Expansions for semiclassical conformal blocks

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Abstract: We propose a relation the expansions of regular and irregular semiclassical conformal blocks at different branch points making use of the connection between the accessory parameters of the BPZ decoupling equations to the logarithm derivative of isomonodromic tau functions. We give support for these relations by considering two eigenvalue problems for the confluent Heun equations obtained from the linearized perturbation theory of black holes. We first derive the large frequency expansion of the spheroidal equations, and then compare numerically the excited quasi-normal mode spectrum for the Schwarzschild case obtained from the large frequency expansion to the one obtained from the low frequency expansion and with the literature, indicating that the relations hold generically in the complex modulus plane.

Keywords: Conformal Field Theory, Semiclassical Conformal Blocks, Isomonodromy, Black Holes
1 Introduction

Conformal blocks are special functions that arise in the computation of four point functions in conformal field theories. As such, they are naturally dependent on the conformal dimensions of the operators entering the correlation function. Less trivially, they also depend on the internal dimension, sometimes called channel, as well as the conformal modulus $t$ and the central charge of the Virasoro algebra. Conformal blocks are then the building blocks of generic correlation functions in CFTs.

The AGT correspondence [1] related the conformal blocks to instanton partition functions of 4-dimensional supersymmetric gauge theories. The latter had had a well-established representation in terms of Nekrasov functions [2], and the following years saw a great surge of activity in understanding how the correspondence works and applications of conformal blocks outside of CFTs.

Let us mention two of those, with singular interest. First, we single the formulation of the expansion of isomonodromic tau functions, originally defined in [3], in terms of $c = 1$ conformal blocks $\mathcal{B}$. As a representative example, let us present the Kyiv formula for the Painlevé VI tau function, first derived in [4]

$$\tau_{VI}(\theta_k; \sigma, \eta; t) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \eta} \mathcal{B}(\theta_k; \sigma + 2n; t),$$

$$\mathcal{B}(\theta_k; \sigma; t) = N_{\theta_k, \sigma} \mathcal{M}_{\theta_k} \left( \frac{1}{2}(\sigma^2 - \theta_0^2 - \theta_1^2)(1-t) \right)^{\frac{3}{2} \theta_k, \sigma} \sum_{\lambda, \mu \in Y} \mathcal{B}_{\lambda, \mu}(\theta_k, \sigma)t^{\lambda |+| \mu},$$

(1.1)
where the sum on $Y$ are over pairs $(\lambda, \mu)$ of Young diagrams, and

\[
\mathcal{B}_{\lambda, \mu}(\theta_k, \sigma) = \prod_{(i,j) \in \lambda} \frac{((\theta_1 + \sigma + 2(i-j))^2 - \theta_0^2)((\theta_2 + \sigma + 2(i-j))^2 - \theta_0^2)}{16h^2_{\theta}(i,j)(\lambda'_j - i + \mu_j - j + 1 + \sigma)^2} \times \prod_{(i,j) \in \mu} \frac{((\theta_1 - \sigma + 2(i-j))^2 - \theta_0^2)((\theta_2 - \sigma + 2(i-j))^2 - \theta_0^2)}{16h^2_{\theta}(i,j)(\mu'_j - i + \lambda_i - j + 1 - \sigma)^2},
\]

(1.2)

and

\[
\mathcal{N}^{\phi_2}_{\phi_3, \phi_1} = \prod_{\epsilon = \pm} G(1 + \frac{1}{2}(\phi_3 + \epsilon(\phi_1 + \phi_2)))G(1 - \frac{1}{2}(\phi_3 + \epsilon(\phi_1 - \phi_2))) \times \frac{\Gamma(1 - \theta_0)\Gamma(\theta_2)}{\Gamma(\frac{1}{2}(1 + \theta_2 - \theta_0 + \theta_3))\Gamma(\frac{1}{2}(1 + \theta_2 - \theta_0 - \theta_3))} \exp \left[ \frac{\partial \mathcal{W}}{\partial \theta_2} - \frac{\partial \mathcal{W}}{\partial \theta_0} \right],
\]

(1.3)

where $G(z)$ is the Barnes’ function. Similar expansions are also available for Painlevé V and III [5], which can be seen as confluent limits of (1.1), making use of irregular conformal blocks [6], as explained in [7, 8].

More recently, semiclassical conformal blocks have also been proposed as solutions to the connection problem of solutions to the Heun equation [9], in the Trieste formula [10]

\[
\mathcal{C}_{01} = \frac{\Gamma(1 - \theta_0)\Gamma(\theta_2)}{\Gamma(\frac{1}{2}(1 + \theta_2 - \theta_0 + \theta_3))\Gamma(\frac{1}{2}(1 + \theta_2 - \theta_0 - \theta_3))} \exp \left[ \frac{\partial \mathcal{W}}{\partial \theta_2} - \frac{\partial \mathcal{W}}{\partial \theta_0} \right],
\]

(1.4)

which makes extensive use of the relation between semiclassical conformal blocks $\mathcal{W}$ and accessory parameters [11, 12]. Here $\mathcal{C}_{01}$ is the connection coefficients between Frobenius solutions of the Heun equation constructed at $z = 0$ and $z = 1$. Similar relations involving accessory parameters for the same equation have been derived using different methods in [13].

Surprisingly, both $c = 1$ and semiclassical conformal blocks are related through the accessory parameter of Heun equations. As demonstrated in a variety of manners [14–18], we have schematically that the accessory parameter is given by

\[
c = \frac{\partial \mathcal{W}}{\partial t} = \frac{\partial}{\partial t} \log \tau(t),
\]

(1.5)

where the logarithmic derivative is defined on the zero locus of an associated tau function, as made explicit in the references and revised in (2.9) and (3.8) below. In its latest development, this relation hints at an equivalence between the semiclassical conformal block and the logarithm of the tau function, computed at its zero locus. This seems to be a consequence of the blow-up formulas in gauge theory [19], but we are unaware of a treatment that covers the conformal blocks entering the Painlevé VI and V tau functions.

On the other hand, if one takes the relation as true, one can use known braid transformations of the Painlevé tau functions to derive expansions for the conformal blocks at different singular points. Usually, these singular points are related to the collision of vertex operators [12], and fusion rules can be used to relate the expansions. For the irregular case, these are not known explicitly.

In their own right, accessory parameters of Fuchsian differential equations are a central problem in mathematical physics in applied mathematics with a multitude of applications. As examples, we
single black hole quasi-normal mode calculations [20–24], constructive conformal mapping [15, 25, 26],
the Rabi model in quantum optics [27] and generic remarks on the accessory parameter problem in
[28, 29]. In all of these, the role of the monodromy parameters in formulating the expansion of the
accessory parameter is stressed.

In this paper, we explore the symplectic structure behind the isomonodromic tau functions to
relate expansions of the conformal block at different singular points, both for the regular and irregular
rank 1 cases. While the structure is quite symmetric in the regular case (2.35), (2.36) and (2.37), as
it is expected, the structure is less so in the irregular case (3.18), (3.30). We also present a continued
fraction algorithm to compute these conformal blocks and use the relation between conformal blocks
and accessory parameter of the confluent Heun equation to study spheroidal eigenvalues and quasi-
normal modes for the Kerr black hole. We stress the importance of better understanding the global
analytical property of conformal blocks, which are relevant not only to the applications listed above,
but also in their characterization as special functions in their own right.

2 Conformal blocks: Ward Identities

Let us consider chiral Liouville Field Theory [11] with central charge $C = 1 + 6Q^2$, $Q = b^{-1} + b$. Let
$V_\alpha(z)$ be the local field operator corresponding to the classical field $e^{(Q+\alpha)e(z)}$. The Hilbert space of
this theory decomposes into Verma modules, generated from the action of the Virasoro generators on
a primary state resulting from the action of $V_\alpha(0)$ on the $sl_2$-invariant state $|0\rangle$. The primary state $|\alpha\rangle$
thus generated has conformal weight $\Delta(\alpha) = (Q^2 - \alpha^2)/4$. With the state-operator correspondence,
we can identify correlation functions with matrix elements of a product of primary operators between
two primary states. These correlation functions admit the partial wave decomposition

$$
\langle \alpha_{n-1}|V_{(1,2)}(z)V_{\alpha_{n-2}}(1)\ldots V_{\alpha_1}(z_1)|\alpha_0\rangle = \sum_{\{\beta_k\}} \langle \alpha_{n-1}|V_{(1,2)}(z)V_{\alpha_{n-2}}(1)\Pi_{\beta_{n-2}}\ldots\Pi_{\beta_1}V_{\alpha_1}(z_1)|\alpha_0\rangle,
$$

$$
= \sum_{\{\beta_k\}} \mathcal{C}(\alpha_{n-1},\alpha_{n-2},\beta_{n-2})\ldots \mathcal{C}(\beta_1,\alpha_1,\alpha_0)\mathcal{G}_k(z,z_k;\alpha_k,\beta_k),
$$

(2.1)

where $\Pi_{\beta_k}$ is the projection operator on the Verma module built from $|\beta_k\rangle$ and $\mathcal{C}(\alpha_i,\alpha_j,\alpha_k)$ is the
Liouville structure constant. The function $\mathcal{G}_k(z,z_k;\alpha_k,\beta_k)$ defined in the left hand side of (2.1) are,
up to normalization, the Virasoro conformal blocks.

It has long been known that for special values of $\alpha = \alpha_{n,m} = -nb^{-1} - mb$, $n, m = 1, 2, \ldots$, the
associated Verma module corresponds to degenerate representations of the Virasoro algebra. For
instance, for $\alpha_{(1,2)} = -b^{-1} - 2b$, there is a null state

$$
\frac{1}{b^2} \frac{\partial^2}{\partial z^2}V_{(1,2)}(z) + : T(z)V_{(1,2)}(z) := 0,
$$

(2.2)

which decouples from correlation functions with local operators. This in turn means that regular
conformal blocks involving $V_{(1,2)}(z)$ will satisfy second order differential equations of the Fuchsian
type. After using the global conformal group to fix the position of three of those primary operators
to \( z = 0, 1, \) and \( \infty \), these equations will have the form

\[
\left( \frac{1}{b^2} \frac{\partial^2}{\partial z^2} - \frac{1}{z} + \frac{1}{z - 1} \right) \frac{\partial}{\partial z} + \frac{\Delta_0}{z^2} + \frac{\Delta_{n-2}}{(z - 1)^2} + \sum_{k=1}^{n-3} \frac{\Delta_k}{(z - z_k)^2} + \frac{\Delta_{n-1} - \Delta_{(1, 2)}}{z(z - 1)} - \sum_{k=0}^{n-2} \frac{z_k(z_k - 1)}{z(z - 1)(z - z_k)} \frac{\partial}{\partial z_k} \right) \mathcal{G}_k(z, z_k; \alpha_k, \beta_k) = 0, \tag{2.3}
\]

where \( z_0 = 0 \) and \( z_{n-2} = 1 \).

In the semiclassical limit of Liouville Field Theory, defined as

\[
\alpha_k = \frac{\theta_k}{b}, \quad \beta_k = \frac{\sigma_k}{b}, \quad b \to 0, \tag{2.4}
\]

the conformal blocks are expected to exponentiate [12, 30, 31]:

\[
\mathcal{G}_k(z, z_k; \theta_k, \sigma_k) = \psi(z, z_k; \theta_k, \sigma_k) \exp \left[ \frac{1}{b^2} \mathcal{W}(z_k; \theta_k, \sigma_k) \right] (1 + \mathcal{O}(b^2)). \tag{2.5}
\]

The wave functions \( \psi(z, z_k; \theta_k, \sigma_k) \) now satisfy an ordinary differential equation with regular singular points similar to (2.3), where the accessory parameters

\[
c_k(\sigma_k, z_k) = \frac{\partial}{\partial z_k} \mathcal{W}(z_k; \theta_k, \sigma_k), \tag{2.6}
\]

with the \( \theta_k \) dependence omitted, replace the derivatives with respect to \( z_k \) in the BPZ decoupling equation (2.3),

\[
\left( \frac{\partial^2}{\partial z^2} + \sum_{k=0}^{n-2} \delta_k \right) + \delta_{n-1} \sum_{k=0}^{n-2} \delta_k + \sum_{k=1}^{n-3} \frac{z_k(z_k - 1)c_k}{z(z - 1)(z - z_k)} \right) \psi(z, z_k; \theta_k, \sigma_k) = 0, \tag{2.7}
\]

with \( \delta_k = \lim b^2 \Delta_k = (1 - \theta_k^2)/4 \).

The equation (2.7) is the most general second order Fuchsian differential equation with \( n \) singular points. The relation between accessory parameters \( c_k \) and semi-classical Virasoro conformal blocks is known [12], and it is deeply tied to the monodromy problem of finding \( c_k \) and \( z_k \) in (2.7) such that \( \psi(z, z_k; \theta_k, \sigma_k) \) has prescribed monodromy properties.

The solution to the general monodromy problem was outlined in [31] and made explicit in [26] in terms of logarithm derivative of the isomonodromic Jimbo-Miwa-Ueno tau function, introduced in [3], and whose generic expansion was given in [32]. The relation between these functions and the semiclassical conformal blocks is deep, only really understood from the AGT correspondence in gauge field language [19]. Our purpose in mentioning the monodromy problem is more direct, having to do with a global definition of the semiclassical conformal block \( \mathcal{W} \). The monodromy property stems from
the OPE between $V_{(1,2)}(z)$ and a generic primary

$$V_{(1,2)}(z)V_{\alpha}(w) = A_+(z-w)^{(Q+\alpha)/2}(V_{\alpha+b}(w) + O(z-w)) +$$

$$A_-(z-w)^{(Q-\alpha)/2}(V_{\alpha-b}(w) + O(z-w)),$$  \hspace{1cm} (2.8)

where the structure constants $A_{\pm}$ can be computed from the DOZZ formula, see for instance [33]. Since both expansions in (2.8) are analytic for $z$ sufficiently close to $w$, one can read the monodromy properties of $\theta_k$ from the sets of $\alpha_k$ and $\beta_k$. We propose that this suffices to define the semiclassical conformal block $\mathcal{W}$ globally with respect to the $z_k$ variables.

Let us illustrate this construction for 4 regular singular points, where (2.7) is known as the Heun equation [34]. Let us call the conformal modulus and the accessory parameter $z_1 = t$ and $c_1 = c$, and $\mathcal{W}$ is a function of $t$, $\theta_k$ and a single $\sigma$, which parametrizes the momentum of the intermediate channel. There are a variety of methods of computing $\mathcal{W}$ as a perturbative series in $t$ in this case, from Zamolodchikov’s recursion formula [12], the Nekrasov-Shatashvili limit of the Nekrasov function [2, 35]. More recently [14–16, 18], the following equations involving (1.1) and $c_k$:

$$c(\sigma, t) = \frac{\partial}{\partial t} \log \tau_{VI}(\theta^{-}_{k}; \sigma, \eta; t) + \frac{\theta_2 - 1}{2t} + \frac{\theta_1 - 1}{2(t-1)}, \quad \tau_{VI}(\theta_k; \sigma + 1, \eta; t) = 0,$$  \hspace{1cm} (2.9)

where $\theta^{-}_{k} = \{\theta_0, \theta_1 - 1, \theta_2, \theta_3 + 1\}$. These equations hint at the relation between $\mathcal{W}$ and the logarithm derivative of the tau function, at least on its zero locus. The second equation has many different representations [36] and it is sometimes called Toda equation. The introduction of the $\eta$ parameter seems like an unnecessary complication, but it helps to give the Toda equation a geometrical interpretation.

Borrowing from the isomonodromy construction of the tau function, we will think of a configuration space parametrized by $\sigma, \eta$ and $t$. In turn, the $\sigma$ and $\eta$ parameters are (local) coordinates on the manifold of monodromy data. The zero locus condition above then defines a Lagrangian submanifold of the isomonodromic flow in a manner independent of the local coordinates chosen to parametrize the manifold of monodromy data. The introduction of $\eta$ as an independent variable allows us to take (2.9) as a global definition for $c$, and therefore of $\mathcal{W}$.

The tau function (1.1) has the Painlevé property [37]: apart from the fixed branch points at $t = 0, 1$ and $\infty$, it is an analytic function of $t$. Given the definition of the tau function in terms of isomonodromy flow, the transformation laws can be readily established [38]

$$\tau_{VI}(\theta_k; \sigma, \eta; t) = N_{01} \tau_{VI}(\hat{\theta}_k; \bar{\sigma}, \bar{\eta}; 1-t), \quad \hat{\theta}_k = \{\theta_2, \theta_1, \theta_0, \theta_3\},$$

$$= N_{0\infty} \tau_{VI}(\bar{\theta}_k; \bar{\sigma}, \bar{\eta}; 1/t), \quad \bar{\theta}_k = \{\theta_3, \theta_1, \theta_2, \theta_0\},$$  \hspace{1cm} (2.10)

where the connection coefficients at $t$ independent and were computed in [39]. The new monodromy variables $\bar{\sigma}, \bar{\eta}$ and $\bar{\sigma}$ and $\bar{\eta}$ are related to the monodromy of $C = 1$ conformal blocks, but here we can reinterpret them in terms of semiclassical fusion rules. At any rate, we note that equations (2.9) are invariant under these transformations and then one can use the transformations (2.10) to derive expansions for the accessory parameters – and hence for $\mathcal{W}$ – at different branching points once the
transformation laws for the monodromy parameters are obtained. We will list the relations between the monodromy parameters below.

Let us now return to (2.8). The OPE between the degenerate field and one of the primaries will result in two series with different local monodromies. These series can be identified, up to normalization, with local Frobenius solutions of the Heun equation \( \psi_{k,\pm}(z) \) at \( z_k \). One can in principle work out which particular solution of (2.7) is realized by the conformal block involving \( V_{(1,2)}(z) \), but that will not be necessary to our purposes.

Let us call \( M_k \) the monodromy matrix associated to the analytic continuation of a generic pair of solutions \( \psi_{\pm}(z) \) around \( z_k \). If one picks the pair to be the Frobenius solutions built on \( z_k \), then the monodromy matrix will be diagonal \( M_k = \text{diag}(-e^{\pi i \theta_k}, -e^{-\pi i \theta_k}) \), but of course this need not be the case if one chooses a monodromy path around a different singular point, or indeed a different pair of solutions.

To associate the intermediate momentum parameter \( \sigma \) with monodromy, let us recall that, for the small \( t \) expansion of \( W, \beta = \sigma b^{-1} \) comes from the OPE between \( V_{\alpha_0}(0) \) and \( V_{\alpha_1}(t) \). In the radial quantization picture, the substitution makes sense for \( |z| > |t| \), so then the monodromy related to \( \sigma \) is associated to a path encompassing both \( t \) and \( 0 \). In terms of monodromy matrices,

\[
2 \cos \pi \sigma = - \text{Tr} M_1 M_0. \tag{2.11}
\]

By the same token, expansions at \( t = 1 \) and \( t = \infty \) are related to the OPEs between \( V_{\alpha_1}(t) \) and \( V_{\alpha_2}(1) \) and \( V_{\alpha_3}(\infty) \), respectively, so we conclude

\[
2 \cos \pi \tilde{\sigma} = - \text{Tr} M_2 M_1, \tag{2.12}
\]
\[
2 \cos \bar{\pi} \sigma = - \text{Tr} M_3 M_1. \tag{2.13}
\]

The \( \eta \) coordinate is the canonical conjugate to \( \sigma \) in the symplectic structure of the moduli space of flat meromorphic connections on a 4-punctured Riemann surface, as explained in [40, 41]. We will use the more direct explanation from [42]. Consider a pair of Floquet solutions to (2.7)

\[
\psi_{F,\pm}(z, t; \theta_k, \sigma) = z^{\frac{1}{2}(1+\theta_1 \pm \sigma)}(z - t)^{\frac{1}{2}(1-\theta_1)}(z - 1)^{\frac{1}{2}(1-\theta_2)} \sum_{n \in \mathbb{Z}} a_n z^n, \tag{2.14}
\]

where we assume that the Laurent series converges to an analytic function in an annulus containing \( 0 \) and \( t \), but not 1. In this basis, the monodromy matrix around \( 0 \) and \( t \) is diagonal, so we can write

\[
M_1 M_0 = -e^{\pi i \sigma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = M_2^{-1} M_3^{-1}, \tag{2.15}
\]

where the last equality is the result of the contractibility of the loop circling around all four punctures. The problem of finding two unimodular matrices with known traces that multiply to a diagonal matrix can be solved algebraically (see Eqns. (3.32) and (3.33) in [42]). The solution, however, is not unique: conjugation by a diagonal matrix \( (M_k \to s^{\sigma_3} M_k s^{-\sigma_3}, \text{with} \sigma_3 = \text{diag}(1, -1)) \) leaves the product – itself a diagonal matrix – invariant.
This ambiguity introduces two parameters \( s_i \) and \( s_e \) to the explicit parametrization of the \( M_k \). Given a solution to the equation \( \mathbf{M}_B(\phi_1, \phi_2, \phi_3) \mathbf{M}_A(\phi_1, \phi_2, \phi_3) = -e^{i\phi_3\sigma_3} \), we can then define

\[
\begin{align*}
M_0 &= s_i^2 \mathbf{M}_A(\theta_0, \theta_1, \sigma) s_i^{-2}, \\
M_1 &= s_i^2 \mathbf{M}_B(\theta_0, \theta_1, \sigma) s_i^{-2}, \\
M_2 &= s_e^2 \mathbf{M}_A(\theta_2, \theta_3, -\sigma) s_e^{-2}, \\
M_3 &= s_e^2 \mathbf{M}_B(\theta_2, \theta_3, -\sigma) s_e^{-2}.
\end{align*}
\] (2.16)

From this explicit representation we can see that only the ratio \( s_e / s_i \) is invariant by change of basis. We thus define \( \eta \) as

\[
e^{\pi i \eta} = \frac{s_i}{s_e}.
\] (2.17)

With (2.16) one can compute \( \tilde{\sigma} \) and \( \tilde{\sigma} \) from (2.13), arriving at

\[
\sin^2 \pi \sigma (\cos \pi \tilde{\sigma} + \cos (\theta_1 + \theta_2)) =
\begin{align*}
&2 \cos \frac{\pi}{2} (\sigma - \theta_2 + \theta_3) \cos \frac{\pi}{2} (\sigma - \theta_2 - \theta_3) \cos \frac{\pi}{2} (\sigma - \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma - \theta_1 - \theta_0)(e^{2\pi i \eta} - 1) + \\
&2 \cos \frac{\pi}{2} (\sigma + \theta_2 + \theta_3) \cos \frac{\pi}{2} (\sigma + \theta_2 - \theta_3) \cos \frac{\pi}{2} (\sigma + \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma + \theta_1 - \theta_0)(e^{-2\pi i \eta} - 1).
\end{align*}
\] (2.18)

and

\[
\sin^2 \pi \sigma (\cos \pi \tilde{\sigma} + e^{\pi i (\theta_1 + \theta_2)} \cos \pi \sigma + e^{\pi i \theta_1} \cos \pi \theta_3 + e^{\pi i \theta_2} \cos \pi \theta_0) =
\begin{align*}
&2 \cos \frac{\pi}{2} (\sigma - \theta_2 + \theta_3) \cos \frac{\pi}{2} (\sigma - \theta_2 - \theta_3) \cos \frac{\pi}{2} (\sigma - \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma - \theta_1 - \theta_0)e^{-\pi i \sigma}(e^{2\pi i \eta} - 1) + \\
&2 \cos \frac{\pi}{2} (\sigma + \theta_2 + \theta_3) \cos \frac{\pi}{2} (\sigma + \theta_2 - \theta_3) \cos \frac{\pi}{2} (\sigma + \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma + \theta_1 - \theta_0)e^{\pi i \sigma}(e^{-2\pi i \eta} - 1).
\end{align*}
\] (2.19)

We note that the braid transformations (2.10)

\[
\sigma \to \tilde{\sigma}, \quad \theta_0 \leftrightarrow \theta_2, \quad \text{and} \quad \sigma \to \tilde{\sigma}, \quad \theta_0 \leftrightarrow \theta_3,
\] (2.20)

allows us to define canonically conjugate variables \( \tilde{\eta} \) and \( \tilde{\eta} \) analogously to (2.17).

The symplectic structure is illustrated by defining the Fricke-Jimbo polynomial

\[
W(p_{kl}) = p_0 p_1 p_2 p_3 - p_0 p_1 p_2 p_3 + (p_0 p_1 + p_2 p_3)p_0 + (p_1 p_2 + p_0 p_3)p_1 + (p_0 p_2 + p_1 p_3)p_2 + p_0^2 + p_1^2 + p_2^2 + 4.
\] (2.21)

where \( p_k = -2 \cos \pi \theta_k \) and \( p_{kl} = -2 \cos \pi \sigma_{kl} \), with \( \sigma_{01} = \sigma, \sigma_{12} = \tilde{\sigma} \) and \( \sigma_{02} = \tilde{\sigma} \) are trace coordinates. It can be shown that for any monodromy matrices \( M_k \), \( W(p_{kl}) = 0 \), so only two of the \( p_{kl} \) are independent. This space can be furnished with a symplectic structure [43]:

\[
\{p_\mu, p_\nu\} = -\frac{\partial W}{\partial p_\rho}.
\] (2.22)

where \( \mu, \nu, \rho \) cycle around the three pairs 01, 12, and 02. It is now a straightforward exercise (see Lemma 3.12 in [42]) to show that \( \sigma \) and \( \eta \) are canonical Darboux coordinates of the monodromy parameter space, \( \{\eta, \sigma\} = 1 \). Again in complete analogy we have \( \{\tilde{\eta}, \tilde{\sigma}\} = 1 \) as a result of the
braid transformations (2.10).

Being the canonical conjugate coordinate to \( \sigma \) means that \( \eta \) is not in fact independent, as hinted from (2.9). The key for computing it from knowledge of \( \sigma \) and \( t \) is the zero of \( \tau_{VI} \). The structure of \( \tau_{VI} \) is such that it can be seen as an analytic function of \( t \) and meromorphic in \( \chi = t^\sigma e^{2\pi i \eta} \) [38]. We can then formally invert the expansion \( \tau_{VI} = 0 \) and write a series

\[
\chi = \Theta(\sigma, \theta_2, \theta_3)\Theta(\sigma, \theta_1, \theta_0) (1 + \chi_1 t + \chi_2 t^2 + \ldots),
\]

and compute \( \chi_k \) from the conformal blocks, see [21] for the first coefficients. The function

\[
\Theta(\sigma, \phi_1, \phi_2) = \frac{\Gamma(1 + \sigma) \Gamma\left(\frac{1}{2}(1 + \phi_1 + \phi_2 - \sigma)\right) \Gamma\left(\frac{1}{2}(1 + \phi_1 - \phi_2 - \sigma)\right)}{\Gamma(1 - \sigma) \Gamma\left(\frac{1}{2}(1 + \phi_1 + \phi_2 - \sigma)\right) \Gamma\left(\frac{1}{2}(1 + \phi_1 - \phi_2 + \sigma)\right)},
\]

comes about from the ratio of the Barnes’ \( G \)-functions in the \( \mathcal{N} \) coefficients in (1.1).

Since \( \eta \) is canonically conjugated to \( \sigma \), we expect that its \( t \) dependence to be given by semiclassical conformal block through its \( \sigma \) derivative. Given the relation (2.6) and the \( t \)-independent term in the definition of \( \eta \), we can conjecture

\[
\eta(\sigma, t) = \frac{1}{2\pi i} \log \Theta(\sigma, \theta_1, \theta_0) \Theta(\sigma, \theta_2, \theta_3) + \frac{1}{\pi i} \frac{\partial}{\partial \sigma} \int_0^t d\sigma' c(\sigma, t) = \frac{1}{\pi i} \frac{\partial \mathcal{W}_s}{\partial \sigma},
\]

where we can fix the term independent of \( t \) by comparing with the expansion for \( \eta(\sigma, t) \) obtained from the condition of zero of the tau function. With this in mind, now define the small \( t \) expansion of the conformal block \( \mathcal{W}_s \) with the \( t \)-independent term as

\[
\Omega(k, \sigma) = \frac{1}{2} \int_0^\sigma d\sigma' \log \Theta(\sigma', \theta_1, \theta_0) \Theta(\sigma', \theta_2, \theta_3),
\]

and we note that this integral can be expressed in terms of Barnes’ \( G \)-functions – see, for instance, (5.17.4) in NIST’s Digital Library for Mathematical Functions.

The strategy to solving (2.27) is as follows. First divide it by \( a_n \) to find

\[
v_n = \frac{a_n}{a_{n+1}} = \frac{t D_n}{B_n + t C_n - A_n v_{n-1}},
\]

with

\[
A_n = (\sigma + 2n - 1 - \theta_2)^2 - \theta_3^2, \quad B_n = (\sigma + 2n)^2 - \theta_0^2 - \theta_1^2 + 1,
\]

\[
C_n = (\sigma + 2n + \theta_1 - \theta_2)^2 + \theta_1^2 - \theta_3^2 - 1 - 4(t - 1)c, \quad D_n = (\sigma + 2n + 1 + \theta_1)^2 - \theta_0^2.
\]
which allows to compute $v_0$ recursively from the knowledge of $v_n$ for $n > 0$. Given the factors of $t$, this expression can be used to give a relation between $c(t)$ and $\sigma$ valid for small $t$. Following the same procedure for $u_n = \frac{a_{n-1}}{a_n}$, we consider (2.27) for $n = 0$:

$$
t A_0 D_{-1} - t A_{-1} D_{-2} + t A_1 - t A_{-1} D_2 B_{-1} + t C_{-1} + t A_{-2} D_{-3} B_{-2} + t C_{-2} - \frac{t D_0 A_1}{B_1 + t C_1} - \frac{t D_1 A_2}{B_2 + t C_2} = B_0 + t C_0.
$$

(2.30)

Assuming a small $t$ expansion for $c(\sigma, t)$,

$$
c(\sigma, t) = c_{-1}(\sigma)t^{-1} + c_0(\sigma) + c_1(\sigma)t + \ldots + c_n(\sigma)t^n + \ldots
$$

(2.31)

we can truncate (2.30) to the $n$-th approximant to compute $c_n$. The first terms are

$$
c_{-1}(\sigma) = \delta_\sigma - \delta_0 - \delta_1,
$$

(2.32)

$$
c_0(\sigma) = (\delta_\sigma - \delta_0 + \delta_1)(\delta_\sigma - \delta_3 + \delta_2),
$$

(2.33)

$$
c_1(\sigma) = \frac{(\delta_\sigma - \delta_0 + \delta_1)(\delta_\sigma - \delta_3 + \delta_2) - 2\delta_\sigma}{2\delta_\sigma} - \frac{(\delta_0 - \delta_1)^2}{8\delta^2_\sigma}((\delta_2 - \delta_3)^2 - \delta_2^2)
$$

$$
+ \frac{(\delta_2^2 + 2\delta_\sigma(\delta_0 + \delta_1) - 3(\delta_0 - \delta_1)^2)(\delta_2^2 + 2\delta_\sigma(\delta_3 + \delta_2) - 3(\delta_3 - \delta_2)^2)}{32\delta^2_\sigma(\delta_\sigma + \frac{3}{4})},
$$

(2.34)

where $\delta_k = (1 - \theta_k^2)/4$ and $\delta_\sigma = (1 - \sigma^2)/4$. These agree with the expansion found in the literature, as, for instance in [31]. Given the condition (2.25), we then see that the small $t$ expansion of the conformal block $\mathcal{W}(\sigma, t)$ to be the primitive of $c(\sigma, t)$, with the $t$-independent term given by $\Omega(\sigma)$:

$$
\mathcal{W}_s(\sigma, t) = c_{-1}(\sigma)\log t + \Omega(\sigma) + c_0(\sigma)t + \frac{1}{2}c_1(\sigma)t^2 + \ldots + \frac{1}{n}c_{n-1}(\sigma)t^n + \ldots
$$

(2.35)

By the same token, given the intermediate channel parameters $\tilde{\sigma}$ and $\tilde{\sigma}$, we can define the expansion near $t = 1$:

$$
\mathcal{W}_i(\tilde{\sigma}, t) = \tilde{c}_{-1}(\tilde{\sigma})\log(1-t) + \Omega(\tilde{\sigma}) + \tilde{c}_0(\tilde{\sigma})(1-t) + \ldots + \frac{1}{n}\tilde{c}_{n-1}(\tilde{\sigma})(1-t)^n + \ldots
$$

(2.36)

where $\tilde{c}_n(\tilde{\sigma})$ is obtained from $c_n(\sigma)$ by the interchange $\delta_0 \leftrightarrow \delta_2$ and $\sigma \rightarrow \tilde{\sigma}$. And we have the expansion for large $t$ as

$$
\mathcal{W}_i(\sigma, t) = -\tilde{c}_{-1}(\tilde{\sigma})\log t + \Omega(\tilde{\sigma}) + \tilde{c}_0(\tilde{\sigma})t^{-1} + \ldots + \frac{1}{n}\tilde{c}_{n-1}(\tilde{\sigma})t^{-n} + \ldots
$$

(2.37)

where now $\tilde{c}_n(\tilde{\sigma})$ is obtained from $c_n(\sigma)$ by $\delta_0 \leftrightarrow \delta_3$ and $\sigma \rightarrow \tilde{\sigma}$.

In order to understand the relation between the three expansions, let us take a geometrical view of the parameters. Firstly, let us repeat that $\{\sigma, \eta\}$ are coordinates for the character manifold of the
four-punctured sphere. With the expansions $\mathcal{W}_t$ and $\mathcal{W}_u$ we can define two variables

$$\tilde{\eta} = \frac{1}{\pi i} \frac{\partial \mathcal{W}_t}{\partial \tilde{\sigma}}, \quad \bar{\eta} = \frac{1}{\pi i} \frac{\partial \mathcal{W}_u}{\partial \bar{\sigma}},$$

(2.38)

in such a way that two new sets of coordinates are defined, namely $\{\tilde{\sigma}, \tilde{\eta}\}$ and $\{\bar{\sigma}, \bar{\eta}\}$. All three parametrize the same character manifold.

On the other hand, each of the functions $\mathcal{W}_s$, $\mathcal{W}_t$ and $\mathcal{W}_u$ can be interpreted as generating functions for the symplectomorphism between the accessory parameter space of the Heun equation, parametrized by $\{t, c\}$ and the manifold of monodromy data, labelled by the appropriate pair of coordinates of the character manifold defined in the preceding paragraph. As such, they are linked by the canonical transformations

$$\{t, \sigma\} \longrightarrow \{1 - t, \tilde{\sigma}\}, \quad \{1 - t, \tilde{\sigma}\} \longrightarrow \{1/t, \bar{\sigma}\},$$

(2.39)

between $\mathcal{W}_s$ and $\mathcal{W}_t$, and $\mathcal{W}_t$ and $\mathcal{W}_u$, respectively. They are in principle different solutions of the Hamilton-Jacobi equation associated to the Painlevé VI system. In the complex $t$ plane, we can put conditions on the monodromy parameters in such a way that the expansions for $\mathcal{W}$ above have overlapping domains of convergence\(^1\), in these domains, if for the same value of $t$, the monodromy parameters are equivalent in the sense that they represent the same point in the character manifold, we can say that they are equivalent in the sense that their accessory parameter are the same, after a simple transformation of variables. This fact means that the canonical transformation linking any two expansions are functions of the monodromy parameters alone. For instance, $d\mathcal{W}_s = d\mathcal{W}_t + \pi i(\eta d\sigma - \tilde{\eta} d\tilde{\sigma})$, with the extra term being the gradient of a function of the monodromy variables $\sigma, \tilde{\sigma}$ alone, due to the symplectic structure of the monodromy manifold itself. Even though the restriction of this function to the Lagrangian submanifold defined by the zero of the tau function (2.9) will introduce a complicated $t$ dependence for the conformal block, we will consider in this case that the three expansions describe the asymptotic dependence on the $t$ variable of a single globally defined function $\mathcal{W}$.

In the regular case, the three patches of the expansion for the conformal block have been motivated by the relatively simple transformation law of the Painlevé VI tau function (2.10), which is in itself due to the $D_4$ symmetry group of poles of the Heun equation. The symplectic transformations (2.39) can be thought of as the action of the braid transformations restricted to the Lagrangian submanifold of parameters of the tau function defined by the zero condition of $\tau_{VI}$ in (2.9). Although this condition does not have a straightforward CFT interpretation – at least one that we know of – it was checked, both algebraically (up to order $t^6$) and numerically (to order $t^{64}$) in [44]. Specifically, we checked that the derivative of $\mathcal{W}$ defined in the procedure above is the value of $\eta$ obtained by formally inverting the condition $\tau_{VI} = 0$. As we will see in the following, this property carries on to confluent limits of $\tau_{VI}$ and similar results can be extracted for the irregular versions of $\mathcal{W}$.

\(^1\)For the semiclassical limit of conformal blocks in unitary CFTs, mutual convergence seems to follow from the usual arguments of OPEs between primary operators. Alternatively, one could think of the implicit definition given by (2.31).
3 The irregular semiclassical conformal block

Confluent limits of the Heun equation are obtained in the semiclassical limit of irregular conformal blocks [45]. Let us define the rank 1 irregular operator

\[ V_{\alpha,\gamma}(z) = \lim_{\Lambda \to \infty} \left( \frac{\gamma}{\Lambda} \right)^{\frac{1}{2}(Q+\alpha_+)(Q+\alpha_-)} V_{\alpha_+}(z + \gamma/\Lambda) V_{\alpha_-}(z), \]  

where \( \alpha_{\pm} = \pm \Lambda + \alpha/2 - Q/2 \). Starting from this definition we can find that the state \( |\alpha, \gamma\rangle = V_{\alpha,\gamma}(0)|0\rangle \) belongs to a Whittaker module with

\[ L_0|\alpha, \gamma\rangle = \left( \Delta(\alpha) + \gamma \frac{\partial}{\partial \gamma} \right)|\alpha, \beta\rangle, \quad L_1|\alpha, \gamma\rangle = \frac{1}{2}\gamma(Q - \alpha)|\alpha, \gamma\rangle, \quad L_2|\alpha, \gamma\rangle = -\frac{1}{2}\gamma^2|\alpha, \gamma\rangle, \]  

and \( L_n|\alpha, \gamma\rangle = 0 \) for \( n > 2 \). In this expression and below we will take \( \Delta(\alpha) = \frac{1}{4}(Q^2 - \alpha^2) \). We note that, unlike \( \alpha, \gamma \) is not invariant under global transformations, so the eigenvalue of the BPZ dual \( |\alpha, \gamma\rangle \) under \( L_{-2} \) will not be \(-\frac{1}{4}\gamma^2\).

The analogue of the BPZ identity (2.3) for one Whittaker operator and \( n - 1 \) primaries is now

\[
\begin{align*}
\left( \frac{1}{b^2} \frac{\partial^2}{\partial z^2} - \left( \frac{1}{z + 1} \right) \frac{\partial}{\partial z} + \frac{\Delta_0}{z^2} + \frac{\Delta_{n-2}}{(z - 1)^2} + \sum_{k=1}^{n-3} \frac{\Delta_k}{(z - z_k)^2} + \sum_{k=1}^{n-3} \frac{z_k(z_k - 1)}{z(z - 1)(z - z_k)} \frac{\partial}{\partial z_k} + \frac{\Delta(\alpha) - \Delta(1,2)}{z(z - 1)} \sum_{k=0}^{n-2} \Delta_k \right) &+ \frac{\gamma_0(Q - \alpha)}{2z} - \frac{\gamma_0^2}{4} + \frac{\gamma_0}{z(z - 1)} \frac{\partial}{\partial \gamma_0} \right) G_n^e(z, z_k; \alpha_k, \beta_k, \gamma_0) = 0, \tag{3.3}
\end{align*}
\]

where \( \gamma_0 \) is proportional to \( \gamma \), after using the global transformations to fix the position of the Whittaker operator at \( z = \infty \).

We start by considering \( G_n^e \), the irregular analogue of the conformal block involving one degenerate operator solved in [33]. Its solutions are given in terms of confluent hypergeometric functions

\[
G_n^e = (z - 1)^{2\Delta(1,2)} \left[ A^{-\frac{1}{2}(Q+\alpha)}_e \ {}_1F_1 \left( \frac{1}{2}b(\alpha_1 + \alpha - b); b(Q + \alpha - b); b\gamma_0z \right) + A^\frac{1}{2}(Q-\alpha)_e \ {}_1F_1 \left( \frac{1}{2}b(\alpha_1 - \alpha - b); b(Q - \alpha - b); b\gamma_0z \right) \right] \exp \left( -\frac{1}{2}b\gamma_0z \right). \tag{3.4}
\]

Again, using the OPE for the null field (2.8), we find the constants \( A^e \) in terms of the two-point functions involving Whittaker operators. These were constructed in [45] and [9] from the DOZZ formula.

The irregular BPZ equation (3.3) has a similar semiclassical limit to (2.3)

\[
\alpha_k = \frac{\theta_k}{b}, \quad \beta_k = \frac{\sigma_k}{b}, \quad \gamma_0 = \frac{t}{b}, \quad b \to 0, \tag{3.5}
\]
and $G^c_n$ is again expected to exponentiate

$$G^c_n(z, z_k; \alpha_k, \beta_k, \gamma_0) = \psi^c(z, z_k; \alpha_k, \beta_k, \gamma_0) \exp \left[ \frac{1}{b^2} W^c(z, z_k; \theta_k, \sigma_k, t) \right] (1 + \mathcal{O}(b^2)). \quad (3.6)$$

For the conformal block with one Whittaker operator $- V_{\alpha, \gamma}$ and two primary operators $- V_{\alpha_0}$ and $V_{\alpha_0}$ in the semiclassical limit $- \alpha = \theta_*/b$ and $\gamma_0 = t/b$ — the BPZ identity results in the confluent Heun equation [17]

$$\left( \frac{\partial^2}{\partial z^2} + \frac{\delta_0}{z^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_0 - \delta_1 + tc^c}{z(z - 1)} + t(1 - \theta_*) - \frac{t^2}{4} \right) \psi^c(z, z_k; \theta_k, \sigma_k, t) = 0, \quad (3.7)$$

with $\delta_* = (1 - \theta_0^2)/4$ and

$$c^c = \frac{\partial W^c}{\partial t} = - \frac{\partial}{\partial t} \log \tau_V(\theta_k; \sigma, \eta, t) + \frac{\delta_0 - \delta_*}{4t} - \frac{(1 - \theta_1)^2}{4t} \quad (3.8)$$

where the arguments $\theta_k = \{\theta_0, \theta_1, \theta_* - 1\}$ and $\tau_V$ is obtained from the Kyiv formula (1.1) by a confluence limit [5]

$$\theta_3 = \Lambda + \frac{\theta_* - 1}{2}, \quad \theta_2 = -\Lambda + \frac{\theta_* - 1}{2}, \quad t \to \frac{t}{\Lambda}, \quad \Lambda \to \infty, \quad (3.9)$$

and $\sigma$ and $\eta$ are kept fixed. The result is that the accessory parameter of the confluent Heun equation is given by the logarithmic derivative of the confluent tau function (3.8), with again $\eta$ determined from the zero locus condition

$$\tau_V(\theta_k; \sigma + 1, \eta; t) = 0, \quad \theta_k = \{\theta_0, \theta_1, \theta_* - 1\}. \quad (3.10)$$

Once more the conditions (3.8) and (3.10) define the analytical properties of $W^c$ from the Painlevé property. Moreover, the small $t$ expansion of $W^c$ follows the same logic as the non-confluent case (2.35). First let us make a change of variables to (3.7)

$$z = \frac{z'}{t}, \quad \psi^c(z) = z'^{\frac{1}{2}(1 - \theta_0)}(z' - t)^{\frac{1}{2}(1 - \theta_1)} y(z'), \quad (3.11)$$

which brings the equation to a standard form (as seen in Eqn. (4) in [24]). We then postulate a Floquet solution of the kind

$$y(z') = e^{\frac{1}{2}z'} z'^{\frac{1}{2}(\sigma + \theta_0 + \theta_1 - 1)} \sum_{n \in \mathbb{Z}} a_n z'^n, \quad (3.12)$$

where again $\sigma$ parametrizes the Liouville momentum of the intermediate channel. Substitution into (3.7) yields another three-term recursion relation

$$A_n a_{n-1} - (B_n + t C_n) a_n + t D_n a_{n+1} = 0 \quad (3.13)$$
with
\[ A_n = 2\sigma + 4n - 2\theta_1, \quad B_n = - (\sigma + 2n)^2 + \theta_1^2, \quad C_n = 2\sigma + 4n + 2\theta_1 - 2\theta_2 + 2 - 4c^\varepsilon, \]
\[ D_n = - (\sigma + 2n + \theta_0 + \theta_1 + 1)(\sigma + 2n - \theta_0 + \theta_1 + 1). \]  
(3.14)

Going through the same procedure leading to (2.31), we find the expansion for \( c^\varepsilon \),
\[ c^\varepsilon(\sigma, t) = c^\varepsilon_{-1}(\sigma)t^{-1} + c^\varepsilon_0(\sigma) + c^\varepsilon_1(\sigma)t + \ldots + c^\varepsilon_n(\sigma)t^n + \ldots \]  
(3.15)
where the first coefficients are
\[ c^\varepsilon_{-1}(\sigma) = \delta_\sigma - \delta_\star, \quad c^\varepsilon_0(\sigma) = \frac{(1 - \theta_\star)(\delta_\sigma - \delta_0 + \delta_1)}{4\delta_\sigma}, \]
\[ c^\varepsilon_1(\sigma) = \frac{\delta_\sigma - \delta_0 - \delta_1}{12} - \frac{(4(1 - \theta_\star)^2 - \delta^2_\sigma)((\delta_0 - \delta_1)^2 - \delta^2_\delta)}{8\delta^2_\delta} + \frac{(\delta^2_\sigma - \frac{3\theta_0}{16}(1 - \theta_\star)^2)(\delta^2_\delta + 2\delta_\star(\delta_0 + \delta_1) - 3(\delta_0 - \delta_1)^2)}{24\delta^2_\delta(\delta_\sigma + \frac{3}{4})}. \]  
(3.16)

The small \( t \) expansion of the semiclassical irregular conformal block \( \mathcal{W}'^\varepsilon \) is then defined as the primitive of \( c^\varepsilon(\sigma, t) \):
\[ \mathcal{W}'^\varepsilon(\sigma, t) = c^\varepsilon_{-1}(\sigma) \log t + \Omega^\varepsilon(\sigma) + c^\varepsilon_0(\sigma)t + \frac{1}{2} c^\varepsilon_1(\sigma)t^2 + \ldots + \frac{1}{n} c^\varepsilon_{n-1}(\sigma)t^n + \ldots, \]  
(3.17)
with the \( t \)-independent term to be determined below.

Because of the confluent limit (3.9), however, the structure of the expansion of \( \mathcal{W}'^\varepsilon \) at large \( t \) is quite different. We can still follow the same argument used for the Painlevé VI tau function, though: since the tau function in (3.8) is globally well-defined, we can use the expansion of the accessory parameter to define \( \mathcal{W}'^\varepsilon \) globally.

In parallel to the regular case, the first step is introduce the \( \eta \) parameter
\[ \eta = \frac{1}{2\pi i} \log \Theta(\sigma, \theta_1, \theta_0) \Theta^\varepsilon(\sigma, \theta_\star) + \frac{1}{\pi i} \int^t d\tau' c^\varepsilon(\sigma, \tau') = \frac{1}{\pi i} \frac{\partial \mathcal{W}'^\varepsilon}{\partial \sigma}, \]  
(3.18)
where \( \Theta \) is as (2.24), and
\[ \Theta^\varepsilon(\sigma, \theta_\star) = \frac{\Gamma(1 + \sigma) \Gamma \left( \frac{1}{2}(\theta_\star - \sigma) \right)}{\Gamma(1 - \sigma) \Gamma \left( \frac{1}{2}(\theta_\star + \sigma) \right)}. \]  
(3.19)
which, just as the regular case, can be written in terms of Barnes’ \( G \)-function. This parameter can be seen to correspond to a zero of \( \tau_\varepsilon \), (3.10), [17], and has a similar interpretation as in the non-confluent case (2.17). This sets the \( t \)-independent term of (3.18) as
\[ \Omega^\varepsilon(\sigma) = \frac{1}{2} \int^\sigma d\sigma' \log \Theta(\sigma', \theta_1, \theta_0) \Theta^\varepsilon(\sigma', \theta_\star). \]  
(3.20)
As with the regular case, the conditions (3.8) and (3.10) have a geometrical meaning in the sense
that they do not depend on the coordinates chosen to parametrize the monodromy manifold. In order to describe the quantities relevant for the large $t$ expansion of the conformal block let us define, following [8],

$$X_+ = \lim_{\Lambda \to -i\infty} 2e^{\pi i \Lambda} \cos \pi \sigma,$$

$$X_- = \lim_{\Lambda \to +i\infty} 2e^{\pi i \Lambda} \cos \pi \tilde{\sigma}.$$

which, when written explicitly in terms of $\sigma$ and $\eta$, find

$$\sin^2 \pi \sigma (X_+ \mp ie^{\pm \pi i (\theta_1 + \frac{1}{4} \theta_\tau}) =$$

$$2 \cos \frac{\pi}{2} (\sigma - \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma - \theta_1 - \theta_0) \sin \frac{\pi}{2} (\sigma - \theta_\tau) e^{\pm \frac{\pi i}{2} \sigma (e^{2 \pi i \eta} - 1)} +$$

$$+ 2 \cos \frac{\pi}{2} (\sigma + \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma + \theta_1 - \theta_0) \sin \frac{\pi}{2} (\sigma + \theta_\tau) e^{\pm \frac{\pi i}{2} \sigma (e^{-2 \pi i \eta} - 1)}. \quad (3.23)$$

We will now define the parameter $\nu$ as $e^{2 \pi i \nu} = X_-$, which will play the role of spectral parameter for the classical conformal blocks in the $t \to \infty$ limit. The eigenvalue $c^c(\nu; t)$, now seen as a function of $\nu$, can be computed by considering the following expansion for the solution of (3.7)

$$y(z') = \sum_{n \in \mathbb{Z}} a_n e^{-\frac{\pi i}{2} z^0} F_n (z'), \quad F_n (z') = \frac{\Gamma(\nu + n + 1) \Gamma(b - a - n) \Gamma(b)}{(n! \Gamma(b - a))} e^{\pi i (b - a)}(e^{\pi i \eta} - 1) \sin \frac{\pi}{2} (\sigma - \theta_\tau) e^{\pm \frac{\pi i}{2} \sigma (e^{2 \pi i \eta} - 1)} +$$

$$+ 2 \cos \frac{\pi}{2} (\sigma + \theta_1 + \theta_0) \cos \frac{\pi}{2} (\sigma + \theta_1 - \theta_0) \sin \frac{\pi}{2} (\sigma + \theta_\tau) e^{\pm \frac{\pi i}{2} \sigma (e^{-2 \pi i \eta} - 1)}. \quad (3.24)$$

This Ansatz comes from the expression of the irregular conformal block involving the degenerate operator (3.4). The $\nu$ parameter represents the shift of the formal monodromy to the irregular operator due to the OPE with the primary, as described in [8]. From the analytical perspective, the expansion parameter $\nu$ is related to the Stokes parameter of the expansion, pending of course the thorny question of its convergence. From the connection formula for confluent hypergeometric equation,

$$1F_1(a; b; z) = e^{-\pi i a} \frac{\Gamma(b)}{\Gamma(b - a)} U(a; b; z) + e^{\pm \pi i (b - a)} \frac{\Gamma(b)}{\Gamma(a)} e^{\pi i \eta} U(b - a; b; e^{\pm \pi i z}), \quad (3.25)$$

where $U(a; b; z)$ is the Tricomi function, defined as the solution of the confluent hypergeometric differential equation with asymptotics $\propto z^{-a}$ for $z \to \infty$ in the sector $|\arg z| < \frac{3\pi}{2}$. It is a straightforward exercise to use this connection formula to show that $\nu$ parametrizes the corresponding connection formula for the solution of the confluent Heun equation parametrized by (3.24).

We follow through the BPZ equation (3.7), using the confluent hypergeometric equation to replace the second derivative of $1F_1$, arriving at

$$\sum_{n \in \mathbb{Z}} a_n \left[ (1 - \theta_1) z \frac{\partial}{\partial z} F_n (z) - \left( \nu + n + \frac{1}{2} (1 + \theta_1) - \frac{1}{4} \theta_\tau \right) z F_n (z) -$$

$$- \left( t \left( \nu + n - \frac{1}{4} (1 - \theta_\tau) - e^\eta \right) + \frac{1}{4} \theta_\tau^2 - \frac{1}{4} (1 + \theta_0 - \theta_1)^2 \right) F_n (z) \right] = 0. \quad (3.26)$$
using the following properties of Kummer’s function
\[
z \frac{\partial}{\partial z} F_1(a; b; z) = a F_1(a + 1; b; z) - F_1(a; b; z),
\]
\[
z F_1(a; b; z) = a F_1(a + 1; b; z) - (2a - b) F_1(a; b; z) + (a - b) F_1(a - 1; b; z),
\]
we arrive at the 3-term recursion
\[
A_n a_{n-1} - (B_n + C_n) a_n + D_n a_{n+1} = 0,
\]
\[
A_n = (2\nu + 2n - \theta_1 - \frac{1}{2}(3 - \theta_*))(2\nu + 2n + \theta_0 - \frac{1}{2}(1 + \theta_*)),
\]
\[
B_n = 8(\nu + n)^2 - \theta_0^2 - \theta_1^2 + \frac{1}{2}(1 + \theta_*)^2,
\]
\[
C_n = -4\nu - 4n - 4c^\nu + 1 - \theta_*,
\]
\[
D_n = (2\nu + 2n + \theta_1 + \frac{1}{2}(1 + \theta_*))(2\nu + 2n - \theta_0 + \frac{1}{2}(3 - \theta_*)).
\]
The recursion equation is again solved by continued fractions. Using (2.30) once more, we obtain the first terms of the expansion for the accessory parameter as
\[
c^\nu(t) = e_0^\nu(\nu) + e_1^\nu(\nu) e^{-t} + e_2^\nu(\nu) e^{-2t} + \ldots,
\]
\[
e_0^\nu(\nu) = -\nu + \frac{1}{2}(1 - \theta_0),
\]
\[
e_1^\nu(\nu) = 2\nu^2 + \delta_0 + \theta_1 + \frac{1}{8}(1 + \theta_*)^2 - \frac{1}{2},
\]
\[
e_2^\nu(\nu) = 4\nu^3 + (2\delta_0 + 2\delta_1 - \frac{1}{2}(1 - \theta_*)^2)\nu - \frac{1}{8}(1 - \theta_*)(\delta_0 - \delta_1),
\]
\[
e_3^\nu(\nu) = 20\nu^4 + (4 + 12\delta_0 + 12\delta_1 - \frac{3}{2}(1 - \theta_*)^2)\nu^2 - 4(\delta_0 - \delta_1)(1 - \theta_*)\nu - \frac{1}{4}(1 - \theta_*)^2(1 - \delta_0 - \delta_1) - 4\delta_0\delta_1 + \frac{1}{64}(1 - \theta_*)^4,
\]
which can be seen to agree with the semiclassical limit of the confluent conformal block of the second kind presented in [8].

We can now define the expansion for the semiclassical confluent conformal block itself
\[
\mathcal{W}^c_\nu(\nu; t) = e_0^\nu(\nu) t + \tilde{\Theta}^c(\nu) + e_1^\nu(\nu) \log t + \ldots - \frac{1}{n} e_{n+1}^\nu(\nu) t^{-n} + \ldots
\]
For completeness, we can determine the monodromy parameter conjugate to \( \nu \)
\[
\rho = -\frac{1}{\pi i} \log \tilde{\Theta}^c(\nu, \theta_*, \theta_0) \tilde{\Theta}^c(\nu, -\theta_*, \theta_1) + \int t \, c^\nu(\nu, t') = \frac{1}{\pi i} \frac{\partial \mathcal{W}^c_\nu}{\partial \nu},
\]
where
\[
\tilde{\Theta}^c(\nu, \theta_*, \phi) = \frac{1}{2\pi i} \Gamma\left(\frac{1}{4}(1 - 2\phi - \theta_0 + 4\nu)\right) \Gamma\left(\frac{1}{4}(1 + 2\phi - \theta_0 + 4\nu)\right)
\]
and \( \rho \) is related to monodromy data by \( e^{\pi i \rho} = 1 - X_+ X_- \). We can verify that \( \rho \) computed from (3.31) satisfies
\[
\tau_\nu(\theta_k; \nu, \rho; t) = 0, \quad \theta_k = \{0, \theta_1, \theta_* - 1\},
\]
in the large imaginary \( t \) expansion presented in [8]. This allows us to find the \( t \)-independent term in (3.30) as
\[
\tilde{\Theta}^c(\nu) = \int d\nu' \, \log \tilde{\Theta}^c(\nu', \theta_*, \theta_0) \tilde{\Theta}^c(\nu', -\theta_*, \theta_1),
\]
which again can be expressed in terms of Barnes’ $G$-functions. In parallel with the regular case, we then conjecture that the expansions of $\mathcal{W}^c$ are related by a canonical transformation in the Lagrangian submanifold of monodromy data determined by $\tau_V = 0$ (3.10), parametrized by $\{t, \sigma\}$ for small $t$ and by $\{1/t, \nu\}$ for large $t$. The two functions $\mathcal{W}_s^c$ and $\mathcal{W}_u^c$ solve the same Hamilton-Jacobi equation for the isomonodromic problem and have the same initial conditions when $\{\sigma, \eta\}$ and $\{\nu, \rho\}$ represent the same point in the character manifold. Furthermore, since the accessory parameter derived from them coincide in the overlapping regions of convergence, the canonical transformation relating them is a function of the monodromy parameters alone.

As a simple test, we recover from (3.30) the accessory parameter (eigenvalue) expansion for the Mathieu equation by taking $\theta_0 = \theta_1 = -1/2$, $\theta_s = 1$, $t = -8h$ and $\nu = -s/4$ in the notation of NIST’s Digital Library for Mathematical Functions,

$$-t \mathcal{C}_{\text{Mathieu}}(h) = 2sh - \frac{1}{2^3}(s^2 + 3) - \frac{1}{2^4h}(s^3 + 3s) - \frac{1}{2^{12}h^2}(5s^4 + 34s^2 + 9) + \mathcal{O}(h^{-3}) \tag{3.35}$$

which can be compared the large $t$ expansion is listed at §28.16 through $\lambda(h) = \frac{1}{4} - 2h^2 - t \mathcal{C}_{\text{Mathieu}}(h)$, (see also [46]). The extra factor of $\frac{1}{4} - 2h^2$ comes about by bringing the Mathieu equation into the standard Heun format (3.7).

A few extra comments about (3.30) are in order. First, the large $t$ expansion of the Painlevé V tau function has only been constructed so far along the rays $\arg t = 0, \frac{\pi}{2}$. The existence of Stokes sectors for the large $t$ expansions of the confluent hypergeometric functions make us suspect that any expansion like (3.30) is at best asymptotic. Like the regular case, convergence can be inferred from OPE arguments in the case where the parameters can be realized from unitary CFTs. For generic $\nu$ and $\delta_k$, however, there is no such argument.$^2$

The asymptotic character of the expansion (3.30) notwithstanding, we have found, however, numerical evidence that the implicit definition of $c^c$ provided by (3.28) has a finite radius of validity for generic monodromy parameters. First and foremost, remark that the coefficient of the highest term in $\nu$ is the $R(\nu)$ polynomial, defined as the formal power series with constant term 0 defined by

$$\nu = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n}^2 R^{n+1}(\nu) = R(\nu) \, _2F_1(1/2, 1/2, 2; 16R(\nu)),$$  

which appears on rooted planar Eulerian orientations, see [47] for details and an application to the six-vertex model, where the corresponding conformal block appears as the partition sum of the system. Given that the elliptic functions that are usually related to the hypergeometric functions are in fact analytic, it seems that so is the series (3.30), at least in this limit. From the AGT perspective, keeping the highest term in $\nu$ and $\theta_k$ corresponds to taking the “Seiberg-Witten” limit of the Nekrasov functions, in the semiclassical (Nekrasov-Shatashvili) limit. In a straightforward exercise, one can show that the first WKB approximation for the accessory parameter is expressible in terms of elliptic

$^2$We thank Referee 1 for studying the expansion and showing that the large-order behavior of the coefficients of (3.30) display the factorial growth associated to asymptotic series.
integrals\(^3\).

Last, but not least, an aspect of resurgence (see [48] for a review focused in Painlevé transcendents) is also present here. The expansion for (3.30) is actually buried in the small \(t\) recursion relation. If one takes (3.13) with a large \(t\) expansion for \(c^c(\sigma, t)\):

\[
\tilde{c}^c(\sigma, t) = \tilde{c}^c_{-1}(\sigma)t + \tilde{c}^c_0(\sigma) + \tilde{c}^c_1(\sigma)t^{-1} + \ldots + \tilde{c}^c_n(\sigma)t^{-n} + \ldots,
\]

(3.37)

one recovers the large \(t\) expansion (3.29) by the substitution

\[
\sigma \rightarrow -2\nu - \theta_1 - \frac{1}{2}(1 - \theta).
\]

(3.38)

This correspondence between the coefficients \(\tilde{c}^c_n(\nu) = \tilde{c}^c_n(\nu)\) has been verified to order \(t^{-6}\) for generic monodromy parameters\(^4\).

4 Examples: Angular Spheroidal Harmonics and Teukolsky radial equation

It has long been known that linear perturbation theory of the Kerr black hole yields a system of separable equations of the confluent Heun type [49]. Eigenvalue problems are natural to consider from the monodromy properties of the solutions, see [17, 24] for details in this context. Actual scattering coefficients can be computed using the Trieste formula (1.4) and the expressions given here for \(\mathcal{W}\), using the map between the accessory parameters of the differential equations and the monodromy parameter \(\sigma\). We will, however, defer this analysis to future work, and restrict ourselves to the monodromy considerations.

| \(a\omega\) | \(-2\lambda_{20}\) | \(-2\lambda_{20}^\text{lit}\) | \(-2\lambda_{21}\) | \(-2\lambda_{21}^\text{lit}\) | \(-2\lambda_{22}\) | \(-2\lambda_{22}^\text{lit}\) |
|---|---|---|---|---|---|---|
| 3.0 | -1.6055512 | -1.6135162 | -6.6147648 | -6.6144134 | -11.7605013 | -11.7605089 |
| 4.0 | -6.7445626 | -6.748527 | -23.7187995 | -23.7187929 | -29.7168143 | -29.7168144 |
| 5.0 | -13.8102496 | -13.8102593 | -22.7797029 | -22.7797028 | -31.8521549 | -31.8521549 |
| 6.0 | -22.8488578 | -22.8488580 | -33.8191725 | -33.8191725 | -44.8762232 | -44.8762232 |
| 7.0 | -33.8743420 | -33.8743420 | -46.8467287 | -46.8467287 | -59.8936159 | -59.8936159 |
| 8.0 | -46.8924439 | -46.8924439 | -61.8670293 | -61.8670293 | -76.9067523 | -76.9067523 |
| 9.0 | -61.9059737 | -61.9059737 | -78.8825961 | -78.8825961 | -95.9170161 | -95.9170161 |
| 10.0 | -78.9164728 | -78.9164728 | -97.8949078 | -97.8949078 | -116.9252528 | -116.9252528 |
| 15.0 | -193.9463772 | -193.9463772 | -222.9310799 | -222.9310799 | -251.9500731 | -251.9500731 |
| 20.0 | -358.9605057 | -358.9605057 | -397.9487361 | -397.9487361 | -436.9625304 | -436.9625304 |

Table 1. The comparison between angular eigenvalue for \(s = -2\), \(\ell = 2\) and \(m = 0, 1, 2\) as the value of \(a\omega\) increases. These results were obtained assuming the quantization condition (4.6), while the values for \(\lambda^\text{lit}\) are from [50].

\(^3\)We thank O. Lisovyy for this remark.

\(^4\)We thank Referee 1 for verifying this relation to order \(t^{-150}\).
4.1 Angular Spheroidal Harmonics

The angular equation for a frequency $\omega$, azimuthal and magnetic quantum numbers $\ell, m$, spin $s$ perturbation of a black hole with angular momentum parameter $a = J/M$ is

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{dS}{d\theta} \right] + \left[ a^2 \omega^2 \cos^2 \theta - 2a \omega s \cos \theta - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s + s \lambda_{\ell, m} \right] S(\theta) = 0, \tag{4.1}
\]

whose solutions are known as spheroidal harmonics. The associated problem is to write the allowed values of $s \lambda_{\ell,m}$ that correspond to regular functions at both North and South poles $\theta = 0, \frac{\pi}{2}$.

The single monodromy parameters can be read by recasting (4.1) in the form (3.7)

\[
\begin{align*}
\theta_0 &= -(m + s), & \theta_1 &= m - s, & \theta_* &= 1 - 2s, & t &= -4a\omega \tag{4.2}
\end{align*}
\]

with accessory parameter

\[
tc^e = -s \lambda_{\ell,m} - 2s - 2sa\omega - a^2 \omega^2. \tag{4.3}
\]

The regular singular points at $z = 0$ and $z = 1$ correspond to the North and South poles.

As discussed in [17], the condition that there are solutions regular at both $z = 0$ and $z = 1$ can be cast in terms of monodromy data

\[
\sigma_\ell = \theta_0 + \theta_1 + 2\ell + 3, \quad \ell \in \mathbb{N}, \tag{4.4}
\]

where $2\cos \pi \sigma_\ell = -\text{Tr} M_1 M_0$. A straightforward exercise from their definition (3.23) leads us to the values of $X_{\pm}$:

\[
X_{\pm} = \pm ie^{\pm i(\theta_1 + 4\theta_*)} + 2i \frac{\sin \pi \theta_1 \cos \frac{\pi}{2} (\theta_* + \sigma) \sin \pi (\theta_1 + \theta_0)}{\sin \pi (\theta_1 + \theta_0)} e^{\pm \frac{3}{2} i\sigma (e^{-2\pi i\eta} - 1)}, \tag{4.5}
\]

which simplifies considerably for the parameters (4.2), given that $s$ and $m$ are half integers. Considering $2s$ to be an integer before simplifying, and using the same strategy as the small frequency expansion, the $\eta$ parameter can be found to be

\[
\nu_\ell = -\frac{1}{2} \theta_1 - \frac{1}{2} (\theta_* + 1) + \ell \tag{4.6}
\]

with the integer chosen so to reproduce the expansion in [51]. Substituting the terms directly in the
accessory parameter expansion (3.29), we find

\[ s\lambda_{\ell,m} = -a^2 \omega^2 + 4q\omega + \frac{1}{2}(m^2 - 2s - 1) - \frac{q^2}{2} - \frac{1}{a\omega} \left( \frac{q^3}{8} - \frac{1}{8} q \left( m^2 + 2s^2 - 1 \right) - \frac{mqs^2}{4} \right) \]

\[ + \frac{1}{a^2\omega^2} \left( \frac{1}{32} q^2 (3m^2 + 6s^2 - 5) + \frac{1}{64} (4m^2s^2 - m^4 + 2m^2 + 4s^2 - 1) + \frac{1}{4} mqs^2 - \frac{5q^4}{64} \right) \]

\[ - \frac{1}{a^3\omega^3} \left( \frac{1}{128} ms^2 (m^2 + 2s^2 - 13) + \frac{1}{512} (-36m^2s^2 + 13m^4 - 50m^2 + 8s^4 - 100s^2 + 37) q \right) \]

\[ - \frac{33}{128} ms^2q^2 - \frac{1}{256} (23m^2 + 46s^2 - 57) q^3 + \frac{33q^5}{512} \right) + \mathcal{O}(a^{-4}\omega^4), \quad (4.7) \]

where \( q = 2(\ell + s) - m + 1 \).

Numerically, the expansion (3.15) is also consistent with the results in the literature [50], as can be checked in Table 1. We remark that this is actually a test of the irregular conformal block expansion at large negative values of \( t \), with good agreement (6 significant digits) obtained for \( t \approx -20 \) and below.

### 4.2 The radial equation

The radial equation for linear perturbations is

\[ \Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{dR(r)}{dr} \right) + \left( \frac{K^2(r) - 2is(r - M)K(r)}{\Delta} + 4is\omega r - s\lambda_{\ell,m} - a^2\omega^2 + 2am\omega \right) R(r) = 0, \quad (4.8) \]

where

\[ K(r) = (r^2 + a^2)\omega - am, \quad \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_+). \quad (4.9) \]

The equation can be brought to the canonical form (3.7), with parameters

\[ \theta_0 = s - i\frac{\omega - m\Omega_-}{2\pi T_-}, \quad \theta_1 = s + i\frac{\omega - m\Omega_+}{2\pi T_+}, \quad \theta_* = 1 + 2s - 4iM\omega, \]

\[ 2\pi T_\pm = \frac{r_+ - r_-}{4Mr_\pm}, \quad \Omega_\pm = \frac{a}{2Mr_\pm}, \quad (4.10) \]

where \( r_\pm \) are the radial positions of the inner and outer event horizons, corresponding to the regular singularities of the differential equation.

\[ t = 2i(r_+ - r_-)\omega, \]

\[ te^c = -s\lambda_{\ell,m} - 2s + is(r_+ - r_-)\omega + 2i(1 - 2s)M\omega - r_- r_+ \omega^2 + 2(r_+ - r_-)M\omega^2 + 4M^2\omega^2. \quad (4.11) \]

The quasi-normal modes (QNM) problem for (4.8) poses a suitable test for the convergence of the formulas (3.18) and (3.30), since generically \( \omega - \) and thus \( t - \) are complex numbers. Given that there are no natural small parameter for the problem, actual QNM frequencies of interest are at finite, specific values of \( t \), so the survey has to be done numerically. One notes that low-lying modes and/or the near extremal case \( r_- \to r_+ \) can be suitably studied with the small \( t \) expansion [24]. The large \( t \) expansion (3.15) will, on the other hand, be useful for higher excited modes, as well as small rotation
| $n$ | $M_{-2\omega_{20}}$ – large $t$ | $M_{-2\omega_{20}}$ – small $t$ |
|-----|------------------------------|------------------------------|
| 1   | 0.37367168441804 – 0.08896231568894i | 0.37367168441804 – 0.08896231568894i |
| 2   | 0.34671099687916 – 0.27391487529123i | 0.34671099687916 – 0.27391487529123i |
| 3   | 0.30105345461236 – 0.47827698322307i | 0.30105345461236 – 0.47827698322307i |
| 4   | 0.25150496218559 – 0.70514820243349i | 0.25150496218559 – 0.70514820243349i |
| 5   | 0.20751457981306 – 0.94684489086635i | 0.20751457981306 – 0.94684489086635i |
| 6   | 0.16929940309304 – 1.19560805413585i | 0.16929940309304 – 1.19560805413585i |
| 7   | 0.13325234042519 – 1.44791062616204i | 0.13325234042519 – 1.44791062616204i |
| 8   | 0.09282233367020 – 1.70384117220614i | 0.09282233367020 – 1.70384117220614i |
| 9   | 0.06326350512560 – 2.30264476515854i | 0.06326350512560 – 2.30264476515854i |
| 10  | 0.07655346288598 – 2.56082661738151i | 0.07655346288598 – 2.56082661738151i |

Table 2. First overtones for $s = -2 \ell = 2$ in the Schwarzschild black hole. To the left, we list the values for $M_{s\omega_{1,m}}$ found from (3.30), using the quantization condition (4.13). To the right, we present the values obtained from the small $t$ expansion of $\mathcal{Y}^c$ (3.18) with the quantization condition (4.12).

parameter. One notes that for the actual solutions, $t$ in (4.11) lie in the third quadrant $0 < \arg t < \frac{1}{2} \pi$, where the expansion of $\tau_V$ may not have either the forms given in [8].

As before, the first step is to translate the boundary conditions to monodromy data. For the parameters $\sigma$, $\eta$, this was found to be [17]

$$e^{2\pi i \eta} = e^{-\pi i \sigma} \cos \left( \frac{\pi}{2} (\sigma + \theta_1 + \theta_0) \right) \cos \left( \frac{\pi}{2} (\sigma + \theta_1 - \theta_0) \right) \sin \left( \frac{\pi}{2} (\sigma + \theta_*) \right) \cos \left( \frac{\pi}{2} (\sigma - \theta_1 - \theta_0) \right) \sin \left( \frac{\pi}{2} (\sigma - \theta_1 + \theta_0) \right) \sin \left( \frac{\pi}{2} (\sigma - \theta_*) \right).$$

(4.12)

and another straightforward exercise using (3.23) shows that the value of $\nu$ is determined:

$$\nu_n = -\frac{1}{2} \theta_1 - \frac{1}{2} (\theta_0 + 1) + n, \quad n \in \mathbb{Z},$$

(4.13)

which now is independent of $\rho$, regardless of the values of the $\theta_k$. The numerical calculation for the first ten overtones for the Schwarzschild case $a/M = 0$ can be checked in Table 2, and match those obtained in the small $t$ expansion obtained previously (3.18) and the values accepted in the literature from [52]. We present the contour plot for (2.30) in the large $t$ limit with the radial parameters in Fig. 1, where we can see that the zeros are simple, a consequence of the Painlevé property.

In order to compute the values, we used a simple root finding algorithm with the implicit relation between the accessory parameter and $t$. We truncate (2.30) to the $N_c$-th level of recursion – the $N_c$-th “approximant” – for $A_n, \ldots, D_n$ parameters given by (3.28), using the monodromy parameters (4.10) and the quantization condition (4.13). As an indication of the convergence of the procedure, we give the eigenfrequency computed as a function of $N_c$ in Table 3.

5 Discussion

Conformal blocks are special functions important to the characterization of conformal field theories. They are also relevant in a variety of applications, of which we have focussed in the accessory parameter of Fuchsian differential equations in the complex plane and their confluent limits. In this paper we
presented a method of relating the conformal block expansions at different branch points which relies on the monodromy information. We gave an explicit construction in the regular and rank one irregular cases. We illustrate the construction by recovering known large frequency expansions for spheroidal harmonics and showing that both expansions agree numerically on the overtone spectrum for the Schwarzschild black hole.

Our goal was to help characterize conformal blocks in the whole of complex plane for the $t$ variable. Monodromy parameters manifest the symmetry of the expansions in the regular case, and help

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**Table 3.** First and second overtones for $s = -2 \ell = 2$ in Schwarzschild black hole found from the implicit relation (2.30) as a function of the approximant $N_c$. In the last line we display the common digits of the largest $N_c$ obtained from the small and large $t$ formulas. In the lines above, we underline the digits in common with the value in the last line.
identifying the suitable variables of the coefficients in the irregular case. We have found that using the symplectic structure to that end makes for a clearer formulation, at the expense of computing an extra monodromy parameter – such as $\eta$ in the small $t$ expansion. More direct methods have been proposed [10], where one can see that $\mathcal{W}$ and $\mathcal{W}^c$ have a rather complicated singularity structure for large $t$. In the relations given here, these branch points and poles can be seen to arise from the transformation between the monodromy parameters – for instance between $\sigma, \eta$, which are suitable near $t = 0$, to $\tilde{\sigma}, \tilde{\eta}$ which are suitable for $t = \infty$ expansions of $\mathcal{W}$. It would be interesting to investigate whether the singularity structure can be completely described from the relations proposed here. A related but important problem is to use these expansions to efficiently compute the Nekrasov-Shatashvili limit of the Nekrasov functions entering the expansion of the semiclassical conformal blocks.

The fact that this extra monodromy parameter, such as $\eta$ in (2.25), can also be computed from the semiclassical conformal block expansions may have interesting repercussions for the isomonodromic tau functions themselves. We can state one simple such repercussion by noting that $\eta$ thus defined as a function of $\sigma, t$ and the $\theta_k$ correspond to a zero of $\tau_{VI}$ as can be seen in, for instance, (2.9). Due to the structure of the expansion of the tau function, it seems natural to define $\eta$ by formally inverting the Kyiv formula, and thus $t$ and $\sigma$ are natural parameters for the Lagrangian submanifold of monodromy parameters, at least for small $t$. When the inversion is performed, we make use of the quasi-periodicity $\tau(\sigma + 2n; t) \propto \tau(\sigma; t)$ of the tau function and choose a branch of $\sigma$, where the proportionality factor is independent of $t$. One can in principle then define an infinite number of monodromy parameters $\eta_n = \eta(\sigma + 2n; t)$ corresponding to multiple branches of $\sigma$. Each of these $\eta_n$ correspond to a different zero of the tau function. Furthermore, we can construct similar zeros from the quasi-periodicity with respect to $\tilde{\sigma}$ and $\bar{\sigma}$. The natural question to be considered is whether all zeros of the isomonodromic tau function can be obtained from this procedure.

In the application front, monodromy variables are also relevant parameters for a number of properties of solutions of the Heun equation as well as their confluent limits [9]. Expressions for black hole properties such as greybody factors and Love numbers have also a clean definition in terms of $\sigma$ [53, 54], so a natural question that arises is whether $\nu$ plays a similar role for high frequency – which translates to large $t$. More abstractly, we can transpose the construction to irregular conformal blocks involved in the Painlevé III transcendent to shed light on the work of [46]. At any rate, the applications of conformal blocks to the accessory parameters display a variety of relevant physical phenomena which can hopefully shed light on the global structure of these important special functions, including the generic number of points and higher genus cases.

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