How much can supergravity teach us about the microscopic features of BPS black holes?

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Abstract

We review recent results in the study of regular four dimensional BPS black holes in toroidally compactified type II (or M) theory. We discuss the generating solution for this kind of black holes, its microscopic description(s), and compute the corresponding microscopic entropy. These achievements, which provide a description of the fundamental degrees of freedom accounting for the entropy of any regular BPS black hole in the theory under consideration, are inscribed within a research project aimed to the study of the microscopic properties of this kind of solutions in relation to U–duality invariants computed on the corresponding macroscopic (supergravity) description.

1 Introduction

One of the main issues of the “second string revolution” (1995) is the concept of string dualities which provided a new insight into the non–perturbative side of the known superstring theories. These dualities are mappings between regimes of different superstring theories (some of them have been verified while other just conjectured). Their existence naturally induces to consider the known superstring theories as perturbative realizations on different backgrounds of a fundamental theory of gravity (FTG) whose general formulation however is still missing. It is known that the low energy limit of superstring theory is described by supergravity. Although supergravity in this picture is regarded just as a \textit{macroscopic} theory, it is expected to possess important informations about the FTG. Indeed, it has been argued \cite{1} that the largest (continuous) global symmetry group $U$ of the supergravity
field equations and Bianchi identities at classical level should encode the definition, as a suitable discrete group $U(Z)$, of the conjectured superstring $U$–duality, namely the ultimate duality connecting all superstring theories realized on various backgrounds. This duality is thus expected to be an exact symmetry of the FTG. Unfortunately not much is known about the group $U(Z)$, starting from the very definition and its action on superstring states. On the other hand the action of the group $U$ on the supergravity solutions is, in principle, known.

A fundamental role in probing superstring dualities has been played so far by the BPS black hole solutions of supergravity. These solutions are characterized by the property of preserving a fraction of the original supersymmetries, and this feature protects their physical quantities, to a certain extent, from quantum corrections. As a consequence of their supersymmetry, BPS black holes in supergravity are expected to correspond to exact solutions of superstring theory. The BPS condition moreover is $U$–duality invariant. This allows to characterize these supergravity solutions within orbits of the continuous $U$–duality group, defined by a certain number of $U$–invariants $\{I_k\}$ (e.g. the entropy). All the physical properties of the BPS solutions entering the same $U$–duality orbit are expected to be encoded in the corresponding generating solution. The generating solution of BPS black holes is defined, within a certain supergravity theory, as the solution depending on the least number of parameters such that the invariants $\{I_k\}$ are free for a certain choice of the boundary conditions. As a consequence of its definition, by acting on the generating solution by means of $U$ one recovers the whole $U$–duality orbit. A suitable discrete set of points within this orbit should correspond to superstring black holes (non–perturbative solutions) connected by the action of $U(Z)$ and which therefore represent different descriptions of a same solution within the FTG (see figure 1). The microscopic degrees of freedom described by the FTG are indeed related to invariants of the group $U(Z)$. Pinpointing the exact correspondence between the macroscopic (supergravity) and microscopic descriptions (e.g. in terms of D–branes in a suitable regime) of a generating solution, one would in principle be able to study systematically the microscopic realization of a generic solution in the same orbit. Moreover this could be the first step in order to unravel the action of $U(Z)$ on stringy objects in higher dimensions and to ultimately deduce their fundamental degrees of freedom.

Here we review some recent results achieved in [2, 3, 4] where a macroscopic (supergravity) starting point was adopted for a systematic microscopic analysis of regular BPS (static, spherically symmetric) black holes within type II (M) theory compactified down to four dimensions on tori, and whose zero modes are described by $N = 8$ four dimensional supergravity.

The paper is organized as follows. In section 2 we shall start addressing the question: how much can we learn at classical supergravity level about the microscopic description of a BPS solution? A possible answer will lead us to discuss the mathematical analysis carried out in [4] which provides an intrinsic group theoretical characterization of the scalar and vector fields in the $D = 4, N = 8$ theory in
terms of dimensionally reduced type II fields. The geometrical framework so defined turns out to provide the convenient “laboratory” in which to systematically study the microscopic descriptions of BPS solutions and their duality relations. Using these tools one can then characterize R–R charged generating solutions of regular BPS black holes as elements of a suitable equivalence class defined with respect to the action of $S$ and $T$ dualities. This result is discussed in section 3 and allows us to formulate the precise correspondence, worked out in [3], between the parameters defining R–R charged (D–brane) microscopic descriptions of black hole solutions and the supergravity quantities related to its macroscopic descriptions. This will be used, in section 4, for providing a type IIA/IIB/M–theory description of the generating solution of regular BPS black holes, and a prediction on the expression of the macroscopic entropy (at tree level) in terms of microscopic parameters. Finally, in section 5, focusing on the M-theory description of the generating solution, the same expression for the entropy will be retrieved from a counting of BPS micro–states, extending the analysis in [3] to the toroidal case. This last result was achieved in [4]. From the very definition of generating solution, this analysis accounts for the microscopic entropy of the most general black hole solution of this kind.

2 Supergravity Laboratory

The only prediction which may be drawn at classical supergravity level on the microscopic description of a BPS solution is clearly limited to the background fields which couple to it. This can be done for instance by associating each superstring
scalar and vector zero–mode with quantities intrinsic to the $U$–duality group of the
low–energy supergravity $[3, 2]$.

The $D = 4, N = 8$ supergravity is a maximally extended supersymmetric theory, i.e. it has 32 supercharges. Its bosonic sector consists of the graviton, 70 scalar fields, spanning the homogeneous manifold $M_{\text{scal}} = E_7(7)/SU(8)$, and 28 vector fields. The latter are related to a vector of 56 quantized charges $(p^\Lambda, q_\Sigma)$, which transforms in the $Sp(56)$ of $E_7(7)$, and a central charge matrix $Z_{AB}$ entering the local realization on the moduli space of the supersymmetry algebra and transforming in the 28 of $SU(8)$. The former charges are moduli–independent and should be regarded just as supergravity parameters, while the latter are moduli dependent and are related to the physical charges, i.e. the actual charges one would measure in the asymptotically flat radial infinity of a black hole solution.

The $U$–duality group of the classical theory is $U = E_7(7)$ $[4]$. It acts as a generalized electro–magnetic duality, i.e. it has a non–linear action on the scalar fields and a linear (symplectic) action on the vector of quantized charges. As previously mentioned, the $D = 4, N = 8$ theory describes the low–energy limit of type II superstring theory on $T_6$ (or M–theory on $T_7$). The first step towards a group theoretical characterization of the ten–dimensional origin of the scalars and charges in this supergravity model is to use a linear algebraic description of the scalar fields. This is achieved by adopting the solvable Lie algebra (SLA) parameterization of the scalar manifold $[4, 6, 7, 8]$, which consists in describing the scalar fields as local parameters of a solvable Lie algebra which generates (globally) the scalar manifold as a solvable Lie group. Homogeneous non–compact manifolds of symmetric type like $M_{\text{scal}}$ do admit such a representation:

$$M_{\text{scal}} = \text{Exp}(\text{Solv}(U)) \quad (1)$$

The algebra $\text{Solv}(U)$ is defined by the Iwasawa decomposition of $E_7(7)$ and can be written as $\text{Solv}(U) = C \oplus N$, where $C$ is the Cartan subalgebra of $E_7(7)$ while $N$ is the nilpotent subalgebra of $E_7(7)$ generated by all the shift generators corresponding to positive roots. In this framework a one to one correspondence between the scalar fields and the generators of $\text{Solv}$ is defined.

Two relevant duality groups for our discussion are the $S = SL(2, R)$ and $T = O(6, 6)$ subgroups of $U$, defined as the continuous counterparts at the classical level of the discrete $S$ and $T$ superstring dualities $[4]$. Since these dualities are the largest preserving the R–R and NS–NS identities of the fields, decomposing $\text{Solv}(U)$ with respect to $\text{Solv}(S) \times \text{Solv}(T)$ one may achieve an intrinsic characterization of the R–R and NS–NS fields at classical supergravity level. On the other hand the dimensional reduction of type II superstring to four dimensions may be performed through intermediate steps which define, in the low–energy limit, higher dimensional interactions.

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1In our formalism, this $S$–duality has not to be confused with the self–duality of the type IIB theory. Indeed in four dimensions it acts only on the effective dilaton and the four dimensional axion deriving from the NS-NS Kalb–Ramond field. The present $SL(2, R)$ has just a $O(1, 1)$ intersection with the ten dimensional type IIB $SL(2, R)$ symmetry group.
maximal supergravities, with their own $U$–duality group $U_{D>4}$ at tree level. Fixing then the embedding of $\text{Solv}(U_{D>4})$ within $\text{Solv}(U)$ for various $D > 4$ allows to identify in a consistent way the scalar fields of the $N = 8$ theory, as associated with the corresponding generators of $\text{Solv}(U)$, with dimensionally reduced type II zero–modes.

On the vector field side, it is convenient to work with a set of physical charges $(y^A, x_\Sigma)$ (transforming under $SU(8)$) which are expressed in the same basis of weights $\{\lambda\}$, generating the 56 of $U$, as the quantized charges $(p, q)$. These charges are obtained from the vector $(\text{Re}Z_{AB}, \text{Im}Z_{AB})$ through a suitable rotation and are related to the quantized charged $(p, q)$ by a moduli–dependent symplectic transformation which makes them quantized as well [2]. Decomposing the weight basis $\{\lambda\}$ with respect to the action of the higher dimensional $U$–dualities $U_{D>4}$ it was possible to associate consistently with each weight $\lambda$ a one–form electric or magnetic potential in four dimensions deriving from suitable ten dimensional zero–modes.

As a result of this first group theoretical analysis an $N = 8$ algebraic dictionary [2] could be established on the weight lattice $\Lambda_W(U)$ of $U$ in which the directions (namely Cartan generators in $\mathcal{C}$) and the positive roots are associated with scalar fields (through the SLA parameterization) and the weights $\{\lambda\}$ with electric and magnetic one–form potentials, each of these fields having a specific ten dimensional characterization.

3 Regular BPS black holes with R–R charge

Regular BPS black holes are BPS solutions having a finite horizon area. They where shown to preserve $1/8$ of the original $N = 8$ supersymmetries and to interpolate between an $N = 2$ vacuum of the form $AdS_2 \times S^2$ near the horizon ($r \to 0$) and a Minkowski vacuum at radial infinity ($r \to \infty$) [11]. The physical charges of a BPS black hole solution, as previously mentioned, are related to the (antisymmetric) central charge matrix $Z_{AB}$ which depends on the point on the moduli space $\phi_0$, representing the boundary condition at radial infinity of the scalar fields, as well as on the quantized charges. The $U$–duality invariants $\{I_k\}$ of the solution are given by all the $SU(8)$ invariants which can be built out of $Z_{AB}$. Indeed, acting by means of a $U$–duality transformation on the scalar fields and the quantized charges, the central charge matrix will transform under a corresponding $SU(8)$ transformation. These invariants are five and on the orbit of regular BPS black holes they are independent parameters. A way of expressing them is in terms of the norm of the central charge skew–eigenvalues $Z_\alpha$ ($\alpha = 0, \ldots, 3$) and their overall phase, i.e. $\{I_k\} = \{|Z_\alpha|, \Theta\}$. By suitably combining them it is possible to obtain a moduli–independent invariant, namely the quartic invariant $J(x, y)$ of the 56 of $E_7(7)$ (the orbits of BPS black holes have $J(x, y) \geq 0$ [11]). This is the only invariant characterizing the near–horizon geometry of the solution. The area of the horizon is $A = 4\pi \sqrt{J(x, y)}$ and, using
Bekenstein–Hawking formula, the tree level entropy turns out to be [12]:

\[ S = \mathcal{A}/4 = \pi \sqrt{J(x, y)} \]  

The generating solution is defined by a choice of the bosonic vacuum at infinity \( \phi_0 \) and by the minimum number (i.e. five) of charges in terms of which the invariants \( \{I_k\} \), computed on \( \phi_0 \), are independent functions. This solution can be described within a \( STU \) model, which is characterized as the smallest consistent truncation of the \( N = 8 \) theory on which the four \( Z_\alpha \) are independent [13]. The \( STU \) model is an \( N = 2 \) supergravity coupled to three vector multiplets, its classical \( U \)–duality group \( U_{STU} = SL(2, \mathbb{R})^3 \subset U \) is defined by the isometry group of the scalar manifold \( \mathcal{M}_{STU} = U_{STU}/SO(2)^3 \). The latter, in the SLA formalism, may be described as a solvable Lie group generated by a solvable Lie algebra \( \text{Solv}_{STU} \) which is parametrized by just three dilaton fields \( b_i \) and three axions \( a_i \). This model moreover has four vector fields which give rise to eight quantized charges \( (p^\alpha, q_\beta) \) and eight physical charges \( (y^\alpha, x_\beta) \). In the light of the previously defined algebraic dictionary, different microscopic descriptions of the generating solution can be put in correspondence with different embeddings of the \( STU \) model within the \( N = 8 \) one (defined by the embedding of the corresponding solvable Lie algebras and charge weights\(^2\)).

Dualities relating different embeddings of the \( STU \) model are naturally described in terms of the action on \( \Lambda_W(U_{STU}) \) of automorphisms (\( \text{Aut} \)) of the relevant duality algebra [2]. In order to characterize the generating solution as charged with respect to R–R or NS–NS fields, we would need then to consider the action of the \( S \times T \) dualities through their authomorphism group (\( \text{Aut}(S \times T) \)). The Dynkin diagram of the \( T \) algebra is \( D_6 \). It has inner and outer automorphisms, the latter being related, through Weyl transformations, to the only symmetry of \( D_6 \) (for a study of Weyl duality transformations in supergravity see [14]). These outer automorphisms are particularly interesting since they are not a symmetry and can be thought of as relating two different descriptions of the same theory, namely the type IIA and type IIB ones. Indeed, using the SLA representation it was shown in [2] that the outer automorphisms of \( T \) correspond to a “large ↔ small radius” \( T \)–dualities along an odd number of directions inside \( T_6 \). Taking into account the action of these outer automorphisms the \( N = 8 \) algebraic dictionary was consistently enlarged to accommodate both type IIA and IIB descriptions of the \( N = 8 \) theory (see tables 2 and 3 of [2]).

Within the mathematical framework defined above, two \( T \)–dual embeddings \( STU_1, \ STU_2 \) of the \( STU \) model, for which the charges were related to suitable R–R one–forms, were worked out in [2]. The corresponding two descriptions of the fields in the \( STU \) model in terms of \( E_7(7) \) weights are mapped into each other through an outer automorphism of \( T \), which is interpreted, in the SLA formalism, as a “large ↔ small radius” duality along the directions \( x^5, x^7, x^9 \) of \( T_6 \) (in our notation the

\(^2\)In other words, by the embedding of the weight lattices: \( \Lambda_W(U_{STU}) \subset \Lambda_W(U) \).
compact directions are $x^4, \ldots, x^9$ while the non-compact are $x^0, \ldots, x^3$). One embedding ($STU_1$) can be indeed consistently described in the type IIA setting while the other ($STU_2$) in the type IIB one. In particular, from the $N = 8$ algebraic dictionary, it is possible to characterize the axions of the $STU_1$ embedding as deriving from the antisymmetric tensor $B_{MN} (\{a_i\} = \{B_{45}, B_{67}, B_{89}\})$ while those in $STU_2$ as deriving from the metric $G_{MN} (\{a_i\} = \{G_{45}, G_{67}, G_{89}\})$. As far as the vector fields are concerned, in an analogous way the charges $(y^\alpha, x^\beta)$ in the type IIA embedding $STU_1$ are associated with 1–form (magnetic and electric) potentials deriving from the following components of the ten dimensional R-R fields $A_M, A_{MNP}$:

\[
(y^\alpha) \leftrightarrow (A_\mu 456789, A_\mu 6789, A_\mu 4567) \\
(x^\beta) \leftrightarrow (A_\mu, A_\mu 45, A_\mu 67, A_\mu 89)
\]  

(3)

while for the type IIB embedding $STU_2$ this correspondence between charges and components of the R–R forms $A_{MN}, A_{MNPQ}$ reads:

\[
(y^\alpha) \leftrightarrow (A_\mu 468, A_\mu 568, A_\mu 4578) \\
(x^\beta) \leftrightarrow (A_\mu 579, A_\mu 479, A_\mu 569, A_\mu 578)
\]  

(4)

From this background field prediction and from the values of the physical charges of the generating solution at infinity (for a suitable choice of the boundary conditions), two $T$–dual D–brane descriptions, corresponding to the embeddings discussed above, can be consistently worked out and precise relations established between the parameters defining the macroscopic (supergravity) and microscopic (D–brane) descriptions of the generating solution $\mathcal{E}$. Finally, acting on $STU_{1,2}$ by means of $Aut(S \times T)$ one could define an equivalence class of R–R charged embeddings of the $STU$ model (yielding all the R–R charged generating solutions) within the $N = 8$ theory.

4 Generating solution in type IIA/IIB/M–theory

The machinery reviewed in the previous section was used in $\mathcal{E}$ to construct out of the macroscopic description of the generating solution, possible IIA, IIB and M-theory microscopic realizations. Let us briefly summarize its structure. The ansatze for the generating solution in terms of metric, scalar fields $z^i = a_i + i b_i$ and the four vector field strengths is:

\[
ds^2 = e^{2\mu(r)} dt^2 - e^{-2\mu(r)} dx^2 \quad (r^2 = \vec{x}^2) \\
 z^i(x) = z^i(r) \\
 F^\Lambda(r) = \frac{p_\Lambda}{2r^3} \epsilon_{krs} x^k dx^r \wedge x^s - \frac{l_\Lambda(r)}{r^3} e^{2\mu(r)} dt \wedge \vec{x} \cdot d\vec{x}, \quad \Lambda = 0, 1, 2, 3
\]  

(5)

$l_\Lambda$ being the moduli–dependent electric charges defined in $\mathcal{E}$. The solution of both field equations and the first order differential equations representing the BPS
condition can be expressed in terms of harmonic functions. In particular the generating solution we are interested in will depend only on five charges chosen in such a way that the three charges which are set to zero break completely the $SO(2)^3$ local symmetry of the model. A possible choice for these charges is $p^1, p^2, p^3, q_0, q_1$. Secondly we choose the following boundary condition for the scalar fields at radial infinity:

$$\langle \phi_0 \rangle \equiv \begin{cases} a_1 = a_2 = 0; a_3 = g \\ b_i = -1 \end{cases} \quad g = \frac{q_1}{p^1 + p^2}$$

(6)

This allows to write the physical charges $(x, y)$ in terms of the moduli–independent ones $(p, q)$ and to use the former to describe the solution. The symplectic transformation connecting the two sets of charges on our solution is:

$$y_0 = 0, y_i = p_i, x_0 = q_0, x_1 = -x_2 = g p^1.$$ Let us now introduce the following five harmonic functions:

$$H_i(r) = 1 + \sqrt{2} \frac{y_i}{r} \quad \text{with} \quad i = 1, 2, 3$$

$$H_0(r) = 1 + \sqrt{2} \frac{x_0}{r} \quad \text{and}$$

$$H_1(r) = g \left( H^1(r) + H^2(r) - 1 \right)$$

(7)

where the constant $g$, which is fixed by supersymmetry, is given in eq.(6) and can be alternatively expressed in terms of the physical charges as $g = x_1/y_1$. The solution as far as the scalar fields and metric are concerned has the following form:

$$a_1 = \frac{-H_1 H^1 + g H^2}{2 H^2 H^3}, \quad b_1 = -\sqrt{\frac{H_0 H^1}{H^2 H^3} - \frac{1}{4} \left( \frac{H_1 H^1 - g H^2}{H^2 H^3} \right)^2}$$

$$a_2 = \frac{H_1 H^1 - g H^2}{2 H^1 H^3}, \quad b_2 = -\sqrt{\frac{H_0 H^2}{H^1 H^3} - \frac{1}{4} \left( \frac{H_1 H^1 - g H^2}{H^1 H^3} \right)^2}$$

$$a_3 = \frac{H_1 H^1 + g H^2}{2 H^1 H^3}, \quad b_3 = -\sqrt{\frac{H_0 H^3}{H^1 H^2} - \frac{1}{4} \left( \frac{H_1 H^1 - g H^2}{H^1 H^2} \right)^2}$$

$$U = -\frac{1}{4} \ln \left( H_0 H^1 H^2 H^3 - \frac{1}{4} (H_1 H^1 - g H^2)^2 \right)$$

(8)

We can see from the above expressions that the parameter $x = x_1 = -x_2$ has a special role: switching it off the axion fields $a_i$ become identically zero and the solution reduces to a four parameter purely dilatonic one. We shall comment in the sequel on the microscopic interpretation of this fifth parameter.

Starting from the two IIA and IIB $T$–dual embeddings previously defined, it was possible to characterize two microscopic descriptions of the generating solution where all the four one–form potentials derived from R–R ten dimensional fields.
In this way the solution could be described in the weak string coupling limit in terms of bound states of D–branes wrapped on $T_6$. On the type IIB front the microscopic system consists of $N_0$, $N_1$, $N_2$, $N_3$ D3–branes arranged within $T_6$ in such a way as to preserve $N = 1$ supersymmetry and this requires the relative rotation between each couple of D3–brane to be a $SU(3)$ rotation. The corresponding $T$–dual type IIA system consists of a set of D0–branes and three sets of D4–branes along the four–cycles (6789), (4589) and (4567). In addition, there is a magnetic flux switched on the world volume of the latter (i.e. along (4567)) which is proportional to a rational number $\gamma = m/n$, where the integers $m, n$ are related to the non–trivial angle $\theta$ characterizing the type IIB configuration by the condition: $n \sin \theta = m \cos \theta$. This flux induces effective D0 and D2 charges via Chern-Simons couplings [16].

The eleven dimensional $S$–dual (M–theory) correspondent of the type IIA configuration is summarized in table 1 and consists of a set of three bunches of M5–branes intersecting on a (compact) line and with non–trivial 3–form field strength $h^{(3)} = db^{(2)}$ switched–on on their world volume. In this phase the compact space has an extra dimension, of course, $T_6 \times S_1$. Besides the M5–branes, which are $N_1, N_2, N_3 n^2$ respectively, there are $N_0 + N_3 m^2$ units of KK momentum along the spatial $10^{th}$ direction and the magnetic flux, related to non–trivial 3–form field strength excited on the M5–brane, is proportional to $\gamma$. This is of course the same flux present in the $S$–dual type IIA description.

| Brane       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------------|---|---|---|---|---|---|---|---|---|---|----|
| $P_L$       |   |   |   |   |   |   |   |   |   |   | x  |
| $M5$        |   |   |   |   |   | * |   | * | * | * | x  |
| $M5$        |   |   |   |   |   | * |   | * | * | * | x  |
| $M5 + h^{(3)}$ |   | * |   | * | * | * |   | * | * | * | x  |

Table 1: The M–theory generating configuration.

The precise relation between the macroscopic charges $(y, x)$ as related to the effective charges along the various cycles of $T_6 \times S_1$ and the microscopic parameters $\{N_\alpha, p, q\}$, is given in table 2.

One of the crucial issues in the stringy description of supergravity black holes is the precise characterization of the parameters entering the solution in terms of microscopic quantities. In particular, the interpretation of the fifth parameter of the generating solution, which is related to a non trivial overall phase $\Theta$ of the central charge skew–eigenvalues, is rather tricky to be dealt with [17, 18]. Thanks to the construction reviewed in the previous section we can clearly understand its role. It is clear from eq.(8) that the non vanishing of this parameter is related to our solution being an axionic one. Switching it off indeed, as previously noticed, one gets back a pure (four parameter) dilatonic solution. The number of independent harmonic functions is four, in both cases. This is an expected feature and is related to the fact that within the five invariants of the $U$–duality group four are
Table 2: The relation between M/IIA/IIB microscopic parameters and the macroscopic charges.

moduli–dependent and one is moduli–independent (namely the J(x, y) polynomial related to the entropy). The conclusion which can be drawn is thus that the generating solution is intrinsically axionic, i.e., in order to recover the full U–duality orbit the pure dilatonic solution is not sufficient. According to table 2 one can also easily understand the microscopic interpretation of this parameter: it is related to the non-trivial magnetic flux switched-on on the D4 or M5-brane world-volumes, in the type IIA and M–theory phases, respectively. From a type IIB viewpoint, the same quantity is related to a non–trivial SU(3) rotation between couples of intersecting D3-branes. Switching it off one gets back a configuration of orthogonally intersecting D3–branes or a type IIA/M brane configuration without any non-trivial world–volume field. This is consistent with the interpretation of the axion fields in the corresponding two embeddings ST U₂ and ST U₁ discussed in section 3: in the former they are related to non–diagonal components of the ten dimensional metric tensor, in the latter to internal components of the Kalb–Ramond field.

Using Bekenstein–Hawking formula on the generating solution, eq.s (5)–(8), and expressing the charges at infinity in terms of the microscopic parameters by means of table 2, the following expression for the macroscopic entropy (tree level) in terms of the microscopic quantities can be derived:

\[
S = 2 \pi \sqrt{N_1 N_2 N_3 n^2 \left[ N_0 + m^2 N_3 - \frac{1}{4} m^2 N_3 \left( \frac{N_1 + N_2}{N_1 N_2} \right)^2 \right]} \quad (9)
\]

In the following, we shall review the derivation of the expression (9) from a microscopic BPS state counting.

5 Microscopic entropy counting

In [4] the M–theory description of the generating solution was considered and its microscopic entropy computed. Since the physical quantities related to the BPS
microstates that we are interested in are insensitive to smooth deformations of the background moduli, the latter can be chosen in such a way as to make the microscopic entropy counting feasible. The choice made in \cite{4} corresponds to the regime in which the dynamics of the branes decouple from the bulk and their “thickness” is much smaller than all the other length scales in the theory (in M–theory, the Planck length and the size of the internal manifold $T_7$) and therefore the supergravity description of the solution cannot in general be trusted. In this limit the low energy effective theory on the world volume of each M5–brane in table \ref{4} in the background of the other two is a $(0, 2)$ SCFT. The quantization of the eleven dimensional system is therefore performed in the framework of M–theory on $\mathbb{R}^{1,3} \times T_7$ extending to the toroidal case the results of \cite{3} for the case of M–theory on $\mathbb{R}^{1,3} \times CY_3 \times S_1$.

A fruitful strategy for performing a microscopic entropy counting on a BPS solution has been so far to restrict to a particular background on which the low energy dynamics of the system is actually described by a $1 + 1$ SCFT \cite{19, 20, 5}. In this limit, the asymptotic value of the degeneracy of states for a high excitation level is given by the Cardy formula \cite{24}, which, if we restrict only to the left–movers (see below), has the form:

$$\rho(n) \approx e^{2\pi \sqrt{c_L n / 6}}$$  \hspace{1cm} (10)

$\rho(h)$ being the state degeneracy for the left–moving excitation level $h$ while $c_L$ is the central charge of the left–mover sector (the above formula holds in the limit $h \gg c_L$). Using Boltzman equation and eq. (10) the microscopic entropy can be expressed as:

$$S = \ln \rho(h) \approx 2\pi \sqrt{c_L h / 6}$$  \hspace{1cm} (11)

As far as the M–brane system in table \ref{4} is concerned the effective low energy description on terms of a $1 + 1$ SCFT is obtained in the limit in which the radius $R$ of the eleventh dimension $S_1$ is much larger than the linear size of the orthogonal $T_6$. In particular, the M5 branes can be described as wrapped on $P \times S_1$, $P$ being an holomorphic cycle of $T_6$ (seen as a complex 3–manifold). As the size of $P$ shrinks with respect to R the low energy dynamics of the M5 brane is described by the dimensional reduction of the $(0, 2)$ SCFT to $S_1 \times \mathbb{R}$, which is a $(0, 4)$ SCFT in $1 + 1$ dimensions. The bosonic fields of the latter theory are the moduli of the cycles $P$ and of the chiral two form $b^{(2)}$. BPS states in this framework are annihilated by all the four right–moving supercharges and therefore are characterized only by the left–moving excitation level $h$: $|BPS> = |h\rangle_L \otimes |0\rangle_R$.

In order to use equation (11) for computing the entropy, we need therefore to determine $c_L$ and $h$ in terms of the charges $(x, y)$ characterizing our black hole.\footnote{For a review on microscopic entropy counting see \cite{21, 22, 23} and references therein.}
5.1 Computation of $c_L$

The general expression for $c_L$ is $c_L = N^B_L + N^F_L / 2$, where $N^B_L$ and $N^F_L$ are the number of bosonic and fermionic degrees of freedom in the left–moving sector, respectively. The former consists of the left–moving moduli $d_P$ of $P$ and of the moduli associated to the form $b^{(2)}$. Let us outline how to evaluate their contribution.

We may associate $P$ with its fundamental class $[P] \in H^{(2)}(T_6, \mathbb{Z})$. In our configuration $[P]$ is expanded in a system of three 2–cycles $\alpha_i$ dual to each four cycle in $P$ as in table 4, $(\alpha_i \equiv dx^a \wedge dx^b, \{(a, b)\} \equiv \{(4, 5), (6, 7), (8, 9)\})$ and the corresponding coefficient is the integer magnetic charge $y^i$: $[P] = \sum_i y^i \alpha_i$. The intersection matrix restricted to $(\alpha_i)$ is denoted by $D_{ijk} = (\int_{T_6} \alpha_i \wedge \alpha_j \wedge \alpha_k) / 6$ and is a symmetric matrix whose only non vanishing entry is $D_{123} = 1 / 6$. The volume of $P$ is therefore $\text{Vol}(P) = \int_{T_6} [P]^3 = 6 D_{ijk} y^i y^j y^k = 6 y^1 y^2 y^3$. The moduli of $P$ are, roughly speaking, the number of ways $P$ can be deformed leaving the magnetic charges (i.e. $[P]$) fixed. Using tools of algebraic geometry, the number of these holomorphic deformations can be exactly determined, assuming $P$ to be a very ample divisor, and turns out to be in our case: $d_P = \text{Vol}(P) / 3 - 2$.\(^4\)

As far as the moduli of $b^{(2)}$ are concerned, indicizing by $a$ the coordinates on $P$ and by $b$ those on $S_1 \times \mathbb{R}$, the dimensional reduction of $b^{(2)}$ yields the non trivial fields $b_{ab}$ and $b_{a\beta}$ in the $1 + 1$ theory. The former split into self–dual $b_{ab}^+$ and anti–self–dual components $b_{ab}^-$ on $P$ (spanning the spaces $b^\pm$ respectively) which, as a consequence of the selfduality of $h^{(3)}$ are associated with the left–moving and right–moving moduli, respectively. The dimensions of $b^\pm$ can be easily determined using the Hodge index theorem (again under the hypothesis of $P$ very ample). The components $b_{a\beta}$, on which we shall comment in a moment, are non–dynamical vector fields spanning a space of dimension $b_1(P) = 2 h^{(1,0)}(P)$.

The number of the left–moving and right–moving fermionic moduli, $N^F_L$ and $N^F_R$, can be shown to be related to the dimensions of the cohomology groups $H^{(2r+1,0)}(P)$ and $H^{(2r,0)}(P)$. The final expression for the various quantities cited so far turns out to be the following one:

\[
N^B_L = \int_{T_6} [P]^3 + 2 h_{(1,0)} \left\{ -2 h_{(1,0)} = \int_{T_6} [P]^3 \right\}
\]

\[
N^F_L = 4 h_{(1,0)} \left\{ -4 h_{(1,0)} = 0 \right\}
\]

\[
N^B_R = \frac{2}{3} \int_{T_6} [P]^3 + 2 h_{(1,0)} \left\{ -2 h_{(1,0)} = \frac{2}{3} \int_{T_6} [P]^3 \right\}
\]

\(^4\) The divisor $P$ may be indeed characterized as the zero locus of a holomorphic section of a line bundle $L$ on $T_6$. This section is defined up to multiplication by a non vanishing complex number. Any other holomorphic section of $L$ will define through its zero–locus a different divisor $P'$ (a deformation of $P$) with the same fundamental class: $[P'] = [P]$. Therefore the space of all the holomorphic deformations of $P$ with this property is a linear space (complete linear system) which coincides with the projectivization of the space of holomorphic sections of $L$: $\mathbb{P} [H^{(0)}(T_6, O(L))] \sim \mathbb{CP}^{d_P^2 / 2}$. Assuming $P$ to be a very ample divisor the higher order cohomology groups $H^{(n)}(T_6, O(L))$ become trivial and $d_P$ can be computed as an index, yielding the above result.
\[ N_R^F = 4h_{(2,0)} + 4 \left\{ -4h_{(1,0)} \right\} \]  \hspace{1cm} (12)

where the terms in the curly brackets represent the effect of a left–right symmetric gauging of the \( b_1 \) non–dynamical gauge fields. The coupling of the two sectors to these vector fields reduces indeed the scalar degrees of freedom by \( b_1 \) and the fermionic ones by \( 2b_1 \). This gauging is necessary in order to restore supersymmetry on the right–moving sector, which otherwise would not hold [26], as it can be easily checked from eqs. (12) using the property \( \int_{T_6} [\mathcal{P}]^3 = 6(h_{(2,0)}(P) - h_{(1,0)}(P) + 1) \).

From the above results the central charge is easily computed on our solution to be

\[ c_L = \text{Vol}(P) = 6y^1y^2y^3 = 6n^2N_1N_2N_3 \]  \hspace{1cm} (13)

5.2 Computation of \( h \).

The excitation level \( h \) is clearly the non–zero mode contribution to the total momentum \( L_0 - \mathcal{T}_0 \) along \( S_1 \), denoted by \( x_0 \) in table 2. We may therefore write \( h = x_0 - \Delta x_0 \), where \( \Delta x_0 \) is the zero–mode contribution to the same momentum and is the quantity which remains to be computed.

It is instructive to express \( \Delta x_0 \) in terms of type IIA quantities, going into a regime \((R^6 \ll \text{Vol}(T_6))\) in which the low energy dynamics is no more described by the 1+1 SCFT but by the zero modes of the open strings attached to the three sets of \( N_1, N_2, q^2 N_3 \) D4–branes deriving from the dimensional reduction of the M5–branes and to the \( x_0 = N_0 + m^2 N_3 \) D0–branes on top of them, see table 2. Before performing the dimensional reduction, let us shift the eleven dimensional metric \( G_{MN} \) from the Minkowski background by an infinitesimal symmetric matrix whose only non vanishing entries are those along the directions \((0,10)\): \( G_{MN} = \eta_{MN} + \delta G_{MN} \). In the low energy action on the M5–brane world volume the term \( \delta S \propto \int_{M5} T_{010} \delta \hat{G}_{010} \) would appear, representing the contribution to the action of the momentum along \( S_1 \). In the following we shall be interested in the contribution to the energy–momentum tensor \( T_{nm} \) associated with the form \( b^{(2)} \), namely \( T_{mn}(h) \propto h^{(3)}_{mkl} h^{(3)}_{nkl} \), which encodes the zero mode contribution to be evaluated. After compactifying on \( S_1 \) we obtain the type IIA configuration: the deformation \( \delta G_{10N} \) becomes the R–R one–form \( C_N^{(1)} \) coupled to the D0–brane and the components \( h_{10kl}^{(3)} \) give rise to the vector field strength components on the D4 brane world volumes, namely \( \mathcal{F}_{kl}/(2\pi) \).

The same action term \( \delta S \) on the D4–brane world volume can be shown to reduce to: \( \delta S = -1/2(2\pi)^2 \left[ \int_{P \times \mathbb{R}^4} \mathcal{F} \wedge \mathcal{F} \wedge C^{(1)} \right] \). This is the Chern–Simons term defining the effective D0–brane charge induced by a magnetic flux on the world volumes of the D4–branes.

In the regime of validity of the low energy description of the system in terms of the 1 + 1 SCFT the zero mode contribution to \( \delta S \) corresponds in the type IIA setting to a magnetic flux \( \mathcal{F}^{(0)} \) along the cycles \((4,5), (6,7), (8,9)\) equal on the intersecting D4–branes along their common directions:

\[ \Delta x_0 = \frac{-1}{2(2\pi)^2} \int_P \mathcal{F}^{(0)} \wedge \mathcal{F}^{(0)} \]  \hspace{1cm} (14)
The field strength \( F^{(0)} \) can be expanded along the cycles \( \alpha_i \) previously introduced: 
\[
F^{(0)} = F^{(0)|i} \alpha_i.
\]
The values of \( F^{(0)|i} \) are determined in terms of the electric four dimensional charges \( x_i \) from the Chern–Simons couplings of the magnetic flux to the R–R form \( C^{(3)} \) in the D4–brane world volume theory, and from the requirement that the three matter vector potentials \( A_i^\mu \) in the STU model derive from the reduction of \( C^{(3)} \) on \( \alpha_i \): 
\[
C^{(3)} = \sum_i A_i^\mu dx^\mu \wedge \alpha_i.
\]
The equations for \( F^{(0)|i} \) are: 
\[
x_i = 6 D_{ijk} y^j F^{(0)|k}/(2\pi).
\]
Solving them we are able to express \( F^{(0)|i} \), and therefore \( \Delta x_0 \), using eq. (14), in terms of \( y^i \) and \( x_i \): 
\[
\Delta x_0 = m^2 N_3 (N_1 + N_2)^2/(4 N_1 N_2).
\]
The excitation level for our configuration is therefore:

\[
h = x_0 - \Delta x_0 = N_0 + m^2 N_3 - m^2 N_3 (N_1 + N_2)^2/4 N_1 N_2.
\] (15)

implementing eqs. (13), (14) in eq. (11) we obtain precisely the expression for the macroscopic entropy in eq. (9)! This result provides a statistical interpretation of Bekenstein–Hawking area law for the thermodynamical entropy whose validity, for the very definition of generation solution, automatically extends to the most general black hole of the same kind.

6 Discussion

There are still many things to be understood on the front of BPS black holes. We have focused on a particular class of these solutions, namely the regular four dimensional BPS black holes in the \( N = 8 \) theory and addressed the question of how much may be learned already at the classical supergravity level on their microscopic description. This naturally led to the study of the corresponding \( U \)–duality orbit and its generating solution. One of the main goals of our analysis in [2, 3, 4] is to have clarified the meaning of the five parameters characterizing this orbit (and therefore encoding all the microscopic degrees of freedom of a generic solution in it) both macroscopically and microscopically. The first macroscopic descriptions of the generating solution were given in [27, 28, 17] within the framework of the heterotic string theory, nevertheless the microscopic interpretation of its parameters remained obscure (in particular of the parameter which lifted the solution from a purely dilatonic black hole to a generating solution). In [18] a suitable D–brane configuration was conjectured to describe the generating black hole, but its microscopic description as a R–R charged solution was missing. We have filled this gap by providing both the macroscopic and microscopic description of a particular generating solution [3] and defining a framework in which to study all possible microscopic realizations of the solution and their duality relations at a classical supergravity level [3]. In the light of these results the mysterious fifth parameter, which in the description of table 2 is proportional to the parameter \( m \), turns out to be related to a conserved quantity (\( T_{10,m} \) is indeed a conserved current on the M5 world-volume SCFT at least in a flat background) associated with the \( b^{(2)} \) degrees of freedom in
the M–theory picture or to a non trivial magnetic flux on the D4–brane world volumes in the type IIA picture. This charge allows the system to couple non–trivially to the axionic fields, which are naturally interpreted in the type IIA setting as the components along the compact directions of the Kalb–Ramond field and in type IIB framework as off–diagonal components of the internal metric. Finally a further step along the microscopic analysis of the generating solution was achieved in [4] by computing its entropy from a microscopic state counting. This achievement, besides providing a statistical interpretation of the Bekenstein–Hawking entropy of the most general regular BPS black hole in the theory under consideration, shed some light on how the five parameters enter the quantum low energy dynamics on the generating solution and therefore the expression of the microscopic entropy itself.

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