Continuous-state branching processes, extremal processes and super-individuals

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Consider a continuous-state branching process \( (X_t(x), t \geq 0) \) starting from \( x \).

Questions

- Can we characterize the growth rate of a supercritical CSBP with infinite mean (no Malthusian parameter)?
- Can we characterise the decay rate of a subcritical CSBP with infinite variation?

Consider a continuous branching population encoded by a flow of CSBPs (as Bertoin Le Gall 2000).

Questions

How are organized the growth and the decay locally in the population? Do all families evolve at the same scale or some initial individuals (super-individuals) have progenies growing faster than all the others?
Definition (CSBP)

A positive Markov process \((X_t(x), t \geq 0)\) with \(X_0(x) = x \geq 0\) is a CSBP if for any \(y \in \mathbb{R}_+\)

\[
(X_t(x + y), t \geq 0) \overset{d}{=} (X_t(x), t \geq 0) + (\tilde{X}_t(y), t \geq 0)
\]

where \((\tilde{X}_t(y), t \geq 0)\) is an independent copy of \((X_t(y), t \geq 0)\).

This ensures the existence of a map \(t \mapsto v_t(\lambda)\) s.t.

\[
\mathbb{E}[e^{-\lambda X_t(x)}] = \exp(-xv_t(\lambda)) \text{ and } v_{s+t}(\lambda) = v_s \circ v_t(\lambda),
\]

Theorem (Jirina (58), Lamperti (67))

There exists \(\Psi\) of the form

\[
\Psi(q) = \frac{\sigma^2}{2}q^2 + \gamma q + \int_0^{+\infty} \left( e^{-qx} - 1 + qx1_{\{x \leq 1\}} \right) \pi(dx)
\]

such that \(\frac{dv_t(\lambda)}{dt} = -\Psi(v_t(\lambda))\).
Asymptotic behaviors

**Proposition (Grey 74)**

- **Supercritical case:** \( \Psi'(0) \in [-\infty, 0[ , \)
  - The largest root of \( \Psi \), called \( \rho \), is in \((0, +\infty]\)
  - \( X_t(x) \xrightarrow{t \to \infty} 0 \) with probability \( e^{-x\rho} \)
  - \( X_t(x) \xrightarrow{t \to \infty} +\infty \) with probability \( 1 - e^{-x\rho} \)
  - Non-explosion: \( \mathbb{P}(\forall t, X_t(x) < \infty) = 1 \iff \int_0^\infty \frac{dq}{|\Psi(q)|} = +\infty \).

- **Subcritical case:** \( \Psi'(0) \geq 0 \),
  - The largest root of \( \Psi \) is 0
  - \( X_t(x) \xrightarrow{t \to \infty} 0 \) a.s.
  - Persistence: \( \mathbb{P}(\forall t; X_t(x) > 0) = 1 \iff \int^{+\infty} \frac{dq}{\Psi(q)} = +\infty \)
Continuous population model

Definition (Flow of CSBPs: Bertoin Le Gall 2000, Dawson Li 2012, Duquesne Labbé 2014)

Let $N_\Psi$ a measure on $\mathcal{D}(\mathbb{R}^*_+, \mathbb{R}^*_+)$ s.t. $N_\Psi(\cdot) = \lim_{x \to 0} \frac{1}{x} \mathbb{P}^{\Psi}_x(\cdot)$.

Consider $\mathcal{N} = \sum_{i \in I} \delta_{(x_i, X_i)}$ a PPP over $\mathbb{R}^*_+ \times \mathcal{D}$ with intensity $dx \otimes N_\Psi(dX)$. For all $x \geq 0$, let $X_0(x) = x$ and for all $t \geq 0$,

$$X_t(x) = \sum_{x_i \leq x} X^i_t$$

- for all $t \geq 0$ $(X_t(x), x \geq 0)$ is a ( càdlàg ) subordinator with Laplace exponent $\lambda \mapsto v_t(\lambda)$
- for any $y \geq x$, $(X_t(y) - X_t(x), t \geq 0)$ is a CSBP($\Psi$) started from $y - x$, independent of $(X_t(x), t \geq 0)$.

$^a$in the infinite variation case.
$^b$In the finite variation case, there is an other Poisson representation
The flow \( (X_t(x), t \geq 0, x \geq 0) \) provides a continuous population:

- The individual \( y \) is a descendant at time \( t \) of the individual \( x \) living at time 0 if
  \[
  X_t(x-) < y < X_t(x)
  \]

- \( \Delta X_t(x) = X_t(x) - X_t(x-) \) is the progeny of \( x \) at time \( t \).
Super-individuals

In a non-explosive CSBP with **infinite mean** \( \Psi(u)/u \to -\infty \) and in a persistent CSBP with **infinite variation** \( \Psi(u)/u \to +\infty \), some individuals have a progeny that overwhelms the total progeny of all individuals below them.

**Definition**

The individual \( x \) is a **super-individual** if \( \lim_{t \to +\infty} \frac{\Delta X_t(x)}{X_t(x^-)} = +\infty \) a.s.

Denote by \( S \) the set of super-individuals

\[
S := \left\{ x > 0; \lim_{t \to +\infty} \frac{\Delta X_t(x)}{X_t(x^-)} = +\infty \right\}.
\]

⚠️ There is an order between super-individuals: if \( x_1, x_2 \in S \) and \( x_1 \leq x_2 \), then \( \frac{\Delta X_t(x_1)}{\Delta X_t(x_2)} \leq \frac{X_t(x_2^-)}{\Delta X_t(x_2)} \to 0 \) as \( t \to \infty \).
**Supercritical CSBP with finite mean**

**Definition (Bertoin et al. 2008)**

An individual $x$ is **prolific** if $\Delta X_t(x) \xrightarrow{t \to \infty} +\infty$ a.s.

$$\mathcal{P} := \{ x > 0; \Delta X_t(x) \xrightarrow{t \to \infty} +\infty \}$$

**Proposition (Grey 74 + Bertoin et al. 2008 + Duquesne Labbé 2014)**

Assume $\Psi'(0+) \in (-\infty, 0)$. Almost-surely, for all $x > 0$,

$$v_{-t}(\lambda)X_t(x-) \xrightarrow{t \to \infty} W^\lambda_x \text{ and } v_{-t}(\lambda)X_t(x) \xrightarrow{t \to \infty} W^\lambda_x$$

where $(W^\lambda_x, x \geq 0)$ is a càdlàg subordinator

1. $\lambda \mapsto v_{-t}(\lambda)$, the inverse of $\lambda \mapsto v_t(\lambda)$
2. $\mathcal{P} = \{ x > 0; \Delta W^\lambda_x > 0 \}$ and $S \cap \mathcal{P}$ is degenerate.
Supercritical CSBP with infinite mean

Theorem (preliminary version, Grey 77, F. Ma 16)

Suppose $\Psi'(0+) = -\infty$ and $\int_0^\infty \frac{du}{\Psi(u)} = -\infty$. Fix $\lambda_0 \in (0, \rho)$, and define $G(y) := \exp \left(- \int_y^{\lambda_0} \frac{du}{\Psi(u)} \right)$ for $y \in (0, \rho)$. Then, for all $x \geq 0$, almost-surely

$$e^{-t} G \left( \frac{1}{X_t(x)} \wedge \rho \right) \xrightarrow{t \to +\infty} Z_x.$$

- $G$ is decreasing and slowly varying at 0
- $\{Z_x = 0\} = \{X_t(x) \xrightarrow{t \to \infty} 0\}$
- $\mathbb{P}(Z_x \leq z) = \exp(-xG^{-1}(z))$ with $G^{-1}(z) = v_{\log(1/z)}(\lambda_0)$.

Example (Neveu’s mechanism)

$\Psi(u) = u \log u$ for which $\rho = 1$. Fix $\lambda_0 = \frac{1}{e}$, $G(z) = \log(1/z)$. 
Question

What is the nature of the process \((Z_x, x \geq 0)\)?

Definition (extremal process=”subordinator for the max operator”)

A process \((Z_x, x \geq 0)\) is an extremal-\(F\) process if

\[
\begin{align*}
\mathbb{P}(Z_x \leq z) &= F(z)^x \\
Z_{x+y} &= Z_x \lor Z'_y \text{ a.s. where } Z'_y \perp (Z_u)_{0 \leq u \leq x} \text{ and } Z'_y \overset{d}{=} Z_y
\end{align*}
\]

Lemma

\((Z_x, x \geq 0)\) is an extremal-\(F\) process with \(F(z) = e^{-\nu \log(1/z)}(\lambda_0)\).

Proof.

\[
\begin{align*}
X_t(x+y) &= X'_t(y) + X_t(x) \text{ with } X'_t(y) = X_t(x+y) - X_t(x). \\
\implies e^{-t}G\left(\frac{1}{X_t(x+y)}\right) &\geq e^{-t}G\left(\frac{1}{X_t(x)}\right) \lor e^{-t}G\left(\frac{1}{X'_t(y)}\right) \text{ thus} \\
Z_{x+y} &\geq Z_x \lor Z'_y \text{ a.s. but } Z_{x+y} \overset{d}{=} Z_x \lor Z'_y
\end{align*}
\]
Fact ("Lévy-Itô decomposition" of extremal processes)

Consider a PPP with intensity \(dx \otimes \mu\) over \(\mathbb{R}_+ \times \mathbb{R}\). The process of its records is a càdlàg extremal-F process with \(F(z) = e^{-\bar{\mu}(z)}\).

Theorem (F. Ma 2016, supercritical part)

There exists \(\mathcal{M} := \sum_{i \in I} \delta_{(x_i, Z_i)}\) a PPP(\(dx \otimes \mu(dz)\)) with \(\bar{\mu}(z) = \nu \log(1/z)(\lambda_0)\) such that almost-surely, for all \(x \geq 0\)

\[
e^{-t} G \left( \frac{1}{X_t(x)} \wedge t \rho \right) \xrightarrow{t \to +\infty} Z_x = \sup_{x_i \leq x} Z_i
\]

\(\mu\) has total mass \(\bar{\mu}(0) = \rho \in (0, +\infty]\) and has no atom.

Moreover,

Proposition

Almost-surely, if \(Z_i > Z_j\) then \(\Delta X_t(x_j)/\Delta X_t(x_i) \xrightarrow{t \to \infty} 0\),

\(\mathcal{P} = \{x_i; Z_i > 0, i \in I\}\) and \(\mathcal{S} \cap \mathcal{P} = \{x > 0; \Delta Z_x > 0\}\) a.s.
Poisson representation

○ denotes \((x_l, Z_l)\) such that \(Z_l = 0\) (non prolific)

× denotes \((x_i, Z_i)\) not a partial record (prolific non superprolific)

● denotes \((x_i, Z_i)\) partial record: (superprolific)
Assume $\Psi'(0+) \geq 0$. Since for all $x \geq 0$, $X_t(x) \to 0$ a.s. there is no prolific individual in the population. Recall

$$S := \left\{ x > 0; \lim_{t \to +\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty \right\}.$$ 

A super-individual is an individual whose decay is much slower than the decay of all individuals below it.
Subcritical CSBP with finite variation

Definition (variation)

For any $\Psi$, 
\[
\lim_{u \to +\infty} \frac{\Psi(u)}{u} =: d = +\infty 1_{\sigma > 0} + \gamma + \int_{0}^{1} x\pi(dx) \in \mathbb{R} \cup \{+\infty\}.
\]

Proposition (Grey 74 + Duquesne Labbé 2014)

Assume $d \in \mathbb{R}$. For all $x$, $(X_{t}(x), t \geq 0)$ is persistent and almost-surely, for all $x \geq 0$

\[
v_{-t}(\lambda)X_{t}(x) \xrightarrow{t \to +\infty} V_{x}^{\lambda} \quad \text{and} \quad v_{-t}(\lambda)X_{t}(x-) \xrightarrow{t \to +\infty} V_{x-}^{\lambda}.
\]

where $(V_{x}^{\lambda}, x \geq 0)$ is a càdlàg subordinator. Thus $S$ is degenerate.
Subcritical process with infinite variation

Theorem (F. Ma 2016, subcritical part)

Suppose $d = +\infty$ and $\int^\infty \frac{d\mu}{\Psi(u)} = +\infty$. Fix $\lambda_0 \in (0, +\infty)$ and define $G(y) := \exp \left( - \int^{y}_{\lambda_0} \frac{d\mu}{\Psi(u)} \right)$ on $(0, +\infty)$.

There exists $M := \sum_{i \in I} \delta_{(x_i, Z_i)}$ a PPP($d\mu \otimes \mu(dz)$) with $\bar{\mu}(z) = G^{-1}(z) = \nu\log(z)(\lambda_0)$ such that almost-surely

$$e^t G \left( \frac{1}{X_t(x)} \right) \xrightarrow{t \to +\infty} Z_x = \sup_{x_i \leq x} Z_i$$

for all $x \geq 0$.

⚠️ $\mu(0, \infty) = \infty$ and $\mu$ has no atom.

Proposition

$S = \{x > 0; \Delta Z_x > 0\}$ a.s.
Corollary (Supercritical case (Grey 79 for GW chains) )

Consider two independent CSBPs \( (X_t(x), t \geq 0), (Y_t(y), t \geq 0) \) non-explosive with infinite mean and same mechanism.

Conditionally on \( \{X_t(x) \overset{t \to +\infty}{\longrightarrow} +\infty\} \cap \{Y_t(y) \overset{t \to +\infty}{\longrightarrow} +\infty\} \),

\[
\frac{X_t(x)}{Y_t(y)} \overset{t \to +\infty}{\longrightarrow} \begin{cases} 
+\infty & \text{with probability } \frac{x}{x+y} \\
0 & \text{with probability } \frac{y}{x+y}.
\end{cases}
\]

Corollary (Subcritical case)

Consider two independent subcritical persistent CSBPs \( (X_t(x), t \geq 0), (Y_t(y), t \geq 0) \) with infinite variation and same mechanism

\[
\frac{X_t(x)}{Y_t(y)} \overset{t \to +\infty}{\longrightarrow} \begin{cases} 
+\infty & \text{with probability } \frac{x}{x+y} \\
0 & \text{with probability } \frac{y}{x+y}.
\end{cases}
\]
Duquesne and Labbé (2014) have considered the following question:

**Question**

*Does the population (encoded by a flow of CSBPs) concentrates on the progeny of a single individual? In other words, fix the initial size $x$, is there an individual $e \in [0, x]$ (the Eve), such that*

$$\frac{\Delta X_t(e)}{X_t(x)} \xrightarrow{t \to \infty} 1 \text{ a.s.}$$

**Corollary (Duquesne and Labbé 2014)**

*In the case of infinite variation and infinite mean, the population has an Eve. In our framework, the Eve corresponds to the last super-individual in $[0, x]$.***
Consider \((X_t(x), t \geq 0)\) a CSBP of Neveu. It is non-explosive with infinite mean and persistent with infinite variation. A well-known result, attributed to Neveu (1992) (shown in Fleischmann, Sturm (2004)), states that for any fixed \(x\)

\[
e^{-t} \log X_t(x) \xrightarrow{t \to +\infty} Z_x \text{ a.s.}
\]

where \(Z_x\) has a Gumbel law over \(\mathbb{R}\). By combining our results, we get:

**Proposition**

*Almost-surely for all \(x \geq 0\),*

\[
e^{-t} \log X_t(x) \xrightarrow{t \to +\infty} Z_x
\]

*where \((Z_x, x \geq 0)\) is an extremal-\(\Lambda\) process with \(\Lambda(z) = e^{-e^{-z}}\) for \(z \in \mathbb{R}\).*
In the non-persistent and explosive cases, extremal processes still arise, but through the times of explosion and absorption.

When the reproduction has infinite mean, the infinite divisibility of the flow \((X_t(x), t \geq 0, x \geq 0)\) becomes the \(\text{max}\)-infinite divisibility of the process \((Z_x, x \geq 0)\). Less clear in the subcritical setting...

- Bertoin, Le Gall, *The Bolthausen-Sznitman coalescent and the genealogy of CSBPs*, PTRF (2000)

- Duquesne, Labbé, *On the Eve property for CSBP*, Electron. J. Probab. (2014)

- Neveu, *A CSBP in relation with the GREM model of spin glass theory*, Rapport interne (unprinted:-( Ecole Polytechnique, 1992.)
D. R. Grey, *Asymptotic behaviour of CSBPs*, JAP (1974),

D. R. Grey, *Almost-sure convergence in Markov branching process with infinite mean*, J. Appl. Probability 14 (1977), 702–716.

D. R. Grey, *On regular branching processes with infinite mean*, SPA (1978/79),

Thank you!!
**Sketch of proof**

Fix \( x \).

1. For all \( \lambda \in (0, \rho) \), \( v_{-t}(\lambda)X_t(x) \xrightarrow{t \to \infty} W_x^\lambda \in \{0, \infty\} \) a.s. and \( \mathbb{P}(W_x^\lambda = \infty) = e^{-x\lambda} \).

2. If \( \lambda' \geq \lambda \) then \( W_x^{\lambda'} \geq W_x^\lambda \).

3. \( \Lambda_x := \inf\{\lambda \in (0, \rho) \cap \mathbb{Q}; W_x^\lambda = +\infty\} \) is a random variable!

4. Let \( \lambda' \in (0, \rho) \). If \( \lambda < \Lambda_x < \lambda' \), then for large \( t \):

\[
v_{-t}(\lambda) \leq 1/X_t(x) \text{ and } v_{-t}(\lambda') \geq 1/X_t(x)
\]

\[
\implies G(v_{-t}(\lambda)) \geq G(1/X_t(x)) \geq G(v_{-t}(\lambda'))
\]

\[
\implies G(\lambda) \leq e^{-t} G(1/X_t(x)) \leq G(\lambda')
\]

5. If \( \Lambda_x = \rho \) then \( X_t(x) \xrightarrow{t \to \infty} 0 \).

This yields the a.s convergence: \( e^{-t} G(1/X_t(x) \land \rho) \xrightarrow{t \to \infty} Z_x \).