Noether’s Theorem in Multisymplectic Geometry

Jonathan Herman
Department of Pure Mathematics, University of Waterloo
j3herman@uwaterloo.ca

Abstract

We extend Noether’s theorem to the setting of multisymplectic geometry by exhibiting a correspondence between conserved quantities and continuous symmetries on a multi-Hamiltonian system. We show that a homotopy co-momentum map interacts with this correspondence in a way analogous to the moment map in symplectic geometry.

We apply our results to generalize the theory of the classical momentum and position functions from the phase space of a given physical system to the multisymplectic phase space. We also apply our results to manifolds with a torsion-free $G_2$ structure.

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1 Introduction

In this paper, we generalize concepts from Hamiltonian mechanics to multisymplectic geometry. In particular, we are interested in the relationship between continuous symmetries and conserved quantities on multi-Hamiltonian systems. Recall that a symmetry on a symplectic Hamiltonian system \((M, ω, H)\) is a symplectic vector field whose flow preserves the Hamiltonian. A conserved quantity is a function that is constant along the motions of the given physical system.

Noether’s theorem sets up a correspondence between conserved quantities and continuous symmetries. For example, under a conservative system in \(\mathbb{R}^3\), the flows of each vector field coming from the Lie algebra action of \(so(3)\) all leave the total energy, \(H\), invariant. Thus, each of the infinitesimal generators from \(so(3)\) are continuous symmetries. An example of a conserved quantity would be the angular momentum of a particle moving in this physical system. Noether’s theorem sees conservation of angular momentum as a consequence of the rotational symmetry.

In a multi-Hamiltonian system \((M, ω, H)\), with \(ω \in Ω^{n+1}(M)\) and \(n ≥ 1\), the Hamiltonian \(H\) is now a ‘Hamiltonian form’ of degree \(n−1\). Here Hamiltonian means that there exists a unique vector field \(X_H\) satisfying \(X_H ω = −dH\). The flow of \(X_H\) gives the dynamics of the given physical system.

A notion of conserved quantity on a multisymplectic manifold was given in [16]. They defined three types; a differential form \(α\) is called a local, global, or strict conserved quantity if \(L_{X_H}α\) is closed, exact or zero respectively. In our work we modify their definition by requiring that a conserved quantity \(α\) is also Hamiltonian, meaning that \(X_α ω = −dα\) for some multivector field \(X_α\). By adding in this requirement, we are then able to study how the extended ‘Poisson’ bracket, \(\{·, ·\}\), from [3] interacts with the conserved quantities.

We find that, analogous to the case of Hamiltonian mechanics, the Poisson bracket of two conserved quantities is always strictly conserved. That is,

**Proposition 1.1.** Let \(α\) and \(β\) be two (local, global or strict) conserved quantities on a multi-Hamiltonian system \((M, ω, H)\). Then \(\{α, β\}\) is strictly conserved, meaning \(L_{X_H}\{α, β\} = 0\).

From this proposition we will show that the conserved quantities, modulo closed forms, constitute a graded Lie algebra. We will also show that when restricted to a certain subspace, namely the Lie \(n\)-algebra of observables (see definition 3.6), the conserved quantities form an \(L_∞\)-algebra.

Similarly, we will see that our continuous symmetries also generate a graded Lie algebra. As an extension from Hamiltonian mechanics, we define a symmetry to be a Hamiltonian multivector field with respect to which the Lie derivative of the Hamiltonian has a specific form. Just as for the conserved quantities, we have three types of continuous symmetry. Namely, a multivector field \(X\) is a local, global, or strict symmetry on \((M, ω, H)\) if \(L_X ω = 0\) and \(L_X H\) is closed, exact, or zero respectively. A generalization from Hamiltonian mechanics is

**Proposition 1.2.** Given any two (local, global, strict) continuous symmetries \(X\) and \(Y\), their Schouten bracket \([X, Y]\) is a continuous symmetry of the same type.

From this proposition we will see that the continuous symmetries, modulo elements in the kernel of \(ω\), form a graded Lie algebra.

Our first generalization of Noether’s theorem says that there is a correspondence between these notions of symmetry and conserved quantity on a multisymplectic manifold.
Theorem 1.3. If $\alpha$ is a (local or global) conserved quantity, then every corresponding Hamiltonian multivector field $X_{\alpha}$ is a (local or global) continuous symmetry. Conversely, if $X$ is a (local or global) continuous symmetry, then every corresponding Hamiltonian form is a (local or global) conserved quantity.

As in symplectic geometry, this correspondence is not one-to-one. Indeed, for a Hamiltonian form, any two of its corresponding Hamiltonian multivector fields differ by an element in the kernel of $\omega$. Conversely, any two Hamiltonian forms corresponding to a Hamiltonian multivector field differ by a closed form:

Let

$$\Omega_{\text{Ham}}(M) = \{ \alpha \in \Omega^\bullet(M); \alpha = X_{\omega} \text{ for some } X \in \Gamma(\Lambda^\bullet(TM)) \}$$

denote the graded vector space of Hamiltonian forms, and let $\tilde{\Omega}_{\text{Ham}}(M)$ denote the quotient of $\Omega_{\text{Ham}}(M)$ by closed forms. Similarly, we let

$$X_{\text{Ham}}(M) = \{ X \in \Gamma(\Lambda^\bullet(TM)); X_{\omega} \text{ is exact} \}$$

denote the graded vector space of multi Hamiltonian vector fields and $\tilde{X}_{\text{Ham}}(M)$ denote the quotient of $X_{\text{Ham}}(M)$ by elements in the kernel of $\omega$. We will then show, as in [3], that $\{\cdot, \cdot\}$ descends to a well defined graded Poisson bracket on $\tilde{\Omega}_{\text{Ham}}(M)$. Then we show that

Theorem 1.4. There is a natural isomorphism of graded Lie algebras between $(\tilde{X}_{\text{Ham}}(M), \{\cdot, \cdot\})$ and $(\tilde{\Omega}_{\text{Ham}}(M), \{\cdot, \cdot\})$.

As a consequence of this theorem, we will then show that our symmetries and conserved quantities, after appropriate quotients, are in one-to-one correspondence. In particular, we let $C_{\text{loc}}(X_H)$, $C(X_H)$, $C_{\text{str}}(X_H)$ denote the spaces of local, global, and strict conserved quantities respectively, and $\tilde{C}_{\text{loc}}(X_H)$, $\tilde{C}(X_H)$ and $\tilde{C}_{\text{str}}(X_H)$ their quotients by closed forms. Similarly, we let $S_{\text{loc}}(H)$, $S(H)$ and $S_{\text{str}}(H)$ denote the space of local, global, and strict continuous symmetries respectively, and $\tilde{S}_{\text{loc}}(H)$, $\tilde{S}(H)$ and $\tilde{S}_{\text{str}}(H)$ their quotient by elements in the kernel of $\omega$. We obtain:

Theorem 1.5. There is an isomorphism of graded Lie algebras from $(\tilde{S}(H), \{\cdot, \cdot\})$ to $(\tilde{C}(X_H), \{\cdot, \cdot\})$ and from $(\tilde{S}_{\text{loc}}(H), \{\cdot, \cdot\})$ to $(\tilde{C}_{\text{loc}}(X_H), \{\cdot, \cdot\})$. Moreover, there is an injective graded Lie algebra homomorphism from $(S_{\text{str}}(H), \{\cdot, \cdot\})$ to $(\tilde{C}(X_H), \{\cdot, \cdot\})$ and from $(\tilde{C}_{\text{str}}(X_H), \{\cdot, \cdot\})$ to $(\tilde{S}(H), \{\cdot, \cdot\})$.

Furthermore, we will show that under certain assumptions for a group action on $M$, a homotopy co-momentum map $(f)$ (see definition [3,10]) gives rise to a whole family of conserved quantities and continuous symmetries. A group action on a multi-Hamiltonian system $(M, \omega, H)$ is called locally, globally, or strictly $H$ preserving if the Lie derivative of $H$ under each infinitesimal generator from $\mathfrak{g}$ is closed, exact, or zero respectively.

In [16] it was shown that if the group locally or globally preserves $H$, then for any $p \in \mathcal{P}_{\mathfrak{g},k}$, the $k$-th Lie kernel (see definition [3,8]), $f_k(p)$ is locally conserved, and if the group strictly preserves $H$ then $f_k(p)$ is globally conserved. We add to this result by showing that under the above assumptions $V_p$ is a local or global continuous symmetry. In particular, let $S_k = \{ V_p; p \in \mathcal{P}_{\mathfrak{g},k} \}$ denote the infinitesimal generators coming from the Lie kernel. Then $S = \oplus S_k$ is a differential graded Lie algebra. Let $C_k = \{ f_k(p); p \in \mathcal{P}_{\mathfrak{g},k} \}$ denote the image of the moment map. We set $C = \oplus C_k$ and show that $C \cap L_\infty(M, \omega)$ is an $L_\infty$-subalgebra of $L_\infty(M, \omega)$, the Lie $n$-algebra of observables. We then obtain

Theorem 1.6. For a (locally, globally or strictly) preserving $H$-action, a homotopy co-momentum map induces an $L_\infty$-morphism from $S$ to $C \cap L_\infty(M, \omega)$.
We finish by giving two applications of our results:
Recall that in symplectic geometry, any vector field on the given manifold induces a vector field on the phase space, called the corresponding classical momentum function. Moreover, any function on the base manifold pulls back to a function on the phase space giving the so-called classical position functions. These momentum and position functions on the phase space satisfy specific commutation relations that form the bridging gap from classical to quantum mechanics (see chapter 5.4 of [1]). As an application of our results we will replace vector fields and functions with arbitrary multivector fields and differential forms to obtain ‘position and momentum forms’. We will show that these position and momentum satisfy higher order bracket relations generalizing the bracket relations in symplectic geometry.

Lastly, we finish by applying our work to manifolds with \( G_2 \)-structure. In particular, we extend Example 6.7 of [11] by obtaining a homotopy co-momentum map for a \( T^2 \) action on a torsion-free \( G_2 \)-manifold.

2 Differential Graded Lie Algebras and \( L_\infty \)-algebras

We start by recalling some basic properties of differential graded Lie algebras.

2.1 Differential Graded Lie Algebras

We first need to recall the definition of a differential graded Lie algebra and a Gerstenhaber algebra.

**Definition 2.1.** A differential graded Lie algebra is a \( \mathbb{Z} \)-graded vector space \( L = \oplus_{i \in \mathbb{Z}} L_i \) together with a bracket \([\cdot,\cdot] : L_i \otimes L_j \to L_{i+j}\) and a differential \( d : L_i \to L_{i-1}\). The bilinear map \([\cdot,\cdot]\) is graded skew symmetric:
\[
[x, y] = -(-1)^{|x||y|} [y, x],
\]
and satisfies the graded Jacobi identity:
\[
(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.
\]
Lastly, the differential and bilinear map satisfy the graded Leibniz rule:
\[
d[x, y] = [dx, y] + (-1)^{|x|}[x, dy].
\]
Here we have let \( x, y, \text{ and } z \) be arbitrary homogeneous elements in \( L \) of degrees \( |x|, |y| \text{ and } |z| \) respectively.

**Definition 2.2.** A Gerstenhaber algebra is a \( \mathbb{Z} \)-graded commutative algebra \( A = \oplus_{i \in \mathbb{Z}} A_i \) with a bilinear map \([\cdot, \cdot] : A \otimes A \to A\) satisfying the following properties:

- \( ||a, b|| = |a| + |b| - 1 \) (the bilinear map has degree \(-1\)),
- \( [a, bc] = [a, b]c + (-1)^{|a|-1}|b||a||c|[a, c] \) (the bilinear map satisfies the Poisson identity),
- \( [a, b] = -(-1)^{|a|-1}|b|-1|b|[b, a] \) (the bilinear map is antisymmetric),

and lastly, the bilinear map satisfies the Jacobi identity:
\[
(-1)^{|a|-1}|c|-1[a, [b, c]] + (-1)^{|b|-1}|a|-1[b, [c, a]] + (-1)^{|c|-1}|b|-1[c, [a, b]] = 0.
\]
Here we have let \( |a| \) denote the order of \( a \in A \), and \( ab \) the product of \( a \) and \( b \) in \( A \).
Let \((V, [\cdot, \cdot])\) be a Lie algebra. The Schouten bracket turns \(\Lambda^\bullet V\), the exterior algebra of \(V\), into a Gerstenhaber algebra. We quickly recall some properties of the Schouten bracket. A more detailed discussion can be found in [13].

**Proposition 2.3.** The Schouten bracket is the unique bilinear map \([\cdot, \cdot] : \Lambda^\bullet V \times \Lambda^\bullet V \rightarrow \Lambda^\bullet V\) satisfying the following properties:

- If \(\deg X = k\) and \(\deg Y = l\) then \(\deg([X, Y]) = k + l - 1\).
- \([X, Y] = -(-1)^{(k+1)(l+1)}[Y, X]\).
- It coincides with the Lie bracket on \(V\).
- For \(X, Y \in \Gamma(\Lambda^k V)\) and \(Z \in \Gamma(\Lambda^l V)\),
  \([X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(k-1)l} Y \wedge [X, Z]\).
- It satisfies the graded Jacobi identity: For \(X, Y\) and \(Z\) of degree \(k, l\) and \(m\) respectively,
  \[
  \sum \text{cyclic} (-1)^{(k-1)(m-1)}[X, [Y, Z]] = 0.
  \]

**Proof.** This is Proposition A.1 of [4].

On decomposable multivectors \(X = X_1 \wedge \cdots \wedge X_k \in \Lambda^k V\) and \(Y = Y_1 \wedge \cdots \wedge Y_l \in \Lambda^l V\) the Schouten bracket is given by

\[
[X, Y] := \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [X_i, Y_j] X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l.
\]

We now let \(M\) be a manifold and consider the Gerstenhaber algebra \((\Gamma(\Lambda^\bullet (TM)), \wedge, [\cdot, \cdot])\).

**Definition 2.4.** For a decomposable multivector field \(X = X_1 \wedge \cdots \wedge X_k \in \Gamma(\Lambda^k (TM))\) and a differential form \(\tau\), we define the contraction of \(\tau\) by \(X\) to be

\[
X \lrcorner \tau := X_k \lrcorner \cdots \lrcorner X_1 \lrcorner \tau,
\]

and extend by linearity to all multivector fields. We define the Lie derivative of \(\tau\) in the direction of \(X\) to be

\[
\mathcal{L}_X \tau := d(X \lrcorner \tau) - (-1)^k X \lrcorner d \tau.
\]

(2.1)

Note that this is the usual Lie derivative when \(k = 1\).

Throughout the paper we will make extensive use of the following propositions.

**Proposition 2.5.** Let \(X \in \Gamma(\Lambda^k (TM))\) and \(Y \in \Gamma(\Lambda^l (TM))\) be arbitrary. For a differential form \(\tau\), the following hold:

\[
d\mathcal{L}_X \tau = (-1)^{k+1} \mathcal{L}_X d \tau
\]

(2.2)

\[
[X, Y] \lrcorner \tau = (-1)^{(k+1)l} \mathcal{L}_X (Y \lrcorner \tau) - Y \lrcorner (\mathcal{L}_X \tau)
\]

(2.3)

\[
\mathcal{L}_{[X, Y]} \tau = (-1)^{(k+1)(l+1)} \mathcal{L}_X \mathcal{L}_Y \tau - \mathcal{L}_Y \mathcal{L}_X \tau
\]

(2.4)

\[
\mathcal{L}_{X \wedge Y} \tau = (-1)^l Y \lrcorner (\mathcal{L}_X \tau) + \mathcal{L}_Y (X \lrcorner \tau)
\]

(2.5)
Proof. This is Proposition A.3 of [4].

Another formula for the interior product by the Schouten bracket is given by the next proposition.

**Proposition 2.6.** For $X \in \Gamma(\Lambda^k(TM))$ and $Y \in \Gamma(\Lambda^l(TM))$ we have that interior product with their Schouten bracket satisfies

$$i[X,Y] = [-i(Y),d,i(X)],$$

where the bracket on the right hand side is the graded commutator. Written out fully, this says that for an arbitrary form $\tau$,

$$[X,Y] \tau = -Y \lrcorner d(X \lrcorner \tau) + (-1)^{kl+k+l} X \lrcorner Y \lrcorner d\tau - (-1)^{kl+k+l} X \lrcorner d(Y \lrcorner \tau) \quad (2.6)$$

**Proof.** This is Proposition 4.1 of [13]. It can also be derived directly from equations (2.1) and (2.3). We state it as a separate proposition because equation (2.6) will be used frequently in the rest of the paper.

Next we recall the Chevalley-Eilenberg complex. We start with a Lie algebra $\mathfrak{g}$ and its exterior algebra $\Lambda \mathfrak{g}$. The Gerstenhaber algebra $(\Lambda \mathfrak{g}, \wedge, [\cdot,\cdot])$ is turned into a differential algebra by the following differential.

**Definition 2.7.** For a Lie algebra $\mathfrak{g}$, consider the differential

$$\partial_k : \Lambda^k \mathfrak{g} \to \Lambda^{k-1} \mathfrak{g}, \quad \xi_1 \wedge \cdots \wedge \xi_k \mapsto \sum_{1 \leq i < j \leq k} (-1)^{i+j} [\xi_i,\xi_j] \wedge \xi_1 \wedge \cdots \hat{\xi_i} \wedge \cdots \hat{\xi_j} \wedge \cdots \wedge \xi_k$$

for $k \geq 1$, and extend by linearity to non-decomposables. Define $\Lambda^{-1} \mathfrak{g} = \{0\}$ and $\partial_0$ to be the zero map. It follows from the graded Jacobi identity that $\partial^2 = 0$. The differential Gerstenhaber algebra $(\Lambda \mathfrak{g}, \wedge, \partial, [\cdot,\cdot])$ is called the Chevalley-Eilenberg complex.

**Definition 2.8.** We follow the terminology and notation of [11] and call $P_{\mathfrak{g},k} = \ker \partial_k$ the $k$-th Lie kernel, which is a vector subspace of $\Lambda^k \mathfrak{g}$. Let $P_{\mathfrak{g}}$ denote the direct sum of all the Lie kernels:

$$P_{\mathfrak{g}} = \bigoplus_{k=0}^{\text{dim} \mathfrak{g}} P_{\mathfrak{g},k}.$$

Note that if the group is abelian then $P_{\mathfrak{g},k} = \Lambda^k \mathfrak{g}$.

A straightforward computation gives the following lemma.

**Lemma 2.9.** For arbitrary $p \in \Lambda^k \mathfrak{g}$ and $q \in \Lambda^l \mathfrak{g}$ we have that

$$\partial (p \wedge q) = \partial (p) \wedge q + (-1)^k p \wedge \partial (q) + (-1)^{k+l} [p,q].$$

From this lemma we get the following.

**Proposition 2.10.** We have $(P_{\mathfrak{g}}, \partial, [\cdot,\cdot])$ is a differential graded subalgebra of the Chevalley-Eilenberg complex, with $\partial = 0$. 
Proof. The only nontrivial thing we need to show is that the Schouten bracket preserves the Lie kernel. While this follows immediately from the fact that \( \partial \) is a graded derivation of the Schouten bracket, we can also show it using Lemma 2.9. Indeed, for \( p \in \mathcal{P}_\mathfrak{g},l \) and \( q \in \mathcal{P}_\mathfrak{g},l \) we have that
\[
\partial(p \wedge q) = \partial(p) \wedge q + (-1)^{k}p \wedge \partial(q) + [p,q]
\]
Hence, \([p,q]\) is exact and therefore closed. \( \square \)

We now define a differential graded Lie algebra consisting of multivector fields.

Let \( G \) be a connected Lie group acting on a manifold \( M \). For \( \xi \in \mathfrak{g} \) let \( V_\xi \in \Gamma(TM) \) denote the infinitesimal generator of the induced action on \( M \) by the one-parameter subgroup of \( G \) generated by \( \xi \). For decomposable \( p = \xi_1 \wedge \cdots \wedge \xi_k \) in \( \Lambda^k \mathfrak{g} \) we introduce the notation \( V_p := V_{\xi_1} \wedge \cdots \wedge V_{\xi_k} \) for the infinitesimal generator of \( p \). Let
\[
S_k = \{ V_p : p \in \mathcal{P}_{\mathfrak{g},k} \}
\]
and set
\[
S = \bigoplus_{k=0}^{\dim(\mathfrak{g})} S_k.
\]

**Proposition 2.11.** We have that \((S, [\cdot, \cdot])\) is a graded Lie algebra. Moreover, for \( p \in \Lambda^k \mathfrak{g} \), we have that \( \partial V_p = -V_{\partial p} \). Note that we have abused notation and let \( \partial \) denote the Chevalley-Eilenberg differentials for both the Lie algebras \((\Gamma(TM), [\cdot, \cdot])\) and \((\mathfrak{g}, [\cdot, \cdot])\).

**Proof.** We first show that \( V_{[p,q]} = -[V_p, V_q] \). Let \( p = \xi_1 \wedge \cdots \wedge \xi_k \) and \( q = \eta_1 \wedge \cdots \wedge \eta_l \). Then we have that
\[
V_{[p,q]} = \sum_{i,j} (-1)^{i+j} V_{[\xi_i,\eta_j]} \wedge V_{\xi_1} \wedge \cdots \wedge \hat{V}_{\xi_i} \wedge \cdots \wedge \hat{V}_{\eta_j} \wedge \cdots \wedge V_{\eta_l}
\]
\[
= \sum_{i,j} (-1)^{i+j} [V_{\xi_i}, V_{\eta_j}] \wedge V_{\xi_1} \wedge \cdots \wedge \hat{V}_{\xi_i} \wedge \cdots \wedge \hat{V}_{\eta_j} \wedge \cdots \wedge V_{\eta_l}
\]
\[
= -[V_p, V_q].
\]

Here we used the standard result of group actions that \([V_{\xi_i}, V_{\eta_j}] = -V_{[\xi_i, \eta_j]}\). The first claim now follows since \((\mathcal{P}_\mathfrak{g}, [\cdot, \cdot])\) is a graded Lie algebra by Proposition 2.10. Moreover, we have that
\[
\partial V_p = \partial(V_{\xi_1} \wedge \cdots \wedge V_{\xi_k})
\]
\[
= \sum_{1 \leq i < j \leq k} (-1)^{i+j} [V_{\xi_i}, V_{\xi_j}] \wedge V_{\xi_1} \wedge \cdots \wedge \hat{V}_{\xi_i} \wedge \cdots \wedge \hat{V}_{\xi_j} \wedge \cdots \wedge V_{\xi_k}
\]
\[
= - \sum_{1 \leq i < j \leq k} (-1)^{i+j} V_{[\xi_i, \xi_j]} \wedge V_{\xi_1} \wedge \cdots \wedge \hat{V}_{\xi_i} \wedge \cdots \wedge \hat{V}_{\xi_j} \wedge \cdots \wedge V_{\xi_k}
\]
\[
= -V_{\partial p}
\]
In particular then, if \( p \) is in the Lie kernel, we have that \( \partial V_p = -V_{\partial p} = 0 \). \( \square \)

The last lemma of this section will be used repeatedly in the rest of the paper. We remark that it holds for arbitrary multivector fields; however, for our purposes it will suffice to consider the restriction to elements of \( S \).
Lemma 2.12. (Extended Cartan Lemma) For decomposable $p = \xi_1 \wedge \cdots \wedge \xi_k$ in $\Lambda^k g$ and differential form $\tau$ we have that
\[
(-1)^k d(V_p \triangledown \tau) = V_{\partial_p} \triangledown \tau + \sum_{i=1}^k (-1)^i (V_{\xi_1} \wedge \cdots \wedge \hat{V}_{\xi_i} \wedge \cdots \wedge V_{\xi_k}) \triangledown V_{\xi_i} \tau + V_p \triangledown d\tau.
\]

Proof. This is Lemma 3.4 of [11] or Lemma 2.18 of [16].

While in this paper we will mostly be concerned with differential graded Lie algebras, we will also have the need to consider the more general structure of an $L_\infty$-algebra.

2.2 $L_\infty$-algebras

We only state the definition of an $L_\infty$-algebra and do not go into detail. More detail can be found in [14], for example.

Definition 2.13. An $L_\infty$-algebra is a graded vector space $L = \bigoplus_{i=-\infty}^{\infty} L_i$ together with a collection of graded skew-symmetric linear maps $\{l_k : L^{\otimes k} \rightarrow L \ ; \ k \geq 1\}$, with $\deg(l_k) = k - 2$, satisfying the following identity for all $m \geq 1$:
\[
\sum_{\sigma \in \text{Sh}(m+1)} (-1)^{\sigma} \epsilon(\sigma)(-1)^{(i-j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(m)})) = 0.
\]

Here $\sigma$ is a permutation of $m$ letters, $(-1)^{\sigma}$ is the sign of $\sigma$, and $\epsilon(\sigma)$ is the Koszul sign. The subset $\text{Sh}(p,q)$ of permutations on $p + q$ letters is the set of $(p,q)$-unshuffles. A permutation $\sigma$ of $p + q$ letters is called a $(p,q)$-unshuffle if $\sigma(i) < \sigma(i+1)$ for $i \neq p$.

An $L_\infty$-algebra $(L, \{l_k\})$ is called a Lie $n$-algebra if $L_i = 0$ for $i \geq n$ and $i < 0$.

Since any differential graded Lie algebra is a Lie $n$-algebra (indeed, just take $l_1 = \partial$, $l_2 = [\cdot, \cdot]$ and $l_k = 0$ for $k \geq 3$), Propositions 2.10 and 2.11 show that the spaces $S$ and $P_g$ have $L_\infty$-algebra structures.

3 Multisymplectic Geometry

We first recall some concepts from multisymplectic geometry.

3.1 Multi-Hamiltonian Systems

Definition 3.1. A manifold $M$ equipped with a closed $(n+1)$-form $\omega$ is called a pre-multisymplectic (or pre-$n$-plectic) manifold. If in addition the map $T_p M \rightarrow \Lambda^n T_p^* M$, $V \mapsto V \triangledown \omega$ is injective, then $(M, \omega)$ is called a multisymplectic or $n$-plectic manifold.

We will provide examples of multisymplectic manifolds in future sections, but for now give an example which comes up frequently in the rest of the paper.

Example 3.2. (Multisymplectic Phase Space) Let $N$ be a manifold and let $M = \Lambda^k (T^* N)$. Then $\pi : M \rightarrow N$ is a vector bundle over $N$ with canonical $k$-form $\theta \in \Omega^k(M)$ defined by
\[
\theta_{\mu_x}(Z_1, \ldots, Z_k) := \mu_x(\pi_*(Z_1), \ldots, \pi_*(Z_k)),
\]
for $x \in N$, $\mu_x \in \Lambda^k(T^*_x N)$, and $Z_1, \ldots, Z_k \in T_{\mu_x} M$. The $(k + 1)$-form $\omega \in \Omega^{k+1}(M)$ defined by $\omega = -d\theta$ is the canonical $(k + 1)$-form. The pair $(M, \omega)$ is a $k$-plectic manifold.

**Definition 3.3.** If for $\alpha \in \Omega^{n-1}(M)$ there exists $X_\alpha \in \Gamma(TM)$ such that $d\alpha = -X_\alpha \omega$ then we call $\alpha$ a Hamiltonian $(n-1)$-form and $X_\alpha$ its corresponding Hamiltonian vector field. We let $\Omega^{n-1}_{\text{Ham}}(M)$ denote the space of Hamiltonian $(n-1)$-forms.

**Remark 3.4.** If $\omega$ is $n$-plectic then the Hamiltonian vector field $X_\omega$ is unique. If $\omega$ is pre-$n$-plectic then Hamiltonian vector fields are unique up to an element in the kernel of $\omega$. Also, notice that in the 1-plectic (i.e. symplectic) case, every function is Hamiltonian.

**Definition 3.5.** In analogy to Hamiltonian mechanics, for a fixed $n$-plectic form $\omega$ and Hamiltonian $(n-1)$-form $H$, we call $(M, \omega, H)$ a multi-Hamiltonian system. We denote the Hamiltonian vector field of $H$ by $X_H$.

There are many examples of multi-Hamiltonian systems and we refer the reader to Section 3.1 of [16] for some results on their existence.

In [14] it was shown that to any multisymplectic manifold one can associate the following $L_\infty$-algebra.

**Definition 3.6.** The Lie $n$-algebra of observables, $L_n(M, \omega)$ is the following $L_\infty$-algebra. Let $L = \oplus_{i=0}^n L_i$ where $L_0 = \Omega^{n-1}_{\text{Ham}}(M)$ and $L_i = \Omega^{n-i-1}(M)$ for $1 \leq i \leq n-1$. The maps $l_k : L^\otimes k \to L$ of degree $k - 2$ are defined as follows. The map $l_1$ is defined to be the exterior derivative on elements of positive degree, and 0 on Hamiltonian $(n-1)$-forms. For $k > 1$ the maps $l_k$ are defined to be 0 on elements with positive degree and to be $l_k(\alpha_1, \ldots, \alpha_k) := \zeta(k)X_{\alpha_1} \cdots X_{\alpha_k} \omega$ if all of $\alpha_1, \ldots, \alpha_k$ are Hamiltonian $(n-1)$-forms. Here $\zeta(k)$ is defined to equal $(-1)^{k-1}$. We introduce this notation as this sign comes up frequently.

**Remark 3.7.** It is easily verified that $\zeta(k) \zeta(k+1) = (-1)^{k+1}$. For future reference we also note that $\zeta(k) \zeta(l) \zeta(k+l-1) = (-1)^{k+l+kl}$ and $\zeta(k) \zeta(k) = (-1)^{k+1} \zeta(k+l)$.

The following lemma from [14] will be useful later on in the paper.

**Lemma 3.8.** Let $\alpha_1, \ldots, \alpha_m \in \Omega^{n-1}_{\text{Ham}}(M)$ be arbitrary Hamiltonian $(n-1)$-forms on a multisymplectic manifold $(M, \omega)$. Let $X_1, \ldots, X_m$ denote the associated Hamiltonian vector fields. Then

$$d(X_m \cdots X_j X_i \omega) = (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} X_m \cdots \hat{X}_j \cdots \hat{X}_i \cdots X_j X_i \omega.$$  

**Proof.** This is Lemma 3.7 of [14].

Lastly we recall the terminology for group actions on a multisymplectic manifold.

**Definition 3.9.** A Lie group action $\Phi : G \times M \to M$ is called multisymplectic if $\Phi^*_\xi \omega = \omega$. A Lie algebra action $\mathfrak{g} \times \Gamma(TM) \to \Gamma(TM)$ is called multisymplectic if $L_{\mathfrak{g}} \omega = 0$ for all $\xi \in \mathfrak{g}$. We remark that a multisymplectic Lie group action induces a multisymplectic Lie algebra action. Conversely, a multisymplectic Lie algebra action induces a multisymplectic group action if the Lie group is connected. Moreover, as in [16], we will call a Lie group action on a multi-Hamiltonian system $(M, \omega, H)$ locally, globally, or strictly $H$-preserving if it is multisymplectic and if $L_{\mathfrak{g}} H$ is closed, exact, or zero respectively, for all $\xi \in \mathfrak{g}$.  

9
3.2 Homotopy Co-Momentum Maps

For a group acting on a symplectic manifold $M$, a moment map is a Lie algebra morphism between $(\mathfrak{g}, [[\cdot, \cdot]])$ and $(C^\infty(M), \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$ is the Poisson bracket. In multisymplectic geometry, the $n$-plectic form no longer provides a Lie algebra structure on the space of smooth functions. However, as we saw in the previous section, the $n$-plectic structure does define an $L\infty$-algebra, namely the Lie $n$-algebra of observables. A homotopy co-momentum map is an $L\infty$-morphism from $\mathfrak{g}$ to the Lie $n$-algebra of observables. We explain what this means in the following definition, while refer the reader to [2] for further information on $L\infty$-morphisms.

For the rest of this section, we assume a multisymplectic action of a Lie algebra $\mathfrak{g}$ on $(M, \omega)$.

**Definition 3.10.** A (homotopy) co-momentum map is an $L\infty$-morphism $(f)$ between $\mathfrak{g}$ and the Lie $n$-algebra of observables. This means that $(f)$ is a collection of maps $f_1 : \Lambda^1 \mathfrak{g} \to \Omega_{\text{Ham}}^{n-1}(M)$ and $f_k : \Lambda^k \mathfrak{g} \to \Omega^{n-k}(M)$ for $k \geq 2$ satisfying,

$$df_1(\xi) = -V_{\xi} \omega$$

for $\xi \in \mathfrak{g}$ and

$$-f_{k-1}(\partial p) = df_k(p) + \zeta(k) V_p \omega,$$

(3.1)

for $p \in \Lambda^k \mathfrak{g}$ and $k \geq 1$. A co-momentum map is called equivariant if each component $f_i : \Lambda^i \mathfrak{g} \to \Omega^{n-i}(M)$ is equivariant with respect to the adjoint and pullback actions respectively.

For reasons to be discussed in detail in the next section, we will mostly be concerned with the restriction of co-momentum maps to the Lie kernel. Under this assumption, equation (3.1) reduces to

$$df_k(p) = -\zeta(k) V_p \omega.$$  

(3.2)

**Definition 3.11.** A weak (homotopy) co-momentum map $(f)$ is a collection of maps $f_k : P_{\mathfrak{g},k} \to \Omega_{\text{Ham}}^{n-k}(M)$, where $1 \leq k \leq n$, satisfying equation (3.2).

**Remark 3.12.** By restricting the domain of a homotopy co-momentum map to the Lie kernel, we see that the multi-moment maps of Madsen and Swann (in [11] and [12]) are given precisely by the $n$-th component of our weak homotopy co-momentum maps.

We conclude this section with some examples of weak co-momentum maps:

In Hamiltonian mechanics, the phase space of a manifold $M$ is the symplectic manifold $(T^*M, \omega = -d\theta)$. The next example generalizes this to the setting of multisymplectic geometry.

**Example 3.13. (Multisymplectic Phase Space)** As in Example 5.2, let $N$ be a manifold and let $M = \Lambda^k(T^*N)$, with $\pi : M \to N$ the projection map. Let $\theta$ and $\omega = -d\theta$ denote the canonical $k$ and $(k+1)$-forms respectively. Let $G$ be a group acting on $N$ and lift this action to $M$ in the standard way. Such an action on $M$ necessarily preserves $\theta$. We define a weak homotopy co-momentum map by

$$f_l(p) := -\zeta(l+1)V_p \omega,$$

for $p \in P_{\mathfrak{g},l}$.

We now show that $(f)$ is a weak homotopy co-momentum map. For $l \geq 1$, first consider a decomposable element $p = A_1 \wedge \cdots \wedge A_l$ in $\Lambda^l \mathfrak{g}$. Then,

$$df_l(p) = -\zeta(l+1)d(V_p \omega)$$

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\[ = -\zeta(l + 1)(-1)^l \left( \partial V_p \theta + \sum_{i=1}^{l} (-1)^i A_1 \wedge \cdots \wedge \widehat{A}_i \wedge \cdots \wedge A_l \mathcal{L}_{A_i} \theta - V_p \omega \right) \] 

by Lemma 2.12

\[ = \zeta(l) (\partial V_p \theta - V_p \omega) \]

since \( G \) preserves \( \theta \)

By linearity, we thus see that this equation holds for an arbitrary element in \( \Lambda^l g \). That is, for all \( p \in \Lambda^l g \),

\[ df_l(p) = \zeta(l) (\partial V_p \theta - V_p \omega). \]

If we assume now that \( p \in \mathcal{P}_{g,l} \), it then follows from Proposition 2.11 that

\[ df_l(p) = -\zeta(l) V_p \omega. \]

Thus by equation (3.2) we see \((f) \) is a weak homotopy co-momentum map.

Remark 3.14. In symplectic geometry, symmetries on the phase space \( T^*M \) have an important relationship with the classical momentum and position functions (see Chapter 4.3 of [1]). These momentum and position functions satisfy specific commutation relations which play an important role in connecting classical and quantum mechanics. Once we extend the Poisson bracket to multisymplectic manifolds and discuss a generalized notion of symmetry, we will come back to this multisymplectic phase space and give a generalization of these classical momentum and position functions (see Section 6.1).

The next two examples will be used when we look at manifolds with a torsion-free \( G_2 \) structure.

Example 3.15. (\( \mathbb{C}^3 \) with the standard holomorphic volume form) Consider \( \mathbb{C}^3 \) with standard coordinates \( z_1, z_2, z_3 \). Let \( \Omega = dz^1 \wedge dz^2 \wedge dz^3 \) denote the standard holomorphic volume form. Let

\[ \alpha = \text{Re}(\Omega) = \frac{1}{2}(dz^1 \wedge dz^2 \wedge dz^3 + d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3). \]

It follows that \( \alpha \) is a 2-plectic form on \( \mathbb{C}^3 \). We consider the diagonal action by the maximal torus \( T^2 \subset SU(3) \) given by \( (e^{i\theta}, e^{i\eta}) \cdot (z_1, z_2, z_3) = (e^{i\theta}z_1, e^{i\eta}z_2, e^{-i(\theta + \eta)}z_3) \). We have \( t^2 = \mathbb{R}^2 \) and that the infinitesimal generators of \((1,0)\) and \((0,1)\) are

\[ A = \frac{i}{2} \left( z_1 \frac{\partial}{\partial z_1} - z_3 \frac{\partial}{\partial z_3} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_3 \frac{\partial}{\partial \bar{z}_3} \right) \]

and

\[ B = \frac{i}{2} \left( z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_3 \frac{\partial}{\partial \bar{z}_3} \right) \]

respectively.

A computation then shows that

\[ A \mathcal{J} \alpha = \frac{1}{2} d(\text{Im}(z_1 z_3 dz^2)) \]

and

\[ B \mathcal{J} \alpha = \frac{1}{2} d(\text{Im}(z_1 z_3 dz^1)). \]

Moreover,

\[ B \mathcal{J} A \mathcal{J} \alpha = -\frac{1}{4} d(\text{Re}(z_1 z_2 z_3)) \]

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Since $G = T^2$ is abelian we have that $\mathcal{P}_{g,2} = \Lambda^2 g$. Thus, by equation (3.2) we see a weak homotopy co-momentum map is given by

$$f_1(A) = \frac{1}{2}(\text{Im}(z_1 z_3 d\bar{z}^2)) \quad f_1(B) = \frac{1}{2}(\text{Im}(z_1 z_3 dz^1))$$

and

$$f_2(A \wedge B) = \frac{1}{4}(\text{Re}(z_1 z_2 z_3)).$$

If instead of $\text{Re}(\Omega)$ we were to consider $\text{Im}(\Omega)$ then in the above expressions for $f_1$ and $f_2$ we would just swap the roles of $\text{Re}$ and $\text{Im}$.

**Example 3.16. (Cubic with the standard Kahler form)** Working with the same set up as Example 3.15, now consider the standard Kahler form $\omega = i^2 (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$. This is a 1-plectic (i.e. symplectic) form on $\mathbb{C}^3$. A computation shows that

$$A_\omega = -\frac{1}{4} d(|z_1|^2 - |z_3|^2)$$

and

$$B_\omega = -\frac{1}{4} d(|z_2|^2 - |z_3|^2).$$

Thus, by equation (3.2) a weak homotopy co-momentum map is given by

$$f_1(A) = -\frac{1}{4}(|z_1|^2 - |z_3|^2) \quad f_1(B) = -\frac{1}{4}(|z_2|^2 - |z_3|^2).$$

For our last example of this section, we consider a multi-Hamiltonian system which models the motion of a particle, with unit mass, under no external net force.

**Example 3.17. (Motion in a conservative system under translation)** Consider $\mathbb{R}^3$ with the standard metric $g$ and standard coordinates $q^1$, $q^2$, $q^3$. Let $q^1$, $q^2$, $q^3$, $p_1$, $p_2$, $p_3$ denote the induced coordinates on $T^*\mathbb{R}^3 = \mathbb{R}^6$. The motion of a particle in $\mathbb{R}^3$, subject to no external force, is given by a geodesic. That is, the path $\gamma$ of the particle is an integral curve for the geodesic spray $S$, a vector field on $T\mathbb{R}^3 = \mathbb{R}^6$. Using the metric to identity $T\mathbb{R}^3$ and $T^*\mathbb{R}^3$, the geodesic spray is given by

$$S = g^{kj} p_j \frac{\partial}{\partial q^k} - \frac{1}{2} g^{ij} p_i p_j \frac{\partial}{\partial p_k},$$

as shown in Example 5.21 of [6]. Since we are working with the standard metric, the geodesic spray is just

$$S = \sum_{i=1}^3 p_i \frac{\partial}{\partial q^i}.$$

Let $M = T^*\mathbb{R}^3 = \mathbb{R}^6$ and consider the multi-Hamiltonian system $(M, \omega, H)$ where

$$\omega = \text{vol} = dq^1 dq^2 dq^3 dp_1 dp_2 dp_3$$

is the canonical volume form, and

$$H = \frac{1}{2} ((p_1 q_2 dq^3 - p_1 q_3 dq^2) - (p_2 q_1 dq^3 - p_2 q_3 dq^2) + (p_3 q_1 dq^2 - p_3 q_2 dq^1)) dp_1 dp_2 dp_3.$$

Then

$$S_\omega = dH$$
so that the \( X_H = S \). That is, the Hamiltonian vector field in this multi-Hamiltonian system is the geodesic spray. Consider the translation action of \( G = \mathbb{R}^3 \) on \( \mathbb{R}^3 \) and pull this back to an action on \( M \). The infinitesimal generators of \( e_1, e_2, e_3 \) on \( M \) are \( \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3} \) respectively. We compute the moment map for this action:

Since \( \frac{\partial}{\partial q^1} \omega = dq^2 dq^3 dp_1 dp_2 dp_3 \) it follows that \( f_1(e_1) = \frac{1}{2} (q^2 dq^3 - q^3 dq^2) dp_1 dp_2 dp_3 \) satisfies \( df(e_1) = V_{e_1} \omega \). Similar computations show that the following is a homotopy co-momentum map for the translation action on \((M, \omega, H)\):

\[
\begin{align*}
  f_1(e_1) &= \frac{1}{2} (q^2 dq^3 - q^3 dq^2) dp_1 dp_2 dp_3, \\
  f_1(e_2) &= \frac{1}{2} (q^1 dq^3 - q^3 dq^1) dp_1 dp_2 dp_3, \\
  f_1(e_3) &= \frac{1}{2} (q^1 dq^2 - q^2 dq^1) dp_1 dp_2 dp_3, \\
  f_2(e_1 \wedge e_2) &= q^3 dp_1 dp_2 dp_3, \\
  f_2(e_1 \wedge e_3) &= q^2 dp_1 dp_2 dp_3, \\
  f_2(e_2 \wedge e_3) &= q^1 dp_1 dp_2 dp_3, \\
  f_3(e_1 \wedge e_2 \wedge e_3) &= \frac{1}{3} (p_1 dp_2 dp_3 + p_2 dp_3 dp_1 + p_3 dp_1 dp_2).
\end{align*}
\]

**Remark 3.18.** In Section 5.3 we will come back to Example 3.17 and consider the multisymplectic symmetries and conserved quantities coming from this homotopy co-momentum map.

### 4 Multisymplectic Symmetries and Conserved Quantities

In this section we give a definition of conserved quantities and continuous symmetries on multisymplectic manifolds. In symplectic geometry, the Poisson bracket plays a large role in the discussion of conserved quantities. To that end, we first try to generalize the Poisson bracket to multisymplectic geometry.

#### 4.1 A Generalized Poisson Bracket

We first extend the notion of a Hamiltonian \((n-1)\)-form to arbitrary forms of degree \( \leq n - 1 \).

**Definition 4.1.** We call

\[
\Omega^{n-k}_{\text{Ham}}(M) := \{ \alpha \in \Omega^{n-k}(M); \text{ there exists } X_\alpha \in \Gamma(\Lambda^k(TM)) \text{ with } d\alpha = -X_\alpha \lrcorner \omega \}
\]

the set of Hamiltonian \((n-k)\)-forms. For a Hamiltonian \((n-k)\)-form \( \alpha \), we call \( X_\alpha \) a corresponding Hamiltonian \( k \)-vector field (or multivector field if \( k \) is not explicit).

We call

\[
\mathfrak{X}^k_{\text{Ham}}(M) := \{ X \in \Gamma(\Lambda^k(TM)); X \lrcorner \omega \text{ is exact} \}
\]

the set of Hamiltonian \( k \)-vector fields. We will refer to a primitive of \( X \lrcorner \omega \) as a corresponding Hamiltonian \((n-k)\)-form.

Of course, given a Hamiltonian \((n-k)\)-form, it does not necessarily have a unique associated Hamiltonian multivector field. Moreover, a Hamiltonian \( k \)-vector field doesn’t necessarily have a unique corresponding Hamiltonian \((n-k)\)-form. However, the following is clear:
Proposition 4.2. For $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$, any two of its Hamiltonian $k$-vector fields differ by something in the kernel of $\omega$. Conversely, for $X \in X^k_{\text{Ham}}(M)$, any two of its Hamiltonian forms differ by a closed form.

Proposition 4.2 motivates consideration of the following spaces. Let $\tilde{X}^k_{\text{Ham}}(M)$ denote the quotient space of $X^k_{\text{Ham}}(M)$ by elements in the kernel of $\omega$. Let $\tilde{\Omega}^{n-k}_{\text{Ham}}(M)$ denote the quotient of $\Omega^{n-k}_{\text{Ham}}(M)$ by closed forms. We let

$$\Omega_{\text{Ham}}(M) = \bigoplus_{k=0}^{n-1} \Omega^k_{\text{Ham}}(M)$$

and

$$\tilde{\Omega}_{\text{Ham}}(M) = \bigoplus_{k=0}^{n-1} \tilde{\Omega}^k_{\text{Ham}}(M).$$

Similarly, we let

$$X_{\text{Ham}}(M) = \bigoplus_{k=0}^{n-1} X^k_{\text{Ham}}(M)$$

and

$$\tilde{X}_{\text{Ham}}(M) = \bigoplus_{k=0}^{n-1} \tilde{X}^k_{\text{Ham}}(M).$$

It is clear that the map from $\tilde{\Omega}^{n-k}_{\text{Ham}}(M)$ to $\tilde{X}^k_{\text{Ham}}(M)$ given by $[\alpha] \mapsto [X_\alpha]$ is a bijection.

Proposition 4.3. The spaces $\tilde{\Omega}^{n-k}_{\text{Ham}}(M)$ and $\tilde{X}^k_{\text{Ham}}(M)$ are isomorphic.

Later on, we will see that there are graded Lie brackets on the spaces $\tilde{\Omega}_{\text{Ham}}(M)$ and $\tilde{X}_{\text{Ham}}(M)$ making them isomorphic as graded Lie algebras.

The next proposition will be used to show that certain statements about a Hamiltonian form are independent of the choice of the corresponding Hamiltonian multivector field.

Proposition 4.4. If $\kappa$ is in the kernel of $\omega$, then for any $X \in X^k_{\text{Ham}}(M)$ we have $[X, \kappa] \omega = 0$.

Proof. Using equation (2.3) together with the fact that $L_X \omega = 0$ and $\kappa \omega = 0$ we have

$$[X, \kappa] \omega = (-1)^{k(k+1)} L_X (\kappa \omega) - \kappa L_X \omega = 0.$$

The next proposition shows that, as in symplectic geometry, any Hamiltonian multivector field preserves $\omega$.

Proposition 4.5. For $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$ we have that $L_{X_\alpha} \omega = 0$ for all Hamiltonian multivector fields $X_\alpha$ of $\alpha$.

Proof. Let $X_\alpha$ be a Hamiltonian multivector field. We have that

$$L_{X_\alpha} \omega = L_{X_\alpha} \omega$$

$$= d(X_\alpha \omega) - (-1)^k X_\alpha d\omega$$

by equation (2.1)

$$= d(X_\alpha \omega)$$

since $d\omega = 0$

$$= -d(d\alpha)$$

by definition

$$= 0.$$

\[\square\]
We now put in a structure analogous to the Poisson bracket in Hamiltonian mechanics, which has graded analogous properties.

Given \( \alpha \in \Omega^{p-2k}_{\text{Ham}}(M) \) and \( \beta \in \Omega^{q-l}_{\text{Ham}}(M) \), a first attempt would be to define their generalized bracket to be
\[
\{ \alpha, \beta \} := X_\beta \llcorner X_\alpha \llcorner \omega,
\]
mimicking the Poisson bracket in symplectic geometry. However, we can see right away that this bracket is not graded anti-commutative since \( \{ \alpha, \beta \} = (-1)^{kl} \{ \beta, \alpha \} \). Hence, we modify our grading of the Hamiltonian forms, following the work done in [3].

**Definition 4.6.** Let \( \mathcal{H}^p(M) = \Omega^{p-2k+1}_{\text{Ham}}(M) \). That is, we are assigning the grading of \( \alpha \in \Omega^{n-k}_{\text{Ham}}(M) \) to be \(|\alpha| = k + 1 \). For \( \alpha \in \Omega^{n-k}_{\text{Ham}}(M) \) and \( \beta \in \Omega^{n-l}_{\text{Ham}}(M) \) (i.e. \( \alpha \in \mathcal{H}^{k+1}(M) \) and \( \beta \in \mathcal{H}^{l+1}(M) \)) we define their (generalized) Poisson bracket to be
\[
\{ \alpha, \beta \} := (-1)^{|\beta|} X_\beta \llcorner X_\alpha \llcorner \omega = (-1)^{l+1} X_\beta \llcorner X_\alpha \llcorner \omega.
\]
Notice that this bracket is well defined follows directly from Proposition 4.2.

With this new grading, the generalized Poisson bracket is graded commutative.

**Proposition 4.7.** Let \( \alpha \) be a form of grading \(|\alpha| = k + 1 \) and \( \beta \) a form of grading \(|\beta| = l + 1 \). That is, \( \alpha \in \Omega^{n-k}_{\text{Ham}}(M) \) and \( \beta \in \Omega^{n-l}_{\text{Ham}}(M) \). Then we have that
\[
\{ \alpha, \beta \} = (-1)^{|\alpha||\beta|} \{ \beta, \alpha \}.
\]

**Proof.** By definition,
\[
\{ \alpha, \beta \} = (-1)^{l+1} X_\beta \llcorner X_\alpha \llcorner \omega
\]
\[
= (-1)^{l+1} (-1)^{kl} X_\alpha \llcorner X_\beta \llcorner \omega
\]
\[
= (-1)^{(l+1)(k+1)+1} X_\alpha \llcorner X_\beta \llcorner \omega
\]
\[
= (-1)^{|\alpha||\beta|} (-1)^{k+1} X_\alpha \llcorner X_\beta \llcorner \omega
\]
\[
= (-1)^{|\alpha||\beta|} \{ \beta, \alpha \}.
\]
\( \square \)

The next lemma shows that the bracket of two Hamiltonian forms is Hamiltonian. In symplectic geometry, we have \( X_{\{f,g\}} = [X_f, X_g] \) (or \( X_{\{f,g\}} = -[X_f, X_g] \) if the defining equation for a Hamiltonian vector field is \( X_\omega = d\alpha \)). In multisymplectic geometry we have

**Lemma 4.8.** For \( \alpha \in \Omega^{n-k}_{\text{Ham}}(M) \) and \( \beta \in \Omega^{n-l}_{\text{Ham}}(M) \) their bracket \( \{ \alpha, \beta \} \) is in \( \Omega^{n+1-k-l}_{\text{Ham}}(M) \). That is, \( \{ \alpha, \beta \} \) is a Hamiltonian form with grading \(|\{ \alpha, \beta \}| = k + l - 2 \). More precisely, we have that \([X_\alpha, X_\beta]\) is a Hamiltonian vector field for \( \{ \alpha, \beta \} \).

**Proof.** We have that
\[
[X_\alpha, X_\beta] \llcorner \omega = -X_\beta \llcorner d(X_\alpha \llcorner \omega) + (-1)^l d(X_\beta \llcorner X_\alpha \llcorner \omega)
\]
\[
+ (-1)^{kl} X_\alpha \llcorner X_\beta \llcorner dw - (-1)^{kl+1} X_\alpha \llcorner d(X_\beta \llcorner \omega)
\]
\[
= (-1)^l d(X_\beta \llcorner X_\alpha \llcorner \omega)
\]
\[
= -d(\{ \alpha, \beta \}).
\]
\( \square \)
We now investigate the Jacobi identity for this bracket. In [3] it was mentioned that the graded Jacobi identity holds up to a closed form. We now show that the graded Jacobi identity holds up to an exact term.

**Proposition 4.9. (Graded Jacobi.)** Fix $\alpha \in \Omega^{n-k}_\text{Ham}(M)$, $\beta \in \Omega^{n-l}_\text{Ham}(M)$ and $\gamma \in \Omega^{n-m}_\text{Ham}(M)$. Let $X_\alpha, X_\beta$ and $X_\gamma$ denote arbitrary Hamiltonian multivector fields for $\alpha, \beta$ and $\gamma$ respectively. Then we have that

$$
\sum_{\text{cyclic}} (-1)^{|\alpha||\gamma|} \{\alpha, \{\beta, \gamma\}\} = (-1)^{|\beta||\gamma|+|\beta||\alpha|+|\alpha|} d(X_\alpha \lrcorner X_\beta \lrcorner X_\gamma \lrcorner \omega).
$$

**Proof.** By definition, we have that

$$
\{\alpha, \beta\} = (-1)^{|\beta|} X_\beta \lrcorner X_\alpha \lrcorner \omega = (-1)^{|\beta|+1} X_\beta \lrcorner d\alpha.
$$

Since $X_\beta$ is in $\Lambda^{|\beta|+1}(TM)$, by (2.1) it follows that

$$
\{\alpha, \beta\} = (-1)^{|\beta|+1} (-1)^{|\beta|+1} (d(X_\beta \lrcorner \alpha) - \mathcal{L}_{X_\beta} \alpha)
= d(X_\beta \lrcorner \alpha) - \mathcal{L}_{X_\beta} \alpha.
$$

(4.1)

Thus,

$$
\{\alpha, \{\beta, \gamma\}\} = (-1)^{|\beta||\gamma|+1} \{\alpha, \{\gamma, \beta\}\}
= (-1)^{|\beta||\gamma|+1+|\beta|+|\gamma|} \{\{\gamma, \beta\}, \alpha\}
= (-1)^{|\beta||\gamma|+1+|\beta|+|\gamma|} d(X_\alpha \lrcorner \{\gamma, \beta\}) - \mathcal{L}_{X_\alpha} \{\gamma, \beta\}\}
= (-1)^{|\beta||\gamma|+|\beta||\alpha|+|\gamma|+|\alpha|} (d(X_\alpha \lrcorner \{\gamma, \beta\}) - \mathcal{L}_{X_\alpha} \{d(X_\beta \lrcorner \gamma)\} + \mathcal{L}_{X_\alpha} \mathcal{L}_{X_\beta} \gamma).
$$

Hence,

$$
(-1)^{|\alpha||\gamma|} \{\alpha, \{\beta, \gamma\}\} = (-1)^{|\beta||\gamma|+|\beta||\alpha|} (d(X_\alpha \lrcorner \{\gamma, \beta\}) - \mathcal{L}_{X_\alpha} d(X_\beta \lrcorner \gamma) + \mathcal{L}_{X_\alpha} \mathcal{L}_{X_\beta} \gamma).
$$

(4.2)

Similarly, since $|\{\gamma, \alpha\}| = |\gamma| + |\alpha| - 2$, we have that

$$
\{\beta, \{\gamma, \alpha\}\} = (-1)^{|\gamma|+|\alpha|} \{\{\gamma, \alpha\}, \beta\}
= (-1)^{|\gamma|+|\alpha|+|\beta|+1} (d(X_\beta \lrcorner \{\gamma, \alpha\}) - \mathcal{L}_{X_\beta} d(X_\alpha \lrcorner \gamma) + \mathcal{L}_{X_\beta} \mathcal{L}_{X_\alpha} \gamma).
$$

(4.3)

Hence,

$$
(-1)^{|\beta||\alpha|} \{\beta, \{\gamma, \alpha\}\} = (-1)^{|\gamma|+|\beta|+1} (d(X_\beta \lrcorner \{\gamma, \alpha\}) - \mathcal{L}_{X_\beta} d(X_\alpha \lrcorner \gamma) + \mathcal{L}_{X_\beta} \mathcal{L}_{X_\alpha} \gamma).
$$

(4.4)

Lastly, using Lemma 4.8 and (4.1), we have that

$$
(-1)^{|\gamma||\beta|} \{\gamma, \{\alpha, \beta\}\} = (-1)^{|\gamma||\beta|} (d([X_\alpha, X_\beta] \lrcorner \gamma) - \mathcal{L}_{[X_\alpha, X_\beta]} \gamma).
$$

(4.4)

Now we notice that by (2.4) the terms involving $\mathcal{L}_{X_\alpha} \mathcal{L}_{X_\beta} \gamma$ from (4.2), $\mathcal{L}_{X_\beta} \mathcal{L}_{X_\alpha} \gamma$ from (4.3) and $\mathcal{L}_{[X_\alpha, X_\beta]} \gamma$ from (4.4) add to zero. Hence we now consider the term $(-1)^{|\gamma||\beta|} d([X_\alpha, X_\beta] \lrcorner \gamma)$ from (4.4). We have that
\[d([X_{\alpha}, X_{\beta}]d\gamma) = d\left((-1)^{|\alpha||\beta|+1}\mathcal{L}_{X_{\alpha}}(X_{\beta}d\gamma) - X_{\beta}\mathcal{L}_{X_{\alpha}}d\gamma\right)\]  
\[= (-1)^{|\alpha||\beta|+1}d(\mathcal{L}_{X_{\alpha}}(X_{\beta}d\gamma)) - d(X_{\beta}\mathcal{L}_{X_{\alpha}}d\gamma) + (-1)^{|\alpha|+1}d(X_{\beta}\mathcal{L}_{X_{\alpha}}d\gamma)\]  
\[= (-1)^{|\alpha||\beta|}\mathcal{L}_{X_{\alpha}}d(X_{\beta}d\gamma) - \mathcal{L}_{X_{\beta}}d(X_{\alpha}d\gamma) + (-1)^{|\alpha|}d(X_{\beta}\mathcal{L}_{X_{\alpha}}d\gamma).\]  
\[
(4.5)
\]

Thus

\[-1)^{|\alpha||\gamma|}\{\{\alpha, \beta, \gamma\} + (-1)^{|\beta||\gamma|}\{\{\beta, \gamma\}, \alpha\} + (-1)^{|\gamma||\beta|}\{\{\gamma, \alpha\}, \beta\}
= (-1)^{|\beta||\gamma|+|\gamma||\beta|+|\beta||\alpha|}d(X_{\alpha}d\gamma) + (-1)^{|\gamma||\beta|+|\beta||\alpha|}d(X_{\beta}d\gamma)
+ (-1)^{|\gamma||\beta|+|\beta||\alpha|}d(X_{\beta}\mathcal{L}_{X_{\alpha}}d\gamma)
= (-1)^{|\beta||\gamma|+|\gamma||\beta|+|\beta||\alpha|}d(X_{\alpha}d\gamma) + (-1)^{|\gamma||\beta|+|\beta||\alpha|}d(X_{\beta}d\gamma)
+ (-1)^{|\gamma||\beta|+|\beta||\alpha|}d(X_{\beta}\mathcal{L}_{X_{\alpha}}d\gamma).
\]

Summing up the results of this section we have confirmed Theorem 4.1 of [3]:

**Proposition 4.10.** With the above grading, \((\widetilde{\Omega}_{\text{Ham}}(M), \{\cdot, \cdot\})\) is a graded Lie algebra.

**Proof.** The bracket is well defined on \(\widetilde{\Omega}_{\text{Ham}}(M)\) since if \(\gamma\) is closed then \(\{\gamma, \alpha\} = (-1)^{k}X_{\alpha}d\gamma = 0\). Clearly the bracket is bilinear. Proposition 4.11 shows that the bracket is skew graded and Proposition 4.13 shows that it satisfies the Jacobi identity.

### 4.2 Conserved Quantities and their Algebraic Structure

We now turn our attention towards conserved quantities. In symplectic geometry, a conserved quantity is a 0-form \(\alpha\) that is preserved by the Hamiltonian, i.e. satisfying \(\mathcal{L}_{X_{H}}\alpha = 0\). A generalization of this definition to multisymplectic geometry was given in [16]; however, we add the requirement that a conserved quantity is also Hamiltonian. By adding in this requirement, we can now take the generalized Poisson bracket of two conserved quantities, as in symplectic geometry.

We work with a fixed multi-Hamiltonian system \((M, \omega, H)\) with \(\omega \in \Omega^{n+1}(M)\) and \(H \in \Omega_{\text{Ham}}^{n-1}(M)\), and let \(X_{H}\) denote the corresponding Hamiltonian vector field.

**Definition 4.11.** A Hamiltonian \((n - k)\)-form \(\alpha\) in \(\Omega_{\text{Ham}}^{n-k}(M)\) is called

- locally conserved if \(\mathcal{L}_{X_{H}}\alpha = 0\)
- globally conserved if \(\mathcal{L}_{X_{H}}\alpha = 0\)
- strictly conserved if \(\mathcal{L}_{X_{H}}\alpha = 0\).

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As in [16], we denote the space of locally, globally and strictly conserved forms by $C_{\text{loc}}(X_H)$, $C(X_H)$ and $C_{\text{str}}(X_H)$ respectively. We will let $\widetilde{C}_{\text{loc}}(X_H)$, $\widetilde{C}(X_H)$ and $\widetilde{C}_{\text{str}}(X_H)$ denote the conserved quantities modulo closed forms. Note that $C_{\text{str}}(X_H) \subset C(X_H) \subset C_{\text{loc}}(X_H)$ and $\widetilde{C}_{\text{str}}(X_H) \subset \widetilde{C}(X_H) \subset \widetilde{C}_{\text{loc}}(X_H)$.

The next lemma is a generalization of Lemma 1.7 in [16].

**Lemma 4.12.** Fix a Hamiltonian $(n-k)$-form $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$. If $\alpha$ is a local conserved quantity then $[X_\alpha, X_H] \lhd \omega = 0$, for some (or equivalently every) Hamiltonian multivector field $X_\alpha$ of $\alpha$. Conversely, if $[X_\alpha, X_H] \lhd \omega = 0$ then $\alpha$ is locally conserved.

**Proof.** Let $X_\alpha$ be an arbitrary Hamiltonian multivector field of $\alpha$. We have that

$$[X_\alpha, X_H] \lhd \omega = -X_H \lhd d(X_\alpha \lhd \omega) - d(X_H \lhd X_\alpha \lhd \omega) + X_\alpha \lhd (d(X_H \lhd \omega)) + X_H \lhd X_\alpha \lhd d\omega$$

by Prop 2.6

$$= -d(X_H \lhd X_\alpha \lhd \omega)$$

$$= -\mathcal{L}_{X_H}(X_\alpha \lhd \omega)$$

by (2.1)

$$= d\mathcal{L}_{X_H} \alpha$$

by (2.2).

Recall the following standard result from Hamiltonian mechanics: If $H$ is a Hamiltonian on a symplectic manifold and $f$ and $g$ are two strictly conserved quantities, i.e. $\{f, H\} = 0 = \{g, H\}$, then $\{f, g\}$ is strictly conserved. This is because $\mathcal{L}_{X_H} \{f, g\} = \{\{f, g\}, H\} = 0$ by the Jacobi identity. Moreover, if $f$ and $g$ are local or global conserved quantities (meaning that their bracket with $H$ is constant) then again $\{f, g\}$ is strictly conserved by the Jacobi identity together with the fact that the Poisson bracket with a constant function vanishes.

The next proposition generalizes these results to multisymplectic geometry.

**Proposition 4.13.** The bracket of two (local, global, or strict) conserved quantities is a strictly conserved quantity.

**Proof.** Let $\alpha \in \Omega^{n-k}_{\text{Ham}}(M)$ and $\beta \in \Omega^{n-l}_{\text{Ham}}(M)$ be any two (local, global or strict) conserved quantities. Let $X_\alpha$ and $X_\beta$ denote arbitrary Hamiltonian multivector fields corresponding to $\alpha$ and $\beta$ respectively. By definition,

$$\mathcal{L}_{X_H} \{\alpha, \beta\} = (-1)^{|\beta|} \mathcal{L}_{X_H} X_\beta \lhd X_\alpha \lhd \omega.$$

By (2.3) together with Lemma 4.12 we see that we can commute the Lie derivative and interior product. Hence,

$$\mathcal{L}_{X_H} \{\alpha, \beta\} = (-1)^{|\beta|} X_\alpha \lhd X_\beta \lhd \mathcal{L}_{X_H} \omega.$$

The claim now follows since $\mathcal{L}_{X_H} \omega = 0$, by Proposition 4.5.

As a consequence, we obtain:

**Proposition 4.14.** The spaces $(\tilde{C}_{\text{loc}}(X_H), \{\cdot, \cdot\})$, $(\tilde{C}(X_H), \{\cdot, \cdot\})$ and $(\tilde{C}_{\text{str}}(X_H), \{\cdot, \cdot\})$ are graded Lie subalgebras of $(\tilde{\Omega}_{\text{Ham}}(M), \{\cdot, \cdot\})$.

**Proof.** Proposition 4.13 shows that each of these spaces is preserved by the bracket. The claim now follows from Proposition 4.10.
We conclude this section by showing that the Hamiltonian forms constitute an $L_\infty$-subalgebra of the Lie $n$-algebra of observables. Moreover, restricting a homotopy co-momentum map to the Lie kernel gives an $L_\infty$-morphism into this $L_\infty$-algebra:

Let $\hat{L}_\infty(M, \omega) = (\hat{L}, \{l_k\})$ denote the graded vector space $\hat{L}_i = \Omega^{n-i}_\text{Ham}(M)$ for $i = 0, \ldots, n - 1$, together with the maps $l_k$ from the Lie $n$-algebra of observables.

**Theorem 4.15.** The space $(\hat{L}, \{l_k\})$ is an $L_\infty$-subalgebra of $(L, \{l_k\})$.

**Proof.** We note that $l_1$ preserves $\hat{L}$ since closed forms are Hamiltonian. For $k > 1$, since $l_k$ vanishes on elements of positive degree we need only consider $l_k(\alpha_1, \ldots, \alpha_k) = -(-1)^{\frac{k(k+1)}{2}} X_{\alpha_k} \mathcal{J} \cdots \mathcal{J} X_{\alpha_1} \mathcal{J} \omega$, where $\alpha_1, \ldots, \alpha_k$ are Hamiltonian $(n - 1)$-forms. By Lemma 3.8 we see that $l_k(\alpha_1, \ldots, \alpha_k)$ is a Hamiltonian $(n + 1 - k)$-form.

**Proposition 4.16.** The spaces $\mathcal{C}(X_H) \cap \hat{L}$, $\mathcal{C}_{\text{loc}}(X_H) \cap \hat{L}$ and $\mathcal{C}_{\text{str}}(X_H) \cap \hat{L}$ are $L_\infty$-subalgebras of $\hat{L}_\infty(M, \omega)$.

**Proof.** The proof is analogous to the proof of Proposition 1.15 in [16]. Since the proof is short, we include it here. From Theorem 4.15 we see that each of the spaces $\mathcal{C}(X_H) \cap \hat{L}$, $\mathcal{C}_{\text{loc}}(X_H) \cap \hat{L}$ and $\mathcal{C}_{\text{str}}(X_H) \cap \hat{L}$ are closed under each $l_k$. It remains to show that for Hamiltonian $(n - 1)$-forms $\alpha_1, \ldots, \alpha_k$ which are (locally, globally, strictly) conserved, that $l_k(\alpha_1, \ldots, \alpha_k)$ is (locally, globally, strictly) conserved. Indeed,

$$\mathcal{L}_{X_H} l_k(\alpha_1, \ldots, \alpha_k) = \mathcal{L}_{X_H} X_{\alpha_k} \mathcal{J} \cdots \mathcal{J} X_{\alpha_1} \mathcal{J} \omega.$$ 

Using equation (2.3) together with Lemma 1.12 we see that we can commute the Lie derivative and interior product. The claim then follows since $\mathcal{L}_{X_H} \omega = 0$. \hfill \Box

### 4.3 Continuous Symmetries and their Algebraic Structure

Fix a multi-Hamiltonian system $(M, \omega, H)$. Our motivation for the definition of a continuous symmetry comes from Hamiltonian mechanics; we directly generalize the definition. As is the case with conserved quantities, we define three types of continuous symmetry.

**Definition 4.17.** We say that a Hamiltonian multivector field $X \in \mathfrak{X}_\text{Ham}(M)$ is

- a local continuous symmetry if $\mathcal{L}_X H$ is closed,
- a global continuous symmetry if $\mathcal{L}_X H$ is exact,
- a strict continuous symmetry if $\mathcal{L}_X H = 0$.

Note that a continuous symmetry automatically preserves $\omega$ by Proposition 4.5. We denote the space of local, global, and strict continuous symmetries by $\mathcal{S}_{\text{loc}}(H)$, $\mathcal{S}(H)$, and $\mathcal{S}_{\text{str}}(H)$ respectively. Moreover, we let $\mathcal{S}_{\text{loc}}(H), \mathcal{S}(H)$, and $\mathcal{S}_{\text{str}}(H)$ denote the quotient by the kernel of $\omega$.

We will say that a multivector field $X$ is a weak (local, global, strict) continuous symmetry if $\mathcal{L}_X \omega = 0$ and $\mathcal{L}_X H$ is closed, exact, or zero respectively. That is, a weak continuous symmetry is not necessarily Hamiltonian.

**Proposition 4.18.** We have that $(\mathfrak{X}_\text{Ham}(M), [\cdot, \cdot])$ is a graded Lie subalgebra of $(\Gamma(\Lambda^*(TM)), [\cdot, \cdot])$.

**Proof.** By equation (2.6) we see that $[X, Y] \mathcal{J} \omega = (-1)^{|X|} d(X \mathcal{J} Y \mathcal{J} \omega)$. Hence the space of Hamiltonian multivector fields is closed under the Schouten bracket. \hfill \Box
Proposition 4.19. The spaces $S_{loc}(H)$, $S(H)$, and $S_{str}(H)$ are graded Lie subalgebras of $(\mathfrak{X}_{Ham}(M), [\cdot, \cdot])$.

Proof. We see that each of $S_{loc}(H)$, $S(H)$, and $S_{str}(H)$ are closed under the Schouten bracket directly from equations (2.2) and (2.4).

The next lemma generalizes Lemma 2.9 (ii) of [16].

Lemma 4.20. Let $Y \in \Gamma(\Lambda^k(TM))$. If $Y$ is a local symmetry, then $[Y, X_H] \omega = 0$. Conversely, if $[Y, X_H] \omega = 0$ and $L_Y \omega = 0$, then $Y$ is a local symmetry.

Proof. We have that $[Y, X_H] \omega = (-1)^{k+1} L_Y (X_H \omega) - X_H (L_Y \omega) = (-1)^k L_Y dH$ since $L_Y \omega = 0$ and $X_H \omega = -dH$ by (2.2).

Recall that for a group $G$ acting on a manifold $M$ we had defined in equation (2.7) the set $S_k := \{ V_p ; p \in \mathcal{P}_{g, k} \}$. Proposition 2.11 showed that $S = \bigoplus S_k$ was a graded Lie algebra. We now get the following.

Proposition 4.21. The spaces $S_{loc}(H) \cap S$, $S(H) \cap S$, and $S_{str}(H) \cap S$ are graded Lie subalgebras of $S$.

Proof. By Proposition 4.19 we have that the spaces of symmetries are preserved by the Schouten bracket. The claim now follows by Proposition 2.11.

5 Noether’s Theorem in Multisymplectic Geometry

In this section we show how Noether’s theorem extends from symplectic to multisymplectic geometry. To see this generalization explicitly, we first recall how Noether’s theorem works in symplectic geometry.

5.1 Noether’s Theorem in Symplectic Geometry

Let $(M, \omega, H)$ be a Hamiltonian system. That is $(M, \omega)$ is symplectic and $H$ is in $C^\infty(M)$. Noether’s theorem gives a correspondence between symmetries and conserved quantities. If $f \in C^\infty(M)$ is a (local, global) conserved quantity then $X_f$ is a (local, global) continuous symmetry. Conversely, if a vector field $X_f$ is a (local, global) continuous symmetry, then $f$ is a (local, global) conserved quantity. Note that in the symplectic case, local and strict symmetries and conserved quantities are the same thing.

If $X$ is only a weak (local, global) continuous symmetry, then $\mathcal{L}_X \omega = 0$ so that by the Cartan formula around each point there is a neighbourhood $U$ and a function $f \in C^\infty(U)$ such that $X = X_f$ on $U$. This function $f$ is a (local, global) conserved quantity in the Hamiltonian system $(U, \omega|_U, H|_U)$.

If we only consider the symmetries and conserved quantities coming from a co-momentum map $\mu : \mathfrak{g} \to C^\infty(M)$ then, under the assumption of an $H$-preserving group action, each symmetry $\xi$ has corresponding global conserved quantity $\mu(\xi)$ and vice versa.

The rest of this subsection formalizes this, and the following sections will generalize it to multisymplectic geometry.
Recall that an equivariant co-momentum map gives a Lie algebra morphism between \((g, [\cdot, \cdot])\) and \((C^\infty(M), \{\cdot,\cdot\})\).

**Proposition 5.1.** Let \(\mu : g \rightarrow C^\infty(M)\) be a momentum map. For \(\xi, \eta \in g\) we have that \(\mu([\xi, \eta]) = \{\mu(\xi), \mu(\eta)\} + \text{constant}\). If the moment map is equivariant then \(\mu([\xi, \eta]) = \{\mu(\xi), \mu(\eta)\}\).

**Proof.** See Theorem 4.2.8 of \([1]\). \(\square\)

**Remark 5.2.** The constant in the above proposition actually has a specific form. It is given by a specific Lie-algebra cocycle. It turns out that this cocycle has a generalization to multisymplectic geometry, a topic being explored by the author in \([7]\).

As stated above, it is clear that in the symplectic case \(C_{\text{loc}}(X_H) = C_{\text{str}}(X_H)\) and \(S_{\text{loc}}(H) = S_{\text{str}}(H)\). It is easily verified that the map \(\alpha \mapsto X_\alpha\) is a Lie algebra morphism from \((C(X_H), \{\cdot, \cdot\})\) to \((\tilde{S}(H), [\cdot, \cdot])\) and from \((C_{\text{loc}}(X_H), \{\cdot, \cdot\})\) to \((\tilde{S}_{\text{loc}}(H), [\cdot, \cdot])\). However, under the quotients this map turns into a Lie algebra isomorphism.

**Proposition 5.3.** The map \(\alpha \mapsto X_\alpha\) is a Lie algebra isomorphism from \((\tilde{C}(X_H), \{\cdot, \cdot\})\) to \((\tilde{S}(H), [\cdot, \cdot])\) and \((\tilde{C}_{\text{loc}}(X_H), \{\cdot, \cdot\})\) to \((\tilde{S}_{\text{loc}}(H), [\cdot, \cdot])\).

As a consequence of this proposition, we can now see how a momentum map sets up a Lie algebra isomorphism between the symmetries and conserved quantities it generates. Let \(C = \{\mu(\xi); \xi \in g\}\) and \(S = \{V_\xi; \xi \in g\}\). Let \(\tilde{C}\) be the quotient of \(C\) by constant functions. Let \(\tilde{S}\) denote the quotient of \(S\) by the kernel of \(\omega\). Since the kernel of \(\omega\) is trivial, \(S = \tilde{S}\). Then we get an induced well defined Poisson bracket on \(\tilde{C}\) and an induced well defined Lie bracket on \(\tilde{S}\). We thus get a Lie algebra isomorphism:

**Proposition 5.4.** The map between \((\tilde{C}, \{\cdot, \cdot\})\) and \((\tilde{S}, [\cdot, \cdot])\) that sends \([V_\xi]\) to \([\mu(\xi)]\) is a Lie-algebra isomorphism.

With our newly defined notions of symmetry and conserved quantity on a multisymplectic manifold, we now exhibit how these concepts generalize to the setup of multisymplectic geometry.

### 5.2 The Correspondence between Multisymplectic Conserved Quantities and Continuous Symmetries

We first examine the correspondence between symmetries and conserved quantities on multi-Hamiltonian systems. We will make repeated use of the following equations. Fix \(\alpha \in \Omega_H^{n-k}(M)\). By definition we have that

\[
\{\alpha, H\} = -X_H \lrcorner X_\alpha \lrcorner \omega = X_H \lrcorner d\alpha = \mathcal{L}_{X_H} \alpha - d(X_H \lrcorner \alpha).
\]

But we also know that \(\{\alpha, H\} = -\{H, \alpha\}\), since \(|H| = 2\). Thus, by definition of the Poisson bracket and equation \((2.1)\) we have that

\[
-\{H, \alpha\} = (-1)^k X_\alpha \lrcorner X_H \lrcorner \omega = (-1)^{k+1} X_\alpha \lrcorner dH = \mathcal{L}_{X_\alpha} H - d(X_\alpha \lrcorner H).
\]

Putting these together we obtain

\[
\mathcal{L}_{X_\alpha} H = d(X_\alpha \lrcorner H) + \mathcal{L}_{X_H} \alpha - d(X_H \lrcorner \alpha) \tag{5.1}
\]

and

\[
\mathcal{L}_{X_H} \alpha = d(X_H \lrcorner \alpha) + \mathcal{L}_{X_\alpha} H - d(X_\alpha \lrcorner H). \tag{5.2}
\]
Theorem 5.5. If \( \alpha \in \Omega_{\text{Ham}}^{n-k}(M) \) is a (local, global) conserved quantity then any corresponding Hamiltonian \( k \)-vector field is a (local, global) continuous symmetry. Conversely, if \( A \in \Gamma(\Lambda^k(TM)) \) is a (local, global) continuous symmetry, then any corresponding Hamiltonian form is a (local, global) conserved quantity.

Proof. Consider \( \alpha \in \Omega_{\text{Ham}}^{n-k}(M) \). Let \( X_\alpha \) be an arbitrary Hamiltonian multivector field. Then, by equation (5.1) we have that

\[
\mathcal{L}_{X_\alpha} H = d(X_\alpha \lrcorner H) + \mathcal{L}_H \alpha - d(X_H \lrcorner \alpha).
\]

Thus, if \( \alpha \) is a (local or global) conserved quantity then \( X_\alpha \) is a (local or global) continuous symmetry.

Conversely, suppose that \( A \) is a (local or global) continuous symmetry and let \( \alpha \) be a corresponding Hamiltonian form. Following the same argument above, we have by equation (5.2)

\[
\mathcal{L}_{X_H} \alpha = d(X_H \lrcorner \alpha) + \mathcal{L}_\alpha H - d(X_\alpha \lrcorner H)
\]



The correspondence between strictly conserved quantities and strict continuous symmetries is a little bit different. We have that

Corollary 5.6. If \( \alpha \in \Omega_{\text{Ham}}^{n-k}(M) \) is a strictly conserved quantity then \( X_\alpha \) is a global continuous symmetry. Conversely, if \( A \) is a strict continuous symmetry then the corresponding Hamiltonian \( (n-k) \)-form \( \alpha \) is a global conserved quantity.

Proof. This follows from the proof of the above theorem.

Remark 5.7. If we were to consider weak continuous symmetries in the above theorem, then by the Poincaré lemma, a continuous symmetry would still give a conserved quantity, but only in a neighbourhood around each point of the manifold.

The following simple example exhibits the correspondence.

Example 5.8. Consider \( M = \mathbb{R}^3 \) with volume form \( \omega = dx \wedge dy \wedge dz \), \( H = -xdy \) and \( \alpha = zdx \).

Then \( dH = -dx \wedge dy \) so that \( X_H = \frac{\partial}{\partial z} \). Also, \( d\alpha = dz \wedge dx \) and so \( X_\alpha = \frac{\partial}{\partial y} \). By the Cartan formula, we have that

\[
\mathcal{L}_{X_\alpha} H = -dx + dx = 0,
\]

which means that \( X_\alpha \in \mathcal{S}_{\text{str}}(H) \subset \mathcal{S}(H) \). We also have that

\[
\mathcal{L}_{X_H} \alpha = d(X_H \lrcorner \alpha) + \{\alpha, H\} = d(X_H \lrcorner \alpha) - d(X_\alpha \lrcorner H) = dx.
\]

That is, \( \alpha \in \mathcal{C}(X_H) \). Thus \( \alpha \) is a global conserved quantity and \( X_\alpha \) is a global continuous symmetry.

5.3 A Homotopy Co-Momentum Map as a Morphism

We work with a fixed multi-Hamiltonian system \( (M, \omega, H) \) with acting symmetry group \( G \). By definition, a co-momentum map is an \( L_\infty \)-morphism between the Chevalley-Eilenberg complex and the Lie \( n \)-algebra of observables. Recall that in Section 4 we had defined the \( L_\infty \)-algebra \( \hat{L}(M, \omega) \), where \( \hat{L} \) consisted entirely of Hamiltonian forms: \( \hat{L} = \bigoplus_{i=0}^{n-1} \Omega_{\text{Ham}}^{n-1-i}(M) \).

Proposition 5.9. A weak homotopy co-momentum map is an \( L_\infty \)-morphism from \( (\mathcal{P}_g, \partial, \{[\cdot, \cdot]\}) \) to \( (\hat{L}, \{l_k\}) \).
Proof. Equation (3.2) shows that a homotopy co-momentum map sends each element of the Lie kernel to a Hamiltonian form. Hence the claim follows from Proposition 2.10 and Theorem 4.15.

Next we study how a weak homotopy co-momentum map interacts with the generalized Poisson bracket on the space of Hamiltonian forms. In particular, to make a connection with Proposition 5.1 from symplectic geometry, we compare the difference of $f_{k+l-1}(\{p, q\})$ and $\{f_k(p), f_l(q)\}$.

Let $G$ be a Lie group acting on a multi-Hamiltonian system $(M, \omega, H)$. Let $(f)$ be a weak homotopy co-momentum map. By equation (5.2) we see that under this restriction the image of the co-momentum map is contained in the $L^\infty$-algebra $\tilde{L}(M, \omega)$ of Hamiltonian forms. Moreover, we obtain that every element in the image is a conserved quantity. This was one of the main points of [16]. Indeed, in [16] Propositions 2.12 and 2.21 say:

**Proposition 5.10.** If the group locally or globally preserves $H$, then $f_k(p)$ is a local conserved quantity for all $p \in \mathcal{P}_{g,k}$. If the group strictly preserves $H$ then $f_k(p)$ is a globally conserved quantity for all $p \in \mathcal{P}_{g,k}$.

Thus, by restricting a homotopy co-momentum map to the Lie kernel, we see that, under assumptions on the group action, every element is a conserved quantity, analogous to the setup in symplectic geometry.

As a consequence of Theorem 5.5 we see that the moment map also gives a family of continuous symmetries.

**Proposition 5.11.** If the group locally or globally preserves $H$, then $V_p$ is a local continuous symmetry for all $p \in \mathcal{P}_{g,k}$. If the group strictly preserves $H$ then $V_p$ is a global continuous symmetry for all $p \in \mathcal{P}_{g,k}$.

**Example 5.12. (Motion in a conservative system under translation)**

Recall that in Example 3.17 we considered the translation action of $\mathbb{R}^3$ on $(M, \omega, H)$ where $M = T^*\mathbb{R}^3 = \mathbb{R}^6$, $\omega = \text{vol}$ and

$$H = \frac{1}{2}((p_1q^2dq^3 - p_1q^3dq^2) - (p_1q^3dq^2 - p_2q^3dq^1) + (p_3q^1dq^2 - p_3q^2dq^1)) dp_1dp_2dp_3,$$

where $q^1, q^2, q^3$ are the standard coordinates on $\mathbb{R}^3$ and $q^1, q^2, q^3, p_1, p_2, p_3$ are the induced coordinates on $T^*\mathbb{R}^3$. It is easy to check that each of $\mathcal{L}_{\frac{\partial}{\partial q^i}} H$ are exact for $i = 1, 2, 3$. That is, the group action globally preserves $H$. Hence, by Proposition 5.10 each of the differential forms, computed in Example 3.17,

$$f_1(e_1) = \frac{1}{2}(q^2dq^3 - q^3dq^2) dp_1dp_2dp_3,$$

$$f_1(e_2) = \frac{1}{2}(q^3dq^1 - q^1dq^3) dp_1dp_2dp_3,$$

$$f_1(e_3) = \frac{1}{2}(q^1dq^2 - q^2dq^1) dp_1dp_2dp_3,$$

$$f_2(e_1 \wedge e_2) = q^3 dp_1 dp_2 dp_3, \quad f_2(e_1 \wedge e_3) = q^2 dp_1 dp_2 dp_3, \quad f_2(e_2 \wedge e_3) = q^1 dp_1 dp_2 dp_3,$$

and

$$f_3(e_1 \wedge e_2 \wedge e_3) = \frac{1}{3}(p_1 dp_2 dp_3 + p_2 dp_3 dp_1 + p_3 dp_1 dp_2).$$

are all globally conserved. Thus, by Example 3.17 the Lie derivative of these differential forms by the geodesic spray are all exact.

Moreover, by Proposition 5.11 each of $\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial q^3}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_3}$ and $\frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial q^2}$ are global continuous symmetries in this multi-Hamiltonian system.
Proof. By definition of the co-momentum map (equation (3.2)) we have that $X_\alpha - \zeta(k)V_p$ and $X_\beta - \zeta(l)V_q$ are in the kernel of $\omega$. Hence, by Proposition 4.4 we have that

$$[X_\alpha - \zeta(k)V_p, X_\beta - \zeta(l)V_q] \mathcal{I} \omega = 0.$$ 

Proposition 4.4 also shows that

$$[X_\alpha - \zeta(k)V_p, X_\beta - \zeta(l)V_q] \mathcal{I} \omega = ([X_\alpha, X_\beta] + \zeta(k)\zeta(l)[V_p, V_q]) \mathcal{I} \omega.$$ 

Thus

$$[X_\alpha, X_\beta] \mathcal{I} \omega = -\zeta(k)\zeta(l)[V_p, V_q] \mathcal{I} \omega.$$ 

The claim now follows from Proposition 4.8.

Our generalization of Proposition 5.1 to multisymplectic geometry is:

**Proposition 5.14.** For $p \in \mathcal{P}_{g,k}$ and $q \in \mathcal{P}_{g,l}$ we have that $\{f_k(p), f_l(q)\} - (-1)^{k+l+kl} f_{k+l-1}([p, q])$ is a closed $(n + 1 - k - l)$-form.

**Proof.** By definition of co-momentum map (equation (5.1)) we have that

\[
d(f_{k+l-1}([p, q])) = -f_{k+l-2}(\partial[p, q]) - \zeta(k + l - 1)V_{[p, q]} \mathcal{I} \omega = -\zeta(k + l - 1)V_{[p, q]} \mathcal{I} \omega
\]

by Proposition 2.10

\[
= \zeta(k + l - 1)[V_p, V_q] \mathcal{I} \omega
\]

by Proposition 2.11

\[
= \zeta(k + l - 1)\zeta(k)\zeta(l)[X_{f_k(p)}, X_{f_l(p)}] \mathcal{I} \omega
\]

by Proposition 5.13

\[
= -(-1)^{k+l+kl} X_{\{f_k(p), f_l(q)\}} \mathcal{I} \omega
\]

by Proposition 4.8 and Remark 3.7

\[
= (-1)^{k+l+kl} d(\{f_k(p), f_l(q)\})
\]

by definition.

**Remark 5.15.** In the symplectic case we have that $\mu([\xi, \eta]) - \{\mu(\xi), \mu(\eta)\}$ is closed (i.e. constant) and if $\mu$ is equivariant then this constant is zero. In the multisymplectic set up, the above proposition shows that $f_{k+l-1}([p, q]) - \{f_k(p), f_l(q)\}$ is a closed form. It can be shown that if the co-momentum map $(f)$, restricted to the Lie kernel, is equivariant, then this difference is exact (this is the content of ongoing research in \cite{7}). We thus have obtained the following generalization from symplectic geometry: A homotopy co-momentum map, restricted to the Lie kernel, is equivariant if and only if $\{f_k(p), f_l(q)\} = f_{k+l-1}([p, q])$ in deRham cohomology.

From this proposition we see that a co-momentum map does not necessarily preserve brackets; however, we now show that once we pass to certain cohomology groups then it will. Moreover, the co-momentum map will give an isomorphism of graded Lie algebras, generalizing Proposition 5.4.

Recall that we had defined $\mathfrak{X}_{\text{Ham}}^k(M)$ to be the quotient of $\mathfrak{X}_k(M)$ by the kernel of $\omega$ restricted to $\Lambda^k(TM)$. We set $\tilde{\mathfrak{X}}_{\text{Ham}}(M) = \oplus \tilde{\mathfrak{X}}^k_{\text{Ham}}(M)$.

**Proposition 5.16.** The Schouten bracket on $\mathfrak{X}_{\text{Ham}}(M)$ descends to a well defined bracket on $\tilde{\mathfrak{X}}_{\text{Ham}}(M)$. 
Proof. This follows directly from Proposition 4.4.

Similarly, we let $\tilde{\Omega}_n^{\cdot-k}(M)$ denote the quotient of $\Omega_n^{\cdot-k}(M)$ by the closed forms of degree $k$ and set $\tilde{\Omega}_{\text{Ham}}(M) = \oplus \Omega_n^{\cdot-k}(M)$. Recall that Proposition 4.10 showed that $(\tilde{\Omega}_{\text{Ham}}(M), \{\cdot, \cdot\})$ was a well defined graded Lie algebra.

Theorem 5.17. The map $\alpha \mapsto X_\alpha$ is an isomorphism of graded Lie algebras from $(\tilde{\Omega}_{\text{Ham}}(M), \{\cdot, \cdot\})$ to $(\tilde{\mathfrak{X}}_{\text{Ham}}(M), [\cdot, \cdot])$.

Proof. The map is well defined since the Hamiltonian multivector field of a closed form is the zero vector field. The map is clearly surjective. It is injective since if $X_\alpha = X_\beta$ then $d\alpha = d\beta$. Lastly, by Lemma 4.8, we have that $X_{\{\alpha, \beta\}} = [X_\alpha, X_\beta]$ in the quotient space.

We have now obtained a generalization of Proposition 5.3 from symplectic geometry.

Corollary 5.18. The map $\alpha \mapsto X_\alpha$ is a graded Lie algebra isomorphism from $(\tilde{\mathcal{C}}(X_H), \{\cdot, \cdot\})$ to $(\tilde{\mathcal{S}}(H), [\cdot, \cdot])$ and from $(\tilde{\mathcal{C}}_{\text{loc}}(X_H), \{\cdot, \cdot\})$ to $(\tilde{\mathcal{S}}_{\text{loc}}(H), [\cdot, \cdot])$.

Proof. We know from Proposition 4.13 that each of the spaces of conserved quantities are closed under the Poisson bracket. Similarly, by Proposition 4.19, the spaces of continuous symmetries are all closed under the Schouten bracket. The claim now follows from Theorem 5.17.

We now give a generalization of Proposition 5.4 to multisymplectic geometry: We let $C_k$ denote the image of the Lie kernel under the co-momentum map. That is, let $C_k = f_{n-k}(P_{g,n-k})$. Let $\tilde{C}_k$ denote the quotient of $C_k$ by closed forms and set $\tilde{C} = \oplus \tilde{C}_k$. Recall that we had defined $\tilde{S}_k$ to be the set $\{V_p; p \in P_{g,k}\}$. Let $\tilde{S}_k$ denote the quotient of $\tilde{S}_k$ by elements in the Lie kernel and set $\tilde{S} = \oplus \tilde{S}_k$. Our generalization of Proposition 5.4 is given by the following corollaries.

Corollary 5.19. A momentum map induces an $L_\infty$-algebra morphism from $\tilde{S}$ to $\tilde{C} \cap \tilde{\mathcal{L}}$ given by $V_p \mapsto f_k(p)$.

Proof. We see by Proposition 5.14 that the Poisson bracket preserves $\tilde{C}$. The claim follows since by definition a homotopy co-momentum map is an $L_\infty$-morphism.

Corollary 5.20. For a group action that is (locally, globally or strictly) $H$-preserving, an equivariant homotopy co-momentum map induces an isomorphism of graded Lie algebras between $(\tilde{S}, [\cdot, \cdot])$ and $(\tilde{C}, \{\cdot, \cdot\})$. Explicitly, the map is given by $[V_p] \mapsto [f_k(p)]$.

Proof. The Lie algebra isomorphism given in Theorem 5.17 is precisely the co-momentum map. Indeed, if $\alpha = f_k(p)$ for $p \in P_{g,k}$, then $X_{f_k(p)} = V_p$ in $\tilde{\mathfrak{X}}_{\text{Ham}}(M)$, since both are Hamiltonian vector fields for $\alpha$. Proposition 5.14 now shows that the co-momentum map preserves the Lie brackets on these quotient spaces.

6 Applications

We first apply the generalized Poisson bracket to extend the theory of classical momentum and position functions on the phase space of a manifold to the multisymplectic phase space.
6.1 Classical Multisymplectic Momentum and Position Forms

Recall the following notions from Hamiltonian mechanics:

Let \( N \) be a manifold and \((T^*N, \omega = -d\theta)\) the canonical phase space. Given a group action on \( N \) we can extend this to a group action on \( T^*N \) that preserves both the tautological forms \( \theta \) and \( \omega \). It is easy to check that a co-momentum map for this action is given by \( f : g \to C^\infty(N), \xi \mapsto V_\xi \circ \theta \). From this co-momentum map we can introduce the classical momentum functions, as discussed in Propositions 4.2.12 and 5.4.4 of [1]. Given \( X \in \Gamma(TN) \), its classical momentum function is \( P(X) \in C^\infty(T^*N) \) defined by \( P(X)(\alpha_q) := \alpha_q(X_q) \). Corollary 4.2.11 of [1] then shows that

\[
\{ P(X), P(Y) \} = P([X,Y])
\]

Next, given \( h \in C^\infty(N) \) define \( \tilde{h} \in C^\infty(T^*N) \) by \( \tilde{h} = h \circ \pi \). The function \( \tilde{h} \) is referred to as the corresponding position function. The following Poisson bracket relations between the momentum and position functions are then obtained in Proposition 4.2.12 of [1]:

\[
\{ \tilde{h}, \tilde{g} \} = 0
\]

and

\[
\{ \tilde{h}, P(X) \} = \widetilde{X} \!(\tilde{h}).
\]

These bracket relations are the starting point for obtaining a quantum system from a classical system.

Remark 6.1. In [1] their first bracket relation actually reads \( \{ P(X), P(Y) \} = -P([X,Y]) \). This is because their defining equation for a Hamiltonian vector field is \( dh = X_h \lrcorner \omega \), as compared to our \( dh = -X_{h \lrcorner \omega} \).

The goal of this subsection is to show how these concepts generalize to the multisymplectic phase space. As in Example 3.2 let \( N \) be a manifold and \((M, \omega)\) the multisymplectic phase space. That is, \( M = \Lambda^k(T^*N) \) and \( \omega = -d\theta \) is the canonical \((k+1)\)-form on \( M \). Let \( \pi : M \to N \) denote the projection map. In Example 3.13 we showed that

\[
f_\ell : \mathcal{P}_{g,\ell} \to \Omega^{k-\ell}_{\text{Ham}}(M), \quad p \mapsto -\zeta(\ell+1)V_{p\lrcorner \omega}
\]

was a weak homotopy co-momentum map for the action on \( M \) induced from the action on \( N \).

Definition 6.2. Given decomposable \( X = X_1 \wedge \cdots \wedge X_l \) in \( \Gamma(\Lambda^l(TN)) \) we define its momentum form \( P(X) \in \Omega^{k-\ell}(M) \) by

\[
P(X)(\mu_x)(Z_1, \ldots, Z_{k-\ell}) := -\zeta(\ell+1)\mu_x(X_1, \cdots, X_l, \pi_*Z_1, \cdots, \pi_*Z_{k-\ell}),
\]

where \( \mu_x \) is in \( M \), and then extend by linearity to non-decomposables. Moreover, given \( \alpha \in \Omega^{k-\ell}(N) \), we define the corresponding position form to be \( \pi^*\alpha \), a \((k-\ell)\)-form on \( M \). Notice that in symplectic case \((l=1)\) this definition coincides with the classical momentum and position functions.

Definition 6.3. Given a vector field \( Y \in \Gamma(TN) \) with flow \( \theta_t \), the complete lift of \( Y \) is the vector field \( Y^2 \in \Gamma(TM) \) whose flow is \((\theta_t^2)^{-1} \). For a decomposable multivector field \( Y = Y_1 \wedge \cdots \wedge Y_l \in \Gamma(\Lambda^l(TN)) \) we define its complete lift \( Y^2 \) to be \( Y_1^2 \wedge \cdots \wedge Y_l^2 \) and then extend by linearity.
For \( \xi \in \mathfrak{g} \), let \( V_\xi \) denote its infinitesimal generator on \( N \) and let \( V_\xi^p \) denote its infinitesimal generator on \( M \). Similarly, for \( p = \xi_1 \land \cdots \land \xi_l \) in \( \mathcal{P}_{\mathfrak{g},l} \) let \( V_p \) denote \( V_{\xi_1} \land \cdots \land V_{\xi_l} \) and \( V_p^e \) denote \( V_{\xi_1}^e \land \cdots \land V_{\xi_l}^e \). Notice that by definition, we are not abusing notation by letting \( V_p^e \) denote both the complete lift of \( V_p \) and the infinitesimal generator of \( p \) under the induced action on \( M \).

Lastly, note that by the equivariance of \( \pi : M \to N \), we have
\[
\pi_*(V_p^e) = V_p \circ \pi.
\]

We now examine the bracket relations between our momentum and position forms. We first rewrite the momentum form in a different way:

**Proposition 6.4.** For \( Y \in \Gamma(\Lambda^1(TN)) \) we have that \( P(Y) = -\zeta(l + 1)Y^\sharp \theta \).

**Proof.** Let \( Y = Y_1 \land \cdots \land Y_l \) be an arbitrary decomposable element of \( \Gamma(\Lambda^l(TN)) \). Let \( Z_1, \cdots, Z_{k-l} \) be arbitrary vector fields on \( M \). Fix \( \mu_x \in M \). Then
\[
(Y^\sharp \theta)_{\mu_x}(Z_1, \cdots, Z_{k-l}) = \theta_{\mu_x}(Y^\sharp_1, \cdots, Y^\sharp_l, Z_1, \cdots, Z_{k-l}) \\
= \mu_x(\pi_*Y^\sharp_1, \cdots, \pi_*Y^\sharp_l, \pi_*Z_1, \cdots, \pi_*Z_{k-l}) \\
= \mu_x(Y_1, \cdots, Y_l, Z_1, \cdots, Z_{k-l}) \\
= -\zeta(l + 1)P(Y)_{\mu_x}(Z_1, \cdots, Z_{k-l}).
\]

\( \square \)

As a corollary to the above proposition, we obtain a generalization of \( (6.1) \) to multisymplectic geometry:

**Corollary 6.5.** For \( p = \xi_1 \land \cdots \land \xi_l \) in \( \mathcal{P}_{\mathfrak{g},l} \) we have
\[
P(V_p) = f_l(p).
\]

**Proof.** This follows immediately from Proposition 6.4 since \( V_p^e \) is the infinitesimal generator of \( p \) on \( M \).

\( \square \)

In the symplectic case, given \( Y \in \Gamma(TN) \) the complete lift \( Y^\sharp \in \Gamma(T^*(TN)) \) preserves the tautological forms \( \theta \) and \( \omega \). Hence \( d(Y^\sharp \theta) = Y^\sharp \omega \) showing that each momentum function is Hamiltonian with Hamiltonian vector field the complete lift of the base vector field.

In the multisymplectic case, it is no longer true that \( \mathcal{L}_{Y^\sharp} \theta = 0 \) for a multivector field \( Y \). Instead, we need to restrict our attention to multivector fields in the Lie kernel, which we defined in Definition 2.8. We quickly recall this definition and some terminology and notation introduced in \([12]\).

Any degree \( l \)-multivector field is a sum of multivectors of the form \( Y = Y_1 \land \cdots \land Y_l \). We consider the differential graded Lie algebra \((\Gamma(\Lambda^*(TN)), \partial)\) where \( \partial_l : \Gamma(\Lambda^l(TN)) \to \Gamma(\Lambda^{l-1}(TN)) \) is given by
\[
\partial_l(Y_1 \land \cdots \land Y_l) = \sum_{1 \leq i \leq j \leq l} [Y_i, Y_j] \land Y_1 \land \cdots \land \widehat{Y}_i \land \cdots \land \widehat{Y}_j \land \cdots \land Y_l.
\]

As in \([12]\), for a differential form \( \tau \), let
\[
(\mathcal{L}_Y \tau) = \sum_{i=1}^l Y_1 \land \cdots \land \widehat{Y}_i \land \cdots \land Y_l \mathcal{L}_Y \tau.
\]

A more general version of Lemma 2.12 is given by Lemma 3.4 of \([12]\):
Lemma 6.6. For a differential form \( \tau \) and \( Y = Y_1 \wedge \cdots \wedge Y_l \in \Gamma(\Lambda^l(TN)) \) we have that
\[
Y \lrcorner d\tau - (-1)^l d(Y \lrcorner \tau) = (\mathcal{L}_Y \tau - \partial_t(Y) \lrcorner \tau).
\]

Definition 6.7. As in Definition 2.3, we call \( \mathcal{P}_l = \ker \partial_t \) the \( l \)-th Lie kernel.

Proposition 6.8. For an \( l \)-multivector field in the Lie kernel, \( Y \in \mathcal{P}_l \), we have that \( P(Y) \) is in \( \Omega^{k-l}_{\text{Ham}}(M) \). More precisely, \( \zeta(l)Y^2 \) is a Hamiltonian multivector field for \( P(Y) \).

Proof. Abusing notation, let \( \partial_t \) denote the differential on both \( \Gamma(\Lambda^*(TN)) \) and \( \Gamma(\Lambda^*(TM)) \). By definition, we have \( \partial_t(Y) = 0 \). It follows that \( \partial_t(Y^2) = 0 \). Now, since the action on \( M \) preserves \( \theta \), we have that \( \mathcal{L}_Y \tau \theta = 0 \). Thus, by Proposition 6.4 and Lemma 6.6, we have that
\[
\begin{align*}
d(P(Y)) &= -\zeta(l+1)d(Y^2 \lrcorner \theta) \\
&= -\zeta(l+1)(-1)^l(Y^2 \lrcorner d\theta) \\
&= \zeta(l+1)(-1)^l(Y^2 \lrcorner \omega) \\
&= -\zeta(l)Y^2 \lrcorner \omega,
\end{align*}
\]
where in the last equality we used Remark 3.7.

Remark 6.9. In the setup of classical Hamiltonian mechanics, the phase space of \( N \) is just \( T^*N \), and so \( k = l = 1 \). Since \( \mathcal{P}_1 = \Gamma(TN) \) we see that we are obtaining a generalization from Hamiltonian mechanics.

We now arrive at our generalization of equation (6.2):

Proposition 6.10. For \( Y_1 \in \mathcal{P}_s \) and \( Y_2 \in \mathcal{P}_t \) we have that
\[
\{P(Y_1), P(Y_2)\} = -(-1)^{s+t+1}P([Y_1, Y_2]) - \zeta(s+1)(s+1)d(Y_1^s \lrcorner Y_2^t \lrcorner \theta).
\]

Proof. Using Proposition 6.8, Remark 3.7, and the definition of the bracket, we have
\[
\begin{align*}
\{P(Y_1), P(Y_2)\} &= (-1)^{t+1}\zeta(t)(s+1)d(Y_1^s \lrcorner Y_2^t \lrcorner \omega) \\
&= (-1)^{s+t+1}\zeta(s+t)d(Y_1^s \lrcorner Y_2^t \lrcorner \omega). \tag{6.5}
\end{align*}
\]

On the other hand, by Proposition 6.4 and Remark 3.7, we have
\[
P([Y_1, Y_2]) = -\zeta(s+t)[Y_1^s, Y_2^t] \lrcorner \theta.
\]

By Proposition 6.8 and Remark 3.7, we have that
\[
d(Y_1^s \lrcorner \theta) = (-1)^s Y_1^s \lrcorner \omega
\]
and
\[
d(Y_2^t \lrcorner \theta) = (-1)^t Y_2^t \lrcorner \omega.
\]

Using these two equations and equation (2.6), we have that
\[
[Y_1^s, Y_2^t] \lrcorner \theta = Y_2^t \lrcorner (d(Y_1^s \lrcorner \theta)) + (-1)^s d(Y_2^t \lrcorner Y_1^s \lrcorner \theta) - (-1)^s Y_1^s \lrcorner Y_2^t \lrcorner \omega - (-1)^{s+t} Y_1^s \lrcorner d(Y_2^t \lrcorner \theta))
\]
\[
= (-1)^s(Y_2^t \lrcorner Y_1^s \lrcorner \omega) + (-1)^t d(Y_2^t \lrcorner Y_1^s \lrcorner \theta) - (-1)^s Y_1^s \lrcorner Y_2^t \lrcorner \omega - (-1)^{s+t}(-1)^{t+1} Y_1^s \lrcorner Y_2^t \lrcorner \omega
\]
\[
= (-1)^s Y_2^t \lrcorner Y_1^s \lrcorner \omega + (-1)^t d(Y_2^t \lrcorner Y_1^s \lrcorner \theta) - (-1)^s Y_1^s \lrcorner Y_2^t \lrcorner \omega - (-1)^{s+t} Y_1^s \lrcorner Y_2^t \lrcorner \omega.
\]

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Abusing notation, it follows that
\[
\pi \star (Y_2^{\sigma} \lrcorner Y_1^{\sigma} \lrcorner \omega) + (-1)^t d(Y_2^{\sigma} \lrcorner Y_1^{\sigma} \lrcorner \theta) - (-1)^{s+t+s} Y_1^{\sigma} \lrcorner Y_2^{\sigma} \lrcorner \omega + (-1)^{s+t+s} Y_1^{\sigma} \lrcorner Y_2^{\sigma} \lrcorner \theta.
\]
Thus,
\[
P([Y_1^{\sigma}, Y_2^{\sigma}]) = -\zeta(s + t)(-1)^s (Y_2^{\sigma} \lrcorner Y_1^{\sigma} \lrcorner \omega) - \zeta(s + t)(-1)^t d(Y_2^{\sigma} \lrcorner Y_1^{\sigma} \lrcorner \theta). \tag{6.6}
\]
Equating equations (6.5) and (6.6) and using Remark 3.7 gives the result.

To generalize (6.3) and (6.4) to the multisymplectic phase space, we need the following lemma:

**Lemma 6.11.** Let \( \alpha \) be an arbitrary \((k - l)\)-form on \( N \) and let \( \pi^* \alpha \) be the corresponding classical position form in \( \Omega^{k-l}(M) \). Then \( \pi^* \alpha \) is Hamiltonian and \( \pi_*(X_{\pi^* \alpha}) = 0 \).

**Proof.** Let \( q^1, \ldots, q^n \) denote coordinates on \( N \), and let \( \{p_{i_1 \cdots i_k}; 1 \leq i_1 < \cdots < i_k \leq n\} \) denote the induced fibre coordinates on \( M \). In these coordinates we have that
\[
\theta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} p_{i_1 \cdots i_k} dq^{i_1} \wedge \cdots \wedge dq^{i_k}
\]
so that
\[
\omega = -d\theta = \sum_{1 \leq i_1 < \cdots < i_k \leq n} -dp_{i_1 \cdots i_k} \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_k}.
\]
An arbitrary \((k - l)\)-form \( \alpha \) on \( N \) is given by
\[
\alpha = \alpha_{i_1 \cdots i_{k-l}} dq^{i_1} \wedge \cdots \wedge dq^{i_{k-l}}.
\]
Abusing notation, it follows that
\[
\pi^* \alpha = \alpha_{i_1 \cdots i_{k-l}} dq^{i_1} \wedge \cdots \wedge dq^{i_{k-l}}.
\]
Thus,
\[
d\pi^* \alpha = \frac{\partial \alpha_{i_1 \cdots i_{k-l}}}{\partial q^j} dq^j \wedge dq^{i_1} \wedge \cdots \wedge dq^{i_{k-l}}.
\]
An arbitrary \( l \)-vector field on \( M \) is of the form
\[
X = a^{i_1 \cdots i_l} \frac{\partial}{\partial q^{i_1}} \cdots \frac{\partial}{\partial q^{i_l}} + a_j^{i_1 \cdots i_{l-1}} \frac{\partial}{\partial q^{i_1}} \cdots \frac{\partial}{\partial q^{i_{l-1}}} \wedge \frac{\partial}{\partial p^j} + \cdots + a_{j_1 \cdots j_l} \frac{\partial}{\partial p^{j_1}} \cdots \frac{\partial}{\partial p^{j_l}}.
\]
Now, the multivector field \( X_{\pi^* \alpha} \) we are looking for satisfies \( X_{\pi^* \alpha} \lrcorner \omega = d\pi^* \alpha \). An exercise in combinatorics shows that there always exists an \( l \)-vector field \( X \) satisfying \( X \lrcorner \omega = d\pi^* \alpha \), proving that \( \pi^* \alpha \) is Hamiltonian. Note we can see directly from the equality \( X \lrcorner \omega = d\pi^* \alpha \) that necessarily
\[
a^{i_1 \cdots i_l} = 0.
\]
Thus \( \pi_*(X_{\pi^* \alpha}) = 0 \) as desired.

Our generalization of (6.3) is:

**Proposition 6.12.** For \( \alpha \in \Omega^{k-i}(N) \) and \( \beta \in \Omega^{k-j}(N) \) we have that
\[
\{\pi^* \alpha, \pi^* \beta\} = 0.
\]
Proof. Let $Z_1, \ldots, Z_{k+1-i-j} \in \Gamma(TM)$ be arbitrary. Then,

$$\{\pi^*\alpha, \pi^*\beta\}(Z_1, \cdots, Z_{k+1-i-j}) = (-1)^{j+1}X_{\pi^*\beta}X_{\pi^*\alpha}d\omega(Z_1, \cdots, Z_{k+1-i-j})$$

$$= (-1)^jX_{\beta}X_{\alpha}d\omega(Z_1, \cdots, Z_{k+1-i-j})$$

$$= (-1)^j\pi^*d\alpha(X_\beta, Z_1, \cdots, Z_{k+1-i-j})$$

$$= (-1)^j\pi^*d\alpha(0, \pi_\sigma Z_1, \cdots, \pi_\sigma X_{k+1-i-j})$$

by Lemma 6.11

$$= 0.$$

\[\Box\]

Our generalization of (6.4) is:

**Proposition 6.13.** For $\alpha \in \Omega^{k-i}(N)$ and $Y \in \mathcal{P}_j$, we have that

$$\{\pi^*\alpha, P(Y)\} = -\zeta(j)\pi^*(Y \lrcorner d\alpha).$$

**Proof.** Let $Z_1, \cdots, Z_{k+1-i-j} \in \Gamma(TM)$ be arbitrary. Then,

$$\{\pi^*\alpha, P(Y)\}(Z_1, \cdots, Z_{k+1-i-j}) = (-1)^{j+1}X_{P(Y)\lrcorner}X_{\pi^*\alpha}d\omega(Z_1, \cdots, Z_{k+1-i-j})$$

$$= (-1)^{j+1}\zeta(j)Y^\sharp X_{\pi^*\alpha}d\omega(Z_1, \cdots, Z_{k+1-i-j})$$

by Lemma 6.11

$$= (-1)^j\zeta(j+1)Y^\sharp \pi^*d\alpha(Z_1, \cdots, Z_{k+1-i-j})$$

$$= -\zeta(j)Y^\sharp \pi^*d\alpha(Z_1, \cdots, Z_{k+1-i-j})$$

by Remark 3.7

$$= -\zeta(j)d\alpha(\pi_\sigma Y^\sharp, \pi_\sigma Z_1, \cdots, \pi_\sigma Z_{k+1-i-j})$$

$$= -\zeta(j)d\alpha(Y, \pi_\sigma Z_1, \cdots, \pi_\sigma Z_{k+1-i-j})$$

$$= -\zeta(j)\pi^*(Y \lrcorner d\alpha)(Z_1, \cdots, Z_{k+1-i-j}).$$

\[\Box\]

### 6.2 Torsion-Free $G_2$ Manifolds

We first recall the standard $G_2$ structure on $\mathbb{R}^7$. More details for the material in this section can be found in [8]. Let $x^1, \cdots, x^7$ denote the standard coordinates on $\mathbb{R}^7$ and consider the three form $\varphi_0$ defined by

$$\varphi_0 = dx^{123} + dx^1(dx^{45} - dx^{67}) + dx^2(dx^{46} - dx^{75}) - dx^3(dx^{47} - dx^{56})$$

where we have ommitted the wedge product signs. The stabilizer of this three form is given by the Lie group $G_2$. For an arbitrary 7-manifold we define a $G_2$ structure to be a three form $\varphi$ which has around every point $p \in M$ local coordinates with $\varphi = \varphi_0$, at the point $p$.

The three form induces a unique metric $g$ and volume form, $vol$, determined by the equation

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6g(X, Y)vol.$$

From the volume form we get the Hodge star operator and hence a 4-form $\psi := \ast \varphi$. We will refer to the data $(M^7, \varphi, \psi, g)$ as a manifold with $G_2$ structure. We remark that the $G_2$ form $\varphi$ is more than just non-degenerate:
Proposition 6.14. The $G_2$ form $\varphi$ is fully nondegenerate. This means that $\varphi(X,Y,\cdot)$ is non-zero whenever $X$ and $Y$ are linearly independent.

Proof. See Theorem 2.2 of [12].

We will call a manifold with $G_2$ structure torsion-free if both $\varphi$ and $\psi$ are closed. A theorem of Fernandez and Gray shows that this happens precisely when $\varphi$ is parallel with respect to the induced metric $g$. Thus we see that a torsion-free $G_2$ structure is an example of a multisymplectic manifold.

Remark 6.15. All of the results in this section will only use the fact that $\varphi$ is closed so that, in particular, all of our results holds if the $G_2$ structure is torsion-free.

We now quickly recall some first order differential operators on a $G_2$ manifold, while referring the reader to section 4 of [10] for more details. Given $X \in \Gamma(TM)$ we will let $X^b = X \downarrow g$. Conversely, given $\alpha \in \Omega^1(M)$, let $\alpha^\sharp$ denote the metric dual vector field. Recall that given $f \in C^\infty(M)$ its gradient is defined by

$$\text{grad}(f) = (df)^\sharp.$$ 

From the metric and the three form we can define the cross product of two vector fields. Given $X,Y,Z \in \Gamma(TM)$ the cross product $X \times Y$ is defined by the equation

$$\varphi(X,Y,Z) = g(X \times Y,Z).$$

Equivalently, the cross product is defined by

$$(X \times Y)^l = X^i Y^j \varphi_{ijkl} g^{kl}.$$ 

In coordinates, this says that

$$(X \times Y)^l = X^i Y^j \varphi_{ijkl} g^{kl} \quad (6.7)$$

The last differential operator we will consider is the curl of a vector field. We first need to recall the following decomposition of two forms on a $G_2$ manifold.

Proposition 6.16. The space of 2-forms on a $G_2$ manifold has the $G_2$ irreducible decomposition

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M),$$

where

$$\Omega^2_7(M) = \{ X \downarrow \varphi; X \in \Gamma(TM) \}$$

and

$$\Omega^2_{14}(M) = \{ \alpha \in \Omega^2(M); \psi \wedge \alpha = 0 \}.$$

The projection maps: $\pi_7 : \Omega^2(M) \to \Omega^2_7(M)$ and $\pi_{14} : \Omega^2(M) \to \Omega^2_{14}(M)$ are given by

$$\pi_7(\alpha) = \frac{\alpha - * (\varphi \wedge \alpha)}{3} \quad (6.8)$$

and

$$\pi_{14}(\alpha) = \frac{2\alpha + * (\varphi \wedge \alpha)}{3} \quad (6.9)$$

Proof. See Section 2.2 of [9].
We can now define the curl of a vector field. Given \( X \in \Gamma(TM) \) its curl is defined by
\[
(\text{curl}(X))^\flat = *(dX^\flat \wedge \psi).
\] (6.10)
This is equivalent to saying that
\[
\pi_7(dX^\flat) = \text{curl}(X) \lrcorner \varphi.
\] (6.11)
In coordinates,
\[
\text{curl}(X)^l = (\nabla a X^b)g^bi \varphi_{ijk}g^{kl},
\] (6.12)
where \( \nabla \) is the Levi-Civita connection corresponding to \( g \). This is reminiscent of the fact that in \( \mathbb{R}^3 \) the curl is given by the cross product of \( \nabla \) with \( X \). Again, we refer the reader to Section 4.1 of [10] for more details.

We now translate our definition of Hamiltonian forms and vector fields into the language of \( G_2 \) geometry. By definition, we see that a 1-form is Hamiltonian if and only if its differential is in \( \Omega^2(M) \). That is,
\[
\Omega^1_{\text{Ham}}(M) = \{ \alpha \in \Omega^1(M); \pi_{14}(d\alpha) = 0 \}.
\]
Similarly,
\[
\mathfrak{x}^1_{\text{Ham}}(M) = \{ X \in \Gamma(TM); X = \text{curl}(\alpha^\sharp) \text{ and } \pi_{14}(d\alpha) = 0 \text{ for some } \alpha \in \Omega^1(M) \}.
\]
Note that if \( M \) is compact, then it follows from (6.8) and Hodge theory that there are no non-zero Hamiltonian 1-forms.

**Proposition 6.17.** If \( \alpha \) is a Hamiltonian 1-form then its corresponding Hamiltonian vector field is \( \text{curl}(\alpha^\sharp) \).

**Proof.** Since a Hamiltonian 1-form satisfies \( \pi_{14}(\alpha) = 0 \), this follows immediately from equation (6.11). \( \square \)

From Proposition 6.17 and equation (6.7) we see that the generalized Poisson bracket is given by the cross product:
\[
\{ \alpha, \beta \} = \text{curl}(\alpha^\sharp) \times \text{curl}(\beta^\sharp),
\] (6.13)
for \( \alpha, \beta \in \Omega^1_{\text{Ham}}(M) \).

Proposition 4.5 showed that a Hamiltonian vector field preserves the \( n \)-plectic form. In the language of \( G_2 \) geometry this gives:

**Proposition 6.18.** Given \( \alpha \in \Omega^1(M) \) with \( \pi_{14}(d\alpha) = 0 \), the curl of \( \alpha^\sharp \) preserves the \( G_2 \) structure. That is,
\[
\mathcal{L}_{\text{curl}(\alpha^\sharp)}\varphi^\flat = 0.
\]

**Proof.** This follows immediately from Propositions 6.17 and 4.5. \( \square \)

As a consequence of Proposition 4.8, we get the following:

**Proposition 6.19.** Let \( \alpha \) and \( \beta \) be in \( \Omega^1(M) \) with \( \pi_{14}(d\alpha) = 0 = \pi_{14}(d\beta) \). Then
\[
\pi_{14}(\text{curl}(\alpha^\sharp) \times \text{curl}(\beta^\sharp)) = 0.
\]
Moreover,
\[
\text{curl}(\text{curl}(\alpha^\sharp) \times \text{curl}(\beta^\sharp)) = [\text{curl}(\alpha^\sharp), \text{curl}(\beta^\sharp)].
\]
Proof. By equation (6.13) and Lemma 4.8 we see that \( d(\text{curl}(\alpha^2) \times \text{curl}(\beta^2)) = [X_\alpha, X_\beta] \omega \). Thus, \( \text{curl}(\alpha^2) \times \text{curl}(\beta^2) \) is in \( \Omega^2(M) \), showing that \( \pi_{14}(\text{curl}(\alpha^2) \times \text{curl}(\beta^2)) = 0 \). Moreover, we have that

\[
\text{curl}(\text{curl}(\alpha^2) \times \text{curl}(\beta^2)) \varphi = \text{curl}(\{\alpha, \beta\}) \varphi = d(\{\alpha, \beta\}) \varphi = [X_\alpha, X_\beta] \varphi = [\text{curl}(\alpha^2), \text{curl}(\beta^2)] \varphi
\]

by (6.13)

The proposition now follows since \( \varphi \) is non-degenerate.

We now consider the definition of a homotopy co-momentum map in the setting of a \( G_2 \) manifold. The equations defining the components of a homotopy co-momentum map, i.e. (3.2), reduce to finding functions \( f_1 : \mathfrak{g} \to \Omega^1(M) \) and \( f_2 : \mathcal{P}_g \to C^\infty(M) \) satisfying

\[
\pi_{14}(d(f_1(\xi))) = 0 \text{ and } \text{curl}(f_1(\xi)^2) = V_\xi,
\]

\[
V_\xi \times V_\eta = -(d(f_2(\xi \wedge \eta)))^3.
\]

We finish this section by computing a homotopy co-momentum map in the following set up, extending Example 6.7 of [11].

Consider \( \mathbb{R}^7 = \mathbb{R} \oplus \mathbb{C}^3 \) with standard 3-form given by

\[
\varphi = \frac{1}{2} \left( dz^1 \wedge dz^2 \wedge dz^3 + d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 \right) - \frac{i}{2} \left( dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3 \right) \wedge dt.
\]

In terms of \( t, x^1, x^2, x^3, y^1, y^2, y^3 \) this is

\[
\varphi = dx^1 dx^2 dx^3 - dx^1 dy^3 dy^3 - dy^1 dx^2 dy^3 - dy^1 dy^2 dx^3 - dt dx^1 dy^1 - dt dx^2 dy^2 - dt dx^3 dy^3,
\]

where we have omitted the wedge signs. Equivalently,

\[
\varphi = \Omega_3 - dt \wedge \omega_3
\]

where \( \Omega_3 \) is the standard holomorphic volume and \( \omega_3 \) is the standard Kahler form on \( \mathbb{C}^3 \). That is, \( \Omega_3 = dz^1 \wedge dz^2 \wedge dz^3 \) and \( \omega_3 = \frac{i}{2} (dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \).

As in Examples 3.13 and 3.16 we consider the standard action by the diagonal maximal torus \( T^2 \subset SU(3) \) given by \( (e^{i\theta}, e^{i\eta}) \cdot (t, z_1, z_2, z_3) = (t, e^{i\theta} z_1, e^{i\eta} z_2, e^{-i(\theta+\eta)} z_3) \). We have \( t^2 = \mathbb{R}^2 \) and that the infinitesimal generators of \((1, 0)\) and \((0, 1)\) are

\[
A = \frac{i}{2} \left( z_1 \frac{\partial}{\partial z_1} - z_3 \frac{\partial}{\partial z_3} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_3 \frac{\partial}{\partial \bar{z}_3} \right)
\]

and

\[
B = \frac{i}{2} \left( z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_3 \frac{\partial}{\partial \bar{z}_3} \right)
\]

respectively.

By Example 3.15 it follows that

\[
A \varphi = A \cdot (\Omega_3 - dt \wedge \omega_3) = \frac{1}{2} d(\text{Im} z_1 z_3 dz^2) - \frac{1}{4} dt \wedge d(|z_1|^2 - |z_3|^2)
\]

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\[
= \frac{1}{2} d \left( \text{Im}(z_1 z_3 dz^2) - \frac{1}{2}(|z_1|^2 - |z_3|^2) dt \right).
\]

Similarly,
\[
B \mathbf{\hat{\nabla}} \varphi = \frac{1}{2} d \left( \text{Im}(z_1 z_2 dz^3) - \frac{1}{2}(|z_1|^2 - |z_2|^2) dt \right).
\]

It follows that
\[
f_1((1,0)) = \frac{1}{2} \text{Im}(z_1 z_3 dz^2) - \frac{1}{4} (|z_1|^2 - |z_3|^2) dt
\]
and
\[
f_1((0,1)) = \frac{1}{2} \text{Im}(z_1 z_2 dz^3) - \frac{1}{4} (|z_1|^2 - |z_2|^2) dt
\]
give the first component of a homotopy co-momentum map. Plugging in \(f_1((1,0))\) and \(f_1((0,1))\) into (6.9) shows that
\[
\pi_{14}(f_1((1,0))) = 0 = \pi_{14}(f_1((0,1))).
\]
Moreover, using (6.12) one can directly verify that
\[
\text{curl}(f_1((1,0)))^2 = A
\]
and
\[
\text{curl}(f_1((0,1)))^2 = B,
\]
confirming (6.14).

Using Example 3.16 it follows that
\[
B \mathbf{\hat{\nabla}} A \mathbf{\hat{\nabla}} \varphi = B \mathbf{\hat{\nabla}} A \mathbf{\hat{\nabla}} (\Omega_3 - dt \wedge \omega_3)
= B \mathbf{\hat{\nabla}} A \mathbf{\hat{\nabla}} \Omega_3
= \frac{1}{4} d(\text{Re}(z_1 z_2 z_3)).
\]

Thus the second component of the homotopy co-momentum map is given by
\[
f_2(A \land B) = \frac{1}{4} \text{Re}(z_1 z_2 z_3),
\]
in accordance with Example 6.7 of [11]. Written out in the coordinates \(t, x^1, x^2, x^3, y^1, y^2, y^3\), the infinitesimal vector fields coming from the torus action are
\[
A = \frac{1}{2} \left( -y^1 \frac{\partial}{\partial x^1} + y^3 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial y^1} - x^3 \frac{\partial}{\partial y^3} \right),
\]
\[
B = \frac{1}{2} \left( -y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial y^2} - x^3 \frac{\partial}{\partial y^3} \right).
\]

Using the metric to identify 1-forms and vector fields, equation (6.7) gives the cross product of \(A\) and \(B\) to be
\[
4(A \times B) = (y^2 y^3 - x^2 x^3) \frac{\partial}{\partial x^1} + (y^1 y^3 - x^1 x^3) \frac{\partial}{\partial x^2} + (y^1 y^2 - x^1 x^2) \frac{\partial}{\partial x^3} + (x^2 y^2 + x^3 y^3) \frac{\partial}{\partial y^1} + (x^3 y^1 + x^1 y^3) \frac{\partial}{\partial y^2} + (x^1 y^2 + x^2 y^1) \frac{\partial}{\partial y^3}
\]
\[
= d(x^1 x^2 x^3 - x^1 y^2 y^3 - y^1 x^2 y^3 - y^1 y^2 x^3)
= d(\text{Re}(z_1 z_2 z_3))
\]
confirming equation (6.15). We thus have extended Example 6.7 of [11] by obtaining a full homotopy co-momentum map for the diagonal torus action on \(\mathbb{R}^7\) with the standard torsion-free \(G_2\) structure.
7 Concluding Remarks

This work poses many natural questions for future research. The following are just a few ideas:

- The existence and uniqueness of homotopy co-momentum maps has been studied in [15] and [5], for example. However, in this paper we were mostly concerned with co-momentum maps restricted to the Lie kernel, i.e. weak co-momentum maps. It would thus be desirable to have results on the existence and uniqueness of these restricted maps.

It is clear that a co-momentum map restricts to a co-momentum map on the Lie kernel. That is, if a collection of maps satisfies equation (3.1), then it satisfies equation (3.2). However, are there examples of weak co-momentum maps which do not come from the restriction of a full co-momentum map? As mentioned throughout the paper, this question is currently being investigated in [7].

- In our work, we provided a few examples of multi-Hamiltonian systems. What are some examples of other physical or interesting multi-Hamiltonian systems to which this work could be applied?

- In Section 6.1 we generalized the classical momentum and position functions on the phase space of a manifold to momentum and position forms on the multisymplectic phase space. Since, as discussed in [1], the classical momentum and position functions play an important role in connecting classical and quantum mechanics, a natural question is if there is an analogous application of our more general theory to quantum mechanics?

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