Quasi-complete intersections in $\mathbb{P}^2$ and syzygies

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Abstract

Let $C \subset \mathbb{P}^2$ be a reduced, singular curve of degree $d$ and equation $f = 0$. Let $\Sigma$ denote the jacobian subscheme of $C$. We have $0 \to E \to 3, \mathcal{O} \to \mathcal{I}_\Sigma(d - 1) \to 0$ (the surjection is given by the partials of $f$). We study the relationships between the Betti numbers of the module $\mathcal{H}^0(E)$ and the integers, $d, \tau$, where $\tau = \text{deg}(\Sigma)$. We observe that our results apply to any quasi-complete intersection of type $(s, s, s)$.

Keywords Quasi complete intersections · Vector bundle · Syzygies · Global Tjurina number · Plane curves

Mathematics Subject Classification Primary 14H50; Secondary 14M06, 14M07, 13D02

1 Introduction

Let $C \subset \mathbb{P}^2$ be a reduced, singular curve, of degree $d$, of equation $f = 0$. The partials of $f$ determine a morphism: $3, \mathcal{O} \xrightarrow{\partial f} \mathcal{O}(d - 1)$, whose image is $\mathcal{I}_\Sigma(d - 1)$, where according to our assumptions, $\Sigma \subset \mathbb{P}^2$, is a closed subscheme of codimension two. The subscheme $\Sigma$, whose support is the singular locus of $C$, is called the jacobian subscheme of $C$. We denote by $\tau$ its degree, it is the global Tjurina number of the plane curve $C$.

We have:

$$0 \to E \to 3, \mathcal{O} \to \mathcal{I}_\Sigma(d - 1) \to 0$$

(1)

where $E$ is a rank two vector bundle with Chern classes $c_1 = 1 - d, c_2 = (d - 1)^2 - \tau$ (see for instance [11] and references therein). The bundle $E$ is the sheaf of logarithmic vector fields along $C$, also denoted $\text{Der}(-\log C)$ [5,14,15]. A particular case of this situation is when $C$ is an arrangement of lines [8,13,17]. This is a very active field of research with a huge literature.

In [9], using techniques of the theory of singularities, du Plessis and Wall gave sharp bounds on $\tau$ in function of $d$ and $d_1$, the least twist of $E$ having a section. Observe that
\(H^0(E)\) is the module of syzygies between the partials. This result has been extended (see [11]) to the case of quasi-complete intersections (q.c.i.), using vector bundles techniques.

In this note, inspired by [6], instead of considering only \(d_1\), the minimal degree of a generator of \(H^0(E)\), we consider the full minimal resolution of this module. So we will assume that \(H^0(E)\) is minimally generated by \(m\) elements of degree \(d_1 \leq d_2 \leq \cdots \leq d_m\).

The \(m\)-uple \((d_1, \ldots, d_m)\) is the exponent of \(C\). We have \(m \geq 2\), with equality if and only if \(E\) splits. In this case one says that \(C\) is a free divisor [1,14] or, equivalently, that \(\Sigma\) is an almost complete intersection. The case \(m = 3\) is handled in [6]. Here we deal with the general case \(m \geq 3\).

Starting from the minimal free resolution of \(H^0(E)\) we show how to get a free (non necessarily minimal) resolution of \(I_E\). With this we show (Corollary 6) that if \(\Sigma\) is a complete intersection, then \(m \leq 4\). Then (Theorem 8) we prove that \(2d - 4 \geq d_i, \forall i\) and that the inequality is sharp if and only if \(\Sigma\) is a point (\(\tau = 1\)). Finally we prove: \(d_m = d - 1\) or \(2d - m \geq d_m\).

Then (Theorem 13), shows that \(d_3 \leq d - 1\) and characterizes the q.c.i. realizing the lower bound, \((d - 1)(d - 1 - d_1) = \tau\), in du Plessis–Wall theorem: this happens if and only if \(\Sigma\) is a point (\(\tau = 1\)). Finally we prove: \(d_m = d - 1\) or \(2d - m \geq d_m\).

Finally, in the setting of q.c.i., we answer to a conjecture raised in [7] (Proposition 15) and describe the sub-maximal case (see Proposition 17).

The exact sequence (1) presents \(\Sigma\) as a quasi-complete intersections (q.c.i.) of type \((d - 1, d - 1, d - 1)\). In our proofs we will never use the fact that the three curves giving the q.c.i. are the partials of a polynomial \(f\). So setting \(s = d - 1\), all our results are true for q.c.i. of type \((s, s, s)\). Actually, after appropriate changes in notations (see [11]) they should hold for all q.c.i. (i.e. of any type \((a, b, c)\)). Observe that to determine the minimal free resolution (m.f.r.) of \(H^0(E)\) amounts to determine the m.f.r. of the (nonsaturated if \(m > 2\)) q.c.i. ideal \(J = (F_1, F_2, F_3)\). For a purely algebraic approach to q.c.i. see for example [16].

As the first version of this paper was finished I received the preprint [7] containing some overlaps. This obliged me to revisit my text. This version contains some improvements (so thank you to the authors of [7]!), but overlaps are still present. However, since the methods are different, it could be useful to see how geometric techniques apply in this context.

I thank Alexandru Dimca for useful discussions, in particular about (i) of Theorem 13.

### 2 Setting, notations

Following [6] we have:

**Definition 1** We will say that \(C\) is a \(m\)-syzygy curve if \(H^0(E)\) is minimally generated by \(m\) elements of degree \(d_1 \leq d_2 \leq \cdots \leq d_m\). The \(m\)-uple \((d_1, \ldots, d_m)\) is the exponent of \(C\).

**Remark 2** We have \(m \geq 2\). Moreover \(m = 2\) if and only if \(E\) is the direct sum of two line bundles.

In the sequel we will always assume \(m \geq 3\).

For any \(i\), \(E(d_i)\) has a section vanishing in codimension two.

Besides the exact sequence (1) we will also consider the following ones:

\[
0 \to \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j) \to \bigoplus_{i=1}^{m} \mathcal{O}(-d_i) \to E \to 0
\] (2)
The minimal presentation of $H^0_*(E)$ yields $\bigoplus_{i=1}^m \mathcal{O}(-d_i) \to E \to 0$, the kernel; $K$, is locally free of rank $m - 2$ with $H^1_*(K) = 0$, hence $K$ is a direct sum of line bundles.

$$0 \to \mathcal{O} \to E(d_1) \to I_Z(2d_1 + 1 - d) \to 0 \quad (3)$$

Here $Z \subset \mathbb{P}^2$ is a locally complete intersection (l.c.i.), zero-dimensional subscheme of degree

$$\deg(Z) = c_2(E(d_1)) = d_1(1 - d) + (d - 1)^2 - \tau + d_1^2 \quad (4)$$

### 3 Resolutions

Starting from (2) we can get the minimal free resolution of $H^1_*(E)$ and $H^0_*(I_Z)$, more precisely:

**Lemma 3** Let $E$ be a rank two vector bundle on $\mathbb{P}^2$ and let $Z = (s)_{0}, s \in H^0(E(d_1))$, where $d_1 = \min\{k \mid h^0(E(k)) \neq 0\}$.

1. The following are equivalent:
   a. $H^0_*(E)$ is minimally generated by $m$ elements
   b. $H^1_*(E)$ is minimally generated by $m - 2$ elements
   c. $H^0_*(I_Z)$ is minimally generated by $m - 1$ elements.

   Assume the minimal free resolution of $H^0_*(E)$ is given by (2) and that $c_1(E) = 1 - \tau$, then:

2. The minimal free resolution of $H^1_*(E)$ is

   $$0 \to \bigoplus_{j=1}^{m-2} S(-b_j) \to \bigoplus_{i=1}^m S(-d_i) \to \bigoplus_{i=1}^m S(d_i + 1 - d) \to \bigoplus_{j=1}^{m-2} S(b_j + 1 - d) \to H^1_*(E) \to 0 \quad (5)$$

3. The minimal free resolution of $H^0_*(I_Z)$ is:

   $$0 \to \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j + d - 1 - d_1) \to \bigoplus_{i=2}^m \mathcal{O}(-d_i + d - 1 - d_1) \to I_Z \to 0 \quad (6)$$

**Proof** Let $E$ be a rank two vector bundle on $\mathbb{P}^2$ and assume that $H^0_*(E)$ is minimally generated by $m$ elements. We have $G_1 \to E \to 0$, with $G_1 = \bigoplus_{j=1}^m \mathcal{O}(-d_j)$. As explained before the kernel, $G_2$, is a direct sum of line bundles: $G_2 = \bigoplus \mathcal{O}(-b_j)$. By dualizing the exact sequence: $0 \to G_2 \to G_1 \to E \to 0$, we get: $0 \to E^{\ast} \to G_1^{\ast} \to G_2^{\ast} \to 0$. Taking into account that $E^{\ast} \simeq E(-c_1)$ ($c_1 = c_1(E)$) because $E$ has rank two, we get: $0 \to E \to G_1^{\ast}(c_1) \to G_2^{\ast}(c_1) \to 0$. Taking cohomology this yields: $0 \to H^0_*(E) \to G_1^{\ast}(c_1) \to G_2^{\ast}(c_1) \to H^1_*(E) \to 0$. This is the beginning of a minimal free resolution of $H^1_*(E)$. We conclude with (2). This proves (ii) and also (a) $\Rightarrow$ (b) in (i). By uniqueness of the minimal free resolution this also proves (b) $\Rightarrow$ (a) in (i).
We have:

\[
\begin{array}{ccc}
0 & \to & \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j + d_1) \\
\downarrow & & \downarrow \\
\mathcal{O} & = & \mathcal{O} \\
\downarrow & & \downarrow \\
0 & \to & \bigoplus_{j=1}^{m-2} \mathcal{O}(-b_j + d_1) \oplus \mathcal{O}(-d_1 + d_1) \oplus \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O} & \to & \mathcal{O} \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{O} \\
\end{array}
\]

which proves (iii) and also (a) ⇔ (c) in (i) (observe that we have \(0 \to S \xrightarrow{f} H_*^0(E(d_1)) \to H_*^0(\mathcal{I}_Z(2d_1 - d + 1)) \to 0\), where, by assumption, the image of \(f\) yields a minimal generator of \(H_*^0(E(d_1))\).

\[\square\]

4 Resolution of \(H_*^0(\mathcal{I}_\Sigma)\)

Starting from the resolution of \(H_*^0(E)\) it is also possible to get a resolution of \(H_*^0(\mathcal{I}_\Sigma)\) but this resolution is not necessarily minimal:

**Proposition 4** We have the following free resolution

\[
0 \to \bigoplus_{i=1}^{m} \mathcal{O}(d_i - 2d + 2) \to \bigoplus_{j=1}^{m-2} \mathcal{O}(b_j - 2d + 2) \oplus 3.\mathcal{O}(1 - d) \to \mathcal{I}_\Sigma \to 0 \tag{7}
\]

This resolution is minimal up to cancellation of \(\mathcal{O}(1 - d)\) terms with some \(\mathcal{O}(d_i - 2d + 2)\) (in this case \(d_i = d - 1\)).

**Proof** Since \(\mathcal{I}_\Sigma(d - 1)\) is generated by global sections we can link \(\Sigma\) to a zero-dimensional subscheme \(T\) by a complete intersection of type \((d - 1, d - 1)\). From the exact sequence (1), by mapping cone, we get that \(T\) is a section of \(E(d - 1)\). So we have an exact sequence: \(0 \to \mathcal{O}(1 - d) \to E \to \mathcal{I}_T \to 0\). From (2) we get a surjection: \(\bigoplus_{i=1}^{m} \mathcal{O}(-d_i) \to \mathcal{I}_T \to 0\).

Using (2) we can build a commutative diagram and by the snake lemma we get:

\[
0 \to \bigoplus_{i=1}^{m-2} \mathcal{O}(-b_j) \oplus \mathcal{O}(1 - d) \to \bigoplus_{i=1}^{m} \mathcal{O}(-d_i) \to \mathcal{I}_T \to 0
\]

This resolution is minimal unless the section of \(E(d - 1)\) yielding \(T\) is a minimal generator of \(H_*^0(E)\). From the above resolution, by mapping cone, we get the desired resolution of \(\mathcal{I}_\Sigma\). Again this resolution is minimal unless one curve (resp. both curves) of the complete intersection \((d - 1, d - 1)\) linking \(T\) to \(\Sigma\) is a minimal generator (resp. both curves are minimal generators) of \(\mathcal{I}_T\).

On the other hand, by minimality of the resolution (2) no term \(\mathcal{O}(b_j - 2d + 2)\) can cancel.

\[\square\]

**Remark 5** Cancellations can occur. Let \(C = X \cup L\), where \(X\) is a smooth curve of degree \(d - 1\), \(d \geq 3\), and where \(L\) is a line intersecting \(X\) transversally. Clearly \(\Sigma\) is a set of \(d - 1\) points on the line \(L\). The minimal free resolution of \(\mathcal{I}_\Sigma\) is: \(0 \to \mathcal{O}(-d) \to \mathcal{O}(-1) \oplus \mathcal{O}(1 - d) \to \mathcal{O} \oplus \mathcal{O}(d + 1) \to \mathcal{O} \to 0\).
\( I_2 \to 0 \). Comparing with (4) we see that \( m = 3 \) and that two terms \( \mathcal{O}(1 - d) \) did cancel. So we have \( d_1 = d - 2, d_2 = d_3 = d - 1 \).

See Remark 9 for another example.

**Corollary 6** If \( m \geq 5 \), \( \Sigma \) can’t be a complete intersection.

**Proof** Indeed \( \Sigma \) is a complete intersection if and only if the minimal free resolution of \( I_2 \) starts with two generators. According to Proposition 4 we have certainly \( m - 2 \) minimal generators of degrees \( 2d - 2 - b_j \) in the minimal free resolution of \( I_2 \).

Before to go on we recall a basic fact about zero-dimensional subscheme of \( \mathbb{P}^2 \):

**Lemma 7** Let \( X \subset \mathbb{P}^2 \) be a zero-dimensional subscheme with minimal free resolution:

\[
0 \to \bigoplus_{i=1}^{t} \mathcal{O}(-b_j) \to \bigoplus_{i=1}^{t+1} \mathcal{O}(-a_i) \to I_X \to 0
\]

Then \( a_i \geq t, \forall i \).

In particular if \( h^0(I_X(n)) \neq 0 \), then \( H^0_*(I_X) \) can be generated by \( n + 1 \) elements.

**Proof** This should be well known (see for example [10], Corollary 3.9), but for the convenience of the reader we give a proof. We work by induction on \( t \). The case \( t = 1 \) is clear. Assume the statement for \( t - 1 \). Let \( a_1 \leq \cdots \leq a_{t+1} \). Since \( I_X(a_{t+1}) \) is generated by global sections we can always perform a liaison of type \((a_1, a_{t+1})\). By mapping-cone the linked scheme, \( T \), has the following resolution:

\[
0 \to \bigoplus_{i=2}^{t} \mathcal{O}(a_i - a_1 - a_{t+1}) \to \bigoplus_{i=1}^{t} \mathcal{O}(b_j - a_1 - a_{t+1}) \to I_T \to 0
\]

This resolution is minimal and by the inductive assumption we get: \( a_1 + a_{t+1} - b_j \geq t - 1 \), hence \( a_1 \geq b_j - a_{t+1} + t - 1 \). We have \( b_j - a_{t+1} \geq 0, \forall j \) (they are the degrees of the elements of the last row of the matrix \( M \)). If \( b_j - a_{t+1} = 0, \forall j \), then, by minimality, the last row of \( M \) is zero. By the Hilbert–Buch Theorem (see [10], Theorem 3.2) the maximal minors of \( M \) yield a minimal set of generators of the ideal \( I(X) := H^0_*(I_X) \). If \( M \) has a row of zeroes, we get only one non-zero generator, this is impossible. It follows that \( a_1 \geq t \).

**Theorem 8** (i) With notations as in Sect. 2, if \( d \geq 3 \), then \( 2d - 4 \geq d_i, \forall i \).

(ii) Moreover, if \( d > 3 \), we have equality (i.e. \( d_m = 2d - 4 \)) if and only if \( \tau = 1 \).

(iii) We have \( d_m = d - 1 \) (hence \( d_i \leq d - 1, \forall i \)) or \( d_i \leq 2d - m, \forall i \).

**Proof** (i) This is clear if \( d_i = d - 1 \), so we may assume that the term \( \mathcal{O}(d_i - 2d + 2) \) really appears in (7) even after possible cancellations. This implies \( 2d - 2 - d_i \geq 2 \).

(ii) We have \( \min\{2d - d_i - 2\} = 2d - d_m - 2 \). Assume \( 2d - d_m - 2 = 2 \). For \( d > 3 \), the term \( \mathcal{O}(d_m - 2d + 2) \simeq \mathcal{O}(-2) \) really appears in the minimal free resolution of \( I_2 \). This implies that there are two generators of degree one, hence \( \Sigma \) is a point.

Conversely if \( \Sigma \) is a point, let \( T \) be linked to \( \Sigma \) by a complete intersection \((d - 1, d - 1)\).

Then using the minimal free resolution of \( I_2 \), by mapping-cone, we have: \( 0 \to 2\mathcal{O}(1 - d) \to \mathcal{O}(2d + 3) \to 2\mathcal{O}(1 - d) \oplus \mathcal{O}(-2d + 4) \to I_T \to 0 \). But using instead the resolution (1) we see that \( T \) is a section of \( E(d - 1) \), so we have \( 0 \to \mathcal{O}(1 - d) \to E \to I_T \to 0 \). Using the above resolution of \( I_T \), we get after some diagram-chasing: \( 0 \to 2\mathcal{O}(2d + 3) \to 3\mathcal{O}(1 - d) \oplus \mathcal{O}(-2d + 4) \to E \to 0 \). This resolution is clearly minimal. It follows that \( m = 4 \) and \( d_m = 2d - 4 \).
(iii) Assume \( d_m \neq d - 1 \), then, according to Proposition 4, the term \( \mathcal{O}(d_m - 2d + 2) \) appears in the minimal free resolution of \( I_\Sigma \). Let \( 2d - 4 - u = d_m \). We have \( u \geq 0 \) by (i). Since there is a relation of degree \( u + 2 \), there are at least two minimal generators of degree \( \leq u + 1 \) in the minimal free resolution of \( I_\Sigma \). So \( h^0(I_\Sigma(u + 1)) \neq 0 \) and \( I_\Sigma \) can be generated by \( u + 2 \) elements (Lemma 7). This implies (see 7) that \( m - 3 \leq u + 1 \), hence \( d_m \leq 2d - m \). \( \Box \)

Remark 9 (i) Point (i) was known by different methods (see [4,7]).

(ii) The proof of (iii) above shows the following: if \( d \neq 4 \) and if \( d_m = 2d - 5 \), then \( \tau \leq 4 \) or \( h^0(I_\Sigma(1)) = 0 \) but \( \Sigma \) contains a subscheme of length \( \tau - 1 \) lying on a line.

(iii) If \( \Sigma = \{ p \} \), then for any \( d \geq 3 \) we can present \( \Sigma \) as a q.c.i. of type \((d - 1, d - 1, d - 1)\) and, clearly, the term 3.\( \mathcal{O}(1 - d) \) will cancel in (7).

Example 10 We can have \( m = 4 \) and \( \Sigma \) a complete intersection, so the bound of Corollary 6 is sharp.

From the point of view of the jacobian ideal to get a curve \( C \) with \( \tau = 1 \) we may argue as follows. Let \( \mathbb{P} \) denote the blowing-up of \( \mathbb{P}^2 \) at a point. We have \( \mathbb{P} = \mathbb{P}_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \) (see for ex. [2]). Denote by \( h, f \) the classes of \( \mathcal{O}_{\mathbb{P}^1}(1) \) and of a fiber in \( Pic(\mathbb{P}_1) \). We have \( h^2 = 1 = hf, f^2 = 0 \). The exceptional divisor is \( E = h - f \). For any \( a \geq 1 \), the linear system \(|ah + 2f| \) contains a smooth irreducible curve, \( C' \), such that \( C', E = 2 \). The image of \( C' \) in \( \mathbb{P}^2 \) is a curve, \( C \), of degree \( a + 2 \) with \( \tau(C) = 1 \) (for \( a = 1 \) \( C \) is a nodal cubic).

Other examples with \( m = 4 \) and \( \Sigma \) complete intersection can be obtained by taking \( C = A \cup B \) where \( A, B \) are smooth curves, of degrees \( a, b \), intersecting transversally. We have \( d = a + b, \tau = ab \) and \( \Sigma \) is a complete intersection \((a, b)\). Assume \( a \geq 2 \) then, arguing as above, we get \( d_1 = d - 2, d_2 = d_3 = d_4 = d - 1, b_1 = d + a - 2, b_2 = d + b - 2 \) and the corresponding resolution of \( H^0_*(E) \) is minimal.

Another consequence of Lemma 7:

Corollary 11 With notations as in Sect. 2 (in particular \( m \geq 3 \), see Remark 2) we have:

(i) \( d_1 + d_7 \geq d + m - 3, \forall i \geq 2 \)

(ii) \( Z \) is a complete intersection if and only if \( m = 3 \). In that case \( Z \) is a complete intersection of type \((d_1 + d_2 - d + 1, d_1 + d_3 - d + 1)\).

Proof (i) This follows from (6) and Lemma 7.

(ii) Follows from (iii) of Lemma 3. \( \Box \)

Remark 12 Part (i) is proved also in [7] and (ii) is Prop. 3.1. of [6]. The proofs are different.

If \( m = 3 \) and \( d_1 + d_2 = d \), following [6] one says that \( C \) is a plus one generated curve. We see that \( C \) is a plus one generated curve if and only if \( Z \) (of degree \( d_3 - d_2 + 1 \)) is contained in a line. We recover the fact that \( C \) is nearly free (i.e. \( Z \) is a point) if, moreover, \( d_3 = d_2 \).

5 Around the extremal cases in du Plessis–Wall’s theorem

We recall the bound given by du Plessis–Wall ([9], see [11] for a different proof, valid also for q.c.i.): \((d - 1)(d - 1 - d_1) \leq \tau \leq (d - 1)(d - 1 - d_1) + d_1^2 \).

Theorem 13 With notations as in Sect. 2 (in particular \( m \geq 3 \)).
(i) We have $d_1 \leq d_2 \leq d_3 \leq d - 1$.

(ii) We have $d + 1 \geq m$.

(iii) We have $(d - 1)(d - 1 - d_1) = \tau$ if and only if $\Sigma$ is a complete intersection of type $(d - 1, d - 1 - d_1)$. In this case $m = 3$ and $d_2 = d_3 = d - 1$.

(iv) Assume $\tau = (d - 1)(d - 1 - d_1) + 1$. If $\tau > 1$, then $m = 4$ and $d_1 = \{d_1, d - 1, d - 1, d - 3 + d_1\}$ or $d_1 = 1, m = 2$ and $E$ splits like $O(-1) \oplus O(d - 2)$.

**Proof** (i) Let us denote by $g_1, g_2, g_3$ the generators of degrees $d_1, d_2, d_3$ of $H^0_*(E)$. We will consider the $g_i$’s as relations among the partials.

Consider the Koszul relations: $K_z = (f_y, -f_x, 0), K_y = (f_z, 0, -f_x), K_x = (0, f_z, -f_y)$. We have:

$$f_z K_z - f_y K_y + f_x K_x = 0$$

(9)

The relations $K_x, K_y, K_z$ correspond to sections $s_x, s_y, s_z$ of $E(d - 1)$. It follows that $d_1 \leq d - 1$. We also clearly have $d_2 \leq d - 1$. Indeed otherwise $K_x, K_y, K_z$ are multiple of $g_1 = (u_1, v_1, w_1)$, which is impossible ($P(u_1, v_1, w_1) = (f_y, -f_x, 0)$ implies $w_1 = 0$ and going on this way we get $g_1 = 0$). If $d_3 \geq d$, these sections are combinations of $g_1, g_2$ only. Now (9) yields a relation involving only $g_1$ and $g_2$. We claim that this relation is non trivial.

Indeed let $s_x = a_1 g_1 + b g_2, s_y = a' g_1 + b' g_2, s_z = a'' g_1 + b'' g_2$. Then (9) becomes: $g_1(a f_x - a' f_y + a'' f_z) + g_2(b f_y - b' f_y + b'' f_z) = 0$. Assume $a f_x - a' f_y + a'' f_z = 0$ and $b f_y - b' f_y + b'' f_z = 0$. Then $\alpha = (a, -a', a'')$ determines a section of $E(d - 1 - d_1)$ and $\beta = (b, -b', b'')$ a section of $E(d - 1 - d_2)$. Since $d - 1 - d_2 \leq d - 1$ (Corollary 11), we get $\beta = 0$, hence $b = b' = b'' = 0$. Since $d - 1 - d_1 \leq d - 1$ (Corollary 11), we see that $\alpha$ is a multiple of $g_1$: $(a, -a', a'') = P(u_1, v_1, w_1)$. It follows that $a = Pu_1$. Moreover $s_x = (0, f_z, -f_y) = a g_1 = (P u_1^2, P u_1 v_1, P u_1 w_1)$ and it follows that $P u_1 = 0 = a$, hence $s_x = 0$, which is impossible.

So we have a non trivial relation $Ag_1 = B g_2$. We may assume $(A, B) = 1$ (otherwise just divide by the common factors). It follows that $B$ divides every components $u_1, v_1, w_1$ of $g_1$ and we get a relation $(u_1', v_1', w_1')$ of degree $< d_1$, against the minimality of $d_1$. We conclude that $d_3 \leq d - 1$.

(ii) From (i) we have $2d - 2 \geq d_1 + d_3$. We conclude with Corollary 11.

(iii) Assume $\tau = (d - 1)(d - 1 - d_1)$. Since $I_{\Sigma}(d - 1)$ is generated by global sections we can link $\Sigma$ to a subscheme $\Gamma$ by a complete intersection $F \cap G$ of type $(d - 1, d - 1)$. Clearly $\deg(\Gamma) = (d - 1)^2 - \tau = d_1(d - 1)$. By mapping cone we have (after simplifications): $0 \to O \to E(d - 1) \to I_{\Gamma}(d - 1) \to 0$. Twisting by $1 - d + d_1$ we get: $0 \to O(1 - d + d_1) \to E(d_1) \to I_{\Gamma}(d_1) \to 0$. Since $\tau > 0, d_1 < d - 1$, hence $h^0(I_{\Gamma}(d_1)) \neq 0$. It follows that $\Gamma$ is contained in a complete intersection $(d_1, d - 1)$. Indeed the base locus of the linear system of curves of degree $d - 1$ containing $\Gamma$ has dimension zero (consider $F \cap G$) and $d_1 < d - 1$. For degree reasons $\Gamma$ is a complete intersection $(d_1, d - 1)$ and we have $0 \to O(1 - d - d_1) \to O(-d_1) \oplus O(1 - d) \to I_{\Gamma} \to 0$. By mapping cone again: $0 \to O(1 - d) \oplus O(d_1 - 2d_2 + 2) \to O(d_1 + 1 - d) \oplus 2O(1 - d) \to I_{\Sigma} \to 0$. We claim that we can cancel the repeated term $O(1 - d)$. Indeed, since $\dim(F \cap G) = 0$, we may assume that $F$ or $G$ is not a multiple of $S$, the curve of degree $d_1$ containing $\Gamma$, hence $F$ or $G$ is a minimal generator of $H^0_*(I_{\Gamma})$. It follows that $\Sigma$ is a complete intersection. We conclude with Proposition 4.

Conversely if $\Sigma$ is a complete intersection $(d - 1, d - 1 - d_1)$, from Proposition 4 we get $m = 3$ and $d_2 = d_3 = d - 1$.

(iv) We argue as above. The assumption $\tau > 1$ makes sure that $h^0(I_{\Gamma}(d_1)) \neq 0$. This time we find that $\Gamma$ is linked to one point by a complete intersection $(d - 1, d_1)$. By mapping cone we get: $0 \to 2O(-d - d_1 + 2) \to O(-d - d_1 + 3) \oplus O(-d_1) \oplus O(-d + 1) \to I_{\Gamma} \to 0$. 

\[ \text{Springer} \]
This resolution is minimal except if \( d_1 = 1 \) in which case we have: \( 0 \to O(1 - d) \to O(2 - d) \oplus O(-1) \to \mathcal{I}_Z \to 0 \). As we have seen above \( \Gamma = (s) \) where \( s \in H^0(E(d - 1)) \). If \( s \) is a minimal generator of \( H^0(E) \), then \( H^0(I_Z) \) has \( m - 1 \) minimal generators, otherwise it has \( m \) minimal generators. So if \( d_1 > 1, 3 \leq m \leq 4 \). By mapping cone we go back to \( \Sigma \). If \( d_1 > 1 \) we get: \( 0 \to O(-d + d_1 - 1) \oplus O(-2d + 2 + d_1) \to 2O(-d + d_1) \oplus O(1 - d) \to \mathcal{I}_Z \to 0 \). From Proposition 4 we conclude that \( m = 4 \) and \( \{ d_i \} = \{ d_1, d - 1, d_1, d - 3 + d_1 \} \). If \( d_1 = 1 \), by mapping cone we get \( 0 \to O(-2d + 3) \oplus O(-d) \to 3O(1 - d) \to \mathcal{I}_Z \to 0 \). This resolution is minimal. Hence \( m = 2 \) and \( E \) splits like \( O(-d + 2) \oplus O(-1) \).

\[ \square \]

**Remark 14** See [6] for a different proof of part (i). Point (ii) is proved in [7].

Since the minimal free resolution of sets of points of low degree are known (see for example [12] for a list), the analysis above can be extended to the cases \( \tau = (d - 1)(d - 1 - d_1) + x \), for small \( x \).

It is easy to show that if \( \tau \) reaches the upper-bound in the first part of du Plessis–Wall’s Theorem, then \( E \) splits (because \( c_2(E(d_1)) = 0 \) and \( h^0(E(d_1)) \neq 0 \) i.e. \( \Sigma \) is an almost complete intersection (or \( C \) is a free curve). However there is a second part in du Plessis–Wall’s theorem: under the assumption \( 2d_1 + 1 > d \) (which amounts to say that \( E \) is stable), we have a better upper-bound: \( \tau \leq \tau_+ := (d - 1)(d - 1 - d_1) + d_1^2 - \frac{1}{2}(2d_1 + 1 - d)(2d_1 + 2 - d) \). Notice that this holds true also for q.c.i. [11].

In [7] Thm. 3.1, the authors prove that this bound is reached if and only if we have:

\[ 0 \to (m - 2)O(-d_1 - 1) \to mO(d_1) \to E \to 0 \tag{10} \]

with \( m = 2d_1 - d + 3 \).

This can be proved as follows. From the exact sequence (3) we have \( h^0(I_Z(2d_1 - d)) = 0 \) (observe that \( Z \neq \emptyset \) because \( 2r + 1 > d \)). It follows that \( \deg(Z) \geq h^0(O(2d_1 - d)) \). The assumption \( \tau = \tau_+ \) implies [use (4)] that we have equality: \( \deg(Z) = h^0(O(2d_1 - d)) \). This implies \( h^1(I_Z(2d_1 - d)) = 0 \). It follows (Castelnuovo–Mumford’s lemma or numerical character) that the minimal free resolution of \( I_Z \) is: \( 0 \to sO(-s - 1) \to (s + 1)O(-s) \to I_Z \to 0 \), with \( s = 2d_1 - d + 1 \). We conclude with Lemma 3.

Conversely if we have (10), by Lemma 3 we get that \( I_Z \) has a linear resolution and \( \deg(Z) = h^0(O(2d_1 - d)) \). This implies \( \tau = \tau_+ \).

Then the authors ask ([7] Conjecture 1.2) if for any integer \( d \geq 3 \) and for any integer \( r, d/2 \leq r \leq d - 1 \), there exists \( \Sigma \) with \( d_1 = r \) and \( \tau = \tau_+ \). I don’t know the answer in general but, in the framework of q.c.i., the answer is yes:

**Proposition 15** With notations as above, for every \( d \geq 3 \) and for every integer \( r, d/2 \leq r \leq d - 1 \), there exists a q.c.i. subscheme \( \Sigma \subset \mathbb{P}^2 \), of degree \( \tau_+ \), with \( d_1 = r \)

**Proof** We recall that a general set of \( s(s + 1)/2 \) points has a linear resolution:

\[ 0 \to sO(-s - 1) \to (s + 1)O(-s) \to I_Z \to 0 \tag{11} \]

Actually to have such a resolution is equivalent to have \( h^0(I_Z(s - 1)) = 0 \). Since the Cayley–Bacharach condition \( CB(s - 3) \) (see for instance [3]) is obviously satisfied we may associate a rank two vector bundle to \( I_Z(s) \): \( 0 \to O \to E \to I_Z(s) \to 0 \). We have \( c_1(E) = s \) and \( c_2(E) = s(s + 1)/2 = \deg(Z) \). Since \( h^1(O) = 0 \) and \( I_Z(s) \) and \( O \) are globally generated, \( E \) also is globally generated. For \( a \geq 0 \) let us consider a section of \( E(a) \): \( 0 \to O \to E(a) \to I_Z(2a + s) \to 0 \). For \( k \geq a + s \), \( I_Z(k) \) is globally generated and we can link \( \Gamma \) to \( \Sigma \) by a complete intersection of type \((k, k)\). By mapping cone we get, if \( k = 2a + s \):

\[ 0 \to E(-3a - 2s) \to 3O(-2a - s) \to I_Z \to 0 \tag{12} \]
We have \( c_2(\mathcal{E}(a)) = as + s(s + 1)/2 + a^2 = \deg(\Gamma) \). It follows that \( \tau := \deg(\Sigma) = 3a^2 + 3as + s(s − 1)/2 \). Since \( d_1 = a + s \) (\( E := \mathcal{E}(−a − s) \)), it is easy to check that \( \tau = \tau_+ \).

Let \( d \) be an integer. Assume \( d \) odd, \( d = 2\delta + 1 \). For \( 1 \leq \rho \leq \delta \), set \( a = \delta − \rho, s = 2\rho, d_1 = a + s \) and \( d = 2a + s + 1 \). Then the construction above yields \( \Sigma \) of degree \( \tau_+, \) q.c.i. of three curves of degree \( d − 1 \), with \( d_1 = a + s \). We have \( \delta + 1 \leq d_1 \leq 2\delta \).

If \( d = 2\delta \), for \( 0 \leq \rho \leq \delta − 1 \), set \( a = \delta − \rho − 1 \) and \( s = 2\rho + 1 \) (\( d_1 = a + s \)). \( \Box \)

**Remark 16** It is not clear at all that there are examples with \( \Sigma \) a jacobian set. For some partial results see [7], Section 4.

More generally to characterize the zero-dimensional subschemes that are jacobian sets seems quite a challenge.

It is possible to give a little improvement, namely:

**Proposition 17** Assume \( 2d_1 + 1 > d \) and \( \tau = \tau_+ − 1 \). Set \( s := 2d_1 − d \). Then we have two possibilities:

(a) The minimal free resolution of \( \mathcal{I}_Z \) is:

\[
0 → \mathcal{O}(−s − 2) ⊕ (s − 2) \mathcal{O}(−s − 1) → s \mathcal{O}(−s) → \mathcal{I}_Z → 0
\]

In this case \( m = 2d_1 − d + 1 \) and \( d_i = d_1, \forall i \).

(b) The minimal free resolution of \( \mathcal{I}_Z \) is:

\[
0 → \mathcal{O}(−s − 2) ⊕ (s − 1) \mathcal{O}(−s − 1) → \mathcal{O}(−s − 1) ⊕ s \mathcal{O}(−s) → \mathcal{I}_Z → 0
\]

In this case \( m = 2d_1 − d + 2 \) and \( d_i = d_1, 2 ≤ i < m, d_m = d_1 + 1 \).

**Proof** Arguing exactly as above this time we have \( \deg Z = h^0(\mathcal{O}(s − 1)) + 1, h^0(\mathcal{I}_Z(s − 1)) = 0, \) hence \( h^1(\mathcal{I}_Z(s − 1)) = 1 \). Let \( 0 → \bigoplus \mathcal{O}(−\beta_j) → \bigoplus \mathcal{O}(−\alpha_i) → \mathcal{I}_Z → 0 \) denote the minimal free resolution of \( \mathcal{I}_Z \). Since \( \beta^+ > \alpha^+ (\beta^+ = \max \{ \beta_j \} \) and the same for \( \alpha^+ \) \) and since \( \beta^+ − 3 = \max \{ k \mid h^1(\mathcal{I}_Z(k)) \neq 0 \} \), we see that \( \beta^+ = s + 2 \) (with coefficient equal to \( 1 \) because \( h^1(\mathcal{I}_Z(s − 1)) = 1 \)). It follows that \( \mathcal{I}_Z(\mathcal{I}_Z) \) is generated in degrees \( s + 1 \). Of course we have \( s \) minimal generators of degree \( s \) and in general nothing else (it is easy to produce examples for any \( s \)). We conclude that in this case the resolution is like in (a).

What about generators of degree \( s + 1 \)? If there at least two such generators, then the matrix of the resolution has two rows of the form \( (L, 0, \ldots, 0) \). By erasing another row, we get a maximal minor which is zero, but this is impossible (the maximal minors are the generators). So there is at most one generator of degree \( s + 1 \). In this case the resolution is like in (b). Examples exist for any \( s \): take \( s + 1 \) points on a line and the remaining ones in general position. \( \Box \)

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