PSEUDOCOMPACTNESS, PRODUCTS AND TOPOLOGICAL BRANDT
$\lambda^0$-EXTENSIONS OF SEMITOPOLOGICAL MONOIDS

OLEG GUTIK AND OLEKSANDR RAVSKY

ABSTRACT. In the paper we study the preservation of pseudocompactness (resp., countable compactness, sequential compactness, $\omega$-boundedness, totally countable compactness, countable pracoCOMPACTNESS, sequential pseudocompactness) by Tychonoff products of pseudocompact (and countably compact) topological Brandt $\lambda^0$-extensions of semitopological monoids with zero. In particular we show that if $\{(B_\lambda^0(S_i),\tau_{B(S_i)}) : i \in \mathcal{I}\}$ is a family of Hausdorff pseudocompact topological Brandt $\lambda^0$-extensions of pseudocompact semitopological monoids with zero such that the Tychonoff product $\prod \{S_i : i \in \mathcal{I}\}$ is a pseudocompact space then the direct product $\prod \{(B_\lambda^0(S_i),\tau_{B(S_i)}) : i \in \mathcal{I}\}$ endowed with the Tychonoff topology is a Hausdorff pseudocompact semitopological semigroup.

1. INTRODUCTION AND PRELIMINARIES

Further we shall follow the terminology of [5, 7, 11, 25, 29]. By $\mathbb{N}$ we shall denote the set of all positive integers.

A semigroup is a non-empty set with a binary associative operation. A semigroup $S$ is called inverse if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such the element $y$ in $S$ is called inverse to $x$ and is denoted by $x^{-1}$. The map assigning to each element $x$ of an inverse semigroup $S$ its inverse $x^{-1}$ is called the inversion.

For a semigroup $S$ by $E(S)$ we denote the subset of idempotents of $S$, and by $S^1$ (resp., $S^0$) we denote the semigroup $S$ the adjoined unit (resp., zero) (see [7, Section 1.1]). Also if a semigroup $S$ has zero $0_S$, then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$.

For a semilattice $E$ the semilattice operation on $E$ determines the partial order $\leq$ on $E$:

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$ 

This order is called natural. An element $e$ of a partially ordered set $X$ is called minimal if $f \leq e$ implies $f = e$ for $f \in X$. An idempotent $e$ of a semigroup $S$ without zero (with zero) is called primitive if $e$ is a minimal element in $E(S)$ (in $(E(S))^+)$.

Let $S$ be a semigroup with zero and $\lambda \geq 1$ be a cardinal. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\}$ we define a semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If $S$ is a monoid, then the semigroup $B_\lambda(S)$ is called the Brandt $\lambda$-extension of the semigroup $S$ [14]. Obviously, $\mathcal{J} = \{0\} \cup \{(\alpha, \emptyset, \beta) : \emptyset$ is the zero of $S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S) / \mathcal{J}$ and we shall call $B_\lambda^0(S)$ the Brandt $\lambda^0$-extension of the semigroup $S$ with zero [15]. Further, if $A \subseteq S$ then we shall denote $A_{\alpha, \beta} = \{(\alpha, s, \beta) : s \in A\}$ if $A$ does not contain zero, and $A_{\alpha, \beta} = \{(\alpha, s, \beta) : s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in \lambda$. If $\mathcal{I}$ is a trivial semigroup (i.e., $\mathcal{I}$ contains only one element), then by $\mathcal{I}^0$ we denote the semigroup $\mathcal{I}$ with the adjoined zero. Obviously, for any $\lambda \geq 2$ the Brandt $\lambda^0$-extension of
the semigroup $T^0$ is isomorphic to the semigroup of $\lambda \times \lambda$-matrix units and any Brandt $\lambda^0$-extension of a semigroup with zero contains the semigroup of $\lambda \times \lambda$-matrix units. Further by $B_\lambda$ we shall denote the semigroup of $\lambda \times \lambda$-matrix units and by $B_\lambda^0(1)$ the subsemigroup of $\lambda \times \lambda$-matrix units of the Brandt $\lambda^0$-extension of a monoid $S$ with zero.

A semigroup $S$ with zero is called $0$-simple if $\{0\}$ and $S$ are its only ideals and $S^2 \neq \{0\}$, and completely $0$-simple if it is $0$-simple and has a primitive idempotent [7]. A completely $0$-simple inverse semigroup is called a Brandt semigroup [23]. By Theorem II.3.5 [25], a semigroup $S$ is a Brandt semigroup if and only if $S$ is isomorphic to a Brandt $\lambda$-extension $B_\lambda(G)$ of a group $G$.

A non-trivial inverse semigroup is called a primitive inverse semigroup if all its non-zero idempotents are primitive [25]. A semigroup $S$ is a primitive inverse semigroup if and only if $S$ is an orthogonal sum of Brandt semigroups [25, Theorem II.4.3].

In this paper all topological spaces are Hausdorff. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then by $cl_Y(A)$ and $int_Y(A)$ we denote the topological closure and interior of $A$ in $Y$, respectively.

A subset $A$ of a topological space $X$ is called regular open if $int_X(cl_X(A)) = A$.

We recall that a topological space $X$ is said to be

- semiregular if $X$ has a base consisting of regular open subsets;
- compact if each open cover of $X$ has a finite subcover;
- sequentially compact if each sequence $\{x_i\}_{i \in \mathbb{N}}$ of $X$ has a convergent subsequence in $X$;
- $\omega$-bounded if every countably infinite set in $X$ has the compact closure [13];
- totally countably compact if every countably infinite set in $X$ contains an infinite subset with the compact closure [12];
- countably compact if each open countable cover of $X$ has a finite subcover;
- countably compact at a subset $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point $x$ in $X$;
- countably pracompact if there exists a dense subset $A$ in $X$ such that $X$ is countably compact at $A$ [2];
- sequentially pseudocompact if for each sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of the space $X$ there exist a point $x \in X$ and an infinite set $S \subseteq \mathbb{N}$ such that for each neighborhood $U$ of the point $x$ the set $\{n \in S : U_n \cap U = \emptyset\}$ is finite [21];
- $H$-closed if $X$ is Hausdorff and $X$ is a closed subspace of every Hausdorff space in which it is contained [1];
- pseudocompact if each locally finite open cover of $X$ is finite.

According to Theorem 3.10.22 of [11], a Tychonoff topological space $X$ is pseudocompact if and only if each continuous real-valued function on $X$ is bounded. Also, a Hausdorff topological space $X$ is pseudocompact if and only if every locally finite family of non-empty open subsets of $X$ is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is pseudocompact (see [2]). We observe that pseudocompact spaces in topological literature also are called lightly compact or feebly compact (see [3, 10, 27]).

We recall that the Stone-Čech compactification of a Tychonoff space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\beta f : \beta X \to Y$ [11].

A (semi)topological semigroup is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an inverse topological semigroup. A topological inverse semigroup is an inverse topological semigroup with continuous inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [3, Proposition II.1]). A Hausdorff topology $\tau$ on a (inverse) semigroup $S$ is called (inverse) semigroup if $(S, \tau)$ is a topological (inverse) semigroup. A paratopological (semitopological) group is a Hausdorff topological space with a jointly (separately) continuous group operation. A paratopological group with continuous inversion is a topological group.
Let \( \mathcal{STSG}_0 \) be a class of semitopological semigroups.

**Definition 1.1** ([14]). Let \( \lambda \geq 1 \) be a cardinal and \( (S, \tau) \in \mathcal{STSG}_0 \) be a semitopological monoid with zero. Let \( \tau_B \) be a topology on \( B_\lambda(S) \) such that

a) \( (B_\lambda(S), \tau_B) \in \mathcal{STSG}_0; \)

b) for some \( \alpha \in \lambda \) the topological subspace \( (S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}}) \) is naturally homeomorphic to \( (S, \tau) \).

Then \( (B_\lambda(S), \tau_B) \) is called a *topological Brandt \( \lambda \)-extension of \( (S, \tau) \) in \( \mathcal{STSG}_0 \).

**Definition 1.2** ([15]). Let \( \lambda \geq 1 \) be a cardinal and \( (S, \tau) \in \mathcal{STSG}_0 \). Let \( \tau_B \) be a topology on \( B_0^\lambda(S) \) such that

a) \( (B_0^\lambda(S), \tau_B) \in \mathcal{STSG}_0; \)

b) the topological subspace \( (S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}}) \) is naturally homeomorphic to \( (S, \tau) \) for some \( \alpha \in \lambda \).

Then \( (B_0^\lambda(S), \tau_B) \) is called a *topological Brandt \( \lambda^0 \)-extension of \( (S, \tau) \) in \( \mathcal{STSG}_0 \).

Later, if \( \mathcal{STSG}_0 \) coincides with the class of all semitopological semigroups we shall say that \( (B_\lambda(S), \tau_B) \) (resp., \( (B_0(S), \tau_0) \)) is called a *topological Brandt \( \lambda \)-extension (resp., a topological Brandt \( \lambda^0 \)-extension) of \( (S, \tau) \).

Algebraic properties of Brandt \( \lambda^0 \)-extensions of monoids with zero, non-trivial homomorphisms between them, and a category whose objects are ingredients of the construction of such extensions were described in [23]. Also, in [18] and [23] a category whose objects are ingredients in the constructions of finite (resp., compact, countably compact) topological Brandt \( \lambda^0 \)-extensions of topological monoids with zeros were described.

Gutik and Repovš proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt \( \lambda \)-extension \( B_\lambda(H) \) of a countably compact topological group \( H \) in the class of all topological inverse semigroups for some finite cardinal \( \lambda \geq 1 \) [22]. Also, every 0-simple pseudocompact topological inverse semigroup is topologically isomorphic to a topological Brandt \( \lambda \)-extension \( B_\lambda(H) \) of a pseudocompact topological group \( H \) in the class of all topological inverse semigroups for some finite cardinal \( \lambda \geq 1 \) [19]. Next Gutik and Repovš showed in [22] that the Stone-Čech compactification \( \beta(T) \) of a 0-simple countably compact topological inverse semigroup \( T \) has a natural structure of a 0-simple compact topological inverse semigroup. It was proved in [19] that the same is true for 0-simple pseudocompact topological inverse semigroups.

In the paper [1] the structure of compact and countably compact primitive topological inverse semigroups was described and was showed that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

Comfort and Ross in [8] proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact topological groups is a pseudocompact topological group. Also, they proved there that the Stone-Čech compactification of a pseudocompact topological group has a natural structure of a compact topological group. Ravsky in [26] generalized Comfort–Ross Theorem and proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact primitive topological inverse semigroups is pseudocompact. Also, there is proved that the Stone-Čech compactification of a pseudocompact primitive topological inverse semigroup has a natural structure of a compact primitive topological inverse semigroup.

In the paper [20] we studied the structure of inverse primitive pseudocompact semitopological and topological semigroups. We find conditions when a maximal subgroup of an inverse primitive pseudocompact semitopological semigroup \( S \) is a closed subset of \( S \) and described the topological structure of such semiregular semigroup. Also there we described structure of pseudocompact topological Brandt \( \lambda^0 \)-extensions of topological semigroups and semiregular (quasi-regular) primitive inverse topological semigroups. In [20] we shown that the inversion in a quasi-regular primitive inverse pseudocompact topological semigroup is continuous. Also there, an analogue of Comfort–Ross Theorem is proved for
such semigroups: the Tychonoff product of an arbitrary non-empty family of primitive inverse semiregular pseudocompact semitopological semigroups with closed maximal subgroups is a pseudocompact space, and we described the structure of the Stone-Čech compactification of a Hausdorff primitive inverse countably compact semitopological semigroup $S$ such that every maximal subgroup of $S$ is a topological group.

In this paper we study the preserving of Tychonoff products of the pseudocompactness (resp., countable compactness, sequential compactness, $\omega$-boundedness, totally countable compactness, countable pracomactness, sequential pseudocompactness) by pseudocompact (and countably compact) topological Brandt $\lambda^0$-extensions of semitopological semitopological monoids with zero. In particular we show that if $\{(B^0_{\lambda}(S_i),\tau^0_{B(S_i)}): i \in \mathcal{I}\}$ is a family of Hausdorff pseudocompact topological Brandt $\lambda^0$-extension of pseudocompact semitopological monoids with zero such that the Tychonoff product $\prod \{S_i: i \in \mathcal{I}\}$ is a pseudocompact space, then the direct product $\prod \{(B^0_{\lambda}(S_i),\tau^0_{B(S_i)}): i \in \mathcal{I}\}$ with the Tychonoff topology is a Hausdorff pseudocompact semitopological semigroup.

2. Tychonoff products of pseudocompact topological Brandt $\lambda^0$-extensions of semitopological semigroups

Later we need the following theorem from [17]:

**Theorem 2.1** ([17, Theorem 12]). For any Hausdorff countably compact semitopological monoid $(S, \tau)$ with zero and for any cardinal $\lambda \geq 1$ there exists a unique Hausdorff countably compact topological Brandt $\lambda^0$-extension $(B^0_{\lambda}(S),\tau^0_{B(S)})$ of $(S, \tau)$ in the class of semitopological semigroups, and the topology $\tau^0_{B(S)}$ is generated by the base $\mathcal{B}_B = \bigcup \{\mathcal{B}(t): t \in B^0_{\lambda}(S)\}$, where:

(i) $\mathcal{B}(t) = \{(U(s) \setminus \{0_s\})_{\alpha, \beta}: U(s) \in \mathcal{B}(s)\}$, where $t = (\alpha, s, \beta)$ is a non-zero element of $B^0_{\lambda}(S)$, $\alpha, \beta \in \lambda$;

(ii) $\mathcal{B}(0) = \{U(0) = \bigcup_{(\alpha, \beta) \in (\lambda, \lambda) \setminus A} S_{\alpha, \beta} \cup \bigcup_{(\gamma, \delta) \in A} (U(0_s))_{\gamma, \delta}: A$ is a finite subset of $\lambda \times \lambda$ and $U(0_s) \in \mathcal{B}(S_0)\}$, where $0$ is the zero of $B^0_{\lambda}(S)$, and $\mathcal{B}(s)$ is a base of the topology $\tau$ at the point $s \in S$.

**Lemma 2.2.** For any Hausdorff sequentially compact semitopological monoid $(S, \tau)$ with zero and for any cardinal $\lambda \geq 1$ the Hausdorff countably compact topological Brandt $\lambda^0$-extension $(B^0_{\lambda}(S),\tau^0_{B(S)})$ of $(S, \tau)$ in the class of semitopological semigroups is a sequentially compact space.

**Proof.** In the case when $\lambda < \omega$ the statement of the lemma follows from Theorems 3.10.32 and 3.10.34 from [11].

Next we suppose that $\lambda \geq \omega$. Let $\mathcal{A}(\lambda)$ be the one point Alexandroff compactification of the discrete space of cardinality $\lambda$. Then $\mathcal{A}(\lambda)$ is scattered because $\mathcal{A}(\lambda)$ has only one non-isolated point, and hence by Theorem 5.7 from [40] the space $\mathcal{A}(\lambda)$ is sequentially compact. Since cardinal $\lambda$ is infinite without loss of generality we can assume that $\lambda = \lambda \cdot \lambda$ and hence we can identify the space $\mathcal{A}(\lambda)$ with $\mathcal{A}(\lambda \times \lambda)$. Then by Theorem 3.10.35 from [11] the space $\mathcal{A}(\lambda \times \lambda) \times S$ is sequentially compact. Later we assume that $a$ is non-isolated point of the space $\mathcal{A}(\lambda \times \lambda)$. We define the map $g: \mathcal{A}(\lambda \times \lambda) \times S \to B^0_{\lambda}(S)$ by the formulae

$$g(a) = 0 \quad \text{and} \quad g((\alpha, \beta, s)) = \begin{cases} (\alpha, s, \beta), & \text{if } s \in S \setminus \{0_s\}; \\ 0, & \text{if } s = 0_s. \end{cases}$$

Theorem 2.1 implies that so defined map $g$ is continuous and hence by Theorem 3.10.32 of [11] we get that the topological Brandt $\lambda^0$-extension $(B^0_{\lambda}(S),\tau^0_{B(S)})$ of $(S, \tau)$ in the class of semitopological semigroups is a sequentially compact space.

Lemma 2.2 and Theorem 3.10.35 from [11] imply the following theorem:

**Theorem 2.3.** Let $\{B^0_{\lambda}(S_i): i \in \omega\}$ be a countable family of Hausdorff countably compact topological Brandt $\lambda^0$-extension of sequentially compact Hausdorff semitopological monoids. Then the direct product
Theorem 2.4. Let \( \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\} \) be a non-empty family of Hausdorff countably compact topological Brandt \( \lambda^0 \)-extension of countably compact Hausdorff semitopological monoids such that the Tychonoff product \( \prod \{S_i : i \in \mathcal{I}\} \) is a countably compact space. Then the direct product \( \prod \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\} \) with the Tychonoff topology is a Hausdorff sequentially compact semitopological semigroup.

Proof. For every infinite cardinal \( \lambda_i, i \in \mathcal{I} \), we shall repeat the construction proposed in the proof of Lemma 2.2. Let \( \mathcal{A}(\lambda_i) \) be the one point Alexandroff compactification of the discrete space of cardinality \( \lambda_i \). Since cardinal \( \lambda_i \) is infinite without loss of generality we can assume that \( \lambda_i = \lambda_i \cdot \lambda_i \) and hence we can identify the space \( \mathcal{A}(\lambda_i) \) with \( \mathcal{A}(\lambda_i \times \lambda_i) \). Later we assume that \( \alpha_i \) is non-isolated point of the space \( \mathcal{A}(\lambda_i \times \lambda_i) \).

We define the map \( g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i) \) by the formulae

\[
    g_i(a_i) = 0_i \quad \text{and} \quad g_i((\alpha_i, \beta_i, s_i)) = \begin{cases} 
    (\alpha_i, s_i, \beta_i), & \text{if } s_i \in S \setminus \{0_S\}; \\
    0_i, & \text{if } s = 0_S,
    \end{cases}
\]

where \( 0_i \) and \( 0_S \) are zeros of the semigroup \( B^0_{\lambda_i}(S_i) \) and the monoid \( S_i \), respectively. Theorem 2.1 implies that so defined map \( g_i \) is continuous.

In the case when cardinal \( \lambda_i, i \in \mathcal{I}, \) is finite we put \( \mathcal{A}(\lambda_i \times \lambda_i) \) is the discrete space of cardinality \( \lambda_i^2 + 1 \) with the fixed point \( a_i \in \mathcal{A}(\lambda_i \times \lambda_i) \). Next we define the map \( g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i) \) by the formulae (1), where \( 0_i \) and \( 0_S \) are zeros of the semigroup \( B^0_{\lambda_i}(S_i) \) and the monoid \( S_i \), respectively. It is obviously that such defined map \( g_i \) is continuous. Then the space \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \) is homeomorphic to \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i \in \mathcal{I}} S_i \) and hence by Theorem 3.2.4 and Corollary 3.10.14 from [11] the Tychonoff product \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \) is countably compact. Later we define the map \( g : \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to \prod_{i \in \mathcal{I}} B^0_{\lambda_i}(S_i) \) by putting \( g = \prod_{i \in \mathcal{I}} g_i \). Since for any \( i \in \mathcal{I} \) the map \( g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \to B^0_{\lambda_i}(S_i) \) is continuous, Theorem 2.1 and Proposition 2.3.6 of [11] imply that \( g \) is continuous too. Therefore by Theorem 3.10.5 from [11] we obtain that the direct product \( \prod \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\} \) with the Tychonoff topology is a Hausdorff countably compact semitopological semigroup.

Lemma 2.5. For any Hausdorff totally countably compact semitopological monoid \((S, \tau)\) with zero and for any cardinal \( \lambda \geq 1 \) the Hausdorff countably compact topological Brandt \( \lambda^0 \)-extension \((B^0_{\lambda}(S), \tau^0_B)\) of \((S, \tau)\) in the class of semitopological semigroups is a totally countably compact space.

Proof. In the case when \( \lambda < \omega \) the statement of the lemma is trivial. So we suppose that \( \lambda \geq \omega \).

Let \( A \) be an arbitrary countably infinite subset of \((B^0_{\lambda}(S), \tau^0_B)\). Put \( \mathcal{J} = \{ (\alpha, \beta) \in \lambda \times \lambda : A \cap S_{\alpha, \beta} \neq \emptyset \} \). If the set \( \mathcal{J} \) is finite then total countable compactness of the space \((S, \tau)\) and Lemma 2 of [17] imply the statement of the lemma. So we suppose that the set \( \mathcal{J} \) is infinite. For each pair of indices \( (\alpha, \beta) \in \mathcal{J} \) choose a point \( a_{\alpha, \beta} \in A \cap S_{\alpha, \beta} \) and put \( K = \{0\} \cup \{a_{\alpha, \beta} : (\alpha, \beta) \in \mathcal{J}\} \). Then the definition of the topology \( \tau^0_B \) on \( B^0_{\lambda}(S) \) implies that \( K \) is a compact subset of the \((B^0_{\lambda}(S), \tau^0_B)\) and \( K \cap A \) is infinite. This completes the proof of the lemma.

Lemma 2.5 and Theorem 4.3 from [12] imply the following theorem:

Theorem 2.6. Let \( \{B^0_{\lambda_i}(S_i) : i \in \omega\} \) be a countable family of Hausdorff countably compact topological Brandt \( \lambda^0 \)-extension of totally countably compact Hausdorff semitopological monoids. Then the direct product \( \prod \{B^0_{\lambda_i}(S_i) : i \in \omega\} \) with the Tychonoff topology is a Hausdorff totally countably compact semitopological semigroup.

Theorem 2.7. Let \( \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\} \) be a non-empty family of Hausdorff countably compact topological Brandt \( \lambda^0 \)-extension of Hausdorff totally countably compact semitopological monoids such that the Tychonoff product \( \prod \{S_i : i \in \mathcal{I}\} \) is a totally countably compact space. Then the direct product \( \prod \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\} \) with the Tychonoff topology is a totally countably compact semitopological semigroup.
Proof. Let for every \( i \in \mathcal{I} \), \( \mathcal{A}(\lambda_i \times \lambda_i) \) be a space and \( g_i: \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \rightarrow B^0_{\lambda_i}(S_i) \) be a map defined in the proof of Theorem 2.4. Also, Theorem 2.1 implies that the map \( g_i \) is continuous for every \( i \in \mathcal{I} \). Since the space \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \) is homeomorphic to \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i \in \mathcal{I}} S_i \) and Theorem 4.3 from [12] we see that the Tychonoff product \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \) is a totally countably compact space. Then by Theorem 2.1 and Proposition 2.3.6 of [11] the map \( g = \prod_{i \in \mathcal{I}} g_i \) is continuous. Simple verification imply that a continuous image of a totally countably compact space is a totally countably compact space too. Hence the direct product \( \prod \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) with the Tychonoff topology is a totally countably compact semitopological semigroup. \( \square \)

Similarly to the proof of Lemma 2.5 we can prove the following

Lemma 2.8. For any Hausdorff \( \omega \)-bounded semitopological monoid \((S, \tau)\) with zero and for any cardinal \( \lambda \geq 1 \) the Hausdorff countably compact topological Brandt \( \lambda^0 \)-extension \((B^0_{\lambda}(S), \tau^S_B)\) of \((S, \tau)\) in the class of semitopological semigroups is an \( \omega \)-bounded space.

Since by Lemma 4 of [13] the Tychonoff product of an arbitrary non-empty family of \( \omega \)-bounded spaces is an \( \omega \)-bounded space, similarly to the proof of Theorem 2.7 we can prove the following

Theorem 2.9. Let \( \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) be a non-empty family of Hausdorff countably compact topological Brandt \( \lambda^0 \)-extension \((B^0_{\lambda}(S), \tau^S_B)\) of \((S, \tau)\) in the class of semitopological semigroups. Then the direct product \( \prod \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) with the Tychonoff topology is an \( \omega \)-bounded semitopological semigroup.

Theorems 2.1 and 2.9 imply the following:

Corollary 2.10. Let \( \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) be a non-empty family of Hausdorff totally countably compact topological Brandt \( \lambda^0 \)-extension \((B^0_{\lambda}(S), \tau^S_B)\) of \((S, \tau)\) in the class of semitopological semigroups. Then the direct product \( \prod \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) with the Tychonoff topology is an \( \omega \)-bounded semitopological semigroup.

Later we shall use the following theorem from [17]:

Theorem 2.11 ([17, Theorem 15]). For any semiregular pseudocompact semitopological monoid \((S, \tau)\) with zero and for any cardinal \( \lambda \geq 1 \) there exists a unique semiregular pseudocompact topological Brandt \( \lambda^0 \)-extension \((B^0_{\lambda}(S), \tau^S_B)\) of \((S, \tau)\) in the class of semitopological semigroups, and the topology \( \tau^S_B \) is generated by the base \( \mathcal{B}_B = \bigcup \{ \mathcal{B}_B(t) : t \in B^0_{\lambda}(S) \} \), where:

(i) \( \mathcal{B}_B(t) = \{ (U(s) \setminus \{ 0(s) \})_{\alpha, \beta} : U(s) \in \mathcal{B}_S(s) \} \), where \( t = (\alpha, s, \beta) \) is a non-zero element of \( B^0_{\lambda}(S) \), \( \alpha, \beta \in \lambda \);

(ii) \( \mathcal{B}_B(0) = \{ U_A(0) = \bigcup_{(\alpha, \beta) \in (\lambda \times \lambda) \setminus A} S_{\alpha, \beta} \cup \bigcup_{(\gamma, \delta) \in A} (U(0))_{\gamma, \delta} : A \) is a finite subset of \( \lambda \times \lambda \) and \( U(0) \in \mathcal{B}_S(0) \} \), where 0 is the zero of \( B^0_{\lambda}(S) \), and \( \mathcal{B}_S(0) \) is a base of the topology \( \tau \) at the point \( s \in S \).

Theorem 2.12. Let \( \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) be a non-empty family of semiregular pseudocompact topological Brandt \( \lambda^0 \)-extension of semiregular pseudocompact semitopological monoids such that the Tychonoff product \( \prod \{ S_i : i \in \mathcal{I} \} \) is a pseudocompact space. Then the direct product \( \prod \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) with the Tychonoff topology is a semiregular pseudocompact semitopological semigroup.

Proof. Since by Lemma 20 from [20] the Tychonoff product of regular open sets is regular open we obtain that Tychonoff product of semiregular topological spaces is semiregular.

Let for every \( i \in \mathcal{I} \), \( \mathcal{A}(\lambda_i \times \lambda_i) \) be a space and \( g_i: \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \rightarrow B^0_{\lambda_i}(S_i) \) be the map defined in the proof of Theorem 2.4. Theorem 2.11 implies that the map \( g_i \) is continuous for every \( i \in \mathcal{I} \). Since the space \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \) is homeomorphic to \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i \in \mathcal{I}} S_i \) and Corollary 3.3 of [20] imply that the Tychonoff product \( \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \) is a pseudocompact space. Then by Theorem 2.11 and Proposition 2.3.6 of [11] the map \( g = \prod_{i \in \mathcal{I}} g_i \) is continuous, and hence the direct product \( \prod \{ B^0_{\lambda_i}(S_i) : i \in \mathcal{I} \} \) with the Tychonoff topology is a semiregular pseudocompact semitopological semigroup. \( \square \)
Proposition 2.13. Let $X, Y$ be Hausdorff countably pracompact spaces. Then the product $X \times Y$ is countably pracompact provided $Y$ is a $k$-space or sequentially compact.

Proof. Let the space $X$ be countably pracompact at its dense subset $D_X$ and the space $Y$ be countably compact at its dense subset $D_Y$. The set $D_X \times D_Y$ is a dense subset of the space $X \times Y$. We claim that the space $X \times Y$ is countably compact at the set $D_X \times D_Y$. Indeed, let $A = \{(x_s, y_s) : s \in S\}$ be an infinite subset of the set $D_X \times D_Y$ such that $(x_s, y_s) \neq (x_{s'}, y_{s'})$ provided $s \neq s'$. Assume that the set $A$ has no accumulation point in the space $X \times Y$.

If $Y$ is a $k$-space then Lemma 3.10.12 from [11] implies that there exists an infinite subset $S_0 \subset S$ such that either the set $\{x_s : s \in S_0\}$ or the set $\{y_s : s \in S_0\}$ has no accumulation point. This set is finite. Without loss of generality, we can assume that there is a point $x \in X$ and an infinite subset $S_1$ of the set $S_0$ such that $x_s = x$ for each index $s \in S_1$. Since the space $Y$ is countably compact at the set $D_Y$, there exists an accumulation point $y \in Y$ of the set $\{y_s : s \in S_1\}$. Then the point $(x, y)$ is an accumulation point of the set $\{(x_s, y_s) : s \in S_1\}$, a contradiction.

If $Y$ is a sequentially compact space then the proof of the claim is similar to the proof of Theorem 3.10.36 from [11]. □

Proposition 2.13 implies the following corollary:

Corollary 2.14. The product $X \times Y$ of Hausdorff countably pracompact space $X$ and compactum $Y$ is countably pracompact.

Theorem 2.15. Let $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ be a non-empty family of semiregular countably pracompact topological Brandt $\lambda^0$-extension of countably pracompact semiregular semitopological monoids such that the Tychonoff product $\prod \{S_i : i \in \mathcal{I}\}$ is a countably pracompact space. Then the direct product $\prod \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a semiregular countably pracompact semitopological semigroup.

Proof. Let for every $i \in \mathcal{I}$, $\mathcal{A}(\lambda_i \times \lambda_i)$ be a space and $g_i : \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \rightarrow B^0_{\lambda_i}(S_i)$ be a map defined in the proof of Theorem 2.4. Theorem 2.11 implies that the map $g_i$ is continuous for every $i \in \mathcal{I}$. Since the space $\prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is homeomorphic to $\prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times \prod_{i \in \mathcal{I}} S_i$, Theorem 3.2.4 from [11] and Corollary 2.13 imply that the Tychonoff product $\prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i$ is a countably pracompact space. Then by Theorem 2.11 and Proposition 2.3.6 of [11] the map $g : \prod_{i \in \mathcal{I}} \mathcal{A}(\lambda_i \times \lambda_i) \times S_i \rightarrow \prod_{i \in \mathcal{I}} B^0_{\lambda_i}(S_i)$ defined by the formula $g = \prod_{i \in \mathcal{I}} g_i$ is continuous, and since by Lemma 8 from [17] every continuous image of a countably pracompact space is countably pracompact, we see that the direct product $\prod \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a semiregular pseudocompact semitopological semigroup.

Since for any semitopological monoid $(S, \tau)$ with zero and for any finite cardinal $\lambda \geq 1$ there exists a unique topological Brandt $\lambda^0$-extension $(B^0_{\lambda}(S), \tau_B)$ of $(S, \tau)$ in the class of semitopological semigroups, the proof of the following theorem is similar to the proofs of Theorems 2.12 and 2.15.

Theorem 2.16. Let $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ be a non-empty family of Hausdorff pseudocompact (countably pracompact) topological Brandt $\lambda^0$-extension of Hausdorff pseudocompact (countably pracompact) semitopological monoids such that the Tychonoff product $\prod \{S_i : i \in \mathcal{I}\}$ is a Hausdorff pseudocompact (countably pracompact) space and every cardinal $\lambda_i, i \in \mathcal{I}$, is non-zero and finite. Then the direct product $\prod \{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ with the Tychonoff topology is a Hausdorff pseudocompact (countably pracompact) semitopological semigroup.

By Theorem 2.19 of [20] we have that a topological Brandt $\lambda^0$-extension $(B^0_{\lambda}(S), \tau_B)$ of a topological monoid $(S, \tau_S)$ with zero in the class of Hausdorff topological semigroups is pseudocompact if and only if cardinal $\lambda$ is finite and the space $(S, \tau_S)$ is pseudocompact. Hence Theorem 2.16 and Theorem 2.19 of [20] imply the following:

Theorem 2.17. Let $\{B^0_{\lambda_i}(S_i) : i \in \mathcal{I}\}$ be a non-empty family of Hausdorff pseudocompact (countably pracompact) topological Brandt $\lambda^0$-extension of Hausdorff pseudocompact (countably pracompact)
topological monoids in the class of Hausdorff topological semigroups such that the Tychonoff product
\[ \prod \{ S_i : i \in \mathcal{I} \} \]

is a Hausdorff pseudocompact (countably pracompact) space. Then the direct product
\[ \prod \{ B^0_B(S_i) : i \in \mathcal{I} \} \]

with the Tychonoff topology is a Hausdorff pseudocompact (countably pracompact) topological semigroup.

The following lemma describes the main property of a base of the topology at zero of a Hausdorff pseudocompact topological Brandt \( \lambda^0 \)-extension of a Hausdorff pseudocompact semitopological monoid in the class of Hausdorff semitopological semigroups.

**Lemma 2.18.** Let \( (B^0_B(S), \tau^S_B) \) be any Hausdorff pseudocompact topological Brandt \( \lambda^0 \)-extension of a pseudocompact semitopological monoid \( (S, \tau) \) with zero in the class of semitopological semigroups. Then for every open neighbourhood \( U(0) \) of zero 0 in \( (B^0_B(S), \tau^S_B) \) there exist at most finitely many pairs of indices \( (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in \lambda \times \lambda \) such that \( S^*_{\alpha_i, \beta_i} \not\subseteq \text{cl}_{B^0_B(S)}(U(0)) \) for every positive integer \( i \).

**Proof.** Suppose to the contrary: there exist an open neighbourhood \( V(0) \) of zero 0 in \( (B^0_B(S), \tau^S_B) \) and infinitely many pairs of indices \( (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n), \ldots \in \lambda \times \lambda \) such that \( S^*_{\alpha_i, \beta_i} \not\subseteq \text{cl}_{B^0_B(S)}(U(0)) \) for every positive integer \( i \). Then by Proposition 1.1.1 of [11] for every positive integer \( i \) there exists a non-empty open subset \( W_{\alpha_i, \beta_i} \) in \( (B^0_B(S), \tau^S_B) \) such that \( W_{\alpha_i, \beta_i} \subseteq S^*_{\alpha_i, \beta_i} \) and \( V(0) \cap W_{\alpha_i, \beta_i} = \emptyset \). Hence by Lemma 3 of [17] we have that \( \{ W_{\alpha_i, \beta_i} : i = 1, 2, 3, \ldots \} \) is an infinite locally finite family in \( (B^0_B(S), \tau^S_B) \) which contradicts the pseudocompactness of the space \( (B^0_B(S), \tau^S_B) \). The obtained contradiction implies the statement of our lemma.

Given a topological space \((X, \tau)\) Stone [28] and Katětov [24] consider the topology \( \tau_r \) on \( X \) generated by the base consisting of all regular open sets of the space \((X, \tau)\). This topology is called the regularization of the topology \( \tau \). It is easy to see that if \((X, \tau)\) is a Hausdorff topological space then \((X, \tau_r)\) is a semiregular topological space.

**Example 2.19.** Let \((S, \tau)\) be any semitopological monoid with zero. Then for any infinite cardinal \( \lambda \) we define a topology \( \tau^S_B \) on the Brandt \( \lambda^0 \)-extension \( (B^0_B(S), \tau^S_B) \) of \((S, \tau)\) in the following way. The topology \( \tau^S_B \) is generated by the base \( \mathcal{B}_B = \bigcup \{ \mathcal{B}_B(t) : t \in B^0_B(S) \} \), where:

(i) \( \mathcal{B}_B(t) = \{ (U(s) \setminus \{ 0_s \})_{\alpha, \beta} : U(s) \in \mathcal{B}_S(s) \} \), where \( t = (\alpha, s, \beta) \) is a non-zero element of \( B^0_B(S) \), \( \alpha, \beta \in \lambda \);

(ii) \( \mathcal{B}_B(0) = \{ U_A(0) = \bigcup_{(\alpha, \beta) \in (\lambda \times \lambda) \setminus \lambda} S_{\alpha, \beta} \cup \bigcup_{(\gamma, \delta) \in A} (U(0_s))_{\gamma, \delta} : A \text{ is a finite subset of } \lambda \times \lambda \text{ and } U(0_s) \in \mathcal{B}_S(0_s) \} \), where 0 is the zero of \( B^0_B(S) \), and \( \mathcal{B}_S(s) \) is a base of the topology \( \tau \) at the point \( s \in S \).

We observe that the space \((B^0_B(S), \tau^S_B)\) is Hausdorff (resp., regular, Tychonoff, normal) if and only if the space \((S, \tau)\) is Hausdorff (resp., regular, Tychonoff, normal) (see: Propositions 21 and 22 in [17]).

**Proposition 2.20.** Let \( \lambda \) be any infinite cardinal. If \((S, \tau)\) is a Hausdorff semitopological monoid with zero then \((B^0_B(S), \tau^S_B)\) is a Hausdorff semitopological semigroup. Moreover, the space \((S, \tau)\) is pseudocompact if and only if so is \((B^0_B(S), \tau^S_B)\).

**Proof.** The Hausdorffness of the space \((B^0_B(S), \tau^S_B)\) follows from Proposition 21 from [17].

Let \( a \) and \( b \) be arbitrary elements of \( S \) and \( W(ab) \), \( U(a) \), \( V(b) \) be arbitrary open neighborhoods of the elements \( ab \), \( a \) and \( b \), respectively, such that \( U(a) \cdot b \subseteq W(ab) \) and \( a \cdot V(b) \subseteq W(ab) \). Then we have that the following conditions hold:

(i) \( (U(a))_{\alpha, \beta} \cdot (\beta, b, \gamma) \subseteq (W(ab))_{\alpha, \gamma} \);

(ii) \( (\alpha, a, \beta) \cdot (V(b))_{\beta, \gamma} \subseteq (W(ab))_{\alpha, \gamma} \);

(iii) if \( \beta \neq \gamma \) then \( (U(a))_{\alpha, \beta} \cdot (\gamma, b, \delta) = \{ 0 \} \subseteq W_A(0) \) and \( (\alpha, a, \beta) \cdot (V(b))_{\gamma, \delta} = \{ 0 \} \subseteq W_A(0) \) for every finite subset \( A \) of \( \lambda \times \lambda \) and every \( W(0_s) \in \mathcal{B}_S(0_s) \);

(iv) \( W_A(0) \cdot 0 = \{ 0 \} \subseteq W_A(0) \) and \( 0 \cdot W_A(0) = \{ 0 \} \subseteq W_A(0) \) for every finite subset \( A \) of \( \lambda \times \lambda \) and every \( W(0_s) \in \mathcal{B}_S(0_s) \);
\((v)\quad (U(a))_{\alpha, \beta} \cdot 0 = \{0\} \subseteq W_A(0)\) and \(0 \cdot (V(b))_{\beta, \gamma} = \{0\} \subseteq W_A(0)\) for every finite subset \(A\) of \(\lambda \times \lambda\) and every \(W(0_S) \in \mathcal{B}_S(0_S)\);

\((vi)\quad (\alpha, a, \beta) \cdot V_B(0) \subseteq W_A(0)\) for every finite subset \(\{\alpha_1, \ldots, \alpha_k\} \subseteq \lambda\) and every \(W(0_S) \in \mathcal{B}_S(0_S)\), where \(A = \{\alpha, \alpha_1, \ldots, \alpha_k\} \times \{\alpha_1, \ldots, \alpha_k\}\) and \(B = \{\beta, \alpha_1, \ldots, (\beta, \alpha_k)\}\);

\((vii)\quad V_B(0) \cdot (\alpha, a, \beta) \subseteq W_A(0)\) for every finite subset \(\{\alpha_1, \ldots, \alpha_k\} \subseteq \lambda\) and every \(W(0_S) \in \mathcal{B}_S(0_S)\), where \(A = \{\alpha_1, \ldots, \alpha_k\} \times \{\beta, \alpha_1, \ldots, \alpha_k\}\) and \(B = \{(\alpha_1, \alpha), \ldots, (\alpha_k, \alpha)\}\),

for each \(\alpha, \beta, \gamma, \delta \in \lambda\). This completes the proof of separate continuity of the semigroup operation in \((B^0_\lambda(S), \tau^0_B)\).

The implication \((\Rightarrow)\) of the last statement follows from Lemma 9 of [17]. To show the converse implication assume that \(\{U_i: i \in \mathcal{I}\}\) is any locally finite family of open subsets of \((B^0_\lambda(S), \tau^0_B)\). Without loss of generality we can assume that \(0 \notin U_i\) for any \(i \in \mathcal{I}\). Then the definition of the base of the topology \(\tau^0_B\) at zero implies that there exists a finite family of pairs of indices \(\{(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)\} \subseteq \lambda \times \lambda\) such that almost all elements of the family \(\{U_i: i \in \mathcal{I}\}\) are contained in the set \(S^*_{\alpha_1, \beta_1} \cup \cdots \cup S^*_{\alpha_k, \beta_k}\).

Since a union of a finite family of pseudocompact spaces is pseudocompact, \(S_{\alpha_1, \beta_1} \cup \cdots \cup S_{\alpha_k, \beta_k}\) with the topology induced from \((B^0_\lambda(S), \tau^0_B)\) is pseudocompact space. This implies that the family \(\{U_i: i \in \mathcal{I}\}\) is finite.

\(\square\)

**Example 2.21.** Let \(\lambda\) be any infinite cardinal. Let \((S, \tau_S)\) be a Hausdorff pseudocompact semitopological monoid with zero \(0_S\) and \((B^0_\lambda(S), \tau^0_{B_S})\) be a pseudocompact topological Brandt \(\lambda^0\)-extension of \((S, \tau_S)\) in the class of Hausdorff semitopological semigroups.

For every open neighbourhood \(U(0)\) of zero in \((B^0_\lambda(S), \tau^0_{B_S})\) we put

\[
A_{U(0)} = \{(\alpha, \beta) \in \lambda \times \lambda: S_{\alpha, \beta} \not\subseteq \text{cl}_B^0(\lambda)(U(0))\}.
\]

Let \(\pi_{B_S}: B_\lambda(S) \rightarrow B^0_\lambda(S) = B_\lambda(S)/\mathcal{I}\) be the natural homomorphisms, where

\[
\mathcal{I} = \{0\} \cup \{(\alpha, 0_S, \beta): 0_S\text{ is zero of }S\}
\]

is an ideal of the semigroup \(B_\lambda(S)\).

We generate a topology \(\tau_{B_S}\) on the Brandt \(\lambda\)-extension \(B_\lambda(S)\) by a base \(\mathcal{B}_B = \bigcup \{\hat{\mathcal{B}}_B(t): t \in \mathcal{B}_\lambda(S)\}\), where:

\(i)\quad \hat{\mathcal{B}}_B(\alpha, s, \beta) = \{(U(s))_{\alpha, \beta}: U(s) \in \mathcal{B}_S(s)\}, \text{ for all } s \in S \text{ and } \alpha, \beta \in \lambda;

(ii) \quad \hat{\mathcal{B}}_B(0) = \left\{U_\pi(0) = \pi^{-1}(U(0)) \bigcup_{(\alpha, \beta) \in A_{U(0)}} S_{\alpha, \beta}: U(0) \text{ is an element of a base of the topology } \tau^0_{B_S} \text{ at zero of } B^0_\lambda(S) \right\},

and \(\mathcal{B}_S(s)\) is a base of the topology \(\tau\) at the point \(s \in S\).

**Proposition 2.22.** Let \(\lambda\) be any infinite cardinal. Let \((B^0_\lambda(S), \tau^0_{B_S})\) be a pseudocompact topological Brandt \(\lambda^0\)-extension of a Hausdorff pseudocompact semitopological monoid with zero \((S, \tau_S)\) in the class of Hausdorff semitopological semigroup. Then \((B_\lambda(S), \hat{\tau}_{B_S})\) is a Hausdorff semitopological semigroup. Moreover, the space \((S, \tau)\) is pseudocompact if and only if so is \((B_\lambda(S), \hat{\tau}_{B_S})\).

**Proof.** We observe that simple verifications show that the natural homomorphism \(\pi_{B_S}: (B_\lambda(S), \hat{\tau}_{B_S}) \rightarrow (B^0_\lambda(S), \tau^0_{B_S})\) is a continuous map.

Let \(a\) and \(b\) be arbitrary elements of the semitopological semigroup \((S, \tau_S)\) and \(W(ab), U(a), V(b)\) be arbitrary open neighborhoods of the elements \(ab, a\) and \(b\), respectively, such that \(U(a) \cdot b \subseteq W(ab)\) and \(a \cdot V(b) \subseteq W(ab)\). Then the following conditions hold:

\(i)\quad (U(a))_{\alpha, \beta} \cdot (\beta, b, \gamma) \subseteq (W(ab))_{\alpha, \gamma};

(ii) \quad (\alpha, a, \beta) \cdot (V(b))_{\beta, \gamma, \delta} \subseteq (W(ab))_{\alpha, \gamma};

(iii) \quad \text{if } \beta \neq \gamma \text{ then } (U(a))_{\alpha, \beta} \cdot (\gamma, b, \delta) = \{0\} \subseteq U_\pi(0) \text{ and } (\alpha, a, \beta) \cdot (V(b))_{\gamma, \delta} = \{0\} \subseteq U_\pi(0) \text{ for every open neighbourhood } U(0) \text{ of zero in } (B^0_\lambda(S), \tau^0_{B_S});

(iv) \quad U_\pi(0) \cdot 0 = \{0\} \subseteq U_\pi(0) \text{ and } 0 \cdot U_\pi(0) = \{0\} \subseteq U_\pi(0) \text{ for every open neighbourhood } U(0) \text{ of zero in } (B^0_\lambda(S), \tau^0_{B_S});

\(\square\)
(v) \((U(a))_{\alpha, \beta} \cdot 0 = \{0\} \subseteq U_\pi(0)\) and \(0 \cdot (V(b))_{\beta, \gamma} = \{0\} \subseteq U_\pi(0)\) for every open neighbourhood \(U(0)\) of zero in \((B^0_\lambda(S), S^0_{BS})\).

(vi) \((\alpha, a, \beta) \cdot U_\pi(0) \subseteq W_\pi(0)\) in \((B_\lambda(S), \hat{T}_{BS})\) for \(U_\pi(0), W_\pi(0) \in \hat{T}_{BS}(0)i\) where \(U(0)\) and \(W(0)\) are elements of a base of the topology \(\tau^0_{BS}\) at zero of \(B^0_\lambda(S)\) such that \((\alpha, a, \beta) \cdot U(0) \subseteq W(0)\).

(vii) \(U_\pi(0) \cdot (\alpha, a, \beta) \subseteq W_\pi(0)\) in \((B_\lambda(S), \hat{T}_{BS})\) for \(U_\pi(0), W_\pi(0) \in \hat{T}_{BS}(0)\) where \(U(0)\) and \(W(0)\) are elements of a base of the topology \(\tau^0_{BS}\) at zero of \(B^0_\lambda(S)\) such that \(U(0) \cdot (\alpha, a, \beta) \subseteq W(0)\), for each \(\alpha, \beta, \gamma, \delta \in \lambda\).

The proof of the last statement is similar to the proof of the second statement of Proposition 2.20.

Remark 2.23. Also, we may consider the semitopological semigroup \((B_\lambda(S), \hat{T}_{BS})\) as a topological Brandt \(\lambda^0\)-extension of a Hausdorff pseudocompact semitopological monoid \(T = S \sqcup 0_T\) with “new” isolated zero \(0_T\).

Theorem 2.24. Let \(\{ (B^0_\lambda(S_i), \tau^0_{BS} (S_i)) : i \in \mathcal{I} \}\) be a non-empty family of Hausdorff pseudocompact topological Brandt \(\lambda^0\)-extension of Hausdorff pseudocompact semitopological monoids with zero such that the Tychonoff product \(\prod \{ S_i : i \in \mathcal{I} \}\) is a pseudocompact space. Then the direct product \(\prod \{ (B^0_\lambda(S_i), \tau^0_{BS} (S_i)) : i \in \mathcal{I} \}\) with the Tychonoff topology is a Hausdorff pseudocompact semitopological semigroup.

Proof. We consider two cases: 1) \(\lambda_i\) is finite cardinal, and 2) \(\lambda_i\) is infinite cardinal, \(i \in \mathcal{I}\).

1) Let \(i \in \mathcal{I}\) be an index such that \(\lambda_i\) is infinite cardinal. Then we put \(\hat{T}_{BS} (S_i)\) is the topology on the Brandt \(\lambda^0\)-extension \(B_\lambda(S_i)\) defined in Example 2.21 Then by Proposition 2.22 \((B_\lambda(S_i), \hat{T}_{BS}(S_i))\) is a Hausdorff pseudocompact semitopological semigroup. By Remark 2.23 we have that the semitopological semigroup \((B_\lambda(S_i), \hat{T}_{BS}(S_i))\) is a topological Brandt \(\lambda^0\)-extension of a Hausdorff pseudocompact semitopological monoid \(T_i = S \sqcup 0_T\) with isolated zero \(0_T\). By \(\tau_i\) we denote the topology of the space \(T_i\). Let \(\tau^T_{BS}\) be the topology on the Brandt \(\lambda^0\)-extension \((B^0_\lambda(T_i), \tau^T_{BS})\) of \((T_i, \tau_i)\) defined in Example 2.19. Next we algebraically identify the semigroup \(B^0_\lambda(T_i)\) with the Brandt \(\lambda^0\)-extension \(B_\lambda(S_i)\) and the topology \(\tau^T_{BS}\) on \(B_\lambda(S_i)\) we shall denote by \(\tau^S_{BS}\).

2) Let \(i \in \mathcal{I}\) be an index such that \(\lambda_i\) is finite cardinal. We put \(T_i = S \sqcup 0_T\) with isolated zero \(0_T\). It is obvious that the semitopological semigroup \(T_i\) is pseudocompact if and only if so is the space \(S_i\). Then by Theorem 7 from [17] there exists the unique topological Brandt \(\lambda^0\)-extension \((B^0_\lambda(T_i), \hat{T}_{BS}(T_i))\) of the semitopological monoid \(T_i\) in the class of semitopological semigroups. Also, Theorem 7 from [17] implies that the topological space \((B^0_\lambda(T_i), \hat{T}_{BS}(T_i))\) is homeomorphic to the topological sum of topological copies of the space \(S_i\) and isolated zero, and hence we obtain that the space \((B^0_\lambda(T_i), \hat{T}_{BS}(T_i))\) is pseudocompact if and only if so is the space \(S_i\). Next we algebraically identify the semigroup \(B^0_\lambda(T_i)\) with the Brandt \(\lambda^0\)-extension \(B_\lambda(S_i)\) and the topology \(\tau_{BS}(T_i)\) on \(B_\lambda(S_i)\) we shall denote by \(\tau_{BS}(S_i)\). Also in this case (when \(\lambda_i\) is a finite cardinal) we put \(\tau^S_{BS}(T_i) = \tau_{BS}(S_i)\).

Then the definitions of topologies \(\tau_{BS}(S_i)\) and \(\tau^S_{BS}\) on \(B_\lambda(S_i)\) imply that for every index \(i \in \mathcal{I}\) the identity map \(\hat{id}_i : (B_\lambda(S_i), \tau_{BS}(S_i)) \to (B_\lambda(S_i), \tau^S_{BS})\) is continuous. Let \(\tau^R_{BS}(S_i)\) be the regularization of the topology \(\tau_{BS}(S_i)\) on \(B_\lambda(S_i)\). Then the definition of the topology \(\tau^S_{BS}\) on \(B_\lambda(S_i)\) implies that the identity map \(\hat{id}^R_\lambda : (B_\lambda(S_i), \tau^S_{BS}) \to (B_\lambda(S_i), \tau^R_{BS}(S_i))\) is continuous. Since the pseudocompactness is preserved by continuous maps we obtain that \((B_\lambda(S_i), \tau^R_{BS}(S_i))\) is a semiregular pseudocompact space (which is not necessarily a semitopological semigroup). Also, repeating the proof of Theorem 2.12 for our case, we get that the Tychonoff product \(\prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau^S_{BS}(S_i))\) is a pseudocompact space. Then the Cartesian products \(\prod_{i \in \mathcal{I}} \hat{id}_i : \prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau_{BS}(S_i)) \to \prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau^S_{BS}(S_i))\) and \(\prod_{i \in \mathcal{I}} \hat{id}^R_\lambda : \prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau^S_{BS}) \to \prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau^R_{BS}(S_i))\) are continuous maps. This implies that \(\prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau^R_{BS}(S_i))\) is a pseudocompact space. Then by Lemma 20 of [26] the regularization of the product \(\prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau_{BS}(S_i))\) coincides with \(\prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau^R_{BS}(S_i))\) and hence by Lemma 3 of [26] we have that the space \(\prod_{i \in \mathcal{I}} (B_\lambda(S_i), \tau_{BS}(S_i))\) is pseudocompact.
Let $\pi_{B_{0}^\lambda(S_i)}: B_{\lambda}(S_i) \to B_{\lambda}^0(S_i) = B_{\lambda}(S_i)/\mathcal{I}$ be the natural homomorphism, where $\mathcal{I} = \{0\} \cup \{(\alpha, 0_{S_i}, \beta) : 0_{S_i} \text{ is zero of } S_i\}$ is an ideal of the semigroup $B_{\lambda}(S_i)$. Then the natural homomorphism $\pi_{B_{0}^\lambda(S_i)} : (B_{\lambda}(S_i), \tau_{B(S_i)}) \to (B_{\lambda}^0(S_i), \tau_{B(S_i)})$ is a continuous map. This implies that the product $\prod_{i \in \mathcal{I}} \pi_{B_{0}^\lambda(S_i)} : \prod_{i \in \mathcal{I}} (B_{\lambda}(S_i), \tau_{B(S_i)}) \to \prod_{i \in \mathcal{I}} (B_{\lambda}^0(S_i), \tau_{B(S_i)})$ is a continuous map, and hence we get that the Tychonoff product $\prod_{i \in \mathcal{I}} (B_{\lambda}(S_i), \tau_{B(S_i)})$ is a pseudocompact space. □

**Proposition 2.25.** Each $H$-closed space is pseudocompact.

**Proof.** Let $X$ be an $H$-closed space. Assume that the space $X$ is not pseudocompact. Then there exists an infinite locally finite family $\mathcal{U}$ of non-empty open subsets of the space $X$. Since the family $\mathcal{U}$ is locally finite, each point $x \in X$ has an open neighbourhood $U_x$ intersecting only finitely many members of the family $\mathcal{U}$. Since the space $X$ is $H$-closed and $\{U_x : x \in X\}$ is an open cover of the space $X$, by Exercise 3.12.5(4) from [1] (also see [1] Chapter III, Theorem 4) there exists a finite subset $F$ of the space $X$ such that $X = \bigcup_{x \in F} \{c_{X(U_x)} : x \in F\}$. But then the set $X$, as the union of the finite family $\{c_{X(U_x)} : x \in F\}$ intersects only finitely many members of the family $\mathcal{U}$, a contradiction. □

Let $\lambda$ be any cardinal $\geq 1$ and $S$ be any semigroup. We shall say that a subset $\Phi \subset B_{\lambda}^0(S)$ has the $\lambda$-finite property in $B_{\lambda}^0(S)$, if $\Phi \cap S_{\alpha,\beta}^\infty$ is finite for all $\alpha, \beta \in \lambda$ and $\Phi \neq 0$, where 0 is zero of $B_{\lambda}^0(S)$.

**Example 2.26.** Let $\lambda$ be an infinite cardinal and $\mathcal{T}$ be the unit circle with the usual multiplication of complex numbers and the usual topology $\tau_{\mathcal{T}}$. It is obvious that $(\mathcal{T}, \tau_{\mathcal{T}})$ is a topological group. The base of the topology $\tau_{\mathcal{T}}^\infty$ on the Brandt semigroup $B_{\lambda}(\mathcal{T})$ we define as follows:

1) for every non-zero element $(\alpha, x, \beta)$ of the semigroup $B_{\lambda}(\mathcal{T})$ the family

$$\mathcal{B}_{(\alpha, x, \beta)} = \{(\alpha, U(x), \beta) : (\alpha, x, \beta) \in B_{\lambda}(\mathcal{T})\}$$

where $\mathcal{B}_{(\alpha, x, \beta)}$ is a base of the topology $\tau_{\mathcal{T}}$ at the point $x \in \mathcal{T}$, is the base of the topology $\tau_{\mathcal{T}}^\infty$ at $(\alpha, x, \beta) \in B_{\lambda}(\mathcal{T})$;

2) the family

$$\mathcal{B}_0 = \{U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; F) : \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \lambda, n \in \mathbb{N}, F \text{ has the } \lambda\text{-finite property in } B_{\lambda}^0(S)\}$$

is the base of the topology $\tau_{\mathcal{T}}^\infty$ at zero $0 \in B_{\lambda}(\mathcal{T})$.

Simple verifications show that $(B_{\lambda}(\mathcal{T}), \tau_{\mathcal{T}}^\infty)$ is a non-semiregular Hausdorff pseudocompact topological space for every infinite cardinal $\lambda$. Next we shall show that the semigroup operation on $(B_{\lambda}(\mathcal{T}), \tau_{\mathcal{T}}^\infty)$ is separately continuous. The proof of the separate continuity of the semigroup operation in the cases $0 \cdot 0$ and $(\alpha, x, \beta) \cdot (\gamma, y, \delta)$, where $\alpha, \beta, \gamma, \delta \in \lambda$ and $x, y, \in \mathcal{T}$, is trivial, and hence we only consider the following cases:

$$(\alpha, x, \beta) \cdot 0 \quad \text{and} \quad 0 \cdot (\alpha, x, \beta).$$

For arbitrary $\alpha, \beta \in \lambda$ and $\Phi \subset B_{\lambda}(\mathcal{T})$ we denote $\Phi^{\alpha,\beta} = \Phi \cap \mathcal{T}_{\alpha,\beta}$ and put $\Phi_{\mathcal{T}}(\alpha, \beta)$ is a subset of $\mathcal{T}$ such that $(\Phi_{\mathcal{T}}(\alpha, \beta))_{\alpha,\beta} = \Phi \cap \mathcal{T}_{\alpha,\beta}$.

Fix an arbitrary non-zero element $(\alpha, x, \beta) \in B_{\lambda}(\mathcal{T})$. Let $\Phi \subset B_{\lambda}^0(S)$ be an arbitrary subset with the $\lambda$-finite property in $B_{\lambda}^0(S)$. Since $\mathcal{T}$ is a group, there exist subsets $\Upsilon, \Psi \subset B_{\lambda}^0(S)$ with the $\lambda$-finite property in $B_{\lambda}^0(S)$ such that

$$(x \cdot \Upsilon(\beta, \gamma))_{\alpha,\gamma} = \Phi \cap \mathcal{T}_{\alpha,\gamma} \quad \text{and} \quad (\Psi_{\mathcal{T}}(\gamma, \alpha) \cdot x)_{\gamma,\beta} = \Phi \cap \mathcal{T}_{\gamma,\beta}$$

Then we have that

$$(\alpha, x, \beta) \cdot U(\beta, \beta_1; \ldots; \beta_n; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Upsilon) \subseteq \{0\} \cup \bigcup \{\mathcal{T}_{\alpha,\gamma} \setminus ((\alpha, x, \beta) \cdot \Upsilon^{\beta,\gamma}) : \gamma \in \lambda \setminus \{\beta_1, \ldots, \beta_n\}\} \subseteq \{0\} \cup \bigcup \{\mathcal{T}_{\alpha,\gamma} \setminus (x \cdot \Upsilon(\beta, \gamma))_{\alpha,\gamma} : \gamma \in \lambda \setminus \{\beta_1, \ldots, \beta_n\}\} \subseteq U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Phi)$$
and similarly
\[ U(\alpha_1, \alpha; \ldots; \alpha_n, \alpha; \beta_1; \ldots; \alpha_n, \beta_n; \Psi) \cdot (\alpha, x, \beta) \subseteq \{0\} \cup \bigcup \{T_{\gamma, \beta} \setminus (\Psi_{\gamma, \alpha} \cdot (\alpha, x, \beta)) : \gamma \in \lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\} \subseteq \{0\} \cup \bigcup \{T_{\gamma, \beta} \setminus (\Psi_{\gamma, \alpha} \cdot (\alpha, x, \beta))_{\gamma, \beta} : \gamma \in \lambda \setminus \{\alpha_1, \ldots, \alpha_n\}\} \subseteq U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Phi), \]

for every \( U(\alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; \Phi) \in \mathcal{B}_0 \). This completes the proof of separate continuity of the semigroup operation in \((B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})\).

Next we shall show that the space \((B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})\) is not countably pracompact. Suppose to the contrary: there exists a dense subset \( A \) in \((B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})\) such that \((B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})\) is countably compact at \( A \). Then the definition of the topology \( \tau_B^{\text{fin}} \) implies that \( A \cap \mathbb{T}_{\alpha, \beta} \) is a dense subset in \( \mathbb{T}_{\alpha, \beta} \) for all \( \alpha, \beta \in \lambda \). We construct a subset \( \Phi \subset B_\lambda(\mathbb{T}) \) in the following way. For any \( \alpha, \beta \in \lambda \) we fix an arbitrary point \( (\alpha, x^A_{\alpha, \beta}, \beta) \in A \cap \mathbb{T}_{\alpha, \beta} \) and put \( \Phi = \{(\alpha, x^A_{\alpha, \beta}, \beta) : \alpha, \beta \in \lambda\}. \) Then \( \Phi \) is the subset with the \( \lambda \)-finite property in \( B_\lambda^0(S) \), and the definition of the topology \( \tau_B^{\text{fin}} \) on \( B_\lambda(\mathbb{T}) \) implies that \( \Phi \) has no an accumulation point \( x \) in \((B_\lambda(\mathbb{T}), \tau_B^{\text{fin}})\), a contradiction.

Example 2.26 shows that there exists a Hausdorff non-semiregular pseudocompact topological Brandt \( \lambda^0 \)-extension of a Hausdorff compact topological group with adjoined isolated zero which is not a countably pracompact space. Also, Example 18 from [17] shows that there exists a Hausdorff non-semiregular pseudocompact topological Brandt \( \lambda^0 \)-extension of a countable Hausdorff compact topological monoid with adjoined isolated zero which is not a countably compact space. But, as a counterpart for the \( H \)-closed case or the sequentially pseudocompact case we have the following

**Proposition 2.27.** Let \( S \) be semitopological monoid with zero which is an \( H \)-closed (resp., a sequentially pseudocompact) space. Then every Hausdorff pseudocompact topological Brandt \( \lambda^0 \)-extension \( B_\lambda^0(S) \) of \( S \) in the class of Hausdorff semitopological semigroup is an \( H \)-closed (resp., a sequentially pseudocompact) space.

**Proof.** First we consider the case when \( S \) an \( H \)-closed space. Suppose to the contrary that there exists a Hausdorff pseudocompact topological Brandt \( \lambda^0 \)-extension \( (B_\lambda^0(S), \tau_B) \) of \( S \) in the class of Hausdorff semitopological semigroup such that \((B_\lambda^0(S), \tau_B)\) is not an \( H \)-closed space. Then there exists a Hausdorff topological space \( X \) which contains the topological \((B_\lambda^0(S), \tau_B)\) as a non-closed subspace. Without loss of generality we may assume that \((B_\lambda^0(S), \tau_B)\) is a dense subspace of \( X \) such that \( X \setminus B_\lambda^0(S) \neq \emptyset \). Fix an arbitrary point \( x \in X \setminus B_\lambda^0(S) \). Then we have that \( U(x) \cap B_\lambda^0(S) \neq \emptyset \) for any open neighbourhood \( U(x) \) of the point \( x \) in \( X \). Now, the Hausdorffness of \( X \) implies that there exist open neighbourhoods \( V(x) \) and \( V(0) \) of \( x \) and zero 0 of the semigroup \( B_\lambda^0(S) \) such that \( V(x) \cap V(0) = \emptyset \). Then by Lemma 2.18 we obtain that there exist at most finitely many pairs of indices \((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in \lambda \times \lambda\) such that \( S_{\alpha_i, \beta_i}^* \not\subseteq \text{cl}_{B_\lambda^0(S)}(V(0)) \) for any \( i = 1, \ldots, n \). Hence by Corollary 1.1.2 of [13], the neighbourhood \( V(x) \) intersects at most finitely many subsets \( S_{\alpha, \beta}, \alpha, \beta \in \lambda \). Then by Lemma 2 of [17] we get that \( S_{\alpha, \beta} \) is a closed subset of \( X \) for all \( \alpha, \beta \in \lambda \), and hence \( B_\lambda^0(S) \) is a closed subspace of \( X \), a contradiction.

Next we suppose that \( S \) is a sequentially pseudocompact space. Let \( \{U_n : n \in \mathbb{N}\} \) be any sequence of non-empty open subsets of the space \( B_\lambda^0(S) \). If there exists finitely many pairs of indices \((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in \lambda \times \lambda\) such that \( \bigcup \{U_n : n \in \mathbb{N}\} \subseteq S_{\alpha_1, \beta_1} \cup \cdots \cup S_{\alpha_n, \beta_n} \) the sequential pseudocompactness of \( S \) and Lemma 2 from [17] imply that there exist a point \( x \in S_{\alpha_1, \beta_1} \cup \cdots \cup S_{\alpha_n, \beta_n} \) and an infinite set \( S \subset \mathbb{N} \) such that for each neighborhood \( U(x) \) of the point \( x \) the set \( \{n \in S : U_n \cap U(x) = \emptyset\} \) is finite. In the other case by Lemma 2.18 we get that there exists an infinite set \( S \subset \mathbb{N} \) such that for each neighborhood \( U(0) \) of zero 0 of the semigroup \( B_\lambda^0(S) \) the set \( \{n \in S : U_n \cap U(0) = \emptyset\} \) is finite. This completes the proof of our lemma. \( \square \)
Since by Theorem 3 from [6] (see also Problem 3.12.5(d) in [11]) the Tychonoff product of the non-empty family non-empty $H$-topological spaces is $H$-closed, and by Proposition 2.2 from [21], the Tychonoff product of a non-empty family of non-empty sequentially pseudocompact spaces is sequentially pseudocompact Proposition 2.27 implies the following

**Corollary 2.28.** Let \( \{ (B^0_{\lambda}(S_i), \tau^0_{B(S_i)}): i \in \mathcal{I} \} \) be a non-empty family of Hausdorff pseudocompact topological Brandt $\lambda^0$-extension of Hausdorff $H$-closed (resp., a sequentially pseudocompact) semitopological monoids with zero. Then the direct product \( \prod \{ (B^0_{\lambda}(S_i), \tau^0_{B(S_i)}): i \in \mathcal{I} \} \) with the Tychonoff topology is a Hausdorff $H$-closed (resp., a sequentially pseudocompact) semitopological semigroup.

**References**

[1] P. Alexandroff and P. Urysohn, *Mémoire sur les espaces topologiquement compacts*, Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1, 14(1):1–96, 1929.

[2] A. V. Arkhangel’skii, *Topological Function Spaces*, Kluwer Publ., Dordrecht, 1992.

[3] R. W. Bagley E. H. Connolly, and J. D. McKnight, Jr. *On Properties Characterizing Pseudocompact Spaces*, Proc. Amer. Math. Soc. 9 (1958), 500–506.

[4] T. Berezovski, O. Gutik, and K. Pavlyk, *Brandt extensions and primitive topological inverse semigroups*, Int. J. Math. Math. Sci. 2010 (2010) Article ID 671401, 13 pages, doi:10.1155/2010/671401.

[5] J. H. Carruth, J. A. Hildebrant, and R. J. Koch, *The Theory of Topological Semigroups*, Vol. I, Marcel Dekker, Inc., New York and Basel, 1983; Vol. II, Marcel Dekker, Inc., New York and Basel, 1986.

[6] C. Chevalley and O. Frink, Jr., *Bicompleteness of Cartesian products*, Bull. Amer. Math. Soc. 47:8 (1941), 612–614.

[7] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vols. I and II, Amer. Math. Soc. Surveys 7, Providence, R.I., 1961 and 1967.

[8] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacif. J. Math. 16:3 (1966), 483–496.

[9] C. Eberhart and J. Selden, *On the closure of the bicyclic semigroup*, Trans. Amer. Math. Soc. 144 (1969), 115–126.

[10] M. Fernández and M. G. Tkachenko, *Subgroups of paratopological groups and feebly compact groups*, Appl. Gen. Topol. 15:2 (2014), 235–248.

[11] R. Engelking, *General Topology*, 2nd ed., Heldermann, Berlin, 1989.

[12] Z. Frolík, *The topological product of two pseudocompact spaces*, Czech. Math. J. 10:3 (1960), 339–349.

[13] S. L. Gulden, W. M. Fleischman and J. H. Weston, *Linearly ordered topological spaces*, Proc. Amer. Math. Soc. 24:1 (1970), 197–203.

[14] O. V. Gutik, *On Howie semigroup*, Mat. Metody Phis.-Mech. Polya. 42:4 (1999), 127–132 (in Ukrainian).

[15] O. V. Gutik and K. P. Pavlyk, *On Brandt $\lambda^0$-extensions of semigroups with zero*, Mat. Metody Phis.-Mech. Polya. 49:3 (2006), 26–40.

[16] O. V. Gutik and K. P. Pavlyk, *Pseudocompact primitive topological inverse semigroups*, Mat. Metody Phis.-Mech. Polya. 56:2 (2013), 7–19; reprinted version in: J. Math. Sc. 203:1 (2014), 1–15.

[17] O. Gutik and K. Pavlyk, *On pseudocompact topological Brandt $\lambda^0$-extensions of semitopological monoids*, Topological Algebra and its Applications 1 (2013), 60–79.

[18] O. Gutik, K. Pavlyk, and A. Reiter, *Topological semigroups of matrix units and countably compact Brandt $\lambda^0$-extensions*, Mat. Stud. 32:2 (2009), 115–131.

[19] O. V. Gutik, K. P. Pavlyk, and A. R. Reiter, *On topological Brandt semigroups*, Math. Methods and Phys.-Mech. Fields 54:2 (2011), 7–16 (in Ukrainian); English Version in: J. Math. Sc. 184:1 (2012), 1–11.

[20] O. Gutik and O. Ravsky, *On feebly compact inverse primitive (semi)topological semigroups*, Preprint (arXiv:1310.4313).

[21] O. Gutik and O. Ravsky, *On sequentially pseudocompact spaces*, Preprint.

[22] O. Gutik and D. Repovš, *On countably compact 0-simple topological inverse semigroups*, Semigroup Forum 75:2 (2007), 464–469.

[23] O. Gutik and D. Repovš, *On Brandt $\lambda^0$-extensions of monoids with zero*, Semigroup Forum 80:1 (2010), 8–32.

[24] M. Katětov, *On $H$-closed extensions of topological spaces*, Časopis Pěst. Mat. Fys. 72:1 (1947), 17–32.

[25] M. Petrich, *Inverse Semigroups*, John Wiley & Sons, New York, 1984.

[26] A. Ravsky, *Pseudocompact paratopological groups*, Preprint (arXiv:1003.5343v5).

[27] M. Sanchis and M. Tkachenko *Feebly compact paratopological groups and real-valued functions*, Monatsh. Math. 168:3 (2012), 579–597.

[28] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41:3 (1937), 375–481.

[29] W. Ruppert, *Complex Semitopological Semigroups: An Intrinsic Theory*, Lect. Notes Math., 1079, Springer, Berlin, 1984.
[30] J. E. Vaughan, *Countably compact and sequentially compact spaces*, in K. Kunen, J. E. Vaughan (eds.), Handbook of Set-Theoretic Topology, Elsevier, 1984, P. 569—602.

 Faculty of Mathematics, National University of Lviv, Universytetska 1, Lviv, 79000, Ukraine
 *E-mail address*: o_gutik@franko.lviv.ua, ovgutik@yahoo.com

 Pidstrygach Institute for Applied Problems of Mechanics and Mathematics of NASU, Naukova 3b, Lviv, 79060, Ukraine
 *E-mail address*: oravsky@mail.ru