Space-time analogy and its application to design schemes borrowed from Fourier optics for processing ultrafast optical signals

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Abstract
Square-wave pulse generation with a variable duty ratio can be realized with the help of the ideas of Talbot array illuminators formulated for binary phase gratings. A binary temporal phase modulation of continuous wave (CW) laser field propagating through a group-delay-dispersion circuit of the fractional Talbot length \(P/Q\) results in a well defined sequence of square-wave-form pulses. When \(P = 1\) a duty ratio of the pulses \(D\) is \(1/2\) for \(Q = 4\) and \(1/3\) for \(Q = 3\) and 6. Maximum intensity of the pulses doubles and triples compared to the CW intensity for \(D = 1/2\) and \(1/3\), respectively. These pulses can be used for return-to-zero laser field modulation in optical fiber communication. For \(D = 1/3\) extra features between the pulses are found originating from a finite rise and drop time of phase in a binary phase modulation. A similar effect to the benefit of the time-space analogy is predicted for binary phase gratings and interpreted as gleams produced by imperfect edges of the components of the rectangular phase gratings.

Keywords: fiber optics, interfaces, parallel circuits, integrated circuits, optical signal processing

(Some figures may appear in colour only in the online journal)

1. Introduction
The space-time analogy is a useful concept that gives ideas for designing new schemes for optical signal processing in time domain borrowed from Fourier optics and developing in a new domain known as temporal optics, see review [1]. For example, a periodic phase modulated continuous wave (CW) light, which is transmitted through a group-delay-dispersion (GDD) circuit, has a spatial analogue corresponding to the wave field after Fresnel diffraction by one-dimensional periodic objects in space. Flat-top pulse generation with a duty ratio of 50% is one of the examples of the space-time analogy application at a quarter of the Talbot condition. These pulses are obtained by sinusoidal [2–4] and binary [4–6] electrooptic phase modulations of a CW laser field after transmission of the field through an appropriate GDD circuit. This method can be used in ultrahigh-speed optical fiber communications for energy preserving modulation of the laser field intensity with high extinction ratio.

Mode-locked lasers are widespread sources of ultrashort optical pulses. However, the generated pulses suffer from instability problems resulting in jitters of the peak-power and duration. It is also difficult to control the pulse width, shape, repetition rate, and position in a time slot for fine tuning and synchronization with other electrical signals in the integrated circuit. These problems can be solved by use of electrooptic phase modulators, which allow generation of ultrashort pulses from the CW light emitted from a narrowband stabilized laser. Electrooptic modulation transforms CW light to phase-modulated light with ultrawide optical sidebands on the order
of terahertz. Then, after transmission through a GDD circuit this chirped CW light is compressed into trains of ultrashort pulses in the sub-picosecond range [7]. Meanwhile, the sinusoidally phase-modulated light with a large phase-modulation index is not fully compressed into pulses [8]. Half of the field energy is left in a dc floor level between pulses [4]. Therefore, conditions for return-to-zero laser field modulation without losses need to be analyzed.

In the space domain, periodic phase gratings are capable of converting a uniformly wide beam losslessly into an array of periodic spots of concentrated light [9–14]. This is so called an array illuminator (AIL), which provides illumination for microcomponents such as optical logic gates or bistable elements in a 2D discrete parallel processor [15]. AIL has applications in multiple imaging [16] and optical interconnectors [17]. In AIL a uniformly wide beam is transformed by a laterally-periodic phase grating under paraxial approximation into the Fresnel diffraction pattern at a distance \( z = (P/Q)Z_T \), where \( P \) and \( Q \) are coprime integers and \( Z_T \) is the Talbot length. This is known as the spatial fractional Talbot effect. With a binary phase grating as the input one can create arrays of many periodic light spots with varying spot shapes at the fractional imaging distances [9–14]. In this paper a binary phase modulation of CW laser field and successive phase treatment by a GDD circuit is considered. Here, a phase modulation substitutes the phase grating, while the GDD circuit simulates Fresnel diffraction. The fractional Talbot effect, discussed in this paper, gives a wide variety of possibilities to generate and control ultrashort optical pulses by integrated optical components.

2. Binary phase-modulated CW field

We consider the CW radiation field \( E_M(t) = E_0 \exp[-i\omega_0 t + ikz + i\varphi(t)] \) with a periodic binary phase modulation, which is

\[
\varphi(t) = \sum_{n=-\infty}^{+\infty} \phi(t-nT)
\]

where

\[
\phi(t) = \Delta [\theta(t-T_-) - \theta(t-T_+)].
\]

Here, \( E_0 \) is the constant field amplitude, \( \omega_0 \) and \( k \) are the carrier frequency and the wave number of the field, respectively, \( \theta(t) \) is the Heaviside step function, \( \Delta \) is the maximum phase shift induced by an electrooptic modulator, \( T_\pm = (T \pm T_p)/2 \), \( T \) is a modulation period, and \( T_p \) is a duration of the phase shift. Below we will consider three cases when \( T_p/T = 1/2, 1/3, \) and \( 2/3 \), see figure 1(a). These cases have been previously explored in the field of Fourier Optics to design binary phase gratings known as AILs that produce a two-dimensional array of high contrast bright spots in the spatial irradiance distribution at the fractional Talbot length without losses [12, 13]. In this paper the findings in Fourier Optics (spatial domain) will be applied to generate short pulses in Temporal Optics (time domain) producing the irradiance concentration in well defined time slots without losses.

![Figure 1](image_url)
This is in conflict with the definition of the phase modulated field $E_M(t) = E_0 \exp[-i \omega t + i k z + i \varphi(t)]$, which gives constant intensity $|E_M(t)|^2 = E_0^2$ for any $t$ and any phase modulation function $\varphi(t)$. To avoid this inconsistency we introduce linear rise and drop of the phase substituting stepwise phase changes in binary phase modulation. In this case the function $\varphi(t)$ in equation (1) is replaced by

$$
\varphi(t) = \Delta \left\{ \frac{1 - T_+}{\tau} \theta(t - T_+) - \frac{T_+ - t}{\tau} \theta(t - T_+) + \left( 1 - \frac{1 - T_-}{\tau} \right) \theta(t - T_- - \tau) - \left( 1 + \frac{1 - T_+}{\tau} \right) \theta(t - T_+ + \tau) \right\},
$$

where $\tau$ is a short time interval when the phase rises or drops, see figure 1(b). For such a phase evolution the coefficients in the Fourier series expansion (4) are modified for $n \neq 0$ as

$$
C_n = \frac{(-1)^n}{d_n} \left[ e^{i \Delta} \sin b_n - \sin c_n - ig_n \left( e^{i \Delta} \cos b_n - \cos c_n \right) \right],
$$

and for $n = 0$ as

$$
C_0 = 1 + a \frac{T_+}{T} - \frac{2 \tau}{\pi} \left[ 1 + a \left( 1 + \frac{i}{\Delta} \right) \right],
$$

where $b_n = \pi n (T_p - 2 T) / T$, $c_n = \pi n T_p / T$, $d_n = \pi n \left[ 1 - \left( \frac{2 \pi \tau}{\Delta^2} \right)^2 \right]$, and $g_n = 2 \pi n \frac{\tau}{\Delta^2}$. Then, the field amplitude $E_M(t)$ does not experience a drop at $t = (2m + 1)T/4$. To avoid time dependence of the intensity $I_M(t)$, one has to take at least $2T/\tau$ terms in the sum (4). For example, for $T/\tau = 10$, it is necessary to sum 21 terms in the interval $(-n, n)$ where $n = 10$.

In numerical simulations we should avoid fulfilment of the condition $n^2 = (\Delta T / 2 \pi \tau)^2$ when $d_n = 0$ making one of the coefficients $C_n$ in equations (4) and (8) infinite. Meanwhile, this singularity is false since according to the definition (3) all the coefficients $C_n$ are finite.

### 3. Space-time analogy

In this section we show that spatial diffraction of light from periodic structure is analogous to the propagation of the phase modulated light through the GDD circuit. The results obtained previously for the binary phase grating are applied for generating rectangular pulses with variable duty ratio from the phase modulated CW field.

It is well known that a uniform plane wave can be converted by binary phase grating into many concentrated light spots in a controllable way with almost no energy loss [9–14]. If the grating is located at $z = 0$, the plane wave $E_S(x, z)$ propagating along $z$-direction is transformed immediately behind the grating for the 1D case as

$$
E_S(x, z) / E_0 = 1 + \sum_{n=\infty}^{n=\infty} A_n e^{-\beta z n / D},
$$

where $A_n = [\exp(i \Psi) - 1] (W/D) \frac{\sin(\pi n W / D)}{\pi n W / D}$ is the $n$th Fourier coefficient representing the $n$th complex amplitude of the grating angular spectrum, $D$ is a grating period along $x$-direction, which is perpendicular to the propagation direction $z$. $\Psi$ is a phase step, and $W$ is a width of the phase step in the binary phase grating. Mathematically $E_S(x, 0)$ is fully equivalent to the periodically phase modulated field $E_M(t)$, equation (4). This can be shown by substituting $x \rightarrow t$, $\Psi \rightarrow \Delta$, $D \rightarrow T$, $W \rightarrow T \Psi$. The only difference is in $(-1)^n$ present in the coefficients $C_n$ and originating from the definition of a spatial grating center. This difference is easily removed by the center shift on a half of the grating period.

The field amplitude at a plane $z \geq 0$ is transformed due to Fresnel diffraction under the paraxial approximation as

$$
E_S(x, z) / E_0 = 1 + \sum_{n=-\infty}^{n=\infty} A_n e^{-\beta z n^2 / 2 \pi} \frac{\exp(-i \pi n / \Psi)}{\Psi},
$$

where $Z_T = 2D^2 / \lambda$ is the Talbot length and $\lambda$ is the wavelength of the field [10–14]. The Fresnel field of the grating in the plane located at the fractional Talbot length $Z_{T \nu} / \Psi$, where $P$ and $Q$ are coprime integers, is analyzed in [9–14]. To find conditions producing AIL with binary distribution of the field $E_S(x, z)$, it was expressed in [13] as binary, i.e.,

$$
E_S(x, z) = h E_0 \sum_{n=-\infty}^{n=\infty} c_n \exp(-i \pi n x / D),
$$

where $h$ is a complex quantity, $c_n = (W/D) \frac{\sin(\pi n W / D)}{\pi n W / D}$, is the period of the light spots, $W$ is the width of the individual light spot not necessarily equal to $W$. Analysis in [13] showed that binary field distribution is realized for the following values of the parameters (i) $W/D = W'/D = 1/2$, $\Psi = \pi / 2$, $z = Z_T / 4$, (ii) $W/D = W'/D = 1/3$, $\Psi = 2 \pi / 3$, $z = Z_T / 3$, and (iii) $W/D = 2/3$, $W'/D = 1/3$, $\Psi = 2 \pi / 3$, $z = Z_T / 6$.

Below we apply these findings in time domain and develop a method of generating rectangular pulses with variable duty ratio. We derive simple analytical expressions with the help of different approach developed to calculate the defocussed pattern for an arbitrary periodic grating in space [18]. We also show that imperfect edges of the binary (rectangular) phase grating result in appearance of gleams, which are unwanted narrow light spots between designed spots in the binary light pattern in space and time domains.

The phase modulated field, $E_M(t)$, which propagates inside the GDD circuit, is transformed at distance $z$ as, see for example [1],

$$
E_M(t) = E(t) \sum_{n=-\infty}^{n=\infty} C_n e^{-\beta z n^2 / 2 \pi} \frac{\exp(-i \pi n / \Psi)}{\Psi},
$$

where $E(t) = E_0 e^{-i \omega t + i k z}$, $\Phi_1$ is the group delay due to the reduced group velocity, $\Phi_2 = \beta_2 z$ is the GDD coefficient, $\beta_2$ is the second-order dispersion coefficient, $z$ is the propagation distance, $\Phi_{2 T} = T^2 / \pi$ is the Talbot dispersion, and $f = |\Phi_2| / \Phi_{2 T}$. Omitting for simplicity $\Phi_1$, one can find that equations (11) and (12) are equivalent if $\Phi_{2 T}$ is taken as the Talbot length and $\Phi_2$ is a propagation distance, both are normalized. Below we consider the normalized fractional Talbot length $|\Phi_2| / \Phi_{2 T} = P / Q$ with $P = 1$. 


According to [18, 19] for \( P = 1 \), integer \( Q \), and negative \( \beta_2 \) the field \( E_f(t) \) is reduced to

\[
E_f(t) = E(t) \sum_{m=1}^{Q} e^{i \left[ \frac{\pi}{2} \frac{m}{Q} (t+\frac{1+\epsilon}{\epsilon}) \right]} \frac{[1+\epsilon^Q(-1)^m]}{\sqrt{2Q}}. \tag{13}
\]

(i) In time domain this case is realized for the following values of the parameters: \( T_p/T = 1/2 \), \( \Delta = \pi/2 \), and \( P/Q = 1/4 \), known as the one-quarter Talbot condition. For \( f = 1/4 \) the field \( E_f(t) \), equation (12), is simplified as follows (see [18, 19])

\[
E_1(t) = E(t) \frac{e^{-i \frac{\pi}{2} + i \psi(t)} + e^{i \frac{\pi}{2} + i \psi(t+T/2)}}{\sqrt{2}}. \tag{14}
\]

Then, it is easy to calculate very simple expression for the field intensity, \( I_f(t) = |E_f(t)|^2 \), which is

\[
I_1(t) = I_{CW} [1 + \sin \psi_0(t)], \tag{15}
\]

where \( \psi_0(t) = \varphi(t) - \varphi(t + T/2) \). This expression was also derived by different method in [4, 12].

For binary phase modulation, shown in figure 1(b), the phase \( \psi_0(t) \) jumps stepwise between two values, \( \Delta \) and \( -\Delta \). If \( \Delta = \pi/2 \), then according to equation (15), the field intensity varies stepwise between \( 2I_{CW} \) and zero, see figure 2(a), demonstrating flat-top pulses with a 50% duty ratio \( (D = 1/2) \) and complete extinction between pulses. The energy of the phase modulated field is distributed between pulses with intensity \( 2I_{CW} \) and dark windows with zero intensity not violating the law of conservation of energy. Pulses are produced due to constructive interference of two replicas of the field, which are relatively shifted to each other in phase and time, see equation (14). Dark windows appear due to destructive interference of these replicas. Experimental implementation of this case is reported in [5]. These pulses with 50% duty ratio are proposed for return-to-zero amplitude modulation in optical fiber communication.

One can obtain multilevel temporal pattern if \( T_p \neq T/2 \). Example of the case when \( T_p = T/3 \) is shown in figure 3(a). Rectangular shape pulses of duration \( T_p \) and intensity \( 2I_{CW} \) sitting on another rectangular pulse of duration \( T - T_p \) are formed. Intensity of the shoulders of the wide rectangular pulses is \( I_{CW} \). The pulses are separated by dark windows with zero intensity and duration \( T_p \).

(ii) This case is realized for \( T_p/T = 1/3 \), \( \Delta = 2\pi/3 \), and \( f = 1/3 \). The last condition is known as the one-third Talbot condition, which gives for equation (13) the following expression [19]

\[
E_2(t) = E(t) \frac{e^{-i \frac{\pi}{2} + i \psi(t)} + e^{i \frac{\pi}{2} + i \psi(t+T)} + e^{i \frac{\pi}{2} + i \psi(t-T)}}{\sqrt{3}}. \tag{16}
\]

Interference of three phase and time shifted replicas of the field in equation (16) gives the following expression for the field intensity

\[
I_2(t) = \frac{I_{CW}}{3} [1 + 8 \cos \psi_0(t) \cos \psi_2(t) \cos \psi_3(t)], \tag{17}
\]

**Figure 2.** Time evolution of the field intensity after propagating the GDD circuit with (a) \( f = 1/4 \) and (b) \( f = 1/3 \). The field is binary phase modulated with (a) \( T_p = T/2 \), \( \Delta = \pi/2 \) and (b) \( T_p = T/3 \), \( \Delta = 2\pi/3 \). Rise and drop time of the phase is \( \tau = 0.05T \) in both plots.

where \( \psi_1(t) = [\varphi(t + T/3) - \varphi(t - T/3)]/2 \), \( \psi_2(t) = [\varphi(t) - \varphi(t + T/3) - 2\pi/3]/2 \), and \( \psi_3(t) = [\varphi(t) - \varphi(t - T/3) - 2\pi/3]/2 \). Within time windows \( 1/3 + nT < t < (2/3 + n)T \) when phase \( \varphi(t) \) takes value \( 2\pi/3 \) for \( \tau \to 0 \) and an integer \( n \), we have \( \psi_1(t) = \psi_2(t) = \psi_3(t) = 0 \), which gives \( I_2(t) = 3I_{CW} \). Then, in time windows \( (2/3 + n)T < t < (1 + n)T \) when phase \( \varphi(t) \) is zero, we have \( \psi_1(t) = \psi_2(t) = -\pi/3 \) and \( \psi_3(t) = -2\pi/3 \), which gives \( I_2(t) = 0 \). In the next time window \( (1 + n)T < t < (4/3 + n)T \) we have \( \psi_1(t) = \pi/3, \psi_2(t) = -2\pi/3, \) and \( \psi_3(t) = -\pi/3, \) which gives again \( I_2(t) = 0 \).

Thus, binary phase modulated field with \( T_p/T = 1/3 \) and \( \Delta = 2\pi/3 \) is transformed by the GDD circuit with \( f = 1/3 \) to the sequence of flat-top pulses with intensity \( 3I_{CW} \), 33% duty ratio \( (D = 1/3) \), and complete extinction between pulses, see figure 2(b).

If the rise and drop time \( \tau \) of phase \( \varphi(t) \) is finite, some extra features appear in time dependence of the field intensity \( I_2(t) \). They originate from the changes of phases \( \psi_1(t), \psi_2(t), \) and \( \psi_3(t) \), which take place in the vicinity of \( t = nT \), i.e. \( \psi_1(t) \) rises from \( -\pi/3 \) to \( \pi/3 \), \( \psi_2(t) \) drops from \( -\pi/3 \) to \( -2\pi/3 \), and \( \psi_3(t) \) rises from \( -2\pi/3 \) to \( -\pi/3 \).

For the linear time dependencies of the field phase \( \varphi(t) \), which substitute stepwise changes in equations (1) and (2), phases \( \psi_1(t), \psi_2(t), \) and \( \psi_3(t) \) change linearly within time intervals specified below. Phase \( \psi_1(t) = \Delta (t - nT)/2\tau \) rises in a time interval \( (nT - \tau, nT + \tau) \), phase \( \psi_2(t) = -\Delta (nT + \tau)/2\tau \) drops in a time interval \( (nT, nT + \tau) \), and phase \( \psi_3(t) = -\Delta (t - nT)/2\tau \) rises in a time interval \( (nT - \tau, nT + \tau) \).
Time dependence of the field intensity is

$$I_\pm(t) = \frac{I_{CW}}{3} \left[ 1 + 8 \cos \psi_4(t) \cos \psi_5(t) \cos \psi_6(t) \right],$$

(19)

where $\psi_4(t) = [\varphi(t + T/6) - \varphi(t - T/6)]/2$, $\psi_5(t) = [\varphi(t + T/6) + \varphi(t - T/6)]/2$, and $\psi_6(t) = [\varphi(t + T/2) - \varphi(t - T/6) + 2\pi/3]/2$. This binary phase modulated field with $T_p/T = 2/3$ and $\Delta = 2\pi/3$ is transformed by GDD circuit with $f = 1/6$ to the same sequence of flat-top pulses as in case (ii), which is shown in figure 2(b). It is interesting to notice that phases $\psi_4(t)$, $\psi_5(t)$, and $\psi_6(t)$ are very close to those, which are introduced in case (ii), i.e. $\psi_4(t) \approx \psi_1(t)$, $\psi_5(t) \approx -\psi_3(t)$, and $\psi_6(t) \approx -\psi_2(t)$. There are small time shifts between these functions in particular time intervals. However, they take place such that time shifts with delay and advance compensate each other in equation (19) giving exactly the same time dependence as for $I_\pm(t)$.

Experimental implementation of this case ($T_p/T = 2/3$, $\Delta = 2\pi/3$, and $f = 1/6$) is reported in [5] for $\tau \approx 0.17$. Bell shaped pulses originating from finite rise-fall time of binary phase modulation are clearly seen between flat-top pulses in the experiment. It should be noted that parameter $q$ defined in [5] is related to $Q$ as $q = Q/2$.

In spatial domain these extra features between designed spots of binary form can be considered as a glare produced by edges of the rectangular phase grating. If the phase linearly grows or decreases at these edges between zero and $\Psi$, light spots of triangular form appear between rectangular spots. Maximum intensity of the spots equals to the intensity of the incident radiation. The spot size equals to the sum of the edge lengths of the individual phase grating element.

Multilevel temporal pattern also appears in the case $f = 1/6$ if $T_p \neq 2T/3$. For example, if $T_p = T/2$ this pattern is identical to that considered in (ii).

To estimate the parameters of the generated pulses we consider commercial arbitrary waveform generators with bandwidth ranging from 40 to 100 GHz. They are capable to produce rectangular phase modulation with rise/drop time ranging from 25 to 10 ps. This limits the repetition rate of the pulses in a window ranging from 2 to 5 GHz. Minimum pulse duration in this case can reach 66 ps. Meanwhile, Gunn diode [20] and IMPATT diode [21] are capable to discharge with the rate up to 1 THz producing sharp drop of the edges in the rectangular phase modulation with duration around 1 ps. Then, all the parameters of the pulses estimated above can be improved in the order of magnitude.

4. Conclusion

Square-wave pulse generation with variable duty ratio based on the temporal fractional Talbot effect is analyzed for binary phase modulated fields. Simple expressions for the field intensity after propagating through GDD circuit are presented for $P/Q$ fractional Talbot condition with $P = 1$, $Q = 3$, 4, and 6. Space-time analogy helps to find conditions when rectangular pulses with variable duty ratio are generated. This method allows energy preserving conversion of CW to pulse

![Figure 3](image-url)
trains with repetition rates in the gigahertz range and sub-nanosecond pulse width. It is shown that finite time of rise and drop in binary phase modulation results in short, low intensity pulses between pulses of designed sequence for fractional Talbot effect with $P/Q = 3$ and 6. Similar effect takes place in space domain. Generation of the multilevel temporal pattern of pulses with high extinction ratio is proposed.

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