A MAXIMUM PRINCIPLE FOR SELF-SHRINKERS AND SOME CONSEQUENCES

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Abstract. Using a maximum principle for self-shrinkers of the mean curvature flow, we give new proofs of a rigidity theorem for rotationally symmetric compact self-shrinkers and a result about the asymptotic behavior of self-shrinkers. This comparison argument also implies a linear bound for the second fundamental form of self-shrinking surfaces under natural assumptions. As a consequence, translating solitons can be related to these self-shrinkers.

A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker for the mean curvature flow, normalized so that it shrinks in unit time to the origin $0 \in \mathbb{R}^{n+1}$, if it satisfies the following equation:

$$H = \frac{\langle x, \nu \rangle}{2}$$

where $\nu$ is the outer normal unit vector, $H = \text{div}(\nu)$ denotes the mean curvature and $x$ is the position vector. Such a surface will be called more briefly a self-shrinker. Equivalently, one can say that the set of hypersurfaces $\{\sqrt{-t} \Sigma; t < 0\}$ is a solution of the mean curvature flow, i.e. verifies the equation:

$$\left( \partial_t x \right)^\perp = -H \nu.$$  

Self-shrinkers provide models for blow-ups at singularities of mean curvature flow: consider a family of hypersurfaces evolving by the mean curvature flow and starting from a closed embedded hypersurface and focus on a point of singularity, then rescalings yield a subsequence converging weakly to a "tangent flow", which satisfies [I] (see [II], [20], [I]). The classification of embedded self-shrinkers proved to be a difficult problem. The simplest examples are given by cylinders $S^k \times \mathbb{R}^{n-k}$ where $S^k$ is the $k$-sphere of radius $\sqrt{2k}$. If $n \geq 2$, Huisken, then Colding and Minicozzi showed that those hypersurfaces were the only ones with polynomial volume growth and whose mean curvature $H$ is nonnegative ([3], [9], [4] Theorem 0.17). When $n = 1$, straight lines passing through the origin and the circle of radius $\sqrt{2}$ are the only self-shrinking embedded curves [I], but as soon as the dimension is greater than or equal to 2, there are non trivial self-shrinkers, as
the Angenent torus \([2]\). See also \([17]\) and \([18]\) for construction of complete embedded self-shrinkers of high genus.

The aim of this note is to present some applications of a maximum principle adapted to self-shrinking hypersurfaces viewed locally as graphs (Proposition \([6]\), Corollary \([8]\)). It will be used to rule out some hypersurfaces from the set of self-shrinkers. The usual maximum principle states for example that the distance between two compact hypersurfaces moving by the mean curvature flow is non-decreasing in time. Here, we prove a maximum principle for graphs, the advantage being that within a hypersurface, sometimes one can find two subsets forming graphs of two functions whose difference achieves a minimum, whereas a minimal distance (even a local one) between the two subsets is not achieved.

Using this remark (see Proposition \([6]\), Corollary \([8]\)), we prove that the only embedded rotationally symmetric compact self-shrinkers are either a \(S^1 \times S^{n-1}\) or a round sphere with the appropriate radius (this fact is part of a more general statement covering the non compact case proven by Kleene and Møller \([13]\)):

**Theorem 1 (\([13]\)).** Let \(\Sigma^n \subset \mathbb{R}^{n+1}\) be a compact embedded rotationally symmetric hypersurface. If \(\Sigma\) is a self-shrinker then either it is the sphere of radius \(\sqrt{2n}\) centered at the origin or a \(S^1 \times S^{n-1}\).

Then, with the same argument, we show that self-shrinkers are weakly asymptotic to cones, a result known by Ilmanen (\([12]\) p.8, though I didn't find his proof). In the two-dimensional case, some additional information is given.

**Theorem 2 (\([12]\)).** If \(\Sigma \subset \mathbb{R}^{n+1}\) is a complete properly immersed self-shrinker, then there exists a cone \(C\) such that

\[
\lambda \Sigma \to C \text{ when } \lambda \to 0^+
\]

locally in the Hausdorff metric on closed sets.

In the case \(n = 2\), if the number of connected components of \(S(0,r) \cap \Sigma\) is bounded above, then \(C \cap S^2\) has 2-dimensional Lebesgue measure 0.

Finally in the case of surfaces, our result in Section \([3]\) gives a linear bound for the norm of the second fundamental form \(|A|\) of self-shrinking surfaces, under the assumption that the "curvature concentration" is bounded, i.e. that these surfaces satisfy a local integral bound for \(|A|\) (see Definition \([17]\)). This is a natural assumption in the context of weak blowups (see Proposition \([18]\)). It is proved that:

**Theorem 3.** If \(\Sigma \subset \mathbb{R}^3\) is a complete properly embedded self-shrinker of finite genus \(g\) such that "the curvature concentration is bounded by \(\kappa\)", then

\[
\exists C, \forall x \in \Sigma, |A(x)| \leq C(1 + |x|).
\]
Moreover for such a surface with ends that are "δ-separated at infinity", the constant $C$ only depends on $g$, $δ$, $κ$ and on a "bound for the topology of $Σ"$.

For more precise statements of this result, see Definition 7, Definition 25, Theorem 19 and Theorem 27.

As in [5], the surfaces in the second part of the theorem are supposed to be homeomorphic to closed surfaces with finitely many disjoint disks removed. The proof is based on a blow-up argument, the compactness for self-shrinkers [5], the maximum principle and some results about classical minimal surfaces. Consequently, it is pointed out that translating solitons of the mean curvature flow can model regions of $Σ$ far from the origin (Corollary 24).

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1. A maximum principle for embedded self-shrinkers

The mean curvature flow equation for graphs is given by ([15] p.10):

**Lemma 4.** Suppose that $φ_t : M \rightarrow \mathbb{R}^{n+1}$ are smooth hypersurfaces moving by mean curvature and are graphs on the open subset $Ω$ of the hyperplane $\langle e_1, ..., e_n \rangle \subset \mathbb{R}^{n+1}$, that is, there exists a smooth function $f : Ω \times [0,T) \rightarrow \mathbb{R}$ with

$$φ_t(p) = (x_1(p),...,x_n(p),f(x_1(p),...,x_n(p),t)),$$

then

$$\partial_t f = \Delta f - \frac{\text{Hess} f(∇f, ∇f)}{1 + |∇f|^2} = √1 + |∇f|^2 \text{div} (\frac{∇f}{√1 + |∇f|^2}).$$

Conversely, if $f$ satisfies this equation, then it corresponds to hypersurfaces moving by mean curvature.

Next we give the maximum principle for self-shrinkers:

**Lemma 5.** Let $f, g : Ω \times [0,T) \subset \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be two functions satisfying (3), where $Ω$ is an open subset of $\mathbb{R}^n$. Suppose that there exists a compact set $K \subset Ω$ such that for all $t' \in [0,T)$, the minimum of $(f - g)(.,t')$ is attained at least at one point of $K$.

Define $u(t) = \min_{p \in Ω} (f - g)(p,t)$. Then $u$ is a locally Lipschitz function, hence differentiable almost everywhere and if it exists, the differential is nonnegative.
Proof. By Hamilton’s trick ([13] p.26), $u$ is a locally Lipschitz function, hence differentiable almost everywhere and where it makes sense:

$$\frac{du(t)}{dt} = \frac{\partial(f - g)(p, t)}{\partial t},$$

$p \in K$ being a point where the minimum of $(f - g)$ is attained. But at such a point $p$, $\nabla f = \nabla g$, the Hessian of $f - g$ is nonnegative and $\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}$ is a vector whose Euclidian norm is less than 1, hence the lemma. \hfill \Box

To show that a hypersurface $S$ is not a self-shrinker, one can thus try to apply this maximum principle to two graphs given by two parts of $S$. Next we give a proposition implementing this strategy and which will be used in various forms throughout this note:

**Proposition 6.** Let $S_1$ and $S_2$ two disjoint complete hypersurfaces of $\mathbb{R}^{n+1}$, which can be noncompact and have boundaries. Define the function $h : S_1 \to \mathbb{R} \cup \{\infty\}$:

$$h(a) = \min\{a_{n+1} - b_{n+1}; (a_1, ..., a_n, b_{n+1}) \in S_2 \text{ and } b_{n+1} < a_{n+1}\}$$

if $\{b_{n+1}; b_{n+1} < a_{n+1} \text{ and } (a_1, ..., a_n, b_{n+1}) \in S_2\} \neq \emptyset$

$$h(a) = \infty \text{ otherwise.}$$

If $h$ achieves a local finite minimum at $a \in S_1$ and $b \in S_2$, two points not in the respective boundaries, and if $\langle \nu_a, e_{n+1} \rangle \neq 0$ where $\nu$ is the outward normal vector of $S$, then $S_1$ or $S_2$ is not a part of a self-shrinker.

Proof. Suppose that the two hypersurfaces are self-shrinkers. Under these hypotheses, we can locally write $S_1$ and $S_2$ as graphs of $f$ and $g$ which satisfy the hypotheses of lemma [5]. If $u(t) = \min(f - g)(p, t)$ then $\frac{d\min(u(t))}{dt} \geq 0$ almost everywhere. On the other hand, by the definition of self-shrinker, the solution $S_i(t)$ ($i = 1, 2$) of the mean curvature flow with $S_i(0) = S_i$ is given by $S_i(t) = \sqrt{-t}S_i$ ($-1 \leq t \leq 0$), which would mean $\frac{d\min(u(t))}{dt} < 0$. This is the desired contradiction. \hfill \Box

Next, we give a specialized form of this proposition, but before let’s define hypersurfaces with $\delta$-separated ends.

**Definition 7.** Let $\delta > 0$. $S \subset \mathbb{R}^{n+1}$ is a properly immersed hypersurface. Suppose that $S_1$ and $S_2$ are two disjoint open connected subsets of $S$ with $\overline{S_1} \cup \overline{S_2} = S$ ($\overline{S_i}$ being the closure of $S_i$ in $S$) and $\partial \overline{S_1} = \partial \overline{S_2}$. Suppose that this boundary $\partial \overline{S_1}$ is bounded.

If for every such partition $\overline{S_1} \cup \overline{S_2} = S$, for $r$ sufficiently large, one of the $(\frac{1}{r}S_i) \setminus B(0, 1)$ is empty or

$$\inf\{||x_1 - x_2||; x_i \in (\frac{1}{r}S_i) \setminus B(0, 1)\} \geq \delta$$

then $S$ is said to have $\delta$-separated ends.
Corollary 8. Let $\delta > 0$. Let $S \subset \mathbb{R}^{n+1}$ be a complete properly immersed hypersurface with $\delta$-separated ends.

Suppose that there exists an embedding $\alpha : S^{n-1} \to S$ such that the two disjoint connected components of $S \setminus \alpha(S^{n-1})$ are called $S^+$ and $S^-$, and a set $A \subset \mathbb{R}^n$ of non zero $n$-dimensional Lebesgue measure with the following properties: if $\bar{e} = (e_1, ..., e_{n+1})$ is an orthonormal base of $\mathbb{R}^{n+1}$ with $e_{n+1} \in A$ then in the coordinates determined by $\bar{e}$

(1) there exists $x = (x_1, ..., x_n, x_{n+1}) \in S^-$ and $y = (x_1, ..., x_n, y_{n+1}) \in S^+$ such that $y_{n+1} < x_{n+1}$,

(2) there exists a neighborhood $\mathcal{V}$ of $\alpha(S^{n-1})$ in the closure of $S^-$ verifying that if $a = (z_1, ..., z_n, a_{n+1}) \in \mathcal{V}$ then there is a point $u = (z_1, ..., z_n, u_{n+1}) \in S^-$ with $a_{n+1} > u_{n+1}$ such that there is no $v = (z_1, ..., z_n, v_{n+1}) \in S^+$ with $a_{n+1} > v_{n+1} > u_{n+1}$,

(3) if $z = (z_1, ..., z_{n+1}) \in \alpha(S^{n-1})$ and $b = (z_1, ..., z_n, b_{n+1}) \in S^-$ satisfy $b_{n+1} > z_{n+1}$, then there is a $w = (z_1, ..., z_n, w_{n+1}) \in S^+$ such that $b_{n+1} > w_{n+1} > z_{n+1}$.

Then $S$ is not a self-shrinker.

Remark 9. This corollary covers the case where $S$ is compact (see Figure 7) because such a surface automatically has $\delta$-separated ends. The existence of $A$ will always be easy to check in our applications. Besides, although we will only need the situation of the corollary, where the common boundary is given by an $(n-1)$-sphere $\alpha(S^{n-1})$, one can actually consider more general boundaries.

Proof. To prove this corollary, we find two parts of $S$, the first one in $S^+$, the other one in $S^-$, to which is applied the Maximum Principle. Then the conclusion will follow by Proposition 6.

Suppose that $S$ is a self-shrinker. As in Proposition 6 define $h : S^- \cup \alpha(S^{n-1}) \to \mathbb{R} \cup \{\infty\}$ by

$$h(a) = \min \{a_{n+1} - b_{n+1}; (a_1, ..., a_n, b_{n+1}) \in S^+ \cup \alpha(S^{n-1}) \text{ and } b_{n+1} < a_{n+1}\}$$

if $a \in S^-$ and if such a $b$ exists,

$$h(a) = \infty \text{ if } a \in S^- \text{ and if such a } b \text{ doesn't exist},$$

$$h(a) = \lim_{n \to \infty} \inf \{h(x); x \in S^- \text{ and } |x - a| < \frac{1}{n}\}$$

if $a \in \alpha(S^{n-1})$.

By Proposition 6 we just need to show that $h$ attains a finite minimum at $a \in S^-$ and $b \in S^+$ with $\langle v_a, e_{n+1} \rangle \neq 0$.

This function is not constantly $\infty$ by the first condition and is lower semicontinuous: it attains its finite minimum on the set $S^- \cup \alpha(S^{n-1})$ because of the $\delta$-separation hypothesis. In fact, it can’t be achieved in $\alpha(S^{n-1})$ because of the second condition. So we can find $a = (a_1, ..., a_{n+1}) \in S^-$ and
Figure 1. Some typical configurations where Corollary 8 applies, but where the usual maximum principle is inefficient. Embeddings of $S^{n-1}$ are represented in bold line.

$b = (b_1, ..., b_{n+1}) \in \mathcal{S}$ such that $a_{n+1} - b_{n+1} = \min b$. The third condition gives $b \in \mathcal{S}^+$. Moreover $\nu_a = \nu_b$. Then, the Sard Lemma and the fact that the three conditions are true for $e_{n+1} \in A$ imply that we can suppose $\langle \nu_a, e_{n+1} \rangle \neq 0$: indeed the set of vectors $e \in \mathbb{S}^n$ for which there exists $a \in \mathcal{S}^-$ and $b \in \mathcal{S}^+$ with $\nu_a = \nu_b$ and $\langle \nu_a, e_{n+1} \rangle = 0$ is of Lebesgue measure zero (they are critical values of $(a, b) \in \mathcal{S}^- \times \mathcal{S}^+ \mapsto (a - b)/||a - b||$). The conclusion follows from Proposition 6.

\[\square\]

2. SOME APPLICATIONS

By "maximum principle for self-shrinkers", one usually means that the distance between two hypersurfaces moving by mean curvature is non-decreasing. Here, Corollary 8 gives a maximum principle for graphs and one has to choose the axis $\mathbb{R}e_{n+1}$: this non canonical choice enables more flexibility, as we will see with the following paragraphs.

2.1. Rotationally symmetric compact self-shrinkers. Let $u$ be a vector of $\mathbb{R}^{n+1}$, $\mathbb{S}^{n-1}$ is identified with the unit sphere of the hyperplane orthogonal to $u$. Consider a simple curve $\gamma : [a, b] \to \mathbb{R} \times \mathbb{R}^+$. Let $\Sigma_{\gamma}$ be the image of an embedding $\varphi : \mathbb{S}^{n-1} \times [0, 1] \to \mathbb{R}^{n+1}$ which can be written as
\( \varphi(\omega, s) = x(s)u + r(s)\omega \), where \( s \mapsto (x(s), r(s)) \) is a parametrization of \( \gamma \). \( \Sigma_\gamma \) is then said to be the rotationally symmetric hypersurface generated by \( \gamma \): it is obtained by rotating \( \gamma \) around the axis \( \mathbb{R}u \). Here is a result proved in [13] (see also [7]).

**Theorem 10 ([13]).** If \( \Sigma^n \subset \mathbb{R}^{n+1} \) is an embedded compact rotationally symmetric self-shrinker, then \( \Sigma^n \) is:

1. either the sphere \( S^n \) of radius \( \sqrt{2n} \) centered at the origin,
2. or an embedded \( S^1 \times S^{n-1} \).

**Proof.** Suppose that \( \Sigma^n \) is generated by \( \gamma : [a, b] \to \mathbb{R} \times \mathbb{R}^+ \). By compactness and embeddedness of \( \Sigma \), \( \gamma \) is the disjoint union of simple closed curves of \( \mathbb{R} \times \mathbb{R}^+ \) and simple curves whose ends are in \( \mathbb{R} \times \{0\} \). The usual maximum principle implies that this union has only one element.

If \( \gamma \) is a simple closed curve in \( \mathbb{R} \times \mathbb{R}^+ \) then \( \Sigma \) is a torus \( S^1 \times S^{n-1} \).

Suppose now that \( \gamma(0) \) and \( \gamma(1) \) are in \( \mathbb{R} \times \{0\} \). \( \Sigma \) is necessarily diffeomorphic to a sphere and we want to show that it is in fact the sphere of radius \( \sqrt{2n} \) centered at the origin. Because of results proven in [8], we just have to show that \( H \geq 0 \) on \( \Sigma \). Suppose that this is not the case: \( \{s; H(\gamma(s)) < 0\} \neq \emptyset \). \( u \) is the horizontal unit vector (identified with \( (1, 0) \in \mathbb{R} \times \mathbb{R}^+ \)), let \( v \) be the vertical unit vector \( (0, 1) \in \mathbb{R} \times \mathbb{R}^+ \). Write \( (x(s), r(s)) \) for \( (x(\gamma(s)), r(\gamma(s))) \). Denote by \( \theta(s) \) the angle between \( (x(s), r(s)) \) and \( \gamma(s) \) for all the points \( \gamma(s) \) different from the origin. The mean curvature \( H \) vanishes at \( \gamma(s) \) if and only if \( \theta(s) = 0[\pi] \) or \( x(s) = r(s) = 0 \) and its sign is given by the sign of \( \sin(\theta(s)) \). By changing the parametrization, we can suppose that \( x(0) > x(1) \). \( \Sigma \) being smooth, \( \gamma'(0) \) is parallel to \( v \) and \( \nu(\gamma(0)) \) is parallel to \( u \).

**Lemma 11.** Suppose that \( \Sigma \) is a compact self-shrinker but \( \{s; H(\gamma(s)) < 0\} \neq \emptyset \). Then, by changing \( u \) to \( -u \) if necessary, there would be \( s_1 < t < s_2 \in [0, 1] \) such that the following properties are satisfied:

1. \( \gamma[s_1, s_2] \) is the graph of a function \( f : [x(s_1), x(s_2)] \to \mathbb{R} \) over the \( x \)-axis,
2. \( \langle v, \nu(\gamma(s)) \rangle < 0 \) for all \( s \in [s_1, s_2] \),
3. \( f \) attains its maximum at \( x(t) \in [x(s_1), x(s_2)] \) and \( f(x(s_i)) < f(x(t)) \) for \( i = 1, 2 \).

**Proof.** (of the lemma) With a small abuse of notation, we write \( \nu(s) \) for \( \nu(\gamma(s)) \). Let \( a \) be such that \( H(\gamma(a)) < 0 \). By changing \( u \) to \( -u \) if necessary, we can suppose by continuity that \( x(a) < 0 \). We can also suppose that \( \nu(a) \neq \pm u \). Define the functions \( \rho_1(a) = \sup\{s < a; \nu(s) = \pm u\} \) and \( \rho_2(a) = \inf\{s > a; \nu(s) = \pm u\} \). Let’s distinguish two cases, depending on the sign of \( \langle v, \nu(a) \rangle \).

If \( \langle v, \nu(a) \rangle > 0 \) then, the fact that \( x(a) < 0 \) and the usual maximum principle imply that \( \nu(\rho_2(a)) = u \) (if not, consider a plane touching locally \( \Sigma \) between \( \gamma(a) \) and \( \gamma(\rho_2(a)) \)). Once again by this argument, there exists \( a' \)
greater than \( \rho_2(a) \) such that \( x(\rho_1(a')) < x(a') \) and \( \nu(\rho_1(a')) = u \). Now either \( \nu(\rho_2(a')) = -u \) or \( \nu(\rho_2(a')) = u \). In the first case, let
\[
s_1 = \rho_1(a'), s_2 = \rho_2(a')
\]
and take \( t \in ]s_1, s_2[ \) such that \( r(t) \) is maximal when \( t \in ]s_1, s_2[ \): the properties of the lemma are indeed satisfied. In the second case (the two normal vectors have the same direction, see figure 2): consider the rotationally symmetric hypersurface \( \mathcal{B} \) generated by \( \gamma_{[\rho_1(a'), \rho_2(a')]} : [\rho_1(a'), \rho_2(a')] \to \mathbb{R} \times \mathbb{R}^+ \). Define the function
\[
\tau : z \in \mathcal{B} \to \langle \nu_z, u \rangle.
\]
On the boundary \( \partial \mathcal{B} \), \( \tau \) is equal to its greater possible value 1, so this function attains its minimum inside \( \mathcal{B} \). This minimum is strictly less than 1 as \( s \in [\rho_1(a'), \rho_2(a')] \to x(\gamma(s)) \) is not constant. Besides, if \( A \) denotes the second fundamental form, \( |A|^2 \neq 0 \) as soon as \( \tau \neq 1 \) or \( -1 \) because of the rotational symmetry. Using the self-shrinker equation \( \Box \) and the Codazzi equations, we compute in local charts:
\[
\nabla \tau = A.u^T
\]
than
\[
\Delta \tau = g^{ij} \nabla_i \nabla_j \langle \nu, u \rangle
= g^{ij} \nabla_i \langle \nabla_j \nu, u \rangle
= g^{ij} \nabla_i \langle h_{ji} g^{lm} \frac{\partial X}{\partial x^m}, u \rangle
= \langle g^{ij} \nabla_i h_{ji} g^{lm} \frac{\partial X}{\partial x^m}, u \rangle + \langle h_{ji}, g^{lm} g^{ij} \nabla_i \frac{\partial X}{\partial x^m}, u \rangle
= \langle \nabla H, u \rangle - |A|^2 \tau
= \langle X, A.u^T \rangle - |A|^2 \tau.
\]
The function \( \tau \) is thus strictly negative at a point of minimum, which makes it possible to find \( s_1 < t < s_2 \) as in the lemma (recall that \( x(\rho_1(a')) < x(a') \)).

If \( \langle v, \nu(a) \rangle < 0 \), then either \( \langle u, \nu(a) \rangle > 0 \), or \( \langle u, \nu(a) \rangle < 0 \). Consider the first case, the second one being similar. If \( \nu(\rho_2(a)) = -u \) then the lemma is verified. Suppose that \( \nu(\rho_2(a)) = u \). If \( \nu(\rho_1(a)) = u \), by arguing as in the preceding case where the two normal vectors considered have the same direction. Finally, there is the case \( \nu(\rho_2(a)) = u \) and \( \nu(\rho_1(a)) = -u \): a fortiori, \( x(\rho_1(a)) < x(\rho_2(a)) \). Remind that \( x(0) > x(1) \). The lemma is then verified by considering a neighborhood of the point where
\[
\min \{ \max \{ r(s); s \in [0, \rho_1(a)] \}; \max \{ r(s); s \in [\rho_2(a), 1] \} \}
\]
is achieved.

\( \Box \)
To conclude the proof of the theorem, let $s_1$, $s_2$, $a$ be as in the lemma. Of course we can suppose that $r(s_1) = r(s_2)$ and that there is no $s$ between $s_1$ and $s_2$ such that $r(s) = r(s_1)$ and $x(s) < x(s_1) < x(s_2)$. If \( \min\{\max\{r(s); s \in [0, s_1]\}; \max\{r(s); s \in [s_2, 1]\}\} \) is attained at $s \in [0, s_1]$ then define $S^-$ as the hypersurface generated by $\gamma_{[0, s_1]}$, if not define $S^-$ as generated by $\gamma_{[s_2, 1]}$. Let $S^+$ be the complement of the closure of $S^-$ in $\Sigma$. Apply \( \ref{prop:maximum_principle} \) with $e_{n+1} \approx u$ in the first case, $e_{n+1} \approx -u$ in the second case where the sign $\approx \pm u$ means that $e$ is chosen very close to $\pm u$ (essentially $\Sigma$ looks like the left of Figure \( \ref{fig:geometric} \)).

\[ \square \]

2.2. Asymptotic behavior and cones. We now apply Proposition \( \ref{prop:maximum_principle} \) to get a simple proof of a result of Ilmanen (see \cite{Ilmanen} Lecture 2, B, remark on p.8).

**Definition 12.** Let $K$ be a compact subset of $S^n$. The set

\[ \{r x | r > 0 \quad x \in K\} \cup \{0\} \subset \mathbb{R}^{n+1} \]

is called the cone generated by $K$.

As usual, $S(0, r)$ (resp. $B(0, r)$) denotes the sphere (resp. the ball) of radius $r$ centered at the origin.

**Theorem 13** (\cite{Ilmanen}). Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a complete properly immersed self-shrinker. Then there exists a cone $C \subset \mathbb{R}^{n+1}$ generated by a compact set $K \subset S^n$ such that:

\[ \lambda \Sigma \to C \text{ as } \lambda \to 0^+, \]

locally for the Hausdorff metric.
Moreover in the case \( n = 2 \), if the number of connected components of \( S(0, r) \cap \Sigma \) is bounded when \( r \to \infty \), then the compact \( K \) is of 2-dimensional Lebesgue measure 0.

**Proof.** Denote by \( d_H \) the Hausdorff distance for non-empty compact sets in \( \mathbb{R}^{n+1} \) and \( \Sigma_\lambda = \lambda \Sigma \) for \( \lambda > 0 \). Let \( r > 0 \) fixed. Define
\[
K_\lambda^r = \Sigma_\lambda \cap S(0, r).
\]
It is sufficient to show that \( K_\lambda^r \) converges in the Hausdorff metric to a compact set \( K_r \subset S(0, r) \) when \( \lambda \) goes to 0.

Let’s introduce some other notations: if \( 0 < \lambda < r/\sqrt{2n} \) we write
\[
L_\lambda^r = \{ p \in S(0, r); d(p, K_\lambda^r) \leq \sqrt{2n} \lambda \}
\]
where \( d \) is the distance between a point and a closed set,
\[
M_\lambda^r = \cap_{\lambda \leq \mu \leq \frac{r}{\sqrt{2n}}} L_\mu^r,
\]
\[
Y_r = \Sigma \setminus B(0, r)
\]
and
\[
Z^\lambda_r = \{ p \in \mathbb{R}^{n+1}; ||p|| \geq r \text{ and } r \frac{p}{||p||} \in L_\lambda^r \}.
\]

Firstly, let’s show a kind of monotonicity relation:
\[
(4) \quad \forall \lambda, \mu \in ]0, \frac{r}{\sqrt{2n}}[, \quad \mu \leq \lambda \Rightarrow K_\mu^r \subset L_\lambda^r.
\]

Note that if \( \lambda = \frac{r}{\sqrt{2n}} \), then \( K_r^1 \neq \emptyset \). Indeed, by homogeneity, this is tantamount to saying that \( K_{\sqrt{2n}}^1 \neq \emptyset \), which is true by the Maximum Principle. Likewise, to prove 4, we need to show
\[
\forall r > \sqrt{2n} \quad Y_r \subset Z^1_r.
\]

To prove this, suppose that there is \( a \in S(0, r) \setminus L_1^r \) and \( b \in Y_r \) such that \( r \frac{b}{||b||} = a \). Choose \( e_{n+1} \) "very close" to \( -\frac{a}{||a||} \), in a sense precised below. Consider then the two following hypersurfaces \( S_1 \) and \( S_2 \): \( S_1 \) is the spherical self-shrinker of radius \( \sqrt{2n} \) centered at the origin and \( S_2 = Y_r \). Now, for \( e_{n+1} \) well chosen near \( -\frac{a}{||a||} \), the function \( h : S_1 \to \mathbb{R} \cup \{ \infty \} \) is not \( \infty \) everywhere and attains its minimum on points not on the boundary of \( S_2 \) because \( d(a, K_1^r) > \sqrt{2n} \). We conclude with Proposition 6. In particular, \( M_\lambda^1 \) is not empty if \( K_\lambda^1 \) is not.

Suppose now that \( \Sigma \) is unbounded, otherwise there is nothing to prove. This means that \( K_\lambda^r \neq \emptyset \) for \( \lambda \leq \frac{r}{\sqrt{2n}} \). Define
\[
K_r = \cap_{\lambda \leq \frac{r}{\sqrt{2n}}} L_\lambda^r.
\]
The \( M_\lambda^r \) being non empty for \( \lambda \leq \frac{r}{\sqrt{2n}} \) and included one in the other, \( K_r \) is a non empty compact set. Let’s show that \( K_\lambda^r \) converges to \( K_r \). As \( K_\lambda^r \subset M_\lambda^r \),
\[
\sup_{x \in K_\lambda^r} d(x, M_\lambda^r) = 0.
\]
Then by the definition of $M^\lambda_r$, if $y \in M^\lambda_r$, there exists $s \in K^\lambda_r$ such that $d(x, y) \leq \sqrt{2n}\lambda$, so
\[
\sup_{y \in M^\lambda_r} d(y, K^\lambda_r) \leq \sqrt{2n}\lambda.
\]
Now, $K^\lambda_r$ is the intersection of the $M^\lambda_r$ which constitute a decreasing nested sequence of non empty compact sets so $M^\lambda_r$ converge to $K^\lambda_r$. Finally,
\[
d_H(K^\lambda_r, K_r) \leq d_H(K^\lambda_r, M^\lambda_r) + d_H(M^\lambda_r, K_r) \to 0,
\]
which is the desired convergence.

In what follows, suppose that $n = 2$, that $\Sigma$ is properly immersed and that the number of connected components of $S(0, r) \cap \Sigma$ is bounded when $r \to \infty$. The $d$-dimensional Lebesgue measure of a submanifold of $\Sigma$ its $d$-volume, denoted by $\text{Vol}_d(M)$. We have

\[
(5) \quad \text{Vol}_2(L^\lambda_r) \to 0 \text{ as } \lambda \to 0^+.
\]

To show this, we use the Euclidean volume growth for properly immersed self-shrinkers ([6]). Because $K^\lambda_r$ is the intersection of the $L^\lambda_r$, it is sufficient to prove that there is a sequence $(\lambda_k)_{k \in \mathbb{N}}$ converging to 0 with $\text{Vol}_2(L^\lambda_k) \to 0$. The Sard Lemma implies that for almost all $\lambda > 0$, $K^\lambda_r$ is a 1-dimensional submanifold of $S(0, r)$. Let $\Lambda$ be the set of those $\lambda$. Suppose that there is an $\epsilon > 0$ and a $\lambda_0 > 0$ such that

\[
(6) \quad \forall \lambda < \lambda_0, \text{Vol}_2(L^\lambda_r) > \epsilon,
\]
we want to find a contradiction.

**Lemma 14.** Take $r > 0$, $0 < \eta < 1$. Suppose that $\gamma \subset S(0, r) \subset \mathbb{R}^3$ is an immersed closed curve of length $l$. Then there exist $C_1, C_2$ two constants independent of $\gamma$, $r$ and $\eta$ such that
\[
\text{Vol}_2(\{x \in S(0, r); d(x, \gamma) \leq \eta\}) \leq \eta. (C_1 l + C_2 \eta).
\]

**Proof.** (of the lemma)
Let $k$ be the maximal number of discs of radius $\eta$ (in $S(0, r)$) such that they are disjoint and centered on a point of $\gamma$. As $\gamma$ is connected,
\[
(k - 1) \leq \frac{l}{2\eta}.
\]
Let $\mathcal{F}$ be a family of such discs, with $k$ elements. It is not empty (i.e. $k \neq 0$) and the distance between $\bigcup_{D \in \mathcal{F}} D$ and a point of $\gamma$ which is not in $\bigcup_{D \in \mathcal{F}} D$ is less than $\eta$; otherwise one could add to $\mathcal{F}$ the disc centered on this point, which would contradict the maximality of $\mathcal{F}$. Consequently,
\[
\text{Vol}_2(\Gamma) \leq \tilde{C} k. \pi(3\eta)^2 \leq \tilde{C}(\pi(3\eta)^2 + \frac{l}{2\eta} \pi(3\eta)^2) \leq \tilde{C}'(\pi(3\eta)^2 + \frac{9}{2} \pi l. \eta)
\]
where $\tilde{C}$ depends on $r$ and the lemma is proved. \qed
This lemma and the assumption \([6]\) imply that
\[
\text{Vol}_1(K^\lambda_r) \to \infty
\]
when \(\lambda \in \Lambda\) goes to 0. Equivalently, one can write that
\[
\text{Vol}_1(K^1_r) = \sigma(r)r,
\]
where \(\sigma\) is a strictly positive function defined for almost all \(r\) and converging to \(\infty\). But, if \(N\) is the norm function of \(\mathbb{R}^{n+1}\), the co-area formula gives:

\[
\text{Vol}_2(\Sigma \cap \{\sqrt{2} \leq N \leq R\}) = \int_{\Sigma \cap \{2 \leq N \leq R\}} d\text{Vol}_2
\geq \int_{\Sigma \cap \{2 \leq N \leq R\}} ||\nabla N|| d\text{Vol}_2
= \int_{2}^{R} \int_{K^1_r} d\text{Vol}_1 dr
= \int_{2}^{R} \sigma(r)r dr
\geq \frac{1}{2} (R^2 - R_0^2) \min_{[R_0, R]} \sigma
\]

where \(R > R_0\) are two real numbers greater than 2, and \(\{2 \leq N \leq R\}\) denotes the set \(B(0, R) \setminus B(0, 2)\). As \(\sigma \to \infty\), this computation contradicts the quadratic volume growth of \(\Sigma\), so in fact \(\text{Vol}_2(K_r) = 0\).

Next, we give two propositions which are based on a refined form of the argument used previously to show the monotonicity relation, which was the key argument for proving Theorem 13.

For a point \(q \in \mathbb{R}^{n+1}\) and a vector \(v \in \mathbb{R}^{n+1}\), we denote by \(L(q, v)\) the half-line beginning at \(q\) and whose direction is given by \(v\). Recall also the notation \(K^1_r = \Sigma \cap S(0, r)\) where \(\Sigma\) is a complete properly immersed self-shrinker.

**Proposition 15.** Let \(r > 0\) and \(p_0\) be a point in \(S(0, r)\) (also considered as a vector in \(\mathbb{R}^{n+1}\)). Define

\[
U_r(p_0) = \{q \in \mathbb{R}^{n+1} \setminus B(0, r); L(q, -p_0) \cap S(0, \sqrt{2n}) \neq \emptyset\},
\]

\[
V_r = \bigcup_{p \in S(0, r) \text{ and } U_r(p) \cap K^1_r = \emptyset} U_r(p).
\]

Consider the sets

\[
Y_r = \Sigma \setminus B(0, r),
\]

\[
X_r = \mathbb{R}^{n+1} \setminus (B(0, r) \cup V_r).
\]

Then

\[
\forall r > \sqrt{2n} \quad Y_r \subset X_r.
\]
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Figure 3. In this figure, $\Sigma$ cannot be a self-shrinker because the point $p$ is out of $X_r$.

**Proof.** Suppose the contrary: there is $p \in V_r \cap Y_r$. Apply Proposition 6 to the following hypersurfaces: $S_1$ is the self-shrinking sphere, and $S_2$ is $Y_r$. Let $\Pi$ be the orthogonal projection on the vectorial hyperplane orthogonal to $p$, defined on the half-space $E = \{x \in \mathbb{R}^{n+1}; \langle x, p \rangle > 0\}$. Then, by the definition of $V_r$, $\Pi(B(0, \sqrt{2n}) \cap \Pi(E \cap K^1_r)) \neq \emptyset$. Thanks to the Sard Lemma, Proposition 6 is then applied with $e_{n+1}$ near $-p/\|p\|$ (see figure 3).

The following proposition illustrates the fact that some bound on the curvature of the trace $K^1_r = \Sigma \cap S(0, r)$ gives a bound on the mean curvature of $\Sigma$ in the case where $n = 2$.

**Proposition 16.** Suppose $n = 2$ and let $\Sigma_0$ be an end of $\Sigma \subset \mathbb{R}^3$, i.e. a connected component of $\Sigma \setminus B(0, r_0)$ for a $r_0 > 0$. Let $\epsilon > 0$. If $\Sigma_0$ intersects $S(0, r)$ transversally for all $r > r_0$ (in particular $\Sigma_0 \cap S(0, r)$ a union of closed simple curves) and if the curvature of $\Sigma_0 \cap S(0, r)$ is bounded by $\frac{1}{2} - \epsilon$ then the mean curvature of $\Sigma_0$ is bounded by 2 if $r_0$ is chosen large enough.

**Proof.** Define $I_r = \Sigma_0 \cap S(0, r)$. For $p \in S(0, r)$, denote by $b_r(p)$ the intersection $B(q, 2) \cap S(0, r)$ where $q$ is parallel to $p$ with the same direction and
maximizes the 2-volume of this intersection. As \( r_0 \) increases, \( S(0, r) \ (r > r_0) \) becomes flat and the curve \( \partial b_r(p) \) has curvature larger than \((1 - \epsilon)/2\).

Consequently, if \( x_0 \in I_r \), there exists \( p_1, p_2 \in S(0, r) \) such that the \( b_r(p_i) \) are tangent to \( I_r \) at \( x_0 \) (one on each side of \( x_0 \)) and \( I_r \) is outside \( b_r(p_1) \cup b_r(p_2) \) near \( x_0 \). To prove the proposition, it is sufficient to show that a neighborhood of \( x_0 \) in \( \Sigma \setminus B(0, r) \) is outside \( U_r(p_1) \cup U_r(p_2) \) (see notation in Proposition [15]). Indeed, the bound on the mean curvature will then follow from the self-shrinker equation (1). Suppose the contrary: for instance there are points \( y_k \in \Sigma \setminus B(0, r) \) converging to \( x_0 \) with \( y_k \in U_r(p_1) \). Consider \( C \), the connected component of \( (\Sigma \setminus B(0, r)) \cap U_r(p_1) \) containing \( y_k \), well defined for \( k \) large. Note that because of the bound on the curvature, the boundary \( \partial C \) only intersects \( b_r(p_1) \) at \( x_0 \). This can be seen as follows: suppose that \( \partial C \) intersects \( b_r(p_1) \) at another point so that for \( r' \) slightly bigger than \( r \), \( C \cap S(0, r') \cap U'_r(p_1) \) has more than one component. As \( r' \) increases from \( r \), by connectivity of \( C \), two of these components of \( C \cap S(0, r') \cap U'_r(p_1) \) have to meet smoothly for an \( r' > r \). But this can only happen on the boundary \( \partial b_r(p) \), which would contradict the bound on the curvature of \( \Sigma_0 \cap S(0, r) \). Then we apply once again Lemma [6] to \( S(0, 2) \) and \( C \), with the Sard Lemma to guarantee that the function \( h \) achieves a minimum. \( \square \)

3. A linear bound for the second fundamental form of some self-shrinkring surfaces

Let’s define the curvature concentration:

**Definition 17.** Let \( \kappa > 0 \). Let \( S \) be a surface in \( \mathbb{R}^3 \). The curvature concentration of \( S \) is bounded by \( \kappa \) if

\[
\forall x \in S \setminus B(0, 1), \quad \int_{S \cap B(x, \rho(x))} |A|^2 < \kappa,
\]

where

\[
\rho(x) = \frac{1}{2||x||}.
\]

Requiring that a surface has bounded curvature concentration seems quite restrictive, but it is in fact natural in the context of Brakke limit flows:

**Proposition 18.** Let \( \{M\}_{t \in [0, T]} \) be a family of embedded surfaces flowing smoothly in \( \mathbb{R}^3 \) and beginning at a closed surface \( M_0 \). If \( S \) is a self-shrinker produced by a weak blowup at \( T \) then there exists a constant \( \kappa(M_0) > 0 \) depending only on \( M_0 \), such that the curvature concentration of \( S \) is bounded by \( \kappa(M_0) \).

**Proof.** Let’s recall the definition of a weak blowup: consider the family \( \{M\}_{t \in [0, T]} \) as above. Let \( \lambda_i > 0 \) be a sequence converging to 0. Rescale
the flow parabolically about \((y, T)\) for some \(y\) by defining

\[ M^i_t = \lambda_t^{-1}(M_{T + \lambda^2_t} - y), \quad t \in [-T/\lambda^2_t, 0). \]

Ilmanen [11] and White [20] proved that by taking a subsequence if necessary, \(\{M^i_t\}\) converges to a limiting Brakke self-shrinking flow \(\{\sqrt{-t}\mathcal{S}\}_{t \in (-\infty, 0)}\) in the following sense:

1. for all \(t < 0\), \(M^i_t \to \sqrt{-t}\mathcal{S}\) in the sense of Radon measures,
2. for a.e. \(t < 0\), there is a subsequence \(\{i_k\}\) depending on \(t\) such that \(M^i_t \to \sqrt{-t}\mathcal{S}\) as varifolds.

Moreover, Ilmanen showed that in dimension 3, the limit flow is smooth. This procedure is called a weak blowup. Then Ilmanen gives the following integral curvature estimate (Theorem 4 in [11]): for every \(B(x, r) \times [t - r^2, t) \subset \mathbb{R}^3 \times [0, T)\),

\[
(7) \quad r^{-2} \int_{t - r^2}^t \int_{M^i_t \cap B(x, r)} |A|^2 \leq C(M_0).
\]

By lower semicontinuity of the integral of the squared norm of the generalized second fundamental form (see [10], Theorem 5.3.2), it yields the first part of the proposition. Indeed, without loss of generality suppose that \(y = 0\), and consider \(z \in \mathcal{S} \subset \mathbb{R}^3\) whose norm is bigger than 1. Applying (7) to \(t = T - \lambda_i^2 ||z||^2, x = \lambda_i ||z||z\) and \(r = \lambda_i\) for all \(i\), we obtain a subsequence \(i_k\) and a time \(t' \in [-||z||^2 - 1, -||z||^2]\) such that

1. \(M^i_k \cap B(||z||z, 1)\) converge to \(\sqrt{-t'}\mathcal{S} \cap B(||z||z, 1)\) as varifolds,
2. \(B(\sqrt{-t'}z, \sqrt{-t'}/(2||z||)) \subset B(||z||z, 1)\),
3. \(\int_{M^i_k \cap B(||z||z, 1)} |A|^2\) is bounded by a constant depending only on \(M_0\).

Consequently, after rescaling, we get that \(\int_{S \cap B(z, \rho(z))} |A|^2\) is bounded by a constant depending only on \(M_0\), where \(\rho(z) = \frac{1}{2||z||}\).

The main theorem of this section gives a linear bound for \(|A|\):

**Theorem 19.** If \(\Sigma \subset \mathbb{R}^3\) is a complete properly embedded self-shrinker of finite genus \(g\) such that the curvature concentration is bounded by \(\kappa\), then

\[ \exists C = C(\Sigma), \forall x \in \Sigma, |A(x)| \leq C(1 + |x|). \]

**Remark 20.** Self-shrinkers can be viewed as minimal surfaces under a conformal change of metric but, as noted in [5], this kind of result does not follow for instance from Choi-Schoen’s compactness theorem [3] mainly because the new metric can not even be extended to a complete metric.

**Proof.** Let \(K = (H^2 - |A|^2)/2\) denote the Gauss curvature. Suppose that the conclusion is not verified along a sequence of points \(p_k \in \Sigma\). Define \(\mu_k = |A(p_k)|\). Then, because of the self-shrinker equation (1):
Moreover, \(-K(p_k)/|A(p_k)|^2 \to 1/2\). By modifying the \(p_k\) if necessary, \(\mu_k. (\Sigma - p_k)\) is a sequence of surfaces whose second fundamental form is locally uniformly bounded and whose mean curvature goes to zero. Indeed, we have

**Lemma 21.** Under the hypotheses of the theorem, there is a sequence \(p_k\) such that

\[
\forall \alpha > 0, \quad |A(p_k)|/(1 + ||p_k||) \to \infty \quad \text{and} \quad \max\{|A(q)|/\mu_k; q \in \Sigma \cap B(p_k, \alpha/\mu_k)\} \leq 2 \quad \text{for large } k.
\]

**Proof.** (of the lemma) The proof goes by induction. If \(p \in \Sigma, |A(p)| > 0\) and \(\alpha > 0\), define

\[
m(p, \alpha) = \max\{|A(q)|/|A(p)|; q \in \Sigma \cap B(p, \alpha/|A(p)|)\}.
\]

Let \(n \in \mathbb{N}\). Denote by \(\mathcal{P}(n)\) the following assertion:

There is a sequence \(p_k \in \Sigma\) such that

\[
|A(p_k)|/(1 + ||p_k||) \to \infty \quad \text{and} \quad \forall k \geq n, m(p_k, n) \leq 2.
\]

Because \(\mathcal{P}(0)\) is trivially true, the lemma will ensue by a diagonal extraction argument from the following claim: if \(\mathcal{P}(n)\) is verified then by modifying \(p_k\) if necessary (for \(k \geq n + 1\), we have \(\mathcal{P}(n + 1)\).

Let’s check this claim: suppose that \(\mathcal{P}(n)\) is true for the sequence \((x_k)\). Fix a \(k\) bigger than or equal to \(n + 1\). One can suppose that \((n+1)/|A(x_k)| \leq 1\).

If \(m(x_k, n + 1) > 2\) then there is a point \(x_k^1 \in \Sigma \cap B(x_k, 1/2)\) with

\[
|A(x_k^1)|/|A(x_k)| > 2.
\]

Likewise, if \(m(x_k^1, n + 1) > 2\), we can find \(x_k^2 \in \Sigma \cap B(x_k^1, 1/2^2)\) such that

\[
|A(x_k^2)|/|A(x_k^1)| > 2.
\]

This construction goes on as long as \(m(x_k^l, n + 1) > 2\). In fact, it has to stop because \(d(x_k^l, x_k^{l+1}) \leq 1/2^{l+1}\) and \(|A(x_k^l)| \geq 2^l|A(x_k)|\). Let’s call \(p_k\) the last \(x_k^l\) constructed and of course define \(x_k = p_k\) for \(k \leq n\). The sequence \((p_k)\) verifies \(\mathcal{P}(n+1)\).

\[\square\]

Define now \(S_k = \mu_k. (\Sigma - p_k)\). The previous lemma shows that \(S_k\) is a sequence of surfaces such that at the origin, the Gauss curvature is \(-1/2\) and the second fundamental form is locally uniformly bounded. Besides, (8) implies that for all \(a > 0\) and all \(x_k \in \Sigma \cap B(p_k, a/\mu_k)\), the quantity \(H(x_k)/\mu_k\) goes to 0. In other words, the mean curvature of \(S_k\) converges locally uniformly to 0. The local uniform bound on \(|A|\) means that in any ball of \(\mathbb{R}^3\) small enough (the radius depends only on this bound), for \(k\) large, \(S_k\) is the graph of a function with bounded gradient and Hessian. Consequently, these functions satisfy uniformly elliptic equations with uniformly controlled
coefficients. Thus by Schauder estimates and standard elliptic theory, for all $r > 0$, the intrinsic balls $B^S(0, r)$ converge subsequently in the $C^m$ topology to a smooth embedded surface with boundary called $T_r$, which is in fact minimal. Define $S$ to be the union of the $T_r$ for $r > 0$. It’s an embedded complete minimal surface. Here the surfaces considered are all oriented embedded, the integral of $|A|^2$ is bounded uniformly and $S$ is non-flat. Consequently the convergence has in fact multiplicity one, i.e. $B^S(0, r)$ converge smoothly with multiplicity one to $B^S(0, r) = T_r$. Besides, note that

$$(9) \quad \forall r > 0 \quad \int_{S \cap B(0,r)} |A|^2 = 2 \int_{S \cap B(0,r)} |K| \leq \limsup_{k \to \infty} \int_{S_k \cap B(0,r)} |A|^2 \leq \kappa.$$ 

Now, let’s cite two results which will imply that $S$ is necessarily a catenoid.

The first well-known theorem of Osserman describes minimal surfaces with finite total curvature.

**Theorem 22.** [19] Let $M \subset \mathbb{R}^3$ be a complete oriented minimal surface with Gauss curvature $K$ such that $\int_M |K| < \infty$. Then there exists a closed Riemann surface $\tilde{M}$, a finite set of points $\{p_1, \ldots, p_k\} \in \tilde{M}$ and a conformal diffeomorphism between $M$ and $\tilde{M} \setminus \{p_1, \ldots, p_k\}$. Moreover, the Gauss map extends meromorphically across the punctures.

The following theorem, proved by López and Ros, then Meeks and Rosenberg, classifies the properly embedded minimal surfaces with finite topology and genus 0:

**Theorem 23** ([14], [16]). The only properly embedded minimal surfaces of $\mathbb{R}^3$, with finite topology and genus 0 are planes, helicoids and catenoids.

By invariance under dilatations of integrals like $\int_{S_k \cap B(0,r)} |A|^2$, the first of the theorems above, the assumption on the curvature concentration and (9) show that $S$ has finite topology. Note that moreover, the normal vectors are well defined at each end, which implies that $S$ is proper. Because the norm of the second fundamental form of $\Sigma$ at $p_k$ goes to infinity, $||p_k|| \to \infty$. Hence, $S$ has genus 0 (the convergence has multiplicity one). The second theorem, plus the fact that the Gauss curvature of $S$ at the origin is $-1/2$, imply that $S$ is necessarily a catenoid. Note that, as the multiplicity of the convergence is one, for $k$ large $\Sigma_k$ really looks like a small catenoid near $p_k$.

The end of the proof consists in finding a contradiction with the fact that $S$ is a catenoid. Note that a posteriori, we could have chosen $p_k$ to have the origin on the closed simple geodesic of the catenoid $S$, and also to make $p_k/||p_k||$ converge to a vector, say $v$. Let $(e'_1, e'_2, e'_3)$ be a basis of $\mathbb{R}^3$ such that $e'_3$ gives the rotation axis of $S$.

Let $c \subset S$ be the closed simple geodesic which encircles the neck of $S$. Let $(c_k)$ be a sequence of curves in $\Sigma$ corresponding to embedded curves in $S_k$ converging smoothly to $c \subset S$. Let $\Sigma_k^+$ and $\Sigma_k^-$ be the two open connected
components separated by $c_k$, $\Sigma_k^+$ being the component whose rescaling converge to $\{x \in S; \langle x, e'_3 \rangle > 0\}$ (we can suppose that there are two distinct components because the genus of $\Sigma$ is finite and $||p_k|| \to \infty$). Note that seen from afar, a catenoid is like two superposed planes. So for $k$ large, $\Sigma_k^+$ and $\Sigma_k^-$ are two surfaces nearly parallel and flat around $p_k$ and glued together along $c_k$.

There are two cases: either $v$ is parallel to $e'_3$ or it isn’t. Suppose that $v$ is parallel to $e'_3$, say $v = e'_3$. Then by applying Proposition 15, we see that, for $k$ large, $\Sigma_k^+$ should be contained in a set looking like a long tube with boundary $c_k$ going to infinity and as narrow as $c_k$ near this little neck. But $\Sigma_k^+$ is more like a plane orthogonal to $e'_3 = p_k/||p_k||$ with a small half neck at $p_k$. This is the desired contradiction.

Suppose otherwise that $v$ is not parallel to $e'_3$. The latter can be chosen so that $\langle e'_3, v \rangle > 0$. In this case, by the usual maximum principal, for all $k$ large

$$\Sigma_k^\pm \cap S(0, 2) \neq \emptyset.$$  

This observation plus the fact that the genus of $\Sigma$ is finite would imply that $\Sigma$ is not properly embedded, which is absurd.

\[ \square \]

This theorem implies that translating solitons can help to understand regions of $\Sigma$ far from the origin:

**Corollary 24.** Let $\Sigma \subset \mathbb{R}^3$ be as in the previous theorem. Consider a sequence $x_k \in \Sigma$ of points whose norm goes to infinity. Define $S_k = ||x_k||(\Sigma - x_k)$. Then subsequently, for all $d > 0$, $B^{S_k}(0, d)$ converge smoothly with multiplicity one to a surface with boundary $T_d$ and the union $T = \bigcup T_d$ is a complete embedded translating soliton of the mean curvature flow with genus 0.

**Proof.** One can suppose that $x_k/||x_k||$ converge to a vector $u$. Theorem 19 implies that the norm of the second fundamental form of $\Sigma$ grows at most linearly. Hence, the second fundamental form of the rescalings $S_k$ is bounded. Then the convergence of $B^{S_k}(0, d)$ to a $T_d$ (up to a subsequence) is shown as in the proof of the previous theorem. Moreover, each limit $T_d$ satisfies

$$H = \frac{\langle u, v \rangle}{2}$$

which is the equation of translating solitons. Note that this time the integral of $|A|^2$ is uniformly bounded only on fixed compact sets, that is

$$\forall s > 0, \exists c = c(s), \forall k, \int_{S_k \cap B(0, s)} |A|^2 \leq c.$$  

Take a $d'$ small enough (depending on the constant $C(\Sigma)$ in the theorem) so that $B^{S_k}(0, d')$ are graphs. Then if $B^{S_k}(0, d')$ converge to a non flat surface, the multiplicity of the convergence of $B^{S_k}(0, d)$ for all $d > 0$ has to be
one because of (10). If $B^{S_k}(0, d')$ converge to a flat surface, then by unique continuation $B^{S_k}(0, d)$ converge to a flat surface for all $d > 0$ (remember that each $T_d$ is a piece of translating soliton, which is a minimal surface after a conformal change of metric). These arguments show that the convergence of $B^{S_k}(0, d)$ to $T_d$ has to be one for all $d$. The genus of $T$ is zero because the convergence has multiplicity one and because $\|x_k\| \to \infty$.

□

For the end of this section, we give a result analogous to Theorem 19, but for surfaces with $\delta$-separated ends, so that the constant this time won’t depend on the surface itself but on some geometric parameters mentioned in the following definitions:

**Definition 25.** Let $R > 0$. A complete properly embedded surface $S \subset \mathbb{R}^3$ with finite genus $g(S)$ is said to have topology bounded by $R$ if the genus of $B(0, R) \cap S$ is equal to $g(S)$.

**Definition 26.** Define $\mathcal{F}_{\delta, \kappa, g, R}$ the family of complete self-shrinking surfaces properly embedded in $\mathbb{R}^3$ with finite topology such that:

1. The genus is less than $g$,
2. The topology is bounded by $R$,
3. The curvature concentration is bounded by $\kappa$,
4. The ends are $\delta$-separated.

The proof of the following theorem is similar to the previous one but the new point is that the assumption on the ends of the self-shrinking surfaces enables us to use Corollary 8. More precisely, small catenoid-like necks on self-shrinking surfaces with $\delta$-separated ends are ruled out by this corollary.

**Theorem 27.** There exists a constant $C = C(\delta, \kappa, g, R)$ depending only on, $\delta$, $\kappa$, $g$ and $R$ such that

$$\forall S \in \mathcal{F}_{\delta, \kappa, g, R}, \forall x \in S, |A(x)| \leq C(1 + \|x\|).$$

**Proof.** The beginning of the proof is nearly identical to the previous one, except that this time we consider $p_k \in S_k \in \mathcal{F}_{\delta, \kappa, g, R}$ hypothetically contradicting the theorem, define $S_k = \mu_k(S_k - p_k)$ and use this lemma:

**Lemma 28.** Under the hypotheses of the theorem, there is a sequence $(p_k)$ with $p_k \in S_k \in \mathcal{F}_{\delta, \kappa, g, R}$ such that

$$\forall \alpha > 0, |A(p_k)|/(1 + \|p_k\|) \to \infty$$

and

$$\max\{|A(q)|/\mu_k; q \in S_k \cap B(p_k, \alpha/\mu_k)\} \leq 2$$

for large $k$.

As before, we get a limit $S$. To prove that it is once again a catenoid, the same arguments work if we can prove that the genus of $S$ is 0, which was clear when only one surface was considered. We use a theorem of compactness for self-shrirkers by Colding and Minicozzi.
Theorem 29 ([5]). Let \( g \) be a integer. The space \( \mathcal{F}_g \) of properly embedded self-shrinkers without boundary, with finite topology and genus less than \( g \) is compact for the topology of \( \mathcal{C}^m \) convergence on compacts.

Remark 30. Colding and Minicozzi initially supposed that the volume growth was quadratic, with is a natural assumption in the context of limit surfaces for mean curvature flow flowing from embedded closed surface [1]. This hypothesis is in fact always true as long as the self-shrinker is proper, as shown by Ding and Xin [6].

From this theorem and the fact that \( S_{q,R}^{k} \subset \mathcal{F}_g \), we deduce that \( ||p_k|| \to \infty \). The topology being uniformly bounded by \( R \), \( S \) has indeed genus 0 (remind that the convergence of \( B^{S_k}(0,r) \) to \( B^S(0,r) \) has multiplicity one).

The end of the proof is based on Corollary 8, as explained previously. A posteriori, we could have chosen \( p_k \) to have the origin on the closed simple geodesic of the catenoid \( S \), and also to make \( p_k/||p_k|| \) converge to a vector, say \( v \). Let \((e'_1, e'_2, e'_3)\) be a basis of \( \mathbb{R}^3 \) such that \( e'_3 \) gives the rotation axis of \( S \). We denote by \( P \) the plane containing \( e'_1 \) and \( e'_2 \).

We will distinguish two cases: either the limit \( v \) is not in the plane \( P \), or \( v \) is in \( P \) (i.e. \( v \) is orthogonal to \( e'_3 \)).

Case 1. Suppose that \( v \) is not in the plane \( P \). Let \( c \subset S \) be the closed simple geodesic which encircles the neck of \( S \). Note that because of the choice of \( p_k \) in Lemma 28 and because \( S \) and \( \tilde{S} \subset \mathbb{R}^3 \) necessarily intersect if \( S \) is a plane or a catenoid, one can suppose that \( B(0,k) \cap S_k \) has only one connected component for all \( k \). Let \((c_k)\) be a sequence of curves in \( S_k \) corresponding to embedded curves in \( S_k \) converging smoothly to \( c \subset S \). We can suppose that \( \langle e'_3,v \rangle > 0 \). Let \( S^+_k \) and \( S^-_k \) be the two open connected components separated by \( c_k \), \( S^+_k \) being the component whose rescaling converge to \( \{x \in S; \langle x,e'_3 \rangle > 0 \} \) (we can suppose that there are two distinct components because the topology of the \( S_k \) is bounded and \( ||p_k|| \to \infty \)). Note that seen from afar, a catenoid is like two superposed planes. So for \( k \) large, \( S^+_k \) and \( S^-_k \) are two surfaces nearly parallel and flat around \( p_k \) and glued together along \( c_k \).

Suppose first that \( S^+_k \) intersects the self-shrinking sphere \( S(0,2) \): this is in fact always the case if \( v \neq e'_3 \) by the usual maximum principle, because then for \( k \) large there are points in \( S^+_k \) which are closer to the origin than any points on \( c_k \). Then, we can use Corollary 8. Indeed, take \( y \in S^+_k \cap S(0,2) \). By changing the canonical basis if necessary, we can suppose that \( e_3 = (p_k-y)/||p_k-y|| \). Let’s check for instance the second hypothesis of Corollary 8. For every point \( q \) in a tubular neighborhood of \( c_k \), for \( k \) large, the half-line \([q,y]\) first touches \( S^-_k \) before arriving at \( y \) (recall that \( B(0,k) \cap S_k \) has only one connected component for all \( k \)). By the same kind of arguments, the hypothesis of Corollary 8 are all verified. So \( S_k \) is not a self-shrinker, which is absurd.
Suppose now that $S^+_k$ does not intersect the self-shrinking sphere $S(0, 2)$: necessarily $v = e'_3$. By applying Proposition 15, we see that, for $k$ large, $S^+_k$ should be contained in a set looking like a long tube with boundary $c_k$ going to infinity and as narrow as $c_k$ near this little neck. But $S^+_k$ is more like a plane orthogonal to $e'_3 = p_k / ||p_k||$ with a small half neck at $p_k$. This is the desired contradiction.

Case 2.

Suppose that $v \in P$. Choose a sequence of embedded curve $(c_k) \subset S_k$ as before, let $S^+_k$ and $S^-_k$ be the two open connected components separated by $c_k$, $S^+_k$ being the component whose rescaling converge to $\{ x \in S; \langle x, e'_3 \rangle > 0 \}$. By Theorem 13, we know that $\lambda S^+_k$ (resp. $\lambda S^-_k$) converge (when $\lambda \to 0$) to a cone whose intersection with $S(0, 1)$ is called $K^+_k$ (resp. $K^-_k$).

By $\delta$-separation of the ends of $S_k$, $d(K^+_k, K^-_k) \geq \delta$. Define $\Theta_{e_0, \theta}$ the linear rotation with axis $Re$ and angle $\theta$. Consider the axis $e_0 = v \land e'_3$. Because $S_k$, $d(K^+_k, K^-_k) \geq \delta$, for $\theta$ sufficiently small independent of $k$, there is a $r$ depending on $k$ such that if $r > r$, 

$$\Theta_{e_0, \theta}(S^+_k) \setminus B(0, r) \cap S^-_k \setminus B(0, r) = \emptyset.$$ 

Now, fix an angle $0 < \theta_0 < \pi$ sufficiently small. With these choices,

$$\forall k \text{ large, } \Theta_{e_0, \theta_0}(S^+_k) \cap S^-_k = \emptyset.$$ 

Indeed, if it is not true for $k'$, then consider the first $\theta$ between $0$ and $\theta_0$ such that $\Theta_{e_0, \theta}(S^-_{k'})$ touches $S^+_{k'}$. Then these surfaces are in contact at a point in the ball $B(0, \overline{r})$ and not outside. But that contradicts the usual maximum principle. We can finally conclude, because for $k$ large,

$$\inf\{ ||x - y||; x \in \Theta_{e_0, \theta_0}(S^+_k), y \in S^-_k \}$$ 

will be achieved since $\theta_0$ is independent of $k$: once again the usual maximum principle is violated so $S_k$ can’t be self-shrinkers for $k$ large, which is absurd.

□

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