New Solutions to the Yang–Baxter Equation from Two–Dimensional Representations of $U_q(sl(2))$ at Roots of Unit

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Abstract: We present particularly simple new solutions to the Yang–Baxter equation arising from two–dimensional cyclic representations of quantum $SU(2)$. They are readily interpreted as scattering matrices of relativistic objects, and the quantum group becomes a dynamical symmetry.

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1. Introduction: $U_\epsilon(sl(2))$ with $\epsilon^4 = 1$

Quantum groups at roots of unit enjoy a beautiful representation theory \[1, 2, 3, 4, 5, 6\] which has been applied successfully to the understanding of the chiral Potts model \[7, 8, 2, 9, 10, 11\]. In this letter, we apply the general formalism of cyclic representations of $U_\epsilon(sl(2))$ to the somewhat degenerate case of $\epsilon^4 = 1$, i.e. $q^2 = 1$. It complements the work for $U_q(sl(2))$ in \[3\] of the $p = 3$ case and in \[5\] of the limit $p \to \infty$, as well as the case $p = 2$ of $U_q(sl(3))$ in \[12\].

When $q^p = 1$, the center of $U_\epsilon(sl(2))$ contains not only the standard quadratic Casimir

$$C = K^{-1}FE + \frac{1}{q - 1}\left(q^{-1}K + K^{-1}\right)$$  \hspace{1cm} (1.1)

but also the elements $F^p$, $E^p$ and $K^p$. We work in the contour basis \[13\] with co-multiplication

$$\Delta E = E \otimes 1 + K \otimes E$$
$$\Delta F = F \otimes 1 + K \otimes F$$
$$\Delta K = K \otimes K$$ \hspace{1cm} (1.2)

The generators $E$, $F$ and $K$ of $U_\epsilon(sl(2))$ satisfy the standard relations:

$$EF - qFE = 1 - K^2$$
$$EK - qKE = 0$$
$$FK - q^{-1}KF = 0$$ \hspace{1cm} (1.3)

Let us denote by $\xi = (x, y, z)$ the eigenvalues of $(E^p, F^p, K^p)$, and introduce the notation

$$\mu = \frac{x}{1 - z}$$
$$\nu = \frac{y}{1 - z}$$ \hspace{1cm} (1.4)

Specializing to the case of interest, namely $p = 2$ ($q = -1$), and letting $z = \lambda^2$, the Casimir eigenvalue is then

$$c = \frac{(2 - \sqrt{1 - 4\mu\nu})}{2}\left(\lambda - \lambda^{-1}\right)$$ \hspace{1cm} (1.5)

The spectrum of $U_\epsilon(sl(2))$ with $\epsilon = e^{i\pi/2}$ consists of a three-dimensional continuum of two-dimensional representations, labelled by $\xi$, with a singular (orbifold) point at $z = 1$, which corresponds to the only regular representation in this theory, namely the identity.
The irreducible representations of the theory consist thus the identity operator and the manifold of doublet cyclic representations. The latter constitute an intrinsically quantum generalization of the customary and useful doublet irrep of $SU(2)$, and the purpose of this letter is to analyze some of their physical properties from the purely algebraic point of view.

Let $e_r(\xi) \ (r = 0,1)$ be the basis for the cyclic representation $\pi_\xi = \pi(x,y,z=\lambda^2)$, defined as follows $(z = \lambda^2 \neq 1)$:

\begin{align*}
    Ke_0 &= \lambda e_0 & F e_0 &= \sqrt{\nu}(1-\lambda) e_1 & E e_0 &= \frac{1-\sqrt{1-4\mu\nu}}{2\sqrt{\nu}}(1-\lambda)e_1 \\
    Ke_1 &= -\lambda e_1 & F e_1 &= \sqrt{\nu}(1+\lambda) e_0 & E e_1 &= \frac{1+\sqrt{1-4\mu\nu}}{2\sqrt{\nu}}(1+\lambda)e_0
\end{align*}

(1.6)

In our choice of basis, we have implicitly assumed that $\nu \neq 0$. The special class of representations with $\mu = 0$ are called semi-cyclic: for them, $e_0$ is a highest weight ($E e_0 = 0$) and yet $e_1$ is not a lowest weight ($F e_1 \neq 0$).

The seasoned reader is certainly struck by the appearance of anti-commutators in the relations among the generators. Quite simply, when $q^2 = 1$, $q$-commutators become anti-commutators. For future reference, let us introduce a “fermionic” basis for $U_\epsilon(s\ell(2))$ when $\epsilon^4 = 1$:

\begin{align*}
    b &= \frac{1}{1 + KE} \\
    b^\dagger &= \frac{1}{1 + KF}
\end{align*}

(1.7)

The quantum algebra is then generated by $K$, $b$ and $b^\dagger$, with the following anticommutation relations:

\begin{align*}
    \{b, K\} &= \{b^\dagger, K\} = 0 \\
    \{b, b^\dagger\} &= 1 \\
    \{b, b\} &= 2\mu \\
    \{b^\dagger, b^\dagger\} &= 2\nu
\end{align*}

(1.8)

($\mu$ and $\nu$ are c-numbers or, equivalently, operators proportional to the identity). We may rescale the quadratic Casimir by $(K^2 - 1)$ to get the quantum relative of the number operator:

\[Q = K \left(b^\dagger b - \frac{1}{2}\right)\]

(1.9)

which commutes with $K$, $b$ and $b^\dagger$. A cyclic irreducible representation is characterized in
this language by $e_0 = |\rangle\rangle$ and $e_1 = |\rangle\rangle$ with
\[
K |\pm\rangle = \pm \lambda |\pm\rangle \\
b |\pm\rangle = \frac{1 + \sqrt{1 - 4\mu\nu}}{2\sqrt{\nu}} |\mp\rangle \\
b^\dagger |\pm\rangle = \sqrt{\nu} |\mp\rangle
\]
Again, note that in the semi-cyclic case, $\mu = 0$, we may think of $|\rangle\rangle$ as akin to the ground state (it is annihilated by $b$).

2. Intertwiners and Yang–Baxter equation: general solution

The intertwiner $R(\xi_1, \xi_2)$ between two cyclic representations effects a braiding and can be thought of as a $2 \to 2$ scattering matrix:
\[
e_{r_1}(\xi_1) \otimes e_{r_2}(\xi_2) = R_{r_1 r_2}^{r'_1 r'_2}(\xi_1, \xi_2)e_{r'_1}(\xi_2) \otimes e_{r'_2}(\xi_1) \tag{2.1}
\]
Quasi–triangularity requires $[R, \Delta] = 0$, i.e.
\[
R_{r_1 r_2}^{r''_1 r''_2}(\xi_1, \xi_2)\Delta_{\xi_1, \xi_2}(g)^{r'_1 r'_2}_{r''_1 r''_2} = \Delta_{\xi_2, \xi_1}(g)^{r''_1 r''_2}_{r'_1 r'_2}R_{r_1 r_2}^{r'_1 r'_2}(\xi_1, \xi_2) \tag{2.2}
\]
for any $g \in U_q(s\ell(2))$. Specializing to $g = E^2$ and $g = F^2$, we see that if the intertwiner $R(\xi_1, \xi_2)$ is to exist, then $\xi_1$ and $\xi_2$ are constrained to lie on the same spectral variety [2]:
\[
\frac{x_1}{1 - \lambda_1^2} = \frac{x_2}{1 - \lambda_2^2} = \mu \\
\frac{y_1}{1 - \lambda_1^2} = \frac{y_2}{1 - \lambda_2^2} = \nu \tag{2.3}
\]
with arbitrary $\mu, \nu \in \mathbb{C}$ and $\lambda_i^2 \neq 1$.

The main result we discuss in this letter is that, for $\xi_1, \xi_2, \xi_3$ on the same spectral variety there exists an $R$–matrix $R(\xi_i, \xi_j)$ satisfying (2.2) and the Yang–Baxter equation
\[
\sum_{s_1, s_2, s_3} R_{s_1 s_3}^{r'_1 r''_1}(\xi_2, \xi_3)R_{s_2 s_3}^{s_3 r'_2}(\xi_1, \xi_3)R_{r_1 r'_2}^{s_1 s_2}(\xi_1, \xi_2) = \sum_{s_1, s_2, s_3} R_{s_1 s_3}^{r'_1 r''_1}(\xi_1, \xi_2)R_{r_1 s_2}^{s_1 s_3}(\xi_1, \xi_3)R_{r_2 r'_3}^{s_2 s_3}(\xi_2, \xi_3) \tag{2.4}
\]
The explicit form of the $R$–matrix intertwiner satisfying the Yang–Baxter equation is
the following (we show only the non–zero entries):

\[ R_{00}^{11}(\xi_1, \xi_2) = 1 \]
\[ R_{01}^{01}(\xi_1, \xi_2) = \frac{(1 - \lambda_1)(1 + \lambda_2)}{\Omega_2 - \lambda_1 \lambda_2 \Omega_1} \]
\[ R_{01}^{10}(\xi_1, \xi_2) = \frac{\lambda_1 \Omega_2 - \lambda_2 \Omega_1}{\Omega_2 - \lambda_1 \lambda_2 \Omega_1} \]
\[ R_{10}^{10}(\xi_1, \xi_2) = \frac{\lambda_2 \Omega_2 - \lambda_1 \lambda_2 \Omega_1}{\Omega_2 - \lambda_1 \lambda_2 \Omega_1} \]
\[ R_{11}^{11}(\xi_1, \xi_2) = \frac{\Omega_1 - \lambda_1 \lambda_2 \Omega_2}{\Omega_2 - \lambda_1 \lambda_2 \Omega_1} \]

Here, \( \Omega_i = \Omega(\xi_i) \) are the values of an arbitrary function of the labels of the cyclic representation. Note that what really appears is only the ratio \( \Omega_1/\Omega_2 = \Omega(\lambda_1)/\Omega(\lambda_2) \).

The \( R \)-matrix (2.5) with arbitrary \( \Omega(\xi) \) enjoys two remarkable properties. First, it is normalized:

\[ R_{r_1 r_2}^{r'_1 r'_2}(\xi, \xi) = \delta_{r_1}^{r'_1} \delta_{r_2}^{r'_2} \] (2.6)

Second, it is unitary:

\[ \sum_{r'_1 r'_2} R_{r_1 r_2}^{r'_1 r'_2}(\xi_1, \xi_2) R_{r'_1 r'_2}^{r_1 r_2}(\xi_2, \xi_1) = \delta_{r_1}^{r'_1} \delta_{r_2}^{r'_2} \] (2.7)

Let us stress some important differences between the above solution for \( q^2 = 1 \) and the generic \( (q^p = 1, \ p \geq 3) \) semi-cyclic \( (y = 0) \) situation [3, 5]. Firstly, this is the only case in which an arbitrary function \( \Omega \) appears, i.e. there is a whole family of spectral–dependent \( R \)-matrices. Secondly, the \( R \)-matrix (2.5) does not involve the parameters \( \mu = x_i/(1 - z_i) \) nor \( \nu = y_i/(1 - z_i) \) explicitly, although of course \( \xi_1 \) and \( \xi_2 \) must lie on the same spectral variety, i.e. share common values for \( \mu \) and \( \nu \), for \( R(\xi_1, \xi_2) \) to exist at all. (A dependence of \( R \) on \( \mu \) and \( \nu \) could be introduced through \( \Omega \).) Thirdly, the \( R \)-matrix (2.5) conserves the “quantum isospin” exactly

\[ R_{r_1 r_2}^{r'_1 r'_2}(\xi_1, \xi_2) \neq 0 \implies r_1 + r_2 = r'_1 + r'_2 \] (2.8)

instead of just modulo 2. In particular, \( R_{11}^{00} \) and \( R_{01}^{11} \) are always zero, so if we interpret the \( R \)-matrix entries as Boltzmann weights, we are dealing with a six–vertex model. These three observations are not unrelated, as will become clearer below.
The arbitrary function \( \Omega \) is quite akin to an affinization parameter. This interpretation is reinforced by the fact that only the ratio \( \Omega_1/\Omega_2 \) appears in the \( R \)-matrix, which can be decomposed as follows

\[
R(\lambda_1, \lambda_2; \Omega_1/\Omega_2) = \frac{\left( \lambda_1 \lambda_2 \Omega_1 \right)^{-1/2} R(\lambda_1, \lambda_2; 0) - \left( \lambda_1 \lambda_2 \Omega_1 \right)^{1/2} R(\lambda_1, \lambda_2; \infty)}{\left( \lambda_1 \lambda_2 \Omega_2 \right)^{-1/2} - \left( \lambda_1 \lambda_2 \Omega_2 \right)^{1/2}} \tag{2.9}
\]

or, less symmetrically but more succinctly, as

\[
R(\lambda_1, \lambda_2; \Omega_1/\Omega_2) = \frac{1}{1 - \lambda_1 \lambda_2 \Omega_2} \left[ R(\lambda_1, \lambda_2; 0) - \lambda_1 \lambda_2 \frac{\Omega_1}{\Omega_2} R(\lambda_1, \lambda_2; \infty) \right] \tag{2.10}
\]

We may thus view \( R(\lambda_1, \lambda_2; 0) \) as the basic \( R \)-matrix from which the above family is built, because \( R(\lambda_1, \lambda_2; \infty) = R(\lambda_1, \lambda_2; 0)^{-1} \). Remarkably, the \( R \)-matrix \( R(\lambda_1, \lambda_2; 0) \) can be obtained with the help of contour techniques in the semi-cyclic case \((\mu = 0)\) supplemented with the rule that \( F^2 = 0 \). It is thus apparent that \( R(\lambda_1, \lambda_2; 0) \) is a slight generalization of the usual \( \frac{1}{2} \pi \), the \( R \)-matrix for the regular doublet representation of quantum \( SU(2) \) with \( q^p = 1 \) \((p \geq 3)\). In fact, \( R(\lambda_1, \lambda_2; 0) \) has been considered in [4]. Let us conclude this detour by noting that we may not give a contour representation of the general \( R(\lambda_1, \lambda_2; \Omega) \): since it is a linear combination of \( R(\lambda_1, \lambda_2; 0) \) and \( R(\lambda_1, \lambda_2; 0)^{-1} \), it consists of a piece due to braiding by \( \pi \) and another by \(-\pi\).

3. Clebsch–Gordan coefficients and crossing symmetry

We may consider the tensor product of two cyclic representations \( \pi_{\xi_1}, \pi_{\xi_2} \) on the same spectral variety, parametrized by \((\mu, \nu)\). The result is a direct sum of two cyclic representations \( \pi_{\xi_{\pm}} \) again on the same variety, with

\[
\lambda_{\pm} = \pm \lambda_1 \lambda_2
\]

and

\[
x_{\pm} = \mu (1 - \lambda_{\pm}^2) = x_1 + x_2 \lambda_1^2 = x_2 + x_1 \lambda_2^2
\]

and similarly for \( y \). It is easy to check that

\[
\begin{align*}
e_0(\lambda_1 \lambda_2) &= e_0(\lambda_1) \otimes e_0(\lambda_2) \\
e_1(\lambda_1 \lambda_2) &= \frac{-\lambda_1 (1 + \lambda_2)}{1 - \lambda_1 \lambda_2} e_1(\lambda_1) \otimes e_0(\lambda_2) + \frac{1 + \lambda_1}{1 - \lambda_1 \lambda_2} e_0(\lambda_1) \otimes e_1(\lambda_2) \\
e_0(-\lambda_1 \lambda_2) &= \frac{1 - \lambda_1}{1 - \lambda_1 \lambda_2} e_1(\lambda_1) \otimes e_0(\lambda_2) + \frac{\lambda_1 (1 - \lambda_2)}{1 - \lambda_1 \lambda_2} e_0(\lambda_1) \otimes e_1(\lambda_2) \\
e_1(-\lambda_1 \lambda_2) &= e_1(\lambda_1) \otimes e_1(\lambda_2)
\end{align*}
\]
Thus, the non–zero quantum Clebsch–Gordan coefficients are

\[
\begin{align*}
K^0_{\lambda_1 \lambda_2} &= 1 \\
K^0_{\lambda_1 \lambda_2} &= \frac{1 + \lambda_1}{1 - \lambda_1 \lambda_2} \\
K^0_{\lambda_1 \lambda_2} &= \frac{-\lambda_1 (1 + \lambda_2)}{1 - \lambda_1 \lambda_2} \\
K^0_{\lambda_1 \lambda_2} &= \frac{\lambda_1 (1 - \lambda_2)}{1 - \lambda_1 \lambda_2} \\
K^0_{\lambda_1 \lambda_2} &= \frac{1 - \lambda_1}{1 - \lambda_1 \lambda_2} \\
K^0_{\lambda_1 \lambda_2} &= 1 \\
\end{align*}
\]

(3.2)

For completeness, we note also the non–zero inverse quantum Clebsch–Gordan coefficients:

\[
\begin{align*}
\tilde{K}^0_{\lambda_1 \lambda_2} &= 1 \\
\tilde{K}^0_{\lambda_1 \lambda_2} &= \frac{1 - \lambda_1}{1 + \lambda_1 \lambda_2} \\
\tilde{K}^0_{\lambda_1 \lambda_2} &= \frac{-\lambda_1 (1 - \lambda_2)}{1 + \lambda_1 \lambda_2} \\
\tilde{K}^0_{\lambda_1 \lambda_2} &= \frac{\lambda_1 (1 + \lambda_2)}{1 + \lambda_1 \lambda_2} \\
\tilde{K}^0_{\lambda_1 \lambda_2} &= \frac{1 + \lambda_1}{1 + \lambda_1 \lambda_2} \\
\tilde{K}^0_{\lambda_1 \lambda_2} &= 1 \\
\end{align*}
\]

(3.3)

Imagine now decomposing \(e_i(\pm \lambda_1 \lambda_2)\) with \(K^{j_1, j_2 \pm \lambda_1 \lambda_2}_{\lambda_1 \lambda_2} i\), and then braiding the result with \(R(\lambda_1, \lambda_2; \Omega_1/\Omega_2)\). We may compare the result of these two actions, \(RK^{12}_{\lambda_1 \lambda_2}\), with the single immediate decomposition into \(e_{j_1}(\lambda_2) \otimes e_{j_2}(\lambda_1)\) with the help of \(K^{j_1, j_2 \pm \lambda_1 \lambda_2}_{\lambda_2 \lambda_1} i\), i.e. \(K^{21}_{\lambda_1 \lambda_2}\). The two operations are related by a factor, depending only on the label of the composed representation, \(\pm \lambda_1 \lambda_2\), but not on any of the quantum group indices \(i, j \in \mathbb{Z}_2\):

\[
R(\lambda_1, \lambda_2; \Omega_1/\Omega_2)^{j_1, j_2 \pm \lambda_1 \lambda_2}_{i_1, i_2} K^{i_1, i_2 \pm \lambda_1 \lambda_2}_{\lambda_1 \lambda_2} k = \Phi_{\pm}(\lambda_1, \lambda_2) K^{j_1, j_2 \pm \lambda_1 \lambda_2}_{\lambda_2 \lambda_1} k
\]

(3.4)

and

\[
\Phi_{\pm}(\lambda_1, \lambda_2) = 1 \quad , \quad \Phi_{-}(\lambda_1, \lambda_2) = R^{11}_{11}(\lambda_1, \lambda_2; \Omega_1/\Omega_2)
\]

(3.5)

It is interesting to note that a single unique particular choice of the arbitrary function \(\Omega\) allows us to set \(\Phi_{-} = 1\), namely

\[
\Omega_1/\Omega_2 = 1
\]

(3.6)
Then, the $R$–matrix $R(\lambda_1, \lambda_2; 1)$ is given merely by the mismatch in the decompositions of $e(\lambda_1) \otimes e(\lambda_2)$ and $e(\lambda_2) \otimes e(\lambda_1)$ into cyclic irreps, without any extra phase factors:

$$R_{r_1 r_2}^{r'_1 r'_2}(\lambda_1, \lambda_2; 1) = \sum_{\lambda_3 r_3} K_{r_1 r_2 \lambda_3}^{\lambda_1 \lambda_2} K_{\lambda_2 r_1 \lambda_3}^{r'_1 r'_2}$$

(3.7)

Equation (3.7) embodies the fulfillment of the bootstrap program. This $R$–matrix is the only one, among the one–parameter family of semi-cyclic intertwiners (2.5) satisfying the Yang–Baxter equation, which does enjoy the “crossing”–symmetry

$$R_{r_1 r_2}^{r'_1 r'_2}(\xi_1, \xi_2; 1) = R_{r_2 r_1}^{r'_2 r'_1}(\xi_2, \xi_1; 1)$$

(3.8)

This noteworthy property is crucial for the interpretation of $R$ as a scattering matrix.

Compare, again, with the case of $q^p = 1$ ($p \geq 3$): there, the unique intertwiner satisfying Yang–Baxter always has crossing symmetry. The crossing–symmetric $R$–matrix (3.7) with the particular choice $\Omega(\xi) = 1$ is thus the natural extension to $p = 2$ of the general semi-cyclic solutions. The family of $R$–matrices affinized by $\Omega(\xi)$ is peculiar to $q = -1$, but we shall now study in detail the particular crossing–symmetric $R$–matrix with $\Omega = 1$.

4. The particular solution: soliton interpretation

The non–zero entries of the crossing–symmetric $R(\xi_1, \xi_2; 1) = R(\xi_1, \xi_2)$ are, explicitly,

$$
R_{00}^{00}(\xi_1, \xi_2) = 1 \\
R_{01}^{01}(\xi_1, \xi_2) = \frac{(1 - \lambda_1)(1 + \lambda_2)}{1 - \lambda_1 \lambda_2} \\
R_{01}^{10}(\xi_1, \xi_2) = \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \\
R_{10}^{01}(\xi_1, \xi_2) = \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \\
R_{10}^{10}(\xi_1, \xi_2) = \frac{(1 + \lambda_1)(1 - \lambda_2)}{1 - \lambda_1 \lambda_2} \\
R_{11}^{11}(\xi_1, \xi_2) = 1
$$

(4.1)

Due to the crossing symmetry (3.8), in addition to the already noted unitarity and normalization properties (2.6,2.7), the “semi-cyclic” $R$–matrix admits a clear interpretation as a solitonic $S$–matrix: we may picture the two states $e_0(\lambda)$ and $e_1(\lambda)$ as localized around each one of the two potential minima, with $\lambda$ a label very much like relativistic velocity.
Introduce the “relative velocity”

\[ u_{12} = \frac{\lambda_1 - \lambda_2}{1 - \lambda_1 \lambda_2} \]

in terms of which the “semi-cyclic” intertwiner reads as

\[
R_{12} = \begin{pmatrix}
00 & 01 & 10 & 11 \\
00 & 1 & 0 & 0 & 0 \\
01 & 0 & 1 - u_{12} & u_{12} & 0 \\
10 & 0 & -u_{12} & 1 + u_{12} & 0 \\
11 & 0 & 0 & 0 & 1
\end{pmatrix}
\]  \quad (4.2)

and the Yang–Baxter equation becomes

\[
R_{12}(u)R_{13} \left( \frac{u + v}{1 + uv} \right) R_{23}(v) = R_{23}(v)R_{13} \left( \frac{u + v}{1 + uv} \right) R_{12}(u)
\]  \quad (4.3)

Note that only when \( \Omega = 1 \) can we parametrize the whole \( R \)-matrix in terms of a single quantity \( u_{12} \). In the usual trigonometric solutions to the Yang–Baxter equations, the rapidities \( u \) and \( v \) add up linearly, to \( u + v \). Here the rapidities add up like relativistic velocities! It thus turns out that the labels of the irreps under the quantum group may be identified with kinematical parameters: the two–dimensional Poincaré group is thus a manifestation of an internal quantum symmetry. This situation at \( p = 2 \) is very similar to the limit \( p \to \infty \) of the general semi-cyclic intertwiner [5].

The braid group limit of the \( R \)-matrix (4.2) is obtained when \( u \to \pm 1 \), i.e. in the extreme relativistic regime. Letting \( R_{\pm} = \lim_{u \to \pm 1} R(u) \), we find

\[
R_+ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad R_- = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  \quad (4.4)

Let us concentrate on one of them, say \( R_- \) (the analysis is identical for \( R_+ \)). It can be viewed as a particular case of the more general

\[
R_{(b,c)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - bc & b & 0 \\
0 & c & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  \quad (4.5)

which satisfies Yang–Baxter without spectral parameter and is thus a good starting point for the construction of an extended Yang–Baxter system and hence a link invariant [14].
Indeed, we find that \( \mu_0 = bc \) and \( \mu_1 = 1 \) satisfy

\[
\begin{align*}
(\mu_i\mu_j - \mu_k\mu_\ell) R_{ij}^{k\ell} &= 0 \\
\sum_j R_{ij}^{kj} \mu_j &= AB\delta_i^k \\
\sum_j (R^{-1})_{ij}^{kj} \mu_j &= A^{-1}B\delta_i^k
\end{align*}
\]

with \( A^{-1} = B = \sqrt{bc} \). Accordingly, if \( \alpha \in B_n \) is a word of the braid group, the link invariant associated with its closure \( \hat{\alpha} \) is

\[
T(\hat{\alpha}) = A^{-w(\alpha)}B^{-n}\text{tr} (\rho(\alpha) \otimes \mu^\otimes n)
\]

(4.7)

where \( w(\alpha) \) is the wraith number of \( \alpha \) and \( \rho \) is the representation of the braid group assigning to each generator \( \sigma_i^{\pm 1} \) the matrix \( R_{(b,c)}^{\pm 1} \) acting on the \( i \)-th and \( (i + 1) \)-th strands. The link invariant satisfies the skein rule

\[
A^2P_+ - A^{-2}P_- = \left(A - A^{-1}\right)P_0
\]

(4.8)

and is so normalized that \( N \) disconnected unknots are assigned the polynomial \( (A + A^{-1})^N \). This is just Jones polynomial in \( t = A^2 \). This is a nice result, although the particular value \( bc = -1 \) in (4.4) is in fact singular: \( T(1,-1)(\hat{\alpha}) = 0 \) for all \( \alpha \in B_n \).

5. Other cyclic solutions to Yang–Baxter

Have we found all the \( R \)-matrices which satisfy Yang–Baxter and intertwine among cyclic representations of \( U_q(s\ell(2)) \) with \( q^2 = -1 \)? This is interesting because it would represent a major, if not final, step towards the classification of all the \( 4 \times 4 \) \( R \)-matrices satisfying Yang–Baxter. We have not been able to prove that the family \( R(\Omega) \) exhausts the solutions, although we strongly suspect that this is indeed the case except for exceptional situations. For example, a notable curiosity of \( U_q(s\ell(2)) \) with \( q = -1 \) is that there exists a parameter region \( 4\mu\nu = 1 \) for which the raising and lowering generators coincide, up to a proportionality factor: \( F = 2\nu E \). The quadratic Casimir is then zero. Fixing furthermore \( \nu = 1 \) (thus \( \mu = 1/4 \)) allows us to find a different \( R(\lambda_1, \lambda_2) \), namely

\[
R(\lambda_1, \lambda_2) = R(u_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{12} & 1 - u_{12} & 0 \\ 0 & 0 & 1 & 0 \\ 1 - u_{12} & 0 & 0 & u_{12} \end{pmatrix}
\]

(5.1)
with
\[ u_{12} = \frac{(1 - \lambda_1) \lambda_2}{\lambda_1 (1 - \lambda_2)} \] (5.2)

This $R$–matrix (or its transpose, another independent solution) is fundamentally different from those found above in that the conservation of quantum isospin holds only modulo 2: $R_{00}^{11} \neq 0$. Interpreting the $R$–matrix as Boltzmann weights, we have found thus a special case of the eight–vertex model.

We see also that the dependence of the “relative velocity” $u_{12}$ on the quantum group labels $\lambda_1$ and $\lambda_2$ is now quite bizarre, although the composition of spectral parameters in the Yang–Baxter equation is simply multiplicative: $u_{13} = u_{12} u_{23}$. The braid limit of this solution is singular.

6. Conclusions and outlook

It is of course tantalizing to speculate on the dynamical origin of space–time symmetries, albeit in the simplest framework of 1+1 integrable systems. It is clear from our analysis that the irrep label $\lambda$ can be understood as the velocity of the state. Since the $R$–matrix (4.2) intertwines only irreps sharing the same values of $\mu$ and $\nu$, these should be associated with some extensive variables of the dynamical system. A clear interpretation of them is likely to arise from the study of the (vertex– or IRF–like) statistical–mechanical model based on the $R$–matrices of this letter, which we shall address elsewhere. A physical interpretation for the affinization parameter $\Omega$ is also lacking.

The physics of the $p = 2$ case is the fermionic analogue of the bosonic $p \to \infty$ limit considered in [5], and we expect the intermediate cases $2 < p < \infty$ to be associated with anyonic statistics. We hope that the properties of the simple quantum group studied in this letter will guide us towards a deeper physical understanding of quantum groups at roots of unit and, eventually, towards a classification of all solutions to the Yang–Baxter equation.

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