EVENLY DIVISIBLE RATIONAL APPROXIMATIONS OF QUADRATIC IRRATIONALITIES

DAN CARMON

Abstract. In a recent paper of Blomer, Bourgain, Radziwiłł and Rudnick [1], the authors proved the existence of small gaps between eigenvalues of the Laplacian in a rectangular billiard with sides $\pi$ and $\pi/\sqrt{\alpha}$, i.e. numbers of the form $\alpha m^2 + n^2$, whenever $\alpha$ is a quadratic irrationality of certain types. In this note, we extend their results to all positive quadratic irrationalities $\alpha$.

1. Introduction

In [1], the authors investigated the minimal gaps between energy levels of the eigenvalues of the Laplacian of a billiard in a rectangle with width $\pi/\sqrt{\alpha}$ and height $\pi$. These eigenvalues are of the form $\alpha m^2 + n^2$, where $m, n \geq 1$ are integers. For positive irrational $\alpha$, these levels belong to a simple spectrum $0 < \lambda_1 < \lambda_2 < \cdots$, with growth $\lambda_N \sim \frac{4\sqrt{\pi}}{\pi} N$. The minimal gap function is then defined as

$$\delta_{\min}(N) = \min(\{\lambda_{i+1} - \lambda_i : 1 \leq i < N\}).$$

The authors prove several lower and upper bounds on the growth rate of $\delta_{\min}(N)$, for various families and sets of irrational $\alpha$. Amongst these is [1, Theorem 6.1]:

**Theorem 1.1.** For all positive real quadratic irrationalities of the form

$$\alpha = \alpha(x; a, b, \epsilon, r) = r \cdot \left(\frac{x + \sqrt{x^2 + 4\epsilon}}{2}\right)^a \cdot \left(\sqrt{x^2 + 4\epsilon}\right)^b$$

with

$$a \in \mathbb{Z}, \quad b = 0, 1, \quad x \in \mathbb{Z} \setminus \{0\}, \quad \epsilon = \pm 1, \quad r \in \mathbb{Q}^\times,$$

we have $\delta_{\min}^{(\alpha)}(N) \ll_{\alpha, \epsilon} N^{-1+\epsilon}$ infinitely often, for any $\epsilon > 0$.

Furthermore, the authors remark that when $b = 0$ and $a$ is even, they have in fact obtained the stronger result that $\delta_{\min}^{(\alpha)}(N) \ll_{\alpha} N^{-1}$ for all $N$. We refer the reader to [1] for a more detailed introduction to the problem, as well as the motivation for these upper bounds.

Date: October 26, 2018.

The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 320755.
Our goal in this paper is to generalize the result above to any positive quadratic irrationality. Here on, we shall denote by $K = \mathbb{Q}[\sqrt{D}]$ – a real quadratic number field, $\mathcal{O}_K$ – its ring of integers, and for any $\omega \in K$, we shall denote its conjugate in $K$ by $\overline{\omega}$, and its norm and trace (over $\mathbb{Q}$) by $N(\omega) = \omega \overline{\omega}$, $\text{tr}(\omega) = \omega + \overline{\omega}$. In section 3, we prove:

**Theorem 1.2.** Let $K = \mathbb{Q}[\sqrt{D}]$ be any real quadratic number field, where $D \in \mathbb{N}$ is square-free. Let $0 < \alpha \in K$ be a positive element of $K$. Then $\delta_{\min}^{(\alpha)}(N) \ll_{\alpha,\varepsilon} N^{-1+\varepsilon}$ infinitely often, for any $\varepsilon > 0$. Furthermore, if $\alpha \in \mathbb{Q}_>0 \cdot (K^\times)^2$ then $\delta_{\min}^{(\alpha)}(N) \ll N^{-1}$ for all $N$.

One of the key lemmas in [1] relates small gaps between the $\lambda_i$ to finding good rational approximations to $\alpha$ with both numerator and denominator being evenly divisible (or strongly even divisible): We call a sequence $m_n, n_m$ of numbers evenly divisible if there exist divisors $d_n \mid m_n$ such that $\min(d_n, m_n/d_n) \gg \varepsilon^1 m_n^{1/2-\varepsilon}$, for all $\varepsilon > 0$. We call the sequence strongly evenly divisible if $\min(d_m, m_n/d_n) \gg m_n^{1/2}$. Then by [1, Lemma 3.1], we have:

**Lemma 1.3.** If $\alpha > 0$ has infinitely many rational approximations $p_n/q_n$ satisfying

\begin{equation}
|q_n\alpha - p_n| \ll \frac{1}{q_n} 
\end{equation}

with the sequences $\{p_n\}, \{q_n\}$ being evenly divisible (resp. strongly evenly divisible), then $\delta_{\min}^{(\alpha)}(N) \ll N^{-1+\varepsilon}$ for all $\varepsilon > 0$ (resp. $\delta_{\min}^{(\alpha)}(N) \ll N^{-1}$) infinitely often.

If in addition $q_n \gg q_{n+1}$ for all $n$, then these inequalities hold for all $N$.

The authors of [1] were able to find evenly divisible approximations satisfying (1.2) for quadratic irrationalities of the form (1.1). However, the condition (1.2) can be significantly weakened; namely, if we replace it with the condition

\begin{equation}
|q_n\alpha - p_n| \ll \frac{1}{q_n^{1-\varepsilon}}, \text{ for all } \varepsilon > 0 
\end{equation}

The conclusion of Lemma 1.3 that $\delta_{\min}^{(\alpha)}(N) \ll N^{-1+\varepsilon}$ for all $\varepsilon > 0$ remains valid. This is essentially the content of [1, Lemma 3.2], and the form of the lemma that will be more useful to us.

---

1Given any $\alpha \in K$, this condition can be easily checked, as it is equivalent to $\alpha$ satisfying $N(\alpha) \in (\mathbb{Q}^\times)^2$ and $\text{tr}(\alpha) > 0$. We do not use this equivalence in this paper, and leave its proof as a simple exercise to the reader.
2. Divisibility properties

In this section we establish some divisibility properties for traces and twisted traces of powers of a quadratic algebraic integer. These play similar roles to the Chebyshev polynomials featured in [1], and in fact generalize them.

2.1. Divisibility properties of traces. Let \( \omega \in \mathcal{O}_K \) be an algebraic integer with \( \text{tr} \omega \neq 0 \). Note that for any odd positive integer \( \ell, \omega^\ell \in \mathcal{O}_K \), hence \( \text{tr}(\omega^\ell) \) is an integer. Furthermore, it is divisible by \( \text{tr}(\omega) \); this follows from the fact that \( x^\ell + y^\ell \) is a symmetric polynomial in \( \mathbb{Z}[x, y] \), and substituting \( \omega, \bar{\omega} \) for \( x, y \). More generally, for any positive integer \( L \) which is odd and square-free, we have the decomposition

\[
\text{tr}(\omega^L) = \Phi_L(\omega) \Psi_L(\omega),
\]

where \( \Phi_L(\omega), \Psi_L(\omega) \in \mathbb{Z} \) are both integers, defined by

\[
\Phi_L(\omega) = \prod_{\ell | L} \text{tr}(\omega^{L/\ell})^{\mu(\ell)}, \quad \Psi_L(\omega) = \prod_{\ell | L; \ell \geq 1} \text{tr}(\omega^{L/\ell})^{-\mu(\ell)}. \tag{2.2}
\]

Indeed, both \( \Phi_L, \Psi_L \) are symmetric polynomials in \( \mathbb{Z}[\omega, \bar{\omega}] \), hence their values must be in \( \mathbb{Z} \) for \( \omega \in \mathcal{O}_K \).

In order to estimate the sizes of the factors \( \Phi_L, \Psi_L \), assume further that \( |\omega/\bar{\omega}| \leq 1/2 \). Note that

\[
e^{-2|x|} \leq 1 - |x| \leq |1 + x| \leq 1 + |x| \leq e^2|x|
\]

for all \( |x| \leq 1/2 \), and therefore for any \( m \geq 1 \),

\[|
\text{tr}(\omega^m)| = |\omega^m + \bar{\omega}^m| = |\omega^m| \left(1 + \left(\frac{\bar{\omega}}{\omega}\right)^m\right) \in |\omega^m| \cdot [e^{-21-m}, e^{21-m}],
\]

hence,

\[
|\text{tr}(\omega^L)| \asymp |\omega|^L, \quad |\Phi_L(\omega)| \asymp |\omega|^\phi(L), \quad |\Psi_L(\omega)| \asymp |\omega|^{L-\phi(L)}, \tag{2.3}
\]

where the implied constants are all between \( e^{-2} \) and \( e^2 \). It immediately follows that

\[
|\Phi_L(\omega)| \asymp |\text{tr}(\omega^L)|^{\phi(L)/L}, \quad |\Psi_L(\omega)| \asymp |\text{tr}(\omega^L)|^{1-\phi(L)/L}. \tag{2.4}
\]

2.2. Divisibility properties of twisted traces. For \( \omega \in K \), we define the twisted trace of \( \omega \) as

\[
\tilde{\text{tr}}(\omega) := \text{tr}(\sqrt{D} \cdot \omega) = (\omega - \bar{\omega})\sqrt{D} \in \mathbb{Q}. \tag{2.5}
\]

Suppose \( \omega \in \mathcal{O}_K \), and \( \tilde{\text{tr}}(\omega) \neq 0 \) (i.e. \( \omega \notin \mathbb{Q} \)). Then for all positive integers \( \ell \), \( \tilde{\text{tr}}(\omega^\ell) \) is an integer and divisible by \( \text{tr}(\omega) \), since \( \frac{x^\ell + y^\ell}{x + y} \) is a symmetric

---

2 Specifically, \( \Phi_L \) is the homogenized reflected \( L \)-th cyclotomic polynomial, and \( \Psi_L \) is the corresponding cofactor.
polynomial in \(\mathbb{Z}[x, y]\) and \(\sqrt{D}\omega^L \in \mathcal{O}_K\). As above, this generalizes to any square-free \(L \in \mathbb{N}\), by
\[
\tilde{\tr}(\omega^L) = \tilde{\Phi}_L(\omega)\tilde{\Psi}_L(\omega),
\]
where
\[
(2.7) \quad \tilde{\Phi}_L(\omega) = \prod_{\ell \mid L} \tilde{\tr}(\omega^{L/\ell})^\mu(\ell), \quad \tilde{\Psi}_L(\omega) = \prod_{\ell | L; \ell > 1} \tilde{\tr}(\omega^{L/\ell})^{-\mu(\ell)}.
\]
are again integers, and whenever \(|\sqrt[3]{/}\omega| < 1/2 \text{ and } L > 1\), we have
\[
|\tilde{\tr}(\omega^L)| \asymp \sqrt{D}|\omega|^L, \quad |\tilde{\Phi}_L(\omega)| \asymp |\omega|^{\phi(L)}, \quad |\tilde{\Psi}_L(\omega)| \asymp \sqrt{D}|\omega|^{L-\phi(L)},
\]
and consequently
\[
(2.9) \quad |\tilde{\Phi}_L(\omega)| \asymp D^{-\frac{\phi(L)}{2L}}|\tilde{\tr}(\omega^L)|^\frac{\phi(L)}{L}, \quad |\tilde{\Psi}_L(\omega)| \asymp D^{-\frac{\phi(L)}{2L}}|\tilde{\tr}(\omega^L)|^{1-\frac{\phi(L)}{L}}.
\]
The major difference between the divisibility properties for \(\tr(\omega^L)\) and \(\tilde{\tr}(\omega^L)\) is that in the second case \(L\) may be even – and specifically, we may choose \(L = 2\), which is the only number with \(\phi(L)/L = 1/2\), capable of generating strongly evenly divisible sequences, rather than just evenly divisible.

3. Proof of Theorem 1.2

We first note that we may assume \(\alpha \in \mathcal{O}_K\). Indeed, for any \(\alpha \in K\) there is some denominator \(A \in \mathbb{N}\) such that \(A \cdot \alpha \in \mathcal{O}_K\). If we are then able to find (strongly) evenly divisible \(p_n, q_n\) such that \(A \cdot \alpha\) is well-approximated by \(p_n/q_n\), then the sequence \(A \cdot q_n\) is also (strongly) evenly divisible and \(\alpha\) is well-approximated by \(p_n/(A \cdot q_n)\).

Let \(\zeta \in \mathcal{O}_K\) be an algebraic integer with \(N(\zeta) = 1\) and \(|\zeta| > 1 > |\sqrt[3]{/}\zeta|\). Such \(\zeta\) may constructed e.g. as \(\zeta = x + y\sqrt{D}\), where \((x, y) \in \mathbb{N}^2\) is a non-trivial integral solution to the Pell equation \(x^2 - Dy^2 = 1\). Let \(\varepsilon > 0\) be arbitrarily small.

3.1. A symmetric construction. Let \(\{\ell_i\}_{i=1}^\infty, \{\ell'_i\}_{i=1}^\infty\) be two disjoint sub-sequence of the odd primes, with
\[
\prod_{i=1}^\infty \left(1 - \frac{1}{\ell_i}\right) = \prod_{i=1}^\infty \left(1 - \frac{1}{\ell'_i}\right) = 1/2,
\]
the existence of which easily follows from \(\prod_p (1 - 1/p) = 0\). For a fixed \(t \in \mathbb{N}\) write
\[
L = \prod_{i=1}^t \ell_i, \quad L' = \prod_{i=1}^t \ell'_i
\]
and suppose \(t = t(\varepsilon)\) is sufficiently large so that \(\frac{\phi(L)}{L}, \frac{\phi(L')}{L'} \in (1/2, 1/2 + \varepsilon)\).

---

\[^3\]We may choose \(\zeta\) with \(N(\zeta) = -1\) instead, in fields where such integers exist. Note that the number \(\frac{x + \sqrt{2x^2 + 4t}}{2}\) appearing in (1.1) is always an example of an appropriate \(\zeta\) for the appropriate field.
Let $M = M(\varepsilon) \in \mathbb{N}$ be the smallest positive integer such that $M + 1$ is divisible by $L$ and $M$ is divisible by $L'$, and write $M + 1 = m_1 L, M = m_2 L'$. Define $N = nLL'$, where $n = n(\varepsilon)$ is any sufficiently large number such that the following inequalities are satisfied:

\begin{align}
|\zeta/\alpha|^{nL'} &> 2|\alpha/\alpha|^m, \\
|\zeta/\alpha|^{nL} &> 2|\alpha/\alpha|^{m_2}, \\
|\zeta/\alpha|^N &> 4|\alpha - \alpha||N(\alpha)|^M|\alpha|^{-\varepsilon M}.
\end{align}

Such $n$ clearly exist, as $\alpha$ is fixed, $m_1, m_2, L$ and $L'$ depend only on $\varepsilon$, and $|\zeta/\alpha| > 1$, by the definition of $\zeta$. Finally, we define

\begin{align}
P &= P(\varepsilon) = \text{tr}(\alpha^{M+1}\zeta^N) = \text{tr}((\alpha^{m_1}\zeta^{nL'})^L), \\
Q &= Q(\varepsilon) = \text{tr}(\alpha^{M}\zeta^N) = \text{tr}((\alpha^{m_2}\zeta^{nL'})^L).
\end{align}

We first show that $P/Q$ is a good approximation to $\alpha$. Note that (3.1) implies $|Q|$ is approximately $|\alpha^M\zeta^N|$, or, more explicitly,

\begin{equation}
|Q|/|\alpha^M\zeta^N| \in [1 - 2^{-L'}, 1 + 2^{-L'}] \subset [1/2, 2].
\end{equation}

It then follows that

\[ |\alpha Q - P| = |\alpha - \alpha||\alpha^M\zeta^N| = \frac{|\alpha - \alpha||N(\alpha)|^M|N(\zeta)|^N}{|\alpha^M\zeta^N|} \leq \frac{C(\alpha, \varepsilon)}{|Q|} \leq \frac{1}{|Q|^{1-\varepsilon}}, \]

where $C(\alpha, \varepsilon) = 2|\alpha - \alpha||N(\alpha)|^{M(\varepsilon)}$ depends only on $\alpha$ and $\varepsilon$. The two final steps then follow immediately from (3.5), (3.2) and $|N(\zeta)| = 1$.

Next, we show that $P$ and $Q$ are evenly divisible. By their definitions, the inequalities (3.1) and section 2.1, it is evident that the factorization $P = \Phi_L(\alpha^{m_1}\zeta^{nL'})\Psi_L(\alpha^{m_1}\zeta^{nL'})$ satisfies

\[ \min(|\Phi_L(\alpha^{m_1}\zeta^{nL'})|, |\Psi_L(\alpha^{m_1}\zeta^{nL'})|) \times \min(|P|^{1/2}, |P|^{1/2}) \gg |P|^{1/2-\varepsilon}, \]

and similarly $Q = \Phi_L(\alpha^{m_2}\zeta^{nL'})\Psi_L'(\alpha^{m_2}\zeta^{nL})$ with

\[ \min(|\Phi_L(\alpha^{m_2}\zeta^{nL'})|, |\Psi_L'(\alpha^{m_2}\zeta^{nL})|) \gg |Q|^{1/2-\varepsilon}. \]

We conclude that for any sequence $\varepsilon_k \searrow 0$, the sequences $P(\varepsilon_k), Q(\varepsilon_k)$ are evenly divisible and provide good approximations to $\alpha$, which allows us to conclude via Lemma 1.3.

3.2. Alternative constructions. Instead of defining $P$ and $Q$ via traces of integral powers, we may use twisted traces instead, and define

\[ P = \tilde{\text{tr}}((\alpha^{m_1}\zeta^{nL'})^L), \quad Q = \tilde{\text{tr}}((\alpha^{m_2}\zeta^{nL'})^L). \]

Repeating the same computations for the same values of $L, L'$ would lead to the same estimates as in the first construction. However, since we are now using skew-traces, we may define $L$ (resp. $L'$) to be equal to 2 instead of a product of odd primes, which will then correspond to $P$ (resp. $Q$) having two factors of size $\gg |P|^{1/2}$, i.e. the sequence $P(\varepsilon_k)$ (resp. $Q(\varepsilon_k)$) will be strongly evenly divisible. However, we cannot in general do this for both $P$
and $Q$ simultaneously, since it is impossible for both $M + 1$ and $M$ to be even.

Suppose now that $\alpha \in \mathbb{Q}_{>0} \cdot (K^\times)^2$. By multiplying by a proper natural denominator $A$, we may assume that $\alpha \in \mathcal{O}_K^2$, i.e. $\alpha = \beta^2$ for some $\beta \in \mathcal{O}_K$. Now, for all sufficiently large $n$, define

$$P_n := \tilde{\text{tr}}(\alpha \zeta^{2n}) = \tilde{\text{tr}}((\beta \zeta^n)^2) = \tilde{\text{tr}}(\beta \zeta^n) \cdot \text{tr}(\zeta^n),$$

$$Q_n := \tilde{\text{tr}}(\zeta^{2n}) = \tilde{\text{tr}}((\zeta^n)^2) = \tilde{\text{tr}}(\zeta^n) \cdot \text{tr}(\zeta^n).$$

The sequences $P_n$ and $Q_n$ are therefore both strongly evenly divisible, since for large $n$ we have

$$|\tilde{\text{tr}}(\beta \zeta^n)| \asymp \sqrt{D} |\beta \zeta^n| \asymp \sqrt{D} |\text{tr}(\beta \zeta^n)|,$$

$$|\tilde{\text{tr}}(\zeta^n)| \asymp \sqrt{D} |\zeta^n| \asymp \sqrt{D} |\text{tr}(\zeta^n)|.$$}

Furthermore, their ratios approximate $\alpha$ well, as

$$|Q_n \alpha - P_n| = |\alpha - \overline{\alpha}| \overline{|\zeta|}^{2n} = \frac{|\alpha - \overline{\alpha}| \sqrt{D}}{|Q_n|},$$

and the denominators grow geometrically - $|Q_n| \asymp |\zeta^{-2}||Q_{n+1}| \gg |Q_{n+1}|$.

Therefore for such $\alpha$, we have $\delta_{\text{min}}^{(\alpha)}(N) \ll N^{-1}$ for all large $N$, via Lemma 1.3.

### 3.3. Comparison to previous results.

For completeness, we present the constructions used in the original proof of Theorem 1.1, in terms of our notation. As mentioned above, the term $\sqrt{x^2 + 4\epsilon}$ is simply our $\zeta$. The optional term $\sqrt{x^2 + 4\epsilon}$ belongs to $\sqrt{D} \cdot \mathbb{Q}^\times$. Therefore the numbers in $K$ covered by Theorem 1.1 are those of the form

$$\alpha(\zeta; r, a, b) = r \cdot \zeta^a \cdot \sqrt{D}^b$$

where $\zeta \in \mathcal{O}_K^\times$, $r \in \mathbb{Q}^\times$, $a \in \mathbb{Z}$ and $b = 0, 1$. The term $r = c/d$ is dealt with by multiplying all numerators by $c$ and denominators by $d$, and we reduce to the case $r = 1$ (and thus $\alpha \in \mathcal{O}_K$). These $\alpha$ have the special property that the sequences of traces $\text{tr}(\alpha \zeta^N) = \text{tr}(\sqrt{D}^b \zeta^{N+a})$, and $\text{tr}(\zeta^N)$ (as well as $\tilde{\text{tr}}(\alpha \zeta^N)$ and $\tilde{\text{tr}}(\zeta^N)$) can have useful divisibility properties, for good choices of $N$, and their ratios provide good approximations to $\alpha$. For general $\alpha$, we can generalize this method and generate good approximations to $\alpha$ as ratios of the sequences of $\text{tr}(\alpha \omega \zeta^N)$ and $\text{tr}(\omega \zeta^N)$, for any $\omega \in \mathcal{O}_K$. The quality of the approximation is slightly worse: the upper bound on the error grows by the norm of $\omega$, but this is not a problem if $\omega$ is fixed and $N$ is large. In order to have good divisibility properties for these sequences we want $\alpha \omega \zeta^N$ and $\omega \zeta^N$ to be $L$-th and $L'$-th powers, respectively. We achieve this by setting $\omega = \alpha^M$ and choosing appropriate values of $M$ and $N$ – but other choices for $\omega$ and $N$ might also be possible.
References

[1] Blomer, V., Bourgain, J., Radziwiłł, M., and Rudnick, Z. Small gaps in the spectrum of the rectangular billiard. arXiv:1604.02413v3. To appear in Ann. Sci. École Norm. S..

RAYMOND AND BEVERLY SACKLER SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL