ON THE FOURIER ANALYSIS OF MEASURES WITH MEYER SET SUPPORT

NICOLAE STRUNGARU

Abstract. In this paper we show the existence of the generalized Eberlein decomposition for Fourier transformable measures with Meyer set support. We prove that each of the three components is also Fourier transformable and has Meyer set support. We obtain that each of the pure point, absolutely continuous and singular continuous components of the Fourier transform is a strong almost periodic measure, and hence is either trivial or has relatively dense support. We next prove that the Fourier transform of a measure with Meyer set support is norm almost periodic, and hence so is each of the pure point, absolutely continuous and singular continuous components. We complete the paper by discussing some applications to the diffraction of weighted Dirac combs with Meyer set support.

1. Introduction

The discovery of quasicrystals in the 1980’s [39] emphasized the need for a better understanding of the mathematics behind the physical process of diffraction and of aperiodic order in general.

A large class of mathematical models for quasicrystals is represented by Meyer sets and weighted Dirac combs with Meyer set support. It was believed that the Meyer property was the reason behind the unusual diffraction spectrum of quasicrystals [14], and this was proven to be the case: indeed Meyer sets show a large Bragg diffraction spectrum which is highly ordered [40, 41, 43].

The pure point Bragg spectrum of Meyer sets is now pretty well understood [2, 12, 33, 40, 41, 42, 43]. We also know a little about the Bragg continuous spectrum [2, 40, 42, 43] but nothing is known in general about the absolutely continuous and singular continuous Bragg spectrum of Meyer sets. It is our goal in this project to study these two components of the diffraction spectrum of Meyer sets via a systematic study of the Fourier analysis of measures with Meyer set support.

As it was introduced by Hof [17], the physical diffraction of a solid can be viewed as the Fourier transform \( \hat{\gamma} \) of the autocorrelation measure \( \gamma \) of the structure. The measure \( \gamma \) is positive definite, and often also positive,
and therefore it is Fourier transformable as a measure [13 22 34], and its Fourier transform $\hat{\gamma}$ is a positive measure. As any measure, $\hat{\gamma}$ has a Lebesgue decomposition

$$\hat{\gamma} = (\hat{\gamma})_{pp} + (\hat{\gamma})_{ac} + (\hat{\gamma})_{sc}$$

into a pure point, absolutely continuous and singular continuous component. We will refer to these three components as the pure point, absolutely continuous and singular continuous diffraction spectra, respectively, of our structure.

Structures with pure point diffraction, that is structures for which the absolutely and singular continuous diffraction spectra are absent, are now relatively well understood (see for example [6 9 10 11 12 23 25 24 27 28 30 31 43]). Of particular interest among pure point diffraction point sets are model sets, which are constructed by projecting points in a strip from a higher dimensional lattice (see Definition 2.6 below for the exact definition). If the window which produces the strip is regular, then the model set is pure point diffraction [16 38 12 35]. It was recently shown that the regularity of the window can be replaced by a weaker natural condition on the density of the pointset [9 18 19]. Recent work [6 35] has also shown that the diffraction formula for (regular) model sets is just the Poisson summation formula for the underlying lattice, an idea which seems to have been first suggested by Lagarias (see [5 Page 9]). This emphasizes that the long range order exhibited by model sets is a consequence of the periodicity of the lattice in the .

Moving beyond the pure point case, not too much is known in general. For 1-dimensional substitution tilings, some recent progress towards understanding the nature of the diffraction spectrum has been made [4], but much still needs to be done.

A large class of point sets with long range order are Meyer sets. They have been introduced by Y. Meyer in [29], and studied in [20 29 31 43]. They can be characterized via completely different properties: as Delone subsets of model sets, as almost lattices, as Delone sets with a uniform discrete Minkowski difference $\Lambda - \Lambda$ (in $\mathbb{R}^d$ it suffices for $\Lambda - \Lambda$ to be uniformly discrete), as harmonious sets and as Delone sets with relatively dense sets of $\epsilon$-dual characters (see [43] for the full characterization in second countable LCAG, or [20 29 31] for the characterization for $G = \mathbb{R}^d$).

As arbitrary subsets of model sets, one should expect Meyer sets to exhibit some of the inherited order: we expect a large pure point (or Bragg) spectrum and potentially some continuous diffraction spectrum, a consequence of the generic randomness introduced by going to arbitrary subsets.
This is indeed the case: the diffraction pattern of a Meyer set has a relatively dense set of Bragg peaks \([40]\), which are highly ordered \([41]\). The pure point diffraction measure is a strongly almost periodic measure \([40]\), which is also sup almost periodic \([41]\). The continuous diffraction measure \((\hat{\gamma})_c = (\hat{\gamma})_{ac} + (\hat{\gamma})_{sc}\) is a strong almost periodic measure \([41]\), and hence the continuous diffraction spectrum is either non-existent or has a relatively dense support. All these results have been generalized to weighted Dirac combs with Meyer set support \([42, 43]\). One would expect that the almost periodicity of the continuous spectrum would imply that both the singular continuous and absolutely continuous spectrum are highly ordered, but this could not be proved so far.

It is our goal in this paper to solve this problem viz a systematical study the Fourier transform of measures with Meyer set support. The main issue we face is the enigmatic nature of strong almost periodicity, which seems not to be compatible with the Lebesgue decomposition. To solve this problem we show that the class of Fourier transformable measures with Meyer set support admits a generalized notion of Eberlein decomposition (see \([22, 34]\) for definitions and properties): In Theorem 4.1 we show that every Fourier transformable measure \(\gamma\) supported inside a Meyer set \(\Lambda\) has an unique decomposition
\[
\gamma = \gamma_s + \gamma_{0a} + \gamma_{0s}
\]
into three Fourier transformable measures, each supported inside a Meyer set, such that
\[
\hat{\gamma}_s = (\hat{\gamma})_{pp} \quad ; \quad \hat{\gamma}_{0a} = (\hat{\gamma})_{ac} \quad ; \quad \hat{\gamma}_{0s} = (\hat{\gamma})_{sc}.
\]
We obtain this result by exploiting the Meyer property of the support via an embedding into a model set, and by using the Fourier analysis of certain weighted Dirac combs defined by the underlying cut and project scheme (CPS). It is worth pointing out that the Fourier theory we use for the larger weighed Dirac cobs is a consequence of the Poisson summation formula of the Lattice \(\mathcal{L}\) in the CPS. Therefore, similarly to regular model sets, the existence of the generalized Eberlein decomposition \((1.1)\), as well as all the strong consequences of its existence, are a result of the high long-range order of the lattice \(\mathcal{L}\) in the CPS.

Next we provide an upperbound for the support of each of the three components in the generalized Eberlein decomposition. We prove in Theorem 4.1 that, if \(\Lambda(W)\) is any closed model set containing the support of \(\gamma\), then the generalized Eberlein decomposition of \(\gamma\) stays within the class of measures supported inside \(\Lambda(W)\). This result generalizes some results of \([2, 26, 37, 40, 42, 44]\). This type of decomposition was used to derive purity results for 1-dimensional Pisot substitution tilings (see \([4, 8]\)).
As an immediate consequence of Theorem 4.1 we get the strong almost periodicity of each of the spectral components \((\hat{\gamma})_{pp}\), \((\hat{\gamma})_{ac}\) and \((\hat{\gamma})_{sc}\), respectively. As a simple consequence to diffraction, we obtain that each of the spectral components in the diffraction of Meyer sets is either trivial or has relatively dense support.

We continue our Fourier analysis by studying the norm-almost periodicity of the Fourier transform \(\hat{\gamma}\). We prove in Theorem 7.1 that, if \(\gamma\) is Fourier transformable and has Meyer set support, then \(\hat{\gamma}\) is a norm almost periodic measure. In particular we obtain that, under the same conditions, each of the measures \((\hat{\gamma})_{pp}\), \((\hat{\gamma})_{ac}\) and \((\hat{\gamma})_{sc}\), respectively, is norm almost periodic. Since norm almost periodicity is a stronger notion than strong almost periodicity, this improves our previous result, as well as some of the results of \([40, 41, 43]\).

All the results in this paper are then collected in Theorem 8.1.

We complete the introduction by presenting the general strategy for the proof of two main theorems: Theorem 4.1 and Theorem 7.1.

Consider any fixed Meyer set \(\Lambda\). In Proposition 3.2 we construct a twice Fourier transformable measure \(\omega\) supported inside a larger Meyer set \(\Gamma\) such that \(\omega(\{x\}) = 1\) for all \(x \in \Lambda\) and \(\hat{\omega}\) is pure point. This is done by embedding the Meyer set into a model set, and by picking a nice function which is 1 on the window of the model set.

Next, we show that \(\omega(\{x\}) = 1\) for all \(x \in \Lambda\), and that \(\omega\) has uniformly discrete support implies that \(\hat{\omega}\) "dominates" \(\hat{\gamma}\) in the following sense: there exists a finite measure \(\nu\) such that

\[
\hat{\gamma} = \hat{\omega} \ast \nu.
\]

This phenomena is similar to the case of measures with lattice support \([37]\).

Therefore, using the uniqueness of the Lebesgue decomposition, we get

\[
(\hat{\gamma})_{pp} = \hat{\omega} \ast \nu_{pp} \quad ; \quad (\hat{\gamma})_{ac} = \hat{\omega} \ast \nu_{ac} \quad ; \quad (\hat{\gamma})_{sc} = \hat{\omega} \ast \nu_{sc}.
\]

We can use now the twice Fourier transformability of \(\omega\) to show that each of the three measures in (1.2) has an inverse Fourier transform, which is supported inside \(\Gamma = \text{supp}(\omega)\), yielding the desired generalized Eberlein decomposition.

To prove norm almost periodicity, we show that the CPS and function \(h\) used to produce the measure \(\omega\) above can be chosen in such a way that \(\hat{\omega}\) is a norm almost periodic measure. As the convolution with finite measures preserves norm-almost periodicity (see Proposition 6.2) the conclusion of Theorem 7.1 follows.
2. Definitions and Notations

We start by briefly reviewing some basic definitions and properties. For a more detailed overview of these concepts we recommend the monographs [6, 7], as well as [1, 13, 22, 34, 43].

Throughout the paper, $G$ denotes a second countable locally compact Abelian group (LCAG). By $C_u(G)$ we denote the space of uniformly continuous and bounded functions on $G$. This is a Banach space with respect to the sup norm $\| \cdot \|_\infty$. As usual, we denote by $C_0(G)$ the subspace of $C_u(G)$ consisting of functions vanishing at infinity, and by $C_c(G)$ the subspace of compactly supported continuous functions. Note that $C_c(G)$ is not complete in $(C_u(G), \| \cdot \|_\infty)$.

As usual for diffraction theory, by measure we will understand a Radon measure. Via the Riesz Representation Theorem, a Radon measure is simply a linear functional on $C_c(G)$, equipped with the inductive topology (see [36, Appendix] for more details).

For a function $f$ on $G$ we denote by $f^\dagger$ its reflection, that is
\[ f^\dagger(x) := f(-x). \]
Same way, for a measure $\mu$ we denote by $\mu^\dagger$ the reflection $\mu^\dagger(f) := \mu(f^\dagger)$.

Finally, we denote by $K_2(G) := \text{Span}\{f \ast g : f, g \in C_c(G)\}$.

Let us recall here the definition of the Fourier transform of measures. For a more detailed review of the subject, we recommend [34].

**Definition 2.1.** A measure $\mu$ on $G$ is called **Fourier transformable** if there exists a measure $\hat{\mu}$ on $\hat{G}$ such that, for all $g \in K_2(G)$ we have
\begin{align*}
(\text{i}) & \quad \hat{g} \in L^1(\{\hat{\mu}\}), \\
(\text{ii}) & \quad \langle \mu, g \rangle = \langle \hat{\mu}, \hat{g} \rangle.
\end{align*}

Recall that any positive definite measure $\mu$ (see [13, 34] for definition and properties) is Fourier transformable and its Fourier transform $\hat{\mu}$ is positive.

Next, let us recall translation boundedness for measures.

**Definition 2.2.** A measure $\mu$ on $G$ is called **translation bounded** if for all compact sets $K \subseteq G$ we have
\[ \|\mu\|_K := \sup_{t \in G} |\mu|(t + K) < \infty, \]

We will denote by $\mathcal{M}^\infty(G)$ the space of translation bounded measures on $G$.

**Remark 1.** (i) A measure $\mu$ is translation bounded if and only if $\mu \ast f \in C_u(G)$ for all $f \in C_c(G)$ [1 Thm. 1.1].
(ii) \( \| \cdot \|_K \) is a norm on \( \mathcal{M}^\infty (G) \) [12] and \( (\mathcal{M}^\infty (G), \| \cdot \|_K) \) is a Banach space [37].

(iii) If \( K_1, K_2 \) are two compact sets with non-empty interior, then the norms \( \| \cdot \|_{K_1} \) and \( \| \cdot \|_{K_2} \) are equivalent [12]. In particular, \( \mu \) is translation bounded if and only if \( \| \mu \|_K < \infty \) for one compact set \( K \) with non-empty interior.

We complete the section by reviewing almost periodicity for measures.

Definition 2.3. A measure function \( f \in C_u(G) \) is called **Bohr almost periodic** if the hull \( C_f := \{ T_t f | t \in G \} \) of all its translates has compact closure in \( (C_u(G), \| \cdot \|_\infty) \). We denote by \( \text{SAP}(G) \) the subspace of \( C_u(G) \) of all Bohr almost periodic functions.

A measure \( \mu \) is called **strong almost periodic** if for all \( f \in C_c(G) \) we have \( \mu * f \in \text{SAP}(G) \). We denote the space of all strong almost periodic measures by \( \text{SAP}(G) \).

A measure \( \mu \) is called **norm almost periodic** if for each \( \epsilon > 0 \) the set \( P_\epsilon^K (\mu) := \{ t \in G : \| T_t \mu - \mu \|_K < \epsilon \} \) of \( \epsilon \)-norm almost periods is relatively dense, where \( K \) is any fixed compact set with non-empty interior.

Remark 2.

(i) Since different compact sets with non-empty interior define equivalent norms, norm almost periodicity is independent of the choice of the compact \( K \).

(ii) Norm almost periodic pure point measures were studied and classified in terms of CPS \( (G, H, L) \) and continuous functions \( h \in C_0(H) \) in [13].

(iii) A norm almost periodic measure \( \mu \) is strong almost periodic. Moreover, if \( \text{supp}(\mu) - \text{supp}(\mu) \) is uniformly discrete then \( \mu \) is norm almost periodic if and only if \( \mu \) is strong almost periodi [12].

It is easy to see that \( \text{SAP}(G) \subseteq \mathcal{M}^\infty (G) \). The importance of the space \( \text{SAP}(G) \) in the study of the Lebesgue decomposition of Fourier transform is emphasized by the following two results.

Theorem 2.4. [22 Cor. 11.1] Let \( \gamma \) be a Fourier transformable measure. Then \( \gamma \) is pure point if and only if \( \widehat{\gamma} \in \text{SAP}(G) \).

Theorem 2.5. [31 Cor. 4.10.13] Let \( \gamma \) be translation bounded Fourier transformable measure. Then \( \widehat{\gamma} \) is pure point if and only if \( \gamma \in \text{SAP}(G) \).

We complete this section with a brief review of Meyer sets. For more details, we recommend [6, 29, 32, 31, 43].
Definition 2.6. By a cut and project scheme (or CPS) we understand a triple \((G, H, \mathcal{L})\), with \(H\) a LCAG, and a lattice \(\mathcal{L} \subseteq G \times H\) such that

(a) \(\pi_H(\mathcal{L})\) is dense in \(H\).

(b) the restriction \(\pi_G|\mathcal{L}\) of the first projection \(\pi_G\) to \(\mathcal{L}\) is one to one.

Given a CPS \((G, H, \mathcal{L})\) we will denote by \(L := \pi_G(\mathcal{L})\). Then, \(\pi_G\) induces a group isomorphism between \(\mathcal{L}\) and \(L\). Composing the inverse of this with the second projection \(\pi_H\) we get a mapping

\[ \star : L \to H \]

which we will call the \(\star\)-mapping. We can then write

\[ \mathcal{L} = \{(x, x^*) : x \in L\}. \]

We can summarise a CPS in a simple diagram:

\[ \begin{array}{ccc}
G & \xrightarrow{\pi_G} & G \times H \\
\cup & \cup & \cup \\
L & \xrightarrow{\pi_H} & H \\
\end{array} \]

\[ L \xleftarrow{1-1} \mathcal{L} \xrightarrow{\text{dense}} L^* \]

Given a CPS \((G, H, \mathcal{L})\), we can define

\[ \mathcal{L}^0 := \{(\chi, \psi) \in \hat{G} \times \hat{H} : \chi(x)\psi(x^*) = 1 \forall x \in L\}. \]

Then, \((\hat{G}, \hat{H}, \mathcal{L}^0)\) is a CPS \([9, 31, 32, 43]\). We will refer to this as the CPS dual to \((G, H, \mathcal{L})\).

We can now introduce the definition of a Meyer set.

Definition 2.7. A Delone set \(\Lambda \subseteq G\) is called a Meyer set if there exists a cut and project scheme \((G, H, \mathcal{L})\) and a compact set \(W \subseteq H\) such that \(\Lambda \subseteq \Lambda(\mathcal{L})\).

A Delone set is Meyer if and only if \(\Lambda - \Lambda - \Lambda\) is closed and discrete, or equivalently, if and only if there exists a finite set \(F\) such that \(\Lambda - \Lambda \subseteq \Lambda + F\) \([21, 20, 29, 31, 43]\).

3. A Ping-Pong Lemma for measures with Meyer set support

Next, we prove the following Lemma, which will play an important role in our approach.
Lemma 3.1. Let $H$ be a LCAG, $W \subseteq H$ be a compact set and $y \in H$ be any point such that $y \notin W$. Then, there exists some $h \in K_2(H)$ such that $h \equiv 1$ on $W$ and $h(y) = 0$.

In particular, there exists some $h \in K_2(H)$ such that $h \equiv 1$ on $W$.

Proof. Since $W$ is closed and $y \notin W$, there exists some open set $0 \in U = -U$ such that $(y + U) \cap W = \emptyset$. We can then pick some precompact open set $0 \in V = -V$ such that $V + V + V \subseteq U$.

As $0 \in V$ is open, we have $V \subseteq V + V$. It follows that $W + V$ is a compact subset of the open set $W + V + V$. Then, by the Urysohn’s Lemma, there exists some $f \in C_c(H)$ such that $1_{W + V} \leq f \leq 1_{W + V + V}$.

Next, pick some $g \in C_c(H)$ such that $g \geq 0$, supp$(g) \subseteq V$ and $\int_H g(t)dt = 1$. Set $h = f \ast g$. We claim that this $h$ has the desired properties.

By construction $f, g \in C_c(H)$ and hence $h \in K_2(H)$. Since $f = 1$ on $W + V$ and supp$(g) \subseteq V$, it is easy to see that $h(s) = \int_H g(t)dt = 1$ for all $s \in W$.

Finally, as supp$(f) \subseteq W + V + V$ and supp$(g) \subseteq V$ we have supp$(h) \subseteq W + V + V + V \subseteq W + U = W - U$. Therefore $h(y) = 0$ as $y \notin W - U$.

The last claim is immediate. Indeed, if $W \neq H$ then the claim follows from the above by picking some $y \notin W$. Otherwise, $H$ is compact and $h = 1_H$ works.\[\square\]

As a consequence we get:

Proposition 3.2. Let $G$ be a second countable group, and $\Lambda \subset G$ be a Meyer set. Let $(G, H, \mathcal{L})$ be any cut and project scheme and $W \subseteq H$ compact be such that $\Lambda \subseteq \Lambda(W)$.

Then, there exists some $h \in K_2(H)$ such that, the measure

$$\omega_h := \sum_{(x, x^*) \in \mathcal{L}} h(x^*)\delta_x$$

has the following three properties:

(i) $\omega_h(\{x\}) = 1 \forall x \in \Lambda$.

(ii) $\omega_h$ is twice Fourier transformable and

$$\tilde{\omega}_h = \text{det}(\mathcal{L})\omega_h = \text{det}(\mathcal{L}) \sum_{(y, y^*) \in \mathcal{L}^0} \tilde{h}(y^*)\delta_y.$$

(iii) $\Gamma := \text{supp}(\omega_h)$ is a Meyer set.

Moreover, for each $z \in G \setminus \Lambda(W)$, we can pick $h$ in such a way that $z \notin \Gamma = \text{supp}(\omega_h)$.

Proof. By Lemma 3.1, there exists some $h \in K_2(H)$, such that $h = 1$ on $W$.\[\square\]
Since \( h \in K_2(H) \), the measure \( \omega_h \) is twice Fourier transformable [30, Thm. 4.10, Thm. 4.12] and

\[
\widehat{\omega}_h = \det(L) \omega_h = \det(L) \sum_{(y,y^*) \in L^0} \tilde{h}(y^*) \delta_y.
\]

Next, the set \( U := \{ x \in H | h(x) \neq 0 \} \) is open and precompact, and hence \( \Gamma := \wedge(U) = \text{supp}(\omega_h) \) is a Meyer set.

It follows that \( \omega_h \) satisfies properties (i), (ii) and (iii).

Finally, let \( z \in G \backslash \wedge(W) \) be arbitrary. If \( z \notin \pi_G(L) \), then we have \( z \notin \Gamma = \wedge(U) \) for all choices of \( U \).

If \( z \in \pi_G(L) \), then \( z \notin \wedge(W) \) implies that \( z^* \notin W \). Then, by Lemma 3.1, we can pick \( h \in K_2(H) \) such that \( h(z^*) = 0 \), and hence, by the above construction, \( z \notin \Gamma \). \( \square \)

Recall that given a Fourier transformable measure \( \mu \) supported inside a lattice \( L \subset G \), its Fourier transform \( \hat{\mu} \) is fully periodic under the lattice \( L^0 \subset \hat{G} \) dual to \( L \) [1, 13]. It follows [37] that there exists a finite measure \( \nu \) such that

\[
\hat{\mu} = \delta_{L^0} * \nu.
\]

Below we prove that a similar result holds for Fourier transformable measures with Meyer set support. Starting with a measure \( \gamma \) with Meyer support, this result will allow us to move to the Fourier space, take certain decompositions and return to the original space.

**Lemma 3.3** (Ping-Pong Lemma for Meyer sets). Let \( \Lambda \subset G \) be a Meyer set, \( \omega \) a twice Fourier transformable measure with the following two properties:

(a) \( \Gamma := \text{supp}(\omega) \) is \( U \)-uniformly discrete for some open set \( 0 \in U \);

(b) \( \omega(\{x\}) = 1 \) for all \( x \in \Lambda \).

Let \( f \in C_c(G) \) be any function such that \( \text{supp}(f \hat{f}) \subseteq U \).

Then, the following hold:

(i) If \( \gamma \) is any Fourier transformable measure with \( \text{supp}(\gamma) \subseteq \Lambda \), then

\[
\hat{\gamma} = \hat{\omega} * \nu.
\]

(ii) If \( \nu \) is a finite measure on \( \hat{G} \), then \( \hat{\omega} * \nu \) is Fourier transformable and \( \gamma := \left( \hat{\omega} * \nu \right)^\dagger \) is supported inside \( \Gamma \). Moreover, if \( \gamma \) is Fourier transformable, then

\[
\hat{\gamma} = \hat{\omega} * \nu.
\]
Proof. (i)

Since \( \gamma \) is Fourier transformable, for each \( f \in C_c(G) \) the measure \( \nu := \hat{\gamma} \lvert f \rvert^2 \) is finite, and has Fourier transform as finite measure [1 Prop 2.3], [34, Lemma 4.9.24] given by

\[
\hat{\nu} = \hat{\gamma} \lvert f \rvert^2 = \left( \gamma * f * \hat{f} \right)^\dagger.
\]

Next, since \( \hat{\omega} \) is translation bounded (see [1, Thm. 2.5] or [34, Thm. 4.9.23]) \( \hat{\gamma} \lvert f \rvert^2 \) is finite, the measures \( \hat{\omega} \) and \( \nu \) are convolvable [1, Thm. 1.2], [34, Lemma 4.9.19]. Moreover, as \( \hat{\omega} \) is Fourier transformable and \( \nu \) is finite, their convolution is Fourier transformable and [34, Lemma 4.9.26]

\[
\hat{\omega} \ast \nu = \hat{\omega} \hat{\nu} = \hat{\gamma} \lvert f \rvert^2 = \left( \gamma * f * \hat{f} \right)^\dagger \omega^\dagger = \left( \left( \gamma * f * \hat{f} \right) \omega \right)^\dagger.
\]

We claim that

\[
\left( \gamma * f * \hat{f} \right) \omega = \gamma.
\]

Indeed,

\[
\text{supp} \left( \left( \gamma * f * \hat{f} \right) \omega \right) \subseteq \text{supp} \left( \gamma * f * \hat{f} \right) \cap \text{supp} \left( \omega \right) \subseteq (\Lambda + U) \cap \Gamma = \Lambda.
\]

Moreover, for each \( x \in \Lambda \), since \( \omega(\{x\}) = 1 \) and \( \Gamma \) is \( U \)-uniformly discrete, we have

\[
\left( \left( \gamma * f * \hat{f} \right) \omega \right)(\{x\}) = \gamma * f * \hat{f}(\{x\}) = \sum_{y \in \Lambda} \left( f * \hat{f} \right)(x - y) \gamma(\{y\}).
\]

The last sum is zero unless \( x - y \in U \) and \( y \in \Lambda \), which means \( y \in \Lambda \cap (x - U) = \{x\} \). Therefore

\[
\left( \gamma * f * \hat{f} \right) \omega(\{x\}) = \left( f * \hat{f} \right)(0) \gamma(\{x\}) = \gamma(\{x\}).
\]

Now, since \( \gamma \) is Fourier transformable, so is \( \gamma^\dagger = \hat{\omega} \left( \hat{\gamma} \lvert f \rvert^2 \right) \). Therefore, by applying [34, Thm. 4.9.28] to \( \hat{\omega} \) we have

\[
\hat{\gamma}^\dagger = \hat{\omega} \left( \hat{\gamma} \lvert f \rvert^2 \right) = \left( \hat{\omega} \left( \hat{\gamma} \lvert f \rvert^2 \right) \right)^\dagger.
\]

Reflecting this relation we get the claim.

(ii) Exactly as before, since \( \omega \) is twice Fourier transformable, \( \hat{\omega} \) is translation bounded and Fourier transformable. Then, as \( \nu \) is finite, \( \hat{\omega} \ast \nu \) is well defined, Fourier transformable, and

\[
\gamma = \left( \hat{\omega} \ast \nu \right)^\dagger = \left( \hat{\omega} \hat{\nu} \right)^\dagger = \left( \hat{\nu} \right)^\dagger \omega.
\]

Since \( \nu \) is finite, we have \( \hat{\nu} \in C_u(G) \), and hence
supp(\gamma) \subseteq \text{supp}(\omega) \subseteq \Gamma.

If \gamma is Fourier transformable, it follows that \hat{\omega} * \nu is twice Fourier transformable. Then, by [22, Thm. 3.4], [34, Thm. 4.9.28] we get

\begin{equation*}
\overline{(\hat{\omega} * \nu)^\dagger} = \hat{\omega} * \nu.
\end{equation*}

This gives \hat{\gamma} = \hat{\omega} * \nu.

□

Remark 3. In the theory of mathematical diffraction the measures \hat{\gamma} will always be positive. It follows that all measures \gamma appearing in Lemma 3.3 (ii) will be positive definite, and hence Fourier transformable.

Remark 4. If \Lambda = L is a lattice, then one can take \omega = \delta_L, and hence \Gamma = L. Moreover, any \textit{L}\textsuperscript{0}-periodic measure is a Fourier transform [37].

In this case, since \Gamma = L = \Lambda, Lemma 3.3 gives the well known result (compare [3, 37]) that a Fourier transformable measure \gamma is supported inside \textit{L} if and only if there exists a finite measure \nu such that

\begin{equation*}
\hat{\gamma} = \delta_{\textit{L}0} * \nu.
\end{equation*}

Remark 5. Let \textit{L} \subset \textit{G} be a lattice, let \textit{F} \subset \textit{G} be a finite set, and set \Lambda = \textit{L} + \textit{F}.

Now, if \textit{F}' is a minimal subset of \textit{F} with the property that \Lambda = \textit{L} + \textit{F}', then we have \delta_{\Lambda} = \delta_{\textit{F}'} * \delta_{\textit{L}}. We can then pick \omega := \delta_{\Lambda}, and hence \Gamma = \Lambda.

Lemma 3.3 then yields that a Fourier transformable measure \gamma is supported inside \Lambda if and only if there exists a finite measure \nu such that

\begin{equation*}
\hat{\gamma} = \left( \sum_{\chi \in \textit{L}^k} \left( \sum_{f \in \textit{F}'} \chi(f) \delta_{\chi} \right) \right) * \nu.
\end{equation*}

4. On the Generalized Eberlein decomposition for measures with Meyer set support

We can now prove the existence of the generalized Eberlein decomposition for Fourier transformable measures with Meyer set support.

Theorem 4.1 (Existence of the generalized Eberlein decomposition). Let \textit{G} be a second countable LCAG, and let \Lambda \subset \textit{G} be a Meyer set.

Then, for each Fourier transformable measure \gamma with supp(\gamma) \subseteq \Lambda, there exist unique Fourier transformable measures \gamma_8, \gamma_0, \gamma_0 a supported inside a
**Meyer set, such that**

\[
\gamma = \gamma_s + \gamma_0s + \gamma_0a
\]

\[
\widehat{\gamma}_s = (\widehat{\gamma})_{pp}
\]
\[
\widehat{\gamma}_0s = (\widehat{\gamma})_{sc}
\]
\[
\widehat{\gamma}_0a = (\widehat{\gamma})_{ac}.
\]

Moreover, if \((G,H,\mathcal{L})\) is any CPS and \(W \subseteq H\) any compact set such that \(\Lambda \subseteq \wedge (W)\), then

\[
\text{supp}(\gamma_s), \text{supp}(\gamma_0s), \text{supp}(\gamma_0a) \subseteq \wedge (W).
\]

**Proof.** Now, under the notations from Lemma 3.3, by Lemma 3.3 there exists a finite measure \(\nu\) such that

\[
\gamma = \widehat{\omega} \ast \nu.
\]

Since \(G\) is metrisable, \(\hat{G}\) is \(\sigma\) compact. Therefore, the pure point measure \(\widehat{\omega}\) has (at most) countable support.

As \(\hat{G}\) is \(\sigma\)-compact, the measure \(\nu\) admits a Lebesgue decomposition

\[
\nu = \nu_{pp} + \nu_{ac} + \nu_{sc}.
\]

Since \(\widehat{\omega}\) is pure point measure, the measures \(\widehat{\omega} \ast \nu_{pp}, \widehat{\omega} \ast \nu_{ac}, \widehat{\omega} \ast \nu_{sc}\) are pure point, absolutely continuous and singular continuous, respectively. Moreover, we have

\[
\gamma = \widehat{\omega} \ast \nu = \widehat{\omega} \ast \nu_{pp} + \widehat{\omega} \ast \nu_{ac} + \widehat{\omega} \ast \nu_{sc}
\]

and therefore, by the uniqueness of the Lebesgue decomposition, we have

\[
(\widehat{\gamma})_{pp} = \widehat{\omega} \ast \nu_{pp}, \quad (\widehat{\gamma})_{ac} = \widehat{\omega} \ast \nu_{ac}, \quad (\widehat{\gamma})_{sc} = \widehat{\omega} \ast \nu_{sc}.
\]

By Lemma 3.3 (ii), the measures \(\widehat{\omega} \ast \nu_{pp}, \widehat{\omega} \ast \nu_{ac}, \widehat{\omega} \ast \nu_{sc}\) are Fourier transformable, and the measures

\[
\gamma_s := (\widehat{\omega} \ast \nu_{pp})^\dagger
\]
\[
\gamma_0a := (\widehat{\omega} \ast \nu_{ac})^\dagger
\]
\[
\gamma_0s := (\widehat{\omega} \ast \nu_{sc})^\dagger
\]

are supported inside \(\Gamma\).

We next show that \(\gamma_s, \gamma_0s\) and \(\gamma_0a\) are Fourier transformable, or equivalently that \(\widehat{\omega} \ast \nu_{pp}, \widehat{\omega} \ast \nu_{ac}, \widehat{\omega} \ast \nu_{sc}\) are twice Fourier transformable. By [35, Thm. 3.10] this is equivalent to the integrability of \(\tilde{\gamma}\) with respect to each of
these measures, for all $g \in K_2(G)$. We show below that this is an immediate consequence of the Fourier transformability of $\gamma$.

Since $\gamma$ is Fourier transformable, for each $g \in K_2(G)$ we have $\tilde{g} \in L^1(\hat{\gamma})$, and hence $\tilde{g} \in L^1((\hat{\gamma})_{pp}^\dagger), \tilde{g} \in L^1((\hat{\gamma})_{sc}^\dagger)$ and $\tilde{g} \in L^1((\hat{\gamma})_{ac}^\dagger)$. Therefore, since $(\hat{\gamma})_{pp} = \hat{\omega} * \nu_{pp}, (\hat{\gamma})_{ac} = \hat{\omega} * \nu_{ac}$ and $(\hat{\gamma})_{sc} = \hat{\omega} * \nu_{sc}$ are Fourier transformable, by [35, Thm. 3.10] we get that $\hat{\omega} * \nu_{pp}, \hat{\omega} * \nu_{ac}, \hat{\omega} * \nu_{sc}$ are twice Fourier transformable.

This proves the first claim.

Let now $(G, H, \mathcal{L})$ be any CPS and $W \subseteq H$ be any compact set such that $\Lambda \subseteq \Lambda(W)$. We show that

$$\text{supp}(\gamma_s), \text{supp}(\gamma_{0s}), \text{supp}(\gamma_{0a}) \subseteq \Lambda(W).$$

Indeed, for each $z \in G \setminus \Lambda(W)$, by Proposition 3.2 we can pick $\omega$ in Lemma 3.3 such that $\omega(\{z\}) = 0$.

Then, in the above we have $z \notin \Gamma$ and hence

$$\gamma_s(\{z\}) = \gamma_{0s}(\{z\}) = \gamma_{0a}(\{z\}) = 0.$$

This proves the claim.

**Definition 4.2.** We say that a measure $\gamma$ admits a **generalized Eberlein decomposition** if we can find Fourier transformable measures $\gamma_s, \gamma_{0s}, \gamma_{0a}$ such that

$$\gamma = \gamma_s + \gamma_{0s} + \gamma_{0a},$$

$$\tilde{\gamma}_s = (\hat{\gamma})_{pp},$$

$$\tilde{\gamma}_{0s} = (\hat{\gamma})_{sc},$$

$$\tilde{\gamma}_{0a} = (\hat{\gamma})_{ac}.$$

The injectivity of the Fourier transform gives that, whenever the generalized Eberlein decomposition exists, it is unique. In Theorem 4.1 above we showed that Fourier transformable measures with Meyer set support always admit generalized Eberlein decomposition.

We complete the section by listing an interesting consequence of Theorem 4.1.

**Corollary 4.3.** Let $G$ be a second countable LCAG, and let $\Lambda \subset G$ be a Meyer set.

If $\gamma$ is a Fourier transformable measure with $\text{supp}(\gamma) \subseteq \Lambda$, then

$$(\hat{\gamma})_{pp}, (\hat{\gamma})_{sc}, (\hat{\gamma})_{ac} \in \mathcal{SA}(G).$$
By Theorem 4.1, \( \gamma \) admits a generalized Eberlein decomposition. Since \( \gamma_s, \gamma_0, \gamma_0 a \) are pure point, their Fourier transforms are strongly almost periodic measures by Theorem 2.4.

5. On the norm-almost periodicity of a class of measures

In the remaining of this paper we prove that given a Fourier transformable measure \( \gamma \) with Meyer set support, then its Fourier transform \( \hat{\gamma} \) is a norm almost periodic measure.

We start by proving that, given a cut and project scheme of the form \( (G, \mathbb{R}^d \times H, \mathcal{L}) \) for LCAG \( H \) and some function \( h = \phi \otimes \psi \in C_c(\mathbb{R}^d \times H) \) where \( \phi \in \mathcal{S}(\mathbb{R}^d) \cap K_2(\mathbb{R}^d), \psi \in C_c(H) \) then, \( \omega_h \) is norm-almost periodic. Here, the product \( \phi \otimes \psi \) is defined via

\[
\phi \otimes \psi(x, y) := \phi(x)\psi(y).
\]

We will complete the paper by showing that, given a Meyer set, we can find a measure \( \omega \) satisfying the conditions in Lemma 3.3 such that, by the results in this section, \( \hat{\omega} \) is norm-almost periodic and that norm-almost periodicity is preserved by convolution with finite measures.

First let us recall a result of [43] which we need in this section.

**Proposition 5.1.** [43] Thm. 5.5.2 Let \( (G, H, \mathcal{L}) \) be a CPS and \( h \in C_c(H) \). Then, \( \omega_h \) is norm almost periodic.

In the Lemma 5.2 below, we show that for CPS with \( H = \mathbb{R}^d \times H_1 \) and \( h = \phi \otimes \psi \) for some Schwartz function \( \phi \in \mathcal{S}(\mathbb{R}^d) \) and some \( \psi \in C_c(H_1) \) then \( \omega_h \) is a measure, and we give an upperbound for its norm \( \| \omega_h \|_K \).

**Lemma 5.2.** Let \( (G, \mathbb{R}^d \times H_1, \mathcal{L}) \) be a CPS. Then,

(i) Let \( \phi \in \mathcal{S}(\mathbb{R}^d), \psi \in C_c(H_1) \) and let \( h = \phi \otimes \psi \). Then, \( \omega_h \) is a measure.

(ii) For each compact \( K \subseteq G \) and compact \( W \subseteq H_1 \), there exists some constant \( C = C(K, W) \) such that, for all \( \phi \in \mathcal{S}(\mathbb{R}^d), \psi \in C_c(H_1) \) with \( \text{supp}(\psi) \subseteq W \), we have

\[
\| \omega_{\phi \otimes \psi} |_K \| \leq C \| \psi \|_\infty \| (1 + x^{2d}) \phi(x) \|_\infty.
\]

**Proof.** Let \( W \subseteq H_1 \) be any compact set, and let \( \psi \in C_c(H) \) be any function such that \( \text{supp}(\psi) \subseteq W \). Set

\[
C(W, K) := \left( \| \delta_L \|_{K \times [-\frac{1}{2}, \frac{1}{2}]^d \times W} \right) \left( \sum_{n \in \mathbb{Z}^d} \sup_{z \in n + \left[-\frac{1}{2}, \frac{1}{2}\right]^d} \frac{1}{1 + |z|^{2d}} \right) < \infty.
\]
To prove that $\omega_h$ is a measure, we show that for each compact set $K$ we have
\[
\sum_{x \in K} |\omega_h(\{x\})| < \infty .
\]

A simple computation yields
\[
\sum_{x \in K} |\omega_h(\{x\})| = \sum_{(x,x^*) \in \mathcal{L}} |1_K(x)h(x^*)| = \sum_{(x,x^*) \in \mathcal{L}} |1_K(x) (\phi \otimes \psi)(x^*)| .
\]

Now, consider the canonical projections $\pi_{\mathbb{R}^d} : \mathbb{R}^d \times H_1 \to \mathbb{R}^d$ and $\pi_{H_1} : \mathbb{R}^d \times H_1 \to H_1$, respectively. For each $(x,x^*) \in \mathcal{L}$ we will denote, for simplicity, by $x_1^* := \pi_{\mathbb{R}^d}(x^*)$ and $x_2^* := \pi_{H_1}(x^*)$.

Then
\begin{equation}
(5.1)
\end{equation}

\[
\sum_{x \in K} |\omega_h(\{x\})| = \sum_{(x,x^*) \in \mathcal{L}} |1_K(x)\phi(x_1^*)\psi(x_2^*)| \\
= \sum_{(x,x^*) \in \mathcal{L}} 1_K(x) |\phi(x_1^*)| |\psi(x_2^*)| \leq \sum_{(x,x^*) \in \mathcal{L}} 1_K(x) |\phi(x_1^*)| \|\psi\|_{\infty} 1_W(x_2^*) \\
\leq \sum_{m \in \mathbb{Z}^d} \left( \sum_{(x,x^*) \in \mathcal{L}} 1_K(x) |\phi(x_1^*)| \|\psi\|_{\infty} 1_W(x_2^*) \right) \\
= \|\psi\|_{\infty} \sum_{m \in \mathbb{Z}^d} \left( \sum_{(x,x^*) \in \mathcal{L}} 1_K(x) 1_{m + [-\frac{1}{2}, \frac{1}{2})^d}(x_1^*) |\phi(x_1^*)| 1_W(x_2^*) \right) \\
\leq \|\psi\|_{\infty} \|\phi(s)(1 + |s|^{2d})\|_{\infty} \sum_{m \in \mathbb{Z}^d} \left( \sum_{(x,x^*) \in \mathcal{L}} 1_K(x) 1_{m + [-\frac{1}{2}, \frac{1}{2})^d}(x_1^*) \right) \left( \sum_{z \in \mathbb{Z}^d} \frac{1}{1 + |z|^{2d}} 1_W(x_2^*) \right) \\
= \|\psi\|_{\infty} \|\phi(s)(1 + |s|^{2d})\|_{\infty} \left( \sum_{m \in \mathbb{Z}^d} \sup_{x \in \mathbb{Z}^d \in [-\frac{1}{2}, \frac{1}{2})^d} \frac{1}{1 + |z|^{2d}} \left( \sum_{(x,x^*) \in \mathcal{L}} 1_K(x) 1_{m + [-\frac{1}{2}, \frac{1}{2})^d}(x_1^*) 1_W(x_2^*) \right) \right) \\
\leq \left( \sum_{m \in \mathbb{Z}^d} \sup_{x \in \mathbb{Z}^d \in [-\frac{1}{2}, \frac{1}{2})^d} \frac{1}{1 + |z|^{2d}} \right) \delta_{\mathcal{L}}((t + K) \times (m + [-\frac{1}{2}, \frac{1}{2})^d) \times W) \|\psi\|_{\infty} \|\phi(s)(1 + |s|^{2d})\|_{\infty}
\]

This shows that
\[
\sum_{x \in K} |\omega_h(\{x\})| < \infty ,
\]
and proves (i).

We now show (ii).

For each compact set $K$ we have

$$\|\omega_h\|_K = \sup_{t \in G} |\omega_h|(t + K)$$

$$\leq \|\psi\|_\infty \|\phi(s)(1 + |s|^d)\|_\infty \left(\sum_{m \in \mathbb{Z}^d} (\sup_{z \in m + [-\frac{1}{2}, \frac{1}{2}]^d} \frac{1}{1 + |z|^2d}) (\delta_L((t + K) \times (m + [-\frac{1}{2}, \frac{1}{2}]^d) \times W))\right)$$

$$\leq \left(\sum_{m \in \mathbb{Z}^d} \sup_{z \in m + [-\frac{1}{2}, \frac{1}{2}]^d} \frac{1}{1 + |z|^2d}\right) \|\psi\|_\infty \|\phi(s)(1 + |s|^d)\|_\infty \|\delta_L\|_{K \times [-\frac{1}{2}, \frac{1}{2}]^d \times W}$$

$$\leq C(K, W) \|\psi\|_\infty \|\phi(s)(1 + |s|^d)\|_\infty .$$

This completes the proof.

Lemma 5.3. Let $(G, \mathbb{R}^d \times H_1, \mathcal{L})$ be a CPS. Let $\phi \in \mathcal{S}(\mathbb{R}^d), \psi \in C_c(H_1)$ and set $h = \phi \otimes \psi$. Then $\omega_h$ is norm almost periodic.

Proof. Fix compact sets $K \subseteq G$ and $W \subseteq H$ such that supp$(\psi) \subseteq W$.

Let $\epsilon > 0$, and let $C = C(K, W)$ be the constant from Lemma 5.2 (ii).

Since $\phi \in \mathcal{S}(\mathbb{R}^d)$ we can write it as $\phi = \phi_1 + \phi_2$, with $\phi_1 \in C_c(\mathbb{R}^d)$ and

$$\|(1 + |s|^d)\phi_2(s)\|_\infty \leq \frac{\epsilon}{1 + 3C \|\psi\|_\infty \|\delta_L\|_{K \times [-\frac{1}{2}, \frac{1}{2}]^d \times W}} .$$

Then, by Lemma 5.2 we have

$$\|\omega_{\phi_2 \otimes \psi}\|_K < \frac{\epsilon}{3} .$$

Next, by Proposition 5.1 the measure $\omega_{\phi_1 \otimes \psi}$ is norm almost periodic.

Therefore, the set

$$P := \{t \in G \|T_t \omega_{\phi_1 \otimes \psi} - \omega_{\phi_1 \otimes \psi}\|_K < \frac{\epsilon}{3}\}$$

is relatively dense.

Let $t \in P$. Then

$$\|T_t \omega_h - \omega_h\|_K = \|T_t \omega_{\phi_1 \otimes \psi} + T_t \omega_{\phi_2 \otimes \psi} - \omega_{\phi_1 \otimes \psi} - \omega_{\phi_2 \otimes \psi}\|_K$$

$$\leq \|T_t \omega_{\phi_1 \otimes \psi} - \omega_{\phi_1 \otimes \psi}\|_K + \|T_t \omega_{\phi_2 \otimes \psi}\|_K + \|\omega_{\phi_2 \otimes \psi}\|_K$$

$$\leq \|T_t \omega_{\phi_1 \otimes \psi} - \omega_{\phi_1 \otimes \psi}\|_K + \|\omega_{\phi_2 \otimes \psi}\|_K + \|\omega_{\phi_2 \otimes \psi}\|_K < \epsilon$$

This shows that each $t \in P$ is an $\epsilon$-norm almost period for $\omega_h$, from which our claim follows. □
6. Norm almost periodicity and convolution

In this section we show that norm almost periodicity is preserved under convolution with finite measures. We start by proving the following preparation Lemma.

**Lemma 6.1.** Let \( K, K' \) be compact sets such that \( K \subset \text{Int}(K') \). If \( \nu \) is a finite measure and \( \mu \) is any translation bounded measure, then

\[
|\mu * \nu|(K) \leq \|\mu\|_{K'}(|\nu|(G)).
\]

**Proof.** Since \( K \subset \text{Int}(K') \), there exists a non-negative \( f \) such that \( 1_K \leq f \leq 1_{K'} \).

Let \( \epsilon > 0 \) be arbitrary.

By the definition of the total variation, there exists some \( g \in C_c(G) \) with \( |g| \leq f \) such that

\[
|\mu * \nu|(f) \leq |\mu * \nu|(g) + \epsilon.
\]

Then, we have

\[
|\mu * \nu|(K) \leq |\mu * \nu|(f) \leq |\mu * \nu|(g) + \epsilon = \left| \int_G \int_G g(s + t) d\nu(s) d\mu(t) \right| + \epsilon
\]

\[
\leq \int_G \int_G |g(s + t)| d|\nu|(s) d|\mu|(t) + \epsilon = \int_G \left( \int_G |g(s + t)| d|\mu|(t) \right) d|\nu|(s) + \epsilon.
\]

Now, for each \( s \in G \) we have \( g(s + t) = 0 \) for all \( t \notin -s + K' \). Therefore

\[
\int_G |g(s + t)| d|\mu|(t) = \int_{-s + K'} |g(s + t)| d|\mu|(t)
\]

\[
\leq \int_{-s + K'} 1 d|\mu|(t) = |\mu|(-s + K') \leq \|\mu\|_{K'}.
\]

Therefore

\[
|\mu * \nu|(K) \leq \int_G \|\mu\|_{K'} d|\nu|(s) + \epsilon = \|\mu\|_{K'}(|\nu|(G)) + \epsilon.
\]

As \( \epsilon > 0 \) was arbitrary, we are done. \qed

We can now prove the desired result.

**Proposition 6.2.** If \( \nu \) is a finite measure and \( \mu \) is a norm almost periodic, then \( \mu * \nu \) is norm almost periodic.

**Proof.** Fix compact set \( K_1 \subset G \) with non-empty interior. Pick \( K_1' \) any compact set such that \( K_1 \subseteq \text{Int}(K_1') \).
Let $\epsilon > 0$. Since $\mu$ is norm almost periodic, the set
\[ P := \{ t \in G \mid \| T_t \mu - \mu \|_{K_1} < \frac{\epsilon}{1 + 2 (|\nu| (G))} \} \]
is relatively dense. Then, for all $t \in P$ and $s \in G$ we have
\[ |T_t \mu * \nu - \mu * \nu| (s + K_1) = |(T_t \mu - \mu) * \nu| (s + K_1) \]
Now, since $s + K_1 \subseteq \text{Int}(s + K'_1)$, by applying Lemma 6.1 for $K = s + K_1$, we get
\[ |T_t \mu * \nu - \mu * \nu| (s + K_1) \leq (|\nu| (G)) \left( \| T_t \mu - \mu \|_{s + K'_1} \right) \]
\[ \leq (|\nu| (G)) \left( \| T_t \mu - \mu \|_{K'_1} \right) < \frac{\epsilon}{2}. \]
Now, taking the supremum over all $s$, we get that for all $t$ in the relatively dense set $P$ we have
\[ \| T_t \mu * \nu - \mu * \nu \|_{K_1} \leq \frac{\epsilon}{2} < \epsilon. \]
□

7. On the norm almost periodicity of $\hat{\gamma}$ for measures $\gamma$ with Meyer set support

We can finally prove the following theorem.

**Theorem 7.1.** Let $\gamma$ be a Fourier transformable measure supported inside a Meyer set. Then $\hat{\gamma}$ is norm almost periodic.

**Proof.** Let $\Lambda$ be any Meyer set containing the support of $\gamma$. By [43, Cor. 5.9.20] there exists a CPS $(G, H, L)$ with $H$ compactly generated, and some compact set $W \subseteq H$ such that
\[ \Lambda \subseteq \text{Int}(W). \]
Since $H$ is compactly generated, by the structure Theorem we have $H \simeq \mathbb{R}^d \times \mathbb{Z}^n \times \mathbb{K}$ for some $d, n$ and compact group $\mathbb{K}$. We can replace then $H$ in the CPS by $\mathbb{R}^d \times \mathbb{Z}^n \times \mathbb{K}$ and chose the Haar measure on $\mathbb{Z}^d$ to be the counting measure, and the Haar measure on $\mathbb{K}$ to be a probability measure.

Next, since $W$ is compact in $\mathbb{R}^d \times \mathbb{Z}^n \times \mathbb{K}$, there exists some compact $W_0 \subseteq \mathbb{R}^d$ and some finite set $F \subseteq \mathbb{Z}^d$ such that
\[ W \subseteq W_0 \times F \times K. \]
It follows that
\[ \Lambda \subseteq \text{Int}(W_0 \times F \times \mathbb{K}). \]
Now, pick some $g \in C_c^{\infty} (\mathbb{R}^d) \cap K_2 (\mathbb{R}^d)$ such that $g(x) = 1 \forall x \in W_0$. 

Let \( h : \mathbb{Z}^d \times \mathbb{K} \) be \( h = 1_F \otimes 1_K \). It is obvious that \( h \in C_c(\mathbb{Z}^d \times \mathbb{K}) \) and that

\[
h = h * (1_0 \otimes 1_K)
\]

which yields \( h \in \mathcal{K}_2(\mathbb{Z}^d \times \mathbb{K}) \).

Then, by [35], the measure \( \omega = \omega_g \otimes h \) is Fourier transformable and

\[
\hat{\omega} = \overline{\omega_g \otimes h} = \text{dens}(\mathcal{L}) \overline{\omega_g \otimes h}.
\]

Moreover, by construction

\[
\omega(\{x\}) = 1 \forall x \in \Lambda.
\]

Then, by Lemma [5.3], there exists a finite measure \( \nu \) such that

\[
\hat{\gamma} = \hat{\omega} \ast \nu.
\]

Let \( \phi = \tilde{g} \) and \( \psi = \tilde{h} \). Then, as \( g \in C^\infty_c(\mathbb{R}^d) \) we have \( \phi \in \mathcal{S}(\mathbb{R}^d) \). Moreover, recalling that \( \hat{Z}^d \) is a compact group and that \( \hat{\mathbb{K}} \) is discrete, we have

\[
\psi = \tilde{h} = 1_F \overline{\otimes 1_K} = 1_F \overline{\otimes 1_0} \in C_c(\mathbb{Z}^d \times \hat{\mathbb{K}}).
\]

Therefore, by Lemma [5.3], the measure \( \hat{\omega} = \omega_{\phi \otimes \psi} \) is norm almost periodic.

As \( \hat{\omega} \) is norm almost periodic, and \( \nu \) is finite, it follows now from Proposition [6.2] that \( \hat{\gamma} \) is norm almost periodic.

\[\Box\]

As an immediate consequence we get:

**Corollary 7.2.** Let \( \gamma \) be a Fourier transformable measure supported inside a Meyer set. Then each of \((\hat{\gamma})_{pp}, (\hat{\gamma})_{ac}\) and \((\hat{\gamma})_{sc}\) is norm almost periodic.

**Proof.** We know by Theorem [4.1] that the generalized Eberlein decomposition

\[
\gamma = \gamma_s + \gamma_0 + \gamma_0s
\]

exists within the class of Fourier transformable measures with Meyer set support. The claim now follows by applying Theorem [7.1] to each component.

\[\Box\]

**Remark 6.** It is also easy to prove that, if a measure \( \mu \) is norm almost periodic, then so is each of \( \mu_{pp}, \mu_{ac}, \mu_{sc} \). This gives an alternate way to deduce Corollary [7.2] from Theorem [7.1].
8. Applications

Let us first summarise all the results we got in the paper.

**Theorem 8.1.** Let $G$ be a second countable LCAG, and let $\gamma$ be a Fourier transformable measure supported inside some Meyer set $\Lambda$. Then,

(i) $\gamma$ admits an unique decomposition into three Fourier transformable measures $\gamma_s, \gamma_{0s}, \gamma_{0a}$ such that

$$\gamma = \gamma_s + \gamma_{0s} + \gamma_{0a}$$

$$\tilde{\gamma}_s = (\tilde{\gamma})_{pp}$$

$$\tilde{\gamma}_{0s} = (\tilde{\gamma})_{sc}$$

$$\tilde{\gamma}_{0a} = (\tilde{\gamma})_{ac}$$

(ii) $\gamma_s, \gamma_{0s}, \gamma_{0a}$ are supported inside a common Meyer set.

(iii) $(\tilde{\gamma})_{pp}, (\tilde{\gamma})_{sc}, (\tilde{\gamma})_{ac}$ are norm almost periodic.

(iv) If $(G, H, L)$ is any CPS and $W \subseteq H$ any compact set such that $\Lambda \subseteq \Lambda(W)$, then

$$\text{supp}(\gamma_s), \text{supp}(\gamma_{0a}), \text{supp}(\gamma_{0s}) \subseteq \Lambda(W).$$

Next, we list some applications to the diffraction.

**Corollary 8.2.** Let $\omega$ be weighted Dirac comb supported inside a Meyer set, and let $\gamma$ be an autocorrelation of $\omega$. Then each of the diffraction spectral measures $(\tilde{\gamma})_{pp}, (\tilde{\gamma})_{ac}$ and $(\tilde{\gamma})_{sc}$ is norm almost periodic.
Corollary 8.3. Let $\omega$ be weighted Dirac comb supported inside a Meyer set, and let $\gamma$ be an autocorrelation of $\omega$. Then each of diffraction spectral measure $(\hat{\gamma})_{pp}$, $(\hat{\gamma})_{ac}$ and $(\hat{\gamma})_{sc}$ is either trivial or has relatively dense support.

Remark 7. In [40] we have proven that given a Meyer set $\Lambda$, each of $(\hat{\gamma})_{pp}$, $(\hat{\gamma})_{ac}$ of the diffraction spectra of $\Lambda$ is strong almost periodic and hence, it is either trivial or has relatively dense support. These results are an immediate consequence of Corollary 8.2 and Corollary 8.3.

Remark 8. In [41] we have proven that given a be weighted Dirac comb supported inside a Meyer set, its pure point spectrum $(\hat{\gamma})_{pp}$ is a sup-almost periodic measure (see [43, Sect. 5.3] for the definition and properties of sup-almost periodicity) and that, for each $0 < a < \hat{\gamma}({\{0\}})$ the set

$$I(a) := \{ \chi \in \hat{\mathcal{G}} | \hat{\gamma}({\{\chi\}}) \geq a \}$$

is a Meyer set.

Both of these are immediate consequences of Corollary 8.2 and some results of [43]. Indeed, since $(\hat{\gamma})_{pp}$ is norm-almost periodic, it is sup-almost periodic by [43, Lemma 5.3.6]. Next, since $(\hat{\gamma})_{pp}$ is a sup-almost periodic pure point measure, by [43, Thm. 5.2] there exists a CPS $(\hat{G}, \hat{H}, \mathcal{L})$ and some $h \in C_0(\hat{H})$ such that

$$(\hat{\gamma})_{pp} = \omega_h.$$ 

Since $\gamma$ is positive definite, $\hat{\gamma}$ and hence $(\hat{\gamma})_{pp}$ is positive. Then $h$ is also positive, by the denseness of the second projection.

Therefore

$$I(a) = \wedge (W_a),$$

where $W_a := \{ y \in \hat{H} | h(y) \geq a \}$ is compact and has non-empty interior, as $h \in C_0(\hat{H})$ and $h \neq 0$.

It follows that for all $0 < a < \hat{\gamma}({\{0\}})$ the sets $I(a)$ are model sets in the same CPS.

Corollary 8.4. Let $\omega$ be weighted Dirac comb supported inside a Meyer set, and let $\gamma$ be an autocorrelation of $\omega$. Then, for each pre-compact Borel set $B \subseteq \hat{G}$ there exists a relatively dense set $P \subseteq \hat{G}$ such that, for all $t \in P$ we
have
\[ |\hat{\gamma}(t + B) - \hat{\gamma}(B)| < \epsilon \]
\[ |(\hat{\gamma})_{pp}(t + B) - (\hat{\gamma})_{pp}(B)| < \epsilon \]
\[ |(\hat{\gamma})_{ac}(t + B) - (\hat{\gamma})_{ac}(B)| < \epsilon \]
\[ |(\hat{\gamma})_{sc}(t + B) - (\hat{\gamma})_{sc}(B)| < \epsilon \]

Proof. Let \( K \) be any compact set with non-empty interior such that \( B \subseteq K \).
Let us first observe that if \( \mu \) is a norm almost periodic measure, and \( \epsilon > 0 \), then the set of common almost periods of \( \mu, \mu_{pp}, \mu_{ac} \) and \( \mu_{sc} \) is relatively dense.
Indeed, by [15, Thm. 14.22], we have
\[ |\mu| = |\mu_{pp}| + |\mu_{ac}| + |\mu_{sc}|. \]
It follows immediately from here that, for each \( \alpha \in \{pp, ac, sc\} \) we have
\[ \|\mu_\alpha\|_K \leq \|\mu\|_K \leq \|\mu_{pp}\|_K + \|\mu_{ac}\|_K + \|\mu_{sc}\|_K. \]
This implies that, for each \( \alpha \in \{pp, ac, sc\} \) we have
\[ P^K_\epsilon(\mu) \subseteq P^K_\epsilon(\mu_\alpha). \]
Thus, as \( \mu \) is norm almost periodic, the relatively dense set \( P^K_\epsilon(\mu) \) is a common set of \( \epsilon \) norm-almost periods for \( \mu, \mu_{pp}, \mu_{ac} \) and \( \mu_{sc} \).

Now, for each \( -t \) in this relatively dense set we have
\[ |\hat{\gamma}(t + B) - \hat{\gamma}(B)| \leq |T_{-t}\hat{\gamma} - \hat{\gamma}|(B) \leq |T_{-t}\hat{\gamma} - \hat{\gamma}|(K) \leq \|T_{-t}\hat{\gamma} - \hat{\gamma}\|_K < \epsilon, \]
and similarly,
\[ |(\hat{\gamma})_\alpha(t + B) - (\hat{\gamma})_\alpha(B)| \leq \|T_{-t}(\hat{\gamma})_\alpha - (\hat{\gamma})_\alpha\|_K < \epsilon \quad \forall \alpha \in \{pp, ac, sc\}. \]

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References

[1] L. N Argabright, J. Gil de Lamadrid, *Fourier analysis of unbounded measures on locally compact abelian groups*, Memoirs Amer. Math. Soc., Vol 145, 1974.

[2] J.-B. Aujogue, *Pure Point/Continuous Decomposition of Translation-Bounded Measures and Diffraction*, preprint, 2015. arXiv:1510.06381.

[3] M. Baake, *Diffraction for weighted lattice subsets*, Can. Math. Bulletin, vol 45 (4), 483–498, 2002. arXiv:math/0106111.

[4] M. Baake, F. Gähler, *Pair correlations of aperiodic inflation rules via renormalisation: Some interesting examples*, Topol. Appl. 205, 4–27, 2016. arXiv:1511.00885.

[5] M. Baake, U. Grimm, *Kinematic Diffraction from a Mathematical Viewpoint*, Z. Krist. 226, 711–725, 2011. arXiv:1105.0095.

[6] M. Baake, U. Grimm, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge University Press, Cambridge, 2013.

[7] M. Baake, U. Grimm, *Aperiodic Order. Vol. 2: Crystallography and Almost Periodicity*, Cambridge University Press, Cambridge, 2017.

[8] M. Baake, U. Grimm, *Squirals and beyond: Substitution tilings with singular continuous spectrum*, Ergod. Th. & Dynam. Sys. 34, 1077–1102, 2014. arXiv:1205.1384

[9] M. Baake, C. Huck, N. Strungaru, *On weak model sets of extremal density*, Indag. Math. 28(1), 3–31, 2017. arXiv:1512.07129v2

[10] M. Baake, D. Lenz, *Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra*, Ergod. Th. & Dynam. Syst. 24, 1867–1893, 2004. arXiv:math.DS/0302231.

[11] M. Baake, D. Lenz, *Deformation of Delone dynamical systems and pure point diffraction*, J. Fourier Anal. Appl., vol 11 (2), 125–150, 2005. arXiv:math/0404155.

[12] M. Baake, R.V. Moody, *Weighted Dirac combs with pure point diffraction*, J. reine angew. Math. (Crelle) 573, 61–94, 2004. arXiv:math.MG/0203030.

[13] C. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin, 1975.

[14] I. Blech, J. W. Cahn, D. Gratias, *Reminiscences About a Chemistry Nobel Prize Won with Metallurgy: Comments on D. Shechtman and I. A. Blech ; Metall. Trans. A , 1985, vol. 16A, pp. 100512, Metallurgical and Materials Transactions A 43(10), 3411-3422, 2012.

[15] E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis*, Springer, Berlin, 1963.

[16] A. Hof, *Uniform distribution and the projection method*. In: *Quasicrystals and Discrete Geometry*, ed. J. Patera, Fields Institute Monographs 10, AMS, Providence, RI, pp. 201–206, 1988.

[17] A. Hof, *On diffraction by aperiodic structures*, Commun. Math. Phys. 169, 25–43, 1995.

[18] G. Keller, C. Richard, *Dynamics on the graph of the torus parametrisation*, Ergod. Th. & Dynam. Syst., 38, 1048–1085, 2018. arXiv:1511.06137.

[19] G. Keller, C. Richard, *Periods and factors of weak model sets*, preprint, to appear in Israel J. Mathe., 2017. arXiv:1702.02383.

[20] J. Lagarias, *Meyer’s concept of quasicrystal and quasiregular sets*, Commun. Math. Phys. 179, 365–376, 1996.

[21] J. C. Lagarias, *Mathematical quasicrystals and the problem of diffraction*. In: *Directions in Mathematical Quasicrystals* eds. M. Baake and R.V Moody , CRM Monograph Series, Vol 13, AMS, Providence, RI, pp. 61–93, 2000.
[22] J. Gil. de Lamadrid, L. N Argabright, *Almost Periodic Measures*, Memoirs Amer. Math. Soc., Vol 85, No. 428, 1990.

[23] D. Lenz, *Aperiodic order and pure point diffraction*, Philos. Mag. 88, 2059–2071, 2008. (Special Issue: Quasicrystals: the silver jubilee)

[24] D. Lenz, C. Richard, *Pure Point Diffraction and Cut and Project Schemes for Measures: the Smooth Case*, Math. Z. 256, 347–378, 2007. math.DS/0603453.

[25] D. Lenz, N. Strungaru, *Pure point spectrum for measurable dynamical systems on locally compact Abelian groups*, J. Math. Pures Appl. 92, 323–341, 2009. arXiv:0704.2498.

[26] D. Lenz, N. Strungaru, *On weakly almost periodic measures*, preprint, to appear in Trans. Amer. Math. Soc., 2017. arXiv:1609.08219.

[27] N. Lev, A. Olevskii, *Quasicrystals and Poisson’s summation formula*, Invent. Math. 200, 585–606, 2015. arXiv:1312.6884

[28] N. Lev, A. Olevskii, *Fourier quasicrystals and discreteness of the diffraction spectrum*, Advances in Mathematics 315, 1-26, 2017. arXiv:1312.6884

[29] Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland, Amsterdam, 1972.

[30] Y. Meyer, *Measures with locally finite support and spectrum*, PNAS , 113 (12), 3152–3158, 2016.

[31] R. V. Moody, *Meyer sets and their duals*. In: *The mathematics of long-range aperiodic order*, ed. R. V. Moody, NATO ASI Series , Vol C489, Kluwer, Dordrecht, pp. 403–441, 1997.

[32] R. V. Moody, *Model sets: A Survey*. In: *From Quasicrystals to More Complex Systems*, eds. F. Axel, F. Dénoyer and J. P. Gazeau, EDP Sciences, Les Ulis, and Springer, Berlin , pp. 145–166, 2000. arXiv:math.MG/0002020.

[33] R. V. Moody, N. Strungaru, *Point sets and dynamical systems in the autocorrelation topology*, Canad. Math. Bull. Vol. 47 (1), 82-99, 2004.

[34] R.V. Moody, N. Strungaru, *Almost Periodic Measures and their Fourier Transforms*. In: *Aperiodic Order. Vol. 2. Crystallography and Almost Periodicity*, eds. M. Baake and U. Grimm, Cambridge University Press, Cambridge, pp. 173–270, 2017.

[35] C. Richard, N. Strungaru, *Pure point diffraction and Poisson Summation*, Ann. H. Poincaré 18, 3903-3931, 2017. arXiv:1512.00912.

[36] C. Richard, N. Strungaru, *A short guide to pure point diffraction in cut-and-project sets*, J. Phys. A: Math. Theor. 50, no 15, 2017. arXiv:1606.08831.

[37] C. Richard, N. Strungaru, *On the Fourier transformability of measures supported inside lattices*, in preparation.

[38] M. Schlottmann, *Generalized model sets and dynamical systems*. In: *Directions in mathematical quasicrystals*, eds. M. Baake, R.V. Moody, CRM Monogr. Ser., AMS, Providence, RI, pp. 143–159, 2000.

[39] D. Shechtman, I. Blech, D. Gratias, J. W. Cahn, *Metallic phase with long-range orientational order and no translation symmetry*, Phys. Rev. Lett. 53, 183–185, 1984.

[40] N. Strungaru, *Almost periodic measures and long-range order in Meyer sets*, Discr. Comput. Geom. vol. 33(3), 483–505, 2005.

[41] N. Strungaru, *On the Bragg Diffraction Spectra of a Meyer Set*, Can. J. Math. 65, no. 3, 675–701, 2013. arXiv:1003.3019.

[42] N. Strungaru, *On Weighted Dirac Combs Supported Inside Model Sets*, J. Phys. A: Math. Theor. 47, 2014. arXiv:1309.7947.
[43] N. Strungaru, *Almost Periodic Pure Point Measures*. In: *Aperiodic Order. Vol. 2. Crystallography and Almost Periodicity*, eds. M. Baake and U. Grimm, Cambridge University Press, Cambridge, pp. 271–342, 2017. arXiv:1501.00945.

[44] N. Strungaru, V. Terauds, *Diffraction theory and almost periodic distributions*, J. Stat. Phys. 164, no. 5, 1183–1216, 2016. arXiv:1603.04796.

Department of Mathematical Sciences, MacEwan University, 10700 – 104 Avenue, Edmonton, AB, T5J 4S2, Phone: 780-633-3440, and, Institute of Mathematics “Simon Stoilow”, Bucharest, Romania

E-mail address: strungarun@macewan.ca

URL: http://academic.macewan.ca/strungarun/