KPZ IN A MULTIDIMENSIONAL RANDOM GEOMETRY OF MULTIPLICATIVE CASCADES

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Abstract. We show in this note how the one-dimensional KZP formula obtained by Benjamini and Schramm in [BS09] can be extended to a multidimensional setting.

1. Hausdorff dimension in a nested measure space

1.1. Dimension. Let \((S, \mathcal{S}, \mu)\) be a measure space and suppose given a nested family of countable \(\sigma\)-algebras \(\mathcal{S}_n = \sigma(A^i_n; i \geq 1)\), with \(A^i_n \in \mathcal{S}\) disjoint up to \(\mu\) null sets, and \(\mu(A^i_n) > 0\) for each \(i \geq 1\) and \(n \geq 1\). Suppose further that \(\epsilon_n := \sup \mu(A^i_n)\) decreases to 0 as \(n\) goes to infinity. Given \(s \geq 0\) and \(\delta > 0\), set for any measurable \(E \in \mathcal{S}\)

\[ H^s_\delta(E) = \inf \sum \mu(A^i_{\alpha_n})^s, \]

where the infimum is over the set of coverings \(E \subset \bigcup_{\alpha \in A} A^i_{\alpha_n}\) of \(E\), indexed by a subset \(A\) of \(\mathbb{N}^* \times \mathbb{N}^*\), and such that \(\epsilon_{\alpha_n} \leq \delta\) for all \(\alpha \in A\). The quantity \(H^s_\delta(K)\) increases as \(\delta\) decreases to 0. Set

\[ H^s(E) = \lim_{\delta \downarrow 0} H^s_\delta(E). \]

Like in the usual definition of the Hausdorff dimension of a set, it is easy to see that if

- \(H^{s_0}(E) < \infty\) then \(H^t(K) = 0\) for any \(s_0 < t\),
- \(H^{s_0}(E) = \infty\) then \(H^t(K) = 0\) for any \(s < s_0\),

so it makes sense to define the dimension \(\zeta_\mu(E)\) of \(E\) as \(\{s \geq 0; H^s(E) = \infty\} = \inf \{t \geq 0; H^t(E) = 0\}\). As \(H^1\) coincides with \(\mu\), it follows that \(\zeta_\mu(E) \leq 1\), for any \(E \in \mathcal{S}\). So only sets with null \(\mu\)-measure have a dimension smaller than 1.

Open question. Let us work in the space \(S = C([0,1], \mathbb{R})\), with its Borel \(\sigma\)-algebra and Wiener measure. Define \(A_n^{(j,k)}\) as \(\{\omega \in C([0,1], \mathbb{R}); \omega((j + 1)2^{-n}) - \omega(j2^{-n}) \in [k2^{-n}, (k + 1)2^{-n})\}\), for \(0 \leq j \leq 2^n - 1\) and \(k \in \mathbb{Z}\), and set \(\mathcal{S}_n = \sigma(A_n^{(j,k)}; 0 \leq j \leq 2^n - 1, k \in \mathbb{Z})\). Let us call \textbf{Wiener-Hausdorff dimension} the above dimension of a measurable subset of \(C([0,1], \mathbb{R})\). Compute the Wiener-Hausdorff dimension of the set of \(\alpha\)-Hölder continuous paths, for \(\alpha > \frac{1}{2}\).

1.2. Frostman lemma. If \((S, \mathcal{S})\) is \(\mathbb{R}^d\) with its Borel \(\sigma\)-algebra, and \(\mathcal{S}_n\) is the \(\sigma\)-algebra generated by the dyadic cubes of side \(2^{-n}\), then the above definition of dimension coincides with the usual Hausdorff dimension, up to a multiplicative constant \(\frac{1}{d}\); see section 2.4, Chap. 2, in [Fal03]. We adopt the above definition of dimension for the sequel. Like its classical counterpart, the above set function \(H^s(\cdot)\) can be shown to be an \((\mathbb{R}_+ \cup \{\infty\})\)-valued measure.
on \((\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))\). The Euclidean background will not appear anymore except under the form of the nested family \((S_n)_{n \geq 0}\).

Given two points \(x, y \in \mathbb{R}^d\), define the ball \(B(x, y)\) as the smallest dyadic cube containing \(x\) and \(y\), and define their “distance” as \(\mu(B(x, y))\). Define accordingly the ball \(B_r(x) = \{y \in \mathbb{R}^d; \mu(B(x, y)) \leq r\}\). Working exactly as in theorem 4.10 and proposition 4.11 in [Fal03], one can prove the following proposition.

**Proposition 1.** For any Borel set \(E\) with \(0 < \mathcal{H}^s(E) < \infty\), there exists a constant \(c\) and a compact set \(K \subset E\) with \(\mathcal{H}^s(K) > 0\) such that
\[
\mathcal{H}^s(K \cap B_r(x)) \leq cr^s
\]
for all \(x \in \mathbb{R}^d\) and \(r > 0\).

It follows classically that the following version of Frostman lemma holds in our setting. Given any non-negative measure \(\nu\) on \((\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))\), define its \(s\)-energy as
\[
I_s(\nu) = \int \int \nu(dx)\nu(dy) \mu(B(x, y))^{\frac{s}{d}}.
\]

**Theorem 2.** If \(E\) is a Borel set with \(0 < \mathcal{H}^s(E)\), then there exists a non-negative measure \(\nu\) with support in \((\text{a compact subset of})\ E\) such that \(I_t(\nu) < \infty\), for all \(t < s\). This is in particular the case if \(s < \zeta_\mu(E)\).

**Remark.** The work [RV10] contains in section 5.1 a similar, though different, notion of dimension in a metric measure space.

2. A dimension-free KPZ formula

Let \(\mathcal{D}_n = \bigcup_{k=1}^{2^m} A^m_k\) be the dyadic “partition” of the unit cube of \(\mathbb{R}^d\) by closed dyadic cubes of side length \(2^{-n}\). Given \(m < n\), each \(A^m_k\) is a subset of a unique \(A^{m^2}_{k(m)}\). Let \(W\) be a positive real-valued random variable with \(\mathbb{E}[W] = 1\), and let \(\{(W^m_i)_{i=1}^{2^m}; n \geq 1\}\) be an iid sequence of random variables with common law the law of \(W\). Define the measure \(\mu_n\) by its density \(w_n(x)\) with respect to Lebesgue measure. It is constant, equal to \(\prod_{m=0}^{\infty} W^m_{k(m)}\), on each \(A^m_k\). We adopt as in [BS09] the notation \(\ell\) for \(\mu([0, 1]^d)\). It has expectation no greater than 1.

**Proposition 3.** Almost-surely, the measures \(\mu_n\) converge weakly to some random measure \(\mu\), which does not charge any dyadic hyperplane. It is almost-surely non-null if \(\mathbb{E}[W \log W] < d\).

**Proof –** The proof works exactly as in the 1-dimensional proof, with \(2^d\) independent copies of \(\ell\) rather than only two.

The next result generalizes Benjamini and Schramm’s result [BS09] obtained in a one-dimensional setting.

**Theorem 4.** Let \(E\) be any Borel set of \([0, 1]^d\). Denote by \(\zeta_0\) its dimension as defined above using Lebesgue measure, and let \(\eta\) be its dimension using the random measure \(\mu\). Suppose that \(\mathbb{E}[W \log W] < d\), and \(\mathbb{E}[W^{-s}] < \infty\), for all \(s \in [0, 1]\). Then \(\zeta\) is almost-surely a constant and satisfies the identity
\[
2^{\zeta_0} = \frac{2^\zeta}{\mathbb{E}[W^\zeta]}
\]
The above conditions are satisfied by an exponential of Gaussian with a small enough variance.

**Proof** – The proof mimicks word by word the proof of [BS09]. Write $|A|$ for the Lebesgue measure of a Borel set $A$. Set, for $s \in [0,1]$, $\phi(s) = s - \ln_2 \mathbb{E}[W^s]$. Note that since the notion of dimension introduced in section [1] is no greater than 1 the function $\phi$ is an increasing homeomorphism from $[0,1]$ to itself.

**a)** Lemma 3.3 becomes here: $\mathbb{E}[\mu(B(x,y))^s] \leq |B(x,y)|^{\phi(s)}$, for all $x, y \in [0,1]^d$.

Note that the balls $B(x,y)$ are always dyadic balls; suppose the given ball belongs to $\mathcal{D}_n$, so $|B(x,y)| = 2^{-nd}$. Then, we have by the independence in the construction of $\mu$

$$\mathbb{E}[\mu(B(x,y))^s] = 2^{-nd} \mathbb{E}[W^s]^{nd} \mathbb{E}[\ell^s] \leq \{2^{-nd}\}^{\phi(s)} = |B(x,y)|^{\phi(s)},$$

as $0 \leq s \leq 1$, so $\mathbb{E}[\ell^s] \leq \mathbb{E}[\ell^s] = 1$. It follows directly that we have almost-surely $\phi(\zeta) \leq \zeta_0$.

**b)** The proof that $\phi(\zeta) \geq \zeta_0$, theorem 3.5, works identically, replacing the usual energy of a measure by its above modification, and using the version of Frostman lemma provided in theorem [2]. A straightforward adaptation of the proof that $\mathbb{E}[\ell^{-s}] < \infty$ if $\mathbb{E}[W^{-s}] < \infty$, given in [BS09], gives the same result in our setting. Note also that a different choice of Hölder coefficient is needed to prove that the sequence $\nu_n([0,1])$ is uniformly bounded in some $L^p$.

Note that the above theorem does not come as a surprise and should actually hold on much more general state spaces than $[0,1]^d$. It should be interesting in particular to investigate what happens on random trees like Galton-Watson trees, and tree-like objects like random fractals.

**References**

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