Violation of Bell’s inequality: criterion for quantum communication complexity advantage

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We prove that for every Bell’s inequality and for a broad class of protocols, there always exists a multi-party communication complexity problem, for which the protocol assisted by states which violate the inequality is more efficient than any classical protocol. Moreover, for that advantage Bell’s inequality violation is a necessary and sufficient criterion. Thus, violation of Bell’s inequalities has a significance beyond that of a non-optimal-witness of non-separability.

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Entanglement is the essential feature, which distinguishes the quantum from the classical [1]. On one hand, entangled states violate Bell inequalities, and thus rule out local realistic explanation of quantum mechanics [2]. On the other hand, they enable certain communication and computation tasks to have an efficiency not achievable by the laws of classical physics [1].

Intuition suggests that these two aspects, the fundamental one, and the applicational one, could be intimately linked. Specifically, one could expect, that only the quantum communication protocol which makes use of an entangled state which violates some Bell’s inequality can have efficiency larger than any classical protocol. Otherwise one might expect that the efficiency of the protocol could be explainable by a local realistic model, and thus achievable in classical physics. This intuitive reasoning is supported by the result of Ref. [4] where it was shown that violation of Bell’s inequality is a condition for the security of quantum key distribution protocols. Here we give another result which supports the intuitive reasoning: the violation of Bell’s inequalities is a necessary and sufficient criterion for the quantum communication complexity protocol to be more efficient than any classical one.

We shall discuss the following version of the communication complexity problems (such problems were introduced in Ref. [3]). Some input data are distributed over $n$ separated parties. Every party knows the local data, but not the data of the others. The party $i$ obtains an input string $z_i$. The goal is for each of them to determine the value of some function $f(z_1, ..., z_n)$, while exchanging a restricted amount of information. This restriction, in general, enables the parties to compute the function only with an error. Then the goal for all parties is to compute the function correctly with as high a probability as possible. An execution is considered successful, if the values determined by all parties are correct. Before they start the protocol, the parties are allowed to share (classically) correlated random strings, or any other data, which might improve the success of the protocols. They are allowed to process their data locally in whatever way.

The general question is whether and to what extent entanglement can be of advantage for solving such problems. It was shown that entanglement can improve the probability of success in communication complexity protocols beyond the limits which are classically possible [6, 7, 8, 9, 10]. Specifically, Buhrman, Cleve and van Dam [6] found a two-party communication complexity problem, that can be solved with a higher probability of success than in any classical protocol, if the parties share entanglement. The quantum protocol was based on violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [11] by two-qubit maximally entangled state. Similarly, the quantum protocols of multi-party problems of Ref. [4, 5, 12] were based on an application of the GHZ-type [12] argument against local realism for multi-qubit maximally entangled states. In Ref. [13] an equivalence between the CHSH and GHZ tests for three particles and the two- and three-party quantum protocols of Ref. [12], respectively, was shown. All these results indicate that there is a link between the quantum communication complexity protocols and the violation of Bell’s inequality.

However, the problems: (a) which quantum states are needed to achieve an improvement of the probability of success over any classical strategy and (b) what are the classes of functions for which this improvement is possible, are still open. We address the question (a) and (b) for a class of $n$-party communication complexity problems. We prove, that for any Bell’s inequality for $n$ qubits one can formulate at least one communication complexity problem with the following property. The success of the quantum protocol (i.e. which uses entangled states) is higher than in any classical protocol if and only if the $n$ qubits violate the Bell inequality. As exemplary applications we use the complete set of $2^2n$ of $n$-qubit Bell’s inequalities for correlation functions [14, 15]. For this example we find a family of functions for which the improvement of the probability of success over classical strategies is possible. We also present extensions of these results.

Let us define the general $n$-party communication complexity problems to be considered:
• The $i$-th party receives a two-bit input string $(x_i, y_i)$. (For convenience the values of the bits are encoded as follows $x_i = 0$ or 1, and $y_i = -1$ or 1.)

• The distributed values for $y_i$ are chosen randomly and for the $x_i$’s in accordance with a certain probability distribution $Q(x_1, ..., x_n)$. Thus the inputs $x_1, ..., x_n$ can be (classically) correlated.

• After receiving the input strings each party is allowed to broadcast only one bit of information (denoted as $e_i$). It may reveal e.g. a part of the received string, or some locally produced result of computation or measurement.

• Finally each party attempts to give a value for the function $f(x_1, ..., x_n, y_1, ..., y_n)$, with $f = \pm 1$. The execution of the protocol is successful when all parties arrive at the correct value of $f$. Their joint task is to maximize the probability of success.

We shall consider a specific sub-class of the above problems, for which there exists a real-valued function $g(x_1, ..., x_n)$, such that

$$Q(x_1, ..., x_n) = \frac{|g(x_1, ..., x_n)|}{\sum_{x_1, ..., x_n = 0} |g(x_1, ..., x_n)|}, \quad (1)$$

and

$$f = y_1 \cdot y_2 \cdot \ldots \cdot y_n \cdot S[g(x_1, x_2, ..., x_n)], \quad (2)$$

where $S[g] = g/|g| = \pm 1$ is the sign function of $g$.

We shall prove that for any Bell’s inequality for qubits there exists at least one problem from the above class, such that the probability of success in the quantum protocol (i.e. which uses an entangled state) is higher than in any classical one. This is so if, and only if, the entangled state used violates the Bell inequality for correlation functions

$$\sum_{x_1, ..., x_n = 0} g(x_1, ..., x_n)E(x_1, ..., x_n) \leq B(n), \quad (3)$$

(this general form includes also inequalities not known yet). In Ineq. (3) $E(x_1, ..., x_n)$ is a shorthand notation for the Bell-type correlation function $E(O^1_{x_1}, ..., O^m_{x_n})$, for measurements on $n$ particles, which involve, at each local measurement station $i$, two alternative dichotomic observables $O^0_i$ and $O^1_i$, each of spectrum $\pm 1$.

We now present a broad class of classical protocols which will be considered here:

1. Each party $i$ calculates (e.g., with help of a computer) locally any function $a_i(x_i, \lambda_i)$, where $\lambda_i$ is any other parameter, or a set of parameters, on which the function $a_i$ may additionally depend. For example, $\lambda_i$ can be a random string of variables shared among the parties before they start the protocol. Each party $i$ broadcasts $e_i = a_i \cdot y_i$.

2. After the broadcast all parties put as the value of $f$ the number $y_1 \cdot \ldots \cdot y_n \cdot a_1 \cdot \ldots \cdot a_n$, which is equal to the actual value of function $f$ for a certain fraction of cases (see below).

Let us calculate the probability of success achievable for the considered class of classical protocols. It is equal to the probability, $P$, that the product $a_1 \cdot \ldots \cdot a_n$ of the locally computed functions is equal to $S[g]$:

$$P = \sum_{x_1, ..., x_n = 0} Q(x_1, ..., x_n) \cdot P_{x_1 ... x_n} (a_1 \cdot \ldots \cdot a_n = S[g]), \quad (4)$$

where $P_{x_1 ... x_n} (a_1 \cdot \ldots \cdot a_n = S[g])$ is the probability that $a_1 \cdot \ldots \cdot a_n = S[g(x_1, ..., x_n)]$ if parties receive inputs $x_1, ..., x_n$.

Next, we introduce a quantum competitor of the class of classical protocols considered above. The parties share $n$ entangled qubits. Each of them can perform measurements on the local qubit. The quantum protocol reads (Fig. 1):

1. If party $i$ receives $x_i = 0$, she will measure her qubit with the local apparatus, which is set to measure a dichotomic observable $O^n_0$. Otherwise, i.e. for $x_i = 1$, she measures a different observable $O^n_1$. We ascribe to the outcomes of the measurements the two values $\pm 1$. The actual value obtained by party $i$ will be denoted as $a_i$. It will serve the same role as the result of local computation in the classical protocol. Each party $i$ broadcasts $e_i = a_i \cdot y_i$.

2. After the broadcast all parties put as the value of $f$ the number $y_1 \cdot \ldots \cdot y_n \cdot a_1 \cdot \ldots \cdot a_n$, which is equal to the actual value of function $f$ for a certain fraction of cases.

The probability of success in the quantum protocol is the probability, $P$, that the product $\prod_{i=1}^n a_i$ of the local measurement results is equal to $S[g]$. Thus, it can also be expressed by Eq. (3) where now $P_{x_1 ... x_n} (a_1 \cdot \ldots \cdot a_n = S[g])$ is the probability that $\prod_{i=1}^n a_i = S[g(x_1, ..., x_n)]$ if parties measure their qubits with the local apparatus set at $O^{x_1}_1, ..., O^{x_n}_1$.

It is essential to realize that the classical protocols introduced above are equivalent to a local realistic model of the quantum protocol because $\lambda$’s can be considered as local hidden variables, which can be shared between parties. We will now show that the combination of probabilities on the right-hand side of Eq. (3), in the case of a classical protocol, is bounded by the limits imposed by local realistic models. That is, the combination of probabilities in Eq. (3) satisfies a Bell-type inequality.

Note that the correlation function is given by

$$E(x_1, ..., x_n) = S[g] (2P_{x_1 ... x_n} (a_1 \cdot \ldots \cdot a_n = S[g]) - 1). \quad (5)$$
FIG. 1: Multi-party quantum communication complexity protocol which is based on the Bell experiment with $n$ qubits. Every party $i$ receives an input string $(x_i, y_i)$ where both $x_i$ and $y_i$ are bit values. Depending on the values of $x_i$ party $i$ chooses to measure between two different two-values observables $O^i_l$ or $O^i_t$. The actual measurement result obtained by party $i$ is denoted by $a_i$. Each party broadcasts the product $y_i \cdot a_i$.

Using this one easily shows that the right hand side of Eq. (4) is proportional to the left hand side of Ineq. (3). One obtains

$$\sum_{x_1, \ldots, x_n} Q(x_1, \ldots, x_n) \cdot P_{x_1 \ldots x_n} (a_1 \cdots a_n = S[g]) \leq \frac{1}{2} \left( 1 + \frac{B(n)}{\sum |g(x_i_1, \ldots, x_i_n)|} \right).$$

If Ineq. (3) is violated, so is Ineq. (4), and vice versa. That is, the entanglement assisted protocol can result in a higher probability of a success than any classical one of the considered class, if and only if the respective Bell’s inequality is violated [13].

Let us now present some examples. The set of $2^n$ Bell’s inequalities of the form (3) was obtained in Refs. [14, 15]. There the class of functions $g$ is given by

$$g(x_1, \ldots, x_n) = \sum_{s_1, \ldots, s_n = -1} S(s_1, \ldots, s_n) \cdot s_1^{x_1} \cdot \cdots \cdot s_n^{x_n},$$

where $S(s_1, \ldots, s_n) = \pm 1$ is a sign function and the bound is $B(n) = 2^n$. There are $2^n$ different sign functions and correspondingly $2^n$ different functions $g$. The functions, which can lead to a greater success probability in the case of quantum protocols, are those which are associated with non-trivial Bell inequalities of the form (3), that is such ones which are violated by quantum predictions. Note that factorable functions, $S(s_1, \ldots, s_n) = S_1(s_1) \cdot \cdots \cdot S_n(s_n)$ are therefore excluded from this family. In the following we give two explicit functions $g$ of the class (3).

We only give the final results, as these follow from the general proof given above.

Consider $S_{\text{odd}} = \sqrt{2} \cos \left( (s_1 + \cdots + s_n) \frac{\pi}{4} \right)$ for $n$ odd and $S_{\text{even}} = \cos \left( (s_1 + \cdots + s_n) \frac{\pi}{4} \right)$ for $n$ even. This implies for $n$ odd

$$g_{\text{odd}} = \sqrt{2^{n+1}} \cos \left( \frac{\pi}{2} (x_1 + x_2 + \cdots + x_n) \right).$$

whereas for $n$ even one has

$$g_{\text{even}} = \sqrt{2^n} \cos \left( \frac{\pi}{2} (x_1 + x_2 + \cdots + x_n) \right).$$

The probability distribution $Q(x_1, \ldots, x_n)$ is such that with equal probability only the input strings $x_i$ which satisfy the condition that $x_1 + \cdots + x_n$ is even are distributed. This specific type of problem was first considered by Buhrman et al. [16, 17]. The quantum protocol rests on the violation of an inequality, which is equivalent to the Mermin inequality [17, 18]. The maximal probability of success in the classical protocol is $P_{\text{Cmax}}^{\max} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2^{n-1}}} \right)$ for $n$ odd, and $P_{\text{Cmax}}^{\max} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2^n}} \right)$ for $n$ even. In the quantum case, with the use of $n$ qubits in the maximally entangled (GHZ) state, the task can be done with certainty for both cases, i.e. $P_{Q}^{\text{max}} = 1$. Note that in both cases in the limit $n \rightarrow \infty$ the probability of success $P_{\text{Cmax}}^{\max} \rightarrow \frac{1}{2}$ as by a simple random choice, which drastically contrasts the certainty in the quantum protocol.

Next, suppose that the number $n$ of parties is even and consider $g_{\text{even}}' = \sqrt{2} \cos \left( \frac{\pi}{4} (s_1 + \cdots + s_n) \right)$. This implies

$$g_{\text{even}}' = \sqrt{2^{n+1}} \cos \left( \frac{\pi}{2} (x_1 + x_2 + \cdots + x_n) + \frac{\pi}{4} \right).$$

The success rate in the quantum protocol is now based on violation of an inequality equivalent to the one of Ardehali [18, 19]. The maximal probability of success in a classical protocol is $P_{C}^{\text{max}} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2^{n-1}}} \right)$, whereas in the quantum protocol with the use of the maximally entangled state the probability reads $P_{Q}^{\text{max}} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2^n}} \right)$. Thus in this case one does not have certainty. However, because the Bell inequality defined by $g_{\text{even}}'$ is violated by the GHZ states by a higher factor than the one defined by $g_{\text{even}}$ (by $\sqrt{2^{n-1}}$, instead of $\sqrt{2^{n-2}}$) the quantum protocol is more resistant to the possible admixture of noise to the GHZ states.

The advantage of the quantum protocol associated with a given Bell inequality is a new measure of the strength of such an inequality. The last example shows that this measure favors GHZ-type contradictions for perfect correlations $(g_{\text{even}})$, making the factor by which the inequality is violated less important (compare $g_{\text{even}}'$.)

Let us look at generalizations. Consider $g$ as given by $\cos (x_1 + \cdots + x_n)$ where inputs $x_i$ belong to a continuous set $[0, 2\pi]$ (while $y_i$’s remain bits). The quantum protocol should now be adopted such that each party has a choice to measure her qubit in a continuous range of the settings of the apparatus. The protocol is based on violation of the functional Bell inequality for continuous range of the settings of the local apparatuses (for the derivation and magnitude of violation see [20]):

$$\int_{x_1, \ldots, x_n = 0}^{2\pi} \cos (x_1 + \cdots + x_n) E(x_1, \ldots, x_n) \leq 4^n. \quad (11)$$
One can apply the general proof obtained above if one replaces \( \sum_{x_1, \ldots, x_n=0}^{2^n} \) by \( \int_{x_1 \ldots x_n=0}^{2^n} \) in the previous expressions. Thus, if and only if the state violates the functional Bell inequality (1) the quantum protocol will have a higher success rate than the classical one. The maximal probability of success in the classical protocol is \( P_C^{max} = \frac{1}{2} (1 + (\frac{\pi}{2})^{-1}) \), whereas in the quantum protocol with the use of \( n \) qubits in maximally entangled state this probability is \( P_Q^{max} = \frac{1}{2} (1 + \frac{\pi}{2}) \). Similar results can be obtained for an arbitrary discrete number of settings at each side.

For a next generalization, suppose that the inputs \( y_i \) have \( d \) possible values and that the two-valued function \( S(g(x_1, \ldots, x_n)) \) is replaced with a function which has \( d \) possible values. Then the quantum protocol should be based on the violation of Bell’s inequalities for \( d \)-dimensional quantum systems (see [21]). Recently such a two-party protocol was proposed [22].

We end with a remark. There are non-separable quantum states which do not violate any Bell’s inequality directly [22]. This is often interpreted as implying that the violation of Bell’s inequalities is just a, even not optimal, entanglement witness, without any significant importance for the implementation in quantum information tasks. One cannot agree with such an interpretation (see also Ref. [26]). The states which only after local operations and classical communication (LOCC) violate Bell’s inequalities [22], cannot be in any way useful in the communication complexity problems considered here. Simply, any LOCC transformation requires more communication than it is permitted by the problems [26].

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