The open unsymmetrical stadium billiard

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Abstract. In open billiards a particle can escape from the cavity through a leak. This type of systems have received special attention because of their applications to a wide variety of physical phenomena ranging from hydrodynamics to quantum chaos and astronomy. Chaotic leaked billiards are characterized by a so called transient behavior, i.e. by the presence of chaotic motion with a finite life time impossible to be studied just through the analysis of its asymptotic behavior. Under this scenario, we consider the quarter stadium billiard to study the influence of leaking marginal unstable periodic orbits. A rigorous statistical analysis of the survival probability is presented, setting up the classical trajectories’ solution in such a way that the system only depends on its partial separability and construct the Birkhoff map. The possibility of more than one leak into a billiard is also considered.

1. Introduction

Inside billiard systems constructions, a unitary mass point particle moves freely, hold to mirror reflections when it hits the rigid billiards’ edges. Indeed, the particle long time dynamical behavior is strongly influenced by the billiards’ boundaries. Particularly, for 2-Dim-billiard geometries like rectangles, ellipses and circles are typical examples of regular dynamical behavior. The stadium billiard, a system configured by two parallel straight edges with two semicircular curves on either side, has been introduced as an standard example of a fully chaotic regime [1]. Thus, billiards are simple systems useful to model the behavior of chaotic systems in classical, quantum mechanics and also in the semiclassical limit [2]. Alternatively, these have been used as theoretical model useful to understand experiments measuring the conductivity of quantum dots [3], in experimental research on micro wave cavities, optical and acoustical resonances [4].

In an open chaotic billiard a limited region in phase space is allowed to leak classical trajectories, with some chaotic saddles points ruling the motion of the long-lasting transient behavior [5]. Some different issues have been matter of discussion using these constructions, ranging from geometrical acoustics, ergodicity, controlling chaos, cosmology, conductance fluctuations in heterostructure devices [6].

Our physical system is an open unsymmetrical stadium billiard, that preserves a $C_{2v}$ symmetry, a fundamental domain with a single real positive parameter. It is partially separable, and in the square region the dynamics is described by zigzagging free motion. In the radial symmetry region, the dynamics is better described in polar coordinates, thus the angular motion is free motion and the complementary radial motion is governed by the centrifugal potential. Therefore, the complete classical dynamics is the composition of both zigzagging and rotating motion [7].

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We probe the response of the marginal unstable periodic orbits to the survival trajectories probability, for the open unsymmetrical stadium billiard. In section 2 we present the billiards’ leaking theory on the transient chaos theory scheme. In section 3 we construct the Birkhoff map for the quarter stadium billiard, and describe the Sprinkler method. In section 4 numerical results are presented, and finally in section 5 our conclusions are presented.

2. Leaking theory
Let us to consider a 2-Dim cavity with area $A$ and perimeter $L$, supposed to be filled with $N$ particles. The problem is to know how the cavity is emptied when there is a small leak of length $\Delta L$ on its boundary. Following [8], a kinetic theory scheme, we define a phase space density of particles $g(\mathbf{v}, t)$ at a given velocity $\mathbf{v}$ and time $t$, then $\int g(\mathbf{v}, t) d^2\mathbf{v} = N(t)/A$. The rate of particles leaving the cavity is $dN(t)/dt = -\Delta L \int \mathbf{v} \cdot \mathbf{n} g(\mathbf{v}, t) d^2\mathbf{v}$, where $\mathbf{n}$ is the external normal vector to the leaking boundary. We assume the phase space to be isotropic and homogeneous. Then, for a small hole and for a time scale shorter than the average lifetime, there is a quasi-equilibrium distribution of particles sharing the same closed system’s properties. Here, $d^2\mathbf{v} = 2\pi v dv$ and $\mathbf{v} \cdot \mathbf{n} = v \cos \phi, \phi \in (-\pi/2, \pi/2)$, leading to $N(t)/A = 2\pi \int g(\mathbf{v}, t) v dv$, so the probability density for the velocity modulus is given by $\omega(v) dv = 2\pi g(\mathbf{v}, t) v dv A/N(t)$ and $dN(t)/dt = -N(t) \Delta L \langle v \rangle / \pi A$, where $\langle \mathbf{v} \rangle = \int \omega(v) v dv$. Defining the average lifetime as $\langle \tau \rangle = 1/\kappa = \pi A / \Delta L \langle v \rangle$, it is found an exponential survival probability $P(t) = N(t)/N(0) = \exp(-\kappa t)$. The escape time probability density is given by $p(t) dt = -dP(t)$. Since the collision time $<t_{coll}>$ can be measured experimentally, we can write $\kappa = \mu(I) / <t_{coll}>$, where $\mu(I) = \Delta L / L$ is a Lebesgue measure and $<t_{coll}> = \pi A / L \langle v \rangle$ (this measure simply state $<t_{coll}> / <\tau> = \Delta L / L$).

We emphasize that the above considerations are valid under the following conditions: i) the phase space is unchanged due to the small size of the leak and ii) the correlation time decays exponentially due to the particles’ chaotic dynamic nature. Generally, these conditions are experimentally atypical, however the exponential behavior of the survival probability would remain valid for a finite time (for the finite time leak case). Therefore, it becomes necessary to improve the escape rate estimate. We introduce a new conditional invariant measure, $\mu_c : \mu(I) \to \mu_c(I)$, taking into account long living particles. So $<t_{coll}> \to <t_{coll}>_c = \int t_{coll}(\mathbf{x}) d\mu_c$, with $\mathbf{x} = (s, p_s)$ the phase space coordinates along the cavity boundary.

Transient chaos theory emerges from a set of non-attracting chaotic saddle in phase space. In chaotic leaky billiards, the saddle possesses a stable and an unstable manifold causing an exponential decay of the orbit survival probability. Meanwhile, regions with regular motion are composed by stable periodic orbits and also by quasi-periodic KAM tori [9]. It is well known that in closed Hamiltonian systems, both can coexist together with chaotic motion regions. For

![Figure 1](image-url). The unsymmetrical stadium billiard of unitary parameter, where are plotted six different types of unstable periodic orbits classified by similarity.
Figure 2. Figure left-side shows the behavior of the temporal evolution for some orbits’ survival probabilities and on the right-side a typical Birkhoff phase space portrayal, for a billiard with a hole’s width $\delta_1 \approx 0.056$ located on its upper boundary at $s = -0.025$.

instance, any trajectory into the chaotic region approximating some KAM surface, must travel for a long time before exiting the surface, this is the so called ‘stickiness’ behavior. In turn, stickiness and weak chaos are driven by trajectories of measure zero. Therefore, the existence of sticky regions must modify the asymptotic decay of the survival probability, i.e. from an exponential ($e^{-\kappa t}$) to a power law ($t^{-z}$) behavior [8]. As the escape rate $\kappa$ is a property of the saddle, the exponent $z$ is related to the chaotic region’s properties close to the regular regions of measure zero. Thus, we can expect a cross over between both different temporal behaviors and a critical time $t_c$ characterizing the first trajectory approaching the sticky region.

3. Trajectories’ evolution, the Birkhoff map and the sprinkler method
As discussed in [7], we consider the square region with unitary (say cm.) parameter denoted by $a$, with initial conditions $(0, y')$ for the position and with momentum $(-p_{x'}$, $\pm p_{y'})$. Next, the zigzagging particle travels a distance of 2 units along the x-direction until it reaches the point $(0, \tilde{y})$, with $\tilde{y} = \text{sign}(p_{\tilde{y}}) [\text{sign}(p_{y'}) y' + 2 \{\Lambda_T\}]$; with the fractional-part-function $\{\Lambda_T\} = \frac{|p_{y'}|}{p_{x'}} - \left\lfloor \frac{|p_{y'}|}{p_{x'}} \right\rfloor$ and the momentum’s modulus $|p_{y'}| = |p_{\tilde{y}}| = \sqrt{2 - p_{x'}^2}$. Thus, inside the quarter of circle with unitary radius, the initial conditions are $(0, \tilde{y})$ and $(+\nu, \pm p_{\tilde{y}})$, where the angular momentum is $\nu = \tilde{y} p_{x'}$. If the particle rotates an angular distance $\pi$ in order to reach the position $(0, y'' \rangle$, where $\frac{\dot{\nu}}{y''} = \text{sign}(p_{y''}) \left[ \cos (2\theta) - \text{sign}(p_{\tilde{y}}) \frac{p_{y''}}{p_{x'}} \sin (2\theta) \right]$, $\theta = \{\Lambda_R\} \cos^{-1} (\nu / \sqrt{2})$.

Figure 3. Left side figure shows a log-log plot of the time evolution of the orbits’ survival probabilities. Beside, the semi-logarithmic plot displays the behavior of the survival orbit’s mean collision time on the billiard’s upper boundary at $s = -0.975$, for one hole’s width $\delta_1 = 0.056$. 
and \( \{ A_R \} = \frac{\pi/2}{\cos^{-1}(\nu/\sqrt{2})} - \frac{\pi/2}{\cos^{-1}(\nu/\sqrt{2})} \) finally we have \( p_{ext} = -\frac{\mu}{2\kappa} p_{ext} \).

In order to construct the Birkhoff map, we chase the particle’s time-evolution using the above dynamical equations, finding the position \( s \) and momentum \( p_s \) at the \( i-th \) and the \( j-th \) contact points, at the upper billiard boundary. Then, in the square region, we have a displacement \( s_i \rightarrow s_j \), a collision time \( \Delta t_{ij} = \frac{\nu}{p_s^{\nu}} \) with \( s_j = s_i \pm |p_s| \Delta t_{ij}; \ s_j \in (-1, 0) \) and \( |p_s_j| = |p_s| \in (0, \sqrt{2}) \). Finally, in the circular region, we have a re-scaled angular displacement \( s_i \rightarrow s_j \), a collision time \( \Delta t_{ij} = \sqrt{2 - \nu^2} \) and \( s_j = s_i \pm \frac{2}{\pi} \cos^{-1}(\frac{\nu}{\sqrt{2}}); \) with \( s_j \in (0, 1) \) and \( |p_s_j| = |\nu| \in (0, \sqrt{2}) \).

On the Sprinkler method, in the spirit of transient theory [10], this alternative method allows for fast calculations. We consider \( N_0 \gg 1 \) uniformly distributed trajectories on phase space, and we chose an appropriate time \( t^* \gg 1/\kappa \). This initial set of trajectories evolves reaching time \( t^* \), and the subgroup of no escaping trajectories \( (\approx N_0 e^{-\kappa t^*}) \) are kept. Particularly, the set of trajectories with long life time must come closer to the neighborhood of the saddle, following their dynamical evolution pathway. Therefore, the complementary set submitted to an escape processes are distributed along the unstable manifold. Pumping new trajectories compensate the loss due to the escape process, then the density is multiplied by a factor \( e^{\kappa t} \), remaining invariant [10]. This stationary distribution is known to be a conditionally invariant measure.

4. Numerical Results

We consider the unsymmetrical stadium billiard with parameters \( a = 1 \text{ cm}, \ \phi_0 = \pi/2 \text{ rad}, \) billiard’s length \( L = 5.571 \text{ cm} \) and area \( A = 1.785 \text{ cm}^2 \). We set up different hole sizes on the billiard’s upper boundary, i.e. \( \delta_1 = 0.01 \text{ L} \) and \( \delta_2 = 0.005 \text{ L} \), both localized at \( s = \pm 0.975, \pm 0.5 \) and \( \pm 0.025 \). We consider a particular set of marginal periodic orbits classified by similarity, i.e. the type \( O_{1,3,4} \) and type \( O_{2,5,6} \); as displayed in Figure 1. Next, following the Sprinkler method, we consider a number of \( 10^4 \) orbits with initial conditions starting at \( x = 0 \) and nearby to \( O_1 : y = 0.95, \ \phi = 2.4 \phi_0, O_2 : y = 0.61, \ \phi = 1.9 \phi_0, O_3 : y = 0.05, \ \phi = 1.6 \phi_0, O_4 : y = 0.05, \ \phi = 1.15 \phi_0, O_5 : y = 0.95, \ \phi = 1.9 \phi_0; \) and \( O_6 : y = 0.05, \ \phi = 1.9 \phi_0 \) uniformly distributed into a range \( \delta y = \delta p_y = 0.01 \), known \( p_y = \sqrt{2} \sin \phi \), to evolve according to the dynamical equations developed in section 3.

Figure 2 shows the typical behavior of different orbit types survival probability as a function of the time evolution and a typical Birkhoff phase space portrayal, in figure’s left-side and right-side: respectively. Here, a hole is set up at \( s = -0.025 \) and width size \( \delta_1 \approx 0.056 \). In order
to analyze the survival probability temporal behavior, a log-log plot portrayed is displayed on the left-side Figure 3, for different types of marginal periodic orbits with one hole of width $\delta_1$ located at $s = -0.975$. As discussed in section 2, transient behavior happens at a critical time, for $O_3: t_c \approx 0.015$, $O_{1,4}: t_c \approx 0.1$ and $O_{2,5,6}: t_c \approx 0.06$; in units of $10^3s$. Moreover, into the framework of the leaking theory, we compute numerically the parameters: for $O_3$: $\kappa \approx 0.048$, $O_{1,4}$: $\kappa \approx 0.168$, $O_{2,5,6}$: $\kappa \approx 0.086$ and for any orbit $z \approx 0.49$. Analogously, for two holes with width size $\delta_2 \approx 0.028$ located respectively at $s = \pm 0.975$, the survival probability temporal evolution log-log plot is displayed in Fig 4 left-side. We numerically compute the parameters for $O_{1,3}$: $\kappa \approx 0.03$, $O_4$: $\kappa \approx 0.338$, $O_{2,5,6}$: $\kappa \approx 0.02$ and for any orbit $z \approx 0.51$; respectively.

5. Conclusions
We have used the Sprinkler method [10] to study the temporal behavior of a set of marginal periodic orbit survival probabilities, in an open unsymmetrical stadium billiard. A transient behavior was verified according to leak theory [8]. In order to analyze the computed parameters is important to consider the mean collision time, respectively plotted in Figures 3 and 4 right-side. For different orbits, we found variations on the collision time mean values before the critical time is reached, justifying the different numerical values for the escape rate. Then, using the naive approximation, we computed for one hole: $O_3: k \approx 0.0475$, $O_{1,4}: k \approx 0.1$, $O_{2,5,6}: k \approx 0.0702$. A comparison between these escape rate values with those numerically computed, looks reasonable. Now, for the two holes case, we found that the escape rate can take values close to those for the one-hole case, thus the escape rate is not additive. Finally, it is important to discuss the parameter $z$. We found that it converges to the value $1/2$. In reference [11], the authors found numerically the value one, and the same value is computed asymptotically in reference [12]. We noted both works do not make reference to an important parameter, namely the ratio hole size and the the billiard’s size. We have considered marginal periodic orbits, they evolve inside the leaking billiard. When subjected to a small perturbation they spread on phase space, and a subset survives until a critical time when they excited. This critical time and the hole’s size are important parameters to consider when discussing accurate values for both the escape rate and the power law exponent. Work is in progress in order to discuss these issues among others concerning the leaking soft stadium system.

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