NONTRIVIAL ALGEBRAIC CYCLES IN THE JACOBIAN VARIETIES OF SOME QUOTIENTS OF FERMAT CURVES

YUUKI TADOKORO

Abstract. We obtain the trace map images of the values of certain harmonic volumes for some quotients of Fermat curves. These provide the algorithm showing that the algebraic cycles called by $k$-th Ceresa cycles are not algebraically equivalent to zero in the Jacobian varieties. We apply the method for the prime $N < 1000, k = 1$ and $N = 7, 13, k \leq (N - 3)/2$.

1. Introduction

Let $X$ be a compact Riemann surface of genus $g \geq 2$ and $J(X)$ its Jacobian variety. By the Abel-Jacobi map $X \to J(X)$, $X$ is embedded in $J(X)$. Let $X_k$ be the $k$-th symmetric product of $X$ and $W_k$ its image of the Abel-Jacobi map. The algebraic $k$-cycle $W_k - W_k^-$ in $J(X)$, called by $k$-th Ceresa cycle, is homologous to zero. Here we denote by $W_k^-$ the image of $W_k$ under the multiplication map by $-1$. If $X$ is hyperelliptic, $W_k = W_k^-$ in $J(X)$. For the rest of this paper, suppose $g \geq 3$. We put $X - X^+ = W_1 - W_1^-$. B. Harris [8] studied the problem whether the cycle $X - X^+$ in $J(X)$ is algebraically equivalent to zero or not. Roughly speaking, it can be “continuously” (algebraically) deformed into the zero cycle or not. See [5] for example.

Faucette [4] also studied a sufficient condition that the algebraic cycle $W_k - W_k^-$ is not algebraically equivalent to zero in $J(X)$. We remark that Weil [17, pp. 331] mentioned the homologous zero cycle $W_k - W_k^-$ in question.

Let $N$ be a prime number such that $N = 1$ modulo $3$ and $m$ be an integer $m^2 + m + 1 = 0$ modulo $N$. For the quotient of Fermat curve $C_N = C_N^{1,m}$, we denote $f(N,k)$ by a value of the harmonic volume which is defined later using the special values of the generalized hypergeometric function $3F_2$. Using Otsubo’s result [11], we obtain the main theorem

**Theorem 1.1.** For the quotient of Fermat curve $C_N$ and an integer $k$ such that $1 \leq k \leq (N - 3)/2$, if the value $f(N,k)$ is not integer, then $W_k - W_k^-$ is not algebraically equivalent to zero in $J(C_N)$.

The harmonic volume $I$ for $X$ was introduced by Harris [7], using Chen’s iterated integrals [3]. Let $H$ denote the first integral homology group $H_1(X;\mathbb{Z})$ of $X$. The harmonic volume is defined to be a homomorphism $(H^{\otimes 3})' \to \mathbb{R}/\mathbb{Z}$. Here $(H^{\otimes 3})'$ is a certain subgroup of $H^{\otimes 3}$. The twice $2f$ factors through the third exterior product $\wedge^3 H$, and we call it the harmonic volume similarly. See Section [2] for the definition. Let $F_N$ denote the Fermat curve for $N \in \mathbb{Z}_{\geq 4}$. Using $I$, Harris [8, 9] proved that the algebraic cycle $F_4 - F_4^-$ is not algebraically equivalent to zero in $J(F_4)$. Ceresa [2] showed that
$W_k - W_k^-$ is not algebraically equivalent to zero for a generic $X$. For the Klein quartic $C_7^{1,2}$ and Fermat sextic $F_6$, we [15] [16] computed the harmonic volume using the special values of the generalized hypergeometric function $3F_2$ and showed that the algebraic cycle $X - X^-$ is not algebraically equivalent to zero in $J(X)$. Recently, Otsubo [11] ably extended Harris’ and our results, using a primitive $N$-th root of unity and the trace map for the Fermat curve $F_N$. He obtained the algorithm showing that the algebraic $k$-cycle $W_k - W_k^-$ is not algebraically equivalent to zero in $J(F_N)$. We find the above condition for $N$ and another algorithm showing that $W_k - W_k^-$ is not algebraically equivalent to zero in $J(C_N)$. For a complex algebraic variety $V$, we define the $p$-th Griffith group $\text{Griff}^p(V)$ which is generated by all the algebraic cycles of codimension $p$ in $V$ homologically equivalent to zero modulo algebraic equivalence. We also prove the Griffiths group of $J(X)$ is nontrivial. Furthermore, Bloch [1] studied the Fermat quartic $F_4$ by means of $L$-functions.

We give our method to prove the algebraic cycle $C_N - C_N^-$ is not algebraically equivalent to zero in $J(C_N)$, which is similar to Otsubo’s one. See Hodge’s letter [18, pp. 533–534]. Let $\eta_m$ be a third exterior product of holomorphic 1-forms on $C_N$. If the cycle $C_N - C_N^-$ is algebraically equivalent to zero in $J(C_N)$, then the trace map image $f(N, 1) \in \mathbb{R}/\mathbb{Z}$ of the harmonic volume at $\eta_m$ are zero modulo $\mathbb{Z}$. In order to prove the cycle $C_N - C_N^-$ is not algebraically equivalent to zero, we have only to show the above values are not zero. Similarly we obtain the method that $W_k - W_k^-$ is not algebraically equivalent to zero.

Now we describe the contents of this paper briefly. In Section 2 we introduce the harmonic volume and relation between it and the Ceresa cycle. Section 3 is devoted to definition of the Fermat curve and the trace map. In Section 4 we define some quotients of Fermat curve and recall Otsubo’s method. Using an algebraic condition, we obtain the harmonic volume $f(N, k)$ of $C_N$. We carry the numerical computation of the value by means of the special values of the generalized hypergeometric function $3F_2$.

Acknowledgements. The author would like to thank Noriyuki Otsubo for his useful comments. This work is supported by Grant-in-Aid for Young Scientists (B).

Contents

1. Introduction 1
2. The harmonic volume and the algebraic cycle $X - X^-$ 3
3. The Fermat curve 4
4. Some values of the harmonic volume for the quotient of Fermat curve 5
  4.1. Some quotients of Fermat curve 6
  4.2. Hypergeometric functions and numerical computation 6
References 8
2. THE HARMONIC VOLUME AND THE ALGEBRAIC CYCLE $X - X^-$

We recall the harmonic volume [7] for a compact Riemann surface $X$ of genus $g \geq 3$. We identify the first integral homology group $H_1(X; \mathbb{Z})$ of $X$ with the first integral cohomology group by Poincaré duality, and denote it by $H$. The Hodge star operator $*$ on the space of all the 1-forms $A^1(X)$ is locally given by $*(f_1(z) dz + f_2(z) d\bar{z}) = -\sqrt{-1} f_1(z) dz + \sqrt{-1} f_2(z) d\bar{z}$ in a local coordinate $z$ and depends only on the complex structure and not on the choice of Hermitian metric. We identify $H$ with the space of all the real harmonic 1-forms on $X$ with integral periods. Let $(H^{\otimes 2})'$ be the kernel of the intersection pairing $(\cdot, \cdot): H \otimes_{\mathbb{Z}} H \to \mathbb{Z}$. For the rest of this paper, we write $\otimes = \otimes_{\mathbb{Z}}$, unless otherwise stated. For any $\sum_{i=1}^{n} a_i \otimes b_i \in (H^{\otimes 2})'$, there exists a unique $\eta \in A^1(X)$ such that $d\eta = \sum_{i=1}^{n} a_i \wedge b_i$ and $\int_X \eta \wedge * \alpha = 0$ for any closed 1-form $\alpha \in A^1(X)$. Here $a_i$ and $b_i$ are regarded as real harmonic 1-forms on $X$. Choose a point $x_0 \in X$.

**Definition 2.1.** (The pointed harmonic volume [14])

For $\sum_{i=1}^{n} a_i \otimes b_i \in (H^{\otimes 2})'$ and $c \in H$, the pointed harmonic volume $I_{x_0}$ is the homomorphism $(H^{\otimes 2})' \otimes H \to \mathbb{R}/\mathbb{Z}$ defined by

$$I_{x_0}\left(\sum_{i=1}^{n} a_i \otimes b_i \otimes c\right) = \sum_{i=1}^{n} \int_{\gamma} a_i b_i - \int_{\gamma} \eta \mod \mathbb{Z}. $$

Here $\eta \in A^1(X)$ is associated to $\sum_{i=1}^{n} a_i \otimes b_i$ in the way stated above and $\gamma$ is a loop in $X$ with the base point $x_0$ whose homology class is equal to $c$. The integral $\int_{\gamma} a_i b_i$ is Chen’s iterated integral [3], that is, $\int_{\gamma} a_i b_i = \int_{0 \leq t_1 \leq t_2 \leq 1} f_i(t_1) g_i(t_2) dt_1 dt_2$ for $\gamma^* a_i = f_i(t) dt$ and $\gamma^* b_i = g_i(t) dt$. Here $t$ is the coordinate in the interval $[0, 1]$.

The harmonic volume is given as a restriction of the pointed harmonic volume $I_{x_0}$. We denote by $(H^{\otimes 3})'$ the kernel of a natural homomorphism

$$H^{\otimes 3} \to H^{\otimes 3}, a \otimes b \otimes c \mapsto ((a, b) c, (b, c) a, (c, a) b).$$

The *harmonic volume* $I$ for $X$ is a linear form on $(H^{\otimes 3})'$ with values in $\mathbb{R}/\mathbb{Z}$ defined by the restriction of $I_{x_0}$ to $(H^{\otimes 3})'$, i.e., $I = I_{x_0}|_{(H^{\otimes 3})'}$. Harris [7] proved that the harmonic volume $I$ is independent of the choice of the base point $x_0$. We denote $\wedge^3 H$ by the third exterior power of $H$ and $(\wedge^3 H)'$ by the kernel of a homomorphism

$$\wedge^3 H \to H; a \wedge b \wedge c \mapsto (a, b) c + (b, c) a + (c, a) b.$$ 

Then the natural map $(H^{\otimes 3})' \to (\wedge^3 H)'$ and $2I$ factors through

$$2I: (\wedge^3 H)' \to \mathbb{R}/\mathbb{Z}$$

[7].

Let $J = J(X)$ and $X_k$ be the Jacobian variety and $k$-th symmetric product of $X$ respectively. By the Abel-Jacobi map $X \to J(X)$, $X_k$ is embedded in $J$. The image of $X_k$ is denoted by $W_k$. The algebraic $k$-cycle $W_k - W_k^-$ in $J$ is homologous to zero. Here we denote by $W_k^-$ the image of $W_k$ under the multiplication map by $-1$. The cycle $W_k - W_k^-$ is called the $k$-th Ceresa cycle. We put $W_1 - W_1^- = X - X^-$. We say the an algebraic cycle 1-cycle $C$ is *algebraically equivalent to zero in $J$* if there exists a topological 3-chain $W$ such that $\partial W = C$ and $W$ lies on $S$, where $S$ is an algebraic
Proposition 2.2. If \( X - X^- \) is algebraically equivalent to zero in \( J \), then \( 2I(\omega) = 0 \) modulo \( \mathbb{Z} \) for any \( \omega \in \wedge^3 H \cap (\wedge^3 H^{1,0} + \wedge^3 H^{0,1}) \).

If the value \( 2I(\omega) \) is nonzero modulo \( \mathbb{Z} \) for some \( \omega \in \wedge^3 H \cap (\wedge^3 H^{1,0} + \wedge^3 H^{0,1}) \), then \( X - X^- \) is not algebraically equivalent to zero in \( J \).

Generally, if \( W_k - W_k^- \) is algebraically equivalent to zero in \( J \) and satisfying algebraic conditions. Then a constant multiple of \( 2I(\omega) \) is equal to 0 modulo \( \mathbb{Z} \) for any \( \omega \in \wedge^3 H \cap (\wedge^3 H^{1,0} + \wedge^3 H^{0,1}) \). See Faucette [4] and Otsubo [11]. In particular, Otsubo studied the good condition for the Fermat curve \( F_N \).

3. The Fermat curve

For \( N \in \mathbb{Z}_{\geq 4} \), let \( F_N = \{(X : Y : Z) \in \mathbb{C}P^2 \mid X^N + Y^N = Z^N \} \) denote the Fermat curve of degree \( N \), which is a compact Riemann surface of genus \((N - 1)(N - 2)/2\).

Let \( x \) and \( y \) denote \( X/Z \) and \( Y/Z \) respectively. The equation \( X^N + Y^N = Z^N \) induces \( x^N + y^N = 1 \). Here \( \zeta \) denotes \( \exp(2\pi \sqrt{-1}/N) \). Holomorphic automorphisms \( \alpha \) and \( \beta \) of \( F_N \) are defined by \( \alpha(X : Y : Z) = (\zeta X : Y : Z) \) and \( \beta(X : Y : Z) = (X : \zeta Y : Z) \) respectively. Let \( \mu_N \) be the group of \( N \)-th roots of unity in \( \mathbb{C} \). We have that \( \alpha \beta = \beta \alpha \) and the subgroup of the holomorphic automorphisms of \( F_N \) generated by \( \alpha \) and \( \beta \) is isomorphic to \( \mu_N \times \mu_N \). We denoted it by \( G \). Let \( \gamma_0 \) be a path \([0,1] \ni t \mapsto (t, \sqrt[3]{1 - t^N}) \in F(N) \), where \( \sqrt[3]{1 - t^N} \) is a real nonnegative analytic function on \([0,1]\). A loop in \( F_N \) is defined by

\[ \kappa_0 = \gamma_0 \cdot (\beta \gamma_0)^{-1} \cdot (\alpha \beta \gamma_0) \cdot (\alpha \gamma_0)^{-1}, \]

where the product \( \ell_1 \cdot \ell_2 \) indicates that we traverse \( \ell_1 \) first, then \( \ell_2 \). We consider a loop \( \alpha^i \beta^j \kappa_0 \) as an element of the first homology group \( H_1(F_N; \mathbb{Z}) \) of \( F_N \). It is a known fact that \( H_1(F_N; \mathbb{Z}) \) is a cyclic \( G \)-module [Appendix in [6]].

Let \( I \) be an index set \( \{(a, b) \in (\mathbb{Z}/N\mathbb{Z})^\oplus 2 \mid a, b, a + b \neq 0 \} \). For \( a \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\} \), we denote its representative \( \langle a \rangle \in \{1, 2, \ldots, N - 1\} \). A differential 1-form on \( F_N \) is defined by

\[ \omega_0^{a,b} = x^{(a)-1}y^{(b)-1}dx/y^{N-1} \]

Set \( I_{\text{holo}} = \{(a, b) \in I ; \langle a \rangle + \langle b \rangle < N \} \). It is well known that \( \{\omega_0^{a,b}\}_{I_{\text{holo}}} \) is a basis of \( H^{1,0} \) of \( F(N) \). See Lang [10] for example. It is clear that

\[ \int_{\alpha^i \beta^j \gamma_0} \omega_0^{a,b} = \zeta^{ai+bj} \int_{\gamma_0} \omega_0^{a,b} = \zeta^{ai+bj} B((a)/N, (b)/N). \]

The beta function \( B(u, v) \) is defined by \( \int_0^1 t^{u-1}(1 - t)^{v-1}dt \) for \( u, v > 0 \). We denote \( B_{a,b}^N = B((a)/N, (b)/N) \). The integral of \( \omega_0^{a,b} \) along \( \alpha^i \beta^j \kappa_0 \) is obtained as follows.

Proposition 3.1 (Appendix in [6]). We have

\[ \int_{\alpha^i \beta^j \kappa_0} \omega_0^{a,b} = B_{a,b}^N (1 - \zeta^a)(1 - \zeta^b) \zeta^{ai+bj}/N. \]
We denote the 1-form \( N\omega_{0}^{a,b}/B_{a,b}^{N} \) by \( \omega^{a,b} \). This implies \( \int_{\alpha^{i}\beta^{j}\alpha^{a}}\omega_{0}^{a,b} \in \mathbb{Z}[\zeta] \).

Let \( K = \mathbb{Q}(\mu_{N}) \) be the \( N \)-cyclotomic field, \( \mathcal{O} \) be its integer ring and fix a primitive \( N \)-th root of unity \( \xi \). For a \( \mathbb{Z} \)-module \( M \), we denote the \( \mathcal{O} \)-module \( \mathcal{O}_{M} = M \otimes \mathcal{O} \). For each embedding \( \sigma : K \hookrightarrow \mathbb{C} \), we may consider the 1-form \( \omega^{a,b} \) as an element of \( H_{\sigma}^{1} \) depending on the relation of \( \sigma(\xi) \) and \( \zeta \).

The harmonic volume naturally extends to
\[
2I_{\sigma} : (N^{3}H)^{\sigma}_{\mathcal{O}} \to (\mathcal{O} \otimes \mathbb{R})/\mathcal{O}.
\]
We have the natural isomorphism
\[
\mathcal{O} \otimes \mathbb{R} \cong \left[ \prod_{\sigma : K \hookrightarrow \mathbb{C}} \mathbb{C} \right]^{+}
\]
where \( \sigma \) runs through the embedding of \( K \) into \( \mathbb{C} \) and \( + \) denotes the fixed part by the complex conjugation acting the set \( \{\sigma\} \) and \( \mathbb{C} \) at the same time. Let \( 2I_{\sigma} \) denote the \( \sigma \)-component of \( 2I \). Let \( \text{Tr} : (\mathcal{O} \otimes \mathbb{R})/\mathcal{O} \to \mathbb{R}/\mathbb{Z} \) be the trace map. We obtain \( \text{Tr} \circ 2I_{\sigma} = \sum_{\sigma : K \hookrightarrow \mathbb{C}} 2I_{\sigma} \). In order to prove the nontriviality of \( 2I_{\sigma} \), it is enough to prove that of \( \text{Tr} \circ 2I_{\sigma} \).

4. Some values of the harmonic volume for the quotient of Fermat curve

4.1. Some quotients of Fermat curve. For a prime number \( N \) such that \( N \geq 5 \), we define the quotient of Fermat curve \( C_{N}^{a,b} \) as projective curve whose affine equation is
\[
C_{N}^{a,b} := \{(u,v) \in \mathbb{C}^{2}; v^{N} = u^{a}(1-u)^{b}\}.
\]
Here the integers \( a, b \) are coprime and satisfy \( 0 < a, b < N \). It is a compact Riemann surface of genus \((N-1)/2\). We denote by \( \tau : F_{N} \to C_{N} \) the \( N \)-fold unramified covering \( \pi(x,y) = (u,v) = (x^{N},x^{a}y^{b}) \). For any integer \( h \in \{1,2,\ldots,N-1\} \), there is a unique 1-form \( \eta^{(ha),(hb)} \) such that \( \pi^{*}\eta^{(ha),(hb)} = \omega^{ha,hb} \). Then we have \( \{\eta^{(ha),(hb)}\}_{(ha)+(hb)<N} \) is a basis of \( H^{1,0} \) of \( C_{N} \). See Lang [10] for example.

For the rest of this paper, we assume that the prime number \( N \) satisfies \( N = 1 \) modulo 3. There exists an integer \( 1 < m < N - 1 \) such that \( m^{2} + m + 1 = 0 \) modulo \( N \).

Set \((a_{1},b_{1}) = (1,m),(a_{2},b_{2}) = (m,m^{2}) \), and \((a_{3},b_{3}) = (m^{2},1) \).

Lemma 4.1. The above \((a_{i},b_{i}) \)'s satisfy the assumption 4.4 in [11]. Furthermore, the conditions \((ha_{i},hb_{i})\in I_{\text{holo}},i = 1,2,3 \) are equivalent.

Proof. Note that \( h + \langle hm \rangle + \langle hm^{2} \rangle = N \) or \( 2N \). We obtain that \( h + \langle hm \rangle + \langle hm^{2} \rangle = N \) if only and if \((ha_{i},hb_{i})\in I_{\text{holo}} \) for each \( i \).

From now on, we put \( C_{N} = C_{N}^{1,m} \). Since \( \pi \) is an \( N \)-fold unramified covering, we obtain \( N\eta^{(ha),(hb)} \in H_{\mathcal{O}} \) of \( C_{N} \). In order to compute the harmonic volume of \( C_{N} \), it is enough to substitute \( N\eta^{ha,hb} \) for \( \varphi^{a,b} \) in [11]. Set
\[
\eta_{m} = \frac{N\eta_{m}^{1,m} \wedge N\eta_{m}^{m,(m^{2})} \wedge N\eta_{m}^{(m^{2}),1}}{(1 - \xi^{-m^{2}})(1 - \xi^{-1})}.
\]
From Proposition 3.1, it is easy to show that \( \eta_m \) is an element of \( (\wedge^3 H_\O)^* \) of \( C_N \). We have the equation
\[
I_\O(\eta_m) = NI_\O(\pi^*\eta_m) \mod \O.
\]
Here the harmonic volume of LHS is on \( C_N \), and that of RHS is on \( F_N \). Theorem 3.7 in \[11\] gives us

**Proposition 4.2.** We obtain the value of the harmonic volume for \( C_N \)
\[
\text{Tr} \circ 2I_\O(\eta_m) = N^6 \sum_{k_0} \omega^{\,h,\,m_\nu,\,h,\,m} \int_0^\infty \omega^{\,h,\,m_\nu,\,h,\,m} \cdot \omega^{\,h,\,m_\nu,\,h,\,m}^2,
\]
where the sum is taken over \( h \in (\mathbb{Z}/N\mathbb{Z})^* \) such that \( (ha_i, hb_i) \in I_{\text{holo}} \).

**Remark 4.3.** The conditions \( (ha_i, hb_i) \in I_{\text{holo}} \) and \( h + \langle hm \rangle + \langle hm^2 \rangle = N \) are equivalent. Otsubo defined the embedding \( \sigma : K \hookrightarrow \mathbb{C} \) such that \( \sigma(\xi) = \xi^h \).

### 4.2. Hypergeometric functions and numerical computation.

For the numerical calculation, we recall the generalized hypergeometric function \( {}_3F_2 \). We denote the gamma function \( \Gamma(\tau) = \int_0^\infty e^{-t}\tau^{-1} \, dt \) for \( \tau > 0 \) and the Pochhammer symbol \( (\alpha, n) = \Gamma(\alpha + n)/\Gamma(\alpha) \) for any nonnegative integer \( n \). For \( x \in \{ z \in \mathbb{C} ; |z| < 1 \} \) and \( \beta_1, \beta_2 \notin \{ 0, -1, -2, \ldots \} \), the generalized hypergeometric function \( {}_3F_2 \) is defined by
\[
{}_3F_2 \left( \begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1, n)(\alpha_2, n)(\alpha_3, n)}{(\beta_1, n)(\beta_2, n)(1, n)} x^n.
\]
If \( \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 > 0 \), then the generalized hypergeometric function \( {}_3F_2 \) converges when \( |x| = 1 \). See \[12\] for example. We denote
\[
\Gamma^N \left( \begin{array}{c} a_1, a_2, \ldots, a_n \\ b_1, b_2, \ldots, b_m \end{array} \right) = \frac{\Gamma(a_1/N)\Gamma(a_2/N)\cdots\Gamma(a_n/N)}{\Gamma(b_1/N)\Gamma(b_2/N)\cdots\Gamma(b_m/N)}.
\]
Using proposition 5.3 in \[11\], we have

**Proposition 4.4.**
\[
\int_{h \cdot \langle hm \rangle + \langle hm^2 \rangle = N} \omega^{\,h,\,m_\nu,\,h,\,m} = \Gamma^N \left( \begin{array}{c} N - \langle hm \rangle, N - \langle hm^2 \rangle \\ \langle hm \rangle \end{array} \right)^2 {}_3F_2 \left( \begin{array}{c} h/N, \langle h \rangle/N, \langle hm^2 \rangle/N \\ 1, 1 \end{array} ; 1 \right)
\]
for an integer \( h \) such that \( h + \langle hm \rangle + \langle hm^2 \rangle = N \).

**Theorem 4.5.** For the quotient of Fermat curve \( C_N \), if the value
\[
2N^6 \sum_{0 < h < N, h + \langle hm \rangle + \langle hm^2 \rangle = N} \int_{h \cdot \langle hm \rangle + \langle hm^2 \rangle = N} \omega^{\,h,\,m_\nu,\,h,\,m} \omega^{\,h,\,m_\nu,\,h,\,m}^2
\]
is not equal to zero modulo \( \mathbb{Z} \). Then, the algebraic cycle \( C_N - C_N^- \) is not algebraically equivalent to zero in \( J(C_N) \).

This value is independent of the choice of \( m \), we denote it by \( f(N, 1) \). Furthermore, we set \( f(N, k) = k! \cdot N^{4k-4} \cdot f(N, 1) \) for a positive integer \( k \). Using Corollary 4.9 in \[11\], it is to show

**Theorem 4.6.** For the quotient of Fermat curve \( C_N \) and an integer \( k \) such that \( 1 \leq k \leq (N - 3)/2 \), if the value \( f(N, k) \) is not equal to zero modulo \( \mathbb{Z} \). Then, the algebraic cycle \( W_k - W_k^- \) is not algebraically equivalent to zero in \( J(C_N) \).
We show the table of the computation of $f(N, 1)$ and Mathematica program 19 of $f(N, k)$.

| $N$ | $m$ | $f(N, 1)$ | $N$ | $m$ | $f(N, 1)$ | $N$ | $m$ | $f(N, 1)$ |
|-----|-----|---------|-----|-----|---------|-----|-----|---------|
| 7   | 2   | 0.64692 | 283 | 44  | 0.97789 | 631 | 43  | 0.50662 |
| 13  | 3   | 0.30390 | 307 | 17  | 0.66173 | 643 | 177 | 0.72852 |
| 19  | 7   | 0.15972 | 313 | 98  | 0.96320 | 661 | 296 | 0.43828 |
| 31  | 5   | 0.68272 | 331 | 31  | 0.88040 | 673 | 255 | 0.20495 |
| 37  | 10  | 0.53833 | 337 | 128 | 0.61843 | 691 | 253 | 0.58775 |
| 43  | 6   | 0.94719 | 349 | 122 | 0.57242 | 709 | 227 | 0.79285 |
| 61  | 13  | 0.10498 | 367 | 83  | 0.70289 | 727 | 281 | 0.38854 |
| 67  | 29  | 0.67834 | 373 | 88  | 0.55905 | 733 | 307 | 0.12451 |
| 73  | 8   | 0.67715 | 379 | 51  | 0.13144 | 739 | 320 | 0.44354 |
| 79  | 23  | 0.70081 | 397 | 34  | 0.54575 | 751 | 72  | 0.78711 |
| 97  | 35  | 0.67120 | 409 | 53  | 0.59176 | 757 | 27  | 0.10544 |
| 103 | 46  | 0.20164 | 421 | 20  | 0.86406 | 769 | 360 | 0.62163 |
| 109 | 45  | 0.21967 | 433 | 198 | 0.085557| 787 | 379 | 0.10082|
| 127 | 19  | 0.75140 | 439 | 171 | 0.20173 | 811 | 130 | 0.17690 |
| 139 | 42  | 0.89455 | 457 | 133 | 0.055143| 823 | 174 | 0.22898 |
| 151 | 32  | 0.20776 | 463 | 21  | 0.24695 | 829 | 125 | 0.86872 |
| 157 | 12  | 0.65104 | 487 | 232 | 0.82059 | 853 | 220 | 0.57350 |
| 163 | 58  | 0.47898 | 499 | 139 | 0.89265 | 859 | 260 | 0.89417 |
| 181 | 48  | 0.68643 | 523 | 60  | 0.12188 | 877 | 282 | 0.70117 |
| 193 | 84  | 0.65697 | 541 | 129 | 0.20975 | 883 | 337 | 0.26719 |
| 199 | 92  | 0.53788 | 547 | 40  | 0.13131 | 907 | 384 | 0.49691 |
| 211 | 14  | 0.92477 | 571 | 109 | 0.86328 | 919 | 52  | 0.47589 |
| 223 | 39  | 0.14653 | 577 | 213 | 0.83477 | 937 | 322 | 0.94337 |
| 229 | 94  | 0.48453 | 601 | 24  | 0.16953 | 967 | 142 | 0.71751 |
| 241 | 15  | 0.77552 | 607 | 210 | 0.27883 | 991 | 113 | 0.94086 |
| 271 | 28  | 0.95322 | 613 | 65  | 0.91661 | 997 | 304 | 0.79227 |
| 277 | 116 | 0.88313 | 619 | 252 | 0.91440 |       |      |           |

**Figure 1.** Table of the $f(N, 1)$

The table shows that the algebraic cycle $C_N - C_N$ is not algebraically equivalent to zero in $J(C_N)$ for $N < 1000$ satisfying the condition.
\[ hv[n_, m_] := \\
2 \cdot n^6 * \\
\text{Sum}[\text{If}[h + \text{Mod}[h \cdot m, n] + \text{Mod}[h \cdot m^2, n] == n, 1, 0] * \\
\text{Gamma}[1 - \text{Mod}[h \cdot m, n]/n] \cdot 2 \cdot \text{Gamma}[1 - \text{Mod}[h \cdot m^2, n]/n] \cdot 2 / \\
(\text{Gamma}[\text{Mod}[h \cdot m, n]/n] \cdot 2) * \\
\text{HypergeometricPFQ}[\{h/n, \text{Mod}[h \cdot m, n]/n, \text{Mod}[h \cdot m^2, n]/n\}, \\
\{1, 1\}, 1], \\
\{h, 1, n - 1\}] \\
g[n_] := \text{Catch}[\text{Do}[\text{If}[\text{Mod}[1 + m + m^2, n] == 0, \text{Throw}[m], \{m, 2, n - 1\}]]] \\
f[n_, k_] := \text{If}[\text{Mod}[n, 3] == 1, \\
\{n, g[n], \\
\text{N}[\text{FractionalPart}[k! \cdot n^{(4 \cdot k - 4)} \cdot hv[n, g[n]]]], 5] \\
}, \{n, F\}] \\
\]

**Figure 2.** Numerical calculation program of \( f(N, k) \)

**References**

[1] Bloch, Spencer: *Algebraic cycles and values of \( L \)-functions*. J. Reine Angew. Math. 350 (1984), 94–108.

[2] Ceresa, G.: *C is not algebraically equivalent to \( C^\sim \) in its Jacobian*. Ann. of Math. (2) 117 (1983), no. 2, 285–291.

[3] Chen, Kuo Tsai: *Algebras of iterated path integrals and fundamental groups*. Trans. Amer. Math. Soc. 156 1971 359–379.

[4] Faucette, William M.: *Harmonic volume, symmetric products, and the Abel-Jacobi map*. Trans. Amer. Math. Soc. 335 (1993), no. 1, 303–327.

[5] Fulton, William: *Intersection theory. Second edition*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998.

[6] Gross, Benedict H.: *On the periods of abelian integrals and a formula of Chowla and Selberg. With an appendix by David E. Rohrich*. Invent. Math. 45 (1978), no. 2, 193–211.

[7] Harris, Bruno: *Harmonic volumes*. Acta Math. 150 (1983), no. 1-2, 91–123.

[8] Harris, Bruno: *Homological versus algebraic equivalence in a Jacobian*. Proc. Nat. Acad. Sci. U.S.A. 80 (1983), no. 4 i., 1157–1158.

[9] Harris, Bruno: *Iterated integrals and cycles on algebraic manifolds*. Nankai Tracts in Mathematics, 7. World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

[10] Lang, Serge: *Introduction to algebraic and abelian functions. Second edition*. Graduate Texts in Mathematics, 89. Springer-Verlag, New York-Berlin, 1982.

[11] Otsubo, Noriyuki: *On the Abel-Jacobi maps of Fermat Jacobians*. Preprint [arXiv:1003.0357](http://arxiv.org/abs/1003.0357), to appear in Math. Z.

[12] Paranjape, K. H.; Srinivas, V. *Algebraic cycles*, Current trends in mathematics and physics, 71–86, Narosa, New Delhi, 1995.

[13] Slater, Lucy Joan: *Generalized hypergeometric functions*. Cambridge University Press, Cambridge 1966.

[14] Pulte, Michael J.: *The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles*. Duke Math. J. 57 (1988), no. 3, 721–760.

[15] Tadokoro, Yuuki: *A nontrivial algebraic cycle in the Jacobian variety of the Klein quartic*. Math. Z. 260 (2008), no. 2, 265–275.

[16] Tadokoro, Yuuki: *A nontrivial algebraic cycle in the Jacobian variety of the Fermat sextic*. Tsukuba J. Math. 33 (2009), no. 1, 29–38.

[17] Weil, Andre: *Foundations of algebraic geometry*. American Mathematical Society, Providence, R.I. 1962.
[18] Weil, Andre: *Scientific works. Collected papers. Vol. II (1951–1964)*. Springer-Verlag, New York-Heidelberg, 1979.

[19] Wolfram, Stephen: *The Mathematica® book. Fourth edition*. Wolfram Media, Inc., Champaign, IL; Cambridge University Press, Cambridge, 1999.

**Natural Science Education, Kisarazu National College of Technology, 2-11-1 Kiyomidai-Higashi, Kisarazu, Chiba 292-0041, Japan**

*E-mail address: tado@nebula.n.kisarazu.ac.jp*