Let $\Gamma$ be a group. A virtual endomorphism of $\Gamma$ is a homomorphism $\varphi: \Lambda \to \Gamma$, where $\Lambda$ is a finite index subgroup in $\Gamma$. An invariant normal subgroup of $\varphi$ is a normal subgroup $N < \Gamma$ contained in $\Lambda$, so that $\varphi(N) < N$. We will be interested in the question

**Question 1.** When does $\varphi$ have a nontrivial invariant normal subgroup?

We will only consider the case of lattices $\Gamma$ in algebraic (linear) Lie groups $G$. Therefore, without loss of generality, the subgroups $\Lambda$ can be taken torsion-free. In particular, existence of a nontrivial invariant normal subgroup is equivalent to the existence of an infinite invariant normal subgroups. Our main result is

**Theorem 2.** Let $\Gamma$ be an irreducible lattice in a semisimple algebraic Lie group $G$. Then the following are equivalent:

a. $\Gamma$ is virtually isomorphic to an arithmetic lattice in $G$, i.e., contains a finite index subgroup isomorphic to such arithmetic lattice.

b. $\Gamma$ admits a virtual endomorphism $\varphi: \Lambda \to \Gamma$ without nontrivial invariant normal subgroups.

We note that conjugacy classes of faithful self-similar actions of a group $\Gamma$ on a rooted tree (of finite valency) are in 1-1 correspondence with conjugacy classes of virtual endomorphisms of $\Gamma$ which contain no nontrivial invariant normal subgroups [7, 8]. Therefore,

**Corollary 3.** Let $\Gamma$ be an irreducible lattice in a semisimple algebraic Lie group $G$. Then the following are equivalent:

a. $\Gamma$ is virtually isomorphic to an arithmetic lattice in $G$.

b. $\Gamma$ admits an irreducible faithful self-similar action on a regular rooted tree (of finite valency).

The only nontrivial ingredient in the proof of Theorem 2 is the following theorem of Margulis [6]:

**Theorem 4.** An irreducible lattice $\Gamma$ in an algebraic semisimple Lie group $G$ is arithmetic if and only if its commensurator $\text{Comm}(\Gamma)$ in $G$ is non-discrete.

We start our proof of Theorem 2 with some generalities about arithmetic groups.

Let $F$ be an (algebraic) number field. Let $F_\infty$ denote the completion of $F$ with respect to the valuation given by the embedding $F \to \mathbb{C}$. Let $\mathcal{O}$ be the ring of integers
of $F$. Given a prime ideal $p$ in $\mathcal{O}$ we let $F_p$ denote the completion of $F$ with respect to the nonarchimedean valuation on $F$ determined by $p$.

Let $G(F)$ be a semisimple algebraic group defined over $\mathcal{O}$. Then the group $G = G(F_\infty)$ is a real or complex semisimple Lie group. We will regard $G$ as the isometry group of a symmetric space $X$ (the action has finite kernel coming from the finite center of $G$). Similarly, the group $G_p := G(F_p)$ acts as a group of isometries of a Euclidean building $X_p$. The group of integer points $G(\mathcal{O}_p)$ is the stabilizer in $G_p$ of a special vertex $o_p \in X$.

Consider an irreducible arithmetic group $\Gamma \subset G(\mathcal{O})$ in $G$. Recall [6] that the commensurator $\text{Comm}(\Gamma)$ of $\Gamma$ is $G(F)$, the group of $F$-points in $G$.

Given an element $\alpha \in \text{Comm}(\Gamma)$ we let $\varphi = \varphi_\alpha$ denote the automorphism of $G$ induced by conjugation

$$\varphi(g) = \alpha g \alpha^{-1}.$$ 

Thus, $\alpha$ induces virtual endomorphisms of $\Gamma$ defined by taking sufficiently deep finite index subgroups $\Lambda \subset \Gamma$, so that $\varphi(\Lambda) \subset \Gamma$, and then restricting $\varphi$ to $\Lambda$. (For instance, sufficiently deep congruence subgroups will work.)

Since $G(F)$ is dense in $G_p$, we can choose a prime ideal $p$ and $\alpha \in \text{Comm}(\Gamma)$ so that the corresponding isometry $\alpha : X_p \to X_p$ has unbounded orbits (i.e. is hyperbolic).

**Remark 5.** Moreover, if $\alpha \in G(F)$ induces an elliptic automorphism of every Euclidean building $X_p$, then it belongs to a finite extension of $\Gamma$. However, we do not need this fact.

**Lemma 6.** Given $\alpha$ as above, for every finite index subgroup $\Gamma' \subset \Gamma$, we have $\varphi(\Gamma') \neq \Gamma'$.

**Proof.** If $\alpha \Gamma' \alpha^{-1} = \Gamma'$, then $\Gamma'$ and $\alpha$ generate a discrete subgroup $\hat{\Gamma}' \subset G$, which (since $\Gamma'$ is a lattice) necessarily has $\Gamma$ as a finite index subgroup. Therefore, the group $\hat{\Gamma}$ generated by $\Gamma$ and $\hat{\Gamma}'$ is also a finite extension of $\Gamma$. Since $\Gamma$ fixes the point $o_p \in X_p$, it follows that orbit $\hat{\Gamma} \cdot o_p \subset X_p$ is bounded. This contradicts the fact that $\gamma : X_p \to X_p$ is hyperbolic. 

**Theorem 7.** Given $\Gamma, \Lambda, \alpha$ as above, if $N < \Lambda$ is a normal subgroup of $\Gamma$ so that $\varphi(N) \subset N$, then $N$ is finite.

**Proof.** We let $M := X/N$. Then for every $N$ as above, we obtain an isometric covering

$$q = \bar{\alpha} : M \to M$$

induced by the endomorphism $\varphi$ of the fundamental group of $M$. Note that the group $\Gamma'$ acts on $M$ isometrically, we use the notation $\bar{\gamma}$ for an isometry of $M$ induced by $\gamma \in \hat{\Gamma}$.

1. First, consider the case when $G$ has rank $\geq 2$. Then, by a theorem of Margulis [6], every normal subgroup of $\Gamma$ is either finite or has finite index in $\Gamma$. Assuming $N$ has finite index in $\Gamma$, the manifold $M$ has finite volume and, since $q$ preserves the volume form, it has to be a diffeomorphism. Thus, $\varphi(N) = N$, which contradicts our choice of $\alpha$. 

2. Let $G$ be a simple algebraic group defined over $\mathcal{O}$, and let $F$ be a nonarchimedean local field. Then the group $G(F)$ is a semisimple Lie group. We will regard $G$ as the isometry group of a symmetric space $X$ (the action has finite kernel coming from the finite center of $G$). Similarly, the group $G_p := G(F_p)$ acts as a group of isometries of a Euclidean building $X_p$. The group of integer points $G(\mathcal{O}_p)$ is the stabilizer in $G_p$ of a special vertex $o_p \in X$.
2. We now consider the more interesting case when $G$ has rank 1. We assume that $N$ is an infinite group. Therefore, it is Zariski dense in $G$.

Consider the iterations $q^k$ of the isometric endomorphism $q : M \to M$. Let $M_{\text{thick}}, M_{\text{thin}}$ denote the thick and thin parts of $M$ with respect to the Margulis constant of $X$. Pick a connected compact subset $C \subset M_{\text{thick}}$ whose fundamental group maps onto a Zariski dense subgroup $H$ of $N$ and which contains a fundamental domain for the action $\Gamma \curvearrowright M_{\text{thick}}$.

Then, by the Kazhdan-Margulis lemma, for every $k \geq 1$, $q^k(C)$ is never contained in $M_{\text{thin}}$. Therefore, for each $k$ there exists $x_k \in C$ so that $q^k(x_k) \in M_{\text{thick}}$. Moreover, since $q$ is isometric, there exists $\epsilon > 0$ so that for every $k \in \mathbb{N}$, the injectivity radius of $M$ on $q^k(C)$ is bounded from below by $\epsilon$.

Since the action $\Gamma \curvearrowright M_{\text{thick}}$ is cocompact (as $X/\Gamma$ has finite volume), for every $k$ there exists $\gamma_k \in \Gamma$ so that $\overline{\gamma_k \circ q^k(x_k)} \in C$.

Therefore, the sequence of isometries $\overline{\beta_k} := \overline{\gamma_k \circ q^k}$ is precompact.

**Lemma 8.** The set $\overline{I} := \{\overline{\beta_k}, k \in \mathbb{N}\}$ is finite.

**Proof.** If not, then there will be arbitrarily large $k, m$ so that $\overline{\beta_k} \neq \overline{\beta_m}$ and the restrictions $\beta_k|C, \beta_m|C$ are arbitrarily close in the sup-metric. In particular, for large $k, m$, they are homotopic and, hence, induce the same (up to conjugation in $N$) map $H \to N$ given by $h \mapsto \beta_k h \beta_k^{-1}, h \mapsto \beta_m h \beta_m^{-1}$.

Since $H$ is Zariski dense in $G$, its centralizer in $G/Z(G)$ is trivial and, thus we have the equality of the cosets $\beta_k \cdot N \cdot Z(G) = \beta_m \cdot N \cdot Z(G)$.

Since the center of $G$ acts trivially on $X$, we obtain that $\overline{\beta_k} = \overline{\beta_m}$. Contradiction. \(\square\)

We continue with the proof of Theorem 7. Let $I \subset G(F)$ denote the finite set of representatives of lifts of the isometries $\beta_k \in \overline{I}$. Hence, for each $k \in \mathbb{N}$,

$$\gamma_k \cdot \alpha^k \cdot N \subset I \cdot N,$$

and, thus,

$$\alpha^k \in \gamma_k^{-1} \cdot I \cdot N.$$

We now consider the action of the isometries in the above equation on the building $X_p$: the group $\Gamma$ fixes a point $o \in X_p$, the images $I \cdot o$ form a finite set. Therefore, the set

$$\gamma_k^{-1} \cdot I \cdot N \cdot o = \gamma_k^{-1} \cdot I \cdot o$$

is bounded in $X$. However, by our assumption, the orbit $\alpha^k \cdot o, k \in \mathbb{N}$ is unbounded. Contradiction. \(\square\)

**Corollary 9.** Suppose that $\Gamma$ has trivial center. Then, if $N < \Lambda$ is a normal subgroup of $\Gamma$ so that $\varphi(N) \subset N$, then $N$ is trivial.

**Proof.** Since $\Gamma$ has trivial center, its only finite normal subgroups are trivial. \(\square\)

We now consider virtual endomorphisms of non-arithmetic lattices.
Proposition 10. Let $G$ be a rank 1 semisimple Lie group (with finitely many components), which is not locally isomorphic to $SL(2, \mathbb{R})$ and $\Gamma < G$ be a non-arithmetic lattice with trivial center. Then for every virtual endomorphism $\varphi$ of $\Gamma$, there exists an infinite normal subgroup $N < \Gamma$ which is $\varphi$-invariant.

Proof. 1. Suppose that $\varphi : \Lambda \to \Gamma$ is not injective. Let $K$ denote the kernel of $\varphi$. Note that this subgroup is not necessarily normal in $\Gamma$. Our assumption that $\Gamma$ has trivial center implies that $\Gamma$ acts faithfully on the symmetric space $X$ associated with $G$ and, hence, the group $K$ is necessarily infinite.

We first consider the case when $\Lambda$ is normal in $\Gamma$. Let $\gamma_1, ..., \gamma_n$ be the generators of $\Gamma$. Consider the conjugates

$$K_i := \gamma_i K \gamma_i^{-1} \subset \Gamma, \quad K_0 := K.$$ 

Then

$$K' := \bigcap_{i=0}^{n} K_i$$

is a normal subgroup in $\Gamma$. This subgroup is the kernel of the homomorphism

$$\Phi : \Lambda \to \prod_{i=0}^{n} \Gamma,$$

$$\Phi = (\varphi_0, ..., \varphi_n),$$

where

$$\varphi_i : \Lambda \to \Gamma$$

is given by

$$\varphi_i(g) = \varphi(\gamma_i g \gamma_i^{-1}), \quad i = 1, ..., n; \quad \varphi_0 := \varphi.$$ 

We set let $\Gamma_i$ denote the $i$-th factor of the product group $\prod_{i=0}^{n} \Gamma$.

If $K$ is infinite, it contains a free nonabelian subgroup $H$. Thus, each group $K_i$ contains a free nonabelian subgroup $H_i, \ i = 0, ..., n$. Assume, for a moment, that $K'$ is finite.

The group $\Lambda/K'$ embeds in the product group $\prod_{i=0}^{n} \Gamma$. Therefore, the intersections

$$\Phi(\Lambda) \cap \Gamma_i$$

contain free nonabelian subgroups (isomorphic to $H_i), \ i = 0, ..., n$. Hence, the group $\Lambda/K'$ contains a direct product of free nonabelian subgroups. This is impossible since the group $\Lambda/K'$ is isomorphic to a discrete subgroup of isometries of the negatively curved symmetric space $X$. Contradiction. Thus, $K'$ is infinite and we obtain an infinite normal subgroup $N = K' < \Lambda$ of the group $\Gamma$, so that $\varphi(N) = 1 \subset N$.

We now consider the case when $\Lambda$ is not necessarily normal in $\Gamma$. If $\varphi : \Lambda \to \Gamma$ is a virtual endomorphism with infinite kernel, we find a finite index subgroup $\Lambda' \subset \Lambda$ which is normal in $\Gamma$. Then the restriction $\varphi' = \varphi|\Lambda'$ still has infinite kernel and we obtain a contradiction as above.

2. Suppose now that $\varphi$ is injective. Then, by Mostow rigidity theorem, the homomorphism $\varphi$ is induced by conjugation via some $\alpha \in Comm(\Gamma)$. Recall that, by the Margulis theorem, $Comm(\Gamma)$ is discrete. Therefore, since $\Gamma$, is a lattice,
\[ \hat{\Gamma} := \text{Comm}(\Gamma) \] is a finite extension of \( \Gamma \). Therefore, \( \Lambda \) has finite index in \( \hat{\Gamma} \) and, hence, contains a finite index subgroup \( N < \Lambda \) which is normal in \( \hat{\Gamma} \). In particular,

\[ \alpha N \alpha^{-1} = N \]

and \( N < \Gamma \) is a normal subgroup. Clearly, \( N \) is an infinite group as required by the proposition. \( \square \)

If \( G \) is locally isomorphic to \( SL(2, \mathbb{R}) \) then the abstract commensurator of a lattice \( \Gamma < G \) is not isomorphic to the commensurator of \( \Gamma \) in \( G \). However, \( \Gamma \) will contain a finite index torsion-free subgroup \( \Lambda \) which is isomorphic to an arithmetic subgroup of \( G \). Therefore, \( \Gamma \) will have a virtual endomorphism \( \varphi \) without nontrivial invariant normal subgroups.

By combining the above results we obtain Theorem 2. \( \square \)

We observe that our results should, in principle, generalize to Gromov-hyperbolic groups which are not lattices. The problem, however, is that:

1. Among hyperbolic groups \( \Gamma \), (if we ignore torsion) only Poincaré duality groups (fundamental groups of closed aspherical manifolds) are known to be weakly cohopfian, in the sense that if \( \Lambda < \Gamma \) is a finite index subgroup and \( \varphi : \Lambda \to \Gamma \) is an injective homomorphism, then the image of \( \varphi \) is a finite index subgroup of \( \Gamma \). Hyperbolic groups which act geometrically on rank 2 hyperbolic buildings provide good candidates for weakly cohopfian groups [1]. On the other hand, there are no known examples of 1-ended hyperbolic groups which are not weakly cohopfian.

2. Among hyperbolic groups \( \Gamma \) which are Poincaré duality groups, the only known examples where the abstract commensurator \( \text{Comm}(\Gamma) \) is a finite extension of \( \Gamma \), are the non-arithmetic lattices. There are few more classes of hyperbolic groups with small abstract commensurators: Fundamental groups of compact hyperbolic \( n \)-manifolds with totally-geodesic boundary, \( n \geq 3 \), surface-by-free groups [2], as well as some rigid examples constructed in [3]. (In all these examples, the whole quasi-isometry group of \( \Gamma \) is a finite extension of \( \Gamma \).) Conjecturally, the fundamental groups of Gromov-Thurston manifolds [4] should also have small abstract commensurators.

Question 11. Does the action of an arithmetic group \( \Gamma \) on a rooted tree constructed in Theorem 2 ever correspond to a finite-state automaton? Zoran Sunic computed one example (an index 3 subgroup in \( PSL(2, \mathbb{Z}) \)) where he proved that the number of states is infinite. See however the examples constructed by Glasner and Mozes in [3].

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