MHD FLOW OF FRACTIONAL NEWTONIAN FLUID EMBEDDED IN A POROUS MEDIUM VIA ATANGANA-BALEANU FRACTIONAL DERIVATIVES

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Abstract. The novelty of this research is to utilize the modern approach of Atangana-Baleanu fractional derivative to electrically conducting viscous fluid embedded in porous medium. The mathematical modeling of the governing partial differential equations is characterized via non-singular and non-local kernel. The set of governing fractional partial differential equations is solved by employing Laplace transform technique. The analytic solutions are investigated for the velocity field corresponding with shear stress and expressed in term of special function namely Fox-$H$ function, moreover a comparative study with an ordinary and Atangana-Baleanu fractional models is analyzed for viscous flow in presence and absence of magnetic field and porous medium. The Atangana-Baleanu fractional derivative is observed more reliable and appropriate for handling mathematical calculations of obtained solutions. Finally, graphical illustration is depicted via embedded rheological parameters and comparison of models plotted for smaller and larger time on the fluid flow.

1. Introduction. It is well established fact that the idea of fractional calculus has diverted the attention of scientists and researchers because of its crucial applications for the descriptions of complex system and in different academic disciplines. The modeling via integer-order derivatives does not provide better prediction in comparison to modeling via fractional-order derivatives in the real-world problems. The modelling of physical phenomena via fractional-order derivatives is significant for the control theory [44], pharmacokinetics [42], electrical engineering [22], [25], anomalous diffusion [14], [2], [19], [23], fluids [10], [26], [40], [41], electromagnetism [11], [27], [32], [28], [33], heat transfer [13], [3], [34]. Although fractional derivatives are more suitable than ordinary derivatives to describe a physical phenomenon yet

2010 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Atangana-Baleanu fractional derivative, magnetohydrodynamics, porous medium, rheological effects.

The author Kashif Ali Abro is highly thankful and grateful to Mehran University of Engineering and Technology, Jamshoro, Pakistan, for generous support and facilities of this research work.

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some discrepancies in the application of Capotu-Fabrizio, Caputo and Riemann-Liouville fractional derivatives have been vividly pointed out by Atangana et al. [20], [21]. Keeping the drawbacks and pitfalls raised by Atangana et al. [20], [21], Atangana-Baleanu fractional derivatives with the non-singular and non-local kernel introduced a new definition [4]. Inspiring from this new approach to fractional derivatives namely Atangana-Baleanu fractional derivative, the authors noted the key points as: (i) The Caputo-Fabrizio derivative is non-Markovian and the well-known Riemann-Liouville derivative is just Markovian. While, Atangana-Baleanu fractional derivative has at the same time Markovian and Non-Markovian properties. (ii) Caputo-Fabrizio is only exponential decay and Riemann-Liouville derivative is only power law. While Atangana-Baleanu fractional derivative waiting time is at the same time power law, stretched exponential and Brownian motion. (iii) Riemann-Liouville is just power law and scale-invariant, while Atangana-Baleanu fractional derivative’s mean square displacement is a crossover from usual diffusion to sub-diffusion. This means the Atangana-Baleanu fractional derivative is able to describe real-world problems with different scales. For instance, movement of pollution within fractured aquifers, the flow of water within heterogeneous aquifers, the spread of cancer. Muzaffar et al. [35] investigated the helical effects for the fractionalized viscoelastic fluid in helically moved cylinder using Caputo fractional derivative in which study of Newtonian and non-Newtonian fluids is presented for rotational and oscillating flows of circular pipe. Zafar and Fetecau [43] investigated the flow of viscous fluid using Caputo-Fabrizio fractional derivative, here they used integral transforms (Laplace and Fourier sine transforms). Nadeem et al. [36] observed the effects of magnetic field for the flow of a second-grade fluid via Caputo-Fabrizio fractional derivative in presence of radiative heat transfer. Atanagan and Baleanu [15] explored the effects of Groundwater Flow within Confined Aquifer by implementing the Caputo-Fabrizio fractional derivative. Nehad and Ilyas [39] investigated the thermal analysis of a second-grade fluid by using Caputo-Fabrizio fractional derivative. In this study they explored that the heat transfer is caused by the buoyancy force induced by temperature differences between the plate and the fluid. Atangana [16] investigated nonlinear Fisher’s reaction-diffusion by employing the Caputo-Fabrizio derivative and presented the stability and Some numerical simulations. Qasem et al. [37] presented Walter’s Liquid Model-B by applying newly defined approach of Caputo-Fabrizio fractional derivative (CFFD) in which combined analysis of heat and mass transfer together with magnetohydrodynamic (MHD) flow embedded in a porous medium over a vertically oscillating plate is investigated. On the other hand, Algahtani [9] presented the comparison of the model by Allen-Cahn with both Atanagan-Baleanu and Caputo-Fabrizio derivatives in order to see their difference in real world problem. Kashif et al. [29] investigated the Stoke’s second problem for nanofluids by employing newly suggested Atanagan-Baleanu fractional derivative. They considered the model of homogeneous type with nanosized copper (Cu) particles suspended in ethylene glycol (EG). In brevity, the studies can be continued on fractional derivatives but we end here by citing few recent studies [5], [6], [17], [18], [7], [12], [8], [24], [30], [38]. Inspiring and motivating from the above discussed studies, the purpose of this research work is to implement a novel definition of Atangana-Baleanu fractional derivative on electrically conducting viscous fluid embedded in porous medium. The mathematical modeling of governing partial differential equations is characterized via non-singular and non-local kernel. The analytic solutions are investigated for velocity field and shear
stress by invoking Laplace transform satisfying imposed conditions and presented in terms of Fox-H function. The Atangana-Baleanu fractional derivative is observed more reliable and appropriate for handling mathematical calculations of obtained solutions either in presence or absence of magnetic field and porous medium. Finally, graphical illustration is depicted via embedded rheological parameters and comparison of models plotted for smaller and larger time on the fluid flow.

2. Statement of the problem. Assuming an oscillating plate lying in the $xy$ plane in a fixed cartesian coordinate system in which positive $y$-axis is in the upward direction, a generalized incompressible, electrically conducting, viscous fluid having porous medium is at rest over it at $t = 0$. For time $t = 0^+$, the plate begins to displace in its plane with a time dependent oscillating velocity $u(0, t) = R_0 \sin(\omega t)$ where $R_0$ is a constant with dimension of dependent oscillating velocity. Due to the tangential stress the fluid is also moved and its velocity is of the form of

$$v = v(y, t) = (u(y, t), 0, 0),$$

For such motions, the continuity equation is identically verified while the motion and constitutive relations generate the suitable governing equations as

$$\frac{\partial u(y, t)}{\partial t} = \nu \frac{\partial^2 u(y, t)}{\partial y^2} - \left( \frac{\phi \nu}{k} - \frac{B_0^2 \sigma}{\rho} \right) u(y, t),$$

where, $\nu, \phi, k, B_0, \sigma, \rho$ are kinematic viscosity, porosity, permeability of the medium, applied magnetic field, electrical conductivity, constant density of the fluid respectively. In this continuation, the imposed initial and boundary conditions are

$$u(0, t) = R_0 \sin(\omega t), u(y, 0) = 0, u(y, t) \to 0, \frac{\partial u(y, t)}{\partial t} \to 0 \text{ as } y \to \infty,$$

In order to investigate the dimensionless of equations (2 – 3), we implement the below dimensionless parameters in which $\lambda$ has dimension of time. The suitable dimensionless parameters are

$$\gamma = \frac{\nu}{R_0}, \lambda = \frac{\nu}{R_0^2}, y^* = \frac{y}{\gamma}, t^* = \frac{t}{\lambda}, u^* = \frac{u}{R_0}, \tau^* = \frac{\tau}{R_0^2 \rho},$$

Using equation (4) on equations (2 – 3) with newly defined Atangana-Baleanu time fractional operator, we obtain the final expression of governing fractional differential equations with the imposed initial and boundary conditions as $[43]$

$$\mathcal{A}_B \left( \frac{\partial^\alpha u(y, t)}{\partial t^\alpha} \right) + K_{eff} \ u(y, t) = \nu \frac{\partial^2 u(y, t)}{\partial y^2},$$

$$\tau(y, t) = \mu \frac{\partial u(y, t)}{\partial y},$$

$$u(0, t) = R_0 \sin(\omega t), u(y, 0) = 0, u(y, t) \to 0, \frac{\partial u(y, t)}{\partial t} \to 0 \text{ as } y \to \infty,$$

Where, $\frac{1}{K} = \frac{\lambda \phi \nu}{k}, M = \frac{\lambda B_0^2 \sigma}{\rho}, K_{eff} = \left( \frac{1}{k} + M \right)$ and $\mathcal{A}_B \left( \frac{\partial^\alpha u(y, t)}{\partial t^\alpha} \right)$ is the Atangana-Baleanu fractional differential operator of order defined as $[1]$

$$\mathcal{A}_B \left( \frac{\partial^\alpha u(y, t)}{\partial t^\alpha} \right) = \int_0^t E_\alpha \left( -\alpha (z - t)^\alpha \right) \frac{\partial^\alpha u(y, t)}{1 - \alpha} dt, \ 0 < \alpha < 1.$$
3. Investigation of the problem.

3.1. Calculation of velocity field via Atangana-Baleanu fractional differential operator. Taking Laplace transform on Atangana-Baleanu fractional differential equation of velocity field (5) having in mind the imposed conditions (3), we arrive

$$
\left( \frac{\partial^2}{\partial y^2} - \frac{(\nu - K_{eff})s^\alpha - \alpha \beta K_{eff}}{\nu (s^\alpha + \alpha \beta)} \right) \tilde{u}(y, s) = 0,
$$

(9)

Applying the initial and boundary conditions say equations (3), the solution of differential equation (9) is obtained as

$$
\tilde{u}(y, s) = \frac{R_0 \omega}{s^2 + \omega^2} \exp \left( -y \sqrt{\frac{(\nu - K_{eff})s^\alpha - \alpha \beta K_{eff}}{\nu (s^\alpha + \alpha \beta)}} \right),
$$

(10)

developing equation (10) into equivalent form by using series expansion, we have suitable expression as

$$
\tilde{u}(y, s) = \sum_{s_1=1}^{\infty} \sum_{s_2=0}^{\infty} \frac{(-\alpha \beta)^{s_1} \Gamma \left( \frac{s_1}{2} + 1 \right) \Gamma \left( \frac{s_2}{2} + s_3 \right)}{s_3 \Gamma \left( \frac{s_3}{2} - s_2 + 1 \right) \Gamma (s_3 s_2)} \left( \frac{\alpha s_2}{\nu - K_{eff}} \right)^{s_2}.
$$

(11)

inverting equation (11) by Laplace transform and expressing the final version of velocity field in terms of Fox-H function and product of convolution, we have

$$
u(y, t) = R_0 \sin(\omega t) + R_0 \sum_{s_1=1}^{\infty} \left( -y \sqrt{\frac{\nu - K_{eff}}{\nu}} \right)^{s_1} \sum_{s_2=0}^{\infty} \frac{1}{s_2!} \left( \frac{\alpha s_2}{\nu - K_{eff}} \right)^{s_2}
\times \int_0^t \sin \omega(t - z) H_{2,4}^{1,2} \left[ -\alpha \beta \left( \frac{-z}{\nu}, 0, (1 - \frac{z}{\nu}, 1) \right), \left( 1, 1, (1 - \frac{z}{\nu}, 0), (s_2 - \frac{z}{\nu}, 0), (1 - \alpha s_2, \alpha) \right) \right]
\times t^{\alpha (s_3 + s_2) - 1} dz,
$$

(12)

where, the property of Fox-H function is defined as in literature [31]

$$
\sum_{s_3=0}^{\infty} \frac{(-Z)^{q \Pi_{h=1}^{1} \Gamma (a_h + A_h q)}}{q! \Pi_{h=1}^{1} \Gamma (b_h + B_h q)} = H_{j, k+1}^{1, f} \left[ Z \right]_{(a_1, A_1), (1 - a_1, A_1), \ldots, (1 - a_f, A_f), (1 - b_1, B_1), \ldots, (1 - b_g, B_g)}.
$$

(13)

3.2. Calculation of shear stress via Atangana-Baleanu fractional differential operator. Taking Laplace transform on equation of shear stress (6) having in mind the imposed conditions (3) and differentiating the final expression of velocity field with respect to “y” partially, we arrive

$$
\tau(y, s) = -\frac{\mu R_0 \omega}{s^2 + \omega^2} \sqrt{\frac{(\nu - K_{eff})s^\alpha - \alpha \beta K_{eff}}{\nu (s^\alpha + \alpha \beta)}}
\times \exp \left( -y \sqrt{\frac{(\nu - K_{eff})s^\alpha - \alpha \beta K_{eff}}{\nu (s^\alpha + \alpha \beta)}} \right),
$$

(14)
developing equation (14) into equivalent form by using series expansion, we have suitable expression as

\[
\tau(y,s) = -\frac{\mu_0\omega(\sqrt{\nu\beta - K_{eff}})^\alpha}{\sqrt{\nu}(s^2 + \omega^2)} \sum_{s_1=0}^{\infty} \frac{1}{s_1!} \left( -y(\sqrt{\nu\beta - K_{eff}})^\alpha \right)^{s_1} \\
\times \sum_{s_2=0}^{\infty} \frac{1}{s_2!} \left( \frac{K_{eff}\alpha\beta}{\nu\beta - K_{eff}} \right)^{s_2} \sum_{s_3=0}^{\infty} \frac{(-\alpha\beta)^s_3\Gamma\left(\frac{\alpha}{\beta} + \frac{3}{2}\right)\Gamma\left(\alpha + \frac{s_3 + \frac{1}{2}}{2}\right)}{s_3!\Gamma\left(\frac{s_3 + \frac{3}{2}}{2}\right)} y^{s_3+s_2},
\]

(15)

inverting equation (15) by Laplace transform and expressing the final version of shear stress in terms of Fox-H function and product of convolution, we have

\[
\tau(y,t) = -\frac{\mu_0(\sqrt{\nu\beta - K_{eff}})^\alpha}{\sqrt{\nu}} \sum_{s_1=0}^{\infty} \frac{1}{s_1!} \left( -y(\sqrt{\nu\beta - K_{eff}})^\alpha \right)^{s_1} \\
\times \sum_{s_2=0}^{\infty} \frac{1}{s_2!} \left( \frac{K_{eff}\alpha\beta}{\nu\beta - K_{eff}} \right)^{s_2} \int_0^t \sin\omega(t-z) \left( t^\alpha(s_3 + s_2) - 1 \right) dz.
\]

(16)

Indeed, the equations (12) and (16) represent the general solutions of velocity field and corresponding shear stress with non-singular and non-local kernel respectively which satisfies initial and boundary conditions as well. In this continuity, the equations (12) and (16) can immediately generate some unconcealed computations when \(K_{eff} = 0\) is taken into the equations (12) and (16), such solutions have been obtained by [43]. In brevity, the expression of the equations (12) and (16) can also be retrieved in the absence of Atangana-Baleanu fractional differential operator by letting \(\alpha = 1\).

4. Parametric results. In this section, the critical observation is performed to understand the flow behavior of electrically conducting viscous fluid embedded in porous medium with several similarities and differences. The effects of velocity field are analyzed for viscous flow in terms of embedded rheological parameters which are enumerated below

Effects of Atangana-Baleanu fractional differential operator on the velocity field

In order to get some information for the effects of Atangana-Baleanu fractional differential operator on the velocity field, different flow parameters are fixed and Atangana-Baleanu fractional differential operator is varied in Fig. 1. The velocity field is increasing function with respect to increasing Atangana-Baleanu fractional parameter \(\alpha = 0.2, 0.4, 0.6\). Physically, the smaller the values of \(\alpha\), the more rapidly the velocity does not decay. This may be due to fact of fractional order derivatives which examines the complete description of the memory effectively.

Effects of porous medium on the velocity field

Fig. 2 is depicted to shows the influence of porous medium on the velocity field. Here porous medium plays a significant and physical role on the velocity field in presence of magnetic field. It is seen from Fig. 2 that the effect of the porosity is to reduce the velocity field over the boundary. Vividly, increase in the porosity decreases the behavior of fluid flow.
Effects of magnetic field on the velocity field

Figs. 3 illustrate the influence of magnetic field parameter on the velocity field. It is observed that increasing values of the magnetic field parameter declines the velocity field. This is due to fact of Lorentz force because due to the drag force developed by Lorentz force decelerates the fluid flow. This leads to the conclusion that the magnetic field effectively controls flow of fluid over an infinite oscillating plate. On the other hand, reciprocal trend has been observed for fluid flow in comparison with porous flow effects.

Comparison of ordinary and Atangana-Baleanu fractional models

The comparison of analytical solutions of ordinary and Atangana-Baleanu (AB) fractional differential operator is performed on the velocity field for three different times, i.e., smaller time $t = 0.02s$, unit time $t = 1s$ and larger time $t = 5s$ as depicted in Figs. 4, 5 and 6. For the smaller time $t = 0.02s$, the ordinary model moves faster in comparison with Atangana-Baleanu (AB) fractional differential operator. While, for the larger time $t = 5s$, the Atangana-Baleanu (AB) fractional differential operator model moves faster in comparison with ordinary model. It is also noted that an interesting fact is achieved when both ordinary and Atangana-Baleanu (AB) fractional differential operator are analyzed for unit time $t = 1s$, then the both operator have coincident behavior. Meanwhile, the same phenomenon can also be analyzed for shear stress as well.

5. Conclusion. The main findings of this article are enumerated as below

(i). The modeled governing partial differential equations of magnetohydrodynamic viscous fluid in porous have been transformed in dimensionless form using modern method of Atangana-Baleanu (AB) fractional derivatives. The general analytic solutions are investigated for velocity field and shear stress by employing Laplace transform with inversion.

(ii). The velocity field is increasing function with respect to increasing Atangana-Baleanu fractional parameter and magnetic field.

(iii). The effect of the porosity is to reduce the velocity field over the boundary. Moreover, increase in the porosity decreases the behavior of fluid flow.

(iv). The comparison of analytical solutions of ordinary and Atangana-Baleanu (AB) fractional differential operator is performed on the velocity field for three different times, i.e., smaller time $t = 0.02s$, unit time $t = 1s$ and larger time $t = 5s$, in which Atangana-Baleanu (AB) fractional model is swiftest for larger time in comparison with ordinary fluid model.

Acknowledgments. The author Kashif Ali Abro is highly thankful and grateful to Mehran University of Engineering and Technology, Jamshoro, Pakistan, for generous support and facilities of this research work.

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Figure 1. Profile of velocity field via Atangana-Baleanu fractional differential operator for fractional parameter.

Figure 2. Profile of velocity field via Atangana-Baleanu fractional differential operator for porous medium.
Figure 3. Profile of velocity field via Atangana-Baleanu fractional differential operator for magnetic field.

Figure 4. Comparative analysis of velocity field via Atangana-Baleanu fractional differential operator verses ordinary differential operator for short time.
Figure 5. Comparative analysis of velocity field via Atangana-Baleanu fractional differential operator versus ordinary differential operator for unit time.

Figure 6. Comparative analysis of velocity field via Atangana-Baleanu fractional differential operator versus ordinary differential operator for larger time.
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Received May 2018; revised September 2018.

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