COMPARISON OF KE-THEORY AND KK-THEORY

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Abstract. We show that the character from the bivariant K-theory $KE^G$ introduced by Dumitraşcu to $E^G$ factors through Kasparov’s $KK^G$ for any locally compact group $G$. Hence $KE^G$ contains $KK^G$ as a direct summand.

1. Introduction

K-theory may be generalised in several ways to a bivariant theory. One such bivariant K-theory is Kasparov’s KK (see [5]), another is the E-theory of Connes and Higson [1]. Both theories have equivariant versions with respect to second-countable locally compact groups (see [4, 6]). These theories are related by a natural transformation $KK^G(A,B) \rightarrow E^G(A,B)$ because of the universal property of $KK^G$.

Dumitraşcu defines another equivariant bivariant K-theory $KE^G(A,B)$ in his thesis [2], which has the same formal properties as $KK^G$ and $E^G$; in particular, it has an analogue of the Kasparov product and the exterior product. He also constructs an explicit natural transformation

$$KK^G(A,B) \rightarrow KE^G(A,B) \rightarrow E^G(A,B).$$

Hence this also makes the transformation $KK^G \rightarrow E^G$ explicit.

The construction of a new bivariant K-theory is always laden with technical difficulties, especially the construction of a product. KK-theory and E-theory involve differenttechnicalities, and KE-theory needs a share of both kinds of technicalities. When I was asked to referee the article [3] by Dumitraşcu, I therefore wanted to clarify whether $KE^G$ is really a new theory or equivalent to $KK^G$ or $E^G$. I expected $KE^G$ to be equivalent to either $KK^G$ or $E^G$ because the latter two theories are very close. This is made very clear by Thomsen’s description of $KK^G(A,B)$ through asymptotic morphisms with extra properties in [8]. Thomsen’s description of $KK^G(A,B)$ differs from the definition of $E^G(A,B)$ only in the technical detail that asymptotic morphisms are required to be linear, completely positive and $G$-equivariant. This leaves little room for a new theory in between $KK^G$ and $E^G$. I came up quickly with a sketch of an argument why $KE^G$ should be equivalent to $KK^G$, which I communicated to Dumitraşcu, asking him whether he could complete this sketch to a full proof. After a while it became clear that I had to complete the argument myself, which resulted in this note. Its purpose is the following theorem:

**Theorem 1.1.** Let $G$ be a second countable locally compact group and let $A$ and $B$ be separable $G$-C*-algebras. The natural map $KE^G(A,B) \rightarrow E^G(A,B)$ factors through a map $KE^G(A,B) \rightarrow KK^G(A,B)$.

I expect $KE^G(A,B) \cong KK^G(A,B)$, but proving this requires more work.

In Section 2 we recall Dumitraşcu’s definition of cycles for $KE^G(A,B)$ and show that we may strengthen his conditions slightly without changing the set of
homotopy classes. In Section 3, we show how to get completely positive equivariant asymptotic morphisms from the KE$^G$-cycles satisfying our stronger conditions.

2. The definition of KE-theory

Throughout this article, $G$ is a second countable, locally compact topological group; $A$ and $B$ are separable C*-algebras with continuous actions of $G \times \mathbb{Z}/2$. An action of $G \times \mathbb{Z}/2$ is the same as a $\mathbb{Z}/2$-grading together with an action of $G$ by grading-preserving automorphisms; we will frequently combine a $\mathbb{Z}/2$-grading and a $G$-action in this way.

We first recall the definition $KE^G(A, B)$. Mostly, we need a set of cycles for $KE^G(A, B)$. Cycles for $KE^G(A, C([0, 1], B))$ provide homotopies of cycles, and $KE^G(A, B)$ is defined as the set of homotopy classes of cycles for $KE^G(A, B)$.

Let $L := [1, \infty)$ and $BL := C_0(L, B)$. A continuous field of $G \times \mathbb{Z}/2$-equivariant Hilbert $BL$-module $\mathcal{E}$ with a $G \times \mathbb{Z}/2$-equivariant *-homomorphism $\varphi: A \to \mathcal{L}(\mathcal{E})$, where $\mathcal{L}(\mathcal{E})$ denotes the C*-algebra of adjointable operators on $\mathcal{E}$ with its canonical action of $G \times \mathbb{Z}/2$.

Using the evaluation homomorphisms $BL \to B$, $f \mapsto f(t)$, we may view $\mathcal{E}$ as a family of $G \times \mathbb{Z}/2$-equivariant Hilbert $B$-modules $\mathcal{E}_t$; an operator $x \in \mathcal{L}(\mathcal{E})$ is completely determined by a family of operators $x_t \in \mathcal{L}(\mathcal{E}_t)$. Besides the ideal $K(\mathcal{E})$ of compact operators on $\mathcal{E}$, we need the two ideals

$$C(\mathcal{E}) := \{x \in \mathcal{L}(\mathcal{E}) \mid xf \in K(\mathcal{E}) \text{ for all } f \in C_0(L)\},$$

$$I(\mathcal{E}) := \{x \in \mathcal{L}(\mathcal{E}) \mid \lim_{t \to \infty} \|x_t\| = 0\};$$

We have $C(\mathcal{E}) \cap I(\mathcal{E}) = K(\mathcal{E})$.

A cycle for $KE^G(A, B)$ is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is a continuous field of $G \times \mathbb{Z}/2$-equivariant Hilbert $A \times B$-modules and $F$ is an odd, adjointable operator on $\mathcal{E}$ that satisfies the following conditions (for all $a \in A$, $g \in G$):

- **aKm1**: $(F - F^*)\varphi(a) \in I(\mathcal{E})$ for all $a \in A$;
- **aKm2**: $[F, \varphi(a)] \in I(\mathcal{E})$ for all $a \in A$;
- **aKm3**: $\varphi(a)(F^2 - 1)\varphi(a)^* \geq 0$ modulo $C(\mathcal{E}) + I(\mathcal{E})$ for all $a \in A$;
- **aKm4**: $(gF - F)\varphi(a) \in I(\mathcal{E})$ for all $a \in A$, $g \in G$.

We shall need the following strengthenings of these conditions:

- **aKm1s**: $F = F^*$;
- **aKm3s**: $\|F\| \leq 1$ and $(1 - F^2)\varphi(a) \in C(\mathcal{E})$ for all $a \in A$;
- **aKm4s**: $gF = F$ for all $g \in G$.

**Lemma 2.1.** Any cycle for $KE^G(A, B)$ is homotopic to one that satisfies (aKm1s) and (aKm3s). If two cycles satisfying (aKm1s) and (aKm3s) are homotopic, they are homotopic via a homotopy that satisfies (aKm1s) and (aKm3s).

We will treat condition (aKm4s) below in Lemma 2.2.

**Proof.** Let $(\mathcal{E}, F)$ be a cycle for $KE^G(A, B)$. Then $(F + F^*)/2$ is a small perturbation of $F$ and hence gives a homotopic cycle (see [3, Corollary 2.25]) satisfying $F = F^*$.

Now assume $F = F^*$ and (aKm2); then

$$\varphi(a)(1 - F^2)^+ \varphi(a)^* \cdot \varphi(a)(1 - F^2)^- \varphi(a)^*$$

$$\equiv \varphi(a)\varphi(a)^* \varphi(a)(1 - F^2)^+ (1 - F^2)^- \varphi(a)^* \equiv 0 \mod I(\mathcal{E}).$$

Hence $\varphi(a)(F^2 - 1)^+ \varphi(a)^*$ are the positive and negative parts of $\varphi(a)(F^2 - 1)\varphi(a)^*$ in $\mathcal{L}(\mathcal{E})/I(\mathcal{E})$. As a result, (aKm3) is equivalent to $\varphi(a) \cdot (F^2 - 1)^- \varphi(a)^* \in C(\mathcal{E}) + I(\mathcal{E})$ for all $a \in A$. 


Define $\chi: \mathbb{R} \to [-1, 1]$ by $\chi(t) := -1$ for $t \leq -1$, $\chi(t) := t$ for $-1 \leq t \leq 1$, and $\chi(t) := 1$ for $t \geq 1$. Then $\|\chi(F)\| \leq 1$ and $\chi(F)^2 - 1 = (F^2 - 1)$. The reformulation of (aKm3) in the previous paragraph shows that $(\mathcal{E}, \chi(F))$ is again a cycle for $\text{KE}^G(A, B)$ and that the linear path $(\mathcal{E}, sF + (1-s)\chi(F))$ is a homotopy of cycles. Thus any cycle is homotopic to one with $F = F^*$ and $\|F\| \leq 1$.

Next we adapt the standard trick to achieve $F^2 = 1$ for KK-cycles. Let $\mathcal{E}_2 := \mathcal{E} \oplus \mathcal{E}^\text{op}$, where $\text{op}$ denotes the opposite $\mathbb{Z}/2$-grading. Let $A$ act on $\mathcal{E}_2$ by $\varphi_2 := \varphi \oplus 0$. For $s \in [0, 1]$, let

$$F_{2s} := \begin{pmatrix} F & s\sqrt{1-u^2}\sqrt{1-u^2} - F^2 \\ s\sqrt{1-u^2}\sqrt{1-u^2} - F^2 & -F \end{pmatrix},$$

where $u \in \mathcal{L}(\mathcal{E})^{(0)}$ is an even operator as in Lemma [3] Lemma 2.35; that is, $u \in \mathcal{C}(\mathcal{E})$, $[u, F] \in \mathcal{I}(\mathcal{E})$, $[u, \varphi(a)] \in \mathcal{I}(\mathcal{E})$ for all $a \in A$, $(1 - u^2)((\varphi(a))(F^2 - 1)\varphi(a^*) - (F^2 - 1)) \in \mathcal{I}(\mathcal{E})$ for all $a \in A$, and $gu - u \in \mathcal{I}(\mathcal{E})$ for all $g \in G$. Since $u \in \mathcal{C}(\mathcal{E})$ and $\mathcal{C}(\mathcal{E}) \cap \mathcal{I}(\mathcal{E}) = K(\mathcal{E})$, we even have $[u, F] \in K(\mathcal{E})$, $[u, \varphi(a)] \in K(\mathcal{E})$ for all $a \in A$, and $gu - u \in K(\mathcal{E})$ for all $g \in G$. Since we already achieved $\|F\| \leq 1$, we also have $(1 - u^2)(\varphi(a))(F^2 - 1)\varphi(a^*) - (F^2 - 1) \varphi(a) \in \mathcal{I}(\mathcal{E})$, hence $((1 - u^2)\varphi(aa^*) - (1 - u^2)\varphi(a^*) - (1 - u^2)\varphi(a)) \in \mathcal{I}(\mathcal{E})$. This condition for all $a \in A$ is equivalent to $\sqrt{1-u^2}\varphi(a)\sqrt{1-F^2} \in \mathcal{I}(\mathcal{E})$ for all $a \in A$, and we may change the order of the three factors here arbitrarily. Therefore, $[F_{2s}, \varphi_2(a)] \in \mathcal{I}(\mathcal{E}_2)$ for all $a \in A$. Furthermore, $[u, F] \in K(\mathcal{E})$ implies

$$(1 - F^2)\varphi(a) \equiv \begin{pmatrix} (1 - F^2)(1-s^2 + s^2u^2) & 0 \\ 0 & (1 - F^2)(1-s^2 + s^2u^2) \end{pmatrix} \varphi(a) \mod K(\mathcal{E}_2).$$

Hence $(\mathcal{E}_2, F_{2s})$ is a homotopy of cycles for $\text{KE}^G(A, B)$. For $s = 0$, $(\mathcal{E}_2, F_{20})$ is a direct sum of $(\mathcal{E}, F)$ with a degenerate cycle and hence homotopic to $(\mathcal{E}, F)$. Thus $(\mathcal{E}, F)$ is homotopic to $(\mathcal{E}_2, F_{21})$. The diagonal entries of $1 - F_{21}^2$ are $(1 - F^2)u^2$, which belongs to $\mathcal{C}(\mathcal{E})$ because $u \in \mathcal{C}(\mathcal{E})$. Hence $1 - F_{21}^2 \in \mathcal{C}(\mathcal{E}_2)$. Thus any cycle for $\text{KE}^G(A, B)$ is homotopic to one satisfying (aKm1s) and (aKm3s).

If already $F = F^*$ and $\|F\| \leq 1$, then the canonical homotopy from $F$ to $\chi(F + F^*)/2$ is constant. And if also $1 - F^2 \in \mathcal{C}(\mathcal{E})$, then the homotopy $F_{2s}$ constructed above satisfies $1 - F_{2s}^2 \in \mathcal{C}(\mathcal{E}_2)$ for any choice of $u$. If two cycles $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ satisfying (aKm1s) and (aKm3s) are homotopic, then we may apply the modifications above to a homotopy between them; this provides a homotopy between their modifications that satisfies (aKm1s) and (aKm3s); since the canonical homotopies from $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ to their modifications also satisfy (aKm1s) and (aKm3s), we get a homotopy from $(\mathcal{E}_1, F_1)$ to $(\mathcal{E}_2, F_2)$ satisfying (aKm1s) and (aKm3s). Hence restricting to cycles satisfying (aKm1s) and (aKm3s) does not change $\text{KE}^G(A, B)$.

The standard $G \times \mathbb{Z}/2$-equivariant Hilbert $B$-module is

$$\mathcal{H} = \mathcal{H}_B := L^2(G \times \mathbb{Z}/2) \otimes L^2(\mathbb{N}) \otimes B$$

Kasparov’s Stabilisation Theorem says that for any countably generated $G \times \mathbb{Z}/2$-equivariant Hilbert $B$-module $\mathcal{E}$ there is a $G$-continuous, $\mathbb{Z}/2$-equivariant unitary operator $\mathcal{E} \oplus \mathcal{H} \to \mathcal{H}$. Unless $G$ is compact, we cannot expect this unitary to be $G$-equivariant.

Letting $F_0$ be the canonical isomorphism $\mathcal{H}_+ \leftrightarrow \mathcal{H}_-$ and $\varphi_0 = 0$, we get a degenerate cycle with underlying Hilbert module $\mathcal{H}L$. Adding this cycle, we may replace any $\text{KE}^G$-cycle $(\mathcal{E}, F)$ by $(\mathcal{E} \oplus \mathcal{H}L, F \oplus F_0)$. Therefore, we get the same set of homotopy classes $\text{KE}^G(A, B)$ if we restrict attention to cycles $(\mathcal{E}, F)$ for which there is a $G$-continuous $\mathbb{Z}/2$-grading preserving unitary $u: \mathcal{E} \to \mathcal{H}L$. 

\[ \square \]
Lemma 2.2. We get the same group $\text{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G))$ if we restrict attention to cycles for $\text{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G))$ that satisfy (aKm1s), (aKm3s) and (aKm4s) and with underlying Hilbert module $\mathcal{E} = H_{B \otimes \mathcal{K}(L^2 G)}L$, and homotopies between such cycles with the same properties.

Proof. The main ideas below already appeared in [7] and as Fell’s trick in [2] Lemma 3.3.3. By [3, Theorem 3.21], the exterior product map

$$\text{KE}^G(A, B) \to \text{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G)),$$  

is an isomorphism. So any cycle for $\text{KE}^G(A \otimes \mathcal{K}(L^2 G), B \otimes \mathcal{K}(L^2 G))$ is homotopic to $(\mathcal{E} \otimes \mathcal{K}(L^2 G), F \otimes 1)$ for some cycle $(\mathcal{E}, F)$ for $\text{KE}^G(A, B)$; and if two such cycles are homotopic, there is a homotopy of the same form.

We saw in Lemma 2.1 that we do not change the set of homotopy classes if we assume $(\mathcal{E}, F)$ to satisfy (aKm1s) and (aKm3s). And we saw above that we may assume that there is a $G$-continuous grading-preserving unitary $V : \mathcal{E} \to H_{B L}$.

This defines a $G \times \mathbb{Z}/2$-equivariant unitary

$$V' : L^2(G, E) \to L^2(G, H_{B L}), \quad (V' f)(g) := g(V(f(g))).$$

Similarly, $F \in \mathcal{L}(\mathcal{E})$ defines a $G$-equivariant adjointable operator $F'$ on $L^2(\mathcal{G}, \mathcal{E})$. As a Hilbert module over itself, $\mathcal{K}(L^2 G) \cong L^2 G \otimes (L^2 G)^*$, where $(L^2 G)^*$ is viewed as a Hilbert $\mathcal{K}(L^2 G)$-module. Hence $V'$ induces a $G \times \mathbb{Z}/2$-equivariant unitary $\mathcal{E} \otimes \mathcal{K}(L^2 G) \to H_{B L} \otimes \mathcal{K}(L^2 G) = H_{B \otimes \mathcal{K}(L^2 G)}L$, and $F'$ induces a $G$-equivariant odd operator on $\mathcal{E} \otimes \mathcal{K}(L^2 G)$. Since $gF - F \in \mathcal{T}(\mathcal{E})$, $F'$ is a small perturbation of $F \otimes 1$. Thus $(\mathcal{E} \otimes \mathcal{K}(L^2 G), F \otimes 1)$ is homotopic to a cycle with all the extra conditions we want. □

For the passage from $\text{KE}^G$ to $\mathcal{E}$, it is harmless to stabilise the $C^*$-algebras $A$ and $B$. Hence Lemma 2.2 says that it is essentially no loss of generality to restrict attention to those cycles for $\text{KE}^G$ that satisfy the stronger assumptions (aKm1s), (aKm3s) and (aKm4s). Furthermore, we may assume that $\mathcal{E} = H_{B L}$ is the constant family with fibre the standard $G$-equivariant Hilbert $B$-module $H_B$.

Remark 2.3. If a cycle for $\text{KE}^G(A, B)$ is in the image of $\text{KK}^G(A, B)$, then it satisfies more than (aKm2), namely, $[F, \varphi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$. If $\text{KK}^G$ and $\text{KE}^G$ were equivalent, then any cycle for $\text{KE}^G$ would be homotopic to one with this extra property. I do not know, however, how to prove this.

3. CONSTRUCTING ASYMPTOTIC MORPHISMS FROM KE-CYCLES

Let $S := C_0((-1, 1))$ with the $\mathbb{Z}/2$-grading with grading automorphism $\gamma f(x) = f(-x)$. As a preparation, we approximate the identity map on $S$ by $\mathbb{Z}/2$-equivariant, positively contractible functions of finite rank.

Let $n \in \mathbb{N}$. Let $I_n := \{-2^n + 1, -2^n + 2, \ldots, 2^n - 1\}$. For $k \in I_n$, define $\psi_{n,k} \in S$ by

$$\psi_{n,k}(x) := \begin{cases} \sqrt{2^n x - (k - 1)} & \text{for } k - 1 \leq 2^n x \leq k, \\ \sqrt{k + 1 - 2^n x} & \text{for } k \leq 2^n x \leq k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\psi_{n,k}^2$ is the unique piecewise linear function with singularities in $2^{-n} \cdot \{k - 1, k, k + 1\}$ and $\psi_{n,k}^2(2^{-n}k) = 1$ and $\psi_{n,k}^2(2^{-n}l) = 0$ for $k \neq l$. We have $\gamma(\psi_{n,k}) = \psi_{n,-k}$ for all $k \in I_n$. Define

$$\Psi_n : S \to S, \quad f \mapsto \sum_{k \in I_n} f(2^{-n}k) \cdot \psi_{n,k}^2.$$
Equivalently,
\begin{equation}
\Psi_n f(2^{-n}(k + t)) = (1 - t) \cdot f(2^{-n}k) + t \cdot f(2^{-n}(k + 1))
\end{equation}
for \( k \in \{-2^n, -2^n + 1, \ldots, 2^n - 1\} \), \( t \in [0, 1] \), because \( f(\pm 1) = 0 \).

By construction, \( \Psi_n \) is a completely positive map of finite rank. It is grading-preserving because \( \gamma(\psi_{n,k}) = \psi_{n,-k} \), and contractive because \( \sum_{k \in \mathbb{Z}} \psi_{n,k}^2 \leq 1 \).

The sequence \( \Psi_n(f) \) converges uniformly on \((-1, 1)\) (that is, in norm) to \( f \). This convergence is uniform on uniformly continuous subsets in \( S \).

Now let \( A \) and \( B \) be \( \mathbb{Z}/2 \)-graded \( C^* \)-algebras. Let \( \otimes \) be the graded-commutative tensor product. This is functorial for grading-preserving completely positive contractions. Hence we get a grading-preserving completely positive contraction \( \Psi_n^A = \Psi_n \otimes \text{id}_A : S \otimes A \to S \otimes A \).

To make use of Lemma \( \ref{lem} \) we assume \( A = A_0 \otimes K(L^2G) \) and \( B = B_0 \otimes K(L^2G) \) for some \( \mathbb{Z}/2 \)-graded \( C^* \)-algebras \( A_0 \) and \( B_0 \). Then we get the same group KE\( \mu \)\( (A, B) \) if we use cycles and homotopies that satisfy (aKm\( 1 \)), (aKm\( 2 \)), (aKm\( 3 \)), and (aKm\( 4 \)), where the underlying family of Hilbert modules \( \mathcal{E} \) is the constant family between \( \mathcal{H}_B \) with the standard \( G \)-equivariant Hilbert \( B \)-module \( \mathcal{H}_B \) as its fibre.

(Actually, \( \mathcal{H}_B \) is \( G \)-equivariantly isomorphic to \( (B^\infty) \otimes (B^\infty)^{op} \).)

Let \((\varphi, F)\) be such a special cycle for KE\( \mu \)\( (A, B) \). That is, \( \varphi : A \to \mathcal{L}(\mathcal{H}_B, L) \) is a \( G \times \mathbb{Z}/2 \)-equivariant *-homomorphism and \( F \in \mathcal{L}(\mathcal{H}_B, L) \), such that \( \gamma(F) = -F \), \( F = F^* \), \( \| F \| \leq 1 \), \( g(F) = F \) for all \( g \in G \), \( \lim_{t \to \infty} \|[F_t, \varphi(a)]\| = 0 \) for all \( a \in A \), and (1 - \( F^2 \)\( \varphi(a) \)) \( \in \mathcal{C}(\mathcal{H}_B, L) \). Since \( \|(1 - F^2)\varphi(a)\| \leq \| \varphi(a) \| \leq 1 \), it is equivalent to require (1 - \( F^2 \)\( \varphi(a) \)) \( \in \mathcal{C}(\mathcal{H}_B, L) \) or \( \varphi(a)(1 - F^2) \in \mathcal{C}(\mathcal{H}_B, L) \) for all \( a \in A \). Furthermore, this implies \( h(F)\varphi(a) \in \mathcal{C}(\mathcal{H}_B, L) \) and \( \varphi(a)h(F) \in \mathcal{C}(\mathcal{H}_B, L) \) for all \( h \in S \).

Dumitru˘s maps a cycle \((\mathcal{H}_B, \varphi, F)\) for KE\( \mu \)\( (A, B) \) to an asymptotic morphism from \( S \otimes A \) to \( K(\mathcal{H}_B, L) \) in \( \ref{sec4} \) Section 4.1. For \((\varphi, F)\) as above, we are now going to construct a completely positive contractive \( G \times \mathbb{Z}/2 \)-equivariant representative \( \xi = \xi(id, F) \) of Dumitru˘s’s asymptotic morphism.

The first part of the construction is easier to write down for trivially graded \( A \), so we assume this for a moment to explain our idea. Then \( S \otimes A \cong \mathbb{C}((1, -1), A) \).

Since \( \psi_{n,k} \) is in \( S \), we get \( \psi_{n,k} \varphi(a) \in \mathcal{C}(\mathcal{H}_B, L) \) for all \( n \in \mathbb{N} \), \( k \in I_n \), \( a \in A \).

Hence
\begin{equation}
\xi_n(f) := \sum_{k = -2^n + 1}^{2^n - 1} \psi_{n,k}(f(k \cdot 2^{-n})) \psi_{n,k}(F)
\end{equation}
for \( f : (1, 1) \to A \) continuous with \( f(\pm 1) = 0 \) defines a map \( \xi_n : S \otimes A \to \mathcal{C}(\mathcal{H}_B, L) \).

This map is grading-preserving, completely positive and \( G \)-equivariant because \( F = F^* \), \( \| F \| \leq 1 \) and \( F \) is \( G \)-equivariant. If \( f \geq 0 \), then
\begin{align*}
\xi_n(f) &\leq \sum_{k = -2^n + 1}^{2^n - 1} \psi_{n,k}(F) \cdot \| f(k \cdot 2^{-n}) \| \cdot \psi_{n,k}(F) \leq \| f \|_{\infty} \sum_{k = -2^n + 1}^{2^n - 1} \psi_{n,k}(F)^2 \leq \| f \|_{\infty} \sum_{k = -2^n + 1}^{2^n - 1} \psi_{n,k}(F)^2 \leq \| f \|_{\infty};
\end{align*}
thus \( \xi_n \) is contractive.

Let \( \xi_n = (1 - s)\xi_n + s\xi_{n+1} \) for \( n \in \mathbb{N} \), \( s \in [0, 1] \). These are still grading-preserving, \( G \)-equivariant, completely positive contractions \( \xi_s : S \otimes A \to \mathcal{C}(\mathcal{H}_B, L) \).

We compute the composite map
\begin{equation}
S \otimes A \xrightarrow{\xi_s} \mathcal{C}(\mathcal{H}_B, L) \xrightarrow{\mathcal{I}} \mathcal{C}(\mathcal{H}_B, L) / \mathcal{K}(\mathcal{H}_B, L).
\end{equation}
Since \( \mathcal{I}(\mathcal{H}_B, L) \cap \mathcal{C}(\mathcal{H}_B, L) = \mathcal{K}(\mathcal{H}_B, L) \) and \([F, \varphi(A)] \subseteq \mathcal{I}(\mathcal{H}_B, L)\) by (aKm\( 2 \)), \( A \) and \( h(F) \) for \( h \in S \) commute in \( \mathcal{C}(\mathcal{H}_B, L) / \mathcal{K}(\mathcal{H}_B, L) \). Hence there is a *-homomorphism \( \Xi : S \otimes A \to \mathcal{C}(\mathcal{H}_B, L) / \mathcal{K}(\mathcal{H}_B, L) \) that combines the functional calculus for \( F \) on \( S \).
and \( \varphi \) on \( A \). We see that \( \pi \circ \xi_n = \Xi \circ \Psi_n^A \) for \( n \in \mathbb{N} \) and hence \( \pi \circ \xi_s = \Xi \circ \Psi_s^A \) for \( s \in L \). Of course, \( \xi_s \) depends continuously on \( s \).

The \( * \)-homomorphism \( \Xi \) above may be turned into an asymptotic morphism \( \Xi: S \otimes A \to \mathcal{K}(\mathcal{H}_B) \) by lifting it to a map to \( \mathcal{C}(\mathcal{H}_B L) \). This is the asymptotic morphism used in [3, Section 4.1].

Before we go on, we remove the assumption that \( A \) is trivially graded:

**Lemma 3.3.** There is a sequence of \( G \times \mathbb{Z}/2 \)-equivariant completely positive contractive maps \( \xi_n: S \otimes A \to \mathcal{C}(\mathcal{H}_B L) \) with \( \pi \circ \xi_n = \Xi \circ \Psi_n^A \) for \( n \in \mathbb{N} \), even if \( A \) is \( \mathbb{Z}/2 \)-graded.

Then we may also define a continuous family of maps \( \xi_s \) for \( s \in L \) by convex interpolation between the \( \xi_n \).

**Proof.** We fix \( n \in \mathbb{N} \). To make the proof of complete positivity easy, we directly construct the Stinespring dilation of our map \( \xi_n \). Let

\[
E := \bigoplus_{k=0}^{2^n-1} (\mathcal{H}_B L \oplus (\mathcal{H}_B L)^{\text{op}}).
\]

Let \( A \) act by \( \varphi \oplus \varphi \circ \gamma \) on each summand \( \mathcal{H}_B L \oplus (\mathcal{H}_B L)^{\text{op}} \). Let \( x: E \rightarrow E \) be the operator that acts by

\[
\begin{pmatrix}
0 & 2^{-n}k \\
2^{-n}k & 0
\end{pmatrix}
\]

on the \( k \)th summand. This operator is self-adjoint, and it graded-commutes with the representation of \( A \) because we take \( \varphi \gamma \) for the second summands. Thus the functional calculus for \( x \) provides a \( * \)-homomorphism \( S \rightarrow \mathcal{L}(E) \) that graded-commutes with \( A \). Hence we get a \( G \times \mathbb{Z}/2 \)-equivariant \( * \)-homomorphism \( a: S \otimes A \rightarrow \mathcal{L}(E) \). We let \( \xi_n(f) := V^* \alpha(f)V \) for all \( f \in S \otimes A \), where \( V = (V_k)_{k \in I_n}: \mathcal{H}_B L \rightarrow E \) has the components

\[
2^{-1/2}(\psi_{n,k}(F) + \psi_{n,-k}(F)): \mathcal{H}_B L \rightarrow \mathcal{H}_B L,
\]

\[
2^{-1/2}(\psi_{n,k}(F) - \psi_{n,-k}(F)): \mathcal{H}_B L \rightarrow (\mathcal{H}_B L)^{\text{op}}
\]

for \( k > 0 \), and

\[
\psi_{n,0}(F) = 2^{-1}(\psi_{n,k}(F) + \psi_{n,-k}(F)): \mathcal{H}_B L \rightarrow \mathcal{H}_B L,
\]

\[
0 = 2^{-1}(\psi_{n,k}(F) - \psi_{n,-k}(F)): \mathcal{H}_B L \rightarrow (\mathcal{H}_B L)^{\text{op}}
\]

for \( k = 0 \). Notice that \( V_k \) is grading-preserving because \( \psi_{n,k} + \psi_{n,-k} \) is an even function and \( \psi_{n,k} - \psi_{n,-k} \) is an odd function. Since \( V \) is \( G \)-invariant as well, \( \xi_n \) is \( G \times \mathbb{Z}/2 \)-equivariant. The map \( \xi_n \) is completely positive. Since

\[
V^*V = \left( \psi_{n,0}^2 + \frac{1}{2} \sum_{k=1}^{2^n-1} (\psi_{n,k}^2 + \psi_{n,-k}^2)^2 + (\psi_{n,k} - \psi_{n,-k})^2 \right) (F)
\]

\[
= \left( \psi_{n,0}^2 + \sum_{k=1}^{2^n-1} \psi_{n,k}^2 + \psi_{n,-k}^2 \right) (F) = \left( \sum_{k \in I_n} \psi_{n,k}^2 \right) (F) \leq 1,
\]

the map \( \xi_n \) is completely contractive.

Let \( f \in S \) and \( a \in A \). If \( f \in S \) is even, then

\[
\xi_n(f \otimes a) = \psi_{n,0}(F)f(0)\varphi(a)\psi_{n,0}(F)
\]

\[
+ \sum_{k=1}^{2^n-1} (\psi_{n,k}(F) + \psi_{n,-k}(F)) f(2^{-n}k)\varphi(a)(\psi_{n,k}(F) + \psi_{n,-k}(F))
\]

\[
+ (\psi_{n,k}(F) - \psi_{n,-k}(F)) f(2^{-n}k)\varphi(a)(\psi_{n,k}(F) - \psi_{n,-k}(F));
\]
if \( f \in S \) is odd, then
\[
\ell_n(f \otimes a) = \sum_{k=1}^{2^n-1} (\psi_{n,k}(F) - \psi_{n,-k}(F)) f(2^{-n}k)\varphi(a)(\psi_{n,k}(F) + \psi_{n,-k}(F)) \\
+ (\psi_{n,k}(F) + \psi_{n,-k}(F)) f(2^{-n}k)\varphi(a)(\psi_{n,k}(F) - \psi_{n,-k}(F))
\]

Now we use that \( \pi(F) \) graded-commutes with \( \pi\varphi(A) \) to simplify \( \pi \circ \ell_n(f \otimes a) \). For even \( f \), this is equal to the \( \pi \)-image of
\[
\frac{1}{2} \sum_{k=1}^{2^n-1} (\psi_{n,k} + \psi_{n,-k})(2^{-n}k)\varphi(a) + (\psi_{n,k} - \psi_{n,-k})(2^{-n}k)\varphi(a)
\]
\[
\psi_{n,0}(F)f(0)\varphi(a) = \sum_{k \in I_n} \psi_{n,k}^2(F) f(2^{-n}k)\varphi(a) = \Psi_n^A(f)(F) \cdot \varphi(a),
\]
which is \( \Xi \circ \Psi_n^A(f \otimes a) \). For odd \( f \), \( \pi \circ \ell_n(f \otimes a) \) is equal to the \( \pi \)-image of
\[
\sum_{k=1}^{2^n-1} (\psi_{n,k} + \psi_{n,-k})(2^{-n}k)\varphi(a) \\
= \sum_{k=1}^{2^n-1} (\psi_{n,k}^2 - \psi_{n,-k}^2)(2^{-n}k)\varphi(a) \\
= \sum_{k \in I_n} \psi_{n,k}^2(F) f(2^{-n}k)\varphi(a) = \Psi_n^A(f)(F) \cdot \varphi(a),
\]
which is \( \Xi \circ \Psi_n^A(f \otimes a) \) once again. Thus \( \Xi \circ \Psi_n^A(f \otimes a) = \pi(\Psi_n^A(f)(F) \cdot \varphi(a)) \) for all \( f \in S, a \in A \), as desired. \( \square \)

Now we view \( \ell_s \) as a family of functions \( \ell_s: S \otimes A \to K(H_B) \).

**Lemma 3.4.** For separable \( A \), there is a continuous increasing function \( t_0: L \to L \) with \( \lim_{n \to \infty} t_0(s) = \infty \) such that for all \( t \geq t_0, \ell_{s,t(s)}: S \otimes A \to K(H_B) \) is asymptotically equal to the reparametrisation \( \Xi_{t(s)} \) of \( \Xi \) and hence an asymptotic morphism in the same class as \( \Xi \).

**Proof.** Since \( S \otimes A \) is separable, there is a sequence \( (f_i) \) whose closed linear span is \( S \otimes A \). For the asymptotic equality we need \( \pi \ell_{s,t(s)}(f_i) = \Xi_{t(s)}(f_i) \) for \( i \in \mathbb{N} \). We have norm convergence \( \lim_{n \to \infty} \Psi_n^A(f_i) = f_i \) for all \( i \in \mathbb{N} \). Since \( \|f_i\| \to 0 \) and \( \Psi_n^A \) is uniformly bounded, this convergence is uniform. Hence for each \( n \in \mathbb{N} \) there is \( s_n \in L \) such that \( \|\Psi_n^A(f_i) - f_i\| < 1/n \) for all \( s \geq s_n, i \in \mathbb{N} \). We may assume that the sequence \( (s_n) \) is strictly increasing with \( \lim_{n \to \infty} s_n = \infty \).

Since \( \pi \circ \ell_s = \Xi \circ \Psi_n^A \) and \( \Xi \) is a *-homomorphism, we get \( \|\pi \circ \ell_s(f_i) - \Xi(f_i)\| < 1/n \) for all \( s \geq s_n, i \in \mathbb{N} \). By definition of the quotient norm in \( C(H_B L)/K(H_B L) \), we may find \( t_i(s, t(s,t_i(s)) < 1/n \) for \( s \geq s_n, i \geq t_i(s, t(s, t_i(s))) \). Since \( \|f_i\| \to 0 \) for \( i \to \infty \), there are only finitely many \( i \) with \( \|\ell_s(f_i)\| \geq 1/2n \) and \( \Xi_t(f_i) \geq 1/2n \); hence we may find \( (s, t(s, t(s, s))) \) independent of \( i \) with \( \|\ell_s(f_i) - \Xi_t(f_i)\| < 1/n \) for all \( i \in \mathbb{N}, s \geq s, t \geq t(s, s) \).

Now choose \( t_0(s) \) increasing and continuous with \( \lim_{s \to \infty} t_0(s) = \infty \) and \( t_0(s) \geq t(s, t(s, s)) \) for \( s \in [s_n, s_{n+1}] \). If \( t(s) \geq t_0(s) \) for all \( s \in L \), then \( \|\ell_{s,t(s)}(f_i) - \Xi_{t(s)}(f_i)\| < 1/n \) for all \( s \in [s_n, s_{n+1}] \) and all \( i \in \mathbb{N} \). Thus \( \ell_{s,t(s)} \) and \( \Xi_{t(s)} \) are asymptotically equal. This implies that \( \ell_{s,t(s)} \) is an asymptotic morphism because \( \Xi \) is one. \( \square \)

The asymptotic morphism \( \ell_{s,t(s)} \) from \( S \otimes A \) to \( K(H_B) \) in Lemma 3.4 is also \( G \times \mathbb{Z}/2 \)-equivariant, completely positive and contractive. This gives an element in
$\text{KK}^G(A, B)$, following Thom森’s description of $\text{KK}^G(A, B)$ by asymptotic morphisms in [5]. We cannot directly appeal to [8] because we have replaced the ungraded suspension on both $A$ and $B$ by the graded suspension $S$ on $A$ alone. It is well-known, however, that both approaches give the same definition of equivariant $E$-theory. For the same reason, both approaches with added complete positivity requirements give $\text{KK}^G(A, B)$. Let us make this more explicit.

An asymptotic morphism $(\xi_t)$ from $S \otimes A$ to $K(\mathcal{H}_B)$ gives an extension

$$0 \to C_0(L, \mathcal{K}(\mathcal{H}_B)) \to E \to S \otimes A \to 0,$$

where $E = C_0(L, \mathcal{K}(\mathcal{H}_B)) + \xi(S \otimes A)$; it comes with evaluation homomorphisms $\epsilon_t: E \to K(\mathcal{H}_B)$ for $t \in L$. If the asymptotic morphism is $G \times \mathbb{Z}/2$-equivariant, completely positive and contractive, then the extension above has a $G \times \mathbb{Z}/2$-equivariant, completely positive and contractive cross-section. Hence there is a long exact sequence in $\text{KK}^{G \times \mathbb{Z}/2}$ for this extension. Since the kernel is contractible, we get that the quotient map in the extension is invertible in $\text{KK}^{G \times \mathbb{Z}/2}$. Composing its inverse with the evaluation homomorphism, we get a class in

$$\text{KK}^{G \times \mathbb{Z}/2}(S \otimes A, \mathcal{K}(\mathcal{H}_B)) \cong \text{KK}^{G \times \mathbb{Z}/2}(S \otimes A, B) \cong \text{KK}^G(A, B).$$

Here we use a description of $\text{KK}^G$ for $\mathbb{Z}/2$-graded $C^*$-algebras in terms of $G \times \mathbb{Z}/2$-equivariant Kasparov theory that goes back to Haag in the non-equivariant case and is extended to the equivariant case in [7].

Thus we attach a class in $\text{KK}^G(A, B)$ to a cycle for $\text{KE}^G(A, B)$. Since the same construction applies to homotopies, this construction descends to a well-defined map $\xi: \text{KE}^G(A, B) \to \text{KK}^G(A, B)$. By design, the composite map

$$\text{KE}^G(A, B) \to \text{KK}^G(A, B) \to \text{E}^G(A, B)$$

is the functor $\Xi$ of [3].

The Kasparov product in $\text{KK}^G$ becomes the composition of completely positive equivariant asymptotic morphisms in the above picture. A composite of two completely positive equivariant asymptotic morphisms is again completely positive and equivariant. So the same argument as in [2] shows that $\xi$ is a functor.

**Proposition 3.5.** The composite map

$$\text{KK}^G(A, B) \to \text{KE}^G(A, B) \to \text{KK}^G(A, B)$$

is the identity on $\text{KK}^G(A, B)$.

**Proof.** This clearly holds on the class in $\text{KK}^G(A, B)$ of a grading-preserving equivariant $\gamma$-homomorphism $f: S \otimes A \to B$. If this $f$ is a $\text{KK}^G$-equivalence, then $[f]^{-1}$ is mapped to $[f]^{-1}$ as well by functoriality. Hence any composite of such classes is mapped to itself by functoriality. Any class in $\text{KK}^G$ may be written as such a composition of classes of $[f]$ and $[f]^{-1}$. This follows from the Cuntz picture for $\text{KK}^G(A, B) \cong \text{KK}^{G \times \mathbb{Z}/2}(S \otimes A, B)$ in [7].

The transformation $\text{KE}^G \to \text{KK}^G$ shows that a computation in $\text{KE}^G$ gives results in $\text{KK}^G$. For instance, if $G$ has the analogue of a $\gamma$-element in $\text{KE}^G$ or if $\gamma = 1$ in $\text{KE}^G$, then the same follows in $\text{KK}^G$.

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