Chromomagnetic stability of the three flavor Larkin-Ovchinnikov-Fulde-Ferrell phase of QCD

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Abstract

We compute in the Ginzburg-Landau approximation the gluon Meissner masses for the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) phase of QCD with three flavors in the kinematical range where it is energetically favored. We find real Meissner masses and therefore chromomagnetic stability.

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I. INTRODUCTION

The attractive interaction between quarks in the color antisymmetric channel leads at high densities and small temperatures to their Cooper pairing, see [1, 2] and for reviews [3, 4]; in particular, for three flavors at asymptotically high densities, the Color-Flavor-Locking (CFL) phase is the ground state of the theory, characterized by a spinless color- and flavor-antisymmetric diquark condensate [5].

The physically more interesting pre-asymptotic phases, where the mass of the strange quark and the chemical potential differences $\delta \mu$ due to $\beta$ equilibrium cannot be neglected, are subject of intense study at this moment. Proposals include the 2SC phase [2], the gapless phases g2SC [6], and gCFL [7, 8]. Unfortunately the appearance of imaginary gluon Meissner masses (for g2SC see [9], for gCFL see [10]) makes the gapless phases unstable, and also the 2SC phase shows instability [9] (in [11, 12] possible antidotes to cure the chromo-magnetic instability are discussed).

However, for appropriate values of $\delta \mu$, quarks may form pairs with non-vanishing total momentum: $p_1 + p_2 = 2q \neq 0$, see [13] and for a review [14], which leads to a Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) [15] phase. For two flavors it has been shown [16] that the 2SC instability implies that the LOFF phase is energetically favored; however, it is not clear if the neutral LOFF phase can cure the chromomagnetic instability of the two flavor superconductive quark matter [17, 18].

LOFF with three flavors is however the much more difficult but physically interesting case at intermediate densities. A first study of the problem was carried out in a Ginzburg-Landau (GL) approximation [19]. It was found that condensation of the pairs $u - s$ and $d - u$ is possible in the form of the inhomogeneous LOFF pairing.

The problem which comes next is then to find out whether such a phase is chromomagnetic stable. This is the subject of the present note. Within the GL expansion we find that this is indeed the case. This result is another indication that the LOFF phase of QCD plays a major role at intermediate hadronic densities. Since these pre-asymptotic densities are probably relevant for the cores of compact stars, a whole field of investigation is opened up, with items ranging from transport properties, cooling processes and the glitches in the pulsar rotational frequency to the possible implications for gravitational radiation during collisions of compact objects and the mechanisms of gamma ray bursts. Besides, given the universality
which is peculiar to the phenomena occurring at the Fermi surface, one expects important indications to come from the study of the similar problems with cold fermionic atoms, mainly because of the greater parameter flexibility possible in laboratory experiments.

The plan of the paper is as follows. In section II we discuss the model and review the results of [19]; in section III we present the computation of the gluon Meissner masses; in section IV we discuss our results and in section V we give our conclusions. The concluding appendix contains some useful formulas.

II. THE THREE FLAVOR LOFF PHASE OF QCD

In the CFL phase of QCD all the eight gluons acquire real Meissner masses, while introducing the strange quark mass and the neutrality constraints (which leads to the gCFL phase) some of these masses become imaginary, which is a signal of instability. In this paper we compute gluon Meissner masses in the three flavor inhomogeneous case (LOFF phase). We will work in the GL approximation [19]. For a system of two massless (u and d) and one massive (s) quarks the QCD action is:

\[ I = \int d^4x \bar{\psi}_{i\alpha}(x) \left( i D_{\alpha\beta}^{ij} - M_{\alpha\beta}^{ij} + \mu_{ij} \gamma_0 \right) \psi_{\beta j}(x). \]  

Here \( \alpha, \beta, \) are color indexes, \( i, j \) flavor indexes; \( M_{\alpha\beta}^{ij} = \delta_{\alpha\beta} \text{diag}(0, 0, M_s) \) is the quark mass matrix and \( D_{\alpha\beta}^{ij} = \partial_\alpha \delta_{\alpha\beta} \delta_{ij} + igA_a T_\alpha^{ij} \delta_{ij}; \) \( \mu_{ij} \) is the matrix of the chemical potentials, see below. It depends on \( \mu \) (the average quark chemical potential), \( \mu_e \) (the electron chemical potential), and \( \mu_3, \mu_8 \), related to color [7, 21].

We treat the strange quark mass at its leading order, i.e. by a shift in the strange quark chemical potential \( \mu_s \rightarrow \mu_s - M_s^2/2\mu \), and we adopt the High Density Effective Theory (HDET) [22] (see [4] and [23] for reviews). HDET uses the fact that, for \( T \rightarrow 0 \) and at weak coupling, the relevant modes of the QCD action are those near the Fermi surface. It is useful therefore to decompose the quark momentum as follows: \( p = \mu n + \ell; \) \( \mu n \) is the large component (\( n \) a unit vector representing the quark Fermi velocity) and \( \ell \) is the small residual momentum that one can take in the direction of \( n \) using reparameterization invariance, so that \( \ell = \xi n \). Moreover one introduces velocity dependent fields \( \psi_n \) and \( \Psi_n \) corresponding to positive and negative energy solutions of the Dirac equation:

\[ \psi(x) = \int \frac{dn}{4\pi} e^{i\mu_n x} \left( \psi_n(x) + \Psi_n(x) \right). \]
Substituting (2) in (1) and integrating out the negative energy components, one gets in momentum space, at the next-leading-order in $1/\mu$:

$$
L = \psi_{n,\alpha}^\dagger (\ell) \left( V \cdot \ell_{\alpha \beta} + \bar{\mu}_{\alpha \beta} + P_{\mu \nu} \left[ \frac{\ell_{\mu} \ell_{\nu}}{V \cdot \ell + 2\mu_{i j}} \right] \right) \psi_{n,\beta} (\ell),
$$

where $(\ell_{\mu})_{i j}^{\alpha \beta} = \ell_{\mu} \delta_{i j} \delta_{\alpha \beta} - g A_{a} T_{a}^{\alpha \beta} \delta_{i j}$, with $\ell^\mu = (p^0, \xi n)$. Moreover $V^\mu = (1, n)$, $\bar{V}^\mu = (1, -n)$, $\bar{\mu}_{ij}^{\alpha \beta} = \mu_{ij}^{\alpha \beta} - M^2/(2\mu) \delta_{ij} \delta_{\alpha \beta}$ and $P_{\mu \nu} = g_{\mu \nu} - (V^\mu \bar{V}^\nu + \bar{V}^\mu V^\nu)/2$.

In the color-flavor and Nambu-Gorkov (NG) basis introduced in [10] the free propagator reads

$$
S_0 = \begin{pmatrix}
[S_0^{11}]_{AB} & 0 \\
0 & [S_0^{22}]_{AB}
\end{pmatrix} = \delta_{AB} \begin{pmatrix}
(p_0 - \xi + \bar{\mu}_A)^{-1} & 0 \\
0 & (p_0 + \xi - \bar{\mu}_A)^{-1}
\end{pmatrix},
$$

where $\bar{\mu}_A = (\bar{\mu}_{ru}, \bar{\mu}_{gd}, \bar{\mu}_{bs}, \bar{\mu}_{rd}, \bar{\mu}_{ga}, \bar{\mu}_{rs}, \bar{\mu}_{bu}, \bar{\mu}_{gs}, \bar{\mu}_{bd})$. Writing the propagator as in Eq. (4) one doubles the fermion modes, which is compensated by an extra factor $1/2$ in momentum integration.

To keep into account quark condensation we add to the QCD action the bilinear quark term

$$
\mathcal{I}_\Delta = -\frac{1}{2} \int d^4 x \int \frac{d n}{4\pi} \Delta_{i j}^{\alpha \beta} (x) \psi_{\alpha i, -n}^T (x) C \gamma_5 \psi_{j \beta, n} (x) + h.c.
$$

where the gap function is given by

$$
\Delta_{i j}^{\alpha \beta} (x) = \sum_{l=1}^{3} \Delta_{i j} \exp \left\{ 2i q_{i l} \cdot x \right\} \epsilon_{i j}^{\alpha \beta} \epsilon_{i j}.
$$

Eq. (6) corresponds to assume a Fulde-Ferrell ansatz for each inhomogeneous pairing; $2q_{i l}$ represents the momentum of the Cooper pair. The inverse fermion propagator in the superconductive phase becomes

$$
S_0^{-1} = \begin{pmatrix}
[S_0^{11}]^{-1}_{AB} & -\Delta_{AB} \\
-\Delta_{AB}^* & [S_0^{22}]^{-1}_{AB}
\end{pmatrix}, \quad \Delta_{AB} = -\sum_{l=1}^{3} \Delta_{l} \text{Tr} \left[ F_{A}^{T} \epsilon_{T} F_{B} \epsilon_{l} \right].
$$

We assume in this paper that global color neutrality is reached with almost vanishing color chemical potentials, $\mu_3 \approx \mu_8 \approx 0$. This approximation is justified because the phase transition from the superconductive to the normal state is second order (see below), and the color chemical potentials are suppressed by inverse powers of $\mu$. Therefore

$$
\mu_{i j}^{\alpha \beta} = (\mu_{i j} - \mu_{e Q_{i j}}) \delta_{i j} \delta_{\alpha \beta} = \mu_{i} \delta_{i j} \delta_{\alpha \beta},
$$

(8)
where $Q$ is the quark electric-charge matrix.

This formalism was applied in [19], where a GL expansion of the free energy $\Omega$ was performed. We briefly review here the results of this paper. From the ansatz [19] it is clear that $\Omega$ depends on 10 parameters, ie the three gaps, three $q$’s and the electron chemical potential. The energetically favored state must be a global minimum of $\Omega$ in the space of $\Delta$’s and $q$’s, and has to be electrically neutral. As a consequence, one has to solve

$$\frac{\partial \Omega}{\partial \mu_e} = 0, \quad \frac{\partial \Omega}{\partial \Delta_I} = 0, \quad \frac{\partial \Omega}{\partial q_I} = 0.$$  \hspace{1cm} (9)

The electrical neutrality condition leads to the result $\mu_e \approx M_s^2/(4\mu)$, which corresponds to a symmetric splitting of the Fermi surfaces of $d$ and $s$ around the $u$ Fermi sphere. As a consequence $\delta \mu_{du} = \delta \mu_{us} \equiv \delta \mu$ and $\delta \mu_{ds} = 2\delta \mu$. Moreover, for the computation of the gap parameters, it is sufficient to consider the condition $\partial \Omega/\partial q_I = 0$ at the $O(\Delta^2)$, which leads to the well known relation $q = 1.1997|\delta \mu|$. This implies that $\Delta_2 = \Delta_3$ and $q_2 = q_3$, which can be interpreted as the consequence of a symmetric splitting of the Fermi surfaces, and $\Delta_1 = 0$, due to the larger mismatch among the $d$ and $s$ Fermi surfaces. These results hold in the range $(128 - 150)$ MeV for the parameter $M_s^2/\mu$ [19]. For smaller values the LOFF free energy is higher than the gCFL energy; for higher values the normal phase is favored.

For this solution the GL density of free energy reads

$$\Omega = \Omega_n + \frac{\alpha_2 \Delta_2^2 + \alpha_3 \Delta_3^2}{2} + \frac{\beta_2 \Delta_2^4 + \beta_3 \Delta_3^4}{4} + \frac{\beta_{23} \Delta_2^2 \Delta_3^2}{2},$$  \hspace{1cm} (10)

where $\Omega_n$ is the usual normal contribution to the free energy; the explicit formulas for the coefficients can be found in the appendix. The transition to the normal phase at $M_s^2/\mu \approx 150$ MeV is second order, which justifies the GL expansion.

**III. MEISSNER MASSES IN GINZBURG-LANDAU APPROXIMATION**

In the effective theory there are two vertices describing the coupling of gluons and quarks. Either one gluon couples to a quark and an antiquark (three-body vertex, coupling $\sim g$) or two quarks couple to two gluons (four-body vertex, coupling $\sim g^2$). The two vertices come from terms in Eq. [3] with one or two momenta $\ell$. At the order of $g^2$ the four-body coupling gives rise to the contribution $g^2 \mu^2/(2\pi^2)$ identical for all the eight gluons; this result for the LOFF phase is identical to those of the normal or the CFL case (this term is independent
of the gap parameters). The three-body coupling gives rise to the polarization tensor:

\[ i\Pi_{\mu\nu}^{ab}(x,y) = - Tr[i S(x,y) i H_\mu^a i S(y,x) i H_\nu^b] \]  \hspace{1cm} (11)

where the trace is over all the internal indexes; \( S(x,y) \) is the quark propagator, and \( H_\mu^a \) is the vertex matrix in the HDET formalism which can be read from Eq. (3). The quark propagator \( S \) has NG components \( S_{ij} \) (\( i, j = 1, 2 \)), see Eq. (7). At the fourth order in \( \Delta \):

\[ S_{11} = S_{11}^0 + S_{11}^0 \Delta \left[ S_{22}^{22} \Delta^* \left( S_{0}^{11} + S_{11}^{11} \Delta S_{0}^{22} \Delta^* S_{0}^{11} \right) \right], \]  \hspace{1cm} (12)

\[ S_{21} = S_{22}^{22} \Delta^* \left( S_{0}^{11} + S_{11}^{11} \Delta S_{0}^{22} \Delta^* S_{0}^{11} \right), \]  \hspace{1cm} (13)

where \( S_{0}^{ij} \) can be read in Eq. (4); \( S_{12} \) and \( S_{22}^{22} \) are obtained by the changes \( 11 \leftrightarrow 22 \) and \( \Delta \leftrightarrow \Delta^* \).

\[ I = \]
\[ J = \]
\[ K = \]
\[ L = \]
\[ M = \]
\[ N = \]

FIG. 1: Diagrams appearing in the GL expansion of the gluon self-energy in Eq. (11). Wavy lines denote gluons, solid lines are either \( S_{0}^{11} \) and \( S_{0}^{22} \). Full and empty circles (squares) denote respectively the insertion of \( \Delta_2 \) (\( \Delta_3 \)) and \( \Delta^*_2 \) (\( \Delta^*_3 \)). Each \( \Delta^* \) is preceded by \( S_{0}^{22} \) and followed by \( S_{0}^{11} \). For a detailed description see the text.

The different contributions from the GL expansion are proportional to \( \Delta^0, \Delta^2, \Delta^4 \). The contribution independent of \( \Delta \) is equal to \(-g^2\mu^2/(2\pi^2)\) and therefore cancels out the term from the four-body vertex. The \( \mathcal{O}(\Delta^2) \), \( \mathcal{O}(\Delta^4) \) terms arise from classes of diagrams with different topologies. Some of them are depicted in Fig. 1. The remaining diagrams are obtained first by duplicating all the diagrams by the exchanges \( (S_{0}^{11}, \Delta) \leftrightarrow (S_{0}^{22}, \Delta^*) \), then
duplicating the diagrams presenting both the $\Delta_2$ and the $\Delta_3$ insertions by the exchanges $\Delta_2 \leftrightarrow \Delta_3$, and the diagrams with unequal number of insertions on the quark lines by the exchange of the upper and lower lines of the fermion loop.

The Meissner masses are tensors with spatial components and a nontrivial color structure:

$$ (M^2)^{ij}_{ab} = -\Pi_{ab}^{ij}(p_0 = 0, p = 0) , $$

where $i, j$ are spatial (adjoint color) indices. We find

$$ M^2_{ij,11} = 2 I_{ij} \Delta_2^2 + 2 J_{ij} \Delta_4^2 + \Delta_3^2 \Delta_2^2 (3 K_{ij} + M_{ij} - 2 N_{ij}) , $$

$$ M^2_{ij,66} = 2 I_{ij} (\Delta_2^2 + \Delta_3^2) + 2 J_{ij} (\Delta_4^2 + \Delta_3^4) + \Delta_3^2 \Delta_2^2 (K_{ij} + 2 M_{ij}) , $$

$$ M^2_{ij,33} = 2 I_{ij} \Delta_2^2 + 3 J_{ij} \Delta_4^2 + \Delta_3^2 \Delta_2^2 (3 K_{ij} + M_{ij} - 2 N_{ij}) , $$

$$ M^2_{ij,38} = -\frac{2}{\sqrt{3}} I_{ij} \Delta_2^2 - \sqrt{3} J_{ij} \Delta_4^2 + \Delta_3^2 \Delta_2^2 \left( \sqrt{3} K_{ij} - \frac{1}{\sqrt{3}} M_{ij} - \frac{2}{\sqrt{3}} N_{ij} + \frac{1}{\sqrt{3}} L_{ij} \right) , $$

$$ M^2_{ij,88} = I_{ij} \left( \frac{2}{3} \Delta_2^2 + \frac{8}{3} \Delta_3^2 \right) + J_{ij} (\Delta_4^2 + 4 \Delta_3^4) + \Delta_3^2 \Delta_2^2 \left( K_{ij} + \frac{5}{3} M_{ij} - \frac{2}{3} N_{ij} - \frac{2}{3} L_{ij} \right) , $$

(19)

The expressions of $I$, $J$, $K$, $L$, $M$ and $N$, represented in Fig. 1 are given in the appendix; moreover the following identities hold: $M^2_{ij,22} = M^2_{ij,11}$, $M^2_{ij,77} = M^2_{ij,66}$, $M^2_{ij,44} = M^2_{ij,11} (\Delta_2 \leftrightarrow \Delta_3)$, $M^2_{ij,55} = M^2_{ij,44}$. Since the mass matrix of the gluons 3 and 8 is not diagonal in the indices of the adjoint representation, one introduces the mass eigenstates $\tilde{A}_{i3} = \cos \theta_i A_{i3} + \sin \theta_i A_{i8}$, $\tilde{A}_{i8} = -\sin \theta_i A_{i3} + \cos \theta_i A_{i8}$; the physical masses are the corresponding eigenvalues.

IV. RESULTS

A. The limit cases $\Delta_2 = 0$ or $\Delta_3 = 0$

The limit $\Delta_2 = 0$ is equivalent to the two flavor LOFF phase considered in [16, 17], with

$$ \langle \psi_{\alpha i} C \gamma_5 \psi_{\beta j} \rangle \propto \Delta_3 \exp \{ 2i \mathbf{q}_3 \cdot \mathbf{r} \} \epsilon_{\alpha \beta 3} \epsilon_{ij3} . $$

(20)
In this limit we reproduce the results of [17], confirming that for small gap parameters the Meissner masses of the screened gluons are real. First of all, for $\Delta_2 = 0$ one has $M_{ij,11}^2 = M_{ij,22}^2 = M_{ij,33}^2 = 0$, which is expected because there is an unbroken $SU(2)$ color subgroup. Moreover $M_{ij,44}^2 = M_{ij,55}^2 = M_{ij,66}^2 = M_{ij,77}^2$, where $(\Delta_3 = \Delta)$:

$$
M_{xx,44}^2 = \frac{g^2 \mu^2}{96 \pi^2} \frac{\Delta_4}{(q_c^2 - \delta \mu)^2}, \quad M_{zz,44}^2 = \frac{g^2 \mu^2}{8 \pi^2} \frac{\Delta_2}{(q_c^2 - \delta \mu^2)},
$$

(21)

and $q_c = 1.1997 \delta \mu$ given by $\alpha'(q_c) = 0$. Finally, as in Eqs. (93) and (94) of [17],

$$
M_{xx,88}^2 = 0, \quad M_{zz,88}^2 = \frac{g^2 \mu^2}{6 \pi^2} \frac{\Delta_2}{(q_c^2 - \delta \mu^2)}.
$$

(22)

In the same way one can treat the limit $\Delta_3 = 0$: there are 3 massless and 5 massive gluons (with the same masses as before). The massless particles correspond to the generators $T_4$, $T_5$, $T = T_3/2 + \sqrt{3}T_8/2$ of the unbroken $SU(2)$ subgroup.

**B. The three flavor LOFF case**

In the three flavor case, for $\Delta_2 = \Delta_3$, we obtain the results shown in Fig. 2. For the numerical evaluation we have used the results of Ref. [19] for the gap $\Delta_2 = \Delta_3$.

![Graph](image-url)

**FIG. 2:** On the left: longitudinal squared Meissner masses, in units of the CFL Meissner mass, vs $M_s^2/\mu$; from top to bottom the lines refer to the gluons $\tilde{A}_3$, $A_6$, $\tilde{A}_8$ (long dashed), and $A_1$ (dotted line). On the right: transverse squared Meissner masses; from top to bottom the lines refer to the gluons $A_6$, $A_1$, $\tilde{A}_3$, and $\tilde{A}_8$.

On the left panel in Fig. 2 we report the longitudinal (ie $zz$) components of the squared Meissner masses against $M_s^2/\mu$, in units of the CFL squared mass [24, 25] at the $O(\Delta^4)$ for
the representative gluons; on the right panel the results for the transverse (ie $xx$) squared Meissner masses are given. In both cases we obtain positive squared Meissner masses for all the gluons. The masses not reported are obtained from those displayed, according to the discussion at the end of section I.V.

From the figure it is clear that for each value of the strange quark mass, the transverse mass of a gluon is smaller than the longitudinal one. This is so because the transverse mass is zero at $\mathcal{O}(\Delta^2)$ (see the appendix). As a consequence, it gets contribution only from the $\mathcal{O}(\Delta^4)$ and is therefore suppressed as $\Delta^2/\delta\mu^2$ in comparison with the longitudinal one. This behavior is analogous to the two flavor LOFF phase considered in [17]. Moreover, we find that the transverse mass of $\tilde{A}_8$, although positive, is almost zero, being three order of magnitude smaller than the other ones.

We conclude therefore that the LOFF phase with three flavors in the Ginzburg-Landau limit has no chromomagnetic instability.

V. DISCUSSION AND CONCLUSIONS

Some final comments are in order. First of all, we have computed the Meissner masses only for the single plane wave Fulde-Ferrell (FF) structure, see Eq. (6). However from the two flavor case we know that more complicated crystalline structures have a lower free energy than the FF state and we expect the same to be true in the three flavor case [26]. In the general case one should replace (6) with

$$\Delta(x) = \sum_{i=1}^{N} \sum_{I=1}^{3} \Delta_I \exp \left\{ q_I^i \cdot x \right\} \epsilon_{ijI} \epsilon^{\alpha\beta I}$$

where $q_I^i$ ($i = 1, \ldots, N$) are the momenta which define the LOFF crystal; the geometry of the structure and the number $N$ of plane waves should be determined by minimization of the free energy. Once the optimal structure is found, one should compute the Meissner masses. If this structure contains at least three linearly independent momenta, the Meissner tensor should be positive definite for small values of $\Delta$, since it is additive with respect to different terms of (6) to order $\Delta^2$ [17]. These considerations suggest that a LOFF crystal can remove the chromo-magnetic instability of the homogeneous superconductive phases of QCD, resulting as the true vacuum of the theory. A second comment is that the results should be extended beyond the GL expansion. Finally, in a recent paper [27] it has been
found that, at strong coupling, gapless phases can be magnetic stable, and this direction also needs to be explored.

In conclusion we have computed the gluon Meissner masses in the three flavor LOFF phase of QCD using the High Density Effective Theory and the Ginzburg-Landau approximation. The use of the GL expansion is justified in the LOFF window, $128 \text{ MeV} < M_s^2/\mu < 150 \text{ MeV}$ \[19\]. In this region we find that all the squared gluon Meissner masses are positive and therefore the LOFF phase of three flavor QCD is free from chromomagnetic instability.

**APPENDIX A: DEFINITION OF INTEGRALS**

In section \[III\] we have introduced the following integrals:

\[
I_{ij} = -\frac{i g^2 \mu^2}{4\pi^3} \int \frac{d\mathbf{n}}{4\pi} n_i n_j \int \frac{dp_0 d\xi}{(p_0 - \xi + \bar{\mu}_d)^3 \cdot (p_0 + \xi - \bar{\mu}_u + 2\mathbf{q} \cdot \mathbf{n})},
\]

(A1)

\[
J_{ij} = -\frac{i g^2 \mu^2}{4\pi^3} \int \frac{d\mathbf{n}}{4\pi} n_i n_j \int \frac{dp_0 d\xi}{(p_0 - \xi + \bar{\mu}_d)^4 \cdot (p_0 + \xi - \bar{\mu}_u + 2\mathbf{q} \cdot \mathbf{n})^2},
\]

(A2)

\[
K_{ij} = -\frac{i g^2 \mu^2}{4\pi^3} \int \frac{d\mathbf{n}}{4\pi} n_i n_j \int \frac{dp_0 d\xi}{(p_0 - \xi + \bar{\mu}_u)^4 \cdot (p_0 + \xi - \bar{\mu}_s + 2\mathbf{q} \cdot \mathbf{n}) \cdot (p_0 + \xi - \bar{\mu}_d + 2\mathbf{q} \cdot \mathbf{n})},
\]

(A3)

\[
L_{ij} = -\frac{i g^2 \mu^2}{4\pi^3} \int \frac{d\mathbf{n}}{4\pi} n_i n_j \int \frac{dp_0 d\xi}{(p_0 + \xi - \bar{\mu}_u - 2\mathbf{q} \cdot \mathbf{n})^2 \cdot (p_0 - \xi + \bar{\mu}_d)^2 \cdot (p_0 - \xi + \bar{\mu}_s)^2},
\]

(A4)

\[
M_{ij} = -\frac{i g^2 \mu^2}{4\pi^3} \int \frac{d\mathbf{n}}{4\pi} n_i n_j \int \frac{dp_0 d\xi}{(p_0 - \xi + \bar{\mu}_d)^3 \cdot (p_0 + \xi - \bar{\mu}_u + 2\mathbf{q} \cdot \mathbf{n})^2 \cdot (p_0 - \xi + \bar{\mu}_s)},
\]

(A5)

\[
N_{ij} = -\frac{i g^2 \mu^2}{4\pi^3} \int \frac{d\mathbf{n}}{4\pi} n_i n_j \int \frac{dp_0 d\xi}{(p_0 + \xi - \bar{\mu}_u + 2\mathbf{q} \cdot \mathbf{n})^3 \cdot (p_0 - \xi + \bar{\mu}_d)^2 \cdot (p_0 - \xi + \bar{\mu}_s)},
\]

(A6)

depicted in Fig. \[1\] To evaluate these integrals first we perform a Wick rotation $p_0 = ip_4$. The integral in $\xi$ is computed by a contour integration; to evaluate the integral in $p_4$ we note that the $\xi$ integration introduces the sign factor sign($p_4$) and the integrands depend only on $ip_4$, therefore one can use

\[
\int_{-\infty}^{\infty} dp_4 \text{sign}(p_4) F(ip_4) = 2i\Im m \int_{0^+}^{\infty} dp_4 F(ip_4),
\]

(A7)
the remaining angular integrals are performed trivially. The prescription embodied in Eq. (A7) follows from the limit $T \to 0$ once one passes from finite to zero temperature.

The integral $I_{ij} = I_{ij}(q, \delta \mu)$ has to be handled with care. In the case of $I_{xx}$ one has

$$I_{xx} = \frac{g^2}{64} \frac{1}{q} \frac{\partial \alpha(q, \delta \mu)}{\partial q},$$

(A8)

with $\alpha$ given by 19

$$\alpha(q, \delta \mu) = -\frac{4\mu^2}{\pi^2} \left( 1 - \frac{\delta \mu}{2q} \log \frac{|q + \delta \mu|}{|q - \delta \mu|} - \frac{1}{2} \log \left| \frac{4(q^2 - \delta \mu^2)}{\Delta^2_0} \right| \right);$$

(A9)

the condition $\partial \Omega/\partial q = 0$ at $O(\Delta^2)$ reads $\alpha'(q) = 0$: thus $I_{xx}$ vanishes at $O(\Delta^2)$. Defining $q_c$ the value of $q$ satisfying the equation $\alpha'(q_c) = 0$ and expanding around $q_c$ one finds

$$I_{xx} \approx \frac{g^2}{64} \frac{q - q_c}{q_c^2} \alpha''(q_c);$$

(A10)

using $\partial \Omega/\partial q = 0$ at the second order in $\Delta^2$ one gets

$$\alpha''(q_c)(q - q_c) = -\frac{\Delta^2}{2} (\beta'(q_c) + 2\beta_{23}'(q_c))$$

(A11)

where 19 $\beta = \mu^2/[\pi^2(q^2 - \delta \mu^2)]$, and

$$\beta_{23} = -\frac{2\mu^2}{\pi^2} \text{Re} \int \frac{dn}{4\pi} \frac{1}{(2q_3 \cdot n + \mu_u - \mu_d - i\epsilon)(2q_2 \cdot n + \mu_u - \mu_d - i\epsilon)}.$$

(A12)

In (A11) the derivative of $\beta_{23}$ is in the variable $q_3$, for fixed $q_2$; at the end one puts $q_2 = q_3$.

At the lowest non-vanishing order in $\Delta^2$ one has

$$I_{xx} = -\frac{g^2}{64} \frac{\Delta^2}{2q_c} (\beta'(q_c) + 2\beta_{23}'(q_c)).$$

(A13)

The integral in the longitudinal case, $I_{zz}$, is treated in a similar way,

$$I_{zz}(q) \approx I_{zz}(q_c) + (q - q_c) I'_{zz}(q_c).$$

(A14)

As a consequence, the integrals $I_{ij}$ give contributions at the second and fourth order in $\Delta$.

[1] J. C. Collins and M. J. Perry, Phys. Rev. Lett. 34 (1975) 1353; B. Barrois, Nucl. Phys. B129 (1977) 390; S. Frautschi, Proceedings of workshop on hadronic matter at extreme density, Erice 1978; D. Bailin and A. Love, Phys. Rept. 107 (1984) 325.
[2] M. G. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B 422 (1998) 247 [arXiv:hep-ph/9711395]; R. Rapp, T. Schäfer, E. V. Shuryak and M. Velkovsky, Phys. Rev. Lett. 81 (1998) 53, [arXiv:hep-ph/9711396]; D. T. Son, Phys. Rev. D 59 (1999) 094019 [arXiv:hep-ph/9812287]; R. D. Pisarski and D. H. Rischke, Phys. Rev. D 61 (2000) 074017 [arXiv:nucl-th/9910056].

[3] K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333; M. G. Alford, Ann. Rev. Nucl. Part. Sci. 51 (2001) 131 [arXiv:hep-ph/0102047]. T. Schäfer, [arXiv:hep-ph/0304281] M. Huang, Int. J. Mod. Phys. E 14 (2005) 675 [arXiv:hep-ph/0409167]; I. A. Shovkovy, Found. Phys. 35 (2005) 1309 [arXiv:nucl-th/0410091]; T. Schaefer, arXiv:hep-ph/0509068.

[4] G. Nardulli, Riv. Nuovo Cim. 25N3 (2002) 1 [arXiv:hep-ph/0202037].

[5] M. G. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B 537 (1999) 443 [arXiv:hep-ph/9804403].

[6] I. Shovkovy and M. Huang, Phys. Lett. B 564 (2003) 205 [arXiv:hep-ph/0302142]; M. Huang and I. Shovkovy, Nucl. Phys. A 729 (2003) 835 [arXiv:hep-ph/0307273].

[7] M. Alford, C. Kouvaris and K. Rajagopal, Phys. Rev. Lett. 92, 222001 (2004) [arXiv:hep-ph/0311286].

[8] M. Alford, C. Kouvaris and K. Rajagopal, Phys. Rev. D 71 (2005) 054009 [arXiv:hep-ph/0406137]; K. Fukushima, C. Kouvaris and K. Rajagopal, Phys. Rev. D 71 (2005) 034002 [arXiv:hep-ph/0408322].

[9] M. Huang and I. A. Shovkovy, Phys. Rev. D 70 (2004) 051501 [arXiv:hep-ph/0407049]; M. Huang and I. A. Shovkovy, Phys. Rev. D 70 (2004) 094030 [arXiv:hep-ph/0408268].

[10] R. Casalbuoni, R. Gatto, M. Mannarelli, G. Nardulli and M. Ruggieri, Phys. Lett. B 605 (2005) 362 [Erratum-ibid. B 615 (2005) 297] [arXiv:hep-ph/0410401]; K. Fukushima, Phys. Rev. D 72 (2005) 074002 [arXiv:hep-ph/0506080]; M. Alford and Q. h. Wang, J. Phys. G 31 (2005) 719 [arXiv:hep-ph/0501078]; K. Fukushima, [arXiv:hep-ph/0510299].

[11] M. Huang, Phys. Rev. D 73 (2006) 045007 [arXiv:hep-ph/0504235]; D. K. Hong, [arXiv:hep-ph/0506097] M. Alford and Q. h. Wang, J. Phys. G 32 (2006) 63 [arXiv:hep-ph/0507269]; A. Kryjevski, [arXiv:hep-ph/0508180]; T. Schafer, [arXiv:hep-ph/0508190].

[12] E. V. Gorbar, M. Hashimoto and V. A. Miransky, Phys. Lett. B 632 (2006) 305 [arXiv:hep-ph/0507303].
[13] M. G. Alford, J. A. Bowers and K. Rajagopal, Phys. Rev. D 63, 074016 (2001) arXiv:hep-ph/0008208; J. A. Bowers, J. Kundu, K. Rajagopal and E. Shuster, Phys. Rev. D 64 (2001) 014024 arXiv:hep-ph/0010167; A. K. Leibovich, K. Rajagopal and E. Shuster, Phys. Rev. D 64 (2001) 094005 arXiv:hep-ph/0104073; J. A. Bowers and K. Rajagopal, Phys. Rev. D 66, 065002 (2002) arXiv:hep-ph/0204079; R. Casalbuoni, M. Ciminale, M. Mannarelli, G. Nardulli, M. Ruggieri and R. Gatto, Phys. Rev. D 70 (2004) 054004 arXiv:hep-ph/0404090.

[14] R. Casalbuoni and G. Nardulli, Rev. Mod. Phys. 76 (2004) 263 arXiv:hep-ph/0305069.

[15] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 47 (1136) 1964 ( Sov. Phys. JETP 20 (1965) 762); P. Fulde and R. A. Ferrell, Phys. Rev. 135 (1964) A550.

[16] I. Giannakis and H. C. Ren, Phys. Lett. B 611 (2005) 137 arXiv:hep-ph/0412015.

[17] I. Giannakis and H. C. Ren, Nucl. Phys. B 723 (2005) 255 arXiv:hep-th/0504053; I. Giannakis, D. F. Hou and H. C. Ren, Phys. Lett. B 631 (2005) 16 arXiv:hep-ph/0507306.

[18] E. V. Gorbar, M. Hashimoto and V. A. Miransky, Phys. Rev. Lett. 96 (2006) 022005 arXiv:hep-ph/0509334.

[19] R. Casalbuoni, R. Gatto, N. Ippolito, G. Nardulli and M. Ruggieri, Phys. Lett. B 627 (2005) 89 arXiv:hep-ph/0507247.

[20] C. A. Regal et al., Phys. Rev. Lett. 92, 040403 (2004); M. Bartenstein et al., Phys. Rev. Lett. 92, 120401 (2004); M. W. Zwierlein et al., Phys. Rev. Lett. 92, 120403 (2004); J. Kinast et al., Phys. Rev. Lett. 92, 150402 (2004); T. Bourdel et al., Phys. Rev. Lett. 93, 050401 (2004); M. W. Zwierlein et al., Science 311, 492 (2006); G. B. Partridge et al., Science 311, 503 (2006).

[21] M. Buballa and I. A. Shovkovy, Phys. Rev. D 72 (2005) 097501 arXiv:hep-ph/0508197.

[22] D. K. Hong, Phys. Lett. B 473 (2000) 118 arXiv:hep-ph/9812510; D. K. Hong, Nucl. Phys. B 582 (2000) 451 arXiv:hep-ph/9905523; S. R. Beane, P. F. Bedaque and M. J. Savage, Phys. Lett. B 483 (2000) 131 arXiv:hep-ph/0002209.

[23] T. Schafer, eConf, C030614, 038 (2003) arXiv:hep-ph/0310176.

[24] D. T. Son and M. A. Stephanov, Phys. Rev. D 62 (2000) 059902 arXiv:hep-ph/0004095.

[25] D. H. Rischke, Phys. Rev. D 62 (2000) 054017 arXiv:nucl-th/0003063.

[26] K. Rajagopal, talk given at the workshop Pairing in Fermionic Systems: Beyond the BCS Theory, Seattle, USA, September 19 - 23, 2005.

[27] M. Kitazawa, D. H. Rischke and I. A. Shovkovy, arXiv:hep-ph/0602065.