Finite Index Subgroups of R. Thompson’s Group $F$

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ABSTRACT: The authors classify the finite index subgroups of R. Thompson’s group $F$. All such groups that are not isomorphic to $F$ are non-split extensions of finite cyclic groups by $F$. The classification describes precisely which finite index subgroups of $F$ are isomorphic to $F$, and also separates the isomorphism classes of the finite index subgroups of $F$ which are not isomorphic to $F$ from each other; characterizing the structure of the extensions using the structure of the finite index subgroups of $Z \times Z$.

1 Introduction

In this paper, we classify the finite index subgroups of R. Thompson’s group $F$. By this classification, we are able to answer Victor Guba’s Question 4.5 in the problems report [7].

The group $F$ was introduced by R. Thompson in the late 1960’s as part of a family of groups $F \leq T \leq V$. It has been the object of much study, and it’s theory has impacted various fields of mathematics, including not only the theory of infinite groups, but also low-dimensional topology, simple homotopy theory, measure theory, and even category theory. An introductory reference to the theory of $F$, $T$, and $V$ is the survey paper [6].

The main characterization of $F$ that we will use is that it is the group of all piecewise-linear, orientation-preserving homeomorphisms of the unit interval which admit finitely many breaks in slope, where these breaks are restricted to occur over the diadic rationals $Z[1/2]$, and where all slopes of affine segments of the graphs of these elements are integral powers of two. We will also use some of the standard presentations for $F$ in our analysis, which presentations will be given in the next section.

We will use the notation $FIF$ to represent the set of all finite index subgroups of $F$.

In order to state our results in full, we will need to build a specific homomorphism.

Given $f \in F$, we will denote the derivative of $f$ at $x$ by $f'(x)$, if it exists. We will also define $f'(0)$ to be the derivative from the right at 0, and $f'(1)$
to be the derivative from the left at 1. Note that these last two derivatives always exist as elements of $F$ are affine near 0 and 1 in $(0,1)$.

We now define a well known homomorphism $\phi : F \to \mathbb{Z}^2$ by the rule

$$\phi(f) = (\log_2(f'(0)), \log_2(f'(1)))$$

for all $f \in F$.

By a standard fact in the literature of $F$ (Theorem 4.1 from \cite{6}), the group commutator subgroup $F'$ of $F$ consists of precisely the elements in $F$ with leading and trailing slopes one, that is, $F'$ is the kernel of the map $\phi$.

Given two positive integers $a$ and $b$, we define $\tilde{K}_{(a,b)} = \langle (a,0), (0,b) \rangle \leq \mathbb{Z}^2$. We now define

$$K_{(a,b)} = \phi^{-1}(\tilde{K}_{(a,b)}).$$

In particular, $K_{(a,b)}$ can be thought of as the group of all elements in $F$ with graphs having slopes near zero as integral powers of $2^a$ while at the same time having slopes near one as integral powers of $2^b$. We will call any such $K_{(a,b)} \leq F$ a rectangular subgroup of $F$, or simply a rectangular group. We will also refer to the groups $\tilde{K}_{(a,b)} \leq \mathbb{Z}^2$ as rectangular groups, where the context will make clear which sort of rectangular groups we are referring to.

We are now ready to give an explicit list of our results. Our first theorem is a corollary of our last theorem, but as we will prove it earlier in the paper in a direct fashion, we will list it here as a stand-alone result.

**Theorem 1.1** Let $H \in FIF$. $H$ is isomorphic to $F$ if and only if $H = K_{(a,b)}$ for some positive integers $a$ and $b$.

Given positive integers $a$ and $b$, it is immediate that $F/K_{(a,b)} \cong \mathbb{Z}_a \times \mathbb{Z}_b$, in particular we have the following theorem.

**Theorem 1.2** Given any positive integers $a$ and $b$, $F$ can be regarded as a non-split extension of $\mathbb{Z}_a \times \mathbb{Z}_b$ by $F$. In particular, there are maps $\iota$ and $\tau$ so that the following sequence is exact.

$$1 \longrightarrow F \overset{\iota}{\longrightarrow} F \overset{\tau}{\longrightarrow} \mathbb{Z}_a \times \mathbb{Z}_b \longrightarrow 1.$$ 

We will prove two further theorems. Before stating them, we mention some key lemmas, and build some language that will help with the statements of the theorems.

**Lemma 1.3** If $H \in FIF$ then $F' \leq H$. 

2
Once the previous lemma is established, it is not hard to come to the following lemma.

**Lemma 1.4** If $H \in FIF$ then $H \trianglelefteq F$.

Now, the above lemmas assure us that we can analyze all of the finite index subgroups of $F$ by considering the finite index subgroups of $Z^2$.

We have one further lemma, which will assist us in our statements below.

**Lemma 1.5** Suppose $H \in FIF$. There exist rectangular groups $\text{Inner}(H)$ and $\text{Outer}(H)$ so that $\text{Inner}(H)$ is a unique maximal rectangular subgroup in $H$ and $\text{Outer}(H)$ a unique minimal rectangular group containing $H$.

In particular, if $H \in FIF$, then we have the following list of containments (where the first two are equalities in the case that $H$ is a rectangular group).

$$\text{Inner}(H) \trianglelefteq H \trianglelefteq \text{Outer}(H) \trianglelefteq F$$

We are now in a position to state our next theorem.

**Theorem 1.6**  
1. The map $\phi$ induces a one-one correspondence between the finite index subgroups of $F$ and the finite index subgroups of $Z^2$.

2. Suppose $H$ is a finite index subgroup of $F$, with image $\tilde{H} = \phi(H) \leq Z^2$, and that $a$ and $b$ are positive integers so that $\text{Inner } H = K(a,b) \leq H$. If $Q = \tilde{H}/\tilde{K}(a,b)$, then $Q$ is finite cyclic, and there are maps $\iota$, $\rho$, $\tilde{\iota}$ and $\tilde{\rho}$ so that the diagram below commutes with the two rows being exact:

$$\begin{array}{ccccccccc}
& & & F & \cong & & & & \\
& & & \downarrow & & & & & \\
1 & \rightarrow & K(a,b) & \rightarrow & H & \rightarrow & Q & \rightarrow & 1 \\
& & & \phi|_{K(a,b)} & & & & & \\
& & & \downarrow & \phi|_H & & & & \\
1 & \rightarrow & \tilde{K}(a,b) & \rightarrow & \tilde{H} & \rightarrow & \tilde{Q} & \rightarrow & 1.
\end{array}$$

The essence of the above theorem is that in $Z^2$, each finite index subgroup $\tilde{H}$ is a finite cyclic extension (by $Q$ above) of the maximal rectangular subgroup of $\tilde{H}$. The extension pulls back, so that the finite index subgroup $H$ of $F$ can be seen as a finite cyclic extension of the maximal rectangular group $K(a,b)$ in $H$ by the same group $Q$. Whenever $Q$ is non-trivial, the
resulting extension is non-split and results in a group that is not isomorphic with $F$.

We will give several examples at the end of the paper where $Q$ above is non-trivial, that is, examples of finite index subgroups of $F$ which are not isomorphic to $F$.

We define $\text{Res}: \mathbb{F} \rightarrow N$, where we use the rule $H \mapsto n$, where $n$ is the cardinality of $H/\text{Inner}(H)$. We will call the value $n$ in the last sentence the residue of $H$.

It turns out the relationship between a finite index subgroup of $F$ and its maximal rectangular subgroup is very special. We show the following lemma.

**Lemma 1.7** Suppose $H, H' \in \mathbb{F}$, $K = \text{Inner}(H)$, $K' = \text{Inner}(H')$, and $\xi : H \rightarrow H'$ is an isomorphism. Then

1. $\xi(K) = K'$
2. $K$ is characteristic in $H$ and $K'$ is characteristic in $H'$, and
3. $\text{Res}(H) = \text{Res}(H')$.

Note that in the above, the second two points follow easily from the first.

For convenience, given $a, b$ positive integers, let us fix a particular isomorphism $\tau_{(a,b)} : K_{(a,b)} \rightarrow F$, so that if $f \in K_{(a,b)}$ with $\phi(f) = (as, bt)$ then $\phi(\tau_{(a,b)}(f)) = (s, t)$. (We note that these are the precise sorts of isomorphisms which we build in the proof of Theorem 1.1.) We also need to name the isomorphism $\text{Rev} : F \rightarrow F$ which is obtained if we conjugate the elements of $F$ by the orientation-reversing map $\text{rev} : [0,1] \rightarrow [0,1]$ defined by the equation $\text{rev}(x) = 1 - x$. We are now ready to state our final theorem.

**Theorem 1.8** Suppose $H, H'$ are finite index subgroups of $F$. Let $a, b, c, d$ be positive integers so that $K_{(a,b)} = \text{Outer}(H)$, $K_{(c,d)} = \text{Outer}(H')$. $H$ is isomorphic with $H'$ if and only if $\tau_{(a,b)}(H) = \tau_{(c,d)}(H')$ or $\tau_{(a,b)}(H) = \text{Rev}(\tau_{(c,d)}(H'))$.

These investigations were started when Jim Belk asked the first author if he knew whether or not [the group we call $K_{(2,2)}$] is isomorphic to $F$. The approach taken in this paper was motivated by the proof of Brin’s ubiquity result (see [2]), where Brin shows that a subgroup of the full group of piecewise-linear, orientation-preserving homeomorphisms of $[0,1]$ contains a copy of R. Thompson’s group $F$ if certain weak geometric conditions are satisfied.
We are unaware of any published results relating to our own work here. However, Burillo, Cleary and Röver, in the course of their investigations into the abstract commensurator of $F$, and using techniques different from our own, have also understood the one-one correspondence between the finite index subgroups of $Z^2$ and the finite index subgroups of $F$. Also, they have the result that the rectangular subgroups of $F$ are isomorphic to $F$. See [5].

The authors would like to thank Matt Brin for interesting discussions of these results, and also for some observations and questions which helped us to refine the results. Also, the first author would like to thank Jim Belk for asking the initial question that lead to this work, and to thank Mark Brittenham, Ken Brown, Ross Geoghegan, Susan Hermiller, and John Meakin for interesting conversations about these results.

2 Definitions and Notation

Richard Thompson’s Group $F$ can also described by the following presentations.

\[ F \cong \langle x_0, x_1, x_2, \ldots \mid x_j^{x_i} = x_{j+1} \text{ for } i < j \rangle \]

\[ F \cong \langle x_0, x_1 \mid [x_0x_1^{-1}, x_1^{x_0}] = [x_0x_1^{-1}, x_1^{x_2}] = 1 \rangle \]

where $a^b = b^{-1}ab$ and $[a, b] = aba^{-1}b^{-1}$.

In these presentations, the generators $x_0$ and $x_1$ can be realized as piecewise-linear homeomorphisms of the unit interval with breaks in slope occurring over the diadic rationals, and with all slopes being integral powers of two (that is, as elements of $F$ using the definition of $F$ as a group of homeomorphisms of the unit interval). We establish the mechanism of specifying any such function by listing the points in its graph where slope changes. We will call such points breaks, so that we will specify an element of $F$ by listing its set of breaks.

Let $f_0$ be the element with breaks $\{(1/4, 1/2), (1/2, 3/4)\}$ and let $f_1$ be the element with breaks $\{(1/2, 1/2), (5/8, 3/4), (3/4, 7/8)\}$. The functions $f_0$ and $f_1$ play the roles of $x_0$ and $x_1$ in the presentations above. Here are the graphs of these functions.
Note that in the above, composition and evaluation of functions in $F$ will be written in word order. In other words, if $f, g \in F$ and $t \in [0,1]$, then, $tf = f(t)$, $fg = g \circ f$, and $f^{-1} = f \circ g^{-1}$.

One can check, using the convention above, that $f_0 \sim x_0$ and $f_1 \sim x_1$ satisfy the relevant relations from the second presentation. It is well known that the second presentation is derived from the first (see [6] Theorem 3.4). The fact that $f_0$ and $f_1$ generate all of the claimed functions in $F$ (as a group of homeomorphisms) is Corollary 2.6 in [6]. (Note that our functions $f_0$ and $f_1$ are the inverses of the homeomorphisms they use.)

Given a homeomorphism $f : [0,1] \to [0,1]$, we will denote by Supp$(f)$ the support of $f$, where we take this set to be the set of all points in $[0,1]$ which are moved by the action of $f$. That is

$$\text{Supp}(f) = \{ x \in [0,1] | xf \neq f \}.$$ (Note that this is different from the definition used in analysis, where a closure is taken.)

The fact that elements of $F$ have piecewise-linear graphs that admit only finitely many breaks in slope immediately implies that if $f \in F$, then Supp$(f)$ is a finite union of disjoint open intervals. We will call each of these disjoint open intervals an orbital of $f$.

### 3 Previous Results

Here we mention several lemmas necessary for our proof whose statements and proofs are spread throughout the literature. (If we do not give an indication of where a lemma may be found in the literature, then the lemma is standard and simple, and its proof may be taken as an exercise for the reader.)
Lemma 3.1 If $f$ and $g$ are set functions where the support of $f$ is disjoint from the support of $g$, then $f$ and $g$ commute.

Lemma 3.2 Let $g, f \in F$. Let $H$ be the subgroup of $F$ that is generated by $f$ and $g$ and define

$$\text{Supp}(H) = \{x \in [0, 1] | xh \neq x \text{ for some } h \in H\}.$$

Then, $\text{Supp}(H) = \text{Supp}(g) \cup \text{Supp}(f)$.

The first point in the following lemma is essentially standard from the theory of permutation groups. It is stated (basic fact (1.1.a)) in a general form in [4]. The second point is Remark 2.3 in [1].

Lemma 3.3 Let $f \in F$ and let $g \in \text{Homeo}([0, 1])$ be any homeomorphism of the unit interval. Further suppose that $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ are the orbitals of $f$. Under these assumptions

1. the orbitals of $f^g$ are exactly $(a_1 g, b_1 g), (a_2 g, b_2 g), \ldots, (a_n g, b_n g)$, and

2. if $g$ is orientation-preserving and piecewise-linear then for every $i$, the derivative from the right of $f$ at $a_i$ equals the derivative from the right of $f^g$ at $a_i g$ and the derivative from the left of $f$ at $b_i$ equals the derivative from the left of $f^g$ at $b_i g$.

The first part of the following lemma is immediate from the definitions, while the second part is essentially a restatement of Lemma 3.4 in [4].

Lemma 3.4 If $(a, b)$ is an orbital of $f \in F$ and if $c \in (a, b)$, then

1. for all $m \in \mathbb{Z}$, $cf^m \in (a, b)$ and

2. for any $\varepsilon > 0$, there is an $n \in \mathbb{Z}$ so that both $a < cf^n < a + \varepsilon$ and $b - \varepsilon < cf^{-n} < b$.

Given a group $G$ of orientation preserving homeomorphisms of $[0, 1]$, a set $X \subset [0, 1]$, and a positive integer $k$, we say that $G$ acts $k$-transitively over $X$ if given any two sets $x_1 < x_2 < \ldots < x_k$ and $y_1 < y_2 < \ldots < y_k$ of points in $X$, there is a $g \in G$ so that $x_i g = y_i$ for all indices $i$.

The following are restatements of Lemma 4.2 and Theorem 4.3 from [6].
Lemma 3.5  $R$. Thompson’s group $F$ acts $k$-transitively over the diadic rationals in $(0,1)$, for all positive integers $k$.

Lemma 3.6  $F$ has no proper non-abelian quotients.

In particular, if we can find an $f$ and $g$ where

$$[fg^{-1}, g^f] = [fg^{-1}, g^{f^2}] = 1$$

and $f$ and $g$ do not commute, then $f$ and $g$ generate a group that is isomorphic to $F$. We need two more standard facts about $F$ (this is a combination of Theorems 4.1 and 4.5 in [6]).

Lemma 3.7  The group $F' = [F, F]$, the commutator subgroup of $F$, is simple. Furthermore, $F'$ consists of all of the functions $f \in F$ such that both $f'(0) = 1$ and $f'(1) = 1$.

The final lemma is contained in the second author’s thesis [9].

Lemma 3.8  If $G \leq F$ and $G \cong F$, then there are generators $g_0, g_1 \in G$ such that $\langle g_0, g_1 \rangle = G$, and for every orbital $A$ of $g_0$, if $B$ is an orbital of $g_1$, then either $A \cap B = \emptyset$ or $B \subseteq A$.

Furthermore, for the same functions $g_0$ and $g_1$ as described above, if $A = (a_1, a_2)$ is an orbital of $g_0$ but not $g_1$ and $A$ is not disjoint from the support of $g_1$, then there is an $\varepsilon > 0$ such that either

1. $g_0$ and $g_1$ are equal in the interval $(a_1, a_1 + \varepsilon)$ and $(a_2 - \varepsilon, a_2)$ is disjoint from the support of $g_1$, or

2. $g_0$ and $g_1$ are equal in the interval $(a_2 - \varepsilon, a_2)$ and $(a_1, a_1 + \varepsilon)$ is disjoint from the support of $g_1$.

4  Properties of the finite index subgroups of $F$

Here we derive some nice properties of the finite index subgroups of $F$. In particular, we explore their relationships with $F'$, and we examine the extent of their supports.

We begin with a simple lemma about infinite simple groups.

Lemma 4.1  Infinite simple groups do not admit proper subgroups of finite index.
Proof:
Let $G$ be an infinite simple group and let $H$ be a finite index subgroup of $G$. The right cosets $\{He, Hg_2, ..., Hg_n\}$ form a set that $G$ acts on by multiplication on the right (here we are denoting the identity of $G$ by $e$).

The action induces a homomorphism from $G$ to the symmetric group on $n$ letters. Since the codomain of this homomorphism is a finite group, the kernel must be non-trivial. Since $G$ is simple the kernel must be all of $G$. Now, if $n \neq 1$, then we can assume that $Hg_2 \neq He = H$. But now $H = He = H \cdot (g_2g_2^{-1}) = (Hg_2) \cdot g_2^{-1} = Hg_2$ (the last equality follows as the action is trivial). Thus, $n = 1$ and $G = H$.

Here we have the first lemma from the introduction.

**Lemma 1.3** If $H \leq F$ is a finite index subgroup of $F$, then $F' \leq H$.

**Proof:**
Let $H$ be a finite index subgroup of $F$. The group $H \cap F'$ must be finite index in $F'$, which is an infinite simple group by Lemma 3.7. Now, by Lemma 4.1 $F' \subseteq H$.

We can now prove Lemma 1.4.

**Lemma 1.4** Suppose $H$ is a finite index subgroup of $F$, then $H \trianglelefteq F$.

**Proof:**
Suppose that $H$ is not normal in $F$. Then there is an $f \in F$ so that $f^{-1}Hf \neq H$. In particular, there is an $h \in H$ so that $f^{-1}hf \notin H$. This last implies that $h^{-1}(f^{-1}hf) \notin H$. But $h^{-1}f^{-1}hf = [h^{-1}, f^{-1}] \in F'$. Since Lemma 1.3 assures us that $F' \subseteq H$, we have a contradiction.

Also, we are in a good position to prove the following.

**Lemma 1.5**
Suppose $H$ is a finite index subgroup of $F$. Then there exists a unique maximal rectangular subgroup $\text{Inner}(H)$ of $H$ and a unique minimal rectangular group $\text{Outer}(H)$ containing $H$.

**Proof:**
Let $H$ be a finite index subgroup $F$ and suppose $K_{(a,b)} \leq H$ and $K_{(c,d)} \leq H$. Let $r = \gcd(a, c)$ and $s = \gcd(b, d)$. We can use a finite product of elements from $K_{(a,b)}$ and $K_{(c,d)}$ to build an element $f$ with $\phi(f) = (r, 0)$, and likewise, we can build an element $g$ with $\phi(g) = (0, s)$. Now, using Lemma 1.3 it is immediate that $K_{(r,s)} \leq H$. In particular, any finite index subgroup of $F$ has a unique, maximal rectangular subgroup.
\[ F = K_{(1,1)} \] is a rectangular subgroup of \( F \) which contains \( H \), and it is easy to see that the intersection of any two rectangular subgroups of \( F \) is again a rectangular subgroup of \( H \); in particular, the intersection of all of the rectangular subgroups of \( F \) which contain \( H \) produces a unique minimal rectangular group containing \( H \).

\[ \diamond \]

We now pass to some further useful lemmas not mentioned in the introduction.

**Lemma 4.2** If \( H \) is finite index in \( F \), then

1. \( \text{Supp}(H) = (0, 1) \), and
2. there are \( h_1, h_2 \in H \) so that \( \text{Supp}(h_1) = (b, 1) \) and \( \text{Supp}(h_2) = (0, a) \), for some \( 0 < a \leq 1 \) and \( 0 \leq b < 1 \).

**Proof:**

(1) By the proof of Lemma 1.4, \( F' \leq H \), and \( \text{Supp}(F') = (0, 1) \).

(2) Suppose that for all \( h \in H \), \( h'(1) = 1 \). Then, for all \( g_k \in H F^k_0 \), \( (g_k)'(1) = \frac{1}{2k} \). In particular, we have just found infinitely many distinct right cosets of \( H \) in \( F \).

A similar argument shows there is an \( h \in H \) with \( \text{Supp}(h) = (0, a) \).

\[ \diamond \]

**5 Finite Index Subgroups of \( F \) that are Isomorphic to \( F \)**

Consider the functions \( g_0 \) and \( g_1 \) specified by their sets of breaks as follows:

\[
g_0 \text{ has breaks } \left\{ \left( \frac{3}{8}, \frac{3}{8} \right), \left( \frac{1}{2}, \frac{5}{8} \right), \left( \frac{5}{8}, \frac{3}{4} \right), \left( \frac{7}{8}, \frac{7}{8} \right) \right\}
\]

\[
g_1 \text{ has breaks } \left\{ \left( \frac{3}{8}, \frac{3}{8} \right), \left( \frac{7}{16}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{9}{16} \right), \left( \frac{5}{8}, \frac{5}{8} \right) \right\}
\]

These functions have graphs as below.
Lemma 5.1 Let $g_0$ and $g_1$ be the functions in $F$ that are defined above. Then $\langle g_0, g_1 \rangle$

1. consists of every element of $F$ whose support is contained in the interval $\left[\frac{3}{8}, \frac{7}{8}\right]$, and

2. is isomorphic with $F$.

Proof:

We only need show the first point. The second point will then follow since by Lemma 4.4 in [6] the subset of elements of $F$ with support in $[a,b]$ where $a$ and $b$ are diadic rationals with $b-a$ an integral power of two is conjugate by a linear homeomorphism of $R$ to produce exactly $F$.

We explicitly build the linear conjugator of Lemma 4.4 in [6].

Consider the homeomorphism $\omega : R \to R$ defined by $t \mapsto \left( \frac{8t-3}{4} \right)$. This homeomorphism sends $\left[\frac{3}{8}, \frac{7}{8}\right]$ linearly to $[0,1]$, and it induces an isomorphism $\psi : \langle g_0, g_1 \rangle \to H$ for some subgroup $H \leq \text{Homeo}(R)$. (Here, we are considering elements of $F$ to be homeomorphisms from $R$ to $R$, by using the unique extension of any element of $F$ by the identity map away from $[0,1]$).

The function $\psi$ can be thought of as a restriction of the inner automorphism of $\text{Homeo}(R)$ produced by conjugation by $\omega$.

From here out, we will refer to $\langle g_0, g_1 \rangle$ as $\Gamma$.

If we restrict $\psi$ less (potentially, depending on the size of $\Gamma$), and take the preimage of $F$ under $\psi^{-1}$, then Lemma 4.4 in [6] tells us that $\psi^{-1}(F) = \Upsilon \cong F$, where $\Upsilon$ consists of all graphs of $F$ with support in $[3/8,7/8]$.

Since $\omega$ is linear, we can understand $\psi$ by considering how the map $\omega$ moves the breaks of any element in $\langle g_0, g_1 \rangle$. If $(p_i, q_i), 1 \leq i \leq n$, are the breaks of $g \in \langle g_0, g_1 \rangle$, then $g \omega$ is the unique piecewise-linear element of $\text{Homeo}(R)$ whose breaks are $\left( \frac{8p_i-3}{4}, \frac{8q_i-3}{4} \right)$ which acts as the identity near $\pm \infty$. 

11
Now, one can check directly that \( g_0 \psi = f_0 \) and \( g_1 \psi = f_0^2 f_1^{-1} f_0^{-1} \). So
\[
\langle g_0 \psi, g_1 \psi \rangle = \langle f_0, f_0^2 f_1^{-1} f_0^{-1} \rangle = \langle f_0, f_0^{-2} (f_0^2 f_1^{-1} f_0^{-1}) f_0 \rangle = \langle f_0, f_0^{-1} \rangle = \langle f_0, f_1 \rangle = F.
\]
In particular, \( \psi(\Gamma) = F \), hence \( \Upsilon = \Gamma \), and \( \Gamma \cong F \).

We are now ready to prove the first of our main theorems. For the following, we need to recall the \( K_{(a,b)} \) groups:

\[
K_{(a,b)} = \left\{ h \in F \mid \exists m, n \in \mathbb{Z} \text{ s.t. } h'(0) = (2^a)^n \text{ and } h'(1) = (2^b)^m \right\}
\]

where both \( a \) and \( b \) are non-zero integers.

**Theorem 1.1** Let \( H \) be a finite index subgroup of \( F \). \( H \) is isomorphic to \( F \) if and only if \( H = K_{(a,b)} \) for some \( a, b \in \mathbb{N} \).

**Proof:** \( (\leftarrow \rightarrow) \): Fix \( a \) and \( b \) in \( \mathbb{N} \). We will build generators \( y_0 \) and \( y_1 \) for \( K_{(a,b)} \). First, we will define \( y_0 \in K_{(a,b)} \) over a finite collection of points as follows:

- If \( a = 1 \), then let \( (a_1, b_1) = \left( \frac{1}{16}, \frac{1}{8} \right) \). If \( a \neq 1 \), then let \( (a_1, b_1) = \left( \frac{1}{2^a}, \frac{1}{2^a} \right) \).
- Let \( (a_2, b_2) = \left( \frac{1}{8}, \frac{3}{8} \right) \). Let \( (a_3, b_3) = \left( \frac{5}{8}, \frac{7}{8} \right) \). If \( b = 1 \), then let \( (a_1, b_4) = \left( \frac{7}{16}, \frac{15}{16} \right) \).
- If \( b \neq 1 \), then let \( (a_4, b_4) = \left( 1 - \frac{1}{2^b}, 1 - \frac{1}{2^b} \right) \).

Filling in the definition of \( y_0 \).

Extend the definition of \( y_0 \) by making it linear from \((0,0)\) to \((a_1, b_1)\), affine and with slope one from \((a_2, b_2)\) to \((a_3, b_3)\), and affine from \((a_4, b_4)\) to \((1,1)\). All slopes involved so far are integral powers of two, and the set of \( a_i \)'s and \( b_i \)'s are all diadic rationals, so \( y_0 \) still has the potential to be extended to an element of \( F \).

We can now pick some diadic rational pairs \((c_1, d_1)\) and \((c_2, d_2)\) with \( a_1 < c_1 < c_2 < a_2 \) and \( b_1 < d_1 < d_2 < b_2 \) so that the ratios
\[
\frac{d_1 - b_1}{c_1 - a_1} \quad \text{and} \quad \frac{b_2 - d_2}{a_2 - c_2}
\]
both produce integral powers of 2 (not equal to the values of the slope of
\( y_0 \) near zero, or to the value 1, the slope of \( y_0 \) over \((a_2, a_3)\)), and where the
line segments from \((a_1, b_1)\) to \((c_1, d_1)\) and from \((c_2, d_2)\) to \((a_2, b_2)\) (which
we will be adding to the definition of \( y_0 \)) do not cross the line \( y = x \).
We can now extend the definition of \( y_0 \) from 0 to \( c_1 \) and from \( c_2 \) to \( a_3 \) so
that over each interval, \( y_0 \) admits precisely one breakpoint (over \( a_1 \) and \( a_2 \)
respectively), and the graph of \( y_0 \) determined so far stays well above the line
\( y = x \).

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that over each interval, \( y_0 \) admits precisely one breakpoint (over \( a_1 \) and \( a_2 \)
respectively), and the graph of \( y_0 \) determined so far stays well above the line
\( y = x \). By Lemma 3.5, there is an element \( \zeta \) of \( F \) which sends the list of
points \((0, a_1, c_1, c_2, a_3, a_4, 1)\) to the list \((0, b_1, d_1, b_2, b_3, b_4, 1)\). Assume
we have previously expanded these lists as necessary with many diadic points
imbetween \( c_1 \) and \( c_2 \) and correspondingly many diadic points between \( d_1 \) and
\( d_2 \), (all new points roughly evenly spaced out) so that the graph of \( \zeta \) cannot
intersect the line \( y = x \). We can now define \( y_0 \) over the interval \((c_1, c_2)\) to
agree with \( \zeta \). The element \( y_0 \) is now defined over the intervals \((0, a_3)\) and
\((a_4, 1)\).

We can fill in the definition of \( y_0 \) with similar care over the region \((a_3, a_4)\)
(choose diadics \( c_3 \) and \( c_4 \) with \( a_3 < c_3 < c_4 < a_4 \) in a fashion similar to our
choices of \( c_1 \) and \( c_2 \), then connect over the region \((c_3, c_4)\) by some random
appropriate element of \( F \) which does not touch the line \( y = x \)) to finally get
an element \( y_0 \) in \( F \) which

1. is linear over \((0, a_1)\), \((a_2, a_3)\), and \((a_4, 1)\), and

2. has breakpoints including \((a_1, b_1)\), \((a_2, b_2)\), \((a_3, b_3)\), and \((a_4, b_4)\), and

3. does not intersect the line \( y = x \).

Note that while \( y_0 \) is defined everywhere, it is not completely determined
over \((c_1, c_2)\), and it is not completely determined over the similar interval
\((c_3, c_4)\) in \((a_3, a_4)\) (although it is roughly controlled in both locations).

Construct \( y_1 \) as follows:

\[
ty_1 = \begin{cases} 
  t & : t \leq 3/8 \\
  2t - (3/8) & : 3/8 \leq t \leq 5/8 \\
  ty_0 & : 5/8 \leq t \leq 1 
\end{cases}
\]

**Sub-claim 1.1.1:** \( K_{(a,b)} \subsetneq F \).

**Proof of Sub-claim 1.1.1:** Let \( g, h \in K_{(a,b)} \). Suppose \( g'(0) = (2^a)^m \) and \( h'(0) = (2^b)^n \). Since all elements of \( F \) are linear in a neighborhood of 0, then the
chain rule for derivatives from the right applies. In particular \((gh)'(0) = (2^a)^m \cdot (2^b)^n = \cdot (2^{a+b})^{m+n} \).
\((2^a)^{m+n}\). Similarly, \((gh)'(1) = (2^b)^{p+q}\), where \(g'(1) = (2^b)^p\) and \(h'(1) = (2^b)^q\). So \(K_{(a,b)}\) is a subgroup of \(F\).

Let \(f \in F\). From Lemma 3.3 it must be the case that \((g^f)'(0) = (2^a)^m\) and \((g^f)'(1) = (2^b)^p\). So \(g^f \in K_{(a,b)}\). Thus \(K_{(a,b)} \leq F\).

\textbf{SUB-CLAIM 1.1.2:} \(K_{(a,b)}\) is a finite index subgroup of \(F\).

\textbf{Proof of 1.1.2:} Let \(f \in F\). The slope of \(f\) near 0 is \(2^p\) and the slopes of elements of \(K_{(a,b)}\) near 0 is \(2^{an}\) for \(n \in \mathbb{Z}\). Then the slopes of elements of \(fK_{(a,b)}\) near 0 is \(2^{an+p}\). The division algorithm gives us that since \(n \in \mathbb{Z}\), there are exactly \(a\) different cosets of \(K_{(a,1)}\). Similarly, there are exactly \(b\) different cosets for \(K_{(1,b)}\). Since \(K_{(a,b)} = K_{(a,1)} \cap K_{(1,b)}\), then are at most \(ab\) distinct cosets for \(K_{(a,b)}\) in \(F\).

\textbf{SUB-CLAIM 1.1.3:} \(Y = \langle y_0, y_1 \rangle \cong F\).

\textbf{Proof of 1.1.3:} \(y_0\) and \(y_1\) have been constructed specifically to have orbitals of certain products of these functions to be disjoint. Since \(y_0|_{[\frac{5}{8}, 1]} = y_1|_{[\frac{5}{8}, 1]}\), and as both functions have graphs above the line \(y = x\) in this region, it must be the case that \(\text{Supp}(y_0y_1^{-1}) = (0, \frac{5}{8})\). By Lemma 3.3 \(\text{Supp}(y_1^{y_0}) = (\frac{5}{8}, 1)\) and \(\text{Supp}(y_1^{y_0^2}) = (\frac{7}{8}, 1)\). By Lemma 3.3 \(\langle y_0y_1^{-1}, y_1^{y_0} \rangle = 1\) and \([y_0y_1^{-1}, y_1^{y_0^2}] = 1\). \(y_0\) and \(y_1\) do not commute because \(\frac{7}{8}y_0^{-1}y_1 - \frac{5}{8}y_0^{-1}y_1^{-1} = \frac{3}{8}y_1^{-1} = \frac{3}{8} \neq \frac{1}{4}\). So then by Lemma 3.6 \(Y \cong F\).

\textbf{SUB-CLAIM 1.1.4:} \(Y = \langle y_0, y_1 \rangle = K_{(a,b)}\).

\textbf{Proof of 1.1.4:} Note that \(y_1'(0) = 1, y_1'(1) = y_0'(1)\).

\[
y'_0(0) = \begin{cases} 
(1/8) & \text{if } a = 1 \\
(1/2^a) & \text{if } a > 1
\end{cases}
\]

and \(y'_0(1) = \begin{cases} 
(1/16) & \text{if } b = 1 \\
(1/2^b) & \text{if } b > 1
\end{cases}\).

So \(y_0\) and \(y_1\) are both in \(K_{(a,b)}\) and \(Y = \langle y_0, y_1 \rangle \subseteq K_{(a,b)}\).

We have carefully constructed \(y_0\) and \(y_1\) in such a way that even though there are two intervals over which \(y_0\) is not explicitly known, the commutator function \([y_0, y_1]\) is completely determined. Let us demonstrate this point.

Since \(y_0|_{[5/8, 1]} = y_1|_{[5/8, 1]}\), then \(y_0y_1^{-1}y_1^{-1}|_{[5/8, 1]} = 1\). Since \(y_1|_{[0, 3/8]} = 1\) and \(\frac{3}{8}y_0 = \frac{3}{8}\), then \(y_0y_1^{-1}y_1^{-1}|_{[0, 1/8]} = 1\).

The following line segments are taken linearly to each other.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & y_0 & 3 & 1 & y_1 & 3 & 3 \\
8 & 3 & 8 & 2 & 8 & 8 & & \end{array}
\]

14
This guarantees that if \( 2 \), then the chain rule gives \((h)_{orbital of h} \). There exists functions \( (2 \times 1, \ldots, 2 \times n) \), then by Lemma 5.1, \( K \) is a fixed point of \( (2 \times 1) \). By Lemma 3.4, since \( Supp(\[5 \times 1\]) = (0,1) \), then \( Supp(\[5 \times 1\]) \) to \( (1,1) \).

Since \( Supp([y_0, y_1]) = (1, \frac{3}{5}) \) and \( y_0 \) is explicitly known in the interval \((\frac{1}{3}, \frac{5}{8})\), then we can explicitly find \([y_0, y_1]_{y_0} \). Also, since \( Supp([y_0, y_1]_{y_0}) = (\frac{2}{3}, \frac{7}{8}) \) and \( y_1^{-1} \) is explicitly known on \((\frac{3}{5}, \frac{7}{8})\), then \([y_0, y_1]_{y_0y_1^{-1}} \) can also be computed. This computation gives that \([y_0, y_1]_{y_0} = g_0 \) and \([y_0, y_1]_{y_0y_1^{-1}} = g_1 \), where \( g_0 \) and \( g_1 \) are the functions defined in the beginning of Section 5. So then by Lemma 5.1, \([g_0, g_1]\) contains every element of \( F \) that has support inside the interval \((\frac{3}{5}, \frac{3}{2})\).

Since \( g_0 \) and \( g_1 \) are products of the functions \( y_0, y_1 \in K_{(a,b)} \), then \( Y \) and \( K_{(a,b)} \) both contain every element of \( F \) whose support is contained in the interval \((\frac{3}{5}, \frac{3}{2})\).

Let \( h \in F' \). By Lemma 3.7, there exists a \( c, d \in (0,1) \) so that \( Supp(h) \subseteq (c,d) \). By Lemma 3.4, since \( Supp(y_0) = (0,1) \), there is an \( n \in Z \) so that \( Supp(h^{y_0^n}) \subseteq (\frac{2}{3}, dy_0^n) \), where \( \frac{2}{3} < cy_0^n < dy_0^n < 1 \). By Lemma 3.4, since \( Supp(y_1) = (\frac{2}{3}, 1) \), then there is an \( m \in Z \) so that \( \frac{2}{3} = \frac{2}{3}y_1^m < cy_0^my_1^m < dy_0^my_1^m < \frac{5}{8} \) and \( Supp(h^{y_0^n})^{y_1^{-m}} \subseteq (\frac{3}{5}, \frac{7}{8}) \). By the previous argument, it must be the case that \( h^{y_0^n}y_1^{-m} \in Y \). So then \( h = (h^{y_0^n}y_1^{-m})^{y_1^{-m}y_0^{-n}} \in Y \). So \( F' \subseteq Y \).

Let \( w \in K_{(a,b)} \). There is an \( n,m \in Z \) so that \( w'(0) = 2^m \) and \( w'(1) = 2^m \). Since \( w, y_0, y_1 \) are all linear functions in a neighborhoods of \( 0 \) and \( 1 \), then the chain rule gives \((w^{y_0^{-n}})'(0) = (2^m)(2^n)^{-n} = 1, (w^{y_0^{-n}})'(1) = (2^m)(2^n)^{-n} = 2^{b(n+m)} \), \((w^{y_0^{-n}})'(0) = (1)(1)^{m+n} = 1, and (w^{y_0^{-n}})'(1) = 2^{b(n+m)}(2^{-b})^{m+n} = 1 \). So \( w^{y_0^{-n}}y_1^{m+n} \in F' \subseteq Y \Rightarrow w \in Y \).

Thus \( K_{(a,b)} = Y \cong F \).

\((\Rightarrow)\): Assume that \( H \) is a finite index subgroup of \( F \) and \( H \cong F \).

By Lemma 3.4, \( H \cong F \). By Lemma 3.5, \( F' \cong H \) so \( Supp(H) = (0,1) \). There exists functions \( h_0 \) and \( h_1 \) so that \( H = \langle h_0, h_1 \rangle \) that satisfy the conditions listed in Lemma 3.8. One condition in Lemma 3.8 is if \( A \) is an orbital of \( h_0 \) and \( B \) is an orbital of \( h_1 \), then either \( B \subseteq A \) or \( B \cap A = \emptyset \). This guarantees that if \( p \) is a fixed point of \( h_0 \), then \( p \) is also a fixed point.
of $h_1$. So then the point $p$ will be a fixed point of the group $H$. So $p \notin \text{supp}(H) = (0,1)$. So either $p = 0$ or $p = 1$ and $\text{Supp}(h_0) = (0,1)$.

Since $h_0$ is not the identity near either 0 or 1, then there exist non-zero integers $a$ and $b$ so that $h_0'(0) = 2^a$ and $h_0'(1) = 2^b$. Lemma 3.8 also guarantees that either $h_1'(0) = 1$ and $h_1'(1) = 2^b$ or $h_1'(0) = 2^a$ and $h_1'(1) = 1$. Without loss of generality, assume $h_1'(0) = 1$ and $h_1'(1) = 2^b$. We want to show that $H = K(a,b)$.

$(\subseteq)$: $h_0 \in K(a,b)$ and $h_1 \in K(a,b)$, so $H = \langle h_0, h_1 \rangle \subseteq K(a,b)$.

$(\supseteq)$: Let $f \in K(a,b)$. So $f'(0) = 2^{an}$ and $f'(1) = 2^{bm}$ for some $n, m \in Z$. Then, by the chain rule, $(f h_0^{-n})(0) = (2^{an})(2^a)^{-n} = 1$ and $(f h_0^{-n})(1) = (2^{bm})(2^b)^{-n} = 2^{b(m-n)}$. Also, $(f h_0^{-n} h_1 n-m)(1) = 1(1)^{n-m} = 1$ and $(f h_0^{-n} h_1 n-m')(1) = 2^{b(m-n)}(2^b)^{n-m} = 1$. So then by Lemma 3.7, $f h_0^{-n} h_1 n-m \in F'$. Since $H \trianglelefteq F$, then $F' \subseteq H$. So $f h_0^{-n} h_1 n-m \in H$. So then $f = f h_0^{-n} h_1 n-m h_1^{-n} h_0 \in H$. So $H = K(a,b)$.

Theorem 1.2

Given any positive integers $a$ and $b$, $F$ can be regarded as a non-split extension of $Z_a \times Z_b$ by $F$. In particular, there are maps $\iota$ and $\tau$ so that the following sequence is exact.

$$1 \longrightarrow F \xrightarrow{\iota} F \xrightarrow{\tau} Z_a \times Z_b \longrightarrow 1.$$

Proof: This theorem is actually an immediate corollary to Theorem 1.1; simply take $\iota$ to be the composition of the isomorphism from $F$ to $K(a,b)$ with the inclusion map of $K(a,b)$ into $F$.

To prove Theorem 1.6, we will need to produce some analysis of the finite index subgroups of $Z^2$.

6 Finite index subgroups of $Z^2$

In this section we will prove two statements about the finite index subgroups of $Z^2$. While both of these statements could be taken as straightforward exercises in an entry level graduate course in group theory, we will include the proofs for completeness.

Lemma 6.1 Suppose $H$ is a finite index subgroup of $Z^2$. Then there are minimal positive integers $a$ and $b$ so that $K(a,b) \leq H$. Further, if $K(c,d) \leq H$ then $K(c,d) \leq K(a,b)$. 

16
Proof:

$H$ is normal in $\mathbb{Z}^2$ since $\mathbb{Z}^2$ is abelian. In particular, since $H$ has finite index in $\mathbb{Z}^2$, the group $T = \mathbb{Z}^2/H$ is finite. Therefore, there is a minimal positive integer $a$ so that $(a,0) \in H$ and a minimal positive integer $b$ so that $(0,b) \in H$. It is now immediate that $\tilde{K}_{(a,b)} \leq H$.

Suppose $\tilde{K}_{(c,d)} \in H$. Then $(c,0) \in H$. The Euclidean Algorithm now shows that $(j,0) \in H$, where $j = \gcd(a,c)$. If $a \nmid c$ we must have that $j < a$, which contradicts our choice of $a$. In particular, $a \mid c$ and $(c,0) \in \langle(a,0) \rangle \leq \tilde{K}_{(a,b)}$. A similar argument shows that $(0,d) \in \tilde{K}_{(a,b)}$. Since $\tilde{K}_{(c,d)}$ is generated by $(c,0)$ and $(0,d)$, we have that $\tilde{K}_{(c,d)} \leq \tilde{K}_{(a,b)}$.

\[\diamond\]

In the above lemma, we will call the group $\tilde{K}_{(a,b)}$ the maximal $\tilde{K}$ group in $H$.

Lemma 6.2 Suppose $H$ is a finite index subgroup in $\mathbb{Z}^2$ with maximal $\tilde{K}$ group $\tilde{K}_{(a,b)}$. The group $Q \cong H/\tilde{K}_{(a,b)}$ is finite cyclic.

Proof: Thinking of $\mathbb{Z}^2$ as a planar lattice, the points in $H$ not in $\tilde{K}_{(a,b)}$ are the points which will survive under modding $H$ out by $\tilde{K}_{(a,b)}$ to become non-trivial elements of $Q$. Thus, we can find $Q$ as a subgroup of points in the finite rectangular lattice $L = \mathbb{Z}_a \times \mathbb{Z}_b$. Furthermore, as $a$ and $b$ are minimal positive so that $(a,0) \in H$ and $(0,b) \in H$, we must have that the only intersection $Q$ will have with the vertical axis in $L$ (the points of the form $(0,r)$) or with the horizontal axis in $L$ (the points of the form $(r,0)$) is at the point $(0,0)$.

In particular, suppose $(r,s)$ and $(t,u)$ are points in $Q$. If $j \equiv \gcd(r,t)$, then we can again exploit the Euclidean Algorithm to find integers $p$ and $q$ so that $p(r,s)+q(t,u) = (j,m)$ so that $j$ divides both $r$ and $t$. In $\mathbb{Z}_a \times \mathbb{Z}_b$ the point $(j,m) \in \mathbb{Z}^2$ becomes $(j,m_b)$. Now there are positive integers $x$ and $y$ so that $x(j,m_b) = (r,xm_b)$ and $y(j,m_b) = (t,ym_b)$. If $xm_b \not\equiv s \mod b$ then $Q$ has an intersection with the vertical axis in $\mathbb{Z}_a \times \mathbb{Z}_b$ away from $(0,0)$ and if $ym_b \not\equiv u \mod b$ then $Q$ has an intersection with the vertical axis of $\mathbb{Z}_a \times \mathbb{Z}_b$ away from $(0,0)$. Since neither of these intersections can exist, by the definitions of $a$ and $b$, we see that $(r,s)$ and $(t,u) \in \langle(j,m_b) \rangle$ in $Q$. In particular, after a finite induction we see that $Q$ is cyclic.

\[\diamond\]

7 The structure of the extension

We have now done enough work so that Theorem 1.6 is transparent.
Theorem 1.6

1. The map \( \phi \) induces a one-one correspondence between the finite index subgroups of \( F \) and the finite index subgroups of \( \mathbb{Z}^2 \).

2. Let \( H \) be a finite index subgroup \( H \) of \( F \), with image \( \tilde{H} = \phi(H) \leq \mathbb{Z} \times \mathbb{Z} \). There exist smallest positive integers \( a \) and \( b \) with \( K_{(a,b)} \leq H \).

Furthermore, if \( Q = \tilde{H}/\tilde{K}_{(a,b)} \), then \( Q \) is finite cyclic, and there are maps \( \iota, \rho, \tilde{\iota} \) and \( \tilde{\rho} \) so that the diagram below commutes with the two rows being exact:

\[
\begin{array}{c}
F \\
\cong \\
1 \\
\phi|_K \\
\phi|_H \\
1 \\
1 \\
\end{array}
\begin{array}{c}
K \\
\iota \\
\phi|_K \\
\phi|_H \\
\tilde{\iota} \\
\tilde{\rho} \\
\end{array}
\begin{array}{c}
H \\
\rho \\
Q \\
\end{array}
\begin{array}{c}
1 \\
\end{array}
\]

Proof: The first point follows from Lemma 1.3 and the fact that the kernel of \( \phi \) is \( F' \).

The second point follows from a conglomeration of lemmas.

The existence of minimal positive integers \( a \) and \( b \) (so that \( K_{(a,b)} \) is maximal in \( H \)) follows from the existence of a maximal \( \tilde{K}_{(a,b)} \) in \( \tilde{H} \), which is lemma 6.1.

The fact that \( Q \) is finite cyclic comes from Lemma 6.2.

The isomorphism from \( F \) to \( K = K_{(a,b)} \) comes from Theorem 1.1.

The map \( \iota \) is the inclusion map of \( K_{(a,b)} \) into \( H \). The map \( \tilde{\iota} \) is induced from the projection \( \phi \). The map \( \tilde{\rho} \) is the natural quotient onto \( Q \) of the image of \( \iota \) in \( \tilde{H} \). The bottom row is thus exact. \( \rho \) is the composition of the natural quotient of \( H \) by the image of \( \iota \) followed by the isomorphism from \( H/\iota(K) \) to \( \tilde{H}/\tilde{\iota}(\tilde{K}) = Q \), thus, the top row is exact, and the diagram commutes.

To prove Lemma 1.7 we will make use of Rubin’s Theorem. The version we will quote is Theorem 2 in Brin’s paper [3]. That version is itself derived from Theorem 3.1 in the paper [8] of Rubin, where in the statement of the theorem, a technical hypothesis is inadvertently missing (see the discussion of this in [3]).

In order to state Rubin’s Theorem, we will need to define some terminology. In this, we generalize the language of the definition of locally
dense given in Brin's [3]. Our generalization will have no impact on the content of our statement of Rubin's theorem. Suppose $X$ is a topological space and $H(X)$ is its full group of homeomorphisms. Suppose further that $K \leq H(X)$. Given $W \subset X$, we will say $K$ acts locally densely over $W$ if for every $w \in W$ and every open $U \subset W$ with $w \in U$, the closure of

$$\{ w\kappa | \kappa \in K, \kappa |_{(W-U)} = 1_{(W-U)} \}$$

contains some open set in $W$. In particular, for each open $U$ in $Z$, the subgroup of elements fixed away from $U$ has every orbit in $U$ dense in some open set of $W$ in $U$.

We are now ready to state Rubin's theorem. We give essentially the statement given in [3], although we recast it in the language of right actions.

**Theorem 7.1 (Rubin)** Let $X$ and $Y$ be locally compact, Hausdorff topological spaces without isolated points, Let $H(X)$ and $H(Y)$ be the self homeomorphism groups of $X$ and $Y$, respectively, and let $G \subseteq H(X)$ and $H \subseteq H(Y)$ be subgroups. If $G$ and $H$ are isomorphic and both act locally densely over $X$ and $Y$, respectively, then for each isomorphism $\varphi : G \rightarrow H$ there is a unique homeomorphism $\gamma : X \rightarrow Y$ so that for each $g \in G$, we have $g\varphi = \gamma^{-1}g\gamma$.

In our case, and to apply Rubin's theorem to $F$ or subgroups of $F$, we need to consider these groups to be groups of homeomorphisms of $(0,1)$, instead of $[0,1]$. This comes from the simple fact that $F$ does not move 0 or 1 to produce a dense image in any open set!

Having made that (temporary) change to our definition of $F$ and its subgroups, we are ready to apply Rubin’s theorem to any such subgroup, as long as it is locally dense in its action on $(0,1)$.

In the discussion which follows, given $X \subset R$, we will use the notation $D_X$ to denote the set $Z[1/2] \cap X$ of diadic rationals in $X$.

**Lemma 7.2** Finite index subgroups of $F$ act locally densely on $(0,1)$.

**Proof:**

Suppose $H$ is finite index in $F$, and $x \in (0,1)$ and $U$ an open neighborhood of $x$ in $(0,1)$. Let $d_1$ and $d_2$ be two diadic rationals in $U$ with $d_1 < x < d_2$, let $K$ be the subgroup of $F$ consisting of all the elements of $F$ with support in $(d_1, d_2)$. Let $\alpha : R \rightarrow R$ be any piecewise-linear homeomorphism which is the identity near $\pm \infty$ and which has all slopes integral powers of 2, and with all breaks occuring over the diadic rationals, and that
maps \(d_1\) to 0 and \(d_2\) to 1. It is easy to build such a map, and the reader may check that the inner automorphism of \(\text{Homeo}(\mathbb{R})\) generated by conjugation by \(\alpha\) will take \(K\) isomorphically to \(F\).

Now by an induction argument (for instance, as carried out in the first paragraph of Section 3.1. in [6]), it is easy to see that \(\alpha\) takes \(D(d_1,d_2)\) to \(D(0,1)\) in an order preserving fashion. In particular, as \(F\) is \(k\)-transitive on \(D(0,1)\) for any positive integer \(k\) (recall Lemma 3.5), we see that \(K\) is \(k\)-transitive on \(D(d_1,d_2)\) for any positive integer \(k\).

Now, if \(x\) is a diadic rational, then the orbit of \(x\) under \(K\) is dense in \((d_1,d_2)\), as \(K\) acts transitively over \(D(d_1,d_2)\), and \((d_1,d_2)\) is dense in \((d_1,d_2)\).

If \(x\) is not diadic rational, then given any \(\varepsilon > 0\), and any \(y\) in \((d_1,d_2)\), we can find four diadic rationals \(x_1,x_2,y_1,y_2 \in (d_1,d_2)\) so that \(x_1 < x < x_2\) and \(y_1 < y < y_2\), and where the \(y_i\) are chosen epsilon-close to \(y\). Now there is some element \(\kappa\) in \(K\) which throws \(x_1\) to \(y_1\) and \(x_2\) to \(y_2\) (since \(K\) is 2-transitive over \(D(d_1,d_2)\)). In particular, \(|y - x\kappa| < \varepsilon\). Hence, the orbit of \(x\) is dense in \((d_1,d_2)\).

We will use Rubin’s theorem to prove the final lemma from the introduction.

**Lemma 1.7**

Suppose \(H, H' \in FIF\), \(K = \text{Inner}(H)\), \(K' = \text{Inner}(H')\), and \(\xi : H \to H'\) is an isomorphism. Then

1. \(\xi(K) = K'\)

2. \(K\) is characteristic in \(H\) and \(K'\) is characteristic in \(H'\), and

3. \(\text{Res}(H) = \text{Res}(H')\).

**Proof:**

First, let us suppose \(\vartheta : H \to H'\) is an isomorphism.

By Lemma 7.2, \(H\) and \(H'\) both act locally densely on \((0,1)\). In particular, Rubin’s theorem tells us that there is a homeomorphism \(\gamma : (0,1) \to (0,1)\) so that for any \(h \in H\), \(\vartheta(h) = \gamma^{-1}h\gamma \in H'\).

Now by Lemma 3.5, we see that the collection of orbitals of \(h' = \vartheta(h)\) is in bijective correspondence with the orbitals of \(h\).

Further, if \(\gamma\) is orientation-preserving, any orbital of \(h\) which has end \(e \in \{0,1\}\) becomes (under the action of \(\gamma\)) an orbital of \(h'\) with end \(e\). If \(\gamma\) is orientation-reversing, then any orbital of \(h\) with end \(e \in \{0,1\}\) becomes an orbital of \(h'\) with end \(f \neq e\), where \(f \in \{0,1\}\).
Since \( \vartheta \) is a homomorphism, a consequence of the above paragraph is if \( K = K_{(a,b)} \) for some positive integers \( a \) and \( b \), then there are positive integers \( c \) and \( d \) with \( \vartheta(K) = K_{(c,d)} \leq K' \leq H' \).

The correspondence theorem now tells us that the maximal rectangular groups of \( H \) and \( H' \) are mapped precisely to each other by \( \vartheta \), and we have point (1).

The second two points follow immediately.

Note that this argument provides a second proof that amongst the finite index subgroups of \( F \), only the rectangular groups are actually isomorphic to \( F \).

The lemma above provides the key ingredients for the proof of our final theorem.

Recall the isomorphisms \( \tau_{(a,b)} : K_{(a,b)} \to F \) from the introduction (elements of \( K_{(a,b)} \) with slope \( (2^a)^s \) near zero are taken to elements of \( F \) with slope \( 2^s \) near zero, and elements of \( F \) with slope \( (2^b)^t \) near one are taken to elements of \( F \) with slope \( 2^t \) near one), and the map Outer which, given a finite index subgroup \( H \) of \( F \), produces the smallest rectangular subgroup of \( F \) that contains \( H \). With these maps in mind, and with the above lemma in hand, we are finally ready to prove our last theorem.

**Theorem 1.8:**

Suppose \( H, H' \) are finite index subgroups of \( F \). Let \( a, b, c, d \) be positive integers so that \( K_{(a,b)} = \text{Outer}(H) \), \( K_{(c,d)} = \text{Outer}(H') \). \( H \) is isomorphic with \( H' \) if and only if \( \tau_{(a,b)}(H) = \tau_{(c,d)}(H') \) or \( \tau_{(a,b)}(H) = \text{Rev}(\tau_{(c,d)}(H')) \).

*Proof:*

Suppose that \( \vartheta : H \to H' \) is an isomorphism. Lemma 1.7 assures us that there is a well defined positive integer \( n \) so that \( \text{Res}(H) = \text{Res}(H') = n \), and \( \vartheta(K) = K' \). Let us further suppose that \( K_{(r,s)} = \text{Inner}(H) \) and \( K_{(t,u)} = \text{Inner}(H') \).

Let \( \tilde{H} = \phi(H) \), and \( \tilde{H'} = \phi(H') \). Consider the translations of \( \mathbb{Z}^2 \) generated by \( (r,0) \) and \( (0,s) \). Since \( \tilde{H} \) is a group, the sets \( \tilde{H}, \tilde{H} + (r,0), \) and \( \tilde{H} + (0,s) \) are the same. In particular, we can consider the image in the lattice \( \mathbb{Z}^2 \) of \( \tilde{H} \), restricting our view to the rectangle \( R \) of points with integer coordinates where the horizontal coordinates range from 0 to \( r - 1 \) and the vertical coordinates range from 0 to \( s - 1 \), and understand everything about the group \( \tilde{H} \). \( K_{(r,s)} \) only intersects \( R \) at \( (0,0) \), while there are \( n \) total intersections of \( \tilde{H} \) with \( R \), all obtained by translating different powers of some particular vector \( (p,q) \) into \( R \) (using \( (r,0) \) and \( (0,s) \)). Let \( j = \gcd(p,r) \). So, the lowest column number that the image of the translated powers of
(p, q) in R will appear in is column j. Since \( \tilde{H} \) intersects R exactly n times, it must be that case that \( nj = r \) and the images of the translated powers of \((p, q)\) in R will occur in columns 0, \( r/n \), \( 2r/n \), \( (n-1)r/n \). Similarly, \( \gcd(q, s) = s/n \) and the images of the translated powers of \((p, q)\) in R will occur in rows 0, \( s/n \), \( 2s/n \), \( (n-1)s/n \). Now, as \( \tilde{K}_{(a,b)} \) is the smallest rectangular group to contain \( \tilde{H} \), we see that \( a = r/n \) and \( b = s/n \). A similar discussion shows that \( c = t/n \) and \( d = u/n \). Stated another way, we have

\[
\frac{r}{a} = \frac{s}{b} = \frac{t}{c} = \frac{u}{d} = n.
\]

Now, consider the image of \( H \) and \( H' \) under the respective maps \( \tau_{(a,b)} \) and \( \tau_{(c,d)} \). The subgroups \( K_{(r,s)} = \text{Inner}(H) \) and \( K_{(t,u)} = \text{Inner}(H') \) are both taken to \( K_{(n,n)} \). We will now assume that this is how \( H \) and \( H' \) started out, and do all remaining work in these scaled versions of \( H \) and \( H' \).

The isomorphism \( \vartheta \) which is carrying \( H \) to \( H' \) must now preserve the maximal rectangular subgroup \( K_{(n,n)} \), by Lemma 1.7. By Lemma 7.2, both \( H \) and \( H' \) act locally densely on \((0,1)\), so by Rubin’s theorem there is a homeomorphism \( \gamma \) so that for any \( h \in H \), \( \vartheta(h) = \gamma^{-1} h \gamma \in H' \).

Note that as \( \gamma \) need not be piecewise-linear, we should be concerned that conjugating by \( \gamma \) might change slopes, as well as potentially swapping coordinates.

By Lemma 1.7 we know that \( K_{(n,n)} = \text{Inner}(H) = \text{Inner}(H') \) is being brought isomorphically to itself by \( \vartheta \). Suppose \( h \in H \) has an orbital \( A \). Denote by \( E_A \) the set of ends of \( A \) which are in the set \( \{0,1\} \). Now consider \( h' = \vartheta(h) \). The element \( h' \) has an orbital \( B = \gamma(A) \) by point (1) of Lemma 8.6. Denote by \( E_B \) the ends of \( B \) that are actually in the set \( \{0,1\} \). Then as \( \gamma \) preserves the set \( \{0,1\} \) we see that the cardinalities of \( E_A \) and \( E_B \) must be the same.

Now, by the result of the previous paragraph, and using the fact that the \( \vartheta \) takes \( K_{(n,n)} \) isomorphically to itself, we see that if \( \gamma \) is orientation-preserving, we must have that \( \gamma \) will send \( (n,0) \) to \( (n,0) \) and \( (0,n) \) to \( (0,n) \) in the induced map from \( \phi(H) \to \phi(H') \) (note that \( (n,0) \) will not be taken to \( (-n,0) \), as conjugation by an orientation-preserving \( \gamma \) will preserve the local directions that points near zero and one move under the action of \( h \)). Similarly, if \( \gamma \) is orientation-reversing, then the reader can check that the action of \( \gamma \) will send \( (n,0) \) to \( (0,n) \) and \( (0,n) \) to \( (n,0) \), again considering the induced map from \( \phi(H) \to \phi(H') \).

If \( \gamma \) is orientation-reversing, then replace \( H' \) by the isomorphic copy \( Rev(H') \), so that from here out we only need to argue the case where our
Suppose $f \in H$. $\phi(f) = (v, w)$, if and only if $\phi(\vartheta(f)) = (v, w)$. But now as $H$ and $H'$ both contain the commutator subgroup $F'$, and as they each contain an element which has slopes $2^v$ and $2^w$ near zero and one respectively, we see that both $H$ and $H'$ contain all of the elements of $F$ with $\phi(k) = (v, w)$. It is now immediate that $H = H'$.

Now let us suppose that $H$ and $H'$ are finite index subgroups of $F$, and that $K_{(a,b)} = \text{Outer}(H)$ and $K_{(c,d)} = \text{Outer}(H')$. Let us further suppose that the scaling maps $\tau_{(a,b)}$ and $\tau_{(c,d)}$ have the property that $\tau_{(a,b)}(H) = \tau_{(c,d)}(H')$ or $\tau_{(a,b)}(H) = \text{Rev}(\tau_{(c,d)}(H'))$. Since the $\tau_{(a,b)}$ maps are isomorphisms, and the map Rev is an isomorphism, we immediately see that $H$ and $H'$ are isomorphic.

8 Some examples

In this section, we give some examples of finite index subgroups of $F$, and consider them from the perspective of this paper.

Example 1:
Let $H = \{ f \in F \mid f'(0) = 2^{2n+5m} \text{ and } f'(1) = 2^{7n+11m} \text{ for some } m, n \in \mathbb{Z} \}$. $H$ is a finite index subgroup of $F$ but $H$ is not isomorphic to $F$.

Let $\tilde{H} = \phi(H)$. Since 3, 5, 7, and 11 are all odd, the only possible elements of $\tilde{H}$ are (even, even) and (odd, odd). If $n = 5$ and $m = -3$, then $(15 - 15, 35 - 33) = (0, 2) \in \tilde{H}$. If $n = -11$ and $m = 7$, then $(-33 + 35, -77 + 77) = (2, 0) \in \tilde{H}$. So then every (even, even) is in $\tilde{H}$. If $n = -3$ and $m = 2$, then $(1, 1) \in \tilde{H}$. Since $(1, 1)$ and all (even, even) are in $\tilde{H}$, then all (odd, odd) are also in $\tilde{H}$. So $\tilde{H}$ is index 2 in $\mathbb{Z} \oplus \mathbb{Z}$ and $H$ is index 2 in $F$.

To show that $H$ is not isomorphic to $F$, it is enough to show that $H \neq K_{(a,b)}$ for any non-zero integers $a$ and $b$.

Assume that for some non-zero integers $a$ and $b$, $H = K_{(a,b)}$. If $h \in H$, there are integers $p$ and $q$ such that $\phi(h) = (ap, bq)$.

There is an $f \in H$ so that $\phi(f) = (3, 7)$. There is a $g \in H$ so that $\phi(g) = (5, 11)$. Now, there must be integers $p_1$ and $p_2$ so that $ap_1 = 3$ and $ap_2 = 5$. Thus $a = 1$. Also, there must be integers $q_1$ and $q_2$ so that $bq_1 = 7$ and $bq_2 = 11$. So $b = 1$. But $K_{(1,1)} = F \neq H$, so $H$ can not be isomorphic to $F$.

In the above example, note that the maximal $\tilde{K}_{(a,b)}$ group was $\tilde{K}_{(2,2)}$.  

which was proper in $\tilde{H}$, and that the quotient $\tilde{H}/\tilde{K}_{(2,2)} \cong Z_2$. In particular, $H$ is isomorphic to a non-split extension of $F$ by $Z_2$, where the structure of the extension is described by the structure of $\tilde{H}$ as an extension of $\tilde{K}_{(2,2)}$ by $Z_2$.

**Example 2:**
Let $l, r, f, f' \in F$, so that $\phi(l) = (15, 0)$, $\phi(r) = (0, 15)$, $\phi(f) = (3, 3)$, and $\phi(f') = (3, 6)$. Suppose that $H = \langle F', l, r, f \rangle$ while $H' = \langle F', l, r, f' \rangle$.

We see immediately that the maximal rectangular subgroups of $H$ and $H'$ are $K_{(15,15)}$. The minimal rectangular subgroups of $F$ containing $H$ and $H'$ are the same, namely $K_{(3,3)}$. The residues of $H$ and $H'$ are both 5, but $H$ and $H'$ are not isomorphic, as $\tau_{(3,3)}(H) \neq \tau_{(3,3)}(H')$ and $\tau_{(3,3)}(H) \neq \text{Rev}(\tau_{(3,3)}(H'))$.

Below is included a diagram of the rectangle $R$ in $Z^2$ which demonstrates this non-equality.

**Example 3:**
Let $l_1, l_2, r_1$ and $r_2 \in F$ so that $\phi(l_1) = (10, 0)$, $\phi(l_2) = (35, 0)$, $\phi(r_1) = (0, 15)$, and $\phi(r_2) = (0, 20)$. Further, let $g_1, g_2 \in F$ so that $\phi(g_1) = (2, 6)$ and $\phi(g_2) = (14, 4)$. Let $H = \langle F', l_1, r_1, g_1 \rangle$ and let $H' = \langle F', l_2, r_2, g_2 \rangle$.

It is immediate that Inner $H = K_{(10,15)}$, Outer $H = K_{(2,3)}$, Inner $H' = K_{(35,20)}$, and Outer $H' = K_{(7,4)}$. Both $H$ and $H'$ have residue 5. If we apply $\tau_{(2,3)}$ to Outer $H$ and $\tau_{(7,4)}$ to Outer $H'$, and draw our fundamental $5 \times 5$ rectangle in $Z^2$, we get the following diagram. (Below, we are considering $H$ and $H'$ after the rescaling.)
The scaled version of $H$ is $Rev$ of the scaled version of $H'$, so that $H \cong H'$. 
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