THE LARGE SIEVE WITH POWER MODULI FOR $\mathbb{Z}[i]$

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Abstract. We establish a large sieve inequality for power moduli in $\mathbb{Z}[i]$, extending earlier work by L. Zhao and the first-named author on the large sieve for power moduli for the classical case of moduli in $\mathbb{Z}$. Our method starts with a version of the large sieve for $\mathbb{R}^2$. We convert the resulting counting problem back into one for $\mathbb{Z}[i]$ which we then attack using Weyl differencing and Poisson summation.

1. Introduction

The classical large sieve inequality with additive characters asserts that

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}} q \left| \sum_{M<n \leq M+N} a_n e \left( \frac{n \cdot a}{q} \right) \right|^2 \leq (Q^2 + N - 1) \sum_{M<n \leq M+N} |a_n|^2,$$

where $Q, N \in \mathbb{N}$ and $M \in \mathbb{Z}$. This inequality has numerous applications in analytic number theory, in particular, in sieve theory and to questions regarding the distribution of arithmetic functions in arithmetic progressions.

The large sieve with restricted sets of moduli $q$, in particular power moduli, was considered in a series of papers by Baier, Zhao and Halupczok (see [1], [3], [4], [7] and [11]), and these results turned out to be useful tools for applications (see [6] and [5], for example). In the case of square moduli, it was first established by Zhao [11] that

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}} q^2 \left| \sum_{M<n \leq M+N} a_n e \left( \frac{n \cdot a}{q^2} \right) \right|^2 \ll \varepsilon (QN)^\varepsilon \left( Q^3 + N \sqrt{Q} + \sqrt{N}Q^2 \right) \sum_{M<n \leq M+N} |a_n|^2.$$

This was improved in [1], where the term $N \sqrt{Q}$ on the right-hand side of (1) was replaced by $N$. A further improvement was obtained in [7], where [1] with $N + \min \{ N \sqrt{Q}, \sqrt{N}Q^2 \}$ in place of $N \sqrt{Q} + \sqrt{N}Q^2$ was established. In [11], Zhao conjectured that the bound

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}} q^k \left| \sum_{M<n \leq M+N} a_n e \left( \frac{n \cdot a}{q^k} \right) \right|^2 \ll \varepsilon (QN)^\varepsilon \left( N + Q^{k+1} \right) \sum_{M<n \leq M+N} |a_n|^2$$

conjectured
should hold for $k$-th power moduli ($k \in \mathbb{N}$ arbitrary but fixed). This conjecture is still open for every $k \geq 2$. In the same paper \cite{zh1}, he established that

$$
\sum_{q \leq Q} \sum_{a=1}^{q^k} \left| \sum_{M < n \leq M+N} a_n e \left( \frac{n \cdot a}{q^k} \right) \right|^2 \ll (QN)^k \left( Q^{k+1} + NQ^{1-1/k} + N^{1-1/k} Q^{1+k/k} \right) \sum_{M < n \leq M+N} |a_n|^2,
$$

where $k = 2^{k-1}$, thus generalizing \cite{firstls}. Improvements of this result have been established in \cite{bz2} and \cite{hal}.

The large sieve for additive characters was extended to number fields by Huxley. In the case of the number field $\mathbb{Q}(i)$ it takes the form

$$
\sum_{q \in \mathbb{Z}[i] \setminus \{0\} \mod q \atop \mathcal{N}(q) \leq Q} \sum_{r \mod q \atop (r,q) = 1} \left| \sum_{n \in \mathbb{Z}[i] \atop \mathcal{N}(n) \leq N} a_n \cdot e \left( \mathcal{R} \left( \frac{nr}{q} \right) \right) \right|^2 \ll \left( Q^2 + N \right) \sum_{n \in \mathbb{Z}[i] \atop \mathcal{N}(n) \leq N} |a_n|^2.
$$

Here as in the following $\mathcal{N}(q)$ denotes the norm of $q \in \mathbb{Z}[i]$, given by

$$
\mathcal{N}(q) := \mathcal{R}(q)^2 + \mathcal{I}(q)^2.
$$

The large sieve with square norm moduli for the number field $\mathbb{Q}(i)$ was investigated in \cite{squarenorm}, where an analogue of \cite{kls} was established, namely the inequality

$$
\sum_{q \in \mathbb{Z}[i] \setminus \{0\} \mod q \atop \mathcal{N}(q) \leq Q^2} \sum_{r \mod q \atop (r,q) = 1} \left| \sum_{n \in \mathbb{Z}[i] \atop \mathcal{N}(n) \leq N} a_n \cdot e \left( \mathcal{R} \left( \frac{nr}{q} \right) \right) \right|^2 \ll (QN)^2 \left( Q^3 + Q^2 \sqrt{N} + \sqrt{Q} N \right) \sum_{n \in \mathbb{Z}[i] \atop \mathcal{N}(n) \leq N} |a_n|^2.
$$

In this paper, we go a step further and prove an analogue of \cite{kls} for $\mathbb{Q}(i)$, i.e. a large sieve inequality with $k$-th power moduli for $\mathbb{Q}(i)$. Our approach will be more elegant than the previous one in \cite{squarenorm} where the double large sieve and lattice point counting in $\mathbb{R}^2$ were used. Here our method starts with a version of the large sieve for $\mathbb{R}^2$. Then we convert the resulting counting problem back into one for $\mathbb{Z}[i]$ which can be attacked along similar lines as in \cite{zh1} using Weyl differencing and Poisson summation. We begin with square moduli, for which we obtain the essentially same bound as for square norm moduli in \cite{squarenorm}. Then we generalize our method to $k$-th power moduli.

### 2. Statement of main results

We shall establish the following large sieve inequality for square moduli in $\mathbb{Z}[i]$. 
Theorem 1. Let $Q, N \geq 1$ and $(a_n)_{n \in \mathbb{Z}[i]}$ be any sequence of complex numbers. Then
\[
\sum_{q \in \mathbb{Z}[i]\setminus\{0\}} \sum_{r \text{ mod } q^2 \atop N(q) \leq Q \atop (r, q) = 1} \sum_{n \in \mathbb{Z}[i]\atop N(n) \leq N} a_n \cdot e \left( \Re \left( \frac{nr}{q^2} \right) \right)^2 \ll (QN)^\varepsilon \left( Q^3 + Q^2 \sqrt{N} + \sqrt{QN} \right) \sum_{n \in \mathbb{Z}[i]\atop N(n) \leq N} |a_n|^2,
\]
where $\varepsilon$ is any positive constant, and the implied $\ll$-constant depends only on $\varepsilon$.

Theorem 1 will then be generalized to $k$-th power moduli, for which we establish the following.

Theorem 2. Let $k \in \mathbb{N}$, $Q, N \geq 1$ and $(a_n)_{n \in \mathbb{Z}[i]}$ be any sequence of complex numbers. Set $\kappa := 2^{k-1}$. Then
\[
\sum_{q \in \mathbb{Z}[i]\setminus\{0\}} \sum_{r \text{ mod } q^k \atop N(q) \leq Q \atop (r, q) = 1} \sum_{n \in \mathbb{Z}[i]\atop N(n) \leq N} a_n \cdot e \left( \Re \left( \frac{nr}{q^k} \right) \right)^2 \ll (QN)^\varepsilon \left( Q^{k+1} + NQ^{1-1/\kappa} + N^{1-1/\kappa}Q^{1+k/\kappa} \right) \sum_{n \in \mathbb{Z}[i]\atop N(n) \leq N} |a_n|^2,
\]
where $\varepsilon$ is any positive constant, and the implied $\ll$-constant depends only on $k$ and $\varepsilon$.

3. LARGE SIEVE FOR $\mathbb{R}^d$

We shall employ the following version of the large sieve for $\mathbb{R}^d$ (in fact, we shall need it for the case $d = 2$ only).

Theorem 3. Let $R, N \in \mathbb{N}$, $N \geq 2$, $x_1, \ldots, x_R \in \mathbb{R}^d$ and $(a_n)_{n \in \mathbb{Z}^d}$ be any $d$-fold sequence of complex numbers. Then
\[
\sum_{i=1}^R \left| \sum_{n \in \mathbb{Z}^d \atop ||n||_2 \leq N^{1/d}} a_n \cdot e(n \cdot x_i) \right|^2 \ll K NZ,
\]
where
\[
K := \max_{1 \leq i \leq R} \left\{ j \in \{1, \ldots, R\} : \min_{z \in \mathbb{Z}^d} ||x_j - x_i - z||_2 \leq \sqrt{dN^{-1/d}} \right\}
\]
and
\[
Z := \sum_{n \in \mathbb{Z}^d \atop ||n||_2 \leq N^{1/d}} |a_n|^2.
\]
(6)
Here as in the following, \(|s|_2\) denotes the Euclidean norm of \(s \in \mathbb{R}^d\), given by
\[ |(s_1, s_2, ..., s_d)|_2 = \sqrt{s_1^2 + s_2^2 + \cdots + s_d^2}. \]

To prove Theorem 1, we use the duality principle and the Poisson summation formula for \(\mathbb{R}^d\).

**Duality Proposition 1** (Duality principle, Theorem 288 in [HLP]). Let \(C = [c_{mn}]\) be a finite matrix with complex entries. The following two statements are equivalent:

1. For any complex numbers \(a_n\), we have
   \[ \sum_{m} \left| \sum_{n} a_n c_{mn} \right|^2 \leq \Delta \sum_{n} |a_n|^2. \]
2. For any complex numbers \(b_m\), we have
   \[ \sum_{n} \left| \sum_{m} b_m c_{mn} \right|^2 \leq \Delta \sum_{m} |b_m|^2. \]

**Poisson Proposition 2** (Poisson summation formula, see [StW]). Let \(f : \mathbb{R}^d \to \mathbb{C}\) be a smooth function of rapid decay and \(\Lambda\) be a lattice of full rank in \(\mathbb{R}^d\). Then
\[
\sum_{y \in \Lambda} f(y) = \frac{1}{\text{Vol}(\mathbb{R}^d/\Lambda)} \cdot \sum_{x \in \Lambda'} \hat{f}(x),
\]
where \(\Lambda'\) is the dual lattice and \(\hat{f}\) is the Fourier transform of \(f\), defined as
\[
\hat{f}(x) = \int_{\mathbb{R}^2} f(y) e(-x \cdot y) \, dy.
\]

Here as in the following, by rapid decay we mean that the function \(f : \mathbb{R}^d \to \mathbb{C}\) in question satisfies the bound
\[ f(y) \ll (1 + |y|_2)^{-C} \]
for some \(C > d\).

By a linear change of variables, we immediately deduce the following more general version of the Poisson summation formula for shifted lattices from Proposition 2.

**Poisson Proposition 3.** Let the conditions and notations of Proposition 2 be kept and assume that \(B > 0\) and \(a \in \mathbb{R}^d\). Then
\[
\sum_{y \in a + \Lambda} f\left(\frac{y}{B}\right) = \frac{B^d}{\text{Vol}(\mathbb{R}^d/\Lambda)} \cdot \sum_{x \in \Lambda'} e(a \cdot x) \hat{f}(Bx).
\]

**Proof of Theorem 1.** We first note that
\[
K = \max_{1 \leq i \leq R} \# \{j \in \{1, ..., R\} : \min_{z \in \mathbb{Z}^d} ||x_j - x_i - z||_2 \leq \sqrt{d}N^{-1/d}\}
\]
\[
\geq \max_{1 \leq i \leq R} \# \{j \in \{1, ..., R\} : \min_{z \in \mathbb{Z}^d} \max_{1 \leq k \leq d} |x_j^{(k)} - x_i^{(k)} - z^{(k)}| \leq N^{-1/d}\}
\]
\[
= \max_{1 \leq i \leq R} \# \{j \in \{1, ..., R\} : \max_{1 \leq k \leq d} ||x_j^{(k)} - x_i^{(k)} - z^{(k)}|| \leq N^{-1/d}\} =: K'.
\]
where \( \|u\| \) is the distance of \( u \in \mathbb{R} \) to the nearest integer and we write
\[
x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \quad \text{and} \quad z = (z^{(1)}, \ldots, z^{(d)})
\]
for \( i = 1, \ldots, R \).

Now let \( S = \{x_1, x_2, \ldots, x_R\} \). Taking Proposition 4, the duality principle, into account, it suffices to prove that
\[
\sum_{n \in \mathbb{Z}^d \atop \|n\| \leq N^{1/d}} \left| \sum_{x \in S} b_x \cdot e(n \cdot x) \right|^2 \ll KN \sum_{x \in S} |b_x|^2
\]
for any complex numbers \( b_x \). To this end, for \( x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{R}^d \), we define
\[
\phi(x) = \prod_{k=1}^d \left( \frac{\sin \left( \frac{\pi x^{(k)}}{2} \right)}{\frac{\pi x^{(k)}}{2}} \right)^2
\]
and note that \( \phi(x) \) is non-negative and satisfies \( \phi(x) \geq 1 \) if \( |x^{(k)}| \leq 1/2 \) for \( k = 1, \ldots, d \).
Moreover, the Fourier transformation of \( \phi(x) \) equals
\[
\hat{\phi}(s) = \left( \frac{n^2}{4} \right)^d \prod_{k=1}^d \max \left\{ 1 - |s^{(k)}|, 0 \right\},
\]
where we write
\[
s = (s^{(1)}, \ldots, s^{(d)}).
\]
Hence,
\[
\sum_{n \in \mathbb{Z}^d \atop \|n\| \leq N^{1/d}} \left| \sum_{x \in S} b_x \cdot e(n \cdot x) \right|^2 \leq \sum_{n \in \mathbb{Z}^d} \phi \left( \frac{n}{2N^{1/d}} \right) \sum_{x \in S} |b_x|^2 \cdot e(n \cdot x)
\]
\[
= \sum_{x, x' \in S} b_x \overline{b_{x'}} \cdot V(x - x'),
\]
where
\[
V(y) = \sum_{n \in \mathbb{Z}^d} \phi \left( \frac{n}{2N^{1/d}} \right) \cdot e(n \cdot y).
\]
Using Proposition 3, the Poisson summation formula, we transform \( V(y) \) into
\[
V(y) = 2^d N \cdot \sum_{\alpha \in \mathbb{Z}^d \atop \alpha \neq y} \hat{\phi} \left( 2N^{1/d} \alpha \right) = 2^d N \cdot \sum_{\alpha \in \mathbb{Z}^d \atop \alpha \neq y} \hat{\phi} \left( 2N^{1/d} \alpha \right)
\]
\[
= \frac{\pi^{2d}}{2^d} \cdot N \cdot \prod_{k=1}^d \max \left\{ 1 - |2N^{1/d}y^{(k)}|, 0 \right\},
\]
where \( \hat{\phi} \) is the inverse Fourier transform and \( \hat{\phi} \) is the Fourier transform of \( \phi \). Therefore,
\[
\sum_{n \in \mathbb{Z}^d \atop \|n\| \leq N^{1/d}} \left| \sum_{x \in S} b_x \cdot e(n \cdot x) \right|^2 \leq \frac{\pi^{2d}}{2^d} \cdot N \cdot \sum_{x, x' \in S \atop \|x^{(i)} - x'^{(i)}\| \leq N^{-1/d} \text{ for } i = 1, \ldots, d} |b_x||b_{x'}|.
\]
(8)
Now we observe that
\[
|b_x||b_{x'}| \leq \frac{1}{2} \left( |b_x|^2 + |b_{x'}|^2 \right).
\]
It follows that
\[
\sum_{n \in \mathbb{Z}^d} \left| \sum_{x \in S} b_x \cdot e(n \cdot x) \right|^2 \ll K'N^2 \leq KN^2,
\]
where we use (8) and (11). This completes the proof. □

4. Conversion into a counting problem

Now we return to the large sieve for \( \mathbb{Q}(i) \). We begin with restricting the moduli \( q \) to an arbitrary multiset \( S \) of elements of \( \mathbb{Z}[i] \setminus \{0\} \). We shall also restrict the norms of these moduli to dyadic intervals, which is for technical reasons. Thus, we are interested in estimating the quantity

\[
T := \sum_{q \in S} \left| \sum_{r \mod q \atop Q/2 < \mathcal{N}(q) \leq Q} \sum_{n \in \mathbb{Z}[i] \atop \mathcal{N}(n) \leq N} a_n \cdot e\left( \frac{nr}{q} \right) \right|^2.
\]

We shall later confine ourselves to squares or, more generally, \( k \)-th powers.

Our first step is to re-write \( T \) in the form

\[
T = \sum_{q \in S} \left| \sum_{r \mod q \atop Q/2 < \mathcal{N}(q) \leq Q} \sum_{n \in \mathbb{Z}[i] \atop \mathcal{N}(n) \leq N} a_n \cdot e\left( \frac{nr}{q} \right) \right|^2,
\]

where
\[
q = u + iv, \quad r = x + iy, \quad n = s + ti.
\]

To bound \( T \), we employ Theorem 4 for the case \( d = 2 \), which immediately gives us the following.

Corollary 1. For \( T \) as defined in (8), we have the bound

\[
T \ll KNZ,
\]

where \( Z \) is defined as in (9) and

\[
K := \max_{r_1, q_1} \left\{ (r_2, q_2) : \min_{z \in \mathbb{Z}^2} \left\| \left( \frac{x_2 u_2 + y_2 v_2}{\mathcal{N}(q_2)}, \frac{x_2 v_2 - y_2 u_2}{\mathcal{N}(q_2)} \right) - \left( \frac{x_1 u_1 + y_1 v_1}{\mathcal{N}(q_1)}, \frac{x_1 v_1 - y_1 u_1}{\mathcal{N}(q_1)} \right) \right\|_2 \leq 2N^{-1} \}\right\}
\]

with the conventions that, for \( j = 1, 2 \),
\[
q_j \in S, \quad Q/2 < \mathcal{N}(q_j) \leq Q,
\]
\( \{r_j\} \) forms a system of representatives of reduced residue classes modulo \( q_j \) and
\[
q_j = u_j + iv_j, \quad r_j = x_j + iy_j.
\]

Thus, we have converted the problem into a counting problem.
5. Switching back to \( \mathbb{Z}[i] \)

Now we observe that

\[
\frac{r_j}{q_j} = \frac{x_j - iy_j}{u_j - iv_j} = \frac{x_ju_j + y_jv_j}{\mathcal{N}(q_j)} + \frac{x_jv_j - y_ju_j}{\mathcal{N}(q_j)}i.
\]

It follows that

\[
K = \max_{r_1, q_1} \left\{ (r_2, q_2) : \min_{z \in \mathbb{Z}[i]} \left| \frac{r_2}{q_2} - \frac{r_1}{q_1} - z \right|^2 \leq 2N^{-1} \right\}
\]

\[
= \max_{q_1 \in S, q_2 \in S, |q_2| \leq |q_1|} \left\{ (r_2, q_2) : q_2 \in S, \frac{q_2}{q_1} = 1, \right\}
\]

\[
\left| \frac{r_2}{q_2} - \frac{r_1}{q_1} \right|^2 \leq 2N^{-1}
\}

Further,

\[
\left| \frac{r_2}{q_2} - \frac{r_1}{q_1} \right|_2 \leq 2N^{-1} \iff \mathcal{N}(r_1q_2 - r_2q_1) \leq 2N^{-1}\mathcal{N}(q_1)\mathcal{N}(q_2).
\]

We deduce that

\[
K \leq \max_{q_1 \in S, \frac{|q_2|}{\mathcal{N}(q_1)} \leq \frac{|q_1|}{\mathcal{N}(q_2)}} \sum_{b \in \mathbb{Z}[i]} \sum_{q_2 \in S, \frac{|q_2|}{\mathcal{N}(q_2)} \leq \frac{|q_1|}{\mathcal{N}(q_1)}} \sup_{r_1q_2 \equiv b \mod q_1} 1
\]

\[
\ll \max_{q_1 \in S, \frac{|q_2|}{\mathcal{N}(q_1)} \leq \frac{|q_1|}{\mathcal{N}(q_2)}} \sum_{b \in \mathbb{Z}[i]} \sum_{q_2 \in S, \frac{|q_2|}{\mathcal{N}(q_2)} \leq \frac{|q_1|}{\mathcal{N}(q_1)}} \Phi_1\left( \mathcal{N}\left( \frac{b\sqrt{N}}{q_1\sqrt{2Q}} \right) \right) \sum_{r_1q_2 \equiv b \mod q_1} \Phi_2\left( \mathcal{N}\left( \frac{q_2}{q_1\sqrt{2Q}} \right) \right)
\]

\[
= \max_{q_1 \in S, \frac{|q_2|}{\mathcal{N}(q_1)} \leq \frac{|q_1|}{\mathcal{N}(q_2)}} \sum_{q_2 \in S} \Phi_2\left( \mathcal{N}\left( \frac{q_2}{q_1\sqrt{2Q}} \right) \right) \sum_{b \equiv r_1q_2 \mod q_1} \Phi_1\left( \mathcal{N}\left( \frac{b\sqrt{N}}{q_1\sqrt{2Q}} \right) \right),
\]

where, for \( i = 1, 2 \), \( \Phi_i : \mathbb{R} \to \mathbb{R}^+ \) are any smooth functions with rapid decay such that \( \Phi_i(x) \gg 1 \) if \( |x| \leq 1 \). We shall fix \( \Phi_i \) later suitably. Let \( \Psi_i : \mathbb{C} \to \mathbb{R}^+ \) be given by

\[
\Psi = \Phi \circ \mathcal{N} \quad \text{for } i = 1, 2.
\]

Then the above inequality for \( K \) turns into

\[
K \ll \max_{q_1 \in S, \frac{|q_2|}{\mathcal{N}(q_1)} \leq \frac{|q_1|}{\mathcal{N}(q_2)}} \sum_{q_2 \in S} \Psi_2\left( \frac{q_2}{q_1\sqrt{2Q}} \right) \sum_{b \equiv r_1q_2 \mod q_1} \Psi_1\left( \frac{b\sqrt{N}}{|q_1\sqrt{2Q}} \right),
\]
6. Application of Poisson summation

To transform the inner-most sum over \( b \), we use Poisson summation again. The complex numbers \( a \equiv 0 \mod q_1 \) form a square lattice

\[
\Lambda = \left\{ x \left( \begin{array}{c} u_1 \\ v_1 \end{array} \right) + y \left( \begin{array}{c} -v_1 \\ u_1 \end{array} \right) : (x, y) \in \mathbb{Z}^2 \right\} \subset \mathbb{R}^2
\]

with volume \( N(q_1) \) when regarded as vectors in \( \mathbb{R}^2 \). The dual lattice turns out to be

\[
\Lambda' = \frac{1}{N(q_1)} \cdot \Lambda,
\]

which corresponds to the set

\[
\left\{ \frac{a}{N(q_1)} : a \equiv 0 \mod q_1 \right\}
\]

in \( \mathbb{C} \). Hence, Proposition poisson gives

\[
\sum_{b \equiv r_1q_2 \mod q_1} \Psi_1 \left( \frac{b \sqrt{N}}{|q_1| \sqrt{Q}} \right) = \frac{2Q}{N} \cdot \sum_{a \equiv 0 \mod q_1} e \left( -\frac{a}{N(q_1)} \right) \hat{\Psi}_1 \left( \frac{a|q_1| \sqrt{2Q}}{N(q_1) \sqrt{N}} \right)
\]

\[
= \frac{2Q}{N} \cdot \sum_{j \in \mathbb{Z}[i]} e \left( \frac{j}{q_1} \cdot r_1q_2 \right) \hat{\Psi}_1 \left( \frac{j \sqrt{2Q}}{\sqrt{N}} \right),
\]

where for \( x, z \in \mathbb{C} \), we write

\[
\vec{x} = \left( \Re(z) \right) \quad \text{and} \quad \hat{\Psi}(x) = \int_{\mathbb{C}} \Psi(y)e(-\vec{y} \cdot \vec{x}) \, dy_2 dy_1
\]

with \( y_1 := \Re(y) \) and \( y_2 := \Im(y) \). Combining (11) and (12), and re-arranging summation, we deduce that

\[
K \ll \frac{Q}{N} \cdot \max_{q_1 \in S, Q/2 < N(q_1) \leq Q} \sum_{j \in \mathbb{Z}[i]} \hat{\Psi}_1 \left( \frac{j \sqrt{2Q}}{\sqrt{N}} \right) \cdot \sum_{q_2 \in S} \Psi_2 \left( \frac{q_2}{\sqrt{Q}} \right) \cdot e \left( \frac{j \cdot r_1q_2}{q_1} \right).
\]

(13) Kgen

We observe that for \( a, b \in \mathbb{C} \),

\[
e \left( \vec{a} \cdot \vec{b} \right) = e \left( \Re(ab) \right).
\]

(14) obs

Hence, upon a change of variables \( j \rightarrow \vec{j} \), we arrive at

\[
K \ll \frac{Q}{N} \cdot \max_{q_1 \in S, Q/2 < N(q_1) \leq Q} \sum_{j \in \mathbb{Z}[i]} \hat{\Psi}_1 \left( \frac{j \sqrt{2Q}}{\sqrt{N}} \right) \sum_{q_2 \in S} \Psi_2 \left( \frac{q_2}{\sqrt{Q}} \right) \cdot e \left( \Re \left( \frac{j r_1}{q_1} \cdot q_2 \right) \right).
\]

(15) Kgen1

7. The case of square moduli

Now we restrict ourselves to the case when \( S \) is the set of non-zero squares in \( \mathbb{Z}[i] \). We write \( Q_0 = \sqrt{Q} \) and replace \( q_i \) by \( q_i^2 \) \((i = 1, 2)\). Throughout the following, we assume that \( Q_0 > N^{1/4} \) for otherwise the desired result follows from (11) upon
extending the set of moduli to all non-zero Gaussian integers. We deduce from (15) that
\[ K \ll \frac{Q_0^2}{N} \cdot \max_{Q_0/\sqrt{2} < N(q_1) \leq Q_0} \sum_{j \in \mathbb{Z}[i]} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) \cdot \sum_{q_2 \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^2}{Q_0} \right) \cdot e \left( \Re \left( \frac{j r_1}{q_1^2} \cdot q_2^2 \right) \right) \]
\[ \ll \frac{Q_0^3}{N} + \frac{Q_0^2}{N} \cdot \max_{Q_0/\sqrt{2} < N(q_1) \leq Q_0} \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) S(q_1, r_1, j), \]
where
\[ S(q_1, r_1, j) := \sum_{q_2 \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^2}{Q_0} \right) \cdot e \left( \Re \left( \frac{j r_1}{q_1^2} \cdot q_2^2 \right) \right). \]

Here we use the estimate
\[ \sum_{q_2 \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^2}{Q_0} \right) \ll Q_0 \]
to bound the contribution of \( j = 0 \).

7.1. Weyl differencing. Now we employ Weyl differencing in the setting of \( \mathbb{Z}[i] \).
Using the Cauchy-Schwarz inequality, we deduce that
\[ K \ll \frac{Q_0^3}{N} + \frac{Q_0^2}{N} \cdot \max_{Q_0/\sqrt{2} < N(q_1) \leq Q_0} \left( \sum_{j \in \mathbb{Z}[i]} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) \right)^{1/2} \times \]
\[ \left( \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{jQ_0}{\sqrt{N}} \right) \cdot |S(q_1, r_1, j)|^2 \right)^{1/2}, \]
\[ \ll \frac{Q_0^3}{N} + \frac{Q_0}{\sqrt{N}} \cdot \max_{Q_0/\sqrt{2} < N(q_1) \leq Q_0} \left( \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) \cdot |S(q_1, r_1, j)|^2 \right)^{1/2}, \]
where we use the estimate
\[ \sum_{j \in \mathbb{Z}[i]} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) \ll \frac{N}{Q_0^2}. \]

Multiplying out the square, we get
\[ |S(q_1, r_1, j)|^2 = \sum_{q_2, q \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^2}{Q_0} \right) \cdot \Psi_2 \left( \frac{q_2^2}{Q_0} \right) \cdot e \left( \Re \left( \frac{j r_1}{q_1^2} \cdot (q_2^2 - q^2) \right) \right). \]
We now set
\[ \alpha := q_2 - q \]
so that
\[ q_2^2 - q^2 = \alpha^2 + 2\alpha q \quad \text{and} \quad q_2 = \alpha + q. \]
Then \( |S(q_1, r_1, j)|^2 = \sum_{\alpha \in \mathbb{Z}[i]} e \left( \Re \left( \frac{jr_1 \alpha^2}{q_1^2} \right) \right) \times \sum_{q \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q^2}{Q_0} \right) \cdot \Psi_2 \left( \frac{(\alpha + q)^2}{Q_0} \right) \cdot e \left( \Re \left( \frac{2j\alpha r_1}{q_1^2} \cdot q \right) \right) \).

7.2. Poisson summation. We shall apply Proposition [7.2] with \( \hat{g} \) to transform the inner-most over \( q \) on the right-hand side of \((19)\). For \( z = (z_1, z_2) \in \mathbb{R}^2 \), we set

\[
g(z) := \Psi_2 \left( \left( z_1 + iz_2 \right)^2 \right) \cdot \Psi_2 \left( \left( \frac{\alpha}{\sqrt{Q_0}} + z_1 + iz_2 \right)^2 \right).
\]

Then using \((19)\) with

\[
a = q \quad \text{and} \quad b = \frac{2j\alpha r_1}{q_1^2},
\]

we deduce that

\[
\sum_{q \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q^2}{Q_0} \right) \cdot \Psi_2 \left( \frac{(\alpha + q)^2}{Q_0} \right) \cdot e \left( \Re \left( \frac{2j\alpha r_1}{q_1^2} \cdot q \right) \right) = \sum_{x \in \mathbb{Z}^2} e \left( \overline{b} \cdot x \right) g \left( \frac{x}{\sqrt{Q_0}} \right).
\]

Now applying Proposition [7.2] with \( B := \frac{1}{\sqrt{Q_0}} \) and \( f := \hat{g} \), the inverse Fourier transform of \( g \), to the right-hand side of \((20)\), we get

\[
\sum_{x \in \mathbb{Z}^2} e \left( \overline{b} \cdot x \right) g \left( \frac{x}{\sqrt{Q_0}} \right) = Q_0 \cdot \sum_{y \in \mathbb{Z}^2} \hat{g} \left( \sqrt{Q_0} y \right) = Q_0 \cdot \sum_{y \in -B + \mathbb{Z}^2} \hat{g} \left( \sqrt{Q_0} y \right).
\]

It follows that

\[
|S(q_1, r_1, j)|^2 = Q_0 \cdot \sum_{\alpha \in \mathbb{Z}[i]} e \left( \Re \left( \frac{jr_1 \alpha^2}{q_1^2} \right) \right) \cdot \sum_{\beta \in \mathbb{Z}[i]} \hat{g} \left( \sqrt{Q_0} \cdot \left( \beta - \frac{2j\alpha r_1}{q_1^2} \right) \right).
\]

At this point, we specify our choice of \( \Psi_2 \) and compute the Fourier transform of \( g \).

We set

\[
\Phi_2(t) := \exp \left( -\frac{\pi}{2} \cdot \sqrt{|t|} \right)
\]

so that

\[
\Psi_2(z) = \Phi_2(\mathcal{N}(z)) = \exp \left( -\frac{\pi}{2} \cdot \sqrt{\mathcal{N}(z)} \right).
\]

It follows that

\[
g(z) = \exp \left( -\frac{\pi}{2} \cdot \left( z_1^2 + z_2^2 + \left( \frac{\alpha_1}{\sqrt{Q_0}} + z_1 \right)^2 + \left( \frac{\alpha_2}{\sqrt{Q_0}} + z_2 \right)^2 \right) \right),
\]

where \( \alpha_1 := \Re(\alpha) \) and \( \alpha_2 := \Im(\alpha) \). Completing the squares, it follows that

\[
g(z) = \exp \left( -\frac{\pi}{4Q_0} \cdot \left( \alpha_1^2 + \alpha_2^2 \right) \right) \cdot \exp \left( -\pi \cdot \left( z_1 + \frac{\alpha_1}{2\sqrt{Q_0}} \right)^2 + \left( z_2 + \frac{\alpha_2}{2\sqrt{Q_0}} \right)^2 \right).
\]
The Fourier transform of this function equals

\[
\hat{g}(z) = \exp \left( -\frac{\pi}{4Q_0} \cdot (\alpha_1^2 + \alpha_2^2) \right) \cdot e \left( -\frac{\alpha_1 z_1 + \alpha_2 z_2}{2\sqrt{Q_0}} \right) \cdot \exp \left( -\pi \left( z_1^2 + z_2^2 \right) \right)
\]

Plugging (22) into (21), using the triangle inequality and bounding all terms of the form \( e(\gamma) \) trivially by \( |e(\gamma)| \leq 1 \), we get

\[
|S(q_1, r_1, j)|^2 \leq Q_0 \cdot \sum_{\alpha \in \mathbb{Z}[i]} \exp \left( -\frac{\pi}{4Q_0} \cdot \mathcal{N}(\alpha) \right) \times \sum_{\beta \in \mathbb{Z}[i]} \exp \left( -\pi Q_0 \mathcal{N} \left( \beta - \frac{2j\alpha r_1}{q_1^2} \right) \right).
\]

7.3. Counting. The contributions of \( \beta \)'s with

\[
\mathcal{N} \left( \beta - \frac{2j\alpha r_1}{q_1^2} \right) > Q_0^{1+\varepsilon}
\]

and of \( \alpha \)'s with

\[
\mathcal{N}(\alpha) > Q_0^{1+\varepsilon}
\]
to the right-hand side of (23) are negligible. Therefore, it follows from (23) that

\[
|S(q_1, r_1, j)|^2 \ll \left( \frac{Q_0^2}{N} \right)^{2018} + Q_0 \sum_{\mathcal{N}(\alpha) \leq Q_0^{1+\varepsilon}} \sum_{\mathcal{N}(\beta - \frac{2j\alpha r_1}{q_1^2}) \leq Q_0^{\varepsilon-1}} 1
\]

\[
\ll \left( \frac{Q_0^2}{N} \right)^{2018} + Q_0 \sum_{\mathcal{N}(\alpha) \leq Q_0^{1+\varepsilon}} \sum_{\|\frac{2j\alpha r_1}{q_1^2}\| \leq Q_0^{\varepsilon-1/2}} 1
\]

where \( ||z|| \) is the distance of \( z \in \mathbb{C} \) to the nearest Gaussian integer.

Now we want to bound the term in the maximum on the right-hand side of (25). To this end, we choose \( \Psi_1 \) in a suitable way so that \( \hat{\Psi}_1 \) decays exponentially. We set

\[
\Phi_1(t) := \exp \left( -\pi |t| \right)
\]

Since \( \Psi_1 = \Phi_1 \circ \mathcal{N} \), it follows that

\[
\Psi_1(z) = \exp \left( -\pi \mathcal{N}(z) \right) = \hat{\Psi}_1(z).
\]

Hence, using (24), we obtain

\[
\sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{\sqrt{2jQ_0}}{\sqrt{N}} \right) \cdot |S(q_1, r_1, j)|^2 \ll 1 + Q_0 \sum_{0 < \mathcal{N}(j) \leq NQ_0^{-2}} \sum_{\mathcal{N}(\alpha) \leq Q_0^{1+\varepsilon}} \sum_{\|\frac{2j\alpha r_1}{q_1^2}\| \leq Q_0^{\varepsilon-1/2}} 1
\]

upon noting that the contribution of \( \mathcal{N}(j) > NQ_0^{-2} \) is negligible. The contribution of \( \alpha = 0 \) to the right-hand side of (26) is obviously bounded by

\[
\ll \frac{N}{Q_0^{-\varepsilon}}.
\]
Writing $d = 2j\alpha$ and noting that the number of divisors of $d \in \mathbb{Z}[i] \setminus \{0\}$ in the Gaussian integers is bounded by $O(N(d)^{1/2})$, we deduce that

$$\sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) \cdot |S(q_1, r_1, j)|^2 \ll \frac{N}{Q_0^{1-\varepsilon}} + N^\varepsilon Q_0 \sum_{N(d) \leq 4NQ_0^{2^\varepsilon - 1}} \sum_{||dr_1/q_1^2|| \leq Q_0^{\varepsilon - 1/2}} 1,$$

where $r_1$ is a multiplicative inverse of $r_1$ modulo $q_1^2$, i.e. $r_1 r_1^{-1} \equiv 1 \mod q_1^2$. The number of residue classes modulo $q_1^2$ is $N(q_1^2) \leq Q_0^2$, and hence

$$\sum_{N(d) \leq 4NQ_0^{2^\varepsilon - 1}} 1 \ll 1 + \frac{N}{Q_0^{2^\varepsilon - 2\varepsilon}}. \tag{28}$$

Further,

$$\sum_{l \in \mathbb{Z}[i]} 1 \leq \sum_{l \in \mathbb{Z}[i]} 1 \ll Q_0^{1+2\varepsilon}. \tag{29}$$

Combining (27), (28) and (29), we obtain

$$\sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0}{\sqrt{N}} \right) \cdot |S(q_1, r_1, j)|^2 \ll \left( Q_0N \right)^{4\varepsilon} \left( Q_0^2 + \frac{N}{Q_0} \right), \tag{30}$$

and combining (11) and (30), we arrive at

$$K \ll \frac{Q_0^3}{N} + \left( Q_0N \right)^{2\varepsilon} \left( \frac{Q_0^2}{N^{1/2}} + Q_0^{1/2} \right), \tag{31}$$

our final bound for $K$.

Now the statement in Theorem 8.1 follows immediately from Corollary 8.1 and (31), after dividing the moduli into dyadic intervals and replacing $Q_0$ by $Q$.

**8. THE CASE OF $k$-TH POWER MODULI**

Now we take $S$ as the set of non-zero $k$th-powers in $\mathbb{Z}[i]$. We write $Q_0 = Q_1/k$ and replace $q_i$ by $q_i^k$ ($i = 1, 2$). Throughout the following, we assume that $Q_0 > N^{1/(2k)}$ for otherwise the desired result follows from (11) upon extending the set of moduli to...
all non-zero Gaussian integers. We deduce from (30) that

\[ K \ll \frac{Q^k}{N} \cdot \max_{q_0/\mathbb{Z}\subset N(q_1)\leq Q_0} \sum_{j=1}^{\infty} \left| \hat{\psi}_1 \left( \frac{\sqrt{2j}Q_0^{k/2}}{\sqrt{N}} \right) \right| \times \]

\[ \sum_{q_0 \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_0^k}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1} \cdot q_0^k \right) \right) \]

\[ \ll \frac{Q^{k+1}}{N} + \frac{Q^k}{N} \cdot \max_{q_0/\mathbb{Z}\subset N(q_1)\leq Q_0} \sum_{j=1}^{\infty} \left| \hat{\psi}_1 \left( \frac{\sqrt{2j}Q_0^{k/2}}{\sqrt{N}} \right) \right| \cdot |S_k(q_1, r_1, j)|, \]

where

\[ S_k(q_1, r_1, j) = \sum_{q_2 \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^k}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1} \cdot q_2^k \right) \right). \]

Here we use the estimate

\[ \sum_{q_2 \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^k}{Q_0^{k/2}} \right) \ll Q_0 \]

to bound the contribution of \( j = 0 \).

8.1. **Weyl differencing.** Multiplying out the square and setting \( \alpha_1 = q_2 - q \), we obtain

\[ |S_k(q_1, r_1, j)|^2 \]

\[ = \sum_{q_2, q \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q_2^k}{Q_0^{k/2}} \right) \cdot \Psi_2 \left( \frac{q^k}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1} \cdot (q_2^k - q^k) \right) \right), \]

\[ = \sum_{\alpha_1, q \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q^k}{Q_0^{k/2}} \right) \cdot \Psi_2 \left( \frac{(\alpha_1 + q)^k}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1} \cdot ((\alpha_1 + q)^k - q^k) \right) \right). \]

We observe that the contribution of \( \alpha_1 \)'s with \( N(\alpha_1) > Q_0^{1+\epsilon} \) is negligible and write

\[ P_{k-1, \alpha_1}(q) = (\alpha_1 + q)^k - q^k = \binom{k}{1} \cdot \alpha_1 q^{k-1} + \binom{k}{2} \cdot \alpha_1^2 q^{k-2} + \cdots + \binom{k}{k} \cdot \alpha_1^k, \]

thus obtaining

\[ |S_k(q_1, r_1, j)|^2 \ll \sum_{N(\alpha_1) \leq Q_0^{1+\epsilon}} S_{k-1}(q_1, r_1, j, \alpha_1), \]

where

\[ S_{k-1}(q_1, r_1, j, \alpha_1) := \sum_{q \in \mathbb{Z}[i]} \Psi_2 \left( \frac{q^k}{Q_0^{k/2}} \right) \cdot \Psi_2 \left( \frac{(\alpha_1 + q)^k}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1} \cdot P_{k-1, \alpha_1}(q) \right) \right). \]

If \( k > 2 \), we apply the Cauchy-Schwarz inequality again to obtain

\[ |S_k(q_1, r_1, j)|^4 \ll Q_0^{4+\epsilon} \sum_{\alpha_1 \in \mathbb{Z}[i]} N(\alpha_1) \leq Q_0^{4+\epsilon} |S_{k-1}(q_1, r_1, j, \alpha_1)|^2. \]
Multiplying out the square, changing variables and truncating the resulting sums in a similar way as above, we now obtain

\[ |S_{k-1}(q_1, r_1, j, \alpha)|^2 \leq \sum_{\alpha_2 \in \mathbb{Z}[i]} \sum_{q \in \mathbb{Z}[i] \cap N(\alpha_2) \leq Q_0^{k+\varepsilon}} \Psi_2 \left( \frac{q^k}{Q_0^{k/2}} \right) \cdot \Psi_2 \left( \frac{(\alpha_1 + q)^k}{Q_0^{k/2}} \right) \Psi_2 \left( \frac{(\alpha_2 + q)^k}{Q_0^{k/2}} \right) \times \psi_2 \left( \frac{(\alpha_1 + \alpha_2 + q)^k}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1^k} \cdot P_{k-2,\alpha_1,\alpha_2}(q) \right) \right), \]

where

\[ P_{k-2,\alpha_1,\alpha_2}(q) = k(k - 1)\alpha_1\alpha_2q^{k-2} + \cdots \]

is a polynomial of degree \( k - 2 \) in \( q \). We continue this process of repeated use of Cauchy-Schwarz and differencing until we have reached a polynomial of degree 1. Eventually, after combining all inequalities obtained in this way, we get

\[ |S_k(q_1, r_1, j)|^\kappa \leq Q_0^{-k+\varepsilon} \times \sum_{\alpha \in \mathbb{Z}[i]^k} \prod_{q \in \mathbb{Z}[i] \cap (0,1)^{k-1}} \Psi_2 \left( \frac{u \cdot \alpha + q}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1^k} \cdot P_{1,\alpha}(q) \right) \right), \tag{33} \]

where we write \( \kappa = 2^{k-1} \), \( \alpha = (\alpha_1, ..., \alpha_{k-1}) \) and \( u = (u_1, ..., u_k) \), \( u \cdot \alpha \) is the standard inner product, and \( P_{1,\alpha}(q) \) takes the form

\[ P_{1,\alpha}(q) = k!\alpha_1 \cdots \alpha_{k-1} \cdot \left( q + \frac{1}{2} \cdot (\alpha_1 + \cdots + \alpha_{k-1}) \right). \]

8.2. **Poisson summation.** Again, we shall apply Proposition [1] with \( d = 2 \) to transform the inner-most over \( q \) on the right-hand side of (33). For \( z = (z_1, z_2) \in \mathbb{R}^2 \), we set

\[ g(z) := \prod_{u \in \{0,1\}^{k-1}} \psi_2 \left( \left( z_1 + iz_2 + \frac{u \cdot \alpha}{\sqrt{Q_0}} \right)^k \right). \]

Then using [14] with

\[ a = q \quad \text{and} \quad b = \frac{k!\alpha_1 \cdots \alpha_{k-1}jr_1}{q_1^k}, \]

we deduce that

\[ \sum_{q \in \mathbb{Z}[i] \cap (0,1)^{k-1}} \prod_{u \in \{0,1\}^{k-1}} \psi_2 \left( \frac{u \cdot \alpha + q}{Q_0^{k/2}} \right) \cdot e \left( \Re \left( \frac{jr_1}{q_1^k} \cdot P_{1,\alpha}(q) \right) \right) \]

\[ = \sum_{x \in \mathbb{Z}^2} e \left( \overrightarrow{b} \cdot x \right) g \left( \frac{x}{\sqrt{Q_0}} \right). \tag{34} \]
Now applying Proposition \ref{boissongen} with \( B := \frac{1}{\sqrt{Q_0}} \) and \( f := \hat{g} \), the inverse Fourier transform of \( g \), to the right-hand side of \eqref{20}, we get

\[
\sum_{x \in \mathbb{Z}^2} e \left( \frac{x}{\sqrt{Q_0}} \right) g \left( \frac{x}{\sqrt{Q_0}} \right) = Q_0 \cdot \sum_{y \in \mathbb{Z}^2} \hat{g} \left( \sqrt{Q_0} y \right) = Q_0 \cdot \sum_{y \in -b + \mathbb{Z}^2} \hat{g} \left( \sqrt{Q_0} y \right).
\]

It follows that

\[
|S_k (q_1, r_1, j)|^\kappa \ll Q_0^{-k+\varepsilon} \times \sum_{\alpha \in \mathbb{Z}[i]^k} \sum_{\beta \in \mathbb{Z}[i]} \hat{g} \left( \sqrt{Q_0} \cdot \left( \beta - \frac{k! \alpha_1 \cdots \alpha_{k-1} j r_1}{q_1^k} \right) \right). \tag{35} \]

Here we set

\[
\Phi_2 (t) := \exp \left( -\frac{\pi}{\kappa} \cdot \sqrt{|t|} \right)
\]

so that

\[
\Psi_2 (z) = \Phi_2 (\mathcal{N} (z)) = \exp \left( -\frac{\pi}{\kappa} \cdot \sqrt{\mathcal{N} (z)} \right).
\]

It follows that

\[
g(z) = \exp \left( -\frac{\pi}{\kappa} \cdot \sum_{u \in \{0, 1\}^k-1} \left( z_1 + \frac{u \cdot \alpha^{(1)}}{\sqrt{Q_0}} \right)^2 + \left( z_2 + \frac{u \cdot \alpha^{(2)}}{\sqrt{Q_0}} \right)^2 \right),
\]

where

\[
\alpha^{(1)} := \left( \alpha_1^{(1)}, \ldots, \alpha_{k-1}^{(1)} \right) := (\Re (\alpha_1), \ldots, \Re (\alpha_{k-1}))
\]

and

\[
\alpha^{(2)} := \left( \alpha_1^{(2)}, \ldots, \alpha_{k-1}^{(2)} \right) := (\Im (\alpha_1), \ldots, \Im (\alpha_{k-1})).
\]

Completing the squares, it follows that

\[
g(z) = \exp \left( -\frac{\pi}{4Q_0} \cdot \sum_{i=1}^{2k-1} \sum_{v=1}^{2k-2} \left( \alpha^{(i)}_k \right)^2 \right) \cdot \exp \left( -\pi \cdot \sum_{i=1}^{2k-1} \left( z_i + \frac{\sum_{v=1}^{2k-1} \alpha^{(i)}_k}{2\sqrt{Q_0}} \right)^2 \right).
\]

The Fourier transform of this function satisfies

\[
\hat{g}(z) = \exp \left( -\frac{\pi}{4Q_0} \cdot \sum_{i=1}^{2k-1} \sum_{v=1}^{2k-2} \left( \alpha^{(i)}_k \right)^2 \right) \cdot e \left( -\frac{\sum_{i=1}^{2k-1} \sum_{v=1}^{2k-2} \alpha^{(i)}_k}{2\sqrt{Q_0}} \right) \cdot \exp \left( -\pi \left( z_1^2 + z_2^2 \right) \right) \tag{36} \]

\[
\ll \exp \left( -\pi \mathcal{N} (z_1 + iz_2) \right).
\]

Plugging \eqref{finalafterpoiss1} into \eqref{finalafterpoiss}, we get

\[
|S_k (q_1, r_1, j)|^\kappa \ll Q_0^{-k+\varepsilon} \times \sum_{\alpha \in \mathbb{Z}[i]^k} \sum_{\beta \in \mathbb{Z}[i]} \exp \left( -\pi Q_0 \mathcal{N} \left( \beta - \frac{k! \alpha_1 \cdots \alpha_{k-1} j r_1}{q_1^k} \right) \right) \tag{37} \]
8.3. Counting. Now we want to bound the term in the maximum in (32). We choose \( \hat{\Psi}_1 \) as in (25). Then

\[
\sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \hat{\Psi}_1 \left( \frac{\sqrt{2}jQ_0^{k/2}}{\sqrt{N}} \right) \cdot |S_k(q_1, r_1, j)| \ll 1 + \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} |S_k(q_1, r_1, j)|
\]

\[
\ll 1 + \left( \frac{N}{Q_0^{-\varepsilon}} \right)^{1-1/\kappa} \left( \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} |S_k(q_1, r_1, j)|^\kappa \right)^{1/\kappa},
\]

(38) countbeg

where the second line follows from Hölder’s inequality. Using (35) and taking into account that the contributions of \( \beta \)'s with \( N \left( \beta - \frac{k!a_1 \cdots a_{k-1}jr_1}{q_1^k} \right) > Q_0^{-1} \)

is negligible, we deduce that

\[
\sum_{j \in \mathbb{Z}[i] \setminus \{0\}} |S_k(q_1, r_1, j)|^\kappa
\]

\[
\ll Q_0^{-k+1+\varepsilon} \cdot \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \sum_{\alpha \in \mathbb{Z}[i]^k} \sum_{N(j) \leq NQ_0^{-k}} 1
\]

\[
\ll Q_0^{-k+1+\varepsilon} \cdot \sum_{j \in \mathbb{Z}[i] \setminus \{0\}} \sum_{\alpha \in \mathbb{Z}[i]^k} \sum_{N(j) \leq NQ_0^{-k}} \frac{1}{N(\alpha_1, \ldots, \alpha_{k-1}) \leq Q_0^{1+\varepsilon}} N(NQ_0^{k-1} \leq |dr_1/q_1| \leq Q_0^{-1/2})
\]

\[
\ll (NQ_0)^{(k+1)\varepsilon} \cdot Q_0^{-k+1+\varepsilon} \cdot \left( \frac{N}{Q_0^k} + \sum_{d \in \mathbb{Z}[i] \setminus \{0\}} 1 \right)
\]

\[
\ll (NQ_0)^{(k+1)\varepsilon} \cdot (NQ_0^{k-1} + Q_0^{-k+1+\varepsilon}) \cdot \sum_{l \in \mathbb{Z}[i]} \sum_{l/|q_1^k| \leq |l|/|q_1^k| \leq Q_0^{-1/2}} 1
\]

(39) super1

where we recall that \(|z|\) is the distance of \( z \in \mathbb{C} \) to the nearest Gaussian integer. In the above, we have set \( d = k!a_1 \cdots a_{k-1}j \) if \( a_1, \ldots, a_{k-1} \neq 0 \).

The number of residue classes modulo \( q_1^k \) is \( N(q_1^k) \leq Q_0^k \), and hence

\[
\sum_{N(d) \leq N^2 Q_0^{k-1}} 1 \ll 1 + \frac{N}{Q_0^{k+1-k^2}}.
\]

(40) res1
Further
\[ \sum_{l \in \mathbb{Z} \mid \frac{|l|}{q_0} \leq Q^k + \frac{1}{2} \} \leq Q_0^k - 1. \]  
Combining (39), (40) and (41), we obtain
\[ \sum_{j \in \mathbb{Z} \mid \frac{|j|}{N(j)} \leq Q_0^k} |S_k(q_1, r_1, j)|^\kappa \ll (Q_0N)^{(2k+3)\varepsilon} \left( Q_0^\kappa + NQ_0^\kappa - 1 \right), \]  
and combining (32), (38) and (42), and changing \( \varepsilon \) suitably, we arrive at
\[ K \ll Q_0^k N + (Q_0N)^\varepsilon \left( \frac{Q_0^{1+k/\kappa}}{N^{1/\kappa}} + Q_0^{1-1/\kappa} \right). \]  
Now the statement in Theorem 4 follows immediately from Corollary 1 and (43) after dividing the moduli into dyadic intervals and replacing \( Q_0 \) by \( Q \).

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