Inference on the Parameters of the Weibull Distribution Using Records

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Abstract

The Weibull distribution is a very applicable model for the lifetime data. In this paper, we have investigated inference on the parameters of Weibull distribution based on record values. We first propose a simple and exact test and a confidence interval for the shape parameter. Then, in addition to a generalized confidence interval, a generalized test variable is derived for the scale parameter when the shape parameter is unknown. The paper presents a simple and exact joint confidence region as well. In all cases, simulation studies show that the proposed approaches are more satisfactory and reliable than previous methods. All proposed approaches are illustrated using a real example.

Keywords: Coverage probability; Generalized confidence interval; Generalized p-value; Records; Weibull distribution.

MSC2000: 62F03; 62E15; 62-04.

1 Introduction

The Weibull distribution is a well-known distribution that is widely used for lifetime models. It has numerous varieties of shapes and demonstrates considerable flexibility that enables it to have increasing and decreasing failure rates. Therefore, it is used for many applications for example in hydrology, industrial engineering, weather forecasting and insurance. The Weibull distribution with parameters \( \alpha \) and \( \beta \), denoted by \( W (\alpha, \beta) \), has a cumulative distribution function (cdf)

\[
F(x) = 1 - e^{-(\frac{x}{\beta})^{\alpha}}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0,
\]

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and probability density function (pdf)

\[ f(x) = \frac{\beta^\alpha}{\alpha^\beta} x^{\beta - 1} e^{-\left(\frac{x}{\alpha}\right)^\beta}, \quad x > 0. \]

The Weibull distribution is a generalization of the exponential distribution and Rayleigh distribution. Also, \( Y = \log(X) \) has the Gumbel distribution with parameters \( b = \frac{1}{\beta} \) and \( a = \log(\alpha) \), when \( X \) has a Weibull distribution with parameters \( \alpha \) and \( \beta \).

Let \( X_1, X_2, \ldots \) be an infinite sequence of independent identically distributed random variables from the same population with the cdf \( F_\theta \), where \( \theta \) is a parameter. An observation \( X_j \) will be called an upper record value (or simply a record) if its value exceeds that of all previous observations. Thus, \( X_j \) is a record if \( X_j > X_i \) for every \( i < j \). An analogous definition deals with lower record values. The record value sequence \( \{R_n\} \) is defined by

\[ R_n = X_{T_n}, \quad n = 0, 1, 2, \ldots. \]

where \( T_n \) is called the record time of \( n \)th record and is defined as \( T_n = \min\{j : X_j > X_{T_{n-1}}\} \) with \( T_0 = 1 \).

Let \( R_0, \ldots, R_n \) be the first \( n + 1 \) upper record values from the cdf \( F_\theta \) and the pdf \( f_\theta \). Then, the joint distribution of the first \( n + 1 \) record values is given by

\[ f_R(r) = f_\theta(r_n) \prod_{i=0}^{n-1} \frac{f_\theta(r_i)}{1 - F_\theta(r_i)}, \quad r_0 < r_1 < \cdots < r_n, \quad (1.1) \]

where \( r = (r_0, r_1, \ldots, r_n) \) and \( R = (R_0, R_1, \ldots, R_n) \) (for more details see Arnold et al., 1998).

Chandler (1952) launched a statistical study of the record values, record times and inter record times. Record values and the associated statistics are of interest and importance in the areas of meteorology, sports and economics. Ahsanullah (1995) and Arnold et al. (1998) are two good references about records and their properties.

Some papers considered inference on the Weibull distribution based on record values: Dallas (1982) discussed some distributional results based on upper record values. Balakrishnan and Chan (1994) established some simple recurrence relations satisfied by the single and the product moments, and derived the BLUE of the scale parameter when the shape parameter is known. Chan (1998) provided a conditional method to derive exact intervals for location and scale parameters of location-scale family that can be used to derive exact intervals for the shape parameter. Wu and Tseng (2006) provided some pivotal quantities to test and establish confidence interval of the shape parameter based on the first \( n + 1 \) observed upper record values. Soliman et al. (2006) derived the Bayes estimates based on record values for the parameters.
with respect to squared error loss function and LINEX loss function. Asgharzadeh and Abdi (2011b) proposed joint confidence regions for the parameters. Teimouri and Gupta (2012) computed the coefficient of skewness of upper/lower record statistics. Teimouri and Nadarajah (2013) derived exact expressions for constructing bias corrected maximum likelihood estimators (MLE’s) of the parameters for the Weibull distribution based on upper records. Gouet et al. (2012) obtained the asymptotic properties for the counting process of δ-records among the first n observations.

In this paper, we consider inference about the parameters of Weibull distribution based on record values. First, we will propose a simple and exact method for constructing confidence interval and testing the hypotheses about the shape parameter β. Then using the concepts of generalized p-value and generalized confidence interval, a generalized approach for inference about the scale parameter α will be derived. Tsui and Weerahandi (1989) introduced the concept of generalized p-value, and Weerahandi (1993) introduced the concept of generalized confidence interval. These approaches have been used successfully to address several complex problems (see Weerahandi, 1995) such as confidence interval for the common mean of several log-normal distributions (Behboodian and Jafari, 2006), confidence interval for the mean of Weibull distribution (Krishnamoorthy et al., 2009), inference about the stress-strength reliability involving two independent Weibull distributions (Krishnamoorthy and Lin, 2010), and comparing two dependent generalized variances (Jafari, 2012).

We also present an exact joint confidence region for the parameters. Our simulation studies show that the area of our joint confidence region is smaller than those provided by other existing methods.

The rest of this article is organized as follows: A simple method for inference about shape parameter and a generalized approach for inference about the scale parameter are proposed in Section 2. Furthermore, a simulation study is performed and a real example is proposed in this Section. We also present a joint confidence region for the parameters α and β in Section 3.

2 Inference on the parameters

Suppose \( R_0, R_1, \ldots, R_n \) are the first \( n + 1 \) upper record values from a Weibull distribution with parameters α and β. In this section, we consider inference on the parameters α and β. From
The joint distribution of these record values can be written as

\[ f_R(r) = \frac{\beta^{n+1}}{\alpha^\beta(n+1)} e^{-\left(\frac{r_n}{\alpha}\right)^\beta} \prod_{i=0}^{n} r_i^{\beta-1} \quad 0 < r_0 < r_1 < \cdots < r_n. \] (2.1)

Therefore, \((R_n, \sum_{i=0}^{n} \log (R_i))\) is sufficient statistic for \((\alpha, \beta)\). Moreover, it can be easily shown that the MLE’s of the parameters \(\alpha\) and \(\beta\) are

\[ \hat{\beta} = \frac{n+1}{\sum_{i=0}^{n} \log \left(\frac{R_n}{R_i}\right)}, \quad \hat{\alpha} = \frac{R_n}{(n+1)^{\frac{1}{\beta}}}. \] (2.2)

**Theorem 2.1.** Let \(R_0, R_1, ..., R_n\) be the first \(n + 1\) upper record values from a Weibull distribution. Then

i. \(U = 2\beta \sum_{i=0}^{n} \log \left(\frac{R_i}{R_{i-1}}\right)\) has a chi-square distribution with \(2n\) degrees of freedom.

ii. \(V = 2 \left(\frac{R_n}{\alpha}\right)^\beta\) has a chi-square distribution with \(2n + 2\) degrees of freedom.

iii. \(U\) and \(V\) are independent.

**Proof.**

i. Define

\[ Q_m = \frac{R_m}{R_{m-1}}, \quad m = 1, 2, \ldots, n. \] (2.3)

From [Arnold et al. (1998)](1998), page 20, \(Q_m\)'s are independent random variables with

\[ P(Q_m > q) = q^{-\beta m}, \quad q > 1, \]

and

\[ 2\beta m \log(Q_m) = 2\beta m \log\left(\frac{R_m}{R_{m-1}}\right) \sim \chi^2(2). \]

Therefore,

\[
U = 2\beta \sum_{i=0}^{n} \log\left(\frac{R_n}{R_i}\right) = 2\beta \sum_{i=0}^{n-1} \log\left(\frac{R_n}{R_{n-1}} \cdot \frac{R_{n-1}}{R_{n-2}} \cdots \frac{R_{i+1}}{R_i}\right)
\]

\[
= 2\beta \sum_{i=0}^{n-1} \sum_{m=i+1}^{n} \log\left(\frac{R_m}{R_{m-1}}\right) = 2\beta \sum_{m=1}^{n} \sum_{i=0}^{m-1} \log(Q_m) = \sum_{m=1}^{n} 2\beta m \log(Q_m),
\]

has a chi-square distribution with \(2n\) degrees of freedom.

ii. Define

\[ Y = \left(\frac{X}{\alpha}\right)^\beta, \]

where \(X\) has a Weibull distribution with parameters \(\alpha\) and \(\beta\). Then, \(Y\) has an exponential distribution with parameter one. Therefore, we can conclude that \(V\) has a chi-square distribution with \(2n + 2\) degrees of freedom (see [Arnold et al. (1998)](1998), page 9).
iii. Let $\beta$ be known. Then, it can be concluded from (2.1) that $R_n$ is a complete sufficient statistic for $\alpha$. Also, $Q_m$'s in (2.3) are ancillary statistics. Therefore, $R_n$ and $Q_m$'s are independent, and the proof is completed.

2.1 Inference on the shape parameter

Here, we consider inference on the shape parameter, $\beta$ from a Weibull distribution based on record values, and propose a simple and exact method for constructing a confidence interval and testing the one-sided hypotheses

$$H_0 : \beta \leq \beta_0 \quad \text{vs.} \quad H_1 : \beta > \beta_0, \quad (2.4)$$

and the two-sided hypotheses

$$H_0 : \beta = \beta_0 \quad \text{vs.} \quad H_1 : \beta \neq \beta_0, \quad (2.5)$$

where $\beta_0$ is a specified value.

Based on Theorem 2.1, $U = 2\beta \sum_{i=0}^{n} \log \left( \frac{R_n}{R_i} \right)$ has a chi-square distribution with $2n$ degrees of freedom. Therefore, a $100(1 - \gamma)\%$ confidence interval for $\beta$ can be obtained as

$$\left( \frac{\chi^2_{2n, \gamma/2}(R_n, R_i)}{2 \sum_{i=0}^{n} \log \left( \frac{R_n}{R_i} \right)}, \frac{\chi^2_{2n, 1-\gamma/2}(R_n, R_i)}{2 \sum_{i=0}^{n} \log \left( \frac{R_n}{R_i} \right)} \right), \quad (2.6)$$

where $\chi^2_{(k), \gamma}$ is the $\gamma$th percentile of the chi-square distribution with $k$ degrees of freedom. Also, for testing the hypotheses in (2.4) and (2.5), we can define the test statistic

$$U_0 = 2\beta_0 \sum_{i=0}^{n} \log \left( \frac{R_n}{R_i} \right),$$

and the null hypothesis in (2.4) is rejected at nominal level $\gamma$ if

$$U_0 > \chi^2_{(2n, 1-\gamma)},$$

and the null hypothesis in (2.5) is rejected if

$$U_0 < \chi^2_{(2n), \gamma/2} \quad \text{or} \quad U_0 > \chi^2_{(2n), 1-\gamma/2}.$$
for inference about the shape parameter, and showed that \( W(\beta) \) is an increasing function with respect to \( \beta \). Also, its distribution does not depend on the parameters \( \alpha \) and \( \beta \). In fact, \( W(\beta) \) is distributed as

\[
W^* = \frac{\sum_{i=0}^{n} R^*_i}{(n + 1)(\prod_{i=0}^{n} R^*_i)^{\frac{1}{n+1}}},
\]

where \( R^*_i \) is the \( i \)th record from the exponential distribution with parameter one. However, its exact distribution is very complicated, and Wu and Tseng (2006) obtained the percentiles of \( W(\beta) \) using Monte Carlo simulation. The confidence limits for \( \beta \) are obtained by solving the following equations numerically as

\[
W(\beta) = W^*_1 - \frac{\gamma}{2}, \quad W(\beta) = W^*_\frac{\gamma}{2}, \quad (2.7)
\]

where \( W^*_\delta \) is the \( \delta \)th percentile of the distribution of \( W^* \).

### 2.2 Inference on the scale parameter

Here, we consider inference about the scale parameter, \( \alpha \) for a Weibull distribution based on record values, and propose an approach for constructing a confidence interval and testing the one-sided hypotheses

\[
H_0 : \alpha \leq \alpha_0 \quad \text{vs.} \quad H_1 : \alpha > \alpha_0, \quad (2.8)
\]

and the two-sided hypotheses

\[
H_0 : \alpha = \alpha_0 \quad \text{vs.} \quad H_1 : \alpha \neq \alpha_0, \quad (2.9)
\]

where \( \alpha_0 \) is a specified value.

We did not find any approach in literature for inference about \( \alpha \) based on record values when the shape parameter is unknown. Here, we use the concepts of generalized \( p \)-value and generalized confidence interval introduced by Tsui and Weerahandi (1989), and Weerahandi (1993), respectively. In appendix, we briefly review these concepts, and refer readers to Weerahandi (1995) for more details.

Let

\[
T = r_n \left( \frac{2 C_r}{V} \right)^{\frac{2 C_r}{V}} = r_n \left( \frac{\alpha}{R_n} \right)^{\frac{C_r}{\sum_{i=0}^{n} \log \left( \frac{R_i}{R_n} \right)}} \quad (2.10)
\]

where \( C_r = \sum_{i=0}^{n} \log \left( \frac{r_i}{R_i} \right) \), and \( r_i, i = 0, 1, \ldots, n \) is the observed value of \( R_i, i = 0, 1, \ldots, n \), and \( U \) and \( V \) are independent random variables that are defined in Theorem 2.1. The observed value of \( T \) is \( \alpha \), and distribution of \( T \) does not depend on unknown parameters \( \alpha \) and \( \beta \).
Therefore, $T$ is a generalized pivotal variable for $\alpha$, and can be used to construct a generalized confidence interval for $\alpha$.

Let

$$T^* = T - \alpha = r_n \left( \frac{2C_r}{V} \right)^{\frac{1}{2}} - \alpha.$$  

Then, $T^*$ is a generalized test variable for $\alpha$, because i) the observed value of $T^*$ does not depend on any parameters, ii) the distribution function of $T^*$ is free from nuisance parameters and only depends on the parameter $\alpha$, and iii) the distribution function of $T^*$ is an increasing function with respect to the parameter $\alpha$, and so, the distribution of $T^*$ is stochastically decreasing in $\alpha$. Therefore, the generalized $p$-value for testing the hypotheses in (2.8) is given as

$$p = P \left( T^* < 0 \mid H_0 \right) = P \left( T < \alpha_0 \right),$$  

and the generalized $p$-value for testing the hypotheses in (2.9) is given as

$$p = 2 \min \{ P \left( T > \alpha_0 \right), P \left( T < \alpha_0 \right) \}.$$  

The generalized confidence interval for $\alpha$ based on $T$, and the generalized $p$-values in (2.11) and (2.12) can be computed using Monte Carlo simulation (Weerahandi, 1995; Behboodian and Jafari, 2006) based on the following algorithm:

**Algorithm 2.1.** For given $r_0, r_1, \ldots, r_n$,

1. Generate $U \sim \chi^2(2n)$ and $V \sim \chi^2(2n+2)$.
2. Compute $T$ in (2.10).
3. Repeat steps 1 and 2 for a large number times, (say $M = 10000$), and obtain the values $T_1, \ldots, T_M$.
4. Set $D_l = 1$ if $T_l < \alpha_0$ else $D_l = 0$, $l = 1, \ldots, M$.

The 100 $(1 - \gamma)$% generalized confidence interval for $\alpha$ is $[T_{(\gamma/2)}, T_{(1-\gamma/2)}]$, where $T_{(\delta)}$ is the $\delta$th percentile of $T_l$'s. Also, the generalized $p$-value for testing the one-sided hypotheses in (2.11) is obtained by $\frac{1}{M} \sum_{l=1}^{M} D_l$.

**2.3 Real example**

Roberts (1979) gave monthly and annual maximal of one-hour mean concentration of sulfur dioxide (in pphm) from Long Beach, California, for 1956 to 1974. Chan (1998) showed that the
Weibull distribution is a reasonable model for this data set. Wu and Tseng (2006) also study this data set. The upper record values for the month of October from the data are 26, 27, 40, 41.

The 95% confidence interval for the scale parameter $\alpha$ based on our generalized confidence interval with $M = 10000$ is obtained as $(5.4869, 39.9734)$. The 95% confidence interval for the shape parameter $\beta$ in (2.6) is obtained as $(0.6890, 8.0462)$, and based on Wu and Tseng’s method in (2.7) is obtained as $(0.6352, 7.7423)$. Also, the generalized p-value equals to 0.0227 for testing the hypotheses in (2.8) with $\alpha_0 = 5$. Therefore, the null hypothesis is rejected.

Table 1: Empirical coverage probabilities and expected lengths of the generalized confidence interval for the parameter $\alpha$ with confidence level 0.95.

| $\beta$ | $\alpha$ | n  | 0.5 | 1.0 | 1.2 | 1.5 | 2.0 | 3.0 | 5.0 |
|--------|---------|----|-----|-----|-----|-----|-----|-----|-----|
|        |         |    |     |     |     |     |     |     |     |
| Empirical Coverage | 1.0 | 3  | 0.951 | 0.949 | 0.953 | 0.947 | 0.947 | 0.948 | 0.948 |
|          | 7  | 0.952 | 0.949 | 0.950 | 0.950 | 0.951 | 0.953 | 0.952 | 0.952 |
|          | 9  | 0.951 | 0.948 | 0.953 | 0.951 | 0.948 | 0.949 | 0.950 | 0.950 |
|          | 14 | 0.945 | 0.949 | 0.950 | 0.950 | 0.954 | 0.952 | 0.952 | 0.952 |
|          | 2.0 | 3  | 0.949 | 0.952 | 0.947 | 0.949 | 0.951 | 0.950 | 0.953 |
|          | 7  | 0.948 | 0.953 | 0.950 | 0.946 | 0.954 | 0.948 | 0.951 | 0.951 |
|          | 9  | 0.952 | 0.948 | 0.953 | 0.950 | 0.952 | 0.953 | 0.954 | 0.954 |
|          | 14 | 0.950 | 0.946 | 0.949 | 0.951 | 0.951 | 0.952 | 0.952 | 0.955 |
| Expected Length | 1.0 | 3  | 16.740 | 3.581 | 2.804 | 2.155 | 1.653 | 1.211 | 0.847 |
|          | 7  | 13.575 | 3.198 | 2.477 | 1.942 | 1.475 | 1.041 | 0.686 | 0.686 |
|          | 9  | 13.505 | 3.138 | 2.469 | 1.918 | 1.446 | 1.008 | 0.651 | 0.651 |
|          | 14 | 13.122 | 3.082 | 2.403 | 1.854 | 1.376 | 0.943 | 0.596 | 0.596 |
|          | 2.0 | 3  | 33.516 | 7.187 | 5.579 | 4.341 | 3.323 | 2.427 | 1.704 |
|          | 7  | 27.960 | 6.344 | 4.999 | 3.899 | 2.960 | 2.080 | 1.364 | 1.364 |
|          | 9  | 27.342 | 6.304 | 4.935 | 3.831 | 2.890 | 2.016 | 1.302 | 1.302 |
|          | 14 | 26.626 | 6.129 | 4.779 | 3.705 | 2.757 | 1.886 | 1.191 | 1.191 |

2.4 Simulation study

We performed a simulation study in order to evaluate the accuracy of proposed methods for constructing confidence interval for the parameters of Weibull distribution. For this purpose, we generated $n + 1$ record values from a Weibull distribution, and considered $\alpha = 1, 2$. For the simulation with 10000 runs and different values of the shape parameter $\beta$, the empirical coverage probabilities and expected lengths of the methods with the confidence coefficient 0.95 were obtained. The results of our generalized confidence interval for inference on $\alpha$ using the
algorithm with $M = 10000$ are presented in Table 1 and the results of our exact method (E) and the Wu method (W) for inference on $\beta$ are given in Table 2. We can conclude that

i. The empirical coverage probabilities of all methods are close to the confidence level 0.95.

ii. The expected lengths of E and W increase when the parameter $\beta$ increases. Additionally, the expected length of E is smaller than W especially when $\beta$ is large.

iii. The expected length of our generalized confidence interval for $\alpha$ decreases when the parameter $\beta$ increases. Moreover, it is very large when $\beta$ is small.

iv. The expected lengths of all methods decrease when the number of records increases.

v. The empirical coverage probabilities and expected lengths of W and E do not change when the parameter $\alpha$ changes.

3 Joint confidence regions for the parameters

Suppose $R_0, R_1, \ldots, R_n$ are the first $n + 1$ upper record values from a Weibull distribution with parameters $\alpha$ and $\beta$. In this section, we presented a joint confidence region for the parameters $\alpha$ and $\beta$. This is important because it can be used to find confidence bounds for any function of the parameters such as the reliability function $R(t) = \exp\left(-\left(\frac{t}{\alpha}\right)^\beta\right)$. For more references about the joint confidence region based on records, reader can see Asgharzadeh and Abdi (2011a,b) and Asgharzadeh et al. (2011).

3.1 Asgharzadeh and Abdi method

Asgharzadeh and Abdi (2011b) present exact joint confidence regions for the parameters of Weibull distribution based on the record values using the idea presented by Wu and Tseng (2006). The following inequalities determine $100(1 - \gamma)\%$ joint confidence regions for $\alpha$ and $\beta$:

$$A_j = \left\{ \begin{array}{l}
\log \left( \frac{(n-j+1)k_1+1}{\log (R_n/R_{j-1})} \right) < \beta < \log \left( \frac{(n-j+1)k_2+1}{\log (R_n/R_{j-1})} \right) \\
R_n \left( \frac{2}{\chi^2(2n+2),(1+\sqrt{1-\gamma})/2} \right)^{1/\gamma} < \alpha < R_n \left( \frac{2}{\chi^2(2n+2),(1-\sqrt{1-\gamma})/2} \right)^{1/\gamma},
\end{array} \right. \quad (3.1)$$

for $j = 1, \ldots, n$, where

$$k_1 = F_{(2n-2j+2,2j),(1-\sqrt{1-\gamma})/2} \quad k_2 = F_{(2n-2j+2,2j),(1+\sqrt{1-\gamma})/2},$$
Table 2: Empirical coverage probabilities and expected lengths of the methods for constructing confidence interval for the parameter $\beta$ with confidence level 0.95

| $\alpha$ | $n$ | Method | 0.5 | 1.0 | 1.2 | 1.5 | 2.0 | 3.0 | 5.0 |
|----------|-----|--------|-----|-----|-----|-----|-----|-----|-----|
| 1.0      | 3   | W      | 0.950 | 0.952 | 0.952 | 0.949 | 0.949 | 0.953 | 0.947 |
|          |     | E      | 0.949 | 0.953 | 0.953 | 0.950 | 0.948 | 0.953 | 0.946 |
| 7        | W   | 0.951 | 0.949 | 0.950 | 0.948 | 0.949 | 0.949 | 0.953 | 0.953 |
|          |     | E      | 0.951 | 0.950 | 0.948 | 0.950 | 0.953 | 0.953 | 0.950 |
| 9        | W   | 0.949 | 0.949 | 0.945 | 0.949 | 0.947 | 0.949 | 0.950 | 0.950 |
|          |     | E      | 0.948 | 0.950 | 0.948 | 0.948 | 0.949 | 0.952 | 0.949 |
| 14       | W   | 0.946 | 0.949 | 0.951 | 0.951 | 0.953 | 0.949 | 0.951 | 0.951 |
|          |     | E      | 0.947 | 0.948 | 0.950 | 0.951 | 0.953 | 0.952 | 0.952 |
| 2.0      | 3   | W      | 0.954 | 0.955 | 0.948 | 0.950 | 0.950 | 0.952 | 0.949 |
|          |     | E      | 0.953 | 0.953 | 0.947 | 0.949 | 0.949 | 0.952 | 0.950 |
| 7        | W   | 0.950 | 0.955 | 0.947 | 0.948 | 0.952 | 0.947 | 0.948 | 0.950 |
|          |     | E      | 0.949 | 0.952 | 0.951 | 0.948 | 0.952 | 0.947 | 0.950 |
| 9        | W   | 0.953 | 0.948 | 0.956 | 0.950 | 0.953 | 0.951 | 0.952 | 0.952 |
|          |     | E      | 0.952 | 0.948 | 0.951 | 0.951 | 0.953 | 0.951 | 0.953 |
| 14       | W   | 0.948 | 0.947 | 0.949 | 0.950 | 0.951 | 0.951 | 0.952 | 0.952 |
|          |     | E      | 0.950 | 0.947 | 0.949 | 0.949 | 0.953 | 0.951 | 0.955 |

| $\alpha$ | $n$ | Method | 1.704 | 3.431 | 4.194 | 5.279 | 6.913 | 10.396 | 17.479 |
|----------|-----|--------|------|------|------|------|------|--------|--------|
| 1.0      | 3   | E      | 1.630 | 3.285 | 4.013 | 5.041 | 6.611 | 9.937 | 16.722 |
| 7        | W   | 0.932 | 1.879 | 2.224 | 2.808 | 3.752 | 5.578 | 9.276 |
|          |     | E      | 0.853 | 1.716 | 2.038 | 2.574 | 3.437 | 5.115 | 8.499 |
| 9        | W   | 0.806 | 1.603 | 1.928 | 2.412 | 3.222 | 4.797 | 8.024 |
|          |     | E      | 0.730 | 1.450 | 1.748 | 2.185 | 2.928 | 4.352 | 7.267 |
| 14       | W   | 0.625 | 1.262 | 1.509 | 1.888 | 2.505 | 3.766 | 6.263 |
|          |     | E      | 0.558 | 1.125 | 1.343 | 1.685 | 2.236 | 3.353 | 5.590 |
| 2.0      | 3   | W      | 1.713 | 3.458 | 4.156 | 5.277 | 6.856 | 10.307 | 16.998 |
|          |     | E      | 1.638 | 3.306 | 3.967 | 5.053 | 6.560 | 9.859 | 16.266 |
| 7        | W   | 0.934 | 1.866 | 2.208 | 2.822 | 3.738 | 5.629 | 9.392 |
|          |     | E      | 0.854 | 1.710 | 2.026 | 2.589 | 3.419 | 5.151 | 8.581 |
| 9        | W   | 0.808 | 1.600 | 1.923 | 2.418 | 3.189 | 4.798 | 8.032 |
|          |     | E      | 0.733 | 1.451 | 1.743 | 2.193 | 2.890 | 4.347 | 7.276 |
| 14       | W   | 0.628 | 1.260 | 1.496 | 1.888 | 2.523 | 3.768 | 6.302 |
|          |     | E      | 0.560 | 1.124 | 1.338 | 1.688 | 2.250 | 3.366 | 5.624 |
and $F_{a,b},\gamma$ is the $\gamma$th percentile of the F distribution with $a$ and $b$ degrees of freedom. Note that for each $j$, we have a joint confidence region for $\alpha$ and $\beta$. Asgharzadeh and Abdi (2011b) found that in most cases $A_{\lfloor \frac{n+1}{5} \rfloor}$ and $A_{\lfloor \frac{n+1}{5} + 1 \rfloor}$ provide the smallest confidence areas, where $\lfloor x \rfloor$ is the largest integer value smaller than $x$.

### 3.2 A new joint confidence region

From Theorem 2.1, $U = 2\beta \sum_{i=0}^{n} \log \left( \frac{R_{n}}{R_{n-i}} \right)$ has a chi-square distribution with $2n$ degrees of freedom and $V = 2(\frac{R_{n}}{R_{n-i}})^{\beta}$ has a chi-square distribution with $2n + 2$ degrees of freedom, and $U$ and $V$ are independent. Therefore, an exact joint confidence region for the parameters $\alpha$ and $\beta$ of Weibull distribution based on the record values can be given as

$$B = \left\{ \left( \alpha, \beta \right) : \begin{array}{l}
\frac{\chi^2_{(2n),(1-\sqrt{1-\gamma})/2}}{2 \sum_{i=0}^{n} \log \left( \frac{R_{n}}{R_{n-i}} \right)} < \beta < \frac{\chi^2_{(2n),(1+\sqrt{1-\gamma})/2}}{2 \sum_{i=0}^{n} \log \left( \frac{R_{n}}{R_{n-i}} \right)} \\
R_{n} \left( \frac{2}{\chi^2_{(2n+2),(1+\sqrt{1-\gamma})/2}} \right)^{\frac{1}{\beta}} < \alpha < R_{n} \left( \frac{2}{\chi^2_{(2n+2),(1-\sqrt{1-\gamma})/2}} \right)^{\frac{1}{\beta}}
\end{array} \right\} \quad (3.2)$$

**Remark 3.1.** All record values are used in the proposed joint confidence region in (3.2) but not in the proposed joint confidence regions in (3.1).

### 3.3 Real example

Here, we consider the upper record values in the example given in Section 2.3. Therefore, the 95% joint confidence regions for $\alpha$ and $\beta$ based on Asgharzadeh and Abdi (2011b) in (3.1) are

- $A_{1} = \left\{ (\alpha, \beta) : 0.5826 < \beta < 11.9955, \quad 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\}$
- $A_{2} = \left\{ (\alpha, \beta) : 0.1646 < \beta < 6.4905, \quad 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\}$
- $A_{3} = \left\{ (\alpha, \beta) : 0.1720 < \beta < 58.9824, \quad 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\}$

and the 95% joint confidence region for $\alpha$ and $\beta$ in (3.2) is

$$B = \left\{ (\alpha, \beta) : 0.5305 < \beta < 9.0277, \quad 41(0.1029)^{\frac{1}{\beta}} < \alpha < 41(1.1318)^{\frac{1}{\beta}} \right\}.$$

The plot of all joint confidence regions are given in Figure 1. Also, the area of the joint confidence regions $A_{1}$, $A_{2}$, $A_{3}$, and $B$ are 194.9723, 166.7113, 369.7654, and 172.5757, respectively.
Figure 1: The plot of the joint confidence regions $A_1$, $A_2$, $A_3$, and $B$.

3.4 Simulation study

We performed a similar simulation given in Section 2.4 with considering $\alpha = 1$, in order to compare the joint confidence regions proposed by Asgharzadeh and Abdi (2011b) and our joint confidence region ($B$) in (3.2). Here, we consider the confidence areas $A_{\left\lfloor \frac{n+1}{5} \right\rfloor}$ and $A_{\left\lfloor \frac{n+1}{5} \right\rfloor + 1}$ because the coverage probabilities of all $A_i$'s are close to the confidence coefficient and Asgharzadeh and Abdi (2011b) found that in most cases these two confidence areas provide the smallest confidence areas. The empirical coverage probabilities and expected areas of the methods for the confidence coefficient 95% are given in Table 3. We can conclude that

1. The coverage probabilities of the all methods are close to the confidence coefficient 0.95.

2. The expected area of our method is smaller than the expected areas of the proposed methods by Asgharzadeh and Abdi (2011b).

3. The expected areas of all methods decrease when the number of records increases.

4. The expected areas of all methods decrease when the parameter $\beta$ increases.
Table 3: Empirical coverage probabilities of the methods for constructing joint confidence region for the parameters $\alpha$ and $\beta$ with $\gamma = 0.05$.

| $n$ | Region | Coverage Probability |
|-----|--------|----------------------|
| 4   | $A_1$  | 0.950 0.949 0.949 0.951 0.954 0.946 0.950 |
|     | $A_2$  | 0.949 0.951 0.950 0.951 0.953 0.949 0.952 |
|     | $B$    | 0.949 0.950 0.949 0.952 0.954 0.949 0.950 |
| 6   | $A_1$  | 0.951 0.951 0.952 0.952 0.955 0.949 0.950 |
|     | $A_2$  | 0.952 0.948 0.950 0.948 0.953 0.948 0.952 |
|     | $B$    | 0.953 0.949 0.951 0.951 0.953 0.950 0.951 |
| 9   | $A_2$  | 0.953 0.949 0.950 0.955 0.949 0.948 0.953 |
|     | $A_3$  | 0.951 0.951 0.951 0.956 0.949 0.948 0.949 |
|     | $B$    | 0.950 0.952 0.951 0.953 0.949 0.948 0.952 |
| 14  | $A_3$  | 0.947 0.950 0.954 0.948 0.954 0.951 0.952 |
|     | $A_4$  | 0.946 0.951 0.952 0.948 0.951 0.950 0.952 |
|     | $B$    | 0.948 0.949 0.952 0.947 0.951 0.953 0.952 |
| 29  | $A_6$  | 0.951 0.953 0.950 0.950 0.950 0.948 0.950 |
|     | $A_7$  | 0.950 0.952 0.951 0.949 0.948 0.948 0.950 |
|     | $B$    | 0.953 0.955 0.951 0.953 0.950 0.948 0.952 |
|     |        | Expected Area |
| 4   | $A_1$  | 27.787 8.548 7.339 6.331 5.725 5.330 5.203 |
|     | $A_2$  | 30.020 8.976 7.651 6.682 5.989 5.593 5.504 |
|     | $B$    | 22.985 7.371 6.388 5.596 5.099 4.792 4.713 |
| 6   | $A_1$  | 21.062 6.036 5.213 4.576 4.102 3.783 3.701 |
|     | $A_2$  | 20.035 5.824 5.046 4.436 3.985 3.701 3.648 |
|     | $B$    | 14.551 4.714 4.144 3.714 3.399 3.192 3.162 |
| 9   | $A_3$  | 14.631 4.081 3.533 3.059 2.756 2.574 2.510 |
|     | $A_4$  | 14.651 4.086 3.545 3.077 2.767 2.585 2.541 |
|     | $B$    | 9.639 3.137 2.774 2.471 2.275 2.160 2.133 |
| 14  | $A_3$  | 9.436 2.702 2.320 2.035 1.816 1.701 1.661 |
|     | $A_4$  | 9.388 2.686 2.304 2.035 1.812 1.707 1.668 |
|     | $B$    | 5.784 1.999 1.763 1.599 1.471 1.405 1.385 |
| 29  | $A_6$  | 4.244 1.298 1.124 0.988 0.905 0.849 0.828 |
|     | $A_7$  | 4.202 1.291 1.118 0.985 0.904 0.850 0.828 |
|     | $B$    | 2.380 0.932 0.838 0.761 0.719 0.689 0.678 |
Appendix. Generalized $p$-value and generalized confidence interval

Let $X$ be a random variable whose distribution depends on a parameter of interest $\theta$, and a
nuisance parameter $\lambda$. Let $x$ denote the observed value of $X$. A generalized pivotal quantity
for $\theta$ is a random quantity denoted by $T(X; x; \theta)$ that satisfies the following conditions:
(i) The distribution of $T(X; x; \theta)$ is free of any unknown parameters.
(ii) The value of $T(X; x; \theta)$ at $X = x$, i.e., $T(x; x; \theta)$ is free of the nuisance parameter $\lambda$.

Appropriate percentiles of $T(X; x; \theta)$ form a confidence interval for $\theta$. Specifically, if $T(x; x; \theta) = \theta$, and $T_\delta$ denotes the 100$\delta$ percentage point of $T(X; x; \theta)$ then $(T_\gamma/2, T_{1-\gamma/2})$
is a $1-\gamma$ generalized confidence interval for $\theta$. The percentiles can be found because, for a
given $x$, the distribution of $T(X; x; \theta)$ does not depend on any unknown parameters.

In the above setup, suppose we are interested in testing the hypotheses

$$H_0 : \theta \leq \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0,$$  \hspace{1cm} (A.1)

for a specified $\theta_0$. The generalized test variable, denoted by $T^*(X; x; \theta)$, is defined as follows:
(i) The value of $T^*(X; x; \theta)$ at $X = x$ is free of any unknown parameters.
(ii) The distribution of $T^*(X; x; \theta)$ is stochastically monotone (i.e., stochastically increasing or
stochastically decreasing) in $\theta$ for any fixed $x$ and $\lambda$.
(iii) The distribution of $T^*(X; x; \theta)$ is free of any unknown parameters.

Let $t^* = T^*(x; x; \theta_0)$, the observed value of $T^*(X; x; \theta)$ at $(X; \theta) = (x; \theta_0)$. When the
above conditions hold, the generalized $p$-value for testing the hypotheses in (A.1) is defined as

$$p = P(T^*(X; x; \theta_0) \leq t^*)$$  \hspace{1cm} (A.2)

where $T^*(X; x; \theta)$ is stochastically decreasing in $\theta$. The test based on the generalized $p$-value
rejects $H_0$ when the generalized $p$-value is smaller than a nominal level $\gamma$. However, the size
and power of such a test may depend on the nuisance parameters.

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References

Ahsanullah, M. (1995). *Record statistics*. Nova Science Publishers Commack, New York.

Arnold, B., Balakrishnan, N., and Nagaraja, H. (1998). *Records*. John Wiley & Sons Inc, New York.

Asgharzadeh, A. and Abdi, M. (2011a). Confidence intervals and joint confidence regions for the two-parameter exponential distribution based on records. *Communications of the Korean Statistical Society*, 18(1):103–110.

Asgharzadeh, A. and Abdi, M. (2011b). Joint confidence regions for the parameters of the Weibull distribution based on record. *ProbStat Forum*, 4:12–24.

Asgharzadeh, A., Abdi, M., and Kuş, C. (2011). Interval estimation for the two-parameter pareto distribution based on record values. *Selçuk Journal of Applied Mathematics*, Special Issue:149–161.

Balakrishnan, N. and Chan, P. S. (1994). Record values from Rayleigh and Weibull distributions and associated inference. *National Institute of Standards and Technology Journal of Research, Special Publication, Proceedings of the Conference on Extreme Value Theory and Applications*, 866:41–51.

Behboodian, J. and Jafari, A. A. (2006). Generalized inference for the common mean of several lognormal populations. *Journal of Statistical Theory and Applications*, 5(3):240–259.

Chan, P. S. (1998). Interval estimation of location and scale parameters based on record values. *Statistics and Probability Letters*, 37(1):49–58.

Chandler, K. N. (1952). The distribution and frequency of record values. *Journal of the Royal Statistical Society. Series B (Methodological)*, 14(2):220–228.

Dallas, A. C. (1982). Some results on record values from the exponential and Weibull law. *Acta Mathematica Academy of Sciences of Hungary*, 40(3):307–311.

Gouet, R., López, F. J., and Sanz, G. (2012). On δ-record observations: asymptotic rates for the counting process and elements of maximum likelihood estimation. *Test*, 21(1):188–214.

Jafari, A. A. (2012). Inferences on the ratio of two generalized variances: independent and correlated cases. *Statistical Methods and Applications*, 21(3):297–314.
Krishnamoorthy, K. and Lin, Y. (2010). Confidence limits for stress-strength reliability involving Weibull models. *Journal of Statistical Planning and Inference*, 140(7):1754–1764.

Krishnamoorthy, K., Lin, Y., and Xia, Y. (2009). Confidence limits and prediction limits for a Weibull distribution based on the generalized variable approach. *Journal of Statistical Planning and Inference*, 139(8):2675–2684.

Roberts, E. (1979). Review of statistics of extreme values with applications to air quality data: Part ii. applications. *Journal of the Air Pollution Control Association*, 29(7):733–740.

Soliman, A. A., Abd Ellah, A. H., and Sultan, K. S. (2006). Comparison of estimates using record statistics from Weibull model: Bayesian and non-Bayesian approaches. *Computational Statistics and Data Analysis*, 51(3):2065–2077.

Teimouri, M. and Gupta, A. K. (2012). On the Weibull record statistics and associated inferences. *Statistica*, 72(2):145–162.

Teimouri, M. and Nadarajah, S. (2013). Bias corrected MLEs for the Weibull distribution based on records. *Statistical Methodology*, 13:12–24.

Tsui, K. W. and Weerahandi, S. (1989). Generalized p-values in significance testing of hypotheses in the presence of nuisance parameters. *Journal of the American Statistical Association*, 84(406):602–607.

Weerahandi, S. (1993). Generalized confidence intervals. *Journal of the American Statistical Association*, 88(423):899–905.

Weerahandi, S. (1995). *Exact Statistical Methods for Data Analysis*. Springer Verlag, New York.

Wu, J. W. and Tseng, H. C. (2006). Statistical inference about the shape parameter of the Weibull distribution by upper record values. *Statistical Papers*, 48(1):95–129.