Cops and robber on subclasses of $P_5$-free graphs

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Abstract

The game of cops and robber is a turn based vertex pursuit game played on a connected graph between a team of cops and a single robber. The cops and the robber move alternately along the edges of the graph. We say the team of cops win the game if a cop and the robber are at the same vertex of the graph. The minimum number of cops required to win in each component of a graph is called the cop number of the graph. Sivaraman [Discrete Math. 342(2019), pp. 2306-2307] conjectured that for every $t \geq 5$, the cop number of a connected $P_t$-free graph is at most $t-3$, where $P_t$ denotes a path on $t$ vertices. Turcotte [Discrete Math. 345 (2022), pp. 112660] showed that the cop number of any $2K_2$-free graph is at most 2, which was earlier conjectured by Sivaraman and Testa. Note that if a connected graph is $2K_2$-free, then it is also $P_5$-free. Liu showed that the cop number of a connected $(P_t, H)$-free graph is at most $t-3$, where $H$ is a cycle of length at most $t$ or a claw. So the conjecture of Sivaraman is true for $(P_5, H)$-free graphs, where $H$ is a cycle of length at most 5 or a claw. In this paper, we show that the cop number of a connected $(P_5, H)$-free graph is at most 2, where $H \in \{C_4, C_5, \text{diamond, paw, } K_4, 2K_1 \cup K_2, K_3 \cup K_1, P_3 \cup P_1\}$.

Keywords: Cops and Robber; cop number; forbidden induced subgraphs; $P_5$-free graphs.

1 Introduction

All the graphs in this paper are finite, simple, and undirected. The complete graph, cycle, and path on $n$ vertices are denoted by $K_n, C_n$, and $P_n$, respectively. The disjoint union of two vertex-disjoint graphs $G$ and $H$, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a positive integer $r$, $rG$ denotes the disjoint union of $r$ copies of $G$. The game of cops and robber was introduced by Quilliot [19] in 1978 and independently by Nowakowski and Winkler [14] in 1981. The game is played on a connected graph by a team of cops and a robber.
In the first turn, all the cops are placed on the vertices of the graph (multiple cops can be placed on a single vertex). In the second turn, the robber chooses a vertex. Then the cops and the robber take their moves in alternative turns, starting with the cops. A valid move for a cop is to stay at its current position or to move to an adjacent vertex. A valid move for the robber is similar to the cops. A round of moves consists of two consecutive turns in which the cops have the first turn and then the robber has its turn. We say that a cop captures the robber if both of them are on the same vertex of the graph. The cops win if after a finite number of rounds, one of the cops captures the robber. The robber wins if the cops cannot win in a finite number of rounds. The cop number of a graph \( G \), denoted by \( \text{cop}(G) \), is defined as the minimum number of cops required such that the cops win in each component of \( G \).

The characterization of graphs with the cop number one was studied by Quilliot [19] and independently by Nowakowski and Winkler [14]. The study of graphs with higher cop number was initiated by Aigner and Fromme [1]. Several characterizations of graphs with cop number \( k \) have been studied by Clarke and MacGillivray [6]. Berarducci and Intrigila [4] gave an \( O(n^{O(k)}) \)-algorithm to decide whether \( \text{cop}(G) \leq k \) for a \( n \)-vertex graph \( G \). Fomin et al. [7] showed that it is NP-hard to determine the cop number of a graph. Moreover, it is \( W[2] \)-hard to determine whether \( \text{cop}(G) \leq k \), where \( k \) is the parameter [7]. Meyniel conjectured that the cop number of a connected graph on \( n \) vertices is at most \( O(\sqrt{n}) \). The conjecture was made by Meyniel in a personal communication with Frankl in 1985 and is mentioned in [8]. Frankl showed that \( \text{cop}(G) = o(n) \) for any connected \( n \)-vertex graph \( G \).

One can ask whether there exists \( \epsilon > 0 \) such that the cop number of any connected \( n \)-vertex graph \( G \) is \( O(n^{1-\epsilon}) \)? This problem is also open. For more details, see the survey by Baird and Bonato [3]. The best known upper bound on the cop number of a graph on \( n \) vertices is \( n2^{-(1+o(1))\sqrt{\log_2 n}} \) (see [12, 15]). Lu and Peng [12] showed that if \( G \) is a graph of diameter at most 2 or a bipartite graph of diameter at most 3, then \( \text{cop}(G) \leq 2\sqrt{n} - 1 \); thus validating the Meyniel’s conjecture for such graphs. Later, Wagner [20] improved the result of Lu and Peng on the same class of graphs by showing that such graphs satisfy \( \text{cop}(G) \leq \sqrt{2n} \). Lu and Peng used random arguments to prove the bound whereas Wagner’s proof did not include randomness.

The cop number of a family of graphs \( \mathcal{G} \) is the minimum integer \( k \) such that \( \text{cop}(G) \leq k \) for any graph \( G \) in \( \mathcal{G} \). If we cannot find such an integer \( k \), then we say \( \mathcal{G} \) is a family with unbounded cop number (or the cop number of \( \mathcal{G} \) is not bounded). Aigner and Fromme [1] proved that the cop number of a graph with girth \( \ell \) is at least its minimum degree, when \( \ell > 4 \). Thus the cop number of the class of all graphs is not bounded. Andreae [2] showed the cop number of the class of all \( d \)-regular graphs is not bounded. Aigner and Fromme [1] proved that the cop number of the class of all planar graph is 3. Moreover, dodecahedron is a planar graph whose cop number is 3. Clarke [5] proved that the cop number of any outerplanar graph is at most 2. The cop number of the class of all \( k \)-chordal graphs is at most \( k - 1 \) [10].

Let \( \mathcal{F} \) be a family of graphs. A graph \( G \) is said to be \( \mathcal{F} \)-free if no \( H \in \mathcal{F} \) is isomorphic to an induced subgraph of the graph \( G \). When \( \mathcal{F} = \{H\} \) (resp. \( \{H_1, H_2, \ldots, H_k\}, k \geq 2 \)), we use \( H \)-free (resp. \( (H_1, H_2, \ldots, H_k) \)-free) graphs to denote the \( \mathcal{F} \)-free graphs. Joret et al. [9] proved that the cop number of the class of \( H \)-free graphs is bounded if and only if every component of \( H \) is a path. Furthermore, suppose that \( \mathcal{G} \) be a family of graphs such that the diameter of any \( G \in \mathcal{G} \) is at most
Let $k$ for some natural number $k$. Masjoodi and Stacho [13] showed that the class of all $G$-free graphs has bounded cop number if and only if $G$ contains a path or $G$ contains a generalized claw and a generalized net. It is known that the cop number of a connected $P_t$-free graphs is at most $t - 2$ [9]. Sivaraman [16] gave a shorter proof for the same by using Gyárfás path argument, where he conjectured the following.

**Conjecture 1.1 ([16]).** The cop number of a connected $P_t$-free graph is at most $t - 3$ for $t \geq 5$.

Conjecture 1.1 was verified for a subclass of $P_t$-free graphs; the class of $(P_t,C_\ell)$-free graphs, where $t \geq 5$ and $3 \leq \ell \leq t$ [11]. It is known that the cop number of a $P_t$-free graph is at most 2 [9]. Note that if a graph is $2K_2$-free, then it is also $P_5$-free. Sivaraman and Testa [17] showed that the cop number of a connected $2K_2$-free graph $G$ is at most 2 if the diameter of $G$ is 3 or it is $C_\ell$-free for some $\ell \in \{3,4,5\}$. In the same paper, they conjectured that the class of $2K_2$-free graphs has cop number 2. Later, this conjecture was proved by Turcotte [18].

### 1.1 Our results

Liu [11] showed that the cop number of a connected $(P_t,H)$-free graph is at most $t - 3$, where $t \geq 5$ and $H$ is a cycle of length at most $t$ or a claw. First we give a short proof for these results for the case $t = 5$ by using Gyárfás path argument (see Lemma 2.1 of Section 2). Then by using that, we find the cop number of different subclasses of $P_5$-free graphs. In Section 3, we first show that the cop number of a connected $(P_5,\text{paw})$-free graph is at most 2. So if a $(P_5,K_4)$-free graph is also paw-free, then its cop number is at most 2. We study the structure of a connected $(P_5,K_4)$-free graph, that has an induced paw, around an induced paw. By using the structural properties, we show that the cop number of a connected $(P_5,K_4)$-free graph is at most 2 in Section 3. We show that the cop number of every connected $(P_5,K_3 \cup K_1)$-free graph and every connected $P_3 \cup P_1$-free graph is at most 2 in Section 4 and Section 5, respectively. In Section 6, we show that the cop number of every connected $(P_5,\text{diamond})$-free graph and every connected $(P_5,2K_1 \cup K_2)$-free graph is at most 2. We prove these results by showing the non-existence of any minimum counterexample. Again, $C_4$ and $C_5$ are $(P_4,P_3 \cup P_1,2K_2)$-free and $(P_5,K_4,\text{diamond},\text{paw},C_4,\text{claw},K_3 \cup K_1,2K_1 \cup K_2)$-free graphs, respectively with the cop number 2. Therefore, the cop number of the class of $(P_5,H)$-free graphs is 2, where $H$ is a graph on 4 vertices and has at least one edge.

### 2 Notations, terminologies, and preliminary results

Let $G$ be a graph and $x$ be a vertex of $G$. The **neighborhood** of $x$ in $G$, denoted by $N(x)$, is the set of all the neighbors of $x$ in $G$. The **closed neighborhood** of $x$ in $G$, denoted by $N[x]$, is the set $\{x\} \cup N(x)$. For a set $S \in V(G)$, we define $N(S) = \{v \in V(G) \setminus S \mid N(v) \cap S \neq \emptyset\}$ and $N[S] = S \cup N(S)$. A set $D \subseteq V(G)$ is called a **dominating set** of $G$ if $N[D] = V(G)$. For two disjoint sets of vertices $S$ and $T$, $[S,T]$ denotes the set $\{xy \in E(G) \mid x \in S, y \in T\}$. We say that $[S,T]$ is **complete** if every vertex in $S$ is adjacent to every vertex of $T$ in $G$. For a set of vertices $S$, $G[S]$ and
$G - S$ denote the subgraphs induced by $S$ and $V(G) \setminus S$ in $G$, respectively. The length of a path is the number of edges in it. The distance of a vertex $u$ from a set $S$ is the least length of $u - v$ paths for every vertex $v \in S$. We use Cop 1 and Cop 2 as the names of the two cops throughout the paper. We refer to Figure 1 for some special graphs mentioned in this paper.

Recall that in the game of cops and robber, a \textit{round} of moves consists of two consecutive turns in which the cops have the first turn and then the robber has its turn. Moreover, a cop captures the robber if they are at the same vertex of the graph. This happens only if in the robber’s turn, the robber stays or moves to a position that belongs to the closed neighborhood of the position of one of the cops. Equivalently, we can say that a cop at $x$ captures the robber at $y$ if after any round of moves of the cops and the robber, we have $y \in N[x]$.

Liu [11] proved that the cop number of any connected $(P_4, C_5)$-free graph is at most $t - 3$ for any natural number $t \geq 5$ and $\ell \leq t$. Moreover, it also has been shown that the cop number of a connected $(P_5, \text{claw})$-free graph is at most $t - 3$ for $t \geq 5$. To make the paper self-contained, we prove these results for $t = 5$.

\textbf{Lemma 2.1} ([11]). \textit{Let $G$ be a connected $(P_5, H)$-free graph, where $H \in \{C_3, C_4, C_5, \text{claw}\}$. Then cop($G$) $\leq 2$.}

\textit{Proof.} We prove the lemma by arguing that two cops capture the robber after a finite number of turns. Let $v_1$ be a vertex of $G$. In the first turn, we place both the cops at $v_1$. To avoid immediate capture, the robber must choose a vertex $y$ that is not a neighbor of $v_1$. Let $X$ be the component of the graph induced by $V(G) \setminus N[v_1]$ such that $y \in X$. Since $G$ is connected, we have $N(v_1) \cap N(X) \neq \emptyset$. In the next turn, Cop 1 stays at $v_1$ and Cop 2 moves to a vertex $v_2 \in N(v_1) \cap N(X)$. If the robber moves to a vertex of $N(v_1)$, then the cop at $v_1$ captures the robber. So to avoid capture, the robber should stay in $X$. Let the robber choose $x \in N[y] \cap X$ as its position. If $x$ is a neighbor of $v_2$, then the robber gets captured by Cop 2. So we may assume that $x$ is at distance at least 2 from $v_2$. Since $v_2$ has a neighbor in $X$ and $X$ is a component of the graph induced by $V(G) \setminus N[v_1]$, there exists an induced path $P$ between $v_2$ and $x$ of length at least 2 such that $V(P) \setminus \{v_2\} \subseteq V(X)$. This implies that $V(P) \cup \{v_1\}$ induces a path of length at least 3. Since $G$ is $P_5$-free, the length of the path induced by $V(P) \cup \{v_1\}$ is 3. This implies that $v_2$ is at distance 2 from $x$. Let $v_3 \in V(P) \cap X$ be a common neighbor of $v_2$ and $x$. Note that $\{v_1, v_2, v_3, x\}$ induces a $P_4$ in $G$. Define $S := \{u \in V(G) \mid N(u) \cap \{v_1, v_2, v_3, x\} = \{v_1, x\}\}$ and $T := \{u \in V(G) \setminus \{v_3\} \mid N(u) \cap \{v_1, v_2, v_3, x\} = \{v_2, x\}\}$.}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{special_graphs.png}
\caption{Some special graphs}
\end{figure}
We first prove that if \( S = \emptyset \) or \( T = \emptyset \), then \( \text{cop}(G) \leq 2 \). Suppose that \( S = \emptyset \). Then in the next turn, Cop 1 and Cop 2 move to \( v_2 \) and \( v_3 \), respectively. To avoid immediate capture, the robber must move. Suppose that the robber moves to a vertex \( r \). Since \( S = \emptyset \) and \( \{v_1, v_2, v_3, x, r\} \) does not induce a \( P_5 \), \( r \) is adjacent to \( v_2 \) or \( v_3 \). Hence the robber gets captured by Cop 1 or Cop 2 implying that \( \text{cop}(G) \leq 2 \). Now suppose that \( T = \emptyset \). In the next turn, Cop 1 stays at \( v_1 \) and Cop 2 moves to \( v_3 \), respectively. To avoid immediate capture, the robber must move to a vertex \( r \). Since \( T = \emptyset \) and \( \{v_1, v_2, v_3, x, r\} \) does not induce a \( P_5 \), \( r \) is adjacent to \( v_1 \) or \( v_3 \). Hence the robber gets captured by Cop 1 or Cop 2 implying that \( \text{cop}(G) \leq 2 \). Note that if \( H \in \{C_4, \text{claw}\} \), then \( T = \emptyset \). Again note that if \( H = C_5 \), then \( S = \emptyset \). So \( \text{cop}(G) \leq 2 \) if \( H \in \{C_4, C_5, \text{claw}\} \).

Now assume that \( S \neq \emptyset \) and \( T \neq \emptyset \). Then it is clear that \( H = C_3 \). Let \( s \) be a vertex of \( S \). In the next turn, Cop 1 stays at \( v_1 \) and Cop 2 moves to \( v_3 \). To avoid immediate capture, the robber must move to a vertex \( r \) that is not adjacent to \( v_1 \) and \( v_3 \). Note that \( r \in T \); otherwise \( \{v_1, v_2, v_3, x, r\} \) induces a \( P_5 \). Again, \( r \) is not adjacent to \( s \); otherwise \( \{s, x, r\} \) induces a \( C_3 \). In the next turn, Cop 1 moves to \( v_2 \) and Cop 2 stays at \( v_3 \). To avoid immediate capture, the robber must move to a vertex \( r' \) that is not adjacent to \( v_2 \) and \( v_3 \). Since \( \{r', x, r\} \) does not induce a \( C_3 \), \( r' \) is not adjacent to \( x \). Again since \( \{s, v_1, v_2, r, r'\} \) does not induce a \( P_5 \), \( r' \) is adjacent to \( s \) or \( v_1 \). If \( r' \) is adjacent to \( s \), then \( \{r', s, x, v_3, v_2\} \) induces a \( P_5 \) which is a contradiction. So \( r' \) is adjacent to \( v_1 \). Now \( \{r', v_1, v_2, x, v_3\} \) induces a \( P_5 \), again a contradiction. So such a vertex \( r' \) does not exist. Hence the robber cannot escape from \( r \) and gets captured implying that \( \text{cop}(G) \leq 2 \). \( \blacksquare \)

Note that by Lemma 2.1, every \((P_5, C_4)\)-free graph has the cop number at most 2. A more careful observation on the proof of Lemma 2.1 leads to a result on a superclass of the class of \((P_5, C_4)\)-free graphs. Note that in the proof of Lemma 2.1, if \( T \neq \emptyset \), then for any \( t \in T \), \( \{v_1, v_2, v_3, x, t\} \) induces a banner in \( G \). So the following corollary holds.

**Corollary 2.2.** Let \( G \) be a connected \((P_5, \text{banner})\)-free graph. Then \( \text{cop}(G) \leq 2 \).

We now conclude this section by describing a partition around an induced paw of a graph to use that later. The partition is described as follows.

- **Partition around an induced paw:** Let \( G \) be a graph and \( P \) induce a paw in \( G \) with vertex set \( P = \{v_1, v_2, v_3, v_4\} \) and edge set \( \{v_1v_2, v_2v_3, v_3v_4, v_2v_4\} \). Define the following sets.

\[
egin{align*}
A_i &:= \{v \in N(P) \mid N(v) \cap P = \{v_i\}\}, 1 \leq i \leq 4 \\
B_{ij} &:= \{v \in N(P) \mid N(v) \cap P = \{v_i, v_j\}\}, 1 \leq i < j \leq 4 \\
T_i &:= \{v \in N(P) \mid N(v) \cap P = P \setminus \{v_i\}\}, 1 \leq i \leq 4 \\
D &:= \{v \in N(P) \mid P \subseteq N(v)\} \\
X &:= V(G) \setminus (P \cup N(P))
\end{align*}
\]

Let \( A = \bigcup_{1 \leq i \leq 4} A_i, B = \bigcup_{1 \leq i < j \leq 4} B_{ij} \) and \( T = \bigcup_{1 \leq i \leq 4} T_i \). Note that \((A, B, D, P, T, X)\) is partition of \( V(G) \) when the structure of \( G \) is considered around an induced paw.
3 On the class of $(P_5, \text{paw})$-free graphs and $(P_5, K_4)$-free graphs

In this section, we first show that the cop number of a $(P_5, \text{paw})$-free graph is at most 2 (Lemma 3.1). Then we use this result to show that the cop number of any connected $(P_5, K_4)$-free graph $G$ is at most 2. The idea of the proof is as follows. If $G$ is paw-free, then by Lemma 3.1, $\text{cop}(G) \leq 2$. If $G$ contains an induced paw, then we consider the presence of an induced co-banner in $G$. If $G$ contains an induced co-banner, then we consider the partition of $V(G)$ as defined in $P_1$ around the paw contained in an induced co-banner of $G$. Then we have $A_1 \neq \emptyset$ which help us to prove that the robber gets captured by two cops (Lemma 3.4). On the other hand, if $G$ is co-banner-free, then we consider the presence of an induced butterfly in $G$. If $G$ contains an induced butterfly, then we consider the partition of $V(G)$ as defined in $P_1$ around a paw contained in an induced butterfly of $G$. Then we have $B_{12} \neq \emptyset$ which help us to prove that the robber gets captured by two cops (Lemma 3.5). If $G$ is butterfly-free, then we consider the partition of $V(G)$ as defined in $P_1$ around any induced paw of $G$. Since $G$ is (co-banner, butterfly)-free, we have $A_1 = B_{12} = \emptyset$. In fact, the intuition of the proof is to gradually prove that $A_1 = B_{12} = \emptyset$ while taking the stronger hypothesis that $G$ is co-banner-free and butterfly-free. Finally we show that the robber gets captured by two cops (Lemma 3.8).

Lemma 3.1. Let $G$ be a connected $(P_5, \text{paw})$-free graph. Then $\text{cop}(G) \leq 2$.

Proof. If $G$ is $C_3$-free, then by Lemma 2.1, $\text{cop}(G) \leq 2$. So we may assume that $G$ contains a $C_3$, say with vertex set $K = \{u_1, u_2, u_3\}$. If for any vertex $x \in N(K)$, $|N(x) \cap K| = 1$, then $K \cup \{x\}$ induces a paw, a contradiction. So every vertex of $N(K)$ is adjacent to at least two vertices of $K$. Now we show that every vertex of $G$ has a neighbor in $K$. If possible, then let $y$ be a vertex at distance 2 from $K$. Let $w$ be a neighbor of $y$ in $N(K)$. Now since $w \in N(K)$, $w$ is adjacent to at least two vertices of $K$. Without loss of generality, we may assume that $w$ is adjacent to $u_1$ and $u_2$. Then $\{u_1, u_2, w, y\}$ induces a paw, a contradiction. So we may conclude that such a vertex $y$ does not exist. Thus no vertex of $G$ is at distance 2 from $K$. Since $G$ is connected, every vertex of $G$ has a neighbor in $K$. Notice that $\{u_1, u_2\}$ is a dominating set of $G$. Therefore, $\text{cop}(G) \leq 2$. \hfill \Box

Let $G$ be a connected $(P_5, K_4)$-free graph containing an induced paw, say with vertex set $P = \{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_4\}$. Define sets $A_i, B_{ij}, T_i, D$, and $X$ around $P$ as defined in $P_1$ for every $1 \leq i, j \leq 4$ and $i < j$. Since $G$ is $K_4$-free, it can be noticed that $T_1 = D = \emptyset$. So we have the following observation.

Observation 3.2. $T_1 = D = \emptyset$.

Observation 3.3. The following hold.

(a) $N(X) \cap N(P) \subseteq A_2 \cup B_{12} \cup B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4$.
(b) For any vertex $v \in N(P)$ and any component $H$ of $G[X]$, either $[\{v\}, V(H)]$ is complete or $[\{v\}, V(H)] = \emptyset$.
(c) Every vertex in $X$ is at distance 2 from $P$. 

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Proof. (a) Let $u$ be an arbitrary vertex of $N(X) \cap N(P)$ and $x$ be a neighbor of $u$ in $X$. Since 
\{x, u, v_1, v_2, v_3\} does not induce a $P_5$, $u \notin A_1 \cup A_3 \cup B_{14} \cup B_{34}$. Again since 
\{x, u, v_4, v_2, v_1\} does not induce a $P_5$, we have $u \notin A_1 \cup B_{13}$. By Observation 3.2, $T_1 = D = \emptyset$. Therefore, $u \in A_2 \cup B_{12} \cup B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4$. Since $u$ is an arbitrary vertex of $N(X) \cap N(P)$, (a) holds.

(b) If $\{v, V(H)\} = \emptyset$, then we are done. Suppose that $\{v, V(H)\} \neq \emptyset$. Let $x$ be a neighbor of $v$ in $H$. First we show that $v$ is adjacent to every vertex of $N(x) \cap X$. For the sake of contradiction, let $x' \in N(x) \cap X$ such that $x'$ is not adjacent to $v$. Since $v \in N(X) \cap N(P)$, by (a), $v \in A_2 \cup B_{12} \cup B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4$. Note that there exist distinct vertices $v_i, v_j \in P$ for $i, j \in \{1, 2, 3, 4\}$ such that $vv_i, v_i v_j \in E(G)$ and $vv_j \notin E(G)$. Then $\{v_j, v_i, v, x, x'\}$ induces a $P_5$, a contradiction. So $v$ is adjacent to every vertex of $N(x) \cap X$. Now we show that $v$ is adjacent to every vertex of $H$. Let $y$ be an arbitrary vertex of $V(H) \setminus \{x\}$. If $y$ is adjacent to $x$, then, since $v$ is adjacent to every vertex of $N(x) \cap X$, $v$ is adjacent to $y$. If $y$ is not adjacent to $x$, then let $xx_1x_2 \ldots x_ky; k \geq 1$ be an induced path between $x$ and $y$ in $H$. By our argument, we have that $v$ is adjacent to $x_1$. Again by our argument for $x_1$, we have that $v$ is adjacent to $x_2$. Following this way, we conclude that $v$ is adjacent to $y$. Now since $y$ is an arbitrary vertex of $V(H) \setminus \{x\}$, $v$ is adjacent to every vertex of $H$. Thus (b) holds.

(c) Let $H$ be a component of $G[X]$. Since $G$ is connected, we have $N(V(H)) \cap N(P) \neq \emptyset$. Let 
$v \in N(V(H)) \cap N(P)$. By (b), every vertex of $H$ is adjacent to $v$. This implies that every vertex of 
$H$ is at distance 2 from $P$. Since $H$ is an arbitrary component of $G[X]$, (c) holds.

In the following lemma, we show that the cop number of a connected $(P_5, K_4)$-free graph that has 
an induced co-banner is at most 2.

Lemma 3.4. Let $G$ be a connected $(P_5, K_4)$-free graph. If $G$ has an induced co-banner, then 
cop$(G) \leq 2$.

Proof. Suppose that \{a_1, v_1, v_2, v_3, v_4\} induces a co-banner in $G$ with edge set \{a_1v_1, v_1v_2, v_2v_3, 
v_3v_4, v_4v_2\}. Note that $P = \{v_1, v_2, v_3, v_4\}$ induces a paw in $G$. Define the sets 
$A_i, B_{ij}, T_i, D$, and $X$ around $P$ as defined in $P_1$ for every $1 \leq i, j \leq 4$ and $i < j$. Note that 
a_1 \in A_1$. Again, $A_3 = \emptyset$; otherwise for any $u \in A_3$, either \{a_1, v_1, v_2, v_3, u\} or 
u, a_1, v_1, v_2, v_4\} induces a $P_5$. Similarly, we can show that 
$A_4 = \emptyset$. In the first turn, we place Cop 1 and Cop 2 at $v_1$ and $v_2$, respectively. To avoid 
immediate capture, the robber should choose a vertex $x$ that is not adjacent to $v_1$ and $v_2$. Since 
$A_3 = A_4 = \emptyset$, we have $x \in B_{34} \cup X$. To proceed further, we first prove a series of claims.

Claim 1. $N(X) \cap N(P) \subseteq A_2 \cup B_{12} \cup T_2 \cup T_3 \cup T_4$.

Proof of Claim 1. Let $u$ be an arbitrary vertex of $N(X) \cap N(P)$ and $w \in X$ be a neighbor of $u$. 
By Observation 3.3(a), $a_1$ has no neighbor in $X$. In particular, $a_1$ is not adjacent to $w$. Now if 
u \in B_{23} \cup B_{24}$, then either \{a_1, v_1, v_2, u, w\} or \{v_3, v_4, u, a_1, v_1\} induces a $P_5$ which is a contradiction. 
So $u \notin B_{23} \cup B_{24}$. Now by Observation 3.3(a), $u \in A_2 \cup B_{12} \cup T_2 \cup T_3 \cup T_4$. Since $u$ is an arbitrary vertex of $N(X) \cap N(P)$, we have $N(X) \cap N(P) \subseteq A_2 \cup B_{12} \cup T_2 \cup T_3 \cup T_4$. 
\qed
Claim 2. If $x \in X$ and $N(x) \cap T_2 = \emptyset$, then the robber gets captured.

Proof of Claim 2. Suppose that $x \in X$ and $N(x) \cap T_2 = \emptyset$. By Observation 3.3(c), $x$ is at distance 2 from $P$ and hence $x$ has a neighbor in $N(P)$, say $y$. Since $N(x) \cap T_2 = \emptyset$, we have $y \notin T_2$; thus $y$ is adjacent to $v_2$ by Claim 1. Moreover, since $y$ is arbitrary, any neighbor of $x$ in $N(P)$ is a neighbor $v_2$. In the next turn, Cop 1 moves to $v_2$ and Cop 2 moves to $y$. So if the robber moves to a vertex of $N(P)$, then it gets captured by Cop 1. Hence the robber should stay in $X$. Now by Observation 3.3(b), $y$ is adjacent to every vertex of $N[x] \cap X$. So the robber gets captured by Cop 2.

Claim 3. If $x \in X$, $N(x) \cap T_2 \neq \emptyset$, and every vertex of $N(x) \cap T_2$ has a non-neighbor in $N(x) \cap A_2$, then the robber gets captured.

Proof of Claim 3. Suppose that $x \in X$, $N(x) \cap T_2 \neq \emptyset$, and every vertex of $N(x) \cap T_2$ has a non-neighbor in $N(x) \cap A_2$. Recall that $a_1 \in A_1$. Since $x \in X$, by Claim 1, $a_1$ is not adjacent to $x$. In the next turn, Cop 1 and Cop 2 move to $a_1$ and $v_1$, respectively. The robber can stay in $X$ or move to a vertex $y \in N(x) \cap N(P)$ that is not adjacent to $a_1$ and $v_1$ if exists. Suppose that such a vertex $y$ exists. Since $y \in N(x) \cap N(P)$ and $y$ is not adjacent to $v_1$, by Claim 1, $y \in A_2$. Now since $a_1$ is not adjacent to $y$, $\{a_1, v_1, v_2, y, x\}$ induces a $P_5$, a contradiction. So such a vertex $y$ does not exist and hence the robber should stay in $X$. By Observation 3.3(b), without loss of generality, we may assume that it stays at $x$. In the next turn, Cop 1 stays at $a_1$ and Cop 2 moves to a vertex $c \in N(x) \cap T_2$. Such a vertex $c$ exists since $N(x) \cap T_2 \neq \emptyset$. To avoid immediate capture, the robber should move to a vertex $r \in X \cup N(P)$ that is not adjacent to $a_1$ and $c$. By Observation 3.3(b), $c$ is adjacent to every vertex of $N[x] \cap X$. So $r \notin X$ and hence $r \in N(P)$. Since $r$ is adjacent to $x$, $r \in N(x) \cap N(P)$. Then by Claim 1, $r \in A_2 \cup B_{12} \cup T_2 \cup T_3 \cup T_4$. Now we show that $a_1$ is adjacent to every vertex of $N(x) \cap T_2$. For the sake of contradiction, suppose that $w \in N(x) \cap T_2$ is not adjacent to $a_1$. By our assumption in the claim, $w$ has a non-neighbor in $N(x) \cap A_2$, say $a_2$. Recall that $a_1$ is not adjacent to $x$. Then either $\{a_1, a_2, x, w, v_3\}$ or $\{a_1, v_1, v_2, a_2, x\}$ induces a $P_5$, a contradiction. So $a_1$ is adjacent to every vertex of $N(x) \cap T_2$. In particular, $a_1$ is adjacent to $c$. Now since $a_1$ is adjacent to every vertex of $N(x) \cap T_2$ and not adjacent to $r$, we have $r \notin T_2$. So $r \in A_2 \cup B_{12} \cup T_3 \cup T_4$. Then $\{v_2, r, x, c, a_1\}$ induces a $P_5$, a contradiction. So such a vertex $r$ does not exist. Hence the robber cannot escape from $x$ and gets captured by Cop 2.

Claim 4. If $x \in X$ and there exists a vertex in $N(x) \cap T_2$ that is adjacent to every vertex of $N(x) \cap A_2$, then the robber gets captured.

Proof of Claim 4. Let $x \in X$ and $c$ be a vertex of $N(x) \cap T_2$ that is adjacent to every vertex of $N(x) \cap A_2$. In the next turn, Cop 1 moves to $c$ and Cop 2 moves to $v_1$. To avoid immediate capture, the robber should move to a vertex $r$ that is not adjacent to $c$ and $v_1$. By Observation 3.3(b), $c$ is adjacent to every vertex of $N(x) \cap X$. So $r \notin X$ and hence $r \in N(x) \cap N(P)$. Since $x \in X$, we have $r \in N(x) \cap N(P)$. Now by Claim 1, $r \in A_2 \cup B_{12} \cup T_2 \cup T_3 \cup T_4$. Again since $r$ is not adjacent to $v_1$, we have $r \in A_2$. This is a contradiction since $c$ is adjacent to every vertex of $N(x) \cap A_2$. So such a vertex $r$ does not exist. Hence the robber cannot escape from $x$ and gets captured by Cop 1.
Claim 5. If \( x \in B_{34} \) and \( B_{13} \neq \emptyset \) or \( B_{14} \neq \emptyset \), then the robber gets captured.

Proof of Claim 5. Suppose that \( x \in B_{34} \) and \( B_{13} \neq \emptyset \). Let \( b \) be any vertex of \( B_{13} \). Since \( \{b, v_1, v_2, v_4, x\} \) does not induce a \( P_5 \), \( b \) is adjacent to \( x \). Again, \( b \) is adjacent to \( a_1 \); otherwise \( \{v_3, v_4, b, v_1, a_1\} \) induces a \( P_5 \). First we show that \( B_{14} = \emptyset \). For the sake of contradiction, let \( b \) be any vertex of \( B_{14} \). Note that \( b' \) is a neighbor of \( a_1 \); otherwise \( \{v_3, v_4, b', v_1, a_1\} \) induces a \( P_5 \). If \( b \) is not adjacent to \( b' \), then \( \{b, a_1, b', v_4, v_2\} \) induces a \( P_5 \), a contradiction. So \( b \) is adjacent to \( b' \). Again since \( \{a_1, v_1, v_2, v_3, x\} \) does not induce a \( P_5 \), \( a_1 \) is adjacent to \( x \). Then either \( \{a_1, b, b', x\} \) induces a \( K_4 \) or \( \{b, v_1, v_2, v_3, x\} \) induces a \( P_5 \) which is a contradiction. Thus \( B_{14} = \emptyset \).

In the next turn, Cop 1 and Cop 2 moves to \( b \) and \( v_1 \), respectively. Recall that \( b \) is adjacent to \( x \). To avoid immediate capture, the robber must move to a vertex \( r \) that is not adjacent to \( v_1 \) and \( b \). By Observation 3.2, \( r \notin T_1 \). Since \( x \in B_{34} \) and \( r \) is a neighbor of \( x \), by Claim 1, \( r \notin X \). Recall that \( A_3 = \emptyset \) and \( A_4 = \emptyset \) and hence \( r \notin A_3 \cup A_4 \). Again since \( r \) is not adjacent to \( v_1 \) and \( b \), we have \( r \in \{v_3, v_4, x\} \cup A_2 \cup B_{23} \cup B_{24} \cup B_{34} \). Since \( r \notin A_2 \), and every vertex in \( A_2 \), \( r \notin \{v_4, v_3, x\} \cup B_{24} \). Recall that \( b \) is adjacent to \( a_1 \). If \( r \in A_2 \), then \( \{a_1, b, v_3, v_2, r\} \) or \( \{b, a_1, r, v_2, v_4\} \) induces a \( P_5 \), a contradiction. So \( r \notin A_2 \) and hence \( r \notin \{v_4, v_3, x\} \cup B_{23} \). In the next turn, Cop 1 moves to \( v_3 \) and Cop 2 moves to \( v_2 \). If \( r = v_4 \), then, since \( A_4 = B_{14} = \emptyset \), we have \( N[r] \subseteq N[v_2] \cup N[v_3] \) implying that the robber gets captured. So we may assume that \( r \in B_{23} \). Note that \( r \) does not have any neighbor in \( A_1 \); otherwise for any neighbor \( a \in A_1 \) of \( r \), \( \{v_1, a, r, v_3, v_4\} \) induces a \( P_5 \). Since \( A_4 = B_{14} = \emptyset \), to avoid immediate capture, the robber should move to a vertex \( r' \in X \). Since \( b \in B_{13} \), by Claim 1, \( r' \) is not adjacent to \( b \). Then \( \{r', r, v_2, v_1, b\} \) induces a \( P_5 \) which is a contradiction. So such a vertex \( r' \) does not exist. Hence the robber cannot escape from \( r \) and gets captured. Similarly, we can show that the robber gets captured if \( B_{14} \neq \emptyset \) (due to symmetry).

Claim 6. If \( x \in B_{34} \), \( B_{13} = B_{14} = \emptyset \), and every vertex in \( N(x) \cap A_2 \) has a non-neighbor in \( A_1 \), then the robber gets captured.

Proof of Claim 6. Suppose that \( x \in B_{34} \), \( B_{13} = B_{14} = \emptyset \), and every vertex in \( N(x) \cap A_2 \) has a non-neighbor in \( A_1 \). In the next turn, Cop 1 stays at \( v_1 \) and Cop 2 moves to \( v_3 \). To avoid immediate capture, the robber must move to a vertex \( r \) that is not adjacent to \( v_1 \) and \( v_3 \). Since \( x \in B_{34} \), by Claim 1, \( r \notin X \). Again since \( r \) is not adjacent to \( v_1 \) and \( v_3 \), we have \( r \in A_2 \cup A_4 \cup B_{24} \). Recall that \( A_3 = A_4 = \emptyset \). So \( r \notin A_4 \) and hence \( r \notin A_2 \cup B_{24} \).

First assume that \( r \in B_{24} \). In the next turn, Cop 1 moves to \( v_2 \) and Cop 2 stays at \( v_3 \). To avoid immediate capture, the robber must move to a vertex \( r' \) that is not adjacent to \( v_2 \) and \( v_3 \). Since \( A_4 = B_{14} = \emptyset \), we have \( r' \in A_1 \cup X \). Again since \( \{v_3, v_4, r, r', v_1\} \) does not induce a \( P_5 \), \( r' \notin A_1 \). So \( r' \in X \). By Claim 1, \( a_1 \) is not adjacent to \( r' \). Note that \( a_1 \) is not adjacent to \( r \); otherwise \( \{v_3, v_4, r, a_1, v_1\} \) induces a \( P_5 \). Then \( \{a_1, v_1, v_2, r, r'\} \) induces a \( P_5 \) which is a contradiction. So such a vertex \( r' \) does not exist. Hence the robber cannot escape from \( r \) and gets captured by Cop 1.

Now suppose that \( r \in A_2 \). Since \( r \in N(x) \cap A_2 \), due to our assumption in the claim, \( r \) has a non-neighbor in \( A_1 \), say \( a' \). Recall that Cop 1, Cop 2, and the robber are at \( v_1, v_3 \), and \( r \), respectively. In the next turn, Cop 1 moves to \( v_2 \) and Cop 2 stays at \( v_3 \). To avoid immediate capture, the robber
must move to a vertex \( r' \) that is not adjacent to \( v_2 \) and \( v_3 \). Since \( A_4 = B_{14} = \emptyset \), we have \( r' \in A_1 \cup X \).

Suppose that \( r' \in X \). Note that by Claim 1, \( a' \) is not adjacent to \( r' \). Then \( \{ a', v_1, v_2, r, r' \} \) induces a \( P_5 \) which is a contradiction. So we may assume that \( r' \in A_1 \). In the next turn, Cop 1 and Cop 2 move to \( r \) and \( v_2 \), respectively. To avoid immediate capture, robber must move to a vertex \( r'' \) that is not adjacent to \( r \) and \( v_2 \). Since \( r' \in A_1 \), by Claim 1, \( r'' \notin X \). Again since \( A_3 = A_4 = B_{13} = B_{14} = \emptyset \) and \( r'' \) is not adjacent to \( v_2 \), we have \( r'' \in A_1 \cup B_{34} \cup T_2 \). Note that \( r'' \notin A_1 \); otherwise \( \{ r'', r', r, v_2, v_3 \} \) induces a \( P_5 \). Recall that \( a' \in A_1 \) is a non-neighbor of \( r \). Since neither \( \{ a', v_1, v_2, v_3, r'' \} \) nor \( \{ r, v_2, v_1, a', r'' \} \) induces a \( P_5 \), we have \( r'' \notin B_{34} \). So \( r'' \notin T_2 \). Then either \( r'', x, v_3, v_4 \) induces a \( K_4 \) or \( \{ r, x, v_3, r'', v_1 \} \) induces a \( P_5 \) which is a contradiction. So such a vertex \( r'' \) does not exist. Hence the robber cannot escape from \( r' \) and gets captured by Cop 1.

We now return to the proof of Lemma 3.4. Recall that Cop 1, Cop 2, and the robber are at \( v_1, v_2 \), and \( x \in B_{34} \cup X \), respectively. First assume that \( x \in X \). If \( N(x) \cap T_2 = \emptyset \), then by Claim 2, the robber gets captured. So we may assume that \( N(x) \cap T_2 \neq \emptyset \). If every vertex of \( N(x) \cap T_2 \) has a non-neighbor in \( N(x) \cap A_2 \), then by Claim 3, the robber gets captured. So we may assume that there exists a vertex in \( N(x) \cap T_2 \) that is adjacent to every vertex of \( N(x) \cap A_2 \). Then by Claim 4, the robber gets captured.

Now assume that \( x \in B_{34} \). If \( B_{13} \neq \emptyset \) or \( B_{14} \neq \emptyset \), then by Claim 5, the robber gets captured. So assume that \( B_{13} = B_{14} = \emptyset \). Now if every vertex of \( N(x) \cap A_2 \) has a non-neighbor in \( A_1 \), then by Claim 6, the robber gets captured. Note that Claim 6 also includes the case \( N(x) \cap A_2 = \emptyset \). So we may assume that there exists a vertex \( a_2 \in N(x) \cap A_2 \) such that \( a_2 \) is adjacent to every vertex of \( A_1 \). Recall that the cops are at \( v_1 \) and \( v_2 \) whereas the robber is at \( x \in B_{34} \). In the next turn, Cop 1 and Cop 2 move to \( v_2 \) and \( a_2 \), respectively. Now if the robber does not move, then, since \( a_2 \) is adjacent to \( x \), it gets captured by Cop 2. To avoid immediate capture, it must move to a vertex \( r \) that is not adjacent to \( v_2 \) and \( a_2 \). Since \( x \in B_{34} \), by Claim 1, \( x \) does not have any neighbor in \( X \) implying that \( r \notin X \). Recall that \( A_3 = A_4 = B_{13} = B_{14} = \emptyset \) and \( a_2 \) is adjacent to every vertex of \( A_1 \). So \( r \in B_{34} \cup T_2 \). Then \( \{ x, r, v_3, v_4 \} \) induces a \( K_4 \) which is a contradiction. So such a vertex \( r \) does not exist. Hence the robber cannot escape from \( x \) and gets captured implying that \( \text{cop}(G) \leq 2 \). This completes the proof of Lemma 3.4.

In the following lemma, we show that the cop number of any connected \((P_5, K_4, \text{co-banner})\)-free graph that has an induced butterfly is at most 2. The idea of the proof is similar to the proof of Lemma 3.4.

**Lemma 3.5.** Let \( G \) be a connected \((P_5, K_4, \text{co-banner})\)-free graph that has an induced butterfly. Then \( \text{cop}(G) \leq 2 \).

**Proof.** Let \( \{ b, v_1, v_2, v_3, v_4 \} \) induce a butterfly in \( G \) with edge set \( \{ bv_1, v_1v_2, bv_2, v_2v_3, v_3v_4, v_4v_2 \} \). Note that \( P = \{ v_1, v_2, v_3, v_4 \} \) induces a paw in \( G \). Define the sets \( A_i, B_{ij}, T_i, D \), and \( X \) around \( P \) for every \( 1 \leq i, j \leq 4 \) and \( i < j \) as defined in \( P_1 \). Note that \( b \in B_{12} \). Again note that \( A_1 = \emptyset \); otherwise \( G[P \cup A_1] \) contains an induced co-banner. If \( B_{14} \neq \emptyset \), then for any \( b' \in B_{14}, \{ b, v_1, b', v_4, v_3 \} \) induces a \( P_5 \) or a co-banner, a contradiction. So \( B_{14} = \emptyset \). In the first turn, we place Cop 1 and Cop 2 at
Claim 7. If a vertex \( w \) has a neighbor in \( \{v_1\} \cup A_2 \cup B_{12} \), then \( w \) is adjacent to \( v_2 \) or \( v_3 \).

**Proof of Claim 7.** Let \( w \) be a vertex that is not adjacent to \( v_2 \) and \( v_3 \). Note that \( w \) is not a vertex of the butterfly induced by \( \{b, v_1, v_2, v_3, v_4\} \). To prove the claim, it is sufficient to show that \( w \) does not have any neighbor in \( \{v_1\} \cup A_2 \cup B_{12} \). For the sake of contradiction, suppose that \( w \) has a neighbor \( u \) in \( \{v_1\} \cup A_2 \cup B_{12} \). Note that \( w \) is adjacent to \( v_4 \); otherwise \( \{w, u, v_2, v_3, v_4\} \) induces a co-banner. If \( w \) is adjacent to \( v_1 \), then \( w \in B_{14} \), a contradiction to the fact that \( B_{14} = \emptyset \). So \( w \) is not adjacent to \( v_1 \). Recall that \( b \in B_{12} \). Note that \( w \) is not adjacent to \( b \); otherwise \( \{v_1, b, w, v_4, v_3\} \) induces a \( P_5 \). Now \( \{v_1, b, v_2, v_4, w\} \) induces a co-banner, a contradiction. Therefore, \( w \) does not have any neighbor in \( \{v_1\} \cup A_2 \cup B_{12} \). \( \square \)

Claim 8. \( N(P) \cap N(X) \subseteq N(b) \cap (B_{23} \cup B_{24} \cup T_2) \).

**Proof of Claim 8.** Let \( x \) be an arbitrary vertex of \( X \). By Observation 3.3(c), \( x \) is at distance 2 from \( P \) and hence \( N(x) \cap N(P) \neq \emptyset \). Let \( u \in N(x) \cap N(P) \). Since \( \{v_3, v_4, v_2, u, x\} \) does not induce a co-banner, \( u \notin A_2 \cup B_{12} \). Again since neither \( \{b, u, v_1, v_2\} \) induces a \( K_4 \) nor \( \{v_4, v_3, u, v_1, b\} \) induces a \( P_5 \), we have \( u \notin T_3 \cup T_4 \). Hence by Observation 3.3(a), \( u \in B_{23} \cup B_{24} \cup T_2 \) implying that \( N(x) \cap N(P) \subseteq B_{23} \cup B_{24} \cup T_2 \). In particular, \( b \) is not adjacent to \( x \). Since \( \{b, v_1, v_2, u, x\} \) does not induce a co-banner, \( b \) is a neighbor of \( u \). So we have \( N(x) \cap N(P) \subseteq N(b) \) and hence \( N(x) \cap N(P) \subseteq N(b) \cap (B_{23} \cup B_{24} \cup T_2) \). Since \( x \) is an arbitrary vertex of \( X \), we have \( N(P) \cap N(X) \subseteq N(b) \cap (B_{23} \cup B_{24} \cup T_2) \). \( \square \)

We now return to the proof of Lemma 3.5. First suppose that \( r \in \{v_1\} \cup A_2 \cup B_{12} \). In the next turn, Cop 1 stays at \( v_3 \) and Cop 2 moves to \( v_2 \). Now by Claim 7, every vertex of \( N[r] \) is adjacent to \( v_2 \) or \( v_3 \). So wherever the robber moves, it gets captured. Hence we may assume that \( r \in X \). By Observation 3.3(c) and Claim 8, there exists a vertex \( u \in N(P) \) that is a common neighbor of \( r \) and \( v_i \) for some \( i \in \{3, 4\} \). In the next turn, Cop 1 moves to \( v_2 \) and Cop 2 chooses \( v_i \) as its position. By Claim 8, every vertex of \( N(P) \cap N(r) \) is adjacent to \( v_2 \) or \( v_i \). So if the robber moves to \( N(P) \), then it gets captured and hence the robber should stay in \( X \). By Observation 3.3(b), without loss of generality, we may assume that the robber stays at \( r \). In the next turn, Cop 1 and Cop 2 move to \( b \) and \( u \), respectively. To avoid immediate capture, the robber must move to a vertex of \( X \cup N(P) \). By Observation 3.3(b), \( u \) is adjacent to every vertex of \( N[r] \cap X \). So if the robber stays in \( X \), then it gets captured by Cop 2. Hence the robber must move to a vertex of \( N(P) \). Now by Claim 8, \( b \) is adjacent to every vertex of \( N(X) \cap N(P) \). So the robber gets captured by the cop at \( b \). Therefore, \( \text{cop}(G) \leq 2 \). This completes the proof of Lemma 3.5. \( \square \)

In the following lemma, we prove that the robber gets captured in a connected \((P_5, K_4, \text{co-banner, butterfly})\)-free graph if at the end of a round, the two cops and the robber are at some specific vertices of the graph.

**Lemma 3.6.** Suppose that \( G \) is a connected \((P_5, K_4, \text{co-banner, butterfly})\)-free graph and \( P \) induces a paw in \( G \). If at the end of a round in the game of cops and robber, two cops are at two distinct
degree 2 vertices of the graph $G[P]$ and the robber is at a vertex at distance 2 from $P$, then the robber gets captured.

Proof. Let $P = \{v_1, v_2, v_3, v_4\}$ and the edge set of the paw induced by $P$ be $\{v_1v_2, v_2v_3, v_3v_4, v_4v_2\}$. Define the sets $A_i, B_i, T_i, D_i$ and $X$ around $P$ as defined in $P_1$ for every $1 \leq i, j \leq 4$ and $i < j$. We may assume that Cop 1 is at $v_3$, Cop 2 is at $v_4$, and the robber is at $r \in X$ at the end of a round in the game of cops and robber. Note that it is now cops’ turn to move. Again note that $A_1 = B_{12} = \emptyset$; otherwise $G[P \cup A_1]$ or $G[P \cup B_{12}]$ contains an induced co-banner or butterfly. To proceed further, we prove a series of claims.

**Claim 9.** $N(X) \cap N(P) \subseteq B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4$.

**Proof of Claim 9.** Let $y$ be an arbitrary vertex of $N(X) \cap N(P)$ and $x$ be a neighbor of $y$ in $X$. Since $B_{12} = \emptyset$, by Observation 3.3(a), $y \in A_2 \cup B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4$. Again since $\{v_3, v_4, v_2, y, x\}$ does not induce a co-banner, we have $y \notin A_2$. So $y \in B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4$. □

**Claim 10.** If $N(r) \cap B_{23} \neq \emptyset$, $T_3 \neq \emptyset$, and $T_4 \neq \emptyset$, then the robber gets captured.

**Proof of Claim 10.** Let $b \in N(r) \cap B_{23}, c \in T_3,$ and $c' \in T_4$. In the next turn, Cop 1 and Cop 2 move to $b$ and $v_3$, respectively. Since $b$ is adjacent to $r \in X$, by Observation 3.3(b), $b$ is adjacent to every vertex of $N[r] \cap X$. So if the robber stays in $X$, then it gets captured by Cop 1. So to avoid immediate capture, the robber must move to a vertex $r' \in N(r) \cap N(P)$ that is not adjacent to $b$ and $v_3$. Since $r' \in N(X) \cap N(P)$ and $r'$ is not adjacent to $v_3$, by Claim 9, $r' \in B_{24} \cup T_3$. Suppose that $r' \in B_{24}$. Since $\{b, v_3, v_2, c'\}$ does not induce a $K_4$, $b$ is not adjacent to $c'$. Again since $\{r', v_4, v_2, c\}$ and $\{c, v_1, v_2, c'\}$ do not induce any $K_4$, $c$ is not adjacent to $r'$ and $c'$. Note that $r'$ is adjacent to $c'$; otherwise $\{r', v_4, v_3, c', v_1\}$ induces a $P_5$. Similarly, $b$ is adjacent to $c$; otherwise $\{b, v_3, v_4, c, v_1\}$ induces a $P_5$. Then $\{b, c, v_1, c', r'\}$ induces a $P_5$ which is a contradiction to our assumption that $r' \in B_{24}$. So we have $r' \in T_3$. Now since $r'$ is not adjacent to $b$, $\{b, v_3, v_4, r', v_1\}$ induces a $P_5$, a contradiction. So such a vertex $r'$ does not exist. Hence the robber cannot escape from $r'$ and gets captured. □

**Claim 11.** If $T_4 = \emptyset$ and $N(r) \cap T_3 \neq \emptyset$, then the robber gets captured.

**Proof of Claim 11.** Let $T_4 = \emptyset$ and $c \in N(r) \cap T_3$. In the next turn, Cop 1 and Cop 2 move to $v_4$ and $c$, respectively. So if the robber stays in $X$, then it gets captured. So we may assume that the robber moves to a vertex $r' \in N(r) \cap N(P)$ that is not adjacent to $v_4$ and $c$. Now since $T_4 = \emptyset$ and $r'$ is not adjacent to $v_4$, by Claim 9, $r' \in B_{23}$. Then $\{v_1, c, v_4, v_3, r'\}$ induces a $P_5$, a contradiction. So such a vertex $r'$ does not exist. Hence the robber cannot escape from $r'$ and gets captured. □

**Claim 12.** If $N(r) \cap (T_3 \cup T_4) = \emptyset$, $N(r) \cap B_{23} \neq \emptyset$, $N(r) \cap B_{24} \neq \emptyset$, and $B_{13} \neq \emptyset$ or $B_{14} \neq \emptyset$, then the robber gets captured.
Proof of Claim 12. Let \( N(r) \cap (T_3 \cup T_4) = \emptyset \) and \( b \in N(r) \cap B_{23} \). First assume that \( B_{13} \neq \emptyset \). Let \( b_1 \in B_{13} \). In the next turn, Cop 1 stays at \( v_3 \) and Cop 2 moves to \( v_2 \). Note that by Claim 9, every vertex in \( N(r) \cap N(P) \) is adjacent to \( v_2 \) or \( v_3 \). So if the robber moves to a vertex of \( N(P) \), then it is captured. Hence the robber should stay in \( X \). By Observation 3.3(b), without loss of generality, we may assume that the robber stays at \( r \). In the next turn, Cop 1 and Cop 2 move to \( b \) and \( b_1 \), respectively. Since \( b \) is adjacent to \( r \in X \), by Observation 3.3(b), \( b \) is adjacent to every vertex of \( N[r] \cap X \). So if the robber stays in \( X \), then its gets captured by Cop 1. To avoid immediate capture, robber must move to a vertex \( r' \in N(r) \cap N(P) \) that is not adjacent to \( b \) and \( b_1 \). Since \( r' \in N(X) \cap N(P) \), by Claim 9, \( r' \in B_{23} \cup B_{24} \cup T_2 \cup T_3 \cup T_4 \). Since \( b_1 \in B_{13} \) and \( r \in X \), by Claim 9, \( b_1 \) is not adjacent to \( r \). Now since \( \{b_1, v_1, v_2, r', r\} \) does not induce a \( P_5 \), we have \( r' \notin B_{23} \cup B_{24} \). So \( r' \in T_2 \cup T_3 \cup T_4 \). Again since \( N(r) \cap (T_3 \cup T_4) = \emptyset \), we have \( r' \in T_2 \). Note that \( b \) is adjacent to \( b_1 \); otherwise \( \{b, v_1, v_2, b, r\} \) induces a \( P_5 \). Then \( \{b, b, r, r', v_4\} \) induces a \( P_5 \) which is a contradiction. So such a vertex \( r' \) does not exist. Hence the robber cannot escape from \( r \) and gets captured. Now if \( B_{14} \neq \emptyset \), then, since \( N(r) \cap B_{24} \neq \emptyset \), we may conclude that the robber gets captured (due to symmetry).

Claim 13. If \( T_4 = N(r) \cap T_3 = B_{14} = \emptyset \) and \( N(r) \cap B_{23} \neq \emptyset \), then the robber gets captured.

Proof of Claim 13. Let \( T_4 = N(r) \cap T_3 = B_{14} = \emptyset \) and \( b \in N(r) \cap B_{23} \). First assume that \( r \) has no neighbor in \( T_2 \). In the next turn, Cop 1 and Cop 2 move to \( b \) and \( v_2 \), respectively. Since \( b \) is adjacent to \( r \in X \), by Observation 3.3(b), \( b \) is adjacent to every vertex of \( N[r] \cap X \). So if the robber stays in \( X \), then its gets captured by Cop 1. So the robber must move to a vertex of \( N(r) \cap N(P) \). Now since \( r \in X \) and \( r \) has no neighbor in \( T_2 \), by Claim 9, the robber must move to a vertex of \( B_{23} \cup B_{24} \cup T_3 \cup T_4 \). Then the cop at \( v_2 \) captures the robber.

So we may assume that \( r \) has a neighbor in \( T_2 \), say \( c_1 \). In the next turn, Cop 1 moves to \( c_1 \) and Cop 2 stays at \( v_1 \). Since \( c_1 \) is adjacent to \( r \in X \), by Observation 3.3(b), \( c_1 \) is adjacent to every vertex of \( N[r] \cap X \). So if the robber stays in \( X \), then its gets captured by Cop 1. So to avoid immediate capture, the robber must move to vertex \( r' \in N(r) \cap N(P) \) that is not adjacent to \( c_1 \) and \( v_4 \). Since \( T_4 = \emptyset \) and \( r' \) is not adjacent to \( v_4 \), by Claim 9, we have \( r' \in B_{23} \). In the next turn, Cop 1 stays at \( c_1 \) and Cop 2 moves to \( v_3 \). Again to avoid immediate capture, the robber must move to a vertex \( r'' \) that is not adjacent to \( v_3 \) and \( c_1 \). Since \( A_1 = B_{12} = B_{14} = \emptyset \) and \( r'' \) is not adjacent to \( v_2 \), we have \( r'' \in A_2 \cup A_3 \cup B_{24} \cup T_3 \cup X \). If \( r'' \in X \), then \( \{r'', r', v_2, v_4, c_1\} \) induces a \( P_5 \), a contradiction. Thus \( r'' \notin X \). Since \( \{r'', v_2, r', v_1, c_1\} \) does not induce a co-banner, we have \( r'' \notin A_2 \cup B_{24} \). If \( r'' \in T_3 \), then by our assumption in the claim that \( N(r) \cap T_3 = \emptyset \), we conclude that \( r'' \) is not adjacent to \( r \). Now \( \{r'', v_2, v_4, c_1, r\} \) induces a co-banner, a contradiction. So \( r'' \notin T_3 \). Hence \( r'' \in A_4 \). Since \( r \in X \) and \( r'' \in A_4 \), by Claim 9, \( r'' \) is not adjacent to \( r \). Then \( \{r'', r', r, c_1, v_1\} \) induces a \( P_5 \) which is a contradiction. So such a vertex \( r'' \) does not exist. Hence the robber cannot escape from \( r' \) and gets captured.

Claim 14. If \( N(r) \cap B_{23} = N(r) \cap B_{24} = \emptyset \), then the robber gets captured.

Proof of Claim 14. Suppose that \( N(r) \cap B_{23} = N(r) \cap B_{24} = \emptyset \). So by Claim 9, \( N(r) \cap N(P) \subseteq T_2 \cup T_3 \cup T_4 \). In the next turn, Cop 1 moves to \( v_2 \) and Cop 2 stays at \( v_4 \). Note that every vertex
of $N(r) \cap N(P)$ is adjacent to $v_2$ or $v_4$. So to avoid capture, the robber must stay in $X$. By Observation 3.3(b), without loss of generality, we may assume that the robber stays at $r$. In the next turn, Cop 1 and Cop 2 move to $v_1$ and $v_2$, respectively. Since $N(r) \cap N(P) \subseteq T_2 \cup T_3 \cup T_4$, every vertex of $N(r) \cap N(P)$ is adjacent to $v_1$. So if the robber moves to a vertex of $N(P)$, then it gets captured by Cop 1 and hence the robber should stay in $X$. By Observation 3.3(b), without loss of generality, we may assume that the robber stays at $r$. Since $r$ is at distance 2 from $P$, $N(r) \cap N(P) \neq \emptyset$. Let $x \in N(r) \cap N(P)$. In the next turn, Cop 1 and Cop 2 move to $x$ and $v_1$, respectively. Since $x \in N(X) \cap N(P)$, by Observation 3.3(b), $x$ is adjacent to every vertex of $N[r] \cap X$. So if the robber stays in $X$, then Cop 1 captures it and hence the robber should move to a vertex of $N(r) \cap N(P)$. Then the robber gets captured by the cop at $v_1$ since every vertex of $N(r) \cap N(P)$ is adjacent to $v_1$. □

We now return to the proof of Lemma 3.6. First assume that $N(r) \cap B_{23} \neq \emptyset$ and $N(r) \cap B_{24} \neq \emptyset$. If $T_3 \neq \emptyset$ and $T_4 \neq \emptyset$, then by Claim 10, the robber gets captured. So we may assume that $T_3 = \emptyset$ or $T_4 = \emptyset$. Due to symmetry, we may assume that $T_4 = \emptyset$. Now if $N(r) \cap T_3 \neq \emptyset$, then by Claim 11, the robber gets captured. So we may further assume that $N(r) \cap T_3 = \emptyset$. Now if $B_{13} \neq \emptyset$ or $B_{14} \neq \emptyset$, then by Claim 12, the robber gets captured. So we may assume that $B_{13} = B_{14} = \emptyset$. Now by Claim 13, the robber gets captured. This implies that the robber gets captured if $N(r) \cap B_{23} \neq \emptyset$ and $N(r) \cap B_{24} \neq \emptyset$. So we may assume that $N(r) \cap B_{23} = \emptyset$ or $N(r) \cap B_{24} = \emptyset$. If $N(r) \cap B_{23} = \emptyset$ and $N(r) \cap B_{24} = \emptyset$, then by Claim 14, the robber gets captured. So we may assume that exactly one of the sets $N(r) \cap B_{23}$ and $N(r) \cap B_{24}$ is empty. Due to symmetry, we may further assume that $N(r) \cap B_{23} = \emptyset$ and $N(r) \cap B_{24} = \emptyset$. Let $b \in N(r) \cap B_{24}$. Recall that Cop 1, Cop 2, and the robber are at $v_3, v_4$, and $r \in X$, respectively. In the next turn, Cop 1 and Cop 2 move to $v_4$ and $b$, respectively. Since $b$ is adjacent to $r \in X$, by Observation 3.3(b), $b$ is adjacent to every vertex in $N[r] \cap X$. So if the robber stays in $X$, then it gets captured by Cop 2. To avoid immediate capture, the robber must move to a vertex $r' \in N(r) \cap N(P)$ that is not adjacent to $b$ and $v_4$. Since $N(r) \cap B_{23} = \emptyset$, $r' \notin B_{23}$. Again since $r'$ is adjacent to $r \in X$ and is not adjacent to $v_4$, by Claim 9, $r' \in T_4$. Then $\{b, v_4, v_3, r', v_1\}$ induces a $P_5$ which is a contradiction. So such a vertex $r'$ does not exist. Hence the robber cannot escape from $r$ and gets captured. This completes the proof of Lemma 3.6. □

In the following lemma, we show that if $G$ is a connected $(P_5, K_4, \text{co-banner, butterfly})$-free graph that has an induced kite and the two cops and the robber are at some specific vertices of $G$ at the end of a round, then the robber gets captured.

**Lemma 3.7.** Suppose that $G$ is a connected $(P_5, K_4, \text{co-banner, butterfly})$-free graph that has an induced kite. Let $P$ induce a paw in $G$ that is contained in an induced kite of $G$. If at the end of a round in the game of cops and robber, two cops are at two distinct degree 2 vertices of the graph $G[P]$ and the robber is at the degree 1 vertex of $G[P]$, then the robber gets captured.

**Proof.** Let $P = \{v_1, v_2, v_3, v_4\}$ and the edge set of the paw induced by $P$ be $\{v_1v_2, v_2v_3, v_3v_4, v_4v_2\}$. We may assume that Cop 1, Cop 2, and the robber are at $v_3, v_4$, and $v_1$, respectively at the end of a round in the game of cops and robber. Note that it is now cops’ turn to move. Define the sets $A_i, B_{ij}, T_i, D$, and $X$ around $P$ for every $1 \leq i, j \leq 4$ and $i < j$ as defined in $\mathcal{P}_1$. Since the
paw induced by \( P \) is contained in an induced kite of \( G \), we have \( B_{34} \neq \emptyset \). Let \( b \in B_{34} \). Note that \( A_1 = B_{12} = \emptyset \); otherwise for any \( u \in A_1 \cup B_{12}, P \cup \{u\} \) induces a co-banner or a butterfly in \( G \).

Claim 15. If \( A_3 \cup (B_{23} \setminus N(b)) \neq \emptyset \) or \( A_4 \cup (B_{24} \setminus N(b)) \neq \emptyset \) or \( [B_{34}, A_2] \neq \emptyset \), then the robber gets captured.

Proof of Claim 15. Let \( a \in A_3 \cup (B_{23} \setminus N(b)) \). First we show that every neighbor of \( v_1 \) is adjacent to \( v_4 \) or \( b \). For the sake of contradiction, let \( x \) be a neighbor of \( v_1 \) such that \( x \) is not adjacent to \( v_4 \) and \( b \). Clearly, \( x \neq v_2 \) and hence \( x \) is not a vertex of the kite induced by \( \{v_1, v_2, v_3, v_4, b\} \). Then \( \{b, v_4, v_2, v_1, x\} \) induces a \( P_5 \) or a co-banner, a contradiction. So every neighbor of \( v_1 \) is adjacent to \( v_4 \) or \( b \). In the next turn, \( \text{Cop 1} \) moves to \( b \) and \( \text{Cop 2} \) stays at \( v_4 \). Since every neighbor of \( v_1 \) is adjacent to \( v_4 \) or \( b \), to avoid immediate capture, the robber must stay at \( v_1 \). Now we show that \( a \) and \( b \) are not adjacent. If \( a \in A_3 \), then, since \( \{a, b, v_4, v_2, v_1\} \) does not induce a \( P_5 \), \( a \) and \( b \) are not adjacent. Again if \( a \in B_{23} \setminus N(b) \), then \( a \) is not adjacent to \( b \). Note that \( P' = \{a, v_3, b, v_4\} \) induces a paw in \( G \). At the end of this round, the cops are at \( b \) and \( v_4 \) that are degree 2 vertices of \( G[P'] \) and the robber is at \( v_1 \) that is at distance 2 from \( P' \). So by Lemma 3.6, the robber gets captured. Similarly, we can show that the robber gets captured if \( A_4 \cup (B_{24} \setminus N(b)) \neq \emptyset \).

Now suppose that \( [B_{34}, A_2] \neq \emptyset \). Let \( a \in A_2 \) and \( b' \in B_{34} \) such that \( a \) is adjacent to \( b' \). Note that \( P'' = \{a, b', v_3, v_4\} \) induces a paw. Again note that the cops are at \( v_3 \) and \( v_4 \) that are degree 2 vertices of \( G[P''] \) and the robber is at \( v_1 \) that is at distance 2 from \( P'' \). So by Lemma 3.6, the robber gets captured.

We now return to the proof of Lemma 3.7. If \( A_3 \cup (B_{23} \setminus N(b)) \neq \emptyset \) or \( A_4 \cup (B_{24} \setminus N(b)) \neq \emptyset \), then by Claim 15, the robber gets captured. So we may assume that \( A_3 = A_4 = \emptyset \) and \( b \) is adjacent to every vertex of \( B_{23} \cup B_{24} \). If \( [B_{34}, A_2] \neq \emptyset \), then by Claim 15, the robber gets captured. So we may assume that \( [B_{34}, A_2] = \emptyset \). In the next turn, \( \text{Cop 1} \) stays at \( v_3 \) and \( \text{Cop 2} \) moves to \( v_2 \). To avoid immediate capture, the robber must move to a vertex \( r' \) that is not adjacent to \( v_3 \) and \( v_2 \). Since \( A_1 = \emptyset \), we have \( r' \in B_{14} \). Note that \( b \) is adjacent to every vertex of \( B_{14} \); otherwise for any non-neighbor \( y \in B_{14} \) of \( b \), \( \{b, v_3, v_2, v_1, y\} \) induces a \( P_5 \). In particular, \( b \) is adjacent to \( r' \). In the next turn, \( \text{Cop 1} \) stays at \( v_3 \) and \( \text{Cop 2} \) moves to \( v_1 \). To avoid immediate capture, the robber must move to a vertex \( r'' \) that is not adjacent to \( v_3 \) and \( v_2 \). Since \( A_1 = \emptyset \), we have \( r'' \in B_{14} \) by Observation 3.3(a). \( r'' \notin X \). Since \( A_4 = \emptyset \), we have \( r'' \in A_2 \cup B_{24} \). If \( r'' \in B_{24} \), then, since \( b \) is adjacent to every vertex of \( B_{14} \cup B_{24} \), \( \{b, v_4, r', r''\} \) induces a \( K_4 \) in \( G \) which is a contradiction. So \( r'' \notin B_{24} \) and hence \( r'' \in A_2 \). In the next turn, \( \text{Cop 1} \) moves to \( v_2 \) and \( \text{Cop 2} \) stays at \( v_1 \). To avoid immediate capture, the robber must move to a vertex \( r_1 \) that is not adjacent to \( v_1 \) and \( v_2 \). Since \( A_3 = A_4 = \emptyset \) and \( [B_{34}, A_2] = \emptyset \), we have \( r_1 \in X \). Then \( \{v_3, v_4, v_2, r'', r_1\} \) induces a co-banner which is a contradiction. So such a vertex \( r_1 \) does not exist. Hence the robber cannot escape from \( r'' \) and gets captured. This completes the proof of Lemma 3.7.

We now show that the cop number of any connected \((P_5, K_4, \text{co-banner, butterfly})\)-free graph is at most 2.

Lemma 3.8. Let \( G \) be a connected \((P_5, K_4, \text{co-banner, butterfly})\)-free graph. Then \( \text{cop}(G) \leq 2 \).
Proof. If $G$ is paw-free, then by Lemma 3.1, $\text{cop}(G) \leq 2$. So we may assume that $G$ contains an induced paw, say with vertex set $P = \{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_4\}$. Define sets $A_i, B_j, T_i, D_i$ and $X$ around $P$ for every $1 \leq i, j \leq 4$ and $i < j$ as defined in $P_1$. Since $G$ is (co-banner, butterfly)-free, we have $A_1 = B_{12} = \emptyset$. In the first turn, we place Cop 1 and Cop 2 at $v_3$ and $v_4$, respectively. To avoid immediate capture, the robber must choose a vertex $r$ that is not adjacent to $v_3$ and $v_4$. Since $A_1 = B_{12} = \emptyset$ and $r$ is not adjacent to $v_3$ and $v_4$, we have $r \in \{v_1\} \cup A_2 \cup X$. If $r \in X$, then by Lemma 3.6, the robber gets captured. So we may assume that $r \in \{v_1\} \cup A_2$. Due to symmetry, we may further assume that $r = v_1$. Now if $B_{34} \neq \emptyset$, then by Lemma 3.7, the robber gets captured. So assume that $B_{34} = \emptyset$.

Claim 16. If $A_3 = \emptyset$ or $A_4 = \emptyset$, then the robber gets captured.

Proof of Claim 16. Due to symmetry, it is sufficient to show that the robber gets captured if $A_3 = \emptyset$. Let $A_3 = \emptyset$. In the next turn, Cop 1 stays at $v_3$ and Cop 2 moves to $v_2$. To avoid immediate capture, the robber should move to a vertex $r'$ that is not adjacent to $v_2$ and $v_3$. Since $A_1 = \emptyset$, we have $r' \in B_{14}$. In the next turn, Cop 1 and Cop 2 move to $v_2$ and $v_1$, respectively. To avoid immediate capture, the robber must move to a vertex $r''$ that is not adjacent to $v_1$ and $v_2$. Since $r'' \in B_{14}$, by Observation 3.3(a), $r'' \notin X$. Again since $B_{34} = A_3 = \emptyset$, we have $r'' \in A_4$. Then $\{r'', r', v_1, v_2, v_3\}$ induces a $P_5$ which is a contradiction. So such a vertex $r''$ does not exist. Hence the robber cannot escape from $r'$ and gets captured.

Claim 17. If there exists a vertex in $B_{13}$ that is adjacent to every vertex of $T_3$, then the robber gets captured.

Proof of Claim 17. Let there be a vertex $b^* \in B_{13}$ such that $b^*$ is adjacent to every vertex of $T_3$. If $A_3 = \emptyset$ or $A_4 = \emptyset$, then by Claim 16, the robber gets captured. So assume that $A_3 \neq \emptyset$ and $A_4 \neq \emptyset$. Let $a \in A_3$ and $a' \in A_4$. Recall that Cop 1, Cop 2, and the robber are at $v_3, v_4,$ and $v_1$, respectively. In the next turn, Cop 1 and Cop 2 move to $b^*$ and $v_3$, respectively. To avoid immediate capture, the robber must move to a vertex $r'$ that is not adjacent to $b^*$ and $v_3$. Since $A_1 = B_{12} = \emptyset$, we have $r' \in B_{14} \cup T_3$. Since $b^*$ is adjacent to every vertex of $T_3$, $r' \notin T_3$. Note that this includes the case $T_3 = \emptyset$. So $r' \in B_{14}$. Now since $\{a', a, v_3, v_2, v_1\}$ does not induce a $P_5$, $a$ is not adjacent to $a'$. Note that $b^*$ is not adjacent to $a$; otherwise $\{a, b^*, v_1, v_2, v_4\}$ induces a $P_3$. Again, $b^*$ is adjacent to $a'$; otherwise $\{a', v_4, v_3, b^*, v_1\}$ induces a $P_5$. Similarly, we can show that $r'$ is adjacent to $a$ and not adjacent to $a'$. Now since $r'$ is not adjacent to $b^*$, $\{a', b^*, v_1, r', a\}$ induces a $P_5$, a contradiction. So such a vertex $r'$ does not exist. Hence the robber cannot escape from $v_1$ and gets captured.

We now return to the proof of Lemma 3.8. If there exists a vertex in $B_{13}$ that is adjacent to every vertex of $T_3$, then by Claim 17, the robber gets captured. So we may assume that $B_{13} = \emptyset$ or every vertex of $B_{13}$ has a non-neighbor in $T_3$. Recall that Cop 1, Cop 2, and the robber are at $v_3, v_4,$ and $v_1$, respectively. In the next turn, Cop 1 moves to $v_3$ and Cop 2 stays at $v_4$. To avoid immediate capture, the robber must move to a vertex that is not adjacent to $v_2$ and $v_4$. Hence the robber must move to a vertex in $A_1 \cup B_{13}$. Recall that $A_1 = \emptyset$. Now if $B_{13} = \emptyset$, then the robber cannot escape from $v_1$ and gets captured. So assume that $B_{13} \neq \emptyset$ and the robber moves to a vertex $r_1 \in B_{13}$. Due
to our assumption, \( r_1 \) has a non-neighbor in \( T_3 \), say \( t \). Note that \( \{ t, v_2, v_4, v_3, r_1 \} \) induces a kite and the cops and the robber are at the positions such that the hypothesis of Lemma 3.7 holds. So by Lemma 3.7, the robber gets captured implying that \( \text{cop}(G) \leq 2 \). \(\)

By using Lemma 3.4-3.8, we now show that the cop number of any connected \((P_5, K_4)\)-free graph is at most 2.

**Theorem 3.9.** Let \( G \) be a connected \((P_5, K_4)\)-free graph. Then \( \text{cop}(G) \leq 2 \).

**Proof.** If \( G \) contains an induced co-banner, then by Lemma 3.4, \( \text{cop}(G) \leq 2 \). So we may assume that \( G \) is co-banner-free. Now if \( G \) contains an induced butterfly, then by Lemma 3.5, \( \text{cop}(G) \leq 2 \). So we may further assume that \( G \) is butterfly-free. Now since \( G \) is \((P_5, K_4, \text{co-banner, butterfly})\)-free, by Lemma 3.8, \( \text{cop}(G) \leq 2 \). \(\)

4 On the class of \((P_5, K_3 \cup K_1)\)-free graphs

Aigner and Fromme \cite{1} proved that for any natural number \( k \), there exists a \( C_3 \)-free graph with the cop number at least \( k \). Therefore, the class of \( K_3 \cup K_1 \)-free graphs also has unbounded cop number. However, the cop number of a connected \( K_3 \cup K_1 \)-free graph that contains a \( C_3 \), is at most 3 since every \( C_3 \) dominates the graph. In the following theorem, we show that if a connected \( K_3 \cup K_1 \)-free graph \( G \) is also \( P_5 \)-free, then \( \text{cop}(G) \leq 2 \).

**Theorem 4.1.** Let \( G \) be a connected \((P_5, K_3 \cup K_1)\)-free graph. Then \( \text{cop}(G) \leq 2 \).

**Proof.** If \( G \) is paw-free, then by Lemma 3.1, \( \text{cop}(G) \leq 2 \). So we may assume that \( G \) has an induced paw, say with vertex set \( P = \{ v_1, v_2, v_3, v_4 \} \) and edge set \( \{ v_1v_2, v_2v_3, v_3v_4, v_4v_2 \} \). Define the sets \( A_i, B_{ij}, T_i, D, \) and \( X \) around \( P \) as defined in \( \mathcal{P}_1 \) for every \( 1 \leq i, j \leq 4 \) and \( i < j \). Since \( G \) is \( K_3 \cup K_1 \)-free, every \( C_3 \) of \( G \) is a dominating cycle of \( G \). So there is no vertex at distance at least 2 from \( P \) and hence \( X = \emptyset \). Note that \( A_1 = \emptyset \); otherwise for any \( a \in A_1, \{ v_2, v_3, v_4, a \} \) induces a \( K_3 \cup K_1 \). Moreover, \( B_{34} = \emptyset \); otherwise for any \( b \in B_{34}, \{ v_1, v_3, v_4, b \} \) induces a \( K_3 \cup K_1 \). We divide the proof into the following two cases depending on whether \( B_{12} \) is empty or not. In each case, we show that the robber gets captured by two cops after a finite number of turns.

**Case 1:** \( B_{12} \neq \emptyset \).

Let \( b \) be a vertex of \( B_{12} \). If \( A_3 \neq \emptyset \), then for any \( a \in A_3 \), either \( \{ b, v_1, v_2, a \} \) induces a \( K_3 \cup K_1 \) or \( \{ v_4, v_3, a, b, v_1 \} \) induces a \( P_5 \), a contradiction. So \( A_3 = \emptyset \). Due to symmetry, we have \( A_4 = \emptyset \). Now if \( B_{14} \cup T_3 \neq \emptyset \), then for any \( u \in B_{14} \cup T_3 \), either \( \{ v_3, v_4, u, v_1, b \} \) induces a \( P_5 \) or \( \{ u, b, v_1, v_3 \} \) induces a \( K_3 \cup K_1 \), a contradiction. So \( B_{14} \cup T_3 = \emptyset \). In the first turn, we place Cop 1 and Cop 2 at \( v_3 \) and \( v_4 \), respectively. To avoid immediate capture, the robber should choose a vertex \( x \) that is not adjacent to \( v_3 \) and \( v_4 \). Since \( A_1 = X = \emptyset \), we have \( x \in A_2 \cup \{ v_1 \} \cup B_{12} \). In the next turn, Cop 1 stays at \( v_3 \) and Cop 2 moves to \( v_2 \). If the robber stays at \( x \), then it gets captured by Cop 2. Since
that contains $C$ is a complete graph. In particular, $x$ exists; otherwise we have $y$.

In the next turn, Cop 1 and Cop 2 move to $v_2$ and $v_1$, respectively. Note that $x$ does not have any neighbor in $G$; otherwise for any neighbor $y$ of $x$, $\{x, y, v_1, v_2, v_4\}$ induces a $P_5$ in $G$. So the robber cannot move to a vertex of $G$. Moreover, since $A_1 = X = \emptyset$, to avoid immediate capture, the robber should stay in $A_1$. At the end of this round, suppose that the robber is at $x' \in A_1$. Note that $x'$ may be equal to $x$. In the next turn, Cop 1 moves to $v_3$ and Cop 2 stays at $v_1$. To avoid immediate capture, the robber must move to a vertex $r$ that is not adjacent to $v_3$ and $v_1$. Since $A_1 = B_{12} = X = \emptyset$, we have $r \in A_2$. In the next turn, Cop 1 and Cop 2 move to $x'$ and $v_3$, respectively. To avoid immediate capture, the robber must move to a vertex $r'$ that is not adjacent to $v_3$ and $v_1$. Since $A_1 = B_{12} = X = \emptyset$, we have $r' \in A_3 \cup A_4 \cup B_{14} \cup B_{24} \cup T_3$. Since neither $\{r', r, v_3, v_4\}$ nor $\{x', v_3, v_4, r', v_1\}$ induces a $P_5$, we have $r' \notin A_2 \cup B_{14} \cup T_3$. Again since $\{r', v_2, v_4, x'\}$ does not induce a $K_3 \cup K_1$, we have $r' \notin B_{24}$ and hence $r' \in A_4$. In the next turn, Cop 1 and Cop 2 move to $r$ and $v_2$, respectively. To avoid immediate capture, the robber must move to a vertex $r''$ that is not adjacent to $r$ and $v_2$. Since $A_1 = B_{34} = X = \emptyset$ and $r''$ is not adjacent to $v_2$, we have $r'' \notin A_3 \cup A_4 \cup B_{13} \cup B_{14} \cup T_2$. Since $\{r'', r', r, v_2, v_1\}$ does not induce a $P_3$, we have $r'' \notin A_3 \cup A_4$. Again since $\{r'', v_3, v_4, r\}$ induces a $P_5$ or $\{r'', v_3, v_4, v_1\}$ induces a $K_3 \cup K_1$, we have $r'' \notin B_{14} \cup T_2$ and hence $r'' \in B_{13}$. Note that $x'$ is not adjacent to $r''$; otherwise $\{x', r'', v_1, v_2, v_4\}$ induces a $P_3$. Now $\{x', r, r', r'', v_1\}$ induces a $P_5$ which is a contradiction. So such a vertex $r''$ does not exist. Therefore, the robber cannot escape from $r'$ and gets captured by Cop 1.

5 On the class of $P_3 \cup P_1$-free graphs

Let $G$ be a connected $P_3 \cup P_1$-free graph. If $G$ has no induced $P_3$, then $G$ is isomorphic to a complete graph and hence the domination number of $G$ is 1. Suppose that $G$ has an induced $P_3$. Then, since $G$ is $P_3 \cup P_1$-free, the set of vertices of any induced $P_3$ of $G$ is a dominating set of $G$ and hence the domination number of $G$ is at most 3. This implies that the cop number of $G$ is at most 3. In the following theorem, we show that two cops are sufficient to capture the robber in any connected $P_3 \cup P_1$-free graph.

**Theorem 5.1.** Let $G$ be a connected $P_3 \cup P_1$-free graph. Then $\text{cop}(G) \leq 2$.

**Proof.** Let $v \in V(G)$. In the first turn, we place both Cop 1 and Cop 2 at $v$. To avoid immediate capture, the robber must choose a vertex $x$ of $G - N[v]$ if exists. We may assume that such a vertex $x$ exists; otherwise we have $V(G) = N[v]$ and hence $\text{cop}(G) = 1$. Let $C$ be the component of $G - N[v]$ that contains $x$. Since $G$ is $P_3 \cup P_1$-free, $G - N[v]$ is $P_3$-free and hence every component of $G - N[v]$ is a complete graph. In particular, $C$ is a complete graph. Since $G$ is connected, $N(C) \cap N(v) \neq \emptyset$. 

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Let \( r \in N(C) \cap N(v) \). Note that while one of the cops stays at \( v \), the robber cannot move to a vertex of \( G - V(C) \) in order to avoid immediate capture. So for the next few turns, Cop 1 stays at \( v \) and Cop 2 goes to a vertex of \( C \) through the vertex \( r \) and captures the robber.

6 On the class of \((P_5, \text{diamond})\)-free graphs and \(2K_1 \cup K_2\)-free graphs

Let \( H \) be an induced subgraph of a graph \( G \). A **retraction** from \( G \) to \( H \) is a homomorphism from \( G \) onto \( H \) that maps every vertex of \( H \) to itself. Formally, a retraction from \( G \) to \( H \) is a mapping \( \phi : V(G) \to V(H) \) such that: (1) \( \phi(u) = u \) for every \( u \in V(H) \) and (2) if \( xy \in E(G) \), then either \( \phi(x) = \phi(y) \) or \( \phi(x)\phi(y) \in E(H) \). We say that \( H \) is a retract of \( G \) if there exists a retraction from \( G \) to \( H \). Note that if \( G \) is a connected graph and \( H \) is a retract of \( G \), then \( H \) is connected.

In this section, we prove that the cop number of any \((P_5, \text{diamond})\)-free graph and any \((P_5, 2K_1 \cup K_2)\)-free graph is at most 2. We use the following lemma to prove these results. We note that the lemma is an implication of a result of Berarducci and Intrigila [4]. To make the paper self-contained, we give a proof of it.

**Lemma 6.1** ([4]). Let \( H \) be a retract of a connected graph \( G \) such that every component of \( G - V(H) \) is a complete graph. If \( \text{cop}(H) \leq 2 \), then \( \text{cop}(G) \leq 2 \).

**Proof.** Let \( \phi \) be a retraction from \( G \) to \( H \) and \( \text{cop}(H) \leq 2 \). We show that two cops can capture the robber in the graph \( G \) after a finite number of turns. At a turn, we say that the robber’s image is at a vertex \( u \) if the image of the robber’s position is \( u \) under \( \phi \).

In the first turn, we place both the cops at some vertex of \( H \). In the next turn, the robber chooses a vertex of \( G \) as its position. The cops first try to capture the robber’s image by playing in the graph \( H \). Note that since \( \phi \) is a homomorphism, for any move of the robber in \( G \), its image has a valid move in \( H \), that is it either moves to an adjacent vertex in \( H \) or stays at the same vertex of \( H \). Since \( H \) is connected and the cop number of \( H \) is at most 2, the cops capture the robber’s image after a finite number of turns. So without loss of generality, we may assume that one of the cops, say Cop 1, is at the robber’s image, the other cop is at any vertex of \( H \), and it is now robber’s turn to move. Since \( \phi \) is an identity mapping on \( H \), the robber is already captured if it is in \( H \). So we may assume that the robber is in \( G - V(H) \). Now Cop 1 follows the robber’s image, that is after every turn of cops’, Cop 1 is at the robber’s image. So to avoid immediate capture, the robber must stay in \( G - V(H) \). Hence the robber must move within a component of \( G - V(H) \), say \( C \). Then Cop 2 follows a path from its current position to a vertex of \( C \). After a finite number of turns, Cop 2 and the robber are in \( C \) and Cop 1 is at the robber’s image. Recall that every component of \( G - V(H) \) is a complete graph; in particular, \( C \) is a complete graph. So the robber gets captured by Cop 2 implying that \( \text{cop}(G) \leq 2 \). \( \square \)
6.1 \((P_5, \text{diamond})\)-free graphs

In the following theorem, we prove that the cop number of any connected \((P_5, \text{diamond})\)-free graph is at most 2.

**Theorem 6.1.1.** Let \(G\) be a connected \((P_5, \text{diamond})\)-free graph. Then \(\text{cop}(G) \leq 2\).

**Proof.** For the sake of contradiction, assume that there exists a counterexample of the theorem. Let \(G\) be a minimum counterexample of the theorem, that is \(G\) is a connected \((P_5, \text{diamond})\)-free graph with minimum number of vertices such that \(\text{cop}(G) > 2\). To proceed further, we first prove a series of claims.

**Claim 18.** If \(u\) is a vertex of \(G\), then every component of \(G[N(u)]\) is a complete graph. Moreover, \(G[N(u)]\) is a disconnected graph.

**Proof of Claim 18.** Let \(u\) be a vertex of \(G\). Since \(G\) is diamond-free, \(G[N(u)]\) is \(P_3\)-free and hence every component of \(G[N(u)]\) is a complete graph. Now for the sake of contradiction, assume that \(G[N(u)]\) is connected, that is it has only one component, say \(G'\). Note that for any \(w \in V(G')\), \(N[u] \subseteq N[w]\) since \(G'\) is a complete graph. Define a mapping \(\phi : V(G) \to V(G) \setminus \{u\}\) that maps \(u\) to \(w\) for some \(w \in V(G')\) and maps every vertex of \(G - \{u\}\) to itself. Note that \(\phi\) is a retraction from \(G\) to \(G - \{u\}\), that is \(G - \{u\}\) is a retract of \(G\). Moreover, since \(G\) is a minimum counterexample, \(\text{cop}(G - \{u\}) \leq 2\). So by taking \(H = G - \{u\}\) in Lemma 6.1, we have \(\text{cop}(G) \leq 2\), a contradiction. Hence \(G[N(u)]\) is a disconnected graph. \(\square\)

**Claim 19.** If \(u\) is a vertex of \(G\), then for any vertex \(v\) other than \(u\), \(N(v) \setminus N[u] \neq \emptyset\).

**Proof of Claim 19.** Let \(u\) be a vertex of \(G\). For the sake of contradiction, assume that there exists a vertex \(v\) other than \(u\) such that \(N(v) \setminus N[u] = \emptyset\), that is \(N(v) \subseteq N[u]\). Define a mapping \(\phi' : V(G) \to V(G) \setminus \{v\}\) that maps \(v\) to \(u\) and maps every vertex of \(G - \{v\}\) to itself. Note that \(\phi'\) is a retraction from \(G\) to \(G - \{v\}\), that is \(G - \{v\}\) is a retract of \(G\). Moreover, since \(G\) is a minimum counterexample, \(\text{cop}(G - \{v\}) \leq 2\). So by taking \(H = G - \{v\}\) in Lemma 6.1, we have \(\text{cop}(G) \leq 2\), a contradiction. Hence \(N(v) \setminus N[u] \neq \emptyset\). \(\square\)

**Claim 20.** If \(uv\) is an edge of \(G\), then there exist vertices \(y, z \in V(G) \setminus N[u]\) such that \(\{u, v, y, z\}\) induces a \(P_4\) in \(G\).

**Proof of Claim 20.** Let \(uv\) be an edge of \(G\). By Claim 19, \(N(v) \setminus N[u] \neq \emptyset\). Let \(G^*\) be a component of the graph induced by \(N(v) \setminus N[u]\). By Claim 18, every component of the graph \(G[N(v)]\) is a complete graph. Hence every component of the graph induced by \(N(v) \setminus N[u]\) is a complete graph; in particular, \(G^*\) is a complete graph.

To prove the claim, it is sufficient to show the existence of a vertex \(y \in V(G^*)\) that has a neighbor \(z\) such that \(z \notin N[\{u, v\}]\). For the sake of contradiction, assume that every neighbor of every vertex of \(V(G^*)\) is in the set \(N[\{u, v\}]\). Then \(N(V(G^*)) \subseteq N[\{u, v\}]\). Since \(G^*\) is a component of the graph
induced by \( N(v) \setminus N[u] \), we have \( N(V(G^*)) \subseteq N[u] \). Define a mapping \( \phi'': V(G) \rightarrow V(G) \setminus V(G^*) \) that maps every vertex of \( G^* \) to \( u \) and maps every vertex of \( G - V(G^*) \) to itself. Note that \( \phi'' \) is a retraction from \( G \) to \( G - V(G^*) \), that is \( G - V(G^*) \) is a retract of \( G \). Moreover, since \( G \) is a minimum counterexample, \( \text{cop}(G - V(G^*)) \leq 2 \). Since \( G^* \) is a complete graph, by taking \( H = G - V(G^*) \) in Lemma 6.1, we have \( \text{cop}(G) \leq 2 \), a contradiction. Hence the claim holds. \( \square \)

Now we return to the proof of Theorem 6.1.1. If \( G \) is \( K_4 \)-free, then by Theorem 3.9, \( \text{cop}(G) \leq 2 \), a contradiction. Hence \( G \) must contain a subgraph isomorphic to \( K_4 \). Let \( u \) be a vertex of any induced \( K_4 \) of \( G \). Then there exists a component \( G' \) of \( G[N(u)] \) such that \( |V(G')| \geq 3 \). By Claim 18, the graph \( G[N(u)] \) is disconnected and every component of \( G[N(u)] \) is a complete graph. So there exists a component \( G'' \) of \( G[N(u)] \) other than \( G' \) and both \( G' \) and \( G'' \) are complete graphs. Let \( v \) be a vertex of \( G'' \). By Claim 20, there exist vertices \( y, z \in V(G) \setminus N[u] \) such that \( \{u, v, y, z\} \) induces a \( P_4 \) in \( G \). Note that \( y \) has at most one neighbor in \( V(G') \); otherwise \( G[V(G') \cup \{y, u\}] \) contains an induced diamond. Similarly, we can show that \( z \) has at most one neighbor in \( V(G') \). Since \( G' \) and \( G'' \) are different components of \( G[N(u)] \) and \( v \) is a vertex of \( G'' \), \( v \) does not have any neighbor in \( V(G') \). Now since \( |V(G')| \geq 3 \), there exists a vertex \( x \in V(G') \) such that \( x \) is not adjacent to \( v, y, \) and \( z \). Then \( \{x, u, v, y, z\} \) induces a \( P_5 \) in \( G \), a contradiction. So we may conclude that such a graph \( G \) does not exist. This completes the proof of Theorem 6.1.1. \( \square \)

6.2 \( 2K_1 \cup K_2 \)-free graphs

Let \( G \) be a connected \( 2K_1 \cup K_2 \)-free graph and \( uv \) be an edge of \( G \). If \( \{u, v\} \) is a dominating set of \( G \), then \( \text{cop}(G) \leq 2 \). If \( \{u, v\} \) is not a dominating set of \( G \), then there exists a vertex \( z \) that is not adjacent to \( u \) and \( v \). Now since \( G \) is \( 2K_1 \cup K_2 \)-free, \( \{u, v, z\} \) is a dominating set of \( G \). So the cop number of \( G \) is at most 3. Therefore, the cop number of any \( 2K_1 \cup K_2 \)-free graph is at most 3. Turcotte [18] showed the existence of \( 2K_1 \cup K_2 \)-free graphs having the cop number 3 with computer aided graph search. So the cop number of the class of \( 2K_1 \cup K_2 \)-free graphs is 3. Note that any complete multipartite graph is \( K_1 \cup K_2 \)-free. Now consider a \( K_1 \cup K_2 \)-free graph \( G \). Since \( G^* \) is \( P_3 \)-free, \( G^* \) is a disjoint union of complete graphs. So \( G \) is a complete multipartite graph. Hence we may conclude that a graph \( G \) is \( K_1 \cup K_2 \)-free if and only if \( G \) is a complete multipartite graph.

**Theorem 6.2.1.** Let \( G \) be a connected \((P_5, 2K_1 \cup K_2)\)-free graph. Then \( \text{cop}(G) \leq 2 \).

**Proof.** For the sake of contradiction, assume that there exists a counterexample of the theorem. Let \( G \) be a minimum counterexample of the theorem, that is \( G \) is a connected \((P_5, 2K_1 \cup K_2)\)-free graph with minimum number of vertices such that \( \text{cop}(G) > 2 \). Let \( u \) be a vertex of \( G \) and \( G' \) be the graph \( G - N[u] \). Since \( G \) is \( 2K_1 \cup K_2 \)-free, \( G' \) is a \( K_1 \cup K_2 \)-free graph. Hence \( G' \) is a complete multipartite graph, say with \( k \) parts. We first show that \( k \geq 2 \). If possible, then let \( k \leq 1 \). If \( k = 0 \), then \( \{u\} \) is a dominating set of \( G \) implying that \( \text{cop}(G) = 1 \), a contradiction. So \( k = 1 \). Then \( G' \) consists of only isolated vertices. We show that two cops can capture the robber to obtain a contradiction to the fact that \( \text{cop}(G) > 2 \). We place both the cops at \( u \). The robber must choose a vertex of \( G' \) to avoid immediate capture. While one cop stays at \( u \), the robber cannot move since \( G' \) consists of only
isolated vertices. Then the second cop can go and capture the robber in the graph \( G' \). Hence we have \( k \geq 2 \). Now if there exists a partite set \( S \) of \( G' \) such that \( |S| = 1 \), then \( \{u\} \cup S \) is a dominating set of \( G \) implying that \( \text{cop}(G) \leq 2 \), a contradiction. So we may assume that every partite set of \( G' \) has cardinality at least 2.

**Claim 21.** If \( v \) is a neighbor of \( u \), then \( v \) has at most one non-neighbor in \( V(G') \).

**Proof of Claim 21.** Let \( v \) be a neighbor of \( u \). For the sake of contradiction, assume that \( v \) has two non-neighbors in \( V(G') \), say \( x \) and \( y \). Since \( \{u, v, x, y\} \) does not induce a \( 2K_1 \cup K_2 \) in \( G \), \( x \) and \( y \) are in different partite sets of the complete multipartite graph \( G' \). Recall that every partite set of \( G' \) has cardinality at least 2. Let \( x' \) be a vertex of \( G' \) other than \( x \) such that \( x \) and \( x' \) are in the same partite set of \( G' \). Note that \( v \) is adjacent to \( x' \); otherwise \( \{u, v, x, x'\} \) induces a \( 2K_1 \cup K_2 \) in \( G \). Since \( y \) and \( x' \) are in different partite sets of the complete multipartite graph \( G' \), \( y \) is adjacent to \( x' \). Similarly, \( y \) is adjacent to \( x \). Then \( \{u, v, x', y, x\} \) induces a \( P_5 \) in \( G \), a contradiction. So \( v \) has at most one non-neighbor in \( V(G') \). \( \Box \)

**Claim 22.** If \( z \) is a vertex of \( G' \), then \( z \) has a non-neighbor in \( N(u) \).

**Proof of Claim 22.** Let \( z \) be a vertex of \( G' \). For the sake of contradiction, assume that \( z \) is adjacent to every vertex of \( N(u) \), that is \( N(u) \subseteq N(z) \). Define a mapping \( \phi : V(G) \rightarrow V(G) \setminus \{u\} \) that maps \( u \) to \( z \) and maps every vertex of \( G \setminus \{u\} \) to itself. Note that \( \phi \) is a retraction from \( G \) to \( G \setminus \{u\} \). Moreover, since \( G \) is minimum counterexample, \( \text{cop}(G \setminus \{u\}) \leq 2 \). By taking \( H = G \setminus \{u\} \) in Lemma 6.1, we have \( \text{cop}(G) \leq 2 \), a contradiction. So \( z \) has non-neighbor in \( N(u) \). \( \Box \)

Now we show that two cops can capture the robber in \( G \). Let \( v \) be a neighbor of \( u \). In the first turn, we place Cop 1 at \( u \) and Cop 2 at \( v \). To avoid immediate capture, the robber must choose a vertex \( x \) that is not adjacent to \( u \) and \( v \). Since \( x \notin N[u], x \in G' \). Again since \( x \) is not adjacent to \( v \), by Claim 21, \( x \) is the only non-neighbor of \( v \) in \( V(G') \), that is \( v \) is adjacent to every vertex of \( V(G') \setminus \{x\} \). Since every partite set of \( G' \) has cardinality at least 2, there exists a vertex \( x' \) other than \( x \) such that \( x \) and \( x' \) are in the same partite set of \( G' \). By Claim 22, \( x' \) has a non-neighbor \( r \) in \( N(u) \). Again by Claim 21, \( r \) is adjacent to every vertex of \( G' \) except \( x' \). In particular, \( r \) is adjacent to \( x \).

In the next turn, Cop 1 moves to \( r \) and Cop 2 moves to \( u \). To avoid immediate capture, the robber must move to a vertex that is not adjacent to \( r \) and \( u \). Since \( x' \) is the only vertex that is not adjacent to \( r \) and \( u \), the robber must move to \( x' \). This is not possible since \( x \) and \( x' \) are in the same partite set of the multipartite graph \( G' \). So the robber cannot escape from \( x \) and gets captured implying that \( \text{cop}(G) \leq 2 \), a contradiction. So such a graph \( G \) does not exist. This completes the proof of Theorem 6.2.1. \( \Box \)
7 Conclusion

In this paper, we obtained strategies using two cops to capture the robber in a \((P_5, H)\)-free graph, where \(H \in \{C_4, C_5, \text{claw, diamond, paw, } K_4, 2K_1 \cup K_2, K_3 \cup K_1, P_3 \cup P_1\}\). On the other hand, \(P_4\)-free graphs and \(2K_2\)-free graphs are already known to have the cop number at most 2. By including these results, we conclude that the cop number of \((P_5, H)\)-free graphs is at most 2, where \(H\) is any graph on 4 vertices with at least one edge. Moreover, \(C_4\) and \(C_5\) have the cop numbers 2 and at least one of these belongs to the class of \((P_5, H)\)-free graphs. Thus the cop number of this class is 2. Note that Conjecture 1.1 remains open even for \(t = 5\). Even the question whether 2 cops are sufficient to capture the robber in a \(P_5\)-free graph with independence number at most 3, remains open. Although we have focused on the subclasses of \(P_5\)-free graphs, the methods we have used may be applied to obtain the cop number of some other graph classes with forbidden induced subgraphs.

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