Half-integer quantized response in strongly driven quantum systems

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A spin strongly driven by two incommensurate tones can pump energy from one drive to the other at a quantized average rate, in close analogy with the quantum Hall effect. The quantized pumping is a pre-thermal effect with a lifetime that diverges as the drive frequencies approach zero. We study the transition between the pumping and non-pumping pre-thermal states. The transition is sharp at zero frequency and is characterized by a Dirac point in the instantaneous band structure parametrized by the drive phases. We show that the pumping rate is half-integer quantized at the transition and present universal Kibble-Zurek scaling functions for energy transfer processes in the low frequency regime. Our results identify qubit experiments to measure the universal linear and non-linear response of a Dirac point.

Introduction: A spin driven by a time-dependent magnetic field $\vec{B}(t)$ is a quintessential problem [1–9]. At low drive frequencies $\omega$, the spin’s direction is locked to the instantaneous field direction up to an “unlock time” $t_u$, after which it effectively heats to infinite temperature.

Recent work has shown that a spin-\(\frac{1}{2}\) driven by two circularly polarized magnetic fields with incommensurate frequencies ($\omega_1$ and $\omega_2$) pumps energy from drive 1 to drive 2 at a quantized average rate [10,11,12]

$$[P_{IT}]_0 = C_g P_Q \equiv C_g \frac{\omega_1 \omega_2}{2\pi}, \quad (1)$$

where $h = 1$, the spin is prepared in the instantaneous ground state, and $C_g$ equals the Chern number of the instantaneous ground state band as a function of the drive phases. In the trivial regime, $C_g = 0$ and the spin doesn’t pump. In the topological regime however, $C_g$ is a non-zero integer, and the spin pumps energy as long as it is locked to the direction of the field [11,12]. The lifetime of the pumping is exponentially long in $1/\omega$ in the low-frequency regime and can be further extended by either introducing relaxation processes [11], or by counter-diabatic driving [12,13].

In this letter, we show that $C_g$, and thus the power, is quantized to a half-integer at the dynamical transition between the topological and trivial regimes. The transition is controlled by a sign change of the mass at a Dirac point in the instantaneous band structure [12,15,17]. However, for times $t \ll t_u$, the spin trajectory is generally far from the Dirac point and the pumping rate is set by the integrated Berry curvature of the ground state band excluding the Dirac point.

The half-integer quantized pumping is nontrivial to observe as (i) the spin exchanges energy with the drives when it unlocks, and (ii) an unlocked spin pumps energy at a rate less than in Eq. (1). The unlock time $t_u$ at the transition follows from Kibble-Zurek arguments [15,30],

$$t_u \sim \sqrt{B_0/\omega^3}, \quad (2)$$

where $B_0$ is the typical scale of the magnetic field acting on the spin.

We show that the average power has two universal contributions as $t/t_u \to 0$: [31]

$$[P_1]_0 = \frac{1}{2} P_Q + \frac{B_0}{t_u}. \quad (3)$$

The second term, due to spin excitation, dominates the topological term as $\omega \to 0$. However, drawing intuition from the action of time-reversal on Chern insulators, we isolate the topological and excitation terms and their associated universal Kibble-Zurek scaling terms using time evolution with the Hamiltonian and its complex conjugate (see Fig. [1]). Experimentally, complex conjugation corresponds to reversing the circular polarization of one
of the drives. Incommensurately driven few-level quantum systems thus offer a unique window into the universal properties of the topological phase transitions of band insulators.

Model: For concreteness, we work with the same “half-BHZ model” as Refs. [10–12]. The Hamiltonian is

$$H(\vec{\theta}_t) = -\frac{1}{2} \vec{B}(\vec{\theta}_t) \cdot \vec{\sigma}$$

$$\vec{B}(\vec{\theta}_t) = B_0(\sin \theta_{11}, \sin \theta_{12}, 2 + \delta - \cos \theta_{11} - \cos \theta_{12}),$$

(4)

where $\vec{\theta}_t = (\theta_{11}, \theta_{12})$ is the vector of drive phases, $\theta_{1i} = \omega_i t + \theta_{0i}$ for $i = 1,2$ and the ratio of the drive frequencies $\omega_2/\omega_1$ is an irrational number. We assume that the ratio of drive frequencies is order one, so that $\omega = \sqrt{\omega_1^2 + \omega_2^2}$ is the single frequency-scale on which $H$ varies. The spin is prepared in the instantaneous ground state at $t = 0$.

Instantaneous band structure: The instantaneous band structure consists of bands with energies $\pm \frac{1}{2} |\vec{B}(\vec{\theta})|$ over the torus $\vec{\theta} \in [0, 2\pi)^2$. At each $\vec{\theta}$, the spin in the ground (excited) band is aligned (anti-aligned) with $\vec{B}(\vec{\theta})$. This band structure is identical to that of the half-BHZ model in momentum space [15, 32, 33]. The ground state band has Chern number $C_g = 1$ for $-2 < \delta < 0$, and $C_g = 0$ for $\delta > 0$.

In the vicinity of the transition at $\delta = 0$, there is a massive Dirac point in the band structure at $|\vec{\theta}| = 0$:

$$H(\vec{\theta}) = -\frac{B_0}{2}(\theta_1 \sigma_x + \theta_2 \sigma_y + \delta \sigma_z) + O(|\vec{\theta}|^2).$$

(5)

Fig. 2a is a density plot of the Berry curvature of the ground state band in the topological regime, close to the transition. The Berry curvature has two contributions: a piece that is smooth in $\delta$ and $\vec{\theta}$ and integrates to $\pi$; and a singular piece which concentrates into a delta function at $\vec{\theta} = 0$ and integrates to $-\text{sgn}(\delta)\pi$ [15, 32].

The drive phases follow trajectories of constant slope in $\vec{\theta}$-space (shown in blue in Fig. 2a). Consider $t \ll t_u$. Away from the Dirac point, the Berry curvature is small, and the spin state along the trajectory approximately follows the instantaneous ground state [34]. In the low frequency limit and with $\omega_2/\omega_1$ irrational, the trajectory uniformly samples this Berry curvature over time.

We show below that the average power is set by the integrated Berry curvature of the instantaneous ground state band before unlock. Sufficiently far away from the transition, the trajectory samples the entire Berry curvature. At the transition, however, the spin unlocks before sampling the singular component associated with the Dirac point. Thus, the integrated Berry curvature in Eq. (1) is given by

$$C_g = \begin{cases} 
1 \quad \delta < 0 \\
1/2 \quad \delta = 0 \\
0 \quad \delta > 0.
\end{cases}$$

(6)

Pump power: The instantaneous rate of energy transfer from drive 1 is given by [10]

$$P_1 \equiv \omega_1 \langle \theta_{01} H \rangle,$$

(7)

with a corresponding expression for $P_2$. As the spin cannot absorb energy indefinitely, the net energy flux into the system time-averages to zero, $|P_{\text{tot}}| = |P_1| + |P_2| = 0$. Throughout, $[\cdot]_x$ denotes averaging with respect to variable $x$.

In the low frequency limit, $P_1$ is a sum of two terms, one analytic and one non-analytic in $\omega$. The analytic term is completely determined by the instantaneous values of $\vec{\theta}_t$, while the non-analytic terms depend on the entire history of the trajectory. As in the Landau-Zener problem, the analytic terms describe the perturbative “dressing” of the spin state over the instantaneous ground state [34, 35]. The non-analytic terms capture the non-adiabatic excitation processes between the dressed states. Below we refer to the leading order analytic term as $P_1^T$, as it is of topological origin, and non-analytic term as $P_{1E}$, which is due to excitations.

Topological contribution to pumping for $t \ll t_u$. We now derive Eq. (1) with $C_g$ given by Eq. (6). Let $\tilde{g}(\vec{\theta}_t)$
be the spin state dressed to order $\omega$ above the instantaneous ground state. Thus,
\[ \frac{d\hat{\mathbf{g}}}{dt} = H(\hat{\mathbf{g}}) + O(\omega^2) \]
\[ \Rightarrow H(\hat{\mathbf{g}}) = i\omega_1|\partial_{\mathbf{g}}\hat{\mathbf{g}}\rangle + i\omega_2|\partial_{\mathbf{g}_2}\hat{\mathbf{g}}\rangle + O(\omega^2). \]  
where we have suppressed the time dependence of $\hat{\mathbf{g}}$ for brevity. Using the product rule, we obtain
\[ P_{IT} = \omega_1(\mathbf{g}|\partial_{\mathbf{g}}H(\mathbf{g})\rangle = \omega_1|\partial_{\mathbf{g}_1}\langle H(\mathbf{g}) - \langle H(\mathbf{g})|\partial_{\mathbf{g}_1}\rangle \rangle = \omega_1|\partial_{\mathbf{g}_1}\langle g|H(\mathbf{g})\rangle + \omega_1\omega_2\Omega(\mathbf{g}), \]  
where $\Omega(\mathbf{g}) = 2\text{Im}(\partial_{\mathbf{g}_1}\langle g|\partial_{\mathbf{g}_2}\hat{g}\rangle)$ is the Berry curvature of the dressed spin state.

The instantaneous power varies with the initial phase vector $\hat{\mathbf{g}}_0$. Universal results about the spin dynamics at each $t$ follow upon initial phase averaging
\[ [P_{IT}]_{\hat{\mathbf{g}}_0} = [\partial_{\mathbf{g}_1}\langle g|H(\mathbf{g})\rangle]_{\hat{\mathbf{g}}_0} + \omega_1\omega_2[\Omega(\mathbf{g})]_{\hat{\mathbf{g}}_0} = C_g P_0. \]  
\[ \Rightarrow \text{Kibble-Zurek estimate for } t_u: \quad \text{The probability to transition to the dressed instantaneous excited state follows from the Landau-Zener result} \]  
\[ P_{exc} \sim \max_t \exp \left( -\frac{\pi|\partial_{\hat{\mathbf{g}}_0}|^2}{|\partial_{\hat{\mathbf{g}}_0}B|} \right). \]  
\[ \text{Deep in the topological or trivial regimes, the spin’s evolution thus remains adiabatic for an exponentially long time-scale } \sim \exp(B_0/\omega). \]  
\[ \Rightarrow \text{Eq. (11) predicts that the spin unlocks from the field when the instantaneous gap squared becomes comparable or smaller than the rate of change of the field.} \]  
\[ |\hat{\mathbf{g}}_1| \lesssim \theta^* = \sqrt{\omega/B_0}. \]  
\[ \text{This relation defines the “excitation region” within the dashed circle in Fig. 2}. \]  
A typical spin trajectory enters the excitation region for the first time after $2\pi/\theta^*$ periods. We thus obtain the scaling of the unlock time $t_u \sim (\omega\theta^*)^{-1} \sim \sqrt{B_0/\omega^2}$ previously stated in Eq. 2.

In the vicinity of the transition, the spin trajectory encounters small gaps of order $B_0\delta$ near $\theta = 0$ (Fig. 2). If the frequency is sufficiently small
\[ \omega < \omega^* \sim B_0\delta^2, \]  
then the dynamics is adiabatic through the small gaps. The pumped power for $t \ll t_u$ is then set by the $\omega \to 0$ ‘phases’. However, when $\omega > \omega^*$, the transition controls the spin dynamics. The parabola in Fig. 1 separates these two regimes and defines the critical “fate”.

Topological contribution to pumping for $t \gg t_u$: At times much longer than the unlock time, non-adiabatic processes heat the spin. In the initial phase ensemble, the populations in the (dressed) instantaneous ground and excited states thus become equal. As the Chern numbers of the ground and excited state bands sum to zero, the ensemble averaged power $[P_{IT}]_{\hat{\mathbf{g}}_0} \to 0$ as $t/t_u \to \infty$.

Excitation contribution to pumping: The non-adiabatic excitation of the spin results in a distinct contribution to the power $[P_{IE}]_{\hat{\mathbf{g}}_0}$. As the spin absorbs order $B_0$ energy from the drives over a time-scale $t_u$
\[ [P_{IE}]_{\hat{\mathbf{g}}_0} \sim B_0/t_u \propto \omega^{3/2}, \quad t \lesssim t_u. \]
\[ \Rightarrow \text{Unlike the topological contribution, the power due to excitation is non-analytic in } \omega. \]  
\[ \text{The total pumped power is the sum of the topological and excitation contributions. Using Eqs. (10) and (14), we obtain the result Eq. (3) quoted in the introduction.} \]  
\[ \text{A constant rate of excitation results in a linear increase of the excited state population in the initial phase ensemble at small } t/t_u. \]  
\[ \text{At late times, the populations become equal, and statistically the spin ceases to absorb energy from the drives. Thus, } [P_{IE}]_{\hat{\mathbf{g}}_0} \to 0 \text{ as } t/t_u \to \infty. \]
\[ \text{Kibble-Zurek scaling functions: Within the Kibble-Zurek (KZ) scaling limit, the non-equilibrium dynamics of the spin becomes universal even beyond the unlock time. The KZ scaling limit involves taking } \omega, \delta \to 0 \text{ which measuring time in units of the diverging unlock time } t_u, \text{ and the drive frequency in units of the vanishing scale } \omega^* \text{ previously stated in Eq. (2).} \]  
\[ \text{In this limit, the radius of the excitation region } \theta^* \text{ becomes small and the Hamiltonian of the massive Dirac cone (Eq. (5)) controls the excitation of the spin, and hence the decay of the topological component of the power. The topological and excitation components of the power then take the following scaling forms} \]  
\[ [P_{IE}(t;\omega,\delta)]_{\hat{\mathbf{g}}_0} \sim \omega^{3/2} \mathcal{P}_{IE} \left( t^{3/2}; \delta \omega^{-1/2} \right), \]  
\[ [P_{IT}(t;\omega,\delta)]_{\hat{\mathbf{g}}_0} \sim \omega^2 \frac{2\pi}{\mathcal{P}_{IT}} \left( t^{3/2}; \delta \omega^{-1/2} \right). \]  
\[ \Rightarrow \text{Above, } \mathcal{P}_{IE} \text{ and } \mathcal{P}_{IT} \text{ are scaling functions determined solely by the universality class of the transition in the instantaneous band structure. They capture the universal cross-over from the pre-thermal regime to the late-time infinite-temperature regime.} \]  
\[ \text{Scaling functions for the Dirac transition: We now numerically extract the scaling forms } \mathcal{P}_{IE}, \mathcal{P}_{IT} \text{ for the Dirac transition in the half-BHZ model using complex-conjugation of the Hamiltonian } H^*(\hat{\mathbf{g}}) = (H(\hat{\mathbf{g}}))^*. \]  
\[ \Rightarrow \text{Physically, this is achieved by flipping the chirality of the circular polarization of drive 2.} \]  
\[ \text{The Supplemental Material obtains the same scaling functions for a different microscopic model with a Dirac transition, demonstrating universality.} \]
In the scaling limit, the following relations hold

\[ \langle P_{1E} \rangle_{\theta_0} = \frac{1}{2} [\langle P_1 \rangle_{\theta_0} + \langle P_1' \rangle_{\theta_0}] \]
\[ \langle P_{1T} \rangle_{\theta_0} = \frac{1}{2} [\langle P_1 \rangle_{\theta_0} - \langle P_1' \rangle_{\theta_0}] \]

where \( P_1 = \omega_1 \langle \psi_1 | \partial_\phi | H'(|\theta_0|) | \psi_1' \rangle \), the conjugated system \( i \partial_\phi | \psi_1' \rangle \). \( H'(|\theta_0|) | \psi_1' \rangle \) is initialized in its instantaneous ground-state band \( | \psi_0' \rangle = (| \psi_0 \rangle)^* \).

Eq. (16) is obtained as follows. As the probability of excitation in Eq. (11) is invariant under complex conjugation, the ensemble populations of the instantaneous eigenstates are the same at each \( t/t_u \), for time evolution under \( H \) and \( H' \). Thus, \( [P_{1E}]_{\theta_0} = [P_{1E}]_{\theta_0}^* \). The topological piece however changes sign, \( [P_{1T}]_{\theta_0} = -[P_{1T}]_{\theta_0}^* \) as the Berry curvature changes sign under complex conjugation.

Fig. 3 shows the scaling functions across the transition for both contributions as a function of \( \delta/\omega^{1/2} \) for small \( t/t_u \). They confirm the scaling collapse of the (two-dimensional) data from Eq. (14) and the opposite sign of the pumping by the excited population. Finally, despite the negative turn at \( t \omega^{3/2} \approx 10 \) in the inset, we find that \( \mathcal{P}_{1T} \) approaches zero for \( t/t_u \gg 1 \), consistent with the main figure.

**Discussion:** We have demonstrated that a simple quasiperiodically driven spin-\( \frac{1}{2} \) exhibits universal scaling behaviors characteristic of extended classical or quantum equilibrium systems in the vicinity of continuous phase transitions. Our results serve as a new example of the tantalising correspondence emerging between equilibrium systems and systems subject to periodic or
quasi-periodic driving $[10][12][42][78]$.

The KZ scaling functions also provide the universal non-linear response of a clean Dirac material in an electric field $[79]$. On Fourier transforming the $\theta$ coordinates, the model in Eq. (4) is equivalent to the real-space half-BHZ model with an additional electric field $(\omega_1, \omega_2)$ $[10][12]$. If we identify $\omega$ with the magnitude of the electric field, then the topological component of the power is the Hall response, while the exciton component measures the population in the excited band due to dielectric breakdown when the insulator is initially at zero temperature.

Driven few-level systems can access other topological phase transitions in static systems using different driving protocols. Moreover, the KZ scaling theory can be extended to include the effects of dissipation $[11]$, or counter-diabatic driving $[12]$. Both effects increase the unlock time $t_u$, and may simplify experimental access to the half-quantized response in solid-state and quantum optical platforms that host qubits $[43][70][80][88]$.

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SUPPLEMENTAL INFORMATION

Universality of scaling functions at the Dirac transition

The Hamiltonian used in the main text (Eq. (4)) is a well known model \[15,17,32\], which has several special symmetries. Universality requires that the scaling functions \(\mathcal{P}_{1E}, \mathcal{P}_{1T}\) do not depend on these symmetries. Here we repeat the numerical analysis shown in the main text using a model which lacks these symmetries. We obtain scaling collapse in the KZ scaling limit and identical scaling functions to that in Fig. 5.

Symmetries of the Hamiltonian \[4\] relate non-trivial actions in \(\vec{\theta}\)-space to rotations of the spin:

\[
H(\vec{\theta}) = \sigma_z H(\vec{\theta}) \sigma_z \\
-H(\vec{\theta})^* = \sigma_y H(\vec{\theta}) \sigma_y \\
H(\vec{\theta}_2, -\vec{\theta}_1) = U_{\pi/2} H(\vec{\theta}_1, \vec{\theta}_2) U^\dagger_{\pi/2}.
\]

Here \(U_\phi = \exp(-i\phi \sigma_z/2)\). Furthermore, in the vicinity of the Dirac point at \(\delta = 0\),

\[
H(\vec{\theta}) = -B_0 \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \cdot \vec{\sigma} - B_0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \vec{\sigma} + O(|\vec{\theta}|^3) \]

and the final relation in Eq. \[17\] holds for any rotation angle

\[
H(R_\phi \vec{\theta}_1) = U_\phi H(\vec{\theta}_1) U^\dagger_\phi + O(|\vec{\theta}_1|^3).
\]

Above \(R_\phi\) is a rotation by an angle \(\phi\) in \(\vec{\theta}\)-space

\[
R_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}.
\]

To break the symmetries in Eq. \[17\], consider a spin-\(1/2\) driven by the magnetic field \(\vec{B}(\vec{\theta})\)

\[
\vec{B}(\vec{\theta}) = B_0 \begin{pmatrix} \sin(n_1 \theta_1 - \phi) + \sin \phi \\ \sin(n_2 \theta_2 - \phi) + \sin \phi \\ 2 \delta - \cos \theta_1 - \cos \theta_2 \end{pmatrix}.
\]

For \(\cos \phi \neq 0, n_1 \neq n_2\) there is an asymmetric Dirac point at \(\delta = 0, \vec{\theta} = 0\).

\[
H(\vec{\theta}) = -\frac{B_0 \cos \phi}{2} \begin{pmatrix} n_1 \theta_1 \\ n_2 \theta_2 \end{pmatrix} \cdot \vec{\sigma} - \frac{B_0}{4} \begin{pmatrix} n_1^2 \theta_1^2 \sin \phi \\ n_2^2 \theta_2^2 \sin \phi \end{pmatrix} \cdot \vec{\sigma} + O(|\vec{\theta}|^3).
\]

Fig 5 shows the scaling collapse obtained for the topological and trivial contributions to the power. In rescaled units, the scaling functions are identical to those obtained in the main text (Fig 3).