Abstract

Several nonlinear stochastic differential equations have been proposed in connection with self-organized critical phenomena. Due to the threshold condition involved in its dynamic evolution an infinite number of nonlinearities arises in a hydrodynamic description. We study two models with different noise correlations which make all the nonlinear contribution to be equally relevant below the upper critical dimension. The asymptotic values of the critical exponents are estimated from a systematic expansion in the number of coupling constants by means of the dynamic renormalization group.

*E-mail: albert@ulyses.ub.es
I. INTRODUCTION

Recently a lot of attention has been paid to the phenomenon known as self-organized criticality (SOC). Bak, Tang, and Wiesenfeld (BTW) [1] studied a cellular automaton model as a paradigm for the explanation of two widely occurring phenomena in nature: $1/f$ noise and fractal structures. Both have in common a lack of characteristic scales. This scale invariance suggests that the system is critical in analogy with classical equilibrium critical phenomena, but in self-organized criticality one deals with dynamical nonequilibrium statistical properties. On the other hand the system evolves naturally to the critical state without any tuning of external parameters.

Several cellular automata models exhibiting self-organized criticality have been reported in the literature. In the original sandpile model of Bak. *et al.* [1] the system is perturbed externally by a random addition of sand grains, once the slope between two contiguous cells have reached a threshold condition sand is transferred to its neighbors in a fixed amount. Taking this model as a reference, different dynamical rules have been investigated leading to a wide variety of universality classes: directed flows [2], threshold condition imposed on the height, on the gradient or even on the laplacian [3], continuous variables with a full transfer from a cell instead of a fixed discrete amount [4–7], deterministic perturbations in a nonconservative system [8–11], anisotropy [11], as some examples.

On the other hand some authors have attempted to connect these cellular automata models showing SOC to nonlinear stochastic differential equations [12–14]. These continuous descriptions are built according to the symmetry rules obeyed by the discrete models. Anomalous diffusion equations with singularities in the diffusion coefficient have been considered in order to study the dynamics of the avalanches generated in the critical state [15–16].

In this paper we study two nonlinear stochastic differential equations whose deterministic parts keep the symmetries of the usual sandpile BTW models [7,17]. However, they have different noise correlations from which one gets different critical behavior. Both models
are treated independently and the noise correlations are obtained by different limits of a
general Ornstein-Uhlenbeck process and in some way they represent different aspects of
SOC. For both models an infinite number of nonlinearities is relevant below their upper
critical dimension $d_c = 4, 2$; by taking a finite number of nonlinearities we estimate the
values of the critical exponents from an extrapolation. For one of the models we get the
dynamical exponent obtained in the numerical simulations of systems exhibiting SOC.

II. STOCHASTIC DIFFERENTIAL EQUATIONS

The nonlinear stochastic differential equation we are going to study is

$$\frac{\partial E(\vec{r}, t)}{\partial t} = D\nabla^2 E(\vec{r}, t) + \sum_{n=1}^{\infty} \lambda_{2n+1} \nabla^2 E^{2n+1}(\vec{r}, t) + \eta(\vec{r}, t)$$

where $D$ is the bare diffusion coefficient and $\lambda_i$ are the coupling constants accounting for the
nonlinearities of the model. The variable $E$ which we call energy can have different physical
interpretations [5]. The difference among both models lies in the noise correlation

$$< \eta(\vec{r}, t) \eta(\vec{r}', t') > = 2 \Gamma T \delta^{d}(\vec{r} - \vec{r}').$$

$$< \eta(\vec{r}, t) \eta(\vec{r}', t') > = 2 \Gamma \delta(t - t') \delta^{d}(\vec{r} - \vec{r}').$$

$\Gamma$ being the intensity of the noise and $T$ is a macroscopic time to be discussed in the last
section.

From Eq. (1) we can see that both models are translational and rotational invariant
and that the energy is conserved in its deterministic part. An additional symmetry ($E \rightarrow -E$ invariance) has been introduced since in [17] it is showed that it is generated by a
renormalization group procedure when Eq. (1) is derived from an equation involving a
threshold condition.
III. DYNAMIC RENORMALIZATION GROUP

The scaling behavior of the system described by the nonlinear stochastic differential equation is obtained from the correlation function

\[ <(E(\vec{r}_0, t_0) - E(\vec{r}_0 + \vec{r}, t_0 + t))^2 >^{1/2} \sim r^\chi f(t/r^z) \]  

where \( \chi \) and \( z \) are the 'roughening exponent' and the dynamical exponent respectively. The relevance of the different coupling constants can be checked by naive dimensional analysis: a change of scale \( \vec{r} \rightarrow e^l \vec{r} \) is followed by \( t \rightarrow e^{zl} t \) and \( E \rightarrow e^{\chi l} E \) in order to (3) be satisfied. Under this scaling transformation \( z \) and \( \chi \) are chosen to keep the linear equation invariant. With these values one can see that all coupling constants are of the same order in \( \varepsilon = d_c - d \) and hence all nonlinear terms are equally relevant for \( d < d_c \). The upper critical dimension \( d_c \) are 4 and 2 for models (2a) and (2b), respectively.

The dynamic-renormalization-group procedure consists of two steps: an elimination of the faster short wave-length modes followed by a rescaling of the remaining modes [18,19]. The recursion relations after an infinitesimal transformation for the coefficients are [17]

\[ \frac{dD}{dl} = D \left[ z - 2 + 3 \frac{\Gamma A^{d-4} \lambda_3}{D^3} \right] \]  

\[ \frac{d(\Gamma/T)}{dl} = \Gamma \left[ 2z - 2\chi - d \right] \]  

\[ \frac{d\lambda_n(l)}{dl} = \lambda_n [(n - 1)\chi + z - 2] + \sum_j \gamma_j \Gamma A^{d-4} D^{a_j} \prod_{i=2}^{n+2} \lambda_i^{b_{ij}} \]  

and

\[ \frac{dD}{dl} = D \left[ z - 2 + 3 \frac{\Gamma A^{d-2} \lambda_3}{D^2} \right] \]  

\[ \frac{d\Gamma}{dl} = \Gamma [z - 2\chi - d] \]  

\[ \frac{d\lambda_n(l)}{dl} = \lambda_n [(n - 1)\chi + z - 2] + \sum_j \gamma_j \Gamma A^{d-2} D^{a_j} \prod_{i=2}^{n+2} \lambda_i^{b_{ij}} \]
for models (2a) and (2b), respectively. In the previous equations \( \gamma_j \) are numerical factors and \( b_{ij} \) are natural numbers accounting for the vertices \( \lambda_i \) entering the renormalization of the vertex \( \lambda_n \) in the \( j \)-th term of the sum.

Since all terms in this expansion are equally relevant below the respective upper critical dimension a calculation of the critical exponents would involve all the contributions. Nevertheless the exponents can be obtained to first order in \( \varepsilon = d_c - d \) as a series in the number of coupling constants which are taken into account. For details of the calculations for model (2a) see [17] where one obtains

\[
\begin{align*}
z &= 2 - 0.32(4 - d); \quad \chi = 0.18(4 - d) \text{ for } d \leq d_c = 4 \\
&= 2; \quad \chi = \frac{4-d}{2} \text{ for } d \geq d_c = 4
\end{align*}
\]

(6a)

On the other hand the results for model (2b) can be obtained in an equivalent way to be

\[
\begin{align*}
z &= 2 - 0.33(2 - d); \quad \chi = 0.34(2 - d) \text{ for } d \leq d_c = 2 \\
&= 2; \quad \chi = \frac{2-d}{2} \text{ for } d \geq d_c = 2
\end{align*}
\]

(6b)

IV. RELATION WITH SELF-ORGANIZED CRITICALITY

Up to now it is not clear yet the relation between the cellular automata models exhibiting self-organized criticality and the nonlinear stochastic differential equations analyzed in the present work. In the original sand-pile model, and in others derived from it, the system is perturbed externally until a site reaches a threshold condition and then an avalanche is triggered. During the evolution of the avalanche there is no external perturbation onto the system. This introduces two time scales in the model: a fast time scale (the diffusion scale) and a slow one (the noise scale).

In order to be suitable for an analytical treatment one can assume the external noise to be described by a Gaussian process with zero mean and correlation function given by

\[
< \eta(\vec{r}, t) \eta(\vec{r}', t') > = \frac{2\Gamma}{\tau} e^{-|t-t'|/\tau} \delta^d(\vec{r} - \vec{r}').
\]

(7)
This is an Ornstein-Uhlenbeck process, with $\tau$ the correlation time. This general case can describe two different limits depending on whether $\tau$ is a microscopic or a macroscopic time. In the former case we get (2b) whereas in the later one obtains (2a) with a intensity of the noise that scales with $1/\tau$.

When analyzing the SOC models within a mesoscopic time scale we deal with the dynamics of single avalanches. In this case model (2a) is more appropriate since the noise scale is larger than the time scale we are interested in. A quenched noise of low intensity does not affect the evolution of the avalanche since it only depends on whether a site is above its threshold. This is the reason for which the estimation of the dynamical exponent (5a) agrees with the values obtained in the numerical simulations and from scaling arguments [4] concerning the evolution of single avalanches. Nevertheless it can modify the roughness of the interface since it can make two distant sites to be correlated in an artificial way; however, the effect of the nonlinearities is to lower the value of the roughening exponent with respect to the linear model.

SOC models can be analyzed from a different point of view by considering the avalanches to be instantaneous. In this case one is mainly interested in the form of the interface after the relaxation events (avalanches) have taken place and not in its dynamical evolution. This situation is properly described by Eqs. (1) and (2b) but the parameters entering the deterministic evolution diverge since they are related to the diffusion coefficient. Only finite values of the parameters make the problem suitable of an analytical treatment; on doing so we make the two time scales discussed above to be of the same order [14]. Although it is not the situation one deals with in the simulations it is interesting to notice that the roughening exponent $\chi$ is lowered with respect to the previously discussed model. Actually, one should expect a roughening exponent $\chi \leq 0$ due to the fact that the energy at each site is bounded after the avalanches are finished. But as the two dynamics (diffusion and noise) overlap we can still get a rough interface, at least for low dimensionality.

To summarize, we have analyzed two nonlinear stochastic differential equations by means of the dynamic renormalizaton group. Both equations share the deterministic evolution but
differ in the noise correlation. Due to the symmetries of the cellular automata models they are derived from, an infinite number of relevant nonlinearities must be taken into account. By taking a finite number of these nonlinear couplings we extrapolate to infinite and get an estimation of the critical exponents. These two models permit to describe different, and complementary, characteristics of cellular automata models exhibiting SOC. The quenched noise model reproduces the dynamical exponent obtained in the evolution of single avalanches but introduces additional correlations in the interface profile. On the other hand, the delta-correlated noise model gives a roughening exponent which is closer to what one would expect in the profile of the interface after the avalanches have taken place.

ACKNOWLEDGMENTS

This work has been supported by CICYT of the Spanish Government, grants #PB89-0233 and #PB92-0863.
REFERENCES

[1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. A 38 , 364 (1988).

[2] L.P. Kadanoff, S.R. Nagel, L. Wu and S. Zhou, Phys. Rev. A 39 , 6524 (1989).

[3] S.S. Manna, Physica 179A , 249 (1991).

[4] Y.-C. Zhang, Phys. Rev. Lett. 63 , 470 (1989).

[5] L. Pietronero, P. Tartaglia and Y.-C. Zhang, Physica 173A , 22 (1991).

[6] Z. Fodor and I.M. Janosi, Phys. Rev. A 44 , 1386 (1991).

[7] A. Díaz-Guilera, Phys. Rev. A 45 , 8551 (1992).

[8] H.J.S. Feder and J. Feder, Phys. Rev. Lett. 66 , 2669 (1991).

[9] K. Christensen, Z. Olami and P. Bak, Phys. Rev. Lett. 68 , 2417 (1992).

[10] Z. Olami and K. Christensen, Phys. Rev. A 46 , R172 (1992).

[11] I.M. Janosi, Phys. Rev. A 42 , 769 (1990).

[12] T. Hwa and M. Kardar, Phys. Rev. Lett. 62 , 1813 (1989).

[13] G. Grinstein, D.-H. Lee and S. Sachdev, Phys. Rev. Lett. 64 , 1927 (1990).

[14] T. Hwa and M. Kardar, Phys. Rev. A 45 , 7002 (1992).

[15] J.M. Carlson, J.T. Chayes, E.R. Grannan and G.H. Swindle, Phys. Rev. Lett. 65 , 2547 (1990).

[16] P. Bantay and I.M. Janosi, Phys. Rev. Lett. 68 , 2058 (1992).

[17] A. Diaz-Guilera (preprint).

[18] D. Forster, D.R. Nelson and M.J. Stephen, Phys. Rev. A 16 , 732 (1977).

[19] E. Medina, T. Hwa, M. Kardar, Y.-C. Zhang, Phys. Rev. A 39 , 3053 (1990).