Knot Theory With The Lorentz Group

João Faria Martins
Departamento de Matemática, Instituto Superior Técnico,
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
jmartins@math.ist.utl.pt

Abstract

We analyse the perturbative expansion of the knot invariants defined from the unitary representations of the Quantum Lorentz Group in two different ways, namely using the Kontsevich Integral and weight systems, and the $R$-matrix in the Quantum Lorentz Group defined by Buffenoir and Roche. The two formulations are proved to be equivalent; and they both yield $\mathbb{C}[[h]]$-valued knot invariants related with the Melvin-Morton expansion of the Coloured Jones Polynomial.

2000 Mathematics Subject Classification: 57M27, 17B37, 20G42

Introduction

The main aim of this article is to show what a possible path to define knot invariants out of the infinite dimensional representations of the Lorentz Group is.

Let $\mathcal{A}$ be a Hopf algebra, its category of finite dimensional representations is therefore a compact monoidal category. Let $q$ be a complex number not equal to 1 or $-1$. Suppose $\mathcal{A} = U_q(\mathfrak{g})$ is the Drinfeld Jimbo algebra attached to the semisimple Lie algebra $\mathfrak{g}$. Even though $\mathcal{A}$ is not a ribbon Hopf algebra, it possesses a formal $R$-matrix and a formal ribbon element. These elements make sense when applied to finite dimensional representations of $\mathcal{A}$, and thus its category of finite dimensional representations is a ribbon category. This means we have a knot invariant attached to any finite
A similar situation happens in the case of the Quantum Lorentz Group $\mathcal{D}$ as defined by Woronovicz and Podleś in [PoW]. We shall use especially the further developments in its theory by Buffenoir and Roche, see [BR1] and [BR2]. Despite the fact $\mathcal{D}$ is not a Drinfeld Jimbo algebra, its structure of a quantum double, namely $\mathcal{D} = \mathcal{D}(U_q(\mathfrak{su}(2)), \text{Pol}(SU_q(2)))$ with $q \in (0, 1)$, makes possible the definition of a formal $R$-matrix on it. Also, it is possible to define a heuristic ribbon element. The category of finite dimensional representations of $\mathcal{D}$ can be proved to be a ribbon category, and thus we can define knot invariants out of it. In fact, as observed in [BR2], it is possible to prove that it is ribbon equivalent to the category of finite dimensional representations of $U_q(\mathfrak{su}(2) \otimes_{R^{-1}} U_q(\mathfrak{su}(2)))$. This last bialgebra equals $U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$ as an algebra but has a coproduct twisted by $R^{-1}$, the inverse of the $R$-matrix of $U_q(\mathfrak{su}(2))$. This equivalence relates the knot invariants obtained with the Coloured Jones Polynomial in a nice way.

Such splitting of $\mathcal{D}$ is not, however, the most natural when considering unitary infinite dimensional representations of it. The general classification of the unitary representations of the Quantum Lorentz Group is due to Pusz, cf [Pu]. In this case, as well as in the case of harmonic analysis, its definition as a quantum double is usually easier to deal with. A fact observed in [BR1] is that it is possible to describe the action of the formal $R$-matrix of the Quantum Lorentz Group in a class of infinite dimensional representations of it. For this reason, it is natural to ask whether there exists a knot theory attached to the infinite representations of $\mathcal{D}$. See also [C]. We shall see the answer is affirmative at least in the perturbative level. Since we are working with infinite dimensional representations the general formulation of Reshetikhin and Turaev for constructing knot invariants cannot be directly applied. It is possible, though, given a knot diagram, or to be more precise a connected $(1,1)$-tangle diagram, to make a heuristic evaluation of the Reshetikhin-Turaev functor on it. This yields an infinite series for any knot diagram. This method was also elucidated in [NR]. Unfortunately, at least for unitary infinite dimensional representations, these infinite series do not seem to converge at least for some simple knot diagrams. However, they converge $h$-adically for $q = \exp(h/2)$, since the expansions of their terms as power series in $h$ starts increasing in degree. Therefore these evaluations do define $\mathbb{C}[[h]]$-valued knot invariants. This article aims to define these invariants from the Kontsevich Integral and weight systems.
Let \( \mathfrak{g} \) be a semisimple Lie algebra. The \( h \)-adic variant of Drinfeld Jimbo algebras, that is the algebras \( U_h(\mathfrak{g}) \), is usually more practical to deal with if one wants to define knot invariants out of the infinite representations \( \mathfrak{g} \). Let us be given a knot \( K \). The fact that \( U_h(\mathfrak{g}) \) is a ribbon Hopf algebra, and not merely a formal ribbon Hopf algebra, makes it possible that a central element of it can be defined out of \( K \), or to be more precise out of a 2-dimensional diagram for it. See for example [LM]. This central element is well defined and is a knot invariant. The centre of \( U_h(\mathfrak{g}) \) is isomorphic, through a canonical isomorphism, with the algebra of formal power series on the centre of the universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \). This means that given a Lie algebra \( \mathfrak{g} \) we have a knot invariant taking its values on the algebra of formal power series over the centre of \( U(\mathfrak{g}) \). This invariant can be described out of the Kontsevich Integral. If we have an irreducible finite dimensional representation \( V \) of \( \mathfrak{g} \) each of the terms of the formal power series associated with the knot \( K \) will then act in \( V \) as a multiple of the identity. Therefore we can transform a formal power series on the centre of \( U(\mathfrak{g}) \) into a formal power series over \( \mathbb{C} \). If the representations are finite dimensional, these power series have a non zero radius of convergence and their value at \( h = 2 \log(q) \) is the value of the (rescaled) knot invariant associated with \( U_q(\mathfrak{g}) \), as long as we use the representation of \( U_q(\mathfrak{g}) \) that quantises the representation \( V \) of \( \mathfrak{g} \) with which we are working.

Notice that nothing says that the same framework cannot be applied to an infinite dimensional representations of \( \mathfrak{g} \), as long as any central element of \( U(\mathfrak{g}) \) acts in it as a multiple of the identity. Representations of this kind appear frequently in Lie algebra theory, and are commonly known as representations which admit a central character. Some examples are the irreducible cyclic highest weight representations of \( \mathfrak{g} \), for \( \mathfrak{g} \) semisimple, which, in the \( \mathfrak{sl}(2,\mathbb{C}) \) context, are simply constructed by perturbing the spin representation in such a way that we admit arbitrary complex spins. In this case this yields a knot invariant which is in some sense an analytic continuation of the Coloured Jones Polynomial.

Other examples of infinite dimensional representations that admit a central character are the representations of the Lie algebra \( L \) of the Lorentz Group which correspond to the representations of the Lorentz Group in the principal series. These are the classical counterpart of the representations of the Quantum Lorentz Group considered in [BR2]. Therefore we would expect the knot power series invariants that come out of their use to relate somehow with the knot invariants that come from the infinite dimensional
representations of the Quantum Lorentz Group. A main result of this article is that the answer is yes. Of course the way we construct central elements of the enveloping algebra of $L$ must be specified. Notice that the Quantum Lorentz Group is not the Drinfeld-Jimbo algebra associated with the Lorentz Algebra $L$. For the reasons pointed out before, one solution is to define the $h$-adic quantised universal enveloping algebra of $L$ in a non standard way as $U_h(\mathfrak{su}(2)) \otimes_{R^{-1}} U_h(\mathfrak{su}(2))$. Another solution, which is equivalent, is to use the Kontsevich Universal Knot Invariant. Using it, we can associate to a knot a series in the centre of the universal enveloping algebra of $L$, as long as we specify an $L$ invariant, non degenerate, symmetric bilinear form in $L$. These series only depend on the knot isotopy class. Such a bilinear form can be chosen so that the construction of central elements is coherent with the algebraic structure of the Quantum Lorentz Group. The algebraic properties of this kind of knot invariants will be a main topic of this article.

We are mainly interested in the definition of numerical, rather than perturbative, knot invariants from infinite dimensional representations of the Quantum Lorentz Group. We expect our expansions to relate with them, if we can define any, as their perturbation series at the origin. These issues will be dealt with in a separate work, namely [FM], where the convergence properties of the power series obtained is analysed. A major result therein is that even though the power series can have a zero radius of convergence, they are, at least in some cases Borel-Gevrey summable. This indicates that some precise numerical knot invariants may be defined.

I finish referring to the main motivation of this work, namely its possible applications to Quantum Gravity. For an example of the use of the unitary representations of the Lorentz Group in the construction of spin foam models for Quantum General Relativity we refer to [BC]. See also [NR] for its quantised counterpart.

Contents

1 Preliminaries 6
  1.1 Chord Diagrams 6
  1.2 The Kontsevich Integral 8
  1.3 Infinitesimal $R$-matrices 9
    1.3.1 Constructing Infinitesimal $R$-matrices 11
    1.3.2 A Factorisation Theorem 12
| Section                                                                 | Page |
|------------------------------------------------------------------------|------|
| 1.4 The Coloured Jones Polynomial                                      | 13   |
| 1.4.1 A Representation Interpretation of the \( z \)-Coloured Jones Polynomial | 15   |
| 2 Lorentz Group                                                        | 17   |
| 2.1 The Lorentz Algebra                                                | 19   |
| 2.1.1 The Irreducible Balanced Representations of the Lorentz Group    | 23   |
| 2.2 The Lorentz Knot Invariant                                         | 25   |
| 2.2.1 Finite Dimensional Representations                               | 26   |
| 2.2.2 Relation With the Coloured Jones Polynomial                      | 27   |
| 3 The Approach with the Framework of Buffenoir and Roche              | 29   |
| 3.1 Representations of the Quantum Lorentz Group and \( R \)-Matrix     | 29   |
| 3.2 Associated Knot Invariants                                         | 33   |
| 3.2.1 Some Heuristics                                                  | 33   |
| 3.2.2 Finite Dimensional Representation                                | 35   |
| 3.2.3 The series Are Convergent \( h \)-Adically                       | 39   |
| 3.2.4 The Series Define a \( \mathbb{C}[[h]] \)-Valued Knot Invariant  | 40   |
1 Preliminaries

1.1 Chord Diagrams

We recall the definition of the algebra of chord diagrams, which is the target space for the Kontsevich Universal Knot Invariant. For more details see for example [B] or [K]. A chord diagram is a finite set \( w = \{c_1, ..., c_n\} \) of cardinality two, non-intersecting, subsets of the oriented circle, modulo orientation preserving homeomorphisms. The subsets \( c_k \) are called chords. We usually specify a chord diagram by drawing it as in figure 1. In all the pictures we assume the circle oriented counterclockwise.

For each \( n \geq 2 \), let \( V_n \) be the free \( \mathbb{C} \) vector space on the set of all chord diagrams with \( n \) chords. That is the set of formal finite linear combinations \( w = \sum_i \lambda_i w_i \), where \( \lambda_i \in \mathbb{C} \) and \( w_i \) is a chord diagram with \( i \) chords for any \( i \). Consider the sub vector space \( 4T_n \) of \( V_n \) which is the subspace generated by all linear combinations of chord diagrams of the form displayed in figure 2. The 3 intervals considered in the circle can appear in an arbitrary order in \( S^1 \). Define for each \( n \in \mathbb{N}_0 = \{0, 1, 2, ..\} \), the vector space \( A_n = V_n / 4T_n \). We consider \( A_0 = V_0 \) and \( A_1 = V_1 \).

For any pair \( m, n \in \mathbb{N}_0 \), there exists a bilinear map \( \# : A_n \otimes A_m \to A_{m+n} \), called the connected sum product. As its name says, it is performed by doing the connected sum of chord diagrams as in figure 3.

Obviously the product is not well defined in \( V_m \otimes V_n \) since it depends
on the points in which we break the circles. The connected sum makes sense only in $A_m \otimes A_n$, since we are considering the 4-term relations. It is associative, commutative and it has a unit: the chord diagram without any chord. For more details see [B].

The vector space $A_m \otimes A_m$ is mapped via the connected sum product to $A_{m+n}$. Therefore the direct sum $A_{\text{fin}} = \bigoplus_{n \in \mathbb{N}_0} A_n$ has a commutative and associative graded algebra structure. This permits us to conclude that the vector space

$$A = \prod_{n \in \mathbb{N}_0} A_n$$

has a structure of abelian algebra over the field of complex numbers. Call it the algebra of chord diagrams. The algebra $A$ is the target space for the Kontsevich Universal Knot Invariant.

There exist also coproduct maps $\Delta : A_m \to \bigoplus_{k+l=m} A_k \otimes A_l$ which have the form of figure 4 on chord diagrams. They extend to a linear map $\Delta : A \to A \hat{\otimes} A$. Here $A \hat{\otimes} A$ is the vector space

$$\prod_{m \in \mathbb{N}_0} \bigoplus_{k+l=m} A_k \otimes A_l.$$

Notice that $A \otimes A$ is a proper sub vector space of $A \hat{\otimes} A$. 

Figure 3: Connected Sum Product.

Figure 4: Coproduct Maps.
An element \( w \in A \) is called group like if \( \Delta(w) = w \hat{\otimes} w \). That is, if writing \( w = \sum_{n \in \mathbb{N}_0} w_n \) with \( n \in A_n, \forall n \in \mathbb{N}_0 \) we have
\[
\Delta(w_n) = \sum_{l+k=n} w_k \otimes w_l.
\]
For example, \( \exp(\ominus) \) is a group like element. Here \( \ominus \) is the unique chord diagram with only one chord. This is a trivial consequence of the fact \( \Delta(\ominus) = \ominus \otimes 1 + 1 \otimes \ominus \). We have put 1 for the chord diagram without chords.

1.2 The Kontsevich Integral

We skip the definition of the (framed) Kontsevich Integral \( Z \), for which we refer for example to [K], [LM] or [W]. See also [B, CS] for the definition of the unframed version of the also called Kontsevich Universal Knot invariant.

We take the normalisation of the Kontsevich Integral for which the value of the unknot is the wheels element \( \Omega \) of [BLT]. That is \( Z(O) = Z(\infty) \), cf [B] pp 447. This is a different normalisation of the one used in [B]. We now gather the properties of the Kontsevich integral which we are going to use in the sequel:

**Theorem 1** There exists a (oriented and framed) Knot invariant \( K \mapsto Z(K) \), where \( Z(K) \) is in the algebra \( A \) of chord diagrams. Given a framed knot \( K \), \( Z(K) \) satisfies:

1. \( Z(K) \) is grouplike, cf [B]

2. If \( K' \) is obtained from \( K \) by changing its framing by a factor of 1 then \( Z(K') = Z(K) \# \exp(\ominus) \), cf [LM].

3. If \( K^* \) is the mirror image of \( K \), and writing \( Z(K) = \sum_{n \in \mathbb{N}_0} w_n \) with \( \omega_n \in A_n, \forall n \in \mathbb{N}_0 \) we have \( Z(K^*) = \sum_{n \in \mathbb{N}_0} (-1)^n w_n \), cf [CS].

4. If \( K^- \) is the knot obtained from \( K \) by reversing the orientation of it then \( Z(K^-) = \sum_{n \in \mathbb{N}_0} S(w_n) \). Here \( S : A_n \to A_n \) is the map that reverses the orientation of each chord diagram, cf [CS].
Suppose we are given a family of linear maps (weights) $W_n : A_n \to \mathbb{C}$, $n \in \mathbb{N}_0$. A knot invariant whose value on each knot is a formal power series with coefficients in $\mathbb{C}$ is called canonical if it has the form

$$K \mapsto \sum_{n \in \mathbb{N}_0} W_n(w_n)h^n.$$ 

As usual we write $Z(K) = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in A_n, \forall n \in \mathbb{N}_0$.

### 1.3 Infinitesimal R-matrices

Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{C}$. An infinitesimal R-matrix of $\mathfrak{g}$ is a symmetric tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$ such that $[\Delta(X), t] = 0, \forall X \in \mathfrak{g}$. The commutator is taken in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. The map $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the standard coproduct in $U(\mathfrak{g})$. It verifies $\Delta(X) = X \otimes 1 + 1 \otimes X$ if $X \in \mathfrak{g}$.

Suppose we are given an infinitesimal R-matrix $t$. Write $t = \sum_i a_i \otimes b_i$. We will then have:

$$\sum_{i,j} a_j a_i \otimes b_i \otimes b_j - a_i a_j \otimes b_i \otimes b_j + a_i \otimes a_j b_i \otimes b_j - a_i \otimes b_i a_j \otimes b_j = 0,$$

which resembles the 4T relations considered previously. Given a chord diagram $w$ and an infinitesimal R-matrix $t = \sum_i a_i \otimes b_i$ it is natural thus to construct an element $\phi_t(w)$ of $U(\mathfrak{g})$ in the following fashion: Start in an arbitrary point of the circle and go around it in the direction of its orientation. Order the chords of $w$ by the order with which you pass them as in figure 5. Each chord has thus an initial and an end point. Then go around the circle again and write (from the right to the left) $a_{i_k}$ or $b_{i_k}$ depending on whether you got to the initial or final point of the $k^{th}$ chord. Then sum over all the $i_k$’s. For example for the chord diagram of figure 5 the element $\phi_t(w)$ is:

$$\sum_{i_1,i_2,i_3} b_{i_2}b_{i_3}b_{i_1}a_{i_3}a_{i_2}a_{i_1}.$$ 

See [K] or [CV] for more details. It is possible to prove that $\phi_t(w)$ is well defined as an element of $U(\mathfrak{g})$, that is it does not depend on the starting point in the circle. Moreover:

**Theorem 2** Let $\mathfrak{g}$ be a Lie algebra and $t \in \mathfrak{g} \otimes \mathfrak{g}$ be an infinitesimal R-matrix. The linear map $\phi_t : V_n \to U(\mathfrak{g})$ satisfies the 4T relations, therefore it descends to a linear map $\phi_t : A_n \to U(\mathfrak{g})$. Moreover:
Figure 5: Enumerating the Chords of a Chord Diagram.

1. The image of $\phi_t$ is contained in $\mathcal{C}(U(\mathfrak{g}))$, the centre of $U(\mathfrak{g})$.

2. The degree of $\phi_t(w)$ in $U(\mathfrak{g})$ with respect to the natural filtration of $U(\mathfrak{g})$ is not bigger than twice the number of chords of $w$.

3. Given $w \in \mathcal{A}_m$ and $w' \in \mathcal{A}_n$ we have $\phi_t(w\#w') = \phi_t(w)\phi_t(w')$.

4. Consider the map $\phi_{t,h} : \mathcal{A} \to \mathcal{C}(U(\mathfrak{g}))[h]$ such that if $w = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}_0$ we have
   \[
   \phi_{t,h} = \sum_{n \in \mathbb{N}_0} \phi_t(w_n)h^n.
   \]
   Then $\phi_{t,h}$ is a $\mathbb{C}$-algebra morphism.

Recall that the Kontsevich integral is a sum of the form $Z(K) = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in \mathcal{A}_n$, $\forall n \in \mathbb{N}_0$. Therefore, given an infinitesimal $R$ matrix $t$ in a Lie algebra $\mathfrak{g}$, we can obtain a knot invariant $Z_t$ which is defined as being
   \[
   Z_t(K) = \phi_{t,h}(Z(K)) = \sum_n \phi_t(w_n)h^n.
   \]
   The target space of $Z_t$ is therefore the $\mathbb{C}$-algebra of formal power series over the centre of $U(\mathfrak{g})$.

Suppose we are given a morphism $f : \mathcal{C}(U(\mathfrak{g})) \to \mathbb{C}$. Then composing it with $Z(K)$ we obtain a canonical knot invariant $f \circ Z_t$. That is:
   \[
   (f \circ Z_t)(K) = \sum_n f(\phi_t(w_n))h^n.
   \]
It is not difficult to examine the conditions whereby this kind of knot invariants are unframed. Let \( t = \sum a_i \otimes b_i \) be an infinitesimal \( R \) matrix in a Lie algebra. Define \( C_t = \sum a_i b_i = -\phi_t(\otimes) \). It is a central element of the universal enveloping algebra of \( g \). Call it the quadratic central element associated with \( t \). The infinitesimal \( R \)-matrix \( t \) can be recovered from \( C_t \) by the formula
\[
t = \frac{\Delta(C_t) - 1 \otimes C_t - C_t \otimes 1}{2}.
\]

A morphism \( f : \mathcal{C}(U(g)) \to \mathbb{C} \) is said to be \( t \)-unframed if \( f(C_t) = 0 \). From theorem 1 and theorem 2, it is straightforward to conclude that:

**Theorem 3** Let \( g \) be a Lie algebra with an infinitesimal \( R \)-matrix \( t \). Consider also a morphism \( f \) from the centre of \( U(g) \) to \( \mathbb{C} \). Then the knot invariant \( f \circ \mathcal{Z}_t \) is unframed if and only if the morphism \( f \) is \( t \)-unframed.

Notice the Kontsevich integral of each knot is invertible in \( A \). This is because the term \( w_0 \in A_0 \) is the unit of \( A \).

### 1.3.1 Constructing Infinitesimal \( R \)-matrices

There exists a standard way to construct infinitesimal \( R \)-matrices in a Lie algebra \( g \). Suppose we are given a \( g \)-invariant, non degenerate, symmetric bilinear form \( \langle , \rangle \) in \( g \). Here \( g \)-invariance means that we have \( [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \forall X, Y, Z \in g \). If \( g \) is semisimple the Cartan-Killing form verifies the properties above. Take a basis \( \{X_i\} \) of \( g \) and let \( \{X^i\} \) be the dual basis of \( g^* \). Then it is easy to show that for any \( \lambda \in \mathbb{C} \) the tensor \( t = \lambda \sum_i X_i \otimes X^i \) is an infinitesimal \( R \)-matrix of \( g \). We are identifying \( g^* \) with \( g \) using the nondegenerate bilinear form \( \langle , \rangle \).

Suppose \( g \) is a semisimple Lie algebra and let \( t = \sum a_i \otimes b_i \) be an infinitesimal \( R \)-matrix in \( g \). Let also \( \langle , , \rangle \) denote the Cartan-Killing form on \( g \). Then the map \( g \to g \) such that \( X \mapsto \sum_i \langle X, a_i \rangle b_i \) is an intertwiner of \( g \) with respect to its adjoint representation. Therefore if \( g \) is simple it is a multiple \( \lambda \) of the identity. This permits us to conclude that \( t = \lambda X_i \otimes X^i \).

Let us now look at the case \( g \) is semisimple. Then \( g \) has a unique decomposition of the form \( g \cong g_1 \oplus ... \oplus g_n \), where each \( g_i \) is a simple Lie algebra. The Cartan-Killing form in each \( g_i \) will yield an infinitesimal \( R \)-matrix \( t_i \) in each \( g_i \). Obviously each linear combination \( t = \lambda_1 t_1 + ... + \lambda_n t_n \) is an infinitesimal \( R \)-matrix for \( g \). An argument similar to the one before proves that any infinitesimal \( R \)-matrix in \( g \) is of the form above.
It should be said that in the case in which an infinitesimal $R$-matrix in a Lie algebra $\mathfrak{g}$ comes from a non-degenerate, symmetric and $\mathfrak{g}$-invariant bilinear form then our construction of central elements yields the same result of [B], cf [CV].

1.3.2 A Factorisation Theorem

Suppose the Lie algebra $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of two Lie algebras. If $t_1$ and $t_2$ are infinitesimal $R$-matrices in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then $t = t_1 + t_2$ is also an infinitesimal $R$-matrix in $\mathfrak{g}$. It is easy to prove that given a chord diagram $w$ we have the following identity: cf (B)

$$\phi_t(w) = (\phi_{t_1} \otimes \phi_{t_2})\Delta(w).$$

We are obviously considering the standard isomorphism $U(\mathfrak{g}) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ such that $(X, Y) \mapsto X \otimes 1 + 1 \otimes Y$ for $(X, Y) \in \mathfrak{g}$.

If we are given two algebra morphisms $f_i : C(U(\mathfrak{g}_i)) \to \mathbb{C}, i = 1, 2$, then $f = f_1 \otimes f_1$ is an algebra morphism $C(U(\mathfrak{g})) \cong C(U(\mathfrak{g}_1)) \otimes C(U(\mathfrak{g}_2)) \to \mathbb{C}$. It thus makes sense to consider the knot invariant $f \circ Z_t$. It expresses in a simple form in terms of $f_i \circ Z_{t_i}, i = 1, 2$. In fact: see [BG]

**Theorem 4** Given any (oriented and framed) knot $K$ we have:

$$(f \circ Z_t)(K) = (f_1 \circ Z_{t_1})(K) \times (f_2 \circ Z_{t_2})(K),$$

as formal power series

**Proof.**

Let $K$ be a knot, write $Z(K) = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in A_n, \forall n \in \mathbb{N}_0$. We have:

$$ (f \circ Z_t)(K) = \sum_{n \in \mathbb{N}_0} (f \circ \phi_t)(w_n)h^n $$

$$ = \sum_{n \in \mathbb{N}_0} (f_1 \otimes f_2) \circ (\phi_{t_1} \otimes \phi_{t_2})(\Delta(w_n))h^n $$

$$ = \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} (f_1 \otimes f_2) \circ (\phi_{t_1} \otimes \phi_{t_2})(w_k \otimes w_l)h^n $$

$$ = \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} [(f_1 \circ \phi_{t_1})(w_k)] [(f_2 \circ \phi_{t_2})(w_l)] h^{k+l} $$

$$ = (f_1 \circ Z_{t_1})(K) \times (f_2 \circ Z_{t_2})(K). $$

This proof appears in [BG].
1.4 The Coloured Jones Polynomial

Let \( g \) be a semisimple Lie algebra over \( \mathbb{C} \). It is a well known result, see for example [V], that any algebra morphism \( \mathbb{C}(U(g)) \rightarrow \mathbb{C} \) is the central character of some representation of \( g \), which can be infinite dimensional. Recall that \( \mathbb{C}(U(g)) \) stands for the centre of \( U(g) \). To be more precise, let \( g \) be any Lie algebra and \( \rho \) a representation of \( g \) in the vector space \( V \). Then \( \rho \) is said to admit a central character if every element of \( \mathbb{C}(U(g)) \) acts on \( V \) as a multiple of the identity. In this case there exists an algebra morphism \( \lambda_\rho : \mathbb{C}(U(g)) \rightarrow \mathbb{C} \) such that \( \rho(a)(v) = \lambda_\rho(a)v, \forall a \in \mathbb{C}(U(g)), v \in V. \) The algebra morphism \( \lambda_\rho \) is called the central character of the representation \( \rho \). In particular, if \( g \) is a Lie algebra with an infinitesimal \( R \)-matrix \( t \) then given any representation \( \rho \) of \( g \) with a central character, we can construct the knot invariant \( (\lambda_\rho \circ Z_t) \).

The Coloured Jones Polynomial is, up to normalisation, a particular example of this construction. Let \( t \) be the infinitesimal \( R \)-matrix of \( \mathfrak{sl}(2, \mathbb{C}) \) corresponding to the bilinear form in it which is minus the Cartan-Killing form. Consider for any \( \alpha \in \frac{1}{2}\mathbb{N}_0 \) the representation \( \hat{\rho} \) of \( \mathfrak{sl}(2, \mathbb{C}) \) with spin \( \alpha \), thus \( \hat{\rho} \) admits a central character which we denote by \( \lambda_\alpha \). Given a framed knot \( K \) Let \( J_\alpha(K) \) denote the framed Coloured Jones Function of it. Notice we ”colour” the Jones polynomial with the spin of the representation, rather than with the dimension of it. The last one is the usual convention. We have:

\[
\frac{J_\alpha(K)}{2\alpha + 1} = (\lambda_\alpha \circ Z_t)(K), \forall \alpha \in \frac{1}{2}\mathbb{N}_0.
\]

Write

\[
\frac{J_\alpha(K)}{2\alpha + 1} = \sum_{n \in \mathbb{N}_0} J_n^\alpha(K)h^n.
\]

It is a known result that given a knot \( K \) then \( J_n^\alpha(K) \) is a polynomial in \( \alpha \) with degree at most \( 2n \), cf [MM], [C]. This is a consequence of the fact that the centre of \( U(\mathfrak{sl}(2, \mathbb{C})) \) is generated by the Casimir element of it, together with 2 of Theorem 2 Therefore we can write:

\[
\frac{J_\alpha(K)}{2\alpha + 1} = \sum_{n \in \mathbb{N}_0} \sum_{k=0}^{2n} a_k^{(n)}(K)\alpha^k h^n.
\]

For any complex number \( z \) it thus makes sense to consider the \( z \)-Coloured
Jones Function. That is:

$$\frac{J^z(K)}{2z + 1} = \sum_{n \in \mathbb{N}_0} P^n(K)(z)h^n.$$  

This yields thus a knot invariant whose value in a knot is a formal power series in two variables:

$$K \mapsto \sum_{m,n \in \mathbb{N}_0} a_k^{(n)}(K)z^k h^n,$$

with $a_k^{(n)}(K) = 0$ for $k > 2n$. It is an interesting task to investigate whether or not this kind of series defines an analytic function in two variables. As we mentioned in the introduction they have in general a zero radius of convergence, so this can only be made precise in a perturbation theory point of view, cf [FM]. This relates to the question of whether it is possible to define numerical knot invariants out of the infinite dimensional representations of the Lorentz Group. Notice it is known that if $\alpha$ is a half integer then:

$$\frac{J^\alpha(K)}{2\alpha + 1} = \sum_{m \in \mathbb{N}_0} \left(\sum_{k=0}^{2n} a_k^{(n)}(K)\alpha^k\right) h^n,$$

defines an analytic function in $h$.

For the unknot $O$ the series $J^z(O)/(2z + 1)$ has a non zero radius of convergence at any point $z \in \mathbb{C}$. The proof is not very difficult for we can have an explicit expression for it. Define, for each $z \in \mathbb{C}$ the meromorphic function:

$$F_z(h) = \frac{1}{2z + 1} \frac{\sinh((2z + 1)h/2)}{\sinh(h/2)}.$$

Thus for each $\alpha \in \frac{1}{2}\mathbb{N}_0$ we have

$$F_\alpha(h) = \frac{J^\alpha(O)}{2\alpha + 1} = \sum_{n \in \mathbb{N}_0} J_n^\alpha(O)h^n.$$

Consider the expansion:

$$F_z(h) = \sum_{n \in \mathbb{N}_0} c(z)_n h^n.$$
It is not difficult to conclude that each \( c(z)_n \) is a polynomial in \( z \) for fixed \( n \). Moreover \( c(\alpha)_n = J_n^\alpha(O) \), \( \forall \alpha \in \frac{1}{2}\mathbb{N}_0 \). This implies
\[
\frac{J^z(O)}{2z+1} = F_z(h),
\]
also as power series in \( h \). In particular the power series for the unknot are convergent. This means it makes sense to speak about the quantum dimension of the representations of spin \( z \), which are going to be defined later. To be more precise we made sense of the quantum dimension of them divided by their dimension as vector spaces. But notice the dimension of a representation of spin \( z \) with \( z \notin \frac{1}{2}\mathbb{N}_0 \) is infinite. For some more explicit examples see [FM].

1.4.1 A Representation Interpretation of the \( z \)-Coloured Jones Polynomial

We can give an interpretation of the \( z \)-Coloured Jones Polynomial in the framework of central characters. To this end, define the following elements of \( \mathfrak{sl}(2, \mathbb{C}) \):
\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Then the infinitesimal \( R \)-matrix which we are considering in \( \mathfrak{sl}(2, \mathbb{C}) \) expresses in the form:
\[
t = -\frac{1}{4} \left( E \otimes F + F \otimes E + \frac{H \otimes H}{2} \right).
\]

Notice that \( t \) is defined out of the inner product in \( \mathfrak{sl}(2, \mathbb{C}) \) which is minus the Cartan-Killing form. In particular, the Casimir element \( C \) of \( \mathfrak{sl}(2, \mathbb{C}) \) is equal to \( -C_t \), where \( C_t \) is the quadratic central element associated with \( t \). Recall subsection 1.3.

Given a half integer \( \alpha \), the representation space \( \tilde{V}^\alpha \) of the representation of spin \( \alpha \) has a basis of the form \( \{v_0, ..., v_{2\alpha}\} \). The action of the elements \( E, F \) and \( H \) of \( \mathfrak{sl}(2, \mathbb{C}) \) in \( \tilde{V}^\alpha \) is:
\[
Hv_k = (k - \alpha)v_k, \\
Ev_k = (2\alpha - k)v_{k+1},
\]
and

\[ Fv_k = kv_{k-1} \]

For an arbitrary complex number \( z \notin \frac{1}{2}\mathbb{N}_0 \), it makes sense also to speak about the representation \( \tilde{\rho} \) of spin \( z \). Consider \( \tilde{V} \) as being the infinite dimensional vector space which has the basis \( \{v_{2z}, v_{2z-1}, v_{2z-2}, \ldots \} \). Then the representation \( \tilde{\rho} \) of spin \( z \) can be defined in the form:

\[
Hv_k = (k - z)v_k; k = 2z, 2z - 1, ...
\]

\[
Ev_k = (2z - k)v_{k+1}; k = 2z, 2z - 1, ...
\]

\[
Fv_k = kv_{k-1}; k = 2z, 2z - 1, ...
\]

The representations of spin \( z \notin \frac{1}{2}\mathbb{N}_0 \) have a central character \( \lambda_z \), since it is easily proved that each intertwiner \( \tilde{\rho} \rightarrow \tilde{\rho} \) must be a multiple of the identity. But see [\cite{V}, 4.10.2.], namely they are the unique irreducible cyclic highest weight representations with maximal weight \( z \), this relative to the usual Borel decomposition of \( \mathfrak{sl}(2, \mathbb{C}) \). Consider, given \( z \in \mathbb{C} \), the framed knot invariant \( (\lambda_z \circ Z_t) \). Where, if \( \alpha \) is half integer, \( \lambda_\alpha \) is the central character of the usual representation of spin \( \alpha \). Given a framed knot \( K \) it has the form:

\[
(\lambda_z \circ Z_t)(K) = \sum_{n \in \mathbb{N}_0} R^z_n(K)h^n,
\]

where, by definition:

\[
R^z_n(K) = (\lambda_z \circ \phi_t)(w_n) = \sum_{n \in \mathbb{N}_0} \lambda_z(\phi_t(w_n))h^n,
\]

for

\[
Z(K) = \sum_{n \in \mathbb{N}_0} w_n, w_n \in A_n, \forall n \in \mathbb{N}_0.
\]

Also

\[
\frac{J^\alpha(K)}{2z + 1} = \sum_{n \in \mathbb{N}_0} R^\alpha_n(K)h^n, \forall \alpha \in \frac{1}{2}\mathbb{N}.
\]

Suppose \( w \) is a chord diagram with \( n \) chords. Let us have a look at the dependence of \( \lambda_z(\phi_t(w)) \) in \( z \). It is not difficult to conclude that it is a polynomial in this variable of degree at most \( 2n \). This is a trivial consequence
of the definition of the central element \( \phi_t(w) \) as well as the kind of action of the terms appearing in the infinitesimal \( R \)-matrix \( t \) in \( \hat{V} \). See also [V] or [FM]. In particular if \( K \) is a framed knot, \( R^z_n(K) \) is a polynomial in \( z \). Since we also have \( R^\alpha_n(K) = J^\alpha_n(K), \forall \alpha \in \frac{1}{2}\mathbb{N}_0, \) we can conclude:

\[
\frac{J^z(K)}{2z + 1} = (\lambda_z \circ Z)(K),
\]

which gives us an equivalent definition of the \( z \)-Coloured Jones Polynomial.

The central characters of the representations of imaginary spin are actually the infinitesimal characters, cf [Kir], of the unitary representations of \( SL(2, \mathbb{R}) \) in the principal series, cf [L], with the same parameter. Notice however that the derived representation of them in \( \mathfrak{sl}(2, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C}) \) is not any of the representation of imaginary spin just defined. This is the point of view considered in [FM].

2 Lorentz Group

Let \( \mathfrak{g} \) be a semisimple Lie Algebra. As proved by Drinfel’d in [D], there is a one to one correspondence between gauge equivalence classes of quantised universal enveloping algebras \( \mathcal{H} \) of \( \mathfrak{g} \) over \( \mathbb{C}[[h]] \), cf [K], and infinitesimal \( R \)-matrices in \( \mathfrak{g} \). Let us be more explicit about this. It is implicit in the definition of a quantised universal enveloping algebra \( \mathcal{H} \) that there exists a \( \mathbb{C} \)-algebra morphism \( f : \mathcal{H}/h\mathcal{H} \to U(\mathfrak{g}) \). Having chosen such morphism, the canonical 2-tensor of \( \mathcal{A} \) is defined as \( t = f((R_{21}R - 1)/h) \). It is an infinitesimal \( R \)-matrix of \( \mathcal{A} \). Here \( R \) denotes the universal \( R \)-matrix of \( \mathcal{H} \). If \( \mathcal{H} \) quantises the pair \( (\mathfrak{g}, r) \) where \( r \) is a classical \( r \)-matrix in \( \mathfrak{g} \), see [CP], then it is the symmetrisation of \( r \). Each quantised universal enveloping algebra can be given a structure of ribbon quasi Hopf algebra, cf [AC], and therefore there is a knot invariant attached each finite dimensional representation of it, or what is the same, of \( \mathfrak{g} \). These knot invariants take their values in the ring of formal power series over \( \mathbb{C} \). If the representation used is finite dimensional and irreducible then it has a central character. In particular the framework of last section can be applied, using for example the infinitesimal \( R \) matrix \( t \) which is the canonical 2 tensor of \( \mathcal{H} \). It is a deep result that with these choices the two approaches for knot invariants are the same, up to division by the dimension of the representation considered. To be more
precise we need also to change the sign of the infinitesimal $R$-matrix $t$, cf \cite{K}.

In the case in which we consider a $q$-deformation $\mathcal{A}$ of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, then no such classification of gauge equivalence classes of quantised universal enveloping algebras exists. But sometimes it is possible to make sense of the formula for $t$. This is because we have a $q$-parametrised family of braided Hopf algebras that tends to the universal enveloping algebra of $\mathfrak{g}$ as $q$ goes to 1, or alternatively because $\mathcal{A}$ quantises the pair $(\mathfrak{g}, r)$ where $r$ is an $r$-matrix in $\mathfrak{g}$.

As mentioned in the introduction, despite the fact that the $q$-Drinfeld-Jimbo quantised universal enveloping algebras $U_q(\mathfrak{g})$ of semisimple Lie algebras are not ribbon Hopf algebras, their category of finite dimensional representations is a ribbon category. That is they have formal $R$-matrices and ribbon elements, which make sense when acting in their finite dimensional representations. The target space for the knot invariants in this context is the complex plane. These numerical knot invariants can be obtained, apart from rescaling, by summing the powers series which appear in context of $h$-adic Drinfeld-Jimbo algebras. In other words by summing the power series that come out of the approach making use the Kontsevich Integral and using the infinitesimal $R$-matrix which is the heuristic canonical 2-tensor $t$ of $U_q(\mathfrak{g})$.

Let us pass now to the Quantum Lorentz Group $\mathcal{D}$ as defined in \cite{BR1} and \cite{BR2}. It is a quantum group depending on a parameter $q \in (0, 1)$. As said in the introduction, we wish to analyse the question of whether or not there exists a knot theory attached to the infinite dimensional representations of it. The situation is more or less the same as the case of $q$-Drinfeld-Jimbo algebras. Namely we have an heuristic $R$-matrix which comes from its structure of a quantum double as well as a heuristic ribbon element. It is possible to describe how they act in the unitary representations of $\mathcal{D}$. The situation is simpler if the minimal spin of the representation is equal to zero, in which case the representation is said to be balanced. Representations of this kind are called simple in \cite{NR}. In this context, the ribbon element acts as the identity and therefore the knot invariants obtained will be unframed. These invariants express out of an infinite sum as we will see in section 3.

One natural thing to do would be analysing whether the "derivatives" of these sums define or not Vassiliev invariants, or whether is possible to make sense of them, in the framework of Kontsevich Universal Invariant. It is not difficult to find an expression for the heuristic canonical 2 tensor
of the quantum Lorentz Group. Also the unitary representations of the quantum Lorentz Group in the principal and complementary series have a classical counterpart. They are infinite dimensional representations of the Lie algebra of the Lorentz Group which admit a central character and therefore the framework of the last section can be used. This is the program we wish to consider now.

2.1 The Lorentz Algebra

Consider the complex Lie group $SL(2, \mathbb{C})$. Its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is a complex Lie algebra of dimension 3. A basis of $\mathfrak{sl}(2, \mathbb{C})$ is $\{\sigma_X, \sigma_Y, \sigma_Z\}$ where

$$
\sigma_X = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \sigma_Y = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_Z = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

The commutation relations are:

$$
[\sigma_X, \sigma_Y] = \sigma_Z, \quad [\sigma_Y, \sigma_Z] = \sigma_X, \quad [\sigma_Z, \sigma_X] = \sigma_Y.
$$

We can also consider a different basis $\{H_+, H_-, H_3\}$, where

$$
H_+ = i\sigma_X - \sigma_Y, \quad H_- = i\sigma_X + \sigma_Y, \quad H_3 = i\sigma_Z,
$$

the new commutation relations being:

$$
[H_+, H_3] = -H_+, \quad [H_-, H_3] = H_-, \quad [H_+, H_-] = 2H_3.
$$

Restricting the ground field with which we are working to $\mathbb{R}$, we obtain the 6 dimensional real Lie algebra $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$, the realification of $\mathfrak{sl}(2, \mathbb{C})$. It is isomorphic with the Lie algebra of the Lorentz Group.

**Definition 5** The Lorentz Lie Algebra $L$ is defined as being the complex Lie algebra which is the complexification of $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$. That is $L = \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. It is therefore a complex Lie algebra of dimension 6. The Lorentz Algebra is the complex algebra $\mathcal{U}(L)$ which is the universal enveloping algebra of the complex Lie algebra $L$.

The set $\{\sigma_X, B_X = -i\sigma_X, \sigma_Y, B_Y = -i\sigma_Y, \sigma_Z, B_Z = -i\sigma_Z\}$ is a real basis.
of \( \mathfrak{sl}(2, \mathbb{C})_\mathbb{R} \), and thus a complex basis of \( L \). The commutation relations are:
\[
[\sigma_X, \sigma_Y] = \sigma_Z, \quad [\sigma_Y, \sigma_Z] = \sigma_X, \quad [\sigma_Z, \sigma_X] = \sigma_Y,
\]
\[
[\sigma_Z, B_X] = B_Y, \quad [\sigma_Y, B_X] = -B_Z, \quad [\sigma_X, B_X] = 0,
\]
\[
[\sigma_Z, B_Y] = -B_X, \quad [\sigma_Y, B_Y] = 0, \quad [\sigma_X, B_Y] = B_Z,
\]
\[
[\sigma_Z, B_Z] = 0, \quad [\sigma_Y, B_Z] = B_X, \quad [\sigma_X, B_Z] = -B_Y,
\]
\[
[B_X, B_Y] = -\sigma_Z, \quad [B_Y, B_Z] = -\sigma_X, \quad [B_Z, B_X] = -\sigma_Y.
\]

We can also consider the basis \( \{ H_+, H_-, H_3, F_+, F_-, F_3 \} \) of \( L \), where:
\[
H_+ = i\sigma_X - \sigma_Y, \quad H_- = i\sigma_X + \sigma_Y, \quad H_3 = i\sigma_Z,
\]
\[
F_+ = iB_X - B_Y, \quad F_- = iB_X + B_Y, \quad F_3 = iB_Z.
\]

The new commutation relations being:
\[
[H_+, H_3] = -H_+, [H_-, H_3] = H_-, [H_+, H_-] = 2H_3,
\]
\[
[F_+, H_+] = [H_-, F_-] = [H_3, F_3] = 0,
\]
\[
[H_+, F_3] = -F_+, [H_-, F_3] = F_-,
\]
\[
[H_+, F_-] = -[H_-, F_+] = 2F_3,
\]
\[
[F_+, H_3] = -F_+, [F_-, H_3] = F_-,
\]
\[
[F_+, F_3] = H_+, [F_-, F_3] = -H_-, [F_+, F_-] = -2H_3.
\]

The following simple theorem will be one of the most important in our discussion.

**Theorem 6** There exists one (only) isomorphism of complex Lie algebras
\( \tau : \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \to L \cong \mathfrak{sl}(2, \mathbb{C})_\mathbb{R} \otimes \mathbb{R} \mathbb{C} \) such that:
\[
\sigma_X \oplus 0 \mapsto \frac{\sigma_X - i\sigma_X \otimes i}{2} = \frac{\sigma_X + iB_X}{2},
\]
\[
0 \oplus \sigma_X \mapsto \frac{\sigma_X + i\sigma_X \otimes i}{2} = \frac{\sigma_X - iB_X}{2},
\]
\[
\sigma_Y \oplus 0 \mapsto \frac{\sigma_Y - i\sigma_Y \otimes i}{2} = \frac{\sigma_Y + iB_Y}{2},
\]
\[
0 \oplus \sigma_Y \mapsto \frac{\sigma_Y + i\sigma_Y \otimes i}{2} = \frac{\sigma_Y - iB_Y}{2},
\]
\[
\sigma_Z \otimes 0 \mapsto \frac{\sigma_Z - i\sigma_Z \otimes i}{2} = \frac{\sigma_Z + iB_Z}{2},
\]
\[
0 \otimes \sigma_Z \mapsto \frac{\sigma_Z + i\sigma_Z \otimes i}{2} = \frac{\sigma_Z - iB_Z}{2}.
\]

And thus we have also a Hopf algebra isomorphism
\[
\tau : U(\mathfrak{sl}(2, \mathbb{C})) \otimes U(\mathfrak{sl}(2, \mathbb{C})) \to U(L).
\]

**Proof.** Easy calculations

Given \(X \in \mathfrak{sl}(2, \mathbb{C})\), define \(X^l = \tau(X \otimes 0), X^r = \tau(0 \otimes X)\). And analogously for \(X \in U(\mathfrak{sl}(2, \mathbb{C}))\). We have:

\[
H^l_+ = \frac{H_+ + iF_+}{2}, \quad H^l_- = \frac{H_- + iF_-}{2}, \quad H^l_3 = \frac{H_3 + iF_3}{2},
\]
\[
H^r_+ = \frac{H_+ - iF_+}{2}, \quad H^r_- = \frac{H_- - iF_-}{2}, \quad H^r_3 = \frac{H_3 - iF_3}{2}.
\]

Consider also \(C^l = \tau(C \otimes 1)\) and \(C^r = \tau(1 \otimes C)\), where \(C\) is the Casimir element of \(\mathfrak{sl}(2, \mathbb{C})\) defined in 1.4. The elements \(C^l\) and \(C^r\) are called Left and Right Casimirs and their explicit expression is:

\[
4C^l = \frac{H^2_3 - F^2_3}{2} + \frac{H_3 F_3}{2} + i \frac{F_3 H_3}{2}
\]
\[
+ \frac{H_+ H_-}{4} + i \frac{H_+ F_+}{4} + i \frac{F_+ H_-}{4} - \frac{F_+ F_-}{4}
\]
\[
+ \frac{H_- H_+}{4} + i \frac{H_- F_-}{4} + i \frac{F_- H_+}{4} - \frac{F_- F_+}{4},
\]
\[
4C^r = \frac{H^2_3 - F^2_3}{2} - \frac{H_3 F_3}{2} - i \frac{F_3 H_3}{2}
\]
\[
+ \frac{H_+ H_-}{4} - i \frac{H_+ F_+}{4} - i \frac{F_+ H_-}{4} - \frac{F_+ F_-}{4}
\]
\[
+ \frac{H_- H_+}{4} - i \frac{H_- F_-}{4} - i \frac{F_- H_+}{4} - \frac{F_- F_+}{4}.
\]

We can also consider the left and right image under \(\tau \otimes \tau\) of the infinitesimal R-matrix of \(U(\mathfrak{sl}(2, \mathbb{C}))\). We take now \(t \in \mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})\) as being the
infinitesimal $R$-matrix coming from the Cartan-Killing form. That is minus the one considered in [1.4]. These left and right infinitesimal $R$-matrices are:

$$4t^l = \frac{H_3 \otimes H_3}{2} - \frac{F_3 \otimes F_3}{2} + i \frac{H_3 \otimes F_3}{2} + i \frac{F_3 \otimes H_3}{2}$$

$$+ \frac{H_+ \otimes H_-}{4} + i \frac{H_+ \otimes F_-}{4} + i \frac{F_+ \otimes H_-}{4} - i \frac{F_+ \otimes F_-}{4}$$

$$+ i \frac{H_- \otimes H_+}{4} + i \frac{H_- \otimes F_+}{4} + i \frac{F_- \otimes H_+}{4} - \frac{F_- \otimes F_+}{4},$$

$$4t^r = \frac{H_3 \otimes H_3}{2} - \frac{F_3 \otimes F_3}{2} - i \frac{H_3 \otimes F_3}{2} - i \frac{F_3 \otimes H_3}{2}$$

$$+ \frac{H_+ \otimes H_-}{4} - i \frac{H_+ \otimes F_-}{4} - i \frac{F_+ \otimes H_-}{4} - i \frac{F_+ \otimes F_-}{4}$$

$$+ i \frac{H_- \otimes H_+}{4} - i \frac{H_- \otimes F_+}{4} - i \frac{F_- \otimes H_+}{4} - \frac{F_- \otimes F_+}{4}.$$}

Any linear combination $at^l + bt^r$ of the left and right infinitesimal $R$-matrices is an infinitesimal $R$ matrix for $L$. We wish to consider the combination $t_L = t^l - t^r$. That is

$$t_L = i \frac{1}{4} H_3 \otimes F_3 + \frac{1}{4} i F_3 \otimes H_3 + \frac{i}{8} H_- \otimes F_+ + \frac{i}{8} F_- \otimes H_+ + \frac{i}{8} H_+ \otimes F_- + \frac{i}{8} F_- \otimes H_+.$$

Notice another expression of it:

$$t_L = \frac{i}{8} (B_X \otimes \sigma_X + \sigma_X \otimes B_X + B_Y \otimes \sigma_Y + \sigma_Y \otimes B_Y + B_Z \otimes \sigma_Z + \sigma_Z \otimes B_Z).$$

The quadratic central element of $U(L)$ associated with $t_L$ is:

$$C_L = C_{t_L} = i \frac{H_3 F_3}{4} + i \frac{F_3 H_3}{4} + i \frac{H_+ F_-}{8} + i \frac{F_+ H_-}{8} + i \frac{H_- F_+}{8} + i \frac{F_- H_+}{8}.$$

The reason why we consider this particular combination of the left and right infinitesimal $R$-matrices is because it corresponds to the heuristic canonical two tensor of the Quantum Lorentz Group considered in [BR2]. Notice it is the symmetrisation of the classical $r$-matrix of $\mathfrak{sl}(2, \mathbb{C})_R$, see [BNR] page 19. See also [FM]. We shall see later (theorem 23) that it is the right one.
2.1.1 The Irreducible Balanced Representations of the Lorentz Group

Let us be given a complex number \( p = |p| e^{i\theta} \), \( 0 \leq \theta < 2\pi \) different from zero. We define once for all the square root \( \sqrt{p} \) of \( p \) as being \( \sqrt{|p|} e^{i\frac{\theta}{2}} \).

For \( m \in \mathbb{Z} \) define the set \( W_m = \{ p \in \mathbb{C} : |p| \notin N_{|m|+1} \} \), where, in general, \( N_m = \{ m, m+1, \ldots \} \), for any \( m \in \mathbb{N} \). Consider the set \( \mathcal{P} = \{ (m, p) : m \in \mathbb{Z}, p \in W_m \} \). Define, for any \( \alpha \in \mathbb{N} \) and \( (m, p) \in \mathcal{P} \):

\[
C_\alpha(m, p) = \frac{i}{\alpha} \sqrt{\frac{(\alpha^2 - p^2)(\alpha^2 - m^2)}{4\alpha^2 - 1}},
\]

\[
B_\alpha(m, p) = \frac{ipm}{\alpha(\alpha + 1)}.
\]

Thus \( C_\alpha(m, p) \neq 0, \alpha = |m| + 1, |m| + 2, \ldots, \forall p \in W_m \).

Consider the complex vector space

\[
V(m) = \bigoplus_{\alpha \in N_{|m|}} \bar{V}
\]

where \( \bar{V} \) denotes the representation space of the representation of \( \mathfrak{sl}(2, \mathbb{C}) \) of spin \( \alpha \). The set \( \{ \bar{v}_i, i = -\alpha, -\alpha + 1, \ldots, \alpha; \alpha \in N_{|m|} \} \) is a basis of \( V(m) \). Consider the inner product in \( V(m) \) that has the basis above as an orthonormal basis. Define also \( \bar{V}(m) \) as being the Hilbert space which is the completion of \( V(m) \).

Given \( (m, p) \in \mathcal{P} \), consider the following linear operators acting on \( V(m) \):

\[
H^-_k \bar{v}_k = \sqrt{(\alpha + k)(\alpha - k + 1)} \bar{v}_{k+1}, \quad k = -\alpha, -\alpha + 1, \ldots, \alpha
\]

\[
H^+_k \bar{v}_k = \sqrt{(\alpha + k + 1)(\alpha - k)} \bar{v}_{k-1}, \quad k = -\alpha, -\alpha + 1, \ldots, \alpha
\]

\[
F^+_k = C_\alpha(m, p) \sqrt{(\alpha - k)(\alpha - k - 1)} v^{-1}_{k+1}
\]

\[
- B_\alpha(m, p) \sqrt{(\alpha + k + 1)(\alpha - k)} \bar{v}_{k+1}
\]

\[
+ C_{\alpha+1}(m, p) \sqrt{(\alpha + k + 1)(\alpha + k + 2)} v^{\alpha+1}_{k+1}, \quad k = -\alpha, -\alpha + 1, \ldots, \alpha, \alpha \in N_{|m|},
\]

23
\[ F^{-}\hat{v}_k = -C_\alpha(m,p)\sqrt{(\alpha + k)(\alpha + k - 1)}\frac{\alpha^{-1}}{v}k_{-1} \]

\[ - B_\alpha(m,p)\sqrt{(\alpha - k + 1)(\alpha + k)}\hat{v}_{k-1} \]

\[ - C_{\alpha+1}(m,p)\sqrt{(\alpha + 1)^2 - k^2} \frac{\alpha^{+1}}{v}k_{-1}, \]

\[ k = -\alpha, -\alpha + 1, \ldots, \alpha, \alpha \in \mathbb{N}_{|m|}. \]

\[ F_{\alpha}^{\alpha}_k = C_\alpha(m,p)\sqrt{\alpha^2 - k^2}\frac{\alpha^{-1}}{v}k_{-1} - B_\alpha(m,p)\hat{v}_{k} \]

\[ - C_{\alpha+1}(m,p)\sqrt{(\alpha + 1)^2 - k^2} \frac{\alpha^{+1}}{v}k_{-1}, \]

\[ k = -\alpha, -\alpha + 1, \ldots, \alpha, \alpha \in \mathbb{N}_{|m|}. \]

Obviously we are considering \( \hat{v}_k = 0 \) if \( k > \alpha \) or \( k < -\alpha \). We have the following theorem, whose proof can be found in [GMS].

**Theorem 7** If \( (m,p) \in \mathcal{P} \), the operators \( H_-, H_+, H_3, F_-, F_+, F_3 \) define an infinite dimensional representation of the Lorentz Algebra.

Notice that the representations \((m,p)\) and \((-m,-p)\) are equivalent. This has a trivial proof.

Denote the representations above by \( \{\rho(m,p) : (m,p) \in \mathcal{P}\} \). One can prove with no difficulty that they have a central character, for any intertwiner \( V(m) \rightarrow V(m) \) needs to send each space \( V \) to itself and act on it has a multiple of the identity. Considering the action of \( F_+^{\alpha} \), for example, we conclude that the multiples are the same in each space \( V \). Therefore

**Theorem 8** For any \( (m,p) \in \mathcal{P} \) the representation \( \rho(m,p) \) of \( L \) has a central character \( \lambda_{m,p} \).

For any \( (m,p) \in \mathcal{P} \) the representation \( \rho(m,p) \) of \( L \) can always be integrated to a representation \( R(m,p) \) of the Lorentz Group in the completion \( \hat{V}(m) \) of \( V(m) \), or to be more precise of its connected component of the identity. The representation is unitary if and only if \( p \) is purely imaginary, for any \( m \in \mathbb{N}_0 \), in which case the representation is said to belong to the principal series, or if \( m = 0 \) and \( p \in [0,1) \) in which case the representation is said to belong to the complementary series. The vector space \( V(m) \) is
contained in the space of smooth vectors, cf [Kir], of $V(m)$; thus $\lambda_{m,p}$ is the infinitesimal character of $R(m,p)$, in the unitary case. This unifies the approach here with the approach in [FM].

The parameter $m$ is called the minimal spin of the representation. A representation is called balanced if the minimal spin of it is 0. Balanced representations depend therefore on a parameter $p \in W_0$. Denote them by $\{\rho_p, p \in W_0\}$. Two balanced representations $\rho_p$ and $\rho_q$ of $L$ are equivalent if and only if $p = q$ or $p = -q$. These representations were used in [BC] for the construction of a spin foam model for Quantum Gravity. The extension of that work for their quantised counterpart was dealt with in [NR].

Since the representations $\{\rho(m,p) : (m,p) \in \mathcal{P}\}$ have a central character, the left and right Casimirs defined in 2.1 act on $V(m)$ as multiples of the identity. This multiples are, as a function of $m$ and $p$ the following: $\frac{p^2+2mp+m^2-1}{8}$ for $C^l$ and $\frac{p^2-2mp+m^2-1}{8}$ for $C^r$. Therefore:

**Proposition 9** If the infinitesimal $R$-matrix on $U(L)$ is the tensor $t_L$ defined in 2.1 then the central characters $\{\lambda_p, p \in W_0\}$ of the balanced representations are $t_L$-unframed. Recall the nomenclature introduced before theorem 3.

This can obviously be proved without using the explicit expression of the action of the Casimir elements.

Notice also that we can consider the minimal spin of the representations considered to be also to be an half integer, making the obvious change in the form of the representation. These kind of representations cannot be integrated to representations of the Lorentz Group, even though they define representations of $SL(2,\mathbb{C})$. They are called two-valued representations of the Lorentz Group in [GMS].

### 2.2 The Lorentz Knot Invariant

Consider again the infinitesimal $R$-matrix $t_L = t^l - t^r$ of the Lorentz Lie Algebra. We consider for each $(m,p) \in \mathcal{P}$ the representation $\rho(m,p)$ of $L$. It has a central character $\lambda_{m,p}$. We propose to consider the framed knot invariants $\{X(m,p) : (m,p) \in \mathcal{P}\}$, such that for any knot:

$$K \mapsto X(m,p,K) = (\lambda_{m,p} \circ Z_t)(K).$$

Recall the notation of 1.3. Notice $X(m,p) = X(-m,-p)$ for the representations $\rho_{m,p}$ and $\rho_{-m,-p}$ are equivalent.
The value of $X(m, p)$ in a framed knot $K$ is therefore a formal power series with coefficients in $\mathbb{C}$. It is a difficult task to analyse the analytic properties of such power series. We expect they will be perturbation series for some numerical knot invariants that can be defined, cf [FM].

As we have seen, if $m = 0$, that is in the case of balanced representations, the central character $\lambda_p$ is $t_L$-unframed. This is also the case for $p = 0$. Notice we have an explicit expression for the action of the left and right Casimir elements of $L$. Therefore

**Theorem 10** The knot invariant $X(m, p)$ with $(m, p) \in \mathcal{P}$ is unframed if and only if $m = 0$ or $p = 0$.

Obviously, for different combinations of the left and right infinitesimal $R$-matrices, the representations which have unframed central characters with respect to it are different. This gives us a way to define an unframed knot invariant out of any $(m, p) \in \mathcal{P}$. But notice this can be done without changing the infinitesimal $R$-matrix $t_L$ of $L$, since we know how the invariants behave with respect to framing, cf theorem 1.

### 2.2.1 Finite Dimensional Representations

Let us now analyse the knot invariants that come out of the finite dimensional representations of the Lorentz Group. We are mainly interested in the representations which are irreducible.

Since we have the isomorphism $U(L) \cong U(\mathfrak{sl}(2, \mathbb{C})) \otimes U(\mathfrak{sl}(2, \mathbb{C}))$, the finite dimensional irreducible representation of $U(L)$, or what is the same of $L$, are classified by a pair $(\alpha, \beta)$ of half integers. That is each finite dimensional irreducible representation of $L$ is of the form $\hat{\rho} \otimes \hat{\rho}$ as a representation of $U(L) \cong U(\mathfrak{sl}(2, \mathbb{C})) \otimes U(\mathfrak{sl}(2, \mathbb{C}))$. There is an alternative way to construct these finite dimensional representations that shows their close relation with the infinite dimensional representations, [GMS]. Let us explain how the process goes. It is very similar to the $\mathfrak{sl}(2, \mathbb{C})$ case.

Consider $m = \alpha - \beta$ and $p = \alpha + \beta + 1$. Notice that now $C_\alpha(m, p) \neq 0$ if $\alpha \in |m|, |m| + 1, \ldots, p$, and $C_p(m, p) = 0$. The underlying vector space for the representation with spins $(\alpha, \beta)$ is $V(m, p) = \bigotimes_{\nu}^{|m|} \bigotimes_{\nu}^{|m| + 1} \ldots \bigotimes_{\nu}^{p-1}$, and the form of it is given exactly by the same formulae of the infinite dimensional representations. The equivalence of the representations is a trivial consequence of the Clebsh-Gordan formula. This construction gives us a finite
dimensional representation $\rho(m, p)$ for each pair $(m, p)$ with $m, p \in \mathbb{Z}/2$ and $p - |m| \in \mathbb{N}_1$. It makes also sense for $|p| - |m| \in \mathbb{Z}$, making the appropriate changes. As before we have the equivalence $\rho(m, p) \cong \rho(-m, -p)$.

Since we completed the sets $W_m$ defined at the beginning of 2.1.1, we have a representation $\rho(m, p)$ of the Lorentz Algebra for each pair $(m, p)$ with $m \in \mathbb{Z}/2$ and $p \in \mathbb{C}$. All them have a central character $\lambda_{m,p}$, since the new representations considered are finite dimensional and irreducible. The finite dimensional representations give us framed knot invariant $X_{\text{fin}}(m, p)$ for each pair $m, p \in \mathbb{Z}/2$ with $|p| - |m| \in \mathbb{N}_1$. This invariant is independent of the framing if and only if $m = 0$, that is if $\alpha = \beta$.

Consider now the algebra morphisms $\lambda_{m,p} \circ \phi_{tl} : \mathcal{A} \to \mathbb{C}$, where $m \in \mathbb{Z}$ and $p \in \mathbb{C}$. The argument is now similar to the one in 1.4.1. If we look at the expression of the representations $\rho_{m,p}$, it is easy to conclude that given any chord diagram $w$ with $n$ chords, the evaluation of $\lambda_{m,p} \circ \phi_{tl}(w)$ is for a fixed $m$ a polynomial in $p$ of degree at most $2n$. Notice that any factor of the form $C_\alpha(m, p)$ appears in the expression for $\lambda_{m,p} \circ \phi_{tl}(w)$ an even number of times. For the case of balanced representations, that is $m = 0$, we can also prove that it is a polynomial in $p^2$. Also the value of the polynomials in $p = 1$ is zero if $n > 0$ for the pair with $m = 0$ and $p = 1$ yields the trivial one dimensional representation of $L$. We have proved:

**Theorem 11** Consider the framed knot invariants $\{X(m, p), m \in \mathbb{N}_0, p \in \mathbb{C}\}$. If we fix $m \in \mathbb{N}_0$ then the term of order $n$ in the expansion of $X(m, p, K)$ as a power series is polynomial of degree at most $2n$ in $p$. Here $K$ is any framed knot. If $m = 0$ then only the even terms of it are non zero. Moreover the polynomials attain zero at $p = 1$ for $n > 0$.

Therefore, if we know the value of $X(m, p, K)$ for the finite dimensional representations, that is if $|p| - |m| \in \mathbb{N}$ we can determine it for any value of the parameter $p$. This is similar to the $\mathfrak{sl}(2, \mathbb{C})$ case.

### 2.2.2 Relation With the Coloured Jones Polynomial

The relation between the Lorentz knot invariants that come out from finite dimensional and infinite dimensional representations remarked after theorem 11 gives us a way to relate the Coloured Jones Polynomial with the Lorentz Group invariants. In fact:
Theorem 12 Let $K$ be some oriented framed knot, $K^*$ its mirror image. Then for any $z, w \in \mathbb{C}$ with $z - w \in \mathbb{Z}$ we have:

$$\frac{J^z(K^*)}{2z + 1} \times \frac{J^w(K)}{2w + 1} = X(z - w, z + w + 1, K),$$

as formal power series over $\mathbb{C}$.

Proof. For any $m \in \mathbb{Z}/2$ and $x \in \mathbb{C}$, let $z(x, m) = m + x$ and $w(x, m) = -m + x$. Thus each pair $(z, w) \in \mathbb{C}^2$ with $z - w \in \mathbb{Z}$ is of the form $(z(x, m), w(x, m))$ for some $m$ and $x$. Fix $m \in \mathbb{Z}/2$. We want to prove:

$$\frac{J^{(m+x)}(K^*)}{2m + 2x + 1} \times \frac{J^{(-m+x)}(K)}{-2m + 2x + 1} = X(m, 2x + 1, K), \forall x \in \mathbb{C}$$

Each term of the formal power series at both sides of the equality is a polynomial in $x$, thus we only need to prove that the equality is true if both $x - m$ and $x + m$ are half integers. That is if $x - m, x + m \in \frac{1}{2}\mathbb{N}_0$.

Let $t$ the infinitesimal $R$ matrix in $\mathfrak{sl}(2, \mathbb{C})$ coming out of the Cartan-Killing form. Notice it is minus the one considered in 1. Let $\alpha$ be a half integer. Recall that for a framed knot $K$ we have:

$$\frac{J^\alpha(K)}{2\alpha + 1} = (\lambda_\alpha \circ Z_{-t})(K).$$

Therefore by Theorem 3:

$$\frac{J^\alpha(K^*)}{2\alpha + 1} = (\lambda_\alpha \circ Z_t)(K),$$

since $\phi_t(w) = (-1)^n \phi_{-t}(w)$ if $w$ is a chord diagram with $n$ chords.

Let $K$ be a framed knot and $x$ be such that $\alpha = x - m$ and $\beta = x + m$ are half integers. We have by theorem

$$\frac{J^\alpha(K^*)}{2\alpha + 1} \times \frac{J^\beta(K)}{2\beta + 1} = (\lambda_\alpha \circ Z_t)(K) \times (\lambda_\beta \circ Z_{-t})(K)$$

$$= (\lambda_\alpha \circ Z_{\phi_t})(K) \times (\lambda_\beta \circ Z_{-\phi_t})(K)$$

$$= ((\lambda_\alpha \otimes \lambda_\beta) \circ Z_{t_L})(K).$$

Recall $t_L = t^l - t^r$. 
Now, $\lambda_\alpha \otimes \lambda_\beta$ is the central character of the representation $\rho_\alpha \otimes \rho_\beta$ of $U(L) \cong U(s(2, \mathbb{C})) \otimes U(s(2, \mathbb{C}))$. As we have seen before, this representation is equivalent to $\rho(\alpha - \beta, \alpha + \beta + 1) = \rho(m, 2x + 1)$. Thus their central characters are the same. This proves

\[
((\lambda_\alpha \otimes \lambda_\beta) \circ \mathcal{Z}_{t_L})(K) = (\lambda_{m,2x+1} \circ \mathcal{Z}_{t_L})(K)
\]

if both $x - m$ and $x + m$ are half integers, and the proof is finished. ■

We have the following simple consequences.

**Corollary 13** Given a framed knot $K$, then the term of order $n$ in the power series of $X(m, z, K)$ is a polynomial in $m$ and $z$.

**Corollary 14** If $O$ is the unknot, then $X(m, p, O)$ is a convergent power series.

**Corollary 15** For balanced representations, that is if $m = 0$, the invariant $X(0, p)$ does not distinguish a knot from its mirror image.

**Corollary 16** The framed knot invariants $X(m, p)$ are unoriented.

3 The Approach with the Framework of Buffenoir and Roche

The aim of this section is to give a sketch of how the Buffenoir and Roche description of the infinite dimensional unitary representations of the Quantum Lorentz Group relates with our approach. The Quantum Lorentz Group was originally defined by Woronowicz and Podleś in [PoW]. The classification of the irreducible unitary representations of it appeared first in [Pu].

For an expanded treatment of the issues considered in this section, we refer the reader to [PhD].

3.1 Representations of the Quantum Lorentz Group and $R$-Matrix

We now follow [BR1]. Other good references are [BR2] and [BNR]. These references contain all the notation and conventions we use. The Quantum Lorentz Group $\mathcal{D}$ at a point $q \in (0, 1)$ is defined as the quantum double...
$D(U_q(\mathfrak{su}(2)), \text{Pol}(SU_q(2)))$. Notice that both $U_q(\mathfrak{su}(2))$ and $\text{Pol}(SU_q(2))^{\text{cop}}$ are sub Hopf algebras of $D$. The Quantum Lorentz Group thus have a formal $R$-matrix coming from its quantum double structure. Even though it is defined by an infinite sum, it is possible to describe its action in any pair of infinite dimensional irreducible representations of $D$ in the principal series. See [BR1, BR2] for a description of them. For the dual counterpart of the theory, in other words for the theory of corepresentations of the algebra of function in the Quantum Lorentz Group $SL_q(2, \mathbb{C})$ we refer to [PuW].

Let us describe what the situation is in the case the two representations are the same. Suppose also the minimal spin $m$ of them is zero. Similarly with the classical case described above, representations $\rho(p)$ of this kind will be called balanced. They depend a parameter $p \in \mathbb{C}$. If $p \in i\mathbb{R}$ then the representations $\rho(p)$ can be made unitary. Choosing $p \in [0, \frac{2\pi}{\hbar}]$, where $q = e^{\hbar/2}$, parametrises all the unitary representations in the principal series which have minimal spin equal to zero. These last ones are called simple representations in [NR].

Similarly with the classical case, the underlying vector space for the balanced representations $\rho(p), p \in \mathbb{C}$ of the Quantum Lorentz Group is

$$V = V(p) = \bigoplus_{\alpha \in \mathbb{N}_0} V_{\alpha},$$

where

$$\rho_{\alpha} : U_q(\mathfrak{su}(2)) \to L(\alpha)$$

is the irreducible representation of $U_q(\mathfrak{su}(2))$ with spin $\alpha$. A basis of $V_{\alpha}$ is thus given by the vectors $\{v_i, i = -\alpha, -\alpha + 1, \ldots, \alpha\}$. Any element $x$ of $U_q(\mathfrak{su}(2))$ acts in $V$ in the fashion:

$$\prod_{\alpha \in \mathbb{N}_0} \rho_{\alpha}(x).$$

The group like element of the Lorentz Group is given by $G = q^{2J_z}$. The heuristic ribbon element of the Quantum Lorentz Group is easily proved to act as the identity in the balanced representations. See [NR].

Define, given half integers $A, B, C$ and $D$, the complex numbers:

$$\Lambda_{AD}(\rho) = \sum_{\sigma} \left( \begin{array}{cc} 0 & C \\ A & -\sigma \end{array} \right) q^{2\sigma \rho} \left( \begin{array}{cc} -\sigma & D \\ B & C \end{array} \right).$$
For the correct definition of the phases of the Clebsch-Gordan coefficients see [BR2]. We display their explicit expression later. The formal universal $R$-matrix of the Quantum Lorentz Group is: (see [BR1])

$$R = \sum_{\alpha \in \frac{1}{2}\mathbb{N}_0, -\alpha \leq i_a, j_a \leq \alpha} \hat{X}_{ja} \otimes \hat{g}_{i_a},$$

its inverse being:

$$R^{-1} = \sum_{\alpha \in \frac{1}{2}\mathbb{N}_0, -\alpha \leq i_a, j_a \leq \alpha} X_{ja} \otimes S^{-1}(\hat{g}_{i_a}).$$

This antipode $S$ is the one of $\text{Pol}(SU_q(2))^{\text{cop}} \subset \mathcal{D}$, which is the inverse of the one in $\text{Pol}(SU_q(2))$, thus

$$S^{-1}(\hat{g}_{i_a}) = q^{-i_a + j_a}(-1)^{-j_a + i_a} \hat{g}_{-j_a}.$$

See [BR2], equation (25). The action of $\hat{g}_{i_a}$ in the space $V(p)$ is given by

$$\frac{\alpha{i}_a_{\beta}}{\hat{g}_{i_a}} = \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma} \sum_{D, \gamma, x \frac{1}{2}\mathbb{N}_0} \sum_{\gamma \leq i_a \leq i_a} \sum_{D \leq x} \gamma_{\gamma} \left(\begin{array}{ccc} i_\gamma & i_a & D \\ \gamma_a & \alpha & j_a \\
\end{array}\right) \left(\begin{array}{ccc} x & \alpha & \beta \\
D & j_a & i_\beta \\
\end{array}\right) \Lambda_{\alpha \beta}.$$  

Note it is a finite sum. The constants $\mathcal{F}_\alpha$ are defined in [BR2], proposition 1. They will not be used directly. In fact their values are (almost) arbitrary and they only appear to ensure that the representations $\rho(p)$ are unitary for $p \in i\mathbb{R}$, and the natural inner product in $V$. Their appearance does not change the representation itself, therefore not affecting the calculations of knot invariants.

The coefficients $\Lambda_{AL}^{AE}(p)$ are originally defined in [BR2] from an analytic continuation of $6j$-symbols, and at the end proved to coincide with (1). One can show directly that (2) does define a representation of the Quantum Lorentz Group, for any $p \in \mathbb{C}$ since equation (76) of [BR2] holds. See [PhD].

In some particular cases, equation (2) simplifies to:

$$\frac{\alpha{j}_a_{0}}{\hat{g}_{j_a}} = \frac{\mathcal{F}_0}{\mathcal{F}_\gamma} \sum_{\gamma, i_\gamma} \left(\begin{array}{ccc} i_\gamma & i_a & x \\
\gamma & \alpha & j_a \\
\end{array}\right) \Lambda_{\gamma 0}^{\alpha \gamma} \gamma_{\gamma},$$

31
and

\[
\langle \varphi_{\alpha \beta} \rangle = \frac{F_{\beta}}{F_{0}} \left( \frac{i_{\alpha}}{\alpha} \right) \Lambda_{\alpha \beta}^0.
\]

All these formulae are consequences of well known symmetries of Clebsch-Gordan Coefficients listed for example in [BR2]. With them we can also prove \( \Lambda_{\alpha \alpha}^0 = 1 \), from which follows:

\[
\varphi_{\alpha \beta}^{00} = \varphi_{\alpha \beta}.
\]

The elements

\[
X_{\alpha \alpha}^{\alpha} \in \text{Pol}(SU_q(2))^*, \alpha \in \frac{1}{2} \mathbb{N}_0, i_{\alpha} = -\alpha, \ldots, \alpha
\]

act simply as matrix elements, that is:

\[
X_{\alpha \alpha}^{\alpha} = \delta(\alpha, \beta) \delta(i_{\alpha}, i_{\beta}) \varphi_{\alpha \beta}.
\]

Notice \( U_q(\mathfrak{su}(2)) \) is naturally embedded in \( \text{Pol}(SU_q(2))^* \). Moreover any finite dimensional irreducible representation of \( U_q(\mathfrak{su}(2)) \) induces one of \( \text{span} \{ X_{\alpha \alpha}^{\alpha} \} \subset \text{Pol}(SU_q(2))^* \) which has exactly this form, cf [PoW], theorem 5.1.

The action of the group like element \( G \) is

\[
G \varphi_{\alpha \beta} = q^{2i_{\alpha} \beta} \varphi_{\alpha \beta}.
\]

It is easy to compute how \( R \) acts, namely:

\[
R \left( \varphi_{\alpha \beta} \otimes \varphi_{\alpha \beta} \right) = \sum_{D, x, y, i_{\gamma}, j_{\gamma}} \left( \frac{i_{\gamma}}{\gamma} \right) \left( \frac{j_{\alpha}}{\alpha} \right) \left( \frac{x}{\beta} \right) \left( \frac{D}{\gamma} \right) \frac{F_{\beta}}{F_{\gamma}} \Lambda_{\beta \gamma}^\alpha \left( \varphi_{\alpha \beta} \otimes \varphi_{\alpha \beta} \right).
\]

See [BR1], proposition 13. The domain of the sum is the obvious one. The action of \( R \) in \( V \otimes V \) is thus well defined. Note we are considering the algebraic, rather than topological, tensor product. Moreover \( R \) defines a braid group representation. Denote it by \( b \in B(n) \mapsto R_b \in L(V^\otimes) \). Here \( B(n) \) denotes the \( n \)-strand braid group and \( L(V^\otimes) \) the vector space of linear maps \( V^\otimes \to V^\otimes \). Notice that the braiding operators \( R_b \) extend to unitary operators if \( p \in i\mathbb{R} \) since \( R^{\ast \ast} = R^{-1} \), where \( \ast \) is the star structure on the Quantum Lorentz Group, see [BNR], since \( \rho(p) \) is unitary in this case.
3.2 Associated Knot Invariants

We now use the framework just introduced to define quantum lorentzian knot invariants. As we will see they relate to our approach before.

3.2.1 Some Heuristics

Let $q \in (0,1)$ and $p \in \mathbb{C}$. Suppose we are given a braid $b$ with $n+1$ strands. There is attached to it a map $R_b : V^\otimes(n+1) \to V^\otimes(n+1)$. Consider the map $A_b = (\text{id} \otimes G \otimes \cdots \otimes G) R_b$. Suppose the closure of the braid $b$ is a knot. If the representations we are considering were finite dimensional, then the partial trace $T^1(A_b) : V \to V$ of $A_b$ over the last $n$ variables would be an intertwiner and thus a multiple of the identity, since the representations which we are considering are irreducible. Moreover this multiple of the identity would be a knot invariant, which would have the form:

\begin{equation}
S_b(q,p) = \sum_{\alpha_1,\ldots,\alpha_n \in \mathbb{Z}_0^+} \langle \theta^0 \otimes \theta^{\alpha_1 i_1} \otimes \cdots \otimes \theta^{\alpha_n i_n}, A_b \left( \theta^0 \otimes \theta^{i_1}_{\alpha_1} \otimes \cdots \otimes \theta^{i_n}_{\alpha_n} \right) \rangle.
\end{equation}

(8)

Even though the sums above may be not convergent, the assignment of one sum of this kind to a braid whose closure is a knot is not ambiguous. In fact suppose $b$ has $m+1$ strands and $n$ crossings. We can always express this sum in a more suggestive way, namely as:

\begin{equation}
S_b(q,p) = \sum_{\alpha_1,\ldots,\alpha_n \in \mathbb{Z}_0^+, -\alpha_k \leq i_k, j_k \leq \alpha_k, k=1,\ldots,n} \langle \theta^0, \prod_{l=1}^{2n+m} T(\alpha, i, j, l) \theta^0 \rangle,
\end{equation}

(9)

where if $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\underline{i} = (i_1, \ldots, i_n)$ and $\underline{j} = (j_1, \ldots, j_n)$, then $T(\alpha, i, j, l)$ can be either a term of the form $\tilde{g}^{\alpha_k i}$ or $X^{\alpha_k i}$, for some $k \in \{1,\ldots,n\}$ or $G$; and moreover for any $k$ there exists an $l$ such that $T(\alpha, i, j, l)$ is a $X^{\alpha_k i}$, and the same for $\tilde{g}^{\alpha_k i}$. The two examples below should clarify what we mean. We obviously need to suppose that the closure of $b$ is a knot for this to hold. Notice that the transition from (8) to (9) is totally clear if the representations are finite dimensional. We take (9) as the definition of $S_b(q,p)$ if $b$ is a braid whose closure is a knot.
Let us look to the sums above in a bit more detail. We consider the Left and Right Handed Trefoil knots displayed in figure 6. Call the two braids we have chosen to represent them $T_+$ and $T_-$. The sum for the Right Handed Trefoil Knot is:

$$S_{T_+}(q, p) = \sum_{\alpha, \beta, \gamma \in \frac{1}{2} \mathbb{N}_0} \left\langle \begin{array}{c} \gamma^0, \beta^\alpha \alpha^\beta \gamma \end{array}, X_{j_{\alpha}} g_{i_{\alpha}} G X_{j_{\beta}} g_{i_{\beta}} X_{j_{\gamma}} \right| \delta_{00} \right\rangle.$$

Whereas for the Left Handed Trefoil is:

$$S_{T_-}(q, p) = \sum_{\alpha, \beta, \gamma \in \frac{1}{2} \mathbb{N}_0} \left\langle \begin{array}{c} \gamma^0, X_{j_{\alpha}}, S^{-1}(g_{i_{\alpha}}) X_{j_{\beta}} G S^{-1}(g_{i_{\beta}}) X_{j_{\gamma}} S^{-1}(g_{i_{\gamma}}) \end{array}, v_0 \right\rangle.$$

Many of the terms will be zero in the expressions above. Let us look at $S_{T_-}$. We only want the $0 \rightarrow 0$ matrix element, and $\left\langle \begin{array}{c} \gamma^0, X_{j_{\alpha}} \end{array}, v \right\rangle = \delta(\alpha, 0) \left\langle v, v \right\rangle$.

Thus we can make $\gamma = 0$, and then note that $\delta_{00}$ acts as the identity. We obtain: (we skip unnecessary indices)

$$S_{T_-}(q, p) = \sum_{\alpha, \beta \in \frac{1}{2} \mathbb{N}_0} q^{-i_{\alpha} + j_{\alpha} - i_{\beta} + j_{\beta}} (-1)^{i_{\alpha} + j_{\beta} - j_{\alpha} - j_{\beta}} \left\langle \begin{array}{c} \gamma^{-i_{\beta}}, \alpha^\alpha \beta^\beta \end{array}, X_{j_{\beta}} G X_{j_{\alpha}} \right| \delta_{00} \right\rangle.$$

Figure 6: Right and Left Handed Trefoil Knots.
From (6) and (7) follows $\alpha = \beta$ and $i_\alpha = j_\beta$. By (3) and (4) we can conclude:

$$S_{T_-}(q, p) = \sum_{\alpha \in \mathbb{N}_0} \sum_{i_\beta, j_\beta, j_\alpha = -\alpha} q^{j_\alpha - i_\beta + 2j_\beta} (-1)^{i_\beta - j_\beta} \left( \begin{array}{ccc} -i_\beta & \alpha & \alpha \\ \alpha & -j_\beta & j_\alpha \end{array} \right) \left( \begin{array}{ccc} i_\beta & -j_\beta & \alpha \\ \alpha & \alpha & -j_\alpha \end{array} \right) \Lambda^{\alpha \alpha}_{00} \Lambda^{\alpha \alpha}_{00}.$$  

Using the standard symmetries of the Clebsch-Gordan Coefficients, we can express this as:

$$S_{T_-}(q, p) = \sum_{\alpha \in \mathbb{N}_0} \sum_{i_\beta, j_\beta, j_\alpha = -\alpha} q^{2j_\beta} \left( \begin{array}{ccc} -i_\beta & -j_\alpha & \alpha \\ \alpha & \alpha & -j_\beta \end{array} \right) \left( \begin{array}{ccc} -j_\beta & \alpha & \alpha \\ \alpha & -i_\beta & -j_\alpha \end{array} \right) \Lambda^{\alpha \alpha}_{00} \Lambda^{\alpha \alpha}_{00}.$$  

Notice $\Lambda^{\alpha \alpha}_{00}$ is zero unless $\alpha$ is integer. Therefore the final expression for the sum is:

$$S_{T_-}(q, p) = \sum_{\alpha \in \mathbb{N}_0} \left( q^{2\alpha + 1} - q^{-2\alpha - 1} \right) \Lambda^{\alpha \alpha}_{00} \Lambda^{\alpha \alpha}_{00}.$$  

Notice that the series $S(T_-)$ seems to be divergent due to the presence of the $d_\alpha$ term in it. Therefore this sums do not seem to define $\mathbb{C}$-valued knot invariants. This tells us the method of Borel re-summation sketched in [FM] is perhaps more powerful.

3.2.2 Finite Dimensional Representation

Let $Y(\alpha, \beta, \gamma) = 1$ if $\hat{\rho}$ is in the decomposition of $\hat{\rho} \otimes \hat{\rho}$ in term of irreducible representations of $U_q(\mathfrak{su}(2))$ and zero otherwise, where $\alpha, \beta, \gamma \in \frac{1}{2}\mathbb{Z}$. Let also
\[ Y(\alpha, i, \alpha) = 1 \text{ if } i, \alpha \in \{-\alpha, \ldots, \alpha\} \text{ and zero otherwise. We have (see [BR2]):} \]

\[ (m \atop I, J)_{p}^{n} = Y(I, m)Y(J, n)Y(K, p)\delta(m + n, p)Y(I, J, K) \]

\[ q^{m(p+1)+\frac{1}{2}(J(J+1)-I(I+1)-K(K+1))}e^{i\pi(I-m)} \]

\[ \sqrt{\frac{[2K+1][I+J-K]![I-m]![J+n]![K-p]![K+p]!}{[K+J-I]![I+K-J]![I+J+K+1]![I+m]![J+n]!}} \]

\[ \times \sum_{V=0}^{K-p} \frac{q^{V(K+p+1)}e^{i\pi V}}{[V]![K-p-V]![I-m-V]![J-K+m+V]}. \]

Let \( p \in \mathbb{C} \). We thus have an infinite dimensional representation \( \rho(p) \) of the Quantum Lorentz Group given by the constants \( \Lambda_{AD}^{BC}(p) \). Its representation space is by definition \( V = V(p) = \bigoplus_{\alpha \in \mathbb{N}_0} \mathbb{V}_\alpha \). From equation (10), we can easily calculate the coefficients \( \Lambda_{AD}^{BC}(p) \) if \( B = 1/2 \). In fact

**Lemma 17** Let \( C \geq 0 \) be an integer. We have

\[ \Lambda_{CC}^{1/2C-1/2}(p) = \frac{q^{C}(q^p + q^{-p})}{q^{2C} + 1}, \]

\[ \Lambda_{CC}^{1/2C+1/2}(p) = -\frac{q^{C+1}(q^p + q^{-p})}{q^{2C+2} + 1}, \]

\[ \Lambda_{CC+1}^{1/2C+1/2}(p) = \frac{q^{2C+2}q^{-p} - q^p}{q^{2C+2} + 1}, \]

\[ \Lambda_{CC+1}^{1/2C+2}(p) = \frac{q^{2C+2}q^{-p} - q^p}{q^{2C+2} + 1}. \]

Notice that all the other \( \Lambda_{AD}^{BC}(p) \) coefficients with \( B = 1/2 \) are zero. See also [BR2], proof of theorem 3.

In particular, if \( p \in \mathbb{N} \), then the representation \( \rho(p) \) has a finite dimensional subrepresentation \( \rho(p)_{\text{fin}} \) in \( V(p)_{\text{fin}} = V_0 \bigoplus V_1 \oplus \ldots \oplus V_{p-1} \). Compare with 2.2.1 Using Schur’s lemma as in [BR2], proof of theorem 3, one proves these representations are irreducible.

Notice that \( v_0 \in V(p)_{\text{fin}} \). Therefore, looking at (9) we conclude:

**Lemma 18** If \( p \in \mathbb{N} \) and \( b \) is a braid whose closure is a knot, then the infinite sum \( S_b(q, p) \) truncates to a finite sum for any \( q \in (0, 1) \).
As we referred in the introduction, the category of finite dimensional representations of the Quantum Lorentz Group is (almost) ribbon equivalent to the category of finite dimensional representations of $U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$ where $R$ is the $R$-matrix of $U_q(\mathfrak{su}(2))$. Let us explain what this means. We follow [BR2] and [BNR] closely. The ribbon Hopf algebra $U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$ is isomorphic with $U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$ as an algebra, but has a coalgebra structure of the form:

$$\Delta(a \otimes b) = R_{23}^{-1} a' \otimes b' \otimes a'' \otimes b'' R_{23},$$

whereas the antipode is defined as:

$$S(a \otimes b) = R_{21} S(a) \otimes S(b) R_{21}^{-1}.$$

This Hopf algebra has an $R$-matrix given by

$$\hat{R} = R_{14}^{(-)} R_{21}^{(-)} R_{13}^{(+)} R_{23}^{(+)},$$

where $R^{(+)} = R$ and $R^{(-)} = R_{21}^{-1}$. The algebra $U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$ is a ribbon Hopf algebra with group like element $G \otimes G$, where $G = q^{2j_z}$ is the group like element of $U_q(\mathfrak{su}(2))$. See [BNR] page 20.

The irreducible finite dimensional representations $\hat{\rho}_w$ of $U_q(\mathfrak{su}(2))$ are parametrised by an $\alpha \in \frac{1}{2} \mathbb{N}_0$ and an $w \in \{1, -1, i, -i\}$, the level of the representation. See [KS], theorem 13. The irreducible representations $\hat{\rho}$ of spin $\alpha$ are the ones for which $w = 1$. They are the natural quantisation of the representations of $SU(2)$ of spin $\alpha$. The action of the $R$-matrix of $U_q(\mathfrak{su}(2))$ is, apriori, only defined on pairs of representations of level 1, or direct sums of them. Nevertheless, the category of finite dimensional representations of $U_q(\mathfrak{su}(2))$ of this kind is a ribbon category. Let $\hat{\rho}$ and $\hat{\beta}$ be two finite dimensional irreducible representations of $U_q(\mathfrak{su}(2))$ of level 1 which will then generate a representation $\hat{\rho} \otimes \hat{\beta}$ of $U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$. The action of the $R$-matrix of $U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$ is well defined on pairs of representations of this kind. The same is true for the group like element, thus we can define a framed knot invariant $I(\hat{\rho} \otimes \hat{\beta})$. Unpacking the expression of it yields immediately

Lemma 19 For any framed knot $K$ we have

$$I(\hat{\rho} \otimes \hat{\beta})(K) = I(\hat{\rho})(K^*) I(\hat{\beta})(K)$$

where $K^*$ is the mirror image of $K$. Here $I(\hat{\beta})$ is the $U_q(\mathfrak{su}(2))$-framed knot invariant defined from $\hat{\beta}$, in other words the Coloured Jones Polynomial, and the same for $I(\hat{\rho})$. 37
There exists a Hopf algebra morphism $\psi : D \rightarrow U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$. It has the form:

$$\psi : (x, f) \mapsto \sum_{(x)(f)} x'(f'' \otimes \text{id})(R^{(+)}) \otimes x''(f' \otimes \text{id})(R^{(-)})$$

The comultiplications are taken in $U_q(\mathfrak{su}(2))$ and $\text{Pol}(SU_q(2))$. This morphism naturally extends to the elements $X^{\alpha}_{ja} \in \text{Pol}(SU_q(2))^\ast$.

There also exists a morphism $s : \text{Pol}(SU_q(2)) \rightarrow \text{Pol}(SU_q(2))$ such that $s(\tilde{g}^{\text{su}}_{ja}) = (-1)^{2\alpha} g^{\text{su}}_{ja}$. It extends to all the Quantum Lorentz Group provided we define its restriction to $U_q(\mathfrak{su}(2))$ (thus also to span $\left\{ X^{\alpha}_{ja} \right\}_{\alpha, j} \subset \text{Pol}(SU_q(2))^\ast$) to be the identity. A main result of [1], namely theorem 5.4, is the following:

**Theorem 20** Let $\rho$ be an irreducible finite dimensional representation of the Quantum Lorentz Group $D$ which has a structure of a $\text{Pol}(SU_q(2))$-crossed bimodule. In our case this means that the representation $\rho$ restricted to $U_q(\mathfrak{su}(2))$ is a direct sum of representations $\tilde{p}_w$ with $\omega = 1$, which is what happens for the representations $\rho(p)\text{fin}$, $p \in \mathbb{N}$, see [7] proposition 5.1 (These representations define corepresentations of the algebra $S_q L(2, \mathbb{C})$, thus representations of the Quantum Lorentz Group in the sense of [1]). Then there exists $\alpha, \beta \in \frac{1}{2} \mathbb{N}_0$ such that either $\rho = (\tilde{\rho} \otimes \tilde{\beta}) \circ \psi$ or $\rho = (\tilde{\rho} \otimes \tilde{\beta}) \circ \psi \circ s$.

See [BR2], page 507. Therefore if $p \in \mathbb{N}$ and $\alpha = (p - 1)/2$, then either $\rho(p) = (\tilde{\rho} \otimes \tilde{\beta}) \circ \psi$ or $\rho(p) = (\tilde{\rho} \otimes \tilde{\beta}) \circ \psi \circ s$, since the minimal spin of $\rho(p)\text{fin}$ is zero (note that $\psi$ restricted to $U_q(\mathfrak{su}(2))$ is simply the coevaluation $\Delta$).

With a bit more work one can actually prove that it is the second case that holds. See [PhD]

If we consider the action in finite dimensional representations of the form $\rho = (\tilde{\rho} \otimes \tilde{\beta}) \circ \psi$, then $\psi$ transforms the $R$-matrix of the Quantum Lorentz Group into the $R$-matrix of $U_q(\mathfrak{su}(2)) \otimes_{R^{-1}} U_q(\mathfrak{su}(2))$, and analogously for their inverses. This is an easy consequence of the fact $(\Delta \otimes \text{id})(R) = R_{13} R_{23}$ and $(\text{id} \otimes \Delta)(R) = R_{13} R_{12}$. The same is true for the balanced representations $\rho(p) \cong (\tilde{\rho} \otimes \tilde{\beta}) \circ \psi \circ s$ since given that $X^{\alpha}_{ja} \otimes \tilde{g}^{\text{su}}_{ja}$ acts as zero in $V(p)\text{fin} \otimes V(p)\text{fin}$ if $\alpha \in \mathbb{N} + \frac{1}{2}$, it follows that the actions of $(s \otimes s)(R)$ and $R$ in $V(p)\text{fin} \otimes V(p)\text{fin}$ are the same. The detailed calculation appears in [PhD].

The map $\psi$ preserves the group like elements since $\Delta(q^{2J_z}) = q^{2J_z} \otimes q^{2J_z}$. Therefore from lemma [19] we obtain:
Proposition 21 Let $q \in (0,1)$ and $p \in \mathbb{N}$. Let also $\alpha = (p - 1)/2$. Given a braid $b$, let $K_b$ be the closure of $b$ with an arbitrary framing and $K_b^*$ its mirror image. Suppose $K_b$ is a knot. We have:

$$S_b(q, p) = \frac{I(\rho)(K_b^*)I(\rho)(K_b)}{[2\alpha + 1]^2} = X(0, p, K_b)(h) \frac{(2\alpha + 1)^2}{[2a + 1]^2},$$

where $q = \exp(h/2)$. The last equality follows from theorem 12.

Notice also lemma 18 and that $X(0, p, K_b)(h)$ is a convergent power series if $p \in \mathbb{N}$. Therefore the perturbative framework of the previous sections is correct, at least for finite dimensional representations. In the sequel we will generalise this for infinite dimensional representations.

3.2.3 The series Are Convergent $h$-Adicaly

We now define the $h$-adic version of the theory developed by Buffenoir and Roche. Let $q \in (0,1)$ and consider the element $\bar{g}_{j\alpha}^i \in \text{Pol}(SU_q(2))$. For any $p \in \mathbb{C}$, we have a balanced representations $\rho(p)$ of the Quantum Lorentz Group in $V(p)$. The term

$$\left\langle \bar{v}^i \rho(p)(\bar{g}_{j\alpha}^i) | \bar{v}^i_{\gamma} \right\rangle_q$$

can be seen a function of $q$. Due to the fact the building blocks of $\rho(p)$ are Clebsch-Gordan coefficients, it express as a sum of square roots of rational functions of $q$, which extend to a well defined analytic function in a neighbourhood of 1. We can see it for example from (10). In addition we have some terms of the form $q^{\sigma \alpha}$, $\sigma \in \mathbb{Z}$, which after putting $q = \exp(h/2)$ define an analytic function of $h$. Therefore

$$h \mapsto \left\langle \bar{v}^i \rho(p)(\bar{g}_{j\alpha}^i) | \bar{v}^i_{\gamma} \right\rangle_{\exp(h/2)}$$

defines a power series in $h$, uniquely. In particular it follows that if $b$ is a braid then each term of the sum $S_b(\exp(h/2), p)$ defines uniquely a power series in $h$, which converges to the term for $h$ small enough.

Lemma 22 For any $x \in \text{Pol}(SU_q(2))$, the order of:

$$h \mapsto \left\langle \bar{v}^i \rho(p)(x) | \bar{v}^i_{\gamma} \right\rangle_{\exp(h/2)},$$

as a power series in $h$, is bigger or equal to $|\beta - \gamma|$. 

39
Proof. Notice that $\frac{i}{j} g_1$ sends $V^\gamma$ to $V^\gamma - 1 \oplus V^\gamma \oplus V^{\gamma+1}$, in a way such that for $q = 1$ the projection $v$ of $\frac{i}{j} g_1 v_i^\gamma$ in $V^\gamma \oplus V^{-1}$ is zero. We can see this from lemma 17. In particular $v$ has order bigger or equal to one. This lemma is thus a trivial consequence of the fact the elements $\{\frac{i}{j} g_1, -1/2 \leq i, j \leq 1/2\}$ generate $\text{Pol}(SU_q(2))$ as an algebra.

Therefore

Proposition 23 For any braid $b$ whose closure is a knot the infinite sum $S_b(\exp(h/2), p)$ converges in the $h$-adic topology.

Proof. Let $b$ be a braid with $n$ crossings and $m + 1$ strands. Recall equation (9) and comments after. Due to the way the $X_{i,j}^0$ as well as $G$ act in $V(p)$, the previous lemma guaranties that the order of $\langle v^0, \prod_{i=1}^{2n+m} T(\alpha, \dot{k}, \dot{j}, l) v_0 \rangle$ as a power series in $h$ is bigger or equal to $\alpha_k$, for $k = 1, \ldots, n$; and the result follows.

3.2.4 The Series Define a $\mathbb{C}[[h]]$-Valued Knot Invariant

Since we have proved the $h$-adic convergence of the sums $S_b(\exp(h/2), p)$ to a formal power series, we could now use Markov’s theorem and prove that the assignment $b \mapsto S_b(\exp(h/2), p)$ defines a knot invariant. However the best way to prove this is to reduce it to the finite dimensional case, since we already know that it defines a knot invariant and the exact form of it, see proposition 21. Consider a coefficient $\Lambda_{AB}^{BC}(p)$ at $q = \exp(h/2)$, thus it is a power series in $h$ convergent for $h$ small enough. From equation (1), we can see that the dependence of each term in $p$ is polynomial. In particular:

Lemma 24 Let $b$ be a braid, consider the power series $S_b(\exp(h/2), p)$ as a function of $p$, the parameter defining a balanced representation of the Quantum Lorentz Group. Then each term in the expansion of $S_b(\exp(h/2), p)$ as a power series in $h$ is a polynomial in $p$.

Proof. Suppose $A(p) = \sum_{n \in \mathbb{N}_0} A_n(p) h^n$ and $B(p) = \sum_{n \in \mathbb{N}_0} B_n(p) h^n$ are power series whose coefficients depend polynomially in $p$, for example a power series such as $\exp(mph/2)$. Then also the coefficients of their product depend polynomially in $p$. This immediately proves this lemma. Note
that the Clebsch-Gordan coefficients as well as the actions of $G$ and of the elements $X_{\alpha i}$ do not depend on $p$. ■

Therefore

**Theorem 25** Let $p \in \mathbb{C}$ and $b$ be a braid. Let also $K_b$ be the closure of $b$. Let also $K_b^*$ be the mirror image of $K$. We have:

$$S_b(\exp(h/2), p) = \frac{X(0, p, K)(2\alpha + 1)^2}{[2\alpha + 1]^2},$$

where $\alpha = (p - 1)/2$.

Recall that by theorem 10 the knot invariant $X(0, p)$ is unframed.

**Proof.** By lemma 24 we only need to prove this theorem for $p \in \mathbb{N}$. In this case, if $q = (0, 1)$ then $S_b(q, p)$ truncates to a finite sum which from the comments after proposition 21 equals $X(0, p, K)(2\alpha + 1)^2[2\alpha + 1]^{-2}$ at $q = \exp(h/2)$. Recall this power series are convergent if $p$ is integer. Each term of the finite sum $S_b(\exp(h/2), p)$ is a power series in $h$ convergent for $h$ small enough and coinciding with $S_b(q, p)$, for $q \in (0, 1)$ and close enough to 1; thus the result follows. ■

**Acknowledgements**

This work was realised in the course of my PhD in the University of Nottingham under the supervision of Dr John W. Barrett. I was financially supported by the programme “PRAXIS-XXI”, grant number SFRH/BD/1004/2004 of Fundação para a Ciência e a Tecnologia (FCT), financed by the European Community fund Quadro Comunitário de Apoio III, and also by Programa Operacional “Ciência, Tecnologia, Inovação” (POCTI) of the Fundação para a Ciência e a Tecnologia (FCT), cofinanced by the European Community fund FEDER. The last stage of this work was financed by the FCT post-doc grant SFRH/BDP/17552/2004, part of the research project POCIT/MAT/60352/2004 (“Quantum Topology”).

**References**

[AC] Altschuler D., Coste A.: Quasi-Quantum Groups, Knots, Three Manifolds and Topological Field Theory; Commun. Math. Phys., 150, 83-107 (1992).
[B] Bar-Natan D.: On the Vassiliev Knot Invariants, Topology 34 (1995), no. 2, 423–472.

[BG] Bar-Natan D., Garoufalidis S.: On the Melvin-Morton-Rozansky Conjecture, Invent. Math. 125 (1996), no. 1, 103–133.

[BLT] Bar-Natan D., Le T.Q.T., Thurston D.: Two Applications of Elementary Knot Theory to Lie Algebras and Vassiliev Invariants, Geom. Topol. 7 (2003), 1–31.

[BC] Barrett J.W., Crane L.: Relativistic Spin Networks and Quantum gravity, J. Math. Phys. 39 (1998), no. 6, 3296–3302.

[BNR] Buffenoir E., Noui K., Roche Ph: Hamiltonian Quantization of Chern-Simons Theory with SL(2, C) Group, Classical Quantum Gravity 19 (2002), no. 19, 4953–5015.

[BR1] Buffenoir E., Roche Ph.: Tensor Product of Principal Unitary Representations of Quantum Lorentz Group and Askey-Wilson Polynomials, J. Math. Phys. 41 (2000), no. 11, 7715–7751.

[BR2] Buffenoir E., Roche Ph.: Harmonic Analysis on the Quantum Lorentz Group, Commun. Math. Phys. 207 (1999), no. 3, 499–555.

[CP] Chari V., Pressley A.: A Guide to Quantum Groups, Cambridge University Press, Cambridge, 1994.

[C] Chmutov S.: A Proof of the Melvin-Morton Conjecture and Feynman Diagrams, J. Knot Theory Ramifications 7 (1998), no. 1, 23–40.

[CS] Chmutov S., Duzhin S.: The Kontsevich Integral, Acta Appl. Math. 66 (2001), no. 2, 155–190.

[CV] Chmutov S., Varchenko A.: Remarks on the Vassiliev Invariants Coming from sl_2, Topology, Vol 36, No. 1, pp 153-178, 1997.

[D] Drinfeld V.G.: Quasi-Hopf Algebras. (Russian) Algebra i Analiz 1 (1989), no. 6, 114–148; translation in Leningrad Math. J. 1 (1990), no. 6, 1419–1457.
[FM] Faria Martins J.: On the Analytic Properties of the $z$-Coloured Jones Polynomial, QA/0310394, to appear in Journal of Knot Theory and its Ramifications.

[PhD] Faria Martins J.: Quantum Topology and the Lorentz Group, PhD thesis, University of Nottingham, 2004.

[GMS] Gel’fand I.M., Minlos R.A., Shapiro Z.Ya.: Representations of the Rotation and Lorentz groups and their Applications, Oxford Pergamon, 1963.

[G] Gukov S.: Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial, [hep-th/0306165](http://arxiv.org/abs/hep-th/0306165).

[K] Kassel C.: Quantum Groups, Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.

[Kir] Kirillov A.A.: Elements of the Theory of Representations, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin-New York, 1976.

[KS] Klimyk A.; Schmüdgen K.: Quantum groups and their Representations, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1997.

[L] Lang S.: $SL(2; \mathbb{R})$, Addison-Wesley, Reading, MA, 1975.

[LM] Le T.Q.T., Murakami J.: The Universal Vassiliev-Kontsevich Invariant for Framed Oriented Links, Compositio Math. 102 (1996), no. 1, 41–64.

[MM] Melvin P.M., Morton, H.R.: The Coloured Jones Function, Commun. Math. Phys. 169 (1995), no. 3, 501–520.

[NR] Noui K., Roche Ph.: Cosmological Deformation of Lorentzian Spin Foam Models, Classical Quantum Gravity 20 (2003), no. 14, 3175–3213.

[PoW] Podleś P., Woronowicz S.L.: Quantum Deformation of Lorentz Group. Commun. Math. Phys. 130 (1990), no. 2, 381–431.
[Pu] Pusz W.: Irreducible Unitary Representations of Quantum Lorentz Group, Commun. Math. Phys. 152 (1993), no. 3, 591–626.

[PuW] Pusz W., Woronowicz S.L.: Representations of Quantum Lorentz Group on Gelfand Spaces, Rev. Math. Phys. 12 (2000), no. 12, 1551–1625.

[T] Takeuchi M.: Finite-Dimensional Representations of the Quantum Lorentz Group, Commun. Math. Phys. 144 (1992), no. 3, 557–580.

[V] Varadarajan V.S.: Lie groups, Lie Algebras, and their Representations, Graduate Texts in Mathematics, 102, Springer-Verlag, New York, 1984.

[W] Willerton S.: The Kontsevich Integral and Algebraic Structures in The Space Of Diagrams, Knots in Hellas ’98 (Delphi), 530–546, Ser. Knots Everything, 24, World Sci. Publishing, River Edge, NJ, 2000.