Structure Constants and Conformal Bootstrap in Liouville Field Theory

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Abstract

An analytic expression is proposed for the three-point function of the exponential fields in the Liouville field theory on a sphere. In the classical limit it coincides with what the classical Liouville theory predicts. Using this function as the structure constant of the operator algebra we construct the four-point function of the exponential fields and verify numerically that it satisfies the conformal bootstrap equations, i.e., that the operator algebra thus defined is associative. We consider also the Liouville reflection amplitude which follows explicitly from the structure constants.

1. Introduction

Since early 80’s when the two-dimensional Liouville field theory (LFT) was recognized [1] as the effective field theory of the 2D quantum gravity considerable efforts has been directed at this area, especially for its relation to the string theory [1] (see e.g. refs.[2–4]). However despite significant progress in understanding the situation up to now the solution
to LFT is generally lacking. E.g., the structure of the operator algebra and the correlation functions in the general case are still unknown.

Interest in LFT was renewed recently with the development of the alternative matrix model approach to the 2D gravity [5,6]. As the result of this new activity it was shown that LFT is able to reproduce some of the predictions of the matrix model approach, in particular the scaling behavior [7–9], the genus one partition functions [10] and some of the integrated correlation functions [11–15]. We would like to emphasize refs.[12] and [14,15] where the analytic continuation in the order of perturbation theory has been used. This approach is conceptually close to the way we make our guess about the structure constants.

The content below is arranged as follows. In sect.2 the LFT on a sphere is introduced and some notations are defined. In sect.3 we propose an exact expression for the LFT three-point function and discuss some of its properties. The Liouville reflection amplitude is also introduced here. It should be stressed that the arguments of this section have nothing to do with a derivation. These are rather some motivations and we consider the expression proposed as a guess which we try to support in the subsequent sections. This guess appears quite natural\(^1\) and might even be thought obvious to those concerned with the problem. What we believe does promote it to a step forward are the various tests performed in sects.4–7.

Sect.4 contains some calculations in the semiclassical limit. The physical content of the Liouville reflection amplitude is discussed in sect.5 and in sect.6 we make a check of this amplitude by means of the thermodynamic Bethe ansatz for the high-temperature sinh-Gordon model.

The exact LFT structure constants together with an effective numerical algorithm for the conformal block permit us to compute numerically the four-point function of the LFT exponential fields. We perform this in sect.7 and verify that the four-point function satisfies the conformal bootstrap equations, i.e. the necessary conditions of the associativity of the operator algebra. Sect.8 contains some considerations about the classical Liouville action and the related problem of accessory parameters.

2. Liouville field theory

Local properties of the LFT are derived from the Lagrangian density\(^2\)

\[
\mathcal{L} = \frac{1}{4\pi} (\partial_\mu \phi)^2 + \mu e^{2b\phi}
\]  

(2.1)

\(^1\) Compare (3.14) below with the formulae for the structure constants of “minimal CFT”[16] and other “rational CFT”[17] which were taken as “architectural prototypes” for (3.14).

\(^2\) It is conventional to add also another term \( \frac{Q}{4\pi} \hat{R} \sqrt{\hat{g}} \phi \) to the Liouville Lagrangian density, with \( \hat{g} \) and \( \hat{R} \) being arbitrary ”background metric” and associated curvature, and with the parameter \( Q \) adjusted to ensure that all physical quantities be independent of a particular choice of this ”background” (see [8,9]). However, it is always possible to choose a specific background which is flat everywhere except for few selected points; the above term translates then into appropriate ”boundary” terms, as in (2.7) and (2.33) below.
where \( b \) is the dimensionless Liouville coupling constant and the scale parameter \( \mu \) is usually called the cosmological constant. Below we often use the complex euclidean coordinates \( z = x_1 + ix_2; \bar{z} = x_1 - ix_2 \) and denote \( \vartheta = \partial / \partial z; \bar{\vartheta} = \partial / \partial \bar{z} \). The Liouville field \( \phi(z, \bar{z}) \) is not exactly a scalar but varies under holomorphic coordinate transformations \( z \to w(z) \) as

\[
\phi(w, \bar{w}) = \phi(z, \bar{z}) - \frac{Q}{2} \log \left| \frac{dw}{dz} \right|^2
\]

where

\[
Q = b + 1/b
\]

The holomorphic Liouville stress tensor

\[
\begin{align*}
T(z) &= -(\partial \phi)^2 + Q \vartheta^2 \phi \\
\bar{T}(\bar{z}) &= -\bar{\partial} \phi^2 + Q \bar{\vartheta}^2 \phi
\end{align*}
\]

ensures local conformal invariance [18] of LFT with the Liouville central charge

\[
c_L = 1 + 6Q^2
\]

To define LFT globally one has to specify boundary conditions. The LFT on a sphere corresponds to \( \phi \) defined on the whole complex plane with the following asymptotic behavior at \( |z| \to \infty \)

\[
\phi(z, \bar{z}) = -Q \log(z\bar{z}) + O(1) \quad \text{at} \quad |z| \to \infty
\]

For specific calculations it is sometimes convenient to set this asymptotic behavior by considering LFT on a large disk \( \Gamma \) of radius \( R \to \infty \) and adding a boundary term to the Liouville action

\[
A_L = \frac{1}{4\pi} \int_\Gamma \left[ (\partial_a \phi)^2 + 4\pi \mu e^{2b\phi} \right] d^2x + \frac{Q}{\pi R} \int_{\partial \Gamma} \phi dl + 2Q^2 \log R
\]

The last constant term is introduced to make the action finite at \( R \to \infty \). This type of boundary condition is conventionally called the background charge \( -Q \) at infinity.

Exponential Liouville operators

\[
V_\alpha(x) = e^{2\alpha \phi(x)}
\]

are the spinless primary conformal fields of dimensions

\[
\Delta_\alpha = \alpha(Q - \alpha)
\]

Note that the field \( V_{Q-\alpha} \) has the same dimension as (2.8). These two fields are closely related and we shall call \( V_{Q-\alpha} \) the reflection image of \( V_\alpha \) and vice versa. In particular the perturbation operator \( V_b \) in (2.1) have dimension 1 together with its reflection image \( V_{1/b} \). The case \( \alpha = Q/2 \) is degenerate. Here we have two primary fields

\[
V_{Q/2}(x) = e^{Q\phi(x)}
\]
and
\[ U_{Q/2}(x) = \frac{1}{2} \frac{\partial}{\partial \alpha} V_\alpha(x) |_{\alpha=Q/2} = \phi(x)e^{Q\phi(x)} \] (2.11)
of dimension \(Q^2/4\). The last field (2.11) is called sometimes the puncture operator in LFT [11].

The \(n\)-point function of the exponential fields on a sphere is formally defined as a functional integral
\[ G_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \int V_{\alpha_1}(x_1) \ldots V_{\alpha_n}(x_n)e^{-A_L[^{\phi}]D\phi} \] (2.12)
over the fields \(\phi(x)\) with the boundary condition (2.6). Note that in the definition (2.12) we do not divide the right hand side by the zero-point function \(Z_0\).

The scale (\(\mu\)) dependence of any correlation function (2.12) [7–9]
\[ G_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = (\pi\mu)^{(Q-S/\alpha_i)/b} F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) \] (2.13)
(with \(F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)\) independent on \(\mu\)) is easily derived from the action (2.7) and the operator product expansion
\[ \phi(z,\bar{z})V_\alpha(x) = -\alpha \log |z-x|^2 V_\alpha(x) + \ldots \] (2.14)
Sometimes it is convenient to consider the \(n\)-point functions \(G^{(A)}_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)\) with fixed area
\[ A = \int e^{2b\phi} d^2x \] (2.15)
These observables are related to (2.12) as
\[ G_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \int_0^\infty G^{(A)}_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)e^{-\mu A}dA \] (2.16)
so that
\[ G^{(A)}_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) = \left( \frac{A}{\pi} \right)^{(\sum \alpha_i - Q)/b} F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) \frac{\Gamma((Q-S/\alpha_i)/b)}{\Gamma(Q-S/\alpha_i)/b} \] (2.17)

Conformal invariance restricts to some extent the \(x\)-dependence of the correlation functions [18]. In particular the three-point function is specified up to an \(x\)-independent constant \(C(\alpha_1,\alpha_2,\alpha_3)\)
\[ G_{\alpha_1,\alpha_2,\alpha_3}(x_1, x_2, x_3) = |x_{12}|^{2\gamma_1} |x_{23}|^{2\gamma_2} |x_{31}|^{2\gamma_3} C(\alpha_1, \alpha_2, \alpha_3) \] (2.18)
where \(\gamma_1 = \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3}, \gamma_2 = \Delta_{\alpha_3} - \Delta_{\alpha_3} - \Delta_{\alpha_1}, \gamma_3 = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}\) and here and below we denote \(x_{ij} = x_i - x_j\). The four-point function can be reduced to a function of only one coordinate variable, the projective invariant of the four points \(x_1,\ldots,x_4\)
\[ x = \frac{x_{12}x_{34}}{x_{14}x_{32}} \] (2.19)
as follows
\[ G_{\alpha_1,\ldots,\alpha_4}(x_1,\ldots,x_4) = \prod_{i<j} |x_{ij}|^{2\gamma_{ij}} G_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,\bar{x}) \] (2.20)

where \( \gamma_{12} = \gamma_{13} = 0, \gamma_{14} = -2\Delta_{\alpha_1}, \gamma_{24} = \Delta_{\alpha_1} + \Delta_{\alpha_3} - \Delta_{\alpha_2} - \Delta_{\alpha_4}, \gamma_{34} = \Delta_{\alpha_1} + \Delta_{\alpha_2} - \Delta_{\alpha_3} - \Delta_{\alpha_4} \) and \( \gamma_{23} = \Delta_{\alpha_4} - \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3} \). Function \( G_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,\bar{x}) \) satisfies the following symmetries (sometimes called the crossing symmetry relations) [18]
\[ G_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,\bar{x}) = G_{\alpha_1,\alpha_3,\alpha_2,\alpha_4}(1-x,1-\bar{x}) = |x|^{-4\Delta_{\alpha_1}} G_{\alpha_1,\alpha_4,\alpha_3,\alpha_2}(1/x,1/\bar{x}) \] (2.21)

In principle one can separate two pairs of operators in the four-point function, say \( V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) \) and \( V_{\alpha_3}(x_3)V_{\alpha_4}(x_4) \) and represent \( G_{\alpha_1,\ldots,\alpha_4}(x_1,\ldots,x_4) \) as a sum over intermediate physical states [18]. As was established long ago [2,3] the physical LFT space of states \( \mathcal{A} \) consists of a continuum variety of primary states corresponding to operators \( V_{\alpha} \) with
\[ \alpha = \frac{Q}{2} + iP \] (2.22)

\( (P \) is real) and the conformal descendants of these states. The corresponding expression for the four-point function looks as
\[ G_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,\bar{x}) = \frac{1}{2} \int_{-\infty}^{\infty} C(\alpha_1,\alpha_2,Q/2+iP)C(\alpha_3,\alpha_4,Q/2-iP)|F(\Delta_{\alpha_1},\Delta,x)|^2dP \] (2.23)

Here \( C(\alpha_1,\alpha_2,\alpha_3) \) are the structure constants of eq.(2.18) while \( F(\Delta_{\alpha_1},\Delta,x) \) is the so-called conformal block [18] which sums up all the intermediate descendant states of a given primary one. The conformal block is determined completely by the conformal symmetry of the theory. It depends on the corresponding central charge (2.5), on the dimension
\[ \Delta = \frac{Q^2}{4} + P^2 \] (2.24)
of the intermediate primary state and also on the “external” dimensions \( \Delta_{\alpha_i}, i = 1,\ldots,4 \). Unfortunately up to now this function is not known in a closed form. However it is straightforward to evaluate it as a power series in \( x \) [18]. Also there are very efficient algorithms for its numerical computation [19].

One important remark about Eq.(2.23) is in order. As we will see (sect.3) the structure constant \( C(\alpha_1,\alpha_2,\alpha) \) exhibits infinitely many poles in the variable \( \alpha \), their positions being dependent on \( \alpha_1,\alpha_2 \). As one changes \( \alpha_1,\alpha_2 \) and \( \alpha_3,\alpha_4 \) in (2.23) the associated poles of the integrand move around and some of them can cross the integration contour \( \Im mP = 0 \). Therefore the decomposition (2.23) can be taken literally only if \( \alpha_1,\alpha_2 \) and \( \alpha_3,\alpha_4 \) take their values within certain domains, namely
\[ |\alpha_1 - \alpha_2| < Q/2; \quad |Q - \alpha_1 - \alpha_2| < Q/2; \]
\[ |\alpha_3 - \alpha_4| < Q/2; \quad |Q - \alpha_3 - \alpha_4| < Q/2. \] (2.25)
(here we consider the case of real \( \alpha \)'s). Otherwise the four-point function (2.23) has to be understood as analytic continuation away from the domain (2.25). In the course of this continuation some poles of \( C(\alpha_1, \alpha_2, Q/2 + iP) \) and/or of \( C(\alpha_3, \alpha_4, Q/2 + iP) \) cross the real \( P \) axis and the r.h.s. of (2.23) acquires additional “discrete” terms associated with the residues of the integrand at these poles (this seems to agree with qualitative analysis in [11]). We will discuss this phenomenon in greater details elsewhere.

Of course one can pick up another partition of the four operators into pairs and obtain an expression similar to (2.23) summing up over the intermediate states in the corresponding channel. The resulting four-point function must be the same. In other words the four-point function defined as in eq.(2.23) has to satisfy the crossing symmetry relations (2.21). From this point of view these relations appear as a set of non-trivial restrictions for the structure constants \( C(\alpha_1, \alpha_2, \alpha_3) \) which express the associativity of the operator algebra and are known as the conformal bootstrap equations [18]. We discuss more about this point in sect.7. Representations like (2.23) can be written down for higher multipoint functions as well. However they involve the multipoint conformal blocks which are much more complicated objects then the four-point one. Fortunately, the corresponding crossing symmetry relations are not expected to bring up any new restrictions on the structure of the operator algebra [18].

Let us finish this section with few words about the classical limit \( b \rightarrow 0 \) of LFT. Here it is more convenient to use the field

\[
\varphi = 2b \phi \tag{2.26}
\]

which becomes a classical Liouville field in this limit. Its dynamics is governed by the classical action

\[
S_{\text{Liouv}}[\varphi] = b^2 A_L[\phi] \quad \text{at} \quad b \rightarrow 0 \tag{2.27}
\]

Explicitly

\[
S_{\text{Liouv}}[\varphi] = \frac{1}{8\pi} \int_G \left[ \frac{1}{2} (\partial_\alpha \varphi)^2 + 8\pi \mu b^2 e^{\varphi} \right] d^2 x + \varphi_\infty + 2\log R \tag{2.28}
\]

where

\[
\varphi_\infty = \frac{1}{2\pi R} \int_{\partial G} \varphi dl \tag{2.29}
\]

The field \( \varphi(x) \) satisfies the classical Liouville equation

\[
\partial \bar{\partial} \varphi = 2\pi \mu b^2 e^{\varphi} \tag{2.30}
\]

and locally describes a surface of constant negative curvature \(-8\pi \mu b^2\).

The leading (exponential) asymptotic of the \( n \)-point function (2.12) in the classical limit

\[
G_{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n) \sim \exp \left( -\frac{1}{b^2} S^{(cl)} \right) \tag{2.31}
\]

is governed by the classical Liouville action on an appropriate solution to the Liouville equation. Note that the insertion of any \( V_{\alpha_i}(x_i) \) in (2.12) affects the classical field dynamics.
only if the corresponding $\alpha_i$ is “heavy”, i.e. if $\alpha_i = \eta_i/b$ is of order $O(b^{-1})$. Technically one has to distinguish two possibilities. If $\sum_i \eta_i > 1$ a classical solution with negative curvature exists and in (2.30)

$$S^{(cl)}_{\eta_1, \ldots, \eta_n}(x_1, \ldots, x_n) = S_{\text{Liouv}}[\varphi_{\eta_1, \ldots, \eta_n}(x|x_1, \ldots, x_n)]$$

where $\varphi_{\eta_1, \ldots, \eta_n}(x|x_1, \ldots, x_n)$ is a solution to (2.30) with the following boundary conditions

$$\varphi(z, \bar{z}) = -2 \log |z|^2 + O(1) \quad \text{at} \quad |z| \to \infty$$
$$\varphi(z, \bar{z}) = -2 \eta_i \log |z - x_i|^2 + O(1) \quad \text{at} \quad z \to x_i$$

This field configuration is singular at $z \to x_i$. Therefore it is better to cut out small disks of radius $\epsilon_i$ around each point $x_i$ and define the regularized Liouville action on the remaining part $\Gamma$ of the complex plane

$$S_{\text{Liouv}}[\varphi] = \frac{1}{8\pi} \int_{\Gamma} \left[ \frac{1}{2} (\partial_a \varphi)^2 + 8\pi \mu b^2 e^\varphi \right] d^2x + \varphi_\infty + 2 \log R - \sum_i (\eta_i \varphi_i + 2 \eta_i^2 \log \epsilon_i)$$

Here $\varphi_\infty$ is as in eq.(2.29) while the boundary terms with

$$\varphi_i = \frac{1}{2\pi \epsilon_i} \int_{\partial \Gamma_i} \varphi dl$$

are added for each small circle to ensure the behavior (2.33) near $x_i$. Also we include some field independent terms such that $S_{\text{Liouv}}$ is finite and independent on $\epsilon_i$ at $\epsilon_i \to 0$.

At $\sum \eta_i < 1$ there is no solution to (2.30) and (2.33) with negative curvature. In this case it is relevant to consider the $n$-point function (2.17) of fixed area. The leading classical behavior is again governed by the classical action

$$G^{(A)}_{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n) \sim \exp \left( -\frac{1}{b^2} S^{(cl)}_A \right)$$

but now one has to impose the constant area constraint

$$A = \int_{\Gamma} e^\varphi d^2x$$

and solve the positive curvature Liouville equation

$$\partial \bar{\partial} \varphi = \frac{2\pi}{A} \left( \sum \eta_i - 1 \right) e^\varphi$$

with the same boundary conditions (2.33). The corresponding constant area Liouville action is again (2.34) but without the cosmological term $\mu b^2 A$ so that

$$S_{\text{Liouv}}[\varphi] = S^{(cl)}_A[\varphi] + \mu b^2 \int_{\Gamma} e^\varphi d^2x$$
Note that the same fixed area calculation can be performed in the first case \( \sum \eta_i > 1 \) too. The integral over \( A \) in eq.(2.16) is now dominated by the stationary point \( A = \left( \sum \eta_i - 1 \right) / (\mu b^2) \) which corresponds to the classical solution of eq.(2.30).

Suppose now that the multipoint function contains several “heavy” operators with \( \alpha_i = \eta_i / b \) and also a number of “light” ones, i.e. the Liouville exponentials with \( \alpha_j = \sigma_j b \) of the order \( O(b) \). In the classical limit these “light” fields influence neither the classical solution nor the one-loop correction. Therefore one expects that as \( b \to 0 \)

\[
\frac{G_{\sigma_1 b, \ldots, \sigma_l b, \eta_1 / b, \ldots, \eta_n / b}(y_1, \ldots, y_l, x_1, \ldots, x_n)}{G_{\eta_1 / b, \ldots, \eta_n / b}(x_1, \ldots, x_n)} = \prod_{j=1}^{l} e^{\sigma_j \varphi_{\eta_1, \ldots, \eta_l}(y_j|x_1, \ldots, x_n)}
\]

where \( \varphi_{\eta_1, \ldots, \eta_l}(x|x_1, \ldots, x_n) \) is the classical solution for the “heavy” operator configuration. Special care is required if the number of the “heavy” operators is less than 3. In this case the functional integral (2.12) has zero modes which have to be integrated out explicitly even in the classical limit. In particular, if there are no “heavy” operators at all the relevant fixed area classical solution is the 2D sphere metric of area \( A \)

\[
\varphi_0(z, \bar{z}) = \log \frac{A}{\pi(1 + z\bar{z})^2}
\]

This solution has to be integrated over its \( SL(2, C) \) orbit parameterized by four complex numbers \( a, b, c, d \) with the constraint \( ad - bc = 1 \)

\[
\varphi_0(z, \bar{z}|a, b, c, d) = \log \frac{A}{\pi(|az + b|^2 + |cz + d|^2)^2}
\]

This leads to the following expression for the \( n \)-point function of “light” operators in the classical limit [20]

\[
\frac{G_{\sigma_1 b, \ldots, \sigma_n b}(x_1, \ldots, x_n)}{Z_0^{(A)}} = \int \prod_{i=1}^{n} e^{\sigma_i \varphi_0(x_i, \bar{x}_i|a, b, c, d)} d\mu(a, b, c, d)
\]

where \( Z_0^{(A)} \) is the fixed area zero-point function (the partition function of the sphere) and \( d\mu(a, b, c, d) \) stands for the invariant measure on \( SL(2, C) \),

\[
d\mu(a, b, c, d) = 4d^2a \, d^2b \, d^2c \, d^2d \, \delta^{(2)}(ad - bc - 1).
\]

### 3. Three-point function

Naively one can try to expand the \( N \)-point function (2.12) into a perturbative series in the cosmological constant \( \mu \)

\[
G_{\alpha_1, \ldots, \alpha_N}(x_1, \ldots, x_N) = \sum_{n=0}^{\infty} G_{\alpha_1, \ldots, \alpha_N}^{(n)}(x_1, \ldots, x_N)
\]
where
\[ G_{\alpha_1,\ldots,\alpha_N}^{(n)} = \frac{(-\mu)^n}{n!} \int \langle V_{\alpha_1}(x_1) \ldots V_{\alpha_N}(x_N) V_b(u_1) \ldots V_b(u_n) \rangle d^2u_1 \ldots d^2u_n \] (3.2)

and \( \langle \ldots \rangle \) denotes the functional integral over a free field (i.e., (2.1) at \( \mu = 0 \))
\[ \langle V_{\alpha_1}(x_1) \ldots V_{\alpha_N}(x_N) \rangle = \prod_{i=1}^N \int e^{2\alpha_i \phi(x_i)} \exp \left( -\frac{1}{4\pi} \int (\partial_a \phi)^2 d^2x \right) D\phi \] (3.3)

However, it is well known that in LFT expression (3.1) in general does not work. The problem is that the free field functional integral (3.3) for the \( n \)-th term (3.2) matches the spherical boundary condition (2.6) only if
\[ \sum_{i=1}^N \alpha_i = Q - nb \] (3.4)

Following refs.[12–14] we shall interpret eq.(3.4) as a kind of “on-mass-shell” condition. Namely, \( G_{\alpha_1,\ldots,\alpha_N}(x_1, \ldots, x_N) \) exhibit a pole in the variable \( \alpha = \sum \alpha_i \) every time eq.(3.4) is satisfied for \( n = 0, 1, 2, \ldots \), the residue being specified by the corresponding perturbative integral
\[ \sum_{\alpha_i = Q-nb} \text{res} \ G_{\alpha_1,\ldots,\alpha_N}(x_1, \ldots, x_N) = G_{\alpha_1,\ldots,\alpha_N}^{(n)}(x_1, \ldots, x_N) \bigg|_{\sum_{\alpha_i = Q-nb}} \] (3.5)

It seems unlikely that the on-mass-shell condition (3.5) alone is enough to determine the \( N \)-point function. The analysis is hampered by the quite complicated and in general unknown analytic structure of the multipoint perturbative integrals (3.2). The situation is simplified in the three-point case \( N = 3 \). The \( x \)-dependence of the on-mass-shell integrals turns out to be the same as that of the conformal three-point function (2.18)
\[ G_{\alpha_1,\alpha_2,\alpha_3}^{(n)}(x_1, x_2, x_3) \bigg|_{\sum_{\alpha_i = Q-nb}} = |x_{12}|^{2\gamma_3} |x_{23}|^{2\gamma_1} |x_{31}|^{2\gamma_2} I_n(\alpha_1, \alpha_2, \alpha_3) \] (3.6)

with the same \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) as in eq.(2.18), while \( I_n(\alpha_1, \alpha_2, \alpha_3) \) has been carried out explicitly in ref.[21]
\[ I_n(\alpha_1, \alpha_2, \alpha_3) = \left( \frac{-\pi \mu}{\gamma(-b^2)} \right)^n \prod_{j=1}^{n-1} \frac{\gamma(-jb^2)}{\prod_{k=0}^{n-1} [\gamma(2\alpha_1 b + kb^2) \gamma(2\alpha_2 b + kb^2) \gamma(2\alpha_3 b + kb^2)]} \] (3.7)

Here and below the standard notation
\[ \gamma(x) = \Gamma(x)/\Gamma(1-x) \] (3.8)
is used. The on-mass-shell condition now reads

$$\sum_{\alpha_i = Q - nb} C(\alpha_1, \alpha_2, \alpha_3) = I_n(\alpha_1, \alpha_2, \alpha_3)$$  \hspace{1cm} (3.9)$$

At this step we need to introduce some special function $\Upsilon(b, x)$. Below we consider $b$ as a parameter and suppress it in the notation. In the strip $0 < \text{Re} x < Q$ function $\Upsilon(x)$ is defined by the integral representation

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left( \frac{Q}{2} - x \right)}{2 \sinh \frac{t}{2b}} \right]$$  \hspace{1cm} (3.10)$$

From the definition it is clear that

$$\Upsilon(x) = \Upsilon(Q - x)$$
$$\Upsilon(Q/2) = 1$$  \hspace{1cm} (3.11)$$

and that $\Upsilon(x)$ is “self-dual”, i.e. remains unchanged if $b \to 1/b$. Also the following functional relations are derived

$$\Upsilon(x + b) = \gamma(bx) b^{1 - 2bx} \Upsilon(x)$$  \hspace{1cm} (3.12a)$$
$$\Upsilon(x + 1/b) = \gamma(x/b) b^{2x/b - 1} \Upsilon(x)$$  \hspace{1cm} (3.12b)$$

Using (3.10) and (3.12) it is easy to verify that $\Upsilon(x)$ is an entire function of $x$ with zeroes located at $x = -m/b - nb$ and $x = (m + 1)/b + (n + 1)b$, where $m$ and $n$ run over all non-negative integers. Below we also use the notation

$$\Upsilon_0 = \frac{d \Upsilon(x)}{dx} \bigg|_{x=0}$$  \hspace{1cm} (3.13)$$

With the relations (3.12) it is straightforward to verify that

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[ \pi \mu \gamma(b^2) b^{2 - 2b^2} \right]^{(Q - \sum \alpha_i)/b} \times$$

$$\frac{\Upsilon_0 \Upsilon(2\alpha_1) \Upsilon(2\alpha_2) \Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q) \Upsilon(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon(\alpha_3 + \alpha_1 - \alpha_2)}$$  \hspace{1cm} (3.14)$$

satisfies the on-mass-shell condition (3.9). We propose this expression as the exact three-point function in LFT.

As a function of $\alpha = \sum_{i=1}^3 \alpha_i$ expression (3.14) has more poles then predicted by eq.(3.9). They appear at $\alpha = Q - m/b - nb$ and at $\alpha = 2Q + m/b + nb$ for any pair of non-negative integers $m$ and $n$. It seems suggestive to note that the corresponding residues are related to the more general multiple integrals, also evaluated explicitly in ref.[21]. Namely

$$\sum_{\alpha_i = Q - m/b - nb} G_{\alpha_1, \alpha_2, \alpha_3}(x_1, x_2, x_3) =$$

$$\frac{(-\bar{\mu})^m (-\mu)^n}{m! n!} \int \left( \prod_{i=1}^3 V_{\alpha_1}(x_i) \prod_{j=1}^m V_{1/b}(u_j) \prod_{j=1}^n V_b(v_j) \right) \prod_{i=1}^3 d^2 u_i \ldots d^2 u_m d^2 v_1 \ldots d^2 v_n$$  \hspace{1cm} (3.15)$$
where the “dual” cosmological constant $\tilde{\mu}$ is related to $\mu$ as

$$\pi \tilde{\mu} \gamma(1/b^2) = (\pi \mu \gamma(b^2))^{1/b^2}$$

(3.16)

The whole expression (3.14) is self-dual in the sense that it is invariant under the substitution $b \to 1/b$, $\mu \to \tilde{\mu}$.

The three-point function (3.14) reveals some interesting features. In particular the reflection $\alpha \to Q - \alpha$ of each of the three operators introduces the Liouville reflection amplitude $S(P)$

$$C(Q - \alpha_1, \alpha_2, \alpha_3) = C(\alpha_1, \alpha_2, \alpha_3)S(i\alpha_1 - iQ/2)$$

(3.17)

which reads explicitly

$$S(P) = - (\pi \mu \gamma(b^2))^{-2iP/b} \frac{\Gamma(1 + 2iP/b)\Gamma(1 + 2iPb)}{\Gamma(1 - 2iP/b)\Gamma(1 - 2iPb)}$$

(3.18)

This function is discussed more in sect.5. We shall also use the notation

$$G(\alpha) = S(-i\alpha + iQ/2) = \frac{(\pi \mu \gamma(b^2))^{(Q-2\alpha)/b}}{b^2} \gamma(2 - 2\alpha/b + 1/b^2)$$

(3.19)

associating $G(\alpha)$ with the two-point function of operators $V_\alpha(x_1)V_\alpha(x_2)$ in LFT.

There are several different classical limits of the three-point function (3.14) dependent on how the parameters $\alpha_i$ behave as $b \to 0$. We consider here two cases. First, let all the three operators be “heavy”, i.e. $\alpha_i = \eta_i/b$ at $b \to 0$ with $\eta_i$ fixed. The leading asymptotic is

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \exp \left( \frac{1}{b^2} S^{(cl)}(\eta_1, \eta_2, \eta_3) \right)$$

(3.20)

where

$$S^{(cl)}(\eta_1, \eta_2, \eta_3) = (\sum_{i=1}^3 \eta_i - 1) \log(\pi \mu b^2) + F(\eta_1 + \eta_2 + \eta_3 - 1) + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_2 + \eta_3 - \eta_1) + F(\eta_3 + \eta_1 - \eta_2) - F(0) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3)$$

(3.21)

and we have denoted

$$F(\eta) = \int_{1/2}^{\eta} \log \gamma(x)dx$$

(3.22)

Note that $S^{(cl)}(\eta_1, \eta_2, \eta_3)$ vanishes if $\sum_{i=1}^3 \eta_i = 1$.

In the opposite case of three “light” exponentials with $\alpha_i = b\sigma_i$ of the order $O(b)$ it is more relevant to consider the fixed area three-point function (2.17). For this we find at $b \to 0$

$$C^{(A)}(b\sigma_1, b\sigma_2, b\sigma_3) = \frac{A}{\pi} \sum e^{1/b^2} e^{-2C} \frac{\sqrt{2\pi b^4}}{\Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 1)\Gamma(\sigma_1 + \sigma_2 - \sigma_3)\Gamma(\sigma_2 + \sigma_3 - \sigma_1)\Gamma(\sigma_3 + \sigma_1 - \sigma_2)}$$

(3.23)
where $C$ is the Euler’s constant.

We present also the classical limit of the fixed area two-point function $G^{(A)}(\alpha)$, related to (3.19) as in eq.(2.16). In the “heavy” limit $\alpha = \eta/b$, $b \to 0$ the leading exponential asymptotic

$$G^{(A)}(\eta/b) \sim \exp \left( -\frac{1}{b^2} S^{(cl)}(\eta) \right)$$ (3.24)

is governed by the classical action

$$S^{(cl)}(\eta) = (1 - 2\eta) \left( \log \frac{A}{\pi} + \log(1 - 2\eta) - 1 \right)$$ (3.25)

The fixed area two-point function of two “light” operators with $\alpha = b\sigma$ reads at $b \to 0$

$$G^{(A)}(b\sigma) = \left( \frac{A}{\pi} \right)^{2\sigma - 1 - 1/b^2} \frac{e^{1/b^2} e^{-2C}}{\sqrt{2\pi b^3(2\sigma - 1)}}$$ (3.26)

4. Classical limit

The asymptotic (3.20), (3.21) has to be compared with the classical action (2.34) evaluated at the solution $\varphi_{\eta_1, \eta_2, \eta_3}(z|x_1, x_2, x_3)$ of the classical Liouville equation (2.30) with three singular points $x_1, x_2$ and $x_3$. The boundary conditions (2.33) at these points are automatically imposed by the boundary terms in (2.34). In the three-point case the solution $\varphi_{\eta_1, \eta_2, \eta_3}(z|x_1, x_2, x_3)$ can be found explicitly in terms of hypergeometric functions

$$\varphi_{\eta_1, \eta_2, \eta_3}(z|x_1, x_2, x_3) = -2 \log \left[ a_1 \psi_1(z) \psi_1(\bar{z}) + a_2 \psi_2(z) \psi_2(\bar{z}) \right]$$ (4.1)

Here

$$\psi_1(z) = (z - x_1)^{\eta_1} (z - x_2)^{1-\eta_1-\eta_3} (z - x_3)^{\eta_3} {}_2F_1(\eta_1 + \eta_3 - \eta_2, \eta_1 + \eta_2 + \eta_3 - 1, 2\eta_1, x)$$

$$\psi_2(z) = (z - x_1)^{1-\eta_1} (z - x_2)^{\eta_1+\eta_3-1} (z - x_3)^{1-\eta_3} {}_2F_1(1 + \eta_2 - \eta_1 - \eta_3, 2 - \eta_1 - \eta_2 - \eta_3, 2 - 2\eta_1, x)$$ (4.2)

and we have denoted

$$x = \frac{(z - x_1)x_{32}}{(z - x_2)x_{31}}$$ (4.3)

while the constants $a_1$ and $a_2$ read explicitly

$$a_1^2 = \frac{\pi \mu b^2}{|x_{13}|^{4\eta_3 + 4\eta_1 - 2} |x_{12}|^{2\eta_1 - 4\eta_2} |x_{23}|^{2\eta_2 - 4\eta_1} \times \frac{\gamma(\eta_1 + \eta_2 + \eta_3 - 1)\gamma(\eta_1 + \eta_3 - \eta_2)\gamma(\eta_1 + \eta_2 - \eta_3)}{\gamma^2(2\eta_1)\gamma(\eta_2 + \eta_3 - \eta_1)}}$$ (4.4)
and

$$a_2 = -\frac{\pi \mu b^2}{|x_{13}|^2 (1 - 2\eta_1)^2 a_1} \quad (4.5)$$

Now we have to evaluate the integral (2.34). Calculation of the Liouville action can be simplified by the following trick. Since $\varphi_{\eta_1, \ldots, \eta_n}(x|x_1, \ldots, x_n)$ is an extremum of (2.34) we have

$$\frac{\partial}{\partial \eta_i} S_{\text{Liouv}}[\varphi_{\eta_1, \ldots, \eta_n}(x|x_1, \ldots, x_n)] = -\varphi_i - 4\eta_i \log \epsilon_i \quad (4.6)$$

where $\varphi_i$ is defined in eq.(2.35). Near the singular point $x_i$ the solution behaves as

$$\varphi_{\eta_1, \ldots, \eta_n}(z|x_1, \ldots, x_n) = -2\eta_i \log |z - x_i|^2 + X_i + O (|z - x_i|^{-4\eta_i}) \quad (4.7)$$

(we suppose here that all $\eta_i < 1/2$). Therefore in the limit $\epsilon_i \to 0$

$$\frac{\partial}{\partial \eta_i} S_{\text{Liouv}} = -X_i \quad (4.8)$$

This equation implies that the form

$$dS_{\text{Liouv}} = -\sum_{i=1}^n X_i d\eta_i \quad (4.9)$$

can be integrated, defining $S_{\text{Liouv}}$ up to a constant independent on $\eta_i$. To fix this integration ambiguity we note that the Liouville action (2.34) can be explicitly evaluated if $\sum_{i=1}^n \eta_i = 1$

$$S_{\text{Liouv}}|_{\sum \eta_i = 1} = \sum_{i<j} 2\eta_i \eta_j \log |x_i - x_j|^2 \quad (4.10)$$

In the three-point case the coefficients $X_i$ are easily derived from the explicit solution (4.1–5). E.g.

$$X_1 = -(1 - 2\eta_1) \log \frac{|x_{12} x_{13}|^2}{x_{23}} - \log \pi \mu b^2 - \log \frac{\gamma(\eta_1 + \eta_2 + \eta_3 - 1)\gamma(\eta_1 + \eta_2 - \eta_3)\gamma(\eta_1 + \eta_3 - \eta_2)}{\gamma^2(2\eta_1)\gamma(\eta_2 + \eta_3 - \eta_1)} \quad (4.11)$$

while $X_2$ and $X_3$ are obtained from (4.11) by obvious permutations of $\eta_i$ and $x_i$. Integrating the form (4.9) we find

$$S_{\text{Liouv}} = \left( \sum \eta_1 - 1 \right) \log \pi \mu b^2 + (\delta_1 + \delta_2 - \delta_3) \log |x_{12}|^2 + (\delta_2 + \delta_3 - \delta_1) \log |x_{23}|^2 + (\delta_3 + \delta_1 - \delta_2) \log |x_{13}|^2 + F(\eta_1 + \eta_2 + \eta_3 - 1) + F(\eta_1 + \eta_2 - \eta_3) + F(\eta_2 + \eta_3 - \eta_1) + F(\eta_3 + \eta_1 - \eta_2) - F(2\eta_1) - F(2\eta_2) - F(2\eta_3) - F_0(x_i) \quad (4.12)$$
where $\delta_i = \eta_i(1 - \eta_i)$, function $F(\eta)$ is defined by eq. (3.22) and $F_0(x_i)$ is the integration constant independent on $\eta_i$. Comparing with (4.10) we find $F_0(x_i) = F(0)$ in complete agreement with the asymptotic (3.21) of the proposed exact three-point function.

Similar and even more simple calculations with the positive curvature Liouville equation (2.37) support the classical limit (3.25) of the two-point function.

One can also check that in the case of three “light” exponentials asymptotic form (3.23) of the proposed three-point function is in agreement with explicit semiclassical calculation through the equation (2.43). In the case $n = 3$ the $x$-dependence of the integral in (2.43) can be easily isolated with the help of its $SL(2, C)$ transformation properties, so that this integral takes the form

$$|x_{12}|^{2\nu_3}|x_{23}|^{2\nu_1}|x_{31}|^{2\nu_2} I(\sigma_1, \sigma_2, \sigma_3),$$

where $\nu_1 = \sigma_1 - \sigma_2 - \sigma_3$, $\nu_2 = \sigma_2 - \sigma_3 - \sigma_1$, $\nu_3 = \sigma_3 - \sigma_1 - \sigma_2$, and

$$I(\sigma_1, \sigma_2, \sigma_3) = \left(\frac{A}{\pi}\right)^{\sigma_1+\sigma_2+\sigma_3} \times \tilde{I}(\sigma_1, \sigma_2, \sigma_3);$$

$$\tilde{I}(\sigma_1, \sigma_2, \sigma_3) = \int (|b|^2 + |d|^2)^{-2\nu_1}(|a + b|^2 + |c + d|^2)^{-2\sigma_2}(|a|^2 + |b|^2)^{-2\sigma_3}d\mu(a, b, c, d).$$

It is this factor $I(\sigma_1, \sigma_2, \sigma_3)$ which, being multiplied by the partition function $Z_0^{(A)}$ (see (2.43)), is to agree with the asymptotic (3.23). To evaluate the integral $\tilde{I}$ in (4.14) it is convenient to use complex coordinates $\xi_1, \xi_2, \xi_3$ on the group manifold of $SL(2, C)$ related to $a, b, c, d$ as

$$\xi_1 = \frac{b}{d}; \quad \xi_2 = \frac{a + b}{c + d}; \quad \xi_3 = \frac{a}{c}.$$  

(4.15)

In this coordinates the invariant measure on $SL(2, C)$ takes the well known form

$$d\mu(a, b, c, d) = \frac{d^2\xi_1 d^2\xi_2 d^2\xi_3}{|\xi_1 - \xi_2|(|\xi_2 - \xi_3|)(|\xi_3 - \xi_1|)^2},$$

and the integral in (4.14) simplifies as

$$\tilde{I}(\sigma_1, \sigma_2, \sigma_3) = \int d^2\xi_1 d^2\xi_2 d^2\xi_3$$

$$|\xi_{12}|^{-2-2\nu_3}|\xi_{23}|^{-2-2\nu_1}|\xi_{31}|^{-2-2\nu_2}(1 + |\xi_1|^2)^{-2\sigma_2}(1 + |\xi_2|^2)^{-2\sigma_2}(1 + |\xi_3|^2)^{-2\sigma_3}.$$  

(4.17)

Now, it is straightforward to verify that this integral is invariant under the $SU(2)$ subgroup of $SL(2, C)$. Namely, the form of this integral does not change if one substitutes

$$\xi_i \rightarrow \frac{a\xi_i + b}{-b\xi_i + a}, \quad |a|^2 + |b|^2 = 1.$$  

(4.18)

Using this symmetry one can set, say, $\xi_3 = \infty$ and write the integral (4.17) as

$$\tilde{I}(\sigma_1, \sigma_2, \sigma_3) = \pi \int d^2\xi_1 d^2\xi_2 |\xi_1 - \xi_2|^{-2-2\nu_3}(1 + |\xi_1|^2)^{-2\sigma_2}(1 + |\xi_2|^2)^{-2\sigma_2}.$$  

(4.19)
This integral can be evaluated explicitly with the result
\[ \tilde{I}(\sigma_1, \sigma_2, \sigma_3) = \pi^3 \frac{\Gamma(\sigma_1 + \sigma_2 + \sigma_3 - 1) \Gamma(\sigma_1 + \sigma_2 - \sigma_3) \Gamma(\sigma_2 + \sigma_3 - \sigma_1) \Gamma(\sigma_3 + \sigma_1 - \sigma_2)}{\Gamma(2\sigma_1) \Gamma(2\sigma_2) \Gamma(2\sigma_3)}, \]

in agreement with (3.23).

5. Reflection amplitude
Consider LFT on an infinite flat cylinder of circumference $2\pi$ with the cartesian co-
ordinates $x_1, x_2$ (as before $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$) and let us interpret the coordinate
$x_2$ along the cylinder as the (imaginary) time while $x_1 \sim x_1 + 2\pi$ be the space coordinate.
The holomorphic Liouville stress tensor (2.4) allows one to construct two copies (right and left) of Virasoro algebra $Vir$ and $\bar{Vir}$ with the central charge (2.5)

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{cL}{12}(m^3 - m)\delta_{m+n} \]

\[ [\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n} + \frac{cL}{12}(m^3 - m)\delta_{m+n} \]

(5.1)

where the operators $L_n$ and $\bar{L}_n$ appear in the expansion of the stress tensor

\[ T(z) = -\frac{cL}{24} - \sum_{n=-\infty}^{\infty} L_n e^{inz} \]

\[ \bar{T}(\bar{z}) = -\frac{cL}{24} - \sum_{n=-\infty}^{\infty} \bar{L}_n e^{-inz} \]

and act in the space of states $A$ of LFT on the constant time circle $x_2 = \text{const}$. In particular
the Hamiltonian

\[ H = -\frac{cL}{12} + L_0 + \bar{L}_0 \]

(5.3)

generates translations along the time $x_2$.

The space of states $A$ is classified in the highest weight representations of $Vir \otimes \bar{Vir}$

\[ A = \oplus_P A_P \]

(5.4)

Each conformal class $A_P$ contains a primary state $v_P$ which satisfies

\[ L_n v_P = \bar{L}_n v_P = 0 \quad \text{at} \quad n > 0 \]

\[ L_0 v_P = \bar{L}_0 v_P = (Q^2/4 + P^2)v_P \]

(5.5)

(so that the energy of $v_P$ is $2P^2 - 1/12$) and its descendants generated by the action of $L_n$
and $\bar{L}_n$ with $n < 0$ on $v_P$. Right and left generators $L_n$ and $\bar{L}_n$ commute and therefore
$A_P$ has the structure of a direct product of right and left modules.
To get more of an idea about $A$ take the “zero-mode” of the Liouville field $\phi(x)$

$$\phi_0 = \int_0^{2\pi} \phi(x) \frac{dx_1}{2\pi}$$  \hspace{1cm} (5.6)$$

and consider the region $\phi_0 \to -\infty$ in the configuration space. Here one can neglect the exponential interaction term in the LFT action and consider $\phi(x)$ as a free massless field which can be expanded as usual in the free field oscillators

$$\phi(x) = \phi_0 - P(z - \bar{z}) + \sum_{n \neq 0} \left( \frac{i a_n}{n} e^{inz} + \frac{i \bar{a}_n}{n} e^{-inz} \right)$$  \hspace{1cm} (5.7)$$

Here

$$P = -i \frac{\partial}{2 \partial \phi_0}$$  \hspace{1cm} (5.8)$$

is the momentum conjugate to the zero-mode and

$$[a_m, a_n] = \frac{m}{2} \delta_{m+n} \hspace{1cm} \hspace{1cm} [\bar{a}_m, \bar{a}_n] = \frac{m}{2} \delta_{m+n}$$  \hspace{1cm} (5.9)$$

The Virasoro generators are represented at $\phi_0 \to -\infty$ as follows

$$L_n = \sum_{k \neq 0,n} a_k a_{n-k} + (2P + inQ)a_n \quad n \neq 0$$

$$L_0 = 2 \sum_{k>0} a_{-k} a_k + Q^2/4 + P^2$$  \hspace{1cm} (5.10)$$

and the same for $L_n$ with $a_n$ substituted by $\bar{a}_n$. These operators act in the space of states

$$A_0 = L_2(-\infty < \phi_0 < \infty) \otimes \mathcal{F}$$  \hspace{1cm} (5.11)$$

where $\mathcal{F}$ is the Fock space of the oscillators $a_n$, $\bar{a}_n$ with $n \in \mathbb{Z}$, $n \neq 0$. The space $\mathcal{F}$ consists of the Fock vacuum $|0\rangle$ annihilated by all $a_n$, $\bar{a}_n$ with $n > 0$ and the states generated by the action $a_n$, $\bar{a}_n$ with $n < 0$ on $|0\rangle$.

Let $\Psi_s[\phi(x_1)]$ be the wave functional of any state $s \in A$. The $\phi_0 \to -\infty$ asymptotic of this wave functional is naturally associated with an element of (5.11). From (5.10) one observes that the zero-mode plane wave $\exp(2iP\phi_0)$ times the Fock vacuum $|0\rangle$ behaves under (5.11) as the primary state $v_P$. It is clear that the correct wave functional $\Psi_{v_P}[\phi(x_1)]$ of $v_P$ contains at $\phi_0 \to -\infty$ also a reflected wave $\exp(-2iP\phi_0)|0\rangle$, i.e.,

$$\Psi_{v_P}[\phi(x_1)] \sim (\exp(2iP\phi_0 + S(P)e^{-2iP\phi_0})|0\rangle \quad \text{at} \quad \phi_0 \to -\infty \hspace{1cm} (5.12)$$
with some reflection amplitude $S(P)$ dependent on a more complicated dynamics in the region of $\phi_0$ where the exponential interaction term is important (see [11]). More generally, if $s_P$ is some state of $A_P$

$$\Psi_{s_P}[\phi(x_1)] \sim \left(e^{2iP\phi_0} + \hat{S}(P)e^{-2iP\phi_0}\right)|s\rangle, \quad \phi_0 \to -\infty \tag{5.13}$$

where $|s\rangle \in \mathcal{F}$ and $\hat{S}(P)$ is now a unitary operator in $\mathcal{F}$. In particular $S(P)$ is the eigenvalue of $\hat{S}(P)$ on the Fock vacuum $|0\rangle$.

It is important to note that at a given $P$ all the matrix elements of the operator $\hat{S}(P)$ are in fact determined by the conformal symmetry of LFT up to an overall multiplier. For example, applying $L_{-1}$ in the form (5.10) to the wave functional (5.12) we obtain

$$\hat{S}(P)a_{-1}|0\rangle = S(P)\frac{Q-2iP}{Q+2iP}a_{-1}|0\rangle \tag{5.14}$$

For more complicated Fock states the calculations are more involved but always reduce to linear algebra and permit us to restore $\hat{S}(P)$ uniquely up to $S(P)$.

The following features are readily established. First, $\mathcal{F}$ is a tensor product of the right and left modules (generated by $a_n$ and $\bar{a}_n$ respectively) and

$$\hat{S}(P) = S(P)\hat{s}_R(P) \otimes \hat{s}_L(P) \tag{5.15}$$

where $\hat{s}_R(P)$ and $\hat{s}_L(P)$ act independently in the right and left Fock spaces, $\hat{s}_L(P)$ being isomorphic to $\hat{s}_L(P)$ under $a_n \to \bar{a}_n$. Next, $\hat{s}_R(P)$ commutes with $L_0$ and therefore act invariantly at each level (i.e., at every eigenspace of $L_0$). Moreover, it can be argued that $\hat{s}_R(P)$ commutes with the infinite series of the “quantum KdV integrals of motion” [22,23] built of the higher powers of the stress tensor. In other words, it has the same eigenvectors as the quantum KdV transfer matrix constructed in ref.[24].

However, generally $\hat{s}_R(P)$ is not known in a closed form. We quote here its matrix elements at the second level spanned by the Fock states $|1,1\rangle = a_{-1}|0\rangle$ and $|2\rangle = a_{-2}|0\rangle$

$$D(P)\hat{s}_R(P)|1,1\rangle = (8P^3 + (6Q^2 - 2)P + iQ(2Q^2 + 1))|1,1\rangle + 4iPQ|2\rangle$$

$$D(P)\hat{s}_R(P)|2\rangle = -8iPQ|1,1\rangle + (-8P^3 - (6Q^2 - 2)P + iQ(2Q^2 + 1))|2\rangle \tag{5.16}$$

where

$$D(P) = (2P - iQ)(2P - i(b + 2/b))(2P - i(2b + 1/b)) \tag{5.17}$$

The overall factor $S(P)$ in (5.15) is not prescribed by the conformal symmetry and has to be recovered separately. It is easy to evaluate it in the semiclassical limit $b \to 0$. Suppose that $P$ is also small of the order of $O(b)$. Then even in the region of $\phi_0$ where the interaction is significant one can neglect the oscillators in (5.7) and study only the dynamics of the zero-mode $\phi_0$. In this approximation, known as the minisuperspace approach [20] in LFT, the Hamiltonian (5.3) is substituted by

$$H_0 = -\frac{1}{12} - \frac{1}{2} \frac{\partial^2}{\partial \phi_0^2} + 2\pi \mu e^{2b\phi_0} \tag{5.18}$$
and the reflection amplitude (5.12) appears as

\[
S(P) = -\left(\frac{\pi \mu}{b^2}\right)^{-2iP/b} \frac{\Gamma (1 + 2iP/b)}{\Gamma (1 - 2iP/b)} \quad b \to 0
\] (5.19)

Comparing this with (3.18) we find it natural to propose the expression (3.18) as the exact reflection amplitude \(S(P)\) in LFT. In the next section we verify this suggestion numerically using the thermodynamic Bethe ansatz technique.

### 6. Thermodynamic Bethe ansatz

In this section we consider the sinh-Gordon model on a circle of circumference \(R\) with periodic boundary conditions. The problem is defined by the action

\[
A_{\text{shG}} = \int dx_2 \int_0^R dx_1 \left[ \frac{1}{4\pi} (\partial_a \phi)^2 + 2\mu \cosh b\phi \right]
\] (6.1)

where \(\phi(x_1, x_2) = \phi(x_1 + R, x_2)\) is the periodic scalar field, \(\mu \sim [\text{mass}]^{2+2b^2}\) is the dimensional coupling constant and \(b\) is the dimensionless parameter of the model. Due to the scaling properties of the interaction operator \(\exp(2b\phi) + \exp(-2b\phi)\) one can rescale the problem to the circle of circumference \(2\pi\) substituting (6.1) by

\[
S_{\text{shG}} = \int dx_2 \int_0^{2\pi} dx_1 \left[ \frac{1}{4\pi} (\partial_a \phi)^2 + \mu \left( \frac{R}{2\pi} \right)^{2+2b^2} \left( e^{2b\phi} + e^{-2b\phi} \right) \right]
\] (6.2)

We are interested in the ground state energy \(E(R)\) or, more conveniently, the finite-size effective central charge

\[
c_{\text{eff}}(R) = -\frac{6R}{\pi}E(R)
\] (6.3)

in the ultraviolet limit \(R \to 0\). Let \(\Psi_0[\phi(x_1)]\) be the ground state wave functional and define again the zero-mode \(\phi_0\) as in eq.(5.6). At \(R \to 0\) there is a large region \(\log \mu(R/2\pi)^{2+2b^2} < 2b\phi_0 < -\log \mu(R/2\pi)^{2+2b^2}\) in the configuration space where one can neglect the interaction term in (6.2) and consider \(\phi(x)\) as a free massless field (5.7). In this region the ground state wave functional is expected to be a superposition

\[
\Psi_0[\phi(x_1)] \sim \left( c_1 e^{2iP\phi_0} + c_2 e^{-2iP\phi_0} \right) |0\rangle
\] (6.4)

of two zero-mode plane waves with some \(R\)-dependent zero-mode momentum \(P\) times the Fock vacuum \(|0\rangle\) of the oscillators in (5.6). The corresponding effective central charge is determined at \(R \to 0\) mainly by \(P(R)\)

\[
c_{\text{eff}}(R) = 1 - 24P^2 + O(R^2)
\] (6.5)

up to power corrections in \(R\). The momentum \(P(R)\) is quantized due to the right and left potential walls at \(2b\phi_0 \sim \pm \log \mu(R/2\pi)^{2+2b^2}\) in the action (6.2). Consider say the right
wall at $2b\phi_0 \sim -\log \mu (R/2\pi)^{2+2b^2}$. If $R$ is small enough the second exponent $\exp(-2b\phi)$ in the potential term of (6.2) is small in this region and does not affect the dynamics which is therefore expected to be essentially the same as in LFT. Thus for the reflected wave we have

$$\Psi_0[\phi(x_1)] \sim (e^{2iP\phi_0} + (R/2\pi)^{-4iPQ}S(P)e^{-2iP\phi_0})|0\rangle$$  \hspace{1cm} (6.6)

where $Q$ is again given by eq.(2.3) and $S(P)$ is the same reflection amplitude (3.18) as in LFT (5.12). Factor $(R/2\pi)^{-4iPQ}$ appears due to the rescaling of $\mu$ in (6.2). A similar consideration about the left wall reflection leads to the following quantization condition

$$(R/2\pi)^{-8iPQ} S^2(P) = 1$$  \hspace{1cm} (6.7)

For the ground state momentum this equation reads

$$\delta(P) = \pi + 4PQ \log(R/2\pi)$$  \hspace{1cm} (6.8)

where we have introduced the reflection phase $\delta(P)$

$$S(P) = -\exp(i\delta(P))$$  \hspace{1cm} (6.9)

Together with (6.5) equation (6.8) determines the most important part of the $R \to 0$ asymptotic of $c_{\text{eff}}(R)$. For example, using the regular expansion of the reflection phase in the odd powers of $P$

$$\delta(P) = \delta_1(b) P + \delta_3(b) P^3 + \delta_5(b) P^5 + \ldots$$  \hspace{1cm} (6.10)

one can develop $c_{\text{eff}}(R)$ systematically in $1/\log R$

$$c_{\text{eff}}(R) = 1 - \frac{24\pi^2}{l^2} + \frac{48\pi^4\delta_3(b)}{l^5} + \ldots$$  \hspace{1cm} (6.11)

where we have denoted

$$l = \delta_1(b) - 4Q \log(R/2\pi)$$  \hspace{1cm} (6.12)

On the other hand the sinh-Gordon model is integrable and its factorizable scattering matrix is known [25]. This allows us to compute the same effective central charge $c_{\text{eff}}(R)$ by the thermodynamic Bethe ansatz (TBA) technique [26,27]. In the TBA framework it is evaluated as the integral

$$c_{\text{eff}}(R) = \frac{3mR}{\pi^2} \int \cosh \theta \log \left(1 + e^{-\varepsilon(\theta)} \right) d\theta$$  \hspace{1cm} (6.13)

where $\varepsilon(\theta)$ is the solution to the following non-linear integral equation

$$mR \cosh \theta = \varepsilon(\theta) + \int \varphi(\theta - \theta') \log \left(1 + e^{-\varepsilon(\theta')} \right) \frac{d\theta'}{2\pi}$$  \hspace{1cm} (6.14)
In (6.13) and (6.14) \(m\) is the mass of the physical particle in the sinh-Gordon spectrum. It is related to the coupling constant \(\mu\) as [28]

\[
-\frac{\pi \mu}{\gamma(-b^2)} = \left[ \frac{m}{4\sqrt{\pi}} \Gamma \left( \frac{1}{2} + \frac{b^2}{2 + 2b^2} \right) \Gamma \left( 1 + \frac{b^2}{2 + 2b^2} \right) \right]^{2+2b^2} \tag{6.15}
\]

The kernel \(\varphi(\theta)\) in eq.(6.14) contains the information about the sinh-Gordon scattering and reads explicitly

\[
\varphi(\theta) = \frac{4 \sin \frac{\pi b^2}{1+b^2} \cosh \theta}{\cosh 2\theta - \cos \frac{2\pi b^2}{1+b^2}} \tag{6.16}
\]

It is straightforward to solve eq.(6.14) numerically. At \(R\) small enough eqs.(6.5) and (6.8) can be interpreted as a parametric representation of the “experimental” reflection phase \(\delta^{(TBA)}(P)\)

\[
P = \sqrt{\frac{1 - c_{\text{eff}}(R)}{24}} \delta^{(TBA)} = \pi + 4PQ \log(R/2\pi) ,
\]

\(R\) being the parameter. According to (6.5) we expect \(\delta^{(TBA)}(P)\) to reproduce the Liouville phase \(\delta(P)\) up to exponentially small in \(1/P\) corrections

\[
\delta(P) = \delta^{(TBA)}(P) + O \left( \exp \left( -\frac{\pi}{2PQ} \right) \right) \tag{6.18}
\]

In particular in the expansion

\[
\delta^{(TBA)}(P) = \delta_1^{(TBA)}(b)P + \delta_3^{(TBA)}(b)P^3 + \delta_5^{(TBA)}(b)P^5 + \ldots \tag{6.19}
\]

all the coefficients are the same as in eq.(6.10).

We have solved eq.(6.14) numerically at different values of the parameter \(b\) and estimated the few first coefficients \(\delta_{2k+1}^{(TBA)}(b)\) of (6.19). In Table 1 the numbers are compared with the corresponding \(\delta_{2k+1}(b)\) in the expansion (6.10) of the exact Liouville amplitude (3.18)

\[
\delta_1(b) = \frac{4}{b} \log b^2 - 4Q \log \frac{m\Gamma \left( \frac{1}{2} + \frac{b^2}{2 + 2b^2} \right) \Gamma \left( 1 + \frac{b^2}{2 + 2b^2} \right)}{4\sqrt{\pi}} + C \tag{6.20}
\]

\[
\delta_3(b) = \frac{16}{3} \zeta(3)(b^3 + b^{-3})
\]

\[
\delta_5(b) = \frac{64}{5} \zeta(5)(b^5 + b^{-5})
\]

e tc. Here \(C\) is the Euler’s constant and in \(\delta_1(b)\) the cosmological constant \(\mu\) is substituted in terms of \(m\) by means of eq.(6.15). In the numerical calculations we set \(m = 1\).
We consider the content of Table 1 as impressive evidence in support of the exact Liouville reflection amplitude suggested in sect. 3. The same TBA analysis can be applied also for LFT with $c_L < 25$. Parameter $b$ is complex in this case and the sinh-Gordon scattering theory has to be replaced by the staircase model [29]. We hope to say more about this interesting relation in a future publication.

The exact Liouville reflection amplitude can be used in the opposite direction in the analysis of the sinh-Gordon (or the staircase) model itself. Subtracting the leading $R \to 0$ asymptotic of $c_{\text{eff}}(R)$ predicted by eqs. (6.5) and (6.8) from the TBA numerical data one can separate the power-like corrections in (6.5). Perhaps this would allow us to clarify their nature. Work along this line is now in progress.

7. Conformal bootstrap

In this section we study numerically the conformal bootstrap equations (2.21) using the representation (2.23) of the four-point function in LFT. The structure constants $C(\alpha_1, \alpha_2, \alpha_3)$ are proposed explicitly in sect. 3. The conformal block $F(\Delta_{\alpha_1}, \Delta, x)$ which also enters eq. (2.23) is not known generally in an analytic form. In refs. [19] the following convenient representation has been derived

$$F(\Delta_{\alpha_1}, \Delta, x) = (16q)^{P^2} x^{Q^2/4-\Delta_{\alpha_1}-\Delta_{\alpha_2}} (1-x)^{Q^2/4-\Delta_{\alpha_1}-\Delta_{\alpha_3}} \times$$

$$[\theta_3(q)]^{3Q^2-4(\Delta_{\alpha_1}+\Delta_{\alpha_2}+\Delta_{\alpha_3}+\Delta_{\alpha_4})} H(\lambda_i^2, \Delta|q)$$

Here

$$\theta_3(q) = \sum_{n=\infty}^{\infty} q^n$$

and

$$q = e^{i\pi \tau}$$

is related to $x$ by the equation

$$\tau = i \frac{K(1-x)}{K(x)}$$

where

$$K(x) = \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-xt)}}$$

Function $H(\lambda_i^2, \Delta|q)$ is better parameterized in terms of the variables

$$\lambda_i^2 = \frac{Q^2}{4} - \Delta_{\alpha_i} = \left(\frac{Q}{2} - \alpha_i\right)^2$$

instead of the external dimensions $\Delta_{\alpha_i}$. It satisfies the following recurrence relation

$$H(\lambda_i^2, \Delta|q) = 1 + \sum_{m,n} \frac{q^{mn} R_{m,n}(\lambda_i)}{\Delta - \Delta_{m,n}} H(\lambda_i^2, \Delta_{m,n} + mn|q)$$

21
Here the sum is over all pairs \((m, n)\) of positive integers and

\[
\Delta_{m,n} = \frac{Q^2}{4} - \frac{(m/b + nb)^2}{4}
\]  

(7.8)

are the dimensions of degenerate representations of the Virasoro algebra with the central charge (2.5). With the notation

\[
\lambda_{m,n} = (m/b + nb)/2
\]  

(7.9)

the multipliers \(R_{m,n}(\lambda_i)\) read explicitly

\[
R_{m,n}(\lambda_i) = 2 \prod_{r,s} (\lambda_1 + \lambda_2 - \lambda_{r,s})(\lambda_1 - \lambda_2 - \lambda_{r,s})(\lambda_3 + \lambda_4 - \lambda_{r,s})(\lambda_3 - \lambda_4 - \lambda_{r,s}) \over \prod'_{k,l} \lambda_{k,l}
\]  

(7.10)

The products in (7.10) are over the following sets of integers \((r, s)\) and \((k, l)\)

\[
r = -m + 1, -m + 3, \ldots, m - 3, m - 1
\]  

(7.11)

\[
s = -n + 1, -n + 3, \ldots, n - 3, n - 1
\]  

and

\[
k = -m + 1, -m + 2, \ldots, m - 1, m
\]  

(7.12)

\[
l = -n + 1, -n + 2, \ldots, n - 1, n
\]

while the prime sign near the last product symbol \(\prod'_{k,l}\) means that the two pairs \((k, l) = (0, 0)\) and \((m, n)\) are missing. From the numerical point of view the expression (7.7) is “almost analytic” in the sense that it permits us to compute \(H(\lambda^2_i, \Delta|q)\) very fast and precisely.

In view of eq.(7.1) we find it more convenient to write down the four-point function (2.20) as

\[
G_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(x, \bar{x}) = (x\bar{x})^{Q^2/4 - \Delta_{\alpha_1} - \Delta_{\alpha_2}} \left[(1 - x)(1 - \bar{x})\right]^{Q^2/4 - \Delta_{\alpha_1} - \Delta_{\alpha_3}} \times
\]

\[
[\theta_3(q)\theta_3(\bar{q})]^3Q^2 - 4 \sum_i \Delta_{\alpha_i} g_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(x, \bar{x})
\]  

(7.13)

and study the crossing properties of the reduced function

\[
g_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(\tau, \bar{\tau}) =
\]

\[
\frac{1}{2} \int C(\alpha_1, \alpha_2, Q/2 + iP)C(\alpha_3, \alpha_4, Q/2 - iP) \left|(16q)^{P^2} H(\lambda^2_i, Q^2/4 + P^2|q)\right|^2 dP
\]  

(7.14)

From eqs.(2.21) it follows that

\[
g_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(\tau, \bar{\tau}) = g_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(\tau + 1, \bar{\tau} + 1)
\]  

(7.15a)

\[
g_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(\tau, \bar{\tau}) = |\tau|^{Q^2 - 4} \sum_i \Delta_{\alpha_i} g_{\alpha_1, \alpha_3, \alpha_2, \alpha_4}(-1/\tau, -1/\bar{\tau})
\]  

(7.15b)
The first equation (7.15a) is identically satisfied by (7.14) due to the following property of the function \( H(\lambda^2_1, \lambda^2_2, \lambda^2_3, \Delta|q) \)

\[
H(\lambda^2_1, \lambda^2_2, \lambda^2_3, \Delta|q) = H(\lambda^2_1, \lambda^2_2, \lambda^2_3, \Delta|-q) \tag{7.16}
\]

which is easily derived from the relation (7.7) and eq.(7.10). Equation (7.15b) still remains a non-trivial condition for the structure constants which is believed to contain the whole information about the associativity of the operator algebra.

For numerical calculations we have chosen the correlation function of four puncture operators (2.11)\(^3\)

\[
g(\tau, \bar{\tau}) = \frac{1}{16} \frac{\partial^4 g_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(\tau, \bar{\tau})}{\partial \alpha_1 \partial \alpha_2 \partial \alpha_3 \partial \alpha_4} \bigg|_{\alpha_1=\alpha_2=\alpha_3=\alpha_4=Q/2} \tag{7.17}
\]

After separating some irrelevant overall factors

\[
g(\tau, \bar{\tau}) = \frac{\Upsilon^8}{\pi^2} \left[ \pi \mu \gamma(b^2) b^{2-2b^2} \right]^{-Q/b} \left( \frac{\tau - \bar{\tau}}{2i} \right)^{-Q^2/2} f(\tau, \bar{\tau}) \tag{7.18}
\]

we have to verify the relation

\[
f(\tau, \bar{\tau}) = f(-1/\tau, -1/\bar{\tau}) \tag{7.19}
\]

for the function

\[
f(\tau, \bar{\tau}) = \frac{1}{2} \left( \frac{\tau - \bar{\tau}}{2i} \right)^{Q^2/2} \int r(P) \left| (16q)^{Q^2/4} + P^2|q \right|^2 dP \tag{7.20}
\]

Here

\[
r(P) = \frac{\pi^2 \Upsilon(2iP) \Upsilon(-2iP)}{\Upsilon^2 \Upsilon^8(Q/2 + iP)} = \frac{\Upsilon^2 \Upsilon^8(Q/2 + iP)}{\Upsilon^2 \Upsilon^8(Q/2 + iP)} \tag{7.21}
\]

We have computed \( f(\tau, \bar{\tau}) \) for several values of \( b \) and found that eq.(7.19) is satisfied up to high numerical accuracy. In figs.1 and 2 functions \( f(\tau, \bar{\tau}) \) and \( f(-1/\tau, -1/\bar{\tau}) \) are compared for purely imaginary \( \tau = it \) at \( b = 0.8 \) and \( b = (1 + i)/\sqrt{2} \). This last complex value corresponds to \( cL = 13 \).

\(^3\) The corresponding values of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are well within the domain (2.25) and so no “discrete terms” are expected to appear in (2.23) and (7.14).
8. Accessory parameters

The classical Liouville action (2.34) with \( n \) singular points is closely related to the classic problem of uniformization of Riemann surfaces [30] and in particular to the so-called problem of accessory parameters [31]. The last is basically formulated as follows. Consider the ordinary linear differential equation

\[
\partial^2 \psi(z) + \sum_{i=1}^{n} \left( \frac{1}{4(z-x_i)^2} + \frac{C_i}{z-x_i} \right) \psi(z) = 0
\] (8.1)

with \( n \) regular parabolic singular points \( x_i \). The complex infinity \( z = \infty \) is supposed to be a regular point of (8.1) so that the accessory parameters \( C_i \) are restricted by the relations

\[
\sum_{i=1}^{n} C_i = 0
\]

\[
\sum_{i=1}^{n} (x_i C_i + 1/4) = 0
\] (8.2)

\[
\sum_{i=1}^{n} (x_i^2 C_i + x_i/2) = 0
\]

The problem is to tune these parameters in such a way that the monodromy group of eq.(8.1) is a Fuchsian one. It was proven in ref.[32] that the problem is solved by the \( \eta_i \rightarrow 1/2 \) version of the Liouville action (2.34). In the case \( \eta_i \rightarrow 1/2 \) one should be more careful since the boundary conditions (2.33) for the Liouville equation (2.30) are slightly more complicated

\[
\varphi(z, \bar{z}) = -2 \log |z|^2 + O(1) \quad \text{at} \quad |z| \rightarrow \infty
\]

\[
\varphi(z, \bar{z}) = - \log |z - x_i|^2 - 2 \log \log |z - x_i|^2 + O(1) \quad \text{at} \quad |z - x_i| \rightarrow 0
\] (8.3)

and more subtractions are needed to regularize the Liouville action (see ref.[32] for more details). If \( S^{(cl)}(x_1, \ldots, x_n) \) is the classical Liouville action of the solution \( \varphi(z, \bar{z})|_{x_1, \ldots, x_n} \) to (2.29) with the boundary conditions (8.3) then the accessory parameters

\[
C_i = - \frac{\partial}{\partial x_i} S^{(cl)}(x_1, \ldots, x_n)
\] (8.4)

solve the above problem.

From the LFT point of view the action \( S^{(cl)}(x_1, \ldots, x_n) \) can be considered as the leading classical asymptotic \( b \rightarrow 0 \) of the \( n \)-point function

\[
\frac{1}{2^n} \frac{\partial^n G_{\alpha_1, \ldots, \alpha_n}(x_1, \ldots, x_n)}{\partial \alpha_1 \ldots \partial \alpha_n} \bigg|_{\alpha_1 = \ldots = \alpha_n = Q/2} \sim \exp \left( - \frac{1}{b^2} S^{(cl)}(x_1, \ldots, x_n) \right)
\] (8.5)

of the puncture operators (2.11). In this section we use the representation (2.23) of the four-point function to get some information about \( S^{(cl)}(x_1, \ldots, x_n) \) and therefore about
the accessory parameters in the case \( n = 4 \). Note that due to (8.2) in this case there is only one independent accessory parameter \( C(x, \bar{x}) \) and eq.(8.1) can be reduced to

\[
\partial^2 \psi(z) + \left( \frac{1}{4z^2(1-z)^2} + \frac{1}{4(z-x)^2} + \frac{x(1-x)C(x, \bar{x})}{z(1-z)(z-x)} \right) \psi(z) = 0 \tag{8.6}
\]

where \( x \) is defined in eq.(2.19). Eq.(8.5) is reduced to

\[
\frac{1}{16} \left. \frac{\partial^4 G_{\alpha_1, \alpha_2, \alpha_3, \alpha_4}(x, \bar{x})}{\partial \alpha_1 \partial \alpha_2 \partial \alpha_3 \partial \alpha_4} \right|_{\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = Q/2} \sim \exp \left( -\frac{1}{b^2} S^{(cl)}(x, \bar{x}) \right) \tag{8.7}
\]

while

\[
C(x, \bar{x}) = -\frac{\partial S^{(cl)}(x, \bar{x})}{\partial x} \tag{8.8}
\]

Let us now consider the classical limit \( b \to 0 \) of eq.(2.23) with four “heavy” operators \( \alpha_i = \eta_i / b \). It is convenient to rescale the integration variable \( P = p/b \). The structure constants behave as in eq.(3.20) in this limit while the conformal block \( F(\Delta_{\alpha_1}, \Delta, x) \) has a similar \( b \to 0 \) asymptotic

\[
F(\Delta_{\alpha_1}, \Delta, x) \sim \exp \left( \frac{1}{b^2} f(\eta_i, p, x) \right) \tag{8.9}
\]

Function \( f(\eta_i, p, x) \) (which is sometimes called the classical conformal block) is again generically unknown in a closed form but can be straightforwardly developed in a power series in \( x \)

\[
f(\eta_i, p, x) = (\delta - \delta_1 - \delta_2) \log x + \frac{(\delta + \delta_1 - \delta_2)(\delta + \delta_3 - \delta_4)}{2\delta} x + O(x^2) \tag{8.10}
\]

where we have used the notations \( \delta = p^2 + 1/4 \) and \( \delta_i = \eta_i(1 - \eta_i) \). At \( b \to 0 \) the integral in eq.(2.23) is determined by a saddle point \( p = p_s \), i.e., by the minimum of the function

\[
S_{\eta_1, \eta_2, \eta_3, \eta_4}(p|x, \bar{x}) = S^{(cl)}(\eta_1, \eta_2, 1/2 + ip) + S^{(cl)}(\eta_3, \eta_4, 1/2 - ip) - f(\eta_i, p, x) - f(\eta_i, p, \bar{x}) \tag{8.11}
\]

where \( S^{(cl)}(\eta_1, \eta_2, \eta_3) \) is given explicitly by eq.(3.21). Therefore the classical asymptotic of the four-point function is

\[
G_{\alpha_1/b, \ldots, \alpha_4/b}(x, \bar{x}) \sim \exp \left( -\frac{1}{b^2} S^{(cl)}_{\eta_1, \ldots, \eta_4}(x, \bar{x}) \right) \tag{8.12}
\]

where

\[
S^{(cl)}_{\eta_1, \eta_2, \eta_3, \eta_4}(x, \bar{x}) = S_{\eta_1, \eta_2, \eta_3, \eta_4}(p_s|x, \bar{x}) \tag{8.13}
\]

and \( p_s \) is determined by the equation

\[
\frac{\partial}{\partial p} S_{\eta_1, \eta_2, \eta_3, \eta_4}(p|x, \bar{x}) = 0 \tag{8.14}
\]
With the exact formula (3.21) it reads explicitly

\[ 2\pi - i \log S_{\eta_1, \eta_2}(p) - i \log S_{\eta_3, \eta_4}(p) = -\frac{\partial}{\partial p} \left( f(\eta_i, p, x) + f(\eta_i, p, \bar{x}) \right) \]  

(8.15)

where

\[ S_{\eta_1, \eta_2}(p) = \frac{\Gamma^2(1 - 2ip)\gamma(\eta_1 + \eta_2 - 1/2 + ip)\gamma(1/2 + \eta_1 - \eta_2 + ip)}{\Gamma^2(1 + 2ip)\gamma(\eta_1 + \eta_2 - 1/2 - ip)\gamma(1/2 + \eta_1 - \eta_2 - ip)} \]  

(8.16)

The accessory parameter (8.8) corresponds to the special case \( \eta_1 = \eta_2 = \eta_3 = \eta_4 = 1/2 \) of eqs.(8.13–16)

\[ C(x, \bar{x}) = \frac{\partial}{\partial x} f(1/2, p_s, x) \]  

(8.17)

At small \( x \) one can keep only the leading \( x \to 0 \) term in the classical block (8.10) so that

\[ xC(x, \bar{x}) = (p_s^2 - 1/4) (1 + O(x)) \]  

(8.18)

where \( p_s \) is determined (up to power corrections in \( x \)) by the equation

\[ p \log x\bar{x} + 4i \log \frac{\Gamma(1 - 2ip)\Gamma^2(1/2 + ip)}{\Gamma(1 + 2ip)\Gamma^2(1/2 - ip)} = \pi \]  

(8.19)

One also can systematically pick up the power corrections in \( x \) using the expansion

\[ f(1/2, p, x) = \left( p^2 - \frac{1}{4} \right) \log x + \left( p^2 + \frac{1}{4} \right) \frac{x}{2} + \left( \frac{13p^2}{16} + \frac{9}{32} + \frac{1}{256(p^2 + 1)} \right) \frac{x^2}{4} + \left( \frac{23p^2}{24} + \frac{19}{48} + \frac{1}{128(p^2 + 1)} \right) \frac{x^3}{8} + \ldots \]  

(8.20)

### 9. Conclusion

The expression (3.14) for the three-point function, together with the supporting evidence in sects.4–8, is the main result of this paper. Although (3.14) is a conjecture we find the evidence convincing enough to take it as the starting point in addressing some intriguing questions in 2D Quantum Gravity.

The structure constants (3.14) allow one to construct (in principle) the multipoint correlation functions of LFT through the decompositions similar to (2.23). This gives access to multipoint correlation functions of Quantum Gravity (coupled to a matter theory) at fixed conformal moduli; this is in contrast with the integrated correlation functions considered in [11–15]. This moduli dependence of the correlation functions is expected to add some insight on the nature of physical states in 2D Quantum Gravity.

One can also try to analyze the physics of non-minimal CFT (perhaps non-unitary one to maintain \( c_M < 1 \), to begin with) or non-conformal matter QFT coupled to Quantum Gravity.
The most interesting question is exactly what is it that happens to 2D Quantum Gravity when the central charge $c_M$ of matter theory ($c_M = 26 - c_L$) exceeds 1. We hope that the structure constants (3.14) could be an appropriate vehicle to enter this still rather mysterious domain.

And of course it is important to extend the above analysis to incorporate SUSY.

We hope to return to this questions in future.

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Note Added

After this work was completed we have learned that the conjecture (3.14) already exists in the literature. In ref.[34,35] the expression for the Liouville three-point function apparently equivalent to (3.14) is proposed with the motivations very similar to those discussed in sect.3. We are grateful to H.Dorn for bringing the papers [34,35] to our attention.
References

1. A.Polyakov. Phys.Lett., B103 (1981) 207.
2. T.Curtright and C.Thorn. Phys.Rev.Lett., 48 (1982) 1309; E.Braaten, T.Curtright and C.Thorn. Phys.Lett., B118 (1982) 115; Ann.Phys., 147 (1983) 365.
3. J.-L.Gervais and A.Neveu. Nucl.Phys., B238 (1984) 125; B238 (1984) 396; B257[FS14] (1985) 59.
4. E.D’Hoker and R.Jackiw. Phys.Rev., D26 (1982) 3517.
5. V.Kazakov. Phys.Lett., 150 (1985) 282; F.David. Nucl.Phys., B257 (1985) 45; V.Kazakov, I.Kostov and A.Migdal. Phys.Lett., 157 (1985) 295.
6. E.Brézin and V.Kazakov. Phys.Lett., B236 (1990) 144; M.Douglas and S.Shenker. Nucl.Phys., B335 (1990) 635; D.Gross and A.Migdal. Phys.Rev.Lett., 64 (1990) 127.
7. V.Knizhnik, A.Polyakov and A.Zamolodchikov. Mod.Phys.Lett., A3 (1988) 819.
8. F.David. Mod.Phys.Lett., A3 (1988) 1651.
9. J.Distler and H.Kawai. Nucl.Phys., B321 (1989) 509.
10. M.Bershadsky and I.Klebanov. Phys.Rev.Lett.65(1990)3088.
11. J.Polchinski. Remarks on Liouville Field theory, in “Strings 90”, R.Arnowitt et al, eds, World Scientific, 1991; Nucl.Phys. B357 (1991) 241.
12. M.Goulian and M.Li. Phys.Rev.Lett., 66 (1991) 2051.
13. A.Polyakov. Mod.Phys.Lett. A6 (1991) 635.
14. P.Di Francesco and D.Kutasov. Phys.Lett. B261 (1991) 385.
15. Vl.Dotsenko. Mod.Phys.Lett. A6 (1991) 3601.
16. Vl.Dotsenko and V.Fateev. Phys.Lett. 154B (1985) 291.
17. V.Fateev and A.Zamolodchikov. Sov.Phys.JETP 82(1985)215; Sov.J.Nucl.Phys. 43 (1986)657.
18. A.Belavin, A.Polyakov and A.Zamolodchikov. Nucl.Phys. B241 (1984) 333.
19. Al.Zamolodchikov. Commun.Math.Phys., 96 (1984) 419; Theor.Math.Phys., 73 (1987) 1088.
20. N.Seiberg. Notes on Quantum Liouville Theory and Quantum Gravity, in “Random Surfaces and Quantum Gravity”, ed. O.Alvarez, E.Marinari, P.Windey,
Plenum Press, 1990.

21. Vl. Dotsenko and V. Fateev. Nucl. Phys., B251 (1985) 691.

22. R. Sasaki and I. Yamanaka. Adv. Stud. in Pure Math., 16 (1988) 271.

23. T. Eguchi and S.-K. Yang. Phys. Lett., B224 (1989) 373.

24. V. Bazhanov, S. Lukyanov and A. Zamolodchikov. Integrable structure of conformal field theory, quantum KdV theory and thermodynamic Bethe ansatz. RU-94-98.

25. I. Arefyeva and V. Korepin. Pisma v ZhETF, 20 (1974) 680.

26. C. N. Yang and C. P. Yang. J. Math. Phys., 10 (1969) 1115.

27. A. Zamolodchikov. Nucl. Phys., B342 (1990) 695.

28. A. Zamolodchikov. Mass scale in sin-Gordon and its reductions. LPM-93-06.

29. A. Zamolodchikov. Resonance factorized scattering and roaming trajectories. ENS-LPS-335, 1991.

30. A. Poincare. J. Math. Pures Appl. (5) 4 (1898) 157.

31. F. Klein. Math. Ann. 21 (1883) 201; A. Poincare. Acta Math. 4 (1884) 201.

32. P. Zograf and L. Takhtajan. Functional Anal. Appl. 19 (1986) 219.

33. L. Takhtajan. Semi-Classical Liouville Theory, Complex Geometry of Moduli Spaces, and Uniformization of Riemann Surfaces, in "New Symmetry Principles in Quantum Field Theory", eds. J. Frolich et al, Plenum Press, 1992.

34. H. Dorn, H.-J. Otto. Phys. Lett. B291 (1992) 39.

35. H. Dorn, H.-J. Otto. Nucl. Phys. B429 (1994) 375.
Figure Captions

Fig.1. Reduced four-point function $f(\tau, \bar{\tau})$ of eq.(7.20) (continuous line) at $b = 0.8$ and real $t = -i\tau$. It is compared with $f(-1/\tau, -1/\bar{\tau})$ (points) to verify the crossing symmetry relation (7.19).

Fig.2. The same as in fig.1 but at $b = (1 + i)/\sqrt{2}$ corresponding to $c_L = 13$. 
Table 1. First three coefficients $\delta^{(\text{TBA})}$ in the expansion (6.19) obtained by numerical analysis the “experimental” reflection phase (6.17) in comparison with the corresponding “exact” ones $\delta^{(\text{LFT})}$ given by eq.(6.20). The uncertainty in the “experimental” numbers is $\pm 1$ in the last (bracketed) digit.

| $\frac{b^2}{1+b^2}$ | $\delta_1^{(\text{TBA})}$ | $\delta_1^{(\text{LFT})}$ | $\delta_3^{(\text{TBA})}$ | $\delta_3^{(\text{LFT})}$ | $\delta_5^{(\text{TBA})}$ | $\delta_5^{(\text{LFT})}$ |
|----------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.1                  | -16.61723                 | 173.3336                  | -3225.315                 |                           |                           |                           |
| 0.2                  | -4.74(4)                  | 52.0(5)                   | -42(4)                    | -425.1404                 |                           |                           |
| 0.3                  | -0.671(3)                 | 24.64(3)                  | -111.(2)                  | -111.9785                 |                           |                           |
| 0.4                  | 1.043(1)                  | 15.26(8)                  | -41.4(0)                  | -41.39168                 |                           |                           |
| 0.5                  | 1.533(5)                  | 12.81(9)                  | -26.5(6)                  | -26.54535                 |                           |                           |