Abstract

Considering the cubic theory of gravity which has been constructed in [6], we study the existence of Lifshitz and hyper scaling violating Lifshitz solutions. We firstly extend the black hole solution of [6] and find that such extended solutions are valid for any value of dynamical exponent $z$.

Next, we examine the existence of the AdS black hole solution with non-zero hyperscaling-violating exponent $\theta$ and general dynamical exponent $z$. We find that the solutions do exist only for $\theta = 0, 3$ in 4 dimension and $\theta = 0, 4$ in 5 dimension. In particular, when $\theta = 3(4)$ in 4(5) dimension, we have Lifshitz and Schwarzschild-AdS black hole solution solution for $z = \{0, 1\}$.  

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1 Introduction

Recently, extended theories of gravity has attracted serious attentions to study their roll in quantization of metric field, black hole physics, AdS/CFT [1], cosmology and etc. Several intelligent approaches have been constructed such as extended theories with higher curvature, Horava-Lifshitz gravity, bi-gravity, theories with higher spin fields, gravity with higher dimensional effects and etc. See for example [2]. Usually, the analysis of these theories are based on Lagrangian formalism. Accordingly, One should firstly find the Lagrangian of the fields presented in the theory. Specially, for higher derivative theories, different prescriptions was used for adding higher curvature terms to the Einstein-Hilbert action. For example, in conformal gravity [3], one adds a Weyl tensor at quadratic order \( \int W_{\mu\nu\lambda\theta} W^{\mu\nu\lambda\theta} d^4x \). The resultant action is an action which is invariant under the conformal transformation. Besides of some prior ideas and works on higher curvature gravity [1], a beautiful way was recently suggested in [5] for finding higher order curvature action using a concept which is called "Holographic c-Theorem". In fact, considering a simple holographic c-theorem, the author of [5] was able to obtain constrains on new terms with higher order in curvature. By generalizing this idea to higher dimensions("Holographic a-theorem"), the action was calculated at cubic and quartic order in curvature for arbitrary dimension [6, 7]. Also, another approach introduced for calculating third order gravity in [8] where it was used the general properties of Wyle tensor. After finding these new actions, it was studied various aspects of these new extended theories of gravity such as its black hole solution and etc [6, 7](see also [9] as some recent works). There are some other approaches to study higher curvature theories of gravity, see for example [10].

The problem of finding the solution of equation of motion for complete cubic theory would be important and interesting. A class of very important and attractive solutions are Lifshitz geometry. The background metric of such geometry is given by [11]

\[
ds^2 = -\frac{r^{2z}}{L^2} dt^2 + \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} d\vec{x}^2,
\]

where \( z \) is dynamical exponent which exhibits space and time scale differently in Lifshitz geometry. In fact, Such geometry admits anisotropic scaling property

\[
t \mapsto \lambda^z t, \quad r \mapsto \lambda^{-1} r, \quad \vec{x} \mapsto \lambda \vec{x}.
\]
A generalization of this geometry is conformally Lifshitz geometry with the following line element [12]

\[ ds^2 = r^{-2\frac{\theta}{d-1}} \left( -\frac{r^{2z}}{L^2} dt^2 + \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} d\vec{x}^2 \right). \] (3)

Here \( \theta \) is called the hyperscaling violation exponent. In the context of AdS/CFT, the non-zero hyperscaling violation exponent means that in dual field theory the hyperscaling violates and the entropy scales as \( T^{d-\theta-1} \). These solutions and their properties have been studied in several papers for the both Lifshitz and conformally Lifshitz geometries, see for example [13, 14]. It is noteworthy that (conformally) Lifshitz geometries may also be generated in theories where the gravity and matter coupled to each other [15]. Let us define \( w = \frac{\theta}{d-1} \) in the rest of the paper.

Considering the facts that a curvature cubed theory of gravity which was obtained in [6] involves the standard Einstein term, cosmological constant term, Gauss-Bonnet term and three parameters family of curvature cubed terms potentially can produce anisotropy in space and time, we attempt to find (conformally)Lifshitz and asymptotically (conformally)Lifshitz-black hole solution for such curvature cubed theory of gravity. We shall firstly extend the black hole solution of [6] and find that such extended solutions are valid for any value of dynamical exponent \( z \) and are also degenerate. We shall also examine the existence of the Schwarzschild-AdS black hole solution with non-zero hyper scaling violating exponent \( \theta \). We will find that the solutions do exist only for \( w = 0, 1 \). In particular, when \( w = 1 \), we have Lifshitz solution for \( z = \{0, 1, 6\} \) and Schwarzschild-AdS black hole solution for \( z = \{0, 1\} \) with certain constrains on parameters of the theory.

The paper organized as follows. In the next section, we will briefly introduce curvature cubed gravity. In section 3, we consider the solution where was found in [6] for complete cubic theory of gravity in 5 dimension. In section 4, we study (conformally)Lifshitz and (conformally)Lifshitz-black hole solution of cubic gravity in 4 and 5 dimensions. Finally, we end up the paper by concluding remarks.

### 2 Action of Cubic Gravity

To begin, we firstly briefly review curvature cubed theory of gravity which was introduced in [6]. The action of such cubic order in curvature can be
written as following

\[ I = \frac{1}{2^{d-1}} \int d^{d+1}x \sqrt{-g} \left( \frac{d(d-1)}{L^2} \alpha + R + L^2 \mathcal{X} + L^4 \mathcal{Z} \right), \quad (4) \]

where \( \mathcal{X} \) and \( \mathcal{Z} \) contain interactions at quadratic and cubic order of curvature. Hereafter, we will consider \( L = 1 \). In [6, 8], it was shown that \( \mathcal{X} \) is the four-dimensional Euler density [4],

\[ \mathcal{X}_4 = R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \quad (5) \]

For curvature-cubed interactions, it was also argued that a three-parameter family of unitary \( R^3 \) interactions would exist. It can be described in terms of a basis of three independent interactions. The first of these is the cubic Love-lock interaction which is proportional to the six-dimensional Euler density \( \mathcal{X}_6 \) and is given by

\[ \mathcal{X}_6 = \frac{1}{8} \varepsilon_{abcdef} \varepsilon^{ghijkl} R_{ab}^{gh} R_{cd}^{ij} R_{ef}^{kl} \]

\[ = 4 R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} - 8 R_{a b}^{c d} R_{c d}^{e f} R_{e f}^{a b} - 24 R_{abcd} R^{abcd} R^{de} + 3 R_{abcd} R^{abcd} R + 24 R_{abc} R_{abd}^{cd} R^{bde} + 16 R_{a b}^{c d} R_{c d}^{a b} R + 12 R_{a b}^{c d} R_{c d}^{a b} R + R^3 \]

For \( d = 5 \), it does not contribute to the equations of motion. For \( d \leq 4 \), this term vanishes.

The second basis interaction is the quasi-topological interaction \( \mathcal{Z}_{d+1} \),

\[ \mathcal{Z}_{d+1} = R_a^{c d} R_c^{d f} R_e^{a b} + \frac{1}{2(d-1)(d-3)} \left( \frac{3(d-5)}{8} R_{abcd} R^{abcd} R ight. \\
- 3(d-1)R_{abcd} R^{abcd} R^{de} + 3(d-1)R_{abcd} R^{abcd} R^{bd} \\
+ 6(d-1)R_a^{b c} R_b^{c a} R - \left( \frac{3(d-1)}{2} R_a^{b c} R_b^{c a} R + \frac{3(d-1)}{8} R^3 \right). \quad (7) \]

This term was only constructed for \( d = 4 \) and \( d \geq 6 \). So, we don’t have \( \mathcal{Z}_{d+1} \) as a basis interaction in \( d \leq 3 \) or \( d = 5 \). Two other basis for the cubic interaction are constructed from Weyl tensor

\[ \mathcal{W}_1 = W_a^{c d} W_c^{d f} W_e^{a b}, \quad \mathcal{W}_2 = W_{ab}^{cd} W_{cd}^{ef} W_{ef}^{ab}. \quad (8) \]

where

\[ W_{abcd} = R_{abcd} - \frac{2}{d-1} \left( g_{a[c} R_{d]b} - g_{b[c} R_{d]a} \right) + \frac{2}{d(d-1)} R g_{a[c} g_{d]b} \quad (9) \]
is the Weyl Tensor in \(d+1\) dimensions. In fact, these terms do not change the linearized equations of motion. But, as it was noted in [6], such interactions would effect some other properties of the boundary QFT. For example, they would contribute in calculation of the correlation function of the stress tensor.

The above interactions are not all independent for \(d \geq 6\). In fact, we have the following relation between the basis interactions

\[
Z_{d+1} = \mathcal{W}_1 + \frac{3d^2 - 9d + 4}{8(2d-1)(d-3)(d-4)} (\mathcal{X}_6 + 8\mathcal{W}_1 - 4\mathcal{W}_2).
\]  

(10)

Hence, one can use any three of the above interactions as basis for the curvature-cubed interactions when \(d \geq 6\). For \(d = 5\), \(Z_6\) is not defined and so the basis are \(X_6\), \(W_1\) and \(W_2\). For \(d < 5\), Schouten identities reduce the number of possible interactions with \(X_6 = 0\) and \(W_1 = W_2\) [16]. Thus, for \(d = 4\), we have a two parameter family of interactions with \(Z_5\) and \(W_1\). For \(d = 3\), \(Z_4\) is also not defined and so we have a one parameter interaction with only \(W_1\).

Henceforth in this paper, we focus our attention to \(d = 3\) and \(d = 4\) dimensions. The generalization to higher dimensions is straightforward. Our aim is to find static spherically symmetric hyperscaling violated-Lifshitz solutions of full third order action (4). So, let us rewrite the action (4) as

\[
I = \frac{1}{2l_p^{d-1}} \int d^{d+1}x \sqrt{-g} \left( d(d-1)\alpha + R + \lambda \mathcal{X}_4 + \mu Z_{d+1} + \beta \mathcal{W} \right),
\]

(11)

where \(\alpha\) is cosmological constant and \(\lambda, \mu, \beta\) are three arbitrary real parameters and \(\mathcal{W} = \mathcal{W}_1 = \mathcal{W}_2\). We will also consider the following static spherically symmetric ansatz,

\[
ds^2 = -f^2(r)dt^2 + \frac{dr^2}{k^2(r)} + j^2(r)dl_k^2,
\]

(12)

where \(dl_k^2\) is the line element for spherical, flat and hyperboloid geometry as

\[
dl_k^2 = dx_1^2 + k^{-1}sin^2(\sqrt{k}x_1) \left( dx_2^2 + \sum_{i=3}^{d-1} \prod_{j=2}^{i-1} sin^2 \theta_j dx_i^2 \right).
\]

(13)

2Here, it was used the standard notation \(X_{[ab]} = \frac{1}{2} (X_{ab} - X_{ba})\).

3In literatures, the coefficients \(\lambda\) and \(\mu\) usually are taken to be \(\lambda \rightarrow \frac{8(d-1)}{(d-3)(d-2)(d^2 - 2d + 4)}\) and \(\mu \rightarrow -\frac{8(2d-1)}{(d-3)(d-2)(d^2 - 2d + 4)}\).
For the above metric, one can easily show that all curvature dependent functions can be rewritten using the following four independent variables

\[ m = g^{rr} g^{ii} R_{rir}, \quad n = g^{tt} g^{rr} R_{trr}, \]
\[ p = g^{tt} g^{ii} R_{tit}, \quad q = g^{ii} g^{jj} R_{ijj}. \]

(14)

where indices \( i \) stands for \( x_i \) coordinates. In particular, one can obtain

\[ R = 2n + 2(d - 1)m + 2(d - 1)p + (d - 1)(d - 2)q \]

(15)

\[ \mathcal{X}_4 = (d - 1)(d - 2) (8mp + 4nq + 4(d - 3)mq + 4(d - 3)qp + (d - 3)(d - 4)q^2) \]

(16)

\[ Z_{d+1} = \frac{(d^2 - 3d + 2)(3d^2 - 9d + 4)}{8(2d - 1)} \]
\[ \quad (q (6m \ (4p + (d - 5)q) + q (6n + (d - 5) \ (6p + (d - 6)q)))) \]

\[ W_1 = W_2 = - \frac{8(d - 2)(d^3 - 6d^2 + 11d - 4)}{d^2(d - 1)^2} \ (m - n + p - q)^3 \]

(18)

Evaluating the action (4) for the ansatz (12) and varying it with respect to functions \( f(r), h(r) \) and \( j(r) \) gives us three equation of motion for these fields. It is worth noting to recall that whenever \( w = 0 \) we won’t have hyperscaling violation. Therefore, for spherically symmetric background \( j(r) = r \) and we would have two equations for \( f(r) \) and \( h(r) \). The resulting equations of motion are very complicated and tedious. In the next section, we will present reduced form of them and try to find the hyper violating solutions.

3 Hyper Violating Lifshitz Solution

In this section, we will find the equation of motion and its solutions for the full action of cubic gravity (11). For this aim, we perform the computations from two ways. Firstly, in the next section, we generalize the black hole solution where found in [6] for the action (11) with \( \beta \neq 0 \). Secondly, we attempt to find conditions for having a simple hyper violating Lifshitz and Schwarzschild-AdS black hole solution.
3.1 Black Hole Solution

As it was mentioned before, the author of [6] have found 5–dimensional block hole solution using the action (11) with \(\alpha = 1, \beta = 0\) and the ansatz

\[
ds^2 = -(k + r^2 F(r)) N(r)^2 dt^2 + \frac{dr^2}{(k + r^2 F(r))} + r^2 dl_r^2.
\]

The solution is given by

\[
N(r) = \frac{1}{F_\infty}, \quad F_\infty = \lim_{r \to \infty} F(r), \quad \alpha - F + 2\lambda F^2 + \frac{4}{7}\mu F^3 = \frac{\omega^4}{r^4},
\]

where \(\omega\) is an arbitrary constant. After the work of [6], the generalization to Lifshitz solution has been done in [14]. In this case, using (12) with the following replacements

\[
f_2(r) \to r^2 f(r), \quad h_2(r) \to r^2 g(r), \quad j(r) = r,
\]

one can again obtain the algebraic equation (20) for \(\kappa(r)\) where

\[
\kappa(r) = g(r) - \frac{k}{r^2},
\]

and two constraint on parameters

\[
\lambda = 1 - \frac{3}{2}\alpha, \quad \mu = \frac{7}{4}(2\alpha - 1).
\]

Interestingly, the above constraint and equation (20) are enough to solve the equation of motion and thus the function \(f(r)\) remains free and we have degenerate solution. It is easy to see that if \(\omega = 0\) then \(\kappa = 1\) with the conditions (23) would solve (20).

Now, we would like to examine whether such solution is a solution of the full action (11) with \(\beta \neq 0\) or not. So, it is enough to variate the last part of (11) with respect to \(f(r)\) and \(h(r)\) and find the equations which get from this part. Due to third power of curvature, the resulting equations are very complicated but, one can show that the solutions of the following differential equation\(^4\) do solve two equations of motion simultaneously,

\[
2f \left( (z - 1)r^2 (rg' + 2zg') - 2k \right) - r^4 g' f^2 + r^3 f (f' (rg' + 4zg) + 2rgf'') = 0.
\]

\(^4\)In fact, this differential equation comes from the factoring of equation which is the result of variation of the action with respect to \(f(r)\).
where the prime denotes the differentiation with respect to $r$. The solution of the above equation reads as

$$g(r) = \left(\frac{4kr^2(z-1)f(r) \pm c^2}{(rf'(r) + 2(z-1)f(r))} \right) r^{-2z},$$

(25)

where $c$ is an integration constant. As it is clear, up to this stage, the function $f(r)$ again is free and we have degeneracy on the solutions. The next step is to insert $\kappa(r) = g(r) - \frac{k}{r^2}$ into (20) and solve the new nonlinear equation. However, solving this differential equation is very hard but for some simplified versions, one may be able to solve it. In particular, we consider $\kappa = 1$ which means that $g(r) = 1 + \frac{k}{r^2}$. We also consider $\omega = 0$. Then, one can show that

$$f(r) = \frac{1}{4} (c - 2r\tilde{c}) r^{-2z}, \quad k = 0, \quad c > 0$$

(26)

$$f(r) = \left(2 + r^2 \pm 2\sqrt{1 + r^2}\right) r^{-2z}, \quad k = 1, \quad c = 0.$$  

(27)

Note that in the above solutions, for $k = 0$ the $c > 0$ is necessary but for $k = 1$ we set zero all the integration constants for simplicity. One can also find solution for $k = -1$ but the solution is not a real function.

As a summary, the solution (25) with (20-23) is a complete solution of equations of motion for the full action (11) with arbitrary $\beta$.

### 3.2 Schwarzschild-AdS Solution

In this section, our aim is to obtain the simple Schwarzschild-AdS solution. In fact, we want to find hyper scaling violating Lifshitz solution [12, 13] by

$\tilde{c}$

For the specific case where $f(r) = g(r)$, firstly, we solve (25). The result is

$$f(r) = g(r) = \left(-c(z-2)^2 r^2 + 4(r^2 - 2 + \tilde{c}(z-2))\right) \frac{kr^{-2z}}{4(z-2)^2}, \quad k \neq 0,$$

$$f(r) = g(r) = \left(\pm \sqrt{c} r^2 + \tilde{c}(z-2)r^2\right) \frac{r^{-2z}}{(z-2)}, \quad k = 0$$

(28)

where $c, \tilde{c}$ are arbitrary constants. The above solutions are singular at $z = 2$. Inserting $z = 2$ in (25) and solving it with the condition $f(r) = g(r)$ one obtains

$$f(r) = g(r) = \frac{1}{4kr^2} \left(-c + 4k^2 \tilde{c}^2 - 8\tilde{c}k^2 \log r + 4k^2 \log r^2\right)$$

(29)

which is a log solution with singularity in $r = 0$. Then, we should insert these solutions into (20) to find constraint on coefficients.
using the following redefinitions in the ansatz (12),

\[ f(r) = r^{z-w-1}g(r), \quad h(r) = r^w g(r), \quad j(r) = r^{1-w}. \]  

(30)

In this section, we also perform more simplification by considering

\[ g(r) = r^2 - s \]  

(31)

with some arbitrary constant \( s \). By these assumptions, we search the conditions on parameters \( \alpha, \lambda, \mu, \beta \) and \( s \) in which \( h(r) = r^w (r^2 - s) \), \( f(r) = r^{z-w-1}(r^2 - s) \) would be the solutions of equation of motion. We will do the computations for 4 and 5-dimensional cases separately. The generalization to higher dimensions is straightforward.

3.2.1 4-Dimension

First of all, Recall that in 4 dimension, \( W_1 = W_2 \) and \( \mathcal{X}_4 \) is a topological term. Then, varying the action (11), evaluated for the ansatz (12), with respect to \( f(\mathcal{r}) \), \( h(\mathcal{r}) \) and \( j(\mathcal{r}) \) and inserting (30) and (31) into the equations of motion, one finds that the following polynomials should be zero for all range of \( r \),

\[ P_{h,f,j}(r) = \sum_{a,b=0}^{3} P_{ab}^{h,f,j} r^{2a+2bw} = 0, \]  

(32)

where \( P_{ab}^{h,f,j} \) are some constants and are given by\(^6\)

\[ \begin{align*}
    P_{00}^h &= -4\beta k^3, \\
    P_{10}^h &= 0, \\
    P_{20}^h &= -18k, \\
    P_{30}^h &= -54\alpha, \\
    P_{01}^h &= 24\beta k^2 s(1 + w)(-2 + z), \\
    P_{11}^h &= -24\beta k^2 (1 + w)(-1 + z), \\
    P_{21}^h &= -18s(-1 + w)(1 + 3w - 2z), \\
    P_{31}^h &= 18(-1 + w)(-1 + 3w - 2z), \\
    P_{02}^h &= 12\beta ks^2(-2 + z)^3(2 + z), \\
    P_{12}^h &= -24\beta ks(-2 + z)^2(-2 + z + z^2),
\end{align*} \]

\(^6\)Here, \( h, f \) and \( j \) stands for variation of the action with respect to \( h(\mathcal{r}) \) and \( f(\mathcal{r}) \) respectively.
\[ P_{22}^h = 12\beta k(-1 + z)^2 z^2, \]
\[ P_{32}^h = 0, \]
\[ P_{03}^h = -8\beta s^3(-1 + 3w - z)(-2 + z)^5, \]
\[ P_{13}^h = 24\beta s^2(-2 + z)^3(-1 + z)(2 + z - z^2 + w(-2 + 3z)), \]
\[ P_{23}^h = 24\beta s(-2 + z)(-1 + z)^2 z(4w - 3(1 + w)z + z^2), \]
\[ P_{33}^h = 8\beta(3 + 3w - z)(-1 + z)^3 z^2, \]

and

\[ P_{00}^f = -4\beta k^3, \]
\[ P_{10}^f = 0, \]
\[ P_{20}^f = -18k, \]
\[ P_{30}^f = -54\alpha, \]
\[ P_{01}^f = -48\beta k^2 s(1 + w)^2, \]
\[ P_{11}^f = 24\beta k^2(1 + w)(1 + 2w), \]
\[ P_{21}^f = -18s(-1 + w)^2, \]
\[ P_{31}^f = 18(-3 + w)(-1 + w), \]
\[ P_{02}^f = 12\beta ks^2(-2 + z)^2(-4 + (-4 + z)z), \]
\[ P_{12}^f = -24\beta ks(-2 + z)^3(1 + z), \]
\[ P_{22}^f = 12\beta k(-1 + z)^2 z^2, \]
\[ P_{32}^f = 0, \]
\[ P_{03}^f = 8\beta s^3(-2 + z)^4(-2 - 6(-2 + w)w + (-4 + z)z), \]
\[ P_{13}^f = -24\beta s^2(-2 + z)^2(-4 + 3w(-2 + z)^2 + z^2(7 + (-5 + z)z) - 2w^2(4 + 3(-2 + z)z)), \]
\[ P_{23}^f = 24\beta s(-1 + z)z((1 - z)z(2 + (-4 + z)z) - 2w^2(8 + 3(-3 + z)z) - 2w(4 + 3(-2 + z)z)), \]
\[ P_{33}^f = 8\beta(-1 + z)^2 z^2(9 + 3w(5 + 2w) + z - z^2), \]
and

\begin{align*}
P_{j0}^0 &= 4\beta k^3, \\
P_{j1}^0 &= 0, \\
P_{j2}^0 &= 18k, \\
P_{j3}^0 &= 54\alpha, \\
P_{j0}^1 &= -12\beta k^2 s(3 + w - z)(2w + z), \\
P_{j1}^1 &= 12\beta k^2(2(1 + w)^2 - wz - z^2), \\
P_{j2}^1 &= 18s(1 + w - z)^2, \\
P_{j3}^1 &= -18(1 + w^2 + z + z^2 - 2w(1 + z)), \\
P_{j0}^2 &= 12\beta ks^2(-2 + z)^2(-4 + (-2 + z)z), \\
P_{j1}^2 &= -24\beta ks(-2 + z)^2(-2 + z^2), \\
P_{j2}^2 &= 12\beta k(-1 + z)^2 z^2, \\
P_{j3}^2 &= 0, \\
P_{j0}^3 &= -4\beta s^3(-2 + z)^4(8 + 6(-3 + w)w + z + 3wz - z^2), \\
P_{j1}^3 &= -12\beta s^2(-2 + z)^2(-2w^2(4 + 3(-2 + z)z) + \\
&\quad(-2 + z)^2(-2 + z^2) - w(-2 + z)(8 + z(-8 + 3z))), \\
P_{j2}^3 &= 12\beta s(-1 + z)z((-4 + z)(-1 + z)^2z + \\
&\quad wz(-6 + (7 - 3z)z) - 2w^2(8 + 3(-3 + z)z)), \\
P_{j3}^3 &= 4\beta(-1 + z)^2 z^2(6 + 6w^2 - (-4 + z)z + 3w(4 + z)),
\end{align*}

Now, for finding the solutions, a few comments are in order. Firstly, we would like to find the solution by imposing the condition \(\lambda \neq 0\) and \(\mu \neq 0\) and \(\beta \neq 0\). Secondly, as far as one choose a specific value for \(w\), then some powers of \(r\) in (32) would be equal to each other and so the sum of the coefficients of these terms should be set to zero. For example, if \(w = 1\) then the power of \(r\) for \(a = 0, b = 3\) and \(a = 3, b = 0\) are equal and so we should set the sum of \(P_{03}\) and \(P_{30}\) to zero. Finally, noting the coefficients of \(P_{31}^{f,h}\) and \(P_{33}^{f,h}\) implies that

\[ -3 \leq w \leq 3. \]  

By further analysis, one can show the solution with the condition \(\lambda \neq 0\) and \(\mu \neq 0\) and \(\beta \neq 0\) do exist only for the following two cases

\[ w = 0, 1. \]
The case \( w = 0 \) is the usual geometry without scaling violation but for \( w = 1 \) we have hyper scaling violation. In 4 dimension, the \( w = 1 \) case implies that hyper scaling violation exponent \( \theta \) should be equal to 3.

- \( w = 0 \)
  - In this case, the solution depends on the value of \( z \).
    
    \[
    a) \quad s = 0, \quad k = 0, \quad \alpha \neq 0, \quad z \neq \{0, 1, 4\} \\
    \alpha = \frac{18 + 7z + 4z^2 - 2z^3}{9(4 - z)}, \quad \beta = -\frac{3}{2z^2(4 - 5z + z^2)}.
    \]
    
    \[
    b) \quad s = -k, \quad \alpha \neq 0, \quad z = \{1\} \\
    \alpha = 1, \quad \beta = \text{arbitrary}. \tag{38}
    \]
  
  The case \( a \) is the standard Lifshitz geometry \([3]\). The second case with \( s = -k \) gives us the black hole solution but only for \( z = 1 \) and arbitrary \( \beta \).
  
  Here, there is not consistent solution for \( z = \{0, 4\} \).

- \( w = 1 \)
  - The solution depends on \( z \) and whether \( s \) is equal to zero or not.
    
    \[
    a) \quad s = 0, \quad k = 0, \quad \alpha = 0 \quad z = \{0, 1\}; \\
    \beta = \text{arbitrary} \tag{39}
    \]
    
    \[
    b) \quad s \neq 0, \quad k = 0, \quad \alpha \neq 0 \quad z = \{0, 1\} \\
    s = -\frac{(2z + 1)^2}{(z + 1)^3} \alpha, \quad \beta = \frac{27}{4(z - 2)^3} \alpha s^3.
    \]
  
  The case \( a \) again gives us the Lifshitz geometry \([3]\) but only for \( z = \{0, 1\} \). The second case \( b \) gives us a black hole solution but only with flat horizon.

### 3.2.2 5-Dimension

In 5-dimensional, we should also consider the \( \mathcal{A}_4 \) and \( \mathcal{Z}_5 \) terms. The remaining procedure is quite similar to 4 dimensional. Using the ansatz \([30]\) and \([31]\)\(^7\) \note{Note that we always consider \( \beta \neq 0 \).}
in equations of motion for $f(r), h(r)$ and $j(r)$, one finds the coefficients $P_{ab}^{f,h,j}$ as follows

\[ P_{00}^h = 8(27\mu + 7\beta)k^3 \]
\[ P_{10}^h = 0, \]
\[ P_{20}^h = 378k, \]
\[ P_{30}^h = 756\alpha, \]
\[ P_{01}^h = -24(27\mu + 7\beta)k^2s(1 + 3w)(-2 + z), \]
\[ P_{11}^h = 24k(27\mu k(1 + 3w)(-1 + z) + 7\beta k(1 + 3w)(-1 + z) + 63\lambda s(-1 + 5w^2 + z - 3wz)), \]
\[ P_{21}^h = -378(-s(-1 + w)(2w - z) + 4\lambda k(5w^2 + z - 3w(1 + z))), \]
\[ P_{31}^h = -378(-1 + w)(-1 + 2w - z), \]
\[ P_{02}^h = 24ks^2(27\mu(-1 + w)^2(3 + w(14 + 7w - 10z) - 2z) - 7\beta(-2 + z)^3(2w + z)), \]
\[ P_{12}^h = 24s(-54\mu k(-1 + w)^2(2 + w(9 + 7w - 10z) - 2z) + 7(9\lambda s(-1 + w)^3(1 + w - z) + 2\beta k(-2 + z)(-1 + z)(-2 + 2w(-1 + z) + (-2 + z)z))), \]
\[ P_{22}^h = 24(27\mu k(-1 + w)^2(1 + w(4 + 7w - 10z) - 2z) - 7\lambda s(-1 + w)^3(1 + 2w - 2z) + \beta k(-1 + z)^2z(-2 + 2w + z))), \]
\[ P_{32}^h = 1512\lambda(-1 + w)^3(w - z), \]
\[ P_{03}^h = -8s^3(27\mu(-1 + w)^5(4 + 2w - 3z) - 7\beta(1 + 3w - 2z)(-2 + z)^5), \]
\[ P_{13}^h = 24s^2(27\mu(-1 + w)^5(3 + 2w - 3z) + 7\beta(-2 + z)^3(-1 + z)(-2 + w(2 - 3z) + z(-5 + 2z))), \]
\[ P_{23}^h = -24s(27\mu(-1 + w)^5(2 + 2w - 3z) + 7\beta(-2 + z)(-1 + z)^2z(4(1 + w) - 3(3 + w)z + 2z^2)), \]
\[ P_{33}^h = 8(27\mu(-1 + w)^5(1 + 2w - 3z) - 7\beta(9 + 3w - 2z)(-1 + z)^3z^2), \]

and

\[ P_{00}^f = 8(27\mu + 7\beta)k^3, \]
\[ P_{10}^f = 0, \]
\[ P_{20}^f = 378k, \]
\[ P_{30}^f = 756\alpha, \]
\[ P_{01}^f = 24(27\mu + 7\beta)s(k + 3kw)^2, \]
\[ P_{11}' = -72kw(27\mu(k + 3kw) + 7(-12\lambda sw + \beta(k + 3kw))), \]
\[ P_{21}' = -378(-s(-1 + w)^2 + 4\lambda k(1 + w(-3 + 4w))), \]
\[ P_{31}' = -378(-2 + w)(-1 + w), \]
\[ P_{02}' = 24ks^2(27\mu(-1 + w)^2(1 + w(6 + w)) \]
\[-7\beta(-2 + z)^2(2 - 2w(2 + w) + (-4 + z)z), \]
\[ P_{12}' = -48ks(27\mu(-1 + w)^2w(1 + w) \]
\[+7\beta(-2 + z)z - w(-4 + z)z + w^2(4 + z(-5 + 2z))), \]
\[ P_{22}' = 24(27\mu k(-1 + w)^2(-1 + (-4 + w)w) + 7(9\lambda s(-1 + w)^3 \]
\[-\beta k(-1 + z)z(-4 - 2(-3 + w)w + (-1 + z)z)), \]
\[ P_{32}' = -1512\lambda(-1 + w)^3, \]
\[ P_{03}' = 8s^3(27\mu(-1 + w)^6 \]
\[-7\beta(-2 + z)^4(5 - 3(-2 + w)w + 2(-4 + z)z), \]
\[ P_{13}' = -24s^2(27\mu(-1 + w)^5w \]
\[+7\beta(-2 + z)^2(-2(-3 + z)(-1 + z)z \]
\[+ w^2(4 + 3(-2 + z)z + w(-4 + 3z^2))), \]
\[ P_{23}' = 24s(27\mu(-1 + w)^5(1 + w) \]
\[-7\beta(-1 + z)z(-16 + w^2(-8 - 3(-3 + z)z) \]
\[-6w(1 + z(-5 + 2z)) + z(17 + z(3 + 2(-5 + z)z))), \]
\[ P_{33}' = -8(27\mu(-1 + w)^5(2 + w) \]
\[+ 7\beta(-1 + z)^2z^2(36 + 3w(7 + w) - 2(-1 + z)z), \]

and
\[ P_{00}^j = 8(27\mu + 7\beta)k^3, \] \hspace{1cm} (42)
\[ P_{10}^j = 0, \]
\[ P_{20}^j = 378k, \]
\[ P_{30}^j = 756\alpha, \]
\[ P_{01}^j = -8(27\mu + 7\beta)k^2s(-9 + 9w^2 - 3w(-4 + z) + (11 - 3z)z), \]
\[ P_{11}^j = 8k(27\mu k(1 + 9w^2 - 3w(-2 + z) + (2 - 3z)z) + \]
\[7(\beta k(1 + 9w^2 - 3w(-2 + z) + (2 - 3z)z) + \]
\[9\lambda s(4 + 2w(5 + 3w) - 7z - 7wz + 3z^2))), \]

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In this case, analyzing the coefficients $P^{f,h}_{31}$ and $P^{f,h}_{32}$ implies that the allowed values for $w$ are
\[ w = 0, 1. \] (43)
So, the hyper scaling violation exponent is equal to 4 and the solutions are given by
• $w = 0$

\[ a) \quad s = 0, \quad k = 0, \quad z \neq \{0, 1\} \]
\[
\lambda = \frac{-3\alpha(1 + 15z^2 - 18z^3 + 3z^4)}{2(1 + 9z^2 + 16z^3 - 21z^4 - 6z^5 + 2z^6)}
+ \frac{(2 + 27z^2 - 19z^3 - 6z^4 - 3z^5 + z^6)}{2(1 + 9z^2 + 16z^3 - 21z^4 - 6z^5 + 2z^6)}
\]
\[
\mu = \frac{7(2\alpha - 1)}{4(1 + 9z^2 + 16z^3 - 21z^4 - 6z^5 + 2z^6)}
\]
\[
\beta = \frac{27(2\alpha - 1)}{4(1 + 9z^2 + 16z^3 - 21z^4 - 6z^5 + 2z^6)} \quad (44)
\]

\[ b) \quad s = 0, \quad k = 0, \quad z = \{0\} \]
\[
\lambda = 1 - \frac{3\alpha}{2}, \quad \mu = \frac{7}{4}(2\alpha - 1), \quad \beta = \text{arbitrary}, \quad (45)
\]

\[ c) \quad s = 0, \quad k = 0, \quad z = \{1\} \]
\[
\lambda = \frac{1 - \alpha}{2} - \frac{2\mu}{7}, \quad \mu, \beta = \text{arbitrary}, \quad (46)
\]

\[ d) \quad s = -k, \quad s, k \neq 0, \quad z = \{1\} \]
\[
\lambda = \frac{1 - \alpha}{2} - \frac{2\mu}{7}, \quad \mu, \beta = \text{arbitrary}, \quad (47)
\]

\[ e) \quad s \neq k, \quad s, k \neq 0 \quad z = \{1\} \]
\[
\lambda = 1 - \frac{3\alpha}{2}, \quad \mu = \frac{7}{4}(2\alpha - 1), \quad \beta = \frac{27}{4}(2\alpha - 1) \quad (48)
\]

\[ f) \quad s = -\frac{k}{4}, \quad s, k \neq 0, \quad z = \{3\} \]
\[
\lambda = \frac{1}{218}, \quad \mu = -\frac{7}{1308}, \quad \beta = \frac{9}{436} \quad (49)
\]

The solutions $a, b, c$ are the Lifshitz geometry $[3]$. Thus in 5 dimension, we have Lifshitz solution in cubic gravity for any value of $z$. Notice that the case $(c)$ is in agreement with $[28]$. The cases $d, e, f$ demonstrate black hole solution with various horizon geometry. These black hole solutions exist only for $z = 1$ and $z = 3$. Note also that the parameters $\lambda, \mu$ and $\beta$ are exactly fixed for $z = 3$.  

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$w = 1$

\[ a) \quad s = 0, \quad k = 0, \quad \alpha = 0 \quad z = \{0, 1\} \]
\[ \lambda, \mu, \beta = \text{arbitrary}, \quad (50) \]

\[ b) \quad s \neq 0, \quad k = 0, \quad \alpha \neq 0 \quad z = \{0, 1\} \]
\[ s = -\frac{2(2z + 1)}{z + 1} \alpha \quad \beta = \frac{27}{4\beta(z - 2)^{\theta}} \frac{\alpha}{s^3}, \]
\[ \lambda, \mu = \text{arbitrary}. \quad (51) \]

The hyper scaling violating solutions in 5 dimension also contain Lifshitz background in case $a$ but with flat boundary $k = 0$ and exist only for $z = \{0, 1\}$. We also have black hole solution in case $b$. But these solutions do also exist only for $z = \{0, 1\}$ and we have the constraint (b) on parameters of the action $[\Pi]$.

### 4 Conclusion

In this paper, we study the existence of asymptotically Lifshitz and hyper scaling violating asymptotically Lifshitz solutions in full cubic theory of gravity in 4 and 5 dimensions with the action $[\Pi]$. Such cubic action of curvature has been constructed in [6] using some simple ”Holographic c/a theorem” in arbitrary dimensions.

We firstly extend the black hole solution of [6] for full cubic action $[\Pi]$. This solution is valid for any value of dynamical exponent $z$.

Next, we examine the usual Schwarzschild-AdS solution with non-zero hyper scaling violating exponent $-2\frac{\theta}{d-1}$ and general dynamical exponent $z$. We have found that the solutions do exist only for $\theta = 0, 3$ in 4 dimension and $\theta = 0, 4$ in 5 dimension. In particular, when $\theta = 0$, we have Lifshitz solution for any value of $z$ except of $z = \{0, 4\}$ (in 4 dimension). We also have Schwarzschild-AdS black hole solution with $z = \{1\}$ in 4 dimension and with $z = \{1, 3\}$ in 5 dimension. The above solutions exist with certain constrains on parameters of the theory.

Moreover, when $\theta = 3(4)$ in 4(5) dimension, we have Lifshitz and Schwarzschild-AdS black hole solution for $z = \{0, 1\}$.

At the end, we would like to mention that there are various fields of study for theories with cubic action $[\Pi]$ and their solutions. For example, correlation functions in cubic gravity, the thermodynamics of black hole solutions,
the Noether charges and conserved currents of solutions, their CFT dual and many other problems which are open, important and interesting.

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