Mean Field Limit and Propagation of Chaos for a Pedestrian Flow Model

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Abstract  In this paper a rigorous proof of the mean field limit for a pedestrian flow model in two dimensions is given by using a probabilistic method. The model under investigation is an interacting particle system coupled to the eikonal equation on the microscopic scale. For stochastic initial data, it is proved that the solution of the $N$-particle pedestrian flow system with properly chosen cut-off converges in the probability sense to the solution of the characteristics of the non-cut-off Vlasov equation. Furthermore, the result on propagation of chaos is also deduced in terms of bounded Lipschitz distance.

Keywords Probabilistic method · Pedestrian flow · Mean field limit · Vlasov equation · Propagation of chaos

Mathematics Subject Classification 35Q83 · 82C22

1 Introduction

The notable interest of pedestrian flow models can be dated back to four decades ago with a considerable increase in interest since about year 2000. For a general recent overview we refer to [1,2,4–6,9,10,15] and the references therein. Pedestrian models share striking analogies in classical physics such as gases and fluids, but are also applied to the description of opinion formation [18], group dynamics or other social phenomena [13]. Pedestrian flow models are an ideal starting point for the derivation of other or more general quantitative
behavioral models, since the relevant quantities of pedestrian motions are easily measured so that corresponding models are comparable with empirical data [9]. The modelling presented here is based on the idea that behavioral changes are guided by so-called social fields or social forces, which have been suggested by Lewin [12]. Numerical simulations have been recently carried out in [7] on the microscopic and macroscopic level using the finite particle method (FPM). Some interesting spatiotemporal patterns are observed.

This paper provides the detailed derivation from the $N$-particle (pedestrian) Newtonian system to its mean field limit or Vlasov equation. Instead of the formal derivation with the help of the BBGKY hierarchy, which can be found in [7,16], we will rigorously derive the kinetic description by a probabilistic method, which is inspired by Boers and Pickl [3], Hauray and Jabin [8,11], Philipowski [14] and Sznitman [17] and all the references therein.

However, the proposed pedestrian model involves a singularity, which comes from the albeit bounded interaction force and is similar to the one generated by the Coulomb potential in 2-d. While the authors in [3] do not tackle the direct Coulomb potential in 3-d, i.e. they consider the singularity that is a little weaker than for the Coulomb potential, we are capable to deal with the singularity directly due to the compact support of the considered interaction force. Another difficulty lies in the treatment of the dissipative terms since the interaction force depends not only on the position $x$ but also on the velocity $v$. This will lead to extra work on the estimates and is up to our knowledge rarely done before.

We now briefly explain our approach. In order to obtain the convergence between the exact and the mean field dynamics, we mainly split the proof into two parts: Using the Newtonian system with cut-off as a starting point, we show that the Newtonian and the intermediate system (Vlasov flow with cut-off) are close to each other for $N$ being large enough. The next step is to show the intermediate system converges to the Vlasov flow without cut-off. Inbetween we use characteristics as a bridge to connect the Newtonian system and the mean field dynamics.

Additionally, assuming stochastic initial data offers a way to rule out those deterministic dynamics that do not fit into the proper configuration of the Vlasov equation in the sense that those particles have small probability to appear. In doing so, we obtain the convergence in (probability) measure between the exact and the mean field dynamics. As a direct implication of the convergence, we prove the propagation of chaos in terms of bounded Lipschitz distance.

This article is organized as follows: we start with the introduction of the pedestrian flow model in Sect. 2. Then, in Sect. 3 some notations and preliminary work will be introduced. In Sect. 4 we state the main results and present the corresponding proofs. Section 5 is devoted to the propagation of chaos. At this point, we also refer to [17] for other classical results with bounded Lipschitz continuity. Finally, we summarize our results.

### 2 Modeling of Pedestrian Flow

Following the pedestrian flow model originally introduced in [7], we consider a two-dimensional interacting particle system with position $x_i \in \mathbb{R}^2$ and velocity $v_i \in \mathbb{R}^2$, $i = 1, \ldots, N$. The equations of motion read

$$
\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i}{dt} &= \frac{1}{N-1} \sum_{i \neq j} F(x_i - x_j, v_i - v_j) + G(x_i, v_i),
\end{align*}
$$

(2.1)
where $F(x, v)$ denotes the total interaction force and $G(x, v)$ the desired velocity and direction acceleration. More precisely, $F(x, v)$ consists of the interaction force $F_{\text{int}}(x)$ and the dissipative force $F_{\text{diss}}(x, v)$, i.e.,

$$F(x, v) = (F_{\text{int}}(x) + F_{\text{diss}}(x, v))\mathcal{H}(x, v)$$

with

$$F_{\text{int}}(x) = k_n \frac{x}{|x|} (2R - |x|) = 2Rk_n \frac{x}{|x|} - k_n x,$$

$$F_{\text{diss}}(x, v) = F_{\text{diss}}^n(x, v) + F_{\text{diss}}^t(x, v)$$

$$= -\gamma_n \frac{\langle v, x \rangle}{|x|^2} x - \gamma_t \left( v - \frac{\langle v, x \rangle}{|x|^2} x \right)$$

$$= \frac{\langle v, x \rangle}{|x|^2} (\gamma_t - \gamma_n) x - \gamma_t v$$

and

$$\mathcal{H}(x, v) := \mathcal{H}_{2R}(|x|) \cdot \tilde{\mathcal{H}}_{2\tilde{R}}(|v|),$$

where $\mathcal{H}_{2R}(|x|)$ and $\tilde{\mathcal{H}}_{2\tilde{R}}(|v|)$ are smooth functions with compact support that satisfy

$$\mathcal{H}_{2R}(|x|) = \begin{cases} 
0, & |x| > 2R, \\
1, & |x| < R,
\end{cases} \quad \text{and} \quad \tilde{\mathcal{H}}_{2\tilde{R}}(|v|) = \begin{cases} 
0, & |v| > 2\tilde{R}, \\
1, & |v| < \tilde{R}.
\end{cases}$$

Here, $F_{\text{diss}}^n(x, v)$ and $F_{\text{diss}}^t(x, v)$ are the normal dissipative force and the tangential friction force, respectively. Moreover, $k_n$ is the interaction constant and $\gamma_n, \gamma_t$ are suitable positive friction constants.

Remark 2.1 To obtain a realistic behavior of pedestrians, the functions $\mathcal{H}_{2R}(|x|)$ and $\tilde{\mathcal{H}}_{2\tilde{R}}(|v|)$ are used to express that the interaction force and the pedestrian velocity are of finite range. Mathematically, the total force is considered on a bounded domain.

The desired velocity and direction acceleration is given by

$$G(x, v) := G(x, v, \rho) = \frac{1}{T} \left( -U(\rho) \frac{\nabla \Phi(x)}{|\nabla \Phi(x)|} - v \right),$$

where

$$\rho = \rho(x) = \frac{1}{N_{\max}^R} \sum_{j, |x - x_j| < R} 1.$$
The kinetic equation associated with this particle system describes the evolution of the (effective one particle) density $f(t, x, v)$ as

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F \ast f) f] + \nabla_v \cdot (G f) = 0. \quad (2.2)$$

See [7] for more details and the derivation of macroscopic models for different moment closures.

### 3 Notations and Preliminary Work

Now, we consider the pedestrian flow model (2.1) with cut-off of order $N^{-\theta}$ with $0 < \theta < \frac{1}{4}$, i.e.,

$$F^N(x, v) = \begin{cases} 2R_k n \frac{x}{|x|} - k_n x + \frac{\langle v, x \rangle}{|x|^2} (\gamma_t - \gamma_n) x - \gamma_t v \mathcal{H}(x, v), & |x| \geq N^{-\theta}, \\ (2R_k n N^\theta - k_n) x + N^2 \theta \langle v, x \rangle (\gamma_t - \gamma_n) x - \gamma_t v \mathcal{H}(x, v), & |x| < N^{-\theta}. \end{cases} \quad (3.2)$$

In order to present the analytical results in Sect. 4 in a concise and clear manner, we restrict to the following notations.

**Definition 3.1** 1. Let $(X^N t, V^N t)$ be the trajectory on $\mathbb{R}^{4N}$ which evolves according to the Newtonian equation of motion with cut-off, i.e.,

$$\begin{cases} \frac{d}{dt} X^N t = V^N t, \\ \frac{d}{dt} V^N t = \Psi^N (X^N t, V^N t) + \Gamma (X^N t, V^N t), \end{cases} \quad (3.3)$$

where $\Psi^N (X^N t, V^N t)$ denotes the total interaction force with

$$(\Psi^N (X^N t, V^N t))_i = \frac{1}{N-1} \sum_{i \neq j} F^N (x_i^N - x_j^N, v_i^N - v_j^N),$$

while $\Gamma (X^N t, V^N t)$ stands for the desired velocity and direction acceleration with

$$(\Gamma (X^N t, V^N t))_i = G (x_i^N, v_i^N).$$

2. Let $(\overline{X}^N t, \overline{V}^N t)$ be the trajectory on $\mathbb{R}^{4N}$ which evolves according to the Vlasov equation

$$\partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N \ast f^N) f^N] + \nabla_v \cdot (G f^N) = 0, \quad (3.2)$$

i.e.,

$$\begin{cases} \frac{d}{dt} \overline{X}^N t = \overline{V}^N t, \\ \frac{d}{dt} \overline{V}^N t = \overline{\Psi}^N (\overline{X}^N t, \overline{V}^N t) + \overline{\Gamma} (\overline{X}^N t, \overline{V}^N t), \end{cases} \quad (3.3)$$

where $$(\overline{\Psi}^N (\overline{X}^N t, \overline{V}^N t))_i = \int \int F^N (\overline{x}_i^N - y, \overline{v}_i^N - w) f^N (t, y, w) dy dw \quad \text{and} \quad (\overline{\Gamma} (\overline{X}^N t, \overline{V}^N t))_i = G (\overline{x}_i^N, \overline{v}_i^N)$$ represent the total interaction force and the desired velocity and direction acceleration, respectively.
If $N$ is removed from the superscript, then $(X_t, V_t)$ and $(\bar{X}_t, \bar{V}_t)$ denote the particle configurations driven by the force without cut-off. Analogically, if $t$ is removed from the subscript, $(X, V)$ and $(\bar{X}, \bar{V})$ represent the stochastic initial data, which are independent and identically distributed. Note that we always consider the same initial data for both systems, that means $(X, V) = (\bar{X}, \bar{V})$.

Remark 3.1 We also point out several facts for the interaction force $F_N(x, v)$ with cut-off and the acceleration $G(x, v)$. All the properties can be checked by direct computations.

(a) $F_N(x, v)$ is bounded, i.e., $|F_N(x, v)| \leq C$.
(b) $F_N(x, v)$ satisfies the following property

$$|F_N(x, v) - F_N(y, v)| \leq q_N(x, v)|x - y|,$$

where $q_N$ has compact support in $B_{2R} \times B_{2\tilde{R}}$ with

$$q_N(x, v) := \begin{cases} 
C \cdot \frac{1}{|x|} + C, & |x| \geq N^{-\theta}, \\
C \cdot N^\theta, & |x| < N^{-\theta}.
\end{cases}$$

(c) $F_N(x, v)$ is Lipschitz continuous in $v$.
(d) $G(x, v)$ is bounded, i.e., $|G(x, v)| \leq C$.

In this context, we use $C$ as a universal constant that might depend on $k_n, R, \tilde{R}, \gamma_n, \gamma_t$.

Furthermore, if there is a singularity in the velocity $v$ in the interaction potential similar to Remark 3.1(b), i.e.,

$$|F_N(x, v) - F_N(x, w)| \leq \tilde{q}_N(x, v)|v - w|,$$

where $\tilde{q}_N(x, v)$ has compact support in $B_{2R} \times B_{2\tilde{R}}$ with

$$\tilde{q}_N(x, v) := \begin{cases} 
C \cdot \frac{1}{|v|} + C, & |v| \geq N^{-\theta}, \\
C \cdot N^\theta, & |v| < N^{-\theta},
\end{cases}$$

it can be treated by using the same method as above and the results also apply.

4 Mean Field Limit

In this section, we present our key results in full detail. To show the desired convergence, our method can be summarized as follows. First, we start from the Newtonian system with carefully chosen cut-off and meanwhile introduce an intermediate system which involves convolution-type interaction with cut-off, namely (3.2) and (3.3). Then, we show the convergence of the intermediate system to the final mean field limit, where the law of large number comes into play. The crucial point of this method is that we apply stochastic initial data or in other words we consider a stochastic process. It enables us to use the tools from probability theory, which helps to better understand the mean field process. The overall procedure can be summarized as follows:
Theorem 4.1 Let \( \theta \) be the solution to the Vlasov equation (3.2) satisfies Assumptions 4.1(a) and 4.1(b) holds for \( G(x,v) \). Then there exists a constant \( C \) such that

\[
\sup_{0 \leq s \leq t} \left\| \int \frac{1}{|x-y|} f(s, y, v) \, dv \right\|_\infty \leq C,
\]

(b) the function \( G(x,v) \) is Lipschitz continuous both in \( x \) and \( v \), i.e., there exists a constant \( L \) such that

\[
|G(x,v) - G(x',v')| \leq L (|x-x'| + |v-v'|), \quad \forall (x,v),(x',v') \in \mathbb{R}^{4N}.
\]

Definition 4.1 Let \( \alpha \in (0, \frac{1}{2}) \) and \( S_t : \mathbb{R}^{4N} \times \mathbb{R} \to \mathbb{R} \) be the stochastic process given by

\[
S_t = \min \left\{ 1, N^\alpha \sup_{0 \leq s \leq t} \left| (X^N_s, V^N_s) - (\bar{X}^N_s, \bar{V}^N_s) \right|_\infty \right\}.
\]

The set, where \( |S_t| = 1 \), is defined as \( N_\alpha \), i.e.,

\[
N_\alpha := \left\{ (X, V) : \sup_{0 \leq s \leq t} \left| (X^N_s, V^N_s) - (\bar{X}^N_s, \bar{V}^N_s) \right|_\infty > N^{-\alpha} \right\}.
\]
where the convergence rate $r(N) = \max\{N^{-(1-\alpha-4\beta)}$, $N^{\alpha-\beta}$, $N^{-(1-\alpha-4\gamma)\ln^2 N}\}$.

We remark that $f^N(t, x, v) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ is automatically satisfied due to the mass conservation.

With additional assumption on the initial condition $f_0$ for the equations (2.2), (3.2) and on the solution of the Vlasov equation without cut-off, we further extend our result to

**Theorem 4.2** Let $f(t, x, v)$ and $f^N(t, x, v)$ be the solution to the Vlasov equation (2.2) and (3.2) respectively with the same initial data $f_0$. Suppose that Assumption 4.1(b) is satisfied. Moreover, $\nabla f_0$ is integrable and $f(t, x, v) \in L^\infty((0, \infty); L^\infty(\mathbb{R}^2 \times \mathbb{R}^2))$. Then there holds

$$\lim_{N \to \infty} \mathbb{P}_0 \left( \sup_{0 \leq s \leq t} \left| (X_s^N, V_s^N) - (\bar{X}_s^N, \bar{V}_s^N) \right|_\infty > N^{-\alpha} \right) = 0.$$ 

The proofs of both theorems will be presented at the end of this section.

The additional requirement on $f(t, x, v)$ stems from the existence and uniqueness of the solution to the Vlasov equation, which will be shown in another independent work in the near future.

**Definition 4.2** Let $\beta \in (\alpha, \frac{1-\alpha}{4})$, $\gamma \in (0, \frac{1}{4} - \theta)$. The sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ are characterized by

$$\mathcal{N}_\beta := \left\{ (X, V) : \left| \Psi^N(\bar{X}_t^N, \bar{V}_t^N) - \bar{\Psi}^N(\bar{X}_t^N, \bar{V}_t^N) \right|_\infty > N^{-\beta} \right\},$$

$$\mathcal{N}_\gamma := \left\{ (X, V) : \left| Q^N(\bar{X}_t^N, \bar{V}_t^N) - \bar{Q}^N(\bar{X}_t^N, \bar{V}_t^N) \right|_\infty > N^{-\gamma} \right\},$$

where $Q^N(\bar{X}_t^N, \bar{V}_t^N)$ and $\bar{Q}^N(\bar{X}_t^N, \bar{V}_t^N)$ are understood in the sense of

$$(Q^N(\bar{X}_t^N, \bar{V}_t^N))_i := \frac{1}{N-1} \sum_{i \neq j} q^N(\bar{x}_i^N - \bar{x}_j^N, \bar{v}_i^N - \bar{v}_j^N)$$

and correspondingly

$$(\bar{Q}^N(\bar{X}_t^N, \bar{V}_t^N))_i := \int \int q^N(\bar{x}_i^N - y, \bar{v}_i^N - w) f^N(t, y, w) dy dw.$$ 

Next, we will see that the measures of both sets $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ can be arbitrarily small, i.e., the probability of each set tends to 0 as $N$ goes to infinity. We prove the following two lemmas:

**Lemma 4.1** There exists a constant $C < \infty$ such that

$$\mathbb{P}_0(\mathcal{N}_\beta) \leq CN^{-(1-4\beta)}.$$ 

**Proof** First, we let the set $\mathcal{N}_\beta$ evolve along the characteristics of the Vlasov equation

$$\mathcal{N}_{\beta,t} := \left\{ (\bar{X}_t^N, \bar{V}_t^N) : \left| N^\beta \Psi^N(\bar{X}_t^N, \bar{V}_t^N) - N^\beta \bar{\Psi}^N(\bar{X}_t^N, \bar{V}_t^N) \right|_\infty > 1 \right\}.$$
and consider the following fact

$$N^\beta_{\beta,t} \subseteq \bigoplus_{i=1}^{N} N^i_{\beta,t},$$

where

$$N^i_{\beta,t} := \left\{ (\bar{x}^N_i, \bar{v}^N_i) : \right| \left| N^\beta \cdot \frac{1}{N-1} \sum_{i \neq j} F^N(\bar{x}^N_i - \bar{x}^N_j, \bar{v}^N_i - \bar{v}^N_j) - N^\beta (F^N \ast f^N)(t, \bar{x}^N_i, \bar{v}^N_i) \right|_{\infty} > 1 \left\} .$$

We therefore get

$$P_t(N^\beta_{\beta,t}) \leq \sum_{i=1}^{N} P_t(N^i_{\beta,t}) = N P_t(N^1_{\beta,t}),$$

where in the last step we use the symmetry argument in exchanging any two coordinates.

Using Markov inequality gives

$$P_t(N^1_{\beta,t}) \leq \mathbb{E}_t \left[ \left( \frac{N^\beta}{N-1} \right)^4 \mathbb{E}_t \left[ \left( \sum_{j=2}^{N} F^N(\bar{x}^N_i - \bar{x}^N_j, \bar{v}^N_i - \bar{v}^N_j) - (N-1)(F^N \ast f^N)(t, \bar{x}^N_i, \bar{v}^N_i) \right)^4 \right] \right] .$$

Let \( h_j := F^N(\bar{x}^N_i - \bar{x}^N_j, \bar{v}^N_i - \bar{v}^N_j) - \int \int F^N(\bar{x}^N_i - y, \bar{v}^N_i - w) f^N(t, y, w) dy dw \). Then, each term in the expectation (4.4) takes the form of \( \prod_{j=2}^{N} h_j \) with \( \sum_{j=1}^{N} k_j = 4 \), and more importantly, the expectation assumes the value of zero whenever there exists a \( j \) such that \( k_j = 1 \). This can be easily verified by integrating over the \( j \)-th variable first or, in other words, by acknowledging the fact that \( \forall j = 2, \ldots, N \), there holds

$$\mathbb{E}_t \left[ F^N(\bar{x}^N_i - \bar{x}^N_j, \bar{v}^N_i - \bar{v}^N_j) - \int \int F^N(\bar{x}^N_i - y, \bar{v}^N_i - w) f^N(t, y, w) dy dw \right] = 0.$$

Then, we can simplify the estimate (4.4) to

$$P_t(N^1_{\beta,t}) \leq \left( \frac{N^\beta}{N-1} \right)^4 \mathbb{E}_t \left[ \sum_{j=2}^{N} h_j^4 + \sum_{2 \leq m < n} \binom{4}{2} h_m^2 h_n^2 \right] .$$

Since \( F^N \) is bounded and \( \| f^N \|_1 = 1 \), we thus have for any fixed \( j \)

$$|h_j| \leq \left| F^N(\bar{x}^N_i - \bar{x}^N_j, \bar{v}^N_i - \bar{v}^N_j) \right| + \int \int \left| F^N(\bar{x}^N_i - y, \bar{v}^N_i - w) \right| f^N(t, y, w) dy dw \leq C.$$
Therefore $|h_j|$ is bounded to any power and we obtain

$$\mathbb{E}_t \left[ h_m^2 h_n^2 \right] \leq C \quad \text{and} \quad \mathbb{E}_t \left[ h_j^4 \right] \leq C$$

and consequently

$$\mathbb{P}_t(N_{\beta,t}^1) \leq \left( \frac{N^\beta}{N-1} \right)^4 \left( C \cdot (N-1) + C \cdot \frac{(N-1)(N-2)}{2} \right) \leq C \cdot N^{-(2-4\beta)}.$$ 

By noticing the fact that

$$\mathbb{P}_0(N_{\beta}) = \mathbb{P}_t(N_{\beta,t}) \leq N \mathbb{P}_t(N_{\beta,t}^1) \leq N \cdot C \cdot N^{-(2-4\beta)} = C \cdot N^{-(1-4\beta)},$$

we obtain the desired result. \qed

In fact, this result holds for any $\beta$ if we change accordingly the power in the proof to be another even number (depending on $\beta$) greater than four.

Due to the singularity of $\nabla_x F$, which is also the motivation for the cut-off, we exploit a slightly different technique as in Lemma 4.1 to prove

**Lemma 4.2** There exists a constant $C < \infty$ such that

$$\mathbb{P}_0(N_{\gamma}) \leq C \cdot \tilde{r}(N),$$

where $\tilde{r}(N)$ is the convergence rate, which is $N^{-(1-4\gamma)} \ln^2 N$ if $f^N \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ or $N^{-(1-4\theta-4\gamma)}$ otherwise.

**Proof** Let the set $N_{\gamma}$ evolve along the characteristics of the Vlasov equation

$$N_{\gamma,t} := \left\{ (\bar{x}_i^N, \bar{v}_i^N) : \left| N^{\gamma} Q^N((\bar{x}_i^N, \bar{v}_i^N) - N^{\gamma} \overline{Q}^N((\bar{x}_i^N, \bar{v}_i^N)) \right|_\infty > 1 \right\}$$

and consider the fact

$$N_{\gamma,t} \subseteq \bigoplus_{i=1}^N N_{\gamma,t}^i,$$

where

$$N_{\gamma,t}^i := \left\{ (\bar{x}_i^N, \bar{v}_i^N) : \left| N^{\gamma} \cdot \frac{1}{N-1} \sum_{i \neq j} q_i^N (\bar{x}_i^N - \bar{x}_j^N, \bar{v}_i^N - \bar{v}_j^N) 
- N^{\gamma} (q^N * f^N)(t, \bar{x}_i^N, \bar{v}_i^N) \right|_\infty > 1 \right\}.$$ 

Due to the symmetry in exchanging any two coordinates, we get

$$\mathbb{P}_t(N_{\gamma,t}) \leq \sum_{i=1}^N \mathbb{P}_t(N_{\gamma,t}^i) = N \mathbb{P}_t(N_{\gamma,t}^1).$$
Using Markov inequality gives

$$
\mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \leq \mathbb{E}_t \left[ \left( \frac{N^\gamma}{N - 1} \right)^4 \sum_{j=2}^{N} q^N(\bar{x}_1^N - \bar{x}_j^N, \bar{v}_1^N - \bar{v}_j^N) - N^\gamma (q^N \ast f^N)(t, \bar{x}_1^N, \bar{v}_1^N) \right]^4.
$$

$$
= \left( \frac{N^\gamma}{N - 1} \right)^4 \mathbb{E}_t \left[ \left( \sum_{j=2}^{N} q^N(\bar{x}_1^N - \bar{x}_j^N, \bar{v}_1^N - \bar{v}_j^N) - (N - 1)(q^N \ast f^N)(t, \bar{x}_1^N, \bar{v}_1^N) \right)^4 \right].
$$

(4.5)

In order to avoid redundant complexity, we borrow the notation from Lemma 4.1 and also define $h_j := q^N(\bar{x}_1^N - \bar{x}_j^N, \bar{v}_1^N - \bar{v}_j^N) - \int \int q^N(\bar{x}_1^N - y, \bar{v}_1^N - w) f^N(t, y, w) dydw$. With the same argument as in Lemma 4.1, we simplify the estimate (4.5) to

$$
\mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \leq \left( \frac{N^\gamma}{N - 1} \right)^4 \mathbb{E}_t \left[ \sum_{j=2}^{N} h_j^4 + \sum_{2 \leq m < n} 6h_m^2 h_n^2 \right].
$$

On one hand, due to the cut-off, it is clear that

$$
||q^N||_{\infty} \leq C \cdot N^\theta.
$$

On the other hand, by taking out the $L^\infty$-norm of $q^N$ and using the integrability of $f^N$, we achieve

$$
\left| \int \int q^N(\bar{x}_1^N - y, \bar{v}_1^N - w) f^N(t, y, w) dydw \right| \leq C \cdot N^\theta.
$$

Therefore $|h_j|$ is bounded by $C \cdot N^\theta$ and it is now obvious to see that

$$
\mathbb{E}_t \left[ h_j^4 \right] \leq C \cdot N^{4\theta} \quad \text{and} \quad \mathbb{E}_t \left[ h_m^2 h_n^2 \right] \leq C \cdot N^{4\theta}
$$

and consequently

$$
\mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \leq C \cdot \left( \frac{N^\gamma}{N - 1} \right)^4 \left( N^{4\theta} \cdot (N - 1) + N^{4\theta} \cdot \frac{(N - 1)(N - 2)}{2} \right) \leq C \cdot N^{-(2-4\theta-4\gamma)}.
$$

By noticing the fact that

$$
\mathbb{P}_0(\mathcal{N}_{\gamma}) = \mathbb{P}_t(\mathcal{N}_{\gamma,t}) \leq N \mathbb{P}_t(\mathcal{N}^1_{\gamma,t}) \leq C \cdot N^{-(1-4\theta-4\gamma)},
$$

we complete the first part of the lemma.

Furthermore, if $f^N((t, x, v) \in L^\infty((0, \infty); L^1(\mathbb{R}^2 \times \mathbb{R}^2)) \cap L^\infty((0, \infty); L^\infty(\mathbb{R}^2 \times \mathbb{R}^2))$, by applying the inequality $\mathbb{E}[(X - \mathbb{E}[X])^2] \leq \mathbb{E}[X^2]$ for any random variable $X$, we have for any fixed $j$
\[\mathbb{E}_t \left[ \left( q^N(x_1^N - x_j^N, v_1^N - v_j^N) - \int \int q^N(t, y, w) \, dydw \right)^2 \right] \]
\[\leq \mathbb{E}_t \left[ \left( q^N(x_1^N - x_j^N, v_1^N - v_j^N) \right)^2 \right] \]
\[\leq \int \int \left( \int \int_{|z-y|<N^{-\theta}} (C \cdot N^{\theta} + C)^2 f^N(t, y, w) \, dydw \right) f^N(t, z, u) \, dzdu \]
\[\quad + \int \int \left( \int \int_{|z-y|\geq N^{-\theta}} \left( C \cdot \frac{1}{|z-y|} \right)^2 f^N(t, y, w) \, dydw \right) f^N(t, z, u) \, dzdu.\]

We take out the \( L^\infty \)-norm of \( f^N \) in both terms. The integral left inside the first term is bounded by a constant while in the second term the integral can be estimated by
\[\int \int_{|z-y|\geq N^{-\theta}} \left( C \cdot \frac{1}{|z-y|} \right)^2 f^N(t, y, w) \, dydw \leq C + 2\pi \theta \ln N \leq C \cdot \ln N,\]
where we use that \( q^N \) has compact support. Therefore for any fixed \( j \)
\[\mathbb{E}_t \left[ h_j^4 \right] \leq ||h_j||_\infty^2 \mathbb{E}_t \left[ h_j^2 \right] \leq C \cdot N^{2\theta} \ln N,\]
\[\mathbb{E}_t \left[ h_m^2 h_n^2 \right] \leq C \cdot \ln^2 N.\]

Consequently
\[\mathbb{P}_t(\mathcal{N}_t^1) \leq C \cdot \left( \frac{N^\theta}{N-1} \right)^4 \left( N^{2\theta} \ln N \cdot (N - 1) + \ln^2 N \cdot \frac{(N - 1)(N - 2)}{2} \right) \]
\[\leq C \cdot N^{-(2-4\theta)} \ln^2 N.\]

Thus it holds that
\[\mathbb{P}_0(\mathcal{N}_t) = \mathbb{P}_t(\mathcal{N}_t^1) \leq N\mathbb{P}_t(\mathcal{N}_t^1) \leq C \cdot N^{-(1-4\theta)} \ln^2 N.\]

Lemma 4.3 Let \( \mathcal{N}_\alpha, \mathcal{N}_\beta, \mathcal{N}_\gamma \) be defined as in (4.1)–(4.3). Suppose that \( f^N(t, x, v) \) satisfies Assumptions 4.1(a) and 4.1(b) holds for \( G(x, v) \). Then there exists a constant \( C < \infty \) such that
\[\left| \left( V_i^N, \Psi^N(X_i^N, V_i^N) + \Gamma(X_i^N, V_i^N) \right) - \left( V_i^N, \Psi^N(X_i^N, V_i^N) + \Gamma(X_i^N, V_i^N) \right) \right|_\infty \]
\[\leq CS_t(X, V)N^{-\alpha} + N^{-\beta} \]
for all initial data \( (X, V) \in (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c.\)
Next, we estimate term by term.

- Since $(X, V) \notin \mathcal{N}_\alpha$, 
  \[
  |I_1| := \left| V_t^N - \overline{V}_t^N \right|_\infty \leq S_t(X, V) N^{-\alpha}.
  \]

- With the help of $q^N$ which is defined in Remark 3.1 and the fact that $F^N$ is Lipschitz continuous in $v$, we obtain 
  \[
  \left| \frac{1}{N-1} \sum_{i \neq j} F^N(x_i^N - x_j^N, v_i^N - v_j^N) - \frac{1}{N-1} \sum_{i \neq j} F^N(x_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N) \right|
  \leq \frac{1}{N-1} \sum_{i \neq j} \left| q^N(x_i^N - \overline{x}_j^N, \overline{v}_i^N - \overline{v}_j^N) \right| \left( 2|x_i^N - \overline{x}_j^N| + 2|v_i^N - \overline{v}_j^N| \right). \tag{4.6}
  \]

Since $(X, V) \notin \mathcal{N}_\alpha$, it follows in particular for any $1 \leq i \leq N$ 
  \[
  |x_i^N - \overline{x}_i^N| \leq N^{-\alpha} \quad \text{and} \quad |v_i^N - \overline{v}_i^N| \leq N^{-\alpha}.
  \]

So together with (4.6), we have 
  \[
  \left| (\Psi^N(X_i^N, V_i^N))_i - (\Psi^N(\overline{X}_i^N, \overline{V}_i^N))_i \right| \leq 4 \left| (Q^N(\overline{X}_i^N, \overline{V}_i^N))_i \right| N^{-\alpha}.
  \]

On the other hand, because $(X, V) \notin \mathcal{N}_\gamma$, it follows 
  \[
  \left| (Q^N(\overline{X}_i^N, \overline{V}_i^N))_i \right| \leq \|q^N \ast f^N\|_\infty + N^{-\gamma} < C
  \]

and thus 
  \[
  |I_2| := \left| \Psi^N(X_i^N, V_i^N) - \Psi^N(\overline{X}_i^N, \overline{V}_i^N) \right|_\infty \leq CS_t(X, V) N^{-\alpha}.
  \]

- Since $(X, V) \notin \mathcal{N}_\beta$, it follows directly 
  \[
  |I_3| := \left| \Psi^N(\overline{X}_i^N, \overline{V}_i^N) - \Psi^N(\overline{X}_i^N, \overline{V}_i^N) \right|_\infty \leq N^{-\beta}.
  \]

- Since $G(x, v)$ under Assumption 4.1(b) is Lipschitz continuous, we have for each $1 \leq i \leq N$ and $(x_i^N, v_i^N) = ((X_i^N, V_i^N))_i$, $(\overline{x}_i^N, \overline{v}_i^N) = ((\overline{X}_i^N, \overline{V}_i^N))_i$ 
  \[
  \left| G(x_i^N, v_i^N) - G(\overline{x}_i^N, \overline{v}_i^N) \right| \leq L \left| (x_i^N, v_i^N) - (\overline{x}_i^N, \overline{v}_i^N) \right|.
  \]

Together with the fact that $(X, V) \notin \mathcal{N}_\alpha$, there holds 
  \[
  |I_4| := \left| \Gamma(X_i^N, V_i^N) - \Gamma(\overline{X}_i^N, \overline{V}_i^N) \right|_\infty \leq LS_t(X, V) N^{-\alpha}.
  \]
Combining all the four terms, we end up with
\[
\left| (V_t^N, \Psi_t^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)) - (\nabla_t^N, \nabla_t^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)) \right|_{\infty} \\
\leq C S_t(X, V)N^{-\alpha} + N^{-\beta}
\]
for all \((X, V) \in (N_\alpha \cup N_\beta \cup N_\gamma)^c\).
\(\square\)

Using Lemmas 4.1 - 4.3 we can now prove Theorem 4.1 and Theorem 4.2:

**Proof of Theorem 4.1** From the definition of the Newtonian flow (3.1) and the characteristics of the Vlasov equation (3.3), we know
\[
(X_{t+dt}^N, V_{t+dt}^N) = (X_t^N, V_t^N) + (V_t^N, \Psi_t^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N))dt + o(dt),
\]
\[
(\bar{X}_{t+dt}^N, \bar{V}_{t+dt}^N) = (\bar{X}_t^N, \bar{V}_t^N) + (\bar{V}_t^N, \bar{\Psi}_t^N(\bar{X}_t^N, \bar{V}_t^N) + \Gamma(\bar{X}_t^N, \bar{V}_t^N))dt + o(dt).
\]
Thus
\[
\left| (X_{t+dt}^N, V_{t+dt}^N) - (\bar{X}_{t+dt}^N, \bar{V}_{t+dt}^N) \right|_{\infty} \leq \left| (X_t^N, V_t^N) - (\bar{X}_t^N, \bar{V}_t^N) \right|_{\infty}
\]
\[
+ \left| (\bar{V}_t^N, \bar{\Psi}_t^N(\bar{X}_t^N, \bar{V}_t^N) + \Gamma(\bar{X}_t^N, \bar{V}_t^N)) \right|_{\infty} dt + o(dt),
\]
i.e.,
\[
S_{t+dt} - S_t \leq \left| (V_t^N, \Psi_t^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N)) \right|_{\infty} N^\alpha dt + o(dt)
\]
Taking the expectation over both sides yields
\[
\mathbb{E}_0 \left[ S_{t+dt} - S_t \right] = \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid N_\alpha \right] + \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid N_\alpha^c \right]
\]
\[
\leq \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid (N_\beta \cup N_\gamma)^c \backslash N_\alpha \right] + \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid (N_\alpha \cup N_\beta \cup N_\gamma)^c \right]
\]
\[
\leq \mathbb{E}_0 \left[ \left| V_t^N - \bar{V}_t^N \right|_{\infty} \mid (N_\beta \cup N_\gamma)^c \backslash N_\alpha \right] N^\alpha dt
\]
\[
+ \mathbb{E}_0 \left[ \left| \Psi_t^N(X_t^N, V_t^N) - \bar{\Psi}_t^N(\bar{X}_t^N, \bar{V}_t^N) \right|_{\infty} \mid (N_\beta \cup N_\gamma)^c \backslash N_\alpha \right] N^\alpha dt
\]
\[
+ \mathbb{E}_0 \left[ \left| \Gamma(X_t^N, V_t^N) - \Gamma(\bar{X}_t^N, \bar{V}_t^N) \right|_{\infty} \mid (N_\beta \cup N_\gamma)^c \backslash N_\alpha \right] N^\alpha dt
\]
\[
+ \mathbb{E}_0 \left[ S_{t+dt} - S_t \mid (N_\alpha \cup N_\beta \cup N_\gamma)^c \right] + o(dt)
\]
\[
=: J_1 + J_2 + J_3 + J_4 + o(dt),
\]
where in the second step we use \(\mathbb{E}_0(S_{t+dt} - S_t \mid N_\alpha) \leq 0\) and decompose the set \(N_\alpha^c\) into \((N_\beta \cup N_\gamma)^c \backslash N_\alpha\) and \((N_\alpha \cup N_\beta \cup N_\gamma)^c\).
Since \((X, V) \notin N_\alpha^c\), it follows
\[
J_1 = \mathbb{E}_0 \left[ \left| V_t^N - \bar{V}_t^N \right|_{\infty} \mid (N_\beta \cup N_\gamma)^c \backslash N_\alpha \right] N^\alpha dt
\]
\[
\leq \left( \mathbb{P}_0(N_\beta) + \mathbb{P}_0(N_\gamma) \right) dt.
\]
Due to the definition of \(\Psi_t^N, \bar{\Psi}_t^N, \Gamma\) as well as the boundedness of \(F_t^N\), we obtain
\[
J_2 \leq \left( \left\| F_t^N \right\|_{\infty} + \left\| F_t^N * f \right\|_{\infty} \right) \left( \mathbb{P}_0(N_\beta) + \mathbb{P}_0(N_\gamma) \right) N^\alpha dt.
\]
\[ J_3 \leq 2\|G\|_\infty (P_0(N_\beta) + P_0(N_\gamma))N^\alpha dt. \]

Thanks to Lemma 4.1 and Lemma 4.2, we get
\[
J_1 + J_2 + J_3 = [N^{-\alpha} + C] (P_0(N_\beta) + P_0(N_\gamma))N^\alpha dt \\
\leq C \cdot \max(\tilde{r}(N), N^{-(1-4\beta)})N^\alpha dt
\]
where \( \tilde{r}(N) \) is the convergence rate, which is \( N^{-(1-4\gamma)} \ln^2 N \) if \( f^N \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) or \( N^{-(1-4\beta-4\gamma)} \) otherwise. On the other hand, Lemma 4.3 states that
\[
J_4 = E_0[St_{t+dt} - St | (N\alpha \cup N\beta \cup N\gamma)^c] \\
\leq (C \cdot E_0[St] N^{-\alpha} + N^{-\beta}) \cdot N^\alpha dt + o(dt)
\]
\[= C \cdot E_0[St] dt + N^{\alpha-\beta} dt + o(dt). \]

Therefore, we can determine the estimate
\[
E_0[St_{t+dt}] - E_0[St] \leq E_0[St_{t+dt} - St] \\
\leq C \cdot E_0[St] dt + C \cdot \max(\tilde{r}(N)N^\alpha, N^{-(1-4\beta)}, N^{\alpha-\beta}) dt + o(dt).
\]

Equivalently, we have
\[
\frac{d}{dt} E_0[St] \leq C \cdot E_0[St] + C \cdot \max(\tilde{r}(N)N^\alpha, N^{-(1-4\beta)}, N^{\alpha-\beta}).
\]
Gronwall’s inequality yields
\[
E_0[St] \leq e^{Ct} \cdot \max(\tilde{r}(N)N^\alpha, N^{-(1-4\beta)}, N^{\alpha-\beta}).
\]
The proof is completed by the following Markov inequality
\[
P_0\left( \sup_{0 \leq s \leq t} \left| (X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N) \right|_\infty > N^{-\alpha} \right) = P_0(S_t = 1) \leq E_0[St].
\]

\[\square\]

**Proof of Theorem 4.2** Let \( N \in \mathbb{N} \) and
\[
W_t := \sup_{(X, V) \in \mathbb{R}^{4N}} \left| (\overline{X}_t^N, \overline{V}_t^N) - (X_t, V_t) \right|
\]
With the same argument as in the proof of Theorem 4.1, it is not difficult to deduce
\[
W_{t+dt} - W_t \leq \left| (\overline{V}_t^N, \overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N) - (\overline{V}_t, \overline{X}_t, \overline{V}_t) + \Gamma(\overline{X}_t, \overline{V}_t) \right|_\infty dt \\
= o(dt).
\]
Furthermore, with the Lipschitz continuity of \( G(x, v) \), we get
\[
D \leq W_t + \left| \Psi^N_t(X_t^N, V_t^N) - \Psi(X_t, V_t) \right|_{\infty} + \left| \Gamma(X_t^N, V_t^N) - \Gamma(X_t, V_t) \right|_{\infty} \\
\leq C \cdot W_t + \sup_{1 \leq i \leq N} \left| F^N \ast f^N(x_i^N, v_i^N) - F \ast f(x_i, v_i) \right| \\
\leq C \cdot W_t + \sup_{1 \leq i \leq N} \left| F^N \ast f^N(x_i^N, v_i^N) - F^N \ast f(x_i, v_i) \right| \\
+ \sup_{1 \leq i \leq N} \left| F^N \ast f(x_i, v_i) - F \ast f(x_i, v_i) \right|.
\]

By using the integrability of \( \nabla F^N \), we estimate the second term by
\[
\sup_{1 \leq i \leq N} \left| F^N \ast f^N(x_i^N, v_i^N) - F^N \ast f(x_i, v_i) \right| \leq ||\nabla F^N||_1 ||f^N||_{\infty} W_t \leq C \cdot W_t.
\]

Due to the integrability of \( \nabla f_0 \), the third term can be controlled by
\[
\sup_{1 \leq i \leq N} \left| F^N \ast f(x_i, v_i) - F \ast f(x_i, v_i) \right| \leq ||F^N||_{\infty} ||f^N - f||_1 \leq C ||\nabla f_0||_1 W_t.
\]

In the estimates above, the reversibility of both particle trajectories is used. The last term is straightforward to estimate
\[
\sup_{1 \leq i \leq N} \left| F^N \ast f(x_i, v_i) - F \ast f(x_i, v_i) \right| \leq ||f||_{\infty} ||F^N - F||_1 \leq C \cdot N^{-\theta}.
\]

Therefore we arrive at
\[
W_{t+dt} - W_t \leq (C \cdot W_t + C \cdot N^{-\theta}) dt + o(dt),
\]
or equivalently
\[
\frac{d}{dt} W_t \leq C \cdot W_t + C \cdot N^{-\theta}.
\]

Gronwall’s inequality gives
\[
W_t \leq C \cdot N^{-\theta}.
\]

Together with Theorem 4.1, we complete the proof. \( \square \)

5 Propagation of Chaos

We can clearly see as the direct byproduct of the results stated above that chaos indeed propagates, which means the convergence of the one particle marginals of the \( N \)-particle system to the solution of the Vlasov equation in the sense of bounded Lipschitz distance. We illustrate the propagation of chaos also in two steps by using the Vlasov flow with cut-off as an intermediate tool. We present the result in full detail under the conditions of Theorem 4.1.
Definition 5.1 For any two probability densities $\mu, \nu : \mathbb{R}^4 \to \mathbb{R}^+$, the bounded Lipschitz distance is defined by
\[
d_L(\mu, \nu) := \sup_{g \in \mathcal{L}} \left| \int (\mu(x, v) - \nu(x, v)) g(x, v) \, dx \, dv \right|,
\]
where $\mathcal{L} := \{ g : \|g\|_\infty = \|g\|_L = 1 \}$ and $\|g\|_L$ denotes the global Lipschitz constant of $g$.

In order to simplify the notation, we also introduce hereafter $(x_i, -t, v_i, -t)$ and $(\bar{x}_i, -t, \bar{v}_i, -t)$ to be the position and velocity of the $i$-th particle at initial time, which evolves according to the Newtonian and Vlasov flow with cut-off starting from $(x_i, v_i)$ at time $t$, respectively.

Theorem 5.1 Let $f_t^N : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^+$ be the solution to (3.2), $\mu_t : \mathbb{R} \times \mathbb{R}^{4N} \to \mathbb{R}^+$ be the $N$-particle density of the Newtonian flow and the one-particle marginals $\mu_t^{(1)}$ be given by
\[
\mu_t^{(1)}(x_1, v_1) := \int \mu_t(x_1, v_1, \ldots, x_N, v_N) \, dx_2 \, dv_2 \ldots dx_N \, dv_N,
\]
where
\[
\mu_t(x_1, v_1, \ldots, x_N, v_N) := \mu_0(x_1, -t, v_1, -t, \ldots, x_N, -t, v_N, -t).
\]
Assume that initially the one particle marginals converges to the initial probability density $f_0^N$ in the sense of bounded Lipschitz distance, i.e.,
\[
\lim_{N \to \infty} d_L(\mu_0^{(1)}, f_0^N) = 0.
\]
Then under the conditions of Theorem 4.1, there holds
\[
\lim_{N \to \infty} d_L(\mu_t^{(1)}, f_t^N) = 0.
\]

Proof By definition, we have
\[
d_L(\mu_t^{(1)}, f_t^N) = \sup_{g \in \mathcal{L}} \left| \int (\mu_t^{(1)}(x_1, v_1) - f_t^N(x_1, v_1)) g(x_1, v_1) \, dx_1 \, dv_1 \right|
\]
\[
= \sup_{g \in \mathcal{L}} \left| \int (\mu_t(x_1, v_1) - f_t^N(x_1, v_1)) + \prod_{i=1}^N f_i^N(x_i, v_i) g(x_1, v_1) \, dx_1 \, dv_1 \, dx_2 \, dv_2 \ldots dx_N \, dv_N \right|. \tag{5.1}
\]
Since both the Newtonian and Vlasov flow leave the measure invariant, then
\[
(5.1) = \sup_{g \in \mathcal{L}} \left| \int \mu_0(x_1, v_1, \ldots, x_N, v_N) g(x_1, -t, v_1, -t, \ldots, x_N, -t, v_N) \, dx_1 \, dv_1 \ldots dx_N \, dv_N
\]
\[
- \int \prod_{i=1}^N f_i^N(x_i, v_i) g(\bar{x}_1, -t, \bar{v}_1, -t) \, dx_1 \, dv_1 \ldots dx_N \, dv_N \right|
\]
\[
\leq \sup_{g \in \mathcal{L}} \left| \int \mu_0(x_1, v_1, \ldots, x_N, v_N) (g(x_1, -t, v_1, -t)
\]
\[
- g(\bar{x}_1, -t, \bar{v}_1, -t)) \, dx_1 \, dv_1 \ldots dx_N \, dv_N \right|
\]
where

On the other hand, due to the reversibility of both particle trajectories and |

Further we decompose $M_1$ into $M_{11} + M_{12}$, where

and

Under Theorem 4.1, we know

By using the fact that $\|g\|_{L} = 1$, we thus obtain

On the other hand, due to the reversibility of both particle trajectories and $\|g\|_{\infty} = 1$, we have

In summary, $M_1$ converges to zero as $N$ goes to infinity. Meanwhile it is also clear that $M_2$ tends to zero as $N \to \infty$ due to the assumption on the initial probability density. Combining all the terms completes the proof.

\begin{flushright}
\Box
\end{flushright}

**Theorem 5.2** Let $f_t : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^+$ be the solution to (2.2), $\mu_t : \mathbb{R} \times \mathbb{R}^{4N} \to \mathbb{R}^+$ be the $N$-particle density of the Newtonian flow and the one-particle marginals $\mu_t^{(1)}$ be given by

$$
\mu_t^{(1)}(x_1, v_1) := \int \mu_t(x_1, x_1, \ldots, x_N, v_N) \, dx_2 \, dv_2 \ldots dx_N \, dv_N,
$$

where

$$
\mu_t(x_1, v_1, \ldots, x_N, v_N) := \mu_0(x_1, -t, \ldots, x_N, -t, v_1, \ldots, v_N).
$$
Assume that initially the one particle marginals converges to the initial probability density $f_0$ in the sense of bounded Lipschitz distance, i.e.,

$$\lim_{N \to \infty} d_L(\mu^{(1)}_0, f_0) = 0.$$ 

Then under the conditions of Theorem 4.2, there holds

$$\lim_{N \to \infty} d_L(\mu^{(1)}_t, f_t) = 0.$$ 

**Proof** By replacing $f^N_t$ with $f_t$ in the proof of Theorem 5.1 and using the conditions of Theorem 4.2, one will directly get the desired result. But we emphasize that Theorem 5.2 actually implies the convergence of the solution of (3.2) to the solution of (2.2) in the sense of bounded Lipschitz distance. \(\square\)

Note that if the initial one particle marginals converges in a certain rate to the initial probability density in both theorems above, we can also achieve the convergence rate for any fixed time $t$.

**6 Summary**

This paper deals with one core problem: how to derive rigorously the kinetic description of one particle density evolution from the $N$-particle system for $N$ being large enough. Our main results, Theorem 4.1 and Theorem 4.2, state that the trajectories of both the Newtonian system with cut-off and the characteristics of the Vlasov equation are close to each other in the probability sense. Propagation of chaos, as the direct implication of the two theorems, is given in Theorem 5.2. The existence and uniqueness of the $L^\infty$-solution of the Vlasov equation is left for a future independent work.

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**References**

1. Bellomo, N., Dogbé, C.: On the modeling of traffic and crowds: a survey of models, speculations, and perspectives. SIAM Rev. **53**(3), 409–463 (2011)
2. Bellomo, N., Piccoli, B., Tosin, A.: Modeling crowd dynamics from a complex system viewpoint. Math. Models Methods Appl. Sci. **22**, 1230004 (2012)
3. Boers, N., Pickl, P.: On mean field limits for dynamical systems. J. Stat. Phys. **164**(1), 1–16 (2015)
4. Cristiani, E., Piccoli, B., Tosin, A.: Multiscale modeling of granular flows with application to crowd dynamics. Multiscale Model. Simul. **9**(1), 155–182 (2011)
5. Colombo, R., Garavello, M., Lécureux-Mercier, M.: A class of nonlocal models for pedestrian traffic. Math. Models Methods Appl. Sci. **22**(4), 1150023 (2012)
6. Degond, P., Appert-Rolland, C., Moussaid, M., Pettré, J., Theraulaz, G.: A hierarchy of heuristic-based models of crowd dynamics. J. Stat. Phys. **152**(6), 1033–1068 (2013)
7. Etikyala, R., Göttlich, S., Klar, A., Tiwari, S.: Particle methods for pedestrian flow models: from microscopic to nonlocal continuum models. Math. Models Methods Appl. Sci. **24**(12), 2503–2523 (2014)
8. Hauray, M., Jabin, P.E.: N-particles approximation of the Vlasov equations with singular potential. Arch. Ration. Mech. Anal. **183**(3), 489–524 (2007)
9. Helbing, D., Molnar, P.: Social force model for pedestrian dynamics. Phys. Rev. E **51**(5), 4282 (1995)
10. Hughes, R.L.: A continuum theory for the flow of pedestrians. Transp. Res. Part B **36**(6), 507–535 (2002)
11. Jabin, P.E., Hauray, M.: Particles approximations of Vlasov equations with singular forces: propagation of chaos. arXiv preprint (2014). arXiv:1107.3821
12. Lewin, K.: In: D. Cartwright (eds.) Field Theory in Social Science: Selected Theoretical Papers. Harper and Brothers, New York (1951)

13. Naldi, G., Pareschi, L., Toscani, G.: Mathematical Modeling of Collective Behavior in Socio-economic and Life Sciences. Birkhäuser, Boston (2010)

14. Philipowski, R.: Interacting diffusions approximating the porous medium equation and propagation of chaos. Stoch. Process. Appl. 117(4), 526–538 (2007)

15. Piccoli, B., Tosin, A.: Pedestrian flows in bounded domains with obstacles. Contin. Mech. Thermodyn. 21(2), 85–107 (2009)

16. Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer, Berlin (2012)

17. Sznitman, A.S.: Topics in propagation of chaos. In: Ecole d’été de probabilités de Saint-Flour XIX-1989. Springer, Berlin, pp. 165–251 (1991)

18. Toscani, G.: Kinetic models of opinion formation. Commun. Math. Sci. 4(3), 481–496 (2006)