A bound for Hall’s criterion for nilpotence in semi-abelian categories
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Abstract. In this paper, we focus on Hall’s criterion for nilpotence in semi-abelian categories, and we improve the bound of the main theorem of [3, Theorem 3.4] (see Main Theorem). And this bound is best possible.

Key Words: Hall’s criterion for nilpotence; Semi-abelian.

1. Introduction

In [3], Gray has proved a wide generalization of P. Hall’s theorem about nilpotent groups: a group \( G \) is nilpotent if it has a normal subgroup \( N \) such that \( G/[N,N] \) and \( N \) is nilpotent. Gray’s generalization is in a semi-abelian category [8] which satisfies some properties[3, Section 3]. Moreover, Gray’s main theorem gives a bound of the nilpotency class about the similar objects in algebraically coherent semi-abelian category (see [3, Theorem 3.4]). In this note, we improve the bound as follows.

Main Theorem. Let \( C \) be an algebraically coherent semi-abelian category and let \( p : E \to B \) be an extension of a nilpotent object \( B \) in \( C \). If the kernel of \( p \) is contained in the Huq commutator \([N,N]_N\) of a nilpotent normal subobject \( N \) of \( E \), and if \( N \) is of nilpotency class \( c \) and \( B \) is of nilpotency class \( d \), then \( E \) is of nilpotency class at most \( cd + (c-1)(d-1) \).

Here, the definition of algebraically coherent semi-abelian category can be found in [3]. Examples of algebraically coherent semi-abelian categories include the categories of groups, rings, Lie algebra over a commutative ring, and others categories in [9]. And the bound in the categories of groups is found by [11, Theorem 1].

Structure of the paper: After recalling the basic definitions and properties of commutator semi-lattices in Section 2, and we introduce semi-abelian categories and commutators in Section 3. In Section 4, we prove Main Theorem.

2. Jacobi commutator semi-lattices

In this section we collect some known results about commutator semi-lattices. For the background theory of commutator semi-lattices, we refer to [3].

First, let us begin with semi-lattices.

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Definition 2.1. A semi-lattice is a triple \((X, \leq, \lor)\) where \((X, \leq)\) is a poset, and \(\lor\) is a binary operation on \(X\) satisfying:

(a) for each \(a \in X\), \(a \lor a = a\);
(b) the operation \(\lor\) is join-semilattice;
(c) for each \(a, b \in X\), \(a \lor b = b \lor a\);
(d) for each \(a, b, c \in X\), \((a \lor b) \lor c = a \lor (b \lor c)\).

Moreover, a semi-lattice \((X, \leq, \lor)\) is called join-semilattice if

\[ a \leq b \iff a \lor b = b \]

for each \(a, b \in X\).

Definition 2.2. A commutator semi-lattice is a triple \((X, \leq, \cdot)\) where \(X\) is a set, \(\leq\) is a binary relation on \(X\), and \(\cdot\) is a binary operation on \(X\) satisfying:

(a) \((X, \leq)\) is a join-semilattice;
(b) the operation \(\cdot\) is commutative;
(c) for each \(a, b \in X\), \(a \cdot b \leq b\);
(d) for each \(a, b, c \in X\), \(a \cdot (b \lor c) = (a \cdot b) \lor (a \cdot c)\).

Remark 2.3. Let \((X, \leq, \cdot)\) be a commutator semi-lattice, and \(x \in X\). Then the map \(x \cdot - : X \to X\) defined by \(y \mapsto x \cdot y\) is order preserving.

Proof. Let \(y \leq z \in X\), then \(y \lor z = z\). Since \((x \cdot y) \lor (x \cdot z) = x \cdot (y \lor z)\) by above definition (d), we have \((x \cdot y) \lor (x \cdot z) = x \cdot (y \lor z) = x \cdot z\). Hence, \(x \cdot y \leq x \cdot z\). □

Definition 2.4. A commutator semi-lattice \((X, \leq, \cdot)\) is a Jacobi commutator semi-lattice if

(a) for each \(a, b, c \in X\), \(a \cdot (b \lor c) \leq ((a \cdot b) \cdot c) \lor (b \cdot (a \cdot c))\);

Example 2.5. Let \(G\) be a group, and let \(X\) be the set of all normal subgroups of \(G\). For each \(M, N \in X\), we can define that \(N \cdot M = [M,N]\) and \(N \lor M = NM\). It is easy to see that \((X, \leq, \cdot)\) is a Jacobi commutator semi-lattice.

Definition 2.6. A derivation of a commutator semi-lattice \((X, \leq, \cdot)\) is a map \(f : X \to X\) which preserves joins and satisfies:

(a) for each \(a, b \in X\), \(f(a \cdot b) \leq (f(a) \cdot b) \lor (a \cdot f(b))\).

A derivation \(f\) of a commutator semi-lattice \((X, \leq, \cdot)\) is an inner derivation if there exists \(x \in X\) such that \(f = x \cdot -\), that is, for each \(a \in X\), \(f(a) = x \cdot a\).

Remark 2.7. Let \(f\) be a derivation of commutator semi-lattice \((X, \leq, \cdot)\). For each \(a, b \in X\) and \(a \leq b\), then \(f(a) \leq f(b)\).

Proof. Since \(a \leq b\), we have \(a \lor b = b\). Also \(f\) is a derivation, thus \(f\) preserves joins. Hence, \(f(b) = f(a \lor b) = f(a) \lor f(b)\). So \(f(a) \leq f(b)\). □

Proposition 2.8. Let \(g\) be the inner derivation of a Jacobi commutator semi-lattice \((X, \leq, \cdot)\), let \(x\) be an elements of \(X\), and let \(g\) be defined for each \(s \in X\) by \(g(s) = x \cdot s\). Then

\[ g^i(x) \cdot g^j(x) \leq g^{i+j+1}(x) \]

for each each non-negative integers \(i\) and \(j\).
Proof. The proof is by induction on $j$. For $j = 0$, we can see $g^i(x) \cdot g^0(x) = g^i(x) \cdot x = x \cdot g^i(x) = g^{i+1}(x)$.

We can see that
\[
\begin{align*}
g^i(x) \cdot g^j(x) &= (x \cdot (x, \ldots, (x \cdot x))) \cdot (x \cdot (x, \ldots, (x \cdot x))) \\
&\leq ((x, x, \ldots, (x \cdot x))) \cdot (x \cdot (x, \ldots, (x \cdot x))) \\
&\quad \vee x \cdot ((x, x, \ldots, (x \cdot x))) \cdot (x \cdot (x, \ldots, (x \cdot x))) \\
&\leq (x, x, \ldots, (x \cdot x)) \cdot (x, x, \ldots, (x \cdot x)) \\
&\quad \vee x \cdot g^{i+j}(x) \\
&= (x, x, \ldots, (x \cdot x)) \cdot (x, x, \ldots, (x \cdot x)) \vee g^{i+j+1}(x) \\
&= g^{i+1}(x) \cdot g^{j-1}(x) \vee g^{i+j+1}(x) \\
&\leq g^{i+j+1}(x) \vee g^{i+j+1}(x) \\
&= g^{i+j+1}(x).
\end{align*}
\]

Hence, we get the proof. \qed

**Proposition 2.9.** Let $f$ be a derivation of commutator semi-lattice $(X, \leq, \cdot)$. For each $a, b$ in $X$ and for each non-negative integer $n$, we have
\[
f^n(a \cdot b) \leq \bigvee_{i=0}^{n} f^i(a) \cdot f^{n-i}(b).
\]

Proof. The proof is by induction on $n$. For $n = 0$, it follows by $f^0(a \cdot b) = a \cdot b \leq a \cdot b = f^0(a) \cdot f^0(b)$. For $n = 1$, we can see that
\[
f(a \cdot b) \leq (f(a) \cdot b) \vee (a \cdot f(b))
\]
for each $a, b \in X$ by the definition of derivation.

Now, we can assume that the proposition hold for $n - 1$. That means
\[
f^{n-1}(a \cdot b) \leq \bigvee_{i=0}^{n-1} f^i(a) \cdot f^{n-1-i}(b).
\]

By Remark 2.5 and the definition of derivation, we have
\[
f^n(a \cdot b) = f(f^{n-1}(a \cdot b)) \leq f\left(\bigvee_{i=0}^{n-1} f^i(a) \cdot f^{n-1-i}(b)\right) \leq \bigvee_{i=0}^{n-1} f(f^i(a) \cdot f^{n-1-i}(b)).
\]
So, we can see that

\[ f^n(a \cdot b) \leq \bigvee_{i=0}^{n-1} f(f^i(a) \cdot f^{n-1-i}(b)) \]

\[ \leq \bigvee_{i=0}^{n-1} ((f^{i+1}(a) \cdot f^{n-1-i}(b)) \lor (f^i(a) \cdot f^{n-i}(b))) \]

\[ \leq (f^3(a) \cdot f^{n-1}(b)) \lor (f^2(a) \cdot f^{n-2}(b)) \]

\[ \lor (f^1(a) \cdot f^{n-3}(b)) \lor (f^0(a) \cdot f^{n-4}(b)) \]

\[ \vdots \]

\[ \leq \bigvee f^i(a) \lor (f^{n-1}(a) \cdot f(b)) \]

\[ = \bigvee_{i=0}^{n} f^i(a) \cdot f^{n-i}(b). \]

Hence, we get the proof. \( \square \)

**Proposition 2.10.** Let \( f \) be a derivation of a Jacobi commutator semi-lattice \((X, \leq, \cdot)\) bounded above by \( 1_X \), let \( x \) and \( y \) be an elements of \( X \), and let \( g \) be the inner derivation of \((X, \leq, \cdot)\) defined for each \( s \) in \( X \) by \( g(s) = x \cdot s \). If \( x \leq y \) and for some positive integer \( m \), \( f^m(y) \leq g(x) \), then for each positive integer \( k \),

\[ f^{tk}(y) \leq g^k(x) \]

where \( t_k = km + (k - 1)(m - 1) \).

**Proof.** The proof is by induction on \( k \).

**Step 1.** For \( k = 1 \), we can see \( t_1 = m \). So we can see the condition \( f^{tk}(y) \leq g^k(x) \) makes this case holds.

**Step 2.** If \( k > 1 \), then for \( r \leq k - 1 \), we can assume that the proposition hold when \( r \leq k - 1 \). That means that

\[ f^r(y) \leq g^r(x) \]

for each \( 1 \leq r \leq k - 1 \).

**Step 3.** For \( k \), we can see that \( f^{tk}(y) \leq f^{tk-m}(f^m(y)) \leq f^{tk-m}(g(x)) \) by Remark 2.7. And by Proposition 2.9, we have

\[ f^{tk}(y) \leq f^{tk-m}(f^m(y)) \leq f^{tk-m}(g(x)) = f^{tk-m}(x \cdot x) \]

\[ \leq \bigvee_{i=0}^{tk-m} f^i(x) \cdot f^{tk-m-i}(x). \]

Now, we will consider \( f^i(x) \cdot f^{tk-m-i}(x) \) for each \( i \). For each \( i \), there exists \( 1 \leq j \leq k \) such that

\[ 2(j - 1)m - m - (j - 1) + 1 \leq i < 2jm - m - j + 1. \]

**For** \( f^i(x) \), we can see that

\[ f^i(x) \leq f^i(y) \leq f^{2(j-1)m-m-(j-1)+1}(y). \]

Here, \( 2(j - 1)m - m - (j - 1) + 1 = t_{j-1} \). But \( j - 1 \leq k - 1 \), thus

\[ f^{2(j-1)m-m-(j-1)+1}(y) = f^{t_{j-1}}(y) \leq g^{j-1}(x). \]
For $f^{t_k-m-i}(x)$, we can see that
\[ f^{t_k-m-i}(x) \leq f^{t_k-m-i}(y). \]

But
\[
t_k - m - i = 2km - k - 2m + 1 - i - 2jm - j + 2jm + j = 2(k - j)m - (k - j) - m + 1 + jm - m - j - i = t_k - j + jm - m - j - i.
\]

Since $i < 2jm - m - j + 1$, we have $jm - m - j - i \geq 0$. Hence, we have
\[
f^{t_k-m-i}(x) \leq f^{t_k-m-i}(y) = f^{t_k-m-j-i}(y) \leq f^{t_k-i}(f^{jm-m-j-i}(y)) \leq f^{t_k-i}(y) \leq g^{k-j}(x).
\]

Hence, we have
\[
f^i(x) \cdot f^{t_k-m-i}(x) \leq g^{j-1}(x) \cdot g^{k-j}(x) \leq g^{k-j+j-1+1}(x) = g^k(x).
\]

Therefore, we can see
\[
f^k(y) \leq f^{t_k-m}(f^m(y)) \leq f^{t_k-m}(g(x)) = f^{t_k-m}(x \cdot x) \leq \bigvee_{i=0}^{t_k-m} f^i(x) \cdot f^{t_k-m-i}(x) \leq \bigvee_{i=0}^{t_k-m} g^{i-1}(x) \cdot g^{k-j}(x) \leq \bigvee_{i=0}^{t_k-m} g^k(x) = g^k(x)
\]

and we prove this proposition. \(\square\)

The above proof follows [11, Theorem 1].

3. Semi-abelian categories

In this section we collect some known results about semi-abelian categories. For the background theory of semi-abelian categories, we refer to [3].

We introduce the Huq commutator as follows, and we use the notations in [3].

**Definition 3.1.** Let $C$ be a semi-abelian category. Denote by $0$ a zero object in $(C)$, and denote by $0$ each zero morphism, that is, a morphism which factors through a zero object. For each $A,B \in \text{Ob}(C)$, we have a product $A \times B \in \text{Ob}(C)$. By the definition of product, we can write $(1,0) : A \to A \times B$ and $(0,1) : B \to A \times B$ for the unique morphisms with $\pi_1(1,0) := 1_A, \pi_2(1,0) := 0, \pi_1(0,1) := 0$ and $\pi_2(0,1) := 1_B$. A pair of morphisms $f : A \to C$ and $g : B \to C$ commute, if there is a morphism $\varphi : A \times B \to C$ making the diagram commute.

\[
\begin{array}{ccc}
A & \xrightarrow{(1,0)} & A \times B & \xrightarrow{(0,1)} & B \\
\downarrow f & & \downarrow \varphi & & \downarrow g \\
C & \xrightarrow{=} & C & \xrightarrow{=} & C
\end{array}
\]
More generally the Huq commutator of \( f : A \to C \) and \( g : B \to C \) is defined to be the smallest normal subobject \( N \) of \( C \) such that \( qf \) and \( qg \) commute, where \( q : C \to C/N \) is the cokernel of the associated normal monomorphism \( N \to C \).

**Theorem 3.2.** Let \( \mathcal{C} \) be a semi-abelian category. Let \( f : A \to C \) and \( g : B \to C \) be morphisms in \( \mathcal{C} \), then there exists the Huq commutator of \( f \) and \( g \).

**Proof.** See [1] or [3]. \( \square \)

We recall the definition of nilpotent for object in a semi-abelian category \( \mathcal{C} \) as follows.

**Definition 3.3.** For subobjects \( S \) and \( T \) of an object \( C \) in \( \mathcal{C} \), we will write \( [S, T]_C \) for the Huq commutator of the associated monomorphisms \( S \to C \) and \( T \to C \). Recall also that \( C \) is nilpotent if there exists a non-negative integer \( n \) such that \( \gamma^n_C(C) = 0 \), where \( \gamma_C \) is the map sending \( S \) in \( \text{Sub}(C) \) to \( [C, S]_C \) in \( \text{Sub}(C) \). The least such \( n \) is the nilpotency class of \( C \).

### 4. Proof of the main theorem

In this section, we give a proof of the main theorem.

**Theorem 4.1.** Let \( \mathcal{C} \) be an algebraically coherent semi-abelian category and let \( p : E \to B \) be an extension of a nilpotent object \( B \) in \( \mathcal{C} \). If the kernel of \( p \) is contained in the Huq commutator \( [N, N]_N \) of a nilpotent normal subobject \( N \) of \( E \), and if \( N \) is of nilpotency class \( c \) and \( B \) is of nilpotency class \( d \), then \( E \) is of nilpotency class at most \( cd + (c-1)(d-1) \).

**Proof.** Let \( X := \text{NSub}(E) \) (Here, \( \text{NSub}(E) \) means all normal subobjects of \( E \)) and let \( f : X \to X \) and \( g : X \to X \) be the maps defined by \( f(K) = [E, K]_E \) and \( g(K) = [N, K]_E \). Using the proof of [3, Theorem 3.4], we find \( f^c(E) \leq g^c(N) \). So by Proposition 2.10, we get

\[
f^{cd + (c-1)(d-1)}(E) \leq g^c(N).
\]

So, we get the proof because \( N \) is of nilpotency class \( c \). \( \square \)

**Example 4.2.** [11, Section 5, Example] For every pair \( c, d \) of positive integers there is a group \( G \) of class \( cd + (c-1)(d-1) \) which has a normal subgroup \( N \) of class \( c \) such that \( G/[N, N] \) is of class \( d \).

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