Semiclassical analysis of Bose–Hubbard dynamics

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Keywords: Bose–Hubbard, collapse and revival, double well, BEC

Abstract

In this work the two-site Bose–Hubbard model is studied analytically in the limit of weak coupling \( u \) and large number of particles \( N \). In particular, the difference in the occupation between the two sites, where initially all particles are at one site, was calculated analytically. This quantity exhibits collapses and revivals that superimpose rapid oscillations. Excellent agreement with the exact numerical solution was found. The semiclassical approximation where \( N^{-1} \) plays the role of Planck’s constant was used and perturbation theory to order \( u^2 \) was applied. The occupation difference was calculated also for the case where initially both sites are occupied provided that the difference in occupation is sufficiently large. This work provides an analytical description of results that were so far found only numerically. Similar behavior and analysis are expected for a large variety of physical situations.

1. Introduction

The physics of Bose–Einstein condensates (BECs) was extensively studied in the recent years [1–3]. For weakly coupled atoms in a large variety of systems, the Gross–Pitaevskii equation (GPE) [1, 3] describes well the static properties. For the dynamics, the situation is more complicated and it is instructive to study simple paradigmatic systems. The two-site Bose–Hubbard (BH) model is such an extensively studied [4–11] paradigmatic system and will be the subject of the present paper. It consists of \( N \) bosons hopping between two sites. The specific quantity that will be explored is the imbalance in the occupation of the two sites as a function of time given the initial preparation where all atoms occupy one site. In absence of interactions between the bosons, this quantity oscillates with Rabi’s frequency. However, for weak interactions, the oscillations are superimposed by an envelope function that exhibits collapses where it practically vanishes and revivals where it returns to approximately the initial value. The result of this work is an analytic formula for the occupation imbalance that is valid for a large number of bosons \( N \gg 1 \) and for weak interparticle interactions. To the best of our knowledge it is the first explicit analytical expression for the site occupation imbalance for the two-site BH model. It enables us to see in detail the phenomena of collapses and revivals that can be found for a large variety of physical situations.

The two-site BH model can be realized, for example, where the two sites are the degenerate ground states of particles in a harmonic well [12]. It was demonstrated experimentally that at least for short times, this model describes well the experimental results. The BH model is also used to approximate the dynamics of \( N \) interacting bosons in a double well external potential (see figure 1) for sufficiently weak interactions so that the two lowest levels of the double well can be assumed uncoupled of the rest of the levels. In this system, the mean-field dynamics resulting from GPE [13], miss the collapse and revival phenomena and therefore it is accurate only for very short times [6, 7, 11, 14–16]. So far, the dynamics of the double well were experimentally observed in the time regime where GPE is valid [17–20]. During the last years, many theoretical works were dedicated to the BH model [5, 9, 10, 21, 23, 23–30] and many aspects of the double well dynamics were clarified. In particular, collapses and revivals were found in numerical calculations involving some heuristic arguments [6].

For BECs, collapses and revivals were observed in experiments where a condensate of interacting bosons was confined to a lattice [31, 32]. In these experiments, the interference pattern of the matter wave field originating in different lattice sites is controlled by the collapses and revivals of phase coherence in each site as a function of...
time \[4, 33–36\]. For non-interacting particles the phenomenon of collapse and revival is well understood. In quantum optics these were found analytically and numerically in the Jaynes–Cummings model \[37, 38\]. The shape of a function that measures coherence there is the closest to the one found in the present paper for interacting bosons. In optics, this phenomenon is well known as the Talbot effect \[39, 40\] (see also \[41–43\]). We hope that by providing an analytic formula for collapses and revivals in the BH model, our work will stimulate experiments in wider time regimes for systems such as the double well and the degenerate harmonic well \[12\], where this fascinating phenomena can be observed.

In our calculation, we use the semiclassical picture for the two-site BH model that was developed and studied in some detail \[5, 6, 44\]. The dimensionless parameter that controls the corresponding classical behavior is

\[\alpha = \frac{UN}{J}\]  

where \(U\) is the interparticle interaction, \(J\) is the strength of the hopping between the sites, and \(N\) is the number of particles. In this picture, \(\frac{2}{\pi h}\) plays the role of Planck’s constant \(h\) and the thermodynamic limit \(N \to \infty\) plays the role of the classical limit. The various regimes are \[5\]:

1. Rabi regime \(\alpha < 1\)
2. Josephson regime \(1 < \alpha < N^2\)
3. Fock regime \(N^2 < \alpha\).

The Josephson regime is the most extensively studied one \[5, 24, 26, 27, 44\]. It exhibits an interesting phase space, with dynamics related to the experimental observations \[17, 18\]. We confine ourselves to the Rabi regime where the classical behavior is very simple. It enables us to study analytically quantum collapses and revivals that are crucial for the understanding of coherence.

The outline of the paper is as follows: In section 2, the model is defined and the semiclassical picture is presented. In section 3, a transformation to angle action variables is performed and used to find the energies within the WKB approximation to the order \(\alpha^2\). In section 4, the difference in occupation between the two sites is calculated for an initial condition where all atoms are on one site, while in section 5 the initial condition where both sites are occupied is used. The results are discussed in section 6.
2. The two-sites Hubbard model

The two-sites Hubbard model we study is defined by the Hamiltonian

$$H_{BL} = -J \big( a_L^+ a_R^+ + a_R^+ a_L^+ \big) + U \left[ n_L (n_L - 1) + n_R (n_R - 1) \right].$$ (2)

The sites are denoted by $L$ (Left) and $R$ (Right). The creation and annihilation operators on the sites are $a_L^+$, $a_R^+$ and $a_L$, $a_R$. The number operators for the two sites are $n_L = a_L^+ a_L$ and $n_R = a_R^+ a_R$. The commutation relations are $[a_L, a_R^+] = 1$, $[a_R, a_L^+] = 1$, and the units are such that $\hbar = N^{-1}$. It is assumed that the on-site energies on the two sites are identical. The total number of particles $N = n_L + n_R$ is conserved. The first term in (2) represents the hopping between the two sites while the second one is the energy of the interparticle interaction that in the present work is assumed to be small. The Hamiltonian (2) can be written in the form

$$H_{BL} = -J \big( a_L^+ a_R + a_R^+ a_L \big) + U \big( a_L^+ a_R^+ a_R a_L + a_R^+ a_L^+ a_L a_R \big).$$ (3)

By using the angular momentum operators (see for example [5])

$$S_x = \frac{1}{2N^2} \left( a_R^+ a_L + a_L^+ a_R \right)$$
$$S_y = \frac{1}{2N^2} \left( a_R^+ a_L - a_L^+ a_R \right)$$
$$S_z = \frac{1}{2N^2} \left( a_L^+ a_R - a_R^+ a_L \right) = \frac{1}{2N} (n_L - n_R),$$ (4)

the Hamiltonian (3) can be written up to a constant as (shown in appendix A)

$$H' = -2JSx + 2UN^2S_z^2.$$ (5)

Namely,

$$H_{BL} = H' + C_N$$ (6)

where $C_N = \frac{1}{2}N^2U - NU$. The operators (4) satisfy the commutation relations of angular momentum operators

$$\begin{align*}
[ S_x, S_y ] &= \frac{i}{N} S_z \\
[ S_y, S_z ] &= \frac{i}{N} S_x \\
[ S_z, S_x ] &= \frac{i}{N} S_y
\end{align*}$$ (7)

as can be easily verified. These are the standard commutation relations of the angular momentum operators. It is convenient to measure the energy in units of $2JN$, and work with the Hamiltonian

$$H = -S_x + uS_z^2$$ (8)

where $u \equiv \frac{UN}{J}$ (see (2)). Since (4) are angular momentum operators, and the eigenvalues of $NS_z$ are integers $n$ satisfying $-\frac{N}{2} \leq n \leq \frac{N}{2}$, $S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{1}{2N} \left( \frac{N}{2} + 1 \right)$. For large $N$ the semiclassical limit is justified. In the classical limit $N \to \infty$ corresponding to $\hbar \to 0$, the equations of motion can be obtained by replacing $\frac{N}{\hbar} [f, g] \to [f, g]$ where $[f, g]$ are the Poisson’s brackets. These are the Hamilton equations obtained from (8).

As the total number of particles $N$ is conserved, the total angular momentum $S^2$ is conserved as well. Therefore, the vector $\hat{S} = (S_x, S_y, S_z)$ lies on the Bloch sphere of radius $\frac{1}{2}$ and it is possible to write

$$S_y = \frac{1}{\sqrt{4}} - S_z^2 \cos \varphi$$
$$S_z = \frac{1}{\sqrt{4}} - S_z^2 \sin \varphi$$ (9)

where $0 < \varphi < 2\pi$ is an angle circling the $S_z$ axis. Now, the Hamiltonian (8) takes the form

$$H = -S_x + u \left( \frac{1}{4} - S_z^2 \right) \sin^2 \varphi.$$ (10)

For $u < 1$ the classical trajectories encircle the $S_z$ axis on the Bloch sphere as demonstrated in figure 2. There are no turning points since $\varphi \neq 0$ as can be seen from the Hamilton equations corresponding to (10). For this reason there is an advantage to use the pair $\{S_x, \varphi\}$ of conjugate variables.
3. The semi classical calculation of the spectrum

In the absence of interparticle interactions \( u = 0 \), the bosons perform Rabi oscillations and the phase space trajectories circle around the \( S_x \) axis with frequency \( 2J \). We would like to study the dynamics in the Rabi regime \( u < 1 \) (weak inter-particle interactions) by using semi-classical methods. Our aim is to find the spectrum of (10) by using Wentzel–Kramers–Brillouin (WKB) quantization for the action variable (a similar approach was adopted by [5] for the Josephson regime \( < uN^2 \)). \( S_x \) and \( \phi \) are canonically conjugate variables. Their variation is given by Hamilton’s equations generated by \( H \) of (8). It was verified that these are identical to the equations satisfied by the components of \( \vec{S} \). We introduce the action variable via

\[
I = \frac{1}{2\pi} \int_0^{2\pi} S_x \, d\phi.
\]

For this we use the relation between \( S_x \) and \( \phi \) found from (10). The leading order in \( N \) (corresponding to the leading order in Planck’s constant \( \hbar \) in the quantum theory) results in the Bohr Sommerfeld quantization rule [45]

\[
I_n = \frac{n}{N}
\]

where \( n = -\frac{N}{2}, \ldots, \frac{N}{2} \) are integers. This enables us to calculate the levels within the leading order in \( \frac{1}{N} \). The calculation is performed in appendix B. To obtain an explicit expression, we have to expand in powers of \( u \). In the first order in \( u \) we find

\[
E_n^{(1)} \approx -\frac{n}{N} + \frac{1}{8} u - \frac{1}{2N^2} un^2.
\]

In order to compare the energies (13) to the exact spectrum of the BH model (3) (which can be obtained by diagonalizing the Hamiltonian matrix), we should multiply it by \( 2JN \) and add the constants that were omitted in (5) (see (72)), namely,

\[
E_n^{(BH)} = 2JNE_n^{(1)} + C_N \approx 2J \left( -n + \frac{3}{8} uN - \frac{1}{2} u - \frac{1}{2N} un^2 \right).
\]

For \( u < 1 \), this spectrum is a good approximation to the exact BH spectrum (see figure 3).

In second order in \( u \), one finds

\[
E_n^{(2)} \approx -\frac{n}{N} + \frac{1}{8} u - \frac{1}{2N^2} un^2 - \frac{nu^2}{16N} + \frac{n^2 u^2}{4N^3}
\]

leading to

\[
E_n^{(BH2)} \approx 2J \left( -n + \frac{3}{8} uN - \frac{1}{2} u - \frac{1}{2N} un^2 - \frac{1}{16} u^2 n + \frac{1}{4N^2} u^2 n^3 \right).
\]

Numerical calculations (figure 3) verify that the spectrum \( E_n^{(BH2)} \) is indeed closer to the BH spectrum than \( E_n^{(BH1)} \). Although the second order correction is extremely small compared to the first order, it turns out to be of great importance for the dynamics and in particular for the shape of the revival peaks, as will be shown in

\[
\text{Figure 2. Trajectories on Bloch sphere for (a) } u = 0 \text{ and (b) } u = \frac{1}{2}. \text{ The fixed points in the Rabi regime } (u < 1) \text{ are } \{ S_x, S_y, S_z \} = \left\{ 1, 0, 0 \right\} \text{ and } \{ S_x, S_y, S_z \} = \left\{ -\frac{1}{2}, 0, 0 \right\}. \text{ If } u = 0, \text{ the phase space trajectories are circles around these fixed points. For } 0 < u < 1, \text{ the trajectories still circle around the fixed points, but their shape is modified. In the Josephson and Fock regimes (which are not studied in the present work), two additional fixed points appear, leading to a more complicated phase space, see for example [5].}
\]
section 4. Note that the agreement is very good even for $u$ that is not much smaller than 1. Actually, the approximation for the energies works well also for $u = 2$ but not for $u = 4$. There are predictions based on low-order perturbation theory that hold even when the perturbations are not very small [46–49]. For the present work, it is particularly instructive to note equations (A.2) and (A.3) of [49]. The result of (16) is actually not the one of quantum perturbation theory but the expansion to the order $u^2$ of the leading semiclassical result. This expansion is convergent in general for $u < 1$, and for small $n$ used in the paper, it is sufficient that $u < 1$ as can be seen from (73). In appendix C, standard quantum perturbation theory is used and we note that it requires $uN < 1$, a condition that is more restricted than the conditions of the present section.

A natural question is what are the corrections to the leading order in the semiclassical expansion presented here. In appendix F, it is shown that the correction is of order $\frac{1}{N^2}$. Hence it is of the form $\frac{1}{N^2} f(H)$. Then, this term should be added to the RHS of (79) leading to an additional correction to the energy levels. In that appendix, we estimate the correction for representative values of the parameters and find it to be extremely small.

It is worthwhile to note that the choice of the conjugate variables $(S, \varphi)$ is of importance for the simplicity and accuracy of the result since for small $u$ these are approximately the angle action variables. In other works, different pairs of conjugate variables are used. In particular, in [8] different conjugate variables were used, leading to results that are much more involved (no explicit formula) and less accurate than the results of the present work. On the other hand, their choice of conjugate variables enables them to work also in the regime $\ll uN$.  

4. Dynamics

In this section, our aim is to derive an analytic expression for the expectation value of $S_z$ of (4) that is the difference in occupation of the two sites where the initial condition is that all the bosons occupy the site $L$ and $\langle S_z \rangle = \frac{1}{2}$ (it is the north pole of the phase space Bloch sphere, figure 2). In the framework of the BH model (3), it is possible to calculate $S_z(t)$ numerically [6]. The resulting $(S_z(t))$ is a series of collapses and revivals, superimposed on rapid oscillations. We would like to utilize the spectrum (16) in order to study analytically the dynamics in the Rabi regime $u < 1$.

In absence of interparticle interactions ($u = 0$), the operator $S_z$ commutes with the Hamiltonian (8). Hence, for $u < 1$, the eigenstates of (8) can be approximated by the eigenstates of $S_z$ (corrections of higher order will be discussed later), namely by

$$|n\rangle \equiv \frac{1}{\sqrt{\binom{N}{n} \binom{N}{2}^n}} (a_+)^n (a_-)^{N-n} |0\rangle. \quad (17)$$  

Figure 3. The energy spectrum of the BH Hamiltonian for $J = 1$ and $N = 26$. The blue squares are obtained by numerical diagonalization of the Hamiltonian matrix for the Hamiltonian (3). The red dots are analytically calculated to the first order in $u$ (14) and the green stars are analytically calculated to the second order in $u$ (see (16)).
where $a_\pm = \frac{1}{\sqrt{2}} (a_L^+ \pm a_R^+)$ and

$$[a_+, a_-] = 0 \quad (18)$$

$$[a_+, a_-^+] = 1 \quad (19)$$

$$[a_-, a_-^+] = 1 \quad (20)$$

The reason for (17) is that

$$S_\kappa = \frac{1}{2N} (a_+^* a_- - a_-^* a_+) \quad (21)$$

In what follows, we calculate the evolution of the operator

$$\equiv - = + - = \sim + + - (22)$$

for the initial condition

$$|\psi(0)\rangle = \sum_{n=-N/2}^{N/2} c_n |n\rangle \quad (24)$$

where

$$c_n = \frac{1}{\sqrt{N^!}} \cdot \left( \frac{N}{2} + n \right) \cdot \sqrt{ \left( \frac{N}{2} + n \right)! \left( \frac{N}{2} - n \right)! } = \frac{1}{2^{N/2}} \sqrt{\frac{N}{2} + n} \quad (25)$$

For $N \gg 1$ and $n \ll \frac{N}{2}$, the binomial coefficients can be approximated by a Gaussian,

$$c_n \approx \left( \frac{2}{\pi N} \right)^{1/2} e^{-u^2/2N}. \quad (26)$$

We note that the normalized difference between the occupation of the two sites is

$$\Delta(t) = \langle \psi | S_+ | \psi \rangle = \frac{1}{N} \text{Re} \left( \langle \psi | S_+ | \psi \rangle \right) \quad (27)$$

where $S_+ \equiv N \left( S_z - iS_y \right) = \frac{1}{2} \left( a_L^+ + a_R^+ \right) \left(a_L - a_R\right) = a_+^* a_-$. In the basis $|n\rangle$, $S_+$ is a raising operator, therefore,

$$S_+ |n\rangle = \sqrt{ \left( \frac{N}{2} + n + 1 \right) \left( \frac{N}{2} - n \right) } |n + 1\rangle \quad (28)$$

we used (18), (19), and (20). The expectation value of $S_+$ at time $t$ for the initial condition (24) is

$$\langle \psi(t) | S_+ | \psi(t) \rangle = \sum_{n=-N/2}^{N/2} \sqrt{ \left( \frac{N}{2} + n + 1 \right) \left( \frac{N}{2} - n \right) } c_n c_{n+1} e^{-i\left(E_{n}^{(BH2)} - E_{n+1}^{(BH2)}\right)t} \quad (29)$$

First, note that

$$c_n c_{n+1} = \frac{\sqrt{2}}{\sqrt{\pi N}} e^{-2u^2/2N} \quad (30)$$

According to (16),

$$E_{n}^{(BH2)} - E_{n+1}^{(BH2)} = f \left( 2 + \frac{2}{N} u n + \frac{1}{8} u^2 - \frac{3}{2N^2} u^2 n^2 + \frac{u}{N} - \frac{3}{2N^2} u^2 n - \frac{1}{2N^2} u^2 \right) \quad (31)$$

Since $u < 1$, we can neglect the term $\frac{3}{2N^2} u^2 n^2$, which is much smaller than $\frac{2}{N} u n$. For large $N$ and $n \ll \frac{N}{2}$, equation (29) can be written in the form

$$\langle \psi(t) | S_+ | \psi(t) \rangle = \frac{N}{2} \Delta(t) \quad (32)$$
where
\[ \phi = J \left( 2 + \frac{1}{2} u^2 + \frac{u}{N} - \frac{1}{2N^2} u^2 \right) \] (33)
and
\[ \tilde{S} = \frac{\sqrt{2 \pi}}{\sqrt{\pi} N} \sum_{n=-N/2}^{N/2} e^{-\frac{1}{N} \left( 2n^2 + 2n + 1 \right)} e^{-i \frac{f}{N} \left( 2n^2 + \frac{3}{2} n^2 \right)} e^{-i \tilde{S}_n t}. \] (34)

We note that \( e^{-i\phi t} \) is a rapidly oscillating function of \( t \) with a period that is approximately \( \frac{2\pi}{J} \). We turn now to explore the envelope of \( \tilde{S} \). Since \( n \) is an integer, in first order in \( u \), the envelope of the sum (29) is a periodic function of \( t \) with period (revival time) of
\[ T_R = \frac{\pi N}{uJ}. \] (35)

Actually \( T_R \) is the inverse of the coefficient of the linear term in \( n \) on the RHS of (31), namely
\[ \frac{1}{T_R} = J \left( \frac{u}{N} + \frac{3u^2}{2N} \right). \] The estimate (35) assumes \( \frac{u}{N} \ll 1 \). The terms proportional to \( u^2 \) in (31) are ignored for the same reason, taking into account that in what follows only terms where \( n \ll N \) are important. Around the \( m \)th revival, we write \( t = m \cdot T_R + \tau \) with \(- \frac{1}{2} T_R < \tau < \frac{1}{2} T_R \) and write \( \tilde{S} = \sum_{m} \tilde{S}_m \) where
\[ \tilde{S}_m = \frac{\sqrt{2 \pi}}{\sqrt{\pi} N} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{N} \left( 2n^2 + 2n + 1 \right)} e^{-i \frac{f}{N} \left( 2n^2 + \frac{3}{2} n^2 \right)} e^{i \frac{f}{N} \left( mT_R + \tau \right)} \]
\[ = \frac{\sqrt{2 \pi}}{\sqrt{\pi} N} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{N} \left( 2n^2 + 2n + 1 \right)} e^{i \frac{f}{N} \left( mT_R + \tau - \frac{2n^2 + \frac{3}{2} n^2}{N} \right)} \] (36)

We approximate the sum by an integral
\[ \tilde{S}_m \approx \frac{\sqrt{2 \pi}}{\sqrt{\pi} N} \int_{-\infty}^{\infty} e^{-\frac{1}{N} \left( 2n^2 + 2n + 1 \right)} e^{i \frac{f}{N} \left( mT_R + \tau - \frac{2n^2 + \frac{3}{2} n^2}{N} \right)} dn. \] (37)

What enables to approximate the sum over \( n \) by an integral is the fact that in the vicinity of a revival \( \frac{f}{N} \tau + \frac{3}{2N} u^2 \pi t \) is small (while \( \frac{f}{N} nt \) is typically large). The integral was calculated in appendix D (where we should take \( \beta = \gamma = 1 \) and \( m = m \) for the calculations of the present section) using \( \psi \) in the order \( u^0 \). In appendix E, the corrections of the order \( u \) and \( u^2 \) to the eigenstates were added by using quantum perturbation theory. In appendix G, it is verified that the semiclassical wave function gives the same result. The result is
\[ \tilde{S}_m = \frac{\sqrt{2 \pi}}{D_D} e^{D_R} e^{D_D} e^{D_{R}^{\prime}} e^{i (\phi + \phi^\prime)} \] (38)
where \( D_R, D_D, \) and \( \phi \) are given by (96), (98), and (104),
\[ \frac{D_R}{D_D} \approx \frac{-2J^2 u^2 \left( \tau + \frac{3m\pi}{2f} \right)^2 + 2 + \frac{9}{2} u^2 m^2 \pi^2}{N \left[ 4 + \frac{9}{4} u^2 \left( m\pi + \frac{f}{N} \tau \right) \right]^2}, \] (39)
and \( \phi^\prime \) is a phase calculated by taking into account higher orders of the eigenstates, see appendix E. It satisfies
\[ \tan \phi^\prime \approx \frac{u^2}{8} \left( 2 \tau + \frac{3}{2} m\pi \right). \] (40)
For small \( u \), \( \phi^\prime \approx \tan \phi^\prime \approx \frac{u^2}{8} \left( 2 \tau + \frac{3}{2} m\pi \right). \)

The resulting \( \tilde{S}_m \) (38) is approximately a Gaussian of a width
\[ \Delta t_{mR} \approx \frac{\sqrt{2N} \left( 1 + \frac{9}{16} u^2 m^2 \pi^2 \right)}{f_m}. \] (41)
Therefore, \( \tau \leq \frac{2\pi}{J} N \) and \( \frac{1}{N} \tau \) (see denominator on (39)) is of order \( \frac{1}{\sqrt{N}} \) and can be neglected compared to \( m\pi \), as was done in (41) and in the following equations.

For small \( m \), \( \Delta t_{mR} \ll T_R \). However, there is an \( m_{max} \) where the width \( \Delta t_{mR} \) is comparable to \( T_R \) and then the revivals mix and our calculations are not valid. Defining \( m_{max} \) by \( \Delta t_{mR}^{m_{max}} = \frac{1}{2} T_R \), we estimate
\[
m_{\text{max}} = \frac{\sqrt{2(\pi^2 N - 8)}}{3u\pi},
\]

namely, the revivals start to mix at time

\[
T_B = m_{\text{max}} T_R \approx \frac{\pi \sqrt{2} N^2}{3u^2f},
\]

that is the blurring time. For times \( t < m_{\text{max}} T_R \), in the leading order in \( u \), \( \tilde{S} \) can be approximated by

\[
\tilde{S} = \sum_m \frac{1}{1 + \frac{9}{16} u^2 m^2 \pi^2} \exp \left[ -\frac{1}{2} J^2 u^2 \left( t + \frac{3m\pi}{2J} - mT_R \right)^2 + \eta \right] \frac{N}{1 + \frac{9}{16} u^2 m^2 \pi^2} + i\left( \phi_1 + \phi_2 \right).
\]

Therefore, (see (32)),

\[
\langle \psi(t)|\tilde{S}_+|\psi(t)\rangle = \frac{N}{2} \sum_m \frac{1}{1 + \frac{9}{16} u^2 m^2 \pi^2} \exp \left[ -\frac{1}{2} J^2 u^2 \left( t + \frac{3m\pi}{2J} - mT_R \right)^2 + \eta \right] \frac{N}{1 + \frac{9}{16} u^2 m^2 \pi^2} + i\left( \phi_1 - \phi_2 \right),
\]

and (see (27))

\[
\Delta(t) = \frac{1}{2} \sum_m \frac{1}{1 + \frac{9}{16} u^2 m^2 \pi^2} \exp \left[ -\frac{1}{2} J^2 u^2 \left( t + \frac{3m\pi}{2J} - mT_R \right)^2 + \eta \right] \cos \left( \phi_1 - \phi_2 \right),
\]

\[
\langle \psi(t)|\tilde{S}_|-\psi(t)\rangle = \frac{1}{2} \sum_m \frac{1}{1 + \frac{9}{16} u^2 m^2 \pi^2} \exp \left[ -\frac{1}{2} J^2 u^2 \left( t + \frac{3m\pi}{2J} - mT_R \right)^2 + \eta \right] \frac{N}{1 + \frac{9}{16} u^2 m^2 \pi^2} \sin \left( \phi_1 - \phi_2 \right).
\]

where

\[
\eta = -\frac{1}{2} + \frac{9}{8} u^2 m^2 \pi^2
\]

and

\[
\phi \approx f \left( 2 + \frac{1}{8} u^2 + \frac{u}{N} \right).
\]

In the expression for \( \phi \) we neglected \( \frac{1}{N} \) compared to 1 in (33). The other phase variable is

\[
\phi_1 = \phi_2 + \phi_2' \approx \frac{u^2}{8} \left( 2f + \frac{3}{2} m \pi \right) + u \left( \frac{f}{N} + \frac{3}{8} \left[ 1 + \frac{1}{N} \right] \left( m \cdot \pi + \frac{f}{u} \right) \right),
\]

neglecting \( \frac{1}{N} \) compared to 1, one finds

\[
\phi_1 = \phi_2 + \phi_2' \approx \frac{u^2}{8} \left( 2f + \frac{3}{2} m \pi \right) + u \left( \frac{f}{N} + \frac{3}{8} \left( m \cdot \pi + \frac{f}{u} \right) \right).
\]

The evolution of the expectation of the normalized difference in occupation of the two sites is the main result of the present work. In figure 4, it is compared to exact results found by numerical diagonalization of the Hamiltonian (3), for \( u = \frac{1}{2}, N = 100, \) and \( f = 1. \) In figure 4 as well as in figure 5 the expressions (49) and (51) for the phases were used. We checked that if (33) and (50) are used instead, the results cannot be distinguished in the plots. We note remarkable agreement of the envelope with the exact numerical result. The rapid oscillations, exhibit good agreement for short times (figure 4(c)) but it deteriorates for longer times (figure 4(d)).

In figure 5, the evolution of the difference in occupation between the two sites is presented for \( u = \frac{1}{20}, \) \( N = 50, \) and \( f = 1. \) We note also the remarkable agreement between the analytical and numerical
results found for the envelope. The prediction for the rapid oscillations agrees with the exact results for longer times and more revivals than in figure 4.

For short times \( m = 0 \), the dynamics is described by

\[
\Delta(t) = \langle \psi | S_z | \psi \rangle = \frac{1}{2} e^{-\frac{1}{2N}J^2 t^2 i t} \cos \left( \phi t - \phi_1 \right)
\]

and

\[
\langle \psi(t) | S_y | \psi(t) \rangle = -\frac{1}{2} e^{-\frac{1}{2N}J^2 t^2 i t} \sin \left( \phi t - \phi_1 \right).
\]

Both the expectations of \( S_z \) and \( S_y \) oscillate rapidly with the Rabi frequency \( \frac{\phi}{J} \), and at the short time scale have a Gaussian envelope that is

\[
f(t) = \frac{1}{2} e^{-\frac{1}{2N}J^2 t^2 i t}
\]
in the leading order in \( u \) and \( \frac{1}{N} \). Namely, it decays on the time scale

\[
T_c = \frac{\sqrt{2N}}{J_u}.
\]  

(55)

Note that correction term \( J \left( \frac{1}{8} u^2 + \frac{u}{N} \right) \) to the phase in (49) improves the agreement with the exact numerical results compared to Rabi’s phase \( 2Jt \) (see figure 4(c)).

For \( m > m_{\text{max}} \) the revival peaks overlap and the picture presented in figures 4(a) and 5(a) is blurred, as demonstrated in figure 4(b).

How can we understand intuitively the collapses and revivals? Let us focus on the expressions (29), (30), and (31) in the leading order in the interaction \( u \). By (30), typical values of \( n \) for which \( c_n \) is of appreciable amplitude are \(-\sqrt{N} \leq n < \sqrt{N}\). Therefore, in this order, the phases of the terms in (29) are in the interval

\[
2J + \frac{u}{N} - \frac{2Ju}{\sqrt{N}} < \frac{E_n^{(\text{H})}}{Ju} < 2J - \frac{u}{N} + \frac{2Ju}{\sqrt{N}}.
\]

Destructive interference appears when \( 2Ju \approx \pi \), namely, for \( t \approx \frac{\pi Ju}{\sqrt{N}} \). This is of the same order of magnitude as the calculated collapse time (55). On larger time scales, when \( \frac{2Ju}{\sqrt{N}} \approx 2\pi \), revival is found since the phases of the terms in (29) are approximately \( 2\pi n \) (where \( n \) is integer) and constructive interference takes place. Hence, revivals appear at \( T_R \) given by (35) and integer multiples of it. The width of the reviving peaks in this case can be estimated by arguments similar to the ones used to estimate the collapse, as can be seen from (41). The second order in the interaction \( u \) is required to calculate the variation of width of the peaks, as a function of the peak number \( m \). We conclude that the order of

\[
Figure 5. Similar to figure 4 but for \( J = 1 \), \( N = 50 \), and \( u = \frac{1}{20} \). (a) \( \Delta(t) \) for a the time \( t < T_c \). The arrows show the time regimes that are presented in (b)–(d). (b) Short time dynamics. (c) the same as (b) for a time interval near the revival \( m = 2 \). (d) the same as (b) for a time interval near the revival \( m = 3 \), where the analytical result for the phase of (46) no longer agrees with the exact numerical calculation.
magnitude of some of the time scales can be understood and estimated heuristically. The calculation of the present paper is required to obtain the actual values, and the shape of $\Delta(t)$.

We expect that as $u$ increases, the quality of the approximations made in the present work deteriorates. Indeed, from figure 6, we see that our approximation fails for sufficiently large $u$. In particular, for $u > 1$, we enter the Josephson regime where the wavepacket might be self-trapped in one site of an effective double well and our argumentation based on figure 2 does not apply.

5. Initial conditions where both sites are occupied

It is interesting to study the dynamics of a double well where the initial condition is different occupation of the two wells. Such situation is encountered, for example, if a condensate is suddenly separated into two unequal parts, as was done in [18]. The initial condition is of the form

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{N!}} \left( a_n^\dagger \cos \alpha + a_n^\dagger \sin \alpha \right)^N = \frac{1}{2^{N/2} \sqrt{N!}} \left[ (\cos \alpha + \sin \alpha) a_n^\dagger + (\cos \alpha - \sin \alpha) a_n^\dagger \right]^N. \quad (56)$$

Expansion of (56) as a sum $\sum_{n=0}^{N} c_n |n\rangle$ (with $|n\rangle$ given by (17)) yields

$$c_n = \frac{1}{2^{N/2}} \left( \frac{N}{2} + n \right) \left[ (\cos \alpha + \sin \alpha)^N \cos \alpha - \sin \alpha \right]^{N-n}. \quad (57)$$

The coefficients $c_n$ of substantial magnitude are distributed around

$$n_{\text{max}} = \frac{N}{2} \sin (2\alpha) \quad (58)$$

so that

$$c_n = \left( \frac{2\beta}{\pi N} \right)^{1/2} e^{-\frac{1}{2\beta} \left( (n-n_{\text{max}})^2 \right)} \quad (59)$$

and

$$c_n c_{n+1} = \left( \frac{2\beta}{\pi N} \right)^{1/2} e^{-\frac{1}{2\beta} \left( (n-n_{\text{max}})^2 + (n-n_{\text{max}}) + \frac{1}{2} \right)} \quad (60)$$

where

$$\beta = \frac{1}{\cos^2 (2\alpha)}. \quad (61)$$

The expectation value $\Delta(t) = \left\langle S_z(t) \right\rangle$ is calculated in a similar way to what was done in the previous section. The differences are:
1. \( \frac{n_{\text{max}}}{N} \) is not necessarily negligible and therefore \( \tilde{S}_n \left| n \right\rangle = \sqrt{\left( \frac{N}{2} + n + 1 \right) \left( \frac{N}{2} - n \right)} \left| n + 1 \right\rangle \approx \frac{n_{\text{max}}}{N} \left| n + 1 \right\rangle \) and not \( \frac{N}{2} \left| n \right\rangle \) (see for comparison (28)).

2. Due to (60), \( \Delta(t) = \left( S_z(t) \right) \) is multiplied by \( \sqrt{\beta} \).

3. The \( \beta \) in the exponent of (60) affects the result of the integral \( \tilde{S}_m \) of (37), see appendix D.

4. For \( n \approx n_{\text{max}} \), it is possible that \( \frac{3}{2N} u^2 n^2 \) in (31) is not negligible compared to \( \frac{1}{2N} u^2 \). Consequently, the revival time \( T_R \) will be modified as described in what follows. We substitute in (31) \( \Delta = n_{\text{max}} + \Delta n \) and write \( E_n^{(BH)} - E_{n+1}^{(BH)} = J \left( 2 + \frac{1}{2N} u \left( n_{\text{max}} + \Delta n \right) + \frac{1}{2N} u^2 - \frac{3}{2N} u^2 \left( n_{\text{max}} + \Delta n \right)^2 \right) \) for \( n_{\text{max}} \gg 1 \). The first constructive interference is obtained for \( J \left( \frac{2}{N} u \Delta n - \frac{3}{2N} n_{\text{max}}^2 \Delta n \right) T_R = 2\pi \), namely

\[
T_R = \frac{\pi N}{u} \frac{\gamma}{\beta N} \quad (62)
\]

where \( \gamma = \left( 1 - \frac{3}{2} u \frac{n_{\text{max}}}{N} \right)^{-1} = \left( 1 - \frac{3}{4} u \sin (2\alpha) \right)^{-1} \). Therefore, for the initial condition (56), the expectation value \( \Delta(t) \) takes the form (as can be seen by modifying (46)),

\[
\Delta(t) = \frac{\sqrt{\beta}}{2N} \sqrt{\frac{N^2}{4} - n_{\text{max}}^2} \left( \sum_m \frac{1}{\beta^2 + \frac{9}{16} u^2 m^2 \gamma^2 \pi^2} \right)^{1/4} \exp \left[ \frac{-1/2 \beta^2 u^2 \left( t + \frac{3m\pi}{2} - m T_R \right)^2 + \eta}{N \left( \beta^2 + \frac{9}{16} u^2 m^2 \gamma^2 \pi^2 \right)} \right] \cos (\phi_1 - \phi t) \quad (63)
\]

where

\[
\eta = -\frac{\beta^3}{2} + \frac{9}{8N} \beta u^2 m^2 \pi^2,
\]

\[
\phi_1 \approx \frac{u^2}{8\beta} \left( 2\pi \tau + \frac{3}{2} m \gamma \pi \right) + u \left( \frac{1u^2}{N} + \frac{3}{8}\gamma \left( m \gamma \pi + \frac{1}{u^2} \right) \right) \quad (64)
\]

and \( \phi \) is given by (49).

We turn now to estimate the conditions for the validity of the approximation (63). The width of the Gaussian (60) is \( \frac{N}{\beta} \), therefore it is required that

\[
\frac{N}{2} - n_{\text{max}} > \frac{\sqrt{N}}{\beta} \quad (67)
\]

therefore by (58),

\[
\epsilon_1 \equiv \frac{1}{\sqrt{N}} \left( \frac{2}{1 - \sin (2\alpha)} \right) |\cos (2\alpha)| < 1 \quad (68)
\]

The result (68) is demonstrated in figure 7.

Furthermore, the spectrum (16) is more accurate for small values of \( |n| \) (see figure 3) where \( H \) in (73) is small. If one wants to describe the dynamics for large values of \( |n_{\text{max}}| \), higher orders in the expansion of (73) might be needed.

6. Summary and discussion

In the present work the dynamics of the two-site BH model defined by (2) and (3) were analyzed. We analyzed it for weak coupling \( u \) (1) and for large number of particles \( N \). The calculation was performed to order \( u^2 \) and to the leading order in \( \frac{1}{N} \), using a semiclassical method where \( \frac{1}{N} \) plays the role of the Planck’s constant. It is important to note that this is not the standard quantum perturbation theory that requires \( uN < O(1) \) but here it is requires only that \( u \ll O(1) \).

In particular, the normalized difference in the occupation of the sites \( \Delta(t) = \psi(\psi | S_z | \psi(t) \psi(t)) \) as a function of time was calculated in a situation where initially all bosons are on one site leading to (46) with (48), (49), and
It is compared to the exact numerical solution in figures 4 and 5. For the envelope, remarkable agreement with the exact numerical solution is found. The solution exhibits rapid (Rabi) oscillations. The quality of the analytical result for these oscillations is initially very good but it deteriorates with time. The normalized population difference $\Delta(t)$ exhibits three time scales: $T_c$ (55), $T_R$ (35), and $T_B$ (43). Initially, it collapses at a time $T_c$ given by (55). Then, it exhibits revivals at times $mT_R$ with $T_R$ given by (35). These revivals are of increasing width (41). Eventually, at $T_B$ given by (43), this picture is washed away.

Comparison between the approximate result and the exact numerical calculation demonstrates that the result obtained indeed requires the terms in order $u^2$ and $N^{-1}$. The classical approximation (10) reproduces correctly the rapid oscillations for short times. Such a behavior is found also for the GPE in double well [6, 13].

Quantization is essential for the collapses and revivals. The collapse and revival times are predicted correctly by the first order in the interaction $u$, however for the width of the peaks the order $u^2$ is required, since the width depends on $m$ via the combination $m^2u^2$.

Collapses and revivals were found in various situations [4, 6, 31, 33–35, 38–41, 43, 50–52]. To the best of our knowledge, equation (46) is the only complete explicit analytic description of this situation for a specific model of interacting bosons. It is reminiscent of the dynamics of the Jaynes–Cummings model [38]. In particular, the analytic formula (46) enables us to find the shape of the revival peaks and the dependence of their width (41) on time. This enables us to predicts the time $T_B$ where the revival phenomena are blurred. To the best of our knowledge, this was not found so far for the present model.

We studied also the case where initially both sites are populated and found an approximation that is good if the initial difference in occupation is sufficiently large. For finer details, see equation (68) and figure 7. In this case, collapses and revivals are found as well where also here the collapse time $T_c$ and the width of the reviving peaks are proportional to $\sqrt{N}$ and the revival time is proportional to $N$. However, the revival time depends on the initial condition, as is seen from (62).

The generalization to other situations is left for further work.

Acknowledgments

This work resulted from a discussion with D Cohen on [5]. We thank him for motivating this direction of research and many critical discussions and communications. We thank also O Alon, O Alus, I Bloch, E Shimshoni, and J Steinhauer for illuminating and informative discussions. The work was supported in part by the Israel Science Foundation (ISF) grant number 1028/12, by the US-Israel Binational Science Foundation (BSF) grant number 2010132, and by the Shlomo Kaplansky academic chair.

Appendix A. The Hamiltonian

In this appendix, we relate the BH Hamiltonian (3) to the spin Hamiltonian (5). Substituting the definitions (4) in (5), we get

![Figure 7](image-url)
\[ H' = -f \left( a_R^2 a_L + a_R^* a_L^* \right) + \frac{U}{2} \left( a_L a_L^* - a_R a_R^* \right)^2. \]  

(69)

The first term of \( H' \) is identical to the first term of \( H_{BH} \). The second term is

\[
\frac{U}{2} \left( a_L^* a_L - a_R^* a_R \right)^2 \Rightarrow \frac{U}{2} \left( a_R^2 a_R a_R^* a_R + a_R^2 a_L^* a_R^* a_L + a_R^2 a_L a_L^* a_L + a_R^2 a_L^* a_R + a_R^2 a_L^* a_R^* a_L + a_R^2 a_R^* a_L^* a_L \right).
\]  

(70)

We find that

\[
H_{BH} - H' = \frac{U}{2} \left( a_R^2 a_R a_R^* a_R + a_R^2 a_L^* a_R^* a_L + a_R^2 a_L a_L^* a_L - a_R^* a_R \right)
\]

\[
= \frac{1}{2} N^2 U - NU.
\]

(71)

Therefore,

\[ H_{BH} = H' + \frac{1}{2} uN - Ju. \]

(72)

That reduces to (5) up to a constant.

**Appendix B. Details of the calculation of the semi-classical spectrum**

We start the derivation from (11) and use the relation between \( S_x \) and \( \varphi \) found from (10) and given by

\[ S_x = -1 \pm \sqrt{1 + u \sin^2 \varphi \left( u \sin^2 \varphi - 4H \right)} \]

(73)

where only the + solution is consistent with (10) for \( u = 0 \). In the first order in \( u \), the action can be calculated from

\[ S_x \approx -H + \frac{1}{4} u \sin^2 \varphi - uH^2 \sin^2 \varphi \]

(74)

and by (11),

\[ I \approx -H + \frac{1}{8} u - \frac{1}{2} uH^2. \]

(75)

Now, one can write the Hamiltonian in terms of \( I \) as

\[ H \approx \frac{-1 + \sqrt{1 + \frac{1}{4} u^2 - 2ul}}{u} \]

(76)

where only the + solution satisfies (75) for \( u = 0 \). Therefore, to the first order in \( u \),

\[ H \approx -I + \frac{1}{8} u - \frac{1}{2} uH^2. \]

(77)

The action variable \( I \) is quantized [45] so that (12) holds. Note that \( \varphi = -\frac{\partial H}{\partial S_x} \approx -1 \) for small \( u \) and therefore \( \varphi \) never vanishes for \( u < 1 \). Consequently, the Maslov index vanishes. In the first order in \( u \), the spectrum of (8) and (5) is given by (13) and (14), respectively.

In second order in \( u \), one finds

\[ S_x \approx -H + \frac{1}{4} u \sin^2 \varphi - uH^2 \sin^2 \varphi + \frac{1}{2} u^2H \sin^4 \varphi - 2u^2 H^3 \sin^4 \varphi, \]

(78)

which leads to an action variable of the form

\[ I \approx -H + \frac{1}{8} u - \frac{1}{2} uH^2 + \frac{3}{16} u^2H - \frac{3}{4} u^2H^3. \]

(79)

In order to find the corrections to the spectrum (13), we substitute \( H = E_n^{(1)} + u^2 \cdot \delta_n \) in (79) and keep terms up to the second order in \( u \), resulting in
$$\delta_n = -\frac{n}{16N^4} + \frac{n^3}{4N^3}$$  \hspace{1cm} (80)$$

leading to (15) and (16).

**Appendix C. The pertubative calculation of the spectrum**

It is possible to calculate the spectrum of (8) by using standard quantum perturbation theory for small $u$. The perturbation series is likely to converge if $uN < 1$ since the energy differences are of order $\frac{1}{N}$, see (89). In the first order in $u$,

$$E^{(1)} = -\frac{n}{N} + u \langle n | S^2_+ | n \rangle.$$  \hspace{1cm} (81)

The matrix element $\langle k | S^2_+ | n \rangle$ can be calculated easily by using the relation

$$S_z = \frac{1}{2} (\hat{S}_+ + \hat{S}_-)$$  \hspace{1cm} (82)

where $\hat{S}_z = (S_z \mp iS_y)$ given by (4) are ladder operators satisfying

$$S^2_{\pm} \langle n \rangle = \frac{1}{N} \left( \frac{N}{2} \pm n + 1 \right) \left( \frac{N}{2} \mp n \right) \langle n \pm 1 \rangle.$$  \hspace{1cm} (83)

$$S^2_{\pm} \langle n \rangle = \frac{1}{2} \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} \mp \frac{4n}{N^2} \right] \langle n \pm 1 \rangle.$$  \hspace{1cm} (84)

Hence,

$$S^2_z \langle n \rangle = \frac{1}{4} \left( \hat{S}^2_+ + \hat{S}^2_- + \hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+ \right) \langle n \rangle$$

$$= \frac{1}{16} \cdot \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} + \frac{2}{N} \right] \langle n + 2 \rangle$$

$$+ \frac{1}{16} \cdot \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} + \frac{2}{N} \right] \langle n - 2 \rangle$$

$$+ \frac{1}{16} \cdot \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} + \frac{2}{N} \right] \langle n \rangle$$

$$+ \frac{1}{16} \cdot \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} + \frac{2}{N} \right] \langle n \rangle.$$  \hspace{1cm} (85)

Assuming $N \gg 1$, we expand $\langle n | S^2_z | n \rangle$ to the second order in $\frac{1}{N}$ and get

$$\langle n | S^2_z | n \rangle \approx \frac{1}{16} \cdot \left[ 1 + \frac{4}{N} - \frac{8n^2}{N^2} + \frac{4}{N^2} \right]$$

$$+ \frac{1}{16} \cdot \left[ 1 + \frac{4}{N} - \frac{8n^2}{N^2} + \frac{4}{N^2} \right]$$

$$\approx \frac{1}{16} \left[ 2 + \frac{4}{N} - \frac{8n^2}{N^2} \right].$$  \hspace{1cm} (86)

resulting in

$$E^{(1)} = -\frac{n}{N} + u \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} \right].$$  \hspace{1cm} (87)

which is equivalent to the semiclassical correction calculated in (13), if $\frac{1}{N}$ is ignored compared to 1.

The energies to the second order in $u$ are

$$E^{(2)} = -\frac{n}{N} + u \left[ 1 + \frac{2}{N} - \frac{4n^2}{N^2} \right] + u^2 \sum_{k \neq n} \left| \frac{\langle k | S^2_z | n \rangle}{E^{(0)}_n - E^{(0)}_k} \right|^2.$$  \hspace{1cm} (88)
The energy differences are

\[ E_{n}^{(0)} - \tilde{E}_{k}^{(0)} = \frac{k - n}{N} \]  

(89)

and the matrix elements \( \langle k | S_{z}^{2} | n \rangle \) do not vanish only for \( k = n \pm 2 \). Therefore,

\[
\sum_{k \neq n} \left| \langle k | S_{z}^{2} | n \rangle \right|^{2} = \frac{N}{2 \cdot 16^{2}} \left( 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} - \frac{4n}{N} \right) \left( 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} - \frac{12n}{N^{2}} - \frac{8}{N^{2}} \right)
\]

\[
- \frac{N}{2 \cdot 16^{2}} \left( 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} + \frac{4n}{N} \right) \left( 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} + \frac{12n}{N^{2}} - \frac{8}{N^{2}} \right)
\]

\[
= \frac{1}{16^{2}} \left[ -4n \left( 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} \right) - \frac{12n}{N} \left( 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} \right) \right]
\]

\[
= -\frac{4n}{16N} \left[ 4 + \frac{8}{N} - \frac{8}{N^{2}} - \frac{16n^{2}}{N^{2}} \right] = -\frac{n}{16N} \left[ 1 + \frac{2}{N} - \frac{2}{N^{2}} - \frac{4n^{2}}{N^{2}} \right]
\]

(90)

and,

\[ E^{(2)} = -\frac{n}{N} + \frac{u}{8} \left[ 1 + \frac{2}{N} - \frac{4n^{2}}{N^{2}} \right] + u^{2} \left[ -\frac{n}{16N} \left( 1 + \frac{2}{N} - \frac{2}{N^{2}} - \frac{4n^{2}}{N^{2}} \right) \right] \]

(91)

which is equivalent to the semiclassical correction calculated in (15), when \( \frac{1}{N} \) is ignored compared to 1.

### Appendix D. Calculation of integrals required in sections 4, 5

In this appendix, we calculate the integral (37) and the corresponding integral required in section 5, which are of the form

\[ S_{m} = \int_{-\infty}^{\infty} e^{-\left( Ax^{2} + Bx \right)} \, dx = \sqrt{\frac{\pi}{A}} e^{B/4A} \]

(92)

where

\[ S_{m} = \sqrt{2} e^{-\frac{\beta}{\sqrt{\pi N}}} \tilde{S}_{m} \]

(93)

\[ A = \frac{1}{N} \left[ 2\beta - \frac{3}{2} iu \left( m\pi + \frac{1}{N} u\tau \right) \right] \]

\[ B = \frac{2}{N} \left[ \beta + j u\tau \right] \]

\[ m = \gamma m. \]

(94)

In section 4 we consider the case \( \beta = 1, \gamma = 1 \) while in section 5, \( \beta = \frac{1}{\cos(2\alpha)} \) and \( \gamma = [1 - \frac{3}{4} u \sin (2\alpha)]^{-1} \). There, \( \alpha \) determines the initial conditions, see (56). In order to write (92) explicitly, we perform some manipulations where for each order of \( \tau \), only the dominant order in \( u \) is taken into account.

\[ \frac{B^{2}}{4A} = \frac{\frac{1}{N} \left[ \beta^{2} + 2 j u \tau \beta - j^{2} u^{2} \tau^{2} \right]}{2\beta - \frac{3}{2} iu \left( m\pi + \frac{1}{N} u\tau \right)} \]

(95)

After multiplying the numerator and the denominator by the complex conjugate of the denominator,

\[ \frac{B^{2}}{4A} = \frac{\left[ \beta^{2} + 2 j u \tau \beta - j^{2} u^{2} \tau^{2} \right]}{N \left[ 4\beta^{2} + \frac{9}{4} u^{2} \left( m\pi + \frac{1}{N} u\tau \right)^{2} \right]} \]

\[ \left[ 2\beta + \frac{3}{2} iu \left( m\pi + \frac{1}{N} u\tau \right) \right] \]

\[ \left[ 2\beta + \frac{3}{2} iu \left( m\pi + \frac{1}{N} u\tau \right) \right] \]
\[
2\beta^3 + 4i\mu \beta^2 - 2J^2 u^2 \tau^2 \beta + \frac{3}{2} i u \beta^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right)
\]
\[
= \frac{2\beta^3 + 4i\mu \beta^2 - 2J^2 u^2 \tau^2 \beta + \frac{3}{2} i u \beta^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right)}{N \left[4\beta^2 + \frac{9}{4} u^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right)^2\right]}
\]
\[
- 3Ju^2 \tau \beta \left(\tilde{m}\pi + \frac{1}{N} ut\right) + \frac{3}{2} i u^2 \tau^2 \beta \left(\tilde{m}\pi + \frac{1}{N} ut\right)
\]
\[
= \frac{2\beta^3 - 3Ju^2 \tau \tilde{m}\pi \beta - \beta \left(2J^2 u^2 + \frac{3}{N} J^2 u^2 \right) \tau^2}{N \left[4\beta^2 + \frac{9}{4} u^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right)^2\right]}
\]
\[
+ i \left[4i\mu \beta^2 + \frac{3}{2} i u \beta^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right) - \frac{3}{2} Ju^2 \tau^2 \beta \left(\tilde{m}\pi + \frac{1}{N} ut\right)\right]
\]
\[
N \left[4\beta^2 + \frac{9}{4} u^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right)^2\right].
\]

To the leading order in \(u\),
\[
\frac{B^2}{4A} = \frac{D_R + i D_I}{D_D}
\]

where
\[
D_R = \frac{1}{N} \left[2\beta^3 - 3Ju^2 \tau \tilde{m}\pi \beta - 2\beta J^2 u^2 \tau^2\right]
\]
\[
D_I = \frac{1}{N} \left[4i\mu \beta^2 + \frac{3}{2} i u \beta^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right) - \frac{3}{2} Ju^2 \tau^2 \beta \left(\tilde{m}\pi + \frac{1}{N} ut\right)\right]
\]
\[
D_D = 4\beta^2 + \frac{9}{4} u^2 \left(\tilde{m}\pi + \frac{1}{N} ut\right)^2.
\]

\(D_R\) can be written as
\[
D_R = - \frac{2\beta}{N} J^2 u^2 \left(\tau + \frac{3m\pi}{2f}\right)^2 + \frac{2\beta^3}{N} + \frac{9}{2N} u^2 \tilde{m}^2 \tilde{\pi}^2 \beta.
\]

Now we turn to calculate \(\sqrt{\frac{\pi}{N}}\) which appears in (92). According to (98),
\[
A = \sqrt[D_D]{\frac{D_I}{N} e^{i\phi_A}}
\]

where
\[
\tan \phi_A = - \frac{3}{4\beta} i \left(\tilde{m} \cdot \pi + \frac{1}{N} ut\right).
\]

Therefore,
\[
\sqrt{\frac{\pi}{A}} = \sqrt[2]{\frac{D_I}{D_D^2}} e^{-i\phi_A}
\]

and
\[
\tilde{S}_m = \sqrt[2]{\frac{D_I}{D_D^2}} \frac{D_A}{N} e^{-i\phi_T}
\]

where
\[
\phi_T = \frac{D_I}{D_D} - \frac{\phi_A}{2}.
\]
To the order $u^2$,

$$\phi_n \approx \frac{3}{4\beta} u \left( \hat{m} \cdot \pi + \frac{J}{N} u \tau \right)$$  \hspace{1cm} (105)

and

$$\phi_n = u \left( \frac{J}{N} + \frac{3}{8} \left( \frac{1}{\beta} + \frac{1}{N} \right) \left( m \cdot \pi + \frac{J}{N} u \tau \right) \right).$$  \hspace{1cm} (106)

### Appendix E. Corrections to the eigenstates

In this appendix, we calculate the correction to the eigenstates resulting from the fact that for $u \neq 0$, the eigenstates of $H = -S_+ + u S_z^2$ are not identical to the eigenstates of $S_+$. Perturbation theory is justified only for $uN < 1$ because the typical spacing between eigenvalues of $S_z$ is about $\frac{1}{N}$ while the maximum of the perturbation $u S_z^2$ is about $\frac{u}{2}$. However, most of the results presented in sections 4 and 5 are applicable for $u < 1$ (where it is possible that $uN \gg 1$). This is understood in the framework of some aspect of restricted quantum-classical correspondence [46–49]. Let us denote the corrected eigenstates of $H$ by $|n\rangle'$. To the first order in $u$,

$$|n\rangle' = |n\rangle + u \sum_{k \neq n} |k\rangle \cdot \frac{\langle k | S_z^2 | n \rangle}{E_n - E_k}.$$  \hspace{1cm} (107)

The matrix element $\langle k | S_z^2 | n \rangle$ can be calculated easily by using (85) up to the second order in $\frac{1}{N}$. The result is

$$\langle k | S_z^2 | n \rangle \approx \frac{1}{16} \left[ d_2(n) \delta_{k,n+2} + d_{-2}(n) \delta_{k,n-2} + d_0(n) \delta_{k,n} \right]$$  \hspace{1cm} (108)

where

$$d_2(n) = 1 - \frac{4(n + 1)^2}{N^2} + \frac{2}{N},$$

$$d_{-2}(n) = 1 - \frac{4(n - 1)^2}{N^2} + \frac{2}{N},$$

$$d_0(n) = 2 + \frac{4}{N} - \frac{8n^2}{N^2}.$$  \hspace{1cm} (109)

The energy difference is (89) and therefore

$$|n\rangle' = |n\rangle + \frac{uN}{32} \left[ d_2(n) |n + 2\rangle - d_{-2}(n) |n - 2\rangle \right].$$  \hspace{1cm} (110)

For small $n$ and $u$ relevant for the present work, it agrees with the semiclassical result (159). We would like to expand the wavefunction in basis $|n\rangle'$. For this purpose, we define the expansion coefficients

$$c_n' = \sum_k c_k \langle k | n \rangle.$$  \hspace{1cm} (111)

According to (110),

$$c_n' \approx c_n + \frac{uN}{32} \left[ d_2(n) c_{n+2} - d_{-2}(n) c_{n-2} \right]$$  \hspace{1cm} (112)

while the coefficients $c_n$ are given by (59) ($\beta$ and $n_{\text{max}}$ are defined by (61) and (58), in section 4, $\beta = 1$ and $n_{\text{max}} = 0$, resulting in (26)). Therefore,

$$c_n' = \left( \frac{2\beta}{\pi N} \right)^\frac{1}{4} e^{-\frac{1}{N} \beta (n-n_{\text{max}})^2} \left[ 1 + \frac{uN}{32} d_2(n) e^{-\frac{4}{N} \beta (n-n_{\text{max}})} - \frac{uN}{32} d_{-2}(n) e^{\frac{4}{N} \beta (n-n_{\text{max}})} \right].$$  \hspace{1cm} (113)

In the leading order in $\frac{1}{N}$, $n \approx n_{\text{max}}$ and

$$c_n' = \left( \frac{2\beta}{\pi N} \right)^\frac{1}{4} e^{-\frac{1}{N} \beta (n-n_{\text{max}})^2} \left\{ 1 - \frac{u}{4} \left[ \beta \left( 1 - \frac{4n_{\text{max}}}{N^2} \right) (n - n_{\text{max}}) + 2 n_{\text{max}} N \right] \right\}.$$  \hspace{1cm} (114)

However, the coefficients $c_n'$ should be normalized and therefore the last term should be eliminated.

$$c_n' = \left( \frac{2\beta}{\pi N} \right)^\frac{1}{4} e^{-\frac{1}{N} \beta (n-n_{\text{max}})^2} \left\{ 1 - \frac{u}{4} \left[ \beta \left( 1 - \frac{4n_{\text{max}}}{N^2} \right) (n - n_{\text{max}}) \right] \right\}$$ and in the first order in $u$,
\[ e_n^\prime \approx e_n - \frac{2\beta}{\sqrt{\pi N}} \left[ \frac{1}{\beta} + \frac{1}{\beta} \left( 1 - \frac{4n_{\text{max}}^2}{N^2} \right) \left( n - n_{\text{max}} + \frac{1}{2} \right) \right]. \] (115)

The resulting correction to \( S_n \) is of the form

\[ S_n^{(1)} = \frac{\sqrt{2\beta}}{\sqrt{\pi N}} \cdot \frac{u}{\beta} \left( 1 - \frac{4n_{\text{max}}^2}{N^2} \right) \int_{-\infty}^{\infty} x e^{-\left( Ax^2 + Bx \right)} dx - \frac{u\beta}{4} \left( 1 - \frac{4n_{\text{max}}^2}{N^2} \right) S_n \] (116)

where \( A, B \) are presented explicitly in appendix D, equation (94), and \( x = n - n_{\text{max}} \). The integral can be solved by using

\[ \int_{-\infty}^{\infty} x e^{-\left( Ax^2 + Bx \right)} dx = \frac{-B}{2A} e^{-x^2} \sqrt{\frac{\pi}{A}} \] (117)

Therefore,

\[ \tilde{S}_m^{(1)} = -\frac{u}{2} \left[ \beta \left( 1 - \frac{4n_{\text{max}}^2}{N^2} \right) \left( \frac{1}{2} - \frac{B}{2A} \right) \right] S_m. \] (118)

Since \( A \) and \( B \) are of the same order of magnitude (see (94)), this correction is typically small.

Now we calculate the second-order correction for the case \( \beta = 1, n_{\text{max}} = 0 \) relevant for section 4.

\[ |n\rangle' = |n\rangle + u \sum_{k \neq n} \left\{ \frac{k}{E_n - E_k} \right\} S_n + u^2 \sum_{k \neq n} \left\{ \frac{k}{E_n - E_k} \right\} \left\{ \frac{k}{E_n - E_k} \right\} S_n \]

\[ - u^2 \sum_{k \neq n} \left\{ \frac{n}{E_n - E_k} \right\} \left\{ \frac{k}{E_n - E_k} \right\} S_n \frac{1}{4} - \frac{1}{2} u^2 |n\rangle \sum_{k \neq n} \left\{ \frac{n}{E_n - E_k} \right\} \left\{ \frac{k}{E_n - E_k} \right\} S_n \] (119)

According to (108),

\[ |n\rangle' = |n\rangle + \frac{uN}{32} \left[ d_z(n) |n + 2\rangle - d_{-z}(n) |n - 2\rangle \right] \]

\[ + \frac{u^2 N^2}{8 \cdot 16^2} \left[ d_z(n) d_z(n + 2) |n + 4\rangle + d_{-z}(n) d_{-z}(n - 2) |n - 4\rangle \right] \]

\[ + \frac{u^2 N^2}{4 \cdot 16^2} \left[ d_z(n) d_0(n + 2) |n + 2\rangle + d_{-z}(n) d_0(n - 2) |n - 2\rangle \right] \]

\[ - \frac{u^2 N^2}{4 \cdot 16^2} \left[ d_0(n) d_0(n) |n + 2\rangle + d_0(n) d_{-z}(n) |n - 2\rangle \right] \]

\[ - \frac{u^2 N^2}{8 \cdot 16^2} \left[ d_z(n) d_{-z}(n + 2) + d_{-z}(n) d_z(n - 2) \right] |n\rangle \]

(120)

\[ = |n\rangle + \frac{uN}{32} \left[ d_z(n) |n + 2\rangle - d_{-z}(n) |n - 2\rangle \right] \]

\[ + \frac{u^2 N^2}{8 \cdot 16^2} \left[ d_z(n) d_z(n + 2) |n + 4\rangle + d_{-z}(n) d_{-z}(n - 2) |n - 4\rangle \right] \]

\[ + \frac{u^2 N^2}{4 \cdot 16^2} \left[ \left[ d_z(n) \left( d_0(n + 2) - d_0(n) \right) \right] |n + 2\rangle + \left[ d_{-z}(n) \left( d_0(n - 2) - d_0(n) \right) \right] |n - 2\rangle \right] \]

\[ - \frac{u^2 N^2}{8 \cdot 16^2} \left[ d_z(n) d_{-z}(n + 2) + d_{-z}(n) d_z(n - 2) \right] |n\rangle. \]

namely,

\[ |n\rangle' \approx |n\rangle + \frac{uN}{32} \left[ 1 - \frac{4(n + 1)^2}{N^2} + \frac{2}{N} \right] |n + 2\rangle - \left( 1 - \frac{4(n - 1)^2}{N^2} + \frac{2}{N} \right) |n - 2\rangle \] (121)
\[+ \frac{u^2 N^2}{8 \cdot 16^2} \left(1 - \frac{4(n+1)^2}{N^2} - \frac{4(n+3)^2}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right) \left|n+4\right)^2\]
\[+ \frac{u^2 N^2}{8 \cdot 16^2} \left(1 - \frac{4(n-1)^2}{N^2} - \frac{4(n-3)^2}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right) \left|n-4\right)^2\]
\[- \frac{u^2 N^2}{4 \cdot 16^2} \left\{32(n+1) \left|n+2\right)^2 - \frac{32(n-1)}{N^2} \left|n-2\right)^2\right\}\]

Therefore,
\[c'_n \approx \left(1 - \frac{u^2 N^2}{4 \cdot 16^2} \left(1 - \frac{8(n^2+1)}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right)\right) c_n\]

(122)

\[+ \frac{uN}{32} \left(1 - \frac{4(n+1)^2}{N^2} + \frac{2}{N}\right) c_{n+2} - \left(1 - \frac{4(n-1)^2}{N^2} + \frac{2}{N}\right) c_{n-2}\]

(123)

\[+ \frac{u^2 N^2}{8 \cdot 16^2} \left(1 - \frac{4(n+1)^2}{N^2} - \frac{4(n+3)^2}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right) c_{n+4}\]

(124)

\[+ \frac{u^2 N^2}{8 \cdot 16^2} \left(1 - \frac{4(n-1)^2}{N^2} - \frac{4(n-3)^2}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right) c_{n-4}\]

(125)

\[= \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{u^2}{N^4}} \left\{1 + \frac{uN}{32} \left[1 - \frac{4(n+1)^2}{N^2} + \frac{2}{N}\right] e^{\frac{8n+4}{N}} - \left(1 - \frac{4(n-1)^2}{N^2} + \frac{2}{N}\right) e^{\frac{8n-4}{N}}\right\}\]

(126)

Expanding the exponent to the second order in \(\frac{1}{N}\) yields

\[c'_n \approx \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{u^2}{N^4}} \left\{1 - \frac{u^2 N^2}{4 \cdot 16^2} \left(1 - \frac{8(n^2+1)}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right) - \frac{uN}{4}\right\}\]

(127)

\[- \frac{u^2 N^2}{4 \cdot 16^2} \left[\frac{32(n+1)}{N^2} e^{-\frac{8n+4}{N}} - \frac{32(n-1)}{N^2} e^{-\frac{8n-4}{N}}\right] - \frac{u^2 N^2}{4 \cdot 16^2} \left[1 - \frac{8(n^2+1)}{N^2} + \frac{4}{N} + \frac{4}{N^2}\right]\]

Normalization dictates

\[c'_n \approx \left(\frac{2}{\pi N}\right)^{1/2} e^{-\frac{u^2}{N^4}} \left\{1 - \frac{uN}{4} + \frac{u^2 N}{4 \cdot 16^2} \left[1 - \frac{2n^2 - 2}{N}\right]\right\}\]

(128)
and
\[
c_i c_{i+1}' \approx \frac{\sqrt{2}}{\sqrt{\pi N}} e^{-\frac{2\pi^2 + 2\pi^2 + 1}{N}} \left[ 1 - \frac{u}{2} \left( n + \frac{1}{2} \right) \right] + \frac{u^2}{32} \left( -N + 2n^2 + 2n + 2n(n+1) \right).
\]
(129)

The second-order correction to \( \tilde{S}_m \) is of the form
\[
\tilde{S}_m^{(2)} \approx -\frac{u^2 N}{32} \tilde{S}_m + \frac{\sqrt{2}}{\sqrt{\pi N}} \cdot \frac{u^2}{8} \int \left( x^2 e^{-(Ax^2 + Bx)} + x e^{-(Ax^2 + Bx)} + \frac{1}{4} e^{-(Ax^2 + Bx)} \right) dx
\]
where \( A, B \) are defined in (94). The integral can be calculated by using (117) and
\[
\int_{-\infty}^{\infty} x^2 e^{-(Ax^2 + Bx)} dx = \frac{1}{2A} \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}}.
\]
(131)

Therefore,
\[
\tilde{S}_m^{(2)} \approx -\frac{u^2 N}{32} \tilde{S}_m + \frac{u^2}{16A} \left( 1 - \frac{B}{2} \right) \tilde{S}_m + \frac{u^2}{32} \tilde{S}_m
\]
\[
\approx -\frac{u^2 N}{32} \tilde{S}_m + \frac{u^2 N}{32} \left( 1 - \frac{1}{N} \right) \tilde{S}_m + \frac{u^2}{32} \tilde{S}_m
\]
(133)

and taking into account the corrected eigenstates up to second order in \( u \) results in multiplying \( \tilde{S}_m \) by a factor of
\[
1 - \frac{u^2}{2} \left[ 1 - N \left( 1 - \frac{2m}{N} \right) \left( \frac{1}{2} - \frac{B}{2A} \right) \right]
\]
(see (118)). This factor has an absolute value of 1 (in the first orders in \( u \)) and therefore it affects the dynamics by adding a phase \( \phi_s \) to \( \tilde{S}_m \), see equations (38) and (40).

Appendix F. Higher orders in WKB expansion

In this appendix, we calculate higher orders of the WKB expansion and show that its contribution to the spectrum is not important. In the WKB expansion [45], one makes the ansatz
\[
\psi \propto e^{iS(\varphi)/\hbar}
\]
(135)
where \( S(\varphi) \) is the series
\[
S(\varphi) = S_0(\varphi) + \hbar S_1(\varphi) + \hbar^2 S_2(\varphi) + \ldots
\]
(136)
and
\[
\frac{\partial S_0}{\partial \varphi} = S_x,
\]
\[
S_1 = \frac{1}{2} \ln S_x
\]
(138)
and
\[
\frac{\partial S_2}{\partial \varphi} = -\frac{1}{2} \left[ \left( \frac{\partial^2 S_0}{\partial \varphi^2} \right)^2 + \left( \frac{\partial S_1}{\partial \varphi} \right)^2 \right].
\]
(139)

Here, and in the rest of this work, \( \hbar = \frac{1}{N} \). In section 3, we used the Bohr–Sommerfeld quantization, namely, we demanded \( S(\varphi) = S(\varphi + 2\pi) = 2\pi n \hbar \) in order to find the spectrum. In the present work, \( \hbar = \frac{1}{N} \) is understood. There, we replaced \( S \) by \( S_0 \), which is justified only in the leading order in \( \hbar \). Finally, it turned out that the spectrum contains terms of higher orders of \( \hbar \) (16)–(79) and therefore, the effects of \( S_1 \) and \( S_2 \) should be taken into account as well. Fortunately, the contributions of \( S_1 \) and \( S_2 \) are negligible as described in what follows. \( S_x \) is periodic in \( \varphi \) so that \( S_1 \) does not contribute to the spectrum.

In order to find the contribution of \( S_x \), we first calculate the derivatives of \( S_1 \):
\[
\frac{\partial S_1}{\partial \varphi} = \frac{1}{2S_x} \frac{\partial S_x}{\partial \varphi}
\]
(140)
and
\[ \frac{\partial^2 S_1}{\partial \phi^2} = -\frac{1}{2S_x^2} \left( \frac{\partial S_x}{\partial \phi} \right)^2 + \frac{1}{2S_x} \frac{\partial^2 S_x}{\partial \phi^2}. \] (141)

Hence,
\[ \frac{\partial S_2}{\partial \phi} = -\frac{1}{4S_x^2}\left[ -\frac{1}{2S_x} \left( \frac{\partial S_x}{\partial \phi} \right)^2 + \frac{\partial^2 S_x}{\partial \phi^2} \right]. \] (142)

In what follows, all calculations are performed to the order $\hbar^2$. We substitute $S_x$ of (78) and find
\[ \frac{\partial S_x}{\partial \phi} = \left( \frac{1}{4} - H^2 \right) \left[ u \sin (2\phi) + 8u^2H \sin^2 \phi \cos \phi \right]. \] (143)

and
\[ \frac{\partial^2 S_x}{\partial \phi^2} = \left( \frac{1}{4} - H^2 \right) \left[ 2u \cos (2\phi) + 8u^2H \left( 3 \sin^2 \phi \cos^2 \phi - \sin^4 \phi \right) \right]. \] (144)

Therefore, to the second order in $u$,
\[ \frac{\partial S_2}{\partial \phi} = -\frac{1}{4S_x^2} \left( \frac{1}{4} - H^2 \right) \left[ -\frac{u^2}{2S_x} \left( \frac{1}{4} - H^2 \right) \sin^2 (2\phi) + 2u \cos (2\phi) + 8u^2H \left( 3 \sin^2 \phi \cos^2 \phi - \sin^4 \phi \right) \right]. \] (145)

Assuming $u \ll H$ we find \[ \frac{1}{S_x} \approx \frac{1}{\hbar^2} \left( 1 + \frac{2}{H^2} \left( \frac{1}{4} - H^2 \right) \sin^2 \phi \right), \]
leading to
\[ \frac{\partial S_2}{\partial \phi} = -\frac{1}{4H^2} \left( \frac{1}{4} - H^2 \right) \left[ \frac{u^2}{2H} \left( \frac{1}{4} - H^2 \right) \sin^2 (2\phi) + 8 \cos (2\phi) \sin^2 \phi \right] \]
\[ + 2u \cos (2\phi) + 8u^2H \left( 3 \sin^2 \phi \cos^2 \phi - \sin^4 \phi \right). \] (147)

Therefore,
\[ \delta = \frac{H^2}{2\pi} \left( S_2(\phi + 2\pi) - S_2(\phi) \right) = \frac{-3u^2}{16H^2N^2} \left( \frac{1}{4} - H^2 \right)^2. \] (148)

This should be added to the right-hand side of (79), resulting in a contribution of
\[ \delta' = \frac{3u^2N}{16n^4} \left( \frac{1}{4} - \frac{n^2}{N^2} \right)^2. \] (149)

to the spectrum (15). The approximation leading to this term is not valid for small $n$ (see (13)) where $H$ is of order $u$. To find an estimate for the correction in this regime we repeat the calculation for $I = 0$. If $n = 0$,
\[ S_x \approx \frac{-u}{8} + \frac{1}{4} u \sin^2 \phi \] (150)

and the derivatives are \[ \frac{\partial S_x}{\partial \phi} = \frac{1}{4} u \sin (2\phi), \]
\[ \frac{\partial^2 S_x}{\partial \phi^2} = \frac{1}{2} u \cos (2\phi). \]

Therefore, to the second order in $u$,
\[ \frac{\partial S_2}{\partial \phi} = \frac{-1}{4u} \left[ -\frac{1}{8} + \frac{1}{4} \sin^2 \phi \right]^2 \left[ \frac{-1}{2} \left( -\frac{1}{8} + \frac{1}{4} \sin^2 \phi \right) \left( \frac{1}{4} \sin (2\phi) \right)^2 + \frac{1}{2} \cos (2\phi) \right] \]
\[ = \frac{4}{u} \cos (2\phi) \left[ \sin^2 (2\phi) - 2 \cos^2 (2\phi) \right]. \] (152)

This expression is antisymmetric with respect to $2\phi = \frac{\pi}{2} + \alpha \rightarrow 2\phi = \frac{\pi}{2} - \alpha$. Therefore the integral for $S_2$ vanishes. The preceding estimates are only for part of the spectrum. Therefore, we turn to a numerical estimate.

In figure 8, we present the numerically calculated deviations in the spectrum originating from $S_2$ and show that it is small for the parameters of figures 4, 5. The calculation of the spectrum presented in figure 8 was carried out by iterations as described in what follows:

1. For each $n$, $S_n(\phi)$ was calculated according to (73) where $H$ is replaced by the spectrum $E_n^{(2)}$ of (15).
2. $S_2(\phi)$ was found by substitution of $S_n$ in (142) and integration over $\phi$. 


3. The term \( \frac{\hbar}{2\pi} \left( S_z (\varphi_1 + 2\pi) - S_z (\varphi) \right) \) was added to the RHS of (79), which we solved numerically to obtain a corrected spectrum \( \tilde{E}_n^{(2)} \).

4. We repeated steps 1–3 where \( H \) in \( S_x \) is replaced by \( \tilde{E}_n^{(2)} \) until conversion.

5. We multiplied the resulting spectrum by \( JN \) and added the constant \( C_N \) to be able to compare with the exact BH spectrum.

**Appendix G. Semiclassical eigenstates**

In this appendix, we calculate the eigenstates in the semiclassical approximation

\[
|n'\rangle = \frac{1}{\sqrt{2\pi}} e^{ikn\varphi/\hbar}
\]

and show that it can be approximated by the eigenstates of \( S_x \) as was done in sections 4 and 5. According to (137) and (74), in the first order in \( u \),

\[
S_0 = \frac{k}{N} \varphi - \frac{u}{4} \left[ 1 - \frac{k^2}{N^2} \right] \sin (2\varphi).
\]

The eigenstates of \( S_x \) (obtained by substituting (154) with \( u = 0 \) in (153)) are \( e^{iu\varphi} \). These are denoted by \( |n\rangle \) of (17). The overlap between \( |k\rangle \) and \( |n\rangle \) is

\[
\langle n | k \rangle' = \frac{1}{2\pi} \int_0^{2\pi} e^{i\left[ k(n-\bar{C}_2)\sin(2\varphi) \right]} d\varphi
\]

where \( \bar{C}_2 = -\frac{N}{4} \left[ 1 - \frac{1}{N^2} \right] \). In order to solve the integral, we expand to a series of Bessel functions:

\[
e^{-\bar{C}_2 \sin(2\varphi)} = \sum_{l=0}^{\infty} J_l \left( \bar{C}_2 \right) e^{-2il\varphi} + \sum_{l=1}^{\infty} (-1)^l J_l \left( \bar{C}_2 \right) e^{2il\varphi}
\]

and obtain

\[
\langle n | (n+2l) \rangle' = J_l \left( \bar{C}_2 \right)
\]
\[ \langle n \rangle \langle n - 2l \rangle' = (-1)^l J_l \left( \frac{C_2}{2} \right) \]  

for positive integer \( l \). Since \( C_2 \) is small, the Bessel functions can be approximated by \( J_l \left( \frac{C_2}{2} \right) \sim \frac{1}{l!} \left( \frac{C_2}{2} \right)^l \), so that the overlap is substantial only for small values of \( l \) and

\[ |n\rangle' \approx |n\rangle + \frac{1}{2} C_2^2 \left( |n+2\rangle - |n-2\rangle \right). \]

This result reduces to (110) for small \( n \) and contribute only small corrections to the dynamics, as was shown in appendix E.

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