SUBTLETIES AND FANCIES IN
GAUGE THEORY NON TRIVIAL VACUUM *

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(June 6, 1994)

Abstract

The one loop effective potential for a non-Abelian gauge configuration is
analyzed using the background field method. The Savvidy result and the
non-Abelian ansatz, the other alternative possible background that generates
a constant color magnetic field configuration, are compared. This second pos-
sibility is very interesting because it avoids the possible coordinate singularity,
Det$B_\mu^a = 0$, and it is easy to implement in lattice simulations. We empha-
size the interesting dependence of the potential by the gauge fixing parameter
$\alpha$, when the loop expansion is performed around a non trivial background
configuration. Finally, we point out some crucial differences in analyzing the
vacuum structure between non-Abelian gauge theories and the cases of scalar
and Abelian gauge theories.

To appear in Proceeding of the “Workshop on Quantum Infrared Physics”,

*This work is supported in part by funds provided by the U. S. Department of Energy (D.O.E.)
under contracts #DE-FG02-91ER40676
I. INTRODUCTION

Even after many years of intensive efforts, the vacuum structure of non-Abelian gauge theories still remains elusive to us. For a recent article where several aspects of the non-trivial vacuum for these theories have been analyzed see [1], and for a comprehensive review see [2].

As a prelude to truly non-perturbative evaluation of the effective I.R. structure of non-Abelian theories in terms of lattice, the conventional approach is to analyze the effective action using the so-called background field method.

For scalar and Abelian gauge theories, we can assert that we understood very well their I.R. structures. In these cases, the vacuum is not ill-defined and this allows the construction of the complete physical space of the theory by acting with creator operators on the vacuum state. This procedure is not ambiguous and it is formally well understood.

However, the situation of the non-Abelian case is very different, and very little has to do with the Abelian and scalar cases. For non-Abelian theories, there is no simple way to decompose the field into non-interacting “field oscillators”; they are non-linear and coupled together. To be more precise, there are two different regions: for large Euclidean momenta the effective coupling constant $g^2(p)$ is relatively small, and the perturbative picture of the plane gluonic waves is reasonable. Many perturbative results can be achieved in this region. For example, the reason why deep-inelastic scattering gives results which are so clear is not only because in high momenta transfer the gluonic gauge coupling is small, but also because the vacuum is so far away from the measured states with high momenta that the experimental data are independent of the vacuum structure.

The real problem is only with the momenta of the order of 1 GeV or smaller, where the infrared structure of the field theory shows up. For small momenta, the coupling is strong and it is not reasonable to assume that the non-Abelian gauge fields are distributed
in space-time as plane waves. This suggests a classification of the vacuum models. The first model assume that the non-perturbative field is concentrated in some space-time region, which is the instanton-type vacuum. If it is concentrated in space, but on a line along time direction, is called a soliton-type vacuum. If it is concentrated on a two-dimensional surface, it is called a string-type vacuum. However, there is also the possibility that there is a homogeneous-type vacuum that will be referred to as the “non-trivial vacuum” which we will discuss here.

The SU(2) case should not be fundamentally different from other non-Abelian gauge theory. In fact, up to now, there is no reason why the QCD vacuum should not be in the same universality class of the SU(2) case. On the other hand, SU(2) allows explicit calculations that are still inaccessible for the SU(3) case.

Beside lattice computations, the only available method to investigate the properties of Yang-Mills theories vacuum is computation of the effective potential in the presence of a background gauge field. This manifest gauge invariant scheme is based on the observation that the loop expansion corresponds to an expansion in the parameter $\hbar$ which multiplies the entire action. Hence, a shift of the fields, or a redefinition of the division of the Lagrangian into free and interacting parts can be performed at any finite order of the loop expansion without violating the manifest gauge invariance.

Analogous conclusions can be obtained by studying the beta function. This is not surprising because the beta function is dependent on the ultraviolet part of the effective potential.

Further note that this procedure is straightforward only if the effective potential does not contain operators which are irrelevant at the ultraviolet fixed point but become relevant in the infrared. In this case, the $\beta$-function by construction, depends only on the ultraviolet relevant coupling constants and knowledge of it is not sufficient to investigate the infrared properties of the effective potential.

In doing the one-loop calculation, we encountered several technical points, which were known in the literature but not emphasized enough. We specially want to stress the gauge fixing $\alpha$ parameter dependence. Therefore, we decided to briefly go through the standard
background field method in gauge theories with those technical points in mind.

II. EFFECTIVE POTENTIAL FORMALISM

Before we launch an extensive numerical simulation, it is still worthwhile to calculate the one-loop effective potential for the non-Abelian background field Eq.(3.5). Even though it is not totally trustworthy, the loop expansion can still provide indicative information. Furthermore, to examine whether the qualitative feature of the one-loop effective potential strongly depends on the choice of the background field is also interesting. In particular, we would like to find out whether the coordinate singularity has any connection with the existence of the imaginary part in the effective potential. We will prove that the imaginary part cannot be eliminated by avoiding the coordinate singularity of Det$B_i^a = 0$.

We will now outline the basic steps in the background field method in gauge theories [3]. There are two reasons to go through the well-known method. The first is to establish our own notations. The second is to emphasize two technical aspects; the question of whether it is necessary to require the background field to satisfy the classical equation of motion and the gauge choice in the evaluation of the functional integral. When possible we follow the convention of Abbott [3].

The generating functional in the background field method (in Euclidean space) is defined throughout

$$
\tilde{Z}[J, A] = \int[dQ] \det[\frac{\delta G^a}{\delta \omega^b}] \exp\left\{ - \int d^d x [\mathcal{L}(A + Q) + J_\mu^a Q^a_\mu] \right\} \prod_{x,a} \delta[G^a],
$$

(2.1)

where $G^a \equiv \partial_\mu Q^a_\mu + gf^{abc} A^b_\mu Q^c_\mu = D^b_\mu (A) Q^b_\mu$ is the background field gauge condition and $\det[\delta G^a/\delta \omega^b]$ is the corresponding Jacobian. It is clear from the above definition the background field $A^a_\mu$ is fixed and should be regarded as an external parameter in the process of doing the $Q$–integral. Note also that we do not exponentiate the gauge constraint $\prod_{x,a} \delta[G^a]$, in contrast with what was usually done. The necessity of enforcing the gauge constraint with explicit delta-functions in Eq.(2.1) will be discussed in detail.
It is easy to verify that $\tilde{Z}[J, A]$ is invariant under the infinitesimal gauge transformations

$$\delta A^a_\mu = -f^{abc} \omega^b A^c_\mu + \frac{1}{g} \partial_\mu \omega^a; \quad (2.2a)$$

$$\delta J^a_\mu = -f^{abc} \omega^b J^c_\mu. \quad (2.2b)$$

As a consequence of this invariance, the effective action in the background field method, defined as the Legendre transform of $\tilde{Z}$:

$$\tilde{\Gamma}[\tilde{Q}, A] = -\ln \tilde{Z}[J, A] + \int d^d x J^a_\mu \tilde{Q}^a_\mu, \quad (2.3)$$

with $\tilde{Q}^a_\mu = \delta \ln \tilde{Z} / \delta J^a_\mu$, is invariant under

$$\delta A^a_\mu = -f^{abc} \omega^b A^c_\mu + \frac{1}{g} \partial_\mu \omega^a; \quad (2.4a)$$

$$\delta \tilde{Q}^a_\mu = -f^{abc} \omega^b \tilde{Q}^c_\mu. \quad (2.4b)$$

As shown by Abbott [3], when $\tilde{Q}^a_\mu = 0$, $\tilde{\Gamma}[0, A]$, it coincides with the usual effective action. Since Eq.(2.4) is a pure gauge transformation when $\tilde{Q}$ vanishes, $\tilde{\Gamma}[0, A]$ must be a gauge invariant functional of $A^a_\mu$.

A standard way to evaluate $\tilde{Z}[J, A]$ explicitly is to make a loop expansion. For the purpose of convenience, we exponentiate the Jacobian $\det[\delta G^a/\delta \omega^b]$ and gauge constraints $\Pi_{x,a} \delta[G^a]$ by introducing the Faddeev-Popov ghost fields ($\theta^a$ and $\bar{\theta}^a$) and a real scalar auxiliary field $\sigma^a$, respectively,

$$\tilde{Z}[J, A] = \int [dQ][d\theta][d\bar{\theta}][d\sigma] \exp \left\{ -\int d^d x [\mathcal{L}(A + Q) + \bar{\theta}^a D^{ac}_\mu (A) D^{cb}_\mu (A) \theta^b + 2i \sigma^a D^{ab}_\mu (A) Q^b_\mu + J^a_\mu Q^a_\mu] \right\}. \quad (2.5)$$

If we only want to calculate $\tilde{Z}[J, A]$ to one-loop order, then it is equivalent to evaluating Eq.(2.5) in the steepest descent approximation,
\[
\tilde{Z}[J, A] \approx \tilde{Z}_1[J, A] \equiv \int [dQ][d\theta][d\bar{\theta}][d\sigma] \exp \left\{- \int d^4x [\mathcal{L}(A) + \mathcal{L}^{(1)}(A, Q) + \mathcal{L}^{(2)}(A, Q) + \bar{\theta}^a D^{ac}_\mu(A) D^{cb}_\nu(A) \theta^b + 2i \sigma^a D^{ab}_\mu(A) Q^b_\mu + J^a_\mu Q^a_\mu] \right\},
\]

where

\[
\mathcal{L}^{(1)}(A, Q) = -Q^a_\mu D^{ab}_\mu(A) F^{ab}_\mu(A);
\]

\[
\mathcal{L}^{(2)}(A, Q) = Q^a_\mu \left[ -\frac{1}{2} (D_\mu(A) D_\mu(A))^a \delta_{ab} + ig F^{ab}_\mu(A) \right] Q^b_\nu \equiv Q^a_\mu M^{ab}_\mu(A) Q^b_\nu.
\]

In arriving at the expression of \( \mathcal{L}^{(2)}(A, Q) \), the gauge constraint condition \( D^{ab}_\mu Q^b_\mu = 0 \) has been used.

We can now do the \( Q \)-integral in \( \tilde{Z}_1[A, J] \) and make the Legendre transform as in Eq. (2.3). By setting \( \tilde{Q}^a_\mu = 0 \) the final expression for the effective action to one-loop order can be written as \( \Gamma[A] = -\ln \tilde{Z}_1[A] \), with

\[
\tilde{Z}_1[A] = e^{-\int d^4x \mathcal{L}(A) \cdot \det[-D(A) D(A)]} \cdot \int [dQ][d\sigma] \exp \left\{- \int d^4x \left[ Q^a_\mu M^{ab}_\mu(A) Q^b_\nu + 2i \sigma^a D^{ab}_\mu(A) Q^b_\mu \right] \right\} \\
= e^{-\int d^4x \mathcal{L}(A) \cdot \det[-D(A) D(A)] \cdot \left\{ \det M^{ab}_\mu \cdot \det N^{ab}_\mu \right\}^{-1/2},
\]

with \( N^{ab} = -D^{ac}_\mu (M^{-1})^{ce}_\mu D^{eb}_\nu \). The first factor is the classical contribution. The second and third factors are the one-loop quantum corrections due to the ghost field (\( \theta \)) and fluctuation field (\( Q \)), respectively. Notice that the linear term \( \mathcal{L}^{(1)}(A, Q) \) drops out automatically in the calculation process. To remove the linear term there is no need to require that the background field satisfy the classical equation of motion \( D^{ab}_\mu(A) F^{ab}_\mu(A) = 0 \). This independence of the linear term remains true to all orders in loop expansion, because the effective potential is a sum of all one-particle irreducible diagrams with \( A \) fields on the external lines and \( Q \) fields on internal lines (when \( \tilde{Q}^a_\mu = 0 \)). In fact, the linear term is always compensated by the source \( J^a_\mu \). Remember that the physical limit in the background field method is \( \tilde{Q}^a_\mu = 0 \), not \( J^a_\mu = 0 \). On the other hand, it should be pointed out that to require the background field to satisfy the equation of motion, in order to have a good chance to represent the true minimum of the action, is an entirely different matter.
Since the background field is a constant in space-time it is convenient to work in momentum-space. Using the formula \( \det M = \exp(\text{Tr} \ln M) \) combined with Eq. (2.9), we have the expression for the effective potential to one-loop order

\[
V_{\text{eff}}(h) = V_{\text{classical}} - \int \frac{d^d p}{(2\pi)^d} \ln G(h; p) + \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \left\{ \ln M(h; p) + \ln N(h; p) \right\}.
\]  

(2.10)

An eventual non-positive definiteness of \( M(h; p) \) and \( N(h; p) \) for small \( p^2 \) indicates that the background field is only a saddle point in configuration space and therefore is not stable.

Eq. (2.10) must be regularize to eliminate divergent constants. The standard method is to use the regularization of Salam and Strathdee [4], which is a variation of Schwinger’s proper time method [5]. Peculiar tricks have to be implemented depending in how many dimensions the calculation is done. For example, because the three dimensional Yang-Mills theory is super renormalizable, the only divergence we encounter is an overall additive constant. The regularization procedure includes three steps. First, an integral representation for logarithmic function is used. For real \( E \), positive or negative,

\[
\ln(E - i\delta) = \frac{1}{\epsilon} - \frac{i\epsilon}{\epsilon \Gamma(\epsilon)} \int dt \left. t^{\epsilon-1} e^{-it(E-i\delta)} \right|_{t}.
\]

(2.11)

in the limit of \( \epsilon \to 0^+ \). The \( i\delta \) in Eq.(2.11) with \( \delta \to 0^+ \) is to ensure the convergence of the integral. When \( E \) is explicitly complex, Eq.(2.11) can be easily generalized. Then the momentum integration can be done using

\[
\int \frac{d^3 p}{(2\pi)^3} e^{-itp^2} = \frac{1}{8\pi^{3/2}} (it)^{-3/2}.
\]

(2.12)

The remaining \( t \)–integral can be converted into a Gamma function through the contour integral technique. Finally, the limit of \( \epsilon \to 0 \) is taken.

At this point we would like to comment on the implementation of the gauge condition \( \prod_{x,a} \delta[G^a] \) in Eq.(2.1). The standard approach is to exponentiate this factor,

\[
\hat{Z}[J, A; \alpha] = \int [dQ] \det \left[ \frac{\delta G^a}{\delta \omega^\mu} \right] \exp \left\{ - \int d^d x [\mathcal{L}(A + Q) - \frac{1}{2\alpha} (G^a)^2 + J^a_\mu Q^a_\mu] \right\}.
\]

(2.13)

where \( \alpha \) is the gauge fixing parameter. In order to write Eq.(2.1) into the form of Eq.(2.13), one has to generalize the gauge condition as \( \prod_{x,a} \delta[G^a - f^a] \), with \( f^a \) being arbitrary, and
then to integrate out $f^a$ with the weighting factor $\exp[-(f^a)^2/2\alpha]$. If $\tilde{Z}[J,A]$ with the generalized gauge condition were independent of $f^a$, as in the case of zero background field $A^a_\mu = 0$, $\tilde{Z}[J,A;\alpha]$ would be independent of the gauge parameter $\alpha$ and therefore the exact equivalence between Eq.(2.1) and Eq.(2.13). However, in the presence of non-vanishing background field $A^a_\mu$, $\tilde{Z}[J,A]$ in general would depend on $f^a$, as can be easily seen in Eq.(2.5) by adding a term $2i\sigma^a f^a$ to the exponential. Physically, this dependence on $f^a$ is natural, because there would exist correlations between $f^a$ and certain combinations of the background field, such as $(D^{ab}_\mu (A^a) f^b)^2$ when $A^a_\mu$ is non-vanishing. If indeed $\tilde{Z}[J,A]$ depended on $f^a$, then the integral

$$\tilde{Z}[J,A;\alpha] \equiv \int [df] \tilde{Z}[J,A]|_{f^a = f^a} \cdot \exp\left\{-\frac{1}{2\alpha}(f^a)^2\right\}$$ (2.14)

would have to depend on $\alpha$ in general, which in turn implies that Eq.(2.13) could not be equivalent to Eq.(2.1) at all times.

This gauge dependence of the effective action calculation has long been recognized in the literature [6]. It was argued by Vilkovisky that the correct choice of the gauge parameter is to take $\alpha \to 0$ limit, or to choose the Landau background field gauge. Since $\exp[-x^2/2\alpha] \propto \delta(x)$ in the limit of $\alpha = 0$, Eq.(2.13) is formally equivalent to Eq.(2.1) in the Landau background field gauge. It should be emphasized that Eq.(2.13) would still define a gauge invariant effective action with respect to the background field $A^a_\mu$. It is only the functional form of the effective action that depends on the gauge parameter when $\alpha \neq 0$. Since the $\beta$-function is gauge (or $\alpha$) independent in the dimensional regularization and minimal subtraction scheme [7], even though the finite part is explicitly $\alpha$ dependent, this has generated much confusion. Finally, note that in the context of finite temperature QCD, this gauge dependence has been emphasized by Hansson and Zahed [8].

III. SAVVIDY VS NON-ABELIAN

Because the effective potential is the truncation of the effective action at zero momenta, we are seeking solutions that give a constant magnetic field configurations. There are only
two possibilities called, respectively, the Savvidy (or Abelian), and the non-Abelian ansatz. It has been proved \[9\] that there are no other ways to generate a constant magnetic field.

We will first analyze the Savvidy’s case. As in Savvidy’s and Matinyan’s pioneer work \[10\], where they calculated up to one-loop order the effective potential for an Abelian background gauge field:

\[
A_{\mu}^a(x) = \frac{1}{2} H \delta^{a3}(x_1 \delta_{\mu 2} - x_2 \delta_{\mu 1}),
\]

which generates a constant color magnetic field \(B^a_i = H \delta_{i3} \delta_{a3}\) in the SU(2) Yang-Mills theory. It was then found, remarkably, that the vacuum with this background field is energetically favored over the perturbative vacuum (\(H = 0\)). Unfortunately, a more careful analysis \[11\] soon revealed that there exists an imaginary part in the effective potential, indicating that the background field Eq.(3.1) is not a minimum but rather a saddle point in the configuration space in the context of the loop expansion. The subsequent work by the Copenhagen group \[12\], still within the framework of loop expansion, tried to remedy the physically appealing picture of Savvidy’s vacuum by introducing inhomogeneity, leading to the so-called Copenhagen vacuum.

For the Savvidy ansatz, it is easy to solve and find the eigenvalues of the second derivative of the action, because is exactly the same operator of the Landau levels problem, which has similar symmetry. In fact, the corresponding \(M\) matrix is simply the Landau diamagnetic Hamiltonian. Therefore, their eigenvalues are:

\[
\lambda(k) = k^2 + (2n + 1) gH + 2gH S_z \]

where \(k = k_0^2 + k_z^2\), \(n = 0, 1, 2, \ldots\) and \(S_z = \pm 1\). As expected from the symmetry of the problem, the modes in the \(x, y\) directions are quantized. However, the behavior of the modes along the \(z\) axis are labeled by an arbitrary continuous parameter, \(k_z\). Moreover, note the difference from the Landau level where the quantum number \(S_z\) takes value \(S_z = \pm 1/2\), which is the electron polarization; whereas, in this case, we have the polarization of the photon that give \(S_z = \pm 1\), allowing the possibility of negative eigenvalue when \((S_z = -1, n = 0)\).
Using the technique discussed in the previous section, it is possible to evaluate the effective potential. For the reader interested in an accurate discussion of the eigenfunction and of their multiplicity see [13]. For the Savvidy case, in (3+1) space-time dimensions the effective potential up to one loop contributions is given by

\[ V(H) = \frac{1}{2} H^2 + \frac{11}{48\pi^2} g^2 H^2 \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right) - i \frac{1}{8\pi} g^2 H^2 + \ldots. \quad (3.3) \]

It is also interesting to look at the 3 dimensional case. Because the theory in 3 dimensions is superrenormalizable, we should better control the full procedure. Moreover, Lattice simulations can be obviously performed easily, both because of the volume, and because the scaling window is very large due to the polynomial dependence of the renormalizable quantity throughout the \( \beta \) function, in contrast with the well known exponential dependence of the 4 dimensions. Therefore, several authors recently analyzed the 3 dimensional case both analytically and with extensive lattice simulations [13]. In 3 space-time dimensions, for the Savvidy ansatz, we have:

\[ V(H) = \frac{1}{2} H^2 - \frac{1}{2\pi^2} \left[ 1 - \sqrt{2} - \frac{1}{4\pi} \zeta \left( \frac{3}{2} \right) \right] (gH)^{3/2} + i \frac{1}{12\pi} (gH)^{3/2} + \ldots, \quad (3.4) \]

where \( \zeta \) is the Riemann’s Zeta-function.

The non-Abelian ansatz give a similar result in the structure, but different results in the coefficients. Moreover, the physical picture is not the same of Landau levels, but is the same as a system with momenta coupled with spins. The Non-Abelian ansatz is:

\[ A_0^a(x) = 0, \quad \text{and} \quad A_i^a(x) = h\delta_i^a, \quad (3.5) \]

where \( h \) is a constant in space-time. This choice was first considered in a work of Ambjørn, Nielsen and Olesen [14]. In respect to the Savvidy, the non-Abelian has the crucial vantage of being naturally implementable in lattice simulations. Moreover, recently, in an unrelated but very interesting work by Johnson and his collaborators [17], the Schrödinger functional approach in terms of magnetic field strength is applied to the SU(2) Yang-Mills gauge theory. In the explicit Schrödinger functional Hamiltonian they find that there is a factor \( 1/\text{Det}B_i^a \)
in the kinetic energy term, similar to the $1/r$ factor in the quantum mechanics Schrödinger equation in polar coordinate. This factor may signal a potential coordinate singularity for the color magnetic field in the configuration space, similar to that of the wavefunctions in quantum mechanics which have to satisfy certain boundary conditions at $r = 0$. Since the Savvidy ansatz yields $\text{Det}B^a_i = 0$, due to its intrinsic Abelian nature, it is desirable to seek an alternative background field, which avoids this potential coordinate singularity. The corresponding color magnetic field for the non-Abelian ansatz is simply given by $B^a_i = gh^2\delta^a_i$ and obviously has $\text{Det}B^a_i \neq 0$ when $h \neq 0$. It is easy to recognize that the constant magnetic field is generated by Eq.(3.3) through the commutator terms in the field strength, rather than the derivative terms as in the Savvidy case. Therefore, due to the non-trivial correlation between its directions in space and color space, the background field Eq.(3.3) evades the possible coordinate singularity.

In addition, since the non-Abelian background is space-time independent, it is much easier to put it on a lattice than the non-constant background field Eq.(3.1). For example, periodic boundary conditions are automatic.

For the non-Abelian case the second derivative of the action with respect to the gauge fields give the operator

$$M^{ab}_{\mu\nu} = \frac{1}{2} \left[ (p^2 + 2g^2h^2)\delta^{ab}\delta_{\mu\nu} + 2gh (\sigma_{ab})^c p_c \delta_{\mu\nu} - 2g^2h^2 (\sigma^{ab})_c (\sigma_{ce})_{\mu\nu} \right], \quad (3.6)$$

which is a $9 \times 9$ matrix for a given momentum $p$. This matrix can be interpreted as the Hamiltonian of a relativistic spin-1 ($\vec{\sigma}_s$) and color spin-1 ($\vec{\sigma}_c$) boson, with the first term being the free particle part, the second the $\vec{p} \cdot \vec{\sigma}_c$ and the third $\vec{\sigma}_s \cdot \vec{\sigma}_c$. The last two terms can be negative for low momentum, depending on the relative orientations of $\vec{p}$, $\vec{\sigma}_s$ and $\vec{\sigma}_c$.

For the non-Abelian ansatz, in (3+1) dimensions the effective potential is

$$V(H) = \frac{1}{2} H^2 + \frac{21}{48\pi^2} g^2 H^2 \left( \ln \frac{gH}{\mu^2} - \frac{1}{2} \right) + i \frac{3}{50\pi} g^2 H^2 + \ldots, \quad (3.7)$$

and in 3 dimension (see [1,13,18]), we have:

$$V(H) = \frac{1}{2} H^2 - \frac{2}{3\pi^2} (gH)^{3/2} - i \frac{5}{6\pi} (gH)^{3/2} + \ldots. \quad (3.8)$$
IV. INSTABILITY

When for some field configuration there is an imaginary part in the effective action this configuration is called unstable. In this case unfortunately this is what happens for non-Abelian gauge theories where the expansion is around a saddle point in the functional space. The second derivative operator of the action has some negative eigenvalues and, when loop-expanded, it shows a complex effective potential. Therefore, the validity of the loop expansion in these cases has been questioned by Maiani et al [13]. These authors strongly argued that the calculation of the effective potential in the presence of a background field is actually non-perturbative. When the background is not realizing a true minimum there is no possibility for the existence of any reliable perturbative expansion. So far the only known non-perturbative technique in field theory that can be made systematic is the lattice approach. More recently several works with the lattice approach have appeared in the literature, both in four dimensions [14] and three dimensions [15]. Due to various technical reasons, mainly related to the fact that the Savvidy ansatz in Eq.(3.1) is non-uniform, all these lattice works have yet to yield a conclusive result.

Because we expand around a field configuration $\phi_c$ which is a saddle point in the functional space, the second derivative operator of the action will have at least one negative eigenvalue. We use this operator to choose a basis in the vicinity of $\phi_c$ that has as components the eigenvectors of this operator. Let $\phi^s$ and $\phi^u$ designate the stable and the unstable modes; these correspond, respectively, to the positive and negative eigenvalues. Hence, a configuration $\phi$ in the vicinity of $\phi_c$ can be written as:

$$\phi = \phi_c + \sum_i c_i^s \phi^s_i + \sum_j c_j^u \phi^u_j \quad (4.1)$$

where the index $i$ runs over the set of stable modes $\{s\}$, and $j$ runs over the set of unstable modes $\{u\}$.

We can hope that after splitting the stable from the unstable modes, we will be able to perform a perturbative expansion in the stable sector and handle non-perturbatively the
unstable one. Due to the orthogonality of the stable and unstable modes, the measure is
simply defined now as \((d\phi) = (d\phi^s)(d\phi^u)\), and we can write \(W[J]\) as:
\[
e_i\bar{\hbar}W[J] = \int (d\phi) e^{i\bar{\hbar}(S[\phi]+(J,\phi))} = \int (d\phi^s)(d\phi^u) e^{i\bar{\hbar}(S[\phi_c+\phi^s+\phi^u]+(J,\phi_c+\phi^s+\phi^u))} = \int (d\phi^u) e^{i\bar{\hbar}(S_{\text{eff}}[\phi_c+\phi^u]+(J,\phi_c+\phi^u))},
\]
where we have supposed that we were able to handle the integration over the stable modes
sector.

As carefully discussed in reference [13], this is not a correct way to proceed. It is true that
we were formally able to write the connected Green’s functions generating functional \(W[J]\).
However, the Legendre transformation is now different. In fact, now we are in presence of
many solutions, and we must take the Legendre transform, choosing the \(J\) that minimizes
\(W[J] - (J, \phi)\) at fixed \(\phi = \phi_c\). Therefore, the effective potential is defined as:
\[
\Gamma[\phi_c] = \min_{\{J\}} \left[ W[J] - (J, \phi) \right]
\]
This definition reduces to the usual expression for stable configurations, but the choices of
\(J\) may drive the system into a non-perturbative regime for unstable \(\phi_c\).

In the classical limit \(\hbar \to 0\), the integral is still dominated by the saddle point configura-
tion \(\phi_0\). Now, however, the expression above is no longer valid since there are no restoring
forces for the unstable mode. What remains in equation is an effective action
\[
S_{\text{eff}}[\phi_c + \phi^u] = S[\phi_c + \phi^u] + O(\hbar)
\]
which comprises the effects of small fluctuations around \(\phi_c + \phi^u\). This is not what we were
doing in the previous section when we were took into account the small fluctuations around
\(\phi_0\) looking for
\[
S_{\text{eff}} = S[\phi_c + \phi^u + \phi^s] + O(\hbar)
\]
Thus, we can conclude, following [13], that the problems are that, one, \(\phi_0\) will definitely not
dominate \(W[J]\) and, two, the small fluctuations around \(\phi_0\) and \(\phi_c + \phi^u\) have nothing to do
with each other.
A consequence of this is that the Legendre transform, evaluated at \( \phi_c \), deviates from the minimum by a dangerous term which is \( O(\phi^u) \). Hence, we no longer have the possibility of using \( \hbar \) as a parameter to expand the effective potential at \( \phi_0 \). The non-perturbative effects become dominant and, as a result, non-perturbative techniques, such as the lattice regularization, must be used to find what replaces the one-loop analysis which, by definition, was obtained by the help of the perturbation expansion.

If this argument is correct, then the Savvidy solution, which was obtained perturbatively, no longer provides a good picture. Only a non-perturbative approach is reliable.

On the other hand, if the arguments of references \[12\] are correct, the Savvidy solution may be a good approximation, in the sense that even if quantum fluctuations alter the vacuum from the original chromomagnetic configuration, the transition might not be too drastic. If the imaginary part is negligible, in first approximation, the vacuum can be thought as microscopic domains of constant chromomagnetic fields. This is analogous to the case of ferromagnetic domains of iron below the Curie temperature. To be more precise, we cannot prove the existence of domains without studying the balance of the energy contributions of the walls and of the domain bulks. Nevertheless, from the fact that in the effective potential a domain scale of constant chromomagnetic field exists we might expect the existence of such domains. Hence, Monte Carlo simulations will be able to detect \( gH_{\text{min}} \). However, if the fluctuations do occur very drastically, then the configurations will fluctuate so violently that it will be very difficult to collect evidence of a minimum.

A hybrid scenario might also occur. For intense fields, we might have a non-trivial vacuum, and for weak fields, the trivial vacuum, dominates. This hybrid scenario is the most difficult to investigate, because it is not clear quantitatively when the strong field regime is reached. Moreover, for lattice simulations, this is the region in which the samples are most difficult to collect. At the present stage of lattice simulations \[14,15\] we can not exclude consistence with this hybrid scenario, and it will be extremely important to to clarify this possibility with further analysis.

Note, however, that the existence of the imaginary part is also related to the asymptotic
freedom of the pure gauge theory. In fact, in the absence of the imaginary part, the ultraviolet limit of the Savvidy solution implies that the beta function will be the same as the one of a scalar particle of mass $m^2 = 2gH$. It is just the imaginary part which prevents us after regulation from rotating the integration contour in the complex plane and thereby spoiling the asymptotic freedom.

V. COMPARISON WITH COLEMAN AND WEINBERG POTENTIALS

In the discussion of the radiatively induced symmetry breaking, the question of the trustworthiness of the non-trivial vacua is also addressed. In particular, Coleman and Weinberg found a non-trivial minimum in the one-loop effective potential of the $\lambda\phi^4$ theory at

$$\lambda \log \frac{\phi_c^2}{M^2} = -\frac{32}{3} \pi^2 + O(\lambda) \quad (5.1)$$

but they noted immediately that the effective potential is not calculable at this minimum in the loop expansion. In fact, higher loop contributions bring higher powers of $\lambda \log \frac{\phi_c^2}{M^2}$ and then, independently of how small the coupling constant is, the new minimum will lie outside the range of validity of the one-loop approximation.

In the same article, the case of the scalar electrodynamics theory was also addressed. In this case, the effective potential is found to be

$$V = \frac{3}{64\pi^2} e^4 \phi_c^4 \log \left( \frac{\phi_c^2}{<\phi>^2} - \frac{1}{2} \right) \quad (5.2)$$

In this case the non-trivial vacuum must be trusted, the loop expansion can be trusted at the minimum for the appropriate choice of the coupling constants because the higher order terms in $\lambda$ can be balanced by the contributions of the order $e^4 \log \frac{\phi_c}{M}$.

However, there is a fundamental difference between these cases and the non-Abelian gauge theories. For the above cases, the effective potential can be evaluated directly because there is no instability, while the opposite is true of the non-Abelian gauge theory. Consider
for example the $\lambda \phi^4$ theory in the broken phase ($\mu^2 < 0$). A possible source of instability comes from the imaginary part of the contribution of the effective potential, given by:

$$\int dk \ \log(k^2 + V''(\phi_c)) = \int dk \ \log(k^2 + \mu^2 + \frac{\lambda}{2} \phi_c^2) \quad (5.3)$$

when the logarithm is integrated for small $k$. However, because $\phi_c = \phi_c[J]$ it is always possible to find a set of values of $J$ sufficiently large for which there is a region where $\frac{\lambda}{2} \phi_c^2 > \mu^2$ and then it is possible to perform the integration without the imaginary part. Obviously, the existence of this region is not enough: we must reach the limit $J \to 0$ while remaining in the stable region. This is possible here by reaching the minimum via $\phi_c[J_1] > \phi_c[J_2 < J_1] > \phi_c[J = 0]$, while remaining away from the unstable region, that is $\phi < \frac{1}{2} \phi_c[J = 0]$. The lesson of this simple observation is that it is the quartic part, $\phi^4$, of the effective potential which stabilizes the vacuum.

In non-Abelian gauge theories the situation is different. The same analysis might be performed with the vocabulary:

$$-\mu^2 \longleftrightarrow 2gH \quad (5.4)$$

$$\frac{\lambda}{2} \phi_c^2 \longleftrightarrow 6gA_c^2 \quad (5.5)$$

The first identification comes by inspection of the ultraviolet part of the effective potential, or equivalently, by neglecting the imaginary part and comparing the beta function with the one of the scalar theory. The second correspondence comes directly from the quartic part of the action.

For the non-Abelian theory the equivalent contribution of (5.3) is:

$$\log( k^2 - gH ) \quad (5.6)$$

and not the “safe”

$$\log( k^2 - gH + 6gA_c^2 ) \quad (5.7)$$

because the last term is not gauge invariant. Hence, there is no region with stable minimum, as opposed to the scalar field theory. So unstable fluctuations must always be present.
This also teaches us that variational methods will not work for the vacuum properties of non-Abelian gauge theories. The variational method is very useful and powerful in the case of stable configurations. However, in my opinion, to use the variational method to analyze situations where unstable configurations are dominant is extremely hazardous.

The next natural question is if higher order contributions stabilize the instability. Unfortunately, we must perform a partial resummation of the loop expansion to clarify this point for non-Abelian models. At the present there is no evidence for a window for $J$ where the argument of the logarithm could be made positive.

Consider now the case of SU(2) coupled to a fermion. In this case, there is a contribution to the one-loop effective potential from the graph with four gluonic legs connected by a fermion loop. The contribution of this diagram is analogous to the one in the Abelian case (see [20]), and for low momentum transfer, will give an extra term in the effective potential equal to \( \frac{8}{135} \frac{(qe)^4}{m^4} H^4 \), where \( m \) is the mass, and \( (qe) \) is the charge of the fermion. The difference in the coefficient of \( H^4 \) with respect to the Abelian case is due to the fact that for SU(2) there is an extra factor \( f^{bcd} \) at each vertex. Hence, for a sufficiently small, but non zero \( m \), there is a region of \( J \) where the instability is not present. The issue in this case is if it is possible to reach \( J = 0 \) remaining in a stable region. There is a physical argument that might suggest that this is possible: adding gauge bosons at the same space-time point will strongly polarize the vacuum and eventually generate a chromomagnetic domain, but in the presence of fermions, the vacuum will soon create fermion anti-fermion pairs at the expense of the energy of the gluons. Depending on which of these two phenomena occurs first the vacuum structure will be different. It will be interesting to explore this situation in more detail. Unfortunately the realistic lattice simulations to investigate the vacuum structure with quarks are unfeasible by the present techniques.
VI. CONCLUSIONS

To summarize, we have analyzed the effective potential for a non-Abelian background field in SU(2) Yang-Mills theory. The result is found to be qualitatively similar to the Savvidy ansatz, both in real part, which indicates a spontaneous generation of the color magnetic field, and imaginary part, which signals the instability of the background field as vacuum configuration under the loop expansion. Given the qualitative similarity between the three dimensional and four dimensional effective potential in the Savvidy ansatz, we conclude that the effective potential is insensitive to the coordinate singularity, $\text{Det} B^a_i = 0$, if it indeed exists.

Rather, we realize that the presence of the imaginary part in non-Abelian theories is due to the incompatibility in the action between gauge invariance and stabilizing terms.

Technical questions related to the linear term and the gauge choice were illustrated, and in particular, we show the dependence of the one-loop effective potential of the gauge fixing parameter $\alpha$ when the loop expansion is performed around a non-trivial background.

It is important to recognize that the effective potential for these kinds of background would be a well-defined problem if the functional were evaluated non-perturbatively. The appearance of the imaginary part in the effective potential is only caused by the loop expansion. In other words, when we were doing steepest descent approximation, we were expanding at a saddle point. In addition, even if the expansion point were a true minimum, the stationary solution of the effective potential calculated up to a finite loop order could not be trusted quantitatively, due to the fact that at the stationary point of the effective potential the higher order terms become as important as lower order terms and hence the loop expansion breaks down. The final answer has to be settled by a non-perturbative means, such as the lattice simulation as mentioned in the Introduction.

In fact, a lattice generalization of the background field method is rather straightforward. Let us consider

$$Z_L[J_\mu(x), B_\mu(x)] \equiv \int dU_\mu(x) \exp\left\{-S[U_\mu(x)B_\mu(x)] + \text{Tr} J_\mu(x)f[U_\mu(x)]\right\}, \quad (6.1)$$
where $U_\mu(x)$ is the standard link variable, $B_\mu(x)$ is the background link variable, $S$ is the usual Wilson lattice action, $J_\mu(x)$ is a matrix valued external current, and $f[U]$ is an arbitrary function satisfying $gf[U]g^\dagger = f[gUg^\dagger]$ for any unitary matrix $g$. Using the property of the invariance under an unitary transformation for the Haar measure, one can easily verify the following:

$$Z_L[\tilde{J}_\mu(x), \tilde{U}_\mu(x)] \bigg|_{\tilde{U}_\mu(x)=g(x)U_\mu(x)g^\dagger(x+\mu), \tilde{J}_\mu(x)=g(x)J_\mu(x)g^\dagger(x)} = Z_L[J_\mu(x), U_\mu(x)], \quad (6.2)$$

a lattice version of Eq. (2.2). A Legendre transform of $Z_L[J_\mu, U_\mu]$ would lead to a gauge invariant effective action, provided that the induced gauge field is constrained to have zero expectation value by adjusting $J_\mu(x)$, just as in the continuum case. The remarkable thing here is that we do not need to fix the gauge on a lattice and therefore the resulted effective action is unique for a given choice of the functional form of $f$.

As mentioned above the non-Abelian background field ansatz can be conveniently realized on a lattice. The constant nature of the ansatz avoids problems with the boundary condition and non-uniformness of the lattice constant effect due to the linear rising ansatz of Savvidy. The hope is that intensive lattice simulations with the non-Abelian background will clarify the picture. Since the non-Abelian background only involves one parameter $h$ it may not be difficult to find a way to adjust the external current $J_\mu$ to ensure a vanishing of the expectation value for the induced quantum field.

**ACKNOWLEDGMENTS**

This work was possible thanks to the collaboration of Suzhou Huang, Ken Johnson and Janos Polonyi to whom I am very grateful. It is my pleasure to thank Kerson Huang, H. B. Nielsen, H. Trottier, P. van Baal, for many very fruitful discussions.
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