A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR THE STOKES PROBLEM ON POLYTOPAL MESHES

XIU YE∗ AND SHANGYOU ZHANG†

Abstract. A new discontinuous Galerkin finite element method for the Stokes equations is developed in the primary velocity-pressure formulation. This method employs discontinuous polynomials for both velocity and pressure on general polygonal/polyhedral meshes. Most finite element methods with discontinuous approximation have one or more stabilizing terms for velocity and for pressure to guarantee stability and convergence. This new finite element method has the standard conforming finite element formulation, without any velocity or pressure stabilizers. Optimal-order error estimates are established for the corresponding numerical approximation in various norms. The numerical examples are tested for low and high order elements up to the degree four in 2D and 3D spaces.

Key words. Weak gradient, weak divergence, discontinuous Galerkin, finite element methods, the Stokes equations, polytopal meshes.

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

1. Introduction. Consider the Stokes problem: find the velocity \( u \) and the pressure \( p \) such that

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a polygonal or polyhedral domain in \( \mathbb{R}^d \) \((d = 2, 3)\).

In a conforming finite element method for solving above Stokes equations in primary variables \([2, 0, 20, 21, 22, 23]\), such as the Taylor-Hood element, the velocity \( u \) is approximated by continuous piecewise polynomials of degree \( k \) and the pressure \( p \) by continuous/discontinuous piecewise polynomials of degree \( k-1 \), with the following formulation: Find \( u_h \in V_h \subset H^1_0(\Omega) \) and \( p_h \in W_h \subset L^2_0(\Omega) \) such that

\[
\begin{align*}
(\nabla u_h, \nabla v_h) - (\nabla \cdot v_h, p_h) &= (f, v_h) \quad \forall v_h \in V_h, \\
(\nabla \cdot u_h, q_h) &= 0 \quad \forall q_h \in W_h.
\end{align*}
\]

In a discontinuous Galerkin finite element method \([24]\), the velocity \( u \) is also approximated by piecewise polynomials of degree \( k \) but discontinuous, and the pressure \( p \) by discontinuous piecewise polynomials of degree \( k-1 \) (or \( k \)) on polygonal/polyhedral meshes, with the following formulation: Find \( u_h \in V_h \subset L^2(\Omega) \) and \( p_h \in W_h \subset L^2_0(\Omega) \)
such that

\[ \sum_{T \in T_h} \left[ (\nabla u_h, \nabla v_h)_T - (p_h, \nabla \cdot v_h)_T \right] + \sum_{e \in \mathcal{E}_h} \left( \int_e \{\nabla u_h\} \cdot [v_h] + c^e \int_e \{\nabla v_h\} \cdot [u_h] \right) + \int_e \{p_h\} \cdot [v_h] + \sigma^e \int_e [u_h] \cdot [v_h] \right) = (f, v_h) \quad \forall v_h \in V_h, \]

\[ \sum_{T \in T_h} (q_h, \nabla \cdot u_h)_T + \sum_{e \in \mathcal{E}_h} \left( \int_e \{q_h\} \cdot [u_h] + h \int_e [p_h][q_h] \right) = 0 \quad \forall q_h \in W_h. \]

(1.6) 

(1.7)

It is proved that the pressure stabilizer \((h \int [p_h][q_h])\) can be omitted in (1.7) on triangular/tetrahedral meshes. This simplifies the discontinuous Galerkin finite element formulation. We would simplify further the formulation (1.6)-(1.7) by dropping both stabilizers and all boundary integral terms, on general polygonal/polyhedral meshes.

In a conforming discontinuous Galerkin finite element method, the original weak formulation (1.4)-(1.5) of the continuous Galerkin finite element is kept. But the gradient \(\nabla u_h\) of a discontinuous, piecewise polynomial \(u_h\) is no long a Lebesgue measurable function. It can be represented as a function in a dual space, \((\prod_{T \in T_h} H^1(T))^\prime\). We define the \(L^2\) projection of this function in a piecewise polynomial subspace as a weak gradient, i.e., \(\nabla_w u_h \in \prod_{T \in T_h} P_k(T)^d\) such that

\[ \langle \nabla_w u_h, q \rangle = \langle \nabla u_h, q \rangle \quad \forall q \in \prod_{T \in T_h} P_j(T)^d, \]

where \(j \leq k + n + d - 3\) (for \(n\)-faced polygons/polyhedrons) and

\[ \langle \nabla u_h, q \rangle = \sum_{T \in T_h} (u_h, -\nabla \cdot q)_T + \langle \{u_h\}, q \cdot n \rangle_{\partial T} \quad \forall q \in \prod_{T \in T_h} H^1(T). \]

Such a method has been developed for Poisson equations [6 16 17], and for biharmonic equations [5 18]. The definition of weak gradient comes from the weak Galerkin finite element method [6 9 4 7 11 12 13] and the modified weak Galerkin method [4 10 15]. A disadvantage of the conforming discontinuous Galerkin finite element method is its computation of the gradient by higher order polynomials. But this computation is done locally for basis functions only, in advance, before generating and solving the resulting linear systems of equations. It is equivalent to using high-order quadrature formula in the continuous finite element.

In this paper, we propose a new finite element method for the Stokes equations with discontinuous approximations on general polytopal meshes. Our new finite element method uses totally discontinuous \(k\)th degree polynomial for velocity and \((k-1)\)th degree polynomial for pressure. When discontinuous polynomials are employed for both velocity and pressure, stabilizers for velocity or pressure are normally required for the stability of the corresponding finite element formulations, such as (1.6)-(1.7). But in this new conforming discontinuous Galerkin finite element method, we do not have any boundary stabilizer term, neither any other boundary integral term. That is, we find \(u_h \in V_h \subset L^2(\Omega)\) and \(p_h \in W_h \subset L^2_0(\Omega)\) such that

\[ \langle \nabla_w u_h, \nabla_w v_h \rangle_T - (p_h, \nabla \cdot v_h) = (f, v_h) \quad \forall v_h \in V_h, \]

(1.8)

\[ (q_h, \nabla \cdot u_h)_T = 0 \quad \forall q_h \in W_h. \]

(1.9)
To the best of our knowledge, our new method is the only finite element formulation without any velocity or pressure stabilizers among all the methods for the Stokes problem in primary velocity-pressure form with discontinuous approximations on polytopal meshes.

Optimal order error estimates for the finite element approximations are derived in energy norm for the velocity, and $L^2$ norm for both the velocity and the pressure. Numerical examples are tested for the finite elements with different degrees up to $P_4$ polynomials and for different dimensions, 2D and 3D.

2. Finite Element Method. We use standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for $s \geq 0$. When $D = \Omega$, we drop the subscript $D$ in the norm and inner product notation.

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of polygons in two dimensional space or polyhedra in three dimensional space satisfying a set of conditions specified in [13]. Denote by $\mathcal{E}_h$ the set of all flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ be the set of all interior faces.

For $k \geq 1$ and given $\mathcal{T}_h$, define two finite element spaces, for approximating velocity

(2.1) \[ V_h = \{ \mathbf{v} \in L^2(\Omega) : \mathbf{v}|_T \in [P_k(T)]^d, \ \forall T \in \mathcal{T}_h \} \]

and for approximating pressure

(2.2) \[ W_h = \{ q \in L^2(\Omega) : q|_T \in P_{k-1}(T), \ \forall T \in \mathcal{T}_h \} . \]

Let $T_1$ and $T_2$ be two elements in $\mathcal{T}_h$ sharing $e \in \mathcal{E}_h$. For $e \in \mathcal{E}_h$ and $\mathbf{v} \in V_h + H^1_0(\Omega)$, the jump $[\mathbf{v}]$ is defined as

(2.3) \[ [\mathbf{v}] = \mathbf{v} \quad \text{if} \ e \subset \partial \Omega, \quad [\mathbf{v}] = \mathbf{v}|_{T_1} - \mathbf{v}|_{T_2} \quad \text{if} \ e \in \mathcal{E}_h^0. \]

The order of $T_1$ and $T_2$ is not essential.

For $e \in \mathcal{E}_h$ and $\mathbf{v} \in V_h + H^1_0(\Omega)$, the average $\{\mathbf{v}\}$ is defined as

(2.4) \[ \{\mathbf{v}\} = 0 \quad \text{if} \ e \subset \partial \Omega, \quad \{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}|_{T_1} + \mathbf{v}|_{T_2}) \quad \text{if} \ e \in \mathcal{E}_h^0, \]

For a function $\mathbf{v} \in V_h + H^1_0(\Omega)$, its weak gradient $\nabla_w \mathbf{v}$ is a piecewise polynomial tensor such that $\nabla_w \mathbf{v} \in \prod_{T \in \mathcal{T}_h} [P_j(T)]^{d \times d}$ and satisfies the following equation,

(2.5) \[ (\nabla_w \mathbf{v}, \ \tau)_T = (\mathbf{v}, \ -\nabla \cdot \tau)_T + \langle \{\mathbf{v}\}, \ \tau \cdot n \rangle_{\partial T} \quad \forall \tau \in [P_j(T)]^{d \times d} \]
on each $T \in \mathcal{T}_h$, and its weak divergence $\nabla_w \cdot \mathbf{v}$ is a piecewise polynomial such that $\nabla_w \cdot \mathbf{v} \in \prod_{T \in \mathcal{T}_h} P_{k-1}(T)$ and satisfies the following equation,

(2.6) \[ (\nabla_w \cdot \mathbf{v}, \ q)_T = (\mathbf{v}, \ -\nabla q)_T + \langle \{\mathbf{v}\} \cdot n, \ q \rangle_{\partial T} \quad \forall q \in P_{k-1}(T) \]
on each $T \in \mathcal{T}_h$.

Remark 1. The choice of $j$ in (2.3) depends on the number of sides/faces of polygon/polyhedron. For triangular mesh, we can choose $j = k + 1$ [1]. In general, $j = n + k - 1$, where $n$ is the number of edges of polygon [17].

Then we have the following simple penalty free finite element scheme.
Weak Galerkin Algorithm 1. A numerical approximation for (1.1)-(1.3) can be obtained by seeking \( u_h \in V_h \) and \( p_h \in W_h \) such that for all \( v \in V_h \) and \( q \in W_h \),

\[
(\nabla_w u_h, \nabla_w v) - (\nabla_w \cdot v, p_h) = (f, v),
\]

\[
(\nabla_w \cdot u_h, q) = 0.
\]

Let \( Q_h, Q_h \) and \( Q_h \) be the element-wise defined \( L^2 \) projections onto the local spaces \([P_j(T)]^{d \times d}, [P_k(T)]^d\) and \( P_{k-1}(T) \) for \( T \in T_h \), respectively.

Lemma 2.1. Let \( \phi \in H^1_0(\Omega) \), then on \( T \in T_h \)

\[
\nabla_w \phi = Q_h \nabla \phi,
\]

\[
\nabla_w \cdot \phi = Q_h \nabla \cdot \phi.
\]

Proof. Using (2.5) and integration by parts, we have that for any \( \tau \in [P_j(T)]^{d \times d} \)

\[
(\nabla_w \phi, \tau)_T = -(\phi, \nabla \cdot \tau)_T + \langle \{\phi\}, \tau \cdot n \rangle_{\partial T}
\]

\[
= -(\phi, \nabla \cdot \tau)_T + (\phi, \tau \cdot n)_{\partial T}
\]

\[
= (\nabla \phi, \tau) = (Q_h \nabla \phi, \tau)_T,
\]

which implies the desired identity (2.9). Similarly, we can prove (2.10). \( \square \)

For any function \( \varphi \in H^1(T) \), the following trace inequality holds true (see [13] for details):

\[
\|\varphi\|_e^2 \leq C \left( h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla \varphi\|_T^2 \right).
\]

3. Well Posedness. We start this section by introducing two semi-norms \( \|v\| \) and \( \|v\|_{1,h} \) for any \( v \in V_h \) as follows:

\[
\|v\|^2 = \sum_{T \in T_h} (\nabla_w v, \nabla_w v)_T.
\]

\[
\|v\|_{1,h}^2 = \sum_{T \in T_h} \|\nabla v\|_T^2 + \sum_{e \in E_h} h_e^{-1}\|v\|_e^2.
\]

It is easy to see that \( \|v\|_{1,h} \) defines a norm in \( V_h \). But \( \|\cdot\|_e \) also defines a norm in \( V_h \), by the following norm equivalence which has been proved in [1] [17], to each component of \( v \).

\[
C_1\|v\|_{1,h} \leq \|v\| \leq C_2\|v\|_{1,h} \quad \forall v \in V_h.
\]

The inf-sup condition for the finite element formulation (2.7)-(2.8) will be derived in the following lemma.

Lemma 3.1. There exists a positive constant \( \beta \) independent of \( h \) such that for all \( \rho \in W_h \) and \( h \) small enough,

\[
\sup_{v \in V_h} \frac{(\nabla_w \cdot v, \rho)}{\|v\|} \geq \beta \|\rho\|.
\]
Proof. For any given \( \rho \in W_h \subset L^2(\Omega) \), it is known [2] that there exists a function \( \tilde{v} \in H^1_0(\Omega) \) such that

\[
\frac{(\nabla \cdot \tilde{v}, \rho)}{\|\tilde{v}\|_1} \geq C_0\|\rho\|,
\]

(3.5)

where \( C_0 > 0 \) is a constant independent of \( h \). By setting \( v = Q_h \tilde{v} \in V_h \), we claim that the following holds true

\[
\|v\| \leq C\|\tilde{v}\|_1.
\]

(3.6)

It follows from (3.3) and \( \tilde{v} \in H^1_0(\Omega) \),

\[
\|v\|^2 \leq C\|v\|^2_{h} = C(\sum_{T \in T_h} \|\nabla v\|^2_T + \sum_{e \in E_h} h_e^{-1}\|v\|^2_e)
\leq C(\sum_{T \in T_h} \|\nabla Q_h \tilde{v}\|^2_T + \sum_{e \in E_h} h_e^{-1}\|Q_h \tilde{v} - \tilde{v}\|^2_e)
\leq C\|\tilde{v}\|^2_1,
\]

which implies the inequality (3.6). It follows from (2.6) and (3.5) that

\[
|\langle \nabla w, v,\rho \rangle_T| = |-(v, \nabla \rho)_T + \langle \{v, \rho\}, \mathbf{n}\rangle_{\partial T}| = |-(\tilde{v}, \nabla \rho)_T + \langle \{Q_h \tilde{v}, \rho\}, \mathbf{n}\rangle_{\partial T}|
\geq |(\nabla \cdot v, \rho)_T| - C_1 h \|\tilde{v}\|_1 \|\rho\|
\geq (C_0 - C_1 h)\|\tilde{v}\|_1 \|\rho\|.
\]

(3.7)

Using the above equation and (3.6), we have

\[
\frac{|\langle \nabla w, v,\rho \rangle|}{\|v\|} \geq \frac{(C_0 - C_1 h)\|\tilde{v}\|_1 \|\rho\|}{C_0\|\tilde{v}\|_1} \geq \beta \|\rho\|
\]

for a positive constant \( \beta \). This completes the proof of the lemma. \( \Box \)

Lemma 3.2. The weak Galerkin method (2.7)-(2.8) has a unique solution.

Proof. It suffices to show that zero is the only solution of (2.7)-(2.8) if \( f = 0 \). To this end, let \( f = 0 \) and take \( v = u_h \) in (2.7) and \( q = p_h \) in (2.8). By adding the two resulting equations, we obtain

\[
(\nabla w u_h, \nabla w u_h)_T = 0,
\]

which implies that \( \nabla w u_h = 0 \) on each element \( T \). By (3.3), we have \( \|u_h\|_{1,h} = 0 \) which implies that \( u_h = 0 \).

Since \( u_h = 0 \) and \( f = 0 \), the equation (2.7) becomes \( (\nabla w v, p_h) = 0 \) for any \( v \in V_h \). Then the inf-sup condition (3.4) implies \( p_h = 0 \). We have proved the lemma. \( \Box \)

4. Error Equations. In this section, we will derive the equations that the errors satisfy. Let \( e_h = Q_h u - u_h \), \( e_0 = u - u_h \) and \( \varepsilon_h = Q_h p - p_h \).

Lemma 4.1. For any \( v \in V_h \) and \( q \in W_h \), the following error equations hold true,

\[
(\nabla w e_h, \nabla w v)_{T_h} - (\varepsilon_h, \nabla w \cdot v) = \ell_1(u, v) - \ell_2(u, v) - \ell_3(p, v),
\]

(4.1)

\[
(\nabla w \cdot e_h, q) = -\ell_4(u, q),
\]

(4.2)
where

\begin{align*}
\ell_1(u, v) &= \langle (\nabla u - Q_h(\nabla u)) \cdot n, v - \{v\} \rangle_{\partial T_h}, \\
\ell_2(u, v) &= \langle \nabla w(u - Q_h u), \nabla w v \rangle_{T_h}, \\
\ell_3(p, v) &= \langle p - Q_h p, (v - \{v\}) \cdot n \rangle_{\partial T_h}, \\
\ell_4(u, q) &= \langle \{u - Q_h u\} \cdot n, q \rangle_{\partial T_h}.
\end{align*}

Proof. First, we test (1.1) by \(v \in V_h\) to obtain

\begin{equation}
-(\Delta u, v) + (\nabla p, v) = (f, v).
\end{equation}

Integration by parts and the fact \(\langle \nabla u \cdot n, \{v\} \rangle_{\partial T_h} = 0\) give

\begin{equation}
-(\Delta u, v) = \langle \nabla u, \nabla v \rangle_{T_h} - \langle \nabla u \cdot n, v - \{v\} \rangle_{\partial T_h}.
\end{equation}

It follows from integration by parts, (2.5) and (2.9),

\begin{equation}
\langle \nabla u, \nabla v \rangle_{T_h} = (q_h \nabla u, \nabla v)_{T_h} = -(v, \nabla \cdot (q_h \nabla u))_{T_h} + \langle v, q_h \nabla u \cdot n \rangle_{\partial T_h} = (q_h \nabla u, \nabla w v)_{T_h} + \langle v - \{v\}, q_h \nabla u \cdot n \rangle_{\partial T_h} = \langle \nabla w u, \nabla w v \rangle_{T_h} + \langle v - \{v\}, q_h \nabla u \cdot n \rangle_{\partial T_h}.
\end{equation}

Combining (4.8) and (4.9) gives

\begin{equation}
-(\Delta u, v) = \langle \nabla w u, \nabla w v \rangle_{T_h} - \ell_1(u, v).
\end{equation}

Using integration by parts and \(v \in V_h\) and (2.9), we have

\begin{equation}
(\nabla p, v) = -(p, \nabla \cdot v)_{T_h} + \langle p, v \cdot n \rangle_{\partial T_h} = -(Q_h p, \nabla \cdot v)_{T_h} + \langle p, (v - \{v\}) \cdot n \rangle_{\partial T_h} = (\nabla Q_h p, v)_{T_h} - \langle Q_h p, v \cdot n \rangle_{\partial T_h} + \langle p, (v - \{v\}) \cdot n \rangle_{\partial T_h} = -\langle Q_h p, \nabla w v \rangle_{T_h} - \langle Q_h p, (v - \{v\}) \cdot n \rangle_{\partial T_h} + \langle p, (v - \{v\}) \cdot n \rangle_{\partial T_h} = -(Q_h p, \nabla w \cdot v)_{T_h} + \langle p, (v - \{v\}) \cdot n \rangle_{\partial T_h}.
\end{equation}

Substituting (4.10) and (4.11) into (4.7) gives

\begin{equation}
(\nabla w u, \nabla w v)_{T_h} - (Q_h p, \nabla w \cdot v)_{T_h} = (f, v) + \ell_1(u, v) - \ell_3(p, v).
\end{equation}

The difference of (4.12) and (2.7) implies

\begin{equation}
(\nabla_w e_h, \nabla_w v)_{T_h} - (\epsilon_h, \nabla_w \cdot v)_{T_h} = \ell_1(u, v) - \ell_3(p, v) \quad \forall v \in V_h.
\end{equation}

Adding and subtracting \(\nabla_w Q_h u, \nabla_w v)_{T_h}\) in (4.13), we have

\begin{equation}
(\nabla_w e_h, \nabla_w v)_{T_h} - (\epsilon_h, \nabla_w \cdot v)_{T_h} = \ell_1(u, v) - \ell_2(u, v) - \ell_3(p, v),
\end{equation}

which implies (4.11).

Testing equation (1.2) by \(q \in W_h\) and using (2.6) give

\begin{align*}
(\nabla \cdot u, q) &= -(u, \nabla q)_{T_h} + \langle u \cdot n, q \rangle_{\partial T_h} = -(Q_h u, \nabla q)_{T_h} + \langle u \cdot n, q \rangle_{\partial T_h} = (\nabla_w Q_h u, q)_{T_h} + \langle (u - Q_h u) \cdot n, q \rangle_{\partial T_h} = (\nabla_w Q_h u, q)_{T_h} + \ell_4(u, q).
\end{align*}
which implies

\[(4.15) \quad (\nabla w \cdot Q_h u, q)_{T_h} = -\ell_4(u, q).\]

The difference of \[(4.15)\] and \[(2.8)\] implies \[(4.2)\]. We have proved the lemma.

\[\Box\]

### 5. Error Estimates in Energy Norm

In this section, we shall establish optimal order error estimates for the velocity approximation \(u_h\) in \(\| \cdot \|\) norm and for the pressure approximation \(p_h\) in the standard \(L^2\) norm.

It is easy to see that the following equations hold true for \(\{v\}\) defined in \[(2.4)\],

\[(5.1) \quad \|v - \{v\}\|_e = \|v\|_e \quad \text{if } e \subset \partial\Omega, \quad \|v - \{v\}\|_e = \frac{1}{2} \|v\|_e \quad \text{if } e \in \mathcal{E}_h^0.\]

**Lemma 5.1.** Let \((w, p) \in H^{k+1}(\Omega) \times H^k(\Omega)\) and \((v, q) \in V_h \times W_h\). Assume that the finite element partition \(T_h\) is shape regular. Then, the following estimates hold true

\[(5.2) \quad |\ell_1(w, v)| \leq Ch^k|w|_{k+1}||v||,\]

\[(5.3) \quad |\ell_2(w, v)| \leq Ch^k|w|_{k+1}||v||,\]

\[(5.4) \quad |\ell_3(p, v)| \leq Ch^k|\rho||v||,\]

\[(5.5) \quad |\ell_4(w, q)| \leq Ch^k|w|_{k+1}||q||.\]

**Proof.** Using the Cauchy-Schwarz inequality, the trace inequality \[(2.11)\], \[(5.1)\] and \[(3.3)\], we have

\[|\ell_1(w, v)| = \left| \sum_{T \in \mathcal{T}_h} \langle v - \{v\}, \nabla w \cdot n - Q_h(\nabla w) \cdot n \rangle_{\partial T} \right|\]

\[\leq C \sum_{T \in \mathcal{T}_h} \|\nabla w - Q_h \nabla w\|_{\partial T} \|v - \{v\}\|_{\partial T}\]

\[\leq C \left( \sum_{T \in \mathcal{T}_h} h_T ||(\nabla w - Q_h \nabla w)||^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{e \in \mathcal{E}_h} h_e^{-1} ||v||^2_e \right)^{\frac{1}{2}}\]

\[\leq C h^k|w|_{k+1}||v||,\]

It follows from \[(2.5)\], integration by parts, \[(2.11)\] and \[(5.1)\] that for any \(q \in [P_j(T)]^{d \times d}\),

\[|(\nabla w - Q_h w, q)_T| = |(\nabla w - Q_h w, \nabla \cdot q)_T + \langle w - \{Q_h w\}, q \cdot n \rangle_{\partial T}|\]

\[\leq \|\nabla (w - Q_h w)\|_T \|q\|_T + Ch^{-1/2} \|[Q_h w]\|_{\partial T} \|q\|_T\]

\[\leq \|\nabla (w - Q_h w)\|_T \|q\|_T + Ch^{-1/2} \|[w - Q_h w]\|_{\partial T} \|q\|_T\]

\[(5.6) \quad \leq C h^k|w|_{k+1} T ||q||_T.\]

Letting \(q = \nabla w v\) in \[(5.6)\] and taking summation over \(T\), we have

\[|\ell_2(w, v)| = |(\nabla w (w - Q_h w), \nabla w v)_{\mathcal{T}_h}|\]

\[\leq C h^k|w|_{k+1} ||v||.\]
We have proved the lemma. and (5.7). which implies (5.7). The pressure error estimate (5.8) follows immediately from (5.10)

Similarly we have

It follows from the definition of \( Q \) that

\[
|\ell_3(\rho, v)| = |\sum_{T \in \mathcal{T}_h} (\rho - Q_h \rho, v \cdot n - \{v\} \cdot n)_{\partial T}|
\]

\[
\leq C \sum_{T \in \mathcal{T}_h} \|\rho - Q_h \rho\|_{\partial T} \|v - \{v\}\|_{\partial T}
\]

\[
\leq C \left( \sum_{T \in \mathcal{T}_h} h_T \|\rho - Q_h \rho\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|v\|_{\partial T}^2 \right)^{\frac{1}{2}}
\]

\[
\leq Ch^k |\rho|_{k+1} \|v\|.
\]

Similarly we have

\[
|\ell_4(w, q)| = |\sum_{T \in \mathcal{T}_h} (\{w - Q_h w\} \cdot n, q)_{\partial T}|
\]

\[
\leq C \sum_{T \in \mathcal{T}_h} \|w - Q_h w\|_{\partial T} \|q\|_{\partial T}
\]

\[
\leq C \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|w - Q_h w\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T \|q\|_{\partial T}^2 \right)^{\frac{1}{2}}
\]

\[
\leq Ch^k |w|_{k+1} \|q\|.
\]

We have proved the lemma. \( \square \)

**Theorem 5.2.** Let \((u_h, p_h) \in V_h \times W_h\) be the solution of (2.7)-(2.8). Then, the following error estimates hold true

\[
\|Q_h u - u_h\| \leq Ch^k (|u|_{k+1} + |p|_k), \tag{5.7}
\]

\[
\|Q_h p - p_h\| \leq Ch^k (|u|_{k+1} + |p|_k). \tag{5.8}
\]

**Proof.** It follows from \((4.1)\) that for any \( v \in V_h \)

\[
|\langle \varepsilon_h, \nabla \cdot v \rangle_{\mathcal{T}_h}| = |\langle \nabla_w e_h, \nabla \cdot v \rangle_{\mathcal{T}_h} - \ell_1(u, v) + \ell_2(u, v) + \ell_3(\rho, v)\|
\]

\[
\leq C (\|e_h\|_h + h^k |u|_{k+1}) \|v\|_h. \tag{5.9}
\]

Then the estimate \((5.9)\) and \((5.4)\) yield

\[
\|\varepsilon_h\| \leq C (\|e_h\| + h^k |u|_{k+1}). \tag{5.10}
\]

By letting \( v = e_h \) in \((4.1)\) and \( q = \varepsilon_h \) in \((4.2)\) and adding the two resulting equations, we have

\[
\|e_h\|^2 = |\ell_1(u, e_h) - \ell_2(u, e_h) - \ell_3(\rho, e_h) + \ell_4(u, \varepsilon_h)|. \tag{5.11}
\]

It then follows from \((5.2)-(5.5)\) and \((5.10)\) that

\[
\|e_h\|^2 \leq Ch^k (|u|_{k+1} + |p|_k) \|e_h\| + Ch^k |u|_{k+1} \|\varepsilon_h\|
\]

\[
\leq Ch^k (|u|_{k+1} + |p|_k) \|e_h\| + Ch^k |u|_{k+1} (\|e_h\| + Ch^k |u|_{k+1})
\]

\[
\leq Ch^2k (|u|_{k+1}^2 + |p|_k^2) + \frac{1}{2} \|e_h\|^2,
\]

which implies \((5.7)\). The pressure error estimate \((5.8)\) follows immediately from \((5.10)\) and \((5.7)\). \( \square \)
6. Error Estimates in $L^2$ Norm. In this section, we shall derive an $L^2$-error estimate for the velocity approximation through a duality argument. Recall that $e_h = Q_h u - u_h$ and $e_h = u - u_h$. To this end, consider the problem of seeking $(\psi, \xi)$ such that

\begin{align}
(6.1) & \quad -\Delta \psi + \nabla \xi = e_h \quad \text{in } \Omega, \\
(6.2) & \quad \nabla \cdot \psi = 0 \quad \text{in } \Omega, \\
(6.3) & \quad \psi = 0 \quad \text{on } \partial \Omega.
\end{align}

Assume that the dual problem has the $H^2(\Omega) \times H^1(\Omega)$-regularity property in the sense that the solution $(\psi, \xi) \in H^2(\Omega) \times H^1(\Omega)$ and the following a priori estimate holds true:

\begin{equation}
(6.4) \quad \|\psi\|_2 + \|\xi\|_1 \leq C\|e_h\|.
\end{equation}

**Theorem 6.1.** Let $(u_h, p_h) \in V_h \times W_h$ be the solution of (2.7)-(2.8). Assume that (6.4) holds true. Then we have

\begin{equation}
(6.5) \quad \|u - u_h\| \leq Ch^{k+1}(\|u\|_{k+1} + |p|_k).
\end{equation}

**Proof.** Testing (6.1) by $e_h$ gives

\begin{equation}
(6.6) \quad (e_h, e_h) = -(\Delta \psi, e_h) + (\nabla \xi, e_h).
\end{equation}

Using integration by parts and the fact $\langle \nabla \psi \cdot n, \{e_h\}\rangle_{\partial T_h} = 0$, then

\[-(\Delta \psi, e_h) = \langle \nabla \psi \cdot n, e_h \rangle_{\partial T_h} - \langle \nabla \psi \cdot n, e_h - \{e_h\}\rangle_{\partial T_h} = (Q_h \nabla \psi, \nabla e_h)_{T_h} + (\nabla \psi - Q_h \nabla \psi, \nabla e_h)_{T_h} - \langle \nabla \psi \cdot n, e_h - \{e_h\}\rangle_{\partial T_h} = (Q_h \nabla \psi, \nabla e_h)_{T_h} + (\nabla \psi - Q_h \nabla \psi, \nabla e_h)_{T_h} - \langle \nabla \psi \cdot n, e_h - \{e_h\}\rangle_{\partial T_h} = (Q_h \nabla \psi, \nabla e_h)_{T_h} + (\nabla \psi - Q_h \nabla \psi, \nabla e_h)_{T_h} - \ell_1(\psi, e_h).
\]

It follows from (2.9) that

\[(Q_h \nabla \psi, \nabla e_h)_{T_h} = (\nabla w, \nabla e_h)_{T_h} \]

and

\[= (\nabla_w Q_h \psi, \nabla e_h)_{T_h} + (\nabla_w (\psi - Q_h \psi), \nabla w e_h)_{T_h}.
\]

The equation (4.15) implies

\[\\]

\[\]

\[= (\nabla_w Q_h \psi, \nabla e_h)_{T_h} = \ell_1(u, Q_h \psi) - \ell_3(p, Q_h \psi) - \ell_4(\psi, e_h).
\]

Using the equation (4.13) and (6.7), we have

\[\]

\[= \ell_1(u, Q_h \psi) - \ell_3(p, Q_h \psi) - \ell_4(\psi, e_h) + (\nabla_w (\psi - Q_h \psi), \nabla w e_h)_{T_h}
\]

Combining the three equations above imply that

\[\]

\[-(\Delta \psi, e_h) = \ell_1(u, Q_h \psi) - \ell_3(p, Q_h \psi) - \ell_4(\psi, e_h) + (\nabla_w (\psi - Q_h \psi), \nabla e_h)_{T_h}
\]

\[+ (\nabla \psi - Q_h \nabla \psi, \nabla e_h)_{T_h} - \ell_1(\psi, e_h).
\]
It follows from integration by parts and (1.2), (2.6) and (2.8) that

\[
(\nabla \xi, \epsilon_h) = (\nabla \xi, u) - (\nabla \xi, u_h) = -(\nabla \xi, u_b)
= (Q_h \xi, \nabla \cdot u_h)_{T_h} - (\xi, u_h \cdot n - (u_h) \cdot n)_{\partial T_h}
= - (\nabla Q_h \xi, u_h)_{T_h} + (Q_h \xi, u_h \cdot n - (u_h) \cdot n)_{\partial T_h}
= (Q_h \xi, \nabla w \cdot u_h)_{T_h} + (Q_h \xi, u_h \cdot n - (u_h) \cdot n)_{\partial T_h}
= - (\xi, u_h \cdot n - (u_h) \cdot n)_{\partial T_h}
= -\ell_3(\xi, u_h) = \ell_3(\xi, \epsilon_h).
\]

Combining (6.6)-(6.8), we have

\[
\|\epsilon_h\|^2 = \ell_1(u, Q_h \psi) - \ell_3(p, Q_h \psi) - \ell_4(\psi, \epsilon_h) + (\nabla \psi - Q_h \psi, \nabla \epsilon_h)
+ (\nabla \psi - Q_h \nabla \psi, \nabla \epsilon_h)_{T_h} - \ell_1(\psi, \epsilon_h) + \ell_4(\xi, \epsilon_h).
\]

(6.10)

Next we will estimate all the terms on the right hand side of (6.10). Using the Cauchy-Schwarz inequality, the trace inequality (2.11) and the definitions of \(Q_h\) and \(Q_b\) we obtain

\[
|\ell_1(u, Q_h \psi)| \leq |\{(\nabla u - Q_h \nabla u) \cdot n, Q_h \psi - \{Q_h \psi\}\}_{\partial T_h}|
\leq \left( \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} \|Q_h \psi - \{Q_h \psi\}\|_{\partial T}^2 \right)^{1/2}
\leq C \left( \sum_{T \in T_h} h_T^2 \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \|Q_h \psi - \{Q_h \psi\}\|_{\partial T}^2 \right)^{1/2}
\leq C h^{k+1} ||u||_{k+1} \|\psi\|_2.
\]

Similarly, we have

\[
|\ell_3(p, Q_h \psi)| \leq |(p - Q_h p, (Q_h \psi - \{Q_h \psi\}) \cdot n)_{\partial T_h}|
\leq \left( \sum_{T \in T_h} h_T^2 \|p - Q_h p\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \|Q_h \psi - \{Q_h \psi\}\|_{\partial T}^2 \right)^{1/2}
\leq C h^{k+1} \|p\|_k \|\psi\|_2.
\]

It follows from (5.6) and (5.7) that

\[
|(|\nabla w(\psi - Q_h \psi), \nabla \epsilon_h)_{T_h}| \leq C \|\epsilon_h\| \|\psi - Q_h \psi\|
\leq C h^{k+1} (||u||_{k+1} + ||p||_k) \|\psi\|_2.
\]
The estimates (3.3) and (5.7) imply

\[ |(\nabla \psi - Q_h \nabla \psi, \nabla \epsilon_h)_{\Omega_h}| \leq C \left( \sum_{T \in T_h} \| \nabla \epsilon_h \|_T^2 \right)^{1/2} \left( \sum_{T \in T_h} \| \nabla \psi - Q_h \nabla \psi \|_T^2 \right)^{1/2} \]

\[ \leq C \left( \sum_{T \in T_h} (\| (u - Q_h u) \|_T^2 + \| (Q_h u - u_h) \|_T^2) \right)^{1/2} \]

\[ \times ( \sum_{T \in T_h} \| \nabla \psi - Q_h \nabla \psi \|_T^2 )^{1/2} \]

\[ \leq Ch\|\psi\|_2 (h^k|u|_{k+1} + \|e_h\|) \]

\[ \leq Ch^{k+1}(|u|_{k+1} + |p|_k)\|\psi\|_2. \]

Using (3.3) and (5.7), we obtain

\[ |\ell_1(\psi, \epsilon_h)| = |(\langle \nabla \psi - Q_h \nabla \psi \rangle \cdot n, \epsilon_h - \{\epsilon_h\})_{\partial \Omega_h}| \]

\[ \leq \sum_{T \in T_h} h_T^{1/2} \| \nabla \psi - Q_h \nabla \psi \|_{\partial T} h_T^{-1/2} \| \epsilon_h \|_{\partial T} \]

\[ \leq Ch\|\psi\|_2 (\sum_{T \in T_h} h_T^{-1} (\| \epsilon_h \|_{\partial T}^2 + \|u - Q_h u\|_{\partial T}^2 ))^{1/2} \]

\[ \leq Ch\|\psi\|_2 (\| \epsilon_h \| + (\sum_{T \in T_h} h_T^{-1} \|u - Q_h u\|_{\partial T}^2 ))^{1/2} \]

\[ \leq Ch^{k+1}(|u|_{k+1} + |p|_k)\|\psi\|_2. \]

Similarly, we have

\[ |\ell_3(\xi, \epsilon_h)| = |(\xi - Q_h \xi, \epsilon_h - \{\epsilon_h\})_{\partial \Omega_h}| \]

\[ \leq \sum_{T \in T_h} h_T^{1/2} \| \xi - Q_h \xi \|_{\partial T} h_T^{-1/2} \| \epsilon_h \|_{\partial T} \]

\[ \leq Ch^{k+1}(|u|_{k+1} + |p|_k)\|\xi\|_1. \]

Using (5.5) and (5.8), we have

\[ \ell_4(\psi, \epsilon_h) \leq Ch^{k+1}(|u|_{k+1} + |p|_k)\|\psi\|_2. \]

Combining all the estimates above with (6.10) yields

\[ \| \epsilon_h \|^2 \leq Ch^{k+1}(|u|_{k+1} + |p|_k)\|\psi\|_2 + \|\xi\|_1. \]

The estimate (6.5) follows from the above inequality and the regularity assumption (6.4). We have completed the proof. □

7. Numerical Experiments.

7.1. Example 1. Consider problem (1.1)–(1.3) with $\Omega = (0,1)^2$. The source term and the boundary value $g$ are chosen so that the exact solution is

\[ u(x,y) = \left( \begin{array}{c} \sin \pi y \\ \cos \pi x \end{array} \right), \quad p = \sin 2\pi y. \]

In this example, we use uniform triangular grids shown in Figure 7.1. In Table 7.1 we list the errors and the orders of convergence. We can see that the optimal order of convergence is achieved in all finite element methods.
Table 7.1

| Grid | $\|u - u_h\|_0$ rate | $\|u - u_h\|$ rate | $\|p - p_h\|_0$ rate |
|------|----------------|----------------|----------------|
|      | by the $P_2^1-P_0$ finite element | by the $P_2^2-P_1$ finite element | by the $P_2^3-P_2$ finite element |
| 5    | 0.9678E-03 1.90 | 0.5352E-01 1.09 | 0.3859E-01 1.76 |
| 6    | 0.2530E-03 1.94 | 0.2565E-01 1.06 | 0.1257E-01 1.62 |
| 7    | 0.6486E-04 1.96 | 0.1250E-01 1.04 | 0.4897E-02 1.36 |
| 4    | 0.1598E-03 2.94 | 0.1222E-01 1.95 | 0.6123E-02 2.29 |
| 5    | 0.2026E-04 2.98 | 0.3091E-02 1.98 | 0.1411E-02 2.12 |
| 6    | 0.2544E-05 2.99 | 0.7747E-03 2.00 | 0.3384E-03 2.06 |
| 4    | 0.5550E-05 4.01 | 0.5624E-03 3.09 | 0.1731E-02 2.75 |
| 5    | 0.3409E-06 4.03 | 0.6677E-04 3.07 | 0.2267E-03 2.93 |
| 6    | 0.2109E-07 4.01 | 0.8113E-05 3.04 | 0.2881E-04 2.98 |
| 3    | 0.6286E-05 5.24 | 0.4739E-03 4.18 | 0.4781E-03 4.90 |
| 4    | 0.2057E-06 4.93 | 0.3110E-04 3.93 | 0.1493E-04 5.00 |
| 5    | 0.6518E-08 4.98 | 0.1978E-05 3.98 | 0.6207E-06 4.59 |

7.2. Example 2. Consider problem (1.1)-(1.3) with $\Omega = (0,1)^2$. The source term is chosen so that the exact solution is

$$u(x,y) = \frac{-256(x-x^2)^2(y-y^2)(2-4y)}{256(x-x^2)(2-4x)(y-y^2)^3}, \quad p = x + y - 1.$$  

In this example, we use polygonal grids, consisting of dodecagons (12 sided polygons) and heptagons (7 sided polygons), shown in Figure 7.2. In Table 7.2 we list the errors and the orders of convergence. We can see that the optimal order of convergence is achieved in all finite element methods.

7.3. Example 3. Consider problem (1.1)-(1.3) with $\Omega = (0,1)^3$. The source term and the boundary value $g$ are chosen so that the exact solution is

$$u(x,y) = \begin{pmatrix} y^4 \\ x^2 \\ x^2 \end{pmatrix}, \quad p = x - \frac{1}{2}.$$
Fig. 7.2. The first three polygonal grids (consisting of dodecagons and heptagons) for the computation of Table 7.2 (Example 2).

| Grid | $\|u - u_h\|_0$ rate | $\|u - u_h\|$ rate | $\|p - p_h\|_0$ rate |
|------|-------------------|-------------------|-------------------|
|      | by the $P_2^2$-$P_0$ finite element | by the $P_2^2$-$P_1$ finite element | by the $P_2^3$-$P_2$ finite element |
| 4    | 0.3844E-01 1.68   | 0.1046E-01 1.88   | 0.2708E-02 1.95   | 0.4929E-02 3.44   |
| 5    | 0.7406E+00 0.93   | 0.3741E+00 0.99   | 0.1874E+00 1.00   | 0.2366E+00 2.20   |
| 6    | 0.2329E+00 0.90   | 0.9698E-01 1.26   | 0.3982E-01 1.28   | 0.1646E-01 1.95   |
|      | $0.3866E+00$ $2.20$ | $0.2366E+00$ $2.20$ | $0.1646E+00$ $2.20$ |

We use tetrahedral meshes shown in Figure 7.3. The results of the 3D $P_k$-$P_{k+1}$ weak Galerkin finite element methods are listed in Table 7.3. The method is stable and is of optimal order convergence.

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Table 7.3

| Grid | $\|u - u_h\|$ | rate | $\|u - u_h\|_0$ | rate | $\|p - p_h\|_0$ | rate |
|------|--------------|------|-----------------|------|-----------------|------|
| 1    | 0.1845E+00  | 0.00 | 0.1289E-01      | 0.00 | 0.2125E+00      | 0.00 |
| 2    | 0.5331E-01  | 1.79 | 0.2383E-02      | 2.44 | 0.3299E-01      | 2.69 |
| 3    | 0.1422E-01  | 1.91 | 0.3475E-03      | 2.78 | 0.6230E-02      | 2.40 |
|      | by the 3D $P_2^3$-$P_1$ finite element |      |                 |      |                 |      |
| 1    | 0.3237E-01  | 0.00 | 0.1677E-02      | 0.00 | 0.2406E-01      | 0.00 |
| 2    | 0.4013E-02  | 3.01 | 0.1390E-03      | 3.59 | 0.2270E-02      | 3.41 |
| 3    | 0.5015E-03  | 3.00 | 0.9991E-05      | 3.80 | 0.2350E-03      | 3.27 |
|      | by the 3D $P_2^3$-$P_2$ finite element |      |                 |      |                 |      |

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