Quantum $\mathcal{W}$-symmetry in AdS$_3$

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Abstract: It has recently been argued that, classically, massless higher spin theories in AdS$_3$ have an enlarged $\mathcal{W}_N$-symmetry as the algebra of asymptotic isometries. In this note we provide evidence that this symmetry is realised (perturbatively) in the quantum theory. We perform a one loop computation of the fluctuations for a massless spin $s$ field around a thermal AdS$_3$ background. The resulting determinants are evaluated using the heat kernel techniques of arXiv:0911.5085. The answer factorises holomorphically, and the contributions from the various spin $s$ fields organise themselves into vacuum characters of the $\mathcal{W}_N$ symmetry. For the case of the $hs(1,1)$ theory consisting of an infinite tower of massless higher spin particles, the resulting answer can be simply expressed in terms of (two copies of) the MacMahon function.
1. Introduction

One of the remarkable features of AdS spacetimes is the existence of interacting
theories of massless particles with spin greater than two [1]. As is well known, it is
impossible to have such theories in flat spacetime. Typically, the consistency of such
theories in AdS spacetimes requires the introduction of an infinite tower of particles
with arbitrarily high spin. These theories, coupled to gravity, are thus in some sense
intermediate between conventional field theories of gravity involving a finite number
of fields on the one hand, and string theories on the other (see [2] for an introduction
to these matters).
This observation takes on added significance in the context of the AdS/CFT correspondence. Higher spin theories provide an opportunity to understand the correspondence beyond the (super)gravity limit without necessarily having the full string theory under control. In fact, there is the tantalising possibility of a consistent truncation to this subsector within a string theory which might, in itself, be dual to a field theory on the boundary of the AdS spacetime. A conjecture of precisely this nature is that of Klebanov and Polyakov for the dual description of the $O(N)$ vector model in $2+1$ dimensions in terms of a higher spin theory on AdS$_4$ [3, 4]. Recent calculations have provided non-trivial, interesting evidence for this conjecture, see for example [5, 6, 7].

One of the main hurdles in the exploration of this topic has been that, even at the classical level, it is very difficult to perform calculations. In fact, the formulation of higher spin theories is typically quite complicated. Therefore the study of these theories, especially at the quantum level, is in its infancy.

Recently, attention has been drawn [8, 9] to the case of three dimensions where the classical theories are relatively more tractable since it is consistent to consider theories involving a finite number of higher spin fields [10]. Here the massless higher spin fields do not have any propagating degrees of freedom (as we will review in Sec. 1.1), just as in the case of pure gravity. Nevertheless, as we have learnt in the case of theories of gravity on AdS$_3$, there can be interesting quantum dynamics captured by a two dimensional conformal field theory on the boundary. Thus these theories can be a useful stepping stone towards studying such higher spin theories in dimension greater than three.

An important first step was taken in [8, 9] by studying the asymptotic symmetries of these higher spin theories at the classical level. The analysis is the analogue of the Brown-Henneaux study in pure gravity on AdS$_3$ [11]. In the case of theories with massless higher spin particles (of spin $s = 3, \ldots, N$) one finds that the algebra of asymptotic isometries is enlarged from that of two copies (left- and right-moving) of the Virasoro algebra, to two copies of the $\mathcal{W}_N$ algebra. Like in the Brown-Henneaux analysis one finds a central extension of the algebra already at the classical level. The central charge is the same [8, 9] as that of Brown-Henneaux [11], namely

$$c = \frac{3\ell}{2G_N}.$$ (1.1)

In this paper, we study these massless higher spin theories on AdS$_3$ at the quantum level. More specifically, we perform a one loop calculation of the quadratic fluctuations of the fields about a thermal AdS$_3$ background. This requires a careful accounting of the gauge degrees of freedom of these fields. In particular, we show that the partition function reduces to the ratio of two determinants. For a spin $s$ field these involve Laplacians for transverse traceless modes of helicity $\pm s$ as well as
\[ Z^{(s)} = \left[ \det \left( -\Delta + \frac{s(s-3)}{\ell^2} \right) \right]^{\frac{1}{2}} \left[ \det \left( -\Delta + \frac{s(s-1)}{\ell^2} \right) \right]^{\frac{1}{2}}. \quad (1.2) \]

In [12] such determinants were explicitly evaluated, in a thermal background, for arbitrary spin \( s \), using the group theoretic techniques of [13, 14]. By applying the results of [12] we find that the one loop answer factorises neatly into left and right moving pieces

\[ Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2}, \quad (1.3) \]

where \( q = e^{i\tau} \) is the modular parameter of the boundary \( T^2 \) of the thermal background. This generalises the expression for the case of pure gravity \( (s = 2) \) [15], as explicitly checked in [16]. The expression (1.3) is seen to be the contribution to the character for a generator of conformal dimension \( s \). Combining the different fields of spin \( s = 3, \ldots, N \), together with the corresponding expression for the spin two case, one obtains indeed the vacuum character of the \( \mathcal{W}_N \) algebra.

A straightforward generalisation of an argument of Maloney and Witten [15] can now be made to show that this expression (together with the classical contribution \( (q\bar{q})^{-\frac{c}{24}} \)) is one loop exact in perturbation theory. This is to be understood in a particular scheme where the Newton constant is suitably renormalised while keeping \( c \) fixed.

It is interesting that if we consider the Vasiliev higher spin theory with left and right copies of the \( hs(1, 1) \) higher spin algebras, then we find a vacuum character of the \( \mathcal{W}_\infty \) algebra, which can be written as

\[ Z_{hs(1, 1)} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} = \prod_{n=1}^{\infty} \frac{1}{|1 - q^n|^2} \times \prod_{n=1}^{\infty} \frac{1}{|(1 - q^n)^n|^2}. \quad (1.4) \]

It is interesting that the answer can be naturally expressed in terms of the so-called MacMahon function

\[ M(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n}. \quad (1.5) \]

The organisation of this paper is as follows. In the next subsection we review some basic features of the massless higher spin fields. We will find it useful to decompose the fields in terms of transverse traceless modes of various helicities which will enable us to count the physical and gauge degrees of freedom. In Sec. 2, we lay out the basics for the calculation of the quadratic fluctuations, correctly taking into account the redundancy from the gauge modes. In Sec. 3 we obtain the one loop answer for the spin three case in a brute force manner. Sec. 4 shows how the answer
for a general spin can be carried out without having to do too much work. Sec. 5 uses the results of \[12\] to evaluate the determinants (1.2) explicitly in a thermal AdS\(_3\) background to obtain (1.3). Sec. 6 relates these expressions to the vacuum characters of \(\mathcal{W}_N\). We also comment on the case of \(\mathcal{W}_\infty\) and the relation to the MacMahon function. Sec. 7 contains closing remarks while the appendices describe our conventions and spell out some of the details of the spin three calculation.

1.1 Counting Degrees of Freedom

Let us first recall some basic features of massless higher spin theories at the non-interacting level \[17, 18\] (see for example \[19\] for a review and more references). The massless spin \(s\) fields in three dimensions are completely symmetric tensors \(\varphi_{\mu_1\mu_2...\mu_s}\) subject to a double trace constraint

\[
\varphi_{\mu_5...\mu_s\alpha\lambda}^{\alpha\lambda} = 0. \tag{1.6}
\]

This constraint only makes sense if \(s \geq 4\). In addition we have a gauge invariance leading to the identification of field configurations

\[
\varphi_{\mu_1\mu_2...\mu_s} \sim \varphi_{\mu_1\mu_2...\mu_s} + \nabla_{(\mu_1} \xi_{\mu_2...\mu_s)}. \tag{1.7}
\]

The gauge parameter \(\xi_{\mu_2...\mu_s}\) is a symmetric tensor of rank \((s-1)\) which is, in addition, traceless, i.e. \(\xi_{\mu_3...\mu_s\lambda}^{\lambda} = 0\). This last constraint only makes sense for \(s \geq 3\). Our conventions regarding symmetrisation etc. are explained in appendix \[A\].

Now we will count the number of independent components of the fields. Recall that a completely symmetric tensor of rank \(s\) in three dimensions has \(\frac{(s+1)(s+2)}{2}\) independent components. In our case, because of the double trace constraint, many of these components are not actually independent. The constraints are as many in number as those of a symmetric tensor of rank \((s-4)\). Therefore the net number of independent components is given by

\[
\frac{(s + 1)(s + 2)}{2} - \frac{(s - 3)(s - 2)}{2} = 4s - 2. \tag{1.8}
\]

We now argue that half of these are gauge degrees of freedom. Recall that the gauge parameter is given by a traceless rank \((s-1)\) symmetric tensor \(\xi_{\mu_1\mu_2...\mu_{s-1}}\). The number of independent components of \(\xi_{(s-1)}\) is therefore

\[
\frac{s(s + 1)}{2} - \frac{(s - 1)(s - 2)}{2} = 2s - 1. \tag{1.9}
\]

Therefore the non-gauge components are also \((2s - 1)\) in number. This fits in with what we expect for these fields, namely that they have no net propagating degrees of freedom (at least in the bulk of AdS\(_3\)).
Let us now analyse the representation theoretic content of these different modes. This will give us an important clue to the analysis of the one loop answer. We can decompose the field $\varphi(s)$ in the following way

$$
\varphi_{\mu_1 \mu_2 \ldots \mu_s} = \varphi_{\mu_1 \mu_2 \ldots \mu_s}^{\text{TT}} + g_{\mu_1 \mu_2} \tilde{\varphi}_{\mu_3 \ldots \mu_s} + \nabla_{(\mu_1} \xi_{\mu_2 \ldots \mu_s)}.
$$

(1.10)

Here $\varphi_{\mu_1 \mu_2 \ldots \mu_s}^{\text{TT}}$ is the transverse, traceless piece of $\varphi(s)$ and consists of two independent components carrying helicity $\pm s$. $\tilde{\varphi}_{\mu_3 \ldots \mu_s}$ is the spin $(s-2)$ piece which carries all the trace information of $\varphi(s)$. Finally, $\xi_{(s-1)}$ are the gauge parameters. Note that the double trace constraint on $\varphi(s)$ of eq. (1.6) implies that $\tilde{\varphi}_{(s-2)}$ is traceless.

In what follows it will be important for us to make the further decomposition of the gauge field $\xi_{(s-1)}$ into its traceless transverse component $\xi_{(s-1)}^{\text{TT}}$, as well as

$$
\xi_{\mu_1 \ldots \mu_{s-1}} = \xi_{\mu_1 \ldots \mu_{s-1}}^{\text{TT}} + \xi_{\mu_1 \ldots \mu_{s-1}}^{(s)} ,
$$

(1.11)

where $\xi_{(s-1)}^{(s)}$ is the longitudinal component, that can be written as

$$
\xi_{\mu_1 \ldots \mu_{s-1}}^{(s)} = \nabla_{(\mu_1} \sigma_{\mu_2 \ldots \mu_{s-1})} - \frac{2}{(2s-3)} g_{\mu_1 \mu_2} \nabla^\lambda \sigma_{\mu_3 \ldots \mu_{s-1})\lambda ,
$$

(1.12)

with $\sigma_{(s-2)}$ a traceless symmetric tensor. The transverse, traceless component $\xi_{(s-1)}^{\text{TT}}$ carries helicity $\pm (s-1)$.

In order to exhibit the remaining helicity components we can now further decompose $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ into their transverse traceless, as well as their longitudinal spin $(s-3)$ components. The longitudinal pieces that appear in either of these decompositions can then again be decomposed into transverse traceless spin $(s-3)$ components, together with longitudinal components of spin $(s-4)$, etc. In this way we can see that both $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ have helicity components corresponding to all the helicities less or equal to $(s-2)$; this gives rise to $2(s-2) + 1 = 2s-3$ components for each of the two fields. In summary we therefore have

(i) a symmetric transverse traceless field $\varphi_{(s)}^{\text{TT}}$ of spin $s$, with helicities $\pm s$ [2 components]

(ii) a symmetric transverse traceless gauge mode $\xi_{(s-1)}^{\text{TT}}$ of spin $s-1$, with helicities $\pm (s-1)$ [2 components]

(iii) a symmetric traceless (but not necessarily transverse) field $\tilde{\varphi}_{(s-2)}$ of spin $s-2$, with helicities $0, \pm 1, \pm 2, \ldots, \pm (s-2)$ [2s-3 components]

(iv) a symmetric traceless (but not necessarily transverse) gauge field $\sigma_{(s-2)}$ of spin $s-2$, with helicities $0, \pm 1, \pm 2, \ldots, \pm (s-2)$ [2s-3 components]

In particular, there are therefore $2s-1$ non-gauge and $2s-1$ gauge components, in agreement with the above counting. Note that there are precisely as many gauge
components in $\sigma_{(s-2)}$, as there are components in $\tilde{\varphi}_{(s-2)}$. In fact, if we consider the trace part of (1.10), the tracelessness of $\sigma_{(s-2)}$ implies that

$$\varphi_{\mu_1\mu_2...\mu_{s-2}\lambda} = (2s - 1) \tilde{\varphi}_{\mu_1...\mu_{s-2}} + \nabla^\lambda \xi^{(\sigma)}_{\mu_1...\mu_{s-2}}$$

$$= (2s - 1) \varphi_{\mu_1...\mu_{s-2}} + (K\sigma)_{\mu_1...\mu_{s-2}} ,$$

(1.13)

where $K$ is a linear second order differential operator. Thus, at least classically, we can gauge away $\tilde{\varphi}_{(s-2)}$ completely [20]. This therefore suggests that in the calculation of the one loop determinant, the $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ fields will give cancelling contributions. The final answer should therefore only involve the helicity $\pm s$ non-gauge modes of $\varphi_{TT}^{(s)}$, as well as the helicity $\pm (s - 1)$ gauge modes of $\xi^{TT}_{(s-1)}$. This is the intuitive explanation of the answer (1.2). Below we will see explicitly how this happens from a careful consideration of the quadratic functional integral for the $\varphi$-field.

2. The general setup

The quadratic fluctuations we are interested in can be computed from the functional integral

$$Z^{(s)} = \frac{1}{\text{Vol}(\text{gauge group})} \int [D\varphi^{(s)}] e^{-S[\varphi^{(s)}]} .$$

(2.1)

Here $S[\varphi^{(s)}]$ is the action of a spin-$s$ field in a $D = 3$ dimensional AdS background [17] — we are using the conventions and notations of [9]

$$S[\varphi^{(s)}] = \int d^3x \sqrt{g} \varphi^{\mu_1...\mu_s} \left( \tilde{F}_{\mu_1...\mu_s} - \frac{1}{2} g_{(\mu_1\mu_2} \tilde{F}_{\mu_3...\mu_s)\lambda} \right) ,$$

(2.2)

where the symmetric tensor $\tilde{F}_{\mu_1...\mu_s}$ is defined in $D = 3$ as

$$\tilde{F}_{\mu_1...\mu_s} = F_{\mu_1...\mu_s} - \frac{s(s - 3)}{\ell^2} \varphi_{\mu_1...\mu_s} - \frac{2}{\ell^2} g_{(\mu_1\mu_2} \varphi_{\mu_3...\mu_s)} \lambda .$$

(2.3)

Finally, $F_{\mu_1...\mu_s}$ equals

$$F_{\mu_1...\mu_s} = \Delta \varphi_{\mu_1...\mu_s} - \nabla_{(\mu_1} \nabla^\lambda \varphi_{\mu_2...\mu_s)\lambda} + \frac{1}{2} \nabla_{(\mu_1} \nabla_{\mu_2} \varphi_{\mu_3...\mu_s)} \lambda .$$

(2.4)

We should mention in passing that the case $s = 2$ reduces to the familiar gravity action on AdS$_3$, see for example [21].

In order to evaluate the path integral $Z^{(s)}$ in (2.1) it is useful to change variables as

$$[D\varphi^{(s)}] = Z^{(s)}_{gh} [D\varphi^{TT}_{(s)}] [D\tilde{\varphi}_{(s-2)}] [D\xi^{TT}_{(s-1)}] ,$$

(2.5)

1Note that in the last term there is a factor of $\frac{1}{2}$ relative to eq. (2.1) of [9]. This arises because covariant derivatives do not in general commute, and thus we need to sum over both orderings of $\mu_1, \mu_2$. Our conventions regarding symmetrisation are explained in appendix A.
where we use the same decomposition as in (1.10). Here $Z^{(s)}_{gh}$ denotes the ghost determinant that arises from the change of variables.

The gauge invariance of the action, together with the orthogonality of the first terms of (1.10), implies that

$$S[\varphi(s)] = S[\varphi^{TT}(s)] + S[\tilde{\varphi}(s-2)] ,$$

and the first term is simply

$$S[\varphi^{TT}(s)] = \int d^3 x \sqrt{g} \, \varphi^{TT \mu_1 \ldots \mu_s} \left( -\Delta + \frac{s(s-3)}{\ell^2} \right) \varphi_{\mu_1 \ldots \mu_s} . \tag{2.7}$$

Thus the functional integral over the TT modes is easily evaluated to be

$$Z^{(s)} = Z^{(s)}_{gh} \left[ \det \left( -\Delta + \frac{s(s-3)}{\ell^2} \right) \right]^{\frac{1}{2}} \int [D\tilde{\varphi}(s-2)] e^{-S[\tilde{\varphi}(s-2)]} . \tag{2.8}$$

The determination of the functional integral requires therefore that we compute $Z^{(s)}_{gh}$, as well as the quadratic integral over $\tilde{\varphi}(s-2)$. Let us briefly discuss both terms.

### 2.1 The quadratic action of $\tilde{\varphi}(s-2)$

For the component of $\varphi(s)$ proportional to $\tilde{\varphi}(s-2)$, see eq. (1.10), the above action simplifies considerably. Indeed, it follows directly from (2.2) that

$$S[\tilde{\varphi}(s-2)] = -\frac{s(s-1)(2s-3)}{4} \int d^3 x \sqrt{g} \, \tilde{\varphi}^{\mu_1 \ldots \mu_{s-2}} \tilde{F}_{\mu_1 \ldots \mu_{s-2} \lambda} \lambda , \tag{2.9}$$

where $\tilde{F}$ is evaluated on $\varphi(s) = g \tilde{\varphi}(s-2)$. By an explicit computation one finds that

$$\tilde{F}_{\mu_1 \ldots \mu_s} (\varphi) = g(\mu_1 \mu_2) \left[ \Delta \tilde{\varphi}(\mu_3 \ldots \mu_s) - \nabla_{\mu_3} \nabla^{\lambda} \tilde{\varphi}_{\mu_4 \ldots \mu_s} \lambda - \frac{(s^2 + s - 2)}{\ell^2} \tilde{\varphi}_{\mu_3 \ldots \mu_s} \right]$$

$$+ \frac{(2s-3)}{2} \nabla_{(\mu_1} \nabla_{\mu_2} \tilde{\varphi}_{\mu_3 \ldots \mu_s)} . \tag{2.10}$$

Therefore $\tilde{F}_{\mu_3 \ldots \mu_s \lambda} = g^{\mu_1 \mu_2} \tilde{F}_{\mu_1 \ldots \mu_s}$ is given by

$$\tilde{F}_{\mu_3 \ldots \mu_s \lambda} = (2s-1) \left[ \Delta \tilde{\varphi}(\mu_3 \ldots \mu_s) - \nabla_{(\mu_3} \nabla^{\lambda} \tilde{\varphi}_{\mu_4 \ldots \mu_s) \lambda} - \frac{(s^2 + s - 2)}{\ell^2} \tilde{\varphi}_{\mu_3 \ldots \mu_s} \right]$$

$$+ 2 g_{(\mu_3 \mu_4} \nabla^{\lambda} \nabla^{\nu} \tilde{\varphi}_{\mu_5 \ldots \mu_s) \lambda \nu} + \frac{(2s-3)}{2} g^{\lambda \nu} \nabla(\lambda \nabla_{\nu} \tilde{\varphi}_{\mu_3 \ldots \mu_s}) . \tag{2.11}$$

Note that the second but last term in (2.11) will not contribute when put into (2.9) because of the tracelessness condition on $\tilde{\varphi}$. The last term in (2.11) can be evaluated to be

$$g^{\mu_1 \mu_2} \nabla_{(\mu_1} \nabla_{\mu_2} \tilde{\varphi}_{\mu_3 \ldots \mu_s)} = 2 \Delta \tilde{\varphi}(\mu_3 \ldots \mu_s) + 2 \nabla_{(\mu_3} \nabla^{\lambda} \tilde{\varphi}_{\mu_4 \ldots \mu_s) \lambda} + 2 \nabla^{\lambda} \nabla_{(\mu_3} \tilde{\varphi}_{\mu_4 \ldots \mu_s) \lambda}$$

$$= 2 \Delta \tilde{\varphi}(\mu_3 \ldots \mu_s) + 4 \nabla_{(\mu_3} \nabla^{\lambda} \tilde{\varphi}_{\mu_4 \ldots \mu_s) \lambda} - \frac{(s-1)(s-2)}{\ell^2} \tilde{\varphi}_{\mu_3 \ldots \mu_s} . \tag{2.12}$$
Plugging (2.12) and (2.11) into (2.10), the quadratic action for \( \tilde{\varphi} \) finally becomes

\[
S[\tilde{\varphi}_{(s-2)}] = C_s \int d^3x \sqrt{g} \tilde{\varphi}^{\mu_3...\mu_s} \tilde{\varphi}_{\mu_3...\mu_s} \lambda^s
\]

\[
= C_s \int d^3x \sqrt{g} \tilde{\varphi}^{\mu_3...\mu_s} \left[ 4(s-1) \left( \Delta - \frac{s^2 - s + 1}{\ell^2} \right) \right] \tilde{\varphi}_{\mu_3...\mu_s} + (2s - 5) \nabla_{(\mu_3} \nabla^\lambda \tilde{\varphi}_{\mu_4...\mu_s)} \lambda \right] ,
\]

where \( C_s \) is an (unimportant) constant. The path integral over \( \tilde{\varphi}_{(s-2)} \) is now straightforward, and can be expressed in terms of the determinant of the differential operator appearing in (2.13). Notice, however, that \( \tilde{\varphi}_{(s-2)} \) is only traceless, and not transverse. If we want to express this determinant in terms of differential operators acting on transverse traceless operators, more work will be required. This will be sketched for \( s = 3 \) below in section 3.2.

### 2.2 The ghost determinant

For the evaluation of the ghost determinant we shall follow the same strategy as in [21]. This is to say, we write

\[
1 = \int [D\varphi(s)] e^{-\langle \varphi(s); \varphi(s) \rangle} = Z_{gh}^{(s)} \int [D\tilde{\varphi}(s)] [D\tilde{\varphi}_{(s-2)}] [D\xi(s-1)] e^{-\langle \tilde{\varphi}(s); \tilde{\varphi}(s) \rangle} ,
\]

where \( \varphi(s) \equiv \varphi(s)(\varphi_{TT}, \tilde{\varphi}_{(s-2)}, \xi(s-1)) \) as in (1.10). Next we expand out

\[
\langle \varphi(s), \varphi(s) \rangle = \langle \varphi_{TT}(s), \varphi_{TT}(s) \rangle + \langle g \varphi_{(s-2)}, g \varphi_{(s-2)} \rangle + \langle \nabla \xi(s-1), \nabla \xi(s-1) \rangle + \langle g \varphi_{(s-2)}, \nabla \xi(s-1) \rangle + \langle \nabla \xi(s-1), g \varphi_{(s-2)} \rangle .
\]

In order to remove the off-diagonal terms of the last line, we rewrite (1.10) as

\[
\varphi_{\mu_1\mu_2...\mu_s} = \varphi_{TT,\mu_1\mu_2...\mu_s} + g(\mu_1\mu_2\varphi'_{\mu_3...\mu_s}) + \left( \nabla(\mu_1\xi_{\mu_2...\mu_s}) - \frac{2}{2s-1} g(\mu_1\mu_2 \nabla^\lambda \xi_{\mu_3...\mu_s}) \right) .
\]

Then the quadratic term takes the form

\[
\langle \varphi(s), \varphi(s) \rangle = \langle \varphi_{TT}(s), \varphi_{TT}(s) \rangle + \langle g \varphi_{(s-2)}, g \varphi_{(s-2)} \rangle + \langle (\nabla \xi(s-1)) - \frac{2}{2s-1} g \nabla \xi(s-1), (\nabla \xi(s-1)) - \frac{2}{2s-1} g \nabla \xi(s-1) \rangle .
\]

Both the \( \varphi_{TT}(s) \) and the \( \varphi_{(s-2)} \) path integral are now trivial, as is the Jacobian coming from the change of measure in going from \( \tilde{\varphi}_{(s-2)} \) to \( \varphi'_{(s-2)} \). Thus the ghost determinant simply becomes

\[
\frac{1}{Z_{gh}^{(s)}} = \int [D\xi(s-1)] e^{-\langle (\nabla \xi(s-1)) - \frac{2}{2s-1} g \nabla \xi(s-1), (\nabla \xi(s-1)) - \frac{2}{2s-1} g \nabla \xi(s-1) \rangle} .
\]
The exponent can be simplified further by integrating by parts to get

\[ S_\xi = \langle (\nabla \xi_{s-1}) - \frac{2}{2s-1} g \nabla \xi, (\nabla \xi_{s-1}) - \frac{2}{2s-1} g \nabla \xi \rangle \]

\[ = s \int d^3 z \sqrt{g} \left[ \xi_{\mu_1...\mu_{s-1}} \left( -\Delta + \frac{s(s-1)}{\ell^2} \right) \xi_{\mu_1...\mu_{s-1}} - \frac{(s-1)(2s-3)}{2(s-1)} \xi_{\mu_1...\mu_{s-2}} \nabla^\lambda \nabla_\nu \xi_{\mu_1...\mu_{s-2} \lambda} \right]. \]  \hspace{1cm} (2.20)

This is the path integral we have to perform. Before doing the calculation in the general case, it is instructive to analyse the simplest case, \( s = 3 \), first. The impatient reader is welcome to skip the next section and proceed directly to Sec. 4 where we perform the general analysis of the ghost determinant.

3. The Case of Spin Three

As explained in the previous section, see eq. (2.8), the calculation of the 1-loop determinant is reduced to determining the ghost determinant (2.19) as well as the determinant arising from (2.13). We shall first deal with the ghost determinant.

3.1 Calculation of the Ghost Determinant

For \( s = 3 \) the \( \xi \)-dependent exponent of (2.20) is of the form

\[ S_\xi = 3 \int d^3 x \sqrt{g} \left[ \xi_{\nu \rho} \left( -\Delta + \frac{6}{\ell^2} \right) \xi_{\nu \rho} - \frac{6}{5} \xi_{\nu}^{\rho \sigma} \nabla_\rho \nabla_\mu \xi^{\mu \nu} \right]. \]  \hspace{1cm} (3.1)

We shall first do the calculation in a pedestrian manner, following the same methods as in [21]. We shall then explain how our result can be more efficiently obtained. To start with we decompose \( \xi^{\mu \nu} \) as

\[ \xi^{\mu \nu} = \xi^{TT \mu \nu} + \nabla^\mu \sigma T^\nu + \nabla^\nu \sigma T^\mu + \left( \nabla^\mu \nabla^\nu - \frac{1}{3} g^{\mu \nu} \nabla^2 \right) \psi = \xi^{TT \mu \nu} + \nabla(\mu \sigma T^\nu) + \psi^{\mu \nu}, \]  \hspace{1cm} (3.2)

where \( \xi^{TT \mu \nu} \) is the transverse and traceless part of \( \xi^{\mu \nu} \), while \( \sigma T^\nu \) is (traceless) and transverse, i.e.

\[ \nabla_\nu \sigma T^\nu = 0. \]  \hspace{1cm} (3.3)

Plugging (3.2) into (3.1) we obtain after a lengthy calculation — some of the details are explained in appendix B —

\[ S_\xi = \int d^3 x \sqrt{g} \left[ 3 \xi_{\nu \rho}^{TT} \left( -\Delta + \frac{6}{\ell^2} \right) \xi_{TT \nu \rho}^{TT} \right. \]

\[ + \frac{48}{5} \sigma_\nu^T \left( -\Delta + \frac{2}{\ell^2} \right) \left( -\Delta + \frac{7}{\ell^2} \right) \sigma T^\nu \]

\[ + \frac{18}{5} \psi \left( -\Delta \right) \left( -\Delta + \frac{3}{\ell^2} \right) \left( -\Delta + \frac{8}{\ell^2} \right) \psi \right]. \]  \hspace{1cm} (3.4)
The ghost determinant (2.19) is then simply
\[
Z_{gh}^{(s=3)} = J_1^{-1} \left\{ \det \left( -\Delta + \frac{6}{\ell^2} \right)^{\text{TT}} \left( -\Delta + \frac{2}{\ell^2} \right)^{(2)} \right\} \det \left( -\Delta + \frac{7}{\ell^2} \right)^{(1)}
\times \det \left[ (-\Delta) \left( -\Delta + \frac{3}{\ell^2} \right) \left( -\Delta + \frac{8}{\ell^2} \right) \right]^{\frac{1}{2}},
\]
(3.5)
where \( J_1 \) is the Jacobian from the change of measure in going from \( \xi \) to \((\xi^{\text{TT}}, \sigma^T, \psi)\).

This can be calculated from the identity
\[
1 = \int D\xi e^{-\langle \xi, \xi \rangle} = \int J_1 D\xi^{\text{TT}} D\sigma^T d\psi e^{-\langle \xi^{\text{TT}}, \sigma^T, \psi \rangle, \xi^{\text{TT}}, \sigma^T, \psi \rangle}.
\]
(3.6)

Expanding out the exponential, the terms of interest are
\[
\int d^3 x \sqrt{g} \nabla^{(\mu \sigma^T \nu)} = -2 \int \sigma^T \left( \Delta - \frac{2}{\ell^2} \right) \sigma^{T \nu}.
\]
(3.7)
and
\[
\int d^3 x \sqrt{g} \left[ \left( \nabla^{\mu} \nabla^{\nu} - \frac{1}{3} g^{\mu \nu} \Delta \right) \psi \right] \left[ \left( \nabla^{\mu} \nabla^{\nu} - \frac{1}{3} g^{\mu \nu} \Delta \right) \psi \right] = \frac{2}{3} \int d^3 x \sqrt{g} \left[ (-\Delta) \left( -\Delta + \frac{3}{\ell^2} \right) \psi \right].
\]
(3.8)
Thus the \((-\Delta + \frac{2}{\ell^2})\) term is cancelled from the first line of (3.5) and similarly the \((-\Delta)(-\Delta + \frac{3}{\ell^2})\) term from the second line. The complete ghost determinant for \( s = 3 \) therefore equals
\[
Z_{gh}^{(s=3)} = \left\{ \det \left( -\Delta + \frac{6}{\ell^2} \right)^{\text{TT}} \left( -\Delta + \frac{7}{\ell^2} \right)^{(2)} \right\} \det \left( -\Delta + \frac{8}{\ell^2} \right)^{(1)} \left\{ \det \left[ (-\Delta) \left( -\Delta + \frac{3}{\ell^2} \right) \left( -\Delta + \frac{8}{\ell^2} \right) \right]^{\frac{1}{2}} \right\}.
\]
(3.9)

For the following it will be important to observe that this result can be obtained more directly. Indeed, as the above calculation has demonstrated, there are many cancellations between terms arising from \( S_\xi \) and the change of measure \( J_1 \). Actually, it is not difficult to see how this comes about. Consider for example the vector part. The factor by which the second line of (3.4) differs from (3.7) is the eigenvalue of the differential operator
\[
(L^{(3)} \xi)^{\nu} \equiv \left( -\Delta + \frac{6}{\ell^2} \right) \xi^{\nu} - \frac{3}{5} \left( \nabla^{\mu} \nabla_{\mu} \xi^{\nu} + \nabla^{\nu} \nabla_{\mu} \xi^{\rho \mu} \right)
\]
evaluated on the tensors of the form \( \xi^{\nu} = \nabla^{(\nu} \sigma^{T \rho)} \). This reproduces indeed (3.9) since we find
\[
\left( -\Delta + \frac{6}{\ell^2} \right) \nabla^{(\nu} \sigma^{T \rho)} - \frac{3}{5} \left( \nabla^{\rho} \nabla_{\mu} \nabla^{(\nu} \sigma^{T \rho)} + \nabla^{\nu} \nabla_{\mu} \nabla^{(\rho} \sigma^{T \mu)} \right)
= \frac{8}{5} \left[ \nabla^{\nu} \left( -\Delta + \frac{7}{\ell^2} \right) \sigma^{T \rho} + \nabla^{\rho} \left( -\Delta + \frac{7}{\ell^2} \right) \sigma^{T \nu} \right].
\]
(3.11)
Similarly, the action of $\mathcal{L}^{(3)}$ on $\psi^{\mu\nu}$ leads to

\[
\left( -\Delta + \frac{6}{\ell^2} \right) \psi_{\nu\rho} - \frac{6}{10} \left( \nabla_\rho \nabla^\mu \psi_{\mu\nu} + \nabla_\nu \nabla^\mu \psi_{\mu\rho} \right) \nonumber \\
= \frac{9}{5} \nabla_\rho \nabla_\nu \left( -\Delta + \frac{8}{\ell^2} \right) \psi - \frac{1}{3} g_{\rho\nu} \Delta \left( -\Delta + \frac{12}{\ell^2} \right) \psi . \tag{3.12}
\]

While $\psi^{\mu\nu}$ is not an eigenvector of (3.10), the second term proportional to $g_{\rho\nu}$ does not actually matter for the 1-loop calculation since the result is contracted with the traceless tensor $\psi_{\rho\nu}$, see eq. (3.1).

### 3.2 The Quadratic Contribution from $\tilde{\varphi}$

The other piece of the calculation is the quadratic contribution from $\tilde{\varphi}$, which for $s = 3$ takes the form

\[
S[\tilde{\varphi}(1)] = \frac{9}{2} \int d^3 \sqrt{g} \left[ 8 \tilde{\varphi}^\rho \left( -\Delta + \frac{7}{\ell^2} \right) \tilde{\varphi}_\rho - \tilde{\varphi}^\rho \nabla_\rho \nabla^\lambda \tilde{\varphi}_\lambda \right] . \tag{3.13}
\]

In order to express the determinant in terms of those acting on traceless transverse components, we now decompose $\tilde{\varphi}_\rho$ into its transverse and longitudinal component

\[
\tilde{\varphi}_\rho = \tilde{\varphi}_\rho^T + \nabla_\rho \chi . \tag{3.14}
\]

By the usual argument the above quadratic action then becomes

\[
S = -\frac{9}{2} \int d^3 \sqrt{g} \left[ 8 \tilde{\varphi}^{T \rho} \left( \Delta - \frac{7}{\ell^2} \right) \tilde{\varphi}_\rho^T \right. \nonumber \\
\left. - 8 \chi \nabla^\rho \left( \Delta - \frac{7}{\ell^2} \right) \nabla_\rho \chi - \chi \nabla^\rho \nabla_\rho \nabla^\lambda \nabla_\lambda \chi \right] \nonumber \\
= \frac{9}{2} \int d^3 \sqrt{g} \left[ 8 \tilde{\varphi}^{T \rho} \left( -\Delta + \frac{7}{\ell^2} \right) \tilde{\varphi}_\rho^T + 9 \chi \left( -\Delta + \frac{8}{\ell^2} \right) (-\Delta) \chi \right] . \tag{3.15}
\]

The last $(-\Delta)$ factor is removed by the Jacobian that arises because of the change of variables $\tilde{\varphi} \equiv \tilde{\varphi}(\tilde{\varphi}^T, \chi)$; the relevant term there is simply

\[
\int d^3 x \sqrt{g} (\nabla^\rho \chi)(\nabla_\rho \chi) = \int d^3 x \sqrt{g} \chi (-\Delta \chi) . \tag{3.16}
\]

The correction term coming from this part of the calculation is therefore of the form

\[
Z_{\tilde{\varphi}(1)}^{(s=3)} = \left[ \det \left( -\Delta + \frac{7}{\ell^2} \right)_{(3)}^T \det \left( -\Delta + \frac{8}{\ell^2} \right)_{(0)} \right]^{-\frac{1}{2}} . \tag{3.17}
\]

This cancels precisely against two of the factors in $Z_{\tilde{g}h}^{(s=3)}$ of eq. (3.9), as expected from our general considerations above. Combining the different pieces as in eq. (2.8), the total 1-loop determinant for $s = 3$ then equals

\[
Z^{(s=3)} = \left[ \det \left( -\Delta \right)_{(3)}^{TT} \right]^{-\frac{1}{2}} \left[ \det \left( -\Delta + \frac{6}{\ell^2} \right)_{(2)}^{TT} \right]^{\frac{1}{2}} . \tag{3.18}
\]
As expected, only the helicity $s$ and helicity $(s-1)$ terms therefore contribute to this determinant.

4. Quadratic Fluctuations for General Spin

The above calculation is fairly technical, and we cannot hope to generalise it directly to higher spin. However, as explained above, we expect that the contributions of $\tilde{\varphi}_{(s-2)}$ and $\sigma_{(s-2)}$ should cancel each other, and it should therefore be possible to organise the calculation in a way in which this becomes manifest. In the following we shall explain how this can be achieved. In particular, we shall explain that most of the ghost determinant will actually just cancel the quadratic contribution from $\tilde{\varphi}_{(s-2)}$.

4.1 The ghost determinant

Generalising the definition of $L^{(3)}$ in (3.10) let us define $L^{(s)}$ to be the differential operator $L^{(s)}$ appearing in the integral (2.20)

$$(L^{(s)}\xi)_{\mu_1...\mu_{s-1}} = \left(-\Delta + \frac{s(s-1)}{\ell^2}\right) \xi_{\mu_1...\mu_{s-1}} - \frac{(2s-3)}{(2s-1)} \nabla_{(\mu_1} \nabla^\lambda \xi_{\mu_2...\mu_{s-1})\lambda} \ . \quad (4.1)$$

Let us separate $\xi$ into its transverse traceless component as well as $\sigma_{(s-2)}$ as in (1.11), i.e. $\xi_{(s-1)} = \xi_{TT}^{(s-1)} + \xi_{(s-1)}^{(\sigma)}$ with

$$\xi_{\mu_1...\mu_{s-1}}^{(\sigma)} = \nabla_{(\mu_1} \sigma_{\mu_2...\mu_{s-1})} - \frac{2}{(2s-3)} g_{\mu_1\mu_2} \nabla^\lambda \sigma_{\mu_3...\mu_{s-1})\lambda} \ . \quad (4.2)$$

On the transverse traceless components $\xi_{TT}^{(s-1)}$ of $\xi_{(s-1)}$ the second term of $L^{(s)}$ vanishes, and the operator has a simple form, namely

$$L^{(s)} \xi_{TT}^{(s-1)} = \left(-\Delta + \frac{s(s-1)}{\ell^2}\right) \xi_{TT}^{(s-1)} \ . \quad (4.3)$$

In order to determine $L^{(s)} \xi_{(s-1)}$ it therefore remains to calculate $L^{(s)} \xi_{(s-1)}^{(\sigma)}$ as a differential operator on $\sigma_{(s-2)}$. Since the resulting expression will be contracted with $\xi_{(s-1)}^{(\sigma)}$, we only have to evaluate the operator up to ‘trace terms’, (i.e. terms that are proportional to $g_{\mu_i\mu_j}$ for some indices $i, j \in \{1, \ldots, s-1\}$). The calculation can be broken up into different terms. From the first term of $L^{(s)}$ we get

$$\left(-\Delta + \frac{s(s-1)}{\ell^2}\right) \nabla_{(\mu_1} \sigma_{\mu_2...\mu_{s-1})} \cong \nabla_{(\mu_1} \left(-\Delta + \frac{(s-1)(s+2)}{\ell^2}\right) \sigma_{\mu_2...\mu_{s-1})} \ , \quad (4.4)$$

where $\cong$ always denotes equality up to trace terms. The action of $(-\Delta + \frac{s(s-1)}{\ell^2})$ on the second term in (1.12) only produces a trace term.
This leaves us with evaluating the second term of $\mathcal{L}^{(s)}$. The relevant formulae are
\begin{equation}
-\frac{(2s-3)}{(2s-1)} \nabla_{\mu_1} \nabla^\lambda \nabla_{(\mu_2} \sigma_{\mu_3...\mu_{s-1}\lambda)} \cong \frac{(2s-3)}{(2s-1)} \nabla_{(\mu_1} \left(-\Delta + \frac{(s-1)(s-2)}{\ell^2}\right) \sigma_{\mu_2...\mu_{s-1}}
-\frac{(2s-3)}{(2s-1)} \nabla_{(\mu_1} \nabla_{\mu_2} \nabla^\lambda \sigma_{\mu_3...\mu_{s-1}\lambda)} , \tag{4.5}
\end{equation}
as well as
\begin{equation}
-\frac{(2s-3)}{(2s-1)} \nabla_{(\mu_1} \nabla^\lambda \left(-\frac{2}{2s-3}\right) g_{\mu_2\mu_3} \nabla^\nu \sigma_{\mu_4...\mu_{s-1}\lambda})\nu \cong \frac{2}{(2s-1)} \nabla_{(\mu_1} \nabla_{\mu_2} \nabla^\lambda \sigma_{\mu_3...\mu_{s-1}\lambda)} . \tag{4.6}
\end{equation}

Combining (4.4), (4.5) and (4.6) then leads to
\begin{equation}
(\mathcal{L}^{(s)} \xi^{(\sigma)})_{\mu_1...\mu_{s-1}} \cong \frac{1}{(2s-1)} \nabla_{(\mu_1} \left[4(s-1) \left(-\Delta + \frac{s^2-s+1}{\ell^2}\right) \sigma_{\mu_2...\mu_{s-1}}
+(5-2s) \nabla_{\mu_2} \nabla^\lambda \sigma_{\mu_3...\mu_{s-1}\lambda)} \right] . \tag{4.7}
\end{equation}

For the simplest case, $s = 3$, we have also worked out the trace piece; this is described in appendix B.1.

Now the important observation is that the differential operator in the square brackets of (4.7) acts on $\sigma_{(s-2)}$ in precisely the same way as the differential operator in (2.13) acts on $\tilde{\varphi}_{(s-2)}$. Both $\sigma_{(s-2)}$ and $\tilde{\varphi}_{(s-2)}$ are symmetric traceless, but not necessarily transverse tensors of rank $(s-2)$, and thus the eigenvalues of the two operators agree exactly, including multiplicities. As a consequence the contribution to the path integral coming from $\sigma_{(s-2)}$ cancels precisely against that arising from integrating out $\tilde{\varphi}_{(s-2)}$.

In particular, the only contributions that actually survive are those coming from $\varphi_{TT}^{(s)}$, see eq. (2.7), as well as the contribution of $\xi_{(s-1)}^{TT}$ to the ghost determinant, see (4.3). Putting these two contributions together gives the full one-loop amplitude for general $s$ in the simple form
\begin{equation}
Z^{(s)} = \left[\det \left(-\Delta + \frac{s^2-s+1}{\ell^2}\right)^{TT}_{(s)} \right]^{-\frac{3}{2}} \left[\det \left(-\Delta + \frac{s(s-1)}{\ell^2}\right)^{TT}_{(s-1)} \right]^{\frac{1}{2}} , \tag{4.8}
\end{equation}
thus proving (1.2).

5. One loop Determinants and Holomorphic Factorisation

Given the explicit formula for $Z^{(s)}$, it is now straightforward to calculate the one loop determinant on thermal AdS$_3$. As was explained in detail in [12], the relevant determinant is of the form
\begin{equation}
-\log \det \left(-\Delta + \frac{m^2}{\ell^2}\right)^{TT}_{(s)} = \int_0^\infty \frac{dt}{t} K^{(s)}(\tau, \bar{\tau}; t) \ e^{-m^2 t} , \tag{5.1}
\end{equation}
where $K^{(s)}$ is the spin $s$ heat kernel

$$
K^{(s)}(\tau, \bar{\tau}; t) = \sum_{m=1}^{\infty} \frac{\tau_2}{\sqrt{4\pi t} \left| \sin \frac{m\tau}{2} \right|^2} \cos(sm\tau_1) e^{-\frac{m^2\tau^2}{4t}} e^{-(s+1)t}.
$$

(5.2)

Here $q = e^{i\tau}$, with $\tau = \tau_1 + i\tau_2$ the complex structure modulus of the $T^2$ boundary of thermal AdS$_3$. Note that for the case at hand we have for the helicity $s$ component $m^2_s = s(s-3)$, and hence the total $t$-exponent is $s(s-3) + (s+1) = (s-1)^2$, while for the helicity $(s-1)$ component $m^2_{s-1} = s(s-1)$ and we get $s(s-1) + s = s^2$.

Performing the $t$-integral with the help of the identity

$$
\frac{1}{\sqrt{4\pi}} \int_0^{\infty} dt e^{-\frac{\alpha^2}{4t} - \beta^2 t} = \frac{1}{\alpha} e^{-\alpha \beta},
$$

(5.3)

we therefore obtain

$$
-\log \det \left(-\Delta + \frac{s(s-3)}{\ell^2} \right)_{(s)}^{TT} = \sum_{m=1}^{\infty} \frac{1}{m} \cos(sm\tau_1) e^{-m^2\tau^2_{(s-1)}}
$$

$$
= \sum_{m=1}^{\infty} \frac{2}{m} \frac{1}{|1 - q^m|^2} (q^{m\bar{s}} + \bar{q}^{m\bar{s}}),
$$

(5.4)

as well as

$$
-\log \det \left(-\Delta + \frac{s(s-1)}{\ell^2} \right)_{(s-1)}^{TT} = \sum_{m=1}^{\infty} \frac{1}{m} \cos((s-1)m\tau_1) e^{-m\tau_{2s}}
$$

$$
= \sum_{m=1}^{\infty} \frac{2}{m} \frac{q^m \bar{q}^m}{|1 - q^m|^2} (q^{m(s-1)} + \bar{q}^{m(s-1)}).
$$

(5.5)

Hence we find for

$$
-\log Z^{(s)} = \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{|1 - q^m|^2} \left[ (q^{m\bar{s}} + \bar{q}^{m\bar{s}}) - q^m \bar{q}^m (q^{m(s-1)} + \bar{q}^{m(s-1)}) \right]
$$

$$
= \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{|1 - q^m|^2} \left[ q^{m\bar{s}} (1 - q^m) + \bar{q}^{m\bar{s}} (1 - \bar{q}^m) \right]
$$

$$
= \sum_{m=1}^{\infty} \frac{1}{m} \left[ \frac{q^{m\bar{s}}}{1 - q^m} + \frac{\bar{q}^{m\bar{s}}}{1 - \bar{q}^m} \right].
$$

(5.6)

Thus the result is the sum of a purely holomorphic, and a purely anti-holomorphic expression. Expanding the denominator by means of a geometric series and noting that the sum over $m$ just gives the series expansion of the logarithm we hence obtain

$$
Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2},
$$

(5.7)

thus proving (1.3).
6. $\mathcal{W}_N$, $\mathcal{W}_\infty$ and the MacMahon Function

As is explained in [9] it is consistent to consider higher spin gauge theories with only finitely many spin fields. More specifically, the construction of [9] in terms of a Chern-Simons action based on $SL(N) \times SL(N)$ leads to a theory that has, in addition to the graviton of spin $s = 2$, a family of fields of spin $s = 3, \ldots, N$. The quadratic part of its action is just the sum of the actions $S[\varphi(s)]$ with $s = 2, \ldots, N$. The above calculation therefore implies that the corresponding one-loop determinant equals

$$Z_{SL(N)} = \prod_{s=2}^{N} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} = \chi_0(\mathcal{W}_N) \times \bar{\chi}_0(\mathcal{W}_N),$$

where $\chi_0(\mathcal{W}_N)$ is the vacuum character of the $\mathcal{W}_N$ algebra

$$\chi_0(\mathcal{W}_N) = \prod_{s=2}^{N} \prod_{n=s}^{\infty} \frac{1}{(1 - q^n)},$$

see e.g. Sec. 6.3.2 of [22]. Indeed, by the usual Poincare-Birkhoff-Witt theorem (see for example [23]), a basis for the vacuum representation of $\mathcal{W}_N$ is given by

$$W^{(N)}_{-n_1^{(N)}} \cdots W^{(N)}_{-n_{lN}^{(N)}} W^{(N-1)}_{-n_1^{(N-1)}} \cdots W^{(N-1)}_{-n_{l(N-1)}^{(N-1)}} \cdots W^{(2)}_{-n_1^{(2)}} \cdots W^{(2)}_{-n_{l_2}^{(2)}} \Omega,$$

where $W_{n}^{(K)}$ are the modes of the field of conformal dimension $K$, and

$$n_1^{(K)} \geq n_2^{(K)} \geq \cdots \geq n_{l_K}^{(K)} \geq K.$$

Here we have used that $W_{n}^{(K)} \Omega = 0$ for $n \geq -K+1$ — this is the reason for the lower bound in (6.4) — but we have assumed that there are no other null vectors in the vacuum representation; this will be the case for generic central charge $c$. We have furthermore denoted the Virasoro modes by $W_{n}^{(2)} \equiv L_n$. It is then easy to see that (6.2) is just the counting formula for the basis (6.3). Thus our one loop calculation produces the partition function of the $\mathcal{W}_N$ algebra, as suggested by the analysis of [8].

Maloney and Witten [15] have argued that the corresponding answer in the case of pure (super-)gravity should be one loop exact. Essentially, the argument is that for the representation of the (super-)Virasoro symmetry algebra of the theory corresponding to the vacuum character, the energy levels cannot be corrected. One may, in principle, have other states contributing to the partition function. However, we know semi-classically ($c \to \infty$) that there are no propagating gravity states in the bulk. Therefore any additional states that might contribute to the partition function must have energies going to infinity in the semi-classical limit, such as black hole states or other geometries. But these would correspond to non-perturbative corrections from the point of view of the bulk path integral computation.
All the ingredients of this argument are also present in our case of massless higher spin theories. We do not have any propagating states in the bulk, and the only semiclassical physical states are the generalised Brown-Henneaux excitations of the vacuum. These are boundary states and their energies are governed by the $\mathcal{W}_N$ algebra as argued above. Thus any additional contributions would be non-perturbative, and it follows that the above one loop answer is perturbatively exact.

In [8] a classical Brown-Henneaux analysis was also performed for the Blencowe theory based on (two copies of) the infinite dimensional Vasiliev higher spin algebra $hs(1,1)$ \cite{24,25}. This theory possesses one spin field for each spin $s = 2, 3, \ldots$, and thus the one-loop partition function becomes the $N \to \infty$ limit of $Z_{SL(N)}$, i.e.

$$Z_{hs(1,1)} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} = |M(q)|^2 \times \prod_{n=1}^{\infty} |1 - q^n|^2,$$

where $M(q)$ is the MacMahon function \cite{1.3}. Note, in particular, that the MacMahon function is essentially the $\mathcal{W}_\infty$ vacuum character. This connection appears not to be widely known.\footnote{This form of the character for $\mathcal{W}_\infty$ (or rather $\mathcal{W}_{1+\infty}$) appears, for instance, in \cite{26,27}, and the connection also features in the appendix of \cite{28}. We thank B. Szendrői for bringing these references to our attention.}

7. Final Remarks

We have seen that a computation of the leading quantum effects for higher spin theories on AdS$_3$ can be carried out explicitly. Our result suggests strongly that the quantum Hilbert space can be organised in terms of the vacuum representation of the $\mathcal{W}_N$ algebra. This also leads to the conclusion that this answer is perturbatively exact. Thus we have control over the quantum theory, at least to all orders in the power series expansion in Newton’s constant. It would be very interesting to understand whether the full non-perturbative quantum theory is well defined. In the case of pure gravity it was argued in \cite{15} that, under some reasonable looking assumptions, pure gravity on AdS$_3$ does not exist non-perturbatively. It would be very interesting to revisit this question for the higher spin theories considered here. In particular, one may hope that the situation could be different for the $hs(1,1)$ theory with $\mathcal{W}_\infty$ symmetry. We are currently investigating this question and hope to report on it shortly. A positive answer would probably give some impetus to investigations of these symmetries in higher dimensional AdS spacetimes.\footnote{$\mathcal{W}_N$ and $\mathcal{W}_\infty$ algebras have also appeared as spacetime symmetries of non-critical string theories, see e.g. \cite{29,30}. It would be interesting to explore the connection, if any, to the above realisations.}

In this context we find the appearance of the MacMahon function as the $\mathcal{W}_\infty$ vacuum character very significant. The MacMahon function first appeared in string
theory in the non-perturbative investigation of topological strings \[31, 32\]. It was further interpreted in terms of a quantum stringy Calabi-Yau geometry in \[33, 34\]. Perhaps, we should now interpret the ubiquitous appearance of the MacMahon function in the context of topological strings in terms of a hidden $W_\infty$ symmetry. It is also rather suggestive that the MacMahon function (together with the $\eta$-function prefactor of (6.5)) precisely accounts for the contribution of the supergravity modes to the elliptic genus of M-theory on $\text{AdS}_3 \times S^2 \times X_6$ \[33, 36\]. This might provide a concrete link between their appearance in topological strings and in $\text{AdS}_3$.

At a technical level, it might be interesting to redo the analysis of the quadratic fluctuations within the Chern-Simons formulation of the higher spin theories \[24\]. Finally, we should remark that the considerations of this paper should be straightforwardly extendable to the supersymmetric case. The techniques of \[12\] apply equally well to fermionic fields, and for example the one loop answer for $s = \frac{3}{2}$ was already explicitly worked out in \[12\], leading to a super-Virasoro character. Thus one might naturally expect to find supersymmetric versions of $W_N$ and $W_\infty$ vacuum characters for the appropriate supersymmetric higher spin theories \[37, 38\].

**Acknowledgements:** We would like to specially thank Justin David for collaboration in the initial stages of this work as well as for many discussions on related matters. We would also like to thank Dileep Jatkar and Ashoke Sen for helpful discussions. M.R.G. thanks Harvard University and CalTech for hospitality while this work was being done. His work is also supported partially by the Swiss National Science Foundation. R.G.’s work was partly supported by a Swarnajayanthe Fellowship of the Dept. of Science and Technology, Govt. of India and as always by the support for basic science by the Indian people.

**Appendix**

**A. Conventions**

In our conventions the commutator of two covariant derivatives, evaluated on a totally symmetric rank $s$ contravariant tensor, is equal to

$$\left[\nabla_\mu, \nabla_\nu\right] \xi^{\rho_1 \ldots \rho_s} = \sum_{j=1}^{s} R^{\rho_j \mu \nu}_{\delta \mu \nu} \xi^{\hat{\rho}_1 \ldots \hat{\rho}_{j-1} \rho_j \ldots \rho_s \delta}, \quad (A.1)$$

where the notation $\hat{\rho}_j$ means that $\rho_j$ is excluded. The Riemann curvature tensor for $\text{AdS}_3$ is of the form

$$R_{\mu \nu \rho \sigma} = -\frac{1}{\ell^2} \left(g_{\mu \rho}g_{\nu \sigma} - g_{\mu \sigma}g_{\nu \rho}\right). \quad (A.2)$$

The Ricci tensor is then

$$R^\mu_{\nu \mu \sigma} = -\frac{2}{\ell^2} g_{\nu \sigma}. \quad (A.3)$$
We shall also use the conventions of [9] that by an index \((\mu_1 \ldots \mu_s)\) we mean the symmetrised expression without any combinatorial factor, but with the understanding that terms that are obviously symmetric will not be repeated. So for example, the tensor \(\nabla_{(\mu_1} \xi_{\mu_2 \ldots \mu_s)}\) equals

\[
\nabla_{(\mu_1} \xi_{\mu_2 \ldots \mu_s)} = \sum_{j=1}^s \nabla_{\mu_j} \xi_{\mu_1 \ldots \hat{\mu}_j \ldots \mu_s}, \tag{A.4}
\]

if \(\xi_{(s-1)}\) is a symmetric tensor, etc. By \(\Delta\) we always mean the Laplace operator

\[
\Delta = \nabla^\lambda \nabla_\lambda. \tag{A.5}
\]

Because of (A.1), the explicit action depends on the spin \(s\) of the field on which \(\Delta\) acts.

**B. The calculation for \(s = 3\)**

In this appendix we give some of the details of the calculation of section 3. First we explain how to obtain our explicit formula (3.4) for the \(\xi\)-dependent exponent (3.1) of (2.21). To start with we plug (3.2) into (3.1) to obtain

\[
S_\xi = 3 \int d^3x \sqrt{g} \left[ \xi^{TT}_{\nu \rho} \left( -\Delta + \frac{6}{\ell^2} \right) \xi^{TT \nu \rho} 
+ \nabla_{(\nu} \sigma^{T}_{\rho)} \left( -\Delta + \frac{6}{\ell^2} \right) \nabla^{(\nu} \sigma^{T \rho)} - \frac{6}{5} \nabla_{(\nu} \sigma^{T}_{\rho)} \nabla^{\nu} \nabla_{\mu} \nabla^{(\mu} \sigma^{T \nu)} 
+ \psi_{\nu \rho} \left( -\Delta + \frac{6}{\ell^2} \right) \psi^{\nu \rho} - \frac{6}{5} \psi_{\nu \rho} \nabla^{\nu} \nabla_{\mu} \psi^{\mu \nu} \right]. \tag{B.1}
\]

Note that there are no cross-terms between \(\xi^{TT}, \sigma^{T}\) and \(\psi^{\mu \nu}\), simply because any potential index contractions lead to vanishing results on account of the tracelessness and transversality of \(\xi^{TT}\) and \(\sigma^{T \nu}\). We want to simplify the expressions in the second and third line.

First we consider the \(\sigma^{T \nu}\) terms. To this end we observe that

\[
\left( -\Delta + \frac{6}{\ell^2} \right) \nabla_{(\mu} \sigma^{T}_{\nu)} = \nabla_{(\mu} \left( -\Delta + \frac{10}{\ell^2} \right) \sigma^{T}_{\nu)}, \tag{B.2}
\]

as one checks explicitly. It then follows that the first term of the second line of (B.1) leads to

\[
S_{\xi}^{[\sigma,1]} = 3 \int d^3x \sqrt{g} \left[ \nabla_{(\nu} \sigma^{T}_{\rho)} \right] \left[ \left( -\Delta + \frac{6}{\ell^2} \right) \nabla^{(\nu} \sigma^{T \rho)} \right] 
= 6 \int d^3x \sqrt{g} \sigma^{T}_{\nu} \left( -\Delta + \frac{2}{\ell^2} \right) \left( -\Delta + \frac{10}{\ell^2} \right) \sigma^{T \nu}. \tag{B.3}
\]
In order to evaluate the second term of the second line we now calculate

$$\nabla^\mu \nabla_{(\mu} \sigma^T_{\nu)} = \Delta \sigma^T_{\nu} - \frac{2}{\ell^2} \sigma^T_{\nu} , \quad \text{(B.4)}$$

where we have used the transversality of $\sigma^{T\mu}$, (3.3). Using integration by parts we therefore get

$$S_{[\sigma,2]}^{[\xi]} = \frac{18}{5} \int d^3 x \sqrt{g} \sigma^T_{\nu} \left( -\Delta + \frac{2}{\ell^2} \right) \left( \Delta + \frac{2}{\ell^2} \right) \sigma^{Tv} . \quad \text{(B.5)}$$

Putting the two calculations together we thus arrive at the result

$$S_{[\sigma,1]}^{[\xi]} + S_{[\sigma,2]}^{[\xi]} = \frac{48}{5} \int d^3 x \sqrt{g} \sigma^T_{\nu} \left( -\Delta + \frac{2}{\ell^2} \right) \left( \Delta + \frac{7}{\ell^2} \right) \sigma^{Tv} , \quad \text{(B.6)}$$

which is the second line of (3.4).

Next we deal with the $\psi^{\mu\nu}$ terms. The analogue of (B.2) is now

$$\left( -\Delta + \frac{6}{\ell^2} \right) \psi_{\mu\nu} = \left( \nabla_\mu \nabla_\nu - \frac{1}{3} g_{\mu\nu} \Delta \right) \left( \Delta + \frac{12}{\ell^2} \right) \psi . \quad \text{(B.7)}$$

The first term of the third line then leads to

$$S_{[\psi,1]}^{[\xi]} = 3 \int d^3 x \sqrt{g} \left[ \left( \nabla_\nu \nabla_\rho - \frac{1}{3} g_{\nu\rho} \Delta \right) \psi \right] \cdot \left[ \left( \Delta + \frac{6}{\ell^2} \right) \left( \nabla^\nu \nabla^\rho - \frac{1}{3} g^{\nu\rho} \Delta \right) \psi \right]$$

$$= 2 \int d^3 x \sqrt{g} \psi \left( -\Delta \right) \left( \Delta + \frac{12}{\ell^2} \right) \psi . \quad \text{(B.8)}$$

For the second term we calculate

$$\nabla_\rho \nabla^\mu \psi_{\mu\nu} = \frac{2}{3} \nabla_\rho \nabla_\nu \left( \Delta - \frac{3}{\ell^2} \right) \psi , \quad \text{(B.9)}$$

thus leading to

$$S_{[\psi,2]}^{[\xi]} = -\frac{18}{5} \int d^3 x \sqrt{g} \left[ \left( \nabla^\nu \nabla^\rho - \frac{1}{3} g^{\nu\rho} \Delta \right) \psi \right] \cdot \left[ (\nabla_\rho \nabla^\mu) \left( \nabla_\mu \nabla_\nu - \frac{1}{3} g_{\mu\nu} \Delta \right) \psi \right]$$

$$= \frac{8}{5} \int d^3 x \sqrt{g} \psi \left( -\Delta \right) \left( \Delta + \frac{3}{\ell^2} \right) \left( \Delta + \frac{8}{\ell^2} \right) \psi . \quad \text{(B.10)}$$

Putting the two calculations together we therefore get

$$S_{[\psi,1]}^{[\xi]} + S_{[\psi,2]}^{[\xi]} = \frac{18}{5} \int d^3 x \sqrt{g} \psi \left( -\Delta \right) \left( \Delta + \frac{3}{\ell^2} \right) \left( \Delta + \frac{8}{\ell^2} \right) \psi , \quad \text{(B.11)}$$

which is the third line of (3.4).
B.1 The full action of $L^{(3)}$

In this appendix we work out the full action of $L^{(3)}$, including the trace piece. Actually, in order to do this calculation efficiently, it is convenient to modify $L^{(3)}$ by a trace piece so that it maps traceless tensors to traceless tensors. The resulting operator is

$$\left(\hat{L}^{(3)}\xi\right)_{\mu\rho} = \left(-\Delta + \frac{6}{\ell^2}\right)\xi^{\mu\rho} - \frac{3}{5}(\nabla^{\nu}\nabla_\mu\xi^{\mu\rho} + \nabla^{\rho}\nabla_\mu\xi^{\mu\nu}) + \frac{2}{5}g^{\mu\rho}\nabla_\alpha\nabla_\beta\xi^{\alpha\beta}.$$  \hfill (B.12)

Next we consider the action of $\hat{L}^{(3)}$ on the traceless tensor

$$\xi^{(\sigma)\mu\nu} \equiv \nabla^\mu\sigma^\nu + \nabla^\nu\sigma^\mu - \frac{2}{3}g^{\mu\nu}\nabla_\alpha\sigma^\alpha.$$  \hfill (B.13)

After a lengthy calculation one finds

$$\left(\hat{L}^{(3)}\xi^{(\sigma)\mu\nu}\right)_{\rho} = \frac{1}{5}\left[8\nabla^{(\nu}\left(-\Delta + \frac{7}{\ell^2}\right)\sigma^{\rho)} - \nabla^{(\nu}\nabla^{\rho)}\left(\nabla_\lambda\sigma^\lambda\right) - 6g^{\nu\rho}\nabla_\lambda\left(-\Delta + \frac{6}{\ell^2}\right)\sigma^\lambda\right].$$  \hfill (B.14)

The first two terms obviously agree with (4.7) for $s = 3$. To understand how to obtain (B.14) let us look at the various terms of $\hat{L}^{(3)}$ separately: from the first term of $\hat{L}^{(3)}$ one gets

$$\left(-\Delta + \frac{6}{\ell^2}\right)\xi^{(\sigma)\mu\nu} = \nabla^{(\nu}\left(-\Delta + \frac{10}{\ell^2}\right)\sigma^{\rho)} + \frac{2}{3}g^{\nu\rho}\nabla_\lambda\Delta\sigma^\lambda - \frac{20}{3}\ell^2g^{\nu\rho}\nabla_\lambda\left(\nabla_\chi\sigma^\chi\right).$$  \hfill (B.15)

One easily checks that the right-hand side is indeed traceless, as must be. We group the remaining terms into two traceless parts, namely

$$-\frac{3}{5}\left(\nabla^\nu\nabla_\mu\nabla^{(\mu}\sigma^{\rho)} + \nabla^{\rho}\nabla_\mu\nabla^{(\mu}\sigma^{\nu)}\right) + \frac{2}{5}g^{\nu\rho}\nabla_\lambda\nabla_\nu\nabla^{(\lambda}\sigma^{\nu)}$$

$$= \frac{3}{5}\nabla^{(\nu}\left(-\Delta + \frac{2}{\ell^2}\right)\sigma^{\rho)} - \frac{3}{5}\nabla^{(\nu}\nabla^{\rho)}(\nabla_\lambda\sigma^\lambda) + \frac{4}{5}g^{\nu\rho}\nabla_\lambda\Delta\sigma^\lambda,$$ \hfill (B.16)

and

$$-\frac{3}{5}\left(\nabla^\nu\nabla_\mu(-\frac{2}{3})g^{\mu\rho}(\nabla_\lambda\sigma^\chi) + \nabla^\rho\nabla_\mu(-\frac{2}{3})g^{\mu\nu}(\nabla_\lambda\sigma^\lambda)\right) + \frac{2}{5}g^{\nu\rho}\nabla_\lambda\nabla_\tau(-\frac{2}{3})g^{\lambda\tau}(\nabla_\alpha\sigma^\alpha)$$

$$= \frac{2}{5}\nabla^{(\nu}\nabla^{\rho)}\left(\nabla_\lambda\sigma^\lambda\right) - \frac{4}{15}g^{\nu\rho}\nabla_\lambda\left(\Delta + \frac{2}{\ell^2}\right)\sigma^\lambda.$$  \hfill (B.17)

One then easily checks that the sum of (B.13), (B.16) and (B.17) gives indeed the right hand-side of (B.14).

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