ABSTRACT

A new representation for solutions of Maxwell’s equations is derived. Instead of being expanded in plane waves, the solutions are given as linear superpositions of spherical wavelets dynamically adapted to the Maxwell field and well–localized in space at the initial time. The wavelet representation of a solution is analogous to its Fourier representation, but has the advantage of being local. It is closely related to the relativistic coherent–state representations for the Klein–Gordon and Dirac fields developed in earlier work.

Key words: Electrodynamics, Maxwell’s equations, wavelets, coherent states.

§1. Introduction

In previous work [1,2] coherent–state representations have been developed for various relativistic systems, including Klein–Gordon and Dirac fields and their wave functions. The main tool was the analytic–signal transform (AST) [3,4], which gives a canonical extension of fields from real spacetime to complex spacetime. When the field $f(x)$ is free, its extension $f(z)$ is analytic in the double tube domain

$$\mathcal{T} = \{z \equiv (z, z_0) = x - iy \in \mathbb{C}^4 \mid y^2 \equiv y_0^2 - y^2 > 0\} = \mathcal{T}_+ \cup \mathcal{T}_-,$$

where $\pm y_0 > 0$ in $\mathcal{T}_\pm$. The restriction of $f(z)$ to the forward tube $\mathcal{T}_+$ ($y_0 > 0$) then contains only the positive–frequency part of $f$, while its restriction to the backward tube $\mathcal{T}_-$ ($y_0 < 0$) contains only the negative–frequency part. The points of $\mathcal{T}$ parametrize a system of relativistic coherent states $e_z$, with $z \in \mathcal{T}_+$ and $z \in \mathcal{T}_-$ representing particles and antiparticles, respectively, and it was shown that the state $e_z$ labeled by $z = x - iy$ has an expected position $x$ at time $x_0$ and an expected energy–momentum proportional to $y$. Certain six–dimensional submanifolds of $\mathcal{T}$ can therefore be interpreted as phase spaces. Furthermore, when the field has a positive mass, it can be reconstructed from its values on any one of these phase spaces. However, this reconstruction fails when the mass vanishes, due to the divergence of the relevant integrals. (The massless representa-
tion is not square–integrable with respect to the ‘Liouville measure’ on phase space.) In this paper we develop a square–integrable representation for the most common massless system, namely the electromagnetic field. This will be done by reducing the dimensionality of the “phase space” from six to four. Namely, an electromagnetic field will be reconstructed from its values in the Euclidean region (real space–and imaginary time–coordinates), as obtained by applying the AST to the field in real spacetime. The result will be seen to be a multidimensional generalization of wavelet analysis.

That relativistic coherent states behave like wavelets due to Lorentz contractions has been noted in [3]. However, only in the massless case can the correspondence be complete, since a positive mass provides a scale, namely the width of $e_z$ in its rest frame. This led us to the expectation that massless fields (where no rest frames and canonical scales occur) possess a natural wavelet representation [4]. The simplest case is the wave equation in two spacetime dimensions, for which a wavelet representation was indeed constructed in [5]. As shown there, the wavelets are covariant not only under the group of affine transformations (translations and dilations) but also under the larger group of conformal transformations. Here we develop the simplest aspects of a similar construction for the electromagnetic field in four spacetime dimensions. A more detailed analysis of this and related topics, such as conformal invariance and the construction of polarized electromagnetic wavelets, will be dealt with in a forthcoming paper [6]. To the author’s knowledge, the results given in [5] and in the present paper represent the first successful “relativistic coherent–state” formulations of massless fields (in two and more spacetime dimensions, respectively).

It is remarkable that a single function, namely the AST–extended field $f(z)$, combines the concept of phase space (parametrized by position and momentum) with the concept of wavelet space (parametrized by position and scale). This unification is actually the “relativistic dividend” earned from the construction of relativistic coherent states, in the same way as relativity unifies energy with momentum and electric fields with magnetic fields. To see this, note that although $y$ is proportional to the expected energy–momentum of $e_z$, it cannot be an energy–momentum since it has units of length (which scale oppositely to those of momentum). Rather, $y/y_0$ gives the expected velocity, while the imaginary time $y_0$ gives a scale parameter. In the Euclidean region, $y = 0$, which means that $e_z$ represents a spherical wave which first implodes towards a point in space, then explodes away from it. Then $y_0$ measures the scale of this wave by giving its diameter at the instant of maximal localization. Such $e_z$’s will form our system of electromagnetic wavelets in the next section, and the field itself will be constructed from them. The relation of our formalism to the usual one–dimensional wavelet analysis is discussed in Section 3.

We consider solutions of Maxwell’s equations in free spacetime $\mathbb{R}^4$. It will be convenient to unify the electric and magnetic fields $E(x)$ and $B(x)$ into a single complex vector field $F(x,t) \equiv F(x) = E(x) + iB(x)$, which then satisfies (with the speed of light
\( c = 1 \)

\[
\nabla \cdot \mathbf{F} = 0, \quad i\partial_t \mathbf{F} = \nabla \times \mathbf{F}.
\]

Let us review the usual solution using Fourier transforms. The above equations imply that \( \mathbf{F} \) satisfies the wave equation \( \partial_t^2 \mathbf{F} = \nabla^2 \mathbf{F} \), hence it has the form

\[
\mathbf{F}(x) = \int_C d\tilde{p} e^{-ipx} f(p),
\]

where \( p^2 = p_0^2 - \mathbf{p} \cdot \mathbf{x}, \ C = C_+ \cup C_- \) is the double light cone with \( C_\pm = \{ (p, p_0) \mid \pm p_0 = |\mathbf{p}| > 0 \} \) (the origin \( p = 0 \) is excluded) and \( d\tilde{p} \equiv (2\pi)^{-3} d^3 \mathbf{p} / |\mathbf{p}| \) is the Lorentz-invariant measure on \( C \). Maxwell’s equations imply \( \mathbf{p} \cdot f(p) = 0 \) and \( p_0 f(p) = i \mathbf{p} \times f(p) \) for \( p \) in \( C \), which can be solved by introducing a four-potential. A Poincaré-invariant norm (and associated inner product) on solutions is given [7] in momentum space by

\[
\| f \|^2 \equiv \int_C d\tilde{p} \omega^{-2} |f(p)|^2,
\]

where \( \omega \equiv |\mathbf{p}| \), and we denote the Hilbert space of all solutions with finite norm by \( \mathcal{H} \). Gross [7] has shown that a norm unitarily equivalent to the above is also invariant under the fifteen-dimensional conformal group \( \mathcal{C} \) of spacetime, which is generated by the Poincaré group together with uniform dilations \( (x \rightarrow \alpha x, \alpha > 0) \) and inversions in the unit hyperboloid \( (x \rightarrow x/x^2) \). This gives a unitary representation of \( \mathcal{C} \) on \( \mathcal{H} \). (That Maxwell’s equations are invariant under \( \mathcal{C} \) has been known for a long time [8]; the unitary representation gives that invariance a quantum–mechanical flavor, since the elements of \( \mathcal{H} \) may now be regarded as single–photon wave functions.) The new norm has the following non–local expression in terms of the fields at time \( t = 0 \):

\[
\| \mathbf{F} \|^2 = \frac{1}{\pi^2} \int_{\mathbb{R}^6} \frac{d^3 \mathbf{x} d^3 \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^2} \mathbf{F}(\mathbf{x}, 0)^* \cdot \mathbf{F}(\mathbf{y}, 0),
\]

where the asterisk denotes complex conjugation. In the next section we shall find a simpler expression for \( \| \mathbf{F} \|^2 \) in terms of the values \( \mathbf{F}(\mathbf{x}, -is) \) of the field in Euclidean spacetime (real space but imaginary time). This will lead us directly to the wavelet expansion of \( \mathbf{F} \).

\[\text{§2. The Wavelet Representation of Solutions}\]

Our extension of \( \mathbf{F}(\mathbf{x}, t) \) to complex spacetime employs the analytic–signal transform, developed in [1,2,3] and further applied in [4,5]. Given an arbitrary but reasonable function \( f(\mathbf{x}) \) on \( \mathbb{R}^n \) (a smooth function with mild decay is more than sufficiently reasonable), its analytic–signal transform is the function \( f(\mathbf{x} + i\mathbf{y}) \) on \( \mathbb{C}^n \) defined by
\[ f(x + iy) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} f(x + \tau y). \]  

In terms of the Fourier transform \( \hat{f} \) of \( f \),

\[ f(x + iy) = (2\pi)^{-n} \int d^n p \, 2\theta(p \cdot y) e^{ip \cdot (x + iy)} \hat{f}(p), \]

where \( \theta \) is the step function (\( \theta(u) = 1 \) if \( u > 0 \), \( \theta(0) = 1/2 \) and \( \theta(u) = 0 \) if \( u < 0 \)). Although \( f(x + iy) \) is in general not analytic, it can be easily shown to be partially analytic in the direction of \( y \neq 0 \). Furthermore, if \( \hat{f}(p) \) has certain support properties (vanishes outside of a solid double cone \( V \subset \mathbb{R}^n \)), then \( f(x + iy) \) is analytic in a corresponding domain in \( \mathbb{C}^n \) (the double tube over the double cone \( V' \) dual to \( V \)). This will be seen explicitly below, where \( \mathbb{R}^n \) is spacetime.

Application of the AST to the electromagnetic field \( F(x) \) gives

\[ F(x - iy) \equiv \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i} F(x - \tau y) = \int_C d\bar{p} \, 2\theta(py) e^{-ip(x-iy)} f(p). \]

(We write \( x - iy \) instead of \( x + iy \) to conform with the convention used in Streater and Wightman [9].) If \( y \) belongs to the forward solid light cone \( V'_+ \) (i.e., \( y^2 > 0 \) and \( y_0 > 0 \)), then \( \theta(py) \equiv 0 \) on \( C_- \) and \( \theta(py) \equiv 1 \) on \( C_+ \), hence \( F(x - iy) \) is analytic at \( x - iy \) and contains only the positive–frequency part of the field. Similarly, if \( y \) belongs to the backward solid light cone \( V'_- \), then \( F(x - iy) \) is analytic at \( x - iy \) and contains only the negative–frequency part of the field. Thus \( F(x - iy) \) is analytic in the double tube \( \mathcal{T} \) defined in Eq. (1). The jump discontinuity in \( F(x - iy) \) between the future and past imaginary directions is related to the multidimensional Hilbert transform of \( F(x) \) (see [4,5] for details). In this paper, we shall be interested only in the transform of \( F(x, t) \) to complex time \( t \to t - is \) (i.e., \( F(x - iy) \) with \( y = (0, s) \)):

\[ F(x, t - is) = \int_C d\bar{p} \, 2\theta(p_0 s) e^{-ip_0(t-is)+ip \cdot x} f(p) \]
\[ = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} \omega e^{ip \cdot x} \left[ \theta(s) e^{-\omega(s+it)} f(p, \omega) + \theta(-s) e^{\omega(s+it)} f(p, -\omega) \right] \]
\[ = \left\{ \omega^{-1} \left[ \theta(s) e^{-\omega(s+it)} f(p, \omega) + \theta(-s) e^{\omega(s+it)} f(p, -\omega) \right] \right\} \hat{\cdot}(x), \]

where \( \hat{\cdot} \) denotes the inverse Fourier transform in \( \mathbb{R}^3 \). Note that \( F(x, t - is) \) is analytic in \( t - is \) whenever \( s \neq 0 \). The last line in Eq. (9) implies, by Plancherel’s theorem, that for \( s \neq 0 \),
\[
\int d^3x |F(x, t - is)|^2
= \int \frac{d^3p}{(2\pi)^3 \omega^2} \left| \theta(s) e^{-\omega(s+it)} f(p, \omega) + \theta(-s) e^{\omega(s+it)} f(p, -\omega) \right|^2
\]
\[\tag{10}\]
\[
= \int \frac{d^3p}{(2\pi)^3 \omega^2} \left\{ \theta(s) e^{-2\omega s} |f(p, \omega)|^2 + \theta(-s) e^{2\omega s} |f(p, -\omega)|^2 \right\},
\]
and therefore
\[
\int d^3x ds |F(x, t - is)|^2 = \int \frac{d^3p}{(2\pi)^3 2\omega^3} \left\{ |f(p, \omega)|^2 + |f(p, -\omega)|^2 \right\}
\[
= \int_C d\tilde{p} \omega^{-2} |f(p)|^2 = \|f\|^2.
\]
\[\tag{11}\]
Hence we define
\[
\|F\|^2 \equiv \int d^3x ds |F(x, t - is)|^2,
\]
\[\tag{12}\]
so that \(\|F\| = \|f\|\). Note that the new norm \(\|F\|\) and its associated inner product \(\langle G | F \rangle\) are local in the Euclidean spacetime variables \((x, s)\). Define the function \(\hat{e}_{y, t-is}(p)\) on \(C\) by its complex–conjugate as
\[
\hat{e}_{y, t-is}(p, p_0)^* = \omega^2 2\theta(p_0 s) e^{-ip_0(t-is) + i\cdot p \cdot y}.
\]
\[\tag{13}\]
The spacetime function corresponding to \(\hat{e}_{y, -is}\) is
\[
e_{y, -is}(x, t) \equiv \int_C d\tilde{p} e^{-ipx} \hat{e}_{y, -is}(p)
\[
= \int_C d\tilde{p} \omega^2 2\theta(p_0 s) e^{-p_0(s+it) + i\cdot p \cdot (x-y)}.
\]
\[\tag{14}\]
This is a scalar solution of the wave equation which depends only on \(|x - y|\) and \(t - is\), being analytic in the latter variable whenever \(s \neq 0\). Although \(e_{y, t-is}\) does not belong to \(\mathcal{H}\) (since it has no polarization, being a scalar), we shall write Eq. (9) as
\[
F(x, t - is) = \langle \hat{e}_{x, t-is} | f \rangle = \langle e_{x, t-is} | F \rangle,
\]
\[\tag{15}\]
where \(\langle | \rangle\) denotes the inner product in \(\mathcal{H}\), expressed either in momentum space or, equivalently, in Euclidean spacetime. Then it follows from \(\|F\|^2 = \|f\|^2\) that the inner product of two solutions \(G\) and \(F\) in \(\mathcal{H}\) can be written as
\[ \langle G | F \rangle \equiv \int_{R^4} d^3y \, ds \, G(y, is)^* \cdot F(y, is) \]
\[ = \int_{R^4} d^3y \, ds \, \langle G | e_{y, -is} \rangle \langle e_{y, -is} | F \rangle, \]

which gives the continuous resolution of unity
\[ \int_{R^4} d^3y \, ds \, \langle e_{y, -is} | e_{y, -is} \rangle = I_H, \]

where \( I_H \) is the identity operator in the space of solutions and the equality holds in a weak sense. Hence for \( F \) in \( H \) we have
\[ F(x, t - i\sigma) = \langle e_{x, t - i\sigma} | F \rangle = \int_{R^4} d^3y \, ds \, \langle e_{x, t - i\sigma} | e_{y, -is} \rangle \langle e_{y, -is} | F \rangle. \]

A straightforward computation gives
\[ \langle e_{x, t - i\sigma} | e_{y, -is} \rangle = \frac{2\theta(\sigma s)}{\pi^2} \frac{3\tau^2 - r^2}{(\tau^2 + r^2)^3}, \quad \tau \equiv s + \sigma + it, \ r \equiv |x - y|. \]

To obtain the wavelet expansion of \( F(x, t) \), we simply take \( \sigma = 0 \) in Eq. (18):
\[ F(x, t) = \int_{R^4} d^3y \, ds \, \langle e_{x, t} | e_{y, -is} \rangle F(y, -is) \]
\[ = \int_{R^4} d^3y \, ds \, e_{y, -is}(x, t) F(y, -is). \]

As we have seen, \( e_{y, -is}(x, t) \) satisfies the wave equation in \( (x, t) \). Since it depends only on \( r = |x - y| \), it is a spherical wave centered at \( x = y \). Furthermore, \( s \) acts as a scale parameter, since
\[ e_{y, -is}(x, t) = s^{-4} e_{y/s, -i}(x/s, t/s). \]

Hence it suffices to examine any one of the wavelets \( e_{y, -is} \). Figures 1-4 show the behavior of the wavelet centered at the origin with scale \( s = -1 \), i.e. of the basic wavelet
\[ w(r, t) \equiv \pi^2 e_{0, i}(x, t) = \frac{3(1 - it)^2 - r^2}{[(1 - it)^2 + r^2]^3}, \quad r \equiv |x|. \]

These figures confirm that for \( t < 0 \), \( e_{y, -is}(x, t) \) is an incoming spherical wave which builds up rapidly (at the speed of light!) to a well–localized packet in the ball \( |x - y| \leq \sqrt{3}|s| \) and decays rapidly for \( t > 0 \) into an outgoing spherical wave. The characteristics \( t = \pm r \) appear as ripples in figures 1-3. These properties partly justify our use of the term “wavelets.” Further justification is given in the next section.
3. Relation to Wavelet Analysis

The correspondence $F(x,t) \rightarrow F(y,is)$ is analogous to the Fourier transform $F(x,t) \rightarrow f(p)$, and the reconstruction of $F(x,t)$ from $F(y,is)$ is analogous to its reconstruction from $f(p)$ via the inverse Fourier transform. The role of the plane waves $e^{ipx}$ is now played by $e^{-is}$, and the local nature of these functions means that the behavior of $F(y,is)$ is correlated with that of the field $F(x,t)$ in real spacetime, the correlation being strongest when $t = 0$ since $e^{-is}$ is then most localized. [By contrast, the Fourier transform has no such local property since the plane waves extend to all of spacetime; thus a small local perturbation in $F(x,0)$ can cause an unrecognizable change in $f(p)$.] In fact, $F(y,is)$ is a version of $F(x,0)$, blurred to resolution $|s|$. Eq. (20) with $t = 0$ states that $F(x,0)$ is recovered by superposing wavelets with different centers $y$ and scales $s$, with $F(y,is)$ as the coefficient function. This representation of functions as superpositions of their blurred versions is typical of wavelet analysis [10,11,12] and the related multiresolution analysis [13,14]. Moreover, in our case the wavelet representation also gives the time evolution because our wavelets are dedicated [4,5] to the dynamics of the electromagnetic field.

In momentum space, the wavelet with $s = 1$ and $y = 0$ is represented by

$$\hat{e}_{0,-i}(p,p_0) = 2\omega^2\theta(p_0) e^{-p_0}, \quad p \in C.$$  

(23)

This is a multi–dimensional generalization of a basic wavelet which has appeared previously in the literature in connection with the usual (one–dimensional) wavelet theory [15,16], and also in a wavelet analysis of solutions of the wave equation in two space–time dimensions (Ref. [5], Eq. 161). In fact, the AST is a special case of a windowed Radon transform, which in turn has been shown to be a multivariate generalization of the wavelet transform.

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