SINGULAR LIMIT OF ALLEN–CAHN EQUATION WITH CONSTRAINT AND ITS LAGRANGE MULTIPLIER

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Abstract. We consider the Allen–Cahn equation with a constraint. Our constraint is provided by the subdifferential of the indicator function on a closed interval, which is the multivalued function. In this paper we give the characterization of the Lagrange multiplier for our equation. Moreover, we consider the singular limit of our system and clarify the limit of the solution and the Lagrange multiplier for our problem.

1. Introduction. In this paper, for each \( \varepsilon \in (0, 1] \) we consider the following Allen–Cahn equation with a constraint:

\[
\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\varepsilon^2} I_{[-1,1]}(u) + \frac{1}{\varepsilon^2} u^\varepsilon \geq 0 \quad \text{in } Q := (0, T) \times \Omega,
\]

\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma := (0, T) \times \Gamma,
\]

\[
u = 0, \quad \nu = u_0^\varepsilon \quad \text{on } \Gamma,
\]

where \( 0 < T < +\infty \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \( (1 \leq N < +\infty) \) with a smooth boundary \( \Gamma := \partial \Omega \), \( \nu \) is an outward normal vector on \( \Gamma \) and \( u_0^\varepsilon \) is a given initial data. Also, \( \partial I_{[-1,1]}(\cdot) \) is the subdifferential of the indicator function \( I_{[-1,1]}(\cdot) \) on a closed interval \([-1, 1]\) defined by

\[
I_{[-1,1]}(z) := \begin{cases} 
0, & \text{if } z \in [-1, 1], \\
+\infty, & \text{otherwise.}
\end{cases}
\]

More precisely, \( \partial I_{[-1,1]}(\cdot) \) is a set-valued mapping defined by

\[
\partial I_{[-1,1]}(z) := \begin{cases} 
\emptyset, & \text{if } z < -1 \text{ or } z > 1, \\
[0, \infty), & \text{if } z = 1, \\
\{0\}, & \text{if } -1 < z < 1, \\
(-\infty, 0], & \text{if } z = -1.
\end{cases}
\]

The Allen–Cahn equation was proposed to describe the macroscopic motion of phase boundaries. In the physical context, the function \( u^\varepsilon = u^\varepsilon(t, x) \) in \( (P)^\varepsilon \) is the nonconserved order parameter that characterizes the physical structure: \( u^\varepsilon = 1, -1 < u^\varepsilon < 1 \) and \( u^\varepsilon = -1 \) corresponding to pure liquid, mixture and pure solid respectively.

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There are vast literatures of Allen-Cahn equation with or without a constraint. For such works, we refer to \([1, 4, 8, 9, 10, 12, 16, 19]\), for instance. In particular, Chen and Elliott [9] considered the asymptotic behavior of the solution to \((P)\) as \(\varepsilon \to 0\). However, there was no information of an element of \(\partial I_{[-1,1]}(u^\varepsilon)\) in [9].

In this paper, for each \(\varepsilon \in (0, 1]\) we consider an element \(\lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)\), which is called the Lagrange multiplier for \((P)^\varepsilon\) \(\varepsilon \to 0\). Also, we investigate the limiting observation of \(\lambda^\varepsilon\) as \(\varepsilon \to 0\). Namely, we consider the singular limit of our system \((P)^\varepsilon\) and clarify the limit of the solution \(u^\varepsilon\) and the Lagrange multiplier \(\lambda^\varepsilon\) for \((P)^\varepsilon\) as \(\varepsilon \to 0\).

Recently, elliptic and parabolic variational inequalities were considered in connection with Lagrange multipliers (cf. [4, 5, 11, 14]). Note from the constraint that the notion of solution to \((P)^\varepsilon\) is given in variational sense (cf. Remark 2.1 below). Therefore, it is worthy considering the Lagrange multiplier for \((P)^\varepsilon\). Also, we are very interested in the limit of the Lagrange multiplier for \((P)^\varepsilon\) as \(\varepsilon \to 0\).

This present paper aims to consider the Lagrange multiplier \(\lambda^\varepsilon\) and the singular limit of \((P)^\varepsilon\) as \(\varepsilon \to 0\). The main novelties found in this paper are the following:

(i) We give the characterization of the Lagrange multiplier \(\lambda^\varepsilon\) for \((P)^\varepsilon\).
(ii) We show the convergence of the solution \(u^\varepsilon\) and the Lagrange multiplier \(\lambda^\varepsilon\) for \((P)^\varepsilon\) as \(\varepsilon \to 0\).
(iii) We clarify the properties of the limit of \(u^\varepsilon\) and \(\lambda^\varepsilon\) as \(\varepsilon \to 0\).

The plan of this paper is as follows. In Section 2, we state the main results in this paper. In Section 3 we recall the decomposition result of the subdifferential of convex functions. Also, we prove the main result (Theorem 2.1) concerning the existence-uniqueness of solutions to \((P)^\varepsilon\) and properties of the Lagrange multiplier \(\lambda^\varepsilon\) corresponding to the item (i) listed above. In Section 4, we prove Theorem 2.2 corresponding to the items (ii) and (iii) listed above.

**Notations and basic assumptions.** Throughout this paper, for any reflexive Banach space \(B\), we denote \(\| \cdot \|_B\) the norm of \(B\), and denote by \(B^*\) the dual space of \(B\).

In particular, we put \(H := L^2(\Omega)\) with usual real Hilbert space structure, and denote by \((\cdot, \cdot)_H\) the inner product in \(H\). Also, we put \(V := H^1(\Omega)\) with the usual norm

\[\|z\|_V := \left\{ \|z\|^2_H + \|\nabla z\|^2_H \right\}^{1/2}, \quad z \in V,\]

and denote by \(\langle \cdot, \cdot \rangle\) the duality pairing between \(V^*\) and \(V\). By identifying \(H\) with its dual space, we have \(V \subset H \subset V^*\) with compact and dense embeddings; then,

\[\langle u, v \rangle = (u, v)_H \quad \text{for } u \in H \text{ and } v \in V. \tag{1.6}\]

In the proof of Theorem 2.1, we use some techniques of proper (that is, not identically equal to infinity), l.s.c. (lower semi-continuous), convex functions and their subdifferentials, which are useful in the systematic study of variational inequalities. Therefore, let us outline some notations and definitions. For a proper, l.s.c. and convex function \(\psi : H \to \mathbb{R} \cup \{+\infty\}\), the effective domain \(D(\psi)\) is defined by

\[D(\psi) := \{z \in H; \ \psi(z) < \infty\}.\]

The subdifferential of \(\psi\) is a possibly multi-valued operator in \(H\) and is defined by \(z^* \in \partial \psi(z)\) if and only if

\[z \in D(\psi) \quad \text{and} \quad (z^*, y - z)_H \leq \psi(y) - \psi(z) \quad \text{for all } y \in H.\]

For various properties and related notions of the proper, l.s.c., convex function \(\psi\) and its subdifferential \(\partial \psi\), we refer to a monograph by Brézis [6].

Next, let us give an assumption on initial data. Throughout this paper, we assume the following condition (\(A\)):

(A) \(u_0^\varepsilon \in K := \{z \in V; |z| \leq 1 \text{ a.e. in } \Omega\} \) for all \(\varepsilon \in (0, 1]\).
Finally, throughout this paper, $C_i = C_i(\cdot), i = 1, 2, 3, \cdots$, denotes positive (or nonnegative) constants depending only on its arguments.

2. Main results. We begin by giving the rigorous definition of solutions to our problem $(P)^\varepsilon (\varepsilon \in (0, 1])$.

**Definition 2.1.** For each $\varepsilon \in (0, 1]$, a function $u^\varepsilon : [0, T] \to H$ is called a solution to $(P)^\varepsilon$ on $[0, T]$, if the following conditions are satisfied:

(i) $u^\varepsilon \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V)$.

(ii) There is a function $\lambda^\varepsilon \in L^2(0, T; H)$ with $\lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)$ a.e. in $Q$ such that

$$
(u^\varepsilon(t), z)_H + \int_\Omega \nabla u^\varepsilon(t) \cdot \nabla z \, dx + \frac{1}{\varepsilon^2} (\lambda^\varepsilon(t), z)_H = \frac{1}{\varepsilon^2} (u^\varepsilon(t), z)_H
$$

for all $z \in V$ and a.e. $t \in (0, T)$.

(iii) $u^\varepsilon(0) = u^\varepsilon_0$ in $H$.

We call $\lambda^\varepsilon$ in (ii) a Lagrange multiplier for $(P)^\varepsilon$ on $[0, T]$.

**Remark 2.1.** It follows from the constraint $\partial I_{[-1,1]}(\cdot)$ and (ii) of Definition 2.1 that the equation (1.1) is equivalent to the following variational inequality:

$$
\left( u^\varepsilon_t(t) - \frac{1}{\varepsilon^2} u^\varepsilon(t), u^\varepsilon(t) - z \right)_H + \int_\Omega \nabla u^\varepsilon(t) \cdot (\nabla u^\varepsilon(t) - \nabla z) \, dx \leq 0
$$

for all $z \in K$ and a.e. $t \in (0, T)$.

Now, let us mention the first main result in this paper, which is concerned with the existence and basic property of the solution and the Lagrange multiplier for $(P)^\varepsilon$ on $[0, T]$.

**Theorem 2.1.** Assume (A). Then, for each $\varepsilon \in (0, 1]$, there exist a unique solution $u^\varepsilon$ to $(P)^\varepsilon$ on $[0, T]$ and a unique Lagrange multiplier $\lambda^\varepsilon$ in the sense of Definition 2.1 such that

$$
\lambda^\varepsilon(t, x) = \begin{cases} 
\geq 0 & \text{on } \{(t, x) \in Q ; \ u^\varepsilon(t, x) = 1 \}, \\
= 0 & \text{on } \{(t, x) \in Q ; \ -1 < u^\varepsilon(t, x) < 1 \}, \\
\leq 0 & \text{on } \{(t, x) \in Q ; \ u^\varepsilon(t, x) = -1 \}.
\end{cases}
$$

In next Section 3, we give the proof of Theorem 2.1.

Next, we consider the limiting situation of $(P)^\varepsilon$ as $\varepsilon \to 0$. To this end, we use the following energy functional:

$$
\mathcal{F}^\varepsilon(u) := \int_\Omega \left\{ \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} I_{[-1,1]}(u) + \frac{1 - u^2}{2\varepsilon} \right\} \, dx, \ u \in V.
$$

Now we state the second main result in this paper, which is concerned with the singular limit of $(P)^\varepsilon$ as $\varepsilon \to 0$:

**Theorem 2.2.** Assume (A). For each $\varepsilon \in (0, 1]$, let $u^\varepsilon$ be the unique solution to $(P)^\varepsilon$ on $[0, T]$. Also, let $\lambda^\varepsilon$ be the unique Lagrange multiplier for $(P)^\varepsilon$ on $[0, T]$ in the sense of Definition 2.1. Assume that there are the function $u_0 \in L^1(\Omega)$ and a positive constant $M$, independent of $\varepsilon \in [0, 1]$, satisfying $u_0(x) = 1$ or $-1$ for a.e. $x \in \Omega$,

$$
\sup_{\varepsilon \in [0,1]} \mathcal{F}^\varepsilon(u_0^\varepsilon) < M
$$

and

$$
\lim_{\varepsilon \to 0} \int_\Omega |u_0^\varepsilon(x) - u_0(x)| \, dx = 0.
$$
Then, there are a subsequence \( \{ \varepsilon_k \} \) of \( \{ \varepsilon \} \) with \( \varepsilon_k \searrow 0 \) as \( k \to \infty \), two functions \( u \in L^2(0,T;H) \) and \( \lambda^* \in L^2(0,T;V^*) \) and a positive constant \( N_0 \), independent of \( \varepsilon \in (0,1] \), such that \( u(t,x) \) takes only the values 1 or \(-1\) for a.e. \((t,x) \in Q = (0,T) \times \Omega\),

\[
\lim_{k \to \infty} u^{\varepsilon_k}(t,x) = u(t,x), \quad \text{a.e. } (t,x) \in Q, 
\]

\[
\int_\Omega |u(t_1,x) - u(t_2,x)| \, dx \leq N_0 |t_1 - t_2|^{\frac{1}{2}}, \quad \forall t_1, t_2 \in [0,T],
\]

\[
\lim_{t \to 0} u(t,x) = u_0(x), \quad \text{a.e. } x \in \Omega, 
\]

\[
\int_\Omega |\nabla u(t)| \leq N_0, \quad \text{a.e. } t \in (0,T)
\]

and

\[
\lambda^{\varepsilon_k} \rightharpoonup \lambda^* \text{ weakly in } L^2(0,T;V^*) \text{ as } k \to \infty,
\]

where \( \int_\Omega |\nabla u(t)| \) is the total variation measure of \( u(t) \). Moreover, \( \lambda^* - u = 0 \) in \( L^2(0,T;V^*) \), hence,

\[
\lambda^* = 1 \text{ on } \{(t,x) \in Q ; \, u(t,x) = 1\},
\]

\[
\lambda^* = -1 \text{ on } \{(t,x) \in Q ; \, u(t,x) = -1\}.
\]

In Section 4 we prove Theorem 2.2 by using a priori estimates of \( u^\varepsilon \) and \( \lambda^\varepsilon \).

3. Solvability of \((P)^\varepsilon\). In this section we consider \((P)^\varepsilon\) for each \( \varepsilon \in (0,1] \). In fact, we study \((P)^\varepsilon\) by arguments similar to [15, 18], namely by the theory of abstract evolution equations governed by subdifferentials.

Now, we define a functional \( \varphi_0 \) on \( H \) by

\[
\varphi_0(z) := \begin{cases} 
\frac{1}{2} \int_\Omega |\nabla z|^2 \, dx, & \text{if } z \in V, \\
\infty, & \text{otherwise}. 
\end{cases}
\]

Clearly, \( \varphi_0 \) is proper, l.s.c. and convex on \( H \) with \( \partial \varphi_0(z) = -\Delta_N z \) in \( H \), where \( \Delta_N : D(\Delta_N) := \{ z \in H^2(\Omega); \frac{\partial z}{\partial n} = 0 \text{ a.e. on } \Gamma \} \to H \) is the Laplacian with homogeneous Neumann boundary condition (cf. [3, Proposition 2.9]).

Also, we define the proper, l.s.c. and convex functional \( I_{[-1,1]} \) of \( H \) by

\[
I_{[-1,1]}(z) := \int_\Omega I_{[-1,1]}(z) \, dx \quad \text{for any } z \in H,
\]

where \( I_{[-1,1]} \) is the indicator function defined in (1.4).

Next, we consider the functional \( \varphi \) defined by the form:

\[
\varphi(z) = \varphi_0(z) + \frac{1}{\varepsilon^2} I_{[-1,1]}(z) \quad \text{for any } z \in H.
\]

Clearly, \( \varphi \) is proper, l.s.c. and convex on \( H \) with the effective domain \( D(\varphi) = K \), where \( K \) is the set defined in (A).

Here, we recall the following decomposition result of the subdifferential \( \partial \varphi \).

**Proposition 3.1** (cf. [7, Section 3], [18, Theorem 3.1]). The subdifferential \( \partial \varphi \) of \( \varphi \) is decomposed into the following form:

\[
\partial \varphi(z) = \partial \varphi_0(z) + \frac{1}{\varepsilon^2} \partial I_{[-1,1]}(z) \quad \text{in } H \quad \text{for any } z \in H.
\]

Now, we prove Theorem 2.1 by using Proposition 3.1 and applying the abstract theory of nonlinear evolution equations associated with subdifferential \( \partial \varphi \).
Proof of Theorem 2.1. By the similar arguments as in [15, Section 1], we can show the existence-uniqueness of a solution \( u^\varepsilon \) to (P)\(^\varepsilon \) on \([0,T]\) for each \( \varepsilon \in (0,1) \). In fact, we easily prove the uniqueness of solutions to (P)\(^\varepsilon \) on \([0,T]\) by the quite standard arguments: monotonicity and Gronwall’s inequality.

Now, we show the precise proof of the existence of solutions to (P)\(^\varepsilon \). We easily see that the problem (P)\(^\varepsilon \) can be rewritten in an abstract framework of the form:

\[
(\text{CP})^\varepsilon \begin{cases}
\frac{d}{dt}u^\varepsilon(t) + \partial \varphi(u^\varepsilon(t)) - \frac{1}{\varepsilon^2}u^\varepsilon(t) - \frac{1}{\varepsilon}u^\varepsilon(t) \geq 0 \quad \text{in } H, \quad \text{for } t > 0, \\
u^\varepsilon(0) = u^\varepsilon_0 \quad \text{in } H.
\end{cases}
\]

(3.2)

Therefore, applying the Lipschitz perturbation theory of abstract evolution equations (cf. [7, 13, 17]), we can show the existence of a solution \( u^\varepsilon \) to (P)\(^\varepsilon \) on \([0,T]\) for each \( \varepsilon \in (0,1) \) in the variational sense (cf. Remark 2.1).

Also, note from Proposition 3.1 that (CP)\(^\varepsilon \) is equivalent to the following:

\[
\widetilde{(\text{CP})}^\varepsilon \begin{cases}
\frac{d}{dt}v^\varepsilon(t) + \partial \varphi_0(v^\varepsilon(t)) + \frac{1}{\varepsilon^2}\partial I_{[-1,1]}(v^\varepsilon(t)) - \frac{1}{\varepsilon^2}v^\varepsilon(t) \geq 0 \quad \text{in } H, \quad \text{for } t > 0, \\
v^\varepsilon(0) = u^\varepsilon_0 \quad \text{in } H.
\end{cases}
\]

Namely, there are functions \( v^\varepsilon \in L^2(0,T;H) \) and \( \lambda^\varepsilon \in L^2(0,T;H) \) such that \( v^\varepsilon(t) \in \partial \varphi_0(u^\varepsilon(t))\) a.e. in \((0,T)\), \( \lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)\) a.e. in \(Q\) and (3.3) holds in the following sense:

\[
\frac{d}{dt}u^\varepsilon(t) + v^\varepsilon(t) + \frac{1}{\varepsilon^2}\lambda^\varepsilon(t) - \frac{1}{\varepsilon^2}u^\varepsilon(t) = 0 \quad \text{in } H, \quad \text{for } t > 0.
\]

(3.4)

Since \( \partial \varphi_0(z) = -\Delta_N z \) is the single-valued linear operator in \(H\), we easily see that \( u^\varepsilon \) is a unique solution to (P)\(^\varepsilon \) on \([0,T]\) and \( \lambda^\varepsilon \) is the Lagrange multiplier for (P)\(^\varepsilon \) on \([0,T]\) in the sense of Definition 2.1. Also, by (3.4), the uniqueness of solution \( u^\varepsilon \) to (P)\(^\varepsilon \) and the linearity of \( \partial \varphi_0(z) = -\Delta_N z \), we easily show the uniqueness of the Lagrange multiplier \( \lambda^\varepsilon \) for (P)\(^\varepsilon \).

Taking into account of the definition (1.5) of \( \partial I_{[-1,1]}(\cdot) \), we conclude from \( \lambda^\varepsilon \in \partial I_{[-1,1]}(u^\varepsilon)\) a.e. in \(Q\) that the signature result (2.1) of the Lagrange multiplier \( \lambda^\varepsilon \) holds. Thus, the proof of Theorem 2.1 has been completed.

4. Singular limit of (P)\(^\varepsilon \) as \( \varepsilon \to 0 \). In this section we consider the singular limit of (P)\(^\varepsilon \) as \( \varepsilon \to 0 \). Then, we clarify the limit of the solution \( u^\varepsilon \) and the Lagrange multiplier \( \lambda^\varepsilon \) for (P)\(^\varepsilon \) on \([0,T]\).

We begin by giving the uniform estimate of \( u^\varepsilon \) and \( \lambda^\varepsilon \) with respect to \( \varepsilon \in (0,1] \).

Lemma 4.1. Suppose all the same conditions in Theorem 2.2. For each \( \varepsilon \in (0,1] \), let \( u^\varepsilon \) be the unique solution to (P)\(^\varepsilon \) on \([0,T]\). Also, let \( \lambda^\varepsilon \) be the unique Lagrange multiplier for (P)\(^\varepsilon \) on \([0,T]\) in the sense of Definition 2.1. Assume that there is a positive constant \( M \), independent of \( \varepsilon \in (0,1] \), satisfying

\[
\sup_{\varepsilon \in [0,1]} \mathcal{F}^\varepsilon(u^\varepsilon_0) < M.
\]

Then, there is a positive constant \( N_1 > 0 \), dependent on \( M \) and independent of \( \varepsilon \in (0,1] \), such that

\[
\varepsilon \int_0^T |u^\varepsilon_1(\tau)|_H^2 d\tau + \sup_{\tau \in [0,T]} \mathcal{F}^\varepsilon(u^\varepsilon(\tau)) + \int_0^T |\lambda^\varepsilon(\tau)|_V^2 d\tau \leq N_1.
\]

(4.1)

Proof. Multiplying (1.1) by \( \varepsilon u^\varepsilon_1(\tau) \), we get

\[
\varepsilon |u^\varepsilon_1(\tau)|_H^2 + \frac{d}{d\tau} \mathcal{F}^\varepsilon(u^\varepsilon(\tau)) = 0 \quad \text{for a.e. } \tau > 0.
\]

(4.2)
where $\mathcal{F}^\varepsilon(\cdot)$ is the functional defined in (2.2). By integrating (4.2) in $\tau$ over $[0,t]$ ($\subset [0,T]$), we get
\[
\varepsilon \int_0^t |u^\varepsilon_\tau(\tau)|^2 d\tau + \mathcal{F}^\varepsilon(u^\varepsilon(t)) = \mathcal{F}^\varepsilon(u^\varepsilon_0) < M \quad \text{for all } t \in [0,T].
\] (4.3)

Also, taking into account of the constraint $\partial I_{[1-1,1]}(\cdot)$ (cf. (1.5)), we easily see that
\[
|u^\varepsilon| \leq 1, \text{ a.e. in } Q.
\] (4.4)

By (1.6), (4.4) and (ii) of Definition 2.1, we see from the H"older inequality that:
\[
\left| \oint_0^T \langle \lambda^\varepsilon(t), z(t) \rangle dt \right| = \oint_0^T (\lambda^\varepsilon(t), z(t))_H dt 
\leq \oint_0^T |(\sqrt{\varepsilon} u^\varepsilon(t), z(t))_H| dt + \oint_0^T |((u^\varepsilon(t), z(t))_H| dt 
\leq (\varepsilon^2 |u^\varepsilon_\tau|_{L^2(0,T;H)} + \varepsilon^2 \sqrt{T} \sup_{t \in [0,T]} |\sqrt{\varepsilon} u^\varepsilon(t)|_H + \sqrt{T|\Omega|}) |z|_{L^2(0,T;V)}
\] (4.5)
for any $z \in L^2(0,T;V)$, where $|\Omega|$ denotes the volume of $\Omega$. Therefore, from $\varepsilon \in (0,1]$, (2.2), (4.3) and (4.5), we infer that:
\[
|\lambda^\varepsilon|_{L^2(0,T;V')} \leq \sqrt{M} + \sqrt{2MT} + \sqrt{T|\Omega|} \quad \text{for all } \varepsilon \in (0,1].
\] (4.6)

From (4.3) and (4.6), we infer that the uniform estimate (4.1) holds for some positive constant $N_1$. Thus, the proof of Lemma 4.1 has been completed.

\textbf{Corollary 4.1.} Suppose all the same conditions in Lemma 4.1. For each $\varepsilon \in (0,1]$, let $u^\varepsilon$ be the unique solution to (P)$^\varepsilon$ on $[0,T]$. Also, let $N_1 > 0$ be the positive constant obtained in Lemma 4.1. Put
\[
h(s) := \frac{1}{\sqrt{2}} \int_{-1}^s \sqrt{1 - \sigma^2} d\sigma \quad \text{for } s \in [-1,1].
\] (4.7)

Then, the following estimates hold:
\[
\sup_{t \in [0,T]} \int_{\Omega} |\nabla h(u^\varepsilon(t,x))| dx \leq N_1
\] (4.8)
and
\[
\int_{t_1}^{t_2} \int_{\Omega} |h(u^\varepsilon(t,x))_i| dx dt \leq N_1(t_2 - t_1)^{\frac{1}{2}}
\] (4.9)
for all $t_1, t_2$ with $0 \leq t_1 < t_2 \leq T$.

\textbf{Proof.} First, note from (4.7) that $h'(s) = \sqrt{(1 - s^2)/2}$ for $s \in [-1,1]$.

Now, we show the estimate (4.8). By (4.4) and the Schwarz inequality, we have:
\[
\int_{\Omega} |\nabla h(u^\varepsilon(t,x))| dx = \int_{\Omega} |h'(u^\varepsilon(t,x))||\nabla u^\varepsilon(t,x)| dx 
\leq \frac{1}{2\varepsilon} \int_{\Omega} \frac{1 - (u^\varepsilon(t,x))^2}{2} dx + \frac{\varepsilon}{2} \int_{\Omega} |\nabla u^\varepsilon(t,x)|^2 dx 
\leq \mathcal{F}^\varepsilon(u^\varepsilon(t)) 
\leq N_1 \quad \text{for all } t \in [0,T].
\]
Thus, (4.8) holds.
Next, we show (4.9). By (4.4) and the Hölder inequality, we have:
\[
\int_{t_1}^{t_2} \int_{\Omega} |(h(u^\varepsilon(t,x)))_t| dx dt = \int_{t_1}^{t_2} \int_{\Omega} |h'(u^\varepsilon(t,x))||u^\varepsilon_t(t,x)| dx dt \\
\leq \left( \int_{t_1}^{t_2} \int_{\Omega} \frac{1 - (u^\varepsilon(t,x))^2}{2} dx dt \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} |u^\varepsilon_t(t)|^2 H_n dt \right)^{\frac{1}{2}} \\
\leq (t_2 - t_1)^{\frac{1}{2}} \{ \varepsilon \sup_{t \in [t_1,t_2]} F^\varepsilon(u^\varepsilon(t)) \}^{\frac{1}{2}} \frac{1}{\sqrt{\varepsilon}} N_1^{\frac{1}{2}}
\]
for all \( t_1, t_2 \) with \( 0 \leq t_1 < t_2 \leq T \).
Thus, the proof of Corollary 4.1 has been completed.

Now, we prove the main Theorem 2.2, which is concerned with the singular limit of \((P)^\varepsilon\) as \( \varepsilon \to 0 \).

Proof of Theorem 2.2. At first we show the existence of a subsequence \( \{ \varepsilon_k \} \) of \( \{ \varepsilon \} \) and the function \( u \in L^2(0,T;H) \) such that \( u(t,x) \) takes only the values 1 or \(-1\) for a.e. \((t,x) \in (0,T) \times \Omega \) and (2.5) holds.

By the definition of \( h \) (cf. (4.7)), we easily see that the function \( h \) is bounded on \([-1,1] \):
\[
|h(s)| \leq C_h \quad \text{for all } s \in [-1,1] \quad (4.10)
\]
for some positive constant \( C_h > 0 \). Therefore we infer from (4.4) and (4.10) that:
\[
\int_0^T \int_{\Omega} |h(u^\varepsilon(t,x))| dx dt \leq T |\Omega| C_h. \quad (4.11)
\]

Taking into account of (4.8), (4.9) and (4.11), we see that \( \{ h(u^\varepsilon) \} \) is bounded in \( BV((0,T) \times \Omega) \) uniformly in \( \varepsilon \) \( \in \) \( (0,1] \), where \( BV((0,T) \times \Omega) \) is the space of all bounded variation functions on \((0,T) \times \Omega \). Since \( BV((0,T) \times \Omega) \) is compactly embedded into \( L^1((0,T) \times \Omega) \) (cf. [2, Corollary 3.49]), there are subsequence \( \{ \varepsilon_k \} \subset \{ \varepsilon \} \) and the function \( h^* \in BV((0,T) \times \Omega) \) such that \( \varepsilon_k \to 0 \) and
\[
h(u^{\varepsilon_k}) \to h^* \quad \text{in } L^1((0,T) \times \Omega) \quad \text{as } k \to \infty. \quad (4.12)
\]
Therefore, taking a subsequence if necessary, we see that:
\[
h(u^{\varepsilon_k}(t,x)) \to h^*(t,x), \quad \text{a.e. } (t,x) \in (0,T) \times \Omega \quad \text{as } k \to \infty. \quad (4.13)
\]
Since \( h \) is continuous and strictly increasing on \([-1,1] \) (cf. (4.7)), we can find a unique function \( u(t,x) \) such that
\[
h^*(t,x) = h(u(t,x)), \quad \text{a.e. } (t,x) \in (0,T) \times \Omega \quad (4.14)
\]
and
\[
u^{\varepsilon_k}(t,x) \to u(t,x), \quad \text{a.e. } (t,x) \in (0,T) \times \Omega \quad \text{as } k \to \infty, \quad (4.15)
\]
hence (2.5) holds. Clearly, it follows from (4.4) and (4.15) that:
\[
|u| \leq 1, \quad \text{a.e. in } Q. \quad (4.16)
\]

From (4.1), (4.4), (4.15), (4.16) and Lebesgue’s dominated convergence theorem, we infer that:
\[
0 \leq \int_0^T \int_{\Omega} (1 - (u(t,x))^2) dx dt = \lim_{k \to \infty} \int_0^T \int_{\Omega} (1 - (u^{\varepsilon_k}(t,x))^2) dx dt \\
\leq 2 N_1 T \lim_{k \to \infty} \varepsilon_k = 0, \quad (4.17)
\]
which implies that the limit function \( u \) of \( u^{\varepsilon_k} \) takes only the values 1 or \(-1\) for a.e. \((t,x) \in (0,T) \times \Omega \).
Thus, by taking the limit
\[ L \]
under
\[ k \]
Taking the limit in (4.24) as
\[ \lim_{k \to \infty} \int_{\Omega} |h(u^k(t_1, x)) - h(u^k(t_2, x))| \, dx \leq \int_{\Omega} \int_{t_1}^{t_2} |(h(u^k(t, x)))_t| \, dt \, dx \]
\[ \leq N_1(t_2 - t_1)^{\frac{1}{2}} \] (4.18)
for all \( t_1, t_2 \) with \( 0 \leq t_1 < t_2 \leq T \).
Taking the limit in (4.18) as \( k \to \infty \), we infer from (4.4), (4.10), (4.12)–(4.14) and Lebesgue’s dominated convergence theorem that:
\[ \int_{\Omega} |h(u(t_1, x)) - h(u(t_2, x))| \, dx \leq N_1(t_2 - t_1)^{\frac{1}{2}} \] (4.19)
for a.e. \( t_1, t_2 \) with \( 0 \leq t_1 < t_2 \leq T \).

Since the limit function \( u \) of \( u^k \) takes only the values 1 or -1 for a.e. on \((0, T) \times \Omega \), we easily see that:
\[ |h(u(t_1, x)) - h(u(t_2, x))| = \frac{C_3}{2} |u(t_1, x) - u(t_2, x)| \]
with \( C_3 = h(1) - h(-1) \),
a.e. \( x \in \Omega \) and a.e. \( t_1, t_2 \in (0, T) \).

Therefore, we observe from (4.19) and (4.20) that:
\[ \int_{\Omega} |u(t_1, x) - u(t_2, x)| \, dx \leq \frac{2N_1}{C_3} (t_2 - t_1)^{\frac{1}{2}} \] (4.21)
for a.e. \( t_1, t_2 \) with \( 0 \leq t_1 < t_2 \leq T \).

By (4.21), we can redefine the function \( u \) in order that the function \( t \in [0, T] \to u(t) \in L^1(\Omega) \) is continuous. Therefore, (4.21) hold for all \( t_1, t_2 \in [0, T] \) with \( t_2 > t_1 \), thus, (2.6) holds by putting \( N_0 := (2N_1)/C_3 \).

Next, we show (2.7). At first, we note from (2.4) that
\[ u_0^k(x) \to u_0(x), \text{ a.e. } x \in \Omega \text{ as } k \to \infty \] (4.22)
by taking a subsequence if necessary. Therefore, by (2.3), (4.10), (4.22) and the continuity of \( h \), we easily see from Lebesgue’s dominated convergence theorem that:
\[ \lim_{k \to \infty} \int_{\Omega} |h(u_0^k(x)) - h(u_0(x))| \, dx = 0. \] (4.23)

Now, putting \( t_1 = 0 \) in (4.18), we have:
\[ \int_{\Omega} |h(u_0^k(x)) - h(u^k(t_2, x))| \, dx \leq N_1 t_2^{\frac{1}{2}} \text{ for all } t_2 \in (0, T]. \] (4.24)
Taking the limit in (4.24) as \( k \to \infty \), we see from (4.12)–(4.15), (4.23) and (4.24) that:
\[ \int_{\Omega} |h(u_0(x)) - h(u(t_2, x))| \, dx \leq N_1 t_2^{\frac{1}{2}}, \text{ a.e. } t_2 \in (0, T]. \] (4.25)

Since \( u_0(x) \) takes only the values 1 or -1 for a.e. \( x \in \Omega \) and the function \( t \in [0, T] \to u(t) \in L^1(\Omega) \) is continuous, we easily see from (4.20) and (4.25) that:
\[ \int_{\Omega} |u_0(x) - u(t_2, x)| \, dx \leq \frac{2N_1}{C_3} t_2^{\frac{1}{2}} \text{ for all } t_2 \in (0, T]. \] (4.26)
Thus, by taking the limit \( t_2 \to 0 \) in (4.26), we observe that (2.7) holds.

Next, we show (2.8). By (4.8), (4.12) and the lower semicontinuity of the total variation under \( L^1 \)-convergence (cf. [2, Proposition 3.6]), we observe that
\[ \int_{\Omega} |\nabla u^*(t)| \leq N_1, \text{ a.e. } t \in (0, T], \] (4.27)
where $\int_{\Omega} |\nabla h^*(t)|$ is the total variation measure of $h^*(t)$. Since $u(t, x)$ takes only the values 1 or $-1$ for a.e. $(t, x) \in (0, T) \times \Omega$, we infer from (4.14) and (4.20) that

$$\int_{\Omega} |\nabla u(t)| \leq \frac{2N_1}{C_3}, \quad \text{a.e. } t \in (0, T).$$

(4.28)

Hence, (2.8) holds.

Finally, we show (2.9)–(2.11). By the uniform estimate (4.1), we easily see that there is a subsequence of $\{\varepsilon_k\}$ (which we denote $\varepsilon_k$ for simplicity) and a function $\lambda^* \in L^2(0, T; V^*)$ satisfying (2.9).

From (4.4), (4.15) and Lebesgue’s dominated convergence theorem, we infer that:

$$u^{\varepsilon_k} \rightharpoonup u \quad \text{in } L^2(0, T; H) \quad \text{as } k \to \infty.$$

(4.29)

By (1.6), (4.1) and (ii) of Definition 2.1, we have that:

$$\int_0^T \langle \lambda^*(t) - u(t), z(t) \rangle dt = \lim_{k \to \infty} \int_0^T \langle \lambda^{\varepsilon_k}(t) - u^{\varepsilon_k}(t), z(t) \rangle dt$$

$$\leq \int_0^T \left| (\varepsilon_k^2 u^{\varepsilon_k}(t), z(t))_H \right| dt + \int_0^T \left| \varepsilon_k^2 (\nabla u^{\varepsilon_k}(t), \nabla z(t))_H \right| dt$$

$$\leq \varepsilon_k^2 \sqrt{N_1} |z|_{L^2(0, T; H)} + \varepsilon_k^2 \sqrt{2TN_1} |z|_{L^2(0, T; V)}$$

for any $z \in L^2(0, T; V)$.

From (2.9), (4.29) and the inequality as above, we see that

$$\int_0^T \langle \lambda^*(t) - u(t), z(t) \rangle dt = \lim_{k \to \infty} \int_0^T \langle \lambda^{\varepsilon_k}(t) - u^{\varepsilon_k}(t), z(t) \rangle dt \leq 0.$$  

(4.30)

Since $z \in L^2(0, T; V)$ is arbitrary, we infer from (4.30) that

$$\lambda^* - u = 0 \quad \text{in } L^2(0, T; H).$$

(4.31)

Since the function $u \in L^2(0, T; H)$ takes only the values 1 or $-1$ for a.e. in $(0, T) \times \Omega$, we easily conclude from (4.31) that (2.10) and (2.11) hold. Thus, the proof of Theorem 2.2 has been completed.

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