REMARKS ON SYMPLECTIC GEOMETRY

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Abstract. We survey the progresses on the study of symplectic geometry past four decades. We briefly deal with the convexity properties of a moment map, the classification of symplectic actions, the symplectic embedding problems, and the theory of Gromov-Witten invariants.

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1. Introduction

The relatively-new subject, symplectic geometry, has been studied past four decades. In the 1970s, Arnold, Marsden and Weinstein took this subject and began to study symplectic manifolds. In the early 1980s, the convexity properties of the moment map were investigated by Atiyah, Guillemin, Sternberg, Mumford and Kirwan [1, 2, 30, 31, 36].

In 1985 Gromov introduced the pseudo-holomorphic (or J-holomorphic) curve technique into symplectic geometry and the symplectic capacity to prove the famous Nonsqueezing Theorem in his seminal paper, “Pseudoholomorphic curves in symplectic manifolds” [27]. Thereafter pseudo-holomorphic curves motivated the occurrence of the new mathematical subject, symplectic topology which has been developed by

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the McDuff school [45, 46]. Pseudo-holomorphic curves are used as a tool in the four-dimensional symplectic topology, the study of the symplectic embeddings, and in the theory of the Gromov-Witten invariants. Taubes [62, 63] discovered that in four dimensional symplectic manifolds, the Gromov-Witten invariants coincide with the Seiberg-Witten invariants. The theory of Floer homology (cf. [73]) is based on pseudo-holomorphic curves with boundary lying on a Lagrangian submanifold. This led on to the notion of the Fukaya category which is closely related to the homological mirror symmetry formulated by Kontsevich. Later the symplectic capacity which is a notion of monotonic symplectic invariant was pioneered by Ekeland and Hofer [17], and then has been developed by Hofer and his collaborators (cf. [33, 34]) from the angle of dynamical systems and Hamiltonian dynamics. After the middle 1980s there have been many major developments on symplectic geometry, for example, symplectic embedding problems, classification of symplectic and Hamiltonian group actions, the theory of Gromov-Witten invariants, quantum cohomology, Floer theory, symplectic capacities, Fukaya category, homological mirror symmetry etc.

The purpose of this article is to survey briefly various results related on symplectic geometry past four decades. The author had an opportunity to learn recent progresses on symplectic geometry and symplectic topology while he was writing this survey paper. This article is organized as follows. In section 2, we give basic notions, definitions, and examples of symplectic manifolds. We mention Marsden-Weinstein-Meyer theorem and Duistermaat-Heckman theorem. In section 3, we review some fundamental results, which are Darboux theorem, Weinstein Lagrangian neighborhood theorem, Weinstein tubular neighborhood theorem, Gromov’s non-squeezing theorem, Poincaré-Birkhoff theorem and Arnold’s conjecture. In section 4, we review the convexity properties of a moment map obtained by Atiyah, Guillemin, Sternberg and Kirwan. Using the convexity properties of a moment map, Delzant [12] classified all symplectic-toric manifolds in terms of a set of very special polytopes, so-called the Delzant polytopes. We state the Delzant’s work. In the final section we survey various results related to symplectic geometry obtained during past three decades. We briefly deal with the following subjects: the classification problem of symplectic and Hamiltonian actions, the symplectic embedding problems and the theory of Gromov-Witten invariants. Finally we mention the recent work of Shen and Zhou on the Landau-Ginzburg/Calabi-Yau correspondence for elliptic orbifold projective lines.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{Q}^+$ (resp. $\mathbb{R}^+$) denotes the set of all positive rational (resp. real) numbers. We denotes by $\mathbb{Z}_+$ (resp. $\mathbb{Q}_+$, $\mathbb{R}_+$) the set of all non-negative integers (resp. rational numbers, real numbers). $\mathbb{Q}^\times$ (resp. $\mathbb{R}^\times$, $\mathbb{C}^\times$) denotes the group of nonzero rational (resp. real, complex) numbers. The symbol “:=" means that the expression on the right is the definition of that on the left. For two positive integers $k$ and $l$, $F^{(k,l)}$ denotes the
set of all $k \times l$ matrices with entries in a commutative ring $F$. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $\sigma(A)$ denotes the trace of $A$. For any $B \in F^{(k,l)}$, $B^\dagger$ denotes the transpose of $B$. For a positive integer $n$, $I_n$ denotes the identity matrix of degree $n$. For a complex matrix $A$, $\overline{A}$ denotes the complex conjugate of $A$. $\text{diag}(a_1, \cdots, a_n)$ denotes the $n \times n$ diagonal matrix with diagonal entries $a_1, \cdots, a_n$. For a smooth manifold $M$, we denote by $C_c(M)$ (resp. $C_c^\infty(M)$) the algebra of all continuous (resp. infinitely differentiable) functions on $M$ with compact support, and by $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on $M$. $H$ denotes the Poincaré upper half plane and $\mathbb{D}$ denotes the Poincaré disk.

$$H_g = \{ \Omega \in \mathbb{C}^{(g, g)} | \Omega = {}^t\Omega, \quad \text{Im} \Omega > 0 \}$$

denotes the Siegel upper half plane of degree $g$. If $z \in \mathbb{C}$, we put $e(z) := e^{2\pi i z}$. The contraction $i(X)$ of a $k$-form $\alpha$ with a vector field $X$ is defined to be the $(k-1)$-form given by

$$(i(X)\alpha)(X_1, \cdots, X_{k-1}) := \alpha(X, X_1, \cdots, X_{k-1}).$$

2. Basic notions, definitions and examples

Let $(M, \omega)$ be a symplectic manifold of dimension $2n$, that is, a smooth manifold of dimension $2n$ equipped with a closed ($d\omega = 0$), nondegenerate ($\omega^n \neq 0$) 2–form $\omega$. The notion of symplectic structures arose in the Hamiltonian formulation of the theory of classical mechanics. A classical mechanical system can be modelled by the phase space which is a symplectic space. On the other hand, a quantum mechanical system is modelled by a Hilbert space. Each state of the system corresponds to a line in a Hilbert space.

**Definition 2.1.** A diffeomorphism $\phi : M_1 \to M_2$ of two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ is called a symplectomorphism if $\phi^* \omega_2 = \omega_1$. We denote by $\text{Symp}(M, \omega)$ the group of all symplectomorphisms $\phi : (M, \omega) \to (M, \omega)$.

**Definition 2.2.** Let $(M, \omega)$ be a symplectic manifold. A submanifold $N$ of $M$ is called a Lagrangian submanifold if, for each point $p \in N$, $T_pN$ is a Lagrangian subspace of $T_pM$ and $\dim M = 2 \dim N$.

**Definition 2.3.** Let $(M, \omega)$ be a symplectic manifold. An almost complex structure $J = \{ J_p \}$ on $M$ is said to be compatible with $\omega$ (or $\omega$-compatible) if

$$g_J := \{ g_p : T_pM \times T_pM \to \mathbb{R}, \quad p \in M \}$$

defined by

$$g_p(X, Y) := \omega_p(X, J_pY), \quad X, Y \in T_pM, \quad p \in M$$

is a Riemannian metric on $M$. 


The following facts (CJ1)-(CJ3) are well known.

(CJ1) Any symplectic manifold \((M, \omega)\) has a \(\omega\)-compatible almost complex structure.

(CJ2) The set of all \(\omega\)-compatible almost complex structures on a symplectic manifold \((M, \omega)\) is path-connected and contractible.

(CJ3) Let \((M, \omega)\) be a symplectic manifold equipped with a \(\omega\)-compatible almost complex structure \(J\). Then any almost complex submanifold \(N\) of \((M, J)\) is a symplectic submanifold of \((M, \omega)\).

Let \((M, h)\) be a Kähler manifold of dimension \(n\) with hermitian metric \(h = (h_{ij})\). Then its Kähler form \(\omega\) is given by

\[
\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\overline{z}_j, \quad [\omega] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R}).
\]

Then \(\omega\) is a symplectic form on \(M\). Thus \((M, \omega)\) is a symplectic manifold. According to the positivity of \(h\), we see that the symplectic form \(\omega\) satisfies the positivity condition

\[
\omega(X, JX) > 0 \quad \text{for all } X \in \mathfrak{X}(M).
\]

Moreover the complex structure \(J\) satisfies the property \(J^* \omega = \omega\), where \(J^* \omega(X, Y) := \omega(JX, JY)\) for all \(X, Y \in \mathfrak{X}(M)\). The symplectic volume form is

\[
\frac{\omega^n}{n!} = \left(\frac{i}{2}\right)^n \det(h_{ij}) \, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n.
\]

We may say that a Kähler manifold is a symplectic manifold \((M, \omega)\) equipped with an integral \(\omega\)-compatible almost complex structure \(J\).

**Theorem 2.1. (Banyaga)** Let \(M\) be a compact complex manifold. Let \(\omega_1\) and \(\omega_2\) be Kähler forms on \(M\). Assume that \([\omega_1] = [\omega_2] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})\). Then \((M, \omega_1)\) is symplectomorphic to \((M, \omega_2)\).

**Theorem 2.2.** Let \(\omega\) be a closed real \((1,1)\)-form on a complex manifold and let \(p \in M\). Then there exist a neighborhood \(U\) of \(p\) and a Kähler potential \(\varphi \in C^\infty(U; \mathbb{R})\) such that

\[
\omega = \frac{i}{2} \partial \overline{\partial} \varphi \quad \text{on } U.
\]

**Definition 2.4.** Let \(G\) be a connected Lie group and let \((M, \omega)\) be a symplectic manifold. A smooth \(G\)-action \(\phi : G \times M \rightarrow M\) is said to be **symplectic** if \(G\) acts by symplectomorphisms, that is, for each \(g \in G\) the diffeomorphism \(\phi_g : M \rightarrow M\) given by \(\phi_g(p) := \phi(g, p), \ p \in M\) satisfies the condition \(\phi_g^* \omega = \omega\). The triple \((M, \omega, \phi)\) is called a **symplectic \(G\)-manifold**.

**Definition 2.5.** Let \(G\) be a connected Lie group and let \((M, \omega)\) be a symplectic manifold. Let \(\mathfrak{g}\) be the Lie algebra of \(G\). Let \(\phi : G \times M \rightarrow M\) be an action of \((M, \omega)\). For
any $X \in \mathfrak{g}$, we denote by $X^M$ the vector field on $M$ generated by the one-parameter subgroup of global diffeomorphisms $p \mapsto \phi(\exp(tX), p)$, $p \in M$, that is,

$$X^M(p) = \frac{d}{dt} \bigg|_{t=0} \phi(\exp(tX), p), \quad p \in M.$$  

(2.4)

Given a smooth function $H : (M, \omega) \rightarrow \mathbb{R}$, let $X_H$ be the vector field on $M$ defined by Hamilton’s equation

$$\iota(X_H)\omega = \omega(X_H, \cdot) = -dH.$$  

(2.5)

**Definition 2.6.** A smooth vector field $Y$ on a symplectic manifold $(M, \omega)$ is said to be symplectic if its flow preserves the symplectic structure $\omega$, and Hamiltonian if there exist a smooth function $H : M \rightarrow \mathbb{R}$ such that $Y = X_H$.

**Definition 2.7.** Let $\sigma : G \rightarrow \text{Symp}(M, \omega)$ be a symplectic action of a Lie group $G$ on a symplectic manifold $(M, \omega)$. The action $\sigma$ is called a Hamiltonian action if there exists a map $\mu : M \rightarrow \mathfrak{g}^*$ satisfying the following conditions (HA1) and (HA2):

(1) For each $X \in \mathfrak{g}$,

$$d\mu_X = \iota(X^M)\omega,$$

where $\mu_X : M \rightarrow \mathbb{R}$ is a function defined by $\mu_X(p) := \langle \mu(p), X \rangle$ and $X^M$ is the vector field on $M$ generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\}$ of $G$.

(HA2) $\mu$ is equivariant with respect to the given action $\sigma$ and the coadjoint action $\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$, that is,

$$\mu \circ \sigma_g = \text{Ad}^*(g) \circ \mu \quad \text{for all } g \in G.$$  

(2.7)

In this case, the quadruple $(M, \omega, G, \mu)$ is called a Hamiltonian $G$-space and $\mu$ is called the moment map.

**Remark 2.1.** (1) A $G$-action on a symplectic manifold $(M, \omega)$ is symplectic if and only if all the vector fields $X^M (X \in \mathfrak{g})$ are symplectic if and only if all the one forms $\iota(X^M)\omega$ are closed. A $G$-action on a symplectic manifold $(M, \omega)$ is Hamiltonian if and only if all the vector fields $X^M (X \in \mathfrak{g})$ are Hamiltonian if and only if all the one forms $\iota(X^M)\omega$ are exact.

(2) Any symplectic $G$-action on a simply connected symplectic manifold $(M, \omega)$ is Hamiltonian. Indeed, $[\iota(X^M)\omega] \in H^1(M, \mathbb{R}) = 0$ for all $X^M (X \in \mathfrak{g})$ and hence all $\iota(X^M)\omega$ are exact.

**Remark 2.2.** Let $T$ be a torus. The fixed point set in a Hamiltonian $T$-space $(M, \omega, T, \mu)$ is a finite union of connected symplectic submanifolds of $M$.

**Remark 2.3.** Let $T$ be a torus. The $T$-orbits in a Hamiltonian $T$-space $(M, \omega, T, \mu)$ are isotropic, and hence if a symplectic $T$-action has symplectic orbits, it is not Hamiltonian.

Marsden and Weinstein [40], and Meyer [48] proved the following.
Theorem 2.3. (Marsden-Weinstein [1974]-Meyer [1973]) Let \((M, \omega, G, \mu)\) be a Hamiltonian \(G\)-space for a compact Lie group \(G\). Let \(i : \mu^{-1}(0) \hookrightarrow M\) be the inclusion map. Assume that \(G\) acts freely on \(\mu^{-1}(0)\). Then

(a) the orbit space \(M_{\text{red}} := \mu^{-1}(0)/G\) is a manifold,

(b) \(\pi : \mu^{-1}(0) \longrightarrow M_{\text{red}}\) is a principal \(G\)-bundle, and

(c) there exists a symplectic form \(\omega_{\text{red}}\) on \(M_{\text{red}}\) such that \(i^*\omega = \pi^*\omega_{\text{red}}\).

Definition 2.8. Let \((M, \omega)\) be a symplectic manifold of dimension \(2n\). The symplectic measure (or Liouville measure) of a Borel subset \(U\) of \(M\) is defined to be

\[
m_\omega(U) := \int_U \frac{\omega^n}{n!},
\]

where \(\omega^n/n!\) is the symplectic volume form of \(M\).

Definition 2.9. Let \(G\) be a torus of dimension \(n\). Let \((M, \omega, G, \mu)\) be a a Hamiltonian \(G\)-space of dimension \(2n\) such that the moment map \(\mu\) is proper. The Duistermaat-Heckman measure (briefly D-H measure), \(m_{\text{DH}}\), on \(g^*\) is defined to be the push-forward of \(m_\omega\) by \(\mu : M \longrightarrow g^*\). More precisely, for any Borel subset \(W\) of \(g^*\), we have

\[
m_{\text{DH}}(W) = (\mu_*m_\omega)(W) := \int_{\mu^{-1}(W)} \frac{\omega^n}{n!};
\]

where \(\omega^n/n!\) is the symplectic volume form of \(M\).

For a function \(f \in C_c^\infty(g^*)\), we define its integral with respect to the D-H measure to be

\[
\int_{g^*} f \, dm_{\text{DH}} = \int_M (f \circ \mu) \frac{\omega^n}{n!}.
\]

On \(g^* \simeq \mathbb{R}^n\), there is also the Lebesque measure \(m_0\). The relation between \(m_{\text{DH}}\) and \(m_0\) is governed by the Radon-Nikodym derivative, denote by \(\frac{dm_{\text{DH}}}{dm_0}\), which is a generalized function satisfying

\[
\int_{g^*} f \, dm_{\text{DH}} = \int_{g^*} f \, \frac{dm_{\text{DH}}}{dm_0} \, dm_0.
\]

Duistermaat and Heckman [15] proved the following.

Theorem 2.4. (Duistermaat-Heckman [1982]). Let \(G\) be a torus of dimension \(n\). Let \((M, \omega, G, \mu)\) be a a Hamiltonian \(G\)-space of dimension \(2n\) such that the moment map \(\mu\) is proper. The D-H measure \(m_{\text{DH}}\) on \(g^*\) is a piecewise polynomial multiple of the Lebesque measure \(m_0\) on \(g^* \simeq \mathbb{R}^n\), that is, the Radon-Nikodym derivative

\[
\vartheta = \frac{dm_{\text{DH}}}{dm_0}
\]

is a piecewise polynomial. More precisely, for any Borel subset \(W\) of \(g^*\),

\[
m_{\text{DH}}(W) = \int_W \vartheta(x) \, dx,
\]
where \( dx = dm_0 \) is the Lebesgue volume form on \( W \) and \( \vartheta : g^* \rightarrow \mathbb{R} \) is a polynomial on any region consisting of regular values of \( \mu \). In particular, for the standard Hamiltonian action of \( S^1 \) on \((S^2, \omega)\), we have \( m_{DH} = 2\pi m_0 \) with a constant polynomial \( \vartheta = 2\pi \).

**Remark 2.4.** As far as I know, the terminology, **Symplectic Geometry**, first appeared in Siegel’s paper [61] in 1943. In that paper, Siegel studied the geometry of the Siegel upper half plane \( \mathbb{H}_g \) which is biholomorphic to the Hermitian symmetric space \( \text{Sp}(2g, \mathbb{R})/U(g) \). He discovered the explicit fundamental domain with respect to the Siegel modular group \( \text{Sp}(2g, \mathbb{Z}) \). In fact, \( \mathbb{H}_g \) is a symplectic manifold of dimension \( g(g+1) \) because it is a Kähler manifold. Now we roughly state the geometry of \( \mathbb{H}_g \).

For a given fixed positive integer \( g \), we let
\[
\mathbb{H}_g = \{ \Omega \in \mathbb{C}^{(g,g)} | \Omega = \Omega^t, \quad \text{Im} \Omega > 0 \}
\]
be the Siegel upper half plane of degree \( g \) and let
\[
\text{Sp}(2g, \mathbb{R}) = \{ \alpha \in \mathbb{R}^{(2g,2g)} | \alpha J_g \alpha = J_g \}
\]
be the symplectic group of degree \( 2g \), where \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \) for two positive integers \( k \) and \( l \), \( \alpha^t \) denotes the transpose matrix of a matrix \( \alpha \) and
\[
J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.
\]
Then \( \text{Sp}(2g, \mathbb{R}) \) acts on \( \mathbb{H}_g \) transitively by
\[
(2.8) \quad \alpha \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},
\]
where \( \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{R}) \) and \( \Omega \in \mathbb{H}_g \). Let
\[
\Gamma_g = \text{Sp}(2g, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{R}) \mid A, B, C, D \text{ integral} \right\}
\]
be the Siegel modular group of degree \( g \). This group acts on \( \mathbb{H}_g \) properly discontinuously. C. L. Siegel investigated the geometry of \( \mathbb{H}_g \) and automorphic forms on \( \mathbb{H}_g \) systematically. Siegel [61] found a fundamental domain \( \mathcal{F}_g \) for \( \mathbb{H}_g \) with respect to \( \Gamma_g \) and described it explicitly. Moreover he calculated the volume of \( \mathcal{F}_g \). Let \( \mathcal{A}_g = \Gamma_g \setminus \mathbb{H}_g \) be the Siegel modular variety of degree \( g \). In fact, \( \mathcal{A}_g \) is one of the important arithmetic varieties in the sense that it is regarded as the moduli of principally polarized abelian varieties of dimension \( g \).

For \( \Omega = (\omega_{ij}) \in \mathbb{H}_g \), we write \( \Omega = X + iY \) with \( X = (x_{ij}), \ Y = (y_{ij}) \) real. We put \( d\Omega = (d\omega_{ij}) \) and \( d\overline{\Omega} = (d\overline{\omega}_{ij}) \). The Bergman metric \( ds_g^2 \) on \( \mathbb{H}_g \) which is a \( \text{Sp}(2g, \mathbb{R}) \)-invariant Kähler metric is given by
\[
ds_g^2 = \text{tr}(Y^{-1}d\Omega Y^{-1}d\overline{\Omega}) = \sum_{1 \leq i,j \leq g, 1 \leq k,l \leq g} h_{[ij][kl]} d\omega_{ij} d\overline{\omega}_{kl}
\]
and its Kähler form is
\[ \omega_g = \sum_{1 \leq i \leq j \leq g} \sum_{1 \leq k \leq l \leq g} h_{ij[kl]} d\omega_{ij} \wedge d\omega_{kl}. \]

Put \( N = \frac{g(g+1)}{2} \). The volume form is given by
\[ \frac{\omega_g^N}{N!} = \left( \frac{i}{2} \right)^N \det(h_{ij[kl]}) dv_g = \left( \frac{i}{2} \right)^N (\det Y)^{(g+1)} dv_g, \]
where
\[ dv_g = 2^{\frac{g(g-1)}{2}} \wedge_{1 \leq i \leq j \leq g} d\omega_{ij} \wedge d\omega_{kl}. \]

The invariant metric \( ds_g^2 \) is Kähler-Einstein, that is,
\[ \omega_g = \frac{2i}{g+1} \partial \bar{\partial} \log \det(h_{ij[kl]}) = -\frac{2i}{g+1} \partial \bar{\partial} \log(\det Y)^{g+1}. \]

The function \( \Phi(\Omega) := \frac{4}{g+1} \partial \bar{\partial} \log \det(h_{ij[kl]}) \) is the Kähler potential of \( \omega_g \).

A natural question arises as follows:

**Question.** Is there a Hamiltonian action of a group \( G \) on \((\mathbb{H}_g, \omega_g)\) which is important arithmetically and geometrically?

**Examples:** (1) Let \( x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n} \) be linear coordinates on \( \mathbb{R}^{2n} \). Then
\[ \omega_0 := \sum_{i=1}^{n} dx_i \wedge dx_{n+i} \]
is a symplectic form. Thus \((\mathbb{R}^{2n}, \omega_0)\) is a symplectic manifold. Now we consider the case \( n = 1 \). The one-dimensional torus
\[ S^1 = SO(2) = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathbb{R}^{(2,2)} \mid 0 \leq t < 2\pi \right\} \]
acts on \( \mathbb{R}^2 \) by rotations. This is a Hamiltonian action. The Lie algebra of \( S^1 \) is given by
\[ \mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \in \mathbb{R}^{(2,2)} \mid a \in \mathbb{R} \right\} \cong \mathbb{R}. \]

\( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is a basis of \( \mathfrak{so}(2) \). Since
\[ \exp(tJ) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad 0 \leq t < 2\pi, \]
the vector field \( J^x \) on \( \mathbb{R}^2 \) generated by \( J \) is given by
\[ J^x = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \]
The moment map $\mu : \mathbb{R}^2 \rightarrow \mathfrak{so}(2)^*$ satisfies
$$i(J^2)\omega_0 = x \, dx + y \, dy = d(\mu(\cdot), J).$$

Thus we obtain
$$\langle \mu(x, y), J \rangle = \frac{1}{2} (x^2 + y^2), \quad (x, y) \in \mathbb{R}^2.$$  

(2) The standard symplectic form $\omega$ on the two dimensional sphere $S^2$ is induced by
$$\omega_p(X, Y) := \langle p, X \times Y \rangle, \quad p \in S^2, \; X, Y \in T_pS^2 = \{p\}^\perp.$$  

Thus $(S^2, \omega)$ is a symplectic manifold. Let $\omega_{st} = d\theta \wedge dh$ be the standard area form in cylindrical polar coordinates $\theta, h$ ($0 \leq \theta \leq 2\pi$ and $-1 \leq h \leq 1$) on $S^2$. Endow $(S^2, \omega_{st})$ with the rotational $S^1$-action about the $z$-axis. This is a Hamiltonian action with the moment map $\mu : S^2 \rightarrow \mathbb{R}$ given by $\mu(\theta, h) = h$, and the momentum polytope is $\mu(S^2) = [-1, 1]$. The Duistermaat-Heckman polynomial is
$$\vartheta = 2 \pi \chi_{[-1,1]},$$  

where $\chi_{[-1,1]}$ is the characteristic function of $[-1, 1]$. Hence $m_{\text{DH}}([a, b]) = 2\pi(b - a)$ for all $[a, b] \subset [-1, 1]$. But $S^n$ $(n > 2)$ has no symplectic structures.

(3) Let $M$ be a smooth manifold of dimension $n$. Let $x_1, \ldots, x_n$ be local coordinates on an open neighborhood $U$ of $x \in M$. If $\alpha \in T^*_x X$, then
$$\alpha = \sum_{i=0}^n \alpha_i (dx_i)_x, \quad \alpha_i \in \mathbb{R}.$$  

Then $(T^*U, x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n)$ is a local coordinate chart for $T^*M$. We define
$$\Omega_{\text{can}} := \sum_{i=1}^n dx_i \wedge d\alpha_i \quad \text{and} \quad \alpha := \sum_{i=1}^n \alpha_i \, dx_i.$$  

Then it is easily seen that $\alpha$ is intrinsically defined and $\Omega_{\text{can}} = -d\alpha$. Thus $(T^*M, \Omega_{\text{can}})$ is a symplectic manifold of dimension $2n$. The zero section of $T^*M$ given by
$$(T^*M)_0 := \{(x, \alpha) \in T^*M \mid \alpha = 0 \text{ in } T^*_x M \}$$  

is a Lagrangian submanifold of $(T^*M, \Omega_{\text{can}})$. The 1-form $\alpha$ is called the tautological form and $\Omega_{\text{can}}$ is called the canonical symplectic form on $T^*M$.

Suppose the Lie group $G$ acts on a smooth manifold $M$. Then $G$ acts naturally on $TM$ and $T^*M$. We can show that the action of $G$ on $(T^*M, \Omega_{\text{can}})$ is Hamiltonian. The moment map $\mu : T^*M \rightarrow \mathfrak{g}^*$ is given by
$$\langle \mu(\cdot), Y \rangle = -i(Y^g)\alpha \quad \text{for all } Y \in \mathfrak{g},$$  

where $Y^g$ is the vector field on $T^*M$ generated by $Y$.

(4) For any submanifold $Y$ of a smooth manifold $M$, the conormal bundle $N^*Y$ is a Lagrangian submanifold of $(T^*M, \Omega_{\text{can}})$.

(5) Let $(M_1, \omega_1)$ and $(M_2, \omega_2)$ be two symplectic manifolds of dimension $2n$. Let $\pi_i : M_1 \times M_2 \rightarrow M_i$ $(i = 1, 2)$ be the natural projections. For any nonzero real
numbers \(a \) and \(b\), the 2-form \(a \pi_1^* \omega_1 + b \pi_2^* \omega_2\) is a symplectic form on \(M_1 \times M_2\). We consider the twisted product form

\[
\tilde{\omega} := \pi_1^* \omega_1 - \pi_2^* \omega_2.
\]

Let \(\phi : M_1 \longrightarrow M_2\) be a diffeomorphism. Then we can show that \(\phi\) is a symplectomorphism if and only if the graph of \(\phi\) is a Lagrangian submanifold of \((M_1 \times M_2, \tilde{\omega})\).

(6) For any positive integers \(m\) and \(n\), we consider the Heisenberg group

\[
H^{(n,m)}_\mathbb{R} := \{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu \lambda \text{ symmetric} \}
\]

endowed with the following multiplication law

\[
(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda' \mu' - \mu' \lambda').
\]

The Heisenberg group \(H^{(n,m)}_\mathbb{R}\) is embedded in the symplectic group \(Sp(2(m + n), \mathbb{R})\) via the mapping

\[
H^{(n,m)}_\mathbb{R} \ni (\lambda, \mu, \kappa) \longmapsto \left( \begin{array}{ccc} I_n & 0 & 0 \\ \lambda & I_m & \mu \\ 0 & 0 & I_n \\ 0 & 0 & 0 \end{array} \right) \in Sp(2(m + n), \mathbb{R}).
\]

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of smooth compactifications of the Siegel modular variety. In fact, \(H^{(n,m)}_\mathbb{R}\) is obtained as the unipotent radical of the parabolic subgroup of the rational boundary component \(F_n\) (cf. [20] pp. 122-123 or [52] p. 21).

Now we find the coadjoint orbits of the Heisenberg group \(H^{(n,m)}_\mathbb{R}\) and their symplectic forms which are called the Kostant-Kirillov symplectic structures. For brevity, we let \(G := H^{(n,m)}_\mathbb{R}\) as before. Let \(g\) be the Lie algebra of \(G\) and let \(g^*\) be the dual space of \(g\). We observe that \(g\) can be regarded as the subalgebra consisting of all \(2(m + n) \times 2(m + n)\) real matrices of the form

\[
X(\alpha, \beta, \gamma) := \left( \begin{array}{ccc} 0 & 0 & \beta \\ \alpha & 0 & \gamma \\ 0 & 0 & -t^\alpha \end{array} \right), \alpha, \beta \in \mathbb{R}^{(m,n)}, \gamma = t^\gamma \in \mathbb{R}^{(m,m)}
\]

of the Lie algebra \(sp(2(m + n), \mathbb{R})\) of the symplectic group \(Sp(2(m + n), \mathbb{R})\). An easy computation yields

\[
[X(\alpha, \beta, \gamma), X(\delta, \epsilon, \xi)] = X(0, 0, \alpha^t \epsilon + \epsilon^t \alpha - \beta^t \delta - \delta^t \beta).
\]

The dual space \(g^*\) of \(g\) can be identified with the vector space consisting of all \(2(m + n) \times 2(m + n)\) real matrices of the form

\[
F(a, b, c) := \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \\ b & c & -a \end{array} \right), a, b \in \mathbb{R}^{(m,n)}, c = t^c \in \mathbb{R}^{(m,m)}
\]
so that
\[(2.9) \quad \langle F(a, b, c), X(\alpha, \beta, \gamma) \rangle = \sigma(F(a, b, c)X(\alpha, \beta, \gamma)) = 2\sigma'(a\alpha + b\beta) + \sigma(c\gamma).\]

The adjoint representation \( Ad \) of \( G \) is given by \( Ad_G(g)X = gXg^{-1} \) for \( g \in G \) and \( X \in \mathfrak{g} \). For \( g \in G \) and \( F \in \mathfrak{g}^* \), \( gFg^{-1} \) is not of the form \( F(a, b, c) \). We denote by \((gFg^{-1})_*\) the
\[
\begin{pmatrix}
0 & * & 0 \\
0 & 0 & 0 \\
0 & * & 0 \\
* & * & 0
\end{pmatrix}
\]
part of the matrix \( gFg^{-1} \). Then it is easy to see that the coadjoint representation \( Ad^*_G : G \rightarrow GL(\mathfrak{g}^*) \) is given by \( Ad^*_G(g)F = (gFg^{-1})_* \), where \( g \in G \) and \( F \in \mathfrak{g}^* \). More precisely,
\[(2.10) \quad Ad^*_G(g)F(a, b, c) = F(a + c\mu, b - c\lambda, c),\]
where \( g = (\lambda, \mu, \kappa) \in G \). Thus the coadjoint orbit \( \Omega_{a,b} \) of \( G \) at \( F(a, b, 0) \in \mathfrak{g}^* \) is given by
\[(2.11) \quad \Omega_{a,b} = Ad^*_G(G)F(a, b, 0) = \{ F(a, b, 0) \}, \text{ a single point}\]
and the coadjoint orbit \( \Omega_c \) of \( G \) at \( F(0, 0, c) \in \mathfrak{g}^* \) with \( c \neq 0 \) is given by
\[(2.12) \quad \Omega_c = Ad^*_G(G)F(0, 0, c) = \{ F(a, b, c) \mid a, b \in \mathbb{R}^{(m,n)} \} .\]

Therefore the coadjoint orbits of \( G \) in \( \mathfrak{g}^* \) fall into two classes:

(I) The single point \( \{ \Omega_{a,b} \mid a, b \in \mathbb{R}^{(m,n)} \} \) located in the plane \( c = 0 \).

(II) The affine planes \( \{ \Omega_c \mid c = \ell \in \mathbb{R}^{(m,m)}, \ c \neq 0 \} \) parallel to the homogeneous plane \( c = 0 \).

In other words, the orbit space \( O(G) \) of coadjoint orbits is parametrized by
\[
\begin{cases}
\text{c-axis, } c \neq 0, \ c = \ell \in \mathbb{R}^{(m,m)}; \\
(a, b) - \text{plane } \cong \mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}. 
\end{cases}
\]

The single point coadjoint orbits of the type \( \Omega_{a,b} \) are said to be the degenerate orbits of \( G \) in \( \mathfrak{g}^* \). On the other hand, the flat coadjoint orbits of the type \( \Omega_c \) are said to be the non-degenerate orbits of \( G \) in \( \mathfrak{g}^* \).

It is well known that each coadjoint orbit is a symplectic manifold. We will state this fact in detail. For the present time being, we fix an element \( F \) of \( \mathfrak{g}^* \) once and for all. We consider the alternating \( \mathbb{R} \)-bilinear form \( B_F \) on \( \mathfrak{g} \) defined by
\[(2.13) \quad B_F(X, Y) := \langle F, [X, Y] \rangle = \langle \text{ad}^*_F(Y)X, F \rangle, \quad X, Y \in \mathfrak{g},\]
where \( \text{ad}^*_F : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}^*) \) denotes the differential of the coadjoint representation \( Ad^*_G : G \rightarrow GL(\mathfrak{g}^*) \). More precisely, if \( F = F(a, b, c), X = X(\alpha, \beta, \gamma) \), and \( Y =
\( X(\delta, \epsilon, \xi), \) then
\[
B_F(X, Y) = \sigma \{ c(\alpha^t \epsilon + \epsilon^t \alpha - \beta^t \delta - \delta^t \beta) \}.
\]

For \( F \in g^* \), we let
\[
G_F = \{ g \in G \mid \text{Ad}_G^*(g)F = F \}
\]
be the stabilizer of the coadjoint action \( \text{Ad}^* \) of \( G \) on \( g^* \) at \( F \). Since \( G_F \) is a closed subgroup of \( G \), \( G_F \) is a Lie subgroup of \( G \). We denote by \( g_F \) the Lie subalgebra of \( g \) corresponding to \( G_F \). Then it is easy to show that
\[
g_F = \text{rad} B_F = \{ X \in g \mid \text{ad}_g^*(X)F = 0 \}.
\]
Here \( \text{rad} B_F \) denotes the radical of \( B_F \) in \( g \). We let \( \tilde{B}_F \) be the non-degenerate alternating \( \mathbb{R} \)-bilinear form on the quotient vector space \( g/\text{rad} B_F \) induced from \( B_F \). Since we may identify the tangent space of the coadjoint orbit \( \Omega_F \cong G/G_F \) with \( g/g_F = g/\text{rad} B_F \), we see that the tangent space of \( \Omega_F \) at \( F \) is a symplectic vector space with respect to the symplectic form \( \tilde{B}_F \).

Now we are ready to prove that the coadjoint orbit \( \Omega_F = \text{Ad}_G^*(G)F \) is a symplectic manifold. We denote by \( \tilde{X} \) the smooth vector field on \( g^* \) associated to \( X \in g \). That means that for each \( \ell \in g^* \), we have
\[
\tilde{X}(\ell) = \text{ad}_g^*(X) \ell.
\]
We define the differential 2-form \( B_{\Omega_F} \) on \( \Omega_F \) by
\[
B_{\Omega_F}(\tilde{X}, \tilde{Y}) = B_{\Omega_F}(\text{ad}_g^*(X)F, \text{ad}_g^*(Y)F) := B_F(X, Y),
\]
where \( X, Y \in g \).

**Lemma 2.1.** \( B_{\Omega_F} \) is non-degenerate.

**Proof.** Let \( \tilde{X} \) be the smooth vector field on \( g^* \) associated to \( X \in g \) such that \( B_{\Omega_F}(\tilde{X}, \tilde{Y}) = 0 \) for all \( \tilde{Y} \) with \( Y \in g \). Since \( B_{\Omega_F}(\tilde{X}, \tilde{Y}) = B_F(X, Y) = 0 \) for all \( Y \in g \), \( X \in g_F \). Thus \( \tilde{X} = 0 \). Hence \( B_{\Omega_F} \) is non-degenerate. \( \square \)

**Lemma 2.2.** \( B_{\Omega_F} \) is closed.

**Proof.** If \( \tilde{X}_1, \tilde{X}_2, \) and \( \tilde{X}_3 \) are three smooth vector fields on \( g^* \) associated to \( X_1, X_2, X_3 \in g \), then
\[
\begin{align*}
dB_{\Omega_F}(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3) &= \tilde{X}_1(B_{\Omega_F}(\tilde{X}_2, \tilde{X}_3)) - \tilde{X}_2(B_{\Omega_F}(\tilde{X}_1, \tilde{X}_3)) + \tilde{X}_3(B_{\Omega_F}(\tilde{X}_1, \tilde{X}_2)) \\
&= B_{\Omega_F}(\tilde{X}_1, [\tilde{X}_2, \tilde{X}_3]) + B_{\Omega_F}(\tilde{X}_2, [\tilde{X}_1, \tilde{X}_3]) + B_{\Omega_F}(\tilde{X}_3, [\tilde{X}_1, \tilde{X}_2]) \\
&= -\langle F, [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \rangle \\
&= 0 \quad (\text{by the Jacobi identity}).
\end{align*}
\]
Therefore \( B_{\Omega_F} \) is closed. \( \square \)

In summary, \( (\Omega_F, B_{\Omega_F}) \) is a symplectic manifold of dimension \( 2mn \) or \( 0 \). We remark that there is a one-to-one correspondence between the collection of the coadjoint orbits
Let \( G \) be a Lie group. Then the coadjoint orbit \( \mathcal{O}(F) \) of \( F \in \mathfrak{g}^* \) is a symplectic manifold equipped with the Kostant-Kirillov symplectic form \( \omega_F \). Let

\[
\mathcal{O}(F)^- := (\mathcal{O}(F), -\omega_F).
\]

Then the natural product action of \( G \) on \( M \times \mathcal{O}(F)^- \) is Hamiltonian with the following moment map \( \mu_F : M \times \mathcal{O}(F)^- \rightarrow \mathfrak{g}^* \) defined by

\[
\mu_F(p, \xi) := \mu(p) - \xi, \quad p \in M, \xi \in \mathcal{O}(F)^-.
\]

If the above action of \( G \) is free, we obtain a reduced space with respect to the coadjoint orbit \( \mathcal{O}(F) \).

For a positive integer \( n \), let \( T_1 := (\mathbb{R}/\mathbb{Z})^{2n-1} \) and \( T_2 := (\mathbb{R}/\mathbb{Z})^n \) be two tori. It is easily seen that the \( T_1 \)-action on \((M, \omega) = ((\mathbb{R}/\mathbb{Z})^{2n}, \sum_{i=1}^n dx_i \wedge dy_i)\) with coordinates \((x_1, y_1, \ldots, x_n, y_n)\) in \((\mathbb{R}/\mathbb{Z})^{2n}\) by translation on the first \(2n-1\) components \((x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n)\) is free and symplectic. According to Remark 2.3, this action is not Hamiltonian. The \( T_1 \)-orbits are coisotropic submanifolds of \( M \) diffeomorphic to \( T_1 \). The \( T_2 \)-action on \((M, \omega)\) by translations on \((x_1, \ldots, x_n)\) is free, symplectic and hence not Hamiltonian. Its \( T_2 \)-orbits are Lagrangian submanifolds of \( M \) diffeomorphic to \( T_2 \). Here a submanifold \( N \) of a symplectic manifold \((M, \omega)\) is said to be coisotropic if \( T_x N \) is a coisotropic subspace of \((T_x M, \omega_x)\) for all \( x \in N \).

Let \( N := ((\mathbb{R}/\mathbb{Z})^2 \times S^2, dx \wedge dy + d\theta \wedge dh) \) be a 4-dimensional symplectic manifold. The 2-torus \( T := (\mathbb{R}/\mathbb{Z})^2 \) acts freely by translations on the left factor of \( N \). Let \( Z_2 := \mathbb{Z}/2\mathbb{Z} \) act on \( S^2 \) by rotating each point horizontally by \( \pi \) radians, and let \( Z_2 \) act on \((\mathbb{R}/\mathbb{Z})^2\) by the antipodal action on the first circle \( \mathbb{R}/\mathbb{Z} \). Then the diagonal action of \( Z_2 \) on \( N \) fixes \( \text{id} \). So the quotient space \( M := (\mathbb{R}/\mathbb{Z})^2 \times_{Z_2} S^2 \) is a smooth manifold equipped with the symplectic form \( \omega \) and the \( T \)-action inherited from the ones on \( N \). We see easily that the action of \( T \) on \( M \) is symplectic but not free, and the \( T \)-orbits are symplectic tori of dimension 2. The orbit space \( M/T = S^2/Z_2 \) is an orbifold with two singular points of order 2, the south and north poles of \( S^2 \).

Let \((M, \omega) := (\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\) be a 4-dimensional symplectic manifold, where \((x_1, y_1) \in \mathbb{R}^2 \) and \((x_2, y_2) \in (\mathbb{R}/\mathbb{Z})^2 \). Consider the action of \( \mathbb{Z}^2 \) on \( M \) such that \( \mathbb{Z}^2 \) acts on \( \mathbb{R}^2 \) linearly and on \((\mathbb{R}/\mathbb{Z})^2\) by

\[
(a, b) \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 + by_2 \\ y_2 \end{pmatrix}, \quad (a, b) \in \mathbb{Z}^2, \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^2.
\]

The quotient space of \((M, \omega)\) by the diagonal action of \( \mathbb{Z}^2 \) is a compact connected symplectic manifold, called the **Kodaira variety**, \((\mathbb{K}T, \omega) := (\mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\). Consider the action of the 2-torus \( T := \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \) on \((\mathbb{K}T, \omega)\) such that the first circle acts on the \(x_1\)-component and the second circle acts on the \(y_2\)-component. It
is easily checked that this action is symplectic and free. The \( T \)-orbits in \((KT, \omega)\) are Lagrangian orbits.

(11) The projective curve \( M = \mathbb{P}^1(C) = C \cup \{\infty\} \) is a Kähler manifold. Its Kähler metric \( ds^2_{FS} \) on the local chart \( U_0 = \{[z_0, z_1] \in M \mid z_0 \neq 0\} \) is given by

\[
ds^2_{FS} = \frac{dx^2 + dy^2}{(x^2 + y^2 + 1)^2} = \frac{dz \, d\bar{z}}{(|z|^2 + 1)^2},
\]

where \( z_1/z_0 = z = x + iy (x, y \in \mathbb{R}) \) is the usual coordinate on \( C \). Its Kähler form \( \omega_{FS} \) is given by

\[
\omega_{FS} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(|z|^2 + 1)^2}.
\]

The function \( K(z) := \log(|z|^2 + 1) \) is the Kähler potential, that is,

\[
\omega_{FS} = \frac{i}{2} \partial_z \partial_{\bar{z}} \log(|z|^2 + 1).
\]

Thus \((M, \omega_{FS})\) is a two-dimensional symplectic manifold. \( \omega_{FS} \) is called the Fubini-Study symplectic form. The total area of \( M = \mathbb{C} \cup \{\infty\} \) with respect to \( \omega_{FS} \) is given by

\[
\int_M \omega_{FS} = \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2} = \pi.
\]

Since \( M \) is diffeomorphic to \( S^2 \) by stereographic projection, we obtain

\[
\omega_{FS} = \frac{1}{4} \omega_{st}. \quad \text{(see Example (2)).}
\]

(12) Let \( \mathbb{T}^n := (\mathbb{R}/\mathbb{Z})^n \) be the \( n \)-torus. For \( \lambda > 0 \), let \((\mathbb{P}^n(C), \lambda \omega_{FS})\) be the \( n \)-dimensional symplectic complex projective space with the Fubini-Study form \( \lambda \omega_{FS} \). We consider the rotational \( \mathbb{T}^n \)-action on \( \mathbb{P}^n(C) \) induced from the rotational \( \mathbb{T}^n \)-action on the \((n + 1)\)-dimensional complex plane. This is a Hamiltonian action with its moment map \( \mu : \mathbb{P}^n(C) \to \mathbb{R}^n \) given by

\[
(2.19) \quad \mu([z_0 : z_1 : \cdots : z_n]) := \left( \frac{\lambda |z_1|^2}{\sum_{i=0}^n |z_i|^2}, \cdots, \frac{\lambda |z_n|^2}{\sum_{i=0}^n |z_i|^2} \right).
\]

If \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n, \cdots, e_n = (0, \cdots, 0, 1) \in \mathbb{R}^n \), the momentum polytope

\[
(2.20) \quad \mu(\mathbb{P}^n(C)) = \text{Convex Hull} \{0, \lambda e_1, \cdots, \lambda e_n\}.
\]

3. Some fundamental theorems

According to the Darboux’s theorem, a symplectic form (or symplectic structure) \( \omega \) can always be written in the following form

\[
(3.1) \quad \omega = \sum_{k=1}^n dp_k \wedge dq_k
\]
in suitable canonical coordinates \( p_1, \ldots, p_n, q_1, \ldots, q_k \). However these canonical coordinates are not uniquely determined. This theorem says that locally all symplectic forms are the same, i.e., all symplectic structures are indistinguishable. We recall the Moser’s stability theorem which says that if \( \omega_t \ (t \in [0,1]) \) is a smooth path of symplectic forms such that the cohomology class \([\omega_0] = [\omega_t] \in H^2(M, \mathbb{R})\) for all \( t \in [0,1] \), then all these symplectic forms are the same in the sense that one can make them coincide by moving the points of \( M \) in a suitable way. In other words, one cannot change the symplectic form in any way by deforming it, provided that the cohomology class \([\omega] \in H^2(M, \mathbb{R})\) is unchanged.

In 2-dimensions, a symplectomorphism can be characterized as an area preserving diffeomorphism. More precisely, if \( S \) is a region in \( \mathbb{R}^2 \) that is diffeomorphic to a disc \( D \) and has the same area as \( D \), then there exists a symplectomorphism \( \phi : S \rightarrow D \).

**Weinstein Lagrangian Neighborhood Theorem:** Let \( M \) be a smooth manifold of dimension \( 2n \), \( X \) a compact \( n \)-dimensional submanifold, \( i : X \hookrightarrow M \) the inclusion map, and \( \omega_1 \) and \( \omega_2 \) symplectic forms on \( M \) such that \( i^* \omega_1 = i^* \omega_2 = 0 \), i.e., \( X \) is a Lagrangian submanifold of both \((M,\omega_1)\) and \((M,\omega_2)\). Then there exist neighborhoods \( U_1 \) and \( U_2 \) of \( X \), and a diffeomorphism \( \phi : U_1 \rightarrow U_2 \) such that \( i_2 = \phi \circ i_1 \) and \( \phi^* \omega_2 = \omega_1 \), where \( i_k : X \hookrightarrow U_k \) \((k = 1,2)\) are the inclusion maps.

**Proof.** The proof can be found in [66]. \( \square \)

**Weinstein Tubular Neighborhood Theorem:** Let \( M \) be a smooth manifold of dimension \( n \), \( X \) is a submanifold of \( M \), \( NX \) the normal bundle of \( X \) in \( M \), \( i_0 : X \hookrightarrow M \) the inclusion map, and \( i : X \hookrightarrow M \) inclusion. Then there exist neighborhoods \( U_0 \) of \( X \) in \( NX \), \( U_2 \) of \( X \) in \( M \) and a diffeomorphism \( \phi : U_0 \rightarrow U_1 \) such that \( i = \phi \circ i_0 \).

**Proof.** The proof can be found in [67]. \( \square \)

In 1985 using the pseudo-holomorphic curve technique and constructing the Gromov radius, Gromov proved the famous **Gromov’s nonsqueezing theorem** which is also called the principle of the symplectic camel.

**Gromov’s Nonsqueezing Theorem:** Let \((\mathbb{R}^{2n},\omega_0)\) be the symplectic manifold equipped with the standard symplectic form

\[
\omega_0 = \sum_{k=1}^{n} dp_k \wedge dq_k,
\]

where \((p_1, \ldots, p_n, q_1, \ldots, q_n)\) is a coordinate in \( \mathbb{R}^{2n} \). Let

\[
B(R) := \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^{2n} \mid \sum_{i=1}^{n} (p_i^2 + q_i^2) < R^2 \}, \quad R > 0
\]

be the ball of radius \( R \) in \( \mathbb{R}^{2n} \) and let

\[
Z(r) := \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbb{R}^{2n} \mid p_1^2 + q_1^2 < r^2 \}, \quad r > 0
\]
be the cylinder of radius $r$, each of them equipped with the symplectic form $\omega_0$. If we can find a symplectic embedding $\varphi : B(R) \longrightarrow Z(r)$, then $R \leq r$.

Proof. The proof can be found in [27]. □

**Eliashberg’s Principle:** An obstruction to symplectic embeddings (beyond the volume condition) can be described by a $J$-holomorphic curve.

We refer to [18] for more details on Eliashberg’s Principle.

An influential precursor in the study of global aspects of symplectic geometry is Arnold’s conjecture which is a high-dimensional analogue of the classical fixed point theorem of H. Poincaré and G. Birkhoff.

**Theorem 3.1. (Poincaré-Birkhoff [1913])** Suppose $f : S \longrightarrow S$ is an area-preserving diffeomorphism of the closed annulus $S = \mathbb{R}/\mathbb{Z} \times [-1, 1]$ which preserves the two components of the boundary, and twists them in opposite directions. Then $f$ has at least two distinct fixed points.

Proof. The proof can be found in [5]. □

Arnold’s conjecture was proved by Conley-Zehnder, Floer, Hofer-Salamon, Ono, Fukaya-Ono, Liu-Tian using Floer homology which is an infinite dimensional analogue of Morse theory. Now Arnold’s conjecture is described in the form of the following theorem:

**Theorem 3.2.** Let $(M, \omega)$ be a closed symplectic manifold of dimension $2n$ and $\phi : M \longrightarrow M$ an exact symplectomorphism of $(M, \omega)$ with only non-degenerate fixed points. Then the number of fixed points of $\phi$ is at least the sum of Betti numbers of $M$, that is, $\sum_{k=0}^{2n} H^k(M, \mathbb{Q})$.

4. Convexity properties of the moment map

Atiyah, Guillemin and Sternberg [1, 30] proved the following Convexity Theorem for a Hamiltonian $\mathbb{T}^m$-space. Here $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ is an $m$-dimensional torus.

**Theorem 4.1. (Convexity Theorem: Atiyah, Guillemin-Sternberg [1982])** Let $(M, \omega)$ be a compact connected symplectic manifold. Assume that $(M, \omega, \mathbb{T}^m, \mu)$ is a Hamiltonian $\mathbb{T}^m$-space. Then the following properties (M1)-(M3) are satisfied

(M1) the levels of $\mu$ are connected;

(M2) the image $\mu(M)$ of $\mu$ is convex;

(M3) the image of $\mu$ is the convex hull of the images of the fixed points of the action.

Proof. The proof can be found in [1, 30]. We briefly sketch the proof of Atiyah [1] (cf. [6] pp. 169–170).

**Claim 1.** The levels of $\mu$ are connected for any $\mathbb{T}^m$-action, $m = 1, 2, \ldots$. 
We leave the proof of Claim 1 to the reader.

**Claim 2.** The image of $\mu$ are convex for any $\mathbb{T}^m$-action, $m = 1, 2, \ldots$.

Now we prove Claim 2. For a $\mathbb{T}^1$-action, $\mu(M)$ is convex because in $\mathbb{R}$ connectivity is convexity. For a Hamiltonian $\mathbb{T}^m$-space $(M, \omega, \mathbb{T}^m, \mu)$ ($m \geq 2$), we first take a matrix $A \in \mathbb{Z}^{(m,m-1)}$ of rank $m - 1$. Let $\rho_A : \mathbb{T}^{m-1} \rightarrow \text{Symp}(M, \omega)$ be the action of $\mathbb{T}^{m-1}$ on $(M, \omega)$ defined by

$$
\rho_A(t) \cdot p =: (At) \cdot p, \quad t \in \mathbb{T}^{m-1}, \ p \in M.
$$

Then $\rho_A$ is the Hamiltonian $\mathbb{T}^{m-1}$-action on $(M, \omega)$ with its moment map $\mu_A : M \rightarrow \mathbb{R}^{m-1}$ given by

$$
\mu_A(p) := {^t}A\mu(p), \quad p \in M.
$$

For $\xi \in \mathbb{R}^{m-1}$, choose $p_0 \in \mu_A^{-1}(\xi)$. Then

$$
p \in \mu_A^{-1}(\xi) \quad \text{if and only if} \quad {^t}A\mu(p) = \xi = {^t}A\mu(p_0).
$$

So we have

$$
\mu_A^{-1}(\xi) = \left\{ p \in M \mid \mu(p) - \mu(p_0) \in \ker({^t}A) \right\}.
$$

According to Claim 1, $\mu_A^{-1}(\xi)$ is connected. For two points $p_0, p_1 \in \mu_A^{-1}(\xi)$, we take a curve $\alpha : [0, 1] \rightarrow \mu_A^{-1}(\xi)$ with $\alpha(0) = p_0$ and $\alpha(1) = p_1$. Then we obtain a curve $\gamma : [0, 1] \rightarrow \ker({^t}A) \subset \mathbb{R}^{m}$ defined by

$$
\gamma(t) := \mu(\alpha(t)) - \mu(p_0), \quad t \in [0, 1].
$$

Since $\ker({^t}A)$ is a one-dimensional subspace of $\mathbb{R}^m$,

$$
c\mu(p_0) + (1 - c)\mu(p_1) \in \mu(M) \quad \text{for any} \ c \in [0, 1].
$$

Any $p_0, p_1 \in M$ can be approximated arbitrarily by points $q_0$ and $q_1$ in $M$ with $\mu(q_0) - \mu(q_1) \in \ker({^t}A)$ for some matrix $A \in \mathbb{Z}^{(m,m-1)}$ of rank $m - 1$. Taking the limits $q_0 \rightarrow p_0$ and $q_1 \rightarrow p_1$, we see that $\mu(M)$ is convex. This completes the proof of the statement (M2).

Let $N$ be the fixed point set of the $\mathbb{T}^m$-action $\rho$ on $(M, \omega)$. Then $N$ is a finite disjoint union of connected symplectic submanifolds $N_1, \ldots, N_k$. The moment map $\mu$ is constant on each $N_j$ ($1 \leq j \leq k$), say, $\mu(N_j) = c_j \in \mathbb{R}^m$ ($1 \leq j \leq k$). By (M2), the convex hull $\text{ConH}(c_1, \ldots, c_k)$ is contained in $\mu(M)$. Conversely, suppose that $\xi \in \mathbb{R}^m$ and $\xi \notin \text{ConH}(c_1, \ldots, c_k)$. Choose $\zeta \in \mathbb{R}^m$ with rationally independent components such that

$$
\langle \xi, \zeta \rangle > \langle c_j, \zeta \rangle, \quad j = 1, \ldots, k.
$$

Here $\langle \cdot, \cdot \rangle$ is a $\mathbb{T}^m$-invariant positive definite inner product on $\mathbb{R}^m$. By the irrationality of $\zeta$, the set $\{\exp t\zeta \mid t \in \mathbb{R}\}$ is dense in $\mathbb{T}^m$. Hence the zeros of the vector field $X^\zeta$ on $M$ generated by $\zeta$ are the fixed points of the $\mathbb{T}^m$-action $\rho$. Since the function $\langle \mu(\cdot), \zeta \rangle$ attains its maximum value on one of the sets $N_j$, we have the relation

$$
\langle \xi, \zeta \rangle > \sup_{p \in M} \langle \mu(p), \zeta \rangle.
$$
So $\xi \not\in \mu(M)$. Therefore $\mu(M)$ is contained in $\text{ConH}(c_1, \cdots, c_k)$. This completes the proof of the statement (M3).

**Theorem 4.2.** Let $M$ be a connected nonsingular complex projective variety with its Kähler form $\omega$ and $G$ a compact connected subgroup of the group of complex symplectic transformations of $(M, \omega)$. Suppose that for some point $p \in M$, the stabilizer $G_p$ of $p$ in $G$ is finite. Let $t$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of $G$, and let $t^*$ be the subspace of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ corresponding to $t$. Let $t^*_+\alpha$ be a Weyl chamber in $t^*$, and let $\mu : M \to \mathfrak{g}^*$ be the moment map. Then the intersection $\mu(M) \cap t^*_+$ is a convex polytope of dimension equal to the rank of $G$.

**Proof.** The proof can be found in [30, pp. 511-513].

Kirwan [36] generalized the above convexity theorem to the case of a Hamiltonian $G$-space where $G$ is a compact Lie group.

**Theorem 4.3.** (Kirwan [1984]) Let $G$ be a compact connected Lie group. Let $(M, \omega)$ be a compact connected symplectic manifold. Assume that $(M, \omega, G, \mu)$ is a Hamiltonian $G$-space. Let $t^*_+\alpha$ be a positive Weyl chamber in $t^*$ of the Lie algebra $t$ of a maximal torus $T$ of $G$. Then the intersection of $\mu(M) \cap t^*_+$ of the image of the moment map with a positive Weyl chamber in $t^*$ is convex.

**Proof.** We give a sketch of Kirwan’s proof. We first fix a $G$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ and use it to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $t^*$ with $t$. Here $t$ is the Lie algebra of a maximal torus $T$ of $G$. Let $t^*_+$ be a positive Weyl chamber in $t^*$. Let $\| \cdot \|$ be the associated norm on $\mathfrak{g}$.

**Claim 1.** The subset of points of $M$ where $\|\mu\|^2$ takes its minimum value is connected.

We refer to [36 (3.1)] for more details of Claim 1.

For any point $\alpha \in \mathfrak{g}$, there exists a natural $G$-invariant symplectic structure $\tau_\alpha$ on the coadjoint orbit $\mathcal{O}(\alpha) := \text{Ad}(G)\alpha$ of $\alpha$. Then $\mathcal{O}(\alpha)$ has the form $G/H$, where $H$ is the centralizer of $\alpha$ in $G$. The inclusion map

$$\mu_H : G/H \hookrightarrow \mathfrak{g} \cong \mathfrak{g}^*$$

is the moment map on $G/H = \mathcal{O}(\alpha)$. Put $\omega_\alpha = -\tau_\alpha$. Clearly $(\mathcal{O}(\alpha), \omega_\alpha)$ is a symplectic manifold. Then we see that $M \times G/H$ becomes a symplectic manifold and its moment map

$$\mu^{(\alpha)} : M \times G/H \longrightarrow \mathfrak{g}$$

is given by $\mu^{(\alpha)}(p, gH) = \mu(p) - \text{Ad}(g)\alpha, \quad p \in M, \ g \in G$.

**Claim 2.** For any sufficiently small $\varepsilon > 0$, there exists an element $\alpha \in t^*_+$ such that the ball $B(\alpha, \varepsilon)$ of radius $\varepsilon$ and center $\alpha$ meets $\mu(M) \cap t^*_+$ in precisely two points $\alpha_1$ and $\alpha_2$ neither of which lies in the interior of $B(\alpha, \varepsilon)$.

The proof of Claim 2 may be found in [36] pp. 549–551.
Claim 3. The function $\|\mu^{(\alpha)}\|^2$ on $M \times G/H$ takes its minimum value precisely at those points $(x, gH)$ such that

$$\mu(g^{-1}x) = \alpha_j, \quad j = 1, 2.$$ 

Here $\alpha_1$ and $\alpha_2$ are two points in Claim 2.

The proof of Claim 2 may be found in [36, pp. 551–552].

Using Claim 1, Claim 2 and Claim 3, Kirwan proved the above theorem as follows: Suppose $\mu(M) \cap t_+$ is not convex. Let $\|\mu^{(\alpha)}\|^2$ be the function $M \times G/H$ where $\alpha$ satisfies the conditions of Claim 2. According to Claim 3, the set

$$\{(\rho, gH) \in M \times G/H | \mu(g^{-1}\rho) = \alpha_j, \quad j = 1, 2\}$$

is the disjoint union of the following two non-empty closed subsets

$$G(\mu^{-1}(\alpha_1) \times \{H\}) \quad \text{and} \quad G(\mu^{-1}(\alpha_2) \times \{H\}).$$

This contradicts Claim 1. Thus $\mu(M) \cap t_+$ is convex. Hence we complete the proof of the above theorem. \[\Box\]

Though we do not have a classification of symplectic manifolds so far, fortunately we have a classification of symplectic-toric manifolds which are very special Hamiltonian torus-spaces in terms of combinatorial data. We recall that a symplectic-toric manifold is a compact connected symplectic manifold $(M, \omega)$ of dimension $2n$ equipped with an effective Hamiltonian action of an $n$-dimensional torus $T^n$ and with a corresponding moment map $\mu : M \rightarrow \mathbb{R}^n$. It can be seen that every symplectic-toric manifold is simply connected. For instance, according to Example (2) in section 2, $(S^2, d\theta \wedge dh, S^1)$ is a symplectic-toric manifold. In 1988 T. Delzant classified all symplectic-toric manifolds in terms of a set of very special polytopes. We describe his classification roughly.

Definition 4.1. A Delzant polytope $\Delta$ in $\mathbb{R}^n$ is a convex polytope satisfying the following properties (DP1)-(DP3):

(DP1) it is simple, i.e., there are $n$ edges meeting at each vertex;

(DP2) it is rational, i.e., the edges meeting at the vertex $p$ are rational in the sense that each edge is of the form $p + t \alpha_i$, $t \geq 0$, where $\alpha_i \in \mathbb{Z}^n$ $(1 \leq i \leq n)$.

(DP3) it is smooth, i.e., for each vertex, the corresponding $\alpha_1, \ldots, \alpha_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

First we give several definitions for the reader’s convenience. A facet of a polytope $\Delta$ with $\dim \Delta = n$ in $\mathbb{R}^n$ is a $(n - 1)$-dimensional face. Let $\Delta$ be a Delzant polytope with $\dim \Delta = n$ and $d =$ the number of facets of $\Delta$. A lattice vector $v \in \mathbb{Z}^n$ is said to be primitive if it cannot be written as $v = ku$ with $u \in \mathbb{Z}^n$, $k \in \mathbb{Z}$ with $|k| > 1$.

Theorem 4.4. (Delzant [1988]) Symplectic-toric manifolds are classified by Delzant polytopes. More precisely, there is the one-to-one correspondence between the set $\mathcal{A}$ of
all symplectic toric manifolds of dimension $2n$ and the set $\mathcal{B}$ of all Delzant polytopes in $\mathbb{R}^n$ given by
\begin{equation}
\mathfrak{A} \ni (M, \omega, \mathbb{T}^n, \mu) \mapsto \mu(M) \in \mathcal{B}.
\end{equation}

Proof. We give a sketchy proof of the “if” part following [29] (cf. [6]). Let $\Delta$ be a Delzant polytope with $d$ facets. Let $v_i \in \mathbb{Z}^n \ (i = 1, 2, \cdots, d)$ be the primitive outward-pointing normal vectors to the facets. Then
\[(\Delta = \{ x \in (\mathbb{R}^n)^\times \mid \langle x, v_i \rangle \leq \lambda_i, \ 1 \leq i \leq d \} \text{ for some } \lambda_i \in \mathbb{R} \ (1 \leq i \leq d).\]
Let $\{e_1, \cdots, e_d\}$ be the standard basis of $\mathbb{R}^d$. If $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be the map defined by $\pi(e_i) = v_i, \ i = 1, 2, \cdots, d$, then it is easily seen that $\pi$ is surjective and $\pi(\mathbb{Z}^d) = \mathbb{Z}^n$. Thus $\pi$ induces a surjective group homomorphism
\[\theta : \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \longrightarrow \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n\]
between $\mathbb{T}^d$ and $\mathbb{T}^n$. Let $N$ be the kernel of $\theta$ with its Lie algebra $n$. The exact sequence of tori
\[0 \longrightarrow N \longrightarrow \mathbb{T}^d \longrightarrow \mathbb{T}^n \longrightarrow 0\]
induces an exact sequence of Lie algebras
\[0 \longrightarrow n \longrightarrow \mathbb{R}^d \longrightarrow \mathbb{R}^n \longrightarrow 0\]
with its dual exact sequence
\[0 \longrightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}) \longrightarrow \text{Hom}(\mathbb{R}^d, \mathbb{R}) \longrightarrow \text{Hom}(n, \mathbb{R}) \longrightarrow 0.\]
Let $(\mathbb{C}^d, \omega_0, \mathbb{T}^d, \mu)$ be a symplectic toric manifold equipped with the standard Hamiltonian action of $\mathbb{T}^d$ on $\mathbb{C}^n$:
\[(\xi_{t_1}, \cdots, \xi_{t_d}) \cdot (z_1, \cdots, z_d) := (\xi_{t_1}^1 z_1, \cdots, \xi_{t_d}^d z_d), \quad \xi := e^{2\pi i}\]
and
\[\mu : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^* \quad \mu(z_1, \cdots, z_d) := -\pi(|z_1|^2, \cdots, |z_d|^2) + (\lambda_1, \cdots, \lambda_d).\]
We consider the map
\[i^* \circ \mu : \mathbb{C}^d \longrightarrow \text{Hom}(\mathbb{R}^d, \mathbb{R}) \longrightarrow n^* := \text{Hom}(n, \mathbb{R})\]
and put
\[Z := (i^* \circ \mu)^{-1}(0), \quad \text{the zero—level set of } i^* \circ \mu.\]
Then we can show that $Z$ is compact and $N$ acts on $Z$ freely. Thus $p : Z \longrightarrow M_\Delta := Z/N$ is the principal $N$-bundle on $M_\Delta$. According to the Marsden-Weinstein-Meyer theorem, there exists a symplectic form $\omega_\Delta$ on $M_\Delta$ such that
\[p^* \omega_\Delta = j^* \omega_0,\]
where $j : Z \hookrightarrow \mathbb{C}^d$ is the inclusion. Therefore $(M_\Delta, \omega_\Delta)$ is a compact symplectic manifold of dimension $2n$. Furthermore we can show that the torus $\mathbb{T}^n$ acts on $M_\Delta$ in a Hamiltonian fashion and its moment map $\mu_n : M_\Delta \longrightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R})$ such that $\mu_n(M_\Delta) = \Delta$ (we refer to [6 pp.185-186] for the detailed proof). Finally the
quadruple $(M_\Delta, \omega_\Delta, T^n, \mu_n)$ is the required symplectic toric manifold of dimension $2n$ corresponding to $\Delta \subset \text{Hom}(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^*$.

**Remark 4.1.** Guillemin, Miranda, Pires and Scott proved the analogue of Theorem 4.2 for log symplectic-toric manifolds which are defined to be generically symplectic-toric and degenerate along a normal crossing configuration of smooth hypersurfaces. Log symplectic-toric manifolds belong to a class of Poisson manifolds. Most often degeneracy loci for Poisson structures are singular.

**Theorem 4.5.** (Ahara and Hattori, Audin) Suppose $(M, \omega, S^1)$ is a compact connected symplectic 4-dimensional manifold equipped with an effective Hamiltonian $S^1$-action. Then $(M, \omega, S^1)$ is $S^1$-equivariantly diffeomorphic to a complex surface with a holomorphic $S^1$-action which is obtained from $\mathbb{P}^2(\mathbb{C})$, a Hirzebruch surface, or a $\mathbb{P}^1(\mathbb{C})$-bundle over a Riemann surface with appropriate circle actions by a sequence of blowups at the fixed points.

We refer to [3, 4] for some details.

5. Modern theory of symplectic geometry

5.1. The classification problems of symplectic actions

**Definition 5.1.** A symplectic manifold with a Hamiltonian action of a compact Lie group is called a multiplicity-free space if the Poisson brackets of any invariant smooth functions vanish.

**Definition 5.2.** Suppose that a torus $T$ acts effectively and symplectically on a compact connected symplectic manifold. The $T$-action is said to be coisotropic if there is a coisotropic $T$-orbit.

**Definition 5.3.** Suppose that a torus $T$ acts effectively and symplectically on a compact connected symplectic manifold. If there is a $\dim T$-dimensional symplectic $T$-orbit, we say that the $T$-action is a maximal symplectic action.

Over past twenty years the theory of the symplectic actions on symplectic manifolds has been developed by some experts in symplectic geometry. In the good survey article [54], Álvaro Pelayo described classifications on compact connected symplectic manifolds $(M, \omega)$:

(a) “Maximal Hamiltonian case”: Hamiltonian $T$-action, $\dim M = 2 \dim T$.
(b) “$S^1$-Hamiltonian case”: Hamiltonian $T$-action, $\dim M = 4 \quad \dim T = 1$.
(c) “Four-dimensional case”: $\dim M = 4 \quad \dim T = 2$.
(d) “Maximal symplectic case”: there is a $T$-orbit symplectic orbit.
(e) “Coisotropic case” there is a coisotropic $T$-orbit.
Here $T$ denotes a torus. He outlined connections of these works with algebraic geometry, toric varieties, log-symplectic toric geometry, torus bundles over tori, nilpotent Lie groups, integral systems and the classification of semi-toric systems.

Let $S^1 = U(1)$ be a torus of dimension one. A Hamiltonian $S^1$-action on a compact connected symplectic manifold $(M,\omega)$ of dimension $2n$ has at least $n + 1$ fixed points. In fact, the number of fixed points is $\sum_{k=0}^{2n} \text{rank} H^k(M,\mathbb{R})$ and $0 \neq [\omega^k] \in H^{2k}(M,\mathbb{R})$ for $1 \leq k \leq n$. The following natural questions arise:

**Question A.** Under which conditions is a symplectic $G$-action Hamiltonian? Describe the obstruction to being Hamiltonian. Here $G$ is an $m$-dimensional torus, a compact connected Lie group, a non-compact abelian group or a semisimple Lie group.

**Question B.** Are there non-Hamiltonian symplectic $S^1$-actions on compact connected symplectic manifolds with nonempty discrete fixed point sets?

Recently some affirmative answers were given as follows.

**Theorem 5.1.** (Tolman and Weitsman [65]) Let $(M,\omega)$ be a compact connected symplectic manifold equipped with a semi-free symplectic $S^1$-action with isolated fixed points. If there is at least one fixed point, the $S^1$-action is Hamiltonian.

**Theorem 5.2.** (Frankel [23]) Let $(M,\omega)$ be a compact connected Kähler manifold admitting an $S^1$-action preserving the Kähler structure $\omega$. If the the $S^1$-action has some fixed points, then it is Hamiltonian.

**Theorem 5.3.** (McDuff [41]) A symplectic $S^1$-action on a compact connected symplectic 4-manifold with some fixed point is Hamiltonian.

**Theorem 5.4.** (Tolman [64]) There exists a symplectic non-Hamiltonian $S^1$-action on a compact connected symplectic manifold with exactly 32 fixed points.

**Theorem 5.5.** (Giacobbe [26]) An effective symplectic action of an $n$-dimensional torus on a compact connected symplectic $2n$-dimensional manifold with some fixed point must be Hamiltonian.

We present some results on the classification of symplectic actions.

**Theorem 5.6.** (Duistermaat and Pelayo [16]) Compact connected symplectic manifolds $(M,\omega)$ with a coisotropic $T$-action are classified up to $T$-equivariant symplectomorphisms by symplectic invariants: the fundamental form $\omega^k$, the Hamiltonian torus $T_h$ and its associated polytope $\Delta$, the period lattice $P$ of $N = (1/t_h)^*$, the Chern class $c : N \times N \to 1$ of $M_{\text{reg}} \to M_{\text{reg}}/T$, and the holonomy invariant $[\tau : P \to T]_B \in \text{Hom}_c(P,T)/B$. Moreover, for any such list $\mathcal{L}$ of five invariants there exists a compact connected symplectic manifolds $(M,\omega)$ with a coisotropic $T$-action with list of invariants $\mathcal{L}$.

Álvaro Pelayo [54] proposed the following natural classification problem:

**Problem A:** Let $G$ be an $m$-dimensional compact connected Lie group. Construct symplectic invariants and classify, up to equivariant symplectomorphisms, effective
symplectic $G$-actions on compact connected symplectic $2n$-dimensional manifolds $(M, \omega)$ in terms of these invariants.

We propose the following problem:

**Problem B:** Let $G$ be a connected Lie group. Find the examples of the symplectic or Hamiltonian $G$-actions on connected symplectic manifolds $(M, \omega)$ which are important arithmetically and geometrically. Here $G$ and $M$ are noncompact in general.

Pelayo [53, 54] proved the following result.

**Theorem 5.7.** (Pelayo [2000]) Let $(M, \omega)$ be a compact connected symplectic 4-manifold equipped with an effective action of a 2-torus $T$. If the symplectic $T$-action is Hamiltonian, then:

1. $(M, \omega)$ is a symplectic-toric manifold, so classified up to $T$-equivariant symplectomorphisms by the image of the moment map $\mu : M \rightarrow \mathfrak{t}^*$ of the $T$-action.

If the symplectic $T$-action is not Hamiltonian, then one and only one of the following cases occurs:

2. $(M, \omega)$ is equivariantly symplectomorphic to $\mathbb{R}/\mathbb{Z} \times S^2$.

3. $(M, \omega)$ is equivariantly symplectomorphic to $(T \times \mathfrak{t}^*)/Q$ with the induced form and the $T$-action, where $Q \leq T \times \mathfrak{t}^*$ is a discrete cocompact subgroup for the group structure on $T \times \mathfrak{t}^*$.

4. $(M, \omega)$ is equivariantly symplectomorphic to a symplectic orbifold bundle

$$P := \tilde{\Sigma} \times_{\pi_1^{\text{orb}}(\Sigma), p_0} T$$

over a good orbisurface $\Sigma$, with symplectic form and $T$-action induced by the product ones. Here, in order to form the quotient $P$, the orbifold fundamental group $\pi_1^{\text{orb}}(\Sigma)$ acts on $\tilde{\Sigma} \times T$ diagonally, and on $T$ by means of a homomorphism $\mu : \pi_1^{\text{orb}}(\Sigma) \rightarrow T$.

**5.2. The symplectic embedding problems**

Given two open subsets $U$ and $V$ in $\mathbb{R}^n$, we often write $U \hookrightarrow V$ instead of “there exists a symplectic embedding of $U$ into $V$”.

We denote by $D(a)$ the open disk in $\mathbb{R}^2$ of area $a$, centered at the origin, and $P(a_1, \cdots, a_n) = D(a_1) \times \cdots \times D(a_n)$ the open polydisk in $\mathbb{R}^{2n}$. We let $C^{2n}(a) := P(a, \cdots, a)$ be the cube in $\mathbb{R}^{2n}$ and let $Z^{2n}(a) := D(a) \times \mathbb{C}^{n-1}$ be the symplectic cylinder. Let

$$E(a_1, \cdots, a_n) := \left\{ (z_1, \cdots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} < 1 \right\}$$

denote the open ellipsoid whose projection to the $j$-th complex coordinate plane is $D(a_j)$ and let $B^{2n}(a) = E(a, \cdots, a)$ be the ball.
of radius $\sqrt{a/\pi}$. We put $\mathbb{T}^4(A) := \mathbb{T}^2(A) \times \mathbb{T}^2(A)$, where $\mathbb{T}^2(A)$ is the torus $\mathbb{R}^2/(AZ \oplus \mathbb{Z})$ endowed with the symplectic form $dx \wedge dy$ inherited from $\mathbb{R}^2$.

We define the symplectic capacities

$$c_{EZ}(a) := \inf \left\{ A \mid E(1,a) \hookrightarrow Z^4(A) \right\}, \quad a \geq 1,$$

$$c_{EC}(a) := \inf \left\{ A \mid E(1,a) \hookrightarrow C^4(A) \right\}, \quad a \geq 1$$

and

$$c_k(C^4) := \inf \left\{ A \mid \bigsqcup_k B^4(1) \hookrightarrow C^4(A) \right\},$$

where $\bigsqcup_k B^4(1)$ denotes any collection of $k$ disjoint balls $B^4(1)$ in $\mathbb{R}^4$.

**Theorem 5.8.** (Gromov [27]) Let $a \geq 1$. Then $E(1,a) \hookrightarrow Z^4(A)$ if and only if $A \geq 1$. That is, $c_{EZ}(a) = 1$.

**Theorem 5.9.** (Frenkel and Müller [24]) Let $\sigma = 1 + \sqrt{2}$ be the silver ratio. Then the symplectic capacity $c_{EC}(a)$ satisfies the following:

(a) On the interval $[1, \sigma^2]$, the function $c_{EC}(a)$ is given by the Pell stairs.

(b) On the interval $[\sigma^2, 2^{-5}152]$ we have $c_{EC}(a) = \sqrt{a/2}$ except on seven disjoint intervals where $c_{EC}$ is a step made from two segments. The first of these steps has edge at $(6, \frac{1}{4})$ and the last at $(7, \frac{7}{8})$.

(c) $c_{EC}(a) = \sqrt{a/2}$ for all $a \geq 2^{-5}152$.

**Remark 5.1.** $\bigsqcup_k B^4(1) \hookrightarrow C^4(A)$ if and only if $E(1,k) \hookrightarrow C^4(A)$, that is, $c_k(C^4) = c_{EC}(k)$ for all $k \in \mathbb{Z}^+$.

**Theorem 5.10.** (Entov and Verbitsky [19]; Latschev, McDuff and Schlenk [39]) Let $a \geq 1$. Then $E(1,a) \hookrightarrow \mathbb{T}^4(A)$ whenever $\text{Vol}(E(1,a)) < \text{Vol}(\mathbb{T}^4(A))$.

**Theorem 5.11.** (Schlenk [57, p. 154]) If $B^{2n}(a) \bigsqcup B^{2n}(a) \hookrightarrow B^{2n}(A)$, then $2a \leq A$.

Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. The Gromov width is defined to be

$$c_B^n(M, \omega) := \sup \left\{ a \mid B^{2n}(a) \hookrightarrow (M, \omega) \right\}.$$

**Problem:** Compute $c_B^n(M, \omega)$ for $n \geq 2$. Is it finite?

We recall the following principle in section 3.

**Eliashberg’s Principle** [18]: An obstruction to symplectic embeddings (beyond the volume condition) can be described by a J-holomorphic curve.

Surprisingly J-holomorphic curves can be used to construct symplectic embeddings.

For two positive real numbers $a$ and $b$, let

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1 \right\} \subset \mathbb{C}^2$$
be the ellipsoid and $N(a, b)$ be the sequence of real numbers by arranging all the linear combinations $ma + nb$, $m$ and $n$ non-negative integers, in non-decreasing order with repetitions. We say that $N(a, b) \leq N(c, d)$ if, for every $k$, the $k$-th entry of $N(a, b)$ is smaller than or equal to the $k$-th entry of $N(c, d)$. McDuff proved the Hofer conjecture:

**Theorem 5.12.** (McDuff [43]) $E(a, b)$ embeds symplectically into $E(c, d)$ if and only if $N(a, b) \leq N(c, d)$.

She proved the Hofer conjecture using the Taubes-Seiberg-Witten theory and pseudo-holomorphic curves technique. We can show that

$$E(1, a) \hookrightarrow E(A, 2A) \text{ if and only if } E(1, a) \hookrightarrow C^4(A).$$

Combining Theorem 5.12 and (5.1), we obtain

$$c_{EC}(a) = \sup_{k \in \mathbb{Z}^+} \frac{N_k(1, a)}{N_k(1, 2)}$$

**Theorem 5.13.** (Schlenk [57, p. 177]) (1) There exists a symplectic embedding $P(1, \infty, \infty) \hookrightarrow P(2, 2, \infty)$.

(2) There exists a symplectic embedding $P(1, a, a) \hookrightarrow P(2 + \varepsilon, 2 + \varepsilon, \infty)$ for all $a \geq 1$.

(3) There exists a symplectic embedding $P(1, \infty, \infty) \hookrightarrow P(2 + \varepsilon, 2 + \varepsilon, \infty)$ for all $\varepsilon > 0$.

**Theorem 5.14.** (Schlenk [57, p. 178]) For any $n \geq 3$ and for every $\varepsilon > 0$, there exists a symplectic embedding $F : \mathbb{C}^n \hookrightarrow \mathbb{C}^n$ such that

$$\text{vol}_{2k} \left( \pi_k(F(Z^{2n}(1))) \right) < \varepsilon$$

for $k = 2, 3, \ldots, n - 1$, where $\text{vol}_{2k}(U) := (k!)^{-1} \int_U \omega^k_0$ denotes the Euclidean volume of a domain $U$ in $\mathbb{C}^k$ and $\pi_k : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ is the projection given by $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k)$.

For a positive integer $k$, we consider the function defined by

$$c_k(x) := \inf \{ A \mid E(1, x) \times \mathbb{R}^{2k} \hookrightarrow B^4(A) \times \mathbb{R}^{2k} \}.$$

R. K. Hind [32] proved that $c_k(x) \leq \frac{3x}{x+1}$ if $x > \tau^4$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This implies that $c_k(x) < c_0(x)$ for $k \geq 1$. In 2018, D. McDuff showed that Hind’s bound is sharp for certain values of $x$.

**Theorem 5.15.** (McDuff [44]) If $x = 3m - 1$ with $m \in \mathbb{Z}^+$ and $x > \tau^4$, then

$$c_k(x) = \frac{3x}{x+1}$$

**Conjecture:** $c_k(x) = \frac{3x}{x+1}$ if $x > \tau^4$. 

The Gromov width of a symplectic manifold \((M, \omega)\) of dimension \(2n\) is defined to be

\[
GW(M, \omega) := \sup \{ a | B^{2n}(a) \hookrightarrow (M, \omega) \}.
\]

Let \(K\) be a compact connected Lie group and let \(\mathfrak{t}^*\) be the dual of the Lie algebra \(\mathfrak{t}\) of \(K\). Each coadjoint orbit \(O\) in \(\mathfrak{t}\) is equipped with the Kostant-Kirillov-Souriau symplectic form \(\omega\) canonically defined by

\[
\omega_\eta(X^\sharp, Y^\sharp) = \langle \eta, [X, Y] \rangle_8, \quad \eta \in O_\lambda, \ X, Y \in \mathfrak{t},
\]

where \(X^\sharp, Y^\sharp\) are the vector fields on \(\mathfrak{t}^*\) corresponding to \(X, Y \in \mathfrak{t}\) induced by the coadjoint action. Each coadjoint orbit intersects a positive Weyl chamber in a single point. So there is a bijection between the coadjoint orbits and points in the positive Weyl chamber. Points in the interior of the positive Weyl chamber are called regular points. The orbits corresponding to regular points are called generic orbits that are diffeomorphic to \(K/T\) for \(T\) a maximal torus of \(K\). Coadjoint orbits intersecting the positive Weyl chamber at its boundary are called degenerate orbits.

Caviedes Castro [7] proved the following:

**Theorem 5.16.** Let \(K\) be a compact connected Lie group. The Gromov width of a coadjoint orbit \(O_\lambda\) through a point \(\lambda\) lying on some rational line in \(\mathfrak{t}^*\), equipped with the Kostant-Kirillov-Souriau symplectic form \(\omega_\lambda\), can not be greater than the following quantity

\[
(5.5) \quad \min \{ |\langle \lambda, \alpha^\vee \rangle| | \alpha^\vee \text{ is a coroot and } \langle \lambda, \alpha^\vee \rangle \neq 0 \}.
\]

Here \(\mathfrak{t}^*\) is the dual of the Lie algebra of a maximal torus \(T\) of \(K\).

Fang, Littelmann and Pabiniak [21] gave a uniform proof for the conjectured Gromov width of rational coadjoint orbits of all compact connected simple Lie groups by analyzing simplices in Newton-Okounkov bodies.

**Theorem 5.17.** (Fang, Littelmann and Pabiniak [2017]) Let \(K\) be a compact connected Lie group. Then the Gromov width of a coadjoint orbit \(O_\lambda\) through a point \(\lambda\) lying on some rational line in \(\mathfrak{t}^*\), equipped with the Kostant-Kirillov-Souriau symplectic form \(\omega_\lambda\), is equal to

\[
\min \{ |\langle \lambda, \alpha^\vee \rangle| | \alpha^\vee \text{ is a coroot and } \langle \lambda, \alpha^\vee \rangle \neq 0 \}.
\]

Here \(\mathfrak{t}^*\) is the dual of the Lie algebra of a maximal torus \(T\) of \(K\).

Furthermore they proved the following fact in [21].

**Proposition 5.1.** Let \(K\) be a compact connected Lie group, not of type \(G_2, F_4\) or \(E_8\) and let \((O_\lambda, \omega_\lambda)\) be its generic coadjoint orbit through a point \(\lambda\) lying on some rational line in \(\mathfrak{t}^*\), equipped with the Kostant-Kirillov-Souriau symplectic form \(\omega_\lambda\). Then there exists a symplectic embedding of a ball of capacity (2.29) into \((O_\lambda, \omega_\lambda)\).
5.3. The theory of the Gromov-Witten invariants

Let \((M, \omega, J)\) be a symplectic manifold of dimension \(2n\) with a \(\omega\)-compatible almost complex structure \(J\). For two nonnegative integers \(g, k \geq 0\), let \(\mathcal{M}_{g,k}\) be the Deligne-Mumford moduli space of stable curves of genus \(g\) with \(k\) marked points, and \(\mathcal{M}_{g,k,A}\) be the moduli space of stable maps into \(M\) of homology class \(A \in H_2(M, \mathbb{Z})/\mathrm{torsion}\). The elements of \(\mathcal{M}_{g,k,A}\) are of the form

\[
(\Sigma, p_1, \ldots, p_k, f),
\]

where \(\Sigma\) is a (not necessarily stable) curve of genus \(g\) with \(k\) marked points and \(f: \Sigma \rightarrow M\) is a pseudoholomorphic curve. Let

\[
Y := \mathcal{M}_{g,k} \times M^k.
\]

Then we have the evaluation map \(\mathrm{ev}: \mathcal{M}_{g,k,A} \rightarrow Y\) defined by

\[
\mathrm{ev}(\Sigma, p_1, \ldots, p_k, f) = (\mathrm{st}(C, p_1, \ldots, p_k), f(p_1), \ldots, f(p_k)),
\]

where \(\mathrm{st}(C, p_1, \ldots, p_k)\) is the stabilization of \(C\). By the Atiyah-Singer index theorem, we obtain

\[
d := \dim_{\mathbb{R}} \mathcal{M}_{g,k,A} = 2(n-3)(1-g) + 2k + 2c_1(M) \cdot A.
\]

The evaluation map sends the fundamental class of \(\mathcal{M}_{g,k,A}\) to a \(d\)-dimensional rational homology class \(\Phi^A_{g,k} \in H_d(Y, \mathbb{Q})\). The homology class \(\Phi^A_{g,k} \in H_d(Y, \mathbb{Q})\) is called the Gromov-Witten invariant of \(M\) for the data \((g, k, A)\). It is an invariant of the symplectic isotopy class of \((M, \omega)\).

Let us interpret \(\Phi^A_{g,k}\) geometrically. If \(\beta \in H_*(\mathcal{M}_{g,k})\) and \(\alpha_1, \ldots, \alpha_k \in H_*(M)\) such that the sum of the codimensions of \(\beta, \alpha_1, \ldots, \alpha_k\) is equal to \(d\), we define

\[
\Phi^A_{g,k}(\beta, \alpha_1, \ldots, \alpha_k) := \Phi^A_{g,k} \cdot \beta \cdot \alpha_1 \cdots \alpha_k \in H_0(Y, \mathbb{Q}),
\]

where \(\cdot\) denotes the intersection product in \(H_*(Y, \mathbb{Q})\). This is a rational number. Let \(c_i: \mathcal{M}_{g,k,A} \rightarrow M\) \((1 \leq i \leq k)\) be the evaluation map at the \(i\)-th marked point. If \(\delta_1, \ldots, \delta_k \in H^*(M, \mathbb{Z})/\mathrm{torsion}\) such that \(\sum_{i=1}^k \deg(\delta_i) = d\), we put

\[
\Phi^A_{g,k}(\delta_1, \ldots, \delta_k) := \int_{\mathcal{M}_{g,k,A}} e_1^*\delta_1 \wedge \cdots \wedge e_k^*\delta_k.
\]

The number \(\Phi^A_{g,k}(\delta_1, \ldots, \delta_k)\) can roughly be understood as the number of pseudoholomorphic curves of genus \(g\) representing the homology class \(A\) and intersecting \(k\) given cycles \(\mathrm{PD}(\delta_i)\) Poincaré dual to the cohomology classes \(\delta_i\) \((1 \leq i \leq k)\).

E. Witten [69] defined the so-called Gromov–Witten potential \(\Phi^M_\omega: H^*(M, \mathbb{C}) \rightarrow \mathbb{C}\) by

\[
\Phi^M_\omega(\delta_0, \delta_1, \ldots, \delta_{2n}) := \sum_k \sum_A \sum_{i_1, \ldots, i_k} \frac{\exp(-\int_A \omega)}{k!} \Phi^A_{g,k}(\delta_{i_1}, \ldots, \delta_{i_k}),
\]

where \(\delta_i \in H^i(M, \mathbb{C})\) \((0 \leq i \leq 2n)\), \(A\) runs over \(H_2(M, \mathbb{Z})\) and unordered \(i_1, \ldots, i_k \in \{0, 1, \ldots, 2n\}\) with \(\sum_{i=1}^k \deg(\delta_{i_k}) = d\). The convergence problem arises. The fact
that the Gromov-Witten potential $\Phi^M_\omega$ satisfies the WDVV equations was proved by Y. Ruan and G. Tian [55]. Maxim Kontsevich and Yuri Manin [37] described how the WDVV equations yield a potential Dubrovin structure on $H^*(M, \mathbb{C})$, understood as a supermanifold. Therefore each tangent space of $H^*(M, \mathbb{C})$ is endowed with a metric given by Poincaré duality, and a multiplication

$$x \ast y = \sum_{i,j,k} A_{ij}^k x_i y_j \phi_k, \quad A_{ij}^k = \sum_\ell \partial_i \partial_j \partial_\ell \Phi^M_\omega g_{k\ell},$$

where the set $\{\phi_i\}$ is a homogeneous basis of $H^*(X, \mathbb{C})$ with $\phi_i \cdot \phi_j = g_{ij}$, the $(g^{ij})$ denotes the inverse matrix $(g_{ij})$, $x = \sum_i x_i \phi_i$ and $y = \sum_i y_i \phi_i$. This is the quantum deformation of the cup product and the WDVV equations are equivalent to the associativity of the multiplication. The ordinary cup product is the limit of $\Phi^M_\omega$ with $t \to \infty$.

Y. Ruan and G. Tang [55] nicely developed a theory of quantum cohomology which is related to symplectic topology, algebraic geometry, quantum field theory, mirror symmetry, and differential topology in 4-dimensional manifolds. They first constructed the Gromov-Witten theory for semi-Fano symplectic manifolds. Thereafter the Gromov-Witten theory was generalized by other people. The quantum cohomology ring endows the affine space $H^*(X, \mathbb{C})$ with the structure of a Frobenius manifold, namely, a Riemannian manifold with an associative product on the tangent spaces and various compatibilities. C. Taubes [62, 63] proved that in 4-dimensional symplectic manifolds, certain Gromov-Witten invariants coincide with the gauge-theoretic Seiberg-Witten invariants. This relates symplectic topology and differential topology via the Gromov-Witten theory. W. Chen and Y. Ruan [8, 9] defined the Gromov-Witten invariants for compact symplectic orbifolds extending the Gromov-Witten invariants for compact symplectic manifolds. Based on a proposal by E. Witten, H. Fan, T. Jarvis and Y. Ruan [22] introduced and developed the new Gromov-Witten type theory of geometric invariants, known as the FJRW theory that is the mathematically rigorous development of topological gravity coupled with $A$-type topological Landau-Ginzburg matter, as an intersection theory on the moduli space of solutions of the Witten equation. The FJRW theory is believed to be the counterpart of the Gromov-Witten theory in the Landau-Ginzburg model [10, 11, 38, 58]. The relationship between these two theories is referred to as Landau-Ginzburg/Calabi-Yau (briefly LG/CY) correspondence.

Recently Y. Shen and J. Zhou [60] proved the LG/CY correspondence between the Gromov-Witten theories of elliptic orbifold curves $\mathbb{P}_{3,3,3}$, $\mathbb{P}_{4,4,2}$, $\mathbb{P}_{6,3,2}$ and $\mathbb{P}_{2,2,2,2}$ and their FJRW theory counterparts via the theory of quasi-modular forms. We briefly describe this correspondence. Let $W : \mathbb{C}^3 \to \mathbb{C}$ be the weighted homogeneous polynomial with weights $q_1, q_2, q_3$, the so-called superpotential of the LG-model that satisfies the Calabi-Yau condition $q_1 + q_2 + q_3 = 1$. Let

$$G_W := \{(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3 \mid W(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) = W(x_1, x_2, x_3)\}$$
be the group of diagonal symmetries and let $G$ be a subgroup of $G_W$ containing the exponential grading element

\[(5.11) \quad J_W := (\exp(2\pi i q_1), \exp(2\pi i q_2), \exp(2\pi i q_3)), \quad i = \sqrt{-1}.\]

The hypersurface $X_W$ defined by $\{W = 0\}$ is a one-dimensional Calabi-Yau variety in a weighted projective space. Then $G_W$ acts on $X_W$, and $J_W$ acts trivially. Thus we obtain the CY orbifold curve which is a global quotient

\[(5.12) \quad \mathfrak{x}_W := X_W/(G/(\langle J_W \rangle)).\]

The elliptic orbifold curves $\mathfrak{x}_W$ are as follows:

\[(5.13) \quad W = x_1^3 + x_2^3 + x_3^3, \quad G = G_W, \quad \mathfrak{x}_W = \mathbb{P}_{3,3,3};\]
\[(5.14) \quad W = x_1^4 + x_2^4 + x_3^2, \quad G = G_W, \quad \mathfrak{x}_W = \mathbb{P}_{4,4,2};\]
\[(5.15) \quad W = x_1^6 + x_2^3 + x_3^3, \quad G = G_W, \quad \mathfrak{x}_W = \mathbb{P}_{6,3,2};\]
\[(5.16) \quad W = x_1^4 + x_2^4 + x_3^2, \quad G = G_1 \times G_{x^2}, \quad \mathfrak{x}_W = \mathbb{P}_{2,2,2,2},\]

where $G_1 := \{(i, i), (1, -1)\}$.

For the pair $(W, G)$, both GW theory and FJRW theory come with a graded vector space equipped with a non-degenerate pairing, which we denote by

\[\left(\mathcal{H}^{GW}, \eta^{GW}\right), \quad \left(\mathcal{H}^{FJRW}, \eta^{FJRW}\right).\]

Here $\mathcal{H}^{GW}$ is the Chen-Ruan cohomology [8, 9] of $\mathfrak{x}_W$, and $\mathcal{H}^{FJRW}$ is the FJRW state space [22] of $(W, G)$. Let $\mathcal{M}_{g,k}$ be the Deligne-Mumford moduli space of $k$-pointed stable curves of genus $g$ and $\psi_j \in H^2(\mathcal{M}_{g,k}, \mathbb{Q})$ be the $j$-th $\psi$-class. Let $\beta$ be an effective curve class in the underlying coarse moduli of $\mathfrak{x}_W$, $\{\alpha_j\}$ be elements in $\mathcal{H}^{GW}$ and $\{\gamma_j\}$ be elements in $\mathcal{H}^{FJRW}$. Then one can define the ancestor GW invariant $\langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_k \psi_k^{\ell_k}\rangle_{g,k,\beta}^{GW}$ (cf. [60] p. 6 (2.2)) and the FJRW invariant $\langle \gamma_1 \psi_1^{\ell_1}, \cdots, \gamma_k \psi_k^{\ell_k}\rangle_{g,k}^{FJRW}$ (cf. [60] p. 9 (2.13)).

We parametrize a Kähler class $\mathcal{P} \in \mathcal{H}^{GW}$ by $t$ and set $q = e^t$. The Divisor Axiom in GW theory allows us define a GW correlation function as a formal $q$-series

\[(5.17) \quad \langle \langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_k \psi_k^{\ell_k}\rangle_{g,k}^{GW}(q) := \sum_{\beta} \langle \alpha_1 \psi_1^{\ell_1}, \cdots, \alpha_k \psi_k^{\ell_k}\rangle_{g,k,\beta}^{GW} q^f_{\beta} \mathcal{P}.\]

The GW invariants give rise to various structures on $\mathcal{H}^{GW}$. Among them the quantum multiplication $\star_q$ is defined by

\[(5.18) \quad \alpha_1 \star_q \alpha_2 := \sum_{\mu, \nu} \langle \langle \alpha_1, \alpha_2, \mu \rangle \rangle_{\alpha, \beta}^{GW} \eta^{GW}_{\alpha, \beta} \nu.\]

Here both $\mu, \nu$ run over a basis of $\mathcal{H}^{GW}$ and $\eta^{GW}_{\alpha, \beta}$ is the inverse of the pairing $\eta^{GW}_{\alpha, \nu}$. At the large volume limit $t = -\infty$, the quantum multiplication $\star_q$ becomes the Chen-Ruan product.
Similarly we parametrize a degree 2 element \( \phi \in \mathcal{H}^{FJRW} \) by \( u \) and define an FJRW correlation function
\[
(5.19) \quad \langle \langle \gamma_1 \psi_1^{f_1}, \ldots, \gamma_k \psi_1^{f_k} \rangle \rangle_{g,k}^{FJRW} (u) := \sum_{n \geq 0} \frac{u^n}{n!} \langle \gamma_1 \psi_1^{f_1}, \ldots, \gamma_k \psi_1^{f_k}, \phi, \ldots, \phi \rangle_{g,k+n}^{FJRW}.
\]

We have a Frobenius algebra \( (\mathcal{H}^{FJRW}, \bullet) \), where the multiplication \( \bullet \) is defined from the pairing \( \eta_{FJRW} \) on \( \mathcal{H}^{FJRW} \) and the genus zero 3-point invariants through the following formula
\[
(5.20) \quad \eta_{FJRW}^{(\gamma_1, \gamma_2, \cdot \cdot \cdot, \gamma_3)} = \langle \gamma_1, \gamma_2, \cdot \cdot \cdot, \gamma_3 \rangle_{g,k}^{FJRW}.
\]

We define the quantum multiplication \( \bullet_u \) by
\[
(5.21) \quad \gamma_1 \bullet_u \gamma_2 := \sum_{\gamma, \zeta} \langle \langle \gamma_1, \gamma, \gamma \rangle \rangle_{g,3}^{FJRW} \eta_{FJRW}^{(\gamma, \zeta)} \cdot \zeta.
\]

Here both \( \gamma, \zeta \) run over a basis of \( \mathcal{H}^{FJRW} \) and \( \eta_{FJRW}^{(\cdot, \cdot)} \) is the inverse of the pairing \( \eta_{FJRW}^{(\cdot, \cdot)} \). The quantum multiplication \( \bullet_u \) is a deformation of the multiplication \( \bullet \) in (5.25) as \( \bullet_{u=0} = \bullet \).

We refer to [58, 59, 60] for more precise details on \( (\mathcal{H}^{GW}, \eta^{GW}) \) and \( (\mathcal{H}^{FJRW}, \eta_{FJRW}) \). The LG/CY correspondence [69, 22] says that the two enumerative theories should be equivalent under an appropriate transformation.

Let \( \mathbb{H} \) (resp. \( \mathbb{D} \)) be the Poincaré upper half plane (resp. the Poincaré unit disk). We denote by \( \tilde{M}(\Gamma) \) (resp. \( \tilde{M}(\Gamma) \)) the ring of almost-holomorphic modular forms (resp. quasi-modular forms) on \( \mathbb{H} \) for an arithmetic subgroup \( \Gamma \) of \( SL(2, \mathbb{Z}) \). Let \( \mathcal{C}_H^\omega \) (resp. \( \mathcal{C}_D^\omega \)) be the ring of real analytic functions on \( \mathbb{H} \) (resp. \( \mathbb{D} \)). We denote by \( \mathcal{O}_D \) the ring of holomorphic functions on \( \mathbb{D} \). Shen and Zhou [60] introduced the Cayley transform \( \mathcal{C} : \tilde{M}(\Gamma) \subset \mathcal{C}_H^\omega \rightarrow \mathcal{C}_D^\omega \) and its variant \( \mathcal{C}_{hol} : \tilde{M}(\Gamma) \rightarrow \mathcal{O}_D \). They are induced by the Cayley transform \( T : \mathbb{H} \rightarrow \mathbb{D} \) based at a point \( \tau_0 \in \mathbb{H} \) given by
\[
(5.22) \quad T(\tau) := \frac{\tau - \tau_0}{\tau - \tau_0 - \frac{\tau_0}{\tau - \tau_0}}, \quad \tau \in \mathbb{H}.
\]

Using the quasi-modularity of the Gromov-Witten correlation functions and the Cayley transform \( \mathcal{C} \) and its variant \( \mathcal{C}_{hol} \), they proved the following LG/CY correspondence:

**Theorem 5.18.** Let \( (W, G) \) be a pair in (5.12)-(5.16). Then there exists a degree and pairing isomorphism between the graded vector spaces
\[
(5.23) \quad \mathcal{G} : (\mathcal{H}^{GW}, \eta^{GW}) \rightarrow (\mathcal{H}^{FJRW}, \eta_{FJRW})
\]
and the Cayley transform \( \mathcal{C}_{hol} \), based at an elliptic point \( \tau_0 \in \mathbb{H} \), such that for any \( \{\alpha_j\} \subseteq \mathcal{H}^{GW} \),
\[
(5.24) \quad \mathcal{C}_{hol} \left( \langle \langle \alpha_1 \psi_1^{f_1}, \ldots, \alpha_k \psi_1^{f_k} \rangle \rangle_{g,k}^{GW} (q) \right) = \langle \langle \mathcal{G} (\alpha_1) \psi_1^{f_1}, \ldots, \mathcal{G} (\alpha_k) \psi_1^{f_k} \rangle \rangle_{g,k}^{FJRW} (u).
\]

Here \( \psi_j \in H^2(\overline{M}_{g,k}, \mathbb{Q}) \) is the \( j \)-th \( \psi \)-class and \( q = e^t \) where \( t \) was parametrized by a Kähler class \( \mathcal{P} \in \mathcal{H}^{GW} \).
REMARKS ON SYMPLECTIC GEOMETRY

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