Phantasy and Reality - Teleportation versus quantum computing

P Nagy\textsuperscript{1} and P Tasnádi\textsuperscript{2}

\textsuperscript{1}GAMF Faculty, John von Neumann University, Izsáki u. 10., Kecskemét, H-6000 Hungary
\textsuperscript{2}Faculty of Science, Eötvös Loránd University, 1/A, Pázmány Péter, Budapest, H-1117 Hungary

Abstract. Application of quantum theory in informatics and creation of quantum information devices have revolutionary changed our idea about the information systems. It seems to us that the only question is that quantum computers will be introduced in the next decade or in some years later. Anyhow we should prepare to teach a radically new paradigm, the quantum algorithms, in informatics. Quantum algorithm can be demonstrated with quantum simulators. Present paper shows an educational material, which facilitate the teaching of this new field. Besides summarising the basic concepts and methods of quantum informatics the working of a quantum algorithm on a quantum simulator will be also illustrated with the discussion of an exciting quantum physical phenomenon (the teleportation).

1. Introduction
Ever since Einstein, Podolsky and Rosen (EPR) have created their famous and paradoxical “gedankene” experiment to prove that quantum theory cannot give a complete description of the physical reality, many physicists have tried to understand the “spooky action at a distant” of entangled particles. \textsuperscript{1} (As Einstein called the weird behaviour of the entangled particles.\textsuperscript{1a}) Entangled state means that e.g. a pair of photons can be ejected and separated by a beam splitter and sent on different paths where they move away very far from each other. However, when somebody observes the separated photons, they behave as if they are still connected. Einstein did not believe that such type of nonlocality is possible.

The existence of nonlocality can be judged only through its confrontation with the results of measurements. But, from the publication of EPR paper it took a long time to realize an "experimentum crucis". Decisive steps have made by Bohm (1957)\textsuperscript{1b} and Bell (1961) \textsuperscript{1c} toward the solution. Bohm has formulated the EPR paradox in a simpler way and Bell has derived some inequalities that could checked experimentally. The experiment which have clearly proved the nonlocal nature of the entangled particles was made by Aspect (1982) \textsuperscript{1d}. The Aspect experiment has opened new perspectives for the application of quantum physics particularly in informatics and it has also aroused the imagination of the scifi writers. This paper, besides a short discussion of quantum teleportation, presents a possibility of teaching the introductory steps towards the understanding of the basic principles of quantum computers by the use a simulation program, which can improve the motivation of university students to learn modern physics.
2. Teleportation
The idea of the teleportation has excited the imagination of sci-fi novelists for a long time. The astronauts in the adventure series of the Star Trek transfer themselves via radiation to planets and back from there to their spacecraft with a special equipment. This action is commonly called as teleportation.

Figure 1. Teleportation in series Star Trek

Teleportation means that matter is transferring from one place to another without crossing the physical space between the two places. More exactly an extended material body is carried through the space in a bodiless form. It is very important to emphasize that such type of teleportation is absolutely impossible.

However, in quantum physics it has been explored a possibility of a kind of teleportation which is not exactly the same as its commonly known form, but in some way like to it. The quantum teleportation is the destruction of a state of a microphysical system and its later reproduction at a different place by the use of entangled states and classical communication. It should be made clear that in this case material bodies does not cease nowhere and reversely they do not appear somewhere from the nothing. In fact, the state of an existing particle is transferred to another existing particle. Since the two particles can be very far from each other the process was called (quantum) teleportation [1]. In this sense teleportation means the transfer of a quantum state through a classical channel. The name, due to its common meaning, is equivocal, so it has caused many misunderstandings.

3. Summary of quantum-physical background
The quantum bit or qubit is a generalization of the classical bit. The state of a physical system which represents a quantum bit can be the arbitrary superposition of the $|0\rangle$ and $|1\rangle$ orthogonal and normalized basis states: $|q\rangle = a|0\rangle + b|1\rangle$ (where $a$ and $b$ are complex numbers satisfying the requirement $|a|^2 + |b|^2 = 1$). Conventionally the basis states are represented by ket vectors

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while their transposes by bras: $\langle 0| = |0\rangle^T = (1, 0)$ és $\langle 1| = |1\rangle^T = (0, 1)$.

Transformations are operations which convert a quantum state to another one. Transformations can be expressed by multiplication of operators and qubits. The operators can be constructed out of one or more qubits. The most important one qubit operators which will be used are:
\[
\begin{align*}
\bar{I} &= |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow |1\rangle \end{pmatrix} \text{ identity operator}, \\
\bar{X} &= \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} |0\rangle \rightarrow |1\rangle \\ |1\rangle \rightarrow |0\rangle \end{pmatrix} \text{ NOT (Pauli-X) operator}, \\
\bar{Z} &= \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow -|1\rangle \end{pmatrix} \text{ Pauli-Z operator}, \\
\bar{M}_\theta &= \left( \cos \theta \cdot |0\rangle + \sin \theta \cdot |1\rangle \right)\langle 0| + \left( -\sin \theta \cdot |0\rangle + \cos \theta \cdot |1\rangle \right)\langle 1| = \\
&= \begin{pmatrix} \cos \theta & -
\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix} |0\rangle \rightarrow \cos \theta \cdot |0\rangle + \sin \theta \cdot |1\rangle \\
|1\rangle \rightarrow -\sin \theta \cdot |0\rangle + \cos \theta \cdot |1\rangle
\end{pmatrix} \text{ mixing operator}, \\
\bar{H} &= \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right)\langle 0| + \left( |0\rangle - |1\rangle \right)\langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} |0\rangle \rightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \\
|1\rangle \rightarrow \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \end{pmatrix} \text{ Hadamard operator}.
\end{align*}
\]

According to our everyday experiences if parts of a system are so far from each other, that there is no interaction between them than the state of the whole system can be described as the sum of the states of the individual parts. (This is the classical sense.) Systems consisting of more quantum bits are called quantum registers.

In classical physics, the state space of a system is a direct sum of the state space of its subsystems. The dimension of a system is the sum of the dimension of its subsystems \( \dim(R_1 \oplus R_2) = \dim(R_1) + \dim(R_2) \) (e.g. the dimension of the state space of a noble gas of \( N \) atoms is \( 6N \) since that of a gas atom is 6). In contrast, the state space of a quantum system is the direct product of those of its subsystems \( \dim(R_1 \otimes R_2) = \dim(R_1) \cdot \dim(R_2) \). Contrary to the classical systems the dimension of which is increasing linearly with the number of the elements, the dimension of quantum systems depends exponentially on the number of the subsystems.

While the basis of a one-bit quantum system is \( \{ |0\rangle, |1\rangle \} \), the basis of the two-bit quantum register is

\[
\begin{align*}
\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}, \text{ ahol:} \\
|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \quad |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, & \quad |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]
which consists of four-vectors. The most important two qubit transformations are the so-called C controlled one-qubit transformations e.g. the C-NOT transformation, which can entangle and disentangle EPR states:

\[
\overline{C - NOT} = |0\rangle \langle 0| \otimes \overline{I} + |1\rangle \langle 1| \otimes \overline{X} = |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 11| + |11\rangle \langle 10| = \\
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} = \\
\begin{pmatrix}
|00\rangle \rightarrow |00\rangle \\
|01\rangle \rightarrow |01\rangle \\
|10\rangle \rightarrow |11\rangle \\
|11\rangle \rightarrow |10\rangle
\end{pmatrix}
\]

Multiple bit quantum systems exhibit very strange properties. One of them is that the state of the whole system cannot be build up out of the states of its parts. Such types of quantum states are called entangled (EPR) states. As an example, consider the state \( \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \). It is an entangled one, since equation

\[
(a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle) = a_1a_2 |00\rangle + a_1b_2 |01\rangle + b_1a_2 |10\rangle + b_1b_2 |11\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\]
cannot be solved for \( \{a_1, a_2, b_1, b_2\} \).

Applying the Hadamard \( \overline{H} \) operation to every qubits of an n-qubit quantum register one by one a superposed state of the quantum register is created which contains each number between 0 and 2n-1 (a classical register contains only one number of this range). It means that the operations executed on a quantum register happen at the same time with all the number of the register. Recognizing this, we surely begin to feel the enormous efficiency of “quantum parallelism”, and its calculation potential. \( \overline{H} \)

\[
(\overline{H} \otimes \overline{H} \otimes \ldots \otimes \overline{H})(|00\ldots0\rangle) = \overline{H} |0\rangle \otimes \overline{H} |0\rangle \otimes \ldots \otimes \overline{H} |0\rangle = \frac{1}{\sqrt{2^n}} (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \ldots \otimes (|0\rangle + |1\rangle) = \\
= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle.
\]

The operators changing the quantum states of a system can be divided into two groups. Operators of the first group produce the time development of the system and reversibly change its state. Operators of the other group represent the measurements, which can be made on the systems. A measurement on an arbitrary qubit of a superposed state \( |q\rangle = a |0\rangle + b |1\rangle \) pushes the qubit irreversibly and randomly into one of its eigenstate. (The probability that the qubit jumps into the zero state \( |0\rangle \) is \( |a|^2 \) while that it jumps into the state \( |1\rangle \) is \( |b|^2 \). Therefore despite a qubit can be in infinite number of superposed states, due to the destructive property of the measurement (due to a measurement the state “collapses” into an eigenstate) only one bit information can be obtained from it. Theoretical physicist have continued passionate debates on the physical sense, moreover the definition, of the measurement but fortunately the mathematical description of it is relatively simple and calculations independently from the interpretations lead to clear results. For a better understanding of the measurement mechanism, consider a two-qubit register of the superposed state \( a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle \) (a, b, c and d
complex numbers and \[ |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \] and execute a one bit (partial) measurement on the first bit of it in the basis \([0, 1]\) In order to get the result some conversion should be made on the register:

\[
a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle = |0\rangle \otimes (a|0\rangle + b|1\rangle) + |1\rangle \otimes (c|0\rangle + d|1\rangle) =
\]

\[
= u|0\rangle \otimes \left( \frac{a}{u}|0\rangle + \frac{b}{u}|1\rangle \right) + v|1\rangle \otimes \left( \frac{c}{v}|0\rangle + \frac{d}{v}|1\rangle \right),
\]

where \( u = \sqrt{|a|^2 + |b|^2} \) and \( v = \sqrt{|c|^2 + |d|^2} \), therefore the norms of state vectors \( \left( \frac{a}{u}|0\rangle + \frac{b}{u}|1\rangle \right) \) and \( \left( \frac{c}{v}|0\rangle + \frac{d}{v}|1\rangle \) are 1. The formula produces the state as a linear combination of the direct multiplication of the qubit and the unit vectors. The probability distribution of the results of the measurement can be simply read from it. In case of the first qubit the result will be \( |0\rangle \) and \( |1\rangle \) with probability \( |u|^2 \) and \( |v|^2 \), respectively. If the result for the first qubit is \( |0\rangle \) than the second qubit will be in the superposed state \( \left( \frac{a}{u}|0\rangle + \frac{b}{u}|1\rangle \right) \). If the state of the first bit is \( |1\rangle \) the superposed state of the second qubit is \( \left( \frac{c}{v}|0\rangle + \frac{d}{v}|1\rangle \right) \)

In quantum informatics it is suitable to regard the multi qubit measurements as a series of one bits ones. Each of the yes-no type measurements can be considered a one qubit measurement. (Such type of measurement for example the decision of the question whether a particle is on the left or the right side of a plane. Similarly a yes-no type measurement is whether the values of a quantum register fall within a given interval.) The yes-no type measurements can be described by the separation of the space into two orthogonal subspaces. The result of measurements can be easily determined if the qubit investigated is expressed as the linear combination of the normal vectors of the orthogonal subspaces. As the result of the measurement the qubit jumps into its projection to one or the other subspace. The probability of the jump is the square of the proper coefficient in the linear combination.

Measurement can be constructed to decide that two qubit is identical or not, without obtaining any information about their values! This means for us that if we want to make computer simulations on the time development of a quantum systems than the realisation of one qubit measurements are enough, since every other calculation can be led back to it.

4. **Quantum teleportation on a simulator**

To produce quantum teleportation besides a classical communication channel an entangled qubit pair is necessary. Be Q the superposed quantum state (qubit) which should be forwarded, and the sender and receiver be the classical couple, Alice and Bob, respectively. Alice and Bob share an entangled qubit pair (in the following: EPR pair). Alice's qubit is A, and Bob's one is B. Alice makes a two bit measurement on Q and A qubits and sends the result through a classical two bit channel to Bob.
This procedure can be followed with computer simulation. The series of transformations can be resolved into consecutive operations where an operation is executed at a time on a qubit or on a group of qubits. An effective and popular representation of quantum algorithms is the so-called quantum network in analogy with the networks which represents the classical algorithms. In quantum networks the gates correspond to the operators of the algorithm, and wires correspond to the qubits. The succession of the connections determines the time sequence of the operations.

Similarly to the programs of planning and simulating classical networks nowadays many simulators of quantum networks are also available. (e.g. SimQubit, qcad). In this presentation the Jaquzzi simulator is used. (Of course the real quantum parallelism cannot be realised on a classical computer. So this simulation is virtual one like to the so called multitask type calculations.) In the Jaquzzi simulator horizontal wires represents the qubits, the connections between them correspond to the quantum gates. To follow the time sequence of the operations one should go from left to right in the network.

A teleportation algorithm was realized on JaQuzzi (Fig. 3. shows a screen shot of it.) The upper, middle and lower wires represent the qubits \( Q \), \( A \) and \( B \), respectively. The consecutive gates correspond the state changes of the qubits. In the first step \( Q \) is transformed into an arbitrarily chosen superposed state with the M mixing gate:

\[
|Q\rangle = (\cos(\theta)|0\rangle + \sin(\theta)|1\rangle).
\]

In the second step an entangled qubit pair is created by the use of a Hadamard and a C-NOT gate. Then the state of the whole quantum system is the direct product of the \( Q \) superposed state and the state of the entangled EPR pair:

\[
|QAB\rangle = (\cos(\theta)|0\rangle + \sin(\theta)|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \\
= \frac{1}{\sqrt{2}} (\cos(\theta)|000\rangle + \cos(\theta)|011\rangle + \sin(\theta)|100\rangle + \sin(\theta)|111\rangle).
\]
After it Alice (the sender) executes a C-NOT transformation on her EPR bit A (where Q is the controlling bit). Then the state of the system will be:

\[
|QAB\rangle = \frac{1}{\sqrt{2}} \left( \cos(\theta)|000\rangle + \cos(\theta)|011\rangle + \sin(\theta)|110\rangle + \sin(\theta)|101\rangle \right) = \\
= \frac{1}{\sqrt{2}} \left( \cos(\theta)|0\rangle \otimes |00\rangle + \cos(\theta)|0\rangle \otimes |11\rangle + \sin(\theta)|1\rangle \otimes |10\rangle + \sin(\theta)|1\rangle \otimes |01\rangle \right).
\]

Finally Alice carries out an Hadamard transformation on Q therefore we should put a Hadamard gate into the upper wire. So the quantum state of the system:

\[
|QAB\rangle = \frac{\cos(\theta)}{2} \left( |000\rangle + |011\rangle + |100\rangle + |111\rangle \right) + \frac{\sin(\theta)}{2} \left( |010\rangle + |001\rangle - |110\rangle - |101\rangle \right) = \\
= \frac{1}{2} |00\rangle \otimes \left( \cos(\theta)|0\rangle + \sin(\theta)|1\rangle \right) + \frac{1}{2} |01\rangle \otimes \left( \cos(\theta)|1\rangle + \sin(\theta)|0\rangle \right) + \\
+ \frac{1}{2} |10\rangle \otimes \left( \cos(\theta)|0\rangle - \sin(\theta)|1\rangle \right) + \frac{1}{2} |11\rangle \otimes \left( \cos(\theta)|1\rangle - \sin(\theta)|0\rangle \right).
\]

The last equation shows well, that in the final state Bob’s EPR bit (B) carries the full information of the original Q bit which should be transferred. So, if the first two bits were measured then the third will be in a state from which one can restore state Q state if he/she knows the result of the measurement. Therefore Alice sends to Bob the result of a one bit measurement carried out on Q and her EPR bit A (measurements are denoted in Jaquzzi by red-yellow thunderbolt icon) in a classical channel.
On the basis of the last equation we can easily find out how Bob could reproduce the original qubit Q from the results he got from Alice. He should apply a proper transformation to his EPR qubit B.

If the pair is 00, then he should apply the I-transformation (first term), if it is 01, he should apply the X-transformation (second term), if it is 10 he should apply the Z-transformation (third term), and finally if it is 11 then he should apply consecutively X_ and the Z– transformation (fourth term). Bob can realize these operations by putting proper gates into the network.

Summary
The investigation of the entangled states opened new perspectives in both quantum theory and computer physics. Quantum computers provide very promising possibilities for the development of new very quick computers. It is important to make this possibilities familiar with students. The paper presented a motivating tool for the demonstration of a quantum algorithm, which can be taught for students who want to be familiar with this new field of quantum physics.

Acknowledgements
This study was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences.
This research is supported by EFOP-3.6.1-16-2016-00006 "The development and enhancement of the research potential at John von Neumann University" project. The Project is supported by the Hungarian Government and co-financed by the European Social Fund.

References:
1a https://plato.stanford.edu/entries/qt-epr/
1b A. Einstein, B Podolsky and N. Rosen:(1935) Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?, Physical Review, 47, pp 777-780.
1c Bohm, D: "The Paradox of Einstein, Rosen, and Podolsky." Quantum Th., 611-623, 1951.
1d Bell, J. S. On the Einstein-Podolsky-Rosen Paradox, Physics 1, 195-200, 1964.
1e • Aspect, A.; Grangier, P.; and Roger, G. "Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities." Phys. Rev. Let. 49, 91-94, 1982.

[1] http://www.sebastianwill.de/Sebastian_Wills_Homepage/Home_files/quantumteleportation.pdf
[2] https://www.research.ibm.com/ibm-q/
[3] http://davidbkemp.github.io/QuantumComputingArticle/
[4] https://people.cs.umass.edu/~strubell/doc/quantum_tutorial.pdf
[5] http://www.eng.buffalo.edu/~phygons/jaQuzzi/jaQuzzi.html