Non-Abelian vortex dynamics: Effective world-sheet action

Sven Bjarke Gudnason\footnote{e-mail address: gudnason(at)df.unipi.it}, Yunguo Jiang\footnote{e-mail address: jiang(at)df.unipi.it}, Kenichi Konishi\footnote{e-mail address: konishi(at)df.unipi.it}

Department of Physics, University of Pisa, Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy
and

INFN, Sezione di Pisa, Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy

Abstract

The low-energy vortex effective action is constructed in a wide class of systems in a color-flavor locked vacuum, which generalizes the results found earlier in the context of $U(N)$ models. It describes the weak fluctuations of the non-Abelian orientational moduli on the vortex worldsheet. For instance, for the minimum vortex in $SO(2N) \times U(1)$ or $USp(2N) \times U(1)$ gauge theories, the effective action found is a two-dimensional sigma model living on the Hermitian symmetric spaces $SO(2N)/U(N)$ or $USp(2N)/U(N)$, respectively. The fluctuating moduli have the structure of that of a quantum particle state in spinor representations of the GNO dual of the color-flavor $SO(2N)_{C+F}$ or $USp(2N)_{C+F}$ symmetry, i.e. of $SO(2N)$ or of $SO(2N+1)$. Applied to the benchmark $U(N)$ model our procedure reproduces the known $CP^{N-1}$ worldsheet action; our recipe allows us to obtain also the effective vortex action for some higher-winding vortices in $U(N)$ and $SO(2N)$ theories.
1 Introduction

The last several years have witnessed quite an unforeseen progress in our understanding of non-Abelian vortices, i.e. soliton vortex solutions in four (or three-) dimensional gauge theories possessing exact, continuous non-Abelian moduli. These continuous zero-modes arise from the breaking (by the soliton vortex) of an exact color-flavor diagonal symmetry of the system under consideration. The structure of their moduli, the varieties and group-theoretic properties of these modes as well as their dynamics, and the dependence of all these on the details of the theory such as the matter content and gauge groups, etc. turn out to be surprisingly rich. In spite of quite an impressive progress made in the last several years, the full implication of these theoretical developments is as yet to be seen.

In the present work we turn our attention to the low-energy vortex dynamics. In particular our aim is to construct the low-energy effective action describing the fluctuations of the orientational moduli parameters on the vortex worldsheet, generalizing the results found several years ago in the context of $U(N)$ models [1]-[3]. For concreteness and for simplicity, we start our discussion with the case of the $SO(2N) \times U(1)$ and $USp(2N) \times U(1)$ theories, although our method is quite general. In the case of the $SU(N) \times U(1)$ theory our result exactly reduces to the one found earlier; furthermore we shall obtain the effective action for a few other cases with higher-winding vortices in $U(N)$ and $SO(2N)$ theories.

2 Self-dual vortex solutions and the orientational moduli

Our system is a simple generalization of the Abelian Higgs model with quartic scalar potentials

$$
\mathcal{L} = -\frac{1}{4e^2} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu q_f)^\dagger D^\mu q_f - \frac{e^2}{2} \left| q_f^\dagger t^0 q_f - \frac{v^2}{\sqrt{4N}} \right|^2 - \frac{g^2}{2} \left| q_f^\dagger t^a q_f \right|^2,
$$

(1)

to a general class of gauge groups $G' \times U(1)$ where $G'$ is any simple Lie group. To concretize our idea let us consider two classes of theories $G' = SO(2N), USp(2N)$ with any $N \geq 1$. The repeated indices are summed: $a = 1, \ldots, \dim(G')$ labels the generators of $G'$, $0$ indicates the Abelian gauge field, $f = 1, \ldots, N_f$ labels the matter flavors (“scalar quark” fields), all of them in the fundamental representation of $G'$.\footnote{We adopt the convention where the metric $\eta_{\mu\nu} = \text{diag}(+,-,-,-)$.} The covariant derivatives and the field tensors are defined
in the standard manner

\[ \mathcal{D}_\mu q_f = \partial_\mu q_f + iA_\mu q_f , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] , \quad A_\mu = A_\mu^0 t^0 + A_\mu^a t^a , \quad (2) \]

with the normalization as follows

\[ \text{Tr} (t^a t^b) = \frac{1}{2} \delta^{ab} , \quad t^0 \equiv \frac{1}{\sqrt{4N}} . \quad (3) \]

To allow the system to possess a vacuum with the maximally color-flavor locked symmetry, we assume that number of matter flavors is \( N_f = 2N \). The squark fields \( q \) can then conveniently be represented as a color-flavor mixed matrix of dimension \( 2N \times 2N \), the color (flavor) index running vertically (horizontally). The vacuum in which we work is characterized by the squark vacuum expectation value (VEV)

\[ \langle q \rangle = \frac{v}{\sqrt{2N}} 1_{2N} . \quad (4) \]

Performing a Bogomol’nyi completion one obtains the BPS (or self-dual) equations

\[ \bar{\mathcal{D}} q = 0 , \quad (5) \]

\[ F_{12}^0 - \frac{e^2}{4N} (\text{Tr}(qq^\dagger) - v^2) = 0 , \quad (6) \]

\[ F_{12}^a t^a - \frac{g^2}{4} (qq^\dagger - J^\dagger(qq^\dagger)^T J) = 0 , \quad (7) \]

where \( 2\bar{\mathcal{D}} \equiv \mathcal{D}_1 + i\mathcal{D}_2 \) and \( z \equiv x^1 + ix^2 \) is the standard complex coordinate in the transverse plane. A glance at Eq. (1) reveals that the BPS-saturated tension \[4\]

\[ T = \pi v^2 k , \quad k \in \mathbb{Z}_+ . \quad (8) \]

is related to the \( U(1) \) winding only.

This last fact shows that a minimal vortex solution can be constructed \[5\] by letting the scalar field wind (far from the vortex axis) by an overall \( U(1) \) phase rotation with half angle \( (\pi) \), and completing (or canceling) it by a half winding \( ( \pi \) or \( -\pi \)) in each and all of the Cartan subgroups \( U(1) \)^N \( \subset G' \). Depending on which signs are chosen in the \( N \ U(1) \) factors, we find \( 2^N \) distinct solutions.

\(^2\)See Subsec. 3.3 below.
By choosing the plus sign for all of the $U(1)^N \subset G'$ factors, one finds a solution of the form:

$$q = \begin{pmatrix} e^{i\theta} \phi_1(r) 1_N & 0 \\ 0 & \phi_2(r) 1_N \end{pmatrix} = \frac{e^{i\theta} \phi_1(r) + \phi_2(r)}{2} 1_{2N} + \frac{e^{i\theta} \phi_1(r) - \phi_2(r)}{2} T,$$

$$A_i = \frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} [(1 - f(r)) 1_{2N} + (1 - f_{\text{NA}}(r)) T],$$

where

$$T = \text{diag}(1_N, -1_N),$$

and $z, r, \theta$ are cylindrical coordinates. The appropriate boundary conditions are

$$\phi_{1,2}(\infty) = \frac{v}{\sqrt{2N}}, \quad f(\infty) = f_{\text{NA}}(\infty) = 0, \quad \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{\text{NA}}(0) = 1.$$  

(12)

By going to singular gauge,

$$q \to \text{diag}(e^{-i\theta} 1_N, 1_N) \cdot q,$$  

(13)

the vortex takes the form

$$q = \begin{pmatrix} \phi_1(r) 1_N & 0 \\ 0 & \phi_2(r) 1_N \end{pmatrix} = \frac{\phi_1(r) + \phi_2(r)}{2} 1_{2N} + \frac{\phi_1(r) - \phi_2(r)}{2} T,$$

$$A_i = -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} [f(r) 1_{2N} + f_{\text{NA}}(r) T];$$  

(14)

in this gauge the whole topological structure arises from the gauge-field singularity along the vortex axis. The BPS equations (5)-(7) for the profile functions are given (in both gauges) by

$$\partial_r \phi_1 = \frac{1}{2r} (f + f_{\text{NA}}) \phi_1,$$  

$$\partial_r \phi_2 = \frac{1}{2r} (f - f_{\text{NA}}) \phi_2,$$  

(15)

$$\frac{1}{r} \partial_r f = \frac{e^2}{2} \left( \phi_1^2 + \phi_2^2 - \frac{v^2}{N} \right),$$

$$\frac{1}{r} \partial_r f_{\text{NA}} = \frac{g^2}{2} \left( \phi_1^2 - \phi_2^2 \right).$$  

(16)

The above is a particular vortex solution with a fixed $(+ + \ldots +)$ orientation. As the system has an exact $SO(2N)_{C+F}$ or $USp(2N)_{C+F}$ color-flavor diagonal (global) symmetry, respected by

3It is convenient to work with the skew-diagonal basis for the $SO(2N)$ group, i.e. the invariant tensors are taken as

$$J = \begin{pmatrix} 0 & 1_N \\ 1_N & 0 \end{pmatrix},$$  

(9)

where $\epsilon = \pm$ for $SO(2N)$ and $USp(2N)$ groups, respectively.
the vacuum \(4\), which is broken by such a minimum vortex, the latter develops “orientational” zero-modes. Degenerate vortex solutions can indeed be generated by color-flavor \(SO(2N)\) (or \(USp(2N)\)) transformations

\[
q \rightarrow U q U^{-1}, \quad A_i \rightarrow U A_i U^{-1},
\]

as

\[
q = U \begin{pmatrix} \phi_1(r) 1_N & 0 \\ 0 & \phi_2(r) 1_N \end{pmatrix} U^{-1} = \frac{\phi_1(r) + \phi_2(r)}{2} 1_{2N} + \frac{\phi_1(r) - \phi_2(r)}{2} U T U^{-1},
\]

\[
A_i = -\frac{1}{2} \epsilon_{ij} \frac{x^j}{r^2} \left[ f(r) 1_{2N} + f_{NA}(r) U T U^{-1} \right], \quad i = 1, 2.
\]

Actually, the full \(SO(2N)\) (or \(USp(2N)\)) group does not act on the solution, as the latter remains invariant under \(U(N) \subset SO(2N)\) (or \(USp(2N)\)). Only the coset \(SO(2N)/U(N)\) (or \(USp(2N)/U(N)\)) acts non-trivially on it, and thus generates physically distinct solutions. An appropriate parametrization of the coset, valid in a coordinate patch including the above solution, has been known for some time (called the reducing matrix) \([6,4]\),

\[
U = \begin{pmatrix} 1_N & -B^\dagger \\ 0 & 1_N \end{pmatrix} \begin{pmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & Y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1_N & 0 \\ B & 1_N \end{pmatrix} = \begin{pmatrix} X^{-\frac{1}{2}} & -B^\dagger Y^{-\frac{1}{2}} \\ BX^{-\frac{1}{2}} & Y^{-\frac{1}{2}} \end{pmatrix},
\]

where the matrices \(X\) and \(Y\) are defined by

\[
X \equiv 1_N + B^\dagger B, \quad Y \equiv 1_N + BB^\dagger,
\]

in terms of an \(N \times N\) complex matrix \(B\), being antisymmetric for \(SO(2N)\) and symmetric for \(USp(2N)\). Note that the matrix \((19)\) indeed satisfies the defining properties the two groups

\[
U^{-1} = U^\dagger, \quad U^T J U = J,
\]

with the respective invariant tensor \([9]\). The matrix \(B\) parametrizes the “Nambu-Goldstone” modes of symmetry breaking (by the vortex)

\[
SO(2N) \rightarrow U(N), \quad \text{or} \quad USp(2N) \rightarrow U(N),
\]

\(4\)As was studied in detail in Ref. \([4]\), the vortex moduli space in \(SO(2N)\) (or \(USp(2N)\)) theories is a non-trivial complex manifold, requiring at least \(2^{N-1}\) (or \(2^N\)) local coordinate neighborhoods (patches). The moduli space structure is actually richer, as these vortices possess semi-local moduli (related to the size and shape moduli) as well, besides the orientational moduli under consideration here, even with the minimum number of flavors needed for a color-flavor locked phase, in contrast to the original \(U(N)\) model. Here we consider only the orientational moduli related to the exact symmetry of the system.
and the number of independent parameters in $B$, $N(N - 1)$ or $N(N + 1)$, correctly matches the 
(real) dimension of the coset $SO(2N)/U(N)$ or $USp(2N)/U(N)$. The following identities turn 
out to be useful below:

$$B X^m = Y^m B, \quad X^m B^\dagger = B^\dagger Y^m, \quad [X^m, B^\dagger B] = 0, \quad [Y^m, BB^\dagger] = 0, \quad \forall m .$$

(23)

In the next section we shall allow for a $(x^3, x^0)$ dependence in $B$ and determine the effective 
action for these degrees of freedom.

## 3 Remarks

Before proceeding, however, let us briefly comment on a few aspects of our vortex systems.

### 3.1 $\mathcal{N} = 2$ supersymmetry

A point which deserves mention is supersymmetry. Although the main aim of this paper is 
the effective action for the internal degrees of freedom of the bosonic vortex, it is most natural 
to regard our system as a (truncated) bosonic sector of an $\mathcal{N} = 2$ supersymmetric model, as 
in Refs. [1, 2, 3, 4, 5, 7, 8]. There are many reasons for this; the BPS nature of our vortices 
is naturally implied by supersymmetry, as the quartic scalar coupling is related to the gauge 
coupling in the critical way. Furthermore, such a relation in the tree Lagrangian is maintained 
derived renormalization, due to the non-renormalization theorem. The resulting vortex effective 
sigma model will naturally be an $\mathcal{N} = (2, 2)$ supersymmetric sigma model. It is a consistent 
matter of fact that the vortex effective theory found below is a non-linear sigma model on a 
target space which is Kähler in all cases.

### 3.2 Moduli-matrix

A second, more technical issue concerns the moduli-matrix formalism [9 10 11]. The first BPS 
equation (5) can be solved by the Ansatz,

$$q = S^{-1} H_0(z), \quad A = -i S^{-1} \partial S, \quad S \in U(1)^C \times G^C,$$

(24)

where $H_0(z)$ is a holomorphic matrix (the moduli matrix) and $G^C = U(1)^C \times G^C$ denotes the 
complexification of the gauge group. The decomposition above is defined up to an equivalence 
relation

$$(H_0(z), S(z, \bar{z})) \sim V(z) (H_0(z), S(z, \bar{z})) ,$$

(25)
where $V(z)$ is any holomorphic matrix belonging to $G^\mathbb{C}$. $\Omega \equiv SS^\dagger$ satisfies a second-order equation equivalent to the gauge field equations (16).

The $(++\ldots +)$ Ansatz of Eq. (10) (or Eq. (14) in singular gauge), corresponds to the moduli matrix

$$H_0(z) = \begin{pmatrix} z1_N & 0 \\ 0 & 1_N \end{pmatrix}.$$ (26)

In this formalism the vortices of generic orientation (in the local coordinate patch) was constructed in Ref. [4] and is simply expressed by

$$H_0(z) = \begin{pmatrix} z1_N & 0 \\ 0 & 1_N \end{pmatrix} U \sim \begin{pmatrix} z1_N & 0 \\ B & 1_N \end{pmatrix},$$ (27)

where the matrix $U \in G'$ is the color-flavor rotation of Eq. (19) and $\sim$ denotes that we have used an appropriate $V$-transformation. The vortex of a generic orientation of Eq. (18) is nothing but the very same solution associated with the moduli matrix Eq. (27).

Although we shall not make explicit use of the moduli-matrix formalism below, these remarks should be sufficient to illustrate the power of the formalism, which proved in fact to be an indispensable tool for the analysis of the structure of the vortex moduli spaces (i.e., their connectedness, the minimum number of the patches needed, the transition functions, etc.) as complex manifolds [4].

### 3.3 Vacuum degeneracy

As was noted in Ref. [4, 12] and in the footnote of pp. 4 above, a notable fact that distinguishes the $U(N)$ model considered earlier, is that it possesses a unique vacuum in the color-flavor locked phase. This is not the case for other gauge theories and even with the minimum number of flavors needed for a color-flavor locked vacuum, the vacuum degeneracy in general leads to various interesting phenomena, such as “semi-local” vortices with arbitrary transverse size (which interpolate between ANO (“local”) vortices and the sigma model lumps), or fractional vortices [12]. Even though they are of considerable interest in their own right, we focus our attention below on the local (ANO-like) vortices defined in the maximally color-flavor locked vacuum, and to the study of the zero-modes associated with exact global symmetries of the system. After all, there are reasons to believe that these are among the most robust features of the non-Abelian vortices which would survive, for instance, certain non-BPS corrections which could eliminate some or all of the other vortex moduli [13].
In a similar spirit, we study in a later section certain subclasses of vortices among given winding-number solutions, transforming according to some definite irreducible representation of the (dual of the) color-flavor group.

4 Vortex moduli fluctuations: the worldsheet action

As the orientational modes considered in Eq. (18) represent exact Nambu-Goldstone-like zero-modes, nothing can prevent them from fluctuating in the space-time, from one point to another, with an arbitrarily small expenditure of energy. However, they are not genuine Nambu-Goldstone modes, as the vacuum itself is symmetric under $SO(2N)_{C+F}$ or $USp(2N)_{C+F}$: they are massive modes in the 4-dimensional space-time bulk. They propagate freely only along the vortex-axis and in time. To study these excited modes we set the moduli parameters $B$ to be (quantum) fields of the form

$$B = B(x^a), \quad x^a = (x^3, x^0).$$

(28)

When this expression is substituted into the action $\int d^4x \mathcal{L}$, however, one immediately notes that

$$\sum_{\alpha=0,3} \left[ \sum_{f=1}^{2N} |\partial_\alpha q_f|^2 + \sum_{i=1,2} \frac{1}{2g^2} |F_{i\alpha}|^2 \right],$$

(29)

leads to an infinite excitation energy, whereas one knows that the system must be excitable without mass gap (classically)\(^5\)

The way how the system reacts to the space-time dependent change of the moduli parameters, can be found by an appropriate generalization of the procedure adopted earlier for the vortices in $U(N)$ theories. A key observation [1]-[3] is to introduce non-trivial gauge field components, $A_\alpha$, to cancel the large excitation energy from (29). A naïve guess would be

$$A_\alpha = -i \rho(r) U^{-1} \partial_\alpha U,$$

(30)

with $U$ of Eq. (19) and some profile function $\rho$. This however does not work. The problem is that even though

$$i U^{-1} \partial_\alpha U = i \begin{pmatrix} X^{-\frac{1}{2}} B^i \partial_\alpha B X^{-\frac{1}{2}} & -X^{-\frac{1}{2}} \partial_\alpha B^i Y^{-\frac{1}{2}} \\ Y^{-\frac{1}{2}} \partial_\alpha B X^{-\frac{1}{2}} & Y^{-\frac{1}{2}} B \partial_\alpha B^i Y^{-\frac{1}{2}} - \partial_\alpha Y^j B^i Y^{-\frac{1}{2}} \end{pmatrix},$$

(31)

\(^5\)Whereas in the far infrared, we expect that either the world-sheet effective sigma model will by quantum effects develop a dynamic mass gap (as the $CP^{N-1}$ model) or end up in a conformal vacuum – a possibility for $SO,USp$ theories [14].

7
certainly is in the algebra $\mathfrak{g}'$ of $G'$, it in general contains the fluctuations also in the $U(N)$ directions (massive modes). To extract the massless modes, we first project it on directions orthogonal to the fixed matter-field orientation, Eq. (14), that is

$$i \left( U^{-1} \partial_{\alpha} U \right)_\perp \equiv \frac{i}{2} \left( U^{-1} \partial_{\alpha} U - TU^{-1} \partial_{\alpha} UT \right) = i \begin{pmatrix} 0 & -X^{-\frac{1}{2}} \partial_{\alpha} B Y^{-\frac{1}{2}} \\ Y^{-\frac{1}{2}} \partial_{\alpha} B X^{-\frac{1}{2}} & 0 \end{pmatrix},$$

(32)

such that $\text{Tr} [U^{-1} \partial_{\alpha} U]_\perp q^0 = 0$, where $q^0$ indicates the vortex (14). As the quark fields fluctuate in the $SO(2N)$ (or $USp(2N)$) group space, we must keep $A_{\alpha}$ orthogonal to them. The appropriate Ansatz then is

$$A_{\alpha} = i \rho(r) U \left( U^{-1} \partial_{\alpha} U \right)_\perp U^{-1}, \quad \alpha = 0, 3,$$

(33)

together with $q$ and $A_i$ of Eq. (18). One sees that the following orthogonality conditions

$$\text{Tr} \{ A_{\alpha} \} = 0, \quad \text{Tr} \{ A_{\alpha} U T U^{-1} \} = 0, \quad \text{Tr} \{ A_{\alpha} \partial_{\alpha} (U T U^{-1}) \} = 0$$

(34)

are satisfied: the first two hold by construction; the third can easily be checked. The constant BPS tension is independent of the vortex orientation; the excitation above it arises from the following terms of the action

$$\text{Tr} |D_{\alpha} q|^2 = - \frac{\rho^2}{2} \left( \phi_1^2 + \phi_2^2 \right) (1 - \rho) \left( \phi_1 - \phi_2 \right)^2 \text{Tr} \left( (U^{-1} \partial_{\alpha} U)_\perp \right)^2,$$

(35)

$$\frac{1}{g^2} \text{Tr} F^2_{i\alpha} = - \frac{1}{g^2} \left[ (\partial_{\alpha} \rho)^2 + \frac{1}{r^2} f_{2\alpha}^2 (1 - \rho)^2 \right] \text{Tr} \left( (U^{-1} \partial_{\alpha} U)_\perp \right)^2,$$

(36)

where we have used the identity

$$\text{Tr} \left( U^{-1} \partial_{\alpha} (U T U^{-1}) \right)^2 = - \text{Tr} \left( U^{-1} \partial_{\alpha} U - TU^{-1} \partial_{\alpha} UT \right)^2 = -4 \text{Tr} \left( (U^{-1} \partial_{\alpha} U)_\perp \right)^2.$$

(37)

By using Eq. (32) one arrives at the world-sheet effective action

$$S_{1+1} = 2 \beta \int dtdz \text{tr} \left\{ X^{-1} \partial_{\alpha} B Y^{-1} \partial_{\alpha} B \right\}$$

$$= 2 \beta \int dtdz \text{tr} \left\{ (1_N + B B^1)^{-1} \partial_{\alpha} B \right\},$$

(38)

where

$$\beta = \frac{2\pi}{g^2 I}$$

(39)

and the trace tr acts on $N \times N$ matrices. Even though the sigma-model metric reflects the specific symmetry breaking patterns of the system under consideration, the coefficient $I$ turns out to be universal, and indeed is formally identical to the one found for the $U(N)$ model $^6$

$$I = \int_0^\infty dr r \left[ (\partial_{\alpha} \rho)^2 + \frac{1}{r^2} f_{2\alpha}^2 (1 - \rho)^2 + \frac{g^2 \rho^2}{2} \left( \phi_1^2 + \phi_2^2 \right) + g^2 (1 - \rho) (\phi_1 - \phi_2)^2 \right].$$

(40)

$^6$In that case the effective sigma model has a $CP^{N-1}$ target space $^2$; see Subsec. 5.1 below.
The equation of motion for \( \rho \) minimizing the coupling constant \( \beta \) (the Kähler class) of the vortex world-sheet sigma model can be solved accordingly by [2, 3]

\[
\rho = 1 - \frac{\phi_1}{\phi_2},
\]

as can be checked by a simple calculation making use of the BPS equations for the profile functions \( \phi_{1,2}, f_{NA} \). The integral \( I \) turns out to be a total derivative

\[
I = \int_0^\infty dr \, \partial_r \left( f_{NA} \left[ \left( \frac{\phi_1}{\phi_2} \right)^2 - 1 \right] \right),
\]

and by using the boundary conditions (12) the final result is

\[
I = f_{NA}(0) = 1 .
\]

The action found in Eq. (38) is precisely that of the \((1+1)\)-dimensional sigma model on Hermitian symmetric spaces \( SO(2N)/U(N) \) and \( USp(2N)/U(N) \) [6, 15]. The metric is Kählerian, with the Kähler potential given by

\[
K = \text{tr} \log \left( 1_N + BB^\dagger \right), \quad g_{I\bar{J}} = \frac{\partial^2 K}{\partial B^I \partial B^{\dagger \bar{J}}},
\]

where \( I, \bar{J} = \{(i,j) = 1, \ldots, N \mid i \leq j \} \).

In the context of \( \mathcal{N} = 2 \) supersymmetric models, the low-energy effective vortex action is a two-dimensional, \( \mathcal{N} = (2,2) \) supersymmetric sigma model [15]:

\[
S_{1+1}^{\text{susy}} = 2\beta \int dt dz d^2\theta \, d^2\bar{\theta} \, K(B, \bar{B})
\]

in terms of the Kähler potential Eq. (44), where \( B \) now is a matrix chiral superfield (\( \bar{B} \) anti-chiral superfield containing \( B^\dagger \)). The \( \beta \)-functions for these sigma models have been determined in [15]. In the supersymmetric case, the number of quantum vacua is given by the Euler characteristic of the manifold on which the world-sheet action lives [17, 18], which can be found in the mathematical literature [19] and we show the relevant numbers in Table 1.

### 5 Other examples

Our recipe for constructing the effective vortex action appears to be of considerable generality; below a few other examples will be discussed.
moduli space $\mathcal{M}$ | $\chi(\mathcal{M})$
---|---
$\frac{SO(2N)}{U(N)}$ | $2^{N-1}$
$\frac{USp(2N)}{U(N)}$ | $2^N$
$\mathbb{CP}^{N-1} = \frac{SU(N)}{SO(N-1) \times U(1)}$ | $N$
$Gr_{N,k} = \frac{SU(N)}{SU(k) \times U(N-k)}$ | $\left(\begin{array}{c} N \\ k \end{array}\right)$
$Q^{N-2} = \frac{SO(2N)}{SO(2) \times SO(2N-2)}$ | $2^N$

Table 1: Number of quantum vacua for the relevant vortices under consideration which is given by the Euler characteristic $\chi$.

### 5.1 $U(N)$ vortices and the $\mathbb{CP}^{N-1}$ sigma model

For the fundamental (i.e. of the minimum winding) vortex of the $U(N)$ model discussed by Shifman et. al. [2,3], the vortex Ansatz is very similar to Eq. (10) except for changes in the field Ansatz and accordingly the reducing matrix $U$:

$$q = \begin{pmatrix} e^{i\beta} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} 1_{N-1} = \frac{e^{i\beta} \phi_1(r) + \phi_2(r)}{2} 1_N + \frac{e^{i\beta} \phi_1(r) - \phi_2(r)}{2} T, \quad (46)$$

$$A_i = \epsilon_{ij} \frac{2^j}{r^2} \left[ \frac{1}{N} (1 - f(r)) 1_N + \frac{1}{2} (1 - f_{NA}(r)) \left( T - \frac{2}{N} 1_N \right) \right], \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1_{N-1} \end{pmatrix},$$

with the boundary conditions

$$\phi_{1,2}(\infty) = \frac{\nu}{\sqrt{N}}, \quad f(\infty) = f_{NA}(\infty) = 0, \quad \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{NA}(0) = 1. \quad (47)$$

The unitary transformation $U$ (the reducing matrix) giving rise to vortices of generic orientation has the same form as in Eq. (19), except that the matrix $B$ is now an $(N-1)$-component column-vector

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_{N-1} \end{pmatrix}, \quad (48)$$

while $B^\dagger$ is correspondingly a row-vector;

$$X = 1 + B^\dagger B, \quad Y = 1_{N-1} + BB^\dagger, \quad (49)$$

are a scalar and an $(N-1) \times (N-1)$ dimensional matrix, respectively. Going through the same steps as in Sec. [14] the effective worldsheet action in this case is \textit{exactly} given by Eq. (38), includ-
ing the normalization integral of Eqs. (39)-(43), with these replacements. $B = (b_1, \ldots, b_{N-1})^T$ represent the standard inhomogeneous coordinates of $\mathbb{C}P^{N-1}$.

In order to find the relation between the $N$-component complex unit vector $n$ used by Gorsky et. al. [3] and our $B$ matrix, note that

$$\frac{1}{N} U \begin{pmatrix} -(N-1) & 0 \\ 0 & 1_{N-1} \end{pmatrix} U^{-1} = \frac{1}{N} 1_N - n n^\dagger, \quad (50)$$

$$\Rightarrow \quad n n^\dagger = U \begin{pmatrix} 1 & 0 \\ 0 & 0_{N-1} \end{pmatrix} U^{-1} = \begin{pmatrix} X^{-1} & X^{-1} B^\dagger \\ B X^{-1} & B X^{-1} B^\dagger \end{pmatrix}, \quad (51)$$

which allows us to identify

$$n = \begin{pmatrix} X^{-\frac{1}{2}} \\ B X^{-\frac{1}{2}} \end{pmatrix}. \quad (52)$$

By the identification (52), our Ansatz (33) is seen to be equal, after some algebra, to

$$A_\alpha = i \rho(r) \left[ \partial_\alpha n n^\dagger - n \partial_\alpha n^\dagger - 2 n n^\dagger (n^\dagger \partial_\alpha n) \right], \quad (53)$$

which is the one proposed in Ref. [3]. Consequently, our $\mathbb{C}P^{N-1}$ effective action (38) with Eqs. (39)-(43) reduces to the one given by these authors. Our result thus goes some way towards clarifying the meaning of the seemingly arbitrary Ansatz (53) (or better, an Ansatz found by a brilliant intuition, but that cannot easily be applied to other theories) used in Ref. [3].

5.2 Completely symmetric $k$-winding vortices in the $U(N)$ model

Next let us consider the orientational moduli of the coincident $k$-winding vortex in the $U(N)$ model [20, 21, 22, 23]. We consider a vortex solution of a particular, fixed orientation given by

$$q := \begin{pmatrix} e^{ik\theta} \phi_1(r) & 0 \\ 0 & \phi_2(r) 1_{N-1} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1_{N-1} \end{pmatrix}, \quad (54)$$

$$A_i = \epsilon_{ij} \frac{x^j}{r^2} \left[ \frac{1}{N} (k - f(r)) 1_N + \frac{1}{2} (k - f_{NA}(r)) \left( T - \frac{2-N}{N} 1_N \right) \right],$$

with the boundary conditions

$$\phi_{1,2}(\infty) = \frac{v}{\sqrt{N}}, \quad f(\infty) = f_{NA}(\infty) = 0, \quad \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{NA}(0) = k. \quad (55)$$
Being a composition of $k$ vortices of minimum winding in the same orientation, it is obvious that the vortex (54) transforms under the totally symmetric representation:

\[
\begin{array}{c}
\text{\small\textcircled{1}} \\
\vdots \\
\text{\small\textcircled{k}}
\end{array}
\]

of the color-flavor $SU(N)_{C+F}$ group.

The construction of the effective vortex action in this case is almost identical to that in the preceding subsection, in particular the reducing matrix acting non-trivially on the vortex is the same as in the single $U(N)$ vortex case, see Eqs. (48)-(49). The effective vortex action is the same $CP^{N-1}$ model (38). The only difference is in the value of the gauge profile functions at the vortex core, Eq. (55). As a consequence the coefficient (the coupling strength) in front of the action (38) (see Eq. (42)) is now given by

\[
\beta = \frac{2\pi}{g^2} I, \quad I = f_{NA}(0) = k .
\]

**5.3 Completely antisymmetric $k$-winding vortices in the $U(N)$ model**

Consider now a $k$-vortex (with $k < N$) of the form

\[
q := \begin{pmatrix} e^{i\theta} \phi_1(r) 1_k & 0 \\ 0 & \phi_2(r) 1_{N-k} \end{pmatrix}, \quad T = \begin{pmatrix} 1_k & 0 \\ 0 & -1_{N-k} \end{pmatrix},
\]

\[
A_i = \epsilon_{ij} x_j \left[ k N (1 - f(r)) 1_N + \frac{1}{2} (1 - f_{NA}(r)) \left( T - \frac{2k - N}{N} 1_N \right) \right],
\]

with the following boundary conditions

\[
\phi_{1,2}(\infty) = \frac{v}{\sqrt{N}}, \quad f(\infty) = f_{NA}(\infty) = 0, \quad \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{NA}(0) = 1 .
\]

It is invariant under an $SU(k) \times SU(N - k) \times U(1) \subset SU(N)_{C+F}$ subgroup, showing that it belongs to the completely antisymmetric $k$-th tensor representation:

\[
\begin{array}{c}
\text{\small\textcircled{1}} \\
\vdots \\
\text{\small\textcircled{k}}
\end{array}
\]

The color-flavor transformations $U$ acting non-trivially on it belong to the coset

\[
Gr_{N,k} = \frac{SU(N)}{SU(k) \times SU(N - k) \times U(1)} ,
\]
and is again of the standard form of the reducing matrix, Eq. (19), but now the matrix $B$ is a $(N - k) \times k$ complex matrix field, whose elements are the local coordinates of the Grassmannian manifold. The effective action – the world-sheet sigma model – is then simply given by Eq. (38) with the standard normalization, Eqs. (39)-(43) and the Kähler potential is then given by Eq. (44).

5.4 Higher-winding vortices in the $SO(2N)$ model

Let us now consider doubly-wound vortex solutions in the $SO(2N) \times U(1)$ system. They fall into distinct classes of solutions which do not mix under the $SO(2N)$ transformations of the original fields [5]; they are:

$$
k = 2, \quad \begin{pmatrix}
  n_1^+ & n_1^-
  n_2^+ & n_2^-
  \vdots & \vdots
  n_{N-1}^+ & n_{N-1}^-
  n_N^+ & n_N^-
\end{pmatrix} = \begin{pmatrix}
  2 & 0 & 2 & 0 & 2 & 0 & 1 & 1
  2 & 0 & 2 & 0 & 1 & 1 & 1 & 1
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
  2 & 0 & 0 & 2 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

These correspond to different $SO(2N)_{C+F}$ orbits, living in coset spaces $SO(2N)/[U(N - \ell) \times SO(2\ell)]$, where $\ell$ is the number of $(1,1)$ pairs. Analogously vortices with $k \geq 3$ can be constructed. As was explained in Ref. [5], the argument that the minimum vortices transform as two spinor representations implies that the $k = 2$ vortices transform as various irreducible antisymmetric tensor representations of $SO(2N)_{C+F}$, appearing in the decomposition of products of two spinors [24]:

$$
2^{N-1} \otimes 2^{N-1} \quad \text{or} \quad 2^{N-1} \otimes 2^{N-1},
$$

where the spinors of different chiralities are distinguished by the bar. For instance, the last configuration of Eq. (60) is a singlet, the second last is the $2N$ representation, and so on.

The effective action of the

$$
\begin{pmatrix}
  2 & 0 \\
  \vdots & \vdots \\
  2 & 0
\end{pmatrix}
$$

---

Footnote 8: Here we use the notation of [5]. $n_i^\pm = \frac{k}{2} \pm n_i \in \mathbb{Z}$, where $\frac{k}{2}$ is the winding in the overall $U(1)$; $n_i$ is the winding number of the $i$-th Cartan $U(1)$ factor. $n_i \in \mathbb{Z}/2$ are quantized in half integers [4][4]. In this notation the fundamental vortex of Eq. (10) is simply

$$
\begin{pmatrix}
  1 & 0 \\
  \vdots & \vdots \\
  1 & 0
\end{pmatrix}$$
vortex (the first of Eq. (60)) has the same form as that found for the fundamental vortices in Sec. 11; a sigma model in the target space $SO(2N)/U(N)$. The normalization constant in front is however different: it is now given by
\[ \beta = \frac{2\pi}{g^2} \mathcal{I}, \quad \mathcal{I} = f_{\text{NA}}(0) = 2. \]  

As a last nontrivial example, let us consider the vortex solutions belonging to the second last group of (60). The orientational modes of the vortex now live in the coset space
\[ SO(2N)/[SO(2) \times SO(2N-2)], \]  
a real Grassmannian space. The construction of the reducing matrix in this case is slightly more elaborated, but has already been done by Delduc and Valent [6].

The Ansatz for this vortex can be written as
\[ q = \begin{pmatrix} e^{i\theta} \phi_0(r) & 0 & 0 \\ 0 & e^{i2\theta} \phi_1(r) & 0 \\ 0 & 0 & \phi_2(r) \end{pmatrix} = e^{i\theta} \phi_0 1_{2N} + \frac{1}{2} (e^{i2\theta} \phi_1 + \phi_2 - 2e^{i\theta} \phi_0) T_1 + \frac{1}{2} (e^{i2\theta} \phi_1 - \phi_2) T_2, \]
\[ A_i = \epsilon_{ij} \frac{T_j}{T^2} [(1 - f) 1_{2N} + (1 - f_{\text{NA}}) T_2], \]  
where the relevant matrices are
\[ T_1 \equiv \begin{pmatrix} 0_{2N-2} & 1 \\ 1 & 1 \end{pmatrix}, \quad T_2 \equiv \begin{pmatrix} 0_{2N-2} & 1 \\ 1 & -1 \end{pmatrix}, \]  
and the following relations are useful
\[ T_1^2 = T_1, \quad T_2^2 = T_1, \quad T_1 T_2 = T_2 T_1 = T_2. \]  

We will also need the BPS equations for this vortex
\[ \partial_r \phi_0 = \frac{1}{r} f \phi_0, \quad \frac{1}{r} \partial_r f = \frac{e^2}{4N} \left( (2(N-1)) \phi_0^2 + \phi_1^2 + \phi_2^2 - \alpha^2 \right), \]  
\[ \partial_r \phi_1 = \frac{1}{r} (f + f_{\text{NA}}) \phi_1, \quad \frac{1}{r} \partial_r f_{\text{NA}} = \frac{g^2}{4} (\phi_1^2 - \phi_2^2), \]  
\[ \partial_r \phi_2 = \frac{1}{r} (f - f_{\text{NA}}) \phi_2, \]  
with the following boundary conditions
\[ \phi_{0,1,2}(\infty) = \frac{\nu}{\sqrt{2N}}, \quad f(\infty) = f_{\text{NA}}(\infty) = 0, \]
\[ \phi_0(0) = \phi_1(0) = 0, \quad \partial_r \phi_2(0) = 0, \quad f(0) = f_{\text{NA}}(0) = 1. \]
We have furthermore made a basis change such that the invariant rank-two tensor of \( SO(2N) \) is

\[
J = \begin{pmatrix}
1_{2N-2} & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

(72)

The Ansatz for the gauge fields \( A_{0,3} \) is still given by Eq. (33), however the reducing matrix is now [6]:

\[
U = \left( \sqrt{1_{2N-2} - EE^\dagger} \begin{pmatrix}
E \\
\sqrt{1_{2} - E^\dagger E}
\end{pmatrix},
\right)
\]

(73)

where

\[
E = \frac{\sqrt{2}}{D} \left( \varphi \bar{\varphi} \right),
\]

(74)

\[
D \equiv \sqrt{1 + 2\varphi^\dagger \varphi + (\varphi^T \varphi) \bar{\varphi} \varphi^\dagger + (\varphi^\dagger \bar{\varphi}) \varphi \varphi^T}.
\]

(75)

\( E \) is a \((2N - 2) \times 2\)-dimensional matrix and \( \varphi \) is a \((2N - 2)\)-dimensional column vector, while the following matrix expressions are essential for the calculation

\[
\sqrt{1_{2} - E^\dagger E} = \frac{1}{D} \left( \begin{pmatrix}
1 & -\varphi^\dagger \bar{\varphi} \\
-\varphi^T \varphi & 1
\end{pmatrix},
\right)
\]

(76)

\[
\sqrt{1_{2N-2} - EE^\dagger} = 1_{2N-2} - (1 + D) \left( \varphi \varphi^\dagger + \bar{\varphi} \varphi^T \right) + (\varphi^T \varphi) \bar{\varphi} \varphi^\dagger + (\varphi^\dagger \bar{\varphi}) \varphi \varphi^T.
\]

(77)

Now we will follow the recipe of Sec. 4 by going to the singular gauge and rotating with the color-flavor rotation \( U \) of Eq. (73)

\[
q = \varphi_0 1_{2N} + \frac{1}{2} (\varphi_1 + \varphi_2 - 2\varphi_0) UT_1 U^{-1} + \frac{1}{2} (\varphi_1 - \varphi_2) UT_2 U^{-1},
\]

\[
A_i = -\epsilon_{ij} \frac{x_j}{r^2} \left[ f 1_{2N} + f_{NA} UT_2 U^{-1} \right],
\]

(78)

from which together with the Ansatz (33) and

\[
T = 1_{2N} - 2T_1 = \begin{pmatrix}
1_{2N-2} & 0 \\
0 & -1 \\
0 & -1
\end{pmatrix},
\]

(79)

we can calculate the contributions

\[
\text{Tr} \left[ D_{\alpha} q \right]^2 = - \left( (1 - \rho) \left[ (\varphi_1 - \varphi_0)^2 + (\varphi_0 - \varphi_2)^2 \right] + \frac{\rho^2}{2} (2\varphi_0^2 + \varphi_1^2 + \varphi_2^2) \right] \text{Tr} \left[ (1_{2N} - T_1) X_{\alpha} T_1 X_{\alpha} \right],
\]

\[
\frac{1}{g^2} \text{Tr} \left[ F_{\alpha}^2 \right] = - \frac{2}{g^2} \left[ (\partial_r \rho)^2 + \frac{1}{r^2} f_{NA}^2 (1 - \rho)^2 \right] \text{Tr} \left[ (1_{2N} - T_1) X_{\alpha} T_1 X_{\alpha} \right],
\]

(80)
where $X_{\alpha} \equiv U^{-1} \partial_{\alpha} U$ and we have used the following non-trivial relations

$$\text{Tr} [T_1 X_{\alpha} T_1 X_{\alpha}] = \text{Tr} [T_2 X_{\alpha} T_2 X_{\alpha}] , \quad \text{Tr} [(\mathbf{1}_{2N} - T_1) X_{\alpha} T_2 X_{\alpha}] = 0 . \quad (81)$$

Let us use the notation

$$X_{\alpha} = \begin{pmatrix} A_{\alpha} & B_{\alpha} \\ C_{\alpha} & D_{\alpha} \end{pmatrix} . \quad (82)$$

The first relation of Eq. (81) can be proved by showing that $D_{\alpha}$ is indeed diagonal, while the second relation can be proved by showing that $B_{\alpha} r^3 C_{\alpha}$ is antisymmetric, and hence traceless.

The following trace can be rewritten as

$$\text{Tr} [(\mathbf{1}_{2N} - T_1) X_{\alpha} T_1 X_{\alpha}] = \frac{1}{8} \text{Tr} [X_{\alpha} - TX_{\alpha} T]^2 = \frac{1}{2} \text{Tr} [(X_{\alpha})_{\perp}]^2 . \quad (83)$$

After the dust settles one finds the effective world-sheet action

$$S_{1+1} = 2 \beta \int dt \, dz \left\{ \frac{\partial_{\alpha} \varphi^\dagger \partial_{\alpha} \varphi + 2 |\varphi^\dagger \partial_{\alpha} \varphi|^2}{1 + 2 \varphi^\dagger \varphi + |\varphi^\dagger \varphi|^2} - \frac{2 |\varphi^\dagger \partial_{\alpha} \varphi + (\varphi^\dagger \varphi) (\varphi^\dagger \partial_{\alpha} \varphi)|^2}{[1 + 2 \varphi^\dagger \varphi + |\varphi^\dagger \varphi|^2]^2} \right\} , \quad (84)$$

where

$$\beta = \frac{2\pi}{g^2} \mathcal{I} , \quad (85)$$

and the normalizing integral now reads

$$\mathcal{I} = \int_0^\infty dr \left( (\partial_r \rho)^2 + \frac{1}{r^2} f_{\text{NA}}^2 (1 - \rho)^2 \right)$$

$$+ \frac{g^2}{2} (1 - \rho) \left[ (\phi_1 - \phi_0)^2 + (\phi_0 - \phi_2)^2 \right] + \frac{g^2 \rho^2}{4} \left(2 \phi_0^2 + \phi_1^2 + \phi_2^2\right) \right] . \quad (86)$$

The boundary conditions for $\rho(r)$ are

$$\rho(0) = 1 , \quad \rho(\infty) = 0 , \quad (87)$$

while its equation of motion is simply

$$\frac{1}{r} \partial_r (r \partial_r \rho) + \frac{1}{r^2} f_{\text{NA}}^2 (1 - \rho) + \frac{g^2}{4} \left[ (\phi_1 - \phi_0)^2 + (\phi_0 - \phi_2)^2 \right] - \frac{g^2 \rho^2}{4} \left(2 \phi_0^2 + \phi_1^2 + \phi_2^2\right) = 0 . \quad (88)$$

It is non-trivial to find a solution to this non-linear equation. To find the solution, the crucial point is the non-trivial relation

$$\phi_0^2 = \phi_1 \phi_2 . \quad (89)$$
By using this relation, the solution can be expressed in several different forms, which however can be seen all to be equivalent to each other:

\[ \rho = 1 - \frac{\phi_0}{\phi_2} = 1 - \frac{1}{2} \left( \frac{\phi_1 + \phi_0}{\phi_0 + \phi_2} \right) = 1 - \frac{\phi_0 (\phi_1 + \phi_2)}{\phi_0^2 + \phi_2^2}, \]

(90)

To prove the relation (89), we combine the BPS-equations as follows

\[ \partial_r \log \left( \frac{\phi_0^2}{\phi_1 \phi_2} \right) = 0, \]

(91)

from which it follows that this ratio is a constant. This constant is given by the boundary conditions and hence is equal to one.

Now we can plug the result into the normalizing integral and by using the BPS equations again, we find that the integral reduces to

\[ \mathcal{I} = \int_0^\infty dr \, \partial_r \left( f_{NA} \left[ \left( \frac{\phi_0}{\phi_2} \right)^2 - 1 \right] \right) = f_{NA}(0) = 1. \]

(92)

The action (84) is exactly that of the (1 + 1)-dimensional sigma model on the Hermitian symmetric space \( SO(2N)/[SO(2) \times SO(2N - 2)] \). It has a Kähler metric: the Kähler potential is given by

\[ K = \log \left( 1 + 2 \phi^\dagger \phi + |\phi^T \phi|^2 \right). \]

(93)

5.5 The vortex transformations: GNO duality

Let us return to the minimal vortices in \( SO(2N) \times U(1) \) or \( USp(2N) \times U(1) \) theory discussed in Secs. 2 and 4. There are \( 2^N \) such representative solutions with degenerate, minimal tension (see the remarks before Eq. (11), and also Ref. \[5, 4\]). Furthermore in the case of \( SO(2N) \times U(1) \) theory, the minimal vortex solutions fall into two distinct classes \[5, 4\] which do not mix under the \( SO(2N) \) transformations of the original fields. These observations suggest that the vortices transform according to spinor representations of the GNO dual of \( SO(2N) \) or \( USp(2N) \), i.e. as two \( 2^{N-1} \) dimensional spinor representations of \( Spin(2N) \), or as a \( 2^N \)-dimensional representation of \( SO(2N + 1) \), respectively.

That they do so can be checked explicitly. The reducing matrix Eq. (19) shows that the infinitesimal transformations of the vortex are generated by the complex matrices \( B \)

\[ U = 1_{2N} + \begin{pmatrix} 0_N & -B^\dagger \\ B & 0_N \end{pmatrix} + \ldots, \]

(94)
where $B$ is an infinitesimal antisymmetric ($SO(2N)$) or symmetric ($USp(2N)$) $N \times N$ matrix. Transformations around any other point $P$ is generated by the conjugation

$$R \begin{pmatrix} 0 & -B' \dagger \\ B' & 0 \end{pmatrix} R^{-1},$$

(95)

where $R$ is a finite $SO(2N)$ (or $USp(2N)$) transformation of the form of Eq. (19), bringing the origin of the moduli space to $P$.

The spinors can be represented by using a system made of $N$ spin-$\frac{1}{2}$ subsystems: $|s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_N\rangle$. The $SO(2N)$ generators $\Sigma_{ij}$ in the spinor representation can be expressed in terms of the (anti-commuting) creation and annihilation operators $a_i, a_i^\dagger$ in the well-known fashion [24] (see Appendix A). The $k$-th annihilation operators acts as

$$a_k = \frac{1}{2} \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_- \otimes 1 \otimes \cdots \otimes 1,$$

(96)

while $\tau_-$ is replaced by $\tau_+$ in $a_i^\dagger$.

We map the special vortex configurations and the spinor states as follows:

$$(\pm, \cdots, \pm) \sim |s_1\rangle \otimes |s_2\rangle \otimes \cdots |s_N\rangle, \quad |s_j\rangle = |\downarrow\rangle \text{ or } |\uparrow\rangle.$$  

(97)

In particular, the $(++\ldots+)$ vortex solution described by Eq. (10) is mapped to the all-spin-down state

$$(+\ldots+) \sim |\downarrow\ldots\downarrow\rangle.$$  

(98)

An infinitesimal transformation of this spinor state is given by

$$S = e^{i \omega_{ij} \Sigma_{ij}} = 1 + \sum_{i,j=1}^{N} (\omega_{ij} - \omega_{N+i,N+j} - i \omega_{N+i,j} - i \omega_{N+j,i}) a_i^\dagger a_j^\dagger + \ldots,$$

(99)

as the operators $a_j$ annihilate the state $|\downarrow\ldots\downarrow\rangle$. There is thus a one-to-one correspondence between the vortex transformation law (19) and the spinor transformation law, under the identification

$$B_{ij} = \sum_{i,j=1}^{N} (\omega_{ij} - \omega_{N+i,N+j} - i \omega_{N+i,j} - i \omega_{N+j,i}) ,$$

(100)

which are indeed generic antisymmetric, complex $N \times N$ matrices.

Infinitesimal transformations around any other spinor state ($|P\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots |s_N\rangle$) are generated by the conjugation

$$S \left( B_{ij} a_i^\dagger a_j^\dagger \right) S^{-1},$$

(101)
where $S \in \text{Spin}(2N)$ transforms the origin (98) to $|P\rangle$.

We conclude that the connected parts of the vortex moduli space are isomorphic to the orbits of spinor states: they form two copies of $SO(2N)/U(N)$.

The consideration in the case of the $USp(2N)$ vortices is analogous. The (abstract) $SO(2N+1)$ spinor generators can be expressed in terms of the annihilation and creation operators as in Appendix A. We map the $USp(2N)$ vortex solutions and $SO(2N+1)$ spinor states as in Eq. (97), with the origin of the moduli spaces identified as before, i.e. as in Eq. (98).

Both in the vortex and the spinor moduli spaces, in contrast to the $SO(2N)$ case, there is no conserved chirality now: all of the $2^N$ special vortex solutions (spinor states) are connected by $USp(2N)$ ($SO(2N+1)$) transformations. Infinitesimal transformations of the $USp(2N)$ vortices around the origin are generated by a complex, symmetric matrix $B$, Eq. (19). On the other hand, the $SO(2N+1)$ spinors transform as in Eqs. (111)-(112): the origin $|\downarrow \ldots \downarrow\rangle$ is transformed by

$$S = e^{i\omega_{\alpha\beta} \Sigma_{\alpha\beta} + i\omega_{\gamma,2N+1} \Sigma_{\gamma,2N+1}} = 1 + \beta_{ij} a_i^\dagger a_j^\dagger + d_i a_i^\dagger + O(\omega^2) : \quad (102)$$

they describe the coset $SO(2N+1)/U(N)$. The map between the $USp(2N)$ vortex transformation law and the $SO(2N+1)$ spinor transformation law is then

$$\left( \beta_{ij}, d_i \right) \iff B , \quad (103)$$

that is, the infinitesimal neighborhoods of the origin of the vortex and spinor moduli spaces are mapped to each other by the identification of the local coordinates

$$\beta_{ij} = -\beta_{ji} = B_{ij} \quad (i > j) ; \quad d_i = B_{ii} . \quad (104)$$

Both for the vortex and for the spinors, transformations around any other point are generated by the conjugation analogous to Eqs. (95), (101) with appropriate modifications ($B_{\text{anti}} \rightarrow B_{\text{sym}}$; $\beta_{ij} a_i^\dagger a_j^\dagger \rightarrow \beta_{ij} a_i^\dagger a_j^\dagger + d_i a_i^\dagger$). Under such a map, the vortex transformations in the moduli space ($USp(2N)/U(N)$) are mapped to the orbits of the spinor states, $SO(2N+1)/U(N)$.

### 6 Discussion

In this paper we have constructed the low-energy effective action describing the fluctuations of the non-Abelian orientational zero-modes on the vortex worldsheet in a certain class of models, generalizing the $\mathbb{C}P^{N-1}$ action found some time ago in the $U(N)$ model. In the cases of the minimal vortices in $SO(2N) \times U(1)$ and $USp(2N) \times U(1)$ theories, they are given by two-dimensional
sigma models in Hermitian symmetric spaces $SO(2N)/U(N)$ and $USp(2N)/U(N)$, respectively. We have also found the effective action for some higher-winding vortices in $SO(2N) \times U(1)$ as well as in the $U(N)$ theory.

Not much has appeared yet in the literature about the study of orientational moduli and their fluctuation properties in the case of higher-winding vortices \cite{20, 21, 22}. Group-theoretic and dynamical properties of higher-winding vortices in the $U(N)$ model are presently under investigation, taking full advantage of the Kähler quotient construction, and will appear soon \cite{23}. The present paper and this forthcoming work \cite{23} are in many senses complementary.

Our vortex effective actions define the way the vortex orientational modes fluctuate just below the typical mass scales characterizing the vortex solutions, and are somewhat analogous to the bare Lagrangian defining a given four-dimensional (4D) gauge-matter system, at some ultraviolet scale.

In the case of minimal vortices in $SO(2N) \times U(1)$ and $USp(2N) \times U(1)$ theories, their moduli and transformation laws have been found to be isomorphic to spinor orbits in the GNO duals, $Spin(2N)$ and $Spin(2N + 1)$. This could possibly be important in view of the general vortex-monopole connection, implied in a hierarchical symmetry breaking scenario, in which our vortex systems play the role of a low-energy approximation \cite{5, 8, 21, 25}.

On the other hand, the effective vortex sigma models obtained here are, either in the non-supersymmetric version \cite{6} or in a supersymmetric extension \cite{15}, all known to be asymptotically free. They become strongly coupled at mass scales much lower than the typical vortex mass scale. The vortex effective action does not tell immediately what happens at such long distances, just as the form of the bare (ultraviolet) Lagrangian of an asymptotically-free 4D system does not immediately teach us about the infrared behavior of the system (Quantum Chromodynamics being a famous example). Let us note that the infrared behavior of our vortex fluctuations depends on whether or not the system is supersymmetric, or more generally, which other bosonic or fermionic matter fields are present, even though they do not appear explicitly (i.e. these fields are set to zero) in the classical vortex solutions.

In the case of the non-Abelian vortex fluctuations in $\mathcal{N} = 2$ supersymmetric $U(N)$ theory such vortex dynamics has been analyzed carefully by Shifman et. al. \cite{2, 3}. We plan to come back in a separate work to discuss these questions in the context of a more general class of models treated here.

20
Acknowledgments

The authors’ thanks are due to Roberto Auzzi, Minoru Eto, Muneto Nitta, Keisuke Ohashi and Walter Vinci, for useful comments and discussions.

A Spinor representation of $SO(2N + 1)$

The spinor generators of the $SO(2N + 1)$ group $(a, b = 1, 2, \ldots, 2N + 1)$

$$[\Sigma_{ab}, \Sigma_{cd}] = -i (\delta_{bc} \Sigma_{ad} - \delta_{ac} \Sigma_{bd} - \delta_{bd} \Sigma_{ac} + \delta_{ad} \Sigma_{bc}) ,$$

(105)
can be constructed as [24]

$$\Sigma_{2j-1,2N+1} \equiv \frac{1}{2} j-1 \otimes \tau_3 \otimes \tau_1 \otimes 1 , \quad \Sigma_{2j,2N+1} \equiv \frac{1}{2} j-1 \otimes \tau_3 \otimes \tau_2 \otimes 1 , \quad j = 1,2,\ldots,N ,$$

(106)
acting on the $N$-dimensional spin-$\frac{1}{2}$ system

$$|s_1 \rangle \otimes |s_2 \rangle \otimes \cdots |s_N \rangle ,$$

(107)
with the sub-algebra $SO(2N)$ generated by:

$$\Sigma_{\alpha\beta} = -i [\Sigma_{\alpha,2N+1}, \Sigma_{\beta,2N+1}] , \quad \alpha, \beta = 1,2,\ldots,2N .$$

(108)
The annihilation and creation operators are defined by

$$a_k = \frac{1}{\sqrt{2}} (\Sigma_{2k-1,2N+1} - i \Sigma_{2k,2N+1}) = \frac{1}{2} k-1 \otimes \tau_3 \otimes \tau_- \otimes 1 ,$$

$$a_k^\dagger = \frac{1}{\sqrt{2}} (\Sigma_{2k-1,2N+1} + i \Sigma_{2k,2N+1}) = \frac{1}{2} k-1 \otimes \tau_3 \otimes \tau_+ \otimes 1 ,$$

(109)
where

$$\tau_\pm \equiv \tau_1 \pm i\tau_2 \sqrt{2} .$$

(110)
By expressing the generators $\Sigma_{ab}$ in terms of $a_j, a_j^\dagger$ and using \{ $a_j, a_k^\dagger$ \} = $\delta_{jk}/2$, we find that the spinors transform as follows:

$$S = e^{i \omega_{\alpha\beta} \Sigma_{\alpha,2N+1} \Sigma_{\beta,2N+1}}$$

$$= 1 + \alpha_{ij} a_i^\dagger a_j + \beta_{ij} a_i^\dagger a_j^\dagger + \beta_{ij}^\dagger a_i a_j + d_i a_i^\dagger - d_i^\dagger a_i + i\omega_{2i,2i-1} + O (\omega^2) ,$$

(111)
where

\[
\alpha_{jk} \equiv 2 \left( \omega_{2j,2k} + \omega_{2j-1,2k-1} + i \omega_{2j-1,2k} - i \omega_{2j,2k-1} \right),
\]

\[
\beta_{jk} \equiv - \left( \omega_{2j,2k} - \omega_{2j-1,2k-1} + i \omega_{2j-1,2k} + i \omega_{2j,2k-1} \right),
\]

\[
d_j \equiv \frac{1}{\sqrt{2}} \left( \omega_{2j,2N+1} + i \omega_{2j-1,2N+1} \right),
\]

(112)
in terms of the original real rotation parameters \(\omega_{ij}\). \(\alpha_{jk}\) represent the parameters of \(U(N) \subset SO(2N+1)\) which leaves invariant the origin Eq. (98), whereas \(\beta_{jk}\) and \(d_j\) parametrize the coset, \(SO(2N+1)/U(N)\). The imaginary constants in Eq. (111) contribute simply to the complex phase of \(S\). \(\beta_{jk}\) are antisymmetric complex matrices and \(d_j\) is a complex \(N\)-component vector.

By restricting to the \(2N\)-dimensional subspace the discussion above is valid for the \(SO(2N)\) spinors as well.

References

[1] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, “Nonabelian superconductors: Vortices and confinement in N = 2 SQCD,” Nucl. Phys. B 673, 187 (2003) [arXiv:hep-th/0307287].

[2] M. Shifman and A. Yung, “Non-Abelian string junctions as confined monopoles,” Phys. Rev. D 70, 045004 (2004) [arXiv:hep-th/0403149].

[3] A. Gorsky, M. Shifman and A. Yung, “Non-Abelian Meissner effect in Yang-Mills theories at weak coupling,” Phys. Rev. D 71, 045010 (2005) [arXiv:hep-th/0412082].

[4] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi and W. Vinci, “Constructing Non-Abelian Vortices with Arbitrary Gauge Groups,” Phys. Lett. B 669, 98 (2008) [arXiv:0802.1020 [hep-th]]; M. Eto et al., “Non-Abelian Vortices in SO(N) and USp(N) Gauge Theories,” JHEP 0906, 004 (2009) [arXiv:0903.4471 [hep-th]].

[5] L. Ferretti, S. B. Gudnason and K. Konishi, “Non-Abelian vortices and monopoles in SO(N) theories,” Nucl. Phys. B 789, 84 (2008) [arXiv:0706.3854 [hep-th]].

[6] F. Delduc and G. Valent, “Classical And Quantum Structure Of The Compact Kahlerian Sigma Models,” Nucl. Phys. B 253, 494 (1985); F. Delduc and G. Valent, “Renormalizability Of The Generalized Sigma Models Defined On Compact Hermitian Symmetric Spaces,” Phys. Lett. B 148 (1984) 124.
[7] A. Hanany and D. Tong, “Vortices, instantons and branes,” JHEP 0307, 037 (2003) arXiv:hep-th/0306150.

[8] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, “Nonabelian monopoles and the vortices that confine them,” Nucl. Phys. B 686, 119 (2004) arXiv:hep-th/0312233.

[9] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Construction of non-Abelian walls and their complete moduli space,” Phys. Rev. Lett. 93, 161601 (2004) arXiv:hep-th/0404198.

[10] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “All exact solutions of a 1/4 Bogomol’nyi-Prasad-Sommerfield equation,” Phys. Rev. D 71, 065018 (2005) arXiv:hep-th/0405129.

[11] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Moduli space of non-Abelian vortices,” Phys. Rev. Lett. 96, 161601 (2006) arXiv:hep-th/0511088.

[12] M. Eto et al., “Fractional Vortices and Lumps,” Phys. Rev. D 80, 045018 (2009) arXiv:0905.3540 [hep-th].

[13] R. Auzzi, M. Eto, S. B. Gudnason, K. Konishi and W. Vinci, “On the Stability of Non-Abelian Semi-local Vortices,” Nucl. Phys. B 813, 484 (2009) arXiv:0810.5679 [hep-th].

[14] G. Carlino, K. Konishi and H. Murayama, “Dynamical symmetry breaking in supersymmetric SU(n(c)) and USp(2n(c)) gauge theories,” Nucl. Phys. B 590, 37 (2000) arXiv:hep-th/0005076; G. Carlino, K. Konishi, S. P. Kumar and H. Murayama, “Vacuum structure and flavor symmetry breaking in supersymmetric SO(n(c)) gauge theories,” Nucl. Phys. B 608, 51 (2001) arXiv:hep-th/0104064.

[15] A. Y. Morozov, A. M. Perelomov and M. A. Shifman, “Exact Gell-Mann-Low Function Of Supersymmetric Kahler Sigma Models,” Nucl. Phys. B 248, 279 (1984).

[16] K. Higashijima and M. Nitta, “Supersymmetric nonlinear sigma models as gauge theories,” Prog. Theor. Phys. 103, 635 (2000) arXiv:hep-th/9911139.

[17] E. Witten, “Constraints On Supersymmetry Breaking,” Nucl. Phys. B 202, 253 (1982).

[18] K. Hori and C. Vafa, “Mirror symmetry,” arXiv:hep-th/0002222.

[19] C. U. Sánchez, A. L. Calí and J. L. Moreschi, “Spheres in Hermitian Symmetric Spaces and Flag Manifolds,” Geometriae Dedicata 64, 261 (1997).
[20] M. Eto, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, “Non-Abelian vortices of higher winding numbers,” Phys. Rev. D 74, 065021 (2006) [arXiv:hep-th/0607070].

[21] M. Eto, L. Ferretti, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci and N. Yokoi, “Non-Abelian duality from vortex moduli: a dual model of color-confinement,” Nucl. Phys. B 780, 161 (2007) [arXiv:hep-th/0611313].

[22] R. Auzzi, S. Bolognesi and M. Shifman, “Higher Winding Strings and Confined Monopoles in N=2 SQCD,” Phys. Rev. D 81, 085011 (2010) [arXiv:1001.1903 [hep-th]].

[23] M. Eto, T. Fujimori, S. B. Gudnason, Y. Jiang, K. Konishi, M. Nitta, K. Ohashi, in preparation.

[24] H. Georgi, “Lie Algebras in Particle Physics”, Westview, Advanced Book Program (1999).

[25] S. B. Gudnason and K. Konishi, “Low-energy U(1) x USp(2M) gauge theory from simple high-energy gauge group,” Phys. Rev. D 81, 105007 (2010) [arXiv:1002.0850 [hep-th]].