Abstract. We invoke the law of sines to prove Morley’s trisector theorem. Though the sinusoidal function appears, the proof is safe for trigonometric distancing.

1. Telegraphic Introduction.

We will provide a proof of the celebrated Morley trisector theorem, relying on the law of sines for the crux of the argument. Although we employ the sinusoidal function, the proof is deemed safe for those with trigonometric restrictions.

Figure 1. The Morley triangle $T$.

We begin with a triangle $\triangle ABC$ having interior angles $3\alpha, 3\beta, 3\gamma$. By trisecting each interior angle and then joining nearby rays of trisection, we obtain three points inside $\triangle ABC$. These interior points form a triangle $T$ (see Figure 1) and we wish to show that $T$ is equilateral. The triangle $\triangle ABC$, which we’ll call the native triangle, will figure only fractionally in our discussion, until it makes its full return.

2. A Nonlinear Lemma.

Lemma 1. Let $\alpha, \beta$ and $\alpha', \beta'$ be two pairs of proper acute angles. Assume they have the same sum and the same ratio of sines:

$$\alpha + \beta = \alpha' + \beta'; \quad \frac{\sin(\alpha)}{\sin(\beta)} = \frac{\sin(\alpha')}{\sin(\beta')}.$$

Then the corresponding angles are equal:

$$\alpha = \alpha', \quad \beta = \beta'.$$

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Gazing at the hypothesis, we will denote the relation to the left of the semi-colon the *linear relation* and denote the other as the *nonlinear* relation.

**Proof.** Suppose that $\alpha < \alpha'$. By the linear relation in the hypothesis, we then have $\beta > \beta'$. Noting that the sine function is monotone increasing in the domain of acute angles, we have

$$0 < \sin(\alpha) < \sin(\alpha') ; \quad \sin(\beta) > \sin(\beta') > 0.$$  \hfill (1)

Taking the reciprocal version of the right portion of (1) and multiplying, we obtain

$$\frac{\sin(\alpha)}{\sin(\beta)} < \frac{\sin(\alpha')}{\sin(\beta')} ,$$

which contradicts the nonlinear portion of the hypothesis. Next, a similar argument excludes the possibility $\alpha > \alpha'$. So $\alpha = \alpha'$ and hence $\beta = \beta'$. \hfill \square

### 3. Dashed Hopes.

We now consider an abstract equilateral triangle $W$ (in the cloud). Draw $W$ with one side close to horizontal at the top and subtend angles of $\gamma + 60^\circ$ to the left and right vertices of this nearly horizontal side. Form left and right triangles with the subtended angles, so that the new far angles are $\beta$ and $\alpha$ and the new angles at the lower vertex of $W$ are $\alpha + 60^\circ$ and $\beta + 60^\circ$, as in Figure 2. Since $\alpha + \beta + \gamma = 60^\circ$, this is clearly possible.

\[\begin{align*}
&\text{Figure 2. An abstract triangle.} \\
&\begin{array}{c}
\begin{tikzpicture}
\fill[black!20,fill opacity=0.5] (0,0) -- (2,2) -- (4,0) -- cycle; \\
\fill[black!20,fill opacity=0.5] (6,0) -- (4,2) -- (2,0) -- cycle; \\
\fill[black!20,fill opacity=0.5] (0,0) -- (2,2) -- (4,0) -- cycle; \\
\fill[black!20,fill opacity=0.5] (6,0) -- (4,2) -- (2,0) -- cycle; \\
\fill[black!20,fill opacity=0.5] (0,0) -- (2,2) -- (4,0) -- cycle; \\
\fill[black!20,fill opacity=0.5] (6,0) -- (4,2) -- (2,0) -- cycle; \\
\end{tikzpicture}
\end{array}
\end{align*}\]

\[\text{Figure 3. Equality of angle sums.} \]

**Lemma 2.** In Figure \[3\]

$$\alpha + \beta = \alpha' + \beta'. \hfill (2)$$

**Proof.** Adding interior angles in the lowest triangle in Figure 3 we have

$$\alpha' + \beta' + \gamma + 120^\circ = 180^\circ.$$ 

Since $\alpha, \beta, \gamma$ are trisected editions of the interior angles of our native triangle, we have

$$\alpha + \beta + \gamma = 60^\circ.$$
Combining the last two equations we have $\alpha + \beta = \alpha' + \beta'$.

We now sleepily label with $Z$’s the 3 sides of the abstract equilateral triangle. In the lowest triangle of Figure 3, we label with $X$ the side opposite $\alpha'$ and we label with $Y$ the side opposite $\beta'$, obtaining Figure 4.

![Figure 4. Equality of angle pairs.](image)

**Lemma 3.** The corresponding companion angles introduced by the lowest triangle are equal. That is, 

$$\alpha = \alpha'; \quad \beta = \beta'.$$

**Proof.** By the law of sines in the upper-left dashed triangle of Figure 4:

$$\frac{\sin(\beta)}{Z} = \frac{\sin(\gamma + 60^\circ)}{X}.$$  

By the law of sines in the upper-right dashed triangle of Figure 4:

$$\frac{\sin(\alpha)}{Z} = \frac{\sin(\gamma + 60^\circ)}{Y}.$$  

Dividing the last two equations, we obtain $\frac{\sin(\alpha)}{\sin(\beta)} = \frac{X}{Y}$. Making a direct law of sines argument for $\alpha', \beta'$ using the lowest triangle, we obtain

$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{X}{Y} = \frac{\sin(\alpha')}{\sin(\beta')}.$$  

In 2 above we established that $\alpha + \beta = \alpha' + \beta'$. This, together with 3 and Lemma 1 completes the proof.

**Theorem.** The Morley trisector triangle is equilateral.

**Proof.** We have shown that the lowest triangle in Figure 4 is similar to the lowest triangle in the native Morley trisector diagram. Let’s call this triangle an *outer triangle*. With the same ideas we can construct two additional “outer” triangles, emanating from the other two vertices of the abstract equilateral triangle $W$, as in Figure 5.

The three outer triangles “surround” the abstract equilateral triangle $W$ (in a friendly way). Their longest sides form a large triangle with angles $3\alpha, 3\beta, 3\gamma$. These angles are trisected by our dashed lines and the large triangle is similar to our native triangle $\triangle ABC$. The dilatation (zoom) factor that sends our constructed large triangle to the native triangle $\triangle ABC$ also sends the three constructed outer triangles to corresponding triangles in the original Morley picture, and identifies our abstract equilateral triangle $W$ with the native Morley triangle $T$, which, hence, is also equilateral.
We contend that the proof above is trig-light. For the law of sines emanates from basic geometry, as does the monotonicity of the sine function over acute angles. In contrast, see [5] for a (justifiably) trig-heavy proof of an analog of Morley’s theorem. This is not to disparage trig-heavy arguments, which can be surprisingly beautiful [7]. Commentary on mathematical surprise and beauty, in the context of Morley’s theorem, is given in G.-C. Rota’s book [14].

The law of sines, while less ancient than Euclid, is said to be implicit in Ptolemy’s work [6, p. 44] and salient in the work of the 13th century mathematician Mohammed al-Tusi. Interestingly, the MathSciNet review of [6] points to al-Tusi’s result that a pair of angles may be determined from their sum and the ratio of their sines. Our nonlinear Lemma 1 is a nondeterministic uniqueness variant.

Proofs of the Morley trisector theorem abound in the literature. See, for instance [4, pp. 23–25] and [2, 3, 10, 11, 12, 15]. For some time, it was thought to be a difficult theorem to prove. (See, e.g., [2, 3].) Then a plethora of “easy” proofs emerged, and the trend continues.

Yet, this theorem still does not appear in Proofs From the Book [1]. Then again, neither does the far more senior Pythagorean theorem, which has several hundred proofs [9]. Now, the 6th edition of [1] has six “Book” proofs of the infinitude of the primes. Is there a Book proof of the Morley trisector theorem? Perhaps the case of the Pythagorean theorem should be settled first.

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