Increasing distillable key from bound entangled states by using local filtration

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We show the enhancement of distillable key rate for quantum key distribution (QKD), by local filtering, for several bound entangled states. Through our work it becomes evident that the local filtration operations, while transforming one bound entangled state to another, have the potential to increase the utility of the new state for QKD. We demonstrate three examples of ‘one way distillable key rate’ enhancement by local filtering and in this process, discover new bound entangled states which are key distillable.

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I. INTRODUCTION

A perfect cryptography protocol can be set up if one can distribute a private key between the trusted parties interested in secure communication [1]. It has been established that only quantum key distribution (QKD) protocols are fundamentally secure, as they draw their security from the laws of quantum physics, as opposed to their classical counterparts where the security is based on the impossibility of solving certain mathematical problems in polynomial time [2–4]. The QKD protocols are either of the prepare and measure type, such as the BB84 protocol [5, 6] or the entanglement assisted protocols such as the E91 protocol [7]. These two classes of protocols are intimately connected and entanglement is considered a fundamental resource for QKD [8].

Bipartite quantum entanglement involves non-classical correlations between two parties and can occur in pure as well mixed states [9–11]. A powerful tool to identify entanglement is the transpose operation, where entangled quantum states can transform to non-states, when we apply the transpose operation on one of the sub systems [12]. The states whose density operators become negative under such partial transposition are entangled and are called negative under partial transpose (NPT) and those which remain positive and valid are called positive under partial transpose (PPT). From the work of Peres and Horodeckis’, it became clear that partial transpose is necessary and sufficient only for 2 ⊗ 2 or 2 ⊗ 3 bipartite systems, while for higher dimensional systems there can be entangled states which are PPT [13]. Due to the existence of PPT entangled states, the geometry of states in composite dimensions other than 4 and 6 has not been fully understood. The examples and methods to construct classes of PPT entangled states are sparse [14–16], although a plethora of literature exists on the subject [11, 17, 18].

Ideally, quantum information tasks require pure entangled states. Since mixedness caused by the environment is unavoidable, for operational purposes it is necessary to recreate pure copies of the required entangled state via distillation protocols [19, 20]. Entangled states from which we cannot distill any pure entangled states are called bound entangled and their entanglement is called bound entanglement. All pure entangled states are NPT, and distillable entangled states from which maximally entangled states can be distilled, are also NPT. It can be shown that pure entangled states can not be distilled from PPT entangled states and hence PPT entangled states are bound entangled [21]. For a long time it was believed that distillable entanglement from which one can obtain maximally entangled states is an essential resource for QKD protocols [22] and bound entanglement is not such a resource. Thus it came as a surprise when it was shown that PPT bound entangled states may also be useful for QKD [23–25].

A key role is played by a new class of states called private states [23, 24, 26] which can be used to carry out QKD. These states need not be maximally entangled but still can be used to obtain a perfectly correlated key that is completely uncorrelated to any eavesdropper [23, 24]. Further, it was demonstrated that one can find PPT bound entangled states that are arbitrarily close in trace norm to the private states [25]. Thus using the idea of private states it was shown that the PPT bound entangled states can also be used for QKD [23, 24]. The initial class of key distillable bound entangled states presented in [23, 26] were of large dimensions and hence of limited use for experimental applications. A new class of low dimensional bound entangled states which can be used for QKD key was introduced in [27]. These have been further studied in [28–30].

Evaluating a PPT entangled states for its utility for QKD is not always straightforward and there are more than one criteria being used in the literature. The two most important parameters in this context are ‘distillable key rate’ (\(K_D\)) defined in reference [23] and the ‘one
way distillable key rate’ ($K_{DW}^{D}$) given in the Devetak-Winter protocol [31]. As per the definition given in [23], a state is declared useful for QKD if $K_{D} > 0$. However for the states defined in [27], while it has been shown that $K_{D} > 0$, we have $K_{DW}^{D} < 0$ [31]. The situation therefore is delicate and different definitions may not always agree [25]. However, for the cases that we consider in our work we have $K_{D} > K_{DW}^{D}$ and thus we use the ‘one way distillable key rate’ as a lower bound of the available key from a PPT state.

Local filtration is a process where starting with an ensemble of bipartite states, the members are selected or discarded based on the results of certain local measurements, thereby obtaining a new smaller (filtered) ensemble of bipartite states [32]. While local filtration can not change the nature of entanglement (PPT or NPT), the filtered ensemble can have different properties as compared to the original one. Particularly, such a filtration has been used to enhance the Bell violation [33], for detecting the entanglement of PPT entangled states, classical-classical-quantum (ccq) states, the one way distillable key rate and local filters that we will use in the next section for enhancement of the relevant key rates of certain bound entangled states are discussed.

### A. Private states and distillable key

Consider a situation where Alice and Bob want to carry out QKD; Alice has systems $A$ and $A'$ of dimension $d_A$ and $d_{A'}$, respectively, while Bob has systems $B$ and $B'$ of dimension $d_B$ and $d_{B'}$, respectively. The key is contributed by the systems $A$ and $B$, therefore $d_A = d_B = d$. Let a state $\rho_{ABA'B'} \in B(\mathcal{H}_A \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_{B} \otimes \mathcal{H}_{B'})$ be shared between Alice and Bob and the eavesdropper has the standard purifying system $E$. The purification $|\psi_p\rangle_{ABA'B'E}$ of the state $\rho_{ABA'B'}$ is called secure if upon measurement on systems $A$ and $B$ in the basis $\{ |ij\rangle_{AB} \}_{i,j=0}^{d_A-1}$, followed by tracing over the $A'B'$ subsystem, the joint state of the systems $A, B$ and $E$ takes the form of a ‘classical-classical-quantum’ (ccq) state:

$$\rho^{ccq} = \sum_{i,j=0}^{d_A-1} p_{ij} |ij\rangle_{AB} \otimes \rho_E.$$  

The state $\rho_{ABA'B'}$ whose purification is secure is called a private state or a pdit and $\rho^{ccq}$ is called the ccq-state corresponding to the secure state.

The private states are related to maximally entangled states of the system $AB$, and are obtained by applying twisting operations on the tensor product of a maximally entangled state of $AB$ system and some arbitrary state $\sigma_{A'B'}$ of the system $A'B'$ in the following way:

$$\gamma_{ABA'B'} = \frac{1}{d} \sum_{i,j=0}^{d_A-1} |ij\rangle_{AB} \otimes U^{ii} \sigma_{A'B'} U^{jj\dagger}.$$  

The operators $U^{ii}$ are arbitrary unitary transformations on the $A'B'$ system. Twisting does not change the security related properties of a state as the corresponding ccq-state is invariant under the twisting operation. Therefore, the private states are as efficient for QKD as maximally entangled states and have the same amount of key. In fact given a private state we should always be able to find a twisting operation so that we recover the maximally entangled state of the $AB$ system by twisting [23, 26].

Since, private states are analogous to maximally entangled states in entanglement theory, similar to the definition of measure of distillable entanglement, one can define maximal distillable pdits. For any given state $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ let us consider a sequence of LOCC operations $\{P_n\}$ such that $P_n(\rho_{AB}) = \phi_n$, where $\phi_n \in B(\mathcal{H}_A^{(n)} \otimes \mathcal{H}_B^{(n)})$. A set of operations $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ is called a pdit distillation protocol of state $\rho_{AB}$ if there is a pdit $\gamma_{d_n}$ whose key part is of dimension $d_n \times d_n$, satisfying

$$\lim_{n \to \infty} ||\phi_n - \gamma_{d_n}|| = 0,$$  

### II. BACKGROUND

In this section we briefly recapitulate the results that our work builds upon. In this context private states and their role in QKD protocols, classical-classical-quantum (ccq) states, the one way distillable key rate and local filters that we will use in the next section for enhancement...
For a given distillation protocol $P$, the key rate is given by
\[
\mathcal{R}(P) = \limsup_{n \to \infty} \frac{\log d_n}{n}
\] (4)

The distillable key rate of state $\rho_{AB}$ is given by maximizing over all possible protocols.
\[
K_D(\rho_{AB}) = \sup_P \mathcal{R}(P).
\] (5)

If one can transform by LOCC, such as the recurrence protocol [36], sufficiently many copies of a state $\rho_{AB^{A'}}$ into a state close enough to a private state in trace norm, then $K_D(\rho_{AB^{A'}}) > 0$.

It can be shown that there are PPT states for which the key rate $K_D$ is positive [37, 38]. In any case since separable states by construction do not have any distillable key rate and PPT states are always bound entangled, we can conclude that these states are bound entangled states with non-zero distillable key rate. We will consider a few of these states in the next section and the detailed mathematical proofs are available in [26, 27]. It is not always straightforward to calculate $K_D$ for a given quantum state as an optimization over distillation protocols in involved.

B. One way distillable key rate

In the special case when $d = 2$, the private state is called a pbit, and we will restrict to this case in the rest of this paper. In this scenario, consider a state $\rho_{AB^{A'}} \in B(\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_d \otimes \mathcal{H}_d^{A'})$. Expanding this state in the computational basis of the qubits $A$ and $B$ namely, $\{|ij\}$, with $i, j \in \{0, 1\}$:
\[
\rho_{AB^{A'}} = \begin{bmatrix}
\sigma^{0000} & \sigma^{0001} & \sigma^{0010} & \sigma^{0011} \\
\sigma^{0100} & \sigma^{0101} & \sigma^{0110} & \sigma^{0111} \\
\sigma^{1000} & \sigma^{1001} & \sigma^{1010} & \sigma^{1011} \\
\sigma^{1100} & \sigma^{1101} & \sigma^{1110} & \sigma^{1111}
\end{bmatrix},
\] (6)

The term $|00\rangle \langle 11|_{AB} \otimes \sigma^{0011}$ (upper right hand corner in the above equation), with $\sigma^{0011}$ being a density operator for the $A' B'$ sub-system plays an important role. If the state $\rho_{AB^{A'}}$ was such that the $A'B'$ sub-system is in a maximally entangled state $1/\sqrt{2}(|00\rangle + |11\rangle)$ then this term and its hermitian conjugate namely the term $|11\rangle \langle 00|_{AB} \otimes \sigma^{1100}_{A'B'}$ are the only non-zero off diagonal terms in the expansion and the trace norm $\|\sigma^{1100}\| = \|\sigma^{0011}\| = 1/2$. For any private state, by an appropriate twisting operation, $\|\sigma^{1100}\|$ can always be brought to $1/2$. For an arbitrary state $\rho_{AB^{A'}}$, in order to determine its ability to carry out QKD, we carry out an operation called privacy squeezing where we maximize $\|\sigma^{1100}\|$ by applying twisting operations. The state $\rho^{PS}$ that we obtain after such a maximization is called privacy squeezed state corresponding to $\rho_{AB^{A'B'}}$. It has been shown by a more detailed analysis [31] that if $\|\sigma^{0000}\| = \|\sigma^{0011}\| = \|\sigma^{1111}\|$ and $\|\sigma^{0101}\| < \|\sigma^{0011}\|$, then $K_D(\rho_{AB^{A'B'}}) > 0$.

For a specific form of $\rho_{AB^{A'B'}} \in B(\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^d)$
\[
\rho_{AB^{A'B'}} = |\phi^+\rangle \langle \phi^+| \otimes \sigma_0 + |\phi^-\rangle \langle \phi^-| \otimes \sigma_1
\] + $|\psi^+\rangle \langle \psi^+| \otimes \sigma_2 + |\psi^-\rangle \langle \psi^-| \otimes \sigma_3,$ (7)

where $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$ are Bell states in $\mathcal{H}_2 \otimes \mathcal{H}_2$. Then if $\|\sigma_0 - \sigma_1\| > 1/2$ and $\text{tr} (\sigma_0 \sigma_1) = 0$, then $K_D(\rho_{AB^{A'B'}}) > 0$.

One-way key distillation protocol expresses the key distillation rate for the ccq-state in terms of the Devetak-Winter function $K_D^{DW}$ [31]. Devetak-Winter function or the one way distillable key rate, $K_D^{DW}$ is the difference between the mutual informations of the Alice-Bob subsystem and the Alice-Eve subsystem:
\[
K_D^{DW} = I(A : B) - I(A : E)
\] (8)

where the mutual information between $A$ and $B$ is given as $I(A : B) = S(A) + S(B) - S(AB)$ and between $A$ and $E$ is given as $I(A : E) = S(A) + S(E) - S(AE)$, with $S(X)$ being the von-Neumann entropy of the sub-system $X$.

For a state $\rho_{AB^{A'B'}}$ of the form given in Equation (6), let us define the following parameters
\[
x = (\|\sigma^{0000}\| + \|\sigma^{1111}\|)/2 + \|\sigma^{0011}\|,
y = (\|\sigma^{0000}\| + \|\sigma^{1111}\|)/2 - \|\sigma^{0011}\|,
z = (\|\sigma^{0101}\| + \|\sigma^{1010}\|)/2 + \|\sigma^{1011}\|,
w = (\|\sigma^{0101}\| + \|\sigma^{1010}\|)/2 - \|\sigma^{1011}\|.
\] (9)

In terms of these parameters define
\[
S(E) = -x \log_2 x - y \log_2 y - z \log_2 z - w \log_2 w,
\] (10)

The analysis given in [31] proves that
\[
K_D^{DW}(\rho^{PS}_{AB^{A'B'}}) = 1 - S(E)
\] (11)

This is the key formula that we will use to compute the one way distillable key rates for various states in the next section.

For any bipartite state $\rho_{AB^{A'B'}} \in B(\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_d \otimes \mathcal{H}_d^{A'})$, it can be shown that
\[
K_D(\rho_{AB^{A'B'}}) \geq K_D^{DW}(\rho^{PS}_{AB^{A'B'}})
\] (12)

where $[\rho^{PS}_{AB^{A'B'}}]$ is the ccq-state corresponding to the privacy squeezed state $\rho_{AB^{A'B'}}$. Combining Equation (10) with Equation (12) will allow us to prove the positive distillable key rates of various bound entangled states.

C. Local Filters

In a local filtering process that we consider, Alice and Bob perform local POVMs on the state $\rho_{AA'BB'} \in B(\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2^d)$ defined by the operators $L_A$ and
L_B$. They retain the cases when both of them get positive outcome for the operator $L_A$ and $L_B$ and discard the other cases. The state after such an operation changes as follows:

$$
\rho \rightarrow (L \rho L^\dagger)/(\text{Tr} L \rho L^\dagger), \quad L = L_A \otimes L_B
$$

(13)

In our case $L_A$ and $L_B$ are $4 \times 4$ diagonal matrices leading to

$$
L = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & d \\
\end{bmatrix}_A \otimes \begin{bmatrix}
r & 0 & 0 & 0 \\
0 & s & 0 & 0 \\
0 & 0 & t & 0 \\
0 & 0 & 0 & u \\
\end{bmatrix}_B
$$

(14)

The $16 \times 16$ matrix $L$ can also be written as

$$
L = \begin{bmatrix}
L_1 & 0 & 0 & 0 \\
0 & L_2 & 0 & 0 \\
0 & 0 & L_3 & 0 \\
0 & 0 & 0 & L_4 \\
\end{bmatrix}
$$

(15)

where $L_1, L_2, L_3$ and $L_4$ all are $4 \times 4$ diagonal matrices determined in terms of $L_A$ and $L_B$ and we have

$$
L_1 = \begin{bmatrix}
ar & 0 & 0 & 0 \\
0 & as & 0 & 0 \\
0 & 0 & at & 0 \\
0 & 0 & 0 & au \\
\end{bmatrix},
L_2 = \begin{bmatrix}
br & 0 & 0 & 0 \\
0 & bs & 0 & 0 \\
0 & 0 & bt & 0 \\
0 & 0 & 0 & bu \\
\end{bmatrix},
$$

$$
L_3 = \begin{bmatrix}
cr & 0 & 0 & 0 \\
0 & cs & 0 & 0 \\
0 & 0 & ct & 0 \\
0 & 0 & 0 & cu \\
\end{bmatrix},
L_4 = \begin{bmatrix}
dr & 0 & 0 & 0 \\
0 & ds & 0 & 0 \\
0 & 0 & dt & 0 \\
0 & 0 & 0 & du \\
\end{bmatrix}
$$

(16)

The parameters $\{a, b, c, d, r, s, t, u\} \in (0, 1]$ define the local filter. The filtration is typically considered as an operator on a large ensemble of identically prepared states $\rho_{AA'BB'}$ and lead to smaller ensemble of filtered states. The success probability $P$ of a filter is defined as the probability with which both the measurement give the positive result together and the size of the ensemble after filtration is $P$ times the size of the original ensemble.

III. ENHANCEMENT OF DISTILLABLE KEY OF BOUND ENTANGLED STATES BY LOCAL FILTRATION

We are now ready to consider the possibility of enhancing the distillable key rate from bound entangled states by a local filtration process. The starting states that we consider are bound entangled because they are PPT and the local filters cannot change states from PPT to NPT and hence the filtered states are also bound entangled. We then analyze the distillable key rate from the new and old states by specifically computing the one way distillable key rate defined in Devetak-Winter protocol by a procedure described in Section II. In every case the one distillable key rate goes from a negative to a positive value.

![Graphs](image)

**FIG. 1.** Improvement in the value of $K_D^{DW}$ after local filtration operation on the one parameter family of quantum states $\rho_p^{(1)}$. (a) Value of the one way distillable key rate $K_D^{DW}$ for the family of states $\rho_p^{(1)}$ as a function of $p$. (b) The value of $K_D^{DW}$ after optimal local filtration operation on $\rho_p^{(1)}$. (c) The probability of success $P$ of the optimal local filter. (d) The effective one way distillable key rate after filtration obtained by multiplying the value of $K_D^{DW}$ with the success probability of the optimal filter. For all plots the $x$-axis is the parameter $p \in [0, 1/2]$. The enhancement of the one way distillable key rate can be seen by comparing the graphs shown in (a) and (d).

A. Example 1

Consider a one parameter family of states $\rho_p^{(1)} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$ parameterized by a real parameter with $p \in [0, 1/2]$ defined as

$$
\rho_p^{(1)} = \frac{1}{1+2p} \begin{bmatrix}
2p \tau_1 & 0 & 0 & 2p \tau_1 \\
0 & (\frac{1}{2} - p) \tau_2 & 0 & 0 \\
0 & 0 & (\frac{1}{2} - p) \tau_2 & 0 \\
2p \tau_1 & 0 & 0 & 2p \tau_1 \\
\end{bmatrix}
$$

(17)

with matrices $\tau_1$ and $\tau_2$ given as

$$
\tau_1 = \begin{bmatrix}
\frac{1}{6} & 0 & 0 & 0 \\
0 & \frac{1}{6} - \frac{1}{2} \frac{1}{6} & 0 & 0 \\
0 & -\frac{1}{6} & \frac{1}{2} \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{6} \\
\end{bmatrix}
$$

and

$$
\tau_2 = \begin{bmatrix}
\frac{1}{6} & 0 & 0 & 0 \\
0 & \frac{1}{6} & 0 & 0 \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{6} \frac{1}{6} \\
\end{bmatrix}
$$

(18)

This family of state is PPT and hence any entanglement if present, has to be bound entanglement.

Since this family of states belongs to $B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$ as per Equation (12), $K_D$ is always greater than $K_D^{DW}$ and therefore, $K_D^{DW}$ provides us with a lower bound for the distillable key rate. We calculate the value of $K_D^{DW}$ for these states by exploiting the fact that the structure of the family of states is the same as that given in Equation (6), for which we can calculate $K_D^{DW}$. Comparison with Equation (9) reveals that for this family
of states we have \( x = 4 \left| \frac{p}{1 + 2p} \right|, \ y = 0, \ z = \frac{1}{2} \left| \frac{1 + 2p}{1 + 2p} \right| \) and \( w = \frac{1}{2} \left| \frac{1 + 2p}{1 + 2p} \right| \). Therefore, using the formula (10) and Equation (11) we can calculate \( K^{\text{DW}}_D (\rho^{(1)}_p \mid P^{\text{S(eq)}}_{\text{L}}) \). The graph of \( K^{\text{DW}}_D (\rho^{(1)}_p \mid P^{\text{S(eq)}}_{\text{L}}) \) as a function of \( p \) is plotted in Figure 1(a). The value of \( K^{\text{DW}}_D \) is negative up to a certain values of \( p \) and then crosses over to a positive value at approximately \( p = 0.31 \).

Now, we propose local filtration operation on the family of states \( \rho^{(1)}_p \) with a view to raise the value of the corresponding \( K^{\text{DW}}_D (\rho^{(1)}_p \mid P^{\text{S(eq)}}_{\text{L}}) \). Consider the application of the local filter described in Equations (13) on the state \( \rho^{(1)}_p \). The state after filtration is given by

\[
\rho^{(1)}_p \rightarrow \frac{L \rho^{(1)}_p L^\dagger}{\text{Tr}(L \rho^{(1)}_p L^\dagger)} = \rho^{(1)'}_p. \tag{19}
\]

Exploiting the structure of the filtration operation detailed in Equations (14), (15) & (16) we can write \( \rho^{(1)'}_p \) in explicit matrix form as:

\[
\rho^{(1)'}_p = \frac{1}{(1 + 2p)M} \times \begin{bmatrix}
L_1(2p\tau_1)L^\dagger_1 & 0 & 0 & L_1(2p\tau_1)L^\dagger_1 \\
0 & L_2(1 - p)\tau_2L_2 & 0 & 0 \\
0 & 0 & L_3(1 - p)\tau_2L_3 & 0 \\
L_4(2p\tau_1)L^\dagger_1 & 0 & 0 & L_4(2p\tau_1)L^\dagger_1
\end{bmatrix} \tag{20}
\]

Where \( M = \text{Tr}(L \rho^{(1)}_p L^\dagger) \). For calculating the one way distillable key rate we use the following Equation (9) to ascertain the values of the parameters \( x, y, z \) and \( w \) for this state as

\[
x = (\|L_1 2p(\tau_1)L^\dagger_1/(1 + 2p)M\| + \|L_2 2p(\tau_1)L^\dagger_1/(1 + 2p)M\|)/2 \\
y = (\|L_1 2p(\tau_1)L^\dagger_1/(1 + 2p)M\| + \|L_2 2p(\tau_1)L^\dagger_1/(1 + 2p)M\|)/2 \\
z = (\|L_3(1 - p)\tau_2L_3/(1 + 2p)M\| + \|L_4(1 - p)\tau_2L_4/(1 + 2p)M\|)/2, \tag{21}
\]

\[
w = (\|L_1 2p(\tau_1)L^\dagger_1/(1 + 2p)M\| + \|L_2 2p(\tau_1)L^\dagger_1/(1 + 2p)M\| + \|L_3(1 - p)\tau_2L_3/(1 + 2p)M\|)/(2). \]

The value of \( K^{\text{DW}}_D (\rho^{(1)}_p \mid P^{\text{S(eq)}}_{\text{L}}) \) can be again calculated following Equation (10) and (11), as was done before filtering.

A given filter has a success probability \( P \) which is defined as the fraction of cases the filter gives a positive answer. In real terms, for the purposes of using an ensemble of states for QKD we must multiply the key rate with the success probability to get the effective key rate. This is particularly important if we want to make any comparison with the key rate before filtration. A filter may give us a very high key rate, however, if its probability of success is small the effective key rate may in fact be small. For a given value of \( p \) we numerically optimize the product \( K^{\text{DW}}_D P \) to obtain the optimal filter. It turns out that for this family of states the parameters of the optimal filters are

\[
a = d = r = s = t = u = 1 \tag{22}
b = c = \text{(small } p \text{ dependent value)}.
\]

The results are shown in Figure 1. In Figure 1(a) the value of one way distillable key rate \( K^{\text{DW}}_D \) for the family of states \( \rho^{(1)}_p \) is plotted as a function of \( p \in [0, 1/2] \) before filtration. This value is clearly negative for \( p \) values up to a certain value and becomes positive only after \( p \) approximately crosses 0.31. In Figure 1(b) the value of \( K^{\text{DW}}_D \) is plotted as a function \( p \) after optimal local filtration operation. It is noteworthy that we are always able to find a filter such that the value of the one way distillable key rate is equal to its maximum possible value 1. In Figure 1(c) we plot the success probability \( P \) of the optimal local filter as a function of \( p \) which clearly is small for small values of \( p \) but quickly becomes significant as the value of \( p \) increases and approaches 1 as \( p \) approaches 0.5. In Figure 1(d) we plot the effective one way distillable key rate obtained by multiplying \( K^{\text{DW}}_D \) and \( P \) for the optimal filter as a function of \( p \). It is clear form the graphs that the filtration process is able to make the one way distillable key rate positive for all values \( p \). We must compare the values in Figure 1(a) with values in Figure 1(d) to access the enhancement achieved through filtration. The comparison clearly shows that every member of the family of states \( \rho^{(1)}_p \) after filtration has a positive value of \( K^{\text{DW}}_D \). This implies that this family of PPT states have non-zero one way distillable key rate after filtration. In other words, \( \rho^{(1)}_p \) is a family of bound entangled states with a nonzero value of \( K^{\text{DW}}_D \). Since for this class of states \( K_D > K^{\text{DW}}_D \) they thus have a nonzero distillable key rate. Any PPT entangled state remains PPT entangled under a local filtration operation, this implies that all states in the original family(before filtration) are PPT entangled states which can be employed for QKD after filtration.

As it turns out this family of states is closely related to the states considered and analyzed by K. Horodecki et al. [26]. They consider states \( \rho_{(p,d,k)} \) which for \( d = 2, k = 1 \) are defined as

\[
\rho_{(p,2,1)} = \begin{bmatrix}
\frac{p}{2}(\tau_1 + \tau_2) & 0 & 0 & \frac{p}{2}(\tau_1 - \tau_2) \\
0 & (\frac{1}{2} - p)\tau_2 & 0 & 0 \\
0 & 0 & (\frac{1}{2} - p)\tau_2 & 0 \\
\frac{p}{2}(\tau_1 - \tau_2) & 0 & 0 & \frac{p}{2}(\tau_1 + \tau_2)
\end{bmatrix}, \tag{23}
\]
Consider a projection operator $A$ such that

$$A = \frac{1}{2} \begin{bmatrix} I & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & I \end{bmatrix}. \quad (24)$$

Where $I$ is an identity operator on a $2 \otimes 2$ space. The families of states $\rho_p^{(1)}$ and $\rho_{(p,2,1)}$ are connected with each other via the operator $A$ as follows:

$$\rho_p^{(1)} = \frac{A\rho_{(p,2,1)}A^+}{\text{Tr}(A\rho_{(p,2,1)}A^+)} \quad (25)$$

**B. Example 2**

Next we consider examples of families of states considered by Chi et al. [27]. The family of states $\rho_p^{(2)} \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2)$ is defined as follows:

$$\rho_p^{(2)} = \frac{1}{2} \begin{bmatrix} \sigma_0 + \sigma_1 & 0 & 0 & \sigma_0 - \sigma_1 \\ 0 & \sigma_2 + \sigma_3 & \sigma_2 - \sigma_3 & 0 \\ 0 & \sigma_2 - \sigma_3 & \sigma_2 + \sigma_3 & 0 \\ \sigma_0 - \sigma_1 & 0 & 0 & \sigma_0 + \sigma_1 \end{bmatrix}. \quad (26)$$

with

$$\sigma_0 = \frac{1}{2} \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & 2p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & p \end{bmatrix}, \quad \sigma_1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & -p \end{bmatrix},$$

$$\sigma_2 = \frac{1}{2} \begin{bmatrix} 1 - 4p - 2\sqrt{2}p & 0 & 0 & 0 \\ 0 & (\sqrt{2} + 1)p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & (\sqrt{2} - 1)p \end{bmatrix},$$

$$\sigma_3 = \frac{1}{2} \begin{bmatrix} 1 - 4p - 2\sqrt{2}p & 0 & 0 & 0 \\ 0 & (\sqrt{2} - 1)p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & (\sqrt{2} + 1)p \end{bmatrix}. \quad (27)$$

This family of states is determined by single real parameter $p$ in the range $(\frac{1}{8}, \frac{1}{4+2\sqrt{2}})$. It is straightforward to see that the states in the family are PPT as $(\rho_p^{(2)})^\dagger = \rho_p^{(2)}$ and therefore, if they have entanglement it has to be bound entanglement [38]. Since the family of states have a form given in Equation (7) and here we have $\|\sigma_0 + \sigma_1\| = 4p$ and $\|\sigma_0 - \sigma_1\| = 4p > 1/2$, we have $K_D(\rho_p^{(2)}) > 0$ [27].

In order to calculate the one way distillable key rate for this family of states, we first identify the parameters $x, y, z$, and $w$ as per Equation (9) which turn out to be

$$x = 4p, \quad y = 0, \quad z = \frac{1 - 4p + 2\sqrt{2}p}{2} \text{ and } w = \frac{1 - 4p - 2\sqrt{2}p}{2}. \quad (28)$$

Using Equation (10) & (11) we obtain the value of

$$K_D^{\text{BW}}([\rho_p^{(1)}]_{A\rho_p^{(2)}E'}) = 1 + 4p \log_2 4p + \left(\frac{1 - 4p - 2\sqrt{2}p}{2}\right) \log_2 \left(\frac{1 - 4p - 2\sqrt{2}p}{2}\right) + \left(\frac{1 - 4p + 2\sqrt{2}p}{2}\right) \log_2 \left(\frac{1 - 4p + 2\sqrt{2}p}{2}\right). \quad (29)$$

Here $[\rho_p^{(2)}]_{A\rho_p^{(2)}E'}$ is the ccq state for the privacy squeezed state of $\rho_p^{(2)}$. For this class of states the values of $K_D^{\text{BW}}$ is negative for most value of $p$, therefore, although $K_D > 0$ utility of the states for QKD is not clear. The exact plot of $K_D^{\text{BW}}$ as function of $p$ is shown in Figure 2(a).

Next we apply filtration operation on $\rho_p^{(2)}$ as described in Equation (13),

$$\rho_p^{(2)} \rightarrow \frac{L\rho_p^{(2)}L^+}{\text{Tr}L\rho_p^{(2)}L^+} = \rho_p^{(2)}'. \quad (30)$$

By using the form of filtration matrices given in Equation (14) & (15) the density operator after filtration operation can be written as:
Each element in the above expression is a $4 \times 4$ matrix and can be computed by using Equation (16), here $M=\text{Tr}(L_p^{(2)'} L_1^\dagger)$. In order to calculate the value of $K_D^{\text{DW}}(\rho_p^{(2)'PS_{ccq}^{\text{ccq}}_{ABE}^{\text{ccq}}})$, we first map the parameters $x,y,z$ and $w$ as per Equation (9) which turn out to be

$$
\begin{align*}
x &= \left(\|L_1(\sigma_0 + \sigma_1) L_1^\dagger/2M\| + \|L_4(\sigma_0 + \sigma_1) L_4^\dagger/2M\|\right)/2 \\
y &= \left(\|L_1(\sigma_0 - \sigma_1) L_1^\dagger/2M\| + \|L_4(\sigma_0 - \sigma_1) L_4^\dagger/2M\|\right)/2 \\
z &= \left(\|L_2(\sigma_2 + \sigma_3) L_2^\dagger/2M\| + \|L_3(\sigma_2 + \sigma_3) L_3^\dagger/2M\|\right)/2 \\
w &= \left(\|L_2(\sigma_2 - \sigma_3) L_2^\dagger/2M\| + \|L_3(\sigma_2 - \sigma_3) L_3^\dagger/2M\|\right)/2.
\end{align*}
$$

We now calculate the value of $K_D^{\text{DW}}(\rho_p^{(2)'PS_{ccq}^{\text{ccq}}_{ABE}^{\text{ccq}}})$ by once again using Equations (10)(b) and (11).

The filter is numerically optimized to maximize the effective key rate which is the product of the one way distillable key rate $K_D^{\text{DW}}$ and the success probability of the filter $P$. The structure of the filter turns out to be similar to what was obtained for the previous example and is given in Equation (22). The results are displayed in Figure 2, where in Figure 2(a) we plot $K_D^{\text{DW}}$ before filtration, in Figure 2(b) we plot $K_D^{\text{DW}}$ after filtration, in Figure 2(c) we plot the success probability of the filter and in Figure 2(d) we plot the effective key rate which is the quantity $K_D^{\text{DW}} P$, as functions of $p$. It is clear from a comparison of plots Figure 2(a) and Figure 2(d) that for the entire family of states the one way distillable key rate turns positive from negative. By using Devetak-Winter lower bound of distillable key rate for the ccq states we can therefore state that distillable key rate $K_D(\rho_p^{(2)'}) \geq K_D^{\text{DW}}(\rho_p^{(2)'PS_{ccq}^{\text{ccq}}_{ABE}^{\text{ccq}}}) > 0$. This establishes the usefulness of this family of states for QKD and provides a quantitative estimate of the available key.

C. Example 3

We consider another example from the paper of Chi et al. [27] namely the family of states $\rho_p^{(3)}$ given by the density operator:

$$
\rho_p^{(3)} = \frac{1}{2} \begin{bmatrix}
\sigma_0 + \sigma_1 & 0 & 0 & \sigma_0 - \sigma_1 \\
0 & 2\sigma_2 & 0 & 0 \\
0 & 0 & 2\sigma_2 & 0 \\
\sigma_0 - \sigma_1 & 0 & 0 & \sigma_0 + \sigma_1
\end{bmatrix}.
$$

The operators $\sigma_0$ and $\sigma_1$ are defined in Equation (27) and $\sigma_2 = \frac{\sqrt{2}}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|) + (\frac{1}{\sqrt{2}} - (1 + \frac{1}{\sqrt{2}})) p(|00\rangle\langle 00| + |11\rangle\langle 11|)$). The states in this one parameter family of states are well defined and are bound entangled states in the range $\frac{1}{8} \leq p \leq \frac{1}{4} + \frac{\sqrt{2}}{4}$. Again as per [27] for this family of states $K_D(\rho_p^{(3)}) > 0$. However by an explicit calculation it was shown that $K_D^{\text{DW}}(\rho_p^{(3)'PS_{ccq}^{\text{ccq}}_{ABE}^{\text{ccq}}}) < 0$ for $\frac{1}{8} \leq p \leq \frac{1}{4} + \frac{\sqrt{2}}{4}$.

As was done in the earlier cases we apply filtration operation on $\rho_p^{(3)}$ as defined in Equation (13)

$$
\rho_p^{(3)} \rightarrow \frac{L\rho_p^{(3)} L_1^\dagger}{\text{Tr}L\rho_p^{(3)} L_1^\dagger} = \rho_p^{(3)'}
$$

Again the explicit form of the density operator can be written in terms of filter parameters explicitly as: written as:

$$
\rho_p^{(3)'} = \frac{1}{2M} \times \begin{bmatrix}
L_1(\sigma_0 + \sigma_1) L_1^\dagger & 0 & 0 & L_1(\sigma_0 - \sigma_1) L_4^\dagger \\
0 & L_2(2\sigma_2) L_2^\dagger & 0 & 0 \\
0 & 0 & L_3(2\sigma_2) L_3^\dagger & 0 \\
L_4(\sigma_0 - \sigma_1) L_1^\dagger & 0 & 0 & L_4(\sigma_0 + \sigma_1) L_4^\dagger
\end{bmatrix}.
$$
Each element in the above expression is a $4 \times 4$ matrix and can be computed by using Equation (16), here $M = \text{Tr}(L_\rho^{(3)}P^L)$. In order to calculate the value of $K_D^{\text{DW}}(P^{\text{AE}}|_{ABE}^{(3)})$, we first map the parameters $x, y, z$ and $w$ as per Equation (9) which turn out to be

$$x = (\|L_1(\sigma_0 + \sigma_1)L^1_1/2M\| + \|L_4(\sigma_0 + \sigma_1)L^1_4/2M\|)/2$$
$$+ \|L_1(\sigma_0 - \sigma_1)L^1_1/2M\|,$$
$$y = (\|L_1(\sigma_0 + \sigma_1)L^1_2/2M\| + \|L_4(\sigma_0 + \sigma_1)L^1_4/2M\|)/2$$
$$- \|L_1(\sigma_0 - \sigma_1)L^1_1/2M\|,$$
$$z = (\|L_2(\sigma_2)L^2_2/2M\| + \|L_3(\sigma_2)L^3_2/2M\|)/2$$
$$w = (\|L_2(\sigma_2)L^2_2/2M\| + \|L_3(\sigma_2)L^3_2/2M\|)/2.$$

(36)

The one way distillable key rate is calculated for the numerically optimized filter which in this case again has the structure given in Equation (22). The results are displayed in different parts of Figure 3, where $K_D^{\text{DW}}$ before and after filtration operation, the success probability of the filter and the effective key rate ($K_D^{\text{DW}}P$) are plotted as functions of $p$ in the relevant range of the parameter $p$. A comparison of Figure 3(a) and Figure 3(d) make it clear that the one way distillable key rate has turned from negative to positive under filtration and we now have a filtered ensemble ready for QKD.

IV. CONCLUDING REMARKS

In this paper we have explored the role of local filtration operations in enhancing the distillable key rate available from bound entangled states for QKD. The route we took was to calculate the Winter-Devetak function (also called the one way distillable key rate) which provides a lower bound for the distillable key rate for the examples that we considered. Three examples of families of bound entangled states were analyzed from this point of view and in each case a significant enhancement of the key rate was achieved by the filtration process. The filter was optimized so as to maximize the post filtration one way distillable key rate multiplied by the success probability of the filter. The one way distillation key rate in each case turned from negative to positive and the effective key rate was significant. Our results provide quantitative estimates of available key rates for the family of states that we consider.

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