Rings with finite $n$-Weak injective dimension and $(n, k)$-Weak cotorsion modules

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Abstract. Let $R$ be a ring and $n, k$ be two non-negative integers. As an extension of several known notions, we introduce and study $(n, k)$-weak cotorsion modules using the class of right $R$-modules with $n$-weak flat dimensions at most $k$. Various examples and applications are also given.

Keywords: $(n, k)$-Weak cotorsion module; $n$-weak injective module; $n$-weak flat module; $n$-super finitely presented module.

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1 Introduction

Throughout this paper, $R$ will denote an associative (not necessarily commutative) ring with identity. As usual, we denote by $R$-$\text{Mod}$ and $\text{Mod}$-$R$ the category of left $R$-modules and right $R$-modules, respectively. The class of all right (resp. left) $R$-modules of flat dimension at most a
non-negative integer $k$ is denoted by $\mathcal{F}_k(R^{op})$ (resp. $\mathcal{F}_k(R)$) and for a module $M$, $E(M)$ denotes its injective envelope.

In 1984, Enochs introduced the concept of cotorsion modules as a generalization of cotorsion abelian groups [6]. It differs from Matlis’ definition in [16] whose concern was with domains. Since then, the investigation of these modules have become a vigorously active area of research. We refer the reader to [2, 6, 15] for background on cotorsion modules. In 2007, Mao and Ding in [14] introduced the concept of $k$-cotorsion left modules as a new classification of cotorsion modules by using the class $\mathcal{F}_k(R)$.

In 2015, Gao and Wang in [11] introduced the concept of weak injective and weak flat modules by using super finitely presented modules instead of finitely presented modules. Then, they investigated the properties of $R$-modules with weak injective and weak flat dimension at most a non-negative integer $k$. Also, Gao and Huang in [10], investigated the right $\mathcal{WI}$-dimension of modules in terms of left derived functors of Hom and the left $\mathcal{WI}$-resolutions of modules, where $\mathcal{WI}$ is the class of weak injective $R$-modules. This process continues in 2018 when Zhao in [24] introduced another relative derived functor in terms of left or right $\mathcal{WF}$-resolutions to compute the weak flat dimension of modules, where $\mathcal{WF}$ is the class of weak flat $R$-modules. Then, they investigated homological properties of modules with finite weak injective and weak flat dimensions. Let $\mathcal{WI}_k(R)$ and $\mathcal{WF}_k(R^{op})$ be, respectively, the classes of left and right $R$-modules with weak injective and weak flat dimensions at most $k$, respectively. In 2018, Selvaraj and Prabakaran in [20], introduced a particular case of $k$-cotorsion modules and called them $k$-weak cotorsion modules by using the class $\mathcal{WF}_k(R^{op})$ instead of $\mathcal{F}_k(R^{op})$.

Recently in [1], the notion of $n$-super finitely presented left $R$-modules for any non-negative integer $n$ has been introduced as a particular case of super finitely presented. Then, by using $n$-super finitely presented left $R$-modules, a natural extension of the notions of weak injective and weak flat modules have been introduced and studied. They are called the $n$-weak injective and $n$-weak flat modules, respectively. Furthermore, some homological aspects of modules with finite $n$-weak injective and $n$-weak flat dimensions have been also investigated in [1]. Throughout, $\mathcal{WI}^n(R)$, $\mathcal{WF}^n(R^{op})$, $\mathcal{WI}_k^n(R)$ and $\mathcal{WF}_k^n(R^{op})$ will denote respectively the classes of $n$-weak injective left $R$-modules, $n$-weak flat right $R$-modules, left $R$-modules with $n$-weak injective dimensions less than or equal to $k$ and right $R$-modules with $n$-weak flat dimensions less than or equal to $k$.

In this paper, a natural extension of the notion of weak cotorsion modules is introduced and
studied which is called \((n, k)\)-weak cotorsion modules (see Definition 3.1). Every \((n, k)\)-weak cotorsion module is \(k\)-weak cotorsion, \(k\)-cotorsion and cotorsion, and \((0, k)\)-weak cotorsion modules are exactly \(k\)-weak cotorsion modules.

The paper is organized as follows:

In Sec. 2, some fundamental concepts and some preliminary results are stated. Namely, we recall the definitions of \(n\)-super finitely presented, \(n\)-weak injective left \(R\)-modules, and \(n\)-weak flat right \(R\)-modules introduced in [1] (see Definitions 2.1 and 2.3).

In Sec. 3, we introduce \((n, k)\)-weak cotorsion modules. Then, examples are given to show that for \(n > m\geq 0\), \((m, k)\)-weak cotorsion \(R\)-modules need not be \((n, k)\)-weak cotorsion (see Remarks 3.2). Moreover, we show that if \(n\)-\text{wid}_R(R) \leq k \) and \(M\) is an \((n, k)\)-weak cotorsion right \(R\)-module, then \(M\) is an \((n, k)\)-weak cotorsion right \(R\)-module (see Theorem 3.8).

In Sec. 4, we prove some equivalent characterizations of rings with finite \(n\)-super finitely presented dimensions. Namely, for a coherent ring \(R\) such that \(n\)-\text{wid}_R(R) \leq k \), we prove that \(\text{l.n.sp.gl\text{ldim}}(R) \leq k\) if and only if every right \(R\)-module in \(\mathcal{W}_k^{n}(R)^\perp\) is \(n\)-weak flat if and only if every right \(R\)-module in \(\mathcal{W}_k^{n}(R)^\perp\) is \(n\)-weak flat if and only if every left \(R\)-module in \(\mathcal{W}_k^{n}(R)^\perp\) belongs to \(\mathcal{W}_k^{n}(R)\) if and only if every left \(R\)-module in \(\mathcal{W}_k^{n}(R)^\perp\) is \(n\)-weak injective if and only if every left \(R\)-module has a monic \(\mathcal{W}_k^{n}(R)^\perp\)-cover (see Theorem 4.4). Then, we show that \(\text{l.nsp.gl\text{ldim}}(R) \leq k\) if and only if every \((n, k)\)-weak cotorsion right \(R\)-module is injective if and only if every \((n, k)\)-weak cotorsion right \(R\)-module is in \(\mathcal{W}_k^{n}(R^{op})\) (see Theorem 4.5), and if every \((n, k)\)-weak cotorsion right \(R\)-module has a \(\mathcal{W}_k^{n}(R^{op})\)-envelope with the unique mapping property, then \(\text{l.n.sp.gl\text{ldim}}(R) \leq k + 2\) (see Proposition 4.6).

2 Preliminaries

In this section, some fundamental concepts are recalled and notations are stated.

A right \(R\)-module \(M\) is said to be cotorsion [6] if \(\text{Ext}_{R}^{1}(F, M) = 0\) for any flat right \(R\)-module \(F\). For a non-negative integer \(k\), a right \(R\)-module \(M\) is said to be \(k\)-cotorsion [15] if \(\text{Ext}_{R}^{1}(F, M) = 0\) for any right \(R\)-module \(F \in \mathcal{F}_{k}(R^{op})\). A left \(R\)-module \(U\) is called super finitely presented [9] if there exists an exact sequence \(\cdots \to F_{2} \to F_{1} \to F_{0} \to U \to 0\), where each \(F_{i}\) is finitely generated and free. A left \(R\)-module \(M\) is called weak injective [10] if \(\text{Ext}_{R}^{1}(U, M) = 0\)
for any super finitely presented left $R$-module $U$. A right $R$-module $M$ is called weak flat [10] if $\text{Tor}_1^R(M, U) = 0$ for any super finitely presented left $R$-module $U$. For a non-negative integer $k$, a right $R$-module $M$ is called $k$-weak cotorsion [20] if $\text{Ext}_1^R(F, M) = 0$ for any $F \in \mathcal{K}(R^{\text{op}})$. A left $R$-module $M$ is said to be FP-injective [21] if $\text{Ext}_1^R(N, M) = 0$ for any finitely presented left $R$-module $N$.

**Definition 2.1** ([1], Definition 2.1). Let $n$ be a non-negative integer. A left $R$-module $U$ is said to be $n$-super finitely presented if there exists an exact sequence

$$\cdots \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to U \to 0$$

of projective $R$-modules, where each $F_i$ is finitely generated and projective for any $i \geq n$. If $K_i := \text{Im}(F_{i+1} \to F_i)$, then for $i = n - 1$, the module $K_{n-1}$ is called special super finitely presented.

Notice that $\text{Ext}^{n+1}_R(U, M) \cong \text{Ext}^1_R(K_{n-1}, M)$ and $\text{Tor}^R_{n+1}(N, U) \cong \text{Tor}^R_1(N, K_{n-1})$, where $U$ is an $n$-super finitely presented left module with an associated special super finitely presented module $K_{n-1}$. This fact will be used throughout the paper.

0-Super finitely presented modules are just super finitely presented modules. Also for any $m \geq n$, every $n$-super finitely presented left $R$-module is $m$-super finitely presented but not conversely (see Example 2.2 and [1] Examples 2.4 and 2.5).

The finitely presented dimension of an $R$-module $A$ is defined as $\text{f.p.dim}_R(A) = \inf \{m \mid \text{there exists an exact sequence } F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0 \text{ of } R\text{-modules, where each } F_i \text{ is projective, and } F_n \text{ and } F_{n+1} \text{ are finitely generated}\}$. We also define the finitely presented dimension of $R$ (denoted by $\text{f.p.dim}(R)$) as $\sup \{\text{f.p.dim}_R(A) \mid A \text{ is a finitely generated } R\text{-module}\}$.

Also, $R$ is called an $(a, b, c)$-ring if $\text{w.gl.dim}(R) = a$, $\text{gl.dim}(R) = b$ and $\text{f.p.dim}(R) = c$ (see [17]).

**Example 2.2.** Let $R = R_1 \oplus R_2$, where $R_1$ is a ring of polynomials in 4 indeterminates over a field $k$, and $R_2$ is a $(3, 3, 4)$-ring (see, [1] Example 2.4 and [17] Proposition 3.8). Then by [17] Proposition 3.8, $R$ is a coherent $(4, 4, 4)$-ring. Then, $\text{f.p.dim}(R) = 4$ and so there exists a finitely generated $R$-module $U$ with $\text{f.p.dim}_R(U) = 4$. Hence, there exists an exact sequence $F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0 \to U \to 0$, where $F_3$ and $F_5$ are finitely generated and projective $R$-modules. Also, $K_3 := \text{Im}(F_4 \to F_3)$ is a special super finitely presented module,
since $R$ is coherent. So by Definition 2.1 $U$ is 4-super finitely presented. But, every 4-super finitely presented is not 3-super finitely presented otherwise $\text{f.p.dim}(R) = 3$, a contradiction.

**Definition 2.3** ([1], Definition 2.2). Let $n$ be a non-negative integer. Then, a left $R$-module $M$ is called $n$-weak injective if $\text{Ext}^{n+1}_R(U, M) = 0$ for every $n$-super finitely presented left $R$-module $U$. A right $R$-module $N$ is called $n$-weak flat if $\text{Tor}^{R}_{n+1}(N, U) = 0$ for every $n$-super finitely presented left $R$-module $U$.

0-Weak injective modules are just weak injective modules and 0-weak flat modules are just weak flat modules. And for any $m \geq n$, every $n$-weak injective and every $n$-weak flat module is $m$-weak injective and $m$-weak flat respectively, but not conversely (see Example 3.3(1)) and [1] Example 2.5(2)).

The left super finitely presented dimension $\text{l.sp.gl.dim}(R)$ of $R$ is defined as $\text{l.sp.gl.dim}(R) := \sup \{ \text{pd}_R(M) \mid M$ is a super finitely presented left $R$-module}, whereas the $n$-super finitely presented dimension of a ring $R$ is defined as follows: $\text{l.n.sp.gl.dim}(R) := \sup \{ \text{pd}_R(K_{n-1}) \mid K_{n-1}$ is a special super finitely presented left $R$-module} (see [1] Definition 4.9).

It is easy to see that for any $n \geq 0$, $\text{l.n.sp.gl.dim}(R) \leq \text{l.sp.gl.dim}(R)$. Furthermore, Example [3,3.8] shows that we do not have an equality, but if $n = 0$, then clearly $\text{l.n.sp.gl.dim}(R) = \text{l.sp.gl.dim}(R)$.

In [11, Theorem 3.8], it is proved that $\text{l.sp.gl.dim}(R) \leq \text{w.gl.dim}(R)$. Then, it is easy to obtain the following result.

**Proposition 2.4.** For any $n \geq 0$, $\text{l.n.sp.gl.dim}(R) \leq \text{w.gl.dim}(R)$.

Let $\mathcal{F}$ be a class of $R$-modules and $M$ be an $R$-module. Following [7], we say that a morphism $f : F \to M$ is an $\mathcal{F}$-precover of $M$ if $F \in \mathcal{F}$ and $\text{Hom}_R(F', F) \to \text{Hom}_R(F', M) \to 0$ is exact for any $F' \in \mathcal{F}$. An $\mathcal{F}$-precover $f : F \to M$ is said to be an $\mathcal{F}$-cover if every endomorphism $g : X \to X$ such that $fg = f$ is an isomorphism. Dually, we have the definitions of an $\mathcal{F}$-preenvelope and an $\mathcal{F}$-envelope. The class $\mathcal{F}$ is called (pre)covering if each $R$-module has an $\mathcal{F}$-(pre)cover. Similarly, if every $R$-module has an $\mathcal{F}$-(pre)envelope, then we say that $\mathcal{F}$ is (pre)enveloping.

To any given class of right $R$-modules $\mathcal{L}$ and class of left $R$-modules $\mathcal{L}'$, we associate its orthogonal classes as follows:

$$\mathcal{L}^\perp = \text{Ker} \text{Ext}^1_R(\mathcal{L}, -) = \{ C \in R\text{-Mod} \mid \text{Ext}^1_R(L, C) = 0 \text{ for any } L \in \mathcal{L} \},$$
Proposition 2.6. We start with the following result.

\[ \mathcal{I} = \text{Ker} \text{Ext}^1_R(-, \mathcal{L}) = \{ C \in R\text{-Mod} \mid \text{Ext}^1_R(C, L) = 0 \text{ for any } L \in \mathcal{L} \}, \]

\[ \mathcal{L}^\top = \text{Ker} \text{Tor}_1^R(\mathcal{L}, -) = \{ C \in R\text{-Mod} \mid \text{Tor}_1^R(L, C) = 0 \text{ for any } L \in \mathcal{L} \}, \]

\[ ^\top \mathcal{L}' = \text{Ker} \text{Tor}_1^R(-, \mathcal{L}') = \{ C \in R\text{-Mod} \mid \text{Tor}_1^R(C, L) = 0 \text{ for any } L \in \mathcal{L}' \}. \]

A pair \((\mathcal{F}, \mathcal{C})\) of classes of \(R\)-modules is called a cotorsion theory \([7]\) if \(\mathcal{F}^\perp = \mathcal{C}\) and \(\mathcal{F} = ^\perp \mathcal{C}\). A cotorsion theory \((\mathcal{F}, \mathcal{C})\) is called hereditary if whenever \(0 \to F' \to F \to F'' \to 0\) is exact with \(F, F'' \in \mathcal{F}\) then \(F' \in \mathcal{F}\), or equivalently, if \(0 \to C' \to C \to C'' \to 0\) is an exact sequence with \(C, C' \in \mathcal{C}\), then \(C'' \in \mathcal{C}\). A cotorsion theory \((\mathcal{F}, \mathcal{C})\) is called complete \([22]\) if every \(R\)-module has a special \(\mathcal{C}\)-preenvelope; that is, a monic \(\mathcal{C}\)-preenvelope with cokernel in \(\perp \mathcal{C}\) (and a special \(\mathcal{F}\)-precover; that is, an epic \(\mathcal{F}\)-precover with kernel in \(\perp \mathcal{F}\)). A cotorsion theory \((\mathcal{F}, \mathcal{C})\) is called perfect if every \(R\)-module has a \(\mathcal{C}\)-envelope and an \(\mathcal{F}\)-cover (see \([8, 12]\)).

Definition 2.5 \([1]\), Definition 3.1. Let \(n\) be a non-negative integer. Then, the \(n\)-weak injective dimension of a left \(R\)-module \(M\) is defined by:

\[ \text{n-wid}_R(M) := \inf \{ k : \text{Ext}^{k+1}_R(K_{n-1}, M) = 0 \text{ for every special super finitely presented left module } K_{n-1} \}, \]

and the \(n\)-weak flat dimension of a right \(R\)-module \(N\) is defined by:

\[ \text{n-wfd}_R(N) := \inf \{ k : \text{Tor}^{R}_k(N, K_{n-1}) = 0 \text{ for every special super finitely presented left module } K_{n-1} \}. \]

The classes of left and right \(R\)-modules with \(n\)-weak injective and \(n\)-weak flat dimensions at most \(k\) will be denoted respectively by \(\mathcal{W}I^n_k(R)\) and \(\mathcal{W}F^n_k(R\text{op})\). In \([1]\) Theorem 4.4 and 4.5, it is proved that \(\mathcal{W}I^n_k(R)\) and \(\mathcal{W}F^n_k(R\text{op})\) are both covering and preenveloping. Here, we give more properties of these two classes of modules. We start with the following result.

Proposition 2.6. Let \(n\) and \(k\) be non-negative integers. Then, the following assertions hold:

1. If \(M\) is the cokernel of a \(\mathcal{W}I^n_k(R)\)-preenvelope \(X \to F\) of a left \(R\)-module \(X\) with \(F\) flat, then \(M \in ^\top \mathcal{W}F^n_k(R\text{op})\).

2. If \(M\) is the cokernel of a \(\mathcal{W}F^n_k(R\text{op})\)-preenvelope \(A \to F\) of a right \(R\)-module \(A\) with \(F\) flat, then \(M \in \mathcal{W}I^n_k(R\text{op})^\top\).

Proof. (1) Consider the short exact sequence \(0 \to X \to F \to M \to 0\). If \(N \in \mathcal{W}F^n_k(R\text{op})\), then by \([1]\) Proposition 3.6], \(N^* \in \mathcal{W}I^n_k(R)\). Thus, \(\text{Hom}_R(F, N^*) \to \text{Hom}_R(X, N^*) \to 0\) is exact and so \((F \otimes_R N)^* \to (X \otimes_R N)^* \to 0\) is exact. Hence, \(0 \to X \otimes_R N \to F \otimes_R N\) is exact.
and from the exactness of the long exact sequence \( 0 \to \text{Tor}_1^R(M, N) \to X \otimes_R N \to F \otimes_R N \) (because \( F \) is flat), it follows that \( \text{Tor}_1^R(M, N) = 0 \) and so \( M \in \overline{\mathcal{W}F^n_k(R^{op})} \).

(2) Consider the exact sequence \( 0 \to X \to F \to M \to 0 \), where \( X = \text{Im}(A \to F) \). By using similar arguments in (1), we prove that \( M \in \mathcal{W}T^n_k(R)^\top \).

**Corollary 2.7.** Let \( n \) and \( k \) be non-negative integers. Then, \( \mathcal{W}F^n_k(R^{op}) \subseteq \mathcal{W}T^n_k(R)^\top \).

**Proof.** Consider an exact sequence \( 0 \to X \to F \to N \to 0 \), where \( N \in \mathcal{W}F^n_k(R^{op}) \) and \( F \) is projective. Then, \( \text{Ext}_1^R(N, M) = 0 \) for any \( M \in \mathcal{W}F^n_k(R^{op}) \). Hence, \( X \to F \) is a \( \mathcal{W}F^n_k(R^{op}) \)-preenvelope of \( X \), and so \( N \in \mathcal{W}T^n_k(R)^\top \) by Proposition 2.6(2).

**Theorem 2.8.** Let \( n \) and \( k \) be non-negative integers and \( U \) be an \( n \)-super finitely presented left module. If \( \cdots \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to U \to 0 \) is a projective resolution such that for any \( i \geq n \), \( F_i \) is finitely generated and projective and \( K_i = \text{Im}(F_{i+1} \to F_i) \), then the following assertions hold:

(1) If \( n\text{-wid}(R, R) \leq k \), then \( K_{n-1} \in \mathcal{W}T^n_k(R)^\top \) if and only if \( K_n \to F_n \) is a \( \mathcal{W}T^n_k(R)^\top \)-preenvelope.

(2) \( K_{n-1} \in \mathcal{W}T^n_k(R)^\top \) if and only if \( K_n \to F_n \) is a \( \mathcal{W}F^n_k(R^{op}) \)-preenvelope if and only if \( K_{n-1} \in \mathcal{W}F^n_k(R^{op}) \).

**Proof.** (1) Consider the exact sequence \( 0 \to K_n \to F_n \to K_{n-1} \to 0 \) with \( F_n \) is finitely generated and projective. It is clear by [1] Proposition 4.8] that \( F_n \in \mathcal{W}T^n_k(R) \) (because \( n \text{-wid}(R, R) \leq k \)). We show that \( K_n \to F_n \) is a \( \mathcal{T}T^n_k(R) \)-preenvelope. For any \( X \in \mathcal{W}T^n_k(R) \), we have \( X^* \in \mathcal{W}F^n_k(R^{op}) \) by [1] Proposition 3.6(2)]. So \( \text{Tor}_1^R(K_{n-1}, X^*) = 0 \). By [18] Lemma 3.55, we obtain the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & K_n \otimes_R X^* \\
\downarrow f_2 & & \downarrow f_1 \\
\text{Hom}_R(K_n, X)^* & \xrightarrow{\beta} & \text{Hom}_R(F_n, X)^*.
\end{array}
\]

where \( f_1 \) and \( f_2 \) are isomorphisms. We deduce that \( \beta \) is a monomorphism and consequently the sequence \( \text{Hom}_R(F_n, X) \to \text{Hom}_R(K_n, X) \to 0 \) is exact. The converse follows from Proposition 2.6(1).

(2) The proof of the first equivalence is similar to that of (1) using Proposition 2.6(2). The second equivalence is clear using Corollary 2.7(2).
3 \((n, k)\)-Weak cotorsion modules

In this section, we introduce and study \((n, k)\)-weak cotorsion modules. We start with the definition.

**Definition 3.1.** Let \(n\) and \(k\) be non-negative integers. Then, a right \(R\)-module \(M\) is called \((n, k)\)-weak cotorsion if \(\text{Ext}_1^R(N, M) = 0\) for every \(N \in \mathcal{W}F^n_k(R^{\text{op}})\).

**Remark 3.2.**

1. \((0, k)\)-Weak cotorsion right modules are exactly \(k\)-weak cotorsion modules, and \((0, 0)\)-weak cotorsion right modules are exactly weak cotorsion modules.

2. Every \((n, k)\)-weak cotorsion right module is \((m, k)\)-weak cotorsion for any \(n \geq m\), but not conversely. Indeed, if every \((m, k)\)-weak cotorsion right module \(M\) is \((n, k)\)-weak cotorsion, then either \(M\) is injective or for each \(N \in \mathcal{W}F^n_k(R^{\text{op}})\), \(N\) is projective or \(N \in \mathcal{W}F^{n-1}_k(R^{\text{op}})\), and this is impossible (see Example 3.3(2)).

3. The study of the class of \((n, k)\)-weak cotorsion right modules involves the classes of injective, \((n, k)\)-weak cotorsion, \(k\)-weak cotorsion, \(k\)-cotorsion and cotorsion right \(R\)-modules denoted \(\mathcal{I}, \mathcal{WC}^n_k(R^{\text{op}}), \mathcal{WC}_k(R^{\text{op}}), C_k(R^{\text{op}})\) and \(C(R^{\text{op}})\) respectively. In fact,

\[
\mathcal{I} \subseteq \mathcal{WC}^n_k(R^{\text{op}}) \subseteq \mathcal{WC}_k(R^{\text{op}}) \subseteq C_k(R^{\text{op}}) \subseteq C(R^{\text{op}}).
\]

But not every \(k\)-cotorsion right module is \((n, k)\)-weak cotorsion (see Example 3.3(3)).

**Example 3.3.** Let \(R = k[x] \times S\), where \(k[x]\) is a ring of polynomials in one indeterminate over a field \(k\) and \(S\) is a non-Noetherian hereditary von Neumann regular ring. For example, \(S\) is a ring of functions of \(X\) into \(k\) continuous with respect to the discrete topology on \(k\), where \(k\) is a field and \(X\) is a totally disconnected compact Hausdorff space whose associated Boolean ring is hereditary (see examples of [4]). Then, by [17, Proposition 3.10], \(R\) is a coherent \((1, 1, 2)\)-ring. Then, we have:

1. Every \(R\)-module is 1-weak flat, since \(\text{gl.dim}(R) = 1\). But not every \(R\)-module is 0-weak flat, for otherwise each super finitely presented \(R\)-module would be projective. On the other hand, every finitely presented is super finitely presented (because \(R\) is coherent). Thus every finitely presented is flat and by [25, Theorem 3.9], each \(R\)-module is flat and so \(\text{w.gl.dim}(R) = 0\), a contradiction.
(2) Notice that \(\text{l.n.sp.gldim}(R) = 0\) for any \(n \geq 1\), since \(\text{pd}_R(U) \leq 1\) for any \(n\)-super finitely presented \(R\)-module \(U\). So by Theorem 4.5, every \((n, 0)\)-weak cotorsion right \(R\)-module is injective for any \(n \geq 1\). But not every \((0, 0)\)-weak cotorsion right \(R\)-module is injective, for otherwise every \(R\)-module would be in \(\mathcal{W} \mathcal{F}^0_0(R^\text{op})\) and so each 0-super finitely presented would be flat. Similarly to (1), \(\text{w.gl.dim}(R) = 0\), a contradiction.

(3) Also, \(\text{l.n.sp.gldim}(R) = 1\), since \(R\) is coherent and by [11, Theorem 3.8], \(\text{l.n.sp.gldim}(R) = \text{w.gl.dim}(R)\). So \(\text{l.n.sp.gldim}(R) \neq \text{l.n.sp.gldim}(R)\) and it follows by Proposition 3.4 that not every \(k\)-cotorsion is \((n, k)\)-weak cotorsion.

Example 3.3(3) gives a context when \(\text{l.n.sp.gldim}(R) \leq \text{l.n.sp.gldim}(R)\).

Now we show, for non-negative integers \(k\) and \(n\), when every \(k\)-cotorsion module is \((n, k)\)-weak cotorsion.

**Proposition 3.4.** Let \(n\) be a non-negative integer. Then, the following conditions are equivalent:

1. For every non-negative integer \(k\), every \(k\)-cotorsion right module is \((n, k)\)-weak cotorsion.
2. For every non-negative integer \(k\), \(\mathcal{W} \mathcal{F}^n_k(R^\text{op}) = \mathcal{F}_k(R^\text{op})\).
3. \(\text{l.n.sp.gldim}(R) = \text{w.gl.dim}(R)\).

**Proof.** The implication (1) \(\Rightarrow\) (2) follows from the fact that \((\mathcal{W} \mathcal{F}^n_k(R^\text{op})), \mathcal{W} \mathcal{C}^n_k(R^\text{op}))\) and \((\mathcal{F}_k(R^\text{op})), \mathcal{C}_k(R^\text{op}))\) are cotorsion theories. And, the implications (2) \(\Leftrightarrow\) (3) \(\Rightarrow\) (1) are clear.

The following result can be easily obtained.

**Proposition 3.5.** Let \(n\) and \(k\) be non-negative integers. Then, the following assertions hold:

1. If \(\{M_i\}_{i \in I}\) is a family of right \(R\)-modules, then \(\prod_{i \in I} M_i\) is \((n, k)\)-weak cotorsion if and only if each \(M_i\) is \((n, k)\)-weak cotorsion.
2. The class \(\mathcal{W} \mathcal{C}^n_k(R^\text{op})\) is closed under extensions and direct summands.

Now we state one of the main results of this section which characterizes when a module \(M\) is \((n, k)\)-weak cotorsion. We start with the following proposition.

**Proposition 3.6.** Let \(n\) and \(k\) be non-negative integers and let \(M\) be a right \(R\)-module. Then, the following conditions are equivalent:

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(1) $M$ is $(n, k)$-weak cotorsion.

(2) $M$ is injective with respect to every exact sequence $0 \to K \to N \to D \to 0$ of right $R$-modules, where $D \in \mathcal{WF}_{k}^n(R^{op})$.

Moreover if $n$-$\text{wid}(R) \leq k$, then the above conditions are also equivalent to:

(3) For every exact sequence $0 \to M \to E \to D \to 0$ of right $R$-modules, where $E$ is injective, $E \to D$ is a $\mathcal{WF}_{k}^n(R^{op})$-precover of $D$.

(4) $M$ is the kernel of a $\mathcal{WF}_{k}^n(R^{op})$-precover $E \xrightarrow{\phi} D$ where $E$ is injective.

**Proof.** (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (1) Let $D \in \mathcal{WF}_{k}^n(R^{op})$ and consider an exact sequence $0 \to K \to P \to D \to 0$ where $P$ projective. Then, from the long exact sequence $\text{Hom}_R(P, M) \to \text{Hom}_R(K, M) \to \text{Ext}^1_R(D, M) \to \text{Ext}^1_R(P, M)$ we see that $\text{Ext}^1_R(D, M) = 0$ and hence $M$ is $(n, k)$-weak cotorsion.

(1) $\Rightarrow$ (3) For every exact sequence $0 \to M \to E \xrightarrow{\phi} D \to 0$ where $E$ is injective, we have by [1] Proposition 4.8 that $E \in \mathcal{WF}_{k}^n(R^{op})$. So for any $N$ in $\mathcal{WF}_{k}^n(R^{op})$, we have

$$0 \to \text{Hom}_R(N, M) \to \text{Hom}_R(N, E) \to \text{Hom}_R(N, D) \to \text{Ext}^1_R(N, M) = 0.$$ 

It follows that $\phi$ is a $\mathcal{WF}_{k}^n(R^{op})$-precover of $D$.

(3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) By hypothesis, there exists an exact sequence $0 \to M \to E \xrightarrow{\phi} D \to 0$, where $\phi$ is a $\mathcal{WF}_{k}^n(R^{op})$-precover of $D$. Therefore for each $L$ in $\mathcal{WF}_{k}^n(R^{op})$, we have:

$$0 \to \text{Hom}_R(L, M) \to \text{Hom}_R(L, E) \to \text{Hom}_R(L, D) \to \text{Ext}^1_R(L, M) \to \text{Ext}^1_R(L, E),$$

We have $\text{Hom}_R(L, E) \to \text{Hom}_R(L, D)$ is epic and $\text{Ext}^1_R(L, E) = 0$ so $\text{Ext}^1_R(L, M) = 0$ and consequently $M$ is $(n, k)$-weak cotorsion.

Recall that an $R$-module $N$ is said to be reduced [7] if $N$ has no non zero injective submodules. In the next proposition, we characterize reduced $(n, k)$-weak cotorsion modules when $n$-$\text{wid}(R) \leq k$.  

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**Proposition 3.7.** Let \( n \) and \( k \) be non-negative integers and assume that \( n \text{-wid}(R) \leq k \). Then for a right \( R \)-module \( M \), \( M \) is a reduced \((n,k)\)-weak cotorsion if and only if \( M \) is the kernel of a \( \mathcal{WF}_k^n(R^{op}) \)-cover \( N \xrightarrow{\phi} D \) where \( N \) is an injective module.

**Proof.** Suppose that \( M \) is a reduced \((n,k)\)-weak cotorsion right module and consider the exact sequence \( 0 \to M \to E(M) \xrightarrow{\phi} \frac{E(M)}{M} \to 0 \). Then, by Proposition 3.6 \( \phi \) is a \( \mathcal{WF}_k^n(R^{op}) \)-precover. Since \( M \) is reduced, \( E(M) \) has no non zero direct summand \( L \) contained in \( M \). Also by [1, Theorem 4.5], \( \frac{E(M)}{M} \) has a \( \mathcal{WF}_k^n(R^{op}) \)-cover. Then, [23, Corollary 1.2.8] implies that \( \phi \) is a \( \mathcal{WF}_k^n(R^{op}) \)-cover.

Conversely, suppose that \( M \) is the kernel of a \( \mathcal{WF}_k^n(R^{op}) \)-cover \( N \xrightarrow{\phi} D \) where \( N \) is an injective \( R \)-module. Then by Proposition 3.6, \( M \) is \((n,k)\)-weak cotorsion. Let \( N_1 \) be an injective submodule of \( M \). If \( N = N_1 \oplus N_2 \), \( f : N \to N_2 \) is the projection map and \( i : N_2 \to N \) is the inclusion map, then \( \phi(N_1) = 0 \) and \( \phi(if) = \phi \), and hence \( if \) is an isomorphism. So \( i \) is an epimorphism and consequently \( N = N_2 \) and \( N_1 = 0 \). Therefore, \( M \) is reduced. \( \blacksquare \)

Now we are in a position to show the following theorem.

**Theorem 3.8.** Let \( n \) and \( k \) be non-negative integers and assume that \( n \text{-wid}(R) \leq k \). Then for a right \( R \)-module \( M \), \( M \) is \((n,k)\)-weak cotorsion if and only if \( M \) is a direct sum of an injective \( R \)-module and a reduced \((n,k)\)-weak cotorsion right \( R \)-module.

**Proof.** Consider the exact sequence \( 0 \to M \to E(M) \xrightarrow{\phi} \frac{E(M)}{M} \to 0 \), where \( M \) is an \((n,k)\)-weak cotorsion right module. By Proposition 3.7, \( \phi \) is a \( \mathcal{WF}_k^n(R^{op}) \)-cover of \( \frac{E(M)}{M} \). Also by [1, Theorem 4.5], \( \frac{E(M)}{M} \) has a \( \mathcal{WF}_k^n(R^{op}) \)-cover \( N \xrightarrow{\phi} \frac{E(M)}{M} \). Then, there exists the commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & N_1 & \xrightarrow{g_1} & N & \xrightarrow{h_1} & \frac{E(M)}{M} & \xrightarrow{h_2} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M & \xrightarrow{g_2} & E(M) & \xrightarrow{f_1} & \frac{E(M)}{M} & \xrightarrow{f_2} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & N_1 & \xrightarrow{g_3} & N & \xrightarrow{f_3} & \frac{E(M)}{M} & \xrightarrow{f_4} & 0
\end{array}
\]

We have \( E(M) = \ker f_2 \oplus \text{im} h_2 \), since \( f_2 h_2 \) is an isomorphism. Also \( \text{im} h_2 \cong N \) so \( N \) and \( \ker f_2 \) are injective. Thus by Proposition 3.7, \( N_1 \) is reduced and \((n,k)\)-weak cotorsion. On the other
hand, by the Five Lemma, $f_1 h_1$ is an isomorphism. Hence, $M = \ker f_1 \oplus \im h_1$ where $\im h_1 \cong N_1$.

Consider the following commutative diagram:

![commutative diagram]

Consequently $\ker f_1 \cong \ker f_2$ and so $M = N_1 \oplus \ker f_1$, where $\ker f_1$ is injective. The converse follows from Proposition 3.5(1).

We end this section with a result that shows when every right $R$-module is $(n, k)$-weak cotorsion. To this end, we prove first the following proposition.

**Proposition 3.9.** The following assertions hold:

1. If a right $R$-module $M$ is $(n, k)$-weak cotorsion, then $\Ext_R^{j \geq m+1}(N, M) = 0$ for every $N \in \WF_{m+k}^n(R^{\mathrm{op}})$.

2. The $m$th cosyzygy of every $(n, k)$-weak cotorsion right $R$-module is $(n, m+k)$-weak cotorsion.

**Proof.** (1) Let $N \in \WF_{m+k}^n(R^{\mathrm{op}})$. Then, there is an exact sequence:

$$0 \to K_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to N \to 0,$$

where each $P_i$ is projective and $K_m \in \WF_K^n(R^{\mathrm{op}})$. Hence, $\Ext_R^{m+1}(N, M) \cong \Ext_R^{1}(K_m, M) = 0$ (because $M$ is $(n, k)$-weak cotorsion).

(2) Let $M$ be any $(n, k)$-weak cotorsion right $R$-module and $D^m$ the $m$th cosyzygy of $M$. Then by (1), $\Ext_R^{m+1}(N, M) = 0$ for every $N \in \WF_{m+k}^n(R^{\mathrm{op}})$. Thus, $0 = \Ext_R^{m+1}(N, M) \cong \Ext_R^{1}(N, D^m)$ and consequently $D^m$ is $(n, m+k)$-weak cotorsion. 

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**Theorem 3.10.** Let \( n \) and \( k \) be non-negative integers. Then, the following conditions are equivalent:

1. Every right \( R \)-module is \((n, k)\)-weak cotorsion.
2. Every right \( R \)-module in \( \mathcal{WF}^n_k(\text{R}^{\text{op}}) \) is projective.
3. Every right \( R \)-module in \( \mathcal{WF}^n_k(\text{R}^{\text{op}}) \) is \((n, k)\)-weak cotorsion.
4. For any integer \( m \), \( \text{Ext}^i_R(M, N) = 0 \) for all right \( R \)-modules \( N \) and all \( R \)-module \( M \) in \( \mathcal{WF}^n_{m+k}(\text{R}^{\text{op}}) \).
5. Every right \( R \)-module \( M \) has a \( \mathcal{WF}^n_k(\text{R}^{\text{op}}) \)\(-\)envelope with the unique mapping property.
6. Every projective right \( R \)-module is \((n, k)\)-weak cotorsion.
7. \( R \)-\( R \) is \((n, k)\)-weak cotorsion and every right \( R \)-module has a \( \mathcal{WF}^n_k(\text{R}^{\text{op}}) \)\(-\)precover.

**Proof.**

(1) \( \iff \) (2) Clear because \( (\mathcal{WF}^n_k(\text{R}^{\text{op}}), \mathcal{WC}^n_k(\text{R}^{\text{op}})) \) and \( (\text{Proj}, \text{Mod-R}) \) are cotorsion theories.

(1) \( \Rightarrow \) (4) Follows from Proposition 3.9(1).

(4) \( \Rightarrow \) (3) Take \( m = 0 \).

(3) \( \Rightarrow \) (1) Let \( M \) be a right \( R \)-module. Then by [11 Proposition 4.13], there exists an exact sequence \( 0 \to L \to N \xrightarrow{\phi} M \to 0 \), where \( \phi \) is \( \mathcal{WF}^n_k(\text{R}^{\text{op}}) \)\(-\)cover of \( M \). Also by [23 Lemma 2.1.1], \( L \) is in \( \mathcal{WF}^n_k(\text{R}^{\text{op}}) \)\(\perp = \mathcal{WC}^n_k(\text{R}^{\text{op}}) \). Since \( (\mathcal{WF}^n_k(\text{R}^{\text{op}}), \mathcal{WF}^n_k(\text{R}^{\text{op}}) \perp) \) is a hereditary perfect cotorsion pair, we deduce that every right \( R \)-module \( M \) is \((n, k)\)-weak cotorsion.

(1) \( \Rightarrow \) (5) Clear.

(5) \( \Rightarrow \) (3) Let \( N \in \mathcal{WF}^n_k(\text{R}^{\text{op}}) \). Then, the following commutative diagram exists:

\[
\begin{array}{c}
0 \rightarrow N \xrightarrow{\alpha} X \xrightarrow{\beta} D \xrightarrow{\gamma} 0, \\
0 \rightarrow X' \xrightarrow{\gamma} \end{array}
\]
where \( \alpha \) and \( \gamma \) are \( \mathcal{W}F_k^n(R^{op})^\perp \)-envelope. Since \( \gamma \beta \alpha = 0 = 0 \), it follows by assumption that \( \gamma \beta = 0 \). Consequently \( D = \text{im} ( \beta ) \subseteq \ker(\gamma) = 0 \) and so \( D = 0 \). Hence, \( N \) is \((n,k)\)-weak cotorsion.

(1) \( \Rightarrow \) (7) Clear.

(7) \( \Rightarrow \) (6) By \cite{13} Proposition 1], \( \mathcal{W}F_k^n(R^{op})^\perp \) is closed under direct sums. Since \( R \) is \((n,k)\)-weak cotorsion, it follows that every free right \( R \)-module is \((n,k)\)-weak cotorsion. Then by Proposition 3.5(2), we deduce that each projective right \( R \)-module is \((n,k)\)-weak cotorsion.

(6) \( \Rightarrow \) (1) Let \( M \) be a right \( R \)-module. Then by \cite{1} Proposition 4.13], there exists an exact sequence \( 0 \to L \to N \xrightarrow{\alpha} M \to 0 \), where \( \alpha \) is \( \mathcal{W}F_k^n(R^{op}) \)-cover of \( M \) and \( L \in \mathcal{W}F_k^n(R^{op})^\perp \) by \cite{23} Lemma 2.1.1]. By hypothesis, every projective right \( R \)-module is \((n,k)\)-weak cotorsion, and then by Remark 3.2(1), (2], every projective right \( R \)-module is cotorsion. Hence by \cite{3} Corollary 10], \( R \) is perfect. So by \cite{19} Theorem Bass], any flat right \( R \)-module \( Y \) is projective and from the assumption, \( Y \) is \((n,k)\)-weak cotorsion. Hence, \( N \) is \((n,k)\)-weak cotorsion (because \( X_1 \) and \( F_0 \) are \((n,k)\)-weak cotorsion). Similarly, \( M \) is \((n,k)\)-weak cotorsion with respect to the exact sequence \( 0 \to L \to N \xrightarrow{\alpha} M \to 0 \).

\section{Applications}

In this section, we characterize rings with finite \( n \)-super finitely presented dimension in terms of \( n \)-weak injective and \( n \)-weak flat modules and \((n,k)\)-weak cotorsion modules.

First, we start with the following lemmas.

\textbf{Lemma 4.1.} Let \( n \) and \( k \) be non-negative integers. Then, the following conditions are equivalent for a left \( R \)-module \( N \):

\begin{enumerate}
\item \( N \in \mathcal{W}I_k^n(R)^\perp \).
\item \( N^* \in \mathcal{W}I_k^n(R)^\perp \).
\item \( N \in \perp \mathcal{X} \), where \( \mathcal{X} = \{ A^* \mid A \text{ is in } \mathcal{W}I_k^n(R) \} \).
\end{enumerate}
(4) The functor $-\otimes_R N$ preserves the exactness of every exact sequence $0 \to A \to B \to C \to 0$ where $C \in \mathcal{W}_k^n(R)$.

Proof. By [18, Theorem 11.55], for any right $R$-module $D$, $\text{Ext}^1_R(N, D^*) \cong \text{Tor}^1_R(D, N)^* \cong \text{Ext}^1_R(D, N^*)$. So $(1) \iff (2) \iff (3)$ follows. Also, $(1) \iff (4)$ is clear. ■

Lemma 4.2. Assume that $n\text{-wid}(R) \leq k$ where $k \geq 1$. Then,

(1) If $N \in \mathcal{W}_{k-1}^n(R)^\perp$, then there is an exact sequence $0 \to X \to E \to N \to 0$ where $E$ is injective and $X \in \mathcal{W}_k^n(R)^\perp$.

(2) If $N \in \mathcal{W}_{k-1}^n(R)^\top$, then there is an exact sequence $0 \to N \to F \to D \to 0$ where $F$ is flat and $D \in \mathcal{W}_k^n(R)^\top$.

Proof. (1) Consider an exact sequence $0 \to Y \to F \to N \to 0$ of left $R$-modules, where $F$ is projective. Then, we have the following pushout diagram

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & Y & \to & F & \to & N & \to & 0 \\
\downarrow & & \downarrow & \alpha & \downarrow & \uparrow & \downarrow \\
0 & \to & Y & \to & E(F) & \to & L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \to & B \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
$$

where $\alpha$ is an injective envelope of $F$. Let $U$ be an $n$-super finitely presented left $R$-module with associated special super finitely presented module $K_{n-1}$. Then by assumption and from [11, Proposition 3.2], $\text{Ext}^{k+1}_R(K_{n-1}, R) = 0$ so $\text{Ext}^{k+1}_R(K_{n-1}, F) = 0$ and hence $n\text{-wid}_R(F) \leq k$.

Also by [11, Proposition 3.2], it follows that $n\text{-wid}_R(B) \leq k - 1$, and so $B \in \mathcal{W}_k^n(R)$. Hence, $\text{Ext}^1_R(B, N) = 0$, and consequently the sequence $0 \to N \to L \to B \to 0$ splits and then $N$ is a quotient of $E(F)$.

By [10, Theorem 3.1], there is a weak injective cover of left $R$-module $N$. If $\gamma : E \to N$ is a weak injective cover of $N$, then $\gamma$ is epic. Now consider the exact sequence $0 \to X \to E \to
$N \to 0$. By [23, Lemma 2.1.1], $X \in \mathcal{W}_0^0(R)^\perp$. We prove that $X \in \mathcal{W}^n_k(R)^\perp$. For let $L_1$ be in $\mathcal{W}^0_k(R)$ and consider the exact sequence $0 \to L_1 \to E(L_1) \to L_2 \to 0$, where $L_2 \in \mathcal{W}^n_{k-1}(R)$ by [11, Proposition 3.2]. Then we get the induced exact sequence

$$\text{Ext}^1_R(L_2, N) \to \text{Ext}^2_R(L_2, X) \to \text{Ext}^2_R(L_2, E) = 0.$$ 

By hypothesis, we have $\text{Ext}^1_R(L_2, N) = 0$ and so $\text{Ext}^2_R(L_2, X) = 0$. Also, we obtain the induced exact sequence

$$\text{Ext}^1_R(E(L_1), X) \to \text{Ext}^1_R(L_1, X) \to \text{Ext}^2_R(L_2, X).$$

We have $\text{Ext}^2_R(L_2, X) = 0$ and since $X$ is in $\mathcal{W}_0^0(R)^\perp$ and $E(L_1) \in \mathcal{W}^0_0(R)$, it follows that $\text{Ext}^1_R(E(L_1), X) = 0$. Hence, $\text{Ext}^1_R(L_1, X) = 0$ and consequently $X \in \mathcal{W}^n_k(R)^\perp$.

(2) Consider an exact sequence $0 \to K \to F \to N^* \to 0$ of left $R$-modules where $F$ is projective. Then, we have the following pushout diagram

$$\begin{array}{ccccccccc}
0 & \to & K & \to & F & \to & N^* & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & E(F) & \to & L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Z & \to & Z & \to & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & & & & & \\
\end{array}$$

where $\alpha$ is an injective envelope of $F$. By Lemma 4.1, $N^* \in \mathcal{W}^n_k(R)^\perp$. Then similar to the proof of (1), we see $\text{Ext}^1_R(Z, N^*) = 0$, and hence the sequence $0 \to N^* \to L \to Z \to 0$ splits. Thus, $N^*$ is a quotient of $E(F)$ and then we have the exact sequence $0 \to N^{**} \to (E(F))^* \to \text{Ext}^1_R(Z, N^*) = 0$, with $(E(F))^*$ is flat. By [23, Proposition 2.3.5], $N$ is pure in $N^{**}$, and hence $N$ embeds in flat $R$-module.

Let $\beta : N \to F$ be a flat preenvelope of $N$. Then, $\beta$ is monic, and so we have the exact sequence $0 \to N \to F \to D \to 0$. By [23, Lemma 2.1.2], $D \in \mathcal{W}^0_0(R)^\perp$. We show that $D \in \mathcal{W}^n_k(R)^\perp$. Now, let $K_1$ be in $\mathcal{W}^n_k(R)^\perp$. We have $\text{Ext}^1_R(K_1, N) = 0$ and so $\text{Ext}^2_R(K_1, X) = 0$. Also, we obtain the induced exact sequence

$$\text{Ext}^1_R(E(K_1), X) \to \text{Ext}^1_R(K_1, X) \to \text{Ext}^2_R(K_1, X).$$

We have $\text{Ext}^2_R(K_1, X) = 0$ and since $X$ is in $\mathcal{W}_0^0(R)^\perp$ and $E(K_1) \in \mathcal{W}^0_0(R)$, it follows that $\text{Ext}^1_R(E(K_1), X) = 0$. Hence, $\text{Ext}^1_R(K_1, X) = 0$ and consequently $X \in \mathcal{W}^n_k(R)^\perp$.
Then, there exists an exact sequence $0 \to K_1 \to E(K_1) \to K_2 \to 0$, where $K_2 \in \mathcal{W}_n^{l-1}(R)$ by Proposition 3.2. Then we get the exact sequence

$$0 = \text{Tor}^R_2(K_2, F) \to \text{Tor}^R_2(K_2, D) \to \text{Tor}^R_1(K_2, N).$$

We have $\text{Tor}^R_1(K_2, N) = 0$ and so $\text{Tor}^R_2(K_2, D) = 0$. Also, we have the following exact sequence

$$0 = \text{Tor}^R_2(K_2, D) \to \text{Tor}^R_1(K_1, D) \to \text{Tor}^R_1(E(K_1), D).$$

Since $D$ is in $\mathcal{W}_n^0(R)$ and $E(K_1) \in \mathcal{W}_n^0(R)$, we deduce that $\text{Tor}^R_1(E(K_1), D) = 0$. Hence, $\text{Tor}^R_1(K_1, D) = 0$ and then $D \in \mathcal{W}_n^0(R)$.

Lemma 4.3. Assume that $\text{l.nsp.gldim}(R) < \infty$. Then, $\text{l.nsp.gldim}(R) = \text{n-wid}(R)$. 

Proof. By Proposition 3.5, $\text{l.nsp.gldim}(R) = \sup \{ \text{n-wid}_R(M) \mid M \text{ is a left } R\text{-module} \} = \sup \{ \text{n-wfd}_R(M) \mid M \text{ is a right } R\text{-module} \}$. So, we have $\text{n-wid}(R) \leq \text{l.nsp.gldim}(R)$. Now, let $\text{l.nsp.gldim}(R) = k$. We show that $\text{n-wid}(R) \geq k$. For let $U$ be an $n$-super finitely presented left $R$-module with associated special super finitely presented module $K_{n-1}$. So there exists left $R$-module $M$ such that $\text{Ext}^k_R(K_{n-1}, M) \neq 0$. Consider an exact sequence $0 \to L \to F \to M \to 0$ where $F$ is free. Then we have the following exact sequence:

$$\text{Ext}^k_R(K_{n-1}, F) \to \text{Ext}^k_R(K_{n-1}, M) \to \text{Ext}^{k+1}_R(K_{n-1}, L).$$

We have $\text{Ext}^{k+1}_R(K_{n-1}, L)$ (because $\text{l.nsp.gldim}(R) = k$). Therefore $\text{Ext}^k_R(K_{n-1}, R) \neq 0$, otherwise $\text{Ext}^k_R(K_{n-1}, F) = 0$ and so from the exact sequence above, $\text{Ext}^k_R(K_{n-1}, M) = 0$ which is a contradiction. Consequently $\text{n-wid}(R) \geq k$.

The following theorem extends the results of Mao and Ding [14, Theorem 6.4] and of Selvaraj and Prabakaran [20, Theorem 6].

Theorem 4.4. Let $n$ be a non-negative integer and assume that $\text{n-wid}(R) \leq k$ where $k \geq 1$. Then, the following conditions are equivalent:

1. $\text{l.n.sp.gldim}(R) < \infty$.
2. $\text{l.n.sp.gldim}(R) \leq k$.
3. Every left $R$-module in $\mathcal{W}_n^0(R)$ belongs to $\mathcal{W}_n^0(R)$.  

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Every left $R$-module in $\mathcal{W}_k^{n-1}(R)^\perp$ is injective.

(5) Every left $R$-module in $\mathcal{W}_k^{n-1}(R)^\perp$ is $n$-weak injective.

(6) Every left $R$-module in $\mathcal{W}_k^n(R)^\perp$ is injective.

(7) Every left $R$-module in $\mathcal{W}_k^n(R)^\perp$ is $n$-weak injective.

(8) Every left $R$-module in $\mathcal{W}_k^n(R)^\perp$ belongs to $\mathcal{W}_k^n(R)$.

(9) Every left $R$-module (with $M \in \mathcal{W}_k^{n-1}(R)^\perp$) has a monic $\mathcal{W}_k^{n-1}(R)$-cover.

If $R$ is a left coherent ring, then the above conditions are also equivalent to:

(10) Every right $R$-module in $\mathcal{W}_k^{n-1}(R)^\top$ is flat.

(11) Every right $R$-module in $\mathcal{W}_k^{n-1}(R)^\top$ is $n$-weak flat.

(12) Every right $R$-module in $\mathcal{W}_k^n(R)^\top$ is flat.

(13) Every right $R$-module in $\mathcal{W}_k^n(R)^\top$ is $n$-weak flat.

Proof. (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (6) $\Rightarrow$ (7), (4) $\Rightarrow$ (5) $\Rightarrow$ (7), (10) $\Rightarrow$ (11) $\Rightarrow$ (13) and (10) $\Rightarrow$ (12) $\Rightarrow$ (13) are clear.

(1) $\Rightarrow$ (2) Clear by Lemma 4.3.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (9) and (2) $\Rightarrow$ (8) Follow from [11 Proposition 4.10].

(8) $\Rightarrow$ (2) Let $M$ be a left $R$-module. Since $n\text{-wid}(R) \leq k$, it follows that $R \in \mathcal{W}_k^n(R)$. Then by [11 Proposition 4.8], there is a short exact sequence $0 \to K \to F \xrightarrow{h} M \to 0$, where $h$ is an epic $\mathcal{W}_k^n(R)$-cover of $M$. Hence by [23 Lemma 2.1.1] and [11 Proposition 4.12], we deduce that $K \in \mathcal{W}_k^n(R)^\perp$. By assumption, $K \in \mathcal{W}_k^n(R)$, and so [11 Corollary 3.4] implies that $M \in \mathcal{W}_k^n(R)$. Consequently, by [11 Proposition 4.10], $\text{l.n.sp.gldim}(R) \leq k$.

(6) $\Rightarrow$ (4) Clear by Lemma 4.2.1.

(7) $\Rightarrow$ (5) Clear by Lemma 4.2.1 and [11 Corollary 3.4(1)].

(2) $\Leftrightarrow$ (6) Follows from [11 Propositions 4.10 and 4.12 (1)].

(5) $\Rightarrow$ (4) Let $M \in \mathcal{W}_k^{n-1}(R)^\perp$. By assumption, $M$ is also $n$-weak injective. Now consider an exact sequence $0 \to M \to E \to D \to 0$, where $E$ is injective and so also $n$-weak injective. Then by [11 Corollary 3.4], $D$ is $n$-weak injective. By [11 Theorem 2.13(1)], $\text{Ext}_R^k(K_{n-1}, D) = 0$.
for any special super finitely presented \( K_{n-1} \), and hence by [1 Proposition 3.2], \( D \in \mathcal{W}T_{n-1}^k(R) \). Then, \( \operatorname{Ext}^1_R(D, M) = 0 \). So the above exact sequence splits and consequently \( M \) is injective.

(7) \( \Rightarrow \) (6) Similar to the proof of (5) \( \Rightarrow \) (4).

(12) \( \Rightarrow \) (10) Trivial by Lemma 4.2(2).

(13) \( \Rightarrow \) (11) Trivial by Lemma 4.2(2) and [1 Corollary 3.4(2)].

(13) \( \Rightarrow \) (12) Let \( M \) be in \( \mathcal{W}T_{n-k}^k(R) \). Then by (13), \( M \) is \( n \)-weak flat. By Lemma 4.1, \( M^* \in \mathcal{W}T_{n-k}^k(R) \) and is also \( n \)-weak injective by [1 Proposition 2.6]. Similar to the proof of (5) \( \Rightarrow \) (4), we see that \( M^* \in \mathcal{W}T_{n-k}^k(R) \) and is also injective. Therefore by Lemma 4.1 again, \( M \in \mathcal{W}T_{n-k}^n(R) \) and is also flat.

(12) \( \Rightarrow \) (1) Every FP-injective \( R \)-module is \( n \)-weak injective. So, if \( \mathcal{F}_k \) denotes the class of FP-injective right \( R \)-modules with dimension at most \( k \), then \( \mathcal{F}_k \subseteq \mathcal{W}T_{n-k}^n(R) \). By assumption every right \( R \)-module in \( \mathcal{F}_k \) is flat. Hence by [14 Theorem 6.4], \( wD(R) \leq k \), and then by Theorem 2.4 (n, sp, gldim) \( (R) < \infty \).

(6) \( \Rightarrow \) (12) Follows from Lemma 4.3.

**Theorem 4.5.** Let \( n \) and \( k \) be non-negative integers and consider the following assertions:

1. \( \iota_{n, sp, gldim}(R) \leq k \).
2. Every \((n, k)\)-weak cotorsion right \( R \)-module is injective.
3. \( \operatorname{id}_R(M) \leq k \) for every \((n, 0)\)-weak cotorsion right \( R \)-module \( M \).
4. Every \((n, k)\)-weak cotorsion right \( R \)-module is in \( \mathcal{W}F_k^n(R^{op}) \).
5. \( \operatorname{id}_R(M) \leq k' \) for any \( k' \) with \( 0 \leq k' \leq k \) and any \((n, k-k')\)-weak cotorsion right \( R \)-module \( M \).
6. \( \operatorname{id}_R(M) \leq k' \) for some \( k' \) with \( 0 \leq k' \leq k \) and any \((n, k-k')\)-weak cotorsion right \( R \)-module \( M \).
7. \( \operatorname{Ext}^{1+i}_R(M, N) = 0 \) for any \( i \geq m \) and any \((n, k+m)\)-weak cotorsion module \( M \) and \((n, k)\)-weak cotorsion module \( N \).
8. Every right \( R \)-module \( M \) has a \( \mathcal{W}F_k^n(R^{op}) \)-cover with the unique mapping property.

Then, (1) \( \iff \) (2) \( \iff \) (4) \( \iff \) (7) \( \iff \) (8) and (1) \( \Rightarrow \) (5) \( \Rightarrow \) (6) \( \Rightarrow \) (3).
Proof. (1) \iff (2) \text{l.n.sp.gldim}(R) \leq k \text{ if and only if } \text{pd}_R(K_{n-1}) \leq k \text{ for all special super finitely presented module } K_{n-1} \text{ of any } n\text{-special super finitely presented left } R\text{-module } U, \text{ if and only if } \text{ every right } R\text{-module is in } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}) \text{ by [1] Proposition 3.5}, \text{ if and only if } \text{Ext}_R^1(N,C) = 0 \text{ for any } R\text{-module } N \text{ and any } (n,k)\text{-weak cotorsion right } R\text{-module } C.

(1) \implies (4) \text{ Similar to the proof of } (1) \implies (2), \text{ it follows that every right } R\text{-module is in } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}).

(4) \implies (1) \text{ Let } N \text{ be a right } R\text{-module. Then by [1] Proposition 4.13, there exists an exact sequence } 0 \to N \to X \to D \to 0, \text{ where } X \text{ is in } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}})^{\perp} = \mathcal{W}\mathcal{C}_k^n(R^{\text{op}}) \text{ and } D \text{ is in } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}). \text{ By assumption, } X \text{ is in } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}). \text{ Therefore, from [1] Corollary 34}, \text{ } N \in \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}), \text{ and so by [1] Proposition 3.5, it follows that } \text{pd}_R(K_{n-1}) \leq k \text{ for all special super finitely presented module } K_{n-1} \text{ of any } n\text{-special super finitely presented left } R\text{-module } U. \text{ Consequently } \text{l.n.sp.gldim}(R) \leq k.

(4) \implies (7) \text{ Ext}_R^1(M',N) = 0 \text{ for all } (n,k)\text{-weak cotorsion right } R\text{-modules } M' \text{ and } N \text{ by assumption. Let } N \text{ be an } (n,k)\text{-weak cotorsion right } R\text{-module and } D^m \text{ the } m\text{th cosyzygy of } N. \text{ Then, we have } \text{Ext}_R^{m+1}(M',N) \cong \text{Ext}_R^1(M',D^m). \text{ By Proposition 3.9(2), } D^m \text{ is } (n,k+m)\text{-weak cotorsion, and also it easy to see that } D^m \text{ is } (n,k)\text{-weak cotorsion. Since } M' \in \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}), \text{ we deduce that } \text{Ext}_R^1(M',D^m) = 0, \text{ and hence } \text{Ext}_R^{1+i}(M',N) = 0 \text{ for any } i \geq m. \text{ On the other hand, every } (n,k+m)\text{-weak cotorsion } M \text{ is } (n,k)\text{-weak cotorsion. Therefore, } \text{Ext}_R^{1+i}(M,N) = 0 \text{ for any } i \geq m \text{ and for all } (n,k+m)\text{-weak cotorsion modules } M \text{ and } (n,k)\text{-weak cotorsion right } R\text{-modules } N.

(7) \implies (4) \text{ By assumption, } \text{Ext}_R^1(M,N) = 0 \text{ for all } (n,k)\text{-weak cotorsion right } R\text{-modules } M \text{ and } N. \text{ So every } (n,k)\text{-weak cotorsion right } R\text{-module is in } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}}).

(1) \implies (8) \text{ Clear.}

(8) \implies (4) \text{ Let } M \text{ be an } (n,k)\text{-weak cotorsion right } R\text{-module. Then by assumption, we have the following commutative diagram}

\begin{align*}
0 & \longrightarrow M \quad \alpha \quad X \quad \beta \quad D \quad 0, \\
0 & \quad \gamma \beta \quad \gamma \quad \gamma \quad \gamma
\end{align*}

\text{where } \alpha \text{ and } \gamma \text{ are } \mathcal{W}\mathcal{F}_k^n(R^{\text{op}})\text{-envelope. Since } \gamma \beta \alpha = 0, \text{ it follows that } \gamma \beta = 0 \text{ by assumption.}
Consequently $L = \text{im}(\beta) \subseteq \ker(\gamma) = 0$ and so $L = 0$. Hence, $M \cong X$, and so $M$ is in $\mathcal{W}_k^n(R^{op})$.

$(1) \Rightarrow (5)$ Let $N$ be a right $R$-module. By [1, Proposition 3.3], $n.\text{wfd}_R(N) \leq k$, and so there is an exact sequence:

$$0 \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to N \to 0,$$

where any $F_i$ is $n$-weak flat. Since $0 \leq k' \leq k$, the $(k' - 1)$th syzygy of $N \in \mathcal{W}_k^{n,k'}(R^{op})$. If $L$ is the $(k' - 1)$th syzygy of $N$, then $\text{Ext}^{k'+1}_R(N, M) \cong \text{Ext}^1_R(L, M) = 0$ for any $(n, k - k')$-weak cotorsion right $R$-module $M$. Hence, $\text{id}_R(M) \leq k'$.

$(5) \Rightarrow (6)$ Clear.

$(6) \Rightarrow (3)$ Let $M$ be an $(n, 0)$-weak cotorsion right $R$-module. Then, by Proposition 3.9(2), the $(n, k - k')$th cosyzygy $D^{k-k'}$ of $M$ is $(n, k - k')$-weak cotorsion. So, $\text{id}_R(D) \leq k'$ and hence $\text{id}_R(M) \leq k$.

**Proposition 4.6.** Let $n$ and $k$ be non-negative integers. If $R$ satisfies one of the following conditions:

1. Every $(n, k)$-weak cotorsion right $R$-module has a $\mathcal{W}_k^n(R^{op})$-envelope with the unique mapping property.

2. Every finitely presented right $R$-module has a $\mathcal{W}_k^n(R^{op})$-envelope with the unique mapping property.

Then, $\text{l.n.sp.gldim}(R) \leq k + 2$.

**Proof.** Assume (1). Then, by [1 Theorem 4.5], every right $R$-module has a $\mathcal{W}_k^n(R^{op})$-cover. So, by Proposition [3.3] and [23 Lemma 2.1.1], we have exact sequences $0 \to K \xrightarrow{i} F \xrightarrow{\alpha} N \to 0$ and $0 \to K' \xrightarrow{i'} F' \xrightarrow{\alpha'} K \to 0$, where $\alpha$ and $\alpha'$ are $\mathcal{W}_k^n(R^{op})$-covers of $N$ and $K$, respectively. Also $K$ and $K'$ are $(n, k)$-weak cotorsion. Hence, we obtain the exact sequence:

$$0 \to K' \xrightarrow{i'} F' \xrightarrow{i'\alpha'} F \xrightarrow{\alpha} N \to 0.$$

Let $\beta : K' \to L$ be a $\mathcal{W}_k^n(R^{op})$-envelope with the unique mapping property. Then there exists $\gamma : L \to F'$ such that $i' = \gamma\beta$. So $i'\alpha'\gamma = i'\alpha' = 0$, and then $i'\alpha'\gamma = 0$, which implies that
im(\gamma) \subseteq \ker(i\alpha') = \im(i'). Therefore, there exists \delta : L \to K' such that i'\delta = \gamma. Thus, we get the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \overset{i'}{\longrightarrow} & F' & \overset{\delta}{\longrightarrow} & L & \overset{\beta}{\longrightarrow} & \overset{i\alpha'}{\longrightarrow} & F & \overset{\alpha}{\longrightarrow} & N & \longrightarrow & 0,
\end{array}
\]

where i'\beta = i'. So \delta \beta = 1_{K'} (because i' is monic). Thus, K' is isomorphic to a direct summand of L, and hence by using of [1 Proposition 2.9], it follows that \(K' \in WF^n_{k}(R^{op})\). Therefore, we deduce that \(K \in WF^n_{k+1}(R^{op})\) and consequently \(N \in WF^n_{k+2}(R^{op})\). Hence, by [1 Proposition 3.3], n-wfld_{R}(N) \leq k + 2, and so [1 Proposition 3.5] implies that l.n.sp.gldim_{R}(R) \leq k + 2.

For the converse, by [5 Lemma 3.2], every right R-module has a \(WF^n_{k+1}(R^{op})\)-envelope with the unique mapping property (because \(WF^n_{k+1}(R^{op})\) is closed under direct limits).

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