FEKETE SZEGÖ PROPERTIES FOR THE CLASS OF MOCANU FUNCTIONS ASSOCIATED WITH $q$—RUSCHEWEYH OPERATOR

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Abstract. The objective of this paper is to introduce and investigate new subclass of analytic functions involving $q$—derivative Ruscheweyh operator. For functions belonging to this class, we obtain coefficient estimates on Taylor - Maclaurin series and the results on the famous Fekete Szegö inequality.

Keywords: univalent function; subordination; $q$—derivative; Fekete-Szegö problem.

2010 AMS Subject Classification: 30C45, 30C50.

1. INTRODUCTION

The $q$-difference calculus or quantum calculus was initiated at the beginning of 19th century, that was initially developed by Jackson [8, 9]. The $q$—calculus is one of the tool which is used to introduce and investigate many number of subclasses of analytic functions. Basic definitions and properties of $q$-difference calculus can be found in the book mentioned in [10]. The origin of fractional $q$-difference calculus has been found in the works by Al.Salam [3] and Agarwal

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Received November 8, 2020
Due to the application of $q-$calculus in various branches of science, recently, the area of $q$-calculus has attracted the serious attention of researchers. Later, geometrical interpretation of $q-$analysis has been recognised through studies on quantum groups. Mohammed and Darus [14] studied approximation and geometric properties of these $q$-operators for some subclasses of analytic functions in compact disk.

Let $A$ denote the class of all functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U})$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ be the subclass of $A$ consisting of all univalent functions in $\mathbb{U}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, then we say that the function $f(z)$ is subordinate to $g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalence to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

A $q$-analog of the class of starlike functions was first introduced in 1990 [7] by means of the $q$-difference operator $D_qf(z)$ acting on functions $f \in A$ given by (1.1) and $0 < q < 1$, the $q$-derivative of a function $f(z)$ is defined by (see [8, 9])

$$D_qf(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0).$$
\( D_q f(0) = f'(0) \) and \( D_q^2 f(z) = D_q(D_qf(z)) \). From (1.2), we deduce that,

\[
D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}
\]

where

\[
[q]_q = \frac{1 - q^k}{1 - q}.
\]

As \( q \to 1^- \), \([k]_q \to k\). For a function \( h(z) = z^k \), we observe that,

\[
D_q(h(z)) = D_q(z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = [k]_q z^{k-1},
\]

\[
\lim_{q \to 1^-} (D_q(h(z))) = \lim_{q \to 1^-} ([k]_q z^{k-1}) = k z^{k-1} = h'(z),
\]

where \( h' \) is the ordinary derivative.

As a right inverse, Jackson [9] introduced the q-integral

\[
\int_0^z f(t) d_q t = z(1 - q) \sum_{k=0}^{\infty} q^k f(zq^k),
\]

provided that the series converges. For a function \( h(z) = z^k \), we have

\[
\int_0^z h(t) d_q t = \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} \quad (k \neq -1)
\]

\[
\lim_{q \to 1^-} \int_0^z h(t) d_q t = \lim_{q \to 1^-} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t) dt,
\]

where \( \int_0^z h(t) dt \) is the ordinary integral. Note that the q-difference operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [4, 5, 6, 12, 16]). Kanas and Răducanu in [11] used the Ruscheweyh q-differential operator to introduce and study some properties of \((q,k)\) uniformly starlike functions of order \( \alpha \). One can clearly see that \( D_q f(z) \to f'(z) \) as \( q \to 1^- \). This difference operator helps us to generalize the class of starlike functions \( S^* \) analytically.

Ma and Minda [13] unified various subclasses of starlike and convex functions for which either of quantity \( \frac{zf''(z)}{f(z)} \) (or) \( 1 + \frac{zf''(z)}{f'(z)} \) is subordinate to a more general superordinate function. For this purpose, they considered an analytic function \( \phi(z) \) with positive real part in the unit disc \( \mathbb{U} \), with \( \phi(0) = 1, \phi'(0) > 0 \) and \( \phi \) maps \( \mathbb{U} \) onto a region starlike, with respect to the real
axis.

The classes of Ma-Minda starlike functions consists of functions $f(z) \in \mathcal{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$.

2. Preliminaries

In this session, we present some of the known concepts and new definitions defined in the open unit disc $\mathbb{U}$.

**Definition 2.1.** For $0 \leq \alpha \leq 1$; a function $f \in \mathcal{A}$ is in $\mathcal{M}_\alpha(\phi)$ if

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z).$$

Quite recently, Abdullah and Darus in [1] introduced the new differential operator $D_{m,v,q,\mu,\delta,k,\lambda}$ by

$$D_{m,v,q,\mu,\delta,k,\lambda} f(z) = z + \sum_{i=2}^{\infty} \Omega_{k,\lambda,\delta,\mu}^{m,v} \left([i]_q\right) a_i z^i,$$

where

$$(\delta, k, \lambda, \mu \geq 0), k > \lambda, \delta > \mu, m \in \mathbb{N}_0.$$

$$\Omega_{k,\lambda,\delta,\mu}^{m,v} = (k - \lambda)(\delta - \mu) \left([i]_q - 1\right) [v - 1 + i]_q! [v_q]! [i - 1]!.$$ 

**Remark 2.2.** For different values of $v, k, \lambda, \delta, \beta$ and $\mu$, we get various differential operators explained as Remark in [1].

By inspiring the works of Abdullah and Darus [1], we now define the new subclass $\mathcal{M}_{q,v}^{m,v}(\phi)$ of $\mathcal{A}$ associated with the differential operator(2.1).
Definition 2.3. Let \( f \in \mathcal{A} \) and \( 0 \leq \alpha \leq 1 \), then \( f \) is said to be in the class \( A_q^{m,v}(\phi) \) if it satisfies the following subordination condition:

\[
(1 - \alpha) \left( z \partial_q D_q^{m,v} f(z) \right) + \alpha \left( \frac{\partial_q (z \partial_q D_q^{m,v} f(z))}{\partial_q D_q^{m,v} f(z)} \right) \prec \phi(z).
\]

In order to prove the main result, we need the following lemma.

Lemma 2.4. [15] If \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is an analytic function with positive real part in \( U \), then

\[
|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}.
\]

The result is sharp for the function \( p(z) = \frac{1 + z^2}{1 - z^2} \) and \( p(z) = \frac{1 + z}{1 - z} \).

we also need the following results for our investigation.

Lemma 2.5. [13] If \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is an analytic function with positive real part in \( U \), then

\[
|c_2 - \nu c_1^2| \leq \begin{cases} 
-4\nu + 2 & \text{if } \nu \leq 0 \\
2 & \text{if } 0 \leq \nu \leq 1 \\
4\nu - 2 & \text{if } \nu \geq 1
\end{cases}
\]

3. Main Results

The main purpose of this paper is to obtain the Fekete-Szegö inequality for certain class of analytic functions defined by the differential operator involving \( q \)--Ruscheweyh operator.
Theorem 3.1. Let 0 ≤ α ≤ 1. Also let \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \), where the coefficients \( B_n \) are real with \( B_1 > 0 \). If \( f(z) \) given by (1.1) belongs to \( \mathcal{M}_q^{m,v}(\phi) \), then

\[
|a_3 - va_2^2| \leq \frac{B_1}{2(1 - \alpha + [3]_q \alpha) ([3]_q - 1) \Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)} \times \\
\max \left\{ 1; \frac{B_2}{B_1} \left[ 1 + \frac{B_1(1 - \alpha + ([2]_q)^2 \alpha)}{([2]_q - 1)(1 - \alpha + [2]_q \alpha)^2} \right] \right\}
\]

(3.1)

Proof. Observe that the condition in (2.2) can be written as follows:

\[
(1 - \alpha) \left( z \partial_q \phi_q^{m,v}(f(z)) \right) + \alpha \left( \frac{\partial_q (z \partial_q \phi_q^{m,v}(f(z)))}{\partial_q \phi_q^{m,v}(f(z))} \right) = \phi(\omega(z)).
\]

(3.2)

Here, the function \( \omega(z) \) is analytic in \( \mathbb{U} \) with the condition \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \mathbb{U} \).

Let \( h(z) \) be an analytic function defined in \( \mathbb{U} \) with \( \Re\{h(z)\} > 0 \) and \( h(0) = 1 \) be given by

\[
h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{for} \quad z \in \mathbb{U},
\]

(3.3)

Since \( \omega(z) \) is a Schwarz function, we have

\[
\phi(\omega(z)) = \phi \left( \frac{\omega(z) - 1}{\omega(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left( \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \ldots.
\]

(3.4)

Upon computation we get,

\[
(1 - \alpha) \left( z \partial_q \phi_q^{m,v}(f(z)) \right) + \alpha \left( \frac{\partial_q (z \partial_q \phi_q^{m,v}(f(z)))}{\partial_q \phi_q^{m,v}(f(z))} \right) = \\
1 + \left[ ([2]_q - 1) \Omega_{k,\lambda,\delta,\mu}^{m,v}([2]_q) (1 - \alpha + [2]_q \alpha) \right] a_2 z \\
+ \left[ ([3]_q - 1) \Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q) (1 - \alpha + [3]_q \alpha) \right] a_3 \\
+ \left[ (-[2]_q + 1) \Omega_{k,\lambda,\delta,\mu}^{m,v}([2]_q)^2 (1 - \alpha + ([2]_q)^2 \alpha) \right] a_2^2 z^2 + \ldots
\]

From (3.4) and (3.5), we obtain
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\[ a_2 = \frac{B_1c_1}{2(1 - \alpha + [2]q\alpha)([2]q - 1)\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]q)} \]  

and

\[ a_3 = \frac{B_1}{2(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]q)} \]

\[ \left\{ c_2 = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} - \frac{B_1(1 - \alpha + ([2]q)^2\alpha)}{([2]q - 1)(1 - \alpha + [2]q\alpha)^2} \right] c_1^2 \right\}. \]

Therefore,

\[ a_3 - \mu a_2^2 = \frac{B_1}{2(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]q)}(c_2 - \sigma c_1^2). \]

Where

\[ \sigma = \frac{1}{2} \left\{ 1 - \frac{B_2}{B_1} - \frac{B_1(1 - \alpha + ([2]q)^2\alpha)}{([2]q - 1)(1 - \alpha + [2]q\alpha)^2} \right\} \]

\[ \left\{ 1 - \frac{([3]q - 1)(1 - \alpha + [3]q\alpha)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]q)}{([2]q - 1)(1 - \alpha + ([2]q)^2\alpha)\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]q)^2)} \right\}. \]

We get our desired result by applying Lemma 2.4. This completes the proof of Theorem 3.1. □

Remark 3.2. If we set \( \alpha = 0 \) and \( \alpha = 1 \), then we have the results of Theorem 5 and Theorem 6 obtained by Abdullah and Darus [1] respectively.

If we set \( m = 0 \) and \( v = 0 \) in Theorem 3.1, we thus obtain the following:

Corollary 3.3. Let \( \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \), with \( B_1 > 0 \), and if \( f(z) \) given by (1.1) belongs to \( \mathcal{M}_q(\phi) \), then

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{([3]q - 1)(1 - \alpha + [3]q\alpha)\max\left\{ 1; \frac{B_2}{B_1} + \frac{B_1(1 - \alpha + ([2]q)^2\alpha)}{([2]q - 1)(1 - \alpha + [2]q\alpha)^2} \left( 1 - \frac{([3]q - 1)(1 - \alpha + [3]q\alpha)}{([2]q - 1)(1 - \alpha + ([2]q)^2\alpha)\sigma} \right) \right\}}. \]

The result is sharp.
Theorem 3.4. Let $0 \leq \alpha \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where the coefficients $B_n$ are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $\mathcal{M}^{m,v}_{q}(\phi)$, then

$$|a_3 - \nu a_2^2| \leq \begin{cases} F_1 & \text{if } \nu \leq \chi_1, \\ B_2 & \text{if } \chi_1 \leq \nu \leq \chi_2, \\ F_2 & \text{if } \nu \leq \chi_2. \end{cases}$$

(3.10)

Where,

$$F_1 = \frac{B_2}{(1 - \alpha + [3]_q \alpha)([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)}$$

$$+ \frac{B_1^2}{([2]_q - 1)^2} \left[ \frac{(2)_{q - 1}(1 - \alpha + [2]_q^2 \alpha)}{(1 - \alpha + [3]_q \alpha)(1 - \alpha + [2]_q \alpha)^2([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)} - \frac{\nu}{(1 - \alpha + [2]_q \alpha)^2(\Omega_{k,\lambda,\delta,\mu}^{m,v}([2]_q))^2} \right],$$

$$F_2 = \frac{B_1^2}{([2]_q - 1)^2} \left[ \frac{(1 - \alpha + [2]_q \alpha)^2(\Omega_{k,\lambda,\delta,\mu}^{m,v}([2]_q))^2}{(2)_{q - 1}(1 - \alpha + [2]_q^2 \alpha)} - \frac{(2)_{q - 1}(1 - \alpha + [2]_q^2 \alpha)}{(1 - \alpha + [3]_q \alpha)(1 - \alpha + [2]_q \alpha)^2([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)} \right]$$

$$- \frac{B_2}{(1 - \alpha + [3]_q \alpha)([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)}$$

and

$$\chi_1 = \frac{([2]_q - 1)(\Omega_{k,\lambda,\delta,\mu}^{m,v}([2]_q))^2}{([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)} \left[ \frac{([2]_q - 1)(1 - \alpha + [2]_q \alpha)^2(B_2 + B_1) + (1 - \alpha + [2]_q^2 \alpha)B_1^2}{B_1^2(1 - \alpha + [3]_q \alpha)} \right],$$

$$\chi_2 = \frac{([2]_q - 1)(\Omega_{k,\lambda,\delta,\mu}^{m,v}([2]_q))^2}{([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]_q)} \left[ \frac{([2]_q - 1)(1 - \alpha + [2]_q \alpha)^2(B_2 - B_1) + (1 - \alpha + [2]_q^2 \alpha)B_1^2}{B_1^2(1 - \alpha + [3]_q \alpha)} \right].$$

Proof. The Proof is followed by Lemma 2.5.

Using (3.8) and (3.9), we have the following cases:
Case (i): If $\nu \leq \chi_1$,
\[
|a_3 - \nu a_2^2| \leq \frac{B_1}{2(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} [2 - 4\sigma] \\
\leq \frac{B_1}{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} \left[ \frac{B_2}{B_1} + \frac{(1 - \alpha + [2]q^2\alpha)B_1}{(1 - \alpha + [2]q\alpha)^2([2]q - 1)} \right] \\
\left( 1 - \frac{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)}{(1 - \alpha + [2]q\alpha)(([2]q - 1))} \right)^2.
\] (3.11)

Therefore,
\[
|a_3 - \nu a_2^2| \leq \frac{B_2}{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} + \frac{B_1^2}{([2]q - 1)^2} \left( 1 - \frac{(1 - \alpha + [2]q\alpha)(([2]q - 1))}{(1 - \alpha + [2]q\alpha)[(3]q - 1])\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} \right)^2.
\] (3.12)

Case (ii): If $\chi_1 \nu \leq \chi_2$,
\[
|a_3 - \nu a_2^2| \leq \frac{B_1}{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)}.
\]

Case (iii): If $\nu \geq \chi_2$,
\[
|a_3 - \nu a_2^2| \leq \frac{B_1}{2(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} [4\sigma - 2] \\
\leq -\frac{B_1}{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} \left[ \frac{B_2}{B_1} + \frac{(1 - \alpha + [2]q^2\alpha)B_1}{(1 - \alpha + [2]q\alpha)^2([2]q - 1)} \right] \\
\left( 1 - \frac{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)}{(1 - \alpha + [2]q\alpha)(([2]q - 1))} \right)^2.
\] Therefore,
\[
|a_3 - \nu a_2^2| \leq -\frac{B_2}{(1 - \alpha + [3]q\alpha)([3]q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} - \frac{B_1^2}{([2]q - 1)^2} \left( 1 - \frac{(1 - \alpha + [2]q\alpha)(([2]q - 1))}{(1 - \alpha + [2]q\alpha)[(3]q - 1])\Omega_{k,\lambda,\delta,\mu}^{m,v}([3]q)} \right)^2.
\] (3.13)

This completes the proof of the Theorem 3.4.
Remark 3.5. If we set $\alpha = 0$ and $\alpha = 1$, then we have the results of Theorem 10 and Theorem 11 obtained by Abdullah and Darus [1] respectively.

If we set $m = 0$ and $v = 0$ in Theorem 3.4, we thus obtain Fekete Szegő inequality for the subclass $\mathcal{M}_q(\phi)$:

Corollary 3.6. Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots$, with $B_1 > 0$, and if $f(z)$ given by (1.1) belongs to $\mathcal{M}_q(\phi)$, then

$$|a_3 - va_2^2| \leq \begin{cases} 
G_1 & \text{if } v \leq \Psi_1, \\
\frac{B_2}{(1-\alpha + [3]_q \alpha)([3]_q - 1)} & \text{if } \Psi_1 \leq v \leq \Psi_2, \\
G_2 & \text{if } v \leq \Psi_2.
\end{cases} \quad (3.14)$$

Where,

$$G_1 = \frac{B_2}{(1-\alpha + [3]_q \alpha)([3]_q - 1)} + \frac{B_1^2}{(1-\alpha + [2]_q \alpha)^2} \left[ \frac{([2]_q - 1)(1-\alpha + ([2]_q)^2 \alpha)}{(1-\alpha + [3]_q \alpha)(1-\alpha + [2]_q \alpha)^2([3]_q - 1)} \right],$$

$$G_2 = \frac{B_2^2}{(2-\alpha + [2]_q \alpha)^2} \left[ \frac{v}{(1-\alpha + [2]_q \alpha)^2} - \frac{([2]_q - 1)(1-\alpha + ([2]_q)^2 \alpha)}{(1-\alpha + [3]_q \alpha)(1-\alpha + [2]_q \alpha)^2([3]_q - 1)} \right] - \frac{B_2}{(1-\alpha + [3]_q \alpha)([3]_q - 1)}.$$

And

$$\Psi_1 = \frac{([2]_q - 1)}{([3]_q - 1)} \left[ \frac{([2]_q - 1)(1-\alpha + ([2]_q)^2 \alpha)(B_2 + B_1) + (1-\alpha + ([2]_q)^2 \alpha)B_1^2}{B_1^2(1-\alpha + ([3]_q \alpha)} \right],$$

$$\Psi_2 = \frac{([2]_q - 1)}{([3]_q - 1)} \left[ \frac{([2]_q - 1)(1-\alpha + ([2]_q)^2 \alpha)(B_2 - B_1) + (1-\alpha + ([2]_q)^2 \alpha)B_1^2}{B_1^2(1-\alpha + ([3]_q \alpha)} \right].$$

The result is sharp.

\hfill \Box

Conflict of Interests

The author(s) declare that there is no conflict of interests.
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