An extremal problem for a class of entire functions of exponential type

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Abstract

Let \( f \) be an entire function of exponential type whose indicator diagram is contained in the segment \([-i\sigma, i\sigma], \sigma > 0\). Then the upper density of zeros of \( f \) is at most \( c\sigma \) where \( c \approx 1.508879 \) is the positive solution of the equation

\[
\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}.
\]

This estimate is exact.

We consider the class \( E_\sigma, \sigma > 0 \) of entire functions of exponential type whose indicator diagram is contained in a segment \([-i\sigma, i\sigma]\), which means that

\[
h(\theta) := \limsup_{r \to +\infty} \frac{\log |f(re^{i\theta})|}{r} \leq \sigma |\sin \theta|, \quad |\theta| \leq \pi.
\]

(1)

An alternative characterization of such functions follows from a theorem of Pólya [6]:

\[
f(z) = \frac{1}{2\pi} \int_{\gamma} F(\zeta) e^{-i\zeta z} d\zeta,
\]

where \( F \) is an analytic function in \( \mathbb{C}\setminus[-\sigma, \sigma] \), \( F(\infty) = 0 \), and \( \gamma \) is a closed contour going once around the segment \([-\sigma, \sigma]\). In other words, the class of

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entire functions satisfying (1) consists of Fourier transforms of hyperfunctions supported by $[-\sigma, \sigma]$, see, for example, [2] and [3].

Let $n(r)$ be the number of zeros of $f$ in the disc $\{ z : |z| \leq r \}$, counting multiplicity. We are interested in the upper density

$$D = \limsup_{r \to \infty} \frac{n(r)}{r}. \quad (2)$$

If $f$ satisfies the additional condition

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty, \quad (3)$$

then the limit (density) in (2) exists and equals $(2\pi)^{-1} \int_{-\pi}^{\pi} h(\theta) d\theta$. For example, if $f(z) = \sin \sigma z$, then $f \in E_{\sigma}$ and $D = 2\sigma/\pi \approx 0.6366\sigma$ The existence of the limit follows from a theorem of Levinson [5, 6]. Much more precise information about $n(r)$ under the condition (3) is contained in the theorem of Beurling and Malliavin [1].

In the general case, the density might not exist as was shown by examples in [4, 10]. Moreover, it is possible that $D > 2\sigma/\pi$, see [2]. An easy estimate using Jensen’s formula gives $D \leq 2e\sigma/\pi \approx 1.7305\sigma$. This estimate is exact in the larger class of entire functions satisfying the condition $h(\theta) \leq \sigma$, but it is not exact in $E_{\sigma}$.

In this paper we find the best possible upper estimate for the upper density of zeros of functions in $E_{\sigma}$.

**Theorem.** The upper density of zeros of a function $f \in E_{\sigma}$ does not exceed $c\sigma$ where $c \approx 1.508879$ is the unique solution of the equation

$$\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}, \quad \text{on} \quad (0, +\infty). \quad (4)$$

For every $\sigma > 0$ there exist entire functions $f \in E_{\sigma}$ such that $D = c\sigma$.

**Proof.** Without loss of generality we assume that $\sigma = 1$. Moreover, it is enough to consider only even functions. To make a function $f$ even we replace it by $f(z)f(-z)$, which results in multiplication of both the indicator $h$ and the upper density $D$ by the same factor of 2.

Let $t_n \to +\infty$ be such sequence that $\lim n(t_n)/t_n = D$. Consider the sequence of subharmonic functions $v_n(z) = t_n^{-1} \log |f(t_n z)|$. Compactness
Principle for subharmonic functions [3, Theorem 4.1.9] implies that one can choose a subsequence that converges in $D'$ (Schwartz’s distributions). The limit function $v$ is subharmonic in the plane, and satisfies
\[ v(z) \leq |\text{Im } z|, \quad z \in \mathbb{C}, \quad \text{and} \quad v(0) = 0. \tag{5} \]

Let $\mu$ be the Riesz measure of this function. We have to show that
\[ \mu(\{z : |z| \leq 1\}) \leq c. \tag{6} \]

First we reduce the problem to the case that the Riesz measure $\mu$ is supported by the real line. We have
\[ v(z) = \frac{1}{2} \int \log \left| 1 - \frac{z^2}{\xi^2} \right| d\mu_\xi. \]

Let us compare this with
\[ v^*(z) = \frac{1}{2} \int_0^\infty \log \left| 1 - \frac{z^2}{t^2} \right| d\mu^*_t, \]

where $\mu^*$ is the radial projection of the measure $\mu$: it is supported on $[0, +\infty)$ and $\mu^*(a, b) = \mu(\{z : a < |z| < b\})$, $0 \leq a < b$. It is easy to see that
\[ v^*(z) \leq \sigma'|\text{Im } z|, \quad z \in \mathbb{C}, \quad \text{and} \quad v^*(0) = 0 \tag{7} \]
with some $\sigma' > 0$. We claim that one can choose $\sigma' \leq 1$ in (7). Let $\sigma'$ be the smallest number for which (7) holds. Then, by the subharmonic version of the theorem of Levinson mentioned above (see, for example, [9]), the limit
\[ \lim_{r \to \infty} r^{-1} v^*(r z) = \sigma'|\text{Im } z| \]
events in $\mathcal{D}'$ and thus
\[ \lim_{r \to \infty} \frac{1}{r} \int_0^r \frac{n_{v^*}(t)}{t} \, dt = \lim_{r \to \infty} \frac{1}{2\pi r} \int_{-\pi}^{\pi} v^*(r e^{i\theta}) \, d\theta = 2\sigma'/\pi, \]

where
\[ n_{v^*}(r) = \mu^*[0, r] = \mu(\{z : |z| \leq r\}). \tag{8} \]

Similar limits exist for $v$, and we have $n_v = n_{v^*}$, from which we conclude that $\sigma' \leq 1$. 

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From now on we assume that \( v \) is harmonic in the upper and lower half-planes, and that
\[
v(iy) \sim y, \quad y \to +\infty.
\] (9)

Let \( u \) be the harmonic function in the upper half-plane such that \( \phi = u + iv \) is analytic, and \( \phi(0) = 0 \). Then \( \phi \) is a conformal map of the upper half-plane onto some region \( G \) of the form
\[
G = \{x + iy : y > g(x)\},
\] (10)
where \( g \) is an even upper semi-continuous function, \( g(0) = 0 \). Moreover,
\[
\phi(iy) \sim iy, \quad \text{as} \quad y \to +\infty,
\] (11)
which follows from (9), and
\[
\phi(-\bar{z}) = -\overline{\phi(z)},
\] (12)
because both the region \( G \) and the normalization of \( \phi \) are symmetric with respect to the imaginary axis. Finally we have
\[
\mu([0, x]) = \frac{2}{\pi} u(x).
\] (13)

For all these facts we refer to [7].

**Remark.** The function \( \text{Re} \phi(x) = u(x) \) might be discontinuous for \( x \in \mathbb{R} \). We agree to understand \( u(x) \) as the limit from the right \( u(x+0) \) which always exists since \( u \) is increasing.

Inequality (5) implies that \( v(x) \leq 0 \), thus \( g(x) \leq 0 \), in other words, \( G \) contains the upper half-plane.

Thus we obtain the following extremal problem: *Among all univalent analytic functions \( \phi \) satisfying (12) and mapping the upper half-plane onto regions of the form (10) with \( g \leq 0 \), \( g(0) = 0 \) and satisfying \( \phi(0) = 0 \) and (11), maximize \( \text{Re} \phi(1) \).*

We claim that the extremal function \( g \) for this problem is
\[
g_0(x) = \begin{cases}  
-\infty, & 0 < |x| < \pi c/2, \\
0, & \text{otherwise},
\end{cases}
\]
where \( c > 1 \) is the solution of equation (4). The corresponding region is shown in Fig. 1. For the extremal function we have \( \phi_0(1) = \pi c/2 - i\infty \).
To prove the claim, we fist notice that for a given $G$ the mapping function is uniquely defined. Let $a = \phi(1)$, and $b = \text{Re } a$. Next we show that making $g$ smaller on the interval $(0, b)$ results in increasing $\text{Re } \phi(1)$ and making $g$ larger on the interval $(b, +\infty)$ also results in increasing $\text{Re } \phi(1)$. The proofs of both statements are similar. Suppose that $g_1 \leq g$, $g_1 \neq g$, and $g_1(x) = g(x)$ outside of the two intervals $p < |x| < q$, where $0 < p < q < b$. Let $G_1$ be the region above the graph of $g_1$, and $\phi_1$ the corresponding mapping function normalized in the same way as $g$. Then $G \subset G_1$, and the conformal map $\phi^{-1} \circ \phi$ is defined in the upper half-plane and maps it into itself. We have

$$\phi^{-1} \circ \phi(x) = x + 2x \int_0^\infty \frac{w(t)}{t^2 - x^2} dt,$$

where $w \neq 0$ is a non-negative function supported on some interval inside $(0, 1)$. Putting $x = 1$ we obtain

$$\phi^{-1}_1(a) = 1 + 2 \int_0^\infty \frac{w(t)}{t^2 - 1} dt,$$

so $\phi^{-1}_1(a) < 1$, that is $\text{Re } \phi_1(1) > b$. This proves our claim.

It remains to compute the constant $b$ in the extremal domain. We recall that $\phi_0(1) = b - i\infty$ and assume that $b = \phi_0(k)$ for some $k > 1$. Here $\phi_0$ is the extremal mapping function. Then by the Schwarz–Christoffel formula we have

$$\phi_0(z) = \frac{1}{2} \int_0^{z^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} d\zeta. \quad (14)$$

To find $k$, we use the condition that

$$\text{Im } p.v. \int_0^{\pi/2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} d\zeta = 0.$$
Denoting \( c = \sqrt{k^2 - 1} \) and evaluating the integral, we obtain
\[
\log(\sqrt{c^2 + 1} + c) = \sqrt{1 + c^{-2}}.
\]

Finally the jump of the real part of the integral in (14) occurs at the point 1 and has magnitude \( \pi \sqrt{k^2 - 1} = \pi c \). This completes the proof of the upper estimate in Theorem 1.

To construct an example showing that this estimate can be attained, we follow the construction in [2, Sect.9-10]. The role of the subharmonic function \( u_1 \) there is played now by our extremal function \( v_0 = \text{Im} \phi_0 \).

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