GROUP SCHEMES AND LOCAL DENSITIES OF RAMIFIED HERMITIAN LATTICES IN RESIDUE CHARACTERISTIC 2

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Abstract. The obstruction to the local-global principle for a hermitian lattice \((L, H)\) can be quantified by computing the mass of \((L, H)\). The mass formula expresses the mass of \((L, H)\) as a product of local factors, called the local densities of \((L, H)\). The local density formula has known except for a ramified hermitian lattice of residue characteristic 2. In this paper, we finally obtain the local density formula for a ramified hermitian lattice of residue characteristic 2, by constructing a smooth integral group scheme model for an appropriate unitary group. Consequently, this paper, combined with the paper [6] of W. T. Gan and J.-K. Yu, allows the computation of the mass formula for a hermitian lattice \((L, H)\).

1. Introduction

The subject of this paper is old and has intrigued many mathematicians. If \((V, H)\) and \((V', H')\) are two hermitian \(k'\)-spaces (or quadratic \(k\)-spaces), where \(k\) is a number field and \(k'\) is a quadratic field extension of \(k\), then it is well known that they are isometric if and only if for all places \(v\), the localizations \((V_v, H_v)\) and \((V'_v, H'_v)\) are isometric. That is, the local-global principle holds for hermitian spaces and quadratic spaces. It is natural to ask whether or not the local-global principle holds for a hermitian \(R'\)-lattice (or a quadratic \(R\)-lattice) \((L, H)\), where \(R'\) and \(R\) are the rings of integers of \(k'\) and \(k\) respectively. In general, the answer to this question is no. However, there is a way (the mass of \((L, H)\)) to quantify the obstruction to the local-global principle, and an essential tool for this is the mass formula. The mass formula expresses the mass of \((L, H)\) as a product of local factors, called the local densities of \((L, H)\).

Therefore, it suffices to find the explicit local density formula in order to obtain the mass formula and accordingly to quantify the obstruction to the local-global principle.

For a quadratic lattice, the local density formula was first computed by G. Pall [18] (for \(p \neq 2\)) and G. L. Watson [20] (for \(p = 2\)). For an expository sketch of their approach, see [13]. There is another proof of Y. Hironaka and F. Sato [11] computing the local density when \(p \neq 2\). They treat an arbitrary pair of lattices, not just a single lattice, over \(\mathbb{Z}_p\) (for \(p \neq 2\)). J. H. Conway
and J. A. Sloane [4] further developed the formula for any $p$ and gave a heuristic explanation for it. Later, W. T. Gan and J.-K. Yu [6] (for $p \neq 2$) and S. Cho [2] (for $p = 2$) provided a simple and conceptual proof of Conway-Sloane’s formula by observing the existence of a smooth affine group scheme $G$ over $\mathbb{Z}_2$ with generic fiber $\text{Aut}_{\mathbb{Q}_2}(L, H)$, which satisfies $G(\mathbb{Z}_2) = \text{Aut}_{\mathbb{Z}_2}(L, H)$.

There has not been as much work done in computing local density formulas for hermitian lattices as in the case of quadratic lattices. Hironaka obtained the local density formula for an unramified hermitian lattice in the papers [9] and [10]. In addition, M. Mischler computed the formula for a ramified hermitian lattice ($p \neq 2$) under restricted conditions in [15]. Later, in the paper [6] mentioned above, Gan and Yu found a conceptual and elegant proof of the local density formula for an unramified hermitian lattice without any restriction on $p$, and for a ramified hermitian lattice with the restriction $p \neq 2$, by observing the existence of certain smooth affine group scheme models of a unitary group. The local density formula for a ramified hermitian lattice with $p = 2$ has not been computed yet and therefore the mass formula, when the ideal $(2)$ is ramified in $k'/k$, is not known. The difficulties involved in this case (ramified with $p = 2$) are explained in [3].

The main contribution of this paper is to get an explicit formula for the local density, by constructing smooth integral models of a unitary group and investigating their special fibers, when $L$ is a hermitian $B$-lattice, where $B$ is a quadratic ramified extension of $A$ and $A$ is an unramified finite extension of $\mathbb{Z}_2$.

In conclusion, this paper, combined with [6], allows the computation of the mass formula for a hermitian $R'$-lattice $(L, H)$ when the ideal $(2)$ is unramified over $R$. We emphasize that $R'$ is the ring of integers of $k'$ and $k'$ is any quadratic field extension of $k$. That is, the ideal $(2)$ can be ramified over $R'$. As the simplest case, we can compute the mass formula for an arbitrary hermitian lattice explicitly when $k$ is $\mathbb{Q}$ and $k'$ is any quadratic field extension of $\mathbb{Q}$.

This paper is organized as follows. We first state the structural theorem for integral hermitian forms in Section 2. Note that there are two different cases (denoted by $\text{Case 1}$ and $\text{Case 2}$) for a quadratic ramified extension $E/F$, as explained in the first paragraph of Section 2. These two cases should be handled independently. We then give an explicit construction of $G$ (in Section 3) and its special fiber (in Section 4) in $\text{Case 1}$. Finally, we obtain an explicit formula for the local density in Section 5 in $\text{Case 1}$. We repeat the process used in Sections 3-5 to treat $\text{Case 2}$ in Sections 6-8.

As in [6], the smooth group schemes constructed in this paper should be of independent interest.
2. Structural theorem for hermitian lattices and notations

2.1. Notations. Notations and definitions in this section are taken from [2], [6] and [12].

- Let $F$ be a non-Archimedean local field with $A$ its ring of integers and $\kappa$ its residue field.
- Let $E$ be a ramified quadratic field extension of $F$ with $B$ its ring of integers and $\kappa$ its residue field.
- We fix 2 and $\pi$ as a uniformizing element of $A$ and $B$, respectively.
- Let $\sigma$ be the non-trivial element of the Galois group $\text{Gal}(E/F)$.
- The lower ramification groups $G_i$'s of the Galois group $\text{Gal}(E/F)$ satisfy one of the following:
  \[
  \begin{aligned}
  &\text{Case 1:}\ G_{-1} = G_0 = G_1, G_2 = 0; \\
  &\text{Case 2:}\ G_{-1} = G_0 = G_1 = G_2, G_3 = 0.
  \end{aligned}
  \]

We justify the above briefly. Based on Section 6 and Section 9 of [12], there is a suitable choice of a uniformizer $\pi$ as follows. We describe this below without proof. In Case 1, $E = F(\sqrt{1 + 2u})$ for some unit $u$ of $A$ and $\pi = 1 + \sqrt{1 + 2u}$. Then $\sigma(\pi) = \epsilon \pi$, where $\epsilon \equiv 1 \mod \pi$ and $\frac{1}{\pi}$ is a unit in $A$. This induces that $\sigma(\pi) + \pi, \sigma(\pi) \cdot \pi \in (2) \setminus (4)$.

In Case 2, $E = F(\pi)$. Here, $\pi = \sqrt{2\delta}$, where $\delta \in A$ and $\delta \equiv 1 \mod 2$. Then $\sigma(\pi) = -\pi$.

From now on, $\pi$ is as explained above throughout this paper.

The construction of smooth models associated to these two cases are different and we will treat them independently.

- We consider a $B$-lattice $L$ with a hermitian form
  \[ h : L \times L \to B, \]
  where $h(a \cdot v, b \cdot w) = \sigma(a)b \cdot h(v, w)$ and $(w, v) = \sigma(h(v, w))$. We denote by a pair $(L, h)$ a hermitian lattice. We assume that $V = L \otimes_A F$ is nondegenerate with respect to $h$.

- We denote by $(\epsilon)$ the $B$-lattice of rank 1 equipped with the hermitian form having Gram matrix $(\epsilon)$. We use the symbol $A(a, b, c)$ to denote the $B$-lattice $B \cdot e_1 + B \cdot e_2$ with the hermitian form having Gram matrix $\begin{pmatrix} a & c \\ \sigma(c) & b \end{pmatrix}$. Especially, for each integer $i$ the lattice of rank 2 having Gram matrix $\begin{pmatrix} 0 & \pi^i \\ \sigma(\pi^i) & 0 \end{pmatrix}$ is called the hyperbolic plane and denoted by $H(i)$. 

A hermitian lattice \( L \) is the orthogonal sum of sublattices \( L_1 \) and \( L_2 \), written \( L = L_1 \oplus L_2 \), if \( L_1 \cap L_2 = 0 \), \( L_1 \) is orthogonal to \( L_2 \) with respect to the hermitian form \( h \), and \( L_1 \) and \( L_2 \) together span \( L \).

The ideal generated by \( h(x,x) \) as \( x \) runs through \( L \) will be called the norm of \( L \) and written \( n(L) \).

By the scale \( s(L) \) of \( L \), we mean the ideal generated by the subset \( h(L,L) \) of \( E \).

**Definition 2.1.** For a given hermitian lattice \( L \),

a) For any non-zero scalar \( a \), define \( aL = \{ ax | x \in L \} \) which is also a lattice in the space \( L \otimes_B E \). Call a vector \( x \) of \( L \) maximal in \( L \) if \( x \) does not lie in \( \pi L \).

b) \( L \) will be called \( \pi^i \)-modular if the ideal generated by the subset \( h(x,L) \) of \( E \) is \( \pi^i B \) for every maximal vector \( x \) in \( L \).

c) Assume that \( i \) is even. A \( \pi^i \)-modular lattice \( L \) is of parity type I if \( n(L) = s(L) \); otherwise of parity type II. The zero lattice is considered to be of parity type II. Caution that we do not assign parity type to \( L_i \) with \( i \) odd.

**2.2. The Structural Theorem for Integral hermitian Forms.** We state the structural theorem for \( \pi^i \)-modular lattices as follows:

**Theorem 2.2.** Let \( i = 0 \) or 1.

a) Let \( L \) be a \( \pi^i \)-modular lattice of rank at least 3. Then \( L = \bigoplus_{\lambda} H_\lambda \oplus K \), where \( K \) is \( \pi^i \)-modular of rank 1 or 2, and each \( H_\lambda = H(i) \).

b) Assume that \( K \) is \( \pi^i \)-modular of rank 2.

If \( i = 0 \), \( K \cong A(1,2b,1) \) or \( K \cong A(2a,2b,1) \). In particular, \( A(2a,2b,1) = A(2\delta,2b,1) \) in Case 2.

If \( i = 1 \), \( K \cong A(2,2a,\pi) \) (resp. \( K \cong A(2\delta,4a,\pi) \) or \( K \cong A(4\delta,4a,\pi) \) ) in Case 1 (resp. Case 2), where \( a \in A \).

c) If \( K \) is \( \pi^1 \)-modular of rank 1 and \( i = 0 \), then \( K \cong (\epsilon) \) where \( a \equiv 1 \mod 2 \). Notice that \( i \) cannot be an odd integer if the rank of \( L \) is odd.

**Proof.** The theorem is proved in Proposition 10.2 and Proposition 10.3 of [12] except that \( K \cong A(1,2b,1) \) if \( K \) is a \( \pi^0 \)-modular lattice of rank 2 satisfying \( n(L) = s(L) \). Thus we only provide the proof of this below. Based on Proposition 10.2 of [12], for such a lattice \( K \), \( K \cong A(1,a,1) \) with respect to a basis \( (e_1,e_2) \) where the determinant \( a - 1 \) is a unit in \( A \). Since the residue
field \( \kappa \) is perfect, there is a unit element \( \beta \) in \( A \) such that \( a - 1 \equiv \frac{1}{\beta} \mod 2 \). We now choose another basis \((e_1, (1-\beta)e_1 + \beta e_2)\). This completes the proof. \( \square \)

**Remark 2.3.**

a) ((4.5) in [12]) If \( L \) is \( \pi^i \)-modular, then \( \pi^j L \) is \( \pi^{i+j} \)-modular for any integer \( j \).

b) (Section 4 in [12]) For a general lattice \( L \), we have a Jordan splitting, namely \( L = \bigoplus_i L_i \) such that \( L_i \) is \( \pi^i \)-modular. Jordan splittings \( L = \bigoplus_{1 \leq i \leq t} L_i \) and \( K = \bigoplus_{1 \leq i \leq T} K_i \) will be said to be of the same type if \( t = T \), and \( s(L_i) = s(K_i) \), rank \( L_i = \text{rank } K_i \), and \( n(L_i) = n(K_i) \) if and only if \( n(K_i) = s(K_i) \) for each \( i \). Jordan splitting is not unique but canonical in the sense that two Jordan splittings of isometric lattices are always of the same type.

c) If we allow \( L_i \) to be the zero lattice, then we may assume that \( s(L_i) = (\pi^i) \) without loss of generality. We can rephrase the above remark b) as follows: Let \( L = \bigoplus_i L_i \) be a Jordan splitting with \( s(L_i) = (\pi^i) \) for all \( i \geq 0 \). Then the scale, rank and parity type of \( L_i \) depend only on \( L \). We will deal exclusively with a Jordan splitting satisfying \( s(L_i) = (\pi^i) \) from now on.

### 2.3. Lattices

In this subsection, we will define several lattices and corresponding notations. A hermitian lattice \((L, h)\) is given. We denote by \( (\pi^i) \) the scale \( s(L) \) of \( L \).

1. \( A_i, \{x \in L \mid h(x, L) \in \pi^i B\} \).
2. \( X(L) \), the sublattice of \( L \) such that \( X(L)/\pi L \) is the kernel of the symmetric bilinear form \( \frac{1}{\pi^i} h \mod \pi \) on \( L/\pi L \).

Let \( l = 2m \) or \( l = 2m - 1 \). We consider the function defined over \( L \)

\[
\frac{1}{2^m q} : L \rightarrow A, x \mapsto \frac{1}{2^m} h(x, x).
\]

Then \( \frac{1}{2^m q} \mod 2 \) defines a quadratic form \( L/\pi L \rightarrow \kappa \). Assume that \( \frac{1}{2^m q} \mod 2 \) on \( L/\pi L \) is an additive polynomial. Then we define the following lattice:

3. \( B(L) \), the sublattice of \( L \) such that \( B(L)/\pi L \) is the kernel of the additive polynomial \( \frac{1}{2^m q} \mod 2 \) on \( L/\pi L \) and that \( B(L) = L \) if this polynomial is not additive.

We notice that the above function is an additive polynomial if \( l = 2m \), or \( l = 2m - 1 \) and \( K/F \) satisfies Case 2. Otherwise, that is, if \( l = 2m - 1 \) and \( K/F \) satisfies Case 1, this is not additive.

To define a few more lattices, we need some preparation as follows.
Assume $B(L) \subsetneq L$ and $l$ is even. Then the bilinear form $\frac{1}{(\pi \cdot \sigma(\pi))^\frac{l}{2}} h \mod \pi$ on $\kappa$-vector space $L/X(L)$ is nonsingular symmetric and non-alternating. It is well known that there is a unique vector $e \in L/X(L)$ such that $\frac{1}{(\pi \cdot \sigma(\pi))^\frac{l}{2}} h(v,e)^2 = \frac{1}{(\pi \cdot \sigma(\pi))} h(v,v) \mod \pi$ for every vector $v \in L/X(L)$. Let $\langle e \rangle$ denote the 1-dimensional vector space spanned by the vector $e$ and denote by $e^\perp$ the 1-codimensional subspace of $L/X(L)$ which is orthogonal to the vector $e$ with respect to $\frac{1}{(\pi \cdot \sigma(\pi))^\frac{l}{2}} h \mod \pi$. Then

$$B(L)/X(L) = e^\perp.$$ 

If $B(L) = L$, then the bilinear form $\frac{1}{(\pi \cdot \sigma(\pi))^\frac{l}{2}} h \mod \pi$ on $\kappa$-vector space $L/X(L)$ is nonsingular symmetric and alternating. In this case, we put $e = 0 \in L/X(L)$ and note that it is characterized by the same identity.

The remaining lattices we need for our definition are:

4. $W(L)$, the sublattice of $L$ such that

$$\begin{cases} W(L)/X(L) = \langle e \rangle & \text{if } l \text{ is even}; \\ W(L) = X(L) & \text{if } l \text{ is odd}. \end{cases}$$

5. $Y(L)$, the sublattice of $L$ such that $Y(L)/\pi L$ is the kernel of

$$\begin{cases} \text{the alternating bilinear form } \frac{1}{(\pi \cdot \sigma(\pi))^\frac{l}{2}} h \mod \pi \text{ on } B(L)/\pi L & \text{if } l = 2m \text{ in Case 1}; \\ \text{the alternating bilinear form } \frac{1}{(\pi \cdot \sigma(\pi))^{m-\frac{l}{2}}} h \mod \pi \text{ on } B(L)/\pi L & \text{if } l = 2m - 1 \text{ in Case 2}. \end{cases}$$

The last lattice we need to define in this subsection is

6. $Z(L)$, the sublattice of $L$ such that $Z(L)/\pi L$ is the kernel of

$$\begin{cases} \text{the quadratic form } \frac{1}{2^m} q \mod 2 \text{ on } L/\pi L & \text{if } l = 2m - 1 \text{ in Case 1}; \\ \text{the quadratic form } \frac{1}{2^m} q \mod 2 \text{ on } B(L)/\pi L & \text{if } l = 2m \text{ in Case 2}. \end{cases}$$

Remark 2.4. a) We can associate the 6 lattices above to $(A_i, h)$. Denote the resulting lattices by $A_i, B_i, W_i, X_i, Y_i, Z_i$.

b) As $\kappa$-vector spaces, the dimensions of $A_i/B_i, W_i/X_i, Y_i/Z_i$ are at most 1.

Let $L = \bigoplus_i L_i$ be a Jordan splitting. When $i$ is even, we assign a type to each $L_i$ as follows:

$$\begin{cases} I & \text{if } L_i \text{ is of parity type I}; \\ I^o & \text{if } L_i \text{ is of parity type I and the rank of } L_i \text{ is odd}; \\ I^e & \text{if } L_i \text{ is of parity type I and the rank of } L_i \text{ is even}; \\ II & \text{if } L_i \text{ is of parity type II}. \end{cases}$$
When \( i \) is odd, we say that \( L_i \) is
\[
\begin{cases}
\text{of type II} & \text{if } E/F \text{ satisfies Case 1}; \\
\text{of type I} & \text{if } E/F \text{ satisfies Case 2 and } A_i \supseteq B_i; \\
\text{of type II} & \text{if } E/F \text{ satisfies Case 2 and } A_i = B_i.
\end{cases}
\]

In addition, when \( i \) is even, we say that \( L_i \) is
\[
\begin{cases}
\text{bound of type I} & \text{if } L_i \text{ is of type I and either } L_{i-2} \text{ or } L_{i+2} \text{ is of type I}; \\
\text{bound of type II} & \text{if } L_i \text{ is of type II and either } L_{i-1} \text{ or } L_{i+1} \text{ is of type I}; \\
\text{free} & \text{otherwise}.
\end{cases}
\]

When \( i \) is odd, we say that \( L_i \) is
\[
\begin{cases}
\text{bound} & \text{if either } L_{i-1} \text{ or } L_{i+1} \text{ is of type I}; \\
\text{free} & \text{otherwise}.
\end{cases}
\]

Notice that each type of \( L_i \) is determined canonically regardless of the choice of a Jordan splitting.

2.4. Developed Structural Theorem for Integral hermitian Forms. We now develop Theorem 2.2 according to the type of each \( L_i \). For this, we need a series of lemmas.

**Lemma 2.5.** *(Proposition 9.2 in [12])* Let \( L \) be a \( \pi^i \)-modular lattice of rank 2 with \( n(L) = n(H(i)) \). Then \( L \cong H(i) \) in Case 2 with \( i \) odd and in Case 1 with \( i \) even.

**Lemma 2.6.** *(Proposition 4.4 in [12])* A \( \pi^i \)-modular lattice \( L \) has an orthogonal basis if \( n(L) = s(L) \).

**Lemma 2.7.** Assume that \( E/F \) satisfies Case 2. Let \( L = A(4a, 2\delta, \pi) \oplus (2c) \) with respect to a basis \((e_1, e_2, e_3)\), where \( c \equiv 1 \mod 2 \). Then \( L \cong H(1) \oplus (2c') \) where \( c' \equiv 1 \mod 2 \).

*Proof.* We choose a basis \((e_1 - \frac{2a\pi}{2\delta}, e_2 + e_3, \frac{e_1}{2} + e_3)\) of \( L \). With respect to this basis, \( L \cong A(-4a - 16a^2, 2(\delta + c), \pi(1 + 4a)) \oplus (2c(1 - \frac{4a\pi}{2\delta})) \). Moreover, \( n(A(-4a - 16a^2, 2(\delta + c), \pi(1 + 4a))) = n(H(1)) = (4) \). This, by combining with the above lemma, completes the proof. \( \square \)

Combining the above lemmas with Theorem 2.2, we have the following structural theorem:

**Theorem 2.8.** Let \( L = \bigoplus I_i \) be a Jordan splitting of the given lattice \( L \). Let \( i = 0 \) or \( i = 1 \). Then \( L_i = \bigoplus \lambda H_\lambda \oplus K \), where \( K \) is \( \pi^i \)-modular of rank 1 or 2, and each \( H_\lambda = H(i) \).
a) In Case 1,\
\[
K = \begin{cases} 
(a) \text{ where } a \equiv 1 \mod 2 & \text{ if } i = 0 \text{ and } L_0 \text{ is of type } I^c; \\
A(1, 2b, 1) & \text{ if } i = 0 \text{ and } L_0 \text{ is of type } I^o; \\
H(0) & \text{ if } i = 0 \text{ and } L_0 \text{ is of type } II; \\
A(2, 2b, \pi) & \text{ if } i = 1.
\end{cases}
\]

b) In Case 2,\
\[
K = \begin{cases} 
(a) \text{ where } a \equiv 1 \mod 2 & \text{ if } i = 0 \text{ and } L_0 \text{ is of type } I^c; \\
A(1, 2b, 1) & \text{ if } i = 0 \text{ and } L_0 \text{ is of type } I^o; \\
A(2\delta, 2b, 1) & \text{ if } i = 0 \text{ and } L_0 \text{ is of type } II; \\
A(4\alpha, 2\delta, \pi) & \text{ if } i = 1 \text{ and } L_1 \text{ is free of type } I; \\
H(1) & \text{ if } i = 1, \text{ and } L_1 \text{ is bound of type } I \text{ or of type } II.
\end{cases}
\]

From now on, the pair \((L, h)\) is fixed throughout this paper.

3. The construction of the smooth model \(G\) in Case 1

We reproduce the first paragraph of Section 3 in [2] to make contents of this section precise. Let \(G'\) be a naive integral model of the unitary group \(U(V, h)\), where \(V = L \otimes_A F\), such that for any commutative \(A\)-algebra \(R\),
\[
\mathcal{G}'(R) = \text{Aut}_{B \otimes_A R}(L \otimes_A R, h \otimes_A R).
\]

Let \(G\) be the smooth group scheme model of \(U(V, h)\) such that
\[
\mathcal{G}(R) = \mathcal{G}'(R)
\]
for any étale \(A\)-algebra \(R\). Notice that \(\mathcal{G}\) is uniquely determined with these properties by Proposition 3.7 in [6]. For a detailed exposition of the relation between the local density of \((L, h)\) and \(\mathcal{G}\), see Section 3 of [6].

In this section, we give an explicit construction of the smooth integral model \(G\) when \(E/F\) satisfies Case 1. The construction of \(G\) is based on that of Section 5 in [6] and Section 3 in [2]. Let \(K = \text{Aut}_{B \otimes_A A}(L, h) \subset \text{Res}_{K/F}GL_K(V)(F)\), and \(\bar{K} = \text{Aut}_{B \otimes_A A^\text{sh}}(L \otimes_A A^\text{sh}, h)\), where \(n\) is the dimension of \(V\) as a \(B\)-vector space and \(A^\text{sh}\) is the strict henselization of \(A\). To ease the notation, we say \(g \in \bar{K}\) stabilizes a lattice \(M \subseteq V\) if \(g(M \otimes_A A^\text{sh}) = M \otimes_A A^\text{sh}\).
3.1. Main construction. In this subsection, we observe properties of elements of $\bar{K}$ and their matrix interpretation. We choose a Jordan splitting $L = \bigoplus_i L_i$ and a basis of $L$ as explained in Theorem 2.8 and Remark 2.3.a). Let $g$ be an element of $\bar{K}$.

(1) First of all, $g$ stabilizes $A_i$’s for every integer $i$. We interpret this fact in terms of matrices. Let $n_i = \text{rank}_B L_i$, and $n = \text{rank}_B L = \sum n_i$. Assume that $n_i = 0$ unless $0 \leq i < N$. We always divide an $(n \times n)$-matrix $g$ into $(N \times N)$-blocks such that the $(i,j)$-block is of size $n_i \times n_j$. (For simplicity, $(0,0)$-block is $(1,1)$-block in the usual sense.) The fact that $g$ stabilizes $A_i$’s for every integer $i$ means that the $(i,j)$-block has entries in $\pi^{\text{max}\{0,j-i\}} B^h$.

From now on, we write $g = \left( \pi^{\text{max}\{0,j-i\}} g_{i,j} \right)$.

(2) $g$ stabilizes $A_i, B_i, W_i, X_i$ and induces the identity on $A_i/B_i$ and $W_i/X_i$.

We also interpret these facts in terms of matrices as described below:

a) If $i$ is odd, then $A_i = B_i$ and $W_i = X_i$ and so there is no contribution. Hence, we assume that $i$ is even for now.

b) If $L_i$ is of type $P$, the diagonal $(i, i)$-block $g_{i,i}$ is of the form

$$
\begin{pmatrix}
  s_i & \pi y_i \\
  \pi v_i & 1 + \pi z_i
\end{pmatrix} \in \text{GL}_{n_i}(B^h),
$$

where $s_i$ is an $(n - 1) \times (n - 1)$-matrix, etc.

c) If $L_i$ is of type $F$, the diagonal $(i, i)$-block $g_{i,i}$ is of the form

$$
\begin{pmatrix}
  s_i & \pi t_i \\
  \pi y_i & 1 + \pi x_i \\
  v_i & u_i & 1 + \pi w_i
\end{pmatrix} \in \text{GL}_{n_i}(B^h),
$$

where $s_i$ is an $(n - 2) \times (n - 2)$-matrix, etc.

3.2. Construction of $\underline{M^*}$. We define a functor from the category of commutative flat $A$-algebras to the category of monoids as follows. For any commutative flat $A$-algebra $R$, set

$$
\underline{M}(R) \subset \{ m \in \text{End}_{B \otimes A R}(L \otimes_A R) \}
$$

with the following conditions:

(1) $m$ stabilizes $A_i \otimes_A R, B_i \otimes_A R, W_i \otimes_A R, X_i \otimes_A R$ for all $i$.

(2) $m$ induces the identity on $A_i \otimes_A R/B_i \otimes_A R, W_i \otimes_A R/X_i \otimes_A R$ for all $i$. 
Then, by Lemma 3.1 of [2], the functor $M$ is representable by a unique flat $A$-algebra $A(M)$ which is a polynomial ring over $A$ of $2n^2$ variables. Moreover, it is easy to see that $M$ has the structure of a scheme of monoids by showing that $M(R)$ is closed under multiplication.

We stress that the above description of $M(R)$, the set of $R$-points on the scheme $M$, is no longer true when $R$ is a $\kappa$-algebra. Now suppose that $R$ is a $\kappa$-algebra. By choosing a basis for $L$ as in Section 3.1, we describe each element of $M(R)$ formally as a matrix $\begin{pmatrix} 2^{\max\{0,j-i\}}m_{i,j} \end{pmatrix}$, where $m_{i,j}$ is an $(n_i \times n_j)$-matrix with entries in $R$ as described in Section 3.1. To multiply $(m_{i,j})$ and $(m'_{i,j})$, we refer to the description of Section 5.3 in [6].

For any commutative $A$-algebra $R$, set $M^*(R) = \{ m \in M(R) : m^{-1} \in M(R) \}$. Then $M^*$ is represented by a group scheme $M^*$ and $M^*$ is an open subscheme of $M$, with generic fiber $M^* = \text{Res}_{K/F}\text{GL}_K(V)$, and $M^*$ is smooth over $A$. The proof of this is similar to Section 3.2 in [2] and so we will omit it.

**Remark 3.1.** We give another description for the functor $M$. Let us define a functor from the category of commutative flat $A$-algebras to the category of rings as follows:

For any commutative flat $A$-algebra $R$, set $M'(R) = \{ m \in M(R) : s(L,f) \subseteq s(L,h) \}$ and $f$ satisfies the following conditions:

1. $m$ stabilizes $A_i \otimes_A R, B_i \otimes_A R, W_i \otimes_A R, X_i \otimes_A R$ for all $i$.
2. $m$ maps $A_i \otimes_A R, W_i \otimes_A R$ into $B_i \otimes_A R, X_i \otimes_A R$, respectively.

Then the functor $M$ is the same as the functor $1 + M'$, where $(1 + M')(R) = \{ 1 + m : m \in M'(R) \}$.

**3.3. Construction of $H$.** Recall that the pair $(L,h)$ is fixed throughout this paper and the lattices $A_i, B_i, W_i, X_i$ only depend on the hermitian pair $(L,h)$. For any flat $A$-algebra $R$, let $H(R)$ be the set of hermitian forms $f$ on $L \otimes_A R$ (with values in $B \otimes_A R$) such that $s(L, f) \subseteq s(L, h)$ and $f$ satisfies the following conditions:

a) $f(L \otimes_A R, A_i \otimes_A R) \subset \pi_i B \otimes_A R$ for all $i$.
b) Assume that $i = 2m$ is even. Then $B_i \otimes_A R / \pi A_i \otimes_A R$ is contained in the kernel of the additive polynomial $x \mapsto \frac{1}{2m}f(x, x) \mod 2$ on $A_i \otimes_A R / \pi A_i \otimes_A R$. 
c) Assume $B_i \not\subset A_i$ and $i = 2m$ is even. We have seen the existence of the unique vector $e \in A_i/X_i$ such that $(\frac{1}{(\pi \cdot \sigma(\pi))})^m h(v, e))^2 = \frac{1}{(\pi \cdot \sigma(\pi))} h(v, v)$ mod $\pi$ for every vector $v \in A_i/X_i$ in Section 2.3. Then $e \otimes 1 \in A_i \otimes_A R/X_i \otimes_A R$ also satisfies the condition that $(\frac{1}{(\pi \cdot \sigma(\pi))})^m f(v, e \otimes 1))^2 = \frac{1}{(\pi \cdot \sigma(\pi))} f(v, v)$ mod $\pi$ for every vector $v \in A_i \otimes_A R/X_i \otimes_A R$.

d) Let $i = 2m$. $\frac{1}{(\pi \cdot \sigma(\pi))} f(a_i, a_i) \mod 2 = \frac{1}{(\pi \cdot \sigma(\pi))} h(a_i, a_i) \mod 2$, where $a_i \in A_i \otimes_A R$.

e) Let $i = 2m$. $\frac{1}{(\pi \cdot \sigma(\pi))} f(w_i, w_i) - \frac{1}{(\pi \cdot \sigma(\pi))} h(w_i, w_i) \in (2)$, where $w_i \in W_i \otimes_A R$.

f) Let $i = 2m$. $\frac{1}{\pi} f(a_i, w_i) \equiv \frac{1}{\pi} h(a_i, w_i) \mod \pi$, where $a_i \in A_i \otimes_A R$ and $w_i \in W_i \otimes_A R$.

We interpret the above conditions in terms of matrices. For a flat $A$-algebra $R$, $H(R)$ is the set of hermitian matrices

$$\left(\pi^{\max\{i,j\}} f_{i,j}\right)$$

of size $n \times n$ satisfying the following:

1. $f_{i,j}$ is an $(n_i \times n_j)$-matrix with entries in $B \otimes_A R$.
2. If $i$ is even and $L_i$ is of type $I^*$, then $f_{i,i}$ is of the form

$$\begin{pmatrix}
a_i & \pi b_i \\
\sigma(\pi^t \cdot b_i) & 1 + 2c_i
\end{pmatrix}.$$  

Here, the diagonal entries of $a_i$ are $\equiv 0 \mod 2$, where $a_i$ is an $(n_i - 1) \times (n_i - 1)$-matrix, etc.

3. If $i$ is even and $L_i$ is of type $I^*$, then $f_{i,i}$ is of the form

$$\begin{pmatrix}
d_i & \pi e_i \\
\sigma(b_i) & 1 + 2a_i \\
\sigma(\pi \cdot e_i) & \sigma(1 + \pi c_i)
\end{pmatrix}.$$  

Here, the diagonal entries of $d_i$ are $\equiv 0 \mod 2$, where $d_i$ is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

4. Assume that $L_i$ is of type $II$. The diagonal entries of $f_{i,i}$ (resp. $\pi f_{i,i}$) are $0 \mod 2$ if $i$ is even (resp. odd).

It is easy to see that $H$ is represented by a flat $A$-scheme which is isomorphic to an affine space of dimension $2n^2 - \dim U(V, h)$. Note that our fixed hermitian form $h$ is an element of $H(A)$.

3.4. Smooth affine group scheme $G$. 
Theorem 3.2. For any flat $A$-algebra $R$, the group $M^*(R)$ acts on the right of $H(R)$ by $f \circ m = \sigma^t(m) \cdot f \cdot m$. Then this action is represented by an action morphism

$$H \times M^* \to H.$$ 

The proof of this theorem is similar to that of Theorem 3.4 in [2] and so we will omit it.

Theorem 3.3. Let $\rho$ be the morphism $M^* \to H$ defined by $\rho(m) = h \circ m$. Then $\rho$ is smooth of relative dimension $\dim U(V,h)$.

Proof. The theorem follows from Theorem 5.5 in [6] and the following lemma. □

Lemma 3.4. The morphism $\rho \otimes \kappa : M^* \otimes \kappa \to H \otimes \kappa$ is smooth of relative dimension $\dim U(V,h)$.

Proof. Define $T_3(R)$ to be the set of all $(n \times n)$-matrices $y$ over $R \otimes_A B$ with the following conditions:

a) The $(i,j)$-block $y_{i,j}$ of $y$ has entries in $\pi^{\max(i,j)}B \otimes_A R$ so that

$$y = \left( \pi^{\max(i,j)}y_{i,j} \right).$$

Here, the size of $y_{i,j}$ is $n_i \times n_j$.

b) If $L_i$ is of type $P$, $y_{i,i}$ is of the form

$$\begin{pmatrix} s_i & \pi y_i \\ \pi v_i & \pi z_i \end{pmatrix} \in M_{n_i}(R \otimes_A B)$$

where $s_i$ is an $(n_i - 1) \times (n_i - 1)$ matrix, etc.

c) If $L_i$ is of type $P$, $y_{i,i}$ is of the form

$$\begin{pmatrix} s_i & r_i & \pi t_i \\ y_i & x_i & \pi z_i \\ \pi v_i & \pi u_i & \pi w_i \end{pmatrix} \in M_{n_i}(R \otimes_A B)$$

where $s_i$ is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

It is easy to see that the functor $T_3$ is represented by a flat $A$-scheme.

With the functor $T_3$, the rest of the proof is similar to that of Lemma 3.6 in [2] so we will omit it. □

Let $G$ be the stabilizer of $h$ in $M^*$. It is an affine group subscheme of $M^*$, defined over $A$. Thus we have the following theorem.
Theorem 3.5. The group scheme $\tilde{G}$ is smooth, and $\tilde{G}(R) = \text{Aut}_{B \otimes A R}(L \otimes_A R, h)$ for any étale $A$-algebra $R$.

4. The special fiber $\tilde{G}$ in Case 1

In this section, we will determine the structure of the special fiber $\tilde{G}$ of $G$ by observing the maximal reductive quotient and the component group when $E/F$ satisfies Case 1, based on Section 4 of [2]. From this section to the end, the identity matrix is denoted by $id$.

4.1. The reductive quotient of the special fiber. Assume that $i$ is even. Recall that $Y_i$ is the sublattice of $B_i$ such that $Y_i/\pi A_i$ is the kernel of the alternating bilinear form $\frac{1}{(\pi \cdot \sigma(\pi))^2} h \mod \pi$ on $B_i/\pi A_i$. Let $h_i$ denote the nonsingular alternating bilinear form $\frac{1}{(\pi \cdot \sigma(\pi))^2} h \mod \pi$ on $B_i/Y_i$. It is obvious that each element of $G(R)$ fixes $h_i$ for every flat $A$-algebra $R$. Based on this, we have a morphism of algebraic groups

$$\varphi_i : \tilde{G} \longrightarrow \text{Sp}(B_i/Y_i, h_i)$$

defined over $\kappa$. The dimension of $B_i/Y_i$ as a $\kappa$-vector space is as follows:

$$\begin{cases} 
  n_i & \text{if } L_i \text{ is of type II;} \\
  n_i - 1 & \text{if } L_i \text{ is of type I}'; \\
  n_i - 2 & \text{if } L_i \text{ is of type I}''.
\end{cases}$$

We next assume that $i = 2m - 1$ is odd. Recall that $Z_i$ is the sublattice of $A_i$ such that $Z_i/\pi A_i$ is the kernel of the quadratic form $\frac{1}{2m} q \mod 2$ on $A_i/\pi A_i$, where $\frac{1}{2m} q(x) = \frac{1}{2m} h(x, x)$. Let $\tilde{q}_i$ denote the nonsingular quadratic form $\frac{1}{2m} q \mod 2$ on $A_i/Z_i$. It is also obvious that each element of $G(R)$ fixes $\tilde{q}_i$ for every flat $A$-algebra $R$. Based on this, we have a morphism of algebraic groups

$$\varphi_i : \tilde{G} \longrightarrow \text{O}(A_i/Z_i, \tilde{q}_i)^{\text{red}}$$

defined over $\kappa$, where $\text{O}(A_i/Z_i, \tilde{q}_i)^{\text{red}}$ is the reduced subgroup scheme of $\text{O}(A_i/Z_i, \tilde{q}_i)$. Notice that if the dimension of $A_i/Z_i$ is even and positive, then $\text{O}(A_i/Z_i, \tilde{q}_i)^{\text{red}} = \text{O}(A_i/Z_i, \tilde{q}_i)$ is disconnected. In addition, the dimension of $A_i/Z_i$ as a $\kappa$-vector space is $n_i$ (resp. $n_i + 1$) if $L_i$ is free (resp. bound). The integer $n_i$ with $i$ odd is always even.

Theorem 4.1. The morphism $\varphi$ defined by

$$\varphi = \prod_i \varphi_i : \tilde{G} \longrightarrow \prod_{i: \text{even}} \text{Sp}(B_i/Y_i, h_i) \times \prod_{i: \text{odd}} \text{O}(A_i/Z_i, \tilde{q}_i)^{\text{red}}$$
is surjective.

**Proof.** Assume that dimension of \( \tilde{G} \) = dimension of Ker \( \varphi \) + \( \sum_{i: \text{even}} \) (dimension of Sp(\( B_i/Y_i, h_i \))) + \( \sum_{i: \text{odd}} \) (dimension of O(\( A_i/Z_i, \bar{q}_i \))\text{red}). Thus Im \( \varphi \) contains the identity component of \( \prod_{i: \text{even}} \text{Sp}(B_i/Y_i, h_i) \times \prod_{i: \text{odd}} \text{O}(A_i/Z_i, \bar{q}_i)\text{red} \). Here Ker \( \varphi \) denotes the kernel of \( \varphi \) and Im \( \varphi \) denotes the image of \( \varphi \).

Recall that a matrix form of an element of \( \tilde{G}(R) \) for a \( \kappa \)-algebra \( R \) is \( m = (\pi^\max\{0,j-i\} m_{i,j}) \).

Let \( \mathcal{H} \) be the set of odd integer \( i \)'s such that \( O(A_i/Z_i, \bar{q}_i)\text{red} \) is disconnected. Notice that \( O(A_i/Z_i, \bar{q}_i)\text{red} \) is disconnected exactly when \( L_j \) with \( i \) odd is free. For such a lattice \( L_i \), we define the closed subgroup scheme \( H_i \) of \( \tilde{G} \) by the equations \( m_{j,k} = 0 \) if \( j \neq k \), and \( m_{j,j} = \text{id} \) if \( j \neq i \). The group scheme \( H_i \) is isomorphic to the special fiber of the smooth affine group scheme model of the unitary group associated to the hermitian lattice \( L_i \). Moreover, it is easily seen that \( \varphi_i \) is surjective from \( H_i \) to \( O(A_i/Z_i, \bar{q}_i)\text{red} \) and that \( Z_i = X_i \). Consider the morphism

\[
\prod_{i \in \mathcal{H}} H_i \rightarrow \tilde{G},
\]

\((h_i)_{i \in \mathcal{H}} \mapsto \prod_{i \in \mathcal{H}} h_i\). Note that \( H_i \) and \( H_j \) commute with each other in the sense that \( h_i \cdot h_j = h_j \cdot h_i \) for all \( i \neq j \), where \( h_i \in H_i(R) \) and \( h_j \in H_j(R) \) for a \( \kappa \)-algebra \( R \). Thus the product \( \prod_{i \in \mathcal{H}} H_i \) is embedded into \( \tilde{G} \) as a closed subgroup scheme. Since \( \varphi_i |_{H_j} \) is trivial for all \( i \neq j \), the morphism

\[
\prod_{i \in \mathcal{H}} \varphi_i : \prod_{i \in \mathcal{H}} H_i \rightarrow \prod_{i \in \mathcal{H}} O(A_i/Z_i, \bar{q}_i)
\]
is surjective. Therefore, \( \varphi \) is surjective. Now it suffices to establish the assumption made at the beginning of the proof, which is the next lemma. \( \square \)

**Lemma 4.2.** Ker \( \varphi \) is smooth and unipotent of dimension \( l \). In addition, the number of connected components of Ker \( \varphi \) is \( 2\beta \). Here,

- \( l \) is such that \( l + \sum_{i: \text{even}} \) (dimension of Sp(\( B_i/Y_i, h_i \))) + \( \sum_{i: \text{odd}} \) (dimension of O(\( A_i/Z_i, \bar{q}_i \))\text{red}) = dimension of \( \tilde{G} \).
- \( \beta \) is the size of the set of even integer \( j \)'s such that \( L_j \) is of type I and \( L_{j+2} \) is of type II.

The proof is postponed to the Appendix A.
Remark 4.3. We describe $\text{Im } \varphi_i$ as follows.

| Type of lattice $L_i$ and $i$ | $\text{Im } \varphi_i$ |
|-------------------------------|----------------------|
| $II$, $i$ : even              | $\text{Sp}(n_i, h_i)$ |
| $P$, $i$ : even               | $\text{Sp}(n_i - 1, h_i)$ |
| $P^r$, $i$ : even             | $\text{Sp}(n_i - 2, h_i)$ |
| free, $i$ : odd               | $\text{O}(n_i, \bar{q}_i)$ |
| bound, $i$ : odd              | $\text{SO}(n_i + 1, \bar{q}_i)$ |

4.2. The construction of component groups. The purpose of this subsection is to define the surjective morphism from $\tilde{G}$ to $(\mathbb{Z}/2\mathbb{Z})^{\beta}$, where $\beta$ is defined in Lemma 4.2.

Definition 4.4. We define the lattice $L^1$ which is the sublattice of $L$ such that $h(L^1, L) \in (\pi)$. Similarly we define the lattice $L^i$ which is the sublattice of $L^{i-1}$ such that $h(L^i, L^{i-1}) \in (\pi^i)$. For simplicity, put

$$L^0 = L = \bigoplus_{i \geq 0} L_i, 0 \leq i < N.$$ 

The description of $L^i$, when $i = 2m$ is even, is

$$L^{2m} = \pi^m(L_0 \oplus L_1) \oplus \pi^{m-1}(L_2 \oplus L_3) \oplus \cdots \oplus \pi(L_{2m-2} \oplus L_{2m-1}) \oplus \bigoplus_{i \geq 2m} L_i.$$ 

We choose a Jordan splitting for a hermitian lattice $(L^{2m}, \frac{1}{(\pi \sigma(\pi))^m} h)$ as follows:

$$L^{2m} = \bigoplus_{i \geq 0} M_i,$$

where

$$M_0 = \pi^m L_0 \oplus \pi^{m-1} L_2 \oplus \cdots \oplus \pi L_{2m-2} \oplus L_{2m},$$

$$M_1 = \pi^m L_1 \oplus \pi^{m-1} L_3 \oplus \cdots \oplus \pi L_{2m-1} \oplus L_{2m+1}$$

and $M_k = L_{2m+k}$ if $k \geq 2$.

Here, $M_i$ is $\pi^i$-modular.

Definition 4.5. We define $C(L)$ to be the sublattice of $L$ such that

$$C(L) = \{ x \in L \mid h(x, y) \in (\pi) \text{ for all } y \in B(L) \}.$$
We choose any even integer $j$ such that $L_j$ is of type I and $L_{j+2}$ is of type II, and consider a Jordan splitting $\bigoplus_{i \geq 0} M_i$ associated to $L^j$. We stress that $M_0$ is of type I and $M_2 = L_{j+2}$ is of type II. Choose a basis $(\langle e_i, e \rangle)$ (resp. $(\langle e_i, a, e \rangle)$) for $M_0$ based on Theorem 2.8 when the rank of $M_0$ is odd (resp. even). Then $B(L^j)$ is spanned by

$$(\langle e_i, \pi e \rangle) \text{ (resp. } (\langle e_i, \pi a, e \rangle)) \text{ and } M_1 \oplus \bigoplus_{i \geq 2} M_i$$

and $C(L^j)$ is spanned by

$$(\langle \pi e_i, e \rangle) \text{ (resp. } (\langle \pi e_i, \pi a, e \rangle)) \text{ and } M_1 \oplus \bigoplus_{i \geq 2} M_i.$$

We now construct a morphism $\psi_j : \tilde{G} \to \mathbb{Z}/2\mathbb{Z}$ as follows (There are 2 cases depending on whether $M_0$ is of type I or of type II):

1. Firstly, we assume that $M_0$ is of type I. We choose a Jordan splitting for the hermitian lattice $(C(L^j), \frac{1}{(\pi \cdot \sigma(\pi))} h)$ as follows:

$$C(L^j) = \bigoplus_{i \geq 1} M'_i.$$ 

Notice that $M'_2$ is of type II so that $M'_1$ is free. Let $G_j$ denote the special fiber of the smooth affine group scheme associated to the hermitian lattice $(C(L^j), \frac{1}{(\pi \cdot \sigma(\pi))} h)$. We now have a morphism from $\tilde{G}$ to $G_j$. Moreover, since $M'_1$ is free, we have a morphism from $G_j$ to the even orthogonal group associated to $M'_1$ as explained in Section 4.1. Thus, the Dickson invariant of this orthogonal group induces the morphism

$$\psi_j : \tilde{G} \to \mathbb{Z}/2\mathbb{Z}.$$ 

2. We next assume that $M_0$ is of type II. We choose a Jordan splitting for the hermitian lattice $(C(L^j), \frac{1}{(\pi \cdot \sigma(\pi))} h)$ as follows:

$$C(L^j) = \bigoplus_{i \geq 0} M'_i.$$ 

Notice that the rank of the $\pi^0$-modular lattice $M'_0$ is 1 and the lattice $M'_2$ is of type II. If $G_j$ denotes the special fiber of the smooth affine group scheme associated to the hermitian lattice $(C(L^j), \frac{1}{(\pi \cdot \sigma(\pi))} h)$, we have a morphism from $\tilde{G}$ to $G_j$.

We now consider the new hermitian lattice $M'_0 \oplus C(L^j)$. The smooth affine group scheme associated to the hermitian lattice $(C(L^j), \frac{1}{(\pi \cdot \sigma(\pi))} h)$ can be embedded into the smooth affine
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The group scheme associated to the hermitian lattice $M_0' \oplus C(L^j)$ as a closed subgroup scheme. Thus the special fiber $G_j$ of the former group scheme is embedded into the special fiber of the latter group scheme. Since the $\pi^0$-modular lattice $M_0' \oplus M_0'$ is of type I, where $(M_0' \oplus M_0') \oplus \bigoplus_{i \geq 1} M_i'$ is a Jordan splitting of the hermitian lattice $M_0' \oplus C(L^j)$, we have a morphism from the special fiber of the latter group scheme to $\mathbb{Z}/2\mathbb{Z}$ as constructed in the first case. It induces the morphism

$$\psi_j : \tilde{G} \to \mathbb{Z}/2\mathbb{Z}.$$ 

3. Combining all cases, we have the morphism

$$\psi = \prod_j \psi_j : \tilde{G} \to (\mathbb{Z}/2\mathbb{Z})^\beta,$$

where $\beta$ is the size of the set of even integer $j$’s such that $L_j$ is of type I and $L_{j+2}$ is of type II.

We now have the following result.

**Theorem 4.6.** The morphism

$$\psi = \prod_j \psi_j : \tilde{G} \to (\mathbb{Z}/2\mathbb{Z})^\beta$$

is surjective.

Moreover, the morphism

$$\varphi \times \psi : \tilde{G} \to \prod_{i: \text{even}} \text{Sp}(B_i/Y_i, h_i) \times \prod_{i: \text{odd}} \text{O}(A_i/Z_i, \bar{q}_i)^{\text{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta$$

is also surjective.

**Proof.** We first show that $\psi_j$ is surjective. Recall that for such an even integer $j$, $L_j$ is of type I and $L_{j+2}$ is of type II. Notice that we define the closed subgroup scheme $F_j$ of $\tilde{G}$ in the proof of Lemma 9.7 of Appendix A. Based on the description of the Dickson invariant explained in Remark 4.4 of [2], it is easily seen that $\psi_j$ induces a surjective morphism from $F_j$ to $\mathbb{Z}/2\mathbb{Z}$.

The rest of the proof is similar to that of Theorem 4.6 in [2] and so we will omit it.

\[\square\]

4.3. **The maximal reductive quotient of $\tilde{G}$.** We finally have the structural theorem for the algebraic group $\tilde{G}$.

**Theorem 4.7.** The morphism

$$\varphi \times \psi : \tilde{G} \to \prod_{i: \text{even}} \text{Sp}(B_i/Y_i, h_i) \times \prod_{i: \text{odd}} \text{O}(A_i/Z_i, \bar{q}_i)^{\text{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta$$
is surjective and the kernel is unipotent and connected. Consequently,
\[ \prod_{i: \text{even}} \text{Sp}(B_i/Y_i, h_i) \times \prod_{i: \text{odd}} \text{O}(A_i/Z_i, \bar{q}_i)^{\text{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta \]
is the maximal reductive quotient.

**Proof.** We only need to prove that the kernel is unipotent and connected. The kernel of \( \varphi \) is a closed subgroup scheme of the unipotent group \( \tilde{M}^+ \) which is defined in Lemma 9.2 of Appendix A and so it suffices to show that the kernel of \( \varphi \times \psi \) is connected. Equivalently, it suffices to show that the kernel of the restricted morphism \( \psi|_{\text{Ker} \varphi} \) is connected. From Lemma 4.2, the number of connected components of \( \text{Ker} \varphi \) is \( 2^\beta \). Since the restricted morphism \( \psi|_{\text{Ker} \varphi} \) is surjective onto \( (\mathbb{Z}/2\mathbb{Z})^\beta \), we complete the proof by counting the number of connected components. \( \square \)

5. Comparison of volume forms and final formulas in Case 1

This section is based on Section 7 of [6]. In the construction of Section 3.2 of [6], pick \( \omega'_{M} \) and \( \omega'_{H} \) to be such that
\[ \int_{M(A)} |\omega'_{M}| = 1 \text{ and } \int_{H(A)} |\omega'_{H}| = 1. \]
Put \( \omega_{\text{can}} = \omega'_{M}/\rho^* \omega'_{H} \). By Theorem 3.3, we have an exact sequence of locally free sheaves on \( M^* \):
\[ 0 \rightarrow \rho^* \Omega_{H/A} \rightarrow \Omega_{M^*/A} \rightarrow \Omega_{M^*/H} \rightarrow 0. \]
It follows that \( \omega_{\text{can}} \) is of the type discussed in Section 3 of [6].

**Lemma 5.1.** Recall that 2 is a uniformizer of \( A \). Then
\[ \omega_{M} = 2^{N_M} \omega'_{M}, \quad N_M = \sum_{i: \text{even and } L_i: \text{type I}} (2n_i - 1) + \sum_{i<j} (j-i) \cdot n_i \cdot n_j, \]
\[ \omega_{H} = 2^{N_H} \omega'_{H}, \quad N_H = \sum_{i: \text{even and } L_i: \text{type I}} (n_i - 1) + \sum_{i<j} j \cdot n_i \cdot n_j + \sum_{i: \text{even}} i + \frac{2}{2} \cdot n_i + \sum_{i: \text{odd}} \frac{i+1}{2} \cdot n_i + \sum d_i, \]
\[ \omega_{\text{id}} = 2^{N_M - N_H} \omega_{\text{can}}. \]

Here, \( d_i = i \cdot n_i \cdot (n_i - 1)/2. \)

**Theorem 5.2.** Let \( f \) be the cardinality of \( \kappa \). The local density of \( (L, h) \) is
\[ \beta_L = f^N \cdot f^{\dim U(V,h)} \tilde{G}(\kappa), \]
where

$$N = N_H - N_M = \sum_{i<j} i \cdot n_i \cdot n_j + \sum_{i: \text{even}} \frac{i+2}{2} \cdot n_i + \sum_{i: \text{odd}} \frac{i+1}{2} \cdot n_i + \sum_{i: \text{even}} d_i - \sum_{i: \text{odd}} n_i.$$  

**Remark 5.3.**  
(1) In the above local density formula, $\#\tilde{G}(\kappa)$ is computed as follows. We denote by $R_u\tilde{G}$ the unipotent radical of $\tilde{G}$ so that the maximal reductive quotient of $\tilde{G}$ is $\tilde{G}/R_u\tilde{G}$. That is, there is the following exact sequence of group schemes over $\kappa$:

$$1 \to R_u\tilde{G} \to \tilde{G} \to \tilde{G}/R_u\tilde{G} \to 1.$$  

Furthermore, the following sequence of groups

$$1 \to R_u\tilde{G}(\kappa) \to \tilde{G}(\kappa) \to (\tilde{G}/R_u\tilde{G})(\kappa) \to 1.$$  

is also exact by Lemma 9.1. If $m$ is the dimension of $R_u\tilde{G}$, then $\#R_u\tilde{G}(\kappa)$ is $f^m$. Furthermore, since the dimension of $\tilde{G}$ is $n^2$ with $n = \text{rank}_B L$, the dimension of $R_u\tilde{G}$ can be computed explicitly based on Theorem 4.7. In addition, the order of an orthogonal group or a symplectic group defined over a finite field is well known. Thus, one can compute $\#(\tilde{G}/R_u\tilde{G})(\kappa)$ explicitly based on Theorem 4.7. Finally, the order of the group $\tilde{G}(\kappa)$ is identified as follows:

$$\#\tilde{G}(\kappa) = \#R_u\tilde{G}(\kappa) \cdot \#(\tilde{G}/R_u\tilde{G})(\kappa).$$

(2) As in Remark 7.4 of [6], although we have assumed that $n_i = 0$ for $i < 0$, it is easy to check that the formula in the preceding theorem remains true without this assumption.

6. **The construction of the smooth model $G$ in Case 2**

We start with introducing the symbol $\delta_j$ according to the type of $L_i$. We keep this throughout this paper.

$$\delta_j = \begin{cases} 1 & \text{if } L_j \text{ is of type I;} \\ 0 & \text{if } L_j \text{ is of type II.} \end{cases}$$

In this section, we give an explicit construction of the smooth integral model $G$ when $E/F$ satisfies Case 2. The construction of $G$ is based on that of Section 5 in [6] and Section 3 in [2]. Let $K = \text{Aut}_{B \otimes A}(L, h) \subseteq \text{Res}_{K/F}GL_V(F)$, and $\bar{K} = \text{Aut}_{B \otimes A^{sh}}(L \otimes A^{sh}, h)$, where $n$ is the dimension of $V$ as a $B$-vector space and $A^{sh}$ is the strict henselization of $A$. To ease the notation, we say $g \in \bar{K}$ stabilizes a lattice $M \subseteq V$ if $g(M \otimes A^{sh}) = M \otimes A^{sh}$. 


6.1. Main construction. In this subsection, we observe properties of elements of $\tilde{K}$ and their matrix interpretation. We choose a Jordan splitting $L = \bigoplus_i L_i$ and a basis of $L$ as explained in Theorem 2.8 and Remark 2.3.a). Let $g$ be an element of $\tilde{K}$.

(1) First of all, $g$ stabilizes $A_i$’s for every integer $i$. We interpret this fact in terms of matrices.

Let $n_i = \text{rank}_B L_i$, and $n = \text{rank}_B L = \sum n_i$. Assume that $n_i = 0$ unless $0 \leq i < N$. We always divide an $(n \times n)$-matrix $g$ into $(N \times N)$-blocks such that the $(i,j)$-block is of size $n_i \times n_j$. (For simplicity, $(0,0)$-block is $(1,1)$-block in the usual sense.) The fact that $g$ stabilizes $A_i$’s for every integer $i$ means that the $(i,j)$-block has entries in $\pi^{\max\{0,j-i\}} B_{sh}$.

From now on, we write

$$g = \left( \pi^{\max\{0,j-i\}} g_{i,j} \right).$$

(2) Let $i$ be even. $g$ stabilizes $A_i, B_i, W_i, X_i$ and induces the identity on $A_i/B_i$ and $W_i/X_i$.

We also interpret these facts in terms of matrices as described below:

a) If $L_i$ is of type $I$, the diagonal $(i,i)$-block $g_{i,i}$ is of the form

$$\begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} \in \text{GL}_{n_i}(B_{sh}),$$

where $s_i$ is an $(n_i - 1) \times (n_i - 1)$-matrix, etc.

b) If $L_i$ is of type $\bar{I}$, the diagonal $(i,i)$-block $g_{i,i}$ is of the form

$$\begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix} \in \text{GL}_{n_i}(B_{sh}),$$

where $s_i$ is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

(3) Assume that $i = 2m$ is even and that $L_i$ is of type $I$. $g$ stabilizes $Z_i$ and induces the identity on $W_i/(X_i \cap Z_i)$. To prove the latter, we choose an element $w$ in $W_i$. It suffices to show that $gw - w \in X_i \cap Z_i$. By (2), it suffices to show that $gw - w \in Z_i$. This follows from the computation:

$$\frac{1}{2} \cdot \frac{1}{2^{2m}} q(gw - w) = \frac{1}{2}(2 \cdot \frac{1}{2^{2m}} q(w) - \frac{1}{2^{2m}}(h(gw,w) + h(w,gw))) = \frac{1}{2^{2m}}(q(w) - \frac{1}{2}(h(w + x,w) + h(w,w + x))) = \frac{1}{2^{2m}} \cdot \frac{1}{2}(h(x,w) + h(w,x)) = 0 \mod 2,$$

where $gw = w + x$ for some $x \in X_i$. Recall that $\frac{1}{2^{2m}} q(w) = \frac{1}{2^{2m}} h(w,w)$. 

In terms of matrices, we have the following:

\[ z_i + \delta_{i-2}k_{i-2,i} + \delta_{i+2}k_{i+2,i} \in (\pi). \]

Here,

a) \( z_i \) is an entry of \( g_{i,i} \) as described in Step (2).

b) \( k_{i-2,i} \) (resp. \( k_{i+2,i} \)) is the \( (n_i-2, n_i)^{th} \)-entry (resp. \( (n_i+2, n_i)^{th} \)-entry) of the matrix \( g_{i-2,i} \) (resp. \( g_{i+2,i} \)) if \( L_{i-2} \) (resp. \( L_{i+2} \)) is of type \( P \).

c) \( k_{i-2,i} \) (resp. \( k_{i+2,i} \)) is the \( (n_i-2 - 1, n_i) \)-entry (resp. \( (n_i+2 - 1, n_i)^{th} \)-entry) of the matrix \( g_{i-2,i} \) (resp. \( g_{i+2,i} \)) if \( L_{i-2} \) (resp. \( L_{i+2} \)) is of type \( F \).

(4) Assume that \( i \) is odd. \( g \) induces the identity on \( A_i/B_i \).

To interpret this as a matrix, we consider the following \((1 \times n_i)\)-matrix:

\[
\begin{cases}
  v_i \cdot (g_{i,i} - \text{Id}_{n_i}) & \text{if } L_i \text{ is free of type I}; \\
  \delta_{i+1}v_{i+1} \cdot g_{i+1,i} + \delta_{i+1}v_{i+1} \cdot g_{i+1,i} & \text{if } L_i \text{ is bound of type I}.
\end{cases}
\]

Here,

a) \( v_i = (0, \cdots, 0, 1) \) of size \( 1 \times n_i \) and \( \text{Id}_{n_i} \) is the identity matrix of size \( n_i \times n_i \).

b) \( v_{i-1} = (0, \cdots, 0, 1) \) (resp. \( v_{i-1} = (0, \cdots, 0, 1, 0) \)) of size \( 1 \times n_{i-1} \) if \( L_{i-1} \) is of type \( P \) (resp. of type \( F \)).

c) \( v_{i+1} = (0, \cdots, 0, 1) \) (resp. \( v_{i+1} = (0, \cdots, 0, 1, 0) \)) of size \( 1 \times n_{i+1} \) if \( L_{i+1} \) is of type \( P \) (resp. of type \( F \)).

Then each entry of the above matrix lies in the prime ideal \((\pi)\).

(5) Assume that \( i \) is odd. The fact that \( g \) induces the identity on \( A_i/B_i \) is equivalent to the fact that \( g \) induces the identity on \( B_i^+/A_i^+ \).

We give another description of this condition. Since the space \( V \) has a non-degenerate bilinear form \( h \), \( V \) can be identified with its own dual. We define the adjoint \( g^* \) characterized by \( h(gv, w) = h(v, g^*w) \). Then the fact that \( g \) induces the identity on \( B_i^+/A_i^+ \) is the same as the fact that \( g^* \) induces the identity on \( A_i/B_i \).

In terms of matrices, we consider the following \((1 \times n_i)\)-matrix:

\[
\begin{cases}
  v_i \cdot (g_{i,i} - \text{Id}_{n_i}) & \text{if } L_i \text{ is free of type I}; \\
  \delta_{i-1}v_{i-1} \cdot g_{i-1,i} + \delta_{i+1}v_{i+1} \cdot g_{i+1,i} & \text{if } L_i \text{ is bound of type I}.
\end{cases}
\]
Here,  

\( a) \ v_i = (0, \cdots, 0, 1, 0) \) of size \( 1 \times n_i \) and \( \text{Id}_{n_i} \) is the identity matrix of size \( n_i \times n_i \).  

\( b) \ v_i - 1 \) (resp. \( v_i + 1 \)) = \( (0, \cdots, 0, 1) \) of size \( 1 \times n_i - 1 \) (resp. \( 1 \times n_i + 1 \)).

Then each entry of the above matrix lies in the prime ideal \((\pi)\).

6.2. Construction of \( \underline{M}^* \). We define a functor from the category of commutative flat \( A \)-algebras to the category of monoids as follows. For any commutative flat \( A \)-algebra \( R \), set  

\[
\underline{M}(R) \subset \{ m \in \text{End}_{B \otimes_A R}(L \otimes_A R) \}
\]

with the following conditions:

1. \( m \) stabilizes \( A_i \otimes_A R, B_i \otimes_A R, W_i \otimes_A R, X_i \otimes_A R, Z_i \otimes_A R \) for all \( i \).
2. \( m \) induces the identity on \( A_i \otimes_A R/B_i \otimes_A R \) for all \( i \).
3. \( m \) induces the identity on \( W_i \otimes_A R/(X_i \cap Z_i) \otimes_A R \) for all even integer \( i \).
4. \( m \) induces the identity on \( B_i^\perp \otimes_A R/A_i^\perp \otimes_A R \) for all odd integer \( i \).

Then, by Lemma 3.1 of [2], the functor \( \underline{M} \) is representable by a unique flat \( A \)-algebra \( A(\underline{M}) \) which is a polynomial ring over \( A \) of \( 2n^2 \) variables, and \( \underline{M} \) has the structure of a scheme of monoids by showing that \( \underline{M}(R) \) is closed under multiplication.

We stress that the above description of \( \underline{M}(R) \), the set of \( R \)-points on the scheme \( \underline{M} \), is no longer true when \( R \) is a \( \kappa \)-algebra. Now suppose that \( R \) is a \( \kappa \)-algebra. By choosing a basis for \( L \) as in Section 6.1, we describe each element of \( \underline{M}(R) \) formally as a matrix \( \begin{pmatrix} 2^{\max\{0,j-i\}}m_{i,j} \end{pmatrix} \), where \( m_{i,j} \) is an \( (n_i \times n_j) \)-matrix with entries in \( R \) as described in Section 3.1. To multiply \( (m_{i,j}) \) and \( (m'_{i,j}) \), we refer to the description of Section 5.3 in [6].

For any commutative \( A \)-algebra \( R \), set  

\[
\underline{M}^*(R) = \{ m \in \underline{M}(R) : m^{-1} \in \underline{M}(R) \}.
\]

Then \( \underline{M}^* \) is represented by a group scheme \( \underline{M}^* \) and \( \underline{M}^* \) is an open subscheme of \( \underline{M} \), with generic fiber \( M^* = \text{Res}_{K/F}\text{GL}_K(V) \), and \( \underline{M}^* \) is smooth over \( A \). The proof of this is similar to Section 3.2 in [2] so we will omit it.

Remark 6.1. We give another description for the functor \( \underline{M} \). Let us define a functor from the category of commutative flat \( A \)-algebras to the category of rings as follows:
For any commutative flat $A$-algebra $R$, set

$$M'(R) \subset \{ m \in \text{End}_{B \otimes_A R}(L \otimes_A R) \}$$

with the following conditions:

1. $m$ stabilizes $A_i \otimes_A R, B_i \otimes A R, W_i \otimes A R, X_i \otimes A R, Z_i \otimes A R$ for all $i$.
2. $m$ maps $A_i \otimes A R$ into $B_i \otimes A R$ for all $i$.
3. $m$ maps $W_i \otimes A R$ into $(X_i \cap Z_i) \otimes A R$ for all even integer $i$.
4. $m$ maps $B_i^\perp \otimes A R$ into $A_i^\perp \otimes A R$ for all odd integer $i$.

Then the functor $M$ is the same as the functor $1 + M'$, where $(1 + M')(R) = \{ 1 + m : m \in M'(R) \}$.

6.3. **Construction of $H$.** Recall that the pair $(L, h)$ is fixed throughout this paper and the lattices $A_i, B_i, W_i, X_i, Z_i$ only depend on the hermitian pair $(L, h)$. For any flat $A$-algebra $R$, let $H(R)$ be the set of hermitian forms $f$ on $L \otimes A R$ (with values in $B \otimes A R$) such that $s(L, f) \subseteq s(L, h)$ and $f$ satisfies the following conditions:

a) $f(L \otimes A R, A_i \otimes A R) \subseteq \pi^i B \otimes A R$ for all $i$.

b) Let $i = 2m$ or $i = 2m - 1$. Then $B_i \otimes A R/\pi A_i \otimes A R$ is contained in the kernel of the additive polynomial $x \mapsto \frac{1}{2m} f(x, x)$ mod 2 on $A_i \otimes A R/\pi A_i \otimes A R$.

c) Assume $B_i \not\subseteq A_i$ and $i = 2m$ is even. We have seen the existence of the unique vector $e \in A_i/X_i$ such that $(\frac{1}{(\pi \cdot \sigma(\pi))^m}h(v, e))^2 = \frac{1}{(\pi \cdot \sigma(\pi))^m}h(v, v)$ mod $\pi$ for every vector $v \in A_i/X_i$ in Section 2.3. Then $e \otimes 1 \in A_i \otimes A R/X_i \otimes A R$ also satisfies the condition that $(\frac{1}{(\pi \cdot \sigma(\pi))^m}f(v, e \otimes 1))^2 = \frac{1}{(\pi \cdot \sigma(\pi))^m}f(v, v) \mod \pi$ for every vector $v \in A_i \otimes A R/X_i \otimes A R$.

d) Let $i = 2m$ or $i = 2m - 1$. $\frac{1}{(\pi \cdot \sigma(\pi))^m}f(a_i, a_i) \mod 2 = \frac{1}{(\pi \cdot \sigma(\pi))^m}h(a_i, a_i) \mod 2$, where $a_i \in A_i \otimes A R$.

e) Let $i = 2m$. $Z_i \otimes A R/\pi A_i \otimes A R$ is contained in the kernel of the quadratic form $\frac{1}{2} \cdot \frac{1}{\pi^i} f$ mod 2 on $B_i \otimes A R/\pi A_i \otimes A R$. In addition, $\frac{1}{(\pi \cdot \sigma(\pi))^m}f(w_i, w_i) - \frac{1}{(\pi \cdot \sigma(\pi))^m}h(w_i, w_i) \in (4)$, where $w_i \in W_i \otimes A R$.

f) Let $i = 2m$. $\frac{1}{\pi^i} f(a_i, w_i) \equiv \frac{1}{\pi} h(a_i, w_i) \mod \pi$, where $a_i \in A_i \otimes A R$ and $w_i \in W_i \otimes A R$.

g) If $i$ is odd, then $f(B_i, B_i^\perp) \in B$ and $f(a_i, b_i') - h(a_i, b_i') \in B$, where $a_i \in A_i$ and $b_i' \in B_i^\perp$.

We interpret the above conditions in terms of matrices. We represent the given hermitian form $h$ by a hermitian matrix $\left(\pi^i \cdot h_i\right)$ with $\pi^i \cdot h_i$ for the $(i, i)$-block and 0 for remaining blocks.
For a flat $A$-algebra $R$, $H(R)$ is the set of hermitian matrices
\[
\left(\pi^{\max\{i,j\}}f_{i,j}\right)
\]
of size $n \times n$ satisfying the following:

1. $f_{i,j}$ is an $(n_i \times n_j)$-matrix with entries in $B \otimes_A R$.
2. If $i$ is even and $L_i$ is of type $I^*$, then $f_{i,i}$ is of the form
\[
\begin{pmatrix}
a_i & \pi b_i \\
\sigma(\pi \cdot b_i) & \epsilon + 4c_i
\end{pmatrix}.
\]

Here, the diagonal entries of $a_i$ are $\equiv 0 \mod 2$, where $a_i$ is an $(n_i - 1) \times (n_i - 1)$-matrix, etc.

3. If $i$ is even and $L_i$ is of type $I^e$, then $f_{i,i}$ is of the form
\[
\begin{pmatrix}
d_i & b_i & \pi e_i \\
\sigma(b_i) & 1 + 2a_i & 1 + \pi c_i \\
\sigma(\pi \cdot e_i) & \sigma(1 + \pi c_i) & 2b + 4f_i
\end{pmatrix}.
\]

Here, the diagonal entries of $d_i$ are $\equiv 0 \mod 2$, where $d_i$ is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

4. Assume that $i$ is even and $L_i$ is of type $II$. The diagonal entries of $f_{i,i}$ are 0 mod 2.
5. Assume that $i$ is odd. The diagonal entries of $\pi f_{i,i} - \pi h_i$ are 0 mod 4.
6. Assume that $i$ is odd. Consider the following $(1 \times n_i)$-matrix:
\[
\begin{cases}
(0, \cdots, 0, 1, 0) \cdot f_{i,i} + (0, \cdots, 0, 1) & \text{if } L_i \text{ is free of type } I; \\
\delta_{i-1}(0, \cdots, 0, 1) \cdot f_{i-1,i} + \delta_{i+1}(0, \cdots, 0, 1) \cdot f_{i+1,i} & \text{if } L_i \text{ is bound of type } I.
\end{cases}
\]

Then each entry of the above matrix lies in the ideal $(\pi)$.

It is easy to see that $H$ is represented by a flat $A$-scheme which is isomorphic to an affine space of dimension $2n^2 - \dim U(V,h)$. Note that our fixed hermitian form $h$ is an element of $H(A)$.

6.4. Smooth affine group scheme $G$.

**Theorem 6.2.** For any flat $A$-algebra $R$, the group $M^*(R)$ acts on the right of $H(R)$ by $f \circ m = \sigma(m') \cdot f \cdot m$. Then this action is represented by an action morphism
\[
H \times M^* \longrightarrow H.
\]
GROUP SCHEMES AND LOCAL DENSITIES OF RAMIFIED HERMITIAN LATTICES WHEN $p = 2$

The proof of this theorem is similar to that of Theorem 3.4 in [2] and so we will omit it.

**Theorem 6.3.** Let $\rho$ be the morphism $M^* \to H$ defined by $\rho(m) = h \circ m$. Then $\rho$ is smooth of relative dimension $\dim U(V, h)$.

**Proof.** The theorem follows from Theorem 5.5 in [6] and the following lemma. □

**Lemma 6.4.** The morphism $\rho \otimes \kappa : M^* \otimes \kappa \to H \otimes \kappa$ is smooth of relative dimension $\dim U(V, h)$.

**Proof.** Define $T_3(R)$ to be the set of all $(n \times n)$-matrices $y$ over $R \otimes A B$ with the following conditions:

1. The $(i, j)$-block $y_{i,j}$ of $y$ has entries in $\pi^{\max(i,j)} B \otimes_A R$ so that
   \[ y = \left( \pi^{\max(i,j)} y_{i,j} \right). \]
   Here, the size of $y_{i,j}$ is $n_i \times n_j$.

2. a) If $L_i$ is of type $P$, $y_{i,i}$ is of the form
   \[
   \begin{pmatrix}
   s_i & \pi y_i \\
   \pi v_i & \pi z_i
   \end{pmatrix} \in M_{n_i}(R \otimes_A B)
   \]
   where $s_i$ is an $(n_i - 1) \times (n_i - 1)$ matrix, etc.
   b) If $L_i$ is of type $F$, $y_{i,i}$ is of the form
   \[
   \begin{pmatrix}
   s_i & r_i & \pi t_i \\
   y_i & x_i & \pi w_i \\
   \pi v_i & \pi u_i & \pi z_i
   \end{pmatrix} \in M_{n_i}(R \otimes_A B)
   \]
   where $s_i$ is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

3. Assume that $i$ is even and $L_i$ is of type $I$. Then
   \[ z_i + \delta_{i-2} k_{i-2,i} + \delta_{i+2} k_{i+2,i} \in (\pi). \]
   Here,
   a) $z_i$ is the $(n_i \times n_i)^{th}$-entry of $y_{i,i}$ as described in Step (2) above.
   b) $k_{i-2,i}$ (resp. $k_{i+2,i}$) is the $(n_i-2 \times n_i)^{th}$-entry (resp. $(n_i+2 \times n_i)^{th}$-entry) of the matrix $y_{i-2,i}$ (resp. $y_{i+2,i}$) if $L_{i-2}$ (resp. $L_{i+2}$) is of type $I$. 


(4) Assume that $i$ is odd. Consider the following $(1 \times n_i)$-matrix:

\[
\begin{cases}
  v_i \cdot y_{i,i} & \text{if } L_i \text{ is free of type I;} \\
  \delta_{i-1} v_{i-1} \cdot y_{i-1,i} + \delta_{i+1} v_{i+1} \cdot y_{i+1,i} & \text{if } L_i \text{ is bound of type I.}
\end{cases}
\]

Here,

a) $v_i = (0, \cdots, 0, 1, 0)$ of size $1 \times n_i$.

b) $v_{i-1}$ (resp. $v_{i+1}$) = $(0, \cdots, 0, 1)$ of size $1 \times n_{i-1}$ (resp. $1 \times n_{i+1}$).

Then each entry of the above matrix lies in the prime ideal $(\pi)$.

(5) Assume that $i$ is odd. Consider the following $(1 \times n_i)$-matrix:

\[
\begin{cases}
  v_i \cdot t y_{i,i} & \text{if } L_i \text{ is free of type I;} \\
  \delta_{i-1} v_{i-1} \cdot t y_{i-1,i} + \delta_{i+1} v_{i+1} \cdot t y_{i+1,i} & \text{if } L_i \text{ is bound of type I.}
\end{cases}
\]

Here, $v_i, v_{i-1}, v_{i+1}$ are as described in Step (4). Then each entry of the above matrix lies in the prime ideal $(\pi)$.

It is easy to see that the functor $T_3$ is represented by a flat $A$-scheme.

With the functor $T_3$, the rest of the proof is similar to that of Lemma 3.6 in [2] so we will omit it.

□

Let $G$ be the stabilizer of $h$ in $M^*$. It is an affine group subscheme of $M^*$, defined over $A$. Thus we have the following theorem.

**Theorem 6.5.** The group scheme $G$ is smooth, and $G(R) = \text{Aut}_{B \otimes A R}(L \otimes A R, h)$ for any étale $A$-algebra $R$.

7. THE SPECIAL FIBER $\tilde{G}$ IN CASE 2

In this section, we will determine the structure of the special fiber $\tilde{G}$ of $G$ by observing the maximal reductive quotient and the component group when $E/F$ satisfies Case 2, based on Section 4 of [2]. Recall that the identity matrix is denoted by id.
7.1. The reductive quotient of the special fiber. Assume that \( i = 2m \) is even. Recall that \( Z_i \) is the sublattice of \( B_i \) such that \( Z_i/\pi A_i \) is the kernel of the quadratic form \( \frac{1}{2^{m+1}} q \) mod 2 on \( B_i/\pi A_i \), where \( \frac{1}{2^{m+1}} q(x) = \frac{1}{2^{m+1}} h(x, x) \). Let \( \bar{q}_i \) denote the nonsingular quadratic form \( \frac{1}{2^{m+1}} q \mod 2 \) on \( B_i/Z_i \). It is also obvious that each element of \( \mathbf{G}(R) \) fixes \( \bar{q}_i \) for every flat \( A \)-algebra \( R \). Based on this, we have a morphism of algebraic groups

\[
\varphi_i : \tilde{G} \to O(B_i/Z_i, \bar{q}_i)^{\text{red}}
\]

defined over \( \kappa \), where \( O(B_i/Z_i, \bar{q}_i)^{\text{red}} \) is the reduced subgroup scheme of \( O(B_i/Z_i, \bar{q}_i) \). The dimension of \( B_i/Y_i \) as a \( \kappa \)-vector space is as follows:

\[
\begin{align*}
& n_i - 1 \quad \text{if } L_i \text{ is of type } I; \\
& n_i \quad \text{if } L_i \text{ is of type } I \text{ or free of type } II; \\
& n_i + 1 \quad \text{if } L_i \text{ is bound of type } II.
\end{align*}
\]

We next assume that \( i = 2m - 1 \) is odd. Recall that \( Y_i \) is the sublattice \( B_i \) such that \( Y_i/\pi A_i \) is the kernel of the alternating bilinear form \( \frac{1}{\pi} \cdot \frac{1}{(\pi \sigma(\pi))^{m-1}} \langle - , - \rangle \mod \pi \) on \( B_i/\pi A_i \). Let \( h_i \) denote the nonsingular alternating bilinear form \( \frac{1}{\pi} \cdot \frac{1}{(\pi \sigma(\pi))^{m-1}} \langle - , - \rangle \mod \pi \) on \( B_i/Y_i \). It is obvious that each element of \( \mathbf{G}(R) \) fixes \( h_i \) for every flat \( A \)-algebra \( R \). Based on this fact, we have a morphism of algebraic groups

\[
\varphi_i : \tilde{G} \to \text{Sp}(B_i/Y_i, h_i)
\]

defined over \( \kappa \). The dimension of \( B_i/Y_i \) as a \( \kappa \)-vector space is as follows:

\[
\begin{align*}
& n_i \quad \text{if } L_i \text{ is of type } II \text{ or is bound of type } I; \\
& n_i - 2 \quad \text{if } L_i \text{ is free of type } I.
\end{align*}
\]

Theorem 7.1. The morphism \( \varphi \) defined by

\[
\varphi = \prod_i \varphi_i : \tilde{G} \to \prod_{i: \text{even}} O(B_i/Z_i, \bar{q}_i)^{\text{red}} \times \prod_{i: \text{odd}} \text{Sp}(B_i/Y_i, h_i)
\]

is surjective.

Proof. Assume that dimension of \( \tilde{G} = \text{dimension of Ker } \varphi + \sum_{i: \text{odd}} (\text{dimension of Sp}(B_i/Y_i, h_i)) \)

\[+ \sum_{i: \text{even}} (\text{dimension of } O(B_i/Z_i, \bar{q}_i)^{\text{red}}). \] Thus \( \text{Im } \varphi \) contains the identity component of the product of \( \prod_{i: \text{odd}} \text{Sp}(B_i/Y_i, h_i) \times \prod_{i: \text{even}} O(B_i/Z_i, \bar{q}_i)^{\text{red}} \). Here \( \text{Ker } \varphi \) denotes the kernel of \( \varphi \) and \( \text{Im } \varphi \) denotes the image of \( \varphi \).

Recall that a matrix form of an element of \( \tilde{G}(R) \) for a \( \kappa \)-algebra \( R \) is

\[
m = \left( \pi^{\max\{0,j-i\}} m_{i,j} \right).
\]
Let $\mathcal{H}$ be the set of even integer $i$'s such that $O(B_i/Z_i, \bar{q}_i)^{\text{red}}$ is disconnected. Notice that $O(B_i/Z_i, \bar{q}_i)^{\text{red}}$ is disconnected exactly when $L_i$ with $i$ even is free of type II. For such a lattice $L_i$, we define the closed subgroup scheme $H_i$ of $\tilde{G}$ by the equations $m_{j,k} = 0$ if $j \neq k$, and $m_{j,j} = \text{id}$ if $j \neq i$. The group scheme $H_i$ is isomorphic to the special fiber of the smooth affine group scheme model of the unitary group associated to the hermitian lattice $L_i$. Moreover, it is easily seen that $\varphi_i$ is surjective from $H_i$ to $O(B_i/Z_i, \bar{q}_i)$ and that $B_i = A_i, Z_i = X_i$. Note that $H_i$ and $H_j$ commute with each other in the sense that $h_i \cdot h_j = h_j \cdot h_i$ for all $i \neq j$, where $h_i \in H_i(R)$ and $h_j \in H_j(R)$ for a $\kappa$-algebra $R$. Thus the product $\prod_{i \in \mathcal{H}} H_i$ is embedded into $\tilde{G}$ as a closed subgroup scheme. Since $\varphi_i|_{H_j}$ is trivial for all $i \neq j$, the morphism

$$
\prod_{i \in \mathcal{H}} \varphi_i : \prod_{i \in \mathcal{H}} H_i \to \prod_{i \in \mathcal{H}} O(B_i/Z_i, \bar{q}_i)
$$

is surjective. Therefore, $\varphi$ is surjective. Now it suffices to establish the assumption made at the beginning of the proof, which is the next lemma.

\begin{lemma}
\label{lemma:kernel dimension}
$\text{Ker } \varphi$ is smooth and unipotent of dimension $l$. In addition, the number of connected components of $\text{Ker } \varphi$ is $2\beta$. Here,

- $l$ is such that $l + \sum_{i: \text{odd}} \text{(dimension of } \text{Sp}(B_i/Y_i, h_i)) + \sum_{i: \text{even}} \text{(dimension of } O(B_i/Z_i, \bar{q}_i)^{\text{red}}) = \text{dimension of } \tilde{G}$.
- $\beta$ is the size of the finite set $\mathcal{B}$, where $\mathcal{B}$ is the set of an integer $j$ such that $L_j$ is of type I and $L_{j+2}, L_{j+3}, L_{j+4}$ (resp. $L_{j-1}, L_{j+1}, L_{j+2}, L_{j+3}$) are of type II if $j$ is even (resp. odd).

The proof is postponed to the Appendix B.
\end{lemma}

\begin{remark}
\label{remark:image description}
We describe $\text{Im } \varphi_i$ as follows.

| Type of lattice $L_i$ and $i$ | $\text{Im } \varphi_i$ |
|-----------------------------|---------------------|
| II, free, $i : \text{even}$ | $O(n_i, q_i)$        |
| II, bound, $i : \text{even}$| $\text{SO}(n_i + 1, q_i)$ |
| $\mathcal{F}', i : \text{even}$ | $\text{SO}(n_i, q_i)$       |
| $\mathcal{F}', i : \text{even}$ | $\text{SO}(n_i - 1, q_i)$   |
| II, $i : \text{odd}$        | $\text{Sp}(n_i, h_i)$       |
| I, bound, $i : \text{odd}$  | $\text{Sp}(n_i, h_i)$       |
| I, free, $i : \text{odd}$   | $\text{Sp}(n_i - 2, h_i)$   |
\end{remark}
7.2. The construction of component groups. The purpose of this subsection is to define the surjective morphism from $\tilde{G}$ to $(\mathbb{Z}/2\mathbb{Z})^3$, where $\beta$ is defined in Lemma 7.2.

Recall that we have defined the sublattices $L^j$ and $C(L)$ of $L$, and have chosen a Jordan splitting $L^{2m} = \bigoplus_{i \geq 0} M_i$, in Section 4.2. We choose any even integer $j = 2m$ such that $L_j$ is of type I and $L_{j+2}, L_{j+3}, L_{j+4}$ are of type II. We stress that $M_0$ is of type I and $M_2 = L_{j+2}, M_3 = L_{j+3}, M_4 = L_{j+4}$ are of type II. Choose a basis $(\langle e_i, e \rangle)$ (resp. $(\langle e_i, a, e \rangle)$) for $M_0$ based on Theorem 2.8 when the rank of $M_0$ is odd (resp. even). Then $B(L^j)$ is spanned by $(\langle e_i, \pi e \rangle)$ (resp. $(\langle e_i, \pi a, e \rangle)$) and $C(L^j)$ is spanned by $(\langle \pi e_i, e \rangle)$ (resp. $(\langle \pi e_i, \pi a, e \rangle)$) and $M_1 \oplus \bigoplus_{i \geq 2} M_i$.

We now construct a morphism $\psi_j : \tilde{G} \rightarrow \mathbb{Z}/2\mathbb{Z}$ as follows (There are 2 cases depending on whether $M_0$ is of type $I$ or of type $P$):

1. Firstly, we assume that $M_0$ is of type $P$. We choose a Jordan splitting for the hermitian lattice $(C(L^j), \frac{1}{(\pi \cdot (\pi))^{m+1}} h)$ as follows:

   $$ C(L^j) = \bigoplus_{i \geq 1} M_i'. $$

   Notice that $M_1'$ is free of type I and $M_2', M_3', M_4'$ are of type II. Then consider the sublattice $Y(C(L^j))$ of $C(L^j)$ and choose a Jordan splitting for the hermitian lattice $(Y(C(L^j)), \frac{1}{(\pi \cdot (\pi)^{m+1}} h)$ as follows:

   $$ Y(C(L^j)) = \bigoplus_{i \geq 0} M_i''. $$

   Notice that $M_0''$ is free of type II. Let $G_j$ denote the special fiber of the smooth affine group scheme associated to the hermitian lattice $(Y(C(L^j)), \frac{1}{(\pi \cdot (\pi))^{m+1}} h)$). We now have a morphism from $\tilde{G}$ to $G_j$. Moreover, since $M_0''$ is free of type II, we have a morphism from $G_j$ to the even orthogonal group associated to $M_0''$ as explained in Section 7.1. Thus, the Dickson invariant of this orthogonal group induces the morphism

   $$ \psi_j : \tilde{G} \rightarrow \mathbb{Z}/2\mathbb{Z}. $$
2. We next assume that $M_0$ is of type I. We choose a Jordan splitting for the hermitian lattice 
\( (C(L^j), \frac{1}{(\pi \sigma (\pi))} h) \) as follows:
\[
C(L^j) = \bigoplus_{i \geq 0} M'_i.
\]
Notice that the rank of the $\pi^0$-modular lattice $M'_0$ is 1 and the lattice $M'_2, M'_3, M'_4$ are of type II. If $G_j$ denotes the special fiber of the smooth affine group scheme associated to the hermitian lattice 
\( (C(L^j), \frac{1}{(\pi \sigma (\pi))} h) \), we have a morphism from $\tilde{G}$ to $G_j$.

We now consider the new hermitian lattice $M'_0 \oplus C(L^j)$. The smooth affine group scheme associated to the hermitian lattice 
\( (C(L^j), \frac{1}{(\pi \sigma (\pi))} h) \) can be embedded into the smooth affine group scheme associated to the hermitian lattice $M'_0 \oplus C(L^j)$ as a closed subgroup scheme. Thus the special fiber $G_j$ of the former group scheme is embedded into the special fiber of the latter group scheme. Since the $\pi^0$-modular lattice $M'_0 \oplus M'_0$ is of type $\mathcal{P}$, where 
\( (M'_0 \oplus M'_0) \oplus \bigoplus_{i \geq 1} M'_i \) is a Jordan splitting of the hermitian lattice $M'_0 \oplus C(L^j)$, we have a morphism from the special fiber of the latter group scheme to $\mathbb{Z}/2\mathbb{Z}$ as constructed in the first case. It induces the morphism
\[
\psi_j : \tilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}.
\]

3. Choose any odd integer $j$ such that $L_j$ is of type I and $L_{j-1}, L_{j+1}, L_{j+2}, L_{j+3}$ are of type II. Consider the sublattice $L^{j-1}$ of $L$ and a Jordan splitting $L^{j-1} = \bigoplus_{i \geq 0} M_i$. We stress that $M_1$ is of type I and $M_2 = L_{j+1}, M_3 = L_{j+2}, M_4 = L_{j+3}$ are of type II. Assume that $M_0$ is of type II. Choose a Jordan splitting $C(L^{j-1}) = \bigoplus_{i \geq 1} M'_i$. Then $M'_1$ is free of type I and $M'_2, M'_3, M'_4$ are of type II. For such a lattice, we have constructed the desired morphism in the first case. If $M_0$ is of type I, this case is reduced to the first and the second cases. In conclusion, we construct the morphism
\[
\psi_j : \tilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}.
\]

4. Combining all cases, we have the morphism
\[
\psi = \prod_j \psi_j : \tilde{G} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^\beta.
\]

We now have the following result.

**Theorem 7.4.** The morphism
\[
\psi = \prod_j \psi_j : \tilde{G} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^\beta
\]

is surjective.
Moreover, the morphism
\[
\varphi \times \psi : \tilde{G} \to \prod_{i: \text{even}} O(B_i/Z_i, \tilde{q}_i)^{\text{red}} \times \prod_{i: \text{odd}} \text{Sp}(B_i/Y_i, h_i) \times (\mathbb{Z}/2\mathbb{Z})^\beta
\]
is also surjective.

Proof. We first show that \(\psi_j\) is surjective. Recall that we define the closed subgroup scheme \(F_j\) of \(\tilde{G}\) in the proof of Lemma 10.6 of Appendix B. Based on the description of the Dickson invariant explained in Remark 4.4 of [2], it is easily seen that \(\psi_j\) induces a surjective morphism from \(F_j\) to \(\mathbb{Z}/2\mathbb{Z}\).

The rest of the proof is similar to that of Theorem 4.6 in [2] and so we may will omit it. \(\square\)

7.3. The maximal reductive quotient of \(\tilde{G}\). We finally have the structural theorem for the algebraic group \(\tilde{G}\).

**Theorem 7.5.** The morphism
\[
\varphi \times \psi : \tilde{G} \to \prod_{i: \text{even}} O(B_i/Z_i, \tilde{q}_i)^{\text{red}} \times \prod_{i: \text{odd}} \text{Sp}(B_i/Y_i, h_i) \times (\mathbb{Z}/2\mathbb{Z})^\beta
\]
is surjective and the kernel is unipotent and connected. Consequently,
\[
\prod_{i: \text{even}} O(B_i/Z_i, \tilde{q}_i)^{\text{red}} \times \prod_{i: \text{odd}} \text{Sp}(B_i/Y_i, h_i) \times (\mathbb{Z}/2\mathbb{Z})^\beta
\]
is the maximal reductive quotient.

Proof. We only need to prove that the kernel is unipotent and connected. The kernel of \(\varphi\) is a closed subgroup scheme of the unipotent group \(\tilde{M}^+\) which is defined in Lemma 10.1 of Appendix B and so it suffices to show that the kernel of \(\varphi \times \psi\) is connected. Equivalently, it suffices to show that the kernel of the restricted morphism \(\psi|_{\text{Ker}\varphi}\) is connected. From Lemma 7.2, the number of connected components of \(\text{Ker}\varphi\) is \(2\beta\). Since the restricted morphism \(\psi|_{\text{ker}\varphi}\) is surjective onto \((\mathbb{Z}/2\mathbb{Z})^\beta\), we complete the proof by counting the number of connected components. \(\square\)

8. Comparison of volume forms and final formulas in Case 2

This section is based on Section 7 of [6]. In the construction of Section 3.2 of [6], pick \(\omega'_M\) and \(\omega'_H\) to be such that
\[
\int_{M(A)} |\omega'_M| = 1 \quad \text{and} \quad \int_{H(A)} |\omega'_H| = 1.
\]
Put $\omega_{\text{can}} = \omega'_M / \rho^* \omega'_H$. By Theorem 6.3, we have an exact sequence of locally free sheaves on $M^*$:

$$0 \to \rho^* \Omega_{H/A} \to \Omega_{M^*/A} \to \Omega_{M^*/H} \to 0.$$ 

It follows that $\omega_{\text{can}}$ is of the type discussed in Section 3 of [6].

Lemma 8.1. Recall that 2 is a uniformizer of $A$. Then

$$\omega_M = 2^n M \omega'_M, \quad N_M = \sum_{L_i, \text{type I}} 2n_i + \sum_{i < j} (j - i) \cdot n_i \cdot n_j - a,$$

$$\omega_H = 2^n H \omega'_H, \quad N_H = \sum_{L_i, \text{type I}} n_i + \sum_{i < j} i \cdot n_i \cdot n_j + \sum_{i: \text{even}} \frac{i + 2}{2} \cdot n_i + \sum_{i: \text{odd}} \frac{i + 3}{2} \cdot n_i + \sum di - a,$$

$$\omega^{ld} = 2^{N_M - N_H} \omega_{\text{can}}.$$ 

Here,

- $a$ is the total number of $L_i$’s such that $i$ is odd and $L_i$ is free of type I.
- $d_i = i \cdot n_i \cdot (n_i - 1)/2$.

Theorem 8.2. Let $f$ be the cardinality of $\kappa$. The local density of $(L, h)$ is

$$\beta_L = f^N \cdot f^{-\dim U(V, h)} \tilde{G}(\kappa),$$

where

$$N = N_H - N_M = \sum_{i < j} i \cdot n_i \cdot n_j + \sum_{i: \text{even}} \frac{i + 2}{2} \cdot n_i + \sum_{i: \text{odd}} \frac{i + 3}{2} \cdot n_i + \sum di - \sum_{L_i, \text{type I}} n_i.$$ 

Remark 8.3. (1) In the above local density formula, $\tilde{G}(\kappa)$ is computed as follows. We denote by $R_u \tilde{G}$ the unipotent radical of $\tilde{G}$ so that the maximal reductive quotient of $\tilde{G}$ is $\tilde{G}/R_u \tilde{G}$. That is, there is the following exact sequence of group schemes over $\kappa$:

$$1 \to R_u \tilde{G} \to \tilde{G} \to \tilde{G}/R_u \tilde{G} \to 1.$$ 

Furthermore, the following sequence of groups

$$1 \to R_u \tilde{G}(\kappa) \to \tilde{G}(\kappa) \to (\tilde{G}/R_u \tilde{G})(\kappa) \to 1.$$ 

is also exact by Lemma 9.1. If $m$ is the dimension of $R_u \tilde{G}$, then $\sharp R_u \tilde{G}(\kappa)$ is $f^m$. Furthermore, since the dimension of $\tilde{G}$ is $n^2$ with $n = \text{rank}_{B} L$, the dimension of $R_u \tilde{G}$ can be computed explicitly based on Theorem 7.5. In addition, the order of an orthogonal group or a symplectic group defined over a finite field is well known. Thus, one can compute
$\sharp(\tilde{G}/R_u\tilde{G})(\kappa)$ explicitly based on Theorem 7.5. Finally, the order of the group $\tilde{G}(\kappa)$ is identified as follows:

$$\sharp\tilde{G}(\kappa) = \sharp R_u\tilde{G}(\kappa) \cdot \sharp(\tilde{G}/R_u\tilde{G})(\kappa).$$

(2) As in Remark 7.4 of [6], although we have assumed that $n_i = 0$ for $i < 0$, it is easy to check that the formula in the preceding theorem remains true without this assumption.

9. Appendix A : The proof of Lemma 4.2 in Case 1

The proof of Lemma 4.2 is based on Proposition 6.3.1 in [6] and Lemma 4.2 in [2]. For preparation, we state a series of lemmas.

Lemma 9.1. (Lemma 6.3.3. in [6])

Let $1 \to X \to Y \to Z \to 1$ be an exact sequence of group schemes that are locally of finite type over $\kappa$, where $\kappa$ is a perfect field. Suppose that $X$ is smooth, connected, and unipotent. Then $1 \to X(R) \to Y(R) \to Z(R) \to 1$ is exact for any $\kappa$-algebra $R$.

- Let $\tilde{M}$ be the special fiber of $M^*$. Let

$$\tilde{M}_i = \begin{cases} \text{GL}_{B/B}(B_i/Y_i) & \text{if } i \text{ is even;} \\ \text{GL}_{B/B}(A_i/X_i) & \text{if } i \text{ is odd.} \end{cases}$$

For any $\kappa$-algebra $R$, let $m = \left(\pi^{\max\{0,j-i\}}m_{i,j}\right) \in \tilde{M}(R)$. Assume that $i$ is even. Let $s_i$ be a block of $m_{i,i}$ as explained in Section 3.1 if $L_i$ is of type I, and let $s_i = m_{i,j}$ if $L_i$ is of type II. When $i$ is odd, we set $s_i = m_{i,i}$. If we write $s_i = s_i^1 + \pi s_i^2$, then $s_i^1 \in \tilde{M}_i(R)$.

Therefore, we have a surjective morphism of algebraic groups

$$r : \tilde{M} \to \prod \tilde{M}_i,$$

defined over $\kappa$. We now have the following lemma:

Lemma 9.2. The kernel of $r$ is the unipotent radical $\tilde{M}^+$ of $\tilde{M}$, and $\prod \tilde{M}_i$ is the maximal reductive quotient of $\tilde{M}$.

- The morphism $\varphi$ is extended to the morphism

$$\bar{\varphi} : \tilde{M} \to \prod_{i:\text{even}} \text{Aut}_\kappa(B_i/Y_i) \times \prod_{i:\text{odd}} \text{Aut}_\kappa(A_i/Z_i)$$

such that $\bar{\varphi}|_{\tilde{G}} = \varphi$. It is obvious that $\text{Ker} \bar{\varphi}$ is a closed subgroup scheme of $\tilde{M}^+$ and is smooth and unipotent.
Recall that we have defined $M'$ such that $1 + M'$ is the same as $\mathcal{M}$ in Remark 3.1. Notice that $\pi M'(R)$ is a $B/2B$-algebra for any $\kappa$-algebra $R$. Therefore, we consider the subfunctor $\pi M' : R \mapsto \pi M'(R)$ of $\mathcal{M} \otimes \kappa$ and the subfunctor $\tilde{M} : R \mapsto 1 + \pi M'(R)$ of Ker $\tilde{\varphi}$. Then we have the following easy lemma.

**Lemma 9.3.** (i) The functor $\tilde{M}$ is representable by a smooth, connected, unipotent group scheme over $\kappa$. Moreover, $\tilde{M}$ is a closed normal subgroup of Ker $\tilde{\varphi}$.

(ii) The quotient group scheme $\text{Ker } \tilde{\varphi}/\tilde{M}$ represents the functor

$$R \mapsto \text{Ker } \tilde{\varphi}(R)/\tilde{M}(R)$$

by Lemma 9.1 and is smooth, connected, and unipotent.

Let $R$ be a $\kappa$-algebra. From the lemma, we describe the functors of points of the schemes $\tilde{M}$, Ker $\tilde{\varphi}$, and Ker $\tilde{\varphi}/\tilde{M}$ as follows:

- An element of $\tilde{M}(R)$ is

$$m = \left( \pi^{\max\{0, j-i\}} m_{i,j} \right).$$

Here, if $i$ is even and $L_i$ is of type $P$ or of type $F$,

$$m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} \text{ or } \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix},$$

respectively, where, $s_i \in M_{(n_i-1) \times (n_i-1)}(R \otimes B/2B)$ (resp. $s_i \in M_{(n_i-2) \times (n_i-2)}(R \otimes B/2B)$), etc. In other cases, $m_{i,j} \in M_{n_i \times n_j}(R \otimes B/2B)$.

- Ker $\tilde{\varphi}(R)$ is the subgroup of $\tilde{M}(R)$ with the following setting:
  a) If $i$ is even and $L_i$ is of type $I$, set $s_i = \text{id mod } \pi$.
  b) If $i$ is even and $L_i$ is of type $II$, set $m_{i,i} = \text{id mod } \pi$.
  c) If $i$ is odd, set $m_{i,i} = \text{id mod } \pi$ and $\delta_{i-1} e_{i-1} \cdot m_{i-1,i} + \delta_{i+1} e_{i+1} \cdot m_{i+1,i} = 0 \text{ mod } \pi$.

Here, $\delta_j = \begin{cases} 1 & \text{if } L_j \text{ is of type } I; \\ 0 & \text{if } L_j \text{ is of type } II, \end{cases}$

and $e_j = (0, \cdots, 0, 1)$ (resp. $e_j = (0, \cdots, 0, 1, 0)$) of size $1 \times n_j$ if $L_j$ is of type $P$ (resp. of type $F$).

- An element of Ker $\tilde{\varphi}/\tilde{M}^1(R)$ is

$$m = \left( \pi^{\max\{0, j-i\}} m_{i,j} \right).$$
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a) Assume that $i$ is even and $L_i$ is of type I. Then a matrix form of $m_{i,i}$ is as described above such that $s_i \in M_{(n_i-1)\times(n_i-1)}(R)$ (resp. $s_i \in M_{(n_i-2)\times(n_i-2)}(R)$), etc., if $L_i$ is of type $I'$ (resp. of type $I''$). In addition,

(i) If $i$ is even and $L_i$ is of type I, set $s_i = id$.

(ii) If $i$ is even and $L_i$ is of type II, set $m_{i,i} = id$.

(iii) If $i$ is odd, set $m_{i,i} = id$ and $\delta_{i-1}e_{i-1} \cdot m_{i-1,i} + \delta_{i+1}e_{i+1} \cdot m_{i+1,i} = 0$.

Here, $\delta_j$ and $e_j$ are as explained in the description of $\text{Ker } \tilde{\varphi}(R)$.

b) In other cases, $m_{i,j} \in M_{n_i \times n_j}(R)$.

We remark that the above only describes the underlying schemes. The group law is to be interpreted as Section 5.3 in [6].

- This paragraph is based on 6.3.6 in [6]. Recall that there is a closed immersion $\tilde{G} \to \tilde{M}$. Notice that $\text{Ker } \varphi$ is the kernel of the composition $\tilde{G} \to \tilde{M} \to \tilde{M}/\text{Ker } \tilde{\varphi}$. We define $\tilde{G}^1$ as the kernel of the composition

$$\tilde{G} \to \tilde{M} \to \tilde{M}/\tilde{M}^1.$$

Then $\tilde{G}^1$ is the kernel of the morphism $\text{Ker } \varphi \to \text{Ker } \tilde{\varphi}/\tilde{M}^1$ and, hence, is a closed normal subgroup of $\text{Ker } \varphi$. The induced morphism $\text{Ker } \varphi/\tilde{G}^1 \to \text{Ker } \tilde{\varphi}/\tilde{M}^1$ is a monomorphism, and thus $\text{Ker } \varphi/\tilde{G}^1$ is a closed subgroup scheme of $\text{Ker } \tilde{\varphi}/\tilde{M}^1$ by (Exp. VI$_B$, Corollary 1.4.2 in [5]).

**Lemma 9.4.** $\tilde{G}^1$ is connected, smooth, and unipotent.

**Proof.** We state the equations defining $\text{Ker } \varphi$. They are obtained by the matrix equation $\sigma(t^m)hm = h$, where $m$ is an element of $\text{Ker } \tilde{\varphi}(R)$ for a $\kappa$-algebra $R$.

We represent the given hermitian form $h$ by a hermitian matrix $\left(\pi^i \cdot h_i\right)$ with $\left(\pi^i \cdot h_i\right)$ for the $(i,i)$-block and 0 for remaining blocks.

By observing the diagonal $(i,i)$-blocks of $\sigma(t^m)hm = h$, we have the following matrix equation:

$$\sigma(t^{m_{i,i}})h_i + \sigma(t^{m_{i-1,i}})h_{i-1}m_{i-1,i} + \sigma(t^{m_{i+1,i}})h_{i+1}m_{i+1,i} + \pi^2(t^{m_{i-2,i}})h_{i-2}m_{i-2,i} + \sigma(t^{m_{i+2,i}})h_{i+2}m_{i+2,i} = h_i,$$

where $0 \leq i < N$.

By observing the $(i,j)$-blocks of $\sigma(t^m)hm = h$, where $i < j$, we have the following matrix
equation:

\[
\sum_{i \leq k \leq j} \sigma(i m_{k,i}) h_k m_{k,j} + \pi \sigma(i m_{i-1,i}) h_{i-1} m_{i-1,j} + \pi \sigma(i m_{j+1,i}) h_{j+1} m_{j+1,j} = 0,
\]

where 0 ≤ i, j < N.

The equations defining \( \tilde{G}^1 \) are obtained by setting

1. \( m_{i,j} = \pi m'_{i,j} \) if \( i \neq j \),
2. \( m_{i,i} = \text{id} + \pi m'_{i,i} \) if \( i \) is odd, or \( i \) is even and \( L_i \) is of type II,
3. \( m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} = \begin{pmatrix} \text{id} + \pi s'_{i} & \pi^2 y'_i \\ \pi^2 v'_i & 1 + \pi^2 z'_i \end{pmatrix} \) if \( i \) is even and \( L_i \) is of type I,
4. \( m_{i,i} = \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix} = \begin{pmatrix} \text{id} + \pi s'_{i} & \pi r'_{i} & \pi^2 t'_i \\ \pi^2 y'_i & 1 + \pi^2 x'_i & \pi^2 z'_i \\ \pi v'_i & \pi u'_i & 1 + \pi^2 w'_i \end{pmatrix} \) if \( i \) is even and \( L_i \) is of type II.

It is then easy to check that the underlying algebraic variety of \( \tilde{G}^1 \) is an affine space of dimension

\[
\sum_{i<j} n_i n_j + \sum_{i \text{ odd}} \frac{n_i^2 + n_i}{2} + \sum_{i \text{ even}} \frac{n_i^2 - n_i}{2} + 3 \{ i : i \text{ is even and } L_i \text{ is of type I} \}.
\]

This completes the proof. \( \square \)

**Lemma 9.7.** Ker \( \varphi/\tilde{G}^1 \) is isomorphic to \( A^{l'} \times (\mathbb{Z}/2\mathbb{Z})^{\beta} \) as \( \kappa \)-varieties, where \( A^{l'} \) is an affine space of dimension \( l' \).

Here,

- \( l' \) is such that \( l' + \text{dimension of } \tilde{G}^1 = l \). Notice that \( l \) is defined in Lemma 4.2.
- \( \beta \) is the size of the set of even integer \( j \)’s such that \( L_j \) is of type I and \( L_{j+2} \) is of type II.

**Proof.** Lemma 9.1 and Lemma 9.4 imply that Ker \( \varphi/\tilde{G}^1 \) represents the functor \( R \mapsto \text{Ker } \varphi(R) / \tilde{G}^1(R) \). If \( m = \left( \pi \sigma(i \cdot m_{i,j}) \right) \in (\text{Ker } \varphi/\tilde{M}^1)(R) \) is such that \( m \in (\text{Ker } \varphi/\tilde{G}^1)(R) \), then \( m \) satisfies the following equations (which are given as equalities in \( R \)):

1. \( \sum_{i \leq k \leq j} \sigma(i m_{k,i}) h_k m_{k,j} = 0 \) if \( i < j \),
2. \( m_{i,i} = \text{id} \) if \( i \) is odd, or \( i \) is even and \( L_i \) is of type II,
3. Assume that \( i \) is odd. Then \( \delta_i e_{i-1} \cdot m_{i-1,i} + \delta_i e_{i+1} \cdot m_{i+1,i} = 0 \).
4. Assume that \( i \) is even and \( L_i \) is of type I. Then \( s_i = \text{id} \).
Then it is easy to check that

\[ h_i = \begin{pmatrix} h'_i & 0 \\ 0 & h''_i \end{pmatrix}, \]

where \( h'_i \) is \((n_i - 1) \times (n_i - 1)\)–matrix (resp. \((n_i - 2) \times (n_i - 2)\)–matrix) if \( L_i \) is of type \( F^1 \) (resp. of type \( F^2 \)).

If \( i \) is even and \( L_i \) is of type \( F^2 \), \( \pi^{-1}(\pi h'_i y_i + \sigma(\pi) \cdot t v_i) + \mathcal{P}^i_{1,2} = 0. \)

Here, \( \mathcal{P}^i_{1,2} \) is a polynomial with variables \( m_{i-1,i}, m_{i+1,i} \).

If \( i \) is even and \( L_i \) is of type \( F^1 \), \( h'_i r_i + t v_i = 0, \pi^{-1}(\pi h'_i t + \sigma(\pi) \cdot t y + \pi \cdot t v z) + \mathcal{P}^i_{1,3} = 0, \pi^{-1}(\pi \cdot t h'_i t + \pi z + \pi w + \sigma(\pi) x + \pi uz) + \mathcal{P}^i_{2,3} = 0. \)

Here, \( \mathcal{P}^i_{1,3}, \mathcal{P}^i_{2,3} \) are suitable polynomials with variables \( m_{i-1,i}, m_{i+1,i} \).

Assume that \( i \) is even and \( L_i \) is of type \( I \). Then we have

\[
\mathcal{F}_i : \frac{1}{2}(\pi \sigma(\pi)(z_i + z_i^2) + \sigma(t m'_{i-1,i}) \pi h_{i-1} m'_{i-1,i} + \sigma(t m'_{i+1,i}) \pi h_{i+1} m'_{i+1,i} + \sigma(t m'_{i-2,i}) \pi^2 h_{i-2} m'_{i-2,i} + \sigma(t m'_{i+2,i}) \pi^2 h_{i+2} m'_{i+2,i}) = 0.
\]

Here, \( m'_{i,j} \) is the last column vector of the matrix \( m_{j,i} \) and \( z_i \) is an entry of \( m_{i,i} \) as described before.

We now choose an even integer \( j \) such that \( L_j \) is of type \( I \) and \( L_{j+2} \) is of type \( II \). For such a \( j \), there is a non-negative integer \( m_j \) such that \( L_{j-2l} \) is of type \( I \) for every \( l \) with \( 0 \leq l \leq m_j \) and \( L_{j-2(m_j+1)} \) is of type \( II \). Then the sum of equations

\[
\sum_{l=0}^{m_j} \mathcal{F}_{j-2l}
\]

becomes

\[
\sum_{l=0}^{m} (z_{j-2l} + z_{j-2l}^2) = 0.
\]

Let \( G^\dagger \) be the subfunctor of Ker \( \bar{\varphi}/\bar{M}^1 \) consisting of those \((m_{i,j})\) satisfying the above equations. Then it is easy to check that \( G^\dagger \) is represented by a smooth closed subscheme of Ker \( \bar{\varphi}/\bar{M}^1 \) and is isomorphic to \( \mathbb{A}^{l'} \times (\mathbb{Z}/2\mathbb{Z})^\beta \) as \( \kappa \)-varieties, where \( \mathbb{A}^{l'} \) is an affine space of dimension \( l' \). In fact,

\[
l' = \sum_{i<j} n_i n_j - \sum_{i: odd \ and \ L_i: bound} n_i + \sum_{i: even \ and \ L_i: of \ type \ I^e} (n_i - 1) + \sum_{i: even \ and \ L_i: of \ type \ I^o} (2n_i - 2).
\]

For ease of notation, let \( G^\dagger = \text{Ker} \varphi/\bar{G}^1 \). Since \( G^\dagger \) and \( G^\ddagger \) are both closed subschemes of Ker \( \bar{\varphi}/\bar{M}^1 \) and \( G^\dagger(\bar{\kappa}) \subset G^\ddagger(\bar{\kappa}) \), \((G^\dagger)^{\text{red}}\) is a closed subscheme of \((G^\dagger)^{\text{red}} = G^\dagger \). It is easy to check that \( \dim G^\dagger = \dim G^\ddagger \).

Claim \((G^\dagger)^{\text{red}}\) contains at least one (closed) point of each connected component of \( G^\dagger \).
We choose an even integer $j$ such that $L_j$ is of type I and $L_{j+2}$ is of type II. Consider the closed subgroup scheme $F_j$ of $\tilde{G}$ defined by the following equations:

- $m_{i,k} = 0$ if $i \neq k$,
- $m_{i,i} = \text{id}$ if $i \neq j$,
- and for $m_{j,j}$:
  \[
  \begin{cases} 
  s_j = \text{id}, y_j = 0, v_j = 0 & \text{if } L_i \text{ is of type I}; \\
  s_j = \text{id}, r_j = t_j = y_j = v_j = u_j = w_j = 0 & \text{if } L_i \text{ is of type II}. 
  \end{cases}
  \]

Then $F_j$ has exactly two connected components. Notice that $F_j$ and $F_{j'}$ commute with each other for all even integers $j \neq j'$, where $L_j$ and $L_{j'}$ (resp. $L_{j+2}$ and $L_{j'+2}$) are of type I (resp. of type II), in the sense that $f_j \cdot f_{j'} = f_{j'} \cdot f_j$, where $f_j \in F_j$ and $f_{j'} \in F_{j'}$.

Let $F = \prod_j F_j$. Notice that $F$ is smooth and is a closed subgroup scheme of $\text{Ker } \varphi$. Let $F^\dagger$ be the image of $F$ in $(G^\dagger)^\text{red}$. Then $F^\dagger$ contains at least one (closed) point of each connected component of $G^\dagger$ and this justifies our claim.

Combining this fact with $\text{dim } G^\dagger = \text{dim } G^\dagger$, we conclude that $(G^\dagger)^\text{red} \simeq G^\dagger$, and hence, $G^\dagger = G^\dagger$ because $G^\dagger$ is a subfunctor of $G^\dagger$. This completes the proof. \qed

We now prove Lemma 4.2.

**Proof.** We start with the following short exact sequence

$$1 \to \tilde{G}^1 \to \text{Ker } \varphi \to \text{Ker } \varphi/\tilde{G}^1 \to 1.$$  

It is obvious that $\text{Ker } \varphi$ is smooth and unipotent by Lemma 9.4 and Lemma 9.7. Since $\tilde{G}^1$ is connected by Lemma 9.4, the number of connected components of $\text{Ker } \varphi$ is the same as that of $\text{Ker } \varphi/\tilde{G}^1$ by the next lemma. Moreover, the dimension of $\text{Ker } \varphi$ is the sum of dimension of $\tilde{G}^1$ and the dimension of $\text{Ker } \varphi/\tilde{G}^1$. This completes the proof. \qed

**Lemma 9.8.** Assume that there is a short exact sequence of algebraic groups

$$1 \to A \to B \to C \to 1.$$  

Let $\pi_0(B)$ be the component group of $B$ and $\sharp(\pi_0(B))$ be the cardinality of the group $\pi_0(B)$. Then

$$\sharp(\pi_0(B)) \leq \sharp(\pi_0(A)) \cdot \sharp(\pi_0(C)).$$

Moreover, the equality holds if $A$ is connected.
Proof. By definition of a component group, there exists a surjective morphism \( \pi : B \rightarrow \pi_0(B) \) whose kernel is connected. Let \( A' \) be the image of \( A \) under the morphism \( \pi \). Notice that \( A' \) is a normal subgroup of \( \pi_0(B) \). Then the morphism \( \pi \) induces a surjective morphism from \( C \) to \( \pi_0(B)/A' \). This completes the proof. \( \square \)

10. Appendix B : The proof of Lemma 7.2 in Case 2

We repeat the process used in Appendix A. The proof of Lemma 7.2 is based on Proposition 6.3.1 in [6] and Lemma 4.2 in [2].

- Let \( \tilde{M} \) be the special fiber of \( M^* \). Let

\[
\tilde{M}_i = \text{GL}_{B/\pi B}(B_i/Y_i) \quad \text{for all } i.
\]

For any \( \kappa \)-algebra \( R \), let

\[
m = \left( \pi^{\max(0,j-i)} m_{i,j} \right) \in \tilde{M}(R).
\]

Assume that \( i \) is even. Let \( s_i \) be a block of \( m_{i,i} \) as explained in Section 6.1 if \( L_i \) is of type I, and let \( s_i = m_{i,i} \) if \( L_i \) is of type II. If we write \( s_i = s_i^1 + \pi s_i^2 \), then \( s_i^1 \in \tilde{M}_i(R) \) for all even integer \( i \). Now assume that \( i \) is odd. If \( L_i \) is free of type I, then \( m_{i,i} \) is of the form

\[
m_{i,i} = \begin{pmatrix}
s_i & \pi r_i & t_i \\
y_i & 1 + \pi x_i & z_i \\
\pi v_i & \pi u_i & 1 + \pi w_i
\end{pmatrix},
\]

where \( s_i \) is an \((n_i - 2) \times (n_i - 2)\)-matrix, etc. Let \( s_i \) be a block of \( m_{i,i} \) as explained above, and let \( s_i = m_{i,i} \) if \( L_i \) is of type II or bound of type I. If we write \( s_i = s_i^1 + \pi s_i^2 \), then \( s_i^1 \in \tilde{M}_i(R) \) for all odd integer \( i \).

Therefore, we have a surjective morphism of algebraic groups

\[
r : \tilde{M} \rightarrow \prod \tilde{M}_i,
\]

defined over \( \kappa \). We now have the following lemma:

**Lemma 10.1.** The kernel of \( r \) is the unipotent radical \( \tilde{M}^+ \) of \( \tilde{M} \), and \( \prod \tilde{M}_i \) is the maximal reductive quotient of \( \tilde{M} \).

- The morphism \( \varphi \) is extended to the morphism

\[
\tilde{\varphi} : \tilde{M} \rightarrow \prod_{i: \text{even}} \text{Aut}_\kappa(B_i/Z_i) \times \prod_{i: \text{odd}} \text{Aut}_\kappa(B_i/Y_i)
\]
such that $\tilde{\varphi}|_{\tilde{G}} = \varphi$. It is obvious that $\text{Ker } \tilde{\varphi}$ is a closed subgroup scheme of $\tilde{M}^+$ and is smooth and unipotent.

Recall that we have defined $\tilde{M}'$ such that $1 + \tilde{M}'$ is the same as $\tilde{M}$ in Remark 6.1. Notice that $\tilde{M}'(R)$ is a $B/2B$-algebra for any $\kappa$-algebra $R$. Therefore, we consider the subfunctor $\tilde{\pi}M' : R \mapsto \tilde{\pi}M'(R)$ of $\tilde{M}' \otimes \kappa$ and the subfunctor $\tilde{M}^1 : R \mapsto 1 + \tilde{\pi}M'(R)$ of $\text{Ker } \tilde{\varphi}$. Then we have the following easy lemma.

**Lemma 10.2.** (i) The functor $\tilde{M}^1$ is representable by a smooth, connected, unipotent group scheme over $\kappa$. Moreover, $\tilde{M}^1$ is a closed normal subgroup of $\text{Ker } \tilde{\varphi}$.

(ii) The quotient group scheme $\text{Ker } \tilde{\varphi}/\tilde{M}^1$ represents the functor

$$R \mapsto \text{Ker } \tilde{\varphi}(R)/\tilde{M}^1(R)$$

by Lemma 9.1 and is smooth, connected, and unipotent.

Let $R$ be a $\kappa$-algebra. From the lemma, we describe the functors of points of the schemes $\tilde{M}$, $\text{Ker } \tilde{\varphi}$, and $\text{Ker } \tilde{\varphi}/\tilde{M}^1$ as follows:

- An element of $\tilde{M}(R)$ is

$$m = \left( \pi^{\max\{0,j-i\}} m_{i,j} \right)$$

and $m$ satisfies the following:

- If $i$ is even and $L_i$ is of type $\Gamma'$ or of type $\Gamma$,

$$m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi y_i & 1 + \pi z_i \end{pmatrix} \text{ or } \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix},$$

respectively. Here, $s_i \in M_{(n_i-1) \times (n_i-1)}(R \otimes B/2B)$ (resp. $s_i \in M_{(n_i-2) \times (n_i-2)}(R \otimes B/2B)$), etc.

- If $i$ is odd and $L_i$ is free of type $I$,

$$m_{i,i} = \begin{pmatrix} s_i & \pi r_i & t_i \\ y_i & 1 + \pi x_i & u_i \\ \pi y_i & \pi x_i & \pi z_i & 1 + \pi w_i \end{pmatrix},$$

where $s_i \in M_{(n_i-2) \times (n_i-2)}(R \otimes B/2B)$, etc.

- In other cases, $m_{i,j} \in M_{n_i \times n_j}(R \otimes B/2B)$.
d) Assume that \( i \) is even and \( L_i \) is of type I. Then

\[
z_i + \delta_{i-2}k_{i-2,i} + \delta_{i+2}k_{i+2,i} = \pi \tilde{z}_i
\]

such that \( \tilde{z}_i \in R \otimes B/2B \). Here,

(i) \( z_i \) is an entry of \( m_{i,i} \) as described above.

(ii) \( k_{i-2,i} \) (resp. \( k_{i+2,i} \)) is the \( (n_{i-2}, n_i)^{th} \)-entry (resp. \( (n_{i+2}, n_i)^{th} \)-entry) of the matrix \( m_{i-2,i} \) (resp. \( m_{i+2,i} \)) if \( L_{i-2} \) (resp. \( L_{i+2} \)) is of type \( P \).

(iii) \( k_{i-2,i} \) (resp. \( k_{i+2,i} \)) is the \( (n_{i-2} - 1, n_i)^{th} \)-entry (resp. \( (n_{i+2} - 1, n_i)^{th} \)-entry) of the matrix \( m_{i-2,i} \) (resp. \( m_{i+2,i} \)) if \( L_{i-2} \) (resp. \( L_{i+2} \)) is of type \( \mathcal{F} \).

e) Assume that \( i \) is odd and \( L_i \) is bound of type I. Then

\[
\delta_{i-1}v_{i-1} \cdot m_{i-1,i} + \delta_{i+1}v_{i+1} \cdot m_{i+1,i} = \pi \tilde{m}_{i,i}
\]

such that \( \tilde{m}_{i,i} \in M_{1 \times n_i}(R \otimes B/2B) \). Here,

i) \( v_{i-1} = (0, \cdots, 0, 1) \) (resp. \( v_{i-1} = (0, \cdots, 0, 1, 0) \)) of size \( 1 \times n_{i-1} \) if \( L_{i-1} \) is of type \( P \) (resp. of type \( \mathcal{F} \)).

ii) \( v_{i+1} = (0, \cdots, 0, 1) \) (resp. \( v_{i+1} = (0, \cdots, 0, 1, 0) \)) of size \( 1 \times n_{i+1} \) if \( L_{i+1} \) is of type \( P \) (resp. of type \( \mathcal{F} \)).

f) Assume that \( i \) is odd and \( L_i \) is bound of type I. Then

\[
\delta_{i-1}v_{i-1} \cdot {^t}m_{i-1,i} + \delta_{i+1}v_{i+1} \cdot {^t}m_{i+1,i} = \pi \tilde{m}_{i,i}^t
\]

such that \( \tilde{m}_{i,i}^t \in M_{1 \times n_i}(R \otimes B/2B) \). Here, \( v_{i-1} \) (resp. \( v_{i+1} \)) = \( (0, \cdots, 0, 1) \) of size \( 1 \times n_{i-1} \) (resp. \( 1 \times n_{i+1} \)).

\[\text{Ker } \tilde{\varphi}(R) \text{ is the subgroup of } \tilde{M}(R) \text{ with the following setting:}\]

a) If \( i \) is even and \( L_i \) is of type I, set \( s_i = \text{id} \mod \pi \).

b) If \( i \) is even and \( L_i \) is of type II, set \( m_{i,i} = \text{id} \mod \pi \).

c) Let \( i \) be even. Then \( \delta_i v_i \) (resp. \( \delta_i(y_i + bv_i) \)) + \( (\delta_{i-1}e_{i-1} \cdot m_{i-1,i} + \delta_{i+1}e_{i+1} \cdot m_{i+1,i} + \delta_{i-2}e_{i-2} \cdot m_{i-2,i} + \delta_{i+2}e_{i+2} \cdot m_{i+2,i}) \tilde{e}_i = 0 \mod \pi \) if \( L_i \) is of type \( P \) (resp. of type \( \mathcal{F} \)). Here,

i) \( b \) is chosen in Theorem 2.8.b).

ii) \( \delta^*_j = \begin{cases} 1 & \text{if } j \text{ is odd and } L_j \text{ is free of type I;} \\ 0 & \text{otherwise.} \end{cases} \)
iii) When \( j \) is even, \( e_j = (0, \cdots, 0, 1) \) (resp. \( e_j = (0, \cdots, 0, 1, 0) \)) of size \( 1 \times n_j \) if 
\( L_j \) is of type \( P \) (resp. of type \( P \)).
iv) If \( j \) is odd, then \( e_j = (0, \cdots, 0, 1) \) of size \( 1 \times n_j \).

v) \( \tilde{\epsilon}_i = \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} \) of size \( n_i \times (n_i - 1) \) (resp. \( n_i \times (n_i - 2) \)), where \( \text{id} \) is the identity matrix of size \( (n_i - 1) \times (n_i - 1) \) (resp. \( (n_i - 2) \times (n_i - 2) \)) if \( L_i \) is of type \( P \) (resp. of type \( P \)).

vi) \( \tilde{\epsilon}_i = \text{id} \) of size \( n_i \times n_i \) if \( L_i \) is of type II.

d) If \( i \) is odd and \( L_i \) is of type II or bound of type I, set \( m_{i,i} = \text{id} \mod \pi \).
e) If \( i \) is odd and \( L_i \) is free of type I, set \( s_i = \text{id} \mod \pi \).

• An element of \( \ker \tilde{\varphi}/\tilde{M}^1(R) \) is 
\[
m = \left( \pi^{\max\{0,j-i\}} m_{i,j} \right) \text{ with } \tilde{z}_i, \tilde{m}_i, \tilde{m}_i', \tilde{m}_i''.
\]

Here, \( m_{i,j} \in M_{n_i \times n_j}(R) \) if \( i \neq j \).

a) Assume that \( i \) is even and \( L_i \) is of type I. Then a matrix form of \( m_{i,i} \) is as described above such that \( s_i \in M_{(n_i-1)\times(n_i-1)}(R) \) (resp. \( s_i \in M_{(n_i-2)\times(n_i-2)}(R) \)), etc. if \( L_i \) is of type \( P \) (resp. of type \( P \)).
b) If \( i \) is even and \( L_i \) is of type I, set \( s_i = \text{id} \).
c) If \( i \) is even and \( L_i \) is of type II, set \( m_{i,i} = \text{id} \).
d) If \( i \) is even, \( \delta_i v_i (\text{resp. } \delta_i(y_i + bv_i)) + (\delta_{i-1}e_{i-1} \cdot m_{i-1,i} + \delta_{i+1}e_{i+1} \cdot m_{i+1,i} + \delta_{i-2}e_{i-2} \cdot m_{i-2,i} + \delta_{i+2}e_{i+2} \cdot m_{i+2,i})\tilde{\epsilon}_i = 0 \) if \( L_i \) is of type \( P \) (resp. of type \( P \)). Here, each notations follow those of c) in the description of \( \ker \tilde{\varphi}(R) \).
e) If \( i \) is even and \( L_i \) is of type I, \( z_i + \delta(i) k_{i-2,i} + \delta_{i+2} k_{i+2,i} = 0 \). Here, each notations follow those of d) in the description of \( \tilde{M}(R) \).
f) Let \( i \) be even. Then \( \tilde{z}_i \in R \) and \( \tilde{z}_i \) occurs only when \( L_i \) is of type I.
g) Assume that \( i \) is odd and \( L_i \) is free of type I. Then a matrix form of \( m_{i,i} \) is as described above such that \( s_i \in M_{(n_i-2)\times(n_i-2)}(R) \), etc. In addition, set \( s_i = \text{id} \).
h) If \( i \) is odd and \( L_i \) is of type II or bound of type I, set \( m_{i,i} = \text{id} \).
i) If \( i \) is odd, \( \delta_{i-1} v_{i-1} \cdot m_{i-1,i} + \delta_{i+1} v_{i+1} \cdot m_{i+1,i} = 0 \). Here, each notations follow those of e) in the description of \( \tilde{M}(R) \).
j) If \( i \) is odd, \( \delta_{i-1} v_{i-1} \cdot t m_{i-1,i} + \delta_{i+1} v_{i+1} \cdot t m_{i+1,i} = 0 \). Here, each notations follow those of f) in the description of \( \tilde{M}(R) \).
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k) Let $i$ be odd. Then $m_{i,i}, m'_{i,i} \in M_{1 \times n_i}(R)$ and $\tilde{m}_{i,i}, \tilde{m}'_{i,i}$ occur only when $L_i$ is bound of type I.

We remark that the above only describes the underlying schemes. The group law is to be interpreted as Section 5.3 in [6].

- This paragraph is based on 6.3.6 in [6]. Recall that there is a closed immersion $\tilde{G} \to \tilde{M}$. Notice that Ker $\varphi$ is the kernel of the composition $\tilde{G} \to \tilde{M} \to \tilde{M}/\text{Ker } \tilde{\varphi}$. We define $\tilde{G}^1$ as the kernel of the composition

$$\tilde{G} \to \tilde{M} \to \tilde{M}/\tilde{M}^1.$$  

Then $\tilde{G}^1$ is the kernel of the morphism $\varphi \to \ker \tilde{\varphi}/\tilde{M}^1$ and, hence, is a closed normal subgroup of $\ker \varphi$. The induced morphism $\ker \varphi/\tilde{G}^1 \to \ker \tilde{\varphi}/\tilde{M}^1$ is a monomorphism, and thus $\ker \varphi/\tilde{G}^1$ is a closed subgroup scheme of $\ker \tilde{\varphi}/\tilde{M}^1$ by (Exp. VI$_B$, Corollary 1.4.2 in [5]).

Lemma 10.3. $\tilde{G}^1$ is connected, smooth, and unipotent.

Proof. We state the equations defining $\ker \varphi$. They are obtained by the matrix equation $\sigma(t^m)hm = h$, where $m$ is an element of $\ker \tilde{\varphi}(R)$ for a $\kappa$-algebra $R$.

We represent the given hermitian form $h$ by a hermitian matrix $\left(\pi \cdot h_i\right)$ with $(\pi \cdot h_i)$ for the $(i,i)$-block and 0 for remaining blocks.

By observing the diagonal $(i,i)$-blocks of $\sigma(t^m)hm = h$, we have the following matrix equation:

$$\sigma(t^{m_{i,i}})h_im_{i,i} + \pi(\sigma(t^{m_{i-1,i}})h_{i-1}m_{i-1,i} + \sigma(t^{m_{i+1,i}})h_{i+1}m_{i+1,i}) +$$

$$\pi^2(\sigma(t^{m_{i-2,i}})h_{i-2}m_{i-2,i} + \sigma(t^{m_{i+2,i}})h_{i+2}m_{i+2,i}) = h_i,$$

(10.4)

where $0 \leq i < N$.

By observing the $(i,j)$-blocks of $\sigma(t^m)hm = h$, where $i < j$, we have the following matrix equation:

$$\sum_{i \leq k \leq j} \sigma(t^{m_{k,i}})h_km_{k,j} + \pi \sigma(t^{m_{i-1,i}})h_{i-1}m_{i-1,j} + \pi \sigma(t^{m_{j+1,i}})h_{j+1}m_{j+1,j} = 0,$$

(10.5)

where $0 \leq i, j < N$.

The equations defining $\tilde{G}^1$ are obtained by setting

$$m_{i,j} = \pi m'_{i,j} \text{ if } i \neq j,$$

(1)
(2) If \( i \) is even and \( L_i \) is of type I',
\[
m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} = \begin{pmatrix} \text{id} + \pi s_i' & \pi^2 y_i' \\ \pi^2 v_i' & 1 + \pi^2 z_i' \end{pmatrix}.
\]

(3) If \( i \) is even and \( L_i \) is of type I',
\[
m_{i,i} = \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix} = \begin{pmatrix} \text{id} + \pi s_i' & \pi r_i' & \pi^2 t_i' \\ \pi^2 y_i' & 1 + \pi^2 x_i' & \pi^2 z_i' \\ \pi^2 v_i' & \pi u_i' & 1 + \pi^2 w_i' \end{pmatrix}.
\]

(4) If \( i \) is even and \( L_i \) is of type II, \( m_{i,i} = 1 + m_{i,i}' \).

(5) If \( i \) is even and \( L_i \) is of type I,
\[
z_i' + \delta_{i-2} k_{i-2,i} + \delta_{i+2} k_{i+2,i} = 0.
\]
Here, \( k_{i-2,i} = \pi k_{i-2,i}' \) and \( k_{i+2,i} = \pi k_{i+2,i}' \), where \( k_{i-2,i} \) and \( k_{i+2,i} \) are defined in d) of the description of \( \tilde{M}(R) \).

(6) If \( i \) is odd and \( L_i \) is free of type I, then set
\[
m_{i,i} = \begin{pmatrix} s_i & \pi r_i & t_i \\ y_i & 1 + \pi x_i & u_i \\ \pi v_i & \pi z_i & 1 + \pi w_i \end{pmatrix} = \begin{pmatrix} \text{id} + \pi s_i' & \pi^2 r_i' & \pi t_i' \\ \pi^2 y_i' & 1 + \pi^2 x_i' & \pi u_i' \\ \pi^2 v_i' & \pi^2 z_i' & 1 + \pi^2 w_i' \end{pmatrix}.
\]
If \( i \) is odd and \( L_i \) is of type II or bound of type I, then set \( m_{i,i} = \text{id} + \pi m_{i,i}' \).

(7) If \( i \) is odd and \( L_i \) is bound of type I,
\[
\begin{align*}
\delta_{i-1} v_{i-1} \cdot m_{i-1,i}' + \delta_{i+1} v_{i+1} \cdot m_{i+1,i}' = 0; \\
\delta_{i-1} v_{i-1} \cdot t m_{i,i-1}' + \delta_{i+1} v_{i+1} \cdot t m_{i,i+1}' = 0.
\end{align*}
\]
Here, notations follow those of e) and f) in the description of \( \tilde{M}(R) \).

It is then easy to check that the underlying algebraic variety of \( \tilde{G}^1 \) is simply an affine space of dimension
\[
\sum_{i<j} n_i n_j + \sum_{i: \text{even}} \frac{n_i^2 + n_i}{2} + \sum_{i: \text{odd}} \frac{n_i^2 - n_i}{2} + \sharp\{ i : i \text{ is odd and } L_i \text{ is free of type I} \} - \sharp\{ i : i \text{ is even and } L_i \text{ is of type I} \} + \sharp\{ i : i \text{ is even, } L_i \text{ is of type I and } L_{i+2} \text{ is of type II} \}.
\]
This completes the proof. \( \square \)
Lemma 10.6. \( \text{Ker } \varphi/\tilde{G}^1 \) is isomorphic to \( A^\nu \times (\mathbb{Z}/2\mathbb{Z})^\beta \) as \( \kappa \)-varieties, where \( A^\nu \) is an affine space of dimension \( l' \).

Here,
- \( l' \) is such that \( l' + \text{dimension of } \tilde{G}^1 = l \). Notice that \( l \) is defined in Lemma 7.2.
- \( \beta \) is the size of the set \( B \), where \( B \) is the set of an integer \( j \) such that \( L_j \) is of type I and \( L_{j+2}, L_{j+3}, L_{j+4} \) (resp. \( L_{j-1}, L_{j+1}, L_{j+2}, L_{j+3} \)) are of type II if \( j \) is even (resp. odd).

Before proving this lemma, we need some preparation. Assume that \( L_i \) is of type I. Observe the equation (10.4). We define the equations \( F_i \) and \( E_i \) as follows:

\[
(1) \quad F_i = \begin{cases} 
(i, i)^{th\text{-}entry \ of \ (10.4)} & \text{if } i \text{ is even and } L_i \text{ is of type I}; \\
(i-1, i-1)^{th\text{-}entry \ of \ (10.4)} & \text{if } i \text{ is odd and } L_i \text{ is free of type I}; \\
0 & \text{otherwise}.
\end{cases}
\]

\[
(2) \quad E_i = \begin{cases} 
(i-1, i-1)^{th\text{-}entry \ of \ (10.4)} & \text{if } i \text{ is even and } L_i \text{ is of type } \ell; \\
0 & \text{otherwise}.
\end{cases}
\]

Assume that \( i \) is even and \( L_i \) is of type I. Let \( h_i = \begin{pmatrix} h_i' & 0 \\ 0 & h_i'' \end{pmatrix} \), where \( h_i' \) is an \((n_i - 1) \times (n_i - 1)\)-matrix (resp. \((n_i - 2) \times (n_i - 2)\)-matrix) if \( L_i \) is of type \( \ell \) (resp. of type \( \ell' \)). Based on Theorem 2.8, if \( L_i \) is of type \( \ell' \), then \( h_i'' = \pm \begin{pmatrix} 1 & 1 \\ 1 & 2b_i \end{pmatrix} \). We denote \( b_i \mod \pi \) by \( \tilde{b}_i(\in \kappa) \).

For \( j \in B \), there is a non-negative integer \( m_j \leq j \) such that \( L_{m_j} \) is of type I, \( l \notin B \) with \( m_j \leq l < j \), and \( L_{m_j-1}, L_{m_j-2}, L_{m_j-3}, L_{m_j-4} \) (resp. \( L_{m_j-1}, L_{m_j-2}, L_{m_j-3} \)) are of type II if \( m_j \) is even (resp. odd). Let \( \tilde{z}_i = z_i \mod \pi \) if \( i \) is odd and \( L_i \) is free of type I, where \( z_i \) is denoted in b) of the description \( M(R) \). Then the sum of equations

\[
\sum_{l=m_j}^{j} F_l + \sum_{m_j \leq l \leq j, \ L_l \text{ is of type } \ell^c \text{ with } b_l \text{ unit}} \tilde{b}_l \cdot E_l
\]

becomes

\[
\sum_{m_j \leq l \leq j, \ L_l \text{ is of type } I} (\tilde{z}_l + \tilde{z}_l^2) + \sum_{m_j \leq l \leq j, \ L_l \text{ is of type } \ell' \text{ with } b_l \text{ unit}} (\tilde{b}_l u_l + \tilde{b}_l^2 u_l^2) \mod \pi = 0.
\]

Let us provide the proof of Lemma 10.6 below.

Proof. Lemma 9.1 and Lemma 10.3 imply that \( \text{Ker } \varphi/\tilde{G}^1 \) represents the functor \( R \mapsto \text{Ker } \varphi(R)/\tilde{G}^1(R) \). If \( m = \left( \pi^{\max\{0, j-i\}}_{m_{i,j}} \right) \in (\text{Ker } \varphi/\tilde{M}^1)(R) \) is such that \( m \in (\text{Ker } \varphi/\tilde{G}^1)(R) \), then \( m \) satisfies the following equations (which are given as equalities in \( R \)):
\( \sum_{i \leq k \leq j} \sigma(t^i m_{k,i}) h_k m_{k,j} = 0 \) if \( i < j \),

Assume that \( i \) is even.

(2) Choose an even integer \( i \) such that \( L_i \) is of type I and \( L_{i+2} \) is of type II. For such \( i \), there is a non-negative integer \( m'_i \) such that \( L_{i-2l} \) is of type I for every \( l \) with \( 0 \leq l \leq m'_i \) and \( L_{i-2(m'_i+1)} \) is of type II. Then we have the equation \( \sum_{l=0}^{m'_i} F_{i-2l} \).

(3) \( m_{i,i} = id \) if \( L_i \) is of type II, and \( s_i = id \) if \( L_i \) is of type I.

(4) \( \mathcal{E}_i \) if \( L_i \) is of type \( I' \).

(5) \( \delta_i v_i \) (or \( \delta_i (y_i + \tilde{b}_i v_i) \) if \( L_i \) is of type \( I' \)) + \( (\delta_i - 2 \epsilon_i - 2 \cdot \epsilon_i - 2 \cdot \epsilon_i - 2 \cdot m_{i+1,i} + \delta_{i+2} \epsilon_{i+2} \cdot m_{i+2,i}) \tilde{e}_i = 0 \).

Here, notations follow those of c) in the description of \( \tilde{\varphi}(R) \).

(6) If \( L_i \) is of type I, \( z_i + \delta_{i-2} m_{i-2,i} + \delta_{i+2} m_{i+2,i} = 0 \). Here, notations follow those of d) in the description of \( \tilde{M}(R) \).

(7) If \( L_i \) is of type \( I' \), \( \pi^{-1} (\pi h'_i y_i + \sigma(\pi) \cdot t^i v_i) + P^i_{1,2} = 0 \).

Here, \( P^i_{1,2} \) is a polynomial with variables \( m_{i-1,i}, m_{i+1,i} \).

(8) If \( L_i \) is of type \( I' \),

\[
\begin{align*}
\left\{ \begin{array}{ll}
h'_i r_i + t^i v_i &= 0; \\
\pi^{-1} (\pi h'_i t + \sigma(\pi) \cdot t^i y_i + \pi \cdot t^i v z) + P^i_{1,3} &= 0; \\
\pi^{-1} (\pi \cdot t^i h'_i t + \pi z + \pi w + \sigma(\pi) x + \pi u z) + P^i_{2,3} &= 0.
\end{array} \right.
\]

Here, \( P^i_{1,3}, P^i_{2,3} \) are suitable polynomials with variables \( m_{i-1,i}, m_{i+1,i} \).

Assume that \( i \) is odd.

(9) \( m_{i,i} = id \) if \( L_i \) is of type II or bound of type I.

(10) If \( L_i \) is bound of type I,

\[
\begin{align*}
\left\{ \begin{array}{ll}
t^i m_{i,i} &= h_i \cdot \tilde{m}_{i,i}^i; \\
\delta_{i-1} v_{i-1} \cdot m_{i-1,i} + \delta_{i+1} v_{i+1} \cdot m_{i+1,i} &= 0.
\end{array} \right.
\]

Here, notations follow those of e) in the description of \( \tilde{M}(R) \).
(11) If $L_i$ is free of type I, we denote $m_{i,i} = \begin{pmatrix} s_i & \pi r_i & t_i \\ y_i & 1 + \pi x_i & z_i \\ v_i & \pi u_i & 1 + \pi w_i \end{pmatrix}$. Let $h_i = \begin{pmatrix} h'_i & 0 \\ 0 & h''_i \end{pmatrix}$, where $h'_i$ is $(n_i - 2) \times (n_i - 2)$-matrix. Then

\[
\begin{align*}
  s_i &= \text{id}; \\
  h'_i \cdot t_i &= y_i; \\
  h'_i \cdot r_i + t' y_i u_i &= v_i; \\
  t'_i h'_i \cdot t_i + w_i + x_i + u_i z_i &= 0.
\end{align*}
\]

(12) $F_i$ if $L_i$ is free of type I.

Notice that the equations (2), (4), (12) give the following equation:

(13) For $j \in \mathcal{J}$,

\[
\sum_{m_j \leq l \leq j, \text{ $L_i$: of type I}} (\bar{z}_l + \bar{z}_l^2) + \sum_{m_j \leq l \leq j, \text{ $L_i$: of type II \ and \ $b_i \in (2) \setminus (4)$}} (\bar{b}_l u_l + b_i u_i^2) = 0.
\]

Let $G^\dagger$ be the subfunctor of Ker $\tilde{\varphi}/\tilde{M}^1$ consisting of those $(m_{i,j})$ satisfying the above equations. Then it is easy to check that $G^\dagger$ is represented by a smooth closed subscheme of Ker $\tilde{\varphi}/\tilde{M}^1$ and is isomorphic to $A'' \times (\mathbb{Z}/2\mathbb{Z})^\beta$ as $\kappa$-varieties, where $A''$ is an affine space of dimension $l'$. Indeed,

\[
l' = \sum_{i < j} n_i n_j + \sum_{i: \text{even \ and } L_i: \text{of type I}} (n_i - 1) + \sum_{i: \text{odd \ and } L_i: \text{free \ of \ type I}} (2n_i - 2) - \sum_{i: \text{even \ and } L_i: \text{bound \ of \ type II}} n_i + \sharp \{ i : i \text{ is even and } L_i \text{ is of type I} \} - \sharp \{ i : i \text{ is even, } L_i \text{ is of type I and } L_{i+2} \text{ is of type II} \}.
\]

(10.7)

For ease of notation, let $G^\dagger = \text{Ker } \varphi/\tilde{G}^1$. Since $G^\dagger$ and $G^\dagger$ are both closed subschemes of Ker $\tilde{\varphi}/\tilde{M}^1$ and $G^\dagger(\tilde{k}) \subset G^\dagger(\tilde{k})$, $(G^\dagger)^{\text{red}}$ is a closed subscheme of $(G^\dagger)^{\text{red}} = G^\dagger$. It is easy to check that $\dim G^\dagger = \dim G^\dagger$.

Claim that $(G^\dagger)^{\text{red}}$ contains at least one (closed) point of each connected component of $G^\dagger$. For $j \in \mathcal{B}$, consider the closed subgroup scheme $F_j$ of $\tilde{G}$ defined by the following equations:

- $m_{i,k} = 0$ if $i \neq k$,
- $m_{i,i} = \text{id}$ if $i \neq j$, 

and for $m_{j,j}$

\[
\begin{cases}
  s_j = \text{id}, y_j = 0, v_j = 0 & \text{if } i \text{ is even and } L_i \text{ is of type } F_i; \\
  s_j = \text{id}, r_j = t_j = y_j = v_j = u_j = w_j = 0 & \text{if } i \text{ is even and if } L_i \text{ is of type } I_i; \\
  s_j = \text{id}, r_j = t_j = y_j = v_j = 0 & \text{if } i \text{ is odd and if } L_i \text{ is of type } I_i.
\end{cases}
\]

Then $F_j$ has exactly two connected components. Notice that $F_j$ and $F_{j'}$ commute with each other for all pair integers $(j, j')$ such that $j \neq j'$ and $j, j' \in B$, in the sense that $f_j \cdot f_{j'} = f_{j'} \cdot f_j$, where $f_j \in F_j$ and $f_{j'} \in F_{j'}$.

Let $F = \prod_j F_j$. Notice that $F$ is smooth and is a closed subgroup scheme of Ker $\varphi$. Let $F^\dagger$ be the image of $F$ in $(G^\dagger)\text{red}$. Then $F^\dagger$ contains at least one (closed) point of each connected component of $G^\dagger$ and this justifies our claim.

Combining this fact with $\dim G^\dagger = \dim G^\dagger$, we conclude that $(G^\dagger)\text{red} \simeq G^\dagger$, and hence, $G^\dagger = G^\dagger$ because $G^\dagger$ is a subfunctor of $G^\dagger$. This completes the proof. □

We now prove Lemma 7.2.

**Proof.** We start with the following short exact sequence

\[1 \to \tilde{G}^\dagger \to \text{Ker } \varphi \to \text{Ker } \varphi/\tilde{G}^\dagger \to 1.\]

It is obvious that Ker $\varphi$ is smooth and unipotent by Lemma 10.3 and Lemma 10.6. Since $\tilde{G}^\dagger$ is connected by Lemma 10.3, the number of connected components of Ker $\varphi$ is the same as that of Ker $\varphi/\tilde{G}^\dagger$ by Lemma 9.8. Moreover, the dimension of Ker $\varphi$ is the sum of dimensions of $\tilde{G}^\dagger$ and Ker $\varphi/\tilde{G}^\dagger$. This completes the proof. □

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