Generation of modules
and transcendence degree of zero-cycles

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Abstract. We construct an example of a regular algebra over $\mathbb{C}$ of dimension $d$ and a projective module of rank $r$ over this algebra which is not generated by $d + r - 1$ elements. This strengthens Swan’s well-known example over the field of real numbers.

Keywords: modules over rings, Chow groups.

§ 1. Introduction

Let $R$ be a finitely generated commutative unital algebra over a field $k$, and let $M$ be a finitely generated projective $R$-module of rank $r$. Let $d$ be the Krull dimension of $R$. It follows from the Forster–Swan theorem [1], [2] that $M$ is generated by $d + r$ elements over $R$ (the theorem actually holds for much more general rings and modules). In our particular case, this result can also be easily deduced from Bertini’s theorem.

It is natural to ask whether the lower bound $d + r$ on the number of generators is exact. In other words, is it possible to find $R$ and $M$ as above such that $M$ is not generated by any $d + r - 1$ elements? (This question is trivial for $d = 1$.) Swan ([3], Theorem 4) has constructed such an example over $k = \mathbb{R}$ for arbitrary $d$, $r$, using the fact that the field of real numbers is not algebraically closed. To the best of the author’s knowledge, so far there have been no examples of this type over an algebraically closed field $k$. Our aim is to construct such an example.

Using the Chern classes with values in Chow groups, we reduce the initial problem to a question about non-vanishing elements in the Chow group of zero-cycles on an affine variety. This idea closely follows [3], where the Stiefel–Whitney classes are used. Then we use the transcendence degree of zero-cycles, an invariant introduced in [4]. This method of establishing with the aid of the transcendence degree that the elements in the Chow group of zero-cycles are non-vanishing is essentially based on the Bloch–Srinivas decomposition of the diagonal [5].

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It may be possible to use our examples of algebras and modules to construct new non-trivial examples of simple Jordan superalgebras by the method of [6]. However, this would require $R$ and $M$ to satisfy some additional conditions concerning derivations. The actual construction of such examples remains an open problem.

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§ 2. The main result and reduction to zero-cycles

Here is the main result of the paper.

**Theorem 1.** For all $d, r \in \mathbb{N}$ there exist a regular finitely generated algebra $R$ over $\mathbb{C}$ of dimension $d$ and a finitely generated projective $R$-module $M$ of rank $r$ such that $M$ is not generated by any $d + r - 1$ elements over $R$.

Let $k$ be an arbitrary field. We put $U := \text{Spec}(R)$. Suppose that $N$ is a projective $R$-module of rank one, $M = N \oplus R^{\oplus (r-1)}$, and $L$ is the line bundle on $U$ corresponding to the $R$-module $N$. Given any algebraic variety $X$ over $k$, we write $CH^p(X)$ for the Chow group of algebraic cycles of codimension $p$ on $X$. Given any vector bundle $E$ over $X$, we denote the corresponding Chern class by $c_p(E) \in CH^p(X)$. The following lemma is a direct analogue of the method from [3], §2.

**Lemma 2.** Suppose that $M$ is generated by $d + r - 1$ elements. Then $c_1(L)^d = 0$ in the group $CH^d(U)$.

**Proof.** By the hypothesis, there is an exact triple of vector bundles on $U$:

$$0 \to E \to C_U^{\oplus (d+r-1)} \to L \oplus C_U^{\oplus (r-1)} \to 0.$$

By the Whitney formula,

$$1 + c_1(E) + \cdots + c_d(E) = (1 + c_1(L))^{-1}.$$

In particular, $c_d(E) = (-1)^d c_1(L)^d$. On the other hand, since the rank of $E$ is $d - 1$, we have $c_d(E) = 0$. □

Thus, to construct our main example, it would be helpful to establish that certain elements of $CH^d(U)$ are non-vanishing. It is important to consider the Chern classes with values in Chow groups and not in the cohomology groups (Betti, de Rham, or étale) because the cohomology groups of degree $2d$ are trivial for every $d$-dimensional affine variety.

The following is an interpretation of the example in [3], §2, Theorem 4 in terms of Chow groups.

**Example 3.** Suppose that $Q \subset \mathbb{P}^d$ is a smooth projective quadric without rational points over $k$, $U = \text{Spec}(R)$ is the complement to $Q$ in $\mathbb{P}^d$, $L = \mathcal{O}_{\mathbb{P}^d}(1)|_U$, $N$ is the corresponding $R$-module, and $M = N \oplus R^{\oplus (r-1)}$. Then $c_1(L)^d$ is the restriction to $U$ of the class of a point in $CH^d(\mathbb{P}^d) \cong \mathbb{Z}$. If $c_1(L)^d = 0$, then the fact that

$$CH^{d-1}(Q) \to CH^d(\mathbb{P}^d) \to CH^d(U) \to 0$$


is an exact sequence implies that the quadric $Q$ has a zero-cycle of degree one over $k$. By Springer’s well-known result ([7], Ch. VII, Theorem 2.3), this contradicts the absence of $k$-points on $Q$. (When $k = \mathbb{R}$, we also obtain a contradiction by considering the action of complex conjugation on an effective zero-cycle of odd degree.) Hence, by Lemma 2, $M$ is not generated by any $d + r - 1$ elements.

§ 3. Transcendence of zero-cycles and proof of the main result

We recall some notions and facts from [4]. Given a subfield $k_0 \subset k$ and a variety $X_0$ over $k_0$, we denote the extension of scalars of $X_0$ from $k_0$ to $k$ by $(X_0)_k$:

$$(X_0)_k = \text{Spec}(k) \times_{\text{Spec}(k_0)} X_0.$$ 

Let $X$ be a variety over $k$. For simplicity we assume that $X$ is irreducible and has dimension $d$. We suppose that $X$ is the extension of scalars of a variety $X_0$ defined over a subfield $k_0 \subset k$. In other words, we fix an isomorphism $X \cong (X_0)_k$. An open (respectively, closed) subset of $X$ is said to be defined over $k_0$ if it is the extension of scalars of some open (respectively, closed) subset of $X_0$.

A point $P \in X(k) = X_0(k)$ corresponds to a morphism of schemes $P: \text{Spec}(k) \to X_0$. We denote its image by $\xi_P$ and put

$$\text{tr. deg}(P/k_0) := \text{tr. deg}(k_0(\xi_P)/k_0).$$

Equivalently, $\text{tr. deg}(P/k_0)$ is the transcendence degree over $k_0$ of the field generated by the coordinates of $P$ with respect to an arbitrary open affine neighbourhood defined over $k_0$. The transcendence degree $\text{tr. deg}(\alpha/k_0)$ of an element $\alpha \in CH^d(X)$ is defined as the smallest positive integer $n$ for which there is a zero-cycle $\sum_i n_i P_i$ on $X$ over $k$ representing $\alpha$ and satisfying the inequality $\text{tr. deg}(P_i/k_0) \leq n$ for every $i$. The transcendence degree for the elements of $CH^d(X)_{\mathbb{Q}} := CH^d(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined in a similar way (considering all representatives with rational coefficients).

**Remark 4.** In the notation introduced above, we assume that there is an open subset $U \subset X$ defined over $k_0$ such that $\alpha|_U = 0$ in $CH^d(U)$. Then $\alpha$ can be represented by a zero-cycle supported on $Z = X \setminus U$. Since the dimension of $Z$ is strictly less than $d$ and $Z$ is defined over $k_0$, we can see from the definition of the transcendence degree of zero-cycles that $\text{tr. deg}(\alpha/k_0) < d$. This remains valid for Chow groups with rational coefficients.

The following proposition is a direct corollary of [4], Theorem 7.

**Proposition 5.** Let $X$ be an irreducible smooth projective variety of dimension $d$ over a field $k$ of characteristic zero. Assume that $X$ is the extension of scalars of a variety $X_0$ defined over a subfield $k_0 \subset k$ such that $k$ contains every finitely generated field over $k_0$ (equivalently, $k$ is algebraically closed and has infinite transcendence degree over $k_0$). Suppose that there is a point $P \in X(k)$ with $\text{tr. deg}(P/k_0) = d$ such that the class $[P]$ of this point in $CH^d(X)_{\mathbb{Q}}$ satisfies $\text{tr. deg}([P]/k_0) < d$. Then $H^0(X, \Omega_X^d) = 0$.

The idea of the proof of Proposition 5 is to regard $P$ as the restriction of the class of the diagonal in $X_0 \times X_0$ to the subscheme $X_0 \times \text{Spec}(k_0(X_0))$ and to use the Bloch–Srinivas decomposition of the diagonal [5].
Proof of Theorem 1. Let $C$ be a smooth projective curve of positive genus over $\overline{\mathbb{Q}}$. We put $X_0 = C^d$, $X = (X_0)_\mathbb{C}$ and consider a complex point $P = (P_1, \ldots, P_d) \in X(\mathbb{C})$ lying on no subvariety in $X$ defined over $\overline{\mathbb{Q}}$ other than $X$ itself. Such a point exists because there are countably many subvarieties of $X$ defined over $\overline{\mathbb{Q}}$, while $\mathbb{C}$ is uncountable. Explicitly, the condition for $P$ can be stated as follows. Choose an affine chart in $C$ defined over $\overline{\mathbb{Q}}$ whose extension of scalars to $\mathbb{C}$ contains $P_1, \ldots, P_d$. Then the coordinates of $P_1, \ldots, P_d$ with respect to this chart generate a field extension of $\overline{\mathbb{Q}}$ of transcendence degree $d$.

We now consider the following divisors in $X$:

$$D_i := \{(x_1, \ldots, x_d) \in X \mid x_i = P_i\}, \quad 1 \leq i \leq d,$$

and the (reducible) divisor $D = \bigcup_i D_i$. Note that $D$ is not defined over $\overline{\mathbb{Q}}$. Finally, let $U = \text{Spec}(R)$ be any non-empty affine open subset in $X$ defined over $\overline{\mathbb{Q}}$, let $L = \mathcal{O}_X(D)|_U$, let $N$ be the corresponding $R$-module, and let $M = N \oplus R^{\oplus(r-1)}$. Then we have

$$c_1(L)^d = c_1(\mathcal{O}_X(D))^{d}|_U = d! \cdot [P]|_U \in CH^d(U).$$

Suppose that $[P]|_U = 0$ in $CH^d(U)_{\mathbb{Q}}$. By Remark 4, tr. deg[$P|/k_0$] $< d$ in $CH^d(X)_{\mathbb{Q}}$. By Proposition 5, this contradicts the condition

$$H^0(X, \Omega^d_{X}) \cong H^0(C, \Omega^1_{C})_{\mathbb{C}} \otimes^d \neq 0.$$

Hence $[P]|_U \neq 0$ in $CH^d(U)_{\mathbb{Q}}$. Therefore $d! \cdot [P]|_U \neq 0$ in $CH^d(U)$ and, by Lemma 2, $M$ is not generated by any $d + r - 1$ elements over $R$. □

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