LINEARIZABLE 3-WEB S AND THE GRONWALL CONJECTURE

JOSEPH GRIFONE, ZOLTAN MUZSNAY, AND JIHAD SAAB

ABSTRACT. In the article [10] published in 2001 in the journal "Nonlinear Analysis", we studied the linearizability problem for 3-webs on a 2-dimensional manifold. Four years after the publication of our article Goldberg and Lychagin [8] obtained similar results by a different method and criticized our article by qualifying the proofs incomplete. However, they obtained false result on the linearizability of a certain web. We present here the complete version of [10] with computations and explicit formulas, because we deem that the opinion of Goldberg and Lychagin in [8] concerning our work is unjustified.

1. Introduction

In the article [10] published in 2001 in the journal "Nonlinear Analysis", we studied the linearizability problem for 3-webs on a 2-dimensional manifold. Using the integrability theory of over-determined partial differential systems, we computed the obstructions to linearizability and we produced an effective method to test the linearizability of 3-webs in the (real or complex) plane. We showed that, in the non-parallelizable case, there exists an algebraic submanifold \( \mathcal{A} \) of the space of vector valued symmetric tensors \( (S^2T^* \otimes T) \), which can be expressed in terms of the curvature of the Chern connection and its covariant derivatives up to order 6, such that the affine deformation tensor is a section of \( S^2T^* \otimes T \) with values in \( \mathcal{A} \). In particular, we proved that a web is linearizable if and only if \( \mathcal{A} \neq \emptyset \), and there exists at most 15 projectively nonequivalent linearizations of a nonparallelizable 3-web. In order to give a coordinate free and intrinsic presentation of the results we used tensors and covariant derivatives to find the obstructions to the linearization.

Recently Goldberg and Lychagin [8] obtained similar results by a different method. They criticized our article by qualifying the proofs incomplete, without giving any justification or reason for their claim. They claim that "...the main and only example of a linearizable (in their approach) 3-web ... is not linearizable at all..." To prove their statement they apply their theory to this particular web and find that the corresponding algebraic submanifold is empty. However, in the article [14] which appears in arXiv with this present paper, Z. Muzsnay shows by producing an explicit linearization, that in accordance with the claim of [10], this web is linearizable. This proves that something is wrong in their work: either the proofs of [7] and [8] are not correct, or some of their calculations are false.

We are putting on the arXiv preprint archive a detailed version of the article [10] with computations and explicit formulas, because we deem that the opinion of Goldberg and Lychagin in [8] concerning our work is unjustified.

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2. INTRODUCTION TO THE LINEARIZABILITY PROBLEM OF 3-WEBS

Let $M$ be a two-dimensional real or complex differentiable manifold. A 3-web is given in an open domain $D$ of $M$ by three foliations of smooth curves in general position. Two webs $W$ and $\tilde{W}$ are locally equivalent at $p \in M$, if there exists a local diffeomorphism on a neighborhood of $p$ which exchanges them.

A 3-web is called linear (resp. parallel) if it is given by 3 foliations of straight lines (resp. of parallel lines). A 3-web which equivalent to a linear (resp. parallel) web is called linearizable (resp. parallelizable).

A linear connection, called Chern connection and denoted by $\nabla$, can be associated to a 3-web. $\nabla$ preserves the web, i.e. the leaves are auto-parallel curves. It is not difficult to see that a 3-web is parallelizable if and only if the curvature of the Chern connection vanishes. The Graf-Sauer Theorem ([2], page 24) gives an elegant characterization of such webs: a linear web is parallelizable if and only if, its leaves are tangent lines to a curve of degree 3.

The problem to give linearizability criterion is a very natural one. Such criterion is important in nomography (cf. [11]): determining whether some nomogram can be reduced to an alignment chart is equivalent to the problem of determining whether a web is linearizable. The most significant works on this subject are due to Bol ([3], [4]). In [3] he suggested how to find a criterion of linearizability, although he is unable to carry out the computation, which really need the use of computer. He shows that the number of projectively different linear 3-webs in the plane to which a non-hexagonal 3-web is equivalent is finite and less that 17. Bol’s proof consists in to associate to a real 3-web two complex vector fields which play an essential role, so his proof cannot be translated in the complex case. In our computation the web can be real as well as complex.

The formulation of the linearizability problem in terms of Chern connection was suggested by Akivis in a lecture given in Moscow in 1973. Following Akivis idea Goldberg in [6] found all affine connection $\Gamma^*$ relative to which the web leaves are geodesic lines and distinguished a linearizable 3-webs by claiming that the connection $\Gamma^*$ is flat. In this paper we are using this approach to solve the problem.

Denoting by $T$ and $T^*$ the tangent and the cotangent bundle of $M$, a section $L$ of the bundle $S^2T^* \otimes T$ on $M$ is called pre-linearization, if the connection $\nabla^L$ defined by

$$\nabla^L_X Y = \nabla_X Y + L(X, Y)$$

preserves the web, that is the three families of leaves are auto-parallels curves with respect to $\nabla^L$. A pre-linearization $L$ is called linearization if the connection $\nabla^L$ is flat i.e. the curvature of the connection $\nabla^L$ given by equation (1) vanishes. This equation gives us a first order partial differential system on $L$. Two linearizations $L$ and $L'$ are projectively equivalent if the connections $\nabla^L$ and $\nabla^{L'}$ are projectively related. The equivalences classes are called classes of linearizations. They are in one-to-one correspondence with the bases of the linearization which is a simple projective invariant, noted by $s$. The linearizability condition can be reformulate with this object by a second order partial differential system. We show that the system is of finite type, and the obstruction to the linearizability can be expressed in terms of polynomials of $s$, whose coefficients depends only on the curvature tensor of the Chern connection. Our main result is the following:

**Theorem 2.1.** Let $W$ be an analytical 3-web on a 2-dimensional real or complex manifold $M$, whose Chern curvature does not vanish at $p \in M$. Then, there exists an algebraic submanifold $A$ of $E$ over a neighborhood of $p$, expressed in terms of the curvature of the Chern
connection and its covariant derivatives up to order 6, so that the linearizations of \( \mathcal{W} \) are sections of \( E \) with values in \( A \). In particular:

1. The web is linearizable if and only if \( A \neq \emptyset \);
2. There exists at most 15 classes of linearizations.

The explicit expression of the polynomials and its coefficients which define \( A \) can be found in Chapter 6 and 7.

3. Notations and definitions

Let \( \mathcal{W} \) be a differential 3-web on a manifold \( M \) given by a triplet of mutually transversal foliations \( \{ F_1, F_2, F_3 \} \). From the definitions it follows that \( M \) is even dimensional and that the dimension of the tangent distributions of the foliations \( F_1, F_2, F_3 \) is the half of the dimension of \( M \). The foliations \( \{ F_1, F_2, F_3 \} \) are called horizontal, vertical and transversal and their tangent space are denoted by \( T^h, T^v, T^t \).

The following theorem proved by Nagy [13] gives an elegant infinitesimal characterization of 3 webs and their Chern connection.

**Theorem.** A 3-web is equivalent to a pair \( \{ h, j \} \) of \((1,1)\)-tensor fields on the manifold, satisfying the following conditions:

1. \( h^2 = h, j^2 = id \),
2. \( jh = vj \), where \( v = id - h \),
3. \( \text{Ker} h, \text{Im} h \) and \( \text{Ker}(h + id) \) are integrable distributions.

For any 3-web, there exists a unique linear connection \( \nabla \) on \( M \) which satisfies

1. \( \nabla h = 0 \),
2. \( \nabla j = 0 \),
3. \( T(hX, vY) = 0 \), for every \( X, Y \in TM \), \( T \) being the torsion tensor of \( \nabla \).

\( \nabla \) is called Chern connection.

In the sequel, we suppose that the dimension of \( M \) is two.

**Definition 3.1.** Let \( \mathcal{W} \) be a 3-web and \( \nabla \) its Chern connection. A symmetrical \((1,2)\)-tensor field \( L \) is called pre-linearization if the connection

\[
\nabla_X^L Y = \nabla_X Y + L(X, Y)
\]

preserves the web, that is the leaves are auto-parallel curves with respect to \( \nabla^L \). A pre-linearization is a linearization if the connection \( \nabla^L \) is flat i.e. its curvature vanishes. Two pre-linearizations \( L \) and \( L' \) are projectively equivalent if the connections \( \nabla^L \) and \( \nabla^{L'} \) are projectively related, that is there exists \( \omega \in \Lambda^1(M) \), such that

\[
\nabla_X^L Y = \nabla_X^{L'} Y + \omega(X)Y + \omega(Y)X
\]

**Proposition 3.2.** A tensor field \( L \) in \( S^2T^* \otimes T \) is a linearization if and only if

1. \( vL(hX, hY) = 0 \),
2. \( hL(vX, vY) = 0 \),
3. \( L(hX, hY) + jL(jhX, jhY) - hL(jhX, hY)

\[
- hL(hX, jhY) - jvL(jhX, hY) - jvL(hX, jhY) = 0,
\]
4. \( \nabla_X L(Y, Z) - \nabla_Y L(X, Z) + L(X, L(Y, Z)) - L(Y, L(X, Z)) + R(X, Y)Z = 0. \)

holds, for any \( X, Y, Z \in T \), where \( R \) denotes the curvature of the Chern connection.
The proof is a straightforward verification. Properties 1), 2) and 3) means that $L$ is a pre-linearization and follows from the fact that $\nabla^L$ preserves the web, while properties 4) expresses, that the curvature of $\nabla^L$ vanishes.

**Definition 3.3.** Let $M$ be a 2-dimensional manifold, $\mathcal{W}$ a web on $M$ and \{e₁, e₂\} a frame at $p \in M$ adapted to the web, i.e. $e₁ \in T^h_p$, $e₂ = je₁ \in T^v_p$. Let $L$ be a pre-linearization at $p$, whose components are $L^k_{ij}$, that is: $L(e_i, e_j) = L^k_{ij}e_k$, and let us set the tensor-field $s$ represented by the components $2L^1_{12} - L^2_{22}$. The tensor $s$ will be called the base of $L$.

The following proposition is elementary, but it is the key for the proof of our main theorem.

**Proposition 3.4.** Two pre-linearizations $L$ and $L'$ are projectively equivalent if and only if they have the same base, i.e. $s = s'$.

Indeed, if $L$ and $L'$ are two projectively equivalent pre-linearizations, then there exists $ω \in T^*$ such that $L' = L + ω \otimes id$, i.e. in the frame \{e₁, e₂\}:

\[
L^1_{11} = L^1_{11} + 2ω_1, \quad L^2_{22} = L^2_{22} + 2ω_2, \quad L^1_{12} = L^1_{12} + ω
\]

where $ω_1$ and $ω_2$ are the components of $ω$. This system is consistent if and only if $L'^1_{12} - L^1_{12} = \frac{1}{2}(L^2_{22} - L^2_{22})$, i.e. $s = s'$.

4. THE LINEARIZATION OPERATOR

Let $M$ be a 2-dimensional real or complex manifold and $\mathcal{W}$ a 3-web on $M$. $\Lambda^kT^*$ and $S^kT^*$ are the bundles of the $k$-skew-symmetric and symmetric forms. If $B \to M$ is a vector bundle on $M$, then $Sec(B)$ will denote the sheaf of the sections of $B$ and $J_k(B)$ the vector bundle of $k$-jets of the sections of $B$.

In the sequel $E$ will denotes the bundle of the pre-linearizations and $F := \Lambda^2T^* \otimes T$. In order to study the linearizability of $\mathcal{W}$, we will consider the differential operator $P_1 : E \to F$ and study the integrability of the differential system $P_1(L) = 0$, where

\[
(P_1(L))(X, Y, Z) = (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) + L(X, L(Y, Z)) - L(Y, L(X, Z)) + R(X, Y) Z
\]

for every $X, Y, Z \in T$.

We will use the theory of the formal integrability of Spencer ([5], [9]). The notations are those of [9], where is given also an accessible introduction to this theory. In particular, if $P$ is a quasi-linear operator of order $k$ and $p \in M$, then $R_{k,p}$ is the bundle of the formal solutions of order $k$ at $p$, $σ_{k+ℓ}(P)$ or simply $σ_{k+ℓ}$ is the symbol of the $ℓ$-th order prolongation $ϕ(σ_k)$ of $P$. We also denote $g_{k+ℓ} = kerσ_{k+ℓ}$ and $K = Cokerσ_{k+1}$.

Let $L \in E$ a pre-linearization. We introduce the tensors

\[
x, y, z : T^h \otimes T^h \to T^h
\]

defined by

\[
\begin{aligned}
x(hX, hY) &= L(hX, hY) \\
y(hX, hY) &= jL(jhX, jhY) \\
z(hX, hY) &= hL(hX, jhY)
\end{aligned}
\]

(2)
One denotes \( z^2 \) the \((1,3)\) tensor defined by \( x^2 (hX, hY, hZ) = x (x(hX, hY), hZ) \). Similarly, we define the product \( xy, x^3 \) (which is a \((1,4)\) tensor field), etc...

The space of pre-linearizations, \( E \) is a 3-dimensional vector bundle over \( M \), and \( x, y, z \) can be used to parameterize it. However, taking into account some symmetries of the problem and the Proposition 3.4, it is better to introduce the tensors \( s, t : T^h \oplus T^h \rightarrow T^h \) defined by

\[
\begin{align*}
s &= 2z - y \\
t &= \frac{1}{3} (x + y - 2z)
\end{align*}
\]

and parameterize \( E \) by \( s, t, z \) where \( s \) is the base of the web (see Definition 3.4).

In order to simplify the notation, we denote by \( C_1 \) and \( C_2 \) the tensor fields \( (\bigotimes^{p+1} T^{h^*}) \otimes T^h \) defined by

\[
C_1 (hX, hX_1, ..., hX_p) = (\nabla hX C) (hX_1, ..., hX_p)
\]

\[
C_2 (hX, hX_1, ..., hX_p) = (\nabla_j hX C) (hX_1, ..., hX_p)
\]

where \( C \) is a tensor field in \( (\bigotimes^p T^{h^*}) \otimes T^h \). By recursion, we introduce the successive covariant derivatives with the convention that \( C_{i_1i_2} := (C_{i_2})_{i_1} \). Thus, \( x_{i_1,...,i_p} \) is the \((1, p+2)\) tensor defined in an adapted frame by

\[
x_{i_1,...,i_p} (e_1, ..., e_i, hX, hY) = (\nabla \nabla \cdots \nabla x) (e_{i_1}, ..., e_{i_p}, hX, hY).
\]

We denote \( \mathcal{R} \) the tensor \( \mathcal{R} : T^h \oplus T^h \oplus T^h \rightarrow T^h \) defined by

\[
\mathcal{R}(hX, hY) hZ = R(jhX, hY) hZ
\]

where \( R \) is the curvature of the Chern connection. With the above notation we have

\[
(\nabla_i \nabla_j L_{i_1, ..., i_m}) - (\nabla_j \nabla_i L_{i_1, ..., i_m}) = R^k_{ijk} L_{i_1, ..., i_m} - R^k_{ij1} L_{k, ..., i_m} - \cdots - R^k_{ijm} L_{i_1, ..., k}.
\]

In particular

\[
C_{12} - C_{21} = (p - 1) \mathcal{R} C
\]

for a tensor field \( C \in (\bigotimes^p T^{h^*}) \otimes T^h \).

Using these notations, and resolving two equations in \( z_1 \) and \( t_2 \) the system \( P_1(L) = 0 \) can be write as:

\[
\begin{align*}
t_1 &= st + t^2, \\
t_2 &= \frac{1}{3} s_1 - \frac{2}{3} s_2 + zt - \frac{1}{3} \mathcal{R}, \\
z_1 &= \frac{2}{3} s_1 - \frac{1}{3} s_2 + zt + \frac{1}{3} \mathcal{R}, \\
z_2 &= -zs + z^2.
\end{align*}
\]

Note that \( P_1 \) is regular because the symbol and his prolongation are regular maps. The system (7) can be seen as a Frobenius system on the variables \( t \) and \( z \), and \( s \) being a parameter. By the formula (6), the integrability conditions are

\[
\begin{align*}
z_{12} - z_{21} &= \mathcal{R} z, \\
t_{12} - t_{21} &= \mathcal{R} t, \\
s_{12} - s_{21} &= \mathcal{R} s,
\end{align*}
\]
and thus from (1) we can arrive to the system

\[
P_2 = \begin{cases} 
  s_{22} = 2s_{21} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2, \\
  s_{11} = 2s_{21} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1.
\end{cases}
\]

The operator \( P_2 : \sec(E_2) \to F_2 \) is a quasi-linear second order differential operator, where

\[
E_2 = T^{h^*} \otimes T^{h^*} \otimes T^h,
\]

and thus from (7) we can arrive to the system

\[
P_2 = \begin{cases} 
  s_{22} = 2s_{21} - ss_2 + 2ss_1 + \mathcal{R}s + \mathcal{R}_2, \\
  s_{11} = 2s_{21} - 2ss_2 + ss_1 + \mathcal{R}s + \mathcal{R}_1.
\end{cases}
\]

At every \( p \in M \) all \( 2^{rd} \)-order solution at \( p \) of \( P_2 \) can be lifted into a \( 3^{rd} \)-order solution.

Indeed, fixing an adapted base \( \{e_1, e_2 = je_1\} \), the symbol of \( P_2 \) is a map \( \sigma_2 : S^2T \otimes E_2 \to F_2 \) defined by

\[
\sigma_2(A) = (A_{22} - 2A_{21}, A_{11} - 2A_{21}),
\]

where \( A_{ij} = A(e_i, e_j) \). So \( g_2 \) is the kernel of the equations \( A_{22} - 2A_{21} = 0 \) and \( A_{11} - 2A_{21} = 0 \). Since these equations are independent, we have: \( \text{rank } \sigma_2 = 2 \) and \( \text{dim } g_2 = 1 \). On the other hand, for the first prolongation \( \sigma_3 : S^3T^* \otimes E_2 \to T^* \otimes F_2 \) we find that \( g_3 = \text{ker } \sigma_3 \) is defined by the equations

\[
B_{k_{22}} - 2B_{k_{21}} = 0, \quad B_{k_{11}} - 2B_{k_{21}} = 0,
\]

\( k = 1, 2 \). It is easy to verify that these equations are also independent. Therefore \( \text{rank } \sigma_3 = 4 = \text{dim}(T^* \otimes F_2) \), and \( \text{dim } g_3 = 0 \), thus \( \sigma_3 \) is onto i.e. \( \text{Coker } \sigma_3 = 0 \). We have the following exact diagram:

\[
\begin{array}{ccc}
S^2T^* \otimes E_2 & \xrightarrow{\sigma_3} & T^* \otimes F \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
R_3 & \xrightarrow{\pi_2} & J_3(E_2) \xrightarrow{p_1(P_2)} J_1F \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
R_2 & \xrightarrow{\pi_0(P_2)} & J_2(E_2) \xrightarrow{p_0(P_2)} F
\end{array}
\]

Consequently \( \overline{\pi}_3 \) is onto, i.e. every \( 2^{nd} \)-order solution of \( P_2 \) can be lifted into a \( 3^{rd} \)-order solution. \( \square \)

**Proposition 4.2.** The operator \( P_2 \) is not 2-acyclic, i.e. there is a higher order obstruction which arises for the operator \( P_2 \).

Indeed, the sequence \( 0 \to g_{l+1}(P_2) \to g_{l}(P_2) \otimes T^* \xrightarrow{\delta_l(P_2)} g_{l-1}(P_2) \otimes \Lambda^2T^* \to 0 \) is not exact for all \( l \geq 2 \), where \( \delta_l \) denotes the skew-symmetrization in the corresponding variables: for \( l = 3 \) we have \( \text{rank } \delta_3 = 0 < \text{dim}(g_2 \otimes \Lambda^2T^*) = 1 \). \( \square \)

5. The first obstruction

In order to find the higher order obstruction we consider the prolongation of \( P_2 \), i.e. the operator \( P_3 := (P_2, \nabla P_2) \), where \( \nabla P_2 : T^* \otimes E_2 \to T^* \otimes F_2 \) is the covariant derivative of
Lemma 5.1. $P_3$ is involutive. Moreover, any 3$^\text{rd}$-order solution of $P_3$ can be lifted into a 4$^\text{th}$-order solution if and only if $\varphi = 0$, where

$$\varphi(s) := -24Rss_2 - (24Rs + 12R1 - 6R2)s_1 + (24Rs + 6R1 - 8R2)s_2 + 3Rs^3 + (-4R_2 - 3R_2 + 2R_1 + 13R_2 - 3R_1)s + 2R_{112} - R_{112} - R_{112} - 5R_2 - 11R_2$$

Proof. The symbol of $P_3$ is just $\sigma_3(P_3) : S^4T^* \otimes E_2 \rightarrow T^* \otimes F_2$, the first prolongation of the symbol of $P_2$. On the other hand, $\sigma_4(P_4) : S^4T^* \otimes E_2 \rightarrow S^2T^* \otimes F_2$ and $g_4 = \ker \sigma_4$ is defined by the equations

$$D_{ij} := C_{ij2} - 2C_{ij2} = 0, \quad D_{ij} := C_{ij11} - 2C_{ij2} = 0, \quad i, j = 1, 2.$$ 

There is one relation between these equations: $D_{11} - 2D_{12} - D_{22} + 2D_{12} = 0$. Therefore the rank of this system is 5, so if $K_2$ denotes the cokernel of $\sigma_4$, i.e. $K_2 = (S^2T^* \otimes F_2)/\ker \sigma_4$, then $\dim K_2 = 1$. If we define a map $\tau : S^2T^* \otimes F_2 \rightarrow \mathbb{C}$ by $\tau(D) = D_{11} - 2D_{12} - D_{22} + 2D_{12}$, then, the sequence

$$0 \rightarrow S^4T^* \otimes E_2 \xrightarrow{\tau} S^2T^* \otimes F_2 \xrightarrow{\tau} K_2 \rightarrow 0$$

is exact. We can deduce that the obstruction to the integrability of $P_3$ is given by $\varphi_p = 0$, where $\varphi : R_3 \rightarrow K_2$ is defined by

$$\varphi(s) = [\nabla p_0(P_3(s))]_{11} - 2[\nabla p_0(P_3(s))]_{12} - [\nabla p_0(P_3(s))]_{22} + 2[\nabla p_0(P_3(s))]_{22}.$$ 

Using the equations (8) and (9), we obtain

$$\varphi(s) = \nabla_{11}[2s_{11} - s_{11} - s_{11} + 2s_{11} + R_2 - 2s_{22} - 2s_{22} - 2s_{22} + 2s_{11} + R_2 + R_2] - 2\nabla_{12}[2s_{21} - s_{22} - 2s_{22} - 2s_{22} + 2s_{11} + R_2 + R_2] + 2\nabla_{12}[2s_{21} - s_{22} - 2s_{22} + 2s_{11} + R_2 + R_2] - \nabla_{22}[2s_{21} - s_{22} - 2s_{22} + 2s_{11} + R_2 + R_2].$$

By the formula (9) we can eliminate the 4$^\text{th}$-order derivatives and find the expression of $\varphi$. 

\footnote{Sometimes there is a confusion between different terminologies. The involutivity here (and also in the mentioned [5] and [9]) means the involutivity of the symbol i.e. that the Cartan’s test for involutivity holds. It doesn’t mean the integrability, which is the case in some another terminologies.}
We can remark that dim\(g_{3,p} = 0\) and therefore dim\(g_{k,p} = 0\) for every \(k > 3\). It follows that \(P_3\) is involutive.

**Remark.** If \(R = 0\), then \(\varphi = 0\), therefore, all \(3^d\)-order solution of \(P_3\) can be lifted into a \(4^d\)-solution. Since \(P_3\) is involutive \(P_3\) is formally integrable and consequently, it is integrable in the analytical case. We have the following result:

**Corollary 5.2.** If \(W\) is a parallelizable 3-web on the plane, then for all \(L_0 \in E_p\), there exists a germ of linearizations \(L\) which prolongs \(L_0\).

In accord of the Gra̋f-Sauer Theorem, one can deduce that for a parallelizable web, there are non projectively equivalent linearizations. Indeed, it is sufficient to consider \(L_0, L'_0 \in E_p\) with \(s_p \neq s'_p\) and to prolong them in germs of linearization to obtain two non projectively equivalent germs of linearization.

6. **Second obstruction**

In the sequel we will suppose that \(R \neq 0\). In this case the compatibility condition \([10]\) is not satisfied, so we have to introduce into our differential system and consider the second order quasi-linear system \(P_\varphi = 0\):

\[P_\varphi := (P_2, \varphi),\]

where \(P_2\) is defined by \([8]\) and \(\varphi\) is given by the equation \([10]\). The diagram associated to \(P_\varphi\) is:

\[
\begin{array}{ccc}
S^3T^* \otimes E_2 & \xrightarrow{\sigma_1(P_\varphi)} & (T^* \otimes F_2) \oplus (T^* \otimes K_2) \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
R_3 & \xrightarrow{p_1(P_\varphi)} & J_1 F_2 \oplus J_1 K_2 \\
\downarrow \pi & & \downarrow \pi \\
R_2 & \xrightarrow{p_0(P_\varphi)} & F_2 \oplus K_2
\end{array}
\]

**Lemma 6.1.** A \(3^d\)-order formal solution \(j_{2,p}s\) of \(P_\varphi\) at \(p \in M\), can be lifted into a \(3^d\)-order solution if and only if:

\[
\begin{align*}
\psi^1_p s := 24R s^2_2 - 48R s_1 s_2 + \alpha(s)s_1 + \beta(s)s_2 + \gamma(s) = 0 \\
\psi^2_p s := -24R s^2_1 + 48R s_1 s_2 + \hat{\alpha}(s)s_1 + \hat{\beta}(s)s_2 + \hat{\gamma}(s) = 0.
\end{align*}
\]

where \(\alpha, \beta, \hat{\alpha}, \hat{\beta}\) are polynomials in \(s\) of degree 2 with coefficients \(R\) and its derivatives up to order 2, \(\gamma\) and \(\hat{\gamma}\) are polynomials in \(s\) of degree 3 with coefficients \(R\) and its derivatives up to order 4. Their explicit expressions are given in Appendix.

**Proof.** The symbol of differential operator \(\varphi\) is \(\sigma_2(\varphi) : S^2 T^* \otimes E_2 \rightarrow K_2\) and its prolongation \(\sigma_3(\varphi) : S^3 T^* \otimes E_2 \rightarrow T^* \otimes K_2\) are given by:

\[
\sigma_2(\varphi)(A) = -24RA_{21} \quad \text{and} \quad \sigma_3(\varphi)(B)(e_i) = -24RB_{i21}, \quad i = 1, 2
\]

where \(A_{21} := A(e_2, e_1)\) and \(B_{i21} = B(e_i, e_2, e_1)\) are the components of the corresponding tensors with respect to the adapted basis \(\{e_1, e_2\}\).

Note, that we have \(g_2(P_\varphi) = g_2(P_2) \cap \text{Ker} \sigma_2(\varphi) = 0\), therefore for every \(\ell > 2\) we obtain that \(g_\ell(P_\varphi) = 0\), and so \(P_\varphi\) is involutive.
The kernel of the symbol of the first prolongation is $g_3(P_\varphi)$ defined by the system

$$
\begin{align*}
A_1^1 & := B_{122} - 2B_{112} = 0, \\
A_1^2 & := B_{222} - 2B_{122} = 0, \\
A_2^1 & := B_{111} - 2B_{112} = 0, \\
A_2^2 & := B_{112} - 2B_{122} = 0, \\
C_1 & := -24\mathcal{R}B_{112} = 0, \\
C_2 & := -24\mathcal{R}B_{122} = 0.
\end{align*}
$$

(11)

There are two relations in this system (11). Namely $24\mathcal{R}A_1 - 2C_1 + C_2 = 0$ and $24\mathcal{R}B_2 + C_1 - 2C_2 = 0$. So rank $\sigma_3(P_\varphi) = 4$, and

$$
\dim K_3 = \dim((T^* \otimes F_2) \oplus (T^* \otimes K_2) / \text{Im} \sigma_3) = 2.
$$

Moreover, if we define

$$
\tau_3 : (T^* \otimes F_2) \oplus (T^* \otimes K_2) \longrightarrow K_3 \cong \mathbb{C}^2,
$$

by:

$$
D^1 := 24\mathcal{R}A_1 - 2C_1 + C_2, \quad \text{and} \quad D^2 := 24\mathcal{R}B_2 + C_1 - 2C_2,
$$

then the sequence

$$
0 \longrightarrow S^3T^* \otimes E_2 \overset{\sigma_3}{\longrightarrow} (T^* \otimes F_2) \oplus (T^* \otimes K_2) \overset{\tau_3}{\longrightarrow} K_3 \longrightarrow 0
$$

is exact. We can deduce that a $2^{\text{nd}}$ order solution $(j_2s)_p$ of $P_\varphi$ can be lifted into a $3^{\text{rd}}$ order solution if and only if $[\tau_3 \nabla(P_\varphi(s))]_p = 0$. Let

$$
(\psi^1, \psi^2)_p := [\tau_3 \nabla(P_\varphi(s))]_p = \tau_3(\nabla P_2(s), \nabla \varphi)_p.
$$

We have:

$$
\psi^1 = 24\mathcal{R}[\nabla(P_2(s))]_1 + [\nabla(\varphi)]_2 - 2[\nabla(\varphi)]_1
$$

and

$$
\psi^2 = 24\mathcal{R}[\nabla(P_2(s))]_2 + [\nabla(\varphi)]_1 - 2[\nabla(\varphi)]_2.
$$

Using the equations $P_2(s)_p = 0$ and $\varphi(s)_p = 0$ and the permutation formula (3), we find that $\psi^1$ and $\psi^2$ can be written as a function of $s$ and its derivatives up to order 3. Nevertheless, using the formula (3), we can also eliminate the $3^{\text{rd}}$ order derivatives of $s$. On the other hand, with the help of the equation $P_2 = 0$ and $\varphi = 0$ we can express the $2^{\text{nd}}$ order derivatives of $s$ too with the $1^{\text{st}}$ order derivatives of $s$. The calculation carried out with MAPLE gives the formulas.

7. The linearization theorem

Since the compatibility conditions $\psi^1 = 0$ and $\psi^2 = 0$ found in the previous section are not identically satisfied, we have to introduce them into the system $P_\varphi$. We arrive at the system:

$$
P_\psi = (P_2, \varphi, \psi^1, \psi^2).
$$
Differentiating the equations \( \psi^1 = 0 \) and \( \psi^2 = 0 \) with respect to \( e_1 \) and \( e_2 \) we find 4 equations:

\[
\begin{align*}
\psi_1^1 &= 24R_1s_2^2 + (48R_2s_2 - 48R_1s_1 + \beta)s_1 + (\alpha - 48R_2)s_1s_2 + (\alpha_1s_1 + \beta_1s_2 + \gamma_1) \\
\psi_2^1 &= 24R_2s_2^2 + (48R_2s_2 - 48R_1s_1 + \beta)s_2 - 48R_2s_1s_2 + (\alpha - 48R_2)s_2 + \alpha_2s_1 + \beta_2s_2 + \gamma_2 \\
\psi_1^2 &= -24R_1s_1^2 + \alpha s_1 - 48R_1s_2 + 48R_1s_1s_2 + (48R_1 + \tilde{\beta})s_1s_2 + \alpha_1s_1 + \beta_1s_2 + \gamma_1 \\
\psi_2^2 &= -24R_2s_1^2 + (48R_2s_2 - 48R_1s_1 + \tilde{\beta})s_2s_1 + 48R_2s_1s_2 + (48R_1 + \tilde{\beta})s_2s_2 + \alpha_2s_1 + \beta_2s_2 + \gamma_2
\end{align*}
\]

In this expression, we can eliminate the second order derivatives using the equation \( P_2 = 0 \) and \( \varphi = 0 \), and with the help of the equation \( \psi^1 = 0 \) and \( \psi^2 = 0 \), we can express the terms \( s_1^2 \) and \( s_2^2 \) as a function of \( s_1 \), \( s_2 \) and the product \( s_1s_2 \). Therefore the system

\[
P_\psi = 0, \quad \nabla P_\psi = 0, \quad \nabla P_\psi = 0
\]

is equivalent to the system formed by the equation \( P_\psi = 0 \) and the four linear equations in \( s_1 \), \( s_2 \) and \( s_1s_2 \):

\[
S = \begin{cases}
a^1s_1 + b^1s_2 + c^1s_1s_2 = d^1, \\
a^2s_1 + b^2s_2 + c^2s_1s_2 = d^2, \\
a^3s_1 + b^3s_2 + c^3s_1s_2 = d^3, \\
a^4s_1 + b^4s_2 + c^4s_1s_2 = d^4,
\end{cases}
\]

where \( a^i, b^i, i = 1, ..., 4 \) are polynomials in \( s \) of degree 3, whose coefficients are \( R \) and its derivatives up to order 3, \( c^1 \) and \( c^4 \) are polynomials in \( s \) of degree 1 with coefficients \( R \), \( R_1 \) and \( R_2 \), \( c^2 \) and \( c^3 \) can be expressed as a function of \( R \), \( R_1 \) and \( R_2 \), and \( d^1 \) (resp. \( d^2 \) and \( d^3 \)) are polynomials in \( s \) of degree 5 (resp. 4), with coefficients \( R \) and its derivatives up to order 5. Its explicit expressions will be given in Appendix.

The direct computation by MAPLE shows us that the determinant

\[
\begin{vmatrix}
a^1 & b^1 & c^1 & d^1 \\
a^2 & b^2 & c^2 & d^2 \\
a^3 & b^3 & c^3 & d^3 \\
a^4 & b^4 & c^4 & d^4
\end{vmatrix}
\]

is identically null, so that the system \( S \) is compatible. On the other hand, the 3rd-order minors of the system \( S \) are polynomials in \( s \) of degree 7 which are not identically zero. There is an open dense \( U \subset \mathbb{C}^2 \) on which,

\[
D(s) = \begin{vmatrix}
a^1 & b^1 & c^1 \\
a^2 & b^2 & c^2 \\
a^3 & b^3 & c^3
\end{vmatrix} \neq 0.
\]

Solving on \( U \) the system \( S \) for \( s_1 \), \( s_2 \) and \( s_1s_2 \) we obtain:

\[
s_1 = F(s) = \frac{A(s)}{D(s)}, \quad s_2 = G(s) = \frac{B(s)}{D(s)}
\]

and

\[
s_1s_2 = H(s) = \frac{C(s)}{D(s)}.
\]
Finally, the equivalent linearizations.

A parallelizable 3-web on a 2-dimensional manifold

Theorem 7.2.

For a non parallelizable A non-parallelizable $\mathcal{U}$ set $\mathcal{Q}$ neighborhood of a point $\mathcal{P}$ where

and

Computing it explicitly we find that

$s_i(15)$

system must be in the algebraic manifold defined by

are of degrees 8, 8, and 11 respectively.

By $\|\|$ we must find $F(s) G(s) = H(s)$. Thus, the solution of $s$ for the linearization manifold system must be in the algebraic manifold defined by

(15)

$Q_1(s) := AB - CD = 0$.

On the other hand, the compatibility condition of the system $\|\|$ is $s_1 - s_2 = \mathcal{R}s$.

Computing it explicitly we find that $s$ must be in the algebraic manifold defined by

$Q_2(s) = 0$,

where $Q_2$ is polynomial in $s$ of degree 15. Indeed, if $A(s) = \sum_{i=1}^8 A_i s^i, B(s) = \sum_{i=1}^8 B_i s^i,$ and $D(s) = \sum_{i=1}^7 D_i s^i$ where $A_i, B_i$ and $C_i$ are function on $M$, then using $\|\|$ we obtain

$$Q_2(s) = \left( \sum_{i=1}^8 (\nabla_2 B_i s^i) \right) \left( \sum_{i=1}^7 D_i s^i \right) - \left( \sum_{i=1}^8 B_i s^i \right) \left( \sum_{i=1}^7 (\nabla_2 D_i) s^i \right)$$

$$- \left( \sum_{i=1}^8 (\nabla_1 A_i) s^i \right) \left( \sum_{i=1}^7 D_i s^i \right) - \left( \sum_{i=1}^8 A_i s^i \right) \left( \sum_{i=1}^7 (\nabla_1 D_i) s^i \right)$$

$$+ \left( \sum_{i=1}^8 B_i s^{i-1} \right) \left( \sum_{i=1}^8 A_i s^{i} \right) - \left( \sum_{i=1}^8 B_i s^{i} \right) \left( \sum_{i=1}^8 A_i s^{i-1} \right) - \mathcal{R}s D^2.$$}

Moreover, we must impose that $s_1$ and $s_2$ given by $\|\|$ verify the 5 equations of $P_\psi$, this implies 5 polynomial equations $Q_i = 0, i = 3, ..., 7$. Finally, we arrive at the conclusion that if the web is linearizable then $s$ must be in the algebraic manifold $\mathcal{A}$, where $\mathcal{A}$ is defined by the equations $Q_i = 0, i = 1, ..., 7$:

$$\mathcal{A} := \{Q_i = 0 | i = 1, ..., 7\}.$$

So the compatibility system (therefor the linearization system) has a solution in the neighborhood of a point $p \in M$ if and only if the algebraic variety $\mathcal{A}$ is not empty. If $\mathcal{A} \neq \emptyset$, then for all smooth point $s_0 \in \mathcal{A}$, there exists a neighborhood $U$ of $s_0$ so that all $s \in U$ can be prolonged in a germ $\tilde{s}$ as a basis of linearization. The explicit expression of the polynomials $Q_i$, can be computed with the help of MAPLE. The degree of these polynomials $Q_i, i = 1...7$ are 18, 15, 23, 23, 24, 17 and 17 respectively. One obtains the following results:

**Theorem 7.1.** A non-parallelizable 3-web $\mathcal{W}$ is linearizable if and only if there is an open set $U$ of $M$ on which the polynomials $Q_1, ..., Q_7$ have common zeros. Moreover, if this condition is satisfied, then for all $p \in U$ and all pre-linearization $L_0 \in E_p$ whose base is in $\mathcal{A} = \{Q_i = 0 | i = 1, ..., 7\}$, there exists a unique linearization $L$ so that $L_p = L_0$.

Since the lowest degree of the polynomials defining $\mathcal{A}$ is 15 we arrive at the

**Theorem 7.2.** For a non parallelizable 3-web, there exists at most 15 projectively non equivalent linearizations.

Finally, the Gronwall conjecture can be expressed now in the following way: for any non parallelizable 3-web on a 2-dimensional manifold

$$\text{deg}[\text{Rad}(Q_1, ..., Q_7)] = 1,$$
where $Rad$ denotes the radical of the corresponding polynomials and $deg$ is its degree.

**Examples**

1. Consider the web $\mathcal{W}$ defined by $x = cte$, $y = cte$, $f(x,y) = cte$, where $f(x,y) := (x+y)e^{-x}$.
   
   This web is not parallelizable in a neighborhood of $(0,0)$ because the Chern connection is not flat. Indeed, the component of the curvature tensor $\mathcal{R}$ at $(0,0)$ can be computed directly from the function $f$ by the formula
   
   $$\mathcal{R} = \frac{1}{f_x f_y} \left( \frac{f_{xxy}}{f_y} - \frac{f_{xyy}}{f_y} + \frac{f_{xy} f_{yy}}{f_y^2} - \frac{f_{xx} f_{yy}}{f_x^2} \right),$$

   (cf. [1], p. 24). In this example we have $\mathcal{R}(0,0) = -1$. The computation gives that $Rad(Q_1, \ldots, Q_7) = s + 1$ on a neighborhood of $(0,0)$. Thus the web is linearizable in a neighborhood of $(0,0)$ and all the linearizations are projectively equivalent.

2. Let $\mathcal{W}$ be the web defined by $x = cte$, $y = cte$, $f(x,y) = cte$, where

   $$f = \log(x) + \frac{1}{2} \log \left( \frac{x^2 + y^2}{x^2} \right) + \arctg \left( \frac{y}{x} \right).$$

   We have $\mathcal{R}(1,0) = 2$, so $\mathcal{W}$ is not parallelizable at $(1,0)$. On the other hand the resultant of the polynomials $Q_2, Q_6$ is not zero at $(1,0)$. So this web is not linearizable at $(1,0)$.

8. **Appendix**

   $$\alpha = 30 R s^2 - 18 R 2s - \frac{3}{4 R}(-16 R R_{22} + 14 R^2 - 40 R^3 - 56 R_1 R_2 + 40 R R_{12} + 56 R_1^2 - 40 R R_{11}),$$

   $$\beta = \frac{3}{4 R}(24 R R_2 - 24 R R_1) s - 15 R s^2 - \frac{3}{2 R}(70 R_1 R_2 - 44 R R_{12} + 20 R R_{11} + 20 R R_{22} - 28 R_1^2 - 28 R_2^2 - 60 R^3),$$

   $$\gamma = -\frac{3}{4 R}(6 R R_1 + 3 R^2) s^3 - \frac{3}{2 R}(7 R_1 R_{12} + 12 R R_{112} - 14 R_1 R_{12} - 7 R_2 R_{22} + 14 R_1 R_{11} - 8 R R_{111}),$$

   $$\hat{\gamma} = 4 R R_{22} + 47 R_2 R^3 + 14 R_1 R_{22} - 7 R_1 R_2 - 30 R_1 R^2 - 12 R R_{22} s - \frac{3}{4 R}(-7 R_2 R_{112} + 7 R_1 R_{22} + 35 R_1 R_2 R - 38 R_1^2 R - 2 R_2^2 R + 12 R R_{122} R - 8 R R_{112} R - 2 R_1^2 R - 48 R^2 R_{12} + 8 R^2 R_{22} + 8 R^4 - 14 R_1 R_{222} + 14 R_1 R_{112}).$$

   $$\hat{\alpha} = \frac{3}{4 R}(24 R R_2 - 24 R R_1) s^3 - 15 R s^2 - \frac{3}{2 R}(20 R R_{22} + 70 R_1 R_2 - 28 R_1^2 + 60 R^3 - 28 R_2^2 - 44 R R_{21} + 20 R R_{11}),$$

   $$\hat{\beta} = 30 R s^2 + 18 R s - \frac{3}{4 R}(40 R^3 + 16 R R_1 - 56 R_1 R_2 + 56 R_1^2 + 14 R_1^2 - 40 R R_{22} - 40 R R_{11}),$$

   $$\hat{\gamma} = -\frac{3}{4 R}(3 R R_1 - 6 R R_2) s^3 - \frac{3}{2 R}(-7 R_1 R_{22} - 14 R_2 R_1 + 12 R R_{22} + 14 R_1 R_2 + 7 R_1 R_{21} - 12 R R_{21} - 8 R R_{22} + 14 R_1 R_{22} - 7 R_1 R_1 + 4 R R_{11} + 3 R^2 R + 47 R_1 R^3) s - \frac{3}{4 R}(35 R_1 R_2 R - 7 R_1 R_{11} + 7 R_1 R_{22} + 8 R R_{221} + 4 R R_{211} - 12 R R_{211} R + 8 R^2 R_{22} - 2 R_2^2 R + 40 R^2 R_{11} - 48 R^2 R_{21} - 38 R_2^2 R - 8 R^4 + 14 R_2 R_{211} - 14 R_2 R_{221}),$$

   $$a^1 = \frac{363}{4} R_1 R_2 - \frac{259}{4} R^3 + \left( \frac{118}{4} R_2 - 72 R_1 \right) s^2 + \left( \frac{525}{8} R_2 - 102 R_1 \right) R + \frac{1}{4 R}\left( (-36 R_2 - 147 R_{11} + 114 R_{12}) R_1 + \left( \frac{2}{7} (\frac{3}{4} R_1^2 - \frac{102}{4} R_1 R_2 + 96 R_2^2) - \frac{102}{7} R_1^2 - \frac{27}{4} R_{11} - \frac{102}{7} R_{22} \right) R + 15 R_{12} - 33 R_{112} - 3 R_{222} + 36 R_{111} + \left( \frac{3}{4} R_2 + \frac{194}{117} R_{11} - \frac{17}{4} R_{12} \right) R_2 + \frac{1}{4 R} \left( \frac{3}{8} R_1^3 - \frac{22}{4} R_1 R_2^2 + \frac{27}{4} R_1 R_2^2 - \frac{27}{4} R_1^2 R_2 \right),\right)$$

   $$a^2 = -\frac{1}{8} R^3 s^2 + \left( \frac{1}{4} R_1 + \frac{1}{2} R_2 \right) R - 42 R_{122} + 12 R R_{222} + 2 R R_{12}, R (\frac{24 R_2 - \frac{194}{4} R_1 + \frac{194}{4} R_2}{R_2} R_1 + \frac{1}{8} R R_{12} R_2 - \frac{1}{8} R_1 R_2 R_2 - \frac{194}{4} R_1 R_2 + \frac{1}{8} R R_{12} R_2),$$

   $$a^3 = -\frac{1}{8} R^3 s^2 + (-\frac{1}{8} R_1 + \frac{1}{8} R_2) R + \frac{1}{8} R R_{12} R_2 + 48 R_{112} + 6 R_{222} - 30 R_{122} - 18 R_{111} + \frac{1}{8} R R_{12} R_2 - \frac{1}{8} R_1 R_2 R_2 - \frac{1}{8} R R_{12} R_2.$$
\[ b^1 = q_{144} R s^3 + (-63 R_2 + \frac{243}{4} R_1) s^2 + (120 R^2 + 156 R_{22} - 156 R_{12} + 57 R_{11} + \frac{1}{4}(-\frac{189}{2} R^2_1 + \frac{39}{4} R_1 R_2 - 219 R^2_2) s + (-99 R + \frac{279}{4} R_1) R + 39 R_{12} - 21 R_{22} - 15 R_{11} + \frac{1}{4}((-\frac{279}{2} R^2_1 + \frac{159}{4} R_2 - \frac{423}{8} R_1) R_1 + (60 R_{12} - 75 R_{11} - 27 R_{22}) R_2) + \frac{1}{8} R((-\frac{279}{2} R^2_1 + \frac{69}{2} R_2 R^2_2 - 307 R_1 R_2^2 + \frac{29}{2} R_2^3),
\]
\[ b^2 = \frac{1}{4} R s^3 + \frac{1}{4} R_1 R - 18 R_2 s^2 + \frac{1}{4} R^2 R^2 - 3 \frac{1}{4} R_1 R - 3 \frac{1}{2} R_{22} + 124 R_{12} + \frac{1}{4}((-\frac{279}{2} R^2_1 + \frac{423}{8} R_1 R_2 + \frac{279}{8} R_2^2)) s + \frac{1}{8} R (4 R_1 + \frac{279}{4} R_2) R + 48 R_{22} - 30 R_{12} - 18 R_{22} + 6 R_{11} + \frac{1}{4}((\frac{279}{2} R_1 + 3 \frac{279}{4} R_1 R_2 - \frac{33}{8} R_2^2) R_1 + (\frac{9}{2} R_1 - \frac{11}{2} R_2 + \frac{7}{2} R_{22}) R_2) + \frac{1}{8} R((-\frac{279}{2} R^2_1 + \frac{279}{2} R_1 R_2 + \frac{279}{8} R_2^2 - \frac{21}{2} R_2^3),
\]
\[ b^3 = \frac{1}{4} R s^3 + (-\frac{279}{4} R_2 + 18 R_1) s^2 + (-\frac{279}{8} R^2 + 3 R_1 + \frac{279}{4} R_2 - \frac{279}{8} R_1 R_2) s + (\frac{243}{4} R_2 + \frac{1}{4} R) R - 42 R_{22} + 42 R_{12} + 12 R_{11} + \frac{1}{4}((-\frac{279}{2} R_1 + \frac{279}{4} R_1 R_2 - \frac{33}{8} R_2^2) R_1 + (24 R_{11} - \frac{87}{2} R_1 R_2 - \frac{45}{2} R^2_2) R_2) + \frac{1}{8} R((-\frac{279}{2} R_1 - \frac{111}{4} R^2_2 R_2 - \frac{27}{4} R_1 R_2^2 + \frac{63}{8} R_2^3),
\]
\[ c^1 = (234 R s + 18 R_2 + 18 R_1)
\]
\[ c^2 = (36 R_1 - 18 R_2),
\]
\[ c^3 = (18 R_1 - 36 R_2)
\]
\[ d^1 = \frac{4 R^5}{3} s^5 + (-\frac{15}{4} R_1 + \frac{4 R_2}) s^4 + (\frac{4 R_1}{9} - \frac{9}{2} R_2 + \frac{1}{4}((\frac{279}{2} R_1 - \frac{33}{8} R_1 R_2 - \frac{339}{8} R_2^2)) s^3 + (-\frac{279}{8} R_2 - 411 R_1) R - \frac{45}{2} R_{12} + \frac{39}{4} R_{11} - 33 R_{22} - \frac{57}{8} R_{11} + \frac{1}{8}((-\frac{279}{2} R_1 + \frac{339}{8} R_1 R_2 + \frac{339}{8} R_2^2) R_1 + (-\frac{11}{2} R_{11} - \frac{11}{2} R_2 + \frac{1}{4} R_2) R_2)s^2 + (-\frac{279}{4} R_{22} - \frac{339}{8} R_{12} - \frac{211}{8} R_{11} - 36 R_{22} + 135 R_{12} + 6 R_{11} + \frac{1}{4}((-\frac{279}{2} R_1 + \frac{339}{8} R_1 R_2 + \frac{339}{8} R_2^2) R_1 - \frac{87}{2} R_2 - \frac{57}{8} R_2^2) R_1)
\]
\[ d^2 = (\frac{2}{3} R_1 - \frac{9}{16} R_2) s^2 + (-9 R^2 - \frac{9}{2} R_2 + 9 R_{12} + \frac{1}{4}((-\frac{279}{2} R_1 - \frac{339}{8} R_1 R_2 - \frac{339}{8} R_2^2)) R_1 + (\frac{9}{4} R_1 + \frac{9}{4} R_1 R_2)
\]
\[ d^3 = \frac{2}{3} R_1 R_2 + 15 R_{12} - 6 R_2 R_2 + \frac{1}{4} R_1 + \frac{1}{8} R_2 + \frac{1}{8} R_1 R_1 + \frac{1}{8} R((-\frac{279}{2} R_1 + \frac{339}{8} R_1 R_2 - \frac{339}{8} R_2^2) R_1 + (\frac{9}{4} R_1 + \frac{9}{4} R_1 R_2)
\]
\[ d^4 = \frac{2}{3} R_1 R_2 + 15 R_{12} - 6 R_2 R_2 + \frac{1}{4} R_1 + \frac{1}{8} R_2 + \frac{1}{8} R_1 R_1 + \frac{1}{8} R((-\frac{279}{2} R_1 + \frac{339}{8} R_1 R_2 - \frac{339}{8} R_2^2) R_1 + (\frac{9}{4} R_1 + \frac{9}{4} R_1 R_2)
\]
\(d^3 = \left(-\frac{9}{8} R_2 + \frac{9}{16} R_1\right) s^4 + \left(-9 R^2 - \frac{9}{2} R_{11} + 9 R_{12} + \frac{1}{16} \left(\frac{3}{8} R_1^2 - \frac{9}{16} R_1 R_2 - \frac{27}{8} R_2^2\right)\right)s^3 + \left(-\frac{273}{8} R_2 + \frac{265}{16} R_1\right) s^2 + \left(-\frac{36}{8} R_{11} R_{22} - \frac{6}{16} R_{12} + \frac{11}{4} R_{111}\right) s + \left(\frac{36}{8} R_1 R_{112} + \frac{3}{16} R_{1112}\right)\)