MOMENTS OF $q$–NORMAL AND CONDITIONAL $q$–NORMAL DISTRIBUTIONS

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Abstract. We calculate moments and moment generating functions of two distributions: the so called $q$–Normal and the so called conditional $q$–Normal distributions. These distributions generalize both Normal ($q = 1$), Wigner ($q = 0$, $q$–Normal) and Kesten-McKay ($q = 0$, conditional $q$–Normal) distributions. As a by product we get asymptotic properties of some expansions in modified Bessel functions.

1. Introduction

The purpose of this short note is to present exact forms of moments and moment generating functions (i.e. Laplace transforms) of four distributions with densities that are denoted by: $f_h(x|q)$, $f_N(x|q)$ and $f_Q(x|a,b,q)$, $f_{CN}(x|y,\rho,q)$. In fact distributions $f_h$ and $f_N$ are related to one another by the linear transformation of random variables having these distributions. Similarly distributions $f_Q$ and $f_{CN}$ are interrelated. The details will presented below. Two of the presented below distributions are called respectively $q$–Normal ($f_N$) and conditional $q$–Normal ($f_{CN}$). These distributions are the elements of the chain of attempts to generalize Normal distribution that exist in the literature. The first of the considered in this paper distributions ($f_N$) was described first in the noncommutative probability context in [6], later in the classical probability context in [7]. Of course it was not the only one attempt to generalize Gaussian distribution. For others, different see e.g. [24], [23], [22].

Let us mention also that for particular values of the parameter $q$ we get Gaussian (Normal) ($q = 1$), Wigner and generalized Kesten-McKay ($q = 0$) distributions. Let us remind that distribution $f_N$ appears in many models of the so called $q$–oscillators that are considered in quantum physics (to mention only [11], [9], [3], [2]). On the other hand Wigner or semicircle and Kesten-McKay distributions appear as limiting distributions of certain combinatorial considerations and also in the context of random graphs, random matrices and large deviations. By the generalized Kesten-McKay distribution we mean distribution that has density of the form $C\sqrt{a^2-x^2}/Q_2(x)$, where $Q_2(x)$ denotes quadratic polynomial that is positive on $[-a,a]$ and $C$ is some normalizing constant. Formal definition dating back to...
papers of Kesten [10] or McKay [13] concerned very special form of quadratic polynomial \( Q \). For more recent uses of Kesten-McKay distribution see e.g. [12], [11], [14].

The distributions that we are going to recall in this paper have appeared also in the context of stochastic processes allowing generalization of Wiener and Ornstein-Uhlenbeck (see e.g. [18]) processes and also in the context of quadratic harnesses (for review of the rich literature on this subject see [17]).

The paper is organized as follows. In the next section we recall basic notation and basic properties of the analyzed in the paper distributions. In Section 3 we present our results. In Section 4 we collected longer proofs.

2. Notation and basic notions

To present these distributions we will use notation commonly used in the context of the so called \( q \)-series theory. Nice introductions to this theory can be found [1] or [8].

So \( q \) will be a parameter such that \( q \in (0, 1) \). For \( |q| < 1 \) the formulae will be explicit, while the case \( q = 1 \) will sometimes be understood as a limiting case.

We set

\[
[q]_0 = 0, \quad [n]_q = 1 + q + \ldots + q^{n-1}, \quad [n]_q! = \prod_{j=1}^n [j]_q, \quad \text{with} \quad [0]_q! = 1,
\]

\[
[n]_q \quad \begin{cases} [n]_q^{[n]}_0 & \text{if } n \geq k \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

We will use also the so called \( q \)-Pochhammer symbol for \( n \geq 1 : (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \), \((a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^k (a_j; q)_n \), with \((a; q)_0 = 1\).

Often \((a; q)_n\) as well as \((a_1, a_2, \ldots, a_k; q)_n\) will be abbreviated to \((a)_n\) and \((a_1, a_2, \ldots, a_k)_n\), if the base will be \( q \) and if such abbreviation will not cause misunderstanding.

It is easy to notice that for \( q \in (-1, 1) \) we have: \((q)_n = (1 - q)^n [n]_q!\) and that

\[
[n]_{k,q} = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k} & \text{if } n \geq k \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

To support intuition let us notice that:

\[
[n]_{1,q} = \begin{cases} 1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}
\]

\([n]_{0,q} = \begin{cases} 1 & \text{if } n = 0, \\ 1 - a & \text{if } n \geq 1. \end{cases}\]

(1 - \( a \))^n and \([n]_{q} = \begin{cases} 1 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \end{cases}\)

Let us denote for simplicity the following real subsets:

\[
J(q) = \left\{ \begin{array}{ll} -2/\sqrt{1-q} & \text{if } |q| < 1 \\ 2/\sqrt{1-q} & \text{if } q = 1 \end{array} \right\}
\]
The four distributions that we are going to consider in the paper are defined by the densities:

\[ f_h(x|q) = \frac{2(q)^\infty}{\pi} \sqrt{1-x^2} \sum_{j=1}^{\infty} ((1+q^j)^2 - 4q^jx^2) I_{[-1,1]}(x), \]

\[ f_N(x|q) = \frac{1}{2\pi \sqrt{4-(1-q)x^2}} \sum_{k=0}^{\infty} \frac{((1+q^k)^2 - (1-q)x^2q^k)}{w_k(x|a,b,q)} I_{[-1,1]}(x), \]

\[ f_Q(x|a,b,q) = \frac{(q,ab)^\infty}{2\pi \sqrt{1-x^2}} \sum_{k=0}^{\infty} \frac{((1+q^k)^2 - 4q^kx^2)}{w_k(x|a,b,q)} I_{[-1,1]}(x), \]

\[ f_{CN}(x|y,\rho,q) = f_N(x|q) \sum_{k=0}^{\infty} \frac{(1-\rho^2q^k)}{W_k(x,y|\rho,q)} I_{J(q)}(x), \]

where we denoted: \( I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \), and \( w_k \) and \( W_k \) are the following polynomials:

\[ W_k(x,y|\rho,q) = (1-\rho^2q^{2k})^2 - (1-q)\rho q(1+\rho^2q^{2k})xy + (1-q)\rho^2(x^2+y^2)q^{2k}. \]

\[ w_k(x|a,b,q) = (1+a^2q^{2k})(1+b^2q^{2k}) - 2x(a+b)q^k(1+abq^{2k}) + 4x^2abq^{2k}. \]

\( k = 0,1,2,\ldots. \)

Notice that \( \forall k \geq 0 : w_k(x|a,b,q) = w_0(x|aq^k,bq^k,1) \), \( W_k(x,y|\rho,q) = W_0(x,y|\rho q^k,q) \) and that \( W_k(x,y|0,q) = 1. \)

Parameters characterizing these distributions (other than \( q \)) have the following ranges: \( y \in J(q), |\rho| < 1, |a|, |b| < 1. \)

These densities are defined for \( |q| < 1 \) with possibility to extend this range to \( q \in (-1,1) \) for densities \( f_N \) and \( f_{CN} \) but the cases \( q = 1 \) will be understood as limit cases. Thus important special cases can be summed up as follows:

\[ f_h(x|0) = \frac{2}{\pi} \sqrt{1-x^2} I_{[-1,1]}(x), \quad f_N(x|0) = \frac{1}{2\pi} \sqrt{4-x^2} I_{[-2,2]}(x), \quad f_Q(x|a,b,0) = \frac{2(1-ab)}{\pi w_0(x|a,b,1)}, \quad f_{CN}(x|y,\rho,0) = \frac{1}{2\pi W_0(x,y|\rho,1)}, \]

\[ f_{CN}(x|y,\rho,1) = \frac{1}{2\pi W_0(x,y|\rho,1)} \exp \left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right). \]

It is known (see e.g. [8] but also detailed review [21]) that these distributions make the following families of polynomials orthogonal. These families will be defined through their 3-term recurrences:

\[ h_{n+1}(x|q) = 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q), \]

\[ H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q), \]

\[ Q_{n+1}(x|a,b,q) = (2x - (a+b)q^n)Q_n(x|a,b,q) - (1-q^n)(1-abq^{n-1})Q_{n-1}(x|a,b,q), \]

\[ P_{n+1}(x|y,\rho,q) = (x - \rho y q^n)P_n(x|y,\rho,q) - (1-\rho^2q^{n-1})[n]_q P_{n-1}(x|y,\rho,q), \]

with \( h_{-1}(x|q) = H_{-1}(x|q) = Q_{-1}(x|a,b,q) = P_{-1}(x|y,\rho,q) = 0, h_0(x|q) = H_0(x|q) = Q_0(x|a,b,q) = P_0(x|y,\rho,q) = 1 \). Polynomials \( h_n \) and \( H_n \) are called \( q \)-Hermite,
(more precisely continuous $q$–Hermite), while $Q_n$ and $P_n$ Al-Salam–Chihara polynomials.

It is also known that the orthogonal relations have the following form:

\[(2.12) \quad \int_{J(q)} H_m(x|q)H_n(x|q)f_N(x|q)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ |n|_q! & \text{if } n = m \end{cases},\]

\[(2.13) \quad \int_{-1}^{1} h_n(x|q)h_m(x|q)f_h(x|q)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ (q)_n & \text{if } m = n \end{cases},\]

\[(2.14) \quad \int_{-1}^{1} Q_n(x|a,b,q)Q_m(x|a,b,q)f_Q(x|a,b,q)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ (q,ab)_n & \text{if } m = n \end{cases},\]

\[(2.15) \quad \int_{J(q)} P_m(x|y,\rho,q)P_n(x|y,\rho,q)f_{CN}(x|y,\rho,q)\,dx = \begin{cases} 0 & \text{if } n \neq m \\ (\rho^2)_n[n]_q! & \text{if } n = m \end{cases}.

There are interesting special cases: $h_n(x|0) = U_n(x)$, $H_n(x|0) = U_n(x/2)$, $H_n(x|1) = H_n(x)$, $Q_n(x|a,b,0) = U_n(x) - (a+b)U_{n-1}(x) + abU_{n-2}(x)$, $P_n(x|y,\rho,0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2)$, $P_n(x|y,\rho,1) = (1 - \rho^2)^{n/2} H_n((x - \rho y) / \sqrt{1 - \rho^2})$, where we denoted by $U_n$ $n$–th Chebyshev polynomial of the second kind and $H_n(x)$ denotes ordinary Hermite polynomial (so called probabilistic) i.e. monic orthogonal with respect to measure with the density: $\exp(-x^2/2)/\sqrt{2\pi}$.

All the above mentioned facts can be found in [3] but also in more detail in [5], [15], [19].

In the sequel we will need the following two facts:

\[(2.16) \quad \int_{-1}^{1} h_n(x|q)f_Q(x|a,b,q)\,dx = S_n(a,b|q),\]

\[(2.17) \quad \int_{S(q)} H_n(x|q)f_{CN}(x|y,\rho,q)\,dx = \rho^n H_n(y|q),\]

where $S_n(a,b|q) = \sum_{i=0}^{n} [n]_q \, a^i b^{n-i}$. The first of them is shown in [20], the second in [3]. Notice that $S_n(a,b|1) = (a + b)^n$, $S_n(a,b|0) = (a^{n+1} - b^{n+1})/(a - b)$ if $a \neq b$ and $S_n(a,a|0) = (n + 1)a^n$.

3. Moments

We will need the following and the following expansion:

\[(3.1) \quad (1 - q)^{n/2}x^n = \sum_{k=0}^{[n/2]} \binom{n}{k} - \binom{n}{k-1}U_{n-2k}(x\sqrt{1 - q/2}),\]

that can be easily obtained from the relationship:

\[x U_n(x/2) = U_{n+1}(x/2) + U_{n-1}(x/2).\]

This expansion can easily be modified to obtain the following ones:

\[(3.2) \quad 2^n x^n = \sum_{k=0}^{[n/2]} \binom{n}{k} - \binom{n}{k-1}U_{n-2k}(x),\]

\[(3.3) \quad x^n = \sum_{k=0}^{[n/2]} \binom{n}{k} - \binom{n}{k-1}U_{n-2k}(x/2).\]
Let us remark that 
\[ (n - 2k + 1)(n + 1)/\binom{n+1}{k}/(n + 1) = \binom{n}{k} - \binom{n}{k+1}. \]

Basing on these expansions we are able to formulate the following proposition giving expansions of \( x^n \) in the series of \( q \)-Hermite polynomials.

**Proposition 1.** Let us denote 
\[ c_{m,n}(q) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j q^{j(j+1)/2} \binom{n-j}{j} q_{m-j}^{[n-2m+j]}, \]
defined for \( n \geq 1 \), \( m \leq \lfloor n/2 \rfloor \). Then
\[ x^n = \sum_{m=0}^{\lfloor n/2 \rfloor} (1 - q)^{-2m} c_{m,n} H_{n-2m}(x|q), \]
\[ x^n = \frac{1}{2n} \sum_{m=0}^{\lfloor n/2 \rfloor} c_{m,n} h_{n-2m}(x|q). \]

**Proof.** We use (3.2) and (3.3) and the two following expansions first of which was shown in [16] (4.2) and the other is its obvious modification:
\[ U_n \left( x \sqrt{1-q}/2 \right) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j (1 - q)^{n/2 - j} q^{j(j+1)/2} \binom{n-j}{j} q H_{n-2j}(x|q), \]
\[ U_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j q^{j(j+1)/2} \binom{n-j}{j} q H_{n-2j}(x|q). \]

**Proposition 2.** Let \( X \sim f_N \) then i) \( \forall n \geq 1 : \)
(3.4) 
\[ (1-q)^{n/2} \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} y^n f_H(y|q)dy = \begin{cases} 0 & \text{if } n \text{ is odd} \end{cases} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j k_j q^{(k_j-1)/2}(k_j+1) \binom{2j+1}{j-k} \]
(3.5) \( \varphi_N(t|q) = E \exp(tX) = \sqrt{1-q}/t \sum_{k=0}^{\infty} (-1)^k q^{(k+1)/2}(2k+1) I_{2k+1} \left( 2t/\sqrt{1-q} \right) \),
where \( I_k(t) \) is the modified Bessel function of the first kind.

ii) Let \( X \sim f_h \) then
(3.6) 
\[ \int_{-1}^{1} x^n f_H(x|q)dx = \begin{cases} 0 & \text{if } n \text{ is odd} \end{cases} \sum_{k=0}^{n} (-1)^k q^{(k-1)/2}(2k+1) \binom{2j+1}{j-k} \]
\[ \varphi_h(t|q) = E \exp(tX) = \frac{2}{t} \sum_{k=0}^{\infty} (-1)^k q^{(k+1)/2}(2k+1) I_{2k+1}(t). \]

**Proof.** Is shifted to Section 4

**Remark 1.** Setting \( q = 0 \) we get
\[ \int_{-2}^{2} y^{2n} f_H(y|0)dy = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n} \]
i.e. \( n \)-Catalan number. Setting \( q = 1 \) we get \( \forall n \geq 1 : \)
\[ (2n-1)!! = \lim_{q \to 1} \frac{1}{(2n+1)(1-q)^n} \sum_{k=0}^{n} (-1)^k q^{(k+1)/2}(2k+1) \binom{2n+1}{n-k}. \]

As far as the moments of \( f_{CN} \) and \( f_Q \) are concerned we the following result.
Proposition 3. Let \( X \sim f_{\text{CN}} \) with parameters \( y, \rho, q \) then

\[ EX^n = \sum_{m=0}^{[n/2]} (1 - q)^m \rho^{n-2m} H_{n-2m}(y|q)c_{m,n}(q), \]

\[ \varphi_{\text{CN}}(t, y, \rho, q) = E \exp(tX) = \frac{\sqrt{1-q}}{t} \sum_{k=0}^{\infty} \frac{(1-q)^k/2}{[k]_q!} \rho^k H_k(y|q) \]

\[ \times \sum_{j=0}^{\infty} (-1)^j \frac{[k+j]_q!(k+2j+1)}{[j]_q!} q^{j(j+1)/2} I_{2j+k+1}(2t/\sqrt{1-q}). \]

ii) Let \( X \sim f_Q \) with parameters \( a, b, q \) then

\[ EX^n = \frac{1}{2^n} \sum_{j=0}^{[n/2]} c_{j,n} S_{n-2j}(a,b|q), \]

\[ \varphi_Q(t, a, b, q) = \frac{2}{t} \sum_{k=0}^{\infty} S_k(a,b|q) \frac{q^k}{(q)_k} \sum_{j=0}^{\infty} (-1)^j \frac{(q)_{k+j}(k+2j+1)}{(q)_j} q^{j(j+1)/2} I_{2j+k+1}(t). \]

Proof. is shifted to Section 4

As a corollary we get the following relationship.

Corollary 1. i) \( \lim_{q \to 1^{-}} c_{m,n}(q)/(1 - q)^m = \)

\[ \lim_{q \to 1^{-}} \frac{1}{(1 - q)^m} \sum_{j=0}^{m} (-1)^j q^{j(j+1)/2} \left( \binom{n}{m-j} - \binom{n}{m-j-1} \right) \left( \begin{array}{c} n-2m+j \\ j \end{array} \right) q^j \]

\[ = \frac{n!}{2^m m!(n-2m)!}. \]

ii)

\[ \lim_{q \to 1^{-}} \frac{\sqrt{1-q}}{t} \sum_{k=0}^{\infty} (-1)^k q^{k+1/2} (2k+1) I_{2k+1}(2t/\sqrt{1-q}) = \exp(-t^2/2), \]

iii)

\[ \lim_{q \to 1^{-}} \frac{\sqrt{1-q}}{t} \sum_{k=0}^{\infty} \frac{(1-q)^{k/2}}{[k]_q!} \rho^k H_k(y|q) \sum_{j=0}^{\infty} (-1)^j \frac{[k+j]_q!(k+2j+1)}{[j]_q!} q^{j(j+1)/2} I_{2j+k+1}(2t/\sqrt{1-q}) = \exp(tpy + (1-\rho^2)t^2/2). \]

Proof. i) We apply the following well known expansion

\[ x^n = \sum_{m=0}^{[n/2]} \frac{n!}{2^m m!(n-2m)!} H_{n-2m}(x), \]

and the fact that \( \int_{-\infty}^{\infty} H_n(x)f_{\text{CN}}(x|y, \rho, 1)dx = \rho^n H_n(y) \) obtaining \( n \)-th moment of the \( f_{\text{CN}}(x|y, \rho, 1) \) distribution:

\[ \int_{-\infty}^{\infty} x^n f_{\text{CN}}(x|y, \rho, 1)dx = \sum_{m=0}^{[n/2]} \frac{n!}{2^m m!(n-2m)!} \rho^{n-2m} H_{n-2m}(y). \]
Now applying uniqueness of the expansion in orthogonal polynomials and assertion ii) we deduce our limit.

ii) We have \( \int_0^\infty \exp(tx) \exp(-(x-\rho y)^2/(2(1-\rho^2)))dx/\sqrt{2\pi}(1-\rho^2) = \exp(t\rho y + (1-\rho^2)t^2/2). \)

\[ \square \]

4. Proofs

Proof of Proposition 3. i) We use the following expansion following expansion and its obvious modification:

\[
(4.1) f_H(x|q) = \frac{\sqrt{1-q}}{2\pi} \sqrt{4 - (1-q)x^2} \sum_{m=0}^\infty (-1)^m q^{(m+1)2} U_{2m}(x\sqrt{1-q}/2),
\]

\[
(4.2) f_h(x|q) = \frac{2\sqrt{1-q^2}}{\pi} \sum_{m=0}^\infty (-1)^m q^{(m+1)2} U_{2m}(x),
\]

given in [15] Lemma 2, iv. Secondly we apply (3.1) and the fact that polynomials \( U_n(x/2) \) are orthogonal with respect to Wigner distribution.

ii) We have: \( \int_{-\sqrt{1-q}}^{\sqrt{1-q}} \exp(y) f_H(y|q)dy = \)

\[
\sum_{j=0}^\infty \frac{j^{2j}}{2j+1!(1-q)^{j}} \sum_{k=0}^j \frac{(-1)^k q^{(k+1)2} (2k+1) (2/j-k)^{2j}}{j! (j+1)! (1-q)^{j}}
\]

\[
= \frac{\sum_{k=0}^\infty (-1)^k q^{(k+1)2} (2k+1) \sum_{j=k}^\infty j^{2j}}{(j-k)! (j+1)! (1-q)^{j}}
\]

\[
= \frac{\sum_{k=0}^\infty (-1)^k q^{(k+1)2} (2k+1) \sum_{m=0}^\infty \frac{m^{2m+2k}}{m!(2k+m+1)! (1-q)^{m+2k}}}{m!(2k+m+1)! (1-q)^{m+2k}}.
\]

Now it is enough to recall that \( I_\alpha(t) = \sum_{m=0}^\infty \frac{t^{2j}}{m!(m+\alpha+1)}. \)

To show \( \mathbf{30} \) we have: \( \int_{-\sqrt{1-q}}^{\sqrt{1-q}} \exp(x) f_h(x|q)dx = \)

\[
\sum_{j=0}^\infty \frac{j^{2j}}{2j+1!(1-q)^{j}} \sum_{k=0}^\infty (-1)^k q^{(k+1)2} (2k+1) (2/j-k)^{2j} \sum_{j=k}^\infty j^{2j} \sum_{m=0}^\infty \frac{m^{2m+2k}}{m!(2k+m+1)! (1-q)^{m+2k}}.
\]

\[ \square \]

Proof of Proposition 3. ii) We use the following three facts: One is (3.1), the second \( \mathbf{2.17} \) \( \mathbf{2.10} \), the third formulæ \( \mathbf{4.1} \) and \( \mathbf{4.2} \). Using them we have:

\[
(1-q)^{n/2}EX^n = \frac{1}{n+1} \sum_{m=0}^\infty \frac{[n/2]}{(n-2k+1)(n+1)} \int_{-\sqrt{1-q}}^{\sqrt{1-q}} U_{n-2k}(x\sqrt{1-q}/2) f_{CN}(x|y, \rho, q)dx
\]

\[
= \frac{1}{n+1} \sum_{k=0}^\infty \frac{[n/2]}{(n-2k+1)(n+1)} \times \sum_{j=0}^\infty \frac{m^{2m}}{m!(m+\alpha+1)} \rho^{2m} H_{n-2m}(y|q) c_{m,n}(q).
\]

We have further:

\[
\varphi_{CN}(t, y, \rho, q) = \sum_{m=0}^\infty \frac{t^{2m}}{(1-q)^{m}} \sum_{n=2m}^{\infty} \frac{m^{-2m}}{n^{m} \rho^{2m} n^{-m} H_{n-2m}(y|q)c_{m,n}(q)}
\]

\[
= \sum_{k=0}^\infty t^{2m}(1-q)^{m} \sum_{0}^{\infty} \frac{m^{2m}}{m!(m+\alpha+1)} \rho^{2m} H_{k}(y|q)c_{m,k+2m}(q)
\]

\[
= \sum_{k=0}^\infty \frac{k^{2m}}{k!(k+2m)!} \rho^{2m} H_{k}(y|q) \sum_{m=0}^{\infty} \frac{m^{2m}}{m!(m+\alpha+1)} \sum_{j=0}^{\infty} \frac{m^{-2m}}{n^{m} \rho^{2m} n^{-m} H_{n-2m}(y|q)c_{m,n}(q)}
\]

\[
= \sum_{k=0}^\infty \frac{k^{2m}}{k!(k+2m)!} \rho^{2m} H_{k}(y|q) \sum_{m=0}^{\infty} \frac{m^{2m}}{m!(m+\alpha+1)} \sum_{j=0}^{\infty} \frac{m^{-2m}}{n^{m} \rho^{2m} n^{-m} H_{n-2m}(y|q)c_{m,n}(q)}
\]

\[
= \sum_{k=0}^\infty \frac{k^{2m}}{k!(k+2m)!} \rho^{2m} H_{k}(y|q) \sum_{m=0}^{\infty} \frac{m^{2m}}{m!(m+\alpha+1)} \sum_{j=0}^{\infty} \frac{m^{-2m}}{n^{m} \rho^{2m} n^{-m} H_{n-2m}(y|q)c_{m,n}(q)}
\]

\[ \square \]
\[= \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k H_k(y|q) \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j}k^{2j+1}}{[j]_q^{2j+1}} q^j (j+1)/2 \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} q^{n(k+2j+1+n)}\]

\[= \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k H_k(y|q) \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j}k^{2j+1}}{[j]_q^{2j+1}} q^j (j+1)/2 \times (1-q)/k+2+1/2t-2-k-1 I_{2j+k+1} \]

\[= \frac{1}{2^t} \sum_{k=0}^{\infty} \frac{(1-q)^{1/2}}{|q|^{1/2}} \rho^k H_k(y|q) \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j}k^{2j+1}}{[j]_q^{2j+1}} q^j (j+1)/2 I_{2j+k+1} (2t/\sqrt{1-q})\]

\[\text{ii) The proof of the first formula is analogous.}\]

Further we have:

\[\varphi_Q(t, a, b, q) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{[n/2]} \frac{[j]!}{[j]_q^{j+1}} S_{n-2j}(a, b|q)\]

\[= \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k S_k(a, b|q) c_{m,k}\]

\[= \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k S_k(a, b|q) \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} c_{m,k}\]

\[= \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k S_k(a, b|q) \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} c_{m,k}\]

\[= \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k S_k(a, b|q) \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} \sum_{k=0}^{\infty} \frac{t^k}{|q|} \rho^k \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{j=0}^{\infty} \frac{[j]!}{[j]_q^{j+1}} c_{m,k}\]

\[= \frac{2}{t} \sum_{k=0}^{\infty} \frac{S_k(a, b|q)}{|q|} \sum_{j=0}^{\infty} (-1)^j \frac{(q-1)(k+2j+1)}{[j]_q^{j+1}} q^j (j+1)/2 I_{2j+k+1} (t).\]

\[\square\]

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