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Strong-randomness infinite-coupling phase in a random quantum spin chain

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We study the ground-state phase diagram of the Ashkin-Teller random quantum spin chain by means of a generalization of the strong-disorder renormalization group. In addition to the conventional paramagnetic and ferromagnetic (Baxter) phases, we find a partially ordered phase characterized by strong randomness and infinite coupling between the colors. This unusual phase acts, at the same time, as a Griffiths phase for two distinct quantum phase transitions, both of which are of infinite-randomness type. We also investigate the quantum multicritical point that separates the two-phase and three-phase regions, and we discuss generalizations of our results to higher dimensions and other systems.

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I. INTRODUCTION

Random quantum many-particle systems are easiest to understand if both interactions and disorder are weak. In these cases, the system often behaves analogously to a clean noninteracting one, with small perturbative corrections. If, on the other hand, interactions or disorder are strong, qualitatively new behavior can arise. For instance, repulsive interactions induce a new phase, the Mott insulator, in systems of lattice bosons or electrons. Moreover, strong randomness leads to an Anderson insulator in which the quantum wave functions are localized.

Particularly strong disorder and correlation effects can be expected in the vicinity of zero-temperature quantum phase transitions where the fluctuations extend over large length and time scales. Examples include infinite-randomness criticality [1,2], quantum Griffiths singularities [3,4], and smeared phase transitions [5] (for recent reviews see, e.g., Refs. [6,7]).

Disordered quantum spin chains are a paradigmatic class of materials to study these phenomena, both in theory and in experiment. Theoretically, they have been attacked by strong-disorder renormalization group (SDRG) methods [8,9] that give asymptotically exact results for a number of one-dimensional systems. The ground state of the antiferromagnetic spin-1/2 random quantum Heisenberg chain is an exotic random-singlet state controlled by an infinite-randomness renormalization group fixed point [10]. Similarly, the ferromagnetic-antiferromagnetic quantum phase transition of the random transverse-field Ising chain is of unconventional infinite-randomness type and accompanied by power-law quantum Griffiths singularities [2]. Some of these phenomena have been observed in early experiments on organic crystals [11,12] and more recently in MgTiOBO$_3$ [13].

In this paper we investigate the random quantum Ashkin-Teller model, a prototypical disordered spin chain (or ladder) that can be understood as two coupled random quantum Ising chains. In addition to quantum spin systems, versions of the Ashkin-Teller model are used to describe layers of atoms absorbed on surfaces [14], current loops in high-$T_c$ superconductors [15], as well as the elastic response of DNA molecules [16].

We explore the ground-state phase diagram of the random quantum Ashkin-Teller chain by a generalization of the SDRG technique. In addition to the conventional paramagnetic and ferromagnetic phases, we identify an unconventional partially ordered phase characterized by finite but strong randomness and infinite coupling between the two constituent Ising chains (see Fig. 1). It plays the role of a “double Griffiths” phase for two separate quantum phase transitions both of which are of infinite-randomness type. The two-phase region at weak coupling and the three-phase region at strong coupling are separated by a distinct infinite-randomness multicritical point.

The remainder of this paper is organized as follows. We introduce the random quantum Ashkin-Teller model in Sec. II. The SDRG method is developed in Sec. III. Section IV is devoted to the resulting ground-state phase diagram and the properties of the quantum phase transitions between the different phases. In the concluding Sec. V, we discuss generalizations of our results to higher dimensions as well as connections to other random quantum systems.

II. RANDOM QUANTUM ASHKIN-TELLER MODEL

The Hamiltonian of the one-dimensional random quantum Ashkin-Teller model is given by [17–19]

\[
H = - \sum_{a=1}^{2} \sum_{i} \left( J_i S_{a,i}^x S_{a,i+1}^x + h_i S_{a,i}^z \right) - \sum_{i} \left( K_i S_{i,j+1}^z S_{i,j+2}^z + g_i S_{i,j}^x S_{i,j+1}^y \right),
\]

(1)

where $S^x$ and $S^z$ denote the usual Pauli matrices. The model can be understood as two identical random transverse-field Ising chains [first line of Eq. (1)], coupled via their energy densities [second line of Eq. (1)]. The index $\alpha = 1,2$ that distinguishes the two chains is often called the color index. The strength of the coupling between the colors can be parametrized by the ratios $\epsilon_{h,i} = g_i/h_i$ and $\epsilon_{J,i} = K_i/J_i$.

Note that the Hamiltonian (1) is invariant under the duality transformation: $S_{a,i}^- \rightarrow \tau_{a,i}^-$, $S_{a,i}^x \rightarrow \tau_{a,i}^x$, $J_i \equiv h_i$, and $\epsilon_{J,i} \equiv \epsilon_{h,i}$, where $\tau^x$ and $\tau^z$ are the dual Pauli operators.
We take the interactions $J_i$ and transverse fields $h_i$ to be independent random variables. Without loss of generality, we can assume the $J_i$ and $h_i$ to be positive as possible negative signs can be absorbed by local transformations of the spin variables. For now, we assume the (bare) coupling strengths to be uniform, $\epsilon_{h,i} = \epsilon_{J,i} = \epsilon_f \geq 0$ [20]. Effects of random $\epsilon$ are discussed later in Sec. IV C.

The behavior of the random quantum Ashkin-Teller chain (1) in the weak-coupling regime, $\epsilon < \epsilon_c = 1$, has been studied in Refs. [19,21]. In the following, we therefore focus on the strong-coupling case, $\epsilon \geq \epsilon_c = 1$, where these results do not apply. For strong coupling, the terms in the second line of Eq. (1) dominate. It is thus convenient to introduce the product $S_{1,i}^z S_{2,i}^z$ as a new variable. We define

$$\sigma_i^z = S_{1,i}^z S_{2,i}^z, \quad \eta_i^z = S_{1,i}^z, \quad \sigma_i^z \eta_i^z = S_{2,i}^z. \quad (2)$$

The mapping of the Pauli matrices $S_{1,i}^z$ and $S_{2,i}^z$ can be easily worked out by exploring their action on a complete set of basis states in the four-dimensional single-site Hilbert space. This gives

$$\sigma_i^z = S_{2,i}^z, \quad \eta_i^z = S_{1,i}^z S_{2,i}^z, \quad \sigma_i^z \eta_i^z = S_{1,i}^z. \quad (3)$$

Using these transformations, the Hamiltonian (1) can be rewritten as

$$H = -\sum_i (K_i \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z) - \sum_i (J_i \eta_i^z \eta_{i+1}^z + g_i \eta_i^z)$$

$$- \sum_i (J_i \sigma_i^z \sigma_{i+1}^z \eta_i^z \eta_{i+1}^z + h_i \sigma_i^z \eta_i^z). \quad (8)$$

This form immediately gives an intuitive physical picture of the strong-coupling regime $\epsilon \gg 1$ close to self-duality, $h_{2p} \approx J_{2p}$, i.e., close to the horizontal line $\delta = 0$ in Fig. 1. Here, the typical values of the fields and interactions are defined as $\ln h_{2p} = (\ln h)$ and $\ln J_{2p} = (\ln J)$ where $\langle \cdots \rangle$ denotes the disorder average. The behavior of the product variable $\sigma_i$ is dominated by the four-spin interactions $K_i$ while the behavior of the variable $\eta_i$ which traces the original spins is dominated by the two-spin transverse fields $g_i$. Moreover, the coupling terms between $\sigma$ and $\eta$ are weak. Thus, we expect the system to be in a phase in which the product variables $\sigma_i$ develop long-range order while the spins remain disordered.

III. STRONG-DISORDER RENORMALIZATION GROUP

To confirm this intuitive picture and to work out the properties of the product phase and its transitions, we now develop a strong-coupling SDRG. The basic idea of any SDRG consists in identifying the largest local energy scale and perturbatively integrating out the corresponding high-energy degree of freedom. As the random quantum Ashkin-Teller model contains four competing local energies $J_i, K_i, h_i, g_i$, rather than the usual two, we need to generalize the renormalization group (RG) scheme by also considering the second-largest energy in a local cluster. Details of this calculation are outlined in Appendix A. In the strong-coupling regime, $\epsilon > 1$, there are four possible SDRG steps:

(a) If the largest energy in the system is the two-spin field $g_i$, and the second largest energy in the local cluster of sites $i - 1, i$ and $i + 1$ is a four-spin interaction, say $K_i$, the SDRG step decimates the variable $\eta_i$ but merges $\sigma_i$ and $\sigma_{i+1}$ to a new cluster $\tilde{\sigma}_i$. The unperturbed Hamiltonian for this SDRG step reads $H_0 = -K_i \sigma_i^z \sigma_{i+1}^z - g_i \eta_i^z$. We now keep only the ground state of $H_0$ and treat all other terms that contain $\sigma_i, \sigma_{i+1}$ or $\eta_i$ in second-order perturbation theory. The resulting Hamiltonian has the same form as Eq. (8) with one fewer site and renormalized energies arranged as shown in Fig. 2:

$$\tilde{J}_{\text{eff}} = \frac{2J_{i-1,i}}{g_i}, \quad \tilde{h}_{\text{eff}} = \frac{2h_i h_{i+1}}{K_i}. \quad (9)$$

As $\tilde{J}_{\text{eff}}$ and $\tilde{h}_{\text{eff}}$ are renormalized downward while all remaining $K_i$ and $g_i$ are unchanged, the coupling strengths $\epsilon_{h,i}$ and $\epsilon_{J,i}$ increase under renormalization.

(b) The same SDRG step is carried out if the largest energy is the four-spin interaction $K_i$ and the second-largest energy in the cluster of sites $i$ and $i + 1$ is a two-spin field, say $g_i$.

(c) If the largest energy in the system is the two-spin field $g_i$, and the second largest energy in the local cluster of sites $i - 1, i$ and $i + 1$ is the field $h_i$, both $\sigma_i$ and $\eta_i$ are decimated. This is equivalent to decimating both original spins $S_{1,i}$ and

FIG. 1. (Color online) Schematic ground-state phase diagram of the random quantum Ashkin-Teller chain. For $\epsilon < 1$, the paramagnetic and ferromagnetic phases are connected by a direct continuous quantum phase transition. For $\epsilon > 1$, they are separated by a partially ordered “product” phase characterized by strong randomness and renormalization group flow towards infinite coupling. The flow is indicated by arrows on the principal axis, $\delta = 0$ and $\epsilon = 1$, of the multicritical point (MCP).
thus behaves analogously to the random transverse-field Ising chain [1,2]: A paramagnetic phase at large transverse fields $h_i$ and a ferromagnetic phase at large interactions $J_i$ are directly connected by an infinite-randomness critical point at $\delta = \ln h_{\text{typ}} - \ln J_{\text{typ}} = 0$ (transition 1 in Fig. 1) [22].

To understand the strong-coupling regime $\epsilon > 1$, let us first focus on the self-duality line $\delta = \ln h_{\text{typ}} - \ln J_{\text{typ}} = 0$. If the bare $\epsilon_i$ is just slightly above 1, most of the recursions will initially be site and bond decimations [types (c) and (d)]. In these steps, the local coupling strengths $\epsilon_{h,i}$ and $\epsilon_{J,i}$ rapidly increase. When they become larger than the widths of the $J$ and $h$ distributions, the character of the SDRG changes. Now, most steps are “mixed steps” of types (a) and (b). As a result, the product variable $\sigma$ forms larger and larger clusters while the spin variable $\eta$ is decimated.

The system is thus in a “double Griffiths phase”: The $\sigma$ part of the Hamiltonian behaves analogously to an ordered Griffiths phase while the $\eta$ part behaves as in a disordered Griffiths phase. This double Griffiths phase has a nonzero product order parameter or polarization $M_p = \sum \sigma_j^2$ while the spin variable $\eta_j^2$ (and thus $S_j^z$ and $S_j^z$) remains disordered, $M = \sum \eta_j^2 = 0$. Note that this behavior is valid not just on the self-duality line, $\delta = 0$, but also in its vicinity because the RG flow of each of the variables $\sigma$ and $\eta$ is dominated by a single term in the Hamiltonian and does not rely on the balance between interactions and transverse fields. Thus, we have indeed discovered a bulk phase rather than a special line in the phase diagram. Moreover, as the $\epsilon_{h,i}$ and $\epsilon_{J,i}$ flow to infinity, the analysis is asymptotically exact.

To find the extensions of the partially ordered double Griffiths phase we need to locate its transitions to the conventional paramagnetic and ferromagnetic phases. Looking at the first sum in the Hamiltonian (8), it is clear that the long-range order of the product variable will be destroyed if we raise $\delta = \ln h_{\text{typ}} - \ln J_{\text{typ}}$ until the transverse fields $h_i$ compete with the four-spin interactions $J_i$. This leads to a competition between the SDRG steps (a) and (c). From comparing the $h$ recursion in Eqs. (9) with the $K$ recursion in Eqs. (10), we conclude that the phase boundary between the double Griffiths phase and the paramagnetic phase (transition 2 in Fig. 1) is located at $K_{\text{typ}} = 2h_{\text{typ}}$ or equivalently $\delta_c = \ln(\epsilon/2)$ in the limit of large $\epsilon$. Moreover, the transition is governed by an infinite-randomness fixed point in the random transverse-field Ising universality class. The phase boundary to the ferromagnetic phase (transition 3 in Fig. 1) can be found analogously. For large $\epsilon$, it is located at $2J_{\text{typ}} = g_{\text{typ}}$ or equivalently $\delta_c = \ln(2/\epsilon)$ in agreement with the self-duality of the Hamiltonian.

The thermodynamics of the double Griffiths phase is highly unusual. It can be found in the usual way, i.e., by including conjugate fields in the SDRG. Each of the two order parameters, the magnetization $M = \sum \eta_i$ and the polarization $M_p = \sum \sigma_j^2$, displays power-law quantum Griffiths singularities controlled by different Griffiths dynamical exponents $\zeta_m$ and $\zeta_p$, respectively, that vary nonuniversally with $\epsilon$ and $\delta$. The exponent $\zeta_m$ diverges at the transition to the ferromagnetic phase while $\zeta_p$ diverges at the transition to the paramagnetic phase. Duality imposes the relation $\zeta_p(\epsilon, \delta) = \zeta_m(\epsilon, -\delta)$. Thermal quantities such as the entropy and the specific heat pick up contributions from both order parameters. Their Griffiths dynamical exponent $z = \max(\zeta_m, \zeta_p)$ thus displays

IV. PHASE DIAGRAM AND PHASE TRANSITIONS

A. Double Griffiths phase

Based on the SDRG recursions (9)–(11), the phase diagram of the random quantum Ashkin-Teller model shown in Fig. 1 is easily understood. Let us start by recalling that in the weak-coupling regime, $\epsilon < 1$, the local coupling strengths $\epsilon_{h,i}$ and $\epsilon_{J,i}$ decrease without limit under renormalization [19,21]. This implies that the two Ising chains that make up the Ashkin-Teller model decouple in the low-energy limit. Our system

![Diagram](image-url)
FIG. 3. (Color online) Schematic of the Griffiths dynamical exponent $z$ as a function of $\delta$ for fixed $\epsilon > 1$. Rare large magnetization clusters lead to Griffiths singularities associated with the exponent $z_m$ (green dashed line) while the singularities stemming from rare polarization clusters are associated with $z_p$ (red dash-dotted line). Thermal quantities are dominated by the larger of the two exponents $z_p$ (red solid line) or $z_m$ (green solid line), as sketched in Fig. 3.

B. Multicritical point

Finally, we consider the infinite-randomness multicritical point (MCP) located at $\epsilon = 1, \delta = 0$. It has two independent unstable directions, the lines $\delta = 0$ and $\epsilon = 1$. On the line $\epsilon = 1$ that separates the weak-coupling and strong-coupling regimes, all SDRG steps are site and bond decimations [types (c) or (d)]. The recursions (10) and (11) reduce to the well-known transverse-field Ising forms $J_{i\ell} = J_{i-1}/J_i$ and $h_{i\ell} = h_i h_{i+1}/(2J_i)$ [23]. The SDRG flow of the $J$ and $h$ distributions on the line $\epsilon = 1$ is thus identical to the corresponding flow of the random-transverse-field Ising chain. We emphasize, however, that although the SDRG flow of the $J$ and $h$ distributions at $\epsilon = 1$ is identical to the weak-coupling regime $\epsilon < 1$, the fixed-point Hamiltonian differs because the two Ising chains that make up the Ashkin-Teller model do not decouple.

The flow along the line $\epsilon = 1$ can be characterized by the following critical singularities: correlation length $\xi \sim |\delta|^{-\nu}$, magnetization $M \sim |\delta|^{\beta}$, and correlation time $\ln \xi_t \sim \xi^{\phi}$ with exponents

$$\nu = 2, \quad \beta = 2 - (1 + \sqrt{5})/2 = 0.382, \quad \psi = 1/2. \quad (12)$$

In contrast, the SDRG flow on the self-duality line $\delta = 0$ for $\epsilon > 1$ close to the MCP is determined by the evolution of $\epsilon$ under repeated site and bond decimations [steps (c) and (d)]. It can be worked out (see Appendix B) by including $\ln (\epsilon)$ as an auxiliary variable in the SDRG flow of the $J$ and $h$ distributions. We find different critical singularities $\xi \sim (\epsilon - 1)^{-\nu_c}$ and $M_p \sim (\epsilon - 1)^{\beta_c}$ with exponents

$$\nu_c = \frac{8}{1 + \sqrt{7}} = 2.194, \quad \beta_c = \frac{6 - 2\sqrt{5}}{1 + \sqrt{7}} = 0.419. \quad (13)$$

The tunneling exponent $\psi$ remains 1/2. Combining these results to write a scaling form of the polarization gives

$$M_p(\delta, \epsilon - 1) = b^{-\beta_c/\nu_c} M_p(\delta b^{1/\nu_c}, \epsilon - 1 b^{1/\nu}), \quad (14)$$

where $b$ is an arbitrary scale factor. The phase transition between the partially ordered and paramagnetic phases corresponds to a singularity of $M_p$ for $\delta > 0$ and $\epsilon > 1$. Using Eq. (14), we find that the phase boundary behaves as

$$\delta_c \sim (\epsilon - 1)^{\nu_c/\nu} = (\epsilon - 1)^{4/5(1 + \sqrt{7})} = (\epsilon - 1)^{1.097} \quad (15)$$

sufficiently close to the multicritical point. The phase boundary to the ferromagnetic phase can be found analogously.

C. Random coupling strength $\epsilon$

So far, we have considered systems in which the (bare) coupling strengths are uniform $\epsilon_{f,i} = \epsilon_{h,i} = \epsilon$. In the present section, we discuss what changes for random coupling strengths.

If all $\epsilon_{f,i}$ and $\epsilon_{h,i}$ are below the multicritical value of 1, the renormalized values $\bar{\epsilon}$ are also smaller than 1 and decrease under renormalization. Thus, the two Ising chains that make up the Ashkin-Teller model decouple in the low-energy limit, just as in the case of uniform bare $\epsilon$. Conversely, if all $\epsilon_{f,i}$ and $\epsilon_{h,i}$ are above the multicritical value of 1, the renormalized values $\bar{\epsilon}$ are also larger than 1 and increase under renormalization. The system thus flows to the strong-coupling region, also just as in the case of uniform bare $\epsilon$. Consequently, none of our results changes in these two cases, except for unimportant modifications of nonuniversal quantities. This also implies that the three bulk phases shown in Fig. 1 are stable against weak randomness in $\epsilon$. The same holds for the phase transitions (1), (2), and (3) sufficiently far away from the multicritical point discussed in Sec. IV B.

In contrast, the multicritical point at $\delta = 0, \epsilon = 1$ itself is unstable against weak disorder in the $\epsilon_{f,i}$ and $\epsilon_{h,i}$. To show this, we analyze how the width of a narrow $\epsilon$ distribution around $\epsilon = 1$ flows under repeated SDRG site and bond decimations. By including $\ln \epsilon$ as an auxiliary variable in the SDRG and using the methods of Ref. [10], we find

$$\sigma_{\ln \epsilon} \sim \sigma_{\bar{\epsilon}}^{\phi_{\text{sym}}(1/2)} \bar{\epsilon} \sim \phi_{\text{sym}}(1/2) \sim \left(1 + \sqrt{6} \right)^{1/4}. \quad (16)$$

(Note that we need to consider the flow of a symmetrically distributed auxiliary variable; the exponent is therefore denoted as $\phi_{\text{sym}}(1/2)$.) This means that a narrow bare distribution broadens under the SDRG, destabilizing the uniform-$\epsilon$ multicritical point of Sec. IV B.

We have not found an analytic solution of the multicritical behavior in the case of random $\epsilon_{f,i}$ and $\epsilon_{h,i}$. Instead, we implement the SDRG numerically. We study systems with up to $5 \times 10^8$ sites. To place the system on the self-duality line $\delta = \ln h_{\text{typ}} - \ln J_{\text{typ}} = 0$, we employ identical power-law distributions $P_J(J) = J^{-1+1/\nu}/\nu!$ and $P_h(h) = h^{-1+1/\nu}/\nu!$ for the interactions and transverse fields, with $\nu$ being a measure of the disorder. The coupling strengths $\ln \epsilon$ are drawn from a box distribution between $\ln \epsilon_{\text{min}}$ and $\ln \epsilon_{\text{max}}$. The results of a strongly poisoned system ($\nu = 2000$) sample system are summarized in Figs. 4 and 5. We fix $\ln \epsilon_{\text{min}} = -1000$ and tune the multicritical point by varying $\ln \epsilon_{\text{max}}$. The data are averages over 50 different chains of $5 \times 10^7$ sites each. Figure 4 shows how the average $\langle \ln \epsilon \rangle$ and standard deviation $\sigma_{\ln \epsilon}$ of the coupling strength evolve under the SDRG. From the inset, we determine the multicritical point to be located...
versus the SDRG length scale $\ln \epsilon$ parameter $\ln \epsilon$. Interestingly, it is not very slow.

The value of the exponent $\kappa$ (Ref. [24]) fulfills the constraint $\kappa < 1$ derived by Fisher [10]. Interestingly, it is not very different from the value $\phi_{1/2}^{(\text{sym})} \approx 0.8624$ that describes the initial increase of $\sigma_{ln \epsilon}$ near the uniform-$\epsilon$ multicritical point.

In Fig. 5, we study how the distance $|\ln(\epsilon) - \ln \epsilon_c|$ from the multicritical point increases with SDRG length scale $\ell$ in the regime $|\ln(\epsilon) - \ln \epsilon_c| < \sigma_{ln \epsilon}$.

The exponent $\psi(\kappa + \lambda)$ is somewhat harder to determine in the weak-disorder case because the available fit range becomes very narrow. We find $\psi(\kappa + \lambda) = 0.88(4)$ in agreement with the strong-disorder value. Further calculations for even weaker disorder and shorter chains (between $1 \times 10^6$ and $5 \times 10^7$ sites) are less precise but compatible with the values given above.

Once $|\ln(\epsilon) - \ln \epsilon_c| > \sigma_{ln \epsilon}$, almost all $\epsilon$ are on the same side of the multicritical point. The further analysis therefore follows the steps outlined in Appendix B. The resulting multicritical behavior along the self-duality line on the strong-coupling side of the MCP is characterized by the power laws $\xi \sim ((\ln(\epsilon) - \ln \epsilon_c)^{-\nu_\epsilon} )$ and $M_p \sim ((\ln(\epsilon) - \ln \epsilon_c)^{\beta_\epsilon}$ with exponents

$$\nu_\epsilon = \frac{4 - 2\kappa}{\lambda} = 2.48(15),$$
$$\beta_\epsilon = \frac{2 - \kappa}{\lambda} (2 - \phi_0) = 0.474(20).$$

The shape of the phase boundary close to the multicritical point can be found as in Sec. IV B yielding $\delta_c \sim (\ln(\epsilon) - \ln \epsilon_c)^{1.24}$.

V. DISCUSSION AND CONCLUSIONS

We have performed analogous calculations for a number of different parameter sets. For the weaker disorder case of $w = 3$ and $\ln \epsilon_{\text{min}} = -3$, the multicritical point is located at $\ln \epsilon_{\text{max}} \approx 2.424$. In this case, our analysis of 180 chains of $5 \times 10^8$ sites gives the same value as above, $\kappa \psi = 0.434(3)$.

In summary, we have investigated the ground-state phase diagram of the random quantum Ashkin-Teller spin chain. The topology of the phase diagram, shown in Fig. 1, is analogous to that of the clean quantum Ashkin-Teller model (see, e.g., Ref. [26]). However, the properties of the phases and phase transitions are different. In addition to the usual paramagnetic and ferromagnetic phases, we have identified a partially ordered phase characterized by strong randomness and infinite coupling between the colors. This phase acts as a Griffiths phase for two distinct quantum phase transitions leading to an unconventional nonmonotonic variation of the Griffiths dynamical exponent throughout the phase.

We now turn our attention to the phase boundaries between the three phases. The direct transition at weak intercolor coupling between the paramagnetic and ferromagnetic (Baxter) phases [transition (i) in Fig. 1] is in the infinite-randomness universality class of the random transverse-field Ising chain, as was already found in Refs. [19,21]. In contrast, the corresponding phase boundary in the clean quantum Ashkin-Teller chain shows an unusual line of fixed points with continuously varying exponents [18,27]. The quantum phase transitions separating the partially ordered phase from the paramagnetic and ferromagnetic phases [transitions (ii) and (iii)] are also of infinite-randomness type and in the universality class of the random transverse-field Ising chain, while they are in the $(1 + 1)$-dimensional Ising universality class in the clean model.

We have also studied the quantum multicritical point separating the two-phase and three-phase regions. It is in one of two different universality classes (both of infinite-randomness type), depending on whether the intercolor coupling strengths...
$\epsilon$ are uniform or random. This differs from the infinite-order multicritical behavior seen in the clean case [18,27].

Generalizations of the Ashkin-Teller Hamiltonian (1) to $n > 2$ colors have recently reattracted considerable attention because they have been used to analyze the fate of first-order quantum phase transitions under the influence of disorder [21,28,29]. Interestingly, for $n > 4$ colors, the paramagnetic and ferromagnetic phases meet directly at the self-dual line \[\langle h \rangle = \langle J \rangle \] for all coupling strengths $\epsilon \geq 0$. Thus, an analog to the partially ordered strong-coupling phase does not exist. For three and four colors, this question is not yet solved to the best of our knowledge.

The random quantum Ashkin-Teller chain (1) with $N$ sites can be mapped onto a random XXZ quantum spin chain with $2N$ sites [30]. Under this mapping, the transverse fields $h_i$ in the Ashkin-Teller model map onto the even bonds of the XXZ chain while the interactions $J_i$ map onto the odd bonds. The coupling strengths $\epsilon_{i,i+1}$ and $\epsilon_{i,j}$ map onto the local anisotropies of the XXZ chain. Importantly, the mapping is nonlocal as it involves (semi-infinite) chains of operators. Thus, although the energy spectra of the Ashkin-Teller model and the XXZ chain are analogous, their order parameters are not directly related. This explains, for example, why the correlation length exponent $\nu_{\epsilon}$ given in Eq. (13) takes the same value as the exponent that describes the effects of weak anisotropy about the Heisenberg fixed point of the XXZ chain [10]. In contrast, our order parameter exponent $\beta_{\epsilon}$ does not have a direct counterpart in the XXZ chain.

Our study has focused on one space dimension. Let us briefly comment on random quantum Ashkin-Teller models in higher dimensions. The crucial step in our understanding of the strong-coupling regime was the transformation defined in Eqs. (2)–(7) from the original spins to the product variable. This transformation is purely local and can be performed in the same way in any space dimension. We therefore believe that the basic features of the phase diagram in higher dimensions will be similar to the one-dimensional case. In particular, for small $\epsilon$, we expect a direct transition between the ferromagnetic and paramagnetic phases while a partially ordered product phase is expected to intervene between them for large $\epsilon$. Obtaining quantitative results in higher dimensions will be significantly more complicated than in one dimension. First, the Hamiltonian is not self-dual in $d > 1$; thus, the phase diagram is not symmetric under the exchange of transverse fields and interactions. Second, the SDRG can only be implemented numerically in $d > 1$ because the decimation steps change the topology of the lattice. This work remains as a task for the future.

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APPENDIX A: SDRG RECURSION RELATIONS

A single step of the SDRG consists in identifying the largest local energy scale in the Hamiltonian and perturbatively integrating out the corresponding high-energy excitations. This is done using the projection technique described, e.g., in Ref. [31]. The Hilbert space is divided into a low-energy subspace and a high-energy subspace. Any wave function $\psi$ can be decomposed as $\psi = \psi_1 + \psi_2$ with $\psi_1$ in the low-energy subspace and $\psi_2$ in the high-energy subspace. This allows us to write the Schrödinger equation in matrix form, \begin{equation}
\begin{pmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix} = E
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix},
\end{equation}
with $H_{ij} = P_i H P_j$. Here, $P_1$ and $P_2$ project on the low-energy and high-energy subspaces, respectively. Eliminating $\psi_2$ from these two coupled equations gives $H_{12} (E - H_{22})^{-1} H_{21} \psi_1 = E \psi_1$. Thus, the effective Hamiltonian in the low-energy Hilbert space is \begin{equation}
H_{\text{eff}} = H_{11} + H_{12} (E - H_{22})^{-1} H_{21}.
\end{equation}
The second term can now be expanded in inverse powers of the large local energy scale.

The quantum Ashkin-Teller Hamiltonian has four competing local energy scales, viz., $J_i$, $K_i$, $h_i$, and $g_i$, rather than two. We therefore generalize the usual SDRG scheme by considering the largest and second-largest energies in a local cluster to define the SDRG step. In the strong-coupling regime, $\epsilon > 1$, the largest local energy is always either a four-spin interaction or a two-spin field. In total, there are four possible steps.

(a) The largest local energy is a two-spin field $g_i$. The second-largest energy in the three-site cluster of sites $i-1$, $i$, $i+1$ is a four-spin interaction, either $K_{i-1}$ or $K_i$. Let us assume that it is $K_i$ for definiteness. In this case, the low-energy Hilbert space is spanned by states for which $\langle \eta_i, \sigma_i, \sigma_{i+1} \rangle = (\rightarrow, \uparrow, \uparrow, \downarrow)$ or $(\rightarrow, \downarrow, \uparrow, \downarrow)$, $H_{11}$ and $H_{22}$ contain all terms in the Hamiltonian that do not flip the spins $\eta_i, \sigma_i, \sigma_{i+1}$; their leading terms are $-K_i \eta_i^x \sigma_i^x \sigma_{i+1}^x - g_i \eta_i^z$. All terms that flip at least one of the variables $\eta_i, \sigma_i, \sigma_{i+1}$ are contained in $H_{12}$ and $H_{21}$. Specifically, \begin{equation}
H_{12} = P_1 \left[ -J_{i-1} \eta_{i-1}^z \eta_i^z - J_i \eta_i^z \eta_{i+1}^z - h_i \sigma_i^z - h_{i+1} \sigma_{i+1}^z - J_{i-1} \sigma_i^z \sigma_{i+1}^z \eta_{i-1}^z \eta_i^z - J_i \sigma_i^z \sigma_{i+1}^z \eta_i^z \eta_{i+1}^z \right] P_2.
\end{equation}

(b) Exactly the same SDRG step is carried out if the largest local energy is the four-spin interaction $K_i$, and the second-largest energy in the two-site cluster of sites $i$ and $i + 1$ is a two-spin field, either $g_i$ or $g_{i+1}$.

Steps (a) and (b) are the dominant SDRG steps for $\epsilon \gg 1$. More precisely, most steps are of types (a) or (b) if $\epsilon$ is larger
than the width of the $h$ and $J$ distributions (on a logarithmic scale). In the opposite case, strong disorder and not too large $\epsilon$, most SDRG steps are site and bond decimations of types (c) and (d).

The largest energy in the system is the two-spin field $g_i$, and the second-largest energy in the local cluster of sites $i-1, i$, and $i+1$ is the field $h_i$. In this case, the low-energy Hilbert space is spanned by states having $(\eta_i, \sigma_i) = (\downarrow \downarrow \downarrow)$. $H_{11}$ and $H_{22}$ contain all terms in the Hamiltonian that do not flip $\eta_i$ and $\sigma_i$, with the leading terms being $-g_i \eta_i^z - h_i \sigma_i^z - h_i \eta_i^z \sigma_i^z$. All terms that flip $\eta_i$ and/or $\sigma_i$ are part of $H_{12}$ and $H_{21}$. Specifically,

$$H_{12} = P_1 \left[ -K_{i-1} \sigma_{i-1}^z \eta_{i-1}^z - K_i \sigma_i^z \eta_i^z + J_{i-1} \eta_{i-1}^z \sigma_i^z 
- J_i \eta_i^z \eta_{i+1}^z \sigma_{i+1}^z - J_{i+1} \eta_{i+1}^z \sigma_i^z \right]P_2,$$

(A4)

and $H_{21}$ takes the same form but with $P_1$ and $P_2$ exchanged. After inserting this into Eq. (A2) and approximating the denominator $E = H_{12}$ by $-2g_i - 2h_i$ or $-4h_i$ depending on which spins are flipped, site $i$ is eliminated (i.e., both $\sigma_i$ and $\eta_i$ are decimated). The effective interaction energies between the neighboring sites $i-1$ and $i+1$ are given in Eqs. (10).

The largest local energy is a four-spin interaction $K_{ii}$, and the second-largest energy in the cluster consisting of sites $i$ and $i+1$ is the interaction $J_{ii}$. The low-energy Hilbert space is spanned by states having $(\eta_i, \eta_{i+1}, \sigma_i, \sigma_{i+1}) = (\uparrow \uparrow \uparrow \uparrow)$ or $(\downarrow \downarrow \downarrow \downarrow)$ or $(\uparrow \uparrow \downarrow \downarrow)$ or $(\downarrow \downarrow \uparrow \downarrow)$. After projection into the low-energy Hilbert space, the two sites $i$ and $i+1$ can thus be represented by a single site with variables $\sigma_{eff}$ and $\eta_{eff}$. $H_{11}$ and $H_{22}$ contain all terms in the Hamiltonian that do not flip $\eta_i, \eta_{i+1}, \sigma_i$, or $\sigma_{i+1}$. The leading terms are $-K_i \sigma_i^z \eta_i^z - J_i \eta_i^z \eta_{i+1}^z - J_i \sigma_i^z \sigma_{i+1}^z \eta_i^z \eta_{i+1}^z$. In contrast, $H_{12}$ and $H_{21}$ consist of the terms that flip $\eta_i, \eta_{i+1}, \sigma_i$, and/or $\sigma_{i+1}$. This gives

$$H_{12} = P_1 \left[ -h_i \sigma_i^z - h_i \sigma_{i+1}^z - g_i \eta_i^z - g_i \eta_{i+1} \eta_i^z \sigma_i^z - h_i \eta_i^z \sigma_i^z - h_i \eta_{i+1} \eta_i^z \sigma_{i+1}^z \right]P_2.$$

(A5)

Inserting this into the effective Hamiltonian (A2) as before yields the transverse field $h_{eff}$ and two-spin field $g_{eff}$ acting on the cluster variables $\sigma_{eff}$ and $\eta_{eff}$. Their values are given in Eqs. (11).

Note that the SDRG steps (c) and (d) are identical to the site and bond decimations employed in the weak-coupling ($\epsilon < 1$) analysis of Refs. [19,21].

**APPENDIX B: MULTICRITICAL POINT**

The multicritical point separating the two-phase and three-phase regions is located at $\delta = 0, \epsilon = 1$. In this appendix, we sketch the derivation of the SDRG flow on the self-duality line $\delta = 0$ for $\epsilon > 1$ but close to the multicritical point.

Let us begin with a qualitative discussion. For $\epsilon \approx 1$ and strong disorder, initially almost all SDRG steps are site decimations (c) or bond decimations (d); thus, the RG flow is identical to that of the random transverse-field Ising chain. Under these steps, $\epsilon$ increases rapidly. When the typical $\ln \epsilon$ reaches the width of the In $J$ and In $h$ distributions, the character of the SDRG flow changes. Now, most steps are “mixed” decimations of types (a) and (b). Under these steps, the magnetization rapidly drops to zero while the polarization (product order parameter) stops decreasing and reaches a nonzero asymptotic value. Thus, the RG scale at which $\ln \epsilon$ reaches the width of the ln $J$ and ln $h$ distributions determines the correlation length and the polarization.

For a quantitative analysis of this SDRG flow, we start from the recursion relations for the coupling strengths $\epsilon_{J,i}$ and $\epsilon_{h,i}$ defined by Eqs. (10) and (11). Written in terms of logarithms, they read

$$\ln \epsilon_h = \ln \epsilon_{h,i} + \ln \epsilon_{h,i+1} + \ln [(1 + \epsilon_{J,i})/2].$$

(B1)

$$\ln \epsilon_J = \ln \epsilon_{J,i-1} + \ln \epsilon_{J,i} + \ln [(1 + \epsilon_{h,i})/2].$$

(B2)

We follow the $\epsilon$ flow from $\ln \epsilon \ll 1$ to $\ln \epsilon \sim P_0^{-1}, R_0^{-1}$ where $P_0$ and $R_0$ are the inverse widths of the ln $J$ and ln $h$ distributions. Two regimes need to be distinguished, $\ln \epsilon < 1$ and $\ln \epsilon > 1$.

For $\ln \epsilon < 1$, we expand in $\delta^{(e)} = \ln \epsilon \approx \epsilon - 1$, and Eqs. (B1) and (B2) simplify to

$$\delta_{h,i}^{(e)} = \delta_{h,i+1}^{(e)} + \frac{1}{2} \delta_{J,i},$$

(B3)

$$\delta_{J,i}^{(e)} = \delta_{J,i-1}^{(e)} + \frac{\delta_{h,i}^{(e)}}{P_2}. (B4)$$

The recursions can be understood as special cases of the general recursion $x_i = x_{i-1} + x_{i+1} + Y_i$ with $Y = 1/2$ [32]. The flow of variables governed by such recursions close to the infinite-randomness fixed point (of the random transverse-field Ising chain) studied in detail by Fisher [10]. He found that the typical $\epsilon$ scale $\Gamma^{(e)} = \ln [\Omega_2/\Omega_1]^{(e)}$ with decreasing SDRG energy scale $\Omega$. The exponent $\phi_\Omega$ is given by $\phi_\Omega = [1 + (5 + 4Y)^{-1}]^{1/2}$. [In contrast to Eqs. (16), we need to use the “asymmetric” version of Fisher’s results because all our $\delta^{(e)} > 0$.] Thus, in the first regime ($\ln \epsilon < 1$), the typical $\ln \epsilon$ scales as

$$\ln \epsilon_{typ} \approx \Gamma^{(e)}: \ln \epsilon_0, \quad \phi_\Omega/2 = \frac{1}{2} \left(1 + \sqrt{7}\right). (B5)$$

In the second regime, $\ln \epsilon > 1$, we can approximate the recursions (B1) and (B2) for $\delta^{(e)} = \ln \epsilon$ by

$$\delta_{h,i}^{(e)} = \delta_{h,i+1}^{(e)} + \frac{1}{2} \delta_{J,i},$$

(B6)

$$\delta_{J,i}^{(e)} = \delta_{J,i-1}^{(e)} + \delta_{h,i}^{(e)}. (B7)$$

These recursions are of the same type as Eqs. (B3) and (B4), but with $Y = 1$. Thus, in the second regime, $\ln \epsilon$ scales as

$$\ln \epsilon_{typ} \sim \Gamma^{(e)}, \quad \phi_\Omega = 2. (B8)$$

To test the predictions (B5) and (B8), we implemented the strong-disorder renormalization group numerically. Figure 6 shows (ln $\epsilon_{typ}$)$^{1/2}$ as a function of $\Gamma$ for systems located on the self-duality line $h_{typ} = J_{typ}$. We employed identical power-law distributions $P_1(J) = J^{-1/\omega}/\omega$ and $P_2(h) = h^{-1/\omega}/\omega$ for the interactions ($0 < J < 1$) and transverse fields ($0 < h < 1$), with $\omega$ being a measure of the disorder. The coupling strength is uniform and close to the multicritical value $\epsilon_1 = 1$. The figure shows that the data in the range ($\ln \epsilon > 1$) lie on straight lines; i.e., they follow Eqs. (B8) as predicted. For ($\ln \epsilon > 1$) the data curve downward, suggesting a smaller
Let us now combine the two regimes. We consider a (bare) system close to the multicritical point, $0 < \ln \epsilon_I \ll 1$, with strong initial disorder; i.e., the widths of the bare distributions of $\ln J$ and $\ln h$ are large, $P_J^{-1} \approx R_J^{-1} \gg 1$. Under repeated site and bond decimations [SDRG steps (c) and (d)], $P_0^{-1} = R_0^{-1} = P_J^{-1} \Gamma$ while the typical $\ln$ scales as $\ln \epsilon \sim \Gamma^2(\ln \epsilon_I)^{\phi_{1/2}}$ once $\ln \epsilon > 1$. Setting $\ln \epsilon = P_0^{-1}$ gives the crossover SDRG scale

$$\Gamma_x = \frac{1}{P_I(\ln \epsilon_I)^{2/\phi_{1/2}}}.$$  

The correlation length is given by the length scale corresponding to $\Gamma_x$,

$$\xi \sim \xi_x \sim (\ln \epsilon_I)^{-4/\phi_{1/2}} \approx (\epsilon_I - 1)^{-4/\phi_{1/2}}.$$  

The correlation length exponent $\nu_x$ thus takes the value $\nu_x = 4/\phi_{1/2}$ as given in Eqs. (13). The product order parameter (polarization) $M_p$ can be found by noting that $\sigma$ clusters are not decimated anymore once $\Gamma > \Gamma_x$. $M_p$ is thus given by its value at $\Gamma_x$,

$$M_p = n(\Gamma_x) \mu(\Gamma_x) \sim \xi_x^{-2+\phi_0} \sim (\ln \epsilon_I)^{2(2-\phi_0)/\phi_{1/2}}.$$  

where $n(\Gamma')$ and $\mu(\Gamma')$ are the number and moment of clusters surviving at SDRG scale $\Gamma$. Using $\phi_0 = [1 + \sqrt{5}]/2$ yields the order parameter exponent $\beta_\epsilon$ given in Eqs. (13).

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