LOCAL MARCHENKO-PASTUR LAW AT THE HARD EDGE OF SAMPLE COVARIANCE MATRICES

CLAUDIO CACCIAPUOTI, ANNA MALTSEV, AND BENJAMIN SCHLEIN

Abstract. Let $X_N$ be a $N \times N$ matrix whose entries are i.i.d. complex random variables with mean zero and variance $\frac{1}{N}$. We study the asymptotic spectral distribution of the eigenvalues of the covariance matrix $X_N^* X_N$ for $N \to \infty$. We prove that the empirical density of eigenvalues in an interval $[E, E + \eta]$ converges to the Marchenko-Pastur law locally on the optimal scale, $N \eta / \sqrt{E} \gg (\log N)^b$, and in any interval up to the hard edge, $(\log N)^b N^2 \lesssim E \leq 4 - \kappa$, for any $\kappa > 0$. As a consequence, we show the complete delocalization of the eigenvectors.

1. Introduction

Let $X$ be a $N \times M$ matrix with entries $x_{ij} = \text{Re} x_{ij} + i \text{Im} x_{ij}$. We assume that $\text{Re} x_{ij}$ and $\text{Im} x_{ij}$ are independent identically distributed real random variables with mean zero and variance $\frac{1}{2}$ so that

$$E x_{ij} = 0 \quad \text{and} \quad E |x_{ij}|^2 = 1 \quad i = 1, \ldots, N, \; j = 1, \ldots, M,$$

In what follows we shall denote by $X_N$ the scaled matrix

$$(1.1) \quad X_N = X / \sqrt{N}.$$ 

We denote by $\nu$ the probability distribution of $\text{Re} x_{ij}$ and $\text{Im} x_{ij}$. Let $s_{\alpha}$, $\alpha = 1, \ldots, N$, be the eigenvalues of $X_N^* X_N$. Since $X_N^* X_N$ is positive definite we can assume that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_N$.

Marchenko and Pastur showed in [19] the convergence of the density of the eigenvalues $s_1, \ldots, s_N$ towards the Marchenko-Pastur law

$$\rho_{MP}(E) = \frac{1}{2\pi} \sqrt{\frac{(\lambda_+ - E)(E - \lambda_-)}{E^2}}, \quad \text{whenever} \quad E \in [\lambda_-, \lambda_+], \quad \text{and} \quad 0 \text{ otherwise.}$$

In this paper, we will be interested in the case $d = 1$. In this case the Marchenko-Pastur law is supported on the interval $[0, 4]$ and is given by

$$\rho_{MP}(E) = \frac{1}{2\pi} \sqrt{\frac{(4 - E)}{E}}.$$

It has therefore a $E^{-1/2}$ singularity close to the origin $E = 0$. This reflects the fact that the typical distance between eigenvalues is of order $\sqrt{E}/N$ rather than $1/N$, as it is in the bulk; for this reason, $E = 0$ is known as the hard edge of the sample covariance matrix $X_N^* X_N$ (soft edges are instead characterized by the fact that the typical distance between neighbouring eigenvalues is larger than in the bulk). While the result of [19] determines the convergence to (1.2) on intervals of order one, containing typically order $N$ eigenvalues, in the present paper we establish the convergence of the density of states locally, on intervals containing typically a bounded number of eigenvalues, independent of $N$. In particular, we consider intervals close to the hard edge $E = 0$. As a direct consequence of the local validity of the Marchenko-Pastur law, we obtain the complete delocalization of the eigenvectors.

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associated to eigenvalues up to the edge. A further possible application of our results consists in establishing the universality of the local eigenvalue correlations close to the hard edge; this can be obtained following the receipt of [7], making use of the result of [2], in the case of complex entries, or similarly to [12, 13], using the method of the local relaxation flow, for both $X_N$ having real or complex entries. We observe, however, that the universality of the local eigenvalue correlations close to the hard edge (where they can be described in terms of the so called Bessel kernel) has already been established, using a different approach, in [22].

In the last years, a lot of progress was achieved in the spectral analysis of random matrices. Local convergence of the density of states of Wigner matrices to the semicircle law and delocalization of the eigenvectors has been established in [9, 10, 11, 15]. Universality of the local eigenvalue correlations was proven for Wigner ensembles with arbitrary symmetry (real symmetric, hermitian, or quaternion hermitian ensembles) in [12, 15]. This result was obtained by the introduction of the local relaxation flow, a flow for the eigenvalues of the Wigner matrix with the property of fast relaxation to equilibrium (and such that, locally, it remains close to the Dyson Brownian motion described by the eigenvalue when the entries are evolved by independent Brownian motions). For ensembles of hermitian Wigner matrices, universality was proven earlier in [7, 23, 8]. In all these proofs of universality, the local convergence of the density of states was a crucial ingredient. Universality at the edge of Wigner matrices was proven in [21] and more recently in [24, 1, 15, 17]. For sample covariance matrices with $0 < d < 1$, local convergence to the Marchenko-Pastur law and universality of the local eigenvalue correlations were determined in the bulk [13, 25] and at the soft edge [26, 1, 20].

More recently, local convergence of the density of states and delocalization results have also been obtained for more structured ensembles, such as the adjacency matrices of Erdős-Rényi graphs [4, 5] and band matrices [6]. In this paper, we focus on the hard edge of sample covariance matrices, proving the local convergence of the density of states to the Marchenko-Pastur law on the optimal scale (up to logarithmic corrections). As a consequence, we obtain complete delocalization of the eigenvectors associated with eigenvalues close to the hard edge.

After the completion of our work, we learned that, independently from us, Bourgade, Yau and Yin study in [3] the convergence of the density of the eigenvalues of a random matrix $X$ with no symmetry constraints towards the circular law, on optimal scales. The basic ingredient of their proof is the study of the spectrum of the hermitization $(X - z)^*(X - z)$. In particular, for $z = 0$, they obtain results similar to ours for the eigenvalues of sample covariance matrices.

An important object used in the proof of the local validity of the Marchenko-Pastur law is the Stieltjes transform defined for any $\theta \in \mathbb{C}$ with $\text{Im} \, \theta > 0$ by

\begin{equation}
\Delta_N(\theta) = \frac{1}{N} \text{Tr}(X_N^* X_N - \theta)^{-1} = \frac{1}{N} \sum_{\alpha=1}^{N} \frac{1}{s_{\alpha} - \theta}.
\end{equation}

In a similar way, one defines $\Delta(\theta)$ to be the Stieltjes transform of the Marchenko Pastur distribution. In the case $d = 1$ that will be considered in this paper

\begin{equation}
\Delta(\theta) = \int_{\mathbb{R}} \frac{1}{x - \theta} \rho(x) dx = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{\theta}}.
\end{equation}

Local convergence towards the Marchenko-Pastur law follows from the convergence of $\Delta_N$ towards $\Delta$. To simplify our analysis, we will assume that $\nu$ has subgaussian decay, i.e., that there exists $\delta_0 > 0$ such that

\begin{equation}
\int_{\mathbb{R}} e^{\delta_0 x^2} d\nu(x) < \infty.
\end{equation}

This condition is needed to apply a version of a theorem of Hanson and Wright as formulated in [11, Prop. 4.5], see also Proposition 2.2 below. At the price of getting weaker convergence rates, this assumption can be substantially relaxed (existence of sufficiently high moments is sufficient). Furthermore, we assume that probability density function of the real and imaginary parts of the
entries is bounded; this will simplify the study of the eigenvalues located very close to the origin. Also in this case, improvements are certainly possible.

Our first result is a proof of a bound on the number of eigenvalues \( s_\alpha \) in a window \( I = [E, E + \eta] \), valid up to the hard edge and for small \( \eta \) (s.t. \( N\eta/\sqrt{E} \gg (\log N)^b \), \( b > 2 \)). The proof of the following theorem can be found in Section 3.

**Theorem 1.** Let \( X_N \) be a \( N \times N \) matrix as described in (1.1), whose entries satisfy (1.2). Let \( I = [E, E + \eta] \) with \( N\eta/\sqrt{E} \geq (\log N)^b \), for some \( b > 2 \). Denote by \( N_I \) the number of eigenvalues of \( X_N^*X_N \) in \( I \). Then there exist constants \( c, C, K_0 > 0 \) such that, for any \( K \geq K_0 \) and \( N \) large enough,

\[
\mathbb{P} \left( N_I \geq K \frac{N\eta}{\sqrt{E}} \right) \leq e^{-c \sqrt{N \frac{\eta \sqrt{E}}{N}}}.
\]

Using the a priori bound in Theorem 1, we prove the convergence of the Stieltjes transform \( \Delta_N(E + i\eta) \) of the sample covariance matrices towards the Stieltjes transform \( \Delta(E + i\eta) \) of the Marchenko-Pastur law, up to the hard edge, and close to the real axis.

**Theorem 2.** Let \( X_N \) be a \( N \times N \) matrix as described in (1.1), whose entries satisfy (1.2). Assume moreover that the probability density function of the real and imaginary part of the entries is bounded. Moreover, set \( \theta = E + i\eta \), with \( E \leq 4 - \kappa \), \( 0 < \eta < \kappa E \), \( N\eta/\sqrt{E} \geq (\log N)^b \), for some \( b > 4 \) and \( 0 < \kappa < 1 \) (these bounds also imply that \( E \geq (\log N)^{2b}/\kappa^2 N^2 \)). Then there exist \( \varepsilon_0 > 0 \), \( c > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \) and \( N \) large enough,

\[
\mathbb{P} \left( \left| \Delta_N(\theta) - \Delta(\theta) \right| \geq \frac{\varepsilon}{\sqrt{E}} \right) \leq e^{-c \sqrt{N \frac{\eta \sqrt{E}}{N}}} + e^{-c (\log N)^{b/4}}.
\]

The proof of this theorem is in Section 5. The convergence of the Stieltjes transform immediately implies the convergence of the density of states.

**Theorem 3.** Let \( X_N \) be a \( N \times N \) matrix as described in (1.1), whose entries satisfy (1.2). Assume moreover that the probability density function of the real and imaginary part of the entries is bounded. Suppose \( E \leq 4 - \kappa \), \( 0 < \eta < \kappa E \), \( N\eta/\sqrt{E} \geq (\log N)^b \), for some \( b > 4 \) and \( 0 < \kappa < 1 \), and let \( I = [E, E + \eta] \). Then there exist \( \varepsilon_0 > 0 \), \( C, c > 0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \) and \( N \) large enough,

\[
\mathbb{P} \left( \left| \frac{N_I}{N\eta} - \frac{1}{\eta} \int_{E}^{E+\eta} \text{d} s \rho(s) \right| \geq \frac{\varepsilon}{\sqrt{E}} \right) \leq C e^{-c \sqrt{N \frac{\eta \sqrt{E}}{N}}} + C e^{-c (\log N)^{b/4}}.
\]

The proof of Theorem 2 can be obtained from Theorem 1 similarly as in [9, Cor. 4.2]. Finally, Theorem 3 implies complete delocalization of the normalized eigenvectors of \( X_N^*X_N \) associated with eigenvalues in the window \( \left[ \frac{(\log N)^b}{\kappa^2 N^2}, 4 - \kappa \right) \), for any \( \kappa > 0 \).

**Theorem 4** (Delocalization). Let \( X_N \) be a \( N \times N \) matrix as described in (1.1), whose entries satisfy (1.2). Assume moreover that the probability density function of the real and imaginary part of the entries is bounded. Fix \( 0 < \kappa < 1 \), \( b > 4 \). Then there exist constants \( c, C > 0 \) such that and for \( N \) large enough,

\[
\mathbb{P} \left( \exists \ u \text{ s.t. } X_N^*X_N u = s u, \ |u| = 1, \ s \in \left[ \frac{(\log N)^b}{\kappa^2 N^2}, 4 - \kappa \right), \text{ and } |u|_{\infty} \geq C \frac{(\log N)^{b/2}}{N^{b/2}} \right) \leq e^{-c (\log N)^{b/4}}.
\]

2. Basic definitions and results

In this section we collect several definitions and results which will be used to prove the main theorems.
2.1. A formula for $\Delta_N$. The proofs of Theorems 1 and 2 rely on the following formula for the diagonal components of the resolvent $(X_N X_N - \theta)^{-1}$ (see [13])

$$
((X_N X_N - \theta)^{-1})_{kk} = \frac{1}{|w_k|^2 - \theta - w_k^* W_k (W_k W_k^* - \theta)^{-1} W_k^* w_k}
$$

(2.1) 

(2.1) 

$$
= - \frac{1}{\theta \left( 1 + w_k^* (W_k W_k^* - \theta)^{-1} w_k \right)}.
$$

where $w_k = x_k/\sqrt{N}$ is the $k$-th column of the matrix $X_N$ and $W_k$ denotes the $N \times (N-1)$ matrix obtained by removing the $k$-th column from the matrix $X_N$ (notice that $W_k^* W_k$ is the $(N-1) \times (N-1)$ minor of $X_N X_N$, obtained by removing the $k$-th row and the $k$-th column). We used here the well-known identity

$$
W_k (W_k^* W_k - \theta)^{-1} W_k^* = W_k^* W_k (W_k W_k^* - \theta)^{-1}
$$

valid for $\text{Im} \, \theta \neq 0$, which can be proved using the Neumann expansion of the resolvent. Eq. (2.1) gives the following formula for the Stieltjes transform $\Delta_N(\theta)$:

$$
\Delta_N(\theta) = -\frac{1}{N} \sum_{k=1}^N \frac{1}{\theta \left( 1 + w_k^* (W_k W_k^* - \theta)^{-1} w_k \right)}.
$$

2.2. Properties of $\Delta$. We collect here some properties of the Stieltjes transform $\Delta(\theta)$ of the Marchenko-Pastur distribution $\rho_{MP}$, defined by

$$
\Delta(\theta) = \int_{\mathbb{R}} \frac{1}{x - \theta} \rho_{MP}(x) = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{\theta}}
$$

where we use the branch of the square root with $\text{Re} \sqrt{1 - 4/\theta} \geq 0$. We use the fixed point equation

$$
\theta(\Delta(\theta) + 1) = -\frac{1}{\Delta(\theta)}.
$$

Lemma 2.1. Let $\theta = E + i\eta$, with $\eta > 0$. Then

$$
|\Delta(\theta)|^2 \leq \frac{1}{E} \quad \text{and} \quad |1 + \Delta(\theta)|^2 \geq \max \left\{ \frac{E}{E^2 + \eta^2}, \frac{1}{4} \right\}
$$

Moreover

$$
\text{Im} \, \Delta(\theta) \geq C \frac{|E - 4|^{1/2} + \eta^{1/2}}{(E^2 + \eta^2)^{1/4}}
$$

if $E^2 + \eta^2 \leq 4E$ (this condition defines a circle of radius 2 around $(E, \eta) = (2, 0)$).

Proof. From (2.2), taking the imaginary part, we get

$$
\text{Im} \, \Delta(1 - |\Delta|^2 E) = \eta \text{Re} (1 + \Delta)|\Delta|^2.
$$

Eq. (2.2) implies that $\text{Re}(1 + \Delta) > 0$. Since $\text{Im} \, \Delta(E + i\eta) > 0$ for $\eta > 0$, together with (2.6), this implies the first bound in (2.4). To get the second bound in (2.4) we first notice that by (2.2), $\text{Re}(1 + \Delta) > 1/2$. This implies immediately that $|1 + \Delta|^2 > 1/4$. The bound $|1 + \Delta|^2 > E/(E^2 + \eta^2)$ follows instead from (2.3), combined with $|\Delta|^2 \leq 1/E$.

To show (2.5), we observe that, from (2.2),

$$
\text{Im} \, \Delta = \frac{1}{2} \text{Im} \sqrt{1 - \frac{4}{\theta} \geq \frac{1}{4} \sqrt{1 - \frac{4}{\theta}} \geq C \frac{|E - 4|^{1/2} + \eta^{1/2}}{(E^2 + \eta^2)^{1/4}}}
$$

under the assumption that $E^2 + \eta^2 < 4E$. Here we used the fact that $\text{Im} \, \sqrt{z} \geq |z|^{1/2}/\sqrt{2}$, if $\text{Re} \, z \leq 0$ and $\text{Im} \, z \geq 0$. □
2.3. Large deviations of quadratic forms. We will make use of the following inequality for the fluctuations of quadratic forms, due to Hanson and Wright. For the proof of the next proposition we refer to [11], Prop. 4.5, see also [13] App. B and [10].

Proposition 2.2. For \( j = 1, \ldots, N \) let \( x_j = Re \, x_j + i Im \, x_j \), where \( \{Re \, x_j, Im \, x_j\}_{j=1}^N \) is a sequence of \( 2N \) real iid random variables, whose common distribution satisfies (1.5). Let \( A = (a_{ij}) \) be a \( N \times N \) complex matrix. Then there exist constants \( c, C > 0 \) such that, for any \( \delta > 0 \)

\[
\mathbb{P} \left( \sum_{i,j=1}^N a_{ij} (x_i \bar{x}_j - \mathbb{E} x_i \bar{x}_j) \geq \delta \right) \leq C e^{-c \min\{\delta/\sqrt{\mathbb{E} A^*A}, \delta^2/\mathbb{E} A^*A\}},
\]

The following proposition is a consequence of the Hanson-Wright inequality. Its proof can be found, for example, in [11].

Proposition 2.3. Let \( \{v_\alpha\}_{\alpha=1}^m \) be a set of \( m \) orthonormal vectors in \( \mathbb{C}^N \) and \( x = (x_1, \ldots, x_N) \in \mathbb{C}^N \), where \( \{Re \, x_j, Im \, x_j\}_{j=1}^N \) are \( 2N \) real iid random variables satisfying (1.5) such that \( \mathbb{E} x_j = 0 \), and \( \mathbb{E} |x_j|^2 = 1 \). Then there exist two constants \( c, C > 0 \) such that

\[
\mathbb{P} \left( \sum_{\alpha=1}^m |x \cdot v_\alpha|^2 \leq \frac{m}{2} \right) \leq C e^{-c \sqrt{m}}.
\]

3. Upper bound for the number of eigenvalues: Proof of Theorem 1

Recall that \( I = [E, E + \eta] \), and that \( \mathcal{N}_I \) denotes the number of eigenvalues of the matrix \( X_N^*X_N \) in \( I \). We have

\[
\mathcal{N}_I = \sum_{\alpha=1}^N 1(s_\alpha \in [E, E + \eta]) \leq C \sum_{\alpha=1}^N \frac{\eta^2}{(s_\alpha - E)^2 + \eta^2} = C \eta \text{Im} \sum_{\alpha=1}^N \frac{1}{s_\alpha - E - i\eta}
\]

\[
= C \eta \text{Im} \text{Tr} (X_N^*X_N - E - i\eta)^{-1} = C \eta \text{Im} \sum_{k=1}^N (X_N^*X_N - E - i\eta)^{-1}_{kk}
\]

\[
= -C \eta \text{Im} \sum_{k=1}^N \frac{1}{\theta + (W_k^*W_k - \theta)^{-1} w_k}.
\]

where we put \( \theta = E + i\eta \) and we used (2.3). Using the spectral decomposition of \( W_k^*W_k \), we find

\[
\mathcal{N}_I \leq -C \eta \text{Im} \sum_{k=1}^N \frac{1}{\theta - \sum_{\alpha=0}^{N-1} |w_k \cdot v_\alpha^{(k)}|^2 + \sum_{\alpha=1}^{N-1} \frac{s_{\alpha}^{(k)} |w_k \cdot v_\alpha^{(k)}|^2}{s_{\alpha}^{(k)} - \theta}}
\]

\[
\leq C \sum_{k=1}^N \frac{1}{\theta - \sum_{\alpha=0}^{N-1} |w_k \cdot v_\alpha^{(k)}|^2 + \sum_{\alpha=1}^{N-1} \frac{s_{\alpha}^{(k)} |w_k \cdot v_\alpha^{(k)}|^2}{s_{\alpha}^{(k)} - \theta}}
\]

\[
\leq C N^2 \frac{\eta^2}{E} \sum_{\alpha=1}^N \sum_{k=1}^N \sum_{s_{\alpha}^{(k)} \in [E, E + \eta]} N |w_k \cdot v_\alpha^{(k)}|^2
\]

where, in the second inequality, we used the fact that \( |\text{Im} \, (1/z)| \leq 1/|\text{Im} \, z| \). Setting \( K = 2C^{1/2} \), it follows that

\[
\mathcal{N}_I \leq K \frac{N \eta}{\sqrt{E}}
\]

unless there exists \( k \in \{1, \ldots, N\} \), with

\[
\sum_{\alpha: s_{\alpha}^{(k)} \in [E, E + \eta]} N |w_k \cdot v_\alpha^{(k)}|^2 \leq \frac{\mathcal{N}_I}{4}
\]
In other words,
\[ P \left( N_I \geq K \frac{N\eta}{\sqrt{E}} \right) = P \left( N_I \geq K \frac{N\eta}{\sqrt{E}} \text{ and } \exists k \in \{1, \ldots, N\} \text{ with } \sum_{\alpha : s_{\alpha}^{(k)} \in [E, E + \eta]} N |w_k \cdot v_{\alpha}^{(k)}|^2 \leq \frac{N_I}{4} \right) \]

Since \( W_k^* W_k \) is a minor of \( X_N^* X_N \), it follows that its eigenvalues are interlaced between the eigenvalues of \( X_N^* X_N \). This implies that
\[ \left| \{ \alpha : s_{\alpha}^{(k)} \in [E, E + \eta] \} \right| \geq N_I - 1 \geq \frac{N_I}{2} \]
on the event we consider. Proposition 2.3 (applied with \( E = 1, \eta = 0 \)) implies therefore that
\[ P \left( N_I \geq K \frac{N\eta}{\sqrt{E}} \right) \leq C N e^{-c \sqrt{\frac{KN\eta}{\sqrt{E}}}} \leq C e^{-c \sqrt{\frac{KN\eta}{\sqrt{E}}}} \]
after adjusting the constants. This concludes the proof of Theorem 1.

4. AN ESTIMATE FOR THE NUMBER OF EIGENVALUES CLOSE TO ZERO

In this section, we show that, with high probability, there cannot be too many eigenvalues at distances smaller than \( 1/N^2 \) from 0. To this end, we need the boundedness of the probability density function of the entries. We use here the notation \( N[a, b] \) to indicate the number of eigenvalues in \( [a, b] \).

**Proposition 4.1.** Let \( X_N = (x_{ij}/\sqrt{N}) \) be a \( N \times N \) matrix as described in equation (1.1). Assume that the probability density function \( h(x) \) of \( \text{Re} \ x_{ij} \) and \( \text{Im} \ x_{ij} \) is bounded. Then there exists a constant \( C, c > 0 \) such that
\[ P \left( N \left[ 0, K/N^2 \right] \geq L \right) \leq C e^{-L} \]
for all \( L > cK \).

**Proof.** We start with the observation that
\[ P(N[0, K/N^2] \geq L) = P((N[0, K/N^2]/L)^p \geq 1) \leq E(N[0, K/N^2]/L)^p \]
Next, we notice that
\[ N[0, K/N^2] \leq \frac{CK}{N^2} \sum_{k=1}^{N} \text{Im} (X_N^* X_N - \theta)^{-1}_{kk} \]
where we set \( \theta = KN^{-2} + iK N^{-2} \). This implies that
\[ \left( \frac{N[0, K/N^2]}{L} \right)^p \leq \left( \frac{CK}{NL} \right)^p \left( \frac{1}{N} \sum_{k=1}^{N} \text{Im} (X_N^* X_N - \theta)^{-1}_{kk} \right)^p \leq \left( \frac{CK}{NL} \right)^p \left( \frac{1}{N} \sum_{k=1}^{N} (\text{Im} (X_N^* X_N - \theta)^{-1}_{kk})^p \right) \]
by H"older inequality. Hence
\[ P(N[0, K/N^2] \geq L) \leq \left( \frac{CK}{NL} \right)^p E(\text{Im} (X_N^* X_N - \theta)^{-1})^p \]
\[ \leq \left( \frac{CK}{L} \right)^p E \left| \frac{1}{\theta + \theta \sum_{\alpha=0}^{N-1} \frac{|w_{\alpha} \cdot v_{\alpha}^{(1)}|^2}{s_{\alpha}^{(1)} - \theta}} \right|^p \]
\[ \leq \left( \frac{CK}{L} \right)^p E \left| \frac{1}{\sum_{\alpha=0}^{N-1} c_{\alpha} |x_{1} \cdot v_{\alpha}^{(1)}|^2} \right|^p \]
where we neglected the real part of the denominator and we defined \( x_1 = \sqrt{N}w_1 \) (it is a vector in \( \mathbb{C}^N \), whose components have iid real and imaginary parts with zero mean and variance 1/2), and

\[
(4.2) \quad c_\alpha = \frac{1}{(N^2 s_\alpha^{(1)}/K - 1)^2 + 1}
\]

We recall that the eigenvalues \( s_\alpha \) and \( s_\alpha^{(1)} \) are ordered in increasing order. On the event \( \mathcal{N}[0, K/N^2] \geq L \), the interlacing property implies that at least \( L - 1 \) eigenvalues of the minor are in the interval \([0, K/N^2]\); i.e. \( s_\alpha^{(1)} \in [0, K/N^2] \) for \( \alpha = 0, \ldots, L - 1 \). This implies that \( c_\alpha \geq 1/2 \) for all \( \alpha = 1, \ldots, L - 1 \).

Therefore

\[
\mathbb{P}(\mathcal{N}[0, K/N^2] \geq L) \leq \left(\frac{2CK}{L}\right)^p \mathbb{E} \left( \frac{1}{\sum_{\alpha=0}^{L-1} |x_1 \cdot v_\alpha^{(1)}|^2} \right)^p
\]

Next, we take \( p = (L - 2)/2 \). Since the matrix entries are assumed to have a bounded probability density function, Lemma A.1 of [18] implies that

\[
\mathbb{E}_{x_1} \left( \sum_{\alpha=0}^{L-1} \frac{1}{|x_1 \cdot v_\alpha^{(1)}|^2} \right)^{(L-2)/2} \leq C
\]

for \( C > 0 \) independent of \( L \). This concludes the proof of the proposition. \( \square \)

5. Convergence of the Stieltjes transform: Proof of Theorem 2

We start from the formula (2.1), rewritten as

(5.1) \[
\Delta_N = -\frac{1}{N} \sum_{k=1}^{N} \theta \frac{1}{1 + \Delta_N + \frac{\Omega_k}{\sqrt{E}}}
\]

which immediately implies that

(5.2) \[
\Delta_N + \frac{1}{\theta(\Delta_N + 1)} = -\frac{1}{N} \sum_{k=1}^{N} \frac{\Omega_k/\sqrt{E}}{\theta(1 + \Delta_N + \Omega_k/\sqrt{E})(1 + \Delta_N)}
\]

Here we defined the error terms

\[
\Omega_k = \sqrt{E} \left( w_k^* (W_k^* W_k - \theta)^{-1} w_k - \frac{1}{N} \text{Tr} (W_k^* W_k - \theta)^{-1} \right) \]

\[
+ \sqrt{E} \left( \frac{1}{N} \text{Tr} (W_k^* W_k - \theta)^{-1} - \frac{1}{N} \text{Tr} (X_N^* X_N - \theta)^{-1} \right)
\]

Observe that, with probability one,

(5.3) \[
\sqrt{E} \left| \frac{1}{N} \text{Tr}(X_N^* X_N - \theta)^{-1} - \frac{1}{N} \text{Tr}(W_k^* W_k - \theta)^{-1} \right| \leq C \sqrt{E}. \frac{N}{N^{3/2}}
\]

This follows from

\[
\frac{1}{N} \text{Tr}(W_k^* W_k - \theta)^{-1} = -\frac{1}{N^\theta} + \frac{1}{N} \text{Tr}(W_k^* W_k - \theta)^{-1}
\]

and from the interlacing of the eigenvalues of \( W_k^* W_k \) between the eigenvalues of \( X_N^* X_N \). To estimate the first difference in the error \( \Omega_k \), we notice that

\[
\mathbb{E}_{w_k} w_k^* (W_k^* W_k - \theta)^{-1} w_k = \frac{1}{N} \text{Tr} (W_k^* W_k - \theta)^{-1}
\]

Therefore, defining the matrix \( A = (a_{ij}) \), with

\[
a_{ij} = \sqrt{E} \frac{\sum_{\alpha} \overline{v}_\alpha(i) v_\alpha(j)}{s_\alpha - \theta}
\]
and the vector \( x = \sqrt{N} w_k = (x_1, \ldots, x_N) \) (this is a vector in \( \mathbb{C}^N \), whose components are order one random variables), we find
\[
\sqrt{E} \left( w_k^* (W_k W_k^* - \theta)^{-1} w_k - \frac{1}{N} \text{Tr} (W_k W_k^* - \theta)^{-1} \right) = \sum_{ij} a_{ij} (x_i \overline{x}_j - \mathbb{E}_x x_i \overline{x}_j)
\]
and therefore (taking into account also (5.3))
\[
P(|\Omega_k| \geq \varepsilon) \leq \mathbb{P} \left( \sum_{ij} a_{ij} (x_i \overline{x}_j - \mathbb{E}_x x_i \overline{x}_j) \geq \varepsilon \right)
\]
Observe that
\[
\text{Tr} A^* A = \frac{E}{N^2} \sum_{\alpha} \frac{1}{|s_{\alpha}^{(k)} - \theta|^2}
\]
In Lemma 5.1, we show that, up to an event with probability at most \( e^{-c(\log N)^{b/4}} \)
\[
\text{Tr} A^* A \leq C \frac{\sqrt{E}}{N \eta}
\]
Therefore, Proposition 2.2 implies that
\[
P(|\Omega_k| \geq \varepsilon) \leq C e^{-c(\log N)^{b/4}} + C e^{-c\varepsilon \sqrt{\frac{N^2}{E}}}
\]
and thus
\[
(5.4) \quad P \left( \max_{k=1,\ldots,N} |\Omega_k| \geq \varepsilon \right) \leq C e^{-c(\log N)^{b/4}} + C e^{-c\varepsilon \sqrt{\frac{N^2}{E}}}
\]
after adjusting the constants. We restrict now our attention to the event \(|\Omega_k| \leq \varepsilon\) for all \( k = 1, \ldots, N \). To complete the proof of Theorem 2, we use a continuity argument. We fix \( \kappa > 0 \), and consider \( \theta = E + i \eta \in \mathbb{C} \) with \( 0 < \eta < \kappa E \). We connect \( \theta \) with the point \( \theta_0 = 2 + 2i \kappa \). Let \( L \) denote the line segment connecting \( \theta \) and \( \theta_0 \). Note that, on \( L \), \( (E - 2)^2 + \eta^2 \leq 4 \) always holds. Hence, Lemma 2.1 implies that, on \( L \), \( \text{Im} \Delta > C_0/\sqrt{E} \) for a constant \( C_0 \) depending only on \( \kappa \) (\( C_0 \) can be chosen as \( \text{const} \cdot \kappa^{1/2}/(1 + \kappa^2)^{1/4} \)).

We claim now that, if \(|\Delta_N - \Delta| \leq C_0/(2\sqrt{E}) \) somewhere on \( L \), then \(|\Delta_N - \Delta| \leq C_\varepsilon/\sqrt{E} \) where \( C \) depends only on \( \kappa \). In fact, \( \text{Im} \Delta > C_0/\sqrt{E} \) and \( |\Delta_N - \Delta| \leq C_0/(2\sqrt{E}) \) imply that \( \text{Im} \Delta_N > C_0/(2\sqrt{E}) \). This implies (for \( \varepsilon < C_0/4 \)) that \( \text{Im} (\Delta_N + (\Omega_k/\sqrt{E})) \geq C_0/(4\sqrt{E}) \). Hence (5.2) gives
\[
|\Delta_N + \frac{1}{\theta(\Delta_N + 1)}| \leq \frac{8\varepsilon}{C_0^2 \sqrt{E}}
\]
Subtracting the fixed point equation \( \Delta + 1/(\theta(\Delta + 1)) = 0 \), we find
\[
(|\Delta_N - \Delta| \left(1 + \frac{\Delta}{\Delta_N + 1}\right) \leq \frac{8\varepsilon}{C_0^2 \sqrt{E}}
\]
Therefore
\[
|\Delta_N - \Delta| \leq \frac{8\varepsilon}{C_0^2 \sqrt{E}} \left| \frac{|\Delta_N + 1|}{\Delta_N + 1} \right|
\]
\[
\leq \frac{8\varepsilon}{C_0^2 \sqrt{E}} \left[ 1(|\Delta_N + 1| > 2|\Delta|)\frac{|\Delta_N + 1|}{\Delta_N + 1} + 1(|\Delta_N + 1| < 2|\Delta|)\frac{|\Delta_N + 1|}{\Delta_N + 1} \right]
\]
\[
\leq \frac{16\varepsilon}{C_0^2 \sqrt{E}} \left(1 + \frac{1}{C_0} \right)
\]
where we used that, on \( L \), \( \text{Im} \Delta > C_0/\sqrt{E} \) and \( |\Delta| \leq 1/\sqrt{E} \) (see Lemma 2.1). Theorem 2 follows because, from (19), \( |\Delta_N(2 + 2i\kappa) - \Delta(2 + 2i\kappa)| \leq C_0/(2\sqrt{2}) \) for \( N \) large enough. This completes the proof of Theorem 2.
Lemma 5.1. Let $X_N = (x_{ij}/\sqrt{N})$ be an $N \times M$ matrix as defined in (1.1). Denote by $s_\alpha$ the eigenvalues of $X_N^*X_N$. Assume the real and imaginary part of the entries $x_{ij}$ are iid random variables with a common bounded probability density function. Let $\theta = E + i\eta$, with $\eta \leq E$ and $N\eta/\sqrt{E} \geq (\log N)^b$. Then there exist constants $C_1, C_2, c > 0$ with

$$\sum_{k=2}^{k_0} e^{-c\sqrt{2-k/2N}\sqrt{E}} \leq k_0 e^{-c(\log N)^{b/4}} \leq e^{-(c/2)(\log N)^{b/4}}$$

We have

$$\sum_{k=2}^{k_0} e^{-c\sqrt{2-k/2N}\sqrt{E}} \leq k_0 e^{-c(\log N)^{b/4}} \leq e^{-(c/2)(\log N)^{b/4}}$$

for appropriate constants $C, c > 0$. Next, we consider the term II. We have

$$\frac{E}{N^2} \sum_{\alpha:s_\alpha \in[(\log N)^b/N^2,E/2]} \left| s_\alpha - \theta \right|^2 \leq \frac{E}{N^2} \sum_{k=2}^{k_0} \frac{N[2^{-k}E,2^{-k+1}E]}{E^2}$$

where $k_0 > 0$ is chosen as the smallest integer with $2^{-k_0}E \leq (\log N)^b/N^2$. From Theorem I, it follows that, for a sufficiently large $K > 0$,

$$N[2^{-k}E,2^{-k+1}E] \leq CN2^{-k/2}\sqrt{E}$$

up to an event of probability at most $\exp(-c(2^{-k/2}N\sqrt{E})^{1/2})$. This implies that, apart from an event of total probability bounded by

$$\sum_{k=2}^{k_0} e^{-c\sqrt{2-k/2N}\sqrt{E}} \leq k_0 e^{-c(\log N)^{b/4}} \leq e^{-(c/2)(\log N)^{b/4}}$$

we have

$$\frac{E}{N^2} \sum_{\alpha:s_\alpha \in[(\log N)^b/N^2,E/2]} \left| s_\alpha - \theta \right|^2 \leq C \frac{E}{N^2} \sum_{k=2}^{k_0} \frac{2^{-k/2}N\sqrt{E}}{E^2} \leq C \frac{1}{N\sqrt{E}} \leq C \frac{\sqrt{E}}{\eta}$$

where we used the assumption $\eta \leq E$. Finally, we have to control the term III. To this end, we observe that

$$\frac{E}{N^2} \sum_{\alpha:s_\alpha > E/2} \frac{1}{\left| s_\alpha - \theta \right|^2} \leq \frac{E}{N^2} \sum_{\alpha:s_\alpha > E/2} \frac{1}{(s_\alpha - E)^2 + \eta^2} \leq \frac{E}{N^2} \sum_{k=0}^{\mathcal{N}_k} \frac{N_l}{2^{2k}\eta^2}$$
where we set $I_k = J_k \cap [E/2, \infty)$, with $J_k = [E - 2^k \eta, E - 2^{k-1} \eta] \cup [E + 2^{k-1} \eta, E + 2^k \eta]$ for $k \geq 1$ and $J_0 = [E - \eta, E + \eta]$. Observe that, by Theorem 1

$$N I_k \leq C 2^k N \eta$$

up to an event with probability at most

$$e^{-c \sqrt{\frac{2^k N \eta}{N^2}}} \leq e^{-c \sqrt{\frac{N \eta}{N^2}}} \leq e^{-c (\log N)^{b/4}}$$

Therefore,

$$\frac{E}{N^2} \sum_{\alpha, s_\alpha > E/2} \frac{1}{s_\alpha - \theta} \leq \frac{E}{N^2} \sum_{k \geq 0} 2^k N \eta \leq C \frac{\sqrt{E}}{N \eta}$$

apart from an event with probability at most $e^{-c (\log N)^{b/4}}$. This completes the proof of the lemma. \(\square\)

6. Delocalization: Proof of Theorem 4

Denote by $\{s_\alpha\}_{\alpha=1}^N$ the eigenvalues of the matrix $X_N^* X_N$ with $s_1 \leq ... \leq s_N$ and by $\{u_\alpha\}_{\alpha=1}^N$ the corresponding set of orthonormal eigenvalues. From the equation $X_N^* X_N u_\alpha = s_\alpha u_\alpha$ and the condition $\|u_\alpha\|^2 = 1$ it follows that (see also [25, Cor. 25] and [10, 9])

$$|u_\alpha(k)|^2 = \frac{1}{1 + \frac{1}{N} \sum_{\beta=1}^{N-1} s_\beta^{(k)} u_\alpha^{(k)} \cdot x_\beta|^2}$$

where $x_k = \sqrt{N} w_k$ and $w_k$ denotes the $k$-th column of the matrix $X_N$, while $s_\beta^{(k)}$ and $u_\alpha^{(k)}$ are the eigenvalues and the corresponding eigenvectors of the matrix $W_k^* W_k$, where the matrix $W_k$ is obtained by removing the $k$-th column from the matrix $X_N$. For arbitrary $\eta > 0$, we have

$$|u_\alpha(k)|^2 \leq \frac{N \eta^2}{s_\alpha} \sum_{\beta, s_\beta^{(k)} \in [s_\alpha, s_\alpha + \eta]} |u_\beta^{(k)} \cdot x_k|^2$$

Taking $\eta = \sqrt{\frac{b (\log N)^{b/4}}{N}}$, Theorem 4 implies that

$$\left\{ s_\beta^{(k)} \in [s_\alpha, s_\alpha + \eta] \right\} \geq C (\log N)^b$$

up to an event with probability smaller than $e^{-c (\log N)^b}$. Prop. 2.3 implies then that

$$\sum_{\beta, s_\beta^{(k)} \in [s_\alpha, s_\alpha + \eta]} |u_\beta^{(k)} \cdot x_k|^2 \geq \frac{(\log N)^b}{2}$$

apart from an event with probability smaller than $e^{-c (\log N)^b}$. This implies that

$$\mathbb{P} \left( |u_\alpha(k)|^2 \geq \frac{(\log N)^b}{N} \right) \leq C e^{-c (\log N)^b}$$

Taking the maximum over $k$, Theorem 4 follows.

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Hausdorff Center for Mathematics, Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: CACCIAPUOTI@HCM.UNI-BONN.DE
E-mail address: ANNAVMALTSEV@GMAIL.COM
E-mail address: BENJAMINSCHLEIN@HCM.UNI-BONN.DE