Pluripolar graphs are holomorphic

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Abstract. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}$ be a continuous function. We prove that the graph $\Gamma(f)$ of the function $f$ is a pluripolar subset of $\mathbb{C}^{n+1}$ if and only if $f$ is holomorphic.

1. Introduction

A function $\varphi$ defined on a domain $U \subset \mathbb{C}^n$ with values in $[-\infty, +\infty)$ is called plurisubharmonic in $U$ if $\varphi$ is upper semicontinuous and its restriction to the components of the intersection of a complex line with $U$ is subharmonic.

A set $E \subset \mathbb{C}^n$ is called pluripolar if there is a neighbourhood $U$ of $E$ and a plurisubharmonic function $\varphi$ on $U$ such that $E \subset \{ \varphi = -\infty \}$. By a result of B. Josefson [J], the function $\varphi$ in this definition can be chosen to be plurisubharmonic in the whole of $\mathbb{C}^n$ (i.e. $U = \mathbb{C}^n$).

In 1963 T. Nishino raised the following question in connection to his paper [N1]:

Let $\Delta$ be the unit disc in $\mathbb{C}$ and let $f : \Delta \to \mathbb{C}$ be a continuous function such that its graph $\Gamma(f)$ is a pluripolar subset of $\mathbb{C}_z\times \mathbb{C}_w$. Does it follow that $f$ is holomorphic?

The main result of this paper gives a positive answer to Nishino’s question and can be formulated as follows.

Theorem. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $f : \Omega \to \mathbb{C}$ be a continuous function. The graph $\Gamma(f)$ of the function $f$ is a pluripolar subset of $\mathbb{C}^{n+1}$ if and only if $f$ is holomorphic.

As a consequence of Theorem one can easily obtain the following more general statement.

Corollary. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $E$ be a closed subset of $\Omega \times \mathbb{C} \subset \mathbb{C}^{n+1}$ such that the fibers $E(z) = \{ w \in \mathbb{C} : (z, w) \in E \}$ of $E$ are finite and
depend continuously on \( z \in \Omega \) in the Hausdorff metric. Assume that the number \( \#E(z) \) of points in the fiber \( E(z) \) is bounded from above in \( \Omega \). Then \( E \) is a pluripolar subset of \( \mathbb{C}^{n+1}_{z,w} \) if and only if it has the form

\[
E = \{(z, w) \in \Omega \times \mathbb{C}_w : w^m + a_1(z)w^{m-1} + ... + a_m(z) = 0\},
\]

where the functions \( a_1(z), a_2(z), ..., a_m(z) \) are holomorphic in \( \Omega \).

Note that the proof of Theorem can not be directly applied to the set \( E \) described in Corollary. Namely, the topological argument used in the proof of Lemma 3 and based on the fact that the first homology group \( H_1(\Omega \times \mathbb{C}_w \setminus \Gamma(f), \mathbb{Z}) \) is nontrivial does not work in this case. In the last section of the paper we construct an example of a compact subset \( E \) of \( \Delta \times \mathbb{C}_w \subset \mathbb{C}^2_{z,w} \) with finite fibers \( E(z) \) depending continuously on \( z \in \Delta \) in the Hausdorff metric such that \( H_1(\Delta \times \mathbb{C}_w \setminus E, \mathbb{Z}) = 0 \). In particular, there is a neighbourhood \( U(E) \) of \( E \) which does not contain any subset of \( \Delta \times \mathbb{C}_w \) defined by a Weierstrass pseudopolynomial (i.e. defined by the equation (1) with \( a_1(z), a_2(z), ..., a_m(z) \) being continuous functions in \( \Omega \)).

Remark. In the special case when the function \( f \) is assumed to be \( C^1 \)-smooth and its graph \( \Gamma(f) \) is assumed to be complete pluripolar (i.e. \( \Gamma(f) = \{ \varphi = -\infty \} \)) for some function \( \varphi \), plurisubharmonic in a neighbourhood of \( \Gamma(f) \), a positive answer to Nishino’s question was given by Ohsawa [O] using \( L^2 \) estimates for \( \bar{\partial} \). In this case one can also apply Pinchuk’s method adapted to \( C^1 \)-surfaces in [CH, p. 59-62] and construct, to get a contradiction, a one-parameter family of holomorphic disks \( \{D_\alpha\} \) attached to a totally real piece of \( \Gamma(f) \) by an arc on the boundary. Restricting the plurisubharmonic function \( \varphi \) such that \( \Gamma(f) \subset \{ \varphi = -\infty \} \) to each of these disks, we get \( \varphi \equiv -\infty \) on \( D_\alpha \) and, hence, \( \bigcup_\alpha D_\alpha \subset \{ \varphi = -\infty \} \) which gives the desired contradiction, since the set \( \bigcup_\alpha D_\alpha \) has real dimension 3. Note that neither of the methods mentioned here can be applied to prove our Theorem.

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2. Preliminaries

For bounded nonempty sets $E_1$ and $E_2$ in $\mathbb{C}_w$, the Hausdorff distance is defined as

$$d(E_1, E_2) = \sup_{w_2 \in E_2} \inf_{w_1 \in E_1} |w_1 - w_2| + \sup_{w_2 \in E_2} \inf_{w_1 \in E_1} |w_1 - w_2|.$$ 

A family of compact sets $E(z)$ in $\mathbb{C}_w$ parametrized by $z \in \Omega \subset \mathbb{C}_z^n$ is said to be continuously dependent on $z$ in the Hausdorff metric if, for each sequence $\{z_n\}$ of points in $\Omega$ converging to a point $z_0 \in \Omega$, one has $d(E(z_n), E(z_0)) \to 0$ as $n \to \infty$. In particular, if $\Omega$ is a domain in $\mathbb{C}_z^n$ and $E$ is a nonempty closed subset of $\Omega \times \mathbb{C}_w$ with bounded fibers $E(z) = \{w \in \mathbb{C}_w : (z, w) \in E\}$ depending continuously on $z \in \Omega$ in the Hausdorff metric, then each fiber $E(z)$, $z \in \Omega$, is nonempty.

For a compact set $K$ in $\mathbb{C}^n$, the polynomial hull $\hat{K}$ of $K$ is defined as

$$\hat{K} = \{z \in \mathbb{C}^n : |P(z)| \leq \sup_{w \in K} |P(w)| \text{ for all holomorphic polynomials } P \text{ in } \mathbb{C}^n\}.$$ 

The set $K$ is called polynomially convex if $\hat{K} = K$.

The first simple lemma is classical and follows, for example, from the theorem 4.3.4 in [H].

**Lemma 1** A compact set $K$ in $\mathbb{C}^n$ is polynomially convex if and only if for any point $Q \in \mathbb{C}^n \setminus K$ there is a function $\varphi$, plurisubharmonic in $\mathbb{C}^n$, such that

$$\sup_{z \in K} \varphi(z) < \varphi(Q).$$

(2)

**Lemma 2** Let $K$ be a polynomially convex compact set in $\mathbb{C}^n$ and let $E$ be a pluripolar compact set in $\mathbb{C}^n$. Then the set $(\hat{K} \cup \hat{E}) \setminus K$ is pluripolar.

**Proof.** From pluripolarity of the set $E$ it follows that there is a function $\varphi_E$, plurisubharmonic in $\mathbb{C}^n$, such that $E \subset \{ \varphi_E = -\infty \}$. To prove Lemma 2, we shall prove that $(\hat{K} \cup \hat{E}) \setminus K \subset \{ \varphi_E = -\infty \}$.

Assume, by contradiction, that there is a point $Q \in (\hat{K} \cup \hat{E}) \setminus K$ such that $\varphi_E(Q) > -\infty$. Since $Q \notin K$, and since the set $K$ is polynomially convex, it follows from Lemma 1 that there is a function $\varphi_K$, plurisubharmonic in $\mathbb{C}^n$, such that $\sup_{z \in K} \varphi_K(z) < \varphi_K(Q)$. Then, for $\varepsilon$ positive and small enough, one also has that $\sup_{z \in K} (\varphi_K(z) + \varepsilon \varphi_E(z)) < \varphi_K(Q) + \varepsilon \varphi_E(Q)$. Since $\varphi_E(z) = -\infty$ for $z \in E$, it follows that $\sup_{z \in (\hat{K} \cup \hat{E})} (\varphi_K(z) + \varepsilon \varphi_E(z)) < \varphi_K(Q) + \varepsilon \varphi_E(Q)$. By Lemma 1 applied to the function $\varphi_K + \varepsilon \varphi_E$, we get that $Q \notin (\hat{K} \cup \hat{E})$. This gives the desired contradiction. \qed

The next statement was first proved by H. Alexander (see Corollary 1 in [A]). For the convenience of reading we include here its proof.
Lemma 3 Let \( U \) be a bounded domain in \( \mathbb{C}_z \times \mathbb{R}_u \subset \mathbb{C}_{z,w}^2 (w = u + iv) \) and let \( g : bU \to \mathbb{R}_u \) be a continuous function. Then \( U \subset \pi(\Gamma(g)) \), where \( \Gamma(g) \) is the graph of \( g \) and \( \pi : \mathbb{C}_{z,w}^2 \to \mathbb{C}_z \times \mathbb{R}_u \) is the projection.

Proof. Consider an approximation of the domain \( U \) by an increasing sequence \( \{U_n\} \) of domains with smooth boundary. Further, consider a sequence of smooth functions \( \{g_n\} : g_n : bU_n \to \mathbb{R}_u \), which approximate the function \( g \), i.e., \( \Gamma(g_n) \to \Gamma(g) \) in the Hausdorff metric. Then it follows from the definition of polynomial hull that \( \limsup_{n \to \infty} \Gamma(g_n) \subset \Gamma(g) \), where convergence is understood to be in the Hausdorff metric. Hence, it is enough to prove the statement of Lemma 3 in the case where the domain \( U \) has a smooth boundary and the function \( g \) is smooth.

Now we argue by contradiction and suppose that there is a point \( Q \in U \setminus \pi(\Gamma(g)) \). Without loss of generality we may assume that \( Q \) is the origin \( O \) in \( \mathbb{C}_z \times \mathbb{R}_u \). We know by Browder [B] that \( H^2(\Gamma(g), \mathbb{C}) = 0 \) (here \( H^2(\Gamma(g), \mathbb{C}) \) is the second Čech cohomology group with complex coefficients). Then, by Alexander duality (see, for example [S], p.296), we get \( H_1(\mathbb{C}_{z,w}^2 \setminus \Gamma(g), \mathbb{C}) = H^2(\Gamma(g), \mathbb{C}) = 0 \) (here \( H_1(\mathbb{C}_{z,w}^2 \setminus \Gamma(g), \mathbb{C}) \) is the first singular homology group with complex coefficients). On the other hand, since \( O \in U \setminus \Gamma(g) \), it follows that the curve \( \gamma_R \) consisting of the segment \( \{z = 0, u = 0, -R \leq v \leq R\} \) and the half-circle \( \{z = 0, w = \text{Re}^{i\theta}, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\} \) do not intersect the set \( \Gamma(g) \) for \( R \) big enough. Moreover, the linking number of \( \Gamma(g) \) and \( \gamma_R \) is not equal to zero. Therefore, \( H_1(\mathbb{C}_{z,w}^2 \setminus \Gamma(g), \mathbb{C}) \neq 0 \). This is a contradiction and the lemma follows. \( \square \)

Lemma 4 Let \( U \) be a simply connected domain in \( \mathbb{C}_z \) and let \( f(z) = u(z) + iv(z) : U \to \mathbb{C}_w \) be a function such that both \( u(z) \) and \( v(z) \) are harmonic in \( U \). If the graph \( \Gamma(f) \) of the function \( f \) is a pluripolar subset of \( \mathbb{C}_{z,w}^2 \), then \( f \) is holomorphic.

Proof. If \( f \) is not holomorphic, we argue by contradiction and suppose that the set \( \Gamma(f) \) is pluripolar. Then there is a function \( \varphi \), plurisubharmonic in \( \mathbb{C}_{z,w}^2 \), such that \( \Gamma(f) \subset \{\varphi = -\infty\} \). Let \( \tilde{v} \) be the harmonic conjugate function to \( u \) in the domain \( U \) such that \( \tilde{v}(z_0) = v(z_0) \) for some fixed point \( z_0 \in U \). Then the set \( \{z \in U : \tilde{v}(z) + \varepsilon = v(z)\} \) is nonempty and consists of real analytic curves for all \( \varepsilon \) small enough. Therefore, each of the holomorphic curves \( \Gamma_{\varepsilon} = \{(z, w) : z \in U, w = u(z) + i(\tilde{v}(z) + \varepsilon)\} \) intersects the set \( \Gamma(f) \subset \{\varphi = -\infty\} \) in real analytic curves. Since a real analytic curve is not polar (see, for example [T, Th.II.26, p.50]), it follows that \( \Gamma_{\varepsilon} \subset \{\varphi = -\infty\} \) for all \( \varepsilon \) small enough. This implies that \( \varphi \equiv -\infty \) in \( \mathbb{C}_{z,w}^2 \) and gives the desired contradiction. \( \square \)
3. Proof of Theorem and Corollary

Proof of Theorem. If the function $f$ is holomorphic, then the same argument as in the proof of Lemma 4 shows that $\Gamma(f)$ is pluripolar. Namely, the function

$$\varphi(z_1, \ldots, z_{n+1}) = \ln |z_{n+1} - f(z_1, \ldots, z_n)|$$

is plurisubharmonic in $\Omega \times \mathbb{C}$ and $\Gamma(f) = \{\varphi = -\infty\}$. Therefore, the set $\Gamma(f)$ is pluripolar in $\mathbb{C}^{n+1}$.

Suppose now that the graph $\Gamma(f)$ of $f$ is pluripolar. To prove that $f$ is holomorphic we consider two cases.

1. The special case $n = 1$. In this case $\Omega$ is a domain in $\mathbb{C}_z$ and $f(z) = u(z) + iv(z) : \Omega \to \mathbb{C}_w$ is a continuous function such that its graph is pluripolar. Since holomorphicity is a local property, we can restrict ourselves to the case when $\Omega$ is a disc in $\mathbb{C}_z$ and, moreover, to simplify our notations, we can assume without loss of generality that $\Omega = \Delta = \{z : |z| < 1\}$ is the unit disc and that the function $f$ is continuous on its closure $\bar{\Delta}$. It follows from Lemma 4 that either the function $f$ is holomorphic or at least one of the functions $u$ and $v$ is not harmonic. Since both cases can be treated the same way, we can, to get a contradiction, assume that the function $u$ is not harmonic. Denote by $\tilde{u}$ the solution of the Dirichlet problem on $\Delta$ with boundary data $u$. Since $u$ is not harmonic, one has that $\tilde{u} \neq u$ in $\Delta$. Without loss of generality we can assume that

$$u(z_0) < \tilde{u}(z_0)$$

for some $z_0 \in \Delta$. Let

$$C = \max\{\sup_{z \in \bar{\Delta}} |u(z)|, \sup_{z \in \bar{\Delta}} |v(z)|\}.$$

Consider the set

$$K = \{(z, w) \in \bar{\Delta} \times \mathbb{C}_w : \tilde{u}(z) \leq u \leq 3C, |v| \leq C\}.$$

Lemma 5 The set $K$ is polynomially convex.

Proof. To prove polynomially convexity of $K$ we use Lemma 1. Consider an arbitrary point $(z^*, w^*) \in \mathbb{C}^2_{z,w} \setminus K$. If the point $(z^*, w^*)$ belongs to the set

$$A_1 = \{(z, w) \in \mathbb{C}^2_{z,w} : |z| > 1 \text{ or } u > 3C \text{ or } |v| > C\},$$

then inequality (2) will be satisfied for the point $Q = (z^*, w^*)$ and the function

$$\varphi_1(z, w) = \max\{|z| - 1, u - 3C, |v| - C\}$$
plurisubharmonic in $\mathbb{C}^2_{z,w}$.

If the point $(z^*, w^*)$, $w^* = u^* + iv^*$ belongs to the set

$$A_2 = \{(z, w) \in \Delta \times \mathbb{C}_w : u < \tilde{u}(z)\},$$

then $u^* < \tilde{u}(z^*)$. Let $\varepsilon = \frac{1}{3}(\tilde{u}(z^*) - u^*)$ and consider a function $\tilde{u}_\varepsilon$ harmonic on the whole of $\mathbb{C}_z$ such that $\max_{z \in \Delta} |\tilde{u}(z) - \tilde{u}_\varepsilon(z)| < \varepsilon$. Since for $(z, w) \in K$ one has $u \geq \tilde{u}(z) \geq \tilde{u}_\varepsilon(z) - \varepsilon$, and since $u^* = \tilde{u}(z^*) - 3 \varepsilon < \tilde{u}_\varepsilon(z^*) - 2 \varepsilon$, it follows that inequality (2) will be satisfied for the point $Q = (z^*, w^*)$ and the function

$$\varphi_2(z, w) = \tilde{u}_\varepsilon(z) - u$$

plurisubharmonic in $\mathbb{C}^2_{z,w}$.

Since $\mathbb{C}^2_{z,w} \setminus K = A_1 \cup A_2$, we conclude from Lemma 1 that the set $K$ is polynomially convex. This completes the proof of Lemma 5.

Consider now the domain

$$U = \{(z, u) \in \Delta \times \mathbb{R}_u : u(z) < u < u(z) + 2C\}$$

in $\mathbb{C}_z \times \mathbb{R}_u$ and the real-valued function $g(z, u) = v(z)$ on $bU$. Since $\sup_{z \in \Delta} |u(z)| \leq C$, one has $\sup_{z \in \Delta} |\tilde{u}(z)| \leq C$ and hence $\tilde{u}(z) \leq u(z) + 2C \leq 3C$. It then follows from the definitions of $U$ and $g$ that the graph $\Gamma(g)$ of the function $g$ is contained in the set $\Gamma(f) \cup K$. Therefore, we get $\Gamma(\tilde{g}) \subset (\Gamma(f) \cup K)$. Since, by Lemma 3, $\pi(\Gamma(\tilde{g})) \supset U$, we conclude that

$$\pi(\Gamma(f) \cup K) \supset U. \quad (4)$$

Consider the following open subset of $U$:

$$\tilde{U} = \{(z, u) \in \Delta \times \mathbb{R}_u : u(z) < u < \tilde{u}(z)\}.$$ 

Inequality (3) obviously implies that the set $\tilde{U}$ is nonempty. Since, by the definition of the sets $K$ and $\tilde{U}$, $\pi(K) \cap \tilde{U} = \emptyset$, it follows from (4) that

$$\pi((\Gamma(f) \cup K) \setminus K) \supset \tilde{U}. \quad (5)$$

Since, by our assumption, the graph $\Gamma(f)$ of $f$ is pluripolar, we conclude from Lemma 2 and Lemma 5 that the set $(\Gamma(f) \cup K) \setminus K$ is pluripolar, i.e.,

$$(\Gamma(f) \cup K) \setminus K \subset \{\varphi = -\infty\} \quad (6)$$

for some plurisubharmonic function $\varphi$. 

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From (3) one has that there is a neighbourhood $V$ of the point $z_0$ in $\mathbb{C}$ such that
\[
u(z) < \tilde{u}(z)
\]
for all $z \in V$. For each $a \in \mathbb{C}$ consider the complex line $\ell_a = \{(z, w) \in \mathbb{C}^2 : z = a\}$ and the set
\[E_a = ((\Gamma(f) \cup K) \setminus K) \cap \ell_a.
\]
It follows from (5) and (7) that for $a \in V$ the projection of $E_a$ on the real line $\ell_a \cap \{v = 0\}$ contains an open segment. Since a polar set in $\mathbb{C}$ has Hausdorff dimension zero (see, for example [T, Th.III.19, p.65]), it cannot be projected on an open segment in $\mathbb{R}$. Therefore, the set $E_a$ is not polar. It then follows from (6) that $\varphi \equiv -\infty$ on $\ell_a$. Since this argument holds true for all $a \in V$, we conclude that $\varphi \equiv -\infty$ on $\mathbb{C}^2_{z,w}$. This contradiction proves Theorem in the case $n = 1$.

2. The general case. Let $k$ be one of the numbers $1, 2, \ldots, n$. For each $a = (a_1, a_2, \ldots, a_n) \in \Omega$ consider the function
\[f^a_k(z_k) = f(a_1, \ldots, a_{k-1}, z_k, a_{k+1}, \ldots, a_n)
\]
defined on the domain
\[\Omega^a_k = \Omega \cap \{z_1 = a_1, \ldots, z_{k-1} = a_{k-1}, z_{k+1} = a_{k+1}, \ldots, z_n = a_n\} \subset \mathbb{C}_{z_k}.
\]
Since, by our assumptions, the set $\Gamma(f)$ is pluripolar, there is a function $\varphi$, plurisubharmonic in $\mathbb{C}^{n+1}$, such that $\Gamma(f) \subset \{\varphi = -\infty\}$. For all points $a$ except for a pluripolar set in $\mathbb{C}^n$ one obviously has that the function
\[\varphi^a_k(z_k, z_{n+1}) = \varphi(a_1, \ldots, a_{k-1}, z_k, a_{k+1}, \ldots, a_n, z_{n+1})
\]
is not identically equal to $-\infty$ in $\mathbb{C}^2_{z_k, z_{n+1}}$. For all such points $a$ we can use the argument from case 1 and conclude from continuity of the function $f^a_k : \Omega^a_k \to \mathbb{C}_{z_{n+1}}$ and from the inclusion $\Gamma(f^a_k) \subset \{\varphi^a_k = -\infty\}$ that the function $f^a_k$ is holomorphic. Since the complement of a pluripolar set is everywhere dense, it follows from continuity of $f$ that the functions $f^a_k$ are holomorphic for all $a \in \Omega$. This argument holds true for any $k = 1, 2, \ldots, n$, so we conclude from the classical Hartogs theorem on separate analyticity that the function $f$ is holomorphic. The proof of Theorem is now completed.

\[\square\]

**Proof of Corollary.** Since, by our assumption, the number $\#E(z)$ of points in the fiber of $E$ is bounded from above in $\Omega$, we can consider $k = \max_{z \in \Omega} \#E(z)$ and then the open subset $U = \{z \in \Omega : \#E(z) = k\} \subset \Omega$. Let $z_0$ be a point of $U$ and let $h_i(z)$, $i = 1, 2, \ldots, k$, be the functions defining single-valued branches of $E(z)$ in a neighbourhood $U$ of $z_0$. Since, by our assumption, $E(z)$ depends continuously on $z \in \Omega$ in the Hausdorff metric, we conclude from Theorem that
the functions $h_i(z)$ are holomorphic in $U$. Hence, $F(z) = \Pi_i \neq j (h_i(z) - h_j(z))$ is a well defined holomorphic function in $U$ such that for each $z' \in bU \cap \Omega$ one has $F(z) \to 0$ as $z \to z'$, $z \in U$. Then the function

$$\tilde{F}(z) = \begin{cases} F(z), & \text{for } z \in U, \\ 0, & \text{for } z \in \Omega \setminus U \end{cases}$$

is continuous in $\Omega$ and holomorphic in $U = \Omega \setminus \{ z : \tilde{F}(z) = 0 \}$. Therefore, by Radó’s theorem (see, e.g. [Č], p. 302), $\tilde{F}$ is holomorphic in $\Omega$. In particular, the set $\{ z \in \Omega : \tilde{F}(z) = 0 \}$ is an analytic hypersurface.

Consider now the function $\Pi_{i=1}^k (w - h_i(z)) = w^k + a_1(z)w^{k-1} + \ldots + a_k(z)$. Since $a_1(z), a_2(z), \ldots, a_k(z)$ are symmetric functions of $h_1(z), h_2(z), \ldots, h_k(z)$, they are well defined and holomorphic in $U$. Moreover, since $E(z)$ depends continuously on $z \in \Omega$ in the Hausdorff metric, these functions are locally bounded near the set $\Omega \setminus U = \{ z : \tilde{F}(z) = 0 \}$. It follows then from removability of analytic singularities that the functions $a_1(z), a_2(z), \ldots, a_k(z)$ are holomorphic in the whole of $\Omega$. Since, by our construction, $E = \{ (z, w) \in \Omega \times \mathbb{C}_w : w^m + a_1(z)w^{m-1} + \ldots + a_m(z) = 0 \}$, the corollary follows. □

Remark. The statement of Corollary was first proved in [Sh] for sets represented by Weierstrass pseudopolynomials by a different (and more complicated) method. It was later observed independently by the author and by A. Edigarian [E] that the methods of chapter 4 in [N2] give a simpler proof for these sets.

4. Example

We first prove the following simple lemma.

**Lemma 6** Let $f$ and $g$ be holomorphic functions, defined in a neighbourhood $U$ of a point $a \in \mathbb{C}_z$, such that $f(a) = g(a)$ and $f'(a) \neq g'(a)$. Let $r$ be a positive number such that $\Delta_r(a) = \{ z \in \mathbb{C}_z : |z - a| \leq r \} \subset U$ and $f(z) \neq g(z)$ for $z \in \Delta_r(a) \setminus \{ a \}$. Then for all sufficiently small $\varepsilon > 0$ the complex curve $\Sigma \subset \Delta_r(a) \times \mathbb{C}_w$ defined by the equation

$$G(z, w) \overset{\text{def}}{=} (w - f(z))(w - g(z)) - \varepsilon = 0$$

is a branched covering over the disk $\Delta_r(a)$ with two branches and two branching points

$$b^\pm = a \pm \frac{2i}{f'(a) - g'(a)} \sqrt{\varepsilon} + O(\varepsilon).$$
Proof. Equation (8) is quadratic with respect to \( w \), hence \( \Sigma \) is a branched covering over \( \Delta_r(a) \) with two branches. A point \( b \) is a branching point of \( \Sigma \) if for some \( w_b \) one has \( 0 = G'(b, w_b) = 2w_b - f(b) - g(b) \). Therefore, \( w_b = \frac{1}{2}(f(b) + g(b)) \) and then (8) implies that \( -\frac{1}{4}(f(b) - g(b))^2 - \varepsilon = 0 \), i.e.

\[
f(b) - g(b) = \pm 2i\sqrt{\varepsilon}.
\]

Hence, in view of our choice of \( r \), \( b \rightarrow a \) as \( \varepsilon \rightarrow 0 \). Then, using Taylor expansions of \( f \) and \( g \) at the point \( a \), we conclude from (10) and the assumption \( f(a) = g(a) \) that \( (f'(a) - g'(a))(b - a) + O(|b - a|^2) = \pm 2i\sqrt{\varepsilon} \). Finally, the assumption \( f'(a) \neq g'(a) \) implies that

\[
b - a = \pm \frac{2i}{f'(a) - g'(a)}\sqrt{\varepsilon} + O(|b - a|^2) = \pm \frac{2i}{f'(a) - g'(a)}\sqrt{\varepsilon} + O(\varepsilon).
\]

\[
\Box
\]

Construction of the set \( E \). Let \( \rho \) be a smooth real-valued function defined on the segment \([0, 1]\) such that

\[
\rho(t) = \begin{cases} 
1, & \text{for } 0 \leq t \leq \frac{1}{3}, \\
decreasing, & \text{for } \frac{1}{3} < t < \frac{2}{3}, \\
0, & \text{for } \frac{2}{3} \leq t \leq 1.
\end{cases}
\]

Consider the set

\[
E_1 = \{(z, w) \in \bar{\Delta} \times \mathbb{C}_w : w^2 = \rho(|z|)z\},
\]

where, as above, \( \Delta = \{ z \in \mathbb{C}_2 : |z| < 1 \} \) is the unit disk. This set has two branches over the disk \( \Delta_{\frac{2}{3}}(0) \) with one branching point at \( z = 0 \). The branches are glued to each other along the circle \( A = \{|z| = \frac{2}{3}, w = 0\} \) and become one branch \( \{w = 0\} \) for \( \frac{2}{3} \leq |z| \leq 1 \). Consider some points \( A_1 = (a_1, 0) \) and \( A_3 = (a_3, \sqrt{a_3}) \) of \( E_1 \) and a point \( A_2 = (a_2, C) \) with \( a_1, a_2, a_3 \) and \( C \) real and positive such that \( \frac{2}{3} < a_1 < 1, 0 < a_3 < \frac{1}{3} \) and \( a_3 < a_2 < a_1 \). Further, consider the complex line \( \mathcal{L}' \) passing through the points \( A_2 \) and \( A_1 \) and the complex line \( \mathcal{L}'' \) passing through the points \( A_3 \) and \( A_2 \). Let \( a_1, a_2, a_3 \) be already chosen and consider \( C \) so big that the line \( \mathcal{L}'' \) intersects \( E_1 \) in two points \( A_3 \) and \( A'_3 = (a'_3, -\sqrt{a_3}) \), with \( a'_3 \) real such that \( 0 < a'_3 < a_3 \), and the line \( \mathcal{L}' \) intersects \( E_1 \) only at the point \( A_1 \). The set \( E \) will be constructed as a small deformation of the set \( E_1 \cup ((\mathcal{L}' \cup \mathcal{L}'') \cap (\bar{\Delta} \times \mathbb{C}_w)) \) near the points \( A_k, k = 1, 2, 3 \), that creates, as in Lemma 6, two branching points instead of each self-intersection point.

Let \( r > 0 \) be so small that the disks \( \bar{\Delta}_1 = \bar{\Delta}_r(a_1), \bar{\Delta}_2 = \bar{\Delta}_r(a_2) \) and \( \bar{\Delta}_3 = \bar{\Delta}_r(a_3) \) neither intersect each other nor the circle \( \{|z| = \frac{2}{3}\} \) and, moreover, do not contain the point \( a'_3 \). Denote by \( \mathcal{E}_1 \) the set \( (E_1 \cup \mathcal{L}') \cap (\bar{\Delta}_1 \times \mathbb{C}_w) \), by \( \mathcal{E}_2 \) the set \( (\mathcal{L}' \cup \mathcal{L}'') \cap (\bar{\Delta}_2 \times \mathbb{C}_w) \) and by \( \mathcal{E}_3 \) the connected component of the set
The union of the graphs of two holomorphic functions \( \tilde{\varepsilon} \) and, moreover, if \( \varepsilon \) is small enough, we will get branched coverings \( \Sigma_1, \Sigma_2 \) and \( \Sigma_3 \) over the disks \( \Delta_1, \Delta_2 \) and \( \Delta_3 \), respectively, with two branches and two branching points contained in the smaller disks \( \Delta'_1 = \bar{\Delta}'_3(a_1), \Delta'_2 = \Delta'_2(a_2) \) and \( \Delta'_3 = \Delta'_3(a_3) \). Moreover, since for each \( k = 1, 2, 3 \) the derivatives at the centers of the disks \( \Delta_k \) of the functions \( f^j_k \), \( j = 1, 2 \), are real, we conclude from (9) that one of the two branching points contained in \( \Delta'_k \) is contained in the half-disk \( \{ z \in \Delta'_k : \text{Im} \, z > 0 \} \), while the other is contained in the half-disk \( \{ z \in \Delta'_k : \text{Im} \, z < 0 \} \). Since both branching points of each set \( \Sigma_k \) are contained in the respective disk \( \Delta'_k \), the set \( \Sigma_k \cap ((\Delta_k \setminus \Delta'_k) \times \mathbb{C}_w) \) will be the union of the graphs of two holomorphic functions \( \tilde{f}^j_k \), \( j = 1, 2 \), defined on \( \Delta_k \setminus \Delta'_k \) and, moreover, if \( \varepsilon \) is small enough, then each function \( \tilde{f}^j_k \) will be close enough to the corresponding function \( f^j_k \). Define the functions

\[
\tilde{f}^j_k(z) = \rho \left( \left| \frac{z - a_k}{r} \right| \right) f^j_k(z) + \left( 1 - \rho \left( \left| \frac{z - a_k}{r} \right| \right) \right) f^j_k(z),
\]

for \( z \in \Delta_k \setminus \Delta'_k, k = 1, 2, 3, j = 1, 2 \). Let \( \tilde{\Sigma}_k \) be the union of the graphs of \( \tilde{f}^1_k \) and \( \tilde{f}^2_k \). Now we can define the set \( E \) as

\[
E = \left( (E_1 \cup (\mathcal{L}' \cup \mathcal{L}'' \cap (\bar{\Delta} \times \mathbb{C}_w))) \setminus \bigcup_{k=1}^{3} \Sigma_k \right) \cup \bigcup_{k=1}^{3} \left( \tilde{\Sigma}_k \cup (\Sigma_k \cap (\bar{\Delta}'_k \times \mathbb{C}_w)) \right).
\]

Define also the set \( E_{\text{reg}} \) as \( E \) with the circle \( A \), the point \( A'_3 \) of the transversal self-intersection of \( E \) and all the branching points of \( E \) being removed. Then, by our construction, \( E_{\text{reg}} \) is a smooth connected 2-dimensional surface transversal to the \( w \)-direction.

Note that each fiber \( E(z) \) of the set \( E \) has at most four points and that the fibers \( E(z) \) depend continuously on \( z \in \bar{\Delta} \) in the Hausdorff metric.

**Claim 1.** \( H_1(\Delta \times \mathbb{C}_w \setminus E, \mathbb{Z}) = 0 \).

**Proof.** Let \( a \) be a real positive number such that \( a_3 \leq a < \frac{1}{2} \). Consider the point \( A = (a, -\sqrt{a}) \in E \) and a disk \( D_s = \{(z, w) : z = a, w + \sqrt{a} \leq s\} \) so small that it intersects the set \( E \) only at the point \( A \). We first prove that the circle \( C_s = bD_s \) is homological to zero in \( \Delta \times \mathbb{C}_w \setminus E \).

Consider the curve \( z(t) \) in \( C_z \) defined as
If \( \pi_z : \mathbb{C}_z \to \mathbb{C}_z \) is the projection, then the curve \( z(t) \) admits a uniquely defined lifting by \( \pi_z^{-1} \) to the piecewise smooth curve \( \gamma \) in \( E \) with the initial point \( A \).

The curve \( \gamma \) is transversal to the \( w \)-direction and has one point of self-intersection, namely, the endpoint \( (\frac{2}{3}, 0) \), where two smooth curves on the side \( \{|z| < \frac{2}{3}\} \) meet each other.

The geometric description of the curve \( \gamma \) looks as follows. We start from the point \( A = (a, -\sqrt{a}) \) and then, over the segment \( \{a \leq \text{Re} z < \frac{2}{3}, \text{Im} z = 0\} \), the curve \( \gamma \) is contained in the “lower” branch of the set \( E_1 \), while over the segment \( \{\frac{2}{3} \leq \text{Re} z \leq a_1 - r, \text{Im} z = 0\} \), \( \gamma \) is contained in the only branch \( \{w = 0\} \) of \( E_1 \) for \( \{|z| > \frac{2}{3}\} \). Since both branching points of \( \Sigma \) are contained in \( \Delta_1 = \{|z - a_1| < r\} \), and since only one of them is contained in the half-disk \( \{z \in \Delta_1 : \text{Im} z > 0\} \), we conclude that over the segment \( \{a_1 - r \leq \text{Re} z \leq a_1 + r, \text{Im} z = 0\} \) the curve \( \gamma \) will “change from the branch \( E_1 \) to the branch \( \mathcal{L}'' \)”. Then, over the half-circle \( \{|z - a_1| = r, \text{Im} z > 0\} \) and the segment \( \{a_2 + r \leq \text{Re} z \leq a_1 - r, \text{Im} z = 0\} \), \( \gamma \) is contained in \( \mathcal{L}' \). After that, applying the same argument as we used for the segment \( \{a_1 - r \leq \text{Re} z \leq a_1 + r, \text{Im} z = 0\} \), we conclude that, over the segment \( \{a_2 - r \leq \text{Re} z \leq a_2 + r, \text{Im} z = 0\} \), the curve \( \gamma \) will “change from the branch \( \mathcal{L}' \) to the branch \( \mathcal{L}''' \)”. Then, over the segment \( \{a_3 + r \leq \text{Re} z \leq a_2 - r, \text{Im} z = 0\} \) and the half-circle \( \{|z - a_3| = r, \text{Im} z > 0\} \), \( \gamma \) is contained in \( \mathcal{L}''' \). After that, the same argument as above shows that, over the segment \( \{a_3 - r \leq \text{Re} z \leq a_3 + r, \text{Im} z = 0\} \), the curve \( \gamma \) will “change from the branch \( \mathcal{L}''' \) to the branch \( E_1'' \)”.

For each \( z_0 \in \pi_z(\gamma) \) and each \( s > 0 \), consider the sets

\[
\Gamma_s(z_0) = \{ (z_0, w) : \min_{(z_0, w') \in \gamma} |w - w'| = s \}
\]

and

\[
\Omega_s(z_0) = \{ (z_0, w) : \min_{(z_0, w') \in \gamma} |w - w'| < s \}.
\]

Then, for \( s \) small enough, each set \( \Omega_s(z_0) \) is the union of finitely many (at most three) disks in \( \{z_0\} \times \mathbb{C}_w \), which are disjoint if \( z_0 \) is far enough from the circle \( \{|z| = \frac{2}{3}\} \), and is the union of two connected components, one of which is a disk and the other one is the union of two disks having nonempty intersection, if \( \{|z_0| < \frac{2}{3}\} \) and \( z_0 \) is close enough to the circle \( \{|z| = \frac{2}{3}\} \). As \( |z_0| \to \frac{2}{3} \) from the side \( \{|z| < \frac{2}{3}\} \), the centers of the two disks constituting the second connected component of
\( \Omega_s(z_0) \) become closer to each other, and for \( \{ |z_0| \geq \frac{2}{3} \} \) this component becomes just one disk. Each set \( \Omega_s(z_0) \) has a natural orientation induced from \( \mathbb{C}_w \) and, hence, \( \Gamma_s(z_0) = b\Omega_s(z_0) \) has also a natural orientation.

Consider the set
\[
T_s = \bigcup_{z_0 \in \pi_s(\gamma)} \Gamma_s(z_0).
\]

Since the curve \( \gamma \) is piece-wise smooth, it follows from the definition of \( \Gamma_s(z_0) \) that the set \( T_s \) is a piece-wise smooth surface of dimension two in \( \Delta \times \mathbb{C}_w \) with the boundary on the above chosen circle \( \mathcal{C}_s \). Moreover, since \( \gamma \) is oriented, and since each set \( \Gamma_s(z_0) \) is oriented, we can also orient the surface \( T_s \). Topologically, \( T_s \) is a torus with a disk removed, \( \mathcal{C}_s \) being the boundary of this disk. Since the curve \( \gamma \subset E \) is transversal to the \( w \)-direction, we conclude that \( T_s \subset \Delta \times \mathbb{C}_w \setminus E \) for \( s \) sufficiently small. This implies that the homology class \( [\mathcal{C}_s] \) of the circle \( \mathcal{C}_s \) in \( H_1(\Delta \times \mathbb{C}_w \setminus E, \mathbb{Z}) \) is trivial.

Now we observe that, for each point \((z, w) \in E^{\text{reg}}\), the circle \( \mathcal{C}_s(z, w) = \{(z, w') : |w - w'| = s\} \) is homological to zero, if \( s > 0 \) is small enough. Indeed, since the set \( E^{\text{reg}} \) is connected, there is a smooth curve \( \tilde{\gamma} \subset E^{\text{reg}} \) connecting the points \( A \) and \((z, w) \). Then, for \( s > 0 \) small enough, the set \( \mathcal{M}_s = \{(z, w') : |w - w'| = s, (z, w) \in \tilde{\gamma}\} \) is a smooth “cylinder” of dimension two which is contained in \( \Delta \times \mathbb{C}_w \setminus E \) and has its boundary on \( \mathcal{C}_s(z, w) \) and \( \mathcal{C}_s \). Therefore, the circles \( \mathcal{C}_s(z, w) \) and \( \mathcal{C}_s \) represent the same homology class in \( H_1(\Delta \times \mathbb{C}_w \setminus E, \mathbb{Z}) \). Since \( \mathcal{C}_s \) is already proved to be homological to zero in \( \Delta \times \mathbb{C}_w \setminus E \), it follows that \( \mathcal{C}_s(z, w) \) is also homological to zero in \( \Delta \times \mathbb{C}_w \setminus E \).

Finally, let \( \mathcal{C} \) be any smooth closed curve in \( \Delta \times \mathbb{C}_w \setminus E \). Then, there is a 2-dimensional disk \( \mathcal{D} \) smoothly embedded into \( \Delta \times \mathbb{C}_w \) such that \( \mathcal{C} = b\mathcal{D} \). We can assume that the disk \( \mathcal{D} \) is in general position, in particular, that \( \mathcal{D} \) intersects \( E \) in finitely many points \( \{(z_p, w_p)\}_{p=1}^k \) which are contained in \( E^{\text{reg}} \). Without loss of generality, we can also assume that \( \mathcal{D} \) is parallel to the \( w \)-direction in a neighbourhood of each point \((z_p, w_p) \). Then the disks \( \mathcal{D}_s(z_p, w_p) = \{(z_p, w') : |w_p - w'| \leq s\} \) are contained in \( \mathcal{D} \) for \( s > 0 \) small enough. Therefore, \( \mathcal{C} = b\mathcal{D} \) is homological to \( \bigcup_{p=1}^k b\mathcal{D}_s(z_p, w_p) \) in \( \Delta \times \mathbb{C}_w \setminus E \), the homology being \( \mathcal{D} \setminus \bigcup_{p=1}^k \mathcal{D}_s(z_p, w_p) \). Since each circle \( \mathcal{C}_s(z, w) = b\mathcal{D}_s(z_p, w_p) \) is already proved to be homological to zero in \( \Delta \times \mathbb{C}_w \setminus E \), we conclude that \( \mathcal{C} \) is also homological to zero. The proof of the claim is now completed.

As an application of Claim 1 we show the following property of the set \( E \).

**Claim 2.** There exists a neighbourhood \( U(E) \) of the set \( E \) which does not contain any subset of \( \bar{\Delta} \times \mathbb{C}_w \) defined by a Weierstrass pseudopolynomial.

**Proof.** Assume, to get a contradiction, that every neighbourhood \( U(E) \) of \( E \) contains a subset defined by a Weierstrass pseudopolynomial. For \( R \) big enough
consider the circle \( C_R = \{ z = 0, |w| = R \} \subset \Delta \times \mathbb{C}_w \setminus E \) oriented counterclockwise in the \( w \)-variable. Then, in view of Claim 1, there is a 2-chain \( S \) such that \( bS = C_R \) and \( \text{supp} \ S \subset \Delta \times \mathbb{C}_w \setminus E \). The last inclusion implies that there exists a neighbourhood \( U(E) \) of \( E \) such that \( \text{supp} \ S \cap U(E) = \emptyset \). By our assumption, there is a subset \( \tilde{E} \) of \( U(E) \) which is defined by a Weierstrass pseudopolynomial, i.e. it has the form (1) with \( a_1(z), a_2(z), \ldots, a_m(z) \) being continuous functions. Since \( \text{supp} \ S \cap \tilde{E} = \emptyset \), the homology class \( [C_R] \) of the circle \( C_R \) in \( H_1(\Delta \times \mathbb{C}_w \setminus \tilde{E}, \mathbb{Z}) \) is trivial. Consider the continuous map \( \Phi : \Delta \times \mathbb{C}_w \setminus \tilde{E} \to S^1 \) defined by

\[
\Phi(z, w) = \frac{w^m + a_1(z)w^{m-1} + \ldots + a_m(z)}{|w^m + a_1(z)w^{m-1} + \ldots + a_m(z)|}.
\] (11)

Then, on one hand, \( [C_R] = 0 \) in \( H_1(\Delta \times \mathbb{C}_w \setminus \tilde{E}, \mathbb{Z}) \) and, hence, \( \Phi_*([C_R]) = 0 \) in \( H_1(S^1, \mathbb{Z}) \). On the other hand, the term \( w^m \) in the numerator of formula (11) will dominate for \( (z, w) \in C_R \), if \( R \) is big enough. Therefore, the degree of the restriction of \( \Phi \) to \( C_R \) (it is a map from \( S^1 \) to \( S^1 \)) is equal to \( m \). Hence, \( \Phi_*([C_R]) = m[S^1] \neq 0 \) in \( H_1(S^1, \mathbb{Z}) \). This gives the desired contradiction and proves the claim.

\[
\square
\]

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