Analysis of nonlinear dynamic response of spatial flexible tether system by Symplectic difference scheme

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Abstract. This paper transforms the discrete Langrangian formulation of spatial flexible tether systems to Hamiltonian formulism. The resulting Hamiltonian canonical equations are solved by Symplectic difference scheme. The application of Symplectic difference method for solution of nonlinear tether dynamic problems ensures conservation of system energy, momentum and volume. Two numerical examples are conducted to validate the proposed method, one is the free swing pendulum system and the other one is the three-dimensional circular towed system. The simulation results are compared with theoretical and the existing numerical results. The comparisons demonstrate the proposed Symplectic difference integrator for the Hamiltonian nodal position finite element method is numerically accurate and efficient to predict the dynamic response of spatial flexible tether systems.

1. Introduction
In recent years, flexible tether structures are widely applied in engineering, for instance, the tethered satellite systems [1], underwater remote operated vehicles [2], recovering Micro Air Vehicles [3]. Spatial tethered systems may experience large rotation and long-term motion. In general, tethered systems behave strong nonlinear due to the flexible property, dynamic loads and pertinent boundary conditions [4]. The prediction of the nonlinear dynamic responses of spatial flexible tether systems has become an active field of research. Approximate analytical methods and numerical methods are used to analyse the nonlinear dynamics of spatial flexible tether systems.

Generally speaking, the modelling technique of the tether systems can be grouped into four categories: direct integration method, lumped parameters method, finite difference method and finite element method. 1) The direct integration method recasts the boundary value problem as a series of initial value problems, mainly for the limited cases with simplified assumption. Sun et al. [5], Gatti-Bono and Perkins [6] applied this method to transform the dynamic equation of tether systems into quasi-static equations by an implicit direct integration scheme. The final analysis at each time step are then obtained by recombining the partial solutions to satisfy the boundary conditions of tether systems. 2) The lumped parameters method arising from the lumped mass and external applied force at adjacent nodes is widely used. Buckham et al. [7] applied this method to study the dynamics and control of a towed underwater vehicle system. Williams et al. [8] used the lumped mass model and parameter optimization technique to the dynamic analysis of tethered aircraft. However, the solution process is
diverged easily. 3) The finite difference method is commonly applied for its simplicity. Burgess [9] and Park et al. [10] applied this method to analyse the dynamics of marine slender structure and ocean cable system with modified algorithms. However, this method could not be implemented for complex tether geometries, such as the variation of the tether properties along the tether length. 4) The finite element method derives the dynamic governing equation with an integration over the entire element firstly, by assembling all the elemental equations, the complex geometries or the different properties of the continuous tether system can be easily modeled algorithmically [11]. This method have been widely applied to variable length problems of submerged tethered underwater vehicles [12] and the dynamics of aerial refuelling systems [13].

The demerit with traditional finite element method is the calculation error caused by the discretization of a continue system [14]. When the tether system experiences large rigid body rotation coupled with small deformation, this calculation error can be minimized either by finer mesh or using higher order polynomial elements, however at the cost of integration time. Zhu [15] developed a new nodal position finite element model in order to overcome the challenge by traditional finite element method for the dynamic solution of tether systems. Ding [16, 17] developed a Hamiltonian nodal position finite element method to face the challenge that the error accumulation over long-term numerical calculation for dynamic modelling of spatial tether systems. For flexible tether system dynamics, Hamiltonian method is preferred as it preserves the invariant Symplectic structure [18, 19], the corresponding Symplectic integration schemes ensure conservation of energy, momentum and volume of the tether systems [20].

In this work, the dynamic response of spatial tether system is considered. The dynamic governing equation of tether system is derived and formulated by Hamiltonian canonical formulism, Symplectic difference scheme is applied to solve the resulting canonical equations. After the brief introduction for spatial tether system simulations, Section 2 provides a detail description on the Hamiltonian nodal position finite element method of spatial tether system, and the corresponding Symplectic difference solution scheme. In Section 3, we verify the Hamiltonian nodal position finite element method by the simulations of an undamped free swing pendulum system and a circularly towed system numerically. Finally, we concluded the paper in Section 4.

2. Mathematic model of Hamiltonian nodal position finite element method

The Hamiltonian function of differential element is built by Green-Lagrangian strain theory, and linear constitutive relationship of tether is applied. The integration of the differential equation along the element length produces the dynamic governing equation of each tether element. By assembling all the elemental equations, the system dynamic governing equation will be obtained correspondingly. Finally, the resulting system dynamic governing equation will be solved by the first-order Symplectic difference scheme, coupled with the boundary conditions of the tether system.

2.1. Nodal position finite element theory

Assume that the tether is homogeneous and isoparametric, the local coordinate system o-xyz of the tether element is built as following: the origin point o is located at the elemental starting point and ox is along the tether element length. oy is orthogonal to oz, and both oy and oz are in the plane perpendicular to ox axis. By the classical Tait-Bryan transition, the elemental transformation matrix for each tether element is derived as,

\[
T_{g2i} = \begin{bmatrix}
\cos \theta_x \cos \theta_z & \cos \theta_z \sin \theta_x & -\sin \theta_x \\
-\sin \theta_x \sin \theta_z + \cos \theta_x \sin \theta_y & \cos \theta_z \sin \theta_x + \sin \theta_x \sin \theta_y & \sin \theta_x \\
\sin \theta_y \sin \theta_z + \cos \theta_z \cos \theta_x & \sin \theta_y \cos \theta_z + \cos \theta_z \sin \theta_x & \cos \theta_z
\end{bmatrix}
\]  

(1)

where \(T_{g2i}\) is the elemental transformation matrix, and \((\theta_x, \theta_y, \theta_z)\) are the three rotational angles of the Tait-Bryan transition.

Linear shape function \(N(x)\) is applied to interpolate the position \(r(x, t)\), velocity \(v(x, t)\) and acceleration \(a(x, t)\) of an arbitrary point, using the nodal values of the element, such as
\[ \mathbf{r}(x,t) = \mathbf{N}(x) \mathbf{x}_e(t) \quad \mathbf{v}(x,t) = \mathbf{N}(x) \mathbf{v}_e(t) \quad \mathbf{a}(x,t) = \mathbf{N}(x) \mathbf{a}_e(t) \]  

(2)

where \( \mathbf{x}_e(t), \mathbf{v}_e(t) \) and \( \mathbf{a}_e(t) \) are the vectors of elemental position, velocity and acceleration, respectively.

With the assumption \( \xi = x/L(0 \leq \xi \leq 1) \), \( \mathbf{N}(x) \) can be defined as

\[
\mathbf{N} = \begin{bmatrix}
1 - \xi & 0 & 0 & \xi & 0 & 0 \\
0 & 1 - \xi & 0 & 0 & \xi & 0 \\
0 & 0 & 1 - \xi & 0 & 0 & \xi
\end{bmatrix}
\]  

(3)

The Green-Lagrange strain of the strait tether element is defined as

\[ e_L = \frac{L - L_0}{L_0} \]  

(4)

where \( L_0 \) is the initial length of element.

The elemental kinetic energy \( T_e \) can be obtained by the integration in local coordinate system

\[ T_e = \frac{1}{2} \int_0^{t_e} \rho A \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} dx = \frac{1}{2} \mathbf{v}_e^T \mathbf{M}_e \mathbf{v}_e \]  

(5)

and \( \mathbf{M}_e \) is the elemental mass matrix in local coordinate system which has the expression

\[
\mathbf{M}_e = \frac{1}{2} \int_0^{t_e} \rho A \mathbf{N}^T \cdot \mathbf{N} dx = \frac{\rho A L_0}{6} \begin{bmatrix}
2 \mathbf{I}_{3 \times 3} & \mathbf{I}_{3 \times 3} \\
\mathbf{I}_{3 \times 3} & 2 \mathbf{I}_{3 \times 3}
\end{bmatrix}
\]  

(6)

where \( \rho \) and \( A \) are the density and cross-section area of the tether element, \( \mathbf{I}_{3 \times 3} \) is the \( 3 \times 3 \) unit matrix. It should be noticed that the local elemental mass matrix remains constant.

We can express the elemental kinetic energy \( T_e \) in global coordinate system using the elemental transformation matrix

\[ T_e = \frac{1}{2} \mathbf{v}_e^T \mathbf{M}_e \mathbf{v}_e = \frac{1}{2} \mathbf{V}_e^T \mathbf{M}_e \mathbf{V}_e \]  

(7)

and the elemental mass matrix in global coordinate system can be obtained as

\[ \mathbf{M}_e = \mathbf{T}_e^T \mathbf{M}_e \mathbf{T}_e \]  

(8)

The elastic potential energy \( U_e \) can be integrated in local coordinate system along the tether element based on the Green-Lagrange strain theory

\[ U_e = \frac{1}{2} \int_0^{t_e} E A e_L^2 dx = \frac{1}{2} E A \left( \frac{L - L_0}{L_0} \right)^2 \]  

(9)

where \( E \) is the Young's modulus of tether. The resulting \( U_e \) shows that it will be a same expression in global coordinate system since its value is just related to the undeformed and deformed element length.

For tether structures, the self-weight should be taken into consideration. The direction of gravity is aligned with the OZ axis of global coordinate. We set the horizontal plane as the zero potential plane, which includes the end point of the undeformed free-sagging tether configuration (\( Z = -L_{total} \)). By the application of the linear interpolation function \( \mathbf{N}(X) \), the elemental gravitational potential energy \( U_{Ge} \) in global coordinate system will be obtained as

\[ U_{Ge} = \int_0^{t_e} \left[ \begin{bmatrix}
0 & 0 & -L_{total} - \mathbf{X}_e \mathbf{N}_{0}^T \\
0 & 0 & -\rho \mathbf{A} \mathbf{g}
\end{bmatrix} \right] \mathbf{v}_e^T dx = \rho \mathbf{A} L_0 L_{total} - \mathbf{X}_e^T \mathbf{F}_{Ge} \]  

(10)

where \( g, \mathbf{F}_{Ge} \) and \( L_{total} \) are the gravitational acceleration value, elemental gravitational force vector and total length of initial tether structure, respectively. \( \mathbf{X}_e \) is the global elemental nodal position vector.

\[ \mathbf{F}_{Ge} = \int_0^{t_e} \mathbf{N}_{0}^T \begin{bmatrix}
0 & 0 & -\rho \mathbf{A} \mathbf{g}
\end{bmatrix} dx = \frac{\rho \mathbf{A} L_0}{2} \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix} \]  

(11)

2.2. Hamiltonian formulation

The undamped elemental Lagrangian function can be built from the elemental kinetic energy, elastic potential energy and gravitational potential energy derived before, such as
by applying Legendre’s transformation, the conjugate momentum vector of the tether element can be expressed as

$$P_e = \partial L_e \left( X_e, \dot{X}_e \right) / \partial \dot{X}_e = \tilde{M}_e \dot{X}_e$$

thus, the elemental Hamiltonian function will be

$$H_e (P_e, X_e) = P_e^T \dot{X}_e - L \left( X_e, \dot{X}_e \right)$$

Substituting equations (12) and (13) into (14) will lead to the matrix form of Hamiltonian

$$H_e (P_e, X_e) = T_e + U_e + U_{ge} = \frac{1}{2} P_e^T \tilde{M}_e P_e + \frac{1}{2} EA \frac{(L - L_0)^2}{L_0} + \rho A g L L_{total} - X_e^T F_{ge}$$

Subsequently, the corresponding Hamiltonian canonical equation of the tether element will be

$$\dot{P}_e = -\partial H(P_e, X_e) / \partial X_e = -K_e X_e + F_{ge}$$

$$\dot{X}_e = \partial H(P_e, X_e) / \partial P_e = \tilde{M}_e \dot{P}_e$$

where $K_e$ is the elemental stiffness matrix

$$K_e X_e = \frac{\partial U_e}{\partial X_e} = EA \left[ \frac{1}{L_0} \begin{bmatrix} I_{3x3} & -I_{3x3} \\ -I_{3x3} & I_{3x3} \end{bmatrix} \right]$$

The dynamic governing equation of the tether system would be assembled by all the dynamic governing equations of tether elements following the standard finite element assembly procedure based on the nodal degrees of freedom. According to the position equilibrium condition at the conjunction elemental nodes, the undamped dynamic equation of the tether system can be obtained from assembling equation (16) of each element, such as

$$\ddot{P} = -K X + F_G$$

$$\dot{X} = \tilde{M}_e \ddot{P}$$

2.3. Symplectic difference scheme

Symplectic integration algorithm can preserve all the linear and quadratic conservative quantities of the Hamiltonian formalism. The application of Symplectic difference method for solution of non-linear tether problems ensures conservation of energy, momentum and volume of the systems. For the dynamics of undamped spatial flexible tether system, the first-order Symplectic difference scheme \[14\] for $k+1$ step is

$$P^{k+1} = P^k - h H_X \left( P^{k+1}, X^k \right)$$

$$X^{k+1} = X^k + h H_p \left( P^{k+1}, X^k \right)$$

where $h$ is the calculation time step, $H_X$ and $H_p$ denote the partial differentiation of the system Hamiltonian function to system position vector and momentum vector, this Symplectic difference scheme is an implicit solution scheme. Combination of equations (16), (18) and (19) produces the matrix form of the first-order Symplectic difference algorithm for tether system dynamics,

$$P^{k+1} = P^k - h \left( K X^k - F_G \right)$$

$$X^{k+1} = X^k + h \tilde{M}_e^{-1} P^{k+1}$$

Equation (18) are a series of first-order differential equations, the solution scheme (20) will need the position constraints and momentum boundary conditions of the tether system in the numerical analytical process.
3. Numerical validation
The proposed mathematical model is compiled into codes of VC++ environment. In order to validate the accuracy of the proposed model, two numerical simulations will be adopted in this section. In the following part, the proposed Hamiltonian nodal position finite element method solved by the first-order Symplectic difference algorithm is called HNPFS.

3.1. Free swing pendulum
Consider a free swing pendulum system as shown in Figure 1. which moves in O-XY plane. The pendulum system consists of a massless tether and a particle of mass \( m = 1.0 \text{kg} \) attached to the end of the tether. The tether length \( l = 1.0 \text{m} \), cross sectional area \( A = 1.0 \times 10^{-4} \text{m}^2 \), and the Young's modulus \( E = 500 \text{GPa} \). Assuming that the plan \( y = -l \) as zero potential energy plane and gravitational acceleration is set to be \( g = 9.8 \text{m/s}^2 \). In the calculations, the pendulum system is released at the initial included angle \( \theta = \pi / 3 \text{rad} \). The time step is \( \Delta t = 1 \times 10^{-4} \text{s} \) and the simulation period is 8s.

![Figure 1. Free swing pendulum.](image)

The Lagrangian function of the pendulum system will be

\[
L_{\text{lag}} = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \sin \theta) \tag{21}
\]

Thus, the Second Lagrangian formulation for this pendulum system is

\[
ml^2 \ddot{\theta} - mgl \cos \theta = 0 \tag{22}
\]

The theoretical solution for equation (22) is highly nonlinear and the advanced continuous simulation language (ACSL) software is applied to solve the equation over time history.

By introducing the Hamiltonian variable \( p \), the corresponding Hamiltonian canonical equation is

\[
\dot{p} = -\partial H / \partial q = mgl \cos q \\
\dot{q} = \partial H / \partial p = p / ml^2
\]

and the first-order implicit Symplectic difference method is listed as follow,

\[
p^{k+1} = p^k + \Delta t * mgl \cos(q^k) \\
q^{k+1} = q^k + \Delta t * p^{k+1} / ml^2
\]

Figure 2 shows the time histories of the Hamiltonian variables \( p \) and \( q \) retraced by ACSL and HNPFS, respectively. Figure 3 presents the comparison of the pendulum system energy. The ACSL and HNPFS results are identical as expect, HNPFS keeps the system energy constant while ensuring the accuracy of the solution.
3.2. Three-dimensional circularly towed system
Sugiyama [21] studied the dynamic problem of three-dimensional circularly towed system based on absolute nodal coordinate finite element method. As shown in Figure 4, the tether is assumed to be made from an isotropic rubber with Young’s modulus of $1.0 \times 10^6$ Pa, mass density of $1200 \text{kg/m}^3$, initial tether length of 1.0 m and cross-section area of $4.0 \times 10^{-4} \text{m}^2$. The lumped mass is 1.3 kg.

The simulation of the system starts from the static equilibrium position in which the tether static elongation due to the gravity force is obtained. From the static equilibrium position, the tether system is accelerated for a period of $t_0=1.0 \text{s}$ after which it is towed using a constant angular velocity. The tow radius is $R=0.15 \text{m}$. The position boundary conditions of the towing point is

$$X = \begin{cases} -R \sin(\omega_m t^2 / 2 t_0) & 0 \leq t \leq t_0, \\ -R \sin(\omega_m t - \omega_m t_0 / 2) & t > t_0 \end{cases}, \quad Y = \begin{cases} R \cos(\omega_m t^2 / 2 t_0) & 0 \leq t \leq t_0, \\ R \cos(\omega_m t - \omega_m t_0 / 2) & t > t_0 \end{cases}, \quad Z = 0$$

In the simulations of HNPFS, the tether is modelled by 8 elements averagely, the time step is $\Delta t=5 \times 10^{-5} \text{s}$. The trajectory of towing point can be composed by two parts as mentioned above. Similarly to the simulations by Sugiyama [21], two different scenarios are examined using two different angular velocities of $\omega_m=2.0 \text{rad/s}$ and $\omega_m=3.0 \text{ rad/s}$, respectively. Figure 5 shows the rotation radius of the tip point measured in a horizontal plane from the axis of rotation when the tether system reaches to a dynamic equilibrium state, the HNPFS fit the results of Sugiyama [21] exactly.
4. Conclusions
In this paper, a new Symplectic integrator for the Hamiltonian nodal position finite element method is proposed. The solution has great accuracy and numerical stability with low calculated error accumulation for analyzing the dynamic responses of the spatial flexible tether systems, it ensures conservation of energy, momentum and volume of the tether system. Taking the rigid body motion of the towed body coupled with the flexible tether dynamics into consideration, the proposed solution scheme coupled with the Hamiltonian nodal position finite element method may be applied to the tethered satellite system dynamic problems in subsequent researches.

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