ON THE WELL-POSEDNESS OF A GENERALIZED MOMENT PROBLEM AND ITS NUMERICAL SOLUTION

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Abstract. We show that the unique solution to a parametric version of the generalized moment problem depends continuously on the prior function, and thus the problem is well-posed in the sense of Hadamard. Based on this result, the problem is reparametrized via a diffeomorphic change of variables out of a numerical consideration, and then a continuation method is used to get the solution. Numerical aspects such as convergence of the proposed algorithm and certain computational procedures are addressed.

Key words. generalized moment problem, spectral estimation, well-posedness, spectral factorization, numerical continuation method

AMS subject classifications. 30E05, 47A57, 47A68, 65L05, 65L20, 93E12

1. Introduction. The generalized moment problem studied in this paper can be traced back to the (scalar) rational covariance extension problem formulated by Kalman [34], which aims to find an infinite extension of a finite covariance sequence such that the resulted spectral density, i.e., the Fourier transform of the infinite sequence, is a rational function. The problem was solved first partially by Georgiou [23,24], and then completely after a series of works [10–14] by Byrnes, Lindquist, and coworkers. The problem of covariance extension is also closely connected to the analytic interpolation problem of various generality [5, 8, 9, 25, 26, 42], since both can be recast as moment problems [1, 35] with different basis functions. The theory has a wide range of applications in the fields of systems and control, circuit theory, and signal processing; cf. the afore cited papers and references therein.

The current framework of the generalized moment problem was pioneered in [7] in the context of spectral estimation and further developed in [27,28,31]. In order to estimate the unknown spectral density of a process, one first feeds it into a bank of filters and collects the steady output covariance matrix as data. Then the problem is to find a spectral density that is consistent with the covariance data. Similar to the classical moment problem, when a solution exists, there are infinitely many solutions unless in certain degenerate case. Therefore, such a problem is not well-posed in the sense of Hadamard\(^1\). The mainstream approach today to promote uniqueness of the solution is built on calculus of variations and optimization theory. It has two main ingredients. One is the introduction of a prior spectral density function \(\Psi\) as additional data, which represents our “guess” of the desired solution \(\Phi\). The other is a cost functional \(d(\cdot, \cdot)\), which is usually a distance (divergence) between two spectral densities. Then one tries to solve the optimization problem of minimizing \(d(\Phi, \Psi)\) subject to the (generalized) moment equation as a constraint. We mention some works that explore different cost functionals, including the Kullback-Leibler divergence [3, 4, 18, 22, 31, 39] and its matricial extension [30], the Hellinger distance [19, 40, 41], the Itakura-Saito distance [16, 17, 32], and some more general divergence...
families [45–47].

The optimization approach guarantees existence and uniqueness of the solution (identifiability) as cited above. However, continuous dependence of the solution on the data does not seem to have attracted much attention, especially in the multivariate case. For the scalar rational covariance extension problem [6, 10–12, 14], such continuity argument is actually part of the results of well-posedness. More precisely, in [14] the correspondence between the covariance data and the solution vector has been shown to be a diffeomorphism, i.e., a $C^1$ function with a $C^1$ inverse. Not many results in this respect exist in multivariate formulations. We mention [41, 43], where continuous dependence of the solution on the covariance matrix has been shown in the contexts of optimization with the Hellinger distance and a certain parametric formulation, respectively.

The present work can be seen as a continuation of [43]. As one main contribution, we shall here show that, when restricted to a predefined family of spectral densities, the solution to a generalized moment problem depends also continuously on the prior function under a suitable metric topology. The idea is to study the moment map directly in a parametric form. Due to the regularity of the moment map, we can view the solution parameter as an implicit functional of the prior, and then invoke the Banach space version of the implicit function theorem to prove continuity. Based on this continuity result, the rest of the paper is devoted to a numerical procedure to compute the unique solution (parameter) using a continuation method which is a quite standard tool from nonlinear analysis. A conservative bound on the step length of a proposed algorithm is given to ensure convergence, and certain numerical consideration is addressed.

The paper is organized as follows. In section 2, we introduce a parametric family of spectral densities and formulate a generalized moment problem. We show that the solution map is well defined and continuous, and hence the problem is well-posed. Next in section 3, we reformulate the problem in terms of the spectral factor of a certain rational spectral density, whose numerical solution can be achieved with a continuation method presented in section 4. Convergence of the proposed algorithm is investigated in detail. Finally, section 5 contains a computational procedure to implement a subroutine of the afore mentioned algorithm.

**Notations.** Some notations are common as $\mathbb{E}$ denotes mathematical expectation, $\mathbb{C}$ the complex plane, and $\mathbb{T}$ the unit circle $\{z : |z| = 1\}$.

Sets: The symbol $\mathfrak{H}_n$ represents the vector space of $n \times n$ Hermitian matrices, and $\mathfrak{H}_{+,n}$ is the subset that contains positive definite matrices. The space of $\mathfrak{H}_m$-valued continuous functions on $\mathbb{T}$ is denoted with $C(\mathbb{T}; \mathfrak{H}_m)$. The set $C_+(\mathbb{T})$ is consisted of continuous functions on $\mathbb{T}$ that take real and positive values, which is an open subset (under the metric topology) of $C(\mathbb{T}) \equiv C(\mathbb{T}; \mathfrak{H}_1)$. The symbol $\mathfrak{S}_m$ denotes the family of $\mathfrak{H}_{+,m}$-valued functions defined on $\mathbb{T}$ that are bounded and coercive.

Linear algebra: The notation $(\cdot)^*$ means taking complex conjugate transpose when applied to a matrix and $(\cdot)^{-1}$ is a shorthand for $[(\cdot)^{-1}]^*$. When considering a rational matrix-valued function with a state-space realization $G(z) = C(zI - A)^{-1}B + D$, $G^*(z) := B^*(z^{-1}I - A^*)^{-1}C^* + D^*$. Matrix inner product is defined as $\langle A, B \rangle := \text{tr}(AB^*)$ for $A, B \in \mathbb{C}^{m \times n}$, and $\|A\|_F := \sqrt{\langle A, A \rangle}$ is the Frobenius norm. The Euclidean 2-norm of $x \in \mathbb{C}^n$ is $\|x\|_2 := \sqrt{x^*x}$. The subscript is usually omitted and we simply write $\|\cdot\|$. When applied to a matrix $A \in \mathbb{C}^{m \times n}$ or more generally a multilinear function, $\|A\|$ means the induced 2-norm.
2. Well-posedness of a generalized moment problem. A framework proposed in [31] to estimate the unknown spectral density $\Phi(z)$ of a zero-mean wide-sense stationary discrete-time $\mathbb{C}^n$-valued process $y(t)$ is described as follows. First we feed the process into a linear filter with a state-space representation

\begin{equation}
    x(t+1) = Ax(t) + By(t),
\end{equation}

whose transfer function is simply

\begin{equation}
    G(z) = (zI - A)^{-1}B.
\end{equation}

There are some extra conditions on the system matrices. More precisely, $A \in \mathbb{C}^{n \times n}$ is Schur stable, i.e., has all its eigenvalues strictly inside the unit circle and $B \in \mathbb{C}^{n \times m}$ is of full column rank ($n \geq m$). Moreover, the pair $(A, B)$ is required to be reachable.

Next an estimate of the steady-state covariance matrix $\Sigma := \mathbb{E}\{x(t)x(t)^*\}$ of the state vector $x(t)$ is computed. Such structured covariance estimation problem has been discussed in the literature; see [21,37,48]. Here we shall assume that the matrix $\Sigma > 0$ is given and we have

\begin{equation}
    \int G\Phi G^* = \Sigma.
\end{equation}

The integration is carried out on the unit circle $T$ with respect to the normalized Lebesgue measure $\frac{d\theta}{2\pi}$. This simplified notation will be adopted throughout the paper.

In general, an estimated covariance matrix may not be compatible with the filter structure (2.2). In other words, viewing (2.3) as a constraint on the input spectrum, there may not exist a feasible $\Phi$. In this paper we shall always assume such feasibility. Specifically, let us define the linear operator

\begin{equation}
    \Gamma: C(T; \mathcal{H}_m) \to \mathcal{H}_n
\end{equation}

\begin{equation}
    \Phi \mapsto \int G\Phi G^*.
\end{equation}

Then we assume that the covariance matrix $\Sigma \in \text{Range} \Gamma$. Equivalent conditions are elaborated in [27,28]; see also [17–21,40,41,48].

Our problem now is to find a spectral density $\Phi$ satisfying the generalized moment constraint (2.3). It can been seen as a generalization of the classical moment problem due to the next example.

Consider the choice of the matrix pair $(A, B)$:

\begin{equation}
    A = \begin{bmatrix}
        0 & I_m & 0 & \cdots & 0 \\
        0 & 0 & I_m & \cdots & 0 \\
        \vdots & \vdots & \ddots & \vdots & \vdots \\
        0 & 0 & 0 & \cdots & I_m \\
        0 & 0 & 0 & \cdots & 0
    \end{bmatrix},
    \quad
    B = \begin{bmatrix}
        0 \\
        0 \\
        \vdots \\
        0 \\
        I_m
    \end{bmatrix}.
\end{equation}

Here each block in $A$ or $B$ is of $m \times m$ and $n = m(p+1)$ in this case. It is easy to verify that

\begin{equation}
    G(z) = (zI - A)^{-1}B = \begin{bmatrix}
        z^{-p-1}I_m \\
        z^{-p}I_m \\
        \vdots \\
        z^{-1}I_m
    \end{bmatrix}.
\end{equation}
The covariance matrix $\Sigma$ has a block-Toeplitz structure, i.e.,

$$
\Sigma = \begin{bmatrix}
\Sigma_0 & \Sigma_1^* & \cdots & \Sigma_p^* \\
\Sigma_1 & \Sigma_0 & \cdots & \Sigma_{p-1}^* \\
\Sigma_2 & \Sigma_1 & \cdots & \Sigma_{p-2}^* \\
\vdots & \vdots & \ddots & \ddots \\
\Sigma_p & \Sigma_{p-1} & \cdots & \Sigma_1 \\
\end{bmatrix},
$$

where $\Sigma_k := \mathbb{E}\{y(t+k)y(t)^*\} \in \mathbb{C}^{m \times m}$ with a slight abuse of notation. In fact, the constraint (2.3) is equivalent to the set of moment equations

$$
\int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi} = \Sigma_k, \quad k = 0, 1, \ldots, p.
$$

To find a spectral density $\Phi$ satisfying (2.8) is the classical covariance extension problem [33].

As well studied in the literature, the inverse problem problem of finding $\Phi$ such that (2.3) holds is typically not well-posed when feasible, because there are infinitely many solutions. One way to remedy this is to restrict the candidate solution to some particular family of spectral densities, as we shall proceed below.

Let us define the set

$$
\mathcal{L}_+ := \{ \Lambda \in \mathcal{H}_n : G^*\Lambda G > 0, \quad \forall z \in \mathbb{T} \}.
$$

By the continuous dependence of eigenvalues on the matrix entries, one can verify that $\mathcal{L}_+$ is an open subset of $\mathcal{H}_n$. Define next the set $\mathcal{L}_+^\Gamma := \mathcal{L}_+ \cap \text{Range } \Gamma$, and the family of spectral densities

$$
\mathcal{S} := \{ \Phi(\psi, \Lambda) = \psi(G^*\Lambda G)^{-1} : \psi \in C_+(\mathbb{T}), \ \Lambda \in \mathcal{L}_+^\Gamma \}.
$$

Our problem is formulated as follows.

**Problem 2.1.** Given the filter bank $G(z)$ in (2.2) and the matrix $\Sigma \in \text{Range}_+ \Gamma := \text{Range } \Gamma \cap \mathcal{H}_{+,n}$, find all the spectral densities $\Phi$ in the family $\mathcal{S}$ such that (2.3) holds.

The motivation for choosing such a family $\mathcal{S}$ lies in the observation that for a fixed $\psi$, the solution of the optimization problem

$$
\max_{\Phi \in \mathcal{S}_m} \int \psi \log \det \Phi \quad \text{subject to } (2.3)
$$

has exactly that form, where the matrix $\Lambda$ appears in the dual problem

$$
\min_{\Lambda \in \mathcal{L}_+^\Gamma} \mathcal{J}_\psi(\Lambda) = \langle \Lambda, \Sigma \rangle - \int \psi \log \det(G^*\Lambda G).
$$

The scalar function $\psi$ encodes some a priori information that we have on the solution density $\Phi$. This optimization problem has been well studied in [3], which can be seen as a multivariate generalization of the scalar problem investigated in [31]. Here we shall approach Problem 2.1 in a different way.

Consider the map

$$
f : D = C_+(\mathbb{T}) \times \mathcal{L}_+^\Gamma \to \text{Range}_+ \Gamma
$$

$$
(\psi, \Lambda) \mapsto \int G\psi(G^*\Lambda G)^{-1}G^*
$$
Given $\Sigma \in \text{Range}_+ \Gamma$, we aim to solve the equation

\begin{equation}
(2.14) \quad f(\psi, \Lambda) = \Sigma,
\end{equation}

which is in fact equivalent to the stationarity condition $\nabla J(\Lambda) = 0$ of the function in (2.12) when $\psi$ is fixed.

The map $f$ has a nice property. As shown in [43], for a fixed $\psi \in C_+(\mathbb{T})$, the section of the map

\begin{equation}
(2.15) \quad \omega(\cdot) := f(\psi, \cdot) : \mathcal{L}_+^\Gamma \to \text{Range}_+ \Gamma
\end{equation}

is a diffeomorphism\textsuperscript{2}. This means that the map above is (at least) of class $C^1$, and its Jacobian, which contains all the partial derivatives of $f$ w.r.t. its second argument, vanishes nowhere in the set $\mathcal{L}_+^\Gamma$. This implies that the solution map

$s : (\psi, \Sigma) \mapsto \Lambda$

is well defined, and for a fixed $\psi$, the map $s(\psi, \cdot) : \text{Range}_+ \Gamma \to \mathcal{L}_+^\Gamma$ is continuous.

We shall next show the well-posedness in the other respect, namely, continuity of the map

\begin{equation}
(2.16) \quad s(\cdot, \Sigma) : C_+(\mathbb{T}) \to \mathcal{L}_+^\Gamma
\end{equation}

when $\Sigma$ is held fixed. Note that continuity here is to be understood in the metric space setting. Clearly, it is equivalent to consider solving the functional equation (2.14) for $\Lambda$ in terms of $\psi$ when its right-hand side is fixed, which naturally falls in to the scope of the implicit function theorem.

We first show that $f$ is of class $C^1$ on its domain $D$. According to [36, Proposition 3.5, p. 10], it is equivalent to show that the two partial derivatives of $f$ exist and are continuous in $D$. More precisely, the partials evaluated at a point are understood as linear operators between two underlying vector spaces

\begin{align}
(2.17) \quad f'_1 : D &\to L(C(\mathbb{T}), \text{Range} \Gamma) \\
f'_2 : D &\to L(\text{Range} \Gamma, \text{Range} \Gamma).
\end{align}

The symbol $L(X, Y)$ denotes the vector space of continuous linear operators between two Banach spaces $X$ and $Y$, which is itself a Banach space. We need some lemmas. Notice that convergence of a sequence of continuous functions on a fixed interval $[a, b] \subset \mathbb{R}$ will always be understood in the max-norm

\begin{equation}
(2.18) \quad \|f\| := \max_{t \in [a,b]} |f(t)|.
\end{equation}

For $m \times n$ matrix valued continuous functions in one variable, define the norm as

\begin{equation}
(2.19) \quad \|M\| := \max_{t \in [a,b]} \|M(t)\|_F.
\end{equation}

It is easy to verify that convergence in the norm (2.19) is equivalent to element-wise convergence in the max-norm (2.18). The first lemma is simple and stated without proof.

\textsuperscript{2}The word “diffeomorphism” in this paper should always be understood in the $C^1$ sense.
**Lemma 2.2.** For a \( m \times n \) matrix continuous function \( M(\theta) \) on \([-\pi, \pi]\), the inequality holds for the Frobenius norm

\[
\left\| \int M(\theta) \right\|_F \leq \sqrt{mn} \int \|M(\theta)\|_F.
\]

**Lemma 2.3.** If a sequence \( \{\Lambda_k\} \subset \mathcal{L}_+^\Gamma \) converges to \( \Lambda \in \mathcal{L}_+^\Gamma \), then the sequence of functions \( \{(G^*\Lambda_k G)^{-1}\} \) converges to \( (G^*\Lambda G)^{-1} \) in the norm (2.19).

*Proof.* From [43, Lem. 10], there exists \( \mu > 0 \) such that for any \( k \) and \( \theta \in [-\pi, \pi] \), \( G^*\Lambda_k G \geq \mu I \). Hence we have

\[
\| (G^*\Lambda_k G)^{-1} - (G^*\Lambda G)^{-1} \|_F = \| (G^*\Lambda_k G)^{-1} G^* (\Lambda - \Lambda_k) G (G^*\Lambda G)^{-1} \|_F \\
\leq \kappa^2 \mu^{-2} G_{\text{max}}^2 \| \Lambda_k - \Lambda \|_F \rightarrow 0,
\]

where the constant \( G_{\text{max}} = \max_{\theta \in [-\pi, \pi]} \| G(e^{i\theta}) \|_F \), and we have used submultiplicativity of the Frobenius norm and norm equivalence \( \| \cdot \|_F \leq \kappa \| \cdot \|_2 \).

**Proposition 2.4.** The map \( f \) in (2.13) is of class \( C^1 \).

*Proof.* Consider the partial derivative w.r.t. the first argument. Due to linearity, one has

\[
f'_1(\psi, \Lambda) : C(\mathbb{T}) \rightarrow \text{Range } \Gamma
\]

\[
\delta \psi \mapsto \int G \delta \psi (G^*\Lambda G)^{-1} G^*.
\]

Clearly, the operator does not depend on \( \psi \). Let the sequence \( \{(\psi_k, \Lambda_k)\} \subset D \) converge in the product topology to \( (\psi, \Lambda) \in D \), that is, \( \psi_k \rightarrow \psi \) in the max-norm and \( \Lambda_k \rightarrow \Lambda \) in any matrix norm. We need to show that

\[
f'_1(\psi_k, \Lambda_k) \rightarrow f'_1(\psi, \Lambda).
\]

in the operator norm. Indeed, we have

\[
\| f'_1(\psi_k, \Lambda_k) - f'_1(\psi, \Lambda) \|
\]

\[
= \sup_{\| \delta \psi \| = 1} \left\| \int G \delta \psi \left[ (G^*\Lambda_k G)^{-1} - (G^*\Lambda G)^{-1} \right] G^* \right\|_F
\]

\[
\leq nG_{\text{max}}^2 \| (G^*\Lambda_k G)^{-1} - (G^*\Lambda G)^{-1} \| \rightarrow 0,
\]

where we have used the inequality (2.20) and Lemma 2.3.

For the partial derivative of \( f \) w.r.t. the second argument, we have

\[
f'_2(\psi, \Lambda) : \text{Range } \Gamma \rightarrow \text{Range } \Gamma
\]

\[
\delta \Lambda \mapsto - \int G \psi (G^*\Lambda G)^{-1} (G^* \delta \Lambda G)(G^*\Lambda G)^{-1} G^*.
\]

To ease the notation, let \( \delta \Phi(\psi, \Lambda; \delta \Lambda) = \Phi(\psi, \Lambda)(G^* \delta \Lambda G)(G^*\Lambda G)^{-1} \). Through similar computation, we arrive at

\[
\| f'_2(\psi_k, \Lambda_k) - f'_2(\psi, \Lambda) \|
\]

\[
\leq \sup_{\| \delta \Lambda \| = 1} nG_{\text{max}}^2 \| \delta \Phi(\psi_k, \Lambda_k; \delta \Lambda) - \delta \Phi(\psi, \Lambda; \delta \Lambda) \| \rightarrow 0.
\]
The limit tends to 0 because the part
\[
\sup_{\|\delta \Lambda\| = 1} \| \delta \Phi(\psi_k, \Lambda_k; \delta \Lambda) - \delta \Phi(\psi; \Lambda; \delta \Lambda) \|
\]
\[
= \max_{\|\delta \Lambda\| = 1, \theta \in [-\pi, \pi]} \| \delta \Phi(\psi_k, \Lambda_k; \delta \Lambda) - \delta \Phi(\psi; \Lambda; \delta \Lambda) \|_F
\]
\[
= \max_{\|\delta \Lambda\| = 1, \theta \in [-\pi, \pi]} \left\| \delta \Phi(\psi_k, \Lambda_k; \delta \Lambda) - \Phi(\psi_k, \Lambda_k)(G^* \delta \Lambda G)(G^* \Lambda G)^{-1} + \Phi(\psi, \Lambda_k)(G^* \delta \Lambda G)(G^* \Lambda G)^{-1} \right\|_F
\]
\[
\leq \max_{\|\delta \Lambda\| = 1, \theta \in [-\pi, \pi]} \| \Phi(\psi_k, \Lambda_k) - \Phi(\psi, \Lambda) \|_F \| G^* \delta \Lambda G \|_F \| (G^* \Lambda G)^{-1} \|_F
\]
\[
\leq \kappa \mu^{-1} G_{\max}^2 (K_\psi \| (G^* \Lambda G)^{-1} - (G^* \Lambda G)^{-1} \| + \| \Phi(\psi_k, \Lambda_k) - \Phi(\psi, \Lambda) \|)
\]
Note that \( \| \psi_k \| \leq K_\psi \) for some \( K_\psi > 0 \) uniformly in \( k \) because \( \psi_k \to \psi \). Also, \( \Phi(\psi_k, \Lambda_k) \to \Phi(\psi, \Lambda) \) is a simple consequence of Lemma 2.3 and the fact
\[
f_k g_k \to fg \text{ if } f_k \to f, \ g_k \to g.
\]

We are now in a place to state the main result of this section.

**Theorem 2.5.** For a fixed \( \Sigma \in \text{Range}_+ \Gamma \), the implicit function \( s(\cdot, \Sigma) \) in (2.16) is of class \( C^1 \).

**Proof.** The assertion follows directly from the Banach space version of the implicit function theorem; see, e.g., [36, Theorem 5.9, p. 19], because restrictions of \( s(\cdot, \Sigma) \) must coincide with those locally defined, continuously differentiable implicit functions, which exist around every \( \psi \in C_+ (\Gamma) \) following from Proposition 2.4 and the fact that the partial \( f_2^T(\psi, \Lambda) \) is a vector space isomorphism everywhere in \( D \). \( \square \)

**3. Reformulation in terms of the spectral factor.** Problem 2.1 can be reformulated in terms of the spectral factor of \( G^* \Lambda G \) for \( \Lambda \in \mathcal{L}_+^\Gamma \). Though it may appear more complicated, this reformulation is preferred from a numerical viewpoint, as the Jacobian of the new map corresponding to (2.13) will have a smaller condition number when the solution is close to the boundary of the feasible set. This point has been illustrated in [3, 15]; see also later in section 5. We shall first introduce a diffeomorphic spectral factorization.

According to [20, Lemma 11.4.1], given \( \Lambda \in \mathcal{L}_+^\Gamma \), the continuous spectral density \( G^* \Lambda G \) admits a unique right outer spectral factor, i.e.,
\[
G^* \Lambda G = W_\Lambda^* W_\Lambda.
\]
Furthermore, such a factor can be expressed in terms of the matrix \( P \in \mathcal{H}_n \), the unique stabilizing solution of the Discrete-time Algebraic Riccati Equation (DARE)
\[
X = A^* X A - A^* X B (B^* X B)^{-1} B^* X A + \Lambda,
\]
as
\[
W_\Lambda(z) = L^{-1} B^* P A (z I - A)^{-1} B + L,
\]
where \( L \) is the right (lower-triangular) Cholesky factor of the positive matrix \( B^* P B (= L^* L) \).
Next, following the lines of [3], let us introduce a change of variables by letting
\[(3.4)\]
\[C := L^{-*}B^*P.\]

Then it is not difficult to recover the relation \(L = CB\). In this way, the spectral factor \((3.3)\) can be rewritten as
\[(3.5)\]
\[W_{\Lambda}(z) = CA(zI - A)^{-1}B + CB = zCG,\]
where the second equality holds because of the identity \(A(zI - A)^{-1} + I = z(zI - A)^{-1}\).

In view of this, the factorization \((3.1)\) can then be rewritten as
\[(3.6)\]
\[G^*\Lambda G = G^*C^*CG, \quad \forall z \in T.\]

As explained in [3, Section A.5.5], it is possible to build a bijective change of variables from \(\Lambda\) to \(C\) by carefully choosing the set where the “factor” \(C\) lives. More precisely, let the set \(\mathcal{C}_+ \subset \mathbb{C}^{m \times n}\) contain those matrices \(C\) that satisfy the following two conditions:
- \(CB\) is lower triangular with real and positive diagonal entries;
- \(A - B(CB)^{-1}CA\) has eigenvalues in the open unit disk.

Define the map
\[(3.7)\]
\[h : \mathcal{L}_+^T \rightarrow \mathcal{C}_+\]
\[\Lambda \mapsto C \text{ via } (3.4).\]

It has been shown in [3] that the map \(h\) is a \emph{homeomorphism} with an inverse
\[(3.8)\]
\[h^{-1} : \mathcal{C}_+ \rightarrow \mathcal{L}_+^T\]
\[C \mapsto \Lambda := \Pi_{\text{Range } \Gamma}(C^*C),\]
where \(\Pi_{\text{Range } \Gamma}\) denotes the orthogonal projection operator onto \(\text{Range } \Gamma\). This result has been further strengthened in [43], as we quote below.

**Theorem 3.1** ([43]). The map \(h\) of spectral factorization is a \emph{diffeomorphism}.

Now we can introduce the moment map \(g : \mathcal{C}_+(T) \times \mathcal{C}_+ \rightarrow \text{Range } \Gamma\) parametrized in the new variable \(C\) as
\[(3.9)\]
\[g(\psi, C) := f(\psi, h^{-1}(C)) = \int G\psi(G^*C^*CG)^{-1}G^*,\]
and the sectioned map when \(\psi \in \mathcal{C}_+(T)\) is held fixed
\[(3.10)\]
\[\tau := \omega \circ h^{-1} : \mathcal{C}_+ \rightarrow \text{Range } \Gamma,\]
where \(\omega\) has been defined in (2.15). A corresponding problem is formulated as follows.

**Problem 3.2.** Given the filter bank \(G(z)\) in (2.2), the matrix \(\Sigma \in \text{Range } \Gamma\), and an arbitrary \(\psi \in \mathcal{C}_+(T)\), find the parameter \(C\) such that
\[(3.11)\]
\[\tau(C) = \Sigma.\]

The next corollary is an immediate consequence of Theorems 2.5 and 3.1 and is stated without proof.
Corollary 3.3. The map $\tau$ in (3.10) is a diffeomorphism. Moreover, if we fix the matrix $\Sigma$ and allow the prior $\psi$ to vary, then the solution map

\begin{equation}
(3.12) \quad h \circ s(\cdot, \Sigma) : C(\mathbb{T}) \to \mathcal{C}_+ \end{equation}

is of class $C^1$.

Therefore, Problem 3.2 is also well-posed exactly like Problem 2.1. However, unlike the map $f$ defined in the previous section, the equation (3.11) is different from the stationarity condition of the function $J \circ h^{-1}$ for a fixed $\psi$. Therefore, we cannot deal with Problem 3.2 in the way of optimization. Nonetheless, it can be solved using a numerical continuation method, which is the content of the next section.

4. A continuation method to the solution. In this section we shall work with coordinates in the sense explained next since it is convenient for analysis. Specifically, we know from [21, Proposition 3.1] that $\text{Range } \Gamma \subset \mathcal{H}_n$ is a linear space with real dimension $M := m(2n - m)$. At the same time, the set $\mathcal{C}_+$ is an open subset of the linear space

$$\mathcal{C} := \{ C \in \mathbb{C}^{m \times n} : CB \text{ is lower triangular with real diagonal entries} \},$$

whose real dimension coincides with $\text{Range } \Gamma$ (cf. the proof of [3, Theorem A.5.5]). We can hence choose orthonormal bases $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_M\}$ and $\{C_1, \ldots, C_M\}$ for $\text{Range } \Gamma$ and $\mathcal{C}$, respectively, and parameterize $\Lambda \in \mathcal{L}_+^\Gamma$ and $C \in \mathcal{C}_+$ as

\begin{align}
\Lambda(x) &= x_1 \Lambda_1 + x_2 \Lambda_2 + \cdots + x_M \Lambda_M, \\
C(y) &= y_1 C_1 + y_2 C_2 + \cdots + y_M C_M,
\end{align}

for some $x_j, y_j \in \mathbb{R}$, $j = 1, \ldots, M$. We shall then introduce some abuse of notation and make no distinction between the variable and its coordinates. For example, $f(\psi, x)$ is understood as $f(\psi, \Lambda(x))$ defined previously and similarly, $\tau(y)$ means $\tau(C(y))$.

Instead of dealing with one particular equation (3.11), a continuation method (cf. [2]) aims to solve a family of equations related via a homotopy, i.e., a continuous deformation. In our context, there are two ways to construct different homotopies. One is to deform the covariance data $\Sigma$ and study the equation (3.11) for a fixed $\psi$. Such an argument has been used extensively in [29, 30]. Here we shall adopt an alternative, that is, deforming the prior function $\psi$ while keeping the covariance matrix fixed, in the style of [15, Section 4] for the problem of scalar rational covariance extension. An advantage to do so is that we can obtain a family of matrix spectral densities that are consistent with the covariance data.

The set $C_+(\mathbb{T})$ is easily seen to be convex. One can then connect $\psi$ with the constant function $1$ (taking value $1$ on $\mathbb{T}$) via the line segment

\begin{equation}
(4.2) \quad p(t) = (1 - t)1 + t\psi, \quad t \in U = [0, 1],
\end{equation}

and construct a convex homotopy $U \times \mathcal{C}_+ \to \text{Range}_+ \Gamma$ given by

\begin{equation}
(4.3) \quad (t, y) \mapsto g(p(t), y).
\end{equation}

Now let the covariance matrix $\Sigma \in \text{Range}_+ \Gamma$ be fixed whose coordinate vector is $x_\Sigma$, and consider the family of equations

\begin{equation}
(4.4) \quad g(p(t), y) = x_\Sigma
\end{equation}
parametrized by \( t \in U \). By Corollary 3.3, we will have a continuously differentiable solution path in the set \( C_+ \)

\[
y(t) = h(s(p(t), x_\Sigma)).
\]

Moreover, differentiating (4.4) on both sides w.r.t \( t \), one gets

\[
g'_1(p(t), y(t); p'(t)) + g'_2(p(t), y(t); y'(t)) = 0,
\]

where \( p'(t) \equiv \psi - 1 \) independent of \( t \), and the partial derivatives are given by

\[
\begin{align*}
g'_1(\psi, y) &= f'_1(\psi, h^{-1}(y)), \\
g'_2(\psi, y) &= f'_2(\psi, h^{-1}(y)) J_{h^{-1}}(y).
\end{align*}
\]

The symbol \( J_{h^{-1}}(y) \) means the Jacobian matrix of \( h^{-1} \) evaluated at \( y \). Hence the path \( y(t) \) is a solution to the initial value problem (IVP)

\[
\begin{align*}
y'(t) &= -g'_2(p(t), y(t)) \quad \text{subject to } y(0) = y^{(0)},
\end{align*}
\]

Notice that the partial \( g'_2 \) is a finite-dimensional Jacobian matrix which is invertible everywhere in \( D \) since both terms on the right hand side of (4.6b) are nonsingular (cf. [43]). From classical results on the uniqueness of solution to an ODE, we know that the IVP formulation and (4.4) are in fact equivalent.

The initial value \( y^{(0)} \) corresponds to \( \psi = 1 \), and it is the spectral factor of the so-called maximum entropy solution, i.e., solution to the problem

\[
\max_{\Phi \in S_m} \int \log \det \Phi \quad \text{subject to } (2.3).
\]

As has been worked out in [27], the above optimization problem has a unique solution \( \Phi = (G^* \Lambda G)^{-1} \) with

\[
\Lambda = \Sigma^{-1} B (B^* \Sigma^{-1} B)^{-1} B^* \Sigma^{-1},
\]

from which the corresponding spectral factor \( C \) can be computed as

\[
C = L^{-*} B^* \Sigma^{-1},
\]

where \( L \) is the right Cholesky factor of \( B^* \Sigma^{-1} B \). According to [27], such \( C \) is indeed in the set \( C_+ \), i.e., \( CB \) lower triangular and the closed-loop matrix is stable.

At this stage, any numerical ODE solver can in principle be used to solve the IVP and obtain the desired solution \( y(1) \) corresponding to a particular prior \( \psi \). However, this IVP is special in the sense that for a fixed \( t \), \( y(t) \) is a solution to a finite-dimensional nonlinear system of equations, for which there are numerical methods (such as Newton’s method) that exhibit rapid local convergence properties, while a general-purpose ODE solver does not take this into account. Out of such consideration, a method called “predictor-corrector” is recommended in [2] to solve the IVP, which is reviewed next.

Suppose for some \( t \in U \) we have got a solution \( y(t) \) and we aim to solve (4.4) at \( t + \delta t \) where \( \delta t \) is a chosen step length. The predictor step is just numerical integration of the differential equation in (4.7) using e.g., the Euler method

\[
z(t + \delta t) := y(t) + v(t) \delta t,
\]
Algorithm 4.1 Predictor-Corrector

Let \( k = 0, t = 0, \) and \( y^{(0)} \) initialized as in (4.9)

Choose a sufficiently small step length \( \delta t \)

while \( t \leq 1 \) do

  Predictor: \( z^{(k+1)}(t) = y^{(k)} + v(t)\delta t \) the Euler step (4.10)

  Corrector: solve (4.4) at \( t + \delta t \) for \( y^{(k+1)} \) initiated at \( z^{(k+1)} \) using Newton’s method

  Update \( t := \min\{1, t + \delta t\} \), \( k := k + 1 \)

end while

return The last \( y^{(k)} \) corresponding to \( t = 1 \)

where \( v(t) := - [g'_2(p(t), y(t))]^{-1}g'_1(p(t), y(t); p'(t)) \). The corrector step is accomplished by the Newton’s method to solve (4.4) initialized at the predictor \( z(t + \delta t) \). If the new solution \( y(t + \delta t) \) can be attained in this way, one can repeat such a procedure until reaching \( t = 1 \). The algorithm is summarized in the table.

We are now left to determine the step length \( \delta t \) so that the corrector step can converge and the algorithm can return the target solution \( y(1) \) in a finite number of steps. We show next that one can choose a uniformly constant step length \( \delta t \) such that the predictor \( z^{(k)} \) will be close enough to the solution \( y^{(k)} \) for the Newton’s method to converge locally. We shall need the next famous Kantorovich theorem which can be found in [38, p. 421].

Theorem 4.1 (Kantorovich). Assume that \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) is differentiable on a convex set \( D_0 \subset D \) and that

\[
\|f'(x) - f'(y)\| \leq \gamma \|x - y\|, \quad \forall x, y \in D_0
\]

for some \( \gamma > 0 \). Suppose that there exists an \( x^{(0)} \in D_0 \) such that \( \alpha = \beta \gamma \eta \leq 1/2 \), for some \( \beta, \eta > 0 \) meeting

\[
\beta \geq \|f'(x^{(0)})^{-1}\|, \quad \eta \geq \|f'(x^{(0)})^{-1} f(x^{(0)})\|.
\]

Set

\[
\begin{align}
t^* &= (\beta \gamma)^{-1} \left[ 1 - (1 - 2\alpha)^{1/2} \right], \\
t^{**} &= (\beta \gamma)^{-1} \left[ 1 + (1 - 2\alpha)^{1/2} \right],
\end{align}
\]

and assume that the closed ball \( \overline{B}(x^{(0)}, t^*) \) is contained in \( D_0 \). Then the Newton iterates

\[
x^{(k+1)} = x^{(k)} - f'(x^{(k)})^{-1} f(x^{(k)}), \quad k = 0, 1, \ldots
\]

are well-defined, remain in \( \overline{B}(x^{(0)}, t^*) \), and converge to a solution \( x \) of \( f(x) = 0 \) which is unique in \( \overline{B}(x^{(0)}, t^{**}) \cap D_0 \).

In order to apply the above theorem, we need to take care of the locally Lipschitz property. To this end, we shall first introduce a compact set in which we can take extrema of various norms.

Lemma 4.2. There exists a compact set \( K \subset \mathcal{C}_+ \) that contains the solution path \( \{y(t) : t \in U\} \) indicated in (4.5) in its interior.
Proof. We know from previous reasoning that the solution path is contained in the open set \( \mathcal{C}_+ \). By continuity, the set \( \{y(t)\} \) is easily seen to be compact, i.e., closed and bounded, and thus admits a compact neighborhood \( K \subset \mathcal{C}_+ \). Such a neighborhood \( K \) can be constructed explicitly as follows. Let \( B(y(t)) \subset \mathcal{C}_+ \) be an open ball centered at \( y(t) \) such that its closure is also contained in \( \mathcal{C}_+ \). Then the set \( \bigcup_{t \in U} B(y(t)) \) and an open cover of \( \{y(t)\} \), which by compactness, has a finite subcover

\[
\bigcap_{k=1}^{n} B(y(t_k))
\]

whose closure can be taken as \( K \).

Lemma 4.3. For a fixed \( t \in U \), the derivative \( g_2'(p(t), y) \in L(\mathcal{C}, \text{Range } \Gamma) \) is locally Lipschitz continuous in \( y \) in any convex subset of the compact set \( K \) constructed in Lemma 4.2, where \( p(t) \) is the line segment given in (4.2). Moreover, the Lipschitz constant can be made independent of \( t \).

Proof. It is a well known fact a continuously differentiable function is locally Lipschitz. Hence we need to check the continuity of the second-order derivative following from (4.6b)

\[
g_{22}''(\psi, y; \delta y_1, \delta y_2) = f_{22}''(\psi, h^{-1}(y); J_{h^{-1}}(y)\delta y_2, J_{h^{-1}}(y)\delta y_1) \\
+ f_2'(\psi, h^{-1}(y)) \frac{d}{dy} J_{h^{-1}}(y)(\delta y_2, \delta y_1).
\]

Here \( \frac{d}{dy} J_{h^{-1}}(y) \) is the second order derivative of \( h^{-1} \) evaluated at \( y \) which is viewed as a bilinear function \( \mathcal{C} \times \mathcal{C} \to \text{Range } \Gamma \). It is continuous in \( y \) since the function (3.8) is actually smooth (of class \( C^\infty \)). Albeit more tedious computation, one can also show that the second-order partial \( f_{22}''(\psi, x) \) is continuous following the lines in section 2. Therefore, for fixed \( \delta y_1, \delta y_2 \), the differential (4.12) is continuous in \( (\psi, y) \). Consider any convex subset \( D_0 \subset K \) with \( y_1, y_2 \in D_0 \). By the mean value theorem

\[
\|g_2'(\psi, y_2) - g_2'(\psi, y_1)\| \leq \max_{y K} \|g_{22}''(\psi, y)\| \|y_2 - y_1\|
\]

(4.13)

Now let us replace \( \psi \) with \( p(t) \). The local Lipschitz constant can be taken as

\[
\gamma := \max_{t \in U, y K} \|g_{22}''(p(t), y)\|.
\]

(4.14)

Based on precedent lemmas, our main result in this section is stated as follows.

Theorem 4.4. Algorithm 4.1 returns a solution to (4.4) for \( t = 1 \) in a finite number of steps.

Proof. At each step, our task is to solve the equation (4.4) for \( t + \delta t \) from the initial point \( z(t + \delta t) = y(t) + v(t) \delta t \) given in (4.10). The idea is to work in the compact set \( K \) introduced in Lemma 4.2. The boundary of \( K \) is denoted by \( \partial K \) which is also compact.

First, we show that the predictor \( z(t + \delta t) \) will always stay in \( K \) as long as the step length \( \delta t \) is sufficiently small. Define

\[
c_1 := \min_{t \in U} d(y(t), \partial K),
\]

(4.15)

\[
c_2 := \max_{t \in U} \|v(t)\|,
\]

(4.16)
where \( d(x, A) := \min_{y \in A} d(x, y) \) is the distance function from a point \( x \) to a set \( A \). Note that \( c_1 > 0 \) because all the points \( \{y(t)\} \) are in the interior of \( K \). Then we see that the condition

\[
\delta t < \frac{c_1}{c_2} := \delta t_1
\]

is sufficient since in this way

\[
\|v(t)\delta t\| \leq c_2 \delta t < c_1 \leq d(y(t), \partial K), \quad \forall t \in U,
\]

which implies that \( z(t + \delta t) \in K \). The reason is that one can always go from \( y(t) \) in the direction of \( v(t) \) until the boundary of \( K \) is hit.

Secondly, we want to apply the Kantorovich Theorem to ensure convergence of the corrector step, i.e., the Newton iterates. The function \( \psi = p(t + \delta t) \) is held fixed in the corrector step. The uniform Lipschitz constant \( \gamma \) has been given in Lemma 4.3, and there are two remaining points:

(i) We need to take care of the constraint \( \alpha = \beta \gamma \eta \leq 1/2 \). Clearly, we can simply take

\[
\beta = \|g_2'(\psi, y_{in}^{(0)})^{-1}\|, \quad \eta = \|g_2'(\psi, y_{in}^{(0)})^{-1}(g(\psi, y_{in}^{(0)}) - x_\Sigma)\|,
\]

where \( y_{in}^{(0)} = z(t + \delta t) \) is the initialized inner-loop variable. Define

\[
c_3 := \max_{y \in K, \eta \in U} \|g_2'(p(t), y)^{-1}\|,
\]

\[
c_4 := \max_{y \in K, \eta \in U} \|g_2''(p(t), y)\|,
\]

and obviously we have \( \beta \leq c_3, \eta \leq c_3\|g(\psi, y_{in}^{(0)}) - x_\Sigma\| \). Hence a sufficient condition is

\[
\|g(\psi, y_{in}^{(0)}) - x_\Sigma\| \leq \frac{1}{2c_3^2 \gamma},
\]

and we need an estimate of the left hand side. The Taylor expansion of \( g \) in its second argument is

\[
g(\psi, y(t) + v(t)\delta t) = g(\psi, y(t)) + \delta t g_2'(\psi, y(t)) v(t) + \frac{\delta t^2}{2} B[v(t), v(t)],
\]

where \( B \) is the bilinear function determined by the second order partials. Due to linearity and the identity \( \psi = p(t + \delta t) = p(t) + \delta t p'(t) \), the first term

\[
g(p(t + \delta t), y(t)) = g(p(t), y(t)) + \delta t g_1'(p(t), y(t); p'(t)) = x_\Sigma + \delta t g_1'(p(t), y(t); p'(t)).
\]

The matrix in the second term\(^3\)

\[
g_2'(p(t + \delta t), y(t)) = g_2'(p(t), y(t)) + \delta t g_2'(p'(t), y(t))
\]

Substituting these two expressions into (4.20), we obtain a cancellation due to the definition of \( v(t) \) after (4.10) and we have

\[
g(\psi, y(t) + v(t)\delta t) - x_\Sigma = \delta t^2 g_2'(p'(t), y(t)) v(t) + \frac{\delta t^2}{2} B[v(t), v(t)]
\]

\(^3\)Attention: \( g_2'(p'(t), y(t)) \) is an abuse of notation because \( p'(t) = \psi - 1 \) may not be in the domain of the functional. It should be understood as substituting \( \psi \) with \( p'(t) \) in the expression of \( g_2'(\psi, y(t)) \).
whose norm is less than \( \delta t^2 (c_5 c_2 + \frac{1}{2} c_2^2 c_4) \), where

\[
    c_5 := \max_{y \in K} \| g'_2(p(t),y) \|.
\]

We end up having the sufficient condition

\[
    \delta t^2 (c_5 c_2 + \frac{1}{2} c_2^2 c_4) \leq \frac{1}{2c_3^2} \quad \implies \quad \delta t \leq \delta t_2.
\]

(ii) We need to insure that the closed ball \( B(y(0),t^* \in U) \) is also contained in \( K \). Clearly, we only need to make \( t^* \leq \min_{t \in U} d(p(t) + v(t) \delta t_1/2, \partial K) =: r_2 \), where \( \delta t_1 \) is the uniform step determined in (4.17). This can be done since by its definition (4.11a), \( t^* \) tends to 0 when the step length \( \delta t \to 0 \). A sufficient condition is

\[
    1 - \sqrt{1 - 2\alpha} \leq c_6 \gamma r_2 \iff \alpha \leq \frac{1}{2}(1 - (1 - c_6 \gamma r_2)^2),
\]

provided that \( 1 - c_6 \gamma r_2 > 0 \), where

\[
    c_6 := \min_{y \in K, t \in U} \| g'_2(p(t),y)^{-1} \|.
\]

With the bound for \( \beta \) and \( \eta \) in the previous point, a more sufficient condition is

\[
    \delta t^2 c_2 (c_5 c_2 + \frac{1}{2} c_2^2 c_4) \leq \frac{1}{2}(1 - (1 - c_6 \gamma r_2)^2) \quad \implies \quad \delta t \leq \delta t_3.
\]

At last we can just take \( \delta t := \min \{ \delta t_1/2, \delta t_2, \delta t_3 \} \). In this way, the Kantorovich theorem is applicable to ensure local convergence in each inner loop. The reasoning above is independent of \( t \) and hence the step length is uniform. This concludes the proof.  

\[\Box\]

5. Computation of the inverse Jacobian. The coordinate thinking is suitable for theoretical reasoning. However, when implementing the algorithm, it is better to work with matrices directly. In this section, we present a matricial linear solver adapted from [40]; see also [17]. Here we shall assume \( \psi \) is rational and admits a factorization \( \psi = \sigma \sigma^* \) where \( \sigma \) is outer rational and hence realizable. A crucial step in the implementation of the numerical algorithm is the computation of the Newton direction \( g'_2(\psi, y)^{-1} g(\psi, y) \), which amounts to solving the linear equation in \( V \) given \( C \) and \( \psi \)

\[
(5.1) \quad g'_2(\psi, C; V) = g(\psi, C)
\]

where

\[
(5.2a) \quad g(\psi, C) = \int G\psi(G^* C^* C G)^{-1} G^*
\]

\[
(5.2b) \quad g'_2(\psi, C; V) = \int G\psi(G^* C^* C G)^{-1} G^* (V^* C + C^* V) G (G^* C^* C G)^{-1} G^*
\]

\[
(5.2c) \quad = \int G\psi(C G)^{-1} [(G^* C^*)^{-1} G^* V^* + V G(C G)^{-1}] (G^* C^*)^{-1} G^*
\]

The cancellation of one factor \( C G \) from (5.2b) to (5.2c) is precisely why the condition number of the Jacobian \( g'_2 \) is smaller than that of \( f'_2 \) in (2.24) when \( C \) tends to the boundary of \( \mathcal{G}_+ \), i.e., when \( C G(e^{i\theta}) \) tends to be singular for some \( \theta \).
We first need to fix an orthonormal basis \( \{ C_1, \ldots, C_M \} \) of \( \mathcal{C} \) such that 
\[
C_1 = \frac{C}{\|C\|}. \quad \text{(4)}
\]
Then one can obtain a basis \( \{ V_1, \ldots, V_M \} \) of \( \mathcal{C} \) such that for \( k = 1, \ldots, M \)
\[
G^*(z)(V_k^* C + C^* V_k)G(z) > 0, \quad \forall z \in T
\]
by setting \( V_k = C_k + \alpha_k C \) for some \( \alpha_k \geq 0 \). The procedure for solving (5.1) is described as follows:

1) Compute \( Y = g(\psi, C) \) and \( Y_k = g'_2(\psi, C; V_k) \).
2) Find \( \alpha_k \) such that \( Y = \sum \alpha_k Y_k \).
3) Set \( V = \sum \alpha_k V_k \).

In order to obtain the coordinates \( \alpha_k \) in Step 2, one needs to solve a linear system of equations whose the coefficient matrix is consisted of inner products \( \langle Y_k, Y_j \rangle \). The matrix is invertible because \( \{ Y_k \} \) are linearly independent, which is a consequence of the Jacobian \( g'_2(\psi, C) \) being nonsingular.

The difficult part is Step 1 where we need to compute the integrals \( g(\psi, C) \) and \( g'_2(\psi, C; V_k) \). Since we want to avoid numerical integration, we shall need some techniques from spectral factorization. Evaluation of the former integral was essentially done in the proof of [20, Theorem 11.4.3]. More precisely, we have the expression
\[
G(zCG)^{-1} = (zI - \Pi)^{-1}B(CB)^{-1},
\]
where \( \Pi := A - B(CB)^{-1}CA \) is the closed-loop matrix which is stable. With a state-space realization \( (A_1, B_1, C_1) \) of the stable proper transfer function \( \sigma G(zCG)^{-1} \), one then solves a discrete-time Lyapunov equation for \( R \)
\[
R - A_1R A_1^* = B_1B_1^*.
\]
Finally the integral \( g(\psi, C) = C_1 R C_1^* \).

The integral \( g'_2(\psi, C; V_k) \) can be computed similarly. The only difference is that we need to compute a left outer factor \( W(z) \) of
\[
Z^*(z) + Z(z) > 0 \quad \text{on } T
\]
where
\[
Z(z) = zVG(zCG)^{-1} = V z(zI - \Pi)^{-1}B(CB)^{-1} = V \Pi(zI - \Pi)^{-1}B(CB)^{-1} + VB(CB)^{-1}
\]
(5.3)
The factorization involves solving a DARE for the unique stabilizing solution, in terms of which the factor can be expressed. Such a procedure is standard; cf. Appendix A for details. Once we have the factor \( W(z) \), a realization of the transfer function \( \sigma G(zCG)^{-1}W \) can be obtained, and we can just proceed in the same way as computing \( g(\psi, C) \).

6. Conclusions. Together with the previous work [43], we have shown in this paper that the generalized moment problem formulated in a parametric fashion is well-posed with respect to either parameter while the other is held fixed. Although the numerical algorithm seems more complicated than the optimization approach in [3], it serves as a viable alternative with some benefits.

\footnote{This is always possible by adding \( C \) into any set of basis matrices and performing Gram-Schmidt orthonormalization starting from \( C \).}
Moreover, we hope to generalize the results in this work to the open problem left in [43, 44] when the prior function is matrix-valued. Certain technical difficulties in that more general case remain to be tackled.

Appendix A. From additive decomposition to spectral factorization.

Let $Z(z) = H(zI - F)^{-1}G + J$ with $F \in \mathbb{C}^{n \times n}$ stable, $G \in \mathbb{C}^{n \times m}$, $H \in \mathbb{C}^{m \times n}$, and $J \in \mathbb{C}^{m \times m}$. Suppose that $\Phi(z) = Z(z) + Z^*(z) > 0$ for all $z \in \mathbb{T}$. Set $R := J + J^* > 0$. Then one can write

$$
\Phi(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & G^* & (z^{-1}I - F^*)^{-1}H^* \\ G & I \end{bmatrix},
$$

which adding to the identity that holds for any Hermitian $P$

$$
0 = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} FPF^* - P & FPH^* \\ HPF^* & HPH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}
$$

yields

$$
\Phi(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} FPF^* - P & G + FPH^* \\ G^* + HPH^* & R + HPH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.
$$

Consequently, if $P$ is the unique stabilizing solution of the DARE

$$
P = FPF^* - (G + FPH^*)(R + HPH^*)^{-1}(G^* + HPH^*)
$$

such that $R + HPH^* > 0$, then one obtains the factorization

$$
\begin{bmatrix} FPF^* - P & G + FPH^* \\ G^* + HPH^* & R + HPH^* \end{bmatrix} = \begin{bmatrix} G + FPH^* \\ R + HPH^* \end{bmatrix}(R + HPH^*)^{-1}
\times \begin{bmatrix} G^* + HPH^* & R + HPH^* \end{bmatrix}.
$$

Taking $L$ as the Cholesky factor of $R + HPH^* (= LL^*)$, one gets a left outer factor of $\Phi(z)$ in this way

$$
W(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} G + FPH^* \\ R + HPH^* \end{bmatrix} L^{-*}
= H(zI - F)^{-1}(G + FPH^*)L^{-*} + L.
$$

REFERENCES

[1] N. I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis, Oliver & Boyd, Edinburgh, 1965.
[2] E. L. Allgower and K. Georg, Introduction to Numerical Continuation Methods, SIAM, 1990.
[3] E. Avventi, Spectral Moment Problems: Generalizations, Implementation and Tuning, PhD thesis, KTH Royal Institute of Technology, Stockholm, 2011.
[4] G. Baggio, A global convergence analysis of the Pavon–Ferrante algorithm for spectral estimation. Submitted to IEEE Trans. Automat. Control, 2017, https://arxiv.org/abs/1612.03570.
[5] A. Blomqvist, A. Lindquist, and R. Nagamune, Matrix-valued Nevanlinna–Pick interpolation with complexity constraint: An optimization approach, IEEE Trans. Automat. Control, 48 (2003), pp. 2172–2190.
[6] C. I. Byrnes, P. Enqvist, and A. Lindquist, Identifiability and well-posedness of shaping-filter parameterizations: A global analysis approach, SIAM J. Control Optim., 41 (2002), pp. 23–59.
WELL-POSEDNESS OF A GENERALIZED MOMENT PROBLEM

[7] C. I. BYRNE, T. T. GEORGIU, AND A. LINDQUIST, A new approach to spectral estimation: A tunable high-resolution spectral estimator, IEEE Trans. Signal Process., 48 (2000), pp. 3189–3205.

[8] C. I. BYRNE, T. T. GEORGIU, AND A. LINDQUIST, A generalized entropy criterion for Nevanlinna–Pick interpolation with degree constraint, IEEE Trans. Automat. Control, 46 (2001), pp. 822–839.

[9] C. I. BYRNE, T. T. GEORGIU, A. LINDQUIST, AND A. MEGRETSKI, Generalized interpolation in $H^\infty$ with a complexity constraint, Trans. Amer. Math. Soc., 358 (2006), pp. 965–987.

[10] C. I. BYRNE, S. V. GUSEV, AND A. LINDQUIST, A convex optimization approach to the rational covariance extension problem, SIAM J. Control Optim., 37 (1998), pp. 211–229.

[11] C. I. BYRNE, S. V. GUSEV, AND A. LINDQUIST, From finite covariance windows to modeling filters: A convex optimization approach, SIAM Rev., 43 (2001), pp. 645–675.

[12] C. I. BYRNE, H. J. LANDAU, AND A. LINDQUIST, On the well-posedness of the rational covariance extension problem, in Current and Future Directions in Applied Mathematics, Springer, 1997, pp. 83–108.

[13] C. I. BYRNE AND A. LINDQUIST, On the partial stochastic realization problem, IEEE Trans. Automat. Control, 42 (1997), pp. 1049–1070.

[14] C. I. BYRNE, A. LINDQUIST, S. V. GUSEV, AND A. S. MATEEVE, A complete parameterization of all positive rational extensions of a covariance sequence, IEEE Trans. Automat. Control, 40 (1995), pp. 1841–1857.

[15] P. ENQVIST, A homotopy approach to rational covariance extension with degree constraint, Int. J. Appl. Math. Comput. Sci., 11 (2001), pp. 1173–1201.

[16] P. ENQVIST AND J. KARLSSON, Minimal Hara–Sato distance and covariance interpolation, in 47th IEEE Conference on Decision and Control (CDC 2008), IEEE, 2008, pp. 137–142.

[17] A. FERRANTE, C. MASERO, AND M. PAVON, Time and spectral domain relative entropy: A new approach to multivariate spectral estimation, IEEE Trans. Automat. Control, 57 (2012), pp. 2561–2575.

[18] A. FERRANTE, M. PAVON, AND F. RAMPONI, Further results on the Byrnes–Georgiou–Lindquist generalized moment problem, in Modeling, Estimation and Control, Springer Berlin Heidelberg, 2007, pp. 73–83.

[19] A. FERRANTE, M. PAVON, AND F. RAMPONI, Hellinger versus Kullback–Leibler multivariable spectrum approximation, IEEE Trans. Automat. Control, 53 (2008), pp. 954–967.

[20] A. FERRANTE, M. PAVON, AND M. ZORZI, Application of a global inverse function theorem of Byrnes and Lindquist to a multivariable moment problem with complexity constraint, in Three Decades of Progress in Control Sciences, Springer Berlin Heidelberg, 2010, pp. 153–167.

[21] A. FERRANTE, M. PAVON, AND M. ZORZI, A maximum entropy enhancement for a family of high-resolution spectral estimators, IEEE Trans. Automat. Control, 57 (2012), pp. 318–329.

[22] A. FERRANTE, F. RAMPONI, AND F. TICOZZI, On the convergence of an efficient algorithm for Kullback–Leibler approximation of spectral densities, IEEE Trans. Automat. Control, 56 (2011), pp. 506–515.

[23] T. T. GEORGIU, Partial Realization of Covariance Sequences, PhD thesis, University of Florida, Gainesville, 1983.

[24] T. T. GEORGIU, Realization of power spectra from partial covariance sequences, IEEE Trans. Acoust. Speech Signal Process., 35 (1987), pp. 438–449.

[25] T. T. GEORGIU, A topological approach to Nevanlinna–Pick interpolation, SIAM J. Math. Anal., 18 (1987), pp. 1248–1260.

[26] T. T. GEORGIU, The interpolation problem with a degree constraint, IEEE Trans. Automat. Control, 44 (1999), pp. 631–635.

[27] T. T. GEORGIU, Spectral analysis based on the state covariance: the maximum entropy spectrum and linear fractional parametrization, IEEE Trans. Automat. Control, 47 (2002), pp. 1811–1823.

[28] T. T. GEORGIU, The structure of state covariances and its relation to the power spectrum of the input, IEEE Trans. Automat. Control, 47 (2002), pp. 1056–1066.

[29] T. T. GEORGIU, Solution of the general moment problem via a one-parameter imbedding, IEEE Trans. Automat. Control, 50 (2005), pp. 811–826.

[30] T. T. GEORGIU, Relative entropy and the multivariable multidimensional moment problem, IEEE Trans. Inform. Theory, 52 (2006), pp. 1052–1066.

[31] T. T. GEORGIU AND A. LINDQUIST, Kullback–Leibler approximation of spectral density functions, IEEE Trans. Inform. Theory, 49 (2003), pp. 2010–2017.

[32] T. T. GEORGIU AND A. LINDQUIST, Likelihood analysis of power spectra and generalized moment problems, IEEE Trans. Automat. Control, 62 (2017), pp. 4580–4592.
[33] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, California Monographs in Mathematical Sciences, University of California Press, 1958.

[34] R. E. Kálmán, *Realization of covariance sequences*, in Toeplitz Centennial, Springer, 1982, pp. 331–342.

[35] M. G. Krein and A. A. Nudel’man, *The Markov Moment Problem and Extremal Problems*, vol. 50 of Translations of Mathematical Monographs, American Mathematical Society, Providence, Rhode Island, 1977.

[36] S. Lang, *Fundamentals of Differential Geometry*, vol. 191 of Graduate Texts in Mathematics, Springer-Verlag New York, Inc., 1999.

[37] L. Ning, X. Jiang, and T. Georgiou, *On the geometry of covariance matrices*, IEEE Signal Process. Lett., 20 (2013), pp. 787–790.

[38] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, vol. 30 of SIAM’s Classics in Applied Mathematics, SIAM, 2000.

[39] M. Pavon and A. Ferrante, *On the Georgiou–Lindquist approach to constrained Kullback–Leibler approximation of spectral densities*, IEEE Trans. Automat. Control, 51 (2006), pp. 639–644.

[40] F. Ramponi, A. Ferrante, and M. Pavon, *A globally convergent matricial algorithm for multivariate spectral estimation*, IEEE Trans. Automat. Control, 54 (2009), pp. 2376–2388.

[41] F. Ramponi, A. Ferrante, and M. Pavon, *On the well-posedness of multivariate spectrum approximation and convergence of high-resolution spectral estimators*, Systems Control Lett., 59 (2010), pp. 167–172.

[42] M. S. Takyar and T. T. Georgiou, *Analytic interpolation with a degree constraint for matrix-valued functions*, IEEE Trans. Automat. Control, 55 (2010), pp. 1075–1088.

[43] B. Zhu, *On a parametric spectral estimation problem*, submitted to the 18th IFAC Symposium on System Identification (SYSID 2018), 2017, https://arxiv.org/abs/1712.07970.

[44] B. Zhu and G. Baggio, *On the existence of a solution to a spectral estimation problem à la Byrnes-Georgiou-Lindquist*. Accepted for publication in IEEE Trans. Automat. Control, 2017, https://arxiv.org/abs/1709.09012.

[45] M. Zorzi, *A new family of high-resolution multivariate spectral estimators*, IEEE Trans. Automat. Control, 59 (2014), pp. 892–904.

[46] M. Zorzi, *Rational approximations of spectral densities based on the alpha divergence*, Math. Control Signals Systems, 26 (2014), pp. 259–278.

[47] M. Zorzi, *Multivariate spectral estimation based on the concept of optimal prediction*, IEEE Trans. Automat. Control, 60 (2015), pp. 1647–1652.

[48] M. Zorzi and A. Ferrante, *On the estimation of structured covariance matrices*, Automatica J. IFAC, 48 (2012), pp. 2145–2151.