Faddeev-Jackiw quantization of spin-2 field

M. Leclerc*
Section of Astrophysics and Astronomy, Department of Physics,
University of Athens, Greece

December 20, 2006

Abstract
We apply the Faddeev-Jackiw method to the Hamiltonian analysis of the massless spin-two field. As expected, the reduced Hamiltonian contains only the traceless-transverse tensor, while some, but not all of the non-propagating components are determined by the constraints of the theory. In particular, it is concluded that no gauge choice can be imposed on the fields such that only the propagating modes remain in the theory, meaning that for the spin-2 field there is no direct analogue to the Coulomb gauge of electromagnetism. Implications for General Relativity are discussed.

1 Introduction

The Faddeev-Jackiw formalism [1] for constrained Hamiltonian systems is often the quicker alternative to the conventional Dirac method [2], because the introduction of unnecessary momentum variables is avoided and thus the number of constraints is reduced right from the outset. This holds in particular for first order theories, like Einstein-Cartan or Dirac theory. Here, we will take advantage of yet another feature of this formalism, namely the fact that the physical degrees of freedom are clearly identified without the need to impose a gauge fixation, and Poisson brackets are introduced only for those degrees of freedom. The idea behind the Dirac method was in a certain sense just the opposite, namely to treat all fields identically, at least initially, and to reduce the Hamiltonian only with respect to the second class constraints. First class constraints can be directly imposed on the states without leading to inconsistencies. In this way, one can pass to the quantum theory without explicitly solving the constraints. While this is particularly convenient in theories where the constraints cannot be solved, as is the case, e.g., in canonical General Relativity [3], it is nevertheless not very instructive as far as the identification of the physical modes, and thus of the particle contents is concerned. The theory, in the Dirac approach, can only be reduced to the physical modes by explicitly fixing the gauge.

On the other hand, in the Faddeev-Jackiw approach, the constraints are explicitly solved (whenever possible) and the Hamiltonian is reduced to the physical degrees of freedom without choosing a gauge. This is very apparent in the case of the electromagnetic field presented in [1], where the Faddeev-Jackiw method leads straightforwardly to the Coulomb gauge Hamiltonian, which contains only the transverse spin-one field. This shows that the Coulomb gauge is actually the natural choice, albeit not always the most convenient one. This holds as long as we perform the Hamiltonian analysis with respect to a spacelike hypersurface. Other gauges appear, e.g., for a lightcone quantization.

* mleclerc@phys.uoa.gr
That only the physical degrees of freedom are quantized in the Coulomb gauge was known long before the advent of theoretical treatments of constrained Hamiltonian systems. However, in the case of the spin-two field, the situation is not quite as clear. What one would like to see is a gauge choice that leads to a wave equation for the traceless-transverse (3-dimensional) tensor, while the remaining 8 components of the symmetric (4-dimensional) tensor are either eliminated by a Gauss type law (as is $A_0$ in the Coulomb gauge) or put to zero by a convenient gauge choice (as is the longitudinal part of $A_\mu$ in the Coulomb gauge). While this should be possible on the base of a counting argument, the fact is that this has never been completely accomplished in practice. For instance, in the appendix of \cite{4}, we have shown how in the field equations of linearized General Relativity one can make a gauge choice that allows for the elimination of four components by a Gauss type law $\Delta \psi_{0i} = 0$, $i = 0, 1, 2, 3$, where $\psi_{ik}$ is the trace-reversed metric perturbation, while the remaining tensor is transverse, $\psi^{\mu\nu} = 0$, $(\mu, \nu = 1, 2, 3)$, and satisfies a wave equation. The remaining gauge freedom, however, is too restricted to eliminate yet another component of $\psi^{\mu\nu}$ in order to reduce it to the propagating traceless-transverse spin-two field. This can only be done for explicit (e.g., plane) wave solutions. While this is sufficient to establish the nature of the field and to exclude, e.g., the existence of an additional scalar particle, it is nevertheless not really satisfying. In particular, one might wonder whether this is a result of a bad gauge choice, or whether it is truly impossible to exclude all 8 non-propagating components of $h_{ik}$. The Faddeev-Jackiw analysis should shed some light on the issue.

\section{Faddeev-Jackiw reduction}

We start from the special relativistic Fierz-Pauli Lagrangian, which is also the Lagrangian of General Relativity to second order in the metric perturbation $g_{ik} = \eta_{ik} + h_{ik}$,

$$L = \frac{1}{4} h^{il} h_{il}^{\phantom{il},k} - \frac{1}{2} h^{il} h_{kl}^{\phantom{kl},k} - \frac{1}{4} h^{mn} h_{ik}^{\phantom{ik},k} h_{kl}^{\phantom{kl},l} + \frac{1}{2} h^{km} h_{ik}^{\phantom{ik},k} h_{im}^{\phantom{im},m}, \quad (1)$$

where latin subscripts run from 0 to 3 and greek ones from 1 to 3. Our metric convention is $\eta_{ik} = \text{diag}(1, -1, -1, -1)$. Further, we will use the notation $\Box = \partial_\mu \partial^\mu$ and $\Delta = \partial_\mu \partial^{\mu}$. Whenever we write down a Lagrangian $L$ or a Hamiltonian $H$, an integration over three dimensional space is understood. Also, total time derivatives as well as two-dimensional surface terms are added or eliminated without explicitly renaming $L$ (or $H$). We choose as independent fields the components $h_{0\mu}, h_{00}$ and $h_{\mu\nu}$. The corresponding momenta are found in the form

$$p^{00} = -\frac{1}{2} h^{00} \dot{\nu}^{\nu}, \quad p^{0\mu} = -h^{0\mu} \dot{\nu}^{\nu} + \frac{1}{2} h_{\mu}^{\phantom{\mu},0}, \quad (2)$$

$$p^{\mu\nu} = \frac{1}{2} h^{\mu\nu} - \frac{1}{2} \dot{h}_{\mu\nu}^{\phantom{\mu\nu}} + \frac{1}{2} h^\lambda \eta^{\mu\nu}, \quad (3)$$

where $h = h^{\mu}_{\mu}$ is the trace of the 3-dimensional tensor, and the dot denotes partial time derivatives. In the Dirac formalism, the relations \ref{2} would be considered as constraints. Here, they simply indicate that there are no momentum variables conjugated to $h_{0\mu}$ and $h_{00}$, and the symbols $p^{00}$ and $p^{0\mu}$ are only used as an abbreviation for the expressions on the right hand side. We introduce the Hamiltonian

$$\hat{H} = p^{00} \dot{h}_{00} + p^{0\mu} \dot{h}_{0\mu} + p^{\mu\nu} \dot{h}_{\mu\nu} - L, \quad (4)$$

where the velocities $\dot{h}_{\mu\nu}$. are eliminated with the help of \ref{4}. (Obviously, the remaining velocities, $\dot{h}_{0\mu}, \dot{h}_{00}$, cancel out.) The first order Lagrangian therefore is of the form

$$L = p^{00} \dot{h}_{00} + p^{0\mu} \dot{h}_{0\mu} + p^{\mu\nu} \dot{h}_{\mu\nu} - \hat{H}(p_{\mu\nu}, h_{\mu\nu}, h_{0\mu}, h_{00}). \quad (5)$$
Using the explicit expressions for $p_{00}$ and $p_{0\mu}$, performing partial integrations and omitting surface terms and total time derivatives, we can alternatively write

$$L = (p_{\mu\nu} - h_{0(\mu,\nu)} + \frac{1}{2} \eta_{\mu\nu} h_{0\lambda,\lambda}) \dot{h}_{\mu\nu} - \dot{H}(p_{\mu\nu}, h_{\mu\nu}, h_{0\mu}, h_{00})$$

$$= \pi_{\mu\nu} \dot{h}_{\mu\nu} - \dot{H}(\pi_{\mu\nu}, h_{\mu\nu}, h_{0\lambda}, h_{00}),$$

where we have defined the new momenta

$$\pi_{\mu\nu} = p_{\mu\nu} - h_{0(\mu,\nu)} + \frac{1}{2} \eta_{\mu\nu} h_{0\lambda,\lambda}.$$  \hspace{1cm} (6)

Explicitely, the Hamiltonian has the form

$$\dot{H} = \pi_{\mu\nu} \pi_{\mu\nu} - \frac{1}{2} \pi_{\mu\nu} \pi_{\mu\nu} - \frac{1}{4} \eta_{\mu\nu} \pi_{\mu\nu} + \frac{1}{4} \eta_{\mu\nu} \pi_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \pi_{\mu\nu}$$

$$- h_{0\mu}(2\pi_{\mu\nu},_{\nu}) - h_{00}(-\frac{1}{2} \eta_{\mu\nu},_{\mu\nu}, + \frac{1}{2} \eta_{\mu\nu})$$  \hspace{1cm} (8)

The variables $h_{0\mu}$ and $h_{00}$ remain undetermined and play the role of Lagrange multipliers. Changing the notation from $h_{0\mu}$ to $\lambda_{\mu}$ and from $h_{00}$ to $\lambda_{0}$, we can write the Lagrangian in the form

$$L = \pi_{\mu\nu} \dot{h}_{\mu\nu} - H(\pi_{\mu\nu}, h_{\mu\nu}) - \lambda_{\mu} \Phi_{\mu} - \lambda_{0} \Phi_{0},$$

where $H(\pi_{\mu\nu}, h_{\mu\nu})$ is the reduced Hamiltonian, given by the first line in (8), and $\Phi_{\mu}$ and $\Phi_{0}$ are the constraints found in the second line of (8), namely

$$\pi_{\mu\nu} = 0, \ h_{\mu\nu},_{\mu\nu} - \Delta h = 0.$$  \hspace{1cm} (10)

In the Dirac approach, those constraints, together with the constraints (2) (which are easily shown to be all first class, see also [5]) are imposed on the states of the quantum theory, and we could essentially stop the analysis at this stage (after checking that no tertiary constraints arise). This, however, does not really illuminate the situation as far as the physical degrees of freedom are concerned. On the other hand, the Faddeev-Jackiw procedure consists in solving the constraints, thereby further reducing the Hamiltonian.

Let us solve the constraints, at least formally. First, we see that the momentum $\pi_{\mu\nu}$ is transverse. In contrast to the vector case, however, the decomposition into a transverse and a longitudinal part of a symmetric tensor is not unique. It turns out though, that the explicit solution is not required for the moment. We simply solve the constraint by writing

$$\pi_{\mu\nu} = \tilde{\pi}_{\mu\nu},$$

where $\tilde{\pi}_{\mu\nu}$ is transverse. The second constraint can be solved for the trace $h$. Defining $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{3} \eta_{\mu\nu} h$, which is the traceless part of $h_{\mu\nu}$, we find $\Delta h = (3/2)\tilde{h}_{\mu\nu,\mu\nu}$, or

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{2} \frac{\eta_{\mu\nu}}{\Delta} \nabla_{\alpha} \nabla_{\beta} \tilde{h}_{\alpha\beta},$$

where $\nabla_{\mu} \equiv \partial_{\mu}$ denotes partial differentiation. The Lagrangian reduces to

$$L = \tilde{\pi}_{\mu\nu} \dot{\tilde{h}}_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \nabla_{\alpha} \nabla_{\beta} \tilde{h}_{\alpha\beta}$$

$$- \left[ \tilde{\pi}_{\mu\nu} \tilde{\pi}_{\mu\nu} - \frac{1}{4} \tilde{h}_{\mu\nu} \tilde{h}_{\mu\nu} - \frac{1}{2} \tilde{h}_{\mu\lambda} \tilde{h}_{\lambda\alpha} + \frac{1}{4} \tilde{h}_{\mu\nu} \nabla_{\alpha} \nabla_{\beta} \tilde{h}_{\alpha\beta} \right].$$

(12)
We have yet to bring the kinematical term into the canonical form. Obviously, $\pi_{\mu\nu}$ cannot be conjugated to $\dot{h}_{\mu\nu}$, because the number of independent components does not match. Fortunately, the reduction is straightforward. First, we define the traceless-transverse part of a symmetric tensor by

$$\tag{13} \TT_{\mu\nu} = A_{\mu\nu} - \frac{\nabla_\mu \nabla_\nu}{\Delta} A_{\lambda\nu} - \frac{\nabla_\nu \nabla_\lambda}{\Delta} A_{\lambda\mu} + \frac{1}{2} \eta_{\mu\nu} \frac{\nabla_\alpha \nabla_\beta}{\Delta} A^{\alpha\beta} + \frac{1}{2} \frac{\nabla_\mu}{\Delta} A + \frac{1}{2} \frac{\nabla_\nu}{\Delta} A + \frac{1}{2} \frac{\nabla_\alpha}{\Delta} A^{\alpha\beta} - \frac{1}{2} \frac{\eta_{\mu\nu}}{\Delta} A.$$

This expression, in contrast to the merely transverse part, is unique, as is easily shown by considering a general linear combination of the terms occurring in (12) and requiring $\TT_{\mu\nu} = 0$ and $\TT A = 0$. In particular, for $\pi_{\mu\nu}$, which is already transverse, we have

$$\tag{14} \TT \pi_{\mu\nu} = \pi_{\mu\nu} + \frac{1}{2} \frac{\nabla_\lambda}{\Delta} \pi_{\lambda\nu} - \frac{1}{2} \eta_{\mu\nu} \pi.$$

It is not hard to show that, up to a divergence, we have the following identity

$$\tag{15} \TT \pi_{\mu\nu} (\dot{h} + \frac{1}{2} \eta_{\mu\nu} \frac{\nabla_\alpha}{\Delta} \dot{h}_{\alpha\beta} ) = \pi_{\mu\nu} (\dot{h} + \frac{1}{2} \eta_{\mu\nu} \frac{\nabla_\alpha}{\Delta} \dot{h}_{\alpha\beta} ).$$

On the other hand, we also have $\TT \pi_{\mu\nu} A^{\mu\nu} = \pi_{\mu\nu} \TT A^{\mu\nu} = \TT \pi_{\mu\nu} \TT A^{\mu\nu}$ for any symmetric tensor $A^{\mu\nu}$ (up to divergence terms), and moreover, we have $\TT \dot{h}_{\mu\nu} = \TT (\dot{h} + \frac{1}{2} \eta_{\mu\nu} \nabla_\alpha \dot{h}_{\alpha\beta} )$. As a result, the kinetic term in (12) is equivalent to $\TT \pi_{\mu\nu} \TT \dot{h}_{\mu\nu}$, and the Lagrangian can be written as

$$L = \TT \pi_{\mu\nu} \TT \dot{h}_{\mu\nu} - H(\TT \pi_{\mu\nu}, \TT \dot{h}_{\mu\nu}, \lambda),$$

where $\lambda$ denotes collectively the remaining parts of $\TT \pi_{\mu\nu}$ and $\TT \dot{h}_{\mu\nu}$. Those parts could lead to additional constraints and to a further reduction of the Hamiltonian. It turns out, however, that the Hamiltonian does not depend on the non-traceless-transverse components of $\dot{h}_{\mu\nu}$ and $\pi_{\mu\nu}$. Indeed, using (13) as well as

$$\tag{17} \TT \dot{h}_{\mu\nu} = \dot{h}_{\mu\nu} - \frac{\nabla_\lambda}{\Delta} \delta_{\lambda\nu} h_{\mu\lambda} - \frac{\nabla_\nu}{\Delta} h_{\lambda\mu} + \frac{1}{2} \eta_{\mu\nu} \frac{\nabla_\alpha}{\Delta} \dot{h}_{\alpha\beta} + \frac{1}{2} \frac{\nabla_\mu}{\Delta} \frac{\nabla_\nu}{\Delta} \dot{h}_{\alpha\beta},$$

where $\dot{h}_{\mu\nu}$ is already traceless, we find, up to surface terms, the simple expression

$$H = \TT \pi_{\mu\nu} \TT \pi_{\mu\nu} - \frac{1}{4} \TT \dot{h}_{\mu\nu} \TT \dot{h}_{\mu\nu} \lambda.$$

Note that one can also directly check that (13) depends only on the traceless-transverse part of $h_{\mu\nu}$ (except for the expressions contained in the constraints), by checking the invariance of the $h_{\mu\nu}$-terms under $\delta h_{\mu\nu} = \xi_{\mu\nu} + \xi_{\nu\mu}$. The Poisson brackets (at equal times) can be read off from (16) and read

$$\tag{19} [\TT \pi_{\mu\nu}(\vec{x}), \TT h_{\alpha\beta}(\vec{y})] = -\frac{1}{2} (\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha} \delta^{\mu}_{\beta}) \delta(\vec{x} - \vec{y}).$$

The equations of motion derived from (13) are

$$- \TT \pi_{\mu\nu} = [H, \TT \pi_{\mu\nu}] = \frac{1}{2} \Delta \TT h_{\mu\nu}, \quad -\TT \dot{h}_{\mu\nu} = [H, \TT \dot{h}_{\mu\nu}] = -2 \TT \pi_{\mu\nu}. $$
Putting this together leads to the expected equation for the propagating traceless-transverse tensor

\[ \Box \mathcal{T} h^{\mu\nu} = 0. \]  
(21)

As expected, the Faddeev-Jackiw method leads straightforwardly to the identification of the propagating field modes. From (18) and (19), one can directly pass over to the quantum theory. Let us also note that our results are in accordance with those obtained in [5] based on the Dirac method, after imposing convenient gauge conditions and evaluating the corresponding Dirac brackets. It should be noted, however, that in our treatment, no gauge conditions have been imposed.

3 Discussion

While the above procedure was successful inasmuch the propagating components are concerned, a simple counting argument shows that we are missing something concerning the remaining field components. Indeed, according to (8), \( h_{00} \) and \( h_{0\mu} \) remain undetermined. Obviously, this is a result of the four gauge degrees of freedom of the theory (see below) and is a desired feature. Next, the two independent components of \( \mathcal{T} h^{\mu\nu} \) satisfy the dynamical equation (21). Thus, the remaining 4 components, which can be chosen as \( h^{\lambda\mu}_{\ ,\mu} \) and \( h_{\lambda\mu} \), should be completely determined by the constraints (10). Those constraints, however, are not independent of each other and satisfy \( \Phi_i,\delta \Phi_i = 0 \), which is a result of the linearized Bianchi identity, as we will see below. Thus, we have only three independent constraints for four fields, which means that one field component remains undetermined.

To show this more explicitly, consider the field equations derived from the Lagrangian (1),

\[ 0 = G_{00} = \frac{1}{2} \Delta h - h^{\mu\nu}_{\ ,\mu,\nu} \]  
(22)

\[ 0 = G_{0\mu} = \frac{1}{2} \left( -\Delta h_{0\mu} + h^\nu_{\ ,\nu,\mu} + h_{0\nu,\mu} - \dot{h}_{0\mu} \right) \]  
(23)

\[ 0 = G_{\mu\nu} = -\frac{1}{2} \Box h_{\mu\nu} + \frac{1}{2} \left( h^{\lambda}_{\ ,\lambda,\mu} + h^\lambda_{\ ,\lambda,\nu} - h_{\nu,\mu} \right) - \frac{1}{2} \eta_{\mu\nu} \left( -\Box h + h^{\lambda}_{\ ,\lambda,\nu} \right) \]  
(24)

where \( G_{ik} \) corresponds to the linearized Einstein tensor. Before we continue, let us recall that the above equations are invariant under the gauge transformation

\[ \delta h_{ik} = \xi_{i,k} + \xi_{k,i}, \]  
(25)

which can be interpreted as the residual of the diffeomorphism invariance of General Relativity after linearization. In the Dirac approach, where Poisson brackets are defined for the complete set of fields, e.g.,

\[ [\pi_{\mu\nu}(x), h^{\alpha\beta}(y)] = -\delta^{[\alpha}_{\mu} \delta^{\beta]}_{\nu} \delta(x - y), \]

this symmetry is reflected in the occurrence of the constraints (10). For instance, we have

\[ \int \xi_{\mu} 2\pi_{\mu\nu}d^3x, h_{\alpha\beta} = \xi_{\alpha,\beta} + \xi_{\beta,\alpha} = -\delta h_{\alpha\beta}, \text{ while } \left[ \int \xi^{\mu}_{\nu}(-h^{\mu\nu}_{\ ,\nu} + \Delta h)d^3x, \pi_{\alpha\beta} \right] = -\xi^{\mu}_{\alpha,\beta} + \eta_{\alpha\beta} \Delta \xi^{\mu}_{0}, \]  
which is (on-shell) equal to \( \delta\pi_{\alpha\beta} \) derived from (20) using the explicit expression for \( \pi_{\alpha\beta} \) in terms of the fields, equations (3) and (7).

Let us return to the field equations. First, from the explicit expression for \( \pi_{\alpha\beta} \), we recognize in (22) and (23) the constraints (10). Those equations do not contain second time derivatives. Next, with the projection given in (19), we take the traceless-transverse part of (24) and find

\[ 0 = \mathcal{T} G_{\mu\nu} = -\frac{1}{2} \Box \mathcal{T} h_{\mu\nu}, \]  
(26)
which is the equivalent to our Hamiltonian equations, see (21). This leaves us with the longitudinal and trace components of $G_{\mu
u}$. However, as is well known, the field equations satisfy the identity $G^i_{\phantom{i}ik} = 0$ (linearized, contracted Bianchi identity) and therefore, we find $G_{\mu\nu} = -\dot{\epsilon}^{\mu\nu}_{\phantom{\mu\nu}i\lambda} \lambda$, which is zero as a result of (23). Thus, $G_{\mu\nu}$ is already transverse as a consequence of the constraint equations. Finally, we take the trace $G = G_{\mu}^\mu$ and find

$$ G = 0 = G_{00} + \dot{h} + \Delta h_{00} - 2h_{0,0,\lambda}^{0\lambda}, $$

which gives us one more equation

$$ \dot{h} + \Delta h_{00} - 2h_{0,0,\lambda}^{0\lambda} = 0. $$

This equation did not appear in the previous section. Being the only equation that involves $h_{00}$, it is certainly independent of the remaining ones. Moreover, it is immediately clear from (27) that it is gauge invariant, since both $G$ and $G_{00}$ are gauge invariant. It looks like a dynamical equation, since it contains second order time derivatives.

To make thinks very explicit, let us choose the gauge $h_{00} = h_{0\mu} = 0$. It is clear from (24) that the gauge is now quite fixed (although transformations $\xi^\mu$ with $\dot{\xi}^\mu = 0$ are still allowed, but those leave $\dot{\dot{h}}$ invariant). We then have the constraints $G_{00} = C^{00}_{\phantom{00}0} = 0$, which represent four relations between the four non-traceless-transverse components of $h_{\mu\nu}$ (which are not independent though, since $G_{0\mu}^{0\nu} = 0$). Further, we have $\tau T G_{\mu\nu} = 0$, as well as a remaining equation $\dot{h} = 0$, which was missing in our Hamiltonian analysis. As mentioned above, this equation cannot be gauged away. The fact that the constraints are not independent is obviously the reason why four constraints and four gauge choices cannot determine completely the 8 non-propagating components of $h_{ik}$, as long as we miss the equation $\dot{h} = 0$.

Note that in the Dirac approach, where the constraints are not eliminated but rather imposed on the states of the quantum theory, we get the missing relation as dynamical equation of motion from $[H, \pi] = -\Delta h_{00} = G_{00}$, with the Hamiltonian $H(\pi_{\mu\nu}, h_{\mu\nu})$ from (20). So, neither the Lagrangian field equations, which contain $\ddot{h}$, nor the Dirac analysis, which leads to an additional dynamical equation, actually reveal the fact that $\dot{h}$ is a non-propagating field. In other words, it is not obvious in those approaches that our Lagrangian describes only a spin-two field, and not, e.g., an additional scalar particle.

On the other hand, the Faddeev-Jackiw formalism shows straightforwardly the true particle contents of the theory, but the price we have to pay is that one of the remaining field equations is mysteriously lost. One might think that we did something wrong during our calculations. This is not the case, however. The fact is that it is not possible to get the missing equation in the Faddeev-Jackiw approach for the following reason. The formalism consists in reducing the Hamiltonian to a form which contains only the truly dynamical fields, and only for those fields Poisson brackets, and thus, ultimately, commutators, are introduced. In our case, this is undoubtlessly the traceless-transverse tensor $\tau T h_{\mu\nu}$ describing the spin-two particle. Thus, the remaining components of $h_{ik}$ cannot be determined by dynamical equations of the form $[H, A] = -\dot{A}$. The only way we can determine those components is therefore by the constraints of the theory. It is clear, however, that an equation containing second time derivatives (e.g., $\dot{h}$) cannot appear in the form of a constraint, because it necessarily contains velocities (e.g., $\dot{\pi}$). Thus, there is no way, in the framework of the Faddeev-Jackiw formalism, to obtain equation (28), which is seemingly dynamical, but nevertheless relates only non-propagating fields.

The fact that the Faddeev-Jackiw method sometimes leads to a loss of information has been demonstrated by García and Fons in (6). The specific case where this information represents a dynamical equation (and not a constraint) is a subcase of what is referred to as a type 2 problem in (6). The authors relate the failure of the Faddeev-Jackiw method to the occurrence of so-called ineffective constraints. In our case, the situation seems different though. We do not go into the mathematical details of the analysis. Instead, we wish to point out an interesting conjecture put forth in (6). Namely, the authors believe that, when the Dirac conjecture fails,
then the Faddeev-Jackiw method must fail too. Thus, if the Dirac conjecture fails in the spin-two theory, then this not only provides an additional support for the García-Pons conjecture, but moreover, it shows that there are actually natural counterexamples to Dirac’s conjecture, as opposed to the pathological theories whose only purpose is to make the conjecture fail.

It is clear that the loss of information was the result of omitting the constraints in the Hamiltonian \([8]\). By doing though, the remaining Hamiltonian depends only on the traceless-transverse fields, and we loose the equation \([H, \pi] = -\Delta \eta_{00}\) (on the constraint surface). This shows in particular that it is not allowed to replace \(h_{00}\) and \(h_{0\mu}\) in \([5]\) by arbitrary multipliers, i.e., to couple the secondary (in the Dirac scheme) constraints by Lagrange multipliers. Dirac’s conjecture thus seems to be violated, but in a somewhat different sense.

For a detailed investigation, one will have to check that the generator formed as a linear combination (with independent multipliers) of the eight constraints \([2]\) and \([10]\) leaves invariant the physical quantities. Physical quantities are those that are invariant under the gauge transformation \([24]\), in particular the tensor components \(G_{00}, G_{0\mu}\) and \(G_{\mu\nu}\). For a detailed discussion of Dirac’s conjecture in the context of gauge theories, we refer to \([7]\). We find indeed that equation \([27]\) (expressed in terms of \(\pi\)) is not invariant under the action of this generator. (Note that the remaining equations are trivially invariant: The constraint equations are first class, and thus commute with each term of the generator separately, while the propagating equation involves only traceless-transverse components, which are not contained in the constraints, and thus neither in the generator.)

The strange thing, however, is that \([27]\) it is not even invariant under the action of the generator formed merely with the primary constraints \([2]\). The reason is that the only constraint that involves \(\pi_{\mu}\), and thus the only constraint that can induce a transformation on \(h_{0\mu}\), does not generate the transformation dictated by \([27]\). On the other hand, since the propagating fields \(\tau^{\mu}_{\mu\nu}\) and \(\tau_{\nu\mu}\) are indeed invariant under the full generator, in a certain sense, Dirac’s conjecture holds true. In contrast to electrodynamics, however, where both \(\pi_{\mu}\) and \(F_{\mu\nu}\) are invariant \([7]\), and thus the complete tensor \(F_{\mu\nu}\), in our case, there are quantities (albeit non-propagating ones) that are not invariant under the generator formed from the constraints although they are invariant under \([27]\) when expressed in terms of configuration space variables. Such quantities cannot be discarded as unphysical, and we must conclude that, in this somewhat modified sense, the Dirac conjecture is violated. As outlined above, this is the reason why the replacement of \(h_{00}\) and \(h_{0\mu}\) by Lagrange multipliers (or simply the omission of the constraints) in the Hamiltonian \([8]\) is not permitted and leads to a loss of information.

Finally, we wish to stress once again that, although the Faddeev-Jackiw approach has failed in the sense that some (seemingly) dynamical information has been lost, there is no problem concerning the propagating modes. The formalism works perfectly well in order to determine the physical modes and thus is perfectly suited for the transition to the quantum theory. In \([6]\), it has been shown that either constraints or dynamical equations might get lost in the Faddeev-Jackiw analysis. The spin-two case presented here tells us that those dynamical equations are actually not quite as dynamical as they seem, since they involve only non-propagating field components.

We conclude that the answer to the question raised in the introduction section is negative. It is not possible to choose the gauge in a way that for some tensor \(\psi_{ik}\) (which need not be equal to \(h_{ik}\) nor to the trace-reversed tensor \(h_{ik} - \frac{1}{2}\eta_{ik} h_{00}\), but could be an arbitrary combination of components of \(h_{ik}\), we have \(\Delta \psi_{00} = \Delta \psi_{0\mu} = 0\) (i.e., a Gauss type law for the free field case), and such that in addition, \(\psi_{\mu\nu}\) is traceless-transverse and satisfies a wave equation. Independently of the gauge we impose, eliminating thereby 4 components, the resulting field equations will still contain 3 (and not 2) accelerations. Thus, apart from the propagating field modes, there will remain one more equation that cannot be eliminated by a Gauss type law. In the Hamiltonian formulation, this means that 3 (and not 2) pairs of Hamiltonian field equations are required in order to determine the configuration completely. The naive counting argument (10 components, 4 Gauss type eliminations and 4 gauge degrees of freedom, 2 propagating components) does not work in the strict sense. In other words, there is no Coulomb type gauge for the spin-two field. One should not confuse the situation with the corresponding
problems arising in non-abelian gauge theories. In our case, e.g., it is impossible to impose $h^\mu\nu_{\nu} = 0$ together with $h = 0$ directly on the field in order to eliminate unphysical components. On the contrary, in the case of gauge theory, the condition $A^\alpha_{\mu\nu}$ ($\alpha$ is the internal group index) can be imposed, but it does not fix the gauge completely (so-called Gribov ambiguity). This is more of a technical matter, meaning that the transversality condition of the free spin-one field cannot be generalized to the self-interacting case without modifications. The principal situation, namely that we have one constraint and two propagating modes (for each value of the $\alpha$) remains unaffected by this.

Finally, one might come up with the idea to change the Lagrangian, in order to avoid the above problems. A class of alternative spin-two theories has been considered in [9]. The simplest modification of the theory would consist in starting with a traceless field $h_{ik}$ right form the start. This corresponds to the linear approximation of the so-called unimodular General Relativity, see [9]. It still leads to a relativistic spin-two theory, but with a number of field equations that is reduced by one. Hence, there is the hope that the Lagrangian field equations are now equivalent to the set of propagating and constraint equations obtained in the Faddeev-Jackiw reduction. This is not the case, though. Indeed, one finds that the number of constraints is reduced by one too, and there will again be one equation that gets lost during the reduction. The reason is that the gauge transformations $\xi^i$ now have to satisfy $\xi^i_{,i} = 0$ and thus, there are essentially only three gauge degrees of freedom, and hence three constraints.

4 Outlook: General Relativity

Since it is difficult to couple the spin-two field consistently to other fields, in particular to gravity, the main motivation for its study is the fact that it is also the first order approximation to General Relativity. It is therefore of interest to relate the above discussion to the case of the generally covariant theory. Obviously, from a perturbative point of view, there is not much to be expected. Treating gravity as a spin-two field on a flat background will reproduce to lowest order the above results, and only higher order modifications will result from the selfcouplings. The more interesting approach is certainly the canonical theory [8]. Our question is thus the following. Suppose we were able, at least in principle, to solve the constraints of General Relativity, and to perform the Faddeev-Jackiw reduction of the Hamiltonian, eliminating all non-propagating field modes, Would we loose again a dynamical field equation, as in the case of the spin-two field? In other words, is the field completely determined by the constraints and the propagating modes, or is there an additional dynamical equation?

Before we try to gain some insight into that matter, let us consider a simpler example that permits us to see what would be the consequences of one or the other answer to the above question. Covariant theories are characterized by a vanishing Hamiltonian. Their mechanical counterpart are theories invariant under reparameterization. Therefore, let us consider the following special-relativistic point particle theory

$$L = -m \sqrt{\dot{x}^i \dot{x}_i},$$

where the dot means derivation with respect to an unspecified curve parameter $\tau$, and the action is given by $S = \int Ld\tau$. The momenta satisfy $p_i p^i = m^2$, which can be solved for $p_0$ such that the first order Lagrangian takes the form

$$L = p_\mu \dot{x}^\mu + \sqrt{m^2 - p_\mu p^\mu} \dot{x}^0 - H(p_\mu, x^\mu, x^0).$$

To bring this into canonical form, we define

$$X^\mu = x^\mu - \frac{p^\mu x^0}{\sqrt{m^2 - p_\mu p^\mu}}.$$
which leads, up to a total $\tau$-derivative, to
\[ L = p_\mu \dot{X}^\mu - H(p^\mu, X^\mu, x^0). \] (32)

Next, we have to eliminate the non-canonical variable $x^0$ by its equation of motion, $\partial H/\partial x^0 = 0$. However, the Hamiltonian is zero, and therefore, we are left with the equations of motion $\dot{X}^\mu = p_\mu = 0$. In particular, the first equation can be trivially integrated, and from \[ \mu \nu \], we find $x^\mu = \frac{p^\mu x^0}{\sqrt{m^2 - p_\nu p^\nu}} + a^\mu$, with a constant $a^\mu$. Since $p_\mu$ is constant too, this simply means $x^\mu = v^\mu x^0 + a^\mu$ with constant $v^\mu$, corresponding to free particle motion.

What can we learn from this example in relation to General Relativity? First of all, let us point out that, despite the fact that $H = 0$, there is nothing static about the above system. For instance, neither $dx^\mu/d\tau$, nor $dx^\mu/dx^0$ need to be zero. On the other hand, the canonical variables $X^\mu, p^\mu$ are indeed constant. So, let us suppose we can devise a method to solve the constraints of General Relativity, at least formally, and to reduce the Hamiltonian to the physical variables (fields and momenta), which we shall denote by $^TTg_{\mu\nu}$ and $^TT\pi_{\mu\nu}$, without having in mind that they are in some sense transverse, nor traceless (or rather unimodular), such that we have
\[ L = ^TT\pi_{\mu\nu} ^TTg_{\mu\nu} - H(^TTg_{\mu\nu}, ^TT\pi_{\mu\nu}). \] (33)

On the other hand, we know \[ \mu \nu \] that in General Relativity, we have four constraints, which correspond again to the Einstein equations $G^{0i} = 0$, and that, on the constraint surface, the Hamiltonian vanishes. Thus, we have $H = 0$, and consequently, the dynamical equations read $^TTg_{\mu\nu} = 0$ and $^TT\pi_{\mu\nu} = 0$. This apparent static nature of the theory has been studied intensively in literature, see, e.g., \[ \mu \nu \] and \[ \mu \nu \]. In spite of this, we will nevertheless continue to refer to those modes as propagating, whatever this means without reference to a background spacetime. Since they are the only modes that are subject to quantization (assuming that the standard procedures are applicable to General Relativity), we must conclude that they describe the spin-two particle. (Meaning that they contain the propagating modes $^TTh_{\mu\nu}$ previously found in the special-relativistic theory. Without reference to a background space, the notion of a particle does not make sense anyway.)

Thus, if everything works out as planned, our system will be determined by the dynamical equations $^TTg_{\mu\nu} = 0$ and $^TT\pi_{\mu\nu} = 0$, as well as by the four constraints $G^{0i} = 0$. But what, if this is not the case? What, if we have lost a dynamical equation during the Faddeev-Jackiw reduction? In that case, we would have an additional equation, say $\pi = f(g_{ik}, \pi_{ik})$, and things would not be so static after all. In particular, this would mean that the standard approach is not entirely correct. Namely, it is traditionally assumed, since the dynamical equations are trivial as a result of $H = 0$, that the complete information is contained in the four constraints $G^{0i}$, see \[ \mu \nu \] for details. But this might not be the case, as the example of the spin-two particle shows. As we have outlined in the previous section, the critical point is the replacement of $h_{00}$ and $h_{0\mu}$ that appear in front of the constraints, by arbitrary multipliers, or, alternatively, the omission of the constraints. In the same way, however, one deals with $g_{00}$ and $g_{0\mu}$ (or the lapse and shift functions $N, N_{\mu}$, respectively, see \[ \mu \nu \]) in the canonical approach to gravity. It could happen that this is not allowed and leads to a loss of information.

It is not easy to estimate whether we have indeed such a situation, but on the basis of a counting argument, it seems not improbable. First of all, there are again four undetermined components $g_{00}$ and $g_{0\mu}$, which is a result of the general coordinate invariance. Further, we have four constraints $G^{0i}$, related via the Bianchi identity $G^{0i} = 0$, where the semicolon denotes covariant derivation in four dimensions. On the constraint surface, this leads to $\Gamma^0_{\mu\nu}G^{\mu\nu} = 0$, where $\Gamma^i_{kl}$ is the Christoffel connection formed from the four dimensional metric. The remaining part, $G^{\mu\nu}$, contains the dynamical components. On the constraint surface, we find from $G^{0i} = 0$ the relation $\Gamma^\mu_{\nu\rho}G^{\rho\nu} + \Gamma^\nu_{\mu\rho}G^{\rho\nu} + \Gamma^\nu_{\nu\rho}G^{\rho\nu} = 0$. We see that thinks are a lot more interrelated than in the spin-two case, but the basic situation is the same. The relations $G^{\mu\nu} = 0$ contain 3 independent equations, while from the four constraints $G^{0i} = 0$, only 3 are independent. As a result, there must be three equations

9
containing second time derivatives (or, in the Hamiltonian formulation, three canonically conjugated pairs of equations).

That the Einstein equations contain three independent accelerations can be checked explicitly. Consider, e.g., the metric \( ds^2 = dt^2 - g_{\mu\nu} dx^\mu dx^\nu \). Since \( g_{0\mu} \) and \( g_{00} \) are already fixed, only static, spatial coordinate transformations are still allowed. Let us choose \( g_{\mu\nu} = S(x) \tilde{g}_{\mu\nu} \), with \( \tilde{g}_{xx} = 1 - f(x) \), \( \tilde{g}_{yy} = 1 + f(x) \), \( \tilde{g}_{xy} = g(x) \), \( \tilde{g}_{zz} = 1/(1 - f(x)^2 - g(x)^2) \), \( \tilde{g}_{xz} = \tilde{g}_{yz} = 0 \). Note that \( \tilde{g}_{\mu\nu} \) is unimodular and \( S(x) \) is essentially a cosmological function. One can now directly check that \( G_{00} \) and \( G_{0\mu} \) do not involve second time derivatives. On the other hand, \( G_{\mu\nu} \) contains \( \dot{S}(x) \), \( f(x) \) and \( \dot{g}(x) \). Since the coordinate system is already fixed with respect to time, this means that we have indeed three physically relevant functions that enter the field equations by their accelerations. (Note that it cannot be more than three neither, since we can still diagonalize \( g_{\mu\nu} \).) We have chosen the particular form of \( g_{\mu\nu} \) because \( S \) is related to \( \det g_{\mu\nu} \), and thus to the trace \( h \) in the linearized theory. Therefore, if the situation is similar to the linearized theory, we would have to conclude that \( S(x) \) is not a propagating field, and in particular, is not subject to quantization. Such a result is certainly of relevance to quantum cosmology.

The question is therefore reduced to whether after the complete reduction of the Hamiltonian, we still have three propagating modes contained in \( \tau^2 \tilde{g}_{\mu\nu} \), or whether we have only two, as seems to be suggested by the perturbative approach. The second case would result if we loose one equation during the reduction procedure (e.g., the equation containing \( \dot{S} \)). This might appear to be an issue without contents, since the so-called propagating modes satisfy \( \tau^2 \dot{g}_{\mu\nu} = 0 \) anyway and are no more and no less propagating than the remaining components. It is, however, of fundamental relevance, since after all, it should be of importance to know how many degrees of freedom are actually quantized.

Without giving a definite answer to the above question, let us just say that the situation looks problematic in either way. In the second case, where we have only two propagating modes, we have a situation similar to the spin-two theory. That is, the two dynamical equations (which are trivial in the case of General Relativity), together with the four (not independent) constraints, do not describe the system completely. As we have outlined above, this means that the standard approach is not completely correct, i.e., the constraints do not completely determine the dynamics of the quantum system, in contrast to what is claimed, e.g., in [3]. Rather, there is one more dynamical equation, involving, e.g., the time derivative of the momentum conjugated to \( S \). In particular, this could mean that not everything needs to be static after all. (For the precise meaning of the word static in this context, we refer to the interpretation given in [3].)

In the other case, we would have three propagating modes, that is, three independent degrees of freedom are to be quantized. In that case, the conventional treatment would be valid, and, since the dynamical equations are trivial, the theory could be fully described by the constraints alone. It does not make sense to ask what kind of particles are described by such a theory, since one would need a reference background to answer that. The true question is, what happens to the third degree of freedom in the perturbative approach? How can we end up, to lowest order, with a pure spin-two theory, if we start from a theory with three independent physical degrees of freedom?

Having at least a little amount of faith in the perturbative approach, we would rather exclude that last possibility and tend to believe that General Relativity is a spin-two theory. A further hint for the similarity to the linear theory comes from the fact that one can directly construct examples which demonstrate that the theory is not completely described by the constraints and the propagating modes alone. Although we do not have identified the propagating modes exactly, we know (see [3]) that they are contained in the three-dimensional tensors \( g_{\mu\nu} \) and \( \pi^{\mu\nu} \). It is now easy to check, for instance, that the solution \( g_{\mu\nu} = \eta_{\mu\nu}, \pi^{\mu\nu} = 0 \), and \( g_{0\mu} = 0, g_{00} = \exp[-r] \) satisfies the constraints and must also satisfy the (trivial) dynamical equations for the propagating parts contained in \( \pi^{\mu\nu} \) and \( g_{\mu\nu} \) (since \( \dot{g}_{\mu\nu} = \dot{\pi}^{\mu\nu} = 0 \) anyway). Nevertheless, the above is not
5 Additional remarks

Since the appearance of the first version of this paper, objections have been raised that the conclusions obtained above are not correct and are in conflict with the results previously obtained in literature. More precisely, it has been claimed that there is actually no conceptional difference between the case of the spin-two field and the spin-one field (Maxwell theory), and in particular that in both theories, the system is completely specified by the propagating fields and the constraints alone. Therefore, we will try, in this section, to explain in more detail and from a slightly different point of view why we feel that there are fundamental differences, and in particular to clarify the meaning of our conclusion that there is no Coulomb gauge for the spin-two field.

The Hamiltonian reduction of the spin-two theory has been carried out in [10], and generalized to General Relativity in [11]. The word reduction has been put into italics to underline the difference to the Dirac approach (as used in [5] and [3], respectively), where the constraints are imposed on the states.

We will confine ourselves mainly to the linear spin-two theory. The results of [11] supply very strong support for our conclusion, that the situation in the self-interacting case is fundamentally the same. Indeed, the authors find that the theory contains two propagating degrees of freedom, which according to our counting argument of the previous section, means that there should remain one additional equation that is not part of the constraints.

The strong similarity between General Relativity and its linear counterpart with respect to the choice of the canonical (propagating) variables has been very explicitly underlined in section 4.4. of the classical ADM article [12]:

We will now see the usefulness of the linearized theory in suggesting the choice of canonical variables for the full theory. Since the identification is made from the bilinear part of the Lagrangian $\pi^{ij}\partial_t g_{ij}$, which is the same as for the linearized theory, the greater complexity of the full theory, i.e., its self-interaction, is to be found only in the non-linearity of the constraint equations.

Let us therefore turn to the spin-two field. First, it should be noted that the reduction in [10] is based on a first order (Palatini type) formulation of the theory. This does not affect the issues we discuss here, since, after all, the Hamiltonian approach is in fact a first order reformulation of the Lagrangian theory anyway. Greater ease in the reduction process has been obtained in [10] by the use of the decomposition of symmetric tensors $A_{\mu\nu} = TT A_{\mu\nu} + T A_{\mu\nu} + a_{\mu,\nu} + a_{\nu,\mu}$, where the first term is traceless-transverse and the second one transverse.

The authors then solve the second constraint in (10), $h_{\mu\nu,\mu,\nu} - \Delta h = 0$, with $h_{\mu\nu} = TT h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$. This is consistent with our results, namely the solution (11), if we insert the expression for $\tilde{h}_{\mu\nu}$ in terms of $TT h_{\mu\nu}$ that can be obtained from (17) solving for the first term at the r.h.s. This form is particularly useful, since it shows that $h_{\mu\nu}$ is actually traceless-transverse, up to a gauge transformation.

Let us therefore adopt the gauge where $h_{\mu\nu} = TT h_{\mu\nu}$. There are not further transformations possible that involve $\xi^\mu$. Let us further impose the gauge $h_{0\mu,\mu} = 0$, which exhausts the last gauge freedom for $\xi^0$. Let us now consider the field equations (22)-(24). The first is now identically satisfied, since we have already solved the constraint. The second reduces to $\Delta h_{0\mu} = 0$, i.e., $h_{0\mu} = 0$. The traceless-transverse part of (24) remains unchanged and leads to the wave equation for the propagating fields. At this point, we have completely fixed the gauge, we have solved the constraint equations (22) and (23), and we have also obtained the dynamical equation for the radiative field. Thus, if it is true that there is no difference to the case of Maxwell’s theory, then the system should now be completely determined. In other words, $h_{00}$ remains arbitrary. But this is obviously not in accordance with the Lagrangian field equations, since the trace of (24) leads to the additional equation $\Delta h_{00} = 0$, i.e., $h_{00} = 0$. That is why we claim that the theory is not completely described by the
constraints and the propagating parts alone. It is interesting to note that the previously *dynamical* equation 
\[ \dot{h} + \Delta h_{00} - 2h^{0\lambda} \lambda = 0 \]  
(see (25)) has been reduced, in the particular gauge, to a *constraint* equation, \( \Delta h_{00} = 0 \).

This explains why this equation is neither a true constraint, nor a true dynamical equation, because, depending on the gauge one adopts, it appears in one or the other form. For the same reason, it does not turn up in the Faddeev-Jackiw reduction.

What does this have to do with the Coulomb gauge? With the above gauge choice, we have \( \Delta h_{00} = \Delta h_{00} = 0 \), and \( h_{\mu \nu} = \tau h_{\mu \nu} \), which looks like a perfect Coulomb gauge. It is not a Coulomb gauge however, simply because it is not gauge, but it is a combination of a gauge and a solution. In fact, it is the direct analogue of the radiation gauge \( A_i = (0, \tau A_\mu) \) in Maxwell’s theory, which is a combination of the transverse gauge \( A_\mu = \tau A_\mu \) and the solution \( A_0 = 0 \) of the constraint equation \( \Delta A_0 = 0 \). So, what we have actually established above is the existence of the radiation gauge \( h_{00} = h_{00} = 0 \) and \( h_{\mu \nu} = \tau h_{\mu \nu} \). But this is not really exciting though, it has been known at least since the works on weak gravitational waves by Einstein and Rosen and is also shown in almost any textbook on General Relativity. This has nothing to do with the Coulomb gauge.

The difference between Maxwell’s theory and the spin-two field lies in the following. In the Maxwell case, we can solve the constraint equation \( F^{\mu \alpha} = 0 \) with \( A_0 = (1/\Delta)A^\alpha_\mu \). This is completely independent of the gauge choice. In particular, we can impose the transversality condition on \( A^\alpha_\mu \). But obviously, this transversality condition can be imposed independently of whether \( A_\mu \) is a solution of the constraint or not. Why is that of importance? Well, if we have a source, then the solution of the constraint reads \( A_\mu = (1/\Delta)(\dot{A}^\mu_\mu + \rho \delta_\mu) \), but just as before, we can impose the gauge \( A^\mu_\mu = 0 \), meaning that apart from \( A_0 \) determined as above by the constraint, we have only to deal with the transverse part of \( A_\mu \), which is determined by the dynamical equations.

On the other hand, in the spin-two theory, the constraint reads \( h^{\mu \nu, \mu, \nu} - \Delta h = 0 \). As we have seen, this can be solved as \( h_{\mu \nu} = \tau h_{\mu \nu} + \xi_{\mu, \nu} + \xi_{\nu, \mu} \), and it is possible to impose the traceless-transverse condition. But now, this condition can only be imposed on the solutions of the constraint, and not independently of the constraint, as in Maxwell theory. In general, we have \( h^{\mu \nu} = \tau h^{\mu \nu} + h^{\mu \nu} + \xi_{\mu, \nu} + \xi_{\nu, \mu} \), which does not allow to choose a traceless-transverse gauge.

The point is that, as soon as we have source terms, the constraint will change, and so will the solution to the constraint. It is not hard to see that, if we have a source \( T_{00} = \rho \) in the field equation (22), i.e., in the constraint equation, then the solution to the constraint can be brought into the form \( h_{\mu \nu} = \delta_{\mu \nu} \frac{1}{2} \eta^{\alpha \beta} (\nabla_\alpha \nabla_\beta / \Delta) \dot{h}^{\alpha \beta} + \eta_{\mu \nu} \rho / \Delta \), or, inserting the expression for \( h_{\mu \nu} \) in terms of \( \tau h_{\mu \nu} \) with the help of (17), \( h_{\mu \nu} = \tau h_{\mu \nu} + \xi_{\mu, \nu} + \xi_{\nu, \mu} + \eta_{\mu \nu} \rho / \Delta \), where \( \xi_{\mu \nu} \) is given in terms of \( h_{\mu \nu} \). But this is not equal to \( \tau h_{\mu \nu} \) up to a gauge transformation. Thus, we cannot reduce \( h_{\mu \nu} \) to the propagating components only.

To summarize, the traceless-transverse condition cannot be imposed independently of the solutions to the constraint. In other words, there is no traceless-transverse gauge in the strict sense. There are only solutions (namely the purely radiative solutions) that can be brought into that form. This is the reason why in textbooks on General Relativity, explicit reference to wave solutions has to be made in order to show the traceless-transverse nature of gravitational waves. (Quite in contrast to the Maxwell case, where the transversality condition can be trivially imposed.)

From a more mathematical standpoint, we can recognize the difference to Maxwell’s theory in the following. In the Hamiltonian (5), although the terms in \( h_{\mu \nu} \) are gauge invariant, the terms \( \pi^{\mu \nu} \pi_{\mu \nu} - \frac{1}{2} \pi^2 \) are not, as is easily shown from \( \delta \pi_{\mu \nu} = -\xi^{\alpha}_{\mu, \nu} + \eta_{\mu \nu} \Delta \xi^\alpha \) under (25). Otherwise stated, if we use the decomposition \( \pi_{\mu \nu} = \tau \pi_{\mu \nu} + \tau \pi_{\mu \nu} + \rho_{\mu, \nu} + p_{\mu, \nu} \), the Hamiltonian will not be independent of the gauge field \( \tau \pi_{\mu \nu} \). Rather, we find \( \delta (\pi_{\mu \nu} \pi^{\mu \nu} - \frac{1}{2} \pi^2) = -2 \xi^{\alpha} \pi^{\mu \nu} \), which is proportional to the constraint. Thus, the Hamiltonian is invariant only on the constraint surface, quite in contrast to the Maxwell case, where the gauge fields, i.e., the longitudinal modes of \( A_\mu \) do not appear in the Hamiltonian at all. Obviously, the above variation is compensated by the variation of \( h_{00} \) in (5). The fact that both terms are not independently gauge invariant, however, makes clear
that we cannot require the gauge fields not to appear in the Hamiltonian and to have $h_{0\mu}$ and $h_{00}$ as arbitrary Lagrange multipliers at the same time, as seems to be the viewpoint in [10]. It is either one or the other. For the exact same reason, those fields are interrelated by the missing equation.

If this is still not convincing, here is an explicit example that shows that the system of constraints and dynamical (propagating) field equations is under-determined. Consider the solution given by $h_{\mu\nu} = \pi_{\mu\nu} = 0$, $h_{0\nu} = 0$, $h_{00} = \exp[-r]$, with $r = \sqrt{x^2 + y^2 + z^2}$. The constraints $h^{\mu\nu, \mu, \nu} + \Delta h = 0$ and $\pi^{\mu\nu, \mu} = 0$ are trivially satisfied, and so are the dynamical equations (20). Nevertheless, this is not a solution of the field equations (22)-(24). The trace of $G_{\mu\nu}$ does not vanish. In other words, the missing equation $\dot{\pi} = \Delta h_{00}$, or, in configuration space, $\ddot{h} + \Delta h_{00} - 2\dot{h}_{0\mu, \mu} = 0$ (see (28)) is not satisfied, since $\Delta h_{00} \neq 0$.

Those considerations support further our conclusions that there are fundamental differences between the spin-one theory and the spin-two theory, namely the absence of a Coulomb gauge in the latter, as well as the fact that the theory is not completely specified by the constraints and the dynamical (propagating) equations alone. For those reasons, the Faddeev-Jackiw procedure does not lead to the complete set of independent field equations.

References

[1] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988)
[2] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York, 1964)
[3] B. S. DeWitt, Phys. Rev. 160, 1113 (1967)
[4] M. Leclerc, gr-qc/0608096 (2006)
[5] N. S. Baaklini and M. Tuite, J. Phys. A 12, L13 (1979)
[6] J. A. García and J. M. Pons, Int. J. Mod. Phys. A 13, 3691 (1998)
[7] M. E. V. Costa, H. O. Girotto, and T. J. M. Simões, Phys. Rev. D 32, 405 (1985)
[8] A. Komar, Phys. Rev. 153, 1385 (1967)
[9] E. Alvarez, D. Blas, J. Garriga, and E. Verdaguer, hep-th/0606019
[10] R. Arnowitt and S. Deser, Phys. Rev. 113, 745 (1959)
[11] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 117, 1595 (1960)
[12] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: an introduction to current research, L. Witten, ed. (Wiley, New York, 1962), p227-264, (available as: gr-qc/0405109)