REDUCIBILITY OF THE POLYNOMIAL REPRESENTATION OF THE DEGENERATE DOUBLE AFFINE HECKE ALGEBRA

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1. INTRODUCTION

In this note we determine the values of parameters $c$ for which the polynomial representation of the degenerate double affine Hecke algebra (DAHA), i.e. the trigonometric Cherednik algebra, is reducible. Namely, we show that $c$ is a reducibility point for the polynomial representation of the trigonometric Cherednik algebra for a root system $R$ if and only if it is a reducibility point for the rational Cherednik algebra for the Weyl group of some root subsystem $R' \subset R$ of the same rank; such subsystems for any $R$ are given by the well known Borel-de Siebenthal algorithm.

This generalizes to the trigonometric case the result of [DJ2], where the reducibility points are found for the rational Cherednik algebra. Together with the result of [DJ2], our result gives an explicit list of reducibility points in the trigonometric case.

We emphasize that our result is contained in the recent work of I. Cherednik [Ch2], where reducibility points are determined for nondegenerate DAHA. Namely, the techniques of [Ch2], based on intertwiners, work equally well in the degenerate case. In fact, outside of roots of unity, the questions of reducibility of the polynomial representation for the degenerate and nondegenerate DAHA are equivalent, and thus our result is equivalent to that of [Ch2]. However, our proof is quite different from that in [Ch2]; it is based on the geometric approach to Cherednik algebras developed in [E2], and thus clarifies the results of [Ch2] from a geometric point of view. In particular, we explain that our result and its proof can be generalized to the much more general setting of Cherednik algebras for any smooth variety with a group action.

We note that in the non-simply laced case, it is not true that the reducibility points for $R$ are the same in the trigonometric and rational settings. In the trigonometric setting, one gets additional reducibility points, which arise for type $B_n$, $n \geq 3$, $F_4$, and $G_2$, but not for $C_n$. This phenomenon was discovered by Cherednik (in the $B_n$ case, see [Ch3], Section 5); in [Ch2], he gives a complete list of additional reducibility points. At first sight, this list looks somewhat mysterious; our work demystifies it, by interpreting it in terms of the Borel - de Siebenthal classification of equal rank embeddings of root systems.
The result of this note is a manifestation of the general principle that the representation theory of the trigonometric Cherednik algebra (degenerate DAHA) for a root system $R$ reduces to the representation theory of the rational Cherednik algebra for Weyl groups of root subsystems $R' \subset R$. This principle is the “double” analog of a similar principle in the representation theory of affine Hecke algebras, which goes back to the work of Lusztig [L], in which it is shown that irreducible representations of the affine Hecke algebra of a root system $R$ may be described in terms of irreducible representations of the degenerate affine Hecke algebras for Weyl groups of root subsystems $R' \subset R$. We illustrate this principle at the end of the note by applying it to finite dimensional representations of trigonometric Cherednik algebras.

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2. Preliminaries

2.1. Preliminaries on root systems. Let $W$ be a irreducible Weyl group, $\mathfrak{h}$ its (complex) reflection representation, and $L \subset \mathfrak{h}$ a $\mathbb{Z}$-lattice invariant under $W$.

For each reflection $s \in W$, let $L_s$ be the intersection of $L$ with the $-1$-eigenspace of $s$ in $\mathfrak{h}$, and let $\alpha_s^\vee$ be a generator of $L_s$. Let $\alpha_s$ be the element in $\mathfrak{h}^*$ such that $s\alpha_s = -\alpha_s$, and $(\alpha_s, \alpha_s^\vee) = 2$. Then we have

$$s(x) = x - (x, \alpha_s)\alpha_s^\vee, \quad x \in \mathfrak{h}.$$

Let $R \subset \mathfrak{h}^*$ be the collection of vectors $\pm \alpha_s$, and $R^\vee \subset \mathfrak{h}$ the collection of vectors $\pm \alpha_s^\vee$. It is well known that $R, R^\vee$ are mutually dual reduced root systems. Moreover, we have $Q^\vee \subset L \subset P^\vee$, where $P^\vee$ is the coweight lattice, and $Q^\vee$ the coroot lattice.

Consider the simple complex Lie group $G$ with root system $R$, whose center is $P^\vee/L$. The maximal torus of $G$ can be identified with $H = \mathfrak{h}/L$ via the exponential map.

For $g \in H$, let $C_g(g)$ be the centralizer of $g$ in $\mathfrak{g} := \text{Lie}(G)$. Then $C_g(g)$ is a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, and its Weyl group is the stabilizer $W_g$ of $g$ in $W$.

Let $\Sigma \subset H$ be the set of elements whose centralizer $C_g(g)$ is semisimple (of the same rank as $\mathfrak{g}$). $\Sigma$ can also be defined as the set of point strata for the stratification of $H$ with respect to stabilizers. It is well known that the set $\Sigma$ is finite, and the Dynkin diagram of $C_g(g)$ is obtained from the extended Dynkin diagram of $\mathfrak{g}$ by deleting one vertex (the Borel-de Siebenthal algorithm). Moreover, any Dynkin diagram obtained in this way corresponds to $C_g(g)$ for some $g$. 2
2.2. The rational Cherednik algebra. Let $W$ be a Coxeter group with reflection representation $\mathfrak{h}$. For any conjugacy invariant function $c$ on the set of reflections in $W$, one can define the rational Cherednik algebra $H_{1,c}(W,\mathfrak{h})$ (see e.g. [E1], Section 7); we will denote it shortly by $H_c(W,\mathfrak{h})$. This algebra has a polynomial representation $\mathbb{C}[\mathfrak{h}]$, which is defined using Dunkl operators. A function $c$ is said to be singular if the polynomial representation is reducible. Let $\text{Sing}(W,\mathfrak{h})$ be the set of singular $c$. This set is determined explicitly in [DJO].

2.3. The degenerate DAHA. Let $W, L, H$ be as in subsection 2.1. A reflection hypertorus in $H$ is a connected component $T$ of the fixed set $H^s$ for a reflection $s \in W$. Let $c$ be a conjugation invariant function on the set of reflection hypertori.

The degenerate DAHA attached to $W, H$ was introduced by Cherednik, see e.g. [Ch1]. This algebra is generated by polynomial functions on $H$, the group $W$, and trigonometric Dunkl operators. Using the geometric approach of [E2], which attaches a Cherednik algebra to any smooth affine algebraic variety with a finite group action, the degenerate DAHA can be defined as the Cherednik algebra $H_{1,c}(W, H)$ attached to the variety $H$ with the action of the finite group $W$. We will shortly denote this algebra by $H_c(W, H)$.

Note that this setting includes the case of non-reduced root systems. Namely, in the case of a non-reduced root system the function $c$ may take nonzero values on reflection hypertori which don’t go through $1 \in H$.

3. The results

3.1. The main results. The degenerate DAHA has a polynomial representation $M = \mathbb{C}[H]$ on the space of regular functions on $H$. We would like to determine for which $c$ this representation is reducible.

Let $g \in \Sigma$. Denote by $c_g$ the restriction of the function $c$ to reflections in $W_g$; that is, for $s \in W_g$, $c_g(s)$ is the value of $c$ on the (unique) hypertorus $T_{g,s}$ passing through $g$ and fixed by $s$. Denote by $\text{Sing}_g(W, L)$ the set of $c$ such that $c_g \in \text{Sing}(W_g, \mathfrak{h})$.

Remark 3.1. If $c(T) = 0$ unless $T$ contains $1 \in H$ (“reduced case”) then $c$ can be regarded as a function of reflections in $W$, and $c_g$ is the usual restriction of $c$ to reflections in $W_g$.

Our main result is the following.

Theorem 3.2. The polynomial representation $M$ of $H_c(W, H)$ is reducible if and only if $c \in \bigcup_{g \in \Sigma} \text{Sing}_g(W, L)$.

The proof of this theorem is given in the next subsection.

Corollary 3.3. If $c$ is a constant function (in particular, if $R$ is simply laced), then the polynomial representation $M$ of $H_c(W, H)$ is reducible if and only if $c$ is the polynomial representation of the rational Cherednik
algebra $H_c(W,h)$, i.e. iff $c = j/d_i$, where $d_i$ is a degree of $W$, and $j$ is a positive integer not divisible by $d_i$.

Proof. The result follows from Theorem 3.2, the result of [DJO], and the well known fact that for any subgroup $W' \subset W$ generated by reflections, every degree of $W'$ divides some degree of $W$. □

However, if $c$ is not a constant function, the answer in the trigonometric case may differ from the rational case, as explained below.

3.2. Proof of Theorem 3.2. Assume first that the polynomial representation $M$ is reducible. Then there exists a nonzero proper submodule $I \subset M$, which is an ideal in $\mathbb{C}[H]$. This ideal defines a subvariety $Z \subset H$, which is $W$-invariant; it is the support of the module $M/I$. It is easy to show using the results of [E2] (see e.g. [BE]) that $Z$ is a union of strata of the stratification of $H$ with respect to stabilizers. In particular, since $Z$ is closed, it contains a stratum which consists of one point $g$. Thus $g \in \Sigma$. Consider the formal completion $\widehat{M}_g$ of $M$ at $g$. As follows from [E2] (see also [BE]), this module can be viewed as a module over the formal completion $\widehat{H}_{c_g}(W_g,h)_0$ of the rational Cherednik algebra of the group $W_g$ at 0, and it has a nonzero proper submodule $\widehat{I}_g$. Thus, $\widehat{M}_g$ is reducible, which implies (by taking nilpotent vectors under $h^*$) that the polynomial representation $M$ over $\overline{H}_{c_g}(W_g,h)$ is reducible, hence $c_g \in \text{Sing}(W_g)$, and $c \in \text{Sing}_g(W,L)$.

Conversely, assume that $c \in \text{Sing}_g(W,L)$, and thus $c_g \in \text{Sing}(W_g)$. Then the polynomial representation $\overline{M}$ of $\overline{H}_{c_g}(W_g,h)$ is reducible. This implies that the completion $\widehat{M}_g = \overline{\mathbb{C}[H]}_g$ is a reducible module over $\overline{H}_{c_g}(W_g,h)_0$, i.e. it contains a nonzero proper submodule (=ideal) $J$. Let $I \subset \mathbb{C}[H]$ be the intersection of $\mathbb{C}[H]$ with $J$. Clearly, $I \subset M$ is a proper submodule (it does not contain 1). So it remains to show that it is nonzero. To do so, denote by $\Delta$ a regular function on $H$ which has simple zeros on all the reflection hypertori. Then clearly $\Delta^n \in J$ for large enough $n$, so $\Delta^n \in I$. Thus $I \neq 0$ and the theorem is proved.

3.3. Reducibility points in the non-simply laced case. In this subsection we will consider the reduced non-simply laced case, i.e. the case of root systems of type $B_n, C_n, F_4$, and $G_2$. In this case, $c$ is determined by two numbers $k_1$ and $k_2$, the values of $c$ on reflections for long and short roots, respectively.

\footnote{This fact is proved as follows. Let $P_W(t)$ be the Poincaré polynomial of $W$; so $P_W(t) = \prod_i \frac{1 - t^{d_i(W)}}{1 - t}$, where $d_i(W)$ are the degrees of $W$. Then by Chevalley’s theorem, $P_W(t)/P_W'(t)$ is a polynomial (the Hilbert polynomial of the generators of the free module $\mathbb{C}[h]^W$ over $\mathbb{C}[h]^W$). So, since the denominator vanishes at a root of unity of degree $d_i(W')$, so does the numerator, which implies the statement.}

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The set \( \text{Sing}(W, \mathfrak{h}) \) is determined for these cases in [DJO], as the union of the following lines (where \( l \geq 1, u = k_1 + k_2, \) and \( i = 1, 2 \)).

\[ B_n = C_n: \]

\[ 2jk_1 + 2k_2 = l, \ l \neq 0 \mod 2, j = 0, ..., n - 1, \]

and

\[ jk_1 = l, \ (l, j) = 1, \ j = 2, ..., n. \]

\[ F_4: \]

\[ 2k_i = l, \ 2k_i + 2u = l, \ l \neq 0 \mod 2; 3k_i = l, \ l \neq 0 \mod 3; \]

\[ 2u = l, 4u = l, \ l \neq 0 \mod 2; \]

\[ 6u = l, \ l = 1, 5, 7, 11 \mod 12. \]

\[ G_2: \]

\[ 2k_i = l, \ l \neq 0 \mod 2; \ 3u = l, \ l \neq 0 \mod 3. \]

By using Theorem 3.2, we determine that the polynomial representation in the trigonometric case is reducible on these lines and also on the following additional lines:

\[ B_n, \ n \geq 3: \]

\[ (2p - 1)k_1 = 2q, n/2 < q \leq n - 1, p \geq 1, (2p - 1, q) = 1. \]

\[ F_4: \]

\[ 6k_1 + 2k_2 = l, 4k_1 = l, \ l \neq 0 \mod 2. \]

\[ G_2: \]

\[ 3k_1 = l, \ l \neq 0 \mod 3. \]

In the \( C_n \) case, we get no additional lines.

Note that exactly the same list of additional reducibility points appears in [Ch2].

**Remark 3.4.** As explained above, the additional lines appear from particular equal rank embeddings of root systems. Namely, the additional lines for \( B_n \) appear from the inclusion \( D_n \subset B_n \). The two series of additional lines for \( F_4 \) appear from the embeddings \( B_3 \subset F_4 \) and \( A_3 \times A_1 \subset F_4 \), respectively. Finally, the additional lines for \( G_2 \) appear from the embedding \( A_2 \subset G_2 \).

### 3.4. Generalizations.

Theorem 3.2 can be generalized, with essentially the same proof, to the setting of any smooth variety with a group action, as defined in [L2].

Namely, let \( X \) be a smooth algebraic variety, and \( G \) a finite group acting faithfully on \( X \). Let \( c \) be a conjugation invariant function on the set of pairs \( (g, Y) \), where \( g \in G \), and \( Y \) is a connected component of \( X^g \) which has codimension 1 in \( X \). Let \( H_{1,c,0,X,G} \) be the corresponding sheaf of Cherednik algebras defined in [L2]. We have the polynomial representation \( \mathcal{O}_X \) of this sheaf.

Let \( \Sigma \in X \) be the set of points with maximal stabilizer, i.e. points whose stabilizer is bigger than that of nearby points. Then \( \Sigma \) is a finite set. For \( x \in X \), let \( G_x \) be the stabilizer of \( x \) in \( G \); it is a finite subgroup.
of $GL(T_xX)$. Let $c_x$ be the function of reflections in $G_x$ defined by $c_x(g) = c(g, Y)$, where $Y$ is the reflection hypersurface passing through $x$ and fixed by $g$ pointwise. Let $\text{Sing}_x(G, X)$ be the set of $c$ such that $c_x \in \text{Sing}(G_x, T_xX)$ (where $\text{Sing}(G_x, T_xX)$ denotes the set of values of parameters $c$ for which the polynomial representation of the rational Cherednik algebra $H_c(G_x, T_xX)$ is reducible).

Then we have the following theorem, whose statement and proof are direct generalizations of those of Theorem 3.2 (which is obtained when $G$ is a Weyl group and $X$ a torus).

**Theorem 3.5.** The polynomial representation $O_X$ of $H_{1,c,0,X,G}$ is reducible if and only if $c \in \bigcup_{x \in \Sigma} \text{Sing}_x(G, X)$.

Note that this result generalizes in a straightforward way to the case when $X$ is a complex analytic manifold, and $G$ a discrete group of holomorphic transformations of $X$.

### 3.5. Finite dimensional representations of the degenerate double affine Hecke algebra.

Another application of the approach of this note is a description of the category of finite dimensional representations of the degenerate DAHA in terms of categories of finite dimensional representations of rational Cherednik algebras. Namely, let $FD(A)$ denote the category of finite dimensional representations of an algebra (or sheaf of algebras) $A$. Then in the setting of the previous subsection we have the following theorem (see also Proposition 2.22 of [122]).

Let $\Sigma'$ be a set of representatives of $\Sigma/G$ in $\Sigma$.

**Theorem 3.6.** One has

$$FD(H_{1,c,0,X,G}) = \bigoplus_{x \in \Sigma'} FD(H_{c_x}(G_x, T_xX)).$$

**Proof.** Suppose $V$ is a finite dimensional representation of $H_{1,c,0,X,G}$. Then the support of $V$ is a union of finitely many points, and these points must be strata of the stratification of $X$ with respect to stabilizers, so they belong to $\Sigma$. This implies that $V = \bigoplus_{\xi \in \Sigma/G} V_{\xi}$, where $V_{\xi}$ is supported on the orbit $\xi$. Taking completion of the Cherednik algebra at $\xi$, we can regard the fiber $(V_{\xi})_x$ for $x \in \xi$ as a module over the rational Cherednik algebra $H_{c_{\xi}}(G_x, T_xX)$ (see [122] [BE]). In this way, $V$ gives rise to an object of $\bigoplus_{x \in \Sigma'} FD(H_{c_x}(G_x, T_xX))$.

This procedure can be reversed; this implies the theorem. \hfill \square

**Corollary 3.7.** One has

$$FD(H_c(W, H)) = \bigoplus_{g \in \Sigma/W} FD(H_{c_g}(W_g, h)).$$

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2The contents of this subsection arose from a discussion of the author with M. Varagnolo and E. Vasserot.
Remark 3.8. Recall that a representation of $H_c(W, H)$ is said to be spherical if it is a quotient of the polynomial representation. It is clear that the categorical equivalence of Corollary 3.7 preserves sphericity of representations (in both directions). This implies that the results of the paper [VV], which classifies spherical finite-dimensional representations of the rational Cherednik algebras, in fact yields, through Corollary 3.7, the classification of spherical finite dimensional representations of degenerate DAHA, and hence of nongenerate DAHA outside of roots of unity.

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