Critical phenomena for Riemannian manifolds:
Simple Homotopy
and
simplicial quantum gravity

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Abstract

Simplicial quantum gravity is an approach to quantizing gravity by using
as a regularization scheme dynamical triangulations of riemannian manifolds.
In dimension two, such an approach has led to considerable progress in our
understanding of 2D-gravity. In dimension three and four, the current state
of affairs is less favourable, and dynamically triangulated models of quantum
gravity suffer from serious difficulties which can be only partially removed by
the use of computer simulations. In this paper, which to a large extent reviews
some of our recent work, we show how Gromov’s spaces of bounded geometries
provide a general mathematical framework for addressing and solving many of
the issues of 3D-simplicial quantum gravity. In particular, we establish entropy
estimates characterizing the asymptotic distribution of combinatorially inequivalent
triangulated 3-manifolds, as the number of tetrahedra diverges. Moreover, we
offer a rather detailed presentation of how spaces of three-dimensional riemannian
manifolds with natural bounds on curvatures, diameter, and volume can be used to
prove that three-dimensional simplicial quantum gravity is connected to a Gaussian
model determined by the simple homotopy types of the underlying manifolds. This
connection is determined by a Gaussian measure defined over the general linear
group $GL(R, \infty)$.

By exploiting these results it is shown that the partition function of three-
dimensional simplicial quantum gravity is well-defined, in the thermodynamic
limit, for a suitable range of values of the gravitational and cosmological coupling constants. Such values are determined by the Reidemeister-Franz torsion invariants associated with an orthogonal representation of the fundamental groups of the set of manifolds considered. The geometrical system considered shows also critical behavior, and in such a case the partition function is exactly evaluated and shown to be equal to the Reidemeister-Franz torsion. The phase structure in the thermodynamical limit is also discussed. In particular, there are either phase transitions describing the passage from a simple homotopy type to another, and (first order) phase transitions within a given simple homotopy type which seem to confirm, on an analytical ground, the picture of three-dimensional quantum gravity recently suggested by numerical simulations.
1 Introduction

Random surface theory and the strictly related notion of random triangulations have played a key role in our understanding of models of quantum gravity. Advances have been particularly significant in dimension two, where dynamically triangulated models of two-dimensional quantum gravity, (either pure or coupled to suitable matter fields), can be put in correspondence with an appropriate matrix field theory. The relation between two-dimensional random triangulations and matrix models arises from the observation that each vertex of the graph associated with the dual triangulation, $T^*$, (the graph the vertices of which are the centers of the triangles and the edges of which arise from pairs of adjacent triangles), is of order three, (see for a thorough account of such topics). This remark implies that every Feynman diagram in the formal diagrammatic expansion of a $\phi^3$ field theory can be labelled by dual triangulations $T^*$. It is well known that similar mappings into a matrix model, mappings which are instrumental to the success of the two-dimensional theory, are not so well-behaved in higher dimensions. The techniques of matrix field theory are not helpful in such a case, at least apparently, and one must resort to more direct combinatorial constructions, (often computer-assisted), in the theory of triangulated manifolds in order to investigate the properties of three and four-dimensional simplicial quantum gravity models.

At the basis of the difficulties in dealing with these higher dimensional models of simplicial quantum gravity, (difficulties which are present also in the more standard Regge calculus approach), there is a lack of control of the interplay between manifolds topologies and the riemannian invariants which, in their discretized form, provide the action of the theory. To what extent such control is necessary follows by noticing that in order to construct a sensible statistical theory of riemannian manifolds out of random triangulations one needs entropy bounds assuring that the number of triangulated manifolds of given dimension, volume and fixed topology grows with the volume at most at an exponential rate, possibly with a non-trivial subleading asymptotics.

In the case of surfaces the required entropy bounds can be proved either by direct counting arguments, or by quantum field theory techniques as applied to graphs enumeration, a technique, this latter, that has found its way in a number of far reaching applications in surface theory. In higher dimension, the natural generalization of such entropy bounds has, up to now, defied analytical proofs, even if it is known that they hold under some suitable restriction on the way one constructs triangulations, and their validity is confirmed by numerical simulation. It must also be stressed that without any restriction on topology, the situation gets considerably more involved since it can be shown for instance that the number of distinct three-manifolds, with given volume $V$ and arbitrary topological type, grows at least factorially with $V$. Thus a systematic method for understanding and enforcing entropy bounds relating topology to riemannian invariants appears as a major issue to deal with in order to extend to higher dimensions.
dimensions\[21,22,23\], (in particular to \(n = 3\) and \(n = 4\), see for instance\[24,25\] and also\[26,27\]), the results of the two-dimensional theory.

Of particular interest in this line of research is the study of the three-dimensional case, either as a necessary step before moving on to the physical relevant case of dimension four\[28,29,30,31\], or because one hopes in some explicit solvability, as hinted by the Chern-Simons approach fostered by Witten\[32\] in the field theoretic setting.

In order to describe in barest essential such three-dimensional models, let us recall that dynamical triangulations \(\mathcal{T}\) for three-dimensional manifolds are constructed through regular tetrahedra glued together along their two-dimensional faces in such a way so as to form a piecewise linear manifold. The *dynamical variable* of the resulting theory is shifted from the length of the links, used in the usual Regge calculus approach, and here set to one, to the connectivity properties of the manifolds. This has the effect that the discretized version of the continuous Einstein-Hilbert action (with cosmological term), takes on the form

\[
S[\mathcal{T}] := k_3 N^{(3)}(\mathcal{T}) - k_1 N^{(1)}(\mathcal{T})
\]

where \(N^{(3)}(\mathcal{T})\) and \(N^{(1)}(\mathcal{T})\) respectively denote the number of tetrahedra and the number of links in the given triangulation \(\mathcal{T}\), and where \(k_3\) is a bare cosmological constant while \(k_1\) can be related to a gravitational coupling constant.

A simplicial quantum gravity model is realized by considering as statistical sum associated with the action \(S[\mathcal{T}]\) the expression

\[
\sum_{\mathcal{T}} \rho(\mathcal{T}) \exp[-S[\mathcal{T}]]
\]

where the sum is over a suitable class of triangulations, and where the factor \(\rho(\mathcal{T})\) is a weight for each triangulation, weight which is usually assumed equal to one.

A clear account of the geometry and the physics underlying such construction can be found in the quoted papers by Ambjørn\[21\], suffices here to say that in order to make sense of the partition function (2), one has to impose strong restrictions of topological nature on the class of allowable triangulations \(\mathcal{T}\) to be summed over. For instance, a reasonable model\[21,22,23,24\] can be constructed by restricting the topology of the manifolds, simplicially approximated by \(\mathcal{T}\), to the three-sphere topology.

Under such assumptions, it is possible to discuss the general characteristics of the critical behavior of the model, so as to conclude that the partition function (2) is well defined in a convex region \(\mathcal{D}\) in the parameters space \((k_1, k_3)\). Moreover, numerical simulations suggest that a phase transition, presumably of first order, occurs. Such phase transition separates two phases, one, (hot), in which *crumpled* geometries dominate, while
the other, \textit{(cold)}, is characterized by geometries of a more regular nature (at least in the sense that their Hausdorff dimension is three).

By reference to the structure of the action in (2), these results hint toward a mechanism which control, in terms of the gravitational coupling the onset of a regular geometry out of crumbled \textit{quantum} universes. It is not yet clear if one can take any significant continuum limit of the theory in the critical regime, however it seems that a sensible vacuum exists in the cold phase of the model developed by Boulatov and Krzywicki\textsuperscript{25}. Such vacuum is a reasonable candidate for a physically acceptable continuum limit of the theory.

The starting point of this Paper, which reviews and extends some recent work\textsuperscript{37,38,39,40} of ours, is the observation that the ideas and techniques introduced in Riemannian geometry by Gromov’s\textsuperscript{36} coarse grained description of the space of \textit{n}-dimensional riemannian structures provide a natural framework for addressing and, to a large extent, solving the above issues. In particular, we show that the partial results on three-dimensional simplicial quantum gravity recalled above, most of which are based on computer simulations, can analytically recovered and put in a systematic geometrical perspective within a general regularization scheme which exploits the coarse-graining description of riemannian manifolds introduced by Gromov. The rationale which motivates such an approach follows by noticing that the problems encountered in evaluating partitions functions in simplicial quantum gravity are counting problems, typically the numbering of the different topological types of manifolds contributing with a given volume or average curvature to the state of the system. Similar questions appear in disguise in riemannian geometry in the form of finiteness theorems concerning the topology of riemannian manifolds, and the basic observation underlying Gromov’s analysis is that such finiteness theorems do not depend on a detailed local description of the geometry of the manifold, but they are rather connected to controlling geometry in the large by imposing some natural constraints on the size of the manifold. Such constraints concern the diameter, the volume and the sectional curvatures of the manifold, and it is not too difficult to realize, already at this point, that they are the geometric conditions under which (a weak form of) the entropy bounds mentioned above hold true. These entropy bounds, which we prove here as a by-product of a detailed analysis of three-dimensional simplicial gravity, are basically a consequence of Gromov’s compactness theorem (see below). They are one of the main reasons that can make Gromov’s spaces of bounded geometries an appropriate mathematical setting for discussing models of \textit{n}-dimensional quantum gravity.

Other important motivations for adopting Gromov’s coarse grained point of view can be seen at work when we exploit the structure of Gromov’s spaces to the effect of proving the basic fact that three-dimensional simplicial quantum gravity is connected with a gaussian theory describing the random insertion of three-dimensional cells onto the two-dimensional skeleton of suitable triangulations. It is interesting to remark
that, from a geometrical point of view, such role of the two-skeleton does not surprise, since it is known that triangulated \( n \)-manifolds are determined by their \([n/2] + 1\)-skeleta. In particular, for \( n = 3 \), a triangulated three-manifold is determined by the (simplicial) isomorphism class of its two-skeleton. Obviously is to be stressed that the two-skeleton of a triangulation is not, in general, the triangulation of a two-dimensional surface, but it is possible to take care of this difficulty. In this connection, rather than generic triangulations, we use as a regularization framework the simplicial approximations generated by minimal geodesic balls coverings of three-dimensional riemannian manifolds. The riemannian manifolds in question are thought of as points in the infinite dimensional compact space of bounded geometries generated by Gromov-Hausdorff completion of the set of all riemannian three-manifolds with suitable bounds on curvatures, volume and diameter. As already remarked, the choice of such configurational space for the theory, allows for a natural control of the topology of the manifolds involved in the statistical sum, and yields naturally for those entropy bounds which, otherwise are to be imposed \textit{ad hoc}.

It must be noticed that the constraints on curvatures, volume and diameter, (sectional curvatures bounded below by a real constant, volume bounded below and diameter bounded above), can be removed if one carefully perform a direct limit of such Gromov spaces. This is a most delicate issue, (its counterpart in dimension two is connected to summation over all surface genera), however we defer the analysis of this further limiting procedure to a forthcoming paper: even if our treatment is so biased, we trust that the reader will be convinced of the usefulness of Gromov’s spaces of bounded geometries in yielding a unified framework within which to study dynamically triangulated quantum gravity models.

Now we briefly delineate the strategy adopted and the main results obtained.

For the convenience of the reader, we begin by providing a brief review of some elementary aspects of Gromov’s theory of spaces of bounded geometry. We apologize to the expert reader who may skip section 2.

The starting point of our analysis is the observation that geodesic balls coverings of manifolds of bounded geometry naturally yields for simplicial approximations which are strictly related to the topology of the underlying manifold. In particular the homotopy type of the manifold can be labelled by the one-skeleton of the covering. This is a rather non-trivial fact which allows to discuss fluctuations of the homotopy types of manifolds of bounded geometry, in any dimension. Such matters have already been discussed at length in the papers, where they have been applied to the generic \( n \)-dimensional manifold of bounded geometry. Here, by combining reviews of our former work and new results, we specialize to dimension three, and exploit the theorem of Dancis recalled above, theorem which relates the combinatorial isomorphism class of a simplicial three-manifold to its two-
skeleton. In this way, we can put into work the above homotopy classification by writing down, rather than the discrete gravitational action \( (\mathcal{L}) \), the simplicial quantum gravity partition function for the two-skeleton of the geodesic ball coverings of the three-manifolds in question, and by studying the thermodynamical limit of such partition function as the covering becomes finer and finer.

A key point in this part of the paper, is the connection which can be established between the statistical model so defined and the classical statistical mechanics of a lattice gas on a suitable (denumerable) infinite graph. The interplay between lattice gas theory and the compactness properties of Gromov’s space of bounded geometries allow to establish the existence of the thermodynamical limit of the partition function, to understand its phase structure, and in particular to prove the existence of a critical regime of the theory where the role of the topology of the manifolds come to the fore.

The use of a partition function associated with the two-skeleton of a geodesic ball covering for describing a three-dimensional manifold, \( (\text{i.e.}, \) of a two-dimensional statistical theory for describing a three-dimensional object), has some advantages and some drawbacks. The advantages concern a qualitative similarity between the theory described here and the general features of two-dimensional quantum gravity models based on random triangulation of surfaces. The drawbacks come about by noticing that the resulting statistical approach is rather combinatorial and measure-theoretic in spirit, to the effect that geometry plays only a relatively auxiliary role in the theory. In particular, it is difficult to provide a sharp estimate of the entropic contribution to the partition function. Moreover, it is difficult to get information on the existence and on the properties of a sensible continuum limit of the theory, (if any such limit exists). This circumstance does not surprise, since the fact that we are dealing with three-dimensional objects is only indirectly exploited (through the role of the first order intersection pattern of the covering), and topology only enters in a relatively elementary way in the theory.

A more fundamental approach, based on first principles which from the very onset call into play geometry and topology is thus called for. Such an approach is dealt with in the second part of the paper, and its implementation relies on the use of few elementary facts of simple homotopy theory.

Roughly speaking, simple homotopy can be related to the way the tetrahedra of the nerve of the covering are attached to the two-dimensional faces of the two-skeleton. Inequivalent ways of attaching, up to a suitable action of the fundamental group, provide an obstruction to lifting an homotopy equivalence to a topological equivalence, and are of natural interest for simplicial gravity.

In general, the simple homotopy description of the insertion of simplices into a lower dimensional complex is a difficult matter to handle. However, in dimension three, such insertion can be equivalently described by attaching a finite number of two-dimensional and three-dimensional cells to a vertex of the two-dimensional skeleton, (the boundary of
the three-cells being non-trivially pasted onto the two-cells via the action of a suitable incidence matrix).

If one is willing to blend a statistical formalism with such geometrical framework, it is natural to attach such cells according to a Gaussian distribution, (in the sense that, given an orthogonal representation of the fundamental group, the characteristic maps describing the insertion of the cells in the two-skeleton, are distributed according to a Gaussian probability law with variance related to the representation of the incidence matrix). Not surprisingly, the outcome of this construction is that a Reidemeister-Franz representation torsion\(^43\) makes its appearance in the statistical sum; what is less obvious is that such Gaussian pasting reproduces the statistical sum of simplicial three-dimensional quantum gravity, \(^44\).

The Reidemeister-Franz representation torsion is a combinatorial invariant that, in its analytical counterpart, namely as Ray-Singer\(^44\) torsion, also naturally appears in Schartz-Witten’s approach to three-dimensional gravity\(^32\). Thus the combinatorial result we get indicates that one is on the right track for connecting the simplicial theory to the continuous formalism.

As the geodesic ball covering becomes finer and finer, we prove that the three-dimensional partition function obtained by the above construction is well defined if the couplings \((k_1, k_2)\) vary in a convex region whose boundary is determined by the Reidemeister torsions sampled. The rationale underlying this result is an intriguing connection between the thermodynamical limit of the partition function and a group-theoretic construction of a Gaussian measure on the general linear group \(GL(\mathbb{R})\).

The critical regime of the system is also discussed to the effect of confirming the existence of a non-trivial phase structure of the theory. In the thermodynamical limit the system exhibits either a phase transition describing the passage from a simple homotopy type to another, or, within a given simple homotopy type, a first order phase transition. In such a critical regime, the partition function is evaluated and shown to be equal to the corresponding value of the Reidemeister-Franz torsion.

From a physical point of view, since the representation torsions are combinatorial invariant, it follows that at this latter critical point the partition function has lost memory of the details of the simplicial approximation used, (\(viz\), that the given combinatorial approximation was obtained through geodesic balls coverings), and a sort of scaling limit is active in such a case, (obviously, a full scaling limit would require also the removal of the cut-offs on curvatures, diameter and volume which specify the space of bounded geometry in which the statistical sum has been computed. In this way, as already remarked, one can sample all topologies). Such critical point can be rather naturally identified as the transition numerically found by Boulatov and Krzywicki\(^25\), and which appears as a suitable candidate for the vacuum of the theory. As in their case, the critical regime we discuss holds in a cold phase, namely for geometries of finite (Hausdorff) dimension. This follows
by noticing that spaces of bounded geometry are compact, and simplicial approximations associated with minimal geodesic balls coverings yield, as the covering becomes finer and finer, objects which are no more singular than three-dimensional homology manifolds. Moreover, the phase transition in question is of first order, a fact which completely consistent with the numerical analysis quoted above. This first order nature also seems to indicate that there is no reasonable continuum limit of the theory. This lack of existence of a continuum limit is to be seen as statistical counterpart of the fact that in dimension three homology manifolds are the (Gromov-Hausdorff) closure of the space of riemannian manifolds of bounded geometry. The Gromov-Hausdorff topology, which appears as the natural topology to put on the space of riemannian structures in order to implement a statistical formalism of use in (euclidean) simplicial quantum gravity, is so weak that in the thermodynamical limit one gets a large entropic contribution from geometrical object which are so convoluted as to have no control on their topological structure.

In higher dimensions (e.g., in dimension four), the (Gromov-Hausdorff) closure of spaces of bounded geometries is nicer, being generated by topological manifolds, and on such grounds we would expect the existence of a more reasonable continuum limit of the corresponding statistical theory. Here we do not address four-dimensional simplicial quantum gravity, even if some of the results presented in this paper appear to be readily generalizable to dimension four. A rough explanation of this restriction comes about by noticing that our construction based on simple homotopy theory is intrinsically three-dimensional. To be more precise, given an incidence \( \mathbb{Z}(\pi_1) \)-matrix, (see section 5, for definitions), representing the simple homotopy type of a four-dimensional manifold, (or more in general, of a \( n \)-dimensional manifold), then, there always exists a three-dimensional \( CW \)-complex realizing such incidence matrix. Thus, in order to address the case of four dimension, we must put into work something finer than simple homotopy theory. In the concluding remarks of the paper, we provide few, rather non-conclusive, indications in this direction.

2 Geodesic balls coverings of manifolds of bounded geometries

Our starting point is a classical result of M.Gromov according to which the space of all riemannian structures of a given dimension, not necessarily of a given topology, with natural constraints on their diameter and Ricci curvature is characterized by some remarkable compactness properties which make the global characteristics of geometric functionals, (typically their boundedness), very trasparent and natural. The compactness we are referring to can be very effectively brought to surface by introducing a suitable notion of distance, (actually, a metrizable uniform structure), on the space of all riemannian manifolds of dimension \( n \), (possibly with different underlying topologies) considered as elements of the more general category of all compact metric spaces. This
distance, first introduced by M. Gromov [36], generalizes the familiar notion of Hausdorff distance between compact subsets of a given metric space, and even if it looks rather formal, it provides a sort of visual distance between manifolds, a remark, this latter, that will be made more precise below by discussing geodesic balls coverings of manifolds of bounded geometry.

Consider two riemannian manifolds $M_1$ and $M_2$, and let $i_1(M_1)$ and $i_2(M_2)$ denote respectively isometric embeddings of $M_1$ and $M_2$ in some metric space $(A, d)$, (this may be a Euclidean space of sufficiently high dimension, but in full generality we must allow for curved and possibly infinite-dimensional embedding spaces). A Hausdorff distance in $(A, d)$ between $i_1(M_1)$ and $i_2(M_2)$ can be introduced according to

$$d^A_{\text{H}}[i_1(M_1), i_2(M_2)] = \inf \{ \epsilon > 0, U_\epsilon(i_1(M_1)) \supseteq i_2(M_2) \land U_\epsilon(i_2(M_2)) \supseteq i_1(M_1) \}$$

(3)

where $U_\epsilon(i_1(M_1)) = \{ z \in A : d(z, i_1(M_1)) \leq \epsilon \}$ and similarly for $U_\epsilon(i_2(M_2))$. In other words $d^A_{\text{H}}[i_1(M_1), i_2(M_2)]$ is the lower bound of the $\epsilon$ such that $i_1(M_1)$ is contained in the $\epsilon$-neighborhood of $i_2(M_2)$ and vice versa. The Gromov distance between the two riemannian manifolds $M_1$ and $M_2$, $d_G(M_1, M_2)$, is then defined according to the following [36]

**Definition 1** $d_G(M_1, M_2)$ is the lower bound of the Hausdorff distances

$$d^A_{\text{H}}[i_1(M_1), i_2(M_2)]$$

(4)

as $A$ varies in the set of metric spaces and $i_1, i_2$ vary in the set of all isometric embeddings of $M_1$ and $M_2$ in $(A, d)$.

Notice that $d_G$ is not, properly speaking, a distance since it does not satisfy triangle inequality, it rather gives rise to a metrizable uniform structure. For our purposes, we can use it as if it were a distance function. In any case, in working with Gromov distance one gets a sense of a geometric nearness among riemannian structures having to do with a classification of riemannian manifolds according to how they can be covered by small geodesic balls. In particular two riemannian manifolds can be considered near to each other in the Gromov distance if the can both be covered with a collection of small geodesic balls arranged in a similar packing configuration. In order to explain this remark, perhaps the main source of geometrical intuition in working with the Gromov distance, we are naturally led to introduce the following class of riemannian structures [36]

**Definition 2** For $k$ a real number and $D$ a positive real number let $\text{Ric}[n, k, D]$ denote the space of isometry classes of closed connected $n$-dimensional riemannian manifolds $(M, g)$ with $\text{Ric}_M \geq -(n - 1)k$ and $\text{diam}_M \leq D$. 

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Recall that, for a closed \((n\)-dimensional) Riemannian manifold \(M\) we define the diameter of \(M\) as \(\sup_{(p,q) \in M \times M} d(p,q)\) where \(d(\cdot,\cdot)\) denotes the distance function of \(M\). And if \(\text{Ric}(M)\) denotes the Ricci curvature of \(M\), we let
\[
k(x) \equiv \inf \{\text{Ric}(u,u); u \in T_x M, |u_x| = 1\}
\]
thus, the lower bound of the Ricci tensor of a Riemannian manifold \(M\) is defined as the lower bound of \(k(x)\) as \(x\) varies in \(M\).

The point of the introduction of \(R[n,k,D]\) is that for any manifold \(M\) in such class it is possible to introduce suitable coverings by geodesic balls providing a coarse classification of the Riemannian structures occurring in \(Ric[n,k,D]\). In particular for any manifold \(M\) in such class and for any given \(\epsilon > 0\) it is always possible to find an ordered set of points \(\{p_1,\ldots,p_N\}\) in \(M\), so that
\((i)\) the open balls \(B_M(p_i,\epsilon) = \{x \in M|d(x,p_i) \leq \epsilon\}, i = 1,\ldots,N\), cover \(M\); in other words the collection
\[
\{p_1,\ldots,p_N\}
\]
is an \(\epsilon\)-net in \(M\).
\((ii)\) the open balls \(B_M(p_i,\epsilon/2), i = 1,\ldots,N\), are disjoint, \(i.e.,\) \(\{p_1,\ldots,p_N\}\) is a minimal \(\epsilon\)-net in \(M\).

Any such minimal net is also characterized by its intersection pattern, namely by the set
\[
I_\epsilon(M) \equiv \{(i,j)|i,j = 1,\ldots,N|B(p_i,\epsilon) \cap B(p_j,\epsilon) \neq \emptyset\}
\]
Any two manifolds \(M_1\) and \(M_2\) in \(Ric[n,k,D]\) endowed with \(\epsilon\)-minimal nets \(\{p_1,\ldots,p_N\}\) and \(\{q_1,\ldots,q_L\}\), respectively, are said to be equivalent if and only if \(N = L\) and if they have the same intersection pattern. Namely if, up to combinatorial isomorphisms, we can write
\[
N_{(\epsilon)}^{(0)}(M_1) = N_{(\epsilon)}^{(0)}(M_2)
I_{(\epsilon)}(M_1) = I_{(\epsilon)}(M_2)
\]
where we have introduced the filling function \(N_{(\epsilon)}^{(0)}(M)\) of the covering, \(i.e.,\) the function which associates with \(M\) the maximum number of geodesic balls realizing a minimal \(\epsilon\)-net on \(M\).

The filling function and the (first) intersection pattern characterize the 1-skeleton, \(\Gamma_1(M)\), of the geodesic balls covering. Similarly, upon considering the higher order intersection patterns of the set of balls \(\{B_i(\epsilon)\}\), we can define the two-skeleton \(\Gamma_2(M)\), and eventually the nerve \(N\{B_i\}\) of the geodesic balls covering of the manifold \(M\), this is a simplicial complex realized according to the following rules:
\((i)\) the vertices \(p_i^{(0)}\) of \(N\) correspond to the balls \(B_i(\epsilon)\),
(ii) the edges $p_{ij}^{(1)}$ correspond to pairs of geodesic balls \( \{ B_i(\epsilon), B_j(\epsilon) \} \) having a non-empty intersection \( B_i(\epsilon) \cap B_j(\epsilon) \neq \emptyset \),

(iii) the faces $p_{ijk}^{(2)}$ correspond to triples of geodesic balls with non-empty intersection \( B_i(\epsilon) \cap B_j(\epsilon) \cap B_k(\epsilon) \neq \emptyset \),

(iv) the \( k \)-simplexes $p_{i_1,i_2\ldots i_{k+1}}^{(k)}$ correspond to collections of \( k + 1 \) geodesic balls such that \( B_1 \cap B_2 \cap \ldots \cap B_{k+1} \neq \emptyset \).

If \( \epsilon \) is sufficiently small this nerve gives rise to a polytope which approximate the manifold \( M \). In particular, the 1-skeleton is just the graph providing the vertex-edge structure of this approximation. It is to be stressed that since for manifolds in \( \text{Ric}[n,k,D] \) arbitrarily small metric balls need not be contractible, this approximation is rather rough. In any case, what happens is that the inclusion of sufficiently small geodesic balls into suitably larger balls is homotopically trivial. Thus one gets a polytope which is homotopically dominating the underlying manifold. This homotopical approximation yielding for the homotopy finiteness theorem recalled below is all we need for the analysis of simplicial gravity that follows.

The equivalence relation (7) partitions \( \text{Ric}[n,k,D] \) into disjoint equivalence classes whose finite number can be estimated in terms of the parameters \( n, k, D \). In particular, if we set

\[
\mathcal{O}^{(c)}_{(\lambda,\Gamma)} \equiv \{ M \in \text{Ric}[n,k,D] : N^{(0)}_{(c)}(M) = \lambda, I_{(c)}(M) = \Gamma \}
\]

where \( \lambda \) is a given positive integer and \( \Gamma \) is a given graph defined by a collection of \( \lambda \) vertices \( \{ p_1, \ldots, p_\lambda \} \), and by a collection of edges \( \{ p_i, p_j \} \) joining the vertices \( p_i \) and \( p_j \). Then for each given \( \epsilon \) we can write \( \text{Ric}[n,k,D] \) as a disjoint finite union

\[
\text{Ric}[n,k,D] = \bigcup_{(\lambda,\Gamma)} \mathcal{O}^{(c)}_{(\lambda,\Gamma)}
\]

going over all possible choices of \( \lambda \) and \( \Gamma \) accessible to the manifolds in \( \text{Ric}[n,k,D] \).

Thus each equivalence class of manifolds is characterized by the abstract (unlabelled) graph \( \Gamma_{(c)} \) defined by the 1-skeleton of the \( \mathcal{L}(\epsilon) \)-covering. The order of any such graph (i.e., the number of vertices) is provided by the filling function \( N^{(0)}_{(c)} \), while the structure of the edge set of \( \Gamma_{(c)} \) is defined by the intersection pattern \( I_{(c)}(M) \). It is important to remark that on \( \text{Ric}[n,k,D] \) either the filling function or the intersection pattern cannot be arbitrary. The former is always bounded above for each given \( \epsilon \), and the best filling of a riemannian manifolds with geodesic balls of radius \( \epsilon \) is realized on (portions of) spaces of constant curvature. The latter is similarly controlled through the geometry of the manifold to the effect that the average degree, \( d(\Gamma) \), of the graph \( \Gamma_{(c)} \), (i.e., the average number of edges incident on a vertex of the graph), is bounded above by a constant as the radius of the balls defining the covering tend to zero, (i.e., as \( \epsilon \to 0 \)). Such constant is independent from \( \epsilon \), and can be estimated in terms of the parameters
These remarks are the content of three basic propositions which obtain for manifolds in $\text{Ric}[n,k,D]$. The first proposition is an immediate consequence of the Bishop-Gromov volume comparison theorem. It provides the quoted geometrical bound on the maximum number of geodesic balls realizing a minimal $\epsilon$-net.

**Proposition 1** Let $\text{Ric}(g)$ and $d(M)$ respectively denote the Ricci tensor and the diameter of a manifold $M \in \text{Ric}[n,k,D]$. Assume that $\text{Ric}(g) \geq (n-1)kg$ for some real number $k$, and let us consider a geodesic ball, $\bar{B}(d(M))$, of radius $d(M)$ in the model space $\bar{M}_k$ of constant sectional curvature equal to $k$. Denote by $\bar{N}(\epsilon)(k)$ the value that the filling function $N(\epsilon)$ attains on $\bar{B}(d(M))$, viz.

$$N(\epsilon)(k) = \frac{\int_0^{d(M)} \bar{J}(t)^{n-1} dt}{\int_0^{d(M)} \bar{J}(t)^{n-1} dt}$$

where $\bar{J}(t) = t$, $\frac{\sin \sqrt{kt}}{\sqrt{k}}$, $\frac{\sinh \sqrt{-kt}}{-\sqrt{k}}$ for $k = 0, > 0, < 0$ respectively. Then

$$N(\epsilon)(M) \leq \bar{N}(\epsilon)(k)$$

The next proposition provides an a priori estimate on the average degree of the graph $\Gamma(\epsilon)$ describing the intersection pattern for minimal nets.

**Proposition 2** Let $\{p_1, \ldots, p_N\}$ be any $\epsilon$-net in $M \in \text{Ric}[n,k,D]$. Then there is a constant $N_2$ depending on $k$, and $D$, such that for any $x \in M$, the geodesic ball $B_x(\epsilon)$ intersects at most $N_2$ of the balls $B(p_1, \epsilon), \ldots, B(p_N, \epsilon)$, and as $\epsilon \to 0$ we get

$$\limsup_{\epsilon \to 0} N_2(\epsilon) \leq (C_{n,k,D})^{n-1}$$

where $C_{n,k,D}$ is a constant depending only on the parameters $n$, $k$, $D$, (and not from $\epsilon$), which characterize $\text{Ric}[n,k,D]$.

The bound on $d(\Gamma(\epsilon))$ follows by applying the above theorem in particular to each of the balls of the covering. Then $N_2(\epsilon)$ has the meaning of an upper bound to the degree of $\Gamma(\epsilon)$ at the vertex $x = p_i$ considered. Since this bound holds for any vertex $p_i$, $i = 1, \ldots, N(\epsilon)$, we get as well

$$d(\Gamma(\epsilon)) \leq (C_{n,k,D})^{n-1}$$

Finally the third proposition allows us to compare distances in minimal $\epsilon$-nets with the some intersection pattern.
**Proposition 3** Let \( \{p_1, \ldots, p_N\} \) and \( \{q_1, \ldots, q_N\} \) respectively denote minimal \( \varepsilon \)-nets with the same intersection pattern in \( M_1 \) and \( M_2 \in \text{Ric}[n, k, D] \). Then for any \( K \) such that

\[
d_{M_1}(p_i, p_j) < K \cdot \varepsilon
\]  

there is a constant \( N_3(K) \), depending on \( K \), and \( r, D, n \) for which

\[
d_{M_2}(q_i, q_j) < N_3(K) \cdot \varepsilon
\]

In other words, equivalent \( \varepsilon \)-nets on manifolds in \( \text{Ric}[n, k, D] \), are metrically equivalent up to a dilatation. This remark can be made more precise if we recall the notion of Lipschitz distance between two metric spaces.

**Definition 3** Let \( X \) and \( Y \) be two metric spaces and let \( f: X \to Y \) be a (biLipschitz) homeomorphism. The Lipschitz distance between \( X \) and \( Y \), \( d_L\{X, Y\} \), is the lower bound, as \( f \) varies in the collection of the above homeomorphisms, of the \( L \) such that

\[
e^{-L} \leq \frac{d_Y(f(p), f(q))}{d_X(p, q)} \leq e^L
\]

for all \( p \neq q \) in \( X \).

According to this latter definition, minimal \( \varepsilon \)-nets with the same intersection patterns in any given two manifolds \( M_1 \) and \( M_2 \in \text{Ric}[n, k, D] \), are at finite Lipschitz distance from each other; i.e.,

\[
d_L(\{p_1, \ldots, p_N\}, \{q_1, \ldots, q_N\}) < C
\]

where the constant \( C \) is related to \( N_3 \). It can be also shown that the two riemannian manifolds \( M_1 \) and \( M_2 \) are at finite Gromov distance from each other. This circumstance is connected to the following result which relates the notion of Gromov distance between riemannian manifolds to the Lipschitz distance between nets of points on such manifolds.

**Proposition 4** If a sequence \( \{M_i\} \) of riemannian manifolds converges to a space \( M \), (not necessarily a riemannian manifold, as we shall see momentarily), for the Gromov distance, then for every positive \( \varepsilon \) and \( \bar{\varepsilon} > \varepsilon \), every \( \varepsilon \)-net of \( M \) is the limit for the Lipschitz distance of a sequence \( N_i \) where \( N_i \) is an \( \bar{\varepsilon} \)-net of \( M_i \). Conversely, if \( M \) and \( M_i \) are riemannian manifolds in \( \text{Ric}[n, k, D] \), and if for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-net of \( M \) which is the limit for the Lipschitz distance of a sequence of \( \varepsilon \)-nets \( N_i \) of \( M_i \), then \( M_i \) converges to \( M \) for the Gromov distance.
In other words, two Riemannian manifolds $M_1$ and $M_2 \in \text{Ric}[n, k, D]$ are nearby to each other in the Gromov distance if, given a suitably small $\epsilon > 0$, they admit $\epsilon$-nets, $\{p_1, \ldots, p_N\}$ and $\{q_1, \ldots, q_N\}$ respectively, such that their Lipschitz distance is small.

Thus two Riemannian manifolds in $\text{Ric}[n, k, D]$ get closer and closer in the Gromov distance if we can introduce on them finer and finer minimal $\epsilon$-nets of geodesic balls with the same intersection patterns.

As an immediate consequence of proposition 4 one can easily show that any two Riemannian manifolds $M_1$ and $M_2$, such that $d_G(M_1, M_2) = 0$ are necessarily isometric. More in general, the preceding results imply the following approximation theorem 1. For any $\epsilon > 0$ there is a finite collection $\{M^*_a\}$, $(a = 1, \ldots, k(\epsilon))$, of Riemannian manifolds in $\text{Ric}[n, k, D]$, such that

$$d_G(M, \{M^*_a\}) < \epsilon$$

(18)

for any Riemannian manifold $M$ in $\text{Ric}[n, k, D]$.

In other words, the set $\text{Ric}[n, k, D]$ of closed Riemannian manifolds $(M, g)$ of dimension $n$, with Ricci curvature bounded below and diameter bounded above, is precompact in the set of all compact metric spaces endowed with the Gromov distance.

This last theorem implies that we can approximate any $M$ in $\text{Ric}[n, k, D]$ by one of the $\{M^*_a\}$ in the sense that, for a given cut-off length scale $\epsilon$, we can choose a finite number of model Riemannian manifolds $M^*_a$ such that any given manifold in $\text{Ric}[n, k, D]$ is, on length scales sufficiently larger than the cut-off $\epsilon$, metrically indistinguishable from one of the manifolds $M^*_a$.

It must be stressed that for manifolds in $\text{Ric}[n, k, D]$ we have no a priori bounds on the injectivity radius, and small geodesic balls can be topologically quite complicated. This situation is made manifest by the fact that a sequence of Riemannian manifold in $\text{Ric}[n, k, D]$, $\{M_i\}$, may converge, under the Gromov distance, to an object which is no longer a manifold. Namely, the space $(\text{Ric}[n, k, D], d_G)$ is not complete, (this lack of completeness explains why the approximation theorem above is a precompactness result). What happens is that the manifolds in the sequence $\{M_i\}$ may degenerate in the sense that the constraints on the elements of $\text{Ric}[n, k, D]$ are so weak that pathological metrics with sectional curvatures uniformly bounded, diameter uniformly bounded above, but with injectivity radius converging uniformly toward zero everywhere are in $\text{Ric}[n, k, D]$. It follows that metric spaces of a very complex structure may result from the Gromov-convergence of a sequence of Riemannian manifolds in $\text{Ric}[n, k, D]$. In general such singular spaces can be characterized as length spaces, that is metric spaces where the distance between any two points is the lower bound of the length of the curves joining such two points. This is a rather weak characterization and the explicit structure of the boundary points of $\text{Ric}[n, k, D]$ is largely unknown. The reader interested to non-trivial examples
and further analysis of these matters may profitably consult the papers of Fukaya and Pansu.

The situation described above gets considerably simpler if we consider the subspace of $\text{Ric}[n, k, D]$ generated by the class $\text{R}[n, r, D, V]$ of closed riemannian $n$-dimensional manifolds in $\text{Ric}[n, k, D]$ with sectional curvatures, (rather than Ricci curvature), bounded below by $r$, and with volume bounded below by a constant $V$. For such subspace the following results obtain:

**Theorem 2** Let $\mathcal{R}(n, r, D, V)$ denote the Gromov-Hausdorff closure of the set of closed riemannian $n$-manifolds with sectional curvatures bounded below by $r$, diameter bounded above by $D$, and volume bounded below by $V$. Then each length space in $\mathcal{R}(n, r, D, V)$ is either a riemannian $n$-manifold or a $n$-dimensional homology manifold. Moreover, if $n \geq 5$, a $d_G$-limit of a sequence of riemannian manifolds in $\mathcal{R}(n, r, D, V)$ is a topological $n$-manifold.

(For notational convenience in what follows, we shall indiscriminately use the word manifold and the symbol $M$ either to mean a regular riemannian manifold or a metric homology manifold arising as $d_G$-limit of a sequence of riemannian manifolds. When a more explicit characterization is necessary, the specifications riemannian or homology are used).

We conclude this section by recalling the following basic homotopy finiteness result. It provides the topological rationale underlying the use of spaces of bounded geometries in simplicial quantum gravity

**Theorem 3** For any dimension $n \geq 2$, and for $m$ sufficiently large, manifolds in $\mathcal{R}(n, r, D, V)$ with the same 1-skeleton $\Gamma_{(m)}$ are homotopically equivalent, and the number of different homotopy-types of manifolds realized in $\mathcal{R}(r, D, V)$ is finite.

(we recall that two manifolds $M_1$ and $M_2$ are said to have the same homotopy type if there exists a continuous application $\phi$ of $M_1$ into $M_2$ and $f$ of $M_2$ into $M_1$, such that both $f \cdot \phi$ and $\phi \cdot f$ are homotopic to the respective identity mappings, $I_{M_1}$ and $I_{M_2}$. Obviously, two homeomorphic manifolds are of the same homotopy type, but the converse is not true).

Further details and many useful examples of Gromov-Hausdorff convergence of manifolds of bounded geometry can be found in the quoted paper of K.Fukaya

3 Homotopy types of three-manifolds of bounded geometry and lattice gas statistical mechanics

In the following sections we shall explicitly assume that $n = 3$, and henceforth we shall denote by $\mathcal{R}(r, D, V)$ the generic compact metric space of three-dimensional (homology)
manifolds of bounded geometry, characterized by Gromov-Hausdorff completion of closed
riemannian three-manifolds with sectional curvatures bounded below by $k$, diameter
bounded above by $D$, and volume bounded below by $V$. As already stressed, many of
the statements which follows hold true regardless of the dimension, $(n \geq 2)$, and as a
matter of fact one can develop a rather general theory along the line discussed below.

Let us begin by recalling a theorem of Dancis, according to which the simplicial
isomorphism class of a simplicial three-manifold is determined by its underlying two-
skeleton. In its more general setting, the theorem reads (see th.1)

**Theorem 4** Let $M$ and $W$ be compact, triangulated homology $n$-manifolds. Let $k \geq n/2 + 1/2$. Given a simplicial isomorphism $f: \text{Skel}^{(k)}(M) \to \text{Skel}^{(k)}(W)$, there is a simplicial isomorphism $f_k$ of $M$ onto $W$ which is an extension of $f$.

(Where $\text{Skel}^{(k)}(M)$ and $\text{Skel}^{(k)}(W)$ denote the $k$-skeletons of the triangulated homology manifolds $M$ and $W$, respectively).

This result suggests that three-dimensional simplicial quantum gravity may, to some
extent, be described by the (dynamical triangulation) partition function associated with
the two-skeleton of the underlying simplicial approximation scheme adopted.

### 3.1 A two-dimensional partition function for three-manifolds

In order to implement the point of view implied by Dancis’ theorem, let $L(m) \equiv 1/m$, $0 < m < \infty$ denote a cut-off parameter to be interpreted as the radius $\epsilon/2$ of minimal geodesic balls coverings on manifolds of bounded geometry. With this notational remark and within the geometrical framework delineated above, the most suitable form, (at least for our purposes), of a simplicial quantum gravity partition function to be associated with a $L(m)$-geodesic ball two-skeleton is

$$
\Xi(m, z) = \sum_{\Gamma} z^{N(0)(\Gamma)} \exp \left\{ \beta N(1)(\Gamma) \right\}
$$

where $-\ln z \equiv c$, and $\beta$ are constants, and where the (finite) sum is over all inequivalent 1-skeletons $\Gamma_{(m)}$, with order $N^{(0)}(\Gamma)$ and size $N^{(1)}(\Gamma)$, realized by the possible $L(m)$-geodesic balls covering over manifolds in $\mathcal{R}(r, D, V)$, (the size $N^{(1)}(\Gamma)$ is the number of edges in the graph $\Gamma_{(m)}$). Notice that the more general form of such partition function would be

$$
\sum_{\Gamma^{(2)}(m)} \exp \left\{ -[c_0 N^{(0)}(\Gamma) - c_1 N^{(1)}(\Gamma) + c_2 N^{(2)}(\Gamma)] \right\}
$$

where $N^{(2)}(\Gamma)$ is the number of two-faces in the two-skeleton $\Gamma^{(2)}_{(m)}$, and $c_0$, $c_1$, $c_2$ are suitable constants. By exploiting Euler’s relation relating the number of faces, edges and
vertices to the Euler number of the two-skeleton, one can arrange the terms in (20) so as to reproduce (19) or other equivalent expressions. However, it must be stressed that the Euler number of the two-skeleton has no obvious topological meaning for the underlying three-manifold, (contrary to what happens for a surface), and thus there is some ambiguity in the choice of the two-dimensional partition function to be associated with the two-skeleton. Our choice (19) is the simplest partition function yielding for non-trivial results. It directly evidentiates the role of the one-skeleton $\Gamma^{(1)}(m)$ of the geodesic balls covering, rather then of the two-skeleton $\Gamma^{(2)}(m)$. As a consequence of the homotopy finiteness theorem recalled in the previous paragraph, it follows that, at least for manifolds in $\mathcal{R}(r, D, V)$, there is no loss in generality in doing so. We shall came back to the homotopical interplay between the one-skeleton and the two-skeleton later on, when estimating, in terms of the presentation of the fundamental group, the configurational entropy associated with (19).

A rigorous analysis of the properties of (19) follows by noticing that from a thermodynamical point of view, (i.e., when $m \to \infty$), $\Xi(m, z)$ has the structure of a grand-partition function computed for a lattice gas with negative pair interactions evaluated at inverse temperature $\beta$ and for a fugacity $z$. In order to avoid any misunderstanding, it must be stressed that the lattice gas in question evolves on a denumerable graph $\Omega$, (to be defined below), rather than on a regular, say hypercubic, lattice.

To discuss this equivalence more in details we explicitly identify the geodesic balls of a minimal $L(m)$-net on a manifold $M$ of bounded geometry with the vertices $\{p^{(0)}_1, \ldots, p^{(0)}_N\}$, of an abstract graph $\Gamma(M)$. Such $\Gamma(M)$ is defined by connecting the $\{p^{(0)}_i\}$ among themselves with undirected edges $p^{(1)}_{ij} = \{p^{(0)}_i, p^{(0)}_j\}$ if and only if the geodesic balls labelled $p^{(0)}_i$ and $p^{(0)}_j$ have a non-empty intersection when we double their radius. Any two such graph are considered equivalent if, up to the labelling $\{p^{(0)}_i\}$ of the balls, they have the same vertex-edge scheme, viz. we are considering unlabelled graphs associated with the possible one-skeletons of geodesic balls coverings of manifolds of bounded geometries. For any given $m$, any such graph can be topologically imbedded in a larger regular, (i.e., whose vertices have all the same degree), graph $\Omega(m)$, ($\Omega$ for short), having as its connected subgraphs all possible 1-skeletons $\Gamma^{(j)}(m)$ which are realized by $L(m)$-geodesic balls coverings as $M$ varies in $\mathcal{R}(r, D, V)$. One may think of such $\Omega$ as the configuration space for the finite set of possible 1-skeletons that can be realized by minimal nets on the manifolds in $\mathcal{R}(r, D, V)$.

For a given $m$, a graph realizing $\Omega$ can be constructively defined first by labelling the graphs $\Gamma^{(j)}(m)$ and order them by inclusion, i.e., $\Gamma^{(i)}(m) \subset \Gamma^{(j)}(m)$ if $\Gamma^{(i)}(m)$ is a subgraph of $\Gamma^{(j)}(m)$, and then by considering the union $H(m)$ over the prime graphs $\Gamma^{(k)}(m)$, (i.e., over those graphs which do not appear as subgraphs of other $\Gamma^{(j)}(m)$):

$$H(m) \equiv \bigcup_j \Gamma^{(j)}(m)$$  \hspace{1cm} (21)

Let $d(\Omega(m))$ the maximum degree of the vertices of $H$, (or which is the same, the maximum
degree of the vertices among those occurring in the various $\Gamma^{(j)}_{(m)}$). Notice that, for every $m$, we have $d(\Omega) \leq (C_{n,r,D})^{n-1}$, where the constant $C_{n,r,D}$ is the upper bound to the average degree of the graphs $\Gamma_{(m)}$ associated with the $L(m)$-geodesic balls realized in $R(r, D, V)$, (see proposition 2). With these preliminary remarks, we characterize the graph $\Omega_{(m)}$ according to the following

**Lemma 1** The graph $\Omega_{(m)}$ is the regular graph, of degree $d(\Omega_{(m)})$, of the least possible order having as its connected subgraphs all possible one-skeletons $\Gamma_{(m)}$ which are realized by $L(m)$-geodesic balls coverings as $M$ varies in $R(r, D, V)$

**PROOF.** The existence of such graph is assured by a theorem of P.Erdos and P.Kelly, and in order to construct it, we proceed as follows. First add to $H_{(m)}$ a set $I$ of $b_{(m)}$ isolated points, (the number of which, as indicated, is a function of the given $m$), a new graph is formed from $H_{(m)}$ and $I$ by adding edges between pairs of points in $I$ and $H_{(m)}$, (notice that no edges are added between vertices in $H_{(m)}$). The strategy is to give rise, in this way, to a new graph which is regular of degree $d(\Omega_{(m)})$ while keeping the number $b_{(m)}$ of vertices to be added to $H_{(m)}$ as small as possible. As expected, the number $b_{(m)}$ of added vertices depends only on the degree sequence of the graph $H_{(m)}$. In particular if $d_i$ denote the various degrees at the vertices of $H_{(m)}$, then $b_{(m)}$ is the least integer satisfying: (i) $b_{(m)}d(\Omega_{(m)}) \geq \sum_i (d(\Omega) - d_i)$, (ii) $b_{(m)}^2 - (d(\Omega) + 1)b_{(m)} + \sum_i (d(\Omega) - d_i) \geq 0$, (iii) $b_{(m)} \geq \max(d(\Omega) - d_i)$, and (iv) $(b_{(m)} + h_{(m)})d(\Omega)$ is even, where $h_{(m)}$ is the order of the union graph $H_{(m)}$.

It should be noted that in the large $m$ limit, $\Omega_{(m)}$ is a regular graph whose degree, $d(\Omega_{(m)})$, is independent from $m$, being bounded above in terms of $n, r, D$, by $(C_{n,r,D})^{n-1}$.

With these preliminary remarks we can work out the correspondence between the statistical system described by (19) and a lattice gas by identifying the collection of geodesic balls, providing the dense packing of $M$, with a gas of indistinguishable particles. Each particle can occupy at most one site of $\Omega_{(m)}$, (the geodesic balls of radius $L(m) = 1/m$ are disjoint), and the configuration of occupied sites corresponding to the net of balls covering $M$ is defined by the vertices of the graph $\Gamma(M) \subset \Omega_{(m)}$.

The interaction energy between two occupied sites is supposed to be different from zero only if the sites in question are connected by an edge $\{p_{i}^{(0)}, p_{j}^{(0)}\}$ of $\Gamma(M)$, namely if the points of the minimal $L(m)$-net corresponding to the lattice sites in question are in the intersection pattern of the manifold according to the definition recalled above, (the geodesic balls $p_{i}^{(0)}$ and $p_{j}^{(0)}$ have a non-empty intersection when their radius is doubled). More in general, $p_{i}^{(0)}$ and $p_{j}^{(0)}$ will be said to be neighbors if $\{p_{i}^{(0)}, p_{j}^{(0)}\}$ is an edge of the graph $\Omega_{(m)}$. Obviously, $\Omega_{(m)}$ will contain as possible configurations of occupied sites, (i.e., as possible subgraphs), not only those graphs $\Gamma(M)$ representing one-skeleton of
manifolds in $\mathcal{R}(r,D,V)$, but also graphs associated to the $b(m)$ added vertices (and to the corresponding edges), needed in order to construct $\Omega_{(m)}$. In any case, given a configuration of occupied sites in $\Omega_{(m)}$ represented by a graph $\Gamma = \Gamma(M)$ with $N_{(m)}^{(0)}(M)$ vertices, the total interaction energy corresponding to such configuration is given by

$$E(\Gamma(M)) = -\epsilon_0 N_{(m)}^{(1)}(\Gamma(M))$$

where $\epsilon_0$ is a constant (which provide the scale of energy, and which here and henceforth we set equal to one) and $N_{(m)}^{(1)}$ is the number of edges in the graph $\Gamma(M)$, viz., the total number of pairs $(i,j)$ belonging to the intersection pattern of the minimal $L(m)$-net $\{p_1^{(0)}, \ldots, p_N^{(0)}\}$. The corresponding Boltzmann weight (as a lattice gas) is then

$$\exp[-\beta E(\Gamma(M))] = \exp[\beta N_{(m)}^{(1)}(\Gamma(M))]$$

The existence of the thermodynamical limit of the statistical sum (19), which exists owing to the compactness of $\mathcal{R}(r,D,V)$, as we shall see, along with the boundary condition $\lim_{m \to \infty} b(m) = 0$, allows us to identify $\Xi(m,z)$ with the (grand canonical ensemble) partition function of a lattice gas evaluated at a temperature proportional to the average degree of the graph representing the 1-skeleton of the manifolds in $\mathcal{R}(r,D,V)$, viz.,

$$KT \sim \langle N_{(m)}^{(0)} \rangle^{-1} N_{(m)}^{(1)}$$

where $K$ denotes Boltzmann’s constant, and at a chemical potential $i_c \equiv -cKT$ proportional to the average volume, (as measured by $\langle N_{(m)}^{(0)} \rangle \text{Vol}B^e(1)m^{-n}$), of the manifolds in $\mathcal{R}(r,D,V)$, averages being understood with respect to the statistical sum (19).

### 3.2 Gromov-Hausdorff compactness and the thermodynamical limit

Roughly speaking, according to the correspondence just established the configurations of the lattice gas considered are labelled by random graphs representing the possible 1-skeletons $\Gamma_{(m)}$ of manifolds with bounded geometry, and, for a sufficiently large $m$, we get a grand-ensemble of such configurations whose thermodynamical parameters keep track of the average volume and curvature of the manifolds in $\mathcal{R}(r,D,V)$. We can exploit such a correspondence in discussing the thermodynamical limit $m \to \infty$ for (19). From a technical side, one can prove the existence of such limit by exploiting the compactness properties of $\mathcal{R}(r,D,V)$. These properties easily allow to show that $\lim_{m \to \infty} \Xi(m,z)$ exists even if in general this limit is not unique. To be more specific, let us remark that if (19) is truly to be interpreted as the grand-partition function of a lattice gas statistical mechanics on the denumerable graph, $\Omega_\infty$, resulting from $\lim_{m \to \infty} \Omega_{(m)}$, and if this statistics is to be connected to geometry, then we must have a measure, of geometrical origin, on the
possible configuration space of the system: the set of all possible subgraphs of \( \Omega_\infty \), set which we rewrite as \( \{0,1\}^{\Omega_\infty} \), (this is the set of functions from the set of vertices of \( \Omega_\infty \) to \( \{0,1\} \), expressing occupation or not of the vertices).

In order to discuss the existence of such a measure, we start by noticing that since \( \mathcal{R}(r,D,V) \) is compact, we can easily turn it into a measure space, (proofs of results below not explicitly given are simple adaptations of proof that may be found, for example, in the book of K.R.Parthasarathy \( \square \)).

**Definition 4** The smallest \( \sigma \)-algebra of subsets of \( \mathcal{R}(r,D,V) \) which contains all \( d_G \)-open subsets of \( \mathcal{R}(r,D,V) \) is denoted by \( \mathcal{B} \) and is called the Borel \( \sigma \)-algebra of \( \mathcal{R}(r,D,V) \).

**Definition 5** By a measure \( \mu \) on \( \mathcal{R}(r,D,V) \) we shall understand a measure (or the completion of a measure) defined on \( \mathcal{B} \). If the measure is normalized, \( (\mu[\mathcal{R}(r,D,V)] = 1) \), and complete we call it a probability measure over \( \mathcal{R}(r,D,V) \). The set of all measures on \( [\mathcal{R}(r,D,V),\mathcal{B}] \) will be denoted by \( \mathcal{M}(\mathcal{R}(r,D,V)) \).

In connection with the latter definition we recall the if \( C(\mathcal{R}(r,D,V)) \) denotes the Banach space of all the \( d_G \)-continuous real (complex) valued functions on \( \mathcal{R}(r,D,V) \), (with the sup norm), then \( \mathcal{M}(\mathcal{R}(r,D,V)) \) inherits a topology, the weak (or vague) topology, defined by taking as a basis of open neighborhoods for \( \mu \in \mathcal{M}(\mathcal{R}(r,D,V)) \) the sets

\[
\left\{ \nu \in \mathcal{M} : \left| \int f_j d\mu - \int f_j d\nu \right| < \epsilon_j \right\}
\]

where \( j = 1,\ldots,k \), \( \epsilon_j > 0 \) and \( f_j \in C(\mathcal{R}(r,D,V)) \).

In particular, we say that a sequence \( \{P_n : n = 1,2,\ldots\} \) in \( \mathcal{M}(\mathcal{R}(r,D,V)) \) converges weakly to \( P \in \mathcal{M}(\mathcal{R}(r,D,V)) \), and write \( P = \lim_{n \to \infty}^w P_n \) if \( \int f dP_n \to \int f dP \) for every \( f \in C(\mathcal{R}(r,D,V)) \).

An important consequence of the compactness of \( \mathcal{R}(r,D,V) \) is that any measure \( \mu \in \mathcal{M} \) is regular and that \( \mathcal{M} \) itself is a compact space.

**Proposition 5** \( \mathcal{M}(\mathcal{R}(r,D,V)) \) is a compact metrizable space in the weak topology.

Finally for what concerns the characterization of the support of a measure in \( (\mathcal{R}(r,D,V),\mathcal{B}) \) we get

**Definition 6** Let \( \mu \in \mathcal{M} \). The set of all manifolds (homology manifolds) \( M \in \mathcal{R}(r,D,V) \) with the property that \( \mu(U) > 0 \) for any open set \( U \) containing \( M \) is called the support of \( \mu \). A measure \( \mu \) whose support reduces to a single manifold is called a point (or atomic) measure. Conversely a measure \( \mu \) on \( \mathcal{R}(r,D,V) \) is said to be non-atomic (or diffuse) if \( \mu(\{M\}) = 0 \) for all manifolds \( M \in \mathcal{R}(r,D,V) \).
Notice that in the weak topology diffuse measures are prevalent over point measures.

According to Gromov’s precompactness theorem\cite{Gromov87}, (see theorem \[36\]), given the cut-off length scale $L(m) = 1/m$, we can always choose a finite number (depending on $L(m)$) of riemannian manifolds, $\{M^*_i\}_{i = 1, \ldots, k(m) < \infty}$, such that any manifold in $\mathcal{R}(r, D, V)$ is, on length scales larger than the cutoff $L$, metrically similar to one of the model manifolds $M^*_i$.

We blend this basic remark with measure theory on $\mathcal{R}(r, D, V)$, by considering, for any given $m$, the convex combination of point measures defined on $\mathcal{R}(r, D, V)$ by

$$\mu_m(B) \equiv [\Xi(m, z)]^{-1} \sum_{M^*_i} z^{N^{(0)}_m(M^*_i)} \exp[\beta N^{(1)}_m(M^*_i)] \delta(M^*_i)(B)$$

where $\delta(M^*_i)$ denotes the atomic measure based on the generic model manifold $M^*_i$, viz., the measure defined by $\delta(M^*_i)(B) = 1$ if the manifold $M^*_i \in B$, and 0 otherwise, ($B$ being a Borel subset of $\mathcal{R}(r, D, V)$). Obviously, $N^{(0)}_m(M^*_i)$ and $N^{(1)}_m(M^*_i)$ respectively denote the filling function and the size of the one-skeleton $\Gamma_m(M^*_i)$ associated with the minimal $L(m)$-covering of $M^*_i$.

With this choice of a measure on $\mathcal{R}(r, D, V)$, which is the most natural one in the framework we are considering, the following results hold true

**Proposition 6** For any given $m$, let $f: \mathcal{R}(r, D, V) \to \Omega_m, M \mapsto \Gamma_m(M) \subset \Omega_m$, the map that with each $M$ associates its $L(m)$-geodesic balls one-skeleton, (up to combinatorial isomorphisms). Then:

(i) The induced measure on $\{0,1\}^{\Omega_m}$ defined by

$$P_m(A) \equiv \mu_m(f^{-1}(A))$$

(where $A$ is the generic Borel subset of $\{0,1\}^{\Omega_m}$), gives the probability law on $\{0,1\}^{\Omega_m}$ determined by

$$[\Xi(m, z)]^{-1} \sum_{M^*_i} z^{N^{(0)}_m(M^*_i)} \exp[\beta N^{(1)}_m(M^*_i)] \delta(M^*_i)(f^{-1}(A))$$

which is the natural lattice gas grand-ensemble probability law on the graph $\Omega_m$ with the conditioning that the subgraphs $\Gamma \subset \Omega_m$ which are $L(m)$-geodesic balls one-skeletons occur with probability one.

(ii) The marginal functions $P_m$ so defined tend to a (possibly not-unique) limit distribution as $m \to \infty$.

**PROOF** The first part of (i) follows immediately from the definition of induced measure associated with a measurable mapping between two measure spaces.
Consider now the (marginals of the) probability laws yielding for a grand-ensemble statistical mechanical description of a lattice gas on the sequence of graphs $\Omega_{(m)}$, $0 < m < \infty$. These laws are given by the probability measures on $\{0, 1\}^{\Omega_{(m)}}$ provided by

$$P_{\Omega}(A) \equiv [\Xi(m, z)^*]^{-1} \sum_{\Gamma \in A} z^{N^{(0)}_{(m)}(\Gamma)} \exp[\beta N^{(1)}_{(m)}(\Gamma)] \quad (29)$$

where $A$ denotes the generic borel subset in the probability space $\{0, 1\}^{\Omega_{(m)}}$, e.g., any given collection of subgraphs $\Gamma$ of $\Omega_{(m)}$. (Recall that the set $\{0, 1\}$ is topologized by the discrete topology, and $\{0, 1\}^{\Omega_{(m)}}$ by the product topology; the Borel $\sigma$-field of $\{0, 1\}^{\Omega_{(m)}}$ is generated by the open sets of the product topology). The order $N^{(0)}_{(m)}(\Gamma)$ and the size $N^{(1)}_{(m)}(\Gamma)$ have, in Eq.(29), the usual graph-theoretical meaning (and they are not a priori connected with any geodesic balls covering; the use of the same symbols adopted for graphs associated to one-skeletons is maintained, in Eq.(29) and in other similar expressions below, for notational convenience only). Finally, the grand-partition function $\Xi(m, z)^*$ is given by an expression formally analogous to (19)

$$\Xi(m, z)^* \equiv \sum_{\Gamma \subset \Omega} z^{N^{(0)}_{(m)}(\Gamma)} \exp[\beta N^{(1)}_{(m)}(\Gamma)] \quad (30)$$

where the summation is now extended to all subgraphs of $\Omega_{(m)}$, (whilst in (19) the sum is a priori restricted to the subgraphs of $\Omega_{(m)}$ which are one-skeletons of $L(m)$-geodesic balls coverings of manifolds in $R(r, D, V)$).

Let $B \equiv f[R(r, D, V)]$ denote the $P_{\Omega}$-measurable subset of $\Omega_{(m)}$ consisting of graphs, $\Gamma$, which are realized as one-skeletons of $L(m)$-geodesic balls coverings of $M \in R(r, D, V)$. Since $P_{\Omega}(B) > 0$, the conditional probability of $A$ given $B$, $P_{\Omega}(A|B) = P_{\Omega}(A \cap B)/P_{\Omega}(B)$ is provided by

$$P_{\Omega}(A|B) = \frac{\sum_{\Gamma \in A \cap B} z^{N^{(0)}_{(m)}(\Gamma)} \exp[\beta N^{(1)}_{(m)}(\Gamma)]}{\sum_{\Gamma \in B} z^{N^{(0)}_{(m)}(\Gamma)} \exp[\beta N^{(1)}_{(m)}(\Gamma)]} \quad (31)$$

From the definition of the subset $B$ it follows that we can equivalently rewrite the sum $\sum_{\Gamma \in A \cap B}$ as the sum $\sum_{M_i^* \in f^{-1}(A \cap B)} \ldots$, while the sum $\sum_{\Gamma \in B}$ reduces to $\Xi(m, z)$. Thus

$$P_{\Omega}(A|B) = [\Xi(m, z)^*]^{-1} \sum_{M_i^*} z^{N^{(0)}_{(m)}(M_i^*)} \exp[\beta N^{(1)}_{(m)}(M_i^*)] \delta(M_i^*)(f^{-1}(A \cap B)) \quad (32)$$

which proves the second part of (i).

For what concerns statement (iii), which addresses the existance of the limit distribution associated with (28) as $m \to \infty$, we can proceed as follows.

Let $\{m_i\}$ be a numerical sequence with $m_i \to \infty$, and let $\{\mu_{(m_i)}\}$ the corresponding sequence of measures defined by Eq.(23). Since $R(r, D, V)$ is a compact metric space, its
associated space of measures $\mathcal{M}(\mathcal{R}(r, D, V))$ besides being not-empty, it is also (weakly) compact, (see proposition 3). In particular, convex combinations of atomic measures such as $(26)$ are dense in $\mathcal{M}(\mathcal{R}(r, D, V))$. It follows that from the sequence of measures $\{\mu(m_i)\}$ we can extract, (by choosing appropriate subsequences if necessary), a converging subsequence as $m_i \to \infty$. In particular, we may consider the set of limits

$$
F_\mu \equiv \{\mu \in \mathcal{M}(\mathcal{R}(r, D, V)): \mu = \lim_{\Gamma \uparrow \Omega_\infty} \mu(m)\}
$$

(33)

where $\{\Gamma\}$ is any increasing sequence of graphs, yielding for $\Omega_\infty$, as $m_i \to \infty$.

As noticed before, since $\mathcal{M}(\mathcal{R}(r, D, V))$ is weakly compact, $F_\mu$ is non-empty, and its closed convex hull may consist of more than one measure. This latter eventuality will indicate the possible onset of phase transitions in the statistical system of manifolds (coverings) described in $\mathcal{R}(r, D, V)$ by (28). ♣

The result discussed above is a general existence result, and as such it does not accomplish very much since it can not provide us with any detailed information on the structure of the thermodynamical limit of (19). In particular the measure-theoretic proof that the closed convex hull of $F_\mu$ actually contains more than one measure is not so obvious, since we are not working with graphs on a regular integer lattice $\mathbb{Z}^d$; (critical behavior for lattice gases on regular $\mathbb{Z}^d$ lattices is a considerably well-developed part of rigorous statistical mechanics. A reason for such success is Peierls’ argument, in the refined version developed by Pirogov and Sinai, which provides the key tool in addressing the phase analysis of the model, and which is not easily extended to a generic graph).

When the lattice gas evolves on a general (denumerable) graph, phase transition phenomena are still the rule, but critical parameters are strongly graph dependent, and general results are less accessible (at least to the authors). One may still draw interesting consequences by exploiting random graphs theory, which is a well developed field of modern graph theory. Typical problems which are addressed and solved in such field are, (not surprisingly), strongly reminiscent, of the study of phase transitions, (e.g., the existence of threshold functions for the connectivity of a graph, or for finding particular types of subgraphs). But the mathematical techniques adopted are custom-tailored to graph theory, and not easily adapted to lattice gas statistical mechanics.

### 3.3 Some algebraic properties of the partition function

Either from general properties of random graphs or from considerations based on the physics of regular lattice gases, phase transitions for a lattice gas on a general graph (thought of as the union of a monotonically increasing sequence of finite graphs $\{\Gamma_i\}$), are expected to occur when the graphs $\Gamma_i \subset \Gamma_{i+1} \subset \ldots$ have a large percentage of vertices...
on their boundaries, and when the graphs have a sufficiently large number of edges (i.e., the interaction between neighboring vertices is not too weak). The rationale behind such remark being that in this case the configurations of occupied vertices depend very much upon boundary conditions. More explicitly, we consider the generic $\Gamma_i$ as included within $\Gamma_{i+1} \subset \Gamma_{i+2} \subset \ldots$, and fix a priori the occupation values of the vertices $p_j \not\in \Gamma_i$. We modify accordingly the energy of all configurations possible on $\Gamma_i$ and carry out the thermodynamic limit on this conditioned system. If the limit shows sensible dependence upon such boundary conditions then we are in presence of a phase transition regime.

Thus our strategy in addressing the study of possible phase transitions for the system described by (19) is to redefine the law (28) by conditioning the configurations in $\Omega_{(m)} \setminus \Gamma$, and discuss the behavior of the resulting probability distributions when $m \to \infty$.

As a first step in this direction we rewrite the size $N^{(1)}_{(m)}(\Gamma)$ of the generic graph $\Gamma_{(m)} \subset \Omega_{(m)}$ as in the following

**Lemma 2** For any $\Gamma \subset \Omega_{(m)}$ we get

$$N^{(1)}_{(m)}(\Gamma) = \frac{1}{2} \sum_{p \in \Gamma} \left[ \sum_{q \in \Omega \setminus \Gamma, q \neq p} A_{\Omega}(p, q) - \sum_{q \in \Omega \setminus \Gamma} A_{\Omega \setminus \Gamma}(p, q) \right]$$

(34)

where $A_{\{\}}(p, q)$ is the adjacency matrix of the graph indicated, (i.e., the matrix whose entries are 1 if the vertices $p$ and $q$ of the graph in question are connected by an edge, zero otherwise). And where, with a slight abuse of notation, we have denoted by $\Omega \setminus \Gamma$ the graph in $\Omega$ obtained by removing all the edges, (but not the vertices), belonging to $\Gamma$.

**PROOF.** The proof of (34) is trivial, since the first term in the square brackets is simply the number of all edges, in $\Omega_{(m)}$ issuing from $p$, (i.e., the degree of $\Omega$), while the second term provides the number of all edges issuing from $p$ and not belonging to the graph $\Gamma$, (i.e., the degree of $\Omega \setminus \Gamma$ at $p$). The overall summation over all points $p$ in $\Gamma$ is then halved owing to the symmetry underlying the argument, (more explicitly, one exploits the handshaking lemma according to which the sum of the degrees of a graph is twice the size of the graph itself). ♣

Since the graph $\Omega_{(m)}$ is regular, the expression (34) can be put into the form

$$N^{(1)}_{(m)}(\Gamma) = \frac{1}{2} d(\Omega_{(m)}) N^{(0)}_{(m)}(\Gamma) - \frac{1}{2} \sum_{p \in \Gamma} \left[ \sum_{q \in \Omega \setminus \Gamma} A_{\Omega \setminus \Gamma}(p, q) \right]$$

(35)

We can now take advantage of this expression for the size of the graph $\Gamma_{(m)}$ in order to rewrite $\Xi(m, z)$ as the polynomial

$$\Xi(m, z) = \sum_{\Gamma} z^{N^{(0)}_{(m)}(\Gamma)} \prod_{p \in \Gamma} \prod_{q \in \Omega \setminus \Gamma} \exp\left[-\frac{1}{2} \beta A_{\Omega \setminus \Gamma}(p, q)\right]$$

(36)
where we have introduced a normalized fugacity $\hat{z}$ according to

$$\hat{z} \equiv z \exp\left[\frac{1}{2} \beta d(\Omega)\right]$$

(Notice that in general $\hat{z}$ depends from the given $m$ through the expression of the degree $d(\Omega(m))$, however, for $m$ sufficiently large this dependence will eventually disappear). We wish to stress again that the sum (36) is restricted only to those graphs $\Gamma(m)$ which are $L(m)$-geodesic balls one-skeletons for manifolds in $R(r,D,V)$, and it is not extended to all possible subgraphs of $\Omega(m)$. The unrestricted sum $\Xi(m,\hat{z})^*$, already introduced during the proof of proposition 6, is given by

$$\Xi(m,\hat{z})^* = \sum_{\forall \Gamma \subset \Omega} \hat{z}^{N(0)(\Gamma)} \prod_{p \in \Gamma} \prod_{q \in \Omega \setminus \Gamma} \exp\left[-\frac{1}{2} \beta A_{\Omega \setminus \Gamma}(p,q)\right]$$

and we can formally write

$$\Xi(m,\hat{z})^* = \Xi(m,\hat{z}) + \sum_{\Omega \setminus f(R(r,D,V))} \hat{z}^{N(0)(\Gamma)} \prod_{p \in \Gamma} \prod_{q \in \Omega \setminus \Gamma} \exp\left[-\frac{1}{2} \beta A_{\Omega \setminus \Gamma}(p,q)\right]$$

where the sum over $\Omega \setminus f(R(r,D,V))$ indicates summation over all graphs in $\Omega(m)$ which are not $L(m)$-one-skeletons for manifolds in $R(r,D,V)$.

Notice that in the construction of $\Omega(m)$ out of the one-skeleton graphs $\Gamma(m)$ with the Erdos-Kelly algorithm, (see (21)), no edges are introduce between the graphs $\Gamma(m)(M)$, new edges may occur only between the $b(m)$ added vertices and between these latter vertices and the graphs $\Gamma(m)(M)$. From such remarks it follows that the graphs in $\Omega(m) \setminus f(R(r,D,V))$ are defined by the $b(m)$ added vertices and by the possible edges between them and the graphs $\Gamma(m)$.

The relation (39) between the unrestricted, $\Xi^*$, and the restricted statistical sum $\Xi$, is useful for suggesting the boundary conditions which uncover the non-trivial phase structure of $\lim_{m \to \infty} \Xi(m,\hat{z})$.

In order to proceed in this direction, let us notice that the unrestricted statistical sum (36), $\Xi(m,\hat{z})^*$, takes on the classical polynomial Lee-Yang structure. It follows, according to the Lee-Yang circle theorem, that the zeroes of $\Xi(m,\hat{z})^*$, thought of as a function of the complex variable $\hat{z}$, all lie, for each given value of $m$, on the circle $\{\hat{z} : |\hat{z}| = 1\}$ in the plane of the complexified fugacity $\hat{z}$. In other words

**Proposition 7** In the plane of complexified fugacity $\hat{z}$, all the zeros of the polynomial $\Xi(m,\hat{z})^*$ are of module one.

**Proof.** Let $S \equiv \{i_1, \ldots, i_s\}$, and $\bar{S} \equiv \{j_1, \ldots, j_{n-s}\}$ respectively denote the generic subset of $\{1, \ldots, n\}$ and its complement. Also, let $(F_{i,j})_{i \neq j}$ be a family of real numbers
such that \(-1 \leq F_{ij} \leq 1\), \(F_{ij} = F_{ji}\) for \(i, j = 1, \ldots, n\). Define the polynomial (of degree \(n\)) in \(z\) by
\[
\mathcal{P}^n(z) = \sum_S z^{|N(S)|} \prod_{i \in S} \prod_{j \in S} F_{ij}
\]
where \(N(S)\) is the number of elements in \(S\). Then, Lee-Yang circle theorem asserts that the zeros of \(\mathcal{P}^n(z)\) all lie on the circle \(\{z: |z| = 1\}\).

In order to apply this result to \(\Xi(m, \hat{z})^*\) it is sufficient to label the vertices of the graph \(\Omega(m)\) and to identify \(S\) and \(\bar{S}\) with the induced labelling of the vertices of the generic subgraph \(\Gamma \subset \Omega(m)\) and of its complement \(\Omega \setminus \Gamma\), respectively. Finally take \(F_{ij} \equiv \exp[-\frac{1}{2} \beta A_{\Omega \setminus \Gamma}(p_i, q_j)]\). ♠

For each given \(m\), \(\Xi(m, \hat{z})^*\) is a polynomial in the variable \(\hat{z}\) of degree \(\lambda \equiv |\Omega(m)|\). Since its zeroes all lie on the unit circle, and \(\Xi(m, \hat{z})^* \leq \Xi(m, 1)^*\), for \(0 < \hat{z} \leq 1\), it follows that if we choose the determination of \([\Xi(m, \hat{z})^*]^{1/|\Omega(m)|}\) which is real for \(\hat{z} > 0\), then \([\Xi(m, \hat{z})^*]^{1/|\Omega(m)|}\) defines, for \(|\hat{z}| < 1\) and as \(m\) varies, a uniformly bounded family of analytic functions. We can now follow a standard argument, to the effect that if \({m_i}\) is a numerical sequence with \(m_i \to \infty\) then \({[\Xi(m_i, \hat{z})^*]^{1/|\Omega(m_i)|}}\) converges for \(\hat{z} > 0\), and the convergence is uniform on any disk \(\{\hat{z}: |\hat{z}| \leq \alpha, \alpha < 1\}\). If we introduce the thermodynamic potential, (the pressure of the lattice gas),
\[
\Psi(\hat{z}, \beta) = \beta^{-1} \lim_{m_i \to \infty} \left[ \frac{1}{|\Omega(m)|} \ln \sum_{\Gamma \subset \Omega} \hat{z}^{|N(\Gamma)|} \prod_{p \in \Gamma} \prod_{q \in \Omega \setminus \Gamma} \exp[-\frac{1}{2} \beta A_{\Omega \setminus \Gamma}(p, q)] \right]
\]
then, the above remarks show that such \(\Psi(\hat{z}, \beta)\) can be analytically extended form the interval \(0 < \hat{z} < 1\) to the function \(p(\hat{z}, \beta) \equiv \beta^{-1} \ln \{\lim_{m_i \to \infty}[\Xi(m, \hat{z})^*]^{1/|\Omega(m)|}\}\) which is analytic if \(|\hat{z}| < 1\).

Similarly, by exploiting the fact that under the inversion \(\hat{z} \to 1/\hat{z}\), the partition function \(\Xi(m, \hat{z})^*\) goes over into \(\hat{z}^{-|\Omega|}\Xi(m, \hat{z})^*\), we can analytically extend \(\Psi(\hat{z}, \beta)\) from the interval \(1 < \hat{z} < +\infty\) to the function 
\[
p(\hat{z}^{-1}, \beta) + \ln \hat{z}.
\]
Such well-known analysis implies that for the statistical system described by \(\Xi(m, \hat{z})^*\), a phase transition can occur, in the continuum limit \(m \to \infty\), at most for \(\hat{z} = 1\). Under the lattice gas correspondence, the onset of such phase transitions corresponds to the familiar behavior of a lattice gas, with negative pair interaction, for which first-order phase transitions are present for sufficiently small temperatures unless the interaction is essentially one-dimensional.
3.4 The phase structure in the thermodynamical limit

Lee-Yang type results for regular $\mathbb{Z}^d$-lattice gas relates the presence or absence of a phase transition to the analyticity properties of the free energy associated with the grand-partition function $\Xi^*$. Such result depends in a rather delicate fashion upon the way the thermodynamic limit is carried out, i.e., the increasing sequence of finite volume configurations must tend \textit{regularly}\footnote{Regularly towards $\mathbb{Z}^d$.} towards $\mathbb{Z}^d$. Typically one requires that the limit is regular in the sense of van Hove, (roughly speaking, the boundary of each finite volume region does not grow as to display a very irregular geometric structure).

In order to extend such results to our case, we could exploit the particular geometric origin of the one-skeletons graphs $\Gamma_{(m)}(M)$. Roughly speaking, the idea is to show that for $m$ sufficiently large, we can always embed the generic $\Gamma_{(m)}(M)$ in a region of a regular lattice such as $\mathbb{Z}^d$, provided that we add (a finite number of edges) between some of the sites of the regular lattice. It must be stressed that edges are added and not deleted, and that the number of edges to be added does not depend on $m$ as $m \to \infty$.

In this way one can establish a correspondence between the lattice gas described by $\Xi(m, \hat{z})^*$ and a regular lattice gas whose attractive interaction between sites has been here and there enhanced. The existence of phase transitions for this system can be discussed by standard techniques. Obviously, all this would be rather formal, and not particularly illuminating from a geometrical point of view. Thus we proceed differently.

If we fix $m$ then the probability, $P_\Omega(\Gamma_{(m)}(M))$, of occurrence of a one-skeleton graph (of a geodesic balls covering) of a manifold of bounded geometry among all possible subgraphs of $\Omega_{(m)}$ is different from zero. This follows from the very definition of the measure space $(\Omega_{(m)}, P_\Omega)$, and as we let the parameter $\beta$ to vary we can also get configurations whereby a particular one-skeleton graph dominates over another. Both such properties are not so obvious when we carry out the $m \to \infty$ limit, i.e., when $\Omega_{(m)} \uparrow \Omega_{\infty}$. Even if we take this limit with the exterior boundary condition $b_{(m)} = 0$ imposed upon the sequence of $\Omega_{(m)}$, we have no \textit{a priori} reasons of assuming that the limit distribution $P_\Omega$ gives non-zero probability to geodesic balls one-skeleton graphs. Moreover, even if this is the case, the $P_\Omega$-dominance of a particular class of one-skeleton graphs over another, \textit{viz.}, the question of the existence of phase transitions needs to be explicitly answered and related to the geometry of the manifolds in $\mathcal{R}(r, D, V)$. An affirmative answer to such questions is provided by the proposition below. In particular, the nature of its proof suggests a strategy for proving the existence of a non-trivial phase structure for $\Xi(\hat{z})$ and for understanding its geometrical interpretation on $\mathcal{R}(r, D, V)$.

Before stating the result referred to above, few remarks are in order. Let us start by pointing out that, according to the Lee-Yang characterization of the complex zeros of $\Xi^*$, if there is a phase transition in the system described by $\Xi(m, \hat{z})^*$, as $m \to \infty$, then this can occur at most for $\hat{z} = 1$. We can use this condition for characterizing
the value of the chemical potential $i_c \equiv -(c/\beta)$ for which a critical behavior is possible.

From the definition of the normalized fugacity $\hat{z}$, (see (37)), we get $i_c = -(1/2)d(\Omega)$. Geometrically speaking, this condition fixes the average volume of the manifolds sampled out in $\mathcal{R}(r,D,V)$ according to the probability law $P_\Omega$.

A second remark concerns the boundary condition to be imposed on the sequence of graphs $\Omega_{(m)}$ in carrying over the limit $\Omega_{(m)} \uparrow \Omega_\infty$. According to the relation (39) between $\Xi^*$ and $\Xi$, the most natural exterior boundary condition which may significantly affect the nature of the limit distribution $P_\Omega$ as $m \to \infty$ is to keep empty the $b_{(m)}$ added vertices needed to regularize the set $H_{(m)}$, of geodesic balls one-skeleton graphs, (see Lemma [1]). In other words, in what follows we take the limit $m \to \infty$, through subgraphs of $H_{(m)}$, viz., $H_{(m)} \uparrow \Omega_\infty$. For any finite $m$ this choice tends to favor graphs which are geodesic balls one-skeleton graphs rather than generic graphs. However, as remarked above, this enhancement does not trivially extend to $\Omega_\infty$. We shall see that this type of boundary condition is strictly connected with the topology of the manifolds sampled out by $P_\Omega$.

Phase transitions should manifest themselves by a sensible dependence upon the boundary conditions introduced above. The resulting limit distribution $P_\Omega$ then must describe some local distortion of the system around some reference configuration, or, in more geometrical terms, fluctuations around a given reference homotopy type $\pi^*$.

With these preliminary remarks along the way, we can prove the following

**Proposition 8** For any given $m$, (sufficiently large), let $\{\Gamma_{(m)}^*\}$ denote the (finite) collection of one-skeleton graphs associated with minimal $L(m)$-geodesic balls covering of manifolds in $\mathcal{R}(r,D,V)$ of a same homotopy type $\pi^*$. As $m \to \infty$, let us denote respectively by $H_{(m)} = \{\Gamma_{(m)}^*\}$ and $\Omega_{(m)}$ the associated sequence of $L(m)$-geodesic balls one-skeleton graphs and the Erdos-Kelly minimal regular graph generated by such $\{\Gamma^i\}$.

Then, as $H_{(m)} \uparrow \Omega_\infty$ and for the given value $-(1/2)d(\Omega)$ of the chemical potential $i_c$, there is a $\beta_{cr}$ such that

$$\lim_{H_{(m)} \uparrow \Omega_\infty} P_\Omega \{\Gamma_{(m)}^*\} > 0 \quad (42)$$

for all $\beta > \beta_{cr}$.

In other words, at sufficiently low temperatures, a (denumerable) infinite graph $\Gamma^*$ representing the homotopy type $\pi^*$ occurs with non-zero probability among all possible subgraphs of $\Omega_{(m)} \uparrow \Omega_\infty$.

**PROOF. First part: an activity estimate.**

For a given $m$, the Erdos-Kelly graph $\Omega_{(m)}$, (with labelled vertices), contains by construction the set of graphs $\{\Gamma_{(m)}^*\}$ and graphs, (not necessarily describing one-skeletons of geodesic balls coverings), which do not contain $\{\Gamma_{(m)}^*\}$. Probabilities associated with these latter set of graphs can be estimated as follows.
For \( \lambda \) a positive integer, let \( \Omega_m^{(\lambda)} \) be the set of all graphs in \( \Omega_m \), of order \( \lambda \), which do not contain the set of graphs \( \{ \Gamma^*_m \} \). We have

\[
P_{\Omega}\{ \Gamma \subset \Omega_m; \Gamma \not\supset \{ \Gamma^*_m \} \} \leq \sum_{\lambda} \sum_{\Gamma, \Gamma_\lambda \in \Omega(\lambda)} P_{\Omega}\{ \Gamma \subset \Omega_m; \Gamma \supset \Gamma_\lambda \} \quad (43)
\]

Next, we derive an (activity) estimate expressing the fact that if \( \Gamma_\lambda \) is a fixed graph, (in \( \Omega_m^{(\lambda)} \)), of large order, then the probability \( P_{\Omega}\{ \Gamma \subset \Omega_m; \Gamma \supset \Gamma_\lambda \} \) should be controlled by the activity \( z^\lambda \) of the graph \( \Gamma_\lambda \). In more geometrical terms let \( \Gamma_\lambda \) be a geodesic balls one-skeleton graph of a given manifold \( M \). Since a large order for \( \Gamma_\lambda \) means a large number of disjoint geodesic balls packing up in \( M \), i.e., a large volume for \( M \), such an estimate expresses the not surprising fact that the \( P_{\Omega} \)-probability of a manifold of large volume is small.

Thus, given a fixed (labelled) graph \( \Gamma_\lambda \in \Omega_m^{(\lambda)} \), let \( \Lambda_m^{(\lambda)} \) denote the set of graphs \( \Gamma \subset \Omega_m \) containing \( \Gamma_\lambda \). We have

\[
P_{\Omega}\{ \Gamma \subset \Omega_m; \Gamma \supset \Gamma_\lambda \} = \sum_{\Gamma \in \Lambda} z^{N_m^0(\Omega \rightarrow \Gamma_\lambda)} \exp[\beta N_m^1(\Gamma)] \sum_{\Gamma \in \Omega} z^{N_m^0(\Omega \rightarrow \Gamma)} \exp[\beta N_m^1(\Gamma)] \quad (44)
\]

To get the desired estimate, we map the generic graph \( \Gamma \) in \( \Lambda_m^{(\lambda)} \) into a new graph \( \tilde{\Gamma} \) in \( \Omega_m \) obtained by deleting \( \Gamma_\lambda \) out of \( \Gamma \). We can write \( N_m(\Omega \rightarrow \Gamma_\lambda) = N_m(\Omega \rightarrow \tilde{\Gamma}) + N_m(\Gamma_\lambda) \), and \( N_m^1(\Gamma) = N_m^1(\tilde{\Gamma}) + N_m^1(\Gamma_\lambda) + N_m^1(\tilde{\Gamma} \to \Gamma_\lambda) \), where \( N_m^1(\tilde{\Gamma} \to \Gamma_\lambda) \) is the number of edges connecting \( \tilde{\Gamma} \) to \( \Gamma_\lambda \). In order to estimate the number of such edges let us introduce the (average) superficial degree of the graph \( \Gamma_\lambda \)

\[
d_S(\Gamma_\lambda) \equiv \frac{2N_m^1(\Omega \to \Gamma_\lambda)}{N_m^S(\Gamma_\lambda)} \quad (45)
\]

where \( N_m^1(\Omega \to \Gamma_\lambda) \) is the number of edges connecting \( \Omega_m \) to \( \Gamma_\lambda \), and where \( N_m^S(\Gamma_\lambda) \) is the number of boundary vertices of \( \Gamma_\lambda \) in \( \Omega_m \), (viz., the number of vertices in \( \Gamma_\lambda \) which are connected with vertices in \( \Omega_m \)).

Since \( \Gamma \subset \Omega_m \), we get

\[
N_m^1(\tilde{\Gamma} \to \Gamma_\lambda) \leq \frac{1}{2} d_S(\Gamma_\lambda) \frac{N_m^S(\Gamma_\lambda)}{\lambda} \quad (46)
\]

Thus, if we introduce the effective average degree of the graph \( \Gamma_\lambda \) according to

\[
d_{eff}(\Gamma_\lambda) \equiv d(\Gamma_\lambda) + d_S(\Gamma_\lambda) \frac{N_m^S(\Gamma_\lambda)}{\lambda} \quad (47)
\]

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(where \(d(\Gamma_\lambda)\) is the average degree of the graph \(\Gamma_\lambda\)), we can bound the generic term of the sum appearing in the numerator of (44) by

\[
z^\lambda \exp\left[\frac{1}{2} \beta d_{\text{eff}}(\Gamma_\lambda) \lambda\right] z^{N(\hat{\Gamma})} \exp[\beta N^{(1)}_{(m)}(\hat{\Gamma})]
\]

Whence

\[
P_\Omega\{\Gamma \subset \Omega_{(m)}: \Gamma \supset \Gamma_\lambda\} \leq z^\lambda \exp\left[\frac{1}{2} \beta d_{\text{eff}}(\Gamma_\lambda) \lambda\right] \frac{\sum_{\hat{\Gamma} \in \Omega} z^{N^{(0)}_{(m)}(\hat{\Gamma})} \exp[\beta N^{(1)}_{(m)}(\hat{\Gamma})]}{\sum_{\Gamma \in \Omega} z^{N^{(0)}_{(m)}(\Gamma)} \exp[\beta N^{(1)}_{(m)}(\Gamma)]}
\]

Since each graph \(\hat{\Gamma}\) is uniquely associated with a given \(\Gamma\), the ratio appearing in the above expression is not-greater than one. Thus we get

\[
P_\Omega\{\Gamma \subset \Omega_{(m)}: \Gamma \supset \Gamma_\lambda\} \leq z^\lambda \exp\left[\frac{1}{2} \beta d_{\text{eff}}(\Gamma_\lambda) \lambda\right]
\]

which, on introducing the normalized fugacity \(\hat{z}\), can be rewritten as

\[
P_\Omega\{\Gamma \subset \Omega_{(m)}: \Gamma \supset \Gamma_\lambda\} \leq \hat{z}^\lambda \exp\left[-\frac{1}{2} \beta (d(\Omega) - d_{\text{eff}}(\Gamma_\lambda)) \lambda\right]
\]

This provides the required activity estimate.

**Second part: an entropy estimate.**

### 3.5 Entropy estimates for geodesic ball coverings of three-manifolds of bounded geometry

In order to complete the proof of the proposition we need an entropy estimate which would provide an exponential bound to the number of combinatorially inequivalent graphs contained in the set \(\Omega_{(m)}(\lambda)\). To this end, we shall estimate the number of isomorphism classes of graphs in the set \(H_{(m)}\), corresponding to geodesic balls one-skeleton graphs of manifolds of a given homotopy type \(\pi^*\).

Let us consider \(L(m)\)-geodesic balls coverings whose filling function \(N^{(0)}_{(m)}\) takes on the running (integer) value \(\lambda\). Correspondingly let us introduce the function \(B_\lambda(V, \Gamma_{(m)}, \pi(M))\) which, at scale \(L(m)\), counts the number of combinatorially inequivalent 1-skeletons \(\Gamma_{(m)}\) with \(\lambda\) vertices which can be generated by minimal geodesic balls coverings on a manifold \(M\), of given volume \(V\), in the homotopy class \(\{\pi(M)\}\).

If \(m\) is sufficiently large, the function \(B_\lambda(V, \Gamma_{(m)}, \pi(M))\) depends, besides on the volume, only on the fundamental group \(\pi_1(M)\) of \(M\). This remark follows by noticing that there is a relation between the presentation, associated with the geodesic balls
covering, of the fundamental group $\pi_1(M)$ and the fundamental group $\pi_1[(\Gamma)(M)]$ of the graph $\Gamma(m)(M)$. As for any finite connected graph, $\pi_1[(\Gamma)(M)]$ is a free group on $1 + N(1)(\Gamma) - N(0)(\Gamma)$ generators. The inclusion map $\Gamma(m)(M) \rightarrow \Gamma(2)(M)$ induces an epimorphism $\pi_1[\Gamma(M)] \rightarrow \pi_1(\Gamma(2))$ whose kernel is generated by the elements of $\pi_1[\Gamma(M)]$ which are killed off by pasting the faces $p_{ijk}$ of the geodesic ball skeleton, while the further pasting of higher-dimensional simplices has no effect on the presentation, and, as stated, the function $B_\lambda(V, \Gamma(m), \pi(M))$ can depend only on the fundamental groups $\pi_1(M)$ realized by the manifolds in $R(r, D, V)$.

From the algebraic properties of $\Xi(m, \zeta)^*$, (see proposition 7 and the associated remarks) it easily follows that $B_\lambda(V, \Gamma(m), \pi(M))$ can have, at worst, an exponential growth with a possible subleading asymptotics, (this latter being compatible with the possible development of singularities in $\Xi(m, \zeta)^*$, as $\zeta \rightarrow 1$, consequence of the zeros of $\Xi(m, \zeta)^*$ accumulating on the unit circle). This remark would suffice our purposes, however it is largely unsatisfactory since it does not make explicit the $\pi_1$- dependence of $B_\lambda(V, \Gamma(m), \pi(M))$.

In order to make explicit such dependence, let us start by noticing that, by construction, the function $B_\lambda(V, \Gamma(m), \pi(M))$ is continuous under Gromov-Hausdorff convergence, in the sense that if $\{M(i)\}$ is a sequence of manifolds in $R(r, D, V)$ $d_G$-converging to a (homology) manifold $M$ then, for a given $\lambda$ sufficiently large, we have

$$\lim_{M(i) \rightarrow M} B_\lambda(V(M(i)), \Gamma(m)(M(i)), \pi(M(i))) = B_\lambda(V(M), \Gamma(m)(M), \pi(M))$$

(52)

By relaxing the volume constraint in $R(r, D, V)$ so to allow for sequence of manifolds $\{M(i)\}$ with three-dimensional volume going to zero, we may have $\{M(i)\}$ collapsing to a lower dimensional manifold. The classical example in this direction is afforded by the Berger sphere: let $g_{can}$ the standard metric on $S^3$, and consider the Hopf fibration $\pi: S^3 \rightarrow S^2$. Define $g_\epsilon(v, v) = \epsilon \cdot g_{can}(v, v)$ if $\pi_*v = 0$, and $g_\epsilon(v, v) = g_{can}(v, v)$ if the vector $v \in T_{\epsilon}S^3$ is perpendicular to the fibre of $\pi$. It is easily checked that $(S^3, g_\epsilon) \in R(n = 3, D = 1, V = 0)$ for any $\epsilon \leq 1$, and that

$$\lim_{\epsilon \rightarrow 0} d_G[(S^3, g_\epsilon), (S^2, \hat{g})] = 0$$

(53)

where $\hat{g}$ is the round metric on the two-sphere with curvature 4.

In such a case, and more in general when three-dimensional manifolds collapse to surfaces, (see the paper by K.Fukaya quoted in [4] for a very clear account of such topic), the counting function $B_\lambda(V, \Gamma(m), \pi(M_\epsilon))$ approaches, as $\epsilon \rightarrow 0$, the corresponding function on the surface $\Sigma$ resulting from the collapse, namely

$$B_\lambda(\Gamma(m), \pi(\Sigma)) \xrightarrow{\lambda \rightarrow \infty} (\Lambda)^{\lambda(\frac{1}{\alpha})} \cdot \rho(1 + O(\alpha))$$

(54)
where $\Lambda$, $\gamma$, and $\rho$ are suitable constants, (as stressed in the introductory remarks, this asymptotics can be obtained in a number of inequivalent ways 

Notice that the Euler characteristic, in the above expression, characterizes from a topological point of view, the subleading asymptotics of the graph counting function. Its role here is that of providing the homotopy cardinality of the complex determining the surface $\Sigma$, (this point of view on the Euler characteristic, together with a similar remark on the cardinality meaning of the Reidemeister torsion, has been suggested to us by the paper of D.Fried on dynamical systems quoted in 

As a matter of fact, the Euler characteristic of a finite complex is the only homotopy invariant that satisfies $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ for any two sub-complexes $A$, $B$ of $\Sigma$, with the normalization $\chi(\text{point}) = 1$. This remark suggests that one may prove [54] by induction on the construction of $\Sigma$ by joining handles to a sphere. Similarly, we can try to determine the asymptotics of $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$ for three-dimensional manifolds by induction on the subcomplexes giving rise to $M$. To be more precise, we must look for an expression for $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$ which satisfies the following requirements:

(i) its leading asymptotics is exponential in $\lambda$, this requirement follows from Proposition 7;
(ii) upon collapsing of the manifold $M$ to a surface, it must be consistent with the surface asymptotics as expressed by [54];
(iii) it must take into account that homology manifolds are allowed for in $\mathcal{R}(r, D, V)$;
(iv) if $A$ and $B$ are any two subcomplexes, (with $\lambda$ vertices), of the geodesic ball skeleton yielding for $M$ then we must have

$$B_\lambda(M \mid A \cup B)B_\lambda(M \mid A \cap B) = B_\lambda(M \mid A)B_\lambda(M \mid B)$$

where $B_\lambda(M \ldots)$ stands for $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$ evaluated for $M$ restricted to the subcomplex $\ldots$ considered. The rationale underlying such requirement lies in noticing that the counting function $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$ can be interpreted as a measure on the set of graphs considered;

(v) A detailed analysis of the collapsing mechanism [54] underlying the transition from $\{M(i)\}$ to a lower dimensional manifold shows that the manifold resulting from the collapse in general is a quotient of the original three-dimensional manifold by free circle actions acting on the $\{M(i)\}$ by isometries. This is particularly clear in the quoted example of the Berger sphere, where $S^2 = S^3/S^1$, (more general situations illustrating this point can be found in the quoted paper by Fukaya [54]). Thus, the three-dimensional asymptotics for $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$ must take into account not only the homotopic cardinality of the geodesic ball two-skeleton of the manifold $M$, but also the homotopic cardinality of the circle fibers in $M$.
The above requirements suggest the following strategy for characterizing $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$. Let us start by recalling that according to a theorem by Fukaya\cite{Fukaya}, (in particular see Th.11.1), if $\{M(i)\}$ is a sequence of riemannian manifolds in $\mathcal{R}(n = 3, D, V = 0)$ converging to a length space $(X, d)$, then there exist a $C^\infty$ manifold $M$ endowed with a metric $g_M$ of Lipschitz class $C^{1,\alpha}$, on which there is a smooth and isometric action of the orthogonal group $O(n = 3)$ such that

(a) $(X, d)$ is isometric to $(M, g_M)/O(n = 3)$.
(b) For each $p \in M$, the isotropy group $I_p \equiv \{\eta \in O(n) | \eta p = p\}$ is an extension of a torus $T^k$ by a finite group.

This result on the structure of the collapsed boundary points of $\mathcal{R}(r, D, V)$, (with $V = 0$), shows that orthogonal representations of the fundamental group of manifolds in $\mathcal{R}(r, D, V)$ ought to play a basic role in determining an expression of $B_\lambda(V(M), \Gamma(m)(M), \pi(M))$ which is to be consistent with the known surface asymptotics provided by (54). As a matter of fact, given any such representation, $\theta: \pi_1(M) \to O(n = 3)$, there is a topological invariant which exactly counts circle fibers and which satisfies the required cardinality law. This invariant is the Reidemeister-Franz representation torsion of the manifold $M$ associated with the given representation: $\Delta^\theta(M)$. If $A$ and $B$ are subcomplexes of the geodesic ball nerve yielding for $M$, then

$$\Delta^\theta(M|A \cup B) \Delta^\theta(M|a \cap B) = \Delta^\theta(M|A) \Delta^\theta(M|B)$$

(56)

with $\Delta^\theta$ normalized to one over $S^1$, (the definition of the representation torsion is not explicitly needed here, a detailed discussion of its geometrical meaning as far as the notion of simple homotopy is concerned can be found in the quoted references\cite{42, 43, 44}, and in the second part of this paper where its role in 3- dimensional simplicial quantum gravity is discussed in great details).

With these preliminary remarks along the way, let $\theta: \pi_1(M) \to O(n = 3)$ be an orthogonal representation of $\pi_1(M)$ and let $\Delta^{\theta(a)}(M)$, with $a = 1, \ldots$ denote the corresponding (finite) set of representation torsions, (notice that manifolds in $\mathcal{R}(r, D, V)$ realize a finite number of simple homotopy types, this remark implies that, for a given fundamental group, the number of inequivalent representation torsion actually realized is finite).

Given an orthogonal representation, we let $B_\lambda(\Gamma_m(M), \pi(M), \Delta^\theta(M))$ denote the function which counts the number of combinatorially inequivalent one-skeletons $\Gamma_{(m)}^{(1)}$ with $\lambda$ vertices which can be generated by minimal geodesic balls coverings on a manifold $M$, of given volume $V$, and given $R$--torsion in the given representation $\theta: \pi_1(M) \to O(n = 3)$.

We shall prove the asymptotic estimate for the counting function $B_\lambda(V(M), \Gamma_m(M), \pi(M), \Delta^\theta(M))$ by an inductive argument exploiting geodesic ball coverings and the cardinality laws for the Euler characteristic and the representation torsion.

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Let $1/m = 2\varepsilon$, and correspondingly let $\{B_i(p_i)\}_{i=1,\ldots,\lambda}$ a minimal $\varepsilon$-net in $M \in \mathcal{R}(r, D, V)$, with $B_i(p_i) \cap B_j(p_j) = \emptyset$, for every $i, j$.

Let us denote by $M_{(1)} \equiv \bigcup_{i}^{\lambda} B_i(p_i)$, and more in general, for any integer $k \geq 1$ we set

$$M_{(k)} \equiv \bigcup_{i}^{\lambda} B_{k\varepsilon}(p_i) \quad (57)$$

Note that for $k = 2$ the set $\{B_{2\varepsilon}(p_i)\}$ covers $M$, while for $k$ such that $k\varepsilon \geq \text{diam}(M)$ each ball $B_{k\varepsilon}(p_i)$ covers $M$. Thus for $k$ large enough $\chi(B_{k\varepsilon}(p_i)) = \chi(M)$.

Since the balls $\{B_{\varepsilon}(p_i)\}$ are disjoint $\pi_1(M_{(1)}) = \bigoplus_{i} \pi_1(B_{\varepsilon}(p_i))$, with the generic $\pi_1(B_{\varepsilon}(p_i))$ reducing to the unit element only if the corresponding ball is contractible.

Each of the $\lambda$ groups $\pi_1(B_{\varepsilon}(p_i))$ can be thought of as providing a colouring of the corresponding ball, and therefore there are $\lambda!$ ways of distributing the labels $\pi_1(B_{\varepsilon}(p_i))$, over the unlabelled balls $\{B_{\varepsilon}(p_i)\}$ in $M$, (the coordinate labelling of the balls arising from $\{p_i\}$ must be factored out), consequently we set $B_{\lambda}(\Gamma_{(m)}(M_{(1)}), \pi(M_{(1)}), \Delta^\theta(M_{(1)})) \equiv m^n \rho_1 \lambda!$, where $\rho_1$ is a constant determined by the riemannian volume of the balls $\{B_{\varepsilon}(p_i)\}$, and where $\Gamma_{(m)}(M_{(1)})$ is the graph with $\lambda$ vertices and no edges generated by $\{B_{\varepsilon}(p_i)\}$.

On applying Stirling’s formula, we get

$$B_{\lambda}(\Gamma_{(m)}(M_{(0)}), \pi(M), \Delta^\theta(M_{(1)})) = m^n \rho_1 \lambda! \sqrt{2\pi} \exp[-\lambda + \frac{a}{12\lambda}] \lambda^{\lambda + 1/2} \quad (58)$$

where $a$ depends on $\lambda$ but it is such that $0 < a < 1$.

For large $\lambda$ we can rewrite this expression asymptotically in a more geometrical way, (we drop the inessential factor $\exp(a/12\lambda)$).

To begin with, let us remark that if the balls $\{B_{\varepsilon}(p_i)\}$ are contractible then $\chi(B_{\varepsilon}(p_i)) = 1$ and $\lambda = \chi(M_{(1)})$. In general, for manifolds $M \in \mathcal{R}(r, D, V)$, arbitrarily small geodesic balls are not contractible and consequently $\chi(B_{\varepsilon}(p_i)) \neq 1$. Thus, it is natural to introduce a parameter $\gamma_1$ according to

$$\chi(M_{(1)}) = \sum_{i}^{\lambda} \chi(B_{\varepsilon}(p_i)) = 2\lambda/(\gamma_1 - 2) \quad (59)$$

(The particular choice of the ratio $2/(\gamma_1 - 2)$ is for later convenience. Roughly speaking, this ratio measures to what extent the local $\varepsilon$-balls fail to be contractible).

Since the $\pi_1(B_{\varepsilon}(p_i))$ are non-trivial, it follows also that, for a given orthogonal representation $\theta$ of $\pi_1(M_{(1)})$, the corresponding R-torsions are, in general, non-trivial, i.e., $\Delta^\theta(M_{(1)}) \neq 1$. Thus, for any given value of $\Delta^\theta(M_{(1)})$, of such torsions, it is convenient to introduce a parameter, $\Lambda_1$, measuring the deviation from triviality of the torsion considered, and defined according to

$$\Lambda_1 \equiv [e^{\Delta^\theta(M_{(1)})}]^{-1} \quad (60)$$

(the presence of $e$ and the exponent $-1$ are here to provide a symmetric match with Stirling’s formula).
In terms of the parameters $\gamma_1, \Lambda_1$ so introduced we can rewrite the counting function for $M(1)$ as a function of the Euler characteristic and of the R-torsion according to

$$B_\lambda(\Gamma_m(M(1)), \pi(M(1)), \Delta^\theta(M(1))) \xrightarrow{\lambda \to \infty} m^n(\Lambda_1 \Delta^\theta(M(1)))^\lambda \ (\lambda = m^n V)^{(\gamma(M(1))/2)^{(\gamma_1-2)/2}} \cdot \rho_1(1 + O(\frac{1}{n})) \quad (61)$$

We now extend (61) to the generic $M(k)$ by induction on $k$, namely we assume that (61), (suitably adapted), holds true for the geodesic ball covering $M(k-1)$ and then show that it holds true also for the covering $M(k)$. We put

$$B_\lambda(\Gamma_m(M(k)), \pi(M(k))) \xrightarrow{\lambda \to \infty} m^n(\Lambda_k \Delta^\theta(M(k)))^\lambda \ (\lambda = m^n V)^{(\gamma(M(k))/2)^{(\gamma_k-2)/2}} \cdot \rho_k(1 + O(\frac{1}{n})) \quad (62)$$

where now $\Lambda_{k-1}, \gamma_{k-1},$ and $\rho_{k-1}$ are suitable constants.

The inductive step easily follows by rewriting $M(k) = M(k-1) \cup (M(k) \setminus M(k-1))$. A trivial application of the cardinality laws for the Euler characteristic and the torsion immediately yields

$$B_\lambda(\Gamma_m(M(k)), \pi(M(k))) \xrightarrow{\lambda \to \infty} m^n(\Lambda_k \Delta^\theta(M(k)))^\lambda \ (\lambda = m^n V)^{(\gamma(M(k))/2)^{(\gamma_k-2)/2}} \cdot \rho_k(1 + O(\frac{1}{n})) \quad (63)$$

where

$$\gamma_k = \gamma_{k-1}(1 - \frac{\chi(M(k) \setminus M(k-1))}{\chi(M(k))}) + 2 \frac{\chi(M(k) \setminus M(k-1))}{\chi(M(k))} \quad (64)$$

and

$$\Lambda_k = \Lambda_{k-1}(\Delta^\theta(M(k) \setminus M(k-1)))^{-1} \quad (65)$$

while $\rho_k = \rho_{k-1}$.

In order to show that this induction mechanism yields the required asymptotics, we prove that $61$ stabilizes for $k$ large enough, i.e., $62$ does not depend on $M(k)$ any longer but just from the underlying manifold $M$. To this end, it is sufficient to chose any $k > k_c$ with $k_c$ such that $(k_c - 1) \epsilon > diam(M).$ Under such hypothesis, it is sufficient to apply again the cardinality laws for the Euler characteristic and for the representation torsion so as to get

$$\chi(M(k)) = \chi(\cup_i B_{k_c}(p_i)) = \chi(B_{k_c}(p_1)) = \chi(M) = \chi(M(k-1))$$

$$\Delta^\theta(M(k)) = \Delta^\theta(\cup_i B_{k_c}(p_i)) = \Delta^\theta(B_{k_c}(p_1)) = \Delta^\theta(M) = \Delta^\theta(M(k-1)) \quad (66)$$
Moreover
\[ \gamma_k = \gamma_{k-1} = \gamma_{k-1} \equiv \gamma_c \] (67)
and similarly
\[ \Lambda_k = \Lambda_{k-1} = \Lambda_{k-1} \equiv \Lambda_c \] (68)

These relations, once introduced in (63), immediately yield the asymptotics for the one-skeleton graphs counting function on a manifold \( M \in \mathcal{R}(r, D, V) \), namely
\[
B_{\lambda}(\Gamma(m)(M), \pi(M), \Delta^\theta(M)) \xrightarrow{\lambda \to \infty} m^n(\Lambda_c \Delta^\theta(M))^\lambda \ (\lambda = m^nV)^{(\chi(M))/2(\gamma_c - 2) - 1/2} \cdot \rho_c(1 + O\left(\frac{1}{\lambda}\right)) \] (69)

where the added factor \(-1\), (yielding for \(-1/2\)), in the power determining the subleading asymptotics, \((\chi(M))/2(\gamma_c - 2) + 1/2\), comes about by noticing that since
\[ \Gamma(m)(M(k)) = \bigcup_i^\lambda (\Gamma(m)(B_{k\epsilon}(p_i))) = \bigcup_i^\lambda (\Gamma(m)(M)) \] (70)
we get
\[
B_{\lambda}(\Gamma(m)(M(k)), \pi(M), \Delta^\theta(M(k))) = \lambda B_{\lambda}(\Gamma(m)(M), \pi(M), \Delta^\theta(M)) \] (71)

It is easily checked that, up to the natural arbitrariness connected with the parameters \( \Lambda_c, \gamma_c, \) and \( \rho_c \), (69) is consistent with all the requirements (i)-(v).

In order to obtain the full counting function \( B_{\lambda}(\Gamma(m)(M), \pi(M)) \) from (69) one should first sum over all representation torsion associated with a given representation \( \theta \) of the given fundamental group in \( O(n = 3) \), (this sum is finite owing to the simple homotopy finiteness theorem which holds true for spaces of bounded geometry), then we should average over all possible inequivalent representations of \( \pi_1(M) \) in the orthogonal group, this a rather delicate technical point that we do not address here. It will be (partially) settled down only in the final part of the paper, where we show how such representations of the fundamental group are deeply rooted in the structure of three-dimensional simplicial gravity.

With the above remarks in mind, we are now in position for deriving the required entropy estimate for the number of isomorphism classes of graphs in \( \Omega(m)(\lambda) \).

According to the homotopy finiteness theorem recalled above, only a finite number of homotopy types of manifolds are realized in \( \mathcal{R}(r, D, V) \). Thus, by the definition of \( \Omega(m)(\lambda) \), and of the counting function \( B_{\lambda}(V, \Gamma(m), \pi(M)) \), the number, \( |\Omega(m)(\lambda)| \), of isomorphism
classes of graphs of order \( \lambda \), contained in the set \( \Omega_{(m)}(\lambda) \cap H_{(m)} \) is bounded above, for a given \( V \), by, (recall that we are conditioning the allowable configurations in \( \Omega_{(m)} \) by having the \( b_{(m)} \) boundary vertices empty)

\[
|\Omega_{(m)}(\lambda)| \leq \sum_{\pi} B_{\lambda}(\Gamma_{(m)}, \pi(M))
\]

(72)

where \( \sum_{\pi} \) denotes the finite sum over all homotopy types in \( R(r, D, V) \). Thus, in the \( m \to \infty \) limit where \( \lambda \to \infty \), \( |\Omega_{(m)}(\lambda)| \) can, according to (69), be bounded above by

\[
|\Omega_{(m)}(\lambda)| \leq \sum_{\pi} \left( \Lambda_{c} \Delta^{\theta}(M) \right)^{\lambda} (1 + O(1/\lambda))
\]

(73)

which provides the required entropy estimate.

If we put together (13), (51), and (73) we get for the probability of not-sampling \( \{\Gamma_{(m)}^{*}\} \) out of \( \Omega_{(m)} \) the estimate

\[
P_{\Omega}\{\Gamma \subset \Omega_{(m)}: \Gamma \not\supset \{\Gamma_{(m)}^{*}\}\} \leq \sum_{\pi} \sum_{\lambda} z^{\lambda} \exp \left[ \frac{1}{2} \beta d_{eff}(\Gamma_{\lambda}) \right] \rho_{c}(1 + O(1/\lambda))
\]

(74)

It follows that if for each given \( m_{0} \) and each \( \epsilon > 0 \), there exists some \( m > m_{0} \) such that

\[
\beta \left[ \left( \frac{c}{\beta} \right) - \frac{1}{2} d_{eff}(\Gamma_{\lambda}) \right] \geq \log(\Lambda_{c} \Delta^{\theta}(M)) + \epsilon
\]

(75)

for \( \lambda \) sufficiently large, then the sum appearing at the right side of (74) is finite and converges to zero as \( \beta \to \infty \), for \( (c/\beta) = (1/2)d(\Omega) \) and \( d_{eff}(\Gamma_{\lambda}) < d(\Omega) \).

Notice that this latter condition implies that \( \lim_{\lambda \to \infty} d_{S}(\Gamma_{\lambda}) \frac{N_{S}(\Gamma_{\lambda})}{\lambda} \) should be sufficiently small, namely that \( \Gamma_{\lambda} \) is not a thin graph, like a necklace of occupied sites scattered in \( \Omega_{(m)} \). Critical behavior requires graphs \( \Gamma_{\lambda} \) whose volume growth term \( N_{(m)}^{(0)}(\Gamma_{\lambda}) \) dominates over the surface growth term \( N_{(m)}^{S}(\Gamma_{\lambda}) \).

In such a case, given \( 0 < \delta < (1/2) \), the above convergence properties for (74) imply that there exists an inverse temperature \( \beta \), which is independent from \( m \), (since \( d(\Gamma_{\lambda}) \) is bounded above by \( d(\Omega) \) in terms of the parameters \( n, r, D, \) and \( V \) ), such that

\[
P_{\Omega}\{\Gamma \subset \Omega_{(m)}: \Gamma \not\supset \{\Gamma_{(m)}^{*}\}\} \leq \frac{1}{2} - \delta < \frac{1}{2}
\]

(76)

for all \( \beta \geq \beta \).

Now, if we denote by \( E(\{\Gamma_{(m)}^{*}\}) \) the mathematical expectation for the \( P_{\Omega} \) occurrence of the set of one- skeleton graphs \( \{\Gamma_{(m)}^{*}\} \) in \( \Omega_{(m)} \), then we can write

\[
E(\{\Gamma_{(m)}^{*}\}) = P_{\Omega}\{\Gamma \subset \Omega_{(m)}: \Gamma \supset \{\Gamma_{(m)}^{*}\}\} - P_{\Omega}\{\Gamma \subset \Omega_{(m)}: \Gamma \not\supset \{\Gamma_{(m)}^{*}\}\} = 1 - 2P_{\Omega}\{\Gamma \subset \Omega_{(m)}: \Gamma \not\supset \{\Gamma_{(m)}^{*}\}\} > 2\delta
\]

(77)

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where, in the last two lines, we have used the normalization of the probability $P_\Omega$ and the uniform bound $\langle 76 \rangle$. Since the bound $\langle 77 \rangle$ is independent from $m$, we can apply the dominated convergence theorem and the proof of the proposition is completed. ♣.

The above proof directly shows the existence of distinct limit distributions in the limit set $F_\mu$, (see $\langle 33 \rangle$ and the associated remarks), and provides the geometrical meaning of the various phases described by $\lim_{m \to \infty} \Xi (m, \hat{z})$.

**Proposition 9** Let $\pi_*(i)$, (with $i = 1, \ldots, |\pi_*| < \infty$), denote the finite collection of distinct homotopy types realized in $R(r, D, V)$. Then, as $H_{(m)} \uparrow \Omega_\infty$, there exists a critical inverse temperature $\beta_{cr}$ such that for $\beta > \beta_{cr}$ we have $|\pi_*|$ limit distributions in the limit set $F_\mu$, namely

$$\mu^{(i)} = \lim_{H_{(m)} \uparrow \Omega_\infty} E(\{\Gamma_{(m)}^{*}\}^{(i)})$$

(78)

where $\{\Gamma_{(m)}^{*}\}^{(i)}$ denotes the set of $L(m)$-one-skeleton graphs of manifolds with homotopy type $\pi^{*}_{(i)}$, and where $E$ denotes expectation with respect to the measure $P_{\Omega}$.

**PROOF.** This is an immediate consequence of the proof of the previous proposition. ♣

The phase structure of the statistical system described by the partition function $\Xi(m, \hat{z})^*$ results from the classical mechanism expressing the competition between energy and entropy. This provides a first (but incomplete) explanation of how topology enters in describing the various phases: the entropy estimate $\langle 73 \rangle$ is parametrized by the Euler-Poincaré characteristic, to the effect that the number of configurations accessible to a geodesic balls one-skeleton graph depends on the parameter $\gamma_c$. As long as $\gamma_c - 2 > 0$, the largest entropic contribution to the partition function comes from three-manifolds, $(\chi(M) = 0)$, while, if $\gamma_c - 2 < 0$, pseudo-manifolds, $(\chi(M) < 0)$, dominate.

### 4 Simple homotopy types and the nature of three-dimensional lattice quantum gravity

The analysis of the previous paragraphs shows that the thermodynamical limit $m \to \infty$ of the grand-partition function $\Xi(\hat{z}, m)$, associated with geodesic balls two-skeletons, provides a natural framework for discussing fluctuations of the homotopy type of the manifolds in $R(r, D, V)$. As we shall see in the remaining part of this paper, the need of controlling, in a similar statistical way, either the topology or the smoothness of the three-manifolds sampled by $\Xi$, naturally yields to (euclidean) simplicial three-gravity. We wish to stress that now, we do not start by directly introducing the generalization to dimension
three of $\Xi(\hat{z}, m)$, but rather we derive the partition function of simplicial three-gravity
from few elementary first principles connected with a natural statistical reconstruction of
the nerve of a geodesic balls covering out of the presentations of the fundamental group
associated with the geodesic balls two-skeleton.

Instrumental to such a derivation is, again, the use of the appropriate finiteness
theorems which hold true for $\mathcal{R}(r, D, V)$. In this connection let us remark that for $n = 3$,
spaces of bounded geometries such as $\mathcal{R}(r, D, V)$ besides containing at most finitely many
homotopy types, contain also finitely many \textit{simple homotopy} types, (actually, this result
holds true in any dimension). We recall that simple homotopy is a particular homotopy
equivalence between two spaces, obtained through a finite sequence of (elementary)
expansions and collapses of cells, (given any cellular decomposition of the spaces in
question; see below for the technical aspects associated with these definitions).

That simple homotopy ought to play a significant role in simplicial gravity is also
indicated by the structure of the entropy estimate (69), where the \textit{representation torsions}
are connected to the leading asymptotics of the geodesic ball one-skeleton graphs. Such
torsion, as will be explained more in details below, are invariants of simple homotopy
rather than of hommotopic equivalences.

Loosely speaking, we can exploit simple homotopy in order to reconstruct the nerve
of a geodesic ball covering out of its two-skeleton, (within a given homotopy type).
Notice that since the boundary points in $\mathcal{R}(r, D, V)$ obtained by Gromov-Hausdorff
completion are not topological manifolds, (they are $n$-dimensional Homology manifolds,
\textit{i.e.}, $H_*(M, M - p) \simeq H_*(R^n, R^n - 0)$ for all $p \in M$), nothing else can be said in full
generality on how manifolds topologies are distributed in $\mathcal{R}(r, D, V)$. Thus the first step
in order to discuss to what extent we can statistically control manifold topologies in
$\mathcal{R}(r, D, V)$, is to understand how simple homotopy interacts with minimal geodesic balls
coverings.

\section{Formal deformations of geodesic ball skeletons and Whitehead torsions}

With the foregoing preliminary remarks along the way, let us denote by $\{\pi(M)\}$ the
homotopy class associated with the generic manifold $M \in \mathcal{R}(r, D, V)$. Notice that if $m$
is sufficiently large then the corresponding nerve is in the same homotopy class of the
manifold, and one may use such a nerve in order to label the associated homotopy type.

The interaction between simple homotopy theory and the properties of nerves of
coverings of manifolds is naturally framed in the theory of CW-complexes.

We recall that a CW-complex $K$ is a Hausdorff space along with a family, $\{e_\alpha\}$, of
open topological cells of various dimensions such that, if we set $K_j = \cup\{e_\alpha | \text{dim}(e_\alpha) \leq j\}$,
then the following conditions are satisfied :

\begin{enumerate}
  \item $K = \cup_\alpha e_\alpha$ and $e_\alpha \cap e_\beta = \emptyset$ whenever $\alpha \neq \beta$,
  \item for each cell $e_\alpha$ there is a map $\psi_\alpha : Q^n \rightarrow K$, where $Q^n$ is a topological ball,
\end{enumerate}
(homeomorphic to $[0,1]^n$), of dimension $n = \dim(e_\alpha)$, such that $\psi_\alpha$ restricted to the interior of $Q^n$ is a homeomorphism onto $e_\alpha$, while the image, under $\psi_\alpha$ of the boundary $\partial Q^n$ of $Q^n$ is contained in $K_{n-1}$. The map $\psi_\alpha$ is called the characteristic map for the cell $e_\alpha$, while $\psi|_{\partial Q^n}$ is called the attaching map for $e_\alpha$. The CW-complex $K$ is $n$-dimensional if there are no cells of dimension greater than $n$. Finally, a CW-pair $(K, L)$ is a CW-complex $K$ together with a subset $L \subset K$ which is also a CW-complex.

Let us consider the pair $(\mathcal{N}, \Gamma_2)$, where $\mathcal{N}$ and $\Gamma_2$ respectively denote the nerve and the associated two-skeleton of a sufficiently fine geodesic balls covering of the generic manifold $M$ in $\mathcal{R}(r, D, V)$. According to the above remarks, we can consider $(\mathcal{N}, \Gamma_2)$ as a CW-pair generated by adjoining cells to the graph $\Gamma_{(m)}$. If the parameter $m$ is sufficiently large we can further assume that the nerve $\mathcal{N}$ has the same dimension of the (homology) manifold which it approximates, and here we restrict our attention to dimension equal to three, $n = 3$.

Under such assumptions, the CW-pair $(\mathcal{N}, \Gamma_2)$ is homotopically trivial, $\mathcal{N} \simeq \Gamma_2$, namely $\Gamma_2$ is a strong deformation retract of $\mathcal{N}$. This can be easily proven by noticing that the pair $(\mathcal{N}, \Gamma_2)$ is such that $\pi_j(\mathcal{N}, \Gamma_2) = 0$, for each $j \leq \dim(\mathcal{N} - \Gamma_2)$, where $\pi_j(\mathcal{N}, \Gamma_2)$ denotes the relative homotopy groups of $(\mathcal{N}, \Gamma_2)$.

Our first step is to rewrite the generic pair $(\mathcal{N}, \Gamma_2)$ in a simplified form, which, roughly speaking, directly tells us how $\mathcal{N}$ is homotopically built out by adjoining cells to the generic vertex of $\Gamma_2$.

The building block for such construction is a geometrical operation called a formal deformation, which is defined by a finite sequence of either elementary collapses or elementary expansions. One says that $K$ collapses to $J$ if $K = J \cup e^n \cup e^{n-1} = J$ and if the cells $e^n$ and $e^{n-1}$ are not in $J$, and where the characteristic maps of $e^n$ and $e^{n-1}$, respectively $\phi: Q^n \to K$ and $\phi|_{Q^{n-1}} \to K$, are such that $\phi(\partial Q^n - Q^{n-1}) \subset J$.

Conversely, an elementary expansion is the inverse operation which corresponds to attaching a cell to $J$ along a face of the cell by a suitable map, so as to generate a new complex $K$.

If there is formal deformation between two complexes $K$ and $J$, we write $K \not\simeq J$, and $K$ and $J$ are said to have the same simple-homotopy type. If $K$ and $J$ have a common subcomplex $\hat{K}$, whose cells are not removed during the process of formal deformation, then we write $K \not\simeq J$ rel $\hat{K}$.

There is a basic result in simple homotopy theory, (see e.g., the book of Cohen, chap.II), which along with the above notational remarks, can be easily established for the CW-pair $(\mathcal{N}, \Gamma_2)$. Let $h = \dim(\mathcal{N} - \Gamma_2)$, and let $q \geq h - 1$ be an integer. Let $e^0$ be a 0-cell of $\Gamma_2$, (a vertex). Then there is a formal deformation from $\mathcal{N}$ to a new CW-complex $\mathcal{M}$,
which does not alter the underlying $\Gamma_2$, i.e., $N \wedge \setminus \mathcal{M}$, rel $\Gamma_2$, such that

$$
\mathcal{M} = \Gamma_2 \cup \bigcup_{j=1}^{a} e_j^q \cup \bigcup_{i=1}^{a} e_i^{q+1}
$$

(79)

where the $e_j^q$ and $e_i^{q+1}$ have characteristic maps $\psi_j^{(q)}: Q^q \to \mathcal{M}$ and $\psi_i^{(q+1)}: Q^{q+1} \to \mathcal{M}$, respectively. And where $\psi_j^{(q)}(\partial Q^q) = e^0$, namely the $q$-cells are trivially attached at the vertex $e^0$. If in the relation (79), we choose the integer $q$ in such a way that $q \geq 2$, then we say that the pair $(\mathcal{M}, \Gamma_2)$ is in simplified form.

We recall that the relative homotopy groups $\pi_q(X, A, x)$, with $q \geq 2$, of a triple $(X, A, x)$, $(X$ a topological space, $A$ a closed subspace of $X$ containing the point $x)$, are defined by the collection of arcwise-connected components of the function space $\{\phi \in C(X, A) \mid \phi(x) = a\}$, generated by all mappings $f: Q^q \to X$ such that $f(\partial Q^q)$ lies in $A$, and $f[Q^1 \times \partial(Q^{q-1}) \cup (0 \times Q^{q-1})] = x$, (the set inside the square brackets $J^{q-1} \equiv [Q^1 \times \partial(Q^{q-1}) \cup (0 \times Q^{q-1})]$ is the boundary of the hypercube $Q^q$ minus the open top face). It is known that the relative homotopy groups $\pi_q(X, A, x)$ are abelian groups as soon as $q \geq 3$, they can be provided also with a $\mathbb{Z}\pi_1$-module structure. More in detail, if $\alpha$ and $\phi$ respectively represents the homotopy classes $[\alpha]$ and $[\phi]$ in $\pi_1(A, x)$ and $\pi_1(X, A, x)$, then $\pi_1$ acts on the relative homotopy groups by $[\alpha] \cdot [\phi] = [\bar{\phi}]$, where $[\bar{\phi}]$ is generated by dragging $\phi$ along the loop $\alpha^{-1}$. This action yields for the $\mathbb{Z}\pi_1$-module structure of $\pi_q(X, A, x)$ if we define the multiplication by

$$
(\sum_i \lambda_i[\alpha_i])[\phi] = \sum_i \lambda_i([\alpha_i] \cdot [\phi])
$$

(80)

In other words, each class $[\alpha] \in \pi_1$ characterizes an automorphism of the group $\pi_q(X, A, x)$ by mapping the homotopy class $[\phi]$ into $[\bar{\phi}]$.

The relative homotopy groups of interest to us are obviously those generated by the homotopy classes of the characteristic maps $\psi_i^{q+1}$ and $\psi_i^q$, viz., $\pi_{q+1}(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^{a} e_j^q, \Gamma_2)$, respectively. If we consider them as $\mathbb{Z}\pi_1$-modules in the sense recalled above, then it can be shown that the $\mathbb{Z}\pi_1$-modules with bases $\{[\psi_i^{q+1}]\}_{i=1,...,a}$ and $\{[\psi_i^q]\}_{i=1,...,a}$, respectively.

The Whitehead torsion characterizing the simple homotopy type of the pair $(\mathcal{M}, \Gamma_2)$, is read out through a boundary operator. Recall that, for $q \geq 2$, the boundary function $\partial: F_q(X, A, x) \to F_{q-1}(X, A, x)$ defined by $\partial(f)(t_1, t_2, \ldots, t_q) = f(1, t_2, \ldots, t_q)$ induces a homomorphism, (which we denote again by $\partial$), of $\pi_q(X, A, x)$ into $\pi_{q-1}(A, x)$. In our case, we actually get an isomorphism, (since $\mathcal{M}$ retracts on $\Gamma_2$),

$$
0 \to \pi_{q+1}(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^{a} e_j^q) \to \partial \pi_q(\Gamma_2 \cup \bigcup_{j=1}^{a} e_j^q, \Gamma_2) \to 0
$$

(81)
given by
\[ \partial[\psi_{i+1}] = \sum_j w_{ij}[\psi_j] \] (82)

where \( w_{ij} \) is the \((a \times a)\) non-singular \( \mathbb{Z}\pi_1\)-incidence matrix defining associated with the pair \((\mathcal{M}, \Gamma_2)\).

The matrix \( w_{ik} \) is naturally acted upon by a set of operations which consists in: (i) Multiply on the left the \( i \)-th row of the matrix by (plus or minus) an element of the fundamental group \( \pi_1(\Gamma_2, e_0) \); (ii) Add a left group-ring multiple of one row to another; (iii) Expand the matrix by adding a corner identity matrix.

Such operations give rise to a new incidence matrix which generates a simplified pair in the same simple-homotopy class of \((\mathcal{M}, \Gamma_2)\).

The equivalence classes, \( \tau(w_{ik}) \), under the operations (i), (ii), (iii), generated by the non-singular incidence matrices \( w_{ik} \), are the Whitehead torsions associated with the matrices \( w_{ik} \). The collection of such \( \tau(w_{ik}) \) gives rise to a group, the Whitehead group of \( \pi_1(\Gamma_2), Wh(\pi_1(\Gamma_2)) \).

In general, the Whitehead group is an infinite-dimensional (abelian) group, however, similarly to what happens to the number of distinct homotopy types, the number of inequivalent simple homotopy types realized by manifolds in \( \mathcal{R}(r, D, V) \) is finite. Thus, independently from \( m \), there are only a finite number of inequivalent Whitehead torsions \( \tau(w_{ik}) \) realized as we consider the totality of finer and finer geodesic balls coverings of manifolds in \( \mathcal{R}(r, D, V) \).

### 4.2 Gaussian insertion of three-cells and simplicial gravity

With the foregoing remarks along the way, our strategy will be aimed to providing a statistical procedure for generating, a simplified pair \((\mathcal{M}, \Gamma_2)\) out of the presentation of the fundamental group \( \pi_1(\mathcal{M}) \) associated with the generic \( L(m) \)-geodesic balls two skeleton \( \Gamma_2 \). To this end, let \( \tau_{(i)}(\pi_1, w) \), with \( i = 1, \ldots, k < \infty \), denote the distinct Whitehead torsions realized by the inequivalent, (in the simple homotopy sense), pairs \((\mathcal{M}, \Gamma_2)\) introduced above. As recalled, any such torsion is described, (modulo the transformations described by (i), (ii), (iii)), by a non-singular \( a(m) \times a(m) \) matrix, \( w_{jl}(\pi_1(\Gamma_2, e_0)) \), whose entries are elements of the group ring \( \mathbb{Z}[\pi_1(\Gamma_2, e_0)] \).

Consider an orthogonal representation, \( \theta \), of \( \pi_1(\Gamma_2, e_0) \), say by orthogonal \( p \times p \) matrices, turning the \( p \)-dimensional Euclidean space \( \mathbb{R}^p \) into a right \( \mathbb{R}(\pi_1(\Gamma_2, e_0))-\text{module} \). Correspondingly, we denote by \( \theta_*[w_{jl}(\pi_1(\Gamma_2, e_0))] \), (if no confusion arises we shall write \( \theta_*(w_{jl}) \)), the image through \( \theta \) of the Whitehead torsion \( w_{jl} \), and with \( \Delta^\theta(\mathcal{M}, \Gamma_2) = \det(\theta_*(w_{jl})) \), the associated Reidemeister-Franz representation torsion in the representation \( \theta \).
Notice that $\theta_*(w)$ is a $a_{(m)} \times a_{(m)}$ matrix with entries in $\text{Mat}_p(\mathbb{R})$, namely a matrix of order $pa_{(m)}$ with real entries, $(\text{Mat}_p(\mathbb{R})$ denotes the ring of all $p \times p$ matrices with real entries).

According to the above remarks, the relative homotopy groups describing the *attachment* of the two-dimensional and three-dimensional cells to $\Gamma_2$, which gives rise to $(\mathcal{M}, \Gamma_2)$, respectively are $\pi_2(\Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2, \Gamma_2)$ and $\pi_3(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2)$, (notice that $\pi_2(\Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2, \Gamma_2)$ is abelian since the two-cells $\varepsilon_j^2$ are trivially attached to $\Gamma_2$).

Through the orthogonal representation $\theta$ of the fundamental group $\pi_1(\Gamma_2, e_0)$, these modules, (thought of as $\mathbb{R}(\pi_1(\Gamma_2))$ modules), give rise to the real vector spaces

\[
\pi_3^\theta(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2) = \mathbb{R}^p \otimes_{\mathbb{R}^{\pi}} \pi_3(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2)
\]

\[
\pi_2^\theta(\Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2, \Gamma_2) = \mathbb{R}^p \otimes_{\mathbb{R}^{\pi}} \pi_2(\Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2, \Gamma_2)
\]

which can be endowed with a preferred base, respectively $x_\alpha \otimes [\varepsilon_k^3]$ and $x_\alpha \otimes [\psi_k^2]$, generated by an orthogonal base of $\mathbb{R}^p$, (here denoted by $x_\alpha$), and the distinguished base of the relative homotopy groups $\pi_3$ and $\pi_2$ consisting of the characteristic maps of the cells generating $\mathcal{M}$ out of $\Gamma_2$.

Such choice of a preferred base can be used to characterize an inner product, in the spaces $\pi_2^\theta$, in which $[\theta_*(w_l)^{tr} \theta_*(w_l)]^{-1}$, (where $\theta_*(w_l)^{tr}$ is the transpose of $\theta_*(w_l)$), is a symmetric positive bilinear form.

The construction of this inner product has a natural physical interpretation that will be significant in what follows.

To begin with, let us remark that the generic element of $\pi_2^\theta(\Gamma_2 \cup \bigcup_{j=1}^{a} \varepsilon_j^2, \Gamma_2)$, say $\xi = \sum_{\alpha, i} \xi^{\alpha(i)}(x_\alpha \otimes [\psi_i^2])$, can be equivalently thought of as a configuration of $a_{(m)}$ ordinary vectors, $\xi^{(i)} = \xi^{\alpha(i)}$ in $\mathbb{R}^p$, each such vectors being associated with a corresponding two-cell $\varepsilon_i^2$. The non-trivial attachment of the three-cells $\varepsilon_k^3$ to $\Gamma_2$ is described by the isomorphism $\theta_*(\partial)^{-1} : \pi_2^\theta \to \pi_3^\theta$ which with each vector $\xi \in \pi_2^\theta$ associates the vector $\theta_*(\partial)^{-1}\xi \in \pi_3^\theta$ given by

\[
\theta_*(\partial)^{-1}\xi = \sum_{\alpha, k, l} \xi^{\alpha(k)} x_\alpha [\theta_*(w_{kl})]^{-1} \otimes [\psi_l^3]
\]

Again, this vector can be thought of as a collection of $a_{(m)}$ vectors in $\mathbb{R}^p$, each such vector, say the one associated with the three-cell $\varepsilon_l^3$, $[\theta_*(\partial)^{-1}\xi]_l$, is obtained from the vectors $\xi^{(k)}$ through the action of the $a_{(m)}$ real valued $p \times p$ matrices $[\theta_*(w_{kl})]^{-1}$, (the cell index $l$ being fixed, while $k = 1, 2, \ldots, a_{(m)}$). Namely

\[
[\theta_*(\partial)^{-1}\xi]_l \equiv \sum_k \xi^{(k)} [\theta_*(w_{kl})]^{-1}
\]
With these preliminary remarks, let us consider two generic vectors \( \xi = \sum_i \xi^{(i)} (x_i \otimes [\psi^2_i]) \) and \( \eta = \sum_j \eta^{(j)} (x_j \otimes [\psi^2_j]) \) in \( \pi_2 (\Gamma_2 \cup \bigcup_{j=1}^p e_j^2, \Gamma_2) \), \( (\alpha, \nu = 1, \ldots, p \) and \( i, j = 1, \ldots, a_{(m)} \); here and in what follows, summation over repeated \( \mathbb{R}^p \)-indexes is assumed, while summation over cell indices is always explicitly stated).

According to the previous remarks, we can define an inner product, \( <\xi|\eta> \), between \( \xi \) and \( \eta \) by setting

\[
<\xi|\eta>_w \equiv \sum_l [\theta_w(\partial)^{-1}\xi](l) \cdot [\theta_w(\partial)^{-1}\eta](l)
\]

where \( \cdot \) denotes the ordinary euclidean inner product in \( \mathbb{R}^p \), between the euclidean vectors \( [\theta_w(\partial)^{-1}\xi](l) \) and \( [\theta_w(\partial)^{-1}\eta](l) \). More explicitly, we get

\[
<\xi|\eta>_w \equiv \sum_{i,l,k} \{[\theta_w(w_i)]^{tr}\theta_w(w_{kl})^{-1}\}_{\alpha\nu}\xi^{(i)}\eta^{(k)}
\]

where, (for each given value of the cell-indices \( i, l, k \), \{\[\theta_w(w_i)]^{tr}\theta_w(w_{kl})^{-1}\}_{\alpha\nu} \) is the \( (\mu, \alpha) \) entry in the \( p \times p \) matrix obtained by matrix product between the \( p \times p \) matrices \( [\theta_w(w_i)]^{tr}^{-1} \) and \( [\theta_w(w_{kl})]^{-1} \).

The geometrical construction just described shows connections with the physics of magnetic systems (or, if you prefer, with the formalism of lattice field theory). As a matter of fact, the simplified pair \( (\mathcal{M}, \Gamma_2) \) is described by a collection of \( \mathbb{R}^p \)-vectors \( \xi^{(i)} \) distributed on a set, \( \Lambda \), of \( a_{(m)} \) sites, with the vector \( \xi^{(i)} \) at the site \( (i) \) and the vector \( \xi^{(k)} \) at the site \( (k) \), interacting through a site-dependent interaction matrix \( A_{(i)(k)}^w \) whose components are given by \( (A_{(i)(k)}^w)_{\mu\nu} = -\{[\theta_w(w_{ii})]^{tr}\theta_w(w_{kl})^{-1}\}_{\mu\nu} \)

With such a choice of the interaction term, it follows that the inner product we have introduced is the total interaction energy, \( H_w(\Lambda) \), associated with the corresponding configuration of the \( \mathbb{R}^p \)-vectors \( \xi^{(i)} \) on the collection of sites \( \Lambda_{(m)} \), namely

\[
H_w(\Lambda)[\xi] \equiv -\frac{1}{2} \sum_{i,j \in \Lambda} \xi^{(i)}(A_{(i)(j)})_{\mu\nu}\xi^{(j)(\nu)} = \frac{1}{2} <\xi|\xi>
\]

This remark suggests that the properties of the collection of cells \( \{e_j^2\} \) can treated statistically by assigning, to each configuration of cells associated with the generic vector \( \xi \) of \( \pi_2^0 (\Gamma_2 \cup \bigcup_{j=1}^p e_j^2, \Gamma_2) \) an equilibrium Boltzmann-Gibbs distribution at (inverse) temperature \( J \), given by

\[
dP_{(m)}^w[\xi] \equiv (Z_{(m)}^w)^{-1} \exp(-JH_w(\Lambda)[\xi]) \prod_{i \in \Lambda} d\mu_i(\xi)
\]

where \( d\mu_i \) is the Lebesgue measure on \( \mathbb{R}^p \), and where the normalizing factor, (the partition function), \( Z_{(m)}^w \) is given by

\[
Z_{(m)}^w \equiv \int_\Lambda \exp[-JH_w(\Lambda)] \prod_{i \in \Lambda} d\mu_i(\xi)
\]

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More explicitly

\[
Z_w^{(m)} \equiv \int_{\pi_2} \exp\left(-\frac{1}{2} J < \xi | \xi >\right) D[\mu(\xi)]
\]  

(91)

where \(D[\mu(\xi)] = \prod_{i=1}^{n(m)} \prod_{\alpha=1}^{p} d\xi^{(i)\alpha}\) is the positive, \(O(p)\)-invariant, measure on \(\pi^q_2(\Gamma_2 \cup \bigcup_{j=1}^{a} e_j^2, \Gamma_2)\) associated with the reference inner product \(< \xi | \eta > = \sum_i \xi^{(i)} \cdot \eta^{(i)}\), corresponding to the trivial pasting (trivial torsion), (thus, \(P_w[\xi]\) is the multinormal distribution with variance \([JA_w^{(i)(k)}]^{-1}\)).

In order to see if, in the thermodynamical limit, the pair \((\mathcal{M}, \Gamma_2)\), as described by \(Z_w^{(m)}\), occurs with a non-vanishing probability, it will be necessary to give a precise mathematical definition of the functional measure

\[
\lim_{m \to \infty} dP_w^{(m)}[\xi]
\]

(92)

and in particular, to discuss the behavior of \(Z_w^{(m)}\) in the \(m \to \infty\) limit.

In this connection, two remarks are in order. First of all, as \(m\) increases, the \(L(m)\)-geodesic ball two-skeleton \(\Gamma_2\) changes, and in particular, the presentation of the fundamental group \(\pi_1(\Gamma_2, e_0)\), within a given homotopy class, is altered to the effect that the Whitehead incidence matrices \(w_{ik}\) vary with \(m\) even if they may represent a same Whitehead torsion \(\tau(w_{ik})\).

Moreover, also the number \(a(m)\) of two-cells and three-cells, (hence the number of \(\mathbb{R}^n\)-vectors \(\xi^{(i)}\)), grows with \(m\), and eventually \(a(m) \to \infty\). These two facts are related to each other, since the generic simplified pair \((\mathcal{M}, \Gamma_2)\) is assumed to come from the formal deformation of the pair \((\mathcal{N}, \Gamma_2)\) generated from a \(L(m)\)-geodesic ball nerve and its two skeleton. This remark, to be made precise below, suggests that in order to take into account, as \(m \to \infty\), all possible (equivalent) presentations of the fundamental group sampled by \(dP_w^{(m)}[\xi]\), we have to pass from a canonical to a grand-canonical point of view. Formally this means that, in order to consider the limit \(m \to \infty\) of \(dP_w^{(m)}[\xi]\), we have to introduce a fugacity \(z\) for the simplified pair \((\mathcal{M}, \Gamma_2)\), and define a grand-canonical partition function by weighting each canonical function \(Z_w^{(m)}\) with a factor \(z^{g^{(m)}}\), where \(g^{(m)}\) denotes the number of generators which occur in the presentation of \(\pi_1(\Gamma_2, e_0)\) associated with the \(L(m)\)-geodesic ball covering considered.

Each \(\pi_1(\Gamma_2)\) is finitely presented, and without loss in generality, \(\pi_1(\Gamma_2)\) can be thought of as given by a certain number of generators \(x_1, \ldots, x_g\) and relations among such generators \(R_i(x_1, \ldots, x_g) = 1, (i = 1, \ldots, R^\pi)\). The number \(g^{(m)}\) and \(R^{(m)}\) of such generators and relations is connected to the particular one-skeleton graph \(\Gamma^{(m)}\) which is contained in \(\Gamma_2\). In particular, the number of generators is provided by \(g^{(m)} = 1 + N^{(1)}_m(\Gamma) - N^{(0)}_m(\Gamma)\). Given such generators, we can associate with the graph \(\Gamma^{(m)}\), the wedge product of circles

\[
\vee (\Gamma^{(m)}) = e^0 \cup (e^1_1 \cup \ldots \cup e^1_g)
\]

(93)
where \( e^0 \) is a given vertex of \( \Gamma_m \).

We can use this presentation of (the free fundamental group of) \( \Gamma_m \) in order to generate a two-dimensional complex which is homotopically equivalent to \( \Gamma_2 \), and which we shall denote by \( \Gamma^*_2 \).

Explicitly, if \( x_j \) is the element of the free group \( \pi_1(\nabla(\Gamma_m), e^0) \) which is represented by a characteristic map for \( e_j^1 \), and if \( \phi_i : \partial Q^2 \to \nabla(\Gamma_m) \) represents the relation \( R_i(x_1, \ldots, x_g) \), then we set

\[
\Gamma^*_2 \equiv \nabla(\Gamma_m) \cup \phi_1 Q^2 \cup \phi_2 \ldots \cup \phi_k Q^2
\]

(94)

With these preliminary remarks, let us assume that we are given with a non-singular \( Z(\pi_1(\Gamma^*_2)) \) matrix, \( w^*_{ij} \), to be thought of as a possible Whitehead torsion labelling the simple homotopy type in \( R(r, D, V) \) defined by the \( L(m) \)-geodesic ball covering \( (N, \Gamma_2) \).

Let us assume that \( w^*_{ij} \) is an \( a \times a \) matrix with \( a(m) \leq N^{(3)}(N) \), where \( N^{(3)}(N) \) is the number of three-dimensional faces \( p^{(3)}_{ijk} \) in the geodesic balls nerve \( N \).

We can always replace the original, \( a(m) \times a(m) \) torsion matrix \( w^*_{ij} \) with an equivalent \( N^{(3)}(m) \times N^{(3)}(m) \) torsion matrix \( w_{ik} \) which has a block form obtained by expanding \( w_{ij} \) by adding an identity matrix \( I \) on the corner diagonal.

According to the definition of simplified pair, this expansion does not alter the Whitehead torsion, (see the operations (i), (ii), (iii), characterising the torsion of an incidence matrix \( w_{ik} \)), and it is also possible to show that there is a \( CW \)-pair in simplified form, that, by abuse of notation, we shall continue to denote by \( (M, \Gamma_2) \), whose torsion is provided by this new incidence matrix \( w_{ik} \).

This pair can be constructed as follows, (see proposition 8.7). Since \( w_{ik} \) is an \( N^{(3)}(m) \times N^{(3)}(m) \) matrix we let \( K_2 \equiv \Gamma^*_2 \cup e^2_1 \cup \ldots \cup e^2_{N^{(3)}} \).

The two-cells, \( e^2_i \), so introduced have characteristic maps \( \psi^2_j \) such that \( \psi^2_j(\partial Q^2) = e^0 \). We denote by \( [\psi^2_j] \) the homotopy class, in the relative homotopy group \( \pi_2(K_2, \Gamma^*_2) \), represented by the map

\[
\psi^2_j : (Q^2, Q^1, [Q^1 \times \partial Q^1 \cup (0 \times Q^1)]) \to (K_2, \Gamma^*_2, e^0)
\]

(95)

Notice that the same map, thought of with range in \( (K_2, e^0) \), can be used so as to represent the homotopy class \( [\psi^2_j] \) in \( \pi_2(K_2, e^0) \). Thus, if \( i : (K_2, e^0) \to (K_2, \Gamma^*_2, e^0) \) is the injection, we get \( i_\ast [\psi^2_j] = [\psi^2_j] \).

If we denote by \( f_j : (Q^2, \partial Q^2) \to (K_2, e_0) \) the map representing \( \sum_q w_{jq} [\psi^2_q] \), let us attach 3-cells to \( K_2 \) so as to get

\[
M \equiv K_2 \cup e^3_1 \cup \ldots \cup e^3_{N^{(3)}}
\]

(96)

where the 3-cells \( e^3_j \) have characteristic maps

\[
\psi^3_j : (Q^3, Q^2, [Q^1 \times \partial Q^2 \cup (0 \times Q^2)]) \to (M, K_2, e^0)
\]

(97)

such that \( \psi^3_j Q^2 = i \circ f_j \). But by definition \( \partial[i\circ f_j] = [\phi^3_j Q^2] \), thus we get

\[
\partial [\psi^3_j] = [i \circ f_j] = i_\ast (\sum_q w_{jq} [\psi^2_q] ) = \sum_q w_{jq} [\psi^2_q]
\]

(98)
which shows that the CW-pair \((M, \Gamma_2)\) has boundary operator provided by the matrix \(w_{ij}\), and it is in the same simple homotopy class of \((N, \Gamma_2)\). (Notice that the fact that the \(M\) constructed with \(w_{ik}\) actually retracts on \(\Gamma_2\), is less trivial and the reader is referred to\(^{42}\) for a detailed argument, (second part of proposition 8.7, p. 34)).

Once established that the incidence matrix of the simplified pair associated with \((N, \Gamma_2)\) can be assumed to be a \(Z(\pi_1)\)-matrix of order \(N^{(3)}(\mathcal{N})\), we can easily formalize the stated relation between the canonical and the grand-canonical point of view.

According to the properties of minimal geodesic ball coverings recalled in the introductory remarks, we can always assume that

\[
c = \limsup_{m \to \infty} \left( \frac{N^{(0)}(m)}{N^{(1)}(m)} \right)
\]

is a function on \(\mathcal{R}(r, D, V)\), bounded above in terms of the parameters \(n, r, D, V\). This implies that the number \(g_{\pi_1}^\pi(m) = 1 + N^{(1)}(m) - N^{(0)}(m)\) of generators of \(\pi_1(\Gamma_2, e_0)\) is such that

\[
g_{\pi_1}^\pi(m) = \mathcal{O}(N^{(1)}(\mathcal{N}))
\]

as \(m \to \infty\).

Thus the natural grand-canonical partition function providing the statistical description of the fields \(\xi \in \pi_2^\theta(M, \Gamma_2 \cup \bigcup_{j=1}^n e_j^3)\), can be written as

\[
Z(m, J, \pi_1) \equiv \sum_{(M, \Gamma_2)(m)} \exp \left( i_c N^{(1)}(m)(\Gamma_2) \right) \int_{\pi_2^\theta} \exp \left( -\frac{1}{2} J \langle \xi | \xi \rangle \right) D[\mu(\xi)]
\]

where \(i_c\) plays the role of a chemical potential determining the average number of generators associated with the presentation of \(\pi_1(\Gamma_2, e_0)(m)\), and where the sum \(\sum_{(M, \Gamma_2)}\) is the sum over the three-dimensional simplified pairs, \((M, \Gamma_2)(m)\), generated by \(L(m)\)-geodesic balls coverings of manifolds in \(\mathcal{R}(r, D, V)\) with isomorphic fundamental groups \(\pi_1(\Gamma_2)\). Correspondingly, the integral \(\int_{\pi_2^\theta} \cdots D[\mu(\xi)]\) is to be evaluated, over the vector space \(\pi_2^\theta(\Gamma_2 \cup \bigcup_{j=1}^n e_j^3, \Gamma_2)\), by using the \(N^{(3)}(m)\times N^{(3)}(m)\) incidence matrices \(w_{ik}\) associated with each given \((M, \Gamma_2)\), \((w_{ik}\) and the corresponding \((M, \Gamma_2)\) being explicitly realized following the construction delineated above).

According to the definition of the scalar product \(\langle \xi | \xi \rangle\) between the fields \(\xi \in \pi_2^\theta(\Gamma_2 \cup \bigcup_{j=1}^n e_j^3, \Gamma_2)\), it immediately follows that the canonical partition function \(Z_{w}^{(m)}(\Gamma_2)\), describing the gaussian statistical pasting of the three-cells \(e_j^3\) into a given \(\Gamma_2\), can be explicitly evaluated.
Proposition 10  For a given $L(m)$-geodesic ball covering presentation of $\pi_1(\Gamma_2, e_0)$, and a corresponding $Z(\pi_1(\Gamma_2, e_0))$ incidence matrix $w_{ik}$, the canonical partition function $Z_w(\Gamma_2)$ is given by

$$Z_w(\Gamma_2) = \left(\frac{J}{2\pi}\right)^{-pN^{(3)}_{(m)}} \Delta^\theta[(\mathcal{N}, \Gamma_2)]$$

where $\Delta^\theta[(\mathcal{N}, \Gamma_2)]$ denotes the Reidemeister-Franz representation torsion of the pair $(\mathcal{N}, \Gamma_2)$ corresponding to the simple homotopy type labelled by $w_{ik}$.

**Proof.** By direct computation of the Gaussian integral

$$\int_{\pi_2^N} \exp[-\frac{1}{2}J \sum_{i,j,k} \{[\theta_*(w_{li})^{tr}\theta_*(w_{kl})]^{-1}\}_{\alpha\nu}\xi^{\alpha(i)}\xi^{\nu(k)}]\prod_{i=1}^{N_{(m)}}\prod_{\alpha=1}^{p} d\xi^{(i)\alpha} =$$

$$= \left(\frac{J}{2\pi}\right)^{-(p/2)N^{(3)}_{(m)}} \{\det[\theta_*(w)](\theta_*(w))^{tr}\}^{-1/2}$$

where we have used the expression of $<\xi|\xi>$ in terms of the representation of the incidence matrix $w_{ik}$ defining the simple homotopy type of $(\mathcal{N}, \Gamma_2)$. Notice that $\theta_*(w)$ is a $N^{(3)}_{(m)} \times N^{(3)}_{(m)}$ matrix with entries in $Mat_p(\mathbb{R})$, namely a matrix of order $pN^{(3)}_{(m)}$ with real entries, ($Mat_p(\mathbb{R})$ denotes the ring of all $p \times p$ matrices with real entries). The determinant resulting from the gaussian integration is trivially reduced to the Reidemeister torsion, (in the orthogonal representation $\theta$ of $\pi_1(\Gamma_2, e_0)$), of the simplified pair $(\mathcal{M}, \Gamma_2)$ corresponding to the incidence matrix $w_{ik}$.

Since the Reidemeister-Franz torsion $\Delta^\theta(\mathcal{M}, \Gamma_2)$ is a combinatorial invariant, and the simplified pair $(\mathcal{M}, \Gamma_2)$ is combinatorially equivalent, (in the simple homotopy sense), to the pair $(\mathcal{N}, \Gamma_2)$, we can write

$$\Delta^\theta[(\mathcal{M}, \Gamma_2)] = \Delta^\theta[(\mathcal{N}, \Gamma_2)]$$

which together with (103) yields the relation (102). ♣

The net effect of such result is that we can rewrite $Z(m,J,\pi_1)$ directly in terms of combinatorial quantities referring to $L(m)$-geodesic balls coverings

**Lemma 3** The statistical sum $Z(m,J,\pi_1)$ describing a Gaussian-pasting of three-cells onto $\Gamma_2$ can be rewritten as

$$Z(m,J,\pi_1) = \sum_{(\mathcal{N}, \Gamma_2)} \Delta^\theta(\mathcal{N}, \Gamma_2) \exp[i_cN^{(1)}_{(m)}(\Gamma_2) - \sigma N^{(3)}_{(m)}(\mathcal{N})]$$

where we have set $\sigma \equiv \frac{p}{2} \ln(\frac{J}{2\pi})$. ♣
It follows from (11) that, up to the torsion invariants, \( Z(m, J, \pi_1) \) has the structure of the partition function of \textit{three-dimensional simplicial quantum gravity} as written down for the simplicial approximation generated by minimal \( L(m) \)-geodesic balls coverings. Namely,

\[
Z(m, J, \pi_1) = \sum_{(\Gamma_2, N)} \rho(\Gamma_2, N) \exp -S(\Gamma_2, N)
\]  

(106)

where the \textit{action} \( S(\Gamma_2, N) \) is given by

\[
S(\Gamma_2, N) \equiv \sigma N^{(3)}_{(m)}(N) - i_r N^{(1)}_{(m)}(\Gamma_2)
\]

(107)

and the \textit{weight} \( \rho(\Gamma_2, N) \) is given by

\[
\rho(\Gamma_2, N) \equiv \Delta^\theta(\mathcal{N}, \Gamma_2)
\]

(108)

Before commenting on this connection between the theory described by \( Z(m, J, \pi_1) \) and three-dimensional simplicial quantum gravity, it is useful to stress that here the theory suggests a natural weight, \( \rho(\Gamma_2, N) \), for the triangulations occurring in the statistical sum. As is known, the attribution of such a weight is a problem in all discretized approaches to quantum gravity. Typically one assumes \( \rho(\Gamma_2, N) = 1 \), basing such choice on universality arguments and on the experience gained in dimension two. The weight we get here is the Reidemeister-Franz torsion, which, obviously depends on the orthogonal representation of the fundamental group that we are considering. This dependence, which is rather manifest in the approach presented here, has interesting consequences on which we shall comment in the concluding remarks.

If we put together all such remarks and take into account the Gaussian nature of the partition function \( Z(m, J, \pi_1) \) the we get as a general result the following

**Proposition 11** Let \( \mathcal{T} \) denote the polytope, (or if you prefer, the dynamical triangulation), associated with the nerve \( N_{(m)} \) of the \( L(m) \)-minimal geodesic ball covering of the generic manifold \( M \in \mathcal{R}(r, D, V) \). Define the corresponding simplicial quantum gravity partition function as

\[
\mathcal{Z}(\mathcal{T}) \equiv \sum_{\mathcal{T}} \rho(\mathcal{T}) \exp [\alpha N^{(1)}_{(m)}(\mathcal{T}) - \gamma N^{(3)}_{(m)}(\mathcal{T})]
\]

(109)
where $\rho(T)$ is a weight for the triangulation $T$, $\alpha$ is a bare gravitational coupling constant, and $\gamma$ is a bare cosmological constant.

Then, for a given orthogonal representation $\theta: \pi_1(T) \to O(p)$, of the fundamental group $\pi_1(T)$, such $Z(T)$ is equivalent to the partition function $Z(m, J, \pi_1)$ which describes the gaussian insertion of three-dimensional cells onto $\Gamma_2(T)$.

**Proof.** The proposition is a formal restatement of the above remarks. The only point to stress is that apparently, we could have dispensed ourselves from considering just manifolds of bounded geometry belonging to some $\mathcal{R}(r, D, V)$. Notice however that in such a case we do not have at our disposal the control on homotopy types and simple homotopy types that we have for manifolds in $\mathcal{R}(r, D, V)$. In particular, the full Whitehead group $W(\pi_1(\Gamma_2))$ comes into play. In general $W(\pi_1(\Gamma_2))$ has an infinite number of inequivalent elements (torsions), and the above equivalence between $Z(T)$ and $Z(m, J, \pi_1)$ would be rather formal, involving, for each $\Gamma_2$, a sum over an infinite number of inequivalent gaussian distributions. We see again that $\mathcal{R}(r, D, V)$ provides a very effective regularization framework for question concerning discrete models of euclidean quantum gravity. ♣

5 The thermodynamical limit of three-dimensional simplicial gravity

As recalled in the introductory remarks, when we implement the random triangulation approach to three-dimensional quantum gravity, we fix a priori the length of the edges of the triangulation, (say, by setting their length to one), and fix our attention to the connectivity properties of the simplicial approximation considered. Then we discuss the resulting dynamical triangulation model at longer and longer distance scales, trying to extract information from the scaling limit. Typically this is done with energy-entropy arguments, by examining the critical behavior of the partition function and of the correlation functions.

When using spaces of bounded geometries and geodesic ball coverings as a regularization scheme, we are actually taking the dual point of view. Here the dynamical triangulation has edges the lengths of which are not equal to one but rather to $2L(m) = 2/m$. Thus, the number of cells of the generic $L(m)$-geodesic balls nerve $N_{(m)}$ diverges as $m \to \infty$. In this sense, the limit $m \to \infty$ actually corresponds to a thermodynamical limit rather than to a scaling limit.

More in details, if we have an algorithm for generating random triangulation, (with fixed edge-length), then the condensation of extended objects requires the critical behavior of the partition function in the scaling regime, and then we have to check that the resulting condensate is a manifold thought of as obtained from a sequence of random triangulations by rescaling lengths by a factor $s \to \infty$. 

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The point of view afforded by the use of the spaces of bounded geometries \( R(r, D, V) \) is different and somehow simpler, in the sense that in such a case we know a priori that the simplicial approximations, arising from geodesic ball coverings, yield for (homology) manifolds as \( m \to \infty \), (convergence is in the Gromov-Hausdorff topology). This is a direct consequence of the compactness of \( R(r, D, V) \).

The problem here is not so much to prove that the triangulation involved in the regularization yields in the limit a (generalized) manifold, but rather to prove that the partition function and the associated probability measure are well defined, as \( m \to \infty \), for a suitable range of values of the couplings.

With these preliminary remarks along the way we prove the following result which allows us to disclose the nature of the \( m \to \infty \) limit of the partition function \( Z(m, J, \pi_1) \),

**Proposition 12** Let \( ||\Delta^\theta||(\pi_1) \) denote the finite number of distinct Reidemeister-Franz representation torsions associated with those manifolds in \( R(r, D, V) \) whose fundamental group is isomorphic to a given group \( \pi_1 \), and let us consider the function over \( R(r, D, V) \) defined by

\[
b(M) \equiv \limsup_{m \to \infty} \left( \frac{N^{(1)}(M)}{N^{(3)}(M)} \right)
\]

Then, for any given value of the ratio \( 0 < b < \infty \) there exists a finite value of the chemical potential \( i_c \),

\[
i_c = -\frac{1}{2b} \]

and a finite number of critical values

\[
J^{(i)}_{\text{crit}} = 2\pi \left( \Delta^\theta_{(i)} \right)^2
\]

of the coupling \( J \), (with \( i = 1, \ldots, ||\Delta^\theta||(\pi_1) \)), corresponding to which the statistical system described by the partition function \( Z(m, J, \pi_1)|_{J=J_{\text{crit}}} \), (obtained by \( Z(m, J, \pi_1) \) by restricting the summation over incidence matrices with a given value of the torsion \( \Delta_{(i)} \), and by setting \( J \) equal to the corresponding value of \( J_{\text{crit}} \), admits a well defined limit as \( m \to \infty \).

Let \( GL(\text{Mat}_p(\mathbf{R})) \) be the infinite general linear group of non-singular matrices over the ring \( \text{Mat}_p(\mathbf{R}) \), of all \( p \times p \) matrices with real entries. If \( E_{(m)} \subset GL(\text{Mat}_p(\mathbf{R})) \) is the set of all possible incidence matrices \( \tilde{w} \equiv \theta_*(w) \), realized over \( R(r, D, V) \) as \( m \) varies, then \( \lim_{m \to \infty} Z(m, J, \pi_1) \) is associated with a unique probability measure \( P_{y}(dw) \) on
As $J$ varies from one critical value to another, the system exhibits a phase transition associated with the changes of the corresponding simple-homotopy types, and the set of manifolds in $\mathcal{R}(r,D,V)$ with isomorphic fundamental groups appears as a mixture of pure phases described by the probability measure on $\text{GL}(_p\text{Mat}(\mathbf{R}))$ given by a convex combination of the above measures

$$P(d\tilde{w}) = \frac{1}{\alpha} \sum_{i=1}^{\alpha} P^{(i)}(d\tilde{w})$$

with $\alpha = ||\Delta^\theta||(\pi_1)$.

Proof. Notice that the function $b$ introduced above provides, roughly speaking, the (inverse of the) number of tetrahedra shared, on the average, by the generic link in the polytope generated by the geodesic ball nerves $\mathcal{N}(m)(M)$. As the notation suggests, $b$ is not constant on $\mathcal{R}(r,D,V)$, and by using it as a parameter in controlling the critical regime of the theory, (basically as a chemical potential), we select among all possible manifolds in $\mathcal{R}(r,D,V)$ with Reidemeister torsion $\Delta^{(i)}$, the class of manifolds for which the critical regime occurs.

The proof of proposition 12 is divided in two parts. First, by exploiting few elementary properties of the generic general linear group, $\text{GL}(q,\mathbf{R})$, we introduce a Gaussian measure on the space $\text{GL}(_p\text{Mat}(\mathbf{R}))$. Then, such measure is connected with the $m \to \infty$ limit of $Z(m,J,\pi_1)|_{J=J_{\text{crit}}}$. 

First part:

5.1 Representation torsions and Gaussian measures on the general linear group $\text{GL}(_p\text{Mat}(\mathbf{R}))$

Let $\text{GL}(a,Z\pi_1(\Gamma_2))$ denote the general linear group of non-singular, (i.e., with two-sided inverse), $a \times a$ matrices over the ring $Z\pi_1(\Gamma_2)$. Under the natural injection of $\text{GL}(a,Z\pi_1(\Gamma_2))$ into $\text{GL}(a+1,Z\pi_1(\Gamma_2))$, one defines the infinite general linear group $\text{GL}(Z\pi_1(\Gamma_2))$ as the direct limit $\lim_{a \to \infty} \cup \text{GL}(a,Z\pi_1(\Gamma_2))$.

Similarly, we denote by $\text{GL}(_p\text{Mat}(\mathbf{R}))$ the infinite general linear group of non-singular matrices over the ring $Mat_p(\mathbf{R})$, of all $p \times p$ matrices with real entries. (Notice that for any $a$ we have $\text{GL}(a,Mat_p(\mathbf{R})) \simeq GL(pa,\mathbf{R}))$. 

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Let $bE_{ij}$ denote the $\mathbb{Z}\pi_1(\Gamma_2)$-matrix with an entry $b$ in the $(i,j)$-th place and zero elsewhere. Let us denote by $E(\mathbb{Z}\pi_1(\Gamma_2)) \subset GL(\mathbb{Z}\pi_1(\Gamma_2))$ the (commutator) subgroup generated by the matrices of the form $(I + bE_{ij})$, (elementary matrices). Also, let us introduce the group $E_T$ generated by $E(\mathbb{Z}\pi_1(\Gamma_2))$ and by the group of the trivial units of $\mathbb{Z}\pi_1(\Gamma_2)$. By means of such subgroups we can algebraically characterize the Whitehead group $W(\pi_1(\Gamma_2))$ as the quotient

$$W(\pi_1(\Gamma_2)) = \frac{GL(\mathbb{Z}\pi_1(\Gamma_2))}{E_T}$$

(115)

A similar characterization holds true for $W(Mat_p(\mathbb{R}))$ which can be written as

$$W(Mat_p(\mathbb{R})) = \frac{GL(Mat_p(\mathbb{R}))}{E(Mat_p(\mathbb{R}))}$$

(116)

where $E(Mat_p(\mathbb{R}))$ is the commutator subgroup generated by all elementary matrices in $GL(Mat_p(\mathbb{R}))$.

Notice that the Whitehead group $W(Mat_p(\mathbb{R}))$ can be identified with the half-line of positive numbers, (thought of as a multiplicative group), and we can turn it into a measure space by considering on it the measure $d\mu(\Delta) = d\Delta/\Delta$, (with $d\Delta$ the Lebesgue measure on $\mathbb{R}^+$), invariant under multiplication. Let us also recall that the Haar measure $\mathcal{H}_q$ on $GL(q,\mathbb{R})$ is given by

$$d\mathcal{H}_q(\tilde{w}) = \prod_{i,j} d\tilde{w}_{ij} |\det \tilde{w}|^q$$

(117)

Thus, given a (suitably normalized) left-invariant Haar measure $\mathcal{S}_q$ on the special linear group $SL(q,\mathbb{R})$ generated by all elementary matrices in $GL(q,\mathbb{R})$, we can write (by abuse of notation)

$$\int_{W(Mat_p(\mathbb{R}))} \left( \int_{SL(q,\mathbb{R})} f(\tilde{w}h) d\mathcal{S}_q(h) \right) d(\mu(\Delta))(\tilde{w}) = \int_{GL(q,\mathbb{R})} f(\tilde{w}) d(\mathcal{H}_q)(\tilde{w})$$

(118)

for every continuous function $f$ with compact support on $GL(q,\mathbb{R})$.

Let $E_{ka}$ denote the $q \times q$ matrix which has the element in the $(k,a)$ place equal to 1 and all other elements equal to 0. If $k \neq a$ and $x^k_{(a)} \in \mathbb{R}$, define the elementry matrix in $GL(q,\mathbb{R})$ as

$$B_{ka}(x^k_{(a)}) \equiv I_q + x^k_{(a)} E_{ka}$$

(no summation over the indexes $k$ and $a$), where $I_q$ denotes the identity matrix in $GL(q,\mathbb{R})$. In terms of such elementary matrices, each matrix $\tilde{w} \in GL(q,\mathbb{R})$ can be represented as

$$\tilde{w} = sD = \prod_{ka} B_{ka} D$$

(119)
where \( \tilde{s} \in SL(q, \mathbb{R}) \) is, as indicated, a finite product of elementary matrices \( B_{ka} \), and \( D \) is a diagonal matrix of the form \( D_{ij} = \delta_{ij} \), with \( i, j = 1, \ldots, q-1 \), and \( D_{qq} = \det \tilde{w} = \Delta^q(\tilde{w}) \), namely \( D \equiv I_q + (\Delta^q - 1)E_{qq} \). (Notice that multiplication on the left such as \( B_{ka}(x)D \), adds \( x \) times the \( a \)-th row to the \( k \)-th row of \( D \).

For each given value of the index \( a = 1, \ldots, q \), let \( \mathcal{G}_{t(q)}(d\mathbf{x}_{(a)}) \) denote the \( q \)-dimensional Gaussian measure of variance \( t > 0 \) defined by

\[
\mathcal{G}_{t(q)}(d\mathbf{x}_{(a)}) = \frac{1}{(2\pi t)^{q/2}} \exp \left[ - \frac{||\mathbf{x}_{(a)}||^2}{2t} \right] d\mathbf{x}_{(a)}
\]

where \( ||\mathbf{x}_{(a)}||^2 \equiv \sum_k (x_{a,k}^k)^2 \), and \( d\mathbf{x}_{(a)} \) means the Lebesgue measure on \( \mathbb{R}^q \).

Let \( C_0(GL(q, \mathbb{R})) \) and \( C_0(W(Mat_p(\mathbb{R}))) \) respectively denote the spaces of continuous functions with compact support on the groups \( GL(q, \mathbb{R}) \) and \( W(Mat_p(\mathbb{R})) \), then by means of the parametrization \( \tilde{w} = \tilde{s}D \) of \( GL(q, \mathbb{R}) \) we can consider the linear mapping from \( C_0(GL(q, \mathbb{R})) \) onto \( C_0(W(Mat_p(\mathbb{R}))) \), defined by gaussian averaging over \( SL(q, \mathbb{R}) \) according to

\[
f(\tilde{w}) \mapsto \hat{f}_\mathcal{G}(t(q), \Delta(\tilde{w})) \equiv \int f(\tilde{w} = \tilde{s}(\mathbf{x}_{(a)})D) \prod_a \mathcal{G}_{t(q)}(d\mathbf{x}_{(a)})
\]

More explicitly, we choose the variance \( t(q) \) according to

\[
2\pi t(q) = \frac{1}{q} \Delta^2
\]

so as to get

\[
\hat{f}_\mathcal{G}(\Delta(\tilde{w})) = \int_{SL(q, \mathbb{R})} f(\tilde{w} = \tilde{s}D)q^{q^2/2} \Delta^{-q^2} \exp \left[ -\pi q \Delta^{-2} ||\mathbf{x}_{(a)}||^2 \right] \prod_k dx_{k,a}^{k(a)}
\]

Notice that the particular choice \( (123) \) of the variance is motivated by the fact that as \( q \) increases, we need to enhance the occurrence (with respect to the given, gaussian measure), of the matrices \( B_{ka}(x_{(a)}^{k}) \) which eventually reduce to the identity matrix, (such matrices generate, up to determinants, the full general linear group). More in details, as \( q \to \infty \), the variance \( t(q) \to 0 \) and the associated Gaussian measure degenerates to the Dirac measure supported on the zero sequence \( \{x_{(a)}^{k} \} = \{0, \ldots, \} \). This yields for \( B_{ka} \) which reduce, almost everywhere with respect to the gaussian measure, to identity matrices, as required.

Notice also that the variables \( x_{(a)}^{k} \) with \( k = a \) do not appear in the factorization \( \tilde{w} = \tilde{s}D \), thus such variables are integrated out to 1 in \( (124) \). This normalization is needed in order to extend the positive linear form

\[
f(\tilde{w}) \mapsto \int_{W(Mat_p(\mathbb{R}))} \hat{f}_\mathcal{G}(\Delta(\tilde{w}))d(\mu(\Delta))
\]
accomplished as follows.

Let \( C(GL(Mat_p(\mathbb{R}))) \) denote the algebra of all continuous real-valued functions on \( GL(Mat_p(\mathbb{R})) \). From the definition of \( GL(Mat_p(\mathbb{R})) \) as the direct limit group of the system \( \{GL(q, \mathbb{R}), i_{q,q+1}\} \), (where \( i_{q,q+1} \) denotes the canonical injection of \( GL(q, \mathbb{R}) \) into \( GL(q+1, \mathbb{R}) \)), it follows that if \( w \) is an element of \( GL(Mat_p(\mathbb{R})) \) then there exists an integer \( q \) and a matrix \( w(q) \in GL(q, \mathbb{R}) \) such that \( i_q(w(q)) = w \), (with \( i_q \) being the injection of \( GL(q, \mathbb{R}) \) into \( GL(Mat_p(\mathbb{R})) \)). Thus if \( f \in C(GL(Mat_p(\mathbb{R}))) \) we can always write \( f(w) = f[i_q(w(q))] \) for some \( w(q) \in GL(q, \mathbb{R}) \).

These remarks show that, as claimed, the measure \([123]\) can be naturally extended to a regular measure \( P(d\bar{w}) \) on \( GL(Mat_p(\mathbb{R})) \) by defining

\[
\int_{GL(Mat_p(\mathbb{R}))} f(\bar{w})dP(\bar{w}) \equiv \int_{W(Mat_p(\mathbb{R}))} (\int_{SL(q, \mathbb{R})} f[i_q(\bar{w}(q))] q^{q^2/2} \Delta^{-q^2} \exp \left[ -\pi q \Delta^{-2} \sum_a ||x(a)||^2 \right] \prod_{j,a}^q d\bar{x}^k_{(a)})(d(\mu(\Delta))) \tag{126}
\]

for every continuous function \( f \) in \( GL(Mat_p(\mathbb{R})) \), and for a \( q \) (possibly depending on \( f \)), sufficiently large.

In order to prove that such an extension is unique (and well-defined in the \( q \to \infty \) limit), let us inject the matrix \( w(q) \in GL(q, \mathbb{R}) \) into \( GL(q+k, \mathbb{R}) \) for \( k > 0 \). Denoting by \( i_{q,q+k}: GL(q, \mathbb{R}) \to GL(q+k, \mathbb{R}) \) the injection mapping, and noticing that

\[
i_{q+k} \circ i_{q,q+k} = i_q \tag{127}
\]

we can write

\[
f[i_q(w(q))] = f[i_{q+k}(w(q+k))] \tag{128}
\]

where we have set \( w(q+k) \equiv i_{q,q+k}(w(q)) \).

If we notice that the factorization \( \bar{w}(q) = \bar{s}_q D = (\prod_{ja}^q B_{ja})D \) yields

\[
\bar{w}(q+k) = \bar{s}_{q+k} D = (\prod_{ja}^{q+k} B_{ja})D \tag{129}
\]

with \( x^j_{(a)} = 0 \) for \( j > q \), and \( a > q \), then we can write

\[
\int_{SL(q, \mathbb{R})} f[i_q(\bar{w}(q))] q^{q^2/2} \Delta^{-q^2} \exp \left[ -\pi q \Delta^{-2} ||x(a)||^2 \right] \prod_{j,a}^q d\bar{x}^k_{(a)} = \int_{SL(q+k, \mathbb{R})} f[i_{q+k}(\bar{w}(q+k))](q+k)^{(q+k)^2/2} \Delta^{-(q+k)^2} \exp \left[ -\pi (q+k) \Delta^{-2} ||x(a)||^2 \right] \prod_{j,a}^{q+k} d\bar{x}^k_{(a)}
\]

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where we have exploited the fact that the extra variables $x^j_{(a)}$ for $j$, $a$ ranging over $q + 1, \ldots , q + k$, not being explicitly present in the function $\int [i_{q+k}(\bar{w}_{(q+k)} = \bar{s}_{q+k}D)]$, integrate out to 1 owing to the Gaussian nature of the integration. According to standard results these remarks imply that $P(d\bar{w})$ is indeed the unique extension of $[123]$ to $GL(Mat_p(R))$.

According to the measure $P(d\bar{w})$, for any $q$, the $q^2$ variables $x^k_{(a)}$ have the multinormal distribution with average 0 and variance matrix $t(q)\delta_{ef}$, ($e$ and $f$ ranging over 1, $\ldots , q^2$). In particular, as $q \to \infty$, the variance goes to 0, and $P(d\bar{w})$ degenerates to the Dirac measure concentrated on the zero sequence in $R^\infty$, the infinite product of real lines. This fact depends on our particular choice, $[123]$, of the variance $t(q)$. Recall that this choice of the variance is motivated by the fact that as $q$ increases, we need to emphasize the role of the matrices which eventually reduces to the identity matrix, so as to generate, in measure, the general linear group.

As a consequence of such remarks, it follows that even if we are considering the $q^2$ variables $x^k_{(a)}$, we should be actually interested in the $q$ random variables

$$y_a = \sum_{k=1}^{q} x^k_{(a)}$$  \hspace{1cm} (130)

with $a = 1, \ldots , q$.

The variables $y_a$ are multinormally distributed with a zero mean, but now with a variance matrix given by

$$qt(q)\delta_{ij} = (2\pi)^{-1}\Delta^2$$  \hspace{1cm} (131)

(with $i,j = 1, \ldots , q$). We shall denote by $P_y(d\bar{w})$ the distribution of the variables $y_a$ with respect to $P(d\bar{w})$.

As follows from standard manipulations of gaussian integrals, this distribution follows, for $q$ generic, by replacing the original set $x^k_{(a)}$ with the equivalent set of variables $(y_a, \hat{y}_i = x^i_{(a)})$ where $i \neq a$, and integrating out to 1 the $q^2 - q$ extra variables $\hat{y}_i$.

Notice that, owing to the presence of the integration over $W(Mat_p(R))$, the measures $P(\bar{w})$ as well as the associated distribution $P_y(\bar{w})$ are not normalized to one.

**Second part:**

### 5.2 The partition function in the thermodynamical limit

Now we establish a connection between the partition function $Z(m, J, \pi_1)$, and the joint distribution $P_y(d\bar{w})$ of the variables $y_a$.

For each given $L(m)$-geodesic balls two-skeleton $\Gamma_2$, the orthogonal representation $\theta$ of the fundamental group $\pi_1(\Gamma_2)$ extend to a unique ring homomorphism
$\theta: \mathbb{Z}_1(\Gamma_2) \rightarrow Mat_p(\mathbb{R})$, which induces a group homomorphism between $GL(\mathbb{Z}_1(\Gamma_2))$ and $GL(Mat_p(\mathbb{R}))$ and between the corresponding Whitehead groups, $\theta_*: W(\pi_1(\Gamma_2)) \rightarrow W(Mat_p(\mathbb{R}))$, given by

$$\theta_*[\tau(w_{ik})] = \tau\theta_*[w_{ik}] = |\det(\theta(w_{ik}))|$$

(132)

Thus, since there are only a finite number of inequivalent simple homotopy types realized in $\mathcal{R}(r, D, V)$, there is only a finite number of inequivalent Reidemeister-Franz representation torsion $\Delta_{(i)}^\theta = |\det(\theta(w_{jk}))|$ realized by manifolds in $\mathcal{R}(r, D, V)$, $(i = 1, \ldots, a < \infty)$.

For a given value of the parameter $m$, let us denote by $\mathcal{I}_m$ the set of matrices which are the image, under the homomorphism $\theta_*$, of the set of incidence matrices realized in $\mathcal{R}(r, D, V)$. In particular, for any given value of $q$, let $\mathcal{E}_{(m, q)}^{(i)}$ be the finite set of matrices $\tilde{w}^{(\alpha)}$ in $\mathcal{I}_m \cap GL(q, \mathbb{R})$ whose Reidemeister torsion is $\Delta_{(i)}^{\theta}$.

Let $U \subset GL(q, \mathbb{R})$ denote a neighborhood of $\mathcal{E}_{(m, q)}^{(i)}$, and let $f \in C^\infty(GL(Mat_p(\mathbb{R})))$ be a function such that the support of $f \circ i_q \downarrow \delta_\varepsilon$, (the Dirac measure supported on $\mathcal{E}_{(m, q)}^{(i)}$), and integrating over $GL(Mat_p(\mathbb{R}))$ with respect to $P_y(d\tilde{w})$ we get

$$\int_{GL(Mat_p(\mathbb{R}))} f(\tilde{w})\phi(\tilde{w})dP_y(\tilde{w}) \rightarrow \sum_{\tilde{w}^{(\alpha)} \in \mathcal{E}_{(m, q)}^{(i)}} \exp[-\pi \Delta_{(i)}^{-2} \sum_{a=1}^{q} ||y_a||^2(\tilde{w}^{(\alpha)})] \Delta_{(i)}^{1-q} \phi(\tilde{w}^{(\alpha)})$$

for any function $\phi \in C^\infty(GL(Mat_p(\mathbb{R})))$ compactly supported in $U$.

More in general, let $\mathcal{E}_{(m, q)}^{(i)} \equiv \bigcup_q \mathcal{E}_{(m, q)}^{(i)}$, as $q$ varies in the finite set of the possible values, over $\mathcal{R}(r, D, V)$, of $p\mathcal{N}_{(m)}^{(3)}$. Such $\mathcal{E}_{(m, q)}^{(i)}$ is the finite set of matrices $\tilde{w}^{(\alpha)}$ in $\mathcal{I}_m$ whose Reidemeister torsion takes on the given value $\Delta_{(i)}$. The foregoing analysis implies that the measure of $\mathcal{E}_{(m, q)}^{(i)}$ with respect to $P_y(d\tilde{w})$ is given by

$$\mu_{(m)}^{(i)}(\mathcal{E}_{(m, q)}^{(i)}) \equiv \int_{GL(Mat_p(\mathbb{R}))} (\tilde{w})dP_y(\delta_\varepsilon(\tilde{w})) =$$

$$\sum_{\tilde{w}^{(\alpha)} \in \mathcal{E}_{(m, q)}^{(i)}} \exp[-\pi \Delta_{(i)}^{-2} \sum_{a=1}^{p\mathcal{N}_{(m)}^{(3)}(\tilde{w}^{(\alpha)})} ||y_a||^2(\tilde{w}^{(\alpha)})] \Delta_{(i)}^{1-p\mathcal{N}_{(m)}^{(3)}(\tilde{w}^{(\alpha)})}$$

(133)

where $dP_y(\delta_\varepsilon(\tilde{w}))$ is the distribution of the counting measure $\delta_\varepsilon(\tilde{w})$ of the set $\mathcal{E}(\tilde{w})$ with respect to $P_y$.

This result for the $P_y(d\tilde{w})$-measure of the set $\mathcal{E}_{(m, q)}^{(i)}$ is rather similar to the expression of the partition function $Z(m, J, \pi_1)$, which according to proposition [1] can be written as

$$\sum_{\Delta_\varepsilon \in W(Mat_p)} \sum_{\tilde{w}^{(\alpha)} \in \mathcal{E}_{(m, q)}^{(i)}} \exp[i_c\mathcal{N}_{(m)}^{(1)}(\Gamma_2)] \left(\frac{J}{2\pi}\right)^{-(p/2)\mathcal{N}_{(m)}^{(3)}(\tilde{w}^{(\alpha)})} \Delta^\theta(\tilde{w}^{(\alpha)})$$

(134)
In order to compare this expression with (133), let us remark that since the random variables \( y_a \) are multinormally distributed with variance \( \Delta^2 / 2\pi \), then according to the strong law of large numbers, we have

\[
\lim_{q \to \infty} \frac{1}{q} \sum_{a=1}^{q} (|y_a|^2) = \frac{\Delta^2}{2\pi}
\]

(135) for \( P_y \)-almost all vectors \( y_a \). It follows that, for \( q \), (and thus \( m \)), sufficiently large (notice that \( q = pN_{(m)}^{(3)} \)), we can write (133), with a slight abuse of notation, as

\[
\mu_{(m)}^{(i)} = \sum_{\tilde{w}^{(a)} \in E_{(m)}^{(i)}} \exp\left[-\frac{p}{2}N_{(m)}^{(3)}(\tilde{w}^{(a)})\right] \Delta_{(i)}^{1-pN_{(m)}^{(3)}(\tilde{w}^{(a)})}
\]

(136) which, upon setting \( N_{(m)}^{(3)} \simeq bN_{(m)}^{(1)} \), bears a strong resemblance with (134). Explicitly, by comparing the expression of the measure \( \mu_{(m)}^{(i)}(E_{(m)}^{(i)}) \), (see (136)), and the right side of (134), (by restricting the sum over the torsions to a given value, say \( \Delta_{(i)} \)), it immediately follows that if the parameter \( J \) takes the value

\[
\frac{J_{\text{crit}}}{2\pi} = \left(\Delta_{(i)}^{\theta}\right)^2
\]

(137) and if

\[
i_{c} = i_{c(m)} \equiv -\frac{p}{2}N_{(m)}^{(3)} / N_{(m)}^{(1)}
\]

(138) then, for \( m \) sufficiently large, and for any given value of the ratio \( N_{(m)}^{(3)}/N_{(m)}^{(1)} \), we can write

\[
Z(m, J, \pi_1)|_{J=J_{\text{crit}}} = \mu_{(m)}^{(i)}(E_{(m)}^{(i)})
\]

(139) where \( Z(m, J, \pi_1)|_{J=J_{\text{crit}}} \) denotes the partition function \( Z(m, J, \pi_1) \) evaluated by restricting the summation over the torsions \( \Delta_{(k)} \) to a given value \( \Delta_{(i)} \), and by setting \( J \) equal to the corresponding value of \( J_{\text{crit}} \).

Notice that, as \( m \to \infty \), we get

\[
i_{c} = -\frac{p}{2}N_{(m)}^{(3)} / N_{(m)}^{(1)} \rightarrow_{m \to \infty} \frac{1}{2} \frac{p}{b}
\]

(140) (where \( b \equiv \limsup_{m \to \infty} (N_{(m)}^{(1)}/N_{(m)}^{(3)}) \)).

Let us denote by \( P_y^{(i)}(d\tilde{w}) \) the measure obtained by restricting the measure \( P_y \) to the subset of matrices in \( GL(Mat_p(R)) \) with torsion \( \Delta_{(i)}^{\theta} \), namely

\[
P_y^{(i)}(d\tilde{w}) \equiv \delta(\Delta_{(i)}) \otimes P_y(d\tilde{w})
\]

(141)
where $\delta(\Delta(i))$ is the Dirac delta, (with respect to the invariant measure $d(\mu(\Delta))$ on $W(\text{Mat}_p(\mathbb{R}))$. Notice that this measure is normalized to $\Delta(i)$ rather than to 1. (The factor $\Delta(i)$ is originated by that $\delta(\Delta(i))$ is a Dirac distribution with respect to the multiplicative measure on $\mathbb{R}^+$.)

According to such remarks it follows that, corresponding to the critical values $J = J_{\text{crit}}$ and $i_c = -p/2b$, we have

$$\lim_{m \to \infty} Z(m, J, i_c, \pi_1)|_{J = J_{\text{crit}}} = \lim_{m \to \infty} \int_{\text{GL}(\text{Mat}_p(\mathbb{R}))} P_{y}^{(i)}(\delta_{\mathcal{E}}(\tilde{w}))$$

(142)

We can explicitly evaluate the above limit and show that $\lim_{m \to \infty} Z(m, J, i_c, \pi_1)|_{J = J_{\text{crit}}} = \Delta(i)$ by noticing that, as $m \to \infty$, the mapping from the set of incidence matrices $\mathcal{E}(m)$ and the subset of matrices in $\text{GL}(\text{Mat}_p(\mathbb{R}))$ with torsion $\Delta^\theta(i)$ is, up to formal deformations, surjective.

In order to prove this latter statement let $\tilde{w}$ be a matrix in $\text{GL}(\text{Mat}_p(\mathbb{R}))$ which is not in $\mathcal{E}(m)$, for some given value of the parameter $m$. This means that for an integer $q$, (depending on $m$), the matrix $\tilde{w}^{(q)} \in \text{GL}(q, \mathbb{R})$, with $i_q(\tilde{w}^{(q)}) = \tilde{w}$, is not in $\mathcal{E}(m)$. However, through formal deformations we can always map such $\tilde{w}^{(q)}$ into a new $\tilde{w}^{(p)}$, in general with $p \neq q$, which is realized as the incidence matrix, (in the given orthogonal representation), of some geodesic ball covering of manifolds in $\mathcal{R}(r, D, V)$. Thus, as $m \to \infty$, and up to formal deformations, we can always assume that a given incidence matrix $\tilde{w}$ is actually realized by some geodesic ball covering for some value, say $m$ of the cut-off parameter. We stress that this is no longer true for a fixed $m$, (since, as recalled above, a formal deformation may require to go from a matrix of order $q$, corresponding to the given $m$, to a matrix of order $p$, with $p \neq q$).

It is natural to define $\mathcal{E}$ as the direct limit of the $\mathcal{E}(m)$ or more explicitly, $\tilde{w} \in \mathcal{E}$ if there exists a value of the cut-off parameter $m$, a corresponding integer $q = q(m)$, and a matrix $\tilde{w}^{(q)} \in \mathcal{E}(m)$ such that $i_q(\tilde{w}^{(q)}) = \tilde{w}$.

According to the above remarks, $\mathcal{E}$ can be identified with the subset of matrices in $\text{GL}(\text{Mat}_p(\mathbb{R}))$ with torsion $\Delta^\theta(i)$, set which supports the measure $P_{y}^{(i)}(d\tilde{w})$. Thus we get

$$\lim_{m \to \infty} Z(m, J, i_c, \pi_1)|_{J = J_{\text{crit}}} = \lim_{m \to \infty} \int_{\text{GL}(\text{Mat}_p(\mathbb{R}))} P_{y}^{(i)}(\delta_{\mathcal{E}_m}(\tilde{w})) =$$

$$= \int_{\text{GL}(\text{Mat}_p(\mathbb{R}))} P_{y}^{(i)}(d\tilde{w}) = \Delta(i)$$

(143)

as stated.

This last results shows, as claimed, that as $m \to \infty$, the statistical system described by $Z(m, J, \pi_1)$ admits a well-defined thermodynamical limit for $i_c = -p/2b$ and $J = J_{\text{crit}}$, (where $J_{\text{crit}}$ is one of the values $([137])$, say $2\pi(\Delta^\theta(i))^2$).
Owing to the combinatorial invariance of the representation torsion, such limit depend only on the particular representation of the fundamental group, and on the simple homotopy type that we are considering. As we shall see in a moment, such limit describes (homology) manifolds $M$ in $\mathcal{R}(r,D,V)$ whose Reidemeister-Franz torsion takes on the given value $\Delta^\theta_{(i)}$, and which are generated, as $m \to \infty$, by rather regular arrays of $L(m)$-geodesic balls.

Moreover, since, if $i \neq k$, we have trivially
\[
\lim_{m \to \infty} \mu_{(m)}^{(i)} \left[ \det \tilde{w} = \Delta_{(k)} \right] = \lim_{m \to \infty} P_y^{(i)}(\mathcal{E}_{(m)}^{(k)}) = 0
\]
this authorizes the conclusion that as $J$ varies from one critical value to another, the system exhibits a phase transition associated with the changes of the corresponding simple-homotopy types, and, through the orthogonal representation $\Theta$, the set of manifolds in $\mathcal{R}(r,D,V)$ with isomorphic fundamental group appears as a mixture of pure phases described by the probability measure on $GL(Mat_p(R))$

\[
P(d\tilde{w}) \equiv \frac{1}{\alpha} \sum_{i=1}^{\alpha} P_y^{(i)}(d\tilde{w})
\]
(with $\alpha = ||\Delta^\theta||(\pi_1)$).

A somewhat surprising feature of the above results is the way in which the gaussian distribution $P_y^{(i)}$ over the general linear group yields for the exact evaluation of the thermodynamical limit of the partition function. Not quite so surprising is the fact that the partition function is provided by the corresponding value of the representation torsion. This latter result, in particular, points toward a direct connection with the Chern-Simons approach to quantize three-gravity, where the corresponding partition function is provided by the Ray-Singer torsion, the analytic counterpart of $\Delta_{(i)}$. There is equality between the two as long as the Reidemeister-Franz torsion is, for a given representation of the fundamental group, known to come from a smooth triangulation.

Either the nature of $P_y^{(i)}$ or the connection with Chern-Simons approach follows from the role that the orthogonal representation of the fundamental group has in our results.

The use of a representation of the fundamental group is forced upon us by the structure of the space of bounded geometries, (recall Fukaya’s theorem on the structure of collapsed manifolds providing the $dG$-boundary points of $\mathcal{R}(n,r,D,V = 0)$). Actually, it is the use of a representation that allows us to label the cells, (to colour them), and thus to implement the rather standard statistical mechanical approach yielding to the results connecting $Z(m,J,\pi_1)$ to simplicial three-gravity. It is clear that different representations, (say on more elementary groups than the orthogonal group), may yield for different
statistical mechanical models, (e.g., Ising-like models; it must be noted that the 3D-Ising model has been recently related to 2 + 1-dimensional quantum gravity), but, as recalled, the orthogonal group suggests itself also because it captures, in an essential way, the non-abelian character of the fundamental group. This is an important point if one does not want to lose geometrical information during the process of (homotopical) reconstruction of the manifold, (through formal deformations), which underlines the definition of $Z(m, J, \pi_1)$.

For a given manifold $M \in \mathcal{R}(r, D, V)$, let $\text{Hom}(\pi_1(M), O(p))$ denote the set of all homomorphisms $\pi_1(M) \rightarrow O(p)$, i.e., the space of all representations of $\pi_1(M)$ into the orthogonal group $O(p)$. For a given $m$, (sufficiently large), the fundamental group $\pi_1(M)$ is actually given, as a finitely generated group, by the presentation associated with the $L(m)$-geodesic ball two skeleton $\Gamma_{(m)}^{(2)}$, as $\pi_1(M) \simeq \pi_1(\Gamma_{(m)}^{(2)})$. More explicitly, as we have seen in a previous paragraph, we can assume that $\pi_1(\Gamma_{(m)}^{(2)})$ has a presentation

$$< \gamma_1, \ldots, \gamma_{l(m)} | R_1(\gamma_1, \ldots, \gamma_i) = \ldots = I > \quad (146)$$

where $\gamma_1, \ldots, \gamma_{l(m)}$ denote the $m$-dependent generators, (with $l(m) = 1 - N_{(m)}^{(0)} + N_{(m)}^{(1)}$) associated with the one-skeleton graph $\Gamma_{(m)}^{(1)}$, and where $R_i = \ldots = I$ denote the corresponding relations (associated with the pasting of the faces $p_{ijk}^{(2)}$ into $\Gamma_{(m)}^{(1)}$). The map $\gamma_i \rightarrow \phi(\gamma_i)$, with $\phi(\gamma_i) \in O(p)$, and $i = 1, \ldots, l(m)$, defines an element of $\text{Hom}(\pi_1(M), O(p))$, and as $m$ varies (and eventually $m \rightarrow \infty$), the $m$-depending presentations $\pi_1(\Gamma_{(m)}^{(2)})$ are associated with a corresponding set of representations in $\text{Hom}(\pi_1(M), O(p))$.

In our approach, leading to simplicial three-gravity, we are actually dealing with the group ring $\mathbb{Z}\pi_1(M)$, and since every linear representation of $\pi_1(M)$ can be extended to a representation of its group ring, it is not too difficult to realize that the measure $P_y^{(i)}(d\tilde{w})$ on $GL(\text{Mat}_y(\mathbb{R}))$ can be interpreted as a measure on the representation space associated with the group ring $\mathbb{Z}\pi_1(M)$.

The space $\text{Hom}(\pi_1(M), O(p))$, or rather, the associated orbit space (under conjugation), $\text{Hom}(\pi_1(M), O(p))/O(p)$, is the deformation space of flat principal $O(p)$-bundles over $M$. In such a sense, $P_y^{(i)}(d\tilde{w})$ induces a measure over such orbit space. Now, the orbit space of flat bundles associated with the space of flat connections, $(SO(2,1)$-connections for 2 + 1-gravity, $SO(3)$-connections for euclidean 3-gravity), is exactly the configuration space over which one functionally integrates in defining Chern-Simons versions of three-gravity. These remarks, which can be made more precise, make less mysterious either the role of $P_y^{(i)}(d\tilde{w})$ or the connection with the standard field-theoretic approach to three-dimensional quantum gravity.

Another non-trivial aspect of the limit $\lim_{m \rightarrow \infty} Z(m, J, \pi_1)$ lies in noticing that such limit is actually well defined in a convex region in the plane of all possible couplings $(\sigma, i_c)$,
and that a first order phase transition occurs precisely when such couplings approach the critical values provided by proposition \[12\]. Such phase transition does not describe the change from a simple homotopy type into another, (since it occurs within a given simple homotopy type), it rather signals the passage from an irregular configuration, (a three-dimensional homology manifold generated by sequences of \(L(m)\)-minimal geodesic ball coverings with a low filling), to a more regular one describing a (homology) manifold generated by more structured packings of \(L(m)\)-geodesic balls, (such nicer packings, are characterized by the statistical domination of configurations with a large filling function \(N^{(0)}_{(m)}\).

In order to prove such statements we proceed as follows.

Let us consider a given value, \(\Delta\), of the Reidemeister-Franz torsion, and rewrite \(Z(m,J,\pi_1)|_\Delta\) as

\[
Z(m,J,\pi_1)|_\Delta = \sum_{(N,\Gamma_2)} \Delta^\theta(N,\Gamma_2) \exp\left[-N^{(3)}_{(m)}(N)\left(\sigma - i_c \frac{N^{(1)}_{(m)}(\Gamma_2)}{N^{(3)}_{(m)}(N)}\right)\right] \tag{147}
\]

For a given \(m\), sufficiently large, and a running integer \(t\), let us denote by \(Q_{(m)}(\Delta, b, t)\) the number of combinatorially inequivalent \(L(m)\)-geodesic balls nerves \(N_{(m)}^{(3)}(N)\) of given torsion \(\Delta\), with \(N_{(m)}^{(3)}(N) = t\) and with a given value of the ratio \(N_{(m)}^{(1)}(\Gamma_2)/N_{(m)}^{(3)}(N) = b\); \((Q_{(m)}(\Delta, b, t)\) is basically the micro-canonical partition function). Thus, for a given value of \(b\), we can write

\[
Z(m,J,\pi_1)|_\Delta = \sum_t \Delta^\theta \left[Q_{(m)}(\Delta, b, t)\right] \exp[-t(\sigma - i_c b)] \tag{148}
\]

With these preliminary remarks, the following result obtains

**Proposition 13** For any given value of \(b\) and for a given value \(\Delta_{(i)}^\theta\) of the torsion, the partition function \(Z(m,J,\pi_1)|_\Delta\) is well defined, as \(m \to \infty\), for all \(i_c\), and \(\sigma\) in the convex region \(\mathcal{D}(\sigma, i_c)\) defined, in the plane of possible couplings \((\sigma, i_c)\), by the condition

\[
\sigma - bi_c \geq \sigma_{crit} + \frac{1}{2} \rho > 0 \tag{149}
\]

where

\[
\sigma_{crit} = \frac{p}{2} \ln\left(\frac{J_{crit}}{2\pi}\right) = p \ln\left(\Delta_{(i)}^\theta\right) \tag{150}
\]

is the critical value of the bare cosmological constant \(\sigma\) corresponding to the given \(\Delta_{(i)}^\theta\). If

\[
g(b) = \lim sup_{t \to \infty} \frac{1}{t} \ln \left[Q_{(m)}(\Delta, b, t)\right] \tag{151}
\]

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denotes the rate of exponential growth in the number of geodesic ball nerves \( N_m \) as a function of \( b \), then \( g(b) \) is a positive concave function of \( b \). Moreover, as \( i_c \to -\frac{1}{2}p/b \), and \( \sigma \to \sigma_{crit} \), the function \( g(b) \) reduces to a constant. Thus, the system described by the partition function \( Z(m, J, \pi_1)|_\Delta \) exhibits, as \( m \to \infty \), a first-order phase transition at \( \sigma = \sigma_{crit} \) and \( i_c(b) = -\frac{1}{2}p/b \).

Proof. For a given value of the parameter \( b \), we can consider (148) as a Dirichlet series of type \( t \), (or, equivalently, as a power series in the variable \( \exp[-\sigma + i_c b] \); there are a number of advantages in considering (148) as a Dirichlet series, advantages which are connected to the use of the variables \( \sigma \) and \( i_c \)). According to proposition 12, such series converges if \( \sigma - i_c b = \sigma_{crit} + \frac{1}{2}p \).

If \( \sigma_{crit} + \frac{1}{2}p > 0 \), or more explicitly, if

\[
\ln(\Delta^\theta(i))^2 + 1 > 0
\]

then this convergence implies that

\[
\sum_t^k Q_{(m)}(\Delta, b, t) = \mathcal{O}\left(\exp[\sigma_{crit} + \frac{1}{2}p]k\right)
\]

as \( t \to \infty \), and conversely it also implies that \( \lim_{m \to \infty} Z(m, J, \pi_1)|_{\Delta,b} \) converges in the half-plane \( \sigma - i_c b > \sigma_{crit} + \frac{1}{2}p \).

Moreover, from (153) it follows that

\[
\sum_t^k Q_{(m)}(\Delta, b, t) < M \exp[\sigma_{crit} + \frac{1}{2}p]k
\]

for some constant \( M \), and \( k = 1, 2, 3, \ldots \). Thus

\[
\frac{\ln \sum_t^k Q_{(m)}(\Delta, b, t)}{k} < \frac{\ln M}{k} + \sigma_{crit} + \frac{1}{2}p
\]

By taking the limit superior of the left-hand side we get the abscissa of convergence of (148), which is thus bounded above by \( \sigma_{crit} + \frac{1}{2}p \). Thus it follows from the foregoing arguments that the function \( g(b) \) defined by (151), exists and it is finite. Its concavity trivially follows from Jensen inequality as applied to the logarithmic function.

We can now show that, when \( i_c = -\frac{1}{2}p/b \) and \( \sigma \to \sigma_{crit} \), an horizontal segment is present in the graph of \( g(b) \).

Again according to proposition 12 as \( m \to \infty \), the partition function \( Z(m, J, \pi_1)|_\Delta \) is well defined for \( \sigma = \sigma_{crit} \) and all \( i_c \) varying on the curve, (in plane \((b,i_c)\)), defined by
\[ i_c(b) = -\frac{1}{2}p/b. \] Corresponding to such values of the couplings we get
\[
\sum_t \left[ Q_m(\Delta, b_1, t) \right] \exp[-t(\sigma_{\text{crit}} - i_c(b_1)b_1)] = \\
\sum_t \left[ Q_m(\Delta, b_2, t) \right] \exp[-t(\sigma_{\text{crit}} - i_c(b_2)b_2)] = \\
\sum_t \left[ Q_m(\Delta, b_1, t) \right] \exp[-t(\sigma_{\text{crit}} - i_c(b_2)b_2)]
\]
(156)
for all pairs \((b_1, i_c(b_1)))\) lying on the curve \(i_c(b) = -\frac{1}{2}p/b\). From (156), it follows that \(Q_m(\Delta, b_1, t) = Q_m(\Delta, b_2, t) = \ldots = Q_m(\Delta, b_{(i)}, t),\) which, by reference to the definition, implies the presence of a horizontal segment in the graph of \(g(b)\). By standard arguments, (see e.g., the book by Ruelle\(\underline{3}\)), the fact that \(g(b)\) reduces to a constant on some interval contained in \([0, b_{\text{max}}]\), implies the existence of a first-order phase transition at \(i_c = -\frac{1}{2}p/b,\) and \(\sigma = \sigma_{\text{crit}}\).

From a geometrical point of view, we recall that \(b\) provides the (inverse) density of tetrahedra for link in the polytope associated with a minimal geodesic ball covering. For each given value of such density, the function \(g(b)\) provides, in the large \(m\) limit, the rate of (exponential) growth in the number of geodesic ball nerves \(N(\chi, \Delta, b, t)\) for all pairs \((\chi, \Delta, b, t)\) associated with the minimal geodesic ball covering. For \(a\) first-order phase transition at \(i_c(b) = -\frac{1}{2}p/b,\) while the density of tetrahedra for link undergoes a finite change.

In order to discuss the geometrical meaning of this phase transition more in details, let us remark that along the same lines of the entropy estimates established in paragraph (3.5), the microcanonical partition function \(Q_m(\chi, \Delta, b, t)\), (for the value \(b = 3\) of the parameter \(b\) which is actually realized, for a given volume \(V\), and for a given value of the Euler characteristic), can be asymptotically estimated as \(t \rightarrow \infty\) according to, (see ??),
\[
Q_m(\chi, \Delta, b, t) \rightarrow \infty \sim \eta(t) n^m \eta_h(1 + O(1/t))
\]
where \(n = 3,\) and \(A_c, \zeta_0,\) and \(\eta_h\) are suitable constants possibly depending on the torsion \(\Delta_{(i)}\).

Upon introducing this expression in \(Z(m, J, \pi_1)\), it follows that as \(m \rightarrow \infty\), the behavior of the partition function is dominated by the subleading power in (157). This subleading power yields to a converging \(Z(m, J, \pi_1)\) as \(m \rightarrow \infty\) only if \((\chi(M) - \zeta_0 - 2) < 0\) which implies that \(\chi(M) < 0\) and \(\zeta_0 - 2 > 0\). While, if \(\chi(M) = 0\) then \(Z(m, J, \pi_1)\) diverges.

Thus pseudo-manifolds dominate the thermodynamical limit of \(Z(m, J, \pi_1)\).  

\[ \clubsuit \]
According to proposition 13 the transition which isolates the pure phase \( P(i) \) among the mixture which is associated with \( \lim_{m \to \infty} Z(m, J, \pi_1) \) geometrically corresponds to a pseudo-manifold.

One is tempted to see in such behavior of the partition function the analytical counterpart of the vacuum, (or rather of the vacua, each vacuum being associated with one of the simple homotopy types realized), whose existence is numerically suggested by more direct computer-assisted approaches \(^{25,26}\).

6 Concluding remarks

The foregoing results show that even if we can define coherently the thermodynamical limit of simplicial three-gravity as described by \( Z(m, J, \pi_1) \), we end up with configurations in which pseudo-manifolds dominate. Obviously, we would be more interested in discussing the possibility for the existence of a continuum limit of the theory, where the cell size, in properly defined physical units, tends to zero, and where manifolds have a non-vanishing probability of occurrence.

It has been argued \(^{25,26}\) that this latter limit presumably does not exists. The first order nature of the transition from the hot phase to the cold phase supports this point of view. In this sense, further analytical support comes also from our results, again owing to the first order nature of the transition accompanying the realization of the regular covering phase out of the phase describing homology manifolds associated with finer and finer geodesic balls coverings.

As we have shown here, euclidean simplicial three-gravity basically appears as a theory which statistically reconstructs an extended three-dimensional object out of the presentation of the fundamental group \( \pi_1(\Gamma_2, e_0) \) associated with a dynamical triangulation. The basic role in the theory is thus played by the fundamental group, and by its orthogonal representations.

We exploited both such roles by resolving the thermodynamic limit of the theory into extended object (homology manifolds) which corresponds to the distinct ways, (up to the action of the fundamental group), one may get homology manifolds through collapses or expansions of cells. The existence of a critical value for \( \sigma \) is associated with the partition of the configuration space of the theory into simple homotopy equivalence classes labelled by the representation torsions \( \Delta^\theta_{(i)} \). While the existence of a critical value for the chemical potential \( i_c \), given a value of the parameter \( b \), is related in a quite non-trivial way to the fact that corresponding to a generic \( L(m) \)-geodesic ball coverings there is a local optimal covering, viz., the minimal \( L(m) \)-geodesic ball covering maximizing the associated filling function \( N^{(0)}_m(M) \). In particular, since homogeneous and isotropic three-manifolds have a strong chemical affinity for a geodesic balls gas, (recall that the best filling is realized on such manifolds), one may expect that (portions of) space forms should be the natural outcome of the phase transition mechanism discussed above. Sometimes this may be
the case, but in general this mechanism is unrealistic. In order to explain this remark, let us notice that the geodesic ball filling mechanism of riemannian manifolds bears a strong analogy with the mechanism governing space-filling procedures through random close packing of small spheres. As is known, random close packing, while not as efficient as crystalline close packing in filling a portion of three-dimensional space, provides a good compromise. It corresponds to a metastable arrangement, (stable against small displacements), associated with a local energy minimum in the configuration space of the system. This stability is so strong that there is no way to deform continuously, through an increase of density, a random close packing into a crystalline close packing.

These remarks suggest that minimal geodesic ball coverings of riemannian manifolds in $\mathcal{R}(r, D, V)$ are the natural counterpart of random close packing of ordinary space, and that (local) crystalline close packing corresponds to the optimal geodesic ball fillings of constant curvature space. In this sense, the statistical behavior of geodesic ball coverings as described by $Z(m, J, \pi_1)$ can be heuristically modelled after that of random close packing. In particular, the difficulties in having a continuum limit of the theory can be related to the known lack of correlation between short-range dense packing and long-range crystalline order which is typical of the dimension three.

The first order phase transition appearing in the thermodynamic limit of the geodesic ball packing may then have its natural counterpart in the sudden deformation of random close packed spheres, (deformation into Wigner-Seitz cells; by definition, a Wigner-Seitz cell surrounding a given site, is the cell containing all points closer to that site than to any other), as they extend to fill a space of fixed volume. This transition is accompanied by a jump of the coordination number between the cells, (number which, roughly speaking, corresponds to our parameter $b$), but it is not associated with a deformation of the random close packing filling into a crystalline close packing, i.e., long range order is not activated.

The above remarks should be compared with the situation in dimension four. Leaving aside the heuristics of random close packings, the point is again in the interplay between homotopy theory and the differentiable structures a manifold can carry.

For spaces of bounded geometries such as $\mathcal{R}(r, D, V)$, there is, as often quoted, a finiteness theorem for the distinct homotopy types that can be realized. In dimension three and four, this is all the topological information we can exploit while keeping ourselves in the Gromov-Hausdorff compactification of the set of riemannian structures considered. This is the price we have to pay so as to be able to exploit the compactness of $\mathcal{R}(r, D, V)$ for measure-theoretical purposes (i.e., for having a thermodynamical limit). As remarked above, the effect of this state of affairs is that in dimension three, the control of $Z(m, J, \pi_1)$ is determined by the fundamental group. However, in dimension four there is a subtle interplay between homotopy theory and differentiable structures. To make a significant example, one knows, by a theorem of Whitehead, that the homotopy type of a compact simply-connected four-manifold is determined by the isomorphism class of the intersection form of the manifold. Moreover, through the results of Freedman,
and Donaldson\textsuperscript{55}, one can establish a direct connection between the algebraic properties of the intersection forms and the topological and differentiable structures that can live on compact (simply-connected) four-manifolds. Thus, one may reasonably try to carry over the previous analysis to the four-dimensional case, hoping to reach for a sensible continuum limit. In the four-dimensional case, the labelling of simple homotopy types through the Reidemeister-Franz representation torsion, does not provide much information. Such torsions are trivial, (\textit{i.e.}, $\Delta^\theta = 1$), for any smooth four dimensional manifolds. However, the homotopical approach, which drives our formalism, can be put to work quite effectively also in dimension four. As recalled, homotopical information is now provided, (say in the simply connected case, for simplicity), by the intersection form, and it is possible to adopt the strategy described here by exploiting the reconstruction of (simply connected) four manifolds out of a wedge of two-spheres on which a four cell is pasted through a suitable attaching map.
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FIGURE CAPTIONS

Fig. 1. A small array of closely packed (geodesic) balls is here shown in three-dimensional Euclidean space. Even in such a case, such packings do not yield for a regular tessellation of the ambient space. In general, a $L(m)$-minimal geodesic ball covering has a rather irregular pattern, also because for manifolds in $\mathcal{R}(r, D, V)$ arbitrarily small metric balls need not be contractible. However, for a given (sufficiently) large value of $m$, either the number of geodesic balls or their intersection pattern, (obtained upon doubling the radius of the balls), embody non-trivial information on the geometry and topology of the underlying riemannian manifold.

Fig. 2. A portion of a 1-skeleton graph corresponding to a $L(m)$-geodesic ball covering. The length of each link is proportional to $1/m$.

Fig. 3. A pictorial representation of the compactness of Gromov’s space. Notice that while the number of equivalence classes $O^{(m)}_\Gamma$ varies with $m$, the number of distinct homotopy types (and of simple homotopy types) is finite and independent from $m$. Such number can be estimated in terms of the parameters $n, r, D, V$ characterizing the particular space of bounded geometries, $\mathcal{R}(r, D, V)$, considered.
Fig. 4. The Erdös-Kelly minimal graph generated by distinct geodesic ball 1-skeletons.

Fig. 5. The lattice gas associated with a geodesic ball covering.

Fig. 6. The exterior boundary condition assumed in carrying out the thermodynamical limit is to keep empty the $b_{(m)}$ added sites associated with the Erdös-Kelly minimal graph generated by the geodesic ball 1-skeletons.
Fig. 7. $K$ collapses to $J$, (across the tetrahedron $e^n$ onto its face $e^{n-1}$). More in general, $K$ collapses to $J$, and we write $K \downarrow J$, if there is a sequence of elementary collapses, as the one described above, from $K$ to $J$. An elementary expansion is the inverse operation, and a sequence of elementary collapses and expansions define a formal deformation from $K$ to $J$. The equivalence relation, in homotopy, associated with such notion of formal deformation yields for the simple homotopy type of a CW-complex.

Fig. 8. The CW-pair $(\mathcal{N}, \Gamma^{(2)})_{(m)}$, (here shown in dimension three), is equivalent, under formal deformation, to the two-skeleton $\Gamma^{(2)}_{(m)}$ at a vertex of which, $e^{(0)}$, one attaches a suitable number of two-cells $Q^2$ and three-cells $Q^3$. Some metrical information concerning the original pair $(\mathcal{N}, \Gamma^{(2)})_{(m)}$ is embodied in the number of three-cells attached.

Fig. 9. Trivially added cells, (here represented by the surfaces of the dotted tetrahedrons), associated with representative elements of $\pi_q(\Gamma_2 \cup \bigcup_{j=1}^{a} e^q_j, \Gamma_2)$, (with $q = 2$, $a = 3$). The fundamental group $\pi_1(\Gamma^{(2)}; e_0)$ acts on $\pi_q(\Gamma_2 \cup \bigcup_{j=1}^{a} e^q_j, \Gamma_2)$. If the loop $s$ represent $[\alpha] \in \pi_1(\Gamma^{(2)}; e_0)$, then $\beta \in \pi_q(\Gamma_2 \cup \bigcup_{j=1}^{a} e^q_j, \Gamma_2)$ is sent into $[\alpha] \beta \in \pi_q(\Gamma_2 \cup \bigcup_{j=1}^{a} e^q_j, \Gamma_2)$. The geometrical idea is to pull the image of $J^{q-1}$ along the path $s$ back to the point $e_0$ with the image of $Q^q$ being dragged in such a way that the image of $Q^{q-1}$ is always in $\Gamma^{(2)}$. In this way, $\pi_1(\Gamma^{(2)}; e_0)$ appears as a gauge group for the spaces $\pi_q(\Gamma_2 \cup \bigcup_{j=1}^{a} e^q_j, \Gamma_2)$ which describes the attachment of new cells to $\Gamma^{(2)}_{(m)}$.

Fig. 10. A representative elements of the group $\pi_{q+1}(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^{a} e^q_j)$, (here with $q = 2$ and $a = 3$), is constructed by filling the tetrahedra in such a way that the resulting three-cell has the surfaces of the tetrahedra as boundary.
Fig. 11. A vector of $\pi_2^\theta(\Gamma_2 \cup \bigcup_{j=1}^a e_j^2, \Gamma_2)$ can be thought of as a collection of euclidean $\mathbb{R}^p$-vectors $\xi^{(i)}$ providing a coloring (labelling) of the two cells representing the elements of $\pi_2(\Gamma_2 \cup \bigcup_{j=1}^a e_j^2, \Gamma_2)$, (cells that we keep on in describing as the boundaries of tetrahedra).

Fig. 12. The vector $[\theta_4(\partial)^{-1}\xi]_{(l)}$, associated with the three-cell $e_3^l$, is obtained from the vectors $\xi^{(k)}$ associated with the two-cells $e_2^k$ through the action of the set of $p \times p$ matrices $[\theta_4(w_{kl})]^{-1}$, ($l$ is fixed while $k$ ranges over the set of vectors $\xi^{(k)}$). Each matrix acts on the corresponding vector, the resulting transformed vectors add up to $[\theta_4(\partial)^{-1}\xi]_{(l)}$.

Fig. 13. In the reciprocal complex associated with $(\mathcal{M}, \Gamma^{(2)})$, the three-cells representing the classes of $\pi_3^\theta(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^a e_j^2)$ are described by the vertices of a graph $\Lambda$ the edges of which corresponds to the two cells of $\Gamma_2 \cup \bigcup_{j=1}^a e_j^2$. Thus, with the vector $\nu \in \pi_3^\theta(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^a e_j^2)$, describing a colouring of the three-cells of $\pi_3^\theta(\mathcal{M}, \Gamma_2 \cup \bigcup_{j=1}^a e_j^2)$, we can associate a magnetic vector model evolving on the graph $\Lambda$ and characterized by an energy $H(\xi)$ given by (88).

Fig. 14. The two-skeleton $(\Gamma^{(2)}_{(m)}, e_0)$ is homotopically equivalent to a bouquet of circles, (originating from $e_0$), and disks.
**Fig. 15.** The $CW$-pair $(N, \Gamma^{(2)})$ is equivalent, in the simple homotopy sense, to a bouquet of circles, disks, and spheres attached at $e_0$. The spheres bound, non-trivially, $N_{(m)}^{(3)}$ three-cells. This bouquet carries all the (simple) homotopical information of the original pair $(N, \Gamma^{(2)})$, while non-trivial metrical information is encoded into the numbers $N_{(m)}^{(3)}$ and $N_{(m)}^{(1)}$ which can be read off from the composition of the bouquet.

**Fig. 16.** According to the Gaussian distribution of the fields $\xi$, the correlations between the colourings of the three-cells of $(M, \Gamma^{(2)})$ decay exponentially.

**Fig. 17.** In the grand-canonical approach, each partition function is weighted by a fugacity term proportional to the number of generators of $\pi_1(\Gamma^{(2)}_m, e_0)$. Upon using the reciprocal complex, this picture corresponds to a vector model whose average number of particles is controlled by the given fugacity.

**Fig. 18.** A pictorial representation of the relations between the Whitehead group $W(\mathbb{R})$, the general linear group $GL(\mathbb{R})$, and the commutator subgroup generated by the elements of $E(\mathbb{R})$.

**Fig. 19.** The Gaussian averaging, described by (124), maps a real-valued function $f$ defined over $GL(q, \mathbb{R})$ into a real-valued function $\hat{f}_G$ defined over $W(\mathbb{R})$. 