Fan affinity laws from a collision model

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Abstract
The performance of a fan is usually estimated using hydrodynamical considerations. The calculations are long and involved and the results are expressed in terms of three affinity laws. In this paper we use kinetic theory to attack this problem. A hard sphere collision model is used, and subsequently a correction to account for the flow behaviour of air is incorporated. Our calculations prove the affinity laws and provide numerical estimates of the air delivery, thrust and drag on a rotating fan.

(Some figures may appear in colour only in the online journal)

Introduction
The fan is the commonest of all aerodynamic devices, being employed in situations as diverse as living rooms, factories and jet engines. Using fluid dynamics [1] to estimate the performance of a fan is surprisingly difficult, however. The performance of a fan is usually expressed in terms of the three affinity laws given below.

- I. The air delivery rate is proportional to its rotation speed and varies as the cube of its diameter.
- II. The total or static pressure is proportional to the square of the diameter, the square of the rotation speed and the density of the air.
- III. The total power is proportional to the cube of the rotation speed, the fifth power of the diameter, and is proportional to the density of air.

These laws follow from the fluid mechanical actuator disc model [2, 3] proposed by Rankine and Froude. In its simplest form, it carries out a one-dimensional flow analysis assuming the air to be incompressible and inviscid. Bernoulli’s principle [4], conservation of mass and conservation of momentum are employed to obtain the fan performance in terms of the flow velocity at the disc. More complex formulations with rotating stream tube assumption can be performed for a refinement of the basic results.

In this paper we attempt to derive simple proofs of the fan affinity laws starting from basic classical mechanics. Fluid mechanics is completely eschewed in our derivation. As such, our treatment is easily accessible for the junior undergraduate and also for the more enterprising high-school student. We also aim to calculate a numerical estimate of the performance of
Figure 1. The relevant fan parameters used in the modelling. Upper panel shows front view of fan with the axis perpendicular to the plane of the paper. Lower panel shows the view along the arrow indicated in the upper panel. The cross section of the blade is visible. The plane of rotation extends into the paper (we see it in edge view) and hence the axis of rotation lies in the plane of the paper.

| Parameter                     | Symbol |
|-------------------------------|--------|
| Fan radius (half of the sweep)| $R$    |
| Blade chord length            | $L$    |
| Number of blades              | $n$    |
| Rotation speed                | $\omega$ |
| Blade pitch angle             | $\alpha$ |
| Density of air                | $\rho$ |
| Viscosity of air              | $\eta$ |

Table 1. Relevant fan parameters.

A fan when given its geometrical parameters. The form of the affinity laws gives rise to the possibility of a derivation from dimensional analysis [5, 6], but we show why such an approach is inconclusive. The relevant parameters can be readily estimated from common sense. They are as in table 1.

Figure 1 shows these parameters for a fan.
Several problems are apparent. Firstly, the dimension of length can refer to either the chord length or the fan radius and it is not possible to differentiate between the two. Secondly, the number of blades and the pitch angle are dimensionless, so our analysis will not be able to provide the dependences on these quantities. Finally, the realization that the entity $\rho \omega R^4/\eta$ is a dimensionless group makes dimensional analysis ineffective.

We now turn to kinetic theory. This theory models a gas as being composed of innumerable small hard molecules that are moving around and undergoing collisions either with each other or with the walls of the confining container. This model has led to successful predictions of material conductivity (Drude model) [7], viscosity and diffusion coefficient [8]. The exact numerical values obtained from such arguments are generally away from the actual values by a factor of 2 or so, but the order of magnitude as well as the dependences of these quantities on various parameters such as temperature are predicted correctly. It is a reasonably standard exercise to calculate the viscous drag on a moving body from kinetic considerations—the problem set in [8] does this for a spacecraft and that of [9] for a sphere. We now extend these procedures to derive the fan affinity laws.

**Derivation**

From a hard ball model viewpoint, the fan action mechanism is through collisions of air molecules with the fan blades. Let us compare the speed of the blades with that of the thermal motion of the air molecules. A typical fan rotates at, say, 1500 rpm and has a diameter of 60 cm. That gives a blade tip speed of about 50 m $s^{-1}$. On the other hand, the thermal velocity of air molecules is given by $(3RT/M)^{1/2}$, where $R$ is the universal gas constant, $T$ is the Kelvin temperature and $M$ is molar mass of air. This evaluates to about 450 m $s^{-1}$ at a room temperature of 300 K. Hence we assume that the timescale of the collisions with the blades is much larger than that of the interparticle collisions. Because of this we can assume that at the time of collision the particle velocities are random and average out to zero. In other words, there is no drift velocity in the laboratory frame prior to the collision.

To get an estimate of the air delivery we want to find the volume of air sweeping through a surface parallel to the plane of the blades per unit time. For this we need only the axial component of the particle velocity as it emerges from the blades; the tangential component plays no role here. We depict the collision in figure 2.
The left panel of figure 2 shows a rotating fan in the laboratory frame. The right panel zooms in on the section of blade marked in orange (online). This window is located at a radius of $r$ and has a negligible extension in the radial direction. It covers the entire blade width though. In the laboratory frame this segment appears to move horizontally with speed $u = \omega r$ to the right. Hence, in a frame moving with the segment the air has a velocity of $u$ towards the left, which has been shown in the figure. Since the blade is infinitely more massive than the air molecule, viewing the collision in the blade frame is easy; the air molecule simply follows the law of reflection like light off a mirror or a ball off a wall. This has been shown in the right panel of figure 2. The velocity component of interest is the one along the rotation axis. Easy geometry yields the angle values in the figure, and the axial component of the emergence velocity is seen to be $u \sin 2\alpha$. This is in the blade frame. Since the motion of the blade is perpendicular to the axial direction, this component of the velocity will remain unchanged in the ground frame. Hence in the laboratory frame, the axial component of the exit velocity will also be $u \sin 2\alpha$.

We now view the fan through a window at position $(r, \theta)$ and area $r \, dr \, d\theta$. This window is shown by the box (orange online) in figure 3. The volume rate $V$ of flow through this window will be

$$dV = (\omega r \, \sin 2\alpha) \, r \, dr \, d\theta.$$  

We note that $\alpha$ might be a function of $r$. In fact, this is the case in most real fans. As the mathematics becomes ‘messy’ if this feature is included, we simplify by assuming $\alpha$ to be constant. This constant is like the average blade pitch. The constant pitch assumption will be valid throughout the rest of our calculations. Equation (2) must be integrated over all windows to obtain the final answer. At each instant of time, there will only be airflow from the region occupied by the blades, with none from the rest of the fan disc. This will set the limits on the integration over $\theta$. The angular span of a blade, i.e. the range of $\theta$ corresponding to each blade is approximately $(L \cos \alpha)/r$. Here the chord length $L$ might be a function of $r$. Then the volume of displaced air is

$$V = n \int_0^R \int_0^{(L \cos \alpha)/r} \omega r^2 \sin 2\alpha \, d\theta \, dr,$$

where $n$ is the number of blades.
Application of this result to actual fans yields results that fall very short of the actual values. The fallacy occurs because it is assumed that the airflow does not occur in the region where there are no blades. This is contrary to experience. If air is suddenly set into motion, say by blowing, the motion persists for some time even after the cause is removed. This phenomenon has to be incorporated into our model. We also note that a single stimulus cannot cause air to flow forever; the flow velocity must die out in time. Combining these two phenomena, we write the expression for airflow induced by a single stimulus (in this case the blade impact) at $t = 0$ as

$$u(t) = (\omega r \sin 2\alpha) e^{-t/\tau}. \quad (4)$$

The time constant can be chosen to be sufficiently large for negligible decay to be assumed to occur in the time interval between successive passages of the blades. This will allow the velocity to be taken the same at all $\theta$. The ‘improved accuracy’ from a more accurate modelling taking the damping into account will be nullified by the fact that the model itself is inaccurate. This renders futile the tedious calculations that result from the incorporation of the damping term. One might wonder why the damping was introduced in the first place—it is just for the sake of not proposing an unphysical concept like that of perpetual flow. Equation (3) now becomes

$$V = \int_0^{\pi} \int_0^{2\pi} (\omega r^2 \sin 2\alpha) \, d\theta \, dr, \quad (5)$$

which evaluates to

$$V = \frac{2\pi}{3} (\sin 2\alpha) \omega R^3. \quad (6)$$

This is nothing but the first affinity law. We apply equation (6) to a Crompton Greaves High Breeze industrial pedestal fan of diameter 450 mm, chord length approximately 6 cm, pitch angle about 18° and a rotating speed of 1430 rpm. This yields the air delivery to be 124 m$^3$ min$^{-1}$, in excellent agreement with the rated value of 125 m$^3$ min$^{-1}$.

Encouraged by this result, we try to calculate the drag and thrust due to a rotating fan. This will be achieved by considering the change in momentum of the air after being struck by the blade. For this we need to calculate the mass of air being handled by the fan per unit time. We will again use rectangular viewing windows, as in figure 3, but this time the window has to be oriented perpendicular to the flow so that the mass can be calculated correctly. This window is shown in figure 4.

One side of the window is along the radial direction, the second side is normal to the airflow. We will call this direction $y$, whereby each window will cover an area $dr \, dy$. Air flows through this window at a speed $\omega r$, hence the total flow volume through the window in a time $dt$ is

$$d\upsilon = \omega r \, dr \, dy \, dt. \quad (7)$$

The mass flow in infinitesimal time is then

$$dm = \rho \omega r \, dr \, dy \, dt, \quad (8)$$

where we have introduced $\rho$, the density of air. For calculating the drag we are interested in the change in momentum in the direction of the blade motion, i.e. normal to the axial direction on account of the collision. This direction will hereafter be called the impingement direction. From the right panel in figure 2, the impingement component of the exit velocity in the blade frame is $u \cos 2\alpha$. The transformation to the laboratory frame involves subtraction of the blade velocity $u$, hence the velocity component as seen from there is $u(-1+\cos 2\alpha)$. Since the sign
Figure 4. Viewing window in orange (online). It extends vertically into the plane of the paper. The red arrows show the area through which airflow takes place. The width of this region is seen to be \( L \sin \alpha \).

is arbitrary, we reverse it, taking care to be consistent later on. Thus the infinitesimal change in momentum

\[
dp = dm (\omega r (1 - \cos 2\alpha)) .
\] (9)

The force is \( dp/dr \), which evaluates to

\[
dF = \rho \omega^2 r^3 (1 - \cos 2\alpha) dy dr
\] (10)

and the torque, which is the product of the force and the radius, evaluates to

\[
d\Gamma = \rho \omega^2 r^3 (1 - \cos 2\alpha) r dy dr.
\] (11)

Now we decide on the limits of integration. The \( r \) limits are straightforward, running the length of the blade, i.e. 0 to \( R \). The \( y \) limits will be determined by the width of the region through which the flow takes place. Figure 4 shows this to be 0 to \( L \sin \alpha \). Then we have, for \( n \) blades,

\[
\Gamma = n \int_0^R \int_0^{L \sin \alpha} \rho \omega^2 (1 - \cos 2\alpha) r^3 dy dr .
\] (12)

For the simple but reasonably common case of \( L \) and \( \alpha \) independent of \( r \), this evaluates to

\[
\Gamma = \frac{n}{4} \rho \omega^2 \sin \alpha (1 - \cos 2\alpha) LR^3 .
\] (13)

The power is the product of the angular velocity and the drag torque. We see that we have almost recovered the third affinity law, except that one \( R \) of the law has been replaced in equation (13) by \( L \). For the Crompton Greaves fan analysed above, this yields a drag torque of 0.15 Nm, which is considerably below the rated value of 0.6 Nm. Both these discrepancies are understandable, as the flow property discussed after equation (3) has been ignored in this calculation. We now incorporate this property.

We assume that the additional change in momentum of this extra flow must also be due to the blades. Since all the airflow is being attributed to the blades, the effect will be the same as if there were blades all over the fan disc, each transferring momentum to the air. Such a continuum of blades is shown in figure 5.

We consider the shape of the blade to be such that it spans a small, constant angular displacement \( \Delta \theta \). Then, geometrical considerations yield

\[
L \cos \alpha = r (\Delta \theta)
\] (14)
and the $r$ integral in equation (12) reduces to

$$\Gamma = \int_0^R n(\Delta \theta) \rho \omega^2 (1 - \cos 2\alpha)(\tan \alpha) r^2 \, dr. \quad (15)$$

Now for the continuum of blades, $n$ tends to infinity and $\Delta \theta$ tends to zero such that their product $n(\Delta \theta) = 2\pi$, whereby

$$\Gamma = \frac{2\pi}{5} \tan \alpha (1 - \cos 2\alpha) \rho \omega^2 R^5. \quad (16)$$

This expression is identical to the third affinity law. For the fan treated previously, the result is 0.96 Nm, which is 50% higher than the observed value. This level of inaccuracy is very common in kinetic theory and is an indicator of accurate modelling.

Analogous to the drag, we can calculate the fan thrust. The calculation for the thrust will mirror that for drag except for the component of velocity under consideration. The drag features the impingement component in equation (9). The thrust must feature the axial component. As discussed after equation (1), this component is $u \sin^2 \alpha$. Incorporating this change and carrying through the formalism leading to equation (16) from equation (9), the thrust evaluates to

$$T = \frac{4\pi}{5}(\sin^2 \alpha) \rho \omega^2 R^5. \quad (17)$$

The second affinity law obtains by recognizing that pressure is force per unit area. The total force is the sum of the drag and thrust components which have the same dependences on the various parameters, and the area is proportional to the square of the radius.

We note that the results derived here are independent of the number of blades of the fan. This is because of the model’s crudeness. In reality, the number of blades does affect fan performance, which is why ceiling fans typically have three blades, exhaust fans four and modern propellers six. A correction to account for this number can be included in our analysis by reconsidering our assumption of $\Delta \theta = 2\pi / n$ made before equation (16). A negative correction of order $1/n^2$ will arise from the fact that the integration treats arcs of circles as straight lines. However, we do not pursue this approach as other factors will play a stronger role in determining the $n$-dependence. The number of blades, and hence the air gap between successive blades, influences parameters such as fan noise and energy dissipation during operation through vortex formation around the blades. Such an analysis cannot be carried out from our model, which, we have to admit, is not of much utility to a fan designer. The model...
should, however, be of considerable interest to the physics student, as it is a demonstration of how kinetic theory can almost trivially throw light on a very difficult problem.

Conclusion

Thus we see that application of kinetic theory to the fan has produced accurate estimates of the air delivery, the thrust and the drag. Unlike the hydrodynamic derivation, our procedure is conceptually and mathematically simple. Nevertheless, the three fan affinity laws have been proved and numerical estimates of fan performance have been obtained. The problem is thus a good demonstration of the strength and simplicity of kinetic theory.

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References

[1] Hamilton J H, Kusian R N and Cope W J 1954 Fan performance and selection using dimensionless ratios Bull. Univ. Utah 45 12
[2] Kulunk E 1970 Aerodynamics of wind turbines Fundamental and Advanced Topics in Wind Power ed R Carriveau (Lyngby: DTU Tryk) http://www.intechopen.com/books/fundamental-and-advanced-topics-in-wind-power/aerodynamics-of-wind-turbines
[3] Mikkelsen R 2003 Actuator disk methods applied to wind turbines PhD Dissertation, Technical University of Denmark
[4] Faulkner B E and Ytreberg F M 2011 Understanding Bernoulli’s principle through simulations Am. J. Phys. 79 214–6
[5] Buckingham E 1914 On physically similar systems; illustrations of the use of dimensional equations Phys. Rev. 4 345–76
[6] Sommerfeld A 1964 Thermodynamics and Statistical Mechanics—Lectures on Theoretical Physics vol 5 (New York: Academic)
[7] Balasubramaniam R 2007 Callister’s Materials Science and Engineering (New Delhi: Wiley)
[8] Reif F 2008 Statistical Physics vol 5 Berkeley Physics Course (New Delhi: McGraw-Hill)
[9] Morin D 2008 Introduction to Classical Mechanics (Cambridge: Cambridge University Press)