a test set from all tuples \((x_1, x_2, y_1)\) and \((x_2, x_1, y_2)\) where \( x_1 = a_n \) and \( x_2 \in [a_n - 3; a_n + 3] \subset \mathbb{Z} \). We only count an extrapolation test as successful if all 14 test cases are correctly solved. We show the success rates in Table 4, averaged over 100 runs with different seeds.

We can see that the NSR outperforms 2NN, which completely fails to learn \( = \) and \( \neq \). Conversely, the NSR also learns these two comparisons reliably. We attribute this to the zero bit, which provides a direct equality signal to the output layer. On the other hand, the NSR also shows strong extrapolation to large unseen numbers. For the test range shown in Figure 4 the NSR generalizes perfectly up to \( a_{10} = 10^6 \). We attach further extrapolation results in Appendix B showing that extrapolation even works up to \( 10^9 \). The cause for failing for higher numbers lies in \( V_1 \) and \( V_2 \), whose entries tend to differ from 0 or 1 by a small error \( \epsilon \). Going to high orders of magnitude, these small errors start playing a significant role. However, we can tackle this \( \epsilon \) through longer training, suggesting this is a numerical rather than a conceptual problem. For example, doubling the number of iterations allows extrapolation to numbers up to 14 orders of magnitude larger. Achieving such extrapolation through single-digit training suggests that the NSR truly captures the underlying arithmetic.

Table 4: Results for binary comparisons. The numbers in the “Train” and following columns (showing extrapolation) denote the success ratio averaged over 100 runs. A success is predicting the training or the extrapolating data perfectly.

| Comparison | Model | Train | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_5 \) | \( a_6 \) | \( a_7 \) | \( a_8 \) | \( a_9 \) | \( a_{10} \) |
|------------|-------|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( > \)    | NSR   | 1.0   | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    |
|            | 2NN   | 1.0   | 1.0    | 0.99   | 0.58   | 0.24   | 0.13   | 0.01   | 0.0    | 0.0    | 0.0    | 0.0    |
| \( < \)    | NSR   | 1.0   | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    |
|            | 2NN   | 1.0   | 1.0    | 1.0    | 0.51   | 0.22   | 0.07   | 0.01   | 0.0    | 0.0    | 0.0    | 0.0    |
| \( \geq \) | NSR   | 1.0   | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    |
|            | 2NN   | 1.0   | 1.0    | 0.97   | 0.47   | 0.28   | 0.12   | 0.01   | 0.0    | 0.0    | 0.0    | 0.0    |
| \( \leq \) | NSR   | 1.0   | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    |
|            | 2NN   | 1.0   | 1.0    | 0.97   | 0.59   | 0.26   | 0.09   | 0.03   | 0.0    | 0.0    | 0.0    | 0.0    |
| \( = \)   | NSR   | 0.99  | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   | 0.99   |
|            | 2NN   | 0.0   | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    |
| \( \neq \) | NSR   | 1.0   | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    | 1.0    |
|            | 2NN   | 0.0   | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    | 0.0    |
3.2 Downstream Learning with Piecewise Functions

In the second experiment, we want to use the NSR to control the execution of downstream layers, in this case Neural Arithmetic Units (NAU) [23], to learn piecewise-defined functions. Therefore, we instantiate two NAUs and gate each with one NSR output neuron. Ideally, each NAU can learn a different function piece and the NSR controls which piece to use. We similarly wire up two NAUs with a 2NN instance. We learn three different functions. First, $\text{abs}(x_1, x_2) = \begin{cases} x_1 - x_2 & \text{if } x_1 > x_2 \\ x_2 - x_1 & \text{else} \end{cases}$ validates downstream control via the NSR outputs. Next, $\text{max}(x_1, x_2, x_3, x_4, x_5) = \begin{cases} x_3 & \text{if } x_3 > x_1 \\ x_2 & \text{else} \end{cases}$ tests the NSR to learn to pick the relevant values out of the input vector $x$. The third function $f(x_1, x_2, x_3, x_4, x_5) = \begin{cases} x_5 + 4 & \text{if } x_1 > x_2 \\ x_4 - x_3 & \text{else} \end{cases}$ is the most challenging and combines both aspects: learning downstream computation and operand selection jointly. Also, $f$ is discontinuous in the integer space, with large gaps. For example $f(1, 1, 1, 1, 1)$ is 0 but $f(2, 1, 1, 1, 1)$ is 5. For $\text{abs}$, we train with all pairs of numbers from the interval $[-9; 9]$. For both $\text{max}$ and $f$, we generate all permutations of five numbers from the interval $[-5; 4]$ as training data for a total of $10^5$ data points. We evaluate extrapolation similar to the previous experiment. We use $a_n$ or a number from $[a_n - 3; a_n + 3]$ for numbers involved in the comparison and draw the remaining numbers randomly out of the interval $[a_n - 10; a_n + 10]$. Table 5 shows the success rates: a success is fitting the training data perfectly or predicting all instances in the extrapolation test set, respectively.

Both models learn the absolute value well. However, 2NN struggles to learn the other two functions. While usually fitting the training set, most solutions do not even extrapolate to $a_2$ and none to $a_5$. On the other hand, the NSR solves the training set more reliable and also extrapolates: almost always to $a_{10}$ for $\text{max}$ and up to $a_7$ for $f$. We refer to Appendix B to investigate the extrapolation space of NSR further. This time, unlike previously, training longer did not directly improve extrapolation for $f$.

3.3 Upstream Learning with MNIST

In this next experiment, we investigate if we can use the NSR as an intermediate layer and still learn upstream layers. For this, we set up an MNIST digit comparison task. We compare two images if the digit shown in the first image is greater than the one in the second image. We feed both MNIST digits through a convolutional neural network (based on the pytorch example). In the last layer, we replace the logsoftmax with a softmax operation. The softmax activations

\footnotesize
\begin{itemize}
  \item 2 https://github.com/pytorch/examples/blob/234bcff4a2d8480f156799e6b9baae06f7ddc96a/mnist/main.py
\end{itemize}
Table 5: Results for piecewise functions. The numbers in the “Train” and following columns (showing extrapolation) denote the success ratio averaged over 100 runs. A success is predicting the training or the extrapolating data perfectly.

| Function | Model | Train | a1 | a2 | a3 | a4 | a5 | a6 | a7 | a8 | a9 | a10 |
|----------|-------|-------|----|----|----|----|----|----|----|----|----|-----|
| abs      | NSR   | 1.0   | 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0 |
|          | 2NN   | 1.0   | 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0| 1.0 |
| max      | NSR   | 1.0   | 1.0| 1.0| 1.0| 1.0| 1.0| 0.99| 0.99| 0.98| 0.98|
|          | 2NN   | 1.0   | 0.44| 0.23| 0.07| 0.01| 0.0| 0.0| 0.0| 0.0| 0.0| 0.0 |
| f        | NSR   | 0.92  | 0.93| 0.93| 0.93| 0.93| 0.92| 0.91| 0.8 | 0.35| 0.01| 0.0 |
|          | 2NN   | 0.78  | 0.61| 0.25| 0.03| 0.02| 0.0 | 0.0| 0.0| 0.0| 0.0| 0.0 |

are multiplied with the digit vector \(d = \[0 1 2 3 4 5 6 7 8 9\]\) and fed into the NSR.

For our task, we split the dataset into batches of 50 images, and training data are all pairs of images per batch. We run without learning rate decay and average over 10 runs. There is no extrapolation in this task, thus we train with the MAE and use no regularization. Figure 6 shows the test accuracy. The shaded region is the one-standard-deviation interval. For reference, we plot the test accuracy for the vanilla digit-recognition task using the pytorch example model.

Figure 6 shows that the NSR learns to compare MNIST images with high accuracy, suggesting the NSR does not deprive upstream layers of learning signals. Not reaching the level of the digit classification task is expected since the supervision signal is much weaker in the comparison task. The same supervision signal can originate from different digits and the same digit can also emit different supervision signals. In the vanilla MNIST task, every class is supervised by has a distinct signal that facilitating learning. For example, every digit showing a four is supervised with the true label 4, whereas in the comparison task, a 4 can be supervised with both a 0 and or 1.

3.4 Studying the Impact of Redundancy

We evaluate the chances of having a problem-solving initialization (winning ticket, refer to Section 2.2) given different levels of redundancy. We test this on the comparisons \(=\), \(\neq\) and on the functions max and \(f\). We rerun these tasks with a varying number of redundant bit constructs ranging from 1 to 15 in increments of 1. For each number, we report the success ratio of fitting the training set averaged over 100 runs after training for \(10^4\) epochs. Figure 7 shows the results. Across all four examples, we see how increasing redundancy leads to higher success ratios. At around 10 redundant constructs, there seems to be a saturation point where adding more redundancy does not straightforwardly equate higher accuracy. On the other hand, we hypothesize, that the increased model complexity caused the accuracy decline. For a fair comparison, we trained all model sizes with the same configuration and for the same time. However, especially larger models
might need a different configuration, in particular a longer training time than $10^4$ epochs. We suggest to closely examine this phenomenon in further study.

4 Handling Floating-Point Comparisons

So far, we looked at computations involving integers. Also physical CPUs use their status register for integers only, as they pass floating-point numbers off to a dedicated unit. Now we want to look if and how we can use the NSR to handle control flow with floats. The most important change here is the minimum difference $\delta$ between two different numbers. In the scope of integers, $\delta$ is 1, but this number can be smaller for floats, causing two problems. First, function values of $\tilde{\text{zero}}$ could be positive for non-equal values. Whereas equality in the integer case can be mainly derived from the sign of $\tilde{\text{zero}}$ — with tuning through $W^\pm$ — equality with small $\delta$ needs to be understood from small distances in values of $\tilde{\text{zero}}$, making learning much harder. Second, function values of $\tilde{\text{sign}}$ approach 0 as $\delta$ approaches 0. In return, the gradient for $W^\pm$ including $\tilde{\text{sign}}$ as a factor starts to vanish (see Equation (1)). This makes training $W^\pm$ slow.

We can tackle both problems by rescaling the function inputs with a constant $\lambda$. These functions are shown as dotted orange lines in Figures 2a and 2b and have the equations:

$$\tilde{\text{sign}}_\lambda(x) = \tanh(\lambda \cdot x) \quad \tilde{\text{zero}}_\lambda(x) = 1 - 2 \cdot \tanh(\lambda \cdot x)^2$$

We examine the efficacy of this approach through an experiment, where we most settings from Section 3.1 with only $10^4$ epochs. Additionally, we independently vary the minimum distance $\delta = 10^{-i}$ and the scaling $\lambda = 10^i$ for $i \in \{0, 1, 2, 3\}$. Tables 8a and 8b show the results.

The previously reported problems manifest themselves in entries below the main diagonal (especially in the first column). In Table 8a we observe that learning $= \text{always fails} \lambda < \delta^{-1}$. These are exactly cases described in the first
problem, where equality and non-equality share the same sign from \( \hat{\text{zero}} \). Table 8b shows the second problem when learning >. In the most bottom-left entries, learning starts becoming less reliable due to vanishing gradients; even failing completely for \( \lambda = 1, \delta = 10^{-3} \).

The rescaling trick remedies both problems, achieving the best results when \( \lambda = \delta^{-1} \). We observe that learning becomes less robust when \( \lambda > \delta^{-1} \). The reasons for this are the derivatives of \( V_1 \) and \( V_2 \). They contain the derivatives for \( \text{sign} \) and \( \hat{\text{zero}} \) as factors (see Equation (3) and (4)). Both derivatives approach 0 exponentially fast, leading to vanishing gradients when the difference is large. At the same time, \( \lambda \) is also a factor in these derivatives but only contribution linearly. Therefore, \( \lambda \) cannot counteract the vanishing gradient. In practice, we observe that learning still works well for > and decently for =, thus leaving a theoretically principled solution for future work.

Table 8: Results for binary comparisons when varying the minimum distance \( \delta \) and scaling factor \( \lambda \). Entries denote the success ratio of perfectly solving the training set, averaged over 100 runs.

| \( \delta \) | \( \lambda \) | 10^0 | 10^1 | 10^2 | 10^3 |
|---|---|---|---|---|---|
| 10^0 | 0.99 | 0.82 | 0.36 | 0.35 |
| 10^{-1} | 1.0 | 1.0 | 0.9 | 0.95 |
| 10^{-2} | 0.97 | 0.76 | 0.41 |
| 10^{-3} | 0.86 | 0.8 |

(a) Learning the = comparison

| \( \delta \) | \( \lambda \) | 10^0 | 10^1 | 10^2 | 10^3 |
|---|---|---|---|---|---|
| 10^0 | 1.0 | 1.0 | 0.9 | 0.95 |
| 10^{-1} | 1.0 | 1.0 | 0.97 | 0.94 |
| 10^{-2} | 0.98 | 1.0 |
| 10^{-3} | 0.98 | 1.0 |

(b) Learning the > comparison

5 Related Work

Neural Mathematics. There are several works that design components allowing mathematical computation. Some recent results were achieved with state of the art sequence-to-sequence models, e.g., Lample and Charton [19], Saxton et al. [26]. Both architectures generalize to an unseen test set but fail extrapolation. Saxton et al. [26] solve various mathematical tasks having explicit extrapolation tests (e.g., larger numbers). Their model performs consistently worse than on the interpolation tests. Lample and Charton [19] use sequence-to-sequence models to solve differential equations or function integration. Their models are sometimes outperforming math solvers but perform worse when predicting test problems that are generated differently than the training set. This is an indicator that the model captures patterns specific to the training data generation but not specific to the problem.
For this reason, other research tackled designing components tailored to extrapolation. Kaiser and Sutskever [15] introduce the neural GPU (improved by Price et al. [24], Freivalds and Liepins [10]) for binary addition and multiplication. Neural GPUs model input numbers as sequences of bits and process the input using a recurrent structure. Neural GPUs achieve very good extrapolation: training from 20 bit long numbers allows to extrapolate to up to 2000 bit long sequences. However, Neural GPUs are unreliable to train as only very few models achieve this extrapolation and chances to train successfully further decrease when using decimal (instead of binary) sequence elements. Thus, Neural GPUs are hard to fit into a larger neural architecture since the combined model would likely be very hard or impossible to train. Therefore, we represent the input in the NSR as real numbers. Computationally, this limits us to the realization of these numbers by the programming language. The experimental results show that the NSR can be learned reliably, also for challenging tasks and especially in conjunction with other layers. The idea of input representations as reals was explored before in Neural Arithmetic Logic Units (NALU) by Trask et al. [31]. NALUs support several arithmetic operations and emphasize on extrapolation performance. However, Madsen and Johansen [23] show that the NALU does not reliably learn, for example, multiplication. They continue to create the Neural Arithmetic Unit (NAU), which has a higher success ratio of learning addition, subtraction and multiplication. NAUs also typically find solutions that extrapolate to their extrapolation test set. However, the extrapolation range is rather small, not even increasing in one order of magnitude. In contrast, the NSR can handle inputs several orders of magnitude larger than those seen during training. At that, the NSR offers control flow, which can complement a computation model such as the NAU for performing more complex computation.

**Differential Storage.** Having differentiable storage is an orthogonal but interesting concept for building a differentiable computer. Such storage was pioneered by Graves et al. [12], building a Neural Turing Machine (NTM). This machine possesses a writable memory tape, managed by a recurrent controller. A new version of the NTM [20] also supports memory for programs. Other works in this field picked up on the idea of Turing Machines [35] or proposed other storage concepts [33, 28, 13], such as queues [14].

**Algorithm Inference.** The above algorithms are often evaluated on simple symbolic tasks, such as copying or reversing an input sequence of symbols. This introduces the field of algorithm inference, which tries to infer whole programs from input-output examples. Zaremba et al. [34] approach this problem with reinforcement learning. Other examples build on ideas from subprogram composition [25, 21, 8], functions and recursion [2, 8], or logic reasoning [7, 9]. Orthogonal to this, Devlin et al. [4] learn to exploit similarities between tasks for higher data efficiency. In the field of algorithm inference, there is limited work on tackling numerical problems, apart from Chen et al. [3], who learn mathematical expressions. Missing are architectures to infer numerical algorithms, where the NSR can become an integral component providing control logic.
6 Discussion and Conclusion

In this paper, we introduced the Neural Status Registers (NSR). The NSR is a neural layer, that allows building control logic into a neural architecture. Inspired from physical circuits, the NSR realizes this logic through numerical comparisons on reals. Through theoretically principled adoptions, the NSR is end-to-end trainable and can learn which numbers to compare and how to compare them. Experiments show that the NSR learns these comparisons reliably, and most importantly: extrapolates. The NSR correctly predicts answers for numbers up to 14 orders of magnitude larger indicating that the NSR finds the true underlying arithmetic. The reliability of the NSR allows for integration with other components. Indeed, further experiments show that the NSR can successfully control downstream Neural Arithmetic Units to compute piecewise functions and train upstream Convolutional Neural Networks to compare MNIST digits. Providing an initially redundant architecture, which is later pruned away, helps to robustly learn complex functions. It remains an open question if redundancy scales indefinitely or if there is a sweet spot. Like its physical counterpart, the NSR performs especially well for integer comparisons but can handle floating-point values as well, although not on the same level of performance. Future extensions could look into improvements in this dimension.

As the main direction for going forward, we want to explore the usage of the NSR for algorithm inference. In the Related Work section we discussed the required components for such an architecture. Follow up work could look at combining the NSR for control flow with other components such as the NAU for computation and the NTM for storage to build a differentiable numeric computer.

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A  Weight assignments for binary comparisons

Figure 9 shows the sample architecture for 2NN with labels for each weight. The following Table 10 shows how to assign weights, such that 2NN can learn to represent any of the six comparisons. Note that the last three comparisons are just the negations of the first three. We can realize these as in Table 3.

![Two-layer feedforward network with one hidden layer using sigmoid activation and an output layer using softmax activation.](image)

Fig. 9: Two-layer feedforward network with one hidden layer using sigmoid activation and an output layer using softmax activation.

| Comparison | $V_{11}$ | $V_{12}$ | $V_{21}$ | $V_{22}$ | $b_1$ | $b_2$ | $W_{1t}$ | $W_{2t}$ | $W_{1f}$ | $W_{2f}$ | $b_t$ | $b_f$ |
|------------|---------|---------|---------|---------|-------|-------|---------|---------|---------|---------|-------|-------|
| $x_1 > x_2$ | 100     | 0       | -100    | 0       | -50   | -50   | 100     | 0       | 0       | 0       | -50   | 0     |
| $x_1 < x_2$ | -100    | 0       | 100     | 0       | -50   | -50   | 100     | 0       | 0       | 0       | -50   | 0     |
| $x_1 \neq x_2$ | 100    | -100    | -100    | -100    | 100   | 50    | 0       | 100     | 0       | 0       | 0     | -50   |
| $x_1 \leq x_2$ | 100     | 0       | -100    | 0       | 50    | -50   | 0       | 100     | 0       | 0       | 0     | -50   |
| $x_1 \geq x_2$ | -100    | 0       | 100     | 0       | 50    | -50   | 0       | 100     | 0       | 0       | 0     | -50   |
| $x_1 = x_2$ | 100     | -100    | -100    | -100    | 100   | 50    | 0       | 100     | 0       | 100     | 0     | 0     | -50   |
B Further Details on NSR Extrapolation

Table 11 shows the ability to extrapolate to further numbers. On average, all comparisons extend to $10^9$ in most cases. However, we can improve this by training longer. The third (>100) and fourth (>1000) column show the difference for training the > comparison for training for 50000 and 100000 epochs, respectively. We can see that training longer directly translates to higher extrapolation capabilities. Table 12 shows further extrapolation ranges for piecewise functions. Here training longer on $f$ does not increase performance.

Table 11: Extrapolation performance of the NSR for comparisons. If present, subscripts indicate training epochs in thousands (default 50000).

| Extrapolation | $>$ | $>$100 | $<$ | $\geq$ | $\leq$ | $=$ | $\neq$ |
|---------------|-----|--------|-----|-------|-------|-----|-------|
| Training      | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^2          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^3          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^4          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^5          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^6          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^7          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^8          | 1.0 | 1.0    | 1.0 | 1.0   | 1.0   | 0.99| 1.0   |
| 10^9          | 0.9 | 1.0    | 0.87| 0.84 | 0.93  | 0.59| 0.7   |
| 10^10         | 0.08| 1.0    | 0.08| 0.04 | 0.14  | 0.11| 0.1   |
| 10^11         | 0.03| 1.0    | 0.04| 0.04 | 0.04  | 0.04| 0.08  |
| 10^12         | 0.03| 1.0    | 0.03| 0.04 | 0.04  | 0.04| 0.07  |
| 10^13         | 0.03| 1.0    | 0.03| 0.04 | 0.04  | 0.04| 0.07  |
| 10^14         | 0.03| 1.0    | 0.03| 0.04 | 0.04  | 0.04| 0.07  |
| 10^15         | 0.03| 1.0    | 0.03| 0.04 | 0.04  | 0.04| 0.07  |
Table 12: Extrapolation performance of the NSR for piecewise functions. If present, subscripts indicate training epochs in thousands (default 50000).

| Extrapolation | abs | max | f   | f_{100} |
|---------------|-----|-----|-----|---------|
| Training      | 1.0 | 1.0 | 0.92| 0.92    |
| $10^2$        | 1.0 | 1.0 | 0.93| 0.93    |
| $10^3$        | 1.0 | 1.0 | 0.93| 0.93    |
| $10^4$        | 1.0 | 1.0 | 0.91| 0.91    |
| $10^5$        | 1.0 | 0.99| 0.35| 0.35    |
| $10^6$        | 1.0 | 0.98| 0.0 | 0.0     |
| $10^7$        | 1.0 | 0.91| 0.0 | 0.0     |
| $10^8$        | 1.0 | 0.66| 0.01| 0.0     |
| $10^9$        | 0.95| 0.2 | 0.01| 0.0     |
| $10^{10}$     | 0.79| 0.0 | 0.0 | 0.0     |
C Derivatives

We assume the loss to be the Mean Absolute Error, and the true target label to be $y_{true} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. First, we compute the loss with respect to the derivation outputs:

$$\frac{\partial L}{\partial y} = \begin{bmatrix} \frac{\partial L}{\partial y_0} \\ \frac{\partial L}{\partial y_1} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

With $y_0, y_1$ being softmax activations, they always fulfill $y_0 < 1$ and $y_1 > 0$. Softmax has the added benefit of avoiding the case $y_{pred} = y_{true}$ when the loss would not be differentiable. We can now get the derivatives of the logits as:

$$\frac{\partial L}{\partial z} = \begin{bmatrix} \frac{\partial L}{\partial y_0} \frac{\partial y_0}{\partial z_0} + \frac{\partial L}{\partial y_1} \frac{\partial y_1}{\partial z_1} \\ \frac{\partial L}{\partial y_0} \frac{\partial y_0}{\partial z_1} + \frac{\partial L}{\partial y_1} \frac{\partial y_1}{\partial z_0} \end{bmatrix} = \begin{bmatrix} -y_0(1 - y_0) - y_0 y_1 \\ y_1(1 - y_1) + y_0 y_1 \end{bmatrix} = 2y_0y_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now we can get the derivatives for the learnable parameters $b, W^\pm, W^0$:

$$\frac{\partial L}{\partial b} = \frac{\partial L}{\partial z} \frac{\partial z}{\partial b} = -2y_0y_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\frac{\partial L}{\partial W^\pm} = \frac{\partial L}{\partial z} \frac{\partial z}{\partial W^\pm} = -2y_0y_1 \begin{bmatrix} -\text{sign} \\ 1 \end{bmatrix}$$

$$\frac{\partial L}{\partial W^0} = \frac{\partial L}{\partial z} \frac{\partial z}{\partial W^0} = -2y_0y_1 \begin{bmatrix} -\text{zero} \\ 1 \end{bmatrix}$$

Similarly, we can get the derivatives with respect to the $\text{sign}$ and $\text{zero}$ functions:

$$\frac{\partial L}{\partial \text{sign}} = \frac{\partial L}{\partial z_0} \frac{\partial z_0}{\partial \text{sign}} + \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial \text{sign}} = 2y_0y_1(W_1^+ - W_0^+)$$

$$\frac{\partial L}{\partial \text{zero}} = \frac{\partial L}{\partial z_0} \frac{\partial z_0}{\partial \text{zero}} + \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial \text{zero}} = 2y_0y_1(W_1^0 - W_0^0)$$

Giving the derivatives with respect to the two operands $o_1$ and $o_2$ as:

$$\frac{\partial L}{\partial o_1} = \frac{\partial L}{\partial \text{sign}} \frac{\partial \text{sign}}{\partial o_1} + \frac{\partial L}{\partial \text{zero}} \frac{\partial \text{zero}}{\partial o_1} = 2y_0y_1(\text{sign}'(W_1^+ - W_0^+) + \text{zero}'(W_1^0 - W_0^0))$$

$$\frac{\partial L}{\partial o_2} = \frac{\partial L}{\partial \text{sign}} \frac{\partial \text{sign}}{\partial o_2} + \frac{\partial L}{\partial \text{zero}} \frac{\partial \text{zero}}{\partial o_2} = -2y_0y_1(\text{sign}'(W_1^+ - W_0^+) + \text{zero}'(W_1^0 - W_0^0))$$

And finally, for $V_1$ and $V_2$, assuming input $x$ has $k$ entries, the derivatives are:

$$\frac{\partial L}{\partial V_1} = \frac{\partial L}{\partial o_1} \frac{\partial o_1}{\partial V_1} = \frac{\partial L}{\partial o_1} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

$$\frac{\partial L}{\partial V_2} = \frac{\partial L}{\partial o_1} \frac{\partial o_1}{\partial V_2} = \frac{\partial L}{\partial o_1} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$