Stability of a non-minimally conformally coupled scalar field in $F(T)$ cosmology

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Abstract In this paper, we introduce a non-minimally conformally coupled scalar field and dark matter in $F(T)$ cosmology and study their dynamics. We investigate the stability and phase space behavior of the parameters of the scalar field by choosing an exponential potential and cosmologically viable form of $F(T)$. We found that the dynamical system of equations admits two unstable critical points; thus no attractor solutions exist in this cosmology. Furthermore, taking into account the scalar field mimicking quintessence and phantom energy, we discuss the corresponding cosmic evolution for both small and large times. We investigate the cosmological implications of the model via the equation of state and deceleration parameters of our model and show that the late-time Universe will be dominated by phantom energy and, moreover, phantom crossing is possible. Our results do not lead to explicit predictions for inflation and the early Universe era.

1 Introduction

Astrophysicists are convinced that the observable universe is in a phase of rapid accelerated expansion. This commonly is termed ‘dark energy’ (DE); it possesses negative pressure and positive energy density. This conclusion has been supported by several astrophysical findings: data for SNe Ia supernovae [1, 2], cosmic microwave background radiation via WMAP [3], galaxy redshift surveys via SDSS [4] and galactic X-ray [5]. Although the phenomenon of dark energy in cosmic history is very recent, $z \sim 0.7$, it has opened new areas in cosmology research. The most elegant and simple resolution to the problem of DE is the cosmological constant [6, 7] but it cannot resolve the fine tuning and cosmic coincidence problem. Hence theorists have looked for alternative models by considering the dynamic nature of dark energy like a quintessence scalar field [8–13], a phantom energy field [14–21] and f-essence [22, 23]. Another interesting set of proposals to resolve the DE puzzle is ‘modified gravity’ (including $F(T)$, $F(R)$, $F(G)$, etc.) which was proposed after the failure of general relativity to explain the DE puzzle. This new set of gravity theories passes several solar system and astrophysical tests [24–32].

A gravitational theory can be constructed on a non-Riemannian (Weitzenbock) manifold where the properties of gravity are determined through torsion of spacetime and not curvature. Some earlier attempts in this direction were made by Einstein himself and other researchers. A recent version of torsion-based gravity is $F(T)$ [33, 34], where $T$ is the torsion scalar constructed from the tetrad. Choosing $F(T) = T$, leads to teleparallel gravity [35–37] and is in good agreement with some standard tests of general relativity in solar system [35, 36]. Numerous features of theoretical interest have been studied in this gravity model already including Birkhoff’s theorem [38], cosmological perturbations [39] and phantom crossing of the state parameter [40]. Moreover, local Lorentz invariance is violated, which henceforth leads to violation of the first law of thermodynamics [41, 42]. Also the entropy–area relation in this gravity model takes a modified form [43]. The Hamiltonian structure of $F(T)$ gravity has been investigated and it was found that there are five degrees of freedom [44]. The torsion-based theory is also an alternative candidate to the mechanism of cosmic inflation [45].

In teleparallel gravity, the equations of motion for any geometry are exactly the same as of general relativity. Due to this reason, teleparallel gravity is termed a ‘teleparallel equivalent of general relativity’. In teleparallel gravity, the dark energy puzzle is studied by introducing a scalar field with a potential. If this field is minimally coupled
with torsion, then this effectively describes quintessence dark energy. However, if it is non-minimally coupled with torsion, then a richer dynamics of the field appears in either quintessence- or phantom-like form, or by experiencing a phantom crossing [46]. Xu et al. [47, 48] investigated the dynamics and stability of a canonical scalar field non-minimally coupled with gravity (arising from torsion). They found that the dynamical system has an attractor solution, and rich dynamical behavior was found. In the context of general relativity, a scalar field non-minimally coupled with gravity has been studied in [49]. We here extend these previous studies by replacing $T$ with an arbitrary function $F(T)$. We found that such a dynamical system possesses no stable equilibrium point, however rich the dynamical behavior of quintessence and phantom energy is that is observed.

We follow the following plan: In Sect. 2 we write the action and equations of motion of our model. In Sect. 3, we give our motivation for choosing particular forms of model functions. In Sect. 4, we write the dynamical system in dimensionless form and discuss its stability and phase space behavior. In Sect. 5, we discuss the cosmological implications of our model related to the present accelerated Universe. We provide conclusions in Sect. 6.

### 2 Basic equations

We are interested in conformally invariant models, i.e. models which remain invariant under conformal transformation. Under the rules $e_\mu' \rightarrow \Omega(x)e'_\mu$ (or $g_{\mu\nu} \rightarrow \Omega(x)g_{\mu\nu}$), $\phi \rightarrow \Omega(x)^{-1}\phi$, the equations of motion and the whole action remain invariant of a conformally invariant model. Here $e'_{\mu}$ is the tetrad (vierbein) basis. In general, for a $D$-dimensional gravity model, the conformal coupling parameter is $\xi = \frac{D_{\text{conf}} - 2}{D_{\text{conf}} - 1}$, so that in a four dimensional theory $\xi = \frac{1}{6}$ [50]. Furthermore, $F(T)$ theories from a dynamical point of view are completely different under conformal transformation, unlike $f(R)$ theory. It is not possible to rewrite the total action of $F(T)$ gravity in the form of teleparallel action plus scalar field [51]. It shows that even the pure $F(T)$ model behaves differently under conformal transformation. We propose an action of $F(T)$ gravity conformally and non-minimally coupled with a scalar field by

$$S = \int d^4x \, e \left[ \frac{F(T)}{2} \left( 1 + \xi \phi^2 \right) + \frac{1}{2} e \dot{\phi}^2 - V(\phi) + \mathcal{L}_m \right],$$

where $e = \det(e'_\mu)$. Here $\xi$ is a conformal coupling parameter, of order unity, while $e = +1, -1$ represents quintessence and phantom energy, respectively. The dynamical quantity of the model is the tetrad and scalar field $\phi$ with a scalar potential $V(\phi)$. $\mathcal{L}_m$ is the matter Lagrangian. It is assumed that both matter and scalar field are distributed as a perfect fluid. The action (1) can be considered as a generalization of a teleparallel gravity non-minimally coupled with a scalar field [47, 48]. We add some more comments on our model: the minimally coupled scalar fields are not conformal invariant, otherwise $F(T)$ gravity is not locally Lorentz invariant. After the Lorentz symmetry breaking, we prefer our model to have conformal symmetry. Just for pure $F(T)$, it is well-known that it may be possible to write the pure $F(T)$ action in a conformal gauge, like $F(R)$. If we couple matter (here scalar field) with $F(T)$, this matter part must have the same (conformal) symmetry, which can be interpreted as a generalized conformal invariance.

We assume a spatially flat Friedmann–Robertson–Walker (FRW) metric as a background spacetime,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

where $a(t)$ is a scale factor and $e'_\mu = (1, a(t), a(t), a(t))$. The equations of motion are obtained by varying the action (1) w.r.t. $a(t)$ and $\phi(t)$, and we get

$$6H^2 F(T) \left( 1 + \xi \phi^2 \right) + \frac{1}{2} \left( 1 + \xi \phi^2 \right) F = \rho_m + \frac{1}{2} \xi \phi^2 + V(\phi),$$

$$\ddot{\phi} + 3H \dot{\phi} - e \left[ \xi \phi F - V'(\phi) \right] = 0,$$

which are the Friedmann and Klein–Gordon equations, respectively.

The second Friedmann equation is

$$\frac{\ddot{a}}{a} = \frac{1}{4a(1 + \xi \phi^2)(-12F_{TT}\dot{a}^2 + a^2F_T)} \times \left[ -8a\dot{a}F_T \left( 1 + \xi \phi^2 \right) + \xi a\phi \right. -48(1 + \xi \phi^2)\dot{a}^2a^{-1}F_{TT} - a^2 \left( (1 + \xi \phi^2)F + e\dot{\phi}^2 - 2V(\phi) \right).$$

We can rewrite (3) as

$$3H^2 = \rho_{\phi}^{\text{eff}} + \rho_m,$$

or

$$1 = \Omega_{\phi}^{\text{eff}} + \Omega_m, \quad \Omega_{\phi}^{\text{eff}} = \frac{\rho_{\phi}^{\text{eff}}}{3H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3H^2},$$

where $\rho_{\phi}^{\text{eff}}$ is the effective energy density of the scalar field, written as

$$\rho_{\phi}^{\text{eff}} = \frac{1}{2} \phi^2 + V(\phi) - \frac{T}{2} + \frac{(1 + \xi \phi^2)}{2} \left( 2T F_T - F \right).$$

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1Here $8\pi G = 1$.  

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Combining Eqs. (5) and (7), we get
\[ \frac{\ddot{a}}{a} + H^2 = -p_{\phi}^{\text{eff}}, \]
(9)
where \( p_{\phi}^{\text{eff}} \) is the effective pressure of scalar field given by
\[ p_{\phi}^{\text{eff}} = \frac{T}{6} - \frac{1}{2a(1 + \xi \phi^2)(-12F_T \dot{a}^2 + a^2 F_T)} \times \left[ -8a \dot{a} F_T ((1 + \xi \phi^2) + \xi a \phi \dot{\phi}) - 48(1 + \xi \phi^2) \dot{a}^4 a^{-1} F_T T - a^3 ((1 + \xi \phi^2) F + \epsilon \dot{\phi}^2 - 2V(\phi)) \right]. \]
(10)
The deceleration parameter for this model is
\[ q = -\frac{\ddot{a}}{a H^2} = \frac{-1}{4a(1 + \xi \phi^2)(-12F_T \dot{a}^2 + a^2 F_T)H^2} \times \left[ -8a \dot{a} F_T ((1 + \xi \phi^2) + \xi a \phi \dot{\phi}) - 48(1 + \xi \phi^2) \dot{a}^4 a^{-1} F_T T - a^3 ((1 + \xi \phi^2) F + \epsilon \dot{\phi}^2 - 2V(\phi)) \right]. \]
(11)

3 Choice of \( F(T) \) and \( V(\phi) \)

We pick a suitable \( F(T) \) expression which contains a constant, is linear, and has a non-linear form of torsion; specifically [52]

\[ F(T) = 2c\sqrt{-T} + aT + C_2, \]
(12)
where \( a, \epsilon \) and \( C_2 \) are arbitrary constants.\(^2\) The first and the third terms (excluding the middle term) have correspondence with the cosmological constant EoS in \( F(T) \) gravity [53]. There are many such kinds of model, reconstructed from different kinds of dark energy model. For example, this form, (12), may be inspired by a model for dark energy from the form proposed for the Veneziano ghost [54]. Recently Capozziello et al. [55] investigated the cosmography of \( F(T) \) cosmology by using data of BAO, Ia supernovae and WMAP. Following their interesting results, we notice that if we choose \( 2c \equiv \sqrt{6}H_0(\Omega_{m0} - 1) \), then one can estimate the parameters of this \( F(T) \) model as a function of Hubble parameter \( H_0 \), the cosmographic parameters, and the value of the matter density parameter. It is interesting to note that reconstruction of the \( F(T) \) model according to holographic dark energy [56] leads to the same model as (12).

\(^2\)Here \( c \) does not represent the speed of light.

Concerning the scalar potential, we choose an exponential function which has numerous implications in cosmological inflation [57–60] and dark energy in the present Universe [61–65],
\[ V(\phi) = V_0 e^{\beta \phi}, \]
(13)
where \( \beta \) and \( V_0 \) are constants. Using (13), it has been shown in the literature [61–65] that a transition of the dark energy state parameter across the cosmological boundary is possible. Also this exponential potential is useful in assisting inflation. Without loss of generality, we assume \( \beta > 0 \). But here we focus only on late-time evolution of the scalar field and we do not give any prediction of our model for inflation. In the limit \( \beta \to 0 \), we recover the constant potential case and therefore the model is continuously connected with \( \Lambda \)CDM. We remark, that our choices for scalar potential and the \( F(T) \) may independently be responsible for DE, but in the present context of non-minimal coupling, both scalar field and \( F(T) \) couple non-minimally to generate the desired effect of cosmic acceleration.

4 Analysis of stability in phase space

We define the dimensionless density parameters by
\[ x \equiv \frac{\phi}{\sqrt{6}H}, \quad y \equiv \frac{\sqrt{\dot{V}}}{\sqrt{3}H}, \quad z \equiv \sqrt{\epsilon} \phi. \]
(14)
Here \( x^2 \) and \( y^2 \) represent the density parameters of the kinetic and potential terms, respectively. We expect the interesting cases to have the scalar field rolling down the slope of the potential, and as \( x > 0 \), we should have \( x > 0 \). The equations in dimensionless variables reduce to
\[
\frac{dx}{dN} = -2x - \frac{\sqrt{6}x^{-1}z}{\epsilon} - \sqrt{6}xz^{-1}\frac{\beta x}{\epsilon} - \frac{1}{2} \frac{\beta \sqrt{6}y^2}{\epsilon} - x \left( -\alpha \left( 1 + 1/2 z^2 \right) - \frac{1}{2} - z^2 \right) - 2 \xi \sqrt{6} \xi^{-1}(\alpha + 1)xz - 6 \epsilon x^2 + \frac{3}{2} y^2 + 2 \xi \epsilon x \sqrt{3} \xi^{-1}yz e^{1/2 \beta \xi e^{-1}} \left( \alpha + 1 \right)^{-1} \left( 1 + z^2 \right)^{-1},
\]
(15)
\[
\frac{dy}{dN} = -\frac{3}{2} \beta \sqrt{6} y x - y \left( -\alpha \left( 1 + 1/2 z^2 \right) - \frac{1}{2} - z^2 \right) - 2 \xi \sqrt{6} \xi^{-1}(\alpha + 1)xz - 6 \epsilon x^2 + \frac{3}{2} y^2 + 2 \xi c x \sqrt{3} \xi^{-1}yz e^{1/2 \beta \xi e^{-1}} \left( \alpha + 1 \right)^{-1} \left( 1 + z^2 \right)^{-1},
\]
(16)
Fig. 1 (Top left) Three dimensional phase portrait of the dynamical system with $\epsilon = +1$. (Top right) Phase space diagram of the dynamical system with $\epsilon = -1$. (Bottom left) Time evolution of dynamical parameters for $\epsilon = +1$. (Bottom right) Time evolution of dynamical parameters for $\epsilon = -1$. In the lower two figures, $x, y, z$ are represented by a solid, dot, and dot-dash line, respectively

\[
\frac{dz}{dN} = \sqrt{6\xi} x, \quad (17)
\]

where $N \equiv \ln a$ is called the e-folding parameter. To discuss the stability of the system (15)–(17), we first obtain the critical points by solving the equations ($\frac{dx}{dN} = 0, \frac{dy}{dN} = 0, \frac{dz}{dN} = 0$). We linearize the system near the critical points up to first order. After constructing a Jacobian matrix of coefficients of the linearized system, we find its eigenvalues. If all eigenvalues are negative, the corresponding critical point is stable (attractor), otherwise we have an unstable point.

In Table 1, we present the critical points, the corresponding eigenvalues, and the stability condition. We notice that the system of dynamical equations admit no stable critical point. The first critical point A is trivial, while point B contains an undetermined component $z_\ast$. For A, $\lambda_1, \lambda_2 > 0, \lambda_3 = 0$, while for B, $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 = 0$; thus, both A and B are unstable points.

5 Cosmological implications

In this section, we will give some cosmological implications of our model by numerically solving the dynamical equa-
Table 1 Critical points and stability conditions

| Point  | $(x, y, z)$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | Stability condition |
|--------|-------------|-------------|-------------|-------------|-------------------|
| A      | (0, 0, 0)   | $\frac{5+4\epsilon}{2(1+\epsilon)}$ | $\frac{2\epsilon+1}{2(1+\epsilon)}$ | 0             | Unstable          |
| B      | (0, 0, $z_*$) | $\frac{z^2(4-5\epsilon+5\epsilon^2)}{2(1+z^2)(\epsilon-1)}$ | $\frac{z^2(\epsilon-2)+2\epsilon-1}{2(1+z^2)(\epsilon-1)}$ | 0             | Unstable          |

Fig. 2 (Top left) Time evolution of scalar fields: phantom energy (red) and quintessence (blue). (Top right) Time evolution of energy densities for small time: phantom energy (blue) and quintessence (black). (Bottom left) Time evolution of energy density of quintessence field for large times. (Bottom right) Time evolution of energy density of phantom energy for large times.

In the top panel of Fig. 1, the three dimensional phase space of $x$, $y$, $z$ is plotted for two different values of $\epsilon$. The top left figure (for $\epsilon = -1$) shows that the trajectory starts from $x = -10 < 0$ causing a negative friction term $-3H\dot{\phi}$ in the Klein–Gordon equation. The friction term gets less dominant, while scalar field and potential energy parameters dominate in the later evolution. It shows that the late stage evolution is determined by the scalar potential alone. This fact is also evident from the bottom left figure; it shows that only the potential term dominates the dynamics, while scalar field and the kinetic term do not contribute in cosmological dynamics. However, putting $\epsilon = +1$ in the dynamical...
cal system reverses the dynamical evolution of the system, as shown in the right panel in Fig. 1. The scalar potential remains vanishing, while the cosmic dynamics is determined by the kinetic term $\dot{\phi}$ and the scalar field. It shows that $\epsilon = +1 (-1)$ leads to regimes dominated by a kinetic term (scalar potential) in the late-time evolution, despite that the system evolves from the same initial conditions ($a(0) = 1$, $\dot{a}(0) = H_0 = 74.2$, $\phi(0) = 1$, $\dot{\phi}(0) = 1$, $\xi = 1/6$).

In Fig. 2 we show the time evolution of scalar fields and logarithmic energy densities for $\epsilon = \pm 1$. The top left figure shows that for phantom energy, the field undergoes successive stages of fluctuations after a time gap of unity. Here the amplitude of phantom scalar field in each successive stage becomes progressively less than the previous stage. In the late-time evolution, it is expected that the field will lose its energy, and fluctuations decrease, as seen in bottom right figure. However, for quintessence, the top left figure shows that the field decays for $t < 0.5$, and the corresponding log energy density stays around $-1$ ($\rho \sim e^{-t}$).

We define the equation of state parameter of the scalar field by

$$w_{\text{eff}} \equiv \frac{p_{\phi}^{\text{eff}}}{\rho_{\phi}^{\text{eff}}},$$

where

$$p_{\phi}^{\text{eff}} = \frac{\epsilon}{2} \phi^2 + V(\phi) - 3H^2[(1 + \alpha)\xi \phi^2 + \alpha],$$

$$\rho_{\phi}^{\text{eff}} = \frac{-2}{(1 + \alpha)(1 + \xi \phi^2)}\left[-\frac{(\alpha + 1)}{2} H^2(1 + \xi \phi^2) - 2\xi(1 + \alpha)\phi \dot{\phi} H - \frac{\epsilon}{4}\phi^2 + \frac{1}{2} V(\phi) + \frac{c\sqrt{6}\xi}{3} \phi \dot{\phi}\right] - \frac{1}{3}\left[\rho_m + \frac{1}{2} \epsilon \dot{\phi}^2 + V(\phi) - 3H^2[(\alpha + 1)\xi \phi^2 + \alpha]\right].$$

In Fig. 3, we plot the time evolution of the effective EoS parameter of the scalar field $w_{\text{eff}}$ for two available values of $\epsilon$. Adopting the same initial conditions as in previous figures, we observe that the state parameters for both models evolve from the same initial value $w_{\text{eff}}(0) \approx -2$, which is a phantom state for the state parameter. However, the later evolution follows an opposite trajectory, i.e., for $\epsilon = -1$, the state parameter goes on to take more negative values with time, thereby evolving to a super-phantom state. For $\epsilon = +1$, the state parameter initially behaves like phantom and slowly evolves to cross the $w_{\text{eff}} = -1$ boundary, while again returning to the phantom regime in the late-time evolution.

6 Conclusion

To summarize, we investigated the stability and phase space of a non-minimally conformally coupled scalar field in $F(T)$ cosmology. We found that the dynamical system of equations admits two unstable critical points; thus no attractor solutions exist in this cosmology. Treating the scalar field as quintessence and phantom energy separately, we found that for phantom energy, the late stage evolution is determined by the potential energy in phase space, while for the quintessence case, it is the kinetic energy term that plays a central role. Furthermore, the time evolution of the phantom scalar field undergoes progressive stages of fluctuations while losing energy density and ending up at the value $e^4$ in the relevant scale. For quintessence, the scalar field remains stable for a long time with a constant energy density. It is interesting to note that our model correctly predicts that the present state of the Universe is dominated by phantom energy, while it will remain so in the far future. Furthermore, we observed that the transition of the state parameter crosses the cosmological boundary twice in the case of a quintessence field only. Also we noted that $\epsilon = +1 (-1)$ leads to regimes dominated by the kinetic term (scalar potential) in the late-time evolution, despite that the system evolves from the same initial conditions.

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