A PRACTICAL APPROACH TO OPTIMIZATION

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Abstract. We present a new approach for finding a minimal value of an arbitrary function assuming only its continuity. The process avoids verifying Lagrange- or KKT-conditions. The method enables us to obtain a Brouwer fixed point (of a continuous function mapping from a cube into itself).

Keywords : convex algorithm · optimization · particle swarm optimization · pattern-search · KKT-conditions · Brouwer fixed points

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1. Introduction

For a given set of continuous functions $f, g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n : \mathcal{C} = \prod_{i=1}^{p}[c_i, d_i] \rightarrow \mathbb{R}$, a minimization problem of the form

$$\min_{x \in \mathcal{C}} f(x)$$

subject to

$$g_i(x) = 0 \ (i = 1, 2, \ldots, m)$$

$$h_i(x) \leq 0 \ (i = 1, 2, \ldots, n).$$

is well known. For the Problem (1.1), $f$ is called the objective function and the equalities (described by $g_i$) and the inequalities (described by $h_i$) are called the constraints. We call the set $\mathcal{A} = \{x \in \mathcal{C} : g_i(x) = 0 \ (i = 1, 2, \ldots, m) \text{ and } h_i(x) \leq 0 \ (i = 1, 2, \ldots, n)\}$ the feasible set of Problem (1.1). If $\mathcal{A}$ is not empty, it is compact since it is a zero set

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of the continuous function $F$ defined below. Consequently, Problem (1.1) always has 
a solution if $\mathcal{A}$ is not empty. The subject is well understood for convex optimization 
with Lagrange multipliers and Karush-Kuhn-Tucker conditions are its familiar main 
tools. It is the purpose of this article to introduce an alternative method in minimiz-
ing a function without using the tools mentioned above. The method transforms the 
constrained Problem (1.1) of $f$ into an unconstrained one of a deformation $f_t$ of $f$. It 
can be considered as a toolkit using for approximating a result by applying any existing 
software. We choose to work on some well-known software to find a decreasing sequence 
$\{f_t(x_n)\}$, namely, particle swarm optimization (PSO), particle-search algorithm, and 
convex optimization. By testing the method over many kinds of objective functions $f$, we believe the method is quite practical. It is found that a problem may work well 
under one software but not under some others. Moreover, the method can be performed 
to obtain a Brouwer fixed point and applied to a vector optimization.

In computational science, particle swarm optimization (PSO) \cite{1–3} is the computa-
tional method that optimization problem by iteratively trying to improve a candidate 
solution with regard to a given measure of quality. A basic variant of the PSO algo-
rythm works by having a population (swarm) of candidate solutions (particles). These 
particles are moved around in the search-space according to a simple formula. The 
movements of the particles are guided by their own best known position in the search-
space. The entire swarm’s best known position. When improved positions are being 
discovered these will then come to guide the movements of the swarm. The process is 
repeated and by doing so it is hoped, but not guaranteed, that a satisfactory solution 
will eventually be discovered.

Pattern search algorithm is a family of numerical optimization methods. It finds a 
sequence of points that approach an optimal point. The value of the objective function 
either decreases or remains the same from each point in the sequence to the next \cite{4–6}.

Convex optimization is a subfield of mathematical optimization that studies the prob-
lem of minimizing convex functions over convex sets. Convex algorithm is a mathemat-
ical method of solving convex optimization \cite{7–9}. The key to the algorithmic success in 
minimizing convex functions is that these functions exhibit a local to global phenome-
on. This local to global phenomenon is that local minimal of convex functions are in 
fact global minimal.

2. Methodology

Put $G_i = |g_i| \ (i = 1, 2, \ldots, m)$, $H_i = |h_i| + h_i \ (i = 1, 2, \ldots, n)$, and $F = \sum_{i=1}^{m} G_i + \sum_{i=1}^{n} H_i$. Clearly, $F$ is continuous and $F(x) = 0$ if and only if $x$ satisfies the constraints 
of Problem (1.1) (i.e., it lies in the feasible set $\mathcal{A}$). For large numbers $K$ and $M$, set for 
t $\in (0, 1)$, $f_t = (1 - t)(f - K) + tMF$.

Since we are going to work on the deformed function $f_t$ for $t$ sufficiently close to 1, we 
therefore take any existing software available. We select 3 softwares, namely Particle 
Swarm Optimization, Pattern-Search, and Convex Algorithm. We let $K$ to be large to 
be certained that the graph of $f - K$ totally lies under the graph of $F$. As for large
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M, we try to make it easy for a software to find a decreasing sequence \( \{ f_t(x_n) \} \). The parameter \( t \) getting close to 1 is to making the iteration point \( x_n \) being closer to or lying in the feasible set \( A \).

**Proposition 2.1.** For any \( t \in (0, 1) \) with \( f_t > 0 \) outside \( A \), \( x \) is a minimizer of Problem (1.1) if and only if \( x \) is a minimizer of \( f_t \).

**Proof.** This is straightforward since \( f_t = (1 - t)(f - K) \) on \( A \).

By the term “minimizer” it is meant to be a minimal element, i.e., a local minimizer.

**Algorithm 1** Example code (PAO our Algorithm)

| Input | Set up problem 1.1 |
|-------|-------------------|
| Parameter | \( K, M, t \) |
| Output | \( x \) |

\[
G_i = |g_i| \quad (i = 1, 2, \ldots, m) \\
H_i = |h_i| + h_i \quad (i = 1, 2, \ldots, n) \\
F = \sum_{i=1}^{m} G_i + \sum_{i=1}^{n} H_i \\
f_t = (1 - t)(f - K) + tMF \\
x = \arg\min_{x \in C} f_t(x)
\]

3. Applications

3.1. Brouwer Fixed Points. The Brouwer fixed theorem says that any continuous mapping \( T = (f_1, \ldots, f_d) : \prod_{i=1}^{d} [a_i, b_i] \rightarrow \prod_{i=1}^{d} [a_i, b_i] \) always has a fixed point. See [10–13] for some new proofs. To find a fixed point of \( T \), set in Problem (1.1), \( f_i(x_1, x_2, \ldots, x_d) = f_i(x_1, x_2, \ldots, x_d) - x_i \) \( (i = 1, 2, \ldots, d) \). (See Example 4.6 and 4.7.)

3.2. Vector Optimization. Given continuous mappings \( f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n : C = \prod_{i=1}^{p} [c_i, d_i] \rightarrow \mathbb{R} \). We need to solve

\[
\min_{x \in C} (f_1(x), f_2(x), \ldots, f_k(x)) \quad \text{(with respect to an order)} \\
\text{subject to } g_i(x) = 0 \quad (i = 1, 2, \ldots, m) \\
h_i(x) \leq 0 \quad (i = 1, 2, \ldots, n).
\]

We consider the problem of the forms:

1. \( \min_{x \in \mathbb{C}} \sum_{i=1}^{k} f_i(x) \). Set \( f = \sum_{i=1}^{k} f_i \) for the objective function in Problem (1.1). (See Example 4.8.)
2. Finding \( x^* = (x_1^*, x_2^*, \ldots, x_p^*) \in C \) such that \( f_i(x^*) \leq c_i \), where \( c_i \leq t_i \) for some thresholds \( t_i \) \( (i = 1, 2, \ldots, k) \). To comply with Problem (1.1), we set \( f = 1 \) as an objective function and additionally define \( h_i = f_i - c_i \) \( (i = n + 1, n + 2, \ldots, n + k) \). (See Example 4.9.)
In practice, if we only want to find a point \( x^* \) with \( f(x^*) \leq c \) for some assigned number \( c \), Problem (1.1) can read as

\[
\begin{align*}
\min_{x \in C} & \quad 1 \\
\text{subject to} & \quad g_i(x) = 0 \quad (i = 1, 2, \ldots, m) \\
& \quad h_i(x) \leq 0 \quad (i = 1, 2, \ldots, n) \\
& \quad f(x) - c \leq 0.
\end{align*}
\]

(3.2)

4. Numerical Examples

We choose \( C = [-10, 10]^p \), \( K = 100 \), \( M = 10000 \) and \( t = 0.95 \). We experiment on nine Examples, and record results in three Tables. The Tables display approximate minimizers and constraint validation.

Example 4.1. [14]

\[
\begin{align*}
\min_{x \in C} & \quad x_1^2 + x_1 x_2 + x_2^2 - 5x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0
\end{align*}
\]

Example 4.2. [14]

\[
\begin{align*}
\min_{x \in C} & \quad -(x_1 - 3)^6 - (x_2 - 4)^6 \\
\text{subject to} & \quad x_1^2 + x_2^2 \leq 25 \\
& \quad x_1 + x_2 \geq 7 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0
\end{align*}
\]

Example 4.3. [14][Geometric Programming]

\[
\begin{align*}
\min_{x \in C} & \quad \frac{1}{x_1 x_2 x_3} + x_1 x_2 \\
\text{subject to} & \quad 0.5x_1 x_3 + 0.25x_1 x_2 \leq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \geq 0
\end{align*}
\]

Example 4.4. [14]

\[
\begin{align*}
\min_{x \in C} & \quad \frac{1}{x_1 x_2 x_3} + x_1 x_2 + x_3^7 \\
\text{subject to} & \quad 0.5x_1 x_3 + 0.25x_1 x_2 \leq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \geq 0
\end{align*}
\]
Example 4.5.

\[
\begin{align*}
\min_{x \in \mathcal{C}} & \quad 4x_1 + 10x_2 + 15x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 3x_3 = 3 \\
& \quad 3x_1 + x_2 + 2x_3 = 7.5 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \geq 0
\end{align*}
\]

Example 4.6.

\[
\begin{align*}
\min_{x \in \mathcal{C}} & \quad 1 \\
\text{subject to} & \quad 0.5(\cos(x_1 + x_2 - x_3 x_5))x_4 - x_1 = 0 \\
& \quad 0.1(|x_1 x_2 + x_3 - x_5| + x_4^2) - x_2 = 0 \\
& \quad (x_1 + x_3 x_4 - (x_2 + x_5)^2)/30 - x_3 = 0 \\
& \quad (x_1 - x_2^2 + x_3 - x_5^2)/12 - x_4 = 0 \\
& \quad (x_1 + x_2 - (x_3 + x_5 + x_4)^2)/40 - x_5 = 0
\end{align*}
\]

Example 4.7.

\[
\begin{align*}
\min_{x \in \mathcal{C}} & \quad 1 \\
\text{subject to} & \quad 0.001((x_1 + 3)^2 + (x_2 - 2)^4 + x_3^2 + x_4^2 + x_5) - x_1 = 0 \\
& \quad 0.01(x_1 + (x_2 + 5)^2 + x_3 + x_4 + (x_5 + 2)) - x_2 = 0 \\
& \quad 0.001(x_4^4 + (x_4 - 3)^2 + (x_5 + 2)^2) - x_3 = 0 \\
& \quad 0.001((x_3 - 3)^4 + x_5^2 + x_1^4) - 1 - x_4 = 0 \\
& \quad 0.01(x_1^2 + x_2 + x_3 - (x_5 - 1)^2) - x_5 = 0
\end{align*}
\]

Example 4.8. [14]

\[
\begin{align*}
\min_{x \in \mathcal{C}} & \quad (x_1^2 - 5x_1 + 7x_2) + (-x_1^2 - x_2^2) + (x_1 - 1)^2 + (x_2 - 5)^2 \\
\text{subject to} & \quad 3x_1 + 4x_2 = 6 \\
& \quad x_1 + x_2 = 2 \\
& \quad 2x_1 + 3x_2 \leq 6 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0
\end{align*}
\]

Example 4.9. [14]

\[
\begin{align*}
& \quad 4x_1^2 + x_2^2 - x_1 - 2 \leq 1 \\
& \quad e^{-x_1} - x_1 - 2x_2 \leq 1 \\
\text{subject to} & \quad 2x_1 + x_2 \leq 1 \\
& \quad x_1^2 \leq 1 \\
& \quad \sqrt{x_1^2 + x_2^2} - x_1^3 \leq 2 \\
& \quad -x_1^3 + 0.5(-x_2 - x_2^3 + |x_2^3 - x_2|) \leq 0, \quad x_1, x_2 \in \mathbb{R}
\end{align*}
\]
Table 1. Particle Swarm Optimization

| Example | initial point | value         | $x$             | $\max_{x \in C} g_i(x)$ | $\max_{x \in C} h_j(x)$ |
|---------|---------------|---------------|------------------|--------------------------|--------------------------|
| 4.1     |               | -4            | (0, 1)           | 0                        | 0                        |
| 4.2     |               | -2            | (4, 3)           | -                        | 0                        |
| 4.3     |               | 0.6325        | (10, 0.0316, 10) | -                        | -0.0316                  |
| 4.4     |               | 2.4397        | (0.1141, 9.9975, 0.7715) | -                        | -0.1141                  |
| 4.5     |               | 12.6          | (2.4, 0.3, 0)    | 0                        | 0                        |
| 4.6     |               |               | (-1.977 x 10^{-11}, 1.02 x 10^{-12}) | 2.546 x 10^{-11} | -                        |
| 4.7     |               | 1             | (0.018, 0.291, 0.019, -0.921, -0.007) | 1.766 x 10^{-12} | -                        |
| 4.8     |               | 16            | (2, 0)           | 0                        | 0                        |
| 4.9     |               | 1             | (0.7312, 1.0271) | -                        | -0.4654                  |

Table 2. Pattern-Search Optimization

| Example | initial point | value         | $x$             | $\max_{x \in C} g_i(x)$ | $\max_{x \in C} h_j(x)$ |
|---------|---------------|---------------|------------------|--------------------------|--------------------------|
| 4.1     | (1, 1)        | -4            | (0, 1)           | 0                        | 0                        |
| 4.2     | (1, 1)        | -94.3669      | (4.6094, 1.9374) | -                        | 0.4532                  |
| 4.3     | (1, 1, 1)     | 0.6325        | (0.6325, 0.5, 10) | -                        | 0                        |
| 4.4     | (1, 1, 1)     | 2.4397        | (1.1385, 1.7715, 1) | -                        | 0.2762                  |
| 4.5     | (1, 1, 1)     | 12.6429       | (2.3571, 0.2143) | 1.5259 x 10^{-5} | 0                        |
| 4.6     | (0, 0, 0, 0)  | 1             | (0, 0, 0, 0)     | 0                        | -                        |
| 4.7     | (0, 0, 0, 0)  | 1             | (0.019, 0.029, 1.93, -0.92, -0.007) | 3.978 x 10^{-6} | -                        |
| 4.8     | (1, 1)        | 18.7777       | (0.6667, 1)      | 1.5259 x 10^{-5} | -0.6667                  |
| 4.9     | (1, 1)        | 1             | (0, 1)           | -                        | 1                        |

Table 3. Convex Algorithm

| Example | initial point | value         | $x$             | $\max_{x \in C} g_i(x)$ | $\max_{x \in C} h_j(x)$ |
|---------|---------------|---------------|------------------|--------------------------|--------------------------|
| 4.1     | (1, 1)        | -3.9694       | (0.0076, 0.9924) | 7.3 x 10^{-9}          | -0.0076                  |
| 4.2     | (1, 1)        | -1.2957       | (3.9302, 3.0698) | -                        | -7.97 x 10^{-13}         |
| 4.3     | (1, 1, 1)     | 0.6325        | (0.5623, 0.5623, 10) | -                        | -0.5623                  |
| 4.4     | (1, 1, 1)     | 2.4397        | (1.0670, 1.0670, 0.7715) | -                        | -0.3038                  |
| 4.5     | (1, 1, 1)     | 12.6392       | (2.3608, 0.0253, 0.1962) | 0.3167 x 10^{-7} | -0.0253                  |
| 4.6     | (0, 0, 0, 0)  | 1             | 10^{-11}, 1.265 x 10^{-10}, 2.282 x 10^{-10}, -3.341 x 10^{-11} | 2.661 x 10^{-10} | -                        |
| 4.7     | (0, 0, 0, 0)  | 1             | (0.018, 0.291, 0.019, -0.921, -0.007) | 8.413 x 10^{-9} | -                        |
| 4.8     | (1, 1)        | 16            | (2, 0)           | 0                        | 0                        |
| 4.9     | (1, 1)        | 1             | (-0.0888, 0.8020) | -                        | -0.8013                  |
5. Discussion

In this paper, we transform a constrained optimization to an unconstrained one. Under our approach, the given objective function $f$ (subjected to some constraints) is replaced by a deformed function $f_t$ (without constraints) for some $t$. We chose to use some software packages to approximate a minimizer of $f_t$. We observe that all outcomes approximately satisfy corresponding constraints. Of course, we may obtain different minimizers from different software. It is challenging to construct a new algorithm for finding a global minimizer even for some special cases.

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