DRINFELD MODULAR CURVE AND WEIL PAIRING

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Abstract. In this paper we describe the compactification of the Drinfeld modular curve. This compactification is analogous to the compactification of the classical modular curve given by Katz and Mazur. We show how the Weil pairing on Drinfeld modules that we defined in earlier work gives rise to a map on the Drinfeld modular curve. We introduce the Tate-Drinfeld module and show how this describes the formal neighbourhood of the scheme of cusps of the Drinfeld modular curve.

1. Introduction

Consider a smooth, projective, geometrically irreducible curve $X$ over $\mathbb{F}_q$, and fix some point $\infty$ on this curve. Let

$$A := \Gamma(X - \infty, \mathcal{O}_X)$$

be the ring of functions on $X$ which are regular outside $\infty$. Let $f \in A \setminus \mathbb{F}_q$ be a non-constant element, and let

$$A_f := A[f^{-1}].$$

The moduli scheme $M^2(f)$ plays an important role in this paper. It represents the functor which associates to every $A_f$-scheme $S$ the set of isomorphy classes of Drinfeld modules with a level $f$-structure over $S$.

In this paper we will address the following problems:

(i) Construct a morphism

$$w_f : M^r(f) \longrightarrow M^1(f).$$

This morphism is induced by the Weil pairing for Drinfeld modules. The Weil pairing is defined in [12].

(ii) Define the Tate-Drinfeld module and describe its universal property, using ideas of G. Böckle in Chapter 2 of [1] and ideas of M. van der Put and J. Top in [22] and [21].

(iii) Describe a compactification $\overline{M^2(f)}$ of $M^2(f)$. The Tate-Drinfeld module will enable us to describe the scheme of cusps

$$\text{Cusps} = (\overline{M^2(f)} - M^2(f))_{\text{red}}.$$

(iv) Compute the number of components of $M^2(f)$ and describe the cusps of the analogue of the classical curve $X_0(N)$.

We would like to point out that another description of the compactification of $M^2(f)$ can be found in T. Lehmkuhl’s ‘Habilitation’ [16]. Our treatment uses the Weil pairing and gives an explicit description of the Tate-Drinfeld
module. In a forthcoming paper this enables us to develop the Néron model of the Tate-Drinfeld module, analogous to the Deligne and Rapoport’s construction of the Néron model of the Tate-elliptic curve in [4]. This will probably give rise to an extension of the functor represented by \( M'(f) \) to a functor represented by the compactification \( \overline{M}(f) \).

Sections 2 and 3 give a brief introduction to the moduli problem and to the moduli schemes. At the end of Section 4 we state the assumptions which are used throughout this paper. In Section 2 we recall the definition of Drinfeld modules over schemes and level structures. In Section 3 we describe the moduli problem that Drinfeld considers in his original paper [5]. The goal of Section 4 is to prove Theorem 4.1, i.e., to construct the morphism \( w_f \) considered in problem (i). In Sections 5, 6, and 7 we discuss problem (ii). In Section 5 we classify the Drinfeld modules of rank 2 with level \( f \)-structure over the quotient field of some complete discrete valuation ring \( V \) which have stable reduction of rank 1. The main result of this section is Theorem 5.8. In Section 6 we define the Tate-Drinfeld module of type \( m \) with level \( f \)-structure. In Section 7 we define the universal Tate-Drinfeld module \( Z \); cf. Theorem 7.4. For the application to problem (iii) we need a weak version of the universal property of the scheme \( Z \) as stated in Theorem 7.8. In Sections 8 and 9 we study problem (iii). In Section 8 we give a compactification of \( \overline{M^2}(f) \). The treatment given here is analogously to the compactification of the classical modular curve given by N.M. Katz and B. Mazur in Chapter 8 of their book [15]. This enables us in Section 9 to identify the formal neighbourhood of the scheme of cusps \( \overline{M^2}(f) - \overline{M^2}(f) \) with the universal Tate-Drinfeld module. The main results are Theorem 9.1 and Corollary 9.2. In Section 10 we compute, using the scheme of cusps, the number of components for every geometric fibre; cf. Theorem 10.1.

2. DRINFELD MODULES OVER SCHEMES

Throughout this paper we will denote the quotient field of any integral domain \( D \) by \( K_D \). We recall the definition of Drinfeld modules over schemes and level structures. There are many texts available for a more extensive account of these definitions; cf. [5], [3], [18], [16] and [20].

2.1. Line bundles and morphisms. Let \( B \) be a commutative \( \mathbb{F}_q \)-algebra with 1 and let \( G_{a,B} \) denote the additive group over \( B \). The ring of \( \mathbb{F}_q \)-linear endomorphisms \( \text{End}_{\mathbb{F}_q}(G_{a,B}) \) of \( G_{a,B} \) is isomorphic to the skew polynomial ring \( B\{\tau\} \). In this skew polynomial ring multiplication is determined by the rule \( \tau b = b^q \tau \) for all \( b \in B \).

This can be generalized to schemes. Let \( S \) be an \( \mathbb{F}_q \)-scheme, and let \( L \rightarrow S \) be a line bundle. As usual, \( L \) is also a group scheme due to its additive group scheme structure. A trivialization of \( L \) is a covering \( \text{Spec}(B) \) of open affines of \( S \) together with isomorphisms \( L|_{\text{Spec}(B_i)} \cong G_{a,B_i} \). By \( \text{End}_{\mathbb{F}_q}(L) \) we denote the \( \mathbb{F}_q \)-linear \( S \)-group scheme endomorphisms of \( L \). Let \( \mathcal{L} \) be the invertible \( O_S \)-sheaf corresponding to \( L \), and let

\[
\tau^i : \mathcal{L} \rightarrow \mathcal{L}^i \quad \text{by} \quad s \mapsto s \otimes \ldots \otimes s.
\]
The ring $\text{End}_{\mathbb{F}_q}(L)$ is isomorphic to the ring of all formal expressions $\sum_i \alpha_i \tau^i$ which are locally finite. Here $\alpha_i : \mathcal{L}_q^i \to \mathcal{L}$ is an $\mathcal{O}_S$-module homomorphism for every $i$. Multiplication in the ring of formal expressions is given by $\alpha_i \tau^i \beta_j \tau^j = \alpha_i \otimes \beta_j^q \tau^{i+j}$. If $\{\text{Spec}(B_i)\}_{i \in I}$ is a trivialization of $L$, then the restriction of $\text{End}_{\mathbb{F}_q}(L)$ to $B_i$ is simply $B_i \{\tau\}$.

Furthermore, we denote by $\partial_0$ the point derivation at 0:

$$\partial_0 : \text{End}_{\mathbb{F}_q}(L) \to \Gamma(S, \mathcal{O}_S) \quad \text{by} \quad \sum_i \alpha_i \tau^i \mapsto \alpha_0.$$

### 2.2 Drinfeld modules over a scheme.

**Definition 2.1.** Let $K$ be an $A$-field equipped with an $A$-algebra structure given by $\gamma : A \to K$. Let $L = \mathbb{G}_{a,K}$. A **Drinfeld module over $K$** is a ring homomorphism $\varphi : A \to \text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,K})$ such that

1. $\partial_0 \circ \varphi = \gamma$;
2. there is an $a \in A$ such that $\varphi_a \neq \gamma(a)$.

A Drinfeld module over a field $K$ has a **rank**, i.e., there is an integer $r > 1$ such that $\deg_r \varphi_a = r \deg(a)$ for all $a \in A$.

**Definition 2.2.** Let $S$ be an $A$-scheme via the morphism $\gamma_S : S \to \text{Spec}(A)$.

A **Drinfeld module of rank $r$ over $S$** is a pair $(L, \varphi)_S$ of a line bundle $L \to S$ and a ring homomorphism $\varphi : A \to \text{End}_{\mathbb{F}_q}(L)$ such that

1. $\partial_0 \circ \varphi = \gamma^S$;
2. For all $a \in A$ the morphism $\varphi_a$ is finite of degree $q^r \deg(a)$.

**Remark 2.3.** The pull-back of a Drinfeld module $(L, \varphi, S)$ along a morphism $\text{Spec}(K) \to S$ for some field $K$ is a Drinfeld module over $K$ in the sense of Definition 2.1.

If $S = \text{Spec}(B)$ and $L$ is isomorphic to $\mathbb{G}_{a,B}$, then we simply write $\varphi$ instead of $(L, \varphi)_S$. The morphism $\gamma_S$ is called the **characteristic** of $(L, \varphi)_S$. An ideal $n \subset A$ is called **away from the characteristic** if $V(n)$ is disjoint with the image of $\gamma_S$.

**Definition 2.4.** A morphism $\xi$ of Drinfeld modules over $S$

$$\xi : (L, \varphi)_S \to (M, \psi)_S$$

is a map $\xi \in \text{Hom}_{\mathbb{F}_q}(L, M)$ such that $\xi \circ \varphi_a = \psi_a \circ \xi$ for all $a \in A$. A morphism $\xi$ is called an **isomorphism**, if it gives an isomorphism between the line bundles $L$ and $M$ over $S$.

An **isogeny** of Drinfeld modules is a finite morphism.

**Remark 2.5.** An isogeny exists only between Drinfeld modules of the same rank.
Remark 2.6. A Drinfeld module \((L, \varphi)_S\) of rank \(r\) and a morphism
\[ f : T \longrightarrow S \]
give by pull-back rise to a Drinfeld module \((f^*L, f^*\varphi)_T\) over \(T\) of rank \(r\).

2.3. Level structures. For any non-zero \(f \in A\), let
\[ \varphi[f] := \ker(\varphi_f : L \longrightarrow L). \]
This is a finite, flat group scheme over \(S\), namely
\[ \ker(\varphi_f : L \longrightarrow L) = L \times_S S \]
where the fibre product is taken over \(\varphi_f : L \longrightarrow L\) and the unit-section \(e : S \longrightarrow L\) of the group scheme \(L\). If \(n \subset A\) is any non-zero ideal, then
\[ \varphi[n] = \prod_{f \in n} \varphi[f] \]
where the product is the fibre product over \(L\). The scheme \(\varphi[n]\) is étale over \(S\) if and only if \(n\) is away from the characteristic.

Remark 2.7. Let \(K\) be an algebraically closed \(A\)-field. If \(S = \text{Spec}(K)\), then
\[ \varphi[n] = \text{Spec}(K[X]/(\varphi_f(X), \varphi_g(X))) \]
with \(n = (f, g)\).

Definition 2.8. Suppose that \(n\) is away from the characteristic, then a level \(n\)-structure \(\lambda\) over \(S\) of \((L, \varphi)_S\) is an \(A\)-isomorphism
\[ \lambda : (A/n)^r \xrightarrow{\sim} \varphi[n](S). \]

Remark 2.9. If \(n\) is not away from the characteristic, then the definition of a level \(n\)-structure is more involved. One can view \(\varphi[n]\) as an effective Cartier divisor on \(L\). By definition a level \(n\)-structure of \((L, \varphi)_S\) is an \(A\)-homomorphism
\[ \lambda : (A/n)^r \longrightarrow L(S) \]
which induces an equality of Cartier-divisors:
\[ \sum_{\alpha \in (A/n)^r} \lambda(\alpha) = \varphi[n]. \]

We will not expand on this. In this paper we restrict to the cases for which Definition 2.8 is enough.

Let the triple \((L, \varphi, \lambda)_S\) denote a Drinfeld module of rank \(r\) over \(S\) with level \(n\)-structure \(\lambda\).

Definition 2.10. A morphism between two triples \((L, \varphi, \lambda)_S\) and \((M, \psi, \mu)_S\) is a morphism \(\xi : (L, \varphi)_S \longrightarrow (M, \psi)_S\) of Drinfeld modules over \(S\) such that \(\xi(S) \circ \lambda = \mu\) where \(\xi(S) : L(S) \longrightarrow M(S)\) is induced by \(\xi\). A morphism is called an isomorphism if \(\xi\) is an isomorphism of Drinfeld modules.
3. The moduli problem

Let $n \subset A$ be a non-zero, proper ideal. In his original paper, Drinfeld considers the following moduli problem for Drinfeld modules. Let

$$\mathcal{F}^r(n) : A - \text{Schemes} \to \text{Sets}$$

be the functor which associates to each $A$-scheme $S$ the set of isomorphism classes of Drinfeld modules over $S$ of rank $r$ with level $n$-structure over $S$. Drinfeld showed the following; cf. Proposition 5.3 and its corollary in [5].

**Theorem 3.1** (V.G. Drinfeld). If $n \subset A$ is an ideal divisible by at least 2 distinct primes, then there exists a fine moduli space $M^r(n) \to \text{Spec}(A)$ representing the moduli problem $\mathcal{F}^r(n)$. Moreover, this scheme has the following properties:

(i) $M^r(n)$ is affine and smooth of dimension $r$;
(ii) $M^r(n) \to \text{Spec}(A)$ is smooth of relative dimension $r - 1$ over $\text{Spec}(A) - V(n)$.

For arbitrary non-zero ideals $n \subset A$, the functor $\mathcal{F}^r(n)$ has in general only a coarse moduli scheme. This coarse moduli scheme will also be denoted by $M^r(n)$. We recall here briefly the construction of this scheme; cf. [20] for a nice exposition of this. Let $n, m \subset A$ be ideals such that $nm$ is divisible by at least two distinct prime ideals, then by Theorem 3.1 there exists an affine scheme $M^r(nm)$ representing the moduli functor $\mathcal{F}^r(nm)$. Let $(L, \varphi, \lambda)_S$ be a Drinfeld module of rank $r$ with full level $mn$-structure $\lambda$ over an $A$-scheme $S$. On these triples the group $\text{Gl}_r(A/rn)$ acts by

$$\sigma(L, \varphi, \lambda) := (L, \varphi, \lambda \circ \sigma) \quad \text{for all} \quad \sigma \in \text{Gl}_r(A/n).$$

Note that $\text{Gl}_r(A/n)$ is isomorphic to the kernel of the mod $m$ reduction map

$$\text{Gl}_r(mn) \to \text{Gl}_r(m).$$

This induces an action of $\text{Gl}_r(A/n)$ on $M^r(mn)$. The coarse moduli scheme of $\mathcal{F}^r(m)$ is defined as

$$M^r(m) := M^r(mn)/\text{Gl}_r(A/n).$$

This quotient exists, because $\text{Gl}_r(A/n)$ is finite. It is, however, not obvious that this scheme is coarse for the given moduli problem. See [20] for a proof of the coarseness of the scheme. The scheme $M^r(m)$ does not depend on the choice of $n$.

### 3.1. Actions on $M^r(n)$

Let $\hat{A} = \varinjlim A/n$, and let $\hat{A}_f = \hat{A} \otimes A \bar{K}_A$. In this subsection we describe the natural action of $\hat{A}_f \cdot \text{Gl}_r(\hat{A})$ on $M^r(n)$; cf. [5] 5D. Using this action, we can define the action of $\text{Gl}_r(A/n)$ and $\text{Cl}(A)$ on $M^r(n)$. To keep this paper self-contained, we recall here the treatment given in Section 3.5 of [16], where the reader can find proofs and details.

A **total level structure** of a Drinfeld module $(L, \varphi)_S$ is a homomorphism

$$\kappa : (K_A/A)^r \to L(S)$$

where $L(S)$ is the Drinfeld module associated to $S$. For arbitrary non-zero ideals $n \subset A$, the functor $\mathcal{F}^r(n)$ has in general only a coarse moduli scheme. This coarse moduli scheme will also be denoted by $M^r(n)$. We recall here briefly the construction of this scheme; cf. [20] for a nice exposition of this. Let $n, m \subset A$ be ideals such that $nm$ is divisible by at least two distinct prime ideals, then by Theorem 3.1 there exists an affine scheme $M^r(nm)$ representing the moduli functor $\mathcal{F}^r(nm)$. Let $(L, \varphi, \lambda)_S$ be a Drinfeld module of rank $r$ with full level $mn$-structure $\lambda$ over an $A$-scheme $S$. On these triples the group $\text{Gl}_r(A/rn)$ acts by

$$\sigma(L, \varphi, \lambda) := (L, \varphi, \lambda \circ \sigma) \quad \text{for all} \quad \sigma \in \text{Gl}_r(A/n).$$

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such that its restriction to \((n^{-1}A/A)^r\) defines a level \(n\)-structure. Let
\[
M^r := \lim \downarrow M^r(n)
\]
where \(n\) runs through the non-zero ideals of \(A\). This is an affine scheme, and
\(M^r\) represents the functor which associates to each \(A\)-scheme \(S\) the set of
isomorphy classes of Drinfeld modules with a total level structure over \(S\).

There is a natural action of \(GL_r(\hat{A}_f)\) on \(M^r\), which is defined as follows. Let
\(S\) be an \(A\)-scheme, and let \((L, \varphi, \kappa)_S\) be a Drinfeld module with a total level
structure over \(S\). Let \(\sigma \in GL_r(\hat{A}_f)\) such that the entries of \(\sigma\) are elements of
\(\hat{A}\), then \(\sigma\) gives rise to a map
\[
\sigma : (K_A/A)^r \longrightarrow (K_A/A)^r.
\]
Let \(H_{\sigma}\) denote the kernel of \(\sigma\). The kernel \(H_{\sigma}\) gives rise to a finite subgroup
scheme of \(L\). We can divide out the pair \((L, \varphi)_S\) by this subgroup scheme.
This gives us a pair \((L', \varphi')_S\). The following diagram equips the pair \((L', \varphi')_S\)
with a total level structure \(\kappa'\):

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_{\sigma} & \longrightarrow & (K_A/A)^r & \longrightarrow & (K_A/A)^r & \longrightarrow & 0 \\
& & \downarrow \kappa & & \sigma & & \downarrow \kappa' & & \\
0 & \longrightarrow & H_{\sigma}(S) & \longrightarrow & L(S) & \longrightarrow & L'(S).
\end{array}
\]

If \(\sigma\) comes from an element in \(A' \setminus \{0\}\), then its action is trivial. This implies
that we get an action of \(GL_r(\hat{A}_f)/K_A^*\) on \(M^r(S)\). As this action is functorial
in \(S\), this defines an action of \(GL_r(\hat{A}_f)/K_A^*\) on \(M^r\).

For the moduli scheme \(M^r(n)\) we have \(M^r(n) = \Gamma(n) \setminus M^r\) with
\[
\Gamma(n) := \ker(GL_r(\hat{A}) \longrightarrow GL_r(A/n)).
\]
The restriction of the universal triple \((L, \varphi, \kappa)\) on \(M^r\) to \(M^r(n)\) gives the
universal pair \((\varphi, \lambda)\) on \(M^r(n)\) (Recall that the line bundle of the universal
Drinfeld module on \(M^r(n)\) is trivial.) As \(\hat{A}_f^* \cdot GL_r(\hat{A})\) commutes with \(\Gamma(n)\)
in \(GL_r(\hat{A}_f)\), it follows that the action of \(GL_r(\hat{A}_f)\) on \(M^r\) defines an action of
\(\hat{A}_f^* \cdot GL_r(\hat{A})\) on \(M^r(n)\). The normal subgroup \(K_A^* \cdot \Gamma(n) \subset \hat{A}_f^* \cdot GL_r(\hat{A})\) acts
trivially on \(M^r(n)\). Let
\[
G := \hat{A}_f^* \cdot GL_r(\hat{A})/K_A^* \cdot \Gamma(n).
\]
As \(\hat{A}_f^*/K_A^* \cdot \hat{A}^* \cong \text{Cl}(A)\), it is not difficult to see that we have the following
exact sequence
\[
0 \longrightarrow GL_r(A/n)/\mathbb{F}_q^* \longrightarrow G \longrightarrow \text{Cl}(A) \longrightarrow 0.
\]

To describe the action of \(GL_r(A/n)/\mathbb{F}_q^*\), let \(\sigma \in GL_r(\hat{A})\) and let \(\hat{\sigma}\) be the
image of \(\sigma\) under the reduction map \(GL_r(\hat{A}) \longrightarrow GL_r(A/n)/\mathbb{F}_q^*\). Then
\[
\sigma : (\varphi, \lambda) \mapsto (\varphi, \lambda \circ \hat{\sigma}^{-1}).
\]
Remark 3.2. However, in the sequel we prefer to drop the inverse. If we talk
about the action of $\sigma \in \text{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$ on $(\varphi, \lambda)$, then we mean the action
given by
$$\sigma : (\varphi, \lambda) \mapsto (\varphi, \lambda \circ \sigma).$$
Consequently, $\text{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$ acts on the right of $M^r(\mathfrak{n})$ and not on the left.

Let $m \in \hat{A} \cap \mathbb{A}^*_f$, then $m$ defines a unique ideal $\mathfrak{m} = (m) \cap A \subset A$. We
suppose that $\mathfrak{m}$ is a non-zero, proper ideal which is relatively prime to $\mathfrak{n}$.
Let $I_r$ denote the identity element in $\text{Gl}_r(\mathbb{A}_f)$, and let $\sigma = m \cdot I_r$. We
describe the action of $\sigma$ on $M^r$. Clearly, $H_\sigma = (m^{-1} A/A)^r$, and $H_\sigma$ maps
to $\varphi[\mathfrak{m}](M^r)$ under $\kappa$. This means that the isogeny
$$\xi_m : (L, \varphi) \mapsto (L', \varphi')$$
defined by $\sigma$ has kernel $\varphi[\mathfrak{m}]$. The total level structure $\kappa'$ is given by
$$\kappa' = \xi_m \circ \kappa \circ m^{-1}.$$

Let $\varphi'$ denote the restriction of $(L', \varphi')$ to $M^r(\mathfrak{n})$. Let $\mathfrak{m}'$ denote the image
of $m$ under the reduction map $\hat{A} \to A/\mathfrak{n}$. As $\mathfrak{m} + \mathfrak{n} = A$, we see that
$\mathfrak{m}' \in (A/\mathfrak{n})^*$. Let $\varphi$ be the restriction of $(L', \varphi')$ to $M^r(\mathfrak{n})$, then the action
of $m$ on the universal pair $(\varphi, \lambda)$ on $M^r(\mathfrak{n})$ is given by
$$m : (\varphi, \lambda) \mapsto (\varphi', \xi_m \circ \lambda \circ \mathfrak{m}'^{-1}).$$
This describes the action of $m$ on $M^r(\mathfrak{n})$.

Let $\mathfrak{m} \subset A$ be a non-zero ideal relatively prime to $\mathfrak{n}$, i.e., $\mathfrak{m} + \mathfrak{n} = A$. Choose
$m \in \hat{A}$ such that $(m) = \mathfrak{m} A$ and $m \equiv 1 \mod \mathfrak{n}$. We define the action of $\mathfrak{m}$
on $(\varphi, \lambda)$ to be the action of $(m)$:
$$\mathfrak{m} : (\varphi, \lambda) \mapsto (\varphi', \xi_m \circ \lambda).$$
This action is well-defined: the chosen element $m$ is unique up to an element
$\alpha \in \hat{A}^*$ with $\alpha \equiv 1 \mod \mathfrak{n}$. For such an element $\alpha$ we have $\alpha I_r \in \Gamma(\mathfrak{n})$.
Consequently, $m \cdot I_r$ and $\alpha m \cdot I_r$ give the same element in $G$.

Using this, we can define the action of $\text{Cl}(A)$ on $M^r(\mathfrak{n})$. First note that
by Lemma 6.4 we can represent every class in $\text{Cl}(A)$ by a non-zero ideal $\mathfrak{m}$
with $\mathfrak{m} + \mathfrak{n} = A$. Namely, suppose that $\mathfrak{m}$ and $\mathfrak{m}'$ are both non-zero ideals relatively prime to $\mathfrak{n}$
which represent the same class in $\text{Cl}(A)$, then there is
an element $x \in K^*_A$ with $\mathfrak{m} = x \mathfrak{m}'$. Let $m, m' \in \hat{A}$ be elements which define
the action of $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Then there is an element $\alpha \in \hat{A}^*$ with
$\alpha \equiv 1 \mod \mathfrak{n}$ such that $m' = x m$ with $x \alpha \cdot I_r \in K^*_A \cdot \Gamma(\mathfrak{n})$. Therefore, $(m)$
and $(m')$ give the same element in $G$.

Remark 3.3. These considerations also imply that
$$G \cong \text{Cl}(A) \times \text{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*. $$

Remark 3.4. Let us once more stretch that we use the convention that the
action of $\sigma \in \text{Gl}_r(A/\mathfrak{n})$ on $M^r(\mathfrak{n})$ is given by
$$\sigma : (\varphi, \lambda) \mapsto (\varphi, \lambda \circ \sigma).$$
So $G = \text{Cl}(A) \times \text{Gl}_r(A/\mathfrak{n})/\mathbb{F}_q^*$ acts on $M^r(\mathfrak{n})$ on the right.
3.2. Assumptions in this paper. In this paper we will make the following two assumptions.

(1) Throughout this paper we will assume that \( n = (f) \) is a non-zero, proper, principal ideal. This simplifies the description of the Tate-Drinfeld module. Dropping this assumption does not seem to give rise to different results.

If \( m \subset A \) is a non-zero proper ideal containing \( f \), then by the previous, we see that \( M^r(n) = M^r(f)/G \) where \( G \) is given by dividing out the action of the kernel

\[
\ker(\text{Gl}_r(A/fA) \to \text{Gl}_r(A/n)).
\]

(2) We will not consider the moduli problem over \( A \)-schemes, but over \( A_f \)-schemes \( S \), i.e., \( f \) is invertible in \( S \). This implies that \( f \) is away from the characteristic of \((L, \varphi)_S\). This assumption is used because the Weil pairing plays an important role in our description, and the Weil pairing is in [12] only defined for \( f \)-torsion which is away from the characteristic.

This assumption implies that a level \( f \)-structure is an isomorphism

\[
\lambda : (A/fA)^r \sim \varphi[f](S).
\]

So in this paper we will be considering the moduli problem

\[
\mathcal{F}^r(f) : A_f - \text{SCHEMES} \to \text{SETS}
\]

which associates to each \( A_f \)-scheme \( S \), the set of isomorphy classes of Drinfeld modules of rank \( r \) with full level \( f \)-structure over \( S \). We write \( M^r(f) \) for the scheme which represents \( \mathcal{F}^r(f) \). It follows from the proof of Theorem 3.1 that the moduli scheme \( M^r(f) \) is a fine moduli scheme if \( f \neq 1 \).

Throughout this paper we will write \( \text{Spec}(R) = M^1(f) \). The ring \( R \) is regular and \( M^1(f) \) is connected. In fact, \( R \) is the integral closure of \( A_f \) in a field extension of \( K_A \). The Galois group of \( K_R/K_A \) is the group

\[
G \cong (A/fA)^* / \mathbb{F}_q^* \times \text{Cl}(A)
\]

that we discussed above; cf. Section 8 in [5].

4. The Weil pairing on the modular schemes.

In this section we will show that the Weil pairing for Drinfeld modules over an \( A_f \)-field \( K \) as defined in the previous paper gives rise to the following theorem:

**Theorem 4.1.** The Weil pairing induces an \( A_f \)-morphism

\[
w_f : M^r(f) \to M^1(f).
\]

The Weil pairing is \( \text{Cl}(A) \times \text{Gl}_r(A/fA) \)-equivariant.

Let \((\varphi, \lambda)\) be a Drinfeld module \( \varphi \) of rank \( r \) over \( K \) with level \( f \)-structure. The Weil pairing is an \( A/fA \)-isomorphism

\[
w_f : \wedge^r \varphi[f](K) \sim \psi[f](K) \otimes_A \Omega^{\otimes r-1}_f.
\]
It is unique up to a unique isomorphism of $\psi$. Once and for all we fix a generator $\omega$ of the $A/fA$-module $\Omega_f$. This gives an $A/fA$-isomorphism

$$w_f : \wedge^r \varphi[f](K) \xrightarrow{\sim} \psi[f](K).$$

The level $f$-structure $\lambda$ induces a canonical isomorphism

$$\wedge^r \lambda : \wedge^r (A/fA)^r \xrightarrow{\sim} \wedge^r \varphi[f](K).$$

Because $\wedge^r (A/fA)^r$ is canonically isomorphic to $A/fA$, $\psi$ comes equipped with a level $f$-structure $\mu$ over $K$ via the following commutative diagram:

$$\begin{array}{ccc}
\wedge^r (A/fA)^r & \xrightarrow{\wedge^r \lambda} & \wedge^r \varphi[f](K) \\
\uparrow & & \downarrow w_f \\
A/fA & \xrightarrow{\mu} & \psi[f](K).
\end{array}$$

Note that if $\xi$ is an isomorphism between $\psi$ and $\psi'$, then also the pairs $(\psi, \mu)$ and $(\psi', \mu')$ are isomorphic via $\xi$. Here $\mu$ and $\mu'$ are defined by the previous diagram by $\psi$ and $\psi'$ respectively. So the pair $(\psi, \mu)$ is unique up to isomorphy. These considerations show the following:

**Lemma 4.2.** The Weil pairing gives for all $A_f$-fields $K$ rise to a map

$$w_K : M^r(f)(K) \longrightarrow M^1(f)(K).$$

This map depends functorially on $K$.

**Proof.** The construction of the map $w_K$ is described above. It associates to each isomorphism class $(\varphi, \lambda)$ of rank $r$ over $K$ a unique isomorphism class $(\psi, \mu)$ of rank 1 over $K$. That this map depends functorially on $K$ follows immediately from the construction in terms of $A$-motives. □

From this lemma, we proceed as follows to prove the existence of the map $w_f$.

**Proof of Theorem 4.1.** Let $(\varphi, \lambda)$ be the universal pair over $M^r(f)$. The moduli scheme $M^r(f)$ is affine and regular over $\text{Spec}(A_f)$. So we may write

$$M^r(f) \cong \prod_{i=0}^n \text{Spec}(S_i)$$

such that each $S_i$ is an integrally closed domain of relative dimension $r - 1$ over $A_f$. Moreover, from Drinfeld’s description we know that

$$M^1(f) = \text{Spec}(R)$$

where $R$ is an integrally closed domain. Let $K_{S_i}$ be the quotient field of $S_i$. Lemma 4.2 gives rise to a unique isomorphy class

$$(\psi, \mu) \in M^1(f)(\prod_i K_{S_i}).$$

This means that there exists a unique $A_f$-ring homomorphism

$$h : R \longrightarrow \prod_j K_{S_j}.$$
Hence, the theorem follows if we can show that \( h(R) \subseteq \prod S_j \).

Let \( p \in \text{Spec}(S_j) \) be a closed point of height 1. By Lemma \[3.3\] it follows that there is a map \( \text{Spec}(S_{j,p}) \to M^1(f) \), inducing the map

\[ \text{Spec}(K_{S_j}) \to M^1(f). \]

Consequently,

\[ h(R) \cap K_{S_j} \subseteq S_{j,p}. \]

As \( S_j = \bigcap_{p \in \text{Spec}(S_j)} S_{j,p} \) where the intersection runs over all primes of height 1 in \( S_j \), one has \( h(R) \cap K_{S_j} \subseteq S_j \).

For the \( \text{Cl}(A) \times \text{Gl}_r(A/fA) \)-equivariance of \( w_f \), note that the \( \text{Gl}_r(A/fA) \)-equivariance is obvious. Let \( m \subset A \) be a non-zero, proper ideal with \( f \notin m \) representing an element in \( \text{Cl}(A) \). For the \( \text{Cl}(A) \)-equivariance we recall its definition in the previous section. Using the notations of the previous section, we see that the action of \( m \) on \( (\varphi, \lambda) \) is given by

\[ (\varphi, \lambda) \mapsto (\varphi', \xi_m \circ \lambda), \]

with \( \xi_m \varphi_a = \varphi'_a \xi_m \) for all \( a \in A \). Let \( (\psi, \mu) \) be the image of \( (\varphi, \lambda) \) under the Weil pairing, and let \( (\psi', \mu') \) be the image of \( (\varphi', \xi_m \circ \lambda) \) under \( w_f \). Let \( F = \prod S_j \). The isogeny \( \xi_m \) induces an isogeny \( \zeta_m : \psi \to \psi' \). The kernel of \( \zeta_m \)

\[ \ker(\zeta_m)(F) = \ker(\xi_m)(F) = \ker \varphi (m)(F) \cong \varphi |m|(F). \]

Therefore, the action of \( \zeta_m \) on \( \psi \) coincides with the action of \( m \) on \( \psi \). So we have

\[ m : (\psi, \mu) \mapsto (\psi', \zeta_m \circ \mu) = (\psi', \mu'). \]

\[ \square \]

**Lemma 4.3.** Let \( S \) be a regular local \( A_f \)-ring, let \( K_S \) be its quotient field, and let \( (\varphi, \lambda) \in M^r(f)(S) \). The unique class \( (\psi, \mu) \in M^1(f)(K_S) \) associated to \( (\varphi, \lambda) \) by the Weil pairing comes from a unique class \( (\psi', \mu') \in M^1(f)(S) \) via the canonical embedding \( S \to K_S \).

**Proof.** Via the ring homomorphism \( S \to K_S \) we may view the pair \( (\varphi, \lambda) \) over \( K_S \), and we can associate via the Weil pairing a pair \( (\psi, \mu) \) of rank 1 over \( K_S \). We want to prove that there is a representing pair in the isomorphism class of \( (\psi, \mu) \) which is defined over \( S \).

Let \( V \) be the set of all height one primes of \( S \). The ring \( S \) is a UFD; cf. Theorem 20.3 in \[17\]. Consequently, every \( p \in V \) is of the form \( p = (h_p) \) for some irreducible \( h_p \in S \); cf. Theorem 20.1 in \[17\]. Let \( v_p \) denote the valuation at \( p \). The invertible elements of \( S \) are given by

\[ S^* = \{ s \in S \mid v_p(s) = 0 \text{ for all } p \in V \}. \]

Let \( S_p \) denote the local ring of \( S \) at \( p \). This is a discrete valuation ring. As the \( f \)-torsion of \( \psi \) is \( K_S \)-rational, it follows that \( \psi \) has good reduction at every \( p \in V \). This implies that for all \( p \in V \) there is an element \( x_p \in K_S \) such that \( x_p \psi x_p^{-1} \) is a Drinfeld module of rank 1 defined over \( S_p \). In fact, we may assume \( x_p = h_p^{m_p} \) for some \( m_p \in \mathbb{Z} \).

There are only finitely many \( m_p \neq 0 \), as we show below. So we can define

\[ x = \prod_{p \in V} h_p^{m_p}. \]

The pair \( (x \psi x^{-1}, x \mu) \) is defined over \( S \). So this is the pair that we are looking for.
To see that there are only finitely many \( m_p \neq 0 \), let \( a \in A \setminus \mathbb{F}_q \) and consider the leading coefficient \( c \) of \( \psi_a \). Under the isomorphism \( x_p \), the leading coefficient becomes \( x_p^{-\deg(a)} c \). As \( v_p(c) = 0 \) for all but finitely many \( p \)'s, it follows that \( m_p = 0 \) for all but finitely many \( p \)'s.

\[ \square \]

**Proposition 4.4.** The morphism \( w_f \) induces the maps \( w_K \) for any \( A_f \)-field \( K \), where \( w_K \) is as in Lemma 4.3.

**Proof.** By the functoriality in \( K \) of the maps \( w_K \), it suffices to prove the statement for algebraically closed fields \( \overline{K} \). So let \( \overline{K} \) be algebraically closed and let

\[ \zeta : \text{Spec}(\overline{K}) \to M^r(f) \]
be a geometric point of \( M^r(f) \). If \( \zeta \) is a generic point of one of the connected components of \( M^r(f) \), then \( w_f \) induces \( w_{\overline{K}} \) by construction. If \( \zeta \) is a closed point we have to do something more. Clearly, \( \zeta \) factors over \( \text{Spec}(k_\zeta) \) where \( k_\zeta \) is the residue field at (the image of) \( \zeta \). To see that \( w_{k_\zeta} \) and consequently \( w_{\overline{K}} \) is induced by \( w_f \), we have to dive into the language of A-motives a little; cf. [12].

Let \( V = \mathcal{O}_\zeta \), then \( V \) is a regular local \( A_f \)-ring and let \((\varphi, \lambda)\) be defined over \( V \) of rank \( r \). Let \( K_V \) be the quotient field of \( V \). Then the construction of A-motives associates to \( \varphi \) the Drinfeld module \( \psi \) of rank 1 for which

\[ M(\psi) \cong \wedge^r M(\varphi). \]

By Lemma 4.3 the pair \((\psi, \mu)\) is also defined over \( V \). Because \( \varphi \) is defined over \( V \), it makes sense to consider \( M^0(\varphi) = V \{ \tau \} \) with the obvious \( V \{ \tau \} \otimes_{\mathbb{F}_q} A \)-action such that

\[ M^0(\varphi) \otimes_V K_V = M(\varphi). \]

Similarly, we can define \( M^0(\psi) \) because \( \psi \) is defined over \( V \). Clearly,

\[ \wedge_{K_V \otimes A} M(\varphi) \cong K_V \otimes_V \wedge_{V \otimes A} M^0(\varphi). \]

Consequently, the \( K_V \{ \tau \} \otimes A \)-isomorphism

\[ M(\psi) \cong \wedge^r M(\varphi) \]

comes from a \( V \{ \tau \} \otimes A \)-isomorphism

\[ M^0(\psi) \cong \wedge_{V \otimes A} M^0(\varphi). \]

This construction can be reduced modulo the maximal of \( V \). This gives us the construction of \( w_{k_\zeta} \). Therefore, \( w_f \) induces \( w_{\overline{K}} \).

\[ \square \]

5. Drinfeld Modules of Rank 2 with Stable Reduction of Rank 1

For this section we fix the following notation. Let \( V \) be a complete discrete valuation ring which is also an \( A_f \)-algebra. Let \( K_V \) be the quotient field of \( V \), and let \( \pi \in V \) be a generator of the maximal ideal of \( V \). Let \( v(x) \) denote the \( \pi \)-valuation of \( x \) for every \( x \in V \).

**Definition 5.1.** Let \( \varphi \) be a Drinfeld module of rank \( r \) over \( K_V \), then \( \varphi \) has stable reduction at \( v \) of rank \( r' \) if \( \varphi \) is isomorphic over \( K_V \) to a Drinfeld module \( \varphi' \) over \( K_V \) such that for all \( a \in A \) each coefficient \( \beta_i(a) \) of the sum \( \varphi'_a = \sum \beta_i(a) r^i \) is an element of \( V \) and the reduction \( \varphi' \mod \pi V \) is a Drinfeld module of rank \( r' \) over \( V/\pi V \).
The Drinfeld module $\varphi$ has potentially stable reduction at $v$ of rank $r'$ if there is a finite field extension $L$ of $K_V$ and a valuation $w$ of $L$ extending $v$ such that $\varphi$ has stable reduction at $w$ of rank $r'$.

Let $\varphi$ be a Drinfeld module of rank 2 over $K_V$ with full level $f$-structure $\lambda$ over $K_V$ such that $\varphi$ has potentially stable reduction of rank 1. The goal of this section is Theorem \ref{thm:classification}, which describes the pairs $(\varphi, \lambda)$ in terms of Drinfeld modules of rank 1 and lattices.

**Lemma 5.2.** Let $\varphi$ be a Drinfeld module with $K_V$-rational $f$-torsion. If $\varphi$ has potentially stable reduction at $(\pi)$, then $\varphi$ has stable reduction at $(\pi)$.

**Proof.** As $\varphi$ has potentially stable reduction, there is an element $k \in \mathbb{Q}$ such that for all $x \in K_V^{\text{sep}}$ with $v(x) = k$ we have the following: \( \tilde{\varphi} := x \varphi x^{-1} \) is a Drinfeld module, \( \tilde{\varphi} a \) has all coefficients in $V$ for all $a \in A$, and $\tilde{\varphi}$ mod $\pi V$ is a Drinfeld module over $V/\pi V$. Cf. Section 4.10 in \cite{8}. As $f \in V^*$, it is not difficult to see that $k$ is the smallest slope of the Newton polygon of $\varphi f(X)$. Therefore,

\[
\begin{align*}
    k &= \max \{v(\alpha) : \alpha \in \varphi[f](K_V) \setminus \{0\}\}.
\end{align*}
\]

As the $f$-torsion of $\varphi$ is $K_V$-rational, we have $k \in \mathbb{Z}$. Therefore, we may choose $x \in K_V$. \hfill \Box

To abbreviate notation, we introduce the following two properties $P$ and $P'$.

- $P(\chi)$ : $\chi$ has stable reduction of rank 1.
- $P'(\chi)$ : $\chi$ has stable reduction of rank 1, $\chi a$ has all coefficients in $V$ for all $a \in A$ and $\chi[f](V) \cong A/fA$.

Suppose that $\kappa$ is a level $f$-structure of $\chi$ over $K_V$. If $r = 1$, then $P'(\chi)$ implies that $(\chi, \kappa)$ is defined over $V$.

By Lemma \ref{lem:classification}, we have $P(\varphi)$ for the pair $(\varphi, \lambda)$. Therefore, we may assume that $\varphi_a$ has all its coefficients in $V$ for all $a \in A$. Moreover, we have that the smallest slope of the Newton polygon of $\varphi f$ equals $v(\varphi f) = 0$. Consequently, $\varphi[f](V) \cong A/fA$. So in the isomorphy class of $(\varphi, \lambda)$, there is a representing element $(\varphi, \lambda)$ with $P'(\varphi)$ and this element is unique up to $V^*$. In fact,

\[
\{(\varphi, \lambda)_{K_V} \text{ with } P'(\varphi)\}/V^* \cong \{(\varphi, \lambda)_{K_V} \text{ with } P(\varphi)\}/K_V.
\]

Note that the Weil pairing equips $V$ with an $R$-structure. The isomorphy class of $(\varphi, \lambda)$ is induced by an $A_f$-morphism $\text{Spec}(K_V) \to M^2(f)$. Composing this morphism with $w_f$ gives rise to an $A_f$-linear ring homomorphism $R \to K_V$. As $R$ is integral over $A_f$, it follows that this ring homomorphism gives an $A_f$-linear ring homomorphism $h : R \to V$.

5.1. **Drinfeld’s bijection without level structure.** To classify the isomorphy classes $(\varphi, \lambda)$ with stable reduction of rank 1, we recall Drinfeld’s classification of Drinfeld modules of rank 2 with potentially stable reduction of rank 1; cf. Proposition 7.2 in \cite{5}.

An $A$-lattice of rank 1 in $K_V^{\text{sep}}$ is a projective $A$-module of rank 1 which lies discretely in $K_V^{\text{sep}}$ and which is invariant under the action of $G_{K_V}$ :=
Gal($K_V^{\text{sep}}/K_V$). Two $A$-lattices $\Lambda_1$ and $\Lambda_2$ are called isomorphic if there is an element $x \in (K_V^{\text{sep}})_V$ such that $xA = \Lambda_2$. Then Drinfeld’s result states the following:

**Theorem 5.3** (V.G. Drinfeld). There is a bijection between the set of isomorphism classes over $K_V$ of Drinfeld modules of rank 2 over $V$ with potentially stable reduction of rank 1 and the set of isomorphism classes over $K_V$ of pairs $(\psi, \Lambda)$, where $\psi$ is a Drinfeld module of rank 1 over $V$ and $\Lambda$ is an $A$-lattice of rank 1 inside $K_V^{\text{sep}}$.

**Sketch of the proof.** Applying Proposition 5.2 in [5] to the ring $V_n = V/(v)^n$ with $n \in \mathbb{Z}_{\geq 1}$ gives us the existence of unique elements $s_n \in V_n\{\tau\}$ such that $s_n \varphi_a s_n^{-1}$ is in standard rank 1 form over $V_n$. Moreover, each $s_n$ has the form

$$s_n = 1 + \sum_{i \geq 1} v_i \tau^i \quad \text{with} \quad v_i \in (v).$$

Let $s = \lim_{\leftarrow} s_n$, then $s$ is an element in $V\{\tau\}$, the set of skew formal power series in $\tau$ over $V$. In the proof of Proposition 7.2 in [5], Drinfeld shows that the homomorphism $s$ is in fact analytic. One has by construction

$$s = 1 + \sum_{i \geq 1} v_i \tau^i \quad \text{with} \quad v_i \in (v).$$

This implies both that $\Lambda := \ker(s)$ is contained in $K_V^{\text{sep}}$. Moreover, each element in $\Lambda \setminus \{0\}$ has strictly negative valuation.

Let $\psi' = s \psi s^{-1}$, then $\psi' \mod (v) = \psi$. We get the following diagram, which is commutative for all $a \in A$:

$$
\begin{array}{c}
0 \longrightarrow \Lambda \longrightarrow K_V^{\text{sep}} \xrightarrow{e_A} K_V^{\text{sep}} \\
\downarrow \psi_a \downarrow \psi_a \downarrow \psi_a \\
0 \longrightarrow \Lambda \longrightarrow K_V^{\text{sep}} \xrightarrow{e_A} K_V^{\text{sep}},
\end{array}
$$

where

$$e_A(z) = z \prod_{\alpha \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{\alpha}\right) = s(z).$$

Let $a \in A \setminus \mathbb{F}_q$, then $\psi_a^{-1} \Lambda$ is mapped surjectively to $\varphi[a]$, hence $\psi_a^{-1} \Lambda/\Lambda \cong (A/aA)^2$ as $A$-module. On the other hand, the kernel of the surjective map

$$\psi_a^{-1} \Lambda/\Lambda \longrightarrow \Lambda/\psi_a \Lambda$$

is isomorphic to $\psi[a](K_V^{\text{sep}})$. So we may conclude that $\Lambda/\psi_a \Lambda \cong A/aA$. This implies that $\Lambda$ is a projective $A$-module of rank 1. We already saw that $\Lambda$ consists of elements with strictly negative valuation, hence $\Lambda$ lies discretely in $K_V^{\text{sep}}$. Finally, for any element $\sigma \in G_{K_V}$, we have $\sigma \circ s(z) = s \circ \sigma(z)$, hence $\Lambda$ is $G_{K_V}$-invariant. We conclude that $\Lambda$ is an $A$-lattice of rank 1 in $K_V^{\text{sep}}$. \hfill $\square$
5.2. The bijection with level $f$-structure. Let $(\psi^{un}, \mu^{un})$ be the universal pair of rank 1 defined over $R$. We will assume that $\mu^{un}(1) = 1$, which is possible because $f$ is invertible in $R$. Drinfeld’s construction in Theorem 5.3 lifts the rank 1 Drinfeld module $\varphi \mod \pi V$ to a unique Drinfeld module $\psi$ of rank 1 defined over $V$. Also, the $f$-torsion of $\psi$ is $V$-rational. We would like to equip $\psi$ with a natural level $f$-structure $\mu$ which comes from $\lambda$. For this we use the ring homomorphism $h: R \rightarrow V$ which arises from the Weil pairing.

Suppose that $\psi$ is equipped with a level $f$-structure $\tilde{\mu}$, then the isomorphy class of $(\psi, \tilde{\mu})$ comes from an $A_f$-linear ring homomorphism $\tilde{h}: R \rightarrow V$, i.e., there is a unique element $v \in V^*$ such that $\tilde{h}((\psi^{un}, \mu^{un})) = (v \psi v^{-1}, v \tilde{\mu})$.

The pair $(\tilde{h}, v)$ uniquely determines $(\psi, \tilde{\mu})$.

As $R$ is integral over $A_f$, there exists an $A_f$-automorphism $g_{\tilde{\mu}}: R \rightarrow R$ with $g_{\tilde{\mu}} \in G = \text{Gal}(R/A_f)$ such that $\tilde{h} = h \circ g_{\tilde{\mu}}$. We described this Galois group in Section 3:

$$G \cong \text{Cl}(A) \times (A/fA)^*/\mathbb{F}_q^*.$$ 

So the element $g_{\tilde{\mu}}$ is given by a pair $(m, \sigma) \in \text{Cl}(A) \times (A/fA)^*/\mathbb{F}_q^*$.

If we have another level $f$-structure $\mu'$, then $\mu' = \alpha \tilde{\mu}$ for some $\alpha \in (A/fA)^*$ and we see that $g_{\mu'}$ corresponds to the pair $(m, \alpha \sigma)$.

We equip $\psi$ with the level $f$-structure $\mu$ such that $g_{\mu}$ is given by the pair $(m, 1)$. More specifically, let $h_{\mu} := h \circ g_{\mu}$. Let $v \in V^*$ be an element with $\psi v v^{-1} = \tilde{h}(\psi^{un})$, then this element $v$ is unique up to $\mathbb{F}_q^*$. Let $(\psi, \mu)$ be the pair determined by $(\tilde{h}, v)$.

The pair $(\psi, \mu)$ that we obtain in this way is unique up to $\mathbb{F}_q^*$. In particular, the isomorphy class of $(\psi, \mu)$ is uniquely determined in this way.

**Remark 5.4.** To make the choice of an element in $\mathbb{F}_q^*$ in the above construction ‘visible’, we consider the set of elements $z$ in $\varphi[f](K_V) \backslash \varphi[f](V)$ with $w_f(z, e_\Lambda(\mu(1))) \in \mathbb{F}_q^* \cdot w_f(\lambda(1, 0), \lambda(0, 1))$.

The set of these elements equals $\mathbb{F}_q^* \cdot z'$ where $z'$ is any element in this set. Fix one of those elements $z$. We can now equip $\psi$ with the unique $\mu$ such that $w_f(z, e_\Lambda(\mu(1))) = w_f(\lambda(1, 0), \lambda(0, 1))$.

In this way, we can associate to a triple $(\varphi, \lambda, z)$ a unique triple $(\psi, \mu, \Lambda)$. 
Our next goal is to prove Theorem 5.8. Drinfeld’s construction gives an $A$-lattice $\Lambda$ of rank 1 such that the pair $(\psi, \Lambda)$ determines $\varphi$ and vice versa. As in Drinfeld’s proof, we can consider $\psi_a$ as a map

$$\psi_a : K_V^{\text{sep}} \rightarrow K_V^{\text{sep}}$$

for every $a \in A$. Because $f \not\in \ker(\psi : K_V \rightarrow K_V)$, it follows that all roots of the equation $\psi_a(X) = a$ lie in $K_V^{\text{sep}}$ for all $a \in A$. And thus we have

$$(\psi_f)^{-1}\Lambda \subset K_V^{\text{sep}},$$

as in Drinfeld’s proof.

**Lemma 5.5.** If the $f$-torsion of $\varphi$ is $K_V$-rational, then $(\psi_f)^{-1}\Lambda \subset K_V$. In particular $\Lambda \subset K_V$.

**Proof.** Cf. Lemma 2.4 in [1]. By definition $\Lambda$ is invariant under $G_{K_V}$. Consequently, also $(\psi_f)^{-1}\Lambda$ is invariant under $G_{K_V}$ as the coefficients of $\psi_f$ lie in $V$. In fact, the action of $G_{K_V}$ on $(\psi_f)^{-1}\Lambda$ splits according to the following splitting exact sequence of $A$-modules:

$$0 \rightarrow \psi[f](K_V) \rightarrow (\psi_f)^{-1}\Lambda \rightarrow ((\psi_f)^{-1}\Lambda)_{\text{proj}} \rightarrow 0.$$  

The $A$-module $((\psi_f)^{-1}\Lambda)_{\text{proj}}$ is the projective part of $(\psi_f)^{-1}\Lambda$, which is isomorphic to $\Lambda$. By Drinfeld’s construction there is an analytic map $e_{\Lambda}$ with the commuting property as in diagram (3), and this map commutes with the action of $G_{K_V}$. Moreover, via this map $\varphi[f]$ is isomorphic to $(\psi_f)^{-1}\Lambda/\Lambda$. By the assumption that $\varphi[f]$ is $K_V$-rational, it follows that $G_{K_V}$ acts trivially on $(\psi_f)^{-1}\Lambda/\Lambda$ and thus on $\psi[f]$. The latter fact implies that the only action of $G_{K_V}$ on $(\psi_f)^{-1}\Lambda$ is on the projective part of $(\psi_f)^{-1}\Lambda$, hence this action gives a subgroup $G$ of $\text{Gl}_1(A) = \mathbb{F}_q^*$. On the other hand, this subgroup $G$ maps injectively into $\text{Aut}_A((\psi_f)^{-1}\Lambda/\Lambda)$. But $G_{K_V}$ acts trivially on $(\psi_f)^{-1}\Lambda/\Lambda$, hence $G$ is trivial. 

So we see that we can associate to a pair $(\varphi, \lambda)$ with $P'((\varphi))$ a triple $(\psi, \mu, \Lambda)$ such that $(\psi, \mu)$ is defined over $V$ and $\Lambda$ is an $A$-lattice with $(\psi_f)^{-1}\Lambda \subset K_V$. This triple is unique up to $\mathbb{F}_q^*$. I.e., the pair $(\varphi, \lambda)$ determines a unique element in $\{(\psi, \lambda) \mid \mathbb{F}_q^*\}$.

Note, however, that the level $f$-structure $\lambda$ is not the unique level structure such that $(\varphi, \lambda)$ is mapped to this unique element in $\{(\psi, \mu, \Lambda) \mid \mathbb{F}_q^*\}$. In fact, if $\sigma \in \text{Gl}_2(A/fA)$, then $(\varphi, \lambda \circ \sigma) \mapsto \{(\psi, \mu, \Lambda) \mid \mathbb{F}_q^*\}$ if and only if $\det(\sigma) \in \mathbb{F}_q^*$. This is due to the fact that the Weil pairing is $\text{Gl}_2(A/fA)$-equivariant. Therefore, if one changes $\lambda$ by $\sigma$, then the morphism $h$ induced by the Weil pairing $w_f$ changes by $\det(\sigma)$. Consequently, the triple $(\psi, \Lambda, \mu)$ changes by $\det(\sigma)$. Let $\Sigma$ denote the subgroup of $\text{Gl}_2(A/fA)$ given by

$$\Sigma = \{\sigma \in \text{Gl}_2(A/fA) \mid \det(\sigma) \in \mathbb{F}_q^*\}.$$  

The previous shows that the map

$$\{(\varphi, \lambda)\}/\Sigma \rightarrow \{(\psi, \mu, \Lambda)\}/\mathbb{F}_q^*$$

is injective.
Remark 5.6. Again, once we have chosen an element $z$ as in Remark 5.4, we get an injective map
$$\{(\varphi, \lambda, z)\}/\text{SL}_2(A/fA) \longrightarrow \{(\psi, \mu, \Lambda)\}.$$  

On the other hand, let a triple $(\psi, \mu, \Lambda)$ be given. We show that there exists a pair $(\varphi, \lambda)$ such that under the above construction $(\varphi, \lambda)$ is mapped to the class $\{(\psi, \mu, \Lambda)\}/\mathbb{F}_q^*$. 

The triple $(\psi, \mu, \Lambda)$ gives rise to the following:

1. a morphism $\tilde{h} : R \longrightarrow V$ which induces $(\psi, \mu)$ on $V$;
2. the pair $(\psi, \Lambda)$ gives a Drinfeld module $\varphi$ of rank 2;
3. as the $f$-torsion of $\varphi$ comes from $(\psi f)^{-1}\Lambda/\Lambda$, the $f$-torsion of $\varphi$ is $K_V$-rational, and thus $\varphi$ has $P'$.  

We equip $\varphi$ with a level $f$-structure as follows: define $\lambda(z, 1) := e_\Lambda(\mu(1))$ and let $\lambda(1, 0) = z$ for some element $z \in \varphi[f](K_V)\setminus\varphi[f](V)$. Any $z$ gives rise to a ring homomorphism $h_z : R \longrightarrow V$ induced by the Weil pairing $w_f$. 

As before, there exists an $A_f$-automorphism $g_z$ of $R$ with $\tilde{h} \circ g_z = h$, and $g_z$ is given by a pair $(m, \sigma) \in \text{Cl}(A) \times (A/fA)^*$. We choose an element $z$ for which $\sigma$ is the identity. As before, $z$ is unique up to a choice of $\mathbb{F}_q^*$. Clearly, the pair $(\varphi, \lambda)$ is mapped to the class $\{(\psi, \mu, \Lambda)\}/\mathbb{F}_q^*$. 

This shows that we have a bijection between the set 
$$\{\text{all pairs } (\varphi, \lambda) \text{ with } P' \}/\Sigma$$

and the set 
$$\{\text{all triples } (\psi, \mu, \Lambda) \text{ with } (\psi, \mu) \text{ over } V \text{ and } (\psi f)^{-1}\Lambda \subset K_V \}/\mathbb{F}_q^*.$$  

Remark 5.7. Similarly, the above argument shows that the injective map of Remark 5.6 is a bijection. 

This bijection can be rephrased in terms of isomorphy classes of pairs $(\varphi, \lambda)$ and triples $(\psi, \mu, \Lambda)$ as follows. We say that a triple $(\psi, \mu, \Lambda)$ is defined over $V$ if the pair $(\psi, \mu)$ is defined over $V$. Two triples $(\psi, \mu, \Lambda)$ and $(\psi', \mu', \Lambda')$ over $V$ are called isomorphic if there exists an element $v \in V^*$ with 
$$(v\psi v^{-1}, v\mu, v\Lambda) = (\psi', \mu', \Lambda').$$

Note that 

$$\{(\psi, \mu, \Lambda) \text{ over } V\}/V^* \overset{\text{bij}}{\cong} \{\psi, \mu, \Lambda\}_{K_V}/K_V^*.$$  

Moreover, if $v \in V^*$, then 
$$v : (\varphi, \lambda) \mapsto (v\varphi v^{-1}, v\lambda)$$

and 
$$v : (\psi, \mu, \Lambda) \mapsto (v\psi v^{-1}, v\mu, v\Lambda).$$

Dividing out the action of $V^*$ and considering the bijections (2) and (4) gives the following theorem.

**Theorem 5.8.** Let $\Sigma = \{\sigma \in \text{GL}_2(A/fA) \mid \det(\sigma) \in \mathbb{F}_q^*\}$. There is a bijection between the following two sets:
(1) Isomorphy classes of pairs \((\varphi, \lambda)\) over \(K_V\) modulo \(\Sigma\) where \(\varphi\) is a Drinfeld module of rank 2 with stable reduction of rank 1 and \(\lambda\) is a full level \(f\)-structure over \(K_V\).

(2) Isomorphy classes of triples \((\psi, \mu, \Lambda)\) over \(K_V\), where \(\psi\) is a rank 1 Drinfeld module over \(K_V\), \(\mu\) is a full level \(f\)-structure over \(K_V\) and \(\Lambda\) is an \(A\)-lattice of rank 1, such that \((\psi_f)^{-1}\Lambda \subset K_V\).

Proof. This theorem follows from the bijection that we have given above. □

6. Tate-Drinfeld modules

We follow the approach of [1, 2.2] and [21] to construct the Tate-Drinfeld module of type \(m\). The Tate-Drinfeld module describes the formal neighbourhood of the cusps of the moduli scheme. At the cusps the universal Drinfeld module with level structure degenerates into a Drinfeld module with stable reduction. Therefore, to define the Tate-Drinfeld module, we use the description of the stable reduction modules as given in the previous section.

Let \((\psi, \mu)\) be the universal Drinfeld module of rank 1 with level \(f\)-structure over \(R\). Because \(f\) is invertible in \(R\), we may assume that the generator \(\mu(1)\) of the \(f\)-torsion of \(\psi\) is an invertible element in \(R\). So we may and will assume that \(\mu(1) = 1\). Then by push-forward via the embeddings

\[ R \rightarrow R[[x]] \rightarrow R((x)) \]

one has a Drinfeld module of rank 1 with level \(f\)-structure on both \(R[[x]]\) and \(R((x))\). For an element \(y = \sum_{i \geq k} r_i x^i \in R((x))\) with \(r_k \neq 0, k \in \mathbb{Z}\), we define its valuation \(v_x(y)\) to be the \(x\)-valuation considered as element in \(K_R((x))\), and \(v_x(y) = k\).

To construct a Drinfeld module of rank 2 over \(R((x))\), we first construct a lattice \(\Lambda_m \subset K_R((x))\), where \(m \subset A\) is an ideal. This lattice turns out to depend only on the class of \(m\) in the class group of \(A\). As before, we can consider \(\psi_f\) as a map

\[ \psi_f : K_R((x))^{\text{sep}} \rightarrow K_R((x))^{\text{sep}}. \]

The lattice \(\Lambda_m\) will be constructed in such a way that \((\psi_f)^{-1}\Lambda_m \subset K_R((x))\). Applying Theorem 5.8 to \((\psi, \mu, \Lambda_m)\) will give us the Tate-Drinfeld module \(\varphi\).

6.1. The construction of the lattice. Let \(m \subset A\) be an ideal. To prepare the construction of the lattice, note the following:

1. There is a unique monic skew polynomial \(P \in K_R\{\tau\}\) with minimal degree such that

\[ \ker(P)(K_R^{\text{sep}}) = \psi[m](K_R^{\text{sep}}). \]

In fact, \(P \in R\{\tau\}\) because the elements in \(\ker(P)(K_R^{\text{sep}})\) are integral over \(R\) and \(R\) is integrally closed.

2. Because \(A_f \hookrightarrow K_R\), the extension \(K_R(\psi[m])/K_R\) is Galois. Moreover, because \(\psi[m](K_R^{\text{sep}}) \cong A/m\), there is an injective representation

\[ \text{Gal}(K_R(\psi[m])/K_R) \rightarrow (A/m)^*. \]
So the Galois group of this extension is a subgroup of \((A/m)^*\).  

3. The field \(K_R(\psi[m])(\langle y \rangle)\) is the splitting field of the equation \(P(y) = \frac{1}{x}\) over \(K_R(\langle x \rangle)\). Then 
\[
K_R(\psi[m])(\langle y \rangle)/K_R(\psi[m])(\langle x \rangle)
\]
is a Galois extension which is totally ramified and its Galois group is isomorphic to \(A/m\). The Galois action is given by \(y \mapsto y + \alpha\) with \(\alpha \in \psi[m](K_R(\psi[m]))\).

Let \(l\) be the following \(A\)-module homomorphism:
\[
l : f^{-1}A \to K_R(\psi[m])(\langle y \rangle) \quad \text{by} \quad f^{-1}a \mapsto \psi_a \left( \frac{1}{y} \right).
\]
We use \(l\) to define the lattice \(\Lambda_m\).

**Lemma 6.1.** The \(A\)-module \(l(f^{-1}m)\) lies inside \(R(\langle x \rangle)\).

*Proof.* For every \(m \in \mathfrak{m}\), there exists a skew polynomial \(Q \in R\{\tau\}\) such that \(\psi_m = Q \cdot P\). Note that we use here that one has division with remainder in the skew ring \(R\{\tau\}\), because the leading coefficients of both \(P\) and \(\psi_m\) are in \(R^*\). Consequently, \(\psi_m \left( \frac{1}{y} \right) = Q \left( \frac{1}{x} \right) \in R(\langle x \rangle)\). \(\square\)

**Remark 6.2.** Let \(m_1, m_2\) generate \(\mathfrak{m}\), then there are elements \(Q_i \in R\{\tau\}\) with \(Q_i \circ P = \psi_{m_i}\). We will use this in the following a few times without further mentioning it.

Define the lattice \(\Lambda\) as follows:
\[
\Lambda := l(\mathfrak{m}), \quad \text{then} \quad (\psi_f)^{-1} \Lambda = l(f^{-1}m) + \psi[f](R) \subset R(\langle x \rangle).
\]

**Lemma 6.3.** The lattice \(\Lambda\) only depends on the class of \(\mathfrak{m}\) in \(\text{Cl}(A)\).

*Proof.* Suppose that \(\mathfrak{m}' \subset A\) is another ideal representing the same class as \(\mathfrak{m}\) in \(\text{Cl}(A)\). Then there exist elements \(b,b' \in A\) with \(b'm' = bm\). So we may reduce to case where \(\mathfrak{m}' = bm\) for some \(b \in A\). The extension \(K_R(\psi[m'])/K_R(\psi[m])\) is given by adding the roots of the polynomial \(\psi_b(X)\) to \(K_R(\psi[m])\). The extension
\[
K_R(\psi[m])(\langle y' \rangle)/K_R(\psi[m])(\langle y \rangle)
\]
is given by adding the roots of the equation \(\psi_b \left( \frac{1}{y} \right) = \frac{1}{y}\). It is not difficult to see that \(\mathfrak{m}\) and \(\mathfrak{m}'\) give the same \(\Lambda\). \(\square\)

By this lemma it makes sense to talk about the type \(\mathfrak{m}\) in \(\text{Cl}(A)\) of the lattice. We will denote \(\mathfrak{m}\) for both the ideal in \(A\) as for the ideal class in \(\text{Cl}(A)\). Moreover, this gives us some freedom in choosing \(\mathfrak{m}\) to construct a lattice of type \(\mathfrak{m}\).

**Lemma 6.4.** Let \(a, \mathfrak{m} \subset A\) be non-zero ideals, then there is an element \(x \in K_A\), the quotient field of \(A\), such that \(x\mathfrak{m} + a = A\).

*Proof.* By Proposition VII.5.9 in [2], there is an element \(x \in K_A\) with \(v_p(x) = -v_p(\mathfrak{m})\) for all primes \(p \subset A\) dividing \(a\) and \(v_p(x) \geq 0\) for all other primes \(p\) of \(A\). Consequently, \(x\mathfrak{m} \subset A\), and there is no prime ideal \(p\) of \(A\) dividing both \(x\mathfrak{m}\) and \(a\). \(\square\)
This lemma shows that we can choose a representative \( \mathbf{m} \) of the class type \([\mathbf{m}]\) of the lattice such that this representative is relatively prime to some chosen ideal \( \mathfrak{a} \subset A \). This gives some help in technical parts of some proofs later on.

We write \( \Lambda_{\mathfrak{m}} \) for the lattice \( \Lambda \) of type \( \mathbf{m} \) that we constructed above. Note that \( K_R((x)) \) is the quotient field of the complete discrete valuation ring \( K_R[[x]] \). By the constructions in Section 5 and Theorem 5.8, we may associate to the triple \( (\psi, \mu, \Lambda_m) \) a unique Drinfeld module \( \varphi \) of rank 2 over \( K_R((x)) \) with stable reduction of rank 1. Moreover, the \( f \)-torsion of \( \varphi \) is \( K_R((x)) \)-rational.

In fact, \( \varphi \) is a Drinfeld module over \( R((x)) \), as we will now show. Using Theorem 5.3, the corresponding exponential map is

\[
e_{\Lambda_m}(z) := z \prod_{\alpha \in \Lambda_m \setminus \{0\}} \left( 1 - \frac{z}{\alpha} \right),
\]

and \( \varphi \) is determined by the following diagram, which commutes for all \( a \in A \):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Lambda_m & \longrightarrow & K_R((x)) & \longrightarrow & K_R((x)) \\
& & \downarrow \psi_a & & \downarrow \psi_a & & \downarrow \varphi_a \\
0 & \longrightarrow & \Lambda_m & \longrightarrow & K_R((x)) & \longrightarrow & K_R((x)) \\
\end{array}
\]

(5)

Note that by construction \( \Lambda_m \subset R((x)) \), each non-zero element of \( \Lambda_m \) has negative \( x \)-valuation and the leading coefficient of this non-zero element is in \( R^* \). Consequently, the map \( e_{\Lambda_m}(z) = 1 + \sum_{i \geq 1} s_i z^i \) has all its coefficients \( s_i \in R[[x]] \). So its inverse exists in \( R[[x]][z] \) and

\[
\varphi_a = e_{\Lambda_m} \circ \psi_a \circ e^{-1}_{\Lambda_m}.
\]

Therefore, \( \varphi_a \) has its coefficients in \( R[[x]] \) for all \( a \in A \).

**Lemma 6.5.** The ring homomorphism \( \varphi \) is a Drinfeld module over \( R((x)) \).

**Proof.** We only need to prove that the leading coefficient of \( \varphi_a \) is an element of \( R((x))^* \). To prove this, we simply copy the computation of Lemma 2.10 in [1]. Note that

\[
\varphi_a(z) = az \prod_{\alpha \in ((\psi_a)^{-1} \Lambda_m/\Lambda_m) \setminus \{0\}} \left( 1 - \frac{z}{e_{\Lambda_m}(\alpha)} \right).
\]

So we need to show that

\[
(*) \prod_{\alpha \in ((\psi_a)^{-1} \Lambda_m/\Lambda_m) \setminus \{0\}} e_{\Lambda_m}(\alpha) = a \cdot u
\]

with \( u \in R((x))^* \).

Every \( \alpha \in ((\psi_a)^{-1} \Lambda_m/\Lambda_m) \setminus \{0\} \) which is not in \( \psi[a] \) can be written uniquely as \( \alpha = \alpha_1 + \alpha_2 \) where \( \alpha_1 \) runs through \( \psi[a] \) and \( \alpha_2 \) runs through a set of representatives in \( \psi_a^{-1} \Lambda_m/\Lambda_m \) of the non-zero elements of \( (\psi_a)^{-1} \Lambda_m/\Lambda_m + \psi[a] \). This set is denoted by \( S_1 \). The remaining elements \( \alpha \) can be written
as \( \alpha = \alpha_1 \) where \( \alpha_1 \) runs through \( \psi[m]\setminus\{0\} \). This set is denoted by \( S_2 \).

By definition
\[
e_{\Lambda_m}(\alpha) = \alpha \prod_{\beta \in \Lambda_m\setminus\{0\}} \left( 1 - \frac{\alpha}{\beta} \right).
\]

Using the rule
\[
\psi_a(z) = \prod_{\alpha_1 \in \psi[a]} (z - \alpha_1),
\]
we see that
\[
\prod_{\alpha_1 \in \psi[a]} e_{\Lambda_m}(\alpha) = \prod_{\alpha_2 \neq 0} \left( \psi_a(\alpha_2) \prod_{\beta \in \Lambda_m\setminus\{0\}} \left( \frac{\psi_a(\beta - \alpha_2)}{\beta \# A/(a)} \right) \right).
\]
This is in fact an element in \( R((x))^* \), which can be seen as follows. Clearly, \( 0 \neq \psi_a(\alpha_2) \in \Lambda_m \), so this element is in \( R((x))^* \). Moreover, the element \( \psi_a(\beta - \alpha_2) \in \Lambda_m \) cannot be 0: if it were 0, then any representative of \( \alpha_2 \) would lie in \( \beta + \psi[a] \subset \Lambda_m + \psi[a] \), i.e., the class \( \alpha_2 = 0 \), contradicting the definition of the set \( S_2 \). So also \( \psi_a(\beta - \alpha_2) \in R((x))^* \). Finally, for almost all \( \beta \) we have that
\[
\frac{\psi_a(\beta - \alpha_2)}{\beta \# A/(a)} \in R[[x]].
\]
So the product exists, and
\[
\prod_{\alpha_1 \in \psi[a]} e_{\Lambda_m}(\alpha) \in R((x))^*.
\]

On the other hand, using the rule
\[
h(z) := \frac{\psi_a(z)}{z} = \prod_{\alpha_1 \in S_2} (z - \alpha_1)
\]
and recalling that \( h(0) = a \), we see that
\[
\prod_{\beta \in \Lambda_m\setminus\{0\}} \left( \frac{h(\beta)}{\beta \# A/(a) - 1} \right),
\]
with each \( \frac{h(\beta)}{\beta \# A/(a) - 1} \) is in \( R[[x]]^* \).

Finally,
\[
\prod_{\alpha \in ((\psi_a)^{-1}\Lambda_m/\Lambda_m)\setminus\{0\}} e_{\Lambda_m}(\alpha) = \prod_{\alpha_1 \in S_1} e_{\Lambda_m}(\alpha) \cdot \prod_{\alpha_2 \in S_2} e_{\Lambda_m}(\alpha).
\]
This finishes the proof. \( \square \)

From the construction of the Drinfeld module \( \varphi \) with \( \theta \) over \( R((x)) \), coming from the triple \((\psi, \mu, \Lambda_m)\), we can deduce at once the following list of properties:

1. \( \varphi : A \longrightarrow R[[x]]\setminus\{\tau\} \) is a ring homomorphism;
2. the \( f \)-torsion of \( \varphi \) is \( R((x)) \)-rational;
3. there is an isomorphism
   \[
   A/fA \overset{\sim}{\longrightarrow} \varphi[f](R[[x]])
   \]
   given by \( 1 \mapsto e_{\Lambda_m}(\mu(1)) \).


(4) \( \varphi \mod xR[x] = \psi \), because \( e_{\Lambda_m}(z) \mod xR[x] \) is the identity map. As in Section 3, the triple \((\psi, \mu, \Lambda_m)\) induces on \( \varphi \) a level \( f \)-structure \( \lambda \) with \( \lambda(0,1) = e_{\Lambda_m}(\mu(1)) \) and \( \lambda(1,0) \) is determined up to \( \mathbb{F}_q^* \) by the Weil pairing.

In this way, we get for every element \( m \in \text{Cl}(A) \) a pair \((\varphi^m, \lambda^m)\). The action of \( \sigma \in \text{Gl}_2(A/fA) \) on this pair is given by

\[
\sigma : (\varphi^m, \lambda^m) \longrightarrow (\varphi^m, \lambda^m \circ \sigma).
\]

The action of the class group of \( A \) is described in the following lemma. Let \( n \in \text{Cl}(A) \) and let \( g_n \) denote the \( A_f \)-linear automorphism of \( R \) which describes the action of \( n \) on \( R \). Let \( \Lambda_{n^{-1}m} \) denote the lattice of type \( n^{-1}m \).

**Lemma 6.6.** The element \( n \) maps the triple \((\psi, \mu, \Lambda_m)\) to the triple \((g_n(\psi), g_n(\mu), g_n(\Lambda_{n^{-1}m}))\).

**Proof.** We choose representatives \( m, n \subset A \) of the classes of \( m \) and \( n \) in \( \text{Cl}(A) \) such that \( n^{-1}m \subset A \), and \( n \) and \( m \) are relatively prime to \( f \).

Write \( \varphi = \varphi^m \). The action of \( n \) on \( \varphi \) is given by a unique monic skew polynomial \( Q \) with minimal degree such that

\[
\ker(Q)(K_R(\langle x \rangle)^\text{sep}) = \varphi[n](K_R(\langle x \rangle)^\text{sep}).
\]

Let \( \varphi' \) be the image under \( Q \):

\[
\varphi \xrightarrow{Q} \varphi'.
\]

Writing \( \Lambda_m = l(m) \) as before, it is not difficult to see that

\[
\ker(Q \circ e_{\Lambda_m})(K_R(\langle x \rangle)^\text{sep}) = l(n^{-1}m) + \psi[n](K_R(\langle x \rangle)^\text{sep}).
\]

The action of \( n \) on \( R \), denoted by \( g_n \), corresponds to a skew polynomial \( P \in R\{\tau\} \) with \( \ker(P)(K_R^\text{sep}) = \psi[n](K_R^\text{sep}) \). Let \( \Lambda' \) be the lattice given by

\[
\Lambda' := P \circ \ker(Q \circ e_{\Lambda_m})(K_R(\langle x \rangle)^\text{sep}) = P \circ l(n^{-1}m)
\]

\[
= \{ g_n(\psi_f \tau) P(\frac{1}{y}) | a \in n^{-1}m \}.
\]

Then \( \Lambda' = g_n(\Lambda_{n^{-1}m}) \), and it is not difficult to see that \( \varphi' \) corresponds to the pair \((g_n(\psi), g_n(\Lambda_{n^{-1}m}))\). \( \square \)

As a corollary to this lemma, we see that \( n \) maps \( \varphi^m \) to \( g_n(\varphi^{n^{-1}m}) \). As the definition of \( \lambda^m \) and \( \lambda^{n^{-1}m} \) depend on a choice of \( \mathbb{F}_q^* \), we cannot say that \( n \) maps \((\varphi^m, \lambda^m)\) to \((\varphi^{n^{-1}m}, \lambda^{n^{-1}m})\). Bearing this in mind, we propose for every type \( m \) the following definition of the pairs \((\varphi^m, \lambda^m)\). For \( m = A \) we define \((\varphi^A, \lambda^A) := (\varphi^m, \lambda^m) \). This pair is unique up to \( \mathbb{F}_q^* \). For the rest of this paper we keep this choice is fixed. In general, \( m^{-1} \) gives rise to an automorphism \( g_{m^{-1}} \) of \( R \). We extend this to an automorphism of \( R[[x]] \) by letting it act trivially on \( x \). We define \((\varphi^{td}, \lambda^{td})\) to be the image of \((\varphi^A, \lambda^A)\) under \( m^{-1} \).

**Definition 6.7.** The pair \((\varphi^m, \lambda^m)\) is called the standard Tate-Drinfeld module of rank 2 and type \( m \) with level \( f \)-structure. A pair \((\varphi^{td}, \lambda^{td})\), where \( \lambda \) is any level \( f \)-structure over \( R((x)) \) is called a Tate-Drinfeld module with level \( f \)-structure.
The pairs \((\xi \varphi_m^{\text{td}} \xi^{-1}, \xi \lambda)\) with \(\xi \in R[[x]]^*\) all belong to the same isomorphy class \((\varphi_m^{\text{td}}, \lambda)\). Every isomorphy class of Tate-Drinfeld modules with level \(f\)-structure comes from a unique morphism
\[
\text{Spec}(R((x))) \longrightarrow M^2(f).
\]
Because we always assume that \(\mu(1) = 1\), the pair \((\varphi_m^{\text{td}}, \lambda_m^{\text{td}})\) is fixed in its isomorphy class. For every \(\lambda\) there is a unique \(\sigma \in \text{Gl}_2(A/fA)\) with \(\lambda = \lambda_m^{\text{td}} \circ \sigma\). Therefore, the pair \((\varphi_m^{\text{td}}, \lambda)\) is fixed in its isomorphy class.

### 7. The Universal Tate-Drinfeld Module

In this section we introduce the universal Tate-Drinfeld module. This is a scheme \(Z\) consisting of the coproduct of a number of copies of \(\text{Spec}(R[[x]])\) and which is equipped with a Tate-Drinfeld structure \((\varphi^{\text{td}}, \lambda^{\text{td}})\). Let \(N \subset \text{Gl}_2(A/fA)\) be the subgroup
\[
N = \left( \begin{array}{cc} \mathbb{F}_q^- & A/fA \\ 0 & (A/fA)^* \end{array} \right).
\]
Choose representatives \(\sigma_1, \ldots, \sigma_n\) of the cosets of \(N\sigma_i\) in \(\text{Gl}_2(A/fA)\) and let \(\sigma_1\) be the identity. As \(\det(N) = (A/fA)^*\), we may assume that the representatives \(\sigma_i\) are elements of \(\text{Sl}_2(A/fA)\). Set
\[
Z = \text{Spec} \left( \bigoplus_{(m, \sigma_i)} R[[x]]_{(m, \sigma_i)} \right),
\]
The pair \((\varphi^{\text{td}}, \lambda^{\text{td}})\) is the Drinfeld module \(\varphi^{\text{td}}\) of rank 2 with level \(f\)-structure \(\lambda^{\text{td}}\) over \(\text{Spec} \left( \bigoplus_{(m, \sigma_i)} R((x))_{(m, \sigma_i)} \right)\) such that the restriction of \((\varphi^{\text{td}}, \lambda^{\text{td}})\) to \(R((x))_{(m, \sigma_i)}\) is equal to \((\varphi_m^{\text{td}}, \lambda_m^{\text{td}} \circ \sigma_i)\) for every pair \((m, \sigma_i)\).

**Definition 7.1.** The universal Tate-Drinfeld module of rank 2 with level \(f\)-structure is the scheme \(Z\) together with the pair \((\varphi^{\text{td}}, \lambda^{\text{td}})\).

**Remark 7.2.** As before, a pair \((m, \sigma_i) \in \text{Cl}(A) \times \text{Gl}_2(A/fA)\) determines a unique action on \((\varphi_m^{\text{td}}, \lambda_m^{\text{td}})\). And by definition we have
\[
(\varphi_m^{\text{td}}, \lambda_m^{\text{td}} \circ \sigma_i) = (m^{-1}, \sigma_i)(\varphi_m^{\text{td}}, \lambda_m^{\text{td}}).
\]
The action of \(\text{Cl}(A) \times \text{Gl}_2(A/fA)\) on the universal Tate-Drinfeld module is given by this action of \((m, \sigma_i)\) and the fact that \(N\) acts trivially.

Clearly, we would like to have a certain universal property for this universal Tate-Drinfeld module. The weak versions of this universal property that we need can be found in Theorems \(\text{TD}3\) and \(\text{TD}4\). In Proposition \(\text{TD}4\) the main work is done for Theorem \(\text{TD}4\). This proposition explains the subgroup \(N\).

**Proposition 7.3.** Let \(\sigma \in \text{Gl}_2(A/fA)\). There exists an \(A_f\)-linear ring homomorphism \(h_\sigma : R[[x]] \longrightarrow R[[x]]\) such that
\[
h_\sigma(\varphi_m^{\text{td}}, \lambda_m^{\text{td}}) \equiv (\varphi_m^{\text{td}}, \lambda_m^{\text{td}} \circ \sigma)
\]
if and only if \(\sigma \in N\). If \(\sigma \in N\), then \(h_\sigma\) is given by \(\det(\sigma) : R \longrightarrow R\) and \(x \mapsto \delta x\) for some \(\delta \in R[[x]]^*\). Moreover, for any \(\sigma \in N\) the image of \((\varphi_m^{\text{td}}, \lambda_m^{\text{td}} \circ \sigma_i)\) under \(h_\sigma\) is isomorphic to \((\varphi_m^{\text{td}}, \lambda_m^{\text{td}} \circ \sigma \circ \sigma_i)\).
Proof. We write $\sigma = (\sigma_{i,j})$. First suppose that $h_\sigma$ exists. Because $h_\sigma$ is determined by $\det(\sigma)$ and $h_\sigma(x)$, it must respect the ordering of the $x$-valuation. This implies that $h_\sigma(\lambda(1,0))$ must have minimal negative valuation and $h_\sigma(\lambda(0,1))$ must have valuation 0. Hence the only $\sigma$’s whose action may come from an $A_f$-linear ring homomorphism $h_\sigma$ are $\sigma \in N$. This shows the ‘only if’.
We prove the ‘if’-part in two steps. Let $\sigma = (\sigma_{i,j}) \in N$.
1. First suppose that $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ with $\alpha \in (A/fA)^*$. The action of $\sigma$ induces an action
   
   $$\alpha = \det(\sigma) : M^1(f) \rightarrow M^1(f),$$
   
i.e., there is an $A_f$-linear ring homomorphism $h_\alpha : R \rightarrow R$ such that
   
   $$(h_\alpha(\psi), h_\alpha(\mu)) = (\xi \psi \xi^{-1}, \xi \mu \alpha) \cong (\psi, \mu \alpha),$$
   
   for some $\xi \in R^*$. We use the notations of Section 6. Recall that the element $\frac{1}{y}$ was used to define the lattice $\Lambda_{m}^{td}$ and that $\frac{1}{x} = P(\frac{1}{y})$. $P \in R\{\tau\}$ is the skew polynomial of minimal degree with
   
   $$\ker(P)(K_{R}^{sep}) = \psi[m](K_{R}^{sep}).$$
   
The map $\frac{1}{y} \mapsto \frac{\xi}{y}$ induces, by applying $P$,
   
   $$\frac{1}{x} \mapsto h_\alpha(P)(\frac{1}{x}) = \delta^{-1} \frac{1}{x} \quad \text{for some} \quad \delta^{-1} \in R[[x]]^*.$$
   
   To see this, note that $h_\alpha(P) = \zeta P \xi^{-1}$ for some $\zeta \in R[x]^*$. (In fact, $\zeta$ is determined by the fact that $h_\alpha(P)$ is monic.)
   
The map $h_\alpha$ is extended to a ring homomorphism
   
   $$h_\sigma : R[x] \rightarrow R[x], \quad x \mapsto \delta x, \quad h_\sigma(r) = h_\alpha(r) \quad \forall r \in R.$$
   
   An easy computation shows that
   
   $$h_\sigma(\Lambda_{m}^{td}) = \xi \Lambda_{m}^{td}, \quad \text{and thus} \quad h_\sigma(\varphi_{m}^{td}) = \xi \varphi_{m}^{td} \xi^{-1}.$$
   
   Using that there is an element $m \in \mathfrak{m}$ with $\lambda(1,0) = e_{\Lambda_{m}^{td}}(\psi_m(\frac{1}{y}))$, we see that
   
   $$h_\sigma \begin{pmatrix} \lambda(1,0) \\ \lambda(0,1) \end{pmatrix} = \begin{pmatrix} e_{\Lambda_{m}^{td}}(\xi \psi_m(\frac{1}{y})) \\ e_{\Lambda_{m}^{td}}(\xi \mu(1)\alpha) \end{pmatrix} = \begin{pmatrix} \xi \lambda(1,0) \\ \xi \lambda(0,1)\alpha \end{pmatrix}.$$
   
   And so indeed,
   
   $$h_\sigma(\varphi_{m}^{td}, \lambda) = (\xi \varphi_{m}^{td} \xi^{-1}, \xi \lambda \circ \sigma) \cong (\lambda^{td}_{m}, \lambda \circ \sigma).$$
   
2. Having dealt with the first case, we may assume that $\sigma \in N$ and $\det(\sigma) \in \mathbb{F}_{q}^*$. As a consequence, the map $h_\sigma$ that we are looking for is $R$-linear and in particular, if $(\psi, \mu, \Lambda)$ is the triple associated to $(\varphi_{m}^{td}, \lambda)$, then $h_\sigma(\psi) = \psi$ and $h_\sigma(\mu) = \mu$.
   
   As always, the action of $\mathbb{F}_{q}^* \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is trivial, so we may assume that $\sigma_{2,2} = 1$. We first give a proof in the simple case $\Lambda \cong A$ and then prove it for general $\Lambda$.  

i. Suppose \( \Lambda \cong A \). Note that in that case, the elements \( e_A(\frac{1}{x}) \) and \( e_A(\mu(1)) \) generate the \( f \)-torsion of \( \varphi^{td} \). In fact, there is a basis transformation
\[
\alpha = \begin{pmatrix}
\alpha_{1,1} & \alpha_{1,2} \\
0 & 1
\end{pmatrix} \quad \text{with} \quad \alpha \begin{pmatrix}
\lambda(1,0) \\
\lambda(0,1)
\end{pmatrix} = \begin{pmatrix}
e_A(\frac{1}{x}) \\
e_A(\mu(1))
\end{pmatrix}
\]
and with \( \alpha_{1,1} \in (A/fA)^* \), \( \alpha_{1,2} \in A/fA \).

Let \( \rho = \alpha \sigma \alpha^{-1} \), then
\[
\rho = \begin{pmatrix}
\rho_{1,1} & \rho_{1,2} \\
0 & 1
\end{pmatrix}
\]
with \( \rho_{1,1} \in \mathbb{F}_q^* \), \( \rho_{1,2} \in A/fA \). We set
\[
\delta^{-1} = \rho_{1,1} + \rho_{1,2}(\mu(1)) \cdot x \in R[[x]]^*,
\]
and we define \( h_{\sigma} \) to be the \( R \)-linear map given by
\[
h_{\sigma} : x \mapsto \delta \cdot x.
\]

Because
\[
\psi_f(\delta^{-1} \frac{1}{x}) = \rho_{1,1} \psi_f(\frac{1}{x}),
\]

it follows that \( h_{\sigma}(\Lambda) = \Lambda \), and thus \( h_{\sigma} \) commutes with \( e_A \). Moreover
\[
h_{\sigma}(\psi, \mu, \Lambda) = (\psi, \mu, \Lambda).
\]

Therefore
\[
h_{\sigma}(\varphi^{td}_m) = \varphi^{td}_m.
\]

So \( h_{\sigma} \) commutes with the \( A \)-action and with \( e_A \). To see what happens on the level structure \( \lambda \) is now an easy computation.
\[
h_{\sigma} \begin{pmatrix}
\lambda(1,0) \\
\lambda(0,1)
\end{pmatrix} = h_{\sigma} \alpha^{-1} \begin{pmatrix}
e_A(\frac{1}{x}) \\
e_A(\mu(1))
\end{pmatrix} = \alpha^{-1} \rho \begin{pmatrix}
e_A(\frac{1}{x}) \\
e_A(\mu(1))
\end{pmatrix} = \sigma \begin{pmatrix}
\lambda(1,0) \\
\lambda(0,1)
\end{pmatrix}.
\]

We conclude that
\[
h_{\sigma}(\varphi^{td}_m, \lambda^{td}_m) = (\varphi^{td}_m, \lambda^{td}_m \circ \sigma).
\]



ii. In general, let \( \Lambda \cong m \) and recall the construction of \( \Lambda \) in Section 5. Using the same notations as in Section 6 let \( m \in m \) be an element such that the image of \( \frac{1}{z} := \psi_m(\frac{1}{y}) \) and \( \mu(1) \) under \( e_A \) generate the \( f \)-torsion of \( \varphi^{td}_m \). We let \( \alpha \) be as in i, i.e.,
\[
\alpha \begin{pmatrix}
\lambda(1,0) \\
\lambda(0,1)
\end{pmatrix} = \begin{pmatrix}
e_A(\frac{1}{x}) \\
e_A(\mu(1))
\end{pmatrix}
\]
and \( \rho = \alpha \sigma \alpha^{-1} \). As in the previous case, we look for an \( h_{\sigma} \) such that
\[
\frac{1}{z} \mapsto (\rho_{1,1} + \rho_{1,2}(\mu(1))z) \cdot \frac{1}{z}.
\]

This can be done as follows. We start by assuming that \( m \in (A/fA)^* \) - we will show below that we may assume this in general. Let \( b \in A \) such that
\[
b \equiv m^{-1} \rho_{1,2} \in A/fA.
\]

Let
\[
\frac{1}{y} \mapsto \rho_{1,1} \frac{1}{y} + \psi_b \mu(1).
\]

Applying \( P \) gives our candidate for \( h_{\sigma} \):
\[
\frac{1}{x} \mapsto \delta^{-1} \frac{1}{x} \quad \text{with} \quad \delta^{-1} = \rho_{1,1} + P(\psi_b(\mu(1))) \cdot x.
\]
By construction
\[ h_\sigma : \frac{1}{z} \mapsto \rho_{1,1} \frac{1}{z} + \psi_m(\psi_b(\mu(1))) = \rho_{1,1} \frac{1}{z} + \rho_{1,2} \mu(1). \]
Note that all elements of \( \Lambda \) have the form \( \psi_f \tilde{m}(\frac{1}{y}) \), with \( \tilde{m} \in \mathfrak{m} \). And clearly,
\[ h_\sigma(\psi_f \tilde{m}(\frac{1}{y})) = \rho_{1,1} \psi_f \tilde{m}(\frac{1}{y}). \]
Because \( \rho_{1,1} \in \mathbb{F}_q^* \), we see that \( h_\sigma(\Lambda) = \Lambda \).

We can conclude the proof in the same way as in the case of \( \Lambda \cong A \).

Finally, it remains to be shown that \( m \), which we used to define \( \frac{1}{z} = \psi_m(\frac{1}{y}) \), is an element of \( (A/fA)^* \). By Lemma 6.3, we may assume that \( \mathfrak{m} \) and \( (f) \) are relatively prime. Furthermore, because \( \frac{1}{z} \) generates one direct summand of the \( f \)-torsion, one has \( \psi_b(\frac{1}{z}) \in \Lambda \) if and only if \( b \in fA \), i.e., \( bm \in f \mathfrak{m} \) if and only if \( b \in fA \). Consequently, \( (f) + (m) = A \).

The ‘moreover’-part of the proposition is obvious.

Using Proposition 7.3, we can immediately prove one weak form of the universal property of the universal Tate-Drinfeld module.

**Theorem 7.4.** For every Tate-Drinfeld module \( (\varphi_{m}, \lambda) \) there is a unique ring homomorphism
\[ h : \bigoplus_{(m', \sigma_i)} R[x]_{(m', \sigma_i)} \rightarrow R[x] \]
such that
\[ h(\varphi_{m}, \lambda) = (\varphi_{m}, \lambda). \]

**Proof.** Let \( \sigma \in \text{GL}_2(A/fA) \) such that \( \lambda_{\sigma} \circ \sigma = \lambda \) and let \( \sigma_k \in \{\sigma_1, \ldots, \sigma_n\} \) such that \( \sigma \in \sigma_k N \). The map \( h \) is defined as follows: \( h \) is the zero-map on \( R[x]_{(m', \sigma_i)} \) if \( m' \neq m \) in the class group of \( A \) or if \( \sigma_k \neq \sigma_i \). On \( R[x]_{(m, \sigma_k)} \) it is the map defined in Proposition 7.3. To show uniqueness, note that any \( A_f \)-linear ring homomorphism
\[ h : R[x]_{(m, \sigma_i)} \rightarrow R[x] \]
induces on \( R[x] \) a Tate-Drinfeld module whose corresponding lattice has type \( m' \). Hence, any such ring homomorphism keeps the type of the Tate-Drinfeld module fixed. Moreover, if there is an \( A_f \)-morphism
\[ R[x]_{(m, \sigma_i)} \rightarrow R[x]_{(m, \sigma_j)} \]
which induces the Tate-Drinfeld structure, then there is a morphism
\[ R[x]_{(m, \sigma_i)} \rightarrow R[x]_{(m, \sigma_j, \sigma_i^{-1})}, \]
and thus by Proposition 7.3, we see that \( \sigma_j \sigma_i^{-1} \in N \). So \( \sigma_i = \sigma_j \). \qed

### 7.1. The weak version of the universal property of \( Z \)

Let, as in Section 5, \( V \) be a complete discrete valuation \( A_f \)-ring, let \( \pi \) be a generator of its maximal ideal, and let \( K_V \) be its field of fractions. Let \( (\varphi, \lambda) \) be a Drinfeld module \( \varphi \) of rank 2 over \( K_V \) with level \( f \)-structure \( \lambda \) such that \( \varphi \) has stable reduction of rank 1 at \( \pi \). In this subsection we discuss the other weak version of the universal property of the universal Tate-Drinfeld
module, which we need in the next section. We prove that there exists a unique ring homomorphism
\[ h_{\varphi,\lambda} : \bigoplus_{(m,\sigma)} R[[x]] \to V \]
such that
\[ h_{\varphi,\lambda}(\varphi,\lambda) \cong (\varphi,\lambda). \]
In Theorem 5.8 we showed that each triple \((\varphi,\lambda,z)\) corresponds modulo \(\Sigma\) to a unique triple \((\psi,\mu,\Lambda)\). Let \(m\) be the type of \(\Lambda\). In trying to avoid confusion, we will write \((\psi,\mu,\Lambda)\) for the triple used in defining the Tate-Drinfeld module of type \(m\) over \(R((x))\), where indeed \((\psi,\mu)\) is the universal Drinfeld module with level \(f\)-structure of rank 1 over \(R\).

The pair \((\psi,\mu)\) over \(V\) comes from an \(A_f\)-linear ring homomorphism, which we called \(\tilde{h} : R[[x]] \to V\).

We show that there exists an extension of \(\tilde{h}\) to \(R[[x]]\)
\[ h_{\psi,\mu} : R[[x]] \to V, \]
where \(R[[x]]\) comes equipped with \(\varphi_m^{td}\), such that \(h_{\psi,\mu}(\varphi_m^{td}) = \varphi\); cf. Proposition 7.6. The main point is showing that there exists a ring homomorphism \(h_{\psi,\mu}\) such that
\[ h_{\psi,\mu}(\psi^{-1}\Lambda_m^{td}) = (\psi_f)^{-1}\Lambda. \]

**Lemma 7.5.** Let \(m_1\) and \(m_2\) generate the ideal \(m\). Then there exists an element \(\zeta \in \overline{K_V}\) such that the projective part of \((\psi_f)^{-1}\Lambda\) is generated as \(A\)-module by the elements \(\psi_{m_1}\) and \(\psi_{m_2}\).

**Proof.** Let \(M = \overline{K_V}\) be the algebraic closure of \(K_V\), and let \(M_{\text{tor}}\) be the set of \(A\)-torsion points in \(M\); the \(A\)-action is given by \(\psi\).

1. The \(A\)-module \(M_{\text{tor}}\) is divisible, i.e., for all \(a \in A\setminus\{0\}\) the map
\[ \psi_a : M_{\text{tor}} \to M_{\text{tor}} \]
is surjective. Namely, if \(x \in M_{\text{tor}}\), then the equation \(\psi_a(z) = x\) has solutions in \(M_{\text{tor}}\). Consequently, \(M_{\text{tor}}\) is an injective module. Cf. Theorem 7.1 in [14], where it is shown that divisibility is the same as injectivity for modules over a PID; it is not difficult to extend this to a theorem over Dedekind domains.

2. \(M/M_{\text{tor}}\) has a natural \(K_A\)-module structure. The following sequence of \(A\)-modules is exact:
\[ 0 \to M_{\text{tor}} \to M \to M/M_{\text{tor}} \to 0. \]

Note that \(\Lambda\) is torsion-free, hence \(\Lambda \oplus M_{\text{tor}} \to M\). Consider the projection map
\[ s : \Lambda \oplus M_{\text{tor}} \to M_{\text{tor}}, \quad by \alpha \oplus m \mapsto m. \]
Because \(M_{\text{tor}}\) is an injective module, it follows that \(s\) extends to a map
\[ s : M \to M_{\text{tor}}. \]
So the exact sequence splits according to \(s\) and \(M \cong M_{\text{tor}} \oplus M/M_{\text{tor}}\).

3. According to 2., we may write \((\psi_f)^{-1}\Lambda = N_1 \oplus N_2\) with \(N_1 = \psi[f](V)\).
being the torsion part and $N_2 \cong m$ being the projective part of $(\psi_f)^{-1}\Lambda$. Let $e_1, e_2$ be generators of $N_2$ such that $\psi_m e_1 = \psi_m e_2$. Then

$$(\psi_m^{-1})^{-1} e_1 \equiv (\psi_m^{-1})^{-1} e_2 \pmod{\text{tor}}.$$ 

Let $\zeta_i \in M$ be the unique element with $\zeta_i \mapsto (\psi_m^{-1})^{-1} e_i \in M/\text{tor}$ and $s(\zeta_i) = 0$. Then $\zeta_1 - \zeta_2 \mapsto 0 \in M/\text{tor}$ and $s(\zeta_1 - \zeta_2) = 0$. Consequently, $\zeta := \zeta_1 = \zeta_2$ is the element we are looking for. \hfill \Box

**Proposition 7.6.** Every rank 2 Drinfeld module $\varphi$ over $V$ with $K_V$-rational $f$-torsion and with stable reduction of rank 1 and type $m$ is induced by $\varphi_m^{\text{td}}$ via the ring homomorphism

$$h_{\psi, \mu} : R[[x]] \longrightarrow V,$$

i.e., $h_{\psi, \mu}(\varphi_m^{\text{td}}) \cong_V \varphi$. Extending $h_{\psi, \mu}$ to

$$R((x)) \longrightarrow K_V$$

maps the $f$-torsion of $\varphi_m^{\text{td}}$ isomorphically to the $f$-torsion of $\varphi$.

**Proof.** In Section 6 we introduced skew polynomials $P, Q_1, Q_2 \in R\{\tau\}$ with $Q_1 \circ P = \psi_m^{\text{un}}$. In particular, we see that $\tilde{h}(P)$ divides $P' := \gcd(\psi_{m_1}, \psi_{m_2}) \in K_V\{\tau\}$. We may assume that $m$ is not contained in the kernel of $A_f \longrightarrow V$. Therefore, $\deg r P' = \deg r h_{\psi, \mu}(P)$. Consequently, there exist elements $\beta_i \in K_V\{\tau\}$ with

$$\tilde{h}(P) = \beta_1 \psi_{m_1} + \beta_2 \psi_{m_2}.$$ 

Let $\zeta$ be the element from Lemma 7.5, and define $\frac{1}{\zeta} := \tilde{h}(P)(\zeta)$. As $\psi_{m_1}(\zeta)$ and $\psi_{m_2}(\zeta)$ generate the projective part of $(\psi_f)^{-1}\Lambda \subset K_V$, we see that $\frac{1}{\zeta} \in K_V$ and $z \in V$.

We extend $h$ to

$$h_{\psi, \mu} : R[[x]] \longrightarrow V \quad \text{by} \quad x \mapsto z.$$ 

One can easily verify that

$$h_{\psi, \mu}((\psi_f^{\text{un}})^{-1}\Lambda_m^{\text{td}}) = (\psi_f)^{-1}\Lambda.$$ 

\hfill \Box

**Remark 7.7.** The proof of Proposition 2.5 in [21] is not entirely complete. One way of completing it, is by adding the construction of the lattice as is done here. This makes sure that the morphism $h$ in Proposition 2.5 in [21] indeed exists.

**Theorem 7.8.** Every pair $(\varphi, \lambda)$ consisting of a rank 2 Drinfeld module $\varphi$ over $V$ with stable reduction of rank 1 and level $f$-structure $\lambda$ is induced by $(\varphi^{\text{td}}, \lambda^{\text{td}})$ via the unique ring homomorphism $h_{\varphi, \lambda}$.

**Proof.** Let $(\psi, \mu, \Lambda)$ be the triple associated to $(\varphi, \lambda)$, and let $m$ be the type of $\Lambda$. Let $h_{\psi, \mu} : R[[x]] \longrightarrow V$ be the morphism such that $h_{\psi, \mu}(\varphi_m^{\text{td}}) = \varphi$. Let $\sigma \in \text{GL}_2(A/fA)$ such that $h_{\psi, \mu}(\lambda_m^{\text{td}} \circ \sigma) = \lambda$. The element $\sigma$ lies in a unique class $N\sigma_i$. Write $\sigma = \tau\sigma_i$ with $\tau \in N$. Let $h_{\tau}$ be the ring homomorphism which is defined in Proposition 7.3. Define

$$h_{\varphi, \lambda} : \oplus R[[x]]_{(m', \sigma_j)} \longrightarrow V$$
as follows: $h_{\varphi,\lambda}$ equals $h_{\psi,\mu} \circ h_{\tau}$ on $R[[x]]_{(m,\sigma)}$ and is zero on the other copies of $R[[x]]$.

The uniqueness follows from the construction and from Theorem 7.4. \(\square\)

8. The compactification of $M^2(f)$

In this section we describe a compactification $\overline{M^2(f)}$ of $M^2(f)$, which is analogous to the compactification of the classical modular curves given by Katz and Mazur in Chapter 8 of [15]. We define the scheme of cusps, which we call $\text{Cusps}$. This is a closed subscheme of $M^2(f)$. Moreover, we consider the formal scheme $\hat{\text{Cusps}}$, which is the completion of $\overline{M^2(f)}$ along $\text{Cusps}$.

In the following section we will use the universal Tate-Drinfeld module as defined in the previous sections to describe the scheme of cusps.

8.1. The morphism $j_{a}$. Let $a \in A\setminus \mathbb{F}_q$. Let $(\varphi, \lambda)$ be the universal Drinfeld module of rank 2 with level $f$-structure over $M^2(f)$. Let $B$ be the ring with $\text{Spec}(B) = M^2(f)$ and write

$$\varphi_a = \sum_{i=0}^{2\deg(a)} b_i \tau^i$$

with $b_i \in B$ for all $i$ and $b_{2\deg(a)} \in B^*$.

Let $j_{a} : M^2(f) \longrightarrow \mathbb{A}^1_{A_f}$ be the morphism given by

$$j_{a}^0 : A_f[j] \longrightarrow B, \quad j \mapsto b_{\deg(a)}^{\deg(a)} + 1 / b_{2\deg(a)};$$

cf. [16, 4.2]. Clearly, $j_{a}$ factors over $M^2(1)$.

Lemma 8.1. The morphism $j_{a}$ is finite and flat.

Proof. (This is the proof of Proposition 4.2.3 in [16].) The morphism $j_{a}$ is of finite type, hence we may use the valuative criterion to prove properness. Suppose that $V$ is a discrete valuation ring, let $K_V$ be its quotient field, and suppose that there are morphisms given that make the following diagram commutative.

$$\begin{array}{ccc}
\text{Spec}(K_V) & \longrightarrow & M^2(f) \\
\downarrow & & \downarrow j_{a} \\
\text{Spec}(V) & \longrightarrow & \mathbb{A}^1_{A_f}.
\end{array}$$

Note that the upper horizontal map gives rise to a (unique) map

$$\text{Spec}(V) \longrightarrow M^2(f)$$

if and only if the pull-back of the pair $(\varphi, \lambda)$ via the upper horizontal map has good reduction at the maximal ideal of $V$. So $j_{a}$ is proper if and only if $j_{a}(x) \in V$ implies that the pull-back $(\varphi', \lambda')$ over $K_V$ has good reduction. If this pull-back does not have good reduction, it has stable reduction of rank 1. Suppose that $(\varphi, \lambda)$ has stable reduction of rank 1, then there exists an element $s \in K_V^*$ such that $s\varphi_a s^{-1}$ has all coefficients in $V$, the $\deg(a)$'th coefficient has valuation 0, and the $2\deg(a)$'th coefficient has strictly positive valuation. This means that the image of $j_{a}(x)$ is not in $V$. We conclude that $j_{a}$ is proper.

Because each connected component of $M^2(f)$ is an affine variety over $\mathbb{F}_q$, it
follows by [11, ex. II.4.6] that \( j_a \) restricted to such a connected component is finite. And thus \( j_a \) is finite.

The finite ring homomorphism \( j_a^\# \) is injective. Let \( \mathfrak{p} \subset B \) be a prime ideal lying above \( \mathfrak{p} \subset A_f[\jmath] \). Then both local rings \( B_{\mathfrak{p}} \) and \( A_f[\jmath]_\mathfrak{p} \) are regular and of equal dimension. By the finiteness of \( j_a \) it follows that \( B_{\mathfrak{p}} \) is a free \( A_f[\jmath]_\mathfrak{p} \)-module, cf. Corollary IV.22 in [19]. Hence, \( j_a \) is flat. \( \square \)

8.2. **The compactification.** The ring \( B \) is a finite \( A_f[\jmath] \)-algebra via \( j_a^\# \). Let \( C \) denote the normalization of \( A_f[\jmath] \) inside the quotient ring of \( B \). Then \( C \) is finite over \( A_f[\jmath] \); cf. Corollary 13.13 in [6].

The compactification \( \overline{M}^2(f) \) of \( M^2(f) \) is defined as the scheme obtained by glueing \( \text{Spec}(B) \) and \( \text{Spec}(C) \) along their intersection. We obtain a finite morphism

\[
\overline{j}_a : \overline{M}^2(f) \rightarrow \mathbb{P}_{A_f}^1.
\]

The following diagram is cartesian

\[
\begin{array}{ccc}
M^2(f) & \rightarrow & \overline{M}^2(f) \\
\downarrow{j_a} & & \downarrow{\overline{j}_a} \\
A^1_{A_f} & \rightarrow & \mathbb{P}_{A_f}^1.
\end{array}
\]

**Remark 8.2.** By Theorem [9.1] it follows that \( \overline{j}_a \) is flat in the points of the boundary of \( \overline{M}^2(f) \); therefore, \( \overline{j}_a \) is flat.

**Lemma 8.3.** The scheme \( \overline{M}^2(f) \) is independent of the chosen element \( a \).

**Proof.** Let \( B' \) be a connected component of \( B \). Let \( a_1, a_2 \in A \backslash \mathbb{F}_q \) and consider the maps \( j_{a_i} : A_f[\jmath_i] \rightarrow B' \). Let \( C_i \) be the integral closure of \( A_f[\jmath_i] \) inside \( K_{B'} \). Let \( X_i \) be the scheme obtained by glueing \( \text{Spec}(C_i) \) and \( \text{Spec}(B') \) along their intersection.

Let \( \mathfrak{p} \in X_1 \backslash \text{Spec}(B') \) be any prime of height one of \( C_1 \), then \( \frac{1}{j_1} \in \mathfrak{p} \). Let \( \nu_{\mathfrak{p}} \) be the valuation of \( K_{B'} \) given by \( \mathfrak{p} \). If \( \nu_{\mathfrak{p}}(\frac{1}{j_2}) \leq 0 \), then \( \mathfrak{p} \) would correspond to a valuation of \( K_A[\jmath_2] \) and therefore to a valuation of \( B' \); cf. Section VII.9 in [2]. Consequently, \( \nu_{\mathfrak{p}}(\frac{1}{j_2}) > 0 \). The same is true if we interchange \( j_1 \) and \( j_2 \). We conclude that the set of valuations \( \nu \) of \( K_{B'} \) with \( \nu(j_i) > 0 \) does not depend on \( i \). Therefore, \( X_1 = X_2 \). \( \square \)

8.3. **The scheme of cusps.** To describe the boundary of \( \overline{M}^2(f) \), we introduce the scheme of cusps, which we call \( \text{Cusps} \). Let \( \tau \) be the intersection of all height 1 primes \( \mathfrak{p} \) containing \( \frac{1}{j} \), i.e., \( \tau = \text{rad}(\frac{1}{j}) \). And \( V(\tau) = \overline{M}^2(f) \backslash M^2(f) \).

Let \( \tilde{C} := \lim_{\leftarrow} C/\mathfrak{p}^n \).

**Lemma 8.4.** The ring \( \tilde{C} \) is normal and a finite \( A_f[\jmath] \)-algebra.

**Proof.** The ring \( B \) is regular. So \( C = \bigoplus_i C_i \) where each \( C_i \) is an integrally closed domain. The ring \( C \) is excellent. By [9, 7.8.3.vii] it follows that \( \tilde{C} \) is normal. As \( C' \) is a finite \( A_f[\jmath] \)-algebra, it follows that \( \tilde{C} \) is a finite \( A_f[\jmath] \)-algebra. \( \square \)
We denote $A_f(\frac{1}{j}) := A_f[\frac{1}{j}][j]$. Furthermore, we define the formal scheme
\[ \widehat{\text{Cusps}} := \text{Spf}(\widehat{C}), \]
which is the formal neighbourhood of $\text{Cusps}$. Let $\mathcal{O}$ denote the structure sheaf of $\widehat{\text{Cusps}}$. The scheme of cusps is defined as
\[ \text{Cusps} := (\widehat{\text{Cusps}}, \mathcal{O}/\mathfrak{v}) = \text{Spec}(C/\mathfrak{v}). \]

**Theorem 8.5.** The $A_f$-morphism $w_f$ given by Theorem 4.1 can be extended to an $A_f$-morphism
\[ w_f : \overline{M}^2(f) \to M^1(f). \]
Its restriction to the scheme of cusps gives a finite $A_f$-morphism
\[ w_f : \text{Cusps} \to M^1(f). \]

**Proof.** The Weil pairing gives an $R$-algebra structure $R \to B$. Because $R$ is integral over $A_f$, $B$ is the integral closure of $A_f[j]$ in the quotient ring of $B$ and $C$ is the integral closure of $A_f[\frac{1}{j}]$ in this quotient ring, it follows immediately, that the ring homomorphism $R \to B$ gives a ring homomorphism $R \to C$. These two maps glue to
\[ w_f : \overline{M}^2(f) \to M^1(f). \]
The restriction of $w_f$ to $\text{Cusps}$ is given by $R \to C \to C/\mathfrak{v}$. As $C$ is finite over $A_f[\frac{1}{j}]$, it follows that $C/\mathfrak{v}$ is finite over $A_f$. As $R$ is finite over $A_f$, we may conclude that $w_f$ restricted to $\text{Cusps}$ is finite. \(\square\)

**9. The cusps and the Tate-Drinfeld module**

In the previous section, we defined the scheme of cusps and the formal scheme $\widehat{\text{Cusps}} = \text{Spf}(\widehat{C})$. In this section, we will relate these schemes to the universal Tate-Drinfeld module, which we introduced in Section 7. In fact, using the universal property of the universal Tate-Drinfeld module and the $\text{Cl}(A) \times \text{GL}_2(A/fA)$-equivariance of the Weil-pairing, we will be able to prove the following theorem. If we write ‘$\oplus_p$’, we mean the direct sum over all minimal primes $p$ containing $\frac{1}{j}$.

**Theorem 9.1.** There exists an $R[\frac{1}{j}]$-linear isomorphism
\[ \widehat{C} \cong \oplus_p \lim \leftarrow C/p^n \sim \oplus R[\frac{x}{m, \sigma_i}], \]
such that
\[ \text{Cusps} \sim \oplus R_{(m, \sigma_i)}. \]

From this theorem we can derive the following important corollary:

**Corollary 9.2.** The compactification $\overline{M}^2(f)$ of $M^2(f)$ is regular, and even smooth over $\text{Spec}(A_f)$. Furthermore, the scheme of cusps is isomorphic to
\[ \text{Cusps} \cong \coprod_{(m, \sigma_i)} M^1(f), \]
where \( \mathfrak{m} \) runs through \( \text{Cl}(A) \) and \( \sigma_i \) runs through the cosets of \( N\backslash\text{GL}_2(A/fA) \) where

\[
N = \left( \begin{array}{cc}
F_q^* & A/fA \\
0 & (A/fA)^*
\end{array} \right) \subset \text{GL}_2(A/fA).
\]

Consequently, the scheme \( \text{Cusps} \) consists of \( h(A)\cdot \text{SL}_2(A/fA) \cdot (A/fA) \cdot (q-1) \) copies of \( M^1(f) \).

**Proof.** By Theorem 9.1 the ring \( C \) is regular in the points above \( \frac{1}{j} \) and thus \( C \) is regular. Consequently, \( \overline{M}^2(f) \) is regular. The description of \( \text{Cusps} \) and the number of its components follows from Theorem 9.1.

To prove smoothness over \( \text{Spec}(A_f) \), note that by the corollary to Proposition 5.4 in [5], the morphism \( \overline{M}^2(f) \rightarrow \text{Spec}(A_f) \) is smooth. So we only need to prove smoothness in the closed points of \( \text{Cusps} \). We have

\[
\hat{C} \cong \oplus R[[x]_{(m,\sigma_i)}],
\]

so \( \hat{C} \) is formally smooth over \( A_f \). This implies by 17.5.1 and 17.5.3 in [10] that the morphism \( \overline{M}^2(f) \rightarrow \text{Spec}(A_f) \) is smooth in the closed points of \( \text{Cusps} \). \( \square \)

9.1. **The proof of Theorem 9.1** The rest of this section is devoted to proving Theorem 9.1. The universal Tate-Drinfeld module over \( \mathbb{Z} \) gives rise to an \( A_f \)-morphism

\[
\text{Z}_{\text{open}} \rightarrow M^2(f),
\]

where \( \text{Z}_{\text{open}} \) denotes the localization of \( \mathbb{Z} \) at \( (x) \), i.e.,

\[
\text{Z}_{\text{open}} = \text{Spec}(\oplus R((x)_{(m,\sigma_i)})).
\]

It follows from Remark 7.2 that this morphism is \( \text{Cl}(A) \times \text{GL}_2(A/fA) \)-equivariant.

Let \( \text{Z}_{x=0} \) denote the scheme

\[
\text{Spec } (\oplus R[[x]_{(m,\sigma_i)}/(x)]).
\]

The line of argument is as follows: in Lemma 9.3 we show how to relate the Tate-Drinfeld module to the study of the cusps, and in Lemma 9.4 we describe the scheme \( \text{Cusps} \). This latter lemma enables us to lift the isomorphism \( \text{Cusps} \rightarrow \text{Z}_{x=0} \) to an isomorphism \( \hat{C} \rightarrow \mathbb{Z} \). Let \( \hat{C}_p \) denote the completion of the local ring of \( C \) at \( p \).

**Lemma 9.3.** Every \( R[[x]_{(m,\sigma_i)}] \) is a finite \( A_f[\frac{1}{j}] \)-algebra. Consequently, the morphism \( \text{Z}_{\text{open}} \rightarrow M^2(f) \) comes from a morphism

\[
h_1 : \text{Z} \rightarrow \text{Spec}(C) \subset \overline{M}^2(f).
\]

Moreover, the universal property of the Tate-Drinfeld module gives rise to a morphism

\[
h_2 : \text{Spec}(\oplus \hat{C}_p) \rightarrow \text{Z}.
\]

The composition \( h_1 \circ h_2 \) is the natural morphism.
Proof. Write $R[x] = R[\{x\}_{(m,\sigma_i)}$ for some pair $(m, \sigma_i)$ and write $(\varphi^{\text{td}}_m)_a = \sum c_i \tau^i$ for the element $a \in A$ which is used to define $j_a$. Then $c_{2 \deg(a)} \in R((x))^*$, because $\varphi^{\text{td}}_m$ is a Drinfeld module over $R((x))$. Moreover,

$$\varphi^{\text{td}}_m \mod (x) = \psi.$$ 

So the coefficient $c_{\deg(a)} \in R[\{x\}]$. This implies by definition, that $\frac{1}{f}$ is mapped to $\alpha \cdot x^k \in R[x]$, with $\alpha \in R[\{x\}]$ and $k \in \mathbb{Z}_{>0}$. From this it follows that $R[x]$ is a finite $R[\{x\}]$-module. The morphism

$$Z_{\text{open}} \longrightarrow M^2(f)$$

comes from a ring homomorphism

$$C[j] \longrightarrow (\oplus R[\{x\}_{(m,\sigma_i)}] \otimes A_f[\frac{1}{j}] A_f[\frac{1}{j}]),$$

Because $C$ is finite over $A_f[\frac{1}{j}]$ and $R[x]$ is finite over $A_f[\frac{1}{j}]$, it follows that the image of $C$ under this ring homomorphism lies in $\oplus R[\{x\}_{(m,\sigma_i)}].$

For the ‘moreover’-part, let $p \subset C$ be a minimal prime ideal containing $\frac{1}{j}$.

The ring $\hat{C}_p$ is a complete discretevaluation ring and comes equipped with a Tate-Drinfeld structure via the morphism

$$\text{Spec}(K_{\hat{C}_p}) \longrightarrow M^2(f).$$

By Theorem \ref{thm:7.8} there exists a unique ring homomorphism

$$\oplus R[\{x\}_{(m,\sigma_i)}] \longrightarrow \hat{C}_p$$

which induces on $\hat{C}_p$ this Tate-Drinfeld structure. This can be done for every minimal prime $p$ containing $\frac{1}{j}$.\hfill \Box

**Lemma 9.4.** The morphism $h_1$ induces an isomorphism

$$Z_{x=0} \overset{\sim}{\longrightarrow} \text{Cusps}.$$  

Every pair $(m, \sigma_i)$ corresponds via this isomorphism to one and only one minimal prime $p \subset C$ containing $\frac{1}{j}$. Consequently,

$$\hat{C} = \oplus_p \lim_{\leftarrow} C/p^n.$$  

Proof. We will first prove that the number of irreducible components of $\text{Cusps}$ equals the number of irreducible components of $Z_{x=0}$. Subsequently, we will show that these components of $\text{Cusps}$ intersect nowhere. Because $\text{Cusps} \cong \text{Spec}(C/(\cap p))$, the irreducible components of $\text{Cusps}$ are in a one-to-one correspondence to the minimal primes containing $\frac{1}{j}$. The morphisms $h_1$ and $h_2$ introduced in Lemma \ref{lem:9.3} give rise to the following maps on the sets of irreducible components:

$$\{\text{irr. comp. of Cusps}\} \xrightarrow{h_2} \{\text{irr. comp. of } Z_{x=0}\} \xrightarrow{h_1} \{\text{irr. comp. of Cusps}\}.$$  

As a morphism of schemes $h_1 \circ h_2$ is the natural map. Therefore, the composition $h_1 \circ h_2$ on the set of irreducible components is the identity. Consequently, $h_2$ on the irreducible components is injective.

Moreover, the set of irreducible components of $Z_{x=0}$ is by definition one orbit under the elements $(m, \sigma_i)$. Clearly, the map $h_1$ on the set of connected
components is equivariant under this group action, and as the image of \( h_2 \) is not empty, it follows that the first map on the irreducible components is also surjective. So we may conclude that the number of irreducible components of \( \text{Cusps} \) equals the number of irreducible components of \( \mathcal{Z}_{x=0} \).

The irreducible components of \( \mathcal{Z}_{x=0} \) intersect nowhere. We will prove that this is also the case for the irreducible components of \( \text{Cusps} \). By the extension of the Weil pairing to \( \overline{M^1(f)} \), the ring \( C \) comes equipped with an \( R \)-algebra structure. Let

\[
\zeta : R \longrightarrow \oplus R[[x]]_{(m, \sigma_i)}
\]

denote the composition \( R \xrightarrow{w_f^\#} C \xrightarrow{h_1^\#} \oplus R[[x]]_{(m, \sigma_i)} \).

Choose any maximal ideal \( n \subset R \) and let \( q \) run over all minimal primes of \( C/\mathfrak{n}C \) containing \( \frac{1}{j} \). We write

\[
\widehat{C/\mathfrak{n}C} = \lim_{\leftarrow} \left( C/\mathfrak{n}C \right)/(1/n^p),
\]

and \( (\widehat{C/\mathfrak{n}C})_q \) for the completion along \( q \) of the local ring \( (C/\mathfrak{n}C)_q \).

In this case we have analogues of the morphisms \( h_1, h_2 \), namely, \( R \)-algebra homomorphisms

\[
\widehat{C/\mathfrak{n}C} \xrightarrow{\tilde{h}_1} \oplus R/\zeta(\mathfrak{n})[x]_{(m, \sigma_i)} \xrightarrow{\tilde{h}_2} \oplus \widehat{C/\mathfrak{n}C}_q.
\]

And now, as before, we define maps on the sets of irreducible components:

\[
\tilde{h}_2 : \{\text{irr. comp. of } \text{Cusps} \times \text{Spec}(R/n)\} \longrightarrow \{\text{irr. comp. of } \mathcal{Z}_{x=0} \times \text{Spec}(R/n)\};
\]

\[
\tilde{h}_1 : \{\text{irr. comp. of } \mathcal{Z}_{x=0} \times \text{Spec}(R/n)\} \longrightarrow \{\text{irr. comp. of } \text{Cusps} \times \text{Spec}(R/n)\}.
\]

The composition of these two maps on the set of irreducible components is the identity. Namely, the composition \( \tilde{h}_1 \circ \tilde{h}_2 \) is the natural map on the rings. Using the same argument as above shows that \( \tilde{h}_2 \) on the irreducible components is a bijection.

We conclude that for every prime \( n \subset R \) the number of irreducible components of

\[
\text{Cusps} \times \text{Spec}(R/n)
\]
equals the number of irreducible components of \( \text{Cusps} \). Recall that by Theorem \([8,5]\) the morphism \( w_f : \text{Cusps} \longrightarrow M^1(f) \) is finite. Therefore, if the irreducible components would intersect above some prime ideal \( n \subset R \), then \( \text{Cusps} \times \text{Spec}(R/n) \) would have less irreducible components. As this is not the case, we conclude that the irreducible components of \( \text{Cusps} \) intersect nowhere.

We write

\[
\text{Cusps} = \text{Spec}(\oplus S_{(m, \sigma_i)})
\]

where \( \text{Spec}(S_{(m, \sigma_i)}) \) are the connected components of \( \text{Cusps} \). For every pair \((m, \sigma_i)\), we get \( R \)-linear ring homomorphisms on the connected components:

\[
S_{(m, \sigma_i)} \xrightarrow{h_1^\#} R \xrightarrow{h_2^\#} S_{(m, \sigma_i)}.
\]
Because the composition is the identity and \( S_{(m,\sigma_i)} \) is finite over \( R \), it follows that
\[
S_{(m,\sigma_i)} \cong R.
\]
For the latter two statements of the lemma, note that the isomorphism implies that the minimal primes \( p \) are relatively prime and, consequently, \( r = \prod p \).

The next step is to lift the isomorphism from Lemma 9.4 to an isomorphism
\[
\hat{C} \xrightarrow{\sim} \bigoplus R[[x]]_{(m,\sigma_i)}.
\]
Let
\[
W := \varprojlim C/p^n
\]
for some minimal prime \( p \) containing \( \frac{1}{j} \).

**Lemma 9.5.** The ring \( W \) is isomorphic to \( R[[x]] \).

**Proof.** By Lemma 8.4, the ring \( W \) is integrally closed and a finite \( A_f[[x]] \)-algebra, and by the isomorphism of Lemma 9.4, the ring \( W \) is a finite \( R[[x]] \)-algebra.

The completion of the local ring \( W_p \) is isomorphic to \( \hat{C}_p \). The morphisms \( h_1 \) and \( h_2 \) give on the completions of the local rings injective maps
\[
\hat{C}_p \xrightarrow{h_1^\#} K_R[[x]] \xrightarrow{h_2^\#} \hat{C}_p.
\]
As \( h_2^\# \circ h_1^\# \) is the identity, there exists an isomorphism \( K_R[[x]] \cong \hat{C}_p \).

We conclude that \( W \) is regular. Therefore, we may assume that \( x \) is an element of \( W \) and that \( W \) is a finite \( R[[x]] \)-algebra. So we get injective \( R[[x]] \)-linear ring homomorphisms
\[
R[[x]] \rightarrow W \rightarrow R[[x]]
\]
where the first map is the \( R[[x]] \)-structure morphism of \( W \) and the second map is \( h_1^\# \). We conclude that \( W \cong R[[x]] \).

This enables us to prove Theorem 9.1

**Proof of Theorem 9.1** By the previous lemma, it follows that
\[
\hat{C} \cong \bigoplus R[[x]]_{(m,\sigma_i)}.
\]
Together with Lemma 9.4, the theorem follows.

10. Components of \( M^2(f) \)

In this section we describe the geometric components of \( \overline{M^2(f)} \) and prove the connectedness of \( M^2(f) \). For a non-zero prime \( \mathfrak{p} \subset R \) we write \( \kappa(\mathfrak{p}) := R/\mathfrak{p} \). The first result is the following:

**Theorem 10.1.** The scheme
\[
\overline{M^2(f)} \times_{A_f} M^1(f)
\]
consists of $h(A) \cdot [(A/fA)^* : \mathbb{F}_q^*]$ connected components, which are all geometrically connected. Moreover, for every non-zero prime ideal $\mathfrak{P} \subset R$ the fibre at $\mathfrak{P}$

$$\overline{M}^2(f) \times \text{Spec}(\kappa(\mathfrak{P})),$$

consists of $h(A) \cdot [(A/fA)^* : \mathbb{F}_q^*]$ connected components, which are all geometrically connected.

**Proof.** Let $K_\infty$ be the completion of the quotient field of $A_f$ along the point $\infty$, and let $C_\infty$ denote the completion of the algebraic closure of $K_\infty$. By the analytic theory, as is shown in [22], we know that

$$\overline{M}^2(f) \times_{A_f} \text{Spec}(C_\infty)$$

consists of $h(A) \cdot [(A/fA)^* : \mathbb{F}_q^*]$ components. Because $R$ is a Galois extension of $A_f$ with Galois group $G$, we have $R \otimes_{A_f} R \cong \oplus G R$. By the Weil pairing one sees that

$$\overline{M}^2(f) \times_{A_f} M^1(f) \xrightarrow{\text{weil}} M^1(f) \times_{A_f} M^1(f)$$

consists of $h(A) \cdot \# G$ connected components. As $\# G = [(A/fA)^* : \mathbb{F}_q^*]$, these components are geometrically connected components.

Consider the fibres over $R$. Let $\mathfrak{P} \subset R$ be a non-zero prime ideal and let $V$ be the completion along $\mathfrak{P}$ of the local ring $R_{\mathfrak{P}}$. Suppose $\overline{M}^2(f) \times R \text{Spec}(\kappa(\mathfrak{P}))$ has more than one connected component, then also $\overline{M}^2(f) \times R \text{Spec}(V/\mathfrak{P})$ has more than one connected component for every $n$ and consequently, both $\overline{M}^2(f) \times \text{Spec}(V)$ and $\overline{M}^2(f) \times R \text{Spec}(K_V)$ consist of more than one component. This, however, contradicts the fact that $\overline{M}^2(f) \times R M^1(f)$ is geometrically connected. So we conclude that $\overline{M}^2(f) \times R \text{Spec}(\kappa(\mathfrak{P}))$ is geometrically connected.

This theorem enables us to say something about the Drinfeld modular curves. Let, as before, $N = \left( \begin{array}{cc} \mathbb{F}_q^* & A/fA \\ A/fA & (A/fA)^* \end{array} \right)$.

**Theorem 10.2.** For every $R$-field $K$ the curve $\overline{M}^2(f) \times R \text{Spec}(K)$ is a smooth, irreducible curve containing $h(A) \cdot [\text{Gl}_2(A/fA) : N]$ cusps.

**Proof.** Clearly, the scheme $\text{Cusps}$ consists of $h(A) \cdot [\text{Gl}_2(A/fA) : N]$ copies of $R$. Consequently,

$$\text{Cusps} \times \text{Spec}(K)$$

consists of $h(A) \cdot [\text{Gl}_2(A/fA) : N]$ points. The irreducibility follows immediately from the proof of Theorem 10.1. □

10.1. **The analogue of $X_0(N)$.** The analogue in the setting of Drinfeld modular curves of the modular curve $X_0(N)$ is the curve

$$X_0(f) := \overline{M}^2(f)/H, \quad \text{where } H = \left( \begin{array}{cc} (A/fA)^* & A/fA \\ 0 & (A/fA)^* \end{array} \right) \subset \text{Gl}_2(A/fA).$$

One may deduce from Theorem 10.1 the following theorem concerning the cusps and the geometric components of $\overline{M}^2(f)/H$. Define $R_0 = R^{(A/fA)^*/\mathbb{F}_q^*}$, i.e., $\text{Spec}(R_0) = M^1(1)$. Write $\text{Cusps}_0$ for the scheme of cusps of $X_0(f)$.
Theorem 10.3. The Weil pairing induces an isomorphism
\[ \text{Cusps}_0 \sim \coprod_{(m,\rho)} M^1(1) \]
where \( \rho \) runs through the double cosets \( \mathbb{N} \backslash \text{GL}_2(A/fA) / H \).
The scheme \( X_0(f) \) is connected, and for any \( \mathbb{R}_0 \)-field \( K \) the scheme \( X_0(f) \times \text{Spec}(K) \) consists of \( h(A) \) geometrically connected components.

Proof. The morphism \( w_f \) gives an isomorphism between \( M^1(f) \) and any connected component of \( \text{Cusps} \). Consequently, \( w_f \) gives an isomorphism \( \text{Cusps} \rightarrow \oplus (m,\sigma_i)M^1(f) \). Recall that \( N \) acts trivially on each copy of \( M^1(f) \). Furthermore, the action of \( \sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \) with \( \alpha \in (A/fA)^* \) on \( \text{Cusps} \) is as follows. Let \( i, j \in \mathbb{N} \) such that \( \sigma_i \circ \sigma \in N \sigma_j \), and consider \( \alpha \) as an element of the Galois group \( \text{Gal}(K_R/K_{R_0}) \cong (A/fA)^*/\mathbb{F}_q^* \), then \( \sigma \) acts as
\[ M^1(f)_{(m,\sigma_i)} \xrightarrow{\alpha} M^1(f)_{(m,\sigma_j)}. \]
Because \( H \) contains the subgroup \( \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \mid \alpha \in (A/fA)^* \right\} \), we see that dividing out \( \text{Cusps} \) by \( H \) gives an isomorphism
\[ \text{Cusps}_0 \sim \oplus_{(m,\rho)} M^1(1) \]
where \( \rho \) runs through the cosets of \( \mathbb{N} \backslash \text{GL}_2(A/fA) / H \).
The number of components follows immediately from Theorem 10.1. \qed

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