Asymptotic Behavior of Minimal-Exploration Allocation Policies: Almost Sure, Arbitrarily Slow Growing Regret

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Abstract

Consider the problem of sampling sequentially from a finite number of $N \geq 2$ populations or ‘bandits,’ where each population $i$ is specified by a sequence of random variables $\{X_i^k\}_{k \geq 1}$, $X_i^k$ representing the reward received the $k^{th}$ time population $i$ is sampled. For each $i$, the $\{X_i^k\}_{k \geq 1}$ are taken to be i.i.d. random variables with finite mean. For any slowly increasing function $g$, subject to mild regularity constraints, we construct two policies (the $g$-Forcing, and the $g$-Inflated Sample Mean) that achieve a measure of regret of order $O\left(g(n)\right)$ almost surely as $n \to \infty$. Additionally, asymptotic probability one bounds on the remainder term are established. In the constructions herein, the function $g$ effectively controls the ‘exploration’ of the classical ‘exploration/exploitation’ tradeoff.

Other contributions of the paper include: i) the derivation of order $g$ lower and upper bounds on the regret for those policies, establishing the minimal value of the corresponding order constants for each policy, and ii) establishing bounds on the remainder terms of the regret, under the additional assumption that the law of iterative logarithm holds.

Keywords: Forcing Actions, Inflated Sample Means, Multi-armed Bandits, Sequential Allocation, UCB policies

1. Introduction and Summary

Consider the problem of sampling sequentially from a finite number of $N \geq 2$ populations or ‘bandits,’ where each population $i$ is specified by a sequence of real-valued i.i.d. random variables, $\{X_i^k\}_{k \geq 1}$, with $X_i^k$ representing the reward received the $k^{th}$ time population $i$ is sampled. In this paper, the distributions $F_i$ of the $X_i^k$ are taken to be unknown; they belong to some collection of distributions $\mathcal{F}$. We restrict these unknown distributions in the following way: each has finite - though unknown - mean $\mu_i = \mathbb{E}[X_i^k] = \int_{-\infty}^{+\infty} x dF_i(x) < \infty$. The purpose of this assumption is to establish for each population $i$ the Strong Law of Large Numbers (SLLN),

$$\mathbb{P}\left(\lim_{k} \bar{X}_i^k = \mu_i\right) = 1.$$  \hspace{1cm} (1)

However, it will turn out that the important distributional properties for the populations are not the i.i.d. structure, but rather Eq. (1) alone. This allows for some relaxation of assumptions, as discussed in Section...
Additionally, we define \( \mu^* = \max_i \mu^i \), and \( N^* \leq N \) to be the number of optimal bandits \( i^* \) such that \( \mu_{i^*} = \mu^* \). It is convenient to define the bandit discrepancies \( \{ \Delta_i \} \) as \( \Delta_i = \mu^* - \mu_i \geq 0 \).

For any adaptive policy \( \pi \), let \( \pi(t) = i \) indicate the event that population \( i \) is sampled at time \( t \), and let \( T^i_\pi(n) = \sum_{t=1}^n 1_{\pi(t)=i} \) denote the number of times \( i \) has been sampled during periods \( t = 1, 2, \ldots, n \), under policy \( \pi \). It is convenient to define \( T^i_\pi(0) = 0 \) for all \( i, \pi \). One is typically interested in maximizing asymptotically the expected value of the sum of the first \( n \) outcomes \( \mathbb{E}S_\pi(n) \) under an adaptive policy \( \pi \),

\[
S_\pi(n) = \sum_{i=1}^N \sum_{k=1}^{T^i_\pi(n)} X^i_k. \tag{2}
\]

To this end we note that if the controller had complete information (i.e., knew the distributions of the \( X^i_k \), \( \forall i \)), she would at every round activate some bandit \( i^* \) with \( \mu_{i^*} = \mu^* \). Three natural measures of the loss due to this ignorance of the distributions, are the quantities below:

\[
R'_\pi(n) = n\mu^* - S_\pi(n) = n\mu^* - \sum_{i=1}^N \sum_{k=1}^{T^i_\pi(n)} X^i_k, \tag{3}
\]

\[
\hat{R}_\pi(n) = n\mu^* - \sum_{i=1}^N \mu_i \left[ T^i_\pi(n) \right] = \sum_{i=1}^N \Delta_i T^i_\pi(n), \tag{4}
\]

\[
R_\pi(n) = n\mu^* - \mathbb{E}S_\pi(n) = \sum_{i=1}^N \Delta_i \mathbb{E} \left[ T^i_\pi(n) \right]. \tag{5}
\]

The functions \( \hat{R}_\pi(n), R'_\pi(n), R_\pi(n) \) have been called in the literature, pseudo-regret, sample regret and regret; for notational simplicity their dependence on the unknown distributions is usually suppressed. Note that:

\[
R_\pi(n) = \mathbb{E}\hat{R}_\pi(n) = \mathbb{E}R'_\pi(n).
\]

Further relationships and forms of pseudo-regret are explored in [Bubeck and Cesa-Bianchi (2012)]. We find the pseudo-regret in some sense more satisfying philosophically to consider than regret by itself, for the reason that, given her ignorance and the inherent randomness, the controller cannot reasonably regret the specific reward gained or lost from an activation of a bandit, as in \( R'_\pi(n) \). She can only reasonably regret the decision to activate that specific bandit, which is captured by \( \hat{R}_\pi(n) \)'s dependence on the \( T^i_\pi(n) \)s alone.

It follows from the above that to maximize \( \mathbb{E}S_\pi(n) \) one needs to minimize \( R_\pi(n) \). Asymptotically, we observe that \( \mathbb{E}S_\pi(n)/n \to \mu^* \), for some policy \( \pi \), if \( R_\pi(n)/n \to 0 \), or \( \mathbb{E}R'_\pi(n)/n \to 0 \), (or \( \mathbb{E}R'_\pi(n)/n \to 0 \)) as \( n \to \infty \).

[Robbins (1952)] first analyzed the problem of maximizing asymptotically the expected value of the sum \( \mathbb{E}S_\pi(n) \). Using only the assumption of the Strong Law of Large Numbers for \( \mathcal{F} \), for \( N = 2 \). Working with the pair \( R_\pi(n), R'_\pi(n) \), he constructed a modified (outside two sparse sequences of forced choices) ‘play the winner’ (greedy) policy, \( \pi_R \), such that with probability one, as \( n \to \infty \), \( S_{\pi_R}(n)/n \to \mu^* \), i.e.,

\[
R'_\pi_R(n) = o(n) \quad (a.s.), \quad \text{as } n \to \infty. \tag{6}
\]

From the above he was able to claim [for the case of Bernoulli bandits] that

\[
R_{\pi_R}(n) = o(n), \quad \text{as } n \to \infty. \tag{7}
\]

In [Lai and Robbins (1985)] the collection of distributions \( \mathcal{F} \) was taken to consist of univariate density functions \( f(x; \theta_i) \) with respect to some measure \( \nu_i \), where \( f(\cdot; \cdot) \) is known and the unknown scalar parameter \( \theta_i \),
is in some known set $\Theta$. Let $\mu_i = \mu(\theta_i) = \mathbb{E}X_i$, $\mu^* = \max\{\mu(\theta_i)\} = \mu(\theta^*)$, $\Delta_i(\theta_i) = \mu(\theta^*) - \mu(\theta_i)$, and let $\mathbb{I}(\theta||\theta') = \int_{-\infty}^{\infty} \ln \frac{f(x;\theta)}{f(x;\theta')} \, dx$ denote the Kullback–Leibler divergence between $f(x;\theta)$ and $f(x;\theta')$. They established, under mild regularity conditions ((1.6), (1.7) and (1.9) therein), that if one requires a policy to have a regret that increases at slower than linear rate:

$$R_\pi(n) = o(n^\alpha), \quad \forall \alpha > 0, \quad \forall \theta_i \in \Theta, \quad \forall i,$$  

then $\pi$ must sample among populations in such a way that its regret satisfies:

$$\liminf_{n \to \infty} \frac{R_\pi(n)}{\ln n} \geq M_{LR}(\theta_1, \ldots, \theta_N), \quad \forall \theta_i \in \Theta, \quad \forall i,$$  

where

$$M_{LR}(\theta_1, \ldots, \theta_N) = \sum_{i: \mu(\theta_i) \neq \mu^*} \Delta_i(\theta_i)/\mathbb{I}(\theta_i||\theta^*).$$

Subsequently, Burnetas and Katehakis (1996) extended this work for the case in which the collection of distribution $\mathcal{F}$ is specified by a known function $f(x;\theta_i)$ that may depend on an unknown vector parameter $\theta_i \in \Theta_i$, (even different populations may have $f(\cdot ; \cdot)$ of different type, e.g., some populations may be normal, some Poisson, etc.) as follows. Let $\bar{\theta}_i = (\bar{\theta}_i, \ldots, \bar{\theta}_N) \in \Theta_i = \Theta_1 \times \cdots \times \Theta_N$, $\mu^* = \mu(\bar{\theta}_i^*) = \max\{\mu(\theta)\}$, $\Delta_i(\bar{\theta}_i) = \mu^* - \mu(\bar{\theta}_i)$. In addition, let $O(\theta) := \{i : \mu(\theta_i) = \mu(\bar{\theta}_i)\}$ be the set of ‘optimal’ populations and let $B(\theta) := \{i : \mu(\theta_i) < \mu(\bar{\theta}_i)\}$. They showed, under certain regularity conditions (part 1 of Theorem 1, therein) that if a policy satisfied Eq. (8), $\forall \theta_i \in \Theta_i$, then it must sample among populations in such a way that its regret satisfies:

$$\liminf_{n \to \infty} \frac{R_\pi(n)}{\ln n} \geq M_{BK}(\bar{\theta}_i), \quad \forall \theta_i \in \Theta_i, \quad \forall i,$$  

where

$$M_{BK}(\bar{\theta}_i) = \sum_{i \in B(\theta)} \Delta_i(\bar{\theta}_i)/\inf\{\mathbb{I}(\bar{\theta}_i, \bar{\theta}_i') : \mu(\bar{\theta}_i') > \mu(\bar{\theta}_i^*)\}.$$  

Further, under certain regularity conditions regarding the estimates $\hat{\theta}_i = \hat{\theta}_i^n(X_i^1, \ldots, X_{T_i}^n)$ of the parameters $\theta_i$, $f(\cdot ; \cdot)$ and $\Theta_i$, they showed that policies which, after taking some small number of samples from each population, always choose the population $\pi^0(n)$ with the largest value of the population dependent index:

$$u_i(\hat{\theta}_i^n) = \sup_{\theta_i \in \Theta_i} \left\{ \mu(\theta_i) : \mathbb{I}(\hat{\theta}_i^n, \theta_i') < \frac{\ln n + o(\ln n)}{T_i^\pi(n)} \right\},$$

are asymptotically efficient (or optimal), i.e.,

$$\limsup_{n \to \infty} \frac{R_{\pi^0}(n)}{\ln n} \leq M_{BK}(\bar{\theta}_i, \ldots, \bar{\theta}_N), \quad \forall \theta_i \in \Theta_i, \quad \forall i.$$  

Policy $\pi^0$ above, was a simplification of a policy first used in Lai and Robbins (1985). Policies that satisfy the requirements of Eq. (7), Eq. (8), and Eq. (8) and Eq. (13) were respectively called uniformly consistent (UC), uniformly fast convergent (UF), and uniformly maximal convergence rate (UM) or simply asymptotically optimal (or asymptotically efficient). The lower bound of Eq. (11) provides a baseline for comparison of the quality of policies and together with Eq. (13) and Eq. (10) provide an alternative way to state the asymptotic optimality of a policy $\pi^0$ as:

$$R_{\pi^0}(n) = M_{BK}(\bar{\theta}_i, \ldots, \bar{\theta}_N) \ln n + o(\ln n), \quad \forall \theta_i \in \Theta_i, \quad \forall i.$$  

1. cf. conditions ‘A1-A3’ in Burnetas and Katehakis (1996).
Policies that achieve this minimal asymptotic growth rate have been derived in Lai and Robbins (1985), Burnetas and Katehakis (1996), Honda and Takemura (2011), Honda and Takemura (2010), Honda and Takemura (2013), Cowan et al. (2015), Cowan and Katehakis (2015a) and references therein. In general it is not always easy to obtain such optimal polices, thus, policies that satisfy the less strict requirement of Eq. (8), ∀θ ∈ Θ, have been constructed, cf. Auer et al. (2002), Audibert et al. (2009), Bubeck and Cesa-Bianchi (2012) and references therein. Such policies usually bound the regret as follows:

\[ R_\pi(n) \leq M^0(\theta) \ln n + M^1(\theta), \quad \text{for all } n \text{ and all } \theta \]

where \( M^0(\theta) \) is often much bigger than \( M_{BK}(\theta) \), for all \( \theta \). The above can also be stated as

\[ R_\pi(n) = O(\ln n), \quad \forall \theta \in \Theta, \text{ and } \forall i. \]

In this paper, we show that the approach of Robbins (1952), of using only the assumption of the Strong Law of Large Numbers for \( F \), to establish the result:

\[ R^'_\pi(n) = o(n), \quad \text{(a.s), as } n \to \infty, \]

can be extended to establish stronger results of the type, that there exist policies \( \pi_g \) such that

\[ \tilde{R}_\pi(n) = O(g(n)), \quad \text{(a.s), as } n \to \infty, \]

for any arbitrarily (slowly) increasing function \( g(n) \) that satisfy mild regularity conditions (such as \( g(n) = \ln \ln \ldots \ln n \)). This will be done with the construction of two \( \pi_g \) policies, \( \text{g-Forcing} \) and \( \text{g-ISM} \) index, both of which achieve (almost surely!) pseudo-regret that grows, asymptotically, at rate \( g \), for any choice of bandit distributions for which the Strong Law of Large Numbers holds.

Thus, we are particularly interested in high probability or guaranteed (almost sure) asymptotic bounds on the growth of the pseudo-regret as \( n \to \infty \). The motivation here is that while the investigator might be pleased to know that the policy she is utilizing has minimal expected regret, the actual regret (as a random variable) may fluctuate considerably around that expectation. Bounds in expectation are essentially in aggregate over the probability space, while the investigator might reasonably be more interested in the specific sample-path she is currently exploring. At an extreme end of this would be a result minimizing regret or pseudo-regret surely (sample-path-wise) or almost surely (with full probability), guaranteeing a sense of optimality independent of outcome. We offer an asymptotic result of this type here in Theorem 2.

This result is in some sense intuitive, in the following way: it will be shown that in the \( \text{g-Forcing} \) and \( \text{g-ISM} \) index policies, the function \( g \) essentially sets the investigator’s willingness to explore and experiment with bandits that do not currently (based on available data) seem to have the highest mean. Even if the controller explores very slowly (i.e., she chose a very slow growing \( g \)), as long as she explores long enough she will eventually develop accurate estimates of the means for each bandit, and achieve very little regret (or pseudo-regret) past that point. We note here that, for the most part, we do not recommend the actual implementation or use of these policies. The cost of this guaranteed asymptotic behavior is that (depending on \( g \) and the bandit specifics), slow pseudo-regret growth is only achieved on impractically large time-scales. We find it interesting, however, that such growth can be guaranteed - independent of the specifics of the bandits! - with as weak assumptions as the Strong Law of Large Numbers. This makes these results fairly broad. Additionally, the \( \text{g-Forcing} \) and \( \text{g-ISM} \) index policies individually capture elements present in many other popular policies, and are suggestive of the almost sure asymptotical behavior of these policies. One takeaway from this is, perhaps, to emphasize that asymptotic behavior by itself is little basis for thinking of a policy as ‘good’. As essentially any asymptotic behavior is possible (through the choice of \( g \)), any useful qualification of a policy must consider not only the asymptotic behavior, but also the timescales over which it is practically achieved.

In the remainder of the paper, we define what it means for a policy to be \( g \)-good (Definition 1), and establish the existence of \( g \)-good policies (Theorem 2) for any \( g \) satisfying mild regularity conditions. The proof is
by example, through the construction of g-Forcing and g-ISM index policies that satisfy its claim. Further, bounds on the corresponding order constants of pseudo-regret growth are established for each policy (Theorems 3 and 5), as well as bounds on the asymptotic remainder terms (Theorems 4 and 8), bounding the remainder from both above and below.

In the attempt to generalize some of these results for the g-ISM index policy, an interesting effect and seeming ‘phase change’ in the resulting dynamics was discovered. Specifically, as discussed in Remark 2, when there are multiple optimal bandits, for $g$ of order greater than $\sqrt{n \ln \ln n}$ all optimal bandits are sampled roughly equally often, while for $g$ of order less than $\sqrt{n \ln \ln n}$, the g-ISM index policy tends to fix on a single optimal bandit, sampling the other optimal bandits much more rarely in comparison.

2. Main Theorems

We characterize a policy by the rate of growth of its pseudo-regret function with $n$ in the following way.

**Definition 1** For a function $g(n)$, a policy $\pi$ is $g$-good if for every set of bandit distributions $\mathcal{F}$, there exists a constant $C_\pi(\mathcal{F}) < \infty$ such that

$$\limsup_n \frac{\hat{R}_\pi(n)}{g(n)} \leq C_\pi(\mathcal{F}) \quad (a.s) \quad \text{as } n \to \infty.$$  

(17)

**Remark 1:** Essentially, a policy is $g$-good if $\hat{R}_\pi(n) = O(g(n)) \quad (a.s), \quad n \to \infty$. Trivially, policies exist that are $n$-good (i.e., $\hat{R}_\pi(n) = O(n) \quad (a.s.)$, for example any policy that samples all populations at constant rate $1/N$.

We next state the following theorem:

**Theorem 2** For $g$, an unbounded, positive, increasing, concave, differentiable, sub-linear function, there exist $g$-good policies.

The proof of this theorem is given by example with Theorems 3, 5 which demonstrate two $g$-good policies: the g-Forcing and the g-ISM index policies.

We note that in the sequel it will be assumed that any $g$ considered is an unbounded, positive, increasing, concave, differentiable, sub-linear function.

2.1 A Class of g-Forcing Policies

Let $g$ be as hypothesized in Theorem 2. We define a g-Forcing policy $\pi_g^F$ in the following way:

**g-Forcing policy:** A policy $\pi_g^F$ that first samples each bandit once, then for $t \geq N$,

$$\pi_g^F(t + 1) = \begin{cases} \arg\max_i \bar{X}_{T_{\pi_g}^i}(t) & \text{if } \min_i T_{\pi_g}^i(t) \geq g(t), \\ \arg\min_i T_{\pi_g}^i(t) & \text{else.} \end{cases}$$  

(18)

Briefly, at any time, if any population has been sampled fewer than $g(t)$ times, sample it. Otherwise, sample from the population with the current highest sample mean. Ties are broken either uniformly at random, or at the discretion of the investigator. In this way, $g$ can be seen as determining the rate of exploration of currently
sub-optimal bandits. This can be viewed as a variant on the policy \( \pi_R \) considered in Robbins (1952), which always sampled the current best population, except on a sparse sequence of times.

It is convenient to define the following constant,

\[
S_\Delta = \sum_{i: \mu_i \neq \mu^*} \Delta_i.
\]  

(19)

The value \( S_\Delta \) in some sense represents the pseudo-regret incurred each time the sub-optimal bandits are all activated once. The next result states that \( g \)-Forcing policies satisfy the conditions of Theorem 2.

**Theorem 3**

For a policy \( \pi_g^F \) as in (18), \( \pi_g^F \) is \( g \)-good, and

\[
P \left( \lim_{n} \frac{\tilde{R}_g^F(n)}{g(n)} = S_\Delta \right) = 1.
\]  

(20)

**Proof** [Theorem 3] This theorem actually follows immediately from the following, much stronger statement, Theorem 4.

**Theorem 4**

For a policy \( \pi_g^F \) as in (18), the following is true: For every \( \epsilon > 0 \), almost surely there exists a \( N_\epsilon < \infty \) such that, for all \( n \geq N_\epsilon \),

\[
g(n)S_\Delta - \epsilon \leq \tilde{R}_g^F(n) \leq \lceil g(n) \rceil S_\Delta.
\]  

(21)

Theorem 4 is considerably stronger than Theorem 3 in that it precludes non-constant \( o(g(n)) \) fluctuations in \( \tilde{R}_g^F(n) \). However, it somewhat obscures the true nature of what is going on: for sufficiently large \( n \), almost surely, sub-optimal bandits \( (i : \mu_i \neq \mu^*) \) are only activated during the ‘forcing’ phase of the policy, when some activations are below \( g \). As a result, since \( g \) increases slowly (e.g. is sub-linear), for large \( n \), \( T_g^F(n) = \lceil g(n) \rceil \) - except for a discrepancy that occurs, for a brief stretch \( (< N) \) of activations, whenever \( g \) surpasses the next integer threshold. At this point, the policy raises the activations of each sub-optimal bandit, restoring the previous equality. Hence, in fact, equality holds in Theorem 4(\( \tilde{R}_g^F(n) = \lceil g(n) \rceil S_\Delta \)) for most large \( n \). Discrepancy occurs increasingly rarely with \( n \), based on the hypotheses on \( g \). If, additionally, the controller specifies a deterministic scheme for tie-breaking, pseudo-regret may be determined explicitly for all sufficiently large \( n \). Leaving ties to the discretion of the controller, Theorem 4 is as strong a statement as can be made. The proof of Theorem 4 is given in Appendix A.

### 2.2 A Class of g-Index Policies

In this section, we consider an index policy related to the classical ’UCB’ index policies. Let \( g \) be as hypothesized. For each \( i \), define an index on \((j, k) \in \mathbb{Z}_+^2\),

\[
u_i(j, k) = \bar{X}_i^j + \frac{g(j)}{k}.
\]  

(22)
**g-ISM index policy:** A policy \( \pi_g \) that first samples each bandit once, then for \( t \geq N \),

\[
\pi_g^O(t + 1) = \arg \max_i u_i(t, T^i_{\pi_g}(t)) = \arg \max_i \left( \bar{X}^i_{T^i_{\pi_g}(t)}(t) + \frac{g(t)}{T^i_{\pi_g}(t)} \right). \tag{23}
\]

Briefly, at any time, the sample means of each bandit are ‘inflated’ by the \( g(t)/T^i_{\pi_g}(t) \) term, and the policy always activates the bandit with the largest inflated sample mean. When unsampled, a bandit’s inflated sample mean increases essentially at rate \( g \), hence \( g \) drives the rate of exploration of current sub-optimal bandits. While this policy is inspired by more traditional ‘Upper Confidence Bound’ policies, we refer to this as an \( \text{ISM} \) policy: in short, at time \( n \), the optimal bandit is unique. Theorem 5 below shows that a \( g \)-ISM index policy satisfies the conditions of Theorem 2, and gives the minimal order constant \( C_{\pi_g} \) for this policy.

**Theorem 5** For a policy \( \pi_g^O \) as in (23), if \( N^* = 1 \),

\[
\mathbb{P} \left( \lim_{n \to \infty} \frac{R_{\pi_g^O}(n)}{g(n)} = N - 1 \right) = 1. \tag{24}
\]

The proof of this theorem depends on the following propositions, the proofs of which are given in Appendix B.

**Proposition 6** For each sub-optimal \( i \), \( \forall \epsilon \in (0, \Delta_i/2) \), \( \exists (a.s.) \) a finite constant \( C^i_{\epsilon} \) such that for \( n \geq N \),

\[
T^i_{\pi_g^O}(n) \leq \frac{g(n)}{\Delta_i - 2\epsilon} + C^i_{\epsilon}. \tag{25}
\]

**Proposition 7** If \( N^* = 1 \), for each sub-optimal \( i \neq \ast \), \( \forall \epsilon \in (0, \min_{j \neq i} \Delta_j/2) \), \( \exists (a.s.) \) some finite \( N' \) such that for \( n \geq N' \),

\[
\frac{g(n)}{(1 + \epsilon)(\Delta_i + 2\epsilon) + 2\epsilon} \leq T^i_{\pi_g^O}(n). \tag{26}
\]

**Proof** [Theorem 5] For each sub-optimal bandit \( i \), as an application of Props. 6, 7, taking the limit of \( T^i_{\pi_g^O}(n)/g(n) \) first as \( n \to \infty \), then as \( \epsilon \to 0 \), gives \( \lim_n T^i_{\pi_g^O}(n)/g(n) = 1/\Delta_i \), almost surely. The theorem then follows similarly, from the definition of pseudo-regret, Eq. (4).

**Remark 2:** In the case that \( N^* > 1 \), it happens that Prop. 6 still holds. It can be shown then that \( \pi_g^O \) remains \( g \)-good in this case, and has a limiting order constant of at most \( N - N^* \). We leave as an open question, however, that of producing a Prop. 7-type lower bound and the verification of \( N = N^* \) as the minimal order constant. The proof of Prop. 7 for \( N^* = 1 \) depends on establishing a lower bound on the activations of the unique optimal bandit: in short, at time \( n \), since the sub-optimal bandits are activated at most \( O(g(n)) \) times (which holds independent of \( N^* \)), it follows from its uniqueness that the optimal bandit is activated at least \( n - O(g(n)) \) times. If, however, \( N^* > 1 \) and the optimal bandit is not unique, while the optimal bandits must
have been activated at least \( n - O(g(n)) \) in total at time \( n \), and the distribution of these activations among the optimal bandits is hard to pin down. Simple simulations seem to indicate a sort of ‘phase change’, in that for \( g \) of order greater than \( \sqrt{n \ln \ln n} \) all optimal bandits are sampled roughly equally often, while for \( g \) of order less than \( \sqrt{n \ln \ln n} \), the policy tends to fix on a single optimal bandit, sampling the other optimal bandits much more rarely in comparison.

We offer the following as a potential explanation of this observed effect: Let us hypothesize, for the moment, that under any circumstances, the optimal bandits are activated linearly with time, that is for any optimal \( i^* \), \( T_{\pi_{i^*}}^g(n) = O(n) \), with the order coefficient depending on the specifics of that bandit. Under policy \( \pi_{i^*}^g \), activations are governed by a comparison of indices. We consider then the fluctuations in value of the two terms of the index, the sample mean \( \bar{X}_{T_{\pi_{i^*}}^g(n)}^{i^*} \) and the inflation term \( g(n)/T_{\pi_{i^*}}^g(n) \). Under the assumption the optimal bandits are activated linearly, and reasonable assumptions on the bandit distributions (to grant the Law of the Iterated Logarithm), the fluctuations in the sample mean over time will be of order \( O(\sqrt{\ln \ln n/n}) \). The fluctuations in the inflation term will be of order \( O(g(n)/n) \). It would seem to follow then that for \( g \) of order less than \( O(\sqrt{n \ln \ln n}) \), when comparing indices of optimal bandits, the sample mean is the dominant contribution to the index, while for \( g \) of order greater than \( O(\sqrt{n \ln \ln n}) \), the inflation term is the dominant contribution to the index. When the inflation term dominates, among the optimal bandits an ‘activate according to the largest index’ policy essentially reduces to a ‘activate according to the smallest number of activations’ policy, which leads to equalization and all optimal bandits being activated roughly equally often. When the sample mean dominates, among the optimal bandits an ‘activate according to the largest index’ policy essentially reduces to a ‘activate according to the highest sample mean’ or ‘play the winner’ policy, which leads to the policy fixing on certain bands for long periods.

This explanation would additionally suggest that on one side of the phase change, when the inflation term dominates, the only properties of the optimal bandits that matter for the dynamics of the problem are their means, that they all have the optimal mean \( \mu^* \). But on the other side of the phase change, when the sample mean dominates, other properties such as the variances \( \{\sigma^2_i\} \) influence the dynamics, through the Law of the Iterated Logarithm. However at this point in time, this remains, while interesting, speculative.

2.2.1 Leveraging the Law of Iterated Logarithms

Based on the results of the previous section, when \( N^* = 1 \), we have the following result:

For each \( i \neq i^* \), \( \forall \epsilon > 0 \), \( \exists \) (a.s.) some finite \( N_\epsilon \) such that for \( n \geq N_\epsilon \),

\[
\frac{1 - \epsilon}{\Delta_i} g(n) \leq T_{\pi_{i^*}}^g(n) \leq \frac{1 + \epsilon}{\Delta_i} g(n). \tag{27}
\]

Similarly, for the optimal bandit \( i^* \),

\[
n - (1 + \epsilon) \sum_{i \neq i^*} \frac{1}{\Delta_i} g(n) \leq T_{\pi_{i^*}}^g(n) \leq n - (1 - \epsilon) \sum_{i \neq i^*} \frac{1}{\Delta_i} g(n). \tag{28}\]

It follows trivially from these that each bandit is activated infinitely often, i.e., almost surely \( \{T_{\pi_{i}}^g(n)\}_{n \geq 1} \) is equivalent to the sequence \( \{0, 1, \ldots\} \) though with some (finite) stretches of term repetition. If in addition to the assumption of Eq. (1) we assume that for all \( i \) there exist finite variance \( \sigma^2_i = \text{Var}(X_i) < \infty \), we may apply the Law of the Iterated Logarithm: For each bandit \( i \),

\[
P \left( \limsup_{k} \pm \frac{X_{\pi_{i}^g}^i - \mu_i}{\sqrt{\ln \ln k/k}} = \sigma_i \sqrt{2} \right) = 1. \tag{29}\]
It follows, based on the above observation about \( \{T_{\pi_g}^i(n)\}_{n \geq 1} \), that
\[
P \left( \limsup_n \frac{\hat{X}_{T_{\pi_g}^i(n)}^i - \mu_i}{\sqrt{\ln \ln T_{\pi_g}^i(n)/T_{\pi_g}^i(n)}} = \sigma_i \sqrt{2} \right) = 1. \tag{30}
\]

This provides greater control over the sample mean of each bandit than what the Strong Law of Large Numbers alone allows, and allows the results of the previous section to be strengthened, as in the following theorem.

**Theorem 8** If \( N^* = 1 \) and \( \pi_g^O \) a policy as in (23), then the following are true:

a) if \( g(n) = o(n/\ln \ln n) \),
\[
P \left( \limsup_n \frac{\hat{R}_{\pi_g}^O(n) - (N-1)g(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq 2\sqrt{2} \sum_{i \neq i^*} \frac{\sigma_i}{\sqrt{\Delta_i}} \right) = 1, \tag{31}
\]
b) if \( g(n) = o(n^{2/3}) \),
\[
P \left( \liminf_n \frac{\hat{R}_{\pi_g}^O(n) - (N-1)g(n)}{\sqrt{g(n) \ln \ln g(n)}} \geq -3\sqrt{2} \sum_{i \neq i^*} \frac{\sigma_i}{\sqrt{\Delta_i}} \right) = 1. \tag{32}
\]

In short, we have that for a \textit{g}-ISM index policy \( \pi_g^O \),
\[
\hat{R}_{\pi_g}^O(n) = (N-1)g(n) + O \left( \sqrt{g(n) \ln \ln g(n)} \right).
\]

It should be observed that, unlike previous results, this theorem is somewhat restrictive in its allowed \( g \). However, since the focus is traditionally on logarithmic regret, i.e., \( g(n) = O(\ln n) \), it is clear that the above restrictions are nothing serious.

This theorem follows trivially from the following refinements of Props. 6, 7, and the definition of pseudo-regret, Eq. (4). Their proofs are given in Appendix C.

**Proposition 9** If \( N^* = 1 \) and \( g(n) = o(n/\ln \ln n) \), for each sub-optimal \( i \neq i^* \), the following holds almost surely:
\[
\limsup_n \frac{\Delta_i T_{\pi_g}^i(n) - g(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq 2\sigma_i \sqrt{\Delta_i}. \tag{33}
\]

**Proposition 10** If \( N^* = 1 \) and \( g(n) = o(n^{2/3}) \), for each sub-optimal \( i \neq i^* \), the following holds almost surely:
\[
\liminf_n \frac{\Delta_i T_{\pi_g}^i(n) - g(n)}{\sqrt{g(n) \ln \ln g(n)}} \geq -3\sigma_i \sqrt{\Delta_i}. \tag{34}
\]

Again, we leave as an open problem that of extending these results to the case of \( N^* > 1 \).

### 3. Comparison between Policies

We have established two policies, \textit{g-Forcing} and \textit{g-ISM index}, that each achieve \( O(g(n)) \) pseudo-regret, almost surely. The question of which policy is ‘better’ is not necessarily well posed. For one thing, the
asymptotic pseudo-regret growth of either policy can be improved by picking a slower $g$. In this sense, there is certainly no ‘optimal’ policy as there will always be a slower $g$. For a fixed $g$, however, the question of which policy is better becomes context specific: for some bandit distributions, the order constant of the $g$-Forcing policy, $S_\Delta$, will be smaller than the order constant of the $g$-ISM index policy, $N - N^*$; for some bandit distributions, the comparison will go the other way.

In terms of the results presented here, the pseudo-regret of the $g$-Forcing policy is much more tightly controlled, Theorem 4 bounding the fluctuations in pseudo-regret around $S_\Delta g(n)$ by at most a constant - indeed, at most $S_\Delta$. The bounds on the $g$-ISM index policy however are $O(\sqrt{g(n) \ln \ln g(n)})$. But, this additional control of the $g$-Forcing policy comes at a cost. It follows from the proof of Theorem 4 that for sub-optimal $i$, for all large $n$,

$$T^i_{\pi g}(n) \approx g(n). \quad (35)$$

However, for the $g$-ISM index policy, following the proof of Theorem 5, for all sub-optimal $i$, and large $n$,

$$T^i_{\pi g}(n) \approx \frac{g(n)}{\Delta_i}. \quad (36)$$

It is clear from this that the $g$-Forcing policy is in some sense the more democratic of the two, sampling all sub-optimal bandits equally, regardless of quality. The $g$-ISM index policy is the more meritocratic, sampling sub-optimal bandits more rarely the farther they are from the optimum. This has the effect of boosting the sampling of bandits near the optimum, but this effect is somewhat counterbalanced as they contribute less to the pseudo-regret.

4. Relaxing Assumptions: i.i.d. Bandits

The assumption that the results from each bandit are i.i.d. is fairly standard - the problem is generally phrased as a matter of knowledge discovery about a set of unknown distributions, though the use of repeated measurements. However, it is interesting to observe that this assumption actually plays no part in the results and proofs present in this paper. The sole distributional property that mattered for establishing the policies as $g$-good was the assumption that for each bandit there existed some finite $\mu_i$ such that $\bar{X}_i^k \to \mu_i$ almost surely with $k$ (though the Law of Iterated Logarithms was utilized to great effect in bounding the remainder terms). In fact, the expected values of the individual $X_i^k$ need not be $\mu_i$, nor must the $X_i^k$ be independent of each other for a given $i$. Further, it is never necessary that the bandits themselves be independent of each other! In that regard, the results herein are actually quite general statements about minimizing pseudo-regret under arbitrary multidimensional stochastic processes that satisfy that strong large number law-type requirement.

However, a word of caution is due: removing the restrictions on $\{X_i^k\}_{k \geq 1}$ in this way, while not influencing the proofs of the results presented here, does somewhat call into question the definition of ‘pseudo-regret’ as given in Eq. (4). The individual sample means freed, it is not necessarily reasonable to define a finite horizon pseudo-regret, $\bar{R}_\pi(n)$, in terms of the infinite horizon means, $\{\mu_i\}$. For instance, it is no longer necessarily true that the optimal, complete knowledge policy on any finite horizon is simply to activate a bandit with infinite horizon mean $\mu^*$ at every point. A more applicable definition of pseudo-regret would have to take into account what is reasonable to know or measure about the state of each bandit in finite time.

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Appendix A. Proof of Theorem 4

Proof To prove Theorem 4 it will suffice to show the following: For all \( i : \mu_i \neq \mu^* \) and all \( \delta > 0 \), \( \exists \) (a.s.) a finite time \( T_\delta < \infty \) such that that,

\[
g(t) - 2\delta \leq T_{g,t}^i (t) \leq \lceil g(t) \rceil \quad \forall t \geq T_\delta. \tag{37}
\]

Theorem 4 follows from this result and Eq. 4, with the appropriate choice of \( \delta \).

Without loss of generality, we may restrict ourselves to \( \delta < 1/2 \).

As a preliminary step: Based on the properties of \( g \), if \( N \) is the total number of bandits, there exists a finite time \( t_\delta \) such that, the following is true:

\[
g(t + N) < g(t) + \delta, \quad \forall t \geq t_\delta. \tag{38}
\]

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This follows from the observation that \( g(t + N) \leq g(t) + g'(t) * N \), and that \( g'(t) \to 0 \). Hence, \( t_\delta \) is not random.

When implementing a \( g \)-Forcing policy \( \pi^F_g \) (hereafter referenced simply as \( \pi \)), there are essentially two alternating phases (or modes) of the policy: ‘catch up’ and ‘play the winner’. During ‘catch up’, some number of bandits have fewer than \( g \) activations (the sub-\( g \) bandits), and they are activated until all bandits have at least \( g \) activations. During ‘play the winner’, each bandit has at least \( g \) activations, and the bandit with the current greatest sample mean is activated. These phases can be seen as governed by the function \( \Delta(t) = g(t) - \min_i T^\pi_i(t) \) so that when \( \Delta(t) > 0 \), the policy is in ‘catch up’ mode, when \( \Delta(t) \leq 0 \), the policy is in ‘play the winner’ mode.

Having activated bandits according to policy \( \pi \) up to time \( t_\delta \), suppose that \( \Delta(t_\delta) > 0 \), hence the policy enters or is in a period of ‘catch up’. Let \( d = d(t_\delta) \) be the number of sub-\( g \) bandits at time \( t_\delta \). Because \( g \) is increasing, and there are \( d \) sub-\( g \) bandits at time \( t_\delta \), it will take at least \( d \) ‘catch up’ activations before the policy enters a period of ‘play the winner’ (\( \Delta \leq 0 \)). Consider activating bandits according to policy \( \pi \) for \( d \) activations. Note, \( d \leq N \), so from Ineq. (38) and increasing property of \( g \) we have: \( g(t_\delta + d) < g(t_\delta) + \delta \) for every bandit realizing the minimum activations will have been activated at least once. It follows that

\[
\Delta(t_\delta + d) = g(t_\delta + d) - \min_i T^\pi_i(t_\delta + d) \\
< g(t_\delta) + \delta - \min_i T^\pi_i(t_\delta) - 1 \\
= \Delta(t_\delta) - (1 - \delta).
\]

Hence, after a period of \( d \) activations from time \( t_\delta \), the spread \( \Delta \) has decreased by at least \( 1 - \delta \). Repeating this argument, based on the number of sub-\( g \) bandits (if any) at time \( t_\delta + d \), it is clear that eventually - in finite time - a time \( T_\Delta < \infty \) is reached such that \( \Delta(T_\Delta) \leq 0 \). At this point, all bandits have been activated at least \( g \) times, and the policy enters a period of ‘play the winner’. We observe the loose, but sample-path-wise, bound that,

\[
T_\Delta \leq t_\delta + N \frac{\Delta(t_\delta)}{1 - \delta} \leq t_\delta + N \frac{g(t_\delta)}{1 - \delta} < \infty,
\]

since \( \Delta(t) \leq g(t) \) always, and at every step the number of sub-\( g \) bandits is at most \( N \). Observe that if in fact \( \Delta(t_\delta) \leq 0 \), then we may take \( T_\Delta = t_\delta \).

Having entered a period of \( \Delta \leq 0 \) or ‘play the winner’ at time \( T_\Delta \), let \( t \geq T_\Delta \) such that \( \Delta(t) \leq 0 \) but \( \Delta(t + 1) > 0 \). That is, in the transition from time \( t \) to \( t + 1 \), \( g \) surpasses the number of activations of some bandits and the policy enters a period of ‘catch up’. At such a point, we have the following relations:

\[
\min_i T^\pi_i(t + 1) < g(t + 1) < g(t) + \delta \leq \min_i T^\pi_i(t) + \delta.
\]

The first inequality is simply that \( \Delta(t + 1) > 0 \), the second following since \( t \geq t_\delta \), and the last since \( \Delta(t) \leq 0 \). However, since the \( T^\pi_i \) are integer valued and non-decreasing, the above yields

\[
\min_i T^\pi_i(t + 1) = \min_i T^\pi_i(t).
\]

Combining Eqns. (41), (42) yields the important relation that \( \Delta(t + 1) < \delta \). Note additionally,

\[
g(t + 1) < g(t) + \delta \leq \min_i T^\pi_i(t) + \delta < \min_i T^\pi_i(t + 1) + 1.
\]

Again noting the \( T^\pi_i \) are integer valued, this implies that while there are sub-\( g \) bandits at time \( t + 1 \), the only sub-\( g \) bandits are those that realize the minimum number of activations \( \min_i T^\pi_i(t + 1) \). All other bandits have activations strictly greater than \( g \). Let the number of sub-\( g \) bandits at time \( t + 1 \) again be denoted \( d = d(t + 1) \). For \( d' < d (\leq N) \) additional activations under \( \pi \), in the ‘catch up’ phase, we have that
By Eq. (44), we have that for all the investigator! Because of the properties of 
Hence we have for each population, for every 
for all 
\[ \Delta(t+1+d') < \Delta(t+1)+\delta < 2\delta. \]
For \( d \) additional activations after time \( t+1 \), each sub-\( g \) bandit has been activated once, raising the minimum number of activations by 1: \( \min_i T^g_i(t+1+d) = \min_i T^g_i(t+1) + 1 \). Additionally, \( g(t+1+d) < g(t+1)+\delta \), hence \( \Delta(t + 1 + d) < \Delta(t + 1) - \delta < 0 \).

We see therefore that after \( T_{\Delta} \), at any point at which \( \Delta \) becomes positive after being at most zero, it is at most \( 2\delta \) for a finite time - the ‘catch up’ phase - before becoming negative. Hence it follows, that for \( t \geq T_{\Delta} \), \( \Delta(t) \leq 2\delta \), or for each \( i \)
\[ g(t) - 2\delta \leq T^g_i(t). \]  
Note, this is true for all \( i \). This acts as justification for the description of \( g \) as the ‘forcing function’, as the policy forces all activations to grow at least at \( g \) asymptotically.

Since \( g \) is unbounded and increasing, all populations are sampled infinitely often over time. Taking the strong law of large numbers to hold, for every \( \epsilon > 0 \) and each \( i \), there exists almost surely some finite \( K_i^\epsilon \) such that \( \bar{X}^i_k \in [\mu_i - \epsilon, \mu_i + \epsilon] \) for all \( k \geq K_i^\epsilon \). It is worth noting here that while such a \( K_i^\epsilon \) exists, it is unknowable to the investigator! Because of the properties of \( g \), we may define a finite \( T^i_\epsilon > T_{\Delta} \) such that \( K_i^\epsilon \leq g(T^i_\epsilon) - 2\delta \).

By Eq. \( 44 \), we have that for all \( t \geq T^i_\epsilon \),
\[ \bar{X}^i_{T^i_\epsilon(t)} \in [\mu_i - \epsilon, \mu_i + \epsilon]. \]  
Hence we have for each population, for every \( \epsilon > 0 \), there exists almost surely a finite time \( T_\epsilon = \max_i T^i_\epsilon < \infty \) past which the sample mean is trapped within the \( \mu_i \pm \epsilon \) interval.

Fix \( \epsilon \) sufficiently small, so as to distinguish \( \mu^* \) from the other means (i.e., \( [\mu^* - \epsilon, \mu^* + \epsilon] \cap [\mu_i - \epsilon, \mu_i + \epsilon] = \emptyset \) for all \( i : \mu_i \neq \mu^* \)). By the previous observations, we have therefore that for all \( t \geq T_\epsilon \), for all sub-optimal \( i \) and any optimal \( i^* \),
\[ \bar{X}^i_{T^i_\epsilon(t)} > \bar{X}^i_{T^i_{\epsilon^*}(t)}. \]  
In short, almost surely there exists a finite time \( T_\epsilon \) past which the sample means of sub-optimal bandits are always inferior to the sample mean of any optimal bandit.

By the structure of the policy \( \pi \), for all \( t \geq T_\epsilon \), sub-optimal populations are only activated during the \( g \)-forced ‘catch up’ periods. If at time \( T_\epsilon \), the number of times a sub-optimal bandit \( i \) has been activated is greater than \( g \) - for instance due to it, at some point, having the largest sample mean during a ‘play the winner’ period - that population will not be sampled again until \( g \) has increased to overcome this temporary excess. As \( g \) is increasing and unbounded, this must occur in finite time. Once this occurs, as observed previously, \( g \) can only exceed \( T^i_\epsilon \) by at most \( 2\delta \) before bandit \( i \) is again activated, raising \( T^i_\epsilon \) above \( g \) once more. As this ‘catch up’ is the only time bandit \( i \) is activated, and \( \delta < 1/2 \), it follows that there exists some finite time \( T^i_{\delta} > T_\epsilon \) such that for \( t \geq T^i_{\delta} \), \( T^i_\epsilon(t) \leq \lfloor g(t) \rfloor \). Taking \( T_{\delta} = \max_{i: \mu_i \neq \mu^*} T^i_{\delta} \), and noting that \( t_\delta \leq T_{\Delta} \leq T_\epsilon \leq T_\delta < \infty, \) we have that for \( t \geq T_{\delta} \), for all sub-optimal \( i \),
\[ g(t) - 2\delta \leq T^i_{\delta}(t) \leq \lfloor g(t) \rfloor. \]  

\section*{Appendix B. Proofs of Propositions \[6,7\]}

In this section, \( \pi \) refers to a \( g \)-ISM index policy as in Eq. \( 23 \). The results to follow depend on the following lemma.
Lemma 11  Under the assumption of Eq. (1), for each $i$, and for any $\epsilon > 0$, the inequality:

$$u_i(j, k) < \mu_i - \epsilon$$

holds for only finitely many $(j, k)$-pairs, almost surely.

Proof  As an application of the strong law, almost surely there is some finite $K^i_\epsilon$ such that $\bar{X}^i_k > \mu - \epsilon/2$, for all $k \geq K^i_\epsilon$. For such $k$, as $g$ is positive, $u_i(j, k) = \bar{X}^i_k + g(j)/k \geq \mu_i - \epsilon$, for all $j$. For any $k < K^i_\epsilon$, the relation $u_i(j, k)\bar{X}^i_k + g(j)/k < \mu_i - \epsilon$ may be true only for finitely many $j$ since $g$ is increasing.

Proof of Proposition 6  For $i \neq i^*$, we define the following quantities. Taking $\epsilon > 0$, and $2\epsilon < \mu^* - \mu_i$, and $n \geq N$,

$$n^i_1(n, \epsilon) = \sum_{t=N}^{n} 1\{\pi(t+1) = i, u_i(t, T^i_{\pi_\epsilon}(t)) \geq \mu^* - \epsilon, \bar{X}^i_{T^i_{\pi_\epsilon}(t)} \leq \mu^i + \epsilon\}$$

$$n^i_2(n, \epsilon) = \sum_{t=N}^{n} 1\{\pi(t+1) = i, u_i(t, T^i_{\pi_\epsilon}(t)) \geq \mu^* - \epsilon, \bar{X}^i_{T^i_{\pi_\epsilon}(t)} > \mu^i + \epsilon\}$$

$$n^i_3(n, \epsilon) = \sum_{t=N}^{n} 1\{\pi(t+1) = i, u_i(t, T^i_{\pi_\epsilon}(t)) < \mu^* - \epsilon\}.$$  (48)

Hence we have the following relationship,

$$T^i_{\pi_\epsilon}(n + 1) = 1 + \sum_{t=N}^{n} 1\{\pi(t+1) = i\} = 1 + n^i_1(n, \epsilon) + n^i_2(n, \epsilon) + n^i_3(n, \epsilon).$$  (49)

The proof proceeds via a pointwise bound on each of the three terms. For the first term,

$$n^i_1(n, \epsilon) \leq \sum_{t=N}^{n} 1\{\pi(t+1) = i, \mu^i + \epsilon + g(t)/T^i_{\pi_\epsilon}(t) \geq \mu^* - \epsilon\}$$

$$= \sum_{t=N}^{n} 1\{\pi(t+1) = i, g(t)/((\mu^* - \mu_i) - 2\epsilon) \geq T^i_{\pi_\epsilon}(t)\}$$

$$\leq \sum_{t=N}^{n} 1\{\pi(t+1) = i, g(t)/((\mu^* - \mu_i) - 2\epsilon) \geq T^i_{\pi_\epsilon}(t)\}$$

$$\leq \frac{g(n)}{((\mu^* - \mu_i) - 2\epsilon) + 1.}$$  (50)

The last inequality comes from viewing $T^i_{\pi_\epsilon}(t)$ as a sum of $1\{\pi(t+1) = i\}$ indicators, and seeing that the condition on it bounds the number of non-zero terms in this sum.
For the second term,

\[ n_2^i(n, \epsilon) \leq \sum_{t = N}^{n} 1\{\pi(t + 1) = i, \bar{X}_k^i(t) > \mu^i + \epsilon\} \]

\[ = \sum_{t = N}^{n} \sum_{k=1}^{t} 1\{\pi(t + 1) = i, \bar{X}_k^i > \mu^i + \epsilon, T_\pi^n(t) = k\} \]

\[ = \sum_{t = N}^{n} \sum_{k=1}^{t} 1\{\pi(t + 1) = i, T_\pi^n(t) = k\} 1\{\bar{X}_k^i > \mu^i + \epsilon\} \]

\[ \leq \sum_{k=1}^{n} 1\{\bar{X}_k^i > \mu^i + \epsilon\} \sum_{t = k}^{n} 1\{\pi(t + 1) = i, T_\pi^n(t) = k\} \]

\[ \leq \sum_{k=1}^{n} 1\{\bar{X}_k^i > \mu^i + \epsilon\}. \]

The last inequality holds as, for a given \(k\), \(\{\pi(t + 1) = i, T_\pi^n(t) = k\}\) may be true for only one \(t\). Taking it one step further, we have

\[ n_2^i(n, \epsilon) \leq \sum_{k=1}^{\infty} 1\{\bar{X}_k^i > \mu^i + \epsilon\}, \]

and since the strong law of large numbers is taken to hold, we have therefore that \(n_2^i(n)\) is almost surely bound by a finite constant, for all \(n \geq N\).

For the third term, note that from the structure of the policy, a population is only sampled if it has the maximal current index. Hence, if \(\pi(t + 1) = i\), it must be true that \(u_{i^*}(t, T_\pi^{u^*}(t)) \leq u_i(t, T_\pi^i(t))\). Hence we have the bound,

\[ n_3^i(n, \epsilon) \leq \sum_{t = N}^{n} 1\{\pi(t + 1) = i, u_{i^*}(t, T_\pi^{u^*}(t)) < \mu^* - \epsilon\} \]

\[ \leq \sum_{t = N}^{n} 1\{u_i(t, T_\pi^i(t)) < \mu^* - \epsilon\} \]

\[ \leq \sum_{t = N}^{\infty} 1\{u_i(t, T_\pi^i(t)) < \mu^* - \epsilon\}. \]

From the prior observation about the form of the index, Lemma[11] we have that \(u_{i^*}(t, T_\pi^{u^*}(t)) < \mu^* - \epsilon\) is true for only finitely many \(t\), almost surely. Hence, from the above bound, \(n_3^i(n)\) is almost surely bound by a finite constant, for all \(n \geq N\).

Combining the above results bounding \(n_1^i, n_2^i, n_3^i\) with Eq.\[49\], and observing too that \(T_\pi^n(n) \leq T_\pi^i(n + 1)\), we have that almost surely there exists some finite \(C_e^i\) such that for all \(n \geq N\),

\[ T_\pi^i(n) \leq \frac{g(n)}{(\mu^* - \mu_i) - 2\epsilon} + C_e^i. \]

**Proof of Proposition[7]** Define a constant \(P_\Delta = \sum_{i \neq i^*} 1/(\mu^* - \mu_i)\). Taking \(\epsilon < \min_{j \neq i^*} (\mu^* - \mu_j)/2\), we may apply Prop.\[6\] to yield for each \(i \neq i^*\), \(\exists\) (a.s.) a finite \(N_e^i\) such that \(T_\pi^i(n) \leq (1 + \epsilon)g(n)/(\mu^* - \mu_i)\)
for all \( n \geq N^i_\epsilon \). Taking \( N_\epsilon = \max_{i \neq i^*} N^i_\epsilon \), summing over these relations and taking \( n \geq N_\epsilon \),

\[
\sum_{i \neq i^*} T^i_n(n) \leq (1 + \epsilon)g(n)P_\Delta. \tag{55}
\]

The sum above equals the number of activations of sub-optimal bandits up to and including time \( n \). As the total number of bandit activations up to time \( n \) is \( n \), we have from the above that \( T^i_n(n) \geq n - O(g(n)) \).

Trivially from this, the optimal bandit \( i^* \) is activated infinitely often, approaching full density of activations as \( n \) increases.

Given this linear lower bound on \( T^i_\pi \), it follows that \( u_{i^*}(n, T^i_\pi(n)) \) converges to \( \mu^* \), almost surely. Hence, almost surely there exists a finite \( \tilde{N}_\epsilon \) such that for \( n \geq \tilde{N}_\epsilon \), \( u_{i^*}(n, T^i_\pi(n)) \leq \mu^* + \epsilon \). As under this policy a bandit is only activated when it has the maximal index, it follows that infinitely often (on the activations of \( i^* \)), the indices of all sub-optimal bandits are at most \( \mu^* + \epsilon \). Given the structure of the indices, it follows that these sub-optimal bandits must be activated infinitely often as well. Hence, almost surely, \( T^i_\pi(n) \) increases without bound, for all \( i \). Applying the strong law here, since there are finitely many bandits being considered, \( \exists \) (a.s.) a finite ‘\( \epsilon \)-trapping time’, \( N^{\text{trap}}_\epsilon \), such that

\[
\tilde{X}^i_{T^i_\pi(n)} \in [\mu_i - \epsilon, \mu_i + \epsilon], \quad \forall n \geq N^{\text{trap}}_\epsilon \text{ and } \forall i.
\]

Let \( \{n_k\}_{k \geq 0} \) be the infinite sequence of times at which bandit \( i^* \) has the current optimal index (and hence is activated next). For a given \( i \neq i^* \), we have that for all sufficiently large \( k \) (\( n_k \geq N^{\text{trap}}_\epsilon \)),

\[
\max_{n_k \leq n \leq n_{k+1}} u_i(n, T^i_\pi(n)) \leq (\mu_i + \epsilon) + \frac{g(n_{k+1})}{T^i_{\pi}(n_k)}
\]

\[
= (\mu_i + \epsilon) + \frac{g(n_{k+1})}{g(n_k)} \frac{g(n_k)}{T^i_{\pi}(n_k)}
\]

\[
= (\mu_i + \epsilon) + \frac{g(n_{k+1})}{g(n_k)} (u_i(n_k, T^i_{\pi}(n_k)) - \tilde{X}^i_{T^i_\pi(n_k)})
\]

\[
\leq (\mu_i + \epsilon) + \frac{g(n_{k+1})}{g(n_k)} (u_i(n_k, T^i_{\pi}(n_k)) - (\mu_i - \epsilon)). \tag{56}
\]

Additionally, however, at time \( n_k \) bandit \( i^* \) has the largest index. For sufficiently large \( k \) (\( n_k \geq \tilde{N}_\epsilon \)), this index must be at most \( \mu^* + \epsilon \). Hence for \( n_k > \max(N_\epsilon, N^{\text{trap}}_\epsilon) \), for \( i \neq i^* \) we have that \( u_i(n_k, T^i_\pi(n_k)) \leq u_{i^*}(n_k, T^i_{\pi}(n_k)) \leq \mu^* + \epsilon \), and

\[
\max_{n_k \leq n \leq n_{k+1}} u_i(n, T^i_\pi(n)) \leq (\mu_i + \epsilon) + \frac{g(n_{k+1})}{g(n_k)} ((\mu^* + \epsilon) - (\mu_i - \epsilon))
\]

\[
= (\mu_i + \epsilon) + \frac{g(n_{k+1})}{g(n_k)} (\mu^* - \mu_i + 2\epsilon). \tag{57}
\]

Since we took \( g \) to be concave, \( g(n_{k+1}) \leq g(n_k) + (n_{k+1} - n_k)g'(n_k) \). The difference \( n_{k+1} - n_k - 1 \) is the number of sub-optimal bandit activations between the \( k \) and \( k + 1 \)-th activations of bandit \( i^* \). This is bound from above by the total number of sub-optimal bandit activations prior to time \( n_{k+1} \), which by Eq. (55) is at most \((1 + \epsilon)g(n_{k+1})P_\Delta \) for all \( n_{k+1} \geq N_\epsilon \). Hence,

\[
g(n_{k+1}) \leq g(n_k) + ((1 + \epsilon)g(n_{k+1})P_\Delta + 1)g'(n_k). \tag{58}
\]

As \( g' \to 0 \), for all sufficiently large \( k \), we have that \((1 + \epsilon)P_\Delta g'(n_k) < 1 \) and

\[
\frac{g(n_{k+1})}{g(n_k)} \leq 1 + \frac{g'(n_k)}{g(n_k)} \leq \frac{1}{1 - (1 + \epsilon)P_\Delta g'(n_k)}. \tag{59}
\]
As \( g \) is taken to be increasing, and \( g' \) is taken to limit to 0, we have from the above that there is some finite \( \bar{N}_g \) such that for all sufficiently large \( k \) (\( n_k \geq \bar{N}_g \)), \( g(n_{k+1})/g(n_k) \leq 1 + \epsilon \). Hence, for \( n_k \geq \max(N_e, \bar{N}_e, \bar{N}_{e^\text{trap}}, \bar{N}_g) \),
\[
\max_{n_k \leq n \leq n_{k+1}} u_i(n, T^i_x(n)) \leq (\mu_i + \epsilon) + (1 + \epsilon)(\mu^* - \mu_i + 2\epsilon).
\] (60)

Let \( N^K_e = \min\{n_k : n_k > \max(N_e, \bar{N}_e, \bar{N}_{e^\text{trap}}, \bar{N}_g)\} < \infty \). As the upper bound above no longer depends on \( k \), we have that for \( n \geq N^K_e \),
\[
u_i(n, T^i_x(n)) \leq (\mu_i + \epsilon) + (1 + \epsilon)(\mu^* - \mu_i + 2\epsilon).
\] (61)

Observing that \( X^i_{T^i_x(n)} \geq \mu_i - \epsilon \), the above yields \( \mu_i - \epsilon + g(n)/T^i_x(n) \leq (\mu_i + \epsilon) + (1 + \epsilon)(\mu^* - \mu_i + 2\epsilon) \), or
\[
g(n)/(1 + \epsilon)(\mu^* - \mu_i + 2\epsilon) + 2\epsilon \leq T^i_x(n).
\] (62)

\section*{Appendix C. Proofs of Propositions [9] [10]}

We present the following preliminary bounds to aid in the proofs of Props. [9][10]. In this section, \( \pi \) is taken to be an \( g \)-ISM index policy as in Eq. (25). Additionally, it is convenient to define
\[
P_\Delta = \sum_{i \neq i^*} \frac{1}{\mu^* - \mu_i} \tag{63}
\]

It follows from Props. [6][7] that for any \( \epsilon > 0 \), \( \exists \) (a.s.) some finite \( N_e \) such that for \( n \geq N_e \), the following holds: for \( i \neq i^* \),
\[
\frac{1 - \epsilon}{\mu^* - \mu_i} g(n) \leq T^i_x(n) \leq \frac{1 + \epsilon}{\mu^* - \mu_i} g(n).
\] (64)

And similarly, for the optimal bandit,
\[
n - (1 + \epsilon)P_\Delta g(n) \leq T^i_x(n) \leq n - (1 - \epsilon)P_\Delta g(n).
\] (65)

To simplify the case for the optimal bandit, slightly, it also holds that for all sufficiently large \( n \), \( T^i_x(n) \geq n/2 \). We’ll also observe here, as an aside, that for some finite \( \bar{N}_e \),
\[
(1 - \epsilon)/(\mu^* - \mu_i)g(n) > 6, \quad \text{for all } n \geq \bar{N}_e, \text{ and } i \neq i^*.
\]

As each bandit is activated infinitely often, \( T^i_x(n) \) increases without bound with \( n \), and hence we may apply the Law of the Iterated Logarithm in the following way: There exists a finite time \( N'_{e^i} \) such that for \( n \geq N'_{e^i} \), for each bandit \( i \),
\[
|X^i_{T^i_x(n)} - \mu_i| \leq \sigma_i \sqrt{2(1 + \epsilon )} \sqrt{\frac{\ln T^i_x(n)}{T^i_x(n)}}.
\] (66)

However, since \( \sqrt{\ln \ln x/x} \) is decreasing for all \( x \geq 6 \), we may apply the above bounds to have that, for \( n \geq \max(N_e, N'_e, \bar{N}_e, 12) \), for \( i \neq i^* \),
\[
|X^i_{T^i_x(n)} - \mu_i| \leq \sigma_i \sqrt{2(1 + \epsilon )} \sqrt{\frac{\ln \left( \frac{1 - \epsilon}{\mu^* - \mu_i} g(n) \right)}{1 - \epsilon}} \leq \sigma_i \sqrt{2(1 + \epsilon )} \sqrt{\frac{\ln \left( \frac{1 - \epsilon}{\mu^* - \mu_i} g(n) \right)}{1 - \epsilon}}.
\] (67)
and for the optimal bandit,

$$|\bar{X}_{i*}^i(n) - \mu^*| \leq \sigma_i \sqrt{2(1+\epsilon) \frac{\ln \ln(n/2)}{n/2}}. \quad (68)$$

**Proof of Proposition 9.** Let $1 > \epsilon > 0$. For $i \neq i^*$, let

$$h_i(t) = \sigma_i \sqrt{2(\mu^* - \mu_i)(1+\epsilon) \frac{\ln \ln g(t)}{g(t)}}. \quad (69)$$

Observe that $h_i \to 0$ from above as $t \to \infty$. Note that there exists a $T_\epsilon < \infty$ such that for $t \geq T_\epsilon$, $g(t)/(\mu^* - \mu_i - 2h_i(t))$ is increasing. The proof proceeds analogously to the proof of Prop. 6 utilizing the improved iterated logarithm bounds above.

For $n \geq T_\epsilon$, define the following functions:

$$\tilde{n}_1^i(n) = \sum_{t=T_\epsilon}^n 1\{\pi(t+1) = i, u_i(t, T_n^i(t)) \geq \mu^* - h_i(t), \bar{X}_{T_n^i(t)}^i \leq \mu_i + h_i(t)\}$$

$$\tilde{n}_2^i(n) = \sum_{t=T_\epsilon}^n 1\{\pi(t+1) = i, u_i(t, T_n^i(t)) \geq \mu^* - h_i(t), \bar{X}_{T_n^i(t)}^i > \mu_i + h_i(t)\} \quad (70)$$

$$\tilde{n}_3^i(n) = \sum_{t=T_\epsilon}^n 1\{\pi(t+1) = i, u_i(t, T_n^i(t)) < \mu^* - h_i(t)\}.$$

Hence, we have the following relationship, that for $n \geq T_\epsilon$,

$$T_n^i(n) \leq T_\epsilon + 1 + \tilde{n}_1^i(n) + \tilde{n}_2^i(n) + \tilde{n}_3^i(n). \quad (71)$$

The proof proceeds as in the proof of Prop. 6 bounding each of the three terms. For the first,

$$\tilde{n}_1^i(n) \leq \sum_{t=T_\epsilon}^n 1\{\pi(t+1) = i, u_i(t) + g(t)/T_n^i(t) \geq \mu^* - h_i(t)\}$$

$$= \sum_{t=T_\epsilon}^n 1\{\pi(t+1) = i, g(t)/(\mu^* - \mu_i - 2h_i(t)) \geq T_n^i(t)\}$$

$$\leq \sum_{t=T_\epsilon}^n 1\{\pi(t+1) = i, g(n)/(\mu^* - \mu_i - 2h_i(n)) \geq T_n^i(t)\}$$

$$\leq \frac{g(n)}{(\mu^* - \mu_i - 2h_i(n))} + 1. \quad (72)$$

As before, the last inequality comes from viewing $T_n^i(t)$ as a sum of $1\{\pi(t+1) = i\}$ indicators, and seeing that the condition on it bounds the number of non-zero terms in this sum. It is also important to observe here that we are explicitly in a regime in which $g(t)/(\mu^* - \mu_i - 2h_i(t))$ is an increasing function with $t$. 

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For the second term,

\[ \tilde{n}_2^i(n) \leq \sum_{t=T_i}^{n} 1 \{ \pi(t + 1) = i, \bar{X}_{T_i}^i(t) > \mu_i + h_i(t) \} \]

\[ \leq \sum_{t=T_i}^{n} 1 \{ \bar{X}_{T_i}^i(t) - \mu_i > h_i(t) \} \]

\[ \leq \sum_{t=T_i}^{n} 1 \left\{ \sigma_i \sqrt{2} \left( 1 + \epsilon \right) \sqrt{\frac{\ln \ln \left( \frac{1-\epsilon}{\mu_2 - \mu_1} g(t) \right)}{\mu_2 - \mu_1 g(t)}} > h_i(t) \right\} \]  

(73)

The last inequality holds, by the iterated logarithm bound in Eq. (67). Taking it one step further, we have

\[ \tilde{n}_2^i(n) \leq \sum_{t=T_i}^{\infty} 1 \left\{ \sigma_i \sqrt{2} \left( 1 + \epsilon \right) \sqrt{\frac{\ln \ln \left( \frac{1-\epsilon}{\mu_2 - \mu_1} g(t) \right)}{\mu_2 - \mu_1 g(t)}} > 1 \right\} . \]  

(74)

Note that as

\[ \lim_{t} \sigma_i \sqrt{2} \left( 1 + \epsilon \right) \sqrt{\frac{\ln \ln \left( \frac{1-\epsilon}{\mu_2 - \mu_1} g(t) \right)}{\mu_2 - \mu_1 g(t)}} = \frac{1}{1 + \epsilon} < 1, \]  

(75)

the event indicated in the above sum bounding \( \tilde{n}_2^i(n) \) may occur only finitely many times, almost surely. Hence, \( \tilde{n}_2^i(n) \) is almost surely bound by a finite constant, for all \( n \geq T_i \).

For the third term, as before, by the structure of the policy, a population is only sampled if it has the maximal current index. Hence, if \( \pi(t + 1) = i \), it must be true that \( u_i^* (t, T_{\pi}^* (t)) \leq u_i^* (t, T_{\pi}^* (t)) \). It follows that

\[ \tilde{n}_3^i(n) \leq \sum_{t=T_i}^{n} 1 \{ \pi(t + 1) = i, u_i^* (t, T_{\pi}^* (t)) < \mu^* - h_i(t) \} \]

\[ \leq \sum_{t=T_i}^{n} 1 \{ u_i^* (t, T_{\pi}^* (t)) < \mu^* - h_i(t) \} \]

\[ = \sum_{t=T_i}^{n} 1 \left\{ \bar{X}_{T_i}^i(t) + \frac{g(t)}{T_{\pi}^* (t)} < \mu^* - h_i(t) \right\} \]

\[ \leq \sum_{t=T_i}^{n} 1 \left\{ -\sigma_i \sqrt{2} \left( 1 + \epsilon \right) \sqrt{\frac{\ln \ln(t/2)}{t/2} + \frac{g(t)}{T_{\pi}^* (t)}} < -h_i(t) \right\} . \]  

(76)

the last equation coming from the iterated logarithm bound for the optimal bandit, Eq. (68). As a final simplification,

\[ \tilde{n}_3^i(n) \leq \sum_{t=T_i}^{\infty} 1 \left\{ -\sigma_i \sqrt{2} \left( 1 + \epsilon \right) \sqrt{\frac{\ln \ln(t/2)}{t/2}} < -h_i(t) \right\} . \]  

(77)

If \( g(n) = o(n/\ln \ln n) \), it is easy to verify that the indicated event in the above sum can only occur for finitely many \( t \). Hence, by the above, there is a finite constant bounding \( \tilde{n}_3^i(n) \) for all \( n \geq T_i \).

Combining the above results, there is a finite constant \( D_i^c \) such that for all \( n \geq T_i \),

\[ T_{\pi}^i(n) \leq \frac{g(n)}{(\mu^* - \mu_i) - 2h_i(n)} + D_i^c. \]  

(78)
We have from this that
\[ (\mu^* - \mu_i) T^*_\pi(n) - g(n) \leq g(n) \frac{2h_i(n)}{(\mu^* - \mu_i) - 2h_i(n)} + (\mu^* - \mu_i) D_i^*. \] (79)

For a fixed \( \epsilon > 0 \), the above yields (taking the limit, given the choice of \( h_i(n) \)),
\[ \limsup_n \frac{(\mu^* - \mu_i) T^*_\pi(n) - g(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq \frac{2\sigma_i \sqrt{2(1 + \epsilon)^2}}{\sqrt{\mu^* - \mu_i}} \] (80)

As the above holds for all \( \epsilon > 0 \), this yields, almost surely,
\[ \limsup_n \frac{(\mu^* - \mu_i) T^*_\pi(n) - g(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq \frac{2\sigma_i \sqrt{2}}{\sqrt{\mu^* - \mu_i}}. \] (81)

**Proof of Proposition 10.** Let \( \epsilon \in (0, 1) \). Recall from the proof of Prop. 7 the infinite sequence \( \{n_k\}_{k \geq 0} \) of times at which the index of the optimal bandit \( i^* \) is maximal. For notational convenience, we will write \( u_i(n) = u_i(n, T^*_\pi(n)) \), and for \( i \neq i^* \), we define
\[ U^i_k = \max_{n_k \leq n \leq n_{k+1}} u_i(n), \] (82)
and
\[ M^i_k = \max_{n_k \leq n \leq n_{k+1}} \bar{X}^i_{T^*_\pi(n)}. \] (83)

We have the following relations,
\[ U^i_k \leq \left( \max_{n_k \leq n' \leq n_{k+1}} \bar{X}^i_{T^*_\pi(n')} \right) + \frac{g(n_{k+1})}{T^*_\pi(n_k)} \]
\[ = M^i_k + \frac{g(n_{k+1})}{g(n_k)} \frac{\mu_i(n_k)}{T^*_\pi(n_k)} \]
\[ = M^i_k + \frac{g(n_{k+1})}{g(n_k)} \left( u_i(n_k) - \bar{X}^i_{T^*_\pi(n_k)} \right) \]
\[ \leq M^i_k + \frac{g(n_{k+1})}{g(n_k)} \left( u_{i^*}(n_k) - \bar{X}^i_{T^*_\pi(n_k)} \right). \] (84)

For \( n \) such that \( n_k \leq n \leq n_{k+1} \), trivially \( u_i(n) \leq U^i_k \). It follows that
\[ \frac{g(n)}{T^*_\pi(n)} \leq \left( M^i_k - X^i_{T^*_\pi(n)} \right) + \frac{g(n_{k+1})}{g(n_k)} \left( u_{i^*}(n_k) - \bar{X}^i_{T^*_\pi(n_k)} \right). \] (85)

Defining the following terms for space,
\[ A_{n,k} = \left( M^i_k - X^i_{T^*_\pi(n)} \right), \]
\[ B_k = \frac{g(n_{k+1})}{g(n_k)} \left( u_{i^*}(n_k) - \mu^* \right), \]
\[ C_k = \frac{g(n_{k+1})}{g(n_k)} \bar{X}^i_{T^*_\pi(n_k)} - \mu_i, \]
\[ \Delta(n) = g(n) - (\mu^* - \mu_i) T^*_\pi(n), \] (86)
The above relation may be rearranged to yield
\[ \Delta(n)/T^*_\pi(n) \leq A_{n,k} + B_k - C_k. \]  
(87)

We may apply the iterated logarithm bounds of Eq. (67), to yield a finite \( K_A \) such that for \( k \geq K_A \),
\[ A_{n,k} \leq 2\sigma_i \sqrt{2}(1+\epsilon) \sqrt{\frac{n}{2} \ln(\frac{1-\epsilon}{\mu^* - \mu_i}) g(n_k)}. \]
(88)

Similarly, there is a finite \( K_B \) such that for \( k \geq K_B \), observing that for sufficiently large \( k \), \( T^*_\pi(n_k) \geq n_k/2 \),
\[ B_k \leq g(n_{k+1}) \left( \mu^* + \sigma_i \sqrt{2}(1+\epsilon) \sqrt{\frac{n}{2} \ln(\frac{1-\epsilon}{\mu^* - \mu_i}) g(n_k)} \right) - \mu^*. \]
(89)

And finally, there is a finite \( K_C \) such that for \( k \geq K_C \),
\[ C_k \geq g(n_{k+1}) \left( \mu_i - \sigma_i \sqrt{2}(1+\epsilon) \sqrt{\frac{n}{2} \ln(\frac{1-\epsilon}{\mu^* - \mu_i}) g(n_k)} \right) - \mu_i. \]
(90)

Rearranging terms for space again, for \( k \geq \max(K_A, K_B, K_C) \) we have
\[ \Delta(n)/T^*_\pi(n) \leq A_{n,k} + B_k - C_k \leq \hat{A}_k + \hat{B}_k + \hat{C}_k + \hat{D}_k, \]
(91)

where
\[ \hat{A}_k = (\mu^* - \mu_i) \left( \frac{g(n_{k+1})}{g(n_k)} - 1 \right) \]
\[ \hat{B}_k = \sigma_i \sqrt{2}(1+\epsilon) \left( 2 + \frac{g(n_{k+1})}{g(n_k)} \right) \sqrt{\frac{n}{2} \ln(\frac{1-\epsilon}{\mu^* - \mu_i}) g(n_k)} \]
\[ \hat{C}_k = \sigma_i \sqrt{2}(1+\epsilon) \frac{g(n_{k+1})}{g(n_k)} \sqrt{\frac{n}{2} \ln(\frac{1-\epsilon}{\mu^* - \mu_i}) g(n_k)} \]
\[ \hat{D}_k = \frac{g(n_{k+1})}{g(n_k)} n_{k+1}/2. \]

Noting that each of the above are positive, we have from Eq. (91),
\[ \frac{\Delta(n)}{\sqrt{g(n)} \ln \ln g(n)} \leq \frac{(\hat{A}_k + \hat{B}_k + \hat{C}_k + \hat{D}_k) T^*_\pi(n)}{\sqrt{g(n)} \ln \ln g(n)}. \]
(93)

Note that, applying Eq. (64) in this case, we have some finite \( K_\epsilon \) such that for \( k \geq K_\epsilon \),
\[ T^*_\pi(n) \leq T^*_\pi(n_{k+1}) \leq \frac{1 + \epsilon}{\mu^* - \mu_i} g(n_{k+1}). \]
(94)

Recall from the proof of Prop. 7 that there is a finite \( K'_\epsilon \) such that for \( k \geq K'_\epsilon \), \( g(n_{k+1}) \leq (1+\epsilon)g(n_k) \). Noting too that \( g(n_k) \leq g(n) \), we have that for \( k \geq \max(K_\epsilon, K'_\epsilon) \),
\[ \frac{\Delta(n)}{\sqrt{g(n)} \ln \ln g(n)} \leq \frac{(\hat{A}_k + \hat{B}_k + \hat{C}_k + \hat{D}_k) (1+\epsilon)^2}{\sqrt{g(n)} \ln \ln g(n_k)} \left( \mu^* - \mu_i \right) g(n_k). \]
(95)
We have
\[
\frac{\hat{D}_k g(n_k)}{\sqrt{g(n_k) \ln \ln g(n_k)}} = \frac{g(n_{k+1})}{g(n_k)} \frac{g(n_k)}{\sqrt{n_k^2}} \frac{g(n_k)}{\sqrt{n_k \ln \ln g(n_k)}} \leq 2(1 + \epsilon) \frac{g(n_k)^{3/2}}{n_k \sqrt{\ln \ln g(n_k)}} = o(1). \tag{96}
\]
The last relationship follows, taking \(g(n) = o(n^{2/3})\).

We have
\[
\frac{\hat{C}_k g(n_k)}{\sqrt{g(n_k) \ln \ln g(n_k)}} = 2\sigma_i (1 + \epsilon) \frac{g(n_{k+1})}{g(n_k)} \sqrt{\frac{\ln \ln(n_k/2)}{n_k \ln \ln g(n_k)}} \leq 2\sigma_i (1 + \epsilon)^2 \frac{\ln \ln(n_k/2)}{n_k \ln \ln g(n_k)} = o(1). \tag{97}
\]
The last relationship follows, taking \(g(n) = o(n/ \ln \ln n)\).

We have
\[
\frac{\hat{B}_k g(n_k)}{\sqrt{g(n_k) \ln \ln g(n_k)}} = \sigma_i \sqrt{2(1 + \epsilon)} \left(2 + \frac{g(n_{k+1})}{g(n_k)}\right) \sqrt{\frac{\ln \left(\frac{1-\epsilon}{\mu - \mu_i} g(n_k)\right)}{\ln \ln g(n_k)}} \leq \frac{\sigma_i \sqrt{2(1 + \epsilon)}}{\sqrt{\frac{1-\epsilon}{\mu^* - \mu_i}}} \frac{\ln \left(\frac{1-\epsilon}{\mu^* - \mu_i} g(n_k)\right)}{\ln \ln g(n_k)} = \sigma_i \sqrt{2(1 + \epsilon)} \frac{(3 + \epsilon)}{\sqrt{\frac{1-\epsilon}{\mu^* - \mu_i}}} (1 + o(1)). \tag{98}
\]
The last relationship follows, taking the \(\{n_k\}_{k \geq 0}\) as infinite and unbounded, and \(g\) as increasing and unbounded.

We have
\[
\frac{\hat{A}_k g(n_k)}{\sqrt{g(n_k) \ln \ln g(n_k)}} = (\mu^* - \mu_i) \left(\frac{g(n_{k+1})}{g(n_k)} - 1\right) \sqrt{\frac{g(n_k)}{\ln \ln g(n_k)}}. \tag{99}
\]
Let \(\delta > 1\) by fixed. We use the bound here that for all positive \(x \leq 1 - 1/\delta\), \(1/(1 - x) \leq 1 + \delta x\). Applying Eq. \((59)\), we have for sufficiently large \(k\),
\[
\frac{g(n_{k+1})}{g(n_k)} - 1 \leq \frac{1 + \frac{g'(n_k)}{g(n_k)}/P_D g'(n_k)}{1 - (1 + \epsilon)P_D g'(n_k)} - 1 \leq \left(1 + \frac{g'(n_k)}{g(n_k)}\right)(1 + \delta(1 + \epsilon)P_D g'(n_k)) - 1 \tag{100}
\]

\[= g'(n_k) (\delta(1 + \epsilon)P_D + o(1)). \]
The last relationship follows, as \( g' \to 0 \) and \( g \to \infty \) with \( n_k \). Applying this to the above bound,

\[
\frac{\hat{A}_k g(n_k)}{\sqrt{g(n_k) \ln \ln g(n_k)}} \leq (\mu^* - \mu_i) \left( \delta (1 + \epsilon) P_{\Delta} + o(1) \right) g'(n_k) \sqrt{\frac{g(n_k)}{\ln \ln g(n_k)}} = o(1).
\]

The last relationship follows, taking \( g(n) = o(n^{2/3}) \).

Applying all of the above to the bound in Eq. (95), this yields

\[
\frac{\Delta(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq \left( \frac{\sigma_i \sqrt{2} (1 + \epsilon) (3 + \epsilon)}{\sqrt{\frac{1 - \epsilon}{\mu^* - \mu_i}}} \right) (1 - o(1)) + o(1) (1 + \epsilon)^2 \frac{\mu^* - \mu_i}{(\mu^* - \mu_i)},
\]

or

\[
\limsup_n \frac{\Delta(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq \left( \frac{\sigma_i \sqrt{2} (1 + \epsilon) (3 + \epsilon)}{\sqrt{\frac{1 - \epsilon}{\mu^* - \mu_i}}} \right) (1 + \epsilon)^2 \frac{\mu^* - \mu_i}{(\mu^* - \mu_i)}.
\]

Taking the limit as \( \epsilon \to 0 \) completes the proof,

\[
\limsup_n \frac{g(n) - (\mu^* - \mu_i) T^i_+(n)}{\sqrt{g(n) \ln \ln g(n)}} \leq \frac{3 \sigma_i \sqrt{2}}{\sqrt{\mu^* - \mu_i}}.
\]

\[\blacksquare\]