GLOBAL EXISTENCE AND UNIQUENESS FOR A NON LINEAR BOUSSINESQ SYSTEM IN DIMENSION TWO

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Abstract. We study the global well-posedness of a two-dimensional Boussinesq system which couples the incompressible Euler equation for the velocity and a transport equation with fractional diffusion of type $|D|^{\alpha}$ for the temperature. We prove that for $\alpha > 1$ there exists a unique global solution for initial data with critical regularities.

1. Introduction

In this paper, we study the two-dimensional Euler-Boussinesq system describing the phenomenon of convection in an incompressible fluid. This system is composed of the Euler equations coupled with a transport-diffusion equation governing the evolution of the density. This system is given by

\begin{equation}
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= F(\theta) \\
\partial_t \theta + v \cdot \nabla \theta + \kappa |D|^{\alpha} \theta &= 0 \\
\text{div} v &= 0 \\
v_{|t=0} &= v^0, \quad \theta_{|t=0} = \theta^0.
\end{aligned}
\end{equation}

Above, $v = v(x,t) \in \mathbb{R}^2$, denotes the velocity vector-field, $p$ is the pressure and $\theta = \theta(x,t)$ is the temperature. The function $F(\theta) = (F_1(\theta), F_2(\theta))$ is a vector-valued function such that $F \in C^5$ and $F(0) = 0$, $\alpha$ is a real number in $[0,2]$ and the nonnegative parameter $\kappa$ denotes the molecular diffusivity. The fractional Laplacian $|D|^{\alpha}$ is defined in a standard way through its Fourier transform by

$$|D|^{\alpha} f(\xi) = |\xi|^{\alpha} \hat{f}(\xi).$$

The system (1.1) generalizes the classical Boussinesq system where $F(\theta) = (0, \theta)$. It is a special case of a class of generalized Boussinesq system introduced in [4].

Before discussing the mathematical aspects of our model with general $F$ we will first focus on the special case $F(\theta) = (0, \theta)$ and review the most significant contributions in the theory of global existence and uniqueness. We note that in space in dimension two the vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies the transport-diffusion equation

$$\partial_t \omega + v \cdot \nabla \omega = \partial_1 \theta.$$

It is well-known that a Beale-Kato-Majda criterion [5] can be applied to our model and thus the control of the vorticity in $L^\infty$ space is a crucial step to get global well-posedness.
results with smooth initial data. Now, by applying a maximum principle we get
\[ \|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} + \int_0^t \|\partial_1 \theta(\tau)\|_{L^\infty} d\tau. \]

The difficulty is then reduced to estimate the quantity \( \int_0^t \|\partial_1 \theta(\tau)\|_{L^\infty} d\tau \) and for this purpose the use the the smoothing effects of the transport-diffusion equation is crucial especially for sub-critical dissipation, that is \( \alpha > 1 \).

For the full viscous equations i.e when \( \kappa > 0 \) and \( \alpha = 2 \), the global well-posedness problem is solved recently in a series of papers \([8, 11, 15]\).

In \([8]\), Chae proved the global existence and uniqueness for initial data \((v^0, \theta^0) \in H^s \times H^s\) with \( s > 2 \). This result was extended in \([14]\) by Hmidi and Keraani to initial data \( v^0 \in B_{p,1}^{\frac{2}{p}+1} \) and \( \theta^0 \in B_{p,1}^{\frac{2}{p}-1} \cap L^r \), with \( r \in ]2, \infty[ \). Recently the study of global existence of Yudovich solutions has been done in \([11]\).

For more weaker dissipation that is \( 1 \leq \alpha < 2 \), global well-posedness results have been recently obtained. Indeed, the subcritical case \( 1 < \alpha \) was solved by Hmidi and Zerguine \([13]\) with critical regularities i.e \( v^0 \in B_{p,1}^{1+\frac{2}{p}} \) and \( \theta^0 \in B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^r \), \( \frac{2}{\alpha-1} < r < \infty \). They used the maximal smoothing effects for a transport-diffusion which can be roughly speaking summarized as follows: for every \( 0 < \varepsilon < 1 \)
\[ \|\theta\|_{L^1_t C^{1-\varepsilon}} \leq C_0(1 + t + \|\omega\|_{L^1_t L^\infty}), \]
with \( C_0 \) a constant depending on the size of the initial data. The critical case \( \alpha = 1 \), is more subtle because the dissipation has the same rate as the possible amplification of the vorticity by \( \partial_1 \theta \). In \([17]\), Hmidi, Keraani and Rousset gave a positive answer for global well-posedness by using a hidden cancellation given by the coupling.

The main goal of this paper is to extend the results of \([13]\) for general source term \( F(\theta) \). Our result reads as follows (see section 2 for the definitions and the basic properties of Besov spaces).

**Theorem 1.1.** Let \((\alpha, p) \in ]1, 2] \times ]1, \infty[, v^0 \in B_{p,1}^{1+\frac{2}{p}}\) be a divergence free vector-field of \( \mathbb{R}^2 \), \( \theta^0 \in B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^\infty \) and \( F \in C^5(\mathbb{R}, \mathbb{R}) \). Then there exists a unique global solution \((v, \theta)\) for the system \((1.1)\) such that
\[ v \in C(\mathbb{R}^+_+; B_{p,1}^{1+\frac{2}{p}}) \quad \text{and} \quad \theta \in L^\infty_{loc}(\mathbb{R}^+_+; B_{p,1}^{-\alpha+1+\frac{2}{p}} \cap L^\infty) \cap L^1_{loc}(\mathbb{R}^+_+; Lip) \]

If we take \( \theta = 0 \), then the system \((1.1)\) is reduced to the well-known 2D incompressible Euler system. It is well known that this system is globally well-posed in \( H^s \) for \( s > 2 \). The main argument is the BKM criterion \([5]\) ensuring that the development of finite-time singularities is related to the blow-up of the \( L^\infty \) norm of the vorticity and in that case the vorticity is only transported by the flow. However the global persistence of critical Besov regularities \( v^0 \in B_{p,1}^{1+\frac{2}{p}} \) can not be derived from BKM criterion from. This problem was solved in \([21]\) by Vishik and his crucial tool is a new logarithmic estimate which can be
formulated as follows:
\[ \| f \circ g^{-1} \|_{B^{0}_{\infty,1}} \leq C(1 + \log(\| \nabla g \|_{L^\infty} \| \nabla g^{-1} \|_{L^\infty})) \| f \|_{B^{0}_{\infty,1}}, \]
with \( f \in B^{0}_{\infty,1} \), \( g \) is a \( C^1 \)-diffeomorphism preserving Lebesgue measure and \( C \) some constant depending only on the dimension \( d \) (see Theorem 4.2 in [21] p-209 for the proof).

Remark 1.2. In the above theorem, we can take \( F \in C^{\lfloor 1 + \frac{2}{p} \rfloor + 2} \) instead of \( F \in C^5 \).

On the other hand if we assume that \( F \in C^5_b \), then we can replace the assumption \( \theta^0 \in L^\infty \) by \( \theta^0 \in L^r, \frac{2}{\alpha - 1} < r < \infty \).

Let us now discuss briefly the difficulties that one has to deal with. The formulation vorticity-temperature of the system (1.1) is described by,

\[
\begin{cases}
\partial_t \omega + v \cdot \nabla \omega = \partial_1(F_2(\theta)) - \partial_2(F_1(\theta)) \\
\partial_t \theta + v \cdot \nabla \theta + |D|^\alpha \theta = 0 \\
\omega|_{t=0} = \omega^0, \quad \theta|_{t=0} = \theta^0.
\end{cases}
\]

Taking the \( L^2 \)-scalar product we get successively

\[ \| \omega(t) \|_{L^2} \leq \| \omega^0 \|_{L^2} + \| \nabla F \|_{L^\infty} \int_0^t \| \nabla \theta(\tau) \|_{L^2} d\tau \]

and

\[ \| \theta(t) \|_{L^2}^2 + \| \theta(t) \|_{L^2 H^{\frac{\alpha}{2}}}^2 \leq \| \theta^0 \|_{L^2}^2. \]

We observe that one can take benefit of these estimates only for \( \alpha = 2 \) in which case we get a bound for \( \omega \) in \( L^\infty_{t,loc}(\mathbb{R}^+, L^2) \) and a bound for \( \theta \) in \( L^\infty_{t,loc}(\mathbb{R}^+, L^2) \cap L^2_{t,loc}(\mathbb{R}^+, \dot{H}^1) \). However for \( 1 < \alpha < 2 \), there is no obvious a priori estimates for the vorticity and we will use the idea developed in [13] consisting in the use of the maximal smoothing effect of the transport-diffusion equation.

The plan of the rest of this paper is organized as follows. In section 2 we detail some basic notions of Littlewood-Paley theory, function spaces and we recall some useful lemmas. We prove in section 3 some smoothing effects about a transport-diffusion equation which we need for the proof of our main result. The proof of our main result is given in section 4. Finally we prove in an appendix the generalized Bernstein inequality.

2. SOME DEFINITION AND TECHNICAL TOOLS

In this preliminary section, we are going to recall the so-called Littlewood-Paley operators and give some of their elementary properties. It will be also convenient to introduce some function spaces and review some important lemmas that will be used later.

We denote by \( C \) any positive constant than will change from line to line and \( C_0 \) a real positive constant depending on the size of the initial data. We will use the following notations:

- For any positive \( A \) and \( B \), the notation \( A \lesssim B \) means that there exists a positive constant \( C \) such that \( A \leq CB \).
- We denote by \( \dot{W}^{1,p} \) with \( 1 \leq p \leq \infty \) the space of distribution \( f \) such that \( \nabla f \in L^p \) (see section 4).
First of all, we define the dyadic decomposition of the full space $\mathbb{R}^d$ and recall the Littlewood-Paley operators (see for example [6]). There exists two nonnegative radial functions $\chi \in D(\mathbb{R}^2)$ and $\varphi \in D(\mathbb{R}^2 \setminus \{0\})$ such that

\begin{align*}
(1) & \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^2, \\
(2) & \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \\
(3) & \quad |p - q| \geq 2 \Rightarrow \text{supp } \varphi(2^{-p} \cdot) \cap \text{supp } \varphi(2^{-q} \cdot) = \emptyset, \\
(4) & \quad q \geq 1 \Rightarrow \text{supp } \chi \cap \text{supp } \varphi(2^{-q} \cdot) = \emptyset.
\end{align*}

Let $h = \mathcal{F}^{-1} \varphi$ and $\bar{h} = \mathcal{F}^{-1} \chi$, the frequency localization operators $\Delta_q$ and $S_q$ are defined by

\begin{align*}
\Delta_q f &= \varphi(2^{-q} D) f = 2^{2q} \int_{\mathbb{R}^2} h(2^q y) f(x - y) \, dy \quad \text{for } q \geq 0, \\
S_q f &= \chi(2^{-q} D) f = \sum_{-1 \leq p \leq q - 1} \Delta_p f = 2^{2q} \int_{\mathbb{R}^2} \bar{h}(2^q y) f(x - y) \, dy, \\
\Delta_{-1} f &= S_0 f, \quad \Delta_q f = 0 \quad \text{for } q \leq -2.
\end{align*}

It may be easily checked that

\[ f = \sum_{q \in \mathbb{Z}} \Delta_q f, \quad \forall f \in \mathcal{S}'(\mathbb{R}^2). \]

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

\[ \Delta_p \Delta_q f = 0 \quad \text{if } \quad |p - q| \geq 2 \]

\[ \Delta_p (S_{q-1} \Delta_q f) = 0 \quad \text{if } \quad |p - q| \geq 5. \]

Let us note that the above operators $\Delta_q$ and $S_q$ map continuously $L^p$ into itself uniformly with respect to $q$ and $p$. We will need also the homogeneous operators :

\[ \forall q \in \mathbb{Z} \quad \hat{\Delta}_q v = \varphi(2^{-q} D) v \quad \text{and} \quad \hat{S}_q v = \sum_{p \leq q - 1} \hat{\Delta}_p v. \]

We notice that $\Delta_q = \hat{\Delta}_q, \forall q \in \mathbb{N}$ and $S_q$ coincides with $\hat{S}_q$ on tempered distributions modulo polynomials.

We now give the way how the product acts on Besov spaces. We shall use the dyadic decomposition.

Let us consider two tempered distributions $u$ and $v$, we write

\[ u = \sum_q \Delta_q u \quad \text{and} \quad v = \sum_{q'} \Delta_{q'} v \]

\[ uv = \sum_{q, q'} \Delta_q u \Delta_{q'} v. \]
Now, let us introduce Bony’s decomposition see [3].

**Definition 2.1.** We denote by $T_u v$ the following bilinear operator:

$$T_u v = \sum_q S_{q-1} u \Delta_q v.$$ 

The remainder of $u$ and $v$ denoted by $R(u, v)$ is given by the following bilinear operator:

$$R(u, v) = \sum_{|q-q'| \leq 1} \Delta_q u \Delta_{q'} v.$$ 

Just by looking at the definition, it is clear that

$$uv = T_u v + T_v u + R(u, v).$$

With the introduction of $\Delta_q$, let us recall the definition of Besov space, see [6].

**Definition 2.2.** Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. The inhomogeneous Besov space $B_{p,r}^s$ is defined by

$$B_{p,r}^s = \{ f \in S'(\mathbb{R}^2) : \| f \|_{B_{p,r}^s} < \infty \}.$$ 

Here

$$\| f \|_{B_{p,r}^s} := \| 2^{qs} \| \Delta_q f \|_{L^p} \| \ell^r \|.$$

We define also the homogeneous norm

$$\| f \|_{B_{p,r}^s} := \| (2^{qs} \| \Delta_q f \|_{L^p})_q \|_{\ell^r(\mathbb{Z})}.$$ 

The definition of Besov spaces does not depend on the choice of the dyadic decomposition. The two spaces $H^s$ and $B_{2,2}^s$ are equal and we have

$$\frac{1}{C|s|+1} \| u \|_{B_{2,2}^s} \leq \| u \|_{H^s} \leq C|s|+1 \| u \|_{B_{2,2}^s}.$$ 

Our study will require the use of the following coupled spaces. Let $T > 0$ and $\rho \geq 1$, we denote by $L_T^\rho B_{p,r}^s$ the space of distributions $f$ such that

$$\| f \|_{L_T^\rho B_{p,r}^s} := \left( \left( 2^{qs} \| \Delta_q f \|_{L^p} \right)_q \right)_T < +\infty.$$ 

Besides the usual mixed space $L_T^\rho B_{p,r}^s$, we also need Chemin-Lerner space $\tilde{L}_T^\rho B_{p,r}^s$ which defined as the set of all distributions $f$ satisfying

$$\| f \|_{\tilde{L}_T^\rho B_{p,r}^s} := \| (2^{qs} \| \Delta_q f \|_{L^p})_q \|_{\ell^r} < +\infty.$$ 

The relation between these spaces are detailed in the following lemma, which is a direct consequence of the Minkowski inequality.

**Lemma 2.3.** Let $s \in \mathbb{R}, \varepsilon > 0$ and $(p, r, \rho) \in [1, +\infty]^3$. Then we have the following embeddings

$$L_T^\rho B_{p,r}^s \hookrightarrow \tilde{L}_T^\rho B_{p,r}^s \hookrightarrow L_T^\rho B_{p,r}^{s-\varepsilon} \quad \text{if} \quad r \geq \rho.$$ 

$$L_T^\rho B_{p,r}^{s+\varepsilon} \hookrightarrow \tilde{L}_T^\rho B_{p,r}^s \hookrightarrow L_T^\rho B_{p,r}^s \quad \text{if} \quad \rho \geq r.$$
A further important result that will be constantly used here is the so-called Bernstein inequalities (for the proof see [6] and the references therein) which are detailed below.

**Lemma 2.4.** There exists a constant $C > 0$ such that for every $q \in \mathbb{Z}$, $k \in \mathbb{N}$ and for every tempered distribution $u$ we have

$$
\sup_{|\alpha|=k} \| \partial^\alpha S_q u \|_{L^b} \leq C^{k/2} 2^q \left( 1 \right)^{ \left( k+2 - \frac{q}{b} \right) } \| S_q u \|_{L^b} \quad \text{for} \quad b \geq a \geq 1
$$

$$
C^{-k} 2^q \| \Delta_q u \|_{L^a} \leq \sup_{|\alpha|=k} \| \partial^\alpha \Delta_q u \|_{L^a} \leq C^{k} 2^q \| \Delta_q u \|_{L^a}.
$$

Notice that Bernstein inequalities remain true if we change the derivative $\partial^\alpha$ by the fractional derivative $|D|^\alpha$. We can find a proof of the next Proposition in [14] which is an extension of [21].

**Proposition 2.5.** Let $(p, r) \in [1, \infty]^2$, $v$ be a divergence free vector-field belonging to the space $L^1_{1, \infty}(\mathbb{R}^+; \text{Lip}(\mathbb{R}^2))$ and let $u$ be a smooth solution of the following transport equation,

$$
\begin{cases}
\partial_t a + v \cdot \nabla a = f \\
a|_{t=0} = a^0.
\end{cases}
$$

If the initial data $a^0 \in B^0_{p, r}$, then we have for all $t \in \mathbb{R}^+$,

$$
\| a \|_{L^\infty t B^0_{p, r}} \lesssim (\| a^0 \|_{B^0_{p, r}} + \| f \|_{L^1 t B^0_{p, r}}) \left( 1 + \int_0^t \| \nabla v(\tau) \|_{L^\infty} d\tau \right).
$$

We now give the following commutator estimate which proved in [14].

**Proposition 2.6.** Let $u$ be a smooth function and $v$ be a smooth divergence-free vector field of $\mathbb{R}^2$ with vorticity $\omega := \text{curl} v$. Then we have for all $q \geq -1$

$$
\| (\Delta_q v \cdot \nabla) u \|_{L^\infty} \lesssim \| u \|_{L^\infty} (\| \nabla \Delta_{-1} v \|_{L^\infty} + (q + 2) \| \omega \|_{L^\infty}).
$$

We recall now the following result of propagation of Besov regularities which is discussed in [6].

**Proposition 2.7.** Let $v$ be a solution of the incompressible Euler system,

$$
\partial_t v + v \cdot \nabla v + \nabla p = f, \quad v|_{t=0} = v^0, \quad \text{div} v = 0.
$$

Then for $s > -1, (p, r) \in ]1, \infty[ \times [1, \infty]$ we have

$$
\| v(t) \|_{B^s_{p, r}} \lesssim C e^{V(t)} \left( \| v^0 \|_{B^s_{p, r}} + \int_0^t e^{-V(\tau)} \| f(\tau) \|_{B^s_{p, r}} d\tau \right),
$$

with $V(t) = \| \nabla v \|_{L^1 t L^\infty}$.

To finish this paragraph we need the following theorem which give the action of smooth functions on the Besov spaces $B^s_{p, r}$, (see [2] for the proof).

**Theorem 2.8.** Let $F \in C[\mathbb{R}^2]$, $s$ a positive real number and $F$ vanishing at 0. If $u$ belongs to $B^s_{p, r} \cap L^\infty$, with $(p, r) \in [1, +\infty]^2$, then $F \circ u$ belongs to $B^s_{p, r}$ and we have

$$
\| F \circ u \|_{B^s_{p, r}} \lesssim C_s \sup_{|x| \leq C \| u \|_{L^\infty}} \| F \|_{L^\infty} \| u \|_{B^s_{p, r}}.
$$
3. AROUND A TRANSPORT-DIFFUSION EQUATION

In this section, we will give some useful estimates for any smooth solution of linear transport-diffusion model given by

\[
\begin{align*}
\partial_t \theta + v \cdot \nabla \theta + |D|^{\alpha} \theta &= f \\
\theta|_{t=0} &= \theta^0.
\end{align*}
\]

(3.1)

We will discuss two kinds of estimates: \( L^p \) estimates and smoothing effects. The proof of the following \( L^p \) estimates can be found in [7].

**Lemma 3.1.** Let \( v \) be a smooth divergence free vector-field of \( \mathbb{R}^2 \) and \( \theta \) be a smooth solution of the equation (3.1). Then for every \( p \in [1, \infty] \)

\[
\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p} + \int_0^t \|f(\tau)\|_{L^p} d\tau.
\]

We give now the following smoothing effects which is proved in [13].

**Proposition 3.2.** Let \( p \in [1, \infty], s > -1 \) and \( v \) be a smooth divergence free vector-field of \( \mathbb{R}^2 \). Let \( \theta \) be a smooth solution of (3.1), then

\[
\|\theta\|_{L_1^\infty B^{s+\alpha}_{p,1}} + \|\theta\|_{L_1^1 B^{s+\alpha}_{p,1}} \leq C e^{C V(t)} \left( \|\theta^0\|_{B^{s}_{p,1}} (1 + t) + \|f\|_{L_1^1 B^{s}_{p,1}} + \int_0^t \Gamma_s(t) d\tau \right),
\]

with,

\[
V(t) \overset{\text{def}}{=} \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau, \quad \Gamma_s(t) \overset{\text{def}}{=} \|\nabla \theta(t)\|_{L^\infty} \|v(t)\|_{B^{s}_{p,1}} 1_{[1,\infty]}(s).
\]

We intend to prove the two following smoothing effects. One is detailed below.

**Proposition 3.3.** Let \( v \) be a smooth divergence free vector-field of \( \mathbb{R}^2 \) with vorticity \( \omega \) and \( \theta \) be a smooth solution of (3.1). Then for every \( r \in [2, \infty] \), there exists a constant \( C \) such that for every \( \rho \geq 1 \), \( q \in \mathbb{N} \) and \( t \in \mathbb{R}_+ \)

\[
2^{\frac{r}{r-2}} \|\Delta_q \theta\|_{L_r^r} \leq C \|\Delta_q \theta^0\|_{L^r} + C \|\theta^0\|_{L^\infty} \|\omega\|_{L_1^1 L^r} + C \|f\|_{L_1^1 L^r}.
\]

Moreover for \( q = -1 \), we have

\[
\|\Delta_{-1} \theta\|_{L_r^r} \leq C \int_0^t \|\Delta_{-1} \theta^0\|_{L^r} + \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^r} d\tau + \int_0^t \|f(\tau)\|_{L^r} d\tau.
\]

**Proof.** We start with localizing in frequencies the equation of \( \theta \): for \( q \in \mathbb{N} \) we set \( \theta_q := \Delta_q \theta \). Then

\[
\partial_t \theta_q + v \cdot \nabla \theta_q + |D|^{\alpha} \theta_q = -[\Delta_q, v \cdot \nabla] \theta + \Delta_q f.
\]

Multiplying the above equation by \( |\theta_q|^{r-2} \theta_q \), integrating by parts and using Hölder inequality, we get

\[
\frac{1}{r} \frac{d}{dt} \|\theta_q\|_{L_r^r} + \int_{\mathbb{R}^2} (|D|^{\alpha} \theta_q) |\theta_q|^{r-2} \theta_q dx \leq \|\theta_q\|_{L_r^{-1}}^{r-1} \|\Delta_q, v \cdot \nabla \|_{L^r} + \|\theta_q\|_{L_r^{-1}}^{r-1} \|\Delta_q f\|_{L^r}.
\]


Using the following generalized Bernstein inequality see the appendix,

$$\forall 1 < r, \quad c2^{q\alpha}\|\theta_q\|^r_{L^r} \leq \int_{\mathbb{R}^d} (|D|^\alpha \theta_q)|\theta_q|^{r-2}\theta_q \, dx,$$

where $c$ depends on $r$. Inserting this estimate in the previous one we obtain

$$\frac{1}{r} \frac{d}{dt}\|\theta_q(t)\|^r_{L^r} + c2^{q\alpha}\|\theta_q(t)\|^r_{L^r} \lesssim \|\theta_q(t)\|^{r-1}_{L^r}\|\Delta_q, v \cdot \nabla\theta(t)\|_{L^r} + \|\theta_q(t)\|^{r-1}_{L^r}\|\Delta_q f(t)\|_{L^r}.$$

This implies that

$$\frac{d}{dt}\|\theta_q(t)\|_{L^r} + c2^{q\alpha}\|\theta_q(t)\|_{L^r} \lesssim \|\Delta_q, v \cdot \nabla\theta(t)\|_{L^r} + \|\Delta_q f(t)\|_{L^r}.$$

We multiply the above inequality by $e^{ct2^{q\alpha}}$, we find

$$(3.3) \quad \frac{d}{dt}\left(e^{ct2^{q\alpha}}\|\theta_q(t)\|_{L^r}\right) \lesssim e^{ct2^{q\alpha}}\|\Delta_q, v \cdot \nabla\theta(t)\|_{L^r} + e^{ct2^{q\alpha}}\|\Delta_q f(t)\|_{L^r}.$$

To estimate the right hand-side, we shall use the following Lemma (see [17] for the proof of this Lemma).

**Lemma 3.4.** Let $v$ be a smooth divergence-free vector field and $\theta$ be a smooth scalar function. Then for all $p \in [1, \infty]$ and $q \geq -1$,

$$\|\Delta_q, v \cdot \nabla\theta\|_{L^p} \lesssim \|\nabla v\|_{L^p}\|\theta\|_{L^\infty}.$$

Combined with (3.3) this lemma yields

$$\frac{d}{dt}\left(e^{ct2^{q\alpha}}\|\theta_q(t)\|_{L^r}\right) \lesssim e^{ct2^{q\alpha}}\|\nabla v(t)\|_{L^r}\|\theta(t)\|_{L^\infty} + e^{ct2^{q\alpha}}\|f(t)\|_{L^r}$$

$$\lesssim e^{ct2^{q\alpha}}\|\omega(t)\|_{L^r}\|\theta^0\|_{L^\infty} + e^{ct2^{q\alpha}}\|f(t)\|_{L^r},$$

we have used in the last line Lemma 3.1 and the classical fact

$$\|\nabla v\|_{L^p} \lesssim \|\omega\|_{L^p} \quad \forall p \in [1, \infty[.$$

Integrating the differential inequality we get

$$\|\theta_q(t)\|_{L^r} \lesssim \|\theta^0_q\|_{L^r} e^{-ct2^{q\alpha}} + \|\theta^0\|_{L^\infty} \int_0^t e^{-c(t-\tau)2^{q\alpha}}\|\omega(\tau)\|_{L^r} d\tau + \int_0^t e^{-c(t-\tau)2^{q\alpha}}\|f(\tau)\|_{L^r} d\tau.$$

By taking the $L^p[0,t]$ norm and using convolution inequalities we obtain

$$2^{q\alpha/p}\|\theta_q\|_{L^p_t L^r} \lesssim \|\theta^0_q\|_{L^r} + \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^r} d\tau + \int_0^t \|f(\tau)\|_{L^r} d\tau$$

$$\lesssim \|\theta^0_q\|_{L^r} + \|\theta^0\|_{L^\infty} \|\omega\|_{L^1_t L^r} + \|f\|_{L^1_t L^r}.$$

This is the desired result for $q \in \mathbb{N}$.

For $q = -1$, we apply the operator $\Delta_{-1}$ to (3.1), we obtain

$$\partial_t \Delta_{-1} \theta + v \cdot \nabla \Delta_{-1} \theta + |D|^{\alpha} \Delta_{-1} \theta = -[\Delta_{-1}, v \cdot \nabla] \theta + \Delta_{-1} f.$$
Taking the $L^r$ norm and using Lemmas 3.1 and 3.4 we get
\[
\|\Delta^{-1}\theta\|_{L^r} \leq \|\Delta^{-1}\theta^0\|_{L^r} + \int_0^t \|\Delta^{-1}, v \cdot \nabla\theta(\tau)\|_{L^r} d\tau + \int_0^t \|\Delta^{-1}f(\tau)\|_{L^r} d\tau
\]
\[
\lesssim \|\Delta^{-1}\theta^0\|_{L^r} + \int_0^t \|\nabla v(\tau)\|_{L^r} \|\theta(\tau)\|_{L^\infty} d\tau + \int_0^t \|f(\tau)\|_{L^r} d\tau
\]
\[
\lesssim \|\Delta^{-1}\theta^0\|_{L^r} + \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^r} d\tau + \int_0^t \|f(\tau)\|_{L^r} d\tau.
\]
By taking the $L^p[0,t]$ norm and using Hölder inequality, we obtain finally
\[
\|\Delta^{-1}\theta\|_{L^p_tL^r} \lesssim t^{\frac{s}{2}} (\|\Delta^{-1}\theta^0\|_{L^r} + \|\theta^0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^r} d\tau + \int_0^t \|f(\tau)\|_{L^r} d\tau).
\]
This is the desired result. \hfill \Box

The second smoothing effect is given by the following Proposition.

**Proposition 3.5.** Let $v$ be a smooth divergence free vector-field of $\mathbb{R}^2$ with vorticity $\omega$. Let $\theta$ be a smooth solution of (3.1). Then we have for $q \geq -1$ and for $t \in \mathbb{R}_+$ with $|f| \equiv 0$,
\[
2^{q\alpha} \int_0^t \|\Delta^{-1}\theta(\tau)\|_{L^\infty} d\tau \lesssim \|\theta^0\|_{L^\infty} \left(1 + t + (q + 2)\|\omega\|_{L^1_tL^\infty} + \|\nabla\Delta^{-1}v\|_{L^1_tL^\infty}\right).
\]

**Proof.** The idea of the proof will be done in the spirit of [12]. First we prove the smoothing effects for a small interval of time depending of vector $v$, but it depend not of the initial data. In the second step, we proceed to a division in time thereby extending the estimate at any time arbitrarily chosen positive.

3.1. Local estimates. We localize in frequency the evolution equation and rewriting the equation in Lagrangian coordinates.

Let $q \in \mathbb{N}$, then the Fourier localized function $\theta_q := \Delta_q \theta$ satisfies
\[
(3.4) \quad \partial_t \theta_q + S_{q-1}v \cdot \nabla \theta_q + |D|^\alpha \theta_q = (S_{q-1}v - v) \cdot \nabla \theta_q - \left[\Delta q, v \cdot \nabla\right] \theta := h_q.
\]

First, we shall estimate the function $h_q$ in the space $L^\infty$, for the first term we have
\[
\|(S_{q-1}v - v) \cdot \nabla \theta_q\|_{L^\infty} \leq \|S_{q-1}v - v\|_{L^\infty} \|
abla \theta_q\|_{L^\infty}
\]
\[
\lesssim \sum_{j \geq q-1} \|\Delta_j v\|_{L^\infty} 2^q \|\theta_q\|_{L^\infty}
\]
\[
\lesssim \|\theta^0\|_{L^\infty} \sum_{j \geq q-1} 2^{q-j} \|\Delta_j \omega\|_{L^\infty}
\]
\[
\lesssim \|\theta^0\|_{L^\infty} \|\omega\|_{L^\infty}.
\]

We have used Hölder inequality, Lemma 2.4 and the result
\[
\|\Delta_j v\|_{L^p} \approx 2^{-j} \|\Delta_j \omega\|_{L^p}, \quad \forall p \in [1, \infty] \quad \text{and} \quad j \in \mathbb{N}.
\]
For \( \|\Delta_q, v \cdot \nabla|\theta|\|_{L^\infty} \), we use Lemma 3.1 and Proposition 2.6 then
\[
\|\Delta_q, v \cdot \nabla|\theta|\|_{L^\infty} \lesssim \|\theta^0\|_{L^\infty} \left( \|\nabla\Delta_{-1}v\|_{L^\infty} + (q + 2)\|\omega\|_{L^\infty} \right).
\]
This implies that
\[
\|h_q(t)\|_{L^\infty} \leq \|(S_q-1v - v) \cdot \nabla\theta_q\|_{L^\infty} + \|\Delta_q, v \cdot \nabla|\theta|\|_{L^\infty}
\lesssim \|\theta^0\|_{L^\infty} \left( \|\nabla\Delta_{-1}v\|_{L^\infty} + (q + 2)\|\omega\|_{L^\infty} \right).
\]
Let us now introduce the flow \( \psi_q \) of the regularized velocity \( S_q-1v \),
\[
\psi_q(t, x) = x + \int_0^t S_q-1v(t, \psi_q(t, x))dt.
\]
We set
\[
\bar{\theta}_q(t, x) = \theta_q(t, \psi_q(t, x)) \quad \text{and} \quad \bar{h}_q(t, x) = h_q(t, \psi_q(t, x)).
\]
Then we have
\[
|D|^\alpha \partial_t \bar{\theta}_q + |D|^\alpha\bar{\theta}_q = \bar{h}_q + |D|^\alpha(\theta_q \circ \psi_q) - (|D|^\alpha \theta_q) \circ \psi_q := \bar{h}_q + h^1_q.
\]
Let us admit the following estimate proven in [13]
\[
|D|^\alpha(\theta_q \circ \psi_q) - (|D|^\alpha \theta_q) \circ \psi_q\|_{L^p} \leq Ce^{CV_q(t)}V_q(t)2^{\alpha q}\|\theta_q\|_{L^p}, \forall \ p \in [1, \infty],
\]
where
\[
V_q(t) = \int_0^t \|\nabla S_q-1v(t)\|_{L^\infty}dt.
\]
Now, since the flow \( \psi_q \) preserves Lebesgue measure then we get by (3.6)
\[
\|\bar{h}_q^1(t)\|_{L^\infty} \leq Ce^{CV_q(t)}V_q(t)2^{\alpha q}\|\theta_q\|_{L^\infty}.
\]
Now, we will again localize in frequency the equation (3.5) through the operator \( \Delta_j \),
\[
\partial_t \Delta_j \bar{\theta}_q + |D|^\alpha \Delta_j \bar{\theta}_q = \Delta_j \bar{h}_q + \Delta_j h^1_q.
\]
Using Duhamel formula
\[
\Delta_j \bar{\theta}_q(t, x) = e^{-t|D|^\alpha} \Delta_j \theta^0_q + \int_0^t e^{-(t-\tau)|D|^\alpha} \Delta_j \bar{h}_q(\tau) d\tau + \int_0^t e^{-(t-\tau)|D|^\alpha} \Delta_j h^1_q(\tau) d\tau.
\]
At this stage we need the following lemma (for \( \alpha \in [0, 2] \) see [16]).

**Lemma 3.6.** There exists a positive constant \( C \) such that for any \( u \in L^p \) with \( p \in [1, +\infty] \),
for any \( t, \alpha \in \mathbb{R}_+ \) and any \( j \in \mathbb{N} \), we have
\[
\|e^{-t|D|^\alpha} \Delta_j u\|_{L^p} \leq Ce^{-ct2^j\alpha} \|\Delta_j u\|_{L^p}.
\]
Where the constants \( C \) and \( c \) depend only on the dimension \( d \).
Combining this estimate with Duhamel formula yields for every $j \in \mathbb{N}$
\[
\|e^{-(t-\tau)D^\alpha}\Delta_j \tilde{h}_q(\tau)\|_{L^\infty} \leq C e^{-c(t-\tau)2^j}\left(\|S_{q-1}v - v\|_{L^\infty} + ||[\Delta_q, v \cdot \nabla]\theta||_{L^\infty}\right).
\]
This implies that
\[
\|e^{-(t-\tau)D^\alpha}\Delta_j \tilde{h}_q(\tau)\|_{L^\infty} \lesssim e^{-c(t-\tau)2^j}\|\theta^0\|_{L^\infty}\left(\|\nabla \Delta_{-1}v\|_{L^\infty} + (q + 2)||\omega||_{L^\infty}\right)
\]
and
\[
\|e^{-(t-\tau)D^\alpha}\Delta_j \tilde{h}_q(\tau)\|_{L^\infty} \lesssim e^{-c(t-\tau)2^j}\|h_q^1(\tau)\|_{L^\infty} \lesssim e^{-c(t-\tau)2^j} e^{CV_q(t)}V_q(t)2^{q\alpha}\|\theta_q\|_{L^\infty}.
\]
Therefore
\[
\|\Delta_j \tilde{\theta}_q(t)\|_{L^\infty} \lesssim e^{-c2^j}\|\Delta_j \tilde{h}_q(\tau)\|_{L^\infty} + 2^{q\alpha} e^{CV_q(t)}V_q(t)\int_0^t e^{-c(t-\tau)2^j}||\tilde{\theta}_q(\tau)||_{L^\infty}d\tau
\]
\[
\qquad + (q + 2)||\theta^0||_{L^\infty}\int_0^t e^{-c(t-\tau)2^j}||\omega(\tau)||_{L^\infty}d\tau
\]
\[
\qquad + ||\theta^0||_{L^\infty}\int_0^t e^{-c(t-\tau)2^j}||\nabla \Delta_{-1}v(\tau)||_{L^\infty}d\tau.
\]
Integrating in time and using Young inequality, we get for all $j \in \mathbb{N}$
\[
\|\Delta_j \tilde{\theta}_q\|_{L^1_t L^\infty} \lesssim 2^{-j\alpha}\left(\|\Delta_j \tilde{h}_q\|_{L^\infty} + (q + 2)||\theta^0||_{L^\infty}||\omega||_{L^1_t L^\infty}\right)
\]
\[
\qquad + ||\theta^0||_{L^\infty}||\nabla \Delta_{-1}v||_{L^1_t L^\infty} + 2^{(q-j)\alpha} e^{CV_q(t)}V_q(t)||\theta_q||_{L^1_t L^\infty}.
\]
Let now $N \in \mathbb{N}$ be a fixed number that will be chosen later. Since the flow $\psi_q$ preserves Lebesgue measure then we write
\[
2^{q\alpha}\|\tilde{\theta}_q\|_{L^1_t L^\infty} = 2^{q\alpha}\|\theta_q\|_{L^1_t L^\infty} \leq 2^{q\alpha}\left(\sum_{|j-q|<N} ||\Delta_j \tilde{\theta}_q\|_{L^1_t L^\infty} + \sum_{|j-q|\geq N} ||\Delta_j \tilde{\theta}_q\|_{L^1_t L^\infty}\right)
\]
\[
\qquad := I + II.
\]
If $q \geq N$, then it follows from (3.7),
\[
I \lesssim \sum_{|j-q|<N} 2^{(q-j)\alpha}\left(||\Delta_j \theta_q\|_{L^\infty} + ||\theta^0||_{L^\infty}\left(\|q + 2||\omega||_{L^1_t L^\infty} + ||\nabla \Delta_{-1}v||_{L^1_t L^\infty}\right)\right)
\]
\[
\qquad + \sum_{|j-q|<N} 2^{(q-j)\alpha} V_q(t)e^{CV_q(t)}2^{q\alpha}\|\theta_q\|_{L^1_t L^\infty}.
\]
Thus
\[
I \lesssim ||\theta^0||_{L^\infty} + 2^{N\alpha}||\theta^0||_{L^\infty}\left((q + 2)||\omega||_{L^1_t L^\infty} + ||\nabla \Delta_{-1}v||_{L^1_t L^\infty}\right) + V_q(t)e^{CV_q(t)}2^{q\alpha}2^{N\alpha}\|\theta_q\|_{L^1_t L^\infty}.
\]
To estimate $II$, we use the following result due to Vishik [21]
\[
||\Delta_j \tilde{\theta}_q||_{L^p} \lesssim 2^{-|q-j|} e^{CV_q(t)}||\theta_q||_{L^p}, \forall p \in [1, \infty].
\]
Second, we choose $N$ to show this, we take first $t$.

Thus

\[
\begin{align*}
\Pi &= 2^{q\alpha} \sum_{|j-q| \geq N} \|\Delta_j \tilde{q}\|_{L^1_t L^\infty} \\
&\lesssim 2^{q\alpha} \sum_{|j-q| \geq N} 2^{-|q-j|} e^{CV_q(t)} \|\theta_q\|_{L^1_t L^\infty} \\
&\lesssim 2^{-N} e^{CV_q(t)} 2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty}.
\end{align*}
\]

We have then

\[
2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty} \lesssim \|\theta^0\|_{L^\infty} + 2^N \|\theta^0\|_{L^\infty} \left( (q + 2) \|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty} \right) \\
+ V_q(t) e^{CV_q(t)} 2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty} + 2^{-N} e^{CV_q(t)} 2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty}.
\]

For low frequencies $q < N$, we have

\[
2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty} \lesssim 2^N \|\theta\|_{L^1_t L^\infty}.
\]

Therefore we get for $q \geq -1$

\[
2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty} \leq C \|\theta^0\|_{L^\infty} + C 2^{2N} \|\theta\|_{L^1_t L^\infty} + C 2^{2N} \|\theta^0\|_{L^\infty} \left( (q + 2) \|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty} \right) \\
+ C \left( V_q(t) e^{CV_q(t)} 2^{N\alpha} + 2^{-N} e^{CV_q(t)} \right) 2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty}.
\]

Now, we claim that there exists two absolute constants $N \in \mathbb{N}$ and $C_1 > 0$ such that if $V_q(t) \leq C_1$, then

\[
V_q(t) e^{CV_q(t)} 2^{N\alpha} + 2^{-N} e^{CV_q(t)} \leq \frac{1}{2C}.
\]

To show this, we take first $t$ such that $V_q(t) \leq 1$, which is possible since $\lim_{t \to 0^+} V_q(t) = 0$. Second, we choose $N$ in order to get $2^{-N} e^C \leq \frac{1}{4C}$. By taking again $V_q(t)$ sufficiently small we obtain that $V_q(t) e^{CV_q(t)} 2^{N\alpha} \leq \frac{1}{4C}$.

Under this assumption $V_q(t) \leq C_1$, we obtain for $q \geq -1$

\[
2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty} \lesssim \|\theta\|_{L^1_t L^\infty} \left( 1 + (q + 2) \|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty} \right).
\]

We use Hölder’s inequality and Lemma \ref{lem:Holder} for $\|\theta\|_{L^1_t L^\infty}$, we get $\|\theta\|_{L^1_t L^\infty} \leq t \|\theta^0\|_{L^\infty}$.

Hence

\[
2^{q\alpha} \|\theta_q\|_{L^1_t L^\infty} \lesssim \|\theta^0\|_{L^\infty} \left( 1 + t + (q + 2) \|\omega\|_{L^1_t L^\infty} + \|\nabla \Delta_{-1} v\|_{L^1_t L^\infty} \right).
\]

Therefore the result is proved for small time.

3.2. Globalization. Let us now see how to extend this for arbitrary large time $T$. We take a partition $(T_i)_{i=0}^M$ of $[0, T]$ i.e $0 = T_0 \leq T_1 \leq \ldots \leq T_M = T$ and such that

\[
\int_{T_i}^{T_{i+1}} \|\nabla S_{q-1} v(t)\|_{L^\infty} dt \approx C_1, \quad \forall \ i \in [0, M].
\]
Reproducing the same argument in (3.8) we find in view of \( \|\theta(T_i)\|_{L^\infty} \leq \|\theta^0\|_{L^\infty} \).

\[
2^{q\alpha} \int_{T_i}^{T_{i+1}} \|\theta_q(t)\|_{L^\infty} dt \lesssim \int_{T_i}^{T_{i+1}} \|\theta(t)\|_{L^\infty} dt + \|\theta^0\|_{L^\infty} + \|\theta^0\|_{L^\infty}(q + 2) \int_{T_i}^{T_{i+1}} \|\omega(t)\|_{L^\infty} dt + \int_{T_i}^{T_{i+1}} \|\nabla \Delta_{-1}v(t)\|_{L^\infty} dt.
\]

Summing these estimates of 0 to M, we get

\[
2^{q\alpha}\|\theta_q\|_{L^1_T L^\infty} \lesssim \|\theta\|_{L^1_T L^\infty} + (M + 1)\|\theta^0\|_{L^\infty} + \|\theta^0\|_{L^\infty}(q + 2)\|\omega\|_{L^1_T L^\infty} + \|\nabla \Delta_{-1}v\|_{L^1_T L^\infty}.
\]

As \( M \approx V_q(t) \), then

\[
2^{q\alpha}\|\theta_q\|_{L^1_T L^\infty} \lesssim \|\theta\|_{L^1_T L^\infty} + (V_q(t) + 1)\|\theta^0\|_{L^\infty} + \|\theta^0\|_{L^\infty}(q + 2)\|\omega\|_{L^1_T L^\infty} + \|\nabla \Delta_{-1}v\|_{L^1_T L^\infty}.
\]

From the definition of the operator \( S_{q-1} \), we have then

\[
S_{q-1}v = \sum_{-1 \leq p \leq q-2} \Delta_p v = \Delta_{-1}v + \sum_{p=0}^{q-2} \Delta_p v.
\]

Thus

\[
\|\nabla S_{q-1}v\|_{L^\infty} \leq \|\nabla \Delta_{-1}v\|_{L^\infty} + \sum_{p=0}^{q-2} \|\nabla \Delta_p v\|_{L^\infty} \leq \|\nabla \Delta_{-1}v\|_{L^\infty} + C \sum_{p=0}^{q-2} \|\Delta_p \omega\|_{L^\infty} \leq \|\nabla \Delta_{-1}v\|_{L^\infty} + C(q - 1)\|\omega\|_{L^\infty}.
\]

We have used Bernstein inequality and the classical result

\[
\|\Delta_q v\|_{L^p} \approx 2^{-q}\|\Delta_q \omega\|_{L^p}, \forall p \in [1, \infty].
\]

Therefore

\[
\|\nabla S_{q-1}v\|_{L^\infty} \lesssim \|\nabla \Delta_{-1}v\|_{L^\infty} + (q + 2)\|\omega\|_{L^\infty},
\]

then inserting this estimate into the previous one

\[
2^{q\alpha}\|\theta_q\|_{L^1_T L^\infty} \lesssim \|\theta^0\|_{L^\infty} \left(1 + T + (q + 2)\|\omega\|_{L^1_T L^\infty} + \|\nabla \Delta_{-1}v\|_{L^1_T L^\infty}\right).
\]
This is the desired result.

Remark 3.7. If the velocity belongs to $L^1_t\text{Lip}$ then the previous estimate becomes
\[
\forall 1 \leq \rho \leq \infty, \quad 2^{\frac{\alpha}{\rho}} \|\Delta \theta\|_{L^\rho_t L^\infty} \lesssim \|\theta^0\|_{L^\infty} \left(1 + t + \|\nabla v\|_{L^1_t L^\infty}\right).
\]

4. Proof of Theorem 1.1

This section is devoted to the proof of theorem 1.1. For conciseness, we shall provide the a priori estimates supporting the claim of the theorem and give a complete proof of the uniqueness, while the proof of the existence part will be shortened and briefly described since it is somewhat contained in [14].

4.1. a Priori Estimates. The important quantities to bound for all times are the $L^\infty$ norm of the vorticity and the Lipschitz norm of the velocity. The main step for obtaining a Lipschitz bound is to give an $L^\infty$-bound of the vorticity. We prove before an $L^p$ estimate with $p < \infty$ and this allows us to bound the vorticity in $L^\infty$. We start then with the following one,

**Proposition 4.1.** Let $(\alpha, p) \in [1, 2] \times [1, \infty]$, $\omega^0 \in L^\infty \cap L^p$, $\theta^0 \in B^{1-\alpha}_{p,1} \cap L^\infty$, and $F \in C^1(\mathbb{R}, \mathbb{R})$. Then any smooth solution of the system (1.1) satisfies
- $\|\theta(t)\|_{L^\infty} \leq \|\theta^0\|_{L^\infty}$,
- $\|\omega(t)\|_{L^\infty \cap L^p} + \|\nabla \theta\|_{L^1_t L^\infty} \leq C_0 e^{C_0 t}$.

**Proof.** The first inequality is a direct consequence of Lemma 3.1. For the second one, we start with the vorticity equation
\[
\partial_t \omega + v \cdot \nabla \omega = F_2'(\theta) \partial_1 \theta - F_1'(\theta) \partial_2 \theta.
\]
Taking the $L^p$ norm we get
\[
\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \int_0^t \left(\sum_{i=1}^2 \|F_i'(\theta)\|_{L^\infty}\right) \|\nabla \theta(\tau)\|_{L^p} d\tau.
\]
Since $\|F_i'(\theta)\|_{L^\infty} = \sup_{x,t} |F_i'(\theta(x,t))|$, then we have
\[
\|F_i'(\theta)\|_{L^\infty} \leq \sup_{|x| \leq \|\theta\|_{L^\infty}} |F_i'(x)| \leq \sup_{|x| \leq \|\theta^0\|_{L^\infty}} |F_i'(x)|.
\]
As $F \in C^1(\mathbb{R}, \mathbb{R})$ and $\theta^0 \in L^\infty$, then
\[
\|F_i'(\theta)\|_{L^\infty} \leq \sup_{|x| \leq \|\theta^0\|_{L^\infty}} \|\nabla F_i\|_{L^\infty} \leq C.
\]
Thus
\[
\|\omega(t)\|_{L^p} \lesssim \|\omega^0\|_{L^p} + \|\nabla \theta\|_{L^1_t L^p}.
\]
Then we have

\[ \| \nabla \theta \|_{L^1_t L^p} \leq \sum_{q \geq 0} 2^q \| \Delta_q \theta \|_{L^1_t L^p} + \| \Delta_{-1} \theta \|_{L^1_t L^p} \]

\[ \leq \sum_{q \geq 0} 2^q \| \Delta_q \theta \|_{L^1_t L^p} + \| \Delta_{-1} \theta \|_{L^1_t L^p} \]

\[ \leq \sum_{q \geq 0} 2^q (1 - \alpha) \| \Delta_q \theta^0 \|_{L^p} + \sum_{q \geq 0} 2^q (1 - \alpha) \| \theta^0 \|_{L^\infty} \| \omega \|_{L^1_t L^p} \]

\[ + t (\| \Delta_{-1} \theta^0 \|_{L^p} + \| \theta^0 \|_{L^\infty} \| \omega \|_{L^1_t L^p}) \]

\[ \lesssim \| \theta^0 \|_{B^{1-\alpha}_{p,1}} + \| \theta^0 \|_{L^\infty} \| \omega \|_{L^1_t L^p} \]

\[ \lesssim \| \theta^0 \|_{B^{1-\alpha} \cap L^\infty} (1 + \| \omega \|_{L^1_t L^p}). \]

Therefore

\[ \| \omega(t) \|_{L^p} \lesssim \| \omega^0 \|_{L^p} + \| \theta^0 \|_{B^{1-\alpha} \cap L^\infty} (1 + \int_0^t \| \omega(\tau) \|_{L^p} d\tau). \]

Using Gronwall’s inequality, we obtain then

\[ (4.1) \quad \| \omega(t) \|_{L^p} \lesssim C_0 e^{C_0 t}. \]

Where \( C_0 \) is a constant depending on the initial data. We can now estimate \( \| \omega(t) \|_{L^\infty} \). For this we take the \( L^\infty \) norm of the velocity equation, we get

\[ \| \omega(t) \|_{L^\infty} \leq \| \omega^0 \|_{L^\infty} + \int_0^t \left( \sum_{i=1}^2 \| F'_{i}(\theta) \|_{L^\infty} \right) \| \nabla \theta(\tau) \|_{L^\infty} d\tau. \]

Then we have

\[ \| \omega(t) \|_{L^\infty} \lesssim \| \omega^0 \|_{L^\infty} + \| \nabla \theta \|_{L^1_t L^\infty}. \]

Using the classical embedding \( B^{1}_{\infty,1} \hookrightarrow \text{Lip} (\mathbb{R}^2) \), we obtain

\[ \| \omega(t) \|_{L^\infty} \lesssim \| \omega^0 \|_{L^\infty} + \| \theta \|_{L^1_t B^{1}_{\infty,1}}. \]

We have from the definition of Besov space and Hölder inequality,

\[ \| \theta \|_{L^1_t B^{1}_{\infty,1}} = \sum_{q \geq 1} 2^q \| \Delta_q \theta \|_{L^1_t L^\infty} \]

\[ = C \| \Delta_{-1} \theta \|_{L^1_t L^\infty} + \sum_{q \in \mathbb{N}} 2^q \| \Delta_q \theta \|_{L^1_t L^\infty} \]

Then we have

\[ \| \theta \|_{L^1_t B^{1}_{\infty,1}} \leq C \| \theta^0 \|_{L^\infty} + \sum_{q \in \mathbb{N}} 2^q \| \Delta_q \theta \|_{L^1_t L^\infty}. \]

We use now Proposition 3.3 we obtain then

\[ \sum_{q \in \mathbb{N}} 2^q \| \Delta_q \theta \|_{L^1_t L^\infty} \leq \sum_{q \in \mathbb{N}} 2^q (1 - \alpha) \| \theta^0 \|_{L^\infty} (1 + t + (q + 2) \| \omega \|_{L^1_t L^\infty} + \| \nabla \Delta_{-1} v \|_{L^1_t L^\infty}). \]
Since $\alpha > 1$, then the series $\sum_{q \in \mathbb{N}} 2^q(1-\alpha)$ and $\sum_{q \in \mathbb{N}} 2^q(1-\alpha)(q+2)$ are convergent. It follows that
\[
\sum_{q \in \mathbb{N}} 2^q ||\Delta_q \theta||_{L_1^1 L_\infty} \lesssim ||\theta^0||_{L_\infty} \left(1 + t + ||\omega||_{L_1^1 L_\infty} + ||\nabla \Delta v||_{L_1^1 L_\infty}\right).
\]
This implies that
\[
||\theta||_{L_1^1 B_{\infty,1}^1} \lesssim ||\theta^0||_{L_\infty} \left(1 + t + ||\omega||_{L_1^1 L_\infty} + ||\nabla \Delta v||_{L_1^1 L_\infty}\right).
\]
Therefore
\[(4.2) \quad ||\theta||_{L_1^1 B_{\infty,1}^1} \lesssim ||\theta^0||_{L_\infty} \left(1 + t + ||\omega||_{L_1^1 (L_\infty \cap L_\infty)}\right).
\]
We have used the classical result $||\nabla v||_{L_p} \approx ||\omega||_{L_p}, \forall p \in ]1, \infty[. \text{ Hence from } (4.1) \text{ and } (4.2), \text{ we get}
\[
||\omega(t)||_{L_\infty} \lesssim ||\omega^0||_{L_\infty} + ||\theta^0||_{L_\infty} \left(1 + t + \int_0^t C_0 e^{C_0 \tau} d\tau + \int_0^t ||\omega(\tau)||_{L_\infty} d\tau\right)
\]
\[
\lesssim ||\omega^0||_{L_\infty} + ||\theta^0||_{L_\infty} \left(1 + t + C_0 e^{C_0 t}\right) + ||\theta^0||_{L_\infty} \int_0^t ||\omega(\tau)||_{L_\infty} d\tau.
\]
According to Gronwall’s inequality, one has
\[(4.3) \quad ||\omega(t)||_{L_\infty} \lesssim C_0 e^{C_0 t}.
\]
Consequently
\[(4.4) \quad ||\omega(t)||_{L_\infty \cap L_p} \lesssim C_0 e^{C_0 t}.
\]
Plugging this estimate into (4.2) gives
\[(4.5) \quad ||\theta||_{L_1^1 B_{\infty,1}^1} \leq C_0 e^{C_0 t}.
\]
This gives in view of Besov embedding
\[
||\nabla \theta||_{L_1^1 L_\infty} \leq C_0 e^{C_0 t}.
\]
This conclude the proof of the proposition.

\[\square\]

We shall now give a bound for the norm Lipschitz of the velocity.

**Proposition 4.2.** Under the same assumptions of Proposition 4.1 and if in addition $\omega^0 \in B_{\infty,1}^0$ and $F \in C^3(\mathbb{R}, \mathbb{R})$, then we have for every $t \in \mathbb{R}_+$
\[
||\omega(t)||_{L_1^\infty B_{\infty,1}^0} + ||\nabla v(t)||_{L_\infty} \leq C_0 e^{\exp C_0 t}.
\]
Proof. We decompose $v$ in frequencies as

$$v(t) = \Delta_{-1}v(t) + \sum_{q \geq 0} \Delta_qv(t)$$

then we have

$$\|\nabla v(t)\|_{L^\infty} \leq \|\nabla \Delta_{-1}v(t)\|_{L^\infty} + \sum_{q \geq 0} \|\nabla \Delta_qv(t)\|_{L^\infty}$$

$$\lesssim \|\nabla \Delta_{-1}v(t)\|_{L^p} + \sum_{q \geq 0} 2^q \|\Delta_qv(t)\|_{L^\infty}$$

$$\lesssim \|\nabla v(t)\|_{L^p} + \sum_{q \geq 0} \|\Delta_q\omega(t)\|_{L^\infty}.$$  

Hence

(4.6) $$\|\nabla v(t)\|_{L^\infty} \lesssim \|\omega(t)\|_{L^p} + \|\omega(t)\|_{\tilde{L}_t^\infty B^0_{\infty,1}}.$$  

Let us now turn to the estimate of $\|\omega\|_{\tilde{L}_t^\infty B^0_{\infty,1}}$. We apply Proposition 2.5 to the vorticity equation,

$$\|\omega\|_{\tilde{L}_t^\infty B^0_{\infty,1}} \lesssim (\|\omega_0\|_{B^0_{\infty,1}} + \int_0^t (\|\partial_1(F_2(\theta(t)))\|_{B^0_{\infty,1}} + \|\partial_2(F_1(\theta(t)))\|_{B^0_{\infty,1}})dt) \left(1 + \|\nabla v\|_{L^1_t L^\infty}\right).$$

By the definition of Besov spaces and Lemma 2.4 we find:

$$\|\partial_1(F_2(\theta))\|_{B^0_{\infty,1}} \leq C \|F_2(\theta)\|_{B^1_{\infty,1}}$$

To estimate $\|F_2(\theta)\|_{B^1_{\infty,1}}$ we use Theorem 2.8

$$\|F_2(\theta)\|_{B^1_{\infty,1}} \leq C \sup_{|x| \leq C\|\theta\|_{L^\infty}} \|F_2^{[1]}(x)\|_{L^\infty} \|\theta\|_{B^1_{\infty,1}}$$

$$\leq C \sup_{|x| \leq C\|\theta_0\|_{L^\infty}} \|F_2^{[3]}(x)\|_{L^\infty} \|\theta\|_{B^1_{\infty,1}}$$

$$\leq C \|\theta\|_{B^1_{\infty,1}}.$$  

Therefore

$$\|\partial_1(F_2(\theta))\|_{B^0_{\infty,1}} \lesssim \|\theta\|_{B^1_{\infty,1}}.$$  

Similarly, we obtain $\|\partial_2(F_1(\theta))\|_{B^0_{\infty,1}} \lesssim \|\theta\|_{B^1_{\infty,1}}$. Finally we get

(4.7) $$\|\omega\|_{\tilde{L}_t^\infty B^0_{\infty,1}} \lesssim (\|\omega_0\|_{B^0_{\infty,1}} + \|\theta\|_{L^1_t B^1_{\infty,1}}) \left(1 + \|\nabla v\|_{L^1_t L^\infty}\right)$$

Putting together (4.1), (4.5), (4.6) and (4.7) and using Gronwall’s inequality, we deduce

(4.8) $$\|\omega\|_{\tilde{L}_t^\infty B^0_{\infty,1}} + \|\nabla v(t)\|_{L^\infty} \leq C_0e^{\exp Ct}.$$  

Now we will describe the last part of the a priori estimates.
Proposition 4.3. Let \((\alpha, p) \in [1, 2] \times [1, \infty], \ v^0 \in B^{-\frac{\alpha+1}{p} + \frac{2}{p}}_{p, 1} \) be a divergence free vector-field of \(\mathbb{R}^2\), \(\theta^0 \in B_{p, 1}^{-\frac{\alpha+1}{p} + \frac{2}{p}} \cap L^\infty\) and \(F \in C^5(\mathbb{R}, \mathbb{R})\). Then for every \(\rho \geq 1\) and for every \(t \in \mathbb{R}_+\),

\[
\begin{align*}
\|\theta\|_{L^1_t B_{p, \infty}^{\frac{\alpha}{p} + \frac{2}{p}}} + \|\theta\|_{L^1_t B_{p, 1}^{\frac{\alpha+1}{p} + \frac{2}{p}}} + \|v\|_{L^\infty_t B_{p, 1}^{\frac{\alpha+1}{p} + \frac{2}{p}}} & \leq C_0 e^{\exp C_0 t}.
\end{align*}
\]

Proof. For the first estimate \(\|\theta\|_{L^1_t B_{p, \infty}^{\frac{\alpha}{p} + \frac{2}{p}}}\), it suffices to combine Remark 3.7 with Lipschitz estimate of the velocity (4.8) as follows

\[
2^{\frac{2}{p}} \|\Delta \theta\|_{L^p_t L^\infty} \lesssim \|\theta^0\|_{L^\infty} \left(1 + t + \|\nabla v\|_{L^1_t L^\infty}\right) \lesssim C_0 e^{\exp C_0 t}.
\]

Hence, it follows that

\[
\|\theta\|_{L^1_t B_{p, \infty}^{\frac{\alpha}{p} + \frac{2}{p}}} \leq C_0 e^{\exp C_0 t}.
\]

In order to prove the second estimate of the Proposition, we need to split the proof in two cases: \(s < 1\) and \(s \geq 1\) with \(s = -\alpha + 1 + \frac{2}{p}\).

- **First case** \(s = -\alpha + 1 + \frac{2}{p} < 1\). We apply Proposition 3.2 to the temperature equation, we get

\[
\|\theta\|_{L^1_t B_{p, 1}^{\frac{\alpha+1}{p} + \frac{2}{p}}} \lesssim \|\theta^0\|_{B_{p, 1}^{-\alpha+1 + \frac{2}{p}}} (1 + t)e^{CV(t)}.
\]

Since

\[
V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \leq \int_0^t C_0 e^{\exp C_0 \tau} d\tau \leq C_0 e^{\exp C_0 t}.
\]

Finally, we obtain for the first case

\[
\|\theta\|_{L^1_t B_{p, 1}^{\frac{\alpha+1}{p} + \frac{2}{p}}} \lesssim C_0 e^{\exp C_0 t}.
\]

- **Second case** \(s = -\alpha + 1 + \frac{2}{p} \geq 1\). Applying once again Proposition 3.2 we get by Hölder’s inequality

\[
\|\theta\|_{L^1_t B_{p, 1}^{\frac{\alpha+1}{p} + \frac{2}{p}}} \lesssim \|\theta^0\|_{B_{p, 1}^{-\alpha+1 + \frac{2}{p}}} (1 + t + \|\nabla \theta\|_{L^1_t L^\infty} + \|v\|_{L^\infty_t B_{p, 1}^{-\alpha+1 + \frac{2}{p}}}) e^{CV(t)}.
\]

Using Proposition 4.1 and (4.8), we obtain easily

\[
\|\theta\|_{L^1_t B_{p, 1}^{\frac{\alpha+1}{p} + \frac{2}{p}}} \leq C_0 e^{\exp C_0 t} \left(1 + \|v\|_{L^\infty_t B_{p, 1}^{-\alpha+1 + \frac{2}{p}}}.\right).
\]

Now, applying Proposition 2.7 to the velocity equation we have

\[
\|v\|_{L^\infty_t B_{p, 1}^{-\alpha+1 + \frac{2}{p}}} \lesssim e^{CV(t)} (\|v^0\|_{B_{p, 1}^{-\alpha+1 + \frac{2}{p}}} + \|F(\theta)\|_{L^1_t B_{p, 1}^{-\alpha+1 + \frac{2}{p}}}).
\]

For the \(\|F(\theta)\|_{B_{p, 1}^{-\alpha+1 + \frac{2}{p}}}\), we use Theorem 2.8 then we have

\[
\|F(\theta)\|_{B_{p, 1}^{-\alpha+1 + \frac{2}{p}}} \leq C \sup_{|x| \leq C \|\theta^0\|_{L^\infty}} \|F[\theta + \frac{2}{p}] + x(x)\|_{L^\infty} \|\theta\|_{B_{p, 1}^{-\alpha+1 + \frac{2}{p}}}.
\]
Since $-\alpha + 1 + \frac{2}{p} \geq 1$, $3 \leq [-\alpha + 1 + \frac{2}{p}]+2 < 4$ and $F \in C^5(\mathbb{R}, \mathbb{R})$, we deduce then
\[ \|F(\theta)\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} \lesssim \|\theta\|_{B_{p,1}^{-\alpha+1+\frac{2}{p}}} . \]
Hence we get
\[ \|v\|_{L_t^\infty B_{p,1}^{-\alpha+1+\frac{2}{p}}} \lesssim C_0 e^{e^{\exp C_0 t}} \left( 1 + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}} \right). \]
Consequently
\[ \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim C_0 e^{e^{\exp C_0 t}} \left( 1 + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}} \right). \]
Iterating this procedure we get for $n \in \mathbb{N}$
\[ \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \leq C_0 e^{e^{\exp C_0 t}} \left( 1 + \|\theta\|_{L_t^1 B_{p,1}^{-\alpha+1+\frac{2}{p}}} \right). \]
To conclude it is enough to choose $n$ such that $-(n+1)\alpha + 1 + \frac{2}{p} < 1$ and then we can apply the first case. Finally we get
\[ (4.9) \quad \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim C_0 e^{e^{\exp C_0 t}}. \]
Applying again Proposition 2.7 to the velocity equation, we obtain
\[ \|v(t)\|_{B_{p,1}^{1+\frac{2}{p}}} \lesssim e^{CV(t)} \left( \|v_0\|_{B_{p,1}^{1+\frac{2}{p}}} + \|F(\theta)\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \right). \]
As above, we use Theorem 2.8 for $\|F(\theta)\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}}$ we obtain
\[ \|F(\theta)\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim C \sup_{|x| \leq C \|\theta\|_{L_t^\infty}} \|F[1+\frac{2}{p}]^2 (x)\|_{L_t^\infty} \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} . \]
Here $[1+\frac{2}{p}]^2 + 2 < 5$, $F \in C^5(\mathbb{R}, \mathbb{R})$ and $\theta^0 \in L^\infty$, then we get
\[ \|F(\theta)\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \lesssim \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} . \]
We obtain from (4.9)
\[ \|v(t)\|_{B_{p,1}^{1+\frac{2}{p}}} \lesssim C_0 e^{e^{\exp C_0 t}} \left( 1 + \|\theta\|_{L_t^1 B_{p,1}^{1+\frac{2}{p}}} \right) \lesssim C_0 e^{e^{\exp C_0 t}} . \]
Therefore
\[ \|v(t)\|_{L_t^\infty B_{p,1}^{1+\frac{2}{p}}} \lesssim C_0 e^{e^{\exp C_0 t}} . \]
4.2. A uniqueness result. In this paragraph, we will establish a uniqueness result for the system \((1.1)\) in the following space.

\[ \mathcal{A}_T = (L^\infty_T L^p \cap L^1_T W^{1,p} \cap L^1_T \text{Lip}) \times (L^1_T L^p \cap L^\infty_T (B_{p,1}^{-\alpha+\frac{2}{p}} \cap L^\infty) \cap L^1_T \text{Lip}), \quad 1 < p < \infty. \]

We take two solutions \(\{(v_j, \theta_j)\}_{j=1}^2\) for \((1.1)\) belonging to the space \(\mathcal{A}_T\), for a fixed time \(T > 0\), with initial data \((v^0_j, \theta^0_j)\), \(j = 1, 2\). We set

\[ v = v_2 - v_1, \quad \theta = \theta_2 - \theta_1 \quad \text{and} \quad p = p_2 - p_1. \]

Then we find the equations

\[
\begin{cases}
\partial_t v + v \cdot \nabla v = -\nabla p - v \cdot \nabla v_1 + F(\theta_1) - F(\theta_2) \\
\partial_t \theta + v \cdot \nabla \theta + |D^{\alpha} \theta| = -v \cdot \nabla \theta_1 \\
v|_{t=0} = v^0, \quad \theta|_{t=0} = \theta^0.
\end{cases}
\]

Taking the \(L^p\) norm of the velocity, we get

\[ \|v(t)\|_{L^p} \leq \|v^0\|_{L^p} + \int_0^t (\|v(\tau)\|_{L^p} \|\nabla v_1(\tau)\|_{L^\infty} + \|\nabla p(\tau)\|_{L^p}) \, d\tau + \|F(\theta_1) - F(\theta_2)\|_{L^1_T L^p}. \]

To estimate the pressure, we write the following identity with the incompressibility condition

\[ \nabla p = \nabla \Delta^{-1} \text{div}(\nabla v_1 + F(\theta_1) - F(\theta_2)) - \nabla \Delta^{-1} \text{div}(v \cdot \nabla v). \]

Since \(\text{div}(v \cdot \nabla v) = \text{div}(v \cdot \nabla v_2)\), then

\[ \nabla p = \nabla \Delta^{-1} \text{div}(-v \cdot \nabla (v_1 + v_2) + F(\theta_1) - F(\theta_2)). \]

Using the continuity of Riesz transform on \(L^p\) with \(1 < p < \infty\), we get

\[ \|\nabla p\|_{L^p} \lesssim \|v\|_{L^p} (\|\nabla v_1\|_{L^\infty} + \|\nabla v_2\|_{L^\infty}) + \|F(\theta_1) - F(\theta_2)\|_{L^p}. \]

Combining this estimate with the \(L^p\) estimate of the velocity we get

\[ \|v(t)\|_{L^p} \lesssim \|v^0\|_{L^p} + \int_0^t (\|v(\tau)\|_{L^p} \|\nabla v_1(\tau)\|_{L^\infty} + \|\nabla v_2(\tau)\|_{L^\infty}) \, d\tau + \|F(\theta_1) - F(\theta_2)\|_{L^1_T L^p}. \]

Let us now estimate \(\|F(\theta_1) - F(\theta_2)\|_{L^1_T L^p}.\) Applying Taylor formula at order 1,

\[ F(\theta_1) - F(\theta_2) = (\theta_1 - \theta_2) \int_0^1 F'(\theta_2 + s (\theta_1 - \theta_2)) \, ds. \]

Taking the \(L^p\) norm yields

\[ \|F(\theta_1) - F(\theta_2)\|_{L^p} \lesssim \|\theta\|_{L^p} \int_0^1 \|F'(\theta_2 + s (\theta_1 - \theta_2))\|_{L^\infty} \, ds. \]

Now we write

\[ \|F'(\theta_2 + s (\theta_1 - \theta_2))\|_{L^\infty} \leq \sup_{|x| \leq C} \sup_{|\theta| \leq L^\infty} |F'(x)| \leq C. \]

Therefore

\[ \|F(\theta_1) - F(\theta_2)\|_{L^1_T L^p} \lesssim \|\theta\|_{L^1_T L^p}. \]
Thus we obtain
\[ \|v(t)\|_{L^p} \lesssim \|v^0\|_{L^p} + \int_0^t \|v(\tau)\|_{L^p} (\|\nabla v_1(\tau)\|_{L^\infty} + \|\nabla v_2(\tau)\|_{L^\infty}) d\tau + \|\theta\|_{L^1_t L^p}. \]

At this stage, we need to split \( \theta \) into two parts \( \theta = \tilde{\theta}_1 + \tilde{\theta}_2 \) where \( \tilde{\theta}_1 \) and \( \tilde{\theta}_2 \) solve respectively the following equations
\[
\begin{align*}
\frac{\partial_t \tilde{\theta}_1 + v_2 \cdot \nabla \tilde{\theta}_1 + |D|^\alpha \tilde{\theta}_1}{\tilde{\theta}_1|_{t=0}} &= -v \cdot \nabla \theta \\
\frac{\partial_t \tilde{\theta}_2 + v_2 \cdot \nabla \tilde{\theta}_2 + |D|^\alpha \tilde{\theta}_2}{\tilde{\theta}_2|_{t=0}} &= 0.
\end{align*}
\]

Taking the \( L^p \) norm of (4.10) we obtain from Lemma 3.1,
\[
\|\tilde{\theta}_1(t)\|_{L^p} \leq \int_0^t \|v \cdot \nabla \theta_1(\tau)\|_{L^p} d\tau \leq \int_0^t \|v(\tau)\|_{L^p} \|\nabla \theta_1(\tau)\|_{L^\infty} d\tau.
\]

Integrating in time, we get
\[
\|\tilde{\theta}_1(t)\|_{L^1_t L^p} \leq \int_0^t \|v(\tau)\|_{L^p} \|\nabla \theta_1(\tau)\|_{L^\infty} d\tau.
\]

Now, we apply the operator \( \Delta_q \) to the equation (4.11) we have
\[
\partial_t \Delta_q \tilde{\theta}_2 + v_2 \cdot \nabla \Delta_q \tilde{\theta}_2 + \Delta_q |D|^\alpha \tilde{\theta}_2 = -(\Delta_q, v_2 \cdot \nabla) \tilde{\theta}_2.
\]

Taking the \( L^p \) norm of the above equation and using Proposition 3.3, Lemma 3.1 and Lemma 3.2, we obtain for \( q \in \mathbb{N} \),
\[
\|\Delta_q \tilde{\theta}_2\|_{L^1_t L^p} \lesssim 2^{-qa} \|\Delta_q \theta^0\|_{L^p} + 2^{-qa} \int_0^t \|\nabla v_2(\tau)\|_{L^p} \|\tilde{\theta}_2(\tau)\|_{L^\infty} d\tau
\]
\[
\lesssim 2^{-qa} \|\Delta_q \theta^0\|_{L^p} + 2^{-qa} \|\theta^0\|_{L^\infty} \int_0^t \|\nabla v_2(\tau)\|_{L^p} d\tau.
\]

Summing these estimates on \( q \geq -1 \) and using Lemma 3.1 and Proposition 3.3, we find
\[
\sum_{q \geq -1} \|\Delta_q \tilde{\theta}_2\|_{L^1_t L^p} \lesssim \sum_{q \geq 0} 2^{-qa} \|\Delta_q \theta^0\|_{L^p} + \sum_{q \geq 0} 2^{-qa} \|\theta^0\|_{L^\infty} \int_0^t \|\nabla v_2(\tau)\|_{L^p} d\tau + \|\Delta_{-1} \tilde{\theta}_2\|_{L^1_t L^p}
\]
\[
\lesssim \sum_{q \geq 0} 2^{-qa} \|\Delta_q \theta^0\|_{L^p} + \|\theta^0\|_{L^\infty} \int_0^t \|\nabla v_2(\tau)\|_{L^p} d\tau
\]
\[
+ t(\|\Delta_{-1} \theta^0\|_{L^p} + \|\theta^0\|_{L^\infty} \int_0^t \|\nabla v_2(\tau)\|_{L^p} d\tau)
\]
\[
\lesssim \|\theta^0\|_{B^{-\alpha}_{p,1}} + \|\theta^0\|_{L^\infty} \|\nabla v_2\|_{L^1_t L^p}.
\]
Therefore
\[ \|\bar{\theta}_2\|_{L^1_t L^p} \leq \sum_{q \geq -1} \|\Delta_q \bar{\theta}_2\|_{L^1_t L^p} \lesssim \|\theta^0\|_{B^{-\alpha+\frac{2}{p}}_{p,1} \cap L^\infty} (1 + \|\nabla v_2\|_{L^1_t L^p}). \]

We have used the Besov embedding \( B^{-\alpha+\frac{2}{p}}_{p,1} \hookrightarrow B^{-\alpha}_{p,1}. \) Now since \( \theta = \bar{\theta}_1 + \bar{\theta}_2, \) then we have
\[ \|\theta\|_{L^1_t L^p} \leq t \int_0^t ||v(\tau)||_{L^p} \|\nabla \theta_1(\tau)\|_{L^\infty} d\tau + \|\theta^0\|_{B^{-\alpha+\frac{2}{p}}_{p,1} \cap L^\infty} (1 + \int_0^t \|\nabla v_2(\tau)\|_{L^p} d\tau). \]

Combining this estimate with the \( L^p \) norm of the velocity, we find
\[ \|v(t)\|_{L^p} \lesssim \|v^0\|_{L^p} + t \int_0^t \|v(\tau)\|_{L^p} \left( \|\nabla v_1(\tau)\|_{L^\infty} + \|\nabla v_2(\tau)\|_{L^\infty} \right) d\tau + t \int_0^t \|v(\tau)\|_{L^p} \|\nabla \theta_1(\tau)\|_{L^\infty} d\tau + \|\theta^0\|_{B^{-\alpha+\frac{2}{p}}_{p,1} \cap L^\infty} (1 + \int_0^t \|\nabla v_2\|_{L^p} d\tau). \]

Finally we get by Gronwall’s inequality,
\[ \|v(t)\|_{L^p} \lesssim e^{C(\|\nabla v_1\|_{L^1_t L^\infty} + \|\nabla v_2\|_{L^1_t L^\infty})} \|v^0\|_{L^p} + \|\theta^0\|_{B^{-\alpha+\frac{2}{p}}_{p,1} \cap L^\infty} \|\nabla v_2\|_{L^1_t L^p}. \]

This gives in turn
\[ \|\theta(t)\|_{L^1_t L^p} \lesssim e^{C(\|\nabla v_1\|_{L^1_t L^\infty} + \|\nabla v_2\|_{L^1_t L^\infty})} \|v^0\|_{L^p} + \|\theta^0\|_{B^{-\alpha+\frac{2}{p}}_{p,1} \cap L^\infty} \|\nabla v_2\|_{L^1_t L^p} (t + 1). \]

The proof of the uniqueness part is now complete.

4.3. **Existence.** Let us now outline briefly the proof of the existence of global solution to (1.1). First we need to the following lemma (see [1] for the proof).

**Lemma 4.4.** Let \( s \in \mathbb{R}, \; (p, r) \in [1, \infty]^2 \) and \( G \in B^s_{p,r}(\mathbb{R}^d). \) Then there exists \( G^n \in \mathcal{S}(\mathbb{R}^d) \) such that for all \( \varepsilon > 0 \) there exist \( n_0 \) such that
\[ \|G^n - G\|_{B^s_{p,r}} \leq \varepsilon, \; \forall \; n \geq n_0. \]

If in addition \( G \in L^\infty(\mathbb{R}^d), \) then
\[ \|G^n\|_{L^\infty(\mathbb{R}^d)} \lesssim \|G\|_{L^\infty(\mathbb{R}^d)}. \]

And if \( \text{div}\; G = 0 \) then \( \text{div}\; G^n = 0. \)

We consider the following system
\[ \begin{align*}
\partial_t v_n + v_n \cdot \nabla v_n + \nabla p_n &= F(\theta_n) \\
\partial_t \theta_n + v_n \cdot \nabla \theta_n + |D|^\alpha \theta_n &= 0 \\
\text{div}\; v_n &= 0 \\
v_n|_{t=0} &= v_{n,0}, \quad \theta_n|_{t=0} = \theta_{n,0}.
\end{align*} \]
Let us now sketch the proof of the continuity in time of the velocity. From the definition \( N(4.14) \) and then we get that \((v_n, \theta_n)\) converges strongly to \((v, \theta)\). Hence it follows that for \( t, T > 0 \) and for \( t, t' \in \mathbb{R}_+ \),

\[
\sum_{q\leq N} 2^{q(1+\frac{2}{p})} \|\Delta_q(v(t) - v(t'))\|_{L^p} + 2 \sum_{q > N} 2^{q(1+\frac{2}{p})} \|\Delta_q v\|_{L^p} \leq C_0 e^{\exp C_0 T} \left( \|v_{n,0} - v_{n,1,0}\|_{L^p} + \|\theta_{n,0} - \theta_{n,1,0}\|_{B_{p,1}^{-\alpha+\frac{2}{p}} \cap L^\infty} \right).
\]

This show that the sequence \((v_n, \theta_n)\) is of a Cauchy in the Banach space \( L^\infty_t L^p \times L^1_t L^p \). Hence it converges strongly to \((v, \theta)\). This allows us to pass to the limit in the system \( (1.1) \) and then we get that \((v, \theta)\) is a solution of the system \( (1.1) \).

Let us now sketch the proof of the continuity in time of the velocity. From the definition of Besov space we have for \( N \in \mathbb{N} , \) \( T > 0 \) and for \( t, t' \in \mathbb{R}_+ \),

\[
\|v(t) - v(t')\|_{B_{p,1}^{1+\frac{2}{p}}} \leq C_2 2^{N(1+\frac{2}{p})} \|v(t) - v(t')\|_{L^p} + 2 \sum_{q > N} 2^{q(1+\frac{2}{p})} \|\Delta_q v\|_{L^p}.
\]

It remains then to estimate \( \|v(t) - v(t')\|_{L^p} \). For this purpose we use the velocity equation

\[
\frac{\partial_t v}{\partial_t} = -\mathcal{P}(v \cdot \nabla v) + \mathcal{P} F(\theta).
\]

Where \( \mathcal{P} \) denotes Leray projector. The solution of this equation is given by Duhamel formula

\[
v(t, x) = v^0(x) - \int_0^t \mathcal{P}(v \cdot \nabla v)(\tau) d\tau + \int_0^t \mathcal{P}(F(\theta(\tau))) d\tau.
\]

Hence it follows that for \( t, t' \in \mathbb{R}_+ \),

\[
v(t, x) - v(t', x) = -\int_{t'}^t \mathcal{P}(v \cdot \nabla v)(\tau) d\tau + \int_{t'}^t \mathcal{P}(F(\theta(\tau))) d\tau.
\]
Taking the $L^p$ norm of the above equation and using the fact that the Leray projector $\mathcal{P}$ is continuously into $L^p$, with $1 < p < \infty$ we get then

$$\|v(t) - v(t')\|_{L^p} \lesssim \int_{t'}^{t} \|(v \cdot \nabla v)(\tau)\|_{L^p} d\tau + \int_{t'}^{t} \|F(\theta(\tau))\|_{L^p} d\tau$$

$$\lesssim \int_{t'}^{t} \|v(\tau)\|_{L^p} \|
abla v(\tau)\|_{L^\infty} d\tau + \int_{t'}^{t} \|F(\theta(\tau))\|_{L^p} d\tau$$

$$\lesssim |t - t'| \|v\|_{L^\infty} \|
abla v\|_{L^\infty} + \int_{t'}^{t} \|F(\theta(\tau))\|_{L^p} d\tau.$$  

(4.16)

We have used Hölder’s inequality and integration by parts for the first term of the above inequality. For the last term we use Taylor formula with $F$ vanishing at 0, 

$$F(\theta) = \theta \int_0^1 F'(s\theta) ds.$$ 

Thus

$$\|F(\theta)\|_{L^p} \leq \|\theta\|_{L^p} \int_0^1 \|F'(s\theta)\|_{L^\infty} ds$$

Now,

$$\|F'(s\theta)\|_{L^\infty} \lesssim \sup_{|y| \leq \|\theta\|_{L^\infty}} |F'(y)| \leq C.$$

Hence it follows that

$$\|F(\theta)\|_{L^p} \lesssim \|\theta\|_{L^p},$$

which yields

$$\int_{t'}^{t} \|F(\theta(\tau))\|_{L^p} d\tau \lesssim \int_{t'}^{t} \|\theta(\tau)\|_{L^p} d\tau \lesssim |t - t'|^{1/2} \|\theta\|_{L_x^2 L^p}.$$

We use now Proposition 3.3 with $\rho = 2$,

$$\|\theta\|_{L_x^2 L^p} \leq \sum_{q \geq -1} \|\Delta_q \theta\|_{L_x^2 L^p}$$

$$\lesssim \sum_{q \geq 0} 2^{-q/2} (\|\Delta_q \theta^0\|_{L^p} + \|\theta^0\|_{L^\infty} \|
abla v\|_{L_x^1 L^p})$$

$$+ t^{1/2} (\|\Delta_{-1} \theta^0\|_{L^p} + \|\theta^0\|_{L^\infty} \|
abla v\|_{L_x^1 L^p})$$

$$\lesssim \|\theta^0\|_{B_{p,1}^{-\alpha + 1 + \frac{2}{p} \cap L^\infty}} + \|\theta^0\|_{L^\infty} \|\omega\|_{L_x^1 L^p}$$

$$\lesssim \|\theta^0\|_{B_{p,1}^{-\alpha + 1 + \frac{2}{p} \cap L^\infty}} (1 + \|\omega\|_{L_x^1 L^p}),$$

we have used the embedding $B_{p,1}^{-\alpha + 1 + \frac{2}{p} \cap L^\infty} \hookrightarrow B_{p,1}^{-\alpha} \cap L^\infty$ (recall that $\alpha \leq 2$). Therefore

$$\int_{t'}^{t} \|F(\theta(\tau))\|_{L^p} d\tau \lesssim |t - t'|^{1/2} \|\theta^0\|_{B_{p,1}^{-\alpha + 1 + \frac{2}{p} \cap L^\infty}} (1 + \|\omega\|_{L_x^1 L^p}).$$
Finally we obtain in (4.16)  
\[ \|v(t) - v(t')\|_{B_{p,1}^{1+\frac{2}{p}}} \lesssim |t - t'| \|v\|_{L^\infty_t L^p} \|\nabla v\|_{L^\infty_t L^\infty} + |t - t'|^{\frac{1}{2}} \|\theta^0\|_{B_{p,1}^{-\alpha + \frac{2}{p}} \cap L^\infty} (1 + \|\omega\|_{L^1_t L^p}). \]

Combining Proposition 4.1 and Proposition 4.2 with the inequalities (4.15) and the previous, we obtain  
\[
\begin{align*}
\|v(t) - v(t')\|_{B_{p,1}^{1+\frac{2}{p}}} &\lesssim 2^{N(1+\frac{2}{p})} \left( |t - t'| \|v\|_{L^\infty_t L^p} C_0 e^{C_0 t} + |t - t'|^{\frac{1}{2}} \|\theta^0\|_{B_{p,1}^{-\alpha + \frac{2}{p}} \cap L^\infty} C_0 e^{C_0 t} \right) \\
& \quad + 2 \sum_{q>N} 2^q (1+\frac{2}{p}) \|\Delta_q v\|_{L^\infty_t L^p}.
\end{align*}
\]

(4.17)

We have by Proposition 4.3 that \( v \in \tilde{L}^\infty_t B_{p,1}^{1+\frac{2}{p}} \), then for \( \varepsilon > 0 \), there exists an integer \( N \) such that  
\[ \sum_{q>N} 2^q (1+\frac{2}{p}) \|\Delta_q v\|_{L^\infty_t L^p} \leq \frac{\varepsilon}{4}. \]

It is enough to choose \( |t - t'| < \eta \) such that  
\[ 2^{N(1+\frac{2}{p})} \left( |t - t'| \|v\|_{L^\infty_t L^p} C_0 e^{C_0 t} + |t - t'|^{\frac{1}{2}} \|\theta^0\|_{B_{p,1}^{-\alpha + \frac{2}{p}} \cap L^\infty} C_0 e^{C_0 t} \right) < \frac{\varepsilon}{2} \]

Finally we find in (4.17) that  
\[ \|v(t) - v(t')\|_{B_{p,1}^{1+\frac{2}{p}}} \leq \varepsilon. \]

This proves the continuity in time of the velocity.

4.4. Appendix: Generalized Bernstein inequality. The generalized Bernstein inequality is proved in [9, 10] for \( 0 < \alpha \leq 2 \) and \( p \geq 2 \). Here we extend this inequality for the remaining case \( p \in ]1, 2] \). More precisely, we have the following proposition.

**Proposition 4.5.** We assume that \( \alpha \in [0, 1] \) and \( p > 1 \). Then we have for every \( G \in \mathcal{S}(\mathbb{R}^2) \) and \( j \in \mathbb{N} \),  
\[ c2^{j\alpha} \|\Delta_j G\|_{L^p} \leq \int_{\mathbb{R}^2} (|D|\alpha |\Delta_j G|) |\Delta_j G|^{p-1} \text{sign} \Delta_j G \, dx, \]

where \( c \) depend on \( p \).

**Proof.** We use the following Corollary (see [19] for the proof).

**Corollary 4.6.** Let \( \alpha \in [0, 1] \) and \( p > 1 \). Then we have,  
\[ \frac{1}{p-1} \frac{1}{p^2} \|D |\frac{\partial}{2} G| |\frac{\partial}{2} G\|^2_{L^2} \leq \int_{\mathbb{R}^2} (|D|\alpha G) |G|^{p-1} \text{sign} G \, dx. \]

It suffices thus to prove,  
\[ c2^{j\alpha} \|\Delta_j G\|_{L^p} \leq \|D |\frac{\partial}{2} (|\Delta_j G| |\frac{\partial}{2} G)\|_{L^2}. \]

Let \( N \in \mathbb{N} \), we define \( \Delta_j G := G_j \), then  
\[ (4.18) \quad \|D |(\Delta_j G)| |\frac{\partial}{2} \|_{L^2} \leq \|S_N |D |(\Delta_j G)| |\frac{\partial}{2} \|_{L^2} + \|(Id - S_N) |D |(\Delta_j G)| |\frac{\partial}{2} \|_{L^2}. \]
Let \( s' > 0 \), then Bernstein inequality gives,
\[
\left\| (Id - S_N)D\left| (\gamma J\frac{\gamma}{2}) \right. \right\|_{L^2} \leq \sum_{k \geq N} \left\| \Delta_k\left| (D\left| (\gamma J\frac{\gamma}{2}) \right. \right) \right\|_{L^2} \\
\lesssim \sum_{k \geq N} 2^{-k s'} 2^{k(1+s')} \left\| \Delta_k\left| (\gamma J\frac{\gamma}{2}) \right. \right\|_{L^2} \\
\lesssim 2^{-N s'} \left\| \gamma J\frac{\gamma}{2} \right\|_{B^{1+s'}_{2,\infty}} \\
\lesssim 2^{-N s'} \left\| \gamma J\frac{\gamma}{2} \right\|_{H^{1+s'}}. 
\]
(4.19)
we have used in the last line the Besov embedding \( H^{1+s'} \hookrightarrow B^{1+s'}_{2,\infty} \).
To estimate \( \left\| \gamma J\frac{\gamma}{2} \right\|_{H^{1+s'}} \), we will use the following Lemma.

**Lemma 4.7.** 1) Let \( \gamma \geq 1 \) and \( s' \in [0, \gamma \cap 0, 2] \). Then
\[
\left\| \left| \gamma J\frac{\gamma}{2} \right| \right\|_{H^{s'}} \lesssim \left\| \gamma J\frac{\gamma}{2} \right\|_{B^{s'}_{2,\infty}}, 
\]
2) For \( 0 < \gamma \leq 1 \), \((p, r) \in [1, \infty]^2\) and \( 0 < s' < 1 + \frac{1}{p} \). Then
\[
\left\| \left| \gamma J\frac{\gamma}{2} \right| \right\|_{B^{s'}_{p,r}} \lesssim \left\| \gamma J\frac{\gamma}{2} \right\|_{B^{s'}_{p,r}}. 
\]
The first estimate is a particular case of a general result due to [9]. The second is established by Sickel [20] (see also Theorem 1.4 of [18]).
We use Lemma 4.7.1 and Bernstein inequality for \( p > 2 \), with \( 0 < s' < \min(\frac{2}{p} - 1, 2) \),
\[
\left\| \left| \gamma J\frac{\gamma}{2} \right| \right\|_{H^{1+s'}} \lesssim \left\| \gamma J\frac{\gamma}{2} \right\|_{B^{1+s'}_{p,2}} \left\| \gamma J\frac{\gamma}{2} \right\|_{B^{1+s'}_{p,2}} \lesssim 2^{j(1+s')} \left\| \gamma J\frac{\gamma}{2} \right\|_{L^p}.
\]
For \( 1 < p \leq 2 \) : we use Lemma 4.7.2) and Bernstein inequality with \( 0 < s' < 1 + \frac{1}{p} \),
\[
\left\| \left| \gamma J\frac{\gamma}{2} \right| \right\|_{H^{1+s'}} \lesssim \left\| \gamma J\frac{\gamma}{2} \right\|_{B^{1+s'}_{p,2}} \lesssim 2^{j(1+s')} \left\| \gamma J\frac{\gamma}{2} \right\|_{L^p}.
\]
We deduce thus from (4.19) and Lemma 4.7.

(4.20)
\[
\left\| (Id - S_N)D\left| (\gamma J\frac{\gamma}{2}) \right. \right\|_{L^2} \leq 2^{-N s' 2^{j(1+s')} \left\| \gamma J\frac{\gamma}{2} \right\|_{L^p}. 
\]
To estimate the first norm of (4.18), we use Bernstein inequality,
\[
\left\| S_N\left| D\left| (\gamma J\frac{\gamma}{2}) \right. \right. \right\|_{L^2} \lesssim \left\| S_N\left| D\left| (\gamma J\frac{\gamma}{2}) \right. \right. \right\|_{L^2} \lesssim 2^{N(1-\frac{\gamma}{2})} \left\| S_N\left| D\left| (\gamma J\frac{\gamma}{2}) \right. \right. \right\|_{L^2} \lesssim 2^{N(1-\frac{\gamma}{2})} \left\| S_N\left| D\left| (\gamma J\frac{\gamma}{2}) \right. \right. \right\|_{L^2} \lesssim 2^{N(1-\frac{\gamma}{2})} \left\| D\left| (\gamma J\frac{\gamma}{2}) \right. \right\|_{L^2}. 
\]
(4.21)
Putting (4.20) and (4.21) into (4.18), we get
\[
\left\| D\left| (\gamma J\frac{\gamma}{2}) \right. \right\|_{L^2} \lesssim 2^{N(1-\frac{\gamma}{2})} \left\| D\left| (\gamma J\frac{\gamma}{2}) \right. \right\|_{L^2} + 2^{-N s' 2^{j(1+s')} \left\| G_J\frac{\gamma}{2} \right\|_{L^p}. 
\]
According to Lemma A.5 of [10], we have for $1 < p < \infty$,
\[ c_p 2^j \| G_j \|_{L_p}^\frac{p}{2} \leq \| D\left(\| G_j \|^{\frac{p}{2}}\right) \|_{L^2}. \]
Combining both last estimates we get,
\[ c_p 2^j \| G_j \|_{L_p}^\frac{p}{2} \leq 2^N (1-\frac{\alpha}{2}) \| D\left(\| G_j \|^{\frac{p}{2}}\right) \|_{L^2} + 2^\alpha (j-N) 2^j \| G_j \|_{L_p}^\frac{p}{2}. \]
Taking $N - j = N_1$ such that $2^{-N_1} \leq \frac{1}{2}$. Therefore
\[ c 2^j \| G_j \|_{L_p}^p \leq \| D\left(\| G_j \|^{\frac{p}{2}}\right) \|_{L^2}^2. \]
with $c$ depend on $p$. This proves the Proposition.

\[ \square \]

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