Research Article

A Numerical Method for Compressible Model of Contamination from Nuclear Waste in Porous Media

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This paper studies and analyzes a model describing the flow of contaminated brines through the porous media under severe thermal conditions caused by the radioactive contaminants. The problem is approximated based on combining the mixed finite element method with the modified method of characteristics. In order to solve the resulting algebraic nonlinear equations efficiently, a two-grid method is presented and discussed in this paper. This approach includes a small nonlinear system on a coarse grid with size $H$ and a linear system on a fine grid with size $h$. It follows from error estimates that asymptotically optimal accuracy can be obtained as long as the mesh sizes satisfy $H = O(h^{1/3})$.

1. Introduction

A compressible nuclear waste disposal contamination problem in porous media is presented by the following coupled systems of partial differential equations. The physical processes can be concreted to be a high-level waste disposal buried in a salt dome, and next the salt dissolves to generate a brine, radioactive elements decay to generate heat, and finally the radionuclides are transported by the flow.

Fluid:

\[
\phi_1 \frac{\partial p}{\partial t} + \nabla \cdot u = -q + R'_i(\bar{c}),
\]

\[
u = \frac{k}{\mu(\bar{c})} \nabla p = -a(\bar{c})^{-1}\nabla p,
\]

where $p$ and $u$ are the fluid pressure and Darcy velocity, respectively, $\phi_1 = \phi c_w$ and $\phi$ is the porosity. $q = q(x,t)$ is a production term, $R'_i(\bar{c}) = [c_i \phi K_i f_i/(1 + c_i)](1 - \bar{c})$ is a salt dissolution term, $k(x)$ is the permeability of the rock, and $\mu(\bar{c})$, the viscosity of the fluid, is dependent upon $\bar{c}$, the concentration of the brine in the fluid.

Brine:

\[
\frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla \cdot (E_c \nabla c) = g(\bar{c}),
\]

where $E_c$ is the diffusion tensor including the molecular diffusion and mechanical diffusion and $E_c = D + D_m I$, $D = (D)_{ij} = (((a_2-u)\delta_{ij} + (a_1 - a_2)u_iu_j)) / |u|$, and $g(\bar{c}) = -\bar{c} [c_i \phi K_i f_i/(1 + c_i)](1 - \bar{c}) - q_c + R_c$. Here, $D_m$ is molecular diffusion. $u_i$ and $u_j$ are two direction cosines of Darcy velocity. $I$ is an identity matrix.

Heat:

\[
d_1(p) \frac{\partial q}{\partial t} + d_2 \frac{\partial T}{\partial t} + c_p u \cdot \nabla T - \nabla \cdot (E_H \nabla T) = Q(u, T, \bar{c}, p),
\]

where $T$ is the temperature of the fluid, $d_1(p) = -\phi c_w [U_0 + (p/p)]$, $d_2 = \phi c_p + (1 - \phi)p_b c_p$, $E_H = Dc_{uw} + K_{uw}$, $K_m = k_m / \rho_0$, and $Q(u, T, \bar{c}, p) = [\nabla U_0 - c_p \nabla T_0] \cdot u + [U_0 + c_p (T - T_0) + (p/p)] [-q - R'_i] - q_L - q_H - q_H$.

Radionuclide (component $i$):

\[
\phi_k \frac{\partial c_i}{\partial t} + u \cdot \nabla c_i - \nabla \cdot (E_c \nabla c_i) + d_3(c_i) \frac{\partial p}{\partial t} = f_i(\bar{c}, c_1, \ldots, c_N), \quad i = 1, \ldots, N,
\]
where \( c_i \) is the trace concentration of the \( i \) th radionuclide, 
\[
d_j(c_i) = \phi c_i, c_i(K_i - 1),
\]
and
\[
f_i(\bar{c}, c_1, \ldots, c_N) = c_i \left( q - \frac{c_i \phi K_i f_i}{1 + c_i} \right) (1 - \bar{c}) - q c_i - q_{ci} \]
\[
+ q_{li} + \sum_{j=1}^{N} k_{ij} \phi c_j - \lambda_i K_i \phi c_i.
\]

(6)

We assume the following:
(1) Zero Neumann boundary conditions for the equations
(2) The initial conditions are assumed given
(3) The medium \( \Omega \) is vertically homogeneous and take \( \Omega \in \mathbb{R}^2 \)
(4) The solutions are smooth and \( \Omega \) periodic
(5) \( k, \phi, \phi_1, d_z, \) and \( K_i \) are bounded below by positive constants, and \( K_i(\bar{c}), \mu(\bar{c}), g(\bar{c}), d_z(c_i), f_i(\bar{c}), \) and \( Q(T) \) are twice continuously differentiable with bounded partial derivatives about the variables in parentheses
(6) \( D = 0 \)

Chou and Li [1], Ewing et al. [2], and Li et al. [3] have presented and studied several numerical methods for system (1)–(5) and its incompressible case. In this paper, we use the mixed finite element method to approximate the fluid problem and treat the brine, heat, and radionuclides by a modified method of characteristic finite element. It is well known that the full discrete approximation scheme is coupled and nonlinear. If simply lagging, the evaluation of the nonlinear items is used to obtain a linear system; it would be inevitable to introduce the constraint conditions about the mesh grid due to the stability requirement. Moreover, it would take an expensive cost to choose the implicit scheme to nonlinear solutions. An efficient method motivated by Xu [4] is considered in this paper. The method is used by Bi et al. [5], Chen et al. [6–10], and Liu et al. [11, 12] for solving some nonlinear problem. We shall relegate all of nonlinear iterations on a coarse grid of size \( H \) much coarser than the original fine grid of size \( h \). According to the error estimates in the context, it obtains the asymptotically optimal accuracy to take \( H = O(h^{1/3}) \).

The remainder of the paper is organized as follows. Notations and approximation assumptions are given in Section 2. A two-grid method is defined and the convergence error estimates are derived in Section 3. In Section 4, we give some conclusions and extensions.

2. Notations and Approximation Results

To analyze the temporal discretization on a time interval \((0, T)\), let \( M \) be a positive integer number, \( \Delta t = T/M \), \( t^n = n \Delta t \) \((0 \leq n \leq M)\), and \( \varphi^n \varphi(\cdot, t^n) \). Let \( L^p(J; W^{1,q}(\Omega)) \) denote the usual set of functions with the norm
\[
\| \psi \|_{L^p(J; W^{1,q}(\Omega))} = \left( \int_J \| \psi(\cdot, t) \|_{W^{1,q}(\Omega)}^p \, dt \right)^{1/p}.
\]
where if \( p = \infty \), the integral is replaced by the essential supremum. Let \( L^p(J; W^{1,q}(\Omega)) \) denote the time discrete analogue to the space \( L^p(J; W^{1,q}(\Omega)) \) with the norm
\[
\| \psi \|_{L^p(\Omega)} = \left( \sum_{n=1}^N \| \psi \|_{W^{1,q}(\Omega)}^p \right)^{1/p}.
\]

Let \( W = L^2(\Omega) \) and \( V = \{ v \in H(\text{div}; \Omega); v \cdot \gamma = 0 \} \). The weak form is presented as follows:

\[
\left( \phi_i, \frac{\partial p}{\partial t}, \omega \right) + (\nabla \cdot u, \omega) = (-q + R_i(\bar{c}), \omega), \quad \forall \omega \in W,
\]
(9)

\[
(a(\bar{c})u, v) - (\nabla \cdot v, p) = 0, \quad \forall v \in V,
\]
(10)

\[
\left( \phi \frac{\partial c_i}{\partial t}, z \right) + (u \cdot \nabla c_i, z) + (g, c_i, z) = (g(\bar{c}), z),
\]
(11)

\[
\left( d_z \frac{\partial T}{\partial t}, z \right) + c_p (u \cdot \nabla T, z) + (E_z \nabla c_i, \nabla z) = (g, c_i, z) + (f_i(\bar{c}, c_1, \ldots, c_N), z),
\]
(12)

\[
\left( \phi K_i \frac{\partial c_i}{\partial t}, z \right) + (u \cdot \nabla c_i, z) + (E_z \nabla c_i, \nabla z) + (f_i(c_i), \frac{\partial p}{\partial t}, z) = (f_i(\bar{c}, c_1, \ldots, c_N), z),
\]
(13)

for \( z \in H^1(\Omega) \) and \( i = 1, \ldots, N \).

Assume that \( V_h \times W_h \) is the Raviart–Thomas space of index at least \( k \) associated with a quasi-triangularization of \( \Omega \) such that the elements have diameters bounded by \( h_p \). Let \( M_h = M_h \) and \( R_h = R_h \) be finite-dimensional subspaces of \( W^{1, \infty}(\Omega) \) for the approximation of concentrations and temperature, respectively, and we take \( M_h \) and \( R_h \) as the piecewise-polynomial space of degree at least \( l \) and \( r \), respectively.
respectively. As in [2, 11, 13], the approximation properties for \( V_h \times W_h \) and \( M_h, R_h \) are given by

\[
\inf_{v_h \in V_h} \| v - v_h \| \leq C \| v \|_{k+1, h},
\]

\[
\inf_{v_h \in V_h} \| \nabla \cdot (v - v_h) \| \leq C (\| v \|_{k+1} + \| \nabla \cdot v \|_{k+1, h}),
\]

for \( v \in V \cap H^{k+1}(\Omega)^2 \) and \( \nabla \cdot v \in H^{k+1}(\Omega) \), and

\[
\inf_{w_h \in W_h} \| w - w_h \| \leq C \| w \|_{k+1, h}, \quad w \in H^{k+1}(\Omega),
\]

\[
\inf_{z_h \in M_h} \| z - z_h \|_{1,q} \leq C \| z \|_{1,q,h}, \quad z \in W^{1,q}(\Omega), 1 \leq q \leq \infty,
\]

\[
\inf_{z_h \in R_h} \| z - z_h \|_{1,q} \leq C \| z \|_{1,q,h}, \quad z \in W^{1,q}(\Omega), 1 \leq q \leq \infty.
\]

If the initial solutions \( \{ p_h^0, u_h^0, c_h^0, T_h^0, \ilde{c}_h^0 \} \in V_h \times W_h \times M_h \times R_h \times M_h^N \), the characteristics-Galerkin and mixed finite element approximation schemes are to find \( \{ p_h^n, u_h^n, c_h^n, T_h^n, \ilde{c}_h^n \} \in V_h \times W_h \times M_h \times R_h \times M_h^N \) satisfying

\[
(\phi^n, \phi^n_p - \phi^n_{h^-1} \Delta t, w) + (\nabla \cdot u^n_h, w) = (\phi^n, \phi^n_p - \phi^n_{h^-1} \Delta t, w), \quad \forall w \in W_h,
\]

\[
(a(\phi^n_h)u^n_h, v) - (\nabla \cdot v, p^n_h) = 0, \quad \forall v \in V_h,
\]

\[
\left( \phi^n_h \frac{\tilde{\phi}_h^n - \phi^n_{h^-1}}{\Delta t}, z \right) + (E_h \nabla \phi^n_h, \nabla z) = (g(\phi^n_h), z), \quad \forall z \in M_h,
\]

\[
\left( d^n_2 \frac{\tilde{T}_h^n - T_{h^-1}}{\Delta t}, z \right) + (E_h \nabla \phi^n_h, \nabla z) + \left( d^n_1 \left( p^n_h \right) \frac{P^n - p^n_{h^-1}}{\Delta t}, z \right) = (Q(u^n_h, T^n_h, \ilde{c}_h^n, P^n_h), z), \quad \forall z \in R_h,
\]

\[
\left( \phi^n_k, \frac{\tilde{c}_h^n - c^n_{h^-1}}{\Delta t}, z \right) + (E_h \nabla \phi^n_h, \nabla z) \left( d^n_3 \left( c^n_{h^-1} \right) \frac{P^n - p^n_{h^-1}}{\Delta t}, z \right) = (f_1(\phi^n_h, c^n_{h^-1}, \ldots, c^n_{Nh}, z), \quad \forall z \in M_h,
\]

where \( i = 1, \ldots, N \) and

\[
\tilde{x}^{n-1} = \tilde{x}^{n-1}(\tilde{x}^{n-1}),
\]

\[
\tilde{x}^{n-1} = x - \tilde{u}_h^n \Delta t \phi^n,
\]

\[
\tilde{c}_h^n = c^n_{h^-1}(\tilde{x}^{n-1}),
\]

\[
\tilde{x}^{n-1} = x - \tilde{u}_h^n \Delta t \phi^n,
\]

\[
\tilde{T}_h^n = T_{h^-1}(\tilde{x}^{n-1}),
\]

\[
\tilde{x}^{n-1} = x - \tilde{c}_h^n \Delta t \phi^n,
\]

\[
\tilde{T}_h^n = T_{h^-1}(\tilde{x}^{n-1}),
\]

Remark. If \( \tilde{x}^{n-1} \) is located outside \( \Omega \), we can join \( \tilde{x}^{n-1} \) with \( Y \in \partial \Omega \) so that \( (\tilde{x}^{n-1} - Y)/\| \tilde{x}^{n-1} - Y \| \) is the outer-normal direction to the boundary \( \partial \Omega \) at \( Y \). Take \( x^* \in \Omega \) so that \( Y - x^* = \tilde{x}^{n-1} - Y \), then we define \( x^{n-1} = x^* + \tilde{x}^{n-1} \).

In order to deduce the error estimates, we employ the elliptic projections by labeling them with tildes:

\[
(\nabla \cdot (u - \tilde{u}), w) + (p - \tilde{P}, w) = 0, \quad w \in W_h,
\]

\[
(a(\tilde{u}) (u - \tilde{u}), v) + (\nabla \cdot v, p - \tilde{P}) = 0, \quad v \in V_h,
\]

\[
(E_h \nabla (\tilde{c} - \tilde{C}), \nabla z) + \tilde{c} (\tilde{c} - \tilde{C}, z) = 0, \quad z \in M_h,
\]

\[
(E_h \nabla (T - \tilde{T}), \nabla z) + \tilde{c} (\tilde{c} - \tilde{C}, z) = 0, \quad z \in R_h,
\]

\[
(E_h \nabla (c_i - \tilde{C}), \nabla z) + \tilde{c} (\tilde{c} - \tilde{C}, z) = 0, \quad z \in M_h,
\]

\[
\eta = p - \tilde{P}, \quad \alpha = p - p_h,
\]

\[
\varphi = u - \tilde{u}, \quad \beta = \tilde{U} - u_h,
\]

\[
\tilde{c}_h = \tilde{c} - \tilde{c}_h,
\]

\[
\tilde{c}_h = \tilde{c} - \tilde{c}_h,
\]

\[
\chi_i = \tilde{c}_i - \tilde{c}_i,
\]

\[
\theta = T - \tilde{T}, \quad \psi = \tilde{T} - T_h.
\]

Subtracting (19) from (9) and taking \( w = d_i x^{n-1} \), we get the error equation about the pressure function as follows:
\[
\begin{align*}
(\phi_1 d_1 a^{n-1}, d_1 a^{n-1}) & + (\nabla \cdot \beta^n, d_1 a^{n-1}) \\
= -\left( \phi_1 \frac{\partial \eta^n}{\partial t}, d_1 a^{n-1} \right) & + \left( \phi_1 \left( \frac{d_1 \bar{p}^{n-1}}{\partial t} - \frac{\partial \bar{p}}{\partial t} \right), d_1 a^{n-1} \right) \\
& + \left( \frac{\partial R'}{\partial c} (\delta_1^n) (\bar{c} - \bar{c}_h^n), d_1 a^{n-1} \right),
\end{align*}
\]

where \(d_1 a^{n-1} = (a^n - a^{n-1})/\Delta t\) and \(\delta_1^n\) is between \(\bar{c}^n\) and \(\bar{c}_h^n\).

Next, combining (20) from (10) at \(t = t^n\) with the test function \(\beta^n\),

\[
(a(\bar{c}_h^n)\beta^n, \beta^n) - (\nabla \cdot \beta^n, a^n) = \left( \frac{\partial a}{\partial c} (\delta_2^n) \bar{U}^{n-1}(c^n - \bar{c}_h^n), \beta^n \right).
\]

Combining (32) with (33), we get

\[
\frac{1}{2} d_1 \left[ (a(\bar{c}_h^n)\beta^{n-1}, \beta^{n-1}) - (\nabla \cdot \beta^n, d_1 a^{n-1}) \right]
= -\frac{1}{2\Delta t} (a(\bar{c}_h^n)\beta^n, \beta^n) + \frac{1}{2\Delta t} (a(\bar{c}_h^n)\beta^n, \beta^n) - \frac{1}{2\Delta t} (a(\bar{c}_h^n)\beta^n, \beta^n)
\]

\[
+ \frac{1}{\Delta t} (a(\bar{c}_h^n)\beta^{n-1}, \beta^n) - \frac{1}{2\Delta t} (a(\bar{c}_h^n)\beta^{n-1}, \beta^n)
- \left( d_1 \left[ \frac{\partial a}{\partial c} (\delta_2^n) \bar{U}^{n-1}(\bar{c}^{n-1} - \bar{c}_h^n), \beta^n \right) \right).
\]

By (31) and (34),

\[
\begin{align*}
&\left( \phi_1 d_1 a^{n-1}, d_1 a^{n-1} \right) + \frac{1}{2} d_1 \left[ (a(\bar{c}_h^n)\beta^{n-1}, \beta^{n-1}) \right] \\
&= \left( \phi_1 \frac{\partial \eta^n}{\partial t}, d_1 a^{n-1} \right) + \left( \phi_1 \left( \frac{d_1 \bar{p}^{n-1}}{\partial t} - \frac{\partial \bar{p}}{\partial t} \right), d_1 a^{n-1} \right) + \left( \frac{\partial R'}{\partial c} (\delta_1^n) (\bar{c} - \bar{c}_h^n), d_1 a^{n-1} \right) \\
&- \frac{1}{2\Delta t} \left[ \left( a(\bar{c}_h^n) - a(\bar{c}_h^{n-1}) \right) \beta^n, \beta^n \right] - \frac{1}{2\Delta t} (a(\bar{c}_h^n)\beta^{n-1}, (\beta^n - \beta^{n-1})) \\
&- \left( d_1 \left[ \frac{\partial a}{\partial c} (\delta_2^n) \bar{U}^{n-1}(\bar{c}^{n-1} - \bar{c}_h^n), \beta^n \right) \right),
\end{align*}
\]

when \(t = t^{n-1}\), we apply the Taylor expansion and obtain that

\[
\begin{align*}
&\left( a(\bar{c}_h^n)\beta^{n-1}, \beta^{n-1} \right) - (\nabla \cdot \beta^n, a^{n-1}) \\
&= \left( a(\bar{c}_h^n)\beta^{n-1}, \beta^n \right) - \left( \frac{\partial a}{\partial c} (\delta_2^n) \bar{U}^{n-1}(\bar{c}^{n-1} - \bar{c}_h^n), \beta^n \right).
\end{align*}
\]

where \(\delta_2\) is between \(\bar{c}\) and \(\bar{c}_h^n\).

Using the deduction as [1, 2], we have
\begin{equation}
\left\| d_{i} \alpha^{n-1} \right\|^2 + \frac{1}{2} d_{i} \left\{ \left( a \left( \tau^{n-1}_{i}, \beta^{n-1}_{i} \right) \right) \right\} \leq \left( \left\| \beta^{n}_{i} \right\|_{\infty} + \varepsilon \right) \left\| d_{i} \hat{x}^{n-1} \right\|^2 + C \left( \left\| \hat{x}^{n-1} \right\|^2 + \left\| \beta^{n} \right\|^2 + h_{p}^{2k+2} + h_{c}^{2l+2} + \Delta t^2 \right),
\end{equation}

After making the induction hypothesis that \( \sup_{1 \leq n \leq M} \left\| \beta^{n} \right\|_{\infty} \rightarrow 0 \), we multiply (36) by \( \Delta t \) and sum over \( 1 \leq n \leq M \) to get

\begin{equation}
\sum_{n=1}^{M} \left\| d_{i} \alpha^{n-1} \right\|^2 \Delta t + (a(\tau^{n-1}_{h}, \beta^{n-1}_{h})) \leq \left( \left\| \beta^{n}_{i} \right\|_{\infty} + \varepsilon \right) \sum_{n=1}^{M} \left\| d_{i} \hat{x}^{n-1} \right\|^2 \Delta t + C \left( \sum_{n=1}^{M} \left\| \hat{x}^{n-1} \right\|^2 + \left\| \beta^{n} \right\|^2 \right) \Delta t + h_{p}^{2k+2} + h_{c}^{2l+2} + \Delta t^2.
\end{equation}

Combining (11) and (21), we get the following equality, in which we choose the test function \( \varepsilon = \hat{x}^{n} - \hat{x}^{n-1} = d_{i} \hat{x}^{n-1} \Delta t \) and sum over \( 1 \leq n \leq M \):

\begin{equation}
\sum_{n=1}^{M} \left( \phi d_{i} \hat{x}^{n-1}, d_{i} \hat{x}^{n-1} \right) \Delta t + \frac{1}{2} \left( a(\tau^{n-1}_{h}, \beta^{n-1}_{h}) \right) - \frac{1}{2} \left( a(\tau^{0}_{h}, \beta^{0}_{h}) \right)
\end{equation}

\begin{equation}
\leq \sum_{n=1}^{M} \left( \phi \frac{\partial \tau^{n}}{\partial t} + u^{n}_{h} \cdot \nabla \tilde{c}^{n} - \phi \frac{\tau^{n} - \tilde{c}^{n}}{\Delta t}, d_{i} \hat{x}^{n-1} \right) \Delta t + \sum_{n=1}^{M} \left( \phi \frac{\tau^{n-1} - \tilde{c}^{n-1}}{\Delta t}, d_{i} \hat{x}^{n-1} \right) \Delta t + \sum_{n=1}^{M} \lambda \left( \tilde{c}^{n}, d_{i} \hat{x}^{n-1} \right) \Delta t
\end{equation}

\begin{equation}
\sum_{n=1}^{M} \left( u^{n} - u^{n}_{h} \right) \cdot \nabla \tilde{c}^{n}, d_{i} \hat{x}^{n-1} \Delta t + \sum_{n=1}^{M} \left( \frac{\partial \tilde{c}^{n}}{\partial t}, \tilde{c}^{n}, d_{i} \hat{x}^{n-1} \right) \Delta t,
\end{equation}

where \( \tau^{n-1} = \hat{x}^{n-1}(\hat{x}^{n-1}), \tilde{c}^{n-1} = \tilde{c}(\tilde{c}^{n-1}), \) and Note that

\begin{equation}
\left| \sum_{n=1}^{M} \left( \phi \frac{\tau^{n-1} - \tilde{c}^{n-1}}{\Delta t}, d_{i} \hat{x}^{n-1} \right) \Delta t \right|
\end{equation}

\begin{equation}
= \left| \left( \phi \left( \hat{x}^{n-1} - \tilde{c}^{n-1} \right), \hat{x}^{n} \right) - \left( \phi \left( \hat{x}^{0} - \tilde{c}^{0} \right), \hat{x}^{0} \right) \right| - \sum_{n=1}^{M-1} \left( \phi d_{i} \left( \tau^{n} - \tilde{c}^{n} \right), \tilde{c}^{n} \right)
\end{equation}

\begin{equation}
= C \Delta t \left( \left\| \tilde{z}^{n-1} \right\|^2 + \left\| \tilde{c}^{n} \right\|^2 + \sum_{n=1}^{M-1} \left( \left\| \tilde{z}^{n-1} \right\|^2 + \left\| d_{i} \tilde{c}^{n} \right\|^2 \right) + \varepsilon \Delta t \left\| \tilde{c}^{0} \right\|^2 \right).
\end{equation}

The reminder of the right side items in (38) is just as [2], that is,
follows:

\[
\sum_{n=1}^{M} \left\| d_{x}^{n-1} \right\|_{\infty}^{2} \Delta t + \left( E_{c} \nabla \tilde{x}_{M}, \nabla \tilde{x}^{M} \right) \leq C \left( \sum_{n=1}^{M} \left[ \left\| \nabla \tilde{x}_{n-1} \right\|_{\infty}^{2} + \left\| \beta_{n} \right\|^{2} \right] \Delta t + h_{p}^{2k+2} + h_{c}^{2k+2} + \Delta t^{2} \right).
\]

It follows from the assumption \( \sup_{M} \left\| \nabla \tilde{x}_{n} \right\|_{\infty} \leq C \) and Gronwall lemma that

\[
\left\| \tilde{x}_{n} \right\|_{L^{\infty}(H^{1})} + \left\| \tilde{x}_{n} \right\|_{L^{2}} + \left\| \beta_{n} \right\|_{L^{\infty}(L^{2})} + \left\| d_{i} \alpha_{n} \right\|_{L^{2}} \leq C(\Delta t + h_{c}^{k+1} + h_{p}^{k+1}).
\]

(40)

Then, by the inverse estimate and (40), we know that the induction hypotheses hold if

\[
\begin{align*}
h_{p}^{k+1} &= o(h_{p}), \\
h_{c}^{k+1} &= o(h_{c}), \\
\Delta t &= o(h_{T}).
\end{align*}
\]

(42)

Finally, from the approximation properties,

\[
\begin{align*}
h_{p}^{k+1} &= o(h_{p}), \\
h_{c}^{k+1} &= o(h_{c}), \\
\Delta t &= o(h_{T}),
\end{align*}
\]

(45)

then there exists a positive constant \( C \) independent of \( h \) and \( \Delta t \), such that

\[
\begin{align*}
\sum_{i=1}^{N} \left\| k_{i} \right\|_{L^{\infty}(H^{1})} + \sum_{i=1}^{N} \left\| d_{x} k_{i} \right\|_{L^{2}} \leq C(\Delta t + h_{c}^{k+1} + h_{p}^{k+1}), \\
\left\| \psi \right\|_{L^{\infty}(H^{1})} + \left\| d_{i} \psi \right\|_{L^{2}} \leq C(\Delta t + h_{c}^{k+1} + h_{p}^{k+1} + h_{T}^{k+1}),
\end{align*}
\]

(44)

where \( h_{p}^{k+1} = o(h_{T}), h_{c}^{k+1} = o(h_{T}), \) and \( \Delta t = o(h_{T}) \) are satisfied.

Note that the time step is limited to be \( o(h) \) due to the theoretical proof.

**Theorem 1.** Define \( \{ p_{n}, u_{n}, \tilde{c}_{n}, \tilde{h}_{n}, \tilde{t}_{n}, \tilde{c}_{th} \} \in V_{h} \times W_{h} \times M_{h} \times R_{h} \times M_{h}^{N} \) for \( n \geq 1 \) by system (19)–(23) and assume that the approximation properties (25)–(29) hold. If

\[
\begin{align*}
\left\| \tilde{c} - \tilde{c}_{h} \right\|_{L^{\infty}(H^{1})} + \left\| d_{i} (\tilde{c} - \tilde{c}_{h}) \right\|_{L^{2}} + \left\| u - u_{h} \right\|_{L^{\infty}(L^{2})} + \left\| d_{i} (p - p_{h}) \right\|_{L^{2}}
\end{align*}
\]

(43)

\[
\begin{align*}
\left\| \tilde{c} - \tilde{c}_{h} \right\|_{L^{\infty}(H^{1})} + \left\| d_{i} (\tilde{c} - \tilde{c}_{h}) \right\|_{L^{2}} + \left\| u - u_{h} \right\|_{L^{\infty}(L^{2})} + \left\| d_{i} (p - p_{h}) \right\|_{L^{2}}
\end{align*}
\]

(46)
Similarly, we can get the error estimates of fine grid scheme in $L^2$ norm.

**Theorem 2.** Define $\{p^n_h, u^n_H, c^n_H, t^n_H, c^n_{ih}\} \in V_h \times W_h \times M_h \times R_h \times M^N_h$ for $n \geq 1$ by system (19)–(23) and assume that the approximation properties (25)–(29) hold. Then, there exists a positive constant $C$ independent of $h, H, \Delta t$, such that

$$
\left\| \tilde{c} - c_h \right\|_{L^\infty(L^2)} + \left\| u - u_h \right\|_{L^2(L^2)} + \left\| p - p_h \right\|_{L^\infty(L^2)} + \sum_{i=1}^N \left\| c_i - c_{ih} \right\|_{L^\infty(L^2)} + \left\| T - T_h \right\|_{L^\infty(L^2)} 
\leq C(\Delta t + h^{l+1}_h + h^{s+1}_T + h^{k+1}_p).
$$

(47)

---

### 3. An Efficient Method

We now use and analyze a two-grid method for iteratively solving the nonlinear problem. The method has two steps.

**Stage 1.** On the coarse grid $\mathcal{T}_H$ with a mesh size $H \in (0, 1)$, solve a small nonlinear system for $\{p^n_H, u^n_H, c^n_H, T^n_H, C^n_{ih}\} \in V_H \times W_H \times M_H \times R_H \times M^N_H$ given by (19)–(23).

**Stage 2.** On the fine grid $\mathcal{T}_h$ with a mesh size $h \in (0, 1) (h \ll H)$, solve the following linear system for $\{p^n_h, u^n_H, c^n_h, T^n_h, C^n_{ih}\} \in V_h \times W_h \times M_h \times R_h \times M^N_h$:

\[
\begin{align*}
\phi_t \left( \frac{p^n_h - p_{n-1}^h}{\Delta t} , w \right) + (\nabla \cdot u^n_h, w) - \left( \frac{\partial R^V}{\partial c} (C^n_{ih}) C^n_{ih}, w \right) \\
= -q + R^T (\tilde{C}^n_H) - \frac{\partial R^T}{\partial c} (\tilde{C}^n_H) C^n_{ih}, w), \quad \forall w \in W_h, \\
\left( a(C^n_{ih}) U^n_h + \frac{\partial a}{\partial c} (C^n_{ih}) U^n_H, C^n_{ih}, v \right) - (\nabla \cdot v, P^n_h) = \left( \frac{\partial a}{\partial c} (C^n_H) U^n_H, C^n_H, v \right), \quad \forall v \in V_h, \\
\left( \phi \frac{\tilde{C}^n_h - \tilde{C}^{n-1}_h}{\Delta t} , z \right) + (U^n_h \cdot \nabla \tilde{C}^n_h, z) + (E_v, \nabla \tilde{C}^n_h, \nabla z) - \left( \frac{\partial q}{\partial c} (C^n_H) C^n_{ih}, z \right) \\
= (U^n_H \cdot \nabla \tilde{C}^n_h, z) + (g(C^n_H) - \frac{\partial q}{\partial c} (C^n_H) C^n_{ih}, z), \quad \forall z \in M_h, \\
\left( d_1 \frac{T^n_h - T^{n-1}_h}{\Delta t} , z \right) + (\tilde{E}_v, \nabla T^n_h, \nabla z) - \left( \frac{\partial Q^H}{\partial T} T^n_h , z \right) \\
= -\left( d_1 \left( \frac{p^n_h - p_{n-1}^h}{\Delta t} \right), z \right) + (Q^H, z) - \left( \frac{\partial Q^H}{\partial T} T^n_h , z \right), \quad \forall z \in R_h, \\
\left( \phi_K \frac{C^n_{ih} - C^{n-1}_{ih}}{\Delta t}, z \right) + (E_v C^n_{ih}, \nabla z) + \left( \frac{\partial d_3}{\partial c_i} (C^n_{ih}) \frac{p^n_h - p_{n-1}^h}{\Delta t} C^n_{ih}, z \right) \\
- \left( \frac{\partial f^H}{\partial c_i} C^n_{ih} + \cdots + \frac{\partial f^H}{\partial c_N} C^n_{ih}, z \right) = -\left( d_3 (C^n_{ih}) \frac{p^n_h - p_{n-1}^h}{\Delta t}, z \right), \\
\left( \frac{\partial d_4}{\partial c_i} (C^n_{ih}) \frac{p^n_h - p_{n-1}^h}{\Delta t} C^n_{ih} + f^I_i - \frac{\partial f^H}{\partial c_i} C^n_{ih} + \cdots - \frac{\partial f^H}{\partial c_N} C^n_{ih}, z \right), \quad \forall z \in M_h,
\end{align*}
\]

where
\[ Q^H = Q(U^n_h, T^n_h, C^n_h, P^n_h), \]
\[ f^H_i = f(C^n_h, C^{nH}, \ldots, C^{nNH}), \]
\[ \bar{C}^{n-1}_h = C^{n-1}_h(\bar{x}^{n-1}), \]
\[ \bar{x}^{n-1} = x - \left( \frac{U^n_h \Delta t}{\phi} \right), \]
\[ \bar{C}^{n-1}_{ih} = C^{n-1}_{ih}(\bar{x}^{n-1}), \]
\[ \bar{x}^{n-1} = x - \left( \frac{U^n_{ih} \Delta t}{\phi} \right), \]
\[ \bar{T}^{n-1}_h = T^{n-1}_h(\bar{x}^{n-1}), \]
\[ \bar{x}^{n-1} = x - \left( \frac{c_2 U^n_{ih} \Delta t}{d_2} \right), \]

and the projection on the fine grid is based on the numerical solutions on coarse grid.

The sequential solution processes are defined as follows. Firstly, we apply the Newton iteration to the coupled system on the coarse grid and obtain \( \{P^n_{ih}, U^n_h, C^n_h, T^n_h, C^{nH}\} \). Next, combining (48)–(50), we get \( P^n_{ih} \) and \( U^n_h \) with \( RT_1 \) and \( C^n_h \) with piecewise linear finite element using a coupled linear system. Finally, from (51) and (52), we can get \( C^n_{ih} \) and \( T^n_h \) by parallel computation.

In order to analyze the linear scheme on the fine grid, we define

\[ \pi = \bar{P} - P_h, \]
\[ \sigma = \bar{U} - U_h, \]
\[ \xi = \bar{C} - C_h, \]
\[ \zeta = C_i - C_{ih}, \]
\[ \omega = \bar{T} - T_h. \]

According to Taylor expansion, there exists a positive \( \delta_3 \) such that

\[ R'(\bar{c}^n) - R'(\bar{C}^n_H) - \frac{\partial R'}{\partial c}(\bar{C}^n_H)(\bar{c}^n - \bar{C}^n_H) \]
\[ = R'(\bar{C}^n_H) + \frac{\partial R'}{\partial c}(\bar{C}^n_H)(\bar{c}^n - \bar{C}^n_H) - R'(\bar{C}^n_{ih}) - \frac{\partial R'}{\partial c}(\bar{C}^n_{ih})(\bar{c}^n - \bar{C}^n_{ih}) \]
\[ + \frac{\partial^2 R'}{\partial c^2}(\delta_3)(\bar{c}^n - \bar{C}^n_H)^2 \]
\[ = \frac{\partial R}{\partial c}(\bar{C}^n_H)(\bar{c}^n - \bar{C}^n_H) + \frac{\partial^2 R'}{\partial c^2}(\delta_3)(\bar{c}^n - \bar{C}^n_H)^2. \]

According to (48), (9), and (25),

\[ \left( \phi_1 d_i, \eta^n_{i-1}, d_i, \eta^{n-1} \right) \]
\[ = \left( \phi_1 \frac{\partial \eta^n}{\partial t}, d_i, \eta^{n-1} \right) + \left( \phi_1 \left( d_i \bar{P}^{n-1} - \frac{\partial \bar{P}^n}{\partial t} \right), d_i, \eta^{n-1} \right) \]
\[ + \left( \frac{\partial R'}{\partial c}(\bar{C}^n_H)(\bar{c}^n - \bar{C}^n_H), d_i, \eta^{n-1} \right) \]
\[ + \left( \frac{\partial R'}{\partial c}(\bar{C}^n_H)(\bar{c}^n - \bar{C}^n_H), d_i, \eta^{n-1} \right) \]

Like the deduction of (34), we see that
\[ \frac{1}{2} d_i \left\{ a \left( \tilde{C}_H^{n-1} \right) \sigma^{n-1} \right\} - (\nabla \cdot \sigma^n, d_i \tilde{\sigma}^{n-1}) \\
= -\frac{1}{2 \Delta t} \left( a \left( \tilde{C}_H^n \right) \sigma^n + \frac{1}{2} d_i \left\{ a \left( \tilde{C}_H^{n-1} \right) \sigma^n \right\} - \frac{1}{2 \Delta t} \left( a \left( \tilde{C}_H^{n-1} \right) \sigma^n \right) \\
+ \frac{1}{\Delta t} \left( a \left( \tilde{C}_H^{n-1} \right) \sigma^{n-1}, \sigma^n \right) - \frac{1}{2 \Delta t} \left( a \left( \tilde{C}_H^{n-1} \right) \sigma^{n-1}, \sigma^n \right) - \left( \frac{\partial a}{\partial c} \left( \tilde{C}_H^{n-1} \right) \tilde{U}^{n-1} \right) \\
+ \frac{\delta a}{\partial c} \left( \tilde{C}_H^n \right) \left( \tilde{U}^{n-1} - U_{H}^{n-1} \right) \left( \tilde{C}_H^{n-1} - \tilde{C}_H^n \right), \sigma^n \right) \\
- \frac{1}{2 \Delta t} \left( \frac{\partial^2 a}{\partial c^2} \left( \delta_i \right) \tilde{U}^{n-1} \left( \tilde{c}^n - \tilde{c}_H^n \right)^2 - \frac{\partial^2 a}{\partial c^2} \left( \delta_i \right) \tilde{U}^{n-1} \left( \tilde{c}^{n-1} - \tilde{c}_H^{n-1} \right)^2, \sigma^n \right) \tag{57} \]

where \( \delta_i \) is between \( \tilde{c} \) and \( \tilde{C}_H \).

Hence, (56) and (57) can give that

\[ \left( \phi \tilde{d}_i, d_i \tilde{\sigma}^{n-1}, \tilde{\sigma}^{n-1} \right) \]

\[
\left( \phi \tilde{d}_i, d_i \tilde{\sigma}^{n-1}, \tilde{\sigma}^{n-1} \right) = \left( \phi \tilde{d}_i, d_i \tilde{\sigma}^{n-1}, \tilde{\sigma}^{n-1} \right) + \left( \phi \tilde{d}_i, d_i \tilde{\sigma}^{n-1}, \tilde{\sigma}^{n-1} \right)
\]

The error equation about \( \tilde{c} \) shows that

\[
\left( \phi \frac{\tilde{c}^n - \tilde{c}_H^{n-1}}{\Delta t}, z \right) + \left( E, \nabla \tilde{c}^n, \nabla z \right)
\]

\[
= \left( \phi \frac{\tilde{c}^n}{\Delta t} + u_H^n \cdot \nabla \tilde{c}^n - \phi \frac{\tilde{c}^n - \tilde{c}_H^{n-1}}{\Delta t}, z \right) + \left( \phi \frac{\tilde{c}^n - \tilde{c}_H^{n-1}}{\Delta t}, z \right)
\]

\[
- \left( \phi \frac{\tilde{c}^n - \tilde{c}_H^{n-1}}{\Delta t}, z \right) + \lambda (\tilde{c}^n, z) - (u^n - U_H^n) \cdot \left( \nabla \tilde{c}^n - \nabla \tilde{C}_H^n \right) + (u^n - U_H^n) \cdot \nabla \tilde{C}_H^n, z
\]

\[
+ \left( \frac{\partial a}{\partial c} \left( \tilde{C}_H^n \right) \left( \tilde{c}_H^n + \tilde{c}_H^n \right), z \right) + \left( \phi \frac{\tilde{c}^n - \tilde{c}_H^{n-1}}{\Delta t}, z \right) \tag{59} \]
where \( \tilde{\xi}^{n-1} = \xi^{n-1}(X^{n-1}), \tilde{\xi}^{n-1} = \xi^{n-1}(X^{n-1}), \) and \( X^{n-1} = x - (U_H^n/\phi(x))\Delta t. \)

Taking the test function \( z = \xi^n - \xi^{n-1} = d_i \xi^{n-1} \Delta t \) and summing over \( 1 \leq n \leq M, \) we have

\[
\sum_{n=1}^{M} \left( \phi d_i \xi^{n-1}, d_i \xi^{n-1} \right) \Delta t + \frac{1}{2} \left( E, V \tilde{\xi}^M, V \tilde{\xi}^M \right) - \frac{1}{2} \left( E, V \tilde{\xi}^0, V \tilde{\xi}^0 \right)
\leq \sum_{n=1}^{M} \left( \phi \frac{\xi^n - \xi^{n-1}}{\Delta t}, d_i \xi^{n-1} \right) \Delta t + \sum_{n=1}^{M} \lambda \left( \xi^n, d_i \xi^{n-1} \right) \Delta t
\]

\[
+ \sum_{n=1}^{M} \left( u^n - U_H^n \right) \cdot \left( \nabla \xi^n - \nabla \tilde{\xi}_H^n \right) + \left( u^n - U_H^n \right) \cdot \nabla \tilde{\xi}_H^n, d_i \xi^{n-1} \right) \Delta t
\]

\[
+ \sum_{n=1}^{M} \left( \frac{\partial g}{\partial c} (\tilde{\xi}_H^n)(\xi^n + \tilde{\xi}_H^n) + \frac{\partial^2 g}{\partial c^2} (\delta)(\xi^n - \tilde{\xi}_H^n, 2, d_i \xi^{n-1}) \right) \Delta t.
\]

Since

\[
\left\| (\tilde{\xi} - \tilde{\xi}_H) \right\| \leq \left\| \tilde{\xi} - \tilde{\xi}_H \right\|_{L^\infty} \leq \left\| \tilde{\xi} - C_H \right\|_{L^\infty} + \left\| C - \tilde{\xi}_H \right\|_{L^\infty}
\leq C \left( \left\| \ln H_c \right\|_{L^\infty} + \left\| H_c (H_{c^1} + H_{c^2}) \right\| \right)
\times \left( \left\| H_{c^1} + H_{c^2} \right\| \right)
\leq C \left( H_{c^2} \left\| \ln H_c \right\| + H_{c^2} \left\| \ln H_c \right\| \right).
\]

(61)

\[
\left\| (u - U_H)(\nabla \tilde{\xi} - \nabla \tilde{\xi}_H) \right\| \leq \left\| u - U_H \right\| \left\| \tilde{\xi} - \tilde{\xi}_H \right\|_{L^\infty}
\leq C \left( H_{c^1} + H_{c^2} \right)
\times \left( \left\| \ln H_c \right\| (H_{c^1} + H_{c^2}) \right)
\leq C \left( H_{c^2} \left\| \ln H_c \right\| + H_{c^2} \left\| \ln H_c \right\| \right).
\]

(62)

At last, we present the error results of the radionuclide equation and heat equation as follows:

\[
\sum_{i=1}^{N} \left\| c_i - c_{ih} \right\|_{L^\infty(H^1)} + \sum_{i=1}^{N} \left\| d_i (c_i - c_{ih}) \right\|_{L^2(\Omega)}
\leq C \left( \Delta t + h_{t^1} + h_{c^1} + H_{c^1} 2 \left\| \ln H_c \right\| + H_{c^2} \left\| \ln H_c \right\| \right),
\]

(65)

\[
\left\| T - T_h \right\|_{L^\infty(H^1)} + \left\| d_i (T - T_h) \right\|_{L^2(\Omega)}
\leq C \left( \Delta t + h_{t^1} + h_{c^1} + H_{c^1} 2 \left\| \ln H_c \right\| + H_{c^2} \left\| \ln H_c \right\| + H_{c^2} \left\| \ln H_c \right\| + H_{c^2} \left\| \ln H_c \right\| \right).
\]
If setting $l = 1, r = 1, k = 1$, $h_{c} = h_{T} = h_{p}$, and $H_{c} = H_{T} = H_{p}$, which accords with the assumption (45), we can get the following theorem of fine grid scheme.

\begin{equation}
\left\|\bar{u} - \bar{C}_{h}\right\|_{L^{\infty}(L^{2}(\tilde{H}))} + \left\|u - U_{h}\right\|_{L^{2}(L^{2})} + \left\|p - P_{h}\right\|_{L^{2}(L^{2})}
+ \sum_{i=1}^{N} \left\|c_{i} - C_{ih}\right\|_{L^{2}(L^{2})} + \left\|T - T_{h}\right\|_{L^{2}(L^{2})}
\leq C(\Delta t + h^{2} + H^{2}) \tag{66}
\end{equation}

**Theorem 4.** Define $\left\{P_{h}^{n}, U_{h}^{n}, \bar{C}_{h}^{n}, T_{h}^{n}, C_{ih}^{n}\right\} \in V_{h} \times W_{h} \times M_{h} \times R_{h} \times M_{h}^{n}$ for $n \geq 1$ by system (48)–(52) and assume that the approximation properties (14)–(18) hold. Then, there exists a positive constant $C$ independent of $h, H, \text{ and } \Delta t$, such that

\begin{equation}
\left\|\bar{u} - \bar{C}_{h}\right\|_{L^{\infty}(L^{2})} + \left\|u - U_{h}\right\|_{L^{2}(L^{2})} + \left\|p - P_{h}\right\|_{L^{2}(L^{2})}
+ \sum_{i=1}^{N} \left\|c_{i} - C_{ih}\right\|_{L^{2}(L^{2})} + \left\|T - T_{h}\right\|_{L^{2}(L^{2})}
\leq C(\Delta t + h^{2} + H^{2}) \tag{67}
\end{equation}

4. Conclusions and Extensions

The two-grid method presented in this paper reduces the complexity of problem. It involves a small nonlinear system on a coarse grid of size $H$ and a linear system on a fine grid of size $h$. It is shown that the coarse space can be extremely coarse and still achieve asymptotically optimal approximation as long as the mesh sizes satisfy $H = O(h^{1/3})$ in $H^{1}$ norm. Compared with the implicit scheme, the two-grid method reduces CPU time. Moreover, the method is suitable to make the large-scale computation and long time duration. The future work is to use the discontinuous finite volume element method [14], block-centered finite difference method [15], higher-order finite volume [16], and SAV method [17, 18] to consider this problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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