1. Introduction

Self-similar solutions arise naturally in the study of mean curvature flow (MCF). Self-shrinkers model the behavior of the singularities of MCF [5]. On the other hand, self-expanders are expected to model the behavior of the MCF that emerges from conical singularities [9]. As a result, the analyses of self-similar solutions attract much attention in the past thirty years.

The starting point of this paper is the work of Lawson and Osserman on the minimal graph system in higher codimensions. While the Simons cone is a famous example of codimension one [8], Lawson and Osserman [6] discovered a 4-dimensional minimal cone in $\mathbb{R}^7$ based on the Hopf fibration:

$$\{(rx, \sqrt{5}r\mathcal{H}(x)) | r \in \mathbb{R}_{\geq 0}, x \in S^3\},$$

where $\mathcal{H} : S^3 \to S^2$ is the Hopf fibration. When equipping $\mathbb{R}^7$ the $G_2$-structure, Harvey and Lawson [3] proved that it is in fact coassociative, and is therefore area-minimizing.

Ding and Yuan [1] resolved the singularity of the Lawson-Osserman cone. They constructed a family of minimal graphs which are asymptotic to the Lawson-Osserman cone, but are smooth at the origin. Recently, Xu, Yang and Zhang [11] generalized the Lawson-Osserman cone and the Ding-Yuan’s minimal graphs by composing suitable maps from $\mathbb{C}P^n$ to $S^m$. Specially, they constructed some maps $\mathcal{L} : S^n \to S^m$ with certain similar properties as the Hopf fibration.

In this paper, we construct self-expanders to the MCF based on the generalized Lawson-Osserman cone. What follows is our main theorem. The technical terms will be explained in section 2.

**Main Theorem.** Suppose that $\mathcal{L} : S^n \to S^m$ is an LOMSE of $(n,p,k)$-type, where $(n,p,k) = (3,2,2), (5,4,2), (5,4,4)$ or $n \geq 7$. Then, there exist positive constants $\varepsilon_0$, $R_0$ and $\varphi_0$ with the following significance: for any $\varepsilon \in (0,\varepsilon_0]$ and $R \in (0,R_0]$, there exist a unique smooth self-expander in $\mathbb{R}^{n+m+2}$ of the form

$$\Sigma = \{(rx, f(r)\mathcal{L}(x)) | r \in \mathbb{R}_{\geq 0}, x \in S^n\}$$
such that \( f : \mathbb{R}_{\geq 0} \to \mathbb{R} \) satisfies the Dirichlet type condition
\[
f(R) = \varepsilon R.
\]
Moreover, \( f \) has the following properties:

(1) \( 0 \leq f < \varphi_0 r, \ 0 \leq f_r \).

(2) \( f \in O(r^k), \ f_r \in O(r^{k-1}) \) as \( r \to 0 \).

(3) \( \lim_{r \to \infty} \frac{f(r)}{r} \) exists. In other words, \( \Sigma \) is asymptotic to the cone
\[
\{(rx, r\varphi_{\infty}L(x)) \mid r \in \mathbb{R}_{\geq 0}, x \in \mathbb{S}^n \},
\]
where \( \varphi_{\infty} = \lim_{r \to \infty} \frac{f(r)}{r} \), as \( r \to \infty \).

The organization of this paper is as follows. In section 3, we use the symmetry to transform the equation of self-similar solutions to a second order ODE. Section 4 contains a stable curve theorem, which is a generalization of the theory of equilibria in the autonomous system. After an appropriate change of variables and using the stable curve theorem, the existence of self-expanders is obtained in section 5. The uniqueness as a Dirichlet problem is established in section 6. Finally, we analyze the asymptotic behavior in section 7.

Acknowledgement. The author is really grateful to Prof. Chung-Jun Tsai for his helpful and inspiring comments. Part of this paper is from the author’s master thesis. He also appreciates Prof. Mao-Pei Tsui’s suggestions on further generalizations and Chin-Bin Hsu’s indications on some deficiencies in an earlier draft.

2. Preliminary

2.1. **Self-similar solutions to mean curvature flow.** We recall some background material of mean curvature flow (MCF).

**Definition.** Let \( \Sigma \) be a smooth submanifold in a Riemannian manifold \( M \). If there exists a family of smooth immersions \( F_t : \Sigma \to M \) satisfying
\[
\begin{cases}
(\frac{\partial F_t}{\partial t}(x))^\perp = H_{\Sigma_t}(x) \\
F_0 = \text{id}
\end{cases}
\]
where \( (\frac{\partial F_t}{\partial t})^\perp \) denotes the projection of the \( \frac{\partial F_t}{\partial t} \) to the normal bundle of \( \Sigma \),
\[
H_{\Sigma_t} := (g_t)^{ij} \nabla_{\frac{\partial F_t}{\partial x^i}} \frac{\partial F_t}{\partial x^j}
\]
denotes the mean curvature vector of \( \Sigma_t \), and
\[
(g_t)_{ij} := \left( \frac{\partial F_t}{\partial x^i}, \frac{\partial F_t}{\partial x^j} \right)_M,
\]
then \( F_t \) is called the mean curvature flow (MCF) of \( \Sigma \).
In geometric flows, singularities are often modelled on soliton solutions. For MCF, there are two types of soliton solutions in Euclidean space that are particularly interested. One is the solutions moving by scaling, and the other one is those moving by translation. In this paper, we focus on the first one.

**Definition.** A submanifold in Euclidean space, $F : \Sigma \to \mathbb{R}^n$, is called a self-similar solution if

$$H_\Sigma \equiv CF^\perp$$

for some constant $C \in \mathbb{R}$, where $H_\Sigma$ is the mean curvature vector of $F : \Sigma \to \mathbb{R}^n$ and $F^\perp$ denotes the projection of the position vector $F$ in $\mathbb{R}^n$ to the normal bundle of $\Sigma$. Moreover, it is called a self-shrinker if $C < 0$ and a self-expander if $C > 0$. When $C = 0$, the submanifold is minimal.

Notice that if $\Sigma$ is a self-similar solution, then $F_t$ defined by

$$F_t = \sqrt{1 + 2Ct}F$$

moves by the MCF.

**Remark 2.1.** After rescaling, only the sign of $C$ matters. It suffices to consider $C = 1, 0, -1$.

2.2. **The generalized Lawson-Osserman cone.** We first recall the definition of the Hopf maps

$$\mathcal{H}_d : S^{2d-1} \to S^d,$$

where $d = 2, 4, 8$.

**Definition.** We identify $\mathbb{R}^d$ with the normed algebra: $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ for $d = 2, 4, 8$, respectively. The Hopf map is defined by

$$\mathcal{H}_d : S^{2d-1} \to S^d, \quad (p, q) \mapsto (\|p\|^2 - \|q\|^2, 2pq^*) .$$

By using the Hopf maps, the Lawson-Osserman cones are as follows.

**Definition.** The Lawson-Osserman cones are

$$C_d = \{(r\mathbf{x}, \kappa_d r\mathcal{H}_d(\mathbf{x})) \mid r \in \mathbb{R}_{\geq 0}, \mathbf{x} \in S^{2d-1}\},$$

where $d = 2, 4, 8$, $\kappa_d = \sqrt{\frac{2d+1}{4(d-1)}}$ and $\mathcal{H}_d : S^{2d-1} \to S^d$ is the Hopf map. The constant $\kappa_d$ is the unique one such that the cone is minimal.

We now explain the notion of generalized Lawson-Osserman cones introduced by Xu, Yang and Zhang [11].
**Definition.** For a smooth map $\mathcal{L} : S^n \to S^m$, if there exists an acute angle $\theta$ such that

$$M_{\mathcal{L}, \theta} := \{(x \cos \theta, \mathcal{L}(x) \sin \theta) \mid x \in S^n\}$$

is a minimal submanifold of $S^{n+m+1}$, then $\mathcal{L}$ is called a Lawson-Osserman map (LOM), $M_{\mathcal{L}, \theta}$ is called the associated Lawson-Osserman sphere (LOS), and the cone $C_{\mathcal{L}, \theta}$ over $M_{\mathcal{L}, \theta}$ is called the corresponding Lawson-Osserman cone (LOC).

Suppose that $\mathcal{L} : S^n \to S^m$ is an LOM whose image is not totally geodesic. Then $\mathcal{L}$ is called an LOMSE if all non-zero singular values $\lambda(x)$ of $\mathcal{L}^* x$ are equal for every $x \in S^n$. Note that $\lambda(x)$ is a continuous function of $x$. For an LOMSE, one can show that $\lambda(x)$ equals a constant $\lambda$ and $\mathcal{L}$ has constant rank $p$. Furthermore, all components of this vector-valued function $\mathcal{L} = (L_1, \cdots, L_{m+1}) \in S^m \subset \mathbb{R}^{m+1}$ are spherical harmonic functions of degree $k \geq 2$ and the singular value $\lambda = \sqrt{\frac{k(n+k-1)}{p}}$ (cf. [11, Theorem 2.8]). Such an $\mathcal{L}$ is called an LOMSE of $(n, p, k)$-type.

It can easily be seen that the Hopf map $H_d$ is exactly the LOMSE of $(2d-1, d, 2)$-type. Here is the classification and all are based on the Hopf maps.

**Proposition 2.2.** [11, Theorem 2.10] There are three families of LOMSEs of $(n, p, k)$-type that generalize the Hopf maps:

1. $(n, p) = (2l + 1, 2l)$ and $k = 2q$, $l, q \in \mathbb{N}$, which generalize the Hopf map $H_2$.
2. $(n, p) = (4l + 3, 4l)$ and $k = 2q$, $l, q \in \mathbb{N}$, which generalize the Hopf map $H_4$.
3. $(n, p) = (15, 8)$ and $k = 2q$, $q \in \mathbb{N}$, which generalize the Hopf map $H_8$.

On the other hand, there are minimal graphs that resolved the singularities of LOCs.

**Proposition 2.3.** [11, Theorem 3.5] Suppose that $\mathcal{L}$ is an LOMSE of $(n, p, k)$-type. Then there is an analytic entire minimal graph of the form

$$M_{\mathcal{L}, r} = \{(r x, \rho(r) \mathcal{L}(x)) \mid r \in \mathbb{R}_{\geq 0}, x \in S^n\}$$

that is asymptotic to $C_{\mathcal{L}, \theta}$. In particular, when $(n, p, k) = (3, 2, 2), (5, 4, 2), (5, 4, 4)$ or $n \geq 7$, $M_{\mathcal{L}, r}$ is below $C_{\mathcal{L}, \theta}$ in the sense that

$$\frac{\rho}{r} \leq \varphi_0 := \tan \theta = \sqrt{\frac{p \lambda^2 - n}{(n-p) \lambda^2}}$$

for all $r$.

---

1Here, a singular value $\lambda(x)$ of $(\mathcal{L}_*)_x$ means the square root of an eigenvalue of the self-adjoint operator $((\mathcal{L}_*)_x)^*(\mathcal{L}_*)_x$. 

2.3. Harmonic maps. Let \((M^m, g_M)\) and \((N^n, g_N)\) be Riemannian manifolds. Let \(f : M \to N\) be a smooth map. Then the energy density of \(f\) at \(x \in M\) is defined to be

\[
e(f) := \sum_{i=1}^{m} g_N(f_*e_i, f_*e_i),
\]

where \(\{e_i\}_{i=1}^{m}\) is an orthonormal basis of \(T_xM\). The energy of \(f\) is defined to be

\[
E(f) := \int_{M} e(f) \, d\text{vol}_{g_M}.
\]

Let \(\nabla^TM\) be the Levi-Civita connection on \(TM\) with respect to \(g_M\). Let \(\nabla f^*TN\) be the connection on \(f^*TN\) that is compatible with \(g_N\). Then the second fundamental form of \(f\) is defined to be

\[
B_{XY}(f) := \nabla_X^{f^*TN}Y - f_*(\nabla_X^TMY).
\]

If \(B(f) \equiv 0\), then \(f\) is called a totally geodesic map. The tension field of \(f\) is defined to be the trace of its second fundamental form under \(g_M\)

\[
\tau(f) := \sum_{i=1}^{m} B_{e_i e_i}(f).
\]

If \(\tau(f) \equiv 0\), then \(f\) is called a harmonic map. Using the first variation, it can be showed that \(f\) is harmonic if and only if \(f\) is a critical point of energy functional \(E(\cdot)\) (cf. [10, Chap. 1.2.3, p. 13–14]).

Given a smooth function \(f : (M, g_M) \to (\mathbb{R}^n, g_{\text{std}})\), its tension field has a simple expression, which is

\[
\tau(f) = \Delta_{g_M} f,
\]

where \(\Delta_{g_M}\) denotes the Laplace-Beltrami operator. Namely, the harmonicity here matches the usual one.

Given an isometric immersion \(\iota : (M, g_M) \hookrightarrow (N, g_N)\), the second fundamental form of \(\iota\) is just the second fundamental of \(M\) as an immersed submanifold of \(N\) (cf. [10, Chap. 1.2.4, p. 15]). Moreover, it implies that the tension field of \(f\) is exactly the mean curvature vector of \(M\) in \(N\). In other words, That \(M\) is minimal in \(N\) is equivalent to \(\iota\) is harmonic in this case.

There is a composition formula for tension fields. Let \(M, N, \tilde{N}\) be Riemannian manifolds. Suppose that \(f : M \to N\) and \(\tilde{f} : N \to \tilde{N}\) are smooth map. Then

\[
\tau(\tilde{f} \circ f) = \tilde{f}_*(\tau(f)) + \sum_{i=1}^{m} B_{f_*e_i f_*e_i}(\tilde{f})
\]

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Suppose that \( \bar{f} \) is an isometric immersion. Combing the discussion in the previous paragraph, we simply write the formula as

\[
\tau(\bar{f} \circ f) = \tau(f) + \sum_{i=1}^{m} h(f^* e_i, f^* e_i),
\]

where \( h \) is the second fundamental form of \( N \) in \( \bar{N} \).

3. Necessary and sufficient conditions for graphical self-similar solutions

Given a smooth function \( f : \mathbb{R}_{\geq 0} \to \mathbb{R} \) and a smooth map \( L : S^n \to S^m \), we consider

\[
F : \mathbb{R}_{\geq 0} \times S^n \to \mathbb{R}^{n+1} \times \mathbb{R}^{m+1}
\]

\[
(r, x) \mapsto (rx, f(r)L(x))
\]

and study when \( \Sigma = F(\mathbb{R}_{\geq 0} \times S^n) \) is a self-similar solution to the MCF in \( \mathbb{R}^{n+m+2} \).

Let \( g \) be the metric on \( \Sigma \) induced by the standard Euclidean metric on \( \mathbb{R}^{n+m+2} \) and \( g_r = I_r^* g \), where

\[
I_r : S^n \to \Sigma \quad x \mapsto (rx, f(r)L(x)).
\]

Let \( \nabla \) be the Levi-Civita connection on \( (\Sigma, g) \). Note that \( \nabla_v r = \nabla_v f = 0 \) for all \( v \in T S^n \).

**Theorem 3.1.** Assume that \( f > 0 \). Then \( \Sigma \) is a self-similar solution to the MCF, i.e. \( H_{\Sigma} = CF^\perp \) where \( C = \pm 1 \), if and only if the following two conditions hold:

1. For each \( r \in \mathbb{R}_{\geq 0} \), \( L : (S^n, g_r) \to (S^m, g_m) \) is harmonic, where \( g_m \) is the standard metric on \( S^m \).
2. For each \( r \in \mathbb{R}_{\geq 0} \),

\[
\Delta_g f - 2e(L)f + \frac{C(rf_r - f)}{1 + f_r^2} = 0
\]

in \( I_r(S^n) \), where \( e(L) \) is the energy density of \( L : (S^n, g_r) \to (S^m, g_m) \).

Moreover, suppose that \( \lambda_1, \cdots, \lambda_n \) are singular values of \( (L_*)_x : (T_x S^n, g_r) \to (T_{L(x)} S^m, g_m) \).

Then condition (2) is equivalent to the equation

\[
\frac{f_{rr}}{1 + f_r^2} + \sum_{i=1}^{n} \frac{rf_r - \lambda_i^2 f}{r^2 + \lambda_i^2 f^2} + C(rf_r - f) = 0.
\]

**Proof.** Let \( \iota_n : (S^n, g_n) \to (\mathbb{R}^{n+1}, g_{std}) \) be the natural isometric immersion. We first write down \( F \) carefully. In fact,

\[
F(r, x) = (rX_1(x), f(r)X_2(x)) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1},
\]

where

\[
X_1 = \iota_n \circ \text{Id}_{S^n} : (S^n, g_r) \to (\mathbb{R}^{n+1}, g_{std})
\]
and

\[ X_2 = \iota_m \circ \mathcal{L} : (\mathbb{S}^n, g_r) \to (\mathbb{R}^{m+1}, g_{\text{std}}). \]

From now on, we do calculations at a fixed point \((r, x) \in \mathbb{R}_{\geq 0} \times \mathbb{S}^n\). Let \(\{v_i\}_{i=1}^n\) be an SVD-basis of \((T_x \mathbb{S}^n, g_r)\) with respect to \((\mathcal{L}_x)_x\) such that each \(v_i\) subjects to the singular value \(\lambda_i\). Then

\[
\frac{\partial}{\partial r} := (X_1, f_r X_2) \\
V_i := (F_*)_{(r, x)}(v_i) = (rv_i, f \cdot (\mathcal{L}_x)_x v_i) \quad \forall \ 1 \leq i \leq n
\]

form an induced orthogonal basis of \((T_{(r, x)} \Sigma, g)\). Note that

\[
g_{00} := \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle = 1 + f_r^2 \\
g_{0i} := \langle \frac{\partial}{\partial r}, V_i \rangle = 0 \\
g_{ij} := \langle V_i, V_j \rangle = (r^2 + \lambda_i^2 f^2) \delta_{ij} \quad \forall \ 1 \leq i, j \leq n
\]

and

\[
\nabla_{\frac{\partial}{\partial r}} V_i - \nabla_{V_i} \frac{\partial}{\partial r} = \frac{\partial}{\partial r}, V_i = 0.
\]

We now derive the expression of \(CF_{\perp}\).

\[
F_{\perp} = F - F^T = F - g_{00} \langle F, \frac{\partial}{\partial r} \rangle \frac{\partial}{\partial r} - \sum_{i=1}^n g^{ii} \langle F, V_i \rangle V_i \\
= (rX_1, f X_2) - \frac{r + f_r f}{1 + f_r^2} (X_1, f_r X_2) \\
= \frac{-rf_r + f}{1 + f_r^2} (-f_r X_1, X_2)
\]

For mean curvature vector \(H_{\Sigma}\), note that

\[
H_{\Sigma} = \Delta g F = (\Delta_g (r X_1), \Delta_g (f X_2)).
\]

We focus on the second component. Since \(f\) is independent of \(x\) and \(X_2\) is independent of \(r\),

\[
\Delta_g (f X_2) = (\Delta_g f) X_2 + f(\Delta_g X_2) + 2(g^{00} f_r \nabla_{\frac{\partial}{\partial r}} X_2 + \sum_{i=1}^n g^{ii} V_i (f) \nabla_{V_i} X_2) \\
= (\Delta_g f) X_2 + f(\Delta_g X_2)
\]
Moreover, in the Riemannian submanifold \((S^n, g_r)\) of \((\Sigma, g)\), we have
\[
\Delta_g X_2 = \Delta_{g_r} X_2 + g^{00} (\nabla_{\frac{\partial}{\partial r}} \nabla_{\frac{\partial}{\partial r}} X_2 - \nabla_{\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}} X_2)
\]
\[
= \Delta_{g_r} X_2 - g^{00} \nabla_{\frac{\partial}{\partial r}} X_2
\]
\[
= \Delta_{g_r} X_2 - g^{00} \sum_{i=1}^{n} g^{ii} (\nabla_{\frac{\partial}{\partial r}} V_i) \nabla_{V_i} X_2
\]
\[
= \Delta_{g_r} X_2 + g^{00} \sum_{i=1}^{n} g^{ii} (\frac{\partial}{\partial r}, \nabla_{V_i} V_i) \nabla_{V_i} X_2
\]
\[
= \Delta_{g_r} X_2 + g^{00} \sum_{i=1}^{n} g^{ii} (\frac{\partial}{\partial r}, \nabla_{V_i} V_i) \nabla_{V_i} X_2.
\]

We also recall that \(L : (S^n, g_r) \to (S^m, g_m)\) and \(X_2 = \iota_m \circ L\), then
\[
\Delta_{g_r} X_2 = \tau(X_2) = \tau(\iota_m \circ L) = \tau(L) = 2e(L)X_2.
\]

Therefore,
\[
\Delta_g (fX_2) = f \cdot \tau(L) + (\Delta_g f - 2e(L)f)X_2.
\]

Notice that \(\tau(L) \in \Gamma(L^*TS^n)\). In other words, \(\langle \tau(L), X_2 \rangle = 0\). Hence, \(H_{\Sigma} = CF^\perp\) implies that
\[
\tau(L) = 0
\]

and
\[
\Delta_g f - 2e(L)f + \frac{C(r f_r - f)}{1 + f_r^2} = 0.
\]

Conversely, if \(\tau(L) = 0\) and \(\Delta_g f - 2e(L)f + \frac{C(r f_r - f)}{1 + f_r^2} = 0\), then
\[
H_{\Sigma} = (\Delta_g (rX_1), \frac{C(-r f_r + f)}{1 + f_r^2} X_2).
\]

Recall that \(H_{\Sigma} \in N\Sigma\), which implies that
\[
0 = \langle \frac{\partial}{\partial r}, H_{\Sigma} \rangle
\]
\[
= \langle (X_1, f_r X_2), (\Delta_g (rX_1), \frac{C(-r f_r + f)}{1 + f_r^2} X_2) \rangle
\]
\[
= \langle X_1, \Delta_g (rX_1) \rangle + f_r \cdot \frac{C(-r f_r + f)}{1 + f_r^2}
\]
and
\[ 0 = \langle V_i, H_{\Sigma} \rangle = \langle rv_i, f \cdot (\mathcal{L}_s)_{x} v_i \rangle, (\Delta_g(rX_1), \frac{C(-rf_r + f)}{1 + f_r^2}X_2) = r\langle v_i, \Delta_g(rX_1) \rangle. \]

Therefore, \( \Delta_g(rX_1) = -f_r \cdot \frac{C(-rf_r + f)}{1 + f_r^2}X_1 \) and
\[ H_{\Sigma} = \frac{C(-rf_r + f)}{1 + f_r^2}(-f_rX_1, X_2) = CF^\perp, \]
\( \Sigma \) is a self-similar solution to the MCF.

Finally, we derive an explicit expression of \( \Delta_g f - 2e(\mathcal{L}) f \). \[ \Delta_g f = f_r \Delta_g r + f_{rr} |\text{grad}_g f|^2 \]
\[ = f_r (g^{00}(\nabla_{\varphi r} \nabla_{\varphi r}^2 r - \nabla_{\varphi r}^2 \nabla_{\varphi v}^2 r) + \sum_{i=1}^n g^{ii}(\nabla_{V_i} \nabla_{V_i}^2 r - \nabla_{V_i}^2 V_i r)) + f_{rr} g^{00} \]
\[ = f_r (-g^{00} \nabla_{\varphi r}^2 \frac{\partial}{\partial r} - \sum_{i=1}^n g^{ii} \nabla_{V_i}^2 V_i r) + \frac{f_{rr}}{1 + f_r^2} \]
\[ = f_r (-g^{00})^2 \nabla_{\varphi r}^2 \frac{\partial}{\partial r} - \sum_{i=1}^n g^{ii} g^{00} (\nabla_{V_i}^2 V_i, \frac{\partial}{\partial r}) + \frac{f_{rr}}{1 + f_r^2} \]
\[ = f_r (-\frac{\nabla_{\varphi r}^2 \frac{\partial}{\partial r}}{2(1 + f_r^2)^2} + \sum_{i=1}^n \nabla_{V_i}^2 \langle V_i, V_i \rangle) + \frac{f_{rr}}{1 + f_r^2} \]
\[ = \frac{f_{rr}}{(1 + f_r^2)^2} + \sum_{i=1}^n \frac{r r f_r + \lambda_2 f_r^2 f}{(1 + f_r^2)(r^2 + \lambda_2^2 f_r^2)} \]
and
\[ 2e(\mathcal{L}) = \sum_{i=1}^n \frac{g_r (\mathcal{L}_s(v_i), \mathcal{L}_s(v_i))}{g_r(v_i, v_i)} = \sum_{i=1}^n \frac{\langle \mathcal{L}_s(v_i), \mathcal{L}_s(v_i) \rangle}{\langle V_i, V_i \rangle} = \sum_{i=1}^n \frac{\lambda_i^2}{r^2 + \lambda_i^2 f_r^2}. \]
It follows that \( \Delta_g f - 2e(\mathcal{L}) f + \frac{C(r f_r - f)}{1 + f_r^2} = 0 \) is equivalent to
\[ \frac{f_{rr}}{1 + f_r^2} + \sum_{i=1}^n \frac{r r f_r - \lambda_2^2 f_r^2}{r^2 + \lambda_2^2 f_r^2} + C(r f_r - f) = 0. \]

\[ \square \]

Now, we consider \( \mathcal{L} \) to be an LOMSE of \((n, p, k)\)-type and obtain a simple version of Theorem 3.1. It has been proved that LOMSEs automatically satisfy condition (1) (cf. [11, Sec. 3.2]). Moreover, for an LOMSE of \((n, p, k)\)-type, it has only two constant singular values
\[ \lambda = \sqrt{\frac{k(n + k - 1)}{p}} \]
and \( 0 \) of constant multiplicities \( p \) and \( n - p \) respectively (cf. [11, Theorem 2.8]). Therefore, we have the following simple version.
Corollary 3.2. Assume that \( f > 0 \) and \( \mathcal{L} \) is an LOMSE of \((n,p,k)\)-type. Then \( \Sigma \) is a self-similar solution to the MCF if and only if \( f \) satisfies
\[
\frac{f_{rr}}{1 + f_r^2} + \frac{(n - p) f_r}{r} + \frac{p (r f_r - \lambda^2 f)}{r^2 + \lambda^2 f^2} + C (r f_r - f) = 0,
\]
where \( \lambda = \sqrt{\frac{k(n+k-1)}{p}} \).

Remark 3.3. Suppose that \( C = 0 \). Then
\[
\frac{f_{rr}}{1 + f_r^2} + \frac{(n - p) f_r}{r} + \frac{p (r f_r - \lambda^2 f)}{r^2 + \lambda^2 f^2} = 0
\]
is a necessary and sufficient condition for \( \Sigma \) to be a minimal graph (cf. [11, Theorem 3.2]). Moreover, for \((n,p,k) = (3,2,2), (7,4,2), (15,8,2)\), the equation was first found by Ding and Yuan (cf. [1, Sec. 2]).

4. An analogous autonomous system

The following section is essentially based on the author’s master thesis. For the sake of completeness, we provide the details here.

Throughout this section, we consider the following system of ODEs:
\[
\begin{aligned}
X(t) &= -\kappa X(t) + f_1(X(t), Y(t)) + e^{-t} g_1(X(t), Y(t)) \\
Y(t) &= \mu Y(t) + f_2(X(t), Y(t)) + e^{-t} g_2(X(t), Y(t))
\end{aligned}
\]
where \( \mu > 0 > -\kappa, \frac{g_2(X,Y)}{\sqrt{X^2+Y^2}} \rightarrow 0 \) as \((X,Y) \rightarrow (0,0)\) and \( g_i \in O(\sqrt{X^2+Y^2}) \) as \((X,Y) \rightarrow (0,0)\) \( \forall \ i = 1,2 \). In this case, \((0,0)\) is an equilibrium.

If we omit the exponential term, then it becomes a classical planar autonomous system. Under that situation, \((0,0)\) is in fact a saddle equilibrium. There is a stable curve theorem for such case (cf. [4, Chap. 8.3, p.169]), which states that we can find an \( \varepsilon > 0 \) and a unique local stable curve of the form \( Y = h(X) \) that is defined for \(|X| < \varepsilon\) and satisfies \( h(0) = 0 \). Moreover, this curve is tangent to the \( X \)-axis and all solutions with initial conditions that lies on this curve tend to the origin as \( t \rightarrow \infty \).

In this section, the goal is to provide a similar stable curve theorem. We first give some notations to clarify the meaning of local. Let \( S_\varepsilon \) be the square bounded by \( \{ X = \pm \varepsilon \} \) and \( \{ Y = \pm \varepsilon \} \). We also define
\[
R_{M,\varepsilon}^+ := \{(X,Y) \in S_\varepsilon \mid |Y| \leq M|X|, X \geq 0\}
\]
and
\[
E_{M,\varepsilon}^+ := R_{M,\varepsilon}^+ \cap \{ X = \varepsilon \}.
\]
Then \( E_{M,\varepsilon}^+ \) is a part of the boundary of \( R_{M,\varepsilon}^+ \).

Now, we state two lemmas about the behavior of the vector field inside \( R_{M,\varepsilon}^+ \).
Lemma 4.1. Given any $M > 0$ and $\delta \in (0, \kappa)$, there exists $\varepsilon > 0$ and $T > 0$ such that $X_t < 0$ in $R_{M,\varepsilon}^+$ whenever $t \geq T$.

Proof. Since $\frac{f_1(X,Y)}{\sqrt{X^2 + Y^2}} \to 0$ as $(X,Y) \to (0,0)$, we may choose $\varepsilon_1 > 0$ so that

$$|f_1(X,Y)| \leq \frac{\delta}{2\sqrt{M^2 + 1}} \sqrt{X^2 + Y^2} \quad \forall \ (X,Y) \in S_{\varepsilon_1}.$$ 

Moreover, since $g_1 \in O(\sqrt{X^2 + Y^2})$, there exist $\varepsilon_2 > 0$ such that

$$|g_1(X,Y)| < \tilde{C} \sqrt{X^2 + Y^2} \quad \forall \ (X,Y) \in S_{\varepsilon_2}$$ 

for some positive constant $\tilde{C}$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Now, we set $T > 0$ so that

$$e^{-t} < \frac{\delta}{2\tilde{C}\sqrt{M^2 + 1}} \quad \forall \ t > T.$$ 

Note that in $R_{M,\varepsilon}^+$, $|Y| \leq MX \implies \sqrt{X^2 + Y^2} \leq \sqrt{M^2 + 1}X$. Therefore,

$$X_t = -\kappa X + f_1(X,Y) + e^{-t}g_1(X,Y)$$

$$\leq -\kappa X + |f_1(X,Y)| + e^{-t}|g_1(X,Y)|$$

$$\leq -\kappa X + \frac{\delta}{2\sqrt{M^2 + 1}} \sqrt{X^2 + Y^2} + \frac{\delta}{2\tilde{C}\sqrt{M^2 + 1}} (\tilde{C} \sqrt{X^2 + Y^2})$$

$$\leq -\kappa X + \frac{\delta}{2\sqrt{M^2 + 1}} \sqrt{M^2 + 1}X + \frac{\delta}{2\tilde{C}\sqrt{M^2 + 1}} (\tilde{C} \sqrt{M^2 + 1}X)$$

$$= - (\kappa - \delta) X < 0$$

whenever $t \geq T$. \hfill \square

Lemma 4.2. Given any $M > 0$, there exists $\varepsilon > 0$ and $T > 0$ such that $Y_t > 0$ on $\{(X,Y) \in R_{M,\varepsilon}^+, |Y| = MX, X > 0\}$ and $Y_t < 0$ on $\{(X,Y) \in R_{M,\varepsilon}^+, |Y| = -MX, X > 0\}$ whenever $t \geq T$.

Proof. Since $\frac{f_2(X,Y)}{\sqrt{X^2 + Y^2}} \to 0$ as $(X,Y) \to (0,0)$, we may choose $\varepsilon_1 > 0$ so that

$$|f_2(X,Y)| \leq \frac{M\mu}{3\sqrt{M^2 + 1}} \sqrt{X^2 + Y^2} \quad \forall \ (X,Y) \in S_{\varepsilon_1}.$$ 

Furthermore, since $g_2 \in O(\sqrt{X^2 + Y^2})$, there exist $\varepsilon_2 > 0$ such that

$$|g_2(X,Y)| < \tilde{C} \sqrt{X^2 + Y^2} \quad \forall \ (X,Y) \in S_{\varepsilon_2}$$ 

for some positive constant $\tilde{C}$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and choose $T > 0$ so that

$$e^{-t} < \frac{M\mu}{3\tilde{C}\sqrt{M^2 + 1}} \quad \forall \ t > T.$$
Then on \( \{(X, Y) \in R_{M,\varepsilon}^+ \mid Y = MX, X > 0\} \),
\[
Y_i = \mu Y + f_2(X, Y) + e^{-t}g_2(X, Y)
\geq \mu Y - |f_2(X, Y)| - e^{-t}|g_2(X, Y)|
\geq \mu Y - \frac{M\mu}{3\sqrt{M^2 + 1}}\sqrt{X^2 + Y^2} - \frac{M\mu}{3C\sqrt{M^2 + 1}}(\tilde{C}\sqrt{X^2 + Y^2})
\geq \mu Y - \frac{M\mu}{3\sqrt{M^2 + 1}}\sqrt{M^{-2} + 1Y} - \frac{M\mu}{3C\sqrt{M^2 + 1}}(\tilde{C}\sqrt{M^{-2} + 1Y})
= \frac{\mu}{3}Y > 0
\]
whenever \( t \geq T \). Similarly, on \( \{(X, Y) \in R_{M,\varepsilon}^+ \mid Y = -MX, X > 0\} \),
\[
Y_i = \mu Y + f_2(X, Y) + e^{-t}g_2(X, Y)
\leq \mu Y + |f_2(X, Y)| + e^{-t}|g_2(X, Y)|
\leq \mu Y - \frac{M\mu}{3\sqrt{M^2 + 1}}\sqrt{M^{-2} + 1Y} - \frac{M\mu}{3C\sqrt{M^2 + 1}}(\tilde{C}\sqrt{M^{-2} + 1Y})
= \frac{\mu}{3}Y < 0
\]
whenever \( t \geq T \).

Using the above lemmas, we are able to show the existence of the stable curve for this analogous autonomous system; however, unlike the classical one, it depends on the initial time.

**Theorem 4.3.** Given a system of ODEs
\[
\begin{align*}
X_i(t) &= -\kappa X(t) + f_1(X(t), Y(t)) + e^{-t}g_1(X(t), Y(t)) \\
Y_i(t) &= \mu Y(t) + f_2(X(t), Y(t)) + e^{-t}g_2(X(t), Y(t))
\end{align*}
\]
where \( \mu > 0 > -\kappa, \frac{f_1(X,Y)}{X+Y} \to 0 \) as \( (X, Y) \to (0, 0) \) and \( g_i \in O(\sqrt{X^2 + Y^2}) \) as \( (X, Y) \to (0, 0) \) \( \forall \) \( i = 1, 2 \). Then for all positive \( M \) and \( \delta \in (0, \kappa) \), there exist positive constants \( \varepsilon_0 \) and \( T_0 \) with the following significance: for any \( \varepsilon \in (0, \varepsilon_0) \) and \( T \in [T_0, \infty) \), there is a solution curve \( (X(t), Y(t)) \) defined on \( t \in [T, \infty) \) with initial condition
\[
\begin{cases}
X(T) = \varepsilon \\
|Y(T)| \leq M\varepsilon
\end{cases}
\]
and \( (X, Y) \to (0, 0) \) as \( t \to \infty \). Furthermore, \( X, Y \in O(e^{-(\kappa-\delta)t}) \) as \( t \to \infty \).

**Proof.** Given any \( M > 0 \) and \( \delta \in (0, \kappa) \), there exists positive constants \( \varepsilon_0 \) and \( T_0 \) such that Lemma 4.1 and Lemma 4.2 hold for all \( \varepsilon \in (0, \varepsilon_0) \) and \( T \in [T_0, \infty) \). In other words, fixing \( \varepsilon \in (0, \varepsilon_0) \) and \( T \in [T_0, \infty) \), Lemma 4.1 shows that those solutions with initial conditions \( (X(T), Y(T)) \in E_{M,\varepsilon}^+ \) strictly decrease in the \( X \)-direction when they remain in \( R_{M,\varepsilon}^+ \). In particular, such solutions can remain in \( R_{M,\varepsilon}^+ \) for all \( t > T \) only if they tend to \( (0, 0) \).
On the other hand, Lemma 4.2 shows that there is a set of initial conditions \( \{(X(T), Y(T))\} \subset E_{M,\varepsilon}^+ \) with solutions that eventually exit \( R_{M,\varepsilon}^+ \) to the top. There also exist another set of initial conditions \( \{(X(T), Y(T))\} \subset E_{M,\varepsilon}^+ \) with solutions that eventually exit \( R_{M,\varepsilon}^+ \) to the below. Due to the smooth dependence of initial conditions, these two sets are both single open intervals. Note that \( E_{M,\varepsilon}^+ \) is connected. We therefore conclude that there exists a nonempty set of initial conditions \( \{(X(T), Y(T))\} \subset E_{M,\varepsilon}^+ \) such that the solutions never leave \( R_{M,\varepsilon}^+ \). That is to say, the solutions tend to \((0,0)\) as \( t \to \infty \).

Moreover, whenever \( t \geq T \), since \( X_t \leq -(\kappa - \delta)X \) by the proof of Lemma 4.1, the Grönwall’s inequality shows that

\[
X(t) \leq X(T)e^{-(\kappa - \delta)(t - T)} = \tilde{C}e^{-(\kappa - \delta)t},
\]

where \( \tilde{C} = \varepsilon e^{(\kappa - \delta)T} \) is a positive constant. That is to say, \( X \in O(e^{-(\kappa - \delta)t}) \) as \( t \to \infty \). It follows from \( |Y| \leq MX \) that \( Y \in O(e^{-(\kappa - \delta)t}) \) as \( t \to \infty \). \( \square \)

**Remark 4.4.** The aforementioned arguments also work on

\[
R_{M,\varepsilon}^- := \{(X, Y) \in S_\varepsilon \mid |Y| \leq M|X|, X \geq 0\}
\]

and

\[
E_{M,\varepsilon}^- := R_{M,\varepsilon}^- \cap \{X = -\varepsilon\}.
\]

In other words, there is also a stable curve in the half plane \( \{X \leq 0\} \).

5. **The existence of self-expanders**

Given a constant \( C = 1 \) or \(-1\), the self-similar solution \((C = 1 \text{ is self-expander and } C = -1 \text{ is self-shrinker})\) is characterized by

\[
\frac{f_{rr}}{1 + f_r^2} + \frac{(n - p)f_r}{r} + \frac{p(rf_r - \lambda^2 f)}{r^2 + \lambda^2 f^2} + C(rf_r - f) = 0.
\]

Let \( t := \log r \), \( \varphi := \frac{f}{r} \), and \( \psi := \varphi_t \). \( f_r = \varphi + \psi \) and \( f_{rr} = \frac{1}{r}(\varphi_t + \psi) = e^{-t}(\psi_t + \psi) \). We can therefore convert the second order ODE to the following system of first order ODEs:

\[
\begin{cases}
    \varphi_t = \psi \\
    \psi_t = -\psi - (n - p + \frac{p}{1 + \lambda^2 \varphi^2} + Ce^{2t})\psi + (n - p + \frac{(1 - \lambda^2 p)}{1 + \lambda^2 \varphi^2})\varphi(1 + (\varphi + \psi)^2).
\end{cases}
\]

Note that this system has exactly two equilibria \((0,0)\) and \((\varphi_0, 0)\), where \( \varphi_0 = \sqrt{\frac{p\lambda^2 - n}{(n - p)\lambda^2}} \), in the half plane \( \{\varphi \geq 0\} \). If we ignore the exponential term, considering the remaining autonomous system, then at \((0,0)\), the linearized system looks like

\[
\begin{pmatrix}
    0 & 1 \\
    p\lambda^2 - n & -n - 1
\end{pmatrix}
\begin{pmatrix}
    0 \\
    k(n + k - 1) - n
\end{pmatrix}
\begin{pmatrix}
    1 \\
    -n - 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    0 & 1 \\
    k(n + k - 1) - n & -n - 1
\end{pmatrix}
\begin{pmatrix}
    0 \\
    -n - 1
\end{pmatrix}
\]
with eigenvalues $\tilde{\lambda}_1 = k - 1, \tilde{\lambda}_2 = -n - k$ and associative eigenvector $V_1 = \begin{pmatrix} 1 \\ k - 1 \end{pmatrix}, V_2 = \begin{pmatrix} 0 \\ 1 - n - k \end{pmatrix}$. It implies that $(0, 0)$ is a saddle. At $(\varphi_0, 0)$, the linearized system looks like

$$\begin{pmatrix} 0 & 1 \\ n(\frac{n}{k} - 1) & -n - 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2n(\frac{n}{k(k+n-1)} - 1) & -n - 1 \end{pmatrix}$$

with eigenvalues $\tilde{\lambda} = -\frac{n+1}{2} \pm \frac{1}{2} \sqrt{n^2 - 6n + 1 + \frac{8n^2}{k(k+n-1)}}$. When $n = 3, k \geq 4$ or $n = 5, k \geq 6$, $\tilde{\lambda}$ are complex numbers with negative real parts. Hence,

1. If $(n, p, k) = (3, 2, 2), (5, 4, 2), (5, 4, 4)$ or $n \geq 7$, then $(\varphi_0, 0)$ is a sink.
2. If $(n, p) = (3, 2), k \geq 4$ or $(n, p) = (3, 2), k \geq 6$, then $(\varphi_0, 0)$ is a spiral sink.

Here, our goal is to construct a compact positive invariant set in the first quadrant.

**Proposition 5.1.** For the self-expander case, i.e. $C = 1$, we have the following:

1. If $(n, p, k) = (3, 2, 2), (5, 4, 2), (5, 4, 4)$, then the compact region $\Delta$ enclosed by $\{\psi = 0\}$ and the graph of $g(\varphi) = \frac{3((1-\lambda)^2 p - (n - p)) \varphi}{2(n - p)}$ is a positive invariant set of the system of ODEs.
2. If $n \geq 7$, then the compact region $\Delta$ enclosed by $\{\psi = 0\}$ and the graph of $g(\varphi) = \frac{2((1-\lambda)^2 p - (n - p)) \varphi}{(n - p)}$ is a positive invariant set of the system of ODEs.

**Proof.** It suffices to check the following two conditions:

1. $\psi_t \geq 0$ on $\{(\varphi, 0) \mid 0 \leq \varphi \leq \varphi_0\}$.
2. $\langle(\varphi_1, \psi_1), (g'(\varphi), -1)\rangle \geq 0$ on $\{(\varphi, g(\varphi)) \mid 0 \leq \varphi \leq \varphi_0\}$, where $(g'(\varphi), -1)$ is the inner normal of the graph of $g$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$.

The first one is clear and the second one is typically based on the results by Xu, Yang and Zhang (cf. [11, Sec. 4.3]). For each case, they have proved that

$$g'(\varphi) > \frac{X_2}{X_1}(\varphi, g(\varphi)) \quad \forall \varphi \in (0, \varphi_0),$$

where

$$\begin{cases} X_1(\varphi, \psi) = \psi \\ X_2(\varphi, \psi) = -\psi - ((n - p + \frac{p}{1 + \lambda^2 \varphi^2}) \psi + (n - p + \frac{(1-\lambda^2) p}{1 + \lambda^2 \varphi^2}) \varphi) (1 + (\varphi + \psi)^2) \end{cases}.$$
Then condition (2) follows from direct calculations:

\[
\langle (\varphi, \psi), (g'(\varphi), -1) \rangle = g'(\varphi)\psi + \psi + (1 + (\varphi + \psi)^2) \\
\cdot ((n - p + \frac{p}{1 + \lambda^2 \varphi^2}) + e^{2t}) \psi + (n - p + (1 - \lambda^2) p) (\varphi) \\
\geq g'(\varphi)\psi + \psi + (1 + (\varphi + \psi)^2) \\
\cdot ((n - p + \frac{p}{1 + \lambda^2 \varphi^2}) + (n - p + (1 - \lambda^2) p) (\varphi)) \\
= g'(\varphi) X_1 - X_2 \\
\geq 0
\]
on \{ (\varphi, g(\varphi)) \mid 0 \leq \varphi \leq \varphi_0 \}.

\[\Box\]

**Remark 5.2.** A tricky point is that we need to find a function \( g \) with \( g'(0) > k - 1 > 0 \) in order to apply Theorem 4.3 near \((0, 0)\) in \( \Delta \).

From now on, we only consider \((n, p, k) = (3, 2, 2), (5, 4, 2), (5, 4, 4) \) or \( n \geq 7 \) and the self-expander case \( C = 1 \).

Let \( X(t) := \frac{n+k}{n+2k-1} \varphi(-t) + \frac{1}{n+2k-1} \psi(-t) \), \( Y(t) := \frac{k-1}{n+2k-1} \varphi(-t) - \frac{1}{n+2k-1} \psi(-t) \). Then the system of first order ODEs changes into the form

\[
\begin{cases}
X_t = (1 - k)X + O(X^2 + Y^2) + e^{-2t} \cdot O(\sqrt{X^2 + Y^2}) \\
Y_t = (n + k)Y + O(X^2 + Y^2) + e^{-2t} \cdot O(\sqrt{X^2 + Y^2})
\end{cases}
\]

which satisfies all the assumptions in Theorem 4.3.

Hence, we can simply choose \( 0 < M < 1, \delta = \frac{1}{2} \) and apply Theorem 4.3 then there exist positive constants \( \delta_0 \) and \( T_0 \) with the following properties: for any \( \delta \in (0, \delta_0] \) and \( T \in [T_0, \infty) \), there is a solution curve \((X(t), Y(t))\) defined on \( t \in [T, \infty) \) with initial condition

\[
\begin{cases}
X(T) = \delta \\
|Y(T)| \leq M \delta
\end{cases}
\]

and \((X, Y) \to (0, 0)\) as \( t \to \infty \). Furthermore, \( X, Y \in O(e^{-(k-\frac{3}{2})t}) \) as \( t \to \infty \).

Since \( \varphi(t) = X(-t) + Y(-t), \psi(t) = (k - 1)X(-t) - (n + k)Y(-t) \), we conclude that there exist positive constants \( \varepsilon_0 \) and \( T_0 \) with the following significance: for any \( \varepsilon \in (0, \varepsilon_0] \) and \( -T \in (-\infty, -T_0] \), there is a solution curve \((\varphi(t), \psi(t))\) defined on \( t \in (-\infty, -T] \) such that \( \varphi(-T) = \varepsilon \), \( (\varphi, \psi) \in \Delta \forall t \in (-\infty, -T] \) and \( (\varphi, \psi) \to (0, 0) \) as \( t \to -\infty \). Moreover, \( \varphi, \psi \in O(e^{(k-\frac{3}{2})t}) \) as \( t \to -\infty \).

Now, Proposition 5.1 shows that \( \Delta \) is a compact positive invariant set. It follows that we can actually extend \((\varphi, \psi)\) to be a global solution (cf. [4, Chap. 7.2, p.146–147]). That is to
say, there is a solution curve \((\varphi(t), \psi(t))\) defined on \(t \in (-\infty, \infty)\) such that \(\varphi(-T) = \varepsilon\) and \((\varphi, \psi) \in \Delta\).

Recall that \(f = r \varphi\) and \(f_r = \varphi + \psi\). Then \(f \geq 0, f_r \geq 0\) follows from \((\varphi, \psi) \in \Delta\). Moreover, since \(\varphi, \psi \in O(e^{(k-\frac{3}{2})t})\) as \(t \to -\infty\), we have that
\[
\begin{align*}
f &\in O(r^{k-\frac{3}{2}}) \\
f_r &\in O(r^{k-\frac{1}{2}})
\end{align*}
\]
as \(r \to 0\). Therefore,

\[
F(r, x) = (r x, f(r) \mathcal{L}(x))
\]
is \(C^1\) near the origin. Applying the classical bootstrapping argument (cf. [7, Theorem 6.8.1] or [2, Theorem 9.13]), which follows from elliptic regularity and Sobolev embedding, \(F\) is actually analytic near \(r = 0\). Combining with the fact that \(k\) is an integer, we further conclude that
\[
\begin{align*}
f &\in O(r^k) \\
f_r &\in O(r^{k-1})
\end{align*}
\]
as \(r \to 0\).

Note that \(f(r)\) is smooth for all \(r > 0\). It follows that \(F(r, x)\) is smooth everywhere. In other words,
\[
\Sigma = F(\mathbb{R}_{\geq 0} \times \mathbb{S}^n) = \{(rx, f(r)\mathcal{L}(x)) \mid r \in \mathbb{R}_{\geq 0}, x \in \mathbb{S}^n\}
\]
is a smooth self-expander.

6. The uniqueness as a Dirichlet problem

Fixing the initial condition \(\varphi(-T) = \varepsilon\), we can also interpret the self-expander equation as a Dirichlet problem for the minimal map equation
\[
g : B(0; e^{-T}) \subset \mathbb{R}_{2n} \to \mathbb{R}^n_2 \times \mathbb{R}^{n+1}_n
\]
with boundary condition
\[
g|_{\partial B(0,e^{-T})} = (\text{id}_{\mathbb{R}^{2n}}, \varepsilon e^{-T} \mathcal{L})
\]
Here, the word minimal is with respect to the conformal metric \(e^{-\frac{3}{2}||x||^2}\delta_{ij}\).

In this regard, we have the following uniqueness property.

**Proposition 6.1.** Given the Dirichlet type condition \(f(R) = \varepsilon R\), where \(\varepsilon \in (0,1]\) and \(R := e^{-T} \in (0, \lambda^{-\frac{2}{n-3}}]\), then there is at most one \(f\) defined on \([0, \infty)\) satisfying the ODE
\[
\frac{f_{rr}}{1 + f_r^2} + \frac{(n - p)f_r}{r} + \frac{p(rf_r - \lambda^2 f)}{r^2 + \lambda^2 f^2} + rf_r - f = 0
\]
and \(f(0) = 0\).
Proof. We first show that there is at most one \( f \) defined on \([0, R]\) satisfying all the conditions. Suppose that both \( f_1, f_0 \) defined on \([0, R]\) satisfy all the conditions. Let \( g(r) = f_1(r) - f_0(r) \). We first notice that \( g \) is continuous on \([0, R]\). Moreover, it is at least \( C^2 \), in fact smooth, on \((0, R)\).

Now,

\[
\frac{(f_1)_r}{1 + (f_1)^2} - \frac{(f_0)_r}{1 + (f_0)^2} = \frac{g_r(1 + (f_0)^2) - g_r((f_0)_r + (f_1)_r)}{(1 + (f_1)^2)(1 + (f_0)^2)},
\]

\[
(n - p)(f_1)_r - (n - p)(f_0)_r = (n - p)\frac{g_r}{r},
\]

\[
(r(f_1)_r - f_1) - (r(f_0)_r - f_0) = rg_r - g.
\]

\[
p(r(f_1)_r - \lambda^2 f_1) - p(r(f_0)_r - \lambda^2 f_0) = \frac{pg_r(r^3 + \lambda^2 r f_0^2)}{(r^2 + \lambda^2 f_1^2)(r^2 + \lambda^2 f_0^2)} - \frac{pg(\lambda^2 r^2 - \lambda^4 f_0 f_1 + \lambda^2 r f_0(f_0 + f_1))}{(r^2 + \lambda^2 f_1^2)(r^2 + \lambda^2 f_0^2)}.
\]

We therefore conclude that \( g \) satisfies the ODE

\[
\frac{g_r(1 + (f_0)^2) - g_r((f_0)_r + (f_1)_r)}{(1 + (f_1)^2)(1 + (f_0)^2)} + \frac{(n - p)g_r}{r} + rg_r - g = 0.
\]

Suppose that \( r_1 \in (0, R) \) is a local maximum point of \( g \). Then \( g_r(r_1) = 0 \) and \( g_{rr}(r_1) \leq 0 \). At \( r_1 \), the ODE of \( g \) becomes

\[
\frac{1 + (f_0)^2}{(1 + (f_1)^2)(1 + (f_0)^2)}g_{rr} = \frac{p(\lambda^2 r_1^2 - \lambda^4 f_0 f_1 + \lambda^2 r_1 (f_0)_r (f_1 + f_0))}{(r_1^2 + \lambda^2 f_1^2)(r_1^2 + \lambda^2 f_0^2)} + 1 \geq 0.
\]

According to the proof of Theorem 4.3 and assumptions about \( \varepsilon, R \), we see that

\[
f_i(r) \leq \varepsilon R^{k - \frac{2}{2}} r^{k - \frac{1}{2}} \leq \lambda^{-1} r^{k - \frac{1}{2}} \quad \forall i = 1, 2 \text{ and } r \in (0, R).
\]

Then \( \lambda^2 r_1^2 - \lambda^4 f_0(f_1(r_1) - f_1(r_1)) \geq \lambda^2 r_1^2(1 - r_1^{2k-3}) \geq 0 \). Combining with \( f_0, f_1, (f_0)_r, (f_1)_r \geq 0 \), we conclude that

\[
\frac{1 + (f_0)^2}{(1 + (f_1)^2)(1 + (f_0)^2)} \geq 0 \text{ and } \frac{p(\lambda^2 r_1^2 - \lambda^4 f_0 f_1 + \lambda^2 r_1 (f_0)_r (f_1 + f_0))}{(r_1^2 + \lambda^2 f_1^2)(r_1^2 + \lambda^2 f_0^2)} + 1 \geq 0.
\]

It follows from the ODE of \( g \) that \( g(r_1) \) must not exceed 0.

If \( r_2 \in (0, R) \) is a local minimum point of \( g \), then similar argument shows that \( g(r_2) \) cannot be less than 0. Combining with \( g(0) = g(R) = 0 \), we conclude that \( g \equiv 0 \). That is to say, \( f_1 \equiv f_0 \).

Finally, the uniqueness of the extension of \( f \) on \([R, \infty)\) follows from the Picard-Lindelöf theorem. Hence, there is at most one \( f \) defined on \([0, \infty)\) satisfying all the conditions. \( \square \)
7. The behavior of the self-expander at infinity

In this section, we investigate the behavior of self-expander we construct at infinity. We first back to the system of first order ODEs

\[
\begin{align*}
\varphi_t &= \psi \\
\psi_t &= -\psi - ((n - p + \frac{p}{1 + \lambda^2 \varphi^2} + e^{2t})\psi + (n - p + \frac{(1 - \lambda^2)p}{1 + \lambda^2 \varphi^2})\varphi)(1 + (\varphi + \psi)^2).
\end{align*}
\]

Note that in the positive invariant set $\Delta$, we have

\[
\psi_t \leq 0
\]

or

\[
(n - p + \frac{p}{1 + \lambda^2 \varphi^2} + e^{2t})\psi + (n - p + \frac{(1 - \lambda^2)p}{1 + \lambda^2 \varphi^2})\varphi < 0.
\]

In other words,

\[
e^{2t} \psi < \varphi((\lambda^2 p - n) - \lambda^2 (n - p)\varphi^2).
\]

The critical point of $\varphi((\lambda^2 p - n) - \lambda^2 (n - p)\varphi^2)$ is $\varphi = \pm \sqrt{\frac{\lambda^2 p - n}{3n(n - p)}} = \pm \frac{\varphi_0}{\sqrt{3}}$. Therefore,

\[
\varphi((\lambda^2 p - n) - \lambda^2 (n - p)\varphi^2) \leq \frac{\varphi_0}{\sqrt{3}}((\lambda^2 p - n) - \lambda^2 (n - p)\frac{\varphi_0^2}{3}) = \frac{2\lambda^2(n - p)\varphi_0^3}{3\sqrt{3}}.
\]

We conclude that $\psi_t > 0$ only if

\[
\psi < \tilde{C}e^{-2t},
\]

where $\tilde{C} = \frac{2\lambda^2(n - p)\varphi_0^3}{3\sqrt{3}}$.

Now, we observe that $\lim_{t \to \infty} \varphi$ exists since $\Delta$ is compact and $\varphi_t = \psi \geq 0$ in $\Delta$.

**Proposition 7.1.** $\lim_{t \to \infty} \psi$ also exists and is equal to 0.

**Proof.** We split into two cases:

1. Suppose that $\exists T > 0$ such that $\psi(t) \neq \tilde{C}e^{-2t} \forall t > T$. Therefore, either
   
   (a) $\psi(t) > \tilde{C}e^{-2t} \forall t > T$ or
   
   (b) $\psi(t) < \tilde{C}e^{-2t} \forall t > T$

   happens.

   For (a), note that $\psi_t(t) < 0 \forall t > T$. Since $\Delta$ is compact, it implies that $\lim_{t \to \infty} \psi$ exists. Moreover, since $\lim_{t \to \infty} \varphi$ exists and $\varphi_t = \psi$, $\lim_{t \to \infty} \psi$ must equal 0.

   For (b), note that $0 \leq \psi(t) < \tilde{C}e^{-2t} \forall t > T$. Then by the squeeze lemma,

   \[
   \lim_{t \to \infty} \psi = 0.
   \]

2. Suppose that $\forall \tilde{T} > 0$, $\exists T > \tilde{T}$ such that $\psi(T) = \tilde{C}e^{-2T}$. We first claim that if $\psi(T) = \tilde{C}e^{-2T}$, then $\psi(t) < \tilde{C}e^{-2T} \forall t > T$. We argue it by contradiction.
Assume that the statement is false, say $\exists t_1 > T$ such that $\psi(t) > \tilde{C}e^{-2T} > \tilde{C}e^{-2t_1}$.

Let 
\[
 g(t) := \psi(t) - \tilde{C}e^{-2t}.
\]

Define 
\[
 S := \{ t \in [T, t_1] \mid g(t) = 0 \}.
\]

Since $S$ is bounded, $\sup S$ exists. Moreover, the continuity of $F$ and the fact that $g(t_1) > 0$ show that $t_0 := \sup S < t_1$. Now, by the Intermediate Value Theorem, 
\[
 \psi(t) > \tilde{C}e^{-2t} \quad \forall t \in (t_0, t_1].
\]

Furthermore, by the Mean Value Theorem, $\exists t_2 \in (t_0, t_1)$ such that 
\[
 \psi_t(t_2) = \frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0} > 0,
\]

which contradicts to the fact that $\psi_t > 0$ only if $\psi < \tilde{C}e^{-2t}$.

Finally, according to the claim, we have a sequence of $\{T_i\}_{i=1}^\infty$ such that $T_i < T_j \quad \forall i < j; \quad T_i \to \infty$ as $i \to \infty$ and $\psi(t) < \tilde{C}e^{-2T_i} \forall t > T_i$. By the squeeze lemma, we conclude that 
\[
 \lim_{t \to \infty} \psi = 0.
\]

Let $\varphi_\infty := \lim_{t \to \infty} \varphi$. Recall that $f = r\varphi$, then the aforementioned discussion shows that the self-expander 
\[
 \Sigma = \{ F(r, x) = (r x, f(r)L(x)) \mid r \in \mathbb{R}_{\geq 0}, x \in S^n \}
\]

is asymptotic to the cone 
\[
 \{(r x, r\varphi_\infty L(x)) \mid r \in \mathbb{R}_{\geq 0}, x \in S^n \}
\]

as $r \to \infty$.

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