Quantum tensor product structures are observable-induced

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It is argued that the partition of a quantum system into subsystems is dictated by the set of operationally accessible interactions and measurements. The emergence of a multi-partite tensor product structure of the state-space and the associated notion of quantum entanglement are then relative and observable-induced. We develop a general algebraic framework aimed to formalize this concept. We discuss several cases relevant to quantum information processing and decoherence control.

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Suppose one is given a four-state quantum system. How does one decide whether such a system supports entanglement or not? In other words, should the given Hilbert space (entanglement or not? In other words, should the given Hilbert space be viewed as bi-partite (\(\otimes C^2 \otimes C^2\)), or irreducible? In the former case there exists a tensor product structure (TPS) that supports two entanglable qubits. In this case one finds a sharp dichotomy between the quantum and classical realms, as perhaps most dramatically exemplified in quantum information processing [1]. In the irreducible case there is no entanglement and hence none of the advantages associated with efficient quantum information processing [2, 3].

Here we propose that a partitioning of a given Hilbert space is induced by the experimentally accessible observables (interactions and measurements) (see also Refs. [1, 4, 5, 6, 7]). Thus it is meaningless to refer to a state such as the Bell state \(\Phi^{\pm} = (|00\rangle \pm |11\rangle) / \sqrt{2}\) as entangled [2], without specifying the manner in which one can manipulate and probe its constituent physical degrees of freedom. In this sense entanglement is always relative to a particular set of experimental capabilities. Before introducing a formalization, let us illustrate these ideas by means of a simple example.

Example 0: Bell basis.— Let \(|x\rangle \otimes |y\rangle \equiv |x,y\rangle\), \((x, y \in \{0, 1\})\) be the standard product basis for a two-qubit system. Each qubit forms a subsystem. With respect to (wrt) this bi-partition the Bell-basis states \(\Phi^{\pm} = (|00\rangle \pm |11\rangle) / \sqrt{2}\) are maximally-entangled. Now note that these can be rewritten as \(|\chi \lambda\rangle := |\chi\rangle \otimes |\lambda\rangle\), where \(\chi = \Phi, \Psi\) and \(\lambda = +, -\). With respect to this new bi-partition the Bell states are by definition product states, and the subsystems are the \(\chi, \lambda\) degrees of freedom. On the other hand the states \(|x,y\rangle\) are now entangled and can be used for entanglement-based quantum information protocols such as teleportation [1]. This striking difference can be highlighted by considering the swap operator \(S\), which is non-entangling in the usual \(x, y\)-bipartition, but, in the \(\chi, \lambda\)-bipartition one has \(S|\chi, \lambda\rangle = (-1)^{\chi \lambda} |\chi, \lambda\rangle\). Thus \(S\) realizes a controlled phase-shift over \(|11\rangle := |\Psi\rangle\), and in the new decomposition \(\text{swap}\) is a maximally-entangling operator. Which then is the correct characterization of the TPS and the associated entanglement? The answer depends on the set of accessible interactions and measurements. In stating that the Bell states are entangled one is implicitly assuming that there is experimental access to (local) observables of the form \(\{\sigma^x \otimes \mathbb{I}, \mathbb{I} \otimes \sigma^y\}\) (where \(\alpha, \beta \in \{x, y, z\}\) and \(\sigma\) are the Pauli matrices). But this assumption may not always be justified. For example, in quantum dot quantum computing proposals utilizing electron spins [8], it is more convenient to manipulate exchange interactions than to control single spins [9, 10]. In such cases the accessible interactions may be non-local, and this is precisely the situation that favors the \(\chi, \lambda\)-bipartition, that then acquires the same operational status as the standard \(x, y\) one.

General framework.— We now lay down a conceptual framework aimed to capture in its generality and relativity the notion of “induced tensoriality” of subsystems. Our definitions will be observable-based and will mostly involve algebraic objects [11]. Let us consider a quantum system with finite-dimensional state-space \(\mathcal{H}\), a subspace \(C \subseteq \mathcal{H}\), and a collection \(\{A_i\}_{i=1}^n\) of subalgebras of End(C) satisfying the following three axioms:

i) Local accessibility: Each \(A_i\) corresponds to a set of observable operators.

ii) Subsystem independence: \(\{A_i, A_j\} = 0\) (\(i \neq j\)).

iii) Completeness: \(\bigvee_{i=1}^n A_i \cong \mathbb{I}, A_i \cong \text{End}(C)\).

Notice that the standard case of \(N\) qubits (d-level systems) \(C = \mathcal{H} = (C^d)^\otimes N\) is the case \(A_i \cong M_d\) acting as the identity over all factors (subsystems) but the \(i\)th one. Now we discuss the physical meaning of the axioms i)–iii).

Axiom i) simply defines the basic algebraic objects at our disposal. These objects are controllable observables (Hamiltonians with tunable parameters, measurements).

Axiom ii) addresses separability. In order to claim that a system is composite it must be possible to perform op-
erations manipulating a well-defined set of degrees of freedom while leaving all the others unaffected. Typically this is achieved by having individually addressable, spatially separated subsystems \( i \) (e.g., a single excess electron per quantum dot [\[3\]), but as we shall see this is certainly not the only possibility.

Axiom iii) is the crucial one in order to ensure that our observable-based definition of multi-partitiveness induces a corresponding one at the state-space level. Its meaning will follow from Proposition 1 below: all the operations not affecting the state of a subsystem (its symmetries), are realized by operators corresponding to non-trivial operations only over the degrees of freedom of the other subsystems. All symmetries are then physical operations and no superselection rules [\[12\]] are present when a suitable state space \( \mathcal{C} \) is chosen. When \( \mathcal{C} \) is a proper subspace of \( \mathcal{H} \), we are dealing with an “encoding”, a notion that has proved useful, e.g., in quantum error correction and avoidance [\[14\]] and encoded universality [\[15\]].

Generalizing Ref. [\[4\]] we have the following central result:

**Proposition 1.** A set of subalgebras \( \mathcal{A}_k \) satisfying Axioms i)-iii) induces a TPS \( \mathcal{C} = \otimes_{k=1}^n \mathcal{H}_k \). We call such a multi-partition an induced TPS.

The proof is given in [\[17\]].

We proceed to explore some consequences of the notion of observable-induced TPSs. We first briefly return to the example of Bell-states discussed above, then continue the discussion at a more general level, and illustrate with examples of unusual and dynamic TPSs.

**Example 1.**— Assume that one is given the following set of independently controllable two-body interactions \( \{\sigma^y \otimes \sigma^y, \sigma^y \otimes \sigma^x, \sigma^x \otimes \sigma^y, \sigma^x \otimes \sigma^x\} \). These interactions generate the following subalgebras: \( \mathcal{A}_\chi := \{1, \sigma^y \otimes 1, 1 \otimes \sigma^y, \sigma^x \otimes \sigma^x\} \), \( \mathcal{A}_\lambda := \{1, 1 \otimes \sigma^y, \sigma^x \otimes \sigma^y, \sigma^x \otimes \sigma^x\} \). These satisfy Axioms i)-iii) (with \( \mathcal{C} = \mathbb{C}^4 \)) and act, respectively, as local identity and Pauli \( x, y, z \) matrices on the \( \chi \) and \( \lambda \) degrees of freedom considered above. Thus by Prop. 1 \( \mathcal{A}_\chi \) and \( \mathcal{A}_\lambda \) induce a TPS \( \mathcal{C}^{\mathcal{A}_\chi} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \), namely, the \( \chi, \lambda \) bi-partition.

**Superselection.**— An important example for which one is led to consider non-standard TPSs is systems exhibiting superselection rules [\[12\]]. There the only allowed physical operations correspond to operators commuting with a set of superselection charges \( \{Q_i\}_{i=1}^M \), e.g., particle-numbers, which generate an abelian algebra \( Q \). Denoting by \( \Pi_Q \) the projector over the commutant of \( Q \), the physically realizable subsystem operations are \( \Pi_Q(A_i) \) (\( i = 1, \ldots, n \)). These projected algebras typically either a) define a new invariant subspace \( C \) with a new induced TPS, or b) do not satisfy axioms ii), iii) anymore and therefore fail to induce a proper TPS. The associated notion of entanglement and entanglement-based protocols then must be reconsidered [\[13\]].

**Irreducible representations.**— A prototypical way for obtaining an encoded bi-partite TPS is to consider the decomposition of \( \mathcal{H} \) into irreducible representations (ir-reps) of a *-subalgebra \( \mathcal{A} \). In that case

\[
\mathcal{H} \cong \oplus_{j} \mathcal{C}^{n_j} \otimes \mathcal{H}_J,
\]

where the \( \mathcal{H}_J \) are the \( d_J \)-dimensional irreps of \( A \) and \( n_j \) their multiplicities. The algebra (commutant) can then be written as \( A \cong \oplus_{j} \mathcal{M}_{n_j} \otimes \mathcal{M}_{A} \). Upon restriction to a particular \( J \)-sector one has \( A \vee \mathcal{A} = \mathcal{M}_{n_j} \otimes \mathcal{M}_{A} \cong \mathcal{A} \vee \mathcal{A} \). Then, according to Prop. 1, \( A \) and \( \mathcal{A} \) induce an (encoded) bi-partite TPS in each irreducible block.

**Example 2: Encoded tensoriality.**— As an example of the above construction, let \( \mathcal{H}_N := (\mathbb{C}^2)^{\otimes N} \) denote an \( N \)-qubit space, \( \mathcal{A}_1 \) the algebra of totally symmetric operators in End(\( \mathcal{H}_N \)), and \( \mathcal{A}_2 \) the algebra of permutations exchanging the qubits. \( \mathcal{A}_1 \) is generated by the collective spin operators, i.e., \( \mathcal{A}_1 = \mathcal{C} \{ S_{\alpha} := \sum_{i=1}^N \sigma_i^{-\alpha} \mid \alpha = x, y, z \} \), and \( \mathcal{A}_2 = \mathcal{A}' \) is generated by Heisenberg exchange interactions: \( \mathcal{A}_2 = \mathcal{C} (\sigma_x, \sigma_y, \sigma_z) \). In the context of decoherence-free subspaces and subsystems [\[13\], \[16\]], \( \mathcal{A}_1 \) is the algebra of error operators (system-bath interactions) and \( \mathcal{A}_2 \) is the algebra of allowed quantum computational operations. Here our perspective is quite different: we view both as algebras of accessible interactions that induce a TPS. This is in fact an encoded TPSs, since one has (for even \( N \)) the Hilbert space decomposition \( \mathcal{H}_N \) with \( J = 0, \ldots, N/2, \mathcal{H}_J = \mathbb{C}^{d_J}, d_J = 2J + 1 \), and \( n_j(N) = (2J + 1)N!/[N/2 + J + 1]! \). Each summand in Eq. \( \mathcal{H} \) is a code subspace with a bi-partite TPS. We stress the unusual feature of this example: the two “qubits” (i.e., subsystems) comprising the TPS need not have the same dimension (though they do for \( J = N/2 - 1 \)), and are manipulable by interactions of a physically distinct nature. The left (right) qubit is manipulated by tuning only Heisenberg exchange couplings (global magnetic fields). This example therefore has implications for spin-based quantum computation [\[3\]], where single-spin addressing is technically very demanding.

**Nested subalgebra chains.**— The commutant construction illustrated above provides a general way to realized an encoded bi-partite TPS. In order to obtain encoded TPSs with more than two subsystems we consider a nested chain of subalgebras:

\[
\mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_n.
\]

We assume that \( \mathcal{B}_0 \) acts irreducibly over \( \mathcal{H} \). Then \( \mathcal{H} \) typically will be reducible w.r.t \( \mathcal{B}_{i+1} \). In particular, w.r.t \( \mathcal{B}_2 \): \( \mathcal{A}^{d_{J_1}} \cong \oplus_{j_2} \mathcal{C}^{n_{j_2}} \otimes \mathcal{C}^{d_{J_2}} \) and \( \mathcal{B}_2 \cong \oplus_{j_1} \mathcal{M}_{n_{j_1}} \otimes \mathcal{M}_{d_{J_2}} \). By iterating over the subalgebra chain one obtains:

\[
\mathcal{H} \cong \oplus_{j_1, \ldots, j_n} \otimes_{k=1}^n \mathcal{A}^{n_{j_k}} \otimes \mathcal{C}^{d_{J_k}}.
\]

This is a sum over code subspaces \( \mathcal{H}(J_1, \ldots, J_n) := \otimes_{k=1}^n \mathcal{C}^{n_{j_k}} \otimes \mathcal{C}^{d_{J_k}} \) with a multi-partite TPS. The nontrivial ones are those for which at least one \( n_{j_k} > 1 \). Note
that while $B_2$ has non-trivial action only on $C^d_{j_2}$, $B_1$ has non-trivial action on $C^d_{j_1} \supset C^d_{j_2}$. So how does one operate on a particular subsystem (qudit), say $C^{d_{j_k}}$? We come to our second main result:

**Proposition 2.** Given a nested subalgebra chain as in Eqs. 4, 5, the subalgebras are given by

$$A_i = B'_i \cap B_{i-1}, \quad (i = 1, \ldots, n).$$

Conversely, when a set of subsystem algebras $\{A_i\}_{i=1}^n$ is given, the nested chain $B_i := \bigvee_{k=i+1}^n A_k, (i = 1, \ldots, n)$ results.

The proof is given in [18]. We now illustrate the notion of a nested subalgebra chain induced-TPS.

**Example 3: Standard TPS.**—The standard qubit-TPS over $H_N$ corresponds to the chain $B_i = 1^2 \otimes M_{2n-i}$, $i = 1, \ldots, n$. In this case all the subalgebras are factors, whence one has a single $H(j_1, \ldots, j_n)$ term in Eq. 8, with multiplicities $n_{j_i} = 2$ and dimensions $d_{j_i} = 2^n - 1$.

**Example 4: Stabilizer codes.**—Consider $N$ qubits and the following chain of nested algebras: $B_0$ acts irreducibly on $(C^2)^{\otimes N}$, $B_1$ acts trivially on the first qubit but irreducibly on the rest, etc. To realize such a chain let $\{X_1, \ldots, X_k\}$ be a set of $N$-qubit, mutually commuting operators, and let $B_i = C'[\{X_1, \ldots, X_i\}]$, $i = 1, \ldots, k$. Further assume that the $X_i$ are unitary, traceless, and square to the identity. Then the corresponding Hilbert space decomposition is $H \cong (C^2)^{\otimes i} \otimes (C^2)^{\otimes (n-i)}$, where the first $i$ factors correspond to the $2^i$ possible eigenvalues of $X_1, \ldots, X_i \in B_i'$. When the $X_i$’s are generators of an abelian subgroup of the Pauli group one recovers the stabilizer codes of quantum error correction [14].

**Example 5: Multi-partite encoded TPS.**—Let us revisit Ex. 2 and show how a multi-partite encoded TPS is induced. Consider $N = n 2^k$ qubits, and the chain $B_0 := C'S_N, B_i := C'(S_{N/2})^\otimes 2^i$, $i = 1, \ldots, K$, where $S$ denotes the symmetric group. Conceptually, we have $2^K$ blocks of $n$ qubits each, and the subalgebra chain corresponds to operating on these blocks with increasing levels of resolution. By Prop. 2 we should find a $K + 1$-partite encoded TPS. To see this, recall that the state-space $H_N \cong (C^2)^{\otimes N}$ of $N$ qubits splits wrt $S_N$ exactly as in the su(2) case (Ex. 2) except that by the duality between $S_N$ and su(2), the role of $n_j$ and $d_j$ is interchanged, while $J$ remains an su(2) irrep label. E.g., for $N = 6$ ($K = 3$ and $n = 3$) we have $H_N \cong \bigotimes_{j=0}^{3} C^d_{j} \otimes C^d_{j} \cong H_0 \otimes C^5 \cong H_1 \otimes C^3 \oplus H_2 \cong C^5 \otimes H_0 \cong C^3$, where now $n_j = 2J + 1$, $d_j = n_j(J)$, and $H_i := (C^2)^{\otimes J+1}$, $J = 0, 1, 2, 3$. The chain then consists of $B_0 = C'S_N$ and $B_i := C'(S_{3^i} \times S_{3^{i-1}})$, i.e., exchanges of the first three $\times$ second three qubits. From Prop. 2 this algebra chain defines the encoded TPSs with algebra subsystems given by $A_0 := B'_0 = \{S_{N/2} \otimes S_{N/2}\}$ operators (recall Ex. 2) and $A_i := B'_i \cap B_0$, where $B'_i$ is block-symmetric operators, so that $A_1 = \text{linear combination of permutations, symmetrized wrt } S_3 \times S_3$. Decomposing the $C^{d_{j_k}}$ factors wrt $S_3 \times S_3$ we find, e.g., for the $H_1 \otimes C^9$ term that it describes a qubit times a qutrit TPS. The operations over the qutrit are provided by the algebra of totally symmetric six-qubits operators. Those over the qubit are realized by operators in $C'S_6$ having the form $X \otimes Y$ where $X$ ($Y$) is a totally symmetric operators over the first (second) three qubits. For example, elements of the form $\sigma_{1+\lambda} \cdot \sigma_{2+3\lambda} \cdot \sigma_{3+3\lambda} \cdot \sigma_{3+3\lambda} \cdot \sigma_{1+3\lambda}$ (i.e., $\lambda = 0, 1$) have trivial action over the qutrit (being a combination of 6 permutations) and a non-trivial one over the qubit [being $su_{1-3}(2) \times su_{4-6}(2)$ elements].

Returning to the case of $K$ blocks, one can see how an encoded multi-partite TPS will emerge. For example, with $n = 3$ and $K = 2$ we have the chain $B_0 := C'S_2 \supset B_1 := C'(S_3 \times S_3) \supset B_2 := C'(S_3 \times S_3 \times S_3)$. By comparing, as in [19], the decompositions of $H_{12}$ wrt $B_1$ and $B_2$ one can identify the tri-partite encoded TPS.

**Example 6: tri-partite hybrid TPS.**—Let us exhibit an unusual example, of a TPS wherein each factor is of a different physical nature. We consider $H := (C^2)^{\otimes 4}$ and $B_1 = 1 \otimes \text{End}(C^2)^{\otimes 3}$ (full operator space over the last three qubits), $B_2 = 1 \otimes C'S_3$ (permutations exchanging the last three qubits). $B_1$ is a factor and one obtains the decomposition $H = C^2 \otimes C^2$. The three-qubit space splits wrt $S_3$ as $C^6 \otimes C^2 \otimes C^2 \otimes C^2$. It follows that $(C^2)^{\otimes 4} \cong (C^2 \otimes C^2) \otimes (C^2 \otimes C^2) \otimes (C^2 \otimes C^2)$. The last term corresponds to a tri-partite system in which the first subsystem is a “standard” qubit, the second is acted upon by collective interactions over the last three “physical” qubits, while the third is acted upon by the algebra of permutations of $S_3$. Interestingly, this hybrid tri-partite system has already been realized experimentally in the context of noiseless-subsystems [21].

**TPS morphing.**—So far we have emphasized kinematics. Next we show that an induced TPS can change dynamically, depending on the algebras of available interactions. Let $\{A_i\}_{i=1}^n$ and $\{\tilde{A}_i\}_{i=1}^n$ define two TPS over $H$. Suppose one has the following Hamiltonian

$$H(\lambda, \mu) = \sum_{i=1}^{n} \lambda_i^a \tilde{H}_i^a + \sum_{i=1}^{n} \mu_i^b \tilde{H}_i^b$$

where $H_i^a \in A_i, \tilde{H}_j^a \in \tilde{A}_i, (i = 1, \ldots, n)$, and all coupling constants $\lambda_i, \mu_i$ are independently tunable. By setting all the $\mu_i (\lambda_i)$ to zero the first (second) TPS is induced. Therefore, dynamical control of the Hamiltonian allows to switch among different induced multipartitions, possibly with a different number of subsystems, in a sort of continuous fashion. We call this “TPS morphing”. For example, consider three qubits with controllable Hamiltonian given by $H(\lambda(t), \mu(t)) = \sum_{j=1}^{3} \lambda_j^a \sigma_j^a + \sum_{\alpha=x,y,z} \frac{\alpha^2}{2} S^\alpha + \sum_{j=1}^{3} \sum_{\beta=x,y,z} \mu_j^\beta \sigma_j^\beta$, where $S^\alpha = \sum_{i=1}^{3} \alpha^a_i (\alpha = x, y, z)$. The first two terms induce the (encoded) TPS described in Ex. 2, whereas the last term induces the bi-partite TPS described in Ex. 2, with a certain structure.
Stroboscopic entanglement.— A TPS can even be switched on and off under appropriate circumstances. Suppose that the algebra of available interactions does not induce a TPS [e.g., since it is \( \cong \text{End}(H) \)]. Now suppose that one can turn on an additional interaction that allows one to refocus (see, e.g., [10]) some of these interactions, so that the remaining interactions do induce a TPS. Then at the end of each refocusing period a TPS will appear. We call this “stroboscopic entanglement”. For instance, and referring back to Ex. 1, suppose that the controllable Hamiltonian is given by

\[ H = \sum_{X \in A_x, Y \in A_y} J_X X + J_Y Y, \]

where the two-body terms are always on and the one-body terms are controllable. This \( H \) mixes the subalgebras \( A_x \) and \( A_y \), so that there is no TPS as long as the two-body terms are present. However, a series of \( \pi \)-pulses in terms of \( \sigma^z \otimes 1 \) (\( 1 \otimes \sigma^z \)) will refocus, i.e., turn off, the two-body terms in \( A_x \) (\( A_y \)) term, thus decoupling the two subalgebras at the end of each refocusing period. In this manner the \( \chi \) and \( \lambda \) factors can be separately manipulated, i.e., the TPS has reappeared.

Conclusions.— We have shown that the TPS of quantum mechanics acquires physical meaning only relative to the given set of available interactions and measurements. These induce a TPS through their algebraic structure. The induced TPS may contain factors (“qudits”) of a different physical nature, and can be dynamical.

A few concluding comments are in order. First, note that while we have given criteria for the appearance of an induced TPS and the associated entanglement, we have deliberately not addressed the issue of efficiency in quantum information processing (QIP) [1], in particular to implement, along with the local operations and classical communication, computations scales exponentially in some resource such as energy levels of a Rydberg atom, while the associated cost of performing a quantum computation scales exponentially in some resource such as spectroscopic resolution [2]. Second, and again in the context of QIP, in order to exploit a given induced TPS for performing quantum computation one has to be able to implement, along with the local operations \( A_i \), at least one entangling transformation \( \mathcal{E} \) in \( \text{End}(C) \cong \mathcal{A}_i \lhd \mathcal{A}_f \). The new set \( \{\mathcal{A}_i, \mathcal{E}\} \), in the prototypical situation of interest in QIP, will be (encoded) universal, i.e., will allow any transformation in \( \text{End}(C) \) to be generated by composition of elementary operations involving \( \{\mathcal{A}_i, \mathcal{E}\} \). This will allow access to other TPSs than the original, induced one (e.g., in the case of Ex. 0 one could argue that access to both the standard and the \( \chi, \lambda \) bipartitions is available once all \( SU(4) \) transformations can be generated). The key point is that there is a hierarchy of TPS: the “natural” one is the one that is induced by the directly accessible observables \( A_i \). The “lower-level” ones are those that are visible only by composition of the elementary observables \( \{\mathcal{A}_i, \mathcal{E}\} \). Third, it is important to emphasize that both interactions and measurements are involved in inducing a TPS, and must be compatible, i.e., induce the same TPS, for this TPS to be both manipulable and observable. 

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Upon restriction to a particular code subspace we have $A^{(1)}_i \cong M_d$. Then $A^{(1)}_i \cap A^{(1)}_{i+1} \subseteq \oplus_{j} \mathbb{I}_{a_j} \otimes \mathbb{I}_{d_j}$. The only decomposition compatible with (A) is when just one irrep appears, i.e., $A^{(1)}_i \cong M_d \otimes \mathbb{I}_{n_j}$. Since $A^{(1)}_i \cup A^{(1)}_j \cong \text{End}(H) \cong M_{n_{d_j}}$ one finds that $A^{(1)}_i$ must be isomorphic to $M_{n_j}$. In summary: \[ \exists H^{(1)}_n \cong \mathbb{C}^{n_j} \cong \mathbb{C}^{n_{d_j}} \text{ such that } C \cong \bigoplus_{k=1}^{n} H_k^{(1)} \text{ and } A^{(1)}_i \cong \text{End}(H_k^{(1)}), \] (i = 1, 2). This argument can be iterated over i for $A^{(1)}_i$ and $A^{(1)}_{i+1}$. The role of $C$ is played by the second state-factor obtained for $i = 1$. Eventually one finds $C \cong \bigoplus_{k=1}^{n} H_k^{(1)}$, with $A_k \cong \text{End}(H_k^{(1)})$. \]

[18] Upon restriction to a particular code subspace we have $B_i \cap B_{i-1} \cong \bigoplus_{k=1}^{i-1} \mathbb{I}_{a_j} \otimes M_{k_{d_j}}$ and $B_i \cong \bigoplus_{k=1}^{n} \mathbb{I}_{a_j} \otimes M_{k_{d_j}} \otimes \mathbb{I}_{d_j}$. Then for $i < n$: \[ B_i \cap B_{i+1} = \bigoplus_{k=1}^{i} \mathbb{I}_{a_j} \otimes M_{k_{d_j}} \otimes \mathbb{I}_{d_k}, \] i.e., has non-trivial action only on the $i$th subsystem. For $i = n$ we have $B_n \cap B_{n-1} = \bigoplus_{k=1}^{n} \mathbb{I}_{a_j} \otimes M_{k_{d_j}} \otimes \mathbb{I}_{d_k}$. For the converse direction, the formula $B_i := \bigvee_{k=i}^{n} A_k$ is immediate. \]

[19] We have to decompose the $\mathbb{C}^{d_j}$ factors wrt $S_3 \times S_3$. Using the $S_3$ decomposition $H_3 \cong H_1 \otimes \mathbb{C}^2 \oplus H_2 \otimes \mathbb{C}^2 \oplus \mathbb{C}$, we have $H_6 \cong (H_3)^{\otimes 2} \cong (H_1 \otimes H_2)^{\otimes 2} \oplus \mathbb{C}^4 \oplus \ldots$. Using the $S_3$ decomposition $H_3 \cong H_1 \otimes H_2 \cong \oplus_{J_1, J_2} \mathbb{C}^{(J_1, J_2)}$ we can collect together the terms in the expansion of $(H_3)^{\otimes 2}$ to match those in $H_6$. Focusing on the $\mathbb{C}^{(J_1 \otimes J_2)} \oplus \mathbb{C}^{(J_1 \otimes J_2)} = H_1 \otimes \mathbb{C}^9$ term in the decomposition of $H_6$, we find that it splits as $H_1 \otimes (\mathbb{C} \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes \mathbb{C}^2)$. The middle two terms represent a two-dimensional $S_3 \times S_3$-irrep appearing with multiplicity two, and can be then written as $\mathbb{C}^{J_1, J_2} \otimes \mathbb{C}^{J_1, J_2} = \mathbb{C}^{J_1} \otimes \mathbb{C}^{J_2}$. The claimed result then follows.

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