ELLIPTIC OPERATORS AND K-HOMOLOGY

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Abstract. If a differential operator $D$ on a smooth Hermitian vector bundle $S$ over a compact manifold $M$ is symmetric, it is essentially self-adjoint and so admits the use of functional calculus. If $D$ is also elliptic, then the Hilbert space of square integrable sections of $S$ with the canonical left $C(M)$-action and the operator $\chi(D)$ for $\chi$ a normalizing function is a Fredholm module, and its $K$-homology class is independent of $\chi$. We provide details to the proof of this fact that have been missing in the literature.

1. Introduction

A differential operator $D$ acting on the sections of a smooth Hermitian vector bundle $S \to M$ over a compact manifold $M$ can be regarded as an unbounded operator on the Hilbert space $L^2(M; S)$ of square integrable sections of $S$. If $D$ is symmetric, then it is automatically essentially self-adjoint and hence we can use functional calculus. If $D$ is also elliptic, then $L^2(M; S)$ with the canonical left $C(M)$-action by multiplication and the operator $\chi(D)$ for $\chi$ a normalizing function turns out to be a Fredholm module over $C(M)$, whose $K$-homology class $[D]$ is independent of the choice of $\chi$. The goal of this paper is to give the details of the proof of [6, Thm. 10.6.5] in the compact case in order to make it more accessible. In particular, we compile the definitions and constructions from [6, § 8, § 9] that are needed to understand the theorem, we elaborate on aspects which are sparse on details, and we provide complete solutions to two crucial steps, namely [6, Exercise 10.9.1] and [6, Exercise 10.9.3].

We start in Section 2 by defining the $K$-homology groups of a $C^*$-algebra $A$. These groups consist of equivalence classes of triples $(\nu, \mathcal{H}, F)$ where $\nu$ is a representation of $A$ on the Hilbert space $\mathcal{H}$ and $F$ is a bounded operator on $\mathcal{H}$ with additional properties. If $A$ is unital, these can be stated as: $F$ is essentially self-adjoint, is essentially unitary, and essentially commutes with the left $A$-action.

In Section 3 we construct the Cayley Transform for densely defined self-adjoint unbounded operators. We conclude that these operators allow the use of functional calculus.

In Section 4 we first survey differential operators and prove some of their properties; for example, what their commutator with a multiplication operator looks like and that we can define their symbol independently of the choice of charts. In Subsection 4.2 we study Sobolov spaces in order to make sense of (but not prove) Gårding’s Inequality. In Subsection 4.3 we prove the existence of normalizing functions whose distributional Fourier Transforms are supported in arbitrarily...
small intervals around 0, as was claimed in [6] Exercise 10.9.3. This is needed to show that $\chi(D)$ essentially commutes with the left action.

At this point, we are equipped to dive into the proof of the main theorem, cf. Theorem 5.1. Afterwards, we quickly review $K$-theory and the index pairing, in order to conclude the existence of a map on $K$-theory determined by the class $[D]$.

The appendix contains a detailed proof of the existence of Friedrichs’ mollifiers, cf. [6] Exercise 10.9.1. This tool is important to show that $D$ is essentially self-adjoint, so that it makes sense to consider $F = \chi(D)$ in the main theorem.

We should point out that the assumption that $D$ be elliptic is needed solely to invoke Gårding’s Inequality. Therefore, we will not dwell upon ellipticity of $D$, despite it being crucial for the construction of the $K$-homology class $[D]$ and despite the title of this paper.

2. Kasparov’s K-homology

2.1. Gradings. The material of this subsection is from [6] Appendix A.

A $\mathbb{Z}/2\mathbb{Z}$-grading of a vector space $V$ is a direct sum decomposition into two subspaces $V = V^+ \oplus V^-$, the vector space’s even and odd part. We will often just say that $V$ is graded. Equivalently, $V$ is equipped with a vector space automorphism $\gamma$ such that $\gamma^2 = \text{id}_V$, and we obtain the decomposition as $V^\pm = \{v \in V \mid \gamma(v) = \pm v\}$. An element $v \in V$ is called homogeneous if it is in one of these two subspaces, and its degree is defined by

$$\partial v = \begin{cases} 0 & \text{if } v \in V^+, \\ 1 & \text{if } v \in V^- . \end{cases}$$

We define $V^{op}$ to be $V$ as vector space but with reversed grading, that is, $(V^{op})^\pm := V^\mp$. For an endomorphism $T$ of $V$, we write $T^{op}$ when we consider it as an endomorphism of $V^{op}$. The direct sum of two graded spaces $V, W$ is equipped with the grading

$$(V \oplus W)^+ := V^+ \oplus W^+ \quad \text{and} \quad (V \oplus W)^- := V^- \oplus W^- .$$

A Hilbert space is graded if it is graded as a vector space, and its even and odd subspaces are closed and mutually orthogonal. Equivalently, the grading automorphism $\gamma$ is a bounded unitary operator.

**Example 1**: A grading of a Hilbert space $\mathcal{H}$ induces a grading on $\mathcal{B}(\mathcal{H})$ by deeming an operator $T$ even (resp. odd) if $T$ preserves (resp. reverses) the two subspaces. In terms of the grading operator $\gamma$, $T$ is even (resp. odd) if and only if $T \circ \gamma = \gamma \circ T$ (resp. $T \circ \gamma = -\gamma \circ T$). If we think of $\mathcal{B}(\mathcal{H})$ as

$$\begin{pmatrix} \mathcal{B}(\mathcal{H}^+) & \mathcal{B}(\mathcal{H}^-, \mathcal{H}^+) \\ \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-) & \mathcal{B}(\mathcal{H}^-) \end{pmatrix},$$

we see that

$$T \text{ even } \iff T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \text{and} \quad T \text{ odd } \iff T = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} .$$

Moreover,

$$\mathcal{B}(\mathcal{H})^+ \circ \mathcal{B}(\mathcal{H})^+ \subseteq \mathcal{B}(\mathcal{H})^+ \quad \text{and} \quad \mathcal{B}(\mathcal{H})^+ \circ \mathcal{B}(\mathcal{H})^- \subseteq \mathcal{B}(\mathcal{H})^- ,$$

and the adjoint preserves the grading: $T$ is even (resp. odd) if and only if $T^*$ is even (resp. odd). This makes $\mathcal{B}(\mathcal{H})$ graded as a $C^*$-algebra.
Example 2 ([6] Def. 11.2.2): Another example of a graded $C^*$-algebra is the complex Clifford algebra: the complex unital $*$-algebra $\mathbb{C}_n$ is generated by $n$ elements $\varepsilon_1, \ldots, \varepsilon_n$ which satisfy
\begin{equation}
\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0 \text{ for } i \neq j, \quad \varepsilon_i^* = -\varepsilon_i, \quad \text{and} \quad \varepsilon_i^2 = 1.
\end{equation}
By deeming the basis $\{\varepsilon_{j_1}\cdots\varepsilon_{j_k} : j_1 < \ldots < j_k, 0 \leq k \leq n\}$ orthonormal, $\mathbb{C}_n$ becomes a Hilbert space. The left-action by multiplication is then a faithful $*$-representation of $\mathbb{C}_n$ on $\mathbb{C}_n$, which makes it a $C^*$-algebra. An element $\varepsilon_{j_1}\cdots\varepsilon_{j_k}$ is regarded as even (resp. odd) if $k$ is even (resp. odd).

2.2. Fredholm modules. For a separable $C^*$-algebra $A$, we recall the following definitions from [6] §8.1, §8.2.

Definition 1: An (ungraded, or odd) Fredholm module over $A$ is a triple $(\nu, \mathcal{H}, F)$ consisting of
\begin{enumerate}
\item a representation $\nu: A \to \mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, and
\item an operator $F \in \mathcal{B}(\mathcal{H})$ such that $\nu(a)(F^* - F), \nu(a)(F^2 - 1), [\nu(a), F]$ are compact for all $a \in A$.
\end{enumerate}
A graded (or even) Fredholm module is a Fredholm module $(\nu, \mathcal{H}, F)$ over $A$ such that
\begin{enumerate}
\item $\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$-graded,
\item the operator $F$ is odd, and all operators $\nu(a)$ are even.
\end{enumerate}

Definition 2: For a non-negative integer $p$, a $p$-graded Fredholm module is a graded Fredholm module $(\nu, \mathcal{H}, F)$ with the additional datum of $p$ multigrading operators: odd operators $\varepsilon_1, \ldots, \varepsilon_p$ that commute with $F$ and with each $\nu(a)$, and that satisfy Property 2 in Example 2. In other words, the Hilbert space is a (right) $\mathbb{C}_p$-module. Notice that 0-graded just means graded, and for convenience, we mean ungraded Fredholm modules when we say $(-1)$-graded.

In the following definition, we will suppress the adjective $p$-graded.

Definition 3: \begin{enumerate}
\item Two Fredholm modules are called unitarily equivalent if there exists a grading preserving unitary isomorphism $U$ between the Hilbert spaces which intertwines the representations of $A$, the distinguished bounded operators, and the multigrading operators.
\item An operator homotopy between Fredholm modules $(\nu, \mathcal{H}, F_0, (\varepsilon_i)_i)$ and $(\nu, \mathcal{H}, F_1, (\varepsilon_i)_i)$ is a family $\{(\nu, \mathcal{H}, F_t, (\varepsilon_i)_i)\}_{t \in [0,1]}$ of Fredholm modules such that $[0,1] \to \mathcal{B}(\mathcal{H}), t \mapsto F_t$, is norm continuous.
\item We say $(\nu, \mathcal{H}, F', (\varepsilon_i)_i)$ is a compact perturbation of $(\nu, \mathcal{H}, F, (\varepsilon_i)_i)$ if the operator $\nu(a)(F' - F)$ is compact for all $a \in A$.
\item A Fredholm module $(\nu, \mathcal{H}, F, (\varepsilon_i)_i)$ is called degenerate if the operators $\nu(a)(F^* - F), \nu(a)(F^2 - 1), [\nu(a), F]$ are zero, and not just compact, for all $a \in A$.
\end{enumerate}
Proposition 2.1: Being a compact perturbation is stronger than being operator homotopic: the straight line from $F$ to its compact perturbation $F'$, given by $F_t := (1 - t)F + tF'$ for $t \in [0, 1]$, does the trick.

2.3. The K-Homology groups. Since the sum of two $p$-graded Fredholm modules (given by the direct sum of Hilbert spaces, of representation, and of operators) is again a $p$-graded Fredholm module, we arrive at the following definition for the K-homology groups $K^-p(A)$:

Definition 4 ([6] Def. 8.2.5): For a separable $C^*$-algebra $A$ and $p \geq -1$, let $K^-p(A)$ be the abelian group with one generator $[x]$ for each unitary equivalence class of $p$-graded Fredholm modules over $A$, subject to the following relations:

1. Two operator homotopic modules give the same class, and
2. if $x, y$ are two $p$-graded Fredholm modules, then $[x \oplus y] = [x] + [y]$.

Remark 1: Being a compact perturbation is stronger than being operator homotopic: the straight line from $F$ to its compact perturbation $F'$, given by $F_t := (1 - t)F + tF'$ for $t \in [0, 1]$, does the trick.

Remark 2 (cf. [6] Prop. 8.2.10, Cor. 8.2.11): It is shown in [6] Prop. 8.2.8 that the additive inverse in $K^-p(A)$ of $[\nu, H, F, (\varepsilon_i)]$ is the class $[\nu^{op}, H^{op}, -F^{op}, (\varepsilon_i^{op})]$. Define the operators

$$ F_1 := \begin{pmatrix} \cos \left( \frac{\pi i}{2} \right) & \sin \left( \frac{\pi i}{2} \right) \\ \sin \left( \frac{\pi i}{2} \right) & -\cos \left( \frac{\pi i}{2} \right) \end{pmatrix} $$

on the Hilbert space $\mathcal{H} \oplus \mathcal{H}^{op}$. Direct calculations show that

$$ x_1 := (\nu \oplus \nu^{op}, \mathcal{H} \oplus \mathcal{H}^{op}, F_1, (\varepsilon_i \oplus \varepsilon_i^{op})) $$

is a $p$-graded Fredholm module. It is an operator homotopy that starts at the class $x_0$, which we want to show is 0, and ends at a class $x_1$ which has the operator $F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $F_1^* = F_1$, $F_1^2 = 1$, and

$$ \begin{bmatrix} \nu(a) & 0 \\ 0 & \nu(a)^{op} \end{bmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0, $$

we see that $x_1$ is degenerate, and so $[x_0] = [x_1] = 0$. As a consequence, every element of $K^-p(A)$ can be represented by a single $p$-graded Fredholm module.

Remark 3 (cf. [6] Lemma 8.3.8): For the definition of the K-homology groups, one can restrict to those Fredholm modules $(\nu, H, F, (\varepsilon_i))$ for which $\nu(A) \mathcal{H}$ is dense in $\mathcal{H}$. In view of Definition 3, we will refrain from calling such Fredholm modules “non-degenerate”. But in the case where $A$ is unital, we can call them unital Fredholm modules, since $\nu(A)\mathcal{H} = \mathcal{H}$ is then equivalent to $\nu(1_A) = \text{id}_\mathcal{H}$.

Proposition 2.1 (Formal periodicity; cf. [6] Prop. 8.2.13, 8.8.5). For any $p \geq -1$, there exists an isomorphism $K^-p(A) \rightarrow K^-p(A)$.

Proof idea. Given $(\nu, H, F, (\varepsilon_i))$, we let

$$ \tilde{H} := H \oplus H^{op}, \quad \tilde{\nu} := \nu \oplus \nu^{op}, \quad \tilde{F} := F \oplus F^{op}, $$

and

$$ \tilde{\varepsilon}_i := \varepsilon_i \oplus (-\varepsilon_i^{op}) \text{ for } i \leq p, \quad \tilde{\varepsilon}_{p+1} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \tilde{\varepsilon}_{p+2} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. $$
Recalling that the grading of $\tilde{H}$ is given by

$$\tilde{H}^+ = H^+ \oplus H^- \quad \text{and} \quad \tilde{H}^- = H^- \oplus H^+,$$

where the first summand comes from the $H$-copy and the second from the $H^{\text{op}}$-copy, one quickly verifies that all $\tilde{\xi}_i$’s and $\tilde{F}$ are odd, and $\tilde{\nu}(a)$ even.

Conversely, given a $(p + 2)$-graded Fredholm module $(\nu, \tilde{H}, \tilde{F}, (\tilde{\xi}_i)_i)$, one can prove that the $+1$-eigenspace of $\tilde{\xi}_{p+1} \tilde{\xi}_{p+2}$ is a $p$-graded Fredholm module when equipped with the restricted operators. Maybe not too surprisingly, the two constructions are inverses of one another. \hfill $\square$

**Remark:** Because of the above proposition, we will without loss of generality concentrate on classes of graded and ungraded Fredholm modules, that is, $\text{K}^0$ and $\text{K}^1$.

For $B$ another separable $C^*$-algebra and a unital $*$-homomorphism $\alpha: B \to A$, we can turn a $(-j)$-graded Fredholm module $(\nu, \mathcal{H}, F)$ ($j = 0, 1$) over $A$ into one over $B$ by considering $(\nu \circ \alpha, \mathcal{H}, F)$. This process respects addition and unitary equivalence, and hence descends to a map on the level of $K$-homology,

$$\text{K}(\alpha) = \alpha^*: \text{K}(A) \to \text{K}(B).$$

It is easily checked that the assignment $\alpha \mapsto \alpha^*$ is a contravariant functor from the category of separable $C^*$-algebras to the category of abelian groups.

### 3. Unbounded operators

3.1. **Terminology.** An *unbounded operator* $D$ on a Hilbert space $\mathcal{H}$ is a linear map from a subspace $\text{dom } D \subseteq \mathcal{H}$ into $\mathcal{H}$. If $\text{dom } D$ is dense, then let

$$\text{dom } D^* := \{ \eta \in \mathcal{H} \mid \text{dom } D \ni \xi \mapsto \langle D\xi | \eta \rangle \text{ is bounded} \}$$

$$= \{ \eta \in \mathcal{H} \mid \exists \chi \in \mathcal{H} : \forall \xi \in \text{dom } D : \langle D\xi | \eta \rangle = \langle \xi | \chi \rangle \}.$$

For $\eta \in \text{dom } D^*$, define $D^*\eta$ to be the unique vector such that $\langle D\xi | \eta \rangle = \langle \xi | D^* \eta \rangle$ for all $\xi \in \text{dom } D$. The operator $D^*$ is linear on its domain, and is called the *adjoint* of $D$.

The operator $D$ is called ...

... *closed* if the graph of $D$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$.

... *closable* if the closure of its graph is the graph of a function. This function is then the closure $\overline{D}$ of $D$.

... an *extension* of an unbounded operator $D'$ if $\text{dom } D' \subseteq \text{dom } D$ and $D = D'$ on $\text{dom } D'$.

... *symmetric* if $\langle D\xi | \eta \rangle = \langle \xi | D\eta \rangle$ for all $\xi, \eta \in \text{dom } D$; in other words, if $D^*$ extends $D$.

... *self-adjoint* if $\text{dom } D^* = \text{dom } D$ and $D^* \xi = D\xi$ for all $\xi \in \text{dom } D$.

... *essentially self-adjoint* if $D$ is symmetric and $\text{dom } D^* = \text{dom } D^\ddag$.

Note that every symmetric operator $D$ is closable, and satisfies $\langle D\xi | \xi \rangle \in \mathbb{R}$ for $\xi \in \text{dom } D$. Moreover, for such $D$, $\text{dom } D^\ddag$ is sometimes called the *minimal domain* of $D$ and $\text{dom } D^*$ the *maximal domain* of $D$. 
3.2. The Cayley Transform and Borel functional calculus.

Lemma 3.1 (cf. [7, Thm. VIII.3]; [1, I.7.3.3]). If $D$ is a symmetric and densely defined unbounded operator on $\mathcal{H}$, then $D$ is self-adjoint if and only if $D \pm i$ are both surjective. Moreover, in that case, $(D \pm i)^{-1}$ is everywhere defined and a bounded operator.

Proof. Regarding the equivalence, we will actually only be interested in the forward implication, so let us disregard the proof of the other direction. We will follow the explanation given in [1, I.7.3.3].

As $D$ is self-adjoint, it is closed and the domains of $(D \pm i)^*$ and $D \pm i$ both coincide with $\text{dom} \, D$.

$$\langle (D \pm i)\xi \mid \eta \rangle = \langle D\xi \mid \eta \rangle \pm i\langle \xi \mid D\eta \rangle = \langle \xi \mid D\eta \rangle \mp \langle \xi \mid i\eta \rangle = \langle \xi \mid (D \pm i)\eta \rangle,$$

we see that $D \mp i$ satisfies the universal property that determines $(D \pm i)^*$ uniquely, so $D \mp i = (D \pm i)^*$.

Claim 1. For $\xi \in \text{dom} \, D$, we have $\| (D \pm i)\xi \| \geq \| \xi \|$, so that $D \pm i$ is bounded below by 1. In particular, $D \pm i$ is injective.

Proof of claim. For $\xi \in \text{dom}(D \pm i) = \text{dom} \, D$, we have because of $D = D^*$

$$\| (D \pm i)\xi \|^2 = \| D\xi \|^2 \pm i\langle \xi \mid D\xi \rangle \pm \langle D\xi \mid i\xi \rangle + \| i\xi \|^2 = \| D\xi \|^2 + \| \xi \|^2. \quad (3)$$

Injectivity is now clear. □

Claim 2. Since $D \pm i$ is bounded below and $D$ is closed, the range of $D \pm i$ is closed.

Proof of claim. A straightforward computation shows that $D \pm i$ is closed because $D$ is. If $T\xi_n \rightarrow \eta$ for $T := D \pm i$ and some $\xi_n \in \text{dom} \, T = \text{dom} \, D$, then $(T\xi_n)_{n=1}^\infty$ is a Cauchy sequence. The previous claim shows

$$\| T(\xi_n - \xi_m) \| \geq \| \xi_n - \xi_m \|,$$

so we see that $(\xi_n)_{n=1}^\infty$ is also Cauchy and hence converges to some $\xi$. As $T$ is closed and $(\xi_n, T\xi_n)_{n=1}^\infty$ is a sequence in its graph that converges, we must have $\xi \in \text{dom} \, T$ and $T\xi_n \rightarrow T\xi$. □

Claim 3. $D \pm i$ has dense range.

Proof of claim. If $\xi \in \text{range}(D \pm i)^{\perp}$, then $\langle (D \pm i)\nu \mid \xi \rangle = 0 = \langle \nu \mid 0 \rangle$ for all $\nu \in \text{dom} \, D$. In particular, $\xi$ is in $\text{dom}(D \pm i)^*$ with $0 = (D \pm i)^*\xi = (D \mp i)\xi$. Thus,

$$\overline{\text{range}(D \pm i)} = \overline{\text{range}(D \pm i)^{\perp}} \supset \ker(D \mp i)^{\perp}.$$

Since $\ker(D \mp i) = \{0\}$ by Claim 1, $D \pm i$ thus indeed has dense range. □

All in all, we have shown that $D \pm i$ is both injective on $\text{dom} \, D$ and surjective. Therefore, there exists a linear map $(D \pm i)^{-1} : \mathcal{H} \rightarrow \text{dom} \, D \in \mathcal{H}$ which is inverse to $D \pm i$. Lastly, since $\| (D \pm i)\xi \| \geq \| \xi \|$, we conclude $\| (D \pm i)^{-1} \| \leq 1$. □
**Definition 5** (cf. [1] I.7.2.5. Def.): For $D$ a densely defined unbounded operator on $\mathcal{H}$, the **spectrum** $\sigma(D)$ of $D$ is defined as

$$\sigma(D) := \mathbb{C} \setminus \{ z \in \mathbb{C} \mid D - z \text{ is injective on } \text{dom}(D - z) = \text{dom } D \text{ with dense range, and } (D - z)^{-1} \text{ is bounded} \}.$$ 

**Remark 4:** Note that the proof of Lemma 3.1 also works for any other $z \in \mathbb{C} \setminus \mathbb{R}$ in place of $i$. Thus we have shown that, if $D$ is self-adjoint, $\sigma(D) \subseteq \mathbb{R}$. Also, it follows from Claim 2 that, if $D$ is closed and $D - z$ is bounded below, then $D - z$ has closed range. So if $z \notin \sigma(D)$, then the range of $D - z$ is all of $\mathcal{H}$.

**Definition 6** (cf. [1] I.7.4.1. ff.): We define

$$c : \mathbb{R} \to \mathbb{S}^1 \setminus \{1\}, \quad c(t) = \frac{t + i}{t - i},$$

with inverse $c^{-1}(z) = \frac{i - z}{z - 1}$.

If $D$ is a densely defined self-adjoint operator on $\mathcal{H}$, then Lemma 3.1 shows that it makes sense to define

$$c(D) := (D + i)(D - i)^{-1} : \mathcal{H} \to \mathcal{H},$$

and that this map is an isomorphism of $\mathcal{H}$. It is called the **Cayley Transform of $D$**. From Equation 3, we see that $\| (D + i)\xi \| = \| (D - i)\xi \|$, so $c(D)$ is even a unitary. Moreover, it does not have 1 in its spectrum: if $c(D)\xi = \xi$, then for $\xi' = (D - i)^{-1}\xi$ we have $(D + i)\xi' = (D - i)\xi'$, that is $i\xi' = -i\xi'$. Thus, $\xi' = 0$ and hence $\xi = 0$, so that we have shown that $c(D) - 1$ is injective. On the other hand, if $\eta \in \mathcal{H}$ is arbitrary, let $\xi := -\frac{1}{2}i(D - 1)\eta$ and compute

$$(c(D) - 1)\xi = (D + i)(D - i)^{-1}\xi - \xi = -\frac{1}{2}i(D + i)\eta + \frac{1}{2}i(D - i)\eta = \eta,$$

so we have shown that $c(D) - 1$ is also surjective.

Conversely, if $U$ is a unitary which does not have 1 as eigenvalue, then $U - 1$ has dense range: if $\xi \in \text{range}(U - 1)^*$, then $\langle (U - 1)\eta | \xi \rangle = 0$ for all $\eta \in \mathcal{H}$, so $(U^* - 1)\xi = 0$. Injectivity of $U - 1$ then implies $\xi = 0$. Therefore, the so-called **inverse Cayley Transform of $U$** defined by

$$c^{-1}(U) := i(U + 1)(U - 1)^{-1} : \text{range}(U - 1) \to \mathcal{H},$$

is densely defined.

**Lemma 3.2.** The inverse Cayley Transform of a unitary which does not have 1 as eigenvalue is a self-adjoint operator.

**Proof.** A quick computation shows that $c^{-1}(U)$ is symmetric, so we only need to check that the domain of its adjoint is contained in $\text{range}(U - 1)$. If $\xi \in \text{dom}(c^{-1}(U))^*$, then there exists $\eta \in \mathcal{H}$ such that for all $\nu' \in \text{range}(U - 1)$, we have

$$\langle c^{-1}(U)\nu' | \xi \rangle = \langle \nu' | \eta \rangle.$$ 

In other words, for every $\nu' = (U - 1)\nu$,

$$\langle i(U + 1)\nu | \xi \rangle = \langle (U - 1)\nu | \eta \rangle.$$ 

Since this holds for every $\nu \in \mathcal{H}$, it follows that $-i(U^* + 1)\xi = (U^* - 1)\eta$. By applying $iU$ to both sides, we get

$$\xi + U\xi = (1 + U)\xi = (1 - U)i\eta = i\eta - Ui\eta.$$
Rearranging and adding $\xi$ to both sides yields
\[ 2\xi = (i\eta - U\eta - U\xi) + \xi = (1 - U)(i\eta + \xi), \]
so $\xi \in \text{range}(U - 1)$ as claimed.

If $1 \not\in \sigma(U)$, then it follows from our comment in Remark 1 that $c^{-1}(U)$ is actually everywhere defined and bounded. One can check that
\[ c^{-1}(c(D)) = D \quad \text{and} \quad c(c^{-1}(U)) = U, \]
so we have found:

**Proposition 3.3.** The Cayley Transform is a bijective map from the densely defined, self-adjoint operators to the unitary operators which do not have 1 as eigenvalue.

The Cayley Transform makes it possible to extend the Borel functional calculus for normal operators to densely defined, self-adjoint operators. It has the following properties:

**Proposition 3.4** (Functional Calculus; [1, I.7.4.5. Thm, I.7.4.7. Def.]). For $D$ a densely defined, self-adjoint operator on $H$, there exists a linear map
\[ h: \mathbb{R} \to C \text{ Borel measurable} \to \{ \text{densely defined unbounded operators on } H \} \]
\[ h \mapsto h(D) \]
with the following properties:

1. $\text{id}_\mathbb{R}(D) = D$.
2. If $h \geq 0$, then $h(D)$ is positive.
3. If $|h| = 1$, then $h(D)$ is unitary.
4. $h(D)^* = \overline{h}(D)$; in particular, if $h$ is real-valued, then $h(D)$ is self-adjoint.
5. If $h$ is bounded and continuous, then
\[ \|h(D)\| = \|h\|_\infty. \]
6. If $h_n$ is a uniformly bounded sequence of functions which converges pointwise to $h$, then $h_n(D) \to h(D)$ strongly.

**Lemma 3.5** (cf. [6, Lemma 10.6.2]). Suppose $D$ is an unbounded, essentially self-adjoint operator on $H$, and $T \in \mathcal{B}(H)$ preserves $\text{dom} D$ and satisfies $TD = -DT$. If $f \in C_b(\mathbb{R})$ is odd, then $Tf(D) = -f(D)T$, and if $f$ is even, then $Tf(D) = f(D)T$.

**Proof.** Let us first set some notation: the decomposition of a function $f$ into its even and odd part is given by
\[ f^e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f^o(x) = \frac{f(x) - f(-x)}{2}, \]
so that $f = f^e + f^o$.

Let us denote by
\[ \tilde{f}(x) := f^e - f^o = f(-x). \]
The claim can now be rephrased to $Tf(D) = \tilde{f}(D)T$. In other words, $T$ graded commutes with $f(D)$ when $C_b(\mathbb{R})$ has the $\mathbb{Z}/2\mathbb{Z}$-grading into even and odd functions.
Claim 1. It suffices to show the claim for elements of $C_0(\mathbb{R})$.

Proof of claim. For $f \in C_0(\mathbb{R})$, take functions $f_n \in C_0(\mathbb{R})$ converging pointwise to $f$. By Property [6] of Functional Calculus, we have strong convergence $f_n(D) \to f(D)$ and also $f_n(D) \to \tilde{f}(D)$, so for every $h \in \mathcal{H}$, we get

$$Tf(D)h = T\left(\lim_{n \to \infty} f_n(D)h\right) = \lim_{n \to \infty} Tf_n(D)h = \lim_{n \to \infty} \tilde{f}_n(D)Th = \tilde{f}(D)Th,$$

where we used the assumption that $Tf_n(D) = \tilde{f}_n(D)T$.

By the Stone-Weierstrass Theorem [2], either of the functions

$$\psi_{\pm}(x) := \frac{1}{1 \pm x} = \psi(x)$$

generate $C_0(\mathbb{R})$ as a $C^*$-algebra.

Claim 2. It suffices to show that $T$ graded commutes with $\psi_{\pm}(D)$.

Proof of claim. For a fixed $f \in C_0(\mathbb{R})$, assume

$$\psi = \sum_{n,k \in \mathbb{N}^*} a_{n,k}\psi_{n}^{\pm}(\psi_{n})^{k} = \sum_{n,k \in \mathbb{N}^*} a_{n,k}\psi_{n}^{\pm}\psi_{n}^{k} \text{ is such that } \|f - \psi\|_{\infty} < \epsilon.$$

The properties of continuous functional calculus shows that, if $T$ graded commutes with $g(D)$, then it also graded commutes with $g^n(D)$ for positive powers of $g$. Thus, we have $T\psi_{\pm}^{n}(D) = (\psi_{\pm})^{n}(D)$ by assumption, which implies

$$\|Tf(D) - \tilde{f}(D)T\| \leq \|Tf(D) - T\psi(D)\| + \|T\psi(D) - \tilde{f}(D)T\|$$

$$= \|Tf(D) - T\psi(D)\| + \|\tilde{\psi}(D)T - \tilde{f}(D)T\|$$

$$\leq \|T\| \cdot \|f - \psi\|_{\infty} + \|\tilde{\psi} - \tilde{f}\|_{\infty} \cdot \|T\| < 2\epsilon \|T\|.$$

Since this is possible for any $\epsilon$, this implies $Tf(D) = \tilde{f}(D)T$ as wanted. As $(\mathbb{1} \pm D)T = T(\mathbb{1} \mp D)$ by assumption, we get

$$T\psi_{\mp}(D) = \psi_{\pm}(D)T. \quad (4)$$

Since $\psi_{\pm}(-x) = \psi_{\mp}(x)$, we can see that

$$\psi_{\pm}^{\circ} = \psi_{\pm}^{\circ} \quad \text{and} \quad \psi_{\pm}^{c} = -\psi_{\pm}^{c}.$$

As a consequence,

$$2(\psi_{\pm} + \psi_{\mp}) = (\psi_{\pm}^{c} + \psi_{\mp}^{c}) + (\psi_{\pm}^{\circ} + \psi_{\mp}^{\circ}) = 2\psi_{\pm}^{c}, \text{ so } \psi_{\pm} + \psi_{\mp} = \psi_{\pm}^{c},$$

and

$$2(\psi_{\pm} - \psi_{\mp}) = (\psi_{\pm}^{c} + \psi_{\mp}^{c}) - (\psi_{\pm}^{\circ} + \psi_{\mp}^{\circ}) = 2\psi_{\pm}^{\circ}, \text{ so } \psi_{\pm} - \psi_{\mp} = \psi_{\pm}^{\circ}.$$

From Equation (4), it thus follows that

$$T\psi_{\pm}^{c}(D) = T(\psi_{\pm} + \psi_{\mp})(D) = (\psi_{\pm} + \psi_{\mp})(D)T = \psi_{\pm}^{c}(D)T$$

and

$$T\psi_{\pm}^{\circ}(D) = T(\psi_{\pm} - \psi_{\mp})(D) = (\psi_{\pm} - \psi_{\mp})(D)T = -\psi_{\pm}^{\circ}(D)T.$$

In other words, $T$ graded commutes with $\psi_{\pm}(D)$.
4. Elliptic operators

**Notation:** We will write $\lambda$ for Lebesgue measure on $\mathbb{R}^n$, $\| \cdot \|_{C^k}$ for the Euclidean norm on $\mathbb{C}^n$, and $\| \cdot \|_2$ for $L^2$-norms.

**Definition:** A vector bundle $S \to M$ over a smooth manifold $M$ is called *smooth* if $S$ is also a manifold and $\pi$ is a smooth map. We write $\Gamma^\infty(M; S)$ for the smooth sections of this bundle, and $\Gamma_c^\infty(M; S)$ for the compactly supported ones.

A smooth vector bundle $S \to M$ is called *Hermitian* if, for each $p \in M$, there is an inner product $(\cdot | \cdot)^S_p$ on the fibre $S_p := \pi^{-1}(p)$, and these inner products *vary smoothly*: for every $u, v \in \Gamma^\infty(M; S)$, the map

$$M \ni p \mapsto \left( u(p) | v(p) \right)^S_p \in \mathbb{C}$$

is smooth.

In the following, we will fix a smooth Hermitian complex vector bundle $S \to M$ of rank $k$ over a smooth manifold $M$ of dimension $n$. Let us denote the norm induced by the inner product $(\cdot | \cdot)^S_p$ on $S_p$ by $\| \cdot \|_{S_p}$. An example to keep in mind is the case where $M$ is spin and $S$ is its spinor bundle.

We further assume that we are given a nowhere-vanishing smooth measure $\mu$ on $M$, that is, $\mu$ is a Borel measure such that for every chart $(U, \varphi)$ of $M$, there exists a smooth function $f : \varphi(U) \to (0, \infty)$ such that $d(\varphi_* \mu_U) = f \, d\lambda_{\varphi(U)}$. This means for a $(\varphi_* \mu_U)$-integrable function $h : \varphi(U) \to \mathbb{C}$ that

$$\int_U h \circ \varphi \, d\mu = \int_{\varphi(U)} h \cdot f \, d\lambda.$$

Moreover, since $f$ does not vanish, we can also consider $g = \frac{1}{f}$ and get for $\lambda_{\varphi(U)}$-integrable $h$

$$\int_U (h \cdot g) \circ \varphi \, d\mu = \int_{\varphi(U)} h \, d\lambda. \quad (5)$$

**Remark 5:** For technical reason, there will be the standing assumption that there exists a number $L$ so that we have for all the above mentioned Radon-Nikodym derivatives the inequality $\|f\|_\infty, \|g\|_\infty \leq L$.

We construct the Hilbert space $L^2(M; S)$ as the completion of $\Gamma^\infty_c(M; S)$ with respect to the norm coming from the inner product

$$\langle u | v \rangle := \int_M (u(p) | v(p))^S_p \, d\mu(p).$$

For a subset $U \subseteq M$, we will write $L^2(U; S)$ for the completion of the smooth sections whose compact support is contained in $U$. Lastly, let

$$M : C_0(M) \to B(L^2(M; S)), \quad g \mapsto M_g,$$

be the representation of $C_0(M)$ which, on the dense subspace $\Gamma^\infty_c(M; S)$, is given by pointwise multiplication.
4.1. Differential operators.

Definition 7: A (first order linear) differential operator acting on the sections of $S$ is a $\mathbb{C}$-linear map

$$D: \Gamma^\infty(M;S) \to \Gamma^\infty(M;S)$$

such that

(a): if $u, v \in \Gamma^\infty(M;S)$ agree on an open set $U$, then $Du, Dv$ also agree on $U$, and

(b): for a coordinate chart of $M$ that also trivializes $S$, say

$$\psi$$

there exist functions $A_1^1, \ldots, A_n, B \in C^\infty(U, M_k(\mathbb{C}))$ such that for all $p \in U$ and all $u \in \Gamma^\infty(M;S)$, we have

$$(Du)(p) = \sum_{j=1}^{n} \Psi^{-1}\left(p, A_j(p) \cdot \partial_j(\psi \circ u \circ \varphi^{-1})|_{\psi(p)}\right) + \Psi^{-1}(p, B(p) \cdot (\psi \circ u)(p)).$$

We will from now on regard such a differential operator as an unbounded operator on $L^2(M;S)$ with dense domain $\Gamma^\infty(M;S)$. By abuse of terminology, we will say “differential operator on $M$,” tacitly assuming a fixed Hermitian bundle $S$.

Lemma 4.1. Let $D$ be a symmetric differential operator on $M$ and let $u \in \text{dom } D^*$ have compact support $K$. Then the support of $D^*u$ is contained in $K$.

Proof. Let $w_k$ be a sequence in $\text{dom } D = \Gamma^\infty(M;S)$ which converges to $u$ in $L^2$-norm. If we take $K = \cap_{k=1}^{\infty} V_k$ for open nested sets $V_{k+1} \subseteq V_k \subseteq M$ (see Lemma 5.13 for a construction), Urysohn gives us smooth $[0,1]$-valued functions $\rho_k$ with $\text{supp}(\rho_k) \subseteq V_k$ which are 1 on $K$. Note that $u_k := \rho_k \cdot w_k$ is also in $\text{dom } D$, and since $u$ is supported in $K$, we see

$$\|u - u_k\|_2^2 = \int_K \|u(p) - u_k(p)\|_{S_p}^2 \, d\mu + \int_{M \setminus K} \|u_k(p)\|_{S_p}^2 \, d\mu$$

$$\leq \int_K \|u(p) - w_k(p)\|_{S_p}^2 \, d\mu + \int_{M \setminus K} \|w_k(p)\|_{S_p}^2 \, d\mu = \|u - w_k\|_2^2,$$

so $u_k$ also converges to $u$. As $u_k$ is supported in $V_k$, we get from Property a) of differential operators that $Du_k$ is supported in $V_k$, too. We know that $Du_k = D^*u_k$ converges to $D^*u$ in $L^2$-norm, so by choosing an appropriate subsequence, we can
assume \((\ast)\) in the following computation:

\[
\frac{1}{k} \left( \| D^* u - D u_k \|_S^2 \right)^2 = \int_{V_k} \| D^* u(p) - D u_k(p) \|_{S_p}^2 \, d\mu + \int_{M \setminus V_k} \| D^* u(p) \|_{S_p}^2 \, d\mu.
\]

\[
\geq \int_{M \setminus V_k} \| D^* u(p) \|_{S_p}^2 \, d\mu.
\]

Now, note that \(V_{k+m} \subseteq V_k\) for any \(m\), and hence

\[
\int_{M \setminus V_k} \| D^* u(p) \|_{S_p}^2 \, d\mu \leq \int_{M \setminus V_{k+m}} \| D^* u(p) \|_{S_p}^2 \, d\mu < \frac{1}{k+m}.
\]

It follows that \(\int_{M \setminus V_k} \| D^* u(p) \|_{S_p}^2 \, d\mu = 0\) for every \(k\), and as \(M \setminus K = \bigcup_k M \setminus V_k\),

\[
\int_{M \setminus K} \| D^* u(p) \|_{S_p}^2 \, d\mu \leq \sum_k \int_{M \setminus V_k} \| D^* u(p) \|_{S_p}^2 \, d\mu = 0.
\]

We conclude that \(D^* u\) is also supported in \(K\).

**Lemma 4.2.** If \(D\) is a differential operator on \(M\) which is locally given by Equation (7), and if \(g \in C^\infty(M)\), then \([D, M_g]\) can locally be written as

\[
[D, M_g] u(p) = \sum_{j=1}^n \partial_j (g \circ \varphi^{-1}) \big|_{\varphi(p)} \cdot \Psi^{-1} \left( p, A^j(p) \cdot (\psi \circ u(p)) \right).
\]  

(8)

In particular, if \(K \subseteq M\) is compact, then \([D, M_g]\) extends to a bounded operator on \(L^2(K; S)\).

**Proof.** It suffices to consider those \(D\) that locally look like only one of the summands in Equation (7). Given a chart \((U, \varphi)\) and a trivialization \(\Psi\) of \(S\), if

\[
(Du)(p) = \Psi^{-1} \left( p, B(p) \cdot (\psi \circ u(p)) \right), \quad B \in C^\infty(U, M_k(C)),
\]

then \(D\) is itself only a multiplication operator (albeit by a matrix), and so it in fact commutes with \(M_g\). So consider the case in which

\[
(Du)(p) = \Psi^{-1} \left( p, A(p) \cdot \partial_j (\psi \circ u \circ \varphi^{-1}) \big|_{\varphi(p)} \right), \quad A \in C^\infty(U, M_k(C))
\]

for some \(1 \leq j \leq n\). We compute for \(u \in \Gamma^\infty(M; S)\) and \(p \in U:\)

\[
[D, M_g] u(p) = D \left( g u \right)(p) - g(p) \left( \frac{\partial}{\partial x} \right)(p)
\]

\[
= \Psi^{-1} \left( p, A(p) \cdot \partial_j (\psi \circ g \circ \varphi^{-1}) \big|_{\varphi(p)} \right)
\]

\[
- g(p) \Psi^{-1} \left( p, A(p) \cdot \partial_j (\psi \circ u \circ \varphi^{-1}) \big|_{\varphi(p)} \right).
\]

As \(g(p)\) is just a scalar and \(\Psi^{-1}(p, \cdot)\) and \(\psi\) are linear, we get

\[
[D, M_g] u(p) = \Psi^{-1} \left( p, A(p) \cdot \partial_j (g \circ \varphi^{-1}) \cdot (\psi \circ u \circ \varphi^{-1}) \big|_{\varphi(p)} \right)
\]

\[
- \Psi^{-1} \left( p, g(p) \cdot A(p) \cdot \partial_j (\psi \circ u \circ \varphi^{-1}) \big|_{\varphi(p)} \right) + g(p) \partial_j (\psi \circ u \circ \varphi^{-1}) \big|_{\varphi(p)}.
\]

By the product rule,

\[
\partial_j (g \circ \varphi^{-1} \cdot (\psi \circ u \circ \varphi^{-1})) \big|_{\varphi(p)} = \partial_j (g \circ \varphi^{-1}) \big|_{\varphi(p)} (\psi \circ u(p)) + g(p) \partial_j (\psi \circ u \circ \varphi^{-1}) \big|_{\varphi(p)},
\]

and similarly for the others. This finishes the proof.

\(\square\)
so we arrive at

$$[D, M_g] u(p) = \Psi^{-1} \left( p, \partial_j (g \circ \varphi^{-1})_{|\varphi(p)} \cdot A(p) \cdot (\psi \circ u(p)) \right).$$  \hfill (9)

**Definition 8:** The symbol $\sigma_D$ of a differential operator $D$ is the $\mathbb{R}$-vector bundle morphism

$$\sigma_D: T^* M \to \text{End}(S)$$

defined as follows: given a cotangent vector $\xi \in T^*_p M$ at $p$, take a chart $(U, \varphi)$ around $p \in M$ and a trivialization of $S_U$ as in Diagram (6). Suppose $D$ locally looks as in Equation (7), and write $\xi = \sum_{j=1}^n \xi_j d\varphi^j_p$, where $(d\varphi^j_p)_j$ denotes the basis of $T^*_p M$ that is dual to the basis $\{\frac{\partial}{\partial \varphi^j}|_p\}_j$ of $T_p M$. Then we define for $\eta \in S_p$,

$$\sigma_D(p, \xi)\eta := \Psi^{-1} \left( p, \sum_{j=1}^n \xi_j A^j(p) \psi(\eta) \right).$$

**Remark:** In Lemma 4.2, we have actually shown that

$$[D, M_g] u(p) = \sigma_D(p, dg_p)(u(p)).$$

**Lemma 4.3.** The definition of $\sigma_D$ does not depend on the choice of $\Psi$ or $\varphi$.

**Proof.** First, assume that $\Omega$ is another trivialization of $S_U$, and let $\omega := \text{pr}_2 \circ \Omega$. Since the fibre maps of both $\Psi$ and $\Omega$ are linear isomorphisms, there exists a smooth map

$$H: U \to \text{GL}_k(\mathbb{C})$$

given by

$$\mathbb{C}^k \xrightarrow{\Omega(p, \cdot)} S_p \xrightarrow{\Psi(p, \cdot)} \mathbb{C}^k.$$

Moreover, we can write $D$ also in the form

$$(Du)(p) = \sum_{j=1}^n \Omega^{-1} \left( p, E^j(p) \cdot \partial_j (\omega \circ u \circ \varphi^{-1})_{|\varphi(p)} \right) + \Omega^{-1} \left( p, E(p) \cdot (\omega \circ u)(p) \right),$$

for all $u \in \Gamma^\infty(M; S)$. By clever choices of $u$ and some use of the product rule, one can conclude that

$$A^j(p) = H(p) E^j(p) H^{-1}(p)$$

for each $1 \leq j \leq n$. Therefore, for any $\eta \in S_p$,

$$A^j(p) \psi(\eta) = H(p) E^j(p) \omega(\eta)$$

and so

$$\Psi^{-1} \left( p, A^j(p) \psi(\eta) \right) = \Psi^{-1} \left( p, H(p) E^j(p) \omega(\eta) \right) = \Omega^{-1} \left( p, E^j(p) \omega(\eta) \right).$$

We see from this that $\sigma_D(p, \xi)$ does not depend on the choice of $\Psi$.

Next, let $\gamma$ be another chart around $p$. Again, we can write $D$ in the form

$$\left( Du \right)(p) = \sum_{l=1}^n \Psi^{-1} \left( p, F^l(p) \cdot \partial_l (\psi \circ u \circ \gamma^{-1})_{|\gamma(p)} \right) + \Psi^{-1} \left( p, F(p) \cdot (\psi \circ u)(p) \right).$$
We get that
\[
\sum_{j=1}^{n} A^j(p) \cdot \partial_j (\psi \circ u \circ \varphi^{-1})_{|\varphi(p)} = \sum_{l=1}^{n} F^l(p) \cdot \partial_l (\psi \circ u \circ \gamma^{-1})_{|\gamma(p)}
\]
and so another clever choice of \( u \) yields
\[
A^j(p) = \sum_{l=1}^{n} \partial_l (\varphi \circ \gamma^{-1})^j_{|\gamma(p)} F^l(p).
\]
Moreover, if \( \xi = \sum_i \nu_i \, d\gamma_i^p \), then
\[
\nu_i = \xi \left( \frac{\partial}{\partial \gamma_i^p} \right) = \sum_{j=1}^{n} \xi_j \partial_l (\varphi \circ \gamma^{-1})^j_{|\gamma(p)}.
\]
Combined, we have for any \( v \in \xi^k \)
\[
\sum_{j=1}^{n} \xi_j A^j(p) v = \sum_{j=1}^{n} \xi_j \left( \sum_{l=1}^{n} \partial_l (\varphi \circ \gamma^{-1})^j_{|\gamma(p)} F^l(p) \right) v = \sum_{l=1}^{n} \nu_l F^l(p) v,
\]
and so we conclude that \( \sigma_D(p, \xi) \) also does not depend on the choice of \( \varphi \).

**Definition 9:** We say that a differential operator is **elliptic** if its symbol \( \sigma_D \) maps each \((p, \xi)\) in \( T^* M \) with \( \xi \neq 0 \) to an invertible endomorphism of \( S_p \).

### 4.2. Sobolev Spaces

We want to construct the Sobolev space associated to our vector bundle. Recall first that for \( f \in C^\infty_c(\mathbb{R}^n, \xi) \), the Sobolev norm is defined by
\[
\| f \|^2_{1, \mathbb{R}^n} := \| f \|^2_2 + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|^2_2.
\]

Take an atlas of \( M \) whose charts are small enough to also allow smooth, fibrewise isometric trivializations as in Diagram (6). For a compact subset \( K \) of \( M \), let \( \{(U_i, \varphi_i)\}_{i=1}^I \) be a subcover of charts, and denote the corresponding trivialisations of \( S \) by \( \Psi_i = \pi \times \psi_i \). These induce maps
\[
\Psi_i^*: \Gamma^\infty(U_i; S_{U_i}) \rightarrow \left( C^\infty(V_i) \right)^k
\]
which send a section \( v: U_i \rightarrow S_{U_i} \) to the map
\[
\mathbb{R}^n \ni V_i \ni x \mapsto \psi_i \left( v(\varphi_i^{-1}(x)) \right).
\]
As explained in Lemma 5.14, we can pick smooth compactly supported functions
\[
\rho_1, \ldots, \rho_l: M \rightarrow [0, 1] \quad \text{such that } \text{supp}(\rho_i) \subseteq U_i \quad \text{and} \quad \sum_{i=1}^{l} \rho_i(p) = 1 \quad \text{for } p \in K.
\]

We define for \( u \in \Gamma^\infty(K; S) \) (that is, sections of the bundle supported in \( K \)):
\[
\| u \|_{1} := \sum_{i=1}^{l} \| \Psi_i^* (\rho_i \cdot u) \|_{1, \mathbb{R}^n}.
\]

Though this norm relies heavily on the choices involved, its equivalence class does not. We define \( L^2_1(K; S) \) to be the completion of \( \Gamma^\infty(K; S) \) with respect to this norm. Let us gather some facts about Sobolev spaces that we will need later:
Lemma 4.4. For \( K \subset M \) compact, there exists a number \( c > 0 \) such that for all \( u \in L^2_0(K; S) \),
\[
\| u \|_2 \leq c \| u \|_1 .
\]

Proof. Since \( \| f \|_{1,R^n} \geq \| f \|_2 \), we get
\[
\| u \|_1 \geq \sum_{i=1}^l \| \Psi_i^*(\rho_i \cdot u) \|_2 .
\]

Let \( f_i, g_i = \frac{1}{n} \) be as in Equation (5) for \( (U_i, \varphi_i) \). Recall that we assumed in Remark 5 that \( \| f_i \|_\infty \leq L \) for some number \( L \) and all \( i \). For \( v \in \Gamma^\infty_c(U_i; S_{\varphi_i}) \), we have
\[
\| \Psi_i^*(v) \|_2^2 = \int_{R^n} \| \Psi_i(v(\varphi^{-1}_i(x))) \|_{C_k}^2 \, d\lambda = \int_{\hat{U}_i} \| \psi_i(v(p)) \|_{C_k}^2 \, (g_i \circ \varphi)(p) \, d\mu,
\]
and since \( \psi_i \) is isometric, we get
\[
\| \Psi_i^*(v) \|_2^2 \geq \frac{1}{L} \int_{\hat{U}_i} \| v(p) \|_{S_{\rho_i}}^2 \, d\mu = \frac{1}{L} \| v \|_2^2 .
\]
Thus,
\[
\| u \|_1 \geq \frac{1}{\sqrt{L}} \sum_{i=1}^l \| \rho_i \cdot u \|_2 .
\]
Furthermore,
\[
\left( \sum_{i=1}^l \| \rho_i \cdot u \|_2 \right)^2 \geq \sum_{i=1}^l \| \rho_i \cdot u \|_2^2 = \sum_{i=1}^l \left( \int_K \rho_i(p)^2 \| u(p) \|_{S_{\rho_i}}^2 \, d\mu \right)
\]
\[
= \int_K \left( \sum_{i=1}^l \rho_i(p)^2 \right) \| u(p) \|_{S_{\rho_i}}^2 \, d\mu
\]
\[
\geq \int_K \frac{1}{l} \left( \sum_{i=1}^l \rho_i(p) \right)^2 \| u(p) \|_{S_{\rho_i}}^2 \, d\mu = \frac{1}{l} \int_K \| u(p) \|_{S_{\rho_i}}^2 \, d\mu = \frac{1}{l} \| u \|_2^2 ,
\]
so that all in all
\[
\| u \|_1 \geq \frac{1}{\sqrt{L \cdot l}} \| u \|_2 .
\]
\[\square\]

Proposition 4.5 ([9], IV.2.2 - without proof). Every differential operator \( D \) on \( M \) has a continuous extension to an operator \( L^2_0(K; S) \to L^2(K; S) \) where \( K \subset M \) is any compact subset.

Corollary 4.6. For \( D \) a symmetric differential operator on \( M \) and \( K \subset M \) compact, \( L^2_0(K; S) \) is contained in the domain of \( D^* \).

Proof. For \( u \in L^2_0(K; S) \), we need to show that there exists \( C > 0 \) such that
\[
|\langle u | Dv \rangle| \leq C \cdot \| v \|_2
\]
for all \( v \in \Gamma^\infty_c(M; S) = \text{dom} \ D \). Let \( (u_n)_n \) be a sequence in \( \Gamma^\infty_c(K; S) \) converging to \( u \) in \( \| \cdot \|_1 \). By Proposition 4.5 \( (Du_n)_n \) converges in \( L^2(M; S) \), so the \( \| \cdot \|_2 \)-norm of
the sequence is bounded by some number \( N \). For \( 0 \neq v, \) take some big enough \( n \) such that \( \|u - u_n\|_1 \leq \frac{|v|}{c(\|Du\|_2 + 1)} \) where \( c > 0 \) is as in Lemma 4.4 and compute
\[
|\langle u | Dv \rangle| \leq |\langle u - u_n | Dv \rangle| + |\langle u_n | Dv \rangle| = |\langle u - u_n | Dv \rangle| + |\langle Du_n | v \rangle| \\
\leq \|u - u_n\|_1 \\|Dv\|_2 + \|Du_n\|_2 \|v\|_2 < (1 + N) \|v\|_2.
\]
\( \square \)

We will need the following propositions later, but we will not prove them here.

**Proposition 4.7** (Rellich Lemma; [6, 10.4.3], [9, IV.1.2] - without proof). For \( K \subseteq M \) compact, the inclusion \( L^2(K; S) \to L^2(K; S) \) is a compact operator.

**Proposition 4.8** (Gårding’s Inequality; [6, 10.4.4] - without proof). Suppose \( M \) is compact. If \( D \) is an elliptic differential operator on \( M \), then there is a constant \( c > 0 \) such that, for all \( u \in L^2(M; S) \),
\[
c \cdot \|u\|_1 \leq \|u\|_2 + \|Du\|_2.
\]

As mentioned in the introduction, to be able to invoke Gårding’s Inequality is the reason why we need to assume ellipticity of \( D \) in our main theorem.

### 4.3. Fourier Transforms and Normalizing functions

Most of the statements below can be found in [4, Chapters 8 and 9].

**Notation:** For \( f: \mathbb{R}^n \to \mathbb{C} \) and \( x, y \in \mathbb{R}^n \), let
\[
\left( \tau_x f \right)(y) := f(y - x) \quad \text{and} \quad \tilde{f}(y) := f(-y).
\]

For \( f \in L^1(\mathbb{R}^n) \), its *Fourier* and *inverse Fourier Transform* are given by
\[
\hat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} f(y) \, dy \quad \text{and} \quad \check{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} f(y) \, dy.
\]  
(10)

If \( f, g \in L^1 \), then
\[
\int_{\mathbb{R}^n} \hat{f}(x) g(x) \, dx = \int_{\mathbb{R}^n} f(x) \hat{g}(x) \, dx.
\]  
(11)

As a consequence, one can show that if \( f, \hat{f} \) are both \( L^1 \), then the *inversion formula* holds: for almost every \( x \in \mathbb{R}^n \), we have
\[
f(x) = \hat{f}(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} \hat{f}(y) \, dy.
\]

**Definition 10:** The *Schwartz space* \( \mathcal{S} \) consists of those smooth functions on \( \mathbb{R}^n \) which have rapidly decaying derivatives. To be more precise, define for \( N \in \mathbb{N} \) and a multi-index \( \alpha \),
\[
\|\phi\|_{N, \alpha} := \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |\partial^\alpha \phi(x)|.
\]  
(12)

Then
\[
\mathcal{S} := \{ \phi \in C^\infty(\mathbb{R}^n) \mid \text{for any } N \in \mathbb{N}, \alpha \text{ multi-index} : \|\phi\|_{N, \alpha} < \infty \}.
\]
When equipped with the seminorms given in Definition 12, \( \mathcal{S} \) becomes a Fréchet space, cf. [4, 8.2. Proposition]. The Fourier Transform then maps \( \mathcal{S} \) continuously into itself and, because of the inversion formula, is hence an isomorphism of \( \mathcal{S} \) (cf. [4, 8.28 Cor.]).

**Definition 11:** A distribution \( F \) is a functional on \( C^\infty_c(\mathbb{R}^n) \). We will denote by \( \langle F, \phi \rangle \) the value of \( F \) at the point \( \phi \in C^\infty_c(\mathbb{R}^n) \), and let \( \mathcal{D}' \) be the space of distributions. The support of \( F \) is the complement of the maximal open subset \( U \subseteq \mathbb{R}^n \) for which

\[
\langle F, \phi \rangle = 0
\]

for all \( \phi \) such that \( \text{supp}(\phi) \subseteq U \). A distribution \( F \) is tempered if it extends continuously to all of \( \mathcal{S} \). As \( C^\infty_c(\mathbb{R}^n) \) is dense in \( \mathcal{S} \) (cf. [4, 9.9 Prop.]), the space of tempered distributions is the dual space \( \mathcal{S}' \) of \( \mathcal{S} \).

**Example 3:** If \( f : \mathbb{R}^n \to \mathbb{C} \) is locally integrable (that is, integrable on compact sets), then it defines a distribution by

\[
\langle f, \phi \rangle := \int f(x)\phi(x) \, dx
\]

for \( \phi \in C^\infty_c(\mathbb{R}^n) \).

If \( \psi \in C^\infty_c(\mathbb{R}^n) \), then

\[
\int \langle f * \psi \rangle(x)\phi(x) \, dx = \int \int f(x)\psi(y-x)\phi(x) \, dy \, dx = \int f(x)(\phi * \tilde{\psi})(x) \, dx,
\]

and so the above Example justifies the following definition:

**Definition 12 ([4], p. 285):** If \( F \in \mathcal{D}' \) and \( \psi \in C^\infty_c(\mathbb{R}^n) \), we define for \( \phi \in C^\infty_c(\mathbb{R}^n) \),

\[
\langle F * \psi, \phi \rangle := \langle F, \phi * \tilde{\psi} \rangle.
\]

One can show (see [4, 9.3 Prop.]) that this distribution is actually given by integration against the function \( F * \psi \) defined by

\[
F * \psi(x) := \langle F, \tau_x \tilde{\psi} \rangle.
\]

**Lemma 4.9.** If \( F \in \mathcal{D}' \) has compact support and if \( \psi \in C^\infty_c(\mathbb{R}^n) \), the function \( F * \psi \) is a smooth compactly supported function.

**Proof.** Regarding smoothness, see [4, 9.3a] Prop.]. If we let \( A \) be the closure of \( \text{supp}(F) + \text{supp}(\psi) \), then \( A \) is compact by assumption. For \( x \notin A \), the function

\[
\tau_x \tilde{\psi} : y \mapsto \psi(x - y)
\]

is supported outside of \( \text{supp}(F) \), so that

\[
F * \psi(x) = \langle F, \tau_x \tilde{\psi} \rangle = 0
\]

**Example 4** (special case of Example 3): If \( f : \mathbb{R} \to \mathbb{C} \) is measurable and bounded, then it defines a *tempered* distribution by

\[
\langle f, \phi \rangle := \int f(x)\phi(x) \, dx
\]
for $\phi \in \mathcal{S}$. Indeed, since
\[
\sup_{x \in \mathbb{R}} (1 + |x|)^2 |\phi(x)| = \|\phi\|_{2,0} < \infty,
\]
we have
\[
\int_{\mathbb{R}} |f(x)\phi(x)| \, dx \leq \int_{\mathbb{R}} \|f\|_{\infty} \frac{|\phi|_{2,0}}{(1 + |x|)^2} \, dx \leq \int_{\mathbb{R}} \frac{\|f\|_{\infty} |\phi|_{2,0}}{1 + |x|^2} \, dx = \|f\|_{\infty} \|\phi\|_{2,0} \pi < \infty,
\]
so the measurable function $f\phi$ is integrable, and $\langle f, \phi \rangle$ is well-defined and continuous.

The advantage of tempered distributions over other distributions is the following definition:

**Definition 13**: If $F$ is a tempered distribution, we define its Fourier and inverse Fourier Transform by
\[
\langle \hat{F}, \phi \rangle := \langle F, \phi \rangle, \quad \text{and} \quad \langle \hat{\hat{F}}, \phi \rangle := \langle \hat{F}, \phi \rangle
\]
for $\phi \in \mathcal{S}$. Because of Equation (11), we see that, if $F$ is integration against an $L^1$ function, then both $\hat{F}$ and $\check{F}$ agree with the definition given in Definition 10. Moreover, we again have the inversion formula $\hat{\hat{F}} = \check{F} = F$.

**Lemma 4.10**: Suppose we are given an even, integrable function $h : \mathbb{R} \to \mathbb{C}$. Then the assignment
\[
\operatorname{pv} \left( \int_{-\infty}^{\infty} \frac{h(t)}{t} \right) : C_c^\infty(\mathbb{R}) \to \mathbb{C},
\]
\[
\varphi \mapsto \lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{\epsilon} \frac{h(t)}{t} \varphi(t) \, dt + \int_{\epsilon}^{\infty} \frac{h(t)}{t} \varphi(t) \, dt \right),
\]
extends continuously to $\mathcal{S}$. Furthermore, the Fourier Transform of this tempered distribution is given by integration against the (well-defined) function
\[
\zeta(x) := \int_{-\infty}^{\infty} \frac{\sin(tx)}{it} \, dh \left( \frac{t}{2\pi} \right) \, dt.
\]

**Proof**: Let $\varphi \in C_c^\infty(\mathbb{R})$. For any $\epsilon > 0$, the following two integrals exist since $h$ is integrable, and are equal because $h$ is even:
\[
\int_{-\infty}^{\epsilon} \frac{h(t)}{t} \varphi(0) \, dt = -\int_{\epsilon}^{\infty} \frac{h(t)}{t} \varphi(0) \, dt.
\]
Therefore, we may rewrite
\[
\left\langle \operatorname{pv} \left( \int_{-\infty}^{\infty} \frac{h(t)}{t} \right), \varphi \right\rangle = \lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{\epsilon} \frac{\varphi(t) - \varphi(0)}{t} h(t) \, dt + \int_{\epsilon}^{\infty} \frac{\varphi(t) - \varphi(0)}{t} h(t) \, dt \right)
\]
\[
= \int_{-\infty}^{\infty} \frac{\varphi(t) - \varphi(0)}{t} h(t) \, dt,
\]
where the last line holds because $t \mapsto \frac{\varphi(t) - \varphi(0)}{t}$ can be smoothly extended at 0 by the value $\varphi'(0)$ by L'Hôpital. We therefore get

$$
\left\| \text{pv} \left( \int \frac{h(t)}{t} \right) , \varphi \right\| \leq \left\| \int_{-\infty}^{\infty} \left| \frac{\varphi(t) - \varphi(0)}{t} \right| h(t) \right\| dt \leq \| h \|_{L^1} \cdot \sup_{t \in \mathbb{R}} \left| \frac{\varphi(t) - \varphi(0)}{t} \right|
$$

by the Mean Value Theorem. In particular, the value is finite for $\varphi \in C^\infty_c(\mathbb{R})$ and in fact also for $\varphi \in \mathcal{S}$. Moreover, given $\varphi_k \in \mathcal{S}$ converging to 0, the above line means that

$$
\left\| \text{pv} \left( \int \frac{h(t)}{t} \right) , \varphi_k \right\| \leq \| h \|_{L^1} \cdot \| \varphi_k \|_{0,1} \rightarrow 0,
$$

so we have shown that our functional extends continuously to $\mathcal{S}$.

Regarding $\zeta$, first note that the (scaled) sinc function $\mathbb{R}^\times \ni t \mapsto \frac{\sin(tx)}{tx}$ can be continuously extended at 0 by assigning it the value $x$, and that it is bounded by $|x|$. Hence, since $h$ is integrable, we see that $\zeta(x)$ is actually a finite number, so $\zeta$ is well-defined. To check that $\zeta$ is the Fourier transform, we can equivalently show that $\zeta = \text{pv} \left( \int \frac{h(t)}{t} \right)$, so consider

$$
\langle \zeta, \varphi \rangle = \int_{-\infty}^{\infty} \zeta(x) \varphi(x) \, dx = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \frac{\sin(tx)}{it} h \left( \frac{t}{2\pi} \right) \, dt \right) \varphi(x) \, dx.
$$

As mentioned above, $\left| \frac{\sin(tx)}{it} \right| \leq |x|$, so since $h$ and $x \varphi$ are integrable (the latter because $\varphi \in \mathcal{S}$), we can use the Dominated Convergence Theorem to get

$$
\langle \zeta, \varphi \rangle = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\epsilon} \frac{\sin(tx)}{it} h \left( \frac{t}{2\pi} \right) \, dt + \int_{\epsilon}^{\infty} \frac{\sin(tx)}{it} h \left( \frac{t}{2\pi} \right) \, dt \right) \varphi(x) \, dx.
$$

Now again, for any $\epsilon > 0$, the following two integrals exist since $h$ is integrable, and are equal because $h$ is even:

$$
\int_{-\infty}^{-\epsilon} \cos(xt) h \left( \frac{t}{2\pi} \right) \, dt = - \int_{\epsilon}^{\infty} \cos(xt) h \left( \frac{t}{2\pi} \right) \, dt.
$$

Therefore, with the previous computation,

$$
\langle \zeta, \varphi \rangle = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{-\epsilon} - \frac{\epsilon e^{itx}}{t} h \left( \frac{t}{2\pi} \right) \, dt + \int_{\epsilon}^{\infty} - \frac{\epsilon e^{itx}}{t} h \left( \frac{t}{2\pi} \right) \, dt \right) \varphi(x) \, dx
$$

$$
= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \left( \int_{-\epsilon}^{\infty} e^{-2\pi itx} h(t) \frac{h(t)}{t} \, dt + \int_{-\infty}^{-\epsilon} e^{-2\pi itx} h(t) \frac{h(t)}{t} \, dt \right) \varphi(x) \, dx.
$$

A standard use of Tonelli’s and Fubini’s Theorem shows that we can interchange the order of integration, so that

$$
\langle \zeta, \varphi \rangle = \lim_{\epsilon \rightarrow 0} \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\epsilon} e^{-2\pi itx} \varphi(x) \, dx \right) \frac{h(t)}{t} \, dt + \int_{\epsilon}^{\infty} \left( \int_{-\infty}^{\infty} e^{-2\pi itx} \varphi(x) \, dx \right) \frac{h(t)}{t} \, dt \right].
$$
Since \( \varphi \in S \), we know that the inversion formula holds: for almost every \( t \), we have
\[
\varphi(t) = \hat{\varphi}(t) = \int_{-\infty}^{\infty} e^{-2\pi i xt} \varphi(x) \, dx.
\]

Therefore,
\[
\langle \hat{\zeta}, \varphi \rangle = \lim_{\epsilon \to 0^+} \left[ \int_{-\infty}^{\epsilon} \varphi(t) \frac{h(t)}{t} \, dt + \int_{\epsilon}^{\infty} \varphi(t) \frac{h(t)}{t} \, dt \right] = \left\langle \text{pv} \left( \int \frac{h(t)}{t} \right), \varphi \right\rangle.
\]

**Definition 14:** A smooth function \( \chi : \mathbb{R} \to [-1, 1] \) is a normalizing function if

1. \( \chi \) is odd,
2. for \( x > 0 \) we have \( \chi(x) > 0 \), and
3. for \( x \to \pm \infty \), we have \( \chi(x) \to \pm 1 \).

**Lemma 4.11.** For every \( \epsilon > 0 \), there exists a normalizing function \( \chi \) whose (distributional) Fourier transform is supported in \((-\epsilon, \epsilon)\).

**Proof.** We will follow the instructions in [6, Exercise 10.9.3].

Fix an even function \( g \in C_c^\infty(\mathbb{R}, \mathbb{R}) \) such that \( g * g(0) = \frac{1}{\pi} \). One could, for example, take a rescaled version of the function
\[
t \mapsto \begin{cases} 
\exp\left(-\frac{1}{1-t^2}\right) & \text{if } |t| < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( f := g * g \), and define
\[
\chi(x) := \int_{-\infty}^{\infty} \frac{\sin(xt)}{t} f(t) \, dt,
\]
which is well-defined (see proof of Lemma 4.10), odd, and smooth.

Now, recall that for any \( a > 0 \), the sinc function is the Fourier transform of a scaled characteristic function, namely
\[
\frac{\sin(2\pi at)}{\pi t} = \hat{1}_{[-a, a]}(t).
\]

Using [4, Lemma 8.25], we can thus rewrite \( \chi \) for positive \( x \) as follows:
\[
\chi(x) = \pi \int_{-\infty}^{\infty} \mathbb{1}_{[-\frac{x}{\pi}, \frac{x}{\pi}]}(t) f(t) \, dt = \pi \int_{-\infty}^{\infty} \mathbb{1}_{[-\frac{x}{\pi}, \frac{x}{\pi}]}(t) \hat{f}(t) \, dt
\]
\[
= \pi \int_{-\frac{x}{\pi}}^{\frac{x}{\pi}} \hat{f}(t) \, dt = \pi \int_{-\frac{x}{\pi}}^{\frac{x}{\pi}} \hat{g}(t)^2 \, dt,
\]
from which we see that \( \chi(x) \geq 0 \). Moreover, since \( g \) is non-zero and smooth with compact support, \( \hat{g} \) does not vanish on any interval (cf. [4, p. 293]). The above equality hence gives \( \chi(x) > 0 \) for \( x > 0 \), so \( \chi \) satisfies Property 2 of normalizing functions. Furthermore,
\[
\pi \int_{-\frac{x}{\pi}}^{\frac{x}{\pi}} \hat{g}(t)^2 \, dt \leq \pi \int_{-\infty}^{\infty} \hat{g}(t)^2 \, dt = \pi \|g\|^2 \overset{(x)}{=} \pi \|g\|^2 = \pi f(0) = 1,
\]
where (*) holds because of the Plancherel Theorem (see [4, 8.29]), so we have shown that \( \chi \) is indeed \([-1,1]\)-valued. Next, the Dominated Convergence Theorem allows us to compute

\[
\lim_{x \to \infty} \chi(x) = \lim_{x \to \infty} \int \frac{\sin(t)}{t} f\left(\frac{t}{x}\right) \, dt \overset{\text{DCT}}{=} \int \frac{\sin(t)}{t} f(0) \, dt = 1,
\]

so we have shown Property 3 of normalizing functions.

From Lemma 4.10 (\cite{4}, Theorem 8.2b), we see that

\[ \hat{\chi} = pv\left( \int \frac{f(2\pi t)}{i t} \right), \]

so

\[ \langle \hat{\chi}, \varphi \rangle = \langle \hat{\chi}, \varphi \rangle = \lim_{\epsilon \to 0} \left( \int_{-\infty}^{\epsilon} \frac{f(2\pi t)}{i t} \varphi(-t) \, dt + \int_{\epsilon}^{\infty} \frac{f(2\pi t)}{i t} \varphi(t) \, dt \right). \]

Thus, if \( \varphi \) has support disjoint from the support of \( t \mapsto f(-2\pi t) = f(2\pi t) \), then \( \langle \hat{\chi}, \varphi \rangle = 0 \). In other words, the support of \( \hat{\chi} \) is contained in \( \frac{1}{2\pi} \text{supp}(f) \), which is compact.

Lastly, out of \( \chi \) with Fourier Transform supported in, say, \((-b,b)\), we want to construct another normalizing function whose Fourier transform is supported in \((-\epsilon,\epsilon)\). Let \( T(x) := \frac{b}{x} \) and \( \chi_2 := \chi \circ T \). As \( \epsilon, a \) are positive, this is again a normalizing function, and we compute for \( \varphi \in \mathcal{S} \),

\[ \langle \hat{\chi}_2, \varphi \rangle = \langle \chi_2, \varphi \rangle = \int_{-\infty}^{\infty} \chi_2(x) \hat{\varphi}(x) \, dx = \int_{-\infty}^{\infty} (\chi \circ T)(x) \hat{\varphi}(x) \, dx \]

\[ = \int_{-\infty}^{\infty} \chi(x)(\hat{\varphi} \circ T^{-1})(x)(T^{-1})'(x) \, dx. \]

From [4, Theorem 8.2b] we know that

\[ (\hat{\varphi} \circ T^{-1}) \cdot (T^{-1})' = (\varphi \circ T). \]

If \( \varphi \) is now supported outside of \((-\epsilon,\epsilon)\), so that \( \varphi \circ T \) is supported outside of \((-b,b)\), then the above computations yield

\[ \langle \hat{\chi}_2, \varphi \rangle = \int_{-\infty}^{\infty} \chi(x)(\varphi \circ T)(x) \, dx = \langle \hat{\chi}, \varphi \circ T \rangle = 0. \]

This proves that \( \hat{\chi}_2 \) is supported in \((-\epsilon,\epsilon)\).

\[ \square \]

**Lemma 4.12** (cf. \cite{8}, Proposition 10.3.5). If \( D \) is an essentially self-adjoint differential operator on \( M \) and \( \psi \) a bounded Borel function on \( \mathbb{R} \) whose Fourier transform has compact support, then for all \( u, v \in \Gamma_c^\infty(M; S) \), we have

\[ \langle \psi(D)u \mid v \rangle = \left( \int \left( e^{2\pi isD}u \mid v \right) ds \right). \]

**Proof.** If we first take \( \psi_1 \in \mathcal{S} \), then \( \hat{\psi}_1 = \hat{\psi}_1 \), so that

\[ \langle \psi_1(D)u \mid v \rangle = \left( \int e^{2\pi isD} \hat{\psi}_1(s) \, ds \right) u \mid v \right) = \int \left( e^{2\pi isD}u \mid v \right) \hat{\psi}_1(s) \, ds. \]
Since for functions in \( L^1(\mathbb{R}) \), the classical Fourier transform coincides with the distributional Fourier transform, the above equation can be rewritten as

\[
\langle \psi_1(D)u \mid v \rangle = \langle \hat{\psi}_1 , g \rangle,
\]

where \( g(s) := \{e^{2\pi isD}u \mid v\} \), which was to be shown. Using the inversion formula for \( \psi_2 = \hat{\psi}_1 \in \mathcal{S} \) once more, we could also write this as

\[
\langle \hat{\psi}_2(D)u \mid v \rangle = \langle \psi_2 , g \rangle
\]

for \( \psi_2 \in \mathcal{S} \) arbitrary. Now let us take a general \( \psi \) as specified in the lemma. As explained in Example 4, \( \psi \) gives rise to a tempered distribution, denoted by \( F \) for now. In particular, it makes sense to speak of its Fourier transform. Fix some \( \phi \in C^\infty_c(\mathbb{R}, \mathbb{R}) \) with \( \int \phi(x) \, dx = 1 \), and define \( \phi_t(x) := \frac{1}{t}\phi\left(\frac{x}{t}\right) \). Since we have assumed \( \hat{F} \) to have compact support, \( \hat{F} \ast \phi_t \in C^\infty_c(\mathbb{R}) \) by Lemma 4.9 so that Equation (13) implies

\[
\langle (\hat{F} \ast \phi_t)(D)u \mid v \rangle = \langle \hat{F} \ast \phi_t , g \rangle.
\]  

If we can now show that

(1) \( \langle \hat{F} \ast \phi_t \rangle = \psi \cdot \hat{\phi}_t \),
(2) \( \lim_{t \to 0} \langle (\hat{F} \ast \phi_t)(D)u \mid v \rangle = \langle \psi(D)u \mid v \rangle \), and
(3) \( \lim_{t \to 0} \langle \hat{F} \ast \phi_t , g \rangle = \langle \hat{F} , g \rangle \),

then

\[
\langle \psi(D)u \mid v \rangle = \lim_{t \to 0} \langle (\hat{F} \ast \phi_t)(D)u \mid v \rangle \leq \langle \hat{F} \ast \phi_t , g \rangle = \langle \hat{F} , g \rangle,
\]

so we would be done.

**ad (1):** By virtue of [4] p. 283, it suffices to check that the functions induce the same distribution: we recall that \( \hat{\phi}(x) = \phi(-x) \), and compute for \( f \in C^\infty_c \),

\[
\langle (\hat{F} \ast \phi_t) , f \rangle = \langle \hat{F} \ast \phi_t \rangle = \langle \hat{F} , \hat{F} \ast \phi_t \rangle = \langle F , (\hat{F} \ast \phi_t)^* \rangle = \langle \hat{F} , (\hat{F} \ast \phi_t)^* \rangle
\]

\[
= \langle F , \hat{f} \ast \hat{\phi}_t \rangle = \int_{-\infty}^{\infty} \psi(x)f(x) \hat{\phi}(x) \, dx = \langle \hat{\psi} \cdot \hat{\phi}_t , f \rangle.
\]

**ad (2):** Using Property (6) of Functional Calculus, it is sufficient to show that \( \| \psi \cdot \hat{\phi}_t \|_\infty \leq \| \psi \|_\infty \cdot \| \hat{\phi} \|_\infty < \infty \).

Secondly, suppose \( \text{supp}(\phi) \subseteq [-a, a] \), and take \( h \in C^\infty_c \) such that \( h_{[\epsilon, \epsilon]} \equiv 1 \), so that \( \phi_t = \phi_t \cdot h \) for \( t \leq 1 \). Since \( \phi_t \rightharpoonup \delta \) in \( \mathcal{D}' \) for \( t \to 0 \) by [4] Prop. 9.1, we let for \( y \in \mathbb{R} \), \( f_y(x) := e^{2\pi ixy}h(x) \in C^\infty_c \) and get

\[
\hat{\phi}_t(y) = \int_{-\infty}^{\infty} e^{2\pi ixy} \phi_t(x) h(x) \, dx = \langle \phi_t , f_y \rangle \rightharpoonup \langle \delta , f_y \rangle = f_y(0) = 1.
\]

**ad (3):** Recall that we defined \( g(s) := \{e^{2\pi isD}u \mid v\} \) for fixed compactly supported sections \( u,v \). Since \( \partial^m g \) is bounded, it follows from [4] Thm. 8.14(c) that

\[
\partial^m(g \ast \phi_t) = (\partial^m g) \ast \phi_t \rightharpoonup \partial^m g.
\]
uniformly on compact sets. Let us take \( h \in C^\infty_c \) such that \( 0 \leq h \leq 1 \)
and \( h|_{\text{supp} \hat{f}} \equiv 1 \), where we use that \( \hat{F} \) is compactly supported. As each \( \partial^j h \)
has compact support, we get
\[
\left\| (\partial^j h) \cdot (\partial^m (g \ast \hat{\phi}_t) - \partial^m g) \right\|_\infty \to 0.
\]
It follows by the product rule that, for any \( k \),
\[
\left\| \partial^k (h \cdot [g \ast \hat{\phi}_t]) - \partial^k (h \cdot g) \right\|_\infty \to 0,
\]
that is, \( h \cdot [g \ast \hat{\phi}_t] \to h \cdot g \) in \( C^\infty_c(\mathbb{R}) \). As \( \hat{F} \) has compact support, it is in
the dual space of \( C^\infty_c(\mathbb{R}) \). Since \( h|_{\text{supp} \hat{f}} \equiv 1 \), we therefore get
\[
\left\langle \hat{F} \ast \phi _t , g \right\rangle = \left\langle \hat{F} , h \cdot [g \ast \hat{\phi}_t] \right\rangle \left\| \hat{F} , h \cdot g \right\| \left\| \hat{F} , g \right\|
\]
which finishes our proof. \( \square \)

5. The Main Theorem

**Theorem 5.1** (cf. [6] Thm. 10.6.5). Let \( D \) be a symmetric elliptic differential
operator on a smooth and compact manifold \( M \). Let \( \mathcal{H} := L^2(M; S) \) and let \( \mathcal{M} \) be
the representation of \( C(M) \) on \( \mathcal{H} \) by multiplication. For \( \chi \) a normalizing function
and \( F := \chi(D) \), the triple \( (\mathcal{M}, \mathcal{H}, F) \) is a Fredholm module. Moreover, its class in
\( K^1(C(M)) \) does not depend on the choice of \( \chi \) and can hence be denoted by \([D]\).

This is the theorem we are set out to prove. As a first step, let us see that we
have functional calculus at our disposal, so that \( \chi(D) \) makes sense.

**Proposition 5.2** (cf. [6] Lemma 10.2.5). Let \( D \) be a symmetric differential operator on a smooth manifold \( M \) and let \( u \in L^2(M; S) \) be compactly supported. Then \( u \in \text{dom} \overline{D} \) if and only if \( u \in \text{dom} D^* \). In particular, if \( M \) is compact, then \( D \) is essentially self-adjoint.

To prove Proposition 5.2 we need the following two lemmas.

**Lemma 5.3** ([6] Lemma 1.8.1] - without proof). If \( D \) is a closable unbounded operator, then \( u \in \text{dom} \overline{D} \) if and only if there exists a sequence \( \{ u_j \}_j \) in \( \text{dom} D \) such that \( u_j \to u \) and \( \{ \| D u_j \| \}_j \) is bounded.

**Lemma 5.4** (cf. [6] Exercise 10.9.1]). For \( K \subseteq M \) compact, there exist for sufficiently small \( \epsilon > 0 \), operators \( F_\epsilon : L^2(K; S) \to L^2(M; S) \) which satisfy
\begin{enumerate}
  \item \( \| F_\epsilon \| \leq C \) for some constant \( C \) and all \( t \),
  \item \( \forall u \in L^2(K; S) : \lim_{t \to 0} F_t u = u \) in \( L^2(M; S) \),
  \item \( \forall u \in L^2(K; S) : F_t u \) is smooth with compact support, and
  \item for any differential operator \( D \) on \( M \), \([D, F_\epsilon]\) extends to a bounded operator \( L^2(K; S) \to L^2(M; S) \), and its norm is bounded independent of \( t \).
\end{enumerate}

We remark that the constant in Property 5.1 is usually supposed to be 1, but
\( C \) is good enough for us. For a proof of the existence of these so-called Friedrichs’
mollifiers, see the appendix on p. 32.
Proof of Proposition 5.2. Since the minimal domain of $D$ is always contained in the maximal domain, let us take $u \in \text{dom} \, D^*$ with compact support (pick any representative). According to Lemma 5.3, we need to find a sequence of $v_n$ in $\text{dom} \, D$ which converges to $u$ in the Hilbert space and such that $\{Dv_n\}_n$ is bounded. Let us take $F_t$ as in Lemma 5.4 for $K := \text{supp}(u)$, let $t_n$ be a sequence converging to 0, and let $v_n := F_{t_n}u$. Since $v_n \in \Gamma_c^\infty(M;S)$ by Property (3) of the mollifiers, it is in the domain of $D$, and by Property (2) $v_n \to u$ in $L^2(M;S)$. It remains to see why the sequence $D(v_n)$ is bounded.

By Lemma 4.1 $D^*u$ is in $L^2(K;S)$, so $F_t(D^*u)$ makes sense. Moreover, by Property (4) of the mollifiers, we also have that $[D, F_t]u$ has a well-defined meaning. All in all, we can therefore write

$$D(v_n) = D^*(F_{t_n}u) = F_{t_n}(D^*u) + [D, F_{t_n}]u.$$ 

Because of Property (1) and Property (4) of $\{F_t\}_t$, there exists $C > 0$ such that for all $t$, we have $\|F_t\|_1 \|D, F_t\| < C$. Hence

$$\|D(v_n)\| \leq \|F_{t_n}(D^*u)\| + \|D, F_{t_n}\|u\| \leq C \cdot (\|D^*u\| + \|u\|),$$

so the sequence is indeed bounded. □

The next proposition will show that the class $[D]$ does not depend on the choice of normalizing function $\chi$, and that $\chi(D)^2 - 1$ is compact.

Proposition 5.5 (cf. [6] Prop. 10.4.5, Lemma 10.6.3). If $D$ is a symmetric elliptic differential operator on a compact manifold $M$, and $\phi \in \mathcal{C}_0(\mathbb{R})$, then $\phi(D) : L^2(M;S) \to L^2(M;S)$ is a compact operator. In particular, if $\chi_1, \chi_2$ are normalizing functions, then the operators $\chi_1(D)$ and $\chi_2(D)$ differ only by a compact operator.

Proof: We first want to show that $\text{dom} \, D = L^2_1(M;S)$ by showing the following containments:

$$\text{dom} \, \overline{D} \subseteq L^2_1(M;S) \subseteq \text{dom} \, D^* = \text{dom} \, \overline{D}.$$ 

By Proposition 5.2, our symmetric operator is essentially self-adjoint (which explains the equality on the right), and Corollary 4.6 gives us $L^2_1(M;S) \subseteq \text{dom} \, D^*$. Now suppose $u \in \text{dom} \, \overline{D}$, that is, $(u, Du) \in \Gamma(D) = \Gamma(D)$. This means there is a sequence $(u_j)_j$ in $D$ such that $u_j \to u$ and $Du_j \to \overline{D}u$ in $L^2(M;S)$. In particular, $(u_j)_j$ is Cauchy in $L^2(M;S)$, so Gårding’s inequality implies that $(u_j)_j$ is Cauchy with respect to $\| \cdot \|_1$ (remember that $M$ is assumed compact). As $L^2_1(M;S)$ is (by definition) complete with respect to this norm, $(u_j)_j$ thus has a $\| \cdot \|_1$-limit in $L^2_1(M;S)$. The Rellich lemma, for example, shows that this limit must coincide with $u$, so we have shown $u \in L^2_1(M;S)$. All in all, $\text{dom} \, \overline{D} = L^2_1(M;S)$.

Now let us focus on the function $\psi(x) = (1 + x)^{-1}$. Since the domain of $D$ is dense and $\overline{D}$ is self-adjoint, Lemma 3.1 implies that $i + \overline{D}$ has full range. Thus, for every $v \in L^2_1(M;S)$, there exists $v \in \text{dom} \, \overline{D} = L^2_1(M;S)$ such that $(i + \overline{D})v = u$. Since $\overline{D}$ is self-adjoint, we know that

$$(i + \overline{D})v = v^2 + \overline{D}v^2,$$

see Equation (3). Hence it follows from Gårding’s inequality and the properties of Functional Calculus that, for some $c > 0$,

$$c \cdot \psi(\overline{D})u \|_1 = c \cdot \|v\|_1 \leq \|v\| + \|\overline{D}v\| \leq \sqrt{2}(i + \overline{D})v = \sqrt{2} \|u\|.$$
In other words, \( \psi(\mathcal{D}) \) is a bounded operator \( L^2(M; S) \rightarrow L^2_1(M; S) \), and thus by the Rellich lemma, it is a compact operator \( L^2(M; S) \rightarrow L^2(M; S) \). Lastly, if we take an arbitrary \( \varphi \in C_0(\mathbb{R}) \), then for any \( \epsilon > 0 \), there are finitely many \( a_{i,j} \in \mathbb{C} \) such that

\[
\left\| \varphi - \sum_{i,j=0}^m a_{i,j} \psi^i \overline{\psi}^j \right\|_\infty < \epsilon,
\]

because \( \psi \) generates \( C_0(\mathbb{R}) \) as a \( C^* \)-algebra. By Property 5.9 of Functional Calculus, we get for \( f := \varphi - \sum_{i,j=0}^m a_{i,j} \psi^i \overline{\psi}^j \) that

\[
\|f(\mathcal{D})\| = \|f\|_\infty < \epsilon.
\]

This means that the operator \( \varphi(\mathcal{D}) \) is approximated by compact operators and is hence itself compact. \( \square \)

The remaining work before the proof of Theorem 5.1 on page 28 will culminate in Proposition 5.9 which says that \( [\chi(D), M/\dot{\lambda}] \) is compact for \( f \in C(M) \).

**Proposition 5.6** (cf. [6] Prop. 10.3.1]). If \( D \) is an essentially self-adjoint differential operator on \( M \), and if \( W \) is an open neighborhood of a compact set \( K \subseteq M \), then there exists \( \epsilon > 0 \) such that

\[
\forall |s| < \epsilon, \forall u \in L^2(K; S) : \quad \text{supp} (e^{isD} u) \subseteq W.
\]

**Proof.** We will follow the proof given in [6]. Let \( g \in C_0^\infty(M, [0,1]) \) be such that

\[
g_{|K} \equiv 1 \quad \text{and} \quad g_{|M-W} \equiv 0.
\]

Pick \( f \in C^\infty(\mathbb{R}, [0,1]) \) non-decreasing such that

\[
\begin{align*}
&\text{for } t < 1 : f(t) < 1, \quad \text{and} \quad \text{for } t \geq 1 : f(t) = 1.
\end{align*}
\]

We have shown in Lemma 4.2 that \( [D, M_g] \) is bounded (even on all of \( L^2(M; S) \) since \( g \) is compactly supported), so let \( c > \|[D, M_g]\| \). We use this to define for \( s \in \mathbb{R}^n \) and \( p \in M \):

\[
h_s(p) := f(g(p) + cs), \quad L_s := \{ p \in M | h_s(p) = 1 \}.
\]

We will deal with positive \( s \) only; for negative \( s \), do the same construction for \( -D \).

**Claim 1.** If \( t \leq s \), then \( L_t \subseteq L_s \).

**Proof of claim.** An element \( p \) is in \( L_t \) exactly if \( f(g(p) + ct) = 1 \). By choice of \( f \), this means \( g(p) + ct \geq 1 \). As \( s \geq t \) and \( c \) is positive, this implies \( g(p) + cs \geq 1 \) also, hence \( f(g(p) + cs) = 1 \). Therefore, \( p \in L_s \). \( \square \)

**Claim 2.** For \( 0 \leq s < \frac{1}{c} \), we have \( K \subseteq L_0 \subseteq L_s \subseteq W \).

**Proof of claim.** For the first inclusion, use \( g_{|K} \equiv 1 \) to see \( f(g(p)) = 1 \) for any \( p \in K \) by choice of \( f \). The second inclusion follows from the above computation. For the last inclusion, recall that, if \( p \notin W \), then \( g(p) = 0 \) by choice of \( g \). Since \( cs < 1 \) by choice of \( s \), we therefore have \( h_s(p) = f(cs) < 1 \) by choice of \( f \). \( \square \)

Let us write \( \hat{h}_s \) to denote

\[
\hat{h}_s(p) := \partial_s (s \mapsto h_s(p)) \big|_{s} = cf'(g(p) + cs)
\]
for \( p \in M \). Since \( c \) is positive and \( f \) is non-decreasing, we have \( \hat{h}_s(p) \geq 0 \) for all \( s \) and \( p \).

Claim 3. \([D, M_{h_s}] = \frac{1}{c} M_{h_s} [D, M_g] \).

Proof of claim. For \( D \) locally as in Equation (7), we have shown in Equation (8) that
\[
\left( [D, M_{h_s}] u \right)(p) = \sum_{j=1}^{n} \partial_j (h_s \circ \varphi^{-1})|_{\varphi(p)} \cdot \Psi^{-1} \left( p, A^j(p) \cdot (\psi \circ u(p)) \right),
\]
and similarly,
\[
\left( \frac{1}{c} M_{h_s} [D, M_g] u \right)(p) = \frac{1}{c} \hat{h}_s(p) \sum_{j=1}^{n} \partial_j (g \circ \varphi^{-1})|_{\varphi(p)} \cdot \Psi^{-1} \left( p, A^j(p) \cdot (\psi \circ u(p)) \right).
\]
If we write \( h_s = f \circ k_s \) where \( k_s(p) := g(p) + cs \), then the chain rule gives
\[
\partial_j (h_s \circ \varphi^{-1})|_{\varphi(p)} = f'(k_s(p)) \partial_j (k_s \circ \varphi^{-1})|_{\varphi(p)} = \frac{1}{c} \hat{h}_s(p) \partial_j (g \circ \varphi^{-1})|_{\varphi(p)}
\]
for each \( 1 \leq j \leq n \), which implies the claim. □

Because of Claim 3 we have
\[
M_{h_s} - i [D, M_{h_s}] = \frac{1}{c} M_{h_s} (c - i [D, M_g]).
\]
By choice of \( c \), we see that
\[
c \cdot 1 \geq \| i [D, M_g] \| \cdot 1 \geq \| i [D, M_g] \|,
\]
so \( c - i [D, M_g] \geq 0 \). By Lemma 4.2 \([D, M_g] \) is a multiplication operator, so it commutes with \( M_{h_s} \). As \( \hat{h}_s \) is non-negative, we have therefore shown that
\[
M_{h_s} - i [D, M_{h_s}] \geq 0. \tag{15}
\]
Since it suffices to prove the proposition for \( u \in \Gamma^\infty(K; S) \), fix such \( u \) and define \( u_s := e^{isD} u \). Since \( (\partial_s u_s)|_s = iDu_s \), we have
\[
\partial_s \left( h_s \cdot u_s | u_s \right)_s = \left( \partial_s (h_s \cdot u_s)_s | u_s \right) + \left( h_s \cdot u_s \right)_s | (\partial_s u_s)_s = \left( h_s \cdot u_s + iDu_s \right)_s | u_s = \left( h_s \cdot u_s + iD\hat{h}_s \cdot Du_s \right)_s | u_s - iD(h_s \cdot u_s)_s | u_s \right)_s \text{ as } D \in D^* \geq 0 \text{ by Equation (15).}
\]
This means that \( \langle h_s \cdot u_s | u_s \rangle \) is an increasing function, and in particular for \( s \geq 0 \)
\[
\langle h_s \cdot u_s | u_s \rangle = \langle h_0 \cdot u_0 | u_0 \rangle = \langle h_0 \cdot u | u \rangle \geq \langle u | u \rangle = \langle u_s | u_s \rangle,
\]
where \( (\ast) \) holds because \( h_0 = f \circ g \) is \( 1 \) on \( K \supseteq \text{supp}(u) \), and the last equality comes from \( e^{isD} \) being a unitary. Since \( 1 \geq h_s \geq 0 \), this means
\[
\| u_s \|^2 \geq \left\| \sqrt{h_s} u_s \right\|^2 = \langle h_s \cdot u_s | u_s \rangle \geq \langle u_s | u_s \rangle = \| u_s \|^2.
\]
Therefore,
\[
\int_M \| u_s(p) \|^2_{S_p} d\mu = \int_M \left\| \sqrt{h_s(p)} u_s(p) \right\|^2_{S_p} d\mu.
\]
Proof. For any \( u \) from \( K \), we then get by Lemma 4.12 that, for all \( p \in M \), we can take a normalizing function \( L \) to use the given functions \( f \). By Lemma 4.11, we can take a normalizing function \( M \) with disjoint supports, and suppose \( \text{supp}(f_1) \) is compact. Then there exists \( \epsilon > 0 \) such that

\[
\forall |s| < \epsilon : \quad M_{f_1} \circ e^{isD} \circ M_{f_2} = 0.
\]

Proof. By assumption, \( K := \text{supp}(f_1) \) is compact. Since the support of \( f_1 \) is disjoint from \( K \), the set \( W := M \setminus \text{supp}(f_1) \) is an open neighborhood of \( K \). By Proposition 5.6, there exists an \( \epsilon > 0 \) such that

\[
\forall |s| < \epsilon, \forall v \in L^2(K;S), \quad \text{supp}(e^{isD}v) \subseteq W.
\]

For any \( u \in L^2(M;S) \), we know that \( M_{f_2}u \) is supported in \( K \), so \( e^{isD}M_{f_2}u \) is supported in \( W \). As \( W = M \setminus \text{supp}(f_1) \), we hence get

\[
M_{f_2}e^{isD}M_{f_2}u = 0
\]

for all \( u \in \Gamma^\infty(M;S) \).

Lemma 5.8 (Kasparov’s lemma; [6] 5.4.7 - without proof). Suppose \( X \) is compact Hausdorff, \( \nu : C(X) \to B(H) \) a non-degenerate representation, and \( T \in B(H) \). If \( \nu(f_1)T
\nu(f_2) \) is compact for every \( f_1, f_2 \in C(X) \) with disjoint support, then \( [T, \nu(f)] \) is compact for every \( f \in C(X) \).

Proposition 5.9 (cf. [6] Lemma 10.6.4]). If \( D \) a symmetric elliptic differential operator on a compact manifold \( M \), \( \chi \) a normalizing function, and \( f \in C(M) \), then \( [\chi(D), M_f] \) is compact.

Proof of Proposition 5.9. Since \( M \) is a non-degenerate representation of \( C(M) \) on \( L^2(M;S) \), Kasparov’s lemma says that it suffices to show that, for all \( f_1, f_2 \in C(M) \) with disjoint supports, \( M_{f_1}\chi(D)M_{f_2} \) is compact. Moreover, because of Proposition 5.5, we can actually show this for any normalizing function, and do not need to use the given \( \chi \).

So let us fix such \( f_1, f_2 \). By Corollary 5.7, there exists \( \epsilon > 0 \) such that

\[
\forall |s| < \epsilon : \quad M_{f_1}e^{2isD}M_{f_2} = 0.
\]

By Lemma 4.11, we can take a normalizing function \( \chi_1 \) with \( \text{supp}(\chi_1) \subseteq (-\epsilon, \epsilon) \). We then get by Lemma 4.12 that, for all \( \tilde{u}, \tilde{v} \in \Gamma^\infty(M;S) \) and \( g(s) := \langle e^{2isD} \tilde{u} | \tilde{v} \rangle \),

\[
\langle \chi_1(D) \tilde{u} | \tilde{v} \rangle = \langle \tilde{u} | g \rangle.
\]

(16)
If we choose \( \hat{u} := f_2 \cdot u \) and \( \hat{v} := \overline{f_1} \cdot v \) for \( u, v \in \Gamma^\infty(M; S) \), then
\[
g(s) = \left\langle e^{2\pi i sD} (f_2 \cdot u), \overline{f_1} \cdot v \right\rangle = \left\langle M_{f_2} \circ e^{2\pi i sD} \circ M_{f_1}(u)v \right\rangle,
\]
so that \( g(s) = 0 \) for \( |s| < \epsilon \) by choice of \( \epsilon \), and hence
\[
\left\langle \chi_1, g \right\rangle = 0 \quad \text{as} \quad \text{supp}(\chi_1) \subseteq (-\epsilon, \epsilon).
\]
Thus, Equation (10) gives \( \left\langle M_{f_1} \chi_1(D) M_{f_2} u | v \right\rangle = 0 \). We conclude that the same even holds for \( u, v \in L^2(M; S) \), so that we have proved \( M_{f_1} \chi_1(D) M_{f_2} = 0 \). \( \square \)

Finally, we can prove Theorem 5.1.

**Proof of Theorem 5.1** F is self-adjoint by Property (4) of Functional Calculus because \( \chi \) is real-valued. Since \( \chi \) is a normalizing function, \( \chi^2 - 1 \in C_0(\mathbb{R}) \), so Proposition 5.5 implies that \((\chi^2 - 1)(D) = F^2 - 1 \) is compact. Proposition 5.9 says \([\chi(D), M_f] = [F, M_f]\) is compact for any \( f \in C(M) \), so we have shown that the properties of a Fredholm module are satisfied. Lastly, if \( \chi_1 \) is another normalizing function, then by Proposition 5.5 again, \( \chi_1(D) \) differs from \( \chi(D) \) only by a compact operator. This means that \((M, \mathcal{H}, \chi_1(D))\) is a compact perturbation of \((M, \mathcal{H}, F)\). Therefore, they determine the same K-homology class by Remark 1. \( \square \)

**Remark 6:** There is an obvious extension of Theorem 5.1 to even K-homology: if \( S \) is equipped with a smooth idempotent vector bundle automorphism \( \gamma_S \) (that is, \( S \) is \( \mathbb{Z}/2\mathbb{Z} \)-graded), then the map
\[
\gamma: \Gamma^\infty_c(M; S) \to \Gamma^\infty_\gamma(M; S), \quad \gamma u(p) := \gamma_S(u(p)),
\]
extends to a grading operator of \( \mathcal{H} = L^2(M; S) \) with respect to which the left \( C(M) \)-action is even. If we further assume that \( D \) is odd, then Lemma 3.5 implies that \( F \) is odd as well, so that the Fredholm module \((M, \mathcal{H}, F)\) is actually graded. Again, the corresponding class in \( K^0(C(M)) \) only depends on \( D \).

### 5.1. The index pairing.
Let us give a quick example of why Theorem 5.1 is of interest. Using the index pairing, which we will construct below, a class \([D]\) for a symmetric elliptic differential operator \( D \) on a compact manifold \( M \) induces a homomorphism from the K-theory of \( M \) to \( \mathbb{Z} \).

**Recall (K-theory - cf. [6] Def. 4.1.1 and 4.8.1):** For a unital \( C^* \)-algebra \( A \), \( K_0(A) \) is the abelian group with one generator \([p]\) for each projection \( p \in M_k(A) \) over \( A \), subject to the following relations:

1. if \( p, q \in M_k(A) \) are homotopic as projections, then \([p] = [q]\),
2. for any projections \( p, q \) over \( A \), we have \([p] + [q] = [p \oplus q]\), and
3. for \( 0 \in M_k(A) \) for any \( k \), we have \([0] = 0\).

If we replace the word “projection” by “unitary” and 0 by 1 \( \in M_k(A) \) in the last condition, then the above is a definition of \( K_1(A) \).

If \( \alpha: A \to B \) is a unital *-homomorphism and \( p \) is a projection over \( A \), then the map \( K_0(\alpha) = \alpha_*: [p] \mapsto [\alpha(p)] \) makes \( K_0 \) a functor from the category of unital \( C^* \)-algebras to the category of abelian groups. A similar construction works for \( K_1 \).

**Proposition 5.10 (cf [6] Def. 7.2.1 and 7.2.3, Prop. 8.7.1 and 8.7.2):** For a separable unital \( C^* \)-algebra \( A \), there are two so-called index pairings between its K-
theory and its K-homology group of the same parity,
\[ \langle \cdot, \cdot \rangle : K_j(A) \times K^j(A) \to \mathbb{Z}, \quad (j = 0, 1) \]
which are bilinear and functorial in the following sense: if \( \alpha : A \to B \) is a \( \ast \)-homomorphism between two separable C\(^\ast\)-algebras, then we have for any \( x \in K_j(A) \) and \( y \in K^j(B) \):
\[ \langle \alpha_\ast(x), y \rangle = \langle x, \alpha^\ast(y) \rangle. \]

Let us construct these maps as done in [6]. For the case \( j = 1 \), given a unitary \( u \in M_k(A) \) and an ungraded unital Fredholm module \( (\nu, \mathcal{H}, F) \) over \( A \), let \( \mathcal{H}^k := \mathbb{C}^k \otimes \mathcal{H} \) and define the following operators on it:
\[ P := 1_k \otimes \frac{1}{2} (1 + F) \quad \text{and} \quad U := (1_k \otimes \nu)(u). \]
Then the operator
\[ PUP - 1 + P : \mathcal{H}^k \to \mathcal{H}^k \]
is Fredholm: since \( P \) essentially commutes with \( U \) and since \( P \) is essentially a projection, \( PUP - 1 + P \) can be shown to be essentially unitary and is hence Fredholm by Atkin’s Theorem (cf. [6, Thm. 2.1.4]). Since the map that assigns the index to a Fredholm operator is continuous, the map
\[ K_1(A) \times K^1(A) \to \mathbb{Z}, \quad ([u], [\nu, \mathcal{H}, F]) \mapsto \text{index}(PUP - 1 + P), \]
is well-defined (that is, the index of \( PUP - 1 + P \) only depends on the equivalence classes of \( u \) and \( (\nu, \mathcal{H}, F) \)).

For the case \( j = 0 \), given a projection \( p \in M_k(A) \) and a graded unital Fredholm module \( (\nu, \mathcal{H}, F) \) over \( A \), write
\[ F = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix} \]
with respect to the decomposition \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \). On the (now graded) Hilbert space \( \mathcal{H}^k = \mathbb{C}^k \otimes \mathcal{H} \), we define
\[ P := (1_k \otimes \nu)(p). \]
Then the operator
\[ P(1_k \otimes U)P : P(\mathbb{C}^k \otimes \mathcal{H}^+) \to P(\mathbb{C}^k \otimes \mathcal{H}^-) \]
is essentially unitary (the choice of domain and codomain are crucial for this to be true), and the map
\[ K_0(A) \times K^0(A) \to \mathbb{Z}, \quad ([p], [\nu, \mathcal{H}, F]) \mapsto \text{index}(P(1 \otimes U)P), \]
is well-defined.

Note that there is a more general construction for non-unital C\(^\ast\)-algebras, but the unital case is all we are interested in for the following corollary:

**Corollary 5.11.** A symmetric elliptic differential operator \( D \) on a smooth and compact manifold \( M \) gives rise to a map on K-theory,
\[ K_1(C(M)) \to \mathbb{Z}, \quad x \mapsto [x, [D]], \]
by pairing a K-theory class with the K-homology class \([D]\) constructed above. Similarly, if the vector bundle \( S \) over \( M \) which is underlying \( D \) is graded and if \( D \) is odd, then we get a map
\[ K_0(C(M)) \to \mathbb{Z}, \quad x \mapsto [x, [D]]. \]
APPENDIX

In order to prove Lemma 5.4 we first need the following version for $\mathbb{R}^n$:

**Lemma 5.12.** There exist operators $\tilde{F}_i : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

(a) $\|\tilde{F}_i\| \leq 1$,
(b) $\forall u \in L^2(\mathbb{R}^n) : \lim_{t \to 0} \tilde{F}_i u = u$ in $L^2(\mathbb{R}^n)$,
(c) $\forall u \in L^2(\mathbb{R}^n) : \tilde{F}_i u$ is smooth,
(d) if $u$ has compact support, then so does $\tilde{F}_i u$, and
(e) for all $1 \leq k \leq n$ and $f \in C^\infty(\mathbb{R}^n)$ with bounded partial derivatives, the operator $\left[ f \cdot \frac{\partial}{\partial x_k}, \tilde{F}_1 \right]$ extends to a bounded operator whose norm is bounded independent of $t$.

**Proof.** Pick a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^+$ with compact support and $\int_{\mathbb{R}^n} \phi \, d\lambda = 1$. Define $\phi_t(x) := t^{-n} \phi(\frac{x}{t})$, which has the same properties as $\phi$. Set $\tilde{F}_i u = \phi_{\ast} u$ for $u \in L^2(\mathbb{R}^n)$, that is:

$$\tilde{F}_i u(x) = t^{-n} \int_{\mathbb{R}^n} \phi \left( \frac{x - y}{t} \right) u(y) \, d\lambda(y).$$

By $[10]$ IV 9.4, we have

$$\|\tilde{F}_i\| \leq \sup\{\|\phi_t\|_{L^1} \cdot \|u\|_2 : \|u\|_2 \leq 1\} = \|\phi_t\|_{L^1} = 1,$$

so Property (a) holds. Moreover, Property (b) and (c) follow from $[10]$ Satz IV 9.5 and $[10]$ Korollar IV 9.7 respectively. It is well known that

$$\text{supp}(\tilde{F}_i u) \subseteq \text{supp}(\phi_t) + \text{supp}(u),$$

so that Property (d) follows from $\phi_t$ having compact support. It remains to check Property (e).

Using integration by parts and the fact that $\phi$ is compactly supported, we can compute for $u \in L^2(\mathbb{R}^n)$

$$\left[ f \cdot \frac{\partial}{\partial x_k}, \tilde{F}_1 \right] u(x) = \int_{\mathbb{R}^n} \left[ \frac{1}{t^{n+1}} \frac{\partial^2 \phi}{\partial x_k^2} \left( f(x) - f(y) \right) + \frac{1}{t^n} \frac{\partial \phi}{\partial x_k} \left( \frac{x - y}{t} \right) \frac{\partial f}{\partial x_k} \right] u(y) \, dy.$$

In other words, $\left[ f \cdot \frac{\partial}{\partial x_k}, \tilde{F}_1 \right]$ is an integral transform with kernel

$$k_t(x, y) = \frac{1}{t^{n+1}} \frac{\partial^2 \phi}{\partial x_k^2} \left( f(x) - f(y) \right) + \frac{1}{t^n} \frac{\partial \phi}{\partial x_k} \left( \frac{x - y}{t} \right) \frac{\partial f}{\partial x_k} \bigg|_y.$$

As stated in $[5]$ Thm. 5.2, the so-called Schur’s test says that, if

$$\sup_{x \in \mathbb{R}^n} \|k_t(x, \cdot)\|_{L^1} \leq \alpha \quad \text{and} \quad \sup_{y \in \mathbb{R}^n} \|k_t(\cdot, y)\|_{L^1} \leq \beta,$$

then the integral transform extends to a bounded operator whose norm is bounded by $\sqrt{\alpha \beta}$. We claim that, if for all $1 \leq j \leq n$ and $\supp(\phi) \subseteq [-a, a]$ we have

$$\left\| \frac{\partial^j f}{\partial x^j} \right\|_\infty < C,$$

then $\alpha = \beta = C(na \|\frac{\partial \phi}{\partial x_k}\|_{L^1} + \|\phi\|_{L^1})$ do the trick.
Lemma 5.13. For $K$ a compact subset of a manifold $M$, we can write $K = \bigcap_{k=1}^{\infty} V_k$ for some open sets $V_k \subseteq V_k \subseteq M$.

Proof. For $K$ contained in some chart $(U, \varphi)$, we have

$$\varphi(K) = \bigcap_{k=1}^{\infty} \tilde{V}_k,$$

where $\tilde{V}_k := \bigcup_{x \in \varphi(K)} B_{\frac{1}{k}}(x)$.

If we then let $\frac{1}{N}$ be smaller than the distance of the compact set $\varphi(K)$ to the closed set $\mathbb{R}^n \setminus \varphi(U)$, then for $k \geq N$ we have $\tilde{V}_k \subseteq \varphi(U)$, and hence

$$K = \bigcap_{k=N}^{\infty} V_k,$$

where $V_k := \varphi^{-1}(\tilde{V}_k)$.

Now, for arbitrary $K$, take finitely many open sets $U_1, \ldots, U_l$ which cover $K$ such that $K \cap U_i$ is contained in a chart. From the above, we get (after re-indexing)

$$K = \bigcap_{k=1}^{N} U_k \cup \ldots \cup \bigcap_{k=1}^{L} U_l = \left( \bigcap_{k=1}^{\infty} V^1_k \right) \cup \ldots \cup \left( \bigcap_{k=1}^{\infty} V^l_k \right) \subseteq \left( \bigcup_{k=1}^{\infty} V^1_k \cup \ldots \cup V^l_k \right).$$

Since each family $\{V^i_n\}_n$ is nested, we also have

$$\left( \bigcap_{k=1}^{\infty} V^1_k \right) \cup \ldots \cup \left( \bigcap_{k=1}^{\infty} V^l_k \right) \supseteq \bigcap_{k=1}^{\infty} \left( V^1_k \cup \ldots \cup V^l_k \right),$$

and hence $K = \bigcap_{k=1}^{\infty} \left( V^1_k \cup \ldots \cup V^l_k \right)$. \qed

Lemma 5.14. For $K$ a compact subset of a manifold $M$ and $\{U_i\}_{i=1}^{l}$ an open cover of $K$ in $M$, there exist smooth compactly supported functions $\rho_1, \ldots, \rho_l : M \to [0,1]$ such that $\text{supp}(\rho_i) \subseteq U_i$ and $\sum_{i=1}^{l} \rho_i(p) = 1$ for all $p \in K$. 

For $x, y \in \mathbb{R}^n$ such that $|x - y| \leq at$, repeated application of the Mean Value Theorem (see the proof of[8, Thm. 5.3.10], for example) gives

$$|f(x) - f(y)| \leq \sum_{j=1}^{n} at \left\| \frac{\partial f}{\partial x_j} \right\|_{\infty} \leq \text{nat} C.$$ 

For these $x, y$, we compute

$$|k_t(x, y)| \leq \frac{1}{t^{n+1}} \left\| \frac{\partial \phi}{\partial x_k} \right\|_{L^\infty} (f(x) - f(y)) + \frac{1}{tn} \left\| \phi \left( \frac{x - y}{t} \right) \frac{\partial f}{\partial x_k} \right\|_{L^\infty} \leq \frac{1}{t^{n+1}} \left\| \frac{\partial \phi}{\partial x_k} \right\|_{L^\infty} \cdot \text{nat} C + \frac{1}{tn} \left\| \phi \left( \frac{x - y}{t} \right) \right\|_{L^\infty} \cdot C$$

and we are done. \qed

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and we are done. \qed
Proof: With $U_0 := M \setminus K$, take a partition of unity $\{\rho_i\}_{i=0}^l$ of $M$ subordinate to the cover $\{U_i\}_{i=0}^l$. Since $M = U_0 \cup U_1 \cup \ldots \cup U_l$, we get from [3 Lemma 1.4.8] that there exists an open cover $\{V_i\}_{i=0}^l$ of $M$ with $\overline{V}_i \subseteq U_i$ for all $0 < i < l$. Now since $K \cap \overline{V}_i \subseteq V_i$ for $i \neq 0$, and $K \cap V_i \subseteq K$ is compact, we know by [4 Prop. 4.31] that there exists a precompact open set $W_i$ such that

$$K \cap V_i \subseteq W_i \subseteq \overline{W}_i \subseteq U_i.$$  

Note that the collection of $W_i$'s covers all of $K$, so that we can take a smooth partition of unity $\{\rho_i\}_{i=0}^l$ of $M$ which is subordinate to $\{M \setminus K\} \cup \{W_i\}_{i=1}^l$. In particular, since $W_i$ is precompact and $\text{supp}(\rho_i) \subseteq W_i$ for $i > 0$, we know that those $\rho$'s have compact support. Moreover, it follows from $\text{supp}(\rho_0) \subseteq M \setminus K$ that for $p \in K$

$$1 = \sum_{i=0}^l \rho_i(p) = \sum_{i=1}^l \rho_i(p).$$  

Lemma (Lemma 5.4). For $M$ and $S$ as specified at the beginning of Section 3 and any $K \subseteq M$ compact, there exist operators $F_i : L^2(K; S) \to L^2(M; S)$ for sufficiently small $\epsilon > t > 0$ which satisfy

1. $\|F_t\| \leq C$ for some constant $C$ and all $t$,
2. $\forall u \in L^2(K; S) : \lim_{t \to 0} F_t u = u$ in $L^2(M; S)$,
3. $\forall u \in L^2(K; S) : F_t u$ is smooth with compact support, and
4. for any differential operator $D$ on $M$, $[D, F_t]$ extends to a bounded operator $L^2(K; S) \to L^2(M; S)$, and its norm is bounded independent of $t$.

Proof of Lemma 5.4

Take an atlas $\mathcal{A}$ of $M$ whose charts are small enough to allow smooth trivializations of $S$ which are isometries on the fibres,

$$\begin{align*}
S_U &\xrightarrow{\nu} U \times \mathbb{C}^k \\
M \ni U &\xrightarrow{\pi} \mathbb{R}^n
\end{align*}$$

Let $\{(U_i, \varphi_i)\}_{i=1}^l$ be finitely many of those charts which cover the compact set $K$, and let $\{\rho_i\}_{i=1}^l$ be as in Lemma 5.14. For our trivializations, we will write

$$\Psi_i : S_{|U_i} \xrightarrow{\nu} U_i \times \mathbb{C}^k, \quad \psi_i := \text{pr}_2 \circ \Psi_i.$$

Moreover, let $f_i, g_i = \frac{1}{f_i} : \mathbb{R}^n \to (0, \infty)$ be such that for all $h \in C_c^\infty(\mathbb{R}^n)$ and $E \subseteq U_i$ Borel, we have

$$\int_E h \circ \varphi_i \, d\mu = \int_{\varphi_i(E)} h \circ f_i \, d\lambda \quad \text{and} \quad \int_E (h \cdot g_i) \circ \varphi_i \, d\mu = \int_{\varphi_i(E)} h \, d\lambda.$$  

We have assumed in Remark 3 that $\|f_i\|_{L^\infty}, \|g_i\|_{L^\infty} \leq L$ for some number $L$. In particular, we have for $v \in \bigoplus_{i}^l L^2_c(\mathbb{R}^n)$ and any $1 \leq i \leq \ell$,

$$\int_{U_i} \|v \circ \varphi_i(p)\|_{L^2_k}^2 \, d\mu = \int_{\mathbb{R}^n} \|v(x)\|_{L^2_k}^2 \cdot f_i(x) \, d\lambda \leq \|v\|_{L^2_k}^2 \cdot L.$$  

(17)
For \( u \in L^2(U_i; S) \), since \( \psi_i \) is isometric we get
\[
\int_{\mathbb{R}^n} \left\| \left( \psi_i \circ u \circ \varphi_i^{-1} \right)(x) \right\|^2_{C^k} \, d\lambda = \int_{\mathbb{R}^n} \left\| \left( u \circ \varphi_i^{-1} \right)(x) \right\|^2_{S_{\varphi_i^{-1}(x)}} \, d\lambda
\]
\[
= \int_{U_i} \left\| u(p) \right\|^2_{S_p} g_i(\varphi_i(p)) \, d\mu \leq \| u \|_2^2 \cdot L. \tag{18}
\]

For \( 1 \leq i \leq l \), we define
\[
\Phi_i^k L^2(\mathbb{R}^n) \quad \text{and} \quad L^2(U_i; S)
\]
\[
\begin{array}{cccc}
F_i^k: & L^2(U_i; S) & \rightarrow & \Phi_i^k L^2(\mathbb{R}^n) \\
& u & \mapsto & \psi_i \circ u \circ \varphi_i^{-1}
\end{array}
\]
\[
\begin{array}{cccc}
& v & \mapsto & \Psi_i^k(\cdot, v \circ \varphi_i)
\end{array}
\]
\[
\Phi_{j=1}^k w_j \mapsto \Phi_{j=1}^k \tilde{F}_i w_j
\]

Notice that, indeed, \( F_i^k \) takes values in \( \Gamma^\infty_c(U_i; S) \): since \( \varphi_i \) is a diffeomorphism, \( \psi_i \circ u \circ \varphi_i^{-1} \) is compactly supported when \( u \) is, and in particular, all of its component functions are compactly supported. By Property (c) of \( F_i \), \( \tilde{F}_i w_j \) is smooth, and by Property (d) it is compactly supported when \( w_j \) is. Since \( \varphi_i \) and \( \Psi_i \) are smooth, so is \( \Psi_i^k(\cdot, v \circ \varphi_i) \) for smooth \( v \), and again, since \( \varphi_i \) is a diffeomorphism, we conclude that \( \Psi_i^k(\cdot, v \circ \varphi_i) \) has compact support for compactly supported \( v \).

Now let
\[
F_i: L^2(K; S) \rightarrow L^2(M; S), \quad F_i u := \sum_{i=1}^{l} F_i^k(\rho_i \cdot u).
\]

By the above explanation, \( F_i \) actually takes values in \( \Gamma^\infty_c(M; S) \) because the \( \rho_i \) are compactly supported. Hence, \( F_i \) satisfies Property (3) and we need to check the other properties. By abuse of notation, we will write \( F_i \) for the operator \( \Phi_i^k \tilde{F}_i \).

ad Property (1) For \( u \in L^2(U_i; S) \), we compute
\[
\left\| F_i^k u \right\|_2^2 = \int_{U_i} \left\| \Psi_i^k(p, \tilde{F}_i(\psi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(p)) \right\|^2_{S_p} \, d\mu
\]
\[
= \int_{U_i} \left\| \tilde{F}_i(\psi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(p) \right\|^2_{C^k} \, d\mu \quad \text{as } \Psi_i(p, \cdot) \text{ is an isometry}
\]
\[
\leq \left\| \tilde{F}_i(\psi_i \circ u \circ \varphi_i^{-1}) \right\|^2_{C^k} \cdot L \quad \text{by Eq. (17)}
\]
\[
\leq \left\| \psi_i \circ u \circ \varphi_i^{-1} \right\|^2_{C^k} \cdot L \leq \| u \|^2 \cdot L^2 \quad \text{since } \| \tilde{F}_i \| \leq 1 \text{ and by Eq. (18)}.
\]

Hence \( \left\| F_i^k u \right\|_2 \leq L, \) so that
\[
\left\| F_i \right\| = \sup_{\| u \|_2 \leq 1} \left\| \sum_{i=1}^{l} F_i^k(\rho_i u) \right\|_2 \leq \sum_{i=1}^{l} \sup_{\| u \|_2 \leq 1} \left\| F_i^k(\rho_i u) \right\|_2 \leq \sum_{i=1}^{l} \left\| F_i^k \right\| \leq l \cdot L =: C.
\]

ad Property (2) As \( \Psi_i(p, \cdot) \) is an isometry, we have for \( u \in L^2(U_i; S) \), \( p \in U_i \):
\[
\left\| (F_i^k u - u)(p) \right\|_{S_p} = \left\| \Psi_i^k(p, \tilde{F}_i(\psi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(p)) - u(p) \right\|_{S_p}
\]
\[
= \left\| \tilde{F}_i(\psi_i \circ u \circ \varphi_i^{-1}) \circ \varphi_i(p) - (\psi_i \circ u \circ \varphi_i^{-1})(\varphi_i(p)) \right\|_{C^k}.
\]
By Equation (17), it therefore follows that
\[
\left\| F_i^t u - u \right\|_2^2 \leq \left\| \hat{F}_i (\psi_i \circ u \circ \varphi_i^{-1}) - (\psi_i \circ u \circ \varphi_i^{-1}) \right\|_2^2 \cdot L,
\]
so that Property (b) of \( \hat{F}_i \) implies that \( \lim_{t \to 0} F_i^t u = u \) in \( L^2 \)-norm. Therefore, for arbitrary \( u \in L^2(K; S) \),
\[
\left\| F_i u - u \right\|_2 = \left\| \sum_{i=1}^{l} (F_i^t(\rho_i u) - \rho_i u) \right\|_2 \leq \sum_{i=1}^{l} \left\| F_i^t(\rho_i u) - \rho_i u \right\|_2 \to 0.
\]

**ad Property (4)** Suppose \( D \) is a differential operator acting on the sections of \( S \). Since \( \rho_i u \) is supported in \( U_i \) for \( u \in \Gamma^{\infty}(M; S) \), we know that \( D(\rho_i u) \in \Gamma^{\infty}(U_i; S) \) by Property a) of differential operators. Therefore, \( F_i^t(D(\rho_i u)) \) also makes sense, and we can write
\[
[F_i, D] u = \sum_{i=1}^{l} F_i^t(\rho_i D_u) - D(F_i^t(\rho_i u))
\]
\[
= \sum_{i=1}^{l} F_i^t(\rho_i D_u) - F_i^t(D(\rho_i u)) + F_i^t(D(\rho_i u)) - D(F_i^t(\rho_i u))
\]
\[
= \sum_{i=1}^{l} F_i^t [M_{\rho_i}, D] u + [F_i^t, D](\rho_i u),
\]
that is,
\[
[F_i, D] = \sum_{i=1}^{l} F_i^t [M_{\rho_i}, D] + [F_i^t, D] M_{\rho_i}.
\] (19)

In order to check that \([F_i, D]\) extends to an operator that is bounded independent of \( t \), we will show that \([F_i^t, M_{\rho_i}, D]\) and \([F_i^t, D] M_{\rho_i}\) do. As was shown in Lemma 4.2 \([M_{\rho_i}, D]\) is a bounded operator on \( L^2(K; S) \), and since \( F_i^t \) is bounded independent of \( t \) (namely by \( L \), as was shown above), so is \([F_i^t, M_{\rho_i}, D]\). It remains to show that \( u \mapsto [F_i^t, D](\rho_i u) \) for a fixed but arbitrary \( 1 \leq i \leq l \) is bounded independent of \( t \). It suffices to consider those \( D \) that (locally) look like only one of the summands in Equation (7). First, recall that for a \( C^k \)-vector valued function \( w \) on \( \mathbb{R}^n \), we have
\[
\psi_i \circ \Psi_i^{-1}(\cdot, w(\cdot)) = w(\cdot),
\]
so for \( u_i := \rho_i u \) and \( p \in U_i \), we compute
\[
(F_i^t D u_i)(p) = \Psi_i^{-1}\left( p, \hat{F}_i (\psi_i \circ (D u_i) \circ \varphi_i^{-1}) \circ \varphi_i(p) \right)
\]
\[
= \Psi_i^{-1}\left( p, \hat{F}_i \left( (A \circ \varphi_i^{-1}) \cdot \partial_j (\psi_i \circ u_i \circ \varphi_i^{-1}) \right) \circ \varphi_i(p) \right)
\]
and
\[
(D F_i^t u_i)(p) = \Psi_i^{-1}\left( p, A(p) \partial_j (\psi_i \circ (F_i^t u_i) \circ \varphi_i^{-1})_{|\varphi_i(p)} \right)
\]
\[
= \Psi_i^{-1}\left( p, A(p) \partial_j \left( \hat{F}_i(\psi_i \circ u_i \circ \varphi_i^{-1}) \right)_{|\varphi_i(p)} \right)
\]
If we write \( \hat{A} := A \circ \varphi_i^{-1} \) and \( v_i := \psi_i \circ u_i \circ \varphi_i^{-1} \), then this means
\[
\left\| [F_i^t, D] u_i(p) \right\|_{S_p} = \left\| \left( \hat{F}_i (\hat{A} \cdot \partial_j v_i) - \hat{A} \cdot \partial_j (\hat{F}_i v_i) \right) \circ \varphi_i(p) \right\|_{C^k}.
\]
Hence by Equation (17):
\[ \| [F_i, D] u_i \|_2^2 \leq \| [\tilde{F}_t, \tilde{A} \cdot \partial_j] v_i \|_2^2 \cdot L. \]

Note that \( v_i \) is supported in the compact set \( \kappa := \varphi_i(\text{supp}(\rho_i)) \subseteq \mathbb{R}^n \). Therefore, Property (c) of \( \tilde{F}_t \) implies that \( [\tilde{F}_t, \tilde{A} \cdot \partial_j] \) extends to an operator on \( \oplus_1^k L^2(\kappa) \) which is bounded by, say, \( c \) independent of \( t \). Since moreover
\[ \| v_i \|_2^2 = \| \psi_i \circ u_i \circ \varphi_i^{-1} \|_2^2 \leq \| u_i \|_2^2 \cdot L, \]
we conclude
\[ \| [F_i, D] u_i \|_2^2 \leq c \cdot \| v_i \|_2^2 \cdot L \leq c \cdot \| u_i \|_2^2 \cdot L^2 \leq c \cdot \| u \|_2^2 \cdot L^2. \]

As neither \( L \) nor \( c \) depend on \( t \), and this holds true for every \( 1 \leq i \leq l \), we are done. \( \square \)

Remark: Note that we do not mind our construction in the proof of Lemma 5.4 to be highly dependent on our choice of atlas and partition of unity.

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