Superstring Gravitational Wave Backgrounds with Spacetime Supersymmetry

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ABSTRACT

We analyse the stringy gravitational wave background based on the current algebra $E_8$. We determine its exact spectrum and construct the modular invariant vacuum energy. The corresponding $N=1$ extension is also constructed. The algebra is again mapped to free bosons and fermions and we show that this background has $N=4$ ($N=2$) unbroken spacetime supersymmetry in the type II (heterotic case).

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Gravitational wave-like backgrounds [1] have special features which render them interesting testing grounds of our understanding of stringy gravity. This subject has been revived recently, [3]-[7, 8] with focus on current algebra generated gravitational wave backgrounds.

In this paper, we will discuss an exact classical solution to string theory found by Nappi and Witten [2], which can be interpreted as a plane gravitational (as well as $B_{\mu\nu}$) wave. It has large symmetry (seven Killing symmetries) and the corresponding 2-d $\sigma$-model is the WZW model for a non semi-simple group, $E_6^c$, the central extension of the 2-d Euclidean group. This model is interesting because it provides us with an exact classical solution to string theory which has at the same time:

- a simple and clear physical interpretation,
- a non-trivial spacetime,
- it can be solved,
- it has unusual features, in particular the spacetime is nowhere asymptotically flat.

Thus, a priori, it is not obvious that a sensible scattering matrix exists in such a background.

In [3] we began solving the model. In particular, among other things, we mapped the current algebra and the representation theory to (almost) free fields. In this note we present a continuation of our effort to understand the physics of the model.

We are interested in considering this background as an exact classical solution to superstring theory. Thus we will supersymmetrize the current algebra and we will again map the model to free bosons and fermions. By studying the $\sigma$-model we will show that the model has a $N = 4$ worldsheet supersymmetry. The associated string theory has a large unbroken spacetime supersymmetry. For the type II string this is $N=4$ spacetime supersymmetry. In fact, this type of 4-d background turns out to be a special case of the class of solutions found recently in [9]. Moreover, we will explicitly construct the $N=4$ superconformal algebra out of the (super) current algebra.

Some of the features though of the model like the potential spectrum and and tree scattering of bosonic states have little difference between the bosonic and the supersymmetric model. Thus we will find the exact spectrum of the associated bosonic string theory and we will compute the (modular invariant) vacuum energy. There are two types of states in the theory, corresponding to different kinds of representations of the current algebra. We will qualitatively describe their scattering.

We will start our discussion by presenting the current algebra symmetry of the background [2]. This is specified by the OPEs

$$J_a(z)J_b(w) = \frac{G_{ab}}{(z-w)^2} + f_{ac}^b \frac{J_c(w)}{(z-w)} + \text{regular}, \quad (1)$$

where $(J_1, J_2, J_3, J_4) \sim (P_1, P_2, J, T)$, the $P_i$ generating the translations and $J$ being the generator of rotations and $T$ being the central element. The only non-zero structure
constants are \( f_{31}^2 = 1 \), \( f_{32}^1 = -1 \) and \( f_{12}^4 = 1 \); \( G_{ab} \) is an invariant bilinear form (metric) of the algebra. \( G \) is a symmetric matrix and the Jacobi identities constrain it so that \( f_{abc} \equiv f_{ab}{}^d G_{dc} \) is completely antisymmetric. The most general invariant bilinear form for \( E_2^c \) is

\[
G = k \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & b & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

where we can assume without loss of generality that \( k > 0, b > 0 \). For \( \hat{E}_2^c \) there is a unique solution to the Master Equation \([10]\), which has all the properties of the Affine Sugawara construction. It is given by:

\[
L_{\text{AS}} = \frac{1}{2k} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -b + \frac{1}{k} \\
0 & 0 & 1 & 0
\end{pmatrix} = \frac{1}{2} G^{-1} + \frac{1}{2k^2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

The \( \sigma \)-model realizing the current algebra above can be constructed in a straightforward fashion, and provides the tool to study string propagation. Its action is \([2]\)

\[
S = \frac{k}{2\pi} \int d^2z \left[ \partial a_i \bar{\partial} a_i + \partial u \bar{\partial} v + \bar{\partial} u \partial v + b \bar{\partial} u \partial u - \varepsilon^{ij} a_i \bar{\partial} a_j \partial u \right]
\]

From now on we will set \( b = 0 \) by an appropriate shift in \( v \). As was shown in \([3]\), from the exact current algebra point of view, once \( b \) is finite it can always rescaled away (assuming that the light-cone coordinates are non-compact). We can also scale \( k \to 1 \) by appropriate scaling of \( a_i, v \). In this coordinate system, the nature of the background is not obvious, however as we will show below, it is in this background that the free field representation of the algebra described in \([3]\) is manifest. It should be also noted that in (4) the last term describes the departure of the background from flat Minkowski space, and its coupling can be made arbitrarily small by a boost of the \( u, v \) coordinates. However, this does not imply that the perturbation is insignificant since the perturbing operator has bad IR behaviour. It should also be mentioned here that the action (4) can be obtained by an \( O(3,3,R) \) transformation of flat space, \([3, 4]\).

The coordinate system where the nature of the background is more transparent is given by \([2]\)

\[
a_1 = x_1 + \cos ux_2, \quad a_2 = \sin ux_2, \quad v \to v + \frac{1}{2} x_1 x_2 \sin u
\]

where the action (4) becomes

\[
S = \frac{1}{2\pi} \int d^2z \left[ \partial x_1 \bar{\partial} x_1 + 2 \cos u \partial x_2 \bar{\partial} x_1 + \partial u \bar{\partial} v + \bar{\partial} u \partial v \right].
\]

In this form we can immediately identify the perturbation from flat space with a graviton+antisymmetric tensor mode given by \( \cos u \partial x_2 \bar{\partial} x_1 \). This is an operator with no IR

\[\text{Effectively we are setting the parameter } Q \text{ of } [3] \text{ to one.}\]
problems, however its coupling is of order one and cannot be rescaled at will. One can however look at the structure of perturbation theory around the flat background, and verify that indeed the current algebra structure described in [3] indeed emerges.

In [3] we described a resolution of the $\hat{E}^c_2$ current algebra in terms of free fields and studied its irreducible representations with unitary base. We will give here a brief summary of the results that we will need and refer the reader to [3] for more details. We will focus for the moment on one copy of the current algebra, and return to the full theory in due time. We introduce four free fields, $x^\mu$ with

$$\langle x^\mu(z)x^\nu(w) \rangle = \eta^{\mu\nu} \log(z-w) \quad \eta^{\mu\nu} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(7)

and define the light-cone $x^\pm = x^0 \pm x^3$ and transverse space $x = x^1 + ix^2, \bar{x} = x^1 - ix^2$ coordinates. Then, the current algebra generators can be uniquely (up to Lorentz transformations) written as

$$J = \frac{1}{2} \partial x^+ \quad T = \partial x^- \quad P^+ = P^1 + iP^2 = ie^{-ix^-}\partial x^+ \quad P^- = P^1 - iP^2 = ie^{ix^-}\partial \bar{x}$$

(8)

(9)

while the affine-Sugawara stress tensor (3) becomes the free field stress tensor

$$T_{AS} = \frac{1}{2} \eta_{\mu\nu} \partial x^\mu \partial x^\nu.$$ 

(10)

The current algebra representations can be built by starting with unitary representations of the $\hat{E}^c_2$ algebra [11] as a base and then acting on them by the negative modes of the currents. They fall into two classes, [3]:

(I) Representations with neither highest nor lowest weight state. The base is generated by vertex operators with $p_- = 0$. The representation is characterized by $p_+ \in [0, 1)$ and transverse momentum $\vec{p}_T$. The base operators are

$$V^I_{p_+,n} = e^{i(p_++n)x^- + i\vec{p}_T \cdot \vec{x}}.$$ 

(11)

(II) Representations with a highest weight state in the base. Their conjugates have a lowest weight. They have $p_- > 0$ (negative for the conjugates). The highest weight operator can be written as

$$V^{II} \sim e^{ip_+x^- + ip_-x^+} H_{p_-}$$

(12)

where $H_{p_-}$ is the generating twist field in the transverse space corresponding to the transformation

$$x(e^{2\pi i}z) = e^{-4\pi ip_-}x(z) \quad \bar{x}(e^{2\pi i}z) = e^{4\pi ip_-}\bar{x}(z).$$ 

(13)

The conformal weight of the operators at the base is $\Delta = -2p_+p_- + p_-(1 - 2p_-)$.

* Except when $\vec{p} = 0$ in which case the representations are one dimensional.
As we will see later on all the representations participate in the modular invariant vacuum energy. Thus we can easily describe the spectrum of physical states. We will assume that the extra 22 dimensions are non-compact although the compact case is as easy to handle.

- For type I states \((p_- = 0)\) we have the physical state conditions \(L_0 = \bar{L}_0 = 1\) which translate to

\[ N = \bar{N} \quad \text{and} \quad \vec{p}^2_T = 2 - 2N \quad N = 0, 1, 2, \ldots \]

where, \(N, \bar{N}\) are the contribution of the transverse currents or oscillators (from the left or right) and \(\vec{p}_T\) is the 24-d transverse momentum. It is obvious that for \(N = 0\) we have \(\vec{p}^2_T = 2\) which is a component of the tachyon, while for \(N = 1\) we have \(\vec{p} = 0\) and we obtain some modes of the massless fields (graviton, antisymmetric tensor and dilaton), which propagate along the wave. For \(N > 1\) there are no physical states.

- For type II states the situation is like in usual string theory, it is just their dispersion relation that changes:

\[ -4p_+p_- + 2p_-(1 - 2p_-) + \vec{p}^2_T = 2 - 2N \]

and \(\vec{p}_T\) now stands for the 22 transverse dimensions. Eq. (15) can also be written after \(p_- \to p_-/4, \quad p_+ \to p_+ - p_-/4 + 1/2\) as

\[ p_+p_- - \vec{p}^2_T = 2(N - 1) \]

which is the flat space spectrum but in 24-d Minkowski space. Thus for almost all states the wave reduces the effective dimensionality by two. As we will see below, this happens because the states are localized in the extra two dimensions.

Eventually we would lie to compute the partition function (vacuum energy). In order to do this we can guess the modular invariant sowing of representations by looking at the wavefunctions on the group. Using the results of [5] these wavefunctions can be computed and we easily deduce that we have the standard diagonal modular invariant.

From these wavefunctions we can see the qualitative behaviour of the fluctuations in this background. Type I states are just plane waves. As for type II states, if we go to the \(x_i\) coordinates (5) where the wave nature is manifest, then we see that the states are localized where \(x_1^2 + x_2^2 + 2x_1x_2\cos(u) = 0\). So for fixed time the state is localized on the two lines \(x_1 = \pm x_2, \quad x_3 = \text{constant}\) where \(u = t - x_3\). This disturbance travels to the left in the \(x_3\) direction with the speed of light (in flat space).

We can now try to construct the vacuum energy by putting together the representations. What can happen at most is the truncation phenomenon as in the compact case. In trying to built the modular invariant partition function we have to overcome the problem that the representations are infinite dimensional and since all the states in the base have the same energy the partition function diverges. We will have to regulate this divergence and ensure that upon the removal of the regulator we will obtain a modular invariant answer. We will start by computing the character formulae for the representations above. In fact what we are interested in is the so called signature character which

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keeps track of the positive and negative norms. We will not worry much about the type I representations (although they are the easiest to deal with) since their contribution is of measure zero. For the type II representations we can easily compute the signature character

\[ \chi_{\rho_+\rho_-}^{II}(q, w) \equiv Tr[(-1)^{\epsilon} q^{L_0 - \frac{c}{6} w J_0}] \]  

(17)

where \( \epsilon = 0, 1 \) for positive (respectively negative) norm states. If \( 2p_- \neq \text{integer} \) then the only non-trivial null vector is the one at the base responsible for the fact that we have a highest (or lowest) weight representation. Thus the character is \( \chi_{\rho_+\rho_-}^{II}(q, w) \)

\[ = i q^{2p_+p_- + p_-(1-2p_-)} \frac{1}{6} w^{p_+ - \frac{1}{2}} \frac{\eta(q)}{\vartheta_1(v, q)} \]  

(18)

where \( \vartheta_1 \) is the standard Jacobi \( \vartheta \)-function and \( w = e^{2\pi i v} \). The infinity we mentioned before can be seen as \( w \to 1 \) as the pole in the \( \vartheta \)-function. In order to see how we must treat this infinity we will calculate this partition function in the quantum mechanical case (that is keeping track only of the zero modes). We will introduce a twisted version of the (Minkowski) amplitude for the particle to go from \( x \) to \( x' \) in time \( \tau \):

\[ < x|x', \tau, v, \tilde{v}> = < x|e^{i\tau H} e^{\zeta J_0 + \bar{\zeta} J_0}|x' > \]  

(19)

The quantum mechanical partition function can be obtained by setting \( x = x', \zeta, \bar{\zeta} \to 0 \) and integrating over \( x \). This last integral gives the overall volume of space that we will drop since we are interested in the vacuum energy per unit volume. The amplitude (19) in the coordinates \( u, v \) and \( a_1 = r \cos \theta, a_2 = r \sin \theta \), can be calculated with the result

\[ < x|x'; \tau, \zeta, \bar{\zeta} > \sim \frac{1}{\tau^2} \frac{(u - u' + \bar{\zeta} - \zeta)}{\sin \left( \frac{u - u' + \bar{\zeta} - \zeta}{2} \right)} \exp \left[ i \frac{(u - u' + \bar{\zeta} - \zeta)^2}{4\tau} - i \frac{(u - u' + \bar{\zeta} - \zeta)(v - v')}{2\tau} \right. \]
\[ \left. - \frac{i}{8\tau} (u - u' + \bar{\zeta} - \zeta) \cot \left( \frac{u - u' + \bar{\zeta} - \zeta}{2} \right) \left( r^2 + r'^2 - 2rr' \cos \left( \theta - \theta' - \frac{u - u' + \bar{\zeta} + \zeta}{2} \right) \right) \right] \]

We can then perform the limits to obtain the finite result

\[ Z(\tau) = < x|x; \tau, \zeta = 0, \bar{\zeta} = 0 > \sim \frac{1}{\tau^2} \]  

(20)

This result may seem surprising since we have only two continuous components of the momentum, namely \( p_+ \) and \( p_- \) but no transverse momentum. However the result is not as surprising as it looks at first, and to persuade the reader to that we will provide with an analogous situation where the answer is obvious. Consider the case of a flat 2-d plane. The zero mode spectrum are the usual plane waves, \( e^{i p \cdot x} \), but we will work in

\[ ^{\dagger} \text{We can also compute the character when } 2p_- \in Z \text{ but this is not needed again for the partition functions since it is of measure zero.} \]
the rotational basis, where the appropriate eigenfunctions corresponding to energy $p^2$ are $e^{i m \theta} J_{|m|} (pr)$. In such a basis, we will have precisely the same problem in calculating the partition function as we had above. Namely, for a given $p$ there are an infinite number of states (numbered by $m \in \mathbb{Z}$) with the same energy. Thus the partition function seems to be infinite. However here we know what to do: go to the (good) plane wave basis, or calculate the propagator and then take the points to coincide in order to get the partition function. This will also give the right result. This is precisely the prescription we have applied above and we found that the quantum mechanical partition function of the gravitational wave zero modes is precisely that of flat Minkowski space. Having found the way to sum the zero mode spectrum, it is not difficult to show using (20) that the vacuum energy of the associated bosonic string theory is equal to the flat case, namely

$$F = \int \frac{d\tau d\bar{\tau}}{Im\tau^2} (\sqrt{Im\tau\bar{\eta}})^{-24}$$

where we have added another 22 flat non-compact dimensions.

We would add here a few comments about scattering. We hope to present the full picture in the future. States corresponding to type I representations scatter among themselves, this is already obvious from their vertex operator expressions. This is dangerous however, since we might have trouble with unitarity. We will check here that in a 4-point amplitude of type I tachyons only physical states appear as intermediate states. Remember that type I tachyons have $p_- = 0$. Thus the four type I tachyons are characterized by their $p_+^I$ and $\vec{p}_T^I$. The 4-point amplitude is then given by the standard $\delta$-functions of $p_+$ and $\vec{p}_T$ multiplied by the Shapiro-Virasoro amplitude with one difference: instead of the invariants $p_i \cdot p_j$ we now have them restricted to transverse space, $\vec{p}_T^i \cdot \vec{p}_T^j$. This is similar to the Shapiro-Virasoro amplitude in 24 Euclidean dimensions (modulo the delta functions) and its analytic structure is different. It can be easily checked that instead of the infinite sequence of poles of the usual amplitude here we have just two poles, one corresponding to intermediate on-shell tachyons and the other to intermediate on-shell massless type I particles. This is in agreement with the fusion rules of the current algebra.

The scattering of the type II states is more complicated and will be dealt with elsewhere.

The bosonic string vacuum as it stands is ill defined in the quantum theory due to the presence of the tachyon in its spectrum. We will construct now the N=1 extension of this background, in order to make a superstring out of it.

In order to do this we will add four free fermions (with Minkowski signature), $\psi_{1,2}$, $\psi_J$, $\psi_T$, normalized as

$$\psi_a(z) \psi_b(w) = \frac{G_{ab}}{(z-w)} + \text{regular}$$

** The extra 22 dimensions could be compactified with no additional effort

†† Scattering for type I states for arbitrary plane waves has been considered in [7] with similar results.
The N=1 supercurrent is given by

\[ G = E^{ab} J_a \psi_b - \frac{1}{6} f^{abc} \psi_a \psi_b \psi_c \]  

(23)

where

\[ E = \frac{1}{k} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -b + 1/2k \end{pmatrix} \]  

(24)

and the group indices are always raised and lowered with the invariant metric \( G_{ab} \). The only non-zero component of \( f^{abc} \) is \( f^{124} = 1/k^2 \). It can be verified that (24) satisfies the superconformal ME \([14]\) and preserves the current algebra structure. In particular

\[ G(z)G(w) = \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{(z-w)} + \text{regular} \]  

(25)

with \( c = 6, T = T^b + T^f \), with \( T^b \) being the affine Sugawara stress tensor and \( T^f \) the free fermionic stress tensor

\[ T^f = -\frac{1}{2} G^{ab} \psi_a \partial \psi_b \]  

(26)

\( G \) is also primary with conformal weight 3/2 with respect to \( T \). The supersymmetric currents in the theory are the superpartners of the free fermions

\[ G(z)\psi_a(w) = \frac{\hat{J}_a(w)}{(z-w)} + \text{regular} \]  

(27)

It is easy to find that

\[ \hat{P}_1 = P_1 - \frac{1}{k} \psi_2 \partial_T , \quad \hat{P}_2 = P_2 + \frac{1}{k} \psi_1 \partial_T \]  

(28)

\[ \hat{J} = J + \frac{1}{2k} T - \frac{1}{k} \psi_1 \partial_2 , \quad \hat{T} = T \]  

(29)

The supersymmetric currents \( \hat{J}_a \) satisfy the same algebra (1) as the bosonic ones \( J_a \). This should be contrasted with the compact case where the level is shifted by the dual Coxeter number. The supercurrent \( G \) has a similar expression in terms of \( \hat{J}_a \) as in the compact case \([13]\)

\[ G = G^{ab} \hat{J}_a \psi_b - \frac{1}{3} f^{abc} \psi_a \psi_b \psi_c \]  

(30)

As in the bosonic case \([3]\) the supersymmetric current algebra can be written in terms of free bosons, \( x^\pm, x, \bar{x} \) and fermions \( \psi^\pm, \psi, \bar{\psi} \) with

\[ < x^+(z)x^-(w) >= - < x(z)\bar{x}(w) >= 2 \log(z-w) \]  

(31)

\[ < \psi(z)\bar{\psi}(w) >= - < \psi^+(z)\psi^-(w) >= \frac{2}{(z-w)} \]  

(32)

all others being zero. Explicitly,

\[ P_1 + iP_2 = i \sqrt{k} e^{-ix^-/Q} \partial x , \quad P_1 - iP_2 = i \sqrt{k} e^{ix^-/Q} \partial \bar{x} \]  

(33)
\[ J = \frac{Q}{2} \partial x^+ + \frac{1}{2} \left( Q + \frac{1}{Q} \right) \partial x^- + \frac{i}{2} \psi \bar{\psi}, \quad T = \frac{Q}{b} \partial x^- \]  

(34)

\[ \psi_1 + i \psi_2 = \sqrt{k} e^{-ix^-/Q} \psi, \quad \psi_1 - i \psi_2 = \sqrt{k} e^{ix^-/Q} \bar{\psi} \]  

(35)

\[ \psi_J = \frac{i}{2} \left( \psi^+ + \psi^- \right), \quad \psi_T = \frac{i}{2} \frac{Q}{b} \psi^- \]  

(36)

where \( Q = \sqrt{kb} \). The fermionic admixture to \( J \) is added to guarantee that \( J_a \) and \( \psi_a \) commute.

It is not difficult to see that the stress tensor and supercurrent are free in the free-field basis

\[ T = \frac{1}{2} \left[ \partial x^+ \partial x^- - \partial x \partial \bar{x} \right] + \frac{1}{4} \left[ \psi^+ \partial \psi^- + \psi^- \partial \psi^+ - \psi \partial \bar{\psi} - \bar{\psi} \partial \psi \right] \]  

(37)

\[ G = \frac{i}{2} \left[ \partial x \bar{\psi} + \partial \bar{x} \psi + \partial x^- \psi^+ + \partial x^+ \psi^- \right] \]  

(38)

In the free field basis it can be easily seen that the theory has an N=2 superconformal algebra. The conventionally normalized U(1) current that determines the "complex structure" is given by

\[ \tilde{J} = \frac{1}{2} \left[ \psi \bar{\psi} - \psi^+ \psi^- \right] \]  

(39)

whereas the second supercurrent is

\[ G^2 = -\frac{1}{2} \left[ \partial x \bar{\psi} - \partial \bar{x} \psi + \partial x^+ \psi^- - \partial x^- \psi^+ \right] \]  

(40)

In conformity with our \( \sigma \)-model discussion later, we will also consider the presence of the dilaton, which here is reflected as background charge in the lightcone directions. We have the following modifications for the N=2 generators

\[ \delta \tilde{J} = -i \left( Q^+ \partial x^- + Q^- \partial x^+ \right), \quad \delta T = \frac{i}{2} \left( Q^+ \partial^2 x^- - Q^- \partial^2 x^+ \right) \]  

(41)

\[ \delta G^1 = Q^+ \partial \psi^- - Q^- \partial \psi^+, \quad \delta G^2 = i \left( Q^+ \partial \psi^- + Q^- \partial \psi^+ \right) \]  

(42)

and \( \delta c = -8Q^+ Q^- \).

The final step is the realization that when \( Q^- = 0 \) the theory has even larger superconformal symmetry, namely N=4. The N=4 superconformal algebra in question contains the stress tensor, \( SU(2)_k \) currents \( J^a \) and four supercurrents that transform as conjugate doublets under the \( SU(2) \). It is defined in terms of the OPEs

\[ J^a(z) J^b(w) = k \frac{\delta^{ab}}{2 (z-w)^2} + i \epsilon^{abc} J^c(w) \frac{J^c(w)}{(z-w)} + \text{regular} \]  

(43)

\[ J^a(z) G^i(w) = \frac{1}{2} \sigma^a_{ji} \frac{G^i(w)}{(z-w)} + \text{regular}, \quad J^a(z) \bar{G}^i(w) = -\frac{1}{2} \sigma^a_{ji} \frac{\bar{G}^j(w)}{(z-w)} + \text{regular} \]  

(44)

\[ G^i(z) G^j(w) = \frac{4 k \delta^{ij}}{(z-w)^2} + 2 \sigma^a_{ji} \frac{\left( 2 J^a(w) \right) \left( z-w \right)}{(z-w)^2} + \text{regular} \]  

(45)
the rest being regular. $J^a, G^i, \bar{G}^i$ are primary with the appropriate conformal weight and $c = 6k$. In our case the SU(2) current algebra has level $k = 1$. The N=4 generators, in the free field basis are

$$G^1 = \frac{i}{\sqrt{2}} \left[ \partial \bar{x} \psi + \partial x^- \psi^- \right], \quad G^2 = -\frac{i}{\sqrt{2}} \left[ \partial \bar{x} \psi^- + \partial x^\alpha \bar{\psi}^- \right] e^{iQ^+ x^-}$$

$$G^1 = \frac{i}{\sqrt{2}} \left[ \partial \bar{x} \psi + \partial x^\alpha \psi^- - 2iQ^+ \partial \psi^- \right], \quad G^2 = \frac{i}{\sqrt{2}} \left[ \partial \bar{x} \psi^- + \partial x^\alpha \bar{\psi} + iQ^+ \psi \psi^- \bar{\psi}^- \right] e^{-iQ^+ x^-}$$

$$J^1 + iJ^2 = \frac{1}{2} \psi \psi^- e^{-iQ^+ x^-}, \quad J^1 - iJ^2 = \frac{1}{2} \bar{\psi} \psi^- e^{iQ^+ x^-}$$

$$J^3 = \frac{1}{4} \left[ \psi \bar{\psi} - \psi^- \bar{\psi}^- - 2iQ^+ \partial x^- \right]$$

and $T$ is given in (37).

One can invert the map to free fields and right these operators in terms of the original (σ-model) currents. Care should be exercised though in normal ordered expressions. What we need here is the appearance, in the N=4 generators, of terms which are non-local in the original currents. These terms are of the form $\exp \left[ \pm \frac{1}{k} (Q^+ - 1) \int T dz \right]$. However, in the sigma model, the current $T$ is given as a total derivative of a free field, $T = \partial u$, with $\partial \bar{\partial} u = 0$. Thus the exponential factors are just $\exp \left[ \pm \frac{1}{k} (Q^+ - 1) u(z) \right]$. Notice also that they disappear at a specific value of the background charge, $Q^+ = 1/Q$.

We have seen above that the $E_2^5$ gravitational wave background has a large (N=4) superconformal symmetry on the worldsheet. We will subsequently show that it belongs in fact to the $N = 4$ class of solutions described in [1], and thus the associated string theory has an N=4 spacetime supersymmetry (in the type II case). We will start in 4-d Euclidean space by introducing two complex coordinates $u$ and $v$. Then the dilaton field is a function of $u, \bar{u}, v, \bar{v}$: $\Phi = \Phi(u, \bar{u}, v, \bar{v})$. To describe the four-dimensional metric and non-constant antisymmetric tensor field background we will use the $N = 2$ superfield formalism of [10] and refer for details to our general analysis [9] on supersymmetric backgrounds in string theory. Specifically, we introduce one $N = 2$ chiral superfield $U$ with $u$ as complex bosonic coordinate and one twisted chiral superfield $V$ with $v$ as complex bosonic coordinate. It follows that the metric and antisymmetric tensor field background can be also entirely derived from a single real function $K(U, \bar{U}, V, \bar{V})$, the so-called quasi Kähler potential:

$$G_{\mu \nu} = \begin{pmatrix} 0 & K_{u \bar{u}} & 0 & 0 \\ K_{u \bar{u}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_{v \bar{v}} \\ 0 & 0 & -K_{v \bar{v}} & 0 \end{pmatrix}, \quad B_{\mu \nu} = \begin{pmatrix} 0 & 0 & 0 & K_{u \bar{v} \bar{v}} \\ 0 & 0 & K_{v \bar{v} \bar{u}} & 0 \\ -K_{v \bar{v} \bar{u}} & 0 & 0 & 0 \end{pmatrix}$$

where $K_{u \bar{u}} = \frac{\partial^2 K}{\partial u \partial \bar{u}}$, etc. The field strength $H_{\mu \nu \lambda}$ can also be expressed in terms of the function $K$ as $H_{u \bar{u} v \bar{v}} = \frac{\partial^2 K}{\partial u \partial \bar{u} \partial v \partial \bar{v}}$, etc.

The field equations, i.e. the the conditions of the vanishing of the (one-loop) $\beta$-functions will now lead to some partial differential equations for the two functions $K$ and
Φ [4]. Before we show that the gravitational wave background is (after some analytic continuation) a solution of these equations, we would first present a general class of solutions with N=4 superconformal symmetry. The condition for N=4 superconformal symmetry is [10]

$$K_{u\bar{u}} = e^\lambda K_{v\bar{v}} , \quad \lambda = \text{const.} \quad (51)$$

and \(\delta c = 0\). As derived in [3], the vanishing of the metric and antisymmetric tensor field \(\beta\)-functions leads to the following set of equations:

$$\partial_u (\log K_{u\bar{u}} - 2\Phi) = C_1 (\bar{u}) K_{u\bar{u}} , \quad \partial_v (\log K_{u\bar{u}} - 2\Phi) = C_2 (\bar{v}) K_{u\bar{u}} \quad (52a)$$

$$\partial_u (\log K_{u\bar{u}} - 2\Phi) = C_1 (u) K_{u\bar{u}} , \quad \partial_v (\log K_{u\bar{u}} - 2\Phi) = C_2 (v) K_{u\bar{u}} \quad (52b)$$

and

$$\partial_u \partial_{\bar{u}} (\log K_{u\bar{u}} - 2\Phi) = e^\lambda \partial_v \log K_{u\bar{u}} \partial_{\bar{v}} (\log K_{u\bar{u}} - 2\Phi) ,$$

$$\partial_v \partial_{\bar{v}} (\log K_{u\bar{u}} - 2\Phi) = e^{-\lambda} \partial_u \log K_{u\bar{u}} \partial_{\bar{u}} (\log K_{u\bar{u}} - 2\Phi)$$

$$\partial_u \Phi \partial_{\bar{u}} \log K_{u\bar{u}} = \partial_u \Phi \partial_{\bar{u}} \log K_{u\bar{u}} , \quad \partial_v \Phi \partial_{\bar{v}} \log K_{u\bar{u}} = \partial_v \partial_{\bar{v}} \log K_{u\bar{u}} . \quad (53b)$$

The central charge deficit, related to the dilaton \(\beta\)-function becomes

$$\delta c = 3\alpha' \frac{1}{K_{u\bar{u}}} \{ 8 \partial_u \Phi \partial_{\bar{v}} \Phi - 2 \partial_u \partial_{\bar{u}} \log K_{u\bar{u}} - 4 \partial_u \log K_{u\bar{u}} \partial_{\bar{u}} \log K_{u\bar{u}}$$

$$- e^{-\lambda} (8 \partial_v \Phi \partial_{\bar{v}} \Phi - 2 \partial_v \partial_{\bar{v}} \log K_{u\bar{u}} - 4 \partial_v \log K_{u\bar{u}} \partial_{\bar{v}} \log K_{u\bar{u}}) \}. \quad (54)$$

From eqs.(52) and (53) we obtain after some algebra that

$$C_1 (u) = Cu + c_1 , \quad \bar{C}_1 (\bar{u}) = C\bar{u} + \bar{c}_1 ,$$

$$C_2 (v) = -e^{-\lambda} Cv + c_2 , \quad \bar{C}_2 (\bar{v}) = -e^{-\lambda} C\bar{v} + \bar{c}_2 . \quad (55)$$

where \(C, c_1, \bar{c}_1, c_2, \bar{c}_2\) are constants. If \(C \neq 0\), \(c_1, c_2, \bar{c}_1, \bar{c}_2\) can be set to zero by simple translations in the coordinates. Then equations (52,53) possess the following unique solution:

$$K_{u\bar{u}} = \frac{a}{u\bar{u} - e^{-\lambda} v\bar{v}} , \quad \Phi = -\frac{1}{2} (1 + Ca) \log (u\bar{u} - e^{-\lambda} v\bar{v}). \quad (56)$$

with \(a\) a constant. For \(\lambda = i\pi\) and \(a = 1\), this background describes the so called semi wormhole space [7, 18]. The corresponding quasi Kähler function can be found in [3].

This background corresponds to the exact conformal field theory, based on the WZW model \(SU(2) \times U(1)\). The central charge deficit \(\delta c\) is zero by choosing an appropriate value for the background charge of the \(U(1)\). Then one can explicitly construct an \(N = 4\) superconformal algebra from the currents of the WZW model [18].

From now on we assume that \(C = 0\).

(i) If \(c_1 = c_2 = \bar{c}_1 = \bar{c}_2 = 0\) we deal with the case already discussed in ref.[3]:

$$\log K_{u\bar{u}} - 2\Phi = \text{const.} \quad (57)$$

This class of solutions describes the so-called axionic instanton background with \(\delta c = 0\).
(ii): If \( c_1 = 0, c_2, \tilde{c}_1, \tilde{c}_2 \neq 0 \) or \( c_1 = \tilde{c}_1 = 0, c_2, \tilde{c}_2 \neq 0 \) it follows that the metric and antisymmetric tensor field backgrounds are trivial: \( U = \text{const} \). The dilaton field may be linear in some of the coordinates.

(iii): If \( c_1, c_2, \tilde{c}_1, \tilde{c}_2 \neq 0 \) eqs(52,53) imply that \( K_{\bar{u} u} = K_{u \bar{u}} \left( \frac{u}{c_1} + \frac{\bar{u}}{\tilde{c}_1} + e^{-\lambda} \left( \frac{v}{c_2} + \frac{\bar{v}}{\tilde{c}_2} \right) \right) \) and \( \Phi = \Phi \left( \frac{u}{c_1} + \frac{\bar{u}}{\tilde{c}_1} + e^{-\lambda} \left( \frac{v}{c_2} + \frac{\bar{v}}{\tilde{c}_2} \right) \right) \) with, in addition, \( 2\Phi' = K_{\bar{u} u} - c_1 \tilde{c}_1 K_{u \bar{u}} \). Assuming \( \Phi = \text{const} \), we obtain the following expression for the background metric \( c_1 = c_2 = \tilde{c}_1 = \tilde{c}_2 = 1 \):

\[
K_{u \bar{u}} = \frac{1}{u + \bar{u} + e^{-\lambda}(v + \bar{v})} \tag{58}
\]

(iv): \( c_1 = c_2 = 0, \tilde{c}_1, \tilde{c}_2 \neq 0 \). This case can be treated similarly to the previous case (iii) and leads to \( K_{u \bar{u}} = K_{u \bar{u}} \left( \frac{a}{c_1} + \frac{\bar{a}}{\tilde{c}_1} \right) \).

(v): \( \tilde{c}_1 = c_2 = \tilde{c}_2 = 0, c_1 \neq 0 \). Then the field equations are uniquely solved by the following background:

\[
\Phi = \text{const}, \quad K_{u \bar{u}} = \frac{a}{u}, \tag{59}
\]

leading to \( \delta c = 0 \). The corresponding metric background is described by the following line element:

\[
ds^2 = \frac{a}{u} \left( dud\bar{u} - e^\lambda d\bar{v}d\bar{v} \right). \tag{60}
\]

This solution follows from the following quasi Kähler function,

\[
K = a \left( \bar{u} \log u + e^\lambda \frac{v\bar{v}}{u} \right), \tag{61}
\]

and the antisymmetric tensor field is given as \( B_{u \bar{u}} = ae^\lambda v/u^2 \).

As we will show now, this supersymmetric background is just the gravitational wave background for \( \lambda = i\pi \) and \( a = -\frac{1}{2} \), performing appropriate changes of coordinates and analytic continuations. Explicitly, we go to the Minkowskian by treating \( u \) and \( \bar{u} = w \) as independent coordinates. Then changing the coordinates as \( u \to -\frac{1}{2} \log(-2u), w \to w + x^2, \theta \to \theta + \frac{\pi}{2} (v = xe^{i\theta}) \) we obtain the following four-dimensional metric from the conformally flat metric eq.(60):

\[
ds^2 = dx^2 + x^2d\theta^2 + x^2du^2 + dudw. \tag{62}
\]

The antisymmetric tensor becomes \( B_{u \bar{u}} = 2x^2 \). Finally, substituting \( u \to iu, w \to iw \) and setting \( x \cos \theta = x_1 + \cos ux_2, x \sin \theta = \sin ux_2 \) and \( w \to w + \frac{1}{2}x_1x_2 \sin u \), the background exactly agrees with the gravitational wave background as determined by the \( \sigma \)-model action (6). However the list of solutions above still share the property that they have \( \text{N=4 spacetime supersymmetry} \) and the presence of a null Killing symmetry. They can be viewed as generalizations of the background studied in this paper.

There are several associated problems which remain open and whose answer will illuminate our present understanding of the physics of stringy gravity. One immediate problem is to finish the calculation of scattering amplitudes in the gravitational wave
background. So far we have been unable to calculate scattering amplitudes in any non-trivial space-time background (i.e. non-compact) of string theory, and we think that such a task will provide some interesting insights into scattering matrices in curved spaces. Eventually we need to understand the GSO projection in the associated superstring theory in order to obtain a tachyon free spectrum. Finally it would be interesting to examine the possibility that such wave profiles could be observed and if they have any characteristic stringy signature which might be measurable.

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