Blow-up Set and Upper Rate Estimate for A Semilinear Heat Equation

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Abstract. This paper is concerned with the blow-up properties for a semilinear heat equation with homogeneous Dirichlet boundary conditions, defined on a ball in $\mathbb{R}^n$. We show that the blow-up in this problem can occur in finite time at only a single point. Moreover, the upper blow-up rate estimate for this problem is derived.

Keywords: Blow-up solution; Blow-up set; Blow-up rate estimate; Semilinear Heat equation; Pointwise estimate; Dirichlet boundary Conditions.

1. Introduction

We consider the Initial-Boundary problem which takes the following form:

$$\begin{cases}
    u_t = \Delta u + \lambda u^p e^u , & (x,t) \in B_R \times (0,T), \\
    u(x,t) = 0 , & (x,t) \in \partial B_R \times (0,T), \\
    u(x,0) = u_0(x) , & x \in B_R 
\end{cases}$$

where $p \geq 1; \quad q, \lambda > 0, \quad B_R$ is a ball in $\mathbb{R}^n$ and $u_0 \in C^2(\mathbb{R}^n)$, nonzero, nonnegative, radially non increasing function, $u_0(x) = 0, \forall x \in \partial B_R$.

The blow-up phenomena in the semilinear heat equation: $u_t = \Delta u + f(u)$, defined on bounded domain has been studied by many authors, see for instance [1-4].

In general, for time-dependent equations, blow-up means some solutions cannot be continued globally in time. In other words, they become unbounded in a finite time, see [2].

i.e. there exists $T > 0$, such that:

$$||u(x,t)||_{\infty} \xrightarrow{t \to T^-} \infty,$$

where $||u(x,t)||_{\infty} = \sup_{x \in \Omega} |u(x,t)|$

In [3], Kaplain showed that, the blow-up occurs, if the function $f$ is convex and satisfies the condition: 

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\[
\int_{u}^{\infty} \frac{d}{f(u)} < \infty, \quad u \geq 1 \tag{2}
\]

Later, in [4], Friedman and McLeod studied the homogenous Dirichlet problem of semilinear heat equations defined on a ball, namely:

\[
\begin{aligned}
    u_t &= \Delta u + f(u), & (x, t) &\in B_R \times (0, T), \\
    u(x, t) &= 0, & (x, t) &\in \partial B_R \times (0, T), \\
    u(x, 0) &= u_0(x), & x &\in B_R 
\end{aligned}
\tag{3}
\]

where \( f \) is a power function or exponential function.

They showed that: for large initial function satisfies the assumptions of (1), blow-up occurs in finite time, and it can only occur at a single point, which is \( x = 0 \), i.e. there exists \( T > 0 \), such that:

\[
u(0, t) \to \infty, \quad a \quad t \to T^-.
\]

For the special case, where \( f(u) = u^p \), \( p > 1 \), it has been shown that:

for a fixed \( t \) and for any \( \alpha \geq \frac{2}{p-1} \), the upper pointwise estimate takes the following form:

\[
    u(x, t) \leq \frac{e}{|x|^\alpha}, \quad x \in B_R/(0), \quad t \in (0, T)
\]

Moreover, in [4], it has been shown that the upper blow-up rate estimate is as follows:

\[
u(0, t) \leq \frac{e}{(T-t)^\beta}, \quad t \in (0, T), \quad \beta = \frac{1}{p-1}, \quad \zeta > 0
\]

For the case, \( f(u) = e^u \), it has been shown that the upper pointwise estimate is as follows:

\[
    u(x, t) \leq \log \zeta + \frac{2}{\alpha} t \left( \frac{1}{|x|} \right), \tag{4}
\]

for \( \zeta > 0, \quad \alpha \in (0, 1) \).

Moreover, there exist \( \zeta > 0 \), such that the upper blow-up rate estimate is as follows:

\[
u(x, t) \leq \log \zeta - \log(T-t), \quad x \in B_R, \quad t \in (0, T).
\tag{5}
\]

In this paper, we aim to extend some of the above blow-up results to problem (1), showing that: the solutions of this problem may blow up in finite time at only a single point and the pointwise estimate of this problem takes the form as in (4). Moreover, we show that the upper rate estimate is independent on \( p \) and takes the following form:

\[
u(0, t) \leq \frac{1}{q} (\log \zeta - \log[q \lambda(T-t)]), \quad \zeta > 0.
\]
2. Preliminaries

Since \( f(u) = \lambda u^p e^q \in \mathcal{L}^1(\Omega) \), the local existence of unique classical solution to problem (1) is guaranteed for some \( T > 0 \), see [5]. On the other hand, the following theorem shows that: for large initial functions the solutions of problem (1), blow up in finite time, which means \( T < \infty \).

**Theorem 1:** Starting with large initial function, the classical solution of problem (1), blows up in a finite time, and blow-up set contains \( \chi = 0 \).

**Proof:** Set \( f(u) = \lambda u^p e^q \), it is easy to show that \( f''(u) \) is positive function in \( (0, \infty) \), so that \( f \) is convex function in \( (0, \infty) \). Moreover, it is clear that \( f \) satisfies condition (2), thus, according to Kaplan, [3], for a large size initial function, the solution of problem (1) blows up in a finite time. Since \( f(u) \geq \epsilon^q \), where \( u \) is large enough, by using the comparison principle [6], the solution of problem (1) can be considered as a super solution to the following problem:

\[
\begin{align*}
&u_t = \Delta u + \epsilon^q, \quad (x, t) \in B_R \times (0, T), \\
&u(x, t) = 0, \quad (x, t) \in \partial B_R \times (0, T), \\
&u(0, 0) = u_0(x), \quad x \in B_R
\end{align*}
\]

But, it is well known that, the solution of problem (6), can blow up in a finite time at only a single point, see [4,7]. Therefore, \( \chi = 0 \) belongs to blow-up set of problem (1).

The next lemma, proved in [6], presents some properties of the solutions of problem (1).

**Lemma 1:** Let \( u \) be a classical solution of problem (1). We can show that

i- \( u \) is positive and radial in \( B_R \times (0, T) \).

\[ i.e. \quad u > 0, \quad u(x, t) = u(r, t), \quad r = |x| = x_1^2 + x_2^2 + \cdots x_n^2 \]

ii- \( u(r, t) \) is decreasing in \( (0, R] \times (0, T) \)

\[ i.e. \quad u_r < 0 \quad \text{in} \quad (0, R] \times (0, T) \]

iii- \( u \) is increasing in time.

\[ i.e. \quad u_t > 0, \quad \text{in} \quad B_R \times (0, T) \]

3. Blow-up Set

This section considers the pointwise estimate of problem (1), showing that the blow-up can only occur at \( \chi = 0 \).

To prove these results, we recall the following lemma, proved in [4,8].

**Lemma 2:** Let \( u \) be a blow-up solution of problem (3), where \( f \in \mathcal{L}^2(\Omega) \), increasing positive function in \( (0, \infty) \), also suppose that

\[ u_{ur}(r) \leq -\alpha, \quad \text{for} \quad 0 < r \leq R, \quad \text{where} \quad \alpha > 0 \quad (7) \]

Consider \( F \in \mathcal{L}^2(0, \infty) \) \( \cap \mathcal{L}[0, \infty) \), such that:

\[ F > 0, \quad F', F'' \geq 0, \quad \text{in} \quad (0, \infty) \quad (8) \]
If the following condition is satisfied
\[ f^I F - f F^I \geq 2\varepsilon F F^I \quad \text{in} \quad (0, \infty) \] (9)

Then the function \( J = \gamma^{r-1} u_r + \varepsilon \gamma^{p-1} F(u) \) is nonpositive in \((0, R) \times (0, T)\) for some \( \gamma > 0 \).

**Theorem 2:** Let \( u \) be a blow-up solution of problem (1), also suppose that \( u_0 \) satisfies (7). Then blow-up can only occur at \( x = 0 \).

**Proof:** Set \( f(u) = \lambda u^\delta e^q \), and \( F(u) = u^\delta e^q \), \( \delta, \alpha \in (0, 1) \)

Clearly, \( F \) satisfies (8).

To prove this theorem, we only need to show that the inequality (9) holds with the above choice of \( F \).

We can calculate the left hand side of inequality (9) as follows:

\[
\begin{align*}
\frac{d}{dt}(u^\delta e^q) &= \lambda[(q - \alpha)u^{p+\delta} + (p - \delta)u^{(p+\delta-1)} e^{2q\alpha}] \\
& \geq 2\varepsilon[au^{2\delta} + \delta u^{(2\delta-1)}]e^{2q\alpha}
\end{align*}
\]

On the other hand, we have

\[
2\varepsilon (u)F^I(u) = 2\varepsilon(au^\delta e^q + \delta u^{(\delta-1)} e^{2q\alpha})(u^\delta e^q)
\]

Clearly,

\[
\lambda[(q - \alpha)u^{p+\delta} + (p - \delta)u^{(p+\delta-1)}]e^{(q+\alpha)u} \geq 2\varepsilon[au^{2\delta} + \delta u^{(2\delta-1)}]e^{2q\alpha}.
\]

provided \( \lambda \geq 2\varepsilon, q \geq 2\alpha \) and \( p \geq 2\delta \).

Thus, under the above assumptions, the condition (9) is satisfied

By Lemma 2, we get
\[
J = \gamma^{r-1} u_r + \varepsilon \gamma^{p-1} u^\delta e^q \leq 0, \quad \text{for} \quad (r, t) \in (0, R) \times (0, T)
\]

Thus \( \gamma^{r-1} u_r + \varepsilon \gamma^{p-1} u^\delta e^q \leq 0 \), for \((r, t) \in (0, R) \times (0, T)\) such that \( u^\delta \geq 1 \).

From above, it follows that:
\[
\frac{-d}{e^q} \geq \varepsilon
\]

By integrating both sides of last inequality, we get
\[
\frac{1}{\alpha} \geq \frac{1}{2} \varepsilon r^2 \quad \text{or} \quad e^q \leq \frac{1}{\alpha} \frac{1}{r^2}.
\]

Thus
\[
u \leq \frac{1}{\alpha} \log(\frac{1}{\alpha}) + \frac{1}{\alpha} \log(\frac{1}{r^2})
\]

So,
\[
u \leq \frac{1}{\alpha} \log(\frac{1}{\alpha}) + \frac{1}{\alpha} \log(1) - \frac{1}{\alpha} \log(r)
\]
From the last inequality, it follows that there exists a positive constant \( C \), such that the upper point wise estimate takes the following form:

\[
u(x, t) \leq \log C + \frac{2}{\alpha} \log \left( \frac{1}{|x|} \right), \quad x \in B_R, \quad t \in (0, T).
\]

Therefore, at any time, the solution is bounded, unless \( x = 0 \). Thus, the blow-up occurs only at a single point.

4. Blow-up Rate Estimate

In this section, we consider the upper bound of the blow-up rate for problem (1). In order to derive this estimate, we use the results of the following lemma, which can be proved, following the procedure used in [4,8].

**Lemma 3:** Let \( u \) be a blow-up solution of problem (3), where \( f \in \mathcal{C}^4(0, \infty) \cap \mathcal{C}[0, \infty) \), increasing function such that \( f, f', f'', f''' \) are positive functions in \((0, \infty)\), also suppose that \( \mathcal{U}_k \) satisfies the condition (7).

Define the function

\[
F(x, t) = u_t - \alpha u, \quad x \in B_R, \quad t \in (0, T).
\]

Then there exists \( \varepsilon \in (0, R), \quad \tau \in (0, T), \quad \text{and} \quad \alpha > 0 \) such that

\[
F(x, t) \geq 0, \quad \text{for} \quad (x, t) \in \overline{B}_{\varepsilon} \times (\tau, T)
\]

**Theorem 3:** Let \( u \) be a blow-up solution of problem (1), with blow-up time \( T > 0 \). Then there is \( C > 0 \) such that:

\[
u(0, t) \leq \frac{1}{q} \left( \log - \log \left[ q \left( T - t \right) \right] \right), \quad \text{for} \quad t \text{ close to} \quad T
\]

**Proof:** Set \( f(u) = \lambda u^p e^q \).

Clearly, \( f \) satisfies all the assumptions of Lemma 3, thus we get

\[
u_t \geq \alpha \lambda u^p e^q, \quad x \in \overline{B}_{\varepsilon}, \quad t \in (\tau, T)
\]

for some \( \varepsilon \in (0, R), \quad \tau \in (0, T), \quad \alpha \in (0,1) \)

By Theorem 2, \( u \) blows up in a finite time \( T \) at only \( x = 0 \), which lead to

\[
u(0, t) \geq 1, \quad \text{for} \quad t \text{ close to} \quad T.
\]

Thus, from (10), we get

\[
u_t(0, t) \geq \alpha \lambda (u(0, t))^{p}\theta \in (\mathcal{U}, \ell) \quad \leq \alpha \lambda e^{q \in (\mathcal{U}, \ell)}, \quad \text{for} \quad t \text{ close to} \quad T.
\]

It follows that:

\[
\frac{d}{\varepsilon q \in (\mathcal{U}, \ell)} \geq \alpha \lambda d
\]

By integrating the last inequality from \( t \) to \( T \), and since
we obtain \( \frac{1}{q} \frac{1}{(t_0 - t)} \geq \alpha (I - t) \),

Thus

\[
\epsilon \leq \frac{1}{q} \frac{1/\alpha}{(I-t)}
\]

Therefore, there is \( \epsilon > 0 \), such that

\[
u(0, t) \leq \frac{1}{q} (\log \epsilon - \log\left( q (I-t) \right)) \text{ for } t \text{ close to } T
\]

5. Conclusion

From this work, it is observed that the blow-up in problem (1) can only occur at a single point. Moreover, the upper blow-up rate estimate is independent on \( p \), which indicates that the power function has no effect on the blow-up profile of problem (1). However, it may have only effect on the blow-up time.

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7. References

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