A real expectation value of the time-dependent non-Hermitian Hamiltonians

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Abstract

With the aim to solve the time-dependent Schrödinger equation associated to a time-dependent non-Hermitian Hamiltonian, we introduce a unitary transformation that maps the Hamiltonian to a time-independent \( \mathcal{PT} \)-symmetric one. Consequently, the solution of time-dependent Schrödinger equation becomes easily deduced and the evolution preserves the \( \mathcal{C}(t)\mathcal{PT} \)-inner product, where \( \mathcal{C}(t) \) is obtained from the charge conjugation operator \( \mathcal{C} \) through a time dependent unitary transformation. Moreover, the expectation value of the non-Hermitian Hamiltonian in the \( \mathcal{C}(t)\mathcal{PT} \) normed states is guaranteed to be real. As an illustration, we present a specific quantum system given by a quantum oscillator with time-dependent mass subjected to a driving linear complex time-dependent potential.

PACS: 03.65.Ca, 03.65.-w

Published in: Physica Scripta 96 (2021) 125265. doi:10.1088/1402-4896/ac3dbd
1 Introduction

It is commonly believed that the Hamiltonian must be Hermitian \( H = H^+ \) in order to ensure that the energy spectrum (the eigenvalues of the Hamiltonian) is real and that the time evolution of the theory is unitary (probability is conserved in time), where the symbol ‘+’ denotes the usual Dirac hermitian conjugation; that is, transpose and complex conjugate. In 1998 this false impression has been challenged by Bender and Boettcher [1] who showed numerically that a few one-dimensional quantum potentials \( V(x) \) may generate bound states \( \psi(x) \) with real energies \( E \) even when the potentials themselves are not real. They show that because \( \mathcal{P(T)} \)-symmetry is an alternative condition to Hermiticity. The central idea of \( \mathcal{P(T)} \)-symmetric quantum theory is to replace the condition that the Hamiltonian of a quantum theory be Hermitian with the weaker condition: the invariance by space-time reflection. This allows one to construct and study many new Hamiltonians that would previously have been ignored.

These two important discrete symmetry operators are parity \( \mathcal{P} \) and time reversal \( \mathcal{T} \). The operators \( \mathcal{P} \) and \( \mathcal{T} \) are defined by their effects on the dynamical variables \( x \) and \( p \). The operator \( \mathcal{P} \) is linear and has the effect of changing the sign of the momentum operator \( p \) and the position operator \( x \): \( p \rightarrow -p \) and \( x \rightarrow -x \). The operator \( \mathcal{T} \) is antilinear and has the effect \( p \rightarrow -p \), \( x \rightarrow x \), and \( i \rightarrow -i \). It is crucial, of course, that when replacing the condition of Hermiticity by \( \mathcal{P(T)} \)-symmetry, we preserve the key physical properties that a quantum theory must have. We see that if the \( \mathcal{P(T)} \)-symmetry of the Hamiltonian is not broken, then the Hamiltonian exhibits all of the features of a quantum theory described by a Hermitian Hamiltonian.

In order to have a coherent and unitary theory, Bender et al. [2] have defined the \( \mathcal{P(T)} \) inner-product associated to \( \mathcal{P(T)} \)-symmetric Hamiltonians as follows

\[
\langle f, g \rangle_{\mathcal{P(T)}} = \int_C dx \left[ \mathcal{P(T)} f(x) \right] g(x),
\]

where \( \mathcal{P(T)} f(x) = f^*(-x) \). The advantage of this inner product is that the associated norm \( (f, f) \), which independent of the global phase of \( f(x) \), is conserved in time. The application of this definition to the eigenfunctions of \( H \) and \( \mathcal{P(T)} \) implies

\[
\langle \psi_m, \psi_n \rangle_{\mathcal{P(T)}} = (-1)^n \delta_{mn},
\]

The situation here (that half of the the eigenfunctions of \( H \) and \( \mathcal{P(T)} \) have positive norm and the other half have negative norm) is analogous to the problem that Dirac encountered in formulating the spinor wave equation in relativistic quantum theory. Following Dirac, Bender et al. [2] constructed a linear operator denoted by \( \mathcal{C} \) and represented in position space as a sum over the energy eigenstates of the Hamiltonian. The operator \( \mathcal{C} \) is the observable that represents the measurement of the signature of the \( \mathcal{P(T)} \) norm of a state. The properties of the new operator \( \mathcal{C} \) resemble those of the charge conjugation operator in quantum field theory. Specifically, if the energy eigenstates satisfy [2], then we have \( \mathcal{C} \psi_n = (-1)^n \psi_n \), \( \mathcal{C} \), called the charge conjugation symmetry with eigenvalues \( \pm 1 \), \( \mathcal{C}^2 = 1 \), such that \( \mathcal{C} \) commutes with the operator \( \mathcal{P(T)} \) but not with the operators \( \mathcal{P} \) and \( \mathcal{T} \) separately, is the operator observable that represents the measurement of the signature of the \( \mathcal{P(T)} \) norm of a state which determines its parity type. We can regard \( \mathcal{C} \) as representing the operator that determines the \( \mathcal{C} \) charge of the state. Quantum states having opposite \( \mathcal{C} \) charge possess opposite parity type.

The introduction of operator \( \mathcal{C} \) permits to formulate a positive \( \mathcal{CPT} \) inner-product

\[
\langle f, g \rangle_{\mathcal{CPT}} = \int_C dx \left[ \mathcal{CPT} f(x) \right] g(x),
\]
thus Eq. (2) becomes
\[ \langle \psi_m, \psi_n \rangle_{\mathcal{CPT}} = \delta_{mn}. \] (4)

The non-Hermitian \( \mathcal{PT} \)-symmetric models have been successfully used for describing several physical systems such as the plasmons in nanoparticle systems [3], the problems related to the quantum information theory [4], nonclassical light [5] and the stability of hydrogen molecules [6].

The generalization to time-dependent non-Hermitian case have been studied in [7–29]. Note that the authors of Ref. [30] emphasize that in nonrelativistic quantum mechanics and in relativistic quantum field theory, the time coordinate \( t \) is a parameter and thus the time-reversal operator \( \mathcal{T} \) does not actually reverse the sign of \( t \). Some authors adopt the fact that the operator \( \mathcal{T} \) changes also the sign of time \( t \to -t \) [7, 31–40], this case could lead sometimes to incorrect results.

In this work, we adopt the following strategy: we introduce a unitary transformation \( F(t) \) which commutes with the parity \( \mathcal{P} \) and maps the solution \( |\psi(t)\rangle \) of the time-dependent Schrödinger equation involving a non-Hermitian Hamiltonian \( H(t) \) to the solution \( |\chi(t)\rangle \) involving a non Hermitian Hamiltonian \( \mathcal{H} \) required to be time-independent and \( \mathcal{PT} \)-symmetric. After performing this transformation, the problem becomes exactly solvable and the evolution preserve the \( \mathcal{CPT} \)-scalar product \( \langle \chi(t)| \chi(t) \rangle_{\mathcal{CPT}} = \langle \chi(t)| \mathcal{CP} |\chi(t)\rangle \). The other essential ingredient of this theory is the construction of a positive-definite inner product with respect to \( H(t) \) being non self-adjoint, so that its time-evolution operator is unitary and we obtain a consistent probabilistic interpretation so that the Hamiltonian under study exhibits real mean values. The most important step towards finding this positive-definite inner product is thus to find a new operator, which we call \( \mathcal{C}(t) = F^+(t) \mathcal{C} F(t) \) such that, we obtain a conserved norm for our original system described by the solution \( |\psi(t)\rangle \) that is \( \langle \psi(t)| \psi(t) \rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle \chi(t)| \chi(t) \rangle_{\mathcal{CPT}}, \) and the mean value of the time-dependent non-Hermitian Hamiltonian \( H(t) \) is real in the new \( \mathcal{C}(t)\mathcal{PT} \)-inner product. This is the main result of this paper.

For this we introduce, in section 2, a formalism based on the time-dependent unitary transformations is given in order to prove that the expectation value of the time-dependent non-Hermitian Hamiltonian \( H(t) \) is real in the new \( \mathcal{C}(t)\mathcal{PT} \)-inner product. In section 3, we illustrate our formalism introduced in the previous section by treating a non-Hermitian time-dependent quantum oscillator with time-dependent mass in linear complex time-dependent potential. On the hand, in the Hermitian case the time-dependent quantum harmonic have been extensively studied in the literature in different ways [41–48]. Finally, section 4 concludes our work.

### 2 Mean value of non-Hermitian time-dependent Hamiltonian

Let us consider a non-hermitian time-dependent Hamiltonian \( H(t) \) where the quantum time evolution of the system is governed by the time-dependent Schrödinger equation (for simplicity we take \( \hbar = 1 \))
\[ \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle. \] (5)

In order to study the evolution of the quantum systems associated to the time-dependent Hamiltonian \( H(t) \), we seek that this Hamiltonian can be converted into a time-independent Hamiltonian by some time-dependent transformations. To this end, we initially perform a unitary transformation \( F(t) \) on \( |\psi(t)\rangle \)
\[ |\chi(t)\rangle = F(t) |\psi(t)\rangle, \] (6)
by inserting (6) in Eq. (5), we obtain the time dependent Schrödinger equation for the state $|\chi(t)\rangle$

$$i\frac{\partial}{\partial t}|\chi(t)\rangle = \mathcal{H}|\chi(t)\rangle,$$  \hspace{1cm} (7)

such that the new Hamiltonian

$$\mathcal{H} = F(t)H(t)F^+ (t) - iF(t)\frac{\partial F^+ (t)}{\partial t},$$  \hspace{1cm} (8)

is time-independent and $\mathcal{P}\mathcal{T}$-symmetric, i.e.;

$$\mathcal{H} = \mathcal{H}_{0}^{\mathcal{P}\mathcal{T}},$$  \hspace{1cm} (9)

its eigenstates $|\chi(t)\rangle$ preserve the $\mathcal{C}\mathcal{P}\mathcal{T}$-inner product

$$\langle \chi(t)|\chi(t)\rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = \langle \chi(t)|\mathcal{C}\mathcal{P}|\chi(t)\rangle,$$  \hspace{1cm} (10)

and in this case the solution of the Schrödinger equation (7) can be written as

$$|\chi(t)\rangle = \exp(-iEt)|\chi\rangle,$$  \hspace{1cm} (11)

where $|\chi\rangle$ is an eigenstate of $\mathcal{H}_{0}^{\mathcal{P}\mathcal{T}}$.

Knowing that our interest is the mean value of the non-Hermitian Hamiltonian $H(t)$, for this aim we calculate firstly the expectation value of the Hamiltonian $\mathcal{H}_{0}^{\mathcal{P}\mathcal{T}}$

$$\langle \mathcal{H}_{0}^{\mathcal{P}\mathcal{T}} \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} = \langle \chi(t)|\mathcal{C}\mathcal{P}\mathcal{H}_{0}^{\mathcal{P}\mathcal{T}}|\chi(t)\rangle = \langle \chi(t)|\mathcal{C}\mathcal{P} \left[ FH(t)F^+ - iF\frac{\partial F^+}{\partial t} \right]|\chi(t)\rangle,$$  \hspace{1cm} (12)

from which we deduce that is

$$\langle \chi(t)|\mathcal{C}\mathcal{P} \left[ FH(t)F^+ \right]|\chi(t)\rangle = \langle \mathcal{H}_{0}^{\mathcal{C}\mathcal{T}} \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} + \langle \chi(t)|\mathcal{C}\mathcal{P} \left[ iF\frac{\partial F^+}{\partial t} \right]|\chi(t)\rangle,$$  \hspace{1cm} (13)

we note that the first term is nothing other than the expectation value of the Hamiltonian $H(t)$ with a new $\mathcal{C}(t)\mathcal{P}\mathcal{T}$-inner product

$$\langle \chi(t)|\mathcal{C}\mathcal{P} \left[ FH(t)F^+ \right]|\chi(t)\rangle = \langle \psi(t)|\mathcal{C}(t)\mathcal{P}H(t)|\psi(t)\rangle = \langle H(t)\rangle_{\mathcal{C}(t)\mathcal{P}\mathcal{T}},$$  \hspace{1cm} (14)

where $[\mathcal{P}, F(t)] = 0$ and the new operator $C(t)$ is defined as $C(t) = F^+(t)CF(t)$, which is similar to the operator $C$ in the sense that verifies the property $C^2(t) = 1$ since $C^2 = 1$.

Finally,

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{P}\mathcal{T}} = \langle \mathcal{H}_{0}^{\mathcal{C}\mathcal{T}} \rangle_{\mathcal{C}\mathcal{P}\mathcal{T}} + \langle \chi(t)|\mathcal{C}\mathcal{P} \left[ iF\frac{\partial F^+}{\partial t} \right]|\chi(t)\rangle,$$  \hspace{1cm} (15)

Indeed, since $\mathcal{H}_{0}^{\mathcal{C}\mathcal{T}}$ is $\mathcal{P}\mathcal{T}$ symmetric and $F$ is unitary, the expectation value $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{P}\mathcal{T}}$ is guaranteed to be real. To our knowledge, this general result is new for explicitly time-dependent non-Hermitian systems.

## 3 Application: non-Hermitian time-dependent mass forced oscillators

Let us consider a class of one dimensional time-dependent harmonic oscillators with variable mass $m(t) = m_0\alpha(t)$ subjected to a driving linear complex time-dependent potential, in the form $i\lambda(t)x$, described by the following non-Hermitian Hamiltonian

$$H(t) = \frac{p^2}{2m_0\alpha(t)} + \alpha(t)\frac{m_0\omega^2(t)}{2}x^2 + ix\sqrt{\alpha(t)},$$  \hspace{1cm} (16)
where \( \alpha(t) \) is a positive real time-dependent function, \( x \) and \( p \) are the canonical conjugates position and momentum operators satisfying \([x, p] = i\). The function \( \lambda(t) \) in the complex potential has been chosen \( \lambda(t) = \sqrt{\alpha(t)} \) in order to obtain in Eq. \((8)\) a time-independent \(PT\)-symmetric Hamiltonian \( \mathcal{H}_0^{PT} \). Without loss of generalities, we choose \( \omega(t) = \omega \) as a constant. The mass \( m_0 \) and the frequency \( \omega \) are the characteristic parameters of the quantum system.

We show that the exact solution of the time-dependent Schrödinger equation \((5)\) can be found by introducing two consecutive unitary transformations. In order to solve the Schrödinger with the Hamiltonian specified by \((16)\), we first try to eliminate the time-dependent parameter \( \alpha(t) \). This can be achieved by the transformation

\[
F_1(t) = \exp \left[-\frac{i}{2} \{x, p\} \ln \left(\sqrt{\alpha(t)}\right)\right] \tag{17}
\]

The unitary operator \( F_1(t) \) has the properties

\[
F_1 x F_1^+ = \frac{x}{\sqrt{\alpha(t)}}, \quad F_1 p F_1^+ = p\sqrt{\alpha(t)}, \quad (18)
\]

In a representation \( x \), the wave function is given by

\[
\langle x | F_1 | \phi \rangle = \alpha^{-\frac{1}{2}} \phi \left(x\alpha^{-\frac{1}{2}}\right). \tag{19}
\]

Suppose that

\[
|\phi(t)\rangle = F_1(t) |\psi(t)\rangle, \tag{20}
\]

Substituting \((20)\) into \((5)\) ruled by the Hamiltonian \((16)\), we find the equation of motion for \(|\phi(t)\rangle\)

\[
i \frac{\partial}{\partial t} |\phi(t)\rangle = H_1(t) |\phi(t)\rangle, \tag{21}
\]

where the Hamiltonian

\[
H_1(t) = F_1(t) H(t) F_1^+(t) - i F_1(t) \frac{\partial F_1^+(t)}{\partial t} \tag{22}
\]

\[
= \frac{p^2}{2m_0} + \frac{m_0 \omega^2}{2} x^2 + i x + \frac{1}{4} \alpha(t) (xp + px) \tag{23}
\]

look like the time-independent harmonic oscillators with variable mass \(m_0\) subjected to a driving linear complex time-independent potential plus a time dependent \((xp + px)\) terms. In order to obtain the usual time-dependent harmonic oscillator with a perturbative linear potential, we remove the cross term in \((23)\) via the transformation

\[
F_2(t) = \exp \left[i \frac{m_0 \dot{\alpha}(t)}{4\alpha(t)} x^2 \right], \tag{24}
\]

where its properties are

\[
F_2 x F_2^+ = x, \quad F_2 p F_2^+ = -\frac{m_0 \dot{\alpha}(t)}{2\alpha(t)} x, \quad (25)
\]

Thus, the following unitary transformation \(F(t) = F_2(t) F_1(t)\)

\[
F(t) = \exp \left[i \frac{m_0 \dot{\alpha}(t)}{4\alpha(t)} x^2 \right] \exp \left[-\frac{i}{2} \{x, p\} \ln \left(\sqrt{\alpha(t)}\right)\right], \tag{26}
\]

transforms the canonical operators \(x\) and \(p\) and their squares \(x^2\) and \(p^2\) as follows

\[
F x F^+ = \frac{x}{\sqrt{\alpha(t)}}, \quad F p F^+ = p\sqrt{\alpha(t)} - \frac{m_0 \dot{\alpha}(t)}{2\sqrt{\alpha(t)}} x, \tag{27}
\]
therefore, the transformed Hamiltonian (8) reads
\[
\mathcal{H} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega^2 x^2 + ix.
\] (28)

where
\[
\Omega^2 = \left( \omega^2 + \frac{1}{4} \frac{\dot{\alpha}^2(t)}{\alpha^2(t)} - \frac{\ddot{\alpha}(t)}{2\alpha(t)} \right)
\] (29)

The central idea in this procedure is to require that the Hamiltonian (28) governing the evolution of $|\chi(t)\rangle$ is time-independent. This is achieved by setting the global time-dependent frequency appearing in (28) equal to a real constant denoted by $\Omega_0^2$ so that its time-derivative leads to an auxiliary equation of the form
\[
\ddot{\alpha} - \frac{\dot{\alpha}^2}{2\alpha} + 2\alpha (\Omega_0^2 - \omega^2) = 0,
\] (30)

the resulting time independent non-Hermitian Hamiltonian
\[
\mathcal{H}_0^{PT} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega_0^2 x^2 + ix,
\] (31)
is $\mathcal{P}\mathcal{T}$-symmetric.

Note that when taking $\alpha(t) = \frac{1}{\rho(t)^2}$, the above auxiliary equation (30) is transformed to the following new auxiliary equation
\[
\dot{\rho} + (\Omega_0^2 - \omega^2) \rho = 0.
\] (32)

which admits the following solutions:

- for $\Omega_0^2 > \omega^2$: $\rho(t) = A \exp \left( it\sqrt{\Omega_0^2 - \omega^2} \right) + B \exp \left( -it\sqrt{\Omega_0^2 - \omega^2} \right)$. For an appropriate choice of the constants: $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2 \cos^2 \left( t\sqrt{\Omega_0^2 - \omega^2} \right)}$.

- for $\Omega_0^2 < \omega^2$: $\rho(t) = A \exp \left( t\sqrt{\omega^2 - \Omega_0^2} \right) + B \exp \left( -t\sqrt{\omega^2 - \Omega_0^2} \right)$. For an appropriate choice of the constants: $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2 \cosh^2 \left( t\sqrt{\omega^2 - \Omega_0^2} \right)}$, and when $B = 0$ and $A \neq 0$ the expression of $\alpha(t)$ is $\alpha(t) = \frac{1}{A^2} \exp \left( -2t\sqrt{\omega^2 - \Omega_0^2} \right)$ and the Hamiltonian $H(t)$ corresponds to the Caldirola-Kanai oscillator [41, 42].

### 3.1 Analysis of the expectation value of the Hamiltonian

The eigen equation of the $\mathcal{P}\mathcal{T}$-symmetric Hamiltonian $\mathcal{H}_0^{PT}$ has the form
\[
\mathcal{H}_0^{PT} |\chi_n(x)\rangle = E_n |\chi_n(x)\rangle,
\] (33)

and the solution of the corresponding Schrödinger equation (7) can be written as
\[
|\chi_n(x,t)\rangle = \exp(-iE_n t) |\chi_n(x)\rangle.
\] (34)

Let us introduce a non unitary transformation of the form
\[
U = \exp \left[ -\frac{p}{m_0\Omega_0^2} \right],
\] (35)
such that

\[ |\chi_n(x)\rangle = U |\varphi_n(x)\rangle. \]  

(36)

The action of \( U \) maps the \( \mathcal{PT} \)-symmetric Hamiltonian \( H_0^{\mathcal{PT}} \) to a Hermitian one as

\[ h = U^{-1} H_0^{\mathcal{PT}} U = \frac{p^2}{2m_0} + \frac{m_0 \Omega_0^2}{2} x^2 - \frac{1}{2m_0 \Omega_0^2}, \]  

(37)

where the eigenfunctions \( |\varphi_n(x)\rangle \) of the Hermitian Hamiltonian \( h \) are

\[ |\varphi_n(x)\rangle = \left[ \sqrt{m_0 \Omega_0} \right]^{1/2} \exp \left( -\frac{m_0 \Omega_0}{2h} x^2 \right) H_n \left[ \frac{m_0 \Omega_0}{h} \right]^{1/2} x. \]  

(38)

Then, the solutions \( |\chi_n(x,t)\rangle \) are obtained as

\[ |\chi_n(x,t)\rangle = \exp(-iE_n t) |\varphi_n(x)\rangle, \]

\[ |\chi_n(x,t)\rangle = \left[ \sqrt{m_0 \Omega_0} \right]^{1/2} \exp(-iE_n t) \exp \left( -\frac{p}{m_0 \Omega_0} \right) \exp \left( -\frac{m_0 \Omega_0}{2h} x^2 \right) H_n \left[ \frac{m_0 \Omega_0}{h} \right]^{1/2} x, \]  

(39)

where the eigenvalues

\[ E_n = \hbar \Omega_0 \left( n + \frac{1}{2} \right) - \frac{1}{2m_0 \Omega_0^2}, \]  

(40)

are real and \( H_n \) is the Hermite polynomial of order \( n \).

A more general way to represent the \( C \) operator is to express it generically in terms of the fundamental dynamical operators \( x \) and \( p : C = e^{Q(x,p)} \). The exact formula of \( C \) associated to the theory described by the Hamiltonian (31) is given as a function of the parity operator \( \mathcal{P} \) as

\[ C = \exp \left[ \frac{2}{m_0 \Omega_0^2} \right] \mathcal{P}, \]  

(41)

such that the operator \( C \) commute with \( \mathcal{PT} \) and \( H_0^{\mathcal{PT}} \), i.e., \( [C, \mathcal{PT}] = [C, H_0^{\mathcal{PT}}] = 0 \).

We can easily verify that the \( C \mathcal{PT} \)-inner product is conserved

\[ \langle \chi_n(x,t) | \chi_n(x,t) \rangle_{C \mathcal{PT}} = \langle \chi_n(x) | \mathcal{C} \mathcal{P} | \chi_n(x) \rangle = \langle \varphi_n | U \mathcal{C} \bar{U} | \varphi_n \rangle = \langle \varphi_n(x) | \varphi_n(x) \rangle = 1. \]  

(42)

Now it is not difficult to calculate the expectation value of the Hamiltonian \( \langle H(t) \rangle_{C(t) \mathcal{PT}} \) defined previously

\[ \langle H(t) \rangle_{C(t) \mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)} \langle \varphi_n(x) | U^{-1} \{ x, p \} U | \varphi_n(x) \rangle + \left( \frac{m_0 \Omega_0}{4 \alpha(t)} \right) \langle \chi_n(x) | \mathcal{C} \mathcal{P} x^2 | \chi_n(x) \rangle, \]  

(43)

thus

\[ \langle H(t) \rangle_{C(t) \mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)} \langle \varphi_n(x) | \{ x, p \} | \varphi_n(x) \rangle \]

\[ + \frac{\dot{\alpha}(t)}{2\alpha(t) m_0 \Omega_0} \langle \varphi_n(x) | p | \varphi_n(x) \rangle + \left( \frac{m_0 \Omega_0}{4} \right) \langle \chi_n(x) | \mathcal{C} \mathcal{P} x^2 | \chi_n(x) \rangle, \]  

(44)

where \( \langle x^2 \rangle_{C \mathcal{PT}} = \langle \chi_n(x) | \mathcal{C} \mathcal{P} x^2 \chi_n(x) \rangle \). By using the following relations

\[ \langle \varphi_n(x) | x | \varphi_n(x) \rangle = \langle \varphi_n(x) | p | \varphi_n(x) \rangle = 0, \]  

(45)

\[ \langle \varphi_n(x) | x^2 | \varphi_n(x) \rangle = \frac{\hbar}{m_0 \Omega_0} \left( n + \frac{1}{2} \right), \]  

(46)

\[ \langle \varphi_n(x) | p^2 | \varphi_n(x) \rangle = m_0 \Omega_0 \hbar \left( n + \frac{1}{2} \right), \]  

(47)
\begin{equation}
\langle \varphi_n(x) | \{x, p\} | \varphi_n(x) \rangle = 0,
\end{equation}
and
\begin{equation}
\langle x^2 \rangle_{CP} = \frac{\hbar}{m_0 \Omega} \left( n + \frac{1}{2} \right) - \frac{1}{(m_0 \Omega_0^2)^2},
\end{equation}
we get the expectation value of \( H(t) \) as
\begin{equation}
\langle H(t) \rangle_{CP} = E_n + \left( \frac{m_0 \alpha(t)}{4 \Omega(t)} \right) < x^2 >_{CP} = E_n + \frac{\dot{\alpha}(t)}{4 \Omega(t)} \left[ \frac{\hbar}{\Omega_0} \left( n + \frac{1}{2} \right) - \frac{1}{(m_0 \Omega_0^2)^2} \right],
\end{equation}
which is real for any positive real time-dependent function \( \alpha(t) \) and more simple than the result given in Eq. (28) in Ref. [7] with less constraints on the parameters of the problem.

### 3.2 Uncertainty relation and probability density

Now, we calculate the expectation values \( \langle x \rangle_{CP} \), \( \langle x^2 \rangle_{CP} \), \( \langle p \rangle_{CP} \) and \( \langle p^2 \rangle_{CP} \) in the states \( \psi_n(x, t) \) of \( H(t) \) defined in Eq. (10). In the same way, using the \( CP \)-inner product [12] and after straightforward calculation we obtain that
\begin{equation}
\langle x \rangle_{CP} = \langle \psi_n(x, t) | F^+ CP F x | \psi_n(x, t) \rangle = - \frac{i}{m_0 \Omega_0^2} \frac{1}{\sqrt{\alpha(t)}},
\end{equation}
\begin{equation}
\langle x^2 \rangle_{CP} = \langle \psi_n(x, t) | F^+ CP F x^2 | \psi_n(x, t) \rangle = \left( n + \frac{1}{2} \right) \frac{\hbar}{m_0 \Omega_0 \alpha(t)} - \frac{1}{\alpha(t)(m_0 \Omega_0^2)^2},
\end{equation}
\begin{equation}
\langle p \rangle_{CP} = \langle \psi_n(x, t) | F^+ CP F p | \psi_n(x, t) \rangle = \frac{i \dot{\alpha}(t)}{2 \Omega_0^2} \frac{1}{\sqrt{\alpha(t)}},
\end{equation}
\begin{equation}
\langle p^2 \rangle_{CP} = \langle \psi_n(x, t) | F^+ CP F p^2 | \psi_n(x, t) \rangle,
\end{equation}
\begin{equation}
\langle p^2 \rangle_{CP} = \hbar \Omega_0 \left( n + \frac{1}{2} \right) m_0 \alpha(t) + \left( \frac{m_0 \dot{\alpha}(t)}{2} \right)^2 \left[ \left( n + \frac{1}{2} \right) \frac{\hbar}{m_0 \Omega_0 \alpha(t)} - \frac{1}{\alpha(t)(m_0 \Omega_0^2)^2} \right].
\end{equation}
We calculate also the position and momentum uncertainties
\begin{equation}
\Delta x = \sqrt{\langle x^2 \rangle_{CP} - \langle x \rangle_{CP}^2} = \left[ \frac{\hbar}{m_0 \Omega_0 \alpha(t)} \left( n + \frac{1}{2} \right) \right]^{1/2},
\end{equation}
\begin{equation}
\Delta p = \sqrt{\langle p^2 \rangle_{CP} - \langle p \rangle_{CP}^2} = \frac{1}{\Delta x} \left[ \left( n + \frac{1}{2} \right)^2 + \left( \frac{m_0 \dot{\alpha}(t)}{2} \right)^2 \Delta x^4 \right]^{1/2}.
\end{equation}
Thus, the uncertainty product is given by
\begin{equation}
\Delta x \Delta p = \left( n + \frac{1}{2} \right) \sqrt{1 + \left( \frac{\hbar \dot{\alpha}(t)}{2 \Omega_0 \alpha(t)} \right)^2},
\end{equation}
it is easy to check that the uncertainty product \( \Delta x \Delta p = \frac{1}{2} \) only for \( n = 0 \) and \( \alpha(t) = \text{constant} \), i.e., for time independent mass oscillators.

Finally, the probability density of the wavefunction \( \psi_n(x, t) \) of \( H(t) \) is in the form
\begin{equation}
|U^{-1} F \psi_n(x, t)|^2 = |U^{-1} \chi_n(x, t)|^2 = |\varphi(x)|^2 = \varphi_n^*(x) \varphi_n(x),
\end{equation}
thus

\[ |U^{-1} F \psi_n(x,t)|^2 = \left[ \frac{\sqrt{m_0 \Omega_0}}{n! 2^n \sqrt{\pi \hbar}} \right] \exp \left( -\frac{m_0 \Omega_0}{\hbar} x^2 \right) \left( H_n \left[ \frac{m_0 \Omega_0}{\hbar} \right]^{1/2} x \right)^2, \tag{59} \]

is the same as the probability density of the eigenstate \( \chi_n(x,t) \) of time independent \( H_0^{PT} \) which is also equal to the probability density of the eigenstate \( \phi_n(x) \) \([38]\) of the standard harmonic oscillator \([37]\). Clearly, \( \phi_n(x) \) are elements from \( L^2(R) \), and therefore the condition \([59]\) yields that

\[ \int |\phi_n(x)|^2 dx = \left[ \frac{\sqrt{m_0 \Omega_0}}{n! 2^n \sqrt{\pi \hbar}} \right] \int \exp \left( -\frac{m_0 \Omega_0}{\hbar} x^2 \right) \left( H_n \left[ \frac{m_0 \Omega_0}{\hbar} \right]^{1/2} x \right)^2 dx = 1 \tag{60} \]

under this observation, we deduce that the probability is finite.

4 Conclusion

The essential ingredient of quantum mechanical non Hermitian theory is the construction of a positive-definite inner product, so that its probability is conserved in time. The operator \( C(t) = F^+(t)CF(t) \) confer to the norm its conservation. The main result of this paper is that the mean value of a time-dependent non-Hermitian Hamiltonian \( H(t) \) is real in the new \( C(t)\mathcal{PT} \)-inner product. For this, we introduced a unitary transformation \( F(t) \) that reduces the study of time-dependent non-Hermitian Hamiltonian \( H(t) \) to the study of time-independent \( \mathcal{PT} \)-symmetric Hamiltonian \( H_0^{PT} \), and derived the analytical solution of the Schrödinger equation of the initial system. Then, we defined a new \( C(t)\mathcal{PT} \)-inner product and showed that the evolution preserves it. Furthermore, we proved that the expectation value of the time-dependent non-Hermitian Hamiltonian \( H(t) \) is real in the \( C(t)\mathcal{PT} \) normed states since the transformation \( F(t) \) is unitary and \([\mathcal{P},F(t)] = 0\). As an illustration, we have investigated a class of quantum time-dependent mass oscillators with a complex linear driving force. The expectation value of the Hamiltonian, the uncertainty relation and probability density have been also calculated.

Acknowledgments One of the authors (M.M) would like to thank Dr. W. Koussa and Dr. N. Mana for their helpful contribution.

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