The Boundedness of some Multilinear operators with rough kernel on the weighted Morrey spaces

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Abstract. In this paper, strong boundedness of $T_{\Omega,\alpha}^A$ and $M_{\Omega,\alpha}^A$, the multilinear fractional integral operators and the corresponding fractional maximal operators, are showed on weighted Morrey spaces with two weights when $D^\gamma A \in \dot{\Lambda}_\beta(|\gamma| = m - 1)$ or $D^\gamma A \in BMO(|\gamma| = m - 1)$. For the multilinear singular integral operators $T_{\Omega}^A$ and the corresponding maximal operators $M_{\Omega}^A$, they are proved to be strong bounded operators on the same spaces if $D^\gamma A \in \dot{\Lambda}_\beta(|\gamma| = m - 1)$; and if $D^\gamma A \in BMO(|\gamma| = m - 1)(m = 1, 2)$, the boundedness of $T_{\Omega}^A$ and $M_{\Omega}^A$ are obtained on weighted Morrey spaces with one weight.

§1 Introduction

Let us consider the following multilinear fractional integral operator:

$$T_{\Omega,\alpha}^A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n - \alpha + m - 1}} R_m(A; x, y) f(y) dy \quad 0 < \alpha < n$$

and the corresponding multilinear fractional maximal operator:

$$M_{\Omega,\alpha}^A f(x) = \sup_{r > 0} \frac{1}{r^{n - \alpha + m - 1}} \int_{|x - y| < r} |\Omega(x - y) R_m(A; x, y) f(y)| dy \quad 0 < \alpha < n$$

where $\Omega \in L^s(S^{n-1})(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$, $A$ is a function defined on $\mathbb{R}^n$ and $R_m(A; x, y)$ denotes the $m$-th order Taylor series remainder of $A$ at $x$ expanded about $y$, that is,

$$R_m(A; x, y) = A(x) - \sum_{|\gamma| < m} \frac{1}{\gamma!} D^\gamma A(y)(x - y)^\gamma$$

$\gamma = (\gamma_1, \cdots, \gamma_n)$, each $\gamma_i (i = 1, \cdots, n)$ is a nonnegative integer, $|\gamma| = \sum_{i=1}^{n} \gamma_i$, $\gamma! = \gamma_1! \cdots \gamma_n!$, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ and $D^\gamma = \sum_{\gamma} \frac{\partial|^\gamma|}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}}$.

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We notice that when $\alpha = 0$, the above operators become the multilinear singular integral operator and the corresponding maximal operator:

$$T^A_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+\alpha}} R_m(A; x, y) f(y) dy$$

$$M^A_{\Omega,\alpha} f(x) = \sup_{r>0} \frac{1}{r^{n+\alpha-1}} \int_{|x-y|<r} |\Omega(x-y) R_m(A; x, y) f(y)| dy$$

For $m = 1$, $T^A_{\Omega,\alpha}$ is obviously the commutator operator, $[A, T^A_{\Omega,\alpha}] f(x) = A(x)T^A_{\Omega,\alpha} f(x) - T^A_{\Omega,\alpha}(Af)(x)$, where $T^A_{\Omega,\alpha}$ is the following fractional integral operator:

$$T^A_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy \quad 0 < \alpha < n$$

The classical Morrey spaces were first introduced by Morrey to study the local behavior of solutions to second order elliptic partial differential equations. From then on, a lot of works concerning Morrey spaces and some related spaces have been done, see [21], [23-28] for details. In 2009, Komori and Shirai [14] considered the weighted Morrey spaces and investigated some concerning Morrey spaces and some related spaces have been done, see [21], [23-28] for details. From the above review, we find that many works concerning the classical singular operators with rough kernels on the weighted Morrey spaces.

In recent years, the multilinear theory have attracted much attentions. In 1967, Bajsanski and Coifman [1] proved the boundedness of the multilinear operator associated with the commutators of singular integrals considered by Calderón. In [2], Cohen studied the $L^p$ boundedness of the operator $T^A_{\Omega,\alpha}$ for $m = 2$. Using the method of good-$\lambda$ inequality, in 1986, Cohen and Gosselin [3] proved that if $\Omega \in Lip_1(S^{n-1})$, satisfies vanishing condition and $D^\gamma A \in BMO(|\gamma| = m - 1)$, then $T^A_{\Omega,\alpha}$ is bounded on $L^p$. In 1994, for $m = 2$, Hofmann [12] proved that $T^A_{\Omega,\alpha}$ is a bounded operator on $L^p(w)$ when $\Omega \in L^\infty(S^{n-1})$ and $w \in A_p$. In 2001, Ding obtained $T^A_{\Omega,\alpha}$ and $M^A_{\Omega,\alpha}$ are both weighted bounded operators from $L^p(w^p)$ to $L^q(w^q)$ with $w \in A(p, q)$ and from $L^p(1 \leq p < \frac{n}{\gamma})$ to $L^\frac{n-\gamma}{n-\gamma}$ with power weight when $D^\gamma A \in L^r(1 < r \leq \infty, |\gamma| = m - 1)$ in [5]. Later, Ding and Lu [8] proved the $(L^p(w^p), L^q(w^q))$ boundedness of $T^A_{\Omega,\alpha}$ and its corresponding maximal operator $M^A_{\Omega,\alpha}$ (the definition of them will be given later). In 2002, Wu and Yang [31] studied if $D^\gamma A \in BMO(|\gamma| = m - 1)$, then $T^A_{\Omega,\alpha}$ is bounded on $L^p$. After that, Lu and Zhang [17] obtained $T^A_{\Omega,\alpha}$ is a bounded operator from $L^p$ to $L^q(1 < p < \frac{n}{\alpha+\beta})$ and from $L^1$ to $L^{-\alpha-\beta, \infty}$ when $D^\gamma A \in \hat{A}_\alpha(|\gamma| = m - 1)$. In the same year, Lu, Wu and Zhang [16] proved $T^A_{\Omega,\alpha}$ is bounded from $L^p$ to $L^q\left(\frac{1}{q} = \frac{1}{p} - \frac{\beta}{\gamma}\right)$ and from $L^1$ to $L^{\frac{n-\beta}{\alpha+\beta}}$ with the bound $C \sum_{|\gamma| = m - 1} \|D^\gamma A\|_{\hat{A}_\beta}$. In [13], for $m = 2$ Jiao showed $T^A_{\Omega,\alpha}$ is bounded on $L^p$ and bounded from $H^1$ to weak $L^1$ and from $L^\infty$ to $BMO$ if $\Omega$ satisfies some conditions. In 2005, Ding [6] obtained the two-weight boundedness of $T^A_{\Omega,\alpha}$ under the condition $D^\gamma A \in L^r(|\gamma| = m - 1)$. In 2008, Han and Lu [11] obtained the $L^{\frac{n}{\gamma}, \infty}$ boundedness of $T^A_{\Omega,\alpha}$ when $D^\gamma A \in BMO(|\gamma| = m - 1)$. Recently, Wu and Tao [32] studied if $D^\gamma A \in BMO(|\gamma| = m - 1)$, then $T^A_{\Omega,\alpha}$ is bounded from $H^1$ to $L^{\frac{n}{\gamma}, \infty}$.

From the above review, we find that many works concerning $T^A_{\Omega,\alpha}$, $M^A_{\Omega,\alpha}$, $T^A_{\Omega}$ and $M^A_{\Omega}$ have
been done on $L^p$ spaces when $D^\gamma A$ belongs to $L^p$, BMO or Lipschitz spaces if $|\gamma| = m - 1$, but there are not any study about these operators on weighted Morrey spaces. So our purpose in this paper is to consider the above operators on weighted Morrey spaces.

The organization of this paper is as follows. We will introduce in next section some definitions and notations that are necessary. The main results will be given in Section 3. In Section 4, we give some requisite lemmas and well-known results that are important in proving theorems. The proof of the theorems will be shown in Section 5.

§2 Definitions and notations

A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight $w$ and a measurable set $E$, we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of $E$ by $|E|$ and the characteristic function of $E$ by $\chi_E$. The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L^p(w)$ with $0 < p < \infty$. We say a weight $w$ satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball $B$, we have $w(2B) \leq Dw(B)$. When $w$ satisfies this condition, we denote $w \in \Delta_2$ for short.

Throughout this paper, $B(x_0, r)$ denotes a ball centered at $x_0$ with radius $r$. $Q$ be a cube with sides parallel to the axes. For $K > 0$, $KQ$ denotes the cube with the same center as $Q$ and side-length being $K$ times longer. When $\alpha = 0$, we will denote $T_{\Omega, \alpha}, T_{\Omega, \alpha}^A, M_{\Omega, \alpha}^A$ by $T_\Omega, T_\Omega^A, M_\Omega^A$ respectively. And for any number $a$, $a'$ is standing for the conjugate of $a$. The letter $C$ is used for various constants, and may changes from one occurrence to another.

To begin with, we introduce the weighted Morrey spaces.

**Definition 2.1.** [14] Let $1 \leq p < \infty$, $0 < \kappa < 1$ and $w$ be a weight. Then a weighted Morrey space is defined by

$$L^{p, \kappa}(w) := \{ f \in L^p_{\text{loc}}(w) : \| f \|_{L^{p, \kappa}(w)} < \infty \}$$

where

$$\| f \|_{L^{p, \kappa}(w)} = \sup_B \left( \frac{1}{w(B)^{\kappa}} \int_B |f(x)|^p w(x)dx \right)^{1/p}$$

and the supremum is taken over all balls $B$ in $\mathbb{R}^n$.

In the case of fractional order, we need to consider a weighted Morrey space with two weights. It is defined as follows:

**Definition 2.2.** [14] Let $1 \leq p < \infty$, $0 < \kappa < 1$, $u, v$ be two weights. The two weights weighted Morrey space is defined by

$$L^{p, \kappa}(u, v) := \{ f : \| f \|_{L^{p, \kappa}(u, v)} < \infty \}$$

where

$$\| f \|_{L^{p, \kappa}(u, v)} = \sup_B \left( \frac{1}{v(B)^{\kappa}} \int_B |f(x)|^p u(x)dx \right)^{1/p}$$

and the supremum is taken over all balls $B$ in $\mathbb{R}^n$. If $u = v$, then we denote $L^{p, \kappa}(u)$ for short.

From Remark 2.2 in [14], we could define the weighted Morrey spaces with cubes instead of balls. So we shall use these two definitions of weighted Morrey spaces appropriate to calculation.
Next, we give the definition of Lipschitz space and $BMO$ space.

**Definition 2.3.** The Lipschitz space of order $\beta$, $0 < \beta < 1$ is defined by

$$\dot{A}_\beta(\mathbb{R}^n) = \{ f : |f(x) - f(y)| \leq C|x - y|^\beta \}$$

and the smallest constant $C > 0$ is the Lipschitz norm $\| \cdot \|_{\dot{A}_\beta}$.

**Definition 2.4.** A locally integrable function $b$ is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \|b\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx < \infty$$

where

$$b_B = \frac{1}{|B|} \int_B b(y) \, dy$$

At last, we shall show the definition of two weight classes.

**Definition 2.5.** A weight function $w$ is in the Muckenhoupt class $A_p$ with $1 < p < \infty$ if there exists $C > 1$ such that for any ball $B$

$$\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C$$

We define $A_\infty = \bigcup_{1 < p < \infty} A_p$.

When $p = 1$, $w \in A_1$ if there exists $C > 1$ such that for almost every $x$,

$$M w(x) \leq C w(x)$$

**Definition 2.6.** [18] A weight function $w$ belongs to $A(p,q)$ for $1 < p < q < \infty$ if there exists $C > 1$ such that

$$\left( \frac{1}{|B|} \int_B w(x)^q \, dx \right)^{\frac{1}{q}} \left( \frac{1}{|B|} \int_B w(x)^{-\frac{q}{p}} \, dx \right)^{\frac{1}{p}} \leq C$$

When $p = 1$, $w$ is in $A(1,q)$ with $1 < q < \infty$ if there exists $C > 1$ such that

$$\left( \frac{1}{|B|} \int_B w(x)^q \, dx \right)^{\frac{1}{q}} \left( \operatorname{ess \ sup}_{x \in B} \frac{1}{w(x)} \right) \leq C$$

**Remark 2.1.** [14] If $w \in A(p,q)$ with $1 < p < q$, then

(a) $w^{\alpha}, w^{-\beta}, w^{-q} \in \Delta_2$.

(b) $w^{-p'} \in A_{p'}$ with $1/t + 1/1' = 1$.

### §3 Main Theorems

Our main results will be stated as follows.

**Theorem 3.1.** If $0 < \alpha + \beta < n$, $\Omega \in L^s(S^{n-1})(s > 1)$ is homogeneous of degree zero, $1 < s' < p < n/(\alpha + \beta), 1/q = 1/p - (\alpha + \beta)/n$, $0 < \kappa < p/q$, $w^\gamma \in A(p/s', q/s')$, $D^\gamma A \in \dot{A}_\beta(|\gamma| = m - 1)$, then

$$\|T_{\Omega,\alpha}^A f\|_{L^{p,q}(\Omega)} \leq C \sum_{|\gamma| = m - 1} \|D^\gamma A\|_{\dot{A}_\beta} \|f\|_{L^{p,q}(w^\gamma)}$$

$$\|M_{\Omega,\alpha}^A f\|_{L^{p,q}(\Omega)} \leq C \sum_{|\gamma| = m - 1} \|D^\gamma A\|_{\dot{A}_\beta} \|f\|_{L^{p,q}(w^\gamma)}$$

(1)
Theorem 3.2. If $0 < \beta < 1$, $\Omega \in L^s(\mathbb{S}^{n-1})(s > 1)$ is homogeneous of degree zero, $1 < s' < p < n/\beta$, $1/q = 1/p - \beta/n$, $0 < \kappa < p/q$, $w \in A(p/s', q/s')$, $D^\alpha A \in \dot{A}_\beta(\gamma = m - 1)$, then
\[
\|T^{A}_{\Omega} f\|_{L^{q/q}(w)} \leq C \sum_{|\gamma| = m-1} \|D^\alpha A\|_{\dot{A}_\beta} \|f\|_{L^{p/(w,q)}}
\]
(3)
\[
\|M^{A}_{\Omega} f\|_{L^{q/q}(w)} \leq C \sum_{|\gamma| = m-1} \|D^\alpha A\|_{\dot{A}_\beta} \|f\|_{L^{p/(w,q)}}
\]
(4)

Theorem 3.3. If $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})(s > 1)$ is homogeneous of degree zero, $1 < s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, $w \in A(p/s', q/s')$, $D^\gamma A \in BMO(\gamma = m - 1)$, then
\[
\|T^{A}_{\Omega,\alpha} f\|_{L^{q/q}(w)} \leq C \sum_{|\gamma| = m-1} \|D^\alpha A\|_{\dot{A}_\beta} \|f\|_{L^{p/(w,q)}}
\]
(5)
\[
\|M^{A}_{\Omega,\alpha} f\|_{L^{q/q}(w)} \leq C \sum_{|\gamma| = m-1} \|D^\alpha A\|_{\dot{A}_\beta} \|f\|_{L^{p/(w,q)}}
\]
(6)

For $T^{A}_{\Omega}$ and $M^{A}_{\Omega}$, we only study the cases when $m = 1$ and $m = 2$. In these two cases, we denote $T^{A}_{\Omega}$, $M^{A}_{\Omega}$ by $[A, T_{\Omega}]$, $[A, M_{\Omega}]$ and $\tilde{T}^{A}_{\Omega}$, $\tilde{M}^{A}_{\Omega}$ respectively in order to distinguish from $T^{A}_{\Omega}$ and $M^{A}_{\Omega}$ that are suitable for any integer $m$. That is,
\[
[A, T_{\Omega}]f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A(x) - A(y)) f(y)dy
\]
\[
[A, M_{\Omega}]f(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|x-y| < r} \Omega(x-y) (A(x) - A(y)) f(y)dy
\]
and
\[
\tilde{T}^{A}_{\Omega} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} (A(x) - A(y) - \nabla A(y)(x-y)) f(y)dy
\]
\[
\tilde{M}^{A}_{\Omega} f(x) = \sup_{r > 0} \frac{1}{r^{n+1}} \int_{|x-y| < r} \Omega(x-y) (A(x) - A(y) - \nabla A(y)(x-y)) f(y)dy
\]
Then for the above operators, we have the following results on weighted Morrey spaces with one weight.

Theorem 3.4. If $\Omega \in L^s(\mathbb{S}^{n-1})(s > 1)$ is homogeneous of degree zero and satisfies the vanishing condition $\int_{\mathbb{S}^{n-1}} \Omega(x') d\sigma(x') = 0$, $1 < s' < p < \infty$, $0 < \kappa < 1$, $w \in A_{p/s'}$, $A \in BMO$, then
\[
\|[A, T_{\Omega}] f\|_{L^{p/(w,q)}} \leq C \|A\|_\ast \|f\|_{L^{p/(w,q)}}
\]
(7)
\[
\|[A, M_{\Omega}] f\|_{L^{p/(w,q)}} \leq C \|A\|_\ast \|f\|_{L^{p/(w,q)}}
\]
(8)

Theorem 3.5. If $\Omega \in L^\infty(\mathbb{S}^{n-1})$ is homogeneous of degree zero and satisfies the moment condition $\int_{\mathbb{S}^{n-1}} \theta \Omega(\theta) d\theta = 0$, $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_{p}$, $\nabla A \in BMO$, then
\[
\|\tilde{T}^{A}_{\Omega} f\|_{L^{p/(w,q)}} \leq C \|\nabla A\|_\ast \|f\|_{L^{p/(w,q)}}
\]
(9)
\[
\|\tilde{M}^{A}_{\Omega} f\|_{L^{p/(w,q)}} \leq C \|\nabla A\|_\ast \|f\|_{L^{p/(w,q)}}
\]
(10)

Here we point out that for $T^{A}_{\Omega}$ and $M^{A}_{\Omega}$, when $D^\gamma A \in BMO(\gamma = m - 1)(m \geq 3)$, the analogous conclusions of Theorem 3.5 is open.
Lemma 4.2. For one of the following conditions.

\[
\int_{R^n} \prod_{i=1}^{k} R_{m_i}(A_i; x, y) \frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}} f(y) dy
\]

\[
M_{\Omega,\alpha}^{A_1,\cdots,A_k} f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha+N}} \int_{|x-y|<r} |\Omega(x-y)| \prod_{i=1}^{k} |R_{m_i}(A_i; x, y)||f(y)| dy
\]

when \(0 < \alpha < n\), it becomes a class of multilinear fractional integral operators. When \(\alpha = 0\), it is a class of multilinear singular integral operators. \(R_{m_i}(A_i; x, y) = A_i(x) - \sum_{|\gamma|<m_i} \frac{1}{i!} D^\gamma A_i(y)(x-y)^\gamma\) \((i = 1, \cdots, k)\), \(N = \sum_{i=1}^{k} (m_i-1)\). Repeating the proofs of theorems above, we will find that, for \(T_{\Omega,\alpha}^{A_1,\cdots,A_k}\) and \(M_{\Omega,\alpha}^{A_1,\cdots,A_k}\) the conclusions of Theorem 3.1 and Theorem 3.2 above with the bounds \(C \prod_{i=1}^{k} (\sum_{|\gamma|=m_i-1} \|D^\gamma A_i\|)\) and Theorem 3.3 with the bounds \(C \prod_{i=1}^{k} (\sum_{|\gamma|=m_i-1} \|D^\gamma A_i\|)\) also hold, respectively.

§4 Lemmas and well-known results

Lemma 4.1. [3] Let \(A(x)\) be a function on \(R^n\) with \(m\)-th order derivatives in \(L^1_{loc}(R^n)\) for some \(l > n\). Then

\[
|R_{m}(A; x, y)| \leq C|x-y|^m \sum_{|\gamma|=m} \left( \frac{1}{|I|^n} \int |D^\gamma A(z)|^n dz \right)^{1/n}
\]

where \(I^m\) is the cube centered at \(x\) with sides parallel to the axes, whose diameter is \(5\sqrt{n}|x-y|\).

Lemma 4.2. [20] For \(0 < \beta < 1, 1 \leq q < \infty\), we have

\[
\|f\|_{\lambda_\beta} \approx \sup_{Q} \frac{1}{|Q|^{1+\beta/n}} \int_{Q} |f(x) - m_Q(f)| dx \approx \sup_{Q} \frac{1}{|Q|^{1+\beta/n}} \left( \frac{1}{|Q|} \int_{Q} |f(x) - m_Q(f)|^q dx \right)^{1/q}
\]

Here \(m_Q(f)\) denotes the average of \(f\) over \(Q\).

For \(q = \infty\), the formula should be interpreted appropriately.

Lemma 4.3. [9] Let \(Q_1 \subset Q_2, g \in \hat{L}_{\beta}(0 < \beta < 1)\). Then

\[
|m_{Q_1}(g) - m_{Q_2}(g)| \leq C|Q_2|^{\beta/n} \|g\|_{\hat{L}_{\beta}}
\]

Theorem 4.1. [7] Suppose that \(0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n\) and \(\Omega \in L^s(S^{n-1})(s > 1)\). Then \(T_{\Omega,\alpha}\) is a bound operator from \(L^p(w^p)\) to \(L^q(w^q)\), if \(s, p, q, \) and \(w\) satisfy one of the following conditions.

\(a) 1 \leq s' < p \) and \(w(x)^{s'} \in A(p/s', q'/s')\)
\(b) s > q \) and \(w(x)^{-s'} \in A(q'/s', p'/s')\)
\(c) s > 1, \alpha/n + 1/s < 1/p < 1/s' \) and there is an \(r\) s.t. \(1 < r < s/(n/\alpha)'\) and \(w(x)^r \in A(p,q)\).

Lemma 4.4. [14] If \(w \in \Delta_2\), then there exists a constant \(D_1 > 1\), s.t.

\[
w(2B) \geq D_1 w(B).
\]

We call \(D_1\) the reverse doubling constant.
Lemma 4.9. Let \( \Omega \in L^s(S^{n-1})(s > 1) \) be homogeneous of degree zero with \( \Omega \in L^s(S^{n-1})(s > 1) \) for \( 1 \leq j \leq k \), \( |j| = m_j - 1 \), \( m_j \geq 2 \), and \( D^\alpha A_j \in BMO(\mathbb{R}^n) \). If the index set \( \{\alpha, p, q, s\} \) satisfies one of the following conditions:

(a) \( s' < p \), \( w(x)^{s'} \in A(p/s', q/s') \);
(b) \( s > q \), \( w(x)^{-s'} \in A(q'/s', p'/s') \);
(c) \( \alpha/n + 1/s < 1/p < 1/s' \), there is an \( r \), \( 1 < r < s/(n/\alpha)' \) such that \( w(x)^{r'} \in A(p, q) \);

then there is a \( C > 0 \), independent of \( f \) and \( A_j \), such that

\[
\left( \int_{\mathbb{R}^n} |T_{\Omega, \alpha}^{A_1, \ldots, A_k} f(x)w(x)|^q dx \right)^{1/q} \leq C \prod_{j=1}^k \left( \sum_{|\gamma|=m_j-1} \|D^\alpha A_j\|_\ast \right) \left( \int_{\mathbb{R}^n} |f(x)w(x)|^p dx \right)^{1/p}
\]

Lemma 4.5. [4] (John-Nirenberg Lemma) Let \( 1 \leq p < \infty \). Then \( b \in BMO \) if and only if

\[
\frac{1}{|Q|} \int_Q |b - b_Q|^p dx \leq C\|b\|_p^p
\]

Lemma 4.6. [22] Assume \( b \in BMO \), then for cube \( Q_1 \subset Q_2 \),

\[
|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|)\|b\|_p
\]

From the above Lemma, we immediately get the following conclusion:

Lemma 4.7. [29] If \( b \in BMO \), then

\[
|b_{2^{j+1}B} - b_B| \leq 2^n(j + 1)\|b\|_p
\]

Theorem 4.3. [15] Suppose that \( \Omega \in L^s(S^{n-1})(s > 1) \) is homogeneous of degree zero and satisfies the vanishing condition \( \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0 \). If \( b \in BMO(\mathbb{R}^n) \) and \( p, q, w \) satisfy one of the following conditions, then \([b, T_{\Omega}]\) is bounded on \( L^p(w) \): (a) \( s' \leq p < \infty \), \( p \neq 1 \) and \( w \in A_{p/s'} \);
(b) \( 1 \leq p \leq s \), \( p \neq \infty \) and \( w^{1-p'} \in A_{p'/s'} \);
(c) \( 1 < p \leq \infty \) and \( w^{1/2-s} \in A_p \).

Theorem 4.4. [12] If \( \Omega \in L^\infty(S^{n-1}) \) is homogeneous of degree zero and satisfies the moment condition \( \int_{S^{n-1}} \theta \Omega(\theta) d\theta = 0 \), \( w \in A_p \), \( 1 < p < \infty \), \( \nabla A \in BMO \), then we have

\[
\|T_{\Omega} A f\|_{L^p(w)} \leq C\|\Omega\|_\infty \|\nabla A\|_\ast\|f\|_{L^p(w)}
\]

Lemma 4.8. [10] The following are true:

1. If \( w \in A_p \) for some \( 1 \leq p < \infty \), then \( w \in \Delta_2 \). More precisely, for all \( \lambda > 1 \) we have

\[
w(\lambda Q) \leq C\lambda^{-p} w(Q)
\]

2. If \( w \in A_p \) for some \( 1 \leq p < \infty \), then there exist \( C > 0 \) and \( \delta > 0 \) such that for any cube \( Q \) and a measurable set \( S \subset Q \),

\[
\frac{w(S)}{w(Q)} \leq C \left( \frac{|S|}{|Q|} \right)^\delta
\]

Lemma 4.9. [19] Let \( w \in A_\infty \). Then the norm of \( BMO(w) \) is equivalent to the norm of
BMO($\mathbb{R}^n$), where
\[
BMO(w) = \left\{ b : \|b\|_{*,w} = \sup_Q \frac{1}{w(Q)} \int_Q |b(x) - m_{Q,w}b| w(x) dx \right\}
\]
and
\[
m_{Q,w}b = \frac{1}{w(Q)} \int_Q b(x) w(x) dx
\]

§5 Proofs of the Main Results

Proof of Theorem 3.1
To prove (1), we give a pointwise estimate of $T_{\Omega,\alpha}^A f(x)$ at first. Set
\[
T_{\Omega,\alpha+\beta} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha-\beta}} |f(y)| dy \quad 0 < \alpha + \beta < n
\]
where $\Omega \in L^s(S^{n-1})(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$. Then we have the following theorem:

Theorem 5.1. If $\alpha \geq 0$, $0 < \alpha + \beta < n$, $D^\gamma A \in \dot{A}_\beta(|\gamma| = m-1)$, then there exists a constant $C$ independent of $f$ such that
\[
|T_{\Omega,\alpha}^A f(x)| \leq C \left( \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{A}_\beta} \right) T_{\Omega,\alpha+\beta} f(x)
\]

Proof. For fixed $x \in \mathbb{R}^n$, $r > 0$, let $Q$ be a cube centered at $x$ and has diameter $r$, $Q_k = 2^k Q$ and set
\[
A_{Q_k}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{Q_k}(D^\gamma A)y^\gamma
\]
where $m_{Q_k}f$ is the average of $f$ on $Q_k$. Then we have when $|\gamma| = m-1$
\[
D^\gamma A_{Q_k}(y) = D^\gamma A(y) - m_{Q_k}(D^\gamma A)
\]
and from [3], we have
\[
R_m(A;x,y) = R_m(A_{Q_k};x,y)
\]
Then,
\[
|T_{\Omega,\alpha}^A f(x)| \leq \sum_{k=-\infty}^{\infty} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|R_m(A_{Q_k};x,y)| \|\Omega(x-y)\|_{\dot{A}_\beta} |f(y)| dy : = \sum_{k=-\infty}^{\infty} T_k
\]
By Lemma 4.1 we get
\[
|R_m(A_{Q_k};x,y)| \leq |R_{m-1}(A_{Q_k};x,y)| + C \sum_{|\gamma|=m-1} |D^\gamma A_{Q_k}(y)||x-y|^{m-1}
\]
\[
\leq C|x-y|^{m-1} \sum_{|\gamma|=m-1} \left( \frac{1}{|I_x^y|} \int_{I_x^y} |D^\gamma A_{Q_k}(z)| dz \right)^\frac{1}{\gamma} + C|x-y|^{m-1} \sum_{|\gamma|=m-1} |D^\gamma A_{Q_k}(y)|
\]
Note that, if $|x-y| < 2^k r$, then $I_x^y \subset 5Q_k$. By Lemma 4.2 and Lemma 4.3 we have when
|γ| = m - 1,
\[
\left(\frac{1}{|I|} \int_{I} |D^\gamma A_{Q_k}(z)|^l dz\right)^{\frac{1}{l}} = \left(\frac{1}{|I|} \int_{I} |D^\gamma A(z) - m_{Q_k}(D^\gamma A)|^l dz\right)^{\frac{1}{l}} \leq \left(\frac{1}{|I|} \int_{I} |D^\gamma A(z) - m_{Q_k}(D^\gamma A)|^l dz\right)^{\frac{1}{l}} + |m_{Q_k}(D^\gamma A) - m_{Q_k}(D^\gamma A)|
\]
\[
\leq C|Q_k|^\frac{2}{q} \|D^\gamma A\|_{\Lambda, \beta} \leq C(2^k)^\beta \|D^\gamma A\|_{\Lambda, \beta}
\]

From Definition 3.3, we obtain when |γ| = m - 1,
\[
|D^\gamma A_{Q_k}(y)| = |D^\gamma A(y) - m_{Q_k}(D^\gamma A)| \leq C|Q_k|^\frac{2}{q} \|D^\gamma A\|_{\Lambda, \beta} \leq C(2^k)^\beta \|D^\gamma A\|_{\Lambda, \beta}
\]

Thus,
\[
|R_m(A_{Q_k}; x, y)| \leq C|x - y|^{m-1}(2^k)^\beta \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\Lambda, \beta}
\]

Therefore,
\[
T_k \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\Lambda, \beta} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{(2^k r)^\beta}{|x-y|^{n-\alpha}} \Omega(x-y)|f(y)|dy
\]
\[
\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\Lambda, \beta} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)|dy
\]

It follows that
\[
|T_{\Omega, \alpha}^A f(x)| \leq \sum_{k=-\infty}^{\infty} \left( C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\Lambda, \beta} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)|dy\right)
\]
\[
\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\Lambda, \beta} \sum_{k=-\infty}^{\infty} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)|dy
\]
\[
= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\Lambda, \beta} \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha-\beta}} |f(y)|dy
\]

Thus we finish the proof of theorem 5.1.

The following theorem is a key theorem in proving (1).

**Theorem 5.2.** Under the same conditions of Theorem 3.1, \(\tilde{T}_{\Omega, \alpha+\beta}\) is bounded from \(L^{p, \kappa}(w^p, w^q)\) to \(L^{q, \kappa/q, r}(w^q)\).

**Proof** Fix a ball \(B(x_0, r_B)\) and We decompose \(f = f_1 + f_2\) with \(f_1 = f\chi_{2B}\). Since \(\tilde{T}_{\Omega, \alpha}\) is a linear operator, we have
\[
\|\tilde{T}_{\Omega, \alpha+\beta} f\|_{L^{q, \kappa/q, r}(w^q)} = \left(\frac{1}{w^q(B)^\kappa q/p} \int_B |\tilde{T}_{\Omega, \alpha+\beta} f(x)|^q w^q(x)dx\right)^{\frac{1}{q}}
\]
\[
\leq \frac{1}{w^q(B)^\kappa q/p} \left(\int_B |\tilde{T}_{\Omega, \alpha+\beta} f_1(x)|^q w^q(x)dx\right)^{\frac{1}{q}} + \frac{1}{w^q(B)^\kappa q/p} \left(\int_B |\tilde{T}_{\Omega, \alpha+\beta} f_2(x)|^q w^q(x)dx\right)^{\frac{1}{q}}
\]
\[
= J_1 + J_2
\]
We estimate $J_1$ at first. From Remark 2.1 (a) we know that $w^q \in \Delta_2$, then by Theorem 4.1 (a) we get,

$$
J_1 \leq \frac{1}{w^q(B)^{\kappa/p}} \|T_{\Omega, \alpha, \beta} f_1\|_{L^q(w^q)}
$$

$$
\leq \frac{C}{w^q(B)^{\kappa/p}} \|f_1\|_{L^p(w^q)} = \frac{C}{w^q(B)^{\kappa/p}} \left( \int_{2B} |f(x)|^p w(x)^p \, dx \right)^{1/p}
$$

$$
\leq C \|f\|_{L^p(w^p, w^q)} w^q(2B)^{\kappa/p} \leq C \|f\|_{L^p(w^p, w^q)}
$$

Now we consider the term $J_2$.

$$
|T_{\Omega, \alpha, \beta} f_2(x)| = \int_{(2B)^c} \frac{|\Omega(x - y)|}{|x - y|^{n-\alpha-\beta}} |f(y)| \, dy = \sum_{j=1}^\infty \int_{2^{j+1}B \setminus 2^j B} \frac{|\Omega(x - y)|}{|x - y|^{n-\alpha-\beta}} |f(y)| \, dy
$$

$$
\leq C \sum_{j=1}^\infty \left( \int_{2^{j+1}B} |\Omega(z')| \, dz' \right)^{\frac{1}{p'}} \left( \int_{2^{j+1}B \setminus 2^j B} \frac{|f(y)|^{qs'}}{|x - y|^{(n-\alpha-\beta)qs'}} \, dy \right)^{\frac{1}{q'}} = C \sum_{j=1}^\infty (I_{1j} I_{2j})
$$

We will estimate $I_{1j}$, $I_{2j}$ respectively. Let $z = x - y$, then for $x \in B$, $y \in 2^{j+1}B$, we have $z \in 2^{j+2}B$. Noticing that $\Omega$ is homogeneous of degree zero and $\Omega \in L^s(S^{n-1})$, we obtain

$$
I_{1j} = \left( \int_{2^{j+2}B} |\Omega(z')| \, dz' \right)^{\frac{1}{p'}} = \left( \int_0^{2^{j+2}B} \int_{S^{n-1}} |\Omega(z')| \, dz' \, r^{n-1} \, dr \right)^{\frac{1}{p'}}
$$

$$
= C \|\Omega\|_{L^s(S^{n-1})} |2^{j+2}B|^{1/p'}
$$

where $z' = z/|z|$. For $x \in B$, $y \in (2B)^c$, $|x - y| \sim |x_0 - y|$, thus

$$
I_{2j} \leq C \left( \frac{1}{|2^{j+1}B|^{1-\frac{n}{ps}}} \int_{2^{j+1}B} |f(y)|^{qs'} \, dy \right)^{\frac{1}{q'}}
$$

By Hölder inequality and $w^{qs'} \in A(p/s', q/s')$, we get

$$
\left( \int_{2^{j+1}B} |f(y)|^{qs'} \, dy \right)^{\frac{1}{q'}} \leq C \left( \int_{2^{j+1}B} |f(y)|^p w(y)^p \, dy \right)^{\frac{1}{p'}} \left( \int_{2^{j+1}B} w(y)^{-\frac{p}{q'}s'} \, dy \right)^{\frac{1}{p'}}
$$

$$
\leq C \|f\|^p_{L^p(w^p, w^q)} w^q(2^{j+1}B)^{\frac{p}{q'}} \left( \int_{2^{j+1}B} w(y)^{-\frac{p}{q'}s'} \, dy \right)^{\frac{1}{p'}}
$$

$$
\leq C \|f\|^p_{L^p(w^p, w^q)} \frac{w^q(2^{j+1}B)^{\frac{p}{q'}}}{w^q(2^{j+1}B)^{\frac{p}{q'}}}
$$

Thus

$$
|T_{\Omega, \alpha, \beta} f_2(x)| \leq C \sum_{j=1}^\infty (I_{1j} I_{2j}) \leq C \sum_{j=1}^\infty \|f\|^p_{L^p(w^p, w^q)} \frac{1}{w^q(2^{j+1}B)^{\frac{p}{q'}} - \frac{p}{q'}}
$$

So we get

$$
J_2 \leq C \|f\|^p_{L^p(w^p, w^q)} \sum_{j=1}^\infty \frac{w^q(B)^{\frac{p}{q'}} - \frac{p}{q'}}{w^q(2^{j+1}B)^{\frac{p}{q'}} - \frac{p}{q'}}
$$

Using Remark 2.1 (a) and Lemma 4.4, we get that $w$ satisfies inequality (11), so the above series converges since the reverse doubling constant is larger than one, as a result,

$$
J_2 \leq C \|f\|^p_{L^p(w^p, w^q)}
$$

Therefore, we have showed the proof of Theorem 5.2.

Here we remark that Theorem 5.2 is essentially verifying the multilinear fractional operator
$T_{\Omega, \alpha}$ is bounded on weighted Morrey spaces.

Now let us turn to prove inequality (11). By Theorem 5.1 and Theorem 5.2, it immediately obtained.

We are now in the place of proving (2) in Theorem 3.1. Set

$$\bar{T}_{\Omega, \alpha} f(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)|dy \quad 0 \leq \alpha < n$$

where $\Omega \in L^s(S^{n-1})(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$. It is easy to see that, for $\bar{T}_{\Omega, \alpha}$, the conclusions of inequality (11) also hold. On the other hand, for any $r > 0$, we have

$$\bar{T}_{\Omega, \alpha} f(x) \geq \int_{|x-y| < r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)| |f(y)|dy$$

Taking the supremum for $r > 0$ on the inequality above, we get

$$\bar{T}_{\Omega, \alpha} f(x) \geq M_{\Omega, \alpha}^A f(x) \quad (15)$$

Thus, we can immediately obtain (2) from (15) and (1).

Before starting proving Theorem 3.2, we give the following theorem at first since this theorem plays an important role in proving Theorem 5.2. Set

$$\bar{T}_{\Omega, \beta} f(x) = \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} f(y)dy$$

where $\Omega \in L^s(S^{n-1})(s > 1)$ is homogeneous of degree zero in $\mathbb{R}^n$.

**Theorem 5.3.** Under the assumptions of Theorem 5.2, $\bar{T}_{\Omega, \beta}$ is bounded from $L^{p, \kappa}(w^p, w^q)$ to $L^{q, \kappa q/p}(w^q)$.

We shall omit the proof for it is the same as that of Theorem 5.2.

Now, let us prove Theorem 3.2. It is not difficult to see that inequality (5) of Theorem 3.2 can be easily obtained from Theorem 5.1 and Theorem 5.3. At the same time, we can immediately arrive at (4) from (15) and (3).

From now on, we are in the place of showing Theorem 3.3. We study (5) at first. Fixing any cube $Q$ whose center is $x$ and diameter is $r$, denote $\bar{Q} = 2Q$ and set

$$A_Q(y) = A(y) - \sum_{|\gamma| = m-1} \frac{1}{|\gamma|!} m_{\bar{Q}}(D^\gamma A)y^\gamma$$

we notice that the above equality is the special case of equality (12) when $k = 1$. Thus the equality (12) and (13) also hold for $A_Q(y)$. We decompose $f$ according to $Q$, that is $f = f_1 + f_2$. Then we have

$$\|T_{\Omega, \alpha}^A f\|_{L^{q, \kappa q/p}(w^q)} \leq \frac{1}{w^q(Q)^{\kappa/p}} \left( \int_Q |T_{\Omega, \alpha}^A f_1(y)|^q w(y)^q dy \right)^{\frac{1}{q}} + \frac{1}{w^q(Q)^{\kappa/p}} \left( \int_Q |T_{\Omega, \alpha}^A f_2(y)|^q w(y)^q dy \right)^{\frac{1}{q}} = I + II$$
From Remark 2.1 (a), we know that \( w^q \in \Delta_2 \), then togethering with Theorem 4.2 (a) we have
\[
I \leq \frac{C}{w^q(Q)^{\kappa/p}} \sum_{|\gamma|=m-1} \|D^\gamma A\|_\sigma \left( \int_Q |f(y)|^p w(y)^p \, dy \right)^{\frac{\kappa}{p}} \\
= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_\sigma \|f\|_{L^p(w^q,w^q)} \left( \frac{w^q(Q)}{w^q(Q)} \right)^{\kappa/p} \\
\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_\sigma \|f\|_{L^p(w^q,w^q)}
\]

Next, we consider the term \( T^{A}_{\Omega,\alpha}f_2(y) \) contained in II. By Lemma 4.1 and equality (13), (14), we have
\[
|T^{A}_{\Omega,\alpha}f_2(y)| \leq \int_{(Q)^c} \frac{|R_m(A_Q,y,z)|}{|y-z|^{m-1}} |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} \, dz \\
\leq C \int_{(Q)^c} \sum_{|\gamma|=m-1} \left( \frac{1}{|I_y|} \int_{I_y} |D^\gamma A_Q(t)| \, dt \right)^{\frac{\kappa}{p}} |\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} \, dz \\
+ C \int_{(Q)^c} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_Q(D^\gamma A)||\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} \, dz = II_1 + II_2
\]

We estimate II_1 and II_2 respectively. By Lemma 4.3 and Lemma 4.6 and then use the similar steps as the proof of Theorem 5.2, we get
\[
II_1 \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_\sigma \|f\|_{L^p(w^q,w^q)} \sum_{j=1}^{\infty} \frac{1}{w^q(2^{j+1}Q)^{\kappa/p}}
\]

For \( y \in Q, z \in (Q)^c \), we have \( |y-z| \sim |x-z| \), so we obtain
\[
II_2 \leq C \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)||\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} \, dz \\
+ C \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \sum_{|\gamma|=m-1} |m_{2^{j+1}Q}(D^\gamma A) - m_Q(D^\gamma A)||\Omega(y-z)| \frac{|f(z)|}{|y-z|^{n-\alpha}} \, dz \\
= II_{21} + II_{22}
\]

By Hölder inequality, we get
\[
II_{21} \leq C \sum_{j=1}^{\infty} \frac{1}{|2^j Q|^{1-\alpha/n}} \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)||\Omega(y-z)||f(z)| \, dz \\
\leq C \sum_{j=1}^{\infty} \frac{1}{|2^j Q|^{1-\alpha/n}} \left( \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{\kappa} |f(z)|^{\kappa} \, dz \right)^{\frac{1}{\kappa}} \\
\left( \int_{2^{j+1}Q} |\Omega(y-z)|^{\kappa} \, dz \right)^{\frac{1}{\kappa}} \\
\leq C \|\Omega\|_{L^\infty(S^{n-1})} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^{\kappa}}{|2^j Q|^{1-\alpha/n}} \left( \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^\gamma A(z) - m_{2^{j+1}Q}(D^\gamma A)|^{\kappa} |f(z)|^{\kappa} \, dz \right)^{\frac{1}{\kappa}}
\]
We can calculate the part of including the function $D^\gamma A$ as follows:

$$\left( \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^{\gamma}A(z) - m_{2^{j+1}Q}(D^{\gamma}A)| \frac{w(z)^{p}}{\gamma^{p}} d\gamma \right)^{\frac{1}{p}} \leq C \left( \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^{\gamma}A(z) - m_{2^{j+1}Q}(D^{\gamma}A)| \frac{w(z)^{p}}{\gamma^{p}} d\gamma \right)^{\frac{1}{p}}$$

$$\leq C \left( \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^{\gamma}A(z) - m_{2^{j+1}Q}(D^{\gamma}A)| \frac{w(z)^{p}}{\gamma^{p}} d\gamma \right)^{\frac{1}{p}}$$

$$+ \sum_{|\gamma|=m-1} m_{2^{j+1}Q; w^{-\frac{w^p}{p-s^p}}}(D^{\gamma}A) - m_{2^{j+1}Q}(D^{\gamma}A) \frac{w^{-\frac{w^p}{p-s^p}}(2^{j+1}Q)}{w^{4}(2^{j+1}Q)^{\frac{1}{p}}} = III + IV$$

For the term $III$, as $w^{s'} \in A(p/s', q/s')$, using Remark 2.4 (b), we obtain that $w^{-\frac{w^p}{p-s^p}} \in A_{s'} \subset A_{\infty}$ ($1/t + 1/t' = 1$), then by Lemma 4.9 that the norm of $BMO(w^{-\frac{w^p}{p-s^p}})$ is equivalent to the norm of $BMO(\mathbb{R}^n)$ and $w^{s'} \in A(p/s', q/s')$ condition, we obtain

$$III \leq C \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{s'} w^{-\frac{w^p}{p-s^p}}(2^{j+1}Q)^{\frac{1}{p}}$$

$$= C \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{s'} \frac{|2^{j+1}Q|^{\frac{1}{p}}}{w^{q}(2^{j+1}Q)^{\frac{1}{p}}}$$

For the term $IV$, from the John and Nirenberg lemma, we have that there exist $C_1 > 0$ and $C_2 > 0$ such that for any cube $Q$ and $s > 0$

$$\left| \left\{ t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} |D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)| > s \right\} \right| \leq C_1 |2^{j+1}Q| e^{-C_2 s/(\sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{s})}$$

since $\sum_{|\gamma|=m-1} (D^{\gamma}A) \in BMO$. Then from Lemma 2.2 (2), we have

$$w\left( \left\{ t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} |D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)| > s \right\} \right) \leq Cw(2^{j+1}Q) e^{-C_2 \delta s/(\sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{s})}$$

for some $\delta > 0$. Hence the inequality implies

$$\sum_{|\gamma|=m-1} m_{2^{j+1}Q; w^{-\frac{w^p}{p-s^p}}}(D^{\gamma}A) - m_{2^{j+1}Q}(D^{\gamma}A)$$

$$\leq \frac{1}{w^{-\frac{w^p}{p-s^p}}(2^{j+1}Q)} \int_{2^{j+1}Q} \sum_{|\gamma|=m-1} |D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)| w^{-\frac{w^p}{p-s^p}}(t) dt$$

$$= C \frac{1}{w^{-\frac{w^p}{p-s^p}}(2^{j+1}Q)} \int_{0}^{\infty} w^{-\frac{w^p}{p-s^p}}(t \in 2^{j+1}Q : \sum_{|\gamma|=m-1} |D^{\gamma}A(t) - m_{2^{j+1}Q}(D^{\gamma}A)| > s) ds$$
The proof of Theorem 3.4

It is not difficult to see that inequality (6) is easy to get from (5) and (15).

Sides of the above inequality, we complete the proof of inequality (5) of Theorem 3.3.

Thus,

\[ IV \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \int_0^\infty w^{\frac{\mu r}{p-r}}(2^{j+1}Q)^{\frac{\mu r}{p-r}} \, ds = C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \]

So,

\[ IV \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \frac{|2^{j+1}Q|^{\frac{\mu r}{p-r}}}{w^n(2^{j+1}Q)^{\frac{n}{p}}} \]

Thus,

\[ II_{21} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,n}(w^n,w)} \sum_{j=1}^{\infty} \frac{1}{w^n(2^{j+1}Q)^{\frac{n}{p}}} \]

For the term \( II_{22} \), using Lemma 4.7 and the analogous steps as the proof of Theorem 5.2 we get

\[ II_{22} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,n}(w^n,w)} \sum_{j=1}^{\infty} \frac{j}{w^n(2^{j+1}Q)^{\frac{n}{p}}} \]

Therefore,

\[ |T_{\Omega,0}^A f_2(y)| \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,n}(w^n,w)} \sum_{j=1}^{\infty} \frac{j}{w^n(2^{j+1}Q)^{\frac{n}{p}}} \]

Consequently,

\[ II \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,n}(w^n,w)} \sum_{j=1}^{\infty} \frac{j}{w^n(2^{j+1}Q)^{\frac{n}{p}}} \]

\[ \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,n}(w^n,w)} \sum_{j=1}^{\infty} \frac{j}{(D_1^{j+1})^{\frac{n}{p}}} \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \|f\|_{L^{p,n}(w^n,w)} \]

where \( D_1 > 1 \) is the reverse doubling constant. Taking supremum over all cubes in \( \mathbb{R}^n \) on both sides of the above inequality, we complete the proof of inequality (5) of Theorem 3.3.

It is not difficult to see that inequality (6) is easy to get from (5) and (10).

The proof of Theorem 3.4

We consider (7) firstly. Let \( Q \) be the same as in the proof of (5) and \( \bar{Q} = 2Q \), we decompose \( f \) according to \( Q \): \( f = f \chi_{\bar{Q}} + f \chi_{(Q)^c} := f_1 + f_2 \). Thus we have

\[ \| [A,T_\Omega] f \|_{L^p(w)} \leq \frac{1}{w(Q)^{\kappa/p}} \left( \int_Q | [A,T_\Omega] f_1(y)|^p w(y)dy \right)^{\frac{1}{p}} + \frac{1}{w(Q)^{\kappa/p}} \left( \int_Q | [A,T_\Omega] f_2(y)|^p w(y)dy \right)^{\frac{1}{p}} = I + II \]

From Theorem 4.3 (a), the \( L^p(w) \) boundedness of \( [A,T_\Omega] \) and Lemma 4.3 (1) that \( w \in \Delta_2 \), we get

\[ I \leq \frac{1}{w(Q)^{\kappa/p}} \| [A,T_\Omega] f_1 \|_{L^p(w)} \leq \frac{C}{w(Q)^{\kappa/p}} \| A \|_* \| f_1 \|_{L^p(w)} \]

So,

\[ IV \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_* \frac{|2^{j+1}Q|^{\frac{\mu r}{p-r}}}{w^n(2^{j+1}Q)^{\frac{n}{p}}} \]
$$= C\|A\|_s \|f\|_{L^p,\kappa(w)} \frac{w(Q)^{s/p}}{w(Q)^{s/p}} \leq C\|A\|_s \|f\|_{L^p,\kappa(w)}$$

For $\|[A, T_\beta]f_2(y)\|$, using Hölder inequality, we obtain

$$\|[A, T_\beta]f_2(y)\| \leq \sum_{j=1}^\infty \int_{2^{j+1}Q} |\Omega(y-z)|^{s} dz (\int_{2^{j+1}Q} |A(y) - A(z)|^s |f(z)| \, dz)^{1/s}$$

$$\leq C \sum_{j=1}^\infty \frac{1}{|2^j|} (\int_{2^{j+1}Q} |\Omega(y-z)|^{s} dz)^{1/s} (\int_{2^{j+1}Q} |A(y) - A(z)|^s |f(z)| \, dz)^{1/s}$$

$$\leq C \sum_{j=1}^\infty \frac{|2^{j+2}Q|}{|2^j|} \int_{2^{j+1}Q} |A(y) - m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A)|^s (\int_{2^{j+1}Q} |f(z)|^{s'} \, dz)^{1/s'}$$

$$+ C \sum_{j=1}^\infty \frac{|2^{j+2}Q|}{|2^j|} (\int_{2^{j+1}Q} |m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A) - A(z)|^s |f(z)|^{s'} \, dz)^{1/s'} = II_1 + II_2$$

We estimate $II_1$ and $II_2$ respectively. By Hölder inequality and the condition that $w \in A_{p/s'}$, we have

$$\frac{1}{w(Q)^{s/p}} \left( \int_{Q} II_1(y)^p w(y) dy \right)^{1/p}$$

$$= C \frac{1}{w(Q)^{s/p}} \sum_{j=1}^\infty \frac{|2^{j+2}Q|}{|2^j|} \left( \int_{Q} |A(y) - m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A)|^p (\int_{2^{j+1}Q} |f(z)|^{s'} \, dz)^{p/s'} w(y) dy \right)^{1/p}$$

$$\leq C \left( \int_{Q} |A(y) - m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A)|^p w(y) dy \right)^{1/p}$$

$$= C \left( \int_{Q} |A(y) - m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A)|^p w(y) dy \right)^{1/p}$$

We calculate the part of including $m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A)$ as follows:

$$\left( \int_{Q} |A(y) - m_{2^{j+1}Q,w - \frac{s'}{p-s'}}(A)|^p w(y) dy \right)^{1/p}$$

$$\leq \left( \int_{Q} |A(y) - m_{2^{j+1}Q,w}^p w(y) dy \right)^{1/p} + \left( \int_{Q} |m_{2^{j+1}Q,w}^p - m_{2^{j+1}Q,w}^p w(y) dy \right)^{1/p}$$

$$= III + IV$$

For term $III$, notice that $w \in A_{p/s'} \subset A_\infty$, thus from Lemma 4.3, we get

$$III \leq w(Q)^{\frac{s}{p}} \|A\|_s$$
Then we estimate $IV$. By Lemma \ref{lemma4.7} and Lemma \ref{lemma4.9}, we have
\[
\left| m_{Q,w}(A) - m_{2^{i+1}Q,w} w^{-\frac{i}{p-q}} (A) \right| \leq \left| m_{Q,w}(A) - m_Q(A) \right| + \left| m_Q(A) - m_{2^{i+1}Q}(A) \right|
\]
\[
+ \left| m_{2^{i+1}Q}(A) - m_{2^{i+1}Q,w} w^{-\frac{i}{p-q}} (A) \right|
\]
\[
\leq \frac{1}{w(Q)} \int_Q |A(t) - m_Q(A)| w(t) dt + 2^n (j+1) \|A\|_*
\]
\[
+ \frac{1}{w^{-\frac{i}{p-q}} (2^{i+1}Q)} \int_{2^{i+1}Q} |A(t) - m_{2^{i+1}Q}(A)| w^{-\frac{i}{p-q}} (t) dt.
\]
\[
\leq C(j+1) \|A\|_*
\]
So,
\[
IV \leq C(j+1) \|A\|_* w(Q)^{\frac{1}{p'}}
\]
As a result,
\[
\frac{1}{w(Q)^{\frac{n}{p'}}} \left( \int_Q II_1(y)^p w(y) dy \right)^{\frac{1}{p}} \leq C \|A\|_* \|f\|_{L_p,\ast}(w) \sum_{j=1}^{\infty} \frac{w(Q)^{\frac{1-p}{p}}}{w(2^{i+1}Q)^{\frac{1}{p'}}}
\]
\[
\leq C \|A\|_* \|f\|_{L_p,\ast}(w)
\]
For $II_2$, by H"older inequality and $w \in A_{p/s'}$, we get
\[
II_2 \leq C \|\Omega\|_{L_p} \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^\frac{1}{p'}}{|2^j Q|} \left( \int_{2^{i+1}Q} |f(z)|^p w(z) dz \right)^{\frac{1}{p'}}
\]
\[
\left( \int_{2^{i+1}Q} \left| m_{2^{i+1}Q,w} w^{-\frac{i}{p-q}} (A) - A(z) \right|^{\frac{p'}{p}} w(z)^{-\frac{i}{p-q}} dz \right)^{\frac{p}{p'}}
\]
\[
\leq C \|\Omega\|_{L_p} \|A\|_* \|f\|_{L_p,\ast}(w) \sum_{j=1}^{\infty} \frac{|2^{j+2}Q|^\frac{1}{p'}}{|2^j Q|} \frac{w(Q)^{\frac{1-p}{p}}}{w(2^{i+1}Q)^{\frac{1}{p'}}} w^{-\frac{i}{p-q}} (2^{j+1}Q)^{\frac{p}{p'}}
\]
\[
\leq C \|A\|_* \|f\|_{L_p,\ast}(w) \sum_{j=1}^{\infty} \frac{1}{w(2^{i+1}Q)^{\frac{1}{p'}}}
\]
Therefore,
\[
\frac{1}{w(Q)^{\frac{n}{p'}}} \left( \int_Q II_2^p w(y) dy \right)^{\frac{1}{p}} \leq C \|A\|_* \|f\|_{L_p,\ast}(w) \sum_{j=1}^{\infty} \frac{w(Q)^{\frac{1-p}{p}}}{w(2^{i+1}Q)^{\frac{1}{p'}}}
\]
\[
\leq C \|A\|_* \|f\|_{L_p,\ast}(w)
\]
So far, we have completed the proof of \eqref{7}.

The inequality \eqref{8} can be immediately obtained from \eqref{15} and \eqref{7}.

The proof of Theorem 3.5: As before, we prove \eqref{9} at first. Assume $Q$ be the same
as in the proof of (5) and $Q = 2Q$, set

$$A_Q(y) = A(y) - m_Q(\nabla A)y$$

We also decompose $f$ according to $\bar{Q}$: $f = f\chi_{\bar{Q}} + f\chi_{\bar{Q}^c} := f_1 + f_2$. Then we get

$$\|\tilde{T}_\Omega^A f\|_{L^p,\kappa(w)} \leq \frac{1}{w(Q)^{\kappa/p}} \left( \int_Q |\tilde{T}_\Omega^A f_1(y)|^p w(y)dy \right)^{\frac{1}{p}}$$

$$+ \frac{1}{w(Q)^{\kappa/p}} \left( \int_Q |\tilde{T}_\Omega^A f_2(y)|^p w(y)dy \right)^{\frac{1}{p}} = I + II$$

For the first term $I$, Theorem 4.4 and Lemma 4.8 (1) implies

$$I \leq \frac{1}{w(Q)^{\kappa/p}} \|\tilde{T}_\Omega^A f_1\|_{L^p(w)} \leq C \frac{\|\Omega\|_{\infty} \|\nabla A\|_\ast \|f_1\|_{L^p(w)}}{w(Q)^{\kappa/p}}$$

$$\leq C \|\nabla A\|_\ast \|f\|_{L^p,\kappa(w)} \leq C \|\nabla A\|_\ast \|f\|_{L^p,\kappa(w)}$$

We will omit the proof for the term $II$ as it is similar to and easier than the part of $II$ in the proof of (5), except under the condition that $w \in AP$, $Ω \in L^\infty(S^{n-1})$, $m = 2$ and $f \in L^{p,\kappa}(w)$. For inequality (10), it can be easily proved by (9) and (15). Thus, we complete the proof of Theorem 3.5.

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