Decoupling of the general scalar field mode and the solution space for Bianchi type I and V cosmologies coupled to perfect fluid sources

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Abstract

The scalar field degree of freedom in Einstein’s plus Matter field equations is decoupled for Bianchi type I and V general cosmological models. The source, apart from the minimally coupled scalar field with arbitrary potential $V(\Phi)$, is provided by a perfect fluid obeying a general equation of state $p = p(\rho)$. The resulting ODE is, by an appropriate choice of final time gauge affiliated to the scalar field, reduced to 1st order, and then the system is completely integrated for arbitrary choices of the potential and the equation of state.

PACS numbers: 04.20.-q, 04.20.Jb, 98.80.Jk
1 Introduction

In the past two decades there has been a growing interest in scalar field cosmological models primarily due to the prominent importance of scalar fields for inflationary scenarios [1]. The importance of the coupling between a scalar field and the gravitational field has been further stressed by Madsen [2], who has shown that it can have non-trivial consequences for the spontaneous breaking of gauge symmetries. A dynamical systems approach has been extensively used in the study of scalar field cosmologies and their asymptotic behavior [3]. See also [4] for a concise review.

As far as exact solutions of scalar field cosmologies are concerned, Burd and Barrow [5] have studied homogeneous but anisotropic Bianchi models of types III and VI (as well as Kantowski-Sachs models) and have found exact solutions. Lidsey, and Aguirregabiria et al [6] have found exact solutions for Bianchi type I models. Feinstein and Ibanez [7] have found exact solutions for Bianchi models of type III and VI, Moss and Wright, Madsen, and Abreut, Crawford and Mimoso [8] have studied exact solutions in the setting of conformal scalar field cosmologies. Paul [9] has obtained exact solutions of a higher derivative theory in the presence of an interacting scalar field. The discovery of the BTZ black hole has motivated the study of analytic solutions in the context of scalar field cosmology in (2+1) dimensions [10], while Russo [11] has obtained the general solution for a scalar field cosmology in d dimensions with exponential potentials and a flat Robertson-Walker metric. An early work on exponential potentials has been done by Salopek and Bond [12], while recent treatments are given by Kehagias and Kofinas [13] and Neupane [14].

Cosmological models containing both a fluid and a scalar field have also been studied. Chimento and Jakubi [15] have given exact solutions of scalar field cosmologies with a perfect fluid and a viscous fluid respectively. Mendez [16] has obtained an exact solution for the case of an imperfect fluid in a FRW spacetime. In the so-called scaling scalar field cosmologies, the energy density due to the scalar field is proportional to the energy density of the perfect fluid. Thus a number of spatially flat, isotropic models in which the energy density of the scalar field is proportional to that of the perfect fluid have been investigated [17]. Billyard, Coley and van den Hoogen [18] have studied the stability of these scaling solutions within the class of spatially homogeneous cosmological models with a barotropic fluid matter content. Furthermore, they have studied the qualitative behavior of spatially homogeneous models with a barotropic fluid and a non-interacting scalar field with an exponential potential in the class of Bianchi type B models [19]. Saha [20] has obtained exact solutions for a Bianchi type I model with a perfect fluid and dark energy content, while Chimento and Cossarini have studied exact solutions in 1+1 dimensions using an isotropic perfect fluid source [21].

In a significant paper, Hawkins and Lidsey [22], have shown that, for a flat FRW geometry, the dynamics of scalar field cosmologies with a perfect fluid matter content can be described by the non-linear, Ermakov-Pinney equation (which, in turn, leads to tantalizing analogies with the dynamics of Bose-Einstein condensates), while an early work of Barrow also deserves mention in this context. Exact solutions have been obtained
in this description [23]. Another kind of decoupling of the scalar field degree of freedom has been initiated in [24], where a Robertson-Walker background geometry minimally coupled to the scalar field has been investigated. By use of integrals of the motion and of the Klein-Gordon equation in the quadratic constraint equation, a single, higher-order non-linear differential equation for the scalar field was obtained.

In the present work we generalize this decoupling for the case of general Bianchi type I and V geometries in the presence of a general perfect fluid source. The advantage is that the resulting ODE is fully integrated and this is achieved for arbitrary choices of the scalar field potential and the fluid equation of state. The paper is organized as follows. In section 2 the geometry and the matter description as well as the governing equations for the system are presented. In section 3 the decoupling of the scalar field degree of freedom is performed for Bianchi type I, the reduction of the resulting ODE is given and the system is completely integrated. The corresponding calculations for Bianchi type V are given in section 4. Finally, the conclusions and a discussion of the results obtained are presented in section 5.

Throughout we use geometrized units, i.e. \( c = 8\pi G = 1 \), while \( g_{\mu\nu} \) has the signature \((-++,++)\).

2 The governing equations and the matter content

Our starting point is the line element for a spatially homogeneous geometry:

\[
ds^2 = [N^\alpha(t)N_\alpha(t) - N^2(t)] dt^2 + 2N_\alpha(t)\sigma^\alpha_i(x) dx^i dt + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x) dx^i dx^j \tag{2.1}
\]

where \( \sigma^\alpha_i(x) \) are the basis one-forms satisfying \( \sigma^\alpha_i(x) - \sigma^\alpha_j(x) = 2C^\alpha_{\beta\gamma}\sigma^\beta_j(x)\sigma^\gamma_i(x) \), (with \( C^\alpha_{\beta\gamma} \) the structure constants of the Lie algebra of the corresponding three-dimensional isometry group of motion acting simply transitively on the spatial hypersurfaces of simultaneity), \( N_\alpha(t) \) is the shift vector, \( N(t) \) is the lapse function, and \( \gamma_{\alpha\beta}(t) \) is the scale factor matrix of the Bianchi type examined.

As explained in [25], there are special general coordinate transformations mixing space and time in the new space coordinates, whose effect on the line element (2.1) is described by

\[
\begin{align*}
\tilde{\gamma}_{\alpha\beta}(t) &= \Lambda^\alpha_\mu\Lambda^\nu_\beta\gamma_{\mu\nu}(t), \\
\tilde{N}(t) &= N(t) \\
\tilde{N}_\alpha(t) &= \Lambda^\beta_\alpha(t) [N_\beta(t) + P^\rho(t)\gamma_{\rho\beta}(t)]
\end{align*}
\tag{2.2}
\]

where \( \Lambda^\mu_\nu(t), P^\rho(t) \) satisfy

\[
\begin{align*}
\Lambda^\alpha_\mu C^\mu_{\beta\gamma} &= \Lambda^\beta_\rho\Lambda^\gamma_\sigma C^\alpha_{\rho\sigma} \\
2P^\mu C^\alpha_{\mu\nu}\Lambda^\nu_\beta &= \dot{\Lambda}^\alpha_\beta
\end{align*}
\tag{2.3}
\tag{2.4}
\]

the overdot denoting differentiation with respect to time. Due to (2.3), \( \Lambda^\beta_\rho(t) \) belongs to the automorphism group of the Bianchi type in question and transformations (2.2)
describe the gauge freedom of the emanating system of Einstein’s field equations in
the case of vacuum. So, the three arbitrary functions of time contained in \( \Lambda_\alpha^\beta(t) \) and
\( P^\alpha(t) \) can be used to simplify the line element (2.1) and, thus, also the aforementioned
equations. Automorphisms induced by general coordinate transformations have also
been considered in [26], while the rigid gauge symmetries have been analyzed in [27].
Time-dependent automorphisms, seen as tangent space transformations, have also been
considered previously [?], [29], while the first (known to us) reference of the relevance of
Automorphisms to a systematic analysis of Bianchi Cosmologies goes back to 1962 [30].

The above result also holds true when the matter content is such that the linear
constraints \( G^0_i \) do not acquire an extra term \( T^0_i \). For orthogonal perfect fluids and scalar
fields depending only on time, \( T^0_i \) is indeed zero and therefore we can use the gauge
freedom (2.2-2.4) to diagonalize \( \gamma_{\alpha\beta}(t) \) without losing generality. Therefore, the scale
factor matrix is taken to be

\[
\gamma_{\alpha\beta}(t) = \text{diag}(a^2, b^2, c^2) \tag{2.5}
\]

while for the shift vector we have \( N_\alpha(t) = 0 \). Our choice of time is specified by the
gauge condition \( N = \sqrt{\det \gamma_{\mu\nu}} \), which is frequently used due to the simplification of the
Einstein tensor and whose importance for the decoupling is essential.

The matter content is a minimally coupled scalar field with an arbitrary potential
\( V(\Phi) \), thus having an energy-momentum tensor

\[
T^{(1)}_{\mu\nu} = \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} (g^{\alpha\lambda} \Phi_{,\alpha} \Phi_{,\lambda} + 2V(\Phi)) \tag{2.6}
\]

and a perfect fluid part

\[
T^{(2)}_{\mu\nu} = (p + \rho) u_\mu u_\nu + pg_{\mu\nu} \tag{2.7}
\]

where \( u^\mu \) is the unit four-velocity vector and a general equation of state \( p = p(\rho) \) is
adopted. The governing Einstein’s Field Equations are taken to be

\[
R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R = T^\mu_\nu \tag{2.8}
\]

where \( T_{\mu\nu} = T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} \), while the scalar field equation (Klein-Gordon) is

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - V'(\Phi) = 0 \tag{2.9}
\]

the prime, from now onwards, denoting differentiation with respect to the argument.

The “equation of motion” for the perfect fluid is the conservation of its energy-
momentum tensor:

\[
T^{(2)}_{\nu(\mu} = 0 \tag{2.10}
\]

(The scalar field energy-momentum tensor is separately conserved by virtue of the Klein-
Gordon equation).
3 Decoupling of the scalar degree and the solution space for Bianchi Type I

The basis one-forms are \( \sigma_i(x) = \delta_i^\alpha \) and, thus, with (2.5), zero shift and the chosen time gauge, the initial form of the metric is given by

\[
g_{\mu\nu} = \begin{pmatrix}
-a^2b^2c^2 & 0 & 0 & 0 \\
0 & a^2 & 0 & 0 \\
0 & 0 & b^2 & 0 \\
0 & 0 & 0 & c^2
\end{pmatrix}
\] (3.1)

Spatial homogeneity implies that \( \Phi = \Phi(t) \) and \( \rho = \rho(t) \). The nonzero components of the Einstein tensor \( G^\mu_{\nu} \), and the energy-momentum tensors \( T^{(1)}_{\mu\nu} \), \( T^{(2)}_{\mu\nu} \) (all multiplied by \( a^2b^2c^2 \)) are given by

\[
G^0_{0} = -\frac{\dot{a}\dot{b}}{ab} - \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc}
\] (3.2)

\[
G^1_{1} = \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}^2}{c^2} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}
\] (3.3)

\[
G^2_{2} = \frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}^2}{c^2} - \frac{\ddot{a}}{a} - \frac{\ddot{c}}{c}
\] (3.4)

\[
G^3_{3} = \frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} - \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b}
\] (3.5)

\[
T^{(1)}_{00} = -a^2b^2c^2 V(\Phi) - \frac{1}{2}\Phi^2
\] (3.6)

\[
T^{(1)}_{11} = T^{(1)2} = T^{(1)3} = -a^2b^2c^2 V(\Phi) + \frac{1}{2}\Phi^2
\] (3.7)

\[
T^{(2)}_{00} = -a^2b^2c^2 \rho(t)
\] (3.8)

\[
T^{(2)}_{11} = T^{(2)2} = T^{(2)3} = a^2b^2c^2 p(\rho(t))
\] (3.9)

Due to the equalities (3.7) and (3.9), we can subtract the corresponding Einstein’s equations \( G^\mu_{\nu} = T^{(1)\mu}_{\nu} + T^{(2)\mu}_{\nu} \), i.e. form the differences (all multiplied by \( a^2b^2c^2 \))

\[
G^2_{2} - G^1_{1} = T^{(1)2}_{2} + T^{(2)2}_{2} - T^{(1)1}_{1} - T^{(2)1}_{1}
\]

and

\[
G^3_{3} - G^1_{1} = T^{(1)3}_{3} + T^{(2)3}_{3} - T^{(1)1}_{1} - T^{(2)1}_{1}
\]

and get the following equations involving only the scale factors

\[
\frac{\dot{a}^2}{a^2} - \frac{\dot{b}^2}{b^2} - \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} = 0
\] (3.10)
\[
\frac{\dot{a}^2}{a^2} - \frac{\dot{c}^2}{c^2} - \frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} = 0
\]  
(3.11)

These equations provide the following integrals of motion:

\[
b(t) = e^{\lambda t} a(t)
\]  
(3.12)

\[
c(t) = e^{\mu t} a(t)
\]  
(3.13)

leaving us with just one undetermined scale factor. At this stage the Klein-Gordon equation becomes

\[-V'(\Phi) - \frac{e^{-2(\lambda + \mu)t} \ddot{\Phi}(t)}{a^6(t)} = 0
\]  
(3.14)

Now, the conservation of the fluid energy-momentum tensor (continuity equation) gives

\[3 \frac{\dot{a}}{a} + \frac{\dot{\rho}}{p(\rho) + \rho} + \lambda + \mu = 0
\]  
(3.15)

It is straightforward to check that the time derivative of the quadratic constraint equation \(G_0^0 = T_{0}^{(1)0} + T_{0}^{(2)0}\) is identically satisfied by virtue of the remaining spatial equation \(G_1^1 = T_{1}^{(1)1} + T_{1}^{(2)1}\), the Klein-Gordon equation and the continuity equation (solved for \(\ddot{a}(t), \ddot{\Phi}(t), \dot{\rho}(t)\)), as expected from the consistency between the aforementioned constraint with the spatial Einstein and the Matter equations. Therefore, the equations to be solved are the Klein-Gordon equation (3.14), the continuity equation (3.15) and the constraint equation which, upon multiplication by \(-2ae^{2(\lambda + \mu)t}\) reads:

\[6 \left( \frac{\dot{a}}{a} \right)^2 + 4(\lambda + \mu) \frac{\dot{a}}{a} + 2\lambda\mu - \ddot{\Phi}^2 - 2e^{2(\lambda + \mu)t}a^6[V(\Phi) + \rho] = 0
\]  
(3.16)

In order to have a closed form for the integral of the continuity eq. (3.15), it is convenient to use the parametrization

\[p(\rho) = \frac{g(\rho)}{g'(\rho)} - \rho
\]  
(3.17)

(the prime denoting differentiation with respect to the argument) through the use of which one obtains the integral

\[\rho = h[\rho_0 a^{-3} e^{-(\lambda + \mu)t}]
\]  
(3.18)

with \(h\) being the inverse function to \(g\), i.e. satisfying \(h(g(x)) = x\).

From the Klein-Gordon equation (3.14) the scale factor and its logarithmic derivative is expressed in terms of the scalar field \(\Phi(t)\) and its derivatives as well as \(V(\Phi)\):

\[a = \left[ -\frac{e^{-2(\lambda + \mu)t} \ddot{\Phi}}{V'(\Phi)} \right]^{1/6}
\]  
(3.19)
\[ \frac{\dot{a}}{a} = \frac{1}{6} \left( \frac{\ddot{\Phi}}{\Phi} - \frac{V''(\Phi)}{V'(\Phi)} \dot{\Phi} - 2(\lambda + \mu) \right) \tag{3.20} \]

The promised decoupling of the scalar field dynamics from the geometry occurs upon inserting (3.18-3.20) into (3.16). The result is the following non-linear, third order ODE for \( \Phi(t) \):

\[ \left( \frac{\ddot{\Phi}}{\Phi} - \frac{V''(\Phi)}{V'(\Phi)} \right)^2 - 4(\lambda + \mu)^2 + 12\lambda\mu - 6\dot{\Phi}^2 \]

\[ + 12 \frac{\ddot{\Phi}}{V'(\Phi)} V(\Phi) + 12 \frac{\dot{\Phi}}{V'(\Phi)} h \left[ \rho_0 \left( -\frac{\ddot{\Phi}}{V'(\Phi)} \right)^{-1/2} \right] = 0 \tag{3.21} \]

At this stage, any solution to this equation determines, through (3.18) and (3.19), a corresponding solution to the full Einstein plus Matter system whose entire space of solution is therefore attained from the solution space of (3.21). Of course, the price paid, for the moment, is the non-linearity in the highest time derivative, which has also been raised to third order. Furthermore, normally one would expect that the arbitrary functions \( h, V \) need first to be specified before hoping to actually get a solution. Nevertheless, it is quite interesting that further reduction of the order of (3.21), and subsequent complete integration of the whole system, is possible. To this end, we first observe that many terms in (3.21), namely all the nontrivial except the 4th and 5th, are functions of the combination \(-\frac{\ddot{\Phi}}{V'(\Phi)}\). We thus define:

\[ \chi = -\frac{\ddot{\Phi}}{V'(\Phi)} \tag{3.22} \]

and write (3.21) as

\[ \left( \frac{\ddot{\chi}}{\chi} \right)^2 - 4(\lambda + \mu)^2 + 12\lambda\mu - 6f(\chi) - 12\chi h(\rho_0 \chi^{-1/2}) = 0 \tag{3.23} \]

where

\[ f(\chi) \equiv \dot{\Phi}^2 + 2\chi V(\Phi) \tag{3.24} \]

implicitly reflects the arbitrariness in choosing \( V(\Phi) \). Now (3.23) can be integrated and, by judicious choices for \( f, h \), even give \( \chi(t) \) in closed form. Suppose \( \chi(t) \) does indeed solve (3.23). Then, multiplying (3.22) by \( 2\dot{\Phi}V'(\Phi) \) and using (3.24) we get the first order linear differential equation for \( f(t) \):

\[ \frac{df(t)}{dt} = 2\dot{\chi}(t)V(t) \tag{3.25} \]

and consequently \( \Phi(t) \) is given by

\[ \frac{d\Phi(t)}{dt} = \pm \sqrt{f(\chi(t)) - 2\chi(t)V(t)} \tag{3.26} \]
Finally, a change of time variable from \( t \) to \( \chi \) (defined in (3.22)) permits the presentation of the entire space of solutions to the system under consideration in closed form. Indeed, considering \( V, f, \Phi \) as functions of \( \chi \) one gets from (3.25-3.26) respectively

\[
\frac{df}{d\chi} = 2V(\chi) \tag{3.27}
\]

\[
\frac{d\Phi(\chi)}{d\chi} \dot{\chi}(t) = \pm \sqrt{f(\chi) - 2\chi V(\chi)} \tag{3.28}
\]

where the time derivative of \( \chi \) is given by the constraint (3.23). The integration of these two equations is trivial, yielding \( f, \Phi \) as:

\[
f(\chi) = \sigma + 2 \int V(\chi) \, d\chi \tag{3.29}
\]

\[
\Phi(\chi) = \kappa \pm \frac{1}{\chi} \sqrt{\frac{f(\chi) - 2\chi V(\chi)}{4(\lambda + \mu)^2 - 12\lambda \mu + 6f(\chi) + 12\chi h(\rho_0 \chi^{-1/2})}} \, d\chi \tag{3.30}
\]

The line element in the new time \( \chi \) reads:

\[
ds^2 = -\frac{d\chi^2}{\chi[4(\lambda + \mu)^2 - 12\lambda \mu + 6f(\chi) + 12\chi h(\rho_0 \chi^{-1/2})]} \tag{3.31}
\]

\[
+ [e^{-2(\lambda+\mu)t(\chi)} \chi]^{1/3} \, dx^2 + [e^{(4\lambda-2\mu)t(\chi)} \chi]^{1/3} \, dy^2 + [e^{(-2\lambda+4\mu)t(\chi)} \chi]^{1/3} \, dz^2
\]

with \( t(\chi) \) given by the integral form of (3.23)

\[
t(\chi) = \pm \frac{1}{\chi} \sqrt{\frac{4(\lambda + \mu)^2 - 12\lambda \mu + 6f(\chi) + 12\chi h(\rho_0 \chi^{-1/2})}} \, d\chi \tag{3.32}
\]

while the density and pressure are given as:

\[
\rho(\chi) = h(\rho_0 \chi^{-1/2}) \quad p(\chi) = \rho_0 \chi^{-1/2} h'(\rho_0 \chi^{-1/2}) - h(\rho_0 \chi^{-1/2}) \tag{3.33}
\]

the prime denoting differentiation with respect to the argument. Quite independently of the way these solutions were obtained, one can straightforwardly check (through, say, a symbolic computing facility) that they do satisfy all ten Einstein’s Equations, the generalized K-G and the continuity equation. Furthermore, since no extra ansatz has been involved in the process of integration of the system (2.8-2.10), equations (3.29-3.33) represent the full space of solutions to the Einstein plus Matter system considered. The functions \( V, h \) can be freely specified to obtain special case solutions.
4 Decoupling of the scalar degree and the solution space for the Bianchi Type V

In this case the basis one-forms are $\sigma^1 = e^{-x}dy$, $\sigma^2 = e^{-x}dz$, $\sigma^3 = dx$. The $G_0^0 = 0$ Einstein equation (due to spatial homogeneity $\Phi = \Phi(t)$ and $\rho = \rho(t)$ and therefore there is no corresponding component of the matter tensor) implies $c = ab$. Thus the initial metric is taken as:

$$g_{\mu\nu} = \begin{pmatrix} -a^3b^3 & 0 & 0 & 0 \\ 0 & ab & 0 & 0 \\ 0 & 0 & a^2e^{-2x} & 0 \\ 0 & 0 & 0 & b^2e^{-2x} \end{pmatrix} \quad (4.1)$$

The nonzero components of the Einstein tensor $G_{\mu\nu}$, and the energy-momentum tensors $T_{\nu}^{(1)\mu}$, $T_{\nu}^{(2)\mu}$, all multiplied by $a^3b^3$, are given by

$$G_0^0 = 3a^2b^2 - \frac{\dot{a}^2}{2a^2} - \frac{2\dot{a}\dot{b}}{ab} - \frac{\ddot{b}^2}{2b^2} \quad (4.2)$$

$$G_1^1 = a^2b^2 + \frac{3\dot{a}^2}{2a^2} + \frac{2\dot{a}\dot{b}}{ab} + \frac{3\ddot{b}^2}{2b^2} - \frac{\dddot{a}}{a} - \frac{\ddot{b}}{b} \quad (4.3)$$

$$G_2^2 = a^2b^2 + \frac{\dot{a}^2}{a^2} + \frac{2\dot{a}\dot{b}}{ab} + \frac{2\ddot{b}^2}{b^2} - \frac{\dddot{a}}{2a} - \frac{3\ddot{b}}{2b} \quad (4.4)$$

$$G_3^3 = a^2b^2 + \frac{2\dot{a}^2}{a^2} + \frac{2\dot{a}\dot{b}}{ab} + \frac{\ddot{b}^2}{b^2} - \frac{3\ddot{a}}{2a} - \frac{\ddot{b}}{2b} \quad (4.5)$$

$$T_0^{(1)0} = -a^3b^3 V(\Phi) - \frac{1}{2} \dot{\Phi}^2 \quad (4.6)$$

$$T_1^{(1)1} = T_2^{(1)2} = T_3^{(1)3} = -a^3b^3 V(\Phi) + \frac{1}{2} \dot{\Phi}^2 \quad (4.7)$$

$$T_0^{(2)0} = -a^3b^3 \rho(t) \quad (4.8)$$

$$T_1^{(2)1} = T_2^{(2)2} = T_3^{(2)3} = a^3b^3 \rho(t) \quad (4.9)$$

The situation is similar to the Type I case, and thus forming the difference (multiplied by $a^3b^3$) $G_2^2 - G_1^1 = T_2^{(1)2} + T_2^{(2)2} - T_1^{(1)1} - T_1^{(2)1}$ we get the following equation involving only the scale factors

$$-\frac{\dot{a}^2}{2a^2} + \frac{\dot{b}^2}{2b^2} + \frac{\ddot{a}}{2a} - \frac{\ddot{b}}{2b} = 0 \quad (4.10)$$

which can be integrated yielding

$$b(t) = e^{\lambda t}a(t) \quad (4.11)$$
The Klein-Gordon equation (2.9) becomes
\[-V'(\Phi) - \frac{e^{-3\lambda t} \ddot{\Phi}(t)}{a^6(t)} = 0 \tag{4.12}\]
while the conservation of the fluid energy-momentum tensor (continuity equation) gives
\[3 \frac{\dot{a}}{a} + \frac{\dot{\rho}}{\rho(p) + \rho} + \frac{3}{2} \lambda = 0 \tag{4.13}\]
Again, the only other equation to be solved is the constraint equation which reads
\[6 \left(\frac{\dot{a}}{a}\right)^2 + 6\lambda \frac{\dot{a}}{a} + \lambda^2 - 6e^{-2\lambda t}a^4 - \ddot{\Phi}^2 - 2e^{-3\lambda t}a^6(V(\Phi) + \rho) = 0 \tag{4.14}\]
where the fourth term is the only non-trivial difference from the corresponding equation (3.16) and its presence is due to the non-vanishing curvature of the spatial slice.

Integrating (4.13) (in the parameterization (3.17)) and solving (4.12) for \(a(t)\), we obtain the following results for the matter density \(\rho(t)\), the scale factor \(a(t)\) and its logarithmic derivative
\[\rho = h \left(\rho_0 a^{-3} e^{-\frac{3}{2} \lambda t}\right) \tag{4.15}\]
\[a = \left(-\frac{e^{-3\lambda t} \ddot{\Phi}}{V'(\Phi)}\right)^{1/6} \tag{4.16}\]
\[\frac{\dot{a}}{a} = \frac{1}{6} \left(\frac{\dddot{\Phi}}{\ddot{\Phi}} - \frac{V''(\Phi)}{V'(\Phi)} \dot{\Phi} - 3\lambda\right) \tag{4.17}\]
Use of these equations in the quadratic constraint equation (4.14) (multiplied by \(-2e^{-3\lambda t}a^6\)) yields
\[\left(\frac{\dddot{\Phi}}{\ddot{\Phi}} - \frac{V''(\Phi)}{V'(\Phi)} \frac{\dddot{\Phi}}{\ddot{\Phi}}\right)^2 - 3\lambda^2 - 36 \left(\frac{\dddot{\Phi}}{V'(\Phi)}\right)^{2/3} - 6\dddot{\Phi}^2 + 12 \frac{\dddot{\Phi}}{V'(\Phi)} V(\Phi) \]
\[+ 12 \frac{\dddot{\Phi}}{V'(\Phi)} h \left(\rho_0 \left(-\frac{\dddot{\Phi}}{V'(\Phi)}\right)^{-1/2}\right) = 0 \tag{4.18}\]
which, with the same definition of \(\chi\), translates into
\[\left(\frac{\dot{\chi}}{\chi}\right)^2 - 3\lambda^2 - 36\chi^{2/3} - 6f(\chi) - 12\chi h(\rho_0\chi^{-1/2}) = 0 \tag{4.19}\]
By arguments completely analogous to the previous Type I case, the final form of the solution is, in this case:
\[f(\chi) = \sigma + 2 \int V(\chi) \, d\chi \tag{4.20}\]
\[ \Phi(\chi) = \kappa \pm \int \frac{1}{\chi} \sqrt{\frac{f(\chi) - 2\chi V(\chi)}{3\lambda^2 + 36\chi^{2/3} + 6f(\chi) + 12\chi h(\rho_0 \chi^{-1/2})}} \, d\chi \quad (4.21) \]

\[ ds^2 = -\frac{d\chi^2}{\chi(3\lambda^2 + 36\chi^{2/3} + 6f(\chi) + 12\chi h(\rho_0 \chi^{-1/2}))} \quad (4.22) \]

\[ + \chi^{1/3} \, dx^2 + e^{-\lambda t(\chi) - 2x} \chi^{1/3} \, dy^2 + e^{\lambda t(\chi) - 2x} \chi^{1/3} \, dz^2 \]

with \( t(\chi) \) given by the integral form of \( (4.19) \)

\[ t(\chi) = \pm \int \frac{1}{\chi \sqrt{3\lambda^2 + 36\chi^{2/3} + 6f(\chi) + 12\chi h(\rho_0 \chi^{-1/2})}} \, d\chi \quad (4.23) \]

while the density and pressure are given as:

\[ \rho(\chi) = h(\rho_0 \chi^{-1/2}) \quad p(\chi) = \rho_0 \chi^{-1/2} h'(\rho_0 \chi^{-1/2}) - h(\rho_0 \chi^{-1/2}) \quad (4.24) \]

The remarks at the end of the previous section apply also here.

### 5 Discussion-Conclusions

We have discussed the dynamics of a scalar field with an arbitrary potential, minimally coupled to a general (anisotropic) Bianchi Type I and V geometry, in the presence of a perfect fluid source obeying a general equation of state. In the case of vacuum, the rich structure of Outer Automorphisms for these two symmetry groups entails the existence of some integrals of motion: Indeed, consider the generators of the rigid symmetries \( (2.3) \), i.e. the vector fields in the space of dependant variables

\[ X_I = \lambda_I^\rho \gamma_{\rho\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \]

where \( \lambda_I^\rho \) satisfy \( \lambda_I^\rho \gamma_{\rho\beta} = \lambda_I^\rho \gamma_{\rho\gamma} \gamma_{\alpha\beta} + \lambda_I^\gamma \gamma_{\rho\beta} C_{\rho\gamma}^{\alpha\beta} \). If one performs the appropriate change of variables \( \gamma_{\alpha\beta} \to z_{\alpha\beta} \) which brings one (or more, if possible) generator into each canonical form, say \( \frac{\partial}{\partial z_{11}} \), then Einstein’s field equations written in the new variables do not explicitly depend on \( z_{11} \). Thus the system becomes of first order in the variable \( \dot{z}_{11} \) and therefore an integral of motion arises.

The initial choice of the time gauge in which the lapse is equal to the determinant of the scale factor matrix has a twofold advantage: Firstly, it enables the corresponding integrals of motion (in the presence of the matter content chosen) to be revealed. Secondly, it makes the Klein-Gordon equation purely algebraic in the scale factor variables. As a result, when the utilization of the integrals of motion has reduced the number of these variables to one, this equation gives this last scale factor as a function of the second derivative of the scalar field, the derivative of the potential with respect to the scalar field, and of time. Substitution of this form of the scale factor into the only equation remaining to be solved, i.e. the quadratic constraint, results in a single ODE for
the scalar field (without any explicit time dependence!). The utilization of a final time
gauge adapted to the scalar field enables the reduction of this equation to first order
and subsequently leads to the complete integration of the entire system of Einstein plus
Matter Field Equations. The description of the space of solutions contains, in an integral
form, the arbitrary functions of the final time \( \chi \), \( V(\chi) \) and \( h(\rho_0 \chi^{-1/2}) \). The presence
of this twofold arbitrariness corresponds to the fact that we have not specified either
the form of the potential (as a function of the scalar field) or the equation of state. It
is evident that prescribing \( V(\chi) \) and \( h(\rho_0 \chi^{-1/2}) \) implicitly corresponds to a choice of
potential form and equation of state. For example, the choice \( h = A(\rho_0 \chi^{-1/2})^{1+\gamma} \)
gives through (4.24) \( p = \gamma \rho \), i.e. the barotropic equation of state. For the scalar field, if we
take the particular case \( \lambda = \mu = \sigma = h(\rho_0 \chi^{-1/2}) = 0 \) (i.e., flat Robertson-Walker with
no fluid) the choice \( V(\chi) = C/\sqrt{\chi} \) corresponds to the functional form \( V(\Phi) = \pm Ce^{\sqrt{3(\kappa-\Phi)}} \).

If, on the other hand, someone insists in prescribing \( V(\Phi) \) and \( p(\rho) := p(h) \) then the
situation must be dealt with in the following manner: As far as the density is concerned,
the equation giving the pressure becomes the holonomic, first order differential equation
\( p(h) = wh'(w) - h(w) \) with \( w \) standing for \( \rho_0 \chi^{-1/2} \) which can be straightforwardly inte-
grated. For example, a barotropic equation \( p = \gamma \rho \) gives \( \rho = A(\rho_0 \chi^{-1/2})^{1+\gamma} \). As far as
the matter field is concerned, the situation is somewhat more complicated as the results
of choosing a particular form \( V(\Phi) \) are influenced by the choice of the density \( h \). As an
example consider again the case \( \lambda = \mu = \sigma = h(\rho_0 \chi^{-1/2}) = 0 \) and an arbitrary potential
form \( V(\Phi) \). Relaxing for a moment (3.30), we get the following Klein-Gordon equation
\[
-\frac{V'(\Phi)}{V(\Phi)} + \frac{6\chi}{6\chi^2 \Phi^2(\chi)} [2\chi \Phi''(\chi) - 6\chi^2 \Phi'3(\chi) + 3\Phi'(\chi)] = 0
\]
which may be difficult to solve depending on the choice of \( V(\Phi) \). This is the price paid
for insisting on prescribing \( V(\Phi) \) and not \( V(\chi) \) in which case the solution would be
given by (3.20) or (3.21) correspondingly. The particular choice \( V = V_0 e^{-\lambda \Phi} \)
corresponds to the case considered in \[11\] and the equation above can be dealt with by
choosing a new time \( \chi = e^{3\tau/2} \), in which the equation becomes the following first order
in the derivative \( \omega \equiv \frac{d\Phi}{d\tau} \):
\[
48 \frac{d\omega}{d\tau} - 96 \omega^3 + 24\lambda \omega^2 - 9\lambda = 0
\]

Finally, there are two cases which, at first sight, need to be separately examined. The
first concerns the case of a constant potential \( V(\Phi) = V_0 \), i.e. a cosmological constant
term. Then, \( V'(\Phi) = 0 \) and the definition of the final time \( \chi \) in equation (8.22) seems
to be precarious. However, the Klein-Gordon eq. (3.14) then implies that also \( \Phi = 0 \)
and, surprisingly enough, \( \chi \) does exist. Indeed, substituting \( V(\Phi) = V_0 \) in the solutions
(3.20) or (3.21) one can see that the solution is still valid. This holds true even
for \( V_0 = 0 \). The second case arises when the ratio \( \frac{\Phi}{V'(\Phi)} = -\frac{1}{e^6} \), for, then, the change
of time from $t$ to $\chi$ is not valid. In this case, the remaining scale factor is given by $a(t) = c e^{-\frac{(\lambda + \mu) t}{3}}$ in the Type I case or $a(t) = c e^{-\frac{\lambda t}{2}}$ in the Type V case. The continuity equation (3.15) or (4.12) respectively) implies that the pressure and density are constants, say, $p_0$ and $\rho_0$. The remaining equations dictate $\Phi(t) = At + B$ and $V(\Phi) = V_0$ and there exists a relation between $c, p_0, \rho_0, A$ and $B$ due to the quadratic constraint.

The general properties of the space of solutions found for both type I and type V cases, e.g. isotropization, attractors and self-similarity, and quintessence, will be examined in a forthcoming paper, in which also particular cases of high interest are to be explicitly elaborated. Possible applications of the method exhibited here can be the cases of $D + 1$ spatially homogeneous spacetimes, in which the Outer Automorphisms are rich enough to provide sufficient integrals of motion. Then the method will be applicable if the matter content is such that the integrals of motion persist.
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