An Extension of the Kadomtsev–Petviashvili Hierarchy and its Hamiltonian Structures

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Abstract

In this note we consider a two-component extension of the Kadomtsev–Petviashvili (KP) hierarchy represented with two types of pseudo-differential operators, and construct its Hamiltonian structures by using the $R$-matrix formalism.

Key words: KP hierarchy; Hamiltonian structure; $R$-matrix

1 Introduction

The Kadomtsev–Petviashvili (KP) hierarchy plays a fundamental role in the theory of integrable systems. There are several ways to define the KP hierarchy, and one contracted way is via Lax equations of pseudo-differential operators. Let

\begin{equation}
L_{KP} = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots, \quad \partial = \frac{d}{dx},
\end{equation}

be a pseudo-differential operator with scalar coefficients $u_i$ depending on the spatial coordinate $x$. The KP hierarchy is composed by the following evolutionary equations

\begin{equation}
\frac{\partial L_{KP}}{\partial t_k} = [(L_{KP})^+_k, L_{KP}], \quad k = 1, 2, 3, \ldots
\end{equation}

Here the subscript “+” means to take the differential part of a pseudo-differential operator. The hierarchy \textsuperscript{1.2} is known to possess a series of bi-Hamiltonian structures, which can be constructed by the $R$-matrix formalism \textsuperscript{22}.

The KP hierarchy \textsuperscript{1.2} has been generalized to multicomponent versions with scalar pseudo-differential operators replaced by matrix-value ones \textsuperscript{10} \textsuperscript{2} \textsuperscript{13}. In such generalizations, the pseudo-differential operators are required to admit certain extra constraints, and it is probably why no Hamiltonian structures underlying have been found. Towards overcoming this difficulty, a step was made by Carlet and Manas \textsuperscript{8}, who solved the constraints in the 2-component case and parameterized the matrix operators with a set of
“free” dependent variables. The study of the 2-component KP hierarchy was also motivated by the development of infinite-dimensional Frobenius manifolds in recent years, especially those associated with the bi-Hamiltonian structures for the Toda lattice and the 2-component BKP hierarchies [6, 28, 29].

Inspired by the Toda lattice hierarchy, we now consider an extension of the KP hierarchy from the viewpoint of scalar pseudo-differential operators rather than matrix-value ones. More exactly, given a pair of scalar operators

\[ P = D + \sum_{i \geq 1} u_i D^{-i}, \quad \hat{P} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 0} \hat{u}_i D^i \]  

(1.3)

with \( D \) being a “derivation” on some differential algebra that contains functions \( u_i \) and \( \hat{u}_i \), we want to define the following commutative flows:

\[ \frac{\partial}{\partial t_k} (P, \hat{P}) = \left( ([P^k]_+, P), ([P^k]_+, \hat{P}) \right), \]  

(1.4)

\[ \frac{\partial}{\partial \hat{t}_k} (P, \hat{P}) = \left( [-(\hat{P}^k)_-, P], [-(\hat{P}^k)_-, \hat{P}] \right) \]  

(1.5)

for \( k = 1, 2, 3, \ldots \). It will be seen that if one takes \( D = \partial - \varphi \) with \( \varphi \) being an unknown function of \( x \), then the hierarchy (1.4)–(1.5) is well defined (see Section 2). Such kind of hierarchies have existed in the work [24] of Szablikowski and Blaszak as a dispersive counterpart of the Whitham hierarchy (see, for example, [25]). Their version in fact involves \( N \) operators of the form \( \hat{P} \) with \( D \) replaced by \( \partial - \varphi_i \) (\( 1 \leq i \leq N \)); however, the convergence property of the operators \( \hat{P}^k \), which can contain infinite many positive powers in \( D \), have not been taken into account before. In this note, we will only consider the case \( N = 1 \), and illustrate that the operators \( \hat{P}^k \) as the so-called pseudo-differential operators of the second type introduced by us [16] in recent years, such that they converge under a suitable topology (see below). As to be seen, the extended KP hierarchy (1.4)–(1.5) can be reduced to the 2-component KP hierarchy [16] under suitable constraints.

Observe that the flows (1.4)–(1.5) are defined on a certain “coupled” Lie algebra \( \mathcal{G}^- \times \mathcal{G}^+ \), with \( \mathcal{G}^+ \) being the algebras of pseudo-differential operators of the first and the second types respectively. It is natural to apply the \( R \)-matrix scheme to search for Hamiltonian structures underlying (1.4)–(1.5), such as what we have done for the Toda lattice and the 2-component BKP hierarchies [26] (see also [5, 27]). We will see that the simple but useful \( R \)-matrix found in [26] is also feasible in the current case, so a series of bi-Hamiltonian structures for the hierarchy (1.4)–(1.5) will be derived. Furthermore, such bi-Hamiltonian structures can be naturally reduced to those for the constrained KP hierarchies [1, 4, 9, 12, 14, 20].

This article is organized as follows. In the next section we recall the notions of pseudo-differential operators of the first and the second types, and then check in detail that the extended KP hierarchy (1.4)–(1.5) is well definition. In Section 3, we review briefly the \( R \)-matrix method for deriving Hamiltonian structures, and apply this method to the hierarchy (1.4)–(1.5). Finally some remarks will be given.
2 An extension of the KP hierarchy

2.1 Preliminary notations

Let $A$ be a differential algebra, and $\partial : A \to A$ be a derivation. The sets of pseudo-differential operators is

$$A((\partial^{-1})) = \left\{ \sum_{i \leq k} f_i \partial^i \mid f_i \in A, k \in \mathbb{Z} \right\},$$

in which the product is defined by

$$f \partial^i \cdot g \partial^j = \sum_{r \geq 0} \binom{i}{r} f \partial^r (g) \partial^{i+j-r}, \quad f, g \in A. \quad (2.1)$$

Clearly one has the commutator $[\partial, f] = \partial(f)$ for any $f \in A$.

From now on we assume $A$ to be a graded algebra $A = \prod_{i \geq 0} A_i$ such that

$$A_i \cdot A_j \subset A_{i+j}, \quad \partial(A_i) \subset A_{i+1}.$$

We write $D^- = A((\partial^{-1}))$, and call it the algebra of pseudo-differential operators of the first type over $A$. In comparison, the algebra of pseudo-differential operators of the second type over $A$ is defined to be

$$D^+ = \left\{ \sum_{i \leq k} \sum_{j \geq \max\{0, k-i\}} a_{i,j} \partial^i \mid a_{i,j} \in A_j, k \in \mathbb{Z} \right\}, \quad (2.2)$$

which is endowed a product defined also by (2.1). Note that an operator in $D^+$ may contain infinitely many positive powers in $\partial$, see [16] for details.

Assume $\varphi \in A_1$ to be an arbitrary element of degree 1. Note $[\partial - \varphi, f] = \partial(f)$ for any $f \in A$. By replacing $\partial$ with $\partial - \varphi$ in $D^+$, we have the following two algebras

$$D^-_\varphi = \left\{ \sum_{i \leq k} f_i (\partial - \varphi)^i \mid f_i \in A, k \in \mathbb{Z} \right\}, \quad (2.3)$$

$$D^+_\varphi = \left\{ \sum_{i \in \mathbb{Z}, j \geq \max\{0, k-i\}} a_{i,j} (\partial - \varphi)^i \mid a_{i,j} \in A_j, k \in \mathbb{Z} \right\}. \quad (2.4)$$

In these algebras the product is defined by (cf. (2.1))

$$f (\partial - \varphi)^i \cdot g (\partial - \varphi)^j = \sum_{r \geq 0} \binom{i}{r} f \partial^r (g) (\partial - \varphi)^{i+j-r}, \quad f, g \in A.$$
In particular,
\[(\partial - \varphi)^{-1} = \partial^{-1}(1 - \varphi \partial^{-1})^{-1} = \partial^{-1} + \partial^{-1} \varphi \partial^{-1} + \partial^{-1} \varphi \partial^{-1} \varphi \partial^{-1} + \ldots \] (2.5)

It is clear to see \(D^-_\varphi \subset D^+_\varphi\); on the other hand, since \(\partial = (\partial - \varphi) + \varphi\) and
\[\partial^{-1} = (\partial - \varphi + \varphi)^{-1} = (\partial - \varphi)^{-1} - (\partial - \varphi)^{-1} \varphi (\partial - \varphi)^{-1} + (\partial - \varphi)^{-1} \varphi (\partial - \varphi)^{-1} \varphi (\partial - \varphi)^{-1} - \ldots,\]
then \(D^+_\varphi \supset D^+_\varphi\). Hence one has \(D^-_\varphi = D^+_\varphi\) indeed.

Given an operator \(A = \sum_{i \in \mathbb{Z}} f_i(\partial - \varphi)^i \in D^-_\varphi\), its differential part is
\[A_+ = \sum_{i \geq 0} f_i(\partial - \varphi)^i, \] (2.6)
while \(A_- = A - A_+\) is the negative part. The residue of \(A\) means
\[\text{res}A = f_{-1}. \] (2.7)

These notations are consistent with the usual notations for \(D^\pm\). For instance, if we write the above operator \(A\) in the form \(A = \sum_{i \in \mathbb{Z}} \tilde{f}_i \partial^i \in D^\pm,\) then \(\tilde{f}_{-1} = f_{-1}\) is the residue as usual.

What is more, on each \(D^+_\varphi = D^+_\varphi\) there is an anti-isomorphism defined by
\[\partial^* = -\partial, \quad f^* = f \text{ with } f \in A.\]

Note that such an anti-isomorphism is an involution.

### 2.2 The extended KP hierarchy

From now on we take the graded algebra \(A = \prod_{i \geq 0} A_i\) to be the set of formal differential polynomials of certain smooth function of a coordinate \(x\) of the loop \(S^1\), on which there is naturally a derivation \(\partial = d/dx\).

With an arbitrary nonvanishing function \(\varphi \in A_1\), one has the algebras \(D^-_\varphi\) of pseudo-differential operators of the first and the second types over \(A\). Introduce
\[P = (\partial - \varphi) + \varphi + \sum_{i \geq 1} u_i (\partial - \varphi)^{-i} \in D^-_\varphi, \quad u_i \in A_0. \] (2.8)

We also introduce a pseudo-differential operator of the second type
\[\hat{P} = (\partial - \varphi)^{-1} \rho + \sum_{i \geq 0} \hat{u}_i (\partial - \varphi)^{i} \in D^+_\varphi \] (2.9)
with \(\rho, \hat{u}_i \in A_0\) and \(\rho \neq 0\).
Definition 2.1 The following hierarchy of evolutionary equations is called the extended KP hierarchy:

\[
\frac{\partial P}{\partial t_k} = [(P_k^k)_+, P], \quad \frac{\partial P}{\partial \hat{t}_k} = [-(\hat{P}_k^k)_-, P], \tag{2.10}
\]

\[
\frac{\partial \hat{P}}{\partial t_k} = [(P_k^k)_+, \hat{P}], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}_k^k)_-, \hat{P}], \tag{2.11}
\]

where \( k = 1, 2, 3, \ldots \).

Remark 2.2 The first equation in line (2.10) gives nothing but the KP hierarchy. Indeed, the operator \( P \) can be recast to

\[
P = \partial + \sum_{i \geq 1} \tilde{u}_i \partial^{-i}, \tag{2.12}
\]

where \( \tilde{u}_i - u_i \in \prod_{i \geq 1} A_i \) (the function \( \varphi \), existing in the definition of \( P \) in (2.8), is not a “real” unknown function in the KP hierarchy). This is partially why we use the name extended KP hierarchies here, though it may not be so appropriate. Another reason is that we would like to distinguish the hierarchy (2.10)–(2.11) with the so called 2-component KP hierarchy \([10, 2, 13]\) represented by matrix-value pseudo-differential operators (by now no connection seems to be found between them, while the second part of [5] implies that there might be some kind of 2-component of extended KP hierarchy).

\[\square\]

Remark 2.3 A hierarchy of the form (2.10)–(2.11) has existed in [24] as a dispersive counterpart of the Whitham hierarchy with only one movable singularity. We hope that the reconstruction of the hierarchy with two types of pseudo-differential operators, as well as the study of its Hamiltonian structures based on an \( R \)-matrix found by us recently in [26], would help to cause more attention to this topic, especially its relation with infinite-dimensional Frobenius manifolds.

\[\square\]

Let us proceed to check that the equations (2.10)–(2.11) are well defined, as pointed out by Szablowski and Blaszcz [24] before. In fact, we only need to check that the negative part on the right hand sides of equations in line (2.11) take a form being consistent with

\[
(\partial - \varphi)^{-1} \frac{\partial \varphi}{\partial t_k} (\partial - \varphi)^{-1} \rho + (\partial - \varphi)^{-1} \frac{\partial \rho}{\partial t_k}, \tag{2.13}
\]

where \( \hat{t}_k \) stand for \( t_k \) or \( \hat{t}_k \). To this end, we need some preparation. Given any function \( f \in A \) and integer \( k > 0 \), one has

\[
f(\partial - \varphi)^k = \left((-\partial - \varphi)^k f\right)^*
\]

\[
= \left(\sum_{r=0}^{k} \binom{k}{r}(-\partial)^r(f)(-\partial - \varphi)^{k-r}\right)^*
\]
\[
= \sum_{r=0}^{i} \binom{k}{r} (\partial - \varphi)^{k-r} (-\partial)^r (f), \quad (2.14)
\]

and
\[
(\partial - \varphi)^{-1} f = f(\partial - \varphi)^{-1} - (\partial - \varphi)^{-1} \cdot \partial(f) \cdot (\partial - \varphi)^{-1}. \quad (2.15)
\]

Suppose that \( Q \) is a differential operator, say
\[
Q = \sum_{i \geq 0} f_i (\partial - \varphi)^i \in D_{\varphi}^+, \quad f_i \in A,
\]
we have
\[
[Q, \hat{P}]_- = [Q, (\partial - \varphi)^{-1} \rho]_-
\]
\[
= f_0(\partial - \varphi)^{-1} \rho - \left((\partial - \varphi)^{-1} \rho \sum_{i \geq 0} f_i (\partial - \varphi)^i \right)_-
\]
\[
= f_0(\partial - \varphi)^{-1} \rho - \left((\partial - \varphi)^{-1} \sum_{i \geq 0} \sum_{r=0}^{i} (-1)^r \binom{i}{r} (\partial - \varphi)^i \partial^r (\rho f_i) \right)_-
\]
\[
= f_0(\partial - \varphi)^{-1} \rho - (\partial - \varphi)^{-1} \sum_{i \geq 0} (-1)^i \partial^i (\rho f_i)
\]
\[
= (f_0(\partial - \varphi)^{-1} - (\partial - \varphi)^{-1} f_0) \rho + (\partial - \varphi)^{-1} \sum_{i \geq 1} (-1)^{i-1} \partial^i (\rho f_i)
\]
\[
= (\partial - \varphi)^{-1} \cdot \partial(f_0) \cdot (\partial - \varphi)^{-1} \rho + (\partial - \varphi)^{-1} \sum_{i \geq 1} (-1)^{i-1} \partial^i (\rho f_i), \quad (2.16)
\]
in the third and the last equalities we have used the formulae \(2.14\) and \(2.15\) respectively. Observe that the form of \( [Q, \hat{P}]_- \) is consistent with \(2.13\), thus we have checked that equations \(2.11\) are indeed well defined.

In particular, for \( \dot{t}_k = t_k, \dot{\gamma}_k \) and \( \hat{P} = P, \hat{P} \), one has
\[
\frac{\partial \varphi}{\partial \dot{t}_k} = \partial \left( \text{res} \hat{P}^k(\partial - \varphi)^{-1} \right), \quad (2.17)
\]
\[
\frac{\partial \rho}{\partial \dot{t}_k} = \sum_{i \geq 1} (-1)^{i-1} \partial^i \left( \rho \text{res} \hat{P}^k(\partial - \varphi)^{-i-1} \right). \quad (2.18)
\]
The right hand side of equation \(2.17\) is a total derivative, which also implies that the assumption \( \varphi \in A_1 \) in the beginning is reasonable.

Finally, one can check that the equations in \(2.10\)–\(2.11\) are compatible, that is, \( \partial/\partial \dot{t}_k \) and \( \partial/\partial \dot{\gamma}_l \) are commutative. Therefore equations \(2.10\)–\(2.11\) compose an integrable hierarchy indeed.
Remark 2.4 Suppose $\varphi = 0$, for the flows $\partial/\partial t_k$ and $\partial/\partial \hat{t}_k$ making sense it is necessary to require
$$\text{res } P^k \partial^{-1} = \text{res } \hat{P}^k \partial^{-1} = 0.$$ 
A nontrivial assumption that fulfills this condition is that $k = 1, 3, 5, 7, \ldots$ and
$$P^* = -\partial P \partial^{-1}, \quad \hat{P}^* = -\partial \hat{P} \partial^{-1}.$$ 
In this way what we obtain is nothing but the Lax representation of the 2-component BKP hierarchy, see [16] and references therein. □

3 Hamiltonian structures for the extended KP hierarchy

For the extended KP hierarchy (2.10)–(2.11), we want to construct its Hamiltonian structures with the $R$-matrix method, which is similar with the algorithm for the Toda lattice hierarchy [26] (see also [5]).

3.1 $R$-matrix and Poisson brackets

Let us recall briefly the $R$-matrix formalism and some relevant results, based on the work [15, 19, 22].

Let $\mathfrak{g}$ be a complex Lie algebra. A linear transformation $R : \mathfrak{g} \to \mathfrak{g}$ is called an $R$-matrix if a Lie bracket is defined by
$$[X, Y]_R = [R(X), Y] + [X, R(Y)], \quad X, Y \in \mathfrak{g}. \quad (3.1)$$
For a linear transformation $R$ being an $R$-matrix, a sufficient condition is that $R$ solves the so-called modified Yang-Baxter equation:
$$[R(X), R(Y)] - R([X, Y]_R) = -[X, Y], \quad X, Y \in \mathfrak{g}. \quad (3.2)$$

Assume $\mathfrak{g}$ to be an associative algebra, whose Lie bracket is defined by the commutator. We also assume that there is a function $\langle \rangle : \mathfrak{g} \to \mathbb{C}$ that induces a non-degenerate symmetric invariant bilinear form (inner product) $\langle , \rangle$ as
$$\langle X, Y \rangle = \langle YX \rangle = \langle XY \rangle, \quad X, Y \in \mathfrak{g}.$$ 
Via this inner product $\mathfrak{g}$ can be identified with its dual space $\mathfrak{g}^*$. Let $T\mathfrak{g}$ and $T^*\mathfrak{g}$ be the tangent and the cotangent bundles of $\mathfrak{g}$, with fibers $T_A\mathfrak{g} = \mathfrak{g}$ and $T^*_A\mathfrak{g} = \mathfrak{g}^*$ respectively at any point $A \in \mathfrak{g}$.

Given an $R$-matrix $R$ on $\mathfrak{g}$, for any $f, g \in C^\infty(\mathfrak{g})$ the following brackets are defined:
$$\{f, g\}_1(A) = \frac{1}{2}\left(\langle [A, df], R(dg) \rangle - \langle [A, dg], R(df) \rangle\right), \quad (3.3)$$
$$\{f, g\}_2(A) = \frac{1}{4}\left(\langle [A, df], R(A \cdot dg + dg \cdot A) \rangle - \langle [A, dg], R(A \cdot df + df \cdot A) \rangle\right), \quad (3.4)$$
\{f, g\}_3(A) = \frac{1}{2} \left( \langle [A, df], R(A \cdot dg \cdot A) \rangle - \langle [A, dg], R(A \cdot df \cdot A) \rangle \right), \quad (3.5)

where \(df, dg \in T^*_A \mathfrak{g}\) are the gradients of \(f, g\) at \(A \in \mathfrak{g}\) respectively. Following [15, 19], the brackets (3.3)–(3.5) are called the linear, the quadratic and the cubic brackets respectively.

Let \(R^*\) denote the adjoint transformation of \(R\) with respect to the above inner product, then the anti-symmetric part of \(R\) is

\[ R_a = \frac{1}{2}(R - R^*). \quad (3.6) \]

**Theorem 3.1** ([15, 19, 22])

1. For any \(R\)-matrix \(R\) the linear bracket is a Poisson bracket.

2. If both \(R\) and its anti-symmetric part \(R_a\) solve the modified Yang-Baxter equation (3.2), then the quadratic bracket is a Poisson bracket.

3. If \(R\) satisfies the modified Yang-Baxter equation (3.2), then the cubic bracket is a Poisson bracket.

Moreover, these three Poisson brackets are compatible whenever all the above conditions are fulfilled.

In the present note, the quadratic bracket is the one to be applied, see below.

### 3.2 Poisson bracket on a coupled Lie algebra

Recall the algebras \(D_{\pm}^\phi\) of pseudo-differential operators of the first and the second types over \(A\). We introduce

\[ D_{\phi} = D_{-}^\phi \times D_{+}^\phi, \quad (3.7) \]

on which the product is defined diagonally as

\[(X, \hat{X}) \cdot (Y, \hat{Y}) = (XY, \hat{X}\hat{Y}), \quad (X, \hat{X}), (Y, \hat{Y}) \in \mathfrak{D}.\]

The algebra \(\mathfrak{D}\) is endowed with an invariant inner product given by

\[ \langle (X, \hat{X}), (Y, \hat{Y}) \rangle = \langle X, Y \rangle + \langle \hat{X}, \hat{Y} \rangle, \quad (3.8) \]

where

\[ \langle A, B \rangle = \int \text{res}(AB) \, dx \in A/\partial(A) \quad (3.9) \]

for any \(A, B \in D_{\pm}^\phi\).

Elements of \(A/\partial(A)\), written in the form \(\int f \, dx\) with \(f \in A\), are called formal functionals on \(\mathfrak{D}\) (viewed as an infinite-dimensional manifold). In this paper we consider only formal functionals whose variational gradients belong to \(\mathfrak{D}\); by the variational gradient
of a formal functional $F$ at $A \in \mathcal{D}$, it means a pair of pseudo-differential operators, denoted as $\delta F/\delta A$, such that $\delta F = \langle \delta F/\delta A, \delta A \rangle$. Note that the variational gradient of a functional is determined up to some kernel part, and usually it is not easy to be written down explicitly.

The algebra $\mathcal{D}$ is naturally a Lie algebra, whose Lie bracket is just the commutator. On the Lie algebra $\mathcal{D}$, it follows from a general result in our previous work [26] that there is an $R$-matrix defined by

$$R(X, \hat{X}) = (X_+ - X_- - 2\hat{X}_-, \hat{X}_+ - \hat{X}_- + 2X_+), \quad (3.10)$$

and this $R$-matrix solves the so-called modified Yang-Baxter equation. Moreover, it can be checked

$$\langle R(X, \hat{X}), (Y, \hat{Y}) \rangle = \langle (X_+ - X_- - 2\hat{X}_-)Y + (\hat{X}_+ - \hat{X}_- + 2X_+)\hat{Y} \rangle$$
$$= \langle (X(Y_+ - Y_- + 2\hat{Y}_-) + \hat{X}(-Y_+ + \hat{Y}_- - 2\hat{Y}_+)) \rangle$$
$$= \langle (X, \hat{X}), -R(Y, \hat{Y}) \rangle;$$

namely, $R$ is anti-symmetric with respect to the inner product (3.8). Thus by using the second assertion of Theorem 3.1, we have the following result.

**Lemma 3.2** On the algebra $\mathcal{D}_\varphi$ there is a Poisson bracket between formal functionals defined by

$$\{F, H\}(A) = \left\langle \frac{\delta F}{\delta A}, P_A \left( \frac{\delta H}{\delta A} \right) \right\rangle, \quad A = (A, \hat{A}) \in \mathcal{D}_\varphi, \quad (3.11)$$

where the Poisson tensor $P : T\mathcal{D}_\varphi^* \to T\mathcal{D}_\varphi$ (the tangent and the cotangent fibers are identified with $\mathcal{D}_\varphi$) is given by

$$P_{(A, \hat{A})}(X, \hat{X}) = \left( -(AX + \hat{A}\hat{X})_- A + A(XA + \hat{X}\hat{A})_-, \right.$$
$$\left. (AX + \hat{A}\hat{X})_+ \hat{A} - \hat{A}(XA + \hat{X}\hat{A})_+ \right). \quad (3.12)$$

### 3.3 Hamiltonian representation for the extended KP hierarchy

Let us start to derive Hamiltonian structures for the extended KP hierarchy (2.10)–(2.11), by reducing the Poisson bracket (3.11) onto some suitable submanifolds of $\mathcal{D}_\varphi$.

Given an arbitrary positive integer $m$, for the operators (2.8) and (2.9) we let

$$A = (A, \hat{A}) = (P^m, \hat{P}). \quad (3.13)$$

More explicitly, one has

$$A = (\partial - \varphi)^m + m \varphi(\partial - \varphi)^{m-1} + \sum_{i \leq m-2} v_i (\partial - \varphi)^i, \quad (3.14)$$
$$\hat{A} = (\partial - \varphi)^{-1} \rho + \sum_{i \geq 0} \hat{v}_i (\partial - \varphi)^i. \quad (3.15)$$
Conversely, suppose \( A \) and \( \hat{A} \) are given as above, then \( P = A^{1/m} \) and \( \hat{P} = \hat{A} \) of the form (2.8)–(2.9) are determined uniquely. Observe that the unknown functions \((v_{m-2}, v_{m-3}, \ldots, \varphi, \rho, \hat{u}_0, \hat{u}_1, \ldots)\) above and \((u_1, u_2, \ldots, \varphi, \rho, \hat{u}_0, \hat{u}_1, \ldots)\) in (2.8)–(2.9) are up to a Miura-type transformation. Thus the extended KP hierarchy (2.10)–(2.11) can be represented equivalently as

\[
\begin{align*}
\frac{\partial A}{\partial t_k} &= \left[ (P^k_+, (P^k_)_+), (A, \hat{A}) \right], \\
\frac{\partial \hat{A}}{\partial \hat{t}_k} &= \left[ (-\hat{P}^k_-), -\hat{(P^k)_-}), (A, \hat{A}) \right].
\end{align*}
\]

(3.16)

(3.17)

Recall the algebras \( D^{\pm}_{\varphi} \) of pseudo-differential operators. We introduce notations like

\[
(D^{\pm}_{\varphi})_{\leq k} = \left\{ \sum_{i \leq k} f_i (\partial - \varphi)^i \in D^{\pm}_{\varphi} \mid f_i \in \mathcal{A} \right\}, \quad k \in \mathbb{Z},
\]

(3.18)

and \((D^{\pm}_{\varphi})_{\geq k}\) means similarly. Then all operators of the form (3.13) compose a subset of \( \mathcal{D}_{\varphi} \) as

\[
\mathcal{U}_m = \left( \partial^m + (D^-_{\varphi})_{\leq m-2} \right) \times \left( (D^+_{\varphi})_{\geq 0} \times \mathcal{M} \right),
\]

(3.19)

where

\[
\mathcal{M} = \{ (\partial - \varphi)^{-1} \rho \mid \varphi, \rho \in \mathcal{A}; \varphi, \rho \neq 0 \}.
\]

(3.20)

We consider \( \mathcal{M} \) as a 2-dimensional manifold with coordinate \((\varphi, \rho)\). This manifold has tangent spaces

\[
T_{\varphi, \rho} \mathcal{M} = \{ (\partial - \varphi)^{-1} a (\partial - \varphi)^{-1} \rho + (\partial - \varphi)^{-1} b \mid a, b \in \mathcal{A} \},
\]

(3.21)

while the cotangent spaces are

\[
T^*_{\varphi, \rho} \mathcal{M} = \mathcal{A} \oplus \mathcal{A}(\partial - \varphi).
\]

(3.22)

For the the subset \( \mathcal{U}_m \), its tangent bundle \( T\mathcal{U}_m \) is composed by the following fibers

\[
T_{\mathcal{A}} \mathcal{U}_m = (D^-_{\varphi})_{\leq m-2} \times \left( (D^+_{\varphi})_{\geq 0} \oplus T_{\varphi, \rho} \mathcal{M} \right).
\]

(3.23)

The cotangent bundle \( T^*\mathcal{U}_m \) has fibers (dual with the tangent spaces)

\[
T^*_{\mathcal{A}} \mathcal{U}_m = (D^-_{\varphi})_{\geq -m+1} \times \left( (D^+_{\varphi})_{\leq -1} \oplus T^*_{\varphi, \rho} \mathcal{M} \right),
\]

(3.24)

where we have used notations (c.f. (3.18))

\[
(D^+_{\varphi})_{\geq k} = \left\{ \sum_{i \geq k} (\partial - \varphi)^i f_k \in D^+_{\varphi} \mid f_i \in \mathcal{A} \right\}.
\]

(3.24)
Lemma 3.3 On the subset $U_m$ consisting of operators of the form \([3.13]\), there is a Poisson tensor $\mathcal{P}^{\text{red}}_A : T^*U_m \rightarrow TU_m$ defined by

$$\mathcal{P}^{\text{red}}_A(X, \hat{X}) = \left( - (AX + \hat{A}\hat{X})_- A + A(XA + \hat{X}\hat{A})_-, \right.$$  
$$\left. (AX + \hat{A}\hat{X})_+ \hat{A} - \hat{A}(XA + \hat{X}\hat{A})_+ \right)$$  
$$+ \frac{1}{m}([f, A], [f, \hat{A}]), \tag{3.25}$$

where $A = (A, \hat{A}) \in U_m$ and

$$f = \partial^{-1} \left( \text{res}([X, A] + [\hat{X}, \hat{A}]) \right) \in \mathcal{A} \tag{3.26}$$

(note that the residue of a commutator is always a total derivative in $x$).

Proof: We recall the Poisson bracket \([3.11]\) on $\mathcal{D}_\varphi$, and want to reduce it onto the subset $U_m$. To this end, let us consider the following decompositions of subspaces:

$$\mathcal{D}_\varphi = T_AU_m \oplus V_A = T^*_AU_m \oplus V^*_A,$$

where

$$V_A = (\mathcal{D}^-_\varphi)_{\geq m-1} \times (\mathcal{D}^+_\varphi)_{\leq 1}/\mathcal{T}_\varphi \mathcal{M},$$

$$V^*_A = (\mathcal{D}^-_\varphi)_{\leq -m} \times \left( \frac{1}{\rho}(\partial - \varphi)(\mathcal{D}^+_\varphi)_{\geq 0}(\partial - \varphi) \right).$$

The Poisson tensor \([3.12]\) can be written as

$$\mathcal{P}_A = \left( \begin{array}{cc} \mathcal{P}^{\mathcal{D}^U}_A & \mathcal{P}^{\mathcal{D}^N}_A \\ \mathcal{P}^{\mathcal{V}^U}_A & \mathcal{P}^{\mathcal{V}^N}_A \end{array} \right) : T^*_AU_m \oplus V^*_A \rightarrow T_AU_m \oplus V_A.$$

More exactly, given $(X, \hat{X}) \in V^*_A$ with

$$X = X_{-m}(\partial - \varphi)^{-m} + X_{-m-1}(\partial - \varphi)^{-m-1} + \ldots, \quad X_i \in \mathcal{A},$$

one has $(AX)_+ = (XA)_+ = X_{-m}$ and $(\hat{A}\hat{X})_- = (\hat{X}\hat{A})_- = 0$. Hence

$$\mathcal{P}^{\mathcal{D}^U}_A(X, \hat{X}) = \left( - (AX - X_{-m})A + A(XA - X_{-m}), \right.$$  
$$\left. (X_{-m} + \hat{A}\hat{X})\hat{A} - \hat{A}(X_{-m} + \hat{X}\hat{A}) \right|_{T_AU_m}$$  
$$= \left( [X_{-m}, A], [X_{-m}, \hat{A}] \right|_{T_AU_m}$$  
$$= \left( [X_{-m}, A] + m \partial(X_{-m})(\partial - \varphi)^{m-1}, [X_{-m}, \hat{A}] \right), \tag{3.27}$$

$$\mathcal{P}^{\mathcal{V}^U}_A(X, \hat{X}) = \left( [X_{-m}, A], [X_{-m}, \hat{A}] \right|_{V_A}$$  
$$= (-m \partial(X_{-m})(\partial - \varphi)^{m-1}, 0), \tag{3.28}$$

$$\text{(note that the residue of a commutator is always a total derivative in $x$).}$$
where we have used the fact
\[
[X_{-m}, \hat{A}]_\ast = X_{-m}(\partial - \varphi)^{-1} \rho - (\partial - \varphi)^{-1} \rho X_{-m} \\
= (\partial - \varphi)^{-1} \partial (X_{-m})(\partial - \varphi)^{-1} \rho \in T_{\varphi, \rho}^\ast \mathcal{M}.
\] (3.29)

On the other hand, for \((X, \hat{X}) \in T_{\hat{A}}^\ast \mathcal{U}_m\), one has
\[
\mathcal{P}_{\hat{A}}^{\text{vl}}(X, \hat{X}) = \left(- \text{res}(AX + \hat{A}\hat{X}) : (\partial - \varphi)^m - 1 + (\partial - \varphi)^m - 1 \text{res}(XA + \hat{X} \hat{A}), \right. \\
\left. (AX + \hat{A}\hat{X})_+ (\partial - \varphi)^{-1} \rho - (\partial - \varphi)^{-1} \rho (XA + \hat{X} \hat{A} + 1) \right|_{\mathcal{V}_{\hat{A}}} \\
= \left( \text{res}(XA + \hat{X} \hat{A} - X - A \hat{X}) \cdot (\partial - \varphi)^m, \right. \\
\left. a(\partial - \varphi)^{-1} \rho - (\partial - \varphi)^{-1} b \right|_{\mathcal{V}_{\hat{A}}} \\
= (\partial(f) \cdot (\partial - \varphi)^m - 1, 0),
\] (3.30)

where
\[a = \text{res}((AX + \hat{A}\hat{X}) (\partial - \varphi)^{-1}) \quad b = \text{res}((\partial - \varphi)^{-1} \rho (XA + \hat{X} \hat{A})),\]

\(f\) is given in (3.29), and the last equality in (3.30) holds for the same reason as (3.29).

According to (3.27)–(3.30), the following Dirac reduction from \(\mathcal{D}_\varphi\) to \(\mathcal{U}_m\) is feasible:
\[
\mathcal{P}_{\hat{A}}^{\text{red}} = \mathcal{P}_{\hat{A}}^{\text{vl}} - \mathcal{P}_{\hat{A}}^{\text{vl}} \circ (\mathcal{P}_{\hat{A}}^{\text{vl}})^{-1} \circ \mathcal{P}_{\hat{A}}^{\text{vl}},
\]
that is, for \((X, \hat{X}) \in T_{\hat{A}}^\ast \mathcal{U}_m\),
\[
\mathcal{P}_{\hat{A}}^{\text{red}}(X, \hat{X}) = \left(- (AX + \hat{A}\hat{X})_+ A + A(XA + \hat{X} \hat{A})_-, \\ (AX + \hat{A}\hat{X})_+ \hat{A} - \hat{A}(XA + \hat{X} \hat{A})_+ \right) \\
- \mathcal{P}_{\hat{A}}^{\text{vl}}(X, \hat{X}) + \frac{1}{m} \left([f, A] + m \partial(f) \cdot (\partial - \varphi)^m, [f, \hat{A}] \right) \\
= \left(- (AX + \hat{A}\hat{X})_+ A + A(XA + \hat{X} \hat{A})_-, \\ (AX + \hat{A}\hat{X})_+ \hat{A} - \hat{A}(XA + \hat{X} \hat{A})_+ \right) \\
+ \frac{1}{m} \left([f, A], [f, \hat{A}] \right). \] (3.31)

The lemma is proved. \(\square\)

By now we have obtained a Poisson tensor \(\mathcal{P}^{\text{red}}\) on \(\mathcal{U}_m\). The following shift transformation
\[
\mathcal{S} : (A, \hat{A}) \mapsto (A + s, \hat{A} + s),
\] (3.32)
where \(s\) is a parameter, induces a push-forward of the Poisson tensor \(\mathcal{P}^{\text{red}}\) as
\[
\mathcal{S}_s \mathcal{P}^{\text{red}} = \mathcal{P}_2 - s \mathcal{P}_1 + s^2 \mathcal{P}_0.
\]

By straightforward calculation, we have \(\mathcal{P}_0 = 0\), and conclude the following lemma.
Lemma 3.4 On the coset $U_m$ there are two compatible Poisson tensors defined as follows:

$P_1(X, \hat{X}) = (-[X_+ + \hat{X}_+, A] + [X, A]_+ + [\hat{X}, \hat{A}]_-

[X_+ + \hat{X}_+, A] - [X, A]_+ - [\hat{X}, \hat{A}]_+)$,

$P_2(X, \hat{X}) = (-AX + \hat{A}X)_- A + A(XA + \hat{A})_-$

$(AX + \hat{A}X)_+ \hat{A} - \hat{A}(XA + \hat{A})_+ - \frac{1}{m} \langle [f, A], [f, \hat{A}] \rangle$ \quad (3.33)

$P_2(X, \hat{X}) = \langle \hat{f} - F \hat{A} \rangle \hat{A} - \hat{F} \hat{A} (\hat{f} - F \hat{A})_+ - \frac{1}{m} \langle [f, A], [f, \hat{A}] \rangle$ \quad (3.34)

with $(X, \hat{X}) \in T^*_A U_m$ at any point $A = (A, \hat{A}) \in U_m$, and $f$ given in \eqref{3.26}.

Finally, let us represent the extended KP hierarchy into a bi-Hamiltonian form.

Theorem 3.5 For any positive integer $m$, let $\{ , \}_1^m$ be the Poisson brackets on $U_m$ given by the tensors $P_{1,2}$ in \eqref{3.33}–\eqref{3.34}. The extended KP hierarchy \eqref{2.10}–\eqref{2.11} can be represented as

$\frac{\partial F}{\partial t_k} = \{ F, H_k \}^m_1 = \{ F, H_k \}^m_2 \quad (3.35)$

$\frac{\partial F}{\partial \hat{t}_k} = \{ F, \hat{H}_{k+1} \}^m_1 = \{ F, \hat{H}_k \}^m_2 \quad (3.36)$

with $k = 1, 2, 3, \ldots$ and Hamiltonians

$H_k = \frac{m}{k} \int \text{res} P^k \, dx, \quad \hat{H}_k = \frac{1}{k} \int \text{res} \hat{P}^k \, dx.$ \quad (3.37)

Proof: The proof is similar to that of Theorem 4.6 in \cite{26}. Recalling $A = (A, \hat{A}) = (P^m, \hat{P})$, since

$\delta H_k = \langle (P^{k-m}, 0), \delta A \rangle, \quad \delta \hat{H}_k = \langle (0, \hat{P}^{k-1}), \delta A \rangle,$

then the gradients of the Hamiltonian functionals are

$\frac{\delta H_k}{\delta A} = (P^{k-m}, 0), \quad \frac{\delta \hat{H}_k}{\delta A} = (0, \hat{P}^{k-1})$ \quad (3.38)

up to a kernel part in $V^*_A$. Note that such a kernel part does not change the following Hamiltonian vector fields

$\mathcal{P}_2 \left( \frac{\delta H_k}{\delta A} \right) = (-\langle P^k \rangle, A + A(\hat{P}^k), (\hat{P}^k) + \hat{A} - \hat{A}(\hat{P}^k),

= \langle (\hat{P}^k), (\hat{P}^k), (A, \hat{A}) \rangle,$ \quad (3.39)

$\mathcal{P}_2 \left( \frac{\delta \hat{H}_k}{\delta A} \right) = (-\langle \hat{P}^k \rangle, A + A(\hat{P}^k), \hat{A} + \hat{A}(\hat{P}^k),

= \langle (-\hat{P}^k), (-\hat{P}^k), (A, \hat{A}) \rangle.$ \quad (3.40)
Namely, by virtue of (3.16)–(3.17) we have
\[
\frac{\partial A}{\partial t_k} = P_2 \left( \frac{\delta H_k}{\delta A} \right), \quad \frac{\partial A}{\partial \hat{t}_k} = P_2 \left( \frac{\delta \hat{H}_k}{\delta A} \right).
\] (3.41)
In the same way, it can be checked
\[
\frac{\partial A}{\partial t_k} = P_1 \left( \frac{\delta H_{k+m}}{\delta A} \right), \quad \frac{\partial A}{\partial \hat{t}_k} = P_1 \left( \frac{\delta \hat{H}_{k+1}}{\delta A} \right).
\] (3.42)
Therefore the theorem is proved. □

Remark 3.6 One can verified that the following 1-form is closed:
\[
\omega = \sum_{k=1,2,3,...} \left( \text{res} P^k \, dt_k + \text{res} \hat{P}^k \, d\hat{t}_k \right).
\]
Hence given a solution of the extended KP hierarchy (2.10)–(2.11) there locally exists a tau function \( \tau = \tau(t_1, t_2, \ldots; \hat{t}_1, \hat{t}_2, \ldots) \) such that
\[
\omega = d(\partial_x \log \tau).
\] (3.43)
In comparison with the KP hierarchy, it is natural to ask whether the extended KP hierarchy can be recast to some bilinear equation of \( \tau \). We hope to study it in the future. □

3.4 Relation to the constrained KP hierarchy

Let us consider a natural reduction of the extended KP hierarchy (2.10)–(2.11), as well as its Hamiltonian structures.

Recall the pseudo-differential operators (3.14)–(3.15). Now assume \( A = \hat{A} \), each equal to
\[
L = (\partial - \varphi)^m + m\varphi (\partial - \varphi)^{m-1} + \sum_{i=0}^{m-2} v_i (\partial - \varphi)^i + (\partial - \varphi)^{-1} \rho.
\] (3.44)
Under this assumption, the extended KP hierarchy (3.16)–(3.17) is reduced to
\[
\frac{\partial L}{\partial t_k} = [(L^{k/m})_{+}, L], \quad k = 1, 2, 3, \ldots,
\] (3.45)
where \( L^{1/m} = P \) takes the form (2.8).

Proposition 3.7 The hierarchy (3.45) can be represented in a bi-Hamiltonian form as
\[
\frac{\partial L}{\partial t_k} = P_1 \left( \frac{\delta H_{k+m}}{\delta L} \right) = P_2 \left( \frac{\delta H_k}{\delta L} \right), \quad k = 1, 2, 3, \ldots,
\] (3.46)
where $\mathcal{P}_{1,2}$ are Poisson tensors given by

\begin{align}
\mathcal{P}_1(Y) &= -[Y_-, L] + [Y, L]_-, \\
\mathcal{P}_2(Y) &= -(LY)_- L + L(YL)_- + \frac{1}{m}[g, L]
\end{align}

with $g = \partial^{-1}(\text{res}[Y, L])$, and the Hamiltonian functionals are

\[ H_k = \frac{m}{k} \int \text{res}L^k/m \, dx. \]

Proof: Let $A_{(A, \hat{A})}$ be the algebra of formal differential polynomials generated by the unknown functions of $(A, \hat{A})$ given in (3.14)–(3.15), and $A_L$ be the differential algebra generated by the unknown function in (3.44). For any functional $F \in A_L/\partial(A_L)$, the embedding $A_L \hookrightarrow A_{(A, \hat{A})}$ due to $L = A = \hat{A}$ implies the following relation of variational gradients

\[ \frac{\delta F}{\delta L} = \frac{\delta F}{\delta A} + \frac{\delta F}{\delta \hat{A}}. \]

Accordingly, it is straightforward to check that the Poisson tensors (3.33)–(3.34) are reduced to (3.47)–(3.48). Therefore the proposition is proved. \qed

Observe that the bi-Hamiltonian structure in the above proposition was obtained by Oevel and Strampp [20] and by Cheng [9], with different methods, respectively. In a recent paper [17], the central invariants for the bi-Hamiltonian structure was calculated based on an explicit formula of the variation derivative $\delta F/\delta L$, which shows that the constrained KP hierarchy (3.45) is the so-called topological deformation of its dispersionless limit (cf. (3.79) below).

Example 3.8 When $m = 1$, we have

\[ L = (\partial - \varphi) + \varphi + (\partial - \varphi)^{-1} \rho = \partial + (\partial - \varphi)^{-1} \rho. \]

The hierarchy (3.45) now reads

\[ \frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad k = 1, 2, 3, \ldots \]

One has $\partial/\partial t_1 = \partial/\partial x$, and according to (2.17)–(2.18),

\[ \frac{\partial \varphi}{\partial t_2} = \partial_x(\varphi^2 + 2\rho + \varphi_x), \quad \frac{\partial \rho}{\partial t_2} = \partial_x(2\varphi \rho - \rho_x). \]

Here the subscript $x$ means the derivative with respect to it.

If we perform the following transformation of variables:

\[ \varphi = \frac{q_x}{q}, \quad \rho = q r, \]

\[ 15 \]
then the equations (3.52) are converted to
\[
\frac{\partial q}{\partial t_2} = 2q^2 r + q_{xx}, \quad \frac{\partial r}{\partial t_2} = -2q r^2 - r_{xx},
\] (3.54)
which form the nonlinear Schrödinger equation. Note that the hierarchy (3.51) together with certain extra flows compose the so-called extended nonlinear Schrödinger hierarchy in [7], which is equivalent to the extended Toda hierarchy under a transformation of variables.

**Example 3.9** When \( m = 2 \), we have
\[
L = (\partial - \varphi)^2 + 2\varphi(\partial - \varphi) + u + (\partial - \varphi)^{-1} p,
\] (3.55)
and the hierarchy (3.45) is the so-called Yajima–Oikawa hierarchy [9, 30]. The first non-trivial equations are
\[
\frac{\partial \varphi}{\partial t_2} = u_x, \quad \frac{\partial \rho}{\partial t_2} = \partial_x (2\varphi \rho - \rho_x), \quad \frac{\partial u}{\partial t_2} = 2\rho_x + 2\varphi u_x + u_{xx}.
\] (3.56)
In fact, these equation are recast to equations (2.20) in [9] via the following transformation:
\[
\varphi = \frac{q_x}{q}, \quad \rho = qr, \quad u = u_1 + \frac{q_{xx}}{q}.
\] (3.57)

**Remark 3.10** Let \( A \) and \( B \) be two differential operators of the form
\[
A = \partial^{m+n} + a_{m+n-1}\partial^{m+n-1} + \cdots + a_1\partial + a_0,
\] (3.58)
\[
B = \partial^n + b_{n-1}\partial^{n-1} + \cdots + b_1\partial + b_0,
\] (3.59)
and \( L = B^{-1}A \) with \( B^{-1} = \partial^{-n} - b_{n-1}\partial^{-n-1} + \ldots \). Recall that a version of constrained KP hierarchy suggested by Aratyn et al [11] and by Bonora et al [4] (see also [12]) is the collection of the following evolutionary equations
\[
\frac{\partial A}{\partial t_k} = -\left( A(L^{k/m})_+ A^{-1} \right)_- A, \quad \frac{\partial B}{\partial t_k} = -\left( B(L^{k/m})_+ B^{-1} \right)_- B,
\] (3.60)
where \( k = 1, 2, 3, \ldots \). In particular, one has
\[
\frac{\partial L}{\partial t_k} = -B^{-1}\frac{\partial B}{\partial t_k} B^{-1}A + B^{-1}\frac{\partial A}{\partial t_k} = B^{-1} \left( B(L^{k/m})_+ B^{-1} - A(L^{k/m})_+ A^{-1} \right)_- A
\]
\[= B^{-1} \left( B(L^{k/m})_+ B^{-1} A \right)_- A = \left[ (L^{k/m})_+, L \right],
\] (3.61)
of which the last equality holds for the fact
\[
\left[ (L^{k/m})_+, L \right] = -\left[ (L^{k/m})_-, L \right]
\]
having order less that \( m \). Hence the hierarchy \((3.45)\) can be regarded as a reduction of the constrained KP hierarchy \((3.60)\) under that assumption

\[
A = \partial^m + a_{m-2} \partial^{m-2} + \cdots + a_1 \partial + a_0, \quad B = \partial - \varphi.
\]  

(3.62)

From this point of view, the second Hamiltonian structure \((3.48)\) is just a reduction of the Hamiltonian structure obtained by Dickey [12] for the constrained KP hierarchy \((3.60)\).

\( \bigcirc \)

3.5 Dispersionless limit

For the extended KP hierarchy, let us study its dispersionless limit and the corresponding Hamiltonian structures.

With a function \( \varphi \in A \) chosen as before, one has two algebras

\[
\mathcal{H}_\varphi^- = A((z - \varphi)^{-1}) = \left\{ \sum_{i \leq k} f_i (z - \varphi)^i \mid f_i \in A, k \in \mathbb{Z} \right\},
\]  

(3.63)

\[
\mathcal{H}_\varphi^+ = A((z - \varphi)) = \left\{ \sum_{i \geq k} f_i (z - \varphi)^i \mid f_i \in A, k \in \mathbb{Z} \right\}
\]  

(3.64)

of Laurent series in \( z - \varphi \) (these Laurent series are defined outside or inside a loop \( \Gamma_\varphi \) surrounding \( z = \varphi \) on the complex plane). Each algebra \( \mathcal{H}_\varphi^\pm \) is endowed with a Lie bracket given by

\[
[a, b] = \frac{\partial a}{\partial z} \frac{\partial b}{\partial x} - \frac{\partial b}{\partial z} \frac{\partial a}{\partial x}.
\]

(3.65)

Moreover, on each \( \mathcal{H}_\varphi^\pm \) there is an invariant inner product

\[
\langle a, b \rangle = \langle a b \rangle, \quad \langle a \rangle = \frac{1}{2\pi \sqrt{-1}} \oint_{S^1} \oint_{\Gamma_\varphi} a(z) \, dz \, dx,
\]

(3.66)

which is invariant with respect to the above Lie bracket.

Introduce two series

\[
p(z) = (z - \varphi) + \varphi + \sum_{i \geq 1} u_i (z - \varphi)^{-i} \in \mathcal{H}_\varphi^-,
\]

(3.67)

\[
\dot{p}(z) = \sum_{i \geq -1} \dot{u}_i (z - \varphi)^i \in \mathcal{H}_\varphi^+
\]

(3.68)

with \( \dot{u}_{-1} = \rho \neq 0 \). The dispersionless extended KP hierarchy (or the Whitham hierarchy of genus zero with one marked point [24, 25], without the logarithm flow) is defined by

\[
\frac{\partial \dot{p}(z)}{\partial t_k} = [(p(z)^k)_+, \dot{p}(z)], \quad \frac{\partial \dot{p}(z)}{\partial t_k} = [-(\dot{p}(z)^k)_-, \dot{p}(z)],
\]

(3.69)
where \( \dot{p}(z) = p(z), \dot{p}(z) \), and \( k = 1, 2, 3, \ldots \). Here the subscripts “±” mean to take the nonnegative and negative part, respectively, of a series in \( z - \varphi \).

Similar as before, let us derive Hamiltonian structures for the dispersionless hierarchy \( (3.69) \). To this end, we introduce a coupled Lie algebra \( \mathcal{H}_\varphi = \mathcal{H}^-_\varphi \times \mathcal{H}^+\varphi \) whose Lie bracket is defined diagonally, and it is equipped with an inner product as follows

\[
((a, \hat{a}), (b, \hat{b})) = \langle a, b \rangle + \langle \hat{a}, \hat{b} \rangle, \quad (a, \hat{a}), (b, \hat{b}) \in \mathcal{H}_\varphi.
\]

Clearly, a map \( R \) of the form \( (3.10) \) defines an \( R \)-matrix that solves the modified Yang–Baxter equation on \( \mathcal{H}_\varphi \).

According to Theorem 3.1, one has the following Poisson bracket

\[
\{F, H\}(a) = \frac{1}{2} \left( \left\langle \left[ a, \frac{\delta F}{\delta a} \right], R \left( a \frac{\delta H}{\delta a} \right) \right\rangle - \left\langle \left[ a, \frac{\delta H}{\delta a} \right], R \left( a \frac{\delta F}{\delta a} \right) \right\rangle \right), \tag{3.70}
\]

where \( F, H \) are arbitrary functionals depending on \( a \in \mathcal{H}_\varphi \), with variational gradients \( \frac{\delta F}{\delta a}, \frac{\delta H}{\delta a} \) respectively (the definition of variational gradient is almost the same as before, so it is omitted here).

Given two arbitrary positive integers \( m \) and \( n \), let

\[
a(z) = (a(z), \hat{a}(z)) = (p(z)^m, \dot{p}(z)^n). \tag{3.71}
\]

All such series form a subset of \( \mathcal{H}_\varphi \) as

\[
U_{m,n} = \left\{ z^m + \sum_{i \leq m-2} v_i (z - \varphi)^i, \sum_{i \geq -n} \hat{v}_i (z - \varphi)^i \in \mathcal{H}^-_\varphi \times \mathcal{H}^+_\varphi \right\}. \tag{3.72}
\]

**Lemma 3.11** On \( U_{m,n} \) there are two compatible Poisson brackets \( \{ , \}_{\nu} \) \((\nu = 1, 2)\) given by the following Poisson tensors:

\[
\mathcal{P}_1(X, \hat{X}) = \left( [X, a]_+ + [\hat{X}, \hat{a}]_+ - [X_+ + \hat{X}_-, a], -[X, a]_+ + [\hat{X}, \hat{a}]_+ + [X_+ + \hat{X}_-, \hat{a}]_+ \right), \tag{3.73}
\]

\[
\mathcal{P}_2(X, \hat{X}) = \left( ([X, a]_+ + [\hat{X}, \hat{a}]_+) a - ([X a + \hat{X} \hat{a}]_-, a], -([X, a]_+ + [\hat{X}, \hat{a}]_+) \hat{a} + ([X a + \hat{X} \hat{a}]_+, \hat{a}) \right)
+ \frac{1}{m} \left( [f, a], [f, \hat{a}] \right), \tag{3.74}
\]

where

\[
f = \partial^{-1} \left( \text{res}_{z=\varphi}([X, a] + [\hat{X}, \hat{a}]) \right), \quad (X, \hat{X}) \in T^*_{(a, \hat{a})} U_{m,n}. \tag{3.75}
\]

**Proof:** The proof is similar with that of Lemma 3.3 with the method of Dirac reduction, so we omit the details here. \( \square \)

Furthermore, in the same way as for Theorem 3.5 we arrive at
Proposition 3.12 For any positive integers \(m\) and \(n\), the dispersionless extended KP hierarchy \([3.69]\) can be represented in a bi-Hamiltonian form as

\[
\frac{\partial F}{\partial t_k} = \{ F, H_{k+m}^n \}, \quad \frac{\partial F}{\partial \hat{t}_k} = \{ F, \hat{H}_{k+n}^m \},
\]

with \(k = 1, 2, 3, \ldots\)

Let us consider reductions of the dispersionless extended KP hierarchy. Assume

\[
p(z)^m = \hat{p}(z)^n = l(z)
\]

with

\[
l(z) = (z - \varphi)^m + m \varphi (z - \varphi)^{m-1} + \sum_{i=-n}^{m-2} v_i (z - \varphi)^i
\]

then the hierarchy \([3.69]\) is reduced to

\[
\frac{\partial l(z)}{\partial t_k} = [(p(z)^k)_{+}, l(z)], \quad \frac{\partial l(z)}{\partial \hat{t}_k} = [-(\hat{p}(z)^k)_{-}, l(z)],
\]

where \(k = 1, 2, 3, \ldots\)

With the same method as for Proposition 3.7 we have

Proposition 3.13 Under the constraint \([3.78]\), the Poisson structures \([3.73]-[3.74]\) are reduced to

\[
\mathcal{P}_1(Y) = [Y, l]_+ - [Y_-, l], \quad \mathcal{P}_2(Y) = [Y, l]_+ - [Y^-]_-, l + \frac{1}{m} [g, l]
\]

with \(g = \partial^{-1} \text{res}_{z=\varphi} [Y, l]\). They give a bi-Hamiltonian structure for the reduced hierarchy \([3.78]\).

4 Concluding remarks

In this article we have considered mainly the extended KP hierarchy defined with scalar pseudo-differential operators, whose dispersionless limit is the Whitham hierarchy with only one marked point. It is clarified how the hierarchy is related to the 2-component BKP hierarchy and the constrained KP hierarchy. For the extended KP hierarchy, we have also constructed a series of bi-Hamiltonian structures. It is natural to expect that
underlying them there would be some infinite-dimensional Frobenius manifolds, along the
line of \([6, 28, 29]\) (see also \([21, 23]\)); we will study it elsewhere.

Another interesting question is whether the Lax equations \((2.10) - (2.11)\) can be rep-
resented into a system of bilinear equations of the tau function given in \((3.43)\), such like
those Hirota equations raised from boson-fermion correspondence \([13, 25]\). The main dif-
ficulty in doing this is the dependence of the operator \(\partial - \varphi\) on \(t\), which is essentially
different from the cases of KP or 2-component BKP hierarchies. We hope that answering
the question would help us to understand generalizations of the KP hierarchy such like
\([8, 13, 25]\).

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