Density of rational curves on $K3$ surfaces

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Abstract Using the dynamics of self rational maps of elliptic $K3$ surfaces together with deformation theory, we prove that the union of rational curves is dense on a very general $K3$ surface and that the union of elliptic curves is dense in the 1st jet space of a very general $K3$ surface, both in the strong topology.

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1 Introduction

1.1 Density of rational curves

The main purpose of this note is to prove that the union of all rational curves on a “very general” projective $K3$ surface $X$ is dense in the usual topology. Here “very general” takes some explanation. It is weaker than the usual sense of being in the complement of countably many closed proper subvarieties.

Let $\mathcal{K}_g$ be the moduli space of $K3$ surfaces of genus $g \geq 2$ and $S_g$ be the universal family over $\mathcal{K}_g$. That is,
\[ K_g = \{(X, L) : X \text{ is a } K3 \text{ surface}, L \in \text{Pic}(X) \text{ is ample primitive and } L^2 = 2g - 2\} \]  
(1.1)

and \( S_g = \{(X, L, p) : (X, L) \in K_g, p \in X\} \).

Let \( C_{g,n} \subset S_g \) be a closed subscheme of \( S_g \) whose fiber over a general point \((X, L) \in K_g\) is the union of all irreducible rational curves in the linear series \(|nL| = \mathbb{P}H^0(X, nL)\). Our main theorem is

**Theorem 1.1** For all \( g \geq 2 \), the set

\[
\bigcup_{n=1}^{\infty} C_{g,n}
\]

(1.2)

is dense in \( S_g \).

Using an elementary topological argument, we can easily conclude the following (actually equivalent) statement.

**Corollary 1.2** For all \( g \geq 2 \), the set

\[
\left\{(X, L) \in K_g : \bigcup_{n=1}^{\infty} C_{X,nL} \text{ is not dense in } X\right\}
\]

(1.3)

is of the first Baire category, i.e., a countable union of nowhere dense subsets in \( K_g \) under the usual topology, where \( C_{X,nL} \) is the fiber of \( C_{g,n} \) over \((X, L)\). Hence the set of \( K3 \) surfaces of genus \( g \) whose rational curves are dense is of the second Baire category.

This partially answers a question raised in [8] (Conjecture 1.2), although we expect that the union of rational curves are dense on every projective \( K3 \) surface, not only the general ones. However, the method here does not lend itself to handle every projective \( K3 \) surface. On the other hand, it is unknown whether the union of rational curves is dense in the Zariski topology on every projective \( K3 \) surface.

**Conjecture 1.3** The union of rational curves is dense in the Zariski topology on every projective \( K3 \) surface \( X \). That is, there are infinitely many rational curves on \( X \).

**Remark 1.4** This was known for a very general \( K3 \) surface using a deformational argument [22]. However, to deal with every projective \( K3 \) surface, some new methods are needed. Recently, some substantial progress has been made on the conjecture. This was proved in [2] for \( g = 2 \) and \( \text{Pic}(X) = \mathbb{Z} \) using characteristic \( p \) reduction. Their method was further developed in [20], where the conjecture was settled in all major cases with the only exception rank \( \text{Pic}(X) = 2 \) (see further remark below). Also the conjecture was known for all elliptic \( K3 \) surfaces [5] (see also [12]). So the only unknown cases are \( K3 \) surfaces of Picard rank two which do not admit an elliptic fibration. However, their method does not seem to apply to the strong topology. Density of rational curves on \( K3 \) surfaces in both Zariski and strong topologies is related to Lang’s conjecture on these surfaces [18].
Remark 1.5 It was pointed out to us by the referee that Zariski density of rational curves is known for “most” $K_3$ surfaces of Picard rank 2 since such a surface $X$ either has an infinite automorphism group or admits an elliptic fibration if $X$ does not contain a $(-2)$-curve. So the only outstanding cases of Conjecture 1.3 are $K_3$ surfaces of Picard rank 2 containing $(-2)$-curves, e.g., a quartic surface with a node.

Although we are unable to prove the density of rational curves on every $K_3$ surface in the strong topology, we can do this for an elliptic $K_3$ surface $X$ as long as the elliptic fibration $X \to \mathbb{P}^1$ admits a rational non torsion (nt) multisection. Here a rational nt multisection $C$ of $\pi: X \to \mathbb{P}^1$ is an irreducible rational curve $C \subset X$ such that $C$ meets the general fiber $X_b$ of $\pi$ at (at least) one point $p$ satisfying $L - mp \not\in J(X_b)_{\text{tors}}$, where $L$ is an ample line bundle on $X$, $m = L \cdot X_b$ and $J(X_b) = \text{Pic}^0(X_b)$ is the Jacobian of the elliptic curve $X_b$ [5].

Using the classical Kronecker’s theorem (see 2.2) together with a study of normal functions associated to an elliptic fibration, we are able to prove that

**Theorem 1.6** The union of rational curves is dense in the strong topology on an elliptic $K_3$ surface $\pi: X \to \mathbb{P}^1$ if there exists a rational nt multisection $C$ of $\pi$.

Remark 1.7 The notion of rational nt multisections is also crucial in the work of Bogomolov–Tschinkel [5] and Hassett–Tschinkel [15]. Originally, we quoted a result in the previously mentioned paper [5, Theorem 1.8] that every elliptic surface $X$ with rank $\text{Pic}(X) \leq 19$ has a rational nt multisection. It was again pointed out to us by the referee that one of the constituent lemmas in that paper [5, Lemma 3.26] is incorrect, unfortunately (see [15] for a counterexample). At the moment, the above statement is still unknown and we cannot yet conclude the density of rational curves on every elliptic $K_3$ surface of Picard rank $\leq 19$.

We want to point out that density of rational curves on elliptic $K_3$ surfaces does not imply the same on a general $K_3$ surface directly despite the fact that elliptic $K_3$ surfaces are dense in the moduli space of $K_3$ surfaces since a rational nt multisection does not deform to a rational curve on a general $K_3$ unless it is a multiple of the polarization divisor.

1.2 Density of elliptic curves

For convenience, we will call a point *Baire general* if it lies in the complement of a countable union of nowhere dense subsets.

Of course, every $K_3$ surface $X$ is covered by one-parameter families of elliptic curves. It is natural to ask whether these curves are dense when lifted to the first jet space $\mathbb{P}T_X$ of $X$. Here the lifting $df: C \to \mathbb{P}T_X$ of a map $f: C \to X$ is induced by the map $f_*: T_C \to f^*T_X$ on the tangent sheaves.

For every $n \in \mathbb{Z}^+$, we let $\mathcal{W}_{g,n}$ be the closure of the subscheme of $\mathbb{P}H^0(S_g, nL)$ whose fiber over a general $(X, L)$ consists of irreducible elliptic curves in $|nL|$ and let

$$\mathcal{E}_{g,n} = \{(X, L, E, p): (X, L, E) \in \mathcal{W}_{g,n}, p \in E \} \subset \mathcal{W}_{g,n} \times_{\mathcal{M}_g} \mathcal{S}_g \quad (1.4)$$

be the universal family over $\mathcal{W}_{g,n}$. 

\footnote{Springer}
**Theorem 1.8** Let $\varphi : E_{g,n} \to \mathbb{P}T_{S_g/\mathcal{K}_g}$ be the rational map induced by the map

$$T\mathcal{E}_{g,n}/\mathcal{W}_{g,n} \to T\mathcal{S}_g/\mathcal{K}_g$$

on the relative tangent sheaves. Then

$$\bigcup_{n=1}^{\infty} \varphi(E_{g,n})$$

is dense in $\mathbb{P}T_{S_g/\mathcal{K}_g}$ for all $g \geq 2$, where $\varphi(E_{g,n})$ is the proper transform of $E_{g,n}$ under $\varphi$.

It follows that the union of $\varphi(E_{X,n,L})$ is dense in $\mathbb{P}T_X$ for a Baire general $(X, L) \in \mathcal{K}_g$, where $E_{X,n,L}$ is the fiber of $E_{g,n}$ over the point $(X, L) \in \mathcal{K}_g$.

### 1.3 Hyperbolic geometry of $K3$ surfaces

One of the reasons we are interested in the elliptic curves on a $K3$ surface $X$ comes from the fact that they are the images of holomorphic maps $\mathbb{C} \to X$. So they are closely related to the hyperbolic geometry of $X$. Let us recall the definition of the Kobayashi–Royden (KR) pseudo-metric on a complex manifold $X$ (cf. [17]): for a point $p \in X$ and a nonzero tangent vector $v \in T_X, p$, we define

$$||v||_\kappa = \inf\{\lambda > 0 : \exists \text{ a holomorphic map } f : \Delta \to X \text{ with } f(0) = p, f_* (\partial/\partial z) = \lambda^{-1} v\}$$

(1.7)

Obviously, if there is a holomorphic map $f : \mathbb{C} \to X$ such that $f(0) = p$ and $f_* (\partial/\partial z) = v$ for some tangent vector $v \in T_X, p$, then $||v||_\kappa = 0$. In particular, if there is holomorphic dominant map $f : \mathbb{C}^n \to X$, then the KR pseudo-metric vanishes everywhere on $X$. Buzzard and Lu [4] classified all the algebraic surfaces that are holomorphically dominable by $\mathbb{C}^2$. They settled every single case except $K3$ surfaces, for which they proved all elliptic and Kummer $K3$ surfaces can be holomorphically dominated by $\mathbb{C}^2$. But it is unknown whether a general $K3$ surface can be dominated by $\mathbb{C}^2$ or has everywhere vanishing KR pseudo-metric, although this is expected to be true.

**Conjecture 1.9** (Buzzard–Lu) Every complex $K3$ surface is holomorphically dominable by $\mathbb{C}^2$. As a consequence, it has everywhere vanishing KR pseudo-metric.

By Theorem 1.8, we know at least that the following holds.

**Corollary 1.10** For $g \geq 2$, a Baire general $(X, L) \in \mathcal{K}_g$ and a Baire general $p \in X$, the set $\{v \in T_X, p : ||v||_\kappa = 0\}$ is dense in $T_X, p$.

The layout of this paper is as follows. We will prove our main theorems in Sect. 2. In Sect. 3, we will re-interpret a key step of our proof in terms of Poincaré normal functions and prove Theorem 1.6.

We are grateful to the referee for suggesting improvements to our paper.
2 Proofs of Theorem 1.1 and 1.8

2.1 Elliptic $K3$ surfaces

Our strategy is to show that rational curves are dense on $X$ for $(X, L)$ in a dense subset of $\mathcal{K}_g$. Then Theorem 1.1 will follow easily. It is well known that Kummer surfaces are dense in the moduli space of polarized $K3$ surfaces. This implies that polarized elliptic $K3$ surfaces are dense. A general projective elliptic $K3$ surface has Picard lattice given by

$$\begin{bmatrix}
2g - 2 & m \\
m & 0
\end{bmatrix}$$

(2.1)

where $m$ is a positive integer.

That is, the Picard group $\text{Pic}(X)$ of $X$ is generated by effective classes $L$ and $F$ satisfying

$$L^2 = 2g - 2, \quad LF = m \quad \text{and} \quad F^2 = 0$$

(2.2)

and the elliptic fibration $\pi : X \to \mathbb{P}^1$ is given by the pencil $|F|$.

Let $\mathcal{P}_{g,m}$ be the moduli space of the triples $(X, L, F)$, where $X$ is a $K3$ surface whose Picard lattice contains (2.1), generated by $L$ and $F$, as a primitive sublattice. Slightly abusing terminology, we sometimes treat $\mathcal{P}_{g,m}$ as a subscheme of $\mathcal{K}_g$; more precisely, it is a finite cover of a subscheme of $\mathcal{K}_g$.

The general theory of $K3$ surfaces tells us that $\mathcal{P}_{g,m}$ is irreducible of codimension 1 in $\mathcal{K}_g$ for each pair $(g, m)$. Also the union of $\mathcal{P}_{g,m}$ is dense in $\mathcal{K}_g$, as mentioned above. In fact, it follows from the standard theory on the periodic domain of $K3$ surfaces (cf. [1]) that we have the slightly stronger statement:

$$\bigcup_{2|m} \mathcal{P}_{g,m} \text{ is dense in } \mathcal{K}_g$$

(2.3)

for all $g \geq 2$.

Remark 2.1 The choice of $m$ being even is purely technical. As we will see, it simplifies the construction of the degeneration of elliptic $K3$ surfaces. It could be removed at the cost of making our later argument more complicated.

2.2 Dynamics under self rational maps

An elliptic $K3$ surface admits self rational maps induced by fiberwise elliptic curve endomorphism (cf. [11]).

Let $(X, L) \in \mathcal{P}_{g,m}$. Fixing $A \in \text{Pic}(X)$ with $AF = a$, we can construct a rational map $\phi_A : X \dasharrow X$ by sending a point $p$ lying on a smooth fiber $X_q = \pi^{-1}(q)$ to the point $A - (a - 1)p$ on $X_q$ using the group structure of the elliptic curve $X_q$, by which we mean that we send $p$ to the unique point $p' \in X_q$ given by
on $X_q$. Obviously, $\phi_A$ is dominant unless $a = 1$. Of course, this construction works for all fibrations of abelian varieties, not just elliptic $K3$’s.

Let $C \subset X$ be an irreducible rational curve which is not contained in a fiber of $\pi$. The proper transform $\phi_A(C)$ of $C$ under $\phi_A$ is also an irreducible rational curve on $X$ not contained in a fiber. Naturally, we expect the following to be true.

**Proposition 2.2** For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2| m$ and a Baire general $(X, L) \in \mathcal{P}_{g,m}$, there exists an irreducible rational curve $C \subset X$ such that the set

$$
\bigcup_{A \in Pic(X)} \phi_A(C) = \bigcup_{n \in \mathbb{Z}} \phi_{nL}(C)
$$

(2.5)

is dense on $X$.

We cannot yet conclude Theorem 1.1 from Proposition 2.2 since $\phi_A(C)$ may not lie on the fiber $\mathcal{C}_{g,n}$ over the point $(X, L)$. Indeed, if $C \in |aL + bF|$, $\phi_{kL}(C) \in |aL + b_kF|$ for some $b_k \in \mathbb{Z}$. As $|k| \to \infty, b_k \to \infty$ since we have only finitely many rational curves in each linear series. Hence $\phi_{kL}(C) \not\sim_{\text{rat}} nL$ for all $n \in \mathbb{Z}$, when $|k|$ is sufficiently large. So the rational curve $\phi_{kL}(C)$ alone cannot be deformed to a rational curve on a general $K3$ surface. But we can find a rational curve $B_k \subset X$ such that $B_k + \phi_{kL}(C) \sim_{\text{rat}} nL$ for some $n \in \mathbb{Z}$ and the union $B_k \cup \phi_{kL}(C)$ can be deformed to an irreducible rational curve on a general $K3$ surface. Namely, we can prove the following.

**Proposition 2.3** For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2| m$, a general $(X, L) \in \mathcal{P}_{g,m}$ and an irreducible rational curve $C \subset X$ such that $C \not\sim_{\text{rat}} lL$ for any $l \in \mathbb{Z}$,

- there exists an irreducible rational curve $B \subset X$ such that $B \cup C$ lies on an irreducible component of $\mathcal{C}_{g,n}$ that dominates $\mathcal{K}_g$;
- there exists an irreducible elliptic curve $B \subset X$ such that $B \cup C$ lies on an irreducible component of $\mathcal{E}_{g,n}$ that dominates $\mathcal{S}_g$.

Clearly, Propositions 2.2 and 2.3 together will give us Theorems 1.1 and 1.8.

Let $X_q$ be a general fiber $\pi$ and $p \in X_q \cap C$. Then $\phi_{nL}$ sends $p$ to the point

$$
\phi_{nL}(p) = nL - (mn - 1)p
$$

(2.6)

and hence
in the Jacobian $\text{Pic}^0(X_q) = J(X_q)$ of the elliptic curve $X_q$.

Proposition 2.2 will follow if we can prove that the subgroup of $J(X_q)$ generated by $L - mp$ is dense. So we naturally ask which points on an elliptic curve, or more generally a compact complex torus, generate a dense subgroup. This is answered by the classical Kronecker’s theorem (cf. [14, Chap. XXIII]). For convenience, we put it in the following form.

**Theorem 2.4 (Kronecker’s Theorem)** Let $A = \mathbb{R}^n / \mathbb{Z}^n$ be a compact real torus of dimension $n$. For a point $p = (x_1, x_2, \ldots, x_n) \in A$, $\mathbb{Z}p = \{kp : k \in \mathbb{Z}\}$ is dense in $A$ if and only if $1, x_1, x_2, \ldots, x_n$ are linearly independent on $\mathbb{Q}$. In particular, the set

$$\{ p \in A : \mathbb{Z}p \text{ is not dense in } A \}$$

is of the first Baire category.

There are two ways we can show that $L - mp$ generates a dense subgroup of $J(X_q)$ using Kronecker’s theorem. One way is via normal functions. This will be done in Sect. 3. The other way is to show that $L - mp$ is general in $J(X_q)$ as $X$ and $q$ vary.

First of all, we have to make what we mean by “general in $J(X_q)$” precise. Let

$$S_{g,m} = S_g \times_{K_g} \mathcal{P}_{g,m}$$

be the pullback of the universal family $S_g$ to $\mathcal{P}_{g,m} \subset K_g$ and let $C_{g,m,A} \subset S_{g,m}$ be the closed subscheme whose fiber over a general point $(X, L) \in \mathcal{P}_{g,m}$ is the union of all irreducible rational curves in $|A|$, where $A \in \text{Pic}(S_{g,m}/\mathcal{P}_{g,m})$. Note that we have an elliptic fibration

$$\pi : S_{g,m} \rightarrow \mathbb{P}^1 \times \mathcal{P}_{g,m}$$

given by the pencil $|F|$. The induced map $C_{g,m,A} \rightarrow \mathbb{P}^1 \times \mathcal{P}_{g,m}$ is generically finite if it is dominant.

Let $p \in C_{g,m,A}$ be a point over a general point $q = \pi(p) \in \mathbb{P}^1 \times \mathcal{P}_{g,m}$. We have a map $p \rightarrow J(E)$ by sending $p$ to $L - mp$, where $E = \pi^{-1}(q) \subset S_{g,m}$ is the fiber of $\pi$ over $q$. Note that $J(E) = \text{Pic}^0(E)$ is the elliptic curve $E$ with a base point 0 corresponding to the trivial bundle $\mathcal{O}_E$. So we have two marked points $(0, L - mp)$ on $J(E)$. Namely, we have a well-defined map

$$\gamma : C_{g,m,A} \rightarrow \overline{\mathcal{M}}_{1,2}$$

sending $p$ to $(J(E), 0, L - mp)$, where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus $g$ with $n$ marked points. By saying $L - mp$ is general, we simply mean that $\gamma$ is dominant.

**Lemma 2.5** For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|m$, there exists an irreducible component of $C_{g,m,A}$ dominating $\mathbb{P}^1 \times \mathcal{P}_{g,m}$ via $\pi$ and dominating $\overline{\mathcal{M}}_{1,2}$ via $\gamma$ for some $A \in \text{Pic}(S_{g,m}/\mathcal{P}_{g,m})$. 

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This, together with Kronecker’s theorem, will give us Proposition 2.2. If \( C_{g,m,A} \) dominates \( \mathbb{P}^1 \times \mathcal{P}_{g,m} \), it obviously dominates \( \overline{\mathcal{M}}_{1,1} \) by

\[
C_{g,m,A} \xrightarrow{\gamma} \overline{\mathcal{M}}_{1,2} \xrightarrow{\tau} \overline{\mathcal{M}}_{1,1}
\]

where \( \tau \) is the forgetting map. So to show that \( \gamma \) is dominant, it suffices to show that the closure of the image of \( \gamma \) contains the boundary component \( \overline{\mathcal{M}}_{0,4} \subset \overline{\mathcal{M}}_{1,2} \). The proof of this fact relies on a degeneration argument.

2.3 Deformation of K3 surfaces

Following the idea in [9], we can deform a K3 surface to a union of two rational surfaces. Let \( R = R_1 \cup R_2 \) be the union of two smooth rational surfaces \( R_1 \) and \( R_2 \) meeting transversely along a smooth elliptic curve \( D = R_1 \cap R_2 \) where \( D = -K_{R_i} \) in \( \text{Pic}(R_i) \) for \( i = 1, 2 \). We see that \( R \) is simply connected and the dualizing sheaf \( \omega_R \) of \( R \) is trivial. So it is expected that \( R \) can be deformed to a K3 surface. The deformation of \( R \) is governed by the map

\[
\text{Ext}(\Omega_R, \mathcal{O}_R) \to H^0(T^1(R)) = H^0(\mathcal{E}xt(\Omega_R, \mathcal{O}_R)) = H^0(\mathcal{O}_D(-K_{R_1} - K_{R_2})).
\]

Then \( R \) can be deformed to a K3 surface only if the image of the above map is base point free in \( H^0(T^1(R)) \). That is, \( R_{\text{sing}} = D \) can be smoothed when \( R \) deforms. This puts some restrictions on \( R_i \). A necessary condition is that \( \mathcal{O}_D(-K_{R_1} - K_{R_2}) \) is base point free. It can be guaranteed if we choose \( R_i \) to be Fano.

A deformation of \( R \) is a complex K3 surface, not necessarily projective. In order to deform \( R \) to a projective K3 surface, in particular, to deform \( R \) to a K3 surface in \( \mathcal{P}_{g,m} \), we need to construct \( R \) in such a way that it has two line bundles \( L \) and \( F \) satisfying (2.2). Let \( L_i = L|_{R_i} \) and \( F_i = F|_{R_i} \) for \( i = 1, 2 \). Then

\[
L_1|_D = L_2|_D \quad \text{and} \quad F_1|_D = F_2|_D.
\]

Indeed, \( R \) is constructed by gluing \( R_1 \) and \( R_2 \) transversely along \( D \) such that

\[
e_1^*L_1 = e_2^*L_2 \quad \text{and} \quad e_1^*F_1 = e_2^*F_2
\]

where \( e_i \) is the embedding \( D \hookrightarrow R_i \) for \( i = 1, 2 \).

As in [7] and [9], we can degenerate every K3 surface of genus \( g \geq 3 \) to a union \( R = R_1 \cup R_2 \) as follows:

- if \( g \geq 3 \) is odd, we let \( R_i \cong \mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( R_1 \cup R_2 \) be polarized by the ample line bundle \( L \) where

\[
L_i = L|_{R_i} = M_i + \frac{g - 1}{2} G_i
\]
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with $M_i$ and $G_i$ being the generators of $\text{Pic}(R_i)$ satisfying

$$M_i^2 = G_i^2 = 0 \quad \text{and} \quad M_i G_i = 1 \quad (2.17)$$

for $i = 1, 2$;

- if $g \geq 4$ is even, we let $R_i \cong \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ and $R_1 \cup R_2$ be polarized by the ample line bundle $L$

$$L_i = L \big|_{R_i} = M_i + \frac{g}{2} G_i \quad (2.18)$$

with $M_i$ and $G_i$ being the generators of $\text{Pic}(R_i)$ satisfying

$$-M_i^2 = M_i G_i = 1 \quad \text{and} \quad G_i^2 = 0 \quad (2.19)$$

for $i = 1, 2$.

Note that this does not cover the case $g = 2$. The genus 2 case will be treated separately in 2.5.

Such $R$ can be deformed to a general $K3$ surface in $\mathcal{K}_g$. In order to deform it to a $K3$ surface in $\mathcal{P}_{g,m}$, we need to have another line bundle $F \in \text{Pic}(R)$ besides $L \in \text{Pic}(R)$.

Here we simply let

$$F_i = F \big|_{R_i} = \frac{m}{2} G_i \quad (2.20)$$

with the “tricky” requirement that

$$\mathcal{O}_D(G_1 - G_2) \in \text{Pic}^0(D) = J(D) \text{ is } (m/2)\text{-torsion.} \quad (2.21)$$

We glue $R_1$ and $R_2$ in such a way that (2.14) are the only relations between $\text{Pic}(R_i)$, with $L_i$ and $F_i$ given by (2.16), (2.18) and (2.20), respectively. More precisely, the kernel $\text{Pic}(R)$ of the map

$$\text{Pic}(R_1) \oplus \text{Pic}(R_2) \xrightarrow{\epsilon_1^* - \epsilon_2^*} \text{Pic}(D) \quad (2.22)$$

is freely generated by $L = L_1 \oplus L_2$ and $F = F_1 \oplus F_2$. Numerically, we have

$$L_i^2 = g - 1, \quad L_i F_i = \frac{m}{2} \quad \text{and} \quad F_i^2 = 0. \quad (2.23)$$

Remark 2.6 It may appear that $F$ is not primitive by (2.20). It actually is since $G_1 - G_2$ is a torsion point of $J(D)$ of order $m/2$ and hence there does not exist $k \in \mathbb{Z}$ such $\mathcal{O}_D(kG_1) = \mathcal{O}_D(kG_2)$ unless $(m/2)|k$. It may also appear that $h^0(F) = m/2 + 1$ by (2.20). Actually, $h^0(F) = 2$, i.e., $|F|$ is a pencil, again by (2.21). Indeed, a member of $|F|$ is a union $N_1 \cup N_2 \cup \cdots \cup N_m$ where
• $N_k \subset R_1$ and $N_k \in |G_1|$ for $k$ odd and $N_k \subset R_2$ and $N_k \in |G_2|$ for $k$ even;
• $\cup N_k$ meets $D$ at points $q_1, q_2, \ldots, q_m$ such that

$$N_k \cdot D = q_k + q_{k+1}$$

(2.24)

for $1 \leq k \leq m$, where we let $q_{m+1} = q_1$.

Obviously, such a union $\cup N_k$ moves in a base point free pencil.

Such $R$ can be deformed to $K3$ surfaces in $P_{g,m}$. That is, there exists a one-parameter family $S$ over the disk $\Delta = \{|t| < 1\}$ and two line bundles $L$ and $F \in \text{Pic}(S/\Delta)$ such that $(S_t, L) \in P_{g,m}$ for $t \neq 0$ and $S_0 = R$ is the union $R$ with $L$ and $F$ constructed as above. The proofs of Lemma 2.5 and Proposition 2.3 both depend on the construction of certain rational curves on the general fibers $S_t$. Our strategy is to produce a reducible rational curve on the central fiber $R$, called a limiting rational curve in [7], and show that it can be deformed to an irreducible rational curve on the general fibers.

**Lemma 2.7** For all $g, m \in \mathbb{Z}^+$ satisfying $g \geq 2$ and $2|\text{m}$ and a general $(X, L) \in P_{g,m}$, there is an irreducible nodal rational curve in $|aL + bF|$ for all $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}$ satisfying

$$\max \left(2a \left\lfloor \frac{g-1}{2} \right\rfloor, a \right) + bm > 0$$

(2.25)

and $b^2 + (g-2)^2 \neq 0$.

**Proof** (when $g \geq 3$). Our construction of limiting rational curves $C_1 \cup C_2$ with $C_i \subset R_i$ is very similar to the construction in [7], but with some added difficulties. Namely, we have to make sure that

there does not exist $C_1' \cup C_2' \subsetneq C_1 \cup C_2$

such that $C_1' \cup C_2' \in |a' C + b' F|$ for some $a', b' \in \mathbb{Z}$;

(2.26)

otherwise, a deformation of $C_1 \cup C_2$ onto a general fiber $S_t$ is not necessarily irreducible. This is a little trickier here due to the fact rank $\text{Pic}(R) = 2$ and the condition (2.21).

The one-parameter family $S$ has sixteen rational double points $p_1, p_2, \ldots, p_{16}$ lying on $D$, which are precisely the zeros of a section in $H^0(T_1(R))$ that is in turn the image of the Kodaira-Spencer class of $S/\Delta$ under the map (2.13). So these sixteen points satisfy

$$\mathcal{O}_D(p_1 + p_2 + \cdots + p_{16}) = \mathcal{O}_D(-K_{R_1} - K_{R_2})$$

$$= \begin{cases} 
\mathcal{O}_D(2M_1 + 2G_1 + 2M_2 + 2G_2) & \text{if } 2 \nmid g \\
\mathcal{O}_D(2M_1 + 3G_1 + 2M_2 + 3G_2) & \text{if } 2 \mid g 
\end{cases}$$

(2.27)

and this is the only relation among $p_1, p_2, \ldots, p_{16}$ for a general choice of $S$. 
We write
\[
(aL + bF) \bigg|_{R_i} = aM_i + \left( a \left\lfloor \frac{g}{2} \right\rfloor + \frac{bm}{2} \right) G_i = aM_i + lG_i.
\] (2.28)

Case 2 \( \nmid g \) and \( a \leq l \). We let
\[
C_i = I_{i1} \cup I_{i2} \cup \cdots \cup I_{i,a-1} \cup J_{i1} \cup J_{i2} \cup \cdots \cup J_{i,a-1} \cup \Gamma_i
\] (2.29)
be the curve on \( R_i \) \((i = 1, 2)\) with irreducible components \( I_{ij} \in |G_i|, J_{ij} \in |M_i| \) and \( \Gamma_i \in |M_i + (l - a + 1)G_i| \) given by
\[
\begin{align*}
I_{11} \cdot D &= p_1 + q_1, \quad J_{21} \cdot D = q_1 + q_2 \\
I_{12} \cdot D &= q_2 + q_3, \quad J_{22} \cdot D = q_3 + q_4 \\
\vdots \\
I_{1,a-1} \cdot D &= q_{2a-4} + q_{2a-3}, \quad J_{2,a-1} \cdot D = q_{2a-3} + q_{2a-2} \\
I_{21} \cdot D &= p_2 + r_1, \quad J_{11} \cdot D = r_1 + r_2 \\
I_{22} \cdot D &= r_2 + r_3, \quad J_{12} \cdot D = r_3 + r_4 \\
\vdots \\
I_{2,a-1} \cdot D &= r_{2a-4} + r_{2a-3}, \quad J_{1,a-1} \cdot D = r_{2a-3} + r_{2a-2}
\end{align*}
\] (2.30)
and
\[
\begin{align*}
\Gamma_1 \cdot D &= p_2 + q_{2a-2} + (2l - 2a + 2)s, \quad \Gamma_2 \cdot D \\
&= p_1 + r_{2a-2} + (2l - 2a + 2)s
\end{align*}
\] (2.32)
where \( q_j, r_j \) and \( s \) are points on \( D \) and we let \( q_0 = p_1 \) and \( r_0 = p_2 \). Intuitively, \( C_1 \cup C_2 \) is the union of two chains of curves, one starting at \( p_1 \) and the other starting at \( p_2 \), consisting of curves in \( |G_i| \) and \( |M_i| \) alternatively and finally “joined” by \( \Gamma_1 \) and \( \Gamma_2 \). We see that (2.26) holds because \( p_1 \) and \( p_2 \) are two general points on \( D \) and \( G_i \) and \( M_{3-i} \) are linearly independent in \( \text{Pic}_Q(D) \) for each \( i = 1, 2 \).

Case 2 \( \nmid g \) and \( a > l \). Note that (2.25) implies \( l > 0 \) when \( g \) is odd. We use the same construction as above for \( a \leq l \) by simply switching \( G_i \) and \( M_i \) and switching \( a \) and \( l \).

Case 2|g. Note that \( l > a \) by (2.25) when \( g \) is even. Let
\[
\alpha = \left\lfloor \frac{a}{2} \right\rfloor \quad \text{and} \quad \beta = \left\lfloor \frac{a - 1}{2} \right\rfloor
\] (2.33)
and let
\[
C_i = I_{i1} \cup I_{i2} \cup \cdots \cup I_{i\alpha} \cup J_{i1} \cup J_{i2} \cup \cdots \cup J_{i\beta} \cup \Gamma_i
\] (2.34)
be the curve on \( R_i \) \((i = 1, 2)\) with irreducible components \( I_{ij}, J_{ik} \in |M_i + G_i|\) and \( \Gamma_i \in |M_i + (l - a + 1)G_i|\) given by

\[
\begin{align*}
I_{11} \cdot D &= p_1 + p_3 + q_1, \quad I_{21} \cdot D = p_2 + q_1 + q_2 \\
I_{12} \cdot D &= p_1 + q_2 + q_3, \quad I_{22} \cdot D = p_2 + q_3 + q_4 \\
&\vdots \\
I_{1\alpha} \cdot D &= p_1 + q_{2\alpha - 2} + q_{2\alpha - 1}, \quad I_{2\alpha} \cdot D = p_2 + q_{2\alpha - 1} + q_{2\alpha}, \\
J_{21} \cdot D &= p_1 + p_4 + r_1, \quad J_{11} \cdot D = p_2 + r_1 + r_2 \\
J_{22} \cdot D &= p_1 + r_2 + r_3, \quad J_{12} \cdot D = p_2 + r_3 + r_4 \\
&\vdots \\
J_{2\beta} \cdot D &= p_1 + r_{2\beta - 2} + r_{2\beta - 1}, \quad J_{1\beta} \cdot D = p_2 + r_{2\beta - 1} + r_{2\beta},
\end{align*}
\] (2.35)

and

\[
\begin{align*}
\Gamma_1 \cdot D &= p_4 + q_{2\alpha} + (\alpha - \beta)p_2 + (2l - 2a - \alpha + \beta + 1)s, \\
\Gamma_2 \cdot D &= p_3 + r_{2\beta} + (\alpha - \beta)p_1 + (2l - 2a - \alpha + \beta + 1)s.
\end{align*}
\] (2.36)

where \( q_j, r_k \) and \( s \) are points on \( D \) and we let \( q_0 = p_3 \) and \( r_0 = p_4 \). Since \( p_1, p_2, p_3, p_4 \) are in general position on \( D \), it is not hard to see that (2.26) holds.

The curve \( C_1 \cup C_2 \) constructed above has the following properties in addition to (2.26):

- every component of \( C_i \) is a smooth rational curve and \( C_i \) has simple normal crossing outside of \( D \);
- if \( C_i \) and \( D \) meet at a point \( q \notin \{p_1, p_2, \ldots, p_{16}\} \), there is only one branch of \( C_i \) locally at \( q \), i.e., \( C_i \) is smooth at \( q \);
- if \( C_i \) and \( D \) meet at a point \( q \in \{p_1, p_2, \ldots, p_{16}\} \), all local branches of \( C_i \) at \( q \) meet transversely with each other and also transversely with \( D \).

Then by the argument in [7], more specifically, by [7, Theorems 2.1 and 2.2], we can show that \( C_1 \cup C_2 \) can be deformed to an irreducible nodal rational curve on the general fibers of \( S/\Delta \). More precisely, there exists a flat family of curves \( \mathcal{C} \subset \mathcal{S} \), after a base change, such that \( \mathcal{C}_0 = C_1 \cup C_2 \) and \( \mathcal{C}_t \) is an irreducible rational curve with only ordinary double points as singularities for \( t \neq 0 \). \( \square \)

**Remark 2.8** The condition (2.25) is trivially satisfied when we take \( a \gg |b| \). Therefore, for every \( C \not\subset \text{rat} \cdot \mathcal{L} \), there is an irreducible nodal rational curve in \( |nL - C| \) for \( n \) sufficiently large. This is what we need for Proposition 2.3.

Now we are ready to prove Lemma 2.5.

**Proof of Lemma 2.5** (when \( g \geq 3 \)). By Lemma 2.7, there is an irreducible component of \( \mathcal{C}_{g,m,A} \) dominating \( \mathbb{P}^1 \times \mathcal{P}_{g,m} \), by setting \( \text{e.g.} \ A = 2L + F \).

Let \( S/\Delta \) be the family of \( K3 \) surfaces constructed above. One may think of \( S \) as the pullback of \( \mathcal{S}_{g,m} \) under a map \( \Delta^* \to \mathcal{P}_{g,m} \). Let \( \mathcal{C} \subset S \) be a family of rational curves constructed in the proof of Lemma 2.7 with \( \mathcal{C}_t \in |2L + F| \). One may think of
\(\mathcal{C}\) as an irreducible component of the pullback of \(C_{g,m,2L+F}\) to \(S\). Correspondingly, we pull back the map \(\gamma\) to \(\overline{\mathcal{C}}\), i.e.,

\[
\gamma : \mathcal{C} \to \overline{\mathcal{M}}_{1,2}
\]

(2.38)

sending \(p \in \mathcal{C}\) to \((J(E_p), 0, L - mp)\), where \(E_p\) is the fiber of the projection \(\pi : S \to \mathbb{P}^1 \times \Delta\) over the point \(\pi(p)\).

It is enough to prove that

\[
\dim(\gamma(\mathcal{C}) \cap \overline{\mathcal{M}}_{0,4}) = 1
\]

(2.39)

where we think of \(\overline{\mathcal{M}}_{0,4}\) as a component of \(\overline{\mathcal{M}}_{1,2} \setminus \mathcal{M}_{1,2}\). Instead of directly studying \(\gamma\), which roughly maps \(p\) to \(L - mp\), we look at the map sending \(p\) to \(p - p'\), where \(p' \neq p\) is another point of intersection between \(E_p\) and \(\mathcal{C}_t\).

More precisely, we let \(T\) be the product \(\mathcal{C} \times \mathbb{P}^1 \times \Delta\) with diagonal removed. We have a well-defined map

\[
\xi : T \to \overline{\mathcal{M}}_{1,2}
\]

(2.40)

sending \((p, p') \in T\) to \((J(E_p), 0, p - p')\). Clearly, \(\gamma\) is dominant if \(\xi\) is; \(\xi\) is dominant if

\[
\dim(\xi(T) \cap \overline{\mathcal{M}}_{0,4}) = 1.
\]

(2.41)

As \(t \to 0\), the fibers \(E_p \in |F|\) of \(\pi : \mathcal{S}_t \to \mathbb{P}^1\) will degenerate to a curve \(N \in |F|\) on the central fiber \(\mathcal{S}_0 = R_1 \cup R_2\) as described in Remark 2.6. That is, \(N\) is a union \(N_1 \cup N_2 \cup \cdots \cup N_m\) given by (2.24). For \(N\) a general member of the pencil \(|F|\), \(N_1\) meets \(C_1\) transversely at two distinct points \(p \neq p' \notin D\), where \(C_0 = C_1 \cup C_2\) is the limiting rational curve constructed in the proof of Lemma 2.7. Clearly, \((p, p') \in T\).

It is not hard to see that \(\xi\) simply sends \((p, p')\) to \((p, p', q, q') \in \overline{\mathcal{M}}_{0,4}\) as four points on \(N_1 \cong \mathbb{P}^1\), where \(N_1 \cap D = \{q, q'\}\). To show (2.41), it suffices to show that the moduli of \((p, p', q, q')\) varies when \(N\) moves in the pencil \(|F|\). From the construction of \(C_1 \cup C_2\), we see that \(C_1\) has a node \(r \notin D\). For \(N\) a general member of \(|F|\), \((p, p', q, q')\) are four distinct points on \(N_1\). When \(N_1\) passes through \(r\), we have \(p = p' = r\) while \(r \neq q\) and \(r \neq q'\). So the moduli of \((p, p', q, q')\) changes as \(N\) varies. We are done.

\(\square\)

2.4 Proof of Proposition 2.3

By Lemma 2.7, we can find an irreducible nodal rational curve \(B \in |nL - C|\) for \(n\) sufficiently large. It also follows that there is an irreducible nodal elliptic curve \(B \in |nL - C|\). It remains to show that we can deform \(B \cup C\) to a rational curve if \(g(B) = 0\) or an elliptic curve if \(g(B) = 1\) on a general \(K3\) surface.
Let us do the case $g(B) = 0$. We fix an intersection $p \in B \cap C$. Let $B^v$ and $C^v$ be the normalizations of $B$ and $C$, respectively, and let

$$\eta : B^v \cup_p C^v \to B \cup C \subset X$$

be a partial normalization of $B \cup C$, where $B^v \cup_p C^v$ is the union of $B^v$ and $C^v$ meeting transversely at a single point over $p$. We want to show that the stable map $\eta : B^v \cup_p C^v \to X$ can be deformed when we deform $X$. This can be done by computing the virtual dimension of the moduli space of stable maps to $K3$ surfaces. However, it cannot be done in a naive way for the following well-known reason: the virtual dimension of $M^{X}_{nL,0}$ is

$$c_1(T_X) \cdot nL + \dim X - 3 = -1$$

which is not the “expected” dimension 0, where $M^{X}_{\gamma,g}$ is the moduli space of stable maps $\eta : A \to X$ of genus $g$ with $[\eta_*A] = \gamma \in H_2(X, \mathbb{Z})$. The remedy for this situation is to replace $X$ by a so-called “twisted” family of $K3$ surfaces, i.e., a complex deformation of $X$. This way we have the right dimension and the stable maps $\eta \in M^{X}_{nL,0}$ only deform onto projective $K3$ surfaces.

Let $\mathcal{S}/\Delta^2$ be a family of complex $K3$ surfaces over the 2-disk $\Delta^2$ with $\mathcal{S}_0 = X$ and the class $L \in H_2(\mathcal{S}, \mathbb{Z})$. Then $\eta \in M^{\mathcal{S}}_{nL,0}$ and the virtual dimension of $M^{\mathcal{S}}_{nL,0}$ is

$$c_1(T_{\mathcal{S}}) \cdot nL + \dim \mathcal{S} - 3 = 1.$$  

Let $V$ be an irreducible component of $M^{\mathcal{S}}_{nL,0}$ containing $\eta$. Then $\dim V \geq 1$. Let

$$W = \{ t \in \Delta^2 : L \in \text{Pic}(\mathcal{S}_t) \}$$

be the subvariety of $\Delta^2$ parameterizing projective $K3$ surfaces polarized by $L$. Obviously, $\dim W = 1$ and $\text{Pic}(\mathcal{S}_t) = \mathbb{Z}$ for $t \neq 0 \in W$ and a general choice of $\mathcal{S}$. Clearly, $V$ maps to $W$ under the projection $\mathcal{S} \to \Delta^2$ and it is obviously flat over $W$ since $\dim V_0 = 0$ and $\dim V \geq \dim W$. Hence $V_t \neq \emptyset$ for $t \neq 0 \in W$.

**Remark 2.9** The above argument on the deformation of reduced stable maps to $K3$ surfaces is well known among the experts. The first author learned it from Jun Li. Please also see [3] and [21, Sect. 2.2].

The case $g(B) = 1$ follows from the same argument. This finishes the proof of Proposition 2.3 and hence Theorem 1.1 follows. We need to say a few things more for Theorem 1.8.

By deformation theory, $B$ moves in a one-parameter family when $g(B) = 1$. A general member of this family is an irreducible nodal elliptic curve meeting $C$ transversely. In addition, the intersection $p \in B \cap C$ moves on $C$ when $B$ varies in the family. Now we let $\mathcal{S}$ be a projective family of $K3$ surfaces polarized by $L$ over $\Delta$ with $\mathcal{S}_0 = X$. We can deform $B \cup C$ to an irreducible elliptic curve on a general fiber $\mathcal{S}_t$ of the family $\mathcal{S}/\Delta$ by the argument above. Namely, there exists a family of curves...
\( \mathcal{C} \subset S \), after a base change, such that \( C_0 = B \cup C \) and \( g(C_t) = 1 \) for \( t \neq 0 \). Let \( \nu : C^v \to \mathcal{C} \) be the normalization of \( \mathcal{C} \). Then \( C^v_0 \) is the union \( B^v \cup_p C^v \) described above. We lift \( \nu : C^v \to S \) to \( d \nu : C^v \to \mathbb{P}T_{S/\Delta} \). Let \( \mu : \tilde{C} \to \mathbb{P}T_{S/\Delta} \) be the stable reduction of the map \( d \nu \) and let \( \tilde{B} \) and \( \tilde{C} \subset \tilde{C}_0 \) be the proper transforms of \( B \) and \( C \), respectively.

Since \( B \) and \( C \) meet transversally at \( p \), the images of the tangent spaces \( T_{B,p} \) and \( T_{C,p} \) in \( T_{X,p} \) differ. Consequently, \( \mu(B) \) and \( \mu(C) \) meet \( \mathbb{P}T_{X,p} \) at two distinct points, where \( \mathbb{P}T_{X,p} \cong \mathbb{P}^1 \) is the fiber of \( \mathbb{P}T_{S/\Delta}/\mathcal{C} \) over the point \( p \in S \). Therefore, \( \tilde{B} \) and \( \tilde{C} \) are disjoint on \( \tilde{C}_0 \) and they must be joined by a tree of rational curves that dominates \( \mathbb{P}T_{X,p} \). That is, \( \mathbb{P}T_{X,p} \subset \mu(\tilde{C}) \) and hence \( \mathbb{P}T_{X,p} \subset \varphi(\mathcal{E}_{g,n}) \). As \( p \) moves on \( C \), we see that

\[
\bigcup_{p \in \mathcal{C}} \mathbb{P}T_{X,p} \subset \varphi(\mathcal{E}_{g,n}).
\] (2.46)

We take \( C \) to be a member of a sequence of rational curves which are dense on \( X \). Hence

\[
\mathbb{P}T_{X} \subset \bigcup_{n=1}^{\infty} \varphi(\mathcal{E}_{g,n})
\] (2.47)

and Theorem 1.8 follows.

2.5 The case \( g = 2 \).

A \( K3 \) surface in \( \mathcal{P}_{2,m} \) can still be degenerated to a union \( R_1 \cup R_2 \) with \( R_i \cong \mathbb{F}_1 \) and \( L_i \) and \( F_i \) given by (2.18), (2.20) and (2.21), just as in the case that \( g \geq 4 \) is even. Let \( S/\Delta \) be the corresponding family of \( K3 \) surfaces with \( S_0 = R_1 \cup R_2 \) and \( (S_i, L_i) \in \mathcal{P}_{2,m} \). Such \( S \) is projective over \( \Delta \) since \( L + nF \) is relatively ample over \( \Delta \) for all \( n > 0 \). However, \( L \) is big and nef but not ample over \( \Delta \) itself. Indeed, the birational map \( \psi : S \to Q \) given by \( |nL| \) for \( n \geq 2 \) contracts the two exceptional curves \( M_i \). The threefold \( Q \) is a family of \( K3 \) surfaces in \( \mathcal{P}_{2,m} \) over \( \Delta \) whose central fiber \( Q_0 = S_1 \cup S_2 \) is a union of \( S_i \cong \mathbb{P}^2 \) meeting transversely along an elliptic curve \( D = S_1 \cap S_2 \). Here we use the same notation \( D \) for both intersections \( R_1 \cap R_2 \) and \( S_1 \cap S_2 \).

The two curves \( M_i \) are contracted by \( \psi \) to two rational double points \( p_{17} \) and \( p_{18} \) of \( Q \) on \( D = S_1 \cap S_2 \). Indeed, \( Q \) has eighteen rational double points \( p_1, p_2, \ldots, p_{16}, p_{17}, p_{18} \) on \( D \) by deformation theory, where \( p_1, p_2, \ldots, p_{16} \) are the images of the rational double points of \( S \) under \( \psi \). Again we use the same notations \( p_1, p_2, \ldots, p_{16} \) for both the rational double points of \( S \) and their images under \( \psi \).

One subtle point is that \( M_i \) are contracted to the same point \( p_{17} = p_{18} \) where \( Q \) has a singularity of the type \( xy = tz^2 \) when \( m = 2 \). Such a singularity can be analyzed in the same way as rational double points. Basically, we have two rational double points “collide” in this special case. However, we can save ourselves some trouble in dealing with this “corner” case by simply assuming that \( m \geq 4 \) since
\[ \bigcup_{\substack{2|m \\ m \geq m_0}} \mathcal{P}_{g,m} \quad (2.48) \]

is obviously dense in \( K_g \) for all \( m_0 \). For our purpose, we may simply assume \( m \) to be sufficiently large. So \( p_{17} \) and \( p_{18} \) are two distinct points on \( D \) and \( R_i \) is the blowup of \( S_i \) at \( p_{16+i} \) for \( i = 1, 2 \), respectively. And \( \psi : S \to Q \) is a small resolution of \( Q \) at \( p_{17} \) and \( p_{18} \). It is well known that there are flops of \( S \) with respect to \( M_i \). Namely, we have the diagram

\[
\begin{array}{ccc}
S & \longrightarrow & S' \\
\psi \downarrow & & \psi' \downarrow \\
Q & & Q
\end{array}
\quad (2.49)
\]

where \( S' \) is the threefold obtained from \( S \) by flops with respect to \( M_1 \) and \( M_2 \). That is, the central fiber \( S'_0 = R'_1 \cup R'_2 \) of \( S' \) is a union of \( R'_i \cong \mathbb{P}_1 \) with \( R'_i \) the blowup of \( S_i \) at \( p_{19-i} \) for \( i = 1, 2 \).

Let \( L' \) and \( F' \in \text{Pic}(S'/\Delta) \) be the proper transforms of \( L \) and \( F \), respectively, and let \( M'_i \) and \( G'_i \) be the generators of \( \text{Pic}(R'_i) \) given in the same way as (2.19). It is not hard to see that

\[
L'_i = L' \bigg|_{R'_i} = M'_i + G'_i \quad \text{and} \quad F'_i = F' \bigg|_{R'_i} = mM'_i + \frac{m}{2} G'_i. \quad (2.50)
\]

So we can work with either \( S \) or \( S' \) to produce rational curves in \( |aL + bF| \) on \( S_i \) or equivalently \( |aL' + bF'| \) on \( S'_i \), depending on the sign of \( b \).

**Proof of Lemma 2.7** when \( g = 2 \). When \( b > 0 \), we have

\[
(aL + bF) \bigg|_{R_i} = aM_i + \left( a + \frac{bm}{2} \right) G_i \quad (2.51)
\]

with \( a + bm/2 > a \). Hence we may use the same construction of limiting rational curves \( C_1 \cup C_2 \) as in the case of \( g \) being even and \( g \geq 4 \).

When \( b < 0 \), we have

\[
(aL' + bF') \bigg|_{R'_i} = (a + bm)M'_i + \left( a + \frac{bm}{2} \right) G'_i \quad (2.52)
\]

where \( a + bm > 0 \) by (2.25) and \( a + bm/2 > a + bm \). So we may use the same construction again by working with \( S' \). \( \square \)

The proof of Lemma 2.5 goes through without any change since we are using the limiting rational curves in \( |2L + F| \) for which no flops are needed.
3 Normal functions associated to elliptic fibrations

Here we will give another proof of Proposition 2.2 via the theory of normal functions. Roughly, we will show that if \( L - mp \) fails to generate a dense subgroup of \( J(X_q) \) for a general point \( q \in \mathbb{P}^1 \), then it has to be torsion for all \( q \). The advantage of this approach is that it does not seem to depend on the general moduli of \( X \), although we do need the fact, which we will prove by degeneration, that the rational curve \( C \subset X \) we start with meets the singular fibers of \( X/\mathbb{P}^1 \) transversely.

Given an elliptic surface \( X/\Gamma \rightarrow \Gamma \), we let \( \Sigma \subset \Gamma \) correspond to the singular fibers of \( \rho/\Gamma : X/\Gamma \rightarrow \Gamma \), with inclusion \( j : U := \Gamma \setminus \Sigma \hookrightarrow \Gamma \). So we have a diagram:

\[
\begin{array}{ccc}
X_U & \hookrightarrow & X/\Gamma \\
\downarrow \rho_U & & \downarrow \rho \\
U & \xrightarrow{j} & \Gamma,
\end{array}
\]

where \( \rho_U \) is smooth and proper. We consider the \( i \)-th Leray direct image sheaves \( R^i\rho_{U,*}\mathcal{C} \) and \( R^i\rho_{*}\mathcal{C} \). The sheaf of invariant “cycles”, i.e. those \( i \)th cohomology classes in the fibers of \( \rho_U \) that are invariant under local monodromy, is given by \( j_*R^i\rho_{U,*}\mathcal{C} \). The local invariant cycle property (see [25], Section 15) gives us a surjection:

\[
R^i\rho_{*}\mathcal{C} \twoheadrightarrow j_*R^i\rho_{U,*}\mathcal{C},
\]

for all \( i \) and hence

\[
H^1(\Gamma, R^1\rho_{*}\mathcal{C}) \simeq H^1(\Gamma, j_*R^1\rho_{U,*}\mathcal{C}).
\]

The Leray spectral sequence for \( \rho_{\Gamma} \) degenerates at \( E_2 \) (see [25], Section 15). This is induced by a Leray filtration: \( H^2(X/\Gamma, \mathbb{Q}) = L^0H^2(X, \mathbb{Q}) \supset L^1H^2(X/\Gamma, \mathbb{Q}) \supset L^2H^2(X/\Gamma, \mathbb{Q}) \supset L^3H^2(X/\Gamma, \mathbb{Q}) = 0. \)

Let \( Gr^i_LH^2(X/\Gamma, \mathbb{Q}) = L^iH^2(X/\Gamma, \mathbb{Q})/L^{i+1}H^2(X/\Gamma, \mathbb{Q}) \). Note that

\[
Gr^2_LH^2(X/\Gamma, \mathbb{Q}) = L^2H^2(X/\Gamma, \mathbb{Q}) = H^2(\Gamma, R^0\rho_{*}\mathcal{Q}) = \mathbb{Q}[F] \simeq \mathbb{Q},
\]

where we use the fact that \( R^0\rho_{*}\mathcal{Q} \simeq \mathbb{Q} \) is the constant sheaf. Further,

\[
Gr^1_LH^2(X/\Gamma, \mathbb{Q}) = H^1(\Gamma, R^1\rho_{*}\mathcal{Q}) \simeq H^1(\Gamma, j_*R^1\rho_{U,*}\mathcal{Q}),
\]

and the kernel of the surjective map

\[
H^2(X/\Gamma, \mathbb{Q}) \twoheadrightarrow Gr^0_LH^2(X/\Gamma, \mathbb{Q}) = H^0(\Gamma, R^2\rho_{*}\mathcal{Q}).
\]
defines $L^1 H^2(X_{\Gamma}, \mathbb{Q})$. There are short exact sequences:

$$
0 \to \mathbb{Q}[F] \to L^1 H^2(X_{\Gamma}, \mathbb{Q}) \to H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}) \to 0,
$$

$$
0 \to H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}) \to \frac{H^2(X_{\Gamma}, \mathbb{Q})}{\mathbb{Q}[F]} \to H^0(\Gamma, R^2 \rho_{\Gamma,*} \mathbb{Q}) \to 0.
$$

There is a commutative diagram

$$
\begin{array}{ccc}
0 & \to & H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{Q}) \\
& & \downarrow \rho_{\Gamma,*}
\end{array}
\begin{array}{ccc}
& & \downarrow \bar{\rho}_{\Gamma,*}
\end{array}
\begin{array}{ccc}
\frac{H^2(X_{\Gamma}, \mathbb{Q})}{\mathbb{Q}[F]} & \to & H^0(\Gamma, R^2 \rho_{\Gamma,*} \mathbb{Q}) \\
& & \to 0
\end{array}
$$

(3.1)

where $\bar{\rho}_{\Gamma,*}$ is induced from $\rho_{\Gamma,*}$. Note that $\ker(\bar{\rho}_{\Gamma,*})$ will involve the components of the bad fibers of $\rho_{\Gamma}$. For instance, if all the fibers of $\rho_{\Gamma}$ are irreducible, then $\bar{\rho}_{\Gamma,*}$ is an isomorphism. Let $F_t := \rho_{\Gamma}^{-1}(t)$. There are holomorphic vector bundles over $U$:

$$
\mathcal{F}^1 := \mathcal{O}_U \left( \prod_{t \in U} H^{1,0}(F_t, \mathbb{C}) \right) \subset \mathcal{F} := \mathcal{O}_U \left( \prod_{t \in U} H^1(F_t, \mathbb{C}) \right);
$$

$$
\mathcal{F}^{1,*} := \mathcal{F}/\mathcal{F}^1 = \mathcal{O}_U \left( \prod_{t \in U} H^{0,1}(F_t, \mathbb{C}) \right),
$$

with canonical extensions

$$
\overline{\mathcal{F}}^1 \subset \overline{\mathcal{F}}, \quad \overline{\mathcal{F}}^{1,*} := \overline{\mathcal{F}}/\overline{\mathcal{F}}^1,
$$

over $\Gamma$ (see [25], Section 3), as well as a short exact sequences of sheaves:

$$
0 \to R^1 \rho_{U,*} \mathbb{Z} \to \mathcal{F}^{1,*} \to \mathcal{J} \to 0 \quad \text{(over $U$)};
$$

$$
0 \to j_* R^1 \rho_{U,*} \mathbb{Z} \to \overline{\mathcal{F}}^{1,*} \to \overline{\mathcal{J}} \to 0 \quad \text{(over $\Gamma$)},
$$

where $\mathcal{J}$, $\overline{\mathcal{J}}$ are the sheaves of germs of normal functions over $U$ and $\Gamma$ respectively. Apart from playing a role in the limiting behavior of normal functions about the singular points $\Gamma \setminus U$, the canonical extensions are useful in calculating the Hodge filtration $\{ F^i H^j(\Gamma, j_* R^1 \rho_{\Gamma,*} \mathbb{C}) \}_{i \geq 0}$ on $H^j(\Gamma, j_* R^1 \rho_{\Gamma,*} \mathbb{C})$. More specifically, from the work of [25], $H^1(\Gamma, j_* R^1 \rho_{\Gamma,*} \mathbb{Z})$ is naturally endowed with a pure Hodge structure of weight $i + 1$; moreover from ([25], Section 9), one has isomorphisms:

$$
H^1(\Gamma, \overline{\mathcal{F}}^{1,*}) \simeq \frac{H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C})}{F^1 H^1(\Gamma, j_* R^1 \rho_{U,*} \mathbb{C})},
$$

$$
H^0(\Gamma, \overline{\mathcal{F}}^{1,*}) \simeq \frac{H^0(\Gamma, R^1 \rho_{\Gamma,*} \mathbb{C})}{F^1 H^0(\Gamma, R^1 \rho_{\Gamma,*} \mathbb{C})}.
$$
Density of rational curves on $K3$ surfaces

(If it is worthwhile pointing out that outside of cases of trivial $j$-invariant, one has $H^0(\Gamma, R^1\rho_{\Gamma,*}\mathbb{C}) = 0$ (see [10], p. 5)). Taking cohomology, one has a short exact sequence:

\[ 0 \to \frac{H^0(\Gamma, \overline{\mathcal{F}}^{1,*})}{H^0(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})} \to H^0(\Gamma, \overline{\mathcal{J}}) \to H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})^{1,1} \to 0, \]

where $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})^{1,1} := \ker(H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z}) \to H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{C})/F^1H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{C})).$

The term $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})^{1,1}$ admits the following interpretation: If we tensor $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})$ with $\mathbb{Q}$, apply diagram (3.1), (which is well-known to be a diagram of Hodge structures), and restrict to algebraic cocycles, we arrive at the short exact sequence:

\[ 0 \to H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Q})^{1,1} \to \frac{H^2_{\text{alg}}(X_\Gamma, \mathbb{Q})}{\mathbb{Q}[F]} \to H^0(\Gamma, R^2\rho_{\Gamma,*}\mathbb{Q}) \to 0, \quad (3.2) \]

where $H^2_{\text{alg}}(X, \mathbb{Q}) \subset H^2(X, \mathbb{Q})$ is the subspace of algebraic cocycles. The intersection pairing involving the components of the bad fibers of $\rho_T$ is well understood in terms of a negative definite property (see Lemma 1.3 of [24]). In particular from (3.2) one can argue that modulo the contribution of a section of $\rho_T$, $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Q})^{1,1}$ can be identified with the quotient of the Néron-Severi group of $X$ (over $\mathbb{Q}$) by the subgroup generated by the irreducible components of the bad fibers of $\rho_T$. The group $H^0(\Gamma, \overline{\mathcal{J}})$ is called the group of normal functions, and for $\nu \in H^0(\Gamma, \overline{\mathcal{J}})$, $\delta(\nu) \in H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})^{1,1}$ is called its topological invariant. We say that $\delta(\nu)$ is nontorsion if $\delta(\nu) \neq 0$ as a class in $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Q})^{1,1}$. We need the following key observation:

**Proposition 3.1** Suppose that $\nu \in H^0(\Gamma, \overline{\mathcal{J}})$ is given such that $\delta(\nu)$ is nontorsion. Then for sufficiently general $t \in U$, the cyclic group generated by $\nu(t)$ is dense in $J^1(E_t)$.

**Proof** A local lifting of the normal function $\nu|_U \in H^0(U, \overline{\mathcal{J}})$ determines an analytic function on a disk $\Delta \subset U$, viz., $\tilde{\nu} \in H^0(\Delta, \mathcal{F}^{1,*}) \simeq H^0(\Delta, \mathcal{O}_\Delta)$, using the fact that $\mathcal{F}^{1,*}$ is a holomorphic line bundle. Further, we have the family of lattices $H^0(\Delta, R^1\rho_{U,*}\mathbb{Z}) \hookrightarrow H^0(\Delta, \mathcal{F}^{1,*})$. Let $\delta_1, \delta_2 \in H^0(\Delta, R^1\rho_{U,*}\mathbb{Z})$ be generators with respective images $[\delta_1], [\delta_2] \in H^0(\Delta, \mathcal{F}^{1,*})$, under the (injective) composite

\[ H^0(\Delta, R^1\rho_{U,*}\mathbb{Z}) \to H^0(\Delta, \mathcal{F}) \to H^0(\Delta, \mathcal{F}^{1,*}). \]

1 At least in the situation of the setting of [24], where there are no exceptional curves of the first kind in the fibers of $\rho_T$, $X_\Gamma$ admits a section, and nonconstant $j$-invariant.

2 Taking into account Remark 2.4 of [24] (and again assuming that $X_\Gamma$ satisfy the assumptions stated in the previous footnote), $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})$ is torsion free, hence the same holds for $H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z})^{1,1} = H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{Z}) \cap F^1H^1(\Gamma, j_*R^1\rho_{U,*}\mathbb{C})$. 

\[ \square \] Springer
Thus we can write
\[ \tilde{v}(t) = x(t)[\delta_{1,t}] + y(t)[\delta_{2,t}], \]
for unique real-valued functions \( x(t), \ y(t), \ t \in \Delta \). Note that \( [\delta_{2,t}] = g(t)[\delta_{1,t}] \) for some holomorphic function \( g(t) \), and likewise \( \tilde{v}(t) = h(t)[\delta_{1,t}] \) for a holomorphic \( h(t) \). Thus \( h(t) = x(t) + y(t)g(t) \), and in particular \( \text{Re}(h(t)) = x(t) + y(t)\text{Re}(g(t)) \), \( \text{Im}(h(t)) = y(t)\text{Im}(g(t)) \). Thus \( x(t) \) and \( y(t) \) are real analytic functions. If the cyclic group generated by \( v(t) \) is not dense in \( J^1(F) \) for uncountably many \( t \in \Delta \), then by a countability and Baire type argument together with Kronecker’s theorem, \( \{1, x(t), y(t)\} \) lie on a hyperplane \( a_1x + a_2y + a_3 = 0 \) in \( \mathbb{R}^2 \), where \( \{a_j\} \in \mathbb{Q} \) are constant with respect to \( t \in \Delta \) and not all zero. Using \( h(t) = x(t) + y(t)g(t) \), one can easily check then that \( a_1x(t) + a_2y(t) + a_3 = 0 \) for all \( t \in \Delta \) implies that \( \tilde{v} \) is constant. More precisely, one can choose the lift \( \tilde{v} \in H^0(\Delta, R^1\rho_{U,*}\mathbb{C}) \) of \( v \) via the composite
\[
H^0(\Delta, R^1\rho_{\Delta,*}\mathbb{C}) \to H^0(\Delta, \mathcal{F}) \to H^0(\Delta, \mathcal{F}/\mathcal{F}^1) = H^0(\Delta, \mathcal{F}^{1,*}), \quad \tilde{v} \mapsto \tilde{v}.
\]
This tells us that the Griffiths’ infinitesimal invariant of \( v \) over \( U \) (see [13], p. 69) is zero. However in this case the Griffiths’ infinitesimal invariant is known to coincide with the topological de Rham invariant (see [23], as well as [19] for a background on this). In the end, this translates to saying that \( \delta(v|_U) = 0 \) as a class in \( H^1(U, R^1\rho_{U,*}\mathbb{Q}) \). Alternatively and more directly, if we assume for the moment that the \( j \)-invariant of the family \( \tilde{X}_U \to U \) is nonconstant, then \( \mathcal{F}^1 \cap R^1\rho_{U,*}\mathbb{C} = 0 \in \mathcal{F} \) and hence \( v|_U \) is induced by a class in \( H^0(U, R^1\rho_{U,*}\mathbb{C}) \). The same conclusion holds, albeit by a tedious argument, if the \( j \)-invariant is constant - the details are left to the reader and involve a generalization of Example 3.2 below. Next by ([25], Section 14), the map \( H^1(\Gamma, R^1\rho_{\Gamma,*}\mathbb{Q}) \hookrightarrow H^1(U, R^1\rho_{U,*}\mathbb{Q}) \) is injective. Hence \( \delta(v) = 0 \in H^1(\Gamma, R^1\rho_{\Gamma,*}\mathbb{Q}) \), a contradiction.

**Example 3.2** Let \( E \) be an elliptic curve and \( Y = E \times E \). We can illustrate Proposition 3.1 rather easily in this situation. With regard to the first projection \( Y \to E \), the sheaf of germs of normal functions \( \mathcal{J} \) is given by the short exact sequences of sheaves over \( E \):
\[
0 \to H^1(E, \mathbb{Z}) \to \mathcal{O}_E(H^{0,1}(E)) \to \mathcal{J} \to 0.
\]
Note that \( H^1(E, \mathcal{O}_E(H^{0,1}(E))) \simeq H^{0,1}(E) \otimes H^{0,1}(E) \), and hence there is the short exact sequence:
\[
0 \to J^1(E) \to H^0(E, \mathcal{J}) \to \{H^1(E, \mathbb{Z}) \otimes H^1(E, \mathbb{Z})\} \cap H^{1,1}(Y) \to 0.
\]
If \( v \in H^0(E, \mathcal{J}) \) has trivial infinitesimal invariant, then
\[
v \in H^0(E, H^{0,1}(E)/H^1(E, \mathbb{Z})) \simeq J^1(E),
\]
\( \circ \) Springer
Density of rational curves on \(K3\) surfaces

and hence \(\delta(\nu) = 0\). For \(n \in \mathbb{N}\), let \(f_n : E \to E\) be given by multiplication by \(n\), and let \(\Xi(n)\) be the graph of \(f_n\) in \(Y\), with Künneth component \([\Xi(n)^{1,1}] \in H^1(E, \mathbb{Z}) \otimes H^1(E, \mathbb{Z})\). It follows rather directly from Kronecker’s theorem that

\[
\bigcup_{n \in \mathbb{N}} \Xi(n) \subset Y,
\]

is dense in \(Y\) in the strong topology. Note however that if \(\nu\) is the normal function associated to \(f_1\), then \(n\nu\) is the normal function associated to \(f_n\). Furthermore \(\delta(n\nu) = [\Xi(n)^{1,1}] \neq 0\), and hence the density also follows from Proposition 3.1. Now let \(X := Y/\pm\) be the corresponding Kummer counterpart with \(C_n\) being the image of \(\Xi(n)\) in \(X\). Then \(C_n\) is a rational curve and

\[
\bigcup_{n \in \mathbb{N}} C_n \subset X,
\]

is likewise dense in \(X\) in the strong topology.

This gives us the density of rational curves in the strong topology on an elliptic \(K3\) surface \(X\) as long as \(\pi : X \to \mathbb{P}^1\) admits a rational nt multisection, i.e., Theorem 1.6. It also gives another proof of Proposition 2.2 as long as we can find a rational nt multisection.

On the other hand, we can produce such a multisection using our deformational argument as follows. Although this is redundant, we keep it here to make our paper self-contained.

Now let us consider a general elliptic \(K3\) surface \((X, L) \in \mathcal{P}_{g,m}\) with \(\rho : X \to \mathbb{P}^1\) the elliptic fibration given by \(|F|\).

Let \(\Gamma_0 \in |L|\) be a rational curve, with desingularization \(\Gamma = \mathbb{P}^1\). Note that \(\rho|_{\Gamma_0} : \Gamma_0 \to \mathbb{P}^1\) has degree \(m\), and hence the corresponding map \(\lambda : \Gamma \to \mathbb{P}^1\) is of degree \(m\). Base change gives us an elliptic surface \(\rho \Gamma : X_\Gamma \to \Gamma\), with section \(\sigma : \Gamma \hookrightarrow X_\Gamma\), (where we can assume after a proper modification, that \(X_\Gamma\) is smooth). Let \(h : X_\Gamma \to X\) be the obvious morphism (of degree \(m\)). Note that

\[
h_* (\sigma(\Gamma)) = \Gamma_0;
\]

moreover we have corresponding classes \(F, h^*(L)\) on \(X_\Gamma\), with \(h_*(F) = F\), and on \(X_\Gamma\):

\[
F^2 = 0, \quad (h^*(L))^2 = m \cdot (2g - 2), \quad (\sigma(\Gamma))^2 = b, \quad \text{(some } b \in \mathbb{Z}),
\]

\[
F \cdot \sigma(\Gamma) = 1, \quad F \cdot h^*(L) = m, \quad \sigma(\Gamma) \cdot h^*(L) = 2g - 2.
\]

Note that \([F, h^*(L), \sigma(\Gamma)]\) are independent over \(\mathbb{Q}\) iff \(m \cdot b \neq 2g - 2\). The independence follows from

**Lemma 3.3** \(b < 0\).

**Proof** Since \(\sigma(\Gamma) = \mathbb{P}^1\), the adjunction formula tells us that
\[-2 = b + K_{X_{\Gamma}} \cdot \sigma(\Gamma).\]

But \(h\) is ramified only along the fibers of \(\rho_{\Gamma}\), i.e. over which \(\lambda\) ramifies, and hence

\[K_{X_{\Gamma}} = h^*(K_X) + k \cdot F,\]

for some integer \(k \geq 0\). But \(X\) a \(K3\) surface implies that \(K_X = 0\), and hence \(b = -(2 + k) < 0\). \(\square\)

Now let us suppose that:

\[\bar{\rho}_{\Gamma,*} \text{ in diagram } (3.1) \text{ is an isomorphism.} \quad (3.3)\]

Then using the fact that \(\rho_{\Gamma,*}(h^*(L)) = m\Gamma\), by (3.1) it follows that

\[[h^*(L) - m\sigma(\Gamma)] \neq 0 \in H^1(\Gamma, j_* R^1 \rho_{\Gamma,*}\mathbb{Q})^{1,1}.\]

The general story (viz., when (3.3) is not satisfied) involves a rational linear combination of the components of the bad fibers of \(\rho_{\Gamma}\) together with \([h^*(L) - m\sigma(\Gamma)]\) (compare for example ([10], Theorem 1.6)). The argument in showing that this gives a nontrivial class in \(H^1(\Gamma, j_* R^1 \rho_{\Gamma,*}\mathbb{Q})^{1,1}\) is similar but more complicated. For our purpose, we can always choose \(\Gamma_0\) such that it meets the singular fibers of \(X/\mathbb{P}^1\) transversely and hence \(X_{\Gamma}\) is smooth and has irreducible fibers over \(\Gamma\) and (3.3) is trivially satisfied. Then \([h^*(L) - m\sigma(\Gamma)]\) determines a normal function \(\nu\) with

\[\delta(\nu) = [h^*(L) - m\sigma(\Gamma)] \in H^1(\Gamma, j_* R^1 \rho_{\Gamma,*}\mathbb{Z})^{1,1},\]

and hence by Proposition 3.1, \(\nu(t)\) has nontrivial dynamics for general \(t \in \Gamma\).

It remains to verify the following.

**Lemma 3.4** For all \(g, m \in \mathbb{Z}^+\) satisfying \(g \geq 2\) and \(2 | m\) and a general \((X, L) \in P_{g,m}\), there is an irreducible nodal rational curve in \(|L|\) that meets all singular curves in \(|F|\) transversely.

**Proof** It is well known that \(X/\mathbb{P}^1\) has 24 nodal fibers. It suffices to figure out where these 24 curves in \(|F|\) go when we degenerate \(X\). Let \(S/\Delta\) be the family of \(K3\) surfaces constructed in 2.3. A curve \(N \in |F|\) on the central fiber \(S_0 = R = R_1 \cup R_2\) is described in Remark 2.6. It is not hard to see that \(N\) is a limit of nodal rational curves in \(|F|\) on the general fibers if one of the following holds:

- \(N\) passes through one of the sixteen rational double points \(p_j\) and there are sixteen such curves;
- \(N\) passes through one of the four points \(\{p \in D : G_1 \sim_{\text{rat}} 2p\ \text{on} \ D\}\) and there are four such curves;
- \(N\) passes through one of the four points \(\{p \in D : G_2 \sim_{\text{rat}} 2p\ \text{on} \ D\}\) and there are four such curves.

One can check that these add up to 24.
Now we let \( C = C_1 \cup C_2 \) with irreducible components \( C_i \in |L_i| \) satisfying

\[
C_1 \cdot D = C_2 \cdot D = (g + 1)q
\]  

(3.4)

for some point \( q \in D \). This is a limiting rational curve and it obviously meets each of the 24 curves \( N \in |F| \) given above transversely. \( \square \)

4 An open question

Suppose that \( X \) is a \( K3 \) surface defined over \( \overline{\mathbb{Q}} \). Let \( \Sigma \subset X \) be the union of all rational curves on \( X \). Rigidity arguments imply that every rational curve in \( X \) is defined over \( \overline{\mathbb{Q}} \). The following was raised by Matt Kerr ([16]):

**Question 4.1** Is \( X(\overline{\mathbb{Q}}) \subset \Sigma(\overline{\mathbb{Q}}) \)?

An affirmative answer to this question would not only imply that \( \Sigma \) is dense in \( X(\mathbb{C}) \) in the usual topology, but this would also provide a nontrivial instance of the Bloch-Beilinson conjecture on the injectivity of Abel-Jacobi maps for smooth projective varieties defined over \( \overline{\mathbb{Q}} \). More specifically, by an application of the connectedness part of Bertini’s theorem, \( \Sigma \) is connected, hence \( \text{CH}^2_{\text{hom}}(X/\overline{\mathbb{Q}}) = 0 \).

It was pointed out to us by the referee that this question actually has a long, albeit poorly documented history. It was posed by Bogomolov [6] as far back as 1981.

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