The Hydrodynamic Representation of the Klein-Gordon Equation with Self-Interacting Field

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Abstract

In this paper the quantum hydrodynamic approach for the KGE owing a self-interaction term is developed both for scalar and charged boson. The model allows to determine the quantum energy impulse tensor density of massive bosons such as the mesons. The generalization of the hydrodynamic Klein-Gordon equation to the non-Euclidean space-time is also derived for a quantum relativistic approach.

Keywords: Quantum hydrodynamic representation; Bhom-Madelung approach; Self-interacting field; Non-Euclidean quantum hydrodynamics

Introduction

Since the introduction of the quantum wave equation by Schrödinger, the quantum hydrodynamic approach (QHA) was presented by Madelung [1]. In this quantum representation, developed by Madelung and then by Bhom, the evolution of a complex variable \( \psi = |\psi| \exp \frac{S}{\hbar} \) is solved as a function of the two real variables, \(|\psi|\) and \(S\) [2-5]. As shown by Weiner et al. [6], the outputs of the quantum hydrodynamic model agree with the outputs of the Schrödinger problem and, more recently, as shown by Koide and Kodama [7], it agrees with the outputs of the stochastic variational method.

Recently, the author has shown that the hydrodynamic approach is strictly correlated to the properties of vacuum on small scale [8].

Moreover, as shown by Bohm and Hiley [9,10] the hydrodynamic approach can be generalized for the description of the quantum fields.

The present work develops the quantum hydrodynamic form of the Klein-Gordon equation (KGE) containing an additional self-interaction term.

The interest in obtaining such a description lies in the fact that such type of KGE can describe the states of bosons, such as mesons. The goal of the paper is to obtain the energy-impulse tensor density of such particles that can be useful in the coupling the field of a meson with the Einstein equation [11]. The paper is organized as follows: in the section 2 the hydrodynamic KGE with a self-interaction term is derived for an uncharged scalar particle as well as the Lagrangean motion equations for the eigenstates and the associated energy impulse tensor density. In the subsection 2.2 the theory is developed for a charged field. In section 3. the formulas are generalized to a non-Euclidean space-time.

The Hydrodynamic KGE with Self-Interacting Field

In this section, the Euclidean hydrodynamic representation of the KGE is derived for a scalar uncharged particle with a self-interaction term that reads

\[
\hat{\partial}_\mu \hat{\partial}^\mu \psi = -\frac{1}{2} \frac{\partial^2}{\partial^2} \psi + j_\psi \psi
\]

where \(j_\psi = -\frac{\partial V(\psi)}{\partial \psi}\) and where, for instance, we assume the quartic renormalizable interaction \(V(\psi) = \frac{\lambda}{4} \psi^4\). Following the procedure given in reference [11,12] (for the ordinary KGE) the hydrodynamic equations of motion are given by the Hamilton-Jacobi type equation

\[
\frac{\partial \hat{S}(q,t)}{\partial q} \hat{\partial}^\mu \hat{\partial}^\mu \psi = -h^2 \left( \hat{\partial}_\mu \hat{\partial}^\mu |\psi| + \lambda |\psi|^4 \right) - m^2 c^2 = 0
\]

coupled to the current equation [2]

\[
\frac{\partial}{\partial q^\mu} \left( \psi^2 \frac{\partial S}{\partial q^\mu} \right) = m \frac{\partial S}{\partial q^\mu} = 0
\]

and where

\[
J\psi = (c \rho, -J_\psi) = \frac{i\hbar}{2m} \left( \frac{\partial \psi}{\partial q^\mu} - \frac{\partial \psi^*}{\partial q^\mu} \right)
\]

Moreover, being the 4-impulse in the hydrodynamic analogy

\[
P\mu = \left( \frac{E}{c}, -p\right) = \frac{\partial \hat{S}}{\partial q^\mu}
\]

it follows that

\[
J\mu = (c \rho, -J_\mu) = -|\psi|^2 \frac{P\mu}{m}
\]

where

\[
\rho = \frac{|\psi|^2}{mc^2} \partial \hat{S} \partial
\]

Moreover, by using (6), equation (2) can be rewritten as

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\[
\frac{\partial S}{\partial q^\mu} \frac{\partial S}{\partial q^\mu} = p_\mu p^\mu = \left( \frac{E^2}{c^2} - p^2 \right) = m^2 c^2 \left( 1 - \frac{V_{qu}(n)}{mc^2} \right)
\]
where
\[
V_{qu} = -\frac{\hbar^2}{m} \left( \frac{\partial \mu (q^\mu | \psi^* | q^\mu | \psi |)^2}{\gamma |q^\mu |} + \lambda |q^\mu |^2 \right)
\]
and where \( P^\nu = P_i P^i \) is the modulus of the spatial momentum.

As shown in reference [11], given the hydrodynamic Lagrangean function
\[
L = \frac{dS}{dt} = \frac{\partial S}{\partial \dot{q}^\mu} \frac{\partial S}{\partial \dot{q}^\mu} - \bar{p}_{ij} \dot{q}_i \dot{q}_j = -P_\mu \dot{q}^\mu
\]
(11)
equation (2) can be expressed by the following system of Lagrangean equations of motion
\[
p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}
\]
(12)
\[
\dot{p}_\mu = \frac{\partial L}{\partial \dot{q}^\mu}
\]
(13)
that for the eigenstates read
\[
p_{n\mu} = -\frac{\partial L_{n\mu}}{\partial \dot{q}^\mu}
\]
(14)
\[
\dot{p}_{n\mu} = -\frac{\partial L_{n\mu}}{\partial \dot{q}^\mu}
\]
(15)
where
\[
L_n = (\pm) - mc^2 \gamma \left[ 1 - \frac{V_{qu}(n)}{mc^2} \right] = (\pm) - mc^2 \gamma \left[ 1 + \frac{\hbar^2}{m^2 c^2} \left( \frac{\partial \mu \partial \mu |q^\mu |^2 + \lambda |q^\mu |^2}{\gamma |q^\mu |^2} \right) \right]
\]
(16)

Generally speaking, for eigenstates, for which it holds \( E = E_\mu = \text{const} \) it follows that:
\[
\left( \frac{E^2}{c^2} - p^2 \right) = m^2 c^2 \left( 1 - \frac{V_{qu}(n)}{mc^2} \right) = m^2 c^2 \left( 1 - \frac{V_{qu}(n)}{mc^2} \right)
\]
(17)

from where it follows that
\[
E_n = \pm mc^2 \sqrt{1 - \frac{V_{qu}(n)}{mc^2}} = \pm mc^2 \sqrt{1 + \frac{\hbar^2}{m^2 c^2} \left( \frac{\partial \mu \partial \mu |q^\mu |^2 + \lambda |q^\mu |^2}{\gamma |q^\mu |^2} \right)}
\]
(18)

(22) where the minus sign stands for antiparticles) and, by using (17), that
\[
p_{n\mu} = \pm mc \frac{\partial L_{n\mu}}{\partial \dot{q}^\mu} = \pm mc \frac{\partial L_{n\mu}}{\partial \dot{q}^\mu} = \pm mc \frac{\partial L_{n\mu}}{\partial \dot{q}^\mu}
\]
(19)

Following the hydrodynamic protocol [11], the eigenstates are represented by the stationary solutions of the hydrodynamic equations of motion obtained by deriving \( P_i (\dot{q}, \dot{q}) \) from (14) and then inserting it into (15) that leads to
\[
\frac{dp_{n\mu}}{ds} = -\gamma \frac{\partial L_{n\mu}}{\partial \dot{q}^\mu} = -\gamma \frac{dL_{n\mu}}{ds c^2} \left( \frac{V_{qu}(n)}{mc^2} \right)
\]
(20)

where
\[
u = \frac{\dot{q}}{c} \frac{\dot{q}^\mu}{c}
\]
and to
\[
\pm mc \left[ 1 - \frac{V_{qu}(n)}{mc^2} \right] \frac{dL_{n\mu}}{ds c^2} = \pm mc \left[ 1 - \frac{V_{qu}(n)}{mc^2} \right] \frac{dL_{n\mu}}{ds c^2}
\]
(21)

where, for eigenstates, the quantum energy-impulse tensor (QET) \( T_{\mu\nu} \) reads [11,12],
\[
T_{\mu\nu} = \left( \frac{\partial L_{n\mu}}{\partial \dot{q}^\mu} - L_{n\mu} \delta_{\mu\nu} \right) = \frac{m^2 c^2}{\gamma} \left( \frac{V_{qu}(n)}{mc^2} \right) \left( u_{\mu\nu} - \delta_{\mu\nu} \right)
\]
(22)
leading to the quantum energy impulse tensor density (QIETD) [11,12],
\[
T_{\mu\nu} \frac{dL_{n\mu}}{ds} = \left( \frac{\partial L_{n\mu}}{\partial \dot{q}^\mu} - L_{n\mu} \delta_{\mu\nu} \right) \left( u_{\mu\nu} - \delta_{\mu\nu} \right)
\]
(23)

where \( L = \psi^* |q^\mu| \psi \) is the (hydrodynamic) Lagrangian density and \( L \) is the hydrodynamic Lagrangian function. Moreover, by using the identity
\[
S_n = \frac{\hbar}{2i} \ln \frac{|q^\mu|}{|q^\mu|^*} \psi \frac{\dot{q}^\mu}{c}
\]
(24)
The QIETD (23) can be written as a function of the wave function as following:
The above equations are coupled to the Maxwell one
\[ F_{\mu\nu} = -4\pi J^\mu \] (38)
where \( J^\mu = \frac{\hbar}{2\imath m} (\psi^* D^\mu - \psi D^\mu \psi^*) \) is the current of the charged particles; \( \psi \) and \( \psi^* \) being the Dirac spinors.\(^{(1)} \)

\( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the electromagnetic field tensor.

**Charged field**

In the case of a charged boson field, equations (1-3) read,

\[ D^\mu D_\mu \psi = -\psi \left( \frac{m^2 c^2}{h^2} + \lambda |\psi|^2 \right) \] (26)

where \( D^\mu = \partial^\mu - \frac{e}{\imath \hbar} A^\mu \),

\[ \frac{\partial S(q, t)}{\partial q} + e A^\mu \frac{\partial S(q, t)}{\partial q^\mu} + e A^\mu = m^2 c^2 + \frac{\hbar^2}{2m} \left( \frac{\partial |\psi|^2}{\partial t} + \lambda |\psi|^2 \right) \] (27)

\[ \frac{\partial J_1}{\partial q} = 0 \] (28)

where the 4-current \( J_1 \) reads

\[ J_1 = (\psi \partial_{\psi} - j_1) = \frac{\hbar}{2\imath m} \left( \psi \left( \partial^\mu - \frac{e}{\imath \hbar} A^\mu \right) \psi - \psi \left( \partial^\mu + \frac{e}{\imath \hbar} A^\mu \right) \psi^* \right) \]

\[ = \frac{\hbar}{2\imath m} \left( |\psi|^2 \partial_{\psi^*} + 2 \frac{\partial |\psi|^2}{\partial t} + 2 \frac{\partial |\psi|^2}{\partial A^\mu} \right) \] (29)

and

\[ \frac{\partial S}{\partial q} = \frac{\partial S}{\partial q^\mu} = p^\mu = \pi^\mu + e A^\mu = \left( \frac{E^\mu}{c}, p^\mu \right) \] (30)

(\( \pi^\mu \) is the mechanical momentum) \( [11,13] \) and where

\[ \mathbf{p} = -\frac{|\psi|^2}{mc^2} \left( \frac{\partial S}{\partial q} + \mathbf{e} \mathbf{\phi} \right) \] (31)

Moreover, analogously to (9, 17-19), from (27) it follows that

\[ \pi_{\mu\nu} = \frac{1}{c^2} \left( \frac{\partial S}{\partial q^\mu} + \mathbf{e} \mathbf{\phi} \right) \dot{q}^\nu \]

\[ = \frac{E_{\nu} - e \mathbf{\phi}}{c^2} \dot{q}^\nu = p_{\nu\mu} - e A_{\nu\mu} \] (32)

that leads to

\[ E_{\nu} - e \mathbf{\phi} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ P_{\mu\nu} - e A_{\mu\nu} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} = E_{\nu} - \dot{q}^\nu \] (33)

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ E_{\nu} - \dot{q}^\nu \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ P_{\mu\nu} - e A_{\mu\nu} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} = E_{\nu} - \dot{q}^\nu \] (34)

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ E_{\nu} - \dot{q}^\nu \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ P_{\mu\nu} - e A_{\mu\nu} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} = E_{\nu} - \dot{q}^\nu \] (35)

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ E_{\nu} - \dot{q}^\nu \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ P_{\mu\nu} - e A_{\mu\nu} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} = E_{\nu} - \dot{q}^\nu \] (36)

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ E_{\nu} - \dot{q}^\nu \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ P_{\mu\nu} - e A_{\mu\nu} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} = E_{\nu} - \dot{q}^\nu \] (37)

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ E_{\nu} - \dot{q}^\nu \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ P_{\mu\nu} - e A_{\mu\nu} = \pm mc^2 \sqrt{1 - \frac{V_{\text{qu}(n)}}{mc^2}} = E_{\nu} - \dot{q}^\nu \] (38)

\[ \pm mc^2 \sqrt{1 + \frac{h^2}{b^2 m^2 c^2} \left( \frac{\partial \mathbf{\phi}^2}{\partial |\psi|^2} + \lambda |\psi|^2 \right)^2} \]

\[ E_{\nu} - \dot{q}^\nu \]
\[ A^1 = (\frac{\dot{q}}{c} - A_1) \]  

(40)

is the potential 4-vector.

**Non-Euclidean Generalization**

The quartic self-interaction is introduced in the KGE in order to describe the states of charged \((\pm 1)\) bosons (e.g., mesons) [15]. The importance of having the hydrodynamic description of bosons [11] lies in the fact that it allows to derive its quantum energy-impulse tensor that can couple them to the Einstein quantum-gravitational equation [11].

The generalization of the quantum hydrodynamic formalism to the non-Euclidean space-time can be obtained by using the General Physics Covariance postulate. By using it, it is possible to derive the non-Euclidean expression of the hydrodynamic model of the KGE

\[
\left( \partial_{\mu} \psi \right)_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \psi \right) = -\psi \left( \frac{m^2 c^2}{h^2} + \lambda |\psi|^2 \right)
\]

(41)

Equations (2-3) in a non-Euclidean space read, respectively,

\[
g^{\mu\nu} \frac{\partial S(q,t)}{\partial q^\nu} = m V_{qu} - m^2 c^2 = 0
\]

(42)

\[
\frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \psi \right) = 0
\]

where

\[
V_{qu} = \frac{h^2}{m} \left( \frac{1}{|\psi|} \sqrt{-g} \partial_\nu \left( \sqrt{g^{\mu\nu} \partial_\nu |\psi|} \right) + \lambda |\psi|^2 \right)
\]

(43)

Moreover, by using the definition of the Lagrangean function

\[ L_n = -\frac{dS}{dt} = -g_{\mu\nu} p^\mu q^\nu \]

(44)

the covariant form of the motion equations (14-15) reads

\[
D_t p_{\mu} = \frac{\partial L_n}{\partial \dot{q}^\mu}
\]

(45)

\[
D_t p_{\mu} = \frac{\partial L_n}{\partial \dot{q}^\mu}
\]

(46)

where

\[ D_t p_{\mu} = \dot{p}_{\mu} - \Gamma^\lambda_{\mu\nu} p_\lambda q^\nu \]

(47)

is the total covariant derivative respect the time and where \(\Gamma^\lambda_{\mu\nu}\) are the Christoffel symbols.

Equations (45-46) leads to the motion equation

\[
D_t \left( -\frac{\partial L_n}{\partial \dot{q}^\mu} \right) = -\frac{\partial L_n}{\partial \dot{q}^\mu}
\]

(48)

where, \(L_n\) reads

\[
L_n = (\pm) \frac{mc^2}{\gamma} \sqrt{1 + \frac{V_{qu}(n)}{mc^2}} = \left( \pm \right) \frac{mc^2}{\gamma} \sqrt{1 + \frac{V_{qu}(n)}{mc^2}}
\]

(49)

From (48) it follows that the motion equation reads

\[
\frac{du_\mu}{dt} = \frac{1}{2} \gamma \frac{g^{\mu\lambda} u_\lambda}{\partial_{\dot{q}^\mu}}
\]

(50)

\[
\frac{c}{\gamma} \frac{\partial \ln \left( \frac{V_{qu}(n)}{mc^2} \right)}{\partial q^\mu} = \frac{c}{\gamma} \frac{\partial \ln \gamma}{\partial q^\mu} = \frac{d}{dt} \ln \left( \frac{V_{qu}}{mc^2} \right)
\]

(51)

where the stationary condition \(\frac{du_\mu}{dt} = 0\), that determines the balance between the "force" of gravity and that one of the quantum potential, leads to the stationary equation for the Eigen states

\[
\frac{1}{2} \gamma \frac{g^{\mu\lambda} u_\lambda}{\partial_{\dot{q}^\mu}} + \frac{c}{\gamma} \frac{\partial \ln \gamma}{\partial q^\mu} = \frac{c}{\gamma} \frac{\partial \ln \left( \frac{V_{qu}(n)}{mc^2} \right)}{\partial q^\mu} = 0
\]

(52)

\[
T_{\mu\nu} = \pm |\psi|^2 mc^2 \left[ g_{\mu\nu} + \frac{V_{qu}(n)}{mc^2} \right] - \gamma \frac{d}{dt} \ln \left( \frac{V_{qu}}{mc^2} \right)
\]

(53)

and where the cosmological energy-impulse density \(\Lambda\) [11], for Eigen states, reads

\[
\Lambda = \frac{8\pi G}{c^4} \left( |\psi|^2 \right) - \frac{L_0}{\gamma}
\]

(54)

where, for scalar uncharged particles leads to

\[
L_0 = \lim_{\hbar \to 0} L = (\pm) - \frac{mc^2}{\gamma}
\]

(55)

Finally, it is worth noting that, as a function of the quantum field, the quantum energy impulse tensor density reads

\[
T_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu} \left( 1 - \frac{V_{qu}(n)}{mc^2} \right) \left( \frac{V_{qu}}{mc^2} \right)
\]

(56)
Charged boson in non-Euclidean space-time

The KGE in non-Euclidean space-time for electromagnetic charged boson

\[ \frac{1}{\sqrt{-g}} D\mu \left( \sqrt{-g} D\nu \psi \right) - A_\mu \partial_\mu \psi + \frac{\hbar^2}{2m^2} \left( \frac{\partial^2 \psi}{\partial x^2} + eA_\mu \frac{\partial \psi}{\partial x^\mu} \right) = -e \psi \left( \frac{2m^2 c^2}{\hbar^2} + \lambda \left| \psi \right|^2 \right) \]  

(57)

leads to the hydrodynamic system of equations

\[ \frac{1}{\sqrt{-g}} \frac{\partial}{\partial q} \sqrt{g} \left( g^{\mu \nu} \left| \psi \right|^2 \left( \frac{\partial S}{\partial q^\nu} + eA_\mu \right) \right) = 0 \]  

(58)

\[ V_{q\mu} = -\frac{2\hbar^2}{m} \left( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial q} \sqrt{g} \left( g^{\mu \nu} \left| \psi \right|^2 \right) \right) + \lambda \left| \psi \right|^2 \]  

(59)

where

\[ V_{q\mu} = -\frac{2\hbar^2}{m} \left( \frac{1}{\sqrt{-g}} \frac{\partial}{\partial q} \sqrt{g} \left( g^{\mu \nu} \left| \psi \right|^2 \right) \right) + \lambda \left| \psi \right|^2 \]  

(60)

Moreover, the Lagrangian motion equations read

\[ p_{n\mu} = -\frac{\partial L_n}{\partial q^\mu} \]  

(61)

\[ D_i p_{n\mu} = -\frac{\partial L_n}{\partial q^\mu} \]  

(62)

where

\[ L_n = (\pm) \left[ \frac{m^2 c^2}{\hbar^2} \frac{q^\mu q^\nu q^\rho q^\delta}{\sqrt{g}} + \lambda \left| \psi \right|^2 \frac{g_{\mu \nu} q^\delta q^\rho}{\sqrt{g}} - e g_{\mu \nu} q^\rho q^\delta \right] \]  

(63)

and to the QIETD

\[ T_{n\mu} = (\pm) - m \left| \psi \right|^2 \left( \frac{2m^2 c^2}{\hbar^2} + \lambda \left| \psi \right|^2 \right) \frac{V_{q\mu}(n)}{mc^2} \]  

\[ \begin{pmatrix} u_\mu + \frac{1}{mc^2} \frac{V_{q\mu}(n)}{mc^2} eA_\mu \end{pmatrix} u^\nu - \left( 1 + \left( \frac{1}{mc^2} \frac{V_{q\mu}(n)}{mc^2} \right) \right) \delta_\mu^\nu \]  

(64)

Conclusion

The hydrodynamic approach allows obtaining the quantum energy-impulse tensor density as a function the field of the particle.

The biunique correspondence between the standard quantum mechanics and the hydrodynamic representation [1-6, 17] warrants that the quantum energy-impulse tensor density can be independently defined by the used formalism.

In this work the quantum energy-impulse tensor, for massive bosons described by a KGE with self-interacting field is derived for defining the coupling with the quantum gravitational equation.

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