Abstract—Under a Bayesian framework, we formulate the fully sequential sampling and selection decision in statistical ranking and selection as a stochastic control problem, and derive the associated Bellman equation. Using a value function approximation, we derive an approximately optimal allocation policy. We show that this policy is not only computationally efficient but also possesses both one-step-ahead and asymptotic optimality for independent normal sampling distributions. Moreover, the proposed allocation policy is easily generalizable in the approximate dynamic programming paradigm.

Index Terms—Bayesian, dynamic sampling and selection, ranking and selection (R&S), simulation, stochastic control.

I. INTRODUCTION

IN THIS paper, we consider a simulation optimization problem of choosing the highest mean alternative from a finite set of alternatives, where the means are unknown and must be estimated by statistical sampling. In simulation, this problem is often called statistical ranking and selection (R&S) problem (see [1]). Applications of R&S include selecting the best alternative from many complex discrete event dynamic systems that are computationally intensive to simulate (see [6]), and finding the most effective drug from different alternatives, where the economic cost of each sample for testing the effectiveness of the drug is expensive (see [40]). Broadly speaking, there are two main approaches in R&S (see [21] and [7]). The first approach allocates samples to guarantee the probability of correct selection (PCS) up to a prespecified level (see [26], [28], and [41]), whereas the second approach maximizes the PCS (or other similar metric) subject to a given sampling budget (see [8], [12], [30], and [31]).

The earliest sampling allocation schemes use two-stage procedures (e.g., [8], [12], and [41]), where unknown parameters are estimated in the first stage. More recently, fully sequential sampling allocation procedures have been developed (see [15], [24], and [27]). In the Bayesian framework, Chen et al. [5], Chick et al. [9], and Frazier et al. [16] proposed sequential algorithms by allocating each replication to maximize the posterior information gains one step ahead; Chick and Frazier [10] and Chick and Gans [11] provided sequential policies analogous to a multiarmed bandit problem and used a continuous-time approximation to solve their Bellman equation; Peng et al. [34] offered a sequential rule achieving the asymptotically optimal sampling rate of the PCS; and Peng and Fu [38] developed a sequential algorithm that possesses both one-step-ahead and asymptotic optimality.

Previous work using the Bayesian framework approached the difficult dynamic R&S problem by replacing the sequential sampling and selection decisions with a more tractable surrogate optimization problem. In this paper, we formulate the R&S problem as a stochastic control problem (SCP) and derive the associated Bellman optimality equation, which requires care due to the interaction between the sampling allocation policy and the posterior distribution. We show that under a canonical condition in R&S, the sampling allocation decision does not affect the Bayesian posterior distributions conditional on the information of sample observations; thus, the SCP is proved to be a Markov decision process (MDP). To the best of our knowledge, this is the first work to study R&S as an SCP and use an MDP to analyze it.

We then analyze the optimal allocation and selection (A&S) policy of the resulting MDP and prove that a commonly used selection policy of selecting the alternative with the largest sample mean is asymptotically consistent with the optimal selection policy under some mild conditions. The size of the state space of the sampling allocation policy for independent discrete sampling distributions is shown to only grow polynomially with respect to the number of allocated replications if the numbers of alternatives and the possible outcomes of the discrete distributions are fixed, but will have an exponential growth rate when the number of alternatives and the number of possible outcomes of the discrete distributions grow together.

Sampling from independent normal distributions is a standard assumption in the R&S literature, so we focus on this setting.

References:

[1] K. B. C. Chen, “Sampling from independent normal distributions is a standard assumption in the R&S literature, so we focus on this setting.”
In contrast to the usual approach of replacing the SCP with a tractable approximate surrogate (static) optimization problem, we address the SCP directly by approximating the value function, as in approximate dynamic programming (ADP) (see [39]). The value function approximation (VFA) using a simple feature of the value function yields an approximately optimal allocation policy (AOAP) that is not only computationally efficient, but also possesses both one-step-ahead and asymptotic optimality. In addition, the VFA approach is easily generalizable in the ADP paradigm. For example, we show how to extend the AOAP to a multistep look-ahead sampling allocation procedure, and how to obtain an efficient sampling algorithm for a low-confidence scenario (see [35]) by implementing an offline learning algorithm.

The rest of the paper is organized as follows: Section II formulates the SCP in R&S, and the associated Bellman equation is derived in Section III. Section IV offers further analysis on the optimal A&S policy, and Section V focuses on the approximations of the optimal A&S policy for normal sampling distributions. Numerical results are given in Section VI. The last section offers conclusions.

II. PROBLEM FORMULATION

Among $k$ alternatives with unknown means $\mu_i$, $i = 1, \ldots, k$, our objective is to find the best alternative defined by

$$
\langle 1 \rangle \triangleq \arg \max_{i=1, \ldots, k} \mu_i
$$

where each $\mu_i$ is estimated by sampling. Let $X_{i,t}$ be the $t$th replication for alternative $i$. Suppose $X_i \triangleq (X_{i,1}, \ldots, X_{i,t})$, $t \in \mathbb{Z}^+$, follows an independent and identically distributed (i.i.d.) joint sampling distribution, i.e., $X_i \sim Q(\cdot; \theta)$, with a density [or probability mass function (p.m.f.)] $q(\cdot; \theta)$, where $\theta \in \Theta$ comprises all unknown parameters in the parametric family. The marginal distribution of alternative $i$ is denoted by $Q_i(\cdot; \theta)$, with a density $q_i(\cdot; \theta)$, where $\theta$ comprises all unknown parameters in the marginal distribution. Generally, $\mu_i \in \Theta$, $i = 1, \ldots, k$, and $(\theta_1, \ldots, \theta_k) \in \Theta$. In addition, we assume the unknown parameter follows a prior distribution, i.e., $\theta \sim F(\cdot; \zeta_0)$, where $\zeta_0$ contains all hyper-parameters for the parametric family of the prior distribution.

We define the two parts of an A&S policy. The allocation policy is a sequence of mappings $A_i(\cdot) = (A_1(\cdot), \ldots, A_k(\cdot))$, where $A_i(\mathcal{E}_{t-1}^i) \in \{1, \ldots, k\}$, which allocates the $t$th sample to an alternative based on information set $\mathcal{E}_{t-1}^i$ collected through all previous steps. The information set at step $t$ is given by

$$
\mathcal{E}_t^i \triangleq \{A_i(\mathcal{E}_{t-1}^i); \mathcal{E}_t^i \}
$$

where $\mathcal{E}_t^i$ contains all sample information and prior information $\zeta_0$. Define $A_{i,t}(\mathcal{E}_{t-1}^i) \triangleq 1 \{A_i(\mathcal{E}_{t-1}^i) = i\}$. The information collection procedure following a sampling allocation policy in R&S problem is illustrated in Fig. 1 for allocating four samples among three alternatives. Given prior information $\zeta_0$, collected information set $\mathcal{E}_t^i$ is determined by the two tables in the figure. The allocation decision represented by the table at the bottom determines the (bold) observable elements in the table on the top. The sampling decision and information flow have an interactive relationship shown in Fig. 2.

The sampling decision and the information set are nested in each other as $t$ evolves. We reorganize the (allocated) sample observations by putting them together and ordering them in chronological arrangement, i.e., $\bar{X}_t = (\bar{X}_{t,1}, \ldots, \bar{X}_{t,t})$, where $t_i \triangleq \sum_{t=1}^{t_{i}} A_{i,t}(\mathcal{E}_{t-1}^i), i = 1, \ldots, k$. Although $t_i$ is also a map from the information set, we suppress the argument for simplicity. For example, in Fig. 1, we specifically illustrate how to reorganize the sample observations in Fig. 3. We have $\bar{X}_t = \{\zeta_0, \bar{X}_{t,1}^i, \ldots, \bar{X}_{t,k}^i\}$.

The selection policy is a map $S(\mathcal{E}_t^p) \in \{1, \ldots, k\}$, which makes the final selection at step $T$ and indicates the best alternative chosen by the A&S algorithm. The final reward for selecting an alternative is a function of $\theta$, given the selection decision, i.e., $V(\theta; i)|_{i \in S}$. In R&S, two of the most frequently used candidates for the final reward are

$$
V_P(\theta; i) \triangleq 1 \{i = \langle 1 \rangle\}, \quad V_E(\theta; i) \triangleq \mu_i - \mu_{(1)}
$$

where the subscripts $P$ and $E$ stand for PCS and expected opportunity cost (EOC), respectively, and $(i)$, $i = 1, \ldots, k$, are order statistics s.t. $\mu_{(1)} > \cdots > \mu_{(k)}$. If the alternative selected as the best is the true best alternative, $V_P$ is one, otherwise $V_P$ is zero; $V_E$ is the difference between the mean of the selected alternative and the mean of the true best alternative, which measures the economic opportunity cost (regret) of the selection decision. Notice that the values of the final rewards $V_P$ and $V_E$ are unknown due to the uncertainty of parameter $\theta$, which
is quantified by the prior distribution of the parameter in the Bayesian framework.

We formulate the dynamic decision in R&S by a SCP as follows. Under the Bayesian framework, the expected payoff for an A&S policy \( (A, S) \), where \( A \triangleq A_T \), in the SCP can be defined recursively by

\[
V_T(\mathcal{E}_T^i; A, S) \triangleq \mathbb{E}[V(\theta; i)|\mathcal{E}_T^i] \big|_{i = S(\mathcal{E}_T^i)} = \int_{\theta \in \Theta} V(\theta; i) F(d\theta|\mathcal{E}_T^i) \big|_{i = S(\mathcal{E}_T^i)}
\]

where \( F(\cdot|\mathcal{E}_T^i) \) is the posterior distribution of \( \theta \) conditioned on the information set \( \mathcal{E}_T^i \), and \( d\cdot \) in \( d\theta \) stands for Lebesgue measure for continuous distributions and the counting measure for discrete distributions. The interaction between sampling allocation policy history, the following theorem establishes that the posterior and predictive distributions at step \( t \) conditioned on \( S_T \) is independent of the information flow before \( q, \) and \( x(\cdot ; \theta) \) are determined by construction of the information set; thus, the variables of the missing replications at step \( t \) in the joint density are integrated out, leaving only the marginal likelihood of the observation at step \( t \). By using the same argument inductively, the second equality in (4) holds. The last equality in (4) holds because the product operation is commutative. With (4), we can denote the likelihood as \( L(\mathcal{E}_i; \theta) \), since the information set \( \mathcal{E}_i \) completely determines the likelihood.

Following Bayes rule, the posterior distribution of \( \theta \) is

\[
F(d\theta|\mathcal{E}_T^i) = \frac{L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}{\int_{\theta \in \Theta} L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}
\]

and

\[
Q_i(dx_{t+1} | \mathcal{E}_T^i) = \frac{\int_{\theta \in \Theta} Q_i(dx_{t+1} | \theta_1) L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}{\int_{\theta \in \Theta} L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}
\]

where \( L(\cdot) \) is the likelihood of the samples. The posterior and predictive distributions for specific sampling distributions will be discussed in the following section. With the formulation of the SCP, we define an optimal A&S policy as

\[
(A^*, S^*) \triangleq \sup_{A, S} V_0(\zeta_0; A, S).
\]

III. R&S as Stochastic Control

In Section III-A, we establish the Bellman equation for SCP (3). In Section III-B, we show that the information set determining the posterior and predictive distributions can be further reduced to hyper-parameters by using conjugate priors.

A. Optimal A&S Policy

To avoid having to keep track of the entire sampling allocation policy history, the following theorem establishes that the posterior and predictive distributions at step \( t \) are determined by \( \mathcal{E}_i \); thus, if we define \( \mathcal{E}_i \) as the state at step \( t \), then SCP (3) satisfies the optimality equation of an MDP.

**Theorem 1:** Under the Bayesian framework introduced in Section II, the posterior distribution (1) of \( \theta \) conditioned on \( \mathcal{E}_T \) and the predictive distribution (2) of \( X_{t+1} \) conditioned on \( \mathcal{E}_i \) are independent of the allocation policy \( A \).

**Proof:** At any step \( t \), all replications except for the replication of the alternative being sampled, \( i = A_t(\mathcal{E}_{t-1}) \), are missing. The likelihood of observations collected by the sequential sampling procedure though \( t \) steps is given by

\[
L(\mathcal{E}_{T}^{i}; \theta) = \int \cdots \int_{X_{t}^{i+1}} \prod_{t=1}^{T} q(x_{t}; \theta) \prod_{i=1}^{k} \{A_{i, t}^{a}(\mathcal{E}_{t-1}) \delta_{X_{t}, i}(dx_{t}, i) + (1 - A_{i, t}^{a}(\mathcal{E}_{t-1})) dx_{t, i}\}
\]

\[
= \left( \sum_{i=1}^{k} A_{i, t}(\mathcal{E}_{t-1}) q_1(X_{t}, i; \theta_i) \right) \int \cdots \int_{X_{t}^{i+1}} \prod_{t=1}^{T} q(x_{t}; \theta) \times \prod_{i=1}^{k} \{A_{i, t}(\mathcal{E}_{t-1}) \delta_{X_{t}, i}(dx_{t}, i) + (1 - A_{i, t}(\mathcal{E}_{t-1})) dx_{t, i}\}
\]

\[
= \prod_{t=1}^{T} \left( \sum_{i=1}^{k} A_{i, t}(\mathcal{E}_{t-1}) q_1(X_{t}, i; \theta_i) \right) = \prod_{i=1}^{k} \prod_{t=1}^{T} q_1(X_{t}, i; \theta_i)
\]

where \( X^* \triangleq X_1 \times \cdots \times X_k \) and \( \delta_x(\cdot) \) is the delta-measure with a mass point at \( x \). The first equality in (4) holds because \( X_t, t \in Z^+ \), are assumed to be i.i.d. and the \( t \)th replication \( X_i \) is independent of the information flow before \( t \) step, i.e., \( \zeta_0 \cup \{X_t \}_{t=1}^{T-1} \), by construction of the information set; thus, the variables of the missing replications at step \( t \) in the joint density are integrated out, leaving only the marginal likelihood of the observation at step \( t \). By using the same argument inductively, the second equality in (4) holds. The last equality in (4) holds because the product operation is commutative. With (4), we can denote the likelihood as \( L(\mathcal{E}_i; \theta) \), since the information set \( \mathcal{E}_i \) completely determines the likelihood.

Following Bayes rule, the posterior distribution of \( \theta \) is

\[
F(d\theta|\mathcal{E}_T^i) = \frac{L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}{\int_{\theta \in \Theta} L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}
\]

which is independent of the allocation policy \( A \), conditioned on \( \mathcal{E}_T \). With (5), we can denote the posterior distribution as \( F(d\theta|\mathcal{E}_T) \), since the information set \( \mathcal{E}_T \) completely determines the posterior distribution. Similarly, the predictive distribution of \( X_{t+1} \) is

\[
Q_i(dx_{t+1} | \mathcal{E}_T^i) = \frac{\int_{\theta \in \Theta} Q_i(dx_{t+1} | \theta_1) L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}{\int_{\theta \in \Theta} L(\mathcal{E}_T^i; \theta) F(d\theta; \zeta_0)}
\]

which is independent of the allocation policy \( A \), conditioned on \( \mathcal{E}_i \). With (6), we can denote the predictive distribution of \( X_{t+1} \) as \( Q_i(dx_{t+1} | \mathcal{E}_i) \), since \( \mathcal{E}_i \) completely determines the predictive distribution.

**Remark:** The interaction between sampling allocation policy and posterior distribution has also been studied by Görder.
and Kolonko [22], but they introduced a monotone missing pattern that is not satisfied by the sequential sampling mechanism assumed in our paper. If the sampling distribution is dependent, i.e., 
\[ Q(x_i; \theta) = \prod_{i=1}^k Q_i(x_{i,t}; \theta), \]
the missing pattern can be fitted into a missing at random (MAR) paradigm studied in incomplete data analysis. MAR means that the missing rule is independent of the missing data, given the observations; see [25, Ch. 2] for a rigorous definition. If the sampling distribution is dependent, the sequential information collection procedure in this paper does not satisfy the classic MAR paradigm. For example, in Fig. 1, we can see that if the sampling distribution is not independent, the missing rule, say \[ A_{1,i}(E_{1}^0) \], could be dependent on the missing data, say \[ X_{3,1} \], since \[ X_{3,1} \] and \[ X_{2,1} \] are dependent and \[ X_{2,1} \in E_{0}^0 \]. Even without the MAR condition, we can still prove our conclusion because of two facts: first, the replications, i.e., \[ X_{i}, t \in \mathbb{Z}^+ \], of the sampling distribution are assumed to be independent; second, the allocation decision at step \[ t \], i.e., \[ A_t(E_{t-1}^0) \], only depends on the information set collected at the step \[ t-1 \] in our setting. We call the special structure of sequential sampling decision in R&S sequentially MAR.

Dependence in the sampling distribution is often introduced by using common random numbers to enhance the efficiency of R&S (see [18] and [32]). Although dependence in the sampling distribution is not a problem, our proof for Theorem 1 does not apply if there is dependence between replications, because \[ X_{i,t} \] and \[ A_{j,t}(E_{t-1}^0) \], \[ j = 1, \ldots, k \], could be dependent in this case. The i.i.d. assumption for replications, assumed in our paper, is a canonical condition in R&S.

1) **Bellman Equation:** With the conclusion of Theorem 1, the R&S problem is an MDP with state \( E_t \), action \( A_{t+1} \) for \( 0 \leq t < T \) and \( S \) for \( t = T \), no reward for \( 0 \leq t < T \) and \( V_T(E_T; S) \) for \( t = T \), and the following transition for \( 0 \leq t < T \):

\[
\begin{align*}
\{ \zeta_0, \bar{X}_1^{(t)} \}, \ldots, \bar{X}_k^{(t)} \} \\
\rightarrow \{ \zeta_0, \bar{X}_1^{(t)} \}, \ldots, \bar{X}_1^{(t)}, X_{i,t+1}, \ldots, \bar{X}_k^{(t)} \}
\end{align*}
\]

where \( X_{i,t+1} \sim Q_i(\cdot | E_t) \), \( i = A_{t+1} \). Then, we can recursively compute the optimal A&S policy \((A^*, S^*)\) of the SCP (3) by the following Bellman equation:

\[
V_T(E_T) \triangleq V_T(E_T; t)|_{t=0}^{T},
\]

where

\[
V_T(E_T; t) \triangleq \mathbb{E} [V(\theta; t)| E_T],
\]

and

\[
S^*(E_T) = \max_{i=1,\ldots,k} V_T(E_T; i)
\]

and for \( 0 \leq t < T \)

\[
V_t(E_t) \triangleq V_t(E_t; i)|_{i=A_{t+1}}^{E_t},
\]

where

\[
V_t(E_t; i) \triangleq \mathbb{E} [V_{t+1}(E_t, X_{i,t+1})| E_t],
\]

and

\[
A^*_{t+1}(E_t) = \arg \max_{i=1,\ldots,k} V_t(E_t; i).
\]

For an MDP, the equivalence between the optimal policy of the SCP, i.e., (3), and the optimal policy determined by the Bellman equation, i.e., (7) and (8), can be established straightforwardly by induction. The equivalence discussion can be found in [2, Proposition 1.3.1].

2) **Conjugacy**

Notice that the dimension of the state space of the MDP in the previous section grows as the step grows. Under the conjugate prior, the information set \( E_t \) can be completely determined by the posterior hyper-parameters, i.e., \( E_t = \zeta_t \). Thus, the dimension of the state space is the dimension of the hyper-parameters, which is fixed at any step. We provide specific forms for the conjugacy of independent Bernoulli distributions and independent normal distributions with known variances.

1) **Bernoulli Distribution:** The Bernoulli distribution is a discrete distribution with p.m.f.: \( q_i(1; \theta_i) = \theta_i \), and \( q_i(0; \theta_i) = 1 - \theta_i \), so the mean of alternative \( i \) is \( \mu_i = \theta_i \). The conjugate prior for the Bernoulli distribution is a beta distribution with density

\[
f_i(\theta; \alpha^{(0)}, \beta^{(0)}) = \frac{\theta_i^{\alpha^{(0)}-1} (1 - \theta_i)^{\beta^{(0)}-1}}{\int_0^1 \theta_i^{\alpha^{(0)}-1} (1 - \theta_i)^{\beta^{(0)}-1} d\theta_i}, \quad \theta_i \in [0, 1], \quad \alpha_i, \beta_i > 0.
\]

With (5) and (6), the posterior distribution of \( \theta_i \) is

\[
F_i(\theta; \alpha^{(t)}, \beta^{(t)}) = f_i(\theta; \alpha^{(t)}, \beta^{(t)}) d\theta_i
\]

where \( \zeta_t i \triangleq (\alpha^{(t)}, \beta^{(t)}) \), and

\[
\alpha_i^{(t)} = \alpha_i^{(0)} + \sum_{t=1}^{t} X_i, \quad \beta_i^{(t)} = \beta_i^{(0)} + \sum_{t=1}^{t} (1 - X_i),
\]

and the predictive p.m.f. of \( X_i, t+1 \) is

\[
q_i(1; \zeta_{t+1} i) = \gamma_i^{(t)}, \quad q_i(0; \zeta_{t+1} i) = 1 - \gamma_i^{(t)}
\]

where

\[
\gamma_i^{(t)} \triangleq \frac{\alpha_i^{(t)}}{\alpha_i^{(t)} + \beta_i^{(t)}}.
\]

Assuming \( \gamma_i^{(0)} = \gamma_j^{(0)}, \) if \( t_i = t_j \) and \( m_i > m_j \), then \( \gamma_i^{(t)} > \gamma_j^{(t)} \), and if \( m_i^{(t)} = m_j^{(t)} \) and \( t_i > t_j \), then \( \gamma_i^{(t)} > \gamma_j^{(t)} \). When \( 0 < \gamma_i^{(0)} < m_i^{(0)} (\gamma_j^{(0)} > m_j^{(0)} \), \( \gamma_i^{(t)} = m_i^{(t)} \), and the prior is a special uninformative prior, which is not a proper distribution, although the posterior distribution can be appropriately defined similarly as the informative prior.

2) **Normal Distribution:** The conjugate prior for the normal distribution \( N(\mu, \sigma^2) \) with unknown mean and known variance is a normal distribution \( N(\mu^{(0)}, \sigma_i^{(0)^2}) \). With (5) and (6), the posterior distribution of \( \mu_i \) is \( N(\mu_i^{(t)}, \sigma_i^{(t)^2}) \), where

\[
\mu_i^{(t)} = (\sigma_i^{(t)^2})^{-1} \left( \frac{\mu_i^{(0)}}{\sigma_i^{(0)^2}} + \frac{t_i m_i^{(t)}}{\sigma_i^{(t)^2}} \right)
\]

\[
\sigma_i^{(t)^2} = \left( \frac{1}{\sigma_i^{(0)^2}} + \frac{t_i}{\sigma_i^{(t)^2}} \right)^{-1}
\]
and the predictive distribution of \(X_{i,t+1}\) is \(N(\mu_{i,t}\bigg|\sigma_i^2 + (\sigma_i^2)^2)\). If \(\sigma_i^{(0)} \to \infty\), \(\mu_i^{(t)} = m_i^{(t)}\), and the prior is the uninformative prior in this case. For a normal distribution with unknown variance, there is a normal-gamma conjugate prior (see [13]).

IV. ANALYSIS OF OPTIMAL A&S POLICY

In Section IV-A, we analyze the properties of the optimal selection policy. For discrete sampling and prior distributions, an explicit form for the optimal A&S policy and its computational complexity are provided in Section IV-B.

A. Optimal Selection Policy

The optimal selection policy is the last step in the Bellman equation. From (5), we know posterior distributions conditioned on \(\hat{E}_T\) are independent when the prior distributions for different alternatives are independent, which will be assumed in this section. For PCS, the optimal selection policy is

\[
S^*(E_T) = \arg \max_{i=1,\ldots,k} P(\mu_i \geq \mu_j, \forall j \neq i|E_T) = \arg \max_{i=1,\ldots,k} \int_{O_i} \prod_{j \neq i} F_j(x|\hat{E}_T) f_i(x|\hat{E}_T) \, dx
\]

where \(O_i\) is the feasible set of \(\mu_i\), \(F_i(\cdot|\hat{E}_T)\) is the posterior distribution of \(\mu_i\) with density \(f_i(\cdot|\hat{E}_T)\), \(i = 1, \ldots, k\), and for EOC, the optimal selection policy is

\[
S^*(E_T) = \arg \max_{i=1,\ldots,k} E[\mu_i|E_T]
\]

and

\[
V_T(\hat{E}_T) = E[\mu_i - \mu(1)|E_T]|_{i=S^*(E_T)}
\]

\[
= E[\mu_i|E_T]|_{i=S^*(E_T)} - E\left[ \sum_{i=1}^k \mu_i 1\{\mu_i > \mu_j, j \neq i\} \mid E_T \right]
\]

\[
= E[\mu_i|E_T]|_{i=S^*(E_T)} - \sum_{i=1}^k E\left[ \mu_i 1\{\mu_i > \mu_j\} \mid \mu_i, E_T \right] E_T
\]

\[
= E[\mu_i|E_T]|_{i=S^*(E_T)} - \sum_{i=1}^k \int_{O_i} \prod_{j \neq i} F_j(x|E_T) f_i(x|E_T) \, dx.
\]

For EOC, the optimal selection policy for the Bernoulli distribution under conjugacy is

\[
S^*(E_T) = \arg \max_{i=1,\ldots,k} \left( m_i^{(T)} + \frac{1}{\alpha_i(T)} \right)
\]

and the optimal selection policy for the normal distribution under conjugacy is

\[
S^*(E_T) = \arg \max_{i=1,\ldots,k} \mu_i^{(T)}.
\]

For PCS, the optimal selection policy depends on the entire posterior distributions rather than just the posterior means. For normal distributions with conjugate priors, Peng et al. [34] showed that except for \(\sigma_i^2 = \cdots = \sigma_k^2\), selecting the largest posterior mean is not the optimal selection policy, which should also incorporate correlations induced by \(\sigma_i^2, \ldots, \sigma_k^2\).

The following theorem establishes that under some mild conditions, the selection policy selecting the alternative with the largest sample mean is asymptotically consistent with the optimal selection policy for EOC, which is analogous to the result for PCS in [34].

**Theorem 2:** Suppose for \(i = 1, \ldots, k\), \(\theta_i = (\mu_i, \xi_i) \in \Omega \times \Xi\), \(X_{i,t} \sim Q_i(\cdot; \theta_i)\), i.i.d., \(t \in \mathbb{Z}^+\), with \(Q_i\) mutually independent, \(\theta_i \sim F_i(\cdot)\) with \(F_i\) mutually independent, and the following conditions are satisfied:

1) \(Q_i(\cdot; \theta) \neq Q_i(\cdot; \theta')\) whenever \(\theta \neq \theta', i = 1, \ldots, k\);
2) \(P(\mu_1 = \cdots = \mu_k) = 0\);
3) \(E[\|\mu_i\|] < \infty, i = 1, \ldots, k\);
4) For any \(B \subset \Omega\) and finite \(T\), \(P(\mu_i \in B|E_T) < 1, i = 1, \ldots, k\).

Then, we have

\[
\lim_{T \to \infty} E\left[ V_E(\theta; i)|E_T^* \right]|_{i=S^*(E_T^*)} = 0, \quad a.s.
\]

where \(S^*(E_T^*) = \arg \max_{i=1,\ldots,k} m_i^{(T)}\) and \(E_T^*\) means the information set obtained by following the optimal allocation policy \(A^*\), and

\[
\lim_{T \to \infty} E\left[ V_E(\theta; i)|E_T^* \right]|_{i\neq S^*(E_T^*)} < 0, \quad a.s.
\]

therefore,

\[
\lim_{T \to \infty} \left[ S^*(E_T^*) - S^*(E_T^*) \right] = 0, \quad a.s.
\]

**Proof:** Denote \(A^*\) as the equal allocation (EA) policy. Following \(A^*\), every alternative will be sampled infinitely often as \(n\) goes to infinity. By the law of large numbers, we know

\[
\lim_{T \to \infty} \max_{i=1,\ldots,k} m_i^{(T)} = \max_{i=1,\ldots,k} \mu_i, \quad a.s.
\]

where \(\mu_i^*\) means the true parameter. In addition, \(\{E[\mu_i|E_T^*]\}\) and \(\{E[\max_{i=1,\ldots,k} \mu_i|E_T^*]\}\) are martingales. With condition (iii), we have

\[
E[\|E[\mu_i|E_T^*]\]| \leq E[\|\mu_i\|] < \infty
\]

\[
E\left[ \max_{i=1,\ldots,k} \mu_i | E_T^* \right] \leq \sum_{i=1}^k E[\|\mu_i\|] < \infty
\]

where \(E_T^*\) means the information set obtained by following \(A^*\). By Doob’s martingale convergence and consistency theorems (see [14] and [45]),

\[
\lim_{T \to \infty} E[\mu_i|E_T^*] = \mu_i^*
\]

\[
\lim_{T \to \infty} \max_{i=1,\ldots,k} \mu_i|E_T^* = \max_{i=1,\ldots,k} \mu_i^*, \quad a.s.
\]

so

\[
\lim_{T \to \infty} E\left[ V_E(\theta; S^*(E_T^*))|E_T^* \right] = 0, \quad a.s.
\]
By definition, we have

\[
0 = \lim_{T \to \infty} \mathbb{E}[V_E(\theta; S^n(\mathcal{E}_T^*))|\mathcal{E}_T^*]
\]

\[
\leq \lim_{T \to \infty} \mathbb{E}[V_E(\theta; S^{*}(\mathcal{E}_T^*))|\mathcal{E}_T^*] \leq 0, \quad \text{a.s.}
\]

Then, we prove that following the optimal policy \( A^* \), every alternative will be sampled infinitely often almost surely as \( T \) goes to infinity. Otherwise, \( \exists \, k_1, k_2 \in \mathbb{Z}^+, \text{s.t.} \, k_1 + k_2 = k \) and

\[
\{i_1, \ldots, i_{k_1} : T_{i_1}^{(T)} \triangleq \lim_{T \to \infty} T_{i_1} < \infty, \, l = 1, \ldots, k_1\} \neq \emptyset
\]

\[
\{j_1, \ldots, j_{k_2} : \lim_{n \to \infty} T_{j_l} = \infty, \, l = 1, \ldots, k_2\} \neq \emptyset.
\]

We have

\[
\mathbb{E}\left[\mathbb{E}\left[\max_{i=1,\ldots,k} \mu_i \mid \mathcal{E}_T, \mu_{i_1}, \ldots, \mu_{i_{k_1}}\right]\right]
\]

\[
\leq \mathbb{E}\left[\max_{i=1,\ldots,k} \mu_i \mid \mathcal{E}_T, \mu_{i_1}, \ldots, \mu_{i_{k_1}}\right] \leq \sum_{i=1}^{k} \mathbb{E}[||\mu_i||] < \infty.
\]

By the dominated convergence theorem (see [43]) and Doob’s martingale convergence and consistency theorems

\[
\lim_{T \to \infty} \mathbb{E}\left[\max_{i=1,\ldots,k} \mu_i \mid \mathcal{E}_T\right] = \lim_{n \to \infty} \mathbb{E}\left[\max_{i=1,\ldots,k} \mu_i \mid \mathcal{E}_T, \mu_{i_1}, \ldots, \mu_{i_{k_1}}\right] = \mathbb{E}\left[\max_{i=1,\ldots,k} \mu_i \mid \mathcal{E}_T\right]
\]

\[
\hat{X}_{i_1}^{(T)}, \ldots, \hat{X}_{i_k}^{(T)}, T_{i_1}^{*}, \ldots, T_{i_k}^{*}, \zeta_0
\]

where the last equality holds almost surely. From the independence condition in the theorem, the conclusion of Theorem 1, and condition (iv), for \( l = 1, \ldots, k_1 \)

\[
P\left(\mu_{i_1} > C \hat{X}_{i_1}^{(T)}, T_{i_1}^{*}, \zeta_0\right) > 0
\]

\[
P\left(\mu_{i_1} < C \hat{X}_{i_1}^{(T)}, T_{i_1}^{*}, \zeta_0\right) > 0
\]

where \( C = \max_{i=1,\ldots,k} \mu_{i_1}^* \), so for \( l = 1, \ldots, k_2 \)

\[
\mu_{j_l}^* = \mathbb{E}\left[\max_{i=1,\ldots,k} \mu_{i_1} \mid i_1, \ldots, k_2 \right]
\]

\[
\hat{X}_{i_1}^{(T)}, \hat{X}_{i_2}^{(T)}, \ldots, \hat{X}_{i_k}^{(T)}, T_{i_1}^{*}, \ldots, T_{i_k}^{*}, \zeta_0
\]

and for \( l = 1, \ldots, k_1 \)

\[
\mathbb{E}\left[\mu_{i_l} \mid \hat{X}_{i_l}^{(T)}, T_{i_l}^{*}, \zeta_0\right] - \mathbb{E}\left[\max \left\{ \max_{i=1,\ldots,k_1} \mu_{i_1}, \max_{l=1,\ldots,k_2} \mu_{j_l}^* \right\} \right] \]

\[
\leq \mathbb{E}\left[\mu_{i_l} \mid \hat{X}_{i_l}^{(T)}, T_{i_l}^{*}, \zeta_0\right] - \mathbb{E}\left[\max \left\{ \mu_{i_l}, \max_{l=1,\ldots,k_2} \mu_{j_l}^* \right\} \right] < 0.
\]

Therefore,

\[
\lim_{T \to \infty} \mathbb{E}[V_E(\theta; S^{*}(\mathcal{E}_T^*))|\mathcal{E}_T^*] = \max_{i=1,\ldots,k} \mathbb{E}[\mu_i|\mathcal{E}_T^*] - \mathbb{E}\left[\max_{i=1,\ldots,k} \mu_i \mid \mathcal{E}_T^*\right] < 0, \quad \text{a.s.}
\]

which is a contradiction. With every alternative sampled infinitely often under \( A^* \), the law of large numbers, and Doob’s martingale convergence and consistency theorems, for \( i = \arg \max_{i=1,\ldots,k} \mu_i^* \)

\[
\lim_{T \to \infty} \mathbb{E}[V_E(\theta; i)|\mathcal{E}_T^*]_{i=S^n(\mathcal{E}_T^*)} = \mu_i^* - \max_{i=1,\ldots,k} \mu_i = 0, \quad \text{a.s.}
\]

and with condition (ii), for \( i \neq \arg \max_{i=1,\ldots,k} \mu_i^* \)

\[
\lim_{T \to \infty} \mathbb{E}[V_E(\theta; i)|\mathcal{E}_T^*]_{i \neq S^n(\mathcal{E}_T^*)} = \mu_i^* - \max_{i=1,\ldots,k} \mu_i < 0, \quad \text{a.s.}
\]

Remark: For independent sampling and prior distributions, the most frequently used conjugate models, including the two models introduced in Section III-B, satisfy the conditions in Theorem 2 (see [13]). In the proof, we can see that under mild regularity conditions, every alternative will be sampled infinitely often following the optimal A&S policy as the simulation budget goes to infinity, which in turn leads to the conclusion of the theorem.

B. Optimal A&S Policy for Discrete Distributions

In the case where the sampling distribution and prior distribution are discrete, in principle, the optimal A&S policy can be calculated. For continuous sampling distribution and prior distribution, discretization can be implemented to reduce to the discrete case as an approximation scheme. Suppose the sampling distribution of \( X_t \) is supported on \( \{y_1, \ldots, y_s\} \), where \( y_j = (y_{1,j}, \ldots, y_{k,j}) \), \( j = 1, \ldots, s \), and the prior distribution of \( \theta \) is supported on \( \{\eta_1, \ldots, \eta_r\} \). From the results in Theorem 1, the p.m.f. for the posterior distribution of \( \theta \) is

\[
f(\eta_j|\mathcal{E}_t) = \frac{\prod_{i=1}^{k} q_i(\hat{X}_{i,t}^j, \eta_j) f(\eta_j; \zeta_0)}{\sum_{j=1}^{r} \prod_{i=1}^{k} q_i(\hat{X}_{i,t}^j, \eta_j')} f(\eta_j'; \zeta_0)
\]
where \( f \) is the p.m.f. of joint distribution \( F \), and the p.m.f. for the predictive distribution of \( X_{i,t+1} \) is

\[
q_i(y_{i,j} | \mathcal{E}_t) = \frac{\sum'_{j=1} q_i(y_{i,j}; \eta_{j'}) \prod_{i=1}^k \prod_{i=1}^{j_i} q_i(\tilde{X}_{i,t}; \eta_{j'}) f(\eta_j; \zeta_0)}{\sum'_{j=1} \prod_{i=1}^k \prod_{i=1}^{j_i} q_i(\tilde{X}_{i,t}; \eta_{j'}) f(\eta_j; \zeta_0)}.
\]

Therefore,

\[
V_T(\mathcal{E}_T; i) = \mathbb{E} \left[ V(\theta; i) | \mathcal{E}_T \right] = \sum_{j=1}^r V(\eta_j; i) f(\eta_j | \mathcal{E}_T)
\]

and for \( 0 \leq t < T \)

\[
V_t(\mathcal{E}_t; i) = \mathbb{E} \left[ V_{t+1}(\mathcal{E}_t, X_{i,t+1}) | \mathcal{E}_t \right] = \sum_{j=1}^s V_{t+1}(\mathcal{E}_t, y_{i,j}) q_i(y_{i,j} | \mathcal{E}_t)
\]

so Bellman equations (7) and (8) can be solved recursively.

1) Bernoulli Distribution: We analyze the size of the state space for calculating the optimal sampling allocation policy for alternatives following independent Bernoulli distributions, which are the simplest discrete distributions. For the Bernoulli distribution, the posterior distribution of alternative \( i \) can be determined by the prior information and \( (M_i^{(t)}, t_i) \), where

\[
M_i^{(t)} = \sum_{t=1}^{t_i} \tilde{X}_{i,t}.
\]

Notice that the number of possible outcomes of \( M_i^{(t)} \) is \( t_i + 1 \), which grows linearly with respect to the number of allocated replications. Fig. 4 provides an illustration for the evolution of \( M_i^{(t)} \).

With given prior information \( \zeta_0 \), \( (M_1^{(t)}, \ldots, M_k^{(t)}, t_1, \ldots, t_k) \) determines the state space of information set \( \mathcal{E}_t \). The size of the state space is

\[
L_{t,k} = \sum_{(t_1, \ldots, t_k): t_1 + \cdots + t_k = t} (t_1 + 1) \times \cdots \times (t_k + 1).
\]

To shed some light on how large the state space is, we provide upper and lower bounds for the summation. It is easy to see that

\[
L_{t,k} \geq \left( \frac{t}{k} \right)^k + 1
\]

and

\[
(t_1 + 1) \times \cdots \times (t_k + 1) \leq (t + 1)^k.
\]

The elements in the set \( \{(t_1, \ldots, t_k) : t_1 + \cdots + t_k = t\} \) are in one-to-one correspondence to possible choices for picking \( k - 1 \) balls from \( t + k - 1 \) balls. From simple combinatorics, the size of the set \( \{(t_1, \ldots, t_k) : t_1 + \cdots + t_k = t\} \) is

\[
C_{t+k-1}^{k-1} = \frac{(t + k - 1)!}{t!(k - 1)!}
\]

thus

\[
L_{t,k} \leq \frac{(t + k - 1)^{2k}}{(k - 1)!}.
\]

For a fixed \( k \), \( L_{t,k} \) grows at a polynomial rate with respect to \( t \). However, if \( t \) and \( k \) grow together and \( t > k \), the lower bound of \( L_{t,k} \) grows at an exponential rate with respect to \( k \). From \( T \) to 1, we can use backward induction to determine the optimal allocation policy \( A_i^t(\cdot) \) for every possible state of \( \mathcal{E}_{t-1} \).

2) General Discrete Distribution: The size of the state space for general independent discrete sampling distributions can be analyzed similarly as in the case of independent Bernoulli distributions. Since the product operation is commutative in the likelihood, a possible outcome \( (\tilde{X}_{i,1}, \ldots, \tilde{X}_{i,k}) \) leading to a distinctive posterior distribution is uniquely determined by the number of elements picked from \( (y_{i,1}, \ldots, y_{i,s_i}) \). The size of the possible outcomes is equivalent to the size of the set \( \{(c_1, \ldots, c_{s_i}) \in \mathbb{N}^{s_i} : c_1 + \cdots + c_{s_i} = t_i\} \), which is

\[
C_{s_i+t_i-1}^{s_i-1} = \frac{(s_i + t_i - 1)!}{t_i!(s_i - 1)!}.
\]

The size of the state space for the information set \( \mathcal{E}_t \) is

\[
L_{t,k} = \sum_{(c_1, \ldots, c_k) \in \mathcal{E}_{t-1}} \prod_{i=1}^k C_{s_i+t_i-1}^{s_i-1}.
\]

Denote \( \bar{s} = \max_{i=1}^k s_i \) and \( \underline{s} = \min_{i=1}^k s_i \). Then, we have

\[
\left( 1 + \frac{t}{k} \right) \left( \frac{\bar{s} + t + k - 1}{\bar{s} - 1} \right)^{k-1} \leq L_{t,k} \leq \frac{(\bar{s} + t + k - 1)^k}{(\bar{s} - 1)!}.
\]

Similarly, for fixed \( k \) and \( \bar{s} \), \( L_{t,k} \) grows at a polynomial rate with respect to \( t \). However, if \( t, k, \) and \( \bar{s} \) grow together and \( t > k(\bar{s} - 1) \), the lower bound of \( L_{t,k} \) grows at an exponential rate with respect to \( k \) and \( \bar{s} \).
V. A&S Policy for Normal Distributions

In this section, we consider the sampling allocation for alternatives following independent normal distributions, which is most frequently assumed in R&S, and we derive an efficient scheme to approximate the optimal A&S policy in an ADP paradigm. More specifically, we do not apply backward induction, but use forward programming by optimizing a VFA one step ahead. Similar to many ADP approaches (see [39]), the proposed procedure in this paper is not guaranteed to achieve the optimal A&S policy, but can alleviate the curse of dimensionality.

We focus on PCS as the final reward of the selection decision and assume a conjugate prior for the unknown means in Section III-B. In this section, we suppose any step $t$ could be the last step. The selection policy is to choose the alternative with the largest posterior mean, which is asymptotically optimal based on analysis in Section IV-A, i.e.,

$$\hat{S}(\mathcal{E}_t) = (1)_t$$

where $\mu^{(t)}_{(1)h} > \cdots > \mu^{(t)}_{(k)h}$. As in ADP, we approximate the value function using some features, specifically, the following VFA is used for selecting the $(1)_t$th alternative:

$$\hat{V}(\mathcal{E}_t; w) = K \left( \sum_{i=1}^{T} w_i g_i(\mathcal{E}_t) \right)$$

(9)

where $g_j(\cdot)$, $j = 1, \ldots, k$, are features of the value function, $w = (w_1, \ldots, w_T)$ are the weights of the features, and $K(\cdot)$ is referred to as the activation function.

A. Approximately Optimal Allocation Policy

We first provide a VFA using one feature in the value function. The value function for selecting the $(1)_t$th alternative is

$$\mathbb{E}_{V_P}(\theta; (1)_t) | \mathcal{E}_t | = P(\mu^{(t)}_{(1)}, \mu^{(t)}_{(j)}, \ldots, \mu^{(t)}_{(k)}) \times P(\mu^{(t)}_{(1)}, \mu^{(t)}_{(j)}, \ldots, \mu^{(t)}_{(k)} | \mathcal{E}_t)$$

which is the posterior integrated PCS (IPCS). Conditioned on $\mathcal{E}_t$, $\mu^{(t)}_{(k)}$ follows a normal distribution with mean $\mu^{(t)}_{(k)}$ and variance $(\sigma^{(t)}_{(k)})^2$, $i = 1, \ldots, k$. Therefore, the joint distribution of vector $(\mu^{(t)}_{(1)}, \mu^{(t)}_{(2)}, \ldots, \mu^{(t)}_{(j)}, \ldots, \mu^{(t)}_{(k)})$ follows a joint normal distribution with mean $(\mu^{(t)}_{(1)}, \mu^{(t)}_{(2)}, \ldots, \mu^{(t)}_{(j)}, \ldots, \mu^{(t)}_{(k)})$ and covariance matrix $\Gamma \in \mathbb{R}^{k \times k}$, where $\Gamma$ indicates the transpose operation, $\Lambda = \text{diag}(\sigma^{(t)}_{(1)}^2, \ldots, \sigma^{(t)}_{(k)}^2)$, and

$$\Gamma = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1
\end{bmatrix}_{k \times (k-1)}$$

By Cholesky decomposition $\Gamma \in \mathbb{R}^{k \times k}$, where $U = [u_{ij}]_{(k-1) \times (k-1)}$ is an upper triangular matrix (i.e., $u_{ij} = 0$, if

By Cholesky decomposition $\Gamma \in \mathbb{R}^{k \times k}$, where $U = [u_{ij}]_{(k-1) \times (k-1)}$ is an upper triangular matrix (i.e., $u_{ij} = 0$, if

Fig. 5. Area of integration for approximation is the circle, where dominant values of integrand $\exp(-(z_1^2 + z_2^2)/2)$ are captured.

$$i > j$$

$$P(\mu^{(t)}_{(i)}, \mu^{(t)}_{(j)}, \ldots, \mu^{(t)}_{(k)}) = 1$$

$$\int \cdots \int_{\mathbb{R}^{k-1}} \exp \left( -\frac{z_k^2}{2} \right) dz_{k-1} \cdots dz_1$$

where $z_i$, $i = 1, \ldots, k-1$, are independent standard normal random variables. The value function (10) is an integration of the density of a $(k-1)$-dimensional standard normal distribution over an area covered by hyperplanes

$$\sum_{j=1}^{k} u_{ij}z_j = \mu^{(t)}_{(i)}, \quad i \neq 1.$$
We use a VFA for the value function (10) given by \( \bar{\mathbb{V}}(\mathcal{E}_i) = d^2(\mathcal{E}_i) \). If the \((t + 1)\)th replication is the last one, a VFA looking one step ahead at step \( t \) by allocating the \( i \)th alternative can be given as follows:

\[
\bar{\mathbb{V}}_i(\mathcal{E}_i; t) = \mathbb{E}
\left[
\left|
\left|
\bar{\mathbb{V}}_{t+1}(\mathcal{E}_i, X_{t+1})\right|
\right|
\right]_{\mathcal{E}_i},
\]

Since the above expectation is difficult to calculate, we use the following certainty equivalence (see [3]) as an approximation: for \( j \neq 1 \)

\[
\bar{\mathbb{V}}_i(\mathcal{E}_i; 1) \triangleq \mathbb{E}\left[\bar{\mathbb{V}}_{t+1}(\mathcal{E}_i, \mathbb{E}\left[X_{t+1}\mid \mathcal{E}_i]\right)\right]\]

\[
= \min_{i \neq 1} \left(\frac{\mu^{(t)}_i - \mu^{(t)}_i}{\sigma^{(t)}_i} \right)^2 \sum_{j \neq 1, j \neq 1} \left(\frac{\mu^{(t)}_i - \mu^{(t)}_i}{\sigma^{(t)}_i} \right)^2 \left(\frac{\mu^{(t)}_i - \mu^{(t)}_i}{\sigma^{(t)}_i} \right)^2.
\]

An AOAP that optimizes the VFA one step ahead is given by

\[
\bar{A}_{t+1}(\mathcal{E}_i) = \arg \max \bar{\mathbb{V}}_i(\mathcal{E}_i; t).
\]

In contrast with knowledge gradient (KG), which uses a surrogate \( \mu^{(t)}_i \) of the derivative of the one-step-ahead optimality (see [16] and [23]), the VFA in AOAP (12) uses an easily interpretable feature (size of hypersphere) that better describes the true value function in our SCP. AOAP (12) possesses the following asymptotic properties.

**Theorem 3:** AOAP (12) is consistent, i.e.,

\[
\lim_{t \to \infty} \bar{S}(\mathcal{E}_i) = (1), \quad a.s.
\]

In addition, the sampling ratio of each alternative asymptotically achieves the optimal decreasing rate of the large deviations of the probability of false selection in [20], i.e.,

\[
\lim_{t \to \infty} r_{i}^{(t)} = r_{i}^{*}, \quad a.s., \quad i = 1, \ldots, k
\]

where \( r_{i}^{*} \triangleq t_{i}/t, \sum_{i=1}^{k} r_{i}^{*} = 1, r_{i}^{*} \geq 0, i = 1, \ldots, k \), and for \( i, j \neq 1 \)

\[
\frac{(\mu_{i} - \mu_{1})^2}{(\sigma_{i})^2} \frac{1}{r_{i}^{*}} + \frac{(\sigma_{i})^2}{r_{i}^{*}} = \frac{(\mu_{j} - \mu_{1})^2}{(\sigma_{j})^2} \frac{1}{r_{j}^{*}} + \frac{(\sigma_{j})^2}{r_{j}^{*}}
\]

\[
r_{i}^{*} = \frac{(\sigma_{i})^2}{\sum_{i=1}^{k} \frac{(\sigma_{i})^2}{r_{i}^{*}}}
\]

**Proof:** We only need to prove that every alternative can be sampled infinitely often almost surely, following AOAP (12), and the consistency will follow by the law of large numbers. Suppose alternative \( i \) is only sampled finitely often and alternative \( j \) is sampled infinitely often. Then,

\[
\lim_{t \to \infty} \bar{\mathbb{V}}_i(\mathcal{E}_i; t) - \bar{\mathbb{V}}_i(\mathcal{E}_i) > 0, \quad \lim_{t \to \infty} \bar{\mathbb{V}}_i(\mathcal{E}_i; j) - \bar{\mathbb{V}}_i(\mathcal{E}_i) = 0, \quad a.s.
\]

which contradicts with the sampling rule that the alternative with the largest VFA is sampled in AOAP (12). Therefore, AOAP (12) must be consistent.

By the law of large numbers, \( \lim_{t \to \infty} \mu^{(t)}_{i} - \mu_{i}, \quad i = 1, \ldots, k \). Because the asymptotic sampling ratios will be determined by the increasing order of \( \bar{\mathbb{V}}_i(\mathcal{E}_i; i) \) with respect to \( t \), we can replace \( \mu^{(t)}_{i} \) and \( (\sigma^{(t)}_{i})^2 = \mu_{i}^2 + \sigma_{i}^2 / t_i \) in \( \bar{\mathbb{V}}_i(\mathcal{E}_i; i) \), \( i = 1, \ldots, k \), for simplicity of analysis. If \( (r_{1}^{*}, \ldots, r_{k}^{*}) \) does not converge to \( (\tilde{r}_{1}, \ldots, \tilde{r}_{k}) \), there exists a subsequence of the former converging to \( (\tilde{r}_{1}, \ldots, \tilde{r}_{k}) \) such that \( \sum_{i=1}^{k} \tilde{r}_{i} = 1, \tilde{r}_{i} \geq 0, i = 1, \ldots, k \), by the Bolzano–Weierstrass theorem (see [42]). Without loss of generality, we can assume \( (r_{1}^{*}, \ldots, r_{k}^{*}) \) converges to \( (\tilde{r}_{1}, \ldots, \tilde{r}_{k}) \). Notice that

\[
\lim_{t \to \infty} \left[ \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / t_i + \sigma_{1}^2 / t_i + 1} \right] = \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{i}}
\]

and

\[
\lim_{t \to \infty} \left[ \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{i} + \sigma_{2}^2 / \tilde{r}_{i}} \right] = \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{i}} + \frac{\mu_{i} - \mu_{1}}{\sigma_{2}^2 / \tilde{r}_{i}}
\]

where

\[
G_{i}(r_{1}, r_{i}) = \sum_{i \neq 1} \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / r_{1} + \sigma_{1}^2 / r_{1}} \quad i \neq 1.
\]

We first claim that \( \tilde{r}_{1} > 0 \). Otherwise, we have \( \tilde{r}_{1} = 0 \) and there exists \( \tilde{r}_{i} > 0, i \neq 1 \). We have

\[
\lim_{t \to \infty} \left[ \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{i}} \right] \left[ \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{i}} \right] > 0
\]

and

\[
\lim_{t \to \infty} \left[ \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{i}} \right] \left[ \frac{\mu_{i} - \mu_{1}}{\sigma_{i}^2 / \tilde{r}_{i} + \sigma_{1}^2 / \tilde{r}_{1}} \right] = 0
\]
which contradicts the sampling rule of AOAP (12); so $\bar{r}_{(1)} > 0$. We next claim that $\bar{r}_{(j)} > 0, j = 1, \ldots, k$. Otherwise, there exists $\bar{r}_{(i)} = 0, i \neq 1$. We have

$$\lim_{t \to \infty} \left( \frac{\sigma_{(i)}}{r_{(i)}} \right)^2 \left( \mu_{(i)} - \mu_{(1)} \right)^2 \left( \frac{\sigma_{(j)}}{r_{(j)}} + \frac{\sigma_{(1)}}{r_{(1)}} \right)^2 = 0$$

$$\lim_{t \to \infty} \left( \frac{\sigma_{(i)}}{r_{(i)}} \right)^2 \left( \mu_{(i)} - \mu_{(1)} \right)^2 \left( \frac{\sigma_{(j)}}{r_{(j)}} + \frac{\sigma_{(1)}}{r_{(1)}} \right)^2 > 0$$

which contradicts the sampling rule of AOAP (12); so $\bar{r}_{(j)} > 0, j = 1, \ldots, k$ If $(\bar{r}_1, \ldots, \bar{r}_k)$ does not satisfy (13), then there exist $i \neq j, i, j \neq 1$ such that

$$G_i(\bar{r}_{(1)}, \bar{r}_{(j)}) > G_j(\bar{r}_{(1)}, \bar{r}_{(j)})$$

If the inequality above holds, then, there exists $T_0 > 0$ such that for all $t > T_0$

$$G_i(\bar{r}_{(1)}^{(t)}, \bar{r}_{(j)}^{(t)}) > G_j(\bar{r}_{(1)}^{(t)}, \bar{r}_{(j)}^{(t)})$$

due to the continuity of $G_i$ and $G_j$ on $(0, 1) \times (0, 1)$. By the sampling rule of AOAP (12), the $(j)$th alternative will be sampled and the $(i)$th alternative will stop receiving replications before the inequality above reverses. This contradicts $(\bar{r}_{(1)}^{(t)}, \ldots, \bar{r}_{(k)}^{(t)})$ converging to $(\bar{r}_1, \ldots, \bar{r}_k)$, so (13) must hold. By the implicit function theorem (see [42]), (13) and $\sum_{i=1}^k \bar{r}_i = 1$ determine implicit functions $\bar{r}_{(i)}(x) = \bar{r}_{(i)}$, $i = 2, \ldots, k$, because

$$\det(\Sigma) = \prod_{i=2}^k \zeta_{i,i} \left\{ \sum_{i=2}^k \zeta_{i,i}^{-1} \right\} > 0$$

where

$$\zeta_{i,i} \triangleq \frac{\partial G_i(\bar{r}_{(1)}^{(t)}, x)}{\partial x} \bigg|_{x = \bar{r}_{(j)}^{(t)}}, \quad i = 2, \ldots, k$$

$$\Sigma \triangleq \begin{pmatrix} \zeta_{2,2} & -\zeta_{3,3} & \cdots & 0 & 0 \\ 0 & \zeta_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \zeta_{k-1,k-1} & -\zeta_{k,k} \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and $R = -\Sigma^{-1} \Upsilon$, where

$$R \triangleq \left( \frac{\partial \bar{r}_2(x)}{\partial x}, \ldots, \frac{\partial \bar{r}_k(x)}{\partial x} \right) \bigg|_{x = \bar{r}_{(j)}^{(t)}},$$

$$\Upsilon \triangleq \begin{pmatrix} \frac{\partial G_2(x, \bar{r}_{(2)})}{\partial x} - \frac{\partial G_3(x, \bar{r}_{(3)})}{\partial x} \\ \vdots \\ \frac{\partial G_{k-1}(x, \bar{r}_{(k-1)})}{\partial x} - \frac{\partial G_k(x, \bar{r}_{(k)})}{\partial x} \end{pmatrix} \bigg|_{x = \bar{r}_{(j)}^{(t)}}.$$  

In addition

$$\frac{\partial G_i(x, \bar{r}_{(1)}^{(t)})}{\partial x} \bigg|_{x = \bar{r}_{(i)}^{(t)}} + \zeta_{i,j} \frac{\partial \bar{r}_{(j)}^{(t)}(x)}{\partial x} \bigg|_{x = \bar{r}_{(i)}^{(t)}} = 0, \quad i \neq 1$$

otherwise, there exists $j \neq 1$ such that the equality above does not hold, say

$$\frac{\partial G_j(x, \bar{r}_{(j)}^{(t)})}{\partial x} \bigg|_{x = \bar{r}_{(i)}^{(t)}} + \zeta_{j,j} \frac{\partial \bar{r}_{(j)}^{(t)}(x)}{\partial x} \bigg|_{x = \bar{r}_{(i)}^{(t)}} > 0.$$

Following the sampling rule of AOAP (12), the $(1)$th alternative will be sampled and the $(j)$th alternative will stop receiving replications before the inequality above is no longer satisfied, which contradicts $(\bar{r}_{(1)}^{(t)}, \ldots, \bar{r}_{(k)}^{(t)})$ converging to $(\bar{r}_1, \ldots, \bar{r}_k)$. Then, $HR = -G$, where

$$G \triangleq \left( \frac{\partial G_2(x, \bar{r}_{(2)})}{\partial x}, \ldots, \frac{\partial G_k(x, \bar{r}_{(k)})}{\partial x} \right) \bigg|_{x = \bar{r}_{(j)}^{(t)}}$$

and

$$H \triangleq \begin{pmatrix} \zeta_{2,2} & 0 & \cdots & 0 \\ 0 & \zeta_{3,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta_{k,k} \end{pmatrix}.$$

Summarizing the above, we have

$$\Upsilon = \Sigma H^{-1} G$$

which leads to

$$\sum_{i=2}^k \frac{\partial G_i(x, \bar{r}_{(i)}^{(t)})}{\partial x} \bigg|_{x = \bar{r}_{(j)}^{(t)}} = 1 \Leftrightarrow (14).$$

Since there is only one solution to (13) and (14) (see [20]), we have $(\bar{r}_1, \ldots, \bar{r}_k) = (r_{(1)}^*, \ldots, r_{(k)}^*)$, which contradicts the premise that $(r_{(1)}^{(t)}, \ldots, r_{(k)}^{(t)})$ does not converges to $(r_{(1)}^*, \ldots, r_{(k)}^*)$. Therefore, $(r_{(1)}^{(t)}, \ldots, r_{(k)}^{(t)})$ converges to $(r_{(1)}^*, \ldots, r_{(k)}^*)$.

Remark: From [20], if $r_{(1)}^* \gg r_{(j)}^{(t)}$, (13) and (14) are equivalent to the optimal computing budget allocation (OCBA) formula in [8], which is derived under a static optimization framework. Many existing sequential sampling allocation procedures such as KG and expected improvement (EI) (see [44]) cannot achieve the asymptotically optimal sampling ratio. Another AOAP that sequentially achieves the OCBA formula can be found in [38], but was derived from a more tractable surrogate optimization formulation. The novelty of AOAP (12) is that it is derived from an ADP framework and due to its analytical form given in (12), it is computationally more efficient than existing sequential sampling allocation procedures, such as KG, EI, and the AOAP in [38].

B. Generalizations in ADP

We can further extend the certainty equivalent approximating scheme (11) to a VFA that looks $b$ steps ahead into the future by the following recursion: for $1 \leq \ell \leq b - 1$

$$\tilde{V}_{t+b-1}^{(b)}(E_{t}; i_1, \ldots, i_b) \triangleq \tilde{V}_{t+b-1}^{(b)}(E_{t}, E[X_{t_1,t+1} | E_{t}], \ldots, E[X_{t_b,t+b} | E_{t}])$$

$$\tilde{V}_{t+b-\ell}^{(b)}(E_{t}; i_1, \ldots, i_{\ell}) = \max_{i_{\ell+1} = 1, \ldots, k} \tilde{V}_{t+\ell}^{(b)}(E_{t}, i_1, \ldots, i_{\ell+1})$$
and the sampling allocation policy looking \( b \) steps ahead is given by

\[
A_{i+1}^{(\mathcal{E}_i)} = \arg \max_{i_1, \ldots, i_b} \bar{V}_i^{(b)}(\mathcal{E}_i; i_1).
\]

For value function (10)

\[
E[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i] = P(\mu_{(i)} > \mu_{(i)}, i \neq 1|\mathcal{E}_i)
\]

\[
= P\left(\frac{\mu_{(i)} - \mu_{(i)} - (\mu_{(i)} - \mu_{(i)})}{\sqrt{(\sigma_{(i)}^2 + (\sigma_{(i)}^2)^2)}} > \frac{1}{\sqrt{(\sigma_{(i)}^2 + (\sigma_{(i)}^2)^2)}}\right)
\]

\[
= P\left(Z_i > -d_i(\mathcal{E}_i), i \neq 1|\mathcal{E}_i\right)
\]

where \( (Z_1, \ldots, Z_k) \) follows a multivariate normal distribution with mean zero, variances all ones, and correlations given by

\[
\rho_{i,j}(\mathcal{E}_i) = \frac{(\sigma_{(i)}^2)^2}{\sqrt{(\sigma_{(i)}^2 + (\sigma_{(i)}^2)^2)} \sqrt{(\sigma_{(i)}^2 + (\sigma_{(i)}^2)^2)}}
\]

which are called the induced correlations in [35], because they are induced by the variance of \( \mu_{(i)} \). From the above rewriting, we know that the value function is a function of \( d_i, i = 2, \ldots, k \), and \( \rho_{i,j}, i, j \neq 1, i \neq j \).

For AOAP (12), we can see the induced correlations are ignored in VFA (11). From Theorem 3, we know that AOAP (12) sequentially achieves the asymptotically optimal sampling rate of the PCS, which implies the induced correlations are not significant factors for the value function (11) when the simulation budget is large enough. However, Peng et al. [33], [35] showed that the induced correlations are significant factors in a low-confidence scenario that are qualitatively described by three characteristics: The differences between means of competing alternatives are small, the variances are large, and the simulation budget is small. Peng et al. [35] provided an efficient sequential algorithm using an analytical approximation of value function (10) for the low-confidence scenarios. Here, we provide an alternative algorithm using a two-factor VFA and a gradient-based Monte Carlo learning (G-MCL) scheme to fit the weights of two factors.

We first establish the general results of the G-MCL scheme for optimally approximating value function (10) by a parametric family of VFAs (9). Assuming that for \( w \in W \triangleq \{w \in \mathbb{R}^\tau : 0 \leq w_j \leq \bar{w}, j = 1, \ldots, \tau\} \), where \( \bar{w} \) is an arbitrarily large constant

\[
E[\bar{V}^2(\mathcal{E}_i; w)] < \infty
\]

an optimal VFA in (9) can be defined by \( \bar{V}(\mathcal{E}_i; w^*) \), where \( w^* \) is the solution of the following least-squares problem (LSP):

\[
w^* = \arg \min_{w \in W} E\left[(\bar{V}(\mathcal{E}_i; w) - E[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i])^2\right]
\]

with \( \mathcal{E}_i \) being the information set generated by a sampling procedure independent of \( w \), e.g., EA. With the optimal VFA \( \bar{V}(\mathcal{E}_i) = \bar{V}(\mathcal{E}_i; w^*) \), we can use the certainty equivalence approximation to derive a one-step look ahead policy or multistep look ahead extension.

Notice that the objective function in (15) involves an expectation of a nonlinear function of a conditional expectation, which is generally computationally intensive to estimate by Monte Carlo simulation. However, for LSP (15), the optimal solution can be efficiently found by the following stochastic approximation (SA) search algorithm (see [29]) with a single-run gradient estimate (see [17]) as an input in each iteration of the SA algorithm

\[
\hat{w}^{(l+1)} = \Pi_W \left(\hat{w}^{(l)} + \lambda_l D(\mathcal{E}_{i}^{(l)}; \hat{w}^{(l)})\right)
\]

where

\[
D(\mathcal{E}_{i}^{(l)}; \hat{w}^{(l)}) \triangleq \left(\bar{V}(\mathcal{E}_{i}^{(l)}; \hat{w}^{(l)}) - \mathbb{E}[\bar{S}(\mathcal{E}_{i}^{(l)}) = \langle 1 \rangle]\right) \nabla_w \bar{V}(\mathcal{E}_{i}^{(l)}; w)|_{w = \hat{w}^{(l)}}
\]

\( \mathcal{E}_{i}^{(l)} \) is the \( l \)th independent realization of the information set \( \mathcal{E}_i \), and \( \Pi_W(\cdot) \) is an operator that projects the argument onto the compact feasible set \( W \). Define

\[
J(w) \triangleq E\left[(\bar{V}(\mathcal{E}_i; w) - \mathbb{1}\{\mathcal{S}(\mathcal{E}_i) = \langle 1 \rangle\})^2\right].
\]

To justify the convergence of SA (16), a set of regularity conditions can be introduced as follows.

1. \( J(w) \) is convex on \( W \).
2. For any \( w \in W \), the gradient estimator \( D(\mathcal{E}_i; w) \) is unbiased and has bounded second moment, i.e.,

\[
\nabla_w J(w)/2 = E[D(\mathcal{E}_i; w)], \quad E[D^2(\mathcal{E}_i; w)] < \infty.
\]

3. The step size sequence \( \{\lambda_l\} \) satisfies the condition:

\[
\sum_{l=1}^{\infty} \lambda_l = \infty \quad \text{and} \quad \sum_{l=1}^{\infty} \lambda_l^2 < \infty.
\]

**Theorem 4:** Under conditions (1)–(3)

\[
\lim_{l \to \infty} \hat{w}^{(l)} = w^*, \quad a.s.
\]

**Proof:** The objective function of LSP (15) is

\[
E\left[\bar{V}(\mathcal{E}_i; w) - E[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i]\right]^2
\]

\[
= E\left[\bar{V}^2(\mathcal{E}_i; w) - 2\bar{V}(\mathcal{E}_i; w)E[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i] + E^2[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i]\right].
\]

Since the last term is a constant independent of \( w \), the solution of LSP (15) is the same as the solution of the following optimization problem:

\[
w^* = \arg \min_{w \in W} E\left[\bar{V}^2(\mathcal{E}_i; w) - 2\bar{V}(\mathcal{E}_i; w)E[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i] + E[V_P(\theta; \langle 1 \rangle)|\mathcal{E}_i]\right].
\]
The objective function of the above optimization problem can be rewritten as
\[
\mathbb{E} \left[ \hat{V}^2(\xi_t; w) - 2\hat{V}(\xi_t; w)\mathbb{E}[V_r(\theta; \langle 1 \rangle_t) | \xi_t] + \mathbb{E}[V_r(\theta; \langle 1 \rangle_t) | \xi_t] \right]
= \mathbb{E} \left[ \hat{V}^2(\xi_t; w) - 2\hat{V}(\xi_t; w) \mathbb{1}\{\hat{S}(\xi_t) = \langle 1 \rangle \} | \xi_t] \right]
+ \mathbb{E} [\mathbb{1}\{\hat{S}(\xi_t) = \langle 1 \rangle \} | \xi_t]
= \mathbb{E} \left[ \left( \hat{V}(\xi_t; w) - 1 \{\hat{S}(\xi_t) = \langle 1 \rangle \} \right)^2 \right] | \xi_t]
= \mathbb{E} \left[ \left( \hat{V}(\xi_t; w) - 1 \{\hat{S}(\xi_t) = \langle 1 \rangle \} \right)^2 \right]
\]
where the first equality is by definition, the second equality is because \(\hat{V}(\xi_t; w)\) is \(\xi_t\)-measurable, and the third equality is due to the law of total expectation. Therefore,
\[
w^* = \arg \min_{w \in W} J(w).
\]
With conditions (1)–(3), the conclusion of the theorem can be proved using standard convergence results of SA (see [29]).

Remark: If \(J(w)\) is not convex, SA (16) might converge to a local minimum. For a linear VFA, we can prove convexity, thus guaranteeing the convergence to the global optimum. Specifically, for a linear VFA, \(K(z) = z\), and assuming the gradient and expectation can be interchanged, which is usually justified by the dominated convergence theorem (see [43])
\[
\nabla_w^2 J(w) = \mathbb{E} \left[ \nabla_w^2 \left( \hat{V}(\xi_t; w) - 1 \{\hat{S}(\xi_t) = \langle 1 \rangle \} \right)^2 \right]
= \mathbb{E} \left[ \nabla_w^2 \left( \sum_{j=1}^{\tau} w_j g_j(\xi_t) - 1 \{\hat{S}(\xi_t) = \langle 1 \rangle \} \right)^2 \right]
= 2\mathbb{E} [g^2(\xi_t) g(\xi_t)]
\]
\]
where \(g(\xi_t) \triangleq (g_1(\xi_t), \ldots, g_\tau(\xi_t))\). It is easy to show \(\nabla_w^2 J(w)\) is positive semidefinite and is positive definite for \(w \in W\) if \(g_j(\xi_t) \geq 0, j = 1, \ldots, \tau\), and are not identically zero.

Then, we propose the following two-factor parametric family of VFAs:
\[
\hat{V}(\xi_t; w) = K_w g_1(\xi_t) + w_2 g_2(\xi_t)
\]
where \(w_1, w_2 \geq 0, w = (w_1, w_2), g_1(\xi_t) \triangleq d^2(\xi_t), \) and
\[
g_2(\xi_t) \triangleq \min_{i,j \neq i, j} p_{i,j}(\xi_t).
\]
The feature \(g_1(\xi_t)\) reflects the mean-variance tradeoff, and \(g_2(\xi_t)\) is a feature including information on the induced correlations. From [33], we know that posterior IPCS (10) is increasing with respect to the values of both \(g_1(\xi_t)\) and \(g_2(\xi_t)\), thus, \(w_1, w_2 \geq 0\) is assumed. Notice that as \(t\) goes to infinity, \(g_1(\xi_t)\) goes to infinity at rate \(O(t)\) whereas \(g_2(\xi_t)\) converges to a constant; thus, the effect of the mean-variance tradeoff reflected in \(g_1(\xi_t)\) will be more and more significant as \(t\) grows. If activation function \(K(\cdot)\) is monotonically increasing, the VFA of AOAP (12) is equivalent to a special case of the AOAP based on the two-factor VFA when \(w_2 = 0\). Besides the linear VFA, we can also choose some nonlinear VFAs such as \(K(z) = 1 - \exp(-z)\).

VI. NUMERICAL RESULTS

We test the proposed AOAPs in Section V for the normal sampling distributions in the low-confidence and high-confidence scenarios. A numerical example to illustrate how to calculate the optimal A&S policy for a simple discrete sampling distribution example can be found in the online appendix (see [37]). The priors for the unknown means of the normal sampling distributions are assumed to be the normal conjugate priors introduced in Section III-B.

Example 1: A High-Confidence Scenario.

In this example, we test the numerical performance of AOAP (12) with ten competing alternatives in a high-confidence scenario with hyper-parameters given by \(\mu_i(0) = 0\), and \(\sigma_i(0) = 1\), and the true variances given by \(\sigma_i = 1, i = 1, \ldots, 10\). The mean and standard deviation of the true mean are
\[
\mathbb{E} [\mu_i] = \mu_i(0) = 0, \quad \text{Std} (\mu_i) = \sigma_i(0) = 1, \quad i = 1, \ldots, 10.
\]
The standard deviation of the true mean controls the dispersion of the true mean following the prior distribution. Statistically speaking, the differences in the true means of competing alternatives would be relatively large, compared with the sampling variances for simulation budget size up to \(T = 400\) in this example, so this example is categorized as a high-confidence scenario. The first 100 replications are equally allocated to each alternative to estimate the sampling variances \(\sigma_i^2, i = 1, \ldots, 10\).

We compare the performance of AOAP (12) with the “most starving” sequential OCBA algorithm in [7], the KG algorithm in [16], and EA. The selection policy is fixed as \(\hat{S}(\xi_t) = \langle 1 \rangle_t\), which selects the alternative with the largest posterior mean. The performance of each sampling procedure is measured by the IPCSs following the sampling procedure to allocate a fixed amount of simulation budget, i.e.,
\[
\text{IPCS} \triangleq \mathbb{E} \left[ \mathbb{1}\{\hat{S}(\xi_t) = \langle 1 \rangle \} \right].
\]
The IPCSs are reported in Fig. 6 as functions of the simulation budget \(t\) up to \(T = 400\). The statistics are estimated from 10⁵ independent macrosimulations.

From Fig. 6, we can see that the three sequential policies have comparable performance that is significantly better than EA. Although OCBA and KG are derived from surrogate optimization problems, the numerical result indicates that for this high-confidence scenario, their performances are quite close to AOAP (12). AOAP, OCBA, KG, and EA take around 1.1, 0.8, 7, and 0.1 s, respectively, to allocate 400 replications. Numerical comparison between AOAP, OCBA, KG, and EA in the presence of correlation can be found in the online appendix (see [37]).

Example 2: Two Low-Confidence Scenarios.

We first test the numerical performance of an AOAP with a two-factor linear VFA. In this example, there are ten competing alternatives in a low-confidence scenario with hyper-parameters
in the prior given by \( \mu_i^{(0)} = 0, \sigma_i^{(0)} = 0.01, i = 1, \ldots, 10 \), and the true variances given by \( \sigma_i = 1, i = 1, \ldots, 10 \). Statistically speaking, the differences in the true means of competing alternatives would be relatively small, compared with the sampling variances for simulation budget size up to \( T = 200 \) in this example, so this example is categorized as a low-confidence scenario.

We use the two-factor linear VFA proposed in Section V-B to take the information of the induced correlation into account. The numerical results for a two-factor nonlinear VFA can be found in the online appendix (see [37]). The weights of the two factors are fitted by the G-MCL scheme. In principle, the weights should be fitted for every step up to \( T = 200 \), because the SCP for R&S is a nonstationary MDP. However, for computational simplicity, we only fit the weights at the final step \( T \), and use the same fitted weights throughout all steps of the allocation decisions. The sampling algorithm generating \( \mathcal{E}_T \) in G-MCL is EA. The step size and the starting point of SA is chosen as \( \lambda_1 = 10 \times \ell^{-\frac{2}{3}} \) and \((w_1^{(0)}, w_2^{(0)}) = (1, 1)\). The trajectory of SA is shown in the online appendix (see [37]), and the final fitting results are \( w_1^* \approx 0.98 \) and \( w_2^* \approx 0.42 \). The rest of the sampling allocation experiment is designed the same as the last example.

From Fig. 7, we can see the IPCS of OCBA decreases as the simulation budget grows, whereas the IPCSs of KG and EA increase with the simulation budget at a slow pace, and the former has a slight edge over the latter; in contrast, the IPCSs of the proposed AOAP using a two-factor VFA increases at a fast rate and is significant larger than the IPCSs obtained by the other methods. This phenomenon is due to the fact that the information of the induced correlations, which is significant in the low-confidence scenario, is ignored or not taken fully into account by OCBA and KG. A detailed theoretical analysis on the property of the PCS and more numerical results on the performances of many existing methods in the low-confidence scenario can be found in [33].

We also provide a scenario that lies between Example 1 and the last example. The hyper-parameters in the prior are set by \( \mu_i^{(0)} = 0, \sigma_i^{(0)} = 0.08, i = 1, \ldots, 10 \), and \( \sigma_i = 1, i = 1, \ldots, 10 \). The true variances are \( \sigma_i = 1, i = 1, \ldots, 10 \). We can see the dispersion of the true means is larger than the last example but much smaller than Example 1. We still use an AOAP with a two-factor linear VFA fitted by the G-MCL scheme. The initialization of the experiment is set the same as the last example. The final fitting results are \( w_1^* \approx 0.22 \) and \( w_2^* \approx 0.53 \). From Fig. 8, we can see that the IPCS of OCBA decreases slightly at the beginning, and then quickly catches up with the IPCS of EA and surpasses the latter at the end. KG has a significant edge over OCBA and EA, but lags behind AOAP.
VII. CONCLUSION

We propose a SCP to formulate the sequential A&S decision for the Bayesian framework of R&S and derive the associated Bellman equation. We further analyze the optimal selection policy and the computational complexity of the optimal A&S policy for discrete sampling and prior distributions. We especially focus on developing efficient techniques to approximate the optimal A&S policy for the posterior IPCS of independent normal sampling distributions. An AOAP using a single feature of the posterior IPCS is proved to be asymptotically optimal. We propose a general G-MCL scheme to optimally fit the posterior IPCS by VFA. A generalized AOAP using a two-factor VFA with the G-MCL scheme can achieve a significant efficiency enhancement in the low-confidence scenario.

The establishment of the Bellman equation of the SCP in this paper relies on the independence of the replications. In practice, the independence assumption might not be always satisfied, e.g., $k$ machines following stationary Markov processes. How to better formulate an SCP for nonindependent replications in the Bayesian framework of R&S is an interesting theoretical question for future research.

Another direction for future research is to incorporate more features to better describe the value function of the PCS payoff under independent normal sampling distributions or even correlated normal sampling distributions. It would be worthwhile to consider efficient approximation schemes for other rewards such as PCS and EOC of subset selection (see [4] and [19]) and the optimal quantile selection (see [36]). How to develop fast learning schemes for nonstationary MDPs in R&S deserves further study. Utilizing the cloud computing platform to solve R&S is also a future research (see [46]).

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