Non-analytic Term in Effective Potential at Finite Temperature for Scalar Field on Compactified Space

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Abstract

We study non-analytic terms, which cannot be written in the form of any positive integer power of field-dependent mass squared, in effective potential at finite temperature in one-loop approximation for a real scalar field on the \(D\)-dimensional space-time, \(S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^{p} S^i_1\). The effective potential can be recast into the integral form in the complex plane by using the integral representation for the modified Bessel function of the second kind and the analytical extension for multiple mode summations. The pole structure of the mode summations is clarified and all the non-analytic terms are obtained by the residue theorem. We find that the effective potential has a non-analytic term when the dimension of the flat Euclidean space, \(D - (p + 1)\) is odd. There appears only one non-analytic term for the given values of \(D\) and \(p\), for which the non-analytic term exists.

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1 Introduction

The effective potential at finite temperature has provided a useful tool to study the phenomena of phase transition in quantum field theory. Dolan and Jackiw \[1\] found that there exists a non-analytic term, which cannot be written in the form of any positive integer power of field-dependent mass squared, in the effective potential at finite temperature for a scalar field. The non-analytic term found by them is proportional to three-halves power of the mass squared, and the term turns out to play a crucial role to trigger the first order phase transition\[2, 3\], for example, in electroweak theories. The magnitude of the term determines the strength of the first order phase transition and thus, it is concerned with the scenario of electroweak baryogenesis \[4\] as well. Hence, the non-analytic term in the effective potential is an important quantity.

Quantum field theory with compactified dimensions has been one of the attractive approaches for physics beyond the standard model. Orbifold compactification, for example, provides an attractive framework for gauge-Higgs unification, where the Higgs field is unified into higher dimensional gauge fields and it is an alternative solution to the gauge hierarchy problem\[5, 6\]. The order of the phase transition in the gauge-Higgs unification at finite temperature has been studied in \[7, 8\], and the first order phase transition can take place due to the term with three-halves power of the field-dependent mass squared in the effective potential. Compactified dimensions also offer the theoretical framework for studying quantum field theory itself. From a point of view of dimensional reduction \[9, 10\], models with several numbers of $S^1$ have been investigated. It has been also shown that the quantum field theory with compactified dimensions (at finite temperature) can possess rich phase structures \[11, 12\].

Taking account of the aforementioned studies, it is important and interesting to investigate non-analytic terms in the effective potential in the presence of extra dimensions at finite temperature. In this paper, we study all the non-analytic terms for a real scalar field on the $D$-dimensional space-time, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^{p} S^1_i$, where $S^1_\tau$ stands for the Euclidean time direction and the spacial directions are compactified on $S^1_i$. The $R^{D-(p+1)}$ is the $D-(p+1)$-dimensional flat Euclidean space. We assume that the scalar field satisfies the periodic boundary condition for the spacial $S^1$ direction.

The effective potential contains the modified Bessel function of the second kind accompanied with multiple mode summations. In addition to the integral representation for the modified Bessel function of the second kind given by the inverse Mellin transformation \[13, 14\], we make use of the analytical extension for the mode summations \[16\] in order to recast the effective potential into the integral form in the complex plane and to obtain the non-analytic terms by the residue theorem.

The analytical extension consists of the products of the gamma, zeta functions and their integrals. We clarify the pole structure of the analytical extension and find that relevant terms in the mode summations for yielding the non-analytic terms satisfy a recurrence relation, which gives a general form for the relevant terms. Then, all the non-analytic terms can be obtained by the residue theorem for the poles of the gamma and zeta functions in the general form, depending on the even/odd $D$ and $n$ ($n = 1, 2, \cdots, p+1$). The positions of the poles that
yield the non-analytic terms turn out not to depend on $D$.

We find that a non-analytic term appears in the effective potential when the dimension of the flat Euclidean space, $D - (p + 1)$ is odd. There is only one non-analytic term for the given values of $D$ and $p$, for which the non-analytic term exists.

This paper is organized as follows. In the next section, we rewrite the effective potential in the integral form in the complex plane and discuss to obtain the non-analytic terms for the case of $p = 1$ explicitly. And we obtain the general form for the relevant terms in the mode summations. We calculate the non-analytic terms by using the general form in the section 3 and obtain the non-analytic terms in the effective potential in the section 4. The final section is devoted to conclusions and discussions. Some details on the pole structure of the analytical extension in the mode summations are given in Appendix A.

2 Effective potential in integral form and non-analytic terms

We study non-analytic terms, which cannot be written in the form of any positive integer powers of the field-dependent mass squared, in the effective potential for a real scalar field at finite temperature on the $D$-dimensional space-time, $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^{p} S^1_i$ in one-loop approximation. We employ the Euclidean time formalism for finite temperature field theory and then the Euclidean time direction is compactified on $S^1_\tau$. The spacial $p$ dimensions are compactified on the $p$ numbers of $S^1_i$. We denote the circumference of each $S^1_i$ as $L_i (i = 0, 1, \cdots, p)$ and $L_0$ stands for the inverse temperature $T^{-1}$.

One needs to evaluate

$$V_{\text{eff}} = (-1)^f \mathcal{N} \frac{1}{2} \left( \prod_{i=0}^{p} \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int d^{D-(p+1)} p_E \times \log \left[ p_E^2 + \left( \frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2 + \sum_{i=1}^{p} \left( \frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 + M^2(\varphi) \right]$$

in order to obtain the effective potential on $S^1_\tau \times R^{D-(p+1)} \times \prod_{i=1}^{p} S^1_i$ in one-loop approximation. The $M^2(\varphi)$ is the field-dependent mass squared of the scalar field $\varphi$. The $p_E$ denotes the $D-(p+1)$-dimensional Euclidean momentum. The $f$ is the fermion number, which is 0 (1) for bosons (fermions). The $\mathcal{N}$ is the on-shell degrees of freedom. The $n_0$ denotes the Matsubara mode at finite temperature and the Kaluza-Klein mode $n_i (i = 1, \cdots, p)$ comes from each $S^1_i (i = 1, \cdots, p)$. The parameter $\eta_0$, which stands for the boundary condition for the $S^1_\tau$ direction, is determined by quantum statistics to be 0 ($\frac{1}{2}$) for bosons (fermions). The parameter $\eta_i (i = 1, \cdots, p)$ specifies the boundary condition for the spacial $S^1_i$ direction.

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3Hereafter, we denote $M(\varphi)$ by $M$ for simplicity.
We employ the zeta-function regularization in order to evaluate Eq.(2.1). Let us define
\[
I(s) \equiv \left( \prod_{i=0}^{p} \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^{D-(p+1)}P}{(2\pi)^{D-(p+1)}} \times \left[ P_{\mathcal{E}}^2 + \left( \frac{2\pi}{L_0} \right)^2 (n_0 + \eta_0)^2 + \sum_{i=1}^{p} \left( \frac{2\pi}{L_i} \right)^2 (n_i + \eta_i)^2 + M^2 \right]^{-s} . \tag{2.2}
\]

Then, the effective potential can be written as
\[
V_{\text{eff}} = (-1)^{f+1} \frac{N^2}{2} \left( -\frac{d}{ds} I(s) \right) \bigg|_{s \to 0} . \tag{2.3}
\]

Using the formula
\[
A^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \ t^{s-1} e^{-At} \tag{2.4}
\]
and the Poisson summation (A.2) in Appendix A with the replacement of \( L_0 \rightarrow \frac{2\pi}{L_i} \), \( m_0 \rightarrow n_i \), \( n_0 \rightarrow m_i \) and \( \eta_0 \rightarrow \eta_i \), we arrive at
\[
V_{\text{eff}} = (-1)^{f+1} \frac{N^2}{2} \left( \frac{\pi D}{(2\pi)^D} \sum_{m_0=-\infty}^{\infty} \cdots \sum_{m_p=-\infty}^{\infty} \int_{0}^{\infty} dt \ t^{\frac{D-1}{2}} e^{-\frac{1}{4t} \left( (m_0 L_0)^2 + \cdots + (m_p L_p)^2 \right) - M^2 t + 2\pi i (m_0 \eta_0 + \cdots + m_p \eta_p)} \right). \tag{2.5}
\]

It is convenient to separate each summation \( \sum_{m_0} \) in Eq.(2.5) into the zero mode \( (m_i = 0) \) and the non-zero ones \( (m_i \neq 0) \), and to express Eq.(2.5) into the form
\[
V_{\text{eff}} = \sum_{n=0}^{p+1} F^{(n)} , \tag{2.6}
\]
where
\[
F^{(n)} = \sum_{0 \leq i_1 < i_2 < \cdots < i_n \leq p} F^{(n)}_{L_{i_1},L_{i_2},\cdots,L_{i_n}} , \tag{2.7}
\]
\[
F^{(n)}_{L_{i_1},L_{i_2},\cdots,L_{i_n}} = (-1)^{f+1} \frac{N^2}{2} \left( \frac{\pi D}{(2\pi)^D} \sum_{m_{i_1}=-\infty}^{\infty} \cdots \sum_{m_{i_n}=-\infty}^{\infty} \int_{0}^{\infty} dt \ t^{\frac{D-1}{2}} e^{-\frac{1}{4t} \left( (m_{i_1} L_{i_1})^2 + \cdots + (m_{i_n} L_{i_n})^2 \right) - M^2 t + 2\pi i (m_{i_1} \eta_{i_1} + \cdots + m_{i_n} \eta_{i_n})} \right). \tag{2.8}
\]

Here, the prime of the summation \( \sum_{m_i=-\infty}^{\infty} \) means that the zero mode \( (m_i = 0) \) is removed.
The $F^{(0)}$ in Eq. (2.6) corresponds to the contribution from all the zero modes $m_0 = m_1 = \cdots = m_p = 0$ in Eq. (2.5) and is given by

$$F^{(0)} = (-1)^{f+1} \frac{\mathcal{N}}{2} \frac{\pi^D}{(2\pi)^D} \int_0^\infty dt \, t^{D/2-1} e^{-M^2 t} = (-1)^{f+1} \frac{\mathcal{N}}{2} \frac{\pi^D}{(2\pi)^D} \Gamma\left(\frac{D}{2}\right)(M^2)^{D/2}. \quad (2.9)$$

On the other hand, by using the formula (A.3) in Appendix A, $F^{(n)}_{L_1 L_2 \cdots L_n}$ ($n \geq 1$) can be obtained as

$$F^{(n)}_{L_1 L_2 \cdots L_n} = (-1)^{f+1} \mathcal{N} \frac{2^n}{(2\pi)^D} \sum_{m_1=1}^\infty \cdots \sum_{m_n=1}^\infty \left( \frac{M^2}{(m_1 L_1)^2 + \cdots + (m_n L_n)^2} \right)^{D/2} \times K_{D/2} \left( \sqrt{M^2\{ (m_1 L_1)^2 + \cdots + (m_n L_n)^2 \} } \right) \cos(2\pi m_1 \eta_1) \cdots \cos(2\pi m_n \eta_n). \quad (2.10)$$

In this paper, we consider a real scalar field ($f = 0, \mathcal{N} = 1$) and take the periodic boundary condition $\eta_j = 0$ ($j = 1, \cdots, p$) for the spatial $S^1_j$ direction.

### 2.1 Non-analytic terms for the case of $S^1_r \times R^{D-2} \times S^1_l$

Let us first consider that the space-time is $S^1_r \times R^{D-2} \times S^1_l$ and study non-analytic terms in the effective potential. Although the results concerning the effective potential in this subsection are not new and, in fact, they are given in [9], this is a simple but nontrivial example and is appropriate to present here the analysis on the non-analytic terms in detail.

The effective potential on $S^1_r \times R^{D-2} \times S^1_l$ takes the form

$$V_{\text{eff}} = F^{(0)} + F^{(1)} + F^{(2)}, \quad (2.11)$$

where $F^{(0)}$ is given by Eq. (2.9) and

$$F^{(1)} = F^{(1)}_{L_0} + F^{(1)}_{L_1}$$

$$= -\frac{2}{(2\pi)^D} \sum_{m_0=1}^\infty \left( \frac{M^2}{m_0^2 L_0^2} \right)^{D/2} K_{D/2} \left( \sqrt{M^2(m_0 L_0)^2} \right)$$

$$-\frac{2}{(2\pi)^D} \sum_{m_1=1}^\infty \left( \frac{M^2}{m_1^2 L_1^2} \right)^{D/2} K_{D/2} \left( \sqrt{M^2(m_1 L_1)^2} \right), \quad (2.12)$$

$$F^{(2)} = F^{(2)}_{L_0 L_1}$$

$$= -\frac{2^2}{(2\pi)^D} \sum_{m_0=1}^\infty \sum_{m_1=1}^\infty \left( \frac{M^2}{(m_0 L_0)^2 + (m_1 L_1)^2} \right)^{D/2} K_{D/2} \left( \sqrt{M^2\{ (m_0 L_0)^2 + (m_1 L_1)^2 \} } \right), \quad (2.13)$$

\(^4\text{It must be understood that } F^{(0)} \text{ is regularized by the dimensional regularization for } D = \text{ even.}\)
which follow from Eq. (2.10). Here, the \( K_\nu(x) \) is the modified Bessel function of the second kind. Let us note that \( F^{(0)} + F^{(1)}_{L_0} \) is the effective potential at finite temperature without the compactified spacial dimension.

For our purpose, let us use the integral representation for the modified Bessel function of the second kind \[13\] in the complex plane\[5\],

\[
K_\nu(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t)\Gamma(t - \nu) \left( \frac{x}{2} \right)^{-2t+\nu}.
\]

The constant \( c \) should be understood to be a point located on the real axis which is greater than all the poles of the gamma functions in the integrand. Then, we deform the integration path in such a way that it encloses all the poles in the integrand and we can perform the \( t \)-integration by the residue theorem.

If we apply Eq. (2.14) to the first term of Eq. (2.12), we have

\[
F^{(1)}_{L_0} = -\frac{2}{(2\pi)^{D/2}} \left( \frac{M^2}{2} \right)^{D/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t)\Gamma(t - D/2)\zeta(2t) \left( \frac{ML_0}{2} \right)^{-2t}.
\]

The zeta function \( \zeta(2t) \) is the consequence of the single mode summation with respect to \( m_0 \). One can obtain all the terms in the effective potential in terms of \( M \) \[2, 15\] by the residue integral for all the poles in the integrand.

Once we obtain the integral form like Eq. (2.15), it is easy to find the non-analytic terms, which cannot be written in the form of any positive integer powers of \( M^2 \). The pole at \( t = D/2 - n \) \((n = 0, 1, 2, \cdots)\) of \( \Gamma(t - D/2) \) yields the mass dependence on \((M^2)^{D/2}M^{-2t} = (M^2)^n\), so that the residue integral for the pole does not produce non-analytic terms. It turns out that any poles depending on \( D \) do not produce non-analytic terms. This observation is crucially used throughout our discussions.

Moreover, we find that the mass dependence on \((M^2)^{D/2}M^{-2t}\) in Eq. (2.15) tells us that for \( D = \) even (odd), half-odd-integer (integer) values of \( t \) can yield non-analytic terms. It follows that a non-analytic term in Eq. (2.15) arises from either a pole of \( \Gamma(t) \) at \( t = 0 \) for \( D = \) odd or that of \( \zeta(2t) \) at \( t = \frac{1}{2} \) for \( D = \) even, whose value of the pole is independent of \( D \), as stated above. One might think that the poles of \( \Gamma(t) \) at \( t = -n \) \((n = 1, 2, \cdots)\) could produce non-analytic terms for \( D = \) odd. This is not, however, the case because of the property \( \zeta(2t) = 0 \) for \( t = -n \) \((n = 1, 2, \cdots)\). This observation is also used throughout our discussions.

For \( D = \) even, the residue theorem for the pole \( t = \frac{1}{2} \) of \( \zeta(2t) \) gives us the non-analytic term given by

\[
F^{(1)}_{L_0} \bigg|_{\text{n.a.}} = -\frac{(-1)^{D/2}}{2^{D/2} \pi^{D/2}} \frac{M^{D-1}}{L_0^{D-1}}.
\]

\[5\]This is the inverse Mellin transformation for the \( K_\nu(x) \)\[14\].
where we have used
\[
\Gamma\left(\frac{1-D}{2}\right)\bigg|_{D=\text{even}} = \left(-1\right)^{\frac{D}{2}} \frac{2^D}{(D-1)!!} \sqrt{\pi}. \tag{2.17}
\]

The abbreviation denoted by “n.a.” in Eq.(2.16) means non-analytic terms. Eq.(2.16) is the famous term found by Dolan and Jackiw [1] for \(D = 4\). The other poles of the gamma functions in the integrand and Eq.(2.9) for \(D = \text{even}\) do not yield non-analytic terms, so that the Dolan-Jackiw term (2.16) is the only possible non-analytic term in the effective potential for \(D = 4\) and \(p = 0\).

For \(D = \text{odd}\), on the other hand, the pole \(t = 0\) of \(\Gamma\left(-\frac{D}{2}\right)\) in Eq.(2.15) gives us
\[
\Gamma\left(-\frac{D}{2}\right)\bigg|_{D=\text{odd}} = \left(-1\right)^{\frac{D+1}{2}} \frac{2^{D+1}}{D!!} \sqrt{\pi}. \tag{2.19}
\]
Eq.(2.18), however, cancels the non-analytic term of Eq.(2.9). Thus, there is no non-analytic term in the effective potential for \(D = \text{odd}\) and \(p = 0\).

Likewise, one can evaluate the second term of Eq.(2.12) and obtains the non-analytic term as
\[
F_{L_1}^{(1)}\bigg|_{\text{n.a.}} = \left\{ \begin{array}{ll}
-\frac{\left(-1\right)^{D+1}}{2^D \pi^{D+2}/(D-1)!!} & \text{for } D = \text{even}, \\
\frac{\left(-1\right)^{D+1} M^{D-1}}{L_1} & \text{for } D = \text{odd}.
\end{array} \right. \tag{2.20}
\]

Let us next proceed to study non-analytic terms in \(F^{(2)} = F^{(2)}_{L_0,L_1}\), which is rewritten in the integral form by using Eq.(2.14) as
\[
F^{(2)}_{L_0,L_1} = -\frac{2^2}{(2\pi)^{D/2}} \left(\frac{M^2}{2}\right)^{D/2} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma\left(t - \frac{D}{2}\right) \left(\frac{M}{2}\right)^{-2t} \times \Gamma\left(t\right) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \left\{ (m_0 L_0)^2 + (m_1 L_1)^2 \right\}^{-t}. \tag{2.21}
\]
One must evaluate the double summations
\[
\Gamma\left(t\right) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \left\{ (m_0 L_0)^2 + (m_1 L_1)^2 \right\}^{-t}. \tag{2.22}
\]
which is known to have an analytical extension [16], as shortly discussed in Appendix A, given by (see Eq. (A.4))

\[
\Gamma(t) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \left\{ (m_0L_0)^2 + (m_1L_1)^2 \right\}^{-t} \\
= -\frac{1}{2} \left[ \Gamma(t) \zeta(2t) + \frac{\sqrt{\pi}}{2} \frac{1}{L_0 L_1} \Gamma(t - \frac{1}{2}) \zeta(2t - 1) \right] \\
+ \frac{1}{\sqrt{\pi}} \left( \frac{\pi}{L_0} \right)^{2t} \frac{1}{2\pi i} \int_{c_1-\infty}^{c_1+i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \zeta(t_1) \left( \frac{L_1}{L_0} \right)^{-2t_1},
\]

(2.23)

where we have first summed over \( m_0 \) and then \( m_1 \).

Inserting (2.23) into (2.21), we have

\[
F_{L_0,L_1}^{(2)} = \frac{1}{(2\pi)^{D/2}} \left( \frac{M^2}{2} \right)^{D/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{D}{2}) \Gamma(t) \zeta(2t) \left( \frac{ML_1}{2} \right)^{-2t} \\
- \frac{1}{(2\pi)^{D/2}} \sqrt{\pi} \left( \frac{M^2}{2} \right)^{D/2} \frac{L_1}{L_0} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{D}{2}) \Gamma(t - \frac{1}{2}) \zeta(2t - 1) \left( \frac{ML_1}{2} \right)^{-2t} \\
- \frac{2}{(2\pi)^{D/2}} \left( \frac{M^2}{2} \right)^{D/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{D}{2}) \left( \frac{M}{2} \right)^{-2t} \frac{1}{\sqrt{\pi}} \left( \frac{\pi}{L_0} \right)^{2t} \\
\times \frac{1}{2\pi i} \int_{c_1-\infty}^{c_1+i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \zeta(2t_1) \left( \frac{L_1}{L_0} \right)^{-2t_1}. \quad (2.24)
\]

We note again that the poles of \( \Gamma(t - \frac{D}{2}) \) do not yield non-analytic terms. This is because the residue integral with the poles of \( \Gamma(t - \frac{D}{2}) \) at \( t = \frac{D}{2} - n \ (n = 0, 1, 2, \ldots) \) gives the mass dependence on \( (M^2)^n \), which are not non-analytic terms for \( n = 0, 1, 2, \ldots \).

Thus, the poles, whose positions depend on \( D \), do not give non-analytic terms, as mentioned before.

We find that non-analytic terms in Eq. (2.24) can arise only from the poles other than those of \( \Gamma(t - \frac{D}{2}) \). In fact, the non-analytic terms are produced by the poles of \( t = 0 \) for \( D = \text{odd} \) and \( t = \frac{1}{2} \) for \( D = \text{even} \) \( (t = \frac{1}{2} \) for \( D = \text{even} \) and \( t = 1 \) for \( D = \text{odd} \) in the first (second) term of Eq. (2.24).

On the other hand, there is no pole that contributes to the residue integral with respect to \( t \) in the third term of Eq. (2.24), other than those of \( \Gamma(t - \frac{D}{2}) \), as shown in the subsection A.1 of Appendix A. This implies that the third term in Eq. (2.24) is irrelevant for the analyses of the non-analytic terms because the poles of \( \Gamma(t - \frac{D}{2}) \) do not produce any non-analytic terms, as stressed above. This is a crucial observation for our study on the non-analytic terms in the effective potential and plays a central role in the discussions.

The non-analytic terms in \( F_{L_0,L_1}^{(2)} \) turn out to come from the first and the second terms in
Eq. (2.24) and we obtain
\[
F^{(2)}_{L_0, L_1} \big|_{n.a.} = \left( -1 \right)^{\frac{D}{2}} \frac{2^{\frac{D}{2} - 1} \pi^{\frac{D}{2} - 2}}{2^{\frac{D+1}{2}} \pi^{\frac{D+1}{2}}} M^{D-1} \left( \frac{1}{L_1} + \frac{1}{L_0} \right) (2.25)
\]
for \( D = \text{even} \). The first (second) term in Eq. (2.25) comes from the pole \( t = \frac{1}{2} \) of \( \zeta(2t) \) \( (t = \frac{1}{2} \) of \( \Gamma(t - \frac{1}{2}) \) \) in the first (second) term of Eq. (2.24). On the other hand, for \( D = \text{odd} \), we obtain the non-analytic terms as
\[
F^{(2)}_{L_0, L_1} \big|_{n.a.} = -\left( -1 \right)^{\frac{D+1}{2}} \frac{2^{\frac{D+1}{2} - 1} \pi^{\frac{D+1}{2} - 2}}{2^{\frac{D+1}{2}} \pi^{\frac{D+1}{2}}} M^{D-2} \left( \frac{1}{L_0 L_1} \right) (2.26)
\]
The first (second) term of Eq. (2.26) comes from the pole \( t = 0 \) of \( \Gamma(t) \) \( (t = 1 \) of \( \zeta(2t - 1) \) \) in the first (second) term of Eq. (2.24).

Collecting the terms we have obtained above, we find that for \( D = \text{even} \) the effective potential has no non-analytic term, i.e.
\[
V_{\text{eff}} \big|_{n.a.} = (F^{(0)} + F^{(1)}_{L_0} + F^{(1)}_{L_1} + F^{(2)}_{L_0, L_1}) \big|_{n.a.} = 0 \quad \text{for} \ D = \text{even}, \quad (2.27)
\]
although \( F^{(1)}_{L_0} \), \( F^{(1)}_{L_1} \) and \( F^{(2)}_{L_0, L_1} \) contain the non-analytic terms. For \( D = \text{odd} \), all of \( F^{(0)}, F^{(1)}_{L_0}, F^{(1)}_{L_1} \) and \( F^{(2)}_{L_0, L_1} \) have the non-analytic terms, but some of them cancel each other. Then, the result is given by
\[
V_{\text{eff}} \big|_{n.a.} = (F^{(0)} + F^{(1)}_{L_0} + F^{(1)}_{L_1} + F^{(2)}_{L_0, L_1}) \big|_{n.a.} = -\left( -1 \right)^{\frac{D+1}{2}} \frac{2^{\frac{D+1}{2} - 1} \pi^{\frac{D+1}{2} - 2}}{2^{\frac{D+1}{2}} \pi^{\frac{D+1}{2}}} M^{D-2} \left( \frac{1}{L_0 L_1} \right) \quad \text{for} \ D = \text{odd}. \quad (2.28)
\]
We conclude that the effective potential for the scalar field on \( S^1_r \times R^{D-2} \times S^1_l \) does not have non-analytic terms for \( D = \text{even} \), while for \( D = \text{odd} \) it has the non-analytic term given by Eq. (2.28).

### 2.2 General form for relevant terms in mode summation

In this subsection, we investigate \( F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} \) \( (n = 1, 2, \ldots, p+1) \) and rewrite it in a tractable form to obtain the non-analytic terms. By use of the formula (2.11), Eq. (2.10) with \( f = 0, N = 1 \) and \( \eta_i = 0 \) can be written as
\[
F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} = -\frac{2^n}{(2\pi)^{\frac{D}{2}}} \left( \frac{M^2}{2} \right)^{\frac{D}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{D}{2}) \left( \frac{M}{2} \right)^{-2t} \times \Gamma(t) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \left\{ (m_0 L_0)^2 + (m_1 L_1)^2 + \cdots (m_{n-1} L_{n-1})^2 \right\}^{-t} = -\frac{2^n}{(2\pi)^{\frac{D}{2}}} \left( \frac{M^2}{2} \right)^{\frac{D}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{D}{2}) \left( \frac{M}{2} \right)^{-2t} S^{(n)}(t; L_0, L_1, \ldots, L_{n-1}), \quad (2.29)
\]
where we have defined
\[ S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) = \Gamma(t) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \left\{ (m_0 L_0)^2 + (m_1 L_1)^2 + \cdots (m_{n-1} L_{n-1})^2 \right\}^{-t}. \quad (2.30) \]

Inserting the formula (A.8) into Eq. (2.29), we have
\begin{align*}
F_{L_0, L_1, \cdots, L_{n-1}}^{(n)} & = - \frac{2^n}{(2\pi)^n} \left( \frac{M^2}{2} \right)^{\frac{\nu}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{\nu}{2}) \left( \frac{M}{2} \right)^{-2t} \\
& \times \left\{ -\frac{1}{2} S^{(n-1)}(t; L_1, L_2, \cdots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0} S^{(n-1)}(t - \frac{1}{2}; L_1, L_2, \cdots, L_{n-1}) \\
& + \frac{1}{\sqrt{\pi} 2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \left( \frac{\pi}{L_0} \right)^{-2t_1+2t} S^{(n-1)}(t_1; L_1, L_2, \cdots, L_{n-1}) \right\}. \quad (2.31)
\end{align*}

Since we are interested in the non-analytic terms of the effective potential, Eq. (2.31) can be expressed as
\begin{align*}
F_{L_0, L_1, \cdots, L_{n-1}}^{(n)} \bigg|_{\text{n.a.}} & = - \frac{2^n}{(2\pi)^n} \left( \frac{M^2}{2} \right)^{\frac{\nu}{2}} \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{\nu}{2}) \left( \frac{M}{2} \right)^{-2t} \\
& \times \left\{ -\frac{1}{2} S^{(n-1)}(t; L_1, L_2, \cdots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0} S^{(n-1)}(t - \frac{1}{2}; L_1, L_2, \cdots, L_{n-1}) \right\}, \quad (2.32)
\end{align*}

where we have dropped the third term in Eq. (2.31) because it is irrelevant to obtain the non-analytic terms in \( F_{L_0, L_1, \cdots, L_{n-1}}^{(n)} \) due to the fact that only the poles of \( \Gamma(t - \frac{\nu}{2}) \) contribute to the residue integral with respect to \( t \) in the third term of Eq. (2.31), as shown in the subsection A.2 of Appendix A, and those of \( \Gamma(t - \frac{\nu}{2}) \) do not produce any non-analytic terms, as stressed before. This is a crucial observation in our analysis.

It follows from Eq. (2.29) and Eq. (2.32) that as long as our considerations are restricted to the non-analytic terms, we can use the recurrence relation
\begin{align*}
S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) & = -\frac{1}{2} S^{(n-1)}(t; L_1, L_2, \cdots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0} S^{(n-1)}(t - \frac{1}{2}; L_1, L_2, \cdots, L_{n-1}) \quad (2.33)
\end{align*}
in Eq. (2.32). The recurrence relation (2.33) is easily solved as
\begin{align*}
S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) & = \frac{(-1)^{n-1}}{2^{n-1}} \sum_{\ell=0}^{n-1} (-1)^\ell \frac{\ell}{2n} \sum_{0 \leq i_1 < i_2 < \cdots < i_{\ell} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_\ell})^{-1} S^{(1)}(t - \frac{\ell}{2}; L_{n-1}). \quad (2.34)
\end{align*}
in terms of $S^{(1)}(t - \frac{\ell}{2}; L_{n-1})$. By direct calculations, $S^{(1)}(t; L)$ is found to be

$$ S^{(1)}(t; L) = \Gamma(t) \sum_{m=1}^{\infty} \left\{ (mL)^2 \right\}^{-t} = \frac{\Gamma(t) \zeta(2t)}{L^{2t}}. $$

Thus, we have found that

$$ F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} \bigg|_{n.a.} = \frac{(-1)^n}{(2\pi)^{\frac{D}{2}}} \left( \frac{M^2}{2} \right)^{\frac{D}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \Gamma(t - \frac{D}{2}) \left( \frac{ML_{n-1}}{2} \right)^{-2t} \times \sum_{\ell=0}^{n-1} (-1)^\ell \frac{\pi^\ell}{2} \sum_{0 \leq i_1 < i_2 < \cdots < i_\ell \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_\ell})^{-1} (L_{n-1})^\ell \Gamma(t - \frac{\ell}{2}) \zeta(2t - \ell) \bigg| \bigg|_{n.a.}, $$

which will be used in the next section.

3 Non-analytic terms in $F^{(n)}_{L_0, L_1, \ldots, L_{n-1}}$

We are ready to calculate the non-analytic terms by using Eq. (2.36) for each case of even/odd $D$ and $n$. We note again that the mass dependence of $(M^2)^{\frac{D}{2}}$ in Eq. (2.36) tells us that $t = \frac{\ell}{2}$ are half-odd-integer (integer) poles of $\Gamma(t - \frac{\ell}{2})$ and $\zeta(2t - \ell)$ yield the non-analytic terms for $D = \text{even}(\text{odd})$.

3.1 $(D, n) = (\text{even}, \text{even})$

Since $D = \text{even}$ in the present case, one needs the poles which are half-odd-integers in the integrand of Eq. (2.36) in order to have the non-analytic terms, that is, the poles of $\Gamma(t - \frac{\ell}{2})$ for $\ell = \text{odd}$ and those of the $\zeta(2t - \ell)$ for $\ell = \text{even}$. Hence, it is convenient to separate the summation over $\ell$ into $\ell = \text{odd}$ and $\ell = \text{even}$, as follows:

$$ F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} \bigg|_{n.a.} = \left( \sum_{j=0}^{\frac{D}{2}} (-1)^{2j} \frac{\pi^{2j}}{2} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1} (L_{n-1})^{2j} \Gamma(t - \frac{j}{2}) \zeta(2t - (2j - 1)) \right) \left( \sum_{j=0}^{\frac{D}{2}-1} (-1)^{2j} \frac{\pi^{2j}}{2} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1} (L_{n-1})^{2j} \Gamma(t - j) \zeta(2t - 2j) \right) \bigg| \bigg|_{n.a.}.$$

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The residue integral for the first term in Eq. (3.1) is performed by the pole \( t = \frac{2j-1}{2} \) of the gamma function, which yields

\[
\sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{D/2} (-1)^{j-1} M^{D-(2j-1)}}{2 \frac{D-2j+1}{2} \pi D-2j} (D - (2j - 1))!! 0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-2.
\] (3.2)

The pole \( t = \frac{2j+1}{2} \) of the zeta function in the second term of Eq. (3.1) gives us

\[
\sum_{j=0}^{\frac{n}{2}-1} \frac{(-1)^{D/2} (-1)^{j} M^{D-(2j+1)}}{2 \frac{D-2j}{2} \pi D-2j} (D - (2j + 1))!! 0 \leq i_1 < i_2 < \cdots < i_{2j} \leq n-2
\] = \[
\sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{D/2} (-1)^{j-1} M^{D-(2j-1)}}{2 \frac{D-2j+1}{2} \pi D-2j} (D - (2j - 1))!! 0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-2.
\] (3.3)

Combining the two results (3.2) and (3.3), we obtain

\[
F_{L_0 L_1 \cdots L_{n-1}}^{(n)} \bigg|_{\text{n.a.}} = \sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{D/2} (-1)^{j-1} M^{D-(2j-1)}}{2 \frac{D-2j+1}{2} \pi D-2j} (D - (2j - 1))!! 0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-1.
\] (3.4)

where we have used the relation

\[
\sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}})^{-1} + \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-2} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-2}} L_{n-1})^{-1}
\] = \[
\sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-1} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}})^{-1}.
\] (3.5)

### 3.2 \((D, n) = (\text{even, odd})\)

Likewise the previous case, one needs the poles of \( \Gamma(t - \frac{\ell}{2}) \) for \( \ell = \text{odd} \) and those of \( \zeta(2t - \ell) \) for \( \ell = \text{even} \) in Eq. (2.36) in order to have the non-analytic terms for \( D = \text{even} \). Then, we write Eq. (2.36) as

\[
F_{L_0 L_1 \cdots L_{n-1}}^{(n)} \bigg|_{\text{n.a.}} = -\frac{1}{(2\pi)^{\frac{D}{2}}} \frac{M^2}{2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \left( t - \frac{D}{2} \right)^{-2t} \Gamma(t - \frac{D}{2}) \left( \frac{ML_n}{2} \right)^{-2t} \times \sum_{j=1}^{\frac{n}{2}} \frac{(-1)^{2j-1}}{2 \pi^{2j-1}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}})^{-1} (L_{n-1})^{2j-1-1} \Gamma(t - \frac{2j-1}{2}) \zeta(2t - (2j - 1))
\] + \[
\sum_{j=1}^{\frac{n+1}{2}} \frac{(-1)^{2j-2}}{2 \pi^{2j-2}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-2} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-2}})^{-1} (L_{n-1})^{2j-2} \Gamma(t - \frac{2j-2}{2}) \zeta(2t - (2j - 2)) \bigg|_{\text{n.a.}}.
\] (3.6)
The residue integral for the pole \( t = \frac{2j-1}{2} \) of the gamma function in the first term of Eq. (3.6) yields

\[
- \sum_{j=1}^{n-1} \frac{(-1)^D}{2^{D-2j+1}} \frac{(-1)^{j-1}M^{D-(2j-1)}}{\pi^{D-2}} (D - (2j - 1))!! \sum_{0 \leq i_1 < \cdots < i_{2j-1} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}})^{-1}, \tag{3.7}
\]

and the residue integral for the pole \( t = \frac{2j-1}{2} \) of the zeta function in the second term in Eq. (3.6) gives

\[
- \sum_{j=1}^{n+1} \frac{(-1)^D}{2^{D-2(j+1)+1}} \frac{(-1)^{j-1}M^{D-(2j-1)}}{\pi^{D-2}} (D - (2j - 1))!! \sum_{0 \leq i_1 < \cdots < i_{2j-2} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-2}} L_{n-1})^{-1}. \tag{3.8}
\]

We combine the two results (3.7) and (3.8) to yield

\[
F_{L_0, L_1, \ldots, L_{n-1}}^{(n)} \big|_{\text{n.a.}} = - \sum_{j=1}^{n+1} \frac{(-1)^D}{2^{D-2(j+1)+1}} \frac{(-1)^{j-1}M^{D-(2j-1)}}{\pi^{D-2}} (D - (2j - 1))!! \sum_{0 \leq i_1 < \cdots < i_{2j-2} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-2}} L_{n-1})^{-1}. \tag{3.9}
\]

### 3.3 \( (D, n) = (\text{odd}, \text{even}) \)

The space-time dimension \( D \) is odd in this case, so that one needs the poles which are integers in the integrand of Eq. (2.36) in order to have the non-analytic terms, that is, the poles of \( \Gamma(\ell - \frac{D}{2}) \) for \( \ell = \text{even} \) and those of \( \zeta(2t - \ell) \) for \( \ell = \text{odd} \). From Eq. (2.36), we have the same expression as Eq. (3.1). The residue integral for the pole \( t = j \) of the zeta function in the first term of Eq. (3.1) yields

\[
\sum_{j=1}^{T} \frac{(-1)^{D-1}}{2^{D-(2j+1)+1}} \frac{(-1)^{j-1}M^{D-2j}}{\pi^{D-2}} (D - 2j)!! \sum_{0 \leq i_1 < \cdots < i_{2j-1} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}} L_{n-1})^{-1}. \tag{3.10}
\]

For the second term in Eq. (3.1), the pole \( t = j \) of the gamma function gives us

\[
\sum_{j=0}^{4} \frac{(-1)^{D-1}}{2^{D-(2j+1)+1}} \frac{(-1)^{j-1}M^{D-2j}}{\pi^{D-2}} (D - 2j)!! \sum_{0 \leq i_1 < \cdots < i_{2j} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1}. \tag{3.11}
\]

We put the two results (3.10) and (3.11) together to yield

\[
F_{L_0, L_1, \ldots, L_{n-1}}^{(n)} \big|_{\text{n.a.}} = \sum_{j=0}^{T} \frac{(-1)^{D-1}}{2^{D-(2j+1)+1}} \frac{(-1)^{j-1}M^{D-2j}}{\pi^{D-2}} (D - 2j)!! \sum_{0 \leq i_1 < \cdots < i_{2j} \leq n-1} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1}. \tag{3.12}
\]

Let us note that the \( j = 0 \) contribution does not have the dependence on the scale of the \( S^1 \), but does the mass scale \( M \) alone.
3.4 \((D, n) = \text{(odd, odd)}\)

Similar to the previous case, one needs the poles of \(\Gamma(t - \frac{j}{2})\) for \(\ell = \text{even}\) and those of \(\zeta(2t - \ell)\) for \(\ell = \text{odd}\) in Eq. (2.36) in order to have the non-analytic terms for \(D = \text{odd}\). Then, we write Eq. (2.36) as

\[
F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} \bigg|_{\text{n.a.}} = - \frac{1}{(2\pi)^{\frac{D}{2}}} \left(\frac{M^2}{2}\right)^{\frac{D}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \left(\frac{ML_{n-1}}{2}\right)^{-2t} \times \left(\sum_{j=1}^{n-1} (-1)^{D-2j} \pi^{\frac{D-2j}{2}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1} (L_{n-1})^{2j-1} \Gamma(t - \frac{2j-1}{2}) \zeta(2t - (2j - 1)) + \sum_{j=0}^{n-1} (-1)^{D-2j} \pi^{j} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1} (L_{n-1})^{2j} \Gamma(t - j) \zeta(2t - 2j) \right) \bigg|_{\text{n.a.}}. \tag{3.13}
\]

The pole \(t = j\) of the zeta function in the first term of Eq. (3.13) gives

\[
- \sum_{j=1}^{n-1} \left(\frac{-1}{2}\right)^{D-2j} \frac{(-1)^{D-2j} \pi^{\frac{D-2j}{2}}}{(D - 2j)!!} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j-1}} L_{n-1})^{-1}. \tag{3.14}
\]

The residue integral for the pole \(t = j\) of the gamma function in the second term of Eq. (3.13) becomes

\[
- \sum_{j=0}^{n-1} \left(\frac{-1}{2}\right)^{D-2j} \frac{(-1)^{D-2j} \pi^{\frac{D-2j}{2}}}{(D - 2j)!!} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j} \leq n-2} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1}. \tag{3.15}
\]

We combine the two results (3.14) and (3.15) to yield

\[
F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} \bigg|_{\text{n.a.}} = - \sum_{j=0}^{n-1} \left(\frac{-1}{2}\right)^{D-2j} \frac{(-1)^{D-2j} \pi^{\frac{D-2j}{2}}}{(D - 2j)!!} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j} \leq n-1} (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1}. \tag{3.16}
\]

Let us note again that the \(j = 0\) contribution does not possess the scale dependence on the \(S^1\), but does the mass scale \(M\).

We have succeeded in obtaining the non-analytic terms in \(F^{(n)}_{L_0, L_1, \ldots, L_{n-1}}\) as Eqs. (3.14), (3.15), (3.12) and (3.16), depending on even/odd \(D\) and \(n\). In order to calculate the non-analytic terms in Eq. (2.7), it is useful to generalize the expressions of \(F^{(n)}_{L_0, L_1, \ldots, L_{n-1}} \bigg|_{\text{n.a.}}\) to \(F^{(n)}_{L_{i_1}, \ldots, L_{i_n}} \bigg|_{\text{n.a.}}\).
Then, we have

\[ (D, n) = \text{(even, even)} : \quad F_{L_1, \ldots, L_n}^{(n)} |_{\text{n.a.}} = \sum_{j=1}^{n} \left( \frac{-1}{2} \right)^{D_j} j^{-1} M^{D-(2j-1)} \frac{2^{D-2j} j!}{\pi^{D-2j} (D - (2j - 1))!!} \times \left\{ \sum_{\{\ell_1, \ell_2, \ldots, \ell_{2j-1}\} \in \{i_1, i_2, \ldots, i_n\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1} \right\}, \quad (3.17) \]

\[ (D, n) = \text{(even, odd)} : \quad F_{L_1, \ldots, L_n}^{(n)} |_{\text{n.a.}} = \sum_{j=1}^{n+1} \left( \frac{-1}{2} \right)^{D_j} j^{-1} M^{D-(2j-1)} \frac{2^{D-2j} j!}{\pi^{D-2j} (D - (2j - 1))!!} \times \left\{ \sum_{\{\ell_1, \ell_2, \ldots, \ell_{2j-1}\} \in \{i_1, i_2, \ldots, i_n\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1} \right\}, \quad (3.18) \]

\[ (D, n) = \text{(odd, even)} : \quad F_{L_1, \ldots, L_n}^{(n)} |_{\text{n.a.}} = \sum_{j=0}^{n} \left( \frac{-1}{2} \right)^{D_j} j^{-1} M^{D-2j} \frac{2^{D-2j} (D - 2j)!}{\pi^{D-2j}} \times \left\{ \sum_{\{\ell_1, \ell_2, \ell_2\} \in \{i_1, i_2, \ldots, i_n\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right\}, \quad (3.19) \]

\[ (D, n) = \text{(odd, odd)} : \quad F_{L_1, \ldots, L_n}^{(n)} |_{\text{n.a.}} = \sum_{j=0}^{n+1} \left( \frac{-1}{2} \right)^{D_j} j^{-1} M^{D-2j} \frac{2^{D-2j} (D - 2j)!}{\pi^{D-2j}} \times \left\{ \sum_{\{\ell_1, \ell_2, \ell_2\} \in \{i_1, i_2, \ldots, i_n\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right\}, \quad (3.20) \]

where the order of the elements in the set should be understood to be \( \ell_1 < \ell_2 < \cdots < \ell_{2j-1} \) and \( i_1 < i_2 < \cdots < i_{n-1} \), etc. Eqs. (3.17) ~ (3.20) will be used in the next section.

### 4 Non-analytic terms in effective potential

Equipped with Eqs. (3.17) ~ (3.20), let us calculate the non-analytic terms in the effective potential (2.6). We omit the abbreviation “n.a.” used in the sections 2 and 3, which stands for the non-analytic terms. One should keep in mind that we are treating the non-analytic terms in this section. We introduce

\[ A_j^D \equiv \left( \frac{-1}{2} \right)^{D_j} j^{-1} M^{D-(2j-1)} \frac{2^{D-2j} j!}{\pi^{D-2j} (D - (2j - 1))!!}, \quad (4.1) \]

\[ B_j^D \equiv \left( \frac{-1}{2} \right)^{D_j} j^{-1} M^{D-2j} \frac{2^{D-2j} (D - 2j)!}{\pi^{D-2j}} \quad (4.2) \]

in Eqs. (3.17) ~ (3.20) in order to write them in compact forms.
4.1 \((D, p + 1) = (\text{even, even})\)

Let us write the \(F^{(n)}\) \((n \geq 1)\) part of the effective potential in Eq. (2.6) for \((D, p + 1) = \text{even, even}\) as

\[
\sum_{n=1}^{p+1} F^{(n)} = \sum_{k=1}^{\frac{p+1}{2}} \left\{ F^{(2k-1)} + F^{(2k)} \right\}
\]

\[
= \sum_{k=1}^{\frac{p+1}{2}} \left\{ \sum_{0\leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} F_{L_{i_1},L_{i_2},\ldots,L_{i_{2k-1}}}^{(2k-1)} + \sum_{0\leq i_1 < i_2 < \cdots < i_{2k} \leq p} F_{L_{i_1},L_{i_2},\ldots,L_{i_{2k}}}^{(2k)} \right\}
\]

\[
= \sum_{k=1}^{\frac{p+1}{2}} \left\{ \sum_{0\leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{j=1}^{k} (-1)^{j} A_{j}^{D} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{j-1}\} \in \{i_1,i_2,\ldots,i_{2k-1}\}} \left( L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{j-1}1-1} \right) \right\} + \sum_{0\leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{j=1}^{k} A_{j}^{D} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{j-1}\} \in \{i_1,i_2,\ldots,i_{2k}\}} \left( L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{j-1}} \right) , \tag{4.3}
\]

where we have used Eqs. (2.7), (3.17) and (3.18). By changing the order of the summations with respect to \(j\) and \(k\), we obtain

\[
\sum_{n=1}^{p+1} F^{(n)} = \sum_{j=1}^{\frac{p+1}{2}} \sum_{k=1}^{\frac{p+1}{2}} \left\{ - \sum_{0\leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{j-1}\} \in \{i_1,i_2,\ldots,i_{2k-1}\}} + \sum_{0\leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{\{\ell_1,\ell_2,\ldots,\ell_{j-1}\} \in \{i_1,i_2,\ldots,i_{2k}\}} \right\}
\times A_{j}^{D} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{j-1}})^{-1} , \tag{4.4}
\]

where we have used the formula

\[
\sum_{k=1}^{\frac{p+1}{2}} \sum_{j=1}^{\frac{p+1}{2}} \sum_{k=1}^{\frac{p+1}{2}} = \sum_{j=1}^{\frac{p+1}{2}} \sum_{k=1}^{\frac{p+1}{2}} \sum_{j=1}^{\frac{p+1}{2}} . \tag{4.5}
\]

There are \(p+1\) \(C_{2j-1}\) numbers of the independent configurations for \((L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{j-1}})^{-1}\) for a given value of \(j\), each of which has the same multiplicity in Eq. (4.4) for a fixed value of \(k\). Hence, it is enough to calculate the multiplicity for the configuration with \(\ell_1 = 0, \ell_2 = 1, \ldots, \ell_{2j-1} = 2j - 2\), that is, \((L_0 L_{1} \cdots L_{2j-2})^{-1}\) for each value of \(k = j, j + 1, \ldots, \frac{p+1}{2}\).

Let us first compute the multiplicity for the case of \(k = j\) with fixed \(j\). Aside from the factor \(A_{j}^{D}\), we have

\[
\left\{ - \sum_{0\leq i_1 < i_2 < \cdots < i_{2j-1} \leq p \in \{i_1,i_2,\ldots,i_{2j-1}\}} + \sum_{0\leq i_1 < i_2 < \cdots < i_{2j} \leq p \in \{i_1,i_2,\ldots,i_{2j}\}} \right\} (L_0 L_{1} \cdots L_{2j-2})^{-1}
\]

\[
= \left\{ 1 - \frac{p+1-(2j-1)}{2} \right\} (L_0 L_{1} \cdots L_{2j-2})^{-1}. \tag{4.6}
\]
Let us next calculate the multiplicity for the case of \( k = j + 1 \). The result is given by

\[
\left\{ - \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j+1} \leq p} \sum_{\ell_1, \ell_2, \cdots, \ell_{2j+1}} (-1)^{A_j} \sum_{\{\ell_{\ell_1}, \ell_{\ell_2}, \cdots, \ell_{\ell_{2j-1}}\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1} \right\}
\]

\[
\times (L_0 L_1 \cdots L_{2j-2})^{-1}
\]

\[
= \left\{ - p+1-(2j-1)C_2 + p+1-(2j-1)C_3 \right\} (L_0 L_1 \cdots L_{2j-2})^{-1}. \tag{4.7}
\]

In the same manner, one can calculate the multiplicity for \( k = j, j + 1, \cdots, \frac{p+1}{2} \). Collecting the terms with \( k = j, j + 1, \cdots, \frac{p+1}{2} \) and denoting \( A \equiv p + 1 - (2j - 1) \), which is odd for the present case, we have

\[
\left\{ -(1 + A)C_2 + AC_2 + \cdots + AC_{A-1} \right\} (L_0 L_1 \cdots L_{2j-2})^{-1}
\]

\[
= \left\{ - \frac{1}{2} \times 2^A + \frac{1}{2} \times 2^A \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} = 0. \tag{4.8}
\]

This holds for any \( j \) satisfying \( 1 \leq j \leq \frac{p+1}{2} \) and the same conclusion \( \text{(4.8)} \) holds for the other configurations of \( L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}} \). Furthermore, Eq. \((2.9)\) is analytic for \( D = \text{even} \). Thus, we conclude that \( V_{\text{eff}} \mid_{n.a.} = 0 \), that is, there is no non-analytic term in the effective potential for \( D = \text{even} \) and \( p + 1 = \text{even} \).

4.2 \((D, p + 1) = (\text{even, odd})\)

We write the \( F^{(n)} \) \((n \geq 1)\) part of the effective potential for this case as

\[
\sum_{n=1}^{p+1} F^{(n)} = \sum_{k=1}^{p} \left\{ F^{(2k-1)} + F^{(2k)} \right\} + F^{(p+1)}
\]

\[
= \sum_{k=1}^{p} \left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{j=1}^{k} (-1)^{A_j} \sum_{\{\ell_{\ell_1}, \ell_{\ell_2}, \cdots, \ell_{\ell_{2j-1}}\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1}
\]

\[
+ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{j=1}^{k} A_j \sum_{\{\ell_{\ell_1}, \ell_{\ell_2}, \cdots, \ell_{\ell_{2j-1}}\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1} \right\}
\]

\[
+ \sum_{j=1}^{\frac{p+1}{2}} (-1)^{A_j} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq p} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1} \right\}. \tag{4.9}
\]
where we have again used Eqs. (2.7), (3.17) and (3.18). The last term in Eq. (4.9) with \( j = \frac{p+2}{2} \) yields \(-A_{D/2}^p(L_0 L_1 \cdots L_p)^{-1}\). Then, we have

\[
\sum_{n=1}^{p+1} F^{(n)} = \sum_{j=1}^{\frac{p}{2}} \left\{ \sum_{k=j}^{\frac{p}{2}} \left[ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{\{\ell_1,\ell_2,\cdots,\ell_{2j-1}\}} + \sum_{0 \leq j_1 < j_2 < \cdots < j_{2k} \leq p} \sum_{\{\ell_1,\ell_2,\cdots,\ell_{2j-1}\}} \right] \right. \\
\left. - \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq p} A_j^p(L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j-1}})^{-1} - A_{D/2}^p(L_0 L_1 \cdots L_p)^{-1}, \right. \tag{4.10}
\]

where we have exchanged the order of the summations with respect to \( j \) and \( k \).

Let us count the multiplicity for the configuration with \( \ell_1 = 0, \ell_2 = 1, \cdots, \ell_{2j-1} = 2j - 2 \). Note that the third term in Eq. (4.10) results one for the configuration. We calculate the multiplicity of the first and the second terms in Eq. (4.10) for \( k = j \). Apart from the factor \( A_j^p \), it is calculated as

\[
\left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq p} \sum_{\{0,1,\cdots,2j-2\}} + \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j-1} \leq p} \sum_{\{0,1,\cdots,2j-2\}} \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} \\
= \left\{ -1 + p+1-(2j-1)C_1 \right\} (L_0 L_1 \cdots L_{2j-2})^{-1}. \tag{4.11}
\]

Likewise, for \( k = j + 1 \), we have

\[
\left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j+1} \leq p} \sum_{\{0,1,\cdots,2j-2\}} + \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j+2} \leq p} \sum_{\{0,1,\cdots,2j-2\}} \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} \\
= \left\{ - p+1-(2j-1)C_2 + p+1-(2j-1)C_3 \right\} (L_0 L_1 \cdots L_{2j-2})^{-1}. \tag{4.12}
\]

Similarly, we can compute the multiplicity for \( k = j, j+1, \cdots, \frac{p}{2} \). Collecting all the terms with \( k = j, j+1, \cdots, \frac{p}{2} \) and including the third term in Eq. (4.10), we have

\[
\left\{ -(1 + A C_2 + A C_4 + \cdots + A C_{A-2}) + (A C_1 + A C_3 + \cdots + A C_{A-1}) - 1 \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} \\
= \left\{ - \left( \frac{1}{2} \times 2^A - A C_A \right) + \frac{1}{2} \times 2^A - 1 \right\} (L_0 L_1 \cdots L_{2j-2})^{-1} = 0, \tag{4.13}
\]

where \( A \equiv p + 1 - (2j - 1) \), which is even in this case. This holds for any value of \( j \) between 1 and \( \frac{p}{2} \). Hence, the last term in Eq. (4.10) alone is left to yield the non-analytic term in the effective potential for \( D = \text{even} \) and \( p + 1 = \text{odd} \), i.e.

\[
V_{\text{eff}} \bigg|_{\text{n.a.}} = -A_{\frac{p+2}{2}}^p(L_0 L_1 \cdots L_p)^{-1} \\
= \frac{(-1)^{\frac{D+p+2}{2}}}{2^{D-2} \pi^{D-\frac{p+2}{2}} (D-(p+1))!!} \frac{M^{D-(p+1)}}{L_0 L_1 \cdots L_p}. \tag{4.14}
\]
4.3 $ (D, p + 1) = (\text{odd, even})$

The $F^{(n)} (n \geq 1)$ part of the effective potential for this case is given by

$$
\sum_{n=1}^{p+1} F^{(n)} = \sum_{k=1}^{\frac{p+1}{2}} \left\{ F^{(2k-1)} + F^{(2k)} \right\} = \sum_{k=1}^{\frac{p+1}{2}} \left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{j=0}^{k-1} B_j^D \sum_{\{\ell_1, \ell_2, \cdots, \ell_{2j}\} \in \{i_1, i_2, \cdots, i_{2k-1}\}} \frac{1}{(L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}} + \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{j=0}^{k} (-1)^j B_j^D \sum_{\{\ell_1, \ell_2, \cdots, \ell_{2j}\} \in \{i_1, i_2, \cdots, i_{2k}\}} \frac{1}{(L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}} \right\}, \tag{4.15}
$$

where we have used Eqs. (2.7), (3.19) and (3.20). Separating the $j = k$ contribution from the second term in Eq. (4.15) and exchanging the summations with respect to $j$ and $k$, we obtain

$$
\sum_{n=1}^{p+1} F^{(n)} = \sum_{j=0}^{\frac{p+1}{2}} \sum_{k=j+1}^{\frac{p+1}{2}} \left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{\{\ell_1, \ell_2, \cdots, \ell_{2j}\} \in \{i_1, i_2, \cdots, i_{2k-1}\}} - \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{\{\ell_1, \ell_2, \cdots, \ell_{2j}\} \in \{i_1, i_2, \cdots, i_{2k}\}} \right\}
\times B_j^D \frac{1}{(L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}} - \sum_{k=1}^{\frac{p+1}{2}} \sum_{0 \leq \ell_1 < \ell_2 < \cdots < \ell_{2k} \leq p} B_k^D \frac{1}{(L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2k}})^{-1}} - B_{\frac{p+1}{2}} D \cdot (L_0 L_1 \cdots L_p)^{-1}. \tag{4.16}
$$

Here, the last term in Eq. (4.16) has been separated from the third term and corresponds to the $k = \frac{p+1}{2}$ contribution of it.

Let us first study the $j = 0$ contribution in the first and the second term in Eq. (4.16), which is given by

$$
\sum_{k=1}^{\frac{p+1}{2}} \left\{ \frac{1}{2} C_{2k-1} - \frac{1}{2} C_{2k} \right\} B_j^D = -\frac{(-1)^{\frac{p+1}{2}}}{2} \sum_{k=1}^{\frac{p+1}{2}} \frac{1}{(2k+1)!} M^D. \tag{4.17}
$$

This exactly cancels the contribution coming from the one-loop correction for the zero modes (2.9) for $D = \text{odd}$. Then, rewriting $k$ as $j$ in the third term in Eq. (4.16), we have

$$
F^{(0)} + \sum_{n=1}^{p+1} F^{(n)} = \sum_{j=1}^{\frac{p+1}{2}} \left\{ \sum_{k=j+1}^{\frac{p+1}{2}} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{\{\ell_1, \ell_2, \cdots, \ell_{2j}\} \in \{i_1, i_2, \cdots, i_{2k-1}\}} - \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{\{\ell_1, \ell_2, \cdots, \ell_{2j}\} \in \{i_1, i_2, \cdots, i_{2k}\}} \right\}
B_j^D \frac{1}{(L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}} - B_{\frac{p+1}{2}} D \cdot (L_0 L_1 \cdots L_p)^{-1}. \tag{4.18}
$$
Let us count the multiplicity for the configuration $\ell_1 = 0, \ell_2 = 1, \ldots, \ell_{2j} = 2j - 1$ for fixed $j$. To this end, we extract all the terms proportional to $L_0 L_1 \cdots L_{2j-1}$ for $k = j+1, j+2, \ldots, \frac{p+1}{2}$ from Eq. (4.18). Apart from the factor $B_j^D$, the result is

$$\left\{ \left( B C_1 + B C_3 + \cdots + B C_{B-1} \right) - \left( B C_2 + B C_4 + \cdots + B C_B \right) - 1 \right\} (L_0 L_1 \cdots L_{2j-1})^{-1}$$

$$= \left\{ \left( \frac{1}{2} \times 2^B \right) - \left( \frac{1}{2} \times 2^B - B C_0 \right) - 1 \right\} (L_0 L_1 \cdots L_{2j-1})^{-1} = 0,$$

(4.19)

where $B = p + 1 - 2j$, which is even in this case. Since Eq. (4.19) holds by any values of $j = 1, 2, \ldots, \frac{p-1}{2}$, only the last term in Eq. (4.18) is left to yield the non-analytic term in the effective potential,

$$V_{\text{eff}} \Big|_{\text{n.a.}} = -B_j^D \frac{(L_0 L_1 \cdots L_p)^{-1}}{2^{D-p} \pi^{D-p/2} (D - (p+1))!!} \frac{M^{D-(p+1)}}{L_0 L_1 \cdots L_p}. \quad (4.20)$$

4.4 \quad $(D, p+1) = (\text{odd, odd})$

Let us proceed the fourth case. The $F^{(n)} \ (n \geq 1)$ part of the effective potential is given by

$$\sum_{n=1}^{p+1} F^{(n)} = \sum_{k=1}^{\frac{p}{2}} \left\{ F^{(2k-1)} + F^{(2k)} \right\} + F^{(p+1)}$$

$$= \sum_{k=1}^{\frac{p}{2}} \left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{j=0}^{k-1} B_j^D \sum_{\{\ell_1, \ell_2, \ldots, \ell_{2j}\} \in \{i_1, i_2, \ldots, i_{2k-1}\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right. $$

$$\left. + \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{j=0}^{k} (-1) B_j^D \sum_{\{\ell_1, \ell_2, \ldots, \ell_{2j}\} \in \{i_1, i_2, \ldots, i_{2k}\}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right\}$$

$$+ \sum_{j=0}^{\frac{p}{2}} B_j^D \sum_{0 \leq \ell_1 < \ell_2 < \cdots < \ell_{2j} \leq p} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}, \quad (4.21)$$
where we have again used Eqs. (2.7), (3.19) and (3.20). We write the above equation as

$$\sum_{n=1}^{p+1} F^{(n)} = \sum_{k=1}^{p} \sum_{j=0}^{k-1} \left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} B_j^D \sum_{\ell_1, \ell_2, \ldots, \ell_{2j}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right. \\
+ \left. \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} (-1)^j B_j^D \sum_{\ell_1, \ell_2, \cdots, \ell_{2j}} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1} \right\} \\
+ \sum_{k=1}^{p} \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} (-1)^j B_j^D (L_{i_1} L_{i_2} \cdots L_{i_{2k}})^{-1} + \sum_{j=0}^{p} B_j^D \sum_{0 \leq i_1 < i_2 < \cdots < i_{2j} \leq p} (L_{\ell_1} L_{\ell_2} \cdots L_{\ell_{2j}})^{-1}. \tag{4.22}$$

Note that the third term in Eq. (4.22) cancels the last one in Eq. (4.22) except for the $j = 0$ term, which cancels the one-loop contribution for the zero modes (2.9) for $D = \text{odd}$.

Then, what is left is the first and the second terms in Eq. (4.22). By exchanging the summations with respect to $j$ and $k$, we obtain

$$F^{(0)} + \sum_{n=1}^{p+1} F^{(n)} = \sum_{j=0}^{p-2} \sum_{k=j+1}^{p} \left\{ \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k-1} \leq p} \sum_{\ell_1, \ell_2, \ldots, \ell_{2j}} - \sum_{0 \leq i_1 < i_2 < \cdots < i_{2k} \leq p} \sum_{\ell_1, \ell_2, \cdots, \ell_{2j}} \right\} \\
\times B_j^D (L_{i_1} L_{i_2} \cdots L_{i_{2j}})^{-1}. \tag{4.23}$$

Let us count the multiplicity for the configuration with $\ell_1 = 0, \ell_2 = 1, \ldots, \ell_{2j} = 2j - 1$ for fixed $j$ by extracting all the terms proportional to $L_0 L_1 \cdots L_{2j-1}$ for $k = j + 1, j + 1, \ldots, \frac{p}{2}$ from Eq. (4.23). Apart from the factor $B_j^D$, the result is given by

$$\left\{ (B C_1 + B C_3 + \cdots + B C_{B-2}) - (B C_2 + B C_4 + \cdots + B C_{B-1}) \right\} (L_0 L_1 \cdots L_{2j-1})^{-1} \\
= \left\{ \left( \frac{1}{2} \times 2^B - B C_B \right) - \left( \frac{1}{2} \times 2^B - B C_0 \right) \right\} (L_0 L_1 \cdots L_{2j-1})^{-1} = 0, \tag{4.24}$$

where $B = p + 1 - 2j$, which is odd in this case. This holds for any $j$ satisfying $0 \leq j \leq \frac{p-2}{2}$, so that we have $V_{\text{eff}}^{\text{D}} \big|_{\text{n.a.}} = 0$, that is, the effective potential does not possess non-analytic terms for $D = \text{odd}$ and $p + 1 = \text{odd}$.

We have calculated the non-analytic terms in the effective potential for any $D$ and $p$. We have found that there is no non-analytic term for $(D, p+1) = (\text{even}, \text{even})$ and $(\text{odd}, \text{odd})$. On the other hand, the non-analytic term appears for $(D, p+1) = (\text{even}, \text{odd})$ and $(\text{odd}, \text{even})$, as shown in Eqs. (4.14) and (4.20), respectively. The results are summarized as

$$V_{\text{eff}}^{\text{D}} \big|_{\text{n.a.}} = \begin{cases} 0 & \text{for } (D, p+1) = (\text{even}, \text{even}), (\text{odd}, \text{odd}) \, , \\
\frac{(-1)^p}{2} \frac{M^{D-(p+1)}}{L_0 L_1 \cdots L_p} & \text{for } (D, p+1) = (\text{even}, \text{odd}), (\text{odd}, \text{even}) \, . \end{cases} \tag{4.25}$$
The famous Dolan-Jackiw term corresponds to the case of $D = 4, p = 0$ in Eq. (4.25). We present some results followed from Eq. (4.25) in the table 1.

We observe that the non-analytic term appears in the effective potential when $D - (p + 1) = \text{odd}$, which corresponds to the odd uncompactified spatial dimensions. It must be noticed that there is only one non-analytic term for the given values of $D$ and $p$, for which the non-analytic term exists.

$$S_1^1 \times R^{D-(p+1)} \times \prod_{i=1}^{p} S_i^1$$

Table 1: Some results on non-analytic terms in the effective potential on the space-time $S_1^1 \times R^{D-(p+1)} \times \prod_{i=1}^{p} S_i^1$. We use $T(=L_0^{-1})$. The “non” in the table means that there is no non-analytic term in the effective potential and the “×” stands for the case, where the condition $D \geq p + 1$ is not satisfied.

| $D$  | $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ | $p = 4$ |
|------|---------|---------|---------|---------|---------|
| 3    | non     | $\frac{1}{2} MT \over L_1$ | non     | ×        | ×        |
| 4    | $\frac{MT}{12\pi}$ | non     | $\frac{1}{2} MT \over L_1L_2$ | non     | ×        |
| 5    | non     | $-\frac{1}{12\pi} M^3T \over L_1$ | non     | $\frac{1}{2} MT \over L_1L_2L_3$ | non |
| 6    | $\frac{M^3T}{120\pi^2}$ | non     | $-\frac{1}{12\pi} M^3T \over L_1L_2$ | non     | $\frac{1}{2} MT \over L_1L_2L_3L_4$ |
| 7    | non     | $\frac{1}{120\pi^2} M^3T \over L_1$ | non     | $-\frac{1}{12\pi} M^3T \over L_1L_2L_3L_4$ | non |

5 Conclusions and Discussions

We have studied the non-analytic terms in the effective potential for the real scalar field at finite temperature in one-loop approximation on the $D$-dimensional space-time, $S_1^1 \times R^{D-(p+1)} \times \prod_{i=1}^{p} S_i^1$. The effective potential is given in terms of the modified Bessel function of the second kind accompanied with the multiple mode summations. We have introduced the integral representation for the modified Bessel function of the second kind and have also made use of the analytical extension for the mode summations. The effective potential is recast into the integral form in the complex plane, and the non-analytic terms are obtained by the residue theorem.

We have clarified the pole structure of the analytical extension for the mode summations and have found the recurrence relation (2.33), from which we have obtained the general form (2.34) for the relevant terms in the mode summations. We have calculated the non-analytic terms, Eqs. (3.17) ~ (3.20) by the residue theorem for the poles of the gamma and zeta functions in Eq. (2.34) with Eq. (2.35), depending on the even/odd $D$ and $n (n = 1, 2, \cdots, p + 1)$. The positions of the poles that yield the non-analytic terms are found to be independent of $D$. Equipped with them, we have calculated the non-analytic terms in the effective potential, which is given by Eq. (4.25), including the famous Dolan-Jackiw term. Some explicit results are summarized in the table 1.
We have found that the effective potential has the non-analytic term when the dimension of the flat Euclidean space, \( D - (p + 1) \) is odd. There is only one non-analytic term for the given values of \( D \) and \( p \), for which the non-analytic term exists.

There are untouched issues in the paper. We have not discussed the physical origin of such the non-analytic term in the effective potential. The paper [11] showed that the famous Dolan-Jackiw term had been emerged through the zero mode of the \( S^1 \) direction, reflecting the infrared dynamics of the theory. It may be important to clarify the physical origin of the non-analytic term found in this paper. Moreover, it may be important to study the physical implication of the non-analytic terms on, for example, the phase transition at finite temperature.

For the case of the fermion, the boundary condition for the \( S^1 \) direction is anti-periodic due to the Fermi statistics, and we have the factor \((-1)^{m\omega}\) in the mode summation. Then, the zeta function \( \zeta(2t) \) in Eq. (2.15) is replaced by the eta function, \(-\eta(2t)\), which does not possess the pole, so that the non-analytic term does not arise for \( D = 4, p = 0 \) [11]. It is expected that the analytical extension for the mode summations is modified to yield different non-analytic terms from those of the scalar field. Moreover, there are degrees of freedom to choose the periodic or anti-periodic boundary condition for each spacial \( S^1_i \) \((i = 1, 2, \cdots, p)\) direction for the fermion (scalar) field. We also expect that the analytical extension for the mode summations is different from that of the case for the periodic boundary condition. Accordingly, we may have a different type of non-analytic terms. These will be reported elsewhere.

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## A Mode summations and analytical extension

The effective potential contains the modified Bessel function of the second kind accompanied with the mode summations. In addition to the integral representation (2.14) in the text for the modified Bessel function of the second kind, one needs an analytical extension for the mode summations in order to recast the effective potential into the integral form and to obtain the non-analytic terms by the residue integral. In this appendix, we present an analytical extension for the multiple mode summations and clarify its pole structure in the residue integral.
A.1 Double mode summations

The double summations (2.22) in the text has an analytical extension [16]. In our analysis, the following formula plays a crucial role:

\[
\Gamma(t) \sum_{m_0=1}^{\infty} \{(m_0 L_0)^2 + c^2\}^{-t} = -1 \frac{1}{2} \frac{\Gamma(t) - \frac{1}{2}}{2 L_0} + \frac{\sqrt{\pi} L_0}{c^{2(t-\frac{3}{2})}} + \frac{2 \pi t}{L_0} \frac{1}{c^{2(t-\frac{1}{2})}} \sum_{n_0=1}^{\infty} n_0^{-\frac{1}{2}} K_{t-\frac{1}{2}} \left( \frac{2 \pi n_0}{L_0} c \right) 
\]

where we have used Eq. (2.14) in the last equality. In obtaining Eq. (A.1), we have made use of the Poisson summation

\[
\sum_{m_0=-\infty}^{\infty} e^{-[(m_0+n_0) L_0]^2 t} = \sum_{n_0=-\infty}^{\infty} \frac{1}{L_0} \left( \frac{\pi}{t} \right)^{-\frac{1}{2}} e^{-\left(2 \pi n_0^2 t\right)^{-1} + 2 \pi i n_0 \eta_0}.
\]

and the formula

\[
K_{\nu}(z) = \frac{1}{2} \left( \frac{z}{2} \right)^{\nu} \int_{0}^{\infty} dt t^{-\nu} e^{-\frac{t}{2} - \frac{1}{2} z^2 t^2}. \quad (A.3)
\]

Setting \( c = m_1 L_1 \) and taking the summation \( \sum_{m_1=1}^{\infty} \), we arrive at

\[
\Gamma(t) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \{(m_0 L_0)^2 + (m_1 L_1)^2\}^{-t} = -1 \frac{1}{2} \frac{\Gamma(t) - \frac{1}{2}}{2 L_1} c^{2(t-\frac{3}{2})} + \frac{\sqrt{\pi} L_0}{c^{2(t-\frac{1}{2})}} + \frac{2 \pi t}{L_0} \frac{1}{c^{2(t-\frac{1}{2})}} \sum_{n_0=1}^{\infty} n_0^{-\frac{1}{2}} K_{t-\frac{1}{2}} \left( \frac{2 \pi n_0}{L_0} c \right) 
\]

This is Eq. (2.23) in the text.

As we will show below, an important observation is that the third term in Eq. (A.1) has the property

\[
\frac{1}{2 \pi i} \int_{c-i\infty}^{c+i\infty} dt f(t) \frac{1}{2 \pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \Gamma(t_1) \zeta(2t_1) \left( \frac{\pi L_1}{L_0} \right)^{-2t_1} = 0,
\]

\[
(A.5)
\]
where \( f(t) \) is any function that has no poles inside the region of the residue integral with respect to \( t \). Eq. (A.5) implies that the poles coming from the third term of Eq. (A.4) do not contribute to the residue integral with respect to \( t \) in Eq. (A.5). A similar property holds for the multiple mode summations, as we will see in the next subsection.

To show Eq. (A.5), we first note that the combination of \( \Gamma(t_1 - \frac{a}{2}) \zeta(2t_1 - a) \) has no poles except for the pole of \( \Gamma(t_1 - \frac{a}{2}) \) at \( t_1 = \frac{a}{2} \) and that of \( \zeta(2t_1 - a) \) at \( t_1 = \frac{a}{2} + \frac{1}{2} \). One might think that \( \Gamma(t_1 - \frac{a}{2}) \zeta(2t_1 - a) \) could have an infinite number of poles at \( t_1 = \frac{a}{2} - n \) \((n = 1, 2, 3, \cdots)\). This is not, however, the case because of the property \( \zeta(-2n) = 0 \) for \( n = 1, 2, 3, \cdots \). This is an important observation, which will be used throughout our analyses.

The \( t_1 \)-integration on the left-hand-side of Eq. (A.5) can be performed as the residue integral and the result is

\[
\frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma(t_1-t+\frac{1}{2}) \zeta(2t_1-2t+1) \Gamma(t_1) \zeta(2t_1) \left( \frac{L_1}{L_0} \right)^{-2t_1} \\
= \zeta(0) \Gamma(t-\frac{1}{2}) \zeta(2t-1) \left( \frac{L_1}{L_0} \right)^{-2(t-\frac{1}{2})} + \frac{1}{2} \Gamma(\frac{1}{2}) \zeta(2t) \left( \frac{L_1}{L_0} \right)^{-2t} \\
+ \zeta(0) \Gamma(\frac{1}{2}-t) \zeta(1-2t) + \frac{1}{2} \Gamma(\frac{1}{2}) \zeta(2-2t) \left( \frac{L_1}{L_0} \right)^{-1},
\]

(A.6)

which leads to Eq. (A.5), as can be confirmed by direct calculations.

### A.2 Multiple mode summations and pole structure

In this section, we generalize the previous analysis of the double mode summations to the multiple ones

\[
S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) \\
\equiv \Gamma(t) \sum_{m_0=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \left\{ (m_0 L_0)^2 + (m_1 L_1)^2 + \cdots + (m_{n-1} L_{n-1})^2 \right\}^{-1}.
\]

(A.7)

By use of the formula (A.1), \( S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) \) can be expressed into a recursive form as

\[
S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) \\
= -\frac{1}{2} S^{(n-1)}(t; L_1, L_2, \cdots, L_{n-1}) + \frac{\sqrt{\pi}}{2L_0} S^{(n-1)}(t-\frac{1}{2}; L_1, L_2, \cdots, L_{n-1}) \\
+ \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 \Gamma(t_1-t+\frac{1}{2}) \zeta(2t_1-2t+1) \left( \frac{L_1}{L_0} \right)^{-2t_1+2t} S^{(n-1)}(t_1; L_1, L_2, \cdots, L_{n-1}),
\]

(A.8)
where we have defined
\[
S^{(n-k)}(t; L_k, L_{k+1}, \ldots, L_{n-1}) = \Gamma(t) \sum_{m_k=1}^{\infty} \sum_{m_{k+1}=1}^{\infty} \cdots \sum_{m_{n-1}=1}^{\infty} \left\{ (m_k L_k)^2 + (m_{k+1} L_{k+1})^2 + \cdots + (m_{n-1} L_{n-1})^2 \right\}^{-t}
\] (A.9)
for \( k = 0, 1, \cdots, n - 1 \).

In the following, we shall show that the third term in Eq.(A.8) has the property
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt f(t) \frac{1}{2\pi i} \int_{c_{1}-i\infty}^{c_{1}+i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \left( \frac{\pi}{L_0} \right)^{-2t_1+2t} \times S^{(n-1)}(t_1; L_1, L_2, \cdots, L_{n-1}) = 0 ,
\] (A.10)
where \( f(t) \) is any function that has no poles inside the region of the residue integral with respect to \( t \). Eq.(A.10) implies that the poles of the third term in Eq.(A.8) with respect to \( t \) do not totally contribute to the residue integral of \( t \), although the third term in Eq.(A.8) has several poles with respect to \( t \), as we will see below.

By repeatedly using the relation (A.8), \( S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) \) can be expressed into the form
\[
S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) = (P_1 + P_2 + P_3)^{n-1} S^{(n)}(t; L_0, L_1, \cdots, L_{n-1})
\]
\[
= \sum_{j_1=1}^{3} \sum_{j_2=1}^{3} \cdots \sum_{j_{n-1}=1}^{3} P_{j_1} P_{j_2} \cdots P_{j_{n-1}} S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) ,
\] (A.11)
where the operation of \( P_j \) \( (j = 1, 2, 3) \) is defined by
\[
P_1 S^{(n-k)}(t; L_k, L_{k+1}, \cdots, L_{n-1}) = -\frac{1}{2} S^{(n-k-1)}(t; L_{k+1}, L_{k+2}, \cdots, L_{n-1}) ,
\] (A.12)
\[
P_2 S^{(n-k)}(t; L_k, L_{k+1}, \cdots, L_{n-1}) = \frac{\sqrt{\pi}}{2L_k} S^{(n-k)}(t - \frac{1}{2}; L_{k+1}, L_{k+2}, \cdots, L_{n-1}) ,
\] (A.13)
\[
P_3 S^{(n-k)}(t; L_k, L_{k+1}, \cdots, L_{n-1}) = \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c_{1}-\infty}^{c_{1}+i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \left( \frac{\pi}{L_k} \right)^{-2t_1+2t} \times S^{(n-k-1)}(t_1; L_{k+1}, L_{k+2}, \cdots, L_{n-1}) .
\] (A.14)

For instance, let us consider the term \((P_1)^{n-\ell-m-1}(P_2)\ell(P_3)^m S^{(n)}(t; L_0, L_1, \cdots, L_{n-1})\), which
is explicitly given by

\[
(P_1)^{n-\ell-m-1}(P_2)^\ell(P_3)^m S^{(n)}(t; L_0, L_1, \ldots, L_{n-1})
= \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c_1+\infty}^{c_1-i\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \left(\frac{\pi}{L_0}\right)^{-2t_1+2t} \times \cdots
\]

\[
\times \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c_2+\infty}^{c_2-i\infty} dt_2 \Gamma(t_2 - t_1 + \frac{1}{2}) \zeta(2t_2 - 2t_1 + 1) \left(\frac{\pi}{L_1}\right)^{-2t_2+2t_1} \times \cdots
\]

\[
\times \frac{1}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{c_m+i\infty}^{c_m-i\infty} dt_m \Gamma(t_m - t_{m-1} + \frac{1}{2}) \zeta(2t_m - 2t_{m-1} + 1) \left(\frac{\pi}{L_{m-1}}\right)^{-2t_m+2t_{m-1}} \times \cdots
\]

\[
\times \frac{\sqrt{\pi}}{2L_m} \left(\frac{\sqrt{\pi}}{2L_{m+1}}\right) \cdots \left(\frac{\sqrt{\pi}}{2\ell L_{m+1}}\right) \left(\frac{1}{2}\right)^{n-\ell-m-1} S^{(1)}(t_m - \frac{\xi}{2}; L_{n-1})
\]

\[
= \frac{(-1)^{n-\ell-m-1} (\sqrt{\pi})^{\ell-m}}{2^{n-m-1} \pi L_0 L_1 \cdots L_{\ell+m-1}} \left(\frac{\pi}{L_0}\right)^{2t} \times \cdots
\]

\[
\times \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+\infty} dt_1 \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \left(\frac{L_0}{L_1}\right)^{2t_1}
\]

\[
\times \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+\infty} dt_2 \Gamma(t_2 - t_1 + \frac{1}{2}) \zeta(2t_2 - 2t_1 + 1) \left(\frac{L_1}{L_2}\right)^{2t_2} \times \cdots
\]

\[
\times \frac{1}{2\pi i} \int_{c_m-i\infty}^{c_m+\infty} dt_m \Gamma(t_m - t_{m-1} + \frac{1}{2}) \zeta(2t_m - 2t_{m-1} + 1)
\]

\[
\times \Gamma(t_m - \frac{\xi}{2}) \zeta(2t_m - \ell) \left(\frac{L_{m-1}}{\pi L_n-1}\right)^{2t_m}.
\]  

(A.15)

In the second equality of Eq. (A.15), we have used

\[
S^{(1)}(t_m - \frac{\xi}{2}, L_{n-1}) = \Gamma(t_m - \frac{\xi}{2}) \sum_{m_n-1=1}^{\infty} (m_{n-1}L_{n-1})^{-2t_{m-1}+\ell} = \frac{\Gamma(t_m - \frac{\xi}{2}) \zeta(2t_m - \ell)}{(L_{n-1})^{2t_m-\ell}}.
\]  

(A.16)

Let \(f(t)\) be any function which has no pole inside the region of the residue integral with respect to \(t\). Then, we can show that

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \cdot f(t) \left[ (P_1)^{n-\ell-m-1}(P_2)^\ell(P_3)^m S^{(n)}(t; L_0, L_1, \ldots, L_{n-1}) \right] = 0
\]  

(A.17)

if \(m \geq 1\).

To show Eq. (A.17), we perform the residue integrals of Eq. (A.15) with respect to \(\{t_m, t_{m-1},

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\[ \cdots, t_2 \} \text{ successively by use of the relations} \]
\[
\frac{1}{2\pi i} \int_{c_j-i\infty}^{c_j+i\infty} dt_j g(t_j) \Gamma(t_j - t_{j-1} + \frac{1}{2}) \zeta(2t_j - 2t_{j-1} + 1) \Gamma(t_j - \frac{k}{2}) \zeta(2t_j - k) \]
\[
= \left[ g(t_{j-1} - \frac{1}{2}) \zeta(0) \Gamma(t_{j-1} - \frac{k+1}{2}) \zeta(2t_{j-1} - (k + 1)) + \frac{1}{2} g(t_{j-1}) \Gamma(\frac{k}{2}) \Gamma(t_{j-1} - \frac{k}{2}) \zeta(2t_{j-1} - k) \right]
\]
\[
+ \left[ g(\frac{k}{2}) \zeta(0) \Gamma(\frac{k+1}{2} - t_{j-1}) \zeta(k + 1 - 2t_{j-1}) + \frac{1}{2} g(\frac{k+1}{2}) \Gamma(\frac{k+2}{2} - t_{j-1}) \zeta(k + 2 - 2t_{j-1}) \right], \tag{A.18} \]
\[
\frac{1}{2\pi i} \int_{c_j-i\infty}^{c_j+i\infty} dt_j g(t_j) \Gamma(t_j - t_{j-1} + \frac{1}{2}) \zeta(2t_j - 2t_{j-1} + 1) \Gamma(\frac{\ell}{2} - t_j) \zeta(\ell - k)
\]
\[
= \left[ g(t_{j-1} - \frac{1}{2}) \zeta(0) \Gamma(\frac{k+1}{2} - t_{j-1}) \zeta(k + 1 - 2t_{j-1}) + \frac{1}{2} g(t_{j-1}) \Gamma(\frac{k}{2}) \Gamma(\frac{k}{2} - t_{j-1}) \zeta(k - 2t_{j-1}) \right]
\]
\[
- \left[ g(\frac{k}{2}) \zeta(0) \Gamma(\frac{k+1}{2} - t_{j-1}) \zeta(k + 1 - 2t_{j-1}) - \frac{1}{2} g(\frac{k+1}{2}) \Gamma(\frac{k}{2}) \Gamma(\frac{k}{2} - t_{j-1}) \zeta(k - 2t_{j-1}) \right], \tag{A.19} \]

where \( g(t_j) \) is any function that has no poles inside the region of the residue integral with respect to \( t_j \). We note that there appears only the combination of the type \( \Gamma(\pm(t_{j-1} - \frac{1}{2})) \zeta(\pm(2t_{j-1} - \ell)) \) for some integer \( \ell \) on the right-hand-side of the formulas (A.18) and (A.19), so that we can perform the residue integrals with respect to \( t_m, t_{m-1}, \ldots, t_2 \), successively.

After performing the residue integrals with respect to \( t_j (j = m, m-1, \cdots, 2) \) by use of the formulas (A.18) and (A.19), Eq.(A.15) can be written as the sum of the terms proportional to the following types of the integrals:
\[
\frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 g_1(t_1) \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1) \Gamma(\pm (t_1 - \frac{\ell}{2})) \zeta(\pm (2t_1 - \ell)), \tag{A.20} \]

where \( \ell \) is some positive integer and \( g_1(t_1) \) is some function, which has no poles inside the region of the residue integral with respect to \( t_1 \).

It is now easy to show Eq.(A.17), which follows from the relation
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt f(t) \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dt_1 g_1(t_1) \Gamma(t_1 - t + \frac{1}{2}) \zeta(2t_1 - 2t + 1)
\]
\[
\times \Gamma(\pm (t_1 - \frac{\ell}{2})) \zeta(\pm (2t_1 - \ell)) = 0 \tag{A.21} \]
with Eqs.(A.18) and (A.19). It should be emphasized that Eq.(A.21) does not mean that the integrand of Eq.(A.17) has no poles with respect to \( t \). In fact, the integrand of Eq.(A.17) has several poles with respect to \( t \) and each pole contributes to the residue integral of \( t \), though the sum of their residues totally cancels each other.

We have proved Eq.(A.17) for a special order of \( P_{j_1}P_{j_2} \cdots P_{j_{n-1}} \). Since \( P_jP_kS^{(n-k)}(t; L_k, L_{k+1}, \cdots, L_{n-1}) \) is identical to the opposite order of \( P_kP_jS^{(n-k)}(t; L_k, L_{k+1}, \cdots, L_{n-1}) \) with the exchange of \( L_k \leftrightarrow L_{k+1} \) (and with a shift of the integration parameter, if necessary), we generally have
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt f(t) \left[ P_{j_1}P_{j_2} \cdots P_{j_{n-1}} S^{(n)}(t; L_0, L_1, \cdots, L_{n-1}) \right] = 0 \tag{A.22} \]

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if some of $j_s$ ($s = 1, 2, \ldots, n - 1$) take the value of 3. This result immediately leads to Eq. (A.10) because Eq. (A.10) corresponds to the case of $j_1 = 3$ in Eq. (A.22).

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