Gauss-Bonnet black holes in a special anisotropic scaling spacetime

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Inspired by the Lifshitz gravity as a theory with anisotropic scaling behavior, we suggest a new \((n+1)\)-dimensional metric in which the time and spatial coordinates scale anisotropically as \((t, r, \theta_i) \rightarrow (\lambda^x t, \lambda^{-z} r, \lambda^z \theta_i)\). Due to the anisotropic scaling dimension of the spatial coordinates, this spacetime does not support the full Schrödinger symmetry group. We look for the analytical solution of Gauss-Bonnet gravity in the context of the mentioned geometry. We show that Gauss-Bonnet gravity admits an analytical solution provided that the constants of the theory are properly adjusted. We obtain an exact vacuum solution, independent of the value of the dynamical exponent \(z\), which is a black hole solution for the pseudo-hyperbolic horizon structure and a naked singularity for the pseudo-spherical boundary. We also obtain another exact solution of Gauss-Bonnet gravity under certain conditions. After investigating some geometrical properties of the obtained solutions, we consider the thermodynamic properties of these topological black holes and study the stability of the obtained solutions for each geometrical structure.

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I. INTRODUCTION

Recently, there has been considerable interest in studying the AdS/CFT correspondence which describes the duality between weakly coupled classical relativistic gravitational systems in anti-de Sitter (AdS) spacetime and certain strongly coupled conformal field theory (CFT) [8]. This correspondence provides a new tool to analyze the systems in some branches of physics such as condensed matter physics [4], QCD quark-gluon plasmas [5] and atomic physics [6, 7]. Over the last several years, the framework of gravity-gauge duality has been generalized to a much wider context than its original formulation and has been extended beyond the relativistic domain. There are a variety of motivations behind this generalization, a number of which are mentioned in what follows. Some real condensed matter systems are described in the vicinity of their critical temperatures by the non-relativistic conformal field theories since they do not have relativistic symmetry in the proximity of this critical point [4, 8]. Another application is using a tool to cold atomic systems in the unitary limit where the system of two-component fermions interacting through a short-ranged potential which is fine-tuned to support a zero-energy bound state and exhibits a non-relativistic conformal symmetry in the limit of zero-range potential [9]. Besides, investigation of the high-temperature superconductors in modern condensed matter physics is one of the targets of gravity-gauge generalization [8]. These attractive topics motivate us to study the non-relativistic version of AdS/CFT duality in order to gain more information about these systems. According to this version of duality, spacetimes with the symmetry group known as Schrödinger group are the gravity makes dual for the non-relativistic scale-invariant systems enjoying Galilean symmetry [10, 11]. In this way, Lifshitz spacetimes, which were originally introduced in [12], are understood as gravitational theories dual to non-relativistic field theories at zero temperature. This kind of gravity model exhibits scaling properties which are anisotropic between space and time directions, i.e. \((t, x) \rightarrow (\lambda^x t, \lambda^z x)\). Such an anisotropic scaling has an effective role in quantum phase transitions in condensed matter systems and ultracold atomic gases [12] and is characterized by the "dynamical critical exponent" \(z\) (where \(z\) measures the degree of anisotropy between space and time directions). Theories with \(z \neq 1\) are invariant under non-relativistic transformations [12, 14] while for \(z = 1\), the metric reduces to the relativistic isotropic scale invariance spacetime corresponding to the AdS geometry.

It is worth mentioning that a metric with Lifshitz geometry is not a solution to the vacuum Einstein field equations with a cosmological constant except for \(z = 1\). For \(z \neq 1\), Einstein’s gravity should be coupled with other fields or modified by considering higher curvature terms investigated in Refs. [14–21]. In other words, since there is nothing to produce an anisotropy in the spacetime in pure Einstein gravity, Einstein’s equations do not allow anisotropic solutions. However, adding some higher-curvature tensors or matter sources to this gravity may lead to anisotropic solutions for field equations.

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A special candidate for the higher curvature corrections to Einstein’s gravity is the Lovelock theory, which is the most general (purely metric) gravitational theory leading to generally covariant field equations in higher dimensions 22. It is worthwhile to note that the higher curvature gravity is also naturally obtained as next-to-leading term in heterotic string effective action in the low-energy limit 23, 27. The simplest Lovelock theory of gravity is well-known as Gauss-Bonnet gravity including Einstein-Hilbert action, cosmological constant, and quadratic curvature terms that is a topological invariant in four and lower dimensions and is not effective in dynamics of the system 22, 28. These curvature-squared terms result in the significant feature that only up to the second-order derivations of metric functions appear in the corresponding field equations 23, 31. Moreover, this theory of gravity is a higher derivative gravity enjoying the absence of ghost modes 23, 24. It is worth mentioning that string theory predicts some additional scalar fields coupled to the Gauss-Bonnet invariant that are important in the appearance of non-singular early time cosmologies and the late-time cosmic acceleration 33, 36. Strictly speaking, considering the heterotic string effective action (loop corrected superstring effective action), one can find the contribution of the Gauss-Bonnet term as well as the existence of a dilatonic field. However, by requiring the dilaton being a constant at the Lagrangian level, one can obtain a pure Gauss-Bonnet term without a scalar field. Nevertheless, there are some important motivations to investigate Gauss-Bonnet gravity without additional scalar/vector fields. For instance, almost the complete study of dilatonic Gauss-Bonnet gravity requires numerical or perturbative evaluation, while one can obtain analytical solutions in the absence of a dilaton field. Besides, from the AdS/CFT correspondence viewpoint, the Gauss-Bonnet term can be viewed as the next-to-leading order corrections of large N expansion of boundary CFTs in the strong coupling limit 37. As a result, various aspects of Gauss-Bonnet gravity and its thermodynamic properties have been addressed in literature 38–42. In the context of holography, the effects of the Gauss-Bonnet gravity on different properties of the system have been explored in some physical contexts such as finite coupling 43, second-order transport 44, entanglement entropy 45, 48 and superconductivity 49, 51.

Here, we consider Gauss-Bonnet gravity in a modified Lifshitz geometry background. To generalize the Lifshitz gravity, one could engineer theories that do not admit Galilean boosts and then experience the anisotropic scaling spatial coordinates. These theories have some applications in condensed matter systems, including optimally doped cuprates and nonfermi liquid metals near heavy electron critical points 52, 53. Besides, there is more evidence that such anisotropies exist in our universe. For instance, it was investigated in Ref. 50 that due to the discrete structure of the spacetime, the first-order quantum corrections may lead to space anisotropy. Such spatial anisotropies will have measurable consequences at short distances if the corrections are between the electroweak and the Planck scale. The mentioned corrections can be incorporated using an anisotropic generalized uncertainty principle (GUP), where the deformation from quantum gravity depends on the direction chosen. One of the main motivations to study the anisotropic GUP is to explain the observed Cosmic Microwave Background (CMB) anisotropies 57, 58. Investigation of the CMB anisotropies has a significant role in developing the modern cosmology and our understanding of the very early universe 59. Another system which is spatially anisotropic is super Yang-Mills (SYM) plasma 60. In fact, the plasma created in a heavy-ion collision can be locally anisotropic for some short time after the collision, $T < T_{iso}$, and then becomes locally isotropic 61. The investigation of anisotropic Lifshitz-like geometries may contribute to the application of holographic methods to this type of system.

Here, motivated by the mentioned observation of anisotropic scaling behaviors, we propose a geometry in which the spatial coordinates also scale anisotropically. We try to investigate the existence of this class of geometry as a solution in Gauss-Bonnet gravity in the vacuum. Since the higher power curvature terms seem to play the role of the desired matter field, we check whether Gauss-Bonnet gravity can support the proposed geometry in the vacuum. Our calculations show that this demand is met under certain conditions. We also explore the properties of the black hole/brane solution in Gauss-Bonnet gravity.

This paper is organized as follows: After the introduction, we briefly review theories with an anisotropic scaling between time and space in Section III. and then we give a particular anisotropic metric where the spatial coordinates scale anisotropically as well. In Sec. III, we consider $n + 1$-dimensional Gauss-Bonnet action and obtain field equations under the mentioned metric ansatz. With these equations in hand, we solve the field equations analytically in vacuum and discuss the main properties of the solution in IV and VI. Thermodynamic behavior and the stability of the black hole solutions are investigated in IV A and VI A. Finally, we end the paper with remarkable results in Sec. VI.

II. BACKGROUND WITH ANISOTROPIC SCALING DIMENSIONS

Taking the problem of renormalizability in UV scale into account, one may consider a Lorentz violation theory in which the higher spatial derivatives are decomposed from higher time one. In this regard, the Horava-Lifshitz approach is a method that assists us in constructing a theory with an anisotropic scaling between time and space as

\[ t \to \lambda^2 t, \quad r \to \lambda r, \]

(1)
where $z$ is the dynamical critical exponent \cite{62}. Generally, a family of backgrounds that geometrically follows this type of scaling symmetry is known as Lifshitz spacetimes, which are a generalization of AdS space and in arbitrary $(n + 1)$-dimensions can be expressed as

$$ ds^2 = -\frac{r^{2z}}{l^{2z}} dt^2 + \frac{r^2 dr^2}{r^2} + r^2 \sum_{i=1}^{n-1} d\theta_i^2, $$

with the following scale transformation

$$ t \to \lambda^z t, \quad \theta_i \to \lambda \theta_i, \quad r \to \lambda^{-1} r, $$

where $z = 1$ gives the usual metric on $AdS_{n+1}$. However, we can consider a situation in which the spatial coordinates also scale anisotropically. To this end, we suggest the following form of the metric

$$ ds^2 = -\frac{r^{2z}}{l^{2z}} dt^2 + \frac{r^2 dr^2}{r^2} + l^2 \sum_{i=1}^{n-1} \frac{r^{2x_i}}{l^{2x_i}} d\theta_i^2, $$

which is invariant under the following scale transformations

$$ t \to \lambda^z t, \quad \theta_i \to \lambda^{x_i} \theta_i, \quad r \to \lambda^{-1} r, $$

where $z$ and $x_i$ play the role of dynamical exponents.

More generally, the metric of an $(n + 1)$-dimensional static spacetime with different geometry that asymptotically (in a special limit) goes to the metric (4) can be written as

$$ ds^2 = -\frac{r^{2z}}{l^{2z}} f(r) dt^2 + \frac{r^2 dr^2}{r^2} + l^2 d\Omega_{n-1,k}^2, $$

where the functions $f(r)$ and $g(r)$ should go to 1 asymptotically ($r \to \infty$). Also, $d\Omega^2$ is the metric of an $(n - 1)$-dimensional hypersurface which can be written as

$$ d\Omega_{n-1,k}^2 = \begin{cases} 
\frac{r^{2z_1}}{l^{2z_1}} d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \frac{r^{2x_i}}{l^{2x_i}} \sin^2 \theta_j d\theta_i^2 & k = 1 \\
\sum_{i=1}^{n-1} \frac{r^{2x_i}}{l^{2x_i}} d\theta_i^2 & k = 0 \\
\frac{r^{2z_1}}{l^{2z_1}} d\theta_1^2 + \sinh^2 \theta_1 \left( \frac{r^{2z_2}}{l^{2z_2}} d\theta_2^2 + \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \frac{r^{2x_i}}{l^{2x_i}} \sin^2 \theta_j d\theta_i^2 \right) & k = -1
\end{cases} $$

For $(x_i, z) > 1$, the scaling is asymmetric between the time and spatial directions and the system is scale invariant without (along with) conformal invariance for $k = \pm 1 (k = 0)$.

It is important to note that Einstein’s gravity with a negative cosmological constant does not admit the metric (4) and at least a matter source is required to engineer the scale invariant anisotropic background. However, in what follows, we show that one may have the geometry (4) as an analytical solution of Gauss-Bonnet gravity without matter, indicating that the higher curvature terms could have the desired effect that matter fields induce.

Before ending this section, it should be mentioned that since we are looking for exact solutions reducing to the metric (4) while $r$ goes to infinity, we always impose the following condition on the solutions throughout the paper

$$ \lim_{r \to \infty} f(r) = \lim_{r \to \infty} g(r) = 1. $$

### III. BLACK HOLE/BRANE SOLUTIONS IN GAUSS-BONNET GRAVITY

The Einstein-Hilbert action with a Gauss-Bonnet term in the presence of cosmological constant can be written down as \cite{29}

$$ S = \frac{1}{16\pi G} \int d^{n+1}x \sqrt{-g} (R - 2\Lambda + \alpha \mathcal{L}_{GB}), $$

where $\mathcal{L}_{GB}$ is the Gauss-Bonnet lagrangian.
where \( \mathcal{L}_{GB} = R^\mu\nu\gamma\delta R_{\mu\nu\gamma\delta} - 4R^\mu\nu R_{\mu\nu} + R^2 \) and \( \alpha \) indicates the Gauss-Bonnet coefficient with dimension \((\text{length})^2\) which is positive according to the phenomenological string theory \[29\]. The gravitational field equation is obtained by variation of the action with respect to the metric, resulting in

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = 2\alpha \left( \frac{1}{4}g_{\mu\nu}\mathcal{L}_{GB} - RR_{\mu\nu} + 2R_{\mu\gamma}R_{\nu}^\gamma + 2R_{\mu\gamma\delta\nu} \right) .
\]  

(10)

According to what mentioned before, we consider the black hole/brane solutions in an \((n + 1)\)-dimensional anisotropic spacetime. To this end, using the metric \[11\], the field equation \[10\] reduces to the following differential equations

\[
E_1 = 2k \left[ \hat{\alpha}'(r)x + \left( \frac{n-2}{2} \right) \hat{\alpha} \right] \left( \frac{r}{l} \right)^{2x} + nx \left[ r^3 \hat{\alpha} g(r) + x^2 f(r) g(r) \right] \left( \frac{r}{l} \right)^{4x} - k^2 \hat{\alpha}(n-4) = 0 ,
\]  

(11)

\[
E_2 = k \left[ x^2 \hat{\alpha} g(r) f'(r) r - \frac{f(r)}{2} \left( l^2 (n-2) - x^2 + 2x(n-3)(n-2) \right) g(r) \right] \left( \frac{r}{l} \right)^{2x} - \frac{1}{2} k^2 \hat{\alpha} f(r)(n-4) + \left\{ \frac{x r^2 \hat{\alpha} g(r) f'(r)}{2} + f(r) \left[ 1 + \frac{x(n-2)}{2z} \right] x z l^2 g(r) - x^2 \hat{\alpha} g(r) \left( \frac{x}{2} (n-4) + 2z \right) + \frac{\Lambda^4}{(n-1)} \right\} \left( \frac{r}{l} \right)^4 = 0 ,
\]  

(12)

\[
E_3 = \frac{1}{2} g(r) f(r) x^2 \left( \frac{1}{2} f'(r) + r f'(r) g(r) \right) + r f'(r) g(r) \left[ \frac{r^2 g(r)}{4} - \frac{\Lambda}{(n-2)x + 2z + 1} \right] \left( \frac{r}{l} \right)^{2x} - 2 g(r) x^2 \hat{\alpha} \left( \frac{(n-4)(n-1)x^2}{4} + (n-2)xz + z^2 \right) + \frac{\Lambda^4}{(n-1)} \left( \frac{r}{l} \right)^{-2x} + k \hat{\alpha} g(r) f(r) f''(r) r^2 - \frac{1}{2} k \hat{\alpha} g(r) f'(r) r^2 + k \alpha f(r) f'(r) \left( \frac{r^2 g(r)}{2} \right) \left( \frac{r}{l} \right)^{2x} - 2 \hat{\alpha} \left( \frac{(n-3)(n-2)x^2}{2} + 2x + 2z \right) \right] = 0 ,
\]  

(13)

where \( \mathcal{Y} = l^2 - 2 \hat{\alpha} g(r) x^2 \), prime denotes the derivative with respect to \( r \) and we have defined \( \hat{\alpha} \equiv (n-2)(n-3) \hat{\alpha} \) for the sake of brevity. It should be mentioned that in order to have a consistent solution, we have to set \( x_i = x \). For convenience, in the next section we have applied this condition in the metric from the beginning.

### A. Vacuum Solutions

We first investigate the possibility of having an \((n + 1)\)-dimensional solution of our proposed metric in the absence of matter field. To this end, one can easily find that for arbitrary \( z \), the following metric function can be obtained

\[
g(r) = 1 + k \frac{\hat{\alpha}_{\text{eff}}}{r^2} ,
\]  

(14)

where

\[
\hat{\alpha}_{\text{eff}} = \hat{\alpha} x \pm \left( \hat{\alpha} x^2 - x^3 \hat{\alpha} \right)^{1/2} \delta_{n,4} ,
\]  

(15)

provided that the following constraints on the cosmological constant and Gauss-Bonnet coefficient are applied

\[
\Lambda = - \frac{n(n-1)x^2}{4l^2} \quad \text{and} \quad \hat{\alpha} = 2x^2 \hat{\alpha} = l^2 ,
\]  

(16)

or equivalently \( \Lambda = - \frac{n(n-1)}{8 \alpha} \hat{\alpha} \). We should note that, here, the value of the cosmological constant and Gauss-Bonnet coefficient depend on the dynamical exponent \( x \) which is the scaling dimension of the \( \theta_i \) coordinates.
According to [15] and dimensional analysis, one can easily find that the dynamical exponent \( x \) must be equal to one for 5—dimensional spacetime and hence, our proposed metric in 5—dimension reduces to that in Lifshitz gravity. However, in higher dimensions, there is no restriction on the value of the parameter \( x \). Therefore, in the following, we will concentrate on the dimensions higher than 5.

Notably, the function of \( f(r) \) cannot be determined by the field equations. In fact, substituting (13) into (11) causes the field equations to become zero, independent of the function of \( f(r) \). The degeneracy of the field equations which leads to the arbitrary function of \( f(r) \) has been previously considered in 5—dimensional Einstein-Gauss-Bonnet gravity with a cosmological constant [63]. However, this degeneracy may be removed in the presence of the matter field [15].

B. General Solutions

Generally, the solutions of our suggested metric (6) in the Einstein-Gauss-Bonnet gravity will be obtained as follows

\[
g(r) = f(r) = 1 + k \frac{\hat{\alpha}^2}{x^2 r^2x} \pm \sqrt{\frac{16\pi G \hat{\alpha} M}{x(n-1)r^nx}}, \tag{17}
\]

providing that the constraints (16) are held and the dynamical exponent \( x \) and \( z \) have the same values. Moreover, in order to correctly reproduce the asymptotic behavior, we have imposed the condition (8) on the solution. Also, in the above relation, \( M \) is an integration constant known as the mass of the solution per unit volume, i.e. \( M = \frac{\hat{\alpha}^2}{2\pi} \), and \( G \) is the gravitational constant. It should be mentioned that the coefficient of the mass term is set by comparing the obtained solution with the Gauss-Bonnet black holes in AdS spacetime provided that \( \hat{\alpha} = \frac{2}{x} [32] \) and considering the fact that the cosmological constant, here, for the case of \( x = 1 \) is half of that in the AdS spacetime. Also, note that this solution has two branches with "+" and "−" signs. Before proceeding further, it is worth mentioning two points: First, one may expect to obtain the usual AdS black hole solutions in Gauss-Bonnet gravity for the case of \( x = 1 \). However, due to the constraints (15), our solution will be equivalent to the AdS solution with \( \hat{\alpha} = -\frac{n(n-1)}{8\Lambda} \) with a difference that, as noted previously, the cosmological constant for the case of \( x = 1 \) in our model is half of that in AdS spacetime. Second, since the null energy condition plays a vitally important role in gravity [64, 65], it is important to explore if the discussed spacetime satisfies it. To this end, we use the geometric form of null energy condition [66]

\[
G_{\mu\nu}N^\mu N^\nu \geq 0,
\]

where \( N^\mu \) is a null vector. Choosing

\[
N^i = \left( \frac{l}{r} \right)^{\frac{2}{x}} \frac{1}{\sqrt{f(r)}}, \quad N^r = \frac{r \sqrt{g(r)}}{l}, \quad N^i = 0, \tag{18}
\]

and making use of (10), one can easily find that \( G_{\mu\nu}N^\mu N^\nu = 0 \) for the solution (17) which means that independent of the dynamical exponent \( x \), these spacetimes meet the null energy condition for three kinds of horizon topology, \( k \).

In what follows, we explore different geometric and thermodynamic properties of ± branches of the solution (17), separately.

IV. NEGATIVE SIGN BRANCH

To investigate some properties of the solution with "−" sign, we have provided some diagrams related to the behavior of the metric function \( f(r) \) in terms of \( r \) for different model parameters. First, we have plotted the function of \( f(r) \) versus \( r \) for \( x = 1 \) with \( k = 0 \) and \( k = \pm 1 \) in Fig. 11. According to this figure, for \( k = 0, -1 \) and \( x = 1 \), the metric functions increase from zero at \( r = r_0 \) to 1 at \( r = \infty \). While, for \( k = 1 \) and \( x = 1 \), the metric function increases to larger than unity at intermediate values of \( r \) and then goes to 1 at spatial infinity.

Next, we try to study the effects of increasing the value of the dynamical exponent \( x \) on the solutions. In this regard, we have provided different panels in Fig. 2 that the behavior of the metric function in terms of \( r \) for each curvature structure is depicted for different values of \( x \). According to this figure, solutions related to \( k = 0 \) and \( k = -1 \) are not very sensitive to the parameter \( x \). However, the solution with \( k = 1 \) is very sensitive to the change of \( x \) and the metric function rapidly increases to values larger than 1 by changing the dynamical exponent \( x \) from 1 to larger values.
Another important point is that the metric function with "−" sign interprets as a black hole with just one horizon for three geometric structures of the horizon ($k = 0, \pm 1$). To confirm our claim, one can calculate $f(r)$ derivatives’ roots, since the number of $f(r)$ derivatives’ roots shows the number of function’s extrema. A straightforward calculation shows that the derivative of the metric function with $k = 0$ and $k = -1$ does not have any real positive roots, independent of the metric parameters. Therefore, these functions do not experience any extrema. However, the derivative of the solution with ($k = 1$) has one root which is related to the maximum point that is shown in Fig. 1.

On the other hand, since
\[
\lim_{r \to \infty} f(r) \to 1, \\
\lim_{r \to 0} f(r) \to -\infty,
\]
the metric function $f(r)$ enjoys at least one root. According to the two mentioned points, we conclude that the metric function with "−" sign is a black hole that meets one horizon for all three geometric structures of the horizon.

We now examine the singularity of the solutions by calculating the scalar curvatures of the spacetime. To do so, we consider some curvature invariants such as the Kretschmann scalar as well as Ricci scalar and investigate their behavior in the presence of large and small radii. No matter the geometric structures of the horizon, both Ricci and Kretschmann invariants diverge as $r \to 0$

\[
K \big|_{r \to 0^+} = \beta(n) \frac{GM\pi}{x\alpha^{n+2} r^{n+2}} + O(r^{-\frac{4n+4}{n+2}}) \quad n \geq 5,
\]

\[
\text{FIG. 1: } f(r) \text{ versus } r \text{ for } n = 5, \ G = 1, \ \alpha = 3, \ M = 5 \text{ and } x = 1 \text{ for } "−" \text{ sign branch.}
\]

\[
\text{FIG. 2: Behavior of } f(r) \text{ versus } r \text{ for } n = 5, \ G = 1, \ \alpha = 3 \text{ and } M = 5 \text{ for } "−" \text{ sign branch.}
\]
and
\[ \mathcal{R}|_{r \to 0^+} = \xi(n, k) \sqrt{\frac{24GM \pi}{2\alpha}} r^{-\frac{n}{x}} + \mathcal{O}(1) \quad n \geq 5, \] (21)

where \( \beta(n) \) and \( \xi(n, k) \) are numbers which vary by changing the dimensions of the spacetime and the value of \( k \). Therefore, this solution has an essential singularity at \( r = 0 \). Besides, the mentioned scalars tend to \( \mathcal{R} = -\frac{n(n+1)}{2\alpha} \) when \( r \) goes to infinity, and thus, the spacetime has a constant curvature at spatial infinity. It is worth mentioning that this is an expected result since the line element (10) goes to the metric (4) based on Eq. (8).

### A. Black Hole/Brane Thermodynamics and Thermal Stability

In this section, we try to investigate the thermodynamic properties of this branch of solution. According to Eq. (17), the mass of black hole per unit volume \( \Sigma_k \) can be expressed in terms of the horizon radius \( r_+ \) as
\[ M = \frac{(n-1)r_+^{(n-2)x} \hat{\alpha}^{x}}{16\pi G x} \left[ k + \frac{k^2 \hat{\alpha}^{x}}{2x^2 r_+^{2x}} + \frac{x^2 r_+^{2x}}{2 \hat{\alpha}^{x}} \right]. \] (22)
It is straightforward to show that for \( x = 1 \), Eq. (22) reduces to the mass of Gauss-Bonnet black holes in AdS spacetime (provided that \( \hat{\alpha} = \frac{l^2}{4x} \)) [32], namely
\[ M = \frac{(n-1)r_+^{n-2} \hat{\alpha}}{16\pi G} \left[ k + \frac{k^2 \hat{\alpha}}{r_+^{2}} + \frac{r_+^{2}}{l^2} \right]. \] (23)
It is also important to note that the difference in the coefficients of the two relations originated from the difference between the cosmological constant in our model and the Gauss-Bonnet black hole in the AdS spacetime.

To explore physical properties, we should determine temperature as the next step. The Hawking temperature of the black holes can be calculated by the surface gravity as
\[ T_H = \frac{1}{4\pi} \sqrt{-g^{tt} g^{rr} g''(r)} \bigg|_{r=r_+}. \] (24)
For the branch with ”−” sign, the temperature will be obtained as
\[ T_H = \frac{4x^2 r_+^{2x} + x^2 r_+^{2x} + k \hat{\alpha}^{x}}{8\pi x r_+^{x} \sqrt{\hat{\alpha}^{x+1}}} (n-4), \] (25)
which considering the points mentioned earlier about the difference between the two models, one can easily check that the temperature for \( x = 1 \) is in agreement with that of the Gauss-Bonnet black hole in the AdS spacetime. In the following, we study the situations in which the expression inside the absolute value is positive. This condition requires that for the case of \( k = -1 \) the horizon radius is larger than \( (\frac{\hat{\alpha}^{x}}{x})^{\frac{-1}{2x}} \). Considering this condition, the temperature relation (25) reduces to the following relation
\[ T_H = \frac{r_+^{2x} nx}{8 \pi \sqrt{x}^{x+1}} \left( 1 + k \frac{n-4}{n} \frac{\hat{\alpha}^{x}}{x^2 r_+^{2x}} \right). \] (26)

Next, we try to compute the entropy of the black hole. For this purpose, we use the fact that as a thermodynamic system, black hole’s entropy must obey the first law of black hole thermodynamics
\[ \delta M = T \delta S. \] (27)
Integrating this relation along with considering the physical assumption that the entropy will vanish if the horizon of the black hole shrinks to zero results that
\[ S = \int_0^{r_+} T^{-1} \left( \frac{\partial M}{\partial r_+} \right) dr_+. \] (28)
Hence, the entropy of the black hole per unit volume $\Sigma_k$ will be obtained by substituting \[22\] and \[26\] into \[28\] as follows

\[
S = \frac{r^\prime(n-1)x^2}{G} \sqrt{\tilde{\alpha}(x^{-1})} \left(1 + \frac{k(n - 1)}{n - 3} \frac{\tilde{\alpha}^x}{x^2r^2_+} \right),
\] (29)

Up to now, we showed that one can regard the obtained black hole as a thermodynamic system. On the other hand, it is necessary to investigate the stability of a thermodynamic system under thermal perturbations. In order to investigate the local stability of black holes, we need to calculate the heat capacity. It is known that the heat capacity of the black hole is defined as

\[
C = T \frac{\partial S}{\partial T},
\] (30)

where for the obtained solution, it gets the following form via the chain rule

\[
C = \frac{x^2(n-1)r^\prime(n-1)x}{G} \sqrt{\tilde{\alpha}(x^{-1})} \left(1 + \frac{k(n - 4)\tilde{\alpha}^x}{nx^2r^2_+} \right) \left(1 - \frac{k(n - 4)\tilde{\alpha}^x}{nx^2r^2_+} \right)^{-1}.
\] (31)

Besides, global stability of the black holes can be explored with the help of free energy of the system, defined as $F = M - TS$. Using \[22\], \[26\] and \[29\], one can obtain the functional form of free energy with the following explicit relation

\[
F = -\frac{r^{nx_+}x^3}{8G\pi\tilde{\alpha}} \left[ \left(1 + \frac{k(n - 1)\tilde{\alpha}^x}{(n - 3)x^2r^2_+} \right) \left(1 + \frac{k(n - 4)\tilde{\alpha}^x}{nx^2r^2_+} \right) n - \left(1 + \frac{\tilde{\alpha}^x}{x^2r^2_+} \right)^2 \right].
\] (32)

So far, we got some thermodynamic quantities of Gauss-Bonnet black holes in our proposed geometry. As can be seen, these quantities strongly depend on the Gauss-Bonnet coefficient $\alpha$, critical exponent $x$, horizon structure $k$, and spacetime dimensions $n$. In the following, we discuss the stability of the solution in more details according to the classification of the horizon structures, $k = 0, k = -1$ and $k = 1$, respectively.

- **Case $k = 0$**:

In the case of the solution with $k = 0$, the thermodynamic quantities are

\[
T = \frac{nxr_+^x}{8\pi\tilde{\alpha}^{x+1}},
\]

\[
S = \frac{r^\prime(n-1)x^2}{G} \sqrt{\tilde{\alpha}(x^{-1})},
\]

\[
C = \frac{x^2(n-1)r^\prime(n-1)x}{G} \sqrt{\tilde{\alpha}(x^{-1})},
\]

\[
F = -\frac{r^{nx_+}x^3}{8G\pi\tilde{\alpha}},
\] (33)

where $r^{nx_+} = \frac{8\pi GM\tilde{\alpha}}{(n-1)x^2}$. Here, we note that as the above relations show, these thermodynamic quantities are dependent on the parameter $\tilde{\alpha}$ owing to the fact that the Einstein gravity has no solution in the proposed geometry and the existence of the Gauss-Bonnet term is necessary to have an appropriate exact solution.

Regarding the stability of the solution, one finds that it is quite locally stable due to the positivity of the temperature and heat capacity for all values of the model parameters. Moreover, the strictly decreasing behavior of the free energy function guarantees the global stability of the solution.

- **Case $k = -1$**:

In this case, calculations show that the solutions are not always stable. To be more clear, we have provided Fig. 3 in which the behavior of temperature, heat capacity, and free energy in terms of $r_+$ are depicted. It is notable that we use different scales for temperature, heat capacity, and free energy to make them comparable with each other.

The temperature of the black hole is always positive. However, the heat capacity is negative for the following horizon radius interval

\[
r_+ \leq r_{+\text{min}},
\] (34)
FIG. 3: Behavior of $T$, $C$ and $F$ versus $r_+$ for $k = -1$, $n = 5$, $x = 2$ and $\alpha = 1$ for ”−” sign branch.

where $r_{+\text{min}} = (\frac{x^2}{x^2})^{\frac{1}{n}}$ is the smallest horizon radius of the stable black hole. The temperature for the smallest stable black hole is

$$T_{\text{min}} = \frac{1}{2\pi\sqrt{\tilde{\alpha}}}.$$  \hfill (35)

Hence, the temperature of the stable black holes starts from $T_{\text{min}}$ at $r_{+\text{min}}$ and monotonically goes to infinity as $r_+ \to \infty$. The existence of a lower limit on the horizon radius and also temperature would be interesting from the information paradox point of view which may be investigated in the future. Moreover, the free energy of the smallest stable black hole is obtained as

$$F_{\text{min}} = \frac{x^3}{\pi G (n-3)\tilde{\alpha}} \left( \sqrt{\frac{\tilde{\alpha} x}{x^2}} \right)^n.$$  \hfill (36)

Our calculation shows that the free energy always experiences a positive maximum value, $F_{\text{max}} = \frac{x^3}{2(n-1)Gx\tilde{\alpha}} \left( \sqrt{\frac{(n-1)\tilde{\alpha} x}{(n-3)x^2}} \right)^n$ at $r_+ = (\frac{(n-1)\tilde{\alpha} x}{(n-3)x^2})^{\frac{1}{n}}$ and then goes to negative infinity as $r_+ \to \infty$. As an example, the behavior of this function in terms of the horizon radius for a set of model parameters is plotted in Fig. 3.

In order to investigate how the free energy behaves by changing the parameters of the model, we have provided Fig. 4 and Fig. 5. Figure 4 indicates that by increasing the value of parameter $\alpha$ parameter $x$ and dimension of spacetime $n$ (and fixed values of the other parameters), the maximum value of this function as well as the radius in which this maximum value occurs increases. In addition, the behavior of the free energy versus $T$ is depicted in Fig. 5. As this figure shows, increasing each of the parameters $\alpha$, $x$ and $n$ causes the maximum value of the free energy function to occur at a lower temperature.

**Case $k = 1$**:

Regarding the solution with $k = 1$, the temperature and the heat capacity have the following form

$$T = \frac{nxr_+^x}{8\pi \sqrt{\tilde{\alpha}^{x+1}}} \left( 1 + \frac{(n-4)\tilde{\alpha} x}{nx^2 r_+^{2x}} \right),$$  \hfill (37)

$$C = \frac{x^2(n-1)r_+^{(n-1)x}}{G} \sqrt{\tilde{\alpha}^{x-1}} \left( 1 + \frac{(n-4)\tilde{\alpha} x}{r_+^{2x} n x^2} \right) \left( 1 + \frac{\tilde{\alpha} x}{x^2 r_+^{2x}} \right) \left( 1 - \frac{(n-4)\tilde{\alpha} x}{r_+^{2x} n x^2} \right)^{-1}.$$  \hfill (38)
As it is clear from the above relations, although the temperature is always positive for each value of model parameters, the heat capacity meets a positive value just for a range of horizon radii. Moreover, the heat capacity function experiences a divergence at

\[ r_{\text{div}} = \left( \frac{(n-4)\tilde{\alpha}^x}{nx^2} \right)^{\frac{1}{x}}, \]  

which its position alters by changing the value of the model parameters. To be more clear, we have provided Fig. 6 in which the behavior of the temperature, heat capacity and free energy are illustrated for a set of model parameters. Hence, if the radius of the event horizon is larger than \( r_{\text{div}} \), the black hole is thermally stable and enjoys some necessary criteria for viable solutions. In addition, Fig. 7 states that increasing the value of parameters \( n, x \) and \( \alpha \) leads to increasing the amount of \( r_{\text{div}} \) and causes the stability interval of the black hole to start from a larger radius.

Similar to the case of \( k = -1 \), the energy function always meets a positive maximum value at \( r_+ = r_{\text{div}} = \left( \frac{\tilde{\alpha}^x(n-4)}{nx^2} \right)^{\frac{1}{x}} \) and tends to negative infinity when \( r_+ \to \infty \) (see Fig. 6). Therefore, the maximum value of the free energy occurs at a horizon radius in which the heat capacity diverges. Furthermore, the maximum value of this function as well as the radius in which this maximum value occurs increases by increasing the value of parameters \( x, n \) and \( \alpha \) (see Fig. 8 for more details).
FIG. 6: Behavior of $C$, $T$ and $F$ with respect to $r_+$ for $k = 1$, $n = 5$, $x = 2$ and $\alpha = 1$ for "−" sign branch.

(a) $C - r_+$ (red), $10^{-2}C - r_+$ (blue) and $10^{-4}C - r_+$ (green) for $\alpha = 1$, $x = 1$ and $G = 1$.

(b) $10C - r_+$ (red), $10^{-4}C - r_+$ (blue) and $10^{-10}C - r_+$ (green) for $n = 5$, $\alpha = 1$ and $G = 1$.

(c) $10C - r_+$ (red), $C - r_+$ (blue) and $10^{-2}C - r_+$ (green) for $n = 5$, $x = 2$ and $G = 1$.

FIG. 7: Behavior of the heat capacity versus $r_+$ for $k = 1$ for "−" sign branch.

V. POSITIVE SIGN BRANCH

Considering the branch of the metric function (17) with "+" sign, we can easily find that the metric functions associated with $k = -1$ have real positive non-zero roots and hence black holes could only exist for the pseudo-hyperbolic horizon (other curvatures result in a naked singularity and do not have black hole interpretation).

Now, we try to investigate physical properties of the solution with the pseudo-hyperbolic horizon by studying the behavior of the obtained solution and looking for the horizons. To this end, we have plotted the function $f(r)$ versus $r$ for different model parameters in Fig. 9. These figures show that, depending on the metric parameters, this solution could represent a black hole with two horizons, an extreme black hole with a degenerate horizon, or a naked singularity. Figure 9(a) indicates that by increasing the value of the Gauss-Bonnet coefficient, $\alpha$, (and fixed values of the other parameters), the number of horizons changes from zero to two. In Fig. 9(b) we consider the effect of the dynamical exponent $x$ on the number of the horizons and find that increasing the value of the dynamical exponent $x$ leads to increasing the number of horizons to two horizons.

To ensure the number of horizons, one can calculate $f(r)$ derivatives’ roots. As mentioned earlier, since the number of $f(r)$ derivatives’ roots shows the number of function’s extrema, the existence of more than one extrema for $f(r)$ indicates that the number of horizons could be more than two. However, the existence of only one extremum represents
that the maximum number of horizons will be two. Concerning our solution, we find that

\[
|r|f'(r)|_{r=0} = \left( \frac{1}{\alpha^2} \right)^{\frac{\alpha M G \pi x}{2(n-1)}}.
\]  

(40)

Since there is only one acceptable extremum (real and positive), independent of the metric parameters \(x\), \(n\), and \(\alpha\), it can be ensured that the maximum number of horizons is two.

Concerning the black hole singularity, our calculations show that the Kretschmann invariant and Ricci scalar follow similar relations to the Eqs. (20) and (21). Therefore, this solution also has an essential singularity at the origin. Besides, considering Fig. 9 it is clear that the singularity is timelike and the behavior of the solutions is similar to charged black holes.

### A. Thermodynamic properties

Now, we study the thermodynamic structure of the solution with "+" sign. Similar to the previous section, we first obtain the thermodynamic quantities and then investigate the local and global stability of the system by calculating heat capacity and free energy. As was already mentioned, for the solution with "+" sign, black holes can be found only for the pseudo-hyperbolic horizon. So, we conduct our investigation for the case of \(k = -1\).
The mass of the black hole is obtained as

\[ M = \left( n - 1 \right) r^2 \frac{(n-2)x}{2x^2 \alpha^x} \left[ -1 + \frac{x^2 \alpha^x}{2x^2 \alpha^x} + \frac{x^2}{2x^2 \alpha^x} \right]. \tag{41} \]

From Eq. (24), the temperature is given by

\[ T_H = \frac{4x^2 r^2}{8\pi r_+ \sqrt{\alpha^{x+1}}} \left| -1 + \left( \frac{(n-4)x^2}{(n-8)x^2} \right)^{\frac{1}{2}} \right| \tag{42} \]

Considering the condition \( r_+ > \left( \frac{\alpha^x}{x^2} \right)^{\frac{1}{2}} \), one finds

\[ T = \frac{(n-8)x^2 r_+}{8\pi \sqrt{x^2 \alpha^{x+1}}} \left( -1 + \left( \frac{(n-4)x^2}{(n-8)x^2} \right)^{\frac{1}{2}} \right). \tag{43} \]

It is worth pointing out that for \( 5 \leq n \leq 8 \), the temperature is always positive. However, for \( n \geq 9 \), it is positive only in special regions, \( \alpha^x \left( \frac{x^2}{x^2} \right)^{\frac{1}{2}} < r_+ < \left( \frac{(n-4)x^2}{(n-8)x^2} \right)^{\frac{1}{2}} \), meaning that in higher dimensions, we have a restriction for the horizon radius of the physical black holes in the theory under consideration.

\[ \text{FIG. 10: Behavior of the heat capacity versus } r_+ \text{ for } G = 1 \text{ and for } ^{+} \text{ sign branch.} \]
Inserting Eqs. (41) and (43) into Eq. (28), one can calculate the entropy as

$$S = \frac{1}{G} \left[ x^{x+1} r_+^{(n-3)x} \sqrt{\alpha^2 (2\alpha)^x - 1} \left( r_+^{2x} - \frac{(n-1)\alpha^x (2x)^{(x-1)}}{(n-3)2x} \right) \right].$$

The important quantity to study the local stability of the system is heat capacity which is determined as

$$C = \frac{(n-1)\alpha^{x+1} (nx^2 r_+^{2x} - (n-4)\alpha^x) (\hat{\alpha}^x - x^2 r_+^{2x})^{(n-2)x}}{\hat{\alpha} G r_+^{x} [(n-8)x^2 r_+^{2x} + (n-4)\alpha^x]}.$$

The behavior of the heat capacity as a function of $r_+$ is depicted in Fig. 10. For up panels of this figure, we fixed the parameter $\alpha$ and investigated the effect of dynamic exponent $x$ on the heat capacity. It can be seen that for all values of $n$ and $x$ the heat capacity is positive, meaning that the black hole is thermally stable all the time. For down panels of this figure, we fixed the dynamic exponent $x$ and examined the influence of $\alpha$ on the heat capacity. Looking at these graphs, one can find that the effect of parameter $\alpha$ on the heat capacity is similar to that of the dynamic exponent $x$. So, a similar discussion can be employed for this case.

FIG. 11: Behavior of the free energy versus $r_+$ for $G = 1$ and for “+” sign branch.

In Fig. 11 we studied the behavior of the free energy under variation of the dynamic exponent $x$ (see up graphs) and variation of the Gauss-Bonnet coefficient $\alpha$ (see down graphs). One can see that for arbitrary dimensions, there are two sets of black holes with small and large horizon radii. According to this figure, small black holes are unstable whereas large black holes enjoy global stability. Besides, the stability region changes as both parameters $x$ and $\alpha$ vary.
VI. CONCLUSION

In this paper, we proposed a noteworthy geometry in which the spatial coordinates scale anisotropically i.e. $(t, r, \theta_1) \to (\lambda^2 t, \lambda^{-1} r, \lambda^{x_1} \theta_1)$. AdS spacetime is one of the trivial subclasses of the mentioned unusual geometry for $z = x_1 = 1$, while it cannot be a nontrivial solution of the pure Einstein gravity without matter field. However, the higher derivative curvature terms could play the role of the desired matter fields, and therefore, we investigated whether Gauss-Bonnet gravity could admit the proposed geometry as a nontrivial solution in the vacuum and understood that such a solution exists in this kind of gravity under certain circumstances (see Eq.(10)) and provided that $x_1 \leq 0$. In addition, we found that there is an exact vacuum solution for $k = \pm 1$, independent of the value of the dynamical exponent $x$, which is a black hole solution for pseudo-hyperbolic horizon structure $k = -1$ while the pseudo-spherically symmetric boundary ($k = 1$) results in a naked singularity. It is also notable that according to our calculations, 5–dimensional spacetime is a special case for which the critical exponent $x$ can only take the value of the unit ($x = 1$). Whereas there is no restriction on the value of the parameter $x$ in higher dimensions ($n > 4$).

Besides, we looked for a general solution of our suggested metric in Einstein-Gauss-Bonnet gravity and found that there is an exact solution provided the parameters of the model (cosmological constant as well as Gauss-Bonnet coefficient) are chosen suitably and the dynamical exponents $x$ and $z$ have the same values. Given the mandatory constraints on the model, the obtained solution is equivalent to the AdS solution with $\lambda = -\frac{n(n-1)}{8\lambda}$ and considering the fact that the cosmological constant in our model for the case of $x = 1$ is half of that in AdS spacetime. Since the obtained solution has two branches with ”+” and ”−” signs, we investigated the properties of each branch of the solution, separately. We found that the metric function with the ”−” sign can be interpreted as a black hole with just one horizon for three types of geometric structures ($k = 0, \pm1$). Moreover, studying the role of varying $x$ on the properties of ”−”, one finds that the solution with $k = 1$ is much more sensitive to the change of $x$ than the solutions with $k = 0$ and $k = -1$.

We also examined the thermodynamic properties of the black hole solutions for the branch with ”−” sign, and then, we studied the stability of the obtained solution for each geometrical structure of the boundary (boundary with $r = cte$ and $t = cte$), separately. The obtained results are

- Concerning thermodynamic behavior of the black hole with $k = 0$ boundary, we found that this solution enjoys local stability due to the positivity of the temperature and heat capacity for all values of the model parameters. Furthermore, strictly decreasing the behavior of the free energy function assures us of the global stability of the solution.

- For the black holes with pseudo-hyperbolic horizon structure ($k = -1$), our calculations indicated that the solutions are not always stable. We calculated the smallest horizon radius of the stable black hole ($r_{+\text{min}}$) and investigated the thermodynamic behavior of the stable system. Our investigation showed that the temperature of the stable black hole starts from $T_{\text{min}}$ at $r_{+\text{min}}$ and monotonically goes to infinity as $r_+ \to \infty$. Also, the free energy always starts from zero, reaches a positive maximum value $F_{\text{max}}$, and then goes to negative infinity as $r_+ \to \infty$.

- Regarding the pseudo-spherically symmetric horizon ($k = +1$), it was found that although the temperature is always positive for each value of model parameters, the heat capacity experiences one divergency at $r_{+\text{div}}$ and meets positive values just for a range of horizon radii which means that the solution is not always stable. We understood that if the radius of the event horizon is larger than $r_{+\text{div}}$, the black hole is thermally stable and enjoys some necessary criteria for viable solutions. Besides, the free energy function always meets a positive maximum value at some $r_+$ and tends to negative infinity when $r_+ \to \infty$.

On the other hand, regarding the branch of metric function with ”+” sign, we understood that black hole solutions only exist for pseudo-hyperbolic horizon ($k = -1$) and other curvature result in a naked singularity. It was also found that, depending on the metric parameters, this solution could represent a black hole with two horizons, an extreme black hole with a degenerate root or a naked singularity. Next, we calculated the thermodynamic quantities in arbitrary dimensions and focus on the thermal behavior of the system. As expected, we found that black holes with large horizon radius is thermally stable.

Regarding this special anisotropic scaling spacetime, it will be interesting to study dynamic stability and related quasinormal modes. Moreover, it is worth to investigate modified dispersion relations in this geometry. Besides, one can generalize the obtained solutions in the presence of matter field. Furthermore, analyzing the effects of higher order curvature terms (such as third order Lovelock gravity) will be interesting. Also, according to obtained results, we found that the geometrical properties of the positive branch with $k = -1$ are more or less similar to the asymptotically flat charged black hole with a spherical horizon. So, it is worth investigating possible difference (or more similarities) as well as the causality of the solution according to the Penrose diagram since the solution enjoys timelike singularity. These topics should be addressed in independent works.
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