Study of Quantum Decoherence in a Finite System.

Three Schrödinger Cats and Crossing of Classical Orbits.

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Nuclei are rather classical systems in a sense. In the old days, their phenomena were roughly explained in classical rules such as the liquid drop model. This fact may be understood that when we see an finite quantum many body system like nucleus, though which is a group of quantum mechanical particles, its any collective degree of freedom has any classicality. Getting a classicality does not depend on spatial scales of objects. It is made by a phenomenon called Quantum Decoherence. We have studied about Quantum Decoherence in a finite system as nucleus. In this paper, at the harmonic three body problem (3 Schrödinger cats), it is shown that one degree of freedom as a sub-system would get classicality because of the other two degrees of freedom. Therefore we can assume that a nuclear collective degree of freedom would get classicality when it couples with any other internal degrees of freedom. In this paper, we also note that there is some relationship between the Quantum Decoherence and Crossing of classical orbits in 3 Schrödinger cats model.

§1. Introduction.

Decoherence is a kind of non-unitary process, which is a disappearance of quantum interferences among different state vectors. Its origin is known as external fluctuations and dissipations. This phenomenon means “the collapse of wave function”. Because the interference is equal to the transition probability to the other quantum states. When the interference vanishes, the system can not be transported

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to any other quantum states, it means the system gets effective classicality. So you can expect that an observation may also cause decoherence. The similarity among fluctuations, dissipations and observations is that these are irreversible processes. Therefore, it is supposed that there is any relation between irreversibility and quantum decoherence.

In classical mechanics, it is known that the irreversibility is made by the coarse graining procedure. For example, the Langevin equation is the Newton equation with damping (dissipation) and random force (fluctuation), which describes irreversible processes such as a motion of a particle in liquid. To derive the irreversible Langevin equation from the reversible Newton equation or Hamilton equation, we need to average out about degrees of freedom of liquid molecules using the projection operators, that is a kind of coarse graining.

デコーヒーレンスは以下に示すような、量子力学における多くのパラドックスを解決すると期待できる。

We can expect that decoherence would solve a lot of quantum mechanical paradoxes as follows.

1.1. Schrödinger’s cat.

"シュレーディンガーの猫"の有名な話は、量子力学の基礎方程式であるシュレーディンガー方程式の、ユニタリ時間発展という性質によるものである。シュレーディンガー方程式の、それぞれの状態ベクトルは決して消える事が無く、それらの状態の重ね合わせは永遠に維持されるように思える。微視的な世界に対しては、私達はこのルールを受け入れる事が出来るかも知れない。しかしシュレーディンガーは2つの微視的量子状態が、私達の世界の2つの巨視的状態のそれぞれに反映するような、有名なシステムを考え出した。

The famous story of “the Schrödinger’s cat” is due to the unitary evolution of the Schrödinger equation which is the basic equation of the Quantum Mechanics. By unitary evolutions, each state vector will never vanish, and the superposition of states will be kept forever. For microscopic world, we may accept this rule. But Schrödinger thought up a famous system in which the two microscopic quantum states reflect the two macroscopic states in our world respectively.

箱の中に、一つの不安定原子核と検知器、ハンマーとガラス瓶に入った毒薬そして一匹の生きた猫が入っている。原子核が崩壊した時には放射線が発生し、検知器がそれを感知するとハンマーが毒薬の入った瓶を壊し、猫は死ぬ。原子核が崩壊していない時には、猫は死なない。そこで、原子核の状態のそれぞれが猫の状態のそれぞれに対応している。

In a box, there are an unstable nucleus, a geiger counter, a hammer, poison in a glass bottle, and a living cat. When the nucleus decays and a radiation occurs, the geiger counter will sense it, then the hammer will crash the poison bottle, and the
cat will die. When the nucleus does not decay, the cat won’t die. There, each of the nuclear states corresponds to each of the cat’s states respectively.

原子核の2つの量子状態はユニタリに発展し、そしてそれらの状態が自発的に消えてしまう事はない。それから、核は2つの異なる状態を同時に持たなくてはならず、そしてそれらの状態もまたユニタリに発展しなくてはならない。

The nuclear two quantum states evolve unitary, and the states will never vanish spontaneously, therefore the cat has to have two different states simultaneously and the states have to evolve unitary, too.

\[
\psi(t) = |\psi_{\text{not decay}}(t)\rangle + |\psi_{\text{decay}}(t)\rangle, \quad (1.1)
\]

Cat’s states:

\[
\phi(t) = |\phi_{\text{alive}}(t)\rangle + |\phi_{\text{dead}}(t)\rangle \quad (1.2)
\]

This story conflicts with the fact that “there has been no cat whose state is superposition of living and dead ever in my life.” In Copenhagen interpretation, the cat’s wave function collapses at the moment someone watches the cat, and the state of cat defines uniquely. But while the cat is in a box, he can’t be observed by anyone and his states can’t be collapsed by observations. Is the cat in the box in the two contradictorily different state at the same time?

Originally, Schrödinger thought this story as a criticism for attempts to apply the quantum mechanics to our macroscopic world, but ironically this story has led us the theory of quantum to classical.

Now, we know decoherence. We can understand this paradox another angle. That is, the states of cat in a box had been already collapsed by environmental effects such as thermal fluctuation by room temperature, regardless of our observations.

1.2. Many Worlds interpretation.

エヴェレットの多世界解釈は量子物理学にとって、コペンハーゲン解釈と同じ位に有名である。コペンハーゲン解釈では、私達は系の現実の状態を選ぶために測定行為を必要とする。言い換えると、私達は量子の古典的理論のために「波動関数の収縮」を必要とする。

Everett’s many worlds interpretation is as famous as the Copenhagen interpretation for quantum physics. In the Copenhagen interpretation, we need measurements to select a real state of the system, in other words, we need a “Collapse of wave
function” for our classical world.

一方で、エヴェレットの解釈では、私達がひとつの物体に興味があるとき、その物体の状態ベクトルと同じ状態を、私達の住む世界が存在すると仮定する。そして私達はただ一つの世界しか感じることができないが、そこでは力学は古典的であり、そして対象となる物体の状態はただ一つに決まっている。しかし他の世界との量子干渉が、物体他の世界に移すことを可能にし、そして量子効果が現れ、それゆえにこの解釈では、私達は波動関数の収縮のメカニズムで心が必要な、さらには、対象となる物体によって古典力学が量子力学を人為的に選ぶ必要も無い。

While, in Everet’s interpretation, when we interested in an object, we assumed that there are our living worlds as many as the number of the object’s state vectors. And we can sense only one world, where the mechanics is classical and the object’s state is defined uniformly. But the quantum interference with the other worlds makes it possible to transfer of the object to the other worlds, then the quantum effects are reproduced. Therefore in this interpretation, we don’t need to concern about the mechanism for collapse of wave function. Moreover, we don’t need any artificial choice of mechanics from classical or quantum depending on the object.

しかしこの解釈は、私達が住んでいる世界とは異なった数多くの量子世界が存在しなくてはならないことを意味している。なぜ私達が他の世界へと移れないのか。という疑問に対しては、例えば「私達は莫大な数の粒子から構成されていて、そして私達の大きさは空間的に大きい。だから私達は他の量子世界に移れないんだ。」と言える事ができるかも知れない。それでも、いくつかの疑問が残る。「なぜ私達は他の量子世界からの情報は何一つ得ることができないのか。」「なぜ私達の意識はこの世界だけにしか存在しないのか。」そして「なぜ私の意識はこの世界の私の身体を選んだのか。」

But this interpretation means that there must be a lot of quantum worlds different from the world we are living in. For a question why we can’t transfer to the other worlds, we may answer for example “Because we are composed of a huge huge numbers of particles, and our sizes are spatially large. Therefore we can’t transfer to the other quantum worlds.” Still, some questions remain, “Why can’t we get any information from the other quantum worlds?”, “Why are our consciousnesses in this world only?” and “Why did my consciousness select my body in this world?”

今日では、私達は量子デコヒーレンス現象を知っている。それは量子状態間の干渉の破壊であり、一種の非ユニタリ過程として知られている。デコヒーレンスはコペンハーゲン解釈での「波動関数の収縮」に対応している。そして多世界解釈では、デコヒーレンスは量子宇宙間の干渉を断ち切る役割を演ずる。デコヒーレンスはどちらの解釈に対しても有効であり、どちらかの解釈を好むわけではない事を記しておく。

Today, we know the quantum decoherence phenomena, which is the destruction of interferences among quantum states, and known to be a kind of non-unitary processes. Decoherence corresponds to “the collapse of wave functions” in Copenhagen interpretation. And in many worlds interpretation, decoherence plays a role of the cutting of interferences among the quantum multiverses. Note that decoherence is valid for both interpretations and does not favor one interpretation of them over another.
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But, the serious problem in the many worlds interpretation remains, “The universe, as the largest Hamiltonian system in our world, must have so many state vectors, and there is no external elements which destroys the interference among state vectors of the universe.”

If we could concern about a Hilbert space for whole particles in the universe, there should be an infinite number of state vectors of the universe. But, remember that we are not interested in the whole microscopic degrees of freedom in our universe. Unfortunately, we are not able to observe the whole degrees of freedom in our universe, therefore we may treat only a small numbers of macroscopic degrees of freedom.

“Macroscopic” doesn’t mean spatial scale, it would mean there is some coarse graining or projection. The macroscopic degrees of freedom would be defined by some reduction of irrelevant microscopic degrees of freedom. Whether consciously or unconsciously, we may neglect most of the microscopic degrees of freedom when we look up at the starry starry sky!

When we interested in the macroscopic universe only, the important question is that, “Is there any factor to cut quantum interferences among the macroscopic quantum universes?”

Case1. There is no such factor.

There are interferences among the macroscopic quantum universes, but their changes by quantum interferences may be very small. Therefore we can’t be aware of them. There is no superposition among widely different states of universe.

Case2. Environmental effects.
Here, the word “environment” means the irrelevant microscopic degrees of freedom neglected. Note that when you wish upon a star, you only care about the macroscopic (collective) degrees of freedom inside our universe.

Case 3. Spontaneous Selection.

The macroscopic degrees of freedom may get classical unity spontaneously without any environments.

Eventually, it is natural that the wavefunction of our macroscopic universe has already collapsed for such reasons above. There wouldn’t be any macroscopic quantum parallel worlds, because we have never been transported to any other quantum universes.

In this paper, an idea like Case 2 is investigated for an isolated quantum system with 3 degrees of freedom. A harmonic oscillator coupled with other 2 harmonic oscillators would get classicality. There the latter 2 oscillators are coarse grained, in other words, they are treated as environments. It is showed that the environmental effect destroys the quantum mechanical property of the main 1 harmonic oscillator.

1.3. Quantum Mine Sweeper and Decoherence at the nuclear fission.

Do the nuclear states “The nuclear fission has done.” and “It has not.” really keeps their superposition until anyone observes the fission products?
Without a direct observation, we can observe only the incoming fission products at far from an unstable nucleus, for example, we are on the planet Pluto and the nucleus is on the Earth. If we observe some fission products, then we will know that the fission has done, and the nuclear state reduces into “The fission has done.” On the other hand, if we can not observe any fission products, the state would also reduce into “It has not.” In the latter case, clearly there is no interaction between the observer and the nucleus. How should we understand this problem?

This question was my motivation for this study. If the collapse of the linear combination of state vectors, so called the Quantum Decoherence, occurs in the nuclear fission process, this problem would be solved. Can we assume that the relevant degrees of freedom at the fission would get a classicality because of the fluctuation from the other (irrelevant) degrees of freedom? Our study in this paper will show you a quantum decoherence in a finite system like nucleus.

♦
§2. About Decoherence.

Quantum decoherence is a kind of non-unitary processes, which means the dissappearance of quantum interference. Its causes are dissipations and random fluctuations. They are the same as the additional terms of the Langevin equation. In fact, it is known that these terms of the quantum mechanical Langevin equation destroys quantum interferences.

In old days, it was a problem that the Langevin equation could not be derived in canonical procedures. The failure to derive the time irreversible Langevin equation from the time reversible Newtonian or Hamiltonian equation is the same as the failure to derive non-unitary “The collapse of the wave function” from the unitary Schrödinger equations.

Remember that energy dissipations, random fluctuations and observations are irreversible processes. Then, it is assumed that there is any relationship between the irreversibility (the break down of the time reversal symmetry), and the quantum decoherence (the break down of unitarity) .

A wave function evolves by $\hat{U}$ as follows.

$$\psi(t) = \hat{U}\psi(0) \quad (2.1)$$

Ordinary, we assume the operator $\hat{U}$ as a unitary operator. But, here we dare to allow the case $\hat{U}$ is not unitary.

While, please imagine the evolution operator $\tilde{U}$ for time reversal world, such as time $(t \rightarrow 0)$ . For example,

$$\tilde{\psi}(0) = \tilde{U}\psi(t) \quad (2.2)$$

Here, the state $\tilde{\psi}(0)$ is the virtual state $\psi(0)$ and same is true.
Here, the state $\bar{\psi}(0)$ is not need to be the same as the original state $\psi(0)$, that is,

$$\bar{\psi}(0) = \bar{U} \hat{U} \psi(0) \quad (2.3)$$

When the Hamiltonian of system is $\hat{H}$, and that of the time reversal world is $\bar{H}$, evolution operators for each are

$$\hat{U} = \exp \left\{ -i \frac{\hbar}{\hbar} \hat{H} t \right\}, \quad \bar{U} = \exp \left\{ +i \frac{\hbar}{\hbar} \bar{H} t \right\} \quad (2.4)$$

Then, using the Baker-Campbell-Hausdorff formula,

$$\bar{U} \hat{U} = \exp \left\{ -i \frac{\hbar}{\hbar} (\hat{H} - \bar{H}) t + \frac{1}{2\hbar^2} [\hat{H}, \bar{H}] t^2 + O(t^3) + \cdots \right\} \quad (2.5)$$

Therefore, the sufficient condition for being ($\bar{U} \hat{U} = 1$) or reversible system is being ($\hat{H} = \bar{H}$).

On the other hand, the complex conjugate of eq.(2.1) and eq.(2.4)

$$\psi^*(t) = \hat{U}^\dagger \psi^*(0), \quad \hat{U}^\dagger = \exp \left\{ +i \frac{\hbar}{\hbar} \hat{H} \right\} \quad (2.6)$$

Then

$$\hat{U}^\dagger \hat{U} = \exp \left\{ -i \frac{\hbar}{\hbar} (\hat{H} - \hat{H}^\dagger) t + \frac{1}{2\hbar^2} [\hat{H}^\dagger, \hat{H}] t^2 + O(t^3) + \cdots \right\} \quad (2.7)$$
それはゆえに，\((\hat{U}^\dagger \hat{U} = 1)\) または ユニタリであるための十分条件は，ハミルトニア
がエルミート（\(H^\dagger = H\)）である事である。

Therefore the sufficient condition for \((\hat{U}^\dagger \hat{U} = 1)\) or unitary，is that the Hamiltonian
is Hermitian，\((\hat{H}^\dagger = \hat{H})\)。

今，私達は不可逆性（時間反転対称性的破れ）とユニタリ性の消滅（デコヒーレンス）の関係を知りたいと思っている。上記の２つの議論を表に示してみよう。

Now we want to know the relationships between the irreversibility (the break down of time reversal symmetry) and the disappearance of unitarity (the quantum decoherence). Let us show the list of two discussions above.

\[
\begin{align*}
\hat{H} = \hat{H} \quad &\text{（time reversal symmetry）} \\
\hat{H}^\dagger = \hat{H} \quad &\text{（Hermitian）}
\end{align*}
\]
\[
\Rightarrow \quad \hat{U} \hat{U} \left( \text{reversibility} \right) \\
\hat{U}^\dagger \hat{U} \left( \text{unitarity} \right)
\]

\[
\{\hat{U} \neq \hat{H}\} \subset \{\hat{H} \neq \hat{H}\}
\]

この表から見れば，可逆性とユニタリ性には関係が無い様に思える。

It seems that there is no relation between the reversibility and the unitarity from this list.

式（2.8）の下の命題は，次の様に書き換えられる。

The lower proposition of eq.(2.8) is rewritten as follows.

\[
\{\hat{U} \neq \hat{H}\} \subset \{\hat{H} \neq \hat{H}\}
\]

ここで，\{…\} はベン図でのそれぞれの事象の表す領域を意味する。ドモルガン
の法則を用いて，私達は次を得る。

Here，\{…\} means the region of each event in Venn diagram. Using the De Morgan’s low，we get

\[
\{\hat{U} \neq \hat{H}\} \subset \{\hat{H} \neq \hat{H}\}
\]

多くの理論的そして実験的研究は，時間反転対称性が破れるような系においてユニ
タリ性が失われる事を示唆している。

Many theoretical and experimental studies imply that unitarity is lost in systems in which
the time reversal symmetry breaks down.

\[
\hat{U}^\dagger \hat{U} \neq 1 \quad \Rightarrow \quad \hat{H} = \hat{H}
\]

すなわち

that is

\[
\{\hat{U} \neq \hat{H}\} \subset \{\hat{H} \neq \hat{H}\}
\]

（この包含関係は散逸系におけるユニタリ性の可能性を考えている。）この関係は
上記の議論からは導けない。

（This inclusion relation is concerned about possibility of unitary dissipation systems．）This relation can’t be derived from discussions above.

しかし，もし私達がエルミート共役の意味を時間反転共役として受け入れるならば，
この難しさは避けられる。
But, if we accept a meaning for Hermitian conjugate as the time reversal conjugate, this difficulty is avoided.

\[
 \hat{H}^\dagger = \overline{\hat{H}}, \quad \hat{U}^\dagger = \overline{\hat{U}} \tag{2.13}
\]

おそらく、私が読んだことがあるいくつかの本で、複素共役と時間反転共役の関係が示唆されていた。例えば、

Probably, the relation between complex conjugate and time reversal conjugate has been implied at some textbooks which I have ever read. For example,

\[
 \psi(t') = \exp \left\{ -\frac{i}{\hbar} \hat{H}(t' - t) \right\} \psi(t) \tag{2.14}
\]

この複素共役は、

Its complex conjugate is

\[
 \psi^*(t') = \exp \left\{ +\frac{i}{\hbar} \hat{H}^\dagger(t' - t) \right\} \psi^*(t) \tag{2.15}
\]

その一方で、その時間反転 \((t' - t \to t - t')\) したものは、

While, the time reversal \((t' - t \to t - t')\) one is

\[
 \bar{\psi}(t') = \exp \left\{ +\frac{i}{\hbar} \bar{\hat{H}}(t' - t) \right\} \bar{\psi}(t) \tag{2.16}
\]

\(\hat{H}\) がエルミートで時間反転対称の場合、式 (2.15) と式 (2.16) はよく似ているため、それゆえにそれらは同一であるかも知れない。

When \(\hat{H}\) is Hermitian and time reversal symmetric, eq.(2.15) and eq.(2.16) are very alike, therefore they may be identical each other.

\[
 \psi^*(t') = \bar{\psi}(t'), \psi^*(t) = \bar{\psi}(t) \tag{2.17}
\]

すなわち、波動関数の複素共役はその時間反転共役を意味するだろう。私達が式 (2.13) を受け入れた時、私達は不可逆系に対する波動関数を定義出来るかも知れない。

That is, the complex conjugate of wave functions would mean its time reversal conjugate. When we can accept eq.(2.13), we may define the wave functions for irreversible systems.

私達が式 (2.13) の関係を受け入れた時、経験的な式 (2.11) は簡単に導出される。

When we accept the relation eq.(2.13), the empirical formula eq.(2.11) is easily derived.

\[
 \hat{U}^\dagger \hat{U} = \exp \left\{ +\frac{i}{\hbar} \hat{H}^\dagger t \right\} \exp \left\{ -\frac{i}{\hbar} \hat{H} t \right\} = \exp \left\{ +\frac{i}{\hbar} \hat{H} t \right\} \exp \left\{ -\frac{i}{\hbar} \hat{H} t \right\} \tag{2.18}
\]

そして

Then,

\[
 \hat{H} = \hat{H} \quad \Rightarrow \quad \hat{U}^\dagger \hat{U} = 1 \tag{2.19}
\]
This means that the sufficient condition for unitarity of system is the time reversal symmetry of Hamiltonian. Taking the contraposition, we get

$$\hat{U} \hat{U}^\dagger \neq 1 \implies \hat{H} \neq \hat{\dot{H}}$$

(2.20)

which means that the necessary condition for quantum decoherence is the breaking down of time reversal symmetry of Hamiltonian.

In classical system, it is known that irreversibility appears by coarse graining. Averaging procedures destroy microscopic informations for system. It is said that Ergodicity also appears by coarse graining.

projection is a kind of Projection procedure into a low resolution world, so we can assume that Projection makes the system irreversible. It is known that the Langevin equation is also derived by the projection method.

Projection is a procedure to extract arbitrary sub system from a original system. When the sub system couples with the other neglected (integrated) degrees of freedom in the original system, the coupling effects would be regarded as “the environmental effects” for the sub system.
§3. A simple model: Asymmetric triangular harmonic oscillators in Schrödinger cat states, “Three Schrödinger cats”.

Here, we will use a simple model to discuss a possibility of quantum decoherence in a finite system. That is three bosonic particles tied up each other with three springs which have different frequencies. For this model, we will apply the Caldeira & Leggett’s technique. Our three particle model is a extreme reduction of their “harmonic oscillator plus reservoir” model.

![Fig. 2. 3cats model. 3 particles are tied up with springs. And each state is Schrödinger cat.]

The main line is, to transform 1-dimensional three particles tied with different springs into two uncoupled harmonic oscillators and one free particle in normal coordinates. And we will derive a propagator for a total wave function of the total system in normal coordinates, and retranslate it into the original system. Then we will be able to get a propagator for harmonically bound three particle. Next, we will prepare the respective initial wave function for each particle as a pair of Gaussian wave packet (the Schrödinger cat state). The initial state of the total system is a product of those states. It will start to turn into an entangle state of three particles by propagator.

In a closed system, it is difficult to suppose that the interference between the
wave functions of the closed system will vanish. But we will pay attention to the particle-1 only and integrate out the degrees of freedom about other two particles, then we will get a reduced density function for particle-1. This procedure corresponds to ignoring the fine informations about other two particles and taking an average. Then, we will observe changes of a pair of Gaussian wave packets of particle-1 and the quantum interference term between them.

3.1. Classical model.

古典的には、この模型（非対称な調和三体問題）は可解、すなわち可積分系である。

Classically, this model (the asymmetric harmonic three body problem) is solvable, that is a integrable system. Its Lagrangean is as follows.

\[
L = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 + \frac{m}{2}\dot{x}_3^2 - \frac{m}{2}\omega_{12}^2(x_1-x_2)^2 - \frac{m}{2}\omega_{13}^2(x_1-x_3)^2 - \frac{m}{2}\omega_{23}^2(x_2-x_3)^2 \tag{3.1}
\]

オイラー・ラグランジュ方程式

Applying the Euler-Lagrange equation

\[
\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{3.2}
\]

to eq.(3.1), we can get three equations of motion.

\[
\frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\omega_{12}^2 + \omega_{13}^2 & \omega_{12}^2 & \omega_{13}^2 \\ \omega_{12}^2 & -\omega_{12}^2 + \omega_{23}^2 & \omega_{23}^2 \\ \omega_{13}^2 & \omega_{23}^2 & -\omega_{13}^2 + \omega_{23}^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.3}
\]

これを書き換えて、

Rewriting this,

\[
\frac{d^2}{dt^2} X(t) = WX(t) \tag{3.4}
\]

そして時間に依存しない行列 \( P \) を用いると、

and using a time-independent matrix \( P \), then,

\[
\frac{d^2}{dt^2} (PX(t)) = PW P^{-1} (PX(t)) \tag{3.5}
\]

ここで、私達は \( PW P^{-1} \) が対角行列になるように選ぶ事が出来て、そして 3 本の連立してない微分方程式を得る。私達はこれを示す、\( W \) の固有値、\( \lambda \) は次の方程式を満たす。

Here, we can set \( P \) in order that \( PW P^{-1} \) is diagonal, and get three uncoupled differential equations. We will show this procedure. Eigen values of \( W \), \( \lambda \) satisfy the equation

\[
- \{ \lambda^3 + 2(\omega_{12}^2 + \omega_{13}^2 + \omega_{23}^2)\lambda^2 + 3(\omega_{12}^2\omega_{13}^2 + \omega_{13}^2\omega_{23}^2 + \omega_{12}^2\omega_{23}^2)\lambda \} \equiv 0 \tag{3.6}
\]
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We call its 3 solutions \( \lambda_1, \lambda_2, \lambda_3 \) and obviously 0 is one solution, then we set it \( \lambda_3 \). And we define

\[
P W P^{-1} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \Lambda \tag{3.7}
\]

\[
PX(t) \equiv Z(t) \tag{3.8}
\]

Then, equation (3.5) comes to

\[
d^2 dt^2 Z(t) = \Lambda Z(t) \tag{3.9}
\]

すなわち，次のようなく3本の独立な微分方程式を得る事ができる，

, that is, we can get three independent differential equations as follows.

\[
\frac{d^2}{dt^2} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ 0 \end{pmatrix} \tag{3.10}
\]

ここで私達は次の定義する。

Here we define

\[
\Delta \omega^2 \equiv \sqrt{\omega_{12}^2 - \omega_{12}^2 \omega_{13}^2 + \omega_{13}^4 - \omega_{13}^2 \omega_{23}^2 + \omega_{23}^4 - \omega_{23}^2 \omega_{12}^2} = \sqrt{\omega_{12}^2 + \omega_{13}^2 + \omega_{23}^2} \tag{3.11}
\]

そして式 (3.6) より，\( \lambda_1, \lambda_2 \) は次の様になる。

and from equation (3.6), \( \lambda_1, \lambda_2 \) are

\[
\begin{cases}
\lambda_1 = -\omega_{12}^2 - \omega_{13}^2 - \omega_{23}^2 + \Delta \omega^2 < 0 \\
\lambda_2 = -\omega_{12}^2 - \omega_{13}^2 - \omega_{23}^2 - \Delta \omega^2 < 0 \tag{3.12}
\end{cases}
\]

それらの符号に気を付けると，私達は式 (3.10) を解くことが出来て，次を得る。

Taking care of their signs, we can solve the equation (3.10) then we get

\[
\begin{align*}
z_{1(t)} &= A_1 \sin \Omega_1 t + B_1 \cos \Omega_1 t \\
z_{2(t)} &= A_2 \sin \Omega_2 t + B_2 \cos \Omega_2 t \\
z_{3(t)} &= C_1 t + C_2 \tag{3.13}
\end{align*}
\]
ここで、

\[ \Omega_1 \equiv \sqrt{-\lambda_1}, \ \Omega_2 \equiv \sqrt{-\lambda_2} \] (3.14)

そして \( A_1, A_2, B_1, B_2, C_1, C_2 \) は積分定数である。それらは \( X \) や \( \dot{X} \) の初期条件に依存する。古典軌道 \( X(t) \) は次の様になる。

\[ X(t) = P^{-1}Z_t \] (3.15)

上記の式を用いて、私達は最終的に \( X(t) \) の古典解を得る。

Using formulae above, we can get the classical solution of \( X(t) \) finally.

\[ X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} \xi_1 z_1(t) + \xi_2 z_2(t) + z_3(t) \\ \eta_1 z_1(t) + \eta_2 z_2(t) + z_3(t) \\ - (\xi_1 + \eta_1) z_1(t) - (\xi_2 + \eta_2) z_2(t) + z_3(t) \end{pmatrix} \] (3.18)

\[ \Delta \equiv \eta_2 \xi_1 - \eta_1 \xi_2 \] (3.19)

を用いて、\( P \) は次の様に書ける。

\[ P = \frac{1}{3 \Delta} \begin{pmatrix} 2 \eta_2 + \xi_2 & -\eta_2 - 2 \xi_2 & -\eta_2 + \xi_2 \\ -2 \eta_1 - \xi_1 & \eta_1 + 2 \xi_1 & \eta_1 - \xi_1 \\ \Delta & \Delta & \Delta \end{pmatrix} \] (3.20)
From the relation
\[ Z(t) = PX(t) \] (3.21)

これらの式は、後の経路積分の計算にとても有用である。

These formulae are very useful for evaluation of path integrals later.

### 3.2. Derivation of a propagator

この節では、私達はこの3体模型に対するファインマンプロパゲーターを導出する。それは微分演算子を用いず、波動関数の時間発展を記述する。元の座標系 \( X \) でのプロパゲーターを導出する事は難しいため、私達は元の系でのラグランジアンを基準座標系 \( Z \) でのラグランジアンに変換する。基準座標系では、2つの独立な振動子と1つの自由粒子が存在する。その変換式は式(3.18)と式(3.22)に与えられている。それらを元のラグランジアンの式(3.1)に代入すると、私達は次を得る。

In this section, we derive the Feynman propagator for this 3 body model. It describes evolution of wave functions without differential operators. It is difficult to derive the propagator in the original coordinates \( X \), therefore we transform the Lagrangean in the original coordinates into the Lagrangean in the normal coordinates \( Z \), where there are two uncoupled harmonic oscillators and a free particle. Its transformation formulae are given in eq.\((3.18)\) and eq.\((3.22)\). We substitute them for the original Lagrangean eq.\((3.1)\), then we get

\[ L = \frac{m_1}{2}z_1^2(t) + \frac{m_2}{2}z_2^2(t) + \frac{m_3}{2}z_3^2(t) - \frac{m_1}{2}\omega_1^2z_1^2(t) - \frac{m_2}{2}\omega_2^2z_2^2(t) \] (3.23)

ここで,

where

\[ m_1 \equiv 2m(\xi_1^2 + \xi_1\eta_1 + \eta_1^2), \quad m_2 \equiv 2m(\xi_2^2 + \xi_2\eta_2 + \eta_2^2), \quad m_3 \equiv 3m \] (3.24)

\[ \begin{align*}
\omega_1^2 &\equiv \frac{m_1}{m_2}\{ w_1^2(2\xi_1^2 - \xi_1\eta_1 - \eta_1^2) + w_2^2(-\xi_1^2 - \xi_1\eta_1 + 2\eta_1^2) + w_3^2(2\xi_1^2 + 5\xi_1\eta_1 + 2\eta_1^2) \} \\
\omega_2^2 &\equiv \frac{m_1}{m_3}\{ w_1^2(2\xi_2^2 - \xi_2\eta_2 - \eta_2^2) + w_2^2(-\xi_2^2 - \xi_2\eta_2 + 2\eta_2^2) + w_3^2(2\xi_2^2 + 5\xi_2\eta_2 + 2\eta_2^2) \} \\
&\quad \times (w_1^2 = \omega_1^2 + \omega_3^2, \quad w_2^2 = \omega_{12}^2 + \omega_{23}^2, \quad w_3^2 = \omega_{13}^2 + \omega_{23}^2) \end{align*} \] (3.25)

である。私達はそれぞれの変数ごとにラグランジアンを分離できて、

we can decouple the Lagrangean with each variable.

\[ L_1(z_1(t), t) \equiv \frac{m_1}{2}\omega_1^2z_1^2(t) - \frac{m_1}{2}\omega_1^2z_1^2(t), \quad L_2(z_2(t), t) \equiv \frac{m_2}{2}\omega_2^2z_2^2(t) - \frac{m_2}{2}\omega_2^2z_2^2(t), \]

\[ L_3(z_3(t), t) \equiv \frac{m_3}{2}\omega_3^2z_3^2(t) : L = L_1 + L_2 + L_3 \] (3.26)
For these Lagrangeans, we can get the classical action integrals summed up from an initial time $t_0$ to an arbitrary time $t$.

$$S^{(cl)}(Z(t), t : Z(t_0), t_0) = \int_{t_0}^{t} L_1(\tau)d\tau + \int_{t_0}^{t} L_2(\tau)d\tau + \int_{t_0}^{t} L_3(\tau)d\tau$$

$$\equiv S_1^{(cl)}(z_1(t), t : z_1(t_0), t_0) + S_2^{(cl)}(z_2(t), t : z_2(t_0), t_0) + S_3^{(cl)}(z_3(t), t : z_3(t_0), t_0)$$

where

$$S_1^{(cl)} = \frac{m_{1}\omega_1}{2\sin\omega_1(t - t_0)} \left\{ \cos\omega_1(t - t_0)(z_1^2(t) + z_1^2(t_0)) - 2z_1(t)z_1(t_0) \right\}$$

$$S_2^{(cl)} = \frac{m_{2}\omega_2}{2\sin\omega_2(t - t_0)} \left\{ \cos\omega_2(t - t_0)(z_2^2(t) + z_2^2(t_0)) - 2z_2(t)z_2(t_0) \right\}$$

$$S_3^{(cl)} = \frac{m_{3}(z_3(t) - z_3(t_0))^2}{2(t - t_0)}$$

Using these formulae, we get the propagator for wave function in the system $Z$.

$$U(Z, t : Z_0, t_0) = \int_{Z(t_0) = Z_0}^{Z(t) = Z} DZ(\tau) \exp \left\{ \frac{i}{\hbar} S(Z, t : Z(\tau), \tau : Z_0, t_0) \right\} \propto \exp \left\{ \frac{i}{\hbar} S^{(cl)}(Z, t : Z_0, t_0) \right\}$$

Here $DZ \equiv Dz_1Dz_2Dz_3$ means path integrals about three variables $(z_1, z_2, z_3)$ in system $Z$. The action integral $S(Z, t : Z(\tau), \tau : Z_0, t_0)$ depends on its integral paths and does not always follow the principle of minimum action. But it is known that the result of path integrals is in proportion to the value of saddle point of their integrands for free particles and for harmonic oscillators, therefore we get equation (3.32). Its proportional factor depends on the initial time $t_0$ and the final time $t$, but here we omit the factor.
Now we assume that the propagator in $Z$ is equivalent to the one in original system $X$, and transform it using equation (3.18). Then we get

$$U(X, t : X_0, t_0) = U(Z, t : Z_0, t_0) \propto \exp \left[ \frac{i}{\hbar} \left\{ A_1 x_1^2(0) + A_2 x_2^2(0) + A_3 x_3^2(0) + B_{12} x_1(0) x_2(0) \\
+ B_{23} x_2(0) x_3(0) + B_{13} x_1(0) x_3(0) + C_1 x_1(0) + C_2 x_2(0) + C_3 x_3(0) + D \right\} \right]$$

(3.33)

Here using elements of the transformation matrix $P$

$$a_1 = \frac{1}{\sqrt{3}} (2\eta_2 + \xi_2) \quad a_2 = \frac{1}{\sqrt{3}} (-\eta_2 - 2\xi_2) \quad a_3 = \frac{1}{\sqrt{3}} (-\eta_2 + \xi_2)$$

$$b_1 = \frac{1}{\sqrt{3}} (-2\eta_1 - \xi_1) \quad b_2 = \frac{1}{\sqrt{3}} (\eta_1 + 2\xi_1) \quad b_3 = \frac{1}{\sqrt{3}} (\eta_1 - \xi_1)$$

(3.34)

$(\Delta = \eta_2 \xi_1 - \eta_1 \xi_2)$ を用いて、式(3.33)内の実数係数 $A_1 - D$ が次のように得られる。

$(\Delta = \eta_2 \xi_1 - \eta_1 \xi_2)$, we get the real coefficients in (3.33) $A_1 - D$ as follows.

$$A_i = \frac{m_1 \omega_1}{2} \cot[\omega_1(t - t_0)] a_i^2 + \frac{m_2 \omega_2}{2} \cot[\omega_2(t - t_0)] b_i^2 + \frac{m_3}{2(t - t_0)} c_i^2 \quad (3.35)$$

$$B_{ij} = m_1 \omega_1 \cot[\omega_1(t - t_0)] a_i a_j + m_2 \omega_2 \cot[\omega_2(t - t_0)] b_i b_j + \frac{m_3}{(t - t_0)} c_i c_j \quad (3.36)$$

$$C_i(X) = -\frac{m_1 \omega_1}{\sin[\omega_1(t - t_0)]}(a_1 x_1 + a_2 x_2 + a_3 x_3) a_i$$

$$-\frac{m_2 \omega_2}{\sin[\omega_2(t - t_0)]}(b_1 x_1 + b_2 x_2 + b_3 x_3) b_i$$

$$-\frac{m_3}{(t - t_0)} (c_1 x_1 + c_2 x_2 + c_3 x_3) c_i \quad (3.37)$$

$$D(X) = \frac{m_1 \omega_1}{2} \cot[\omega_1(t - t_0)](a_1 x_1 + a_2 x_2 + a_3 x_3)^2$$

$$+ \frac{m_2 \omega_2}{2} \cot[\omega_2(t - t_0)](b_1 x_1 + b_2 x_2 + b_3 x_3)^2$$

$$+ \frac{m_3}{2(t - t_0)} (c_1 x_1 + c_2 x_2 + c_3 x_3)^2 \quad (3.38)$$

$(i, j = 1, 2, 3)$ であり、添字 $(t)$ を省略した。すなわち、$(x_i(t) = x_i)$ である。

$(i, j = 1, 2, 3)$, and we omitted index $(t)$, that is $(x_i(t) = x_i)$.

3.3. Derivation of wave function and numerical calculation of reduced density function.

このプロパゲーターを用いて、私達は波動関数の時間発展を次の様に示す事ができる。

Using this propagator, we can write the development of wave function as follows.

$$\psi(X, t) = \int_{-\infty}^{\infty} dX_0 \ U(X, t : X_0, t_0) \ \psi(X_0, t_0)$$

(3.39)
The initial wave function for our 3 body system is a product of wave functions of each particle at the time $t_0$.

$$\psi(X_0, t_0) = \psi_1(x_1(0), t_0) \psi_2(x_2(0), t_0) \psi_3(x_3(0), t_0)$$ (3.40)

This means that there has been no interaction among those 3 particles until the initial time $t_0$. And the initial state is the Schrödinger cat state

$$\psi_1(x_1(0), t_0) = \tilde{N}_1 \left[ \exp \left\{ -\frac{x_1^2(0)}{4\sigma_1^2} \right\} + \exp \left\{ -\frac{(x_1(0) - d_1)^2}{4\sigma_1^2} \right\} \right] + \text{etc...}$$ (3.41)

With equations (3.33), (3.40), (3.41), equation (3.39) comes to

$$\psi(X,t) \propto \int_{-\infty}^{\infty} dx_1(0) \int_{-\infty}^{\infty} dx_2(0) \int_{-\infty}^{\infty} dx_3(0) \times \exp \left\{ i \left( A_1 2_1^2 + A_2 2_2^2 + A_3 2_3^2 + B_{12} x_1(0) x_2(0) + B_{23} x_2(0) x_3(0) + B_{13} x_1(0) x_3(0) + C_1 x_1(0) + C_2 x_2(0) + C_3 x_3(0) + D \right) \right\} \times \left[ \exp \left\{ -\frac{x_1^2(0)}{4\sigma_1^2} - \frac{x_2^2(0)}{4\sigma_2^2} - \frac{x_3^2(0)}{4\sigma_3^2} \right\} + \exp \left\{ -\frac{(x_1(0) - d_1)^2}{4\sigma_1^2} - \frac{x_2^2(0)}{4\sigma_2^2} - \frac{x_3^2(0)}{4\sigma_3^2} \right\} + \text{etc...} \right\}$$ (3.42)

The latter \[\ldots\] of this formula means 8 Gaussian packets in $(x_1, x_2, x_3)$ space. Each packet changes by propagator. For evaluating analytic forms of those packets, integrations with three variables $(x_1(0), x_2(0), x_3(0))$ are needed. With the new complex coefficients $\tilde{A}_1 - \tilde{D}$, eq. (3.42) turns into

$$\psi(X,t) = \sum_{k=0}^{7} \psi^{(k)}(X,t)$$ (3.43)
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\[ \times \exp \left[ -\tilde{A}_1 x_1^2(0) - \tilde{A}_2 x_2^2(0) - \tilde{A}_3 x_3^2(0) + \tilde{B}_{12} x_1(0) x_2(0) + \tilde{B}_{23} x_2(0) x_3(0) \right. 
\left. + \tilde{B}_{13} x_1(0) x_3(0) + \tilde{C}_{1}^{(k)} x_1(0) + \tilde{C}_{2}^{(k)} x_2(0) + \tilde{C}_{3}^{(k)} x_3(0) + \tilde{D}^{(k)} \right] \] (3.44)

ここで \((k = 0 - 7)\) であり、それは私達が解るべきガウス積分である。

\((k = 0 - 7)\), which is the Gaussian integrals we have to solve.

\[ \tilde{A}_1 = \frac{1}{4\sigma_1^2} - \frac{i}{\hbar} A_1, \quad \tilde{B}_{12} = \frac{i}{\hbar} B_{12}, \quad \tilde{C}_{1}^{(k)}(X) = \frac{d_{1}^{(k)}}{2\sigma_1^2} + \frac{i}{\hbar} C_1(X) \text{ etc...} \]

\[ \tilde{D}^{(k)}(X) = -\frac{d_{1}^{(k)^2}}{4\sigma_1^2} - \frac{d_{2}^{(k)^2}}{4\sigma_2^2} - \frac{d_{3}^{(k)^2}}{4\sigma_3^2} + \frac{i}{\hbar} D(X) \] (3.45)

ここで

\[
\begin{pmatrix}
  k = 0 & [ d_{1}^{(k)} & d_{2}^{(k)} & d_{3}^{(k)} ] \\
  1 & [ 0 & 0 & 0 ]
\end{pmatrix}
\[
\begin{pmatrix}
  1 & [ d_{1} & 0 & d_{3} ] \\
  2 & [ 0 & d_{2} & 0 ]
\end{pmatrix}
\[
\begin{pmatrix}
  3 & [ 0 & 0 & d_{3} ]
\end{pmatrix}
\]

このガウス積分を評価することで、私達は次のを得る。

Evaluating this Gaussian integrals, we get

\[ \psi^{(k)}(X, t) \propto \sqrt{\frac{\pi^3}{\Delta}} \exp \left[ \frac{1}{16 \Delta} \tilde{D}^{(k)}(X, t) + \tilde{D}^{(k)}(X, t) \right] \] (3.47)

ここで

\[
\Delta(t) = \tilde{A}_1 \tilde{A}_2 \tilde{A}_3 - \frac{1}{4} (\tilde{A}_2 \tilde{B}_{13}^2 + \tilde{A}_3 \tilde{B}_{12}^2 + \tilde{A}_1 \tilde{B}_{23}^2) - \frac{1}{4} \tilde{B}_{12} \tilde{B}_{13} \tilde{B}_{23} \] (3.48)

\[ \equiv \text{Re} \Delta(t) + i \text{Im} \Delta(t) \]

\[ \text{Re} \Delta(t) = \frac{1}{4 \sigma_1^2 \sigma_2^2 \sigma_3^2} - \frac{1}{4 \hbar^2} \left( \frac{A_2 A_3}{\sigma_1^2} + \frac{A_3 A_1}{\sigma_2^2} + \frac{A_1 A_2}{\sigma_3^2} \right) \]

\[ + \frac{1}{16 \hbar^2} \left( \frac{B_{23}^2}{\sigma_1^2} + \frac{B_{13}^2}{\sigma_2^2} + \frac{B_{12}^2}{\sigma_3^2} \right) \] (3.50)

\[ \text{Im} \Delta(t) = \frac{1}{\hbar^3} A_1 A_2 A_3 - \frac{1}{16 \hbar} \left( \frac{A_3}{\sigma_1^2 \sigma_2^2} + \frac{A_1}{\sigma_2^2 \sigma_3^2} + \frac{A_2}{\sigma_3^2 \sigma_1^2} \right) \]

\[ - \frac{1}{4 \hbar^3} (A_1 B_{23}^2 + A_2 B_{13}^2 + A_3 B_{12}^2) + \frac{1}{4 \hbar^3} B_{12} B_{13} B_{23} \] (3.51)
その一方で、

while

\[
\Phi^{(k)}(X, t) = 4( \dot{A}_2 \dot{A}_3 \dot{C}_2(X) + \ddot{A}_1 \ddot{A}_3 \ddot{C}_2(X) + \dddot{A}_1 \dddot{A}_2 \dddot{C}_2(X) ) \\
- \dddot{B}_2^2 \dddot{C}_2(X) - \dddot{B}_1^2 \dddot{C}_2(X) - \dddot{B}_1^2 \dddot{C}_2(X) \\
+ 2( \dot{B}_{13} \dot{B}_{23} \dot{C}_1(X) \dot{C}_2(X) + \ddot{B}_{12} \ddot{B}_{13} \ddot{C}_2(X) \ddot{C}_3(X) + \dddot{B}_{12} \dddot{B}_{23} \dddot{C}_1(X) \dddot{C}_3(X) ) \\
+ 4( \dot{A}_1 \dot{B}_{23} \dot{C}_2(X) \ddot{C}_3(X) + \ddot{A}_2 \ddot{B}_{13} \ddot{C}_1(X) \ddot{C}_3(X) + \dddot{A}_3 \dddot{B}_{12} \dddot{C}_1(X) \dddot{C}_2(X) )
\]

(3.52)

私達は新しい複素係数を導入する。

We introduce new complex coefficients

\[
(4\dot{A}_2 \dddot{A}_3 - \dddot{B}_2^2) \equiv \lambda_1 \text{ etc.. } , \ (2\dot{B}_{13} \dot{B}_{23} + 4\dddot{A}_3 \dddot{B}_{12}) \equiv \mu_{12} \text{ etc..}
\]

(3.53)

そして \( \Phi^{(k)} \) を次のように書き換える事ができる。

, then we can rewrite \( \Phi^{(k)} \) as follows.

\[
\Phi^{(k)}(X, t) = \sum_{i=1}^{3} \lambda_i \dot{C}_i(X) + \sum_{(i,j)=(1,2) \text{ or } (2,3) \text{ or } (3,1)} \mu_{ij} \dot{C}_i(X) \ddot{C}_j(X)
\]

(3.54)

ここで、私達は \( \lambda_i \) と \( \mu_{ij} \) の実数部分と虚数部分を書く。

Here, we write the real part and the imaginary part of \( \lambda_i \) and \( \mu_{ij} \).

\[
\lambda_i \equiv \Re \lambda_i + i \Im \lambda_i \quad \text{(3.55)}
\]

\[
\Re \lambda_i = -\frac{1}{\hbar^2} \left\{ m_1 \omega_1 \cot[\omega_1(t-t_0)] m_2 \omega_2 \cot[\omega_2(t-t_0)] (a_j b_k - a_k b_j)^2 \\
+ m_2 \omega_2 \cot[\omega_2(t-t_0)] \frac{m_3}{(t-t_0)} (b_j c_k - b_k c_j)^2 \\
+ \frac{m_3}{(t-t_0)} m_1 \omega_1 \cot[\omega_1(t-t_0)] (c_j a_k - c_k a_j)^2 \right\} + \frac{1}{4\sigma_j^2 \sigma_k^2} \quad \text{(3.56)}
\]

\[
\Im \lambda_i = -\frac{1}{2\hbar^2} \left\{ m_1 \omega_1 \cot[\omega_1(t-t_0)] \left( \frac{a_j^2}{\sigma_k^2} + \frac{a_k^2}{\sigma_j^2} \right) \\
+ m_2 \omega_2 \cot[\omega_2(t-t_0)] \left( \frac{b_j^2}{\sigma_k^2} + \frac{b_k^2}{\sigma_j^2} \right) + \frac{m_3}{(t-t_0)} \left( \frac{c_j^2}{\sigma_k^2} + \frac{c_k^2}{\sigma_j^2} \right) \right\} 
\]

(3.57)

そして、

and

\[
\mu_{ij} \equiv \Re \mu_{ij} + i \Im \mu_{ij} \quad \text{(3.58)}
\]
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\[ \Re \mu_{ij} = -\frac{2}{\hbar^2} \left\{ m_1 \omega_1 \cot[\omega_1(t - t_0)] m_2 \omega_2 \cot[\omega_2(t - t_0)] (a_j b_k - a_k b_j)(a_k b_i - a_i b_k) + m_2 \omega_2 \cot[\omega_2(t - t_0)] - \frac{m_3}{t - t_0} (b_j c_k - b_k c_j)(b_k c_i - b_i c_k) + \frac{m_3}{t - t_0} m_1 \omega_1 \cot[\omega_1(t - t_0)] (c_j a_k - c_k a_j)(c_k a_i - c_i a_k) \right\} \]

\[ \Im \mu_{ij} = \frac{1}{\sigma_k^2 \hbar} \left\{ m_1 \omega_1 \cot[\omega_1(t - t_0)] a_i a_j + m_2 \omega_2 \cot[\omega_2(t - t_0)] b_i b_j + \frac{m_3}{t - t_0} c_i c_j \right\} \]

(3.59)

(3.60)

And we introduce new complex factors \( (L_{a_1} - M_{u_0}^{(k)}) \).

\[
L_{a_{dd}} = \sum_{i=1}^{3} \lambda_i \alpha_d^{(i)}^2, \quad M_{u_{dd}} = \sum_{(i,j)=(1,2) \text{ or } (2,3) \text{ or } (3,1)} \mu_{ij} \alpha_d^{(i)} \alpha_d^{(j)} \quad (d = 1, 2, 3) \quad (3.61)
\]

\[
L_{a_{df}} = 2 \sum_{i=1}^{3} \lambda_i \alpha_d^{(i)} \alpha_f^{(i)}, \quad M_{u_{df}} = \sum_{(i,j)} \mu_{ij} \left( \alpha_d^{(i)} \alpha_d^{(j)} + \alpha_d^{(j)} \alpha_d^{(i)} \right) \quad (3.62)
\]

\[
L_{a_d^{(k)}} = \sum_{i=1}^{3} \lambda_i \frac{d_{i}^{(k)}}{\sigma_i^2} \alpha_d^{(i)}, \quad M_{u_d^{(k)}} = 0.5 \sum_{(i,j)} \mu_{ij} \left( \frac{d_{i}^{(k)}}{\sigma_i^2} \alpha_d^{(j)} + \frac{d_{j}^{(k)}}{\sigma_j^2} \alpha_d^{(i)} \right) \quad (3.63)
\]

\[
L_{a_0^{(k)}} = 0.25 \sum_{i=1}^{3} \lambda_i \frac{d_{i}^{(k)}}{\sigma_i^2}^2, \quad M_{u_0^{(k)}} = 0.25 \sum_{(i,j)} \mu_{ij} \frac{d_{i}^{(k)}}{\sigma_i^2} \frac{d_{j}^{(k)}}{\sigma_j^2} \quad (3.64)
\]

この \( \alpha_d^{(i)} \) は実数であり、

この \( \alpha_d^{(i)} \) は実数であり、

\[
C_i(X) = -\alpha_d^{(i)} x_1 - \alpha_d^{(i)} x_2 - \alpha_d^{(i)} x_3 \quad (3.65)
\]

\[
\alpha_d^{(i)} \equiv \left( \frac{m_1 \omega_1}{\sin[\omega_1(t - t_0)]} \right) a_d + \left( \frac{m_2 \omega_2}{\sin[\omega_2(t - t_0)]} \right) b_d + \left( \frac{m_3}{t - t_0} \right) c_d \quad (3.66)
\]

それゆえに、\( (L_{a_1} - M_{u_0}^{(k)}) \) の実数部と座標部は単純に \( (\lambda, \mu_{ij}) \) の実数部、座標部それぞれに対応する。すなわち、
Therefore the real and the imaginary part of \( \lambda_i - Mu_0^{(k)} \) simply correspond to the real part and the imaginary part of \( \lambda_i, \mu_{ij} \) respectively, that is

\[
\Re La_{dd} = \sum_{i=1}^{3} \Re \lambda_i \alpha_d^{(i)2} \quad \text{etc..} \quad (3.67)
\]

Thus \( \Phi^{(k)} \) of the real and imaginary parts are as follows.

Then the real and the imaginary part of \( \Phi^{(k)} \) are

\[
\Phi^{(k)}(X, t) \equiv \Re \Phi^{(k)}(X, t) + i \Im \Phi^{(k)}(X, t) \quad (3.68)
\]

Using the above formulae,

\[
\Re \Phi^{(k)}(X, t) = -\frac{1}{\hbar^2} (\Re La_{11} + \Re Mu_{11}) x_1^2 - \frac{1}{\hbar^2} (\Re La_{22} + \Re Mu_{22}) x_2^2
\]

\[
- \frac{1}{\hbar^2} (\Re La_{33} + \Re Mu_{33}) x_3^2 - \frac{1}{\hbar^2} (\Re La_{12} + \Re Mu_{12}) x_1 x_2
\]

\[
- \frac{1}{\hbar^2} (\Re La_{23} + \Re Mu_{23}) x_2 x_3 - \frac{1}{\hbar^2} (\Re La_{31} + \Re Mu_{31}) x_3 x_1
\]

\[
+ \frac{1}{\hbar} (\Im La^{(k)}_1 + \Im Mu^{(k)}_1) x_1 + \frac{1}{\hbar} (\Im La^{(k)}_2 + \Im Mu^{(k)}_2) x_2
\]

\[
+ \frac{1}{\hbar} (\Im La^{(k)}_3 + \Im Mu^{(k)}_3) x_3 + \Re La^{(k)}_0 + \Re Mu^{(k)}_0 \quad (3.69)
\]

\[
\Im \Phi^{(k)}(X, t) = -\frac{1}{\hbar^2} (\Im La_{11} + \Im Mu_{11}) x_1^2 - \frac{1}{\hbar^2} (\Im La_{22} + \Im Mu_{22}) x_2^2
\]

\[
- \frac{1}{\hbar^2} (\Im La_{33} + \Im Mu_{33}) x_3^2 - \frac{1}{\hbar^2} (\Im La_{12} + \Im Mu_{12}) x_1 x_2
\]

\[
- \frac{1}{\hbar^2} (\Im La_{23} + \Im Mu_{23}) x_2 x_3 - \frac{1}{\hbar^2} (\Im La_{31} + \Im Mu_{31}) x_3 x_1
\]

\[
- \frac{1}{\hbar} (\Re La^{(k)}_1 + \Re Mu^{(k)}_1) x_1 - \frac{1}{\hbar} (\Re La^{(k)}_2 + \Re Mu^{(k)}_2) x_2
\]

\[
- \frac{1}{\hbar} (\Re La^{(k)}_3 + \Re Mu^{(k)}_3) x_3 + \Im La^{(k)}_0 + \Im Mu^{(k)}_0 \quad (3.70)
\]

Therefore \( \bar{D}^{(k)}(X) \) is

And another formula \( \bar{D}^{(k)}(X) \) is

\[
\bar{D}^{(k)}(X) \equiv \Re \bar{D}^{(k)} + i \Im \bar{D}(X) \quad (3.71)
\]

\[
\Re \bar{D}^{(k)} = \frac{d_1^{(k)2}}{4\sigma_1^2} - \frac{d_2^{(k)2}}{4\sigma_2^2} - \frac{d_3^{(k)2}}{4\sigma_3^2} \quad (3.72)
\]

\[
\Im \bar{D}(X) = D(X)/\hbar \quad (3.73)
\]

Next, wave number is (3.47) as follows.
Next, the wave function is from eq.(3.47),

$$
\psi(k)(X, t) \propto \sqrt{\frac{\pi^3}{|\Delta|^2}} \exp \left[ \frac{\Delta^*}{16|\Delta|^2} Q^{(k)}(X, t) + \tilde{D}^{(k)}(X, t) \right]
$$

$$
\equiv Q \exp[\Theta^{(k)}(X, t)]
$$

ここで、

where

$$
\tilde{Q} \equiv \Re\tilde{Q} + i \Im\tilde{Q}
$$

$$
\Re\tilde{Q} = \sqrt{\frac{\pi^3}{|\Delta|^2}} \cos \frac{\phi}{2}, \quad \Im\tilde{Q} = \sqrt{\frac{\pi^3}{|\Delta|^2}} \sin \frac{\phi}{2} : \phi = \arctan (\Im\Delta / \Re\Delta)
$$

そして、

and

$$
\Theta^{(k)}(X, t) \equiv \Re\Theta^{(k)}(X, t) + i \Im\Theta^{(k)}(X, t)
$$

$$
\Re\Theta^{(k)}(X, t) = \frac{1}{16|\Delta|^2} (\Re\Delta \cdot \Re\Phi^{(k)} + \Im\Delta \cdot \Im\Phi^{(k)}) + \Re\tilde{D}^{(k)}
$$

$$
\Im\Theta^{(k)}(X, t) = \frac{1}{16|\Delta|^2} (\Re\Delta \cdot \Im\Phi^{(k)} - \Im\Delta \cdot \Re\Phi^{(k)}) + \Im\tilde{D}(X)
$$

ようすると、波動関数の実数部と虚数部は次のようになる。

Then, the real part and the imaginary part of the wave function are as follows.

$$
\psi^{(k)}(X, t) \equiv \Re\psi^{(k)}(X, t) + i \Im\psi^{(k)}(X, t)
$$

$$
\Re\psi^{(k)}(X, t) \propto \exp[\Re\Theta^{(k)}] \left( \Re\tilde{Q} \cdot \cos[\Im\Theta^{(k)}] - \Im\tilde{Q} \cdot \sin[\Im\Theta^{(k)}] \right)
$$

$$
= \sqrt{\frac{\pi^3}{|\Delta|}} \exp[\Re\Theta^{(k)}] \cdot \cos \left[ \Im\Theta^{(k)} + \frac{\phi}{2} \right]
$$

$$
\Im\psi^{(k)}(X, t) \propto \exp[\Re\Theta^{(k)}] \left( \Re\tilde{Q} \cdot \sin[\Im\Theta^{(k)}] + \Im\tilde{Q} \cdot \cos[\Im\Theta^{(k)}] \right)
$$

$$
= \sqrt{\frac{\pi^3}{|\Delta|}} \exp[\Re\Theta^{(k)}] \cdot \sin \left[ \Im\Theta^{(k)} + \frac{\phi}{2} \right]
$$

ようすると、(k = 0 – 7) について足し合わせた全波動関数は、

Therefore the total wave function summed with (k = 0 – 7) is

$$
\psi^{(total)}(X, t) \equiv \Re\psi^{(total)}(X, t) + i \Im\psi^{(total)}(X, t)
$$

$$
\Re\psi^{(total)} = C \sqrt{\frac{\pi^3}{|\Delta|}} \sum_{k=0}^{7} \exp[\Re\Theta^{(k)}] \cdot \cos \left[ \Im\Theta^{(k)} + \frac{\phi}{2} \right]
$$
\[ \Im m \psi^{(\text{total})} = C \sqrt{\frac{\pi^3}{|\Delta|}} \sum_{k=0}^{7} \exp\left[ \Re e \Theta^{(k)} \right] \cdot \sin\left[ \Im m \Theta^{(k)} + \frac{\phi}{2} \right] \quad (3.86) \]

Here we introduced a real normalization constant \( C \). Then the quantum mechanical probability density function of total system become as follows.

\[
\rho^{(\text{total})}(X,t) = \Re e^2 \psi^{(\text{total})} + \Im m^2 \psi^{(\text{total})}
\]

\[
= C^2 \frac{\pi^3}{|\Delta|} \sum_{k=0}^{7} \sum_{l=0}^{7} \exp\left[ \Re e \Theta^{(k)} + \Re e \Theta^{(l)} \right] \cdot \cos\left[ \Im m \Theta^{(k)} - \Im m \Theta^{(l)} \right]
\]

\[
= C^2 \frac{\pi^3}{|\Delta|} \left( \sum_{k=0}^{7} \exp\left[ 2 \Re e \Theta^{(k)} \right] \right)
\]

\[
+ 2 \sum_{k<l}^{0-7} \exp\left[ \Re e \Theta^{(k)} + \Re e \Theta^{(l)} \right] \cdot \cos\left[ \Im m \Theta^{(k)} - \Im m \Theta^{(l)} \right]
\]

\( (3.87) \)

式 (3.87) の \( \cdots \) 内の最初の項は、初期時刻 \( (t = t_0) \) においてももとガウス型波束であった \( 8 \) つの波束を意味している。その一方で、二番目の項はそれらの「干渉」項であるが、それはこの模型での量子デコヒーレンスによって消える量子干渉とは同じではない。

The first term in \( \cdots \) of equation (3.87) means the eight wave packets which originally are the Gaussian packets at the initial time \( (t = t_0) \). While the second term is their “interference” term, which is not the same as the quantum interference vanishes by quantum decoherence in this model.

私達は全体としての 3 体系についての情報に興味が無い、私達は部分系としての粒子 1 についての情報のみに興味がある事を記しておく、それゆえに、粒子 2 と 3 についての情報を積分して消去する。そうすると私達は粒子 1 のみに関する情報、すなわち粒子 1 の縮約密度関数を得る事ができる。

Note that we are not interested in informations about total 3-body system, we are interested in only the information about particle-1 as a sub-system. Therefore we should integrate out informations about particle-2 and -3. Then we can get the information about particle-1 only, that is, the reduced density function for particle-1.

\[
\rho_1^{(\text{reduced})}(x_1,t) \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \rho^{(\text{total})}(X,t) \quad (3.88)
\]

式 (3.87) を式 (3.88) に代入すると、

We substitute eq. (3.87) into eq. (3.88),

\[
\rho_1^{(\text{reduced})}(x_1,t) = C^2 \frac{\pi^3}{|\Delta|} \left( \sum_{k=0}^{7} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \ \exp[2 \Re e \Theta^{(k)}] \right)
\]
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\[ +2 \sum_{k<l}^{0-7} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \exp(\Ree \Theta^{(k)} + \Ree \Theta^{(l)}) \cdot \cos(\Im \Theta^{(k)} - \Im \Theta^{(l)}) \]  

(3.89)

ここで、「干渉項」にも二種類ある事に注意する。異なる波束間に干渉がある時、それぞれの波束に含まれる粒子の2次元状態は同一であるかも知れない。初期状態を考えるとわかりやすい。式(3.46)から、

Here we notice that there are two kinds of “interference term”. When there is an interference between different packets, macroscopic states of particle-1 included in each packet may be the same. It is simple that we think the initial states. From eq.(3.46),

\[ k = 0, 2, 3, 6 : \text{The packet around } (x_1 = 0) \text{ at initial time } (t = t_0). \]

\[ k = 1, 4, 5, 7 : \text{The packet around } (x_1 = d_1) \text{ at initial time}. \]  

(3.90)

\((x_1, x_2, x_3)\)空間中の8つの波束は、これらの2つのグループに分けられる。同一のグループ内での波束は粒子2と3の状態についての重ね合わせであり、粒子1に対する物では無い。粒子1は同一の状態にいる訳であるから、これらは私達が本当に見たいと思っている、粒子1に対する異なる状態間での本当の干渉ではない。

The 8 packets in the \((x_1, x_2, x_3)\) space are separated into these two groups. The interference between packets in the same group means the transition between the states for particle-2 and 3, not for particle-1. Because the particle-1 is in the same its own state, these are not the true interferences between different states for particle-1 which really we want to see.

今、私達はこれらの8つの波束を2つのグループに分けなくてはならず、そして同じグループ内での「干渉」項を波束部分に加えてはならない。私達はそれらを粒子1にとっての実質的な状態であるとみなす。つまり、

Now we have to separate these 8 packets into two groups above, and we have to add the “interference” terms among the packets in the same group to the packet terms. We regard them as the effective states for the particle-1. That is,

○ The packet around \((x_1 = 0)\) at initial time \(t_0\).

\[ \hat{\rho}_{1\text{eff}}(x_1, t) \equiv C^2 \frac{\pi^3}{|\Delta|} \sum_{k=0, 2, 3, 6} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \exp(2 \Ree \Theta^{(k)}) \right) + 2 \sum_{k<l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \exp(\Ree \Theta^{(k)} + \Ree \Theta^{(l)}) \cdot \cos(\Im \Theta^{(k)} - \Im \Theta^{(l)}) \]  

(3.91)

○ The packet around \((x_1 = d_1)\) at initial time \(t_0\).

\[ \hat{\rho}_{1\text{eff}}(x_1, t) \equiv C^2 \frac{\pi^3}{|\Delta|} \sum_{k=1, 4, 5, 7} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \exp(2 \Ree \Theta^{(k)}) \right) \]
\[+2 \sum_{l=1,4,5,7} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \exp[\Re\Theta^{(k)} + \Re\Theta^{(l)}] \cdot \cos[\Im\Theta^{(k)} - \Im\Theta^{(l)}]\]

(3.92)

◯ Their interference term.

\[
\tilde{\rho}_{1,\text{int}}(x_1, t) = 4C^2 \pi^3 \sum_{k<l}^{k=0,2,3,6,\text{odd}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_2 \, dx_3 \, \exp[\Re\Theta^{(k)} + \Re\Theta^{(l)}] \cdot \cos[\Im\Theta^{(k)} - \Im\Theta^{(l)}]
\]

(3.93)

Finally, we get the reduced density for particle-1 as follows.

\[
\tilde{\rho}_1(x_1, t) = \tilde{\rho}_{1,0}(x_1, t) + \tilde{\rho}_{1,d}(x_1, t) + \tilde{\rho}_{1,\text{int}}(x_1, t)
\]

(3.94)

We used numerical calculation for integrations in eq. (3.88) and final normalization.

3.4. Simulation Result

The case \(\omega_{12} = 0.305, \omega_{13} = 0.1, \omega_{23} = 0.202\) 的場合を以下に示す。左上から右下にかけて、図はそれぞれ時刻

\(t = 0.005, 0.705, 1.405, 2.105, 2.805, 4.205, 5.605, 205.605, 2000.005, 5000.005\) に対応している。見通の通り、それぞれの図において 2 つの波束が存在している。右側の波束 \(\tilde{\rho}_{1,0}^{\text{eff}}\) は初期状態で原点 \((x_1 = 0)\) に存在する。左側の波束 \(\tilde{\rho}_{1,d}^{\text{eff}}\) は初期状態で座標 \((x_1 = d_1)\) に存在している。そして 2 つの波状の線が存在する。下の波線 \(\tilde{\rho}_{1,\text{int}}^{\text{eff}}\) はこれら 2 つの波束間の量子干渉面であり、この消失は古典性の出現を意味している。上の波線 \(\tilde{\rho}_1(x_1, t)\) はそれら全てを合わせた粒子 1 の全密度密度である。それらの横軸は粒子 1 の位置 \(x_1\) である。

The case \(\omega_{12} = 0.305, \omega_{13} = 0.1, \omega_{23} = 0.202\) is showed as follows. From upper left to bottom right, the figures are corresponding to time

\(t = 0.005, 0.705, 1.405, 2.105, 2.805, 4.205, 5.605, 205.605, 2000.005, 5000.005\), respectively. As you can see, there are 2 packets in each figure. The right packet is \(\tilde{\rho}_1^{\text{int}}\) which is in the origin \((x_1 = 0)\) initially. The left packet is \(\tilde{\rho}_1^{\text{d}}\) which is in \((x_1 = d_1)\) initially. And there are 2 wave-like lines. The lower wave line is the quantum interference term \(\tilde{\rho}_{1,\text{int}}^{\text{eff}}\) between these 2 packets, whose disappearance means the emergence of classicality. The upper wave line is their total reduced density for particle 1, \(\tilde{\rho}_1(x_1, t)\). Their horizontal axes are particle-1’s position, \(x_1\).

この場合、しばらくの間は 2 つの波束間の干渉は強いか、しかし時刻 \(t \simeq 5.6\) 以降では、干渉は比較的弱くなる。これはデコヒーレンスが生じた事を意味する。

In this case, momentarily the interference between two packets are strong, but after the time \(t \simeq 5.6\), the interferences are comparatively weakened. It means the quantum decoherence arises.
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Fig. 3. \( \uparrow \) (Left) \( t=0.005 \), (Right) \( t=0.705 \)

Fig. 4. \( \uparrow \) (Left) \( t=1.405 \), (Right) \( t=2.105 \)

Fig. 5. \( \uparrow \) (Left) \( t=2.805 \), (Right) \( t=4.205 \)

今回の3匹のシュレーディンガー方程式によって、3つの自由度を含む閉じた有限系の1体部分系においてデコヒーレンスが生じる事が示された。言い換えると、ただ2つの環境自由度を持つ、閉じた1体系においてデコヒーレンスが生じる事が示された。
By this Three Schrödinger cats model, it was showed that the quantum decoherence would arise in a 1-body sub-system of a closed finite system which consists of three degrees of freedom. In other words, it was showed that decoherence would arise in an opened 1-body system with only two enviromental degrees of freedom.

\[\Diamond\]
§4. Discussion

We showed the possibility of emergence of classicality in a quantum mechanical system with 3 degrees of freedom. What did make this quantum decoherence? In this model, we selected only 1 degree as main system, and regarded other 2 degrees of freedom as environments. Then, the quantum mechanical property of the main system vanished because of the “environmental effects” of other two degrees of freedom. They disturbed the main system and destroyed its quantum interference.

For decoherence, it seems that “randomness” is important. It is known that external random forces make a system decoherence. But remember that the introduction of “randomness” is only an artificial procedure. If we know the time evolution of external forces perfectly, then we can not say they are “random” forces, but the quantum decoherence will arise.

This time, our model is fully deterministic for 3 degrees of freedom as a whole. After we select a degrees of freedom as a main system, we can know when/how the “external” forces from other 2 degrees of freedom work. But the quantum decoherence occurred in our model. Therefore the “randomness” does not destroy the quantum mechanical nature. Maybe, the truly important thing is how the system drives, and the projection procedure.

In this paper, when we select 1 degree of freedom as a main system and regard other 2 degrees of freedom as an environment, we integrate out the environmental 2 degrees of freedom. We call the procedure “projection” here. Both in the Caldeira-Leggett Model and our 3-Schrödinger cats model, the total (main system +
environment) system is treated as a quantum mechanical system. And main system is selected by projection, and they get classicality.

Therefore it is natural that we should think that the projection procedure makes the main system classical. In other words, classicality is the property of subsystems. Because the projection is a method to select some subsystem from a whole system. Therefore we should assume that the decoherence by environmental effects and the one by projection are equivalent.

It is expected that the quantum mechanical property of the system relates to the classical mechanical property of its equivalent classical system. We will show a relationship between the quantum decoherence in our 3-Schrödinger cats model and a behavior of classical orbits in its classical equivalent model as follows.

Fig. 8. Projection makes orbits crossing.

§5. Quantum Decoherence/Irreversibility

The Quantum Decoherence is a kind of non-unitary process, which means disappearance of quantum interference. “The collapse of the wave function” is understood by decoherence. Because of the disappearance of quantum interference, a system can
not do any quantum transitions. Therefore the system gets effective classicality. We know the cause of decoherence such as the dissipation to environment, and the fluctuation from environment. They are corresponding to the damping and the random forces of Langevin equation respectively. So there must be any relationship between the irreversible process and decoherence.

In old days, there was a problem of difficulty to derive the irreversible Langevin equation from the reversible Newton equation or the canonical equations. The situation is the same as the problem of difficulty to derive non-unitary “the collapse of the wave function” by the unitary Schrödinger equation.

Now, it is known that the Langevin equation is derived by the coarse graining procedures using the projection operators. The projection extracts the arbitrary subsystem from the total system. When the subsystem couples with the other neglected part (“environment”), the effect would be considered as the environmental effects. Therefore, the appearance of the irreversibility by projection/coarse graining procedure and that by the environment are equivalent.

A.O.Caldeira and A.J.Leggett showed the disappearance of quantum interference between two Gaussian packets in a harmonic oscillator potential by the heat bath which consists of infinite numbers of harmonic oscillators\(^1\). That is to say, they showed that the system in a heat bath lost its quantum property. They used the Feynman-Vernon’s influence functional method, which is the way of using the propagator with effects of heat bath, and is mathematically equivalent to the projection method. Therefore we can understand the projection make the system classical.

Thus, it is known that the projection procedures make an irreversibility (the break down of time reversal symmetry) and a classicality (the break down of unitarity). But I didn’t know why a projection makes them. So I have tried to make a
qualitative picture for an origin of irreversibility.

Classical orbits within a the phase space for a closed Hamiltonian system do not cross each other. Therefore, if we want to know the past or the future of the system, we can guess them by following an orbit. This guarantees the reversibility and the predictability of motion. Chaos may be expected to make any irreversibility for system. Because when we take a point in phase space with chaos region, we can not guess its past. The indefiniteness of orbits made by chaos increases not only with the time evolution but also with the time reversal.

But, we do not bother to need chaos only for making the classical motion obscure. Anyway to make any indefiniteness of motion, we prepare an external system or an environment. Then the orbit will branch according to probable states of the environment. We may call the effect “the random force”, its origin is the lack of our knowledge of the environment. And in principle there is no need to use any “random seed” for mechanics of the environment.

When we assume the total system, which consists of our main system and the environment, is closed and non-chaotic, then its classical orbits in the total phase space have to be defined uniquely and do not cross each other.

But our main system’s orbits must be branching then. The main system should be defined by a projection procedure, therefore we should realize that the projection procedure itself would make the branch of orbits. Really the branch would be an
intersection of orbits, which is characteristic of figures made by projection. At the intersection, the thing "We can not decide the future uniquely." means the randomness of motion. From the crossing point, there are the some ways not only to the futures but also to the pasts. The thing "We can not decide the past uniquely." means the irreversibility of motion. In this image, the irreversibility and the randomness are the equivalent.

Fig. 9. Crossing makes the system irreversible?

§6. Crossing of Classical Orbits and Quantum Decoherence

The quantum mechanical behavior of quantum system is supposed to relate with the corresponding classical motion. Because decoherence occurs in many "classically" (classical equivalent) irreversible systems. Most observation processes are irreversible. As I noted above, if we assume the irreversibility is made by crossing of classical orbits with different histories, that crossing must affect the quantum systems.

In a classical harmonic three body problem, we can draw the spatial orbit of particle-1 \((x_1 = x_1(t))\) versus time \(t\). Each particle has 2 initial positions and their all initial velocities are set 0. Then we can draw \(2^3 = 8\) lines on \((x_1 - t)\) plane. And we can simulate its equivalent quantum system by the three cats model. Each frequency of potentials \(\omega_{ij}\) is same as the classical model, and 2 initial positions of each particle are expressed by centers of packets of the Schrödinger cat state. Then we will observe the quantum interference term of the reduced density of particle-1.
Fig. 10. The crossing of classical orbits for particle-1. The horizontal axis is time $t$, and the vertical axis is particle-1’s position, $x_1(t)$. Orbits are crossing at time $t=4.0-6.0$.

Fig. 11. The disappearance of quantum interference for particle-1 at three Schrödinger cats model. From left, the time $t=3.5, t=4.0, t=4.5, t=5.0$. The horizontal axis is particle 1’s position $x_1(t)$, and the vertical axis is the reduced probability density for particle-1, $\tilde{\rho}_1(x_1)$. The interference is damped at time $t=4.0-5.0$.

Comparing the quantum system with the classical system, when classical orbits (especially the ones from the different initial points) are crossing, decoherence seems to arise in corresponding quantum system (time $t=4.0-5.0$). This can be understood that the classical crossing make system irreversible, then unitarity of the quantum system breaks down.

古典系と量子系を比較すると、古典軌道（特に異なった初期値からの軌道）が交差している時に対応する量子系でデコヒーレンスが生じている様に思える（時刻 $t=4.0-5.0$）。この事は、古典軌道が交差を不可逆にし、そして量子系のユニタリ性が破れたと解釈できる。

この考えは単純で魅力的で、いくつかの問題点もある。私はこの古典軌道交差が量子デコヒーレンスを関係していると信じる。この研究での私の目的は、原子核物理における古典性と不可逆性の謎を解く事にある。例えば原子核の集団自由度は、それが1粒子動起などのいくつかの自由度と結合している時に古典性を得るのだろうか？また液滴状態などの古典的像が、何故それなりの説得力を持つのだろうか？私はこの研究が科学にとって意味がある事を望みます。
This thought seems to be simple and tempting, and there are some difficulties. I believe that this classical orbits’ crossing relates to quantum decoherence. My goal of this study is to reveal the mystery of classicality and irreversibility in nuclear physics. For example, does a nuclear collective degree of freedom get classicality when it is coupled with some degrees of freedom such as single particle excitation? And why is it valid to use any classical pictures for nuclei such as the liquid drop model? I hope this study is meaningful for science.

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