MHD BOUNDARY LAYERS IN SOBOLEV SPACES WITHOUT MONOTONICITY.
II. CONVERGENCE THEORY

CHENG-JIE LIU, FENG XIE, AND TONG YANG

ABSTRACT. As a continuation of [28], the paper aims to justify the high Reynolds numbers limit for the MHD system with Prandtl boundary layer expansion when no-slip boundary condition is imposed on the velocity field and the perfect conducting boundary condition is given for the magnetic field. Under an assumption that the viscosity and resistivity coefficients are of the same order and the initial tangential magnetic field on the boundary is not degenerate, we justify the validity of the Prandtl boundary layer expansion and give a $L^\infty$ estimate on the error by multi-scale analysis.

1. INTRODUCTION AND MAIN RESULTS

For electromagnetically conducting fluid such as plasmas and liquid metals, the system of magnetohydrodynamics (denoted by MHD) is a fundamental system to describe the movement of fluid under the influence of electro-magnetic field. The study on the MHD was initiated by Hannes Alfvén who showed that the magnetic field can induce current in a moving conductive fluid with a new propagation mechanism along the magnetic field, called Alfvén waves (see [2]). One important problem about Magnetohydrodynamics is to understand the inviscid and vanishing resistivity limit in a domain with boundary. The purpose of this paper is to justify this high Reynolds number limit when the tangential magnetic field is not degenerate along the boundary.

Consider the 2D incompressible viscous MHD equations in the domain \( \{(x, y) \mid x \in \mathbb{T}, y \in \mathbb{R}_+\} \),

\[
\begin{align*}
\partial_t u^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_y) u^\varepsilon + \varepsilon_x p^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) h^\varepsilon &= \mu (\partial_x^2 u^\varepsilon + \partial_y^2 u^\varepsilon), \\
\partial_t v^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_y) v^\varepsilon + \varepsilon_y p^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) g^\varepsilon &= \mu (\partial_x^2 v^\varepsilon + \partial_y^2 v^\varepsilon), \\
\partial_t h^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_y) h^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) u^\varepsilon &= \kappa \varepsilon (\partial_x^2 h^\varepsilon + \partial_y^2 h^\varepsilon), \\
\partial_t g^\varepsilon + (u^\varepsilon \partial_x + v^\varepsilon \partial_y) g^\varepsilon - (h^\varepsilon \partial_x + g^\varepsilon \partial_y) v^\varepsilon &= \kappa \varepsilon (\partial_x^2 g^\varepsilon + \partial_y^2 g^\varepsilon), \\
\partial_x u^\varepsilon + \partial_y v^\varepsilon &= 0, \\
\partial_x h^\varepsilon + \partial_y g^\varepsilon &= 0.
\end{align*}
\]

Here, we assume the viscosity and resistivity coefficients are of the same order in a small parameter \( \varepsilon \). \((u^\varepsilon, v^\varepsilon)\) stands for the velocity field, and \((h^\varepsilon, g^\varepsilon)\) denotes the magnetic field, the tangential variable \( x \in \mathbb{T} \), and the normal variable \( y \in \mathbb{R}_+ \). And the initial data of (1.1) is given by

\[
(u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)|_{t=0} = (u_0, v_0, h_0, g_0)(x, y).
\]

The no-slip boundary condition is imposed on the velocity field

\[
(u^\varepsilon, v^\varepsilon)|_{y=0} = 0,
\]

and the perfectly conducting boundary condition is given for the magnetic field:

\[
(\partial_y h^\varepsilon, g^\varepsilon)|_{y=0} = 0.
\]

The initial-boundary value problem (1.1)-(1.4) with fixed \( \varepsilon > 0 \) has been investigated and its global well-posedness is known, see [6,40] for instance. In this paper, we are concerned with the asymptotic behavior of solutions \((u^\varepsilon, v^\varepsilon, h^\varepsilon, g^\varepsilon)\) to problem (1.1)-(1.4) as \( \varepsilon \to 0 \). Formally, when \( \varepsilon = 0 \), (1.1) becomes the following

---

2000 Mathematics Subject Classification. 76N20, 35A07, 35G31, 35M33.

Key words and phrases. Prandtl boundary layer theory, MHD boundary layer, high Reynolds numbers limit, weighted Sobolev spaces.
incompressible ideal MHD system:

\[
\begin{align*}
\frac{\partial}{\partial t} u^0 + (u^0 \cdot \nabla) u^0 + \nabla p^0 - (h^0 \nabla \cdot \hat{c} + g^0 \nabla \cdot \hat{g}) h^0 &= 0, \\
\frac{\partial}{\partial t} v^0 + (u^0 \cdot \nabla) v^0 + \nabla q^0 - (h^0 \nabla \cdot \hat{c} + g^0 \nabla \cdot \hat{g}) q^0 &= 0, \\
\frac{\partial}{\partial t} h^0 + (u^0 \cdot \nabla) h^0 - (h^0 \nabla \cdot \hat{c} + g^0 \nabla \cdot \hat{g}) v^0 &= 0, \\
\frac{\partial}{\partial t} q^0 + (u^0 \cdot \nabla) q^0 - (h^0 \nabla \cdot \hat{c} + g^0 \nabla \cdot \hat{g}) u^0 &= 0, \\
\frac{\partial}{\partial x} u^0 + \nabla q^0 &= 0, \\
\frac{\partial}{\partial x} h^0 + \nabla q^0 &= 0.
\end{align*}
\] (1.5)

To solve equations (1.5), only the normal component of the velocity and magnetic fields \((v^0, q^0)\) is needed to be imposed on the boundary. Correspondingly, we consider the homogeneous Dirichlet boundary condition for (1.6)

\[
(v^0, q^0)|_{y=0} = 0.
\] (1.6)

Comparing (1.1) with (1.5), there is a mismatch between \((u^*, h^*)\)\((t, x, y)\) and \((u^0, h^0)\)(\(t, x, y)\) on the boundary \(\{y = 0\}\). According to the classical Prandtl boundary theory [35], there is a thin boundary layer of width of the order \(\sqrt{\epsilon}\) in which the boundary layer \((u^0, h^0)(t, x, \sqrt{\epsilon})\) changes significantly from the boundary data of \((u^*, h^*)\) in (1.3) and (1.4) to the outer flow \((u^0, h^0)(t, x, y)\). That is, the solution to the initial-boundary value problem of the incompressible viscous MHD equations (1.1)-(1.4) is expected to satisfy

\[
(u^*, h^*)(t, x, y) = (u^0, h^0)(t, x, y) + (u^0, h^0)(t, x, \frac{y}{\sqrt{\epsilon}}) + o(1), \quad p^* = p^0(t, x, y) + o(1),
\] (1.7)

where the error term \(o(1)\) tends to zero in \(L^\infty\)-norm as \(\epsilon\) tends to zero.

By applying the multi-scale expansion of \((u^*, h^*)\) in Section 2, we will justify the validity of the Prandtl boundary layer theory in (1.7) under the non-degeneracy condition on the tangential magnetic field along the boundary. As it well known that the Alfvén wave propagates along the magnetic field, this result in some sense justifies the physical phenomena that the Alfvén wave along the boundary carries away the energy so that it stabilizes the boundary layer rigorously in mathematics.

The study on fluid around a rigid body with high Reynolds number is an important problem in both physics and mathematics. First of all, there are vast literatures on the MHD system, in particular in the case without boundaries. For this, please refer to the works [3][19][24][47] and the references therein. Since we will focus on the boundary layer problems, we will not give detail of the works in the whole space and those for the compressible fluid. On the other hand, there are much less mathematical results on the plasma boundary layers for MHD system.

To study the plasma boundary layers modeled by MHD, let us first review some key results on the classical Prandtl equations that were derived by Prandtl in 1904 from the incompressible Navier-Stokes equations with no-slip boundary condition. Under the monotone assumption on the tangential velocity in the normal direction, Oleinik firstly obtained the local existence of classical solutions in 2D by using the Crocco transformation, cf. [33]. This result together with some other related works are well written in the classical book [34]. Recently, this well-posedness result was established in the Sobolev spaces by using energy method in [4] and [31] independently. Moreover, by imposing an additional favorable condition on the pressure, a global in time weak solution was obtained in [40]. Some of these results were generalized to 3D with special structure in [27]. For Prandtl system, without the monotonicity condition, boundary separation can be observed. Recently, by considering a perturbation of a shear flow with a non-degenerate critical point, some interesting ill-posedness (or instability) theories to both the linearized and nonlinear Prandtl equations are studied, cf. [8][9][12][14][17][23][29] and the references therein. Hence, the monotonicity assumption on the tangential velocity is essential for well-posedness of the Prandtl equations except for solutions with analyticity or Gevrey regularity, cf. [11][20][28][47][48] and the references therein. The justification of the solution to the Navier-Stokes equations as a superposition of solutions to the Euler and Prandtl systems in the leading order can be obtained in the framework of analytic functions in [37][38]. In 2014, the problem in two-dimensional case was studied by Maekawa in [30] that requires that the initial vorticity of outer Euler flow vanishes in a neighborhood of boundary so that the boundary layer solution is still analytic due to the incompressibility. Recently, the authors in [10] improved the results of Sammartino & Caffarelli [37][38] in Gevrey class. In addition, Guo & Nguyen justified the Prandtl boundary layer expansions for the steady Navier-Stokes flows over a moving plate in [18]. One also refer to [45] for the justification of Prandtl boundary layer theory for linearized compressible Navier-Stokes equations.
For plasma, the boundary layer equations can be derived from the fundamental MHD system and they are more complicated than the classical Prandtl system because of the coupling of the magnetic field with velocity field through the Maxwell equations. On the other hand, in physics, it is believed that the magnetic field has a stabilizing effect on the boundary layer that could provide a mechanism for containment of, for example, the high temperature gas. If the magnetic field is transversal to the boundary, there are extensive discussions on the so called Hartmann boundary layer, cf. [5]. Moreover, there are works on the stability of boundary layers with minimum Reynolds number for flow with different structures to reveal the difference from the classical boundary layers without electro-magnetic field. Recently, the same authors of this paper established the well-posedness of MHD boundary layer equations in weighted Sobolev spaces without monotonicity condition on the velocity in [28]. The key assumption is that initial magnetic field is not degenerate on the boundary. As the continuation of [28], we study the high Reynolds numbers limit problem for (1.1). It should be emphasized that the MHD boundary layer is an important problem in study of plasma with fruitful results, [3, 6, 7, 13, 36, 42, 44].

We are now ready to state main result in this paper as follows.

**Theorem 1.1.** Suppose that the initial data \((u_0, v_0, h_0, g_0)(x, y)\) is smooth, compatible and \(h_0(x, 0) \geq \delta_0\) for some positive constant \(\delta_0\). And assume that the initial data of ideal MHD system \((\ref{2.3})\) is imposed with the same data \((u_0, v_0, h_0, g_0)(x, y)\). Then, there exists \(T_\ast \geq 0\) independent of \(\epsilon\) and a solution \((u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)\) to \((\ref{1.1})\) in the time interval \([0, T_\ast]\), such that for any arbitrarily small \(\sigma > 0\),

\[
\sup_{0 < t < T_\ast} \left\| (u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)(t, x, y) - (u^0, v^0, h^0, g^0)(t, x, y) - (u^0, \sqrt{\epsilon} v^0, h^0, \sqrt{\epsilon} g^0)(t, x, y, \frac{y}{\sqrt{\epsilon}}) \right\|_{L^2_{xy}} \leq C \epsilon^{3/8 - \sigma},
\]

(1.8)

where \((u^0, v^0, h^0, g^0)(t, x, y)\) is the leading order inner flows given by \((\ref{2.1})\)-\((\ref{2.2})\) with the same initial data \((\ref{2.3})\), and \((u^0, v^0, h^0, g^0)(t, x, \frac{y}{\sqrt{\epsilon}})\) is the leading order boundary layers described in \((\ref{2.7})\) with zero initial data.

Finally, we would like to comment why the justification of the high Reynolds numbers limit can be obtained for MHD in the framework of Sobolev space, but the corresponding problem for incompressible Navier-Stokes equations remains open. For this, first note that although some essential cancellations are observed in \([1]\) and \([31]\) for recovering the loss of the derivatives in the classical Prandtl equations for the well-posedness theory, these cancellations destroy the divergence free structure for the newly introduced unknown function so that the estimation on the pressure function becomes a challenging and unsolved problem. However, for MHD system, the newly observed cancellation mechanism for MHD boundary layer equations not only recovers the loss of derivative in the tangential direction, but also preserves the divergence free condition of the newly defined unknown function for the velocity field. In this analysis, the non-degeneracy condition on the magnetic field plays an essential role.

This paper is organized as follows: In Section 2, we will construct a suitable approximation solution and derive some necessary estimates. In Section 3, the error of the approximation is estimated in \(L^2\)-norm for the proof of Theorem 1.1.

## 2. Construction of Approximate Solution

In order to prove Theorem 1.1, we need to construct high order approximate solution to \((\ref{1.1})\). Precisely, we take the forms of the approximate solution to \((\ref{1.1})\) as follows.

\[
\begin{aligned}
&u^\epsilon = u^\epsilon_0(t, x, y) + u^\epsilon_1(t, x, \frac{y}{\sqrt{\epsilon}}) + \sqrt{\epsilon}(u^\epsilon_2(t, x, y) + u^\epsilon_3(t, x, \frac{y}{\sqrt{\epsilon}})), \\
v^\epsilon = v^\epsilon_0(t, x, y) + \sqrt{\epsilon}v^\epsilon_1(t, x, \frac{y}{\sqrt{\epsilon}}) + \sqrt{\epsilon}(v^\epsilon_2(t, x, y) + \sqrt{\epsilon}v^\epsilon_3(t, x, \frac{y}{\sqrt{\epsilon}})), \\
h^\epsilon = h^\epsilon_0(t, x, y) + h^\epsilon_1(t, x, \frac{y}{\sqrt{\epsilon}}) + \sqrt{\epsilon}(h^\epsilon_2(t, x, y) + h^\epsilon_3(t, x, \frac{y}{\sqrt{\epsilon}})), \\
g^\epsilon = g^\epsilon_0(t, x, y) + \sqrt{\epsilon}g^\epsilon_1(t, x, \frac{y}{\sqrt{\epsilon}}) + \sqrt{\epsilon}(g^\epsilon_2(t, x, y) + g^\epsilon_3(t, x, \frac{y}{\sqrt{\epsilon}})), \\
p^\epsilon = p^\epsilon_0(t, x, y) + \sqrt{\epsilon}p^\epsilon_1(t, x, y) + \sqrt{\epsilon}(p^\epsilon_2(t, x, y) + p^\epsilon_3(t, x, \frac{y}{\sqrt{\epsilon}})),
\end{aligned}
\]

(2.1)

where the functions with the subscript \(e\) stand for the inner flow, and the functions with subscript \(b\) denote the boundary layer profile. In the next six subsections, we will give the construction of the profiles in the above approximation \((\ref{2.1})\).
Keep in mind that the fast variable $\eta = \frac{\xi}{\gamma^0},$ and in the following derivation we assume first that for $i = 0, 1,$

$$\lim_{\eta \to +\infty} (u^i_0, v^i_0, h^i_0, g^i_0)(t, x, \eta) = 0, \quad \lim_{\eta \to +\infty} p^i_0(t, x, \eta) = 0, \quad (2. 2)$$

which means the boundary layer profiles decay to zero in the region near the boundary with width of the order $\epsilon^0, \nu < 1/2.$

2.1. Zeroth-order ideal MHD flow. Putting the ansatz (2. 1) into (1. 1) and setting the terms of order $\epsilon^0$

equal to zero, then letting the fast variable $\eta \to +\infty$ yields that the leading order inner flow $(u^0_e, v^0_e, p^0_e, h^0_e, g^0_e)$ satisfies the following ideal MHD equations.

$$\begin{align*}
\hat{c}_t u^0_e + (u^0_e \hat{c}_x + v^0_e \hat{c}_y) u^0_e + \hat{c}_x p^0_e - (h^0_e \hat{c}_x + g^0_e \hat{c}_y) h^0_e = 0,
\hat{c}_t v^0_e + (u^0_e \hat{c}_x + v^0_e \hat{c}_y) v^0_e + \hat{c}_y p^0_e - (h^0_e \hat{c}_x + g^0_e \hat{c}_y) g^0_e = 0,
\hat{c}_t h^0_e + (u^0_e \hat{c}_x + v^0_e \hat{c}_y) h^0_e - (h^0_e \hat{c}_x + g^0_e \hat{c}_y) u^0_e = 0,
\hat{c}_t g^0_e + (u^0_e \hat{c}_x + v^0_e \hat{c}_y) g^0_e - (h^0_e \hat{c}_x + g^0_e \hat{c}_y) v^0_e = 0,
\hat{c}_t u^0_e + \hat{c}_v v^0_e = 0, \quad \hat{c}_x h^0_e + \hat{c}_y g^0_e = 0.
\end{align*}$$

(2. 3)

Similarly, plugging the ansatz (2. 1) into the initial-boundary conditions (1. 2)-(1. 4) yields that, the initial data of ideal MHD equations (2. 3) is taken as the same one as in (1. 2):

$$(u^0_e, v^0_e, h^0_e, g^0_e)|_{t=0} = (u_0, v_0, h_0, g_0)(x, y),$$

(2. 4)

and the following boundary condition is for the normal components of velocity and magnetic field:

$$(v^0_e, g^0_e)|_{y=0} = 0,$$  

(2. 5)

which is the same as (1. 0). It is noted that the boundary condition (2. 3) on the normal component of $(v^0_e, g^0_e)$ is sufficient to solve the initial-boundary value problem (2. 3)-(2. 5). Under the assumption that the initial data $(u_0, v_0, h_0, g_0)(x, y)$ have enough regularity in some Sobolev spaces, the existence and uniqueness of classical solution to (2. 3)-(2. 5) are guaranteed by the results in [39, 43] that can be stated as follows.

**Proposition 2.1.** Let $m > 0$ be a large integer. Suppose the initial data $(u_0, v_0, h_0, g_0)(x, y) \in H^m(T \times \mathbb{R}_+)$ and $h_0(x, 0) \geq \delta_0$ for some positive constant $\delta_0$ satisfies the compatibility conditions up to $(m-1)$-th order for (2. 3)-(2. 5). Moreover, the divergence free conditions hold for the initial data

$$\hat{c}_x u_0 + \hat{c}_y v_0 = 0, \quad \hat{c}_x h_0 + \hat{c}_y g_0 = 0.$$

Then there exists a unique solution $(u^0_e, v^0_e, p^0_e, h^0_e, g^0_e)(t, x, y)$ to (2. 3)-(2. 5) in $[0, T_1],$ which satisfies

$$(u^0_e, v^0_e)(t, x, y) \in C^j([0, T_1]; H^{m-j}(T \times \mathbb{R}_+)),$$

and $(v^0_e, g^0_e)$ is defined through the divergence-free condition and the boundary condition (2. 3).

**Remark 2.1.** In view of the assumption of initial data $h_0$ in Theorem 1.1, from the initial data (2. 3) it is noted that $h^0_e(0, x, y)|_{y=0} = h_0(x, 0) \geq \delta_0 > 0.$ Then, by the properties of the solution $(v^0_e, v^0_e, p^0_e, h^0_e, g^0_e)(t, x, y)$ established in Proposition 2.1 it is not hard to see that there exists a time $T_1 \leq T_1^*$, such that the boundary value $h^0_e(t, x, 0) \geq \frac{\delta_0}{2}$ for all $t \in [0, T_1].$ Moreover, the trace theorem yields

$$\sum_{i=0}^{m-2} \sup_{0 \leq t \leq T_1} \| \hat{c}_t^i (u^0_e, h^0_e, \hat{c}_x p^0_e)(t, x, 0) \|_{H^{m-2-i}(\mathbb{T}_x)} < +\infty.$$  

(2. 6)

After establishing the leading order inner profile $(u^0_e, v^0_e, p^0_e, h^0_e, g^0_e)(t, x, y),$ we now turn to construct the leading order MHD boundary layer profile.
2.2. Zero-order MHD boundary layer. The zero-order MHD boundary layer profile \((u_b^0, v_b^0, h_b^0, g_b^0)(t, x, \eta)\) is given by
\[
\begin{align*}
(u_b^0, h_b^0)(t, x, \eta) &= (u^p_h, h^p(t, x, \eta) - (u_c^0, h_c^0)(t, x, 0), \\
(v_b^0(t, x, \eta) &= \sum_{j=0}^{\frac{|m|}{2} - 2} \delta^i \partial_x u_b^0(t, x, \eta)d\eta, \\
g_b^0(t, x, \eta) &= \sum_{j=0}^{\frac{|m|}{2} - 2} \delta^i \partial_x h_b^0(t, x, \eta)d\eta,
\end{align*}
\] (2. 7)
and \((u^p, h^p)(t, x, \eta)\) can be solved by the following boundary layer system:
\[
\begin{align*}
\partial_t u^p + (u^p \partial_x + v^p \partial_\eta) u^p - (h^p \partial_x + g^p \partial_\eta) h^p &= \mu \partial_x^2 u^p - \partial_x h^p(t, x, 0), \\
\partial_t h^p + (u^p \partial_x + v^p \partial_\eta) h^p - (h^p \partial_x + g^p \partial_\eta) u^p &= \kappa \partial_x^2 h^p, \\
\partial_x u^p + \partial_\eta v^p &= 0, \\
\partial_x h^p + \partial_\eta g^p &= 0, \\
(u^p, v^p, \partial_\eta h^p, g^p)|_{\eta=0} &= 0, \\
\lim_{\eta \to \pm \infty} (u^p, h^p)(t, x, \eta) &= (u_c^0, h_c^0)(t, x, 0), \\
(u^p, h^p)|_{t=0} &= (u_c^0, h_c^0)(0, x, 0) = (0, h_0(x, 0)),
\end{align*}
\] (2. 8)
where we have used in the above initial data, the compatibility conditions of the problem (2. 3)-(2. 5) for \((u_c^0, h_c^0)\). It is noted that we use the assumption of initial data \(h_0\) in Theorem 1.1 to obtain
\[
h^p(0, x, \eta) \geq \delta_0 > 0. 
\] (2. 9)

By the main theorem in [28], we have the local well-posedness theory of solutions to the initial-boundary value problem (2. 8). Before we state the well-posedness theorem, let us introduce some weighted Sobolev spaces used in this subsection. Denote by
\[
\Omega := \{(x, \eta) : x \in \mathbb{T}, \; \eta \in \mathbb{R}_+\}.
\]
For any \(l \in \mathbb{R}\), denote by \(L^2_l(\Omega)\) the weighted Lebesgue space with respect to the spatial variables:
\[
L^2_l(\Omega) := \{f(x, \eta) : \Omega \to \mathbb{R}, \; \|f\|_{L^2_l(\Omega)} := \left(\int_{\Omega} |\langle \eta \rangle|^{2l} |f(x, \eta)|^2 dx d\eta \right)^{\frac{1}{2}} < +\infty\}, \quad \langle \eta \rangle = 1 + \eta,
\]
and then, for any given \(m \in \mathbb{N}\), denote by \(H^m_l(\Omega)\) the weighted Sobolev spaces:
\[
H^m_l(\Omega) := \left\{f(x, \eta) : \Omega \to \mathbb{R}, \; \|f\|_{H^m_l(\Omega)} := \left(\sum_{m_1 + m_2 \leq m} \|\langle \eta \rangle^{l+2m_2} \partial_x^{m_1} \partial_{\eta}^{m_2} f\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} < +\infty\right\}.
\]
Combining Remark 2.1 with the condition (2. 9), we have the following result by the main theorem in [28].

**Proposition 2.2.** Let \((u_c^0, v_c^0, h_c^0, g_c^0)(t, x, \eta)\) be the leading order inner flow, constructed in Proposition 2.7 satisfying \((u_c^0, v_c^0, h_c^0, g_c^0)(t, x, \eta) \in \cap_{j=0}^{m} C^j \left([0, T_1]; H^{m-j}(\mathbb{T} \times \mathbb{R}^+)\right)\) for large integer \(m \geq 0\) and \(T_1 > 0\) given in Remark 2.4. Then, there exist a positive time \(0 < T_2 \leq T_1\) and a unique solution \((u^p, v^p, h^p, g^p)(t, x, \eta)\) to the initial boundary value problem (2. 8), such that
\[
h^p(t, x, \eta) \geq \frac{\delta_0}{2}, \quad \forall (t, x, \eta) \in [0, T_2] \times \Omega
\] (2. 10)
with the constant \(\delta_0 > 0\) given in (2. 9), and for any \(l \geq 0\),
\[
(u^p(t, x, \eta) - u_c^0(t, x, 0), h^p(t, x, \eta) - h_c^0(t, x, 0)) \in \bigcap_{i=0}^{\frac{|m|}{2} - 2} W^{i, \infty} \left(0, T_2; H^{\frac{|m|}{2} - 2 - i}(\Omega)\right),
\] (2. 11)
where \([k], k \in \mathbb{R}\) stands for the largest integer less than or equal to \(k\). Moreover, for the profile \((u_b^0, v_b^0, h_b^0, g_b^0)(t, x, \eta)\) defined by (2. 7) it holds
\[
(u_b^0, h_b^0)(t, x, \eta) \in \bigcap_{i=0}^{\frac{|m|}{2} - 2} W^{i, \infty} \left(0, T_2; H^{\frac{|m|}{2} - 2 - i}(\Omega)\right),
\] (2. 12)
\[
(v_b^0, g_b^0)(t, x, \eta), (\partial_\eta v_b^0, \partial_\eta g_b^0)(t, x, \eta) \in \bigcap_{i=0}^{\frac{|m|}{2} - 3} W^{i, \infty} \left(0, T_2; H^{\frac{|m|}{2} - 3 - i}(\Omega)\right).
\] (2. 13)

**Proof.** First of all, from the local well-posedness theory of the solution \((u^p, v^p, h^p, g^p)(t, x, \eta)\) to problem (2. 8), and the relation (2. 11) has been obtained in [28]. Note that the initial data of \((u^p, h^p)\), given in (2. 8), is independent of normal variable \(\eta\), therefore the index \(l\) of weight with respect to \(\eta\) in (2. 11) can
be arbitrary large. Moreover, (2.12) follows automatically by combining (2.7) with (2.11). Therefore, we only need to show (2.13).

From (2.7) we obtain $(\bar{\varepsilon}_x v_0^0, \bar{\varepsilon}_\eta g_0^0) = -(\bar{\varepsilon}_x u_0^0, \bar{\varepsilon}_x h_0^0)$, and for $\alpha \in \mathbb{N}^2, l \geq 0$,

$$\left| \langle \eta \rangle^l \bar{\varepsilon}_x \bar{\varepsilon}_\eta (t, x, \eta) \right| = \left| \langle \eta \rangle^l \int_\eta \bar{\varepsilon}_x^\alpha \bar{\varepsilon}_x u_0^0(t, x, \eta) d\eta \right| \leq \int_\eta \left| \langle \eta \rangle^l \bar{\varepsilon}_x^\alpha \bar{\varepsilon}_x u_0^0(t, x, \eta) \right| d\eta$$

$$\leq \langle \eta \rangle^{\frac{1}{2} - l_0} \langle \eta \rangle^{l+1} \bar{\varepsilon}_x u_0^0(t, x, \eta) \right\|_{L_2^2}$$

provided $l_0 > \frac{1}{2}$, which implies that

$$\| \bar{\varepsilon}_x^\alpha \bar{\varepsilon}_x u_0^0(t, \cdot) \|_{L_2^2(\Omega)} \lesssim \langle \eta \rangle^{\frac{1}{2} - l_0} \left\| \bar{\varepsilon}_x \bar{\varepsilon}_\eta u_0^0(t, \cdot) \right\|_{L_2^2(\Omega)} \lesssim \| \bar{\varepsilon}_x \bar{\varepsilon}_x u_0^0(t, \cdot) \|_{L_2^2(\Omega)}$$

provided $l_0 > 1$. Similarly, we have that for $l_0 > 1$,

$$\| \bar{\varepsilon}_x^\alpha \bar{\varepsilon}_x h_0^0(t, \cdot) \|_{L_2^2(\Omega)} \lesssim \left\| \bar{\varepsilon}_x \bar{\varepsilon}_x \bar{\varepsilon}_\eta h_0^0(t, \cdot) \right\|_{L_2^2(\Omega)}$$

By using the above two inequalities and combining with (2.12), we get (2.13) immediately.

From (2.7) and the divergence-free conditions in (2.8) we have another expression for $(u_0^0, v_0^0, h_0^0, g_0^0)$:

$$\begin{aligned}
\left\{ \begin{aligned}
u_0^0(t, x, \eta) &= (u^0, h^0)(t, x, \eta) - (u_0^0, h_0^0)(t, x, 0), \\
(v_0^0, g_0^0)(t, x, \eta) &= (v^0, g^0)(t, x, \eta) + \eta(\bar{\varepsilon}_x u_0^0, \bar{\varepsilon}_x h_0^0)(t, x, 0) \\
&\quad + \int_0^\eta \left( \bar{\varepsilon}_x u^p(t, x, \eta) - \bar{\varepsilon}_x u_0^0(t, x, \eta) \right) d\eta
\end{aligned} \right.
\end{aligned}$$

(2.14)

which implies that by virtue of the boundary conditions $(v^p, g^p)_{|\eta=0} = 0$ in (2.8),

$$\begin{aligned}
(v_0^0, g_0^0)(t, x, 0) &= \int_0^\infty \left( \bar{\varepsilon}_x u^p(t, x, \eta) - \bar{\varepsilon}_x u_0^0(t, x, \eta) \right) \bar{\varepsilon}_x h_0^0(t, x, \eta) d\eta.
\end{aligned}$$

(2.15)

Moreover, we can derive the problem of $(u_0^0, v_0^0, h_0^0, g_0^0)(t, x, \eta)$ from (2.8). Indeed, from (2.14) and (2.15) we obtain that

$$\begin{aligned}
\left\{ \begin{aligned}
u_0^0(t, x, \eta) &= (u_0^0, h_0^0)(t, x, \eta) + (u_0^0, h_0^0)(t, x, 0), \\
(v_0^0, g_0^0)(t, x, \eta) &= (v_0^0, g_0^0)(t, x, \eta) + \eta(\bar{\varepsilon}_x u_0^0, \bar{\varepsilon}_x h_0^0)(t, x, 0) - \eta(\bar{\varepsilon}_x u_0^0, \bar{\varepsilon}_x h_0^0)(t, x, 0).
\end{aligned} \right.
\end{aligned}$$

Substituting the above expression into (2.8) and using the notation of $\bar{f}(t, x)$ for the trace of function $f(t, x, y)$ on the boundary $\{y = 0\}$, we obtain that

$$\begin{aligned}
\left\{ \begin{aligned}\ar{\varepsilon}_x u_0^0 + \bar{\varepsilon}_x v_0^0 + \bar{\varepsilon}_x h_0^0 \bar{\varepsilon}_x v_0^0 + (v_0^0 - v_\eta^0 - \eta \bar{\varepsilon}_x u_0^0) \bar{\varepsilon}_\eta u_0^0 - (h_0^0 + h_\eta^0) \bar{\varepsilon}_x h_0^0 - (g_0^0 - g_\eta^0 - \eta \bar{\varepsilon}_x h_0^0) \bar{\varepsilon}_\eta h_0^0 \\
&\quad + \bar{\varepsilon}_x u_0^0 - \bar{\varepsilon}_x h_0^0 = \mu \bar{\varepsilon}_n u_0^0, \\
\bar{\varepsilon}_x h_0^0 \bar{\varepsilon}_x v_0^0 + \bar{\varepsilon}_x h_0^0 + (v_0^0 - v_\eta^0 - \eta \bar{\varepsilon}_x u_0^0) \bar{\varepsilon}_\eta h_0^0 - (h_0^0 + h_\eta^0) \bar{\varepsilon}_x h_0^0 - (g_0^0 - g_\eta^0 - \eta \bar{\varepsilon}_x h_0^0) \bar{\varepsilon}_\eta h_0^0 \\
&\quad + \bar{\varepsilon}_x u_0^0 - \bar{\varepsilon}_x h_0^0 = \kappa \bar{\varepsilon}_n h_0^0, \\
\bar{\varepsilon}_x u_0^0 + \bar{\varepsilon}_x v_0^0 = 0, \\
\bar{\varepsilon}_x h_0^0 + \bar{\varepsilon}_n h_0^0 = 0,
\end{aligned} \right.
\end{aligned}$$

(2.16)

where we have used the equations of $(u_0^0, h_0^0)$ on the boundary $\{y = 0\}$ from the problem (2.3)-(2.5). Also, we have the following initial-boundary values:

$$\left\{ \begin{aligned}
u_0^0 |_{t=0} = 0, \\
\left( u_0^0, h_0^0 \right)_{|\eta=0} = -(u_0^0, h_0^0)(t, x), \\
\lim_{\eta \to +\infty} (u_0^0, h_0^0)(t, x, \eta) = 0.
\end{aligned} \right.$$

(2.17)

and the boundary condition (2.15) for $(v_0^0, g_0^0)$). Moreover, from (2.16) we know that $g_0^0$ satisfies the following equation:

$$\begin{aligned}
\bar{\varepsilon}_x (g_0^0 + \eta \bar{\varepsilon}_x h_0^0) h_0^0 - \bar{\varepsilon}_x v_0^0 + \bar{\varepsilon}_x v_0^0 - \eta \bar{\varepsilon}_x u_0^0) \bar{\varepsilon}_\eta h_0^0 - (h_0^0 + h_\eta^0) \bar{\varepsilon}_x h_0^0 - (g_0^0 - g_\eta^0 - \eta \bar{\varepsilon}_x h_0^0) \bar{\varepsilon}_\eta h_0^0 \\
&\quad + \bar{\varepsilon}_x u_0^0 - \bar{\varepsilon}_x h_0^0 = \kappa \bar{\varepsilon}_n h_0^0.
\end{aligned}$$

(2.18)

After constructing the leading order inner flow $(u_0^0, v_0^0, h_0^0, g_0^0)$ and boundary layer profile $(u_0^0, v_0^0, h_0^0, g_0^0)$, we proceed to construct the next order inner MHD flow.
2.3. First-order ideal MHD flow. Put the ansatz (2.1) into (1.1) and set the terms of order $\epsilon^{1/2}$ equal to zero, then letting $\eta \to +\infty$ yields the first order inner flow $(u_1^e, v_1^e, p_1^e, h_1^e, g_1^e)$ satisfies the following linearized ideal MHD equations.

\[
\begin{cases}
\partial_t u_1^e + (u_0^e \partial_x + v_0^e \partial_y) u_1^e + \partial_x p_1^e - (h_0^e \partial_x + g_0^e \partial_y) h_1^e + (u_1^e \partial_x + v_1^e \partial_y) u_0^e - (h_1^e \partial_x + g_1^e \partial_y) h_0^e = 0, \\
\partial_t v_1^e + (u_0^e \partial_x + v_0^e \partial_y) v_1^e + \partial_y p_1^e - (h_0^e \partial_x + g_0^e \partial_y) g_1^e + (u_1^e \partial_x + v_1^e \partial_y) v_0^e - (h_1^e \partial_x + g_1^e \partial_y) g_0^e = 0,
\end{cases}
\tag{2.19}
\]

\[
\begin{aligned}
\partial_t h_1^e + (u_0^e \partial_x + v_0^e \partial_y) h_1^e - (h_0^e \partial_x + g_0^e \partial_y) u_1^e + (u_1^e \partial_x + v_1^e \partial_y) h_0^e - (h_1^e \partial_x + g_1^e \partial_y) u_0^e = 0, \\
\partial_t g_1^e + (u_0^e \partial_x + v_0^e \partial_y) g_1^e - (h_0^e \partial_x + g_0^e \partial_y) v_1^e + (u_1^e \partial_x + v_1^e \partial_y) g_0^e - (h_1^e \partial_x + g_1^e \partial_y) v_0^e = 0, \\
\partial_x u_1^e + \partial_y v_1^e = 0, \\
\end{aligned}
\]

\[
\partial_x h_1^e + \partial_y g_1^e = 0.
\]

The initial data is chosen to be zero:

\[
(u_1^e, v_1^e, h_1^e, g_1^e)|_{t=0} = 0, \tag{2.20}
\]

and the boundary conditions of $(u_1^e, g_1^e)$ in (2.19) are thus imposed by

\[
(v_1^e, g_1^e)(t, x, 0) = -(v_0^e, g_0^e)(t, x, 0) = \left( -\int_0^x \partial_x u_0^e(t, x, \eta) d\eta, -\int_0^x \partial_x h_0^e(t, x, \eta) d\eta \right), \tag{2.21}
\]

which can be solved by (2.19). Moreover, from Proposition 2.2 we know that

\[
(v_1^e, g_1^e)(t, x, 0) \in \bigcap_{i=0}^{[m/2]-3} W^{i, \infty}(0, T_2; H^{(m/2)-3-i}(\mathbb{T}x)).
\]

By a similar argument as for the Proposition 2.1 for initial-boundary value problem of the linearized ideal MHD equations (2.19)-(2.21), or as a direct consequence of the main results in [32], we also have

**Proposition 2.3.** Let $(u_0^e, v_0^e, h_0^e, g_0^e)(t, x, y) \in \bigcap_{j=0}^m C^j([0, T_1]; H^m-j(\mathbb{T} \times \mathbb{R}^+))$ established in Proposition 2.1 and suppose the boundary data of $(u_1^e, v_1^e, p_1^e, h_1^e, g_1^e)$ to (2.19)-(2.21) in the time interval $[0, T_3]$, such that

\[
(u_1^e, v_1^e, h_1^e, g_1^e)(t, x, y) \in \bigcap_{j=0}^{[m/2]-3} C^j([0, T_3]; H^{[m/2]-3-j}(\mathbb{T} \times \mathbb{R}^+)),
\]

where $0 < T_3 \leq T_2$ is the local lifespan of solution $(u_1^e, v_1^e, p_1^e, h_1^e, g_1^e)$.

Now we consider the following (leading) zero-th order approximation solutions to (1.1):

\[
\begin{cases}
(u_0^0, h_0^0)(t, x, y) = (u_0^e + \sqrt{\epsilon} u_1^e, h_0^e + \sqrt{\epsilon} h_1^e)(t, x, y) + (u_0^e, h_0^e)(t, x, \frac{y}{\sqrt{\epsilon}}), \\
(v_0^0, g_0^0)(t, x, y) = (v_0^e + \sqrt{\epsilon} v_1^e, g_0^e + \sqrt{\epsilon} g_1^e)(t, x, y) + (v_0^e, g_0^e)(t, x, \frac{y}{\sqrt{\epsilon}}), \\
p_0^0(t, x, y) = p_0^e(t, x, y) + \sqrt{\epsilon} p_1^e(t, x, y).
\end{cases}
\]

From the above construction of $(u_i^e, v_i^e, p_i^e, h_i^e, g_i^e)(i=0, 1)$ and $(u_0^e, v_0^e, h_0^e, g_0^e)$, and by a direct calculation we obtain that

\[
\begin{aligned}
\partial_t u_0^0 + (u_0^0 \partial_x + v_0^0 \partial_y) u_0^0 + \partial_x p_0^0 - (h_0^0 \partial_x + g_0^0 \partial_y) h_0^0 - \epsilon \Delta u_0^0 &= R_0^0, \\
\partial_t v_0^0 + (u_0^0 \partial_x + v_0^0 \partial_y) v_0^0 + \partial_y p_0^0 - (h_0^0 \partial_x + g_0^0 \partial_y) g_0^0 - \epsilon \Delta v_0^0 &= R_0^0, \\
\partial_t h_0^0 + (u_0^0 \partial_x + v_0^0 \partial_y) h_0^0 - (h_0^0 \partial_x + g_0^0 \partial_y) u_0^0 - \epsilon \kappa \triangle h_0^0 &= R_0^0, \\
\partial_t g_0^0 + (u_0^0 \partial_x + v_0^0 \partial_y) g_0^0 - (h_0^0 \partial_x + g_0^0 \partial_y) v_0^0 - \epsilon \kappa \triangle g_0^0 &= R_0^0, \\
\partial_x u_0^0 + \partial_y v_0^0 = 0, \\
\partial_x h_0^0 + \partial_y g_0^0 = 0, \\
(u_0^0, v_0^0, h_0^0, g_0^0)|_{x=0} = (u_0, v_0, h_0, g_0)(x, y), \\
(u_0^0, v_0^0, \partial_y h_0^0, g_0^0)|_{y=0} = (\sqrt{\epsilon} u_0^e, 0, \partial_y h_0^e + \sqrt{\epsilon} \partial_y h_1^e, 0)(t, x, 0),
\end{aligned}
\tag{2.22}
\]
where the remainder terms $R_1^0(i = 1 \sim 4)$ are summarized as follows,

\[
\begin{align*}
R_1^0 = & \left( u_0^0 - \bar{u}_0^0 + \sqrt{\epsilon} u_1 \right) \partial_x u_0^0 + u_0^0 \left( \partial_x u_0 - \bar{u}_x u_0 + \sqrt{\epsilon} \partial_x u_0 \right) + \left[ v_0^0 - y \bar{v}_y v_0^0 + \sqrt{\epsilon} (v_1 - \bar{v}_x v_0^0) \right] \partial_y u_0^0 + \sqrt{\epsilon} v_0^0 \partial_y u_0^0 \\
& - \left( h_0^0 + \bar{h}_0^0 + \sqrt{\epsilon} h_1 \right) \partial_y h_0^0 + h_0^0 \left( \partial_y h_0 - \bar{h}_y h_0 + \sqrt{\epsilon} \partial_y h_0 \right) - \left[ g_0^0 - y \bar{g}_y g_0^0 + \sqrt{\epsilon} (g_1 - \bar{g}_x g_0^0) \right] \partial_y h_0^0 \\
& - \sqrt{\epsilon} g_0^0 \partial_y h_0^0 + R_{1h}^0, \\
R_2^0 = & \sqrt{\epsilon} \partial_y u_0^0 + \sqrt{\epsilon} \partial_y u_0^0 + \sqrt{\epsilon} v_0^0 \partial_y u_0^0 + u_0^0 \left( \partial_y h_0 - \bar{h}_y h_0 + \sqrt{\epsilon} \partial_y h_0 \right) + \left[ v_0^0 - y \bar{v}_y v_0^0 + \sqrt{\epsilon} (v_1 - \bar{v}_x v_0^0) \right] \partial_y h_0^0 + \sqrt{\epsilon} v_0^0 \partial_y h_0^0 \\
& - \sqrt{\epsilon} \partial_y h_0^0 + \sqrt{\epsilon} \partial_y h_0^0 + R_{2h}^0, \\
R_3^0 = & \left( u_0^0 - \bar{u}_0^0 + \sqrt{\epsilon} u_1 \right) \partial_x h_0^0 + u_0^0 \left( \partial_x h_0 - \bar{h}_x h_0 + \sqrt{\epsilon} \partial_x h_0 \right) + \left[ v_0^0 - y \bar{v}_y v_0^0 + \sqrt{\epsilon} (v_1 - \bar{v}_x v_0^0) \right] \partial_y h_0^0 + \sqrt{\epsilon} v_0^0 \partial_y h_0^0 \\
& - \left( h_0^0 + \bar{h}_0^0 + \sqrt{\epsilon} h_1 \right) \partial_y u_0^0 + h_0^0 \left( \partial_y u_0 - \bar{u}_y u_0 + \sqrt{\epsilon} \partial_y u_0 \right) - \left[ g_0^0 - y \bar{g}_y g_0^0 + \sqrt{\epsilon} (g_1 - \bar{g}_x g_0^0) \right] \partial_y h_0^0 \\
& - \sqrt{\epsilon} \partial_y h_0^0 + \sqrt{\epsilon} \partial_y h_0^0 + R_{3h}^0, \\
R_4^0 = & \sqrt{\epsilon} \partial_y u_0^0 + \sqrt{\epsilon} \partial_y u_0^0 + \sqrt{\epsilon} v_0^0 \partial_y u_0^0 + u_0^0 \left( \partial_y h_0 - \bar{h}_y h_0 + \sqrt{\epsilon} \partial_y h_0 \right) + \left[ v_0^0 - y \bar{v}_y v_0^0 + \sqrt{\epsilon} (v_1 - \bar{v}_x v_0^0) \right] \partial_y h_0^0 + \sqrt{\epsilon} v_0^0 \partial_y h_0^0 \\
& - \sqrt{\epsilon} \partial_y h_0^0 + \sqrt{\epsilon} \partial_y h_0^0 + R_{4h}^0, \\
\end{align*}
\]

with

\[
\begin{align*}
R_{1h}^0 = & \epsilon \left\{ u_1^0 \partial_x u_2 + (v_1 + v_0^0) \partial_y u_2 - h_1 \partial_x h_1 + (g_1 + g_0^0) \partial_y h_1 - \mu \left[ \triangle (u_0^0 + \sqrt{\epsilon} u_1^0) + \bar{\epsilon} \partial_x u_0^0 \right] \right\}, \\
R_{2h}^0 = & \epsilon \left\{ u_1^0 \partial_x (v_1 + v_0^0) + (v_1 + v_0^0) \partial_y v_1 - h_1 \partial_x (g_1 + g_0^0) + (g_1 + g_0^0) \partial_y g_1 - \mu \left[ \triangle (v_0^0 + \sqrt{\epsilon} v_1^0) + \bar{\epsilon} \partial_x v_0^0 \right] \right\}, \\
R_{3h}^0 = & \epsilon \left\{ u_1^0 \partial_x (h_1 + v_0^0) + (v_1 + v_0^0) \partial_y h_1 - h_1 \partial_x h_1 + (g_1 + g_0^0) \partial_y h_1 - \kappa \left[ \triangle (h_0^0 + \sqrt{\epsilon} h_1^0) + \bar{\epsilon} \partial_x h_0^0 \right] \right\}, \\
R_{4h}^0 = & \epsilon \left\{ u_1^0 \partial_x (g_1 + g_0^0) + (v_1^0 + v_0^0) \partial_y g_1 - h_1 \partial_x (v_1^0 + v_0^0) + (g_1^0 + g_0^0) \partial_y v_1 + \kappa \left[ \triangle (g_0^0 + \sqrt{\epsilon} g_1^0) + \bar{\epsilon} \partial_x g_0^0 \right] \right\}. \\
\end{align*}
\]

It is easy to check that the leading order terms in the remainders $R_1^0, R_2^0$ and $R_3^0$ mainly exist in the boundary layer, and are in fact of order $\sqrt{\epsilon}$. However, thanks to the equation (2.18) for $g_0^0$ we find that the error remainder $R_4^0$ is of order $\epsilon$. Indeed, by virtue of (2.18) we can rewrite $R_4^0$ as

\[
\begin{align*}
R_4^0 = & \sqrt{\epsilon} \left( u_0^0 - \bar{u}_0^0 \right) \partial_x g_0^0 + u_0^0 \left[ \partial_x g_0^0 - y \bar{v}_y g_0^0 \right] + \sqrt{\epsilon} \left[ \partial_x g_0^0 - \bar{v}_x g_0^0 \right] \partial_y g_0^0 \\
& + \sqrt{\epsilon} \left( g_0^0 - \bar{g}_0 \right) \partial_x \left( y \bar{v}_y g_0^0 \right) - \sqrt{\epsilon} \left[ h_0^0 - \bar{h}_0 \right] \partial_x \left( v_0^0 - y \bar{v}_y v_0^0 \right) + \sqrt{\epsilon} \left( v_1 - \bar{v}_x v_0^0 \right) \\
& - \sqrt{\epsilon} \left[ g_0^0 - y \bar{g}_y g_0^0 \right] \partial_y \left( \bar{v}_x v_0^0 \right) + \sqrt{\epsilon} g_0^0 \left[ \partial_y \left( \bar{v}_x v_0^0 \right) \right] + R_{4h}^0. \\
\end{align*}
\]

Thus, as shown in (2.1), we proceed in the next two subsections to construct the leading order boundary layer profile $p_1^0(x, t, \eta)$ of pressure and first order boundary layer profiles $\sqrt{\epsilon} (u_1^0, \sqrt{\epsilon} v_1^0, h_1^0, \sqrt{\epsilon} g_0^0)(x, t, \eta)$ to cancel the leading order terms in $R_2^0$ and $R_1^0, R_3^0$ respectively.

2.4. Leading order boundary layer of pressure. In order to eliminate the leading order terms in $R_2^0$ in (2.23), in other words, the terms of order $\sqrt{\epsilon}$ in (1.1) generated by $(u_0^0, v_0^0, h_0^0, g_0^0)$, we define the leading order boundary layer profile $p_1^0(x, t, \eta)$ for pressure in the following way,

\[
\begin{align*}
p_1^0(x, t, \eta) = & \int_\eta^\infty \left\{ \left[ \partial_x + \left( u_0^0 + \bar{u}_0^0 \right) \partial_x + (v_0^0 + \bar{v}_x v_0^0 + \eta \bar{v}_y v_0^0) \partial_\eta - \mu \partial^2_\eta \right] p_0^0 + \left( \bar{v}_x v_0^0 + \eta \bar{v}_y v_0^0 \right) p_0^0 \right\}(t, x, \tilde{\eta})d\tilde{\eta} \\
& - \int_\eta^\infty \left\{ \left[ \left( h_0^0 + \bar{h}_0^0 \right) \partial_x + (g_0^0 + \bar{g}_x g_0^0 + \eta \bar{g}_y g_0^0) \partial_\eta - \mu \partial^2_\eta \right] g_0^0 + \left( \bar{g}_x g_0^0 + \eta \bar{g}_y g_0^0 \right) h_0^0 \right\}(t, x, \tilde{\eta})d\tilde{\eta}. \\
\end{align*}
\]

(2.26)
By using the above expression (2.26) and combining with (2.12)–(2.13), we can obtain that
\[ p_b^1(t, x, \eta), \quad \partial_\eta p_b^1(t, x, \eta) \in \bigcap_{i=0}^{[m/2]-4} W^{1,\infty}(0, T_2; H^{[m/2]-4-i}_1(\Omega)), \quad \forall t \geq 0. \] (2.27)

2.5. First-order MHD boundary layer. Applying (2.1) into (1.1) and considering the terms of order \( \sqrt{\eta} \), it leads to the first order boundary layer equation for tangential velocity \( u_b^1(t, x, \eta) \):
\[ \partial_t u_b^1 + (u_b^0 + \eta \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x u_b^1 + u_b^1 (\partial_x u_b^1 + \vec{e}_\eta \cdot \vec{h}_b^0) + \left( v_b^0 + \vec{e}_\eta \cdot \vec{g}_b^0 \right) \partial_\eta u_b^1 + v_\eta^0 \partial_\eta h_b^0 \\
- (h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^0 - h_b^1 (\partial_x h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) - \left( g_b^0 + \vec{e}_\eta \cdot \vec{g}_b^0 \right) \partial_\eta h_b^1 - g_\eta^0 \partial_\eta h_b^0 - \kappa \partial_\eta^2 h_b^1 \\
- (w_b^0 + \eta \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x u_b^0 - u_b^0 (\partial_x u_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) - \left( \eta \vec{g}_b^0 \partial_x v_b^0 + \eta^2 \partial_\eta^2 v_b^0 \right) \partial_\eta u_b^0 - v_\eta^0 \partial_\eta u_b^0 \\
+ (h_b^0 + \eta \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^0 + h_b^1 (\partial_x h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) + \left( \eta \vec{g}_b^0 \partial x v_b^0 + \eta^2 \partial_\eta^2 v_b^0 \right) \partial_\eta h_b^0 + g_\eta^0 \partial_\eta h_b^0. \] (2.28)

Similarly, by investigating the terms of order \( \sqrt{\eta} \) in (1.1) we can obtain the following equation for the first order boundary layer \( h_b^1(t, x, \eta) \):
\[ \partial_t h_b^1 + (u_b^0 + \eta \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^1 + u_b^1 (\partial_x h_b^1 + \vec{e}_\eta \cdot \vec{h}_b^0) + \left( v_b^0 + \vec{e}_\eta \cdot \vec{g}_b^0 \right) \partial_\eta h_b^1 + v_\eta^0 \partial_\eta h_b^0 \\
- (h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^0 - h_b^1 (\partial_x h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) - \left( g_b^0 + \vec{e}_\eta \cdot \vec{g}_b^0 \right) \partial_\eta h_b^1 - g_\eta^0 \partial_\eta h_b^0 - \kappa \partial_\eta^2 h_b^1 \\
- (w_b^0 + \eta \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^0 - u_b^0 (\partial_x h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) - \left( \eta \vec{g}_b^0 \partial_x v_b^0 + \eta^2 \partial_\eta^2 v_b^0 \right) \partial_\eta h_b^0 - v_\eta^0 \partial_\eta h_b^0 \\
+ (h_b^0 + \eta \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^0 + h_b^1 (\partial_x h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) + \left( \eta \vec{g}_b^0 \partial x v_b^0 + \eta^2 \partial_\eta^2 v_b^0 \right) \partial_\eta h_b^0 + g_\eta^0 \partial_\eta h_b^0. \] (2.29)

Note that the terms of RHS in (2.28) and (2.29) come from the leading order terms of error terms \( R_{10}^0 \) and \( R_{11}^0 \) given in (2.28) respectively. Of course, we still impose the divergence-free conditions:
\[ \partial_x u_b^1 + \partial_\eta v_b^1 = 0, \quad \partial_x h_b^1 + \partial_\eta g_b^1 = 0, \] (2.30)
and the zero initial data:
\[ (u_b^1, h_b^1)|_{t=0} = 0. \] (2.31)

Moreover, we choose the boundary conditions
\[ (u_b^1, v_b^1, \partial_x h_b^1, g_b^1)|_{\eta=\infty} = 0, \] (2.32)
to homogenize the boundary conditions of approximate solutions. Thereby, we obtain the initial-boundary value problem (2.28)–(2.32) for the first-order boundary layer profile \( (u_b^1, v_b^1, h_b^1, g_b^1)(t, x, \eta) \).

Remark 2.2. From the above construction, the normal components \( v_b^1 \) and \( g_b^1 \) of the first-order boundary layer profile \( (u_b^1, v_b^1, h_b^1, g_b^1) \) are determined by the divergence-free conditions (2.30) and boundary conditions (2.32), in other words,
\[ (v_b^1, g_b^1)(t, x, \eta) = -\int_0^\eta (\partial_x u_b^0, \partial_x h_b^0)(t, x, \eta)d\eta, \]
which shows that in general, \( (v_b^1, g_b^1) \) doesn't decay to zero as \( \eta \to +\infty \). Note that such profile \( (v_b^1, g_b^1) \) is slightly different from the corresponding one given in the ansatz (2.1), which is expected to decay rapidly as \( \eta \to +\infty \). In fact, the difference between \( (v_b^1, g_b^1) \), constructed in this subsection, and the corresponding one in (2.1) is only a function independent of normal variable \( \eta \).

To construct the approximate solutions \( (u^0, v^0, p^0, h^0, g^0) \) of (2.1), it is left to show the well-posedness of the solution \( (u_b^1, v_b^1, h_b^1, g_b^1)(t, x, \eta) \) to (2.28)–(2.32). To this end, we use the energy methods developed in [28]. Specifically speaking, from the divergence-free conditions for \( (u_b^0, v_b^0, h_b^0, g_b^0)(i = 0, 1) \) and \( (u_b^1, v_b^1, h_b^1, g_b^1)(i = 0, 1) \), we rewrite the equation (2.29) as follows.
\[ \partial_t h_b^1 + \partial_\eta \left[ (v_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) \partial_x h_b^1 - (u_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) g_b^0 - \left( g_b^0 + \vec{e}_\eta \cdot \vec{g}_b^0 \right) u_b^1 + (h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) v_b^0 \right] - \kappa \partial_\eta^2 h_b^1 \\
= \partial_\eta \left[ (\eta \vec{g}_b^0 \partial_x v_b^0 + \eta^2 \partial_\eta^2 v_b^0) h_b^0 + (h_b^0 + \vec{e}_\eta \cdot \vec{h}_b^0) v_b^0 \right]. \] (2.33)
Define
\[ \psi(t, x, \eta) = \int_{0}^{1} h_{b}^0(t, x, \eta) d\eta, \]
and it implies that
\[ \partial_{x} \psi(t, x, \eta) = -g_{b}^0(t, x, \eta). \]
Integrating the equation (2.33) over \([0, \eta]\) leads to
\[
\partial_{t} \psi + (u_{b}^{0} + \vec{u}_{b}^{0}) \partial_{x} \psi + (v_{b}^{0} + \vec{v}_{b}^{0} + \eta \vec{\epsilon}_{y} \vec{v}_{b}^{0}) \partial_{\eta} \psi - (b_{b}^{0} + \vec{b}_{b}^{0} + \eta \vec{\epsilon}_{y} \vec{b}_{b}^{0}) u_{b}^{1} + (h_{b}^{0} + \vec{h}_{b}^{0}) v_{b}^{1} - \kappa \vec{\epsilon}_{y} \psi
\]
\[
= \left( \eta \vec{\epsilon}_{y} g_{b}^{1} + \frac{\eta^{2}}{2} \vec{\epsilon}_{y}^{2} \vec{g}_{b}^{0} \right) u_{b}^{0} - (h_{b}^{0} + \eta \vec{\epsilon}_{y} \vec{h}_{b}^{0}) v_{b}^{0} - \left( \eta \vec{\epsilon}_{y} v_{b}^{1} + \frac{\eta^{2}}{2} \vec{\epsilon}_{y}^{2} \vec{v}_{b}^{0} \right) h_{b}^{0} + (u_{b}^{1} + \eta \vec{\epsilon}_{y} u_{b}^{0}) g_{b}^{0}
\]
\[- \vec{h}_{b}^{0} \vec{v}_{b}^{1} + \vec{u}_{b}^{0} \vec{g}_{b}^{0} + \kappa \vec{\epsilon}_{y} \vec{h}_{b}^{0}, \]
(2.34)
where we have used the following boundary conditions
\[ (u_{b}^{0}, v_{b}^{0}, g_{b}^{0})|_{\eta=0} = -(\vec{u}_{b}^{0}, \vec{v}_{b}^{0}, \vec{g}_{b}^{0})(t, x), \quad v_{b}^{1}|_{\eta=0} = 0. \]
Then, for simplicity of presentation, we only give the outline about the applications of the energy estimate method developed in [28] here. First, we derive the \( L^{2} \)-estimates of
\[ \partial_{nx}^{\alpha} \partial_{\eta}^{\alpha} (u_{b}^{1}, h_{b}^{1}), \quad \alpha \in \mathbb{N}^{2}, \quad j \in \mathbb{N}, \quad |\alpha| + i \leq k, \quad |\alpha| \leq k - 1 \]
from the problem (2.28)-(2.32) by standard energy methods. Then, it is left to derive \( L^{2} \)-estimates of \( \partial_{nx}^{\alpha} (u_{b}^{1}, h_{b}^{1}) \) with \( |\alpha| = k \). By introducing the following new quantities:
\[ u_{b}^{0} = \partial_{nx}^{\alpha} u_{b}^{1}, \quad \frac{\partial_{n} u_{b}^{0}}{h_{b}^{0} + \vec{h}_{b}^{0}} \partial_{nx}^{\alpha} \psi, \quad h_{b}^{0} = \partial_{nx}^{\alpha} h_{b}^{1}, \quad \frac{\partial_{n} h_{b}^{0}}{h_{b}^{0} + \vec{h}_{b}^{0}} \partial_{nx}^{\alpha} \psi, \]
and from the equations (2.28), (2.29) and (2.34), we can derive the equations of \( u_{b}^{0} \) and \( h_{b}^{0} \), in which the terms involving \( \partial_{nx}^{\alpha} (v_{b}^{1}, g_{b}^{1}) \) disappear. Therefore, it is possible to obtain the \( L^{2} \)-estimates of \( u_{b}^{0}, h_{b}^{0} \). We need the desired estimates for \( \partial_{nx}^{\alpha} (u_{b}^{1}, h_{b}^{1}) \) by proving the equivalence of \( L^{2} \)-norm between \( (u_{b}^{0}, h_{b}^{0}) \) and \( \partial_{nx}^{\alpha} (u_{b}^{1}, h_{b}^{1}) \), and close the whole energy estimates. Consequently, the well-posedness results of solution \( (u_{b}^{1}, v_{b}^{1}, h_{b}^{1}, g_{b}^{1}) \) to the initial-boundary value problem (2.28)-(2.32) are concluded as follows.

**Proposition 2.4.** Let \( (u_{b}^{0}, v_{b}^{0}, h_{b}^{0}, g_{b}^{0}) \) and \( (u_{b}^{1}, v_{b}^{1}, h_{b}^{1}, g_{b}^{1}) \) be solutions constructed in Propositions 2.7 and 2.8 respectively. Let \( (u_{b}^{0}, v_{b}^{0}, h_{b}^{0}, g_{b}^{0}) \) be the boundary layer profile constructed in Proposition 2.2. Then, there exist a positive time \( 0 < T_{4} \leq T_{3} \) and a unique solution \( (u_{b}^{1}, v_{b}^{1}, h_{b}^{1}, g_{b}^{1}) \) to the initial-boundary value problem (2.28)-(2.32), such that for any \( l \geq 0 \),
\[
(u_{b}^{1}, h_{b}^{1})(t, x, \eta) \in \bigcap_{i=0}^{[m/4]-4} W^{i, \infty}(0, T_{4}; H_{l}^{[m/4]-4-i}(\Omega)),
\]
\[
(v_{b}^{1}, g_{b}^{1})(t, x, \eta) \in \bigcap_{i=0}^{[m/4]-5} W^{i, \infty}(0, T_{4}; L_{x}^{\infty}(\mathbb{R}_{+}; H_{l}^{[m/4]-5-i}(T_{x}))),
\]
\[
(\partial_{\eta} v_{b}^{1}, \partial_{\eta} g_{b}^{1})(t, x, \eta) \in \bigcap_{i=0}^{[m/4]-5} W^{i, \infty}(0, T_{4}; H_{l}^{[m/4]-5-i}(\Omega)).
\]
(2.35)

2.6. **Construction of approximate solutions.** We are in a position to complete the construction of approximate solutions for problem (1.1). Indeed, based on the profiles given in the above five subsections, we
can write down the approximate solutions \((u^a, v^a, h^a, g^a, p^a)(t, x, y)\) used in this paper.

\[
\begin{align*}
  u^a(t, x, y) &= u^a_0(t, x, y) + v^a_0(t, x, y) + \sqrt{\epsilon} u^a_1(t, x, y) + \chi(y) u^a_0(t, x, y) + \sqrt{\epsilon} \chi'(y) \int_0^\infty u^a_b(t, x, \eta) d\eta, \\
  v^a(t, x, y) &= v^a_0(t, x, y) + \sqrt{\epsilon} v^a_1(t, x, y) + \sqrt{\epsilon} \chi'(y) v^a_0(t, x, y) + v^a_0(t, x, y), \\
  h^a(t, x, y) &= h^a_0(t, x, y) + h^a_0(t, x, y) + \sqrt{\epsilon} h^a_1(t, x, y) + \chi(y) h^a_0(t, x, y) + \sqrt{\epsilon} \chi'(y) h^a_0(t, x, \eta) d\eta + \sqrt{\epsilon} \rho(t, x, y), \\
  g^a(t, x, y) &= g^a_0(t, x, y) + \sqrt{\epsilon} g^a_1(t, x, y) + \sqrt{\epsilon} \chi'(y) g^a_0(t, x, y) + \sqrt{\epsilon} \chi'(y) g^a_0(t, x, y) - \epsilon \int_0^\infty \partial_y \rho(t, x, \eta) d\eta, \\
  p^a(t, x, y) &= p^a_0(t, x, y) + \sqrt{\epsilon} p^a_1(t, x, y) + \sqrt{\epsilon} \rho(t, x, y).
\end{align*}
\]

Here the smooth cut-off function \(\chi(y)\) is defined as follows

\[
\chi(y) = \begin{cases}
1, & y \in [0, 1] \\
0, & y \in [2, +\infty)
\end{cases}
\]

and the boundary corrector \(\rho(t, x, y)\) is a smooth function with compact support, which is chosen to satisfy the following two conditions:

\[
\rho(0, x, y) \equiv 0, \quad \partial_y \rho(t, x, 0) = -\partial_y h^a_1(t, x, 0).
\]

It is noted that such function \(\rho(t, x, y)\) exists since the two conditions in (2.38) are compatible from (2.1).

Remark 2.3. We introduce the cut-off function \(\chi(y)\) such that the new boundary layer profiles \(\chi(y) v^a_1(t, x, \frac{y}{\epsilon})\) and \(\chi(y) g^a_1(t, x, \frac{y}{\epsilon})\) have rapid decay for \(y \geq 2\) as \(\epsilon \to 0\), see Remark 2.2. On the other hand, the boundary corrector \(\rho_0\) is used to cancel the boundary value of \(\partial_y h^a_1\) on \(\{y = 0\}\), so that it still holds that \(\partial_y h^a_{|y=0} = 0\) for the approximation 2.39.)

Firstly, direct calculation reads that \((u^a, v^a)\) and \((h^a, g^a)\) satisfy the divergence-free conditions:

\[
\partial_x u^a + \partial_y v^a = 0, \quad \partial_x h^a + \partial_y g^a = 0,
\]

and the following initial-boundary conditions:

\[
(u^a, v^a, h^a, g^a)|_{t=0} = (u_0, v_0, h_0, g_0)(x, y), \quad (u^a, v^a, \partial_y h^a, g^a)|_{y=0} = 0.
\]

Based on the construction in Subsections 2.1 – 2.3, we find that the approximate solution \((u^a, v^a, h^a, g^a, p^a)\) in 2.39 solves the incompressible viscous MHD equations (1.1) with some high order error terms with respect to the small parameter \(\epsilon\). More precisely,

\[
\begin{align*}
  \partial_t u^a + (u^a \partial_x + v^a \partial_y) u^a + \partial_x p^a - (h^a \partial_x + g^a \partial_y) h^a - \mu \epsilon \triangle u^a &= R_1, \\
  \partial_t v^a + (u^a \partial_x + v^a \partial_y) v^a + \partial_y p^a - (h^a \partial_x + g^a \partial_y) g^a - \mu \epsilon \triangle v^a &= R_2, \\
  \partial_t h^a + (u^a \partial_x + v^a \partial_y) h^a - (h^a \partial_x + g^a \partial_y) u^a - \kappa \epsilon \triangle h^a &= R_3, \\
  \partial_t g^a + (u^a \partial_x + v^a \partial_y) g^a - (h^a \partial_x + g^a \partial_y) v^a - \kappa \epsilon \triangle g^a &= R_4, \\
  \partial_x u^a + \partial_y v^a &= 0, \quad \partial_x h^a + \partial_y g^a = 0, \\
  (u^a, v^a, h^a, g^a)|_{y=0} &= 0, \quad (u^a, v^a, h^a, g^a)|_{t=0} = (u_0, v_0, h_0, g_0).
\end{align*}
\]

The expressions of the error terms \(R_i\) \((i = 1 \sim 4)\) caused by the approximation will be given in Appendix. And for \(R_i\) \((i = 1 \sim 4)\) in 2.40 we have the following result, and its proof is also in Appendix.

**Proposition 2.5.** Let the approximate solutions \((u^a, v^a, h^a, g^a, p^a)\) established in 2.39, then the error terms \(R_i\) \((i = 1 \sim 4)\) in 2.40 satisfy the following estimates:

\[
\|R_i\|_{L^\infty(0, T; \mathbb{R})} \leq C\epsilon, \quad \alpha \in \mathbb{N}^2, \quad |\alpha| \leq 3, \quad t \in [0, T],
\]

for some positive constant \(C\) independent of \(\epsilon\).
3. Estimates of the remainder and proof of the main theorem

Recall that in the above section, we have constructed the approximate solution \((u^a, v^a, h^a, g^a, p^a)\), given by \((2.30)\), which satisfies the problem \((2.40)\). Let

\[
(u^a, v^a, h^a, g^a, p^a) = (u^0, v^0, h^0, g^0, p^0) + \epsilon (u, v, h, g, p),
\]

then applying \((2.40)\) and \((3.1)\) in the original problem \((1.1)\), we derive the initial-boundary value problem for the remainder \((u, v, h, g, p)(t, x, y)\):

\[
\begin{align*}
\partial_t u + (u \partial_x + v \partial_y) u + \partial_x p - (h \partial_x + g \partial_y) h + (u \partial_x + v \partial_y) u^a - (h \partial_x + g \partial_y) h^a - \mu \epsilon \Delta u &= r_1^a, \\
\partial_t v + (u \partial_x + v \partial_y) v + \partial_y p - (h \partial_x + g \partial_y) g + (u \partial_x + v \partial_y) v^a - (h \partial_x + g \partial_y) g^a - \mu \epsilon \Delta v &= r_2^a, \\
\partial_t h + (u \partial_x + v \partial_y) h - (h \partial_x + g \partial_y) u + (u \partial_x + v \partial_y) h^a - (h \partial_x + g \partial_y) u^a - \kappa \epsilon \Delta h &= r_3^a, \\
\partial_t g + (u \partial_x + v \partial_y) g - (h \partial_x + g \partial_y) v + (u \partial_x + v \partial_y) g^a - (h \partial_x + g \partial_y) v^a - \kappa \epsilon \Delta g &= r_4^a, \\
\partial_x u + \partial_y v &= 0, \\
\partial_x h + \partial_y g &= 0, \\
(u, v, h, g)|_{y=0} &= 0, \\
(u, v, h, g)|_{t=0} &= 0,
\end{align*}
\]

where \(r_i^a = \epsilon^{-1} R_i, i = 1 \sim 4\) with \(R_i\) given by \((2.40)\). Moreover, from Proposition 2.5 we can achieve that

\[
\|u^a, r_i^a(t, \cdot)\|_{L^2} \leq C, \quad |\alpha| \leq 3, \quad i = 1 \sim 4, \quad t \in [0, T_4]
\]

for some positive constant \(C\) independent of \(\epsilon\).

The key difficulty in the analysis for problem \((3.2)\) in the Sobolev spaces comes from the strong coupling between the vorticity induced by the boundary layer and the remainder terms in the boundary layer of thickness \(O(\epsilon^2)\) even for short time (but independent of \(\epsilon\)). More precisely, consider the following terms in the equations \((3.2)\) and \((3.3)\):

\[
\begin{align*}
\partial_t u^a + (u^a \partial_x + v^a \partial_y) u^a - (h^a \partial_x + g^a \partial_y) h^a + \epsilon \frac{1}{\mu} (h^a \partial_x + g^a \partial_y) u^a - \epsilon \frac{1}{\mu} (h^a \partial_x + g^a \partial_y) h^a &= O(1), \\
\partial_t v^a + (u^a \partial_x + v^a \partial_y) v^a - (h^a \partial_x + g^a \partial_y) g^a + \epsilon \frac{1}{\mu} (h^a \partial_x + g^a \partial_y) v^a - \epsilon \frac{1}{\mu} (h^a \partial_x + g^a \partial_y) g^a &= O(1),
\end{align*}
\]

which cannot be estimated directly by the energy method. Indeed,

\[
\begin{align*}
|\int_{T \times \mathbb{R}_+} u \cdot (v \partial_x u^a - g \partial_y h^a) \, dx \, dy| &\leq O(1) \epsilon \frac{1}{\mu} \|u, v\|_{L^2}^2, \\
|\int_{T \times \mathbb{R}_+} h \cdot (v \partial_x h^a - g \partial_y u^a) \, dx \, dy| &\leq O(1) \epsilon \frac{1}{\mu} \|h, v\|_{L^2}^2,
\end{align*}
\]

and this prevents us to obtain the uniform estimates in \(\epsilon\). Therefore, we need to apply the idea used in existence of solutions to the boundary layer problem \((2.8)\) in \([28]\) to take care of the cancellations between some physical terms according to the structure of the system.

3.1. Key transformation and preliminaries. By the divergence free condition \(\partial_x h + \partial_y g = 0\), there exists a stream function \(\psi(t, x, y)\), such that

\[
h = \partial_y \psi, \quad g = -\partial_x \psi, \quad \psi|_{y=0} = 0, \quad \psi|_{t=0} = 0,
\]

and \(\psi\) satisfies

\[
\partial_t \psi + (u \partial_x + v \partial_y) \psi - g^a v + h^a u - \kappa \epsilon \Delta \psi = \epsilon^{-1} r_5^a \pm r_5^a.
\]

Next, recall the cut-off function \(\chi(y) \in C^\infty(\mathbb{R}_+), 0 \leq \chi(y) \leq 1\) with

\[
\chi(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\
0, & y \geq 2.
\end{cases}
\]

and the boundary layer profiles given in \((2.7)\) and \((2.8)\):

\[
(u^p, h^p)(t, x, y) = (u^0_{b^p}, h^0_{b^p})(t, x) + (u^0_{b^p}, h^0_{b^p})(t, x, \eta).
\]

Taking the positive condition \((2.10)\) for \(h^p\) into account, let us introduce some notations:

\[
a^p(t, x, y) := \chi(y) \frac{u^p(t, x, \sqrt{\frac{y}{\mu}})}{h^p(t, x, \sqrt{\frac{y}{\mu}})}, \quad b^p(t, x, y) := \frac{\partial_y h^p(t, x, \sqrt{\frac{y}{\mu}})}{h^p(t, x, \sqrt{\frac{y}{\mu}})}.
\]

The boundary conditions of \(u^p\) and \(h^p\) in \((2.8)\) yield

\[
a^p(t, x, 0) = b^p(t, x, 0) = 0.
\]
Note that from Proposition 2.2, we know that for \( \alpha \in \mathbb{N}^2, j \in \mathbb{N} \) and any \( l \geq 0 \),

\[
(1 + \eta)^{l+j} \partial_x^j \partial_y^l \bigl( a^p(t, x, \eta) - u^c(t, x, 0), h^p(t, x, \eta) - h^c(t, x, 0) \bigr) = O(1). \tag{3.10}
\]

Therefore, it shows that \( b^p(t, x, y) \) decays rapidly for \( y \gg 2 \) as \( \epsilon \to 0 \), so is the case with \( a^p \) because of the cut-off function \( \chi(y) \) we imposed in the definition of \( a^p(t, x, y) \). Moreover, from (3.10) we obtain that for any \( k \in \mathbb{R}_+ \),

\[
y^k \partial_y^l \partial_x^n a^p(t, x, y) = \begin{cases} O(1), & j = 0, \\ O(\epsilon^{-\frac{k-j}{2}}), & j \geq 1; \end{cases} \quad y^k \partial_y^l \partial_x^n b^p(t, x, y) = O(\epsilon^{-\frac{k-j}{2}}). \tag{3.11}
\]

To overcome the difficulty from (3.1), we introduce the following transformation:

\[
\tilde{u}(t, x, y) := u(t, x, y) - \partial_y(a^p \cdot \psi)(t, x, y), \quad \tilde{v}(t, x, y) := v(t, x, y) + \partial_x(a^p \cdot \psi)(t, x, y), \\
\tilde{h}(t, x, y) := h(t, x, y) - (b^p \cdot \psi)(t, x, y), \quad \tilde{g}(t, x, y) := g(t, x, y). \tag{3.12}
\]

Combining the initial-boundary values of \((u, v, h, g)\) and \(\psi\), given in (3.2) and (3.5) respectively, and using (3.9) we have

\[
(\tilde{u}, \tilde{v}, \tilde{h}, \tilde{g})|_{t=0} = 0, \quad (\tilde{u}, \tilde{v}, \partial_y \tilde{h}, \tilde{g})|_{y=0} = 0.
\]

Denote by

\[
U(t, x, y) := (\tilde{u}, \tilde{v}, \partial_y \tilde{h}, \tilde{g})^T(t, x, y), \tag{3.13}
\]

then the problem (3.2) can be reduced as follows:

\[
\begin{cases}
\partial_t U + A_1(U) \partial_x U + A_2(U) \partial_y U + C(U) U + \psi D + (p_x, p_y, 0, 0)^T - \epsilon B \Delta U = E^*, \\
\partial_x \tilde{u} + \partial_y \tilde{v} = 0, \\
(\tilde{u}, \tilde{v}, \partial_y \tilde{h}, \tilde{g})|_{y=0} = 0, \quad U|_{t=0} = 0. \quad \tag{3.14}
\end{cases}
\]

Here,

\[
A_i(U) = A_i^a + \sqrt{\epsilon} A_i^b + \epsilon \tilde{A}_i(U), \quad i = 1, 2, \tag{3.15}
\]

where

\[
A_1^a = \begin{pmatrix}
(u^a + a^p h^a) I_{2 \times 2} & [(a^p)^2 - 1] h^a I_{2 \times 2} \\
-h^a I_{2 \times 2} & (u^a - a^p h^a) I_{2 \times 2}
\end{pmatrix}, \quad A_2^a = \begin{pmatrix}
(v^a + a^p g^a) I_{2 \times 2} & [(a^p)^2 - 1] g^a I_{2 \times 2} \\
-g^a I_{2 \times 2} & (v^a - a^p g^a) I_{2 \times 2}
\end{pmatrix},
\]

\[
A_1^b = \begin{pmatrix}
0 & -2\mu \sqrt{\epsilon} \partial_x a^p (\mu - \kappa) \sqrt{\epsilon} \partial_y a^p + \eta h b^p \\
0 & -3(\mu - \kappa) \sqrt{\epsilon} \partial_x a^p
\end{pmatrix},
\]

\[
A_2^b = \begin{pmatrix}
0 & -(3\mu - \kappa) \sqrt{\epsilon} \partial_y a^p - (\mu - \kappa) \sqrt{\epsilon} \partial_x a^p \\
0 & 2\mu \sqrt{\epsilon} \partial_y a^p
\end{pmatrix},
\]

and

\[
\tilde{A}_1(U) = \begin{pmatrix}
(u + a^p h) I_{2 \times 2} & [(a^p)^2 - 1] h I_{2 \times 2} \\
-h I_{2 \times 2} & (u - a^p h) I_{2 \times 2}
\end{pmatrix}, \quad \tilde{A}_2(U) = \begin{pmatrix}
(v + a^p g) I_{2 \times 2} & [(a^p)^2 - 1] g I_{2 \times 2} \\
g I_{2 \times 2} & (v - a^p g) I_{2 \times 2}
\end{pmatrix}.
\]

For \( C(U) \):

\[
C(U) = C^a + \epsilon \tilde{C}(U), \tag{3.16}
\]

where

\[
C^a = \begin{pmatrix}
-\partial_y(v^a - a^p g^a) & \partial_y(u^a - a^p h^a) \\
\partial_x(v^a - a^p g^a) & \partial_x(u^a - a^p h^a) \\
\partial_x h^a + b^p g^a & \partial_y h^a - b^p h^a \\
\partial_x g^a & \partial_y g^a
\end{pmatrix},
\]

\[
\tilde{C}(U) = \begin{pmatrix}
C_{13}^{a} & C_{14}^{a} \\
C_{23}^{a} & C_{24}^{a} \\
C_{33}^{a} & C_{34}^{a} \\
C_{43}^{a} & C_{44}^{a}
\end{pmatrix}.
\]
and

\[
\tilde{C}(U) = \begin{pmatrix}
\tilde{c}_y(\tilde{c}_x a^p \cdot \psi) & \tilde{c}_y(\tilde{c}_x a^p \cdot \psi) & \tilde{c}_x a^p(\tilde{c}_x a^p \cdot \psi) & -2\tilde{c}_y(\tilde{c}_x a^p \cdot \psi) & 2\tilde{c}_y(\tilde{c}_x a^p \cdot \psi) & 0 \\
\tilde{c}_x b^p \cdot \psi & \tilde{c}_x b^p \cdot \psi & \tilde{c}_x b^p \cdot \psi & \tilde{c}_x b^p \cdot \psi & \tilde{c}_x b^p \cdot \psi & \tilde{c}_x b^p \cdot \psi \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

with

\[
\begin{align*}
C_{13}^a &= -2a\tilde{c}_y(u^a - a^p g^a) - [(a^p)^2 - 1](\tilde{c}_x g^a - b^p g^a) + [\tilde{c}_t + (u^a + a^p h^a)\tilde{c}_x + (v^a + a^p g^a)\tilde{c}_y - \mu\epsilon\triangle - 2\mu\epsilon^2 \gamma]a^p \\
&\quad - \epsilon\{(\mu - \kappa)b\tilde{c}_y a^p + (\mu - \kappa)a^p[2\tilde{c}_y b^p + (b^p)^2]\}, \\
C_{14}^a &= 2a\tilde{c}_y(u^a - a^p h^a) + [(a^p)^2 - 1](\tilde{c}_x h^a - b^p h^a) - 2\epsilon[(\mu - \kappa)a^p\tilde{c}_x b^p - \mu\epsilon\triangle a^p], \\
C_{23}^a &= 2\tilde{c}_x(v^a - a^p g^a) + [(a^p)^2 - 1]\tilde{c}_x g^a + 2\epsilon[\mu\tilde{c}_x a^p + (\mu - \kappa)\mu\tilde{c}_x a^p], \\
C_{24}^a &= -2a\tilde{c}_x(u^a - a^p h^a) - [(a^p)^2 - 1]h\tilde{c}_x h^a + [\tilde{c}_t + (u^a + a^p h^a)\tilde{c}_x + (v^a + a^p g^a)\tilde{c}_y - \mu\epsilon\triangle - 2\mu\epsilon^2 \gamma]a^p, \\
C_{33}^a &= \tilde{c}_y(v^a - a^p g^a) - (h\tilde{c}_x + g^a\tilde{c}_y) a^p - 2\mu\epsilon\tilde{c}_x b^p, \quad C_{34}^a = -\tilde{c}_y(u^a - a^p h^a) + 2\mu\epsilon\tilde{c}_x b^p, \\
C_{44}^a &= -\tilde{c}_x(v^a - a^p g^a), \quad C_{44}^a = \tilde{c}_x(u^a - a^p h^a) - (h\tilde{c}_x + g^a\tilde{c}_y) a^p,
\end{align*}
\]

Also, for the vector \( D \),

\[
D = D^a + \epsilon pD^p, \tag{3.17}
\]

where \( D^a = (D^a_{1})_{i=1,4} \) and \( D^p = (D^p_{1})_{i=1,4} \) are given by:

\[
\begin{align*}
D_1^a &= -\tilde{c}_x a^p\tilde{c}_y(u^a - a^p h^a) = -(\tilde{c}_x a^p\tilde{c}_y(u^a - a^p g^a) + [(a^p)^2 - 1][(h\tilde{c}_x + g^a\tilde{c}_y)b^p - b^p(g^a\tilde{c}_y - b^p g^a)]) \\
&\quad + [\tilde{c}_t + (u^a + a^p h^a)\tilde{c}_x + (v^a + a^p g^a)\tilde{c}_y - \mu\epsilon\triangle]a^p + b^p[\tilde{c}_t + (u^a + a^p h^a)\tilde{c}_x + (v^a + a^p g^a)\tilde{c}_y - \mu\epsilon\triangle]a^p \\
&\quad - 2\epsilon[b^p\tilde{c}_y a^p + \tilde{c}_x a^p b^p - (\mu - \kappa)\epsilon a^p[\triangle b^p + 3b^p\tilde{c}_x b^p + (b^p)^3] - (\mu - \kappa)\epsilon a^p(\tilde{c}_y b^p + (b^p)^2)], \\
D_2^a &= \tilde{c}_x a^p\tilde{c}_y(u^a - a^p h^a) + (\tilde{c}_x a^p + 2\epsilon b^p)\tilde{c}_x(u^a - a^p g^a) + [(a^p)^2 - 1]b^p\tilde{c}_x g^a \\
&\quad - [\tilde{c}_t + (u^a + a^p h^a)\tilde{c}_x + (v^a + a^p g^a)\tilde{c}_y - \mu\epsilon\triangle]a^p + \epsilon[2\mu b^p\tilde{c}_x a^p + (\mu - \kappa)\tilde{c}_x a^p[\tilde{c}_y b^p + (b^p)^2]], \\
D_3^a &= b^p\tilde{c}_y(v^a - a^p g^a) - \tilde{c}_x[(h\tilde{c}_x + g^a\tilde{c}_y)a^p] + [\tilde{c}_t + (u^a - a^p h^a)\tilde{c}_x + (v^a - a^p g^a - 2\mu\epsilon b^p)\tilde{c}_y - \nu\epsilon\triangle b^p], \\
D_4^a &= -b^p\tilde{c}_x(v^a - a^p g^a) + \tilde{c}_x[(h\tilde{c}_x + g^a\tilde{c}_y)a^p], \\
\end{align*}
\]

and

\[
\begin{align*}
D_1^p &= (\tilde{c}_y a^p + 2a\tilde{c}_y b^p)\tilde{c}_y a^p - \tilde{c}_x a^p \tilde{c}_y a^p + [(a^p)^2 - 1]b^p\tilde{c}_x b^p + 2a\tilde{c}_x a^p(b^p)^2, \\
D_2^p &= -(\tilde{c}_y a^p + 2\epsilon b^p)\tilde{c}_y a^p + \tilde{c}_x a^p \tilde{c}_y a^p, \\
D_3^p &= -b^p\tilde{c}_x a^p - \tilde{c}_x a^p \tilde{c}_y b^p + (b^p)^2] + \tilde{c}_y a^p \tilde{c}_x b^p, \\
D_4^p &= b^p\tilde{c}_x a^p.
\end{align*}
\]

Moreover,

\[
B = \begin{pmatrix}
\mu I_{2\times 2} & (\mu - \kappa)a^p I_{2\times 2} \\
0 & \kappa I_{2\times 2}
\end{pmatrix}, \quad E^x = \begin{pmatrix}
\nu^x - \tilde{c}_y(a^p r_5^x), r_2^x + \tilde{c}_x(a^p r_2^x), r_3^x - b^p r_5^x, r_4^x
\end{pmatrix}^T. \tag{3.18}
\]

By direct calculation and \((3.11)\), it is easy to obtain that for \(|a| \leq 2, \ i = 1, 2,\)

\[
\|c_{10}^a A^p(t, \cdot)\|_{L^\infty}, \quad \|c_{20}^a A^p(t, \cdot)\|_{L^\infty}, \quad \|y^2 c_{10}^a D^p(t, \cdot)\|_{L^\infty}, \quad \|c_{10}^a B(t, \cdot)\|_{L^\infty} = O(1). \tag{3.19}
\]

A key observation is that by direct calculation and \((3.11)\), there is a constant \(C > 0\) independent of \(\epsilon\) such that for \(|a| \leq 2, \ i = 1, 2,\)

\[
\|c_{10}^a C^a(t, \cdot)\|_{L^\infty} + \|y^2 c_{10}^a D^p(t, \cdot)\|_{L^\infty} \leq C, \tag{3.20}
\]

so the difficulty given in \((3.4)\) is absent in the new problem \((3.14)\) for \(U\). The estimates \((3.19)\) and \((3.20)\) are based on the estimates \((3.11)\) and the following facts \((F)\):
Lemma 3.1. There exists a positive constant $C$ such that, by combining (3.21), (3.22) and the Hardy inequality,

$$
\| \partial_y v^p \|_{L^\infty} \leq \| \partial_y v^p \|_{L^\infty} \leq \| v^p \|_{L^\infty},
$$

and similarly,

$$
\| v^p \|_{L^\infty}, \| g^p \partial_y a^p \|_{L^\infty}, \| (\partial_y v^0 - u_0^0) \partial_y v^0 \|_{L^\infty}, \| (h_e^0 - h_e^0) \partial_y v^0 \|_{L^\infty} = O(1), \quad i = 1, 2,
$$

which implies that $\partial_y (v^a - a^p g^a) = O(1)$;

- from the definition (3.8),

$$
\partial_y (u^a - a^p h^a) = \partial_y (u^0_0 - u_0^0 |_{y=0} - a^p (h_e^0 - h_e^0 |_{y=0})) + \partial_y (u^p - a^p h^p) + O(1)
$$

provided that $|\frac{1-\alpha}{\alpha}| \leq 1$, and similarly,

$$
\partial_y h^a b^p = \partial_y h_e^0 - h_e^0 |_{y=0} - b^p (h_e^0 - h_e^0 |_{y=0}) + \partial_y h^p b^p + O(1)
$$

- By utilizing the estimates (3.11) of functions $a^p$ and $b^p$, it is easy to obtain the uniform boundedness of $\| y \partial_y u^p \|_{L^\infty}$. And coupled with the above facts, we can show the uniform boundedness of $\| y \partial_y u^p \|_{L^\infty}$. This is why we impose a cut-off function $\chi(y)$ in the definition of $a^p$ in (3.8).

For the source term $E^\epsilon$ given in (3.18), it follows that by virtue of $r_\epsilon^i = \partial_y^{-1} r_3^i$,

$$
E^\epsilon = (E_i^\epsilon)_{1 \leq i \leq 4} = \begin{pmatrix}
\partial_y (u^a - a^p \cdot r_3^i) - \partial_y v^p \cdot \partial_y^{-1} r_3^i, r_2^i, \partial_x (a^p \cdot \partial_y^{-1} r_3^i), r_3^i - b^p \cdot \partial_y^{-1} r_3^i, r_4^i &
\end{pmatrix}^T,
$$

which implies that by combining (3.8), (3.11) and the Hardy inequality,

$$
\| \partial_y^\alpha E^\epsilon (t, \cdot) \|_{L^2} \lesssim \sum_{1 \leq i \leq 4} \| \partial_y^\alpha r_i^\epsilon (t, \cdot) \|_{L^2} \lesssim C, \quad |\alpha| \leq 2
$$

for some constant $C > 0$ independent of $\epsilon$.

First of all, we have the following lemma to show the estimates on the original unknown $(u, v, h, g)$ in $L^p(1 < p \leq \infty)$ norm by the newly defined function $U$ given by (3.12) and (3.13).

**Lemma 3.1.** There exists a positive constant $C$ independent of $\epsilon$, such that

$$
\| \partial_y a^\alpha (u, v, h, g)(t, \cdot) \|_{L^p} \leq C \sum_{\beta \leq \alpha} \| \partial_y^\beta U(t, \cdot) \|_{L^p}, \quad |\alpha| \leq 2, \quad 1 < p \leq \infty.
$$

**Proof.** Combining (3.8) with (3.12), we have

$$
\frac{\hat{h}(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} = \partial_y \left( \frac{\psi(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} \right), \quad \psi(t, x, y) = h^p (t, x, \frac{y}{\epsilon}) \cdot \partial_y^{-1} \left( \frac{\hat{h}(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} \right),
$$

and then, the Hardy inequality gives that by and the upper-lower bound of $h^p(t, x, \eta)$ given in Proposition 2.2

$$
\| \psi(t, x, y) \|_{L^p} \lesssim \frac{1}{y} \partial_y^{-1} \left( \frac{\hat{h}(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} \right) \|_{L^p} \lesssim \left| \frac{\hat{h}(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} \right|_{L^p} \lesssim C \| \hat{h}(t, \cdot) \|_{L^p}, \quad 1 < p \leq \infty.
$$

By a direct calculation,

$$
\partial_y^\alpha \psi(t, x, y) = \sum_{\beta \leq \alpha} C_{\alpha} \partial_y^\beta \left( \frac{\hat{h}(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} \right) \cdot \partial_y^{-1} \partial_y^\beta \left( \frac{\hat{h}(t, x, y)}{h^p (t, x, \frac{y}{\epsilon})} \right).
$$
and then, the Hardy inequality and the boundedness of $h^p$ yields that for $|\alpha| \leq 2$ and $1 \leq p \leq \infty$,
\[
\|y^{-1} \tilde{\gamma}_t^\alpha \psi(t, x, y)\|_{L^p} \leq \sum_{\beta \leq \alpha} C_{\alpha}^\beta \left\{ \|c_{t,x}^{\alpha-\beta} h^p(t, x, \frac{y}{\sqrt{\epsilon}})\|_{L^\infty} \cdot \|y^{-1} \tilde{\gamma}_t^{\alpha-\beta} \tilde{\psi}(t, x, \frac{y}{\sqrt{\epsilon}})\|_{L^p} \right\}.
\]
\[
\lesssim \sum_{\beta \leq \alpha} \|\bar{h}_t \|_{L^\infty} \frac{\|h_x^p(t, x, \frac{y}{\sqrt{\epsilon}})\|_{L^p}}{\|\bar{h}_t \|_{L^\infty}} \lesssim C_{\alpha} \sum_{\beta \leq \alpha} \|\bar{h}_t^\beta \tilde{h}(t, \cdot)\|_{L^p}. \tag{3.26}
\]
Next, we have
\[
u(t, x, y) = \tilde{v}(t, x, y) + a^p(t, x, \frac{y}{\sqrt{\epsilon}}) \cdot \tilde{h}(t, x, y) + (\tilde{\gamma}_y a^p + \alpha^p b^p)(t, x, \frac{y}{\sqrt{\epsilon}}) \cdot \psi(t, x, y),
\]
\[
h(t, x, y) = \tilde{h}(t, x, y) + b^p(t, x, \frac{y}{\sqrt{\epsilon}}) \cdot \psi(t, x, y), \quad g(t, x, y) = \tilde{g}(t, x, y). \tag{3.27}
\]
Thus, it yields that by using (3.25),
\[
\|u(t, \cdot)\|_{L^p} \leq \|\tilde{u}(t, \cdot)\|_{L^p} + \|a^p(t, \cdot)\|_{L^\infty} \|\tilde{h}(t, \cdot)\|_{L^p} + (\|\gamma a^p(t, \cdot)\|_{L^\infty} + \|a^p b^p(t, \cdot)\|_{L^\infty}) \|y^{-1} \psi(t, \cdot)\|_{L^p}
\]
\[
\lesssim \|\tilde{u}(t, \cdot)\|_{L^p} + C \|\tilde{h}(t, \cdot)\|_{L^p},
\]
and similarly,
\[
\|v(t, \cdot)\|_{L^p} \leq \|\tilde{v}(t, \cdot)\|_{L^p} + C \|\tilde{g}(t, \cdot)\|_{L^p} + C \|\tilde{h}(t, \cdot)\|_{L^p}, \quad \|h(t, \cdot)\|_{L^p} \leq C \|\tilde{h}(t, \cdot)\|_{L^p}, \quad \|g(t, \cdot)\|_{L^p} = \|\tilde{g}(t, \cdot)\|_{L^p}.
\]
Then, we have
\[
c_t^\alpha u(t, x, y) = c_t^\alpha \tilde{u}(t, x, y) + \sum_{\beta \leq \alpha} C_{\alpha}^\beta \left\{ c_t^{\alpha-\beta} a^p(t, x, \frac{y}{\sqrt{\epsilon}}) \cdot \tilde{h}_t \right\}
\]
\[
+ c_t^{\alpha-\beta} (\tilde{\gamma}_y a^p + \alpha^p b^p)(t, x, \frac{y}{\sqrt{\epsilon}}) \cdot \tilde{\psi}(t, x, y),
\]
and then, along with (3.26) and the boundedness of $a^p$ it follows that for $|\alpha| \leq 2$ and $1 \leq p \leq \infty$,
\[
\|c_t^\alpha u(t, \cdot)\|_{L^p} \leq \|c_t^\alpha \tilde{u}(t, \cdot)\|_{L^p} + \sum_{\beta \leq \alpha} C_{\alpha}^\beta \left\{ c_t^{\alpha-\beta} a^p(t, x, \frac{y}{\sqrt{\epsilon}}) \|\tilde{h}_t\|_{L^\infty} \cdot \|c_t^{\alpha-\beta} \tilde{h}(t, \cdot)\|_{L^p}
\]
\[
+ \|\gamma a^p(t, \cdot)\|_{L^\infty} \|\tilde{h}_t\|_{L^\infty} + \|a^p b^p(t, \cdot)\|_{L^\infty} \|y^{-1} \tilde{\psi}(t, x, y)\|_{L^p}\right\}
\]
\[
\lesssim \|c_t^{\alpha} \tilde{u}(t, \cdot)\|_{L^p} + C \sum_{\beta \leq \alpha} \|\tilde{h}_t^\beta \tilde{h}(t, \cdot)\|_{L^p}.
\]
Similarly, we can obtain that
\[
\|c_t^\alpha v(t, \cdot)\|_{L^p} \leq \|c_t^\alpha \tilde{v}(t, \cdot)\|_{L^p} + C \sum_{\beta \leq \alpha} \left\{ \|c_t^{\beta} \tilde{g}(t, \cdot)\|_{L^p} + \|c_t^{\beta} \tilde{h}(t, \cdot)\|_{L^p}\right\}, \quad \|c_t^\alpha h(t, \cdot)\|_{L^p} \leq C \sum_{\beta \leq \alpha} \|\tilde{h}_t^\beta \tilde{h}(t, \cdot)\|_{L^p}.
\]
Moreover, it is nature to get
\[
\|c_t^\alpha g(t, \cdot)\|_{L^p} = \|c_t^\alpha \tilde{g}(t, \cdot)\|_{L^p}.
\]
Combining the above four estimates, we obtain (3.23) immediately.

\[\square\]

Remark 3.1. Note that from the definitions of $\tilde{A}_i(U)$, $i = 1, 2$ and $\tilde{C}(U)$, by through calculation and combining with the relations (3.3), (3.11), a direct consequence of Lemma 3.7 is that for $|\alpha| \leq 2$ and $1 < p \leq \infty$,
\[
\|c_t^\alpha \tilde{A}_i(U)(t, \cdot)\|_{L^p} \leq \sum_{\beta \leq \alpha} \|c_t^{\beta} \tilde{u}(t, \cdot)\|_{L^p} + C \sum_{\beta \leq \alpha} \|c_t^{\beta} U(t, \cdot)\|_{L^p}, \quad i = 1, 2, \tag{3.28}
\]
and
\[
\|c_t^\alpha \tilde{C}(U)(t, \cdot)\|_{L^p} \leq \epsilon^{-\frac{1}{p}} \sum_{\beta \leq \alpha} \|c_t^{\beta} \tilde{h}(t, \cdot)\|_{L^p} + \|y^{-1} c_t^{\beta} \tilde{\psi}(t, \cdot)\|_{L^p} \leq C \epsilon^{-\frac{1}{p}} \sum_{\beta \leq \alpha} \|c_t^{\beta} U(t, \cdot)\|_{L^p}. \tag{3.29}
\]
Now, we will make some preliminary preparation for the problem \([3.14]\) of \(U\). Set
\[
S := \text{diag}(1, 1 - (a^p)^2, 1 - (a^p)^2),
\]
we find that \(S A_i^p, S \tilde{A}_i(U), i = 1, 2\) are symmetric, and
\[
S(A_i^p + \epsilon \tilde{A}_i(U)) = \begin{pmatrix}
(u^r + a^p h^t) I_{2 \times 2} & [(a^p)^2 - 1] h^t I_{2 \times 2} \\
[(a^p)^2 - 1] h^t I_{2 \times 2} & [1 - (a^p)^2](u^r - a^p h^t) I_{2 \times 2}
\end{pmatrix},
\]
Moreover, it follows that
\[
SB = \begin{pmatrix}
\mu & (\mu - \kappa) a^p I_{2 \times 2} \\
0 & \kappa [1 - (a^p)^2] I_{2 \times 2}
\end{pmatrix}.
\]
To ensure that the symmetrizer \(S\) and the matrix \(SB\) in \([3.31]\) are positive definite, we need to impose some restriction on the function \(a^p\). Specifically speaking, by using the local well-posedness results for problem \([2.8]\) obtained in Proposition \([2.2]\) and combining with \(a^p|_{t=0} = 0\) by the initial data of \([2.3]\), we know that for any fixed \(\delta > 0\) sufficiently small, there exists a \(T_\delta : 0 < T_\delta \leq T_\delta\) such that
\[
\sup_{t \in [0, T_\delta]} \|a^p(t, \cdot)\|_{L^\infty} \leq \frac{4(\mu - \delta)(\kappa - \delta)}{(\mu + \kappa)^2 - 4\delta \kappa}.
\]
Then, it is easy to check that under the condition \([3.32]\), \(SB\) given by \([3.31]\) is positive definite, i.e., for any vector \(X = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4\),
\[
SBX \cdot X \geq \delta |X|^2.
\]
Also, we have
\[
1 - (a^p)^2(t, x, y) \geq \frac{(\mu - \kappa)^2 + 4\delta (\mu - \delta)}{(\mu + \kappa)^2 - 4\delta \kappa} \geq c_\delta > 0, \quad t \in [0, T_\delta], \quad (x, y) \in \mathbb{T} \times \mathbb{R}_+,
\]
which, along with \([3.30]\) implies the positive definiteness of \(S\).

**Remark 3.2.** From the definition \([3.8]\) for \(a^p\), we find that the condition \([3.32]\) means that for the leading order boundary layer, the component of tangential velocity is controlled by the component of tangential magnetic field in the time interval \([0, T_\delta]\). This represents in some sense the stabilizing effect of the magnetic field on the velocity field in the boundary layers.

### 3.2. Energy estimates.
This subsection is devoted to the crucial estimates for the solution \(U\) to the problem \([3.14]\).

**Proposition 3.2.** For any fixed small \(\delta > 0\) such that \([3.32]\) holds, there exists a \(0 < T_* \leq T_\delta\) and a unique classical solution \(U(t, x, y)\) to \([3.14]\) on \([0, T_0]\) satisfying the following estimate:
\[
\|U(t, \cdot)\|_{L^2}^2 + \epsilon \|U_x(t, \cdot)\|_{L^2}^2 + \epsilon \int_0^t \epsilon \|\nabla U(s, \cdot)\|_{L^2}^2 + \epsilon \|\nabla U_x(s, \cdot)\|_{L^2}^2) ds \leq C, \quad \forall t \in [0, T_*]
\]
for some constant \(C > 0\) independent of \(\epsilon\).

**Proof.** The local existence and uniqueness of the classical solution \(U\) to problem \([3.14]\), in some time interval \([0, T]\) (\(T\) may depends on \(\epsilon\)), follows from the standard well-posedness result for parabolic problem, so we will only show the estimate \([3.35]\) in the following.

1. **\(L^2\)–estimate for \(U\).** Multiplying \([3.14]_1\) by \(S\) from the left and taking the inner product of the resulting equation and \(U\), it follows that
\[
\frac{d}{2dt} (SU, U) + (SA_1(U)c_x U + SA_2(U)c_y U) + (S(C(U)U + \psi D) - \frac{1}{2} S_t U, U) - \epsilon (SB \triangle U, U) = (S E^\epsilon, U).
\]
Note that we have used the fact:
\[
(S(p_x, p_y, 0, 0)^T, U) = 0,
\]
which can be obtained by integration by parts. The divergence-free condition $\partial_x \tilde{u} + \partial_y \tilde{v} = 0$ and the boundary condition $\tilde{v}|_{y=0} = 0$.

Each term in (3.36) can be treated as follows. First, combining (3.34) with (3.30) yields that
\[
(SU, U) \geq c_{3\delta}\|U(t, \cdot)\|^2_{L^2}.
\] (3.37)

From (3.15), we have
\[
(SA_1(U)\partial_x U + SA_2(U)\partial_y U, U) = (S(A_1^a + \epsilon \hat{A}_1(U))\partial_x U + S(A_2^a + \epsilon \hat{A}_2(U))\partial_y U, U) + \sqrt{\epsilon}(SA_1^a\partial_x U + SA_2^a\partial_y U, U)
\]
\[
\pm I_1 + I_2. 
\] (3.38)

As $SA_i^a, \hat{A}_i(U), i = 1, 2$ are symmetric, and combining with the boundary conditions $SA_2^a|_{y=0} = 0$ and $\hat{A}_2(U)|_{y=0} = 0$, it yields that by integration by parts,
\[
I_1 = -\frac{1}{2} \left( [\partial_x(SA_1^a + \epsilon \hat{A}_1(U)) + \partial_y(SA_2^a + \epsilon \hat{A}_2(U))]U, U \right).
\]

Thus, applying (3.39) and (3.40) in $I_1$ we obtain that
\[
|I_1| \lesssim \|U(t, \cdot)\|^2_{H^1} + \epsilon^\frac{3}{2}\|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|^2_{L^2}.
\] (3.41)

From the Sobolev inequality and interpolation inequality, it follows that
\[
\|U(t, \cdot)\|^2_{H^1} \lesssim \|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{H^1} \lesssim \|U(t, \cdot)\|^2_{L^2} + \|U(t, \cdot)\|^2_{L^2},
\] (3.42)

then, applying (3.23) with $p = 2$ and (3.42) to (3.41) yields that
\[
|I_1| \lesssim \|U(t, \cdot)\|^2_{L^2} + \epsilon^{\frac{3}{2}}\|U(t, \cdot)\|_{L^2}^{\frac{3}{2}}(\|U(t, \cdot)\|_{L^2} + \|U(t, \cdot)\|^2_{L^2})
\]
\[
\leq \frac{\delta \epsilon}{16}\|\nabla U(t, \cdot)\|^2_{L^2} + C(1 + \epsilon^{\frac{3}{2}}\|U(t, \cdot)\|_{L^2}^{\frac{3}{2}}\|U(t, \cdot)\|^2_{L^2})\|U(t, \cdot)\|^2_{L^2}.
\] (3.43)

For the terms $I_2$, it is easy to obtain that by (3.19),
\[
|I_2| \leq C\sqrt{\epsilon}\|\nabla U(t, \cdot)\|_{L^2} + \|U(t, \cdot)\|_{L^2} \leq \frac{\delta \epsilon}{16}\|\nabla U(t, \cdot)\|^2_{L^2} + C\|U(t, \cdot)\|^2_{L^2}.
\] (3.44)

Then, plugging (3.43) and (3.44) into (3.38) we have
\[
(SA_1(U)\partial_x U + SA_2(U)\partial_y U, U) \leq \frac{\delta \epsilon}{8}\|\nabla U(t, \cdot)\|^2_{L^2} + C\left(1 + \|U(t, \cdot)\|^2_{L^2}\right)\|U(t, \cdot)\|^2_{L^2}.
\] (3.45)

From the definitions (3.16), (3.17) and (3.30), it gives
\[
\left(S(C(U)U + \psi D) - \frac{1}{2}S_tU, U\right) = \left(S(C^{\alpha}U + \psi \Delta^{\alpha}) - \frac{1}{2}S_tU, U\right) + \epsilon(\tilde{C}(U)U + \psi \Delta^{\alpha}, SU).
\] (3.46)
Thanks to the estimates \( (3. 20) \), it follows that
\[
\left| \left( S(C^\alpha U + \psi D^\alpha) - \frac{1}{2} S_t U, U \right) \right| \leq \left\| S C^\alpha - \frac{1}{2} S_t \right\|_{L^\infty} \left\| U(t, \cdot) \right\|_{L^2}^2 + \left\| y S D^\alpha \right\|_{L^\infty} \left\| y^{-1} \psi \right\|_{L^2} \left\| U(t, \cdot) \right\|_{L^2} \\
\leq C \left\| U(t, \cdot) \right\|_{L^2}^2. \tag{3. 47}
\]

On the other hand, for the second term on the right-hand side of \( (3. 40) \), we use \( (3. 19) \), \( (3. 25) \) and \( (3. 29) \) with \( p = 4 \), to obtain
\[
\epsilon \left| \left( \tilde{C}(U)U + \psi^2 D^p, SU \right) \right| \leq \epsilon \left\| SU(t, \cdot) \right\|_{L^2}^2 \left( \left\| \tilde{C}(U) \right\|_{L^4} \left\| U(t, \cdot) \right\|_{L^4} + \left\| \psi^2 D^p(t, \cdot) \right\|_{L^2} \left\| y^{-1} \psi(t, \cdot) \right\|_{L^2}^2 \right) \\
\leq \sqrt{\epsilon} \left\| U(t, \cdot) \right\|_{L^2} \left\| U(t, \cdot) \right\|_{L^2}^2 + \epsilon \left\| U(t, \cdot) \right\|_{L^2} \left\| \tilde{h}(t, \cdot) \right\|_{L^2}^2 \\
\leq \sqrt{\epsilon} \left\| U(t, \cdot) \right\|_{L^2} \left\| U(t, \cdot) \right\|_{L^2}^2,
\]
which implies that by \( (3. 12) \),
\[
\epsilon \left| \left( \tilde{C}(U)U + \psi^2 D^p, SU \right) \right| \leq \sqrt{\epsilon} \left\| U(t, \cdot) \right\|_{L^2}^2 \frac{C}{\epsilon} \left( \left\| \tilde{C}(U) \right\|_{L^4} \left\| U(t, \cdot) \right\|_{L^4} \left\| \psi \right\|_{L^2} \right) \left\| \tilde{U}(t, \cdot) \right\|_{L^2}^2. \tag{3. 48}
\]

Substituting \( (3. 47) \) and \( (3. 48) \) into \( (3. 40) \) gives
\[
\left| \left( S(C(U)U + \psi D) - \frac{1}{2} S_t U, U \right) \right| \leq \frac{\delta \epsilon}{8} \left\| \nabla U(t, \cdot) \right\|_{L^2}^2 + C \left( 1 + \left\| U(t, \cdot) \right\|_{L^2}^2 \right) \left\| U(t, \cdot) \right\|_{L^2}^2. \tag{3. 49}
\]

It remains to estimate the term \( -\epsilon(SB\triangle U, U) \). For this, we have that by integration by parts and the boundary conditions given in \( (3. 14) \),
\[
-\epsilon(SB\triangle U, U) = \epsilon(SB\partial_x U, \partial_x U) + \epsilon(SB\partial_y U, \partial_y U) + \epsilon(SB)\partial_x \partial_x U + \epsilon(SB)\partial_y \partial_y U, U,
\]
and note that \( \partial_y (SB) = O(\epsilon^{-\frac{1}{2}}) \), it implies that by \( (3. 11) \) and \( (3. 33) \),
\[
-\epsilon(SB\triangle U, U) \geq \delta \epsilon \left\| \nabla U(t, \cdot) \right\|_{L^2}^2 - C \sqrt{\epsilon} \left\| \nabla U(t, \cdot) \right\|_{L^2} \left\| U(t, \cdot) \right\|_{L^2} \left\| U(t, \cdot) \right\|_{L^2} \geq \frac{3\delta \epsilon}{4} \left\| \nabla U(t, \cdot) \right\|_{L^2}^2 - C \left\| U(t, \cdot) \right\|_{L^2}^2. \tag{3. 50}
\]

Also, it is easy to obtain that
\[
\left( SE^\epsilon, U \right) \leq \left\| E^\epsilon(t, \cdot) \right\|_{L^2} \left\| SU(t, \cdot) \right\|_{L^2} \leq \frac{1}{2} \left\| E^\epsilon(t, \cdot) \right\|_{L^2} + C \left\| U(t, \cdot) \right\|_{L^2}^2. \tag{3. 51}
\]

Now, plugging \( (3. 45), (3. 49), (3. 50) \) and \( (3. 51) \) into \( (3. 36) \), we obtain that
\[
\frac{d}{dt} \left\| SU(t, \cdot) \right\|_{L^2}^2 \leq \left\| E^\epsilon(t, \cdot) \right\|_{L^2}^2 + C \left( 1 + \left\| U(t, \cdot) \right\|_{L^2}^2 \right) \left\| U(t, \cdot) \right\|_{L^2}^2, \tag{3. 52}
\]
therefore, by using \( (3. 37) \) and \( (3. 22) \), there exists a \( 0 < T_* \leq T_3 \) and a constant \( C > 0 \) independent of \( \epsilon \), such that for \( t \in [0, T_3] \),
\[
\left\| U(t, \cdot) \right\|_{L^2}^2 + \epsilon \int_0^t \left\| \nabla U(s, \cdot) \right\|_{L^2}^2 ds \leq C. \tag{3. 53}
\]

(2) \( L^2 \)-estimate for \( \partial_x U \). From the problem \( (3. 14) \), we know that \( U_x \) satisfies the following initial-boundary value problem:
\[
\begin{cases}
\partial_t U_x + A_1(U)\partial_x U_x + A_2(U)\partial_y U_x + \partial_x A_1(U)\partial_x \psi + \partial_x A_2(U)\partial_y \psi + \partial_x \left( C(U)U + \psi D + (p_{xx}, p_{yx}, 0, 0)^T \right) \\
-\epsilon B\triangle U_x - \epsilon \partial_x B\cdot \partial_x U = \partial_x E^\epsilon, \quad \partial_x U_x + \partial_y \tilde{U}_x = 0, \\
\left( \partial_x U_x, \partial_y \tilde{h}_x, \tilde{y}_x \right)_{y = 0} = 0, \quad U_x |_{t = 0} = 0.
\end{cases}
\tag{3. 54}
\]

Multiplying \( (3. 54) \) by \( S \) from the left and taking the inner product of the resulting equation and \( U_x \), it follows that
\[
\frac{d}{2dt} \left( SU_x, U_x \right) + \left( S \left[ A_1(U)\partial_x U_x + A_2(U)\partial_y U_x \right] - \epsilon S B\triangle U_x, U_x \right) + \left( S(p_{xx}, p_{yx}, 0, 0)^T, U_x \right) \\
+ \left( S \partial_x \left( C(U)U + \psi D \right) - \frac{1}{2} S U_x, U_x \right) + \left( S \partial_x A_1(U)U_x + \partial_x A_2(U)U_y, U_x \right) - \epsilon (S \partial_x B\cdot \partial_x U) \\
= (S\partial_x E^\epsilon, U_x). \tag{3. 55}
\]
Now, we will estimate each term in (3.35). First, by similar arguments as given in the above step for $L^2$-norm of $U$, we can obtain that
\begin{equation}
(SU_x, U_x) \geq c_3 |U_x(t, \cdot)|^2_{L^2},
\end{equation}
and then, along with (3.42) we get
\begin{equation}
|\langle S \mathcal{A}_1(U) \partial_x U_x + S \mathcal{A}_2(U) \partial_t U_x, U_x \rangle| \leq \frac{\delta \epsilon}{8} \|\nabla U_x(t, \cdot)\|^2_{L^2} + C \left(1 + \|U(t, \cdot)\|^2_{L^2}\right) \|U_x(t, \cdot)\|^2_{L^2},
\end{equation}
\begin{equation}
- \epsilon(SB \Delta U_x, U_x) \geq \frac{3\delta \epsilon}{4} \|\nabla U_x(t, \cdot)\|^2_{L^2} - C \|U_x(t, \cdot)\|^2_{L^2},
\end{equation}
\begin{equation}
(S(p_{xx}, p_{yx}, 0, 0)^T, U_x) = \frac{1}{2} \|\mathcal{E}_x E^x(t, \cdot)\|^2_{L^2} + C \|U_x(t, \cdot)\|^2_{L^2}.
\end{equation}

Next, we will estimate the other terms in (3.35). For the term $(S \partial_x (C(U) + \psi D) - \frac{1}{2} S_x U_x, U_x)$, we have that by (3.10) and (3.17),
\begin{equation}
(S \partial_x (C(U) + \psi D) - \frac{1}{2} S_x U_x, U_x) = (S \partial_x (C^a U + \psi D^a) - \frac{1}{2} S_x U_x, U_x) + \epsilon (\partial_x \tilde{C}(U) + \psi^2 D^a, SU_x),
\end{equation}
which implies that
\begin{equation}
\left\| (S \partial_x (C(U) + \psi D) - \frac{1}{2} \partial_t SU_x, U_x) \right\| \leq C \|U_x(t, \cdot)\|^2_{L^2} + \|S \partial_x (C^a U + \psi D^a) - \frac{1}{2} \partial_t SU_x\|^2_{L^2} + \epsilon^2 \|\partial_t \tilde{C}(U) + \psi^2 D^a\|^2_{L^2}.
\end{equation}
It is easy to obtain that by virtue of (3.20),
\begin{equation}
\|S \partial_x (C^a U + \psi D^a) - \frac{1}{2} S_x U_x\|^2_{L^2} \leq \|S \mathcal{A}^a - \frac{1}{2} S_x\|^2_{L^2} \|U_x(t, \cdot)\|^2_{L^2} + \|S \partial_x C^a\|^2_{L^2} \|U(t, \cdot)\|^2_{L^2} + \|S \partial_x D^a\|^2_{L^2} \|y^{-1} \psi_x\|^2_{L^2} + \|y S \partial_x D^a\|^2_{L^2} \|y^{-1} \psi\|^2_{L^2} \\
\leq \|U_x(t, \cdot)\|^2_{L^2} + \|U(t, \cdot)\|^2_{L^2} + \|\mathcal{H}_x(t, \cdot)\|^2_{L^2} + \|\tilde{h}(t, \cdot)\|^2_{L^2} \\
\leq \|U_x(t, \cdot)\|^2_{L^2} + \|U(t, \cdot)\|^2_{L^2}.
\end{equation}
On the other hand, by using the estimate (3.28) for $\tilde{C}(U)$ with $p = 4$ it follows that
\begin{equation}
\|\partial_x \tilde{C}(U) + \psi^2 D^a\|^2_{L^2} \leq \|\partial_x \tilde{C}(U)\|^2_{L^2} \|U(t, \cdot)\|^4_{L^4} + \|\tilde{C}(U)\|^2_{L^2} \|U_x(t, \cdot)\|^4_{L^4} + 2 \|\partial_x \tilde{C}(U)^t D^a\|^2_{L^2} \|y^{-1} \psi(t, \cdot)\|^4_{L^4} \\
\leq \epsilon^{-\frac{1}{2}} \|U(t, \cdot)\|^4_{L^4} \left(\|U_x(t, \cdot)\|^4_{L^4} + \|U(t, \cdot)\|^4_{L^4}\right) + \|\tilde{h}(t, \cdot)\|^4_{L^4} \left(\|\mathcal{H}_x(t, \cdot)\|^4_{L^4} + \|\tilde{h}(t, \cdot)\|^4_{L^4}\right) \\
\leq \epsilon^{-\frac{1}{2}} \|U(t, \cdot)\|^4_{L^4} \left(\|U_x(t, \cdot)\|^4_{L^4} + \|U(t, \cdot)\|^4_{L^4}\right),
\end{equation}
and then, along with (3.42) we get
\begin{equation}
\epsilon^2 \|\partial_x \tilde{C}(U) + \psi^2 D^a\|^2_{L^2} \leq \epsilon \|U(t, \cdot)\|^4_{L^4} \left(\|U_x(t, \cdot)\|^4_{L^2} \|U(t, \cdot)\|^4_{H^1} + \|U(t, \cdot)\|^4_{L^2} \|U(t, \cdot)\|^4_{H^1}\right) \\
\leq \frac{\delta \epsilon}{24} \|\nabla U_x(t, \cdot)\|^2_{L^2} + C \left(1 + \epsilon \|U(t, \cdot)\|^2_{L^2} \|U(t, \cdot)\|^2_{H^1}\right) \|U(t, \cdot)\|^2_{L^2} \\
+ C \epsilon \|U(t, \cdot)\|^2_{L^2} \|U(t, \cdot)\|^2_{H^1}.
\end{equation}
Substituting (3.60) and (3.61) into (3.35) yields that
\begin{equation}
\left| (S \partial_x (C(U) + \psi D) - \frac{1}{2} S_x U_x, U_x) \right| \leq \frac{\delta \epsilon}{24} \|\nabla U_x(t, \cdot)\|^2_{L^2} + C \left(1 + \epsilon \|U(t, \cdot)\|^2_{L^2} \|U(t, \cdot)\|^2_{H^1}\right) \|U(t, \cdot)\|^2_{L^2} \\
+ C \epsilon \|U(t, \cdot)\|^2_{L^2} \|U(t, \cdot)\|^2_{H^1}.
\end{equation}
For the term $(S[\partial_x A_1(U) U_x + \partial_x A_2(U) U_y], U_x) = ([\partial_x A_1(U) U_x + \partial_x A_2(U) U_y], SU_x)$, we first get that from the definitions (3.16) of $A_1(U)$,
\begin{equation}
\partial_x A_1(U) U_x + \partial_x A_2(U) U_y = \partial_x \left( A_1^p + \sqrt{\mathcal{A}^p} \right) U_x + \partial_x \left( A_2^p + \sqrt{\mathcal{A}^p} \right) U_y \\
+ \epsilon (\partial_x \tilde{A}_1(U) U_x + \partial_x \tilde{A}_2(U) U_y) \\
= J_1 + J_2.
\end{equation}
Then, it follows from the estimate (3.19) that,
\[ |(J_1, SU_x)| \leq \|J_1\|_{L^2} \|SU_x(t, \cdot)\|_{L^2} \leq \|\nabla U(t, \cdot)\|_{L^2} \|U_x(t, \cdot)\|_{L^2}. \]  
(3.64)

On the other hand, we obtain that by virtue of (3.28) with \( p = 4, \)
\[ |(J_2, SU_x)| \leq \epsilon |SU_x(t, \cdot)|_{L^4} \cdot (|\partial_x \hat{A}_1(U)|_{L^4} + \|\partial_x \hat{A}_2(U)\|_{L^4}) \|\nabla U(t, \cdot)\|_{L^2} \]
\[ \leq \epsilon \|U_x(t, \cdot)\|_{L^4} \cdot (\|U_x(t, \cdot)\|_{L^4} + \|U(t, \cdot)\|_{L^4}) \|\nabla U(t, \cdot)\|_{L^2} \]
\[ \leq \epsilon (\|U_x(t, \cdot)\|_{L^4}^2 + \|U(t, \cdot)\|_{L^4}^2) \cdot \|\nabla U(t, \cdot)\|_{L^2}, \]
and along with (3.42), one deduces that
\[ |(J_2, SU_x)| \leq \epsilon (\|U_x(t, \cdot)\|_{L^4}^2 + \|U(t, \cdot)\|_{L^4}^2) \|\nabla U(t, \cdot)\|_{L^2} \]
\[ \leq \frac{\delta \epsilon}{24} \|\nabla U_x(t, \cdot)\|_{L^2}^2 + C \epsilon \|\nabla U(t, \cdot)\|_{L^2}^2 \|U_x(t, \cdot)\|_{L^2}^2 \]
\[ + C \epsilon \|U(t, \cdot)\|_{L^2}^2 \|U_x(t, \cdot)\|_{L^2}^2. \]  
(3.65)

Collecting (3.64), (3.66) and (3.65), we get that
\[ \left| \left( S[\partial_x A_1(U) U_x + \partial_x A_2(U) U_y], U_x \right) \right| \leq \frac{\delta \epsilon}{24} \|\nabla U_x(t, \cdot)\|_{L^2}^2 + C (1 + \epsilon \|\nabla U(t, \cdot)\|_{L^2}^2) \|U_x(t, \cdot)\|_{L^2}^2 \]
\[ + C (1 + \epsilon \|U(t, \cdot)\|_{L^2}^2) \|U_x(t, \cdot)\|_{L^2}^2. \]  
(3.66)

It remains to control the term \(-\epsilon(S \partial_x B \Delta U, U_x).\) By integration by parts and the boundary conditions, we have
\[ -\epsilon(S \partial_x B \Delta U, U_x) = -\epsilon(S \partial_x B U_x, U_x) + \epsilon(\partial_x (S \partial_x B) U_x, U_x) + \epsilon(S \partial_x B U_y, U_x) + \epsilon(\partial_y (S \partial_x B) U_y, U_x), \]
and note that \( \partial_y (S \partial_x B) = O(\epsilon^{-\frac{1}{2}}), \) it implies that by virtue of (3.19),
\[ \epsilon |S \partial_x B \Delta U, U_x| \leq \epsilon \|\nabla U_x(t, \cdot)\|_{L^2} \|\nabla U(t, \cdot)\|_{L^2} + \sqrt{\epsilon} \|U(t, \cdot)\|_{L^2} \|U_x(t, \cdot)\|_{L^2} \]
\[ \leq \frac{\delta \epsilon}{24} \|\nabla U_x(t, \cdot)\|_{L^2}^2 + C \epsilon \|\nabla U(t, \cdot)\|_{L^2}^2 \|U_x(t, \cdot)\|_{L^2}^2. \]  
(3.67)

Now, plugging (3.57), (3.58), (3.62), (3.66) and (3.67) into (3.53), we obtain that
\[ \frac{d}{dt} (SU_x, U_x) + \delta \epsilon \|\nabla U_x(t, \cdot)\|_{L^2}^2 \leq \|\partial_x E'(t, \cdot)\|_{L^2}^2 + C (1 + \epsilon \|U(t, \cdot)\|_{L^2}^2) \|U(t, \cdot)\|_{H^1}^2 \]
\[ + C [1 + \|U(t, \cdot)\|_{L^2}^2 + \epsilon (1 + \|U(t, \cdot)\|_{L^2}^2) \|U(t, \cdot)\|_{H^1}^2] \|U_x(t, \cdot)\|_{L^2}^2, \]  
(3.68)

Applying Gronwall inequality to the above inequality, and using (3.66) we have
\[ \|U_x(t, \cdot)\|_{L^2}^2 + \epsilon \int_0^t \|\nabla U_x(s, \cdot)\|_{L^2}^2 ds \]
\[ \leq \left( \int_0^t \|\partial_x E'(s, \cdot)\|_{L^2}^2 ds + C \int_0^t \|U(s, \cdot)\|_{H^1}^2 ds \right) \exp \left\{ C \int_0^t (1 + \epsilon \|U(s, \cdot)\|_{H^1}^2) ds \right\} \]
\[ \leq C \epsilon^{-1}, \quad t \in [0, T_*], \]  
(3.69)

where we have used (3.22) and (3.53) again in the second inequality. Thus, we obtain (3.35) by (3.53) and (3.69), and complete the proof.

Next, we want to obtain the estimates for \( U_t. \) Firstly, from (3.14) we know that \( U_t \) satisfies the following initial-boundary value problem:
\[
\begin{align*}
\partial_t U_t + A_1(U) \partial_x U_t + A_2(U) \partial_y U_t + \partial_t A_1(U) \partial_x U + \partial_t A_2(U) \partial_y U + \partial_t (C(U) U + \psi D) + (p_{x1}, p_{y1}, 0, 0)^T \\
-\epsilon B \Delta U_t - \partial_x B \partial U = \phi \partial_x u_t + \phi \partial_y v_t = 0, \\
(\partial_t u_t, \partial_t v_t, \partial_x u_t, \partial_y v_t)|_{y=0} = 0, \\
U_t|_{x=0} = E^0(0, t, x, y) - (p_{x1}, p_{y1}, 0, 0)^T(0, t, x, y).
\end{align*}
\]  
(3.70)
Note that the initial data of \( U_t \) depends on the initial pressure \( p|_{t=0} \), for which we do not have any estimates. Therefore, we need to control the initial data \( U_t|_{t=0} \) first. Actually, we have the following

**Proposition 3.3.** There exists a constant \( C > 0 \) independent of \( \epsilon \), such that
\[
\| U_t(0, \cdot) \|_{L^2} + \| \partial_x U_t(0, \cdot) \|_{L^2} \leq C. \tag{3.71}
\]

**Proof.** Firstly, from the initial data of (3.70) and the definition (3.21) of \( E^t \), it is easy to obtain that for the last two components of \( U_t \),
\[
(\hat{h}_t, \hat{g}_t)(0, x, y) = (E^t_3, E^t_4)(0, x, y) = (r_3^t - b^p \cdot \partial_y^{-1} r_4^t, r_4^t)(0, x, y),
\]
which implies that by virtue of (3.22),
\[
\|(\hat{h}_t, \hat{g}_t)(0, \cdot)\|_{L^2} + \|(\partial_x \hat{h}_t, \partial_x \hat{g}_t)(0, \cdot)\|_{L^2} \leq C. \tag{3.72}
\]
Next, from (3.70) it follows that for the first two component of \( U_t \),
\[
(\bar{u}_t, \partial_x \bar{u}_t)(0, x, y) = (E^t_1, E^t_2)(0, x, y) - (p_x, p_y)(0, x, y). \tag{3.73}
\]
Thus, to estimate \( (\bar{u}_t, \partial_x \bar{u}_t)|_{t=0} \), it remains to estimate \( \nabla p|_{t=0} \).

Thanks to the divergence-free condition \( \partial_x \bar{u}_t + \partial_y \bar{v}_t = 0 \), and the boundary condition \( \bar{v}_1|_{y=0} = 0 \), from (3.73) we obtain that \( p|_{t=0} \) satisfies the following elliptic equation with the Neumann boundary condition,
\[
\Delta p(0, x, y) = (\partial_x E^t_1 + \partial_y E^t_2)|_{t=0} = (\partial_x r_1^t + \partial_y r_2^t)|_{t=0}, \quad p_y(0, x, 0) = E^t_2(0, x, 0) = r_2^t(0, x, 0).
\]
Then, the standard elliptic theory yields that
\[
\| \nabla p|_{t=0} \|_{L^2} + \| \nabla p_y|_{t=0} \|_{L^2} \leq C \left( \| (r_1^t, r_2^t)|_{t=0} \|_{L^2} + \| \partial_x(r_1^t, r_2^t)|_{t=0} \|_{L^2} \right). \tag{3.74}
\]
Combining (3.73) with (3.74) and using (3.22), we know that there is a constant \( C > 0 \) independent of \( \epsilon \), such that
\[
\|(\bar{u}_t, \partial_x \bar{u}_t)(0, \cdot)\|_{L^2} + \|(\partial_x \bar{u}_t, \partial_x \bar{v}_t)(0, \cdot)\|_{L^2} \leq C. \tag{3.75}
\]
Consequently, (3.71) follows immediately from (3.72) and (3.75). \( \square \)

As the estimate on \( U_t|_{t=0} \) has been obtained, we have the following result for \( U_t \).

**Proposition 3.4.** Under the assumptions of Proposition 3.2, we have
\[
\epsilon \| U_t(t, \cdot) \|_{L^2}^2 + \epsilon^2 \| U_{tx}(t, \cdot) \|_{L^2}^2 + \epsilon^2 \int_0^t \left( \| \nabla U_t(s, \cdot) \|_{L^2}^2 + \epsilon \| \nabla U_{tx}(s, \cdot) \|_{L^2}^2 \right) ds \leq C, \quad \forall \ t \in [0, T^\star] \tag{3.76}
\]
for some constant \( C > 0 \) independent of \( \epsilon \).

**Proof.** The desired estimate of \( U_t \) can be obtained in a similar way as the one for \( U_x \), given in the second step of Proposition 3.2. Indeed, we can obtain
\[
\frac{d}{dt}(SU_t, U_t) + \delta \epsilon \| \nabla U_t(t, \cdot) \|_{L^2}^2 \leq \| \partial_t E^t(t, \cdot) \|_{L^2}^2 + C(1 + \epsilon \| U(t, \cdot) \|_{H^1}^2) \| U_t(t, \cdot) \|_{L^2}^2 + C \| U(t, \cdot) \|_{H^1}^2,
\]
and then, applying the Gronwall inequality to the above inequality, one deduces that
\[
\| U_t(t, \cdot) \|_{L^2}^2 + \epsilon \int_0^t \| \nabla U_t(s, \cdot) \|_{L^2}^2 ds \leq \left( \| U_t(0, \cdot) \|_{L^2}^2 + \int_0^t \| \partial_t E^t(s, \cdot) \|_{L^2}^2 ds + C \int_0^t \| U(s, \cdot) \|_{H^1}^2 ds \right) \exp \left\{ C \int_0^t (1 + \epsilon \| U(s, \cdot) \|_{H^1}^2) ds \right\}
\]
\[
\leq Ce^{-\delta \epsilon}, \quad t \in [0, T^\star], \tag{3.77}
\]
where we have used (3.22), (3.53) and (3.71) in the above second inequality.

It remains to obtain the estimate of \( U_{tx} \). From (3.70) we know that \( U_{tx} \) satisfies the following initial-boundary value problem:
\[
\begin{align*}
& \partial_t U_{tx} + A_1(U)\partial_x U_{tx} + A_2(U)\partial_y U_{tx} + (\partial^2_{tx} p_x, \partial^2_{tx} p_y, 0, 0)^T + \partial^2_{tx}(C(U) + \psi D) - \epsilon B \Delta U_{tx} \\
& + [\partial^2_{tx} A_1(U) \partial_x + A_2(U) \partial_y] U - \epsilon [\partial^2_{tx} B] \Delta U = \partial^2_{tx} E^t, \\
& \partial_x U_{tx} + \partial_y U_{tx} = 0, \\
& (\partial_{tx}, \partial_{tx}^2, \partial_y \partial_{tx}^2)|_{y=0} = 0, \\
& U_{tx}|_{t=0} = \partial_x E^t(0, x, y) - (p_{xxx}, p_{yy}, 0, 0)^T(0, x, y),
\end{align*}
\]
where the notation $[\cdot,\cdot]$ stands for the commutator.

Multiplying (3.78) by applying $S$ from the left and taking the inner product of the resulting equations and $U_{tx}$, it follows that

$$\frac{d}{2dt}(SU_{tx},U_{tx}) + \left(S[A_1(U)\partial_xU_{tx} + A_2(U)\delta_yU_{tx}] - \epsilon S B \Delta U_{tx},U_{tx}\right) + \left(S[\tilde{\partial}_{tx}^2(C(U)\psi - D) - \frac{1}{2}S_1 U_{tx}],U_{tx}\right) + \left(S[\tilde{\partial}_{tx}^2, A_1(U)\delta_x + A_2(U)\delta_y]U, U_{tx}\right)$$

$$- \epsilon (S[\tilde{\partial}_{tx}^2,B]\Delta U, U_{tx}) = (\tilde{\partial}_{tx}^2 E^x, U_{tx}).$$

(3.79)

Now, each term in (3.79) can be estimated as follows. Firstly, by a similar argument as given in the above step for $L^2$-norm of $U$, one can obtain that

$$(SU_{tx},U_{tx}) = c_\delta \|U_{tx}(t,\cdot)\|_{L^2}^2,$$

and

$$\left| \left(S[A_1(U)\partial_xU_{tx} + A_2(U)\delta_yU_{tx}, U_{tx}]\right) \right| \leq \frac{\delta \epsilon}{8} \|\nabla U_{tx}(t,\cdot)\|_{L^2}^2 + C(1 + \|U(t,\cdot)\|_{L^2}^2) \|U_{tx}(t,\cdot)\|_{L^2}^2,$$

(3.81)

$$- \epsilon (S B \Delta U_{tx},U_{tx}) \geq \frac{3 \delta \epsilon}{4} \|\nabla U_{tx}(t,\cdot)\|_{L^2}^2 - C\|U_{tx}(t,\cdot)\|_{L^2}^2,$$

$$\left(S[\tilde{\partial}_{tx}^2p_x, \tilde{\partial}_{tx}^2 p_y, 0, 0] U, U_{tx}\right) = 0, \quad \left(S[\tilde{\partial}_{tx}^2 E^x, U_{tx}\right) \leq \frac{1}{2} \|\tilde{\partial}_{tx}^2 E^x(t,\cdot)\|_{L^2}^2 + C\|U_{tx}(t,\cdot)\|_{L^2}^2.$$  

(3.82)

Next, we proceed to estimate the other terms in (3.79). By (3.16), (3.17), it follows that for the term

$$\left(S[\tilde{\partial}_{tx}^2(C(U)\psi + D) - \frac{1}{2}\partial_t S U_{tx}], U_{tx}\right)$$

$$= (\tilde{\partial}_{tx}( C(U)\psi + \psi^2 D^a), SU_{tx})$$

$$= I_1 + I_2.$$

For $I_1$, note that

$$\tilde{\partial}_{tx}^2(C(U)\psi + \psi^2 D^a) = C U_{tx} + \tilde{\partial}_{tx}^2 \psi D^a + \tilde{\partial}_x C U_{tx} + \partial_t \psi \tilde{\partial}_x D^a + \partial_t C U_{tx} + \tilde{\partial}_x \partial_t D^a + \tilde{\partial}_{tx}^2 C U + \psi \tilde{\partial}_{tx}^2 D^a,$$

which, along with (3.20) yields that

$$\|\tilde{\partial}_{tx}^2(C(U)\psi + \psi^2 D^a)\|_{L^2} \leq \|U_{tx}(t,\cdot)\|_{L^2} + y^{-1} \|\tilde{\partial}_{tx}^2 C(U)\psi\|_{L^2} + \|U(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2} + \|\tilde{\partial}_x\psi(t,\cdot)\|_{L^2}$$

$$\leq \|U_{tx}(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2} + \|\tilde{\partial}_x U_{tx}(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2}$$

$$\leq \|U_{tx}(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2} + \|U(t,\cdot)\|_{L^2}.$$

Thus, we obtain

$$|I_1| \leq \|\tilde{\partial}_{tx}^2(C(U)\psi + \psi^2 D^a)\|_{L^2} \|SU_{tx}(t,\cdot)\|_{L^2} + \frac{1}{2}\|S_1\|_{L^2} \|U_{tx}(t,\cdot)\|_{L^2}^2$$

$$\leq \|U_{tx}(t,\cdot)\|_{L^2}^2 + \|U(t,\cdot)\|_{L^2}^2 + \|U(t,\cdot)\|_{L^2}^2 + \|U(t,\cdot)\|_{L^2}.$$

(3.83)

Next, by integration by parts with respect to $x$ the term $I_2$ can be reduced as

$$I_2 = -\epsilon \left(\tilde{\partial}_x(C(U)\psi + \psi^2 D^a), SU_{tx} + S U_{tx}\right),$$

and then, we have

$$|I_2| \leq \frac{\delta \epsilon}{24} \|\tilde{\partial}_x U_{tx}(t,\cdot)\|_{L^2}^2 + C\epsilon \|\tilde{\partial}_x(C(U)\psi + \psi^2 D^a)\|_{L^2}^2 + C\epsilon \|U_{tx}(t,\cdot)\|_{L^2}^2.$$

Similar to (3.61), we can obtain that

$$\epsilon \|\tilde{\partial}_x(C(U)\psi + \psi^2 D^a)\|_{L^2}^2 \leq \|U(t,\cdot)\|_{H^1}^2 + \|U(t,\cdot)\|_{L^2}^2 \|U(t,\cdot)\|_{H^1}^2 + \|U(t,\cdot)\|_{L^2}^2 \|U(t,\cdot)\|_{H^1}^2,$$
and then, combining the above two inequalities yields that

$$|I_2| \leq \frac{\delta \epsilon}{24} \|\partial_x U_{tx}(t, \cdot)\|_{L^2}^2 + C\|U(t, \cdot)\|_{H^1}^2 + C\|U(t, \cdot)\|_{H^1}^2 \|U(t, \cdot)\|_{H^1}^2 (1 + \|U(t, \cdot)\|_{L^2}^2). \quad (3.84)$$

Consequently, we get that by combining (3.83) with (3.84),

$$\left| \left( S[\partial_{tx}^2 (C(U)U + \psi D) - \frac{1}{2}\partial_t S U_{tx}], U_{tx} \right) \right| \leq \frac{\delta \epsilon}{24} \|\partial_x U_{tx}(t, \cdot)\|_{L^2}^2 + C\|U(t, \cdot)\|_{H^1}^2 + C\|U(t, \cdot)\|_{H^1}^2 \|U(t, \cdot)\|_{H^1}^2 (1 + \|U(t, \cdot)\|_{L^2}^2) + C\|U(t, \cdot)\|_{H^1}^2. \quad (3.85)$$

Next, we consider the term \( S[\partial_{tx}^2, A_1(U)\partial_x + A_2(U)\partial_y] U_{tx}, U_{tx} \). By (3.15) and direct calculation, one gets that

$$S[\partial_{tx}^2, A_1(U)\partial_x + A_2(U)\partial_y] U_{tx}, U_{tx} = \left( \left( \partial_{tx}^2, (A^0_x + \sqrt{T}A^0_y)\partial_x + (A^0_x + \sqrt{T}A^0_y)\partial_y \right) U, SU_{tx} \right)$$

$$+ \epsilon \left( \left( \partial_{tx}^2, \tilde{A}_1(U)\partial_x + \tilde{A}_2(U)\partial_y \right) U, SU_{tx} \right) \equiv I_3 + I_4. \quad (3.86)$$

From (3.19), it is easy to have

$$|I_3| \leq \|SU_{tx}(t, \cdot)\|_{L^2} \left( \|\nabla U(t, \cdot)\|_{L^2} + \|\nabla U(t, \cdot)\|_{L^2} + \|\nabla U(t, \cdot)\|_{L^2}^2 \right)$$

$$\leq \|\nabla U(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2. \quad (3.87)$$

We know that \( I_4 \) in (3.86) reads:

$$I_4 = \epsilon \left( \left( \partial_{tx} \tilde{A}_1(U)\partial_x^2 + \partial_x \tilde{A}_2(U)\partial_y^2 \right) U, SU_{tx} \right) U$$

$$+ \epsilon \left( \left( \partial_{tx}^2, \tilde{A}_1(U)\partial_x + \tilde{A}_2(U)\partial_y \right) U, SU_{tx} \right),$$

which implies that by virtue of (3.28),

$$|I_4| \leq \epsilon \|SU_{tx}(t, \cdot)\|_{L^2} \left( \|\nabla U(t, \cdot)\|_{L^2}^2 + \sum_{i=1}^{2} \|\partial_x \tilde{A}_i(U)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 \right)$$

$$+ \|\nabla U(t, \cdot)\|_{L^2} \sum_{i=1}^{2} \|\partial_x \tilde{A}_i(U)\|_{L^2}^2$$

$$\leq \|\nabla U(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 + \epsilon^2 \|U(t, \cdot)\|_{L^2}^2 \left( \|U^2(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 \right)$$

$$+ \epsilon \|\nabla U(t, \cdot)\|_{L^2} \sum_{i=1}^{2} \|\partial_x \tilde{A}_i(U)\|_{L^2}^2 \left( \|U^2(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 \right)$$

$$\leq C \left( \|\nabla U(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 \right)$$

$$+ C \epsilon^2 \|U(t, \cdot)\|_{L^2}^2 \left( \|U^2(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 \right).$$

By (3.42) we have

$$C \epsilon \|\nabla U(t, \cdot)\|_{L^2} \|U_{tx}(t, \cdot)\|_{L^2}^2 \leq \epsilon \|\nabla U(t, \cdot)\|_{L^2} \|U_{tx}(t, \cdot)\|_{L^2} \|U_{tx}(t, \cdot)\|_{H^1}^2$$

$$\leq \frac{\epsilon \delta \epsilon}{48} \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 + C \left( 1 + \epsilon \|\nabla U(t, \cdot)\|_{L^2}^2 \right) \|U_{tx}(t, \cdot)\|_{L^2}^2,$$

and

$$C \epsilon^2 \|U_{tx}(t, \cdot)\|_{L^2}^2 \left( \|U^2(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 \right)$$

$$\leq \epsilon^2 \|U_{tx}(t, \cdot)\|_{L^2}^2 \|U_{tx}(t, \cdot)\|_{H^1}^2 \left( \|U^2(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 + \|U(t, \cdot)\|_{L^2}^2 \right) + \epsilon^3 \|U(t, \cdot)\|_{L^2}^2 \|U_{tx}(t, \cdot)\|_{H^1}^2$$

$$\leq \frac{\epsilon \delta \epsilon}{48} \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 + C \left( 1 + \epsilon^3 \|U_{tx}(t, \cdot)\|_{L^2}^2 \|U_{tx}(t, \cdot)\|_{H^1}^2 + \epsilon^3 \|U(t, \cdot)\|_{L^2}^2 \|U_{tx}(t, \cdot)\|_{H^1}^2 \right) + \epsilon^3 \|U(t, \cdot)\|_{L^2}^2 \|U_{tx}(t, \cdot)\|_{H^1}^2.$$
Collecting the above three inequalities yields that

\[
|I_4| \leq \frac{\delta \varepsilon}{24} \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 + C \left( \|\nabla U_i(t, \cdot)\|_{L^2}^2 + \|\nabla U_x(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 \right)
+ C \|U_{tx}(t, \cdot)\|_{L^2}^2 \cdot \left( 1 + \varepsilon \|\nabla U(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U_x(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right)
+ \varepsilon^3 \|U_i(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right) \cdot \left( 1 + \varepsilon \|\nabla U(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U_x(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right). \tag{3. 88}
\]

Then, substituting (3. 87) and (3. 88) into (3. 88), we have

\[
\left| \left( S[\tilde{c}_{tx}^2, A_1(U)\partial_x + A_2(U)\partial_y] U, U_{tx} \right) \right| \leq \frac{\delta \varepsilon}{24} \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 + C \left( \|\nabla U_i(t, \cdot)\|_{L^2}^2 + \|\nabla U_x(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 \right)
+ C \|U_{tx}(t, \cdot)\|_{L^2}^2 \cdot \left( 1 + \varepsilon \|\nabla U(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U_x(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right)
+ \varepsilon^3 \|U_i(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right). \tag{3. 89}
\]

Now, let us estimate the term \(-\varepsilon \left( S[\tilde{c}_{tx}^2, B] \Delta U, U_{tx} \right)\). Firstly, it follows that by integration by parts,

\[-\varepsilon \left( S[\tilde{c}_{tx}^2, B] \Delta U, U_{tx} \right) = \varepsilon \left( S[\tilde{c}_{tx}^2, B]\partial_x U, \partial_x U_{tx} \right) + \varepsilon \left( S[\tilde{c}_{tx}^2, B]|_{\partial_y U, \partial_y U_{tx}} \right) + \varepsilon \left( S[\tilde{c}_{tx}^2, B]|_{\partial_x U, \partial_x U_{tx}} \right) + \varepsilon \left( S[\tilde{c}_{tx}^2, B]|_{\partial_y U, \partial_y U_{tx}} \right) \equiv I_5 + I_6. \tag{3. 90}
\]

It is easy to obtain by (3. 19) that

\[
|I_5| \leq \varepsilon \|\nabla U_{tx}(t, \cdot)\|_{L^2} \left( \|S[\tilde{c}_{tx}^2, B]\partial_x U\|_{L^2} + \|S[\tilde{c}_{tx}^2, B]|_{\partial_y U}\|_{L^2} \right)
\leq \varepsilon \|\nabla U_{tx}(t, \cdot)\|_{L^2} \left( \|\nabla U_i(t, \cdot)\|_{L^2} + \|\nabla U_x(t, \cdot)\|_{L^2} + \|\nabla U(t, \cdot)\|_{L^2} \right)
\leq \frac{\delta \varepsilon}{24} \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 + C \varepsilon \left( \|\nabla U_i(t, \cdot)\|_{L^2}^2 + \|\nabla U_x(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 \right). \tag{3. 91}
\]

On the other hand, note that \(S_y, B_y = O(\varepsilon^{-\frac{1}{2}})\) and we obtain

\[
|I_6| \leq \sqrt{\varepsilon} \|\nabla U_{tx}(t, \cdot)\|_{L^2} \left( \|\nabla U_i(t, \cdot)\|_{L^2} + \|\nabla U_x(t, \cdot)\|_{L^2} + \|\nabla U(t, \cdot)\|_{L^2} \right)
\leq C \|U_{tx}(t, \cdot)\|_{L^2} + C \varepsilon \left( \|\nabla U_i(t, \cdot)\|_{L^2}^2 + \|\nabla U_x(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 \right). \tag{3. 92}
\]

Thus, plugging (3. 91) and (3. 92) into (3. 90) implies that

\[
|\varepsilon \left( S[\tilde{c}_{tx}^2, B] \Delta U, U_{tx} \right)| \leq \frac{\delta \varepsilon}{24} \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 + C \|U_{tx}(t, \cdot)\|_{L^2}^2 + C \varepsilon \left( \|\nabla U_i(t, \cdot)\|_{L^2}^2 + \|\nabla U_x(t, \cdot)\|_{L^2}^2 + \|\nabla U(t, \cdot)\|_{L^2}^2 \right). \tag{3. 93}
\]

Finally, we substitute (3. 81)-(3. 82), (3. 83), (3. 89) and (3. 93) into (3. 79), to obtain

\[
\frac{d}{dt}(SU_{tx}, U_{tx}) + \delta \varepsilon \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 \leq \|\tilde{c}_{tx}^2 E(t, \cdot)\|_{L^2}^2 + C \left( \|U_i(t, \cdot)\|_{H^1}^2 + \|U_x(t, \cdot)\|_{H^1}^2 + \|U(t, \cdot)\|_{H^1}^2 \right) + C \|U(t, \cdot)\|_{L^2}^2 \cdot \left( 1 + \|U(t, \cdot)\|_{L^2}^2 + \varepsilon \|\nabla U(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U_x(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right)
+ \varepsilon^3 \|U_i(t, \cdot)\|_{L^2}^2 \|U(t, \cdot)\|_{H^1}^2 + C \|U(t, \cdot)\|_{L^2}^2 \|U(t, \cdot)\|_{H^1}^2.
\]

which, along with (3. 35) and (3. 77) implies that

\[
\frac{d}{dt}(SU_{tx}, U_{tx}) + \delta \varepsilon \|\nabla U_{tx}(t, \cdot)\|_{L^2}^2 \leq \|\tilde{c}_{tx}^2 E(t, \cdot)\|_{L^2}^2 + C \left( \|U_i(t, \cdot)\|_{H^1}^2 + \|U_x(t, \cdot)\|_{H^1}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right) \cdot \left( 1 + \varepsilon \|U(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U_x(t, \cdot)\|_{L^2}^2 + \varepsilon^3 \|U(t, \cdot)\|_{H^1}^2 \right) + \varepsilon^3 \|U_i(t, \cdot)\|_{L^2}^2 \|U(t, \cdot)\|_{H^1}^2 + C \|U(t, \cdot)\|_{L^2}^2 \|U(t, \cdot)\|_{H^1}^2.
\]
Consequently, applying the Gronwall inequality to the above inequality, we obtain that
\[
\|U_{tx}(t, \cdot)\|_2^2 + \epsilon \int_0^t \|\nabla U_{tx}(s, \cdot)\|_2^2 ds \\
\leq \left( \|U_{tx}(0, \cdot)\|_2^2 + \epsilon \int_0^t \|\nabla^2 E(s, \cdot)\|_2^2 ds + C \int_0^t (\|U_t(s, \cdot)\|_{H^1}^2 + \|U_x(s, \cdot)\|_{H^1}^2 + \epsilon^{-1}\|U(s, \cdot)\|_{H^1}^2) ds \right) \\
\cdot \exp \left\{ C \int_0^t (1 + \epsilon\|U(s, \cdot)\|_{H^1}^2 + + \epsilon^2\|U_t(t, \cdot)\|_{H^1}^2 + \epsilon^2\|U_{tx}(t, \cdot)\|_{H^1}^2) ds \right\}
\leq C\epsilon^{-2}, \quad t \in [0, T_a],
\] (3. 94)
where we have used (3. 92), (3. 95), (3. 77) and (3. 71) in the second inequality. Thus, from (3. 74) and (3. 94) we obtain (3. 70) and complete the proof.

\[\square\]

3.3. Proof of the main theorem. Now, we are ready to prove our main theorem.

\textbf{Proof of Theorem 1.1.} From (3.1):
\[ (u^e, v^e, h^e, g^e, p^e) = (u^a, v^a, h^a, g^a, p^a) + \epsilon (u, v, h, g, p), \] (3. 95)
and the expression (2. 39) of the approximate solution \((u^e, v^e, h^e, g^e, p^e)\), the local existence in \([0, T_a]\) of the solution \((u^e, v^e, h^e, g^e, p^e)\) to the problem (1. 1) follows from the local existence of \((u, v, h, g)\) given in Proposition 3.2. Also, we know from the expression (2. 36) for \((u^a, v^a, h^a, g^a)\):
\[ (u^a, v^a, h^a, g^a)(t, x, y) = (u_0^a, v_0^a, h_0^a, g_0^a)(t, x, y) + \left( \frac{u_0^a}{\sqrt{\epsilon e}}, \sqrt{\epsilon e^0}, \frac{\sqrt{\epsilon e^0}}{\sqrt{\epsilon}} \right) t, x, y \right) + O(\sqrt{\epsilon}). \] (3. 96)
Therefore, combining (3. 95) with (3. 96), we only need to obtain the \(L^p\)-estimate of \((u, v, h, g)\) to show (1. 8). In addition, with Lemma 3.1 it remains to get the \(L^p\)-estimate of \(U\).

Next, from the estimates (3. 35) and (3. 70) it suffices to get the \(L^p\)-estimate of \(U\). Indeed, by the Sobolev embedding inequality and interpolation inequality with any small \(\lambda > 0\),
\[ \|U\|_{L^p_{txy}} \leq \|U\|_{L^\frac{4}{\lambda} H^\lambda_{txy}} \cdot \|U\|_{L^\frac{4}{\lambda} H^\lambda_{txy}} \leq \epsilon^{-\frac{4}{\lambda}(\frac{4}{\lambda} + \lambda)}. \] (3. 97)
and combining with (3. 35), it follows that
\[ \|U\|_{L^p_{txy}} \leq \|U\|_{L^\frac{4}{\lambda} H^\lambda_{txy}} \cdot \|U\|_{L^\frac{4}{\lambda} H^\lambda_{txy}} \leq \epsilon^{-\frac{4}{\lambda}(\frac{4}{\lambda} + \lambda)}. \] (3. 98)
On the other hand, combining with (3. 70) yields that
\[ \|U\|_{L^p_{txy}} \leq \|U\|_{L^\frac{4}{\lambda} H^\lambda_{txy}} \cdot \|U\|_{L^\frac{4}{\lambda} H^\lambda_{txy}} \leq \epsilon^{-\frac{4}{\lambda}(\frac{4}{\lambda} + \lambda)}. \] (3. 99)
Substituting (3. 98) and (3. 99) into (3. 97), we have
\[ \|U\|_{L^p_{txy}} \leq \epsilon^{-\frac{4}{\lambda}(\frac{4}{\lambda} + \lambda)}, \] (3. 100)
which implies that by virtue of (3. 23),
\[ \|(u, v, h, g)\|_{L^p_{txy}} \leq C \epsilon^{-\frac{4}{\lambda}(\frac{4}{\lambda} + \lambda)}. \] (3. 101)
Therefore, applying (3. 96) and (3. 101) in (3. 95) yields
\[ \|(u^e, v^e, h^e, g^e)(t, x, y) - (u_0^a, v_0^a, h_0^a, g_0^a)(t, x, y) - \left( \frac{u_0^a}{\sqrt{\epsilon e}}, \sqrt{\epsilon e^0}, \frac{\sqrt{\epsilon e^0}}{\sqrt{\epsilon}} \right) t, x, y \right)\|_{L^p_{txy}} \leq C \sqrt{\epsilon} + C \epsilon \|(u, v, h, g)\|_{L^p_{txy}} \leq C \sqrt{\epsilon} + C \epsilon \frac{4}{\lambda} \leq \epsilon \frac{4}{\lambda} \leq \epsilon \frac{4}{\lambda} \leq \frac{4}{\lambda} \leq \frac{4}{\lambda}, \] (3. 102)
provided \(\lambda\) small enough. This ends the proof of Theorem 1.1. \[\square\]
Remark 3.3. From the above proof, we believe that the decay rate in (1.1) with respect to $\epsilon$ can be improved to order $\sqrt{\epsilon}$. Roughly speaking, we need to construct more accurate approximate solution for the problem (1.1), such that the corresponding remainder terms in (2.40) are of order $\epsilon^\gamma, \gamma > \frac{\nu}{2}$. Of course, more regularity requirement on the initial data of (1.1) is needed.

**Appendix A. Expression of error caused by the approximation and its estimates**

Now, we will give the expressions of the remainders $R_i (i = 1 \sim 4)$ in (2.40), which are generated by the approximate solution $v^0 (u^0, v^0, h^0, g^0)$ in (2.30), and then prove Proposition 2.3. For the simplicity of notations, denote by

$$\tau_u(t, x, y) = \chi'(y) \int_0^y u_1^1(t, x, \tilde{y})d\tilde{y}, \quad \tau_h(t, x, y) = \chi'(y) \int_0^y h_1^1(t, x, \tilde{y})d\tilde{y} + \rho(t, x, \frac{y}{\sqrt{\epsilon}}),$$

$$\tau_g(t, x, y) = - \int_0^y \tilde{e}_x \rho(t, x, \tilde{y})d\tilde{y},$$

and then,

$$\begin{align*}
\tilde{u}_1^1(t, x, y) &= \chi(y)u_1^1(t, x, \frac{y}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\chi'(y) \int_0^y u_1^1(t, x, \tilde{y})d\tilde{y}, \\
\tilde{h}_1^1(t, x, y) &= \chi(y)h_1^1(t, x, \frac{y}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\chi'(y) \int_0^y h_1^1(t, x, \tilde{y})d\tilde{y} + \sqrt{\epsilon}\rho(t, x, \frac{y}{\sqrt{\epsilon}}), \\
\tilde{g}_1^1(t, x, y) &= \chi(y)g_1^1(t, x, \frac{y}{\sqrt{\epsilon}}) - \sqrt{\epsilon}\int_0^y \tilde{e}_x \rho(t, x, \tilde{y})d\tilde{y}.
\end{align*}$$

Also from Proposition (2.4) and (2.39) with large $n$, we know that there is a positive constant $C$ independent of $\epsilon$, such that for $|\alpha| \leq 5, 0 \leq i \leq 2$ and $t \in [0, T_0]$,

$$\epsilon^2 \|\tilde{u}^{0}_{\tau_x}(t, \tau_{u}, \tau_{g})(t, \cdot)\|_{L^2} + \|\tilde{h}^{0}_{\tau_x}(t, \tau_{h}, \tau_{g})(t, \cdot)\|_{L^2} \leq C,$$  \hspace{1cm} (A. 1)

and then,

$$\epsilon^2 \|\tilde{g}^{0}_{\tau_x}(t, \tau_{h}, \tau_{g})(t, \cdot)\|_{L^2} + \|\tilde{h}^{0}_{\tau_x}(t, \tau_{h}, \tau_{g})(t, \cdot)\|_{L^2} \leq C.$$ \hspace{1cm} (A. 2)

Next, we find that the remainder terms $R_i (i = 1, 3)$ in (2.40) have the form:

$$R_i = R_i^0 + \sqrt{\epsilon} R_i^1 + \epsilon R_i^H,$$ \hspace{1cm} (A. 3)

and $R_i (i = 2, 4)$ have the form:

$$R_i = R_i^0 + \sqrt{\epsilon} R_i^1 + \epsilon R_i^H.$$ \hspace{1cm} (A. 4)

Each term in the above expressions can be written explicitly as follows. Firstly,

$$R_i^0 = (u_i^0 - h_i^0) \tilde{e}_x u_i^0 + [v_i^0 - y\tilde{e}_y u_i^0 - y^2 \tilde{e}_y^2 u_i^0 + \sqrt{\epsilon}(v_i^0 - v_{i}^0 - y\tilde{e}_y v_{i}^0)] \tilde{e}_y u_i^0$$

$$+ (\tilde{e}_x u_i^0 - \tilde{e}_x v_i^0 - y\tilde{e}_y^2 u_i^0) u_i^0 + \sqrt{\epsilon}(\tilde{e}_y u_i^0 - \tilde{e}_y v_i^0) v_i^0$$

$$- (h_i^0 - h_{i}^0 - y\tilde{e}_y h_i^0) \tilde{e}_x h_i^0 - [g_i^0 - y\tilde{e}_y g_i^0 - y^2 \tilde{e}_y^2 g_i^0 + \sqrt{\epsilon}(g_i^0 - g_{i}^0 - y\tilde{e}_y g_{i}^0)] \tilde{e}_y h_i^0$$

$$- (\tilde{e}_x h_i^0 - \tilde{e}_x v_i^0 - y\tilde{e}_y^2 h_i^0) h_i^0 - \sqrt{\epsilon}(\tilde{e}_y h_i^0 - \tilde{e}_y v_i^0) g_i^0,$$ \hspace{1cm} (A. 5)

$$R_i^3 = (u_i^0 - h_i^0) \tilde{e}_x h_i^0 + [v_i^0 - y\tilde{e}_y u_i^0 - y^2 \tilde{e}_y^2 u_i^0 + \sqrt{\epsilon}(v_i^0 - v_{i}^0 - y\tilde{e}_y v_{i}^0)] \tilde{e}_y h_i^0$$

$$+ (\tilde{e}_x h_i^0 - \tilde{e}_x v_i^0 - y\tilde{e}_y^2 h_i^0) h_i^0 + \sqrt{\epsilon}(\tilde{e}_y h_i^0 - \tilde{e}_y v_i^0) g_i^0$$

$$- (h_i^0 - h_{i}^0 - y\tilde{e}_y h_i^0) \tilde{e}_x h_i^0 - [g_i^0 - y\tilde{e}_y g_i^0 - y^2 \tilde{e}_y^2 g_i^0 + \sqrt{\epsilon}(g_i^0 - g_{i}^0 - y\tilde{e}_y g_{i}^0)] \tilde{e}_y g_i^0$$

$$- (\tilde{e}_x g_i^0 - \tilde{e}_x v_i^0 - y\tilde{e}_y^2 g_i^0) g_i^0 + \sqrt{\epsilon}(\tilde{e}_y g_i^0 - \tilde{e}_y v_i^0) h_i^0,$$ \hspace{1cm} (A. 6)

and

$$R_i^2 = \sqrt{\epsilon}(u_i^0 - h_i^0) \tilde{e}_x v_i^0 + \sqrt{\epsilon}[v_i^0 - y\tilde{e}_y u_i^0 + \sqrt{\epsilon}(v_i^0 - v_{i}^0)] \tilde{e}_y v_i^0 + \sqrt{\epsilon}(\tilde{e}_x v_i^0 - \tilde{e}_x v_i^0 + \sqrt{\epsilon}(\tilde{e}_x v_i^0 - \tilde{e}_x v_i^0)) v_i^0$$

$$+ \sqrt{\epsilon}(\tilde{e}_x v_i^0 - \tilde{e}_x v_i^0) h_i^0 - \sqrt{\epsilon}(g_i^0 - g_{i}^0) \tilde{e}_x g_i^0 - \sqrt{\epsilon}(g_i^0 - g_{i}^0) \tilde{e}_x g_i^0$$

$$- \sqrt{\epsilon}(\tilde{e}_x g_i^0 - \tilde{e}_x v_i^0) g_i^0.$$
\[ R^0_2 = \sqrt{\epsilon (u^e_c - u^g_c)} \partial_x g^0_e + \sqrt{\epsilon v^0_e - y \gamma_c y^0_c} + \sqrt{\epsilon (v^1_c - v^g_c)} \partial_y g^0_e + \sqrt{\epsilon} (\partial_x g^1_e - \partial_x g^0_c) u_b^0 + \sqrt{\epsilon} (\partial_y g^1_e - \partial_y g^0_c) v_b^0 \]

\[ - (\partial_x v^0_e - y \gamma_c y^0_c + \sqrt{\epsilon} (\partial_x v^1_e - \partial_x v^g_c)) h_b^0 - \sqrt{\epsilon} (\partial_y g^0_c - \partial_y y^0_c) g_b^0. \]

Secondly,

\[ R^1_1 = (u^0_e - v^0_e) \partial_x u^b_1 + [u^0_e - y \gamma_c y^0_c + \sqrt{\epsilon} (v^1_e - v^g_c)] \partial_y u^b_1 + (\partial_x u^0_e - \partial_x v^0_e) u_b^1 - (h^0_e - h^g_c) \partial_x h^b_1 - [g^0_e - y \gamma_c y^0_c + \sqrt{\epsilon} (g^1_e - g^g_c)] \partial_y h^b_1 - (\partial_x g^0_e - \partial_x g^0_c) h_b^1, \]

\[ R^1_3 = (u^0_e - v^0_e) \partial_x h^b_1 + [u^0_e - y \gamma_c y^0_c + \sqrt{\epsilon} (v^1_e - v^g_c)] \partial_y h^b_1 + (\partial_x u^0_e - \partial_x v^0_e) u_b^1 - (h^0_e - h^g_c) \partial_x u^b_1 - [g^0_e - y \gamma_c y^0_c + \sqrt{\epsilon} (g^1_e - g^g_c)] \partial_y u^b_1 - (\partial_x g^0_e - \partial_x g^0_c) h_b^1, \]

and

\[ R^2_2 = \sqrt{\epsilon} (\partial_y g^1_e v_b^0 + \partial_x u^0_e v_b^0 - \sqrt{\epsilon} g^0_e g_c b^1 - \partial_x g^0_e h_b^1, \]

\[ R^4_4 = \sqrt{\epsilon} (\partial_y g^1_e v_b^0 + \partial_x u^0_e v_b^0 - \sqrt{\epsilon} g^0_e g_c b^1 - \partial_x g^0_e h_b^1. \]

Thirdly, the error terms \( R^C_c(i = 1, 3) \) caused by the cut-off function are listed as follows.

\[ R^C_1 = (1 - \chi) \left[ (y \gamma_c y^0_c + \sqrt{\epsilon} v^0_c) \partial_x u^b_1 + \left( \frac{v^2}{2} \gamma_c v^0_c + \sqrt{\epsilon} e v^0_c \partial_x v^0_e \right) \partial_y u^b_1 + \sqrt{\epsilon} \partial_x v^0_e v_b^0 \right. \]

\[ - \left. \left( y \gamma_c y^0_c + \sqrt{\epsilon} v^0_c \partial_x h^b_1 + \sqrt{\epsilon} e \gamma_c \partial_x v^0_e \partial_y h^b_1 - \sqrt{\epsilon} \partial_x v^0_e v_b^0 \right) \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_c) - \sqrt{\epsilon} g^0_e \chi^{b_1} \tau_c, \]

\[ R^C_3 = (1 - \chi) \left[ (y \gamma_c y^0_c + \sqrt{\epsilon} v^0_c) \partial_x h^b_1 + \left( \frac{v^2}{2} \gamma_c v^0_c + \sqrt{\epsilon} e v^0_c \partial_x v^0_e \right) \partial_y h^b_1 + \sqrt{\epsilon} \partial_x v^0_e v_b^0 \right. \]

\[ - \left. \left( y \gamma_c y^0_c + \sqrt{\epsilon} v^0_c \partial_x h^b_1 + \sqrt{\epsilon} e \gamma_c \partial_x v^0_e \partial_y h^b_1 - \sqrt{\epsilon} \partial_x v^0_e v_b^0 \right) \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_c). \]

Finally,

\[ R^H_1 = \epsilon \partial_x b^0_e + \epsilon \partial_c \tau_u + (u^1_e + u^1_b) \partial_x (u^1_e + u^1_b) + \epsilon \partial_x \left( u^0_e + u^0_b \right) \partial_u \left( u^1_e + u^1_b + \sqrt{\epsilon} e \partial_y u \right) \]

\[ + \left( v^1_e \partial_y u^0_e + \sqrt{\epsilon} u^0_e \right) (h^1_e + h^1_b) - \partial_x \left( \left( h^0_e + h^0_b \right) \tau_u \right) \]

\[ - \left( g^1_e + g^1_b \right) \partial_y e \partial_x \tau_u + \partial_x \left( h^0_e + h^0_b \right) (u^1_e + u^1_b) + \left( h^0_e + h^0_b \right) \partial_x \tau_u - h^0_e h^0_b \partial_x \tau_u - \partial_x \left( u^0_e + u^0_b \right) \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_c). \]

\[ R^H_3 = \epsilon \partial_c \tau_u + (u^1_e + u^1_b) \partial_x (h^1_e + h^1_b) + \left( u^0_e + u^0_b \right) \partial_x \tau_u + \partial_x \left( h^0_e + h^0_b \right) \partial_x \tau_u + \left( u^0_e + u^0_b \right) \left( h^0_e + h^0_b \right) \partial_x \tau_u - h^0_e h^0_b \partial_x \tau_u - \partial_x \left( u^0_e + u^0_b \right) \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_c). \]

\[ - \left[ \left( h^0_e + h^0_b \right) \partial_x \left( h^1_e + h^1_b \right) + \sqrt{\epsilon} \partial_x \tau_u \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_c). \]

\[ \left[ \Delta \left( h^1_e + h^1_b \right) + \sqrt{\epsilon} \partial_x \tau_u \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_c). \]

and

\[ R^H_2 = \epsilon \partial_x g^1_e + \left( u^1_e + u^1_b \right) \partial_x (v^1_e + v^1_b) + \left( u^1_e + u^1_b \right) \partial_y (v^1_e + v^1_b) + \left( u^1_e + u^1_b \right) \partial_x \tau_u + \left( v^1_e + v^1_b \right) \partial_x \tau_u \]

\[ - \left( h^1_e + h^1_b \right) \partial_x (g^1_e + g^1_b) - \left( g^1_e + g^1_b \right) \partial_y (g^1_e + g^1_b) - h^1_e \partial_x g^1_e - g^1_e \partial_y g^1_e - \partial_x g^0_e \tau_u - g^0_e (x^1_e + \sqrt{\epsilon} e \partial_y x^1_e) \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_u). \]

\[ R^H_4 = \epsilon \partial_x g^1_e + \left( u^1_e + u^1_b \right) \partial_x (g^1_e + g^1_b) + \left( u^1_e + u^1_b \right) \partial_y (g^1_e + g^1_b) + \left( u^1_e + u^1_b \right) \partial_x \tau_u + \partial_x \left( v^1_e + v^1_b \right) \partial_x \tau_u + \partial_x \left( v^1_e + v^1_b \right) \left( h^0_e + h^0_b \right) \partial_x \tau_u - \partial_x \left( h^0_e + h^0_b \right) \right] D(x u^b_1 + \sqrt{\epsilon} \partial_y v^0_c \tau_u). \]

\[ - \left[ \left( h^1_e + h^1_b \right) \partial_x \left( v^0_e + v^0_b \right) + \left( g^1_e + g^1_b \right) \partial_y \left( v^0_e + v^0_b \right) + \left( h^1_e + h^1_b \right) \partial_x \tau_u - h^1_e \partial_x \tau_u - g^1_e \partial_y \tau_u - \partial_x \tau_u \right] \]
Based on the above exact expressions for the error terms \( R_i (i = 1 \sim 4) \), taking into account the estimates of \( (u^*_i, v^*_i, h^*_i, g^*_i) (i = 0, 1) \) in Propositions 2.1 and 2.3 respectively, and the estimates of \( (u^*_j, h^*_j) (j = 0, 1) \) in Propositions 2.2 and 2.4 respectively, we are in a position to prove Proposition 2.5.

**Proof of Proposition 2.5.** We only show the \( L^2 \)-estimate of \( R_1 \) in (2.41). The estimates for other \( R_i (i = 2, 3, 4) \) can be estimated similarly. Moreover, If one applies the tangential derivatives operators \( \partial_{\alpha} \) \((\alpha = 3)\) on the error terms \( R_i (i = 1, 2, 3, 4) \), it does not produce any singular factor \( \frac{1}{\sqrt{\epsilon}} \) in the formulation. Consequently, we can prove (2.41) by direct calculation for \( 1 \leq |\alpha| \leq 3 \).

The estimates of \( R_1 \) will be divided into three parts.

**Part I:** Estimates of \( R_1^0 \) and \( \sqrt{\chi} R_1^1 \).

By Taylor expansion, we have that for \( \eta = \frac{y}{\sqrt{\epsilon}} \) and some \( \theta, \theta, \hat{\theta} \in [0, y] \),

\[
(u^0_c - \overline{u^0_c} - y \overline{\varepsilon u^0_c}) \partial_x u^0_b = \frac{\partial_y^2 u^0_c(t, x, \theta)}{2} \frac{y^2}{\sqrt{\epsilon}} \partial_x u^0_b(t, x, \frac{y}{\sqrt{\epsilon}}),
\]

and combining with the boundary condition \( \overline{v^0_c} = v^0_c |_{y=0} = 0 \),

\[
\int_T \int_{\mathbb{R}^+} \left[ (u^0_c - \overline{u^0_c} - y \overline{\varepsilon u^0_c}) \partial_x u^0_b \right]^2 \, dy dx = \epsilon^{5/2} \int_T \int_{\mathbb{R}^+} \left( \frac{\partial_y^2 u^0_c(t, x, \theta)}{2} \frac{y^2}{\sqrt{\epsilon}} \partial_x u^0_b(t, x, \frac{y}{\sqrt{\epsilon}}) \right)^2 \, dy dx
\]

which implies

\[
\left\| \left( u^0_c - \overline{u^0_c} - y \overline{\varepsilon u^0_c} \right) \partial_x u^0_b \right\|_{L^2} \leq C \epsilon^{5/4}.
\]

Similarly, we have

\[
\left\| \left( v^0_c - y \overline{\varepsilon v^0_c} \right) \partial_y u^0_b \right\|_{L^2} \leq C \epsilon^{5/4}.
\]

Other terms in \( R_1^0 \) and \( R_1^1 \) can be estimated in the same way with Propositions 2.1, 2.2, 2.3 and 2.4. Consequently,

\[
\| R_1^0 (t, \cdot) \|_{L^2} + \sqrt{\chi} \| R_1^1 (t, \cdot) \|_{L^2} \leq C \epsilon^{5/4}.
\]

**Part II:** Estimates of \( R_1^C \).

Note that from the cut-off function \( \chi(y) \), and using Propositions 2.1 and 2.3,

\[
\left\| \left[ (1 - \chi) \left( y \overline{\varepsilon u^0_c} + \sqrt{\varepsilon u^0_c} \right) \partial_x u^0_b \right] \right\|_{L^2}^2
\]

\[
= \epsilon^{5/2} \int_T \int_{\mathbb{R}^+} \left( (1 - \chi(\sqrt{\epsilon})) \left( \eta \overline{\varepsilon u^0_c} + \overline{u^0_c} \right) \partial_x u^0_b \right)^2 \, dy dx
\]

\[
\leq 2 \epsilon^{5/2} \int_T \int_{\mathbb{R}^+} \left( \eta \overline{\varepsilon u^0_c} + \overline{u^0_c} \right)^2 \left( \overline{\varepsilon u^0_c} + \overline{u^0_c} \right)^2 \, dy dx
\]

\[
\leq 4 \epsilon^{5/2} \left( \| \overline{\varepsilon u^0_c} \|_{L^2(\mathbb{T})}^2 \| \partial_x u^0_b \|_{L^2(\mathbb{T})}^2 + \| \overline{u^0_c} \|_{L^2(\mathbb{T})}^2 \| \partial_x u^0_b \|_{L^2(\mathbb{T})}^2 \right)
\]

\[
\leq C \epsilon^{5/2} + l
\]
for any \( l \geq 0 \). Similar argument yields that by using Proposition 2.4,\n\[
\left\| \left( \sqrt{\epsilon} \chi' \epsilon^l u_0 \right)(t, \cdot) \right\|_{L^2(T \times \mathbb{R}_+)}^2 \leq C \epsilon^{2+l} \int_T \int_{1/\sqrt{\epsilon}} \left[ \chi'(\sqrt{\epsilon} \eta) \epsilon^l (t, x, \sqrt{\epsilon} \eta) \cdot \epsilon^l u_0(t, x, \eta) \right]^2 \, d\eta \, dx
\]
\[
\leq C \epsilon^{2+l} \left\| u_0^0 \right\|_{L^2(T \times \mathbb{R}_+)}^2 \left\| u_0^0 \right\|_{L^2(\Omega)}^2 \leq C \epsilon^{2+l}
\]
(A. 15)
for any \( l \geq 0 \). In addition, it follows that by the boundary condition \( \epsilon v_0^0 \partial_y \tau_u = \epsilon \partial_y v_0^0(t, x, \theta_y)y \cdot \partial_y \tau_u(t, x, y) \) for some \( \theta_y \in [0, y] \), which along with\n(A. 14) implies that\n\[
\left\| \left( \epsilon v_0^0 \partial_y \tau_u \right)(t, \cdot) \right\|_{L^2} \leq C \epsilon \| \partial_y v_0^0(t, \cdot) \|_{L^2(\Omega)} \left\| \left( \epsilon \partial_y \tau_u \right)(t, \cdot) \right\|_{L^2} \leq C \epsilon.
\]
(A. 16)
Thus, combining (A. 14), (A. 15) with \( l \geq 1/2 \) and (A. 16), and other terms of \( R_1^C \) can be investigated similarly, we have\n\[
\left\| R_1^C \right\|_{L^2(T \times \mathbb{R}_+)} \leq C \epsilon.
\]
Part III: Estimates of \( R_1^H \).
From Propositions 2.1-2.4 and using (3. 74), (A. 1) and (A. 2), it is easy to check that the \( L^2 \)-norm of each term in \( R_1^H \) is uniformly bounded with respect to \( \epsilon \). As a consequence, we have\n\[
\left\| \epsilon R_1^H(t, \cdot) \right\|_{L^2(T \times \mathbb{R}_+)} \leq C \epsilon.
\]
Combining all estimates in Parts I–III, we have\n\[
\left\| R_1(t, \cdot) \right\|_{L^2} \leq C \epsilon.
\]
(A. 17)

Acknowledgements: The research of the second author was supported by NSFC (Grant No.11571231). The research of the third author is supported by the General Research Fund of Hong Kong, CityU No. 11320016.

References

[1] Alexander, R., Wang, Y-G, Xu, C-J., Yang, T., Well posedness of the Prandtl equation in Sobolev spaces. J. Amer. Math. Soc. 28 (2015), 3, 745-784.
[2] Alfvén H., Existence of electromagnetic-hydrodynamic waves. Nature 150 (1942), 405-406.
[3] Arkhipov, V.N., Influence of magnetic field on boundary layer stability. Dokl. Akad. Nauk SSSR 129 (1959), 1199-1201.
[4] C. Bardos, C. Sulem and P.-L. Sulem, Longtime dynamics of a conductive fluid in presence of a strong magnetic field. Trans. Amer. Math. Soc. 305 (1988), 1, 175-191.
[5] P. A. Davison, An introduction to magnetohydrodynamics, Cambridge Texts in Applied Mathematics, 2001.
[6] Duvant, G., Lions, J. L., Inéquation en thermoélastique et magnétohydrodynamique, Arch. Ration. Mech. Anal. 46 (1972), 241-279.
[7] Drasin, P., Stability of parallel flow in a parallel magnetic field at small magnetic Reynolds number. J. Fluid Mech. 8 (1960), 130-142.
[8] E, W.-N., Engquist, B., Blowup of solutions of the unsteady Prandtl’s equation. Comm. Pure Appl. Math. 50 (1997), 12, 1287-1293.
[9] Gérard-Varet, D., Dormy, E., On the ill-posedness of the Prandtl equations. J. Amer. Math. Soc. 23 (2010), 591-609.
[10] Gérard-Varet, D., Maekawa, Y., Masmoudi, N., Gevrey stability of Prandtl expansions for 2D Navier-Stokes flows. Preprint, 2016, arXiv: 1607.06434.
[11] Gérard-Varet, D., Masmoudi, N., Well-posedness for the Prandtl system without analyticity or monotonicity. Ann. Sci. Éc. Norm. Supér. 48 (2015), 6, 1273-1325.
[12] Gérard-Varet, D., Nguyen, T., Remarks on the ill-posedness of the Prandtl equation. Asymptot. Anal. 77 (2012), 1-2, 71-88.
[13] Gérard-Varet, D., Prestipino, M., Formal Derivation and Stability Analysis of Boundary Layer Models in MHD. Preprint, 2016. arXiv:1612.02641.
[14] Grenier, E., On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* 53 (2000), 9, 1067-1091.

[15] Grenier, E., Guo Y., Nguyen T., Spectral instability of characteristic boundary layer flows. *Duke Math. J.* 165 (2016), 16, 3085-3146.

[16] Grenier, E., Guo Y., Nguyen T., Spectral stability of Prandtl boundary layers: an overview. *Analysis(Berlin)* 35 (2015), 343-355.

[17] Guo, Y., Nguyen, T., A note on the Prandtl boundary layers. *Comm. Pure Appl. Math.* 64 (2011), 1416-1438.

[18] Guo, Y., Nguyen, T., Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate. *Preprint* (2014), arXiv:1411.6984v1.

[19] L.-B. He, L. Xu and P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, arXiv:1603.08205v1.

[20] Ignatova M., Vicol V., Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.* 220 (2016), 2, 809-848.

[21] Kukavica I., Vicol V., On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.* 11 (2013), 1, 269-292.

[22] Kukavica I., Masmoudi N., Vicol V., Wong T.-K., On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.* 46 (2014), 6, 3865-3890.

[23] Li, W.-X., Yang, T., Well-posedness in Gevrey space for the Prandtl equations with non-degenerate points. Preprint, 2016. arXiv:1609.08430.

[24] F. Lin, L. Xu and P. Zhang, Global small solutions of 2-D incompressible MHD system. *J. Diff. Equations* 259 (2015), no. 10, 5440-5485.

[25] Liu, C.-J., Wang Y.-G., Yang T., A well-posedness theory for the Prandtl equations in three space variables. *Adv. Math.* 308 (2017), 1074-1126.

[26] Liu, C.-J., Wang Y.-G., Yang T., On the ill-posedness of the Prandtl equations in three space dimensions. *Arch. Ration. Mech. Anal.* 220 (2016), 1, 83-108.

[27] Liu, C.-J., Wang Y.-G., Yang T., Global existence of weak solutions to the three-dimensional Prandtl equations with a special structure. *Discrete Contin. Dyn. Syst. Ser. S* 9 (2016), 6, 2011-2029.

[28] Liu, C.-J., Xie, F., Yang, T., MHD boundary layers in Sobolev spaces without monotonicity. I. Well-posedness theory. Preprint, 2016. arXiv 1611.05815.

[29] Liu, C.-J., Yang T., Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. In press, *J. Math. Pure Appl.*, DOI: http://dx.doi.org/ 10.1016/j.matpur.2016.10.014.

[30] Maekawa, Y., On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.* 67 (2014), 7, 1045-1128.

[31] Masmoudi, N., Wong, T-K, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.* 68 (2015), 10, 1683-1741.

[32] Ohno, M., Shirotta, T., On the initial-boundary-value problem for the linearized equations of magnetohydrodynamics. *Arch. Rational Mech. Anal.* 144 (1998), no. 3, 259299.

[33] Oleinik, O. A., The Prandtl system of equations in boundary layer theory. *Dokl. Akad. Nauk SSSR* 4 (1963), 583-586.

[34] Oleinik, O. A., Samokhin, V. N., Mathematical Models in Boundary Layers Theory. Chapman and Hall/CRC, 1999.

[35] Prandtl, L., Uber flüssigkeits-bewegung bei sehr kleiner reibung. Verhandlungen des III. Internationalen Mathematiker Kongresses, Heidelberg, Teubner, Leipzig, (1904), 484-491.

[36] Rossow, V.J., Boundary layer stability diagrams for electrically conducting fluids in the presence of a magnetic field, NACA Technical Note 4282 (1958), NACA(Washington).

[37] Sammartino, M., Caflisch, R.-E., Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* 192 (1998), 2, 433-461.

[38] Sammartino, M., Caflisch, R.-E., Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* 192 (1998), 2, 463-491.

[39] Secchi, P., On the equations of ideal incompressible magnetohydrodynamics. *Rend. Sem. Mat. Univ. Padova* 90 (1993), 103119.

[40] Sermange, M., Temam, R., Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* 36 (1983), 635-664.

[41] Wang C., Wang Y., Zhang Z.-F., Zero-Viscosity Limit of the Navier-Stokes Equations in the Analytic Setting. *Arch. Ration. Mech. Anal.* 224 (2017), 555-595.
[42] Wang, S., Xin, Z.-P., Boundary layer problems in the viscosity-diffusion vanishing limits for the incompressible MHD systems. Accepted for publication in Sci. China. 2017.

[43] Yanagisawa, T., Matsumura, A., The fixed boundary value problems for the equations of ideal magnetohydrodynamics with a perfectly conducting wall condition. Comm. Math. Phys. 136 (1991), no. 1, 119140.

[44] Xiao Y.-L., Xin Z.-P., Wu J.-H., Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition. J. Funct. Anal. 257 (2009), 11, 3375-3394.

[45] Xin, Z.-P., Yanagisawa, T., Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane. Comm. Pure Appl. Math. 52 (1999), 479-541.

[46] Xin, Z-P., Zhang, L., On the global existence of solutions to the Prandtl system. Adv. Math. 181 (2004), 88-133.

[47] L. Xu and P. Zhang, Global small solutions to three-dimensional incompressible magnetohydrodynamical system. SIAM. J. Math. Anal. 47(2015), no. 1, 26-65.

[48] Zhang P., Zhang Z.-F., Long time well-posedness of Prandtl system with small and analytic initial data. J. Funct. Anal. 270 (2016), 7, 2501-2615.

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, TAT CHEE AVENUE, KOWLOON, HONG KONG
E-mail address: cjliusjtu@gmail.com

SCHOOL OF MATHEMATICAL SCIENCES, AND LSC-MOE, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, P.R.CHINA
E-mail address: tzxief@sjtu.edu.cn

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, TAT CHEE AVENUE, KOWLOON, HONG KONG
E-mail address: matyang@cityu.edu.hk