Hungry Volterra equation, multi boson KP hierarchy and Two Matrix Models

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October 9, 2018

Abstract

We consider the hungry Volterra hierarchy from the view point of the multi boson KP hierarchy. We construct the hungry Volterra equation as the Bäcklund transformations (BT) which are not the ordinary ones. We call them “fractional” BT. We also study the relations between the (discrete time) hungry Volterra equation and two matrix models. From this point of view we study the reduction from (discrete time ) 2d Toda lattice to the (discrete time ) hungry Volterra equation.
1 Introduction

The $W$ algebra was introduced by Zamolodchikov as an additional symmetry of the conformal field theory. The classical $W_N$ algebra can be constructed from the free fields $r_i$ for $i = 1, 2, \cdots, N$ satisfying the Poisson algebra

$$\{r_i(x), r_j(y)\} = (-\delta_{i,j} + \frac{1}{N})\delta(x-y).$$

(1.1)

The Poisson structure is related $N$-reduced KP hierarchy. The Poisson map from the free fields to the pseudo differential operator $L$ is defined as

$$L = (\partial + r_1(x))(\partial + r_2(x)) \cdots (\partial + r_N(x))$$

(1.2)

with $\partial \equiv \partial/\partial x$, and the constraint

$$\sum_{i=1}^{N} r_i(x) = 0.$$  

(1.3)

Recently deformations of the $W$ algebra received much attention. One is the $q$ deformation of the $W$ algebra. Another deformation of the $W$ algebra is lattice $W$ algebra, which may be related to the $sl_N$ Toda theory. The most famous example is the Lotka-Volterra model. It is well known that the Lotka-Volterra model reduces to the Korteweg-de Vries (KdV) equation in the continuous limit, and that it can be formulated in terms of lattice Virasoro algebra.

In this paper we shall study the hungry Volterra equation (or “Bogoyavlensky lattice”) which is known as an extended Volterra model. It is pointed out that the hungry Volterra equation is a fundamental integrable system of the lattice $W_N$ algebra. We define the $k$-hungry Volterra equation as

$$\frac{dV_n}{dt_1} = V_n(\sum_{i=1}^{k-1} V_{n+i} - \sum_{i=1}^{k-1} V_{n-i}).$$  

(1.4)

In the case $k = 1$ (1.4) becomes the Lotka-Volterra equation.

We consider the hungry Volterra equation as Bäcklund transformations (BT) of the multi boson KP equation which is related to the $sl(k+1,k)$ algebra. We call the BT “fractional BT”. If we repeat the fractional BT $k$ times we can obtain the usual BT. To study relations between the 2d Toda lattice and the hungry Volterra equation we consider the hungry Volterra equation in two matrix models. The most simple model becomes a matrix model for the bi-colored random triangulation. Furthermore we consider the discrete time hungry Volterra equation in two matrix models with Penner type potential. It becomes a simple relation of the partition functions in bilinear form.

This paper is organized as follows. In section 2 we consider the eigenvalue problems of the 3-hungry Volterra equation. In section 3 we define the fractional BT and it is
equivalent to the $k$-hungry Volterra equation. In section 4 we consider the reduction from the 3-hungry Volterra equation to the classical $W_3$ and Virasoro algebra. In section 5 we obtain the hungry Volterra equation and the discrete time hungry Volterra equation in two matrix models. The last section is devoted to the concluding remarks.

## 2 Hungry Volterra equation

We consider the spectral problems,

\[
\lambda^{1/3} \hat{\Psi}_{n+\frac{2}{3}} = \Psi_{n+1} + u_n \Psi_n, \\
\lambda^{1/3} \hat{\Psi}_{n+\frac{1}{3}} = \Psi_{n+\frac{1}{3}} + v_n \Psi_n, \\
\lambda^{1/3} \hat{\Psi}_n = \Psi_{n+\frac{2}{3}} + w_n \Psi_{n-\frac{2}{3}}, \tag{2.1}
\]

From (2.1) we can obtain

\[
\begin{align*}
\lambda \Psi_n &= \Psi_{n+1} + a_0(n) \Psi_n + a_1(n) \Psi_{n-1} + a_2(n) \Psi_{n-2}, \\
\lambda \hat{\Psi}_{n+\frac{1}{3}} &= \hat{\Psi}_{n+\frac{1}{3}} + \hat{a}_0(n) \hat{\Psi}_{n+\frac{1}{3}} + \hat{a}_1(n) \hat{\Psi}_{n-\frac{2}{3}} + \hat{a}_2(n) \hat{\Psi}_{n-\frac{5}{3}}, \\
\lambda \hat{\Psi}_{n+\frac{2}{3}} &= \hat{\Psi}_{n+\frac{2}{3}} + \hat{a}_0(n) \hat{\Psi}_{n+\frac{2}{3}} + \hat{a}_1(n) \hat{\Psi}_{n-\frac{5}{3}} + \hat{a}_2(n) \hat{\Psi}_{n-\frac{8}{3}}, \tag{2.2}
\end{align*}
\]

where

\[
\begin{align*}
a_0(n) &= u_n + v_n + w_n, \quad a_1(n) = v_n u_{n-1} + u_{n-1} w_n + w_n v_{n-1}, \quad a_2(n) = w_n v_{n-1} u_{n-2}, \\
\hat{a}_0(n) &= w_{n+1} + u_n + v_n, \quad \hat{a}_1(n) = u_n w_n + w_n v_n + v_n u_{n-1}, \quad \hat{a}_2(n) = v_n u_{n-1} w_{n-1}, \\
\hat{a}_0(n) &= v_{n+1} + w_{n+1} + u_n, \quad \hat{a}_1(n) = w_{n+1} v_n + v_n u_n + u_n w_n, \quad \hat{a}_2(n) = u_n w_n v_{n-1}. \tag{2.3}
\end{align*}
\]

Using (2.2) we can obtain the first lattice flow $\partial = \partial_x$

\[
\begin{align*}
\partial \Psi_n &= \Psi_{n+1} + a_0(n) \Psi_n, \\
\partial \hat{\Psi}_{n+\frac{1}{3}} &= \hat{\Psi}_{n+\frac{1}{3}} + \hat{a}_0(n) \hat{\Psi}_{n+\frac{1}{3}}, \\
\partial \hat{\Psi}_{n+\frac{2}{3}} &= \hat{\Psi}_{n+\frac{2}{3}} + \hat{a}_0(n) \hat{\Psi}_{n+\frac{2}{3}}. \tag{2.4}
\end{align*}
\]

From (2.2) and (2.4) we can obtain the consistency conditions,

\[
\begin{align*}
\frac{da_0(n)}{dx} &= a_1(n + 1) - a_1(n), \\
\frac{da_1(n)}{dx} &= a_1(n - 1) + \frac{da_0(n - 1)}{dx}, \\
\frac{da_2(n)}{dx} &= a_2(a_0(n) - a_0(n - 2)). \tag{2.5}
\end{align*}
\]

In the same way we can obtain same equations about $\hat{a}(n)$ and $\hat{a}(n)$. These are the generalized Toda equations.
On the other hands, using the consistency condition of (2.4) and (2.4) the equations of motion read

\[ \frac{du_n}{dx} = u_n(-v_n - w_n + v_{n+1} + w_{n+1}), \]
\[ \frac{dv_n}{dx} = v_n(-u_n - w_n + u_{n+1} + w_{n+1}), \]
\[ \frac{dw_n}{dx} = w_n(-u_{n-1} - v_{n-1} + u_n + v_n). \] (2.6)

These are 3-hungry Volterra equations. (2.1) and (2.4) can be cast into the form

\[ \lambda \Psi_n = L_n \Psi_n, \quad \lambda \tilde{\Psi}_{n+\frac{1}{3}} = \tilde{L}_{n+\frac{1}{3}} \tilde{\Psi}_{n+\frac{1}{3}}, \quad \lambda \hat{\Psi}_{n+\frac{2}{3}} = \hat{L}_{n+\frac{2}{3}} \hat{\Psi}_{n+\frac{2}{3}}, \] (2.7)

where

\[ L_n = (\partial - v_{n-1} - u_{n-1})(\partial - v_{n-1} - u_{n-1} - w_n)^{-1}(\partial - u_{n-1} - w_n), \]
\[ (\partial - u_{n-1} - w_n - v_n)^{-1}(\partial - w_n - v_n), \]
\[ \tilde{L}_{n+\frac{1}{3}} = (\partial - u_{n-1} - w_n)(\partial - u_{n-1} - u_{n-1} - v_n)^{-1}(\partial - v_n - w_n), \]
\[ (\partial - v_n - w_n - u_n)^{-1}(\partial - u_n - v_n), \]
\[ \hat{L}_{n+\frac{2}{3}} = (\partial - u_n - v_n)(\partial - w_n - u_n - v_n)^{-1}(\partial - u_n - v_n), \]
\[ (\partial - u_n - v_n - w_{n+1})^{-1}(\partial - u_n - w_{n+1}). \] (2.8)

This is the form of the 4-boson KP hierarchy. Using (2.2) and (2.4) we can obtain the Bäcklund Transformations

\[ L_{n+1} = (\partial - a_0(n))L_{n+1}(\partial - a_0)^{-1}, \]
\[ \tilde{L}_{n+1+\frac{1}{3}} = (\partial - \tilde{a}_0(n))\tilde{L}_{n+1+\frac{1}{3}}(\partial - \tilde{a}_0(n))^{-1}, \]
\[ \hat{L}_{n+1+\frac{2}{3}} = (\partial - \hat{a}_0(n))\hat{L}_{n+1+\frac{2}{3}}(\partial - \hat{a}_0(n))^{-1}. \] (2.9)

### 3 Fractional Bäcklund Transformations

We consider multi boson KP hierarchy:

\[ L^{(k)} = (\partial - q_1)(\partial - \tilde{q}_1)^{-1}(\partial - q_2)(\partial - \tilde{q}_2)^{-1} \cdots (\partial - \tilde{q}_{k-1})^{-1}(\partial - q_k), \] (3.1)

with a Dirac constraint

\[ \sum_{j=1}^{k} q_j - \sum_{l=1}^{k-1} \tilde{q}_l = 0. \] (3.2)

We construct “fractional” Bäcklund transformations (BT),

\[ L_{n+\frac{i+\frac{1}{3}}{k}}^{(k)} = (T_{n+\frac{i+\frac{1}{3}}{k}}^{(k)})^{-1}L_{n+\frac{i+\frac{1}{3}}{k}}^{(k)}T_{n+\frac{i+\frac{1}{3}}{k}}^{(k)}, \] (3.3)
where
\[ T_{n+\frac{1}{k}}^{(k)} = (\partial - q_1^{(n+\frac{1}{k})})(\partial - q_1^{(n+\frac{1}{k})})^{-1}. \] (3.4)

Here we realize that the normal lattice jump \( n \rightarrow n + 1 \) can be given a meaning of the BT,
\[ L_{n+1+\frac{1}{k}}^{(k)} = (\partial - q_1^{(n+1+\frac{1}{k})})(\partial - q_1^{(n+1+\frac{1}{k})})^{-1}(\partial - q_2^{(n+1+\frac{1}{k})})(\partial - q_2^{(n+1+\frac{1}{k})})^{-1} \ldots \]
\[ \times (\partial - q_{k-1}^{(n+1+\frac{1}{k})})^{-1}(\partial - q_k^{(n+1+\frac{1}{k})}), \] (3.5)

and
\[ L_{n+1+\frac{1}{k}}^{(k)} = (T_{n+\frac{1}{k}}^{(k)})^{-1} L_{n+\frac{1}{k}}^{(k)} T_{n+\frac{1}{k}}^{(k)}. \] (3.6)

Using (3.3) and (3.6) we can obtain a relation
\[ T_{n+\frac{1}{k}}^{(k)} = T_{n+1+\frac{1}{k}}^{(k)} T_{n+\frac{1}{k}+1+\frac{1}{k}}^{(k)} \ldots T_{n+\frac{1}{k}}^{(k)}. \] (3.7)

From (3.3) we get
\[ T_{n+\frac{1}{k}}^{(k)} = (\partial - q_1^{(n+\frac{1}{k})})(\partial - q_1^{(n+\frac{1}{k})})^{-1} \]
\[ = (\partial - q_{k-1}^{(n+\frac{1}{k})})^{-1}(\partial - q_k^{(n+\frac{1}{k})}) = (\partial - q_{k-1}^{(n+\frac{1}{k})})^{-1}(\partial - q_1^{(n+\frac{1}{k})}). \] (3.8)

To obtain the last equation we use relations
\[ (\partial - q_{k-1}^{(n+\frac{1}{k})}) = (\partial - q_k^{(n+\frac{1}{k})}) = \ldots = (\partial - q_1^{(n+\frac{1}{k})}). \] (3.9)

To satisfy (3.2) using (3.8) we can set
\[ -q_1^{(n+\frac{1}{k})} + q_1^{(n+\frac{1}{k})} = q_k^{(n+\frac{1}{k})} - q_1^{(n+\frac{1}{k})} \equiv p_{n+\frac{1}{k}}. \] (3.10)

Using (3.10) we can rewrite (3.8)
\[ (\partial - q_1^{(n+\frac{1}{k})})(\partial - q_1^{(n+\frac{1}{k})} - p_{n+\frac{1}{k}})^{-1} = (\partial - q_1^{(n+\frac{1}{k})} - p_{n+\frac{1}{k}})^{-1}(\partial - q_1^{(n+\frac{1}{k})}). \] (3.11)

Then (3.11) becomes an equation
\[ \frac{dp_{n+\frac{1}{k}}}{dx} = p_{n+\frac{1}{k}}(q_1^{(n+\frac{1}{k})} - q_1^{(n+\frac{1}{k})}). \] (3.12)

Using the fractional BT recursively we can obtain
\[ L_{n+1+\frac{1}{k}}^{(k)} = (\partial - q_1^{(n+1+\frac{1}{k})})(\partial - q_1^{(n+1+\frac{1}{k})} - p_{n+1+\frac{1}{k}})^{-1}(\partial - q_2^{(n+1+\frac{1}{k})})(\partial - q_2^{(n+1+\frac{1}{k})} - p_{n+1+\frac{1}{k}})^{-1} \ldots \]
\[ \times (\partial - q_{k-1}^{(n+1+\frac{1}{k})} - p_{n+\frac{i}{k}})^{-1}(\partial - q_k^{(n+1+\frac{1}{k})}). \] (3.13)

From (3.2) we get
\[ q_k^{(n+1+\frac{1}{k})} = \sum_{j=1}^{k-1} p_{n+\frac{i}{k}}. \] (3.14)
If we substitute (3.14) into (3.12), we can obtain

\[ \partial p_{n+\frac{1}{k}} = p_{n+\frac{1}{k}} \left( \sum_{j=1}^{k-1} p_{n+\frac{i+j}{k}} - \sum_{j=1}^{k-1} p_{n-1+\frac{i+j}{k}} \right). \]  

(3.15)

(3.15) is nothing but the k-hungry Volterra equation. Using (3.7) we can obtain the normal lattice BT

\[ T_{n+\frac{1}{k}}^{(k)} = (\partial - \sum_{j=1}^{k} p_{n+\frac{i+j}{k}}). \]  

(3.16)

In the case \( k = 3 \) it corresponds to (2.9).

4 \( W \) algebra and Torus

We can rewrite the Lax operator (2.8) using (2.6) or directly from (2.4),

\[ L_n = (\partial - u_n - v_n - w_n)^{-1} (\partial - u_{n+1} - v_{n+1} - w_{n+1})^{-1} \]
\[ \hat{L}_{n+\frac{1}{3}} = (\partial - u_n - v_n - w_{n+1})^{-1} (\partial - u_{n+1} - v_{n+1} - w_{n+2})^{-1} \]
\[ \hat{L}_{n+\frac{2}{3}} = (\partial - u_n - v_{n+1} - w_{n+1})^{-1} (\partial - u_{n+2} - v_{n+1} - w_{n+1})^{-1} \]

\[ \hat{L}_{n+1} = (\partial - u_n - v_{n+1} - w_{n+1}) (\partial - u_{n+1} - v_{n+1} - w_{n+1}). \]  

(4.1)

We constraint this system under the two-periodic condition

\[ \Psi_{n+2} = \mu \Psi_n, \quad \tilde{\Psi}_{n+2+\frac{1}{3}} = \mu \tilde{\Psi}_{n+\frac{1}{3}}, \quad \tilde{\Psi}_{n+2+\frac{2}{3}} = \mu \tilde{\Psi}_{n+\frac{2}{3}}, \]

(4.2)

where \( \mu \) is an arbitrary constant. Furthermore we set the total energy,

\[ H_1 = - \sum_n (u_n + v_n + w_n) = 0. \]  

(4.3)

Using (4.1) we can obtain

\[ \lambda \mu \Psi_n = L_n^{(1)} \Psi_n, \quad \mu \Psi_n = L_n^{(2)} \Psi_n, \]
\[ \lambda \mu \tilde{\Psi}_{n+\frac{1}{3}} = \hat{L}_{n+\frac{1}{3}}^{(1)} \tilde{\Psi}_{n+\frac{1}{3}}, \quad \mu \tilde{\Psi}_{n+\frac{1}{3}} = \hat{L}_{n+\frac{1}{3}}^{(2)} \tilde{\Psi}_{n+\frac{1}{3}}, \]
\[ \lambda \mu \tilde{\Psi}_{n+\frac{2}{3}} = \hat{L}_{n+\frac{2}{3}}^{(1)} \tilde{\Psi}_{n+\frac{2}{3}}, \quad \mu \tilde{\Psi}_{n+\frac{2}{3}} = \hat{L}_{n+\frac{2}{3}}^{(2)} \tilde{\Psi}_{n+\frac{2}{3}}. \]  

(4.4)

where

\[ L_n^{(2)} = (\partial - u_n - v_n - w_n)(\partial - u_{n+1} - v_{n+1} - w_{n+1}), \]
\[ L_n^{(1)} = (\partial - u_{n+1} - v_{n+1})(\partial - u_n - w_{n+1})(\partial - v_n - w_n), \]
\[ \hat{L}_{n+\frac{1}{3}}^{(2)} = (\partial - u_n - v_n - w_{n+1})(\partial - u_{n+1} - v_{n+1} - w_n). \]
\[
\begin{align*}
\hat{L}_{n+\frac{1}{3}}^{(1)} &= (\partial - u_{n+1} - w_{n})(\partial - v_{n+1} - w_{n+1})(\partial - u_{n} - v_{n}), \\
\hat{L}_{n+\frac{2}{3}}^{(2)} &= (\partial - u_{n} - v_{n+1} - w_{n+1})(\partial - u_{n+1} - v_{n} - w_{n}), \\
\hat{L}_{n}^{(1)} &= (\partial - v_{n} - w_{n})(\partial - u_{n+1} - v_{n+1})(\partial - u_{n} - w_{n+1}).
\end{align*}
\] (4.5)

Then the Lax operators split into two parts. The two parts are the $3$ and $2$ reduced type respectively for the constraint (4.3). It is a torus. But the equations of motion are still 3-hungry Volterra equations.

Furthermore we set
\[
\Psi_{n} = \Psi_{n+1} = \Psi, \quad \tilde{\Psi}_{n+\frac{1}{3}} = \tilde{\Psi}_{n+\frac{4}{3}} = \tilde{\Psi}, \quad \hat{\Psi}_{n+\frac{2}{3}} = \hat{\Psi}_{n+\frac{5}{3}} = \hat{\Psi},
\] (4.6)
then the eigenvalue problems become
\[
\lambda \Psi = L^{(1)} \Psi, \quad \lambda \tilde{\Psi} = \tilde{L}^{(1)} \tilde{\Psi}, \quad \lambda \hat{\Psi} = \hat{L}^{(1)} \hat{\Psi},
\] (4.7)

where
\[
\begin{align*}
L^{(1)} &= (\partial - u - v)(\partial - u - w)(\partial - v - w), \\
\tilde{L}^{(1)} &= (\partial - u - w)(\partial - v - w)(\partial - u - v), \\
\hat{L}^{(1)} &= (\partial - v - w)(\partial - u - v)(\partial - u - w).
\end{align*}
\] (4.8)

From (4.8) we can obtain the classical $W_{3}$ algebra.

On the other hand if we set
\[
\tilde{\Psi} = \hat{\Psi} = 0,
\] (4.9)
the spectral problem becomes
\[
\lambda \mu \Psi_{n} = L_{n}^{(2)} \Psi_{n},
\] (4.10)

where
\[
L_{n}^{(2)} = (\partial - u_{n})(\partial - u_{n+1})
\] (4.11)

(4.11) is nothing but the classical Virasoro algebra.

5 Two Matrix Model and Hungry Volterra equation

5.1 Hungry Volterra equation

We consider the computation of the following integral over two Hermitian matrices $U$ and $V$ of the size $N \times N$ (see [9] for a review)
\[
Z(t_{3k}, \tilde{t}_{3k}, N) = \int dU dV e^{-\text{Tr}V(t, \tilde{t})},
\] (5.12)
with the potential

\[ V(U, V, t_{3k}, \tilde{t}_{3k}) = \sum_{k=1}^{m_1} t_{3k} U_{3k} + \sum_{k=1}^{m_2} \tilde{t}_{3k} V_{3k} + gUV, \]  

(5.13)

and with respect to the Haar measure over Hermitian matrices \( H \)

\[ dH = c_N \prod_{i=1}^{N} dH_{ii} \prod_{1 \leq i < j \leq N} dReH_{ij} dImH_{ij}. \]  

(5.14)

The parameters \( t_{3k}, \tilde{t}_{3k}, g \) are real numbers. We reduce the integral to an integral over the eigenvalues of \( U \) and \( V \), denoted by \( u_i \) and \( v_i \) respectively. In the well known method we can obtain the reduced integral

\[ Z(t, \tilde{t}_{3k}; N) = \int du dv \Delta(u) \Delta(v) e^{-TV(u, v, t_{3k}, \tilde{t}_{3k})}, \]  

(5.15)

where \( \Delta(u) = \prod_{i<j}(u_i - u_j) \) and \( \Delta(v) = \prod_{i<j}(v_i - v_j) \) denote the Vandermonde determinant of matrix \( u \) and \( v \) where \( u = \text{diag}(u_1, \cdots, u_N) \) and \( v = \text{diag}(v_1, \cdots, v_N) \). Here even if we rotate the eigen values

\[ u_i \to e^{\frac{i2\pi}{3}} u_i, \quad v_i \to e^{-\frac{i2\pi}{3}} v_i, \]  

(5.16)

(5.15) does not change from the form of the potential (5.13). As in the standard orthogonal polynomial technique, we introduce two sets of polynomials \( p_n = x^n + \text{lower degree} \) and \( \tilde{p}_n(y) = y^n + \text{lower degree} \), for \( n = 0, 1, 2, \cdots \), which are orthogonal with respect to the one dimensional measure inherited from (5.13), namely

\[ (p_n, \tilde{p}_m) = \int dx dy e^{-V(t_{3k}, \tilde{t}_{3k})} p_n(x) \tilde{p}_m(y) = h_n \delta_{m,n}. \]  

(5.17)

Using the multi-linearity of the determinants, we may rewrite

\[ \Delta(u) = \det[u_i^{j-1}]_{1 \leq i, j \leq N} = \det[p_{j-1}(u_i)]_{1 \leq i, j \leq N}, \]  

\[ \Delta(v) = \det[v_i^{j-1}]_{1 \leq i, j \leq N} = \det[\tilde{p}_{j-1}(v_i)]_{1 \leq i, j \leq N}. \]  

(5.18)

Using the orthogonality relations, the partition function is finally can be written as

\[ Z(t_{3k}, \tilde{t}_{3k}; N) = \text{const.} \prod_{i=0}^{N-1} h_i. \]  

(5.19)

Using the fact (5.16) we can obtain a relation

\[ p_n(x e^{\frac{2\pi i}{3}}) \tilde{p}_n(y e^{-\frac{2\pi i}{3}}) = p_n(x) p_n(y). \]  

(5.20)

The multiplication by \( x \) and \( y \) can be represented by the Jacobi matrixes \( Q \) and \( \tilde{Q} \):

\[ xp_n(x) = \sum_{m=0}^{n+1} Q_{n,m} p_m(x), \quad yp_n(y) = \sum_{m=0}^{n+1} \tilde{p}_m(y) \tilde{Q}_{m,n}, \]  

(5.21)
where $Q_{nl}$ and $\tilde{Q}_{nl}$ are the matrix elements of $Q$ and $\tilde{Q}$.

From the definition of orthogonal polynomials it follows that

$$Q_{n,n+1} = 1, \quad Q_{n,m} = 0, \quad m \geq n + 2, \quad \tilde{Q}_{n+1,n} = 1, \quad \tilde{Q}_{m,n} = 0, \quad m \geq n + 2, \quad (5.22)$$

Define a wave function

$$\Phi_n(t_3, \tilde{t}_3, x) = p_n(x)e^{V(t_3, \tilde{t}_3, x)}. \quad (5.23)$$

Here we introduce the matrix $\bar{Q}$

$$\tilde{Q}_{n,m} = (H\tilde{Q}H^{-1})_{nm}, \quad H_{nm} = h_n\delta_{nm}. \quad (5.24)$$

In what follows it will be convenient to define an explicit parameterization of matrices $Q$ and $\tilde{Q}$. We choose the following parameterization

$$Q_{n,n+1} = 1, \quad Q_{n,n-k} = f_k(n), \quad k = 0, 1, \cdots, 3m_2 - 1, \quad (5.25)$$

$$\tilde{Q}_{n,n-1} = R_n, \quad \tilde{Q}_{n,n+k} = g_k(n)R_{n+1}\cdots R_{n+k}, \quad k = 0, 1, \cdots, 3m_1 - 1, \quad (5.26)$$

where $R_n = h_n/h_{n-1}$. To satisfy the condition (5.20) we can obtain

$$f_{3j}(n) = 0, \quad f_{3j+1}(n) = 0, \quad g_{3j}(n) = 0, \quad g_{3j+1}(n) = 0, \quad j = 0, 1, 2, \cdots. \quad (5.27)$$

We can rewrite (5.21)

$$xp_n = p_{n+1} + \sum_{j=1}^{m_2} f_{3j-1}(n)p_{n+1-3j}, \quad (5.28a)$$

$$y\tilde{p}_n = h_n/h_{n-1} \tilde{p}_{n-1} + \sum_{j=1}^{m_1} g_{3j-1}(n) h_n/h_{n-1+3j} \tilde{p}_{n-1+3j}. \quad (5.28b)$$

From (5.23) we can obtain

$$x\Phi_n = Q\Phi_n, \quad \frac{\partial}{\partial t_r}\Phi_n = Q_r^+\Phi_n, \quad \frac{\partial}{\partial \tilde{t}_s}\Phi_n = -Q_s^-\Phi_n, \quad (5.29)$$

where the subscripts “+” denotes upper triangular plus diagonal parts of the matrix and “-” denotes lower triangular part.

For the first one we can obtain

$$\frac{\partial}{\partial t_3}\Phi_n = \Phi_{n+3} + s_0(n)\Phi_n, \quad (5.30)$$

$$\frac{\partial}{\partial \tilde{t}_3}\Phi_n = -\frac{h_n}{h_{n-3}}\Phi_{n-3},$$

9
where
\[ s_0(n) = f_2(n) + f_2(n + 1) + f_2(n + 2). \] (5.31)

Hereafter we consider the special case \( m_2 = 1 \). In this case the recursion relation (5.28a) becomes
\[ x^3 \Phi_n = \Phi_{n+3} + s_0(n) \Phi_n + s_1(n) \Phi_{n-3} + s_2(n) \Phi_{n-6}, \] (5.32)
where
\[ s_1(n) = f_2(n+1)f_2(n-1) + f_2(n-1)f_2(n) + f_2(n)f_2(n-2), \quad s_2(n) = f_2(n)f_2(n-2)f_2(n-4). \] (5.33)

(5.30) and (5.32) are nothing but the eigenvalue problem (2.4) and (2.1). Then consistency of (5.30) and (5.32) becomes the hungry Vol-Terra equation
\[ \frac{\partial f_2(n)}{\partial t_3} = f_2(n)(f_2(n + 2) + f_2(n + 1) - f_2(n - 1) - f_2(n - 2)). \] (5.34)

To the flows \( t_{3k} \), we can obtain the higher order hierarchy of (5.34).

On the other hand from (5.30) and (5.32)
\[ \frac{\partial R_n}{\partial t_3} = R_n(s_0(n) - s_0(n - 3)), \quad \frac{\partial s_0(n)}{\partial t_3} = R_{n+3} - R_n, \] (5.35)
where \( R_n = h_n/h_{n-3} \). (5.35) is nothing but the 2-dimensional Toda equation.

We consider the most simple case \( m_1 = m_2 = 1 \). We set \( g = N \) and \( t_3 = \tilde{t}_3 = -tN \). If we think of \( t \) as a small parameter, we may at least formally expand the integral \( Z \) in power of \( t \). This expansion is expressible as a sum over Feynmann graphs, made of double lines oriented in opposite directions and carrying a matrix index \( i \in \{1, 2, \ldots, N\} \), involving two types of three-valent vertices, connected by one type of propagator. (<\( UV \) > propagator, weighted by \( 1/N \) ) As usual in matrix integrals, each graph receives a contribution \( N \) per oriented loops and \( tN \) per vertices. We finally obtain the expansion
\[ F(t, N) = \log Z(t, N) = \sum_{\Gamma} t^{V(\Gamma)} N^{\chi(\Gamma)} / |\text{Aut}(\Gamma)|, \] (5.36)
where \( V(\Gamma) \) denotes the number of edges, \( \chi(\Gamma) \) its Euler characteristic, \( |\text{Aut}(\Gamma)| \) denotes the order of the automorphism group of bi-colored connected graphs \( \Gamma \). The bi-colored graphs mean that two adjacent vertices have different colors. We can consider (5.30) as an expansion over dual graphs, with now black and white alternating faces. Then we can finally view \( \log Z \) as the partition function for black and white colored triangurations.

**5.2 Discrete time hungry Volterra equation**

In this subsection we consider following two matrix model
\[ Z(t_{3k}, \tilde{t}_3, l) = \int dU dV e^{-tV(t, \tilde{t}, l)} \] (5.37)
with the potential
\[ V(U, V, t_{3k}, \tilde{t}_3, l) = \sum_{k=1}^{m_1} t_{3k} U^{3k} + \tilde{t}_3 V^3 + gUV - 3l \log U \quad (5.38) \]

We added the “log type potential”\[\text{[11]}\] Note that \( l = 0, 1, 2, \cdots \). In the special case \( l = 0 \), (5.37) becomes the matrix model studied in the previous subsection. The orthogonal polynomials satisfy the orthogonal relations
\[ (p_{n,l}, \tilde{p}_{m,l}) = \int dx dy e^{-V(t_{3k}, \tilde{t}_3, l)p_{n,l}(x)\tilde{p}_{m,l}(y)} = h_{n,l} \delta_{m,n}. \quad (5.39) \]

The partition function (tau function) can be written
\[ Z(t_{3k}, \tilde{t}_3, l; N) = \text{const.} \prod_{i=0}^{N-1} h_{i,l}. \quad (5.40) \]

We can obtain the recursion relations of orthogonal polynomials in the same way
\[ xp_{n,l} = p_{n+1,l} + f_2(n, l)p_{n-2,l}, \quad (5.41a) \]
\[ y\tilde{p}_{n,l} = \frac{h_{n,l}}{h_{n-1,l}} \tilde{p}_{n-1,l} + \sum_{j=1}^{m_1} g_{3j-1}(n, l) \frac{h_{n,l}}{h_{n-1+3j,l}} \tilde{p}_{n-1+3j,l}. \quad (5.41b) \]

Notice that orthogonal polynomials \( p_{n,l} \) and \( \tilde{p}_{n,l} \) and \( f(n, l) \), \( g(n, l) \) and \( h_{n,l} \) are the functions of \( n \) and \( l \). We rewrite \( f_2 \) using the partition function (5.40)
\[ f_2(n, l) \equiv V_{n,l} = \frac{Z_{n+3,l}Z_{n-2,l}}{Z_{n+1,l}Z_{n,l}}. \quad (5.42) \]

Using (5.39) and the condition \( e^{-V(t_{3k}, \tilde{t}_3, l+1)} = x^3 e^{-V(t_{3k}, \tilde{t}_3, l+1)} \) we can get a relation
\[ x^3 p_{n,l+1} = p_{n+3,l} + I_{n,l} p_{n,l}, \quad I_{n,l} = \frac{h_{n,l+1}}{h_{n,l}} = \frac{Z_{n+1,l+1}Z_{n-2,l}}{Z_{n-2,l+1}Z_{n+1,l}}. \quad (5.43) \]

As a compatibility condition of (5.41a) and (5.43) we can obtain
\[ I_{n+1,l} + V_{n,l+1} = V_{n+3,l} + I_{n,l}, \quad I_{n,l}V_{n,l} = V_{n+1,l}I_{n-2,l}. \quad (5.44) \]

Here we change dependent variables \( V_{(n,l)} \rightarrow \tilde{V}_{(n,l)} \). Notice that \( V_{n,l} = f_2(n, l) \) are the dependent variables of the hungry Volterra equation (5.34). We define
\[ V_{n,l} = \tilde{V}_{n,l}(1 + \tilde{V}_{n,l-1})(1 + \tilde{V}_{n,l-2}). \quad (5.45) \]

Using \( \tilde{V}_{n,l} \) we can rewrite
\[ I_{n,l} = (1 + \tilde{V}_{n,l})(1 + \tilde{V}_{n,l-1})(1 + \tilde{V}_{n,l-2}). \quad (5.46) \]

\[ \text{[11]} \]
The equation of motion (5.44) becomes
\[ \frac{\dot{V}_{n, l+1}}{V_{n, l}} = \frac{(1 + \dot{V}_{n-1, l})(1 + \dot{V}_{n-2, l})}{(1 + V_{n-1, l+1})(1 + V_{n-2, l+1})}. \] (5.47)

(5.47) is the discrete hungry Volterra equation. [12]–[14] Using the partition function (5.40) we can rewrite
\[ \dot{V}_{n, l} = \frac{Z_{n+3, l}Z_{n-2, l+1}}{Z_{n, l+1}Z_{n+1, l}}, \quad 1 + \dot{V}_{n, l} = \frac{Z_{n, l}Z_{n+1, l+1}}{Z_{n+1, l}Z_{n, l+1}}. \] (5.48)

(5.47) is cast into the following bilinear equation
\[ Z_{n-2, l+1}Z_{n+3, l} + Z_{n+1, l}Z_{n, l+1} = Z_{n, l}Z_{n+1, l+1}. \] (5.49)

6 Concluding Remarks

In this paper we consider the $k$-hungry Volterra equation as Bäcklund transformations (BT) of the multi boson KP equation which is related to the $sl(k+1, k)$ algebra. We call the BT “fractional BT”. If we repeat the fractional BT $k$ times we can obtain the usual BT which is the Toda lattice.

To study the relations between the 2d Toda lattice and the hungry Volterra equation we consider the hungry Volterra equation in the two matrix model. If we select the time $t_{3k}$ and $\tilde{t}_3$ from the flows of 2d Toda lattice $t = (t_1, t_2, \cdots)$ and $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \cdots)$, we can obtain the 3-hungry Volterra hierarchy which is related to the lattice $W_3$. The most simple case is the matrix model for the bi-colored random triangulation. It is easy to construct the general $k$-hungry Volterra equation in 2d Toda lattice. Using the duality of $t$ and $\tilde{t}$ we can obtain two $k$-hungry Volterra hierarchies in 2d Toda lattice.

Furthermore we consider the discrete time hungry Volterra equation in the two matrix model with the Penner type potential. It is well known that the partition function of the one (multi) matrix model with the Penner type potential satisfies the discrete time Toda equation. [10] [13] [16] We select the flows $l$ and $\tilde{t}_3$ where $l$ is the multiple of 3 of discrete time lattice which belongs to the 2d discrete time Toda equation. If we select the multiple of $k$ of discrete time lattice and $\tilde{t}_k$, we can obtain the discrete time $k$-hungry Volterra equation. Then we can find two discrete time $k$-hungry Volterra equations in 2d discrete time Toda equation.

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