Counting curves on a general linear system with up to two singular points

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Abstract

In this paper we obtain an explicit formula for the number of curves in a compact complex surface $X$ (passing through the right number of generic points), that has up to one node and one singularity of codimension $k$, provided the total codimension is at most 7. We use a classical fact from differential topology: the number of zeros of a generic smooth section of a vector bundle $V$ over $M$, counted with signs, is the Euler class of $V$ evaluated on the fundamental class of $M$.

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1 Introduction

Enumeration of singular curves in $\mathbb{P}^2$ (complex projective space) is a classical problem in algebraic geometry. A more general class of enumerative problem is as follows:

**Question 1.1.** Let $L \to X$ be a holomorphic line bundle over a compact complex surface and $\mathcal{D} := \mathbb{P}H^0(X, L) \approx \mathbb{P}^\delta$ be the space of non-zero holomorphic sections up to scaling. What is $\mathcal{N}(\mathcal{A}_1^\delta X)$, the number of curves in $X$, that belong to the linear system $H^0(X, L)$, passing through $\delta L - (k + \delta)$ generic points and having $\delta$ distinct nodes and one singularity of type $X$, where $k$ is the codimension of the singularity? \(^a\)

The main result of this paper (cf. Theorem 1.5 and Theorem 1.6) is as follows:

**Main Theorem 1.2.** If $\delta \leq 1$ and $\delta + k \leq 7$, then we obtain an explicit formula for $\mathcal{N}(\mathcal{A}_1^\delta X)$, provided $L \to X$ is sufficiently ample.

\(^a\)By codimension we mean the number of conditions having that particular singularity imposes on the space of curves. For example, a node is a codimension one singularity, a cusp is a codimension two singularity, a tachode is a codimension three singularity and so on.
Before giving the formulas for $N(A_k^{\delta}X)$, let us make a few definitions.

**Definition 1.3.** Let $L \to X$ be a holomorphic line bundle over a complex surface and $f \in H^0(X, L)$ a holomorphic section. A point $q \in f^{-1}(0)$ is of singularity type $A_k$, $D_k$, $E_6$, $E_7$, $E_8$ or $X_8$ if there exists a coordinate system $(x, y) : (U, q) \to (\mathbb{C}^2, 0)$ such that $f^{-1}(0) \cap U$ is given by

$$A_k : y^2 + x^{k+1} = 0 \quad k \geq 0,$$

$$D_k : y^2 x + x^{k-1} = 0 \quad k \geq 4,$$

$$E_6 : y^3 + x^4 = 0,$$

$$E_7 : y^3 + xy^3 = 0,$$

$$E_8 : y^3 + x^5 = 0,$$

$$X_8 : x^4 + y^4 = 0.$$

**Definition 1.4.** A holomorphic line bundle $L \to X$ over a compact complex manifold $X$ is sufficiently $k$-ample if $L \approx L_1^{\delta_n} \to X$, where $L_1 \to X$ is a very ample line bundle and $n \geq k$.

The following two theorems are the main results of this paper:

**Theorem 1.5.** Let $X$ be a two dimensional compact complex manifold and $L \to X$ a holomorphic line bundle. Let

$$D := \mathbb{P}H^0(X, L) \approx \mathbb{P}^{\delta_L}, \quad c_1 := c_1(L) \quad \text{and} \quad x_i := c_i(T^*X)$$

where $c_i$ denotes the $i^{th}$ Chern class. Denote $N(X)$ to be the number of curves in $X$, that belong to the linear system $H^0(X, L)$, passing through $\delta_L - k$ generic points and having a singularity of type $X$, where $k$ is the codimension of the singularity $X$. Then

$$N(A_1) = 3c_1^2 + 2c_1 x_1 + x_2,$$

$$N(A_2) = 12c_1^2 + 12c_1 x_1 + 2x_1^2 + 2x_2,$$

$$N(A_3) = 50c_1^2 + 64c_1 x_1 + 17x_1^2 + 5x_2,$$

$$N(A_4) = 180c_1^2 + 280c_1 x_1 + 100x_1^2,$$

$$N(A_5) = 630c_1^2 + 1140c_1 x_1 + 498x_1^2 - 60x_2,$$

$$N(A_6) = 2212c_1^2 + 4515c_1 x_1 + 2289x_1^2 - 406x_2,$$

$$N(A_7) = 7812c_1^2 + 17600c_1 x_1 + 10022x_1^2 - 2058x_2,$$

$$N(D_4) = 15c_1^2 + 20c_1 x_1 + 5x_1^2 + 5x_2,$$

$$N(D_5) = 84c_1^2 + 132c_1 x_1 + 44x_1^2 + 20x_2,$$

$$N(D_6) = 224c_1^2 + 406c_1 x_1 + 168x_1^2 + 28x_2,$$

$$N(D_7) = 720c_1^2 + 1472c_1 x_1 + 720x_1^2,$$

$$N(E_6) = 84c_1^2 + 147c_1 x_1 + 57x_1^2 + 18x_2,$$

$$N(E_7) = 252c_1^2 + 488c_1 x_1 + 217x_1^2 + 42x_2,$$

provided $L$ is sufficiently $C_X$-ample, where

$$C_{A_k} := k + 1, \quad C_{D_k} := k - 1, \quad C_{E_6} = 4, \quad C_{E_7} = 4.$$

**Theorem 1.6.** Let $X$ be a two dimensional compact complex manifold and $L \to X$ a holomorphic line bundle. Let

$$D := \mathbb{P}H^0(X, L) \approx \mathbb{P}^{\delta_L}, \quad c_1 := c_1(L) \quad \text{and} \quad x_i := c_i(T^*X)$$

where $c_i$ denotes the $i^{th}$ Chern class. Denote $N(A_1^{\delta}X)$ to be the number of curves in $X$, that belong to the linear system $H^0(X, L)$, passing through $\delta_L - (k + 1)$ generic points and having one simple
node and one singularity of type $X$, where $k$ is the codimension of the singularity $X$. Then

$$\begin{align*}
\mathcal{N}(A_1A_1) &= -42c_1^2 + 9c_1^4 - 39c_1x_1 + 12c_1^3x_1 - 6x_1^2 + 4c_1^2x_1^2 - 7x_2 + 6c_1x_1x_2 + 4c_1x_1x_2 + x_2^2, \\
\mathcal{N}(A_1A_2) &= -240c_1^2 + 36c_1^4 - 288c_1x_1 + 60c_1^3x_1 - 72x_1^2 + 30c_1^2x_1^2 + 4c_1x_1^3 - 24x_2 + 18c_1^2x_2 \\
&\quad + 16c_1x_1x_2 + 2x_1^2x_2 + 2x_2^2, \\
\mathcal{N}(A_1A_3) &= -1260c_1^2 + 150c_1^4 - 1820c_1x_1 + 292c_1^3x_1 - 596x_1^2 + 179c_1^2x_1^2 + 34c_1x_1^3 - 60x_2 \\
&\quad + 65c_1^2x_1 + 74c_1x_1x_2 + 17x_1^2x_2 + 5x_2^2, \\
\mathcal{N}(A_1A_4) &= -5460c_1^2 + 540c_1^4 - 9240c_1x_1 + 1200c_1^3x_1 - 3740x_1^2 + 860c_1^2x_1^2 + 200c_1x_1^3 + 200x_2 \\
&\quad + 180c_1^2x_2 + 280c_1x_1x_2 + 100x_1^2x_2, \\
\mathcal{N}(A_1A_5) &= -22428c_1^2 + 1890c_1^4 - 43197c_1x_1 + 4680c_1^3x_1 - 20535x_1^2 + 3774c_1^2x_1^2 + 996c_1x_1^3 \\
&\quad + 2754x_2 + 450c_1^2x_2 + 1020c_1x_1x_2 + 498x_1^2x_2 - 60x_2^2, \\
\mathcal{N}(A_1A_6) &= -90468c_1^2 + 6636c_1^4 - 193816c_1x_1 + 17969c_1^3x_1 - 104503x_1^2 + 15897c_1^2x_1^2 \\
&\quad + 4578c_1x_1^3 + 18522x_2 + 994c_1^2x_2 + 3703c_1x_1x_2 + 2289x_1^2x_2 - 406x_2^2, \\
\mathcal{N}(A_1D_4) &= -420c_1^2 + 45c_1^4 - 624c_1x_1 + 90c_1^3x_1 - 196c_1^2x_1 + 55c_1^2x_1^2 + 10c_1x_1^3 \\
&\quad - 100x_2 + 30c_1^2x_2 + 30c_1x_1x_2 + 5x_1^2x_2 + 5x_2^2, \\
\mathcal{N}(A_1D_5) &= -2688c_1^2 + 252c_1^4 - 4564c_1x_1 + 564c_1^3x_1 - 1744x_1^2 + 396c_1^2x_1^2 + 88c_1x_1^3 \\
&\quad - 456x_2 + 144c_1^2x_2 + 172c_1x_1x_2 + 44x_1^2x_2 + 20x_2^2, \\
\mathcal{N}(A_1D_6) &= -8316c_1^2 + 672c_1^4 - 16008c_1x_1 + 1666c_1^3x_1 - 7281x_1^2 + 1316c_1^2x_1^2 + 336c_1x_1^3 \\
&\quad - 546x_2 + 308c_1^2x_2 + 462c_1x_1x_2 + 16x_1^2x_2 + 28x_2^2, \\
\mathcal{N}(A_1E_6) &= -2916c_1^2 + 252c_1^4 - 5400c_1x_1 + 609c_1^3x_1 - 2295x_1^2 + 465c_1^2x_1^2 + 114c_1x_1^3 \\
&\quad - 486x_2 + 138c_1^2x_2 + 183c_1x_1x_2 + 57x_1^2x_2 + 18x_2^2, \\
\end{align*}$$

provided $L$ is sufficiently $C_{A_1X}$-ample, where $C_{A_1X} := 2 + C_X$.

**Remark 1.7.** In the formulas presented in Theorems 1.5 and 1.6, if there is a degree four cohomology class, we mean the cohomology class evaluated on the fundamental class $[X]$. When there is a degree eight cohomology class, we mean the cohomology class evaluated on $[X \times X]$.

## 2 A description of the method used in this paper

Let us first recall the results of our earlier papers. In [1] and [3], we prove the special cases of Theorems 1.5 and 1.6 respectively when $X := \mathbb{P}^2$ and $L := \gamma_{\mathbb{P}^2}$. In [10], we obtain the first three formulas presented in Theorem 1.5. $^b$

We will now give a description of the method we use to obtain the enumerative formulas presented in this paper. Our goal is to enumerate curves in a linear system, passing through the right number of generic points, with certain singularities. A curve having a singularity implies that a certain derivative is zero. We interpret this derivative as the section of some vector bundle. If this section is transverse to the zero set, our desired number is the Euler class of the vector bundle. Computing the Euler class is completely elementary via the splitting principle.

$^b$Actually the results in [10] are slightly more general; we obtain an enumerative formula for hypersurfaces with a node, a cusp or a tacnode.
However, it turns out that there is a subtlety involved very soon in this process. One may look at a simple example to understand what is going on. Consider the complex polynomial

\[ f(z) := z^5(z - 1)(z - 2)(z - 3) \]

and let us ask what is \( n \), i.e.,

\[ n := \left| \{ z \in \mathbb{C} : f(z) = 0, \ z \neq 0 \} \right| = ? \]

A first guess would be 8 by looking at the degree of \( f \). However, this answer is incorrect because \( f \) vanishes when \( z = 0 \). We need to compute the contribution of \( f \) to its degree from the point \( z = 0 \).

To compute this contribution, we smoothly perturb the function \( f \) and count (with a sign) how many zeros are there in a small neighborhood of \( z = 0 \), i.e., we count the signed number of solutions of

\[ f(z) + \nu(z) = 0, \quad |z - 0| < \epsilon \]

where \( \nu \) is a small generic perturbation (i.e., the \( C^0 \)-norm of \( \nu \) is small) and \( \epsilon \) is sufficiently small. This number is 5. Hence

\[ \deg(f) = n + 5 \quad \implies \quad n = 8 - 5. \]

To return to our main discussion of enumerating curves with a singularity, it turns out that once the singularity becomes too degenerate, or we allow the curve to have more than one singular point, the Euler class counts too much. There is a boundary contribution which we have to subtract off from the Euler class. This part is highly non-trivial and challenging.

In [15], [1] and [3] the authors carry out this topological method and obtain the special cases of Theorems 1.5 and 1.6 respectively when \( X := \mathbb{P}^2 \) and \( L := \gamma^d_P \) (when \( d \) is sufficiently high). The bound on \( d \) is required to ensure that the relevant sections are transverse to the zero set. It may seem that one of the potential difficulties of generalizing the formulas to any arbitrary line bundle \( L \rightarrow X \) is that we do not know if these sections are going to be transverse to the zero set. As it turns out there is a very simple criteria to guarantee transversality; if the line bundle is sufficiently ample, then the relevant sections are going to be transverse. Once transversality is achieved, the arguments given in [1] and [3] apply immediately to the more general setup of considering curves in a linear system. As a result, we obtain these formulas in terms of Chern classes.

### 3 Organization of this paper

We now describe the basic organization of this paper. As explained in section 2, the two main aspects of applying our method is: proving transversality and computing the boundary contribution to the Euler class. Towards showing transversality, in [1], [3] and [2], we give a rigorous proof of why the relevant sections are transverse in the special case when \( X := \mathbb{P}^2 \) and \( L := \gamma^d_P \) (provided \( d \) is sufficiently large). In [10], the author shows that in the more general setup, the relevant sections that arise during the computations of \( \mathcal{N}(A_1) \), \( \mathcal{N}(A_2) \) and \( \mathcal{N}(A_3) \) are also going to be transverse provided the line bundle is sufficiently ample. Moreover, the argument that was employed was essentially mimicking the arguments employed in the case of projective space. In particular, it is easy to see that once the line bundle is sufficiently ample, all the arguments presented in [2] to prove transversality can be mimicked for the more general setup of \( L \rightarrow X \). Hence, we have decided to omit the proof of why the relevant sections are transverse from this paper. The reader can can refer to [2] and [10] to see why transversality holds.
We next discuss the more crucial aspect of computing the boundary contribution to the Euler class. As we explained in section 2, the boundary contribution was computed rigorously in [1] and [3] for the special case of the projective space. A little bit of thought shows that as long as the relevant sections are transverse to the zero set, exactly the same arguments apply to the more general setup of $L \to X$ to compute the boundary contribution. Hence, we omit the proof of how we obtain the multiplicities from each boundary component; we explicitly state the multiplicities and show how to obtain the final formula. The reader can refer to [1] and [3] to see how those multiplicities were actually obtained.

Acknowledgements. The second author is grateful to Aleksey Zinger for pointing out [15] and explaining the topological method employed in that paper.

4 A survey of related results in Enumerative Geometry

We now give a brief survey of related results in this area of mathematics, namely Enumerative Geometry of Singular Curves and Hypersurfaces. We start by looking at the results of M.Kazarian. We should mention at the outset that although we are not completely certain, we believe it is very likely that by applying the methods described in [6], the results of Theorem 1.5 and Theorem 1.6 can be recovered. However, Kazarian’s methods are completely different from ours. Furthermore, we believe that our method complements his method very well, since each method has its own advantages and disadvantages.

Let us now explain the method of M.Kazarian. His method works on the principle that there exists a universal formula for these enumerative numbers in terms of the Chern classes. He then goes on to consider enough special cases to find out what that exact combination is. One of the difficulties of this method is to prove the existence of such a universal formula. We also believe it is usually difficult to think of enough special cases in a given situation. However, this method has been successfully applied in many situations and in particular we believe it recovers our results.

The reader who has read section 2 will see immediately that this method is completely different from ours; we do not make any assumption that there is a universal formula in terms of Chern classes. Aside from the results of Kazarian, the rest of the results in this field are either special cases of Theorems 1.5 and 1.6 or a completely different class of enumerative problems.

Let us look at the results of Dmitry Kerner. In his paper [7], Kerner considers the special case of Theorem 1.5, when $X := \mathbb{P}^2$ and $L := \gamma_{\mathbb{P}^2}^d$. Our results are consistent with his in this special case. Kerner also considers in his paper [9] the special cases of Theorem 1.6, when $X := \mathbb{P}_m$ and $L := \gamma_{\mathbb{P}_m}^d$ and obtains three of the formulas we have stated in Theorem 1.6; a formula for $\mathcal{N}(A_1,A_1)$, $\mathcal{N}(A_1,A_2)$ and $\mathcal{N}(A_1,D_4)$.

The crucial difference between the results of Kazarian and our results and those obtained in [7], [8] and [9] is that the author there obtains results only for the special case of the linear system $\gamma_{\mathbb{P}^2}^d \to \mathbb{P}^2$, while Kazarian and our results are for any linear system $L \to X$ that is sufficiently ample.

Next, let us look at the results of I.Vainsencher. He considers a different class of enumerative problems (with the exception of $\mathcal{N}(A_1)$ and $\mathcal{N}(A_2)$; our results are consistent with his). In his paper [14], Vainsencher considers a general linear system $L \to X$ and enumerates curves that have up to six nodes. He also obtains a formula for $\mathcal{N}(A_2)$ in his paper [13].

\footnote{To take a simple example; suppose there is a polynomial of degree $m$. To find out what the polynomial is, we simply have to find the value of the polynomial at enough points.}
Next, let us look at the results S.Klaiman and R.Piene. They do obtain few of the formulas we have obtained in Theorems 1.5 and 1.6; namely \( \mathcal{N}(A_1), \mathcal{N}(D_4), \mathcal{N}(D_6), \mathcal{N}(E_7), \mathcal{N}(A_1 A_1), \mathcal{N}(A_1 D_4) \) and \( \mathcal{N}(A_1 D_6) \). Our results are consistent with theirs. They also study a different class of enumerative questions, namely enumerating curves that have up to eight simple nodes, or one triple point and up to four simple nodes, or one \( D_6 \) node and up to two simple nodes. Our results are of a different nature from theirs; in Theorems 1.5, 1.6 we enumerate curves with up to one node and one arbitrarily degenerate singularity (till a total codimension of seven).

Finally, we note that in their papers [12], [11] and [5], Z. Ran, L.Caporasso and J.Harris have obtained a formula for the number of degree \( d \)-curves in \( \mathbb{P}^2 \) (through the right number of generic points) having \( \delta \)-nodes, for any \( \delta \). However, their results are only for \( \mathbb{P}^2 \). Moreover, the allowed singularities in their cases are simple nodes and not anything more degenerate.

To summarize, aside from the results of Kazarian, all the other results are either special cases of Theorems 1.5 and 1.6 or are enumerative results of a different nature. Our method is completely different from that of Kazarian and complements it very well. Moreover, our method has the potential to go much further beyond codimension seven (just like the method of Kazarian has the potential to go a lot further).

5 Topological computations: one singular point

In this section we will give a proof of Theorem 1.5. Let us first set up some notation. Given a singularity \( \mathcal{X} \) let us define the following spaces

\[
\mathcal{X} := \{([f], q) \in D \times X : f \text{ has a singularity of type } \mathcal{X} \text{ at } q\},
\]

\[
\mathcal{X} := \{([f], l_q) \in D \times \mathbb{P}TX : ([f], q) \in \mathcal{X}\},
\]

\[
\mathcal{P} A_k := \{([f], l_q) \in D \times \mathbb{P}TX : ([f], q) \in A_k, \nabla^2 f(v, \cdot) = 0 \quad \forall v \in l_q\} \quad \text{if } k \geq 2,
\]

\[
\mathcal{P} D_4 := \{([f], l_q) \in D \times \mathbb{P}TX : ([f], q) \in D_4, \nabla^3 f(v, v, v) = 0 \quad \forall v \in l_q\},
\]

\[
\mathcal{P} D_k := \{([f], l_q) \in D \times \mathbb{P}TX : ([f], q) \in D_k, \nabla^3 f(v, v, \cdot) = 0 \quad \forall v \in l_q\} \quad \text{if } k \geq 5,
\]

\[
\mathcal{P} E_k := \{([f], l_q) \in D \times \mathbb{P}TX : ([f], q) \in E_k, \nabla^3 f(v, v, \cdot) = 0 \quad \forall v \in l_q\} \quad \text{if } k = 6, 7, 8.
\]

Next, if \( \mathcal{X} \) is a codimension \( k \) singularity, we define the following two numbers

\[
\mathcal{N}(\mathcal{X}, n_1, m_1, m_2) := \langle c_1^{n_1} x_1^{m_1} x_2^{m_2} y^{\delta L - (n_1 + m_1 + 2m_2 + k)}, \mathcal{X} \rangle,
\]

\[
\mathcal{N}(\mathcal{P} \mathcal{X}, n_1, m_1, m_2, \theta) := \langle c_1^{n_1} x_1^{m_1} x_2^{m_2} \lambda^\theta y^{\delta L - (n_1 + m_1 + 2m_2 + \theta + k)}, \mathcal{P} \mathcal{X} \rangle,
\]

where

\[
c_1 := c_1(L), \quad x_i := c_i(T^*X), \quad \lambda := c_1(\mathcal{X}^*), \quad y := c_1(\gamma^\mathcal{X})
\]

and \( \gamma_D \longrightarrow D \) and \( \mathcal{X} \rightarrow \mathbb{P}TX \) are the tautological line bundles.

We will now give a series of formulas to compute \( \mathcal{N}(A_1, n_1, m_1, m_2) \) and \( \mathcal{N}(\mathcal{P} \mathcal{X}, n_1, m_1, m_2, \theta) \). Note that \( \mathcal{N}(A_1) = \mathcal{N}(A_1, 0, 0, 0) \). In order to compute the remaining \( \mathcal{N}(\mathcal{X}) \) we do the following: if \( \mathcal{X} \) is anything other than \( D_4 \), then we observe that \( \mathcal{N}(\mathcal{X}) = \mathcal{N}(\mathcal{P} \mathcal{X}, 0, 0, 0) \). If \( \mathcal{X} = D_4 \), then we observe that \( \mathcal{N}(D_4) = \mathcal{N}(\mathcal{P} D_k, 0, 0, 0) \).

Note that, Propositions 5.2 to 5.14 and Propositions 6.1 to 6.11 give recursive formulas to compute \( \mathcal{N}(\mathcal{X}) \) and \( \mathcal{N}(A_1 \mathcal{X}) \). We wrote a Mathematica program to obtain the final formulas given in Theorem 1.5 and 1.6. The program can be obtained from our homepage

https://www.sites.google.com/site/ritwik371/home.
Before we prove Theorem 1.5, we also need to define the following spaces; they will come up during the course of our computations:

\[ \hat{A}_1^\#: = \{ ([f], l_q) \in D \times PTX : f(q) = 0, \nabla f|_q = 0, \nabla^2 f|_q(v, \cdot) \neq 0, \forall v \in l_q - 0 \} \]

\[ \hat{D}_4^\#: = \{ ([f], q) \in D \times PTX : f(q) = 0, \nabla f|_q = 0, \nabla^2 f|_q \equiv 0, \nabla^3 f|_q(v, v, v) \neq 0, \forall v \in l_q - 0 \} \]

\[ \hat{D}_k^{\#b} := \{ ([f], l_q) \in D \times PTX : f \text{ has a } D_k \text{ singularity at } q, \nabla^3 f|_q(v, v, v) \neq 0, \forall v \in l_q - 0, \ k \geq 4 \} \]

\[ \hat{X}_8^\#: = \{ ([f], l_q) \in D \times PTX : f(q) = 0, \nabla f|_q = 0, \nabla^2 f|_q \equiv 0, \nabla^3 f|_q = 0, \]

\[ \nabla^4 f|_q(v, v, v, v) \neq 0 \ \forall \ v \in l_q - 0 \}, \quad \forall \ v \in l_q - 0 \text{ and } w \in (T_qX)/l_q - 0, \quad \text{if } k > 4. \]

We will use the following fact from differential topology (cf. [4], Proposition 12.8):

**Theorem 5.1.** Let \( V \to M \) be an oriented vector bundle over a compact oriented manifold \( M \) and \( s : M \to V \) a smooth section that is transverse to the zero set. Then the Poincaré dual of \( [s^{-1}(0)] \) in \( M \) is the Euler class of \( V \). In particular, if the rank of \( V \) is same as the dimension of \( M \), then the signed cardinality of \( s^{-1}(0) \) is the Euler class of \( V \), evaluated on the fundamental class of \( M \).

We are now ready to prove Theorem 1.5. It is to be understood that Propositions 5.2 to 5.14 and Propositions 6.1 to 6.11 are true provided \( L \) is appropriately ample (as stated in Theorems 1.5 and 1.6).

**Proposition 5.2.** The number \( \mathcal{N}(A_1, n_1, m_1, m_2) \) is given by

\[
\mathcal{N}(A_1, n_1, m_1, m_2) = \begin{cases} 
3c_2^2 + 2c_1x_1 + x_2 & \text{if } (n_1, m_1, m_2) = (0, 0, 0), \\
3c_1^2 + c_1x_1 & \text{if } (n_1, m_1, m_2) = (1, 0, 0), \\
c_1^2 & \text{if } (n_1, m_1, m_2) = (2, 0, 0), \\
3c_1x_1 + x_1^2 & \text{if } (n_1, m_1, m_2) = (0, 1, 0), \\
c_1x_1 & \text{if } (n_1, m_1, m_2) = (1, 1, 0), \\
x_1^2 & \text{if } (n_1, m_1, m_2) = (0, 2, 0), \\
x_2 & \text{if } (n_1, m_1, m_2) = (0, 0, 1), \\
0 & \text{otherwise}.
\end{cases} \tag{1}
\]

**Proof:** Let \( \mu \) be a generic pseudocycle representing the homology class Poincaré dual to

\[ c_1^{n_1} x_1^{n_2} m_1^{m_2} y^{d_L-(n_1+m_1+2m_2+1)}. \]

We now define sections of the following two bundles

\[ \psi_{A_0} : (D \times X) \cap \mu \longrightarrow L_{A_0} := \gamma_D^* \otimes L \longrightarrow D \times X, \quad \text{given by} \quad \{ \psi_{A_0}([f], q) \}(f) := f(q) \quad \text{and} \]

\[ \psi_{A_1} : \psi_{A_0}^{-1}(0) \longrightarrow \mathcal{V}_{A_1} := \gamma_D^* \otimes T^* X \otimes L, \quad \text{given by} \quad \{ \psi_{A_1}([f], q) \}(f) := \nabla f|_q. \tag{2} \]

In [10] we show that if \( L \) is sufficiently 2-ample, then these sections are transverse to the zero set.

Hence

\[ \mathcal{N}(A_1, n_1, m_1, m_2) = \langle e(L_{A_0})e(V_{A_1}), [D \times X] \cap [\mu] \rangle. \tag{3} \]

Equation (3) and the Splitting Principle, imply (1). \( \square \)
Proposition 5.3. The number $N(\mathcal{P}A_2, n_1, m_1, m_2, \theta)$ is given by 

$$
N(\mathcal{P}A_2, n_1, m_1, m_2, \theta) = \begin{cases} 
2N(A_1, n_1, m_1, m_2) + 2N(A_1, n_1 + 1, m_2) & \text{if } \theta = 0, \\
+ 2N(A_1, n_1 + 1, m_1, m_2) & \text{if } \theta = 1, \\
+ 3N(A_1, n_1, m_1 + 1, m_2) + 3N(A_1, n_1 + 1, m_1, m_2) & \text{if } \theta = 1, \\
+ 2N(A_1, n_1, m_1 + 2, m_2) & \text{if } \theta = 1, \\
N(\mathcal{P}A_2, n_1, m_1 + 1, m_2, \theta - 1) & \text{if } \theta > 1, \\
-N(\mathcal{P}A_2, n_1, m_1 + 1, \theta - 2) & \text{if } \theta > 1.
\end{cases}
$$

Equation (4)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $e_1^{n_1}x_1^{m_1}x_2^{m_2}\lambda^\theta y^\delta_L-(n_1+m_1+2m_2+2)$.

We now define a section of the following bundle 

$$
\Psi_{\mathcal{P}A_2} : \mathcal{A}_1 \cap \mu \rightarrow \mathbb{V}_{\mathcal{P}A_2} := \hat{\gamma}^* \otimes \gamma_D^* \otimes T^*X \otimes L,
$$

given by 

$$\{\Psi_{\mathcal{P}A_2}([f], l_q)\}(v \otimes f) := \nabla^2 f(v, \cdot) \quad \forall \ v \in l_q.$$

In [10] we show that if $L$ is sufficiently 3-ample, then this section is transverse to the zero set. Hence 

$$
N(\mathcal{P}A_2, n_1, m_1, m_2, \theta) = \left\langle e(\mathbb{V}_{\mathcal{P}A_2}), [\mathcal{A}_1] \cap [\mu] \right\rangle.
$$

Equation (5), the Splitting Principle and the ring structure of $H^*(\mathbb{P}TX)$ imply (4).

Proposition 5.4. The number $N(\mathcal{P}A_3, n_1, m_1, m_2, \theta)$ is given by 

$$
N(\mathcal{P}A_3, n_1, m_1, m_2, \theta) = 3N(\mathcal{P}A_2, n_1, m_1, m_2, \theta + 1) + N(\mathcal{P}A_2, n_1, m_1, m_2, \theta)
+ 3N(\mathcal{P}A_2, n_1 + 1, m_1, m_2, \theta).
$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $e_1^{n_1}x_1^{m_1}x_2^{m_2}\lambda^\theta y^\delta_L-(n_1+m_1+2m_2+3)$.

We now define a section of the following bundle 

$$
\Psi_{\mathcal{P}A_3} : \mathcal{A}_2 \cap \mu \rightarrow \mathbb{L}_{\mathcal{P}A_3} := \hat{\gamma}^* \otimes \gamma_D^* \otimes L,
$$

given by 

$$\{\Psi_{\mathcal{P}A_3}([f], l_q)\}(v \otimes f) := \nabla^3 f(v, v, v) \quad \forall \ v \in l_q.$$

In [10] we show that if $L$ is sufficiently 4-ample, then this section is transverse to the zero set. Hence 

$$
N(\mathcal{P}A_3, n_1, m_1, m_2, \theta) = \left\langle e(\mathbb{L}_{\mathcal{P}A_3}), [\mathcal{A}_2] \cap [\mu] \right\rangle,
$$

which gives us (6).
Proposition 5.5. The number $N(\mathcal{P}D_4, n_1, m_1, m_2, \theta)$ is given by

\[
N(\mathcal{P}D_4, n_1, m_1, m_2, \theta) = 2N(\mathcal{P}A_3, n_1, m_1 + 1, m_2, \theta) - 2N(\mathcal{P}A_3, n_1, m_1, m_2, \theta + 1) + N(\mathcal{P}A_3, n_1, m_1, m_2, \theta) + N(\mathcal{P}A_3, n_1 + 1, m_1, m_2, \theta).
\]  

(8)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

\[
c_1^{n_1} x_1^{m_1} x_2^{m_2} \lambda^\delta y^{(n_1 + m_1 + 2m_2 + \theta + 4)}.
\]

It is shown in [1] that

\[
\mathcal{P}A_3 = \mathcal{P}A_3 \cup \mathcal{P}A_4 \cup \mathcal{P}D_4.
\]

We now define a section of the following bundle

\[
\Psi_{\mathcal{P}D_4} : \mathcal{P}A_3 \cap \mu \rightarrow \mathbb{L}_{\mathcal{P}D_4} := (TX/\hat{\gamma})^\ast \otimes \gamma_D^\ast \otimes L,
\]

given by

\[
\{\Psi_{\mathcal{P}D_4}([f], l_q)\}(w^\otimes \otimes f) := \nabla^2_f|_q(w, w) \quad \forall \ w \in T_qX/l_q.
\]  

(9)

If $L$ is sufficiently 3-ample, then this section vanishes on $\mathcal{P}D_4 \cap \mu$ transversally. Moreover, it does not vanish on $\mathcal{P}A_4 \cap \mu$. Since $\mu$ is generic, we conclude that $\mathcal{P}A_4 \cap \mu = \mathcal{P}A_4 \cap \mu$; hence this section does not vanish on $\mathcal{P}A_4 \cap \mu$. Hence,

\[
N(\mathcal{P}D_4, n_1, m_1, m_2, \theta) = \langle e(\mathbb{L}_{\mathcal{P}D_4}), (\mathcal{P}A_3) \cap [\mu] \rangle.
\]

This proves (8).

Proposition 5.6. The number $N(\mathcal{P}D_5, n_1, m_1, m_2, \theta)$ is given by

\[
N(\mathcal{P}D_5, n_1, m_1, m_2, \theta) = N(\mathcal{P}D_4, n_1, m_1, m_2, \theta + 1) + N(\mathcal{P}D_4, n_1, m_1 + 1, m_2, \theta) + N(\mathcal{P}D_4, n_1 + 1, m_1, m_2, \theta) + N(\mathcal{P}D_4, n_1 + 1, m_1, m_2, \theta).
\]  

(10)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

\[
c_1^{n_1} x_1^{m_1} x_2^{m_2} \lambda^\delta y^{(n_1 + m_1 + 2m_2 + \theta + 5)}.
\]

It is shown in [1] that

\[
\mathcal{P}D_4 = \mathcal{P}D_4 \cup \mathcal{P}D_5 \cup \mathcal{P}D_5^\ast.
\]

We now define a section of the following bundle

\[
\Psi_{\mathcal{P}D_5} : \mathcal{P}D_4 \cap \mu \rightarrow \mathbb{L}_{\mathcal{P}D_5} := \hat{\gamma}^\ast \otimes (TX/\hat{\gamma})^\ast \otimes \gamma_D^\ast \otimes L,
\]

given by

\[
\{\Psi_{\mathcal{P}D_5}([f], l_q)\}(v^\otimes \otimes w \otimes f) := \nabla^2_f|_q(v, v, w) \quad \forall \ v \in l_q, \ w \in T_qX/l_q.
\]  

(11)

If $L$ is sufficiently 4-ample, then the section $\Psi_{\mathcal{P}D_5}$ restricted to $\mathcal{P}D_5 \cap \mu$ vanishes transversally. Moreover, it does not vanish on $\mathcal{P}D_5^\ast \cap \mu$. Hence

\[
N(\mathcal{P}D_5, n_1, m_1, m_2, \theta) = \langle e(\mathbb{L}_{\mathcal{P}D_5}), (\mathcal{P}D_4) \cap [\mu] \rangle.
\]

This proves (10).
Proposition 5.7. The number $N(\mathcal{PE}_6, n_1, m_1, m_2, \theta)$ is given by

$$N(\mathcal{PE}_6, n_1, m_1, m_2, \theta) = 2N(\mathcal{PD}_5, n_1, m_1 + 1, m_2, \theta) - N(\mathcal{PD}_5, n_1, m_1, m_2, \theta + 1) + N(\mathcal{PD}_5, n_1 + 1, m_1, m_2, \theta) + N(\mathcal{PD}_5, n_1, m_1, m_2, \theta). \quad (12)$$

**Proof:** Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^{m_1} x_1^1 x_2^m y^\theta L -(n_1 + m_1 + 2m_2 + \theta + 6).$$

It is shown in [1] that

$$\overline{\mathcal{PD}}_5 = \mathcal{PD}_5 \cup \mathcal{PD}_6 \cup \mathcal{PE}_6.$$ We now define a section of the following bundle

$$\Psi_{\mathcal{PE}_6} : \overline{\mathcal{PD}}_5 \cap \mu \longrightarrow \mathbb{L}_{\mathcal{PE}_6} := \hat{\gamma}^* \otimes (TX/\hat{\gamma})^* \otimes \gamma_D^* \otimes L,$$

given by

$$\{\Psi_{\mathcal{PE}_6}([f], l_q)\}(v \otimes w^2 \otimes f) := \nabla^3 f|_q(v, w, w) \quad \forall \ v \in l_q, \ w \in T_qX/l_q.$$ If the line bundle $L$ is sufficiently 4-ample, then the section $\Psi_{\mathcal{PE}_6}$ vanishes on $\mathcal{PE}_6 \cap \mu$ transversally. Moreover, it does not vanish on $\mathcal{PD}_6 \cap \mu$. Hence

$$N(\mathcal{PE}_6, n_1, m_1, m_2, \theta) = \langle e(\mathbb{L}_{\mathcal{PE}_6}), \overline{\mathcal{PD}}_5 \cap [\mu] \rangle.$$ This proves (12). \hfill \Box

Proposition 5.8. The number $N(\mathcal{PE}_7, n_1, m_1, m_2, \theta)$ is given by

$$N(\mathcal{PE}_7, n_1, m_1, m_2, \theta) = 4N(\mathcal{PE}_6, n_1, m_1, m_2, \theta + 1) + N(\mathcal{PE}_6, n_1, m_1, m_2, \theta) + N(\mathcal{PE}_6, n_1 + 1, m_1, m_2, \theta). \quad (14)$$

**Proof:** Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^{m_1} x_1^1 x_2^m y^\theta L -(n_1 + m_1 + 2m_2 + \theta + 7).$$

It is shown in [1] that

$$\overline{\mathcal{PE}}_6 = \mathcal{PE}_6 \cup \mathcal{PE}_7 \cup \hat{\lambda}^\#.$$ We now define a section of the following bundle

$$\Psi_{\mathcal{PE}_7} : \overline{\mathcal{PE}}_6 \cap \mu \longrightarrow \mathbb{L}_{\mathcal{PE}_7} := \hat{\gamma}^4 \otimes \gamma_D^* \otimes L,$$

given by

$$\{\Psi_{\mathcal{PE}_7}([f], l_q)\}(v^4 \otimes f) := \nabla^4 f|_q(v, v, v) \quad \forall \ v \in l_q.$$ If the line bundle $L$ is sufficiently 5-ample, then the section $\Psi_{\mathcal{PE}_7}$ vanishes on $\mathcal{PE}_7 \cap \mu$ transversally. Moreover, it does not vanish on $\hat{\lambda}^\# \cap \mu$. Hence

$$N(\mathcal{PE}_7, n_1, m_1, m_2, \theta) = \langle e(\mathbb{L}_{\mathcal{PE}_7}), \overline{\mathcal{PE}}_6 \cap [\mu] \rangle.$$ This proves (12). \hfill \Box
Proposition 5.9. The number $N(\mathcal{PD}_6, n_1, m_1, m_2, \theta)$ is given by

$$N(\mathcal{PD}_6, n_1, m_1, m_2, \theta) = 4N(\mathcal{PD}_5, n_1, m_1, m_2, \theta + 1) + N(\mathcal{PD}_5, n_1, m_1 + 1, m_2, \theta) + N(\mathcal{PD}_5, n_1 + 1, m_1, m_2, \theta).$$

(16)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $c_1^{n_1} x_1^{m_1} x_2^{m_2} \lambda^\theta y^L - (n_1 + m_1 + 2m_2 + \theta + 6)$.

It is shown in [1] that

$$\overline{\mathcal{PD}_6} = \mathcal{PD}_5 \cup \overline{\mathcal{PD}_6} \cup \overline{\mathcal{PE}_6}.$$

We now define a section of the following bundle

$$\Psi_{\mathcal{PD}_6} : \overline{\mathcal{PD}_6} \cap \mu \rightarrow \mathbb{L}_{\mathcal{PD}_6} := \hat{\gamma}^4 \otimes \gamma_D^* \otimes L,$$

given by

$$\{\Psi_{\mathcal{PD}_6}([f], \mu)\} \left( v^{\otimes 4} \otimes f \right) := \nabla^4 f|_\gamma(v, v, v, v) \quad \forall \ v \in \hat{\gamma}.$$

(17)

If the line bundle $L$ is sufficiently 5-ample, then the section $\Psi_{\mathcal{PD}_6}$ vanishes on $\mathcal{PD}_6 \cap \mu$ transversally. Moreover, it does not vanish on $\mathcal{PE}_6 \cap \mu$. Hence

$$N(\mathcal{PD}_6, n_1, m_1, m_2, \theta) = \langle e(\mathbb{L}_{\mathcal{PD}_6}), \ (\overline{\mathcal{PD}_6} \cap [\mu]) \rangle.$$

This proves (16).

Let us now make a small abbreviation: if $v \in l_q$ and $w \in TX/l_q$, we define

$$f_{ij} := \nabla^{i+j} f|_q(v, \underbrace{\cdots}_i \cdot v, \underbrace{\cdots}_j \cdot w).$$

Proposition 5.10. The number $N(\mathcal{PD}_7, n_1, m_1, m_2, \theta)$ is given by

$$N(\mathcal{PD}_7, n_1, m_1, m_2, \theta) = 4N(\mathcal{PD}_6, n_1, m_1, m_2, \theta + 1) + 2N(\mathcal{PD}_6, n_1, m_1 + 1, m_2, \theta) + 2N(\mathcal{PD}_6, n_1 + 1, m_1, m_2, \theta).$$

(18)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $c_1^{n_1} x_1^{m_1} x_2^{m_2} \lambda^\theta y^L - (n_1 + m_1 + 2m_2 + \theta + 7)$.

It is shown in [1] that

$$\overline{\mathcal{PD}_6} = \mathcal{PD}_6 \cup \overline{\mathcal{PD}_7} \cup \overline{\mathcal{PE}_7}.$$

(19)

We now define a section of the following bundle

$$\Psi_{\mathcal{PD}_7} : \overline{\mathcal{PD}_6} \cap \mu \rightarrow \mathbb{L}_{\mathcal{PD}_7} := \hat{\gamma}^6 \otimes (TX/\hat{\gamma})^{\otimes 2} \otimes \gamma_D^* \otimes L^\otimes 2,$$

given by

$$\{\Psi_{\mathcal{PD}_7}([f], \mu)\} \left( v^{\otimes 6} \otimes w^{\otimes 2} \otimes f^{\otimes 2} \right) := f_{12}D^f_7 \quad \forall \ v \in \hat{\gamma}, \ w \in TX/\hat{\gamma},$$

(20)

where $D^f_7 := f_{50} - \frac{f_{30}^2}{4}$. If the line bundle $L$ is sufficiently 6-ample, then the section $\Psi_{\mathcal{PD}_7}$, restricted to $\mathcal{PD}_7 \cap \mu$ vanishes transversally. Moreover, the section does not vanish on $\mathcal{PE}_7 \cap \mu$. Hence

$$N(\mathcal{PD}_7, n_1, m_1, m_2, \theta) = \langle e(\mathbb{L}_{\mathcal{PD}_7}), \ (\overline{\mathcal{PD}_6} \cap [\mu]) \rangle.$$

This proves (18).
Proposition 5.11. The number $N(\mathcal{PA}_4, n_1, m_1, m_2, \theta)$ is given by
\[ N(\mathcal{PA}_4, n_1, m_1, m_2, \theta) = 2N(\mathcal{PA}_3, n_1, m_1, m_2, \theta + 1) + 2N(\mathcal{PA}_4, m_1 + 1, m_2, \theta) \]
\[ + 2N(\mathcal{PA}_3, n_1, m_1, m_2, \theta) + 2N(\mathcal{PA}_3, n_1 + 1, m_1, m_2, \theta). \] (21)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $\epsilon_1^n x_1^m x_2^m \lambda^\theta y^\delta L^{-(n_1 + m_1 + 2m_2 + \theta + 4)}$.

It is shown in [1] that
\[ \mathcal{PA}_3 = \mathcal{PA}_3 \cup \mathcal{PA}_4 \cup \mathcal{PD}_4. \] (22)

We now define a section of the following bundle
\[ \Psi_{\mathcal{PA}_4} : \mathcal{PA}_3 \cap \mu \rightarrow \mathbb{L}_{\mathcal{PA}_4} : = \gamma^s \circ (TX/\gamma)^s \circ \gamma^s_2 \circ L^\otimes 2, \]
\[ \{ \Psi_{\mathcal{PA}_4}(f), l_q \} \} (v^\otimes 4 \otimes w^\otimes 2 \otimes f^\otimes 2) := f_{02}A_{4}^f \quad \forall \ v \in l_q, \ w \in T_q X/l_q, \] (23)

where $A_{4}^f := f_{40} - \frac{3f_{20}}{f_{02}}$. If the line bundle $L$ is sufficiently 5-ample, then the section $\Psi_{\mathcal{PA}_4}$, restricted to $\mathcal{PA}_4 \cap \mu$ vanishes transversally. Moreover, the section does not vanish on $\mathcal{PD}_4 \cap \mu$. Hence
\[ N(\mathcal{PA}_4, n_1, m_1, m_2, \theta) = \langle \epsilon(\mathcal{PA}_4), [\mathcal{PA}_3] \rangle. \]

This proves (21). \qed

Proposition 5.12. The number $N(\mathcal{PA}_5, n_1, m_1, m_2, \theta)$ is given by
\[ N(\mathcal{PA}_5, n_1, m_1, m_2, \theta) = N(\mathcal{PA}_4, n_1, m_1, m_2, \theta + 1) + 4N(\mathcal{PA}_4, n_1, m_1 + 1, m_2, \theta) \]
\[ + 3N(\mathcal{PA}_4, n_1, m_1, m_2, \theta) + 3N(\mathcal{PA}_4, n_1 + 1, m_1, m_2, \theta) \]
\[ - 2N(\mathcal{PD}_5, n_1, m_1, m_2, \theta). \] (24)

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $\epsilon_1^n x_1^m x_2^m \lambda^\theta y^\delta L^{-(n_1 + m_1 + 2m_2 + \theta + 5)}$.

It is shown in [1] that
\[ \mathcal{PA}_4 = \mathcal{PA}_4 \cup \mathcal{PA}_5 \cup \mathcal{PD}_5. \] (25)

We now define a section of the following bundle
\[ \Psi_{\mathcal{PA}_5} : \mathcal{PA}_4 \cap \mu \rightarrow \mathbb{L}_{\mathcal{PA}_5} : = \gamma^s x^5 \circ (TX/\gamma)^s \circ \gamma^s_2 \circ L^\otimes 3, \]
\[ \{ \Psi_{\mathcal{PA}_4}(f), l_q \} \} (v^\otimes 5 \otimes w^\otimes 4 \otimes f^\otimes 3) := f_{02}A_{5}^f \quad \forall \ v \in l_q, \ w \in T_q X/l_q, \] (26)

where $A_{5}^f := f_{50} - \frac{10f_{21}f_{31}}{f_{02}} + \frac{15f_{12}f_{21}}{f_{02}}$. If the line bundle $L$ is sufficiently 6-ample, then the section $\Psi_{\mathcal{PA}_5}$, restricted to $\mathcal{PA}_5$ vanishes transversally. It is also shown in [1], that this section vanishes on $\mathcal{PD}_5 \cap \mu$ with a multiplicity of 2. Hence
\[ \langle \epsilon(\mathcal{PA}_5), [\mathcal{PA}_4] \cap \mu \rangle = N(\mathcal{PA}_5, n_1, m_1, m_2, \theta) + 2N(\mathcal{PD}_5, n_1, m_1, m_2, \theta). \]

This proves (21). \qed
Proposition 5.13. The number $N(PA_6, n_1, m_1, m_2, \theta)$ is given by

$$N(PA_6, n_1, m_1, m_2, \theta) = 6N(PA_5, n_1, m_1 + 1, m_2, \theta) + 4N(PA_5, n_1, m_1, m_2, \theta) + 4N(PA_5, n_1 + 1, m_1, m_2, \theta) - 4N(PD_6, n_1, m_1, m_2, \theta) - 3N(PE_6, n_1, m_1, m_2, \theta). \quad (27)$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^n x_1^m y_{- (n_1 + m_1 + 2m_2 + \theta + 6)}.$$

It is shown in [1] that

$$PA_5 = PA_6 \cup PD_6 \cup PE_6. \quad (28)$$

We now define a section of the following bundle

$$\Psi_{PA_6} : \overline{PA_5} \rightarrow \mathbb{L}_{PA_6} := \gamma^6 \otimes (TX/\gamma)^6 \otimes \gamma_{PD}^{\otimes 4}, \quad \text{given by}$$

$$\{\Psi_{PA_5}([f], l_q) \} (v^\otimes 6 \otimes w^\otimes 6 \otimes f^\otimes 4) := f_{02}^{3} A_6^f \quad \forall \, v \in l_q, \quad w \in T_qX/l_q, \quad (29)$$

where

$$A_6^f := f_{00} - \frac{15 f_{21} f_{41}}{f_{02}} - \frac{10 f_{21}^2}{f_{02}} + \frac{60 f_{12} f_{21} f_{31}}{f_{02}^2} + \frac{45 f_{21}^2 f_{22}}{f_{02}^2} - \frac{15 f_{03} f_{21}^3}{f_{02}^3} - \frac{90 f_{12}^2 f_{21}^3}{f_{02}^3}.$$  

If the line bundle $L$ is sufficiently 7-ample, then the section $\Psi_{PA_6}$, restricted to $PA_6$ vanishes transversally. It is shown in [1], that this section vanishes on $PD_6 \cap \mu$ and $PE_6 \cap \mu$ with a multiplicity of 4 and 3 respectively. Hence

$$\langle e(\mathbb{L}_{PA_6}), [\overline{PA_5}] \cap [\mu] \rangle = N(PA_6, n_1, m_1, m_2, \theta) + 4N(PD_6, n_1, m_1, m_2, \theta) + 3N(PE_6, n_1, m_1, m_2, \theta).$$

This proves (21).

Proposition 5.14. The number $N(PA_7, n_1, m_1, m_2, 0)$ is given by

$$N(PA_7, n_1, m_1, m_2, 0) = -N(PA_6, n_1, m_1, m_2, 1) + 8N(PA_6, n_1, m_1 + 1, m_2, 0) + 5N(PA_6, n_1, m_1, m_2, 0) + 5N(PA_5, n_1 + 1, m_1, m_2, 0) - 6N(PD_6, n_1, m_1, m_2, 0) - 7N(PE_6, n_1, m_1, m_2, 0). \quad (30)$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^n x_1^m x_2^m y_{-(n_1 + m_1 + 2m_2 + \theta)}.$$

It is shown in [1] that

$$PA_6 = PA_6 \cup PD_7 \cup PE_7 \cup \mathbb{L}_8. \quad (31)$$

We now define a section of the following bundle

$$\Psi_{PA_7} : \overline{PA_6} \cap \mu \rightarrow \mathbb{L}_{PA_7} := \gamma^7 \otimes (TX/\gamma)^8 \otimes \gamma_{PD}^{\otimes 5} \otimes L^{\otimes 5}, \quad \text{given by}$$

$$\{\Psi_{PA_7}([f], l_q) \} (v^\otimes 7 \otimes w^\otimes 8 \otimes f^\otimes 5) := f_{02}^{3} A_7^f \quad \forall \, v \in l_q, \quad w \in T_qX/l_q, \quad (32)$$
where
\[
A'_j := f_{70} - \frac{21f_{21}f_{51}}{f_{02}} - \frac{35f_{31}f_{41}}{f_{02}} + \frac{105f_{12}f_{21}f_{41}}{f_{02}} + \frac{105f_{21}f_{42}}{f_{02}} + \frac{105f_{32}f_{41}}{f_{02}} + \frac{70f_{12}f_{21}^2}{f_{02}} + \frac{210f_{21}f_{22}f_{31}}{f_{02}} - \frac{105f_{03}f_{21}^2f_{31}}{f_{02}} - \frac{420f_{12}f_{21}f_{31}}{f_{02}} - \frac{630f_{12}f_{21}f_{22}}{f_{02}} - \frac{105f_{13}f_{32}}{f_{02}} - \frac{315f_{03}f_{12}f_{21}^3}{f_{02}} - \frac{630f_{12}f_{21}^3}{f_{02}}.
\]

If the line bundle \( L \) is sufficiently 8-ample, then the section \( \Psi_{\mathcal{P}A_7} \), restricted to \( \mathcal{P}A_7 \cap \mu \) vanishes transversally. It is shown in [1], that this section vanishes on \( \mathcal{PD}_7 \cap \mu \) and \( \mathcal{PE}_7 \cap \mu \) with a multiplicity of 6 and 7 respectively. Assume that it vanishes on \( \widehat{X}_8 \) with a multiplicity of \( \eta \). Hence
\[
\langle e(\mathbb{L}_{\mathcal{P}A_7}), [\mathcal{PA}_6] \cap [\mu] \rangle = N(\mathcal{PA}_7, n_1, m_1, m_2, 0) + 6N(\mathcal{PD}_7, n_1, m_1, m_2, 0) + 7N(\mathcal{PE}_7, n_1, m_1, m_2, 0) + \eta|\widehat{X}_8 \cap \mu|.
\]
Since \( |\widehat{X}_8 \cap \mu| = 0 \), we get (30).

\[\square\]

\section{Topological computations: two singular points}

We will now give a proof of Theorem 1.6. Before that, let us setup some additional notation. Let \( S_1 \) and \( S_2 \) be two subsets of \( D \times X \) and \( T_2 \) be a subset of \( D \times \mathbb{P}TX \). Define
\[
S_1 \circ S_2 := \{(f, q_1, q_2) \in D \times X \times X : ([f], q_1) \in S_1, ([f], q_2) \in S_2, q_1 \neq q_2\},
\]
\[
S_1 \circ T_2 := \{(f, q_1, l_q) \in D \times X \times \mathbb{P}TX : ([f], q_1) \in S_1, ([f], l_q) \in T_2, q_1 \neq q_2\}.
\]

Next, if \( \mathfrak{x} \) is a codimension \( k \) singularity, we define the following numbers:
\[
N(A_1\mathfrak{x}, n_1, m_1, m_2) := \langle e_{1}^{n_1}x_1^{m_1}x_2^{m_2}y^\delta L^{-(n_1+m_1+2m_2+k+1)}, [A_1 \circ \mathfrak{x}] \rangle,
\]
\[
N(A_1\mathcal{P}\mathfrak{x}, n_1, m_1, m_2, \theta) := \langle e_{1}^{n_1}x_1^{m_1}x_2^{m_2}y^\delta L^{-(n_1+m_1+2m_2+\theta+k+1)}, [A_1 \circ \mathcal{P}\mathfrak{x}] \rangle.
\]

Next, if \( S \) and \( T \) are subsets of \( D \times X \) and \( D \times \mathbb{P}TX \) respectively, we define
\[
\Delta S := \{(f, q, q) \in D \times X \times X : ([f], q) \in S\} \quad \text{and} \quad \Delta T := \{(f, l_q) \in D \times X \times \mathbb{P}TX : ([f], l_q) \in T\}.
\]

Finally, we define the following two projection maps
\[
\pi_2 := \text{id} \times \text{proj}_2 : D \times X \times X \longrightarrow D \times X, \quad \pi_2 := \text{id} \times \text{proj}_2 : D \times X \times \mathbb{P}TX \longrightarrow D \times \mathbb{P}TX
\]
where \( \text{proj}_2 \) denotes the projection onto the second factor.

Note that given any bundle over \( D \times X \) (resp. \( D \times \mathbb{P}TX \)), there is an induced bundle over \( D \times X \times X \) (resp. \( D \times X \times \mathbb{P}TX \)) arising from the pullback via \( \pi_2 \). Similarly, given any section of such a bundle, there is a corresponding section on the pullback bundle, via \( \pi_2 \). We will encounter the bundles and sections of these bundles that we defined in section 5; we will encounter them over \( D \times X \times X \) or \( D \times X \times \mathbb{P}TX \) via the pullback of \( \pi_2 \).

We will now give a series of formulas to compute \( N(A_1A_1, n_1, m_1, m_2) \) and \( N(A_1\mathcal{P}\mathfrak{x}, n_1, m_1, m_2, \theta) \). Note that \( N(A_1A_1) = N(A_1A_1, 0, 0, 0) \). In order to compute the remaining \( N(A_1\mathfrak{x}) \) we do the following: if \( \mathfrak{x} \) is anything other than \( D_4 \), then we observe that \( N(A_1\mathfrak{x}) = N(A_1\mathcal{P}\mathfrak{x}, 0, 0, 0) \). If
\(X = D_4\), then we observe that \(\mathcal{N}(A_1D_4) = \frac{\mathcal{N}(A_1P^2_{\mathbb{P}^2,0,0,0})}{4}\).

We are now ready to give a proof of Theorem 1.6. An important notational remark is that in the subsequent proofs we shall use \(\mu\) to denote a homology class Poincaré dual to an element \(c_1^{m_1}x_1^{m_1}x_2^{m_2}\lambda^y\) in \(H^*(D \times X \times \mathbb{P}TX; \mathbb{Z})\). The elements \(\lambda^y\) and \(y^k\) are unambiguously defined via the appropriate pullbacks. But the class \(c_1^{m_1}x_1^{m_1}x_2^{m_2}\) is understood to arise from the second factor in \(X \times \mathbb{P}TX\), i.e., if \(\pi_2 : D \times X \times \mathbb{P}TX \to D \times \mathbb{P}TX\) is the projection map defined earlier, then
\[
c_1 := c_1(\pi_2^*(L)), \quad x_i = \pi_i(\pi_2^*(T^*X)).
\]

We will allow ourselves this abuse of notation.

**Proposition 6.1.** The number \(\mathcal{N}(A_1A_1, n_1, m_1, m_2)\) is given by
\[
\mathcal{N}(A_1A_1, n_1, m_1, m_2) = \mathcal{N}(A_1) \times \mathcal{N}(A_1, n_1, m_1, m_2)
- \left(\mathcal{N}(A_1, n_1, m_1, m_2) + \mathcal{N}(A_1, n_1 + 1, m_1, m_2) + 3\mathcal{N}(A_2, n_1, m_1, m_2)\right).
\]

**Proof:** Let \(\mu\) be a pseudocycle in \(D \times X \times X\) representing the homology class Poincaré dual to
\[
c_1^{m_1}x_1^{m_1}x_2^{m_2}\lambda^y = -(n_1 + m_1 + 2m_2 + 2).
\]

Note that
\[
\overline{A}_1 \times X = \overline{A}_1 \circ (D \times X) = \overline{A}_1 \circ (D \times X) \sqcup \Delta \overline{A}_1.
\]

We show in [3] and [2] that the sections
\[
\pi_2^*\psi_{A_0} : \overline{A}_1 \times X - \Delta \overline{A}_1 \to \pi_2^*L_{A_0}, \quad \pi_2^*\psi_{A_1} : \pi_2^*\psi_{A_0}^{-1}(0) \to \pi_2^*V_{A_1},
\]
are transverse to the zero set (if \(L\) is sufficiently 4-ample). Hence
\[
\langle e(\pi_2^*L_{A_0})e(\pi_2^*V_{A_1}), [\overline{A}_1 \times X \cap \{\mu\}] \rangle = \mathcal{N}(A_1A_1, n_1, m_1, m_2) + C_{\Delta \overline{A}_1 \cap \mu}(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1}),
\]
where \(C_{\Delta \overline{A}_1 \cap \mu}(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1})\) is the contribution of the section \(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1}\) to the Euler class from \(\Delta \overline{A}_1 \cap \mu\). The lhs. of (35), as computed by splitting principle and a case by case check, is
\[
\langle e(\pi_2^*L_{A_0})e(\pi_2^*V_{A_1}), [\overline{A}_1 \times X \cap \{\mu\}] \rangle = \mathcal{N}(A_1) \times \mathcal{N}(A_1, n_1, m_1, m_2).
\]

Next, we compute \(C_{\Delta \overline{A}_1 \cap \mu}(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1})\). Note that \(\overline{A}_1 = A_1 \sqcup \overline{A}_2\). It is shown in [3] that
\[
C_{\Delta \overline{A}_1 \cap \mu}(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1}) = \langle e(\pi_2^*L_{A_0}), [\overline{A}_1 \cap \{\mu\}] \rangle
= \mathcal{N}(A_1, n_1, m_1, m_2) + \mathcal{N}(A_1, n_1 + 1, m_1, m_2),
\]
\[
C_{\Delta \overline{A}_1 \cap \mu}(\pi_2^*\psi_{A_0} \oplus \pi_2^*\psi_{A_1}) = 3\mathcal{N}(A_2, n_1, m_1, m_2).
\]

It is easy to see that (36), (37) and (38) prove (33). \(\square\)
Proposition 6.2. The number $N(A_1 \mathcal{PA}_2, n_1, m_1, m_2, \theta)$ is given by

\[
N(A_1 \mathcal{PA}_2, n_1, m_1, m_2, \theta) = \begin{cases}
2N(A_1A_1, n_1, m_1, m_2) + 2N(A_1A_1, n_1, m_1 + 1, m_2) \\
+ 2N(A_1A_1, n_1 + 1, m_1, m_2) - \left(2N(\mathcal{PA}_3, n_1, m_1, m_2, \theta)\right) & \text{if } \theta = 0, \\
N(A_1A_1, n_1, m_1, m_2) + 2N(A_1A_1, n_1 + 1, m_1, m_2) + N(A_1A_1, n_1 + 2, m_1, m_2) \\
+ 3N(A_1A_1, n_1, m_1 + 1, m_2) + 3N(A_1A_1, n_1 + 1, m_1 + 1, m_2) \\
+ 2N(A_1A_1, n_1, m_1 + 2, m_2) \\
- \left(2N(\mathcal{PA}_3, n_1, m_1, m_2, \theta) + 3N(D_4, n_1, m_1, m_2)\right) & \text{if } \theta = 1, \\
N(A_1 \mathcal{PA}_2, n_1, m_1 + 1, m_2, \theta - 1) & \text{if } \theta > 1.
\end{cases}
\] (39)

Proof: Let $\mu$ be a pseudocycle in $D \times X \times \mathbb{PT}X$ representing the homology class Poincaré dual to

\[c_1^{n_1} x_1^{m_1} x_s^{m_2} A^\theta y^\delta L_{-(n_1+1)\theta+3)}.
\]

We show in [3], that

\[
\overline{A_1} \circ \overline{A_1}^\# = \overline{A_1} \circ \overline{A_1}^\# \cup \overline{A_1} \circ \overline{\mathcal{PA}_2} \cup \Delta \overline{A_3}.
\]

If $L$ is sufficiently 5-ample, then the section

\[\pi_2^* \Psi_{\mathcal{PA}_2} : \overline{A_1} \circ \overline{A_1}^\# \cap \mu \longrightarrow \pi_2^* \mathcal{V}_{\mathcal{PA}_2}
\]

vanishes on $A_1 \circ \mathcal{PA}_2 \cap \mu$ transversely. Hence, the zeros of the section

\[\pi_2^* \Psi_{\mathcal{PA}_2} : \overline{A_1} \circ \overline{A_1}^\# \cap \mu \longrightarrow \pi_2^* \mathcal{V}_{\mathcal{PA}_2},
\]

restricted to $A_1 \circ \mathcal{PA}_2 \cap \mu$ counted with a sign, is our desired number. In other words

\[
\left< e(\pi_2^* \mathcal{V}_{\mathcal{PA}_2}), \overline{[A_1 \circ A_1]}^\# \cap \mu \right> = N(A_1 \mathcal{PA}_2, n_1, m_1, m_2, \theta) + C_{\Delta A_3 \cap \mu} (\pi_2^* \Psi_{\mathcal{PA}_2})
\]

where $C_{\Delta A_3 \cap \mu} (\pi_2^* \Psi_{\mathcal{PA}_2})$ is the contribution of the section to the Euler class from $\Delta \overline{A_3} \cap \mu$. Note that $\pi_2^* \Psi_{\mathcal{PA}_2}$ vanishes only on $\Delta \mathcal{PA}_3 \cap \mu$ and $\Delta \overline{D_4}^\# \cap \mu$ and not on the entire $\Delta \overline{A_3} \cap \mu$. We show in [3] that the contribution from $\Delta \mathcal{PA}_3 \cap \mu$ and $\Delta \overline{D_4}^\# \cap \mu$ are 2 and 3 respectively. Hence

\[
\left< e(\pi_2^* \mathcal{V}_{\mathcal{PA}_2}), \overline{[A_1 \circ A_1]}^\# \cap \mu \right> = N(A_1 \mathcal{PA}_2, n_1, m_1, m_2, \theta) + 2N(\mathcal{PA}_3, n_1, m_1, m_2, \theta) + 3|\Delta \overline{D_4}^\# \cap \mu|.
\]

Since $\overline{D_4}^\# = D_4$, this gives us (39).

Proposition 6.3. The number $N(A_1 \mathcal{PA}_3, n_1, m_1, m_2, \theta)$ is given by

\[
N(A_1 \mathcal{PA}_3, n_1, m_1, m_2, \theta) = 3N(A_1 \mathcal{PA}_2, n_1, m_1, m_2, \theta + 1) + N(A_1 \mathcal{PA}_2, n_1, m_1, m_2, \theta) \\
+ N(A_1 \mathcal{PA}_2, n_1 + 1, m_1, m_2, \theta) - 2N(\mathcal{PA}_5, n_1, m_1, m_2, \theta).
\] (40)
Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^{n_1}c_2^{m_1}p_2^{m_2}x_1^{\theta}x_2^\delta - (n_1 + m_1 + 2m_2 + \theta + 4).$$

We show in [3], that

$$\overline{A_1 \circ P A_2} = \overline{A_1 \circ P A_2} \sqcup \overline{A_1 \circ (PA_3 \cup \hat{D}_4^\#)} \sqcup \overline{\left(\Delta PA_4 \cup \Delta D_5^\#\right)}.$$ 

If $L$ is sufficiently 6-ample, then the section

$$\pi_2^* \Psi_{PA_3}: \overline{A_1 \circ P A_2} \cap \mu \to \pi_2^* \mathbb{L}_{PA_3}$$

vanishes transversely on $A_1 \circ PA_3$. We also show in [3], that the contribution to the Euler class from the points of $\Delta PA_4 \cap \mu$ is 2. This section does not vanish on $A_1 \circ \hat{D}_4^#$ and by definition it also does not vanish on $\Delta D_5^\#$. Hence

$$\left\langle e(\pi_2^* \mathbb{L}_{PA_3}), \overline{[A_1 \circ P A_2]} \cap \mu \right\rangle = N(A_1PA_3, n_1, m_1, m_2) + 2N(\mathcal{P}A_4, n_1, m_1, m_2, \theta)$$

which gives us (40). \qed

Proposition 6.4. The number $N(A_1PA_4, n_1, m_1, m_2, \theta)$ is given by

$$N(A_1PA_4, n_1, m_1, m_2, \theta) = 2N(A_1PA_3, n_1, m_1, m_2, \theta + 1) + 2N(A_1PA_3, n_1, m_1 + 1, m_2, \theta) + 2N(A_1PA_3, n_1 + 1, m_1, m_2, \theta) - 2N(PA_5, n_1, m_1, m_2, \theta).$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^{n_1}c_2^{m_1}p_2^{m_2}x_1^{\theta}x_2^\delta - (n_1 + m_1 + 2m_2 + \theta + 5).$$

We show in [3], that

$$\overline{A_1 \circ P A_3} = \overline{A_1 \circ P A_3} \sqcup \overline{A_1 \circ (PA_4 \cup \hat{PD}_4)} \sqcup \overline{\left(\Delta PA_5 \cup \Delta PD_5^\#\right)}.$$ 

If $L$ is sufficiently 7-ample, then the section

$$\pi_2^* \Psi_{PA_4}: \overline{A_1 \circ P A_3} \cap \mu \to \pi_2^* \mathbb{L}_{PA_4}$$

vanishes transversely on $A_1 \circ PA_4$. It is easy to see that it does not vanish on $A_1 \circ PD_4 \cap \mu$ and $\Delta PD_5^\# \cap \mu$. We also show in [3] that the contribution of this section to the Euler class from the points of $\Delta PA_5 \cap \mu$ is 2. Hence

$$\left\langle e(\pi_2^* \mathbb{L}_{PA_4}), \overline{[A_1 \circ P A_3]} \cap [\mu] \right\rangle = N(A_1PA_4, n_1, m_1, m_2, \theta) + 2N(PA_5, n_1, m_1, m_2, \theta),$$

which gives us (41). \qed
Proposition 6.5. The number $N(A_1 PA_5, n_1, m_1, m_2, \theta)$ is given by

$$N(A_1 PA_5, n_1, m_1, m_2, \theta) = N(A_1 PA_4, n_1, m_1, m_2, \theta + 1) + 4N(A_1 PA_4, n_1, m_1 + 1, m_2, \theta) + 3N(A_1 PA_4, n_1, m_1, m_2, \theta) + 3N(A_1 PA_4, n_1 + 1, m_1, m_2, \theta) - 2N(A_1 PD_5, n_1, m_1, m_2, \theta) - 2N(PA_6, n_1, m_1, m_2, \theta) - 5N(PE_6, n_1, m_1, m_2, \theta). \quad (42)$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^{n_1} x_1^{m_1} x_2^{m_2} y^{\theta - (n_1 + m_1 + 2m_2 + \theta + 6)}.$$

We show in [3], that

$$\overline{A_1 \circ PA_4} = \overline{A_1 \circ PA_4} \cup \overline{A_1 \circ (PA_5 \cup PD_5)} \cup \left( \Delta PA_6 \cup \Delta PD_5 \cup \Delta PE_6 \right),$$

where

$$\Delta PD_7^5 := \{ ([f], q, l_q) \in \overline{A_1 \circ PA_4} : \pi_2^* \Psi_{PD_4}([f], q, l_q) = 0, \pi_2^* \Psi_{PE_6}([f], q, l_q) \neq 0 \}.$$

If $L$ is sufficiently 8-ample, then the section

$$\pi_2^* \Psi_{PA_5} : \overline{A_1 \circ PA_4} \cap \mu \to \pi_2^* \mathcal{L}_{PA_5}$$

vanishes transversely on $A_1 \circ PA_5 \cap \mu$. We also show that this section vanishes on $A_1 \circ PD_5 \cap \mu$, $\Delta PA_6 \cap \mu$ and $\Delta PE_6 \cap \mu$ with a multiplicity of 2, 2 and 5 respectively. Since the dimension of $PD_7$ is one less than the dimension of $[\mu]$ and $\mu$ is generic, $\Delta PD_7 \cap \mu$ is empty. We also show in [3] that $\Delta PD_7^5$ is a subset of $\Delta PD_7$. Hence $\Delta PD_7^5 \cap \mu$ is also empty. Hence

$$\left\langle c(\pi_2^* \mathcal{L}_{PA_5}), [\overline{A_1 \circ PA_4}] \cap [\mu] \right\rangle = N(A_1 PA_5, n_1, m_1, m_2, \theta) + 2N(A_1 PD_5, n_1, m_1, m_2, \theta) + 2N(PA_6, n_1, m_1, m_2, \theta) + 5N(PE_6, n_1, m_1, m_2, \theta),$$

which gives us (42). \hfill \Box

Proposition 6.6. The number $N(A_1 PA_6, n_1, m_1, m_2, \theta)$ is given by

$$N(A_1 PA_6, n_1, m_1, m_2, \theta) = +6N(A_1 PA_5, n_1, m_1 + 1, m_2, \theta) + 4N(A_1 PA_5, n_1, m_1, m_2, \theta) + 4N(A_1 PA_5, n_1 + 1, m_1, m_2, \theta) - 4N(A_1 PD_6, n_1, m_1, m_2, \theta) - 3N(A_1 PE_6, n_1, m_1, m_2, \theta) - 2N(PA_7, n_1, m_1, m_2, \theta) - 6N(PE_7, n_1, m_1, m_2, \theta). \quad (43)$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to

$$c_1^{n_1} x_1^{m_1} x_2^{m_2} y^{\theta - (n_1 + m_1 + 2m_2 + \theta + 5)}.$$

We show in [3], that

$$\overline{A_1 \circ PA_5} = \overline{A_1 \circ PA_5} \cup \overline{A_1 \circ (PA_6 \cup PD_6 \cup PE_6)} \cup \left( \Delta PA_7 \cup \Delta PD_8 \cup \Delta PE_7 \right),$$
A vanishes transversely on \( L \). Let \( \Delta \) giving us \((\cdot,\cdot)\). We also show in [3] that \( \Delta \) is a subset of \( \Delta_6 \). Hence \( \Delta \) is also empty. Hence
\[
\left\langle e(\pi_2^*\mathcal{L}_{\mathcal{A}_6}), [\mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_6}] \right\rangle = \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 0) + 4\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_6}, n_1, m_1, m_2, 0) + 6\mathcal{N}(\mathcal{P}_{\mathcal{A}_7}, n_1, m_1, m_2, 0)
\]
which gives us (43).

**Proposition 6.7.** The number \( \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 0) \) is given by
\[
\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 0) = 2\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{A}_3}, n_1, m_1 + 1, m_2, 0) - 2\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{A}_3}, n_1, m_1, m_2, 1) + \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{A}_3}, n_1, m_1, m_2, 0) + \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{A}_3}, n_1 + 1, m_1, m_2, 0) - 2\mathcal{N}(\mathcal{D}_5, n_1, m_1, m_2, 0).
\]

**Proof:** Let \( \mu \) be a generic pseudocycle representing the homology class Poincaré dual to
\[
c^{n_1} x_1^{m_1} x_2^{m_2} y^{n_1 + m_1 + 2m_2 + 5}.
\]
We show in [3], that
\[
\mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_3} = \mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_4} \cup (\mathcal{P}_{\mathcal{A}_4} \cup \mathcal{D}_4) \cup (\Delta_{\mathcal{P}_{\mathcal{A}_5}} \cup \Delta_{\mathcal{P}_{\mathcal{D}_8}}).
\]
If \( L \) is sufficiently 5-ample, then the section
\[
\pi_2^*\mathcal{P}_{\mathcal{D}_4} : \mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_3} \cap \mu \longrightarrow \pi_2^*\mathcal{L}_{\mathcal{P}_{\mathcal{D}_4}}
\]
vanesishes transversely on \( \mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_4} \cap \mu \). It is easy to see that the section does not vanish on \( \mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_4} \) and \( \Delta_{\mathcal{P}_{\mathcal{A}_5}} \). We also show that the contribution of the section to the Euler class from the points of \( \Delta_{\mathcal{P}_{\mathcal{D}_8}} \cap \mu \) is 2. Hence
\[
\left\langle e(\pi_2^*\mathcal{L}_{\mathcal{D}_4}), [\mathcal{A}_1 \circ \mathcal{P}_{\mathcal{A}_3}] \cap [\mu] \right\rangle = \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 0) + 2\mathcal{N}(\mathcal{D}_5, n_1, m_1, m_2, 0),
\]
giving us (44).

**Proposition 6.8.** The number \( \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 1) \) is given by
\[
\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 1) = \frac{1}{3}\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1, m_1, m_2, 0) + \mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1 + 1, m_1, m_2, 0)
\]
\[
+ \frac{1}{3}\mathcal{N}(\mathcal{A}_1 \mathcal{P}_{\mathcal{D}_4}, n_1 + 1, m_1, m_2, 0).
\]

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Proof: Let \( \mu \) be a generic pseudocycle representing the homology class Poincaré dual to

\[
\langle c_1^{m_1} x_1^{m_1} x_2^{m_2} \lambda \theta L^{-m_1-m_1+2m_2+6} \rangle.
\]

We show in [3], that

\[
\overline{A_1 \circ D_4} = \overline{A_1 \circ D_4} \cup \overline{A_1 \circ PD_4} \cup \left( \overline{\Delta PD_6^{\#}} \right).
\]

If \( L \) is sufficiently 5-ample, then the section

\[
\pi_2^* \Psi_{PD_5} : \overline{A_1 \circ PD_4} \cap \mu \rightarrow \pi_2^* \mathbb{L}_{PD_5}
\]

vanishes transversely on \( A_1 \circ PD_4 \cap \mu \). By definition, the section does not vanish on \( \Delta PD_6^{\#} \cap \mu \).

Hence

\[
\langle e(\pi_2^* \mathbb{L}_{PD_5}), \overline{A_1 \circ PD_4} \cap [\mu] \rangle = N(A_1 PD_4, n_1, m_1, m_2, 1)
\]

which gives us (46).

Proposition 6.9. If \( \theta > 1 \), the number \( N(A_1 PD_4, n_1, m_1, m_2, \theta) \) is given by

\[
N(A_1 PD_4, n_1, m_1, m_2, \theta) = N(A_1 PD_4, n_1, m_1 + 1, m_2, \theta - 1) - N(A_1 PD_4, n_1, m_1 + 1, \theta - 2).
\]

Proof: Follows immediately from the ring structure of \( H^*(\mathbb{P} TX) \).

Proposition 6.10. The number \( N(A_1 PD_5, n_1, m_1, m_2, \theta) \) is given by

\[
N(A_1 PD_5, n_1, m_1, m_2, \theta) = N(A_1 PD_4, n_1, m_1, m_2, \theta + 1) + N(A_1 PD_4, n_1, m_1 + 1, m_2, \theta) + N(A_1 PD_4, n_1 + 1, m_1, m_2, \theta) - 2N(PD_6, n_1, m_2, \theta).
\]

Proof: Let \( \mu \) be a generic pseudocycle representing the homology class Poincaré dual to

\[
\langle c_1^{m_1} x_1^{m_1} x_2^{m_2} \lambda \theta L^{-m_1-m_1+2m_2+\theta+6} \rangle.
\]

We show in [3], that

\[
\overline{A_1 \circ PD_4} = \overline{A_1 \circ PD_4} \cup \overline{A_1 \circ PD_5} \cup \left( \overline{\Delta PD_6^{\#}} \right).
\]

If \( L \) is sufficiently 6-ample, then the section

\[
\pi_2^* \mathbb{L}_{PD_5} : \overline{A_1 \circ PD_4} \cap \mu \rightarrow \pi_2^* \mathbb{L}_{PD_5}
\]

vanishes transversely on \( A_1 \circ PD_5 \). Moreover, it does not vanish on \( \overline{A_1 \circ PD_6^{\#}} \cap \mu \) by definition. We also show that the the contribution of the section to the Euler class from the points of \( \Delta PD_6 \cap \mu \) is 2. The section does not vanish on \( \Delta PD_6^{\#} \cap \mu \) by definition (cf. [3]). Since \( \mu \) is generic, the section does not vanish on \( \Delta PD_6^{\#} \cap \mu \). Hence

\[
\langle e(\pi_2^* \mathbb{L}_{PD_5}), \overline{A_1 \circ PD_4} \cap [\mu] \rangle = N(A_1 PD_5, n_1, m_1, m_2, \theta) + 2N(PD_6, n_1, m_1, m_2, \theta)
\]

which gives us (48).
Proposition 6.11. The numbers $N(A_1 \mathcal{P}D_6, n_1, m_1, m_2, \theta)$ and $N(A_1 \mathcal{P}E_6, n_1, m_1, m_2, \theta)$ are

$$N(A_1 \mathcal{P}D_6, n_1, m_1, m_2, \theta) = 4N(A_1 \mathcal{P}D_5, n_1, m_1, m_2, \theta + 1) + N(A_1 \mathcal{P}D_5, n_1, m_1, m_2, \theta)$$

$$+ N(A_1 \mathcal{P}D_5, n_1 + 1, m_1, m_2, \theta)$$

$$- 2N(\mathcal{P}D_7, n_1, m_1, m_2, \theta) - N(\mathcal{P}E_7, n_1, m_1, m_2, \theta), \quad (49)$$

$$N(A_1 \mathcal{P}E_6, n_1, m_1, m_2, \theta) = 2N(A_1 \mathcal{P}D_5, n_1, m_1 + 1, m_2, \theta) - N(A_1 \mathcal{P}D_5, n_1, m_1, m_2, \theta + 1)$$

$$+ N(A_1 \mathcal{P}D_5, n_1 + 1, m_1, m_2, \theta)$$

$$- N(\mathcal{P}E_7, n_1, m_1, m_2, \theta). \quad (50)$$

Proof: Let $\mu$ be a generic pseudocycle representing the homology class Poincaré dual to $c_1^{n_1}x_1^{m_1}x_2^{m_2}x_3^{\theta}y^{\delta}L^{-(n_1+m_1+2m_2+\theta+7)}$.

We show in [3], that

$$\overline{A_1 \circ \mathcal{P}D_5} = \overline{A_1 \circ \mathcal{P}D_5 \cup A_1 \circ (\mathcal{P}D_5 \cup \mathcal{P}E_6)} \cup \left( \Delta \mathcal{P}D_7 \cup \Delta \mathcal{P}E_7 \right).$$

If $L$ is sufficiently 7-ample or 6-ample, then the sections

$$\pi_2^* \Psi_{\mathcal{P}D_6} : \overline{A_1 \circ \mathcal{P}D_5 \cap \mu} \longrightarrow \pi_2^* \mathcal{L}_{\mathcal{P}D_6}, \quad \pi_2^* \Psi_{\mathcal{P}E_6} : \overline{A_1 \circ \mathcal{P}D_5 \cap \mu} \longrightarrow \pi_2^* \mathcal{L}_{\mathcal{P}E_6}$$

vanish transversely on $A_1 \circ \mathcal{P}D_6 \cap \mu$ and $A_1 \circ \mathcal{P}E_6 \cap \mu$ respectively. Moreover, they do not vanish on $A_1 \circ \mathcal{P}E_6 \cap \mu$ and $A_1 \circ \mathcal{P}D_6 \cap \mu$ respectively. We also show that the contribution of the sections $\pi_2^* \Psi_{\mathcal{P}D_6}$ to the Euler class from the points of $\Delta \mathcal{P}D_7 \cap \mu$ and $\Delta \mathcal{P}E_7 \cap \mu$ are 2 and 1 respectively. We also show that the contribution of the section $\pi_2^* \Psi_{\mathcal{P}E_6}$ from the points of $\Delta \mathcal{P}E_7 \cap \mu$ is 1; moreover it does not vanish on $\Delta \mathcal{P}D_7 \cap \mu$. Hence

$$\left\langle e(\pi_2^* \mathcal{L}_{\mathcal{P}D_6}), [\overline{A_1 \circ \mathcal{P}D_5}] \cap \mu \right\rangle = N(A_1 \mathcal{P}D_6, n_1, m_1, m_2, \theta)$$

$$+ 2N(\mathcal{P}D_7, n_1, m_1, m_2, \theta) + N(\mathcal{P}E_7, n_1, m_1, m_2, \theta),$$

$$\left\langle e(\pi_2^* \mathcal{L}_{\mathcal{P}E_6}), [\overline{A_1 \circ \mathcal{P}D_5}] \cap \mu \right\rangle = N(A_1 \mathcal{P}E_6, n_1, m_1, m_2, \theta) + N(\mathcal{P}E_7, n_1, m_1, m_2, \theta),$$

which give us (49) and (50) respectively. \hfill \Box

References

[1] S. Basu and R. Mukherjee, Enumeration of curves with one singular point. available at http://arxiv.org/abs/1308.2902.

[2] ———, Enumeration of curves with singularities: Further details. available at https://www.sites.google.com/site/ritwik371/home.

[3] ———, Enumeration of curves with two singular points, Bull.Sci.math (2014), http://dx.doi.org/10.1016/j.bulsci.2014.11.006.

[4] R. Bott and L. W. Tu, Differential forms in algebraic topology, vol. 82 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1982.
[5] L. Caporaso and J. Harris, *Counting plane curves of any genus*, Invent. Math., 131 (1998), pp. 345–392.

[6] M. È. Kazarian, *Multisingularities, cobordisms, and enumerative geometry*, Uspekhi Mat. Nauk, 58 (2003), pp. 29–88.

[7] D. Kerner, *Enumeration of singular algebraic curves*, Israel J. Math., 155 (2006), pp. 1–56.

[8] ———, *Enumeration of uni-singular algebraic hypersurfaces*, Proc of London Mathematical Society, 3 (2008), pp. 623–668.

[9] D. Kerner, *On the enumeration of complex plane curves with two singular points*, Int. Math. Res. Not. IMRN, (2010), pp. 4498–4543.

[10] R. Mukherjee, *Enumeration of singular hypersurfaces on arbitrary complex manifolds*. available at http://arxiv.org/abs/1410.4142.

[11] Z. Ran, *On nodal plane curves*, Invent. Math., 86 (1986), pp. 529–534.

[12] Z. Ran, *Enumerative geometry of singular plane curves*, Invent. Math., 97 (1989), pp. 447–465.

[13] I. Vainsencher, *Counting divisors with prescribed singularities*, J. Algebraic Geom., 267 (1981), pp. 399–422.

[14] ———, *Enumeration of n-fold tangent hyperplanes to a surface*, J. Algebraic Geom., 4 (1995), pp. 503–526.

[15] A. Zinger, *Counting plane rational curves: old and new approaches*. available at http://arxiv.org/abs/math/0507105.

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