Dynamics of entropy in bipartite quasi-Hermitian systems and their Hermitian counterparts

Abed Alsalam Abu Moise  Graham Cox  Marco Merkli*

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, NL A1C 5S7
Canada

Abstract

A quasi-Hermitian quantum system can be mapped to a multitude of Hermitian systems by the Dyson map. All Hermitian systems thus obtained are globally unitarily equivalent but the unitary may entangle different parts of the whole system. The choice of the unitary in the Dyson map then leads to physically different Hermitian systems emerging from the same quasi-Hermitian system. We analyze the resulting dependence of the von Neumann entropy for an oscillator coupled to \(N\) other oscillators via a quasi-Hermitian Hamiltonian (a \(PT\)-symmetric Hamiltonian in the symmetry unbroken region). For this model, we explicitly find all Hermitian systems emerging by varying over all unitaries in the Dyson map. We show that the evolution of the entropy of the single oscillator in the Hermitian system depends on the choice of the unitary, but the period of the entropy is universally the same for all choices: it is exactly double that of the entropy of the quasi-Hermitian oscillator. We give a simple explanation of the origin of this phenomenon.

1 Introduction

In recent years there has been much interest in extensions of quantum mechanics that allow for non-Hermitian Hamiltonians. One such extension considers quasi-Hermitian Hamiltonians. For these one can produce associated Hermitian Hamiltonians. Being Hermitian, these associated Hamiltonians can be studied using standard methods, so one hopes that they encode relevant information about the original system. While this is true globally, it is not necessarily true for subsystems of bipartite systems, as we now explain.

*aaabumoise@mun.ca, gcox@mun.ca, merkli@mun.ca
A Hamiltonian $H$ is said to be **quasi-Hermitian** if $H^\dagger = \eta H \eta^{-1}$ for some positive operator $\eta > 0$, called a **metric operator**. One obtains a Hermitian Hamiltonian by $h = S HS^{-1}$, where $S$ is any operator with $S^\dagger S = \eta$. The assignment $H \mapsto h$ is called the **Dyson map**, even though to a single $H$ one can associate many different $h$ for the following reasons. First, there are many different metric operators corresponding to a quasi-Hermitian $H$. Second, once a metric $\eta$ has been fixed, the equation $S^\dagger S = \eta$ will be satisfied by any $S$ of the form $W \sqrt{\eta}$, where $W$ is unitary and $\sqrt{\eta}$ is the unique positive operator that squares to $\eta$. Different choices of $W$ will lead to different Hermitian $h$, which are clearly unitarily equivalent. However, if the system is bipartite with Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ (‘system-bath’) but $W$ is not of product form $W_S \otimes W_B$, then the choice of $W$ will have physical consequences, as it entangles the subsystems.

In the recent literature on $PT$-symmetric quantum theory, it is suggested that instead of studying the evolution of the von Neumann entropy of a subsystem of a quasi-Hermitian system, one may study the entropy of the corresponding subsystem of the associated Hermitian system. However, as the choice of $S$ generically changes entanglement properties of the subsystems, it is not immediately clear in what sense the entropy of the Hermitian subsystem reflects that of the original non-Hermitian subsystem. In fact, we point out that if one allows $S$ to depend on time, as is done in the existing literature, then one has $h = 0$ for a judicious choice of $S$, in which case the dynamics of the Hermitian system is trivial and the entropy is constant, regardless of the actual dynamics of the non-Hermitian system. This ambiguity in determining $h$ is the subject of the present paper.

To clarify the interplay between the non-uniqueness of the Dyson map $H \mapsto h$ and the evolution of the entropy of subsystems, we analyze a simple explicit model, taken from the existing literature. A single quantum harmonic oscillator (the system $S$) interacts with $N$ other ones (the bath $B$) by a $PT$-symmetric Hamiltonian $H$. We consider the symmetry unbroken region of parameters, in which $H$ is quasi-Hermitian. We identify all metric operators $\eta > 0$ associated to $H$ and all possible Hermitian $h_W$ obtained from $H$ by the Dyson map. We find that except for a single (modulo scaling) metric operator $\eta$, the ‘reduced system density matrix’ of the non-Hermitian system $\tilde{\rho}_H(t)$, obtained by taking the trace over the bath degrees of freedom of the total (system plus bath) density matrix, is actually not a density matrix. Rather, it has complex eigenvalues. To be able to make sense of the von Neumann entropy of $\tilde{\rho}_H(t)$, we take this special metric, for which the eigenvalues of $\tilde{\rho}_H(t)$ are positive (and $\tilde{\rho}_H(t)$ is a true density matrix). We then compare the time evolution of the von Neumann entropy of $\tilde{\rho}_H(t)$ with that of $\tilde{\rho}_{h_W}(t)$, the reduced system density matrix of the Hermitian system, for different choices of the unitary $W$.

As the total system of $N+1$ oscillators does not have any continuous spectrum (no scattering states, only bound states), the density matrices and the entropies are periodic functions of time. We show that the periods of $\tilde{\rho}_H(t)$ and $\tilde{\rho}_{h_W}(t)$ are the same. We also show that the period of the von Neumann entropy of the Hermitian system does not depend on the (generic) choice of $W$ and the initial state, and that it is exactly double that of the non-Hermitian system. This gives a robust, universal distinction between the

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1 Every $H$ which has real nondegenerate eigenvalues is quasi-Hermitian, see Appendix A and also [15].
quasi-Hermitian and the Hermitian case, regardless of the initial state and regardless of
the choice of $W$ used in building the Hermitian system from the quasi-Hermitian one.
We give a simple explanation of the doubling of the period in terms of the average values
around which the populations oscillate.

**Motivation and connections with other literature.** The literature on pseudo- and quasi-
Hermitian quantum mechanics and its connections to $PT$-symmetric quantum theory is
huge [1]. It stretches over several domains and flavours of research, from the experimental
to the theoretical to the mathematical. We do not attempt to give an overview (the ref-
ences within our cited papers here contain many of the relevant works, see in particular
[13]). Our present work is inspired by recent studies of the evolution of the von Neumann
entropy of subsystems of $PT$-symmetric (non-Hermitian) models, in particular [9] and
the closely related paper [14] as well as [3]. In [9], the authors examine the entropy of an
oscillator coupled to an ‘environment’ of $N$ oscillators via a $PT$-symmetric Hamiltonian.
As a proxy for studying the von Neumann entropy of the single oscillator in the non-
Hermitian system, they propose to study the corresponding quantity for an associated
Hermitian system constructed by the time-dependent Dyson map [7, 8, 10]. The authors
find a different qualitative behaviour of the entropy of the Hermitian system according to
the $PT$-symmetry phases of the non-Hermitian system. When the non-Hermitian system
is in the $PT$-symmetry unbroken phase, the entropy of the Hermitian one is oscillating,
while in the $PT$-symmetry broken region, that entropy decays to a nonzero value for large
times, and at the critical point it decays to zero. In passing from the non-Hermitian to the
Hermitian system one must make an implicit choice of a transformation – a choice that
affects the evolution of the entropy. The analysis of [9] is done for a specific choice of the
operator $W$ in the Dyson map. Another such choice leads to the entropy to be constant
in time, as we remark at the end of Section 2.2. This raises the question if there is some
universality in the behaviour of entropy, independent of the choice of the Dyson map,
and motivates our current work. We study the same model as in [9] in the $PT$-symmetry
unbroken region and find the evolution of all possible associated Hermitian systems. In
doing so, we uncover such a universal property: regardless of the choice of the unitary in
the Dyson map and regardless of the initial state, the entropy of the Hermitian system
experiences a doubling of the period relative to the quasi-Hermitian one.

## 2 Hermitian and non-Hermitian systems

In Section 2.1 we introduce the physical inner products and density matrices for quasi-
Hermitian systems. In Section 2.2 we explain the Dyson map, linking quasi-Hermitian
to Hermitian systems, and in Section 2.3 we show how the the non-uniqueness of the
Dyson map is parametrized by a unitary. In Section 2.4 we introduce the reduced density
matrices for the quasi-Hermitian and the Hermitian systems. The results and definitions
here are general; in Section 3 all of these objects will be computed for an explicit model.
2.1 Quasi-Hermiticity and the physical inner product

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$. We consider finite-dimensional $\mathcal{H}$ to simplify the exposition. An operator $\eta$ on $\mathcal{H}$ is called a metric operator if it is positive, denoted by $\eta > 0$. This means that $\langle \psi | \eta | \psi \rangle > 0$ for any nonzero $\psi \in \mathcal{H}$ and implies automatically that $\eta = \eta^\dagger$. Here $\dagger$ denotes the adjoint with respect to the inner product $\langle \cdot | \cdot \rangle$, that is, $\langle \phi | A^\dagger | \psi \rangle = \langle \psi | A | \phi \rangle^*$, where $^*$ is the complex conjugate. Any metric operator $\eta$ defines a new inner product on $\mathcal{H}$ by

$$\langle \psi | \phi \rangle_{\eta} = \langle \psi | \eta | \phi \rangle.$$  

(1)

We denote the Hilbert space $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle_{\eta}$ by $\mathcal{H}_{\eta}$, so the original space is $\mathcal{H}_{\perp}$. An operator $H$ on $\mathcal{H}_{\perp}$ is called pseudo-Hermitian if there exists an invertible (not necessarily positive) operator $\eta = \eta^\dagger$ such that

$$H^\dagger = \eta H \eta^{-1}.$$  

(2)

Pseudo-Hermitian Hamiltonians arise in PT-symmetric quantum theory, see for instance [22] and references therein. If (2) holds for a metric operator $\eta$ (so in addition $\eta$ is positive), then $H$ is said to be ($\eta$-) quasi-Hermitian.

The adjoint of an operator $A$ acting on the Hilbert space $\mathcal{H}_{\eta}$, denoted by $A^\dagger$, is given by $\langle \phi | A^\dagger | \psi \rangle_{\eta} = \langle \psi | A | \phi \rangle_{\eta}^*$ or, equivalently,

$$A^\dagger = \eta^{-1} A^\dagger \eta.$$  

It follows that $A$ is $\eta$-quasi-Hermitian if and only if it is Hermitian as an operator on $\mathcal{H}_{\eta}$, meaning $A^\dagger = A$. This latter property implies that all the eigenvalues of a quasi-Hermitian $A$ are real.

Given a quantum system with an $\eta$-quasi-Hermitian Hamiltonian $H$, the space $\mathcal{H}_{\eta}$ is considered the physical Hilbert space, in which $H$ is a physical (Hermitian) observable. The Hamiltonian $H$, however, is not enough to uniquely determine the metric. To see this, let an operator $H$ on $\mathcal{H}$ be given by

$$H = \sum_{n=1}^{N} E_n |\psi_n\rangle \langle \phi_n|,$$  

(3)

where the $E_n \in \mathbb{R}$ are all distinct and $\{ |\psi_n\rangle, |\phi_n\rangle \}_{n=1}^{N}$ are a complete bi-orthonormal family, meaning that $\langle \psi_k | \phi_l \rangle = \delta_{kl}$ and $\sum_{n=1}^{N} |\psi_n\rangle \langle \phi_n| = \mathbb{1}$. We show in Appendix A that for any choice of positive numbers $x_1, \ldots, x_N > 0$,

$$\eta = \sum_{n=1}^{N} x_n |\phi_n\rangle \langle \phi_n|$$  

(4)

is a metric for $H$, i.e., $\eta$ is a metric and $H$ is $\eta$-quasi-Hermitian. Conversely, if $\eta$ is a metric for $H$, then it must be of the form (4) for some positive numbers $x_1, \ldots, x_N$, so we know all the metrics for $H$ of the form (3).
Given this non-uniqueness of the metric, which one should be chosen to define the physical Hilbert space? One answer is that the metric is fixed provided that instead of just $H$, one chooses a whole irreducible family of operators to be physical observables. Namely, it is shown in [21] (see also [17] for the two-dimensional case) that if there is a family of operators $\{A_i\}_i$ on $\mathcal{H}$, and positive operators $\eta, \eta'$ such that $A_i^\dagger = \eta A_i \eta^{-1}$ and $A_i^\dagger = \eta' A_i (\eta')^{-1}$ for all $i$, then

\[ \eta' \text{ is a scalar multiple of } \eta \iff \{A_i\}_i \text{ is an irreducible family of operators on } \mathcal{H}. \]

This means for an irreducible family of quasi-Hermitian operators there is exactly one metric (up to a scalar multiple) that makes those operators Hermitian. The chosen family can then be viewed as physical observables of the theory and the space of pure states is $\mathcal{H}$ with inner product $\langle \cdot | \cdot \rangle_{\eta}$. Examples of irreducible families are the Pauli matrices for a spin, with the Euclidean inner product on $\mathbb{C}^2$, or the position $\hat{x}$ and momentum $\hat{p} = -i\hbar \nabla_x$ for a quantum particle (rather, the bounded Weyl operators generated by them) with the inner product $\langle \psi | \phi \rangle = \int_{\mathbb{R}^3} \bar{\psi}(x) \phi(x) d^3x$.

If interested only in the single observable $H$ (the Hamiltonian), one may keep the $x_n > 0$ in (4) general. This is what we do in the present work. However, for the model in Section 3 we find a different constraint on $\eta$, in terms of the reduced density matrix of a subsystem, that determines it up to a scalar multiple.

Consider now a fixed metric $\eta$, so that the physical Hilbert space is $\mathcal{H}_\eta$ and $H^\dagger = H$. Then $e^{-itH}$ determines the unitary Schrödinger dynamics on $\mathcal{H}_\eta$. The average of an observable $A$ on $\mathcal{H}_\eta$ in the state $\psi \in \mathcal{H}_\eta$ is given by

\[ \langle \psi | A | \psi \rangle_\eta = \langle \psi | \eta A | \psi \rangle = \text{tr}(\psi \langle \psi | \eta A) = \text{tr}(\tilde{\rho} A), \quad (5) \]

where

\[ \tilde{\rho} = |\psi\rangle \langle \psi | \eta \quad (6) \]

is a density matrix (a positive, trace-one operator) on $\mathcal{H}_\eta$. (This is called the ‘generalized density matrix’ in [20].) It is important to point out that the trace in (5) is a purely algebraic quantity: it is the sum of the eigenvalues of the operator, and therefore does not depend on the choice of metric.

### 2.2 From non-Hermitian to Hermitian: the Dyson map

The idea of mapping a non-Hermitian Hamiltonian to a Hermitian one was originally presented by Dyson in the context of the theory of magnetization in [4, 5].

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and let $H$ be an operator on $\mathcal{H}$. Denote by $|\psi(t)\rangle = e^{-itH} |\psi(0)\rangle$ the solution of the evolution equation

\[ i \partial_t |\psi(t)\rangle = H |\psi(t)\rangle, \quad (7) \]
where we have set $\hbar = 1$ throughout. Here, $H$ plays the role of the Hamiltonian, which is not necessarily Hermitian with respect to $\langle \cdot | \cdot \rangle$. Next, let $S(t)$ be any differentiable family of operators on $\mathcal{H}$ such that $S(t)$ is invertible for each $t$, and set

$$|\varphi(t)\rangle = S(t)|\psi(t)\rangle.$$  

(8)

Passing from $|\psi(t)\rangle$ to $|\varphi(t)\rangle$ represents a (possibly time-dependent) change of variables. The evolution equation for $|\varphi(t)\rangle$ is

$$i\partial_t|\varphi(t)\rangle = h(t)|\varphi(t)\rangle,$$

(9)

with

$$h(t) = S(t)HS(t)^{-1} + i\dot{S}(t)S(t)^{-1},$$

(10)

the dot being the time derivative. Conversely, if $|\varphi(t)\rangle$ solves (9) then $|\psi(t)\rangle$ solves (7). Equation (10) is called the time-dependent Dyson equation [9]. By means of $S(t)$, one may thus equivalently solve (7) or (9). If $H$ is not Hermitian, one can look for $S(t)$ such that the resulting $h(t)$ is Hermitian, hence trading a non-Hermitian problem with constant Hamiltonian $H$ for a Hermitian problem with time-dependent Hamiltonian $h(t)$.

One readily sees that

$$h(t)^\dagger = h(t) \iff i\partial_t(S(t)S(t)^{-1}) = H^\dagger S(t)S(t)^{-1} - (S(t)S(t)^{-1})H.$$  

(11)

The operator $\eta(t) = S(t)^\dagger S(t)$ is automatically positive, so $\eta(t)$ is a family of metrics. The equation for $\eta(t)$, according to (11), is

$$i\partial_t\eta(t) = H^\dagger \eta(t) - \eta(t)H.$$

(12)

This is called the quasi-Hermiticity relation in [7]; note that it simplifies to (2) if $\eta$ does not depend on time. It is clear that (12) has a unique solution for any initial condition $\eta(0)$, namely

$$\eta(t) = e^{-itH} \eta(0) e^{itH},$$

(13)

and that $\eta(t)$ is positive for all times if and only if it is positive at some $t_0$.

A strategy to study the dynamics generated by a non-Hermitian $H$ is to find a transformation $S(t)$ such that $h(t)$, as given by (10), is Hermitian, and then analyze the dynamics of this Hermitian system using usual quantum theoretical methods. Finding $S(t)$ for a specific Hamiltonian $H$ is not easy, however. It often involves making a judicious ansatz containing parameters that must solve rather complicated differential equations, which are obtained by imposing the self-adjointness of $h(t)$. This can be done explicitly for some models [2, 7, 9, 16, 19, 11, 12].

The structure of the ansatz for $S(t)$, and the appearance of some undetermined constants in the ensuing differential equations for the parameters, makes it clear that $S(t)$ is not unique. In [19] the authors consider a non-Hermitian oscillator and address the non-uniqueness of (time-independent) $S$ stemming from the non-uniqueness of the metric $\eta$, under the additional constraint that $S$ itself is positive, that is, $S = \sqrt{\eta}$. 


Given $H$, we seek all possible $S(t)$, and the resulting Hermitian Hamiltonians $h(t)$, with the sole requirement that $\eta(t) = S(t)\dagger S(t)$ is positive and satisfies the quasi-Hermiticity relation (12). The solution $\eta(t)$ is uniquely determined by the initial condition $\eta(0)$, which we may choose to be any positive, invertible operator. The most general form of $S(t)$ is thus

$$S(t) = W(t)\sqrt{\eta(0)e^{itH}},$$

(14)

where $W(t)$ is any unitary family and $\sqrt{\eta(0)}$ denotes the unique positive operator squaring to $\eta(0)$. The $h(t)$ associated to (14) by (10) is

$$h(t) = i\dot{W}(t)W(t)\dagger.$$

(15)

Note that we are entirely free to choose $W(t)$. For instance, given an arbitrary $A = A\dagger$, the choice $W(t) = e^{-itA}$ yields $h(t) = A$. This means any time-independent Hermitian $h$ can be obtained from a suitable choice of $S(t)$. A particularly simple choice is $S(t) = e^{itH}$, which results from undoing the dynamics $e^{-itH}$ (going backwards in time) and has $h = 0$.

More generally, suppose $A(t)$ is a continuous family of operators, and let $W(t)$ solve the differential equation

$$i\dot{W}(t) = A(t)W(t).$$

(16)

It is easily shown that if $A(t) = A(t)\dagger$ for all $t$ and the initial condition $W(0)$ is unitary, then the solution $W(t)$ is unitary for all $t$. Choosing this $W(t)$, we find from (15) the Hermitian Hamiltonian $h(t) = i\dot{W}(t)W(t)\dagger = i\dot{W}(t)W(t)^{-1} = A(t)$. This means any time-dependent Hermitian $h(t)$ can also be obtained from a suitable choice of $S(t)$.

### 2.3 Time independent metrics

Given $H$, we now look for time-independent solutions of (10), that is, time-independent, invertible operators $S$ for which

$$h = SHS^{-1}$$

(17)

is Hermitian. This is equivalent to $\eta = S\dagger S$ being a constant solution to the quasi-Hermiticity relation (12), which is satisfied precisely when $H$ is quasi-Hermitian.

Take then a quasi-Hermitian $H$. As discussed in Section 2.1, this does not uniquely determine a metric $\eta$. We take here a fixed $\eta > 0$ such that $H\dagger = \eta H \eta^{-1}$, leaving open the choice of the other observables forming an irreducible set together with $H$.

Given $\eta$, the most general $S$ such that $S\dagger S = \eta$ is

$$S = W\sqrt{\eta},$$

(18)

where $W$ is unitary and $\sqrt{\eta}$ is the unique positive operator whose square equals $\eta$. Once $W$ is chosen, the associated Hermitian $h$, see (10), becomes

$$h_W = W\sqrt{\eta}H\frac{1}{\sqrt{\eta}}W\dagger.$$

(19)
In particular, the $h$ obtained from two different choices of unitaries, say $V$ and $W$, are unitarily equivalent, with $h_V = U h_W U^\dagger$, since $U = VW^\dagger$ is unitary. In this sense, the choice of $W$ is globally immaterial. However, if the Hilbert space has a local structure, say is of bipartite nature $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, then the global unitary $U$ may well change the local properties of the two local subsystems, in which case the choice of $W$ will play a physically relevant role.

### 2.4 Bipartite systems, reduced states, von Neumann entropy

Consider now a bipartite system $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ ('system' and 'bath') and a quasi-Hermitian operator $H$ on $\mathcal{H}$, so that $H^\dagger = \eta H \eta^{-1}$ for some metric $\eta > 0$ on $\mathcal{H}$. To arrive at the Hermitian Hamiltonian, it is necessary to make a choice for the unitary $W$ in (18). The associated Hermitian Hamiltonian is then given by (19).

Let $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$, $|\phi(t)\rangle = e^{-ithW}|\phi(0)\rangle$ be the evolution of the initial states $|\psi(0)\rangle$, $|\phi(0)\rangle$ with respect to $H$ and $h$, respectively. The states are related by $|\phi(t)\rangle = S|\psi(t)\rangle$, $S = W\sqrt{\eta}$, and the density matrices associated to these vector states for the non-Hermitian (see (6)) and the Hermitian systems are $\rho_H(t) = |\psi(t)\rangle\langle\psi(t)|\eta$ and $\rho_{hw}(t) = |\phi(t)\rangle\langle\phi(t)|$, respectively. (We adopt the notation $\rho_{hw}$ and $\rho_H$ for the density matrices on the Hermitian and non-Hermitian sides of the problem from [9].) It is clear from (21) that $\rho_{hw}(t) = S|\psi(t)\rangle\langle\psi(t)|S^\dagger = S|\psi(t)\rangle\langle\psi(t)|(S^\dagger S)S^{-1} = S\rho_H(t)S^{-1}$.

It follows that $\rho_{hw}(t)$ and $\rho_H(t)$ have the same eigenvalues, and hence the same von Neumann entropy, $\mathcal{E}(\rho_{hw}(t)) = \mathcal{E}(\rho_H(t))$, where

$$\mathcal{E}(\rho) = -\text{tr}(\rho \ln \rho) = - \sum \lambda_i \ln \lambda_i$$

and $\{\lambda_i\}$ are the eigenvalues of $\rho$.

Now we consider the reduced states (denoted by an overbar) defined by tracing out the degrees of freedom of the subsystem $\mathcal{H}_B$, $\bar{\rho}_H(t) = \text{tr}_{\mathcal{H}_B}(\rho_H(t))$, $\bar{\rho}_{hw}(t) = \text{tr}_{\mathcal{H}_B}(\rho_{hw}(t))$. In some recent works, the dynamics of a bipartite system generated by a non-Hermitian Hamiltonian $H$ is studied, with particular focus on the von Neumann entropy of the reduced density matrix $\bar{\rho}_H(t)$. The strategy proposed in those works is to examine the entropy of $\bar{\rho}_{hw}(t)$ as a proxy for that of $\bar{\rho}_H(t)$. In this respect, however, one should observe the following facts:
1. The operator $\bar{\rho}_H(t)$ always satisfies $\text{tr}_{H_S}(\bar{\rho}_H(t)) = 1$, but for some choices of $\eta$ the eigenvalues of $\bar{\rho}_H(t)$ can be complex, in which case it is not a valid density matrix.

2. Even if the metric $\eta$ is chosen such that $\bar{\rho}_H(t)$ is a density matrix, for generic choices of $W$ the von Neumann entropies $E(\bar{\rho}_H(t))$ and $E(\bar{\rho}_h(t))$ are not the same. The latter in fact depends on the choice of $W$.

To understand the normalization of the trace mentioned in fact 1. above, we observe (using $1_S$ as the system observable) that

$$\text{tr}_{H_S}(\bar{\rho}_H(t)) = \text{tr}_{H_S}(\bar{\rho}_H(t)1_S) = \text{tr}_{H_S \otimes H_B}(\rho_H(t)(1_S \otimes 1_B)) = \text{tr}_{H_S \otimes H_B}(\rho_H(t)) = 1.$$

If $S = S_S \otimes S_B$, then $\bar{\rho}_h = S_S \bar{\rho}_H S_S^{-1}$ and so the spectra and thus the von Neumann entropies of $\bar{\rho}_h$ and $\bar{\rho}_H$ coincide. However, if $S$ is entangling (not of product form $S_S \otimes S_B$), then the eigenvalues of the two reduced density matrices are not the same in general, and neither are their entropies.

These difficulties are resolved in the next section, where we study the concrete model used in [9]. In particular, we determine for which choices of $\eta$ the reduced operator $\bar{\rho}_H(t)$ is indeed a density matrix, and then we find the von Neumann entropy of $\bar{\rho}_h(t)$ for all possible choices of the unitary $W$.

3 A simple, explicit model

We consider the open, non-Hermitian system used in [9]. An oscillator with creation and annihilation operators $a^\dagger, a$ is coupled to a “bath” of $N$ independent oscillators with creation and annihilation operators $q_i^\dagger, q_i, i = 1, \ldots, N$. The total Hilbert space of the $N + 1$ oscillators is

$$H = H_S \otimes H_B,$$

where $H_S$ is the space of a single oscillator and $H_B$ is that of the other $N$. As in the previous section, we denote the inner product by $\langle \cdot | \cdot \rangle$ and let $\dagger$ denote the adjoint in this inner product. The commutation relations are $[a, a^\dagger] = 1 = [q_i, q_i^\dagger]$, and all operators belonging to different oscillators commute.

The coupled total Hamiltonian is defined to be

$$H = \nu N_{\text{tot}} + (g + \kappa) \sqrt{N} a^\dagger Q + (g - \kappa) \sqrt{N} a Q^\dagger,$$

where $\nu > 0$ and $g, \kappa \in \mathbb{R}$ are parameters and

$$N_{\text{tot}} = a^\dagger a + \sum_{n=1}^{N} q_n^\dagger q_n, \quad Q = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} q_n. \tag{27}$$

Due to the different prefactors of $\kappa$ in the interaction term of (26), $H$ is $\dagger$-Hermitian if and only if $\kappa = 0$. However, for all real $\kappa$ the Hamiltonian $H$ commutes with the antilinear transformation $PT$ on $H$, which satisfies the following relations,

$$PTi = -i PT, \quad PTa^\dagger = -a^\dagger PT, \quad PTq_i^\dagger = -q_i^\dagger PT.$$
Here, the symbol \( ♯ \) indicates that either the adjoint \( † \) is taken or not. In other words, \( H \) is \( PT \)-symmetric. For \( \kappa^2 < g^2 \) the spectrum of \( H \) is real. This corresponds to the \( PT \)-symmetry unbroken regime and \( H \) is quasi-Hermitian, that is, we have \( H^\dagger = H \) for a suitable \( \eta > 0 \). The symmetry broken regime is \( \kappa^2 > g^2 \) \[9\].

The ‘uncoupled’ \((g = \kappa = 0)\) Hamiltonian is simply \( \nu N_{\text{tot}} \), a multiple of the total number operator \( N_{\text{tot}} \). As \( H \) commutes with \( N_{\text{tot}} \), each eigenspace of \( N_{\text{tot}} \), with a fixed number of excitations (in the system plus the bath) is left invariant. Denote by \( |0S0B\rangle \) the (‘vacuum’), zero excitation state, where all oscillators are in the ground state. The single excitation space is defined as

\[
\mathcal{E}_1 = \text{span}\{ |1S0B\rangle, |0S1_1\rangle, |0S1_2\rangle, \ldots, |0S1_N\rangle \},
\]

(28)

where \( |1S0B\rangle = a^\dagger |0S0B\rangle \) and \( |0S1_i\rangle = q_i^\dagger |0S0B\rangle \), for \( i = 1, \ldots, N \). When \( H \) is applied to a vector in \( \mathcal{E}_1 \) then the result is again a vector in \( \mathcal{E}_1 \). Moreover, due to the collective, symmetric nature of the system-bath interaction in (26), \( H \) leaves the even smaller, two-dimensional space

\[
\mathcal{H}_1 = \text{span}\{ |e_S\rangle, |e_B\rangle \}
\]

(29)

invariant, where

\[
|e_S\rangle = |1S0B\rangle, \quad |e_B\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |0S1_n\rangle.
\]

(30)

Those two vectors describe states in which a single excitation is either in \( S \) (the state \( |e_S\rangle \)) or in \( B \), collectively spread over the \( N \) bath oscillators (the state \( |e_B\rangle \)). When \( H \) is applied to a vector in \( \mathcal{H}_1 \) the result is again a vector in \( \mathcal{H}_1 \), so we may view \( H \) as an operator on \( \mathcal{H}_1 \). When we do this we denote it by \( H_1 \), which has the form

\[
H_1 = \nu \mathbb{1} + (g - \kappa)\sqrt{N} |e_B\rangle\langle e_S| + (g + \kappa)\sqrt{N} |e_S\rangle\langle e_B|.
\]

(31)

The eigenvalues of \( H_1 \) are

\[
\omega_{\pm} = \nu \pm \omega, \quad \omega = \sqrt{N\sqrt{g^2 - \kappa^2}},
\]

(32)

which are real for \( \kappa^2 \leq g^2 \) and (purely imaginary) complex conjugates for \( \kappa^2 > g^2 \). \( H_1 \) is diagonalizable except at the transition points defined by \( \kappa^2 = g^2 \neq 0 \), where \( H_1 \) reduces to a Jordan block. Note that the effect of increasing the number \( N \) of oscillators in the bath simply amounts to speeding up the dynamics (the frequency \( \omega \)) by a factor \( \sqrt{N} \).

We consider the ‘\( PT \)-symmetry unbroken regime’ \( \kappa^2 < g^2 \), so that \( \omega_{\pm} \in \mathbb{R} \). For definiteness we take \( g > 0 \) (the case \( g < 0 \) can be dealt with in the same fashion), so

\[
0 \leq |\kappa| < g,
\]

(33)

which is equivalent to \( g + \kappa > 0 \) and \( g - \kappa > 0 \). Then we have \( \omega > 0 \) and

\[
a_1 = \sqrt{g + \kappa} > 0, \quad a_2 = \sqrt{g - \kappa} > 0,
\]

(34)

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where the equalities in (34) define the quantities $a_1$, $a_2$. The two linearly independent (not normalized) eigenvectors of $H_1$ and its adjoint $H_1^\dagger$ are

$$ |v_\pm \rangle \propto a_1 |e_S \rangle \pm a_2 |e_B \rangle \quad \text{and} \quad |v_\pm^* \rangle \propto a_2 |e_S \rangle \pm a_1 |e_B \rangle. $$

They satisfy $H_1 |v_\pm \rangle = \omega_\pm |v_\pm \rangle$ and $H_1^\dagger |v_\pm^* \rangle = \omega_\pm |v_\pm^* \rangle$. The $|v_\pm \rangle$ denote the eigenvectors of $H_1$, not to be confused with the complex conjugates of the eigenvectors $|v_\pm \rangle$ of $H$. We normalize the vectors as

$$ |v_\pm \rangle = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{a_1}{a_2}} |e_S \rangle \pm \sqrt{\frac{a_2}{a_1}} |e_B \rangle \right) \quad \text{and} \quad |v_\pm^* \rangle = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{a_2}{a_1}} |e_S \rangle \pm \sqrt{\frac{a_1}{a_2}} |e_B \rangle \right). \quad (35) $$

Then $\{|v_\pm \rangle, |v_\pm^* \rangle\}$ is a bi-orthonormal basis, satisfying $\langle v_\pm^* | v_\pm \rangle = 0$ and $\langle v_\pm^* | v_\pm \rangle = 1$, and the operator $H_1$ can be written as

$$ H_1 = \omega_+ |v_+ \rangle \langle v_+^*| + \omega_- |v_- \rangle \langle v_-^*|. \quad (36) $$

Using this, one easily finds

$$ e^{-i t H_1} = e^{-i \omega_+ t} |v_+ \rangle \langle v_+^*| + e^{-i \omega_- t} |v_- \rangle \langle v_-^*| $$

$$ = e^{-i t \omega \cos(\omega t) I} - i e^{-i t \nu \sin(\omega t)} \left( \frac{a_1}{a_2} |e_S \rangle \langle e_B| + \frac{a_2}{a_1} |e_B \rangle \langle e_S| \right). \quad (37) $$

### 3.1 Reduced dynamics

We consider initial states which are vectors in $\mathcal{H}_1$, as defined in (29), so the dynamics generated by $H$ is entirely given by the operator $H_1$ from (31). We still consider the regime (33), so that the spectrum of $H_1$ consists of two distinct real eigenvalues. Comparing (3), (4) and (36), we see that $H_1$ is quasi-Hermitian and the set of all associated metrics is

$$ \mathcal{M}_+ = \{ \eta = x_1 |v_+^* \rangle \langle v_+^*| + x_2 |v_-^* \rangle \langle v_-^*| : x_1, x_2 > 0 \}. \quad (38) $$

Fix an $\eta \in \mathcal{M}_+$ and take an initial state of the form

$$ |\psi(0)\rangle = A |e_S \rangle + B |e_B \rangle \quad (39) $$

for some $A, B \in \mathbb{C}$ normalized to have $\|\psi(0)\|_\eta^2 = 1$, that is,

$$ 1 = \left( \frac{x_1 + x_2}{2} \right) \left( \frac{a_2}{a_1} |A|^2 + \frac{a_1}{a_2} |B|^2 \right) + (x_1 - x_2) \text{Re}(\bar{A}B). \quad (40) $$

The dynamics is given by

$$ |\psi(t)\rangle = e^{-itH} |\psi(0)\rangle = e^{-i \omega t} A(t) |e_S \rangle + e^{-i \nu t} B(t) |e_B \rangle, \quad (41) $$

where

$$ A(t) = A \cos(\omega t) - i B \frac{a_1}{a_2} \sin(\omega t), $$

$$ B(t) = B \cos(\omega t) - i A \frac{a_2}{a_1} \sin(\omega t). \quad (42) $$

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The normalization
\[ \|\psi(t)\|^2 = x_1|\langle \psi(t) | v_1^\dagger \rangle|^2 + x_2|\langle \psi(t) | v_2^\dagger \rangle|^2 = 1 \] holds for all \( t \), as \( e^{-iHt} \) is unitary in \( \mathcal{H}_1 \) with the inner product \( \langle \cdot | \cdot \rangle_\eta \). It is the same condition as (40) with \( A \) and \( B \) replaced by \( A(t) \) and \( B(t) \).

We now introduce the reduction of the system to the single oscillator \( (a^+, a) \). The average of a system observable \( O_S \) (observable of the single oscillator) in the state \( |\psi(t)\rangle \) given by (41) evolves according to
\[ \langle \psi(t)|\eta O_S|\psi(t)\rangle = \text{tr}_S(\bar{\rho}_H(t)O_S), \] where the reduced system state is
\[ \bar{\rho}_H(t) = \text{tr}_B \rho_H(t) = \text{tr}_B(|\psi(t)\rangle\langle \psi(t)|\eta). \] For the partial trace we have the identities \( \text{tr}_B|e_S\rangle\langle e_S| = |1_S\rangle\langle 1_S|, \text{tr}_B|e_B\rangle\langle e_B| = |0_S\rangle\langle 0_S| \) and \( \text{tr}_B|e_S\rangle\langle e_B| = 0 = \text{tr}_B|e_B\rangle\langle e_S| \). Using (41) and \( \eta \) of the form (38), we obtain after a calculation
\[ \bar{\rho}_H(t) = \left( \frac{x_1 + x_2}{2} \frac{a_1}{a_2} |B(t)|^2 + \frac{x_1 - x_2}{2} A(t)B(t) \right) |0_S\rangle\langle 0_S| + \left( \frac{x_1 + x_2}{2} \frac{a_2}{a_1} |A(t)|^2 + \frac{x_1 - x_2}{2} A(t)B(t) \right) |1_S\rangle\langle 1_S|. \] This matrix is diagonal in the basis \( \{|0_S\rangle, |1_S\rangle\} \) and the two diagonal entries are its eigenvalues. One checks directly that \( \text{tr}_S(\bar{\rho}_H(t)) = 1 \) (the sum of the diagonal elements equals \( \|\psi(t)\|^2 = 1 \)). However, the eigenvalues of \( \bar{\rho}_H(t) \) are complex, in general, unless the metric is chosen to satisfy \( x_1 = x_2 \). More precisely, choosing real coefficients \( A, B \) in the initial state (39), we get for the imaginary part of the first eigenvalue,
\[ \text{Im} \left( \frac{x_1 + x_2}{2} \frac{a_1}{a_2} |B(t)|^2 + \frac{x_1 - x_2}{2} A(t)B(t) \right) = \frac{x_1 - x_2}{2} \text{Im} (A(t)^*B(t)) = (x_1 - x_2) \left( \frac{1}{x_1 + x_2} - \frac{a_2}{a_1} A^2 \right) \cos(\omega t) \sin(\omega t). \] To arrive at the last equation, we use the equations (42) and write \( B \) as a function of \( A \) according to the normalization condition (40).

Unless \( x_1 = x_2 \) or the initial condition satisfies \( (x_1 + x_2) a_2 = 1 \), the eigenvalues of \( \bar{\rho}_H(t) \) will not be real except at the discrete set of times \( t \) when \( \sin(\omega t) \cos(\omega t) = 0 \). We would like \( \bar{\rho}_H(t) \) to be a density matrix (and in particular be Hermitian) for all times. To do so with a metric that does not depend on the initial conditions we must choose \( x_1 = x_2 \). We thus take
\[ x_1 = x_2 = x \]
for the remainder of the paper, so that $\bar{\rho}_H(t)$, as given by (46), is $\dagger$-Hermitian.

Assuming that the initial state coefficients $A, B$ (see (39)) are real, the reduced density matrix (46) is

$$\bar{\rho}_H(t) = p(t)|0_S\rangle\langle 0_S| + (1 - p(t))|1_S\rangle\langle 1_S|,$$

where

$$p(t) = (1 - \alpha) \cos^2(\omega t) + \alpha \sin^2(\omega t) = \frac{1}{2} + (\frac{1}{2} - \alpha) \cos(2\omega t),$$

and

$$\alpha = x \frac{a_2}{a_1} A^2 \in [0, 1].$$

To arrive at the first expression in (49), we use the normalization constraint $1 = x(\frac{a_2}{a_1} A^2 + \frac{a_1}{a_2} B^2)$, which is (40) for $x_1 = x_2 = x$. The second expression in (49) follows from the identities $\cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t))$ and $\sin^2(\omega t) = \frac{1}{2}(1 - \cos(2\omega t))$.

We see from (49) that $p(t)$ is constant if and only if $\alpha = 1/2$. For $\alpha \neq 1/2$, $p(t)$ oscillates about the value $p_0 = 1/2$ with amplitude $|\frac{1}{2} - \alpha|$ and period $\pi/\omega$. The function $p(t)$ and the corresponding von Neumann entropy

$$\mathcal{E}(\bar{\rho}_H(t)) = -p(t) \ln p(t) - (1 - p(t)) \ln (1 - p(t))$$

are plotted in Figure 1 for various choices of $\alpha$.

![Figure 1: Characteristics of $\bar{\rho}_H(t)$ from (48): The probability $p(t)$ (blue dashed) and the entropy $\mathcal{E}(\bar{\rho}_H(t))$ (orange full) for different values of $\alpha \in [0, 1]$. The time axis is $\omega t$.](image-url)
Next, we turn our attention to the density matrix of the Hermitian system, which according to (23) is

$$\rho_{hw}(t) = S \rho_H(t) S^{-1} = W \sqrt{\eta} \rho_H(t) \frac{1}{\sqrt{\eta}} W^\dagger = W \sqrt{\eta} |\psi(t)\rangle \langle \psi(t)| \sqrt{\eta} W^\dagger.$$  \hspace{1cm} (52)

We keep $W$ in the notation $h_W$ to highlight that the choice of $h$ depends on $W$, see (19). Again choosing a metric $\eta$ of the form (38) with $x_1 = x_2 = x > 0$, we use (41) to obtain

$$\sqrt{\eta} |\psi(t)\rangle = e^{-it\nu} \gamma(t) |e_S\rangle + e^{-it\nu} \delta(t) |e_B\rangle,$$  \hspace{1cm} (53)

where

$$\gamma(t) = \sqrt{\frac{x_2}{x_1}} A(t), \quad \delta(t) = \sqrt{\frac{x_1}{x_2}} B(t).$$  \hspace{1cm} (54)

We then obtain, written in matrix form in the ordered orthonormal basis $\{|e_S\rangle, |e_B\rangle\}$ of $H_1$,

$$\sqrt{\eta} |\psi(t)\rangle \langle \psi(t)| \sqrt{\eta} = \begin{pmatrix} |\gamma(t)|^2 & \gamma(t) \delta(t)^* \\ \gamma(t)^* \delta(t) & |\delta(t)|^2 \end{pmatrix}.$$  \hspace{1cm} (55)

Next we take a general (time-independent) unitary on $H_1$, written in the same basis as

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ac^* + bd^* = 0, \quad |a|^2 + |b|^2 = 1 = |c|^2 + |d|^2.$$  \hspace{1cm} (56)

Then according to (52) (and writing momentarily $\delta, \gamma$ for $\delta(t), \gamma(t)$),

$$\rho_{hw}(t) = \begin{pmatrix} |a\gamma + b\delta|^2 & a^* |\gamma|^2 + bc^* \gamma^* \delta + ad^* \gamma \delta^* + bd^* |\delta|^2 \\ a^* c \gamma \delta^* + a^* d \gamma^* \delta + b^* d |\delta|^2 & |c\gamma + d\delta|^2 \end{pmatrix}.$$  \hspace{1cm}  

Next we calculate the reduced density matrix $\tilde{\rho}_{hw}(t)$ of $S$ by taking the partial trace of $\rho_{hw}(t)$ over $B$,

$$\tilde{\rho}_{hw}(t) = q(t) |0_S\rangle \langle 0_S| + (1 - q(t)) |1_S\rangle \langle 1_S|,$$  \hspace{1cm} (57)

where $q(t) = |c\gamma(t) + d\delta(t)|^2$ and $\gamma(t), \delta(t)$ are given in (54). We get the expression

$$q(t) = x |c \sqrt{\frac{x_2}{x_1}} A(t) + d \sqrt{\frac{x_1}{x_2}} B(t)|^2,$$  \hspace{1cm} (58)

where $A(t)$ and $B(t)$ are given in (42), and the entropy of $\tilde{\rho}_{hw}(t)$ is

$$\mathcal{E}(\tilde{\rho}_{hw}(t)) = -q(t) \ln q(t) - (1 - q(t)) \ln(1 - q(t)).$$  \hspace{1cm} (59)

For $c = 0$ we have $|d| = 1$ (c.f. (56)) and one sees readily from (49) and (58) that $q(t) = p(t)$, which means that $\tilde{\rho}_{hw}(t) = \tilde{\rho}_H(t)$. Similarly, for $d = 0$ we have $|c| = 1$.
and \( p(t) = 1 - q(t) \), so that the density matrices \( \bar{\rho}_{hw}(t) \) and \( \bar{\rho}_H(t) \) are the same up to exchanging \( |0\rangle \leftrightarrow |1\rangle \). This means that

\[
\text{For } c = 0 \text{ or } d = 0 \text{ we have } \mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{hw}(t)) \text{ for all } t. \tag{60}
\]

We show in Appendix B that \( c = 0 \) or \( d = 0 \) is also a necessary condition to have the equality of the entropies stated in (60).

We next consider a small perturbation from the case of equality in (60), plotting the von Neumann entropy of \( \bar{\rho}_{hw}(t) \) for \( c = 0.05 \) in Figure 2.

![Figure 2: Characteristics of \( \bar{\rho}_{hw}(t) \) from (57): The probability \( q(t) \) and the entropy \( \mathcal{E}(\bar{\rho}_{hw}(t)) \) for different values of \( \alpha \in [0, 1] \) and for \( c = 0.05 \). For \( c \neq 0 \) the entropy shows different qualitative features: doubling of period (for \( \alpha \neq 0, \frac{1}{2}, 1 \)) when compared to the entropy of \( \bar{\rho}_H(t) \) in Figure 1, which corresponds to \( c = 0 \); see (60).](image)

Next we increase the value of \( c \) to \( c = 0.5 \) in Figure 3.
Figure 3: Characteristics of $\bar{\rho}_{hw}(t)$: The probability $q(t)$ and the entropy $\mathcal{E}(\bar{\rho}_{hw}(t))$ for different values of $\alpha \in [0, 1]$ and for $c = 0.5$. As in Figure 2, the period of the entropy doubles compared to the non-Hermitian case $c = 0$, Figure 1 (for $\alpha \neq 0, \frac{1}{2}, 1$).

In Figure 4, we compare the von Neumann entropies of the two density matrices $\bar{\rho}_H(t)$ and $\bar{\rho}_{hw}(t)$, directly seeing the doubling of the period.
Figure 4: Comparing the entropies of $\bar{\rho}_{hW}(t)$ and $\bar{\rho}_H(t)$ for different values of $\alpha \in [0, 1]$ and for $c = 0.5$. The doubling of the period for $\alpha \neq 0, 1/2, 1$ is manifest.

4 Period doubling of the von Neumann entropy

In this section we analyze the von Neumann entropies of $\bar{\rho}_{hW}(t)$ and $\bar{\rho}_H(t)$, given in (57) and (48), respectively, explaining the period-doubling phenomenon seen above. For the ease of presentation, we consider the following parameter regime:

$$x = x_1 = x_2 > 0 \quad (\text{metric } \eta, (11)) \quad (61)$$
$$c, d \geq 0 \quad (\text{unitary } W, (56)) \quad (62)$$
$$A, B \geq 0 \quad (\text{initial state } |\psi(0)\rangle, (39)) \quad (63)$$

The relation (61) ensures that $\bar{\rho}_H(t)$ has positive eigenvalues, so the von Neumann entropies of both $\bar{\rho}_H(t)$ and $\bar{\rho}_{hW}(t)$ are well-defined. According to (62) and the unitarity of $W$, we have $d = \sqrt{1 - c^2}$. In the regime (61)–(63), the population $q(t)$ in (58) becomes

$$q(t) = (c\sqrt{\alpha} + \sqrt{1 - c^2}\sqrt{1 - \alpha})^2 \cos^2(\omega t) + \left(\sqrt{1 - c^2}\sqrt{\alpha} + c\sqrt{1 - \alpha}\right)^2 \sin^2(\omega t), \quad (64)$$

where $\alpha = \frac{x_2}{x_1} A^2 \in [0, 1]$ and $c \in [0, 1]$ can be chosen freely (see also (40) and (49)). Then, using the formulas $\cos^2(\omega t) = \frac{1}{2}(1 + \cos(2\omega t))$ and $\sin^2(\omega t) = \frac{1}{2}(1 - \cos(2\omega t))$ we obtain

$$q(t) = q_0 - \Delta \cos(2\omega t) \quad (65)$$
with

\[ q_0 = \frac{1}{2} + 2\sqrt{c^2(1-c^2)}\sqrt{\alpha(1-\alpha)}, \tag{66} \]
\[ \Delta = -\frac{1}{2}(1-2c^2)(1-2\alpha), \tag{67} \]

meaning the population \( q(t) \) oscillates around the average \( q_0 \) with amplitude \( \Delta \). It is constant in time when \( \Delta = 0 \), which happens exactly when \( \alpha = \frac{1}{2} \) or \( c = \frac{1}{\sqrt{2}} \) (\( = d \)). For \( \alpha \neq \frac{1}{2} \) and \( c \neq \frac{1}{\sqrt{2}} \) the function \( p(t) \) is periodic with period \( \pi/\omega \) and oscillates between the extremal values

\[ q_m \equiv q_0 - |\Delta| \leq q(t) \leq q_M \equiv q_0 + |\Delta|. \]

As we will see in Sections 4.1 and 4.2, the value of \( q_0 \) determines the period of the entropy as a function of time: For \( q_0 = \frac{1}{2} \) the period is \( \frac{\pi}{2\omega} \), but for all other values \( q_0 \neq \frac{1}{2} \) the period is doubled to the value \( \frac{\pi}{\omega} \). According to (66), \( q_0 = \frac{1}{2} \) for all initial states (all \( \alpha \)) if and only if \( c = 0 \) or \( c = 1 \). These are precisely the values for which the entropies of \( \bar{\rho}_H(t) \) and \( \bar{\rho}_{hw}(t) \) coincide, see (60).

### 4.1 Entropy of \( \bar{\rho}_H(t) \)

The population \( p(t) \) and \( q(t) \), given in (49) and (65), are the same when \( c = 0 \) (see (60)), which corresponds to \( q_0 = \frac{1}{2} \) and \( \Delta = \alpha - \frac{1}{2} \). We plot the entropy over one period \( \pi/\omega \) of \( p(t) \) in Figure 5.

![Figure 5: Parameters \( q_0 = 0.5, \Delta = 0.2 \). According to (65), \( q(t) \) starts at \( q_m = 0.3 \) when \( t = 0 \) and moves to \( q_M = 0.7 \) at time \( \omega t = \pi/2 \), and then back to \( q_m \) at time \( \omega t = \pi \) (left panel, where \( \mathcal{E}(q) = -q \ln q - (1-q) \ln(1-q) \)), so the value of the entropy \( \mathcal{E}(\bar{\rho}_H(t)) \) evolves through two periods (right panel). In each period, the entropy has three local minima (counting minima at both endpoints of the considered intervals).](image)

### 4.2 Entropy of \( \bar{\rho}_{hw}(t) \)

For \( c \neq 0,1 \) the midpoint \( q_0 \) of the oscillations of \( q(t) \) differs from \( \frac{1}{2} \) (unless \( \alpha = 0,1 \)). This asymmetry leads to a doubling of the period of the entropy, relative to the case \( q_0 = \frac{1}{2} \),
as explained in Figure 6.

Figure 6: Parameters $q_0 = 0.6$, $\Delta = 0.2$. In this case $q(t)$ starts at $q_m = 0.4$ when $t = 0$ and moves to $q_M = 0.8$ at time $\omega t = \pi/2$ and back to $q_m$ at time $\omega t = \pi$ (left panel), so the value of the entropy $\mathcal{E}(q(t))$ evolves through one single period (right panel). In each period, the entropy has three local minima (counting minima at both endpoints).

If the minimum satisfies $q_m > \frac{1}{2}$ then the doubling of the period persists, but the number of local minima of the entropy over a single period is reduced from three to one. We show this in Figure 7.

Figure 7: Parameters $q_0 = 0.8$, $\Delta = 0.2$. In this case $q(t)$ starts at $q_m = 0.6$ when $t = 0$ and moves to $q_M = 1.0$ at time $\omega t = \pi/2$ and back to $q_m$ at time $\omega t = \pi$ (left panel), so the value of the entropy $\mathcal{E}(q(t))$ evolves through one single period (right panel). Since 0.5 is not in the interval $(q_m, q_M)$, the graph of the entropy has only one local minimum in each period (counting minima at both endpoints), instead of three when the interval contains the value 0.5 for $q$. 

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5 Conclusion

The Dyson map assigns to a given quasi-Hermitian quantum system an associated Hermitian system in a non-unique way. We quantify the non-uniqueness by means of a metric operator $\eta$ and a unitary map $W$. Correlations in a multipartite system are affected by the choice of $W$ which can generate entanglement in the Hermitian system. If one allows the Dyson map to depend on time then it can be chosen in such a way that the resulting Hamiltonian of the Hermitian system vanishes. The physical properties of the Hermitian system are therefore critically dependent on the choice of $W$, and it is not obvious how to capture the dynamics of the quasi-Hermitian system from its Hermitian counterpart.

We examine the relation between quasi-Hermitian and Hermitian systems given by a time-independent Dyson map, for an explicit model with a (time-independent) quasi-Hermitian Hamiltonian $H$ describing an oscillator in contact with $N$ other oscillators. This is the Hamiltonian of a $PT$-symmetric system in the symmetry unbroken region, used in the literature. The single oscillator is viewed as the (open) system and the remaining $N$ oscillators act as an environment. There is a unique (up to scaling) metric operator for which the von Neumann entropy is well defined. Using this metric, we construct all Hermitian systems obtained from the quasi-Hermitian one by varying over all $W$. We find that the entropy of the single oscillator of the Hermitian system evolves periodically in time with exactly double the period of the corresponding entropy of the quasi-Hermitian system, independently of $W$. While the shape of the entropy as a function of time depends on the choice of $W$, the doubling of the period is a unifying, robust characteristic of all Hermitian systems stemming from the original quasi-Hermitian system.

A Finding all metrics for a quasi-Hermitian $H$

Suppose $H$ is a quasi-Hermitian operator on $\mathcal{H}$, so $H^\dagger = \eta H \eta^{-1}$ for some $\eta > 0$. Then $(\sqrt{\eta}H\frac{1}{\sqrt{\eta}})^\dagger = \sqrt{\eta}H\frac{1}{\sqrt{\eta}} \equiv K$ is Hermitian, and $H = \frac{1}{\sqrt{\eta}}K\sqrt{\eta}$. Since $K$ is Hermitian, it has the spectral representation $K = \sum_n E_n |\xi_n\rangle\langle \xi_n|$, where $E_n \in \mathbb{R}$ and $\{|\xi_n\rangle\}$ is an orthonormal basis of $\mathcal{H}$. It follows that $H$ is of the form (3), with $|\psi_n\rangle = \frac{1}{\sqrt{\eta}}|\xi_n\rangle$ and $|\phi_n\rangle = \sqrt{\eta}|\xi_n\rangle$.

Let now $H$ be of the form (3) and set

$$\eta = \sum_n x_n |\phi_n\rangle\langle \phi_n|$$

with $x_n > 0$ arbitrary. For any $|\psi\rangle \in \mathcal{H}$, we have $\langle \psi|\eta|\psi\rangle = \sum_n x_n |\langle \phi_n|\psi\rangle|^2$, which is strictly positive except if $|\psi\rangle = 0$, hence $\eta > 0$. Using the normalization $\langle \phi_n|\psi_n\rangle = \delta_{mn}$ and the completeness relation

$$\sum_n |\phi_n\rangle\langle \psi_n| = \sum_n |\psi_n\rangle\langle \phi_n| = \mathbb{1},$$

$$\text{(69)}$$
one readily verifies that $\eta^{-1} = \sum_n x_n^{-1}|\psi_n\rangle\langle\psi_n|$. Furthermore, from (3) we have $\langle\phi_m|H|\psi_n\rangle = \delta_{mn}E_n$, so

$$\eta H \eta^{-1} = \sum_{m,n} \frac{x_m}{x_n} |\phi_m\rangle\langle\phi_m|H|\psi_n\rangle\langle\psi_n| = \sum_n E_n |\phi_n\rangle\langle\psi_n| = H^\dagger.$$ 

It follows that $\eta$ given in (68) is a metric for $H$.

Conversely, suppose that some $\eta$ is a metric for $H$, that is, $\eta > 0$ and $\eta H \eta^{-1} = H^\dagger$. Suppose also that all eigenvalues $E_n$ of $H$ are distinct. We now show that $\eta$ is of the form (68). Using the completeness relation (69) we have

$$\eta = \sum_{m,n} |\phi_m\rangle\langle\psi_m|\langle\psi_n| = \sum_{m,n} \eta_{mn} |\phi_m\rangle\langle\phi_n|,$$

$$\eta_{nn} = \langle\psi_n|\eta|\psi_n\rangle. \tag{70}$$

Then $H^\dagger = \eta H \eta^{-1}$ is equivalent to $H^\dagger \eta = \eta H$. Using (13) this is in turn equivalent to

$$\sum_n E_n |\phi_n\rangle\langle\psi_n| = \sum_n E_n \eta |\psi_n\rangle\langle\phi_n|,$$

or, by (70), to

$$\sum_{m,n} E_n \eta_{mn} |\phi_m\rangle\langle\phi_n| = \sum_{m,n} E_n \eta_{mn} |\phi_m\rangle\langle\phi_n|.$$

Applying both sides to the basis vector $|\psi_k\rangle$, for any $k$, gives

$$\sum_n E_n \eta_{nk} |\phi_n\rangle = \sum_m E_m \eta_{mk} |\phi_m\rangle.$$

Since the $|\phi_j\rangle$ are linearly independent, we have $E_n \eta_{nk} = E_k \eta_{nk}$ for any $n,k$. Since $E_n \neq E_k$ unless $n = k$, this implies that $\eta_{nk} = 0$ whenever $n \neq k$. Using this in (70) yields $\eta = \sum_n |\psi_n\rangle\langle\psi_n|$, since $\eta > 0$ we have $x_n \equiv \langle\psi_n|\eta|\psi_n\rangle > 0$, and hence $\eta$ is of the form (68).

A similar statement and proof also works if the eigenvalues of $H$ are either real or come in complex conjugate pairs (this will be useful elsewhere for the analysis of the $PT$-symmetry broken region, where the eigenvalues of $H$ need not be real; cf. [18, 6]).

B Conditions for $\bar{\rho}_H(t) = \bar{\rho}_{hW}(t)$ and $\mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{hW}(t))$

In this section we assume the metric $\eta$ is of the form (38) with $x_1 = x_2 = x > 0$. Recall the formulas (48) and (57) for the quasi-Hermitian and Hermitian density matrices.

First we ask when the two reduced density matrices coincide. We show that the following statements are equivalent:

1. $\bar{\rho}_H(t) = \bar{\rho}_{hW}(t)$ for all $t$ in an open interval $I \subset \mathbb{R}$ and all $A, B \in \mathbb{C}$;
2. \( \tilde{\rho}_H(t) = \tilde{\rho}_{hw}(t) \) for all \( t \in \mathbb{R} \) and all \( A, B \in \mathbb{C} \);

3. There are two real phases \( \Phi_1, \Phi_2 \), such that

\[
W = \begin{pmatrix} e^{i\Phi_1} & 0 \\ 0 & e^{i\Phi_1} \end{pmatrix}.
\]

1. \( \Rightarrow 3. \) : Assume that \( \tilde{\rho}_H(t) = \tilde{\rho}_{hw}(t) \) for \( t \in I \). Then \( p(t) = q(t) \) for \( t \in I \), where these quantities are given in (49) and (58), respectively. Using the normalization condition (40), the relation \(|c|^2 + |d|^2 = 1\) and the parameter \( \alpha \) defined in (50), one obtains

\[
x_1 + x_2 - 2(1 - 2\alpha)|c|^2 \sin^2(\omega t) + x_4 \cos(\omega t) \sin(\omega t) = 0 \quad \text{for all} \quad t \in I,
\]

with \( x_1 = -2x \) \( AB \) \( \text{Re}(cd) \), \( x_2 = (1 - 2\alpha)|c|^2 \) and \( x_4 = 2(1 - 2\alpha) \) \( \text{Im}(cd) \). Taking the time derivative in (71) gives

\[-2(1 - 2\alpha)|c|^2 \sin(2\omega t) + x_4 \cos(2\omega t) = 0.\]

In particular, \( (1 - 2\alpha)|c|^2 = 0 \), so \( c = 0 \). Then due to (56), \(|d| = 1\) and \( bd = 0 \), so \( b = 0 \), and statement 3. holds.

3. \( \Rightarrow 2. \) : Suppose \( c = 0 \). Then \(|d| = 1\) and from (58) we have \( q(t) = x_{a_1} \alpha |B(t)|^2 \), which equals the population of \( |0_s\) \) in \( \tilde{\rho}_H(t) \), see (46). Therefore 2. holds.

2. \( \Rightarrow 1. \) : Obvious.

This completes the proof of the equivalence of the three statements 1.-3.

Next we ask when the entropies of the two density matrices coincide. We show that the following statements 4.-6. are equivalent:

4. \( \mathcal{E}(\tilde{\rho}_H(t)) = \mathcal{E}(\tilde{\rho}_{hw}(t)) \) for all \( t \) in an open interval \( I \subset \mathbb{R} \) and all \( A, B \in \mathbb{C} \);

5. \( \mathcal{E}(\tilde{\rho}_H(t)) = \mathcal{E}(\tilde{\rho}_{hw}(t)) \) for all \( t \in \mathbb{R} \) and all \( A, B \in \mathbb{C} \);

6. There are two real phases \( \Phi_1, \Phi_2 \) such that \( W \) is of either of the two forms

\[
W = \begin{pmatrix} e^{i\Phi_1} & 0 \\ 0 & e^{i\Phi_1} \end{pmatrix} \quad \text{or} \quad W = \begin{pmatrix} 0 & e^{i\Phi_1} \\ e^{i\Phi_2} & 0 \end{pmatrix}.
\]

4. \( \Rightarrow 6. \) : Start by looking at the function \( \mathcal{E}(q) = -q \ln(q) - (1 - q) \ln(1 - q) \), for \( q \in [0, 1] \). It is clear from the graph of \( \mathcal{E}(q) \) (see the left panel of Figure 5) that \( \mathcal{E}(q) = \mathcal{E}(q') \) exactly if either \( q = q' \) or \( q = 1 - q' \). Consequently, if \( \mathcal{E}(\tilde{\rho}_{hw}(t)) = \mathcal{E}(\tilde{\rho}_H(t)) \) for all \( t \in I \), then for each \( t \in I \) individually, we have either \( p(t) = q(t) \) or \( p(t) = 1 - q(t) \).

We now show that the same alternative must happen for all \( t \in I \).

Suppose first that \( p(t_0) \neq 1 - q(t_0) \) for some \( t_0 \in I \). Then by the continuity of \( p(t) \) and \( q(t) \), we have \( p(t) \neq 1 - q(t) \) for all \( t \) in an open interval \( I_0 \subset I \) around \( t_0 \), so we must have \( p(t) = q(t) \) for \( t \in I_0 \). But this means that \( \tilde{\rho}_H(t) = \tilde{\rho}_{hw}(t) \) for all \( t \in I_0 \). Hence, as 2. and 3. are equivalent, \( W \) is of the diagonal form as given in point 3. above. Similarly, if \( p(t_0) \neq q(t_0) \) for some \( t_0 \in I \), we obtain \( p(t) = 1 - q(t) \) an an interval around \( t_0 \).
Proceeding as in the proof of the implication 1. ⇒ 3. above, this implies that \( a = d = 0 \), so \( W \) is of the off-diagonal form given in statement 6. above.

6. ⇒ 5. : If \( W \) is of the diagonal form, then we already showed that \( p(t) = q(t) \) when we proved 3. ⇒ 2. In the same way, if \( W \) is off-diagonal, then one sees that \( p(t) = 1 - q(t) \). In either case, \( \mathcal{E}(\hat{\rho}_H(t)) = \mathcal{E}(\hat{\rho}_{hw}(t)) \).

5. ⇒ 4. : Obvious.

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