ON CHARACTERIZATIONS OF BLOCH-TYPE, HARDY-TYPE
AND LIPSCHITZ-TYPE SPACES

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Abstract. In this paper, we establish a Bloch-type growth theorem for generalized Bloch-type spaces and discuss relationships between Dirichlet-type spaces and Hardy-type spaces on certain classes of complex-valued functions. Then we present some applications to non-homogeneous Yukawa PDEs. We also consider some properties of the Lipschitz-type spaces on certain classes of complex-valued functions. Finally, we will study a class of composition operators on these spaces.

1. Introduction and main results

For \( a \in \mathbb{C} \), let \( \mathbb{D}(a, r) = \{ z : |z - a| < r \} \). In particular, we use \( \mathbb{D} \) to denote the disk \( \mathbb{D}(0, r) \) and \( \mathbb{D} \), the open unit disk \( \mathbb{D}_1 \). Let \( \Omega \) be a domain in \( \mathbb{C} \), with non-empty boundary. Let \( d_\Omega(z) \) be the Euclidean distance from \( z \) to the boundary \( \partial \Omega \) of \( \Omega \). In particular, we always use \( d(z) \) to denote the Euclidean distance from \( z \) to the boundary of \( \mathbb{D} \).

For a real \( 2 \times 2 \) matrix \( A \), we use the matrix norm \( \| A \| = \sup \{ |Az| : |z| = 1 \} \) and the matrix function \( l(A) = \inf \{ |Az| : |z| = 1 \} \). With \( z = x + iy \in \mathbb{C} \), the formal derivative of the complex-valued functions \( f = u + iv \) is given by

\[
D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},
\]

so that \( \| D_f \| = |f_x| + |f_y| \) and \( l(D_f) = ||f_x| - |f_y|| \). Throughout this paper, we denote by \( \mathcal{C}^n(\mathbb{D}) \) the set of all \( n \)-times continuously differentiable complex-valued function in \( \mathbb{D} \), where \( n \in \{1, 2, \ldots \} \).

Generalized Hardy spaces. For \( p \in (0, \infty) \), the generalized Hardy space \( H^p_g(\mathbb{D}) \) consists of all those functions \( f : \mathbb{D} \to \mathbb{C} \) such that \( f \) is measurable, \( M_p(r, f) \) exists for all \( r \in (0, 1) \) and \( \| f \|_p < \infty \), where

\[
\| f \|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f) & \text{if } p \in (0, \infty), \\ \sup_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty, \end{cases}
\]

and \( M_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \).

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For Definition 1. generalised Bloch-type space denote the
\begin{equation}
|f(z) - f(w)| \leq C|z - w| \quad \text{for all } z, w \in \Omega.
\end{equation}

**Definition 1.** For \( p \in (0, \infty) \), \( \alpha > 0 \), \( \beta \in \mathbb{R} \) and a majorant \( \omega \), we use \( \mathcal{L}_{p,\omega} \mathcal{B}^\beta (\mathbb{D}) \) to denote the generalised Bloch-type space of all functions \( f \in \mathcal{C}^1 (\mathbb{D}) \) with \( \| f \|_{\mathcal{L}_{p,\omega} \mathcal{B}^\beta (\mathbb{D})} < \infty \), where
\begin{equation}
\| f \|_{\mathcal{L}_{p,\omega} \mathcal{B}^\beta (\mathbb{D})} = \begin{cases} 
|f(0)| + \sup_{z \in \mathbb{D}} \left\{ M_p(|z|, \| Df \|) \omega \left( d^\beta (z) \left( \frac{e}{d(z)} \right)^\beta \right) \right\} & \text{if } p \in (0, \infty), \\
|f(0)| + \sup_{z \in \mathbb{D}} \left\{ \| Df(z) \| \omega \left( d^\beta (z) \left( \frac{e}{d(z)} \right)^\beta \right) \right\} & \text{if } p = \infty.
\end{cases}
\end{equation}

It can be easily seen that \( \mathcal{L}_{p,\omega} \mathcal{B}^\beta (\mathbb{D}) \) is a Banach space for \( p \geq 1 \). Moreover, we have the following:

1. If \( \beta = 0 \), then \( \mathcal{L}_{\infty,\omega} \mathcal{B}^0 (\mathbb{D}) \) is called the \( \omega-\alpha \)-Bloch space.
2. If we take \( \alpha = 1 \), then \( \mathcal{L}_{\infty,\omega} \mathcal{B}^\beta (\mathbb{D}) \) is called the logarithmic \( \omega \)-Bloch space.
3. If we take \( \omega(t) = t \) and \( \beta = 0 \), then \( \mathcal{L}_{\infty,\omega} \mathcal{B}^0 (\mathbb{D}) \) is called the generalised \( \alpha \)-Bloch space (cf. [22, 29, 34, 35]).
4. If we take \( \omega(t) = t \) and \( \alpha = 1 \), then \( \mathcal{L}_{\infty,\omega} \mathcal{B}^\beta (\mathbb{D}) \) is called the generalised logarithmic Bloch space (cf. [4, 13, 17, 24, 28, 34]).

Let \( \mathcal{A}(\mathbb{D}) \) be the set of all analytic functions defined in \( \mathbb{D} \). Then \( \mathcal{L}_{\infty,\omega} \mathcal{B}^0 (\mathbb{D}) \cap \mathcal{A}(\mathbb{D}) \) (resp. \( \mathcal{L}_{\infty,\omega} \mathcal{B}^\beta (\mathbb{D}) \cap \mathcal{A}(\mathbb{D})) \) is the \( \alpha \)-Bloch space (resp. logarithmic Bloch space), where \( \omega(t) = t \).

A classical result of Hardy and Littlewood asserts that if \( p \in (0, \infty) \), \( \alpha \in (1, \infty) \) and \( f \) is an analytic function in \( \mathbb{D} \), then (cf. [10, 18, 19])
\[
M_p(r, f') = O \left( \left( \frac{1}{1 - r} \right)^\alpha \right) \quad \text{as } r \to 1
\]
if and only if
\[
M_p(r, f) = O \left( \left( \log \frac{1}{1 - r} \right)^{\alpha - 1} \right) \quad \text{as } r \to 1.
\]

In [15], Girela, Pavlović and Peláez refined the above result for the case \( \alpha = 1 \) as follows. For related investigations in this topic, we refer to [5, 7, 8, 16].

**Theorem A.** ([15, Theorem 1.1]) Let \( p \in (2, \infty) \). For \( r \in (0, 1) \), if \( f \) is analytic in \( \mathbb{D} \), then
\[
M_p(r, f') = O \left( \frac{1}{1 - r} \right) \quad \text{as } r \to 1,
\]
then
\[
M_p(r, f) = O \left( \left( \log \frac{1}{1 - r} \right)^{\frac{1}{2}} \right) \quad \text{as } r \to 1.
\]
**Definition 2.** For $n \in \{1, 2, \ldots\}$, we denote by $\mathcal{H} \mathcal{Z}_n(\mathbb{D})$ the class of all functions $f \in C^n(\mathbb{D})$ satisfying *Heinz’s nonlinear differential inequality* (cf. [20])

$$|\Delta f(z)| \leq a(z)\|Df(z)\| + b(z)|f(z)| + q(z) \quad \text{for } z \in \mathbb{D},$$

where $a(z)$, $b(z)$ and $q(z)$ are real-valued nonnegative continuous functions in $\mathbb{D}$ and $\Delta$ is the usual complex Laplacian operator

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

One of our primary goals is to establish a generalization of Theorem A.

**Theorem 1.** Let $\omega$ be a majorant, $p \in [2, \infty)$, $\alpha > 0$, $\beta \leq \alpha$ and $f \in \mathcal{H} \mathcal{Z}_2(\mathbb{D}) \cap L_{p, \omega}\mathcal{B}_0^\beta(\mathbb{D})$ satisfying sup$_{z \in \mathbb{D}} b(z) < \frac{1}{p}$, sup$_{z \in \mathbb{D}} a(z) < \infty$ and sup$_{z \in \mathbb{D}} q(z) < \infty$. If $\text{Re}(\mathcal{F}\Delta f) \geq 0$, then

$$M_p(r, f) \leq \frac{1}{1 - \frac{pr^2}{4} \sup_{z \in \mathbb{D}} b(z)} \left[ \left( \frac{rp\|f\|_{L_{p, \omega}\mathcal{B}_0^\beta(\mathbb{D})}}{\omega(1)} \right)^2 \int_0^1 \frac{(1 - t) dt}{d^{2\alpha}(rt)} \left( \log \frac{e}{d(rt)} \right)^{2\beta} \right]$$

$$+ \frac{pr^2\|f\|_{L_{p, \omega}\mathcal{B}_0^\beta(\mathbb{D})} \sup_{z \in \mathbb{D}} (a(z))}{\omega(1)} M_p(r, f)$$

$$\times \int_0^1 \frac{(1 - t) dt}{d^{\alpha}(rt)} \left( \log \frac{e}{d(rt)} \right)^{\beta} + |f(0)|^2 + \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (q(z)) M_p(r, f) \right]^{\frac{1}{2}}.$$
Furthermore, if \( \alpha - 1 = \beta = 0 \), \( \lambda(z) \equiv 0 \) is a constant function and \( \omega(t) = t \), then
\[
M_p(r, f) = O \left( \left( \log \frac{1}{1-r} \right)^{\frac{1}{2}} \right) \quad \text{as } r \to 1
\]
and the extremal function \( f(z) = \sum_{n=0}^{\infty} z^{2n} \) shows that the estimate of (1.4) is sharp.

**Proof.** It is easy to see that if \( f \) is a solution to (1.3), then \( f \) satisfies Heinz’s nonlinear differential inequality (1.2). Then Corollary 1 follows from Theorem 1. The sharpness part in (1.4) follows from [16, Theorem 1(b)]. \( \square \)

**Definition 3.** We use \( D_{\gamma,\mu}(\mathbb{D}) \) to denote the Dirichlet-type space consisting of all \( f \in C^1(\mathbb{D}) \) with the norm
\[
\|f\|_{D_{\gamma,\mu}} = |f(0)| + \int_{\mathbb{D}} d^\gamma(z)\|Df(z)\|^\mu d\sigma(z) < \infty,
\]
where \( \gamma > 0 \), \( \mu > 0 \) and \( d\sigma \) denotes the normalized area measure in \( \mathbb{D} \).

It is not difficult to see that if \( \omega(t) = t \), then \( \mathcal{L}_{1,\omega}\mathcal{B}_g^0(\mathbb{D}) \subset D_{\gamma,1}(\mathbb{D}) \).

**Proposition 1.** Let \( f \in C^3(\mathbb{D}) \cap D_{\gamma,2}(\mathbb{D}) \) and \( \text{Re} \left[ (\Delta f)_z \overline{f_z} + (\Delta f)_\overline{z} \overline{f_\overline{z}} \right] \geq 0 \). Then \( f \in \mathcal{L}_{\infty,\omega}\mathcal{B}_{1+\gamma/2}^0(\mathbb{D}) \) with \( \omega(t) = t \).

**Theorem 2.** Let \( f \in HZ_3(\mathbb{D}) \cap D_{\gamma,2}(\mathbb{D}) \) with \( \text{Re} (\overline{f} \Delta f) \geq 0 \) and \( \text{Re} \left[ (\Delta f)_z \overline{f_z} + (\Delta f)_\overline{z} \overline{f_\overline{z}} \right] \geq 0 \), where \( 0 < \gamma \leq 1 \), \( \sup_{z \in \mathbb{D}} a(z) < \infty \), \( \sup_{z \in \mathbb{D}} b(z) < \infty \) and \( \sup_{z \in \mathbb{D}} q(z) < \infty \). If \( a(z) + b(z) + q(z) \) is a non-zero function, then \( f \in H_g^2(\mathbb{D}) \).

The result given below is an easy consequence of Theorem 2.

**Corollary 2.** Let \( 0 < \gamma \leq 1 \), \( f \in C^2(\mathbb{D}) \cap D_{\gamma,2}(\mathbb{D}) \) and satisfy the PDE (1.3), where \( \lambda(z) \) is a nonnegative constant function. Then \( f \in H_g^2(\mathbb{D}) \).

**Bloch-type spaces and weighted Lipschitz functions.** Holland and Walsh [21], and Zhao [33] characterized analytic Bloch spaces and \( \alpha \)-Bloch spaces in terms of weighted Lipschitz functions, respectively. Extended discussions on this topic may be found from [22, 26, 34, 35]. Our next result characterizes generalized \( \alpha \)-Bloch space by using a majorant.

**Theorem 3.** Let \( 0 \leq s < 1 \), \( s \leq \alpha < s + 1 \) and \( \omega \) be a majorant. Then \( f \in \mathcal{L}_{\infty,\omega}\mathcal{B}_g^0(\mathbb{D}) \) if and only if there is a constant \( C_1 > 0 \) such that, for all \( z \) and \( w \) with \( z \neq w \),
\[
\frac{|f(z) - f(w)|}{|z - w|} \leq C_1 \frac{1}{\omega(d^s(z)d^{\alpha-s}(w))}.
\]

We remark that Theorem 3 is indeed a generalization of [21, Theorem 3], [26, Theorem 2] and [22, Theorem A] using a majorant.
Harmonic mappings, Bloch-type spaces and BMO. Let $F$ be an analytic function from $\mathbb{B}^n$ into $\mathbb{D}$, where $\mathbb{B}^n$ denotes the open unit ball in $\mathbb{C}^n$. We say that $F$ has the pull-back property if $f \circ F \in \text{BMOA}(\mathbb{B}^n)$ whenever analytic function $f$ belongs to the Bloch space of $\mathbb{D}$ (cf. [29]).

Open Problem 1.5. ([29, Problem 1]) Let $F$ be an analytic function from $\mathbb{B}^n$ into $\mathbb{C}$. For which $\alpha$ does

\begin{equation}
\sup_{z,w \in \mathbb{B}^n, z \neq w} \frac{|F(z) - F(w)|}{|z - w|^{\alpha}} < \infty
\end{equation}

imply that $F$ has the pull-back property?

It is not difficult to see that $F$ satisfies (1.6) if and only if

$$|\nabla F(z)| = O\left((1 - |z|)^{\alpha - 1}\right),$$

where $\nabla F = (F_{z_1}, \ldots, F_{z_n})$ denote the complex gradient.

A planar complex-valued function $f$ defined in $\mathbb{D}$ is called a harmonic mapping in $\mathbb{D}$ if and only if both the real and the imaginary parts of $f$ are real harmonic in $\mathbb{D}$ (cf. [11]). We consider Problem 1.5 for planar harmonic mappings, and present a characterization on the relationship between $\omega$-$\alpha$-Bloch space and BMO as follows.

Theorem 4. Let $1 \leq \alpha < 2$, $f$ be a harmonic mapping in $\mathbb{D}$ and $\omega$ be a majorant. Then $f \in L^\infty(\mathbb{D}, \omega B_0^\alpha(\mathbb{D}))$ if and only if there is a constant $C_2 > 0$ such that for all $r \in (0, d(z))$,

$$\frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} \left| f(\zeta) - \frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} f(\xi) dA(\xi) \right| dA(\zeta) \leq \frac{C_2 r}{\omega(r^\alpha)},$$

where $dA$ denotes the Lebesgue area measure in $\mathbb{D}$ and $|\mathbb{D}(z,r)|$ denotes the area of $\mathbb{D}(z,r)$.

Theorem 4 gives the following result.

Corollary 3. Let $\alpha = 1$ and $\omega$ be a majorant with $\omega(t) = t$. Then $f \in L^\infty(\mathbb{D}, \omega B_0^\alpha(\mathbb{D}))$ if and only if $f \in \text{BMO}$. 

By Theorems 3 and 4, we also have the following.

Corollary 4. Let $0 \leq s < 1$, $1 \leq \alpha < s + 1$ and $f$ be a harmonic mapping in $\mathbb{D}$. Then the following are equivalent:

1. $f \in L^\infty(\mathbb{D}, \omega B_0^\alpha(\mathbb{D}))$;
2. There exists a constant $C_4 > 0$ such that for all $z, w \in \mathbb{D}$ with $z \neq w$,

$$\frac{|f(z) - f(w)|}{|z - w|} \leq \frac{C_4}{\omega(d^s(z)d^{\alpha-s}(w))};$$

3. There exists a constant $C_5 > 0$ such that for all $r \in (0, d(z))$,

$$\frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} \left| f(\zeta) - \frac{1}{|\mathbb{D}(z,r)|} \int_{\mathbb{D}(z,r)} f(\xi) dA(\xi) \right| dA(\zeta) \leq \frac{C_5 r}{\omega(r^\alpha)}.$$
Definition 4. The little Bloch-type space $\mathcal{L}^{0}_{∞,ω}B_{α}^{β}(\mathbb{D})$ consists of all functions $f ∈ \mathcal{L}^{0}_{∞,ω}B_{α}^{β}(\mathbb{D})$ such that

$$\lim_{|z|\rightarrow 1^{-}}\left\{ \|Df(z)\|ω\left(d^{α}(z)\left(\log \frac{e}{d(z)}\right)^{β}\right)\right\} = 0.$$ 

Our next result provides a characterization for the little Bloch-type space $\mathcal{L}^{0}_{∞,ω}B_{α}^{0}(\mathbb{D})$.

Theorem 5. Let $0 ≤ s < 1$, $s ≤ α < s + 1$ and $ω$ be a majorant. Then $f ∈ \mathcal{L}^{0}_{∞,ω}B_{α}^{0}(\mathbb{D})$ if and only if

$$\lim_{|z|\rightarrow 1^{-}}\sup_{|w| < 1}{\left|\frac{f(z) - f(w)}{z - w}\right|} = 0.$$ 

Composition operators. If $ω(t) = t$, then we denote $LB_{α}^{β}(\mathbb{D}) = A(\mathbb{D}) \cap \mathcal{L}^{∞}_{∞,ω}B_{α}^{β}(\mathbb{D})$. Given an analytic self mapping $φ$ of the unit disk $\mathbb{D}$, the composition operator $C_{φ} : A(\mathbb{D}) \rightarrow A(\mathbb{D})$ is defined by

$$C_{φ}(f) = f ∘ φ,$$

where $f ∈ A(\mathbb{D})$ (cf. [1, 24, 28, 31, 34]).

Theorem 6. Let $α > 0$, $β ≤ α$ and $φ : \mathbb{D} → \mathbb{D}$ be an analytic function. Then the following are equivalent:

1. $C_{φ} : LB_{α}^{β}(\mathbb{D}) \rightarrow H^{2}(\mathbb{D})$ is a bounded operator;
2. $\frac{1}{2π} ∫_{0}^{2π} \frac{1}{|φ(r\epsilon^{iθ})|^{2}} \left(\frac{d}{dφ(r\epsilon^{iθ})}\right)^{-2β} (1 - r) dr dθ < ∞$.

The proofs of Theorems 1 and 2 will be presented in Section 2, and the proofs of Theorems 3, 4 and 5 will be given in Section 3. Theorem 6 will be proved in the last section.

2. Bloch-type growth spaces and applications to PDEs

Green’s theorem (cf. [27]) states that if $g ∈ C^{2}(\mathbb{D})$, then for $r ∈ (0, 1)$,

$$\frac{1}{2π} ∫_{0}^{2π} g(re^{iθ}) dθ = g(0) + \frac{1}{2} ∫_{|z| = r} \Delta g(z) \log \frac{r}{|z|} dσ(z).$$

Lemma 1. Let $f ∈ C^{2}(\mathbb{D})$ such that $\text{Re}(\overline{f}Δf) ≥ 0$. Then for $p ∈ [2, ∞)$, $M_{p}^{α}(r, f)$ is an increasing function of $r$, $r ∈ (0, 1)$.

Proof. First we deal with the case $p ∈ [2, 4)$. In this case, for $n \in \{1, 2, \ldots\}$, we let $F_{n}^{p} = (|f|^{2} + \frac{1}{n})^{\frac{p}{2}}$. Then, by elementary calculations, we have

$$Δ(F_{n}^{p}) = 4\frac{∂^{2}}{∂z∂\overline{z}}(F_{n}^{p})$$

$$= p(p - 2) \left(|f|^{2} + \frac{1}{n}\right)^{\frac{p}{2} - 2} |ff_{z} + f_{\overline{z}}f^{2}|^{2}$$

$$+ 2p \left(|f|^{2} + \frac{1}{n}\right)^{\frac{p}{2} - 1} (|f_{z}|^{2} + |f_{\overline{z}}|^{2}) + p \left(|f|^{2} + \frac{1}{n}\right)^{\frac{p}{2} - 1} \text{Re}(\overline{f}Δf).$$
Let $\tau_n = \Delta(F_n^p)$ and 
\[
\tau = p(p - 2)|f|^{p-2}||D_f||^2 + 2p (|f|^2 + 1)^{p-1} (|f_0|^2 + |f_2|^2) + p (|f|^2 + 1)^{p-1} \text{Re} (\overline{f} \Delta f).
\]
For $r \in (0, 1)$, it is not difficult to see that $\tau_n$ and $\tau$ are integrable in $D_r$, and $\tau_n \leq \tau$.

By (2.1) and Lebesgue’s dominated convergence theorem, we conclude that 
\[
\lim_{n \to \infty} r \frac{d}{dr} M_p^p(r, F_n) = \frac{1}{2} \lim_{n \to \infty} \int_{D_r} \tau_n(z) \, d\sigma(z) \\
= \frac{1}{2} \int_{D_r} \lim_{n \to \infty} \tau_n(z) \, d\sigma(z) \\
= \frac{1}{2} \int_{D_r} \left[ p(p-2)|f(z)|^{p-4}|f(z)f_2(z) + f_\bar{z}(z)f_\bar{\bar{z}}(z)|^2 \\
+ 2p|f(z)|^{p-2}(|f_0(z)|^2 + |f_\bar{z}(z)|^2) \\
+ p|f(z)|^{p-2}\text{Re}(\overline{f(z)} \Delta f(z)) \right] \, d\sigma(z) \\
= r \frac{d}{dr} M_p^p(r, f),
\]
which implies that $M_p^p(r, f)$ is increasing with respect to $r$ in $(0, 1)$.

Next we consider the case $p \in [4, \infty)$. Since 
\[
\Delta(|f|^p) = p(p-2)|f|^{p-4}|f\overline{f_2} + f_\bar{z}\overline{f_\bar{z}}|^2 \\
+ 2p|f|^{p-2}(|f_0|^2 + |f_\bar{z}|^2) + p|f|^{p-2}\text{Re}(\overline{f} \Delta f) \geq 0,
\]
we see that $|f|^p$ is subharmonic in $D$. Hence $M_p^p(r, f)$ is also increasing with respect to $r \in (0, 1)$, and the proof is complete. \qed

**Lemma 2.** Let $f \in C^2(D)$ with $\text{Re}(\overline{f} \Delta f) \geq 0$. Then for $p \in [2, \infty)$, 
\[
\int_{D_r} |f(z)|^p \log \frac{r}{|z|} \, d\sigma(z) \leq \frac{r^2}{2} M_p^p(r, f).
\]

**Proof.** By Lemma 1, we see that 
\[
\int_{D_r} |f(z)|^p \log \frac{r}{|z|} \, d\sigma(z) = \frac{1}{\pi} \int_0^{2\pi} \int_0^r |f(\rho e^{i\theta})|^p \rho \log \frac{r}{\rho} \, d\rho \, d\theta \\
= 2 \int_0^r M_p^p(\rho, f) \rho \log \frac{r}{\rho} \, d\rho \\
\leq 2M_p^p(r, f) \int_0^r \rho \log \frac{r}{\rho} \, d\rho \\
= \frac{r^2}{2} M_p^p(r, f).
\]
The proof of the lemma is complete. \qed

The following lemma easily follows from elementary computations and the monotonicity of the function $\omega(t)/t$. 

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Block-type spaces, Hardy-type spaces and Lipschitz-type spaces
Lemma 3. Suppose that $\alpha > 0$, $\beta \leq \alpha$ and $\omega$ is a majorant. For $r \in (0, 1)$, let
\[ \eta(r) = d^\alpha(r) \left( \log \frac{e}{d(r)} \right)^\beta. \]
Then $\eta(r)$ and $\eta(r)/\omega(\eta(r))$ are decreasing in $(0, 1)$.

Proof of Theorem 1. By Hölder’s inequality, we have
\[
(2.2) \quad \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p-1} \|D_f(re^{i\theta})\| \, d\theta \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{p-1}{p}} \\
\times \left( \frac{1}{2\pi} \int_0^{2\pi} \|D_f(re^{i\theta})\|^p \, d\theta \right)^{\frac{1}{p}} = M_{p-1}(r, f) M_p(r, \|D_f\|),
\]
(2.3) \[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p-2} \|D_f(re^{i\theta})\|^2 \, d\theta \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{p-2}{p}} \\
\times \left( \frac{1}{2\pi} \int_0^{2\pi} \|D_f(re^{i\theta})\|^p \, d\theta \right)^{\frac{2}{p}} = M_{p-2}(r, f) M_p^2(r, \|D_f\|),
\]
and
\[
(2.4) \quad \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{p-1} \, d\theta \leq \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{p-1}{p}} \\
\times \left( \frac{1}{2\pi} \int_0^{2\pi} \, d\theta \right)^{\frac{1}{p}} = M_{p-1}(r, f).
\]
By (1.2), (2.1), (2.2), (2.3), (2.4), Lemmas 2 and 3, and Lebesgue’s dominated convergence theorem, we see that
\[
M_p(r, f) = |f(0)|^p + \frac{1}{2} \int_{E_r} \Delta(|f(z)|^p) \log \frac{r}{|z|} \, d\sigma(z) \\
= |f(0)|^p + \frac{1}{2} \int_{E_r} \left[ p(p-2)|f(z)|^{p-4}|f(z)\overline{f}(z) + f(z)\overline{f}_z(z)|^2 \right. \\
+ 2p|f(z)|^{p-2}(|f_z(z)|^2 + |f(z)|^2) \\
+ p|f(z)|^{p-2} \text{Re}(\overline{f(z)}\Delta f(z)) \left. \right] \log \frac{r}{|z|} \, d\sigma(z) \\
\leq |f(0)|^p + \frac{1}{2} \int_{E_r} \left( p^2 \|f(z)|^{p-2}\|D_f(z)\|^2 \\
+ p|f(z)|^{p-1}|\Delta f(z)| \right) \log \frac{r}{|z|} \, d\sigma(z)
\]
\[
|f(0)|^p + \frac{p}{2} \int_{D^p} \left( p |f(z)|^{p-2} \|Df(z)\|^2 + b(z) |f(z)|^p \\
+ a(z) |f(z)|^{p-1} \|Df(z)\| + q(z) |f(z)|^{p-1} \right) \log \frac{r}{|z|} \, d\sigma(z)
\]

\[
= |f(0)|^p + p^2 \int_0^r \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-2} \|Df(\rho e^{i\theta})\|^2 \, d\theta \right) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ p \sup_{z \in \mathbb{D}} (a(z)) \int_0^r \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-1} \|Df(\rho e^{i\theta})\| \, d\theta \right) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ p \sup_{z \in \mathbb{D}} (b(z)) \int_0^r \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \right) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ p \sup_{z \in \mathbb{D}} (q(z)) \int_0^r \left( \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-1} \, d\theta \right) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
\leq |f(0)|^p + p^2 \int_0^r M_p^{p-2}(\rho, f) M_p^2(\rho, \|Df\|) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ p \sup_{z \in \mathbb{D}} (a(z)) \int_0^r M_p^{p-1}(\rho, f) M_p(\rho, \|Df\|) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (b(z)) M_p^p(\rho, f)
\]

\[
+ p \sup_{z \in \mathbb{D}} (q(z)) \int_0^r M_p^{p-1}(\rho, f) \rho \log \frac{r}{\rho} \, d\rho
\]

which gives

\[
C_p^q(r) M_p^2(r, f) = \left[ 1 - \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (b(z)) \right] M_p^2(r, f)
\]

\[
\leq |f(0)|^2 + p^2 \int_0^r M_p^2(\rho, \|Df\|) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ p \sup_{z \in \mathbb{D}} (a(z)) \int_0^r M_p(\rho, f) M_p(\rho, \|Df\|) \rho \log \frac{r}{\rho} \, d\rho
\]

\[
+ \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (q(z)) M_p(r, f)
\]
\[
= |f(0)|^2 + p^2 \int_0^r M_p^2(\rho, \|D_f\|)(r - \rho) \, d\rho \\
+ \rho \sup_{z \in \mathbb{D}} (a(z)) M_p(r, f) \int_0^r M_p(\rho, \|D_f\|)(r - \rho) \, dt \\
+ \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (q(z)) M_p(r, f)
\]

\[
= |f(0)|^2 + (rp)^2 \int_0^1 M_p^2(tr, \|D_f\|)(1 - t) \, dt \\
+ pr^2 \sup_{z \in \mathbb{D}} (a(z)) M_p(r, f) \int_0^1 M_p(tr, \|D_f\|)(1 - t) \, dt \\
+ \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (q(z)) M_p(r, f)
\]

\[
\leq |f(0)|^2 + (rp) \|f\|_{L_p,\omega}^2 \int_0^1 \frac{d^{2\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{2\beta}}{\omega^2 \left( d^{\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{\beta} \right)} \\
\times \frac{(1 - t) \, dt}{d^{2\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{2\beta}} + pr^2 \|f\|_{L_p,\omega}^2 \sup_{z \in \mathbb{D}} (a(z)) M_p(r, f) \\
\times \int_0^1 \frac{d^{\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{\beta}}{\omega \left( d^{\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{\beta} \right)} \, d^{\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{\beta} \\
+ \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (q(z)) M_p(r, f)
\]

\[
\leq |f(0)|^2 + \left( \frac{rp \|f\|_{L_p,\omega}}{\omega(1)} \right)^2 \int_0^1 \frac{(1 - t) \, dt}{d^{2\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{2\beta}} \\
+ \frac{pr^2 \|f\|_{L_p,\omega}}{\omega(1)} \sup_{z \in \mathbb{D}} (a(z)) M_p(r, f) \int_0^1 \frac{(1 - t) \, dt}{d^{\alpha}(rt) \left( \log \frac{e}{a(rt)} \right)^{\beta}} \\
+ \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (q(z)) M_p(r, f),
\]

where

\[
C_p^\beta(r) = 1 - \frac{pr^2}{4} \sup_{z \in \mathbb{D}} (b(z)).
\]

The desired conclusion follows. \[\square\]

**Lemma 4.** Let \( f \in C^3(\mathbb{D}) \) with \( \text{Re} [(\Delta f) \bar{f}z + (\Delta f) \bar{f}z] \geq 0 \). Then \( F = |f_z|^2 + |f_{\bar{z}}|^2 \) is subharmonic in \( \mathbb{D} \).
Proof. Since $F_z = f_{zz}f_z + f_{z}f_{zz} + f_{z}f_{zz} + f_{zz}f_z$, we see that
$$
\Delta F = 4 \frac{\partial^2 F}{\partial z \partial \overline{z}} = 4(|f_{zz}|^2 + |f_{z}\overline{z}|^2) + \frac{1}{2} |\Delta f|^2 + 2 \Re[(\Delta f)_z f_z + (\Delta f)_z f_{\overline{z}}] \geq 0.
$$
Then $F$ is subharmonic in $\mathbb{D}$. □

Proof of Proposition 1. By Lemma 4, we know that $F = |f_z|^2 + |f_{\overline{z}}|^2$ is subharmonic in $\mathbb{D}$. Then for $r \in [0, d(z))$, we have
$$
F(z) \leq \frac{1}{2\pi} \int_0^{2\pi} F(z + re^{i\theta}) d\theta.
$$
Integration leads to
$$
\frac{d^2(z)F(z)}{4} \leq \int_0^{2\pi} \int_0^{d(z)/2} r |F(z + re^{i\theta})| \frac{dr d\theta}{\pi} = \int_{\mathbb{D}(z, d(z)/2)} F(\zeta) d\sigma(\zeta)
$$
$$
\leq 2^\gamma d^{-\gamma}(z) \int_{\mathbb{D}(z, d(z)/2)} d^{\gamma}(\zeta) F(\zeta) d\sigma(\zeta)
$$
$$
\leq 2^\gamma \|f\|_{D_{\gamma,2}} d^{-\gamma}(z),
$$
which gives
$$
\|D_f(z)\| \leq \sqrt{2F(z)} \leq \frac{C_6}{(d(z))^{1+\gamma/2}},
$$
where $C_6 = 2^{\frac{3+3}{2}} \sqrt{\|f\|_{D_{\gamma,2}}}$. Hence
$$
\sup_{z \in \mathbb{D}} \{(d(z))^{1+\gamma/2}\|D_f(z)\|\} < \infty,
$$
which implies that $f \in \mathcal{L}_{\infty,\omega} B_{1+\gamma/2}^0(\mathbb{D})$, where $\omega(t) = t$. □

The following result is well-known.

Lemma 5. Suppose that $a, b \in [0, \infty)$ and $q \in (0, \infty)$. Then
$$
(a + b)^q \leq 2^{\max\{q-1,0\}}(a^q + b^q).
$$

Proof of Theorem 2. We first prove that
$$
\int_{\mathbb{D}} d(z) \Delta(|f(z)|^{2/\gamma}) d\sigma(z) < \infty.
$$
By (2.5), we have
$$
|f(z)| \leq |f(0)| + \left| \int_{[0, z]} df(\zeta) \right|
$$
$$
\leq |f(0)| + \int_{[0, z]} \|D_f(\zeta)\| |d\zeta|
$$
$$
\leq |f(0)| + \frac{C_7}{(d(z))^{\gamma/2}},
$$
where $C_7 = \left(\frac{2^{1/p}}{\sqrt{\|f\|_{D,2}}}\right)/\gamma$ and $[0, z]$ denotes the line segment from 0 to $z$. Let $p = 2/\gamma$. Then Lemma 5 implies that for $z \in \mathbb{D}$,

$$\tag{2.7} |f(z)|^p \leq \left[|f(0)| + \frac{C_7}{(d(z))^{1/p}}\right]^p \leq 2^{p-1} \left[|f(0)|^p + \frac{C_7^p}{d(z)}\right],$$

$$\tag{2.8} |f(z)|^{p-1} \leq \left[|f(0)| + \frac{C_7}{(d(z))^{1/p}}\right]^{p-1} \leq 2^{p-2} \left[|f(0)|^{p-1} + \frac{C_7^{p-1}}{(d(z))^{(p-1)/p}}\right]$$

and

$$\tag{2.9} |f(z)|^{p-2} \leq \left[|f(0)| + \frac{C_7}{(d(z))^{1/p}}\right]^{p-2} \leq 2^{p-2} \left[|f(0)|^{p-2} + \frac{C_7^{p-2}}{(d(z))^{(p-2)/p}}\right].$$

We divide the remaining part of the proof into two cases, namely $p \in [4, \infty)$ and $p \in (2, 4)$. For the case $p \in [4, \infty)$, easy calculations give

$$\Delta(|f|^p) = 4 \frac{\partial^2}{\partial z \partial \bar{z}}(|f|^p)$$

$$\leq p^2|f|^{p-2}\|D_f\|^2 + p|f|^{p-1}|\Delta f|$$

$$\leq p^2|f|^{p-2}\|D_f\|^2 + pa|f|^{p-1}\|D_f\| + pb|f|^p + pq|f|^{p-1}.$$

Hence we infer from (2.7), (2.8) and (2.9) that for $z \in \mathbb{D}$,

$$d(z) \Delta(|f(z)|^p) \leq p^2d(z)|f(z)|^{p-2}\|D_f(z)\|^2 + pq(z)|f(z)|^{p-1}$$

$$+ p a d(z)|f(z)|^{p-1}\|D_f(z)\| + pb d(z)|f(z)|^p$$

$$= p^2d(z)^{1-\frac{2}{p}}|f(z)|^{p-2}(d(z))^\frac{2}{p}\|D_f(z)\|^2$$

$$+ p \sup_{z \in \mathbb{D}}(a(z))(d(z))^{1-\frac{2}{p}}|f(z)|^{p-1}(d(z))^\frac{2}{p}\|D_f(z)\|$$

$$+ p \sup_{z \in \mathbb{D}}(b(z))d(z)|f(z)|^p + p \sup_{z \in \mathbb{D}}(q(z))d(z)|f(z)|^{p-1}$$

$$\leq C_8(d(z))^{\frac{2}{p}}\|D_f(z)\|^2 + C_9(d(z))^{\frac{2}{p}}\|D_f(z)\| + C_{10},$$

where $C_8 = 2^{p-2}p^2 \left(||f(0)||^{p-2} + C_7^{p-2}\right)$, $C_9 = 2^{p-2}p \sup_{z \in \mathbb{D}}(a(z)) \left(||f(0)||^{p-1} + C_7^{p-1}\right)$ and $C_{10} = 2^{p-1}p \sup_{z \in \mathbb{D}}(b(z)) \left(||f(0)||^p + C_7^p\right) + 2^{p-2}p \sup_{z \in \mathbb{D}}(q(z)) \left(||f(0)||^{p-1} + C_7^{p-1}\right)$. By the Cauchy-Schwarz inequality, we get

$$\tag{2.11} \left(\int_{\mathbb{D}} d\pi(z)\|D(z)\| \, d\sigma(z)\right)^2 \leq \int_{\mathbb{D}} d\pi(z)\|D(z)\|^2 \, d\sigma(z) \int_{\mathbb{D}} d\sigma(z) = \|f\|_{D,2} < \infty.$$
Hence (2.10) and (2.11) imply

\[
\int_{\mathbb{D}} d(z) \Delta(|f(z)|^p) d\sigma(z) \leq \int_{\mathbb{D}} \left[ C_8(d(z))^{\frac{2}{p}} \|Df(z)\|^2 
+ C_9(d(z))^{\frac{1}{p}} \|Df(z)\| + C_{10} \right] d\sigma(z) 
\leq C_8 \|f\|_{\mathcal{D}_{\gamma,2}} + C_9 \|f\|_{\mathcal{D}_{\gamma/2,1}} + C_{10} < \infty.
\]

In the case \( p \in [2, 4) \), we let \( F_{n}^{p} = (|f|^{2} + \frac{1}{n})^{p/2} \) for \( n \in \{1, 2, \ldots\} \). We see that \( \Delta(F_{n}^{p}) \) is integrable in \( \mathbb{D}_r \). Then, by (2.1), (2.10), (2.12) and Lebesgue’s dominated convergence theorem, we have

\[
\lim_{n \to \infty} \int_{\mathbb{D}_r} d(z) \Delta(F_{n}^{p}(z)) d\sigma(z) = 
\int_{\mathbb{D}_r} d(z) \lim_{n \to \infty} \left[ \Delta(F_{n}^{p}(z)) \right] d\sigma(z)
= \frac{1}{2} \int_{\mathbb{D}_r} \left[ p(p-2)|f(z)|^{p-4}|f(z)f_{\bar{z}}(z) + f_{\bar{z}}(z)f(z)|^{2}
+ 2p|f(z)|^{p-2}(|f_{z}(z)|^2 + |f_{\bar{z}}(z)|^2)
+ p|f(z)|^{p-2}\text{Re}\left(f(z)\Delta f(z)\right) \right] d(z) d\sigma(z)
\leq \int_{\mathbb{D}_r} \left[ C_8 d\sigma(z) \|Df(z)\|^2 
+ C_9 d\sigma(z) \|Df(z)\| + C_{10} \right] d\sigma(z) < \infty.
\]

Therefore, (2.6) follows from the two cases.

Next we prove \( f \in H_{g}^{p}(\mathbb{D}) \). As in the proof of Theorem 1.4 in [7], for a fixed \( r \in (0, 1) \), since

\[
\lim_{|z| \to r} \frac{\log r - \log |z|}{r - |z|} = \frac{1}{r},
\]
we see that there is an \( r_0 \in (0, r) \) satisfying

\[
\log r - \log |z| \leq \frac{2}{r}(r - |z|)
\]
for \( r_0 \leq |z| < r \). Then it follows from \( \lim_{\rho \to 0^+} \rho \log(1/\rho) = 0 \) that

\[
\int_{\mathbb{D}_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z) \leq \int_{\mathbb{D}_{r_0}} \Delta(|f(z)|^p) \log \frac{1}{|z|} d\sigma(z)
= \int_{0}^{2\pi} \int_{r_0}^{r} \Delta(|f(\rho e^{i\theta})|^p) \rho \log \frac{1}{\rho} d\rho d\theta < \infty.
\]
Hence, by (2.1), (2.6), (2.13) and (2.14), we obtain

\[
M_p^p(r, f) = |f(0)|^p + \frac{1}{2} \int_{D_r} \Delta(|f(z)|^p) \log \frac{r}{|z|} \, d\sigma(z)
\]

\[
= |f(0)|^p + \frac{1}{2} \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} \, d\sigma(z)
\]

\[
+ \frac{1}{2} \int_{D_r \setminus D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} \, d\sigma(z)
\]

\[
\leq |f(0)|^p + \frac{1}{2} \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} \, d\sigma(z)
\]

\[
+ \int_{D_r \setminus D_{r_0}} \Delta(|f(z)|^p) \left( \frac{r - |z|}{r} \right) \, d\sigma(z)
\]

\[
\leq |f(0)|^p + \frac{1}{2} \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{1}{|z|} \, d\sigma(z)
\]

\[
+ \int_{D \setminus D_{r_0}} d(z) \Delta(|f(z)|^p) \, d\sigma(z)
\]

\[
< \infty,
\]

which implies that \( f \in H^p_g(\mathbb{D}). \)

3. Lipschitz-type spaces

The following simple lemma is useful in the sequel.

**Lemma 6.** Let \( \omega \) be a majorant and \( \nu \in (0, 1] \). Then for \( t \in (0, \infty) \), \( \omega(\nu t) \geq \nu \omega(t) \).

**Proof.** Since \( \omega(t)/t \) is decreasing on \( t \in (0, \infty) \), we see that

\[
\frac{\omega(\nu t)}{\nu t} \geq \frac{\omega(t)}{t}
\]

and the desired conclusion follows. \( \square \)

**Proof of Theorem 3.** We first prove the sufficiency. For \( r \in (0, 1) \) and \( \theta \in [0, 2\pi] \), let \( w = z + re^{i\theta} \). Then

\[
\|D_f(z)\| = \max_{\theta \in [0, 2\pi]} |f_x(z) \cos \theta + f_y(z) \sin \theta|
\]

\[
= \max_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{|f(z + re^{i\theta}) - f(z)|}{r} \right\}
\]

\[
= \max_{\theta \in [0, 2\pi]} \left\{ \lim_{r \to 0^+} \frac{|f(z) - f(w)|}{|z - w|} \right\}
\]

\[
\leq \lim_{r \to 0^+} \frac{C_1}{\omega \left( d^s(z) d^{\alpha-s}(z + re^{i\theta}) \right)}
\]

\[
= \frac{C_2}{\omega(d^\alpha(z))}.
\]
Next we prove the necessity. For \( z, w \in \mathbb{D} \), let \( \chi(t) = zt + (1-t)w \), where \( t \in [0, 1] \).

Since
\[
1 - |\chi(t)| \geq 1 - t|z| - |w| + t|w| \geq (1 - t)(1 - |w|) = (1 - t)d(w)
\]
and similarly, \( 1 - |\chi(t)| \geq td(z) \), we see that
\[
(1 - |\chi(t)|)^{\alpha - s} \geq (1 - t)^{\alpha - s}d^{\alpha - s}(w)
\]
and
\[
(1 - |\chi(t)|)^{s} \geq t^sd(z).
\]

By (3.1) and (3.2), we get
\[
t^s(1 - t)^{\alpha - s}d^s(z)d^{\alpha - s}(w) \leq (1 - |\chi(t)|)^{s},
\]
which implies
\[
\omega \left( t^s(1 - t)^{\alpha - s}d^s(z)d^{\alpha - s}(w) \right) \leq \omega \left( (1 - |\chi(t)|)^{s} \right) = \omega \left( d^s(\chi(t)) \right).
\]

Hence, for \( z, w \in \mathbb{D} \) with \( z \neq w \), by Lemma 6, we know that there is a positive constant \( C \) such that
\[
|f(z) - f(w)| = \left| \int_0^1 \frac{df}{dt}(\chi(t)) \, dt \right| \quad (\zeta = \chi(t))
\]
\[
\leq |z - w| \int_0^1 \|Df(\chi(t))\| \, dt
\]
\[
= |z - w| \int_0^1 \frac{\|Df(\chi(t))\|}{\omega(d^s(\chi(t)))} \omega(d^s(\chi(t))) \, dt
\]
\[
\leq C|z - w| \int_0^1 \frac{dt}{\omega(d^s(\chi(t)))}
\]
\[
\leq \frac{C|z - w|}{\omega(d^s(z)d^{\alpha - s}(w))} \int_0^1 \frac{dt}{(1 - t)^{\alpha - s}t^s}
\]
\[
= \frac{C|z - w|}{\omega(d^s(z)d^{\alpha - s}(w))} B(1 - s, 1 + s - \alpha),
\]
where \( B(\cdot, \cdot) \) denotes the Beta function. Thus, there is a positive constant \( C_1 = CB(1 - s, 1 + s - \alpha) \) such that for all \( z \) and \( w \) with \( z \neq w \),
\[
\frac{|f(z) - f(w)|}{|z - w|} \leq \frac{C_1}{\omega(d^s(z)d^{\alpha - s}(w))}.
\]
The proof of this theorem is complete.
Lemma B. ([6, Lemma 2.2]) Suppose that \( f \) is a harmonic mapping in \( \overline{D}(a, r) \), where \( a \in \mathbb{C} \) and \( r > 0 \). Then
\[
\|D_f(a)\| \leq \frac{2}{\pi r} \int_0^{2\pi} |f(a + re^{i\theta}) - f(a)| \, d\theta.
\]

Proof of Theorem 4. We first prove the sufficiency. By Lemma B, for \( \rho \in (0, d(z)] \),
\[
\|D_f(z)\| \leq \frac{2}{\pi \rho} \int_0^{2\pi} |f(z + \rho e^{i\theta}) - f(z)| \, d\theta,
\]
which gives
\[
\int_0^r \rho^2 \|D_f(z)\| \, d\rho \leq \frac{2}{\pi} \int_0^r \left( \rho \int_0^{2\pi} |f(z + \rho e^{i\theta}) - f(z)| \, d\theta \right) \, d\rho,
\]
where \( r = d(z) \). Then
\[
\|D_f(z)\| \leq \frac{6}{\pi r^3} \int_{D(z,r)} |f(z) - f(\zeta)| \, dA(\zeta) = \frac{6}{r|D(z, r)|} \int_{D(z,r)} |f(z) - f(\zeta)| \, dA(\zeta) \leq \frac{6C_2}{\omega(r^\alpha)}.
\]

Now we prove the necessity. Since \( f \in L_{\infty, \omega} B_0^\alpha(\mathbb{D}) \), we see that there is a positive constant \( C \) such that
(3.3) \[
\|D_f(z)\| \leq \frac{C}{\omega(d^\alpha(z))},
\]
For \( z, w \in \mathbb{D} \) and \( t \in [0, 1] \), if \( d(z) > t|z - w| \), then, by (3.3), we get
\[
|f(z) - f(w)| \leq |z - w| \int_0^1 \|D_f(z + t(w - z))\| \, dt \leq C|z - w| \int_0^1 \frac{dt}{\omega(d^\alpha(z + t(w - z)))} \leq C|z - w| \int_0^1 \frac{dt}{\omega((d(z) - t|z - w|)^\alpha)} = C \int_0^{[z-w]} \frac{dt}{\omega((d(z) - t)^\alpha)},
\]
which implies
\[
\frac{1}{|D(z, r)|} \int_{D(z,r)} |f(z) - f(\zeta)| \, dA(\zeta) \leq \frac{C}{|D_r|} \int_{D_r} \left( \int_0^{[\xi]} \frac{dt}{\omega((d(z) - t)^\alpha)} \right) \, dA(\xi) = \frac{2C}{r^2} \int_0^r \rho \left( \int_0^{\rho} \frac{dt}{\omega((d(z) - t)^\alpha)} \right) \, d\rho.
\]
\[
\leq \frac{2C}{r^{2}} \int_{0}^{r} \left( \int_{t}^{r} \rho \, dp \right) \frac{dt}{\omega \left( (r - t)^{\alpha} \right)} \\
= \frac{2C}{r} \int_{0}^{r} \frac{(r - t)^{\alpha}}{\omega \left( (r - t)^{\alpha} \right)} (r - t)^{1 - \alpha} \, dt \\
\leq \frac{2Cr^{\alpha - 1}}{\omega \left( r^{\alpha} \right)} \int_{0}^{r} (r - t)^{1 - \alpha} \, dt \\
= C_{2} \frac{r}{\omega \left( r^{\alpha} \right)},
\]

where \( C_{2} = \frac{2C}{2 - \alpha} \). The proof of this theorem is complete. \( \square \)

**Proof of Theorem 5.** We first prove the necessity. For \( r \in (0, 1) \), let \( F(z) = f(rz) \).

By the proof of necessity part of Theorem 3, we see that there is a positive constant \( C \) such that

\[
(3.4) \quad \frac{|(F(z) - f(z)) - (F(w) - f(w))| \omega \left( d^{s}(z)d^{\alpha-s}(w) \right)}{|z - w|} \leq C \| f - F \|_{L_{\infty,\omega}B_{0}^{\alpha}(B)}.
\]

Since \( \omega(t)/t \) is non-increasing for \( t > 0 \), we know that there is a positive constant \( C \) such that

\[
\frac{|F(z) - F(w)| \omega \left( d^{s}(z)d^{\alpha-s}(w) \right)}{|z - w|} = \frac{r|F(z) - F(w)| \omega \left( d^{s}(rz)d^{\alpha-s}(rw) \right)}{|rz - rw|} \\
\times \frac{\omega \left( d^{s}(z)d^{\alpha-s}(w) \right)}{\omega \left( d^{s}(rz)d^{\alpha-s}(rw) \right)} \\
\leq C \| f \|_{L_{\infty,\omega}B_{0}^{\alpha}(B)} \frac{\omega \left( d^{s}(z)d^{\alpha-s}(w) \right)}{\omega \left( d^{s}(rz)d^{\alpha-s}(rw) \right)} \\
\times \frac{\omega \left( d^{s}(z)d^{\alpha-s}(w) \right)}{\omega \left( d^{s}(rz)d^{\alpha-s}(rw) \right)} \\
\leq C \| f \|_{L_{\infty,\omega}B_{0}^{\alpha}(B)} \left( \frac{d(z)}{d(rz)} \right)^{s} \left( \frac{d(w)}{d(rw)} \right)^{\alpha-s} \\
\leq C \| f \|_{L_{\infty,\omega}B_{0}^{\alpha}(B)} \left( \frac{d(z)}{d(rz)} \right)^{s}.
\]

By using the triangle inequality, we have

\[
\sup_{z \neq w} \left\{ \frac{|f(z) - f(w)| \omega \left( d^{s}(z)d^{\alpha-s}(w) \right)}{|z - w|} \right\} \leq C \| f - F \|_{L_{\infty,\omega}B_{0}^{\alpha}(B)} + C \| f \|_{L_{\infty,\omega}B_{0}^{\alpha}(B)} \left( \frac{d(z)}{d(rz)} \right)^{s}.
\]

In the above inequality, first letting \( |z| \to 1- \) and then letting \( r \to 1- \), we get the desired result.
Next we begin to prove the sufficiency. Suppose (1.7) holds. For all \( \epsilon > 0 \), there is a \( \delta \in (0, 1) \) such that

\[
\sup_{w \in \mathbb{D}, z \neq w} \left\{ \frac{|f(z) - f(w)| \omega(d^s(z)d^{\alpha-s}(w))}{|z - w|} \right\} < \epsilon,
\]
whenever \( |z| > \delta \). Let \( w \) tend to \( z \) in the radial direction, we obtain

\[
\|D_f(z)\| \omega(d^s(z)) \leq \epsilon
\]
whenever \( |z| > \delta \), which yields \( f \in \mathcal{L}_{\infty, \omega} B^0_\alpha(\mathbb{D}) \).

\[\square\]

4. Composition operators

Given \( f \in \mathcal{A}(\mathbb{D}) \), the Littlewood-Paley \( g \)-function is defined as follows

\[
g(f)(\zeta) = \left( \int_0^1 |f'(r\zeta)|^2(1-r) \, dr \right)^{\frac{1}{2}}, \quad \zeta \in \partial \mathbb{D}.
\]

By [36, Theorems 3.5 and 3.19], we know that \( f \in H^p(\mathbb{D}) \) if and only if \( g(f) \in H^p_g(\mathbb{D}) \) for \( p > 1 \).

**Proof of Theorem 6.** We first prove that (1) \( \implies \) (2). Applying [1, Lemma 1] and Lemma 3, we see that there are two functions \( f_1, f_2 \in \mathcal{L}B^3_\alpha(\mathbb{D}) \) such that for \( z \in \mathbb{D} \),

\[
|f_1'(z)|^2 + |f_2'(z)|^2 \geq d^{-2\alpha}(z) \left( \log \frac{e}{d(z)} \right)^{-2\beta}.
\]

Since for \( k = 1, 2 \), \( C_\phi(f_k) \in H^2(\mathbb{D}) \), by (4.1), we conclude that

\[
\begin{align*}
\infty &> \|g(C_\phi(f_1))\|_2^2 + \|g(C_\phi(f_2))\|_2^2 \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left( |f_1'(\phi(r\zeta))|^2 + |f_2'(\phi(r\zeta))|^2 \right) \left| \phi'(r\zeta) \right|^2 (1-r) \, dr \, d\theta \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| \phi'(re^{i\theta}) \right|^2 \left( \log \frac{e}{d(\phi(re^{i\theta}))} \right)^{-2\beta} (1-r) \, dr \, d\theta,
\end{align*}
\]

which shows that (1) \( \implies \) (2).

Next we prove (2) \( \implies \) (1). For \( f \in \mathcal{L}B^3_\alpha(\mathbb{D}) \) and \( \zeta \in \partial \mathbb{D} \), we get

\[
g^2(C_\phi(f))(\zeta) = \int_0^1 |(C_\phi(f)(r\zeta))'|^2(1-r) \, dr \\
= \int_0^1 |f'(\phi(r\zeta))|^2 |\phi'(r\zeta)|^2 (1-r) \, dr \\
= \int_0^1 |f'(\phi(r\zeta))|^2 d^{2\alpha}(\phi(re^{i\theta})) \left( \log \frac{e}{d(\phi(re^{i\theta}))} \right)^{2\beta} \\
\times |\phi'(r\zeta)|^2 d^{-2\alpha}(\phi(re^{i\theta})) \left( \log \frac{e}{d(\phi(re^{i\theta}))} \right)^{-2\beta} (1-r) \, dr \\
\leq \|f\|_{\mathcal{L}B^3_\alpha(\mathbb{D})}^2 \int_0^{2\pi} \int_0^1 \left| \phi'(re^{i\theta}) \right|^2 \left( \log \frac{e}{d(\phi(re^{i\theta}))} \right)^{-2\beta} (1-r) \, dr,
\]

\[\square\]
which yields (2)⇒(1), whence \( g(C_\phi(f)) \in H_g^2(D) \). The proof of this theorem is complete. □

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