Non-commutative Euclidean structures in compact spaces

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Abstract

Based on results for real deformation parameter $q$ we introduce a compact non-commutative structure covariant under the quantum group $SO_q(3)$ for $q$ being a root of unity. To match the algebra of the $q$-deformed operators with necessary conjugation properties it is helpful to define a module over the algebra generated by the powers of $q$. In a representation where $X^2$ is diagonal we show how $P^2$ can be calculated. To manifest some typical properties an example of a one-dimensional $q$-deformed Heisenberg algebra is also considered and compared with non-compact case.

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1 Introduction

In paper [1] it was shown, how the $q$-deformation of the well-known group $SO(3)$ to quantum group $SO_q(3)$ can be used to define a non-commutative quantum space as a comodule of the quantum group. It is very natural to exploit the $R$ matrix as the main tool. Its decomposition into projectors generates a non-commutative (three-dimensional) Euclidean space of coordinates.

In all papers known to us the non-commutative structure has been defined for real $q$ only. The value of $q$ becomes important when we demand hermiticity for coordinates (and later on for momenta). For general complex $q$ the $R$ matrix looses its hermiticity which requires a new definition of conjugation for the coordinate operators. On the other hand their are at least two reasons why one should investigate the case of complex $q$. 
First, real \( q \) implies always a non-compact coordinate space, while for a compact space we have to admit complex values of \( q \). In context with the fact, that non-commutative geometry [2] is considered to be the result of some deep dynamical principle which may be found e.g. in string theory the case of compactified dimensions is of special interest. We start here the consideration of an example with only compactified coordinates. The more interesting case with compact and non-compact dimensions (which seems to require different \( q \)) is due to further work. Second, we know the quantum group \( SO_q(3) \) for generic \( q \) and especially the case \( q \) being a root of unity, where it demonstrates some peculiarities [3,4]. It is therefore interesting how a non-commutative quantum space can be constructed in that special case. This will be the main aim of our paper.

As we have already mentioned, the key point is the definition of a conjugation for coordinates and momenta, which are later required to be self-adjoint with respect to that conjugation. Different conjugations result in different spaces and hence different physics. The conjugation we will propose below is of course equivalent to ordinary conjugation for real \( q \). We know two ways which are both consistent with \( SO_q(3) \). The choice that fits best with our problem is the one, where \( q \) is left untouched during conjugation. Thus if \( \bar{X} \) is the conjugate of an operator \( X \), the conjugate of \( qX \) is \( q\bar{X} \). This choice has been used already before, e.g. in [3]. To do this in a mathematical correct way we define a right module over the algebra generated by all powers of \( q \) with the additional condition for some power to equal \( -1 \) (see next Chapter). The other way, one may find i.e. in [4], seems to work better in case if one deals with non-hermitean operators having only real eigenvalues, which will not be the case here.

At the first moment our definition looks rather unnatural but in Chap. 2 we shall describe how it works and mention the consequences. The most important one of them is that self-adjoint operators will have (instead of real ones) eigenvalues which are real functions of the parameter \( q \). But this is just what we need, because the scaling operator and its commutation properties force coordinates and momenta to have eigenvalues proportional to powers of \( q \).

The paper is organized as follows. In Chap. 2 we recall the basic formulae for the quantum space of \( SO_q(3) \) and state the modifications for our \( q \). In Chap. 3 we consider a one-dimensional example of a \( q \)-deformed Heisenberg algebra and demonstrate how it works for \( q \) being a root of unity. It is rather helpful to compare our results with earlier ones for real \( q \) with the same example. In our main Chap. 4 the non-commutative space covariant under \( SO_q(3) \) is considered and matrix elements of coordinates and momenta are calculated. The results are presented explicitly and do not contain any divergencies which usually occur if one simply replaces \( q \) in formulae derived earlier for real \( q \) only.
2 Euclidean phase space for $q$ being a root of unity

First we have to recall some basic formulae of the non-commutative space from paper [1] which do not depend on the nature of $q$. The $R$ matrix of $SO_q(3)$ is decomposed like

$$\hat{R} = P_5 - \frac{1}{q^4} P_3 + \frac{1}{q^6} P_1$$

(2.1)

We shall not give the projectors $P_i$ here, because we need only $P_3$. The non-commutative Euclidean space is defined by:

$$P_3XX = 0$$

(2.2)

In the common basis (2.2) looks like:

$$X^3X^+ = q^2X^+X^3$$
$$X^3X^- = q^{-2}X^-X^3$$
$$X^-X^+ = X^+X^- + \lambda X^3X^3$$

(2.3)

here $\lambda = q - \frac{1}{q}$. It is natural to define a metric $g_{AB}$ and an invariant product $X \circ Y$

$$X \circ Y = g_{AB}X^AY^B$$

(2.4)

$$g_{++} = -q, \quad g_{--} = -1/q, \quad g_{33} = 1$$

which let $X \circ X$ commute with $X^A$. $P_3$ can be expressed through a generalized $\epsilon$-tensor

$$P_3^{AB}CD = \frac{1}{1 + q^4} \epsilon^{FAB} \epsilon_{FDC}$$

(2.5)

where its indices are moved according to formulae like

$$\epsilon_{ABC} = g_{CD} \epsilon_{AB}^D$$

(2.6)

$$\epsilon_{+-}^3 = q, \quad \epsilon_{-+}^3 = -q, \quad \epsilon_{33}^3 = 1 - q^2, \quad \epsilon_{+3}^+ = 1, \quad \epsilon_{3+}^+ = -q^2, \quad \epsilon_{-3}^- = -q^2, \quad \epsilon_{3-}^- = 1$$

(2.7)

Eq. (2.3) is then equivalent to

$$X^C X^B \epsilon_{BC}^A = 0$$

(2.8)

and the $R$ matrix can be expressed in the form

$$\hat{R}^{AB}_{CD} = \delta^A_C \delta^B_D - q^{-4} \epsilon^{FAB} \epsilon_{FDC} - q^{-4}(q^2 - 1) g^{AB} g_{CD}$$

(2.9)

Now we come to the definition of conjugation. We still choose

$$\overline{X}^A = g_{AB}X^B \equiv X_A$$

(2.10)
like in paper [1]. But for generic complex \( q \) this is consistent with eqns. (2.3) only if we define \( \bar{q} = q \) which means \( q \) is unchanged under conjugation. This forces us to distinguish between \( q \) (and its functions) and constant complex numbers which are to be conjugated as usual. (We mean e. g. the \( i \) in the Heisenberg relation, s. b.)

That is done best if the vector space the \( q \)-deformed operators act on is considered as a (right) module over an algebra \( A \). This associative (and commutative) algebra \( A \) over the complex numbers is generated by the powers of \( q : q, q^2, \ldots q^{r-1} \) and the condition \( q^r = -1 \). The integer \( r \) is taken larger than 2. Within \( A \) we define an involution * which fulfills the usual conditions

\[
\begin{align*}
a** &= a, (ab)^* = b^*a^* \\
(\alpha a + \beta b)^* &= \bar{\alpha}a^* + \bar{\beta}b^*
\end{align*}
\]  

(2.11)

where \( \alpha, \beta \in C \). Those properties are consistent with the choice \( q^* = q \) which determines the involution for all elements.

As a next step we consider a right module \( M \) over the algebra \( A \). (Since \( A \) is commutative an equivalent approach is given considering a left module.) \( M \) is a complex vector space. For any \( a, b \in A \) and \( \eta, \xi \in M \) we have

\[
\begin{align*}
\eta(ab) &= (\eta a)b \\
\eta(a + b) &= \eta a + \eta b \\
(\eta + \xi) &= \eta a + \xi a
\end{align*}
\]  

(2.12)

and any combination of type \( \eta a \) is again an element of \( M \). For further application we need a hermitean structure which is created by a hermitean inner product. For any pair of elements a bilinear map \( < \eta | \xi > \in A \) is defined with the properties

\[
\begin{align*}
< \eta | \xi >^* &= < \xi | \eta > \\
< \eta a | \xi b > &= a^* < \eta | \xi > b
\end{align*}
\]  

(2.13)

A third property, usually required for a hermitean product, includes the absence of zero norm states. We shall see below that such states cannot be excluded for our choice of \( q \). Therefore, strictly speaking, our structure is not hermitean in the usual sense. Nevertheless we keep this terminology but remember that all unusual properties are connected with the existence of zero norm states. The product allows the definition of the hermitean conjugation \( \overline{O} \) of an operator \( O \)

\[
< \eta | O\xi > = < \overline{O}\eta | \xi >
\]  

(2.14)

We use another symbol not to mix this with the involution in \( A \). Subsequently hermitean and unitary (isometric) operators are defined. Operators in \( M \) can be viewed as matrices with entrances from \( A \), hermitean conjugation is then transposition together with the
involution in $A$ defined above. It is then clear that if $\lambda$ is an eigenvalue of $O$ then $\lambda^*$ is an eigenvalue of $\overline{O}$ and hence $\lambda^* = \lambda$ for all eigenvalues of an operator with $O = \overline{O}$. We shall see below that for our $q$ and the operators we are considering it is not necessary to distinguish between hermitean and self-adjoint operators. Their eigenvalues are real functions of $q$. One can show directly that the eigenvectors are orthogonal (under the product defined above) for different eigenvalues with the usual arguments. If a unitary operator has an eigenstate $\xi$ with eigenvalue $\lambda$ one can easily show

$$< \xi | \xi > = \lambda^* \lambda < \xi | \xi >$$

(2.15)

which gives information about $\lambda$ only for states with non-vanishing norm. This fact becomes important below.

Based on eq (2.10) we can now proceed as in [1] and define a derivative, momentum, angular momentum and the scaling operator $\Lambda$ in the same way. For the components of the momentum we have the analog of (2.8), while for angular momentum

$$L^C L^B \epsilon_{BC}^A = -1/q^2 WL^A$$

(2.16)

and

$$q^4 (q^2 - 1)^2 L \circ L = W^2 - 1$$

$$L^A W = WL^A$$

(2.17)

The scaling operator $\Lambda$ is introduced in the same way with the properties

$$\Lambda^{1/2} X^A = q^2 X^A \Lambda^{1/2}$$

$$\Lambda^{1/2} P^A = q^{-2} P^A \Lambda^{1/2}$$

$$\Lambda^{1/2} L^A = L^A \Lambda^{1/2}$$

$$\Lambda^{1/2} W = W \Lambda^{1/2}$$

(2.18)

Conjugation of vector values is analogous to eq. (2.10), $W$ is self-adjoint and $\Lambda$ is unitary up to normalization:

$$\overline{\Lambda^{1/2}} = q^{-6} \Lambda^{-1/2}$$

(2.19)

Eqns. (2.16) lead to the standard $SO_q(3)$ algebra. The generalized Heisenberg relations are

$$P^A X^B - \hat{R}^{-1} AB_{CD} X^C P^D = -\frac{i}{2} \Lambda^{-1/2} \{(1 + q^{-6}) g^{ABW} - (1 - q^{-4}) \epsilon^{ABC} L_C \}$$

(2.20)

Now we have to study representations of this algebra. For $q$ being a root of unity the physical relevant representations become finite dimensional while for real $q$ they have infinite dimension. Thus there is no difference here between self-adjoint, essentially self-adjoint and hermitean operators.

The representations will be studied in detail in Chap. 4.
3 Representations of a one-dimensional $q$-deformed Heisenberg algebra

We consider now a one-dimensional example of a $q$-deformed Heisenberg algebra. That is neither a projection of the Euclidean space nor based on the deformation of any symmetry group. It is even not non-commutative in the sense of space coordinates because there is only one. Nevertheless it is based on a modified Leibniz rule and has been studied for real $q$ in great detail [5,6]. It reflects very nicely the deep role which is played by the scaling operator $\Lambda$ that one has to introduce in a general non-commutative structure of coordinates and momenta. The algebra looks as follows:

$$\frac{1}{\sqrt{q}}P X - \sqrt{q}XP = -iU$$

$$UP = qPU$$

$$UX = \frac{1}{q}XU$$

(3.1)

Conjugation is given by

$$\bar{P} = P$$

$$\bar{X} = X$$

$$\bar{U} = U^{-1}$$

(3.2)

While there is obviously no problem for real $q$, with our definition of conjugation for operators and involution of algebra elements eq. (3.2) is also consistent with (3.1). To give meaning to operators in our module space we have to enlarge our algebra $A$ to include real functions of $q$ in a straightforward way. We shall consider a representation of the algebra (3.1) based on eigenvectors of $P$. From the second equation it follows that applying $U$ to such an eigenstate we obtain another one with eigenvalue multiplied by $q^{-1}$. Therefore we have

$$P | n >^{\pi_0} = \pi_0 q^n | n >^{\pi_0}$$

(3.3)

where $n$ is integer, $0 \leq n \leq 2r - 1$, and $\pi_0$ is an arbitrary real function of $q$. Further

$$U | n >^{\pi_0} = | n - 1 >^{\pi_0}$$

(3.4)

and according to what was stated in last Chapter

$$\pi_0 < n | m >^{\pi_0} = \delta_{nm}$$

(3.5)

Now we have an example that the self-adjoint operator $P$ has eigenvalues being real functions of $q$. The powers occurring are a consequence of the properties of $U$. For our $q$ chosen we can see that the eigenstate $U | 0 >^{\pi_0}$ has the same eigenvalue as $| 2r - 1 >^{\pi_0}$. Disregarding the case of degeneration we have

$$U | 0 >^{\pi_0} = C(\pi_0) | 2r - 1 >^{\pi_0}$$

(3.6)
where $C$ is a phase factor and different $C$ label different representations. From eqns. (3.4) and (3.6) we have $U^{2r} = C$ for any state in our representation. Now it is straightforward to define another unitary operator $U'$ by

$$U' = U e^{-\frac{i\pi}{2}}$$

(3.7)

where we have put $C = e^{in}$ Then $U^{2r} = 1$ and it is more convenient to work with a new system $|n\rangle'$. From the first equation of (3.1) and its conjugate one can deduce

$$XP = \frac{i}{\lambda} (\sqrt{qU} - \frac{1}{\sqrt{q}} U^{-1})$$

(3.9)

which shows the action of $X$ on the states $|n\rangle'$ states:

$$X |n\rangle' = \frac{i}{q^{|n|\pi_0}} (\sqrt{qe^{\frac{i\alpha}{2}}} |n-1\rangle' - \frac{1}{\sqrt{q}} e^{-\frac{i\alpha}{2r}} |n+1\rangle')$$

(3.10)

This system of $2r$ equations can be solved in principle and the eigenvalues und eigenstates of $X$ can be found. But it is easier to exploit the eigenstates of $U$, as we shall demonstrate below. We start with

$$|\phi_0\rangle = \sum_{n=0}^{2r-1} |n\rangle'$$

(3.11)

$$|\phi_k\rangle = (\pi_0)^{-k} P_k |\phi_0\rangle = \sum_{n=0}^{2r-1} q^{kn} |n\rangle'$$

and integer $0 \leq k \leq 2r - 1$. Obviously

$$U' |\phi_k\rangle = q^k |\phi_k\rangle$$

(3.12)

We mention that for real $q$ those states are non-normalizable which is not the case here.

Before constructing the eigenstates of $X$ we shortly comment on the eigenstates of $U'$ and $U$. Our definition of an adjoint operator in Chap. 2 and the inner product lead to unitary operators with respect to that product which will have properties differing from the usual ones, as we have already seen for self-adjoint operators. The eigenstates of our unitary operators may not be orthogonal and can contain zero norm states. So explicitly

$$<\phi_k | \phi_m> = \sum_{n=0}^{2r-1} q^{n(k+m)}$$

(3.13)

what is non-zero for $m = k = 0$ or $m + k = 2r$. Hence we have only two non-zero norm states for $k = 0$ and $r$ and the eigenstates $|\phi_k\rangle$ and $|\phi_{2r-k}\rangle$ for $k = 1, \ldots, r - 1$ are not orthogonal.
Keeping in mind all that we can still work with the states (3.11) as a basis to construct the $X$ eigenstates.

From the algebra (3.1) follows

$$X | \phi_k > = d_k | \phi_{k-1} >$$

(3.14)

for $1 \leq k \leq 2r - 1$ and

$$X | \phi_0 > = d_0 | \phi_{2r-1} >$$

(3.15)

Next we have to calculate $d_k$. We apply the conjugate of eq. (3.9) to $| \phi_k >$ and find

$$d_k = \frac{i}{\lambda \pi_0} (e^{\frac{\alpha}{2r} q^{k-\frac{1}{2}}} - e^{-\frac{\alpha}{2r} q^{-k+\frac{1}{2}}})$$

(3.16)

This formula works for all $0 \leq k \leq 2r - 1$. We construct the eigenstates the following way

$$X | x_m > = x_m | x_m >$$

$$| x_m > = \sum_{k=0}^{2r-1} a_k | \phi_k >$$

(3.17)

yielding the recursion relation for the coefficients

$$a_{k+1} = \frac{x_m}{d_{k+1}} d_k a_k$$

(3.18)

Consistency requires

$$a_0 = \frac{x_m}{d_0} a_{2r-1}$$

(3.19)

We can put $a_0 = 1$ and the solution of eqs. (3.18) and (3.19) are

$$a_k = (x_m)^k \prod_{l=1}^{2r} d_l^{-1}$$

$$(x_m)^{2r} = \prod_{l=1}^{2r} d_l = \frac{i^{2r}}{\lambda^{2r} \pi_0^{2r}} (-1)^r f^2(q, \alpha)$$

(3.20)

where we have introduced the function

$$f(q, \alpha) = \prod_{k=1}^{r} (q^{k+\frac{1}{2}} e^{\frac{\alpha}{2r}} - q^{-k-\frac{1}{2}} e^{-\frac{\alpha}{2r}})$$

(3.21)

Eq. (3.20) gives (in principle) the possibility to find the eigenvalues of $X$. They depend on $\pi_0$ and the real function $f^2(q, \alpha)$. The fact, that only $(x_m)^{2r}$ is given, reflects the property that due to the unitary equivalence of $X$ and $P x_m$ must be proportional to $q^m$. Thus eq. (3.20) contains no new information we did not have before. The function $f(q, \alpha)$ occurs also in the more realistic three-dimensional case (c. next Chapter).

Now we can compare our results with those for real $q$ obtained in papers [5] and [6]. The main difference is that all our representations have finite dimensions which avoids the mathematical problems of the real case. On the other hand we have to introduce an additional parameter $C$ (or $\alpha$) characterizing the representation. The operators $X$ and $P$ are manifestly equivalent in our representation.
4 $SO_q(3)$ deformation in compact space

In this Chapter we give the representations of the $q$-deformed algebra (2.8), (2.16) - (2.20) for $q' = -1$. We have not written the $L^A X^B$ and $L^A P^B$ relations which are the same as in [1]. We are also not going to repeat the derivations of papers [1] and [7] leading to the $T$-operators and explaining the appearance of the Clebsch-Gordon coefficients because on the algebraic level there are no changes. The changes start as soon as representations are considered, what shall be done now.

We choose $L \circ L$, $L^3$ and $X \circ X$ as a complete set of commuting variables. One can proceed as in the undeformed case and exploit eqs. (2.16) and (2.17). For the angular momentum the eigenvalues are

\[
L \circ L \mid j, m, n > = \frac{q^{-6}}{(q^2 - q^{-2})^2} (q^{4j+2} + q^{-4j-2} - q^2 - q^{-2}) \mid j, m, n >
\]

\[
L^3 \mid j, m, n > = -\frac{q^{-3}}{(q - q^{-1})} (q^{2m} - \frac{q^{2j+1} + q^{-2j-1}}{q + q^{-1}}) \mid j, m, n >
\]

where $j$ and $m$ are integers, $|m| \leq j$ and $0 \leq j \leq j_{max}$. (Note that the sign of $L^3$ is opposite to the usual one, because we have kept the conventions of paper [1].) For $q$ being a root of unity we must remember that there are two types of representations, called types I and II in paper [3]. We allow only type II representations for the construction of the non-commutative space. That the type I representations can be omitted consistently follows from paper [4]. The type II representations behave as for $q = 1$ (and general real $q$) except the fact $j_{max} \leq \frac{r}{2} - 1$. The states are fully determined by the quantum numbers $j$, $m$ and $n$. From the first eq. of (2.13) we read off

\[
X^2 \mid j, m, n > = l^2_0 q^{4n} \mid j, m, n >
\]

It is sufficient to choose the integer $n$ as $0 \leq n \leq r - 1$. The parameter $l_0$ plays the same role as $\pi_0$ in the one-dimensional case.

All our representations are unitary and either irreducible or fully reducible [3]. Irreducible representations are labelled by the integer $j$. Because of eq. (4.2) we deal with finite dimensional irreducible representations like in the one-dimensional case before. That and the existence of a $j_{max}$ are the main differences with respect to real $q$.

The states are normalized in the usual way. The phase factors can be choosen to fulfill

\[
\Lambda^\frac{1}{2} \mid j, m, n > = q^{-3} \mid j, m, n - 1 >
\]

\[
\Lambda^{-\frac{1}{2}} \mid j, m, n > = q^3 \mid j, m, n + 1 >
\]

From eq. (2.16) the matrix elements of $L^\pm$ can be obtained. We mention for further use

\[
W \mid j, m, n > = \frac{\{2j + 1\}}{\{1\}} \mid j, m, n >
\]
where we have introduced the abbreviations
\[
\{ a \} = q^a + q^{-a} \\
[a] = \frac{q^a - q^{-a}}{\lambda}
\]  
(4.5)

In papers [1] and [7] one finds how the $SO_q(3)$ structure can be used to define reduced matrix elements for $X^A$ and $P^A$. For the non-vanishing matrix elements we quote the results

\[
\begin{align*}
< j + 1, m + 1, n | X^+ | j, m, n > &= q^{m-2j} \sqrt{[j+m+1][j+m+2]} < j + 1, n || X^- || j, n > \\
< j - 1, m + 1, n | X^+ | j, m, n > &= q^{m+2j} \sqrt{[j-m][j-m-1]} < j - 1, n || X^- || j, n > \\
< j + 1, m - 1, n | X^- | j, m, n > &= q^m \sqrt{[j+m+1][j+m+2]} < j + 1, n || X^- || j, n > \\
< j - 1, m - 1, n | X^- | j, m, n > &= q^m \sqrt{[j+m][j+m-1]} < j - 1, n || X^- || j, n > \\
< j + 1, m, n | X^3 | j, m, n > &= q^{m-j+5} \sqrt{1+q^2} \sqrt{[j+m+1][j+m+1]} < j + 1, n || X^- || j, n > \\
< j - 1, m, n | X^3 | j, m, n > &= -q^{m+j+5} \sqrt{1+q^2} \sqrt{[j-m][j+m]} < j - 1, n || X^- || j, n >
\end{align*}
\]  
(4.6)

The matrix elements on the r.h.s. are the reduced ones. Using conjugation properties (2.10) we have

\[
< j + 1, n || X^- || j, n > = -q^{2j+2} < j, n || X^- || j + 1, n >
\]  
(4.7)

Therefore only one reduced matrix element has to be determined what is easily obtained from the first eq. of (2.3) and (4.2). We fix the phase by setting

\[
< j + 1, n || X^- || j, n > = \frac{\lambda_0 q^{j+2n}}{\sqrt{[2][j+1][j+3]}}
\]  
(4.8)

By the way, the first eq. of (2.3) also tells us that $< j, n || X^- || j, n >$ must vanish.

Now we come to the matrix elements of $P^A$. Based on eqs. (4.6) and (4.7) they are calculable relying on the matrix elements of the values $X \circ P$ and its conjugate $P \circ X$.

The Heisenberg relation (2.20) with the help of the $R$ matrix (2.9) yields after contraction

\[
P \circ X - q^6 X \circ P = -\frac{i}{2} \lambda^{-\frac{3}{2}}(1 + q^{-6})(q^2 + 1 + q^{-2})W
\]  
(4.9)

Together with its conjugation eq. (4.9) gives

\[
X \circ P = -\frac{i}{2} \frac{(\Lambda^\frac{1}{2} - \Lambda^{-\frac{1}{2}})W}{q^2(q^2 - 1)}
\]
\[
P \circ X = \frac{i}{2} \frac{(q^{-6} \Lambda^{-\frac{1}{2}} - q^6 \Lambda^\frac{1}{2})W}{q^2(q^2 - 1)}
\]  
(4.10)
Therefore $X \circ P$ has matrix elements only between neighbouring $n$. We consider now

$$< j, m, n | X \circ P | j, m, n+1 > = -q^2 [2j+3] [2j+2]$$

$$< j, n \parallel X^- \parallel j+1, n > < j+1, n \parallel P^- \parallel j, n+1 > + [2j] [2j-1]$$

$$< j, n \parallel X^- \parallel j-1, n > < j-1, n \parallel P^- \parallel j, n+1 >$$

$$= -i \frac{W_j}{2q^5(q^2-1)}$$

(4.11)

where the reduced matrix elements of $P^4$ are defined analogous to eqs. (4.6) including the fact that they are no longer diagonal in $n$. Now it is straightforward to take

$$< j, m, n+1 | X \circ P | j, m, n > = -q^2 [2j+3] [2j+2]$$

$$< j, n+1 \parallel X^- \parallel j+1, n+1 > < j+1, n+1 \parallel P^- \parallel j, n > + [2j] [2j-1]$$

$$< j, n+1 \parallel X^- \parallel j-1, n+1 > < j-1, n+1 \parallel P^- \parallel j, n >$$

$$= \frac{i}{2} \frac{W_j q}{q^2-1}$$

(4.12)

We put in eqs. (4.7) and (4.6) and the conjugation relations

$$< j+1, n \parallel P^- \parallel j, n+1 > = -q^{2j+2} < j, n+1 \parallel P^- \parallel j, n >$$

$$< j+1, n+1 \parallel P^- \parallel j, n > = -q^{2j+2} < j, n \parallel P^- \parallel j+1, n+1 >$$

(4.13)

The system (4.11) and (4.12) can be rewritten as two recursion relations in $j$ for the two unknowns, the reduced matrix elements of $P$. An easy way to solve it, is to start with $j = 0$, read off the general formula and prove it by insertion. For clearness, we present all non-vanishing reduced matrix elements

$$< j+1, n \parallel P^- \parallel j, n+1 > = -i q^{-j-6-2n} Z^{-1}, < j, n+1 \parallel P^- \parallel j+1, n+1 > = -i q^{-3j-8-2n} Z^{-1}$$

$$< j+1, n+1 \parallel P^- \parallel j, n > = i q^{3j-2-2n} Z^{-1}, < j, n \parallel P^- \parallel j+1, n+1 > = i q^{j-4-2n} Z^{-1}$$

(4.14)

where the common denominator is

$$Z = 2l_0 \lambda \sqrt{2 [2j+1] [2j+3]}$$

Neither eq. (4.8) nor eq. (4.14) contains any divergencies because of the condition $j_{\text{max}} \leq \frac{r}{2} - 1$. If $j+1$ exceeds $j_{\text{max}}$ the matrix element simply vanishes as it does for $j-1 = -1$.

Our next aim is to calculate the eigenvalues of $P^2 \equiv P \circ P$. We shall follow the lines of Chap. 3 and start with the definition of a unitary operator

$$U = q^3 \Lambda^\frac{1}{2}$$

(4.15)

Going back to eq. (4.3) we have

$$U | n > = | n-1 >$$

(4.16)
where we have omitted all quantum numbers which are unchanged. After
\[ U | 0 > = e^{i\alpha} | r - 1 > \] (4.17)
we introduce
\[ U' = U e^{-i\alpha} \]
\[ U' | n >' = | n - 1 >' \] (4.18)

The eigenstates of the operator \( U' \) are given by
\[ | \phi_k > = \sum_{n=0}^{r-1} q^{2nk} | n >' \]
\[ U' | \phi_k > = q^{2k} | \phi_k > \] (4.19)

Note that the eigenstates for even \( k \) can be produced by the operator \( X^2/l_0^2 \) acting \( k/2 \) times on \( | \phi_0 > \). From the algebra (2.18) follows
\[ P \circ P | \phi_k > = \tilde{d}_k | \phi_{k-2} > \] (4.20)
where we shall calculate \( \tilde{d}_k \) below. For the \( P \)-eigenstates we use the ansatz
\[ P \circ P | p_n > = p_n^2 | p_n > \]
\[ | p_n > = \sum_{k=0}^{r-1} a_k | \phi_k > \] (4.21)

Eq. (4.20) yields the recursion relation
\[ a_{k+2} = \frac{p_n^2}{\tilde{d}_{k+2}} a_k \] (4.22)

Now it is necessary to distinguish between even and odd \( r \). In the first case we obtain two different solutions putting \( a_0 = 1, a_1 = 0 \) and vice versa. They contain either even or odd numbers of \( k \) in the sum (4.21). Consistency gives for the eigenvalues
\[ (p_+^2)^\frac{r}{2} = \prod_{k=0}^{\frac{r}{2}-1} \tilde{d}_{2k} \]
\[ (p_-^2)^\frac{r}{2} = \prod_{k=0}^{\frac{r}{2}-1} \tilde{d}_{2k+1} \] (4.23)

For odd \( r \) the sum (4.21) contains all numbers and hence
\[ (p^2)^r = \prod_{k=0}^{r-1} \tilde{d}_k \] (4.24)
The coefficients $\tilde{d}_k$ are calculated via the matrix elements of $P^2$ between the $|j,n>$ states. We have the same structure as in the first parts of eqs. (4.11) and (4.12), e.g.

$$<j,n+2 | P^2 | j,n> = -q^2 \{|2j+3|2j+2\}$$

$$<j,n+2 \| P^- \| j+1,n+1 > < j+1,n+1 \| P^- \| j,n > + [2j][2j-1]$$

$$<j,n+2 \| P^- \| j-1,n+1 > < j-1,n+1 \| P^- \| j,n >$$

(4.25)

With the results of eq. (4.14) we get

$$<j,n+2 | P^2 | j,n> = -\frac{q^{-4n-10}}{4l^2_0\lambda^2}$$

(4.26)

and the same way

$$<j,n-2 | P^2 | j,n> = -\frac{q^{-4n-2}}{4l^2_0\lambda^2}$$

(4.27)

A little bit more lengthy is the calculation of the diagonal element due to the doubling of terms connected with intermediate states having quantum numbers $n \pm 1$.

$$<j,n | P^2 | j,n> = \frac{q^{-4n-6}}{4l^2_0\lambda^2} \{4j+2\}$$

(4.28)

As soon as the quantum numbers of the r.h.s. ket vector are fixed there are no further non-vanishing matrix elements. Now we consider

$$P^2 | n >' = | n >' < n |' P^2 | n >'$$

$$+ | n+2 >' < n+2 |' P^2 | n >' + | n-2 >' < n-2 |' P^2 | n >'$$

(4.29)

$$= | n >' < n | P^2 | n >$$

$$+ | n+2 >' e^{-2i\alpha} < n+2 | P^2 | n > + | n-2 >' e^{2i\alpha} < n-2 | P^2 | n >$$

From eq. (4.20) follows

$$P^2 | \phi_k > = \sum_{n=0}^{r-1} q^{2nk} | n >' (q^{-4k} e^{-2i\alpha} < n+2 | P^2 | n > + q^{4k} e^{2i\alpha} < n-2 | P^2 | n >$$

$$+ < n | P^2 | n >)$$

$$= \tilde{d}_k \sum_{n=0}^{r-1} q^{2nk-4n} | n >'$$

(4.30)

Substituting eqs. (4.26)-(4.28) we can read off

$$\tilde{d}_k = \frac{q^{-6}}{4l^2_0\lambda^2} (\{4j+2\} - q^{-4k-4} e^{-2i\alpha} - q^{4k+4} e^{2i\alpha})$$

$$= -\frac{q^{-6}}{4l^2_0} [2k+2j+3]_\alpha [2k-2j+1]_\alpha$$

(4.31)
where we have introduced the abbreviation
\[ [a]_\alpha = \frac{q^a e^{i\alpha} - q^{-a} e^{-i\alpha}}{\lambda} \]  

Finally we have for even \( r \)
\[ (p^2)^{\pm}_r = \frac{-ir}{2^r \lambda r l_0^r} (-1)^r f(q, \alpha) f(q, \alpha - \pi r) \]  
and for odd \( r \)
\[ (p^2)^r = \frac{-i2^r}{2^r \lambda 2^r l_0^r} f^2(q, \alpha) f^2(q, \alpha - \pi r) \]

While \( \tilde{d}_k \) depends on \( j \), \( p^2 \), of course, does not. It is remarkable that eqs. (4.33) and (4.34) very much resemble eq. (3.20) derived for the one-dimensional model.

For even \( r \) any eigenvalue is degenerated twice, disregarding the obvious degeneration with respect to \( j \) and \( m \). All eigenvectors (4.21) are orthogonal and normalizable. (Note that this is not true for the \( |\phi_k> \) states.) The eigenvalues of \( P^2 \) are in both cases given by even powers of \( q \) multiplied by the roots of real functions on \( q \). The main difference to real \( q \) is the finiteness in dimension for the eigenvector space.

It would be interesting to know more about the function \( f(q, \alpha) \) esp. it should play a role in a generalized Fourier transformation. We hope to return to this question in our further work. Our experience for finite \( r \) seems to lead to the conjecture that the roots in eqs. (4.33), (4.34) can be easily extracted if one excludes all polynomials which vanish after being multiplied with non-zero combinations of powers of \( q \).

At the end of this section we shall return to the question, how the structure we have found is related to former attempts of combining non-commutative geometry with string theory via a matrix realization [8].

Our operators \( X^i \) can be viewed as matrices acting on vectors with dimension \( \frac{r^3}{4} \) (for even \( r \)) as long as \( l_0 \) is kept fixed. It is natural to ask whether they can be considered as analogues of the \( X_i \) fields \( (0 \leq i \leq 9) \) in the IKKT model [8]. The role played there by \( SO(10, C) \) is here played by \( SO(3) \).

Nevertheless there are substantial differences between the two sets of operators. Even though there is an analogue of their unitary gauge fields \( U_i \) namely the unitary operator \( q^a \Lambda^\frac{1}{2} \) (it is only one), that operator plays a different role. Its ”gauge transformation” induces a multiplicative rescaling while in paper [8] the gauge transformation adds a constant (proportional to the compactification radius). It is that fact which requires infinite matrices in the IKKT model in order to have an infinite trace, while direct calculation yields \( Tr X^i = 0 \) in our case. This discrepancy becomes less important remembering that for the full trace one has to integrate over \( l_0 \) which produces a divergent result.

We propose to solve the remaining problems by taking into account the fact, that in the IKKT model the undeformed group \( SO(10, C) \) was used while we started with the \( q \)-deformed \( SO_q(3) \).
Concluding this remark we state, that we have found a self-consistent structure which is close to become an analogue of some IKKT like matrix model on a non-commutative torus. This problem is under work now.

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