The Korteweg–de Vries Equation

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1 Historical Perspective

In 1895, Diederik Korteweg (1848–1941) and Gustav de Vries (1866–1934) derived a partial differential equation (PDE) that models the “great wave of translation” that naval engineer John Scott Russell had observed in the Union Canal in 1834.

Assuming that the wave propagates in the X-direction, the evolution of the surface elevation $\eta(X,T)$ above the undisturbed water depth $h$ at time $T$ can be modeled by the Korteweg–de Vries (KdV) equation,

$$\eta_t + \sqrt{gh}\eta_x + \frac{3}{2} \frac{\sqrt{gh}}{h} \eta_x \eta_x + \frac{1}{2} h^2 \sqrt{gh} \left( \frac{1}{3} - \frac{T}{pgh^2} \right) \frac{\partial^3 \eta}{\partial X^3} = 0,$$

where $g$ is the gravitational acceleration, $\rho$ is the density, and $T$ is the surface tension. The dimensionless parameter $T/\rho gh^2$, called the Bond number, measures the relative strength of surface tension and the gravitational force. Equation (1) is valid for long waves (or shallow water) of relatively small amplitude, $|\eta|/h \ll 1$.

In dimensionless variables, (1) can be written as

$$u_t + \alpha uu_x + u_{xxx} = 0$$

where subscripts denote partial derivatives. The term $\sqrt{gh} \eta_x$ in (1) has been removed by an elementary transformation. Conversely, a linear term in $u_x$ can be added to (2). The parameter $\alpha$ can be scaled to any real number. Commonly used values are $\alpha = \pm 1$ and $\alpha = \pm 6$.

The term $u_t$ describes the time evolution of the wave. Therefore, (2) is called an evolution equation. The nonlinear term $\alpha uu_x$ accounts for steepening of the wave. The linear dispersive term $u_{xxx}$ describes spreading of the wave.

The KdV equation had already appeared in seminal work on water waves published by Joseph Boussinesq about 20 years earlier.

2 Solitary wave and periodic solution

The balance of the steepening and spreading effects gives rise to a stable solitary wave,

$$u(x,t) = \frac{\omega - 4k^3}{\alpha k} \pm \frac{12k^2}{\alpha} \text{sech}^2(\frac{kx - \omega t + \delta}{\alpha}),$$

where the wave number $k$, the angular frequency $\omega$, and phase $\delta$ are arbitrary constants. Requiring that $\lim_{x \to \pm \infty} u(x,t) = 0$ for all $t$ leads to $\omega = 4k^3$, in which case (3) reduces to

$$u(x,t) = 12k^2/\alpha \text{sech}^2(\frac{kx - 4k^3t + \delta}{\alpha}).$$

This hump-shape solitary wave of finite amplitude $12k^2/\alpha$ travels to the right at constant phase speed $v = \omega/k = 4k^2$. It models the “great wave of translation” that traveled without change of shape over a fairly long distance as observed by Scott Russell.

As shown by Korteweg and de Vries, equation (2) also has a periodic solution,

$$u(x,t) = (\omega - 4k^3(2m - 1))/(\alpha k) + \frac{12(k^2/\alpha) m \text{cn}^2(\frac{kx - \omega t + \delta}{\alpha}; m), \quad (5)$$

where they called the cnoidal wave solution for it involves Jacobi’s elliptic cosine function, $\text{cn}$, with modulus $m$, $0 < m < 1$. In the limit $m \to 1$, $\text{cn}(\xi; m) \to \text{sech}\xi$ and (5) reduces to (3).

3 Modern Developments

The solitary wave was for many years considered an unimportant curiosity in the field of nonlinear waves. That changed in 1965, when Zabusky and Kruskal realized that the KdV equation arises as the continuum limit of a one-dimensional anharmonic lattice used by Fermi, Pasta, and Ulam in 1955 to investigate how energy is distributed among the many possible oscillations in the lattice. Since taller solitary waves travel faster than shorter ones, Zabusky and Kruskal simulated the collision of two waves in a nonlinear crystal lattice and observed that each retains its shape and speed after collision. Interacting solitary waves merely experience a phase shift, advancing the faster and retarding the slower wave. In analogy with colliding particles, they coined the word “solitons” to describe these elastically colliding waves.
To model water waves that are weakly nonlinear, weakly dispersive, and weakly two-dimensional with all three effects being comparable, Kadomtsev and Petviashvili (KP) derived a two-dimensional version of (2) in 1970
\[(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{6}\]
with $\sigma^2 = \pm 1$ and where the $y$-axis is perpendicular to the direction of propagation of the wave (along the $x$-axis).

The KdV and KP equations and the nonlinear Schrödinger (NLS) equation,
\[iu_t + u_{xx} + \kappa |u|^2 u = 0, \tag{7}\]
where $\kappa$ is a constant and $u(x,t)$ is a complex-valued function, are famous examples of so-called completely integrable nonlinear PDEs which can be solved with the inverse scattering transform (IST), a nonlinear analog of the Fourier transform.

The IST is not applied to (2) directly but to an auxiliary system of linear PDEs,
\[
\begin{align*}
\psi_{xx} + (\lambda + \frac{\alpha}{6}u)\psi &= 0, \tag{8} \\
\psi_t + \frac{\alpha}{2} u_x \psi + \alpha u \psi_x + 4\psi_{xxx} &= 0, \tag{9}
\end{align*}
\]
which is called the Lax pair for the KdV equation. Equation (8) is a linear Schrödinger equation for an eigenfunction $\psi$, a constant eigenvalue $\lambda$, and a potential $(-\alpha u)/6$. Equation (9) governs the time evolution of $\psi$. The two equations are compatible, i.e., $\psi_{xxt} = \psi_{txx}$, if and only if $u(x,t)$ satisfies (2). For given $u(x,0)$ decaying sufficiently fast as $|x| \to \infty$, the IST solves (8) and (9) and finally determines $u(x,t)$.

Apart from shallow water waves, the KdV equation is ubiquitous in applied science. It describes, for example, ion-acoustic waves in a plasma, elastic waves in a rod, and internal waves in the atmosphere or ocean. The KP equation models, e.g., water waves, acoustic waves, and magnetoelastic waves in anti-ferromagnetic materials. The NLS equation describes weakly nonlinear and dispersive wave packets in physical systems, e.g., light pulses in optical fibers, surface waves in deep water, Langmuir waves in a plasma, and high-frequency vibrations on a crystal lattice. Equation (7) with an extra linear term $V(x)u$ to account for the external potential $V(x)$ also arises in the study of Bose–Einstein condensates, where it is referred to as the time-dependent Gross–Pitaevskii equation.

### Further Reading

Ablowitz M. J. and Clarkson P. A., Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, U.K., 1991.

Kasman A., Glimpses of Soliton Theory, AMS, Providence, RI, 2010.

Osborne A. R., Nonlinear Ocean Waves and the Inverse Scattering Transform, Academic Press, Burlington, MA, 2010.

### Properties and Applications

Scientists remain intrigued by the rich mathematical structure of completely integrable nonlinear PDEs that can be written as infinite-dimensional bi-Hamiltonian systems. Completely integrable nonlinear PDEs have remarkable features, such as a Lax pair, a Hirota bilinear form, Bäcklund transformations, and the Painlevé property. They have an infinite number of conserved quantities, infinitely many higher-order symmetries, and an infinite number of soliton solutions.