On an Airy matrix model with a logarithmic potential

E. Brézin\textsuperscript{a)}

\textit{and}

S. Hikami\textsuperscript{b),c)}

\textsuperscript{a)} Laboratoire de Physique Théorique, Ecole Normale Supérieure
24 rue Lhomond 75231, Paris Cedex 05, France. e-mail: brezin@lpt.ens.fr\textsuperscript{1}

\textsuperscript{b)} Department of Basic Sciences, University of Tokyo, Tokyo 153, Japan.

\textsuperscript{c)} Okinawa Institute of Science and Technology,
Tancha,Onna-son,Okinawa 904-0412,Japan, e-mail:hikami@oist.jp

Abstract

The Kontsevich-Penner model, an Airy matrix model with a logarithmic potential, may be derived from a simple Gaussian two-matrix model through a duality. In this dual version the Fourier transforms of the n-point correlation functions can be computed in closed form. Using Virasoro constraints, we find that in addition to the parameters $t_n$, which appears in the KdV hierarchies, one needs to introduce here half-integer indices $t_{n/2}$. The free energy as a function of those parameters may be obtained from these Virasoro constraints. The large N limit follows from the solution to an integral equation. This leads to explicit computations for a number of topological invariants.

\textsuperscript{1}Unité Mixte de Recherche 8549 du Centre National de la Recherche Scientifique et de l'École Normale Supérieure.
1 Introduction

In some recent articles [1, 2, 3, 4, 5], we have discussed a relationship between the Airy matrix model (Kontsevich model) and a Gaussian random matrix theory with an external source. The free energy for higher Airy matrix models (degree more than three) is the generating function for the intersection numbers of the moduli space of \( p \)-spin curves [6, 7, 8]. We have shown that the Fourier transform of the n-point correlation function of the Gaussian random matrix is also a generating function of these intersection numbers with n-marked points [3, 4]. The reason for this remarkable agreement is a duality for expectation values of characteristic polynomials [2].

It is interesting to extend this duality to the case of the \( c = 1 \) [9] matrix model, i.e. models dealing with a 1D chain of coupled matrices, and to understand the meaning of the topological invariants. In [4], we have shown that the time dependent random matrix theory, is equivalent to a two-matrix model. These matrices are \( N \times N \) and one considers correlation functions involving \( k_1 \) points for the first matrix, and \( k_2 \) for the second. After duality, we have shown that the edge scaling limit is an Airy matrix model with a logarithmic potential

\[
Z = \int dB e^{-\frac{1}{3} \text{tr} B^3 + k_2 \text{tr} \log B + \text{tr} BA}.
\]

where \( B \) is a \( k_1 \times k_1 \) Hermitian matrix, and \( k_2 \) is the number of characteristic polynomials in the two-matrix model [4]. In section 2, we recall the derivation of (1.1).

This Airy matrix model with a logarithmic potential, the so called Kontsevich-Penner model, and its generalization to higher Airy matrix model (degree larger than three) have been discussed in the literature. For the general \((p + 1)\)-valent vertex model \( V(B) = B^{p+1}/(p + 1) \), the partition function

\[
Z = \int dB e^{\text{tr} V(B) + k \text{tr} \log B + \text{tr} BA}
\]

was considered by Mironov et al [10], in particular for small \( \Lambda \), through character expansions. The parameter \( \Lambda \) separates (i) the strong coupling and (ii) the weak coupling regions. The transition between (i) and (ii) is similar to the Brézin-Gross-Witten transition [11, 12] in the unitary matrix model. Indeed, it can be shown that the partition function \( Z \) in (1.2) with \( p = -2 \) is equivalent to the unitary matrix model (see Appendix B). For \( p = 1 \), it has been studied in [13, 14].

The model (1.2) has been considered with polynomial vertices

\[
V(B) = \sum \tilde{t}_n B^n.
\]

\[1\]
as a generating function of tachyon amplitudes in the c=1 string theory [15, 16, 17]. The existence of a two dimensional black hole [19, 20, 40] has been discussed in related matrix models. We also note that some time evolution problems, such as crystal growth or non-intersecting random walks, share interesting universal features described by the edge singularity of a random matrix theory [21, 22, 23, 24, 25, 26].

In this paper, in view of these interesting applications, we study the Airy matrix model, and higher Airy matrix models, with a logarithmic potential, in detail. We first obtain the Virasoro constraints for the partition function $Z$ of (1.1) ; this gives a series expansion in terms of the parameters $t_n = \text{tr}\Lambda^{-n-\frac{1}{2}}$, constructed from the source matrix $\Lambda$. The parameters $t_n$ characterize the intersection numbers of the moduli space of curves and the KdV hierarchies, following Witten’s well-known conjecture [7, 8]. The Virasoro constraints are derived from the equations of motion for this matrix model. The remarkable difference with the Kontsevich model [30] is that the equation of motion becomes here a third order differential equation, due to the presence of the logarithmic term. This differential equation leads to the appearance of a new series of parameters $t_{n/2}$ ($n/2$ is a half-integer) in addition to the $t_n$ ($n$ is an integer) of the KdV hierarchies. It is interesting to analyze the role of these new variables $t_{n/2}$, absent from the usual KdV hierarchies. These parameters $t_{n/2}$ ($n/2$ half-integer) correspond to the Ramond sector in string theory, decoupled from the Neveu-Schwarz sectors in the Kontsevich model or in the theory of intersection numbers of Riemann surfaces. Parameters similar to $t_{n/2}$ appear also in the antisymmetric Gaussian random matrix theory [27].

In the large $N$ limit the equation of motion leads to a Riemann-Hilbert integral equation. We have verified the consistency of the solution with the results derived from the Virasoro constraints. The free energy is expressed through the parameters $t_n$ and $t_{\frac{n}{2}}$. Starting from the duality relation between the Gaussian two-matrix and the Kontsevich-Penner model, we consider the Fourier transform of the correlation functions in the two-matrix model. The terms $t_{\frac{n}{2}}$ are generated by the correlations between the edge of the spectrum and the bulk. The edge behavior and the bulk behavior obey different scaling laws. This paper is organized as follows; in section 2, we recall how a logarithmic potential is generated from a duality relation for characteristic polynomials in the Gaussian two-matrix model. In section 3, the equations of motion are derived for the Kontsevich model (p=2) with a logarithmic potential. The Virasoro constraints are obtained, as differential equations for $t_0$, $t_{\frac{1}{2}}$ and $t_1$. From these differential equations, we construct the series expansion of the free energy $F$ in terms of the $t_n$ and $t_{\frac{n}{2}}$. In section 4, the integral equation (Riemann-Hilbert problem) is solved for the Airy matrix model with a logarithmic potential. In section 5, we discuss the replica
method which provides explicit results for one marked point. The corresponding correlation function of the two-matrix model is discussed and the result of its Fourier transform is compared with the result of the free energy obtained from Virasoro constraints. The section 6 is devoted to discussions. The formula which expresses the $p$-th derivatives with respect to the external source matrix $\Lambda$ in terms of its eigenvalues is presented in the appendix A. In Appendix B, the equivalence between the unitary matrix model with an external source and $(p = -2)$-higher Airy matrix model with a logarithmic potential is briefly sketched.

2 A logarithmic potential

For the one-matrix model, we have shown that the Kontsevich model is related to a matrix model at the edge of its spectrum, through a duality relation and the replica method. For the mathematical definition of the intersection numbers of the moduli space of curves, we refer to [8]. It involves an integration over the compactified moduli space $\bar{M}_{g,n}$ with genus $g$ and $n$-marked points,

$$<\tau_{d_1} \cdots \tau_{d_n}> = \int_{\bar{M}_{g,n}} c_1(L_1)^{d_1} \cdots c_1(L_n)^{d_n}.$$  \hspace{1cm} (2.1)

where $c_1$ is the first Chern class and $L_i$ is a cotangent line bundle at the $i$-th marked point. This definition of the intersection numbers has been generalized [6] to the moduli space of $p$-spin curves. The intersection numbers have now an additional spin-index like in $\tau_{n,j}$, in which $j$ takes values from 0 to $p - 1$. They are defined by

$$<\tau_{d_1,j_1} \cdots \tau_{d_n,j_n}> = \frac{1}{p^D} \int_{\bar{M}_{g,n}} c_D(V)c_1(L_1)^{d_1} \cdots c_1(L_n)^{d_n}.$$ \hspace{1cm} (2.2)

where $c_D(V)$ is a D-dimensional top Chern class of the vector bundle $V$, $V = H^1(\Sigma, T)$. There is a cover of Riemann surface, and the line bundle $L$ has $p$ roots,

$$L \simeq T^\otimes p$$ \hspace{1cm} (2.3)

where $T$ is isomorphism class. There are $(p - 1)$ roots corresponding to the Neveu-Schwarz sector and one to the Ramond sector in string theory.

These intersection numbers with spin indices may be computed from the $p$-th higher Airy matrix model. Non-vanishing intersection numbers satisfy the condition,

$$\sum_{i=1}^{n} d_i + D = 3g - 3 + n$$ \hspace{1cm} (2.4)
In the limit $p \to -1$ the top Chern class becomes the Euler characteristics with marked points. In this case, the higher Airy matrix model reduces to the Penner model (logarithmic potential), from which one computes the Euler characteristics $\chi_{g,n}$ \cite{37,38}. In this section, we will discuss the duality between the Kontsevich model with a logarithmic potential (1.1) and the Gaussian two-matrix model with an external source.

In previous papers \cite{4,29}, we have discussed the time dependent ($c = 1$) Gaussian random matrix theory. We have shown that the correlation function at two different times is equivalent to the correlation function of a two-matrix model with the following probability distribution,

$$ P(M_1, M_2) = \frac{1}{Z} \exp \left( -\frac{1}{2} \text{tr}M_1^2 - \frac{1}{2} \text{tr}M_2^2 - c\text{tr}M_1M_2 + \text{tr}M_1A_1 + \text{tr}M_2A_2 \right) \tag{2.5} $$

where $Z$ is a normalization constant.

We consider the correlation function $F_{k_1,k_2}$ of characteristic polynomials, which is defined by the $(k_1 + k_2)$-point correlation function,

$$ F_{k_1,k_2} = \left< \prod_{\alpha=1}^{k_1} \det(\lambda_\alpha - M_1) \prod_{\beta=1}^{k_2} \det(\mu_\beta - M_2) \right> \tag{2.6} $$

where the average $\left< \cdots \right>$ is computed with the distribution $P$ in (2.5).

Let us briefly review how this correlation function reduces to the Kontsevich model with a logarithmic potential \cite{4}. This $F_{k_1,k_2}$ is expressed as a Grassmann integral over $\psi_\alpha(\alpha = 1, ..., k_1)$, $\chi_\beta(\beta = 1, ..., k_2)$,

$$ F_{k_1,k_2} = \left< \int d\bar{\psi}d\psi d\bar{\chi}d\chi e^{\bar{\psi}_\alpha (\lambda_\alpha - M_1)\psi_\alpha + \bar{\chi}_\beta (\mu_\beta - M_2)\chi_\beta} \right> \tag{2.7} $$

The integration over the matrices $M_1$ and $M_2$ is then easy with the Gaussian distribution (2.5). It yields quartic terms in $\psi$ and $\chi$. These terms may be expressed as integrals over Hermitian matrices $B_1, B_2$ and complex matrices $D, D^\dagger$. ($k_1 \times k_1$ for the auxiliary matrix $B_1$, $k_2 \times k_2$ for $B_2$, and $k_1 \times k_2$ for the complex matrix $D$).

$$ \exp\left[ -\frac{N}{2(1-c^2)} \bar{\psi}\psi \bar{\psi}\psi \right] = \int dB_1 \exp \left( -\frac{N}{2} \text{tr}B_1^2 + \frac{iN}{\sqrt{1-c^2}} B_1 \bar{\psi}\psi \right) $$

$$ \exp\left[ -\frac{N}{2(1-c^2)} \bar{\chi}\chi \bar{\chi}\chi \right] = \int dB_2 \exp \left( -\frac{N}{2} \text{tr}B_2^2 + \frac{iN}{\sqrt{1-c^2}} B_2 \bar{\chi}\chi \right) $$

$$ \exp\left[ \frac{Nc}{1-c^2} \bar{\psi}\chi \bar{\psi}\chi \right] = \int dDdD^\dagger \exp \left( -N\text{tr}D^\dagger D + \frac{N\sqrt{c}}{\sqrt{1-c^2}} \text{tr}(D\bar{\psi}\chi + D^\dagger \bar{\chi}\psi) \right) \tag{2.8} $$

Note that we have traded the integrations over $N \times N$ matrices by integrals over matrices whose sizes are given by $k_1$ and $k_2$. One can then integrate out
the Grassmann variables $\psi$ and $\chi$, and $F_{k_1,k_2}$ is expressed as

$$F_{k_1,k_2} = \int dB_1dB_2dDdD^\dagger e^{-\frac{N}{2}\text{tr}(B_1^2 + B_2^2 + 2D^\dagger D) + N\text{trlog}(1-X)}$$

(2.9)

where

$$X = \begin{pmatrix}
\frac{i\sqrt{1-c^2}}{a_1-c_2} & \frac{\sqrt{c(1-c^2)}}{a_1-c_2}D \\
\frac{\sqrt{c(1-c^2)}}{a_2-c_1}D^\dagger & \frac{i\sqrt{1-c^2}}{a_2-c_1} \tilde{B}_2
\end{pmatrix}. $$

(2.10)

with

$$(B_1)_{\alpha\alpha'} = (B_1)_{\alpha\alpha'} - i\sqrt{1-c^2}\lambda_\alpha \delta_{\alpha\alpha'}$$

$$(B_2)_{\beta\gamma'} = (B_2)_{\beta\gamma'} - i\sqrt{1-c^2}\mu_\beta \delta_{\beta\gamma'} $$

(2.11)

We have assumed that the external source matrices $A_1$ and $A_2$ are multiple of the identity $A_1 = a_1 \cdot I, A_2 = a_2 \cdot I$. Introducing the diagonal matrices $\Lambda_1$ and $\Lambda_2$

$$\Lambda_1 = \text{diag}(\lambda_1, ..., \lambda_{k_1}), \quad \Lambda_2 = \text{diag}(\mu_1, ..., \mu_{k_2})$$

we obtain

$$F_{k_1,k_2} = e^{\frac{N}{2}(1-c^2)\text{tr}(\Lambda_1^2 + \Lambda_2^2)} \int dB_1dB_2dDdD^\dagger e^{-\frac{N}{2}\text{tr}(\tilde{B}_1^2 + \tilde{B}_2^2 + 2D^\dagger D) + N\text{trlog}(1-X)}$$

$$\times e^{-iN\sqrt{1-c^2}\text{tr}\tilde{B}_1\Lambda_1 - iN\sqrt{1-c^2}\text{tr}\tilde{B}_2\Lambda_2}$$

(2.13)

We now restrict ourselves to $a_2 = 0$. From the expression of $X$, we have

$$\text{tr}X^2 = -\frac{(1-c^2)}{a_1^2}\text{tr}\tilde{B}_1^2 - \frac{1-c^2}{c^2a_1^2}\text{tr}\tilde{B}_2^2 - \frac{2(1-c^2)}{a_1^2}\text{tr}D^\dagger D$$

(2.14)

If we take $a_1^2 = 1-c^2$, the quadratic term in $\tilde{B}_1$ is cancelled. The quadratic term in $\tilde{B}_2$ does not vanish, it becomes $-\frac{N}{2}(1-c^2)\text{tr}\tilde{B}_2^2$. The quadratic term $\text{tr}D^\dagger D$ is also cancelled.

We now denote $\tilde{B}_1, \tilde{B}_2$ by $B_1, B_2$. We find

$$\text{tr}X^3 = -\frac{i(1-c^2)^{3/2}}{a_1^3}\text{tr}B_1^3 - \frac{3i(1-c^2)^{3/2}}{a_1^3}\text{tr}D^\dagger B_1 + \frac{3i(1-c^2)^{3/2}}{ca_1^3}\text{tr}D^\dagger DB_2$$

$$+ \frac{i(1-c^2)^{3/2}}{c^3a_1^3}\text{tr}B_2^3$$

(2.15)

Given the factor $N$ in the exponent, the edge scaling limit under consideration corresponds to

$$B_1 \sim O(N^{-\frac{1}{2}}), B_2 \sim O(N^{-\frac{1}{2}}), D \sim O(N^{-\frac{1}{2}})$$

(2.16)
in the large N limit, since the quadratic term in $B_2$ does not vanish. In this limit most terms disappear; for instance

$$N \text{tr}(D^\dagger D B_2) \sim N^{-\frac{1}{2}}$$

(2.17)
is negligible. Then, in the large N limit (2.16), after dropping the negligible terms, we obtain

$$F_{k_1,k_2} = \int dB_1 dB_2 dD dD^\dagger e^{-iN \text{tr}B_1 A_1 - iN \text{tr}B_2 A_2 + \frac{i}{3} N \text{tr}B_3^3 - \frac{N}{2}(1 - \frac{1}{c^2}) \text{tr}B_2^2 + iN \text{tr}(D D^\dagger B_1)}$$

(2.18)

Since the matrix $B_2$ is decoupled, we can integrate it out. Then, dropping the contribution from the integral over $B_2$, we find

$$F_{k_1,k_2} = \int dB_1 dD dD^\dagger e^{-i \text{tr}B_1 A_1 + \frac{i}{3} \text{tr}B_3^3 + i \text{tr}DD^\dagger B_1}$$

(2.19)

where we have absorbed the powers of $N$ in a rescaling.

We may now integrate out the matrices $D$ and $D^\dagger$ ($D$ is a $k_1 \times k_2$ complex matrix); this yields a one matrix integral with a logarithmic potential,

$$F_{k_1,k_2} = \int dB_1 e^{i \frac{3}{2} \text{tr}B_3^3 - k_2 \text{tr} \log B_1 - i \text{tr}B_1 A_1}.$$

(2.20)

where $B_1$ is a $k_1 \times k_1$ Hermitian matrix. If we chose the replacement $B_1 \to -iB$, we obtain the model (1.1).

3 Virasoro constraints

The Kontsevich model with a logarithmic potential,

$$Z = \int dB e^{\text{tr}(-\frac{i}{2}B^3 + \lambda B + k \log B)},$$

(3.1)
in which $B$ is an Hermitian $P \times P$ matrix (we have replaced $k_1$ by $P$ and $k_2$ by $k$), satisfies the trivial equations of motion,

$$\int dB \frac{\partial}{\partial B_{ba}} e^{\text{tr}(-\frac{1}{2}B^3 + \lambda B + k \log B)} = 0$$

(3.2)
from which one obtains readily

$$\left(-\left(\frac{\partial}{\partial \lambda}\right)_{ab}^3 + \left(\lambda^T \frac{\partial}{\partial \lambda}\right)_{ab} + (P + k) \delta_{ab}\right) Z = 0.$$  

(3.3)
Since $Z$ is a function of the eigenvalues $\lambda_i$ of $\Lambda$, one can trade this for differential equations in terms of these eigenvalues (see appendix A),

$$
\frac{\partial^3 Z}{\partial \lambda_1^3} + \sum_{d \neq c} \frac{1}{\lambda_c - \lambda_d} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_d} \right) (2 \frac{\partial}{\partial \lambda_1} + \frac{\partial}{\partial \lambda_2}) Z
$$

$$
- \sum_{d \neq c} \frac{1}{(\lambda_c - \lambda_d)^2} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_d} \right) Z + 2 \sum_{d \neq c, e} \frac{1}{(\lambda_c - \lambda_e)(\lambda_e - \lambda_d)} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_e} \right)
$$

$$
- \lambda_c \frac{\partial Z}{\partial \lambda_c} - (P + k) Z = 0
$$

(3.4)

The zero-th order contribution for large $\Lambda$, is obtained from the shift $B \rightarrow B + \Lambda^{1/2}$; then keeping only the terms which grow for large $\Lambda$ one finds

$$
Z_0 = \int dB e^{-trB^2\Lambda^{1/2} + \frac{2}{3}tr\Lambda^{3/2} + \frac{1}{3}tr\log\Lambda}
$$

$$
= \frac{1}{\prod_{i,j}(\sqrt{\lambda_i} + \sqrt{\lambda_j})^{3/2}} e^{\frac{2}{3} \sum \lambda_i^{3/2}} \prod \lambda_i^{\frac{2}{3}}
$$

(3.5)

In the limit $\lambda_i \rightarrow \infty$, the partition function reduces to $Z_0$. Therefore, the partition function $Z$ may be expressed as

$$
Z = Z_0 g(\lambda)
$$

(3.6)

where $g$ has an expansion in inverse powers of $\sqrt{\lambda}$:

$$
g = 1 + O\left(\frac{1}{\lambda^{2}}\right)
$$

(3.7)

The Virasoro constraints (3.4) lead to a sequence of equations, which fix the coefficients of the terms $\lambda_c^{-2}$. Let us thus write the Virasoro constraints in terms of the function $g$ of (3.6). For this purpose, we have to substitute $Z_0$ into (3.4). The resulting equations are cumbersome. To avoid complicated and long expressions, we take the simple case of $P = 2$. Although this simple case is manifestly not sufficient to determine the expansion in terms of the $t_n$, it is instructive and useful also for arbitrary $P$ as shown below. Then for $P = 2$, the equations (3.4) become

$$
\left( \frac{\partial^3}{\partial \lambda_1^3} + \frac{1}{\lambda_1 - \lambda_2} \left( \frac{\partial}{\partial \lambda_1} - \frac{\partial}{\partial \lambda_2} \right) (2 \frac{\partial}{\partial \lambda_1} + \frac{\partial}{\partial \lambda_2}) \right)
$$

$$
- \frac{1}{(\lambda_1 - \lambda_2)^2} \left( \frac{\partial}{\partial \lambda_1} - \frac{\partial}{\partial \lambda_2} \right) - \lambda_1 \frac{\partial}{\partial \lambda_1} - (2 + k)
$$

$$
Z = 0
$$

(3.8)

Using $Z = Z_0 g$, we obtain the equations for $g$,

$$
a_1 g + a_2 \frac{\partial g}{\partial \lambda_2} + a_3 \frac{\partial g}{\partial \lambda_1} + a_4 \frac{\partial^2 g}{\partial \lambda_1^2} + a_5 \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} + a_6 \frac{\partial^2 g}{\partial \lambda_2^2} + a_7 \frac{\partial^3 g}{\partial \lambda_1^3} = 0
$$

(3.9)
where

\begin{align*}
a_1 &= \frac{1}{\sqrt{\lambda_1}} \left( \frac{1 - 2k}{4} \right) + \frac{1}{\lambda_1} \left( \frac{1 - 2k}{2\sqrt{\lambda_2}} - \frac{(2k - 1)(2k - 5)}{16\lambda_2^2} \right) \\
&\quad + \frac{1}{\lambda_1^2} \left( \frac{5 - 24k + 12k^2}{16} \right) + \frac{1}{\lambda_1^2} \left( \frac{k - 1}{2\lambda_2} \right) + \frac{1}{\lambda_1^2} \left( \frac{(k - 1)(2k - 3)}{4\lambda_2} \right) \\
&\quad + \frac{1}{\lambda_1^2} \left( \frac{k - 1}{\lambda_2^2} \right) + \frac{1}{\lambda_1^3} \left( \frac{(2k - 1)(2k - 5)(2k - 9)}{64} \right) \\
&\quad \quad - 1 + 16k - 12k^2 \frac{1}{32\lambda_1^2} \frac{1}{\sqrt{\lambda_1 + \sqrt{\lambda_2}}} \\
&\quad \quad \quad \quad \quad (3.10)
\end{align*}

\begin{align*}
a_2 &= \frac{1}{2} \frac{1}{\sqrt{\lambda_1 + \sqrt{\lambda_2}}} - \frac{3}{2} \frac{1}{\lambda_1 - \sqrt{\lambda_2}} + \frac{1 - 2k}{\lambda_1^2 \lambda_2} + \frac{1}{\lambda_1^2 \sqrt{\lambda_2}} \\
&\quad - \frac{3k}{4\lambda_1^2} \frac{1}{\lambda_1 + \sqrt{\lambda_2}} + \frac{1 - \frac{3}{2}k}{\lambda_1^3} \frac{1}{\lambda_1 - \sqrt{\lambda_2}} + \frac{1}{4\lambda_1(\sqrt{\lambda_1} - \lambda_2)^2} (3.11)
\end{align*}

\begin{align*}
a_3 &= 2\lambda_1 + \frac{3k}{\sqrt{\lambda_1}} - \frac{1}{2} \frac{1}{\lambda_1 + \sqrt{\lambda_2}} + \frac{3}{2} \frac{1}{\lambda_1 - \sqrt{\lambda_2}} \\
&\quad + \frac{15 - 36k + 12k^2}{16\lambda_1^2} + \frac{1 - 2k}{4\lambda_1 \lambda_2} \\
&\quad - \frac{3k}{4\lambda_1^2} \frac{1}{\lambda_1 + \sqrt{\lambda_2}} - \frac{1 - \frac{3}{2}k}{\lambda_1^3} \frac{1}{\lambda_1 - \sqrt{\lambda_2}} - \frac{1}{4\lambda_1(\sqrt{\lambda_1} - \lambda_2)^2} (3.12)
\end{align*}

\begin{align*}
a_4 &= 3\sqrt{\lambda_1} - \frac{3(1 - 2k)}{4\lambda_1} - \frac{1}{2\sqrt{\lambda_1(\sqrt{\lambda_1} + \sqrt{\lambda_2})}} + \frac{1}{\sqrt{\lambda_1(\sqrt{\lambda_1} - \lambda_2)}} (3.13)
\end{align*}

\begin{align*}
a_5 &= a_6 = -\frac{1}{\lambda_1 - \lambda_2} \quad (3.14) \\
a_7 &= 1 \quad (3.15)
\end{align*}

We now return to general $P$ (not simply $P = 2$) and define the parameters $t_n$ as

\[ t_n = \sum_{i=1}^{P} \frac{1}{\lambda_i^{n+\frac{1}{2}}} \quad (3.16) \]

in which $n$ takes both integer and half-integer values ($n = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$). Note that only integers appear in the Kontsevich model. The appearance of half-integers is a characteristic of the present model with a logarithmic potential. The derivatives with respect to $\lambda_j$ are replaced by

\[ \frac{\partial}{\partial \lambda_j} = \sum_n \frac{\partial t_n}{\partial \lambda_j} \frac{\partial}{\partial t_n} = -\sum_n \left( n + \frac{1}{2} \right) \frac{1}{\lambda_j^{n+\frac{1}{2}}} \frac{\partial}{\partial t_n} \quad (3.17) \]
\[
\frac{\partial^2}{\partial \lambda_j^2} = \sum_n \frac{(n + \frac{1}{2})(n + \frac{3}{2})}{\lambda_j^{n+\frac{1}{2}}} \frac{\partial}{\partial t_n} + \sum_n \sum_m \frac{(n + \frac{1}{2})(m + \frac{1}{2})}{\lambda_j^{m+n+3}} \frac{\partial^2}{\partial t_n \partial t_m} 
\tag{3.18}
\]

\[
\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} = \sum_n \sum_m \frac{(n + \frac{1}{2})(m + \frac{1}{2})}{\lambda_1^{n+\frac{1}{2}} \lambda_2^{m+\frac{1}{2}}} 
\tag{3.19}
\]

\[
\frac{\partial^3}{\partial \lambda_1^3} = -\sum_n \frac{(n + \frac{1}{2})(n + \frac{3}{2})(n + \frac{5}{2})}{\lambda_1^{n+\frac{1}{2}}} \frac{\partial}{\partial t_n} - \sum_n \sum_m \frac{(n + \frac{1}{2})(m + \frac{1}{2})(2n + m + \frac{9}{2})}{\lambda_1^{n+m+4}} \frac{\partial^2}{\partial t_n \partial t_m} - \sum_n \sum_m \sum_j \frac{(n + \frac{1}{2})(m + \frac{1}{2})(j + \frac{1}{2})}{\lambda_1^{n+m+j+\frac{9}{2}}} \frac{\partial^3}{\partial t_n \partial t_m \partial t_j} 
\tag{3.20}
\]

where \( n, m, j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \).

Returning now to \( P = 2 \), at lowest order in the \( 1/\sqrt{\lambda_1} \) expansion, \( a_1 \) becomes

\[
a_1 \sim \frac{1}{\sqrt{\lambda_1}} \left( \frac{1}{4} \left( \frac{1}{\sqrt{\lambda_1}} + \frac{1}{\sqrt{\lambda_2}} \right)^2 - \frac{k}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) \right) + O\left( \frac{1}{\lambda_1} \right) 
\tag{3.21}
\]

This may be expressed in terms of the \( t_n \) in (3.16) as,

\[
a_1 \sim \frac{1}{\sqrt{\lambda_1}} \left( \frac{1}{4} t_0^2 - \frac{k}{2} t_1 \right) 
\tag{3.22}
\]

\[
a_2 \frac{\partial}{\partial \lambda_2} \sim -\frac{1}{\sqrt{\lambda_1}} \left( \frac{1}{4} t_0 \frac{\partial}{\partial t_0} + \frac{1}{2} \frac{\partial}{\partial t_0} + \cdots \right) 
\tag{3.23}
\]

\[
a_3 \frac{\partial}{\partial \lambda_1} \sim 2 \lambda_1 \frac{\partial}{\partial \lambda_1} \sim -\frac{1}{\sqrt{\lambda_1}} \frac{\partial}{\partial t_0} 
\tag{3.24}
\]

We have considered only the case \( P = 2 \), but this simple calculation is enough to determine correctly the coefficients of \( t_0^2 \) and \( t_1 \), which are defined as the sum of \( \lambda_i \) up to \( i = P \) as (3.16).

The coefficients \( a_4, a_5, a_6, a_7 \) do not appear yet at this order since the multiplications of derivatives in (3.9) give higher orders in \( \lambda_1^{-1} \). Then, we obtain the first equation of order \( \lambda_1^{-1/2} \),

\[
\left( -\frac{\partial}{\partial t_0} + \frac{1}{4} t_0^2 - \frac{k}{2} t_1 + \sum_{n=0, \frac{1}{2}, 1, \ldots} (n + \frac{1}{2}) t_{n+1} \frac{\partial}{\partial t_n} \right) g = 0 
\tag{3.25}
\]

Using \( F = \log g \), it becomes

\[
\frac{\partial F}{\partial t_0} = \frac{1}{4} t_0^2 - \frac{k}{2} t_1 + \sum_{n=0, \frac{1}{2}, 1, \ldots} (n + \frac{1}{2}) t_{n+1} \frac{\partial F}{\partial t_n} 
\tag{3.26}
\]
For the next order $\lambda_1^{-\frac{1}{2}}$, we need to evaluate $a_1$ for $N = 3$, since the $N = 2$ results are not sufficient to determine the coefficients of $t_n$ which appear in the equation. We obtain

$$a_1 = \frac{1}{\sqrt{\lambda_1}} \left( \frac{1}{4} - \frac{k}{2} \right) \left( \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) + \frac{1}{2 \sqrt{\lambda_2 \lambda_3}}$$

$$+ \frac{1}{\lambda_1} \left( \frac{1}{2} - k \right) \left( \frac{1}{\sqrt{\lambda_2}} + \frac{1}{\sqrt{\lambda_3}} \right) + \frac{(-5 + 12k - 4k^2)}{16} \left( \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \right)$$

$$+ \left( -\frac{1}{2} + \frac{k}{2} \right) \left( \frac{1}{\lambda_2 \lambda_3} + \frac{1}{\lambda_3^2 \sqrt{\lambda_3}} + \frac{1}{\lambda_2^2 \sqrt{\lambda_2}} \right)$$

(3.27)

From (3.11), (3.12) and (3.13), we obtain the second equation, involving now a derivative with respect to the $t_{\frac{1}{2}}$,

$$\left( -2 \frac{\partial}{\partial t_{\frac{1}{2}}} - kt_0 + \frac{k}{4} t_{\frac{1}{2}}^2 + \frac{k}{2} t_0 t_{\frac{1}{2}} - \frac{1}{4} t_0^2 t_{\frac{1}{2}} - \frac{1}{16} t_{\frac{1}{2}} - \frac{1}{4} k^2 t_{\frac{1}{2}} \right)$$

$$- \sum_{n=0,\frac{1}{2},1,\ldots} (2n+1) t_{n+\frac{1}{2}} \frac{\partial}{\partial t_n} + k \sum_{n=0,\frac{1}{2},1,\ldots} (n+\frac{1}{2}) t_{n+\frac{1}{2}} \frac{\partial}{\partial t_n}$$

$$- \frac{1}{2} \sum_{-i-j+k=-\frac{1}{2}} (k + \frac{1}{2}) t_i t_j \frac{\partial}{\partial t_k} - \frac{1}{2} \sum_{-i+j+k=-\frac{1}{2}} (j + \frac{1}{2}) (k + \frac{1}{2}) t_j \frac{\partial^2}{\partial t_j \partial t_k} g = 0$$

(3.28)

The next order is proportional to $\lambda_1^{-\frac{3}{2}}$, and we obtain

$$\left( -3 \frac{\partial}{\partial t_{\frac{1}{2}}} - \frac{1}{16} + \frac{3}{4} k^2 + \frac{k}{2} - \frac{1}{16} t_0 t_{\frac{1}{2}} \right)$$

$$- \sum_{n=0,\frac{1}{2},1,\ldots} (\frac{1}{2} + n) t_n \frac{\partial}{\partial t_n} - \sum_{n=0,\frac{1}{2},1,\ldots} (n+\frac{1}{2}) t_{n+\frac{1}{2}} \frac{\partial}{\partial t_n} g = 0$$

(3.29)

These equations determine the free energy $F = \log g$ as up to order $O(\lambda^{-\frac{3}{2}})$,

$$F = \frac{1}{12} t_0^3 + \frac{1}{48} t_1 + \frac{1}{2} kt_0 t_{\frac{1}{2}} + \frac{1}{4} k^2 t_1$$

$$+ \frac{1}{24} t_0^3 t_1 + (\frac{1}{192} + \frac{1}{16} k^2) t_1^2 + \frac{1}{4} k t_0 t_{\frac{1}{2}} t_1 + \frac{1}{24} k t_{\frac{1}{2}}^2 t_1 + (\frac{1}{32} + \frac{3}{8} k^2) t_0 t_2 + \frac{1}{4} k t_0^2 t_{\frac{1}{2}}$$

$$+ \frac{1}{4} k^2 t_{\frac{1}{2}} t_{\frac{1}{2}} + \frac{1}{6} (k + k^2) t_{\frac{1}{2}}$$

$$+ \frac{1}{64} t_0 t_2 + \frac{1}{6} k t_0 t_{\frac{1}{2}} + \frac{1}{48} t_0^3 t_1 + (\frac{5}{128} + \frac{15}{32} k^2) t_0^2 t_3$$

$$+ \frac{3}{16} k t_0^2 t_{\frac{1}{2}} + \frac{1}{4} k t_0 t_1 t_{\frac{1}{2}} + \frac{1}{2} (k + k^3) t_0 t_{\frac{1}{2}} + \frac{1}{12} k^2 t_0 t_{\frac{1}{2}} t_{\frac{1}{2}}$$

10
This expression is consistent with the previous result \[4\]. Note that the parameters \(t_n\) with the half-integers \(n, t_\frac{1}{2}, t_\frac{3}{2}, \ldots\), appear together with the coefficients proportional to \(k\). When \(k\) goes to zero, the free energy \(F\) of (3.30) reduces to the Kontsevich free energy. Another remarkable property of (3.30) is that when \(k\) is of order \(P\), many terms are of the same order in the large \(P\) limit. The leading order is \(P^2\) which gives genus zero contributions.

We will discuss the large \(P\) limit in a later section from a different approach based on integral equations.

To express these equations in compact form, it is convenient to introduce the differential operators \(J_n^{(k)}\), obtained as follows \[31\].

\[
J_m^{(1)}(x) = \frac{\partial}{\partial x_m} - mx_m, \quad (m = \ldots, -2, -1, 0, 1, 2, \ldots) \quad (3.31)
\]

and \(x_m = 0\) for \(x \geq 0\). We define \(J_m^{(k)} (k > 1)\) from \(J_m^{(1)}\) as

\[
J_m^{(2)} = \sum_{i+j=m} : J_i^{(1)} J_j^{(1)} : \quad (3.32)
\]

where \(\cdots\) means normal ordering, i.e. pulling the differential operator to the right. Then we obtain

\[
J_m^{(2)} = \sum_{i+j=k=m} \frac{\partial^2}{\partial x_i \partial x_j} + 2 \sum_{-i-j=k=m} ix_j \frac{\partial}{\partial x_j} + \sum_{-i-j-k=m} (ix_i)(jx_j) \quad (3.33)
\]

\[
J_m^{(3)} = \sum_{i+j+k=m} : J_i^{(1)} J_j^{(1)} J_k^{(1)} : \quad (3.34)
\]

where \(i, j, k = 1, 2, 3, \ldots\)

By setting

\[
x_n = \frac{1}{n} t_n, \quad (3.35)
\]
we find
\[ J_{-4}^{(2)} = 2t_0t_1 + t_\frac{3}{2} + 4 \sum_{n=0, \frac{1}{2}, 1, \ldots} (n + \frac{1}{2})t_{n+2} \frac{\partial}{\partial t_n} \] (3.36)
\[ J_{-2}^{(2)} = t_0^2 + 2 \sum_{n=0, \frac{1}{2}, 1, \ldots} (2n + 1)t_{n+1} \frac{\partial}{\partial t_n} \] (3.37)
\[ J_{-1}^{(2)} = 4 \sum_{n=0, \frac{1}{2}, 1, \ldots} (n + \frac{1}{2})t_{n+\frac{1}{2}} \frac{\partial}{\partial t_n} \] (3.38)
\[ J_0^{(2)} = 4 \sum_{n=0, \frac{1}{2}, 1, \ldots} (n + \frac{1}{2})t_n \frac{\partial}{\partial t_n} \] (3.39)

From (3.34), we have
\[ J_{-4}^{(3)} = 3t_0^2t_\frac{3}{2} + 3 \sum_{-i+j+k=-\frac{3}{2}} (2j+1)(2k+1)t_{i} \frac{\partial^2}{\partial t_j \partial t_k} \]
\[ + 3 \sum_{-i-j+k=-\frac{3}{2}} (k + \frac{1}{2})t_i t_j \frac{\partial}{\partial t_k} \] (3.40)

Then, the first equation for the Virasoro constraints is expressed by
\[ \left( -\frac{\partial}{\partial t_0} + \frac{1}{4} J_{-\frac{3}{2}}^{(2)} - \frac{k}{2} t_\frac{1}{2} \right) g = 0 \] (3.41)

The second equation becomes
\[ \left( -2 \frac{\partial}{\partial t_\frac{1}{2}} - kt_0 - \frac{1}{16} t_\frac{3}{2} - \frac{k^2}{4} t_\frac{3}{2} - \frac{1}{12} J^{(3)} - \frac{k}{4} J^{(2)} - \frac{1}{2} J_{-1}^{(2)} \right) g = 0 \] (3.42)

The third equation is expressed by
\[ \left( -3 \frac{\partial}{\partial t_1} - \frac{1}{16} - \frac{3}{4} k^2 + kt_0 t_\frac{1}{2} - \frac{1}{4} J_0^{(2)} - \frac{1}{4} J_{-3}^{(2)} \right) g = 0 \] (3.43)

The differential operator \( J_{m}^{(3)} \) appears only for the equation of order \( \lambda_{1}^{-n} \) (n=1,2,3,\ldots). This is similar to the p-spin generalized Kontsevich model without logarithmic term, where spin 0 equations are described by \( J_{n}^{(2)} \) and the spin non-zero equation of motion is described by \( J_{m}^{(3)} \) [33].

If we denote the differential operator \( \frac{1}{4} J_{2m}^{(2)} \) as \( L_m \):
\[ L_n = \frac{1}{4} J_{2n}^{(2)} \] (3.44)
then those \( L_n \) have the commutation relations
\[ [L_n, L_m] = (n - m)L_{n+m} \] (3.45)
4 Integral equation for the Airy matrix model

For the unitary matrix model the large N limit may be solved by a Riemann-Hilbert integral equation \[11\]. We apply here the same technique to the Kontsevich model with a logarithmic potential.

When \( k=0 \) (the Kontsevich model), the equation of motion reduces to a simpler second order equation. Let us first consider the \( k = 0 \) case as an exercise, following \[11\].

\[
\frac{\partial^2 Z}{\partial \lambda_c^2} + \sum_d \frac{1}{\lambda_c - \lambda_d} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_d} \right) Z - \lambda_c Z = 0
\]  

(4.1)

Changing to the free energy \( W \)

\[ Z = e^{PW} \]  

(4.2)

(the original Gaussian matrices were \( N \times N \), but the dual matrices are \( P \times P \), and

\[
\frac{\partial Z}{\partial \lambda_c} = P \left( \frac{\partial W}{\partial \lambda_c} \right) Z = PW_c Z.
\]  

(4.3)

Introducing the density of eigenvalues

\[ \rho(x) = \frac{1}{P} \sum_a \delta(x - \lambda_a) \]  

(4.4)

we consider \( W \) as a functional of \( \rho \) from which one obtains \( W_a \) as

\[ W_a = w(x)|_{x=\lambda_a} \]  

(4.5)

with

\[ w(x) = \frac{1}{P} \frac{d}{dx} \frac{\delta W}{\delta \rho(x)} \]  

(4.6)

The second derivative in (4.1) leads to two terms, but in the large \( P \)-limit the leading one is simply \( w(x)^2 \), leading to the integral equation

\[
w^2(x) + \int_a^b dy \rho(y) \frac{w(x) - w(y)}{x - y} = x
\]  

(4.7)

We define \( f \) and \( F \) as

\[ f(z) = \int_a^b dx \frac{\rho(x)}{z - x} \]  

(4.8)

\[ F(z) = \int_a^b dx \frac{\rho(x) w(x)}{z - x}. \]  

(4.9)

Inside the cut \( z \in [a, b] \),

\[ \text{Re} F(z) = w^2(z) + w(x) \text{Re} f(z) - z \]  

(4.10)
and for $z \in [-\infty, \infty]$,

$$\text{Im}F(z) = w(z)\text{Im}f(z) \quad (4.11)$$

We make the ansatz

$$F(z) = w^2(z) + w(z)f(z) - z \quad (4.12)$$

leading to

$$\text{Im}(\text{Re}f + 2\text{Re}w) = 0 \quad z \in [-\infty, \infty], \quad (4.13)$$

$$\text{(Im}w)(\text{Im}w + \text{Im}f) = 0 \quad z \in [a, b] \quad (4.14)$$

From (4.12), in the $z \to \infty$ limit, we find $F \sim 1/z, f(z) \sim 1/z$, and

$$w(z) = \sqrt{z} - \frac{1}{2z} + O(z^{-3/2}) \quad (4.15)$$

Since $\text{Im}w \neq 0$ for $z \in [-\infty, -c]$, we have from (4.13),

$$\text{Re}w = -\frac{1}{2}\text{Re}f \quad (4.16)$$

This is equivalent to

$$\text{Im}(w(z)\sqrt{z + c}) = -\frac{1}{2}f(z)\sqrt{-z - c} \quad (z \in [-\infty, -c]) \quad (4.17)$$

Then by dispersion relation, we get

$$w(z)\sqrt{z + c} = -\frac{1}{2} \int dy \frac{f(y)\sqrt{-y - c}}{z - y}$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{-c} dy \int_a^b dx \frac{\rho(x)\sqrt{-y - c}}{(y - x)(z - y)} \quad (4.18)$$

Noting that

$$\int_{-\infty}^{-c} dy \frac{\sqrt{-y - c}}{(y - x)(z - y)} = \int_c^{\infty} dt \frac{\sqrt{t - c}}{(t + x)(t + z)}$$

$$= \frac{\pi}{\sqrt{z + c} + \sqrt{x + c}} \quad (4.19)$$

Adding the integral constant $z + \frac{c}{2}$, which is determined from the asymptotic behavior of (4.15), we get

$$w(z)\sqrt{z + c} = z + \frac{c}{2} - \frac{1}{2} \int_a^b dy \frac{\rho(y)}{\sqrt{z + c + \sqrt{y + c}}} \quad (4.20)$$

and the parameter $c$ is determined from the condition that there is no pole at $z = -c$,

$$c = -\int_a^b dy \frac{\rho(y)}{\sqrt{y + c}} \quad (4.21)$$
Thus we obtain
\[ w(z) = \sqrt{z + c} + \frac{1}{2} \int_a^b dy \frac{\rho(y)}{(\sqrt{z + c} + \sqrt{y + c})\sqrt{y + c}} \] (4.22)

This function \( w(x) \) is indeed a solution of the integral equation (4.7). The square of the second term, the part of the integration in (4.22), cancels with the second term of (4.7), and the parameter \( c \) is given by (4.21).

By further integration over \( \rho(x) \), we find the free energy \( W \),
\[
W = \frac{2}{3} \int_a^b dz \rho(z)(z + c)^{\frac{3}{2}} - \int_a^b dz \rho(z)\sqrt{z + c} - \int_a^b dz \int_a^b dy \rho(z)\rho(y)\log(\sqrt{z + c} + \sqrt{y + c}) - \frac{1}{12} c^3 \] (4.23)

where we used
\[
\frac{\partial W}{\partial \lambda_c} = \frac{d}{dz} \frac{\delta W(\rho)}{\delta \rho(z)} = w(z) \] (4.24)

The parameter \( c \) satisfies the saddle point equation for \( w \) in (4.23),
\[
\frac{\partial W}{\partial c} = -\frac{1}{4}(c + \sum_d \frac{1}{\sqrt{\lambda_d + c}})^2 = 0 \] (4.25)

and this is consistent with (4.21).

The large \( P \) limit of the free energy \( F \) in (3.30) is obtained by the scaling \( \lambda_i \sim P^{2/3} \) and by taking each sum as order \( P \). The order of \( t_n \) becomes
\[
t_n = \sum \frac{1}{\lambda_i^{n+\frac{1}{2}}} \sim O(P^{\frac{8}{3}(1-n)}) \] (4.26)

The expansion of \( c \) in (4.23) is obtained by the recursive solution with the definition of \( t_n \) \( (t_n = \sum \lambda_i^{-\left(\frac{3}{2}+n\right)}) \),
\[
c = -\sum \frac{1}{\sqrt{\lambda_i + c}} = -t_0 - \frac{1}{2} t_0^2 t_1 - \frac{3}{8} t_0^2 t_2 - \frac{1}{4} t_0^2 t_3 + O(\lambda^{-5}) \] (4.27)

From this equation, \( c \) has to be negative, and \( \lambda > -c \). Therefore, we have only one expansion, the large \( \Lambda \) expansion.

The free energy \( W = \log Z \) in (4.23) is divided into four terms. We expand each term for small \( c \) (\( c \) is a constant expressed by \( t_n \)),
\[
W_1 = \frac{2}{3} \sum (\lambda_i + c)^{\frac{3}{2}}
\]
\[ W_2 = -c \sum_i (\lambda_i + c)^{3/2} \]

\[ W_3 = -\frac{1}{2} \sum_{i,j} \log \left( \sqrt{\lambda_i + c} + \sqrt{\lambda_j + c} \right) \]

\[ W_4 = -\frac{c^3}{12} \]

Inserting the expression of \( c \), we obtain \( W \), which is the sum of these four terms,

\[ W = \frac{2}{3} \lambda_i^{3/2} - \frac{1}{2} \sum_{i,j} \log(\sqrt{\lambda_i} + \sqrt{\lambda_j}) + \frac{1}{4} t_0^3 + \frac{1}{24} t_0^3 t_1 + O\left( \frac{1}{\lambda^3} \right) \]

The first two terms are \( \log Z_0 \) in (3.3) and remainings are consistent with the genus zero part of \( F \) in (3.30). Up to order \( 1/\lambda^3 \), only \( t_0^3 \) and \( t_0^3 t_1 \) are genus zero terms.

The free energy \( F \) is of order \( P^2 \) in the large \( P \) limit. From (3.30), we find in the large \( P \) limit,

\[ u = \frac{\partial^2 F}{\partial t_0^2} = \frac{1}{2} t_0 + \frac{1}{4} t_0 t_1 + \frac{3}{16} t_0^2 t_2 + \frac{1}{8} t_0 t_1^2 + \cdots \]

Thus we find

\[ c = -2u = -2 \frac{\partial^2 F}{\partial t_0^2} \]

Therefore, we understand that \( c \) is the specific heat for the free energy \( F \), when we interpret \( t_0 \) as a temperature.

We now consider the Kontsevich model with a logarithmic term \(( k \neq 0 \)). The equations of motion in (3.4) are expressed as equations for \( W \) and \( W_a \), where

\[ \frac{\partial^3 Z}{\partial \lambda_c^3} = P \left( \frac{\partial^3 W}{\partial \lambda_c^3} \right) Z + 3P^2 \left( \frac{\partial^2 W}{\partial \lambda_c^2} \right) W_c Z + P^3 W_c^3 Z \]

In the large \( P \) limit, the first and the second terms in (4.32) are negligible.
From (3.4), we express it as

\[
\begin{align*}
& w^3(x) - xw(x) + 2 \int \frac{w(x) - w(u)}{(x-u)(u-v)} \rho(u) \rho(v) du dv \\
& - (1 + \frac{k}{P}) + \int du \frac{\rho(u)}{x-u} \left(2w^2(x) - w(x)w(u) - w^2(u)\right) = 0 \quad (4.33)
\end{align*}
\]

From (4.33), we find in the large \(x\) limit,

\[
w(x) \sim \sqrt{x} - \frac{1}{2x} \left(1 + \frac{k}{N}\right) \quad (4.34)
\]

This is a generalization of (4.15) for \(k \neq 0\).

The equation of (4.33) is a cubic equation. If \(w(x)\) has a solution similar to (4.22), the tri-linear terms of \(\rho\) has to be cancelled in this cubic equation of \(w(x)\). First we check that whether the solution of (4.22) satisfies (4.33) when \(k = 0\). We denote

\[
l_x = \sqrt{x} + c, \quad l_y = \sqrt{y} + c, \quad l_z = \sqrt{z} + c, \quad l_s = \sqrt{s} + c \quad (4.35)
\]

The solution for \(k = 0\) is

\[
w(x) = l_x + \frac{1}{2} \int dy \frac{\rho(y)}{(l_x + l_y)l_y} \quad (4.36)
\]

We express the cubic equation of (4.33) in terms of these \(l_x, l_y, l_z, l_s\) by

\[
I_1 + I_2 + I_3 + I_4 + I_5 = 0 \quad (4.37)
\]

\[
I_1 = w^3(x) = l_x^3 + \frac{3}{2} l_x^2 \int dy \frac{\rho(y)}{(l_x + l_y)l_y} + \frac{4}{3} l_x \int dy dz \frac{\rho(y)\rho(z)}{(l_x + l_y)(l_x + l_z)l_y l_z} + \frac{1}{8} \int dy dz ds \frac{\rho(y)\rho(z)\rho(s)}{(l_x + l_y)(l_x + l_z)(l_x + l_s)l_y l_z l_s} \quad (4.38)
\]

\[
I_2 = -xw(x) \quad (4.39)
\]

\[
I_3 = 2 \int dy dz \frac{w(x) - w(y)}{(x-y)(y-z)} \rho(y) \rho(z)
\]

\[
= 2 \int dy dz \frac{l_x - l_y}{(x-y)(y-z)} \rho(y) \rho(z) + \int dy dz ds \frac{\rho(y)\rho(z)\rho(s)}{\rho(s)} \left(\frac{1}{(x-y)(y-z)l_s} - \frac{1}{l_y + l_z l_s}\right) \quad (4.40)
\]
\[ I_4 = -1 \] (4.41)

\[ I_5 = \int dy \frac{\rho(y)}{x - y} \left( 2w^2(x) - w(x)w(y) - w^2(y) \right) \] (4.42)

Up to the first order of \( \rho \), by adding the contribution \( I \) of \( I_1, I_2, I_3, I_4, I_5 \), we have

\[ \Delta I = \frac{c}{2} \int dy \frac{\rho(y)}{(l_x + l_y)l_y} \]
\[ = -\frac{1}{2} \int dy dz \frac{\rho(y)\rho(z)}{(l_x + l_y)l_y l_z} \] (4.43)

where we have used the expression of \( c \) given by (4.21). The summation of above \( \Delta I \) and the second order of \( \rho \) in \( I_1 \) and \( I_5 \) becomes

\[ \Delta I + I_1 + I_5 = 2 \int dy dz \frac{\rho(y)\rho(z)}{(l_x + l_y)(z - y)} \] (4.44)

and this is cancelled by the contribution of \( I_3 \). Thus the contribution up to the second order is cancelled. The terms of the third order of \( \rho \) come from \( I_1, I_3 \) and \( I_5 \). There are triple integrals over \( y, z \) and \( s \). We symmetrize the integrals over these three variables. Before making the symmetrizations, we note that

\[ I_1 = \frac{1}{8} \int dy dz ds \frac{\rho(y)\rho(z)\rho(s)}{(l_x + l_y)(l_x + l_z)(l_y + l_z)l_y l_z l_s} \] (4.45)

(\( I_1 \) has symmetric form).

\[ I_3 = \frac{1}{2} \int dy dz ds \frac{\rho(y)\rho(z)\rho(s)(l_x + l_y + l_z + l_s)}{(l_x + l_y)(l_x + l_y)(l_y + l_s)(l_x + l_z)(l_z + l_s)(l_y + l_z)l_s} \] (4.46)

\[ I_5 = -\frac{1}{4} \int dy dz ds \frac{\rho(y)\rho(z)\rho(s)(l_x + l_s + 2l_y + 2l_z)}{(l_x + l_y)(l_x + l_z)(l_y + l_z)(l_x + l_s)(l_y + l_s)l_x l_s} \] (4.47)

After symmetrization, these three terms cancel completely (\( I_1 + I_3 + I_5 = 0 \)).

Thus, we see that the equation (4.33) is satisfied by the solution of \( w(x) \) in (4.22) when \( k = 0 \).

Since the parameter \( k \) appears only in terms of order zero of \( \rho \) in (4.33), it is easily understood that there is a straightforward solution for \( k \neq 0 \) case, based on the above analysis. Since we have seen the solution of (4.22) satisfies the integral equation of (4.33), we consider the solution (4.22) for \( k \neq 0 \) more carefully, specially the condition for \( c \). Namely, we use the same solution \( w(x) \) as before

\[ w(x) = \sqrt{x + c} + \frac{1}{2} \int_a^b dy \frac{\rho(y)}{(\sqrt{x + c} + \sqrt{y + c})\sqrt{y + c}} \] (4.48)
where we consider that the parameter $c$ is now a function of $x$, $c = c(x)$. We replace all parameters $c$ by $c(x)$ in (4.48). Since $x$ is fixed in the integral equation, this change from a constant to $x$-dependence of $c$ does not make any difference.

Up to first order in $\rho$, by putting this $w(x)$ into the part of first order in $\rho$ of (4.33), we have

$$
\begin{align*}
    w^3(x) - xw(x) - (1 + \frac{k}{P}) + \int dy \frac{\rho(y)}{x-y} \left( 2w^2(x) - w(x)w(y) - w^2(y) \right) \\
    = (x + c)^2 - x\sqrt{x+c} - (1 + \frac{k}{P}) \\
    + (x + \frac{3}{2}c) \int_a^b \frac{\rho(y)}{(\sqrt{x+c} + \sqrt{y+c})\sqrt{y+c}} \\
    + \frac{\int_a^b dy \frac{\rho(y)}{x-y} \left( 2(x + c) - \sqrt{x+c} \sqrt{y+c} - (y+c) \right)}{x-y} \\
    = c\sqrt{x+c} - \frac{k}{P} + \sqrt{x+c} \int_a^b \frac{\rho(y)}{\sqrt{y+c}} + \frac{c}{2} \int_a^b \frac{\rho(y)}{(\sqrt{x+c} + \sqrt{y+c})\sqrt{y+c}}
\end{align*}
$$

(4.49)

where we have used that the integral of $\rho$ is one,

$$\int_a^b \rho(y) dy = 1 \quad (4.50)$$

We now put

$$c = -\int_a^b \frac{\rho(y)}{\sqrt{y+c}} + h \quad (4.51)$$

and r.h.s of (4.49) becomes

$$r.h.s = h\sqrt{x+c} - \frac{k}{P} + \frac{h}{2} \int_a^b \frac{\rho(y)}{(\sqrt{x+c} + \sqrt{y+c})\sqrt{y+c}}$$

$$- \frac{1}{2} \int dz \frac{\rho(z)}{\sqrt{z+c}} \int_a^b \frac{\rho(y)}{(\sqrt{x+c} + \sqrt{y+c})\sqrt{y+c}}$$

(4.52)

The last term is transferred to the part of second order in $\rho$, and we have seen that the second and the third order terms of $\rho$ are cancelled completely. Therefore, we find that

$$h \left( \sqrt{x+c} + \frac{1}{2} \int_a^b \frac{\rho(y)}{(\sqrt{x+c} + \sqrt{y+c})\sqrt{y+c}} \right) - \frac{k}{P}$$

$$= hw(x) - \frac{k}{P} = 0 \quad (4.53)$$
Thus we have the coupled equations,

\[ c = - \int_a^b dy \frac{\rho(y)}{\sqrt{y+c}} + \frac{k}{P} w(x), \]

\[ w(x) = \sqrt{x+c} + \frac{1}{2} \int_a^b dy \frac{\rho(y)}{(\sqrt{x+c} + \sqrt{y+c}) \sqrt{y+c}} \]  \hspace{1cm} (4.54)

This \( w(x) \) satisfies the integral equation \((4.33)\) and in the large \( x \) limit, it satisfies the asymptotic behavior

\[ w(x) \sim \sqrt{x} + \frac{1}{2x} (1 + \frac{k}{P}) \quad (x \to \infty) \]  \hspace{1cm} (4.55)

We obtain the expansion of \( w(x) \) for large \( x \) from the coupled equations in \((4.54)\). By integration over \( \rho(x) \) we obtain the free energy \( W \) for large \( \lambda \) \( (x = \lambda) \),

\[
W = \frac{2}{3} \sum \lambda_i^2 + \frac{k}{2} \sum \log \lambda_i - \frac{1}{2} \sum i,j \log(\sqrt{\lambda_i} + \sqrt{\lambda_j}) \\
+ \frac{1}{12} t_0^2 + \frac{1}{2} t_0 t_1 + \frac{1}{4} k^2 t_1 + \cdots \]  \hspace{1cm} (4.56)

which is consistent with \( \log Z_0 \) in \((3.5)\) and the genus zero part of \( F \) in \((3.30)\).

When \( k \) is sufficient large, we have a solution in which \( c \) is positive in \((4.54)\). In this case, we obtain an expansion for small \( \lambda \).

5 Intersection numbers in a replica limit

In the case of one matrix model, we have used a duality relation between the Kontsevich model and the Gaussian random matrix model at a critical edge point \([1, 2, 3]\). More precisely the Fourier transform of the \( n \)-point correlation function \( U(s_1, \ldots, s_n) \) becomes the generating function of the intersection numbers with \( n \)-marked points. This \( n \)-point correlation function \( U(s_1, \ldots, s_n) \) has a Cauchy integral representation, which is equivalent to the integral of the first Chern class over the moduli space \( \mathcal{M}_{g,n} \).

We have shown in section 2, that there exists a similar duality relation between the partition function of the Kontsevich-Penner model \((1.1)\) and the correlations for the Gaussian distribution \((2.5)\). We want to discuss the origin of the terms \( t_{\frac{j}{2}} \) (half-integer) in this section.

In the expansion of the free energy \( F \) in \((3.30)\), the number of times of appearance of \( t_j \) is the number of marked points according to the definition of the intersection number in \((2.1)\): In \((2.1)\), \( n \) is the number of marked points.
We investigate first the case of one marked point (i.e. a single \( t_j \)). This case is obtained from the replica limit for the matrix \( B \) in (1.1), namely the limit in which its size \( P \) goes to zero. We first make a shift \( B \rightarrow B + \Lambda^{1/2} \) to eliminate the linear term \( \text{tr} BA \),

\[
Z = \int_{P \times P} dB e^{-\frac{1}{3} \text{tr} B^3 - \text{tr} B^2 \Lambda^{1/2} + k \text{tr} \log(\Lambda^{1/2} + B)} \tag{5.1}
\]

where \( B \) is a \( P \times P \) Hermitian matrix. The replica limit for \( B \) means that we take \( P \rightarrow 0 \) limit, selecting thereby the contribution for one marked point \([2, 4]\). In this replica limit, the number of eigenvalues \( \lambda_i \) also goes to zero, and it becomes not necessary to distinguish them. Therefore, we set \( \Lambda = \lambda \cdot I \). Then, the calculation becomes considerably easier. First, we make the rescaling \( B \) to \( B/\lambda_4^{1/4} \). Then the partition function \( Z \) in (5.1) becomes

\[
Z = \int dB e^{-\frac{1}{3} \lambda^{1/2} \text{tr} B^3 - \text{tr} B^2 + k \text{tr} \log(\lambda^{1/2} + B)} \tag{5.2}
\]

We first neglect the cubic vertex \( \text{tr} B^3 \). Then, one recovers for \( Z \) the same model when \( p = 1 \) in (1.2), as was discussed in references \([13, 14]\). We consider here this same model by the duality plus replica method.

\[
Z = \int dB e^{-\text{tr} B^2 + k \text{tr} \log(\lambda^{3/2} + B)} \tag{5.3}
\]

From the duality theorem for characteristic polynomials \([2]\), the above expression has a dual form, which is an integral over a \( k \times k \) Hermitian matrix \( M \):

\[
Z = \int_{k \times k} dM [\det(M + \lambda^{3/2})]^P e^{-\frac{1}{2} \text{tr} M^2} \tag{5.4}
\]

In the limit \( P \rightarrow 0 \), we have \((\hat{\lambda} = \sqrt{2} \lambda^{3/2})\)

\[
\lim_{P \rightarrow 0} \frac{\partial Z}{\partial \lambda} = \int_{k \times k} dM \text{tr} \frac{1}{\hat{\lambda} + M} e^{-\frac{1}{2} \text{tr} M^2} \tag{5.5}
\]

which is the one-particle Green function \( G(\hat{\lambda}) \) for the Gaussian random matrix, and the expansion of the inverse of \( \hat{\lambda} \) is easily obtained as a moment of \( M \) in terms of polynomials of \( k \).

By integration of \( G(\hat{\lambda}) \) about \( \hat{\lambda} \), we obtain \( Z \),

\[
Z = k \log \hat{\lambda} + \sum_{j=1}^{\infty} \frac{1}{(2j)\lambda^{2j}} < \text{tr} M^{2j} > \tag{5.6}
\]
The Gaussian average $\langle \text{tr} M^{2j} \rangle$ is easily evaluated from the integral representation of $U(s)$ [2],

$$
U(s) = \langle \text{tr} e^{sM} \rangle = \frac{1}{s} e^{k^2} \oint \frac{du}{2i\pi} (1 + \frac{s}{u})^k e^{su}
$$

(5.7)

This integral runs over the contour centered at $u = 0$, and it yields

$$
U(s) = k + \frac{k^2}{2} s^2 + \frac{2k^3 + k}{24} s^4 + \frac{k^4 + 2k^2}{144} s^6 + \frac{2k^5 + 10k^3 + 3k}{5760} s^8 + \cdots
$$

(5.8)

From these expressions, we obtain

$$
\langle \text{tr} M^2 \rangle = k^2, \quad \langle \text{tr} M^4 \rangle = 2k^3 + k, \quad \langle \text{tr} M^6 \rangle = 5k^4 + 10k^2,
$$

$$
\langle \text{tr} M^8 \rangle = 14k^5 + 70k^3 + 21k, \quad \cdots
$$

(5.9)

From (5.6), we express $Z$ of (5.6) in terms of $t_j$ ($t_j = \frac{1}{\lambda_j + \frac{1}{2}}$), by noting that $s = \frac{1}{\lambda} = -\frac{1}{\sqrt{2\lambda}}$,

$$
Z = k \log \lambda + \frac{k^2}{4} t_1 + \frac{1}{16} (k + 2k^3) t_2 + \frac{1}{48} (5k^4 + 10k^2) t_4 + \cdots
$$

(5.10)

The term $\frac{1}{4} k^2 t_1$ coincides with the term in $F$ of (3.30).

The tri-valent term, which we have neglected, couples to the logarithmic term, and also make a contribution as polynomials in $k$. The exponent $\exp(-\frac{1}{3} \lambda - \frac{2}{3} \text{tr} B^3)$ is expanded and it gives the contribution in the replica limit $P \to 0$. By the formula of the replica limit, we have nonvanishing average of $\langle \prod_i \text{tr} B^{d_i} \rangle$. This formula is [2]

$$
\lim_{P \to 0} U(s_1, \ldots, s_l) = \lim_{P \to 0} \frac{1}{P} \langle \text{tr} e^{\sigma_1 B} \cdots \text{tr} e^{\sigma_l B} \rangle
$$

$$
= \frac{1}{\sigma^2} \prod_{j=1}^l 2 \sinh \frac{s_j \sigma}{2}
$$

(5.11)

where $\sigma = s_1 + \cdots + s_l$. This provides a generating function for $\langle \prod_i \text{tr} B^{d_i} \rangle$.

From this formula, for instance, we have

$$
\lim_{P \to 0} \frac{1}{P} \langle \text{tr} B^3 \text{tr} B^3 \rangle = 3, \quad \lim_{P \to 0} \frac{1}{P} \langle \text{tr} B^3 \text{tr} B^3 \text{tr} B^2 \rangle = 18, \cdots
$$

(5.12)

Using these values of averages, we are in position to compute the coefficients of the $t_n$ terms. We consider the term $t_5$ in (3.30), which has a coefficient $\frac{2}{3} (k + k^3)$. The partition function $Z$ is

$$
Z = \int dB e^{-\frac{1}{3} \text{tr} B^3 - \text{tr} B^2 \lambda^\frac{1}{2} + k \text{tr} \log (1 + \lambda \cdot \frac{1}{2} B)}
$$

(5.13)
We rescale $B \rightarrow 2^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} B$ and $\lambda^2 \rightarrow 2^{-\frac{1}{4}} \lambda^2$. We have

$$Z = \int dB e^{-\frac{1}{6\lambda^2} \text{tr} B^3 - \frac{1}{k} \text{tr} B^2 + k \text{tr}(1 + \lambda^{-\frac{1}{4}} B)}$$

(5.14)

Expanding then the logarithmic term and $\exp(-\frac{1}{6\lambda^2} \text{tr} B^3)$ term, we find the contributions to order $\frac{1}{\lambda^3}$ from 6 terms. These terms are evaluated by the replica formula ($P \rightarrow 0$ limit) of (5.11),

(i) $\frac{1}{P} < \frac{k}{4\lambda^3} \text{tr} B^4 > = \frac{k}{4\lambda^3}$,

(ii) $\frac{1}{P} < \frac{1}{6\lambda^2} \text{tr} B^3 \cdot \frac{1}{3!} \left( \frac{k}{\lambda^2} \right)^3 \text{tr} B^3 > = \frac{k^3}{6\lambda^3}$

(iii) $\frac{1}{P} < \frac{k}{2} \left( \frac{1}{\lambda^3} \right)^2 \text{tr} B^2 \cdot \frac{1}{2!} \left( \frac{k}{\lambda^2} \right)^2 \text{tr} B > = \frac{k^3}{2\lambda^3}$

(iv) $\frac{1}{P} < \frac{1}{6\lambda^2} \text{tr} B^3 \cdot \frac{1}{3!} \left( \frac{1}{\lambda^2} \right)^3 \text{tr} B^3 > = \frac{k^3}{6\lambda^3}$

(v) $\frac{1}{P} < \frac{1}{2} \left( \frac{1}{6\lambda^2} \right)^2 (\text{tr} B^3)^2 \cdot \frac{k}{2} \left( \frac{1}{\lambda^2} \right)^2 \text{tr} B^2 > = \frac{k^3}{8\lambda^3}$

(vi) $\frac{1}{P} < \frac{k}{\lambda^2} \text{tr} B \cdot \frac{1}{3!} \left( \frac{1}{6\lambda^2} \right) \text{tr} B^3 > = \frac{k}{8\lambda^3}$

(5.15)

Adding these (i)∼(vi) terms, and noting that we have made a scaling of $\lambda$, we obtain as expected the result $\frac{1}{6}(k^3 + k)\lambda^{-3} = \frac{1}{6}(k^3 + k)t_{\frac{3}{2}}$ for (3.30).

We now evaluate the coefficients of $t_j$ from the Fourier transform of one-point correlation function $U(s)$. We have shown that the intersection numbers of n-marked point are obtained from the Fourier transforms of n-point correlation function $U(s_1, ..., s_n)$ \[3, 4\]. The Kontsevich-Penner model involves two interaction terms $\text{tr} B^3$ and $\text{tr} \log B$. For the application of $U(s_1, ..., s_n)$ to this Kontsevich-Penner model, we have to extend the previous duality expression.

Let us return to the duality relation for the one-matrix model, before extending it; in this one-matrix case the duality reads

$$< \prod_{\alpha=1}^{k} \det(\lambda_\alpha - M) >_{M, A} = < \prod_{j=1}^{N} \det(a_j - iB) >_{B, \Lambda}$$

(5.16)

in which the l.h.s. consists of Gaussian average in an external matrix source $A$ for Hermitian $N \times N$ matrices; the r.h.s. is also a Gaussian average for Hermitian $k \times k$ matrices in an external source $\Lambda$ (whose eigenvalues are the $\lambda_\alpha$ of the l.h.s.; the $a_j$ are the eigenvalues of $A$). We also know in closed form the Fourier transform of the n-point correlation function,

$$U(s_1, ..., s_n) = < \prod_{j=1}^{n} \text{tr} e^{s_j M} >_{M, A}$$
$$\oint \prod_{i=1}^{N} \frac{du_i}{2i\pi} e^{N \sum u_i s_i + \frac{1}{2} N \sum s_i^2 \prod_{i=1}^{N} (1 + \frac{s_i}{u_i - a_j}) \det \frac{1}{u_i - u_j + s_i}}$$

(5.17)

Let us consider the one-point function

$$U(s) = \frac{1}{s} \oint \frac{du}{2i\pi} \prod_{j=1}^{N} (1 + \frac{s}{u - a_j}) \exp(Nus + \frac{1}{2} Ns^2)$$

$$= \frac{1}{s} \oint \frac{du}{2i\pi} \exp(-\sum_{m=1}^{\infty} c_m [(u + s)^m - u^m] + Nus + \frac{1}{2} Ns^2)$$

(5.18)

where

$$c_m = \frac{1}{m} \sum_{j=1}^{N} (a_j)^m$$

(5.19)

The r.h.s. of the duality formula (5.16) is also expanded in powers of $B$,

$$\langle \prod \det(a_j - iB) \rangle = \int dB \exp(-\sum_{m=1}^{\infty} c_m (iB)^m - \frac{1}{2} N \text{tr} B^2 + N \text{tr} B \Lambda)$$

(5.20)

From this representation we have investigated the (p,1)-model (the (2,1) corresponds to Kontsevich model), which is obtained by specifying appropriately the $a_j$ [3]. We consider here a more general situation, in view of encompassing the Kontsevich-Penner model. The (p,q)-model is defined by

$$Z = \int dB \exp(-c_{p+1} \text{tr} B^{p+1} - c_{q+1} \text{tr} B^{q+1} + \text{tr} B \Lambda)$$

(5.21)

This is obtained by imposing the following conditions to the $a_j$,

$$\frac{1}{2} \sum_{j=1}^{N} (a_j)^{p+1} = \frac{N}{2}$$

$$\frac{1}{m} \sum_{j=1}^{N} (a_j)^m = 0$$

(5.22)

where $m = 3, 4, ..., q$ and $m \neq p + 1$. These conditions should be understood as holding in the large N limit.

These conditions may also be applied to $U(s)$ in (5.18). Then we have

$$U(s) = \frac{1}{s} \oint \frac{du}{2i\pi} e^{-c_{q+1}((u + s)^q - u^q) - c_{p+1}((u + s)^p - u^p)}$$

(5.23)

For the application to the Kontsevich-Penner model, we have to take the limit $p \to -1$ and $q \to 2$. Since $c_{p+1} = \frac{1}{p+1} \sum (a_j)^{p+1}$, we have

$$Z = \int_{k \times k} dB \exp(-c_3 \text{tr} B^3 - N \text{tr} \log B + \text{tr} B \Lambda)$$

(5.24)
and
\[ U(s) = \frac{1}{s} \oint \frac{du}{2i\pi} e^{-c_3((u+s)^3-u^3)-N\text{trlog}(\frac{u+s}{u})} \] (5.25)

For the n-point correlation functions, we have similarly
\[ U(s_1, ..., s_n) = \oint \prod_{i=1}^{n} \frac{du_i}{2i\pi} e^{-\sum_i c_3((u_i+s_i)^3-u_i^3)-N\sum_i \text{trlog}(\frac{u_i+s_i}{u_i})}\det \frac{1}{u_i - u_j + s_i} \] (5.26)

The intersection numbers for one-marked point, which are obtained from \( Z \) as the coefficients of the linear terms in the \( t_j \)
\[ Z = \sum <\tau_j> t_j + \text{higher degree}, \quad (5.27) \]

are derived from \( U(s) \) as coefficients of the expansion in powers of \( s \).

Apart from notations, in which we have to interchange \( N \) to \( k \), and further \( k \to -k \), and chose \( c_3 = \frac{1}{3} \), \( Z \) is then identical to the Kontsevich-Penner model in (1.1). We then have
\[ U(s) = e^{-\frac{1}{6}s^3} \oint \frac{du}{2i\pi} e^{-su^2-s^2u(u+s)k} \] (5.28)

We now expand \( U(s) \) in powers of \( s \) and \( k \). To that purpose we shift \( u = v - \frac{1}{2}s \), and \( v = sz/2 \), and obtain
\[ U(s) = \frac{1}{2} e^{-\frac{1}{12}s^3} \int \frac{dz}{2i\pi} e^{-\frac{1}{6}s^3z^2}\left(\frac{z+1}{z-1}\right)^k \] (5.29)

This gives an expansion in powers of \( k \), from
\[ \left(\frac{z+1}{z-1}\right)^k = 1 + k\text{log}(\frac{z+1}{z-1}) + \frac{1}{2}k^2[\text{log}(\frac{z+1}{z-1})]^2 \]
\[ + \frac{k^3}{6}[\text{log}(\frac{z+1}{z-1})]^3 + O(k^4) \] (5.30)

The first order of \( k \) leads to the integral
\[ \int_{-\infty}^{\infty} dz e^{-\frac{1}{6}s^3z^2}\text{log}(\frac{z+1}{z-1}) = 2i(\frac{\pi}{s})^{\frac{1}{2}} \text{erf}(\frac{1}{2}sz) \] (5.31)

where \( \text{erf}(x) \) is the error function. For small \( s \) and small \( k \), we obtain
\[ U(s) = k - \frac{1}{6}ks^3 + O(ks^6) \] (5.32)

The integrals for odd powers of the logarithm may be computed analytically. For instance, from (5.29), in the the small \( s \) and \( k \) expansion, we obtain a term of order of \( s^3k^3 \),
\[ I = -\frac{s^3k^3}{2} \int_{-\infty}^{\infty} dz \left(\frac{1}{12} + \frac{z^2}{4}\right)\frac{1}{6}[\text{log}(\frac{z+1}{z-1})]^3 \]
\[ = \frac{1}{6i\pi s^3k^3} \] (5.33)
Therefore we recover from (5.32) and (5.33), the coefficient of $s^3$ as $\frac{1}{6}(k + k^3)s^3$, which yields the expected $\frac{1}{6}(k + k^3)t_2^2$ term in (3.30).

We note that the odd powers of $k$ may be obtained systematically by computing a residue at $z = 1$ in the contour integral (5.29). For the even powers of $k$, we have to integrate the logarithmic terms.

The replica formula (5.11) has been derived with a single logarithmic integral, which has a cut for $-s < u < 0$. This replica formula corresponds to the $k \to 0$ limit of the $(p,q)$-model with $q = 1$ and $p = -1$, which leads to the intersection numbers with one-marked point. In an appendix, we present the two-point function $U(s_1, s_2)$ for $q = 1$ and $p = -1$ as a polynomial in $k$, consistent with the replica formula for $k \to 0$. From this example, we understand that the contours for the $n$-point function should encompass all poles. We now consider the two-point case for the Kontsevich-Penner model with two marked points:

$$U(s_1, s_2) = e^{\frac{k}{3}(s_1^3 + s_2^3)} \oint du_1 du_2 \frac{e^{u_1 s_1 + u_2 s_2}}{(2\pi i)^2 (u_1 - u_2 + s_1)(u_1 - u_2 + s_2)} \frac{k}{u_1} \frac{u_2}{u_1}$$

where the contours for $u_2$ are such that one sums over the three contributions from the poles at $u_2 = 0, u_2 = u_1 + s_1, u_2 = u_1 + s_2$ and then the contour for $u_1$ circles around the origin. This yields

$$U(s_1, s_2) = k^2(s_1^2 s_2 + s_1 s_2^2) + \frac{k^2}{12}(s_1^5 s_2 + s_1 s_2^5) + \frac{k^4}{4} + \frac{7}{12}k^2)(s_1^4 s_2^2 + s_1^2 s_2^4) + \frac{k^4}{4} + \frac{3}{4}k^2)s_1^3 s_2^3 + O(s^9)(5.35)$$

Changing from $s^m$ to $t_{m-\frac{1}{2}}$, we obtain the terms with two marked points of (3.30).

Thus we find that the terms with $n$-marked point for the Kontsevich-Penner model (11) are expressed explicitly by the integral formula of $U(s_1, ..., s_n)$ of (5.26). The Kontsevich-Penner model is the $p = -1, q = 2$ of the $(p,q)$-model of (5.21). Following the same techniques as above we find similarly explicit integral representation of $U(s_1, ..., s_n)$ for the $(p,q)$-model with $p = -1$ and arbitrary $q$.

6 Discussions

In this paper, we have extended the analysis of our previous paper [4] to the Kontsevich-Penner model. We have derived the Virasoro constraints for this model, and we have obtained the large N solution of the corresponding integral equation. The occurrence of parameters $t_n$, with half-integer $n$, is due to the logarithmic potential. Using the correlation functions of the Gaussian
two-matrix model, in a source, with one set of eigenvalues near an edge, and
the other one in the bulk of the spectrum, provides the Kontsevich-Penner
model.

We have used an explicit integral representation for $U(s_1, ..., s_n)$ \[5.26\],
which gives then $k$-dependent coefficients of the free energy $F$ of the Kontsevich-
Penner model. This model turns out to be the special limit $p = -1$ and $q = 2$
of a $(p, q)$-model. The integral representation for $U(s_1, ..., s_n)$ is valid also
for general $q$ with $p = -1$. The details for such cases is left to future work.

In string theory, the $c = 1$ matrix model has attracted considerable in-
terest, renewed recently from the D-brane point of view. The tachyon plays
a central role in the $c = 1$ matrix model. In the present study, the partition
function for the Kontsevich-Penner model (1.1), is derived from a time de-
pendent Gaussian matrix model, and the role of the $t_2$ and $k$-dependence are
clearly understood from the correlation functions of the two-matrix model.
Thus, it may shed a light on the $c = 1$ string theory, FZZT-brane, etc [40].

Acknowledgement S.H. is supported by a Grant-in Aid for Scientific
Research (C) of JSPS.
Appendix A: formula for $p$-th derivatives

The matrix $\Lambda$ has eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding orthonormal eigenfunctions $|\phi_a\rangle$. We now consider a perturbation $d\Lambda$,

$$(\Lambda + d\Lambda)(|\phi > + |d\phi >) = (\lambda + d\lambda)(|\phi > + |d\phi >)$$  \hspace{1cm} (A.1)

and from this equation, we obtain at first order

$$(\Lambda - \lambda_a)|d\phi_a > + (d\Lambda - d\lambda_a)|\phi_a >= 0$$  \hspace{1cm} (A.2)

Multiplying $<\phi_a|$ from the left side, it becomes

$$<\phi_a|d\Lambda|\phi_a >= d\lambda_a$$  \hspace{1cm} (A.3)

In an arbitrary fixed orthonormal basis $|b>$, it becomes

$$d\lambda_a = <\phi_a|b><b|d\Lambda|c><c|\phi_a >$$  \hspace{1cm} (A.4)

Therefore, we obtain the first important formula,

$$\frac{\partial \lambda_a}{\partial \Lambda_{bc}} = <\phi_a|b><c|\phi_a >$$  \hspace{1cm} (A.5)

Note that $<\phi_a|b> = U_{ab}$, where $U$ is a unitary matrix. From (A.2), multiplying by $<\phi_b|(b \neq a)$ the left hand side,

$$<\phi_b|d\phi_a >= \frac{1}{\lambda_a - \lambda_b} <\phi_a|d\Lambda|\phi_a >$$  \hspace{1cm} (A.6)

Therefore, we have

$$|d\phi_a > = \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} |\phi_a > <\phi_b|d\Lambda|\phi_a >$$  \hspace{1cm} (A.7)

from which follows the second formula,

$$\frac{\partial <b|\phi_a >}{\partial \Lambda_{cd}} = \sum_{f \neq a} \frac{1}{\lambda_a - \lambda_f} <b|\phi_f <\phi_f|c><d|\phi_a >$$  \hspace{1cm} (A.8)

The conjugate of this formula is

$$\frac{\partial <\phi_a|b>}{\partial \Lambda_{dc}} = \sum_{f \neq a} \frac{1}{\lambda_a - \lambda_f} <\phi_f|b><c|\phi_f <\phi_a|d >$$  \hspace{1cm} (A.9)
By the chain rule, we obtain the first derivative,
\[ \frac{\partial Z}{\partial \Lambda_{ab}} = \frac{\partial \lambda_c}{\partial \Lambda_{ab}} \frac{\partial Z}{\partial \lambda_c} \]
\[ = < b|\phi_c > < \phi_c|a > \left( \frac{\partial Z}{\partial \lambda_c} \right) \]  \hspace{1cm} (A.10)

The formula for the second derivative is obtained by the use of (A.5) and (A.8).
\[ \left( \frac{\partial^2}{\partial \Lambda^2} \right)_{ab} Z = \left( \frac{\partial}{\partial \Lambda} \right)_{ad} \left( \frac{\partial}{\partial \Lambda} \right)_{db} Z \]
\[ = \frac{\partial}{\partial \Lambda_{ad}} \left< b|\phi_c > < \phi_c|a > \left( \frac{\partial Z}{\partial \lambda_c} \right) \right> \]  \hspace{1cm} (A.11)

Noting that
\[ < \phi_c|d > \frac{\partial}{\partial \Lambda_{ad}} < b|\phi_c > \]
\[ =< \phi_c|d > \left( \frac{\partial Z}{\partial \lambda_c} \right) \sum_f \frac{1}{\lambda_c - \lambda_f} < b|\phi_f > < \phi_f|a > < d|\phi_c > \]
\[ = \sum_d < b|\phi_c > < \phi_c|a > \left( \frac{\partial Z}{\partial \lambda_d} \right) \frac{1}{\lambda_c - \lambda_d} \]  \hspace{1cm} (A.12)
and
\[ < b|\phi_c > \frac{\partial}{\partial \Lambda_{ad}} < \phi_c|d > \]
\[ =< b|\phi_c > \left( \frac{\partial Z}{\partial \lambda_c} \right) \sum_f \frac{1}{\lambda_c - \lambda_f} < \phi_f|d > < d|\phi_f > < \phi_c|a > \]
\[ =< b|\phi_c > < \phi_c|a > \left( \frac{\partial Z}{\partial \lambda_c} \right) \sum_d \frac{1}{\lambda_c - \lambda_d} \]  \hspace{1cm} (A.13)
we obtain
\[ \left( \frac{\partial^2}{\partial \Lambda^2} \right)_{ab} Z = < b|\phi_c > \left( \frac{\partial^2}{\partial \lambda_c^2} + \sum_{d \neq c} \frac{1}{\lambda_c - \lambda_d} \left( \frac{\partial Z}{\partial \lambda_c} - \frac{\partial Z}{\partial \lambda_d} \right) \right) < \phi_c|a > \]  \hspace{1cm} (A.14)

The third order differentiation is obtained by repeating the same procedure. We write \( \Gamma_c \) by
\[ \Gamma_c = \frac{\partial^2}{\partial \lambda_c^2} + \sum_{d \neq c} \frac{1}{\lambda_c - \lambda_d} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_d} \right) \]  \hspace{1cm} (A.15)
\[
\left( \frac{\partial^3}{\partial \Lambda^3} \right)_{ab} = \left( \frac{\partial}{\partial \Lambda} \right)_{pa} \left( \frac{\partial^2}{\partial \Lambda^2} \right)_{ab} \\
= \left( \frac{\partial \lambda_c}{\partial \Lambda} \right) \frac{\partial}{\partial \lambda_c} \left( < b|\phi_c > \Gamma_c < \phi_c |a > \right) \\
= < b|\phi_c >< \phi_c |p >< a|\phi_c > < \phi_c |a > \frac{\partial \Gamma_c}{\partial \lambda_c} \\
+ < \phi_c |p >< a|\phi_c > \Gamma_c < \phi_c |a > \frac{\partial < b|\phi_c >}{\partial \lambda_c} \\
+ < \phi_c |p >< a|\phi_c > < b|\phi_c > \Gamma_c \frac{\partial < \phi_c |a >}{\partial \lambda_c} 
\]

Therefore, we obtain
\[
\left( \frac{\partial^3}{\partial \Lambda^3} \right)_{ab} = < b|\phi_c > \left( \frac{\partial \Gamma_c}{\partial \lambda_c} + \sum_{d \neq c} \frac{1}{\lambda_c - \lambda_d} (\Gamma_c - \Gamma_d) \right) < \phi_c |a > 
\]

(A.16)

Using the identity,
\[
\frac{1}{(\lambda_c - \lambda_d)(\lambda_c - \lambda_e)} + \frac{1}{(\lambda_d - \lambda_c)(\lambda_d - \lambda_e)} + \frac{1}{(\lambda_e - \lambda_c)(\lambda_e - \lambda_d)} = 0 
\]

(A.18)

we obtain the expression in terms of eigenvalues
\[
\left( \frac{\partial^3}{\partial \Lambda^3} \right)_{ab} = \frac{\partial^3}{\partial \lambda_c^3} \\
+ \sum_{d \neq c} \frac{1}{\lambda_c - \lambda_d} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_d} \right) (2 \frac{\partial}{\partial \lambda_c} + \frac{\partial}{\partial \lambda_d}) - \sum_{d \neq c} \frac{1}{(\lambda_c - \lambda_d)^2} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_d} \right) \\
+ 2 \sum_{d \neq e, c \neq c} \frac{1}{(\lambda_c - \lambda_e)(\lambda_e - \lambda_d)} \left( \frac{\partial}{\partial \lambda_c} - \frac{\partial}{\partial \lambda_e} \right) 
\]

(A.19)

If we write
\[
\Gamma_c^{(1)} = \frac{\partial}{\partial \lambda_c} 
\]

(A.20)

we have
\[
\left( \frac{\partial^2}{\partial \Lambda^2} \right)_{ab} = < b|\phi_c > \Gamma_c^{(2)} < \phi_c |a > 
\]

(A.21)

\[
\Gamma_c^{(2)} = \frac{\partial}{\partial \lambda_c} \Gamma_c^{(1)} + \sum_{d} \frac{1}{\lambda_c - \lambda_d} (\Gamma_c^{(1)} - \Gamma_d^{(1)}) 
\]

(A.22)

Repeating this procedure, we obtain
\[
\Gamma_c^{(p+1)} = \frac{\partial}{\partial \lambda_c} \Gamma_c^{(p)} + \sum_{d} \frac{1}{\lambda_c - \lambda_d} (\Gamma_c^{(p)} - \Gamma_d^{(p)}) 
\]

(A.23)
and
\[
\left( \frac{\partial^{p+1}}{\partial \Lambda^{p+1}} \right)_{ab} = < b|\phi_c > \Gamma_c^{(p+1)} < \phi_c|a > \quad (A.24)
\]

Appendix B: Relation to unitary matrix model

We will show that the unitary matrix model with external source \[11\] is equivalent to the higher Airy matrix model with a logarithmic potential for \( p = -2 \) \[3, 10\].

The unitary matrix model (Brezin-Gross model \[11\]) is
\[
Z = \int dU e^{\text{tr}(UA^\dagger + U^\dagger A)} \quad (B.1)
\]
where \( U \) is a unitary matrix and \( A \) is an arbitrary complex matrix. From the unitarity condition, \( UU^\dagger = 1 \), we have
\[
\frac{\partial^2}{\partial A^\dagger \partial A} Z = I \cdot Z \quad (B.2)
\]
Introducing \( \Lambda \) as
\[
\Lambda = AA^\dagger \quad (A_{ij}A_{jk}^\dagger = \Lambda_{ik}) \quad (B.3)
\]
we find
\[
\frac{\partial^2}{\partial A^\dagger_{ij} \partial A_{jk}} = \frac{\partial}{\partial A^\dagger_{ij}} \frac{\partial \Lambda_{qs}}{\partial A_{jk}} \frac{\partial}{\partial \Lambda_{qs}}
\]
\[
= \frac{\partial}{\partial \Lambda_{ks}} \Lambda_{ms} \frac{\partial}{\partial \Lambda_{mi}}
\]
\[
= \Lambda \frac{\partial^2}{\partial \Lambda^2} + N \frac{\partial}{\partial \Lambda} \quad (B.4)
\]
Thus
\[
\left( \Lambda \frac{\partial^2}{\partial \Lambda^2} + N \frac{\partial}{\partial \Lambda} \right) Z = I \cdot Z \quad (B.5)
\]
The equations of motion for the \( p = -2 \) case follows from
\[
\int dM \frac{\partial}{\partial M} e^{\text{tr}M^{-1} + \text{tr}MA + k\text{trlog}M} = 0 \quad (B.6)
\]
Taking two derivatives with respect to \( \lambda \), this leads to
\[
\left( -1 + 2N \frac{\partial}{\partial \Lambda} + \Lambda \frac{\partial^2}{\partial \Lambda^2} + k \frac{\partial}{\partial \Lambda} \right) Z = 0 \quad (B.7)
\]
If we now take \( k = -N \), \( B.7 \) this equation becomes identical to \( B.5 \).
Thus we find that unitary matrix model with an external source is similar to the Kontsevich model with a logarithmic potential. There is a phase transition in this unitary matrix model with a critical point at

$$s = \text{tr} \frac{1}{\sqrt{AA^\dagger}} = \sum_{i=1}^{N} \frac{1}{\lambda_i} = 2.$$  \hspace{1cm} (B.8)
References

[1] E. Brézin and S. Hikami, Vertices from replica in a random matrix theory, J. Phys. A. 40 (2007) 13545. arXiv:0704.2044[math-ph].

[2] E. Brézin and S. Hikami, Intersection theory from duality and replica. Comm. Math. Phys. 283 (2008) 507. arXiv:0708.2210[hep-th].

[3] E. Brézin and S. Hikami, Intersection numbers of Riemann surfaces from Gaussian matrix models. JHEP 10 (2007) 096. arXiv:0709.3378.

[4] E. Brézin and S. Hikami, Computing topological invariants with one and two-matrix models, JHEP 04 (2009) 110, arXiv:0810.1085.

[5] E. Brézin and S. Hikami, Duality and Replicas for a unitary matrix model, JHEP 07 (2010) 067, arXiv:1005.4730.

[6] E. Witten, Algebraic geometry associated with matrix models of two dimensional gravity, in ”Topological Methods in Modern Mathematics”, Publish or Perish INC. 1993, New York. P.235.

[7] E. Witten, On the Kontsevich model and other models of two dimensional gravity, IASSNS-HEP-91/24, (1991).

[8] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. 147, 1 (1992).

[9] E. Brézin, V. Kazakov and Al.B. Zamolodchikov, Scaling violation in a field theory of closed strings in one physical dimension, Nucl. Phys. B338, 673 (1990).

[10] A. Mironov, A. Morozov and G.W.Semenoff, Unitary matrix integrals in the framework of Generalized Kontsevich Model 1. Brézin-Gross-Witten model. arXiv:hep-th/9404005.

[11] E. Brézin and D.J. Gross, The external field problem in the large N limit of QCD. Phys. Lett. B97 (1980) 120.

[12] D. Gross and E. Witten, Phys. Rev. D21 (1980) 446. Possible third-order phase transition in the large-N lattice gauge theory.

[13] L. Chekhov and Yu. Makeenko, A hint on the external field problem for matrix models, arXiv: hep-th/9202006.

[14] I. K. Kostov and M. L. Mehta, Phys. Lett. 189B (1987) 118.
[15] C. Imbimbo and S. Mukhi, The topological matrix model of c=1 string, Nucl. Phys. B449,553 (1995).

[16] S. Mukhi, Topological matrix models, Liouville matrix model and c=1 string theory, arXiv:hep-th/0310287.

[17] J. Ambjorn and L. Chekhov, The NBI matrix model of IIB superstrings, JHEP 9812:007(1998).

[18] S. Mukhi and C. Vafa, Two dimensional black-hole as topological coset model of c=1 string theory. arXiv:hep-th/9301083.

[19] L. Susskind, Matrix theory black holes and the Gross Witten transition, in "Trieste 1998. Nonpertubative aspects of strings, branes and supersymmetry", P. 390 (1998), World Scientific, Singapore.

[20] L.Alvarez-Gaume, P.Basu, M, Marino and S.R.Wadia, Blackhole/String Transition for the small Schwarzschild Blackhole of $AdS_5 \times S^5$ and Critical Unitary Matrix Models, arXiv:hep-th/0605041.

[21] M. Prähofer and H. Spohn, Scale invariance of the PNG droplet and the Airy process, J. Stat. Phys. 108 (2002) 1071.

[22] A. Aptekarev, P. Bleher and A. Kuijlaars, Large n limit of Gaussian random matrices with external source Part II. Commun. Math. Phys. 259 (2005) 367.

[23] C. Tracy and H. Widom, The Pearcey Process, Commun. Math. Phys. 263 (2006) 381.

[24] A. Okounkov and N. Reshetikhin, Random skew plane partitions and the Pearcey process, Commun. Math. Phys. 269 (2007) 571.

[25] E. Brézin and S. Hikami, Universal singularity at the closure of a gap in a random matrix theory, Phys. Rev. B 57 (1998) 4140. arXiv:cond-mat/9804023.

[26] E. Brézin and S. Hikami, Level spacing of random matrices in an external source, Phys. Rev. E58 (1998) 7176. arXiv:cond-mat/9804024.

[27] E. Brézin and S. Hikami, Intersection numbers from antisymmetric Gaussian matrix model, JHEP07(2009)050. arXiv:0804.4531

[28] E. Brézin and S. Hikami, Extension of level-spacing universality. Phys. Rev. E56, 264 (1997). arXiv:cond-mat/9702213.
[29] E. Brézin and S. Hikami, Spectral form factor in a random matrix theory, Phys. Rev. E55, 4067 (1997). arXiv:cond-mat/9608116.

[30] D. Gross and Newman, Unitary and Hermitian Matrices In An External Field II: The Kontsevich Model And Continuum Virasoro Constraints. arXiv: hep-th/9112069.

[31] M. Adler and P. van Moerbeke, A matrix integral solution to two-dimensional $W_p$-gravity, Commun. Math. Phys. 147 (1992) 25.

[32] A. Okounkov, Generating functions for intersection numbers on moduli spaces of curves, Int. Math. Res. Not. 18, 933 (2002).

[33] R. Dijkgraaf, Intersection theory, integrable hierarchies and topological field theory, IASSNS-HEP-91/91 (1991), In New symmetry principles in quantum field theory (Cargèse,1991), page 95-158 Plenum, New York,1992.

[34] C. Itzykson and B.-J.Zuber, Combinatorics of the moduler group II The Kontsevich integrals. Intern. Journ. Mod. Phys. A7 (1992) 5661. arXiv:hep-th/9201001.

[35] Y. Makeenko and G.W.Semenoff, Properties of hermitian matrix models in an external field, Mod. Phys. Lett. A6 (1991) 3455.

[36] E. Brézin and S. Hikami, New correlation functions for random matrices and integrals over supergroups. J. Phys. A 36 (2003) 711.

[37] J.Harer and D.Zagier, The Euler characteristic of the moduli space of curves, Invent. Math. 85 (1986) 457.

[38] R.C.Penner, Perturbative series and the moduli space of Riemann surfaces, J. Diff. Geometry, 27 (1988) 35.

[39] E. Brézin and S. Hikami, Characteristic Polynomials of Random Matrices, Commun. Math. Phys. 214 (2000) 111. arXiv:math-ph/9910005.

[40] A. Mukherjee and S. Mukhi, c=1 matrix models: equivalences and open-closed string duality, arXiv:hep-th/0505180 and refereces there in.