Non-axisymmetric instability of density-stratified Taylor-Couette flow

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Abstract. The stability of density-stratified viscous Taylor-Couette flows is considered. We find that the nonaxisymmetric modes can be the most unstable ones. The nonaxisymmetric modes are unstable also beyond the Rayleigh line. For such modes the instability condition seems simply to be such that angular velocity should decrease with radius as stressed by Yavneh, McWilliams and Molemaker (2001). However, we never found unstable modes for too flat rotation laws fulfilling the condition that linear azimuthal velocity increase with radius. The Reynolds numbers rapidly grow to very high values if this limit is approached. The marginal stability lines for the higher azimuthal numbers do less and less enter the region beyond the Rayleigh line so that we might have to consider the stratorotational instability as a 'low-m instability'. The results application to the accretion disk stability with their strong density stratification and fast rotation is shortly discussed.

1. Introduction

The flow pattern between concentric rotating cylinders with a stable axial density stratification was firstly studied by Thorpe, [1], who concluded that stable stratification stabilizes the flow. The further experimental and theoretical studies [2,3] confirmed the stabilizing role of the density-stratification and showed that i) the critical Reynolds number depends on the buoyancy frequency (or Brunt-Väisälä frequency) of the fluid and ii) the stratification reduces the vertical extension of the Taylor vortices. The computational results [4] have indeed reproduced the experiment results.

Recently, the another condition have been found [6,7]

\[
\frac{d}{dR} \left( R^4 \Omega^2 \right) < 0,
\]

where \( \Omega \) is the angular velocity of the flow. Recently, the another condition have been found [6,7]

\[
\frac{d\Omega^2}{dR} < 0
\]

as the sufficient condition for (nonaxisymmetric) instability. The condition (2) is identical with the condition for magnetorotational instability of Taylor-Couette flow [8]. These results have been
derived by a linear stability analysis for inviscid flow. The numerical results demonstrate, [7], the existence of the hydrodynamic instability also for finite viscosity.

![Figure 1. The geometry of density-stratified Taylor-Couette experiments.](image)

The angular velocity of the Taylor-Couette flow is an arbitrary radius function for ideal flow but a fixed radius function (see (9) and (10)) for viscous flow. The Rayleigh line separates unstable flow region from the stable one. For angular velocity given by (9) and (10) the Rayleigh line is defined by equality \( \hat{\mu} = \hat{\eta}^2 \) (see (11)) according to condition (1).

An instability of density-stratified Taylor-Couette flow beyond the Rayleigh line for nonaxisymmetric disturbances has also been found experimentally [9]. With their wide gap container (\( \hat{\eta} = 0.2 \)) they found for \( \hat{\mu} > 0 \) the stability curve crossing the classical Rayleigh line. The observed instability was reported as nonaxisymmetric. The resulting experimental stability line, however, is very steep for positive \( \hat{\mu} \) (see their Figure 8) and never crosses the line \( \hat{\mu} = \hat{\eta} \) which means that not angular but azimuthal velocity, \( u_\phi = R \Omega \), should decrease with radius to instability

\[
\frac{d(R\Omega)^2}{dR} < 0.
\]

For real viscous flows there are very illustrative results [7]. In the present paper a more comprehensive study of such flows is given. The governing equations and the restrictions of the used Boussinesq approximation are discussed in section 2 while the numerical results are presented in section 3. The summary and final discussion are given in section 4.

2. Equation and basic state

In cylindrical coordinates \((R, \phi, z)\) the equations of incompressible stratified fluid with uniform dynamic viscosity, \( \mu \), are

\[
\frac{\partial u_R}{\partial t} + (u \nabla) u_R - \frac{u_\phi^2}{R} = - \frac{1}{\rho} \frac{\partial P}{\partial R} + \nu \left[ \Delta u_R - \frac{2}{R^2} \frac{\partial u_\phi}{\partial \phi} - \frac{u_R}{R^2} \right],
\]
\[
\frac{\partial u_\phi}{\partial t} + (u \nabla)u_\phi + \frac{u_\phi u_R}{R} = -\frac{1}{\rho R} \frac{\partial P}{\partial \phi} + \nu \left[ \frac{\partial^2 u_\phi}{\partial R^2} + \frac{2}{R^2} \frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2} \right],
\]
\[
\frac{\partial u_z}{\partial t} + (u \nabla)u_z = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g + \nu \Delta u_z,
\]
\[
\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{\rho} \frac{\partial u_R}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0,
\]

where
\[
(u \nabla)u_R = u_R \frac{\partial u_R}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_R}{\partial \phi} + u_z \frac{\partial u_R}{\partial z}
\]

and
\[
\Delta u_R = \frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R} \frac{\partial u_R}{\partial R} + \frac{1}{\rho} \frac{\partial^2 u_R}{\partial \phi^2} + \frac{\partial^2 u_R}{\partial z^2},
\]

\( \rho \) is the density, \( P \) is the pressure, \( g \) is the gravity, and \( \nu = \mu/\rho \) is the kinematic viscosity (the density diffusion term in the mass conservation equation is neglected (see e.g. [3]). The equation which describes the evolution of the density fluctuation moving in the general density field is
\[
\frac{\partial \rho}{\partial t} + (u \nabla)\rho = 0.
\]

We have to formulate the basic state with prescribed velocity profile, \( u = (0, R \Omega(R,0), 0) \) and given density vertical stratification \( \rho = \rho(z) \). The system (4) takes the form
\[
\frac{u_\phi}{R} = \frac{1}{\rho} \frac{\partial P}{\partial \phi} - \frac{\partial u_\phi}{\partial R} + \frac{1}{R} \frac{\partial u_\phi}{\partial \phi} - \frac{u_\phi}{R^2} = 0,
\]
\[
\frac{1}{\rho} \frac{\partial P}{\partial z} = 0.
\]

The second equation defines the angular velocity
\[
\Omega = a \frac{b}{R^2},
\]
where \( a \) and \( b \) are two constants related to the boundary values of the angular velocity, \( \Omega_{in}, \Omega_{out} \), of the inner cylinder with radius \( R_{in} \) and the outer cylinder with radius \( R_{out} \). It follows that
\[
a = \frac{\Omega_{out} \hat{\mu} - \hat{\eta}^2}{1 - \hat{\eta}^2}, \quad b = \frac{\Omega_{in} R_{in}^2}{1 - \hat{\eta}^2},
\]
with
\[
\hat{\mu} = \frac{\Omega_{out}}{\Omega_{in}}, \quad \hat{\eta} = \frac{R_{in}}{R_{out}}.
\]

Differentiating the first equation of the system (8) by \( z \) and the second equation by \( R \), subtracting each other and using the supposed profiles of density and angular velocity one gets
\[
R \Omega^2 \frac{d\rho}{dz} = 0.
\]

After this relation the density can depend only on the vertical coordinate \( z \) in the absence of rotation \( (\Omega = 0) \) and the angular velocity can only depend on radius in the absence of the vertical density stratification \( (d\rho/dz = 0) \). The supposed profiles of the angular velocity, \( \Omega = \Omega(R) \), and the density \( \rho = \rho(z) \) are, therefore, not self-consistent. Thus, we must admit a more general profile for the density \( \rho = \rho(R,z) \) even though the initial stratification for the resting fluid is only vertical. In this case, the condition (12) takes the form
\[
R \Omega^2 \frac{\partial \rho}{\partial z} + g \frac{\partial \rho}{\partial R} = 0.
\]
The fluid transforms under the centrifugal force from the pure vertical stratification at the initial state to mixed (vertical and radial) stratification under the influence of the rotation thereby strongly complicating the problem.

For real experiments the initial (without rotation) vertical stratification as is the ratio of centrifugal acceleration to the vertical gravitation acceleration are small

$$\left| \frac{d \log(\rho)}{d \log(z)} \right| < 1, \quad \left| \frac{R^2 \Omega}{g} \right| < 1,$$

so that after (13) the radial stratification is also small. Let us therefore consider the case of a weak stratification

$$\rho = \rho_0 + \rho_1(R, z), \quad \rho_1 << \rho_0,$$

where \(\rho_0\) is the uniform background density and (13) is fulfilled in zero-order and after (14) in first order. The perturbed state of the flow is described by

$$u_R, \quad u_\phi + R \Omega(R), \quad u_z, \quad \rho_0 + \rho_1(R, z) + \rho, \quad P_0 + P_1(R, z) + P,$$

where \(|P/P_0| << 1\) and \(u_\phi, u_z, P\) and \(\rho\) are the perturbations. Linearizing the system (4), and selecting only the terms of the largest order, the system takes exactly the Boussinesq form

$$\begin{aligned}
\frac{\partial u_R}{\partial t} + \Omega \frac{\partial u_R}{\partial \phi} - 2 \Omega u_\phi &= - \frac{\partial}{\partial R} \left( \frac{P}{\rho_0} \right) + \nu_0 \left[ \Delta u_R - \frac{2}{R^2} \frac{\partial^2 u_R}{\partial \phi^2} - \frac{u_R}{R^2} \right], \\
\frac{\partial u_\phi}{\partial t} + \Omega \frac{\partial u_\phi}{\partial \phi} + \frac{1}{R} \frac{\partial}{\partial R} \left( R^2 \Omega \right) u_R &= - \frac{1}{\rho_0} \frac{\partial}{\partial \phi} \left( \frac{P}{\rho_0} \right) + \nu_0 \left[ \Delta u_\phi + \frac{2}{R^2} \frac{\partial^2 u_R}{\partial \phi^2} - \frac{u_\phi}{R^2} \right], \\
\frac{\partial u_z}{\partial t} + \Omega \frac{\partial u_z}{\partial \phi} &= - \frac{\partial}{\partial z} \left( \frac{P}{\rho_0} \right) - g \frac{\rho_0}{\rho} + \nu_0 \Delta u_z, \\
\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{R} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} &= 0, \\
\frac{\partial}{\partial t} \left( \frac{\rho}{\rho_0} \right) + \Omega \frac{\partial}{\partial \phi} \left( \frac{\rho}{\rho_0} \right) - \frac{N^2}{g} \frac{u_z}{z} &= 0,
\end{aligned}$$

(17)

where \(N\) is the vertical buoyance frequency with

$$N^2 = - \frac{g}{\rho_0} \frac{\partial \rho_1}{\partial z}.$$

(18)

Suppose that the linear vertical density stratification \(\partial \rho_1/\partial z = \text{const}\) and thus \(N^2\) is a constant, too. Then the coefficients of the system (17) only depend on the radial coordinate and we can use a normal mode expansion of the solution \(F=F(R)\exp(i(m \phi + k z + \omega t))\) where \(F\) represents any of the disturbed quantities.

Let \(D=R_{in} - R_{out}\) be the gap between the cylinders. We use \(R=(R_{out} D)^{1/2}\) as the unit of length, the velocity \(\Omega R_0\) as the unit of the perturbed velocity, \(\Omega_0\) as the unit of \(\omega_0, N\) and \(\Omega\). Using the same symbols for normalized quantities and redefining \(\rho\) as the dimensionless density \(\rho g/\rho_0 R_0\), we finally find

$$\begin{aligned}
\frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R} \frac{\partial u_R}{\partial R} &= \frac{u_R}{R^2} - \left( k^2 + \frac{m^2}{R^2} \right) u_R - 2i \frac{m}{R} u_\phi - i \text{Re}(\sigma + m \Omega) u_R + 2 \text{Re} \Omega u_\phi - \text{Re} \frac{\partial P}{\partial R} = 0, \\
\frac{\partial^2 u_\phi}{\partial R^2} + \frac{1}{R} \frac{\partial u_\phi}{\partial R} &= \frac{u_\phi}{R^2} - \left( k^2 + \frac{m^2}{R^2} \right) u_\phi + 2i \frac{m}{R} u_R - i \text{Re}(\sigma + m \Omega) u_\phi - i \text{Re} \frac{m}{R} P - \text{Re} \frac{\partial}{\partial R} \left( R^2 \Omega \right) = 0,
\end{aligned}$$

(19)
\[
\frac{\partial^2 u_z}{\partial R^2} + \frac{1}{R} \frac{\partial u_z}{\partial R} - \frac{u_z}{R^2} - \left( k^2 + \frac{m^2}{R^2} \right) u_z - i \text{Re}(\sigma + m\Omega)u_z - i \text{Re}k\rho - \text{Re}\rho = 0,
\]

\[
\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{i}{R} m u_\phi + ik u_z = 0,
\]

\[
i(\omega + m\Omega)\rho - N^2 u_z = 0
\]

(19)

with the Reynolds number

\[
\text{Re} = \frac{\Omega_{\text{in}} R_{\text{in}} D}{v}.
\]

(20)

The standard no-slip boundary conditions used at the inner and outer cylinder, i.e.

\[
u = u_\phi = u_z = 0,
\]

(21)

complete the classical eigenvalue problem. The same numerical method as in our previous papers about the Taylor-Couette problem [10] is used. Here we use a small negative imaginary part of \( \omega \) to avoid problems with the corotation point \( \omega = m\Omega \) for \( m > 0 \). Thus, the calculated critical Reynolds numbers are not for the marginally stable state but for slightly unstable state. To be sure that the calculated unstable state can be realized in experiments we checked the existence of the transition from stable to unstable state for several arbitrary points.

Figure 2. The marginal stability line for axisymmetric disturbances \((m=0)\) for \( \hat{\eta} = 0.78, \hat{\mu} = 0 \). The dots represent the experimental data, [2].

The code has been tested by computing for \( m=1 \) the critical Reynolds number for the run 2 of [9] with the experimental value 196.2 (with our normalizations) and our computed result 200.6 which we accepted to be in sufficiently good accordance.

3. Results

The imaginary parts of \( \omega \), \( \text{Im}(\omega) \), decrease with increasing Reynolds number. The Reynolds numbers above which the imaginary part of \( \omega \) is smaller than some fixed value depend on the vertical wave number. They have a minimum at a certain wave number for fixed other parameters. This minimum value is called the critical Reynolds number.

In Figure 2 we compare the calculated marginal stability line (i.e. \( \text{Im}(\omega)=0 \) for axisymmetric disturbances with experimental values, [2]. The agreement is rather good except the small values of the Froude number.

The code has been tested by computing for \( m=1 \) the critical Reynolds number for the run 2 of [9] with the experimental value 196.2 (with our normalizations) and our computed result 200.6 which we accepted to be in sufficiently good accordance.
This disagreement may indicate the violation of the Boussinesq approximation. For Kepler disks we find Fr is order of 0.5, [11]. The unstratified fluids possess infinite Froude number.

\[
Fr = \frac{\Omega_{in}}{N}.
\]  

Figure 3. The marginal stability lines for \( m \) and critical Reynolds numbers for \( m > 0 \) for \( \hat{\eta}=0.78 \), and Fr=0.5. The solid vertical line marks \( \hat{\mu} = \hat{\eta}^2 \) limit, the dashed line marks \( \hat{\mu} = \hat{\eta} \) limit and the dotted vertical line marks \( \hat{\mu} = \hat{\eta}^{1.5} \) limit.

The dependence of the critical Reynolds numbers on \( \hat{\mu} \) is given by figure 3 for a narrow gap and in figure 4 for a wide gap. The exact line of marginal stability is plotted only for \( m=0 \). The axisymmetric disturbances are unstable only for \( \hat{\mu} < \hat{\eta}^2 \) in accordance to the Rayleigh condition (1). For \( m>0 \) the slightly unstable lines with \( \text{Im}(\omega) = -10^{-3} \) are given.

The nonaxisymmetric disturbances are unstable also beyond the Rayleigh line (plotted as solid in the figures). The higher the \( m \), however, the more the corresponding instability line approaches the Rayleigh line. Note, therefore, that the ‘stratorotational instability’ (SRI, [11]) only produces low-\( m \) modes. This is an indication that indeed within the short-wave approximation (high-\( m \)) it does not exist (see [12]).

For nonstratified Taylor-Couette flows the nonaxisymmetric modes are only the most unstable disturbances for counter-rotating cylinders (see e.g. [13]). The nonaxisymmetric instability of the stratified Taylor-Couette flow beyond the Rayleigh line (\( \hat{\mu} > \hat{\eta}^2 \)) leads to the existence of some critical value, \( \hat{\mu}_c \), beyond which the nonaxisymmetric disturbances are the most unstable. Our results show that \( \hat{\mu}_c \sim 0.27 \) and almost independent of the Froude number for \( \hat{\eta}=0.78 \) (small gap) and \( \hat{\mu}_c <0 \) for \( \hat{\eta}=0.3 \) (wide gap). It is possible that for some \( \hat{\eta} \) the nonaxisymmetric disturbances are the most unstable for all values of \( \hat{\mu} \) corresponding to unstable flows.
Figure 4. The same as figure 3 but for a wide gap $\hat{\eta}=0.3$.

There is another observation with the figures 3 and 4. Approaching the line $\mu = \hat{\eta}$ the instability lines become more and more steep. We did not find any solution for $\mu > \hat{\eta}$. If this is true one has to change the condition (2) to condition (3). The condition (3) needs much more inclined rotational profiles according to radial coordinate (not so inclined as for nonstratified fluids but also not so flat as for the MRI) to find the modes of the linear SRI. This result seems to be of relevant for the discussion of the stability or instability of Kepler disks.

Figure 5. The same as figure 3 but for vertical wave number (left) and the pattern speed $R(\omega)$ for $m=0$ or $R(\omega)/m$ for $m>0$ (right).

For the narrow gap $\hat{\eta}=0.78$ the critical Reynolds numbers of the nonaxisymmetric modes only slightly depend on $m$. The same is true for the critical vertical wave number and the pattern speed $R(\omega)/m$ (Figure 5). The vertical wave number only weakly depends on $\mu$ and the values $R(\omega)/m$ linearly run with . The situation is changed, however, for the wide gap ($\hat{\eta}=0.3$). All parameters now strongly depend on both $m$ and $\mu$ (Figure 6). The trend for the vertical wave numbers is opposite for $m=3$ to those for $m=1$ and $m=2$. 
The vertical wave numbers for both containers are of order 10 for $m=1$. With our normalization the vertical extent of the Taylor vortices is given by

$$\frac{\xi}{R_{out}} = \frac{\pi}{k} \sqrt{\hat{\eta}(1 - \hat{\eta})}.$$  \hspace{1cm} (23)

With the mentioned value of $k$ it is order of 0.1 for both the small-gap case and the wide-gap case. The cell becomes thus rather flat. For nonstratified TC-flows one finds $\delta z \sim R_{out} - R_{in}$ while the cells under the influence of an axial magnetic field become more and more prolate [10]. The stratification generally reduces the height of the Taylor vortices.

Figure 6. The same as figure 5 but for $\hat{\eta} = 0.3$

Unlike nonstratified Taylor-Couette flow the $\Re(\omega)$ is not zero for stratified Taylor-Couette flow even for axisymmetric disturbances ($m=0$). The onset of instability is thus oscillatory ('overstability'). The question is whether a critical Froude number exists corresponding to the transition from stationary solutions to oscillating solutions? The answer is No. One cannot fulfill Eqs. (19) for marginal stability ($\Im(\omega) = 0$) without a finite real part of $\omega$ for $N^2 \neq 0$. The axially stratified Taylor-Couette flow bifurcates from the purely azimuthal flow through a direct Hopf bifurcation to a wavy regime. Depending on the value of $\hat{\mu}$ and $\hat{\eta}$ this new regime can be either oscillating and axisymmetric or nonaxisymmetric and azimuthally drifting (see figures.3, 4). For both our containers the pattern speeds are negative for positive $m$. The drift of the spirals is thus always in the direction of the cylinder rotation.

Experiments have really demonstrated the oscillating onset of the axisymmetric instability [3]. It would be interesting to design experiments with either a rotating outer cylinder or a wider gap to probe the bifurcation from the overstable oscillating axisymmetric flow pattern to the spiral nonaxisymmetric flow pattern.

As an example for the container with the narrow gap in figure 7 the velocity eigenfunctions are presented for $m=1$ and for $\hat{\mu} = 0.7$ exceeding the value of $\hat{\eta}^2$. The functions are smooth enough and do not suggest that the instability must be explained as a boundary effect.
4. Discussion

It is shown that the Boussinesq approximation yields nonaxisymmetric disturbances with low $m$ of the stratified Taylor-Couette flow as unstable, even beyond the Rayleigh line $\hat{\mu} = \hat{\eta}^2$. Our results, however, also show that the critical Reynolds numbers increase rapidly as the $\hat{\mu} = \hat{\eta}$ line is approached so that now the relation for instability $\hat{\mu} < \hat{\eta}$ seems to appear rather than $\hat{\mu} < 1$ according to [6] and [7]. It is challenging to interpret the line $\hat{\mu} = \hat{\eta}$ with the (galactic) rotation profile $u_\phi = \text{const}$ in the same sense as to interpret the line $\hat{\mu} = \hat{\eta}^2$ with the rotation law for uniform specific angular momentum $R^2 \Omega = \text{const}$. Below we shall consider the line $\hat{\mu} = \hat{\eta}^{1.5}$ as concerning the Kepler flow. Note, however, the general difference of the Kepler rotation law $\Omega \propto R^{-1.5}$ and the Taylor-Couette rotation law (9). Dubrulle et al. [11] for their model of rotating plane Couette flow have found

![Figure 7. The velocity eigenfunctions for $m=1$, $\hat{\eta}=0.78$, $\hat{\mu}=0.7$, $Fr=0.5$ with $Re=567$, $k=10.8$, $\Re(\omega)=-0.825$. The dotted lines are the real part and solid lines are the imaginary part.](image-url)
unstable solutions also beyond the line $\hat{\mu} = \hat{\eta}$ but also in these computations the rotation profile must be steeper than $\mathcal{R}^{2/3}$ (their figure 7).

The SRI also leads to the situation that nonaxisymmetric disturbances can be the most unstable modes not only for counterrotating cylinders but also for corotating cylinders. The characteristic values of $\hat{\mu}$ where the nonaxisymmetric disturbances are the most unstable ones strongly depend on the gap width. For $\hat{\eta} = 0.3$ all positive $\hat{\mu}$ are concerned (see figure 4). It cannot be excluded that the nonaxisymmetric disturbances are the most unstable ones for all $\hat{\mu}$ for $\hat{\eta}$ smaller than some critical value. The instability is always oscillatory ('overstability') but the new regime which appeared after the Hopf bifurcation can be either axisymmetric and oscillating or non-axisymmetric and drifting. The drift is always in the direction of the cylinders rotation.

These results were obtained with the Boussinesq approximation so that two restrictions remain. The vertical density stratification should be sufficiently weak and the rotation should be so slow that the centrifugal acceleration can be neglected in total. If one of these conditions is violated the Boussinesq approximation cannot be used and the situation becomes much more complicated. The disagreement between the calculated and the observed critical Reynolds numbers for small Fr (see figure 2) may already indicate the violation of the Boussinesq approximation for strong stratifications.

We have shown that for not too small negative $d\Omega/dR$, Taylor-Couette flows with vertical density stratification become unstable against nonaxisymmetric disturbances with $m=1$ even if they are stable without density stratification. Kepler flows seem to be concerned by this phenomenon. In the figures 3, 4 the dotted lines represent the limit $\hat{\mu} = \hat{\eta}^{1.5}$ which might mimic the radial shear in Kepler disks. In both figures we find a critical Reynolds number of only about 500 for the lowest ($m=1$) mode. This is indeed a rather small number whose meaning, however, should not be overestimated. Approaching the line $\hat{\mu} = \hat{\eta}$ also these values become increasingly large. Even more important is the finding that in magnetohydrodynamic Taylor-Couette experiments the MRI only needs {em magnetic} Reynolds numbers of $O(10)$. This leads to hydrodynamic Reynolds numbers exceeding $O(10^6)$ only for experiments with liquid metals in terrestrial laboratories. For hot plasma with magnetic Prandtl numbers of order 10, [14], the necessary hydrodynamic Reynolds number is also only $O(1)$ or even smaller!

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