Rationalized evaluation subgroups of mapping spaces between complex Grassmannians

Paul Antony Otieno¹ · Jean Baptiste Gatsinzi² · Vitalis Onyango-Otieno¹

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Abstract
We determine evaluation subgroups of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + 1)$ between complex Grassmannians.

Keywords Evaluation subgroups · Gottlieb group · $G$-sequence

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1 Introduction

Given a pointed topological space $(X, x_0)$, the $n$th Gottlieb group of $X$, also called the evaluation subgroup of $\pi_n(X)$ and denoted by $G_n(X)$, consists of those $\alpha \in \pi_n(X)$ for which there is a map $F : X \times S^n \rightarrow X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X \times S^n & \xrightarrow{F} & X \\
\uparrow f & & \uparrow \nabla \\
X \vee S^n & \xrightarrow{1_x \vee f} & X \vee X,
\end{array}
$$

where $f : S^n \rightarrow X$ is a representative of $\alpha$ and $\nabla$ is the folding map. Thus for every $\alpha \in G_n(X, x_0)$, there exists at least one map $F : X \times S^n \rightarrow X$ such that $F(x_0, s) = f(s)$. We say that $F$ is an affiliated map to $\alpha$ [3]. If $X$ has a base point $x_0$ and $aut X$ denotes the monoid of self homotopy equivalences of $X$ with $ev : aut X \rightarrow X$ the evaluation map at

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Paul Antony Otieno
paotieno@strathmore.edu

Jean Baptiste Gatsinzi
gatsinzij@biust.ac.bw

Vitalis Onyango-Otieno
vonyango@strathmore.edu

1 Strathmore Institute of Mathematical Sciences, Strathmore University, Box 59857, Nairobi, Kenya

2 Department of Mathematics, Botswana International University of Science and Technology, Private Bag 16, Palapye, Botswana
$x_0$, then it follows from the definition that
\[
G_n(X) = \text{im} \left( ev_n : \pi_n(\text{aut } X, 1_X) \rightarrow \pi_n(X, x_0) \right).
\]
Similarly, if $f : X \rightarrow Y$ is a based map between simply connected CW complexes and \(\text{map}(X, Y; f)\), the space of maps from $X$ to $Y$ which are homotopic to $f$, then $G_n(Y, X; f) = \text{im}(ev_n : \text{map}(X, Y; f) \rightarrow \pi_n(Y))$ is the $n$th evaluation subgroup of $f$ [9]. In [10], Woo and Lee defined relative evaluation subgroups $G^r_n(X, Y; f)$ and showed that they fit in a sequence
\[
\cdots \rightarrow G^r_{n+1}(Y, X; f) \rightarrow G_n(X) \rightarrow G_n(Y, X; f) \rightarrow \cdots
\]
called the $G$-sequence of $f$.

We use Sullivan models to compute rational relative Gottlieb groups of the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + 1)$. We refer to [4] for details and work over a field of characteristic zero in this case $\mathbb{Q}$.

**Definition 1** A differential graded algebra (dga) is a graded algebra $A = \bigoplus_{n \geq 0} A^n$ together with a derivation $d, d = d_n : A^n \rightarrow A^{n+1}$ such that $d \circ d = 0$. Then $(A, d)$ is called a cochain algebra. A graded algebra $A$ is commutative if $a \cdot b = (-1)^{\deg a \cdot \deg b} b \cdot a$ for $a, b \in A$ [2, Chap. 3].

**Definition 2** A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$ where $V = \{V^p\}_{p \geq 2}$ and $\wedge V$ denotes the graded free commutative algebra on $V$. A Sullivan model for a commutative cochain algebra $(A, d)$ is a quasi-isomorphism $m : (\wedge V, d) \rightarrow (A, d)$ from a Sullivan algebra $(\wedge V, d)$. A Sullivan algebra is said to be minimal if the differential is decomposable, that is, $\text{im } d \subset \wedge^+ V \cdot \wedge^+ V$. Moreover, if $H^0(A) = \mathbb{Q}$ then $(A, d)$ has a minimal model which is unique up to isomorphism. If $X$ is a nilpotent space and $A_{PL}(X)$ the commutative differential graded algebra (cdgga) of piecewise linear forms on $X$, then a Sullivan model of $X$ is a Sullivan model of $A_{PL}(X)$ [2, Chap. 12].

## 2 Derivation spaces and the rationalized $G$-sequence

Given commutative differential graded algebras $(A, d_A)$ and $(B, d_B)$ and a map $\phi : A \rightarrow B$, define a $\phi$-derivation of degree $n$ to be a linear map $\theta : A^n \rightarrow B^{n-n}$ which satisfies $\theta(xy) = \theta(x)\phi(y) + (-1)^{|x|}\phi(x)\theta(y)$. We only consider derivations of positive degree. Let $\text{Der}_n(A, B; \phi)$ denote the vector space of all $\phi$-derivations of degree $n$ for $n > 0$. Define a linear map $D : \text{Der}_n(A, B; \phi) \rightarrow \text{Der}_{n-1}(A, B; \phi) \oplus D(\theta) = d_B \circ \theta - (-1)^{|\theta|}\theta \circ d_A$.

Then, $(\text{Der}_*(A, B; \phi), D)$ is a chain complex. In case $A = B$ and $\phi = 1_B$, the chain complex of derivations $\text{Der}_*(B, B; 1)$ is just the usual complex of derivations on the commutative differential graded algebra $B$ [6]. If $\phi : (\wedge V, d) \rightarrow (\wedge W, d)$ is a Sullivan minimal model of $f : X \rightarrow Y$, then $H_n(\text{Der}(\wedge V, \wedge W; \phi), D) \cong \pi_n(\text{map}(X, Y; f)) \otimes \mathbb{Q}$; $n \geq 2$ [6], [1]. We note that $\text{Der}(\wedge V, B; \phi) \cong \text{Hom}(V, B)$. If $\{v_i\}$ is a basis of $V$, we denote by $(v_i, b)$, the unique $\phi$-derivation $\theta$ such that
\[
\begin{cases}
\theta(v_i) = b_i & b_i \in B, \\
\theta(v_j) = 0 & i \neq j.
\end{cases}
\]
Pre-composition with $\phi$, respectively post-composition with the augmentation $\varepsilon : B \rightarrow \mathbb{Q}$, gives a map of chain complexes
\[
\phi^* : \text{Der}_*(B, B; 1) \rightarrow \text{Der}_*(A, B; \phi),
\]
respectively
\[ \varepsilon_\ast : \text{Der}_n(A, B; \phi) \longrightarrow \text{Der}_n(A, \mathbb{Q}; \varepsilon). \]

**Definition 3** Let \( \phi : V \longrightarrow W \) be a map of differential graded vector spaces. Define a differential graded vector space, \( \text{Rel}_n(\phi) \), called the mapping cone as follows. \( \text{Rel}_n(\phi) = sV_{n-1} \oplus W_n \) with differential \( \delta \) of degree \(-1\) given by \( \delta(sv, w) = (-sd_V(v), \phi(v) + d_W(w)) \) [7]. There are chain maps \( J : W_n \longrightarrow \text{Rel}_n(\phi) \) and \( P : \text{Rel}_n(\phi) \longrightarrow V_{n-1} \) defined by \( J(w) = (0, w) \) and \( P(sv, w) = v \). These give a short exact sequence of chain complexes
\[
0 \longrightarrow W_n \xrightarrow{J} \text{Rel}_n(\phi) \xrightarrow{P} V_{n-1} \longrightarrow 0,
\]
which leads to a long exact sequence in homology
\[
\cdots \longrightarrow H_{n+1}(\text{Rel}(\phi)) \xrightarrow{H(P)} H_n(V) \xrightarrow{H(\phi)} H_n(W) \xrightarrow{H(J)} H_n(\text{Rel}(\phi)) \longrightarrow \cdots,
\]
whose connecting homomorphism is \( H(\phi) \). We refer to this sequence as the long exact homology sequence of the map
\[ \phi^* : \text{Der}_n(\bigwedge W, \bigwedge W; 1) \longrightarrow \text{Der}_n(\bigwedge V, \bigwedge V; \phi) \]
induced by the minimal model \( \phi : (\bigwedge V, d) \longrightarrow (\bigwedge W, d) \) of the map \( f : X \longrightarrow Y \).

**Definition 4** Given a commutative differential graded algebra map \( \phi : A \longrightarrow B \), we have the following commutative diagram of differential graded vector spaces;
\[
\begin{array}{ccc}
\text{Der}_n(B, B; 1) & \xrightarrow{\phi^*} & \text{Der}_n(A, B; \phi) \\
\downarrow_{\varepsilon_n} & & \downarrow_{\varepsilon_n} \\
\text{Der}_n(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\phi^*} & \text{Der}_n(A, \mathbb{Q}; \varepsilon).
\end{array}
\]
Here, \( \varepsilon \) denotes the augmentation of either \( A \) or \( B \). On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder \((n \geq 2)\),
\[
\begin{array}{ccccc}
\cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_n(\text{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_n(\text{Der}(A, B; \phi)) & \cdots \\
\downarrow_{H(\varepsilon_n, \varepsilon_n)} & & \downarrow_{H(\varepsilon_n)} & & \downarrow_{H(\varepsilon_n)} & & \downarrow_{H(\varepsilon_n)} \\
\cdots & \xrightarrow{H(J)} & H_{n+1}(\text{Rel}(\phi^*)) & \xrightarrow{H(P)} & H_n(\text{Der}(B, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\phi^*)} & H_n(\text{Der}(A, \mathbb{Q}; \varepsilon)) & \cdots.
\end{array}
\]

**Definition 5** Suppose \( \phi : A \longrightarrow B \) is a map of commutative differential graded algebras, we define the evaluation subgroups of \( \phi \) by
\[
G_n(A, B; \phi) = \text{im\{ } H(\varepsilon_n) : H_n(\text{Der}(A, B; \phi)) \longrightarrow H_n(\text{Der}(A, \mathbb{Q}; \varepsilon)).\}
\]
In the special case \( A = B \) and \( \phi = 1_B \), we refer to the Gottlieb group of \( B \), and use the notation \( G_n(B) \). If \( B = (\bigwedge V, d) \) is a model of a simply connected space \( X \), then an element
\( \alpha \in G_n(B) \) is represented by a linear mapping \( f : V^n \rightarrow \mathbb{Q} \) that extends into a derivation \( \theta \) of \( \wedge V \) such that \( \theta \theta = 0 \). Moreover, \( G_n(B) \cong G_n(X_{\mathbb{Q}}) [2, \text{Proposition 29.8}] \).

The \( n \)th relative evaluation subgroup of \( \phi \) is defined by;

\[
G^{	ext{rel}}_n(A, B; \phi) = \text{im}\{H(\varepsilon_\ast, \varepsilon_\ast) : H_n(\text{Rel}(\phi^{\ast})) \rightarrow H_n(\text{Rel}(\hat{\phi}^{\ast}))\}.
\]

Then the image of the upper long sequence in the lower, of the ladder above, gives a sequence

\[
\cdots \rightarrow H(\hat{J}) \rightarrow G^{	ext{rel}}_n(A, B; \phi) \xrightarrow{H(\hat{P})} G_n(B) \xrightarrow{H(\hat{\phi}^{\ast})} G_n(A, B; \phi) \xrightarrow{H(\hat{J})} \cdots
\]

that terminates in \( G_2(A, B; \phi) \).

We refer to this sequence as the \( G \)-sequence of the map \( \phi : A \rightarrow B \). This can be applied to the minimal model \( \phi : (\wedge V, d) \rightarrow (\wedge W, d) \) of the map \( f : X \rightarrow Y \) as stated and proved in [6, Theorem 3.5].

### 3 The inclusion \( Gr(2, n) \hookrightarrow Gr(2, n + 1) \)

Let \( Gr(k, n) \) be the Grassmann manifold of \( k \)-dimensional subspaces of \( \mathbb{C}^n \). The cohomology ring \( H^\ast(Gr(k, n), \mathbb{Q}) \) is generated by the Chern classes \( c_i \in H^{2i}(Gr(k, n), \mathbb{Q}) \), for \( 1 \leq i \leq k \). Further, the cohomology ring has a presentation

\[
H^\ast(Gr(k, n), \mathbb{Q}) = \wedge(c_1, c_2, \ldots, c_k)/(h_{n-k+1}, \ldots, h_n),
\]

as the quotient of the polynomial ring generated by \( c_1, c_2, \ldots, c_k \), \( |c_i| = 2i \), modulo the ideal generated by the elements \( h_j, n-k+1 \leq j \leq n \). Here, \( h_j \) is defined as the \( 2j \)th degree term in the Taylor’s expansion of \( (1 + c_1 + c_2 + c_3 + \cdots + c_k)^{-1} \) where \( (1 + c_1 + c_2 + c_3 + \cdots + c_k) \) is the total Chern class [4].

In particular, the cohomology rings of \( Gr(2, n) \) and \( Gr(2, n + 1) \) are:

\[
H^\ast(Gr(2, n), \mathbb{Q}) = \wedge(y_2, y_4)/(h_{n-1}, h_n) \quad \text{and} \quad H^\ast(Gr(2, n + 1), \mathbb{Q}) = \wedge(x_2, x_4)/(h_n, h_{n+1}) \quad \text{respectively.}
\]

The minimal model of \( Gr(2, n) \) is \( (\wedge(y_2, y_4, y_{2n-3}, y_{2n+1}), d) \) with \( d(y_2) = d(y_4) = 0 \), \( d(y_{2n-3}) = h_{n-1}, d(y_{2n+1}) = h_n \). In the same way, a model of \( Gr(2, n + 1) \) is given by \( (\wedge(x_2, x_4, x_{2n-3}, x_{2n+1}), d) \) with \( dx_2 = dx_4 = 0, dx_{2n-1} = h_n \) and \( dx_{2n+1} = h_{n+1} \).

**Lemma 1** \( h_{n+1} = -x_2 h_n - x_4 h_{n-1} \).

**Proof** Write the Taylor series \( 1 + x_2 + x_4)^{-1} = 1 + h_1 + h_2 + \cdots \) where, \( |h_i| = 2i \). From \((1 + x_2 + x_4)(1 + x_2 + x_4)^{-1} = 1\), one gets the relation \( h_{n+1} = -x_2 h_n - x_4 h_{n-1} \) \( \square \)

In particular, \( h_{n+1} \) is co-boundary in \( (\wedge(x_2, x_4, x_{2n-3}, x_{2n-1}), d) \), that is, there exists \( \alpha \) of degree \( 2n + 1 \) such that \( d\alpha = h_{n+1} \).

**Theorem 1** Let \( B = (\wedge(y_2, y_4, y_{2n-3}, y_{2n-1}), d) \). Then \( G_n(B) = \langle [y^\ast_{2n-3}], y^\ast_{2n-1} \rangle \).

**Proof** Let \( \alpha_{2n-3} = (y_{2n-3}, 1) \) and \( \alpha_{2n-3} = (y_{2n-3}, 1) \). Then \( \delta \alpha_{n-1} = \delta \alpha_{2n-3} = 0 \). Moreover, \( \alpha_{2n-3} \) and \( \alpha_{2n-1} \) can not be boundaries for degree reason. Therefore, \( [\alpha_{2n-3}] \) and \( [\alpha_{2n-1}] \) are non zero homology classes in \( H_n(Der(B, B; 1)) \). Further, \( \varepsilon_\ast(\alpha_{2n-3}) = y^\ast_{2n-3} \) and \( \varepsilon_\ast(\alpha_{2n-1}) = y^\ast_{2n-1} \).

As \( Gr(2, n) \) is a finite CW-complex then \( G_{even}(B) = 0 [2, \text{Pg.379}] \). Hence, \( G_n(B) = \langle [y^\ast_{2n-3}], [y^\ast_{2n-1}] \rangle \).

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The inclusion $Gr(2, n) \hookrightarrow Gr(2, n + 1)$ has a model of the form

$$\phi : \wedge V = \left(\wedge (x_2, x_4, x_{2n-1}, x_{2n+1}), d\right) \rightarrow \left(\wedge (y_2, y_4, y_{2n-3}, y_{2n-1}), d\right) = B$$

where $\phi(x_2) = y_2, \phi(x_4) = y_4, \phi(x_{2n-1}) = y_{2n-1}$ and $\phi(x_{2n+1}) = \alpha$ where $d\alpha = h_{n+1}$ by Lemma 1.

**Theorem 2** Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + 1)$ and $\phi : (\wedge V, d) \rightarrow (B, d)$ its Sullivan model, then $G_*(\wedge V, B; \phi) \cong \langle [x_{2n-1}^*, [x_{2n+1}^*]\rangle$.

**Proof** As $Gr(2, n)$ is formal, $Der(\wedge V, B; \phi) \xrightarrow{\cong} Der(\wedge V, H(B); f \circ \phi)$ where $f : B \xrightarrow{\cong} H(B)$ is a quasi isomorphism. Similarly, since $B$ is formal $Der(B, B; 1) \cong Der(B, H(B); f)$. Define $\theta_{2n-1} = (x_{2n-1}, 1), \theta_{2n+1} = (x_{2n+1}, 1)$ in $Der(\wedge V, H(B); f \circ \phi)$, and $\delta\theta_{2n-1} = \delta\theta_{2n+1} = 0$. Moreover, $\theta_{2n-1}$ and $\theta_{2n+1}$ are nonzero cohomology classes in $H_*(Der(\wedge V, H^*(B); f \circ \phi))$.

We note that, $\theta_2 = (x_2, 1)$ and $\theta_4 = (x_4, 1)$ are not cycles in $Der(\wedge V, H(B); f \circ \phi)$ [8]. Further, $H_*(\theta_{2n-1}) = [x_{2n-1}^*] \in G_{2n-1}(\wedge V, B; f \circ \phi)$. In a similar way, $H_*(\theta_{2n+1}) = [x_{2n+1}^*] \in G_{2n+1}(\wedge V, B; f \circ \phi)$. It then follows that $G_*(\wedge V, B; f \circ \phi) = \langle [x_{2n-1}^*], [x_{2n+1}^*]\rangle$. \hfill \Box

**Theorem 3** Consider the inclusion $Gr(2, n) \hookrightarrow Gr(2, n + 1)$ and

$$\phi : (\wedge V, d) = \left(\wedge (x_2, x_4, x_{2n-1}, x_{2n+1}), d\right) \rightarrow \left(\wedge V(y_2, y_4, y_{2n-3}, y_{2n-1}), d\right) = B$$

its Sullivan model, then $G_*^{rel}(\wedge V, B; \phi) = \langle ([x_{2n-3}^*, 0]), ([0, y_{2n+1}^*])\rangle$.

**Proof** Consider the diagram below [6].

$$
\begin{array}{ccc}
Der(B, H(B); f) & \xrightarrow{\phi^*} & Der(\wedge V, H(B); f \circ \phi) \\
\downarrow{\varepsilon_*} & & \downarrow{\varepsilon_*} \\
Der(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\hat{\phi}^*} & Der(\wedge V, \mathbb{Q}; \varepsilon) \\
& \downarrow{\varepsilon_*} & \downarrow{\varepsilon_*} \\
& \xrightarrow{\hat{\phi}^*} & Rel(\phi^*) \\
& & \downarrow{\varepsilon_*} \\
& & Rel(\phi^*)
\end{array}
$$

Let $\alpha_{2n-1} = (y_{2n-1}, 1), \alpha_{2n-3} = (y_{2n-3}, 1) \in Der(B, H(B); f)$ and $\theta_{2n-1}, \theta_{2n+1} \in Der(\wedge V, H^*(B); \phi)$ as defined above. Then $\phi^*(\alpha_{2n-1}) = \theta_{2n-1}$ and $\phi^*(\alpha_{2n-3}) = 0$.

Further, $D(\alpha_{2n-1}, 0) = (0, \theta_{2n-1}), D(\alpha_{2n-3}, 0) = (0, 0)$ and $D(0, \theta_{2n-1}, 0) = D(0, \theta_{2n+1})$. Therefore, $\langle [\alpha_{2n-3}, 0] \rangle$ and $\langle (0, \theta_{2n+1}) \rangle$ are non zero homology classes in $H_*(Rel(\phi^*))$. Moreover, $H_*(\varepsilon_*, \varepsilon_*)([\alpha_{2n-3}, 0]) = ([x_{2n-3}^*, 0])$ and $H_*(\varepsilon_*, \varepsilon_*)([0, \theta_{2n+1}]) = ([0, y_{2n+1}])$. A straightforward computation shows that $\langle [x_{2n-3}^*, 0] \rangle$ and $\langle (0, y_{2n+1}^*) \rangle$ span $H_*(\varepsilon_*, \varepsilon_*)$. \hfill \Box
The $G$-sequence reduces to

\[
0 \longrightarrow G_{2n+1}(\wedge V, B; \phi) \xrightarrow{H(J)} G_{2n+1}^{rel}(\wedge V, B; \phi) \longrightarrow 0 \cdots
\]

\[
\cdots 0 \longrightarrow G_{2n-1}(B) \xrightarrow{H(\phi^s)} G_{2n-1}(\wedge V, B; \phi) \longrightarrow 0 \cdots
\]

\[
\cdots 0 \longrightarrow G_{2n-2}^{rel}(\wedge V, B; \phi) \xrightarrow{H(P)} G_{2n-3}(B) \longrightarrow 0.
\]

and is exact.

**Example 1** Consider $Gr(2, 4) \rightarrow Gr(2, 5)$. A model of the inclusion is given by

\[\phi : \wedge V = \left( \wedge (x_2, x_4, x_7, x_9), d \right) \longrightarrow \left( \wedge (y_2, y_4, y_5, y_7), d \right) = B,\]

where $dx_2 = dx_4 = 0$, $dx_7 = x_2^2 - 3x_2^2x_4 + x_4^2$, $dx_9 = 4x_2^3x_4 - 3x_2x_4^2 - x_5^2$ $dy_2 = dy_4 = 0$, $dy_5 = 2y_2y_4 - y_2^3$ and $dy_7 = y_4^2 - 3y_2y_4 + y_4^2$.

Moreover, $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_7) = y_7$ and $\phi(x_9) = -y_2y_7 - y_4y_4$.

We compute $G_{s}^{rel}(\wedge V, B; \phi)$. Let $\alpha_7 = (y_7, 1)$, $\alpha_5 = (y_5, 1) \in \text{Der}_s(B, H(B); f)$ where $f : B \cong H(B)$ and $\theta_7 = (x_7, 1)$, $\theta_9 = (x_9, 1) \in \text{Der}(\wedge V, H^s(B); f \circ \phi)$ then $\phi^s(\alpha_7) = \theta_7$ and $\phi^s(\alpha_5) = 0$. Moreover, $D(sa\alpha_7, 0) = (0, \theta_7)$, $D(sa\alpha_5, 0) = (0, 0)$ and $D(0, \theta_9) = (0, 0)$. Hence $[(s\alpha_5, 0)]$ and $[(0, \theta_9)]$ are non zero homology classes. Moreover, $(\epsilon_5, \epsilon_9)(s\alpha_5, 0) = (s\gamma_5^s, 0)$, $(\epsilon_5, \epsilon_9)(0, \theta_9) = (0, y_9^s)$. Therefore

\[G_{s}^{rel}(\wedge V, B; \phi) = \left\{ [(0, x_9^s)], [(s\gamma_5^s, 0)] \right\} .\]

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