On the exact solutions for a type of nonlinear Schrödinger equations with a harmonic potential

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Abstract. The nonlinear Schrödinger equation with harmonic potential (NLSE) plays an important role in quantum mechanics, so the exact solutions of this equation is studied in this paper. The NLSE is transformed into the classical nonlinear Schrödinger equation by a new class of traveling wave transformation. Next, the problem of exact solutions is changed into the solutions of ordinary differential equation (ODE) by the method of undetermined function. Then, through low-order sub-ODE method and hyperbolic function method, we get two classes of solutions of the ODEs. Finally, a series of new exact solutions of the NLSE are obtained.

1. Introduction

In this paper, we study the exact solutions of nonlinear Schrödinger equations with a harmonic potential.

\[ i\dot{\psi} - \frac{\omega^2}{4} x^2 \psi + k \psi = 0, \quad t \geq 0, \quad x \in \mathbb{R} \]

(1)

Where \( \psi = \psi(t, x) \) is wave function, \( t \) and \( x \) represent time and space, respectively, \( \frac{\omega^2}{4} x^2 \) represent the external potential, \( k \neq 0 \) is a real parameter.

The nonlinear Schrödinger equation has received great attention in the recent years, which is applicable to a range of phenomena, such as the description of dipolar quantum gases, the application in laser beam, the Bose Einstein condensation of atoms in lasers [1-8]. In this paper, we are interested in the exact solutions of NLSE, and the work in this direction was strongly stimulated in Optical Solutions [9]. According to our literature research, a few scholars have worked out the exact solutions of NLSE.

So far, many effective methods have been proposed to nonlinear Schrödinger equation [9-19], such as complete discrimination system method [9] and sub-equation method [18]. M. L. Wang et.al, got few exact solutions of nonlinear Schrödinger equation by Sub-ODE method [17,18]. At present, many scholars have studied the exact solutions of the nonlinear Schrodinger equation [20,21]. In this paper, we have more exact solutions of NLSE by low-order Sub-ODE method. K. Hosseini used to have obtain the exact solutions of cubic nonlinear Schrödinger equation based on hyperbolic function method. But we have exact solutions of NLSE with five the power nonlinear term by this method. We thank Rémi
carles with the work in the direction of exact solutions of NLS. According to [10], we can draw the following expression:

$$\psi(t, x) = \frac{1}{\sqrt{\cosh \omega t}} e^{-i \frac{\omega x}{2} \tanh \omega t} \sqrt{\frac{\tanh \omega t}{\cosh \omega t}} v(\omega, x).$$  \quad (2)

The equation (1) is reduced to the following nonlinear Schrödinger equation.

$$iv_t - v_{xx} + k|v|^4 = 0, \quad t \geq 0, \quad x \in \mathbb{R}.$$  \quad (3)

Then, we do the following transformation

$$v(t, x) = \varphi(\tilde{\alpha}) e^{i(\theta(\tilde{\alpha}) + at)},$$  \quad (4)

where $\tilde{\alpha} = x - c_g t$ and $c_g$ is arbitrary constant. From (3) and (4), we obtain a set of equations for $\varphi$ and $\theta$. Assuming that $\theta$ is linear function of the $\tilde{\alpha}$, we have $\theta = -\frac{c_g}{2} \tilde{\alpha}$ (the constant of integral is zero). Then, we assume $\varphi^2 = y$, we can get the following expression:

$$2yy'' - (y')^2 + 4Ay^3 - 4ky^4 = 0.$$  \quad (5)

2. **Low-order Sub-ODE method**

In this section, we solve the nonlinear partial differential by low-order Sub-ODE [17,18]. There are the following four steps.

Step 1 : supposing that the following form of partial differential equation for $y(t,x)$ is given by

$$F(y, y', y'', y''', y'''', \ldots) = 0.$$  \quad (6)

Making traveling wave transformation

$$y(t, x) = y(\xi), \quad \xi = px + qt,$$  \quad (7)

where $p, q$ are constants. And $y = y(\xi)$ can be expressed by

$$y(\xi) = \sum_{i=0}^{n} a_i F^i,$$  \quad (8)

where $a_i (i = 0, 1, 2, \cdots, n)$ are constants which are undetermined later. The traveling wave variable (7) permits reducing partial differential equation (6) to an ODE for $y(\xi)$.

$$F(y, \omega y', ky', \omega^2 y'', k^2 y'''', \cdots) = 0,$$  \quad (9)

where $\omega$ and $k$ are arbitrary constants. Supposing that there is a first order solvable ODE that is called Sub-equation

$$\left( \frac{dF}{d\xi} \right)^2 = c_2 F(\xi)^2 + c_3 F(\xi)^3 + c_4 F(\xi)^4.$$  \quad (10)

From (10), we have

$$\frac{d}{d\xi} = \sqrt{\sum_{j=2}^{4} c_j F^j} \frac{d}{dF}, \quad \frac{d}{d\xi} = \sqrt{\sum_{j=2}^{4} c_j F^j} \frac{d}{dF},$$  \quad (11)
Step 2 : \( n \) can be determined by the homogeneous balance between the highest order derivative of \( F(\xi) \) and the highest order nonlinear term of \( F(\xi) \).

Step 3 : Substituting (8) and (10) into ODE (9), then the left hand side of ODE (9) is converted into a polynomial in \( F \), equating each coefficient of the polynomial to zero yields a set of algebraic equations for \( a_i (i = 0,1,2,\ldots,n) \) and \( c_i (i = 2,3,4) \).

Step 4 : Solving the algebraic equation obtained in step 3, and substituting the results into (8), then we have the traveling wave solutions of (6). According to the step 3, we have

\[
y(\xi) = a_0 + a_1 F(\xi),
\]

where \( a_0 \) and \( a_1 \) are arbitrary constants and cannot be zero together.

According to (5), (10) and (13), the left side of (5) is converted into a polynomial that the terms with \( F'(i=0,1,2,3,4) \) included. Equating the coefficients of \( F'(i=0,1,2,3,4) \) to zero respectively, we reach a system of algebraic equations

\[
\begin{align*}
4Aa_0^2 + 4a_4^4 &= 0, \\
2a_0a_1c_2 + 8Aa_0a_1 + 16a_0^2a_1 &= 0, \\
3a_0a_1c_3 + 2a_1^2c_3 - a_1^2c_2 + 4Aa_1^2 + 24a_0^2a_2 &= 0, \\
4a_0a_1c_4 + 3a_1^2c_3 - a_1^2c_2 + 16a_0a_1^3 &= 0, \\
4a_1^2c_4 - a_1^2c_4 - 4ka_4^4 &= 0.
\end{align*}
\]

By solving algebraic equations (13), we get

\[
c_2 = -4A, c_4 = \frac{4a_1^2k}{3}, a_0 = 0, c_3 = 0,
\]

where \( a_1 \neq 0 \) is constant.

Case 1 : when \( c_2 > 0 \) and \( c_4 > 0 \), in other words, \( \frac{c_2^2}{4} + n < 0 \) and \( k > 0 \).

\[
\psi_1(t,x) = \frac{1}{\sqrt{\cosh(\omega t)}} Q \frac{c_{\text{sech}}^2 \frac{\sqrt{c_2}}{2}(P)}{2c_{\text{sech}}^2 \frac{\sqrt{c_2}}{2}(P)},
\]

\[
\psi_2(t,x) = Q \frac{4c_2(c_2 - c_4)}{4c_2c_4 - (\cosh \sqrt{c_2}(P) + \sinh \sqrt{c_2}(P))^2} \frac{1}{\sqrt{\cosh(\omega t)}}.
\]

\[
\psi_3(t,x) = Q \frac{2c_2 \text{sech} \sqrt{c_2}(P)}{c_2(c_2 - c_4) - c_3(c_2 + c_4) \tanh \sqrt{c_2}(P)} \frac{1}{\sqrt{\cosh(\omega t)}}.
\]

\[
\psi_4(t,x) = Q \frac{c_2(-1 + (\tanh \sqrt{c_2}(P) + \text{sech} \sqrt{c_2}(P))^2)}{2c_2c_4(\tanh \sqrt{c_2}(P) + \text{sech} \sqrt{c_2}(P))} \frac{1}{\sqrt{\cosh(\omega t)}}.
\]
\[ \psi_5(t, x) = \sqrt{a_1 c_z (-1 + \tanh(c_z(P)) - \text{sech}\sqrt{c_z(P)})^2} \frac{1}{2\sqrt{c_z c_4} (\tanh(c_z(P)) - \text{sech}\sqrt{c_z(P)})} \sqrt{\text{csch} \left(\text{csch} \frac{\omega t}{2}\right)}. \] 

\[ \psi_6(t, x) = \frac{1}{\sqrt{\cosh \left(\text{csch} \frac{\omega t}{2}\right)}} \frac{1}{\sqrt{a_1 c_z c_4}} \sqrt{\text{csch} \sqrt{c_z(P)}}. \] 

Case 2: when \( c_2 > 0 \) and \( c_4 < 0 \), in other words, \( \frac{c_2}{4} - n < 0 \) and \( k < 0 \).

\[ \psi_7(t, x) = \frac{1}{\sqrt{\cosh \left(\text{csch} \frac{\omega t}{2}\right)}} \frac{1}{\sqrt{a_1 c_z}} \frac{c_z}{\sqrt{-c_z c_4}} \sqrt{\text{sech} \sqrt{c_z(P)}}. \] 

Case 3: when \( c_2 > 0 \) and \( c_4 > 0 \), in other words, \( \frac{c_2}{4} - n > 0 \) and \( k > 0 \).

\[ \psi_8(t, x) = \frac{1}{\sqrt{\cosh \left(\text{csch} \frac{\omega t}{2}\right)}} \frac{1}{\sqrt{a_1 c_z}} \frac{c_z \sec^2 \sqrt{-c_z(P)}}{2} \sqrt{2\sqrt{-c_z c_4} - \sqrt{-c_z c_4} \sec \sqrt{-c_z(P)}}. \] 

\[ \psi_9(t, x) = \frac{1}{\sqrt{\cosh \left(\text{csch} \frac{\omega t}{2}\right)}} \frac{1}{\sqrt{a_1 c_z}} \frac{c_z \sec^2 \sqrt{-c_z(P)}}{2} \sqrt{-2\sqrt{-c_z c_4} + \sqrt{-c_z c_4} \sec \sqrt{-c_z(P)}}. \] 

\[ \psi_{10}(t, x) = \frac{1}{\sqrt{\cosh \left(\text{csch} \frac{\omega t}{2}\right)}} \frac{1}{\sqrt{a_1 c_z}} \frac{c_z \sec^2 \sqrt{-c_z(P)}}{2} \sqrt{2\sqrt{-c_z c_4} \tan \sqrt{-c_z(P)}}. \] 

\[ \psi_{11}(t, x) = \frac{1}{\sqrt{\cosh \left(\text{csch} \frac{\omega t}{2}\right)}} \frac{1}{\sqrt{a_1 c_z}} \frac{c_z (1 + \tan \sqrt{-c_z(P)} + \sec \sqrt{-c_z(P)})^2}{2\sqrt{-c_z c_4} (\tan \sqrt{-c_z(P)} + \sec \sqrt{-c_z(P)})} \sqrt{\text{csch} \left(\text{csch} \frac{\omega t}{2}\right)}. \] 

\[ \psi_{12}(t, x) = \sqrt{a_1 c_z (1 + \tan \sqrt{-c_z(P)} + \sec \sqrt{-c_z(P)})^2} \frac{1}{2\sqrt{-c_z c_4} (\tan \sqrt{-c_z(P)} - \sec \sqrt{-c_z(P)})} \sqrt{\text{csch} \left(\text{csch} \frac{\omega t}{2}\right)}. \] 

\[ Q = e^{i(-\frac{at}{2} \tanh \omega t - \frac{c_2}{2} \tanh \frac{\omega t}{2} + \frac{c_2 \tanh \omega t}{2} + \frac{\tan \omega t}{\omega})}, \quad P = \frac{x}{\cosh \omega t} - \frac{c_2 \tanh \omega t}{\omega}. \] 

Where \( c_2 < 0 \) and \( c_4 < 0 \), we do not find any exact solutions of (1) now.
3. Hyperbolic function method

The paper [19] provided a sufficient description of principle of hyperbolic function method. In this paper, we construct the exact solutions of ODE (5) by using hyperbolic function method. Then, we construct the exact solutions of equation NLSE from two aspects:

\[ \frac{dw}{d\xi} = \sinh w \quad \text{and} \quad \frac{dw}{d\xi} = \cosh w. \]

3.1. Assumption 1: \( \frac{dw}{d\xi} = \sinh w \).

Considering the homogeneous balance between \( y^{'''} \) and \( y^4 \), we assume that

\[ y(\xi) = b_0 + a \sinh w + b \cosh w, \quad (27) \]

where \( a \) and \( b \) cannot be zero together. According to (5), we derive a system of nonlinear algebraic equations.

\[
\begin{align*}
2b_0b + 8Ab_0 - 4k(12b_0a^3b + 4b_0^3b) &= 0, \\
4b_0a - 4k(12b_0ab^2 + 4b_0^3a) &= 0, \\
4b_0b - 2ab - 4k(12b_0a^2b + 4b_0^3b) &= 0, \\
2ab + 8Aab - 4k(12b_0^2ab + 4a^3b) &= 0, \\
2a^2 + b^2 + 4Aa^2 - 4k(6b_0^2a^2 + 6b_0^3b^2 + 2a^4) &= 0, \\
3a^2 + 3b^2 - 4k(6a^2b^2 + a^4 + b^4) &= 0, \\
6ab - 4k(4ab^3 + 4a^3b) &= 0, \\
4A(b_0^2 + a^2) - 4k(b_0^4 + a^4 + 6b_0^3a^2) &= 0, \\
8A_0a - 4k(4b_0a^3 + 4b_0^3a) &= 0.
\end{align*}
\]

Case 1: \( b_0 = 0, \ a = 0, \ b = -\frac{3}{\sqrt{4k}} \), \( 4A = -1, \ k > 0 \). Therefore, we have the exact solutions of (1)

\[
\psi_{13}(t,x) = \left(\frac{3}{4k}\right)^{\frac{1}{2}} \sqrt{\text{csch}(P)} \frac{1}{\sqrt{\cosh \omega t}} Q, \quad \psi_{14}(t,x) = -\left(\frac{3}{4k}\right)^{\frac{1}{2}} \sqrt{\text{csch}(P)} \frac{1}{\sqrt{\cosh \omega t}} Q. \quad (29)
\]

Case 2: \( b_0 = 0, \ a = 0, \ b = \frac{3}{\sqrt{4k}} \), \( 4A = -1, \ k > 0 \). We obtain the exact solutions of (1)

\[
\psi_{15}(t,x) = i\left(\frac{3}{4k}\right)^{\frac{1}{2}} \sqrt{\text{csch}(P)} \frac{1}{\sqrt{\cosh \omega t}} Q, \quad \psi_{16}(t,x) = -i\left(\frac{3}{4k}\right)^{\frac{1}{2}} \sqrt{\text{csch}(P)} \frac{1}{\sqrt{\cosh \omega t}} Q. \quad (30)
\]

\[ Q = e^{i \left(\frac{a}{4} \tan \omega t - \frac{c_2}{2} \tanh \frac{\omega t}{2} - \frac{c_2}{2} \tan \omega t \right)} P = \frac{x}{\cosh \omega t} - \frac{c_2}{\omega} \tanh \omega t. \]

Where
3.2 Assumption 2: \( \frac{dw}{dz} = \cosh w \).

Case 1: \( b_0 = 0, a = 0, b = \frac{3}{\sqrt{4k}}, 4A = 1, k > 0 \). Therefore, the following exact solutions to the (1) can be gained.

\[
\psi_{17}(t, x) = \left( \frac{3}{4k} \right)^\frac{1}{2} \sqrt{\csc(P)} \frac{1}{\sqrt{\cosh \omega t}} Q, \quad \psi_{18}(t, x) = -\left( \frac{3}{4k} \right)^\frac{1}{2} \sqrt{\csc(P)} \frac{1}{\sqrt{\cosh \omega t}} Q. \tag{31}
\]

Case 2: \( b_0 = 0, a = 0, b = \frac{3}{\sqrt{4k}}, 4A = 1, k > 0 \). Consequently, the following exact solutions to the (1) can be received. Where \( Q = e^{-\frac{\omega}{4} \tanh \omega - \frac{x}{2} \cosh \omega + \frac{c_g}{\alpha} \tanh \omega - \frac{c_g}{\alpha} \tanh \omega}, P = \frac{x}{\cosh \omega t} - \frac{c_g}{\omega} \tanh \omega t \).

4. Numerical simulations

In this section, we select some specific parameters to simulate exact solutions of (1) in three-dimensional space. Fig. 1 (a) — Fig. 3 (a) signify the envelope of \( \psi_7, \psi_8 \) and \( \psi_{10} \), respectively. Fig. 1 (b) — Fig. 3 (b) represent the level curve at different time \( t \) in one dimensional space. Fig. 1 (c) — Fig. 3 (c) represent the level curve at different \( x \) in one dimensional space.

Fig.1 Solution of (1) for \( \psi_7 \) at \( \omega = a_1 = c_4 = c_2 = c_5 = 1 \). (a) Perspective view of the wave. (b) The wave along the \( x \) axis, R : t = 0; G : t = 1 ; B : t = 4. (c) The wave along the \( t \) axis, R : x = 0; G : x = 2 ; B : x = 4.

Fig.2 Solution of (1) for \( \psi_9 \) at \( \omega = a_1 = c_4 = c_5 = 1, c_2 = -4 \). (a) Perspective view of the wave. (b) The wave along the \( x \) axis, R : t = 0; G : t = 1 ; B : t = 2. (c) The wave along the \( t \) axis, R : x = 0; G : x = 1 ; B : x = 2.
Fig. 3 Solution of (1) for $\psi_{19}$ at $\omega = c_g = 1, k = 3$. (a) Perspective view of the wave. (b) The wave along the $x$ axis, R : $t = 0$; G : $t = 1$; B : $t = 2$. (c) The wave along the $t$ axis, R : $x = 0$; G : $x = 1$; B : $x = 2$.

5. Conclusion
We apply some new transformations to solve the exact solutions of this equation by the low-order Sub-ODE method in this paper. The nonlinear Schrödinger equation with a harmonic potential is transformed into the classical nonlinear Schrödinger equation by a new class of traveling wave transformation. Then, the problem of exact solutions is changed into the solutions of ordinary differential equation by the method of undetermined function. Next, we use the low-order Sub-ODE method to solve the exact solutions of this ordinary differential equation, and bring the solutions back in order to obtain the exact solutions of the nonlinear Schrödinger equation with harmonic potential. Finally, these exact solutions are new exact solutions, because we added a new traveling wave transformation. Meanwhile the obtained exact solutions are simulated and the simulation’s results are analysed in detail.

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