Construction of New $D = 3, \mathcal{N} = 4$ Quiver Gauge Theories

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Abstract: In this paper we propose a special class of 3-algebras, called double-symplectic 3-algebras. We further show that a consistent contraction of the double-symplectic 3-algebra gives a new 3-algebra, called an $\mathcal{N} = 4$ three-algebra, which is then identified as the exact gauged three-algebra in the $\mathcal{N} = 4$ quiver gauge theories. A systematic construction is proposed for the 3-brackets and fundamental identities used in building up the $\mathcal{N} = 4$ theories, by starting with two superalgebras whose bosonic parts share at least one simple factor or $U(1)$ factor. This leads to a systematic way of constructing $D = 3, \mathcal{N} = 4$ quiver theories, of which several examples with new gauge groups are presented in detail. The general $\mathcal{N} = 4$ superconformal Chern-Simons matter theories in terms of ordinary Lie algebras can be also re-derived in our new 3-algebra approach.

Keywords: Superalgebras, Symplectic 3-Algebras, Chern-Simons Matter Theories
1. Introduction

In recent years, the construction of extended supersymmetric ($\mathcal{N} \geq 4$) Chern-Simons-matter (CSM) gauge theories has received much attention, because they are candidate low-energy description of multiple M2-branes in M-theory [1]-[15]. There are essentially two different approaches: (1) The three-algebra approach [1, 3, 8, 11, 12, 14, 15], and (2) The ordinary Lie algebra approach [3, 4, 5, 7, 10].

Four years ago, a large class of $\mathcal{N} = 4$ superconformal Chern-Simons-matter (CSM) theories were constructed by Gaiotto and Witten (GW) [3], using a method that enhances
an $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 4$. They also proved that the gauge groups of the GW theory can be classified in terms of classical supergroups. In Ref. [3], the GW theory was generalized to an $\mathcal{N} = 4$ quiver gauge theory by adding additional twisted hyper-multiplets; and the original $\mathcal{N} = 8$ theory was reformulated as generalized GW theory with $SO(4)$ gauge group. The $\mathcal{N} = 4$ theory constructed by Hosomichi, Lee, Lee, and Park in Ref. [3] will be called an $\mathcal{N} = 4$ HLLLP theory. In Ref. [3], some examples of the $\mathcal{N} = 4$ theories were derived by taking a conformal limit of $D = 3$ gauged supergravity theories.

The progress on the $\mathcal{N} \geq 4$ theories mentioned above was made by using mainly ordinary Lie algebras.

The authors of the present paper have been able to develop the 3-algebra approach, which was originally proposed [1, 8] for constructing the $D = 3, \mathcal{N} = 6, 8$ CSM theories, into a unifying approach that can be used to construct all known CSM theories with extended supersymmetries $\mathcal{N} = 4, 5, 6, 8$. Our approach is based on introducing a symplectic form into the underlying 3-algebra, which we called a symplectic 3-algebra.

We observed that the superspace formulation of the general $\mathcal{N} = 4$ CSM quiver gauge theories in our paper [12] is associated with a special class of symplectic 3-algebras. In this paper we present a general formulation of this class of symplectic algebras, and show that its consistent contraction gives a new class of symplectic 3-algebras, which can be used to construct the general $\mathcal{N} = 4$ theories as well. We then construct some $\mathcal{N} = 4$ theories with new gauge groups and recover all known $\mathcal{N} = 4$ theories by using superalgebras to construct the 3-brackets and fundamental identities used in building up the $\mathcal{N} = 4$ theories. It is also demonstrated that the general $\mathcal{N} = 4$ theories in terms of Lie 2-algebras can be rederived in our 3-algebra approach.

Generators of the 3-algebra in this class are a disjoint union of those of two symplectic sub 3-algebras, whose generators are $T_a$ and $T_{a'}$ ($a = 1, \cdots, 2R$ and $a' = 1, \cdots, 2S$), respectively. Hence we call it a double-symplectic 3-algebra. (Its contraction will give an $\mathcal{N} = 4$ three-algebra.) In this paper we allow the number of generators in the two sub 3-algebras to be unequal. The double-symplectic 3-algebra contains six independent 3-brackets whose structure constants satisfy eight independent fundamental identities (FIs) and possess certain symmetry property (see Section 2). The associated matter multiplets, called the un-twisted and twisted multiplets, take values in the two symplectic sub 3-algebras, i.e.

$$\Phi_A = \Phi_A^a T_a \quad \text{and} \quad \Phi_{\hat{A}} = \Phi_{\hat{A}}^{a'} T_{a'},$$

respectively, where $A = 1, 2$ and $\hat{A} = 1, 2$ are indices for fundamental representation of the $SU(2) \times SU(2)$ R-symmetry group.

Then in accordance with the general rules presented in previous papers [1, 12], the general $\mathcal{N} = 4$ theories can be built up by (partially) gauging the symmetry associated with the double-symplectic 3-algebra [12], if the structure constants satisfy certain reality conditions and constraint equations. Specifically, we defined carefully the symmetry transformations that were being gauged, so that after the symmetry transformations $\Phi_A$ ($\Phi_{\hat{A}}$) still take value in the sub 3-algebra spanned by $T_a$ ($T_{a'}$). Under this condition, in Ref. [12] we used only structure constants from four 3-brackets out of six 3-brackets in constructing...
the \( \mathcal{N} = 4 \) Lagrangian, and used only four FIs (see (2.15)) out of eight to demonstrate the closure of the gauged symmetry transformations. In this paper we formalize the gauged symmetry as generated by a new 3-algebra, called the \( \mathcal{N} = 4 \) 3-algebra, as a contraction of the original double symplectic 3-algebra as follows: The \( \mathcal{N} = 4 \) three-algebra has the same set of generators \( T_a \) and \( T_a' \), with the following four 3-brackets remaining unchanged:

\[
\begin{align*}
[T_a, T_b; T_c] &= f_{abc}^d T_d, \\
[T_a, T_b; T_c'] &= f_{a'b'}c'd' T_d', \\
[T_a, T_b'; T_c'] &= f_{a'b'}c'd' T_d', \\
[T_a', T_b'; T_c] &= f_{a'b'}c'd' T_d,
\end{align*}
\]  

(1.2)

and the four FIs (see (2.15)) remaining unchanged as well. But the structure constants of the remaining two 3-brackets of the \( \mathcal{N} = 4 \) three-algebra are set to zero; we call this procedure as contraction of the starting double symplectic 3-algebra. The contraction turns out to be consistent in the sense that the remaining four FIs are automatically satisfied due to the vanishing structure constants. In this way, we identify the \textit{ad hoc} symmetry that was gauged in our previous construction [12] of \( \mathcal{N} = 4 \) theories as a 3-algebra symmetry generated by the resulting \( \mathcal{N} = 4 \) 3-algebra after the consistent contraction.

We briefly describe how to construct the four sets of 3-brackets (1.2) and (1.3) and the four sets of FIs (2.15) in terms of two superalgebras. A symplectic 3-algebra can be realized in terms of a superalgebra [12]. It is natural to introduce \textit{two} superalgebras to realize the \textit{two} symplectic sub 3-algebras. We first introduce a superalgebra \( G \) (see (4.3)) to realize one of the two symplectic sub-superalgebras:

\[
T_a \doteq Q_a, \quad [T_a, T_b; T_c] \doteq \{Q_a, Q_b\}, Q_c].
\]  

(1.4)

where \( Q_a \) are fermionic generators of \( G \). We have identified the generators of the sub 3-algebra with the fermionic generators of \( G \) and constructed the 3-bracket in terms of a double graded commutator on \( G \). In this realization, one can formulate the fundamental identity (FI) of the sub symplectic 3-algebra as the \( M_a M_b Q_c \) Jacobi identity of the superalgebra \( G \) (\( M_a \) are the bosonic generators of \( G \)), and the constraint equation \( f_{(abc)d} = 0 \) required for enhancing the \( \mathcal{N} = 1 \) supersymmetry to \( \mathcal{N} = 4 \) is equivalent to the \( Q_a Q_b Q_c \) Jacobi identity, and the structure constants \( f_{abc}^d \) satisfy the correct symmetry and reality conditions as well. Similarly, we introduce another superalgebra \( G' \) to realize another sub symplectic 3-algebra:

\[
T_{a'} \doteq Q_{a'}, \quad [T_{a'}, T_{b'}; T_{c'}] \doteq \{Q_{a'}, Q_{b'}\}, Q_{c'}].
\]  

(1.5)

Finally, it is natural to construct the 3-brackets (1.3) in terms of double graded commutators defined on the superalgebras \( G \) and \( G' \), for instance,

\[
[T_a, T_b; T_c] \doteq \{\{Q_a, Q_b\}, Q_c].
\]  

(1.6)

In order that there are non-trivial interactions between the twisted and un-twisted multiplets, we must require that

\[
[T_a, T_b; T_c] \neq 0, \quad [T_{a'}, T_{b'}; T_c] \neq 0.
\]  

(1.7)
As we will prove, a sufficient condition for \([\{Q_a, Q_b\}, Q_c] \neq 0\) and \([\{Q_{a'}, Q_{b'}\}, Q_c] \neq 0\) is that the bosonic parts of \(G\) and \(G'\) share at least one simple factor or a \(U(1)\) factor. Two fundamental identities obeyed by the two structure constants of the two 3-brackets in (1.3) (see the second and third equation of (2.15)), follow from the \(M_u M_{a'} Q_{e'}\) and \(M_{a'} M_{e'} Q_{e}\) Jacobi identities, respectively. Here \(M_u\) are bosonic generators of \(G'\). In this realization, the Lie algebra of the gauge group of the \(\mathcal{N} = 4\) quiver theory is the bosonic subalgebras of \(G\) and \(G'\), and the corresponding matter representation is determined by the fermionic generators \(Q_a\) and \(Q_{b'}\). For example, we may choose \(G = OSp(2|2N)\) and \(G' = U(M|1)\), whose bosonic parts \(Sp(2N) \times U(1)\) and \(U(1) \times U(M)\) share a common factor \(SO(2) \cong U(1)\). Then the resulting gauge group is \(Sp(2N) \times U(1) \times U(M)\), and the un-twisted multiplet is in the bi-fundamental representation of \(Sp(2N) \times U(1)\), while the twisted multiplet is in the bi-fundamental representation of \(U(1) \times U(M)\). This example is one of our new \(\mathcal{N} = 4\) theories. Some new \(\mathcal{N} = 4\) theories may contain free parameters; we construct two infinite classes of new theories of this kind in Sec. 5.3 and Sec. 5.4.

Since if we use superalgebras to construct the 3-algebras, the general \(\mathcal{N} = 4\) theories in terms of 3-algebras are equivalent to the \(\mathcal{N} = 4\) HLLLP theories based on conventional Lie 2-algebras, the \(\mathcal{N} = 4\) theories with new gauge groups (see Sec. 5 and 7) can be also derived directly by taking advantage of the classification of gauge groups of the general \(\mathcal{N} = 4\) HLLLP theories. The essentially new things here are only the gauge groups, which were not explicitly discussed in the literature. It is in this sense that we construct new \(\mathcal{N} = 4\) theories.

The superalgebras \(G\) and \(G'\), whose bosonic parts share at least a simple factor or a \(U(1)\) factor, are interested in their own right. In a coming paper, we will propose a procedure to “fuse” certain \(G\) and \(G'\) into a single closed superalgebra.

This paper is organized as follows. In section 2, we introduce the double-symplectic 3-algebra and the \(\mathcal{N} = 4\) three-algebra, and review the \(\mathcal{N} = 4\) theories based on the 3-algebras in section 3. In section 4, we construct the four sets of 3-brackets (1.2) and (1.3) and the four sets of FIs (2.15) in terms of two superalgebras, (We also comment the rest two 3-brackets and four FIs which do not play roles in constructing the \(\mathcal{N} = 4\) theory.) and the subsequent section 5 is devoted to present the construction of a number of new classes of \(\mathcal{N} = 4\) theories. In section 6, the general \(\mathcal{N} = 4\) CSM theories in terms of ordinary Lie algebras are derived from their 3-algebra counterpart. In section 7, we recover all known examples of \(\mathcal{N} = 4\) theories, and produce more new examples. We end in section 8 with conclusions and discussions. We summarize our conventions in Appendix A. The commutation relations of some superalgebras used to construct symplectic sub 3-algebras are given in Appendix B.

2. Double-Symplectic 3-Algebra and \(\mathcal{N} = 4\) Three-Algebra

In this section, we introduce the two classes of 3-algebras mentioned in Section 1. Both of them can be used to construct the general \(\mathcal{N} = 4\) quiver gauge theory. We first introduce the double-symplectic 3-algebra whose generators are the disjoint union of those of two symplectic sub 3-algebras. We then introduce the \(\mathcal{N} = 4\) three-algebra as a consistent
contraction of the double-symplectic 3-algebra. The self-consistency of the $\mathcal{N} = 4$ three-algebras is revealed explicitly.

We denote the generators of the two sub 3-algebra as $T_a$ and $T_{a'}$, respectively, where $a = 1, \cdots, 2R$ and $a' = 1, \cdots, 2S$. To define two symplectic sub 3-algebras, one must introduce two invariant anti-symmetric tensors

$$
\omega_{ab} = \omega(T_a, T_b) \quad \text{and} \quad \omega_{a'b'} = \omega(T_{a'}, T_{b'})
$$

(2.1)

into the two sub 3-algebras, respectively. We denote their inverses as $\omega^{bc}$ and $\omega^{b'c'}$, satisfying

$$
\omega_{ab} \omega^{bc} = \delta^c_a \quad \text{and} \quad \omega_{a'b'} \omega^{b'c'} = \delta^{c'}_{a'}.
$$

$\omega_{ab}$ and $\omega_{a'b'}$ are required to be invariant under the transformations (2.18) and (2.19), respectively. The antisymmetric tensors $\omega$ will be used to lower or raise the indices. Also, we require that the unprimed and primed generators to be symplectic orthogonal, in the sense that

$$
\omega(T_a, T_{b'}) = \omega(T_{b'}, T_a) = 0.
$$

(2.2)

Since $T_a$ and $T_{a'}$ form a complete basis, the general 3-bracket on this 3-algebra can be defined as

$$
[T_I, T_J, T_K] = f_{IJK}^dT_d + f_{IJK}^d'T_{d'}
$$

$$
\equiv g_{IJK}L^T_L,
$$

(2.3)

where $T_I$, $T_J$ and $T_K$ are arbitrary three generators selected from the 3-algebra; each of them can be a primed or an unprimed generator, for instance,

$$
T_I = (T_a \text{ or } T_{a'}).
$$

(2.4)

The basic property of the 3-bracket of symplectic 3-algebra is that it is invariant if we switch the first two generators [11], i.e.

$$
[T_I, T_J, T_K] = [T_J, T_I, T_K].
$$

(2.5)

And we assume that the structure constants satisfy the symmetry condition

$$
\omega([T_I, T_J, T_K], T_L) = \omega([T_K, T_L, T_I], T_J).
$$

(2.6)

The generators are required to satisfy the fundamental identity:

$$
[T_I, T_J; [T_M, T_N; T_K]] = [[T_I, T_J; T_M], T_N; T_K] + [T_M, [T_I, T_J; T_N], T_K] + [T_M, T_N; [T_I, T_J; T_K]].
$$

(2.7)

The FI plays an analogous role as the Jacobi identity of an ordinary Lie algebra. Substituting (2.3) into (2.7), we obtain the FI satisfied by the structure constants:

$$
g_{MNK}^O g_{IJO}^L = g_{IJM}^O g_{ONK}^L + g_{IJN}^O g_{MOK}^L + g_{IJK}^O g_{MNO}^L.
$$

(2.8)

We call the 3-algebra defined by Eqs. (2.1), (2.2), (2.3), (2.7), (2.6) and (2.7) a double-symplectic 3-algebra.
Taking account of (2.4) and (2.5), we see that Eq. (2.3) represents six independent 3-brackets. Since the two sets of generators $T_a$ and $T_{a'}$ span the two sub 3-algebras respectively, the 3-bracket of three unprimed (primed) generators must be a linear combination of unprimed (primed) generators, i.e.

$$ [T_a, T_b; T_c] = f_{abc} d^dT_d \quad \text{and} \quad [T_{a'}, T_{b'}; T_{c'}] = f_{a'b'c'} d'^dT_{d'}. $$

Comparing (2.3) with the general definition of 3-bracket (2.3), we notice that

$$ f_{abc} d = f_{a'b'c'} d' = 0. \quad (2.10) $$

For the rest four 3-brackets, every one contains one primed (unprimed) generator and two unprimed (primed) generators. However, taking account of (2.2), (2.3), (2.6) and (2.10), we find that the four structure constants carrying three unprimed (primed) indices and one primed (unprimed) vanish, i.e.

$$ f_{abc} d = f_{a'b'c'} d' = f_{ab'c} d = f_{ba'c'} d' = 0. \quad (2.11) $$

So these four 3-brackets are given by

$$ [T_a, T_b; T_c] = f_{abc} d^dT_d, \quad [T_{a'}, T_{b'}; T_{c'}] = f_{a'b'c'} d'^dT_d, \quad (2.12) $$

$$ [T_a, T_{b'}; T_c] = f_{abc} d^dT_{d'}, \quad [T_{a'}, T_b; T_{c'}] = f_{ba'c} d'^dT_{d'}. \quad (2.13) $$

Eqs (2.10), (2.11) and (2.6) imply that

$$ f_{ijkl} = f_{klij}, \quad (2.14) $$

where $f_{abcd} = \omega_{de} f_{abc} e$ and $f_{abc'd'} = \omega_{de} f_{abc} e'$. The four subsets of FIs which do not involve $f_{ab'c'd'}$ are given by

$$ f_{abg} f_{fgd} + f_{abf} f_{gd} = \delta_{ef} g f_{abc} = \delta_{f} g f_{abc} = 0, $$

$$ f_{abg} f_{fgd} + f_{abf} f_{gd} = \delta_{ef} g f_{abc} = \delta_{f} g f_{abc} = 0. $$

$$ f_{abg} f_{fgd} = \delta_{ef} g f_{abc} = \delta_{f} g f_{abc} = 0. $$

$$ f_{abg} f_{fgd} = \delta_{ef} g f_{abc} = \delta_{f} g f_{abc} = 0. $$

The other four subsets of FIs involving $f_{ab'c'd'}$ are given by

$$ f_{ac'bd} f_{e'd'g'} = f_{eac'} d f_{bc'd'g'} + f_{eac'} d f_{ad'bg'} + f_{eac'} d f_{ac'd'g'}, $$

$$ f_{ac'bd} f_{e'd'g'} = f_{eac'} d f_{bc'd'g'} + f_{eac'} d f_{ad'bg'} + f_{eac'} d f_{ac'd'g'}, $$

$$ f_{ac'bd} f_{e'd'g'} = f_{eac'} d f_{bc'd'g'} + f_{eac'} d f_{ad'bg'} + f_{eac'} d f_{ac'd'g'}, $$

$$ f_{ac'bd} f_{e'd'g'} = f_{eac'} d f_{bc'd'g'} + f_{eac'} d f_{ad'bg'} + f_{eac'} d f_{ac'd'g'}. $$

In the previous construction of the $\mathcal{N} = 4$ theories [12], we assumed that the variation of a 3-algebra valued superfield $\Phi$ takes the form

$$ \delta_A \Phi = \Lambda^{ab} [T_a, T_b; \Phi] + \Lambda^{a'b'} [T_{a'}, T_{b'}; \Phi], \quad (2.17) $$
where $\Phi$ can be either an untwisted superfield $\Phi = \Phi^a_T a$ or a twisted superfield $\Phi = \Phi^{a'}_T a'$. The infinitesimal parameters $\Lambda^{ab}$ and $\Lambda^{a'b'}$ are independent of superspace coordinates. In terms of components, Eq. (2.17) can be written as

$$
\delta_{\tilde{\Lambda}} \Phi^d_A = \Lambda^{ab} f_{abc} \Phi^c_A + \Lambda^{a'b'} f_{a'b'c} \Phi^c_A, \quad (2.18)
$$

$$
\delta_{\tilde{\Lambda}} \Phi'^d = \Lambda^{ab} f_{abc} \Phi'^c + \Lambda^{a'b'} f_{a'b'c} \Phi'^c. \quad (2.19)
$$

We also assumed that the action is invariant under the symmetry transformation (2.17); and the symmetry will be gauged later [12].

Eq. (2.17) is obviously not the most general possibility, since one may add another term like

$$
\Lambda^{a'b'} [T_a, T_{b'}; \Phi] \equiv \delta_{\tilde{\Lambda}} \Phi
$$

(2.20)

to the right hand side of (2.17). However, substituting $\Phi = \Phi^a_T a$ into (2.20) and using the first equation of (2.13), we obtain

$$
\Lambda^{a'b'} [T_a, T_{b'}; \Phi^a_T a] = \Lambda^{a'b'} f_{abc} T_d \Phi^c_A. \quad (2.21)
$$

The right hand side indicates that $\delta_{\tilde{\Lambda}} \Phi^a_T a$ does not take value in the sub 3-algebra spanned by the unprimed generators anymore, i.e. $\delta_{\tilde{\Lambda}} \Phi^a_T a \neq (\delta_{\tilde{\Lambda}} \Phi)^c_T c$, which conflicts with our basic assumption that $\Phi^a_T a = \Phi^a_T a T_a$. We therefore must exclude this term by setting either $\Lambda^{a'b'} = 0$ or $f_{abc} = 0$.

- If we set $\Lambda^{a'b'} = 0$, we will gauge the symmetry defined by the symmetry transformation (2.17), which is only part of the full symmetry generated by the double-symplectic 3-algebra.

- If we set $f_{abc} = 0$, using (2.0) and (2.3), we obtain $f_{abc} = 0$, implying that $\delta_{\tilde{\Lambda}} \Phi^a_T a = 0$. The new 3-algebra, obtained from the double-symplectic 3-algebra by setting $f_{abc} = f_{abd} = 0$ while keeping the rest structure constants unchanged, will be called an $\mathcal{N} = 4$ three-algebra. It is in this sense that we obtain the $\mathcal{N} = 4$ three-algebra from the double-symplectic 3-algebra by a contraction. This contraction is consistent, since by setting $f_{abc} = f_{abd} = 0$, the other four subsets of FIs (2.14) are automatically satisfied. The only difference between these two 3-algebras is that in the $\mathcal{N} = 4$ three algebra, $f_{abc} = f_{abd} = 0$ strictly vanish, while in the double-symplectic 3-algebra, generally speaking, they do not vanish.

Having defined the symmetry transformations (2.17) and (2.20), we now want to examine the invariance of $\omega_{ab}$ and $\omega_{a'b'}$ under these transformations. Since $T_a$ form a complete basis of the symplectic sub 3-algebra, the antisymmetric tensor $\omega_{cd}$ must be invariant under the transformation:

$$
\delta_{\tilde{\Lambda}} \omega_{cd} = \Lambda^{ab} (f_{abc} \omega_{cd} + f_{abd} \omega_{ce}) = 0, \quad (2.22)
$$

In order that $\omega_{cd}$ is also invariant under the transformation (2.18), one must require that

$$
\delta_{\tilde{\Lambda}_2} \omega_{cd} = \Lambda^{a'b'} (f_{a'b'c} \omega_{cd} + f_{a'b'd} \omega_{ce}) = 0. \quad (2.23)
$$
Eqs (2.22) and (2.23) are equivalent to \( f_{abcd} = f_{abdc} \) and \( f_{a'b'cd} = f_{a'd'bc} \), respectively. Similarly, we obtain \( f_{a'b'c'd'} = f_{a'd'b'c'} \) and \( f_{abcd'} = f_{a'b'd'c'} \) by requiring the transformations to preserve \( \omega_{c'd'} \). These equations are consistent with Eq. (2.6). In the case of the double symplectic 3-algebra, using (2.2), it is not difficult to prove that

\[
\delta_\lambda \omega_{cd} = \delta_\lambda \omega_{c'd'} = 0, \tag{2.24}
\]

where \( \delta_\lambda \) is defined by Eq. (2.20). Eqs. (2.22) \( \sim \) (2.24) indicate that \( \omega_{cd} \) (\( \omega_{c'd'} \)) is not only an invariant tensor on the unprimed (primed) sub 3-algebra, but also an invariant tensor on the double-symplectic 3-algebra. In the case of the \( \mathcal{N} = 4 \) three-algebra, Eqs. (2.24) are satisfied automatically due to the fact that \( f_{ab'd}' \) and \( f_{b'a'd}' \) strictly vanish.

By substituting \( \Phi = \Phi_3 T_a \) and \( \Phi = \Phi_3 T_{a'} \) into (2.17), respectively, we see that the symmetry transformations involve only the four subsets of structure constants \(^1\):

\[
f_{abc}^d, \quad f_{a'b'c'd'}, \quad f_{abc'd'} \quad \text{and} \quad f_{a'b'c'd'}. \tag{2.25}
\]

Therefore, only four subsets of structure constants appear in the action and the supersymmetry transformations (see (2.4) and (3.5)) \(^1\). In summary, these structure constants enjoy the symmetry properties (see (2.5) and (2.6))

\[
\begin{align*}
f_{abcd} &= f_{bacd} = f_{bade} = f_{c dab}, \\
f_{abc'd'} &= f_{bac'd'} = f_{bad'c'} = f_{c' d a b}, \\
f_{a'b'c'd'} &= f_{b'a'c'd'} = f_{b'd'a'c'} = f_{c' d'a'b'},
\end{align*} \tag{2.26}
\]

and satisfy the reality conditions \(^1\)

\[
\begin{align*}
f^{a'b'}_{\ c'd} &= f^b_{\ a c'd}, & f^{a'd'}_{\ b'c} &= f^{b' d'}_{\ a'c}, & f^{a'c'}_{\ b'd'} &= f^{b' c'}_{\ a'd'},
\end{align*} \tag{2.27}
\]

for guaranteeing the positivity of the theory. In addition, in order to close the \( \mathcal{N} = 4 \) super Poincaré algebra, we need to impose the linear constraints on \( f_{abcd} \) and \( f_{a'b'c'd'} \)

\[
f_{(abc)d} = 0 \quad \text{and} \quad f_{(a'b'c')d'} = 0. \tag{2.28}
\]

Since we use the structure constants (2.25) to construct the \( \mathcal{N} = 4 \) theory, they must be invariant under the symmetry transformation (2.11), i.e.

\[
\delta_\lambda f_{abcd} = \delta_\lambda f_{abc'd'} = \delta_\lambda f_{a'b'c'd'} = 0, \tag{2.29}
\]

A short calculation shows that the four subsets of FIs (2.15) are equivalent to Eqs. (2.29). In particular, Eqs. (2.29) do not involve the other four subsets of FIs (2.16) at all.

Thus, to construct the \( \mathcal{N} = 4 \) gauge theory, we only need the four subsets of 3-brackets (2.9) and (2.12), together with the four subsets of FIs (2.15) associated with them. In other words, we need only to gauge the symmetry generated by four sets of 3-brackets (2.9) and (2.12) and the four sets of FIs (2.15). Later we will have chance to comment on the two other subsets of 3-brackets (2.13) which do not appear in the action and supersymmetry transformations, and the involved four sets of FIs (2.16). (See the last paragraph of Section 4).

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\(^1\)Because of the symmetry condition \( f_{ab'c'd'} = f_{c'd'ab} \) (see (2.26)), there are only three independent subsets of structure constants.
3. Construction of $\mathcal{N} = 4$ Theories Based on 3-Algebras

In this section, we will present the $\mathcal{N} = 4$ theories constructed by gauging part of symmetry generated by the double-symplectic 3-algebra, or from another point of view, constructed in terms of the $\mathcal{N} = 4$ three-algebras following the procedure proposed in [12]. The un-twisted multiplets $(Z^a_A, \psi^a_A)$, valued in the sub 3-algebra, satisfy the reality conditions

\begin{equation}
\tilde{Z}^a_A = \omega_{ab} \epsilon^{ABC} Z^b_B, \quad \tilde{\psi}^a_A = \epsilon^{ABC} \omega_{ab} \hat{A}^B Z^b_B,
\end{equation}

where $A, \hat{A} = 1, 2$ transform in the two-dimensional representation of the $SU(2) \times SU(2)$ R-symmetry group. And the twisted multiplets $(Z^a_A, \psi^a_A)$ satisfy similar reality conditions. The gauge fields and the covariant derivatives are defined as

\begin{align*}
D_{\mu} Z^A_a &= \partial_{\mu} Z^A_a - \hat{A}^A_{\mu} c^\mu Z^a_c, \quad \hat{A}^A_{\mu} c^\mu = A^a_{\mu} f^a_{\mu} c^\mu + A'^a_{\mu} f'^a_{\mu} c^\mu, \\
D_{\mu} \tilde{Z}^A_a &= \partial_{\mu} \tilde{Z}^A_a - \hat{A}'^A_{\mu} c'^\mu \tilde{Z}^a_c, \quad \hat{A}'^A_{\mu} c'^\mu = A'^a_{\mu} f'^a_{\mu} c'^\mu + A^a_{\mu} f^a_{\mu} c'^\mu.
\end{align*}

Here $A^a_{\mu}$ and $A'^a_{\mu}$ are independent Hermitian tensors. In this way, we have gauged the symmetry associated with the transformation (2.17), i.e. the symmetry generated by the four subsets of 3-brackets (2.9) and (2.12) and the four subsets of FIs (2.13). The FIs (2.13) can be also derived by requiring that all structure constants are gauge invariant quantities, i.e.

\begin{equation}
D_{\mu} f_{abcd} = D_{\mu} f_{abc'd'} = D_{\mu} f_{a'b'c'd'} = 0.
\end{equation}

The $\mathcal{N} = 4$ Lagrangian, derived from a superspace approach, is given by

\begin{align*}
\mathcal{L} &= \frac{1}{2} (-D_{\mu} \tilde{Z}^A_a D^a_{\mu} Z^a_A - D_{\mu} \hat{Z}^A_{\mu} Z^a_A + \omega_{ab} \epsilon^{ABC} \hat{Z}^a_{ABC} + i \omega_{ab} \epsilon^{ABC} \hat{Z}^a_{ABC} + i \omega_{ab} \epsilon^{ABC} \hat{Z}^a_{ABC}) \\
&- \frac{i}{2} (f_{abcd} Z^a_A Z^b_B \psi^c_B \psi^d_B + f_{a'b'c'd'} Z^a_{A'B'} \psi^c_{B'} \psi^d_{B'}) \\
&+ \frac{i}{2} f_{abc'd'} (Z^a_A Z^b_B \psi^c_B \psi^d_B + Z^a_A Z^b_B \bar{\psi}_{A'B'} + 4Z^a_A Z^b_B \bar{\psi}_{B'B'}) \\
&+ \frac{1}{2} e^{\mu\lambda} (f_{abcd} A_{\mu} \partial_{\nu} A_{\lambda} + \frac{2}{3} f_{abcd} f_{gdef} A_{\mu} A_{g} A_{d} A_{e}) \\
&+ \frac{1}{2} e^{\mu\lambda} (f_{a'b'c'd'} A_{\mu} A_{\nu} A_{\lambda} + \frac{2}{3} f_{a'b'c'd'} f_{gdef} A_{\mu} A_{g} A_{d} A_{e}) + f_{abcd} f_{gdef} A_{\mu} A_{g} A_{d} A_{e} A_{f} A_{\lambda} \\
&+ \frac{1}{12} (f_{abcd} f_{de} f_{gdef} Z^a_A Z^b_B Z^c_C Z^d_D Z^e_{C'} Z^f_{A'} + f_{a'b'c'd'} f_{gdef} f_{gdef} f_{a'b'c'd'} Z^a_{A'B'} Z^b_C Z^c_D Z^d_E Z^e_{F'} Z^f_{A'} Z^g_{C'} Z^h_{D'} Z^i_{F'}) \\
&- \frac{1}{4} (f_{abcd} f_{a'b'c'd'} Z^a_A Z^b_B Z^c_D Z^d_E Z^e_{C'} Z^f_{A'} + f_{a'b'c'd'} f_{a'b'c'd'} Z^a_{A'B'} Z^b_C Z^c_D Z^d_E Z^e_{F'} Z^f_{A'} Z^g_{C'} Z^h_{D'} Z^i_{F'}). 
\end{align*}

(3.4)
And the $\mathcal{N} = 4$ supersymmetry transformations are

\begin{align*}
\delta Z^a_A &= i\epsilon_A \bar{\psi}^a_A, \\
\delta Z^{\alpha'}_A &= i\epsilon_A \bar{\psi}'^{A}_{\alpha'}, \\
\delta \psi^a_A &= -\gamma^\mu D_\mu Z^a_B e^B_A - \frac{1}{3} f^{a' \psi' \psi} Z^a_B Z^{\beta'}_B Z^{\gamma'}_B e^A_{\beta'} + f^{a' \psi' \psi} Z^a_B Z^{\beta'}_B Z^{\gamma'}_B e^A_{\beta'}, \\
\delta \psi^a_A &= -\gamma^\mu D_\mu Z^a_B e^B_A - \frac{1}{3} f_{\alpha \beta \gamma} Z^a_B Z^{B'}_B Z^{C'}_B e^A_{B'} + f_{\alpha \beta \gamma} Z^a_B Z^{B'}_B Z^{C'}_B e^A_{B'}, \\
\delta \tilde{A}^{\mu}_{c d} &= i\epsilon^{AB} \gamma^\mu \psi^{B} Z^a_B f_{a b c}^{\alpha} d + i\epsilon^{AB} \gamma^\mu \psi^{B} Z^a_B f_{a b c}^{\alpha} d', \\
\delta \tilde{A}^{\mu}_{c d'} &= i\epsilon^{AB} \gamma^\mu \psi^{B} Z^a_B f_{a b c}^{\alpha} d' + i\epsilon^{AB} \gamma^\mu \psi^{B} Z^a_B f_{a b c}^{\alpha} d'.
\end{align*}

$\epsilon^A_B$ satisfies the reality condition

\begin{equation}
\epsilon^A_B = -\epsilon^{BC} \epsilon^A_C \epsilon^B.
\end{equation}

We have explicitly verified the closure of the above $\mathcal{N} = 4$ superalgebra \[12\].

If the twisted and untwisted multiplets take values in the same symplectic 3-algebra, for instance, $\Phi_A = \Phi^a_A T_a$ and $\Phi^a_A T_a$, the $\mathcal{N} = 4$ supersymmetry can be promoted to $\mathcal{N} = 5$ (see section \[1\]).

4. Superalgebra Realization

In this section, we will first demonstrate how to use two superalgebras to construct the four sets of 3-brackets (2.9) and (2.12) and the four sets of FIs (2.15); we will then comment on the two other 3-brackets (2.13) and the related four sets of FIs (2.16), though we do not really need (2.13) and (2.16) in constructing the theories.

Let us first briefly review the superalgebra construction of the symplectic 3-algebra in the $\mathcal{N} = 5$ theory \[12\]. In the $\mathcal{N} = 5$ case, we have used the following superalgebra

\begin{equation}
[M^m, M^n] = C^{mn s} M^s, \quad [M^m, Q_R] = -\tau^m_{RS} \omega^{ST} Q_T, \quad \{Q_R, Q_S\} = \tau^m_{RS} k_{mn} M^n, \quad (4.1)
\end{equation}

to realize the symplectic 3-algebra. Here $R = 1, \ldots, 2L$, and $\omega^{ST} = -\omega^{TS}$ and $k_{mn}$ are invariant quadratic forms on the superalgebra. The key idea of the superalgebra realization of 3-algebra is to identify the 3-algebra generators $T_R$ with the fermionic generators $Q_R$, and to construct the 3-brackets in terms of double graded commutators on the superalgebra, i.e.,

\begin{equation}
T_R = Q_R, \quad [T_R, T_S; T_T] = \{Q_R, Q_S\} Q_T. \quad (4.2)
\end{equation}

In this realization, the FI of the 3-algebra can be converted into the $MMQ$ Jacobi identity of the superalgebra, and the constraint equation $f_{(RST)U} = 0$ for enhancing the $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 5$ is equivalent to the $QQQ$ Jacobi identity of the superalgebra. The resulting Lie algebra of the gauge group is just the bosonic subalgebra of the superalgebra \[4.1\], and the corresponding representation is determined by the fermionic generators.

As for the $\mathcal{N} = 4$ case, since both the double-symplectic 3-algebra and the $\mathcal{N} = 4$ three algebra contain two sub symplectic 3-algebras taking the same form as the symplectic
They must satisfy two crucial conditions. First, there is a non-trivial interactions between the twisted and untwisted multiplets, we must require i.e. Eqs (2.10) are indeed satisfied.

We note immediately that \( G \) terms of the two double graded commutators on

\[ [k^{uv}, \omega] = 0, \]

respectively. Here \( a = 1, \ldots, 2R \) and \( a' = 1, \ldots, 2S \); \( \omega^{ab} \) and \( \omega^{a'b'} \) are invariant antisymmetric tensors whose inverses are denoted as \( \omega_{bc} \) and \( \omega_{b'c'} \), satisfying \( \omega^{ab}\omega_{bc} = \delta^a_c \) and \( \omega^{a'b'}\omega_{b'c'} = \delta^{a'}_{c'} \); \( k_{uv} \) and \( k_{u'v'} \) are invariant symmetric forms whose inverses are denoted as \( k^uv \) and \( k^{u'v'} \), satisfying \( k_{uv}k^{uv} = \delta^u_v \) and \( k_{u'v'}k^{u'v'} = \delta^{u'}_{u} \). More explicitly, these bilinear forms are defined as [21]

\[ \omega_{ab} = \kappa(Q_a, Q_b), \quad \omega_{b'a'} = \kappa(Q_{a'}, Q_{b'}), \quad k^{uv} = -\kappa(M^u, M^v), \quad k^{u'v'} = -\kappa(M^{u'}, M^{v'}), \]

which are invariant [3, 21] in the sense that

\[ \kappa([A, B], C) = \kappa(A, [B, C]), \quad \kappa([A', B'], C') = \kappa(A', [B', C']), \]

where \( A = Q_a \) or \( M^u \), and \( A' = Q_{a'} \) or \( M^{u'} \). For example,

\[ \kappa([Q_a, Q_b], M^v) = \kappa(Q_a, [Q_b, M^v]), \quad \kappa([M^u, M^v], M^w) = \kappa(M^u, [M^v, M^w]). \]

(The minus sign on the RHS of the third equation of (4.3) is determined by the first equation of (4.7) and the convention that \( \tau_{ab} = \omega_{ac}\tau_{bc}^{uv} \).

In analogue to (4.2), we construct the 3-brackets in (2.9) as follows

\[ T_a \doteq Q_a, \quad [T_a, T_b; T_c] \doteq \{Q_a, Q_b\}, Q_c = f_{abc}^d Q_d, \]

\[ T_{a'} \doteq Q_{a'}, \quad [T_{a'}, T_{b'}; T_{c'}] \doteq \{Q_{a'}, Q_{b'}\}, Q_{c'} = f_{a'b'c'}^d Q_d. \]

We note immediately that

\[ f_{abc}^d = f_{a'b'c'}^d = 0, \]

i.e. Eqs (2.10) are indeed satisfied.

As we introduced in Section 4, it is natural to construct the two 3-brackets (2.12) in terms of the two double graded commutators on \( G \) and \( G' \):

\[ [T_a, T_b; T_c] \doteq \{Q_a, Q_b\}, Q_c, \quad [T_{a'}, T_{b'}; T_{c'}] \doteq \{Q_{a'}, Q_{b'}\}, Q_{c'}. \]

They must satisfy two crucial conditions. First, there is a physical requirement: To guarantee non-trivial interactions between the twisted and untwisted multiplets, we must require that

\[ f_{abc}^d \neq 0, \quad f_{a'b'c'}^d \neq 0. \]
Secondly, we require that
\[ f_{abc}^d = f_{a'b'}c'^d = 0. \]  
(4.12)

(See Eqs. (2.11).) We will demonstrate that the above two conditions can be satisfied simultaneously by imposing certain conditions on the bosonic parts of \( G \) and \( G' \); then we will be able to prove Eqs. (2.2):
\[ \omega(T_a, T_{b'}) = \kappa(Q_a, Q_{b'}) = 0, \quad \omega(T_{b'}, T_a) = \kappa(Q_{b'}, Q_a) = 0. \]  
(4.13)

Let us first examine \([Q_a, Q_b', Q_{c'}]\). Using (4.3), a short computation gives
\[ [T_a, T_{b'}; T_{c'}] = [Q_a, Q_b', Q_{c'}] = k_{uv} \tau^u_{ab} [M_v, Q_{c'}]. \]  
(4.14)

So requiring that \( f_{abc}^d \neq 0 \) and \( f_{abc}^d = 0 \) is equivalent to requiring that
\[ [M_v, Q_{c'}] = \tau^{v'd'}c'Q_{d'} \]  
(4.15)

with \( \tau^{v'd'}c' \neq 0 \), which means that the set of fermionic generators \( Q_{c'} \) furnish a nontrivial representation of \( M^v \).

It is sufficient for that \( \tau^{v'd'}c' \neq 0 \) if the Lie algebra spanned by \( M^v \) share at least one simple factor or \( U(1) \) factor with the Lie algebra spanned by \( M^{v'} \). To prove this statement, we denote the generators of the common bosonic part as \( M^g \), i.e. schematically,
\[ M^g = M^v \cap M^{v'}. \]  
(4.16)

(It is also allowed that \( M^g \subset M^v \), \( \tilde{M}^g \subset M^{v'} \) and \( \tilde{M}^g = T^{gh}M^h \) with \( T^{gh} \) an invertible complex matrix (see Section 5.2 for an example).) Decompose \( M^v \) and \( M^{v'} \) as
\[ M^v = (M^\alpha, M^g), \quad M^{v'} = (M^{\alpha'}, M^g), \]  
(4.17)

where \([M^\alpha, M^g] = [M^{\alpha'}, M^g] = [M^\alpha, M^{\alpha'}] = 0\). (Here \( \alpha \) is not an index of spacetime spinor.) And we assume that at least one of the two commutators \([M^\alpha, Q_a]\) and \([M^{\alpha'}, Q_{a'}]\) does not vanish, i.e. we exclude the possibility that
\[ [M^\alpha, Q_a] = [M^{\alpha'}, Q_{a'}] = 0. \]  
(4.18)

We can further sharpen Eqs. (4.16) and (4.17) by requiring that
\[ [M^{\alpha'}, Q_a] = 0, \quad [M^\alpha, Q_{a'}] = 0. \]  
(4.19)

Now we assume that the superalgebra \( G' \) is simple\(^2\). If \([M^g, Q_{c'}] = 0\), then the Lie algebra defined by \( M^g \) must be an invariant subalgebra of the superalgebra \( G' \), which

\(^2\)The definition of a simple superalgebra is analogous to that of a simple Lie algebra: A simple superalgebra is a superalgebra without any invariant proper sub-superalgebras. A sub-superalgebra \( \mathcal{I} \) is called invariant if the commutator or anti-commutator of any generator of the whole superalgebra \( \mathcal{S} \) with any generator of the sub-superalgebra is still in \( \mathcal{I} \), i.e., \([X, Y] \subset \mathcal{I}, [X \subset \mathcal{S}, Y \subset \mathcal{I}]\).
contradicts our assumption that $G'$ is a simple superalgebra. We therefore must always have

$$ [M^v, Q_c] = [M^g, Q_c] = \tau^{gd'} c Q_{d'}, \quad \tau^{gd'} c \neq 0. \quad (4.20) $$

If $G'$ is not simple, the right hand side of Eq. (4.13) still does not vanish, provided that the common part $M^g$ is not the center of $G'$.

Similarly, we can demonstrate that the second equation of (4.11) and the second equation of (4.12) are obeyed under the same conditions imposed on $G$ and $G'$. Now Eqs. (4.10) become

$$ [T_a, T_b; T_c] = [(Q_a, Q_b), Q_c] = f_{abc} Q_{d'}, $$

$$ [T_a, T_b; T_c] = [(Q_a', Q_b'), Q_c] = f_{a'b'c} Q_d. \quad (4.21) $$

We see that the conditions (4.11) and (4.12) are satisfied.

We now want to prove that Eqs. (4.13) are also satisfied. Consider the following equation

$$ \kappa([[Q_a, Q_b], Q_c], Q_{d'}) = \kappa([[Q_a, Q_b], Q_{d'}], Q_c). \quad (4.22) $$

A short computation gives

$$ k_{uv} \tau^u_{ab}(\tau^v)^d c \omega_{dd'} = (k_{\alpha\beta} \tau^\alpha_{ab} \tau^\beta_{cd} + k_{gh} \tau^\alpha_{ab}(\tau^h)^d c \omega_{dd'} = k_{gh} \tau^\alpha_{ab}(\tau^h)^d c \omega_{d'c}, \quad (4.23) $$

where $\omega_{dd'} = \kappa(Q_d, Q_{d'})$. We have decomposed $k_{uv}$ into $k_{uv} = (k_{\alpha\beta}, k_{gh})$; in the most right hand-side, we have used the second equation of (4.13). Raising the index $c$, the above equation can be written as

$$ k_{uv} \tau^u_{ab}(\tau^v)^d c \omega_{d'}^d = (k_{\alpha\beta} \tau^\alpha_{ab} \tau^\beta_{cd} + k_{gh} \tau^\alpha_{ab}(\tau^h)^d c \omega_{d'}^d = \omega_{d'c}^d k_{gh} \tau^\alpha_{ab}(\tau^h)^d c \omega_{d'c}), \quad (4.24) $$

where $\omega_{d'}^d = \omega_{d'e} \omega_{ed'}$.  

Recall that we exclude the possibility that the two commutators $[M^\alpha, Q_a]$ and $[M^{\alpha'}, Q_{a'}]$ vanish identically (see Eqs (4.13)). Without loss generality, we assume that $[M^\alpha, Q_c] = (\tau^\alpha)^d c Q_d \neq 0$. In other words, $(\tau^\alpha)^d c \neq 0$, i.e. it is a nontrivial (and irreducible) representation of $M^\alpha$ furnished by the fermionic generators $Q_d$. However, the right hand side of (4.23) indicates that $Q_{d'}$ furnish a trivial representation of $M^\alpha$ in the sense that $(\tau^\alpha)^d c = 0$. In summary, $k_{uv} \tau^u_{ab}(\tau^v)^d c$ and $k_{gh} \tau^\alpha_{ab}(\tau^h)^d c$ are nonequivalent and irreducible representations\(^3\) of $Q_a, Q_b$, furnished by $Q_c$ and $Q_{d'}$, respectively. Apply Schur’s Lemma to equation (4.24), we have immediately $\omega_{d'}^d = 0$. Hence $\omega_{dd'} = 0$ on account of that $\omega_{de}$ is nonsingular. So Eqs. (4.13) are satisfied.

\(^3\)Generally speaking, the set of generators $Q_c = Q_{\hat{k}k}$ furnish a bi-fundamental representation of the anticommutator $Q_a, Q_b = (\tau_\alpha)_{ab} M^\alpha + (\tau_\alpha)_{ab} M^\alpha$, in the sense that $[[Q_a, Q_b], Q_c] = (\tau_\alpha)_{ab} (M^\alpha)^{\hat{k}k} Q_{\hat{k}k}$. Here $k$ and $\hat{k}$ are fundamental indices of $M^g$ and $M^\alpha$, respectively. The two Lie algebras, spanned by $M^\alpha$ and $M^g$ respectively, are distinct and have nothing in common; the fundamental indices $k$ and $\hat{k}$ are distinct as well. So this bi-fundamental representation is irreducible. On the other hand, since $[M^g, Q_{a'}] = 0, Q_{a'}$ only furnish an irreducible representation of $(\tau_\alpha)_{ab} M^g$. (See section 3 and section 4 for many examples.)
Substituting (4.15) into the first equation of (4.10) gives the structure constants

\[ f_{abc'd'} = k_{gh} \tau^g_{ab} \tau^h_{cd'}, \]  

(4.25)

Using (4.3) and (4.4), a short calculation gives the structure constants of 3-brackets in (4.8)

\[ f_{abcd} = k_{uv} \tau^u_{ab} \tau^v_{cd}, \quad f_{a'b'c'd'} = k_{a'a'} \tau^{a'a'}_{d'd'} \tau^v_{cd}. \]

(4.26)

The structure constants (4.25) and (4.26) possess the desired reality and symmetry properties. The \( Q_a Q_b Q_c (Q_d' Q_b' Q_c') \) Jacobi identity implies that \( \tilde{f}_{abc}d = 0 \) (\( \tilde{f}_{a'b'c'd'} = 0 \)) guaranteeing that the supersymmetry can be enhanced from \( N = 1 \) to \( N = 4 \).

The four sets of FIs in (2.13) are equivalent to the \( M^u M^v Q_a, M^u M^v Q_a', M^{u'} M^{v'} Q_a \) and \( M^{u'} M^{v'} Q_{a'} \) Jacobi identities, respectively. For instance, by using Eqs. (4.8) and (4.4), one can easily convert the above equation into the \( M^u M^v Q_{a'} \) Jacobi identity

\[ \tau^u_{ab} \tau^v_{cd} ([M_v, [M_u, Q_{a'}]] - [M_u, [M_v, Q_{a'}]] + [[M_u, M_v], Q_{a'}]) = 0, \]  

(4.27)

where we have used the equation

\[ k_{uw} k_{vx} f^{xy}_{\ y} = k_{uw} k_{yx} f^{xy}_{\ x} = 0, \]  

(4.29)

which is equivalent to the second equation of (1.7) on account of that \( k^{uv} \) is invertible. Therefore the second FI of (2.15) is equivalent to the \( M^u M^v Q_{a'} \) Jacobi identity. Using Eqs. (1.16) and (4.17), Eq. (4.28) becomes

\[ \tau^g_{ab} \tau^h_{cd} ([M_h, [M_g, Q_{a'}]] - [M_g, [M_h, Q_{a'}]] + [[M_g, M_h], Q_{a'}]) = 0, \]  

(4.30)

which is of course obeyed, since the \( M^{u'} M^{v'} Q_{a'} \) Jacobi identity is obeyed and \( M^g \subseteq M^{u'}. \)

In this realization, the un-twisted and twisted multiplets take values in the bosonic subalgebras of the superalgebras (4.3) and (4.4), respectively; and the representations of the bosonic parts (4.3) and (4.4) are determined by the fermionic generators \( Q_a \) and \( Q_{a'} \), respectively.

Here we have to emphasize that so far we have constructed only the four structure constants of the 3-brackets (2.13) and (2.14) in terms of tensor products (1.26) and (1.27) on the superalgebras \( G \) and \( G' \), respectively, and solved the four sets of FIs (2.13) of the double-symplectic 3-algebra in terms of certain Jacobi identities of the superalgebras \( G \) and \( G' \). These constructions are sufficient for the purpose of classifying the gauge groups of the \( \mathcal{N} = 4 \) theory (see Section (7)).

We now would like to comment on the rest two 3-brackets (2.13) and the rest four FIs (2.14), though they do not play any role in constructing the theories. In the case of
double-symplectic 3-algebra, we will demonstrate that at least in the special case of that 
$G$ and $G'$ can be ‘fused’ into a single closed superalgebra (the fusion procedure will be 
introduced in a separated paper [13]), Eqs. (2.13) and (2.16) can be constructed in terms 
of superalgebras as well.

In the case of double-symplectic 3-algebra, under the condition that $G$ and $G'$ can be 
‘fused’ into a closed superalgebra, the rest two 3-brackets Eqs. (2.13) can be constructed 
in analogue to Eqs. (4.8) and (4.10), i.e.

\[ [T_a, T_{b';} T_c] \doteq [(Q_a, Q_{b'}, Q_c), [Q_{a'}, Q_b, Q_c]], \quad [T_{a'}, T_b; T_c] \doteq [(Q_{a'}, Q_b, Q_c), [Q_a, Q_{b'}, Q_c]], \]  
(4.31)

where we have used the fact that the disjoint union of the two sets generators $Q_a$ and $Q_{a'}$
form a complete fermionic basis of the “fused” superalgebra [13]. In summary, we have

\[ T_I \doteq Q_I, \quad [T_I, T_J; T_K] \doteq [(Q_I, Q_J), Q_K], \]  
(4.32)

where the index $I = a$ or $a'$. Using the first equation of Eq. (4.7), it is easy to prove that

\[ \kappa([\{Q_a, Q_b\}, Q_c], Q_d) = \kappa([\{Q_a, Q_b\}, \{Q_c, Q_d\}]). \]  
(4.33)

Generally speaking, the following equations

\[ \kappa([\{Q_I, Q_J\}, Q_K], Q_L) = \kappa([\{Q_K, Q_L\}, \{Q_I, Q_J\}) = \kappa([\{Q_K, Q_L\}, Q_I], Q_J) \]  
(4.34)

hold. The above equations imply that (2.6) is obeyed by the construction

\[ \omega([T_I, T_J; T_K], T_L) \doteq \kappa([\{Q_I, Q_J\}, Q_K], Q_L). \]  
(4.35)

Using (4.9), (4.34) and (4.13), we obtain $f_{abc}d = f_{a'bc}d' = 0$. In summary, we have

\[ f_{abc}d = f_{a'bc}d' = f_{abc'}d = f_{a'b'c}d' = f_{abc}d' = f_{a'b'c}d' = 0, \]  
(4.36)

which are nothing but Eqs. (2.11) and (2.10). Combing (4.34), (4.35) and (4.36), one can 
prove that (2.14) is also satisfied. The first two equations of Eqs. (4.36), i.e. $f_{abc}d = 
f_{a'bc}d' = 0$, imply that the two structure constants associated with the brackets (4.31) are 
given by

\[ [T_a, T_{b';} T_c] \doteq [(Q_a, Q_{b'}, Q_c), Q_c] = f_{abc}d' Q_d', \quad [T_{a'}, T_b; T_c] \doteq [(Q_{a'}, Q_b, Q_c), Q_c] = f_{a'b'c}d Q_d. \]  
(4.37)

Define the anticommutator of $Q_a$ and $Q_{b'}$ as

\[ \{Q_a, Q_{b'}\} = t_{ab'}^{\tilde{a}} M_{\tilde{a}}, \]  
(4.38)

where $M_{\tilde{a}}$ are a set of bosonic generators. We have to emphasis that we have not introduced 
any new generators into the “fused” superalgebra except $M_{\tilde{a}}$ [13]. So the generators of the 
“fused” superalgebra are consisted of by the generators of $G$ and $G'$, as well as $M_{\tilde{a}}$. Every 
Jacobi identity of the “fused” superalgebra is obeyed. In particular, the Jacobi identities

\[ [\{Q_a, Q_{b'}\}, Q_c] + [\{Q_a, Q_c\}, Q_{b'}] + [\{Q_c, Q_{b'}\}, Q_a] = 0, \]  
(4.39)

\[ [\{Q_{a'}, Q_b\}, Q_c'] + [\{Q_{a'}, Q_c'\}, Q_b] + [\{Q_c, Q_b\}, Q_{a'}] = 0 \]  
(4.40)
are satisfied. In other words, we have the identity

\[ f_{ab'c'd'} + f_{cb'ad'} + f_{ac'bd'} = 0 \quad \text{or} \quad \tau_{ab}(\tau_{a})_{cd'} + \tau_{cb}(\tau_{c})_{ad'} + \tau_{ac}(\tau_{c})_{bd'} = 0, \quad (4.41) \]

which follows from (4.38) via (4.39) or (4.40). Therefore these structure constants (or the corresponding 3-brackets) are not independent in this special case.

Also, under the condition that \( G \) and \( G' \) can be ‘fused’ into a single closed superalgebra, it is not difficult to prove that the four FIs (2.14) involving \( f_{ab'c'd'} \) can be converted into certain Jacobi identities of the “fused” superalgebra. These Jacobi identities involve the bosonic generators \( M_{\alpha} \) defined in Eq. (4.38). In this way, the whole double-symplectic 3-algebra can be realized in terms of “fused” superalgebras; however, we are not sure whether it can be realized in terms of \( G \) and \( G' \) or not, if \( G \) and \( G' \) cannot be ‘fused’ into a single closed superalgebra. We therefore leave it as an open question.

In the case of \( N = 4 \) three-algebra, the two structure constants \( f_{ab'c'd'} \) and \( f_{ab'c'd'} \), vanishing identically, cannot be constructed in terms of the tensor products on the superalgebras \( G \) and \( G' \), since the double graded commutators \( [(Q_{a}, Q_{b'}), Q_{c}] = f_{ab'c'd'}Q_{d'} \) and \( [(Q_{a}, Q_{b'}), Q_{c}] = f_{ab'c'd}Q_{d} \) do not vanish. For example, if both \( G \) and \( G' \) are orthosymplectic superalgebras or unitary superalgebras, one can prove that both \( [(Q_{a}, Q_{b'}), Q_{c}] \) and \( [(Q_{a}, Q_{b'}), Q_{c}] \) are not zero by direct calculation [13].

5. Explicit Examples of New \( N = 4 \) Quiver Theories

The classification of gauge groups of the \( N = 4 \) theories can be found in [3], [6] (there is a nice summary in [14]). In this section, however, we are able to construct some new examples which were neglected in previous classification in the literature by using the ideas described in the previous section.

5.1 \( Sp(2N) \times U(1) \times U(M) \)

Here we choose the superalgebras \( G \) and \( G' \) (see (4.3) and (4.4)) as \( OSp(2|2N) \) and \( U(M|1) \) respectively. The common part of the bosonic parts of \( OSp(2|2N) \) and \( U(M|1) \) is \( SO(2) \equiv U(1) \). Some useful \( U(M|1) \) commutation relations are (the commutation relations of \( U(M|N) \) are given by Appendix [3.1])

\[
\begin{align*}
\{ Q_{i'}, Q_{j'} \} &= k'(M_{i'j'} + \delta_{i'j'}M_{U(1)}), \\
[M_{U(1)}, Q_{i'}] &= -\bar{Q}_{i'}, \\
[M_{U(1)}, Q_{j'}] &= Q_{j'}, \\
[Q_{i'}', \bar{Q}_{j'}] &= \delta_{i'j'}Q_{i'}, \\
\{ Q_{i'}', Q_{j'}' \} &= -\delta_{i'j'}Q_{i'}, \\
[M_{i'j'}', Q_{k'}] &= -\delta_{i'j'}Q_{i'}, \\
[M_{i'j'}', Q_{k'}'] &= -\delta_{i'j'}Q_{i'},
\end{align*}
\]

(5.1)

where the subscript index \( i' = 1, \ldots, M \) is the index for the fundamental representation of \( U(M) \), and \( M_{U(1)} \) is the \( U(1) \) generator. We have suppressed the \( U(1) \) indices carried by the fermionic generators. We then identify the \( U(1) \) generator \( M_{U(1)} \) with the \( SO(2) \) generator of \( OSp(2|2N) \) \( M_{ij} \) (the commutation relations of \( OSp(M|2N) \) are given by Appendix [3.2]) by imposing the following equation

\[ \epsilon_{ij}M_{U(1)} = -M_{ij}. \]

(5.2)
Here \(\epsilon_{ij} = -\epsilon_{ji}\) and \(\epsilon_{12} = 1\), with \(i = 1, 2\) an \(SO(2)\) index. With this identification, the commutator of \(M_{U(1)}\) with a fermionic generator of \(OSp(2|2N)\) is given by

\[
[M_{U(1)}, Q_{\bar{ii}}] = \epsilon_{ij} Q_{\bar{ij}}; \tag{5.3}
\]

and the commutators of the \(SO(2)\) generator of \(OSp(2|2N)\) \(M_{\bar{ij}}\) with the fermionic generators of \(U(M|1)\) are given by

\[
[M_{\bar{ij}}, \bar{Q}_{i'}] = \epsilon_{ij} \bar{Q}_{i'}, \quad [M_{\bar{ij}}, Q^{i'}] = -\epsilon_{ij} Q^{i'}. \tag{5.4}
\]

To calculate the structure constants, we define

\[
Q_a = Q_{\bar{ii}} \quad \text{and} \quad Q_{a'} = \left( \frac{\bar{Q}_{i'}}{-Q^{i'}} \right) = Q_{i'} \delta_{1a} - Q^{i'} \delta_{2a}; \tag{5.5}
\]

where \(i = 1, \cdots, 2M\) is an \(Sp(2N)\) fundamental index. In the second equation we have introduced a “spin up” spinor \(\chi_{1a} = \delta_{1a}\) and a “spin down” spinor \(\chi_{2a} = \delta_{2a}\). The structure constants \(f_{abc'd'}\) can be read from the double graded commutator.

\[
[(Q_a, Q_b), Q_{c'}] = f_{abc'd'} Q_{d'}. \tag{5.6}
\]

The double grade commutator can be calculated straightforwardly by using (5.1), (5.4) and Appendix B.2. The structure constants \(f_{abc'd'}\) are given by

\[
f_{abc'd'} = f_{\bar{a}\bar{b}\bar{c}\bar{d}'} = k \omega_{\bar{ij}} \epsilon_{ij} (\delta_{i'j'} \delta_{1a} \delta_{2b} + \delta_{j'i'} \delta_{1b} \delta_{2a}). \tag{5.7}
\]

Similarly, one can calculate \(f_{c'd'ab}\) by using

\[
[(Q_{c'}, Q_{d'}), Q_a] = f_{c'd'a'b} Q_b. \tag{5.8}
\]

By requiring \(f_{abc'd'} = f_{c'd'ab}\), \(k'\) in the anticommutator of (5.1) is determined to be \(k' = -k\), with \(k\) defined in the anticommutator in (B.3).

The structure constants \(f_{abcd}\) can be read off from the double graded commutator \([(Q_a, Q_b), Q_c] = f_{abc'd'} Q_d\); they are given by

\[
f_{abcd} = f_{\bar{a}\bar{b}\bar{c}\bar{d}} = k \delta_{1a} \delta_{1b} \delta_{2c} \omega_{\bar{i}j} \omega_{\bar{k}l} - \delta_{ij \bar{i}j} \delta_{1a} \delta_{2b} \omega_{\bar{i}j} \omega_{\bar{j}k} \omega_{\bar{i}j} \omega_{\bar{j}k}]. \tag{5.9}
\]

Similarly, we have

\[
f_{a'c'd'c'} = f_{j'j'\bar{c}\bar{d}'c'} = f_{j'j'\bar{c}\bar{d}'c'} = f_{j'j'\bar{c}\bar{d}'c'} = f_{j'j'\bar{c}\bar{d}'c'} = f_{j'j'\bar{c}\bar{d}'c'}\]

\[
+ f_{j'j'\bar{c}\bar{d}'c'} = k \delta_{1a} \delta_{1b} \delta_{2c} \omega_{\bar{i}j} \omega_{\bar{j}k} \omega_{\bar{i}j} \omega_{\bar{j}k} \omega_{\bar{i}j} \omega_{\bar{j}k}]. \tag{5.10}
\]

It is straightforward to verify that the structure constants (5.7), (5.9) and (5.10) satisfy the symmetry conditions (2.26) and the reality conditions (2.27), and obey the fundamental

---

4Here \(\alpha\) is not a spacetime spinor index. In this paper, since we have to label many indices, it is unavoidable that some letters will be repeatedly used, but they will be defined explicitly in the sections in which they are used. We hope this will not cause any confusion.
identities \(\text{(2.15)}\). Substituting \(\text{(2.7)}\), \(\text{(2.9)}\) and \(\text{(2.10)}\) into \(\text{(2.2)}\), \(\text{(2.4)}\) and \(\text{(2.5)}\) gives the \(\mathcal{N} = 4\), \(\text{Sp}(2N) \times U(1) \times U(M)\) theory. In this realization, the un-twisted multiplets are in the bifundamental representation of \(\text{Sp}(2N) \times U(1)\), while the twisted multiplets are in the bifundamental representation of \(U(M) \times U(1)\). In Section \(5.4\) we will introduce a more general scheme to identify the \(U(1)\) factors of the even parts of \(\text{OSp}(2|N_2)\) and \(U(N_3|N_4)\). The resulting gauge group will be \(\text{Sp}(2N_2) \times U(1) \times U(N_3) \times U(N_4)\).

**5.2 \(\text{Sp}(2N) \times \text{SU}(2)_R \times \text{SU}(2)_L \times \text{SO}(M)\) and Other Examples**

The superalgebras \(\text{OSp}(4|2N)\) and \(\text{D}(2|1, \alpha)\) (with \(\alpha\) a continuous parameter) are rather special for constructing \(\mathcal{N} = 4\) theories in that both of them contain an \(\text{SO}(4)\) factor in their bosonic parts. A simple observation is the well known decomposition \(\text{SO}(4) \cong \text{SU}(2)_R \times \text{SU}(2)_L\), where the \(\text{SU}(2)_R\) (\(\text{SU}(2)_L\)) generators satisfy an anti-self-duality (self-duality) condition. Since several classes of superalgebras contain a simple factor \(\text{SU}(2)\) (or its isometries \(\text{Sp}(2)\) and \(\text{SO}(3)\)) in their bosonic parts, identifying them with \(\text{SU}(2)_R\) or \(\text{SU}(2)_L\) in the bosonic parts of the superalgebra \(\text{OSp}(4|2N)\) or \(\text{D}(2|1, \alpha)\) will generate new \(\mathcal{N} = 4\) theories.

In this subsection, we present an explicit example to illustrate this idea. We choose \(G = \text{OSp}(4|2N)\) and \(G' = \text{OSp}(M|2)\). (The commutation relations of \(\text{OSp}(M|2N)\) are given by Appendix \(\text{B.3}\).) The bosonic part of the superalgebra \(\text{OSp}(4|2N)\) contains an \(\text{SO}(4)\) factor. Later \(\text{SO}(4)\) will be decomposed into \(\text{SU}(2)_R \times \text{SU}(2)_L\). The bosonic part of \(\text{OSp}(M|2)\) includes an \(\text{Sp}(2)\) factor. It is well known that \(\text{Sp}(2) \cong \text{SU}(2)\), so without loss of generality, we can identify \(\text{Sp}(2)\) with \(\text{SU}(2)_R\). In other words, we choose

\[
\text{Sp}(2) = \text{SU}(2)_R
\]

as the common part of the bosonic parts of \(\text{OSp}(4|2N)\) and \(\text{OSp}(M|2)\). Some useful commutation relations of \(\text{OSp}(M|2)\) are

\[
\{Q_{i\alpha}, Q_{j\beta}\} = k'(\epsilon_{\alpha\beta} M_{ij} + 2\delta_{ij} M_{\alpha\beta}), \quad [2M_{\alpha\beta}, Q_{\gamma\tau}] = \epsilon_{\beta\gamma} Q_{i\alpha} + \epsilon_{\alpha\gamma} Q_{i\beta},
\]

\[
[M_{ij}, Q_{kn}] = \delta_{jk} Q_{i\alpha} - \delta_{ik} Q_{j\alpha},
\]

where in the first line we have introduced a factor 2 for later consistence. Here \(\tilde{i} = 1, \ldots, M\) is an \(\text{SO}(M)\) index, and \(\alpha = 1, 2\) is an \(\text{Sp}(2)\) index, not a spacetime spinor index (we hope this will not cause any confusion). And the useful commutation relations of \(\text{OSp}(4|N)\) are given by

\[
\{Q_{\hat{m}i}, Q_{nj}\} = k(\omega_{ij} M_{mn} + \delta_{mn} M_{ij}), \quad [M_{mn}, Q_{\hat{p}i}] = \delta_{np} Q_{\hat{m}i} - \delta_{mp} Q_{\hat{n}i},
\]

\[
[M_{ij}, Q_{mk}] = \omega_{jk} Q_{\hat{m}i} + \omega_{ik} Q_{\hat{n}j}.
\]

Here \(m = 1, \ldots, 4\) is an \(\text{SO}(4)\) index, and \(\hat{i} = 1, \ldots, 2N\) an \(\text{Sp}(2N)\) index.

To decompose the \(\text{SO}(4)\) generators, we first introduce a set of \(\text{SU}(2)_R \times \text{SU}(2)_L\) \(\sigma\)-matrices as follows

\[
\sigma_{\hat{m}\hat{n}} = (\sigma_1, \sigma_2, \sigma_3, i\vec{l}), \quad \sigma^\dagger_{\hat{m}\hat{n}} = (\sigma_1, \sigma_2, \sigma_3, -i\vec{l}),
\]

\[\text{Strictly speaking, the set of } \text{SU}(2)_R \text{ generators are related to the set of Sp}(2) \text{ generators via a complex linear nonsingular transformation (see Eqs (5.19)).}\]
where \( \sigma_i \) \((i = 1, \ldots, 3)\) are pauli matrices. The \( SU(2)_R \) matrices \( \sigma_{m\alpha} \) and the \( SU(2)_L \) matrices \( \bar{\sigma}_{m\alpha} \) are defined as

\[
\sigma_{m\alpha\beta} = \frac{1}{4}(\sigma_m\sigma_n^\dagger - \sigma_n\sigma_m^\dagger)_{\alpha\beta}, \quad \bar{\sigma}_{m\alpha\dot{\beta}} = \frac{1}{4}(\sigma_m^\dagger\sigma_n - \sigma_n^\dagger\sigma_m)_{\dot{\alpha}\dot{\beta}};
\]

and they satisfy the ‘duality’ conditions

\[
\sigma_{mn} = -\frac{1}{2}\varepsilon_{mnpq}\sigma_{pq}, \quad \bar{\sigma}_{mn} = \frac{1}{2}\varepsilon_{mnpq}\bar{\sigma}_{pq}.
\]

Here \( \varepsilon_{mnpq} \) is the totally antisymmetric tensor.\(^6\) We decompose the \( SO(4) \) generators \( M_{mn} \) by defining

\[
M_{mn}^\pm = \frac{1}{2}(M_{mn} \pm \frac{1}{2}\varepsilon_{mnpq}M_{pq}).
\]

With the above definitions, we have

\[
M_{mn} = M_{mn}^+ + M_{mn}^-, \quad M_{mn}^\pm = \pm \frac{1}{2}\varepsilon_{mnpq}M_{pq}^\pm, \quad [M_{mn}^+, M_{pq}^-] = 0.
\]

Therefore \( M_{mn}^+ \) and \( M_{mn}^- \), satisfying self-duality and anti-self-duality conditions, must be the \( SU(2)_L \) and \( SU(2)_R \) generators, respectively. We now connect the set of \( Sp(2) \) generators \( M_{\alpha\beta} \) of \( OSp(M|2) \) with \( M_{mn}^- \) via the equation \(^7\)

\[
M_{mn}^- = \sigma_{mn}^{\alpha\beta}M_{\alpha\beta}, \quad \text{or} \quad M_{\alpha\beta} = \frac{1}{2}M_{mn}^-\sigma_{mn\alpha\beta}.
\]

The above equations are the precise statement of (5.11). So it may be more appropriate to say that the set of \( SU(2)_R \) generators are related to the set of \( Sp(2) \) generators via a complex linear nonsingular transformation.

One can use the first equation of (5.11) to verify that \( \sigma_{mn}^{\alpha\beta}M_{\alpha\beta} \) also obey the anti-duality condition and satisfy the same commutation relations as \( M_{mn}^- \) do. On the other hand, we connect \( M_{mn}^+ \) with another set of independent \( SU(2) \) generators \( \bar{M}_{\dot{\alpha}\dot{\beta}} \) by the equation

\[
M_{mn}^+ = \bar{\sigma}_{mn}^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}.
\]

In summary, we have

\[
M_{mn} = M_{mn}^- + M_{mn}^+ = \sigma_{mn}^{\alpha\beta}M_{\alpha\beta} + \bar{\sigma}_{mn}^{\dot{\alpha}\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}.
\]

With the non-singular transformations (5.13), we can recast (5.12) into the following form:

\[
\begin{align*}
\{Q_{i\alpha}, Q_{j\beta}\} & = k'\epsilon_{\alpha\beta}M_{ij} + \delta_{ij}M_{mn}^-\sigma_{mn\alpha\beta}, \quad [M_{mn}^-, Q_{\gamma}] = -\sigma_{mn\gamma}^{\alpha\beta}Q_{i\alpha}, \\
[M_{ij}^+, Q_{k\alpha}] & = \delta_{jk}Q_{i\alpha} - \delta_{ik}Q_{j\alpha}.
\end{align*}
\]

\(^6\)Our convention is that \( \varepsilon^{1234} = 1 \) and \( \varepsilon_{mnpq} = \varepsilon^{mnpq} \).

\(^7\)We use the invariant antisymmetric tensor \( \epsilon_{\alpha\beta} \) and \( \epsilon_{\dot{\alpha}\dot{\beta}} \) to lower undotted and dotted indices, respectively. For example, \( \sigma_{m\alpha}^{\dot{\gamma}} = \epsilon_{\alpha\gamma}\sigma_{m\dot{\alpha}} \) and \( \sigma_{m\dot{\alpha}} = \epsilon_{\dot{\gamma}\alpha}\sigma_{m\dot{\gamma}} \). The inverse of \( \epsilon_{\alpha\beta} \) is defined as \( \epsilon^{\dot{\beta}\gamma} \) satisfying \( \epsilon_{\alpha\beta}\epsilon^{\dot{\beta}\gamma} = \delta_{\alpha\gamma} \) (similarly \( \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\gamma}} = \delta^{\dot{\alpha}\dot{\gamma}} \)).
Here it is important to note that we can use either (5.12) or (5.22) to calculate the structure constants $f_{a'b'c'}$ and $f_{a'b'c'd'}$. The final results are the same. In section 3, we will provide an explanation on this.

Now it is straightforward to calculate the structure constants $f_{a'b'c'}$ by using Eq. (4.10):

$$
\{Q_a, Q_b\}, Q_{a'} = \{\{Q_{\tilde{m}}, Q_{\tilde{n}}\}, Q_{1\alpha}\} = [\tilde{\tilde{\omega}}_{ij} M_{mn}, Q_{1\alpha}] = -k\tilde{\tilde{\omega}}_{ij} \sigma_{mna} \beta Q_{1\beta}. 
$$

In the final line we have used the second equation of (5.22). Now $f_{a'b'c'}$ can be read off from the above equation immediately:

$$
f_{a'b'c'} = f_{\tilde{m}, \tilde{n}, \tilde{i}, \alpha, \tilde{j}, \beta} = -k\tilde{\tilde{\omega}}_{ij} \delta_{i\beta} \sigma_{mna} \beta. 
$$

The structure constants $f_{a'b'c'd'}$ can be also read off from $\{\{Q_{a'}, Q_{b'}\}, Q_a\} = f_{a'b'a'b}Q_b$. By requiring that $f_{a'b'c'} = f_{a'b'c'd'}$, $k'$ in (1.12) is determined to be $k' = -\frac{1}{2}k$.

Since both $G$ and $G'$ are ortho-symplectic algebras, the structure constants $f_{abcd}$ and $f_{a'b'c'd'}$ must take the same form as (5.9); they are given by

$$
f_{abcd} = f_{\tilde{m}, \tilde{n}, \tilde{i}, \alpha, \tilde{j}, \beta} = k([\delta_{\mu p} \delta_{\nu q} - \delta_{\mu q} \delta_{\nu p}] \tilde{\tilde{\omega}}_{ij} \tilde{\tilde{\omega}}_{kl} - \delta_{mn} \delta_{pq} (\tilde{\tilde{\omega}}_{ik} \tilde{\tilde{\omega}}_{jl} + \tilde{\tilde{\omega}}_{il} \tilde{\tilde{\omega}}_{jk})),
$$

$$
f_{a'b'c'd'} = f_{\tilde{i}, \tilde{a}, \tilde{j}, \beta, \tilde{k}, \alpha} = -\frac{k}{2}[(\delta_{jk} \delta_{i\beta} - \delta_{i\beta} \delta_{kj}]) \epsilon_{\alpha \beta} \epsilon_{\gamma \delta} - \delta_{ij} \delta_{kl} (\epsilon_{\alpha \gamma} \epsilon_{\beta \delta} + \epsilon_{\alpha \delta} \epsilon_{\beta \gamma})].
$$

It is straightforward to verify that the structure constants (5.24), (5.25) and (5.26) satisfy the symmetry conditions (2.26) and the reality conditions (2.27), and obey the fundamental identities (2.15). In verifying the FLs, we have used the identity

$$
\sigma_{mp\alpha \beta} \sigma_{pm\gamma \delta} - \sigma_{mp\alpha \beta} \sigma_{pm\gamma \delta} = \frac{1}{2} (\epsilon_{\alpha \gamma} \epsilon_{m\beta \delta} + \epsilon_{\beta \gamma} \epsilon_{m\alpha \delta} + \epsilon_{\alpha \delta} \epsilon_{m\beta \gamma} + \epsilon_{\beta \delta} \epsilon_{m\alpha \gamma}).
$$

Substituting (5.24), (5.25) and (5.26) into (3.2), (3.4) and (3.5) gives the $N = 4$, $Sp(2N) \times SU(2)_R \times SU(2)_L \times SO(M)$ theory. In this realization, the un-twisted multiplets are in the bifundamental representation of $Sp(2N) \times SO(4)$, while the twisted multiplets are in the bifundamental representation of $Sp(2) \times SO(M)$.

We now would like to work out the explicit expression of the gauge field $\tilde{A}_{\mu ab}$. Substituting the expressions of $f_{cdab}$ and $f_{a'b'c'd'}$ (see (5.24) and (5.25)) into the second equation of (3.2), the definition of $\tilde{A}_{\mu ab}$, we obtain

$$
\tilde{A}_{\mu ab} = (\tilde{A}_{\mu})_{\tilde{m}, \tilde{n}, \tilde{i},\tilde{j}} = k\tilde{\tilde{\omega}}_{ij} [A_{\mu}^{pq}(\tau_{pq}^{\dagger})_{mn} + (A_{\mu}^{pq} - \frac{1}{2}A_{\mu}^{a'\beta} \sigma_{a'\alpha} \beta)(\tau_{pq}^{\dagger})_{mn}] + k\delta_{mn} A_{\mu}^{k\tilde{i}} (\tau_{kl}^{\dagger})_{ij}.
$$

where we have used $A_{\mu}^{a'd} = A_{\mu}^{p^a, q^d}$ and $A_{\mu}^{a'b'} = A_{\mu}^{\tilde{i}, \tilde{j}, \beta}$, and defined $A_{\mu}^{pq} \equiv \omega_{kl} A_{\mu}^{p^a, q^d}$ and $A_{\mu}^{a'b'} \equiv \delta_{ij} A_{\mu}^{a', \beta}$.

The two sets of matrices $(\tau_{pq}^{\dagger})_{mn}$ are the vector representations of the $SU(2)_L, R$ generators $M_{pq}$, while $(\tau_{kl})_{ij}$ are the fundamental representation of the set of $Sp(2N)$ generators $M_{kl}$; they are given by

$$
(\tau_{pq}^{\dagger})_{mn} = \frac{1}{2}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} + \epsilon_{mnpq}), \quad (\tau_{kl}^{\dagger})_{ij} = -(\omega_{ik}\omega_{jl} + \omega_{il}\omega_{jk}).
$$

(5.29)
If \( f'_{ab'\tau'} \neq 0 \), the theory becomes two copies of independent GW theories, and the gauge field \( \tilde{A}_{\mu ab} \) becomes

\[
(\tilde{A}_{\mu ab})_{f_{ab'\tau'}} = 0 = k \omega_{ij} A^m_{\mu} (\tau^m_{pq})_{mn} + (\tau^-_{pq})_{mn} + k \delta_{mn} A^k_{\mu} (\tau^k_{kl})_{ij},
\]

(5.30)

where the matrices

\[
(\tau^m_{pq})_{mn} + (\tau^-_{pq})_{mn} = (\tau^-_{pq})_{mn} = \delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np}
\]

(5.31)

furnish the familiar vector representation of \( SO(4) \). The RHS of (5.30) indicates that the Chern-Simons levels of \( SU(2)_R \) and \( SU(2)_L \) gauge groups are equal and \textit{same} in sign. Comparing (5.28) and (5.30), we see that the structure constants \( f_{ab'\tau'} \) play a crucial role in constructing the \( N = 4 \) quiver gauge theory. Similarly, we obtain the expression of the gauge field \( \tilde{A}_{\mu ab'} \) defined in the last equation of (3.2):

\[
\tilde{A}_{\mu ab'} = -\frac{k}{2} [\delta_{ij} (A^\gamma_{\mu} - A^\gamma_{\mu} \sigma_{pq} \gamma^\delta) (\tau_{\gamma \delta})_{\alpha \beta} + \epsilon_{\alpha \beta} A^{k \ell}_{\mu} (\tau_{k \ell})_{ij}],
\]

(5.32)

Here \( A^{k \ell}_{\mu} \equiv \epsilon_{\gamma \delta} A^{k \gamma \ell \delta} \), \( (\tau_{\gamma \delta})_{\alpha \beta} \) and \( (\tau_{k \ell})_{ij} \), the fundamental representations of \( M_{\gamma \delta} \) and \( M_{k \ell} \), have similar expressions as that of \( (\tau_{k \ell})_{ij} \) and \( (\tau^-_{pq})_{mn} \) (see (5.29) and (5.31)), respectively.

Although we set \( SU(2)_R = Sp(2) \) by Eq. (5.19), their representations are completely different: The set of fermionic generators \( Q_{mi} \), furnish the vector representation of the \( SU(2)_R \) generators \( M_{mn} \), while \( Q_{\tau_i} \) furnish the fundamental representation of the \( Sp(2) \) generators \( M_{\alpha \beta} \).

Also, If \( f'_{ab'\tau'} = 0 \), the gauge field \( \tilde{A}_{\mu ab'} \) becomes

\[
(\tilde{A}_{\mu ab'})_{f_{ab'\tau'}} = 0 = -\frac{k}{2} [\delta_{ij} A^\gamma_{\mu} (\tau^-_{\gamma \delta})_{\alpha \beta} + \epsilon_{\alpha \beta} A^{k \ell}_{\mu} (\tau_{k \ell})_{ij}],
\]

(5.33)

which is just a gauge field of the GW theory.

Using the same technique, one can also pair \( G = OSp(4|2N) \) with other superalgebras, such as \( OSp(3|2N_1) \). In summary, one can pair \( G = OSp(4|2N) \) with

\[
G' = OSp(M|2), \quad OSp(3|2N_1), \quad PSU(2|2), \quad G_3, \quad SU(2|N_2), \quad F(4), \quad \text{or} \quad D(2|1, \alpha)
\]

(5.34)

by identifying \( SU(2)_R \) or \( SU(2)_L \) factor of \( G \) with a simple factor \( SU(2) \) (or its isometries \( Sp(2) \) and \( SO(3) \)) contained in the bosonic part of \( G' \).

Finally, one can pair \( G = D(2|1, \alpha) \) with

\[
G' = OSp(3|N_1), \quad OSp(M|2), \quad PSU(2|2), \quad G_3, \quad SU(2|N_2), \quad \text{or} \quad F(4)
\]

(5.35)

by identifying \( SU(2)_R \) or \( SU(2)_L \) factor of \( D(2|1, \alpha) \) with a simple factor \( SU(2) \) (or its isometries \( Sp(2) \) and \( SO(3) \)) of the bosonic part of \( G' \).

The pairs of superalgebras \( G \) and \( G' \) in (5.34) and (5.35) can be used to construct the four 3-brackets (2.9) and (2.12) and the four FIs (2.13). The Lie algebras of the gauge groups are just the bosonic parts of \( G \) and \( G' \), and corresponding representations are determined by the fermionic generators of \( G \) and \( G' \).
We note that the bosonic part of $PSU(2|2)$ is $SO(4)$, we therefore can identify it with the $SO(4)$ factor of the bosonic part of $OSp(4|2N)$ or $D(2|1,\alpha)$. The special case of $PSU(2|2)$ paring $OSp(4|2N)$ is interesting in its own right: First, since the Nambu 3-algebra can be constructed in terms of $PSU(2|2)$ [5], so the Nambu 3-algebra can be a sub 3-algebra of a symplectic 3-algebra; secondly, since the set of $SO(4)$ generators of $OSp(4|2N)$ are given by $(M_{mn}^+ + M_{mn}^-) = M_{mn}$ (see (5.21)), while the set of $SO(4)$ generators of $PSU(2|2)$ can be converted into $(M_{mn}^+ - M_{mn}^-) = \bar{M}_{mn}$ [3], so in analogue with (5.19), the two sets of $SO(4)$ generators can be related to each other via the following duality equation:

$$(M_{mn})_{OSp(4|2N)} = \frac{1}{2} \varepsilon_{mnpq}(\bar{M}_{pq})_{PSU(2|2)}. \quad (5.36)$$

Notice also that we can identify the $SO(4)$ factor of the bosonic part of $OSp(4|2N)$ with the $SO(4)$ factor of the bosonic part of $D(2|1,\alpha)$, and the resulting gauge group is different from the one derived by letting that $OSp(4|2N)$ and $D(2|1,\alpha)$ share only one $SU(2)$ factor. The duality equation for identifying two $SO(4)$ factors can be given by

$$\frac{1}{2}(\delta_{mp}\delta_{nq} - \delta_{np}\delta_{mq} + \beta \varepsilon_{mnpq})(M_{pq})_{OSp(4|2N)} = (\bar{M}_{mn})_{D(2|1,\alpha)} \quad (5.37)$$

where $\beta = 2(1 - \alpha)/(1 + \alpha)$. Similarly, the duality equation for identifying the $SO(4)$ factor of $D(2|1,\alpha)$ with the $SO(4)$ factor of $PSU(2|2)$ can be given by

$$\frac{1}{2}[\beta(\delta_{mp}\delta_{nq} - \delta_{np}\delta_{mq}) + \varepsilon_{mnpq}](\bar{M}_{pq})_{PSU(2|2)} = (\bar{M}_{mn})_{D(2|1,\alpha)}. \quad (5.38)$$

We summarize the three pairs as follows

$$(G, G') = (PSU(2|2), OSp(4|2N)), (PSU(2|2), D(2|1,\alpha)), (OSp(4|2N), D(2|1,\alpha)). \quad (5.39)$$

Every pair of superalgebras in the right hand side, whose bosonic parts share the common factor $SO(4)$, can be used to construct the four 3-brackets (2.9) and (2.12) and the four FIs (2.13).

**5.3 $U(N_1) \times U(N_2) \times U(N_3) \times U(N_4)$**

Here we choose the superalgebras $G$ and $G'$ (see (4.3) and (4.4)) as $U(N_1|N_2)$ and $U(N_3|N_4)$ respectively (the commutation relations of $U(M|N)$ are given by Appendix 3.1). We identify the $U(1)$ parts of the bosonic subalgebras of $U(N_1|N_2)$ and $U(N_3|N_4)$ by introducing the following commutators between the generators of $U(N_1|N_2)$ and the generators of $U(N_3|N_4)$:

$$[M_{\hat{u}^i}, Q_{\hat{j}}^{\hat{i}}] = c_1 \delta_{\hat{u}^i}^{\hat{u}^\hat{i}} Q_{\hat{i}}^{\hat{i}}, \quad [M_{\hat{u}^i}, \bar{Q}_{\hat{j}}^{\hat{i}}] = -c_1 \delta_{\hat{u}^i}^{\hat{u}^\hat{i}} \bar{Q}_{\hat{i}}^{\hat{i}},$$

$$[M_{\hat{u}'}^{\hat{j}}, Q_{\hat{i}}^{\hat{i}}] = c_2 \delta_{\hat{u}'}^{\hat{u}' j} Q_{\hat{i}}^{\hat{i}}, \quad [M_{\hat{u}'}^{\hat{j}}, \bar{Q}_{\hat{i}}^{\hat{i}}] = -c_2 \delta_{\hat{u}'}^{\hat{u}' j} \bar{Q}_{\hat{i}}^{\hat{i}},$$

$$[M_{\hat{u}'}^{\hat{j}}, \bar{Q}_{\hat{u}'}^{\hat{i}}] = c_3 \delta_{\hat{u}'}^{\hat{u}' i} \bar{Q}_{\hat{u}'}^{\hat{i}}, \quad [M_{\hat{u}'}^{\hat{j}}, \bar{Q}_{\hat{u}'}^{\hat{i}}] = -c_3 \delta_{\hat{u}'}^{\hat{u}' i} \bar{Q}_{\hat{u}'}^{\hat{i}},$$

$$[M_{\hat{u}'}^{\hat{j}}, Q_{\hat{u}'}^{\hat{i}}] = c_4 \delta_{\hat{u}'}^{\hat{u}' i} Q_{\hat{u}'}^{\hat{i}}, \quad [M_{\hat{u}'}^{\hat{j}}, \bar{Q}_{\hat{u}'}^{\hat{i}}] = -c_4 \delta_{\hat{u}'}^{\hat{u}' i} \bar{Q}_{\hat{u}'}^{\hat{i}}, \quad (5.40)$$
where $Q_{i}^{\hat{i}}$ are the fermionic generators of $U(N_{1}|N_{2})$, with $\hat{i} = 1, \ldots, N_{1}$ fundamental indices of $U(N_{1})$ and $i = 1, \ldots, N_{2}$ anti-fundamental indices of $U(N_{2})$; $M_{\hat{i}}^{\hat{j}}$ and $M_{i}^{j}$ are the bosonic generators of $U(N_{1}|N_{2})$; $Q_{u}^{v}$ are the fermionic generators of $U(N_{3}|N_{4})$, with $u = 1, \ldots, N_{3}$ fundamental indices of $U(N_{3})$ and $v = 1, \ldots, N_{4}$ anti-fundamental indices of $U(N_{4})$; $M_{u}^{\hat{v}}$ and $M_{v}^{\hat{u}}$ are the bosonic generators of $U(N_{3}|N_{4})$; $c_{i} (i = 1, \ldots, 4)$ are arbitrary constants.

Let us now examine the physical meaning of the first commutator of (5.40). Writing the set of $U(N_{3})$ generators $M_{u}^{\hat{v}}$ as

$$
(M_{u}^{\hat{v}})_{U(N_{3})} = (M_{u}^{\hat{v}} - \frac{1}{N_{3}} \delta_{\hat{u}}^{\hat{v}} M_{\hat{u}}^{w} w)_{SU(N_{3})} + (\frac{1}{N_{3}} \delta_{\hat{u}}^{\hat{v}} M_{\hat{u}}^{w} w)_{U(1)},
$$

and using the first commutator of (5.40), we find that $Q_{i}^{\hat{i}}$ commutes with the set of $SU(N_{3})$ generators, while has a nontrivial commutator with the $U(1)$ generator of $U(N_{3})$, i.e.

$$
[M_{u}^{\hat{v}} - \frac{1}{N_{3}} \delta_{\hat{u}}^{\hat{v}} M_{\hat{u}}^{w} w, Q_{i}^{\hat{j}}] = 0, \quad [\frac{1}{N_{3}} \delta_{\hat{u}}^{\hat{v}} M_{\hat{u}}^{w} w, Q_{j}^{\hat{i}}] = c_{1} \delta_{\hat{u}}^{\hat{v}} Q_{i}^{\hat{j}}.
$$

So the first commutator of (5.40) means that the set of fermionic generators of $U(N_{1}|N_{2})$ $Q_{i}^{\hat{i}}$ are charged by the $U(1)$ part of $U(N_{3})$ of the bosonic part of $U(N_{3}|N_{4})$; also, it means that $Q_{i}^{\hat{i}}$ furnish a nontrivial representation of the $U(1)$ part of $U(N_{3})$ of the bosonic part of $U(N_{3}|N_{4})$. The other commutators in (5.40) have a similar interpretation.

On the other hand, let us consider the following commutator of $U(N_{1}|N_{2})$: $[M_{j}^{\hat{k}}, Q_{i}^{\hat{j}}] = \delta_{i}^{j} k Q_{j}^{\hat{i}}$. Contracting on $j$ and $\hat{k}$ gives

$$
[M_{j}^{\hat{k}}, Q_{i}^{\hat{j}}] = \frac{1}{N_{1}} Q_{j}^{\hat{i}}.
$$

Namely, $Q_{i}^{\hat{i}}$ are also charged by the $U(1)$ part of $U(N_{1})$ of the bosonic part of $U(N_{1}|N_{2})$, or $Q_{i}^{\hat{i}}$ furnish a nontrivial representation of the $U(1)$ part of $U(N_{1})$ of the bosonic part of $U(N_{1}|N_{2})$. Similarly, $Q_{i}^{\hat{i}}$ furnish a nontrivial representation of the $U(1)$ part of $U(N_{2})$ of the bosonic part of $U(N_{1}|N_{2})$.

So, after identifying the $U(1)$ parts of the bosonic subalgebras of $U(N_{1}|N_{2})$ and $U(N_{3}|N_{4})$, both $Q_{i}^{\hat{i}}$ and $Q_{u}^{v}$ have nontrivial commutators with all $U(1)$ generators of $U(N_{1}|N_{2})$ and $U(N_{3}|N_{4})$.

We are now ready to calculate the structure constants of the double graded commutators. We define

$$
Q_{a} = \begin{pmatrix} Q_{i}^{\hat{i}} \\ -Q_{i}^{\hat{i}} \end{pmatrix} = \bar{Q}_{i}^{\hat{i}} \delta_{1\lambda} - Q_{i}^{\hat{i}} \delta_{2\lambda},
$$

$$
Q_{a'} = \begin{pmatrix} \bar{Q}_{v}^{\hat{u}} \\ -Q_{u}^{\hat{v}} \end{pmatrix} = \bar{Q}_{v}^{\hat{u}} \delta_{1\alpha} - Q_{u}^{\hat{v}} \delta_{2\alpha},
$$

where $\delta_{1\lambda} = (1, 0)^{T}$ and $\delta_{2\lambda} = (0, 1)^{T}$ are “spin up” spinor (not a spacetime spinor) and “spin down” spinor, respectively. Similarly, $\delta_{1\alpha}$ and $\delta_{2\alpha}$ are another independent pair of spinors. Using $\{Q_{a}, Q_{b}, Q_{a'}\} = f_{aba'}^{\hat{b}} Q_{\hat{b}}$, and the commutation relations of $U(N_{1}|N_{2})$ and (5.40), we obtain

$$
f_{aba'b'} = -k(c_{1} + c_{2}) (\delta_{i}^{\hat{j}} \delta_{j}^{\hat{i}} \delta_{2\lambda} \delta_{1\xi} + \delta_{i}^{\hat{j}} \delta_{j}^{\hat{i}} \delta_{1\lambda} \delta_{2\xi}) (\delta_{v}^{\hat{u}} \delta_{\hat{u}}^{\hat{v}} \delta_{1\alpha} \delta_{2\beta} + \delta_{v}^{\hat{u}} \delta_{\hat{u}}^{\hat{v}} \delta_{1\alpha} \delta_{2\beta}).
$$
Using \([\{Q_a', Q_b'\}, Q_a] = f_{a'b'a'} b Q_b\) to calculate \(f_{a'b'a'b}\) gives the same result as that of \((5.46)\) except that \((c_1 + c_2)\) gets replaced by \((c_3 + c_4)\). In order for that \(f_{a'b'a'b} = f_{a'b'a'b}\), we must set

\[
(c_1 + c_2) = (c_3 + c_4).
\]

(5.47)

It can be seen that this \(\mathcal{N} = 4\) theory contains three free parameters. The second FI and third FI of \((2.15)\) do not impose any further constraint on \((c_1 + c_2)\) or \((c_3 + c_4)\), since for example the summation of the first two terms of the second FI of \((2.15)\) vanishes due to the abelian nature of \(U(1)\). Using \([\{Q_a', Q_b'\}, Q_a'] = f_{a'b'a'd} Q_d\) and the commutation relations of \(U(N_3|N_4)\), one can calculate the structure constants \(f_{a'b'a'd}\) straightforwardly; they are given by

\[
f_{a'b'a'd} = f_{\bar{u} \bar{v} \bar{w}}^{\bar{u}' \bar{v}' \bar{w}'}, k' \bar{u}' \bar{v}' \bar{w}' \bar{y} \delta_{2\alpha_1} \delta_{1\beta_2} \delta_{1\gamma} + f_{\bar{u} \bar{v} \bar{w}}^{\bar{u}' \bar{v}' \bar{w}'}, k' \bar{u}' \bar{v}' \bar{w}' \bar{y} \delta_{2\alpha_1} \delta_{1\beta_2} \delta_{1\gamma} \delta_{2\delta} + f_{\bar{u} \bar{v} \bar{w}}^{\bar{u}' \bar{v}' \bar{w}'}, k' \bar{u}' \bar{v}' \bar{w}' \bar{y} \delta_{1\alpha} \delta_{2\beta} \delta_{1\gamma} \delta_{2\delta}
\]

(5.48)

where

\[
f_{\bar{u} \bar{v} \bar{w}}^{\bar{u}' \bar{v}' \bar{w}'}, k' \bar{u}' \bar{v}' \bar{w}' \bar{y} = k(\delta_{i'y'i'} \delta_{j'y'j'} \delta_{k'y'k'} \delta_{l'y'l'})).
\]

(5.49)

Notice that \((5.49)\) are precisely the structure constants introduced first by Lambert and Bagger \(8\) to construct an \(\mathcal{N} = 6\) theory with \(U(N_3) \times U(N_4)\) gauge group. The relation between \((5.49)\) and \((5.48)\) was first derived in Ref. \(11\). Since both \(U(N_1|N_2)\) and \(U(N_3|N_4)\) are unitary superalgebras, the structure constants \(f_{abcd}\) have a similar expression as that of \((5.48)\); they are given by

\[
f_{\bar{a}\bar{b}\bar{c}\bar{d}} = f_{\bar{a}\bar{b}\bar{c}\bar{d}}^{\bar{a}'\bar{a}'\bar{b}'\bar{b}'}, k' \bar{a}' \bar{b}' \bar{c}' \bar{d}' \delta_{2\alpha_1} \delta_{2\beta_2} \delta_{2\gamma} \delta_{2\delta} + f_{\bar{a}\bar{b}\bar{c}\bar{d}}^{\bar{a}'\bar{a}'\bar{b}'\bar{b}'}, k' \bar{a}' \bar{b}' \bar{c}' \bar{d}' \delta_{2\alpha_1} \delta_{2\beta_2} \delta_{2\gamma} \delta_{2\delta} \delta_{1\delta_2} + f_{\bar{a}\bar{b}\bar{c}\bar{d}}^{\bar{a}'\bar{a}'\bar{b}'\bar{b}'}, k' \bar{a}' \bar{b}' \bar{c}' \bar{d}' \delta_{1\alpha} \delta_{2\beta} \delta_{2\gamma} \delta_{1\delta_2},
\]

(5.50)

where

\[
f_{\bar{a}\bar{b}\bar{c}\bar{d}}^{\bar{a}'\bar{a}'\bar{b}'\bar{b}'}, k' \bar{a}' \bar{b}' \bar{c}' \bar{d}' = k(\delta_{i'y'i'} \delta_{j'y'j'} \delta_{k'y'k'} \delta_{l'y'l'}).\]

(5.51)

We have verified that the structure constants \((5.46), (5.48)\) and \((5.50)\) obey the fundamental identities \((2.15)\), and satisfy the desired symmetry and reality conditions. Substituting them into \((3.2), (4.4)\) and \((3.3)\) gives the \(\mathcal{N} = 4, U(N_1) \times U(N_2) \times U(N_3) \times U(N_4)\) theory. In this realization, the un-twisted multiplets are in the bifundamental representation of \(U(N_1) \times U(N_2)\), while the twisted multiplets are in the bifundamental representation of \(U(N_3) \times U(N_4)\). However, the un-twisted multiplets couple the twisted multiplets nontrivially via the structure constants \(f_{a'b'a'b}\). In the special case of \(c_1 + c_2 = 0\), the structure constants \(f_{a'b'a'b}\) vanish identically. As a result, the action \((5.4)\) becomes two uncoupled GW theories.

\section{5.4 OSp(N2) \times U(1) \times U(N3) \times U(N4)}

In this subsection we choose the superalgebras \(G\) and \(G'\) (see \((1.3)\) and \((4.4)\)) as \(OSp(2|2N)\) and \(U(N_3|N_4)\) respectively. (The commutation relations of \(OSp(M|2N)\) and \(U(M|N)\) are
given by Appendix 3.2 and 3.1 respectively.) In analogy to Section 5.3, we identify the $U(1)$ parts of the bosonic subalgebras of $OSp(2|2N_2)$ and $U(N_3|N_4)$ by introducing the following commutators between their generators:

$$[M^a_{\bar{u}}, Q_{i\bar{i}}] = c_1 \delta^a_{\bar{u}} \epsilon_{ij} Q_{j\bar{i}}, \quad [M^i_{\bar{u}}, Q_{i\bar{i}}] = c_2 \delta^i_{\bar{u}} \epsilon_{ij} Q_{j\bar{i}},$$

$$[M_{\bar{i}j}, Q^i_{i'}] = -c_3 \epsilon_{ij} Q^i_{i'}, \quad [M_{\bar{i}j}, \bar{Q}^i_{i'}] = c_3 \epsilon_{ij} \bar{Q}^i_{i'},$$

(5.52)

where $Q_{i\bar{i}}$ are the set of fermionic generators of $OSp(2|2N_2)$, with $\bar{i} = 1, 2$ fundamental indices of $SO(2)$ and $i = 1, \ldots, N_2$ fundamental indices of $Sp(2N_2)$; $M_{\bar{i}j}$ and $M_{ij}$ are the bosonic generators of $OSp(2|2N_2)$; $Q^i_{i'}$ are the set of fermionic generators of $U(N_3|N_4)$, with $i = 1, \ldots, N_3$ fundamental indices of $U(N_3)$ and $i' = 1, \ldots, N_4$ anti-fundamental indices of $U(N_4)$; $M_{\bar{i}j}$ and $M_{ij}$ are the bosonic generators of $U(N_3|N_4)$; $c_i$ ($i = 1, \ldots, 3$) are arbitrary constants; $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{12} = 1$. The commutators in (5.52) have similar interpretations as that of section 5.3.

To calculate the structure constants $f_{ab'c'd'}$, we define

$$Q_a = Q_{i\bar{i}}, \quad Q_{a'} = \left( \begin{array}{c} Q^i_{i'} \bar{Q}^i_{i'} \\ \end{array} \right) = \bar{Q}^i_{i'} \delta_{1\alpha} - Q^i_{i'} \delta_{2\alpha},$$

(5.53)

where $\delta_{1\alpha} = (1, 0)^T$ and $\delta_{2\alpha} = (0, 1)^T$ are “spin up” spinor (not a spacetime spinor) and “spin down” spinor, respectively. Using $[\{Q_{a'}, Q_{b'}\}, Q_a] = f_{ab'c'}b'Q_b$, and the commutation relations of $U(N_2|N_3)$ and (5.52), we obtain

$$f_{ab'c'd'} = k(c_1 + c_2)(\delta^a_{\bar{u}} \delta^i_{\bar{v}} \delta^j_{1\alpha} \delta^j_{2\beta} + \delta^i_{\bar{u}} \delta^j_{1\alpha} \delta^i_{2\beta} \delta_{1\beta}) \epsilon_{ij} \omega_{ij}.$$  

(5.54)

One can also use $[\{Q_a, Q_b\}, Q_{a'}] = f_{ab'c'}Q_{a'}$ to calculate $f_{ab'c'}$. The final expression is of course the same as that of (5.54) after we set

$$c_1 + c_2 = c_3.$$  

(5.55)

So this $\mathcal{N} = 4$ theory contains two free parameters. By the same reason as that of Section 5.3, the second FI and third FI of (2.13) impose no constraint on $(c_1 + c_2)$ or $c_3$. The structure constants $f_{abcd}$ and $f_{ab'c'd'}$ are the same as (5.3) and (5.48) respectively, except for that here the index $a = 1, \ldots, N_2$. We have also verified that all the structure constants obey fundamental identities (2.18) and satisfy the desired reality and symmetry conditions. Notice that section 7.1 is a special case of this subsection. Actually, if we set $N_4 = 1, N_2 = N$ and $c_3 = 1$, the structure constants $f_{abcd}$, $f_{ab'c'd'}$ and $f_{ab'c'd'}$ are exactly the same as (5.9), (5.7) and (5.10), respectively.

Substituting the structure constants of this subsection $f_{abcd}$, $f_{ab'c'd'}$ and $f_{ab'c'd'}$ into (3.3), (3.4) and (3.5) gives the $\mathcal{N} = 4$, $Sp(2N_2) \times U(1) \times U(N_3) \times U(N_4)$ theory. The un-twisted multiplets are in the bifundamental representation of $Sp(2N_2) \times U(1)$, while the twisted multiplets are in the bifundamental representation of $U(N_3) \times U(N_4)$. 

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6. General $\mathcal{N} = 4$ Theories in Terms of Lie Algebras

In this section, we will derive the general $\mathcal{N} = 4$ theories in terms of Lie algebras from their 3-algebra counterparts. The key point is that we observe that the structure constants of 3-algebras can be expressed in terms of tensor products on the superalgebras $G$ and $G'$. Eqs. (4.26) are two examples.

Recall that the bosonic parts of the two superalgebras (4.3) and (4.4) share at least one simple factor or $U(1)$ factor, and we have decomposed their bosonic generators $M^u$ and $M^{u'}$ into $M^u = (M^\alpha, M^9)^8$ and $M^{u'} = (M^{\alpha'}, \tilde{M}^9)$, respectively. Here $\tilde{M}^9 = T^g_h M^h$ are the generators of the common part, with $T^g_h$ a set of complex non-singular linear transformation matrices. If $T^g_h$ are real and positive definite, the two Lie algebras spanned by $M^9$ and $\tilde{M}^9$ are equivalent. In particular, if $T^g_h = \delta^g_h$, we have $\tilde{M}^9 = M^9$. The independent commutation relations of the bosonic parts of the two superalgebras (4.3) and (4.4) are the following:

$$[M^\alpha, M^\beta] = f^{\alpha\beta\gamma} M^\gamma, \quad [M^f, M^9] = f^{fg} h M^h, \quad [M^{\alpha'}, M^{\beta'}] = f^{\alpha'\beta'\gamma'} M^{\gamma'}. \quad (6.1)$$

Of course, we also have

$$[	ilde{M}^f, \tilde{M}^9] = \tilde{f}^{fg} h \tilde{M}^h. \quad (6.2)$$

However, since this equation can be obtained by transforming the second equation of (6.1) by using $\tilde{M}^9 = T^g_h M^h$, we do not consider it as an independent equation. Accordingly, we decompose the structure constants into

$$f^{uvw} = (f^{\alpha\beta\gamma}, f^{fg} h), \quad f^{w'v'u'} = (f^{\alpha'\beta'\gamma'}, \tilde{f}^{fg} h). \quad (6.3)$$

Now the superalgebra (4.3) is decomposed into

$$[M^\alpha, M^\beta] = f^{\alpha\beta\gamma} M^\gamma, \quad [M^f, M^9] = f^{fg} h M^h,$n
$$[M^\alpha, Q_a] = -\tau^a_{ab} \omega^{bc} Q_c, \quad [M^9, Q_a] = -\tau^a_{ab} \omega^{bc} M^h,$n
$$\{Q_a, Q_b\} = \tau^a_{ab} k^b M^\beta + \tau^a_{ab} k^h M^h. \quad (6.4)$$

Similarly, the superalgebra (4.4) can be written as

$$[M^{\alpha'}, M^{\beta'}] = f^{\alpha'\beta'\gamma'} M^{\gamma'}, \quad [\tilde{M}^f, \tilde{M}^9] = \tilde{f}^{fg} h \tilde{M}^h,$n
$$[M^{\alpha'}, Q_{a'}] = -\tau^{a'}_{a'b'} \omega^{b'c'} Q_{c'}, \quad [M^9, Q_{a'}] = -\tau^{a'}_{a'b'} \omega^{b'c'} Q_{c'},$$n
$$\{Q_{a'}, Q_{b'}\} = \tau^{a'}_{a'b'} k^b M^{\beta'} + \tau^{a'}_{a'b'} k^h M^h. \quad (6.5)$$

Using the non-singular transformation $\tilde{M}^9 = T^g_h M^h$, we are able to recast (6.3) in the form

$$[M^{\alpha'}, M^{\beta'}] = f^{\alpha'\beta'\gamma'} M^{\gamma'}, \quad [\tilde{M}^f, \tilde{M}^9] = \tilde{f}^{fg} h \tilde{M}^h,$n
$$[M^{\alpha'}, Q_{a'}] = -\tau^{a'}_{a'b'} \omega^{b'c'} Q_{c'}, \quad [M^9, Q_{a'}] = -\tau^{a'}_{a'b'} \omega^{b'c'} Q_{c'},$$n
$$\{Q_{a'}, Q_{b'}\} = \tau^{a'}_{a'b'} k^b M^{\beta'} + \tau^{a'}_{a'b'} k^h M^h. \quad (6.6)$$

*Here $\alpha$ is *not* a spacetime index. We hope this will not cause any confusion.*
where
\[
\tau^{ab}_{\alpha \beta} = (T^{-1})^{a}_{b} \tilde{h}_{\tau}^{\gamma} h \hat{f}_{\alpha \beta}, \quad k_{gh} = T_{g}^{f} T_{i}^{k} \tilde{h}_{f i}, \quad f f_{g} h = (T^{-1})^{f}_{i} (T^{-1})^{g}_{j} T^{k}_{h} \tilde{h}_{ij} k. \quad (6.7)
\]

Since \( T_{gh} \) are generally complex nonsingular matrices, the Lie algebra defined by the second equation of (6.6) is generally not equivalent to the one defined by the second equation of (6.3). However, the \( N = 4 \) theories will not be modified if we use (6.6) to construct the 3-brackets. In fact, since we have not transformed the set of fermionic generators \( Q_{a'} \) in (6.3), the double graded brackets \([\{Q_{a'}, Q_{b'}\}, Q_{c'}]\) and \([\{Q_{a}, Q_{b}\}, Q_{a'}]\) will be the same no matter we use (6.3) or (6.6) to construct them. As a result, the structure constants \( f_{a'b'c'd'} \) and \( f_{a'b'ab} \) used to construct the \( N = 4 \) theories will also remain the same. For example, if we use (6.3) and (6.6) to calculate \([\{Q_{a'}, Q_{b'}\}, Q_{c'}] = f_{a'b'c'd'} Q_{d'}\) respectively, we obtain
\[
f_{a'b'c'd'} = \tau^{a'}_{a'b'} k_{\alpha' \beta'} \tau^{b'}_{c'd'} + \tilde{\tau}^{g}_{a'b} \tilde{h}_{g} \tilde{\tau}^{g}_{d'} \quad \text{and} \quad f_{a'b'c'd'} = \tau^{a'}_{a'b'} k_{\alpha' \beta'} \tau^{b'}_{c'd'} + \tilde{\tau}^{g}_{a'b} k_{gh} \tilde{\tau}^{g}_{d'} \quad (6.8)
\]
respectively. But it is not difficult to prove that \( \tilde{\tau}^{g}_{a'b} \tilde{h}_{g} \tilde{\tau}^{g}_{d'} = \tau^{g}_{a'b} k_{gh} \tau^{g}_{d'} \) by using the first two equations of (6.7). So they are indeed the same.

To simplify the expressions of (6.4) and (6.6), we define
\[
M^{m} = (M^{a}, M^{g}, M^{a'}), \quad (6.9)
\]
\[
C^{mn}_{p} = (f^{a \beta' \gamma}, f f_{g} h, f^{a' \beta' \gamma'}), \quad (6.10)
\]
\[
k_{mn} = (k_{\alpha \beta}, k_{gh}, k_{\alpha' \beta'}). \quad (6.11)
\]

We now put the superalgebras (6.4) and (6.6) together:
\[
[M^{m}, M^{n}] = C^{mn}_{p} M^{p}, \quad [M^{m}, Q_{a}] = -\tau^{m}_{a b} \omega^{b c} Q_{c}, \quad [M^{m}, Q_{a'}] = -\tau^{m}_{a b} \omega^{b' c'} Q_{c'}, \quad (6.12)
\]
\[
\{Q_{a}, Q_{b}\} = \tau^{m}_{ab} k_{mn} M^{n}, \quad \{Q_{a}, Q_{b'}\} = \tau^{m}_{a b'} k_{mn} M^{n},
\]
where we have used the equations \( \tau^{a'}_{a b'} = \tau^{a}_{a b} = 0 \) implied by (4.19).

Notice that (6.12) is merely a compact version of (6.4) and (6.6); in particular, it is not necessarily a closed superalgebra due to the set of common generators \( M^{g} \). In fact, because of \( M^{g} \), the double graded commutator \([\{Q_{a}, Q_{b}\}, Q_{c'}] \neq 0\), i.e. \( f_{abc'd'} \neq 0 \) (see the proof in the paragraph containing (4.15)). On the other hand, if (6.12) is a closed superalgebra, then the \( Q_{a} Q_{b} Q_{c'} \) Jacobi identity must be obeyed (see (4.38)), implying that \( \{Q_{a}, Q_{c'}\} \neq 0 \). Therefore the anticommutator (4.38)
\[
\{Q_{a}, Q_{c'}\} = \tilde{\tau}_{a c'}^{\hat{\gamma}} M_{\hat{\gamma}}
\]
must be nontrivial, in the sense that \( \tilde{\tau}_{a c'}^{\hat{\gamma}} \neq 0 \). However, the set of bosonic generators \( M_{\hat{\gamma}} \) are not included in \( M^{m} \) (see (6.9)), and (4.38) is not contained in (6.12). So generally speaking, (6.12) is not a closed superalgebra. But if we introduce \( M_{\hat{\gamma}}, (4.38) \) and some other proper commutation relations into (6.12), it is possible to “fuse” the two superalgebras (6.4) and (6.6) into a single closed superalgebra [13].

\( ^{9}\)Here \( m, n \) and \( p \) are not \( SO(4) \) fundamental indices of Section 5.2. We hope this will not cause any confusion.
With these notations, if we construct the 3-brackets (4.8) and (4.21) by using (6.12), the structure constants (see (4.26) and (4.25)) for take the forms

\[ f_{abcd} = k_{mn} \gamma_{a\beta}^{m} \gamma_{c\gamma}^{n}, \quad f_{a'b'c'd'} = k_{mn} \gamma_{a'b'}^{m} \gamma_{c'd'}^{n}, \quad f_{abc'd'} = f_{c'd'ab} = k_{mn} \gamma_{ab}^{m} \gamma_{c'd'}^{n}. \]  

With the above equations, the four sets of FIs (2.13) can be converted into

\[
\begin{align*}
(k_{np} k_{qm} C_{pnm}^{s} + k_{qm} k_{sp} C_{pnm}^{n}) \tau_{na}^{\alpha} \gamma_{d}^{\tau} \tau_{se}^{f} &= 0, \\
(k_{np} k_{qm} C_{pnm}^{s} + k_{qm} k_{sp} C_{pnm}^{n}) \tau_{na}^{\alpha} \gamma_{d}^{\tau} \tau_{se}^{f} &= 0, \\
(k_{np} k_{qm} C_{pnm}^{s} + k_{qm} k_{sp} C_{pnm}^{n}) \tau_{na}^{\alpha} \gamma_{d}^{\tau} \tau_{se}^{f} &= 0, \\
(k_{np} k_{qm} C_{pnm}^{s} + k_{qm} k_{sp} C_{pnm}^{n}) \tau_{na}^{\alpha} \gamma_{d}^{\tau} \tau_{se}^{f} &= 0.
\end{align*}
\]

They are simply obeyed due to the fact that

\[
k_{np} k_{qm} C_{pnm}^{s} + k_{qm} k_{sp} C_{pnm}^{n} = 0.
\]

With (6.13), the gauge fields (see (3.2)) of the \( N = 4 \) theories become

\[
\begin{align*}
\tilde{A}_{\mu}^{c} &= A_{\mu}^{a} f_{ab} \gamma^{d} + A_{\mu}^{a'b'} f_{a'b'} \gamma^{c} = (A_{\mu}^{ab} + A_{\mu}^{a'b'} \gamma_{d}^{\tau}) k_{mn} \tau_{mc}^{n} \gamma_{d}^{\tau} = A_{\mu}^{m} k_{mn} \tau_{mc}^{n} \gamma_{d}^{\tau}, \\
\tilde{A}_{\mu}^{c'} &= A_{\mu}^{a} f_{ab} \gamma^{d'} + A_{\mu}^{a'b'} f_{a'b'} \gamma^{c'} = (A_{\mu}^{ab} + A_{\mu}^{a'b'} \gamma_{d'}^{\tau}) k_{mn} \tau_{mc'}^{n} \gamma_{d'}^{\tau} = A_{\mu}^{m} k_{mn} \tau_{mc'}^{n} \gamma_{d'}^{\tau}.
\end{align*}
\]

Following Ref. [3], we define the ‘momentum map’ and ‘current’ operators as follows

\[
\begin{align*}
\mu_{AB}^{m} &= \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b}, \\
\mu_{AB}^{m} &= \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}, \\
\tau_{AB}^{m} &= \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}, \quad \tau_{AB}^{m} = \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}, \quad \tau_{AB}^{m} = \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}, \quad \tau_{AB}^{m} = \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}. \quad \tau_{AB}^{m} = \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}. \quad \tau_{AB}^{m} = \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}. \quad \tau_{AB}^{m} = \tau_{ab}^{m} Z_{A}^{a} Z_{B}^{b'}.
\end{align*}
\]

Substituting (6.13) and (6.16) into the \( N = 4 \) Lagrangian (3.4) gives

\[
\begin{align*}
\mathcal{L} &= \frac{1}{2} \epsilon^{\mu\nu\lambda}(k_{mn} A_{\mu}^{m} \partial_{\nu} A_{\lambda}^{n} + \frac{1}{3} \tilde{C}_{mnp} A_{\mu}^{m} A_{\nu}^{n} A_{\lambda}^{p}) \\
&\quad + \frac{1}{2} (-D_{\mu} \tilde{Z}_{a}^{A} D_{\mu} Z_{a}^{A} - D_{\mu} \tilde{Z}_{a}^{A} D_{\mu} Z_{a}^{A} + i \tilde{\psi}_{A}^{a} \gamma_{A} \gamma_{A} D_{\mu} \psi_{A}^{a} + i \tilde{\psi}_{A}^{a} \gamma_{A} \gamma_{A} D_{\mu} \psi_{A}^{a}) \\
&\quad + \frac{i}{2} k_{mn} (j_{AB}^{m} \tau_{AB}^{n} - j_{AB}^{m} \tau_{AB}^{n}) - \frac{1}{4} \tilde{C}_{mnp} (\mu_{AB}^{m} \tau_{AB}^{n} C_{\mu}^{pC} A_{\mu}^{C} + \mu_{AB}^{m} \tau_{AB}^{n} C_{\mu}^{pC} A_{\mu}^{C}) \\
&\quad + \frac{1}{4} k_{mn} k_{np} ((\tau_{AM}^{m} \tau_{AM}^{n} b Z_{A}^{b} Z_{A}^{b} + (\tau_{AM}^{m} \tau_{AM}^{n} a') \psi_{A}^{b} \psi_{A}^{b'} + (\tau_{AM}^{m} \tau_{AM}^{n} a') \psi_{A}^{b} \psi_{A}^{b'}) \tilde{C}_{mnp} (\mu_{AB}^{m} \tau_{AB}^{n} C_{\mu}^{pC} A_{\mu}^{C} + \mu_{AB}^{m} \tau_{AB}^{n} C_{\mu}^{pC} A_{\mu}^{C}) \psi_{A}^{b} \psi_{A}^{b'},
\end{align*}
\]

where \( \tilde{C}_{mnp} = k_{mab} k_{nbc} C_{sp}^{aq}. \) Substituting (6.13) and (6.16) into the \( N = 4 \) supersymmetry transformations (3.3) gives

\[
\begin{align*}
\delta Z_{A} &= i \epsilon_{A}^{ab} \tilde{Z}_{a}^{A}, \\
\delta Z_{A}^{a'} &= i \epsilon_{a'}^{A} \tilde{Z}_{a}^{A}, \\
\delta \psi_{A}^{a} &= -\gamma^{a} D_{\mu} Z_{B}^{a} \epsilon_{A}^{B} - \frac{1}{3} k_{mn} \tau_{ma}^{a'} \psi_{A}^{B} \psi_{A}^{B}, \\
\delta \psi_{A}^{a'} &= -\gamma^{a'} D_{\mu} Z_{B}^{a} \epsilon_{A}^{B} - \frac{1}{3} k_{mn} \tau_{ma}^{a'} \psi_{A}^{B} \psi_{A}^{B}, \\
\delta A_{\mu}^{m} &= i \epsilon_{A}^{AB} \gamma_{\mu} J_{AB}^{m} + i \epsilon_{A}^{AB} \gamma_{\mu} J_{AB}^{m}.
\end{align*}
\]
Here the parameter $\epsilon_A \dot{\beta}$ obeys the reality condition (3.4). The $\mathcal{N} = 4$ Lagrangian (3.18) and supersymmetry transformation law (3.19) are in agreement with those constructed directly in terms of ordinary Lie algebra $\mathfrak{g}$.

If both the twisted and untwisted multiplets take values in the same symplectic 3-algebra spanned by $T_a$, i.e. $\Phi_A = \Phi_A^a$ and $\Phi \dot{A} = \Phi \dot{A}^a$, we need only one superalgebra $G$ to construct the 3-algebra spanned by $T_a$. It follows that both $\Phi_A$ and $\Phi \dot{A}$ are in the same representation of the bosonic subalgebra of $G$; this representation is furnished by the set of fermionic generators $Q_a$. In this case, the $\mathcal{N} = 4$ supersymmetry is promoted to $\mathcal{N} = 5$, as first proved in Ref. [4].

7. Classification of $\mathcal{N} = 4$ Quiver Gauge Theories

After working out the example in section 6, it is not difficult to find out the other gauge groups. We first review all known examples of $\mathcal{N} = 4$ theories. We consider the following pairs of superalgebras [3, 10]:

$$(G, G') = (U(N_1|N_2), (U(N_2|N_3), (OSp(N_1|2N_2), (OSp(N_1|2N_3)),$$

$$(OSp(N_1|2N_2), (OSp(N_3|2N_2)), (OSp(N_1|2N_2), (OSp(2|2N_2)),$$

$$(OSp(2|2N_1), (OSp(2|2N_1)).$$

For every pair, the even parts share at least one common simple factor, hence can be chosen as the Lie algebras of the gauge groups.

It is straightforward to generalize the construction of section 6 by letting that one symplectic 3-algebra contains three symplectic sub 3-algebras. And one can realize the three sub 3-algebras in terms of three superalgebras. Then the gauge group must be the even parts of $(G_1, G_2, G_3)$, where $G_i$ ($i = 1, 2, 3$) is a superalgebra selected from the list

$$U(M|N), \quad OSp(M|2N), \quad OSp(2|2N).$$

(7.1)

Here we assume that the even parts of $G_1$ and $G_2$ share at least one common simple factor, while the even parts of $G_2$ and $G_3$ share at least one common simple factor. For example, one can choose $(G_1, G_2, G_3)$ as $(OSp(N_1|2N_2), OSp(N_1|2N_3), OSp(N_4|2N_3))$. The resulting quiver diagram for gauge groups is

$$Sp(2N_2) - SO(N_1) - Sp(2N_3) - SO(N_4).$$

(7.2)

Or we can set $(G_1, G_2, G_3) = (U(N_1|N_2), U(N_2|N_3), U(N_3|N_4))$, and the resulting quiver diagram for gauge groups is

$$U(N_1) - U(N_2) - U(N_3) - U(N_4).$$

(7.3)

In the general case, one can choose the even parts of $(G_1, \cdots, G_n)$, where $G_i$ ($i = 1, \cdots, n$) is a superalgebra selected from the list (7.2); the even parts of $G_i$ and $G_{i+1}$ ($i = 1, \cdots, n-1$) share at least one common simple factor $\mathfrak{g}$. If the even parts of $G_1$ and $G_n$ (with $n$ an even number) also share at least one common simple factor, then the linear quiver becomes
a closed loop. The linear quiver gauge theories described in this paragraph exhaust all known examples of $\mathcal{N} = 4$ superconformal CMS theories.

As [10] pointed out, if one also takes account of the exceptional superalgebras, and the isomorphisms of the Lie algebras, there are additional possibilities. We will elaborate these ideas by constructing some $\mathcal{N} = 4$ theories with new gauge groups.

Let us first consider the exceptional superalgebras. The even parts of the superalgebras $F(4), G(3)$ and $D(2|1, \alpha)$ (with $\alpha$ a continuous parameter) are $SO(7) \times SU(2)$ ($SO(7)$ is in the spinor representation), $G_2 \times SU(2)$ and $SO(4) \times Sp(2)$, respectively. Now we have the complete list

$$U(M|N), \ OSp(M|2N), \ OSp(2|2N), \ F(4), \ G(3), \ D(2|1; \alpha). \ (7.5)$$

The superalgebras $SU(M|N)$ and $PSU(2|2)$, the cousins of $U(M|N)$, can be also used to realize the symplectic 3-algebra. We therefore may have

$$(G, G') = (F(4), SU(2|N_2)), (G(3), SU(2|N_2)), (G(3), F(4)), \ (OSp(N_1|2), D(2|1, \alpha)), (OSp(7|2N), F(4)), (OSp(4|2N), D(2|1, \alpha)). \ (7.6)$$

Their even parts can be selected as the Lie algebras of the gauge groups.

It also is possible to construct some new $\mathcal{N} = 4$ CMS theories by using the four isomorphisms of the Lie algebras. We know that the Lie algebra of $SO(3)$ is isomorphic to that of $SU(2)$ and $Sp(2)$, the Lie algebra of $SO(5)$ is isomorphic to that of $Sp(4)$, and the Lie algebra of $SO(6)$ is isomorphic to that of $SU(4)$. So the pairs of the superalgebras can be also chosen as

$$(G, G') = (OSp(3|2N_1), OSp(N_2|2)), (OSp(3|2N_1), SU(2|N_2)), (OSp(3|2N_1), F_4), \ (OSp(3|2N_1), D(2|1, \alpha)), (OSp(3|2N_1), G_3), (OSp(N_1|2), SU(2|N_2)), \ (OSp(N_1|2), F_4), (OSp(N_1|2), G_3), (G_3, D(2|1, \alpha)), (F_4, D(2|1, \alpha)), \ (OSp(5|2N_1), OSp(N_2|4)), (OSp(6|N_1), SU(4|N_2)), (D(2|1, \alpha), SU(2|N_2)), \ (7.7)$$

and their even parts can be selected as the Lie algebras of the gauge groups.

Finally, in Section 3 we have constructed two classes of new $\mathcal{N} = 4$ theories by requiring the $U(1)$ parts of the even parts of $G$ and $G'$ are identical; they are given by

$$(G, G') = ((OSp(2|2N_2), U(N_3|N_4)), ((U(N_1|N_2), U(N_3|N_4)). \ (7.8)$$

The other classes of new $\mathcal{N} = 4$ theories in Section 3 are given by the lists (7.34) and (5.35).

In summary, one can use the constructions of the previous paragraphs in the general case $(G_1, \cdots, G_n)$, where any adjacent pair $G_i$ and $G_{i+1}$ ($i = 1, \cdots, n - 1$) is selected from (7.1), (7.6), (7.7), (7.8), (5.34), (5.35) or (5.39). Namely, the even parts of $G_i$ and $G_{i+1}$ share at least one common simple factor or $U(1)$ factor; or at least one simple factor of the even part of $G_i$ is isomorphic to one simple factor of the even part of $G_{i+1}$, even they may
be in different representations. The Lie algebra of the gauge group is just the even parts of
the superalgebras \((G_1, \cdots, G_n)\), and the representations are determined by the fermionic
generators. If \(n\) is even, then \(G_1\) and \(G_n\) may also share one common bosonic parts, i.e.
\(G_1 \sim G_n\) may form a closed loop \([5]\).

8. Conclusions and Discussion

In this paper, we have identified a special class of 3-algebras called double-symplectic 3-
algebras used to construct the general \(\mathcal{N} = 4\) quiver gauge theories, and showed that its
consistent contraction gives a class of 3-algebras called \(\mathcal{N} = 4\) three-algebras, which can
be also used to construct the general \(\mathcal{N} = 4\) theories.

We then have used two superalgebras whose bosonic parts share at least one simple
factor or \(U(1)\) factor to construct the four sets of 3-brackets \((1.2)\) and \((1.3)\) and the four
sets of FIs \((2.15)\) defined on the two sub symplectic 3-algebras in the \(\mathcal{N} = 4\) theories. We
have also generalized the construction to the more general \(\mathcal{N} = 4\) quiver gauge theories in
which more than two sub 3-algebras are used. We have not only rederived all known \(\mathcal{N} = 4\)
theories, but also constructed many classes of \(\mathcal{N} = 4\) quiver gauge theories (see Sec. 5 and
Sec. 7) with new gauge groups. Especially, if the common parts of the bosonic parts of two
superalgebras are \(U(1)\) factors, the resulting \(\mathcal{N} = 4\) theories can contain free parameters.
We have constructed two infinite classes of new theories of this kind (see Sec. 7.3 and Sec.
7.4). It would be interesting to see whether or not there are further constraints imposed
on these free parameters if we quantize these theories.

Taking account of the new \(\mathcal{N} = 4\) theories, we have been able to achieve a new
classification of all possible \(\mathcal{N} = 4\) quiver gauge theories in Sec. 5.4.

Using superalgebras to realize the 3-algebra, we have showed the general \(\mathcal{N} = 4\)
theory in terms of the double-symplectic 3-algebra is equivalent to the previous \(\mathcal{N} = 4\)
theory \([6]\) derived directly from the conventional Lie 2-algebra. Hence the \(\mathcal{N} = 4\) theories
with new gauge groups in Sec. 5 derived in the 3-algebra framework can be also understood
as special examples of the general \(\mathcal{N} = 4\) HLLLP theories in Ref. [6]. Specifically, in 5,
one can choose either \(SO(4) \cong SU(2) \times SU(2)\) or \(U(N) \cong SU(N) \times U(1)\) as the common
bosonic algebra of the two superalgebras; our constructions in Sec. 5 show that one can
also choose only \(SU(2)\) (the indecomposable part of \(SO(4)\)) or \(U(1)\) (the indecomposable
part of \(U(N)\)) as the common bosonic part of the two superalgebras.

Most of gravity duals of these \(N = 4\) quiver gauge theories have not been constructed
yet. It would be nice to study the quantum properties of these \(N = 4\) quiver gauge theories
and to construct and study the corresponding gravity duals.

Finally, we know that it was demonstrated that generic Chern-Simons gauge theories
with or without (massless) matter are conformally invariant even at the quantum level
[16, 17, 18, 19, 20]. It would be interesting to examine the conformal invariance of these
\(\mathcal{N} = 4\) quiver gauge theories at quantum levels.
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A. Conventions and Useful Identities

The conventions and useful identities are adopted from our previous paper [12].

A.1 Spinor Algebra

In 1 + 2 dimensions, the gamma matrices are defined as

\[(\gamma_\mu)_\alpha^\gamma (\gamma_\nu)_\gamma^\beta + (\gamma_\nu)_\alpha^\gamma (\gamma_\mu)_\gamma^\beta = 2\eta_{\mu\nu}\delta_\alpha^\beta.\]  

(A.1)

For the metric we use the (−, +, +) convention. The gamma matrices in the Majorana representation can be defined in terms of Pauli matrices: \[(\gamma_\mu)_\alpha^\beta = (i\sigma_2, \sigma_1, \sigma_3),\] satisfying the important identity \[(\gamma_\mu)_\alpha^\gamma (\gamma_\nu)_\gamma^\beta = \eta_{\mu\nu}\delta_\alpha^\beta + \epsilon_{\mu\nu\lambda}(\gamma_\lambda)_\alpha^\beta.\]  

(A.2)

We also define \[\epsilon_{\mu\nu\lambda} = -\epsilon_{\mu\lambda\nu}.\] So \[\epsilon_{\mu\nu\lambda}\epsilon_{\rho\kappa\lambda} = -2\delta_{\mu\rho}.\] We raise and lower spinor indices with an antisymmetric matrix \[\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha},\] where \[\epsilon_{12} = -1.\] For example, \[\psi_\alpha^\gamma = \epsilon_{\beta\gamma}(\gamma_\mu)_\alpha^\beta,\] and \[\gamma_\alpha^\mu = \epsilon_{\beta\gamma}(\gamma_\mu)_\alpha^\beta,\] where \(\psi_\beta\) is a Majorana spinor. Notice that \(\gamma_\mu^\alpha = (1, -\sigma^3, \sigma^1)\) are symmetric in \(\alpha\beta\). A vector can be represented by a symmetric bispinor and vice versa:

\[A_{\alpha\beta} = A_\mu \gamma^\mu_{\alpha\beta}, \quad A_\mu = -\frac{1}{2} \gamma^\mu_{\alpha\beta} A_{\alpha\beta}.\]  

(A.3)

We use the following spinor summation convention:

\[\psi_\chi = \psi_\alpha^\gamma \chi_\alpha, \quad \psi_\gamma_\mu_\chi = \psi_\alpha^\gamma (\gamma_\mu)_\alpha^\beta \chi_\beta,\]  

(A.4)

where \(\psi\) and \(\chi\) are anti-commuting Majorana spinors. In 1 + 2 dimensions the Fierz transformation reads

\[(\lambda_\chi)\psi = -\frac{1}{2}(\lambda_\psi)\chi - \frac{1}{2}(\lambda_\gamma_\nu_\psi)\gamma^\nu_\chi.\]  

(A.5)

A.2 SU(2) × SU(2) Identities

We define the 4 sigma matrices as

\[\sigma^a_A^B = (\sigma^1, \sigma^2, \sigma^3, i1),\]  

(A.6)

by which one can establish a connection between the SU(2) × SU(2) and SO(4) group. These sigma matrices satisfy the following Clifford algebra:

\[\sigma^a_A^B \sigma^b_B^C + \sigma^b_B^C \sigma^a_A^B = 2\delta^{ab}\delta_A^B,\]  

\[\sigma^a_A^B \sigma^b_B^C \sigma^c_C^D + \sigma^b_B^C \sigma^a_A^D = 2\delta^{ab}\delta_A^D.\]  

(A.7)
We use anti-symmetric matrices

\[ \epsilon_{AB} = -\epsilon^{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{A\dot{B}} = -\epsilon^{A\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]  

(A.9)

to raise or lower un-dotted and dotted indices, respectively. For example, \( \sigma^{a\dot{A}}B = \epsilon^{A\dot{B}}\sigma^{a\dot{A}}B \) and \( \sigma^{a\dot{A}}A = \epsilon^{BC}\sigma^{a\dot{A}}C\dot{A} \). The sigma matrix \( \sigma^a \) satisfies a reality condition

\[ \sigma^{a\dot{A}}B = -\epsilon^{BC}\epsilon_{A\dot{B}}\sigma^{a\dot{A}}C\dot{B}, \quad \text{or} \quad \sigma^{a\dot{A}}B = -\sigma^{aB\dot{A}}. \]  

(A.10)
The antisymmetric matrix \( \epsilon_{AB} \) satisfies an important identity

\[ \epsilon_{AB}\epsilon^{CD} = -(\delta_A^C\delta_B^D - \delta_A^D\delta_B^C), \]  

(A.11)
and \( \epsilon_{A\dot{B}} \) satisfies a similar identity.

The parameter for the \( N = 4 \) supersymmetry transformations is defined as \( \epsilon_{A\dot{B}} = \epsilon_0\sigma^{aA\dot{B}} \).

**B. The Commutation Relations of Superalgebras**

**B.1 U(M|N)**

The commutation relations of \( U(M|N) \) are given by

\[ [M_{\tilde{a}}^\dot{a}, M_{\dot{b}}^\alpha] = \delta_{\tilde{a}}^\dot{a}M_{\dot{b}}^\alpha - \delta_{\dot{b}}^\alpha M_{\tilde{a}}^\dot{a}, \quad [M_{\mu}^{\dot{\alpha}}, M_{\nu}^{\beta}] = \delta_{\mu}^\nu M_{\dot{\alpha}}^{\beta} - \delta_{\nu}^{\dot{\alpha}} M_{\mu}^{\beta} \]
\[ [M_{\tilde{a}}^\dot{a}, Q_{\dot{a}}^{\tilde{a}}] = \delta_{\tilde{a}}^\dot{a}Q_{\dot{a}}^{\tilde{a}}, \quad [M_{\mu}^{\dot{\alpha}}, Q_{\nu}^{\tilde{a}}] = -\delta_{\mu}^{\dot{\alpha}}Q_{\nu}^{\tilde{a}}, \]
\[ [M_{\mu}^{\dot{\alpha}}, Q_{\nu}^{\tilde{a}}] = -\delta_{\mu}^{\dot{\alpha}}Q_{\nu}^{\til{a}}, \quad [M_{\nu}^{\dot{\alpha}}, Q_{\mu}^{\til{a}}] = \delta_{\nu}^{\dot{\alpha}}Q_{\mu}^{\til{a}} \]
\[ \{Q_{\mu}^{\dot{\alpha}}, Q_{\nu}^{\til{a}}\} = k(\delta_{\mu}^{\dot{\alpha}}M_{\nu}^{\til{a}} + \delta_{\nu}^{\dot{\alpha}}M_{\mu}^{\til{a}}), \]  

(B.1)

where \( Q_{\mu}^{\dot{\alpha}} \) carries a \( U(M) \) fundamental index \( \tilde{a} = 1, \ldots, M \) and a \( U(N) \) anti-fundamental index \( \dot{a} = 1, \ldots, N \). Here we have

\[ Q_{\mu}^{\dot{\alpha}} = \begin{pmatrix} \tilde{Q}_{\mu}^{\dot{a}} \\ -\tilde{Q}_{\mu}^{\til{a}} \end{pmatrix} = \tilde{Q}_{\mu}^{\dot{a}}\delta_{1\alpha} - \tilde{Q}_{\mu}^{\til{a}}\delta_{2\alpha}. \]  

(B.2)

In the second equation of (B.2), we have introduced a “spin up” spinor \( \chi_{1\alpha} \) and a “spin down” spinor \( \chi_{2\alpha} \), i.e.,

\[ \chi_{1\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta_{1\alpha} \quad \text{and} \quad \chi_{2\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta_{2\alpha}. \]  

(B.3)

And the anti-symmetric tensor \( \omega_{ab} \) and its inverse read

\[ \omega_{a'b'} = \begin{pmatrix} 0 & \delta_{\tilde{a}}^\dot{a}\delta_{\dot{a}}^{\dot{a}} \delta_{\dot{a}}^{\dot{b}} \\ -\delta_{\tilde{a}}^{\dot{a}}\delta_{\dot{a}}^{\dot{b}} & 0 \end{pmatrix}, \quad \omega^{a'b'} = \begin{pmatrix} 0 & -\delta_{\dot{a}}^{\dot{b}} \delta_{\dot{a}}^{\dot{b}} \delta_{\dot{a}}^{\dot{a}} \\ \delta_{\dot{a}}^{\dot{b}} \delta_{\dot{a}}^{\dot{b}} & 0 \end{pmatrix}. \]  

(B.4)

With (B.2) and (B.4), the superalgebra (B.1) takes the form of (1.3) or (4.4).

---

10Here the index \( \alpha \) is not a spacetime spinor index. We hope this will not cause any confusion.
The superalgebra $OSp(M|2N)$ reads

$$
[M_{ij}, M_{kl}] = \delta_{jk} M_{il} - \delta_{ik} M_{jl} + \delta_{il} M_{jk} - \delta_{jl} M_{ik},
$$

$$
[M_{ij}, M_{\bar{k}\bar{l}}] = \omega_{\bar{m} j} M_{\bar{m} i} + \omega_{\bar{m} i} M_{\bar{m} j} + \omega_{\bar{m} j} M_{\bar{m} i} + \omega_{\bar{m} i} M_{\bar{m} j},
$$

$$
[M_{\bar{i}\bar{j}}, Q_{\bar{k}\bar{l}}] = \delta_{\bar{k}\bar{j}} Q_{\bar{i}\bar{l}} - \delta_{\bar{i}\bar{l}} Q_{\bar{k}\bar{j}},
$$

$$
[M_{\hat{i}\hat{j}}, Q_{\hat{k}\hat{l}}] = \omega_{\hat{k}\hat{j}} Q_{\hat{i}\hat{l}} + \omega_{\hat{i}\hat{l}} Q_{\hat{k}\hat{j}},
$$

$$
\{Q_{\bar{i}}, Q_{\hat{j}}\} = k(\omega_{\hat{i}\hat{j}} M_{\bar{i}\bar{j}} + \delta_{\bar{i}\bar{j}} M_{\hat{i}\hat{j}}),
$$

(E.5)

where $\bar{i} = 1, \cdots, M$ is an $SO(M)$ fundamental index, and $\hat{i} = 1, \cdots, 2N$ an $Sp(2N)$ fundamental index. Here we have

$$
Q_a = Q_{\bar{i}}, \quad \omega_{ab} = \omega_{\bar{i}\hat{j}} = \delta_{\bar{i}\hat{j}} \omega_{\hat{i}\hat{j}}.
$$

(B.6)

Now the superalgebra (B.2) also takes the form of (4.3) or (4.4).

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