Studying the interaction of a reservoir with an ideal compressible fluid with an elastic half-space

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Abstract. We solve in a flat formulation the problem of harmonic oscillations for a basin with an ideal compressible fluid on an elastic half-space exposed to a localized surface vibration load. The problem reduces to an integral equation (IE) of the first kind for the amplitude of the contact hydrodynamic pressure with a kernel that depends on the difference and the sum of arguments. The IE was solved by the factorization method. A semi-analytical method is presented for determining the main parameters of the contact interaction in hydroelastic systems «liquid-soil» taking into account the effect of natural and man-made vibration loads on them. This makes it possible to identify the conditions for the occurrence of dynamic modes that are dangerous for the construction integrity and to estimate their frequencies range depending on defining characteristics of the system.

1. Introduction

The research into the field of interaction between the structures with a fluid and an elastic foundation is currently stimulated by increased requirements for the precision and reliability of assessing the stress-strain state for the created hydraulic structures and technical reservoirs. This, in turn, leads to the need for the models refinement and improvement of the methods for their research. Improving the accuracy of designed schemes and predicting the consequences of vibration effects is especially important in the case when reservoirs, artificial and natural dams and other hydraulic structures are located near settlements. The authors of [1–5] demonstrate various approaches to solving the problems of vibration of tanks with liquid for a wide range of problem statements.

The study of the features for the dynamic interaction of a contacting liquid and elastic media requires a joint solution for interrelated problems of the elasticity theory and hydrodynamics [3, 4]. The authors of [6, 7] study the wave effects in an elastic-fluid medium.

In this paper, the authors investigate the process of vibration excitation for the elastic waves in a geological medium with an artificial reservoir on the surface, the distribution of hydrodynamic pressure inside the basin and contact stresses at the interface between two media. The model is described by a boundary value problem for a system of partial differential equations with mixed boundary conditions, which is reduced by the authors to an integral equation of the first kind with respect to contact stresses in the area of separation between a liquid and elastic media.
2. The plane formulation of the harmonic oscillations problem for a basin with an ideal compressible fluid on an elastic foundation

In the case of mathematical modeling of technical reservoirs and hydraulic structures for various purposes, a limited or semi-limited layer of a compressible fluid is often chosen. The studied hydroelastic system is simulated by a bounded layer of a compressible fluid on an elastic homogeneous semi-bounded foundation. We solve the problem in the plane formulation.

We consider the case of forced steady-state oscillations of a reservoir with an ideal compressible fluid, occupying the volume \( \{0 \leq x \leq d; 0 \leq z \leq h\} \) on a deformable foundation, which is considered as an elastic homogeneous half-plane \( \{-\infty < x < \infty; z < 0\} \). Harmonic oscillations of the system are excited by a vibration source simulated by a load \( F = \{0, q_0(x)\exp(-i\omega t)\} \) distributed in the region \( \{-l - s \leq x \leq -s; z = 0\} \), where \( \omega \) is the vibration frequency. Outside this area, the surface of the medium is free from mechanical stresses. Figure 1 illustrates the geometric representation of the system under consideration in the \( xOz \) coordinate system.

![Figure 1. A reservoir with liquid on an elastic half-space.](image)

The wave field in a liquid is described by the velocity potential \( \varphi^s(x,z,t) = \varphi(x,z)\exp(-i\omega t) \) (\( \varphi(x,z) = \varphi(x,z,\omega) \)). For the considered steady-state nature of oscillations, the factor \( \exp(-i\omega t) \) is omitted everywhere below, the amplitude values of the corresponding functions are used in all ratios. Depending on the vibration mode \( \varphi^s(x,z,t) = \text{Re}[\varphi(x,z,\omega)\exp(-i\omega t)] \) or \( \varphi^s(x,z,t) = \text{Im}[\varphi(x,z,\omega)\exp(-i\omega t)] \), there is no direct indication to the dependence of the complex amplitudes on the frequency in the designations.

Thus the wave equation for the velocity potential takes the form

\[
c_0^2 \Delta \varphi(x,z) = -\omega^2 \varphi(x,z), \quad 0 < x < d, 0 < z < h,
\]
where \( c_0 \) is the sound velocity in the liquid, \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \).

The hydrodynamic pressure is zero on the surface of the liquid, on the bottom of the reservoir there is an elastic foundation, and its vertical walls are impenetrable

\[
\varphi(x,h) = 0, \quad -i\omega \rho_0 \varphi(x,0) = q(x), \quad 0 \leq x \leq d, \quad \frac{\partial \varphi}{\partial x}\bigg|_{x=0} = \frac{\partial \varphi}{\partial x}\bigg|_{x=d} = 0, \quad (0 \leq z \leq h),
\]
where \( \rho_0 \) is the density of the liquid, \( q(x,z) \) is the amplitude of the normal pressure.

The amplitudes of displacements for the deformable foundation \( \mathbf{u} = \{u,w\} \), which are functions of coordinates, are described by the Lame equations [8]

\[
(\lambda + \mu) \text{grad}(\text{div}\mathbf{u}) + \mu \Delta \mathbf{u} = -\omega^2 \rho_0 \mathbf{u}.
\]
Here $\lambda, \mu$ are the elastic constants, $\rho$ is the density of the foundation.

The formulation of the problem is supplemented by the boundary conditions. On the surface of the foundation the following is valid:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad (-\infty < x < \infty, \ z = 0), \quad \left(2\mu\frac{\partial w}{\partial z} + \lambda\frac{\partial u}{\partial x}\right)_{x=0} = \begin{cases} q(x), & 0 \leq x \leq d; \\ q_0(x), & -l - s \leq x \leq -s; \\ 0, & x < -l - s, \ -s < x < 0, \ x > d. \end{cases}$$

Since the elastic half-plane is considered to be the foundation, the condition of decreasing displacements at infinity is satisfied

$$u(x, z) \to 0$$

and the radiation condition is taken from [8].

The ideal contact of the interacting media is determined by the ratio

$$\left(\frac{\partial \phi}{\partial z} + i\omega w\right)_{z=0} = 0, \quad -\infty < x < \infty, \ z = 0.$$  

3. The solution method

We reduce the system of boundary value problems (1), (2) and (3)–(5) with mixed boundary conditions using the method of integral transformation and the subsequent solution of the ordinary differential equation systems to an integral equation of the first kind with respect to the unknown hydrodynamic pressure at the interface between the media.

Thus, we obtain the solution of the boundary value problem (1), (2) by applying the finite cosine Fourier transform [9]

$$\varphi(x, z) = \frac{-i}{d\rho_i\omega} \sum_{n=-\infty}^{\infty} Q_c(n) K_i(n, z) \exp\left(-i\frac{\pi nx}{d}\right),$$

where

$$Q_c(n) = \int_{0}^{d} q(x) \cos\left(\frac{\pi ny}{d}\right) dx, \ K_i(n, z) = \frac{\text{sh}\left(\frac{(n-\epsilon)\pi n}{d} - \frac{\omega\rho}{c_t}\right)}{\text{sh}\left(\frac{\pi n}{d} - \frac{\omega\rho}{c_t}\right)}.$$  

Applying to problem (3)–(5) the elastic half-plane the Fourier transform with respect to $x$, we obtain a system of ordinary differential equations, whose solution gives a representation for the Fourier images of the foundation displacement vector elements

$$U(\alpha, z) = R_0(\alpha, z)(Q(\alpha) + Q_0(\alpha)), \ W(\alpha, z) = R_0(\alpha, z)(Q(\alpha) + Q_0(\alpha)), \quad R_0(\alpha, z) = 2N(\alpha, z)(\Delta(\alpha))^{-1}, \quad R_i(\alpha, z) = 2S(\alpha, z)(\Delta(\alpha))^{-1},$$

$$N(\alpha, z) = i\alpha\left(\eta \exp(\sigma_1 z) - \sigma_1 \sigma_2 \exp(\sigma_2 z)\right), \quad S(\alpha, z) = \sigma_1(\alpha^2 \exp(\sigma_2 z) - \eta \exp(\sigma_1 z)), \quad \Delta(\alpha) = 4\mu(\lambda^2\sigma_1\sigma_2 - \eta^2),$$

$$\sigma_j = \sqrt{\alpha^2 - \kappa_j^2} \quad (j = 1, 2), \quad \kappa_1^2 = \frac{\rho_o^2}{\lambda + 2\mu}, \quad \kappa_2^2 = \frac{\rho_o^2}{\mu}, \quad \eta = \alpha^2 - 0.5\kappa_2^2.$$  

By applying the inverse Fourier transform to the obtained ones, as a result we obtain

$$u(x, z) = \int_{0}^{d} \tilde{q}(\xi) r_i(x - \xi, z) d\xi + v_i(x, z) = \frac{1}{2\pi} \int_{\lambda} U(\alpha, z) \exp(-i\alpha x) d\alpha,$$

$$w(x, z) = \int_{0}^{d} \tilde{q}(\xi) r_2(x - \xi, z) d\xi + w_i(x, z) = \frac{1}{2\pi} \int_{\lambda} W(\alpha, z) \exp(-i\alpha x) d\alpha,$$
where \( R_j(\alpha,z), j=1,2 \) are the components of the last column of the Green's matrix symbol for the half-plane \([8]\). \( Q(\alpha), Q_0(\alpha) \) are the Fourier transforms for the stresses \( q(x) \) and \( q_0(x) \), respectively. The contour \( \Gamma \) deviates from the real axis when going around the real branch points and the poles of the integrand according to the radiation conditions \([8]\).

The obtained representations of the potential for velocities \((7)\) and displacements \((8), (9)\) are related by condition \((6)\). Using this condition, we arrive at the integral equation for the unknown hydrodynamic pressure

\[
\sum_{n=-\infty}^{\infty} \left[ k(x-2nd-\xi)q(\xi)d\xi + \int_0^d k(x-2nd+\xi)q(\xi)d\xi \right] - \rho_0\omega^2 \int_0^d r(x-\xi)q(\xi)d\xi = f(x), \quad 0 < x < d,
\]

where

\[
k(x) = \frac{1}{2\pi i} \int K(\alpha) \exp(-i\alpha x)d\alpha, \quad K(\alpha) = \sigma_j \text{ch}(\alpha h), \quad r(x) = \frac{1}{2\pi i} \int R(\alpha) \exp(-i\alpha x)d\alpha, \quad R(\alpha) = \sigma_j \kappa_j^2 (\Delta(\alpha))^{-1}, \quad \sigma_j = \sqrt{\alpha^2 - \kappa_j^2}, \quad \kappa_j = \frac{\omega}{c_0}.
\]

In \([10]\), we have presented a solution to the integral equation for the case when a strip with a clamped lower edge was chosen as the foundation. Following the same algorithm, we obtain a representation of the integral characteristic \( q(x) \) for the problem under consideration

\[
Q(\alpha) = \frac{1}{2} \left\{ \sum_k F^*(-p_k) \Psi(\alpha, p_k) + \sum_n F(z_n) \frac{\psi'(-z_n)}{\alpha + z_n} \right. \]

\[
+ \sum_n F(-\xi_n) \frac{\psi'(\xi_n,\alpha)}{\alpha + \xi_n} + \left\{ \sum_{j=1}^2 F(u) \Psi(\alpha, u)du \right\} + \frac{1}{2} \exp(i\alpha d) \left\{ \sum_k F^*(-p_k) \Psi(\alpha, p_k) \exp(ip_k d) + \sum_n F(-z_n) \frac{\psi'(-z_n) \exp(i\alpha d)}{\alpha + z_n} \right. \]

\[
+ \sum_n F(-\xi_n) \frac{\psi'(\xi_n,\alpha) \exp(i\alpha d)}{\alpha + \xi_n} + \left\{ \sum_{j=1}^2 F(u) \Psi(\alpha, u) \exp(i\alpha d) \right\},
\]

where

\[
F(\alpha) = R(\alpha)Q_0(\alpha), \quad \psi(\alpha, \beta) = \psi_1(\beta)\psi_2(\alpha), \quad \psi_1(\alpha) = \left( P^*(\alpha) \right)^{-1},
\]

\[
\psi_2(\alpha) = \frac{1}{\sqrt{2}} \left[ I'(\alpha) P^*(\alpha) \exp(-i\alpha d), \quad \Psi(\alpha, \beta) = \frac{\psi(\alpha, \beta) - \psi(\beta, \alpha)}{\alpha - \beta} + \frac{\psi(\alpha, -\beta) - \psi(-\beta, \alpha)}{\alpha + \beta}, \right.
\]

\[
I'(\alpha) = \frac{\exp(i\alpha d) }{2\pi(\alpha+i)^2} \int_{-\infty}^\infty \left\{ \frac{(\xi + i)^0}{P(\xi)} \right\} d\xi, \quad e > 0,
\]

\[
P(\alpha) = P^*(\alpha) \left[ |\alpha| + O(|\alpha|^{-1}) \right], \quad P^*(\alpha) = \rho_0^2 R(\alpha) (K(\alpha) - \rho_0^2 R(\alpha))^{-1}.
\]

In the problem with an elastic foundation in the form of a half-plane, there are the terms in curly brackets that describe the integrals along the sides of the branch cut drawn from \( \kappa_j \) (the branch points), \( j=1,2 \). The symbol «*» corresponds to a residue \( R^*(p_k) = \text{res} R(\alpha), \quad z_k \) and \( \xi_k \) are the real zeroes.
and the poles of \( K(\alpha) \), respectively, \( p_k \) are the real poles of \( R(\alpha) \). The form of the function \( P^r(\alpha) \) for a half-plane is
\[
P^r(\alpha) = c^2 \frac{(\alpha + ib)^{N-N_0}}{\sqrt{\alpha + \xi_k}^{N_0}} \prod_{k=1}^{N_0} (\alpha + z_k) \prod_{k=1}^{N} (\alpha + \xi_k)^{-1},
\]
where \( c^2 = \frac{\rho \omega^2 k_s^2}{4 \mu (k_s^2 - \xi_k^2)} \), \( N \) is the number of positive poles \( \xi_k \), and \( N_0 \) is the number of positive zeros \( z_k \) of the function \( P^r(\alpha) \) \[9\]. The value of the parameter \( b \) determines the order of the terms to be discarded when using the factorization method \[8\].

From ratio (11) we can represent
\[
Q(\alpha) = Q^l(\alpha) + \exp(iad)Q^h(\alpha),
\]
where the functions \( Q^l(\alpha) \) and \( Q^h(\alpha) \) are regular above the contour \( \Gamma \) and have asymptotic \( Q(\alpha) \sim |\alpha|^{1+\varepsilon} \), \( C = \text{const, } \varepsilon > 0 \).

To find the original \( q(x) \) in the plane case, one can find the value of integrals (11) by applying the Jordan lemma, closing the contour in the upper or in the lower half-plane in accordance to a decrease in the integrand. As a result of calculating the integrals of the inverse Fourier transform using the theory of residues, we obtain the formulas for determining the contact stresses at the interface between the media
\[
q(x) = \frac{i}{2} \sum_{n} \psi^*_n(z_n) \left[ \sum_{k} F^r(-p_k) \left( \frac{\psi^*_2(-p_k)}{p_k - z_n} - \frac{\psi^*_2(p_k)}{p_k + z_n} \right) - \frac{F(-z_n)\psi_2(-z_n)}{p_k + z_n} \right] + \sum_{m} F(-\xi_m) \frac{\psi^*_m(\xi_m)}{\xi_m - z_n} + \frac{1}{2\pi i} \sum_{j=1}^{2} \int F(u) \left( \frac{\psi^*_2(u)}{u + z_m} - \frac{\psi^*_2(-u)}{u - z_m} \right) du \right] \exp(iz_nx)
\]
\[
+ \frac{i}{2} \sum_{n} \psi^*_n(z_n) \left[ \sum_{k} F^r(-p_k) \left( \frac{\psi^*_1(p_k)}{p_k + \xi_m} - \frac{\psi^*_1(-p_k)}{p_k - \xi_m} \right) - \frac{F(-\xi_m)\psi_1(\xi_m)}{p_k + \xi_m} \right] + \sum_{m} F(-\xi_m) \frac{\psi^*_m(\xi_m)}{\xi_m - z_n} + \frac{1}{2\pi i} \sum_{j=1}^{2} \int F(u) \left( \frac{\psi^*_1(u)}{u + \xi_m} - \frac{\psi^*_1(-u)}{u - \xi_m} \right) du \right] \exp(i\xi_mx).
\]

Then, applying the inverse Fourier transform to (8), (9), we obtain representations for the displacements of the elastic foundation \( u(x,z) = \{u(x,z), w(x,z)\} \) in the following form:
\[
u(x,z) = \frac{1}{2\pi i} \hat{\mathbf{f}}(a,\alpha) Q^l(\alpha) \exp(-iax) d\alpha + \frac{1}{2\pi i} \hat{\mathbf{f}}(a,\alpha) Q^h(\alpha) \exp(i\alpha(d-x)) d\alpha
\]
\[
+ \frac{1}{2\pi i} \hat{\mathbf{f}}(a,\alpha) \bar{Q}(\alpha) \exp(-i\alpha(x+s)) d\alpha,
\]
\[
\bar{Q}(\alpha) = \left\{ q_0(x-s) \exp(i\alpha x) d\alpha \right\}, \bar{\mathbf{f}}(a,\alpha) = \{ R(x,\alpha), R_1(x,\alpha) \}.
\]

Using relation (7) for the velocity potential in a fluid, we obtain the representation \[10\]
\[
\varphi(x,z) = \frac{d}{\rho_s \omega} \overline{Q}(0) \frac{\sin((h-z)\kappa)}{\sin \kappa} + \frac{2d}{\rho_s \omega} \sum_{n=0}^{\infty} \overline{Q}(n) (h-z) \sigma_{sn} \cos(nz \kappa_3) \right\} \right\} \frac{\sin \kappa_3}{\sin \kappa_3} + \varphi_0 = \sqrt{\frac{\pi n^2 \omega^2}{d^2 - \kappa_3^2}} \sigma_{sn}.
\]
4. Results of the numerical experiments

As illustrations of the application of the obtained calculation formulas, we present the graphs of the distribution for the amplitude of the hydrodynamic pressure with the following dimensionless parameters given: $c_0 = 1.5$; $\rho_0 = 1$; $d = 1$.

Figure 2 shows the graphs of the dependence for the amplitude of the hydrodynamic pressure at the lower corner point of the fluid volume ($x = 0$, $z = 0$) on the dimensionless frequency parameter $\omega = \frac{2\pi v^* \nu}{h}$, where $\nu$ is the frequency in hertz, $h^* =$1000 m, $v^* =$1000 m/s are the characteristic values of the linear size and velocity. The calculated parameters are referred to as characteristic values for the density $\rho^* =$1000 kg/m$^3$.

The distribution of the amplitude for the contact stresses along the interface between the media is shown in figure 3, where the value $|q|$ is given in fractions of 100$q_0$ for the height $h = 0.02; 0.05; 0.1$ of the liquid layer.

![Figure 2](image2.png)  
**Figure 2.** The dependence of the amplitude of the hydrodynamic pressure on the frequency.

![Figure 3](image3.png)  
**Figure 3.** The distribution of the amplitude of contact stresses along the interface.
In the problem for an elastic half-plane as a foundation, an increase in the height of the liquid layer leads to a decrease in the frequency values at which a jump in the amplitude of the contact stresses is observed, while the magnitude of the jump decreases. The maximum amplitude of the contact stresses at the interface between the media for rigid foundations is less than for the soft ones at the same values of geometric and frequency parameters. An increase in the distance between the oscillation source and the reservoir with the liquid entails a decrease in the magnitude of the jump in the amplitude of the contact stresses.

5. Conclusion
In this paper, we have investigated the process of vibration excitation for the elastic waves in a geological environment with an artificial reservoir on the surface.

We have solved the problem of vibration for a reservoir with a liquid of finite dimensions on a deformable semi-bounded foundation in the plane formulation. We have developed an analytical method for solving dynamic contact problems of the joint vibrations between liquid and elastic media, associated with the construction of a solution to an integral equation of the first kind with a kernel of a special form that depends on the sum and difference of arguments. As a result of solving the integral equation, the hydrodynamic pressure at the interface between the elastic and liquid media is presented in the form of superposition of the stress waves, the number of which, as well as the frequency and amplitude, depend on the geometric, physical and mechanical characteristics of the elastic foundation and the volume of the liquid, as well as on the vibration frequency of the applied surface load. We have studied the dependence of the hydrodynamic pressure amplitude at the interface between the media on the coordinate and the exciting vibration frequency, which made it possible to identify the spectrum of «dangerous» frequencies at different thickness values of the liquid layer $h$.

The approach described will make it possible to increase the accuracy of the description for real processes and the reliability of assessment for the possible consequences of a vibration exposure, which is especially important for the operation of technical reservoirs and hydraulic structures for critical industrial purposes near vibration sources. In addition, the study of steady-state oscillations can be considered to be a stage in the construction of a solution for a non-stationary problem.

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