Conformal field theory for annulus SLE: partition functions and martingale-observables

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Abstract
We implement a version of conformal field theory in a doubly connected domain with numerous conformal types to connect it to the theory of annulus SLE of various types, including the standard annulus SLE, the reversible annulus SLE, and the annulus SLE with several force points. This implementation considers the statistical fields generated under the OPE multiplication by the Gaussian free field and its central/background charge modifications with a weighted combination of Dirichlet and excursion-reflected boundary conditions. We derive the Eguchi–Ooguri version of Ward’s equations and Belavin–Polyakov–Zamolodchikov equations for those statistical fields and use them to show that the correlations of fields in the OPE family under the insertion of the one-leg operators are martingale-observables for various annulus SLEs. We find Coulomb gas (Dotsenko–Fateev integral) solutions to the parabolic partial differential equations for partition functions of conformal field theory for the reversible annulus SLE.

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1 Introduction

Since Schramm introduced Schramm–Loewner evolution (SLE) in a simply connected domain [60], SLE theory has successfully produced rigorous proofs of several remarkable conjectures in statistical physics over the past two decades. Examples include the proofs of conformal invariance of scaling limits of some important critical lattice models by employing SLE. Such results have been obtained by the fundamental fact that certain discrete observables converge to conformally covariant martingale processes. For instance, in [64], Smirnov used Cardy’s observables to prove conformal invariance of scaling limit of critical site percolation in a hexagonal lattice. Schramm and Sheffield
made use of a certain bosonic observable to prove that the level lines of discrete Gaussian free field (GFF) with specific height gaps converge to SLE(4), see [62]. Chelkak, Duminil-Copin, Hongler, Kemppaninen, and Smirnov proved the existence of scaling limit of FK Ising model interfaces and its conformal invariance utilizing certain parafermionic observables, see [10] and references therein. In addition, some geometric properties of SLE curves have been obtained through SLE martingale-observables. Examples include the study of Hausdorff dimensions (Beffara’s observables) [6], and left passage probabilities (Schramm’s observables) [61] of SLE.

Beyond the SLE theory in a simply connected domain, there have been several developments of the theory in a multiply connected domain [46] that contains richer geometric structures. In a doubly connected domain, Zhan has made significant results on basic properties of SLE, see e.g., [68, 69, 74]. In particular, in [74], he considered the annulus SLE(κ, Λ) with two marked points and introduced specific parabolic PDE (known as the null-vector equation in the physics literature) of the annulus SLE partition function Z that provides a necessary condition for which the associated SLE trace is reversible, cf. [51, 71]. Here Λ := κ(log Z)' is the drift function of the driving process of SLE. We remark that such an equation for some special cases with κ = 3, 6 also appeared in the literature [16, 32] in connection with the study of discrete lattice models.

In the physics literature, especially in the context of conformal field theory (CFT) [14], it is well known that under the insertion of the ratio of the one-leg operator Ψ to its correlation function EΨ, the correlators in a particular class of fields are martingale-observables for SLEs. In a doubly connected domain, Hagendorf, Bernard, and Bauer used a certain bosonic observable to propose the partition functions for annulus SLE with κ = 4 associated with GFFs, see [29]. Hagendorf presented physical arguments on producing martingale-observables for annulus SLE(κ, Λ) (κ > 0) with Dirichlet boundary condition in [28].

In this paper, we use the framework built in [2, 35–37] to construct a version of CFT in a doubly connected domain with a plan to serve as a solid cornerstone to develop a theory in a general multiply connected domain. The scope of this paper includes Coulomb gas formalism and derivations of some important equations in CFT, such as the Eguchi–Ooguri version of Ward’s equations and Belavin–Polyakov–Zamolodchikov (BPZ) equations.

The simplest fields we consider in this paper form a one-parameter family of GFFs with a weighted (convex) combination of Dirichlet and excursion reflected (ER) boundary conditions, see Fig. 1. In the analysis literature, the Green’s function with ER boundary condition is often called the modified Green’s function [40]. In probability theory, the associated stochastic process, ER Brownian motion, was introduced by Lawler and Drenning [15, 45]. As a characteristic feature of the ER Green’s function, its normal derivatives on the inner boundary component have a vanishing mean, see e.g., [12, 13]. By combining this property with zero Dirichlet boundary condition under a proper weight (of rational type), we introduce a one-parameter family of the Green’s functions interpolating Dirichlet Green’s function with ER Green’s function and consider the associated GFFs.

One of the primary motivations for introducing such general boundary conditions is that this interpolation gives rise to infinitely many linearly independent solutions
to BPZ equation. Our observation affirms the prediction that contrary to a simply connected domain where BPZ equation has a finite-dimensional solution space [23–26], the solution space is no longer finite-dimensional in a doubly connected domain. We also refer to [42, 43, 55] for the analysis of solutions to various equations arising from CFT with a concrete understanding of the underlying algebraic structure.

The SLE martingale observables are constructed from the OPE family $F^{\beta} = F(b, \beta)$ generated by central/background charge modifications $\Phi^{\beta} = \Phi(b, \beta)$ of the GFFs under OPE multiplication. Here, $b$ is a real parameter related to the central charge $c$ and the SLE parameter $\kappa$ as

$$c = 1 - 12b^2, \quad b = \sqrt{\kappa/8 - \sqrt{2/\kappa}}.$$ 

We require that for a divisor $\beta = \sum \beta_j \cdot q_j$ called a background charge, the total sum of the charges should vanish, i.e., $\sum \beta_j = 0$. On a complex torus (of genus one), this requirement is known as the neutrality condition (NC0) in the physics literature, see e.g., [14]. To derive BPZ equations, we need two main ingredients. The first is Ward’s equation and the second is the construction of the one-leg operator $\Psi$ satisfying level two degeneracy equation.

Whereas Ward’s equations in a simply connected domain describe the Virasoro fields in terms of Lie derivative operators within correlations, the statement of those in a doubly connected domain contains the derivative of correlation functions with respect to the modular (Teichmüller) parameter as well. These forms of Ward’s equations are well known in the physics literature, e.g., Ward’s equations in a doubly connected domain [29] and Eguchi–Ooguri’s version of Ward’s equations on a complex torus $M_1$ of genus one [21]. We derive them not from the path integral formalism but the functional equations of classical special functions such as pseudo-addition theorem of
Weierstrass zeta function. See [37] for a mathematical derivation of Eguchi–Ooguri’s version of Ward’s equations on $M_1$.

For given divisor $\tau = \sum_j \tau_j \cdot \xi_j$ satisfying $a + \int \tau = 0$, we construct the one-leg operator $\Psi \equiv \Psi(p, \xi)$ from the GFF as the modified multi-vertex field (or the OPE exponential) with the charge $a = \sqrt{2/\kappa}$ at a given marked point $p$. With this specific choice of the charge at $p$, the one-leg operator $\Psi$ satisfies the desired level two degeneracy equation. The correlation function $E\Psi$ of the one-leg operator (boundary condition changing operator) is called the partition function in CFT (up to a multiplicative constant depending only on a domain), see e.g., [14, Sect. 11.3]. The partition functions have different descriptions in other contexts, such as random matrix theory, statistical mechanics, and SLE theory. For example, Dubédat introduced the partition functions of SLE and the partition functions of the free field with associated boundary data, and then showed that these two definitions coincide, see [20, Theorem 5.3]. Unless stated otherwise, the partition function means the correlation function of the one-leg operator throughout the paper. In some typical examples, we relate the partition function in CFT with the partition of the associated lattice model. We remark that compared to the definition of SLE partition functions in [20], we do not take into account the pre-factor of $\zeta$-regularized determinant, cf. [46]. Indeed, this does not change the drift function of SLE since it is given by the logarithmic derivative with respect to a space variable, see below.

We first study the annulus SLE$(\kappa, \Lambda)$ with

$$\Lambda \equiv \Lambda(p, \xi) = \kappa \partial_{\xi}|_{\xi=p} \log E\Psi(\xi, \xi),$$

see [35] for this type of SLE in a simply connected domain from a viewpoint of Radon-Nikodym derivative (or coordinate change [63]) between the laws of SLEs. Here, the marked boundary points $p$ and $\xi$ play different roles: the point $p$ corresponds to the starting point of SLE$(\kappa, \Lambda)$, whereas $\xi$ are force points. We consider a collection of all correlation functions of fields in the OPE family $F_\beta$ under the insertion of the properly normalized one-leg operator. Then employing BPZ-Cardy equations, we show that this collection forms martingale-observables for SLE$(\kappa, \Lambda)$. In the case of Dirichlet boundary condition, this statement generalizes a result of Izyurov and Kytölä in [34], where they showed that 1-and 2-point functions of GFF with piecewise Dirichlet boundary condition are martingale-observables for associated SLE$(4, \Lambda)$. Here, the neutrality condition coincides with the requirement that jumps of the 1-point function on the boundary should add up to zero.

However, the correlation function $Z := E\Psi$ of this type of SLE$(\kappa, \Lambda)$ with one force point does not satisfy the null-vector equation [see (2.1) in Sect. 2.3] for the reversible annulus SLE partition function (introduced in [74]) unless $\kappa = 4$. To find explicit solutions to the null-vector equation of this type for general $\kappa > 0$ and construct a class of SLE$(\kappa, \Lambda)$ martingale-observables associated with these solutions, we apply the method of screening (also known as the Coulomb gas or Dotsenko–Fateev integrals). If $\kappa > 4$, we can deform a Pochhammer contour in this methodology so that the solution can be represented in terms of the Euler type integral. In the case $\kappa \leq 4$, we perform proper analysis on an analytic continuation (with respect to $\kappa$) of the Pochhammer contour integral to obtain a non-trivial real-valued solution.
Beyond the method of screening, we present further constructions of SLE martingale-observables, which include the implementations of non-atomic background charges and the use of periodization (or chiral bosonization) of conformal fields. These methods allow us to construct martingale-observables for continuum interface curves in statistical physics such as discrete GFF, loop-erased random walk, and critical Ising interfaces. Moreover, we derive some geometric properties such as hitting/left passage probabilities of certain annulus SLEs.

2 Main results

2.1 Notation

Throughout this paper, we use the following symbols and notations: let \( T = \{ z \in \mathbb{C} : |z| = 1 \} \); for \( r > 0 \) let \( \mathbb{A}_r = \{ z \in \mathbb{C} : e^{-r} < |z| < 1 \} \), \( \mathbb{T}_r = \{ z \in \mathbb{C} : |z| = e^{-r} \} \), \( \mathbb{S}_r = \{ z \in \mathbb{C} : 0 < \text{Im} \ z < r \} \), and \( \mathbb{R}_r = \{ z \in \mathbb{C} : \text{Im} \ z = r \} \). We denote by \( \mathbb{C}_r \) the cylinder \( \mathbb{S}_r/\langle \ z \mapsto z + 2\pi \rangle \), and by \( D_r \) a doubly connected domain with modulus \( r \).

A subset \( K \) of a doubly connected domain \( D \) is called a hull in \( D \) if \( D \setminus K \) is a doubly connected domain and \( K \) has positive distance from one boundary component of \( D \). For any hull \( K \) of \( D \), let us define \( \text{cap}_D(K) := \text{mod}(D) - \text{mod}(D \setminus K) \) as the capacity of \( K \) in \( D \), where \( \text{mod}(\cdot) \) denotes the modulus of a doubly connected domain.

Following \([69, 74]\), we write \( \Theta(\rho, z) := \theta(2\pi \rho, \frac{z}{\pi}) \), where \( \theta \) is a Jacobi theta function, see e.g., \([9, \text{p. 63}]\). In Sect. 3.1 we recall the definition of \( \Theta \) and its basic properties. It is convenient to introduce

\[
\Theta_\chi(\rho, z) := \Theta(\rho, z) \exp \left( \frac{z^2}{4(\rho + \chi)} \right), \quad \chi \in [0, \infty].
\]

The parameter \( \chi \) is used to describe a one-parameter family of GFFs with a weighted combination of Dirichlet and excursion reflected (ER) boundary conditions. More precisely, the boundary condition of the Green’s function \( G_r(\zeta, z) \) (that determines 2-point function of the base field \( \Phi(0) \) below) on the inner boundary component \( \gamma \) is given by

\[
\frac{1}{1 + \chi} G_r(\zeta, z) + \frac{\chi}{1 + \chi} \left( \frac{1}{2\pi} \oint_{\gamma} \frac{\partial G_r}{\partial n_z} \, ds_z \right) = 0, \quad z \in \gamma,
\]

where \( \partial n_z \) is the normal derivative and \( ds_z \) is the arc length measure. In the extremal cases when \( \chi = 0 \) and \( \infty \), this boundary condition recovers Dirichlet and ER boundary conditions respectively. This boundary condition will be discussed in more detail in Sect. 3.2.1.

We write \( \zeta_r \) for the Weierstrass zeta function with basic periods \( (2\pi, 2ir) \). In the sequel we sometimes omit the modular parameter \( r \) and write for instance \( \Theta_\chi(\cdot) \equiv \Theta_\chi(r, \cdot) \).
2.2 Basic setup

Let $\Phi_0$ be the GFF in $D_r$ with a weighted combination of Dirichlet and ER boundary conditions, see Sect. 3.2 for details. Its 2-point function is given by

$$E \Phi_0(\zeta) \Phi_0(z) = 2 \log \left| \frac{\Theta_\chi(r, \zeta - \bar{z})}{\Theta_\chi(r, \zeta - z)} \right|$$

in the $C_r$-uniformization. Let us stress here that the GFF $\Phi_0$ depends on the choice of $\chi$. Then the chosen value of $\chi$ propagates through the rest of the formulas generated by the base field $\Phi_0$.

For a real parameter $b$, we define the central charge modification $\Phi_b$ of GFF by adding a non-random pre-pre-Schwarzian (PPS) form as follows:

$$\Phi_b := \Phi_0 - 2b \arg w',$$

where $w$ is a conformal map from $D_r$ onto $C_r$. See Sect. 3.3 for the definition of PPS forms. We remark that $\arg w'$ does not depend on the choice of a conformal map.

Given marked points $q = \{q_k\}$, we call a divisor $\beta = \sum_k \beta_k \cdot q_k$ a background charge if it satisfies the neutrality condition $\int \beta = 0$. We define the background charge modification $\Phi_\beta \equiv \Phi_{\beta, (b)}$ associated with $\beta$ as

$$\Phi_\beta := \Phi_b + \sum \beta_k \arg \left\{ \Theta_\chi(w_k - w_z) \Theta_\chi(\bar{w}_k - w_z) \right\},$$

where $w_z = w(z)$ and $w_k = w(q_k)$. Let us denote by $\mathcal{F}_\beta \equiv \mathcal{F}_{\beta, (b)}$ the OPE family of the $\Phi_\beta$, the algebra (over $\mathbb{C}$) spanned by the generators 1, mixed derivatives of $\Phi_\beta$, those of OPE exponentials $e^{i\alpha/\Phi_\beta} (\alpha \in \mathbb{C})$, and their holomorphic/anti-holomorphic/mixed parts (with certain neutrality conditions) under the OPE multiplication $\ast$, see Sect. 3.2.3 for more details on OPE calculus. In particular, we write $\mathcal{F}_b \equiv \mathcal{F}_{0, (b)}$. We present a more precise definition for OPE exponentials in the following subsection.

We now recall the (covering) annulus SLE with $N$-force points $q_1, \ldots, q_N$. Like the SLE in a simply connected domain, the annulus SLE $(\kappa, \Lambda)$ can be described by the associated Loewner’s differential equation [69]. By definition, the Loewner kernel $S(r, z)$ on $\mathbb{R}_r = \{ z : e^{-r} < |z| < 1 \}$ is given by

$$S(r, z) := \lim_{N \to \infty} \sum_{n=-N}^{N} e^{2nr} + z \over e^{2nr} - z.$$

Let $\xi$ be a (real-valued) continuous function defined on $[0, T)$ for some $0 < T < r$. For each $z \in \mathbb{R}_r$, let $g_t(z)$ be the solution (which exists up to the first time $\tau_z \in (0, T]$ that $g_t(z)$ hits $\xi_t$) of the annulus Loewner equation

$$\partial_t g_t(z) = g_t(z) S(r - t, g_t(z)e^{-i\xi_t}), \quad g_0(z) = z.$$
For each time $t \in [0, T)$, let $K_t := \{ z \in A_r : \tau_z \leq t \}$. Then $K_t$ is a hull in $A_r$ with capacity $t$. Moreover, $g_t$ is a conformal map from $(A_r \setminus K_t, \mathbb{T}_r)$ onto $(A_{r-t}, \mathbb{T}_{r-t})$.

It is convenient to describe the annulus SLE in the covering space $S_r$, where the Loewner kernel $H$ is expressed in terms of $\Theta$ as

$$H(r, z) := -i \, S(r, e^{iz}) = 2 \, \partial_z \log \Theta(r, z).$$

For each $z \in S_r$, we denote by $\tilde{g}_t(z)$ the solution of the equation

$$\partial_t \tilde{g}_t(z) = H(r - t, \tilde{g}_t(z) - \xi_t), \quad \tilde{g}_0(z) = z,$

and set $\tilde{K}_t := \{ z \in S_r : \tau_z \leq t \}$. Then $\tilde{g}_t$ is a conformal map from $(S_r \setminus \tilde{K}_t; \mathbb{R}_r)$ onto $(S_r; \mathbb{R}_r)$. Moreover for each $z \in S_r \setminus \tilde{K}_t$, $e^{i \tilde{g}_t(z)} = g_t(e^{iz})$, and $K_t = \{ e^{iz} \in A_r : z \in \tilde{K}_t \}$. In particular, $\tilde{K}_t$ is $2\pi$-periodic and for each $n \in \mathbb{Z}$, we have $\tilde{g}_t(z + 2n\pi) = \tilde{g}_t(z) + 2n\pi$. We call that $\xi_t$ generates a trace $\gamma$ if the limit $\gamma(t) := \lim_{z \to \tilde{g}_t^{-1}(z)} \tilde{g}_t^{-1}(z)$ exists and if the resulting $\gamma_t$ is a curve. It was proved by Zhan in [69] that $\xi_t := \sqrt{k} \, B_t, (t < r)$ a.s. generates continuous trace, where $B_t$ is a standard one-dimensional Brownian motion.

Suppose $\Lambda$ is a real-valued $C^{0,1}$ function on $(0, \infty) \times (\mathbb{C} \setminus \{ z \in 2\pi \mathbb{Z} + 2ir\mathbb{Z} \})^N$. Let $\xi_t$ be the solution, (which exists up to the first time $T$ such that $\xi_t - \tilde{g}_t(q_j) \in 2\pi \mathbb{Z}_r$ of the SDE

$$d\xi_t = \sqrt{k} \, dB_t + \Lambda(r - t, \xi_t - \tilde{g}_t(q_1), \ldots, \xi_t - \tilde{g}_t(q_N)) \, dt, \quad \xi_0 = p.$$

Applying the Girsanov theorem, it can be shown that up to time $T$, $\xi_t$ a.s. generates continuous trace, see [74, Sect. 3.2] for further details. We call the trace generated by $\xi_t$ the (covering) annulus SLE($\kappa, \Lambda$) trace started from $p$ with force points $q_1, \ldots, q_N$.

Now we recall the notion of SLE martingale-observables. A non-random (conformal) field $f$ in a planar domain $D$ (or on a Riemann surface in general) is an assignment of smooth function $(f \| \phi) : \phi U \to \mathbb{C}$ for each local chart $\phi$. By definition, a non-random field $M$ is a martingale-observable for annulus SLE($\kappa, \Lambda$) if for any $z_1, \ldots, z_n$, the process

$$M_t(z_1, \ldots, z_n) := (M \| \tilde{g}_t^{-1})(z_1, \ldots, z_n), \quad (t < T)$$

(stopped when any $z_j$ or any $q_k$ is swallowed by the Loewner hull $\tilde{K}_t$) is a local martingale on the annulus SLE probability space.

### 2.3 Statement of main results

For a smooth vector field $v$, let us denote by $\mathcal{L}_v^+$ (resp., $\mathcal{L}_v^-$) the $\mathbb{C}$-linear (resp., anti $\mathbb{C}$-linear) part of Lie derivative operator $\mathcal{L}_v$ with respect to $v$, i.e., $\mathcal{L}_v^\pm := (\mathcal{L}_v \mp i \mathcal{L}_i v)/2$. See Sect. 3.3.1 for the definition and local properties for Lie derivative operators. We define a stress tensor $A_\beta \equiv A_{\beta,(b)}$ in terms of the current fields $J_{(b)} := \partial \Phi_{(b)}$ and
\[ J_\beta = \partial \Phi_\beta \] by

\[ A_\beta := -\frac{1}{2} J_{(0)} \odot J_{(0)} + \left( i b \partial - E J_\beta \right) J_{(0)}, \]

where \( \odot \) denotes the Wick’s product. The Virasoro field \( T_\beta \equiv T_{\beta,(b)} \) is given by

\[ T_\beta := -\frac{1}{2} J_\beta \ast J_\beta + i b \partial J_\beta = A_\beta + E T_\beta. \]

See Sect. 3.7 for more details.

We derive the following form of Ward’s equation.

**Theorem A** For any string \( \mathcal{X} \) of fields in the OPE family \( \mathcal{F}_\beta \), we have

\[ 2 \mathcal{E} A_\beta (\zeta) \mathcal{X} = \left( L^+_{v_\zeta} + L^-_{v_\zeta} \right) \mathcal{E} \mathcal{X} + \partial r \mathcal{E} \mathcal{X}, \]

where all fields are evaluated in the identity chart of \( C_r \) and the Loewner vector field \( v_\zeta \) is given by

\[ (v_\zeta \parallel \text{id}_{C_r})(z) := H(r, \zeta - z). \]

**Example (Hadamard’s variational formula)** As a simplest non-trivial example of Theorem A, let us consider the case \( \beta = 0 \) and \( \mathcal{X} = \Phi(z_1) \Phi(z_2) \). Then Ward’s equation is equivalent to the following functional relation of the Green’s function \( G \):

\[ 4 \partial_\zeta G(\zeta, z_1) \partial_\zeta G(\zeta, z_2) + \left( v_\zeta(z_1) \partial z_1 + v_\zeta(z_2) \partial z_2 + v_\zeta(\bar{z}_1) \bar{\partial} z_1 + v_\zeta(\bar{z}_2) \bar{\partial} z_2 + \partial r \right) G(z_1, z_2) = 0. \]

Letting \( \zeta \to p \in \mathbb{R} \), this equation gives rise to Hadamard’s variational formula for annulus Loewner chains:

\[ \frac{d}{dt} G_{\Omega_t}(z_1, z_2) \bigg|_{t=0} = -P(z_1) P(z_2), \quad P(z) := \frac{\partial}{\partial n} \bigg|_{\xi=p} G(\xi, z), \]

where \( \Omega_t := C_r \setminus \tilde{K}_t \). Such a variational formula was studied by Izyurov and Kytölä [34]. We also refer to [30, 41] for various Hadamard’s formula in a planar domain.

We define the one-leg operator \( \Psi \) in terms of a modified multi-vertex field (OPE exponential) below. For this purpose, it is convenient to introduce the formal (1-point) bosonic fields \( \Phi^{\pm}_{(0)} \) in \( D_r \). Although they are not genuine (Fock space) fields, they can be interpreted as the “holomorphic part” and the “anti-holomorphic part” of the...
Given divisors \( \sigma = \sum_{j=1}^{n} \sigma_j \cdot z_j, \sigma_\ast = \sum_{j=1}^{n} \sigma_{\ast j} \cdot z_j \), we set

\[
\Phi_0(\sigma, \sigma_\ast) := \sum_{j=1}^{n} \left( \sigma_j \Phi_0^+ (z_j) - \sigma_{\ast j} \Phi_0^- (z_j) \right).
\]

Then \( \Phi_0(\sigma, \sigma_\ast) \) is a well-defined (Fock space) field if and only if the following neutrality condition \( (NC_0) \) holds: \( \sum_{j=1}^{n} (\sigma_j + \sigma_{\ast j}) = 0 \).

Given a double divisor \( (\sigma, \sigma_\ast) \) for \( z_j \in C_r \) satisfying the neutrality condition \( (NC_0) \), we define the modified multi-vertex field \( (OPE exponentials) \) \( \mathcal{O}_\beta(\sigma, \sigma_\ast) \equiv \mathcal{O}_{\beta,(b)}(\sigma, \sigma_\ast) \) by

\[
\mathcal{O}_\beta(\sigma, \sigma_\ast) := \frac{C(b)[\sigma + \beta/2, \sigma_\ast + \beta/2]}{C(b)[\beta/2, \beta/2]} e^{\partial_i \Phi_0(\sigma, \sigma_\ast)}.
\]

In particular, we write \( \mathcal{O}_{(b)}(\sigma, \sigma_\ast) = \mathcal{O}_{0,(b)}(\sigma, \sigma_\ast) \). (The reason for the terminology “(OPE exponentials)” will become clear due to Proposition 3.6) Here the Coulomb gas correlation function \( C(b)[\sigma, \sigma_\ast] \) is a differential of conformal dimensions \( (\lambda_j, \lambda_{\ast j}) = (\frac{1}{2} \sigma_j^2 - b \sigma_j, \frac{1}{2} \sigma_{\ast j}^2 - b \sigma_{\ast j}) \) at each \( z_j \) and its evaluation in the identity chart of \( C_r \) is given by

\[
C(b)[\sigma, \sigma_\ast] = \Theta'(0) \frac{1}{2} \sum_j (\sigma_j^2 + \sigma_{\ast j}^2) \exp \left( -\frac{(\sum_j \sigma_j z_j + \sum_j \sigma_{\ast j} \bar{z}_j)^2}{4(r + \chi)} \right) \prod_j \Theta(z_j - \bar{z}_j)^{\sigma_j/\sigma_{\ast j}} \times \prod_{j < k} \Theta(z_j - z_k)^{\sigma_j/\sigma_k} \Theta(\bar{z}_j - z_k)^{\sigma_{\ast j}/\sigma_k} \Theta(\bar{z}_j - \bar{z}_k)^{\sigma_{\ast j}/\sigma_{\ast k}}.
\]

Given a divisor \( \tau = \sum \tau_j \cdot \xi_j \) satisfying \( a + \sum \tau_j = 0 \), we define

\[
\Psi_\beta(z) = \Psi_\beta(z, \xi) \equiv \Psi_\beta[a \cdot z + \tau] := \mathcal{O}_\beta[a \cdot z + \frac{1}{2} \tau, \frac{1}{2} \tau].
\]

Then the following form of BPZ equation holds.

**Theorem B** Suppose that \( 2a(a + b) = 1 \). Then for any \( \mathcal{X} \in \mathcal{F}_\beta \), we have

\[
\frac{1}{a^2} \partial_z^2 \mathbf{E} \Psi_\beta(z) \mathcal{X} = \left( \mathcal{L}^+_\nu z + \mathcal{L}^-_{\bar{v}z} + \partial_r + 2h_{1,2} \frac{\xi_r(\pi)}{\pi} + 2\mathbf{E} T_\beta(z) \right) \mathbf{E} \Psi_\beta(z) \mathcal{X},
\]

where all fields are evaluated in the identity chart of \( C_r \). Here, Lie derivative operators do not apply to \( z \) and \( h_{1,2} = a^2/2 - ab \).

To connect CFT with SLE theory, we choose real parameters \( a \) and \( b \) in terms of SLE parameter \( \kappa \) as

\[
a = \sqrt{2/\kappa} , \quad b = \sqrt{\kappa/8} - \sqrt{2/\kappa}.
\]
Let us write
\[ \alpha := a \cdot p + \tau. \]

Now we consider \( \text{SLE}(\kappa, \Lambda) \), where \( \Lambda(p, \xi) \equiv \Lambda_1(p, \xi) \) is defined by
\[
\Lambda(p, \xi) := \kappa \frac{\partial \xi|_{\xi=p} Z(\xi, \xi)}{Z(p, \xi)}, \quad Z(p, \xi) := C(b)[\frac{1}{2} \alpha, \frac{1}{2} \alpha].
\]

We remark that if all the marked points \( \xi_j \)'s are on \( \mathbb{R} \), one can simply take \( Z(p, \xi) = C(b)[\alpha, 0] \).

By the BPZ-Cardy type equations (Proposition 5.7), we construct a family of SLE martingale observables.

**Theorem C** For any string \( \mathcal{X} \) of fields in the OPE family \( \mathcal{F}_\alpha \), a non-random field
\[
M = \mathbb{E}[\mathcal{X}]
\]

is a martingale-observable for \( \text{SLE}(\kappa, \Lambda) \).

**Example (Bosonic observables)** Let \( M(z) := \mathbb{E}\Phi_\alpha(z) \). Then \( M \) is evaluated as
\[
M(z) = -2b \arg w' + 2a \arg \Theta_\chi(r, w_z) + \sum_k \tau_k \arg \left\{ \Theta_\chi(r, w_z - w_k) \Theta_\chi(w_z - \bar{w}_k) \right\}.
\]

In particular when \( \chi = 0 \), the harmonic function \( M \) satisfies piecewise Dirichlet boundary conditions having additional jump of \( 2\pi \alpha \) at \( p \) and \( 2\pi \tau_k \) at \( \xi_k \)'s.

For \( b = 0 \) (thus when \( \kappa = 4 \)) the associated bounded martingale plays an important role in the study of scaling limits of discrete GFF interface [62] and GFF/SLE couplings [20, 34]. Further applications of such observables are presented in Propositions 5.8 and 7.4 in the context of left passage probability and hitting probability of SLE traces.

**Example (Vertex observables)** Let \( \bar{\tau} = \sum \bar{\tau}_j \cdot \bar{\xi}_j \) be a divisor satisfying \( a + \sum \bar{\tau}_j = 0 \) and write \( \bar{\alpha} := a \cdot p + \bar{\tau} \). Consider the vertex observable
\[
M = \mathbb{E}\mathcal{O}_{\bar{\alpha}}[\bar{\alpha} - \alpha] = \frac{C(b)[\frac{1}{2} \bar{\alpha}, \frac{1}{2} \bar{\alpha}]}{C(b)[\frac{1}{2} \alpha, \frac{1}{2} \alpha]}.
\]

This martingale-observable provides the Radon-Nikodym derivative between the law of two annulus SLEs associated with partition functions \( C(b)[\frac{1}{2} \alpha, \frac{1}{2} \alpha] \) and \( C(b)[\frac{1}{2} \bar{\alpha}, \frac{1}{2} \bar{\alpha}] \). Similar martingale-observables also appear in the study of restriction properties, see e.g., [17, 44].

In the physics literature, the construction of martingale-observables is often presented in terms of the one-leg operator
\[
\Psi(z, \xi) := \mathcal{O}(b)[\frac{1}{2} \alpha, \frac{1}{2} \alpha].
\]
The insertion of $\Psi_1$ produces an operator that changes the values of Fock space fields. See [36, Sect. 2.3] for its probabilistic counterpart: insertion of Wick’s exponential $e^{\odot \Phi(f)}$ of the GFF in application to a test function $f$ results in the change of the law of $\Phi(f)$ in terms of the Green potential. We now consider the following insertion procedure:

$$\hat{E}[\mathcal{X}] := \frac{E\Psi(p, \xi)\mathcal{X}}{Z(p, \xi)}.$$  

Then Theorem C can be restated as follows: for any string $\mathcal{X}$ in the OPE family $\mathcal{F}(b)$, a non-random field $M = \hat{E}[\mathcal{X}]$ is a martingale-observable for SLE($\kappa$, $\Lambda$). Based on the spirit of such an insertion procedure, we construct martingale-observables for further classes of annulus SLEs.

We emphasize that except for the ER case when $\chi = \infty$, the drift function $\Lambda$ is not $2\pi$-periodic with respect to space variables. To construct martingale-observables for annulus SLE with $2\pi$-periodic drift function, we use the weighted summation of the chiral fields, see [66] for a similar idea on compact Riemann surfaces. (Cf. see also [74] for implementation of this idea to the annulus SLE partition functions.) To be more precise, for a weight function $\omega: \mathbb{Z} \to \mathbb{R}_+$, set

$$\Psi_\omega(p, \xi) := \sum_{n \in \mathbb{Z}} \omega(n) \Psi(p + 2n\pi, \xi).$$

Here we assume that $\omega$ is suitably chosen so that the summation above converges within correlations. For instance, when $\chi < \infty$, we allow any weight function $\omega$ of polynomial type. We then define

$$Z_\omega(p, \xi) := E\Psi_\omega(p, \xi), \quad \Lambda_\omega(p, \xi) := \kappa \partial_\xi |_{\xi = p} \log Z_\omega(\xi, \xi).$$

Extending Theorem C, we obtain the following.

**Theorem D** For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}(b)$, a non-random field

$$M = \hat{E}[\mathcal{X}] := \frac{E\Psi_\omega(p, \xi)\mathcal{X}}{Z_\omega(p, \xi)}$$

is a martingale-observable for SLE($\kappa$, $\Lambda_\omega$).

**Example** Let us consider the case $\tau = -a \cdot q$ for $q \in \mathbb{R}_r$ and $\omega(n) \equiv 1$. Then up to a multiplicative constant, the partition functions $Z = Z(p, \xi)$ and $Z_\omega = Z_\omega(p, \xi)$ are given by ($x = p - \text{Re} \, q$)

$$Z(x) = \Theta_I(r, x)^{\frac{1}{2}} \exp \left( \frac{x^2}{2\kappa(r + \chi)} \right), \quad Z_\omega(x) = \Theta_I(r, x)^{\frac{1}{2}} \Theta_I \left( \frac{\kappa}{2} (r + \chi), x + \pi \right).$$
where

\[ \Theta_I(r, z) = i \Theta(r, z - ir) \exp \left( -\frac{r}{4} - \frac{iz}{2} \right). \]

This leads to

\[ \Lambda(x) = H_I(r, x) + \frac{x}{r + \chi}, \quad \Lambda^\omega(x) = H_I(r, x) + \frac{\kappa}{2} H_I \left( \frac{\kappa}{2} (r + \chi), x + \pi \right), \]

where \( H_I(r, z) := 2 \partial_z \log \Theta_I(r, z) \). Notice here that \( \Lambda^\omega \) is \( 2\pi \)-periodic, whereas \( \Lambda \) is not.

In particular when \( \chi = 0 \), it can be shown that the associated SLE\((\kappa, \Lambda^\omega)\) trace ends at \( \{ q + 2n\pi : n \in \mathbb{Z} \} \). Then by virtue of vertex observables in the previous example, one can see that for each \( m \in \mathbb{Z} \),

\[ M_m(x) := \frac{Z(x - 2m\pi)}{Z^\omega(x)} = \exp \left( \frac{(x - 2m\pi)^2}{2\kappa r} \right) \Theta_I \left( \frac{\kappa r}{2}, x + \pi \right)^{-1} \]

gives rise to the probability that SLE\((\kappa, \Lambda^\omega)\) trace ends at \( q + 2m\pi \).

We now focus on the chordal type annulus SLE\((\kappa, \Lambda)\) with one force point \( q \) starting from \( p \), where both of the marked points lie on the same boundary component. It was shown in [74] that in order for the SLE process to be reversible, the associated partition function \( Z \) should satisfy the null-vector equation

\[ \partial_r Z = \frac{\kappa}{2} Z'' + H Z' + h_{1,2}(\kappa) H' Z + C(r) Z, \quad h_{1,2}(\kappa) := \frac{6 - \kappa}{2\kappa}, \quad (2.1) \]

where \( C(r) \) is a constant depending only on the modular parameter \( r \).

To implement CFT for such chordal type annulus SLE, we use the method of screening. Due to the local commutations of annulus SLE\((\kappa, \Lambda)\), the partition function should have the same conformal dimensions at \( p \) and \( q \). In the case of chordal SLE in the upper-half plane, it can be achieved by the effective one-leg operator with the total charge \( 2b - a \) (including the background charge \( 2b \)) placed at \( q \) and with the charge \( a \) placed at \( p \). For more details we refer the reader to [35]. However, this choice of partition function cannot be realized in the annulus because the neutrality conditions depend on the connectivity. See [37] for the neutrality conditions for background charges on a compact Riemann surface of genus \( g \). Instead we consider

\[ \Psi_\beta(p, q) := C(\kappa) \int_{\mathcal{P}(p,q)} \mathcal{O}_b[a \cdot p + a \cdot q + s \cdot \zeta + \beta, \, 0] \, d\zeta, \quad (s = -2a), \]

where \( \mathcal{P}(p,q) \) is the Pochhammer contour entwining \( p \) and \( q \), see Fig. 2. Here \( C(\kappa) \) is a universal constant depending only on \( \kappa \) and the background charge is given as

\[ \beta := \beta \cdot q_1 - \beta \cdot q_2, \quad (q_2 - q_1 = 2\pi) \]
for some $\beta \in \mathbb{R}$.

With this choice of $s = -2a$, the modified multi-vertex field $O(b)[a \cdot p + a \cdot q + s \cdot \zeta + \beta, 0]$ is a 1-differential with respect to the “screening” variable $\zeta$ and it satisfies the neutrality condition (NC0). Clearly, $\Psi_{\beta}$ has the same conformal dimensions both at $p$ and $q$. For each $\kappa > 0$, we have the following expression (up to a multiplicative constant) of $Z_{\beta} \equiv Z_{\beta}(r, p - q) := E\Psi_{\beta}(p, q)$ in terms of Jacobi’s theta function:

$$Z_{\beta} = \Theta(p - q)^{r/2} \oint_{P(p, q)} \Theta(p - \zeta)^{-2} \Theta(\zeta - q)^{-2} \exp \left( -\frac{(\Sigma - 2\beta\pi)^{2}}{4(r + \chi)} \right) d\zeta,$$

where $\Sigma = a(p + q - 2\zeta)$. We remark that for $\kappa > 4$, this Pochhammer contour integral simplifies to the integration over the interval between $p$ and $q$. It can be shown by the contour deformation method as we explain in Sect. 6.3. With this choice of the one-leg operator $\Psi_{\beta}(p, q)$, we construct a family of martingale-observables for chordal type SLE$(\kappa, \Lambda_{\beta})$, where $\Lambda_{\beta} = \kappa(\log Z_{\beta})'$.

**Theorem E** For each $\kappa > 0$, the partition function $Z_{\beta} := E\Psi_{\beta}(\cdot, q)$ is a non-trivial (real-valued) solution of the null-vector Eq. (2.1). Moreover, for any string $\mathcal{X}$ of fields in the OPE family $F(b)$, a non-random field

$$M = \hat{E}\mathcal{X} := \frac{E\Psi_{\beta}(p, q)\mathcal{X}}{E\Psi_{\beta}(p, q)}$$

is a martingale-observable for chordal type SLE$(\kappa, \Lambda_{\beta})$.

### 2.4 Organization of the paper and further results

The rest of this paper is organized as follows.

In Sect. 3 we review some standard notions in CFT and describe the conformal Fock space fields generated by central charge modifications of GFF. For the convenience of the reader we borrow the relevant materials from [36] without proofs, thus making our exposition self-contained.

Section 4 is devoted to the study of Eguchi–Ooguri and Ward’s equations (Theorem A).

In Sect. 5 we show the BPZ equations (Theorem B). After relating CFT to SLE theory, we then show the BPZ-Cardy equations (Proposition 5.7) and complete the proof of Theorem C.
In Sect. 6 we present several ways to extend the class of SLE martingale observables and prove Theorems D and E. Moreover, the implementation of non-atomic background charge is explained in Sect. 6.1.

In Sect. 7 based on the theories developed in the previous sections, further examples of SLE martingale-observables are indicated, including those used in the study of continuum limits of discrete models.

In “Appendix A” we present representations of various Green’s functions in terms of special functions.

3 Conformal Fock space fields

We introduce a class of random fields in a doubly connected domain as correlation functional valued maps. All fields that we consider in this paper are constructed from the (modifications of) GFF and their derivatives by means of Wick’s calculus and named after Fock space. We consider Dirichlet and ER boundary conditions for the GFF and their interpolations, see Sect. 3.2 for more details. This subsection also recalls some basic concepts and properties concerning the Fock space fields and OPE products. In Sect. 3.3 we revise the definition of Fock space fields so that their values depend on local coordinates. In Sect. 3.4 we introduce the central charge modifications of GFF and present the associated Virasoro field. Based on the Coulomb gas formalism introduced in the following subsections, in Sect. 3.7 we present a more general theory of background charge modifications of GFF.

3.1 Special functions

In this subsection we compile basic properties of some special functions used to represent the Loewner vector field and the correlation functions of Fock space fields.

For each \( r > 0 \), the Loewner kernel \( H(r, \cdot) \) in \( \mathbb{S}_r \) is given by

\[
H(r, z) := -i \lim_{n \to \infty} \sum_{n \in 2\mathbb{Z}} e^{nr + e^{iz}}.
\]

We also write

\[
H_I(r, z) := H(r, z + ir) + i.
\]

We refer to [74] for fundamental properties of the Loewner kernel \( H(r, \cdot) \). For reader’s convenience, we list some of them as follows:

- a meromorphic function \( H(r, \cdot) \) on \( \mathbb{C} \) has simple poles at \( 2\pi m + 2i rn, (m, n \in \mathbb{Z}) \) with residue 2;
- \( H(r, z + 2\pi) = H(r, z) \) and \( H(r, z + 2ir) = H(r, z) - 2i \);
- \( H(r, \cdot) \) is an odd function;
- \( H(r, \cdot) \) takes real values on \( \mathbb{R} \setminus \{2\pi n : n \in \mathbb{Z}\} \) and \( \text{Im} \ H(r, \cdot) \equiv -1 \) on \( \mathbb{R}_r \);
- \( H(r, \pi) = 0, H(r, ir) = -i, \) and \( H(r, \pi + ir) = -i \).
A holomorphic function

\[
\Theta(r, z) := \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n e^{-r(n+\frac{1}{2})^2} e^{(n+\frac{1}{2})iz}
\]

vanishes only on \(2\pi \mathbb{Z} + 2ir\mathbb{Z}\) and

\[
\Theta_I(r, z) := \sum_{n=-\infty}^{\infty} (-1)^n e^{-rn^2} e^{niz}
\]

is related to \(\Theta\) as

\[
\Theta_I(r, z) = i \Theta(r, z - ir) \exp \left( -\frac{r}{4} - \frac{iz}{2} \right).
\]

Notice that \(\Theta(r, \cdot)\) is odd, whereas \(\Theta_I(r, \cdot)\) is even. The functions \(H\) and \(H_I\) are represented in terms of \(\Theta\) and \(\Theta_I\) as

\[
H(r, z) = 2 \frac{\Theta'(r, z)}{\Theta(r, z)}, \quad H_I(r, z) = 2 \frac{\Theta'_I(r, z)}{\Theta_I(r, z)},
\]

where derivatives are taken with respect to the \(z\)-variable. We remark that \(\Theta\) and \(\Theta_I\) are related to the classical Jacobi theta functions

\[
\theta(z, \tau) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n (n+\frac{1}{2})^2} e^{(2n+1)\pi i z}, \quad \theta_2(z, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2} e^{2n\pi i z}
\]

as

\[
\Theta(r, z) = \theta \left( \frac{z}{2\pi}, \frac{ir}{\pi} \right), \quad \Theta_I(r, z) = \theta_2 \left( \frac{z}{2\pi}, \frac{ir}{\pi} \right).
\]

It is well known that \(\Theta\) and \(\Theta_I\) solve the heat equations

\[
\partial_r \Theta = \Theta'', \quad \partial_r \Theta_I = \Theta''_I.
\]

Moreover, they are represented in terms of fundamental solutions of the heat equation as

\[
\Theta(r, z) = \sqrt{\frac{\pi}{r}} \sum_{n \in \mathbb{Z}} (-1)^n \exp \left( - \frac{(z - \pi + 2n\pi)^2}{4r} \right),
\]

\[
\Theta_I(r, z) = \sqrt{\frac{\pi}{r}} \sum_{n \in \mathbb{Z}} \exp \left( - \frac{(z - \pi + 2n\pi)^2}{4r} \right),
\]
see e.g., [54, Eqs. (20.2.13), (20.13.4)].

To describe correlation functions with weighted boundary conditions indexed by \( \chi \in [0, \infty) \), it is convenient to introduce the following theta function

\[
\Theta_\chi(r, z) := \Theta(r, z) \exp\left(\frac{z^2}{4(r + \chi)}\right).
\]

In the sequel, we also write

\[
H_\chi(r, z) := 2 \log \Theta'_\chi(r, z) = H(r, z) + \frac{z}{r + \chi}.
\]

In particular, we denote \( \tilde{\Theta} := \Theta_0 \) and \( \tilde{H} := H_0 \). It is worth pointing out that \( \tilde{\Theta}(r, z) \in i \mathbb{R}_+ \) if \( z \in \mathbb{R}_r \). Note also that the function \( H_\chi(r, \cdot) \) is real-valued on \( \mathbb{R}_r \) if and only if \( \chi = 0 \). Throughout this paper, we will frequently use the following (quasi) periodicities of theta functions

\[
\Theta(r, z + 2\pi) = -\Theta(r, z), \quad \Theta(r, z + 2ir) = -\Theta(r, z) \exp(r - iz), \tag{3.4}
\]

\[
\Theta_I(r, z + 2\pi) = \Theta_I(r, z), \quad \Theta_I(r, z + 2ir) = -\Theta_I(r, z) \exp(r - iz), \tag{3.5}
\]

The Weierstrass zeta function \( \zeta_r \) with basic periods \((2\pi, 2ir)\)

\[
\zeta_r(z) := \frac{1}{z} + \sum_\eta \eta^* \left( \frac{1}{z - \eta} + \frac{1}{\eta^2} \right) \tag{3.6}
\]

is holomorphic except for the points \( z \in 2\pi \mathbb{Z} + 2ir \mathbb{Z} \), where it has simple poles with residue 1. Here, the sum \( \sum_\eta^* \) in \eqref{eq:zeta} is taken over all \( \eta \) of the form \( 2\pi m + 2irn \) for \((m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \). It is well known that \( \zeta_r(\pi) / \pi \) is represented as

\[
\frac{\zeta_r(\pi)}{\pi} = -\frac{1}{3} \frac{\Theta'''(r, 0)}{\Theta'(r, 0)} = -\frac{1}{3} \partial_r \log \Theta'(r, 0) = \frac{1}{12} - \frac{1}{2} \sum_{k=1}^{\infty} \sinh^{-2}(kr). \tag{3.7}
\]

This value will appear in the formula of the correlation function of the Virasoro field (Proposition 3.2).

The Loewner kernel \( H \) is represented in terms of the Weierstrass zeta function:

\[
H(r, z) = 2 \zeta_r(z) - \frac{2}{\pi} \zeta_r(\pi) z. \tag{3.8}
\]

In particular, we have the following asymptotic behavior of \( H \):

\[
H(r, z) = \frac{2}{z} - \frac{2\zeta_r(\pi)}{\pi} z + O(z^3), \quad \text{as } z \to 0. \tag{3.9}
\]
The following pseudo-addition formula (see [9, p. 57]) for zeta function is a crucial ingredient to derive a version of Eguchi–Ooguri equations (Proposition 4.2).

**Proposition 3.1** If \( x + y + z = 0 \), then

\[
(\zeta_r(x) + \zeta_r(y) + \zeta_r(z))^2 + \zeta'_r(x) + \zeta'_r(y) + \zeta'_r(z) = 0.
\]

(3.10)

By (3.1) and (3.8), one can rewrite (3.10) as follows:

\[
H(z - w)(H(z) - H(w)) = \frac{1}{2}H^2(z - w) + \frac{1}{2}(H(z) - H(w))^2 \\
+ H'(z - w) + H'(z) + H'(w) + \frac{6}{\pi}\zeta_r(\pi).
\]

(3.11)

### 3.2 Correlation functions of Fock space fields

In this subsection we introduce a one-parameter family of GFFs with a weighted combination of Dirichlet and ER boundary conditions in a doubly connected domain and recall some basic concepts in CFT such as Fock space fields and OPE products. As an expansion of the tensor product of two Fock space fields near diagonal, operator product expansion (OPE) gives rise to important binary operations, namely OPE multiplications or OPE products on Fock space fields.

#### 3.2.1 One-parameter family of Gaussian free fields

An excursion reflected Brownian motion (ERBM) \( B^E_{Dr} \) in a doubly connected domain \( D_r (\infty \notin D_r) \) is a strong Markov process stopped at the outer boundary of \( D_r \) that acts like a Brownian motion when it is away from \( \partial D_r \). Also it reflects on the inner boundary of \( D_r \) at a random point chosen according to the harmonic measure from \( \infty \) whenever it hits the inner boundary. It is well known that ERBM is conformally invariant and so is the Green’s function for ERBM. See [15, 46] for more details. We also remark that ERBM is a special case of a more general process called Brownian motion with darning [11], which can be constructed using Dirichlet form theory. See also [7] for another generalization of ERBM, called obliquely reflected Brownian motion.

The Green’s function \( G^{Diri}_{D_r} \) in a doubly connected domain \( D_r \) with zero Dirichlet boundary condition (b.c.) vanishes on \( \partial D_r \). In the identity chart of the annulus \( \mathbb{A}_r \) we have

\[
G^{Diri}_{\mathbb{A}_r}(\zeta, z) = -\frac{\log |\zeta| \log |z|}{r} - \log \left| \sum_{k=-\infty}^{\infty} (-1)^k e^{-(k^2-k)r} (\zeta/z)^k \right|.
\]
see e.g., [31, p. 262]. The Green’s function $G_{\partial_\alpha}^{ER}$ for ERBM in the annulus $\mathbb{A}_r$ is characterized by

$$\mathbb{E}_\xi \left[ \int_0^{\tau_{\partial_\alpha}} 1_E(B_{\partial_\alpha}^{ER})(t) \, dt \right] = \int_E G_{\partial_\alpha}^{ER}(\xi, z) \, dz$$

for any Borel subset $E \subset \mathbb{A}_r$, where $\tau_{\partial_\alpha} := \inf\{ t : B_{\partial_\alpha}^{ER}(t) \in \mathbb{T} \}$. Then by [15, Proposition 5.2 and Lemma 5.5], we have

$$G_{\partial_\alpha}^{ER}(\xi, z) = -\log \left| z \sum_{k=-\infty}^{\infty} (-1)^k e^{-(k^2-k)r} (\xi/z)^k \right| \sum_{k=-\infty}^{\infty} (-1)^k e^{-(k^2-k)r} (\bar{\xi}/z)^k.$$

In the cylinder $C_r$, such Green’s functions $G_r \equiv G_{C_r}$ are represented in terms of Jacobi theta function as

$$G_r(\xi, z) = \begin{cases} \log \left| \frac{\Theta(r, \xi - \bar{z})}{\Theta(r, \xi - z)} \right| & \text{for ER b.c.} \\ \log \left| \frac{\Theta(r, \xi - \bar{z})}{\Theta(r, \xi - z)} \right| - \frac{\Im \xi \Im z}{r} & \text{for Dirichlet b.c.} \end{cases}$$

From the analytic point of view, the Green’s function $G_r$ with ER b.c. satisfies the zero period condition on the inner boundary component $\gamma$. Therefore we have that for $z \in \gamma$,

$$\oint_\gamma \frac{\partial G_r}{\partial n_z} \, ds_z = 0 \quad \text{for ER b.c.}$$

$$G_r(\xi, z) = 0 \quad \text{for Dirichlet b.c.} \quad (3.12)$$

Here $\partial n_z$ denotes the normal derivative and $ds_z$ is the arc length measure. Extending (3.12), we introduce the weighted combination of Dirichlet and ER boundary conditions: for $\chi \in [0, \infty]$,

$$\frac{1}{1+\chi} G_r(\xi, z) + \frac{\chi}{1+\chi} \left( \frac{1}{2\pi} \oint_\gamma \frac{\partial G_r}{\partial n_z} \, ds_z \right) = 0, \quad (3.13)$$

where $z \in \gamma$. One may realize (3.13) as an analog of the well-known Robin boundary condition, in which the Neumann condition is replaced by the ER one. (See Remark in Sect. 4.2 for a field theoretic motivation of such a weighted boundary condition.) It is then straightforward to see that the Green’s function

$$G_\chi(\xi, z) := \log \left| \frac{\Theta_\chi(r, \xi - \bar{z})}{\Theta_\chi(r, \xi - z)} \right| = \log \left| \frac{\Theta(r, \xi - \bar{z})}{\Theta(r, \xi - z)} \right| - \frac{\Im \xi \Im z}{r + \chi} \quad (3.14)$$

satisfies the boundary condition (3.13).
We denote by $\Phi$ the GFF with weighted boundary condition (3.13), see Fig. 1. By definition, the 2-point correlation function of GFF $\Phi$ in $D_r$ is given by

$$E[\Phi(\zeta)\Phi(z)] := 2 G_{D_r}^X(\zeta, z) = 2 \log \frac{\Theta_X(r, w(\zeta) - w(z))}{\Theta_X(r, w(\zeta) - w(z))},$$

(3.15)

where $w$ is a conformal map from $D_r$ onto $C_r$. In general, the $n$-point correlation function of $\Phi$ is given by

$$E[\Phi(z_1) \cdots \Phi(z_n)] = \sum \prod_{k} 2G_{D_r}^X(z_{i_k}, z_{j_k}),$$

(3.16)

where the sum is taken over all partitions of the set $\{1, \ldots, n\}$ into disjoint pairs $\{i_k, j_k\}$.

The current field $J = \partial \Phi$, its conjugate $\bar{J} = \bar{\partial} \Phi$, and higher order derivatives of $\Phi$ are distributional fields (random generalized functions) and their correlators are computed by differentiating those of GFF. For example, for $\zeta \neq z$, in the identity chart of $C_r$, we have

$$E[J(\zeta)\Phi(z)] = \frac{1}{2}(H_X(r, \zeta - \bar{z}) - H_X(r, \zeta - z)).$$

(3.17)

### 3.2.2 Fock space fields

We treat the GFF as a Gaussian (correlation) functional-valued function and view its correlation function as the kernel representing the expectation of the value of GFF on test functions. We also treat the derivatives of GFF as centered (mean zero) Gaussian generalized functions and include Wick’s product of derivatives of GFF in a collection of fields we consider.

By definition, a basic Fock space correlation functional is a (formal form of) Wick’s product $X = X_1(z_1) \circ \cdots \circ X_n(z_n)$ of derivatives $X_j$ of the GFF, and it has zero correlation, i.e., $E X = 0$. Here, points $z_j \in D_r$ are not necessarily distinct and called the nodes of $X$. We write $S_X \equiv S(X)$ for the set of nodes of $X$.

For basic functionals $X_j$ of the form $X_j = X_{j1}(z_{j1}) \circ \cdots \circ X_{jn_j}(z_{jn_j})$ with pairwise disjoint $S(X_j)$ we define the tensor product $X = X_1 \cdots X_m$ by Wick’s formula

$$X_1 \cdots X_m = \sum \prod_{\{v, v’\}} E[X_v(z_v)X_{v’}(z_{v’})] \bigcirc X_{v’’}(z_{v’’}).$$

(3.18)

The sum in the right-hand side of (3.18) is taken over all graphs (Feynman diagrams) with vertices $v$ labeled by functionals $X_{jk}$ such that there are no edges (Wick’s contractions) of vertices with the same $j$, and the Wick’s products are over unpaired vertices.
\[ v"'. \] For example, the Feynman diagram with edges \{1,4\}, \{3,5\} and unpaired vertices 2, 6 corresponds to

\[
(\Phi(z_1) \odot \Phi(z_2) \odot \Phi(z_3))(\Phi(z_4) \odot \Phi(z_5) \odot \Phi(z_6)) = E[\Phi(z_1)\Phi(z_4)]E[\Phi(z_3)\Phi(z_5)]\Phi(z_2) \odot \Phi(z_6). 
\]

The tensor product of correlation functionals is commutative and associative, see [36, Proposition 1.1]. We identify two functionals \(X_1\) and \(X_2\) if \(E[ X_1 Y] = E[ X_2 Y]\) holds for all functionals \(Y\) with \(S_Y \cap (S_{X_1} \cup S_{X_2}) = \emptyset\). The complex conjugation \(\overline{X}\) of \(X\) is a unique correlation functional (modulo trivial functionals) such that \(E[ XY] = E[ \overline{X} Y]\) for all \(Y\)'s of the form \(\Phi(z_1) \odot \cdots \odot \Phi(z_n)\).

### 3.2.3 OPE calculations

The operator product expansion (OPE) of two Fock space fields \(X\) and \(Y\) is an asymptotic expansion of \(X(\zeta)Y(z)\) near diagonal. In particular, if a field \(X\) is holomorphic (i.e., \(\bar{\partial} X = 0\) within correlation), the OPE is defined as a formal Laurent series expansion

\[
X(\zeta)Y(z) = \sum C_n(z)(\zeta - z)^n, \quad \text{as } \zeta \to z. 
\]

More precisely, it means that \(E X(\zeta)Y(z)\mathcal{X} = E \sum C_n(z)(\zeta - z)^n\mathcal{X}\) for each Fock space correlation functional \(\mathcal{X}\) for \(\zeta, z \notin S_X\). In this case, \(*_n\) product is defined as \(X *_n Y := C_n\), the \(n\)-th OPE coefficient. In particular, we write \(*\) for \(*_0\) and call \(X * Y\) the OPE multiplication, or the OPE product of \(X\) and \(Y\). We remark that the \(*_n\) product is neither associative nor commutative. On the other hand, it satisfies Leibniz’s rule,

\[
\partial(X *_n Y) = \partial X *_n Y + X *_n \partial Y. 
\]

We now present some examples of OPEs. As \(\zeta \to z, (\zeta \neq z)\)

\[
\Phi(\zeta)\Phi(z) = \log \frac{1}{|\zeta - z|^2} + 2c(z) + \Phi^{\odot 2}(z) + o(1), \quad (3.19) 
\]

where \(c(z) := u(z, z)\) and \(u(\zeta, z) := G^X_{Dr}(\zeta, z) + \log |\zeta - z|\). By definition, for \(z \in Dr, \Phi * \Phi(z) = 2c(z) + \Phi^{\odot 2}(z)\), where \(c(z)\) is evaluated as

\[
c(z) = \log \left| \frac{\Theta_{\mathcal{X}}(w(z) - \overline{w(z)})}{w'(z)\Theta_{\mathcal{X}}'(0)} \right|. \quad (3.20) 
\]

We remark that the function \(c(z)\) is called the conformal radius for the ER b.c., whereas it is called the domain constant for the Dirichlet b.c., see e.g., [5, Sect. 4].
Differentiating (3.19), we have

\[
J(\zeta)\Phi(z) = -\frac{1}{\zeta - z} + (J \odot \Phi)(z) + \frac{w'(z)}{2} H_\lambda(w(z) - w(z)) - \frac{w''(z)}{2w'(z)} + o(1),
\]

(3.21)

\[
J(\zeta)J(z) = -\frac{1}{(\zeta - z)^2} + (J \odot J)(z) - \frac{1}{6} S_w(z)
- w'(z)^2 \left( \frac{\zeta_r(\pi)}{\pi} - \frac{1}{2(r + \chi)} \right) + o(1).
\]

(3.22)

Here, \( S_h \) is the Schwarzian derivative \( S_h \) of a conformal map \( h \), i.e.,

\[
S_h := N'_h - \frac{1}{2} N_h^2, \quad N_h = \frac{h''}{h'}.
\]

We sometimes use the notation \( \sim \) for the singular part of the OPE, namely,

\[
X(\zeta)Y(z) \sim \sum_{n < 0} C_n(z)(\zeta - z)^n.
\]

For instance, we have

\[
J(\zeta)\Phi(z) \sim -\frac{1}{\zeta - z}, \quad J(\zeta)J(z) \sim -\frac{1}{(\zeta - z)^2}.
\]

(3.23)

### 3.3 Conformal Fock space fields

We treat a stress tensor as a Lie derivative operator to state Ward’s identities. For this purpose, it is needed to consider Fock space fields in a planar domain (e.g., a doubly connected domain) as defined on a Riemann surface. The correlation functions of conformal Fock space fields depend on the choice of local charts at their nodes.

By definition, a non-random conformal field \( f \) on a Riemann surface is a smooth function \( (f \| \phi) : \phi U \to \mathbb{C} \) for each local chart \( \phi \). We define a general conformal Fock space field as \( X = \sum_\alpha f_\alpha X_\alpha \), where \( f_\alpha \)'s are non-random conformal fields and basic fields \( X_\alpha \) are Wick’s products of derivatives of GFF.

We now list some basic non-random fields \( f \) with specific transformation laws between \( f = (f \| \phi) \) and \( \tilde{f} = (f \| \tilde{\phi}) \) for any two overlapping charts \( \phi \) and \( \tilde{\phi} \). A non-random field \( f \) is called a differential of conformal dimension \( (\lambda, \lambda_\ast) \) if

\[
f = (h')^\lambda (\tilde{h'})^{\lambda_\ast} \tilde{f} \circ h,
\]

where \( h \) is the transition map between \( \phi \) and \( \tilde{\phi} \), i.e., \( h = \tilde{\phi} \circ \phi^{-1} : \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U}) \). By definition, pre-pre-Schwarzian forms (PPS-forms), pre-Schwarzian forms (PS-forms) and Schwazian forms (S-forms) of order \( \mu \) are conformal Fock
space fields with transformation laws
\[ f = \tilde{f} \circ h + \mu \log h', \quad f = h' \tilde{f} \circ h + \mu N_h, \quad f = (h')^2 \tilde{f} \circ h + \mu S_h \]
respectively. In general, PPS-forms of order \((\mu, \nu)\) satisfy
\[ f = \tilde{f} \circ h + \mu \log h' + \nu \log h'. \]

3.3.1 Lie derivative of conformal Fock space field

Suppose a non-random smooth vector field \(v\) is holomorphic in some open set \(U \subset M\). For a conformal Fock space field \(X\), the Lie derivative \(L_v X\) in \(U\) is defined as
\[ \frac{d}{dt} \bigg|_{t=0} \left( X\| \phi \circ \psi_t \right), \]
where \(\psi_t\) is a local flow of \(v\), and \(\phi\) is a given chart. For example, we have the followings:
- \(L_v X = (v\partial + \overline{v}\partial + \lambda v' + \lambda_n \overline{v}') X\) for a \((\lambda, \lambda_n)\)-differential \(X\);
- \(L_v X = (v\partial + v') X + \mu v''\) for a PS-form \(X\) of order \(\mu\);
- \(L_v X = (v\partial + 2v') X + \mu v'''\) for an S-form \(X\) of order \(\mu\),

see [36, Proposition 4.1]. For reader’s convenience, we list some basic properties of the Lie derivatives:
- \(E[L_v X] = L_v E[X]\);
- \(L_v (X) = (L_v X)\);
- \(L_v (\partial X) = \partial (L_v X)\);
- Leibniz’s rule holds for tensor, Wick’s, and OPE products.

We define the \(\mathbb{C}\)-linear part \(L_v^+\) and the anti-\(\mathbb{C}\)-linear part \(L_v^-\) of \(L_v\) by
\[ L_v^+ := \frac{L_v - i L_{iv}}{2}, \quad L_v^- := \frac{L_v + i L_{iv}}{2} \]
so that \(L_v\) is decomposed as \(L_v = L_v^+ + L_v^-\). Note that while \(L_v\) depends \(\mathbb{R}\)-linearly on \(v\), \(L_v^+\) (resp., \(L_v^-\)) depends (resp., anti-) \(\mathbb{C}\)-linearly on \(v\).

3.3.2 Stress energy tensor

Here we recall the definition of a stress tensor and review its basic properties. See [36, Sects. 5.2–5.3, Lecture 7, and “Appendix 11’’) for more details.

A stress tensor for a family of conformal Fock space fields is defined as a pair of holomorphic and anti-holomorphic quadratic differentials which represent the Lie derivative operators within correlations of fields in this family. The existence of a stress tensor is a remarkable property of certain families of conformal Fock space fields. For instance, the OPE family of GFFs has a stress tensor in common.
A Fock space field $X$ in $D_r$ is said to have a stress tensor $(A, \tilde{A})$ (or $X \in \mathcal{F}(A, \tilde{A})$) if $A$ is a holomorphic quadratic differential and the residue form of Ward’s identity

$$L^+_v X(z) = \frac{1}{2\pi i} \oint (z) v AX(z), \quad L^-_v X(z) = -\frac{1}{2\pi i} \oint (z) \tilde{v} \tilde{A} X(z)$$

holds on $D_{\text{hol}}(v) \cap U$ for all smooth vector field $v$, where $D_{\text{hol}}(v)$ is the maximal open set where $v$ is holomorphic. By [36, Proposition 5.8], $\mathcal{F}(A, \tilde{A})$ is closed under $\ast_n$ product, in particular, under differentiations. If $X$ is a differential or a form, the residue form of Ward’s identity holds for $X$ if and only if Ward’s OPE for $X$ holds in some or every chart. For example, a $(\lambda, \lambda_*)$-differential $X$ is in $\mathcal{F}(A, \tilde{A})$ if and only if the following singular OPEs hold in every/some chart:

$$A(\zeta) X(z) \sim \frac{\lambda X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z}; \quad A(\zeta) \tilde{X}(z) \sim \frac{\lambda_* \tilde{X}(z)}{(\zeta - z)^2} + \frac{\partial \tilde{X}(z)}{\zeta - z}. \tag{3.24}$$

For any form $X$ of order $\mu$ belongs to $\mathcal{F}(A, \tilde{A})$ if and only if the following singular OPE holds in every/some chart:

$$A(\zeta) X(z) \sim \frac{\mu}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \quad \text{for a PPS-form } X;$$

$$A(\zeta) X(z) \sim \frac{2\mu}{(\zeta - z)^3} + \frac{X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \quad \text{for a PS-form } X;$$

$$A(\zeta) X(z) \sim \frac{6\mu}{(\zeta - z)^4} + \frac{2X(z)}{(\zeta - z)^2} + \frac{\partial X(z)}{\zeta - z} \quad \text{for an S-form } X. \tag{3.25}$$

### 3.4 Central charge modification

In this subsection we introduce the central charge modification $\Phi_{(b)}$ of GFF, where $b$ is the real parameter related to the central charge $c$ as $c = 1 - 12b^2$. In the chordal/radial CFT in a simply connected domain with a marked boundary/interior point $q$, two neutrality conditions are imposed in the construction of OPE exponentials. They have Wick’s part and the correlation part; Wick’s part is Wick’s exponential of the linear combination of bosonic fields, and the correlation part can be expressed in terms of Coulomb gas correlation functions. We require that this linear combination of bosonic fields should be a well-defined Fock space field. This requirement is called a neutrality condition on the coefficients or charges of linear combination. At the same time, conformal invariance of Coulomb gas correlation functions in a simply connected domain gives rise to the other neutrality condition. Because the neutrality conditions are different, we need to place the background charge at $q$ to reconcile these two neutrality conditions. In a doubly connected domain (or on a compact Riemann surface of genus one), no background charge is need to be placed for $\Phi_{(b)}$. It is consistent with the neutrality condition (NC0) that the sum of background charges should vanish due to a version of Gauss–Bonnet theorem. In Sect. 3.7 we introduce the background charge modification $\Phi_\beta$ of GFF for a background charge $\beta$ with NC0.
We express a stress tensor and the Virasoro field of \( \Phi(b) \) in terms of the current field \( J(b) := \partial \Phi(b) \). From now on, the subscript \((0)\) is added to the notation of fields in the OPE family of the GFF, the algebra over \( \mathbb{C} \) spanned by 1 and the derivatives of GFF under OPE multiplication. For instance, \( \Phi(0) \) is the new notation for the GFF and \( J(0) = \partial \Phi(0) \). First, we recall the definition of the Virasoro field.

**Definition** A Fock space field \( T \) is said to be Virasoro field for \( F(A, \bar{A}) \) if

- \( T \in F(A, \bar{A}) \), and
- \( T - A \) is a non-random meromorphic Schwarzian form.

The order of Schwarzian form \( T - A \) is denoted by \( \frac{1}{12} c \), where \( c \) is called the central charge.

For each real parameter \( b \), we define

\[
\Phi \equiv \Phi(b) := \Phi(0) - 2b \arg w', \quad J \equiv J(b) := J(0) + ib \frac{w''}{w'},
\]

where \( w \) is a conformal map from \( D_r \) onto \( C_r \). As explained in Sect. 2.2, \( \arg w' \) does not depend on the choice of \( w \).

**Proposition 3.2** The field \( \Phi(b) \) has a stress tensor

\[
A(b) := -\frac{1}{2} J(0) \circ J(0) + (ib \partial - j(b)) J(0), \quad J(b) := E[J(b)] = ib \frac{w''}{w'},
\]

and its Virasoro field is given by

\[
T(b) = -\frac{1}{2} J(b) \ast J(b) + ib \partial J(b) = A(b) + \frac{c}{12} S_w + (w')^2 \left( \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} \right),
\]

where \( c = 1 - 12b^2 \).

**Proof** First, we prove the case \( b = 0 \). It is obvious that \( A(0) := -\frac{1}{2} J(0) \circ J(0) \) is holomorphic quadratic differential. We claim that \( W = (A(0), \bar{A}(0)) \) is a stress tensor for \( \Phi(0) \). Since \( \Phi(0) \) is a scalar field or a \((0, 0)\)-differential, by (3.24), all we need to verify is the following (Ward’s) OPE in the identity chart of cylinder \( C_r \):

\[
A(0)(\zeta) \Phi(0)(z) \sim \frac{J(0)(z)}{\zeta - z}.
\]  

(3.26)

This is clear from (3.21) and Wick’s calculus.

We now define the Virasoro field \( T(0) \) of \( \Phi(0) \) by \( T(0) := -\frac{1}{2} J(0) \ast J(0) \), equivalently by the following OPE:

\[
J(0)(\zeta) J(0)(z) = -\frac{1}{(\zeta - z)^2} - 2 T(0)(z) + o(1).
\]  

(3.27)
Then it follows from (3.22) that
\[ T(0) = A(0) + \frac{1}{12} S_w + (w')^2 \left( \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} \right). \]

Next, let us prove the general case \( b \neq 0 \). It is easy to see that \( ib \partial J(0) \) and \( J(b) J(0) \) satisfy the following transformation rules:
\[ i b \partial J(0) = i b h'' \tilde{J}(0) \circ h + i b (h')^2 \partial \tilde{J}(0) \circ h; \]
\[ J(b) J(0) = i b \left( \frac{h''}{h'} \right) h' \tilde{J}(0) \circ h + (h')^2 (\tilde{j}(b) \tilde{J}(0)) \circ h. \]

Here, \( h \) is the transition map between the local charts. Thus we readily see that \( A(b) \) is a holomorphic quadratic differential.

Again, we claim that \( W = (A(b), \bar{A}(b)) \) is a stress tensor for \( \Phi_1(b) \). Since \( \Phi_1(b) \) is the real part of a PPS-form, by (3.25), it suffices to show the following (Ward’s) OPE in the identity chart of cylinder \( C_r \):
\[ A(b)(\zeta) \Phi_1(b)(z) = \left( A(0)(\zeta) + (ib \partial - J(b) J(0)(\zeta)) \right) \Phi_1(b)(z) \sim \frac{ib}{(\zeta - z)^2} + \frac{J(b)(z)}{\zeta - z}. \]

The above OPE follows from (3.21), (3.22) and (3.26). Let \( T(b) = -\frac{1}{2} J(b) * J(b) + ib \partial J(b) \). Then by (3.27), we obtain
\[ T(b) = A(b) + \frac{1}{12} S_w + (w')^2 \left( \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} \right) - \frac{1}{2} J(b) + i b \partial J(b) \]
\[ = A(b) + \frac{1 - 2b^2}{12} S_w + (w')^2 \left( \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} \right), \]
which completes the proof.

\[ \Box \]

3.5 Formal bosonic fields and neutrality condition

In this subsection we introduce formal fields and their formal correlation functions. We use them to describe modified multi-vertex fields (OPE exponentials) in the next subsection. We first define the bi-variant field \( \Phi^+ \equiv \Phi^+(b) \) by
\[ \Phi^+(b)(z, z_0) := \left\{ \Phi^+(b)(\gamma) = \int_\gamma J(b)(\zeta) \, d\zeta \right\} = \Phi^+(0)(z, z_0) + ib \log \frac{w'(z)}{w'(z_0)}, \]
where \( \gamma \) is a curve from \( z_0 \) to \( z \) and \( w : D_r \to C_r \) is a conformal map. Note that the values of \( \Phi^+ \) are multivalued functionals.
Define the complex Green’s function in $D_r$ as

$$G^{\chi^+}_r(z, z_1) := \frac{1}{2} \log \left( \frac{\Theta(\chi)(r, w(z) - w(z_1))}{\Theta(\chi)(r, w(z) - w(z_1))} \right).$$

We remark that for the ER case when $\chi = \infty$, the complex Green’s function plays an important role in the theory of conformal mappings between canonical multiply connected domains, see e.g., [12]. Integrating (3.17) over a curve from $z_0$ to $z$, we obtain

$$E[\Phi^+_0(z_0) \Phi^+_0(z_1)] = 2(G^{\chi^+}_r(z, z_1) - G^{\chi^+}_r(z_0, z_1)).$$

Now we introduce formal (1-point) fields $\Phi^\pm_0$ in $D_r$ as centered Gaussian formal fields with (formal) 2-point correlations:

$$E[\Phi^+_0(z_1) \Phi^+_0(z_2)] = -\log \Theta(\chi)(r, w(z_1) - w(z_2)); \quad \quad (3.28)$$

$$E[\Phi^+_0(z_1) \Phi^-_0(z_2)] = +\log \Theta(\chi)(r, w(z_1) - w(z_2)). \quad \quad (3.29)$$

Formal fields $\Phi^\pm_0$ satisfy $\Phi_0 = \Phi^+_0 + \Phi^-_0$ and $\Phi^-_0 = \overline{\Phi^+_0}$ within correlations, e.g.,

$$E[\Phi_0(z_1) \Phi_0(z_2)] = E[\Phi^+_0(z_1) + \Phi^-_0(z_1)](\Phi^+_0(z_2) + \Phi^-_0(z_2)),$$

where $E$ in the left-hand side stands for correlation while we use formal correlations in the right-hand side. We define $\Phi^\pm_{(b)}$ by

$$\Phi^+_0 = \Phi^+_0 + ib \log w’, \quad \Phi^-_0 = \Phi^-_0 - ib \log \overline{w’}.$$

By definition, the following formal rule holds:

$$\Phi^+_0(z, z_0) = \Phi^+_0(z) - \Phi^+_0(z_0).$$

The formal dual boson $\tilde{\Phi}_0$ is defined by

$$\tilde{\Phi}_0 = -i(\Phi^+_0 - \Phi^-_0), \quad \tilde{\Phi}_0(z, z_0) = \tilde{\Phi}_0(z) - \tilde{\Phi}_0(z_0). \quad (3.30)$$

Note that we have the following relations

$$2 \Phi^+_0 = \Phi_0 + i \tilde{\Phi}_0, \quad 2 \Phi^-_0 = \Phi_0 - i \tilde{\Phi}_0.$$

For a finite set of distinct points $\{z_j\}_{j=1}^n$ in $D_r$, let us denote $\sigma = \sum \sigma_j \cdot z_j$, where $\sigma_j$ is a “charge” at each $z_j$. We also consider $\sigma$ as a divisor, i.e., a function $\sigma : D_r \to \mathbb{R}$ which takes the value 0 at all points except for finitely many points, and $\sigma(z_j) = \sigma_j$. 
Sometimes it is convenient to consider $\sigma$ as an atomic measure $\sigma = \sum \sigma_j \cdot \delta_{z_j}$. For example, $\int \sigma = \sum \sigma_j$.

For a double divisor $(\sigma, \sigma_*)$, we define the formal bosonic field $\Phi[\sigma, \sigma_*]$ by

$$\Phi[\sigma, \sigma_*] := \sum \sigma_j \Phi^+(z_j) - \sigma_* \Phi^-(z_j), \quad \Phi^\pm \equiv \Phi^\pm(0).$$

In general, it is not a well-defined Fock space field. The proposition below shows that the neutrality condition

$$\int \sigma + \sigma_* = 0 \quad (3.31)$$

guarantees well-definedness of $\Phi[\sigma, \sigma_*]$ as a Fock space field. For example, bi-variant fields satisfy the neutrality condition.

**Proposition 3.3** If a double divisor $(\sigma, \sigma_*)$ satisfies (3.31), then the formal bosonic field $\Phi[\sigma, \sigma_*]$ can be represented as a linear combination of well-defined Fock space fields.

**Proof** Let us choose any point $z_0 \in D_r$. Then

$$\Phi[\sigma, \sigma_*] = \Phi^+(z_0) \int \sigma - \Phi^-(z_0) \int \sigma_* + \sum \sigma_j \Phi^+(z_j, z_0) - \sigma_* \Phi^-(z_j, z_0).$$

Under the neutrality condition, the first two terms on the right-hand side become the Fock space correlation functional:

$$\Phi^+(z_0) \int \sigma - \Phi^-(z_0) \int \sigma_* = \Phi(z_0) \int \sigma.$$

$\Box$

### 3.6 Coulomb gas correlation function

In this subsection we construct modified multi-vertex fields and compute their correlation functions by the Coulomb gas formalism.

Given divisors $\sigma = \sum \sigma_j \cdot z_j$, $\tau = \sum \tau_k \cdot \xi_k$ for $z_j, \xi_k \in D_r$, set

$$\Phi[\sigma] := \sum \sigma_j \Phi(z_j), \quad \Phi[\tau] := \sum \tau_k \Phi(\xi_k),$$

where $\Phi \equiv \Phi(0)$. Define

$$\mathbf{E}\Phi[\sigma] \ast \Phi[\tau] := \sum_{z_j \neq \xi_k} \sigma_j \tau_k \mathbf{E}\Phi(z_j)\Phi(\xi_k) + \sum_{z_j = \xi_k} \sigma_j \tau_k \mathbf{E}\Phi \ast \Phi(z_j).$$
In particular, set

$$c[\sigma] := \frac{1}{2} E \Phi[\sigma] * \Phi[\sigma]$$

and define the (non-chiral) Coulomb gas correlation function $C_b$ as

$$C_b[\sigma] := e^{-c[\sigma]} \prod (w_j')^{-b\sigma_j (w_j')^{b\sigma_j}},$$

where $w_j := w(z_j)$. By (3.15) and (3.20), if all $z_j$'s are in the bulk, $C_b[\sigma]$ is differential at $z_j$ with conformal dimension $(\frac{1}{2}\sigma_j^2 - b\sigma_j, \frac{1}{2}\sigma_j^2 + b\sigma_j)$, and its evaluation in the identity chart of cylinder $\mathcal{C}_r$ is given as

$$C_b[\sigma] = \Theta'_X(0) \prod \Theta_X(z_j - \bar{z}_j)^{-\sigma_j^2} \prod_{j < k} \left| \frac{\Theta_X(z_j - z_k)}{\Theta_X(z_j - \bar{z}_k)} \right|^{2\sigma_j\sigma_k}.$$ 

We remark that the terminology “Coulomb gas” comes from the resemblance of the 2-point function $E \Phi(z_1) \Phi(z_2)$ with the electric potential energy between two unit charges in a planar Riemann surface (or on a compact Riemann surface). For the genus one case we refer to [27, 38] and references therein.

By definition, a non-chiral vertex field $\mathcal{V}_0[\sigma]$ is given by the OPE exponential

$$\mathcal{V}_0[\sigma] := e^{i\Phi[\sigma]} \equiv \sum_{n=0}^{\infty} \frac{i^n \Phi^{\otimes n}[\sigma]}{n!}.$$ 

Here, $\Phi^{\otimes n}$ denotes the OPE powers, see [36, Sect. 3.3] for their basic properties. Then it is easy to show that

$$\mathcal{V}_0[\sigma] = C_0[\sigma] \mathcal{V}^{\otimes}[\sigma], \quad \mathcal{V}^{\otimes}[\sigma] = e^{i\Phi[\sigma]}.$$ 

see [36, Proposition 3.3].

Let us denote by $(\sigma, \sigma^*) = (\sum \sigma_j \cdot z_j, \sum \sigma^*_j \cdot z_j)$ a double divisor satisfying the neutrality condition ($NC_0$). We define (chiral) Coulomb gas correlation function $C_b[\sigma, \sigma^*]$ as

$$C_b[\sigma, \sigma^*] := \Theta'_X(0) \prod \Theta_X(w_j - \bar{w}_j)^{\sigma_j\sigma^*_j} \prod (w_j')^{\lambda_j (w_j')^{\lambda^*_j}} \times \prod_{j < k} \Theta_X(w_j - w_k)^{\sigma_j\sigma^*_k} \Theta_X(\bar{w}_j - w_k)^{\sigma^*_j\sigma^*_k} \prod_{j < k} \Theta_X(w_j - \bar{w}_k)^{\sigma^*_j\sigma^*_k} \Theta_X(\bar{w}_j - w_k)^{\sigma^*_j\sigma^*_k},$$

where the conformal dimensions are given as

$$(\lambda_j, \lambda^*_j) = (\lambda(\sigma_j), \lambda(\sigma^*_j)), \quad \lambda(x) := \frac{1}{2} x^2 - b x.$$
Notice here that the non-chiral Coulomb gas correlation function is related to chiral one as $C_{(b)}[\sigma] = C_{(b)}[\sigma, -\sigma]$. 

By definition, the multi-vertex field $O[\sigma, \sigma^*] := O_{(b)}[\sigma, \sigma^*] \Vee$ is given by

$$O_{(b)}[\sigma, \sigma^*] := C_{(b)}[\sigma, \sigma^*] V \Vee \sigma, \sigma^* \] = e^{i \Phi_{(b)}[\sigma, \sigma^*]}.$$  

Then the field $O[\sigma, \sigma^*]$ is a differential with conformal dimension $(h_j, h^*_j) = (\lambda(\sigma_j), \lambda(\sigma^*_j))$ at $z_j$. In particular, we have

$$O_{(b)}[\sigma] := O_{(b)}[\sigma, -\sigma] = C_{(b)}[\sigma] e^{i \Phi_{(b)}[\sigma]} = e^{i \Phi_{(b)}[\sigma]}.$$  

(3.34)

Now we show that $O[\sigma, \sigma^*]$ satisfies Ward’s OPE.

**Proposition 3.4** Under the neutrality condition, $O(z) = O[\sigma, \sigma^*](z)$ satisfies Ward’s OPE: as $\xi \to z_j \in D_r$,

$$T(\xi) O(z) \sim h_j \frac{O(z)}{(\xi - z_j)^2} + \partial_j O(z) \xi - z_j, \quad T(\xi) \overline{O}(z) \sim \overline{h}_j \frac{\overline{O}(z)}{(\xi - z_j)^2} + \overline{\partial}_j \overline{O}(z) \xi - z_j.$$  

**Proof** Since vertex fields are differentials, it is enough to prove the proposition in the identity chart of $C_r$. Recall that in the identity chart of $C_r$,

$$J_{(b)} = J_{(0)}, \quad T_{(b)} = -\frac{1}{2} J_{(0)} \odot J_{(0)} + i b \partial J_{(0)} + \left( \frac{\zeta(\pi)}{2\pi} - \frac{1}{4(r + \chi)} \right).$$

By Wick’s calculus, we have

$$\partial_j O = \frac{\partial_j C_{(b)}[\sigma, \sigma^*]}{C_{(b)}[\sigma, \sigma^*]} O + i \sigma_j J_{(b)}(z_j) \odot O.$$  

First, let us show Ward’s OPE in the case $b = 0$. Let us denote

$$F(\xi, z) := E J_{(0)}(\xi) \sum \left( i \sigma_j \Phi_{(0)}^+(z_j) - i \sigma^*_j \Phi_{(0)}^-(z_j) \right) = -\frac{i}{2} \sum \left( \sigma_j H_\chi(r, \xi - z_j) + \sigma^*_j H_\chi(r, \xi - z_j) \right).$$

Then by (3.9), (3.32) and (3.33) as $\xi \to z_j$, we have

$$F(\xi, z) \sim -\frac{i \sigma_j}{\xi - z_j}, \quad F(\xi, z) \xi - z_j \sim -\frac{2h_j}{(\xi - z_j)^2} - \frac{\partial_j C_{(b)}[\sigma, \sigma^*]}{C_{(b)}[\sigma, \sigma^*]} \frac{2}{\xi - z_j}.$$
On the other hand, by Wick’s formula, we have

\[ T(0)(\zeta)O(z) = T(0)(\zeta) \circ O(z) - F(\zeta, z) J(0)(\zeta) \circ O(z) - \frac{1}{2} F(\zeta, z)^2 O(z). \]

Combining all the above equations, our assertion in the case \( b = 0 \) follows. For general \( b \in \mathbb{R} \), we only need to check

\[ \imath b \partial J(0)(\zeta) O(z) \sim -\sigma_j b \frac{O(z)}{(\zeta - z_j)^2}, \text{ as } \zeta \to z_j. \]

Again, this easily follows from Wick’s calculus. We leave it to the reader to verify the proposition for \( O \). \( \square \)

### 3.7 Background charge modifications

Given marked points \( q_k \), we call a divisor \( \beta = \sum_k \beta_k \cdot q_k \) (\( \beta_k \in \mathbb{R} \)) satisfying the neutrality condition \( \int \beta = 0 \) a background charge. By definition, a background charge modification \( \Phi_\beta \equiv \Phi_\beta(b) \) of the GFF \( \Phi \equiv \Phi(b) \) is given as

\[
\Phi_\beta(b) := \Phi(0) + \varphi_\beta, \\
\varphi_\beta(z) := -2b \arg w'_z + \sum \beta_k \arg \left\{ \Theta(x)(w_k - w_z) \Theta(\bar{w}_k - \bar{w}_z) \right\}. 
\]

Here \( w_z = w(z) \) and \( w_k = w(q_k) \). By Wick’s calculus, the non-random harmonic function \( \varphi_\beta \) can also be recognized as

\[
\varphi_\beta(z) = E \Phi_\beta(z) = E[e^{\imath \beta \Phi/2}] \Phi(z). 
\]

The current field \( J_\beta \) is defined in a natural way as

\[
J_{\beta,(b)} \equiv J_\beta := \partial \Phi_\beta = J(0) + J_\beta, \\
J_\beta(\partial) := \partial \varphi_\beta = \imath b \frac{w''_z}{w'_z} + \frac{i w'_z}{4} \sum \beta_k \left\{ H(x)(w_k - w_z) + H(x)(\bar{w}_k - w_z) \right\}. 
\]

Notice that for the trivial divisor \( \beta = 0 \), we have \( \Phi_0 = \Phi(b) \) and \( J_0 = J(b) \).

We now generalize Proposition 3.2.

**Proposition 3.5** The field \( \Phi_\beta \) has a stress tensor

\[
A_\beta := A(0) + \left( \imath b \partial - J_\beta \right) J(0), 
\]
and its Virasoro field is given as

\[ T_\beta := -\frac{1}{2} J_\beta \ast J_\beta + ib \partial J_\beta. \]

In particular, \( ET_\beta(z) \) is evaluated in the identity chart of \( C_r \) as

\[
ET_\beta(z) = \frac{1}{32} \left[ \sum \beta_k \left\{ H_\chi(r, q_k - z) + H_\chi(r, \bar{q}_k - z) \right\} \right]^2 + \frac{b}{4} \sum \beta_k \left\{ H'_\chi(r, q_k - z) + H'_\chi(r, \bar{q}_k - z) \right\} + \frac{\bar{\zeta}_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)}.
\]

**Proof** Throughout the proof, we denote by \( \phi \) the local chart and \( h = \tilde{\phi} \circ \phi^{-1} \) the transition map between \( \phi \) and a chart \( \tilde{\phi} \) overlapping with \( \phi \). We also use the tilde notation for conformal fields evaluated to the chart \( \tilde{\phi} \), e.g., \( \tilde{J}_\beta(0) = (J_\beta(0) \| \tilde{\phi}) \).

We claim that \( A_\beta \) is a \((2, 0)\)-differential and Ward’s OPE

\[
A_\beta(\zeta) \Phi_\beta(z) \sim \frac{ib}{(\zeta - z)^2} + \frac{J_\beta(z)}{\zeta - z} \quad (3.37)
\]

holds. Then the first part of the proposition immediately follows from (3.25).

Recall that \( \partial J_\beta(0) = h'' \tilde{J}_\beta(0) \circ h + (h')^2 \partial \tilde{J}_\beta(0) \circ h \). Also by (3.36),

\[
J_\beta = h' \tilde{J}_\beta \circ h + ib \frac{h''}{h'}. \quad (3.38)
\]

Combining the above equations, one can observe that \((ib\partial - J_\beta)J_\beta(0)\) is a \((2, 0)\)-differential. Note that

\[
A_\beta(\zeta) \Phi_\beta(z) \sim A_{\beta}(0)(\zeta) \Phi_{\beta}(0)(z) + \left( ib\partial - J_\beta \right) J_{\beta}(0)(\zeta) \Phi_{\beta}(0)(z).
\]

Then (3.37) follows from (3.23) and Proposition 3.2.

Next, we prove the second part of the proposition. By Proposition 3.2, we have

\[
T_\beta = A_\beta + ET_\beta(0) - \frac{1}{2} J_\beta^2 + ib \partial J_\beta,
\]

and \( ET_\beta(0) \) is an \( S \)-form of order \( 1/12 \). Moreover, by (3.38), the term \(-\frac{1}{2} J_\beta^2 + ib \partial J_\beta\) is an \( S \)-form of order \(-b^2\), which completes the proof. \( \Box \)

We define holomorphic (resp., anti-holomorphic) function \( \varphi^+_\beta \) (resp., \( \varphi^-_\beta \)) as

\[
\varphi^+_\beta := +ib \log w_z' - \frac{i}{2} \sum \beta_k \log \left\{ \Theta_\chi(w_k - w_z)\Theta_\chi(\bar{w}_k - w_z) \right\},
\]

\[
\varphi^-_\beta := -ib \log \bar{w}_z' + \frac{i}{2} \sum \beta_k \log \left\{ \Theta_\chi(\bar{w}_k - \bar{w}_z)\Theta_\chi(w_k - w_z) \right\}.
\]
The chiral and anti-chiral parts of $\Phi_\beta$ are defined by

$$
\Phi^+ = \Phi^+_{(0)} + \varphi^+_\beta, \quad \Phi^- = \Phi^-_{(0)} + \varphi^-_\beta.
$$

Notice that by definition, $\Phi_\beta = \Phi^+_\beta + \Phi^-_\beta$.

For a double divisor $(\sigma, \sigma^*) = (\sum \sigma_j \cdot z_j, \sum \sigma^*_j \cdot z_j)$ satisfying $\int (\sigma + \sigma^*) = 0$, we write

$$
\varphi_\beta \sigma : = \sum \sigma_j \varphi^+_\beta - \sigma^*_j \varphi^-_\beta.
$$

We define the formal bosonic field $\Phi^\beta \sigma \sigma^*$ associated with the background charge $\beta$ as

$$
\Phi^\beta \sigma \sigma^* := \sum \sigma_j \Phi^+_\beta(z_j) - \sigma^*_j \Phi^-_\beta(z_j) = \Phi_{(0)} \sigma \sigma^* + \varphi_\beta \sigma \sigma^*.
$$

The modified multi-vertex field $O_\beta \sigma \sigma^*$ is defined by

$$
O_\beta \sigma \sigma^* := \frac{C(b)[\sigma + \beta/2, \sigma^* + \beta/2]}{C(b)[\beta/2, \beta/2]} V^\sigma \sigma^*.
$$

In particular, if $\sigma^* = -\sigma$, we simply omit $\sigma^*$ in the notations, e.g.,

$$
\varphi_\beta \sigma : = \sum \sigma_j \varphi_\beta, \quad \Phi_\beta : = \sum \sigma_j \Phi_\beta(z_j).
$$

The modified multi-vertex fields are called the OPE exponentials of GFF. The reason for this terminology is clear due to the following proposition.

**Proposition 3.6** We have $O_\beta \sigma \sigma^* = e^{\varphi_\beta \sigma} \sigma$.

**Proof** By [37, Lemma 5.2] and (3.34),

$$
e^{\varphi_\beta \sigma} \sigma = e^{\varphi_\beta \sigma} e^{\varphi_{(0)} \sigma} = C(0)[\sigma] e^{\varphi_\beta \sigma} e^{\varphi_{(0)} \sigma}.
$$

Therefore all we need to show is

$$
C(0)[\sigma, \sigma^*] e^{\varphi_\beta \sigma \sigma^*} = \frac{C(b)[\sigma + \beta/2, \sigma^* + \beta/2]}{C(b)[\beta/2, \beta/2]}.
$$

Observe that $e^{\varphi_\beta \sigma \sigma^*}$ is a scalar field with respect to each $q_k$ and is a differential with dimension $(-b\sigma_j, -b\sigma^*_j)$ at each $z_j$. Therefore by (3.33), conformal dimensions of both sides of (3.40) are identical. Thus it is enough to show (3.40) in the identity chart of $C_r$. By construction,

$$
e^{\varphi_\beta \sigma \sigma^*} = \prod_{j < k} \Theta_X(r, q_k - z_j) \Theta_X(r, \tilde{q}_k - z_j)^{\frac{\sigma_j \beta_k}{2}}.$$

\[
\prod_{j<k} \left( \Theta \chi(r, \bar{q}_k - \bar{z}_j) \Theta \chi(r, q_k - \bar{z}_j) \right)^{\frac{\sigma_{sj} \beta_k}{2}}.
\]

Now proposition follows from (3.32). \qed

We now generalize Proposition 3.4 for a non-trivial background charge \( \beta \).

**Proposition 3.7** The vertex field \( O_\beta[\sigma, \sigma^*] \) satisfies Ward’s OPE

\[
T_\beta(\zeta)O_\beta[\sigma, \sigma^*] \sim \lambda_j \frac{O_\beta[\sigma, \sigma^*]}{(\zeta - z_j)^2} + \frac{\partial z_j O_\beta[\sigma, \sigma^*]}{\zeta - z_j}, \quad \text{as} \ \zeta \to z_j,
\]

and similar OPE holds for \( \overline{O_\beta}[\sigma, \sigma^*] \). Here \( \lambda_j \) are defined by (3.33).

**Proof** We write \( O_\beta \equiv O_\beta[\sigma, \sigma^*] \) and \( O_{(b)} \equiv O_{(b)}[\sigma, \sigma^*] \) throughout the proof. Since \( O_\beta \) is a differential, it suffices to show the proposition in the identity chart of \( C_r \). By combining Propositions 3.2 and 3.5, we have

\[
T_\beta = T_{(b)} - (J_\beta - J_{(b)})J_{(0)} - \frac{1}{2}(J_{\beta}^2 - J_{(b)}^2) + ib\partial(J_\beta - J_{(b)}).
\]

Then it follows from Proposition 3.4 and \( O_\beta = (EO_\beta/EO_{(b)})O_{(b)} \) that

\[
\begin{align*}
T_\beta(\zeta)O_\beta & \sim \frac{EO_\beta}{EO_{(b)}} T_{(b)}(\zeta)O_{(b)} - (J_\beta - J_{(b)})J_{(0)}(\zeta)O_\beta \\
& \sim \lambda_j \frac{O_\beta}{(\zeta - z_j)^2} + \frac{EO_\beta}{EO_{(b)}} \frac{\partial z_j O_{(b)}}{\zeta - z_j} - (J_\beta - J_{(b)})J_{(0)}(\zeta)O_\beta \\
& \sim \lambda_j \frac{O_\beta}{(\zeta - z_j)^2} + \frac{\partial z_j O_\beta}{\zeta - z_j} - \frac{\partial z_j (EO_\beta/EO_{(b)})}{EO_\beta/EO_{(b)}} \frac{O_\beta}{\zeta - z_j} - (J_\beta - J_{(b)})J_{(0)}(\zeta)O_\beta.
\end{align*}
\]

On the other hand, by Wick’s calculus together with (3.28), (3.29) and (3.9), we have

\[
J_{(0)}(\zeta)O_\beta \sim O_\beta EJ_{(0)}(\zeta) \sum (i\sigma_j \Phi_0^+(z_j) - i\sigma_j \Phi_0^-(z_j)) \sim -i\sigma_j \frac{O_\beta}{\zeta - z_j}.
\]

Moreover by (3.32) and (3.36), we have

\[
\frac{\partial z_j EO_\beta}{EO_\beta} - \frac{\partial z_j EO_{(b)}}{EO_{(b)}} = i\sigma_j (J_\beta - J_{(b)}).
\]

This completes the proof. \qed
4 Eguchi–Ooguri and Ward’s equations

In [21], Eguchi and Ooguri stated Ward’s equation for tensor product $\mathcal{X}$ of primary fields (differentials in the OPE family) on a complex torus of genus one in terms of linear partial differential operators with respect to the nodes of fields and the modular parameter $\tau$. They explained the so-called Teichmüller term $\partial \tau \mathcal{E} \mathcal{X}$ by means of the Virasoro generator $L_0$, the zeroth mode of the Virasoro field. In this section we derive Ward’s equation in a doubly connected domain not from Eguchi and Ooguri’s path integral formalism but some basic identities in special function theory together with Wick’s calculus.

4.1 Ward’s functional and Ward’s identities

We first introduce global Ward’s functional and present the relation between a stress tensor and Ward’s functional.

Let $W = (A_\beta, \bar{A}_\beta)$ be a stress tensor for $\mathcal{F}_\beta$. By Proposition 3.5, it is easy to see that $A = A_\beta$ is continuous and real on the boundary. For a given open set $U \subset D_r$ and a smooth vector field $v$ in $\overline{U}$ continuous up to the boundary, Ward’s functionals $W^\pm (v; U)$ are defined by

$$W^+(v; U) = \frac{1}{2\pi i} \int_{\partial U} vA - \frac{1}{\pi} \int_U (\bar{\partial} v)A, \quad W^-(v; U) = \overline{W^+(v; U)}.$$  

We also write $W(v; U) := 2 \Re W^+(v; U)$. Then for any Fock space functional $\mathcal{X}$ with nodes in $D_r$, $E[W^+(v; U)\mathcal{X}]$ is a well-defined correlation function in $D_{\text{hol}}(v)$.

We will frequently use the following proposition. See [36, Propositions 5.3 and 5.10] for more details. The last equations in the following proposition are known as Ward’s identities.

**Proposition 4.1** The following statements are equivalent:

- For any (local) vector field $v_\zeta$ satisfying $v_\zeta (z) \sim \frac{1}{\zeta - z}$ as $z \to \zeta$, Ward’s OPE

$$\text{Sing}_{\zeta \to z} A(\zeta)X(z) = (\mathcal{L}_{v_\zeta}^{-} X)(z), \quad \text{Sing}_{\zeta \to z} A(\zeta)\bar{X}(z) = (\mathcal{L}_{v_\zeta}^{-} \bar{X})(z)$$

holds for a Fock space field $X$;

- The residue form of Ward’s identity for $X$

$$\mathcal{L}_vX(z) = \frac{1}{2\pi i} \oint_{(z)} vAX(z) - \frac{1}{2\pi i} \oint_{(z)} \bar{v}\bar{A}X(z)$$

holds on $D_{\text{hol}}(v) \cap U$ for all smooth vector field $v$;

- For all $z \in D_{\text{hol}}(v)$,

$$E[V\mathcal{L}_vX(z)] = E[VW(v; U)X(z)]$$

holds for all Fock space functionals $V$ with nodes outside supp$(v)$. 

The definition of Ward’s functionals can be extended to meromorphic vector fields. For a meromorphic vector field \( v \) with poles \( \{ p_k \} \) in \( D_r \), we define

\[
W^+(v) = W^+(v; D_r) := \lim_{\varepsilon \to 0} W^+(v; D_r^\varepsilon),
\]

where \( D_r^\varepsilon = D_r \setminus \bigcup_k B(p_k, \varepsilon) \). Thus, we have

\[
W^+(v) = W^+(v; D_r) = \frac{1}{2\pi i} \int_{\partial D_r} v A - \frac{1}{\pi} \int_{D_r} (\bar{\partial} v) A,
\]

with the interpretation of \( \bar{\partial} v \) in the sense of distributions.

**Proposition 4.2** In the identity chart \( \text{id}_{\mathcal{C}_r} \), we have

\[
2A_{\beta}(\xi) = W^+(v_{\xi}) + W^-(v_{\bar{\xi}}) + \frac{1}{\pi} \int_{[-\pi+ir,\pi+ir]} A_{\beta}(\xi) \, d\xi,
\]

where \( v_{\xi} \| \text{id}_{\mathcal{C}_r} \)(z) = H(r, \xi - z).

**Proof** Recall that a stress tensor \( A_{\beta} \) is a holomorphic quadratic differential in \( \mathcal{C}_r \), and real on the boundary. For \( \xi \in \mathcal{C}_r \), the reflected vector field \( v_{\xi}^\# \) is defined by

\[
v_{\xi}^\#(z) := \overline{v_{\xi}(\bar{z})} = v_{\xi}(z).
\]

Since \( v_{\xi} \) is meromorphic on \( \mathcal{C}_r \) with poles at \( \{ \xi + 2\pi m + 2irn : m, n \in \mathbb{Z} \} \), the reflected vector field \( v_{\xi}^\# \) is holomorphic on \( \mathcal{C}_r \). Then by definition,

\[
W^+(v_{\xi}) = -\frac{1}{\pi} \int_{\mathcal{C}_r} (\bar{\partial} v_{\xi}) A_{\beta} + \frac{1}{2\pi i} \int_{\partial \mathcal{C}_r} v_{\xi} A_{\beta}, \quad W^+(v_{\xi}^\#) = \frac{1}{2\pi i} \int_{\partial \mathcal{C}_r} v_{\xi}^\# A_{\beta}.
\]

Also, since \( A_{\beta} = \bar{A}_{\beta} \) on \( \partial \mathcal{C}_r \), we have

\[
W^-(v_{\bar{\xi}}) = \overline{W^+(v_{\xi})} = \frac{1}{2\pi i} \int_{\partial \mathcal{C}_r} v_{\bar{\xi}}(\bar{\xi}) A_{\beta}(\bar{\xi}) \, d\bar{\xi} = \frac{1}{2\pi i} \int_{\partial \mathcal{C}_r} v_{\xi}(\bar{\xi}) A_{\beta}(\bar{\xi}) \, d\bar{\xi}.
\]

Then it follows from \( \bar{\partial} v_{\xi} = -2\pi \delta_{\xi} \), and \( H(r, z + 2ir) = H(r, z) - 2i \) that

\[
W^+(v_{\xi}) + W^-(v_{\bar{\xi}}) = -\frac{1}{\pi} \int_{\mathcal{C}_r} (\bar{\partial} v_{\xi}) A_{\beta} - \frac{1}{\pi} \int_{[-\pi+ir,\pi+ir]} A_{\beta}(\xi) \, d\xi
\]

\[
= 2A_{\beta}(\xi) - \frac{1}{\pi} \int_{[-\pi+ir,\pi+ir]} A_{\beta}(\xi) \, d\xi.
\]

This completes the proof. \( \square \)
4.2 Eguchi–Ooguri equations

We now state and prove a version of Eguchi–Ooguri equations.

First let us consider the case $\beta = 0$. Recall that $\mathcal{F}(b) = \mathcal{F}_{(b),0}$ and $\mathcal{F}_{(b),\beta}$ is the OPE family generated by $\Phi_{(b),\beta}$.

Lemma 4.3 For any $X \in \mathcal{F}(b)$, in the $C_r$-uniformization,

$$\frac{1}{\pi} \oint_{[-\pi + ir, \pi + ir]} E A(\zeta) X d\zeta = \partial_r E X. \quad (4.1)$$

Proof Let us denote by $W$ the family of Fock space fields satisfying (4.1). We show $\mathcal{F}(b) \subset W$ through the following steps.

Step 1 A tensor product of two GFFs is in $W$.

Step 2 All tensor products of GFFs and their derivatives are in $W$.

Step 3 The family $W$ is closed under linear combinations over $\mathbb{C}$ and OPE products $\ast_n$.

Proof of Step 1. Let us denote $A \equiv A(b)$, $\Phi \equiv \Phi(b)$. Recall that in the identity chart $id_{C_r}$, $A = -\frac{1}{2} J(0) \otimes J(0) + i b \partial J(0)$, and $\Phi = \Phi(0)$. Since $\partial J$ is exact, by the fundamental theorem of calculus, and the fact that all correlation functions lifted to the covering space of $C_r$ are $2\pi$ periodic, it suffices to show (4.1) for $b = 0$.

First, we show this step for the ER boundary condition (i.e., $\chi = \infty$).

For $X = \Phi(z_1)\Phi(z_2)$, by (3.15) and the fact $\partial_r \log \Theta = H'/2 + H^2/4$ [which follows from (3.1) and (3.2)], we have

$$\partial_r E[A(\zeta)\Phi(z_1)\Phi(z_2)] = -I(z_1, z_2) + I(\bar{z}_1, z_2) + I(z_1, \bar{z}_2) - I(\bar{z}_1, \bar{z}_2),$$

where $I(z, w) := H'(z - w)/2 + H^2(z - w)/4$. On the other hand, by Wick’s calculus and (3.17), we have

$$E[A(\zeta)\Phi(z_1)\Phi(z_2)] = -II_{z_1, z_2}(\zeta) + II_{\bar{z}_1, z_2}(\zeta) + II_{z_1, \bar{z}_2}(\zeta) - II_{\bar{z}_1, \bar{z}_2}(\zeta),$$

where $II_{z, w}(\zeta) := H(\zeta - z)H(\zeta - w)/4$. Note that by (3.11), $II_{z, w}(\zeta)$ can be rewritten as

$$II_{z, w}(\zeta) = \frac{1}{2} I(z, w) + \frac{1}{4} III_{z, w}(\zeta) + \frac{1}{8} H^2(\zeta - z) + \frac{1}{8} H^2(\zeta - w) + \frac{3\zeta_r(\pi)}{2\pi}, \quad (4.2)$$

where

$$III_{z, w}(\zeta) := H'(\zeta - z) + H'(\zeta - w) + (H(\zeta - w) - H(\zeta - z))H(w - z).$$

Combining all of the above, we obtain

$$E[A(\zeta)\Phi(z_1)\Phi(z_2)] = \frac{1}{2} \partial_r E[\Phi(z_1)\Phi(z_2)].$$
\[ -\frac{1}{4} \left( III_{z_1,z_2}(\zeta) - III_{\bar{z}_1,\bar{z}_2}(\zeta) - III_{z_1,\bar{z}_2}(\zeta) + III_{\bar{z}_1,z_2}(\zeta) \right). \]

Finally, observe that by (3.1) and (3.4), we have
\[ \oint_{[-\pi + ir, \pi + ir]} III_{z,w}(\zeta) \, d\zeta \equiv 0, \]
which leads to (4.1) for \( \mathcal{X} = \Phi(z_1) \Phi(z_2) \) with ER boundary condition.

We now show this step for the weighted boundary conditions with any \( \chi \in [0, \infty) \).
For notational convenience, we add the superscript "ER" for the field with ER boundary condition, e.g., \( \Phi^{ER}_1 \), \( A^{ER} \) etc. Then by (3.14) and the periodicity of \( H \), we have
\[ \frac{1}{\pi} \int \left( E[A(\zeta) \Phi(z_1) \Phi(z_2)] - E[A^{ER}(\zeta) \Phi^{ER}_1(z_1) \Phi^{ER}_1(z_2)] \right) \, d\zeta \]
\[ = -\frac{(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}{2(r + \chi)^2}. \]

On the other hand,
\[ \partial_r \left( E[\Phi(z_1) \Phi(z_2)] - E[\Phi^{ER}(z_1) \Phi^{ER}(z_2)] \right) = \partial_r \left( \frac{1}{r + \chi} \right) \frac{(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}{2}. \]

Combining all of the above identities, the claim follows.

**Proof of Step 2.** There is nothing to prove for \( \mathcal{X} = \Phi(z_1) \cdots \Phi(z_{2n-1}) \). Therefore it is enough to show (4.1) for \( \mathcal{X} = \Phi(z_1) \cdots \Phi(z_{2n}) \).

For \( j \neq k \), let us denote
\[ \mathcal{X}_{jk} = \Phi(z_1) \cdots \Phi(z_{k-1}) \Phi(z_{k+1}) \cdots \Phi(z_{j-1}) \Phi(z_{j+1}) \cdots \Phi(z_{2n}). \]

Then by Wick’s formula and an induction argument, we have
\[ \frac{1}{\pi} \int E[A(\zeta) \Phi(z_1) \cdots \Phi(z_{2n})] \, d\zeta = \frac{1}{\pi} \int \sum_{j \neq k} E[A(\zeta) \Phi(z_j) \Phi(z_k)] E[\mathcal{X}_{jk}] \, d\zeta \]
\[ = \sum_{j \neq k} \partial_r E[\Phi(z_j) \Phi(z_k)] E[\mathcal{X}_{jk}]. \]

On the other hand, by (3.16) and Leibniz’s rule, we express the right-hand side of (4.1) as
\[ \partial_r E[\Phi(z_1) \cdots \Phi(z_{2n})] = \partial_r \sum_k \prod_k E[\Phi(z_{i_k}) \Phi(z_{j_k})] \]
\[ = \sum \left( \prod_k E[\Phi(z_{i_k}) \Phi(z_{j_k})] \right) \left( \sum_k \frac{\partial_r E[\Phi(z_{i_k}) \Phi(z_{j_k})]}{E[\Phi(z_{i_k}) \Phi(z_{j_k})]} \right). \]
where the outer sum is taken over all partitions of the set \{1, \ldots, 2n\} into disjoint pairs \{i_l, j_l\}. Rearranging the terms, we rewrite the above identity as

\[
\partial_r E[\Phi(z_1) \cdots \Phi(z_{2n})] = \sum_{j \neq k} \left( \partial_r E[\Phi(z_j) \Phi(z_k)] \left( \sum_l E[\Phi(z_{i_l}) \Phi(z_{j_l})] \right) \right).
\]

where the sum is over all partitions of the set \{1, \ldots, 2n\} \\setminus \{j, k\}. Therefore again, by (3.16), we have (4.1) for \(X = \Phi(z_1) \cdots \Phi(z_{2n})\). Differentiating term by term, we complete the proof of Step 2.

**Proof of Step 3.** The first part of this step is obvious. For the second part, we consider the case that the field \(X\) is holomorphic. The other cases can be treated in a similar way. Suppose that we have the following OPE

\[
X(\xi)Y(z) = \sum C_n(z)(\xi - z)^n, \quad \xi \to z.
\]

If both \(X\) and \(Y\) are in \(\mathcal{W}\), then the term by term integration gives

\[
\frac{1}{\pi} \int_{[-\pi + ir, \pi + ir]} E[A(\xi) X(\xi) Y(z)] d\xi
= \sum_n \left( \frac{1}{\pi} \int_{[-\pi + ir, \pi + ir]} E[A(\xi) C_n(z)] d\xi \right) (\xi - z)^n
\]

as \(\xi \to z\). On the other hand, differentiating term by term, we have

\[
\partial_r E[X(\xi)Y(z)] = \sum \partial_r E[C_n(z)](\xi - z)^n.
\]

Therefore, by comparing the coefficients of the two expansions above, we conclude \(C_n \in \mathcal{W}\). \(\square\)

**Remark** Let us consider the GFF \(\Phi\) with 2-point correlation function given by a probabilistic convex interpolation (via Bernoulli distribution with probability \(p \equiv p(r)\)) of Green’s functions with ER and Dirichlet boundary conditions, i.e.,

\[
E[\Phi(\xi) \Phi(z)] = 2 \left( p(r) G_r^{Diri}(\xi, z) + (1 - p(r)) G_r^{ER}(\xi, z) \right)
= 2 \log \left| \frac{\Theta(r, \xi - \bar{z})}{\Theta(r, \xi - z)} \right| - \frac{2 p(r)}{r} \text{Im} \xi \text{Im} z.
\]

Then the conclusion of Lemma 4.3 can be drawn if and only if \(\partial_r (r/p(r)) = 1\). In other words, to derive Eguchi–Ooguri type equation of the form (4.1), it is required that

\[p(r) = r/(r + \chi), \quad (\chi \geq 0).\]

This corresponds to the rational interpolation (3.14).
We have shown Lemma 4.3 for the non-chiral field $X$. To our purpose of implementing Ward’s equation for the OPE family of GFFs associated with the general background charge, we now extend Eguchi–Ooguri equations to a certain family of chiral fields. We write $(\sigma, \sigma^\ast) = (\sum \sigma_j \cdot z_j, \sum \sigma^\ast_j \cdot z_j)$ for a double divisor satisfying $(NC_0)$ and denote by $O \equiv O(b)[\sigma, \sigma^\ast]$ the associated multi-vertex field. Due to the following proposition, we can revise the definition of $F(b)$ so that it includes the fields of the form $OX$.

**Proposition 4.4** Let $\gamma = [-\pi + ir, \pi + ir]$. Then for any $X \in F(b)$,

$$\frac{1}{\pi} \oint_{\gamma} E A(\zeta) OX d\zeta = \partial_r E OX,$$

(4.3)

where all fields are evaluated in the identity chart of $C_r$.

**Proof** We present the proof for ER boundary conditions and leave it to the reader for the general case.

Again it suffices to show (4.3) for $b = 0$. In the sequel, we omit the subscript $(0)$ and write for instance $A \equiv A(0)$, $C \equiv C(b)$. Throughout the proof, we use symbols $I(z, w), II_{z, w}(\zeta)$ and $III_{z, w}(\zeta)$ given in the proof of Lemma 4.3.

By definition of the multi-vertex fields, (4.3) is equivalent to

$$\frac{1}{\pi} \oint_{\gamma} E A(\zeta) V^\circ[\sigma, \sigma^\ast]X d\zeta = \partial_r \frac{C[\sigma, \sigma^\ast]}{C[\sigma, \sigma^\ast]} E V^\circ[\sigma, \sigma^\ast]X + \partial_r E V^\circ[\sigma, \sigma^\ast]X,$$

(4.4)

where $V^\circ[\sigma, \sigma^\ast] = e^{\circ i \Phi[\sigma, \sigma^\ast]}$. Using similar arguments in the proof of Lemma 4.3, it suffices to show the identity (4.4) for tensor products of GFFs.

Let us first show that the identity (4.4) holds for the trivial string $X = 1$. Combining (3.32) with (3.1), (3.2) and (3.7), we have

$$\partial_r \log C[\sigma, \sigma^\ast] = -\frac{3\zeta_r(\pi)}{2\pi} \sum (\sigma_j^2 + \sigma_j^2) + \sum \sigma_j \sigma_k I(z_j, \bar{z}_j)$$

$$+ \sum_{j < k} \sigma_j \sigma_k I(z_j, z_k) + \sigma_j \sigma_k I(\bar{z}_j, \bar{z}_k)$$

$$+ \sigma_j \sigma_k I(z_j, \bar{z}_k) + \sigma_j \sigma_k I(\bar{z}_j, \bar{z}_k).$$

Differentiating (3.28) and (3.29), we have

$$E J(\zeta) \Phi[\sigma, \sigma^\ast] = -\frac{1}{2} \sum \left( \sigma_j H(\zeta - z_j) + \sigma_j H(\zeta - \bar{z}_j) \right).$$

(4.5)
By Wick’s formula and (4.5),
\[
\mathcal{E} \mathcal{A}(\zeta) V^{\circ} [\sigma, \sigma_*] = \frac{1}{8} \sum_j \sigma_j^2 H^2(\zeta - z_j) + \sigma_{*j}^2 H^2(\zeta - \bar{z}_j) + \sum_{j<k} \sigma_j \sigma_k \Pi_{\zeta_j, \zeta_k}(\zeta)
+ \sum_{j<k} \sigma_j \sigma_k \Pi_{\bar{\zeta}_j, \bar{\zeta}_k}(\zeta) + \sigma_{*j} \sigma_{*k} \Pi_{\bar{\zeta}_j, \bar{\zeta}_k}(\zeta).
\]
(4.6)

Then it follows from (4.2) and the neutrality condition (3.31) that
\[
\mathcal{E} \mathcal{A}(\zeta) V^{\circ} [\sigma, \sigma_*] - \frac{1}{2} \partial_r \log C[\sigma, \sigma_*] = \frac{1}{4} \sum_{j<k} \sigma_j \sigma_k \Pi_{\zeta_j, \zeta_k}(\zeta) + \sigma_{*j} \sigma_{*k} \Pi_{\bar{\zeta}_j, \bar{\zeta}_k}(\zeta) + \sigma_{*j} \sigma_{*k} \Pi_{\bar{\zeta}_j, \bar{\zeta}_k}(\zeta).
\]

Now claim follows from the fact that \( \oint_{\gamma} \Pi_{\zeta, w}(\zeta) d \zeta \equiv 0 \).

Next, we show that the identity (4.4) holds for the case \( X = \Phi \). By Wick’s formula, we have
\[
\mathcal{E} V^{\circ} [\sigma, \sigma_*] \Phi(z) = i \mathcal{E} \Phi [\sigma, \sigma_*] \Phi(z)
= i \sum_j \sigma_j \log \frac{\Theta(z_j - \bar{z})}{\Theta(z_j - z)} + \sigma_{*j} \log \frac{\Theta(\bar{z}_j - \bar{z})}{\Theta(\bar{z}_j - z)}.
\]
(4.7)

Therefore by (3.2),
\[
\partial_r \mathcal{E} V^{\circ} [\sigma, \sigma_*] \Phi(z) = i \sum_j \sigma_j \left( I(z_j, \bar{z}) - I(z_j, z) \right) + \sigma_{*j} \left( I(\bar{z}_j, \bar{z}) - I(\bar{z}_j, z) \right).
\]

On the other hand, by (4.6) and (4.7),
\[
\mathcal{E} \mathcal{A}(\zeta) V^{\circ} [\sigma, \sigma_*] \Phi(z) = F(\zeta, z, z) - i G(\zeta, z, z),
\]
where
\[
F(\zeta, z, z) = \mathcal{E} \mathcal{A}(\zeta) V^{\circ} [\sigma, \sigma_*] \mathcal{E} V^{\circ} [\sigma, \sigma_*] \Phi(z),
\]
\[
G(\zeta, z, z) = \mathcal{E} J(\zeta) \Phi(z) \mathcal{E} J(\zeta) \Phi[\sigma, \sigma_*].
\]

Note that
\[
\frac{1}{\pi} \oint_{\gamma} F(\zeta, z, z) d\zeta = \frac{1}{\pi} \oint_{\gamma} \mathcal{E} \mathcal{A}(\zeta) V^{\circ} [\sigma, \sigma_*] d\zeta \mathcal{E} V^{\circ} [\sigma, \sigma_*] \Phi(z)
= \frac{\partial_r C[\sigma, \sigma_*]}{C[\sigma, \sigma_*]} \mathcal{E} V^{\circ} [\sigma, \sigma_*] \Phi(z).
\]
Thus it follows from the neutrality condition (3.31) and (4.5) that

\[
G(\zeta, z, \bar{z}) = \sum \sigma_j \left( \Pi_{\zeta, j}(\zeta) - \Pi_{\zeta, j}(\bar{\zeta}) + \sigma^*_{\ast j} \right) \\
+ \sum \sigma_j \left( \Pi_{\zeta, j}(\bar{z}) - \Pi_{\zeta, j}(\bar{\bar{z}}) + \sigma^*_{\ast j} \right)
\]

Therefore we obtain

\[
\frac{-i}{\pi} \oint_{\gamma} G(\zeta, z, \bar{z}) d\zeta = \partial_r E V^{\odot}[\sigma, \sigma^*] \Phi(\zeta),
\]

which leads to (4.4) for \( \mathcal{X} = \Phi \).

Now Proposition for the tensor products of GFFs follows from a similar argument of Claim 2 in the proof of Lemma 4.3. We leave it to the reader as an exercise.

In the physics literature, instead of a stress tensor \( A \), the Virasoro field \( T \) is often utilized to state Ward’s equation. To describe Eguchi–Ooguri equation in terms of the Virasoro field, let us define the effective multi-vertex field \( O_{\text{eff}} \equiv O_{(b)}^{\text{effective}}[\sigma, \sigma^*] \) as

\[
O_{(b)}^{\text{effective}}[\sigma, \sigma^*] := O_{(b)}[\sigma, \sigma^*] Z_r, \quad Z_r = \Theta'(r, 0)^{-1/3} (r + \chi)^{-1/2}.
\]

We remark that the effective multi-vertex field plays an important role in describing the restriction observables in a doubly connected domain [4, 16, 44, 73].

By Proposition 4.4, we obtain the following corollary.

**Corollary 4.5** Let \( \gamma = [-\pi + ir, \pi + ir] \). Then for any \( \mathcal{X} \in \mathcal{F}(b) \),

\[
\frac{1}{\pi} \oint_{\gamma} E T(\zeta) \mathcal{O}_{\text{eff}} \mathcal{X} d\zeta = \partial_r E O^{\text{effective}} \mathcal{X},
\]

where all fields are evaluated in the identity chart of \( C_r \).

**Proof** By Proposition 4.4, we have

\[
\frac{1}{\pi} \oint_{\gamma} E T(\zeta) \mathcal{O}_{\text{eff}} \mathcal{X} d\zeta = \left( \frac{1}{\pi} \oint_{\gamma} E T(\zeta) \mathcal{O} \mathcal{X} d\zeta \right) Z_r \\
= \left( \frac{1}{\pi} \oint_{\gamma} E A(\zeta) \mathcal{O} \mathcal{X} d\zeta \right) Z_r + \left( \frac{1}{\pi} \oint_{\gamma} E T(\zeta) d\zeta \right) E O \mathcal{X} Z_r \\
= \left( \partial_r E O \mathcal{X} \right) Z_r + 2 E T E O \mathcal{X} Z_r.
\]
Note that by Proposition 3.2, (3.7) and (4.8), we have
\[
ET = -\frac{1}{6} \frac{\Theta'''(r, 0)}{\Theta'(r, 0)} - \frac{1}{4(r + \chi)} = \frac{1}{2} \partial_r \log Z_r.
\]
(Cf. \(ET\) is constant in the identity chart of \(C_r\).) Therefore we conclude that
\[
\frac{1}{\pi} \oint_{\gamma} ET(\zeta) O^{\text{eff}} \mathcal{X} d\zeta = \left( \partial_r E O \mathcal{X} \right) Z_r + E O \mathcal{X} \partial_r Z_r = \partial_r E O^{\text{eff}} \mathcal{X},
\]
which completes the proof.

By (5.7), one can see that the background charge modifications can be treated as an insertion of a multi-vertex field. (see Sect. 5.1 for further details.) Therefore, as a consequence of Proposition 4.4, we extend Lemma 4.3 to the OPE family \(F_\beta\) with non-trivial background charges \(\beta\).

**Corollary 4.6** For any \(\mathcal{X} \in F_\beta\), in the \(C_r\)-uniformization,
\[
\frac{1}{\pi} \oint_{[-\pi + ir, \pi + ir]} EA_\beta(\xi) \mathcal{X} d\xi = \partial_r E \mathcal{X}.
\]

### 4.3 Ward’s equations

In this subsection we derive Ward’s equations (Theorem A) in terms of a stress tensor, Lie derivative operator and the modular parameter. Later, we combine these equations with the level two degeneracy equations for the one-leg operator to derive BPZ equations.

We first prove Theorem A in the case \(\beta = 0\).

**Proof of Theorem A in the case \(\beta = 0\)**

Let us choose \(v(z) = v_\zeta(z) = H(r, \zeta - z)\) with \(\zeta \in C_r\). By Proposition 4.2, Eguchi–Ooguri equations (Corollary 4.6), and Ward’s identities (see Proposition 4.1), we obtain
\[
2EA_\beta(\xi) \mathcal{X} = EW^+(v_\zeta) \mathcal{X} + EW^-(v_\zeta) \mathcal{X} + \partial_r E \mathcal{X} = \left( L^+_{v_\zeta} + L^-_{v_\zeta} \right) E \mathcal{X} + \partial_r E \mathcal{X},
\]
which completes the proof.

For a general background charge \(\beta\), it is required to generalize the residue form of Ward’s identity at the nodes of \(\beta\), see Lemma 5.5. Since this lemma can be more easily obtained by employing the insertion procedure, we defer its proof to Sect. 5.1. Then the proof of Theorem A for a non-trivial background charge can be obtained along the same lines for the trivial background charge with slight modifications.

As a corollary of Theorem A, we derive the following form of Ward’s equations.
Corollary 4.7  For a holomorphic differential $Y \in \mathcal{F}_\beta$ with conformal dimension $h$, and $X = X_1 \cdots X_n$, ($X_j \in \mathcal{F}_\beta$), we have

$$2E_{A_\beta} * Y(z)X = EY(z)L^+_{v_\xi}X + L^-_{v_\xi}EY(z)X + \left(2h\frac{\xi_r(\pi)}{\pi} + \partial_r\right)EY(z)X,$$

where all fields are evaluated in the identity chart of $C_r$.

Proof  By Ward’s OPE, we have the following singular part of the operator product expansion:

$$\text{Sing}_{\xi \to z} A_\beta(\xi)Y(z)X = \frac{h}{(\xi - z)^2} Y(z)X + \frac{\partial_z}{\xi - z} Y(z)X.$$

On the other hand, by Theorem A, we have

$$2E_{A_\beta}(\xi)Y(z)X = (hv_\xi'(z) + v_\xi(z)\partial_z + \partial_r)EY(z)X + EY(z)L^+_{v_\xi}X + L^-_{v_\xi}EY(z)X.$$

Subtracting the singular part from both sides of the above and then taking the limit as $\xi \to z$, we obtain

$$2E_{A_\beta} * Y(z)X = EY(z)L^+_{v_\xi}X + L^-_{v_\xi}EY(z)X + \partial_r EY(z)X + \lim_{\xi \to z} \left\{hv_\xi'(z) - \frac{2h}{(\xi - z)^2} + \left(v_\xi(z) - \frac{2}{\xi - z}\right)\partial_z\right\}EY(z)X.$$

Corollary now follows from the asymptotic behavior (3.9) of the Loewner vector field $v_\xi(z) = H(r, \xi - z)$.

5 Martingale-observables for annulus SLE with atomic background charges

In this section we introduce the notion of the one-leg operator $\Psi(p)$. After explaining the insertion procedure of $\Psi$, we present how the conformal fields generated by the modified GFF can be related to SLE theory.

From the viewpoint of SLE theory, the node $p$ plays the role of growth point, whereas the nodes of background charges play the role of force points. Throughout this section we will restrict our attention to CFT associated with discrete charge distribution. This allows us to construct martingale-observables for annulus SLE with several force points. However, in general, these SLEs are not suitable for describing more general boundary conditions of Dirichlet type and also are not reversible. In the following section we implement CFT associated with continuous charge distribution to address such types of SLEs.

In the first subsection we define the one-leg operator and explain its implementation as a boundary condition changing operator. We then complete the proof of BPZ equa-
tion, Theorem B. Section 5.2 is devoted to the study of general null-vector equations. In Sect. 5.3, we show a version of BPZ-Cardy type equation and prove Theorem C.

### 5.1 One-leg operator and BPZ equations

We introduce the **one-leg operator** $\Psi_\beta$, a specific form of the modified multi-vertex field set to satisfy level two degeneracy equation. Throughout this subsection we denote by $\beta = \sum k \beta_k \cdot q_k$ a given background charge.

By definition, for $n \in \mathbb{Z}$, Virasoro and current generators $L_n, J_n$ are given as

\[
L_n(z) := \frac{1}{2\pi i} \oint (\zeta - z)^{n+1} T_\beta(\zeta) \, d\zeta, \quad J_n(z) := \frac{1}{2\pi i} \oint (\zeta - z)^n J_\beta(\zeta) \, d\zeta,
\]

respectively. Let us recall the following proposition.

**Proposition 5.1** (Cf. Proposition 7.5 [36]) *Let $X$ be a Fock space field. Then any two of the following assertions imply the third one:*

- $X$ belongs to $\mathcal{F}(A, \bar{A})$;
- $X$ is a $[\lambda, \lambda^*]$-differential;
- $L_n X = 0$ for all $n \geq 1$, $L_0 X = \lambda X$, $L_{-1} X = \partial X$, and similar equations hold for $\bar{X}$.

Any field satisfying all conditions in Proposition 5.1 is called a **(Virasoro) primary field** in $\mathcal{F}(A, \bar{A})$. Furthermore, a Virasoro primary field $X$ is called **current primary** if $J_n X = J_n \bar{X} = 0$ for all $n \geq 1$ and $J_0 X = -iq X$, $J_0 \bar{X} = i\bar{q} X$ for some numbers $q$ and $q^*$, which are called charges. One of the main ingredients to connect CFT with SLE theory is the following **level two degeneracy equation**.

**Proposition 5.2** (Cf. Proposition 11.2 [36]) *For a current primary field $V$ with charges $q, q^*$ in $\mathcal{F}_\beta$, we have*

\[
L_{-2} V = \frac{1}{2q^2} L_{-1}^2 V
\]

provided $2q(b + q) = 1$.

Suppose that the real parameters $a$ and $b$ are related to the SLE parameter $\kappa$ as

\[
a = \sqrt{2/\kappa}, \quad b = a(\kappa/4 - 1).
\]

In the sequel, we sometimes use the following Kac labeling of the conformal weights or dimensions:

\[
h_{r,s} = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}.
\]
For a divisor $\tau = \sum_{j=1}^{N} \tau_j \cdot z_j$ satisfying the neutrality condition

$$a + \int \tau = 0,$$ (5.2)

we define the one-leg operator

$$\Psi(z)(\equiv \Psi_{\beta}(z, z) \equiv \Psi_{\beta}[a \cdot z + \tau]) := \mathcal{O}_{\beta}[a \cdot z + \frac{1}{2} \tau, \frac{1}{2} \tau].$$ (5.3)

Notice that $\Psi$ is a scalar field with respect to each node $q_j$ of the background charge $\beta$. On the other hand, by (3.33), the conformal dimensions of $\Psi$ with respect to $z$ and $z_j$ are given by

$$h_z = \frac{6-\kappa}{2\kappa}, \quad h_j = h_{*j} = \frac{1}{8} \tau_j^2 - \frac{1}{2} \tau_j b.$$ 

It follows from (3.39) and (5.3) that

$$E \Psi(z) = \frac{C_{(b)}[a \cdot z + \frac{1}{2} (\tau + \beta), \frac{1}{2} (\tau + \beta)]}{C_{(b)}[\frac{1}{2} \beta, \frac{1}{2} \beta]}.$$ (5.4)

From now on, we call $Z_{\beta} := E \Psi_{\beta}$ the partition function following the terminology in CFT literature, see e.g., [14, Sect. 11.3]. Set $\Lambda_{\beta} \equiv \Lambda_{\beta}(z, z) := \kappa \partial_z \log Z_{\beta}$. By (5.4) and (3.32), the field $\Lambda_{\beta}$ is evaluated in the identity chart of $\mathcal{C}_r$ as

$$\Lambda_{\beta} = \sqrt{\frac{\kappa}{2}} \left[ \sum \frac{\beta_j}{2} \left( H_{\chi}(r, z - z_j) + H_{\chi}(r, z - \bar{z}_j) \right) \right. \\
\left. + \sum \frac{\tau_j}{2} \left( H_{\chi}(r, z - q_j) + H_{\chi}(r, z - \bar{q}_j) \right) \right].$$ (5.5)

We mention that in connection with SLE$(\kappa, \Lambda_{\beta})$, the field $\Lambda_{\beta}$ will be used to describe the dynamics of the driving function $\xi$:

$$d \xi_t = \sqrt{\kappa} \, dB_t + \sqrt{\frac{\kappa}{2}} \text{Re} \left[ \sum \frac{\beta_j}{2} H_{\chi}(r - t, \xi_t - \tilde{g}_t(z_j)) \right. \\
\left. + \sum \frac{\tau_j}{2} H_{\chi}(r - t, \xi_t - \tilde{g}_t(q_j)) \right] \, dt, \quad \xi_0 = p.$$ 

Now we derive the level two degeneracy equation for $\Psi$.

Lemma 5.3 The one-leg operator $\Psi$ is a current primary field with charges $q = a, q_*= 0$ at $z$, i.e.,

$$J_0 \Psi = -ia \Psi, \quad J_0 \overline{\Psi} = 0, \quad J_n \Psi = J_n \overline{\Psi} = 0 \quad (n \geq 1).$$
Proof It follows from the definition that $\Psi$ is Virasoro primary. To show the above equations, it is enough to show that as $\zeta \to z$,

$$J(0)(\zeta)\Psi(z) \sim -ia\frac{\Psi(z)}{\zeta - z}, \quad J(0)(\zeta)\overline{\Psi}(z) \sim 0$$

hold in the identity chart. By Wick’s formula, we have

$$J(0)(\zeta)\Psi(z) = J(0)(\zeta) \odot \Psi(z) + iaE[J(0)(\zeta)\Phi(0)]\Psi(z)$$

$$- \sum \frac{\tau_j}{2} E[J(0)(\zeta)\Phi(z_j)]\Psi(z) - \sum \frac{\beta_j}{2} E[J(0)(\zeta)\Phi(q_j)]\Psi(z),$$

where $\Phi(0)$ is the dual boson in (3.30). It is also easy to see that a similar expression holds for $\overline{\Psi}$. Now lemma follows from (3.28) and (3.9).□

We now obtain the following level two degeneracy equation of $\Psi$.

Proposition 5.4 If $2a(a+b) = 1$, then

$$T_\beta \ast_z \Psi = \frac{1}{2a^2} \partial^2_z \Psi(z). \quad (5.6)$$

Proof By Proposition 5.1, we have $L_{-1} \Psi(z) = \partial_z \Psi(z)$. On the other hand, by definition (5.1) of the Virasoro generator, one can realize that $L_{-2} \Psi = T_\beta \ast_z \Psi$. Therefore the desired identity (5.6) follows from Proposition 5.2. □

We are now ready to show BPZ equation, Theorem B.

Proof of Theorem B By Proposition 5.4, we have

$$\frac{1}{a^2} \partial^2_z E\Psi(z)\mathcal{X} = 2E(T_\beta \ast_z \Psi)\mathcal{X}.$$ 

Then it follows from the basic property $T_\beta = A_\beta + E T_\beta$ that

$$2E(T_\beta \ast_z \Psi)\mathcal{X} = 2E(A_\beta \ast_z \Psi)\mathcal{X} + 2ET_\beta(z)E\Psi(z)\mathcal{X}.$$ 

This completes the proof. □

We now discuss how the one-leg operator $\Psi$ acts on Fock space fields in $D_r$ as a boundary condition changing operator. Given marked boundary points $p \in \mathbb{R}, \{q_j\}$ and $\{\xi_j\}$, set $\beta = \sum \beta_j \cdot q_j$ and $\tau = \sum \tau_j \cdot \xi_j$. By abuse of notation, for $z_j \in C_r$, we sometimes use the same symbol $\tau$ for $\sum \tau_j \cdot z_j$. From now on, we write $w_z = w(z), w_p = w(p), w_j = w(\xi_j)$ and $\tilde{w}_j = w(q_j)$, where $w : D_r \to C_r$ is a conformal transformation. The insertion of field $\Psi/E\Psi$ produces an operator $\mathcal{X} \mapsto \hat{\mathcal{X}}$ on Fock space fields. It is given by

$$\partial \mathcal{X} \mapsto \partial \hat{\mathcal{X}}, \quad \bar{\partial} \mathcal{X} \mapsto \bar{\partial} \hat{\mathcal{X}}, \quad \alpha \mathcal{X} + \beta \mathcal{Y} \mapsto \alpha \hat{\mathcal{X}} + \beta \hat{\mathcal{Y}}, \quad \mathcal{X} \odot \mathcal{Y} \mapsto \hat{\mathcal{X}} \odot \hat{\mathcal{Y}}.$$
and by the formula:

$$\hat{\Phi}_\beta(z) = \Phi_\beta(z) + 2a \text{arg} \Theta_\chi(w_p - w_z) + \sum \tau_j \text{arg} \left\{ \Theta_\chi(w_j - w_z) \Theta_\chi(\tilde{w}_j - w_z) \right\}. \quad (5.7)$$

When $\kappa = 4$, this insertion procedure produces specific height gap of GFF, which gives rise to the coupling with SLE(4, $\Lambda$). For instance, in [29], Hagendorf, Bernard and Bauer studied coupling of GFF with Dirichlet boundary condition and SLE(4, $\Lambda$). For more details, see Section 3.7.9. In the case of multiple force points, the (multivalued) harmonic function $\varphi := E\hat{\Phi}$ in $\tilde{C}_r \setminus \{p, \xi_1, \ldots, \xi_N\}$ has piecewise Dirichlet boundary conditions having additional jump of $2\pi a$ at $p$ and $2\pi \tau_j$ at $\xi_j$'s. Due to the neutrality condition, we require that all jumps should add up to 0. This type of boundary condition of GFF was studied by Izyurov and Kytölä [34].

As in Sect. 2.3, we let

$$\hat{E}[\mathcal{X}] = E[e^{\text{i}a \Phi^+_0(p)} - \sum \frac{i}{2} \Phi'_0(\xi_j) \mathcal{X}].$$

Then it is easy to check that $\hat{E}[\mathcal{X}] = E[\hat{\mathcal{X}}]$ for all $\mathcal{X} \in F_\beta$.

For a deeper discussion of the relation between the insertion procedure and the background charge modifications, we refer the reader to [35].

Examples It is instructive to compare the following examples with those in Sect. 3.7.

- For $J \in F_{(b)}$, the current field $\hat{J}$ is a PS-form of order $ib$. We have

$$\hat{J}(z) = J(z) + \frac{ia w'_z}{2} H_\chi(w_p - w_z) + \frac{i w'_z}{4} \sum \tau_j \left( H_\chi(w_j - w_z) + H_\chi(\tilde{w}_j - w_z) \right).$$

- For $T \in F_{(b)}$, the Virasoro field $\hat{T}$ is an S-form of order $c/12$. We have

$$\hat{T}(z) = -\frac{1}{2} \hat{\mathcal{J}} * \hat{J} + ib \hat{\partial} \hat{J} = T - (\tilde{\mathcal{J}} - \mathcal{J})(0) - \frac{1}{2} (\mathcal{J}^2 - J^2) + ib \hat{\partial} (\tilde{\mathcal{J}} - \mathcal{J})$$

$$= A(0) - \tilde{\mathcal{J}} J(0) + ib \mathcal{J}_0 + \frac{c}{12} S_w + (w'_z)^2 E T - \frac{1}{2} (\mathcal{J}^2 - J^2) + ib \hat{\partial} (\tilde{\mathcal{J}} - \mathcal{J}),$$

where $j := EJ$ and $\tilde{j} := E\hat{J}$. By Proposition 3.2, its evaluation in the identity chart of $\mathcal{C}_r$ is given as

$$E\hat{T}(z) = \frac{1}{8} \left[ a H_\chi(p - z) + \frac{1}{2} \sum \tau_j \left( H_\chi(\xi_j - z) + H_\chi(\tilde{\xi}_j - z) \right) \right]^2.$$
For any \( \mathcal{O} \in \mathcal{F}_b \), the multi-vertex field \( \hat{\mathcal{O}}[\sigma, \sigma_*] \) whose nodes do not intersect with \( \mathcal{S}_\psi \) is a differential with conformal dimension \((h_j, h_{*,j})\). We have

\[
\mathbf{E} \hat{\mathcal{O}}[\sigma, \sigma_*] = C(b)[\sigma, \sigma_*]\mathbf{E} V^\odot[a \cdot p + \tau/2, \tau/2] V^\odot[\sigma, \sigma_*]
= C(b)[\sigma, \sigma_*]\mathbf{e}^{-\mathbf{E} \Phi(0)[a \cdot p + \tau/2, \tau/2] \mathbf{E} \Phi(0)[\sigma, \sigma_*]}
\]

and its evaluation in the identity chart of \( \mathcal{C}_r \) is given as (up to a multiplicative constant)

\[
\mathbf{E} \hat{\mathcal{O}} = \Theta'(0) \sum \frac{1}{2}(\sigma_j^2 + \sigma_k^2) \prod \Theta_\chi(p - z_j)^{a_{\sigma_j}} \Theta_\chi(p - \bar{z}_j)^{a_{\bar{\sigma}_j}} \Theta_\chi(z_j - \bar{z}_j)^{\sigma_j \bar{\sigma}_j} \\
\times \prod_{j,k} (\Theta_\chi(\xi_j - z_k) \Theta_\chi(\bar{\xi}_j - \bar{z}_k))^2 (\Theta_\chi(\xi_j - \bar{z}_k) \Theta_\chi(\bar{\xi}_j - z_k))^2 \\
\times \prod_{j < k} \Theta_\chi(z_j - z_k)^{\sigma_j \sigma_k} \Theta_\chi(z_j - \bar{z}_k)^{\sigma_j \bar{\sigma}_k} \Theta_\chi(z_j - \bar{z}_k)^{\bar{\sigma}_j \sigma_k} \Theta_\chi(\bar{\xi}_j - \bar{z}_k)^{\bar{\sigma}_j \bar{\sigma}_k}.
\]

Suppose \( \beta \) is a background charge with supp \( \beta = \{q_k\} \) \((q_k \in D)\). As discussed in Sect. 4.3, we need the following residue form of Ward’s identity at \( q_k \) to complete the proof of Theorem A. For reader’s convenience, we borrow its proof from [35].

**Lemma 5.5** For any \( \mathcal{X}_\beta \in \mathcal{F}_\beta \) with nodes outside supp \( \beta = \{q_k\} \) and a vector field \( v \) holomorphic in a neighborhood of each of \( q_k \), we have

\[
\frac{1}{2\pi i} \oint_{(q_k)} v \mathbf{E} A_\beta \mathcal{X}_\beta = \mathbf{E} \mathcal{L}^+_v(q_k) \mathcal{X}_\beta.
\]

**Proof** Since \( \mathcal{X}_\beta \) is a scalar with respect to \( q_k \), we have

\[
\mathbf{E} \mathcal{L}^+_v(q_k) \mathcal{X}_\beta = v(q_k) \partial_{q_k} \mathbf{E} \mathcal{X}_\beta.
\]

Moreover, it follows from \( \mathbf{E} \mathcal{X}_\beta = \mathbf{E}[e^{\mathcal{C} i \Phi(\beta/2, \beta/2)} \mathcal{X}_0] \) that

\[
\partial_{q_k} \mathbf{E} \mathcal{X}_\beta = \frac{i \beta_k}{2} \mathbf{E} \mathcal{X}_0 J(q_k) \odot e^{\mathcal{C} i \Phi(\beta/2, \beta/2)}.
\]

On the other hand, by the residue calculus using (3.36), we have

\[
\frac{1}{2\pi i} \oint_{(q_k)} v \mathbf{E} A_\beta \mathcal{X}_\beta = \frac{i \beta_k}{2} v(q_k) \mathbf{E} J(q_k) \mathcal{X}_\beta.
\]
Thus all we need to show is
\[ EJ(q_k)X_\beta = EX_0 J(q_k) \odot e^{\Theta i[\beta/2, \beta/2]}. \] (5.8)

By Wick’s calculus and (3.35), we have that for \( q_k' \neq q_k \),
\[ J(q_k') \odot e^{\Theta i[\beta/2, \beta/2]} = J(q_k')e^{\Theta \beta(q_k')} + (J(q_k') - J\beta(q_k'))e^{\Theta i[\beta/2, \beta/2]}. \]

Therefore by (5.7), we have
\[ E \chi_0 J(q_k') \odot e^{\Theta i[\beta/2, \beta/2]} = E \chi_0 J(q_k')e^{\Theta \beta(q_k')} \]
\[ + (J(q_k') - J\beta(q_k'))E \chi_0 e^{\Theta i[\beta/2, \beta/2]} \]
\[ = E \chi_\beta J\beta(q_k') - J\beta(q_k')E \chi_\beta \]
\[ = E J(q_k').X_\beta. \]

Now (5.8) follows by taking the limit \( q_k' \to q_k \). \( \square \)

### 5.2 Null-vector equations

Throughout this subsection, until further notice, we consider the case that all marked points are on the outer boundary component. This allows us to simply consider the one-leg operator of the form \( \Psi_\beta(z) = \mathcal{O}_\beta[a \cdot z + \tau, 0] \). By (5.4), the associated partition function
\[ Z(p)(\equiv Z_\beta(p, \xi) \equiv Z_\beta[a \cdot p + \tau]) := E \Psi(p) \]
is evaluated in the cylinder \( \mathcal{C}_\tau \) as
\[ Z = C_\tau \prod_j \Theta_\chi(r, p - \xi_j)^{\tau_j} \prod_j \Theta_\chi(r, p - q_j)^{\alpha_j} \]
\[ \prod_{j<k} \Theta_\chi(r, \xi_j - \xi_k)^{\tau_j \tau_k} \prod_{j,k} \Theta_\chi(r, \xi_j - q_k)^{\tau_j \beta_k}, \]
where \( C_\tau = \Theta'(r, 0) \frac{a^2}{\tau} + \sum \frac{\tau^2_j}{2} \). It also follows from the neutrality condition that
\[ Z = C_\tau \exp \left( - \frac{(ap + \int z d\tau + \int z d\beta)^2 - (\int z d\beta)^2}{4(r + \chi)} \right) \]
\[ \times \prod_j \Theta(r, p - \xi_j)^{\tau_j} \prod_j \Theta(r, p - q_j)^{\alpha_j} \]
\[ \prod_{j<k} \Theta(r, \xi_j - \xi_k)^{\tau_j \tau_k} \prod_{j,k} \Theta(r, \xi_j - q_k)^{\tau_j \beta_k}. \] (5.9)
We write $\lambda_p, \lambda_j$ for the (holomorphic) conformal dimension of $\Psi$ with respect to $p$ and $\xi_j$. Then by (3.33), we have

$$\lambda_p = \frac{6 - \kappa}{2\kappa}, \quad \lambda_j = \frac{1}{2} \beta_j^2 - \beta_j b. \quad (5.10)$$

It is worth to pointing out that by the neutrality condition, the sum of all the exponents (of theta functions) in the expression (5.9) vanishes, i.e.,

$$\frac{a^2}{2} + \sum_j \frac{\tau_j^2}{2} + \sum_j a \tau_j + \sum_k a \beta_k + \sum_{j<k} \tau_j \tau_k + \sum_{j,k} \tau_j \beta_k$$

$$= \frac{1}{2} \left( a + \sum_j \tau_j + \sum_k \beta_k \right)^2 - \left( \sum_k \beta_k \right)^2 = 0.$$ 

Therefore it follows from asymptotic behaviors of theta function (see e.g., [9, pp. 65–69])

$$\lim_{r \to \infty} e^{r/4} \Theta(r, x) = 2 \sin \left( \frac{x}{2} \right), \quad \lim_{r \to \infty} e^{r/4} \Theta'(r, 0) = 1$$

that the partition function $Z$ has a non-trivial limit as $r \to \infty$. Indeed, the partition functions degenerate to (up to a multiplicative constant)

$$Z_\infty := \lim_{r \to \infty} Z = \prod_j \sin^{a\tau_j} \left( \frac{p - \xi_j}{2} \right) \prod_j \sin^{a\beta_j} \left( \frac{p - q_j}{2} \right)$$

$$\times \prod_{j<k} \sin^{\tau_j \tau_k} \left( \frac{\xi_j - \xi_k}{2} \right) \prod_{j,k} \sin^{\tau_j \beta_k} \left( \frac{\xi_j - q_k}{2} \right). \quad (5.11)$$

As a consequence of BPZ equation (Theorem B), we obtain a general form of the null-vector equation for the partition function $Z$. Here and subsequently, $\partial$ and $'$ stand for the differentiation with respect to $p$.

**Corollary 5.6** The partition function $Z$ satisfies

$$\partial_r Z = \frac{\kappa}{2} \partial^2 Z + \sum_j \left( \lambda_j H'(r, p - \xi_j) - H(r, p - \xi_j) \partial_{\xi_j} \right) Z$$

$$- \sum_j H(r, p - q_j) \partial_{q_j} Z + C(r, q) Z,$$
where

\[
C(r, q) = -\frac{1}{16} \left[ \sum \beta_k \left( H_\chi(r, q_k - p) + H_\chi(r, \bar{q}_k - p) \right) \right]^2
- \frac{b}{2} \sum \beta_k \left( H'_\chi(r, q_k - p) + H'_\chi(r, \bar{q}_k - p) \right)
- \frac{6 \xi_r(\pi)}{\kappa} + \frac{1}{2(r + \chi)}. \tag{5.12}
\]

**Proof** By applying the trivial string \( X \equiv 1 \) to Theorem B, we have

\[
\partial_r Z = \frac{\kappa}{2} \partial^2 Z - L^+_{v_p} Z + C(r, q) Z, \quad C(r, q) = -2h_{1,2} \frac{\xi_r(\pi)}{\pi} - 2 \mathcal{E} T_\beta.
\]

Now corollary follows from Proposition 3.5 and the fact that

\[
L^+_{v_p} Z = \sum \left( v_p(\xi_j) \partial \xi_j + \lambda_j v'_p(\xi_j) \right) Z + \sum v_p(q_j) \partial q_j Z.
\]

\[\square\]

**Example** Let us consider the simplest case that \( \beta = 0 \) and \( \tau = -a \cdot q \). Write \( \Psi_* := \mathcal{O}[a \cdot p - a \cdot q, 0] \). By (5.9), we have

\[
Z_* := \mathbb{E} \Psi_* = \Theta'_\chi(r, 0)^{\frac{3}{2}} \Theta_\chi(r, p - q)^{-\frac{3}{2}}. \tag{5.13}
\]

For \( \chi = 0 \) and \( \kappa = 4 \), the SLE associated with the partition function (5.13) was studied in [29, 34].

By translation invariance, we have \( \partial_q Z_* = -Z'_* \). Also by (5.10), the conformal dimension \( \lambda_q \) of \( \Psi_* \) with respect to \( q \) is given as

\[
\lambda_q = \frac{1}{2} a^2 + ab = \frac{1}{2} - \frac{1}{\kappa}.
\]

Therefore it follows from Corollary 5.6 that

\[
\partial_r Z_* = \frac{\kappa}{2} Z''_* + H Z'_* + \left( \frac{1}{2} - \frac{1}{\kappa} \right) H'Z_* + C(r) Z_*, \quad C(r) := -\frac{6 \xi_r(\pi)}{\kappa} + \frac{1}{2(r + \chi)}.
\]

Notice that for \( \kappa = 4 \), this corresponds to the null-vector Eq. (2.1).
5.3 BPZ-Cardy equations

In this subsection we prove BPZ-Cardy type equation, a key ingredient to connect CFT with SLE theory, and complete the proof of Theorem C.

Fix a marked point $p = \xi$ on the outer boundary. Recall that $\Psi_\beta$ is given by (5.3).

Let us write

$$Z_\beta(z, z) := \mathbb{E}\Psi_\beta(z, z), \quad \Lambda_\beta(z, z) := \kappa \partial_z \log Z_\beta(z, z).$$

For a Fock space functional $\mathcal{X}$ whose nodes do not intersect marked boundary points, set

$$R_z := \hat{\mathbb{E}}_z[\mathcal{X}] := \frac{\mathbb{E}[\Psi_\beta(z, z)\mathcal{X}]}{Z_\beta(z, z)} = \mathbb{E} \left[ e^{\sum \frac{e_j}{2} \Phi_{(0)}(z_j) + \frac{i}{2} a \Phi_{(0)}(z) + \frac{1}{2} \sum \Phi_{(0)}(\xi_j) + \frac{1}{2} \sum \Phi_{(0)}(\bar{\xi}_j)} \right].$$

Let us denote $\tilde{\mathcal{L}}_{v_0} z := \mathcal{L}_{v_0}(\bar{D}_r\{z\} \cup S_\Psi)$, i.e., $\tilde{\mathcal{L}}_{v_0} \Psi Y(z) X := \Psi Y(z) \tilde{\mathcal{L}}_{v_0} X$ if the nodes of $\mathcal{X}$ do not intersect $z$ and $S_\Psi$.

Now we prove the following BPZ-Cardy equation.

**Proposition 5.7** For any $\mathcal{X} \in \mathcal{F}_\beta$, in the $C_r$-uniformization, we have

$$\frac{1}{a^2} \partial_z^2 Z_\beta(z, z) + \Lambda_\beta(z, z) \partial_z Z_\beta(z, z) = \mathcal{L}^+_{v_0} + \mathcal{L}^-_{v_0}(\xi, z, q) R_z Z_\beta(z, z) + \partial_r R_z Z_\beta(z, z).$$

Here $\partial_\xi = \partial + \bar{\partial}$ is the operator of differentiation with respect to the real variable $\xi$ whereas $\partial_{\xi_j}$ denotes the one with respect to the complex variable $\xi_j$.

**Proof** Let us write $\xi = S\mathcal{X}$. Since $\Psi(z)$ is a holomorphic differential with respect to $z$, we rewrite Theorem B as

$$\frac{1}{a^2} \partial_z^2 R_z Z_\beta(z, z) = \mathcal{L}^+_{v_0} + \mathcal{L}^-_{v_0}(\xi, z, q) R_z Z_\beta(z, z) + \partial_r R_z Z_\beta(z, z).$$

By applying the trivial string $\mathcal{X} \equiv 1$ to the above, we have

$$\frac{1}{a^2} \partial_z^2 Z_\beta(z, z) = \mathcal{L}^+_{v_0} + \mathcal{L}^-_{v_0}(\xi, z, q) Z_\beta(z, z) + \partial_r Z_\beta(z, z).$$

Subtracting the above two equations, we obtain

$$\frac{1}{a^2} \partial_z^2 R_z + \Lambda_\beta(z, z) \partial_z R_z = \mathcal{L}^+_{v_0} + \mathcal{L}^-_{v_0}(\xi, z, q) R_z + \partial_r R_z.$$
We now take the limit $z \to \xi$ and $z_j \to \xi_j$. Since $\partial \xi = \partial + \bar{\partial}$, we obtain
\[
\frac{1}{\alpha^2} \partial^2_{\xi} R_\xi + \Lambda (\xi, \xi) \partial R_\xi = \mathcal{L}_{\nu_\xi} (\xi, \xi) R_\xi + \mathcal{L}_{\nu_{\bar{\xi}}} (\xi) R_\xi + \partial R_\xi.
\]

Now proposition follows from the fact $R_\xi$ is a scalar field with respect to each $\xi_j$. □

Now we are ready to prove Theorem C.

**Proof of Theorem C** Let us denote $R_\xi = \hat{E}_\xi [\mathcal{X}]$. Then
\[
M_t = m \left( r - t, \xi, \tilde{g}_t (\xi), t \right), \quad m (r, \xi, \xi, t) = \left( R (r, \xi, \xi) \| \tilde{g}_t^{-1} \right).
\]

Hence, by Itô’s formula, we have
\[
dM_t = -\partial_r m \ dt + \partial_\xi m \ d\xi_t + \partial^2_{\xi} m \frac{d\langle \xi \rangle_t}{2} + (L_t R \| \tilde{g}_t^{-1} \) \ dt
\]
\[
+ \sum_{j=1}^N \left[ \partial_\xi j m \ d\xi_j (t) + \tilde{\partial}_\xi j m \ d\tilde{\xi}_j (t) \right],
\]
where
\[
L_t := \frac{d}{ds} \bigg|_{s=0} \left( R (r - t, \xi, \tilde{g}_t (\xi)) \| \tilde{g}_t^{-1} \right).
\]
Let $f_{s, t} = \tilde{g}_{t+s} \circ \tilde{g}_t^{-1}$. Then since the time-dependent flow $f_{s, t}$ satisfies the ODE
\[
\frac{d}{ds} f_{s, t} = -v_{\xi_{t+s}} f_{s, t}, \quad v_\xi (z) = H (r, \xi - z),
\]
we obtain
\[
f_{s, t} = \text{id} - s v_{\xi_t} + o(s).
\]

From the fact that the fields in $\mathcal{F}_\beta$ depends smoothly on local charts, it follows
\[
L_t = -\left( \mathcal{L}_{v_{\tilde{g}_t}} R_{\bar{\xi}} \| \tilde{g}_t^{-1} \right).
\]

By the Loewner’s equation $d\xi_j (t) = -v_{\xi_j} (\xi_j (t))$, it follows from Proposition 5.7 that the drift term of $dM_t$ vanishes. This completes the proof of theorem. □
Example The simplest examples of SLE martingale-observables are the 1-point function $M(z)$ and the 2-point function $N(z_1, z_2)$ of the bosonic field $\Phi_\alpha$. For instance, when $\chi = \infty$, for all $\kappa$,

$$\text{d}M_t(z) = -\sqrt{2} \text{Im} \left( H(w_t(z)) \right) dB_t + \left( a + \sum \tau_j \right) \text{Im} \left( \frac{H^2}{2} + H' \right)(w_t(z)) \text{d}t.$$ 

Due to the neutrality condition (5.2), $M_t$ is driftless. For the 2-point function, one can easily show that

$$\text{d}N_t(z_1, z_2) = -\sqrt{2} \left( M_t(z_1) \text{Im} \left( H(w_t(z_2)) \right) + M_t(z_2) \text{Im} \left( H(w_t(z_1)) \right) \right) dB_t + \left( a + \sum \tau_j \right) \left\{ M_t(z_1) \left( \frac{H^2}{2} + H' \right)(w_t(z_1)) \right\} \text{d}t.$$ 

Thus the drifts disappear again by the neutrality condition. We remark that this is equivalent to Hadamard’s variational formula. In [34], Izyurov and Kytölä studied these types of martingale-observables for $\kappa = 4$ with Dirichlet boundary condition.

Example Let us consider the case that there is only one force point $q$ on $\mathbb{R}_r$. By the neutrality condition, we impose the symmetric charge $(-a/2, -a/2)$ at $q$. Here, we set $\beta = 0$. Then by (5.5), the drift function $\Lambda$ is given by

$$\Lambda(p, q) = H_I(r, p - \text{Re} q) + \frac{p - \text{Re} q}{r + \chi}. \quad (5.14)$$

Thus the driving process $\xi_t$ is given by

$$d\xi_t = \sqrt{\kappa} dB_t + \left( H_I(r - t, \xi_t - \text{Re} \tilde{G}_t(q)) + \frac{\xi_t - \text{Re} \tilde{G}_t(q)}{r + \chi - t} \right) \text{d}t, \quad \xi_0 = p.$$ 

Note that in the degenerate limit $r \to \infty$, the associated SLE($\kappa, \Lambda$) converges to the law of radial SLE($\kappa$). We remark that when $\kappa = 4$, the SLE(4, $\Lambda$) corresponds to the scaling limit of level lines of compactified GFF (Cf. [34]), see Fig. 3.
Using the bosonic observables, one can show the following version of the left-passage probability of SLE trace. For this, recall that

\[
\tilde{\Theta}(r, z) := \Theta_0(r, z) = \Theta(r, z)e^{\frac{z^2}{4r}}.
\]

**Proposition 5.8** For \( \chi = 0 \) let \( \eta \) be the trace of \( \text{SLE}(\kappa, \Lambda) \), where \( \Lambda \) is given by (5.14). Then \( \eta \) a.s. ends at the force point \( q \). Furthermore, when \( \kappa = 4 \), the probability \( P(z) \) that \( z \in S_r \) is left to \( \eta \) is given by

\[
P(z) = \frac{1}{\pi} \left( \arg \tilde{\Theta}(r, z - p) - \arg \tilde{\Theta}(r, z - q) \right).
\]

**Proof** By Loewner’s equation, the angle-difference process \( X_t := \xi_t - \text{Re} \g(t)(q) \) satisfies

\[
dX_t = \sqrt{\kappa} dB_t + \frac{X_t dt}{r + \chi - t}, \quad X_0 = p - \text{Re} q.
\]

Therefore the process \( X_t \) is realized as a Brownian bridge starting from \( X_0 \), which reaches 0 at time \( t = r + \chi \), see e.g., [53]. Thus when \( \chi = 0 \), we have \( \lim_{t \uparrow r} X_t = 0 \), which means \( \eta \) a.s. ends at the force point \( q \).

For \( \kappa = 4 \), by Theorem C, the bosonic observable

\[
M_t(z) := \frac{1}{\pi} \left( \arg \tilde{\Theta}(r - t, w_t(z)) - \arg \tilde{\Theta}(r - t, w_t(z) - w_t(q)) \right)
\]

is a (bounded) martingale. On the other hand, note that \( M = M_0 \) is a harmonic function satisfying the boundary condition

\[
M(z) = \begin{cases} 0 & \text{if } z \in (0, p) \cup (ir, q), \\
1 & \text{if } z \in (p, 2\pi) \cup (q, 2\pi + ir). 
\end{cases}
\]

Let us write \( E \) for the event that \( z \) is left to \( \eta \). Then we have

\[
\lim_{t \uparrow r} M_t(z) = \begin{cases} 0 & \text{on } E, \\
1 & \text{otherwise.} 
\end{cases}
\]

Then proposition follows from optional stopping theorem. \( \square \)

### 6 Martingale-observables for annulus SLE with non-atomic charges

We have discussed the use of atomic charges to connect CFT of GFF and annulus SLE with force points. In this section we utilize some intrinsic ideas in CFT such as chiral bosonization or screening method to introduce an implementation
of infinite/continuous charge distribution. Consequently, we construct martingale-observables for annulus SLEs with natural geometric properties. Examples include SLEs with measure-valued boundary conditions, SLEs with periodic drift functions in space variables, and SLEs with reversibility. In particular, some special cases of these SLEs coincide with the scaling limits of certain statistical physics models, which we will discuss in more detail in the following section.

In Sect. 6.1 we explain the use of non-atomic background charges and present the null-vector equation for the associated partition function. In Sect. 6.2 we introduce the notion of periodization (also known as chiral bosonization) and prove Theorem D, which establishes the relation between CFT and annulus SLEs with periodic drift functions. Section 6.3 is devoted to the study of screening method with the goal to provide CFT partition functions and martingale-observables for reversible annulus SLEs (Theorem E).

### 6.1 Non-atomic background charges

In [58], Powell and Wu studied GFF with general Dirichlet boundary conditions and its coupling with SLE in a simply connected domain, see also a recent work [48]. Intending to account for bosonic fields with such boundary conditions from the viewpoint of CFT, we implement general measure-valued background charges beyond the linear combination of the atomic masses.

Let \( \tilde{\beta} \) be a (signed) measure on \( \mathbb{R}_r \) with total mass \(-2b\) and set \( \beta := 2b \cdot q + \tilde{\beta} \) where \( q \in \mathbb{R} \). We write \( \beta_n \) for its discrete counterpart given by

\[
\beta_n := 2b \cdot q + \sum_{k=1}^{n} \frac{\beta_k}{n} \cdot q_k, \quad \left( \sum \beta_k = -2b \right).
\]

Set \( \tau = -a \cdot q \) and define the one-leg operator \( \Psi_\beta \) as the continuum limit of its discrete counterpart, i.e.,

\[
\Psi_\beta[a \cdot p - a \cdot q] := \lim_{n \to \infty} \Psi_{\beta_n}[a \cdot p - a \cdot q].
\]

Let us also write

\[
Z_\beta := E \Psi_\beta[a \cdot p - a \cdot q], \quad Z_{\beta_n} := E \Psi_{\beta_n}[a \cdot p - a \cdot q].
\]

Denoting \( q_k = s_k + ir \), by (6.2) below, we have

\[
Z_\beta = \Theta'(0)^{\frac{1}{2}} \Theta(\sigma_q)^{1-\frac{\kappa}{2}} \exp \left( a \int \log \frac{\Theta_{I}(p - s)}{\Theta_{I}(q - s)} \, d\tilde{\beta}(s) \right) \\
\times \exp \left[ \left( 1 - \frac{6}{k} \right) \frac{(p - q)^2}{4(r + \chi)} + a \frac{p - q}{4(r + \chi)} \int (p + q - 2s) \, d\tilde{\beta}(s) \right] (6.1)
\]

in the identity chart of \( C_r \).
We now derive the null-vector equation for $Z_\beta$.

**Proposition 6.1** The partition function $Z_\beta$ satisfies

$$\partial_r Z_\beta = \frac{\kappa}{2} \partial_p^2 Z_\beta + H(p - q) \partial_p Z_\beta + h_{1,2} H'(p - q) Z_\beta + F_\beta Z_\beta,$$

where

$$F_\beta = -2 \left( \mathcal{E} T_\beta(p) + h_{1,2} \frac{\zeta_r(\pi)}{\pi} \right)$$

$$- \frac{a}{2} \int \left( H(p - q) - H_1(p - s) \right) \left( H_1(p - s) - H_1(q - s) + \frac{p - q}{r + \chi} \right) d\tilde{\beta}(s).$$

Here

$$\mathcal{E} T_\beta(p) = \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} + b^2 \left[ H'(p - q) + \frac{1}{2} \left( H(p - q) + \frac{p - q}{r + \chi} \right)^2 \right]$$

$$+ \frac{1}{8} \left[ \int \left( H_1(p - s) + \frac{p - s}{r + \chi} \right) d\tilde{\beta}(s) \right]^2 + \frac{b}{2} \int H'_1(p - s) d\tilde{\beta}(s).$$

**Proof** By (5.9), we have (up to a multiplicative constant)

$$Z_\beta = \Theta(p - q)^{1 - \frac{6}{r}} \prod_{k=1}^{n} \left( \frac{\Theta_1(p - s_k)}{\Theta_1(q - s_k)} \right)^{\beta_k}$$

$$\times \prod_{k=1}^{n} \exp \left[ \left( 1 - \frac{6}{k} \right) \frac{(p - q)^2}{4(r + \chi)} + a \sum_{k=1}^{n} \beta_k \frac{(p - q)(p + q - 2s_k)}{4(r + \chi)} \right]. \quad (6.2)$$

By Theorem B, the partition function $Z_\beta$ satisfies the following form of the null-vector equation

$$\partial_r Z_\beta = \frac{\kappa}{2} \partial_p^2 Z_\beta - H(p - q) \partial_q Z_\beta + h_{1,2} H'(p - q) Z_\beta$$

$$- 2 \left( \mathcal{E} T_\beta(p) + h_{1,2} \frac{\zeta_r(\pi)}{\pi} \right) Z_\beta$$

$$- \sum H_1(p - s_k) \partial_{s_k} Z_\beta + i \left( \partial_{q_k} - \tilde{\partial}_{q_k} \right) Z_\beta.$$

It follows from the translation invariance $(\partial_p + \partial_q + \sum \partial_{s_k}) Z_\beta = 0$ that this equation is rewritten as

$$\partial_r Z_\beta = \frac{\kappa}{2} \partial_p^2 Z_\beta + H(p - q) \partial_p Z_\beta + h_{1,2} H'(p - q) Z_\beta$$

$$- 2 \left( \mathcal{E} T_\beta(p) + h_{1,2} \frac{\zeta_r(\pi)}{\pi} \right) Z_\beta.$$
\[ + \sum \left( H(p - q) - H_1(p - s_k) \right) \partial_{s_k} Z_{\beta}^n + i (\partial_{q_k} - \bar{\partial}_{q_k}) Z_{\beta}^n. \]

Note that by (6.2) we have
\[ \frac{\partial_{q_k} Z_{\beta}^n}{Z_{\beta}^n} = -\frac{a \beta_k}{2n} \left( H_1(p - s_k) - H_1(q - s_k) + \frac{p - q}{r + \chi} \right). \]

This gives rise to
\[ \partial_r Z_{\beta}^n = \frac{\kappa}{2} \partial_p Z_{\beta}^n + H(p - q) \partial_p Z_{\beta}^n + h_{1,2} H'(p - q) Z_{\beta}^n + F_{\beta} Z_{\beta}^n, \]
where
\[ F_{\beta} = -2 \left( E_T^{\beta} + h_{1,2} \frac{\zeta_r(\pi)}{\pi} \right) \]
\[ - \frac{a}{2} \sum \frac{\beta_k}{n} \left( H(p - q) - H_1(p - s_k) \right) \left( H_1(p - s_k) - H_1(q - s_k) + \frac{p - q}{r + \chi} \right). \]

Furthermore, by Proposition 3.5, we have
\[ E_T^{\beta}(p) = \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} + b^2 \left( H'(p - q) + \frac{H(p - q) + \frac{p - q}{r + \chi}}{2} \right)^2 \]
\[ + \frac{1}{8} \left[ \sum \frac{\beta_k}{n} \left( H_1(p - s_k) + \frac{p - s_k}{r + \chi} \right) \right]^2 + \frac{b}{2} \sum \frac{\beta_k}{n} H'_1(p - s_k). \]

Now proposition follows by letting \( n \to \infty. \)

**Example** Let \( \tilde{\beta} \) be the uniform (signed) measure on \([−\pi, \pi] \) with total mass \( -2b \). In this case, by (6.1) and the periodicity (3.5) of \( \Theta_1(r, \cdot) \), we have
\[ Z_{\beta} = \Theta'(0)^{\frac{1}{2}} \Theta(p - q)^{1 - \frac{6}{k}} \exp \left[ \left( 1 - \frac{6}{k} \right) \frac{(p - q)^2}{4(r + \chi)} \right]. \]

Note also that we have
\[ E_T^{\beta}(p) = \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)} + b^2 \left( H' + \frac{H^2}{2} \right)(p - q) \]
\[ + b^2 \left( \frac{p - q}{r + \chi} H(p - q) + \frac{5}{8} \frac{(p - q)^2}{(r + \chi)^2} \right). \]
On the other hand, it follows from (3.11) that

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( H(p - q) - H_1(p - s) \right) \left( H_1(p - s) - H_1(q - s) + \frac{p - q}{r + \chi} \right) ds \]

\[ = \frac{p - q}{r + \chi} H(p - q) - \frac{1}{2\pi} \int_{-\pi}^{\pi} H_1(p - s) \left( H_1(p - s) - H_1(q - s) \right) ds \]

\[ = \frac{p - q}{r + \chi} H(p - q) + \left( H' + \frac{H^2}{2} \right)(p - q) + \frac{6}{\pi} \zeta_r(\pi). \]

Therefore we obtain (up to an additive constant)

\[ F_\beta = \left( 2 - \kappa \right) \left( 1 - \frac{6}{\kappa} \right) \left[ \frac{H'}{2} + \frac{H^2}{4} \right](p - q) + \frac{p - q}{2(r + \chi)} H(p - q) + \frac{(p - q)^2}{4(r + \chi)^2}. \]

**Example (GFF with general boundary data)** Here we focus on the case when \( b = \chi = 0 \). Namely, we consider the Dirichlet boundary condition and \( \beta \) is a measure defined on the upper boundary component of \( \mathcal{C}_r \) satisfying \( \int \beta = 0 \). Then the associated bosonic field \( \hat{E} \Phi_\beta \) satisfies the boundary condition

\[ \hat{E} \Phi_\beta(z) = \begin{cases} 
-\lambda & \text{if } z \in (0, p) \cup (q, 2\pi), \\
+\lambda & \text{if } z \in (p, q), \\
\left( 1 - \frac{2}{a} \int_0^{\text{Re} z} d\beta(s) \right) \lambda & \text{if } z \in \mathbb{R}_r,
\end{cases} \quad (6.3) \]

where \( \lambda = a\pi = \pi/\sqrt{2} \). Notice that the neutrality condition \( \int \beta = 0 \) is equivalent to the fact that the harmonic function \( \hat{E} \Phi_\beta \) is well defined in the cylinder. To see (6.3), note that if \( \beta \) is a linear combination of atomic measure of the form \( \beta = \sum_{k=1}^n \frac{\beta_k}{n} \cdot \delta_{q_k} \), the correlation function of the bosonic observable has piecewise Dirichlet boundary condition having additional jump of \( 2\pi a \) at \( p \), \( -2\pi a \) at \( q \) and \( 2\pi \beta_k/n \) at \( q_k \)'s.

For a general background charge \( \beta \), we have (up to a multiplicative constant)

\[ Z_\beta(p, q) = \Theta(p - q)^{-\frac{1}{2}} \exp \left( -\frac{(p - q + \sqrt{2} \int s d\beta(s))^2}{8r} \right) \]

\[ \times \exp \left( \frac{1}{\sqrt{2}} \int \log \left( \frac{\Theta_1(p - s)}{\Theta_1(q - s)} \right) d\beta(s) \right). \]

By Proposition 6.1, the null-vector equation for \( Z_\beta \) is given by

\[ \partial_r Z_\beta = 2 \partial_p^2 Z_\beta + H(p - q) \partial_p Z_\beta + \frac{1}{4} H'(p - q) Z_\beta + F_\beta Z_\beta, \]
where
\[
F_\beta = -\frac{3}{2} \frac{\zeta_r(\pi)}{\pi} + \frac{1}{2r} - \frac{1}{4} \left[ \int \left( H_1(p - s) + \frac{p - s}{r} \right) d\beta(s) \right]^2 \\
+ \frac{1}{2\sqrt{2}} \int H_1(p - s) \left( H_1(p - s) - H_1(q - s) + \frac{p - q}{r} \right) d\beta(s).
\]

Moreover, by (3.11), one can also write \( F_\beta \) as
\[
F_\beta = -\frac{3}{2} \frac{\zeta_r(\pi)}{\pi} + \frac{1}{2r} - \frac{1}{4} \left[ \int \left( H_1(p - s) + \frac{p - s}{r} \right) d\beta(s) \right]^2 \\
- \frac{1}{2\sqrt{2}} \int \left( H'_1 - \frac{H_1^2}{2} \right)(p - s) + \left( H'_1 + \frac{H_1^2}{2} \right)(q - s) d\beta(s) \\
+ \frac{1}{2\sqrt{2}} \frac{p - q}{r} \int H_1(p - s) d\beta(s).
\]

**Remark** Note that if \( \beta = \beta \cdot q_1 - \beta \cdot q_2 \) for some \( q_1, q_2 \in \mathbb{R} \), satisfying \( q_2 = q_1 + 2\pi \), the boundary condition (6.3) reduces to (7.3) below. Also it follows from the periodicity of \( H(r, \cdot) \) that
\[
Z(p, q) = \Theta'(0)^\frac{1}{2} \Theta(r, x)^{-\frac{1}{2}} \exp \left( -\frac{x^2 - 2\mu x}{8r} \right), \quad \mu = 2\sqrt{2}\beta \pi, \quad (x = p - q)
\]
and
\[
F_\beta(p, q) = -\frac{3}{2} \frac{\zeta_r(\pi)}{\pi} + \frac{1}{2r} \left( \beta \pi \right)^2.
\]

### 6.2 Periodization

In this section we present an implementation of weighted summation of the chiral conformal fields. We refer to [66] for a similar idea (under the name of *chiral bosonization*) on compact Riemann surfaces. This allows us to construct some important examples of SLE martingale-observables in the next section.

For a suitable weight function \( \omega : \mathbb{Z} \to \mathbb{R}_+ \), let us define
\[
\Psi_\beta^\omega(p, \xi) := \sum_{n \in \mathbb{Z}} \omega(n) \Psi_\beta(p + 2n\pi, \xi)
\]
and write
\[
Z_\beta^\omega(p, \xi) := E \Psi_\beta^\omega(p, \xi), \quad \Lambda_\beta^\omega(p, \xi) := \kappa \left. \partial_\xi \right|_{\xi = p} \log Z_\beta^\omega(\xi, \xi).
\]
We remark that in [74], Zhan used the weight function
\[ \omega(n) := e^{-\frac{2\pi s_0}{\kappa} n}, \quad (s_0 \in \mathbb{R}) \]
to construct partition functions having rotational periodicity.

**Example** Let us consider the case that all \( \xi_j \)'s and \( q_k \)'s are on \( \mathbb{R}_r \). Then up to a multiplicative constant,
\[
Z_\beta(p, \xi) = \exp \left( - \frac{(ap + \sum_j \tau_j \text{Re} \xi_j + \sum_k \beta_k \text{Re} q_k)^2}{4(r + \chi)} \right)
\times \prod_j \Theta_I(r, p - \text{Re} \xi_j)^{\alpha \tau_j} \prod_j \Theta_I(r, p - \text{Re} q_j)^{\alpha \beta_j}
\prod_{j < k} \Theta(r, \xi_j - \xi_k)^{\tau_j \tau_k} \prod_{j, k} \Theta(r, \xi_j - q_k)^{\tau_j \beta_k}.
\]

Note that for \( \chi < \infty \), the partition function \( Z_\beta(p, \xi) \) is not \( 2\pi \)-periodic with respect to the space variable \( p \). Set \( \omega(n) \equiv 1 \). By (3.3), we have
\[
Z_{\beta}^{\omega}(p, \xi) = \Theta_I \left( \frac{k}{2}(r + \chi), p + \sum_j \frac{\tau_j}{a} \text{Re} \xi_j + \sum_k \frac{\beta_k}{a} \text{Re} q_k + \pi \right)
\times \prod_j \Theta_I(r, p - \text{Re} \xi_j)^{\alpha \tau_j} \prod_k \Theta_I(r, p - \text{Re} q_k)^{\alpha \beta_k}
\prod_{j < k} \Theta(r, \xi_j - \xi_k)^{\tau_j \tau_k} \prod_{j, k} \Theta(r, \xi_j - q_k)^{\tau_j \beta_k}
\]
and
\[
\Lambda_{\beta}^{\omega}(p, \xi) = \frac{k}{2} H_I \left( \frac{k}{2}(r + \chi), p + \sum_j \frac{\tau_j}{a} \text{Re} \xi_j + \sum_k \frac{\beta_k}{a} \text{Re} q_k + \pi \right)
+ \sqrt{\frac{k}{2}} \left[ \sum_j \tau_j H_I(r, p - \text{Re} \xi_j) + \sum_k \beta_k H_I(r, p - \text{Re} q_k) \right].
\]

Then it follows from the periodicity of \( H_I \) that \( \Lambda_{\beta}^{\omega} \) is well-defined in the cylinder \( C_r \).

We now prove Theorem D.

**Proof of Theorem D** Let \( \mathcal{X} \) be a string of fields in the OPE family \( \mathcal{F}_\beta \) and set
\[
R_\xi = \hat{E} \mathcal{X} := \frac{E \Psi_\beta^{\omega}(\xi, \xi) \mathcal{X}}{E \Psi_\beta^{\omega}(\xi, \xi)}.
\]
By the argument presented in the proof of Theorem C, it suffices to show the following form of BPZ-Cardy equation

$$
\frac{1}{a^2} \partial_\xi^2 R_\xi + \Lambda^{\omega}_\beta \partial_\xi R_\xi = \mathcal{L}_{v_\xi} R_\xi + \partial_r R_\xi.
$$

(6.4)

By Theorem B, we have

$$
\frac{1}{a^2} \partial_\xi^2 \mathbf{E}_{\Psi_\beta}(\xi, \xi) = \left( \mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2} \frac{\zeta_r(\pi)}{\pi} + 2 \mathbf{E}_T(\xi) \right) \mathbf{E}_{\Psi_\beta}(\xi, \xi).
$$

Therefore we obtain

$$
\frac{1}{a^2} \partial_\xi^2 \mathbf{E}_{\Psi_\beta}(\xi, \xi) = \left( \mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2} \frac{\zeta_r(\pi)}{\pi} \right) \mathbf{E}_{\Psi_\beta}(\xi, \xi)
+ 2 \sum_{n \in \mathbb{Z}} \mathbf{E}_T(\xi + 2n\pi) \mathbf{E}_{\Psi_\beta}(\xi + 2n\pi, \xi).
$$

By Proposition 3.5 and the neutrality condition \( \int \beta = 0 \),

$$
\mathbf{E}_T(\xi) = \frac{1}{32} \left[ \sum_k \beta_k \left\{ H(r, q_k - \xi) + H(r, \bar{q}_k - \xi) + 2 \frac{\text{Re} q_k}{r + \chi} \right\} \right]^2
+ \frac{b}{4} \sum \beta_k \left\{ H'(r, q_k - \xi) + H'(r, \bar{q}_k - \xi) \right\} + \frac{\zeta_r(\pi)}{2\pi} - \frac{1}{4(r + \chi)}.
$$

Thus we have for any \( n \in \mathbb{Z} \),

$$
\mathbf{E}_T(\xi + 2n\pi) = \mathbf{E}_T(\xi),
$$

which leads to

$$
\frac{1}{a^2} \partial_\xi^2 R_\xi Z^{\omega}_\beta(\xi, \xi) = \left( \mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2} \frac{\zeta_r(\pi)}{\pi} + 2 \mathbf{E}_T(\xi) \right) R_\xi Z^{\omega}_\beta(\xi, \xi).
$$

By applying the trivial string \( \mathcal{X} \equiv 1 \) to the above, we have the null-vector equation

$$
\frac{1}{a^2} \partial_\xi^2 Z^{\omega}_\beta(\xi, \xi) = \left( \mathcal{L}_{v_\xi} + \partial_r + 2h_{1,2} \frac{\zeta_r(\pi)}{\pi} + 2 \mathbf{E}_T(\xi) \right) Z^{\omega}_\beta(\xi, \xi).
$$

Subtracting the above two equations, we obtain (6.4), which completes the proof. \( \Box \)
6.3 Screening

From now on, we focus on a reversible SLE($\kappa, \Lambda$) starting from $p$ with one force point $q$. In this subsection we discuss the screening method to construct family of solutions to the null-vector equation

$$\partial_r Z = \frac{\kappa}{2} Z'' + HZ' + \left(\frac{3}{\kappa} - \frac{1}{2}\right) H'Z + C(r)Z$$

(6.5)

for the chordal type annulus partition function and prove Theorem E. The screening is a well-known method in CFT to find solutions of the Knizhnik-Zamolodchikov type equations, see e.g., [22, 65].

In general, the screening method is useful particularly in the theory of multiple SLEs. For instance, in [18], Dubédat found Euler integral representations for solutions to a system of PDEs characterizing commuting SLEs in $\mathbb{H}$ by means of screening. We refer the reader to [25,42,47,57] and references therein for recent studies on solutions to such PDE system and the geometric properties of associated multiple SLEs.

For $q_1, q_2 \in \mathbb{C}$, we set

$$\beta = \beta \cdot q_1 - \beta \cdot q_2.$$  

(6.6)

Here we assume that $q_1, q_2$ are two different fibers of a marked point in annulus satisfying $q_2 = q_1 + 2\pi$. For $p, q \in \mathbb{R}$ and $\zeta \in \mathbb{C}$, write

$$\Psi^\beta_\rho(p, q, \zeta) := \mathcal{O}_\beta[a \cdot p + a \cdot q - 2a \cdot \zeta, 0], \quad Z^\beta_\rho := E\Psi^\beta_\rho.$$  

Then the conformal dimension of $\Psi^\beta_\rho$ at $p, q$ and $\zeta$ are given by

$$\lambda_p = \lambda_q = \frac{6 - \kappa}{2\kappa}, \quad \lambda_\zeta = 1.$$  

(6.7)

By (5.9), we have

$$Z^\beta_\rho(p, q, \zeta) = \Theta'(0)^6 \Theta_\chi(p - q)^2 \Theta_\chi(p - \zeta)^{-4} \Theta_\chi(q - \zeta)^{-4} \times \frac{\Theta_\chi(p - z_0) \Theta_\chi(q - z_0) \Theta_\chi(\zeta - z_1)^2}{\Theta_\chi(p - z_1) \Theta_\chi(q - z_1) \Theta_\chi(\zeta - z_0)^2}.$$  

(6.8)

In particular since $q_2 - q_1 = 2\pi$ and (3.4), this expression reduces to (up to a multiplicative constant)

$$Z^\beta_\rho(p, q, \zeta) = \Theta'(0)^6 \exp\left(\frac{\beta^2 \pi^2}{r + \chi}\right) \Theta(p - q)^2 \Theta(p - \zeta)^{-4} \Theta(q - \zeta)^{-4} \times \exp\left(-\frac{\left(\Sigma - 2\beta \pi \right)^2}{4(r + \chi)}\right).$$  

(6.8)
where \( \Sigma = a(p + q - 2\xi) \). For a general \( \kappa > 0 \), we need to specify a principal branch (as a function of \( \xi \)) in the expression (6.8). In what follows, we use the branch cut \([-\infty, q] \cup [p, \infty]\).

For a closed contour \( \gamma \), set

\[
\Psi_{\beta} \equiv \Psi_{\beta}(p, q) \equiv C(\kappa) \oint_{\gamma} \Psi_{\beta}^\gamma(p, q, \xi) d\xi,
\]

where the normalization constant \( C(\kappa) \) is given by

\[
C(\kappa) = \sin^{-2} \left( \frac{4\pi}{\kappa} \right) \frac{1}{4\Gamma(1 - 4/\kappa)}. \tag{6.9}
\]

Here \( \Gamma \) is the Gamma function. It is easy to see that \( C(\kappa) \) has a simple pole at \( \kappa \) if and only if \( 4/\kappa \) is a positive integer. By (6.8), the partition function

\[
Z_{\beta} \equiv Z_{\beta}(r, p - q) := E\Psi_{\beta} = C(\kappa) \oint_{\gamma} Z_{\beta}^\gamma(p, q, \xi) d\xi \tag{6.10}
\]

is evaluated as (up to a multiplicative constant)

\[
Z_{\beta} = \Theta(p - q)^{\frac{\beta}{2}} \oint_{\gamma} \Theta(p - \xi)^{-\frac{4}{\beta}} \Theta(q - \xi)^{-\frac{4}{\beta}} \exp \left( -\frac{(\Sigma - 2\beta\pi)^2}{4(r + \chi)} \right) d\xi. \tag{6.11}
\]

**Remark** In general, partition functions \( Z_{\beta}(r, \cdot) \) do not have periodic property with respect to \( 2\pi \) except the case \( \chi = \infty \) (i.e., ER b.c.). On the other hand, for \( \chi = 0 \) (i.e., Dirichlet b.c.) \( Z_{\beta} \) has following rotational periodicity with respect to \( 2ir \).

\[
Z_{\beta}(r, x + 2ir) = \exp \left( a(2\beta - a)\pi i \right) Z_{\beta}(r, x). \tag{6.12}
\]

We denote by \( \Pi_1 \equiv \Pi_1(\mathbb{C}\setminus[p, q], \zeta_0) \) the fundamental group of \( \mathbb{C}\setminus[p, q] \) with base point \( \zeta_0 \). If \( 4/\kappa \) is not a positive integer, in order to make the partition function \( Z_{\beta} \) single-valued, the homotopy class \( [\gamma] \) should reside in the commutator subgroup of the \( \Pi_1 \). Moreover, this specific choice (6.9) of \( C(\kappa) \) makes the expression (6.10) non-trivial for all values of \( \kappa > 0 \) after removing possible singularities. We will explain this in Proposition 6.2 below.

The simplest example of such \( \gamma \) is the Pochhammer contour \( \mathcal{P}(p, q) \) entwining \( p \) and \( q \). See Fig. 2 for the description of \( \mathcal{P}(p, q) \), where we indicate the branch cut as a dotted line. It is an elementary but remarkable property that the winding number of \( \mathcal{P}(p, q) \) with respect to each point \( p, q \) is zero. This property allows us to define \( Z^\gamma(p, q, \cdot) \) as a single-valued analytic function on the contour.

Indeed, the Pochhammer contour is the typical example of path contained in the class of *first twisted homology* or *loaded cycle*, see [65, 67]. In other words, for any
(single-valued) function $G(\zeta)$ defined on $\mathcal{P}(p, q)$, we have

$$\oint_{\mathcal{P}(p,q)} \partial_\zeta G(\zeta) \, d\zeta = 0. \quad (6.13)$$

We now present the following version of null-vector equations, which immediately leads to the first assertion of Theorem E.

**Proposition 6.2** For any $\kappa > 0$, $Z_\beta(r, \cdot)$ is a (non-trivial) real-valued solution to

$$\partial_r Z = \frac{\kappa}{2} Z'' + HZ' + \left( \frac{3}{\kappa} - \frac{1}{2} \right) H'Z + C(r)Z, \quad (6.14)$$

where $C(r)$ is given as

$$C(r) = -\frac{6 \, \xi_r(\pi)}{\kappa} + \frac{1}{2(r + \chi)} - \left( \frac{\beta \pi}{r + \chi} \right)^2. \quad (6.15)$$

**Proof** Recall that $Z_\beta^\#(p, q, \zeta) = E\Psi_\beta^\#(p, q, \zeta)$. Combining Corollary 5.6 with (6.7),

$$\partial_r Z_\beta^\# = \frac{\kappa}{2} \partial^2 Z_\beta^\# + \left[ \left( \frac{3}{\kappa} - \frac{1}{2} \right) H' + H'(p - \zeta) \right] Z_\beta^\#$$

$$- \left( H\partial_q Z_\beta^\# + H(p - \zeta)\partial_\zeta Z_\beta^\# \right) + C(r)Z_\beta^\#. \quad (6.16)$$

Here we simplify the term $C(r, q)$ in (5.12) by the $(2\pi)$-periodicity of $H(r, \cdot)$ and the fact $q_2 - q_1 = 2\pi$.

Observe from the evaluation (6.8) that $(\partial + \partial_q + \partial_\zeta)Z_\beta^\#(p, q, \zeta) = 0$. Therefore the above equation is rewritten as

$$\partial_r Z_\beta^\# = \frac{\kappa}{2} \partial^2 Z_\beta^\# + H\partial Z_\beta^\# + \left( \frac{3}{\kappa} - \frac{1}{2} \right) H'Z_\beta^\# + C(r)Z_\beta^\# + \partial_\zeta F(\zeta), \quad (6.16)$$

where

$$F(\zeta) := H(p - q) Z_\beta^\# - H(p - \zeta)Z_\beta^\#. \quad (6.16)$$

By integrating (6.16) with respect to screening variable $\zeta$ along $\mathcal{P}(p, q)$, the desired null-vector equation (6.14) follows from (6.13).

It remains to verify the real-valuedness and non-triviality of $Z_\beta(r, \cdot)$. For $\kappa > 4$, the expression (6.10) can be decomposed as

$$Z_\beta = C(\kappa) \sum_{j=1}^{4} \int_{y_j} Z_\beta^\#(p, q, \zeta) \, d\zeta, \quad (6.17)$$
where the path $\gamma_j$’s are given as in Fig. 4 below.

We denote by $f(\cdot)$ the principal branch of $Z^\pi_\beta(p, q, \cdot)$ defined on $\mathbb{C} \setminus ([-\infty, q] \cup [p, \infty])$. Then one can express $Z^\pi_\beta(p, q, \cdot)$ on $\gamma_j$ as

$$Z^\pi_\beta(p, q, \zeta) = \begin{cases} f(\zeta) & \text{if } \zeta \in \gamma_1, \\ f(\zeta) e^{-2\pi i \frac{4}{\kappa}} & \text{if } \zeta \in \gamma_2, \\ f(\zeta) e^{2\pi i \frac{4}{\kappa}} & \text{if } \zeta \in \gamma_3, \\ f(\zeta) e^{-2\pi i \frac{4}{\kappa}} & \text{if } \zeta \in \gamma_4. \end{cases}$$

Therefore the Pochhammer contour integration (6.17) coincides with the following Euler type integral

$$Z_\beta(r, p - q) = \frac{1}{\Gamma(1 - 4/\kappa)} \int_q^p Z^\pi_\beta(p, q, t) \, dt. \quad (6.18)$$

The integrability condition is satisfied both at $p$ and $q$ if and only if $\kappa > 4$. Note that the integrand $Z^\pi_\beta(p, q, \cdot)$ is a positive function since $\Theta_\chi(r, \cdot)$ is a non-negative function in the interval $[0, 2\pi]$ vanishing only at $0, 2\pi$. Therefore the positivity of $Z_\beta(r, \cdot)$ for $\kappa > 4$ also follows.

We remark that as a function of $\kappa$, $Z_\beta$ can be interpreted as an analytic continuation of the right-hand side of (6.18), which implies the real-valuedness of $Z_\beta$ for $\kappa \leq 4$. Now it remains to check the non-triviality of $Z_\beta$. In other words, we show that $Z_\beta$ cannot be trivial for each value of $\kappa > 0$. By (5.11), we have (up to a multiplicative constant)

$$Z_\infty(p - q) := \lim_{r \to \infty} Z_\beta(r, p - q) = C(\kappa) \sin^2 \left( \frac{p - q}{2} \right) \int_{\mathcal{P}(p, q)} \sin^{-\frac{4}{\kappa}} \left( \frac{p - \zeta}{2} \right) \sin^{-\frac{4}{\kappa}} \left( \frac{\zeta - q}{2} \right) d\zeta.$$
Recall that the regularized hypergeometric function

\[ 2\mathbf{F}_1(a, b; c; z) := \frac{\mathbf{F}_1(a, b; c; z)}{\Gamma(c)} \]

is an entire function of \(a, b, c\). Then by \[54, \text{Eq. (15.6.1)}\], one can express \(Z_\infty\) in terms of \(2\mathbf{F}_1\) as follows:

\[
Z_\infty(x) = \cos^{\frac{3}{2}} \sin^{\frac{1}{2}} \cdot 2\mathbf{F}_1 \left( \frac{1}{2}, 1 - \frac{4}{\kappa}; \frac{3}{2} - \frac{4}{\kappa}, -\tan^2 \left( \frac{x}{4} \right) \right). \tag{6.19}
\]

Therefore by the identity theorem, (6.19) holds not only for \(\kappa > 4\) but also for \(\kappa \leq 4\). Thus we conclude that \(Z_\beta(r, \cdot)\) is a non-trivial function for all \(\kappa > 0\).

\[ \square \]

**Remark** We emphasize that not only the partition functions \(Z_\beta(r, x)\), but also their derivatives with respect to the “variables” \(\beta\) or \(\chi\)

\[
\partial_\beta^m \partial_\chi^n Z_\beta(r, x) \quad (m, n \in \mathbb{Z}^+) \tag{6.20}
\]

satisfy the null-vector equations (6.5). The partition functions of the form (6.20) can be constructed from (6.11) by taking proper non-degenerate limits.

**Example** As a consequence of Proposition 6.2, it is easy to observe that \(Z_\infty\) in (6.19) solves following ordinary differential equation:

\[
0 = \frac{\kappa}{2} Z_\infty'' + \cot \left( \frac{x}{2} \right) Z_\infty' + \left( \frac{3}{\kappa} - \frac{1}{2} \right) \cot \left( \frac{x}{2} \right) Z_\infty - \frac{1}{2\kappa} Z_\infty. \tag{6.21}
\]

The ordinary differential equation (6.21) was introduced by Zhan as a commutation relation for the radial SLE(\(\kappa, \Lambda\)) process, see [74, Sect. 4.5]. In particular for the values of \(\kappa\) such that \(4/\kappa\) is a positive integer, the expression (6.19) is simplified and expressed in terms of trigonometric functions, see e.g., [54, Sect. 15.4]. See for instance Table 1 below, where we write \(\sin_2(x) := \sin(x/2)\) and \(\cot_2(x) = \cot(x/2)\) to lighten notations.

**Example** (Some particular solutions) Observe that if \(4/\kappa\) is a positive integer, \(Z_\beta^\sharp(p, q, \cdot)\) is a well-defined meromorphic function in \(\mathbb{C}\) having poles only at \(p\) and \(q\). Therefore one can choose sufficiently small circle around \(p\) as an integration contour in (6.10). In this case, it is easy to calculate \(Z_\beta\) by residue calculus. We present some particular solutions (up to a multiplicative constant) in Table 2 for ER boundary condition with \(\beta = 0\).

Now we present the proof of Theorem E. Recall that

\[
\Psi_\beta \equiv \Psi_\beta(\xi, q) := C(\kappa) \oint_{\mathcal{P}(\xi, q)} \Psi_\beta^\sharp(\xi, q, \cdot) d\xi.
\]
Table 1  Partition functions $Z_\infty$

| $\kappa$ | $Z_\infty(\cdot)$ |
|---|---|
| 4 | $\sin^2 \frac{1}{2}$ |
| 2 | $\sin^{-1} \cot_2$ |
| 4/3 | $\sin^{-3/2} \left( 3 \cot_2^2 - 2 \cot'_2 + \frac{1}{3} \right)$ |
| 1 | $\sin^{-2} \left( 4 \cot_2^3 - 6 \cot_2 \cot'_2 + \cot''_2 + \cot_2 \right)$ |

Table 2  Partition functions $Z \equiv Z_{\beta=0}$

| $\kappa$ | $Z(r, \cdot)$ |
|---|---|
| 4 | $\Theta^{-1/2}$ |
| 2 | $\Theta^{-1} H$ |
| 4/3 | $\Theta^{-3/2} \left( 3 H^2 - 2 H' + 4 \frac{\xi r(\pi)}{\pi} \right)$ |
| 1 | $\Theta^{-2} \left( 4 H^3 - 6 H H' + H'' + 12 \frac{\xi r(\pi)}{\pi} H \right)$ |

By definition, $E \Psi = Z$ and $\Lambda = \kappa (\log Z)'$.

**Proof of Theorem E** By Proposition 6.2, it remains to prove the second assertion. For any string $\mathcal{X}$ of fields in the OPE family $\mathcal{F}_\beta$, let

$$R_\xi = \hat{E} \mathcal{X} := \frac{E \Psi_\beta(\xi, q) \mathcal{X}}{E \Psi_\beta(\xi, q)}.$$

Note that all we need to show is the following version of BPZ-Cardy equation

$$\frac{1}{a^2} \partial_\xi^2 R_\xi + \Lambda \partial_\xi R_\xi = \mathcal{L}_{v_\xi} R_\xi + \partial_r R_\xi. \quad (6.22)$$

By Theorem B, we have

$$\frac{1}{a^2} \partial_\xi^2 E[\Psi_\beta^\up (\xi, q, \xi) \mathcal{X}] = \mathcal{L}_{v_\xi} E[\Psi_\beta^\up (\xi, q, \xi) \mathcal{X}] + \mathcal{L}_{v_\xi} (\xi) E[\Psi_\beta^\up (\xi, q, \xi) \mathcal{X}] + (\partial_r - C(r)) E[\Psi_\beta^\up (\xi, q, \xi) \mathcal{X}], \quad (6.23)$$

where $C(r)$ is given as (6.15). Here $\mathcal{L}_{v_\xi} = \mathcal{L}_{v_\xi}(q) + \sum \mathcal{L}_{v_\xi}(x_j)$, where $x_j$’s are nodes of $\mathcal{X}$. Note that since $\Psi_\beta^\up$ is a $(1, 0)$-differential with respect to $\xi$, we have

$$\mathcal{L}_{v_\xi}(\xi) E[\Psi_\beta^\up (\xi, q, \xi) \mathcal{X}] = \partial_\xi \left[ v_\xi(\xi) E[\Psi_\beta^\up (\xi, q, \xi) \mathcal{X}] \right].$$
Therefore by (6.13),
\[ \oint_{\mathcal{P}(p,q)} \mathcal{L}_{v_{\xi}}(\xi) E[\Psi_{\beta}^{\pi}(p, q, \xi)X] d\xi = 0. \]

Integrating (6.23) with respect to $\zeta$ along $\mathcal{P}(p,q)$, we obtain
\[ \frac{1}{a^2} \partial_{\xi}^2 \left( Z_{\beta} R_{\xi} \right) = \mathcal{L}_{v_{\xi}} \left( Z_{\beta} R_{\xi} \right) + \left( \partial_r - C(r) \right) \left( Z_{\beta} R_{\xi} \right). \] (6.24)

Combining (6.24) with Proposition 6.2 we conclude (6.22), which completes the proof. □

**Remark** For $\kappa = 4$, let $\gamma$ be a contour around $q$ not encircling $p$ and write
\[ \Psi = O[a \cdot p - a \cdot q], \quad \Psi_{\gamma} = \int_{\gamma} O[a \cdot p + a \cdot q - 2a \cdot \zeta] d\zeta. \]

Then one can easily see that $E\Psi = E\Psi_{\gamma} = \Theta^{-1}_{X}$ up to a multiplicative constant. Thus the insertion of such one-leg operators produces martingale-observables for the same SLE$(4, \Lambda)$, where $\Lambda = -H_{X}$. However, we emphasize that as operators $X \rightarrow \hat{X}$, they are not the same in general.

For instance, let us assume that $\gamma$ encircles the node $z$. Then for ER boundary condition, we have
\[ \frac{EJ(z)\Psi(p, q)}{E\Psi(p, q)} = X(z) := w_{z}^{'} \left( H(w_{p} - w_{z}) - H(w_{q} - w_{z}) \right), \]

whereas
\[ \frac{EJ(z)\Psi_{\gamma}(p, q)}{E\Psi_{\gamma}(p, q)} = X(z) + Y(z), \quad Y(z) := \frac{w_{z}^{'} \Theta^{'}(0)\Theta(w_{p} - w_{q})}{\Theta(w_{p} - w_{z})\Theta(w_{q} - w_{z})}. \]

Combining Theorems C and E, the non-random field $Y(z)$ is also a martingale-observable for SLE$(4, \Lambda)$.

**Example (The classical limit $\kappa \downarrow 0$)** We now briefly discuss the limit $\kappa \downarrow 0$ of the null-vector equation and its solution constructed from CFT. In a simply connected domain, such differential equations for multiple SLE$(0)$ (equivalently, multichordal geodesic) were studied in [1, 56].

For a solution $Z$ to the null-vector Eq. (6.5), let us define
\[ Z := \lim_{\kappa \downarrow 0} Z^{\kappa}, \quad \Lambda_{(0)} := (\log Z)^{'} , \]
if the limit exists. Then it can be easily checked that the functions $Z$ and $\Lambda_{(0)}$ satisfy the following non-linear partial differential equations

$$\partial_r Z = H Z' + 3Hz Z + \frac{(Z')^2}{2Z} + C_0(r)Z,$$

$$\partial_r \Lambda_{(0)} = 3H'' + \Lambda_{(0)}H' + H'\Lambda_{(0)} + \Lambda_{(0)}\Lambda'_{(0)},$$

(6.25)

respectively, where $C_0$ is a constant depending only on the modulus $r$. A special solution

$$\Lambda_{(0)}(r, x) = H(r, x) - 2H(r, x/2)$$

(6.26)

of the Eq. (6.25) was obtained by Zhan in [74, Sect. 8.4].

We shall explain the drift function (6.26) from the viewpoint of a partition function of CFT constructed via screening. Recall that for $\kappa > 0$, we have general solutions (6.11) to the null-vector Eq. (6.5). For $\chi = \infty$, by the method of stationary phase, we formally obtain

$$Z(r, x) = \Theta(r, x)\Theta(r, x/2)^{-8},$$

(6.27)

where we evaluate the critical point $\zeta = x/2$ of the integrand in (6.11). This partition function (6.27) for annulus SLE(0) corresponds to the drift function (6.26). A more general theory on annulus SLE(0) will be addressed in a future work.

7 Examples of annulus SLE martingale-observables

From the statistical physics point of view, among various annulus SLE processes, it would be perhaps particularly interesting to study annulus SLEs that describe scaling limits of certain critical lattice models. However, explicit descriptions of such SLEs in a doubly connected domain are usually much more difficult than those in the case of a simply connected domain since these require non-trivial computations involving elliptic functions and also the reversibility does not uniquely determine the law of annulus SLE. Indeed, such descriptions have been achieved only for continuum interfaces of discrete GFF [29, 34], loop-erased random walk (LERW) [72], and critical Ising model [32]. In other words, some explicit forms of annulus SLE partition functions associated with such discrete models have been found.

In this section we exploit various methods in previous sections to present how the partition functions naturally arising from the above-mentioned lattice models can be realized as special cases of partition functions of CFT. These realizations also allow us to construct martingale observables for annulus SLEs corresponding to the scaling limits of the lattice models. As a consequence, some geometric properties of such SLEs are indicated as well. In particular, Sects. 7.1, 7.2 and 7.3 are devoted to study continuum interfaces of discrete GFF, LERW, and critical Ising model respectively. In Sect. 7.4 we discuss annulus SLEs with one marked point whose special case is known as the standard annulus SLE.
7.1 Bosonic observables

Since Schramm and Sheffield discovered the relation between level lines of discrete GFF and SLE(4) type curves [62], it has played an important role in the study of random conformal geometries, see e.g., [20]. A key ingredient for the coupling of GFF and SLE in the continuum level is the associated bosonic observables. In this spirit, Izyurov and Kytölä proposed a general framework to develop Schramm–Sheffield’s coupling relations in various conformal setups including doubly connected domains with Dirichlet or Dirichlet–Neumann mixed boundary conditions with several force points [34]. See also [39, 59] and references therein for further examples of GFF/SLE couplings in different conformal setups.

For \( \kappa = 4 \), let us consider the one-leg operator \( \Psi_\beta \) of the form

\[
\Psi_\beta(p, q) := \mathcal{O}_\beta[a \cdot p - a \cdot q],
\]

where \( \beta \) is given as (6.6). For the chordal case that \( p, q \in \mathbb{R} \), we have

\[
Z_\beta(r, x) = \Theta(r, x)^{-\frac{1}{2}} \exp \left( -\frac{(x - \mu)^2}{8(r + \chi)} \right), \quad \mu = 2\beta\pi/a.
\]

Here we write \( p = x \) and \( q = 0 \).

**Remark** For the Dirichlet boundary condition (i.e., \( \chi = 0 \)), such GFFs correspond to those studied in [29, 34]. The partition functions (7.2) are also presented in [74, Sect. 8.1] as a specific solutions for the null-vector equation with \( \kappa = 4 \). Here one may find a simple geometric interpretation of \( \beta \in \mathbb{R} \) since the bosonic field \( \hat{\Phi}_\beta \) has a piecewise Dirichlet boundary condition:

\[
E\hat{\Phi}_\beta(z) = \begin{cases} 
-\lambda & \text{if } z \in (0, p) \cup (q, 2\pi), \\
+\lambda & \text{if } z \in (p, q), \\
(1 - \mu/\pi)\lambda & \text{if } z \in \mathbb{R}_r,
\end{cases}
\]

where \( \lambda = a\pi = \pi/\sqrt{2} \).

For SLE(4, \( \Lambda \)) processes associated with partition functions of the form (7.2), we obtain their inner circle hitting probabilities using basic properties of Brownian motions.

**Proposition 7.1** Let \( \eta \) be the trace of SLE(4, \( \Lambda \)) whose partition function is given as (7.2). Then the probability \( P^\mu_\chi(x) \) that \( \eta \) hits the inner boundary component is given by

\[
P^\mu_\chi(x) = \frac{1}{4\pi} \frac{r + \chi}{\chi} \exp \frac{(x - \mu)^2}{8(r + \chi)} \times \int_0^{2\pi} e^{-\frac{(x - s)^2}{8\chi}} \left[ \Theta_I \left( \frac{r - x - s}{2} + \pi \right) - \Theta_I \left( \frac{r + x + s}{2} + \pi \right) \right] ds.
\]
**Proof** By definition and (7.2), the driving process $\xi_t$ of SLE($4\Lambda_1$) is given by

$$d\xi_t = 2 dB_t - \left( H(r-t, \xi_t - q_t) + \frac{\xi_t - q_t - \mu}{r + \chi - t} \right) dt, \quad q_t := \tilde{g}_t(q).$$

We denote by $X_t := \xi_t - q_t$ the angle difference process. Then $P^\mu_{\chi}(x)$ is the probability that $X_t$ stays in the interval $[0, 2\pi]$ up to time $r$.

On the other hand, due to Loewner’s equation, $X_t$ satisfies the following SDE:

$$dX_t := 2 dB_t - \frac{X_t - \mu}{r + \chi - t} dt, \quad X_0 = x.$$ 

Therefore the process $X_t$ is identified as a Brownian bridge starting from $x$, which reaches $\mu$ at time $t = r + \chi$, see e.g., [53]. Thus we obtain

$$P^\mu_{\chi}(x) = \Pr \left\{ 0 < \min_{0 \leq s \leq r} B(s) < \max_{0 \leq s \leq r} B(s) < \pi \quad B(0) = \frac{x}{2}, B(r + \chi) = \frac{\mu}{2} \right\}$$

$$= \sqrt{2\pi(r + \chi)} \exp \left( \frac{(\mu/2 - x/2)^2}{2(r + \chi)} \right) \int_0^\pi \frac{1}{\sqrt{2\pi \chi}} \exp \left( - \frac{(z - \mu/2)^2}{2\chi} \right) d\sigma(z),$$

where $d\sigma$ is given as

$$d\sigma(z) = \Pr \left\{ 0 < \min_{0 \leq s \leq r} B(s) < \max_{0 \leq s \leq r} B(s) < \pi, B(r) \in dz \quad B(0) = \frac{x}{2} \right\}$$

$$= \frac{1}{\sqrt{2\pi r}} \sum_{n \in \mathbb{Z}} \left[ \exp \left( - \frac{(x - 2z + 4n\pi)^2}{2r} \right) 
- \exp \left( - \frac{(x + 2z + 4n\pi)^2}{2r} \right) \right] dz.$$ 

Now proposition follows from (3.3). $\square$

**Remark** Note that for ER boundary condition ($\chi = \infty$), the angle difference process $X_t$ is simply a Brownian motion with speed 2. In this case (7.4) is given as

$$P^\mu_{\infty}(x) = \frac{1}{4\pi} \int_0^{2\pi} \left[ \Theta_I \left( \frac{1}{2}, \frac{x - s}{2} + \pi \right) - \Theta_I \left( \frac{1}{2}, \frac{x + s}{2} + \pi \right) \right] ds,$$

which corresponds to [52, Theorem 7.45]. On the other hand, for Dirichlet boundary condition ($\chi = 0$), it follows from the Gaussian approximation of the Dirac delta measure

$$\frac{1}{\sqrt{8\chi \pi}} e^{-\frac{(\mu - \mu')^2}{8\chi}} \to \delta_\mu(s).$$
The plot displays $P_{\mu}^{\infty}(x)$ with modulus $r = 1$ (dot-dashed line), $r = 2$ (full line) and $r = 4$ (dashed line). The plot displays $P_{\mu}^{0}(x)$ with $r = 2$ for $\mu = \pi/2$ (dot-dashed line), $\mu = \pi$ (full line) and $\mu = 3\pi/2$ (dashed line).

Observe here that the requirement $\mu \in (0, 2\pi)$ for the positive probability is equivalent to the fact that the boundary value of $\hat{\Phi}_\beta$ on the inner circle is between two heights on the outer circle, see (7.3) (Fig. 5).

**Example** For $\chi = 0$, set

$$O^{1,n} := O[a \cdot q - a \cdot q_n], \quad O^{2,n} := O[a \cdot q - a \cdot q_n - 2\beta],$$

where $q_n := q - 4n\pi$. Then one can check that after renormalization, the vertex fields $\hat{O}^{1,n}$, $\hat{O}^{2,n}$ have conformal dimension 0 at $q$ and

$$M^{1,n} := E\hat{O}^{1,n} = \exp\left(\frac{(x - \mu)^2}{8r}\right) \exp\left(-\frac{(x - 4\pi n - \mu)^2}{8r}\right),$$

$$M^{2,n} := E\hat{O}^{2,n} = \exp\left(\frac{(x - \mu)^2}{8r}\right) \exp\left(-\frac{(x - 4\pi n + \mu)^2}{8r}\right),$$

where $x = p - q$. Here notice that by (3.3), we have

$$P_{\mu}^{0}(x) = \sum_{n \in \mathbb{Z}} (M^{1,n} - M^{2,n}).$$

Such martingale-observables (7.5) are utilized in [29] to calculate $P_{\mu}^{0}(x)$. 
**Remark** For the weight function \( \omega(n) := (-1)^{-\frac{n}{2}} \), one can easily observe from (3.3) that (up to a multiplicative constant)

\[
Z_{\beta}^{\omega}(r, x) = \Theta(r, x)^{-\frac{1}{2}} \Theta(2r + 2\chi, x - \mu + \pi).
\]

In particular, for \( \chi = 0 \) and \( \beta = a/2 \), the SLE(4, \( \Lambda \)) associated with this partition function is coupled with GFF having specific height gap on the outer boundary component which obeys Neumann condition on the inner one, see [29, 34].

**Example** (*Crossing case*) Let us consider the case that \( p \in \mathbb{R}, q + ir \in \mathbb{R}_r \). Then the partition function associated with the one-leg operator (7.1) is given as

\[
Z_{\beta}(r, x) = \Theta(2r + 2\chi, x - \mu) = \exp \left( -\frac{(x - \mu)^2}{8(r + \chi)} \right), \quad \mu = 2\beta \pi / a.
\]

In particular, if \( \chi = \beta = 0 \), each partition function \( Z_{\beta}^{\omega}(x) := Z_{\beta}(x + 2n\pi) \) corresponds to the crossing type SLE(4, \( \Lambda \)) connecting two marked points with prescribed winding number, see [74, Sect. 6.3]. Moreover the \( 2\pi \)-periodic partition function

\[
Z_{\beta}^{\omega}(r, x) := \Theta(2r, x + \pi), \quad \omega(n) \equiv 1
\]

agrees with Zhan’s Feynman-Kac expression of the crossing type annulus partition function, see [74, Sect. 8.1].

### 7.2 LERW observables

In this subsection, we present a way of constructing martingale-observables for various continuum LERW. In the physics literature, a version of CFT constructed from sympletic fermions has been proposed for LERW, see e.g., [3, 8, 49] and references therein.

In a doubly connected domain, one may consider various LERW by imposing boundary conditions on the inner boundary components. For instance, let us consider a simple random walk on the lattice approximation of the annulus started from an interior vertex near the initial boundary point conditioned to hit a boundary vertex near the target point. We assume that the random walk continues when it hits the inner circle, and stops when it visits the outer one. Then after the loop-erasing procedure, the resulting curve is the LERW connecting two marked points on the outer boundary component, obeying the Neumann condition on the inner one.

For a general LERW, it is well known that the associated partition function \( Z \) is expressed as

\[
Z(p, q) = \left. \frac{\partial^2}{\partial n_\xi \partial n_\zeta} \right|_{\xi=p, \zeta=q} G(\xi, \zeta), \quad (7.6)
\]

see e.g., [19, Sect. 9], [50] and [68, Sect. 4.5]. Here the Green’s function \( G \) obeys the boundary condition of LERW.
We now present a way of constructing LERW partition functions from the methods introduced in the previous sections. Then by Theorems D and E, the way of constructing the associated martingale-observables also follows.

Recall that for \( 4/\kappa \in \mathbb{Z}_+ \), one can choose the integration contour \( \gamma \) as a circle encircles \( p \) or/and \( q \) in the expression of \( Z_\beta^\omega \). With such a choice of \( \gamma \), by residue calculus, we obtain following two-parameter family of solutions

\[
Z_\beta(r, x) = \Theta(r, x)^{-1} \exp \left( -\frac{(x - 2\beta\pi)^2}{4(r + \chi)} \right) \left( H(r, x) + \frac{x - 2\beta\pi}{r + \chi} \right). 
\]

(7.7)

Using the weight functions \( \omega(n) = 1, (-1)^n \) and (3.3), we obtain following family of solutions having \( 2\pi \)-(anti) periodicity:

\[
Z_\beta^\omega(r, x + 2\pi) = \exp \left( (2\beta - 1)\pi i \right) Z_\beta^\omega(r, x).
\]

(7.8)

7.2.1 LERW partition functions

Let us now focus on the case with \( \chi = 0 \). By (6.12), such \( 2\pi \)-periodic partition functions further satisfy rotational periodicities with respect to \( 2ir \);

\[
Z_\beta^\omega(r, x + 2ir) = \exp \left( (2\beta - 1)\pi i \right) Z_\beta^\omega(r, x).
\]

• LERW with Dirichlet/Riemann–Hilbert mixed boundary conditions. Let \( \omega(n) = 1 \). Then by (7.8), we have

\[
Z_\beta^\omega(r, x) = \Theta(r, x + \pi - 2\beta\pi) \left( H(r, x) - H(r + \chi, x + \pi - 2\beta\pi) \right).
\]

(7.9)

The partition function (7.9) describes the continuous LERW with Riemann–Hilbert boundary condition, i.e., the requirement that the derivative along the oblique direction (indexed by \( \beta \)) vanishes. This follows from (7.6) and the representation of the Green’s function satisfying Riemann–Hilbert boundary condition on the inner boundary component, see “Appendix A”.

We emphasize here that the \( 2\pi \)-periodic partition functions \( Z_\beta^\omega \) are positive for all \( \beta \). Let us also point out that such partition functions for \( \beta \in \{0, b\} \) with \( \chi = 0 \) correspond to those found by Zhan in [74, Sect. 8.2].
It is well known that Riemann–Hilbert b.c. interpolates Neumann and ER conditions, see e.g., [7]. For \( \beta = 0 \), we have an alternative expression

\[
Z(r, x) = H'(2r, x) - H'_I(2r, x).
\]

On the other hand, when \( \beta \to b \), after normalization, we have

\[
Z(r, x) = H'(r, x).
\]

These partition functions (7.10) and (7.11) correspond to the partition functions of LERW with Neumann and ER b.c., respectively.

- **LERW with Dirichlet/Dirichlet boundary condition.** Let \( \beta = b \) and \( \omega(n) = n \), i.e.,

\[
\Psi_\beta^\omega(x) := \sum_{n \in \mathbb{Z}} n \Psi_b(x + 2n\pi).
\]

Then it is straightforward to check that

\[
Z_\beta^\omega(r, x) = \tilde{H}'(r, x) = H'(r, x) + \frac{1}{r}.
\]

This corresponds to the LERW partition function with Dirichlet boundary condition, which appears in [72].

Let us pause here to briefly explain the derivation of (7.12). Note that by (7.7),

\[
E \Psi_b(r, x) = \Theta(r, x)^{-1} \exp \left( -\frac{(x + \pi)^2}{4r} \right) \left( H(r, x) + \frac{x + \pi}{r} \right).
\]

Therefore we have

\[
Z_\beta^\omega(r, x) = \frac{H(r, x)}{\Theta(r, x)} \sum_{n \in \mathbb{Z}} (-1)^n n \exp \left( -\frac{(x + \pi + 2n\pi)^2}{4r} \right) + \frac{1}{\Theta(r, x)} \sum_{n \in \mathbb{Z}} (-1)^n n \left( \frac{x + \pi + 2n\pi}{r} \right) \exp \left( -\frac{(x + \pi + 2n\pi)^2}{4r} \right).
\]

It follows from (3.3) that

\[
\sum_{n \in \mathbb{Z}} (-1)^n n \exp \left( -\frac{(x + \pi + 2n\pi)^2}{4r} \right) = \frac{r}{\pi} \sqrt{\frac{r}{\pi}} \left( \Theta'(r, x) + \frac{x + \pi}{2r} \Theta(r, x) \right).
\]

Combining this identity with (3.1) and (3.2), we obtain (up to a multiplicative constant)

\[
Z_\beta^\omega(r, x) = H(r, x) \left( \frac{\Theta'(r, x)}{\Theta(r, x)} + \frac{x + \pi}{2r} \right) - \left( 2 \frac{\Theta''(r, x)}{\Theta(r, x)} + \frac{1}{r} + \frac{x + \pi}{r} \frac{\Theta'(r, x)}{\Theta(r, x)} \right).
\]
\[
= \left( \frac{1}{2} H^2(r, x) + \frac{x + \pi}{2r} H(r, x) \right) \\
- \left( H'(r, x) + \frac{1}{2} H^2(r, x) + \frac{1}{r} + \frac{x + \pi}{2r} H(r, x) \right),
\]

which leads to (7.12).

**Remark** In parallel, one can construct such partition functions for the crossing case that \( p \in \mathbb{R} \) and \( q \in \mathbb{R}_r \). In particular, for \( \beta = b, \chi = 0 \) with the choice of weight function \( \omega(n) = n \), we obtain \( Z_B^\omega(r, x) = H'_1(r, x) + 1/r \), which corresponds to Zhan’s Feynmann-Kac solution, see [74, Sect. 8.2].

- **LERW aiming at a side arc.** Due to the special feature of conformal dimension

\[
\lambda(a) = \frac{6 - \kappa}{2\kappa} = 1, \quad a = \sqrt{\frac{2}{\kappa}},
\]

if we place the charge \( a \) at the “target point” \( q \) as usual, the node \( q \) can also be utilized as a screening variable. This fact allows us to describe the one-leg operator for LERW aiming at a marked side arc.

For a subset \( I \) of boundary components \( (I \cap p = \emptyset) \), set

\[
\Psi(p, I) := \frac{1}{|I|} \int_I \Psi_B^\omega(p, q) \, dq, \quad \omega(n) = n.
\]

Then the associated partition function \( Z(p, I) = E\Psi(p, I) \) is given by

\[
Z(p, I) := \frac{1}{|I|} \int_I H'(r, p - q) \, dq + \frac{1}{r}.
\]

This corresponds to the partition function of continuous LERW with a target \( I \), see [72].

In particular, when \( I = \mathbb{R}_r \), it describes LERW starting from \( p \), which stops whenever it hits the inner boundary component. By the periodicity, the associated partition function is given by \( Z = \frac{1}{r} \). This can be realized as a partition function for standard SLE(2), see [69, 72].

### 7.2.2 Martingale-observables

Let \( \Psi \) be the one-leg operator associated with various LERWs described above. We present some important martingale-observables.

- **Generalized Poisson kernel.** For simplicity we only consider the case that target set is a boundary point \( q \). Set \( \bigcup := \{a \cdot z - a \cdot q\} \). Let us define

\[
M(z) := \frac{1}{2i} \int_{\xi} \hat{E}\bigcup(z) \, d\xi, \quad \hat{E}\bigcup(z) = \frac{w'(z)Z(r, w(z) - w(p))}{w'(q)Z(r, w(q) - w(p))}.
\]
The scalar field $M$ is called a generalized Poisson kernel. For the Dirichlet boundary condition, Zhan considered the following bounded martingale [Cf. (7.12)]

$$M_t(z) = \frac{1}{w_t'(q)} \frac{\text{Im} \tilde{H}(r-t, w_t(z))}{\tilde{H}'(r-t, w_t(q))}$$

to show the convergence of LERW, see [68, 72].

- **Ending point distribution of standard SLE(2).** Set $I = [q_1 + ir, q_2 + ir] \subset \mathbb{R}_r$.
  Let us write $\psi_1$ for the one-leg operator associated with the standard SLE(2). Set

$$O = \frac{1}{2\pi} \int_I O[a \cdot z - a \cdot q] \, dq.$$

Then the martingale-observable

$$M(z) = \mathbb{E} O = \frac{\mathbb{E} \psi_1 \mathbb{E} O}{\mathbb{E} \psi_1} = \frac{1}{2\pi} \int_{q_1}^{q_2} \left( r H'_1(r, p - q) + 1 \right) \, dq$$

yields the probability that standard SLE(2) ends at $I$, see [70].

### 7.3 Critical Ising observables

In this subsection we focus on the case $\kappa = 3$. As in Sect. 6.1, let us denote by $\tilde{\beta}$ the (signed) uniform measure on $[ir - \pi, ir + \pi]$ with total mass $-2b$ and let $\beta = 2b \cdot q + \tilde{\beta}$. Then we have

$$Z_\beta(r, x) = \mathbb{E} \psi_1 [a \cdot p - a \cdot q] = \Theta(r, x)^{-1} \exp\left(-\frac{x^2}{4r}\right), \quad x = p - q.$$

The null-vector equation for $Z_\beta$ is given as

$$\partial_r Z_\beta = \frac{3}{2} Z''_\beta + H Z'_\beta + \frac{1}{2} H' Z_\beta + F_\beta Z_\beta,$$  \hspace{1cm} (7.13)

where

$$F_\beta(r, x) = -\left(\frac{H'}{2} + \frac{H^2}{4}\right)(r, x) - \frac{x}{2r} H(r, x) - \frac{x^2}{4r^2}.$$  

Using the weight $\omega(n) = (-1)^n$, set

$$\psi(x) := \sum_{n \in \mathbb{Z}} (-1)^n \psi_1 [a \cdot (p + 2n\pi) - a \cdot q].$$
Then by (3.3), we have

\[ Z(r, x) := E\Psi(x) = \sum_{n \in \mathbb{Z}} (-1)^n Z_\beta(r, x + 2n\pi) = \sqrt{\frac{r}{\pi}} \frac{\Theta_1'(r, x + \pi)}{\Theta(r, x)}. \] (7.14)

This corresponds to the partition function for the critical Ising interface studied by Izyurov [32]. Therefore the insertion of the field \( \Psi_1 \) provides martingale-observables for the continuum limit of the interface curve.

Since the interface curve satisfies reversibility, it follows from Izyurov’s convergence result [32] and Zhan’s theory on commuting annulus SLE that the partition function (7.14) satisfies the null-vector Eq. (2.1) with \( \kappa = 3 \). Indeed this also follows from the martingale property of parafermionic observable in the continuum limit. See [33] for verification of BPZ type equations of the partition function in this manner.

Here we present a field theoretical method to show that \( Z \) satisfies (2.1).

**Proposition 7.2** The partition function \( Z \) in (7.14) satisfies the null-vector equation (2.1) with \( \kappa = 3 \).

**Proof** By (7.13), all we need to show is

\[ \frac{1}{Z(r, x)} \sum_{n \in \mathbb{Z}} (-1)^n F_\beta(x + 2n\pi) Z_\beta(r, x + 2n\pi) \] (7.15)

is a constant depending only on the modular parameter \( r \). By differentiating (3.3), we have

\[ \sum_{n \in \mathbb{Z}} \frac{x + 2n\pi}{2r} \exp \left( -\frac{(x + 2n\pi)^2}{4r} \right) = -\sqrt{\frac{r}{\pi}} \frac{\Theta_1'(r, x + \pi)}{\Theta(r, x)} \]

and

\[ \sum_{n \in \mathbb{Z}} \frac{(x + 2n\pi)^2}{4r^2} \exp \left( -\frac{(x + 2n\pi)^2}{4r} \right) = \sqrt{\frac{r}{\pi}} \left( \frac{\Theta_1''(r, x + \pi)}{2r} + \frac{1}{2r} \frac{\Theta_1(r, x + \pi)}{\Theta(r, x)} \right). \]

Using these identities, we obtain that (7.15) simplifies to

\[ \frac{1}{2} H(r, x) H_1(r, x + \pi) - \left( \frac{H_1'}{2} + \frac{H_1^2}{4} \right)(r, x + \pi) \]

\[ - \left( \frac{H_1'}{2} + \frac{H_1^2}{4} \right)(r, x + \pi) - \frac{1}{2r}. \]

Now the proposition follows from (3.11). \( \square \)

**Example** Let \( O_\beta := O_\beta[(2b - a) \cdot z - (2b - a) \cdot q] \). Then the vertex observable \( M(z) := \hat{E}O_\beta \) provides the local martingale

\[ M_1(z) = \frac{w_1'(z) Z(r - t, w_1(z))}{w_1'(q) Z(r - t, w_1(q))}. \]
The discrete counterpart of $M_t$ corresponds to the so-called parafermionic observable in the context of the critical Ising model. This observable plays an important role in determining the law of the continuum Ising interface curve, see [32].

Using a similar idea in previous sections, one can construct further partition functions $Z$ of the form

$$
\frac{\Theta(r, x + \pi - 2\beta\pi)}{\Theta(r, x)}, \quad \frac{\Theta_I(r, x + \pi - 2\beta\pi)}{\Theta(r, x)},
$$

which satisfy the null-vector equation (2.1) and $2\pi$-rotational periodicity. The former has $2\pi$ periodicity, and the latter has $2\pi$-(anti) periodicity. Moreover, the method to construct the chordal type of partition functions in this section can be applied to the crossing case as well.

7.4 Annulus SLE with one marked point

One can use the implementation of the measure-valued charges to construct martingale-observables for annulus SLE with one marked point. A typical example of a such process is called the standard annulus SLE($\kappa$) [68] with driving process $\xi_t = \sqrt{\kappa} B_t$.

For $k \in \{1, \ldots, n\}$, let $q_k := q - \pi + 2\pi \frac{k}{n}$, where $q \in \mathbb{R}$. We then consider

$$
\Psi_n(p) = \mathcal{O}\left( a \cdot p - \sum_{k=1}^{n} \frac{a}{2n} \cdot q_k, -\sum_{k=1}^{n} \frac{a}{2n} \cdot q_k \right).
$$

By (5.5), it follows that the associated drift function $\Lambda_n$ is given by

$$
\Lambda_n(p) = -\frac{1}{2n} \sum_{k=1}^{n} H_{\chi}(r, p - q_k) + H_{\chi}(p - \bar{q}_k)
= -\frac{1}{n} \sum_{k=1}^{n} H_I(r, p - \text{Re} q_k) + \frac{\text{Re} q_k - p}{r + \chi}.
$$

We denote

$$
\Psi(p) := \lim_{n \to \infty} \Psi_n(p) = \mathcal{O}\left( a \cdot p - \frac{a}{2} \cdot q, -\frac{a}{2} \cdot q \right),
\Lambda := \lim_{n \to \infty} \Lambda_n = \frac{\text{Re} q - p}{r + \chi},
$$

where $q$ is the uniform measure on $[q - \pi, q + \pi]$ for $q \in \mathbb{R}$. Here we use the $2\pi$-periodicity of $H_I(r, \cdot)$. By Theorem C, the insertion of the one-leg operator $\Psi(p)$
produces martingale-observables for annulus SLE driven by

\[ d\xi_t = \sqrt{\kappa} dB_t + \frac{\text{Re } q - \xi_t}{r + \chi - t} dt, \quad \xi_0 = p. \]

Notice that this corresponds to the law of the Brownian bridge. In particular, for the ER b.c. case when \( \chi = \infty \) (thus \( \Lambda = 0 \)), it corresponds to the standard SLE(\( \kappa \)). On the other hand, for the Dirichlet b.c. case when \( \chi = 0 \), the SLE trace ends at the point \( q \).

**Example (Level line of GFF)** In the identity chart of \( C_r \), the associated bosonic observable \( M(z) := \hat{E} \hat{\phi}_1(z) \) is evaluated as

\[ M(z) = 2a \text{ arg } \Theta_{\chi}(r, z - p) - a \int \text{ arg } \left\{ \Theta_{\chi}(r, z - q) \Theta_{\chi}(r, z - \bar{q}) \right\} dq. \]

For the Dirichlet b.c., the boundary value of \( M \) has discontinuity \( 2a\pi \) at \( p + 2n\pi \) and has linear growth with speed \( a \) on \( \mathbb{R}_r \). To be precise, \( M(z + 2\pi) = M(z) + 2\lambda \) and

\[ M(z) = \begin{cases} -\lambda & \text{if } z \in (0, p), \\
+\lambda & \text{if } z \in (p, 2\pi), \\
\frac{\text{Re } (z - q)}{\pi} & \text{if } z \in \mathbb{R}_r, \end{cases} \]

where \( \lambda = a\pi = \pi / \sqrt{2} \).

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**Appendix A: Representation of Green’s function**

In this “Appendix” we present an analytic representation of Green’s function in a doubly connected domain with various boundary conditions.

For a parameter \( \beta \in (-1/2, 1/2) \), we say a function \( F \) defined in a domain \( \bar{D} \) satisfies RH\( \beta \) boundary condition (b.c.) on \( l \subset \partial D \) if

\[ \left[ \cos(\beta \pi) \partial_n - \sin(\beta \pi) \partial_t \right] F(\cdot) = 0 \text{ on } l \subset \partial D. \]

Here \( \partial_n \) is the (inwards) normal derivative and \( \partial_t \) is the tangential one. In particular, the case \( \beta = 0 \) corresponds to the Neumann b.c.
Let us define

\[ f_\beta(r, z) := \frac{\Theta'(r, 0)}{\Theta(r, \pi - 2\beta \pi)} \frac{\Theta(r, z + \pi - 2\beta \pi)}{\Theta(r, z)}. \]

By the quasi-periodicities (3.4) of the theta function, we have

\[ f_\beta(r, z + 2\pi) = f_\beta(r, z), \quad f_\beta(r, z + 2ir) = e^{i(2\beta - 1)\pi} f_\beta(r, z). \quad (A.1) \]

Since \( \Theta(r, \cdot) \) is an odd function,

\[ f_\beta(r, z) = -f_{-\beta}(r, -z). \quad (A.2) \]

On the other hand, \( f_\beta \) satisfies

\[ f_\beta(r, z) = \frac{1}{z} + \frac{H(r, \pi - 2\beta \pi)}{2} + O(z), \quad \text{as } z \to 0. \quad (A.3) \]

Thus one can observe that \( f_0 \) has an alternative expression

\[ f_0(r, z) = \frac{1}{2} (H(2r, z) - H(2r, z)). \quad (A.4) \]

It is also easy to show that \( f_\beta(r, \cdot) \) is a conformal map from the cylinder \( C_r \) to the upper half-plane minus a slit with argument \((1/2 - \beta)\pi\). For given \( w \in C_r \), a meromorphic function

\[ g_\beta(z) := f_\beta(r, z - \bar{w}) - f_\beta(r, z - w) \]

satisfies following properties:

- \( g_\beta(z + 2\pi) = g_\beta(z) \), \( g_\beta(r, z + 2ir) = e^{i(2\beta - 1)\pi} g_\beta(r, z) \);
- \( g_\beta(r, \cdot) \) has simple poles at \( w \) (resp., \( \bar{w} \)) with residue \(-1\) (resp., \(1\));
- \( g_\beta(r, \cdot) \) maps \( \mathbb{R} \) to \( i\mathbb{R} \);
- \( g_\beta(r, \cdot) \) maps \( \mathbb{R}r \) to a line passing through \( 0 \) with angle \( \beta \) with \( \mathbb{R} \).

All of these properties are immediate consequences of \((A.1)\), \((A.2)\) and \((A.3)\). For instance, the last property follows from that for \( x \in \mathbb{R} \),

\[ g_\beta(x + ir) = f_\beta(r, x - ir - w) - f_\beta(r, x + ir - w) \]
\[ = e^{i(2\beta - 1)\pi} f_\beta(r, x + ir - w) - f_\beta(r, x + ir - w) \]
\[ = -e^{i\beta\pi} \left( e^{i\beta\pi} f_\beta(r, x + ir - w) + e^{-i\beta\pi} f_\beta(r, x + ir - w) \right) \in e^{i\beta\pi} \mathbb{R}. \]
We denote by $F_{\beta}(r, \cdot)$ a primitive of $f_{\beta}(r, \cdot)$, i.e., $\partial_{z}F_{\beta}(r, z) = f_{\beta}(r, z)$. To our knowledge, for general $\beta$, there is no known expression for $F_{\beta}(r, z)$ in terms of well-known special functions. On the other hand, by (A.4), we have

$$F_{0}(r, z) = \log \left( \frac{\Theta(2r, z)}{\Theta_{1}(2r, z)} \right)$$

(A.5)

up to an additive constant.

We now present Green’s function $G_{\beta}$ in $C_{r}$ with zero Dirichlet b.c. on $\mathbb{R}$ which satisfies RH$_{\beta}$ condition on $\mathbb{R}_{r}$. The associated (non-symmetric) stochastic process is called obliquely reflected Brownian motion (ORBM), see [7] and references therein. We remark that ORBM gives a geometric interpolation between reflected Brownian motion ($\beta = 0$) and ERBM ($\beta \to \pm 1/2$).

We claim that Green’s function $G_{\beta}$ in $C_{r}$ with zero Dirichlet b.c. on $\mathbb{R}$ which satisfies RH$_{\beta}$ condition on $\mathbb{R}_{r}$ is expressed as

$$G_{\beta}(z_{1}, z_{2}) = \text{Re} \left[ F_{\beta}(r, z_{1} - \bar{z}_{2}) - F_{\beta}(r, z_{1} - z_{2}) \right].$$

(A.6)

In particular, by (A.5), Green’s function $G_{0}$ with Dirichlet–Neumann mixed boundary condition is given by

$$G_{0}(z_{1}, z_{2}) = \log \left| \frac{\Theta/\Theta_{1}(2r, z_{1} - \bar{z}_{2})}{\Theta/\Theta_{1}(2r, z_{1} - z_{2})} \right|.$$

To show (A.6), it suffices to check the followings:

- $G_{\beta}(z_{1}, z_{2}) + \log |z_{1} - z_{2}|$ is harmonic function in both variables;
- $G_{\beta}(z_{1}, z_{2}) = 0$ if $z_{1}$ or $z_{2}$ is on $\mathbb{R}$;
- $G_{\beta}(\cdot, z_{2})$ satisfies RH$_{\beta}$ condition on $\mathbb{R}_{r}$.

All of these requirements easily follow from the properties presented above. We remark that $G_{\beta}$ satisfies the asymmetric relation $G_{\beta}(z_{1}, z_{2}) = G_{-\beta}(z_{2}, z_{1})$.

References

1. Alberts, T., Byun, S.-S., Kang, N.-G., Makarov, N.: Pole dynamics and an integral of motion for multiple SLE(0). arXiv:2011.05714v2
2. Alberts, T., Kang, N.-G., Makarov, N.: Conformal field theory for multiple SLEs. in preparation
3. Bauer, M., Bernard, D., Kytölä, K.: LERW as an example of off-critical SLEs. J. Stat. Phys. 132(4), 721–754 (2008)
4. Bauer, R.O.: Restricting SLE(8/3) to an annulus. Stoch. Process. Appl. 117(9), 1165–1188 (2007)
5. Bauer, R.O., Friedrich, R.M.: On radial stochastic Loewner evolution in multiply connected domains. J. Funct. Anal. 237(2), 565–588 (2006)
6. Beffara, V.: The dimension of the SLE curves. Ann. Probab. 36(4), 1421–1452 (2008)
7. Burdzy, K., Chen, Z.-Q., Marshall, D., Ramanan, K.: Obliquely reflected Brownian motion in non-smooth planar domains. Ann. Probab. 45(5), 2971–3037 (2017)
8. Caracciolo, S., Jacobsen, J.L., Saleur, H., Sokal, A.D., Sportiello, A.: Fermionic field theory for trees and forests. Phys. Rev. Lett. 93(8), 080601 (2004)
9. Chandrasekharan, K.: Elliptic Functions. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 281. Springer, Berlin (1985)

10. Chelkak, D., Duminil-Copin, H., Hongler, C., Kemppainen, A., Smirnov, S.: Convergence of Ising interfaces to Schramm’s SLE curves. C. R. Math. Acad. Sci. Paris 352(2), 157–161 (2014)

11. Chen, Z.-Q., Fukushima, M., Rohde, S.: Chordal Komatu–Loewner equation and Brownian motion with darning in multiply connected domains. Trans. Am. Math. Soc. 368(6), 4065–4114 (2016)

12. Crowdy, D., Marshall, J.: Conformal mappings between canonical multiply connected domains. Comput. Methods Funct. Theory 6(1), 59–76 (2006)

13. Crowdy, D., Marshall, J.: Green’s functions for Laplace’s equation in multiply connected domains. IMAJ. Appl. Math. 72(3), 278–301 (2007)

14. Di Francesco, P., Mathieu, P., Sénéchal, D.: Conformal Field Theory. Graduate Texts in Contemporary Physics, Springer, New York (1997)

15. Drenning, S.: Excursion reflected Brownian motion and Loewner equations in multiply connected domains. ProQuest LLC, Ann Arbor, MI, (2011). Thesis (Ph.D.), The University of Chicago

16. Dubédat, J.: Critical percolation in annuli and SLE_6. Commun. Math. Phys. 245(3), 627–637 (2004)

17. Dubédat, J.: SLE(κ, ρ) martingales and duality. Ann. Probab. 33(1), 223–243 (2005)

18. Dubédat, J.: Euler integrals for commuting SLEs. J. Stat. Phys. 123(6), 1183–1218 (2006)

19. Dubédat, J.: Commutation relations for Schramm–Loewner evolutions. Commun. Pure Appl. Math. 60(12), 1792–1847 (2007)

20. Dubédat, J.: SLE and the free field: partition functions and couplings. J. Am. Math. Soc. 22(4), 995–1054 (2009)

21. Eguchi, T., Ooguri, H.: Conformal and current algebras on a general Riemann surface. Nucl. Phys. B 282(2), 308–328 (1987)

22. Etingof, P.I., Frenkel, I.B., Kirillov, A.A., Jr.: Lectures on Representation Theory and Knizhnik–Zamolodchikov Equations. Mathematical Surveys and Monographs, vol. 58. American Mathematical Society, Providence (1998)

23. Flores, S.M., Kleban, P.: A solution space for a system of null-state partial differential equations: Part 1. Commun. Math. Phys. 333(1), 389–434 (2015)

24. Flores, S.M., Kleban, P.: A solution space for a system of null-state partial differential equations: Part 2. Commun. Math. Phys. 333(1), 435–481 (2015)

25. Flores, S.M., Kleban, P.: A solution space for a system of null-state partial differential equations: Part 3. Commun. Math. Phys. 333(2), 597–667 (2015)

26. Flores, S.M., Kleban, P.: A solution space for a system of null-state partial differential equations: Part 4. Commun. Math. Phys. 333(2), 669–715 (2015)

27. Forrester, P.J.: Particles in a magnetic field and plasma analogies: doubly periodic boundary conditions. J. Phys. A 39(41), 13025–13036 (2006)

28. Hagedorn, C.: A generalization of Schramm’s formula for SLE_2. J. Stat. Mech. Theory Exp. 2009(02), P02033 (2009)

29. Hagedorn, C., Bernard, D., Bauer, M.: The Gaussian free field and SLE_4 on doubly connected domains. J. Stat. Phys. 140(1), 1–26 (2010)

30. Hedenmalm, H., Nieminen, P.J.: The Gaussian free field and Hadamard’s variational formula. Probab. Theory Relat. Fields 159(1–2), 61–73 (2014)

31. Henrici, P.: Applied and Computational Complex Analysis. Vol. 3. Pure and Applied Mathematics. Discrete Fourier Analysis—Cauchy Integrals—Construction of Conformal Maps—Univalent Functions. A Wiley-Interscience Publication. Wiley, New York (1986)

32. Izyurov, K.: Critical Ising interfaces in multiply-connected domains. Probab. Theory Relat. Fields 167(1–2), 379–415 (2017)

33. Izyurov, K.: On multiple SLE for the FK-Ising model. Ann. Probab. 50(2), 771–790 (2022)

34. Izyurov, K., Kytölä, K.: Hadamard’s formula and couplings of SLEs with free field. Probab. Theory Relat. Fields 155(1–2), 35–69 (2013)

35. Kang, N.-G., Makarov, N.: Conformal field theory on the Riemann sphere and its boundary version for SLE. arXiv:2111.10057

36. Kang, N.-G., Makarov, N.G.: Gaussian free field and conformal field theory. Astérisque 353, viii+136 (2013)

37. Kang, N.-G., Makarov, N.G.: Calculus of conformal fields on a compact Riemann surface. arXiv:1708.07361 (2017)
38. Katori, M.: Two-dimensional elliptic determinantal point processes and related systems. Commun. Math. Phys. 371(3), 1283–1321 (2019)
39. Katori, M., Koshida, S.: Gaussian free fields coupled with multiple SLEs driven by stochastic log-gases. Adv. Stud. Pure Math. 315–340 (2021)
40. Koebe, P.: Abhandlungen zur theorie der konformen abbildung. Acta Math. 41(1), 305–344 (1916)
41. Krichever, I., Marshakov, A., Zabrodin, A.: Integrable structure of the Dirichlet boundary problem in multiply-connected domains. Commun. Math. Phys. 259(1), 1–44 (2005)
42. Kytölä, K., Peltola, E.: Pure partition functions of multiple SLEs. Commun. Math. Phys. 346(1), 237–292 (2016)
43. Kytölä, K., Peltola, E.: Conformally covariant boundary correlation functions with a quantum group. J. Eur. Math. Soc. (JEMS) 22(1), 55–118 (2020)
44. Lawler, G., Schramm, O., Werner, W.: Conformal restriction: the chordal case. J. Am. Math. Soc. 16(4), 917–955 (2003)
45. Lawler, G.F.: The Laplacian-b random walk and the Schramm–Loewner evolution. Ill. J. Math. 50(1–4), 701–746 (2006)
46. Lawler, G.F.: Defining SLE in multiply connected domains with the Brownian loop measure. arXiv:1108.4364 (2011)
47. Lenells, J., Viklund, F.: Asymptotic analysis of Dotsenko–Fateev integrals. Ann. Henri Poincaré 20(11), 3799–3848 (2019)
48. Lupu, T., Wu, H.: A level line of the Gaussian free field with measure-valued boundary conditions. arXiv:2106.15169 (2021)
49. Majumdar, S.: Exact fractal dimension of the loop-erased self-avoiding walk in two dimensions. Phys. Rev. Lett. 68(15), 2329 (1992)
50. Makarov, N., Smirnov, S.: Off-critical lattice models and massive SLEs. In: XVIth International Congress on Mathematical Physics, pp. 362–371 (2010)
51. Miller, J., Sheffield, S.: Imaginary geometry III: reversibility of SLE for $\kappa \in (4, 8)$. Ann. Math. (2) 184(2), 455–486 (2016)
52. Mörters, P., Peres, Y.: Brownian Motion, vol. 30. Cambridge University Press, Cambridge (2010)
53. Øksendal, B.: Stochastic Differential Equations. An Introduction with Applications. Universitext, 6th edn. Springer, Berlin (2003)
54. Olver, F.W., Lozier, D.W., Boisvert, R.F., Clark, C.W.: NIST Handbook of Mathematical Functions. US Department of Commerce, National Institute of Standards and Technology (2010)
55. Peltola, E.: Basis for solutions of the Benoit & Saint-Aubin PDEs with particular asymptotics properties. Ann. Inst. Henri Poincaré D 7(1), 1–73 (2020)
56. Peltola, E., Wang, Y.: Large deviations of multichordal SLE$_{0+}$, real rational functions, and zeta-regularized determinants of Laplacians. J. Eur. Math. Soc. arXiv:2006.08574 (2020)
57. Powell, E., Wu, H.: Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. Commun. Math. Phys. 366(2), 469–536 (2019)
58. Powell, E., Wu, H.: Level lines of the Gaussian free field with general boundary data. Ann. Inst. Henri Poincaré Probab. Stat. 55(4), 2229–2259 (2017)
59. Qian, W., Werner, W.: Coupling the Gaussian free fields with free and with zero boundary conditions via common level lines. Commun. Math. Phys. 361(1), 53–80 (2018)
60. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math. 118, 221–288 (2000)
61. Schramm, O.: A percolation formula. Electron. Commun. Probab. 6, 115–120 (2001)
62. Schramm, O., Sheffield, S.: Contour lines of the two-dimensional discrete Gaussian free field. Acta Math. 202(1), 21–137 (2009)
63. Schramm, O., Wilson, D.B.: SLE coordinate changes. New York J. Math. 11, 659–669 (2005)
64. Smirnov, S.: Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits. C. R. Acad. Sci. Paris Sér. I Math. 333(3), 239–244 (2001)
65. Varchenko, A.: Special functions, KZ type equations, and representation theory, volume 98 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence (2003)
66. Verlinde, E., Verlinde, H.: Chiral bosonization, determinants and the string partition function. Nucl. Phys. B 288(2), 357–396 (1987)
67. Yoshida, M.: Hypergeometric Functions, My Love: Modular Interpretations of Configuration Spaces, vol. 32. Springer, Berlin (2013)
68. Zhan, D.: Random Loewner chains in Riemann surfaces. PhD Thesis, California Institute of Technology (2004)

69. Zhan, D.: Stochastic Loewner evolution in doubly connected domains. Probab. Theory Relat. Fields 129(3), 340–380 (2004)

70. Zhan, D.: Some properties of annulus SLE. Electron. J. Probab. 11, 1069–1093 (2006)

71. Zhan, D.: Reversibility of chordal SLE. Ann. Probab. 36(4), 1472–1494 (2008)

72. Zhan, D.: The scaling limits of planar LERW in finitely connected domains. Ann. Probab. 36(2), 467–529 (2008)

73. Zhan, D.: Restriction properties of annulus SLE. J. Stat. Phys. 146(5), 1026–1058 (2012)

74. Zhan, D.: Reversibility of whole-plane SLE. Probab. Theory Relat. Fields 161(3–4), 561–618 (2015)

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