Gauge Theories on de Sitter space and Killing Vectors

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Abstract

We provide a general method for studying a manifestly covariant formulation of $p$-form gauge theories on the de Sitter space. This is done by stereographically projecting the corresponding theories, defined on flat Minkowski space, onto the surface of a de Sitter hyperboloid. The gauge fields in the two descriptions are mapped by conformal Killing vectors allowing for a very transparent analysis and compact presentation of results. As applications, the axial anomaly is computed and the electric-magnetic duality is exhibited. Finally, the zero curvature limit is shown to yield consistent results.
1 Introduction

Quantum field theory on the de Sitter space time has a long history beginning from the work of Dirac [1]. Its popularity stemmed from the fact that it is a maximally symmetric example of a curved space time manifold. It is a solution of the (positive) cosmological Einstein’s equations having the same degree of symmetry as the flat Minkowski space time solution. Recently, however, interest in the de Sitter space has increased enormously due to physical consequences when it appeared to have a crucial role in the inflationary cosmological paradigm [2]. Indeed, very recently a non-zero cosmological constant has been proposed to explain the luminosity observations of the farthest supernovae [3]. The de Sitter metric will play an important role if this proposal is validated. These developments indicate that the study of field theories on the de Sitter space deserve attention.

A standard way of constructing field theories on the de Sitter space is to use the coordinate independent approach, also called the ambient formalism, such that there is a close resemblance with the corresponding construction on the flat Minkowski space. For scalar fields this was initially developed in [4, 5] which was later extended to include gauge theories in [6, 7]. An unpleasant feature, which also exists in [1], of this approach is that, whereas the electron wave equation involves the angular momentum operator, the gauge field equation involves the ordinary momentum operator. Now the de Sitter space being a hyperboloid (pseudosphere), the natural operator that should enter the equation of motion is the angular momentum operator, since translations on the de Sitter space correspond to ‘rotations’ on the pseudosphere. This is generally corrected by imposing subsidiary or homogeneity conditions on the gauge potential in order to avoid going off the hypersurface of constant length. All these problems are absent in our approach.

Another approach of obtaining de Sitter theories from flat space is discussed in [8]. It is based on radial dimensional reduction and uses vierbein language suitable for analysing theories on curved manifolds. Such an approach is also helpful for studying totally symmetric tensor fields on constant curvature manifolds [9].

In this paper we discuss a manifestly covariant method of formulating gauge theories on the de Sitter space which illuminates the close connection with the corresponding theories on the flat Minkowski space. The theory on the flat space is mapped to that on the de Sitter space by means of a stereographic projection which is basically a conformal transformation. We show that quantities in the gauge sector (like gauge fields, field strengths etc.) in the two descriptions are related by rules similar to usual tensor analysis, with the conformal Killing vectors playing the role of the metric. The explicit structures of these vectors is derived by solving the Cartan-Killing equation. Using this formalism, results for Yang-Mills theory on the de Sitter space are economically formulated. The equations of motion involve the covariantised angular momentum and subsidiary conditions occurring in the usual ambient space formalism are not needed. Also, vierbein language [8] is not necessary and specific properties related to de Sitter space time simplify the technicalities considerably. The analysis is then extended to the two form gauge theory. As applications of our approach we have computed the axial anomaly and also demonstrated electric-magnetic duality rotations in de Sitter space.

The paper is organised as follows: section 2 analyses the connection between stereographic projection and conformal Killing vectors, including a derivation of the latter from the Cartan-Killing equation; section 3 treats the covariant formulation of Yang-Mills the-
ory; section 4 introduces the matter sector and a computation of the axial $U(1)$ anomaly is given; section 5 reveals the electric-magnetic duality symmetry; section 6 contains an analysis of the two form gauge theory highlighting the appearance of a new gauge symmetry that does not have any analogue in the flat space; section 7 discusses the zero curvature limit and section 8 contains a brief summary.

2 Stereographic Projection and Killing Vectors

Amongst curved spacetimes, the de Sitter and anti-de Sitter spaces are the only possibilities that have maximal symmetry admitting the highest possible number of Killing vectors. The role of these vectors in suitably defining gauge theories on such spaces is crucial to this analysis. We shall restrict our discussion to de Sitter spaces only.

The de Sitter universe is a pseudosphere in a five dimensional flat space with Cartesian coordinates $r^a = (r^0, r^1, r^2, r^3, r^4)$ satisfying,

$$
\eta_{ab} r^a r^b = (r^0)^2 - (r^1)^2 - (r^2)^2 - (r^3)^2 - (r^4)^2 = -l^2
$$

where $l$ is the de Sitter length parameter. The metric of the de Sitter space $dS(4,1)$ is induced from the pseudo Euclidean metric $\eta = \text{diag}(+1, -1, -1, -1, -1)$. It has the pseudo orthogonal group $SO(4,1)$ as the group of motions. Using the mostly negative Minkowski metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ with $\mu, \nu = 0, 1, 2, 3$, we obtain,

$$
-\frac{1}{l^2}g_{\mu\nu} r^\mu r^\nu + (r'^4)^2 = 1
$$

where $r'^4 = \frac{r^4}{l}$ is a dimensionless coordinate.

A useful parametrisation of this space is done by exploiting the stereographic projection. The four dimensional stereographic coordinates ($x^\mu$) are obtained by projecting the de Sitter surface into a target Minkowski space. The relevant equations are [14],

$$
r^\mu = \Omega(x) x^\mu ; \quad \Omega(x) = \left(1 - \frac{x^2}{4l^2}\right)^{-1}
$$

and,

$$
r'^4 = -\Omega(x)(1 + \frac{x^2}{4l^2})
$$

where $x^2 = g_{\mu\nu} x^\mu x^\nu$.

The inverse transformation is given by,

$$
x^\mu = \frac{2}{1 - r'^4} r^\mu
$$

In order to define a gauge theory on the de Sitter space analogous stereographic projections for gauge fields have to be obtained. This is done following the method developed by us [12, 13] in the example of the hypersphere. The point is that there is a mapping of symmetries on the flat space and the pseudosphere (e.g., translations on the former are rotations on the latter) that is captured by the relevant Killing vectors. Furthermore since stereographic projection is known to be a conformal transformation, one
expects that the cherished map among gauge fields would be provided by the conformal Killing vectors. We may write this relation as,

$$\hat{A}_a = K^\mu_a A_\mu + r_a \phi$$  \hspace{1cm} (6)

where the conformal Killing vectors $K^\mu_a$ satisfy the transversality condition,

$$r^a K^\mu_a = 0$$  \hspace{1cm} (7)

and an additional scalar field $\phi$, which is just the normal component of $\hat{A}_a$, is introduced,

$$\phi = -\frac{1}{l^2} r^a \hat{A}_a$$  \hspace{1cm} (8)

The five components of $\hat{A}$ are expressed in terms of the four components of $A$ plus a scalar degree of freedom. To simplify the analysis the scalar field is put to zero. It is straightforward to resurrect it by using the above equations. With the scalar field gone, $\hat{A}$ is now given by,

$$\hat{A}_a = K^\mu_a A_\mu$$  \hspace{1cm} (9)

and satisfies the transversality condition originally used by Dirac \cite{1}

$$r^a \hat{A}_a = 0$$  \hspace{1cm} (10)

The conformal Killing vectors $K^\mu_a$ are now determined. These should satisfy the Cartan-Killing equation which, specialised to a flat four-dimensional manifold, is given by,

$$\partial^\nu K^\mu_a + \partial^\mu K^\nu_a = 2 \partial^\lambda K^\mu_\lambda g^\mu\nu$$  \hspace{1cm} (11)

The most general solution for this equation is given by \cite{16},

$$K^\mu_a = t^\mu_a + \epsilon_a x^\mu + \omega^\mu_\nu x_\nu + \lambda^\mu_a x^2 - 2\lambda^\sigma_\sigma x_\sigma x^\mu$$  \hspace{1cm} (12)

where $\omega^\mu\nu = -\omega^{\nu\mu}$. The various transformations of the conformal group are characterised by the parameters appearing in the above equation; translations by $t$, dilatations by $\epsilon$, rotations by $\omega$ and inversions (or the special conformal transformations) by $\lambda$. Imposing the condition (7) and equating coefficients of terms with distinct powers of $x$, we find,

$$t^\mu_4 = 0$$  \hspace{1cm} (13)

$$x^\nu t^\mu_4 - l\omega^\mu_\nu x_\nu - l\epsilon_4 x^\mu = 0$$  \hspace{1cm} (14)

$$-lx^2 t^\mu_4 + 4l^2 x^\nu \epsilon_\nu x^\mu + 4l^2 x_\nu \omega^{\mu\nu} x^\sigma - 4l^3 \lambda^\nu_4 x^2 + 8l^3 \lambda^\sigma_4 x_\sigma x^\mu = 0$$  \hspace{1cm} (15)

$$4l^2 x^\nu \lambda^\mu_4 x^2 - 8l^2 \lambda^\nu_\sigma x_\sigma x^\mu - lx^2 \epsilon_4 x^\mu - l\omega^\mu_\nu x_\nu x^2 = 0$$  \hspace{1cm} (16)

$$\lambda^\mu_4 x^2 - 2\lambda^\sigma_4 x_\sigma x^\mu = 0$$  \hspace{1cm} (17)

Contracting the above equations (except of course the first one) by $x_\mu$ leads to simplified equations. Using them as well certain symmetry properties it is possible to obtain a solution of the above set. The nonvanishing ones are explicitly written,

$$\epsilon_4 = l^{-1}$$  \hspace{1cm} (18)

$$t^\mu_4 = g^\mu_4$$  \hspace{1cm} (19)

$$\lambda^\mu_4 = \frac{1}{4l^2} g^\mu_4$$  \hspace{1cm} (20)

\footnote{Hat variables are defined on the de Sitter universe while the normal ones are on the flat space}
The basic structures of the Killing vectors, isolating the fourth component, are now written,

\[ K_\mu^\nu = \left( 1 - \frac{x^2}{4l^2} \right) g_\mu^\nu + \frac{x_\nu x^\mu}{2l^2} \]  
\[ K_4^\mu = -K^{4\mu} = \frac{x^\mu}{l} \]  

With the above solution for the Killing vectors, the stereographic projection for the gauge fields (9) is completed leading to,

\[ \hat{A}_\mu = \left( 1 - \frac{x^2}{4l^2} \right) A_\mu + \frac{x^\nu x_\mu}{2l^2} A_\nu \]  
\[ \hat{A}_4 = \frac{x^\mu}{l} A^\mu \]  

The inverse relation is given by,

\[ \left( 1 - \frac{x^2}{4l^2} \right) A_\mu = \hat{A}_\mu - \frac{x_\mu \hat{A}_4}{2l} \]  

Before proceeding to discuss gauge theories some properties of these Killing vectors are summarised. There are two useful relations,

\[ K_a^\mu K^{a\nu} = \left( 1 - \frac{x^2}{4l^2} \right)^2 g^{\mu\nu} \]  
\[ K_a^\mu K_b^\mu = \left( 1 - \frac{x^2}{4l^2} \right)^2 \left( g_{ab} + \frac{r_a r_b}{l^2} \right) \]  

Note that the conformal (jacobian) factor that relates the volume element on the de Sitter space with that in the four-dimensional flat manifold,

\[ d^4x = dx_0 dx_1 dx_2 dx_3 = \left( 1 - \frac{x^2}{4l^2} \right)^4 d\Omega \]  

naturally emerges in (26) and (27). The invariant measure is given by,

\[ d\Omega = \frac{l}{r_4} d^4r = \left( \frac{l}{r_4} \right) dr_0 dr_1 dr_2 dr_3 \]  

Relation (26) shows that the product of the Killing vectors with repeated ‘a’ indices yields, up to the conformal factor, the induced metric. The other relation can be interpreted as the transversality condition emanating from (7). For computing derivatives involving Killing vectors, a particularly useful identity is given by,

\[ K_a^\mu \partial_\mu K^{a\nu} = \left( 1 - \frac{x^2}{4l^2} \right) \frac{x^\nu}{l^2} \]  

The relation (22) shows that the fourth component is just given by the dilatation (scaling), while the other components (given by (21)) involve the special conformal transformations and the translations.
To observe the use of these Killing vectors, let us analyse the generators of the infinitesimal de Sitter transformations. In terms of the host space Cartesian coordinates $r^a$, these are written as,

$$L_{ab} = r_a \frac{\partial}{\partial r^b} - r_b \frac{\partial}{\partial r^a}$$  \hspace{1cm} (31)$$
which satisfy the algebra,

$$[L_{ab}, L_{cd}] = \eta_{bc} L_{ad} + \eta_{ad} L_{bc} - \eta_{bd} L_{ac} - \eta_{ac} L_{bd}$$  \hspace{1cm} (32)$$

In terms of the stereographic coordinates the generator is expressed as,

$$L_{ab} = (r_a K_b^\mu - r_b K_a^\mu) \partial_\mu ; \ \partial_\mu = \frac{\partial}{\partial x^{\mu}}$$  \hspace{1cm} (33)$$
which can be put in a more illuminating form by contracting with $r^a$,

$$r^a L_{ab} = -l^2 K_b^\mu \partial_\mu$$  \hspace{1cm} (34)$$

clearly showing how rotations on the de Sitter space are connected with the translations on the flat space by the Killing vectors.

### 3 Yang-Mills theory on de Sitter space

In this section we discuss the formulation of Yang-Mills theory on the de Sitter space. The theory is obtained by stereographically projecting the usual theory defined on the flat Minkowski space. This is the first use of the generalisation effected by working in terms of the Killing vectors.

The pure Yang-Mills theory on the Minkowski space is governed by the standard Lagrangian,

$$\mathcal{L} = -\frac{1}{4} Tr(F_{\mu\nu}F^{\mu\nu})$$  \hspace{1cm} (35)$$
where the field tensor is given by,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$  \hspace{1cm} (36)$$

To define the field tensor on the de Sitter space we proceed systematically by looking at the gauge symmetries. If the ordinary potential transforms as,

$$A'_\mu = U^{-1}(A_\mu + i\partial_\mu)U$$  \hspace{1cm} (37)$$
then the projected potential transforms as,

$$\hat{A}'_a = K_a^\mu A'_\mu = U^{-1}(\hat{A}_a - \frac{i}{l^2} r_b L_{ba})U$$  \hspace{1cm} (38)$$

obtained by using (33) and (34).

The infinitesimal version of these transformations obtained by taking $U = e^{-i\lambda}$ is then found to be,

$$\delta A_\mu = D_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda]$$  \hspace{1cm} (39)$$
for the flat space while for the de Sitter space it is given by,

\[ \delta \hat{A}_a = -\frac{1}{l^2} r^b L_{ba} \lambda - i[\hat{A}_a, \lambda] \]  

(40)

This is put in a more transparent form by introducing, in analogy with the flat space, a 'covariantised angular derivative' \[13, 15\] on the de Sitter space,

\[ \hat{\mathcal{L}}_{ab} = L_{ab} - i[r_a \hat{A}_b - r_b \hat{A}_a, \ ] \]  

(41)

so that,

\[ \delta \hat{A}_a = -\frac{1}{l^2} r^b \hat{\mathcal{L}}_{ba} \lambda \]  

(42)

The covariantised angular derivative satisfies a relation that is the covariantised version of (34),

\[ r^a \hat{\mathcal{L}}_{ab} = -l^2 K^\mu_b D_\mu \]  

(43)

obtained by using the transversality condition on the gauge fields.

The field tensor \( \hat{F}_{abc} \) on the de Sitter space is now defined. It has to be a fully antisymmetric three index object that transforms covariantly. The covariantised angular derivative is the natural choice for constructing it. We define,

\[ \hat{F}_{abc} = (L_{ab} \hat{A}_c - i r_a [\hat{A}_b, \hat{A}_c]) + c.p. \]  

(44)

where \( c.p. \) stands for the other pair of terms involving cyclic permutations in \( a, b, c \).

To see that \( F_{abc} \) transforms covariantly it is convenient to recast this in a form involving the Killing vectors, analogous to the relation (33). Indeed it is mapped to the field tensor on the flat space by the following relation,

\[ \hat{F}_{abc} \hat{F}^{abc} = -3l^2 (K^\mu_a K^\nu_b K^\lambda_c K^\rho_d) F_{\mu\nu} F_{\lambda\rho} \]  

(45)

so that symmetry properties under exchange of the indices is correctly preserved. To show the equivalence, (31) and (33) are used to simplify (44), yielding,

\[ \hat{F}_{abc} = (r_a K^\mu_b K^\nu_c + r_b K^\mu_c K^\nu_a + r_c K^\mu_a K^\nu_b) F_{\mu\nu} \]  

(46)

The derivatives acting on the Killing vectors sum up to zero on account of the identity,

\[ (r_a K^\mu_b - r_b K^\mu_a) \partial_\mu K^\nu_c + c.p. = 0 \]  

(47)

The derivatives acting on the potentials, together with the other pieces, combine to reproduce (45), thereby completing the proof of the equivalence.

It is now trivial to see, using the above relation (45), that \( F_{abc} \) transforms covariantly simply because \( F_{\mu\nu} \) does. The action for the Yang-Mills theory on the de Sitter space is defined by first considering the repeated product of the field tensors. Taking (13) and using the transversality of the Killing vectors, we get,

\[ \hat{F}_{abc} \hat{F}^{abc} = -3l^2 (K^\mu_a K^\nu_b K^\lambda_c K^\rho_d) F_{\mu\nu} F_{\lambda\rho} \]  

(48)

Finally, using (20), we obtain,

\[ \hat{F}_{abc} \hat{F}^{abc} = -3l^2 \left(1 - \frac{x^2}{4l^2} \right)^4 F_{\mu\nu} F^{\mu\nu} \]  

(49)
Using this identification as well as (28), the actions on the flat space and the de Sitter space are mapped as,

\[ S = -\frac{1}{4} \int d^4 x F_{\mu \nu} F^{\mu \nu} = \frac{1}{12l^2} \int d\Omega \hat{F}_{abc} \hat{F}^{abc}. \]  

(50)

The lagrangian following from this action is given by,

\[ L_\Omega = \frac{1}{12l^2} \hat{F}_{abc} \hat{F}^{abc}. \]

(51)

This completes the construction of the Yang-Mills action which can be taken as the starting point for calculations on the de Sitter space.

4 The U(1) Axial Anomaly

The inclusion of the matter sector is also conveniently done with the help of the Killing vectors. Form invariance of the interaction requires that,

\[ \int dx (j_\mu A^\mu) = \int d\Omega (\hat{j}_a \hat{A}^a) \]

(52)

where \( j_\mu \) and \( \hat{j}_a \) are the currents in the two descriptions. It is clear therefore that the currents are also mapped by a relation similar to (9). However since the measure is given by (28), the currents will involve the conformal factor.

A simple calculation shows that, to satisfy (52), the desired map is provided by,

\[ \hat{j}_a = \left(1 - \frac{x^2}{4l^2}\right)^2 K_\mu^a j_\mu \]

(53)

Naturally the current on the de Sitter space also satisfies the transversality condition which is taken in the literature [1, 4, 6],

\[ r^a \hat{j}_a = 0 \]

(54)

As an application of this formulation it is possible to compute the anomaly on the de Sitter space from a knowledge of the corresponding expression on the flat space. To simplify matters and to avoid a cluttering of notations, we concentrate on the abelian Adler-Bell-Jackiw anomaly. For the axial current, employing a gauge invariant regularisation, the familiar result on the flat space is known to be,

\[ \partial_\mu j^{\mu 5} = \frac{1}{16\pi^2} \epsilon_{\mu \nu \lambda \rho} F^{\mu \nu} F^{\lambda \rho} \]

(55)

Using (34) and the definition of the current (53) (appropriately interpreted for the axial vector currents), it is possible to obtain the identification,

\[ r^a L_{ab} \hat{j}^{b5} = -l^2 \left(1 - \frac{x^2}{4l^2}\right)^4 \partial_\mu j^{\mu 5} \]

(56)

In getting at the final result, use was made of the identity (30).
This provides a map for one side of (55). To obtain an analogous form for the other side, it is necessary to consider the completely antisymmetric tensor $\epsilon_{\mu\nu\lambda\rho}$ whose value is the same in all systems.

In order to provide a mapping among the $\epsilon$-tensors in the two spaces, we adopt the same rule (45) used for defining the antisymmetric field tensor. However there is a slight subtlety. Strictly speaking, this Levi-Civita epsilon is a tensor density. Hence its transformation law is modified by an appropriate conformal (weight) factor,

$$\epsilon_{abcde} = \frac{1}{l} \left(1 - \frac{x^2}{4l^2}\right)^{-4} \left(r_a K^\mu_b K^\nu_c K^\lambda_d K^\rho_e + \text{cyclic permutations in } (a, b, c, d, e)\right) \epsilon_{\mu\nu\lambda\rho} \quad (57)$$

It is possible to verify the above relation by an explicit calculation, taking the convention that both the epsilons are $+1(-1)$ for any even (odd) permutation of distinct entries $(0, 1, 2, 3, 4)$ in that order.

Now the explicit expressions for the anomaly are identified with the minimum of effort. Indeed, using (45) and (57), it follows that,

$$r_a \epsilon_{bcdef} \hat{F}^{abc} \hat{F}^{def} = 3l^3 \left(1 - \frac{x^2}{4l^2}\right)^4 \epsilon_{\mu\nu\lambda\rho} F^{\mu\nu} F^{\lambda\rho} \quad (58)$$

The weight factors cancel out from both sides of the anomaly equation and we obtain,

$$r_a L^{ab} \hat{j}^5 = -\frac{1}{48\pi^2 l^3} r_a \epsilon_{bcdef} \hat{F}^{abc} \hat{F}^{def} \quad (59)$$

This is the desired anomalous current divergence equation in the de Sitter space. It is the exact analogue of the ABJ-anomaly equation in the flat space.

There is another way in which the anomaly equation can be expressed. To see this consider the product of the operator on one side of (59) with the radius vector to get,

$$r_c r_a L^{ab} \hat{j}^5 = -l^2 \left(L_{cb} + r_b K^\mu_c \partial_\mu\right) \hat{j}^b_5 \quad (60)$$

where (34) has been used. The second term on the right hand side is simplified by exploiting (33) and the identity,

$$r^b K^\mu_c \partial_\mu K^a_b = -K^a_c \quad (61)$$

to yield,

$$r_c r_a L^{ab} \hat{j}^5 = l^2 \left(\hat{j}^c_5 - L_{cb} \hat{j}^b_5\right) \quad (62)$$

Thus the anomaly equation (59) takes the form,

$$\hat{j}^c_5 - L_{gb} \hat{j}^b_5 = -\frac{1}{48\pi^2 l^3} r_a \epsilon_{bcdef} \hat{F}^{abc} \hat{F}^{def} \quad (63)$$

Compatibility between the two forms (59) and (63) is easily established by contracting the latter with $r^g$ and using the transversality (34) of the current.

The normal Ward identity for the vector current is obtained by setting the right hand side of either (59) or (63) equal to zero.
5 Duality Symmetry

The well known electric-magnetic duality symmetry swapping field equations with the Bianchi identity in flat space has an exact counterpart on the de Sitter hyperboloid. To see this it is essential to introduce the dual field tensor that enters the Bianchi identity. The dual tensor is defined by,

$$\tilde{F}_{ab} = -\frac{1}{6} \varepsilon_{abcde} \hat{F}^{cde}$$  \hspace{1cm} (64)

Using (45) and (57) together with the properties of the Killing vectors the dual on the de Sitter space is expressed in terms of the dual on the flat space as,

$$\tilde{F}_{ab} = l K^\lambda_a K^\rho_b \hat{F}_{\lambda\rho}$$  \hspace{1cm} (65)

where the flat space dual is given by,

$$\hat{F}_{\lambda\rho} = \frac{1}{2} \varepsilon_{\lambda\rho\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (66)

The Bianchi identity is then given by,

$$r^a L^{ab} \tilde{F}_{bc} = 0$$  \hspace{1cm} (67)

This is confirmed by a direct calculation. Alternatively, it becomes transparent by projecting it on the flat space by means of Killing vectors. Using the basic definitions and the identity,

$$K^\rho_b \partial_\rho (K^{bp} K^{\nu c}) \hat{F}_{\mu\nu} = 0$$  \hspace{1cm} (68)

we obtain,

$$r^a L^{ab} \tilde{F}_{bc} = -l^3 K^{bp} K^\nu_c K^\lambda_b \partial_\mu \hat{F}_{\lambda\rho}$$  \hspace{1cm} (69)

Finally, exploiting (26) we get the desired projection,

$$r^a L^{ab} \tilde{F}_{bc} = -l^3 \left(1 - \frac{x^2}{4l^2}\right)^2 K^\rho_c \partial^\lambda \hat{F}_{\lambda\rho}$$  \hspace{1cm} (70)

which vanishes since $\partial^\lambda \hat{F}_{\lambda\rho} = 0$.

Now the abelian equation of motion following from a variation of the action (50) is given by,

$$L_{ab} \hat{F}^{abc} = 0$$  \hspace{1cm} (71)

The duality transformation is next discussed. Analogous to the flat space rule, $\hat{F} \rightarrow F; F \rightarrow -\hat{F}$ the duality map here is provided by,

$$\tilde{F}_{ab} \rightarrow \frac{r^c}{l} \hat{F}_{abc} \hspace{1cm}; \hspace{1cm} \frac{r^c}{l} \hat{F}_{abc} \rightarrow -\tilde{F}_{ab}$$  \hspace{1cm} (72)

It is easy to check the consistency of this map. The inverse of (64) yields,

$$\hat{F}_{abc} = -\frac{1}{2} \varepsilon_{abcde} \tilde{F}^{dec}$$  \hspace{1cm} (73)

Under the first of the maps in (72), the above relation is transformed as,

$$\hat{F}_{abc} \rightarrow \frac{1}{l} \left(r_a \hat{F}_{bc} + r_b \hat{F}_{ca} + r_c \hat{F}_{ab}\right)$$  \hspace{1cm} (74)
where use was made of (73) at an intermediate step. Contracting the above map by $r_c^c$ immediately leads to the second relation in (72).

Now the effect of the duality map on the equation of motion (71) is considered. Using (74) and the correspondence (53) along with the identity (68) we find,

$$\frac{1}{2} L_{ab} \tilde{F}^{abc} \rightarrow -l^2 \left( 1 - \frac{x^2}{4l^2} \right)^2 K^{\rho \mu} \partial_{\mu} \tilde{F}_c$$  \hspace{1cm} (75)

Finally, using (70) we obtain the cherished mapping,

$$\frac{1}{2} L_{ab} \tilde{F}^{abc} \rightarrow \frac{1}{l} r_a L^{ab} \tilde{F}_{bc}$$  \hspace{1cm} (76)

showing how the equation of motion passes over to the Bianchi identity. Likewise the other map swaps the Bianchi identity to the equation of motion.

It is feasible to perform a continuous $SO(2)$ duality rotations through an angle $\theta$. The relevant transformations are then given by,

$$\frac{r^c}{l} F'_{abc} = \cos \theta \frac{r^c}{l} F_{abc} - \sin \theta \tilde{F}_{ab}$$  \hspace{1cm} (77)

$$\tilde{F}'_{ab} = \sin \theta \frac{r^c}{l} F_{abc} + \cos \theta \tilde{F}_{ab}$$  \hspace{1cm} (78)

The discrete duality transformation corresponds to $\theta = \frac{\pi}{2}$.

### 6 Formulation of Antisymmetric Tensor Gauge Theory

The general formalism developed so far is particularly suited for obtaining a formulation of $p$-form gauge theories. Here we discuss it for the second rank antisymmetric tensor gauge theory. Also, there are some features which distinguish it from the analysis for the vector gauge theory. The extension for higher forms is obvious. Both abelian and nonabelian theories will be considered. To set up the formulation it is convenient to begin with the abelian case which can be subsequently generalised to the nonabelian version. The action for a free 2-form gauge theory in flat four-dimensional Minkowski space is given by [17],

$$S = -\frac{1}{12} \int d^4 x F^{\mu \nu \rho} F_{\mu \nu \rho}$$  \hspace{1cm} (79)

where the field strength is defined in terms of the basic field as,

$$F_{\mu \nu \rho} = \partial_\mu B_{\nu \rho} + \partial_\nu B_{\rho \mu} + \partial_\rho B_{\mu \nu}$$  \hspace{1cm} (80)

The infinitesimal gauge symmetry is given by the transformation,

$$\delta B_{\mu \nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$$  \hspace{1cm} (81)

which is reducible since it trivialises for the choice $\Lambda_\mu = \partial_\mu \lambda$. 
It is sometimes useful to express the action (or the lagrangian) in a first order form by introducing an extra field,

\[ \mathcal{L} = -\frac{1}{8} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} B^{\rho\sigma} + \frac{1}{8} A^\mu A_\mu \]  

(82)

where the \( B \wedge F \) term involves the field tensor,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]  

(83)

Eliminating the auxiliary \( A_\mu \) field by using its equation of motion, the previous form (79) is reproduced. The gauge symmetry is given by (81) together with \( \delta A_\mu = 0 \). The first order form is ideal for analysing the nonabelian theory.

To express the theory on the de Sitter pseudosphere, the mapping of the tensor field is first given. From the previous analysis, it is simply given by,

\[ \hat{B}_{ab} = K_\mu^a K_\nu^b B_{\mu\nu} \]  

(84)

The tensor field with the latin indices is defined on the pseudosphere while those with the greek symbols are the usual one on the flat space. This is written in component notation by using the explicit form for the Killing vectors given in (21) and (22),

\[ \hat{B}_{\mu\nu} = \left(1 - \frac{x^2}{4l^2}\right) \left(1 - \frac{x^2}{4l^2}\right) B_{\mu\nu} + \frac{x^\rho x_\nu}{2l^2} B_{\mu\rho} + \frac{x^\rho x_\mu}{2l^2} B_{\rho\nu} \]  

(85)

and,

\[ \hat{B}_{\mu4} = \frac{1}{l} \left(1 - \frac{x^2}{4l^2}\right) x^\rho B_{\mu\rho} \]  

(86)

These are the analogues of (24). The inverse relation is given by,

\[ \left(1 - \frac{x^2}{4l^2}\right)^4 B^{\mu\nu} = K_\mu^a K_\nu^b \hat{B}_{ab} \]  

(87)

which may also be put in the form,

\[ \left(1 - \frac{x^2}{4l^2}\right)^2 B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{x_\mu \hat{B}_{\nu4}}{2l} - \frac{x_\nu \hat{B}_{\mu4}}{2l} \]  

(88)

which is the direct analogue of (23).

Next, the gauge transformations are discussed. From (81), the defining relation (84) and the angular momentum operator (33), infinitesimal transformations are given by,

\[ \delta \hat{B}_{ab} = -\frac{r^c}{l^2} (K_\mu^a L_{ca} - K_\mu^b L_{cb}) \Lambda_\mu \]  

(89)

In this form the expression is not manifestly covariant. This may be contrasted with (40) which has this desirable feature. The point is that an appropriate map of the gauge parameter is necessary. In the previous example the gauge parameter was a scalar which retained its form. Here, since it is a vector, the required map is provided by a relation like (9), so that,

\[ \hat{\Lambda}_a = K_\mu^a \Lambda_\mu \]  

(90)
Pushing the Killing vectors through the angular momentum operator and using the above map yields, after some simplifications,

$$\delta \hat{B}_{ab} = -\frac{1}{l^2} \left[ r^c \left( L_{ca} \hat{A}_b - L_{cb} \hat{A}_a \right) - r_a \hat{A}_b + r_b \hat{A}_a \right]$$  \hspace{1cm} (91)

It is also reassuring to note that (91) manifests the reducibility of the gauge transformations. Since \( \Lambda_\mu = \partial_\mu \lambda \) leads to a trivial gauge transformation in flat space, it follows from (91) that the corresponding feature should be present in the pseudospherical formulation when,

$$\hat{\Lambda}_a = r^c L_{ca} \lambda$$  \hspace{1cm} (92)

It is easy to check that with this choice, the gauge transformation (91) trivialises; i.e. \( \delta \hat{B}_{ab} = 0 \).

The field tensor on the pseudosphere is constructed from the usual one given in (80). Since the Killing vectors play the role of the metric in connecting the two surfaces, this expression is given by a natural extension of (45),

$$\hat{F}_{abcd} = \left( r_a K_b^\mu K_c^\nu K_d^\rho - r_b K_a^\mu K_c^\nu K_d^\rho + r_c K_a^\mu K_b^\nu K_d^\rho + r_d K_a^\mu K_b^\nu K_c^\rho \right) F_{\mu\nu\rho}$$  \hspace{1cm} (93)

Note that cyclic permutations have to taken carefully since there is an even number of indices.

In terms of the basic variables, the field tensor is expressed as,

$$\hat{F}_{abcd} = \left( L_{ab} \hat{B}_{cd} + L_{bc} \hat{B}_{ad} + L_{ca} \hat{B}_{bd} + L_{da} \hat{B}_{ca} + L_{cd} \hat{B}_{ab} \right)$$  \hspace{1cm} (94)

To show that (93) is equivalent to (94), the same strategy as before, is adopted. Using the definition of the angular momentum (33), (94) is simplified as,

$$\hat{F}_{abcd} = \left( r_a K_b^\mu K_c^\nu K_d^\rho - r_b K_a^\mu K_c^\nu K_d^\rho \right) \partial_\mu \left( K_c^\nu K_d^\rho B_{\nu\rho} \right) + \ldots \ldots \ldots$$  \hspace{1cm} (95)

where the carets denote the inclusion of other similar (cyclically permuted) terms. Now there are two types of contributions. Those where the derivatives act on the Killing vectors and those where they act on the fields. The first class of terms cancel out as a consequence of an identity that is an extension of (17). The other class combines to reproduce (93).

The action on the de Sitter pseudosphere is now obtained by first taking a repeated product of the field tensor (93). Using the properties of the Killing vectors, this yields,

$$\hat{F}_{abcd} \hat{F}^{abcd} = -4l^2 \left( 1 - \frac{x^2}{4l^2} \right)^6 F_{\mu\nu\rho} F^{\mu\nu\rho}$$  \hspace{1cm} (96)

From the definition of the flat space action (79) and the volume element (28), it follows that the above identification leads to the pseudospherical action,

$$S_\Omega = \frac{1}{48l^2} \int d\Omega \left( 1 - \frac{x^2}{4l^2} \right)^{-2} \hat{F}_{abcd} \hat{F}^{abcd}$$  \hspace{1cm} (97)

Thus, up to a conformal factor, the corresponding lagrangian is given by,

$$\mathcal{L}_\Omega = \frac{1}{48l^2} \hat{F}_{abcd} \hat{F}^{abcd}$$  \hspace{1cm} (98)
By its very construction this lagrangian would be invariant under the gauge transformation (91). There is however another type of gauge symmetry which does not seem to have any analogue in the flat space. To envisage such a possibility, consider a transformation of the type

\[
\delta \hat{B}_{ab} = L_{ab} \lambda \quad (99)
\]

which could be a meaningful gauge symmetry operation on the de Sitter space. However, in flat space, it leads to a trivial gauge transformation. To see this explicitly, consider the effect of (99) on (87),

\[
\left(1 - \frac{x^2}{4l^2}\right)^4 \delta B^{\mu \nu} = K_\nu^K_\nu L_{ab} \lambda \quad (100)
\]

Inserting the expression for the angular momentum from (33) and exploiting the transversality (7) of the Killing vectors, it follows that,

\[
\delta B_{\mu \nu} = 0 \quad (101)
\]

thereby proving the statement. To reveal that (99) indeed leaves the lagrangian (98) invariant, it is desirable to recast it in the form,

\[
\mathcal{L}_\Omega = \frac{1}{32} \hat{\Sigma}_a \hat{\Sigma}^a \quad (102)
\]

where,

\[
\hat{\Sigma}_a = \epsilon_{abcde} L_{bc} L_{de} \quad (103)
\]

Under the gauge transformation (99), a simple algebra shows that \(\delta \hat{\Sigma}_a = 0\) and hence the lagrangian remains invariant.

The inclusion of a nonabelian gauge group is feasible. Results follow logically from the abelian theory with suitable insertion of the nonabelian indices. As remarked earlier it is useful to consider the first order form (82). The lagrangian is given by its straightforward generalisation (83), where the nonabelian field strength has already been defined in (36). It is gauge invariant under the nonabelian generalisation of (81) with the ordinary derivatives replaced by the covariant derivatives with respect to the potential \(A_\mu\), and \(\delta A_\mu = 0\). By the help of our equations it is possible to project this lagrangian on the de Sitter space. For instance, the corresponding gauge transformations look like,

\[
\delta \hat{B}_{ab} = -\frac{1}{l^2} \left[ r^c \left( L_{ca} \hat{\Lambda}_b - L_{cb} \hat{\Lambda}_a \right) - r_a \hat{\Lambda}_b + r_b \hat{\Lambda}_a \right] + [\hat{A}_a, \hat{\Lambda}_b] \quad (104)
\]

and so on.

Matter fields may be likewise defined. The fermion current \(j_{\mu \nu}\) will be defined just as the two form field, except that conformal weight factors appear, so that form invariance of the interaction is preserved,

\[
\int dx (j_{\mu \nu} B^{\mu \nu}) = \int d\Omega (\hat{j}_{ab} \hat{B}^{ab}) \quad (105)
\]

quite akin to (52).

\[\text{3Recently such a transformation was considered on the hypersphere [18, 12].}\]
7 Zero Curvature Limit

The null curvature limit (which is also equivalent to a vanishing cosmological constant) is obtained by setting $l \to \infty$. Then the de Sitter group contracts to the Poincare group so that the field theory on the de Sitter space should contract to the corresponding theory on the flat Minkwski space. This is shown very conveniently in the present formalism using Killing vectors. The example of Yang Mills theory with sources will be considered.

The equation of motion in the de Sitter space obtained by varying the action composed of the pieces (50) and (52) is found to be,

$$\frac{1}{2l^2} \tilde{\mathcal{L}}_{ab} \tilde{F}^{abc} - \tilde{j}^c = 0$$

(106)

The operator appearing in the above equation is now mapped to the flat space. The mapping for the usual angular momentum part is first derived,

$$L_{ab} \tilde{F}^{abc} = 2r_a K_b^\mu \partial_\mu \left( [r^a K^{aw} K^{cw} + c.p.] F_{\nu \rho} \right)$$

(107)

Using the transversality condition and the identities among the Killing vectors it is seen that the only nonvanishing contribution comes from the action of the derivative on the field tensor yielding,

$$L_{ab} \tilde{F}^{abc} = -2l^2 \left( 1 - \frac{x^2}{4l^2} \right)^2 K_{c\rho} \partial_\mu F_{\mu \rho}$$

(108)

It is straightforward to generalise this for the covariantised angular momentum and one finds,

$$\tilde{\mathcal{L}}_{ab} \tilde{F}^{abc} = -2l^2 \left( 1 - \frac{x^2}{4l^2} \right)^2 K_{c\rho} D_\mu F_{\mu \rho}$$

(109)

Using the map (53) for the currents, the equation of motion on the de Sitter space finally gets projected on the flat space as,

$$\left( 1 - \frac{x^2}{4l^2} \right)^2 K_{c\rho} \left( D^\mu F_{\mu \rho} + \tilde{j}_\rho \right) = 0$$

(110)

This equation is now multiplied by the Killing vector $K_c^\lambda$. Using the identity among the Killing vectors yields,

$$\left( 1 - \frac{x^2}{4l^2} \right)^4 \left( D^\mu F_{\mu \lambda} + \tilde{j}_\lambda \right) = 0$$

(111)

The zero curvature limit ($l \to \infty$) is now taken. The prefactor simplifies to unity and the standard flat space Yang Mills equation with sources is reproduced.

8 Discussions

We have provided a manifestly covariant formulation of vector and tensor gauge theories on the de Sitter hyperboloid. It was done by mapping the usual forms of these theories, defined on the Minkowski flat surface, onto the hyperboloid by the method of stereographic projection. A distinctive feature was the abstraction of the relevant conformal Killing vectors by solving the Cartan-Killing equation. The importance of these Killing vectors
lay in the fact that tensor forms constructed by taking their products acted like a metric connecting the results between the flat space and the hyperboloid. This essentially new ingredient was crucial for generalisations to include higher form gauge theories. We feel that the present prescription relates theories in flat space time with those in de Sitter space time compactly and elegantly. All technicalities are reduced to simple algebraic properties of the Killing vectors. Although our analysis was shown for the two form case, extension to any $p$-form gauge theory is clear. Also, the present analysis is easily applicable for completely symmetric tensor fields as well as for arbitrary dimensions with obvious changes. These changes entail trivial modifications in the properties of the Killing vectors.

An advantage of this approach is that derivatives like $\frac{\partial}{\partial r^a}$ always occur as the angular momentum $L_{ab} = r_a \partial_b - r_b \partial_a$ in the electromagnetic field strengths or in the equations of motion. In the ambient space approach [1, 2, 3, 4] both types of (linear and angular) derivatives occur. Consequently homogeneity conditions [1, 2, 3, 4] are required in order to avoid going off the hypersurface of constant $r^2$. This is completely avoided in our analysis.

The expressions throughout this paper have the most simple and natural Minkowskian-type structures for obvious reasons. Indeed a chunk of the results on the hypersphere done by us [12, 13] could be appropriately manipulated to yield results on the de Sitter space. This was feasible since the de Sitter space is essentially a pseudosphere that is related to the hypersphere by a Wick-like rotation. The possibility that a hyperspherical analysis could be useful for a de Sitter analysis was already mooted by Adler [20], though it was in a very restrictive sense valid for abelian theories and without the powerful use of Killing vectors.

As applications we have provided new results for the axial anomaly as well as shown the existence of electric-magnetic duality rotations on the de Sitter hyperboloid. Both these phenomenon were obtained by a direct mapping of the known results on the flat Minkowski space. Indeed we feel that our approach gives an intuitive understanding of the closeness of the formulation of gauge field theories on flat and de Sitter spaces. Incidentally, formulating a gauge theory on the de Sitter space that mimics the corresponding formulation on the flat space has been an important objective of several authors [4, 11, 15, 16, 17, 18, 19].

Finally the zero curvature ($l \to \infty$) limit was discussed. The results of de Sitter space gauge theory expectedly contracted to the Minkowski space gauge theory thereby vindicating our analysis. In cosmological terms this limit is essentially the zero cosmological constant limit. The other extreme ($l \to 0$) corresponds to the infinite cosmological constant limit. In this case the de Sitter space tends to the conic spacetime [20]. It would be interesting to construct a gauge theory on this conic space and see whether the results given here pass on to that theory in the ($l \to 0$) limit.

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