NONLINEAR FRAMES AND SPARSE RECONSTRUCTIONS IN BANACH SPACES

QIYU SUN AND WAI-SHING TANG

Abstract. In the first part of this paper, we consider nonlinear extension of frame theory by introducing bi-Lipschitz maps $F$ between Banach spaces. Our linear model of bi-Lipschitz maps is the analysis operator associated with Hilbert frames, $p$-frames, Banach frames, $g$-frames and fusion frames. In general Banach space setting, stable algorithm to reconstruct a signal $x$ from its noisy measurement $F(x) + \epsilon$ may not exist. In this paper, we establish exponential convergence of two iterative reconstruction algorithms when $F$ is not too far from some bounded below linear operator with bounded pseudo-inverse, and when $F$ is a well-localized map between two Banach spaces with dense Hilbert subspaces. The crucial step to prove the later conclusion is a novel fixed point theorem for a well-localized map on a Banach space.

In the second part of this paper, we consider stable reconstruction of sparse signals in a union $A$ of closed linear subspaces of a Hilbert space $H$ from their nonlinear measurements. We create an optimization framework called sparse approximation triple $(A, M, H)$, and show that the minimizer $x^* = \arg\min_{\hat{x} \in M \text{ with } \|F(\hat{x}) - F(x^0)\| \leq \epsilon \|\hat{x}\|_M}$ provides a suboptimal approximation to the original sparse signal $x^0 \in A$ when the measurement map $F$ has the sparse Riesz property and almost linear property on $A$. The above two new properties is also discussed in this paper when $F$ is not far away from a linear measurement operator $T$ having the restricted isometry property.

1. Introduction

For a Banach space $B$, we denote its norm by $\|\cdot\|_B$. A map $F$ from one Banach space $B_1$ to another Banach space $B_2$ is said to have bi-Lipschitz property if there exist two positive constants $A$ and $B$ such that

$$A\|x - y\|_{B_1} \leq \|F(x) - F(y)\|_{B_2} \leq B\|x - y\|_{B_1} \quad \text{for all } x, y \in B_1.$$

Our models of bi-Lipschitz maps between Banach spaces are analysis operators associated with Hilbert frames, $p$-frames, Banach frames, $g$-frames and fusion frames [1, 15, 16, 17, 53]. Our study is also motivated by nonlinear sampling.
theory and phase retrieval, which have gained substantial attention in recent years \cite{2 8 14 21 24 37 48}. The framework developed in the first part of this paper could be considered as a nonlinear extension of frame theory.

Denote by $B(B_1, B_2)$ the Banach space of all bounded linear operators from one Banach space $B_1$ to another Banach space $B_2$. A continuous map $F$ from $B_1$ to $B_2$ is said to be differentiable at $x \in B_1$ if there exists a linear operator, denoted by $F'(x)$, in $B(B_1, B_2)$ such that

$$\lim_{y \to 0} \frac{\|F(x + y) - F(x) - F'(x)y\|_{B_2}}{\|y\|_{B_1}} = 0;$$

and to be differentiable on $B_1$ if it is differentiable at every $x \in B_1$ \cite{20}. For a differentiable bi-Lipschitz map $F$ from $B_1$ to $B_2$, one may easily verify that its derivatives $F'(x), x \in B_1$, are uniformly stable, i.e., there exist two positive constants $A$ and $B$ such that

$$A\|y\|_{B_1} \leq \|F'(x)y\|_{B_2} \leq B\|y\|_{B_1} \text{ for all } x, y \in B_1. \tag{1.2}$$

The converse is not true in general. Then we have the following natural question.

**Question 1:** When does a differentiable map with the uniform stability property (1.2) have the bi-Lipschitz property (1.1)?

We say that a linear operator $T \in B(B_1, B_2)$ from one Banach space $B_1$ to another Banach space $B_2$ is bounded below if

$$\inf_{0 \neq y \in B_1} \frac{\|Ty\|_{B_2}}{\|y\|_{B_1}} > 0. \tag{1.3}$$

For a continuously differentiable map $F$ not too nonlinear, particularly not far away from a bounded below linear operator $T$, a sufficient condition for (1.1) is that for any $0 \neq y \in B_1$, the set $B(y)$ of unit vectors $F'(x)y/\|F'(x)y\|_{B_2}, x \in B_1$, is contained in a ball of radius

$$\beta_{F,T} < 1 \tag{1.4}$$

with center at $Ty/\|Ty\|_{B_2}$, where

$$\beta_{F,T} := \sup_{0 \neq y \in B_1} \sup_{x \in B_1} \frac{\|F'(x)y\|_{B_2}}{\|F'(x)y\|_{B_2}} - \frac{Ty}{\|Ty\|_{B_2}} \|B_2\| \tag{1.5}.$$ 

The above geometric requirement on the radius $\beta_{F,T}$ is optimal in Banach space setting, but it could be relaxed to

$$\beta_{F,T} < \sqrt{2} \tag{1.6}$$

in Hilbert space setting, which implies that for any $0 \neq y \in B_1$, the set $B(y)$ is contained in a right circular cone with axis $Ty/\|Ty\|_{B_2}$ and angle strictly less than $\pi/2$. Detailed arguments of the above conclusions on a differentiable map are given in Appendix [A].

Denote by $F(B_1) \subset B_2$ the image of a map $F$ from one Banach space $B_1$ to another Banach space $B_2$. For a bi-Lipschitz map $F : B_1 \to B_2$, as it is one-to-one, for any $y \in F(B_1)$ there exists a unique $x \in B_1$ such that $F(x) = y$. Our next question is as follows:
**Question 2:** Given noisy observation $z_\epsilon = F(x^0) + \epsilon$ of $x^0 \in B_1$, corrupted by $\epsilon \in B_2$, how to construct a suboptimal approximation $x \in B_1$ such that

\[(1.7) \quad \|x - x^0\|_{B_1} \leq C\|\epsilon\|_{B_2},\]

where $C$ is an absolute constant independent of $x^0 \in B_1$ and $\epsilon \in B_2$?

For a differentiable bi-Lipschitz map $F$ not far away from a bounded below linear operator $T$, define $x_n, n \geq 0$, iteratively with arbitrary initial $x_0 \in B_1$ by

\[(1.8) \quad x_{n+1} = x_n - \mu T^\dagger (F(x_n) - z_\epsilon), \quad n \geq 0,\]

where $T^\dagger$ is a bounded left-inverse of the linear operator $T$, and the relaxation factor $\mu$ satisfies $0 < \mu \leq (\sup_{x \in B_1} \sup_{y \neq 0} \|F'(x)y\|_{B_2}/\|Ty\|_{B_2})^{-1}$. In Theorem 2.1 of Section 2, we show that the sequence $x_n, n \geq 0$, in the iterative algorithm (1.8) converges exponentially to a suboptimal approximation element $x \in B_1$ satisfying (1.7), provided that

\[(1.9) \quad \beta_{F,T} < (\|T\|_{B(\mathbb{B}_1,\mathbb{B}_2)}\|T^\dagger\|_{B(\mathbb{B}_2,\mathbb{B}_1)})^{-1}.\]

The above requirement (1.9) about $\beta_{F,T}$ to guarantee convergence of the iterative algorithm (1.8) is stronger than the sufficient condition (1.4) for the bi-Lipschitz property of the map $F$. In Theorem 2.2 we close that requirement gap on $\beta_{F,T}$ in Hilbert space setting by introducing an iterative algorithm of Van-Cittert type,

\[(1.10) \quad u_{n+1} = u_n - \mu T^* (F(u_n) - z_\epsilon), \quad n \geq 0,\]

where $T^*$ is the conjugate of the linear operator $T$ and $\mu > 0$ is a small relaxation factor.

In the iterative algorithm (1.8), a left-inverse $T^\dagger$ of the bounded below linear operator $T$ is used, but its existence is not always assured in Banach space setting and its construction is not necessarily attainable even it exists. This limits applicability of the iterative reconstruction algorithm (1.8). In fact, for general Banach space setting, a stable reconstruction algorithm may not exist [15, 18]. On the other hand, a stable iterative algorithm is proposed in [18] to find sub-optimal approximation for well-localized nonlinear maps on sequence spaces $\ell^p(\mathbb{Z}), 2 \leq p \leq \infty$. So we have the following question.

**Question 3:** For what types of Banach spaces $B_1$ and $B_2$ and nonlinear maps $F$ from $B_1$ to $B_2$ does there exist a stable reconstruction of $x^0 \in B_1$ from its nonlinear observation $y = F(x^0) \in B_2$?

We say that a Banach space $B$ with norm $\|\cdot\|_B$ is Hilbert-dense (respectively weak-Hilbert-dense) if there exists a Hilbert subspace $H \subset B$ with norm $\|\cdot\|_H$ such that $H$ is dense in $B$ in the strong topology (respectively in the weak topology) of $B$ and

\[\sup_{0 \neq x \in H} \frac{\|x\|_B}{\|x\|_H} < \infty.\]

Our models of the above new concepts are the sequence spaces $\ell^p, 2 \leq p \leq \infty$, for which $\ell^p$ with $2 \leq p < \infty$ are Hilbert-dense and $\ell^\infty$ is weak-Hilbert-dense. For (weak-)Hilbert-dense Banach spaces $B_1$ and $B_2$ and a nonlinear map $F : B_1 \rightarrow B_2$ that has certain localization property, a stable reconstruction of $x^0 \in B_1$ from its nonlinear observation $y = F(x^0) \in B_2$ is proposed in Theorem 3.1 of Section 3.
The crucial step is a new fixed point theorem for a well-localized differentiable map whose restriction on a dense Hilbert subspace is a contraction, see Theorem 3.2. Let \( H_1 \) and \( H_2 \) be Hilbert spaces, and let \( A = \bigcup_{i \in I} A_i \) be union of closed linear subspaces \( A_i, i \in I \), of the Hilbert space \( H_1 \). The second topic of this paper is to study the restricted bi-Lipschitz property of a map \( F : H_1 \to H_2 \) on \( A \), which means that there exist two positive constants \( A \) and \( B \) such that

\[
(1.11) \quad A\|x - y\|_{H_1} \leq \|F(x) - F(y)\|_{H_2} \leq B\|x - y\|_{H_1} \quad \text{for all } x, y \in A.
\]

This topic is motivated by sparse recovery problems on finite-dimensional spaces \([12, 14, 23, 26]\). As we use the union \( A \) of closed linear spaces \( A_i, i \in I \), to model sparse signals, the restricted bi-Lipschitz property of a map \( F \) could be thought as nonlinear correspondence of restricted isometric property of a measurement matrix. So the framework developed in the second part is nonlinear Banach space extension of the finite-dimensional sparse recovery problems.

In the classical sparse recovery setting \([12, 14, 23, 26]\), the set of all \( s \)-sparse signals for some \( s \geq 1 \) is used as the set \( A \). In this case, elements in \( A \) can be described by their \( \ell^0 \)-quasi-norms being less than or equal to \( s \), and the sparse recovery problem could reduce to the \( \ell^0 \)-minimization problem. Due to numerical infeasibility of the \( \ell^0 \)-minimization, a relaxation to (non-)convex \( \ell^q \)-minimization with \( 0 < q \leq 1 \) was proposed, and more importantly it was proved that the \( \ell^q \)-minimization recovers sparse signals when the linear measurement operator has certain restricted isometry property in \( \ell^2 \) \([12, 14, 27, 26, 52]\). This leads to the following question.

**Question 4:** How to create a general optimization framework to recover sparse signals?

Given a Banach space \( M \), we say that a subset \( K \) of \( M \) is proximinal \([10, 38]\) if every element \( x \in M \) has a best approximator \( y \in K \), that is,

\[
\|x - y\|_M = \inf_{z \in K} \|x - z\|_M =: \sigma_{K,M}(x).
\]

Given Hilbert spaces \( H_1 \) and \( H_2 \), a union \( A = \bigcup_{i \in I} A_i \) of closed linear subspaces \( A_i, i \in I \), of \( H_1 \), and a continuous map \( F \) from \( H_1 \) to \( H_2 \), consider the following minimization problem in a Banach space \( M \),

\[
(1.12) \quad x^* = \arg\min_{x \in M} \|F(x) - z\|_M
\]

for any given observation \( z := F(x) \) for some \( x \in A \). To make the above minimization problem suitable for stable reconstruction of \( x \in A \) from its observation \( F(x) \), we introduce the concept of a sparse approximation triple \( (A, M, H_1) \):

(i) (Continuous imbedding property) The Banach space \( M \) contains all elements in \( A \) and it is contained in the Hilbert space \( H_1 \), that is,

\[
(1.13) \quad A \subset M \subset H_1,
\]

and the imbedding operators \( i_A : A \to M \) and \( i_M : M \to H_1 \) are bounded.

(ii) (Proximinality property) The Banach space \( M \) has \( A \) as its closed subset, and all closed subsets of \( M \) being proximinal.
(iii) (Common-best-approximator property) Given any \( i \in I \), a best approximator \( x_{A,M} := \arg \min_{\hat{x} \in A} \| \hat{x} - x \|_M \) of \( x \in M \) in the norm \( \| \cdot \|_M \) is also a best approximator in the norm \( \| \cdot \|_{H_1} \), that is,
\[
(1.14) \quad x_{A,M} = \arg \min_{\hat{x} \in A} \| \hat{x} - x \|_{H_1}.
\]

(iv) (Norm-splitting property) For the best approximator \( x_{A,M} \) of \( x \in M \) in the norm \( \| \cdot \|_M \),
\[
(1.15) \quad \| x \|_M = \| x_{A,M} \|_M + \| x - x_{A,M} \|_M,
\]
and
\[
(1.16) \quad \| x \|_{H_1}^2 = \| x_{A,M} \|_{H_1}^2 + \| x - x_{A,M} \|_{H_1}^2.
\]

(v) (Sparse density property) \( \cup_{k \geq 1} kA \) is dense in \( H_1 \), where
\[
kA := \underbrace{A + A + \cdots + A}_{k \text{ times}} = \{ \sum_{i=1}^k x_i : x_1, \ldots, x_k \in A \}, \quad k \geq 1.
\]

One may easily verify that these five properties are satisfied for the triple \((A, M, H_1)\) in the classical sparse recovery setting, where \( A \) is the set of all \( s \)-sparse vectors, \( M \) is the set of all summable sequences, and \( H_1 \) is the set of all square-summable sequences \([12, 14, 23, 26]\).

In this paper, we rescale the norm \( \| \cdot \|_M \) in the sparse approximation triple \((A, M, H_1)\) so that the imbedding operator \( i_M \) has norm one,
\[
(1.17) \quad \| i_M \|_{B(M,H_1)} = 1,
\]
 otherwise replacing it by \( \| \cdot \|_M \| i_M \|_{B(M,H_1)} \). Next we introduce two quantities to measure sparsity
\[
(1.18) \quad s_A := \| i_A \|_{B(A,M)}^2 = \left( \sup_{0 \neq x \in A} \frac{\| x \|_M}{\| x \|_{H_1}} \right)^2
\]
for signals in \( A \), and sparse approximation ratio
\[
(1.19) \quad a_A := \sup_{0 \neq x \in M} \left( \frac{\| u_{A,M} \|_{H_1}}{\| x_{A,M} \|_M} \right)^2 \leq 1
\]
for elements in \( M \), where \( x_{A,M} \) and \( u_{A,M} \in A \) are the first and second best approximators of \( x \) respectively,
\[
(1.20) \quad \| x - x_{A,M} \|_M = \sigma_{A,M}(x) \text{ and } \| x - x_{A,M} - u_{A,M} \|_M = \sigma_{A,M}(x - x_{A,M}).
\]
The upper bound estimate in (1.19) holds, since
\[
\| u_{A,M} \|_M = \| x - x_{A,M} \|_M - \| x - x_{A,M} - u_{A,M} \|_M \leq \| x - u_{A,M} \|_M - \| x - x_{A,M} - u_{A,M} \|_M \leq \| x_{A,M} \|_M, \quad x \in M,
\]
by the norm splitting property (1.15). In the classical sparse recovery setting with \( A \) being the set of all \( s \)-sparse signals, one may verify that \( s_A = s \) and \( a_A = 1/s \), see Appendix [3] for additional properties of sparse approximation triples.

Having introduced the sparse approximation triple \((A, M, H_1)\), our next question is on optimization approach to sparse signal recovery, see [5, 9, 7, 8, 9, 22, 25, 39] for the classical setting.
Question 5: Given a sparse approximation triple \( (A, M, H_1) \), for what type of maps \( F : H_1 \to H_2 \) does the solution \( x_M^0 \) of the optimization problem

\[
x_M^0 := \arg\min_{x \in M} \|F(x) - F(x^0)\|_{H_2} \quad \text{with} \quad \|F(x) - F(x^0)\|_{H_2} \leq C_1 \sqrt{a_A} \sigma_{A,M}(x)
\]

is a suboptimal approximation to the sparse signal \( x^0 \) in \( A \)?

In this paper, without loss of generality, we assume that \( F(0) = 0 \). We say that \( F : H_1 \to H_2 \) has sparse Riesz property if

\[
\|F(x)|_{H_2} \geq D^{-1}(\|x\|_{H_1} - \beta \sqrt{a_A} \sigma_{A,M}(x)), \quad x \in M,
\]

with \( D, \beta > 0 \), and that \( F \) is almost linear on \( A \) if

\[
\|F(x) - F(y) - F(x - y)\|_{H_2} \leq \gamma_1 \|x - y\|_{H_1} + \gamma_2 \sqrt{a_A}(\sigma_{A,M}(x) + \sigma_{A,M}(y)), \quad x, y \in M,
\]

with \( \gamma_1, \gamma_2 \geq 0 \). Combining the sparse Riesz property and almost linear property of a map \( F \) gives

\[
\|F(x) - F(y)\|_{H_2} \geq (D^{-1} - \gamma_1)\|x - y\|_{H_1} - (D^{-1} + \gamma_2) \sqrt{a_A}(\sigma_{A,M}(x) + \sigma_{A,M}(y)), \quad x, y \in M,
\]

and hence \( F \) has the restricted bi-Lipschitz property on \( A \),

\[
\|F(x) - F(y)\|_{H_2} \geq (D^{-1} - \gamma_1)\|x - y\|_{H_1}, \quad x, y \in A,
\]

when \( \gamma_1 \) and \( D \) satisfy \( D\gamma_1 < 1 \). In Section 4, we show that the solution \( x_M^0 \) of the optimization problem (1.21) is a suboptimal approximation to the signal \( x^0 \) in \( M \), i.e., there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
\|x_M^0 - x^0\|_{H_1} \leq C_1 \sqrt{a_A} \sigma_{A,M}(x^0) + C_2 \epsilon,
\]

provided that \( F \) has the sparse Riesz property (1.22) and almost linear property (1.23) with \( D, \beta, \gamma_1 \) and \( \gamma_2 \) satisfying

\[
1 - 2D\gamma_1 - (D\gamma_1 + D\gamma_2 + \beta) \sqrt{a_A} s_A > 0.
\]

We remark that the approximation error estimate (1.25) implies the sparse Riesz property (1.22) for the map \( F \),

\[
\|F(x)|_{H_2} \geq C_2^{-1}(\|x\|_{H_1} - C_1 \sqrt{a_A} \sigma_{A,M}(x)), \quad x \in M,
\]

which follows from (1.25) by taking \( x^0 = x \) and \( \epsilon = \|F(x^0)\|_{H_2} \).

The sparse Riesz property was introduced in [51] with a different name, sparse approximation property, for the classical sparse recovery setting; and the almost linear property was studied in [28, 34] for bi-Lipschitz maps between Banach spaces. In Section 5, we consider the following question.

Question 6: When does a map \( F : H_1 \to H_2 \) have the sparse Riesz property (1.22) and the almost linear property (1.23)?

We say that a linear operator \( T \in B(H_1, H_2) \) has the restricted isometry property (RIP) on \( 2A \) if

\[
(1 - \delta_2 A(T)) \|z\|_{H_1}^2 \leq \|Tz\|_{H_2}^2 \leq (1 + \delta_2 A(T)) \|z\|_{H_1}^2
\]

for all \( z \in 2A \), where \( \delta_2 A(T) \in [0, 1) \) [12, 14]. A nonlinear map \( F : H_1 \to H_2 \), not far away from a linear operator \( T \) with the restricted isometry property (1.26) in the sense that

\[
\gamma_{F,T}(2A) < \sqrt{1 - \delta_2 A(T)}
\]
has the restricted bi-Lipschitz property (1.11) on $A$, where
\begin{equation}
\gamma_{F,T}(kA) := \sup_{x \in M} \sup_{z \in kA} \|F(x + z) - F(x) - Tz\|_{H^2}, \quad k \geq 1.
\end{equation}

In Section 5, we show that $F$ has the sparse Riesz property (1.22) when
\begin{equation}
\gamma_{F,T}(2A) < \sqrt{\frac{2}{2 - \sqrt{\delta^2}}},
\end{equation}
and the almost linear property (1.23) when $\gamma_{F,T}(4A) < \infty$. Therefore signals $x \in M$ could be reconstructed from their nonlinear measurements $F(x)$ when $F$ is not far from a linear operator $T$ with small restricted isometry constant $\delta^2(T)$, see Theorem 5.4.

2. Iterative Reconstruction Algorithms

For a Banach/Hilbert space $B$, we also denote its norm by $\| \cdot \|$ for brevity. In this section, we establish exponential convergence of the iterative reconstruction algorithms (1.8) and (1.10).

**Theorem 2.1.** Let $B_1$ and $B_2$ be Banach spaces, $F$ be a differentiable map from $B_1$ to $B_2$ with its derivative being continuous and uniformly stable, and let $T \in \mathcal{B}(B_1, B_2)$ be bounded below. Assume that (1.9) holds for some bounded left-inverse $T^\dagger : B_2 \to B_1$ of the linear operator $T$. Given positive relaxation factor $\mu > 0$, an initial $x_0 \in B_1$ and a noisy observation data $z_\epsilon := F(x^0) + \epsilon \in B_2$ for some $x^0 \in B_1$ with additive noise $\epsilon \in B_2$, define $x_n, n \geq 1$, iteratively by (1.8). Then $x_n, n \geq 0$, converges exponentially to some $x_\infty \in B_1$ with
\begin{equation}
\|x_\infty - x^0\| \leq \frac{\|T\|}{1 - \beta_{F,T}\|T\|\|T^\dagger\|} \left( \inf_{x \in B_1} \inf_{0 \neq y \in B_1} \frac{\|F'(x)y\|}{\|Ty\|} \right)^{-1} \|\epsilon\|,
\end{equation}
provided that
\begin{equation}
0 < \mu \leq \left( \sup_{x \in B_1} \sup_{0 \neq y \in B_1} \frac{\|F'(x)y\|}{\|Ty\|} \right)^{-1}.
\end{equation}
Moreover,
\begin{equation}
\|x_n - x_\infty\| \leq \frac{\|T\|\|F(x_0) - z_\epsilon\|}{1 - \beta_{F,T}\|T\|\|T^\dagger\|} \left( \inf_{x \in B_1} \inf_{0 \neq y \in B_1} \frac{\|F'(x)y\|}{\|Ty\|} \right)^{-1} r_0^n, \quad n \geq 1,
\end{equation}
where
\begin{equation}
r_0 = 1 - \mu (1 - \beta_{F,T}\|T\|\|T^\dagger\|) \left( \inf_{x \in B_1} \inf_{0 \neq y \in B_1} \frac{\|F'(x)y\|}{\|Ty\|} \right) \in (0, 1).
\end{equation}
Proof. Set $\alpha_n = \int_0^1 \|F'(x_{n-1} + t(x_n - x_{n-1}))(x_n - x_{n-1})\| dt, n \geq 1$. Then for $n \geq 1$,
\[
\|x_{n+1} - x_n\| = \|x_n - x_{n-1} - \mu T^\dagger (F(x_n) - F(x_{n-1}))\| \\
\leq \|x_n - x_{n-1} - \mu \alpha_n T^\dagger T(x_n - x_{n-1})\| \\
+ \mu \int_0^1 \|F'(x_{n-1} + t(x_n - x_{n-1}))(x_n - x_{n-1})\| \\
\times \|T^\dagger \left( \frac{F'(x_{n-1} + t(x_n - x_{n-1}))(x_n - x_{n-1})}{\|T(x_n - x_{n-1})\|} - \frac{T(x_n - x_{n-1})}{\|T(x_n - x_{n-1})\|} \right) \| dt \\
\leq \left( 1 - \mu \frac{\alpha_n}{\|T(x_n - x_{n-1})\|} \right) \|x_n - x_{n-1}\| + \mu \alpha_n \beta F,T \|T^\dagger\| \\
\leq \left( 1 - \mu (1 - \beta F,T \|T^\dagger\|) \frac{\alpha_n}{\|T(x_n - x_{n-1})\|} \right) \|x_n - x_{n-1}\| \\
\leq r_0 \|x_n - x_{n-1}\|,
\]
by (1.8), (1.9) and (2.2). This proves the exponential convergence of $x_n, n \geq 0$, to its limit $x_\infty \in B_1$.

Taking limit in (1.8) gives
\[
(2.4) \quad T^\dagger (F(x_\infty) - F(x^0)) = T^\dagger \epsilon,
\]
because
\[
\|T^\dagger (F(x_\infty) - z_\epsilon)\| \leq \|T^\dagger (F(x_\infty) - F(x_n))\| + \|x_{n+1} - x_n\|/\mu \to 0, \ n \to \infty.
\]
Then it follows from (1.9), (2.2) and (2.4) that
\[
\left( \int_0^1 \|F'(x_{\infty} + t(x^0 - x_{\infty}))(x^0 - x_{\infty})\| dt \right) \|x^0 - x_{\infty}\| = \|T^\dagger \left( \int_0^1 \|F'(x_{\infty} + t(x^0 - x_{\infty}))(x^0 - x_{\infty})\| \|T(x^0 - x_{\infty})\| dt \right) \|
\leq \beta F,T \|T^\dagger\| \left( \int_0^1 \|F'(x_{\infty} + t(x^0 - x_{\infty}))(x^0 - x_{\infty})\| dt \right) \\
+ \|T^\dagger \int_0^1 F'(x_{\infty} + t(x^0 - x_{\infty}))(x^0 - x_{\infty}) dt \|
\leq \beta F,T \|T^\dagger\| \|T^\dagger\| \left( \int_0^1 \|F'(x_{\infty} + t(x^0 - x_{\infty}))(x^0 - x_{\infty})\| dt \right) \\
\times \|x^0 - x_{\infty}\| + \|T^\dagger\| \|\epsilon\|,
\]
which proves (2.1).

Observe that
\[
\|x_n - x_{\infty}\| \leq \sum_{k=n}^{\infty} \|x_{k+1} - x_k\| \leq \frac{\|x_1 - x_0\|}{1 - r_0} r_n \leq \frac{\mu \|T^\dagger\| \|F(x_0) - z_\epsilon\|}{1 - r_0} r_0.
\]
Then the estimate (2.3) follows. \qed
The iterative algorithm (1.8) in Theorem 2.1 provides a stable reconstruction of $x \in B_1$ from its noisy observation $F(x) + \epsilon \in B_2$ when $\beta_{F,T} < (\|T\|\|T^*\|)^{-1}$, a requirement stronger than $\beta_{F,T} < 1$ that guarantees the bi-Lipschitz property for the map $F$, see Theorem A.3 in Appendix A. Next we close that requirement gap on $\beta_{F,T}$ in Hilbert space setting, cf. Theorem A.7.

**Theorem 2.2.** Let $H_1$ and $H_2$ be Hilbert spaces, $F$ be a differentiable map from $H_1$ to $H_2$ with its derivative being continuous and satisfying (1.2), and let $T \in B(H_1, H_2)$ satisfy (1.3) and (1.6). Given relaxation factor $\mu > 0$, an initial $u_0 \in H_1$, and noisy data $z_\epsilon := F(u^\epsilon) + \epsilon$ for some $u^\epsilon \in H_1$ with additive noise $\epsilon \in H_2$, define $u_n, n \geq 1$, iteratively by (1.10). Then $u_n, n \geq 0$, converges exponentially to some $u_\infty \in H_1$ with

$$
\|u_\infty - u^0\| \leq \frac{2\|T\|}{(2 - \beta_{F,T}^2)(\inf_{\|u\|=1} \|Tu\|)(\inf_{v \in H_1} \inf_{\|u\|=1} \|F'(v)u\|)} \|\epsilon\|,
$$

provided that

$$
0 < \mu < (2 - \beta_{F,T}^2)(\inf_{\|u\|=1} \|Tu\|)(\inf_{v \in H_1} \inf_{\|u\|=1} \|F'(v)u\|) \|T\|^2 (\sup_{v \in H_1} \|F'(v)\|)^2.
$$

**Proof.** Define $S := T^*F$. Observe that

$$
(F'(u)v, Tv) = \|F'(u)v\| \|Tv\| \left(1 - \frac{1}{2} \left(\frac{F'(u)v}{\|F'(u)v\|} - \frac{Tv}{\|Tv\|}\right)^2\right).
$$

Therefore

$$
(F'(u)v, Tv) \geq \frac{2 - (\beta_{F,T})^2}{2} \|F'(u)v\| \|Tv\|.
$$

Therefore

$$
\langle v_1 - v_2, S(v_1) - S(v_2) \rangle = (F(v_1) - F(v_2), T(v_1 - v_2)) = \int_0^1 (F'(v_2 + t(v_1 - v_2))(v_1 - v_2), T(v_1 - v_2))dt \geq 2 - \frac{(\beta_{F,T})^2}{2} \left(\int_0^1 \|F'(v_2 + t(v_1 - v_2))(v_1 - v_2)\|dt\right) \|T(v_1 - v_2)\|
$$

$$
\geq 2 - \frac{(\beta_{F,T})^2}{2} \left(\inf_{\|u\|=1} \|Tu\|\right) \left(\inf_{v \in H_1} \inf_{\|u\|=1} \|F'(v)u\|\right) \|v_1 - v_2\|^2,
$$

where $A$ is the lower stability bound in (1.2). Also one may easily verify that

$$
\|S(v_1) - S(v_2)\| \leq \|T\| (\sup_{u \in H_1} \|F'(v)\|) \|v_1 - v_2\|, \quad v_1, v_2 \in H_1.
$$

Therefore by standard arguments (see for instance [55]), we obtain from (2.8) and (2.9) that

$$
\|u_{n+1} - u_n\|^2 \leq r_1 \|u_n - u_{n-1}\|^2, \quad n \geq 1,
$$

where

$$
r_1 = 1 - \mu(2 - \beta_{F,T}^2)(\inf_{\|u\|=1} \|Tu\|)(\inf_{v \in H_1} \inf_{\|u\|=1} \|F'(v)u\|)
$$

$$
+ \mu^2 \|T\|^2 (\sup_{v \in H_1} \|F'(v)\|)^2 \in (0, 1).
$$
This proves the exponential convergence of \( u_n, n \geq 0 \), in the iterative algorithm (1.10).

Taking limit in the algorithm (1.10) leads to

\[
T^*(F(u_\infty) - u_\epsilon) = 0,
\]

where \( u_\infty \) is the limit of the sequence \( u_n, n \geq 0 \). Thus

\[
2 - \beta_2^2 F(T) \left( \inf_{\|u\|=1} \|Tu\| \right) \left( \inf_{v \in H_1} \inf_{\|u\|=1} \|F'(v)u\| \right) \|u_\infty - u_0\|^2 \leq \langle u_\infty - u_0, T^*(F(u_\infty) - F(u_0)) \rangle = \langle u_\infty - u_\epsilon, T^*\epsilon \rangle \leq \|T\|\|u_0 - u_\infty\|\|\epsilon\|
\]

by (2.8) and (2.10). This proves (2.6) and completes the proof. \( \square \)

3. Iterative algorithm for localized maps

In this section, we develop a fixed point theorem for a well-localized map on a Banach space whose restriction on its dense Hilbert subspace is a contraction, and we establish exponential convergence of the iterative algorithm (1.10) for certain localized maps between (weak-)Hilbert-dense Banach spaces.

To state our results, we recall the concept of differential subalgebras. Given two unital Banach algebras \( A_1 \) and \( A_2 \), \( A_1 \) is said to be a Banach subalgebra of \( A_2 \) if \( A_1 \subseteq A_2 \), \( A_1 \) and \( A_2 \) share the same identity and \( \sup_{0 \neq T \in A_1} \|T\|_{A_2}/\|T\|_{A_1} < \infty \) holds; and a Banach subalgebra \( A_1 \) of \( A_2 \) is said to be a differential subalgebra of order \( \theta \in (0, 1] \) if there exists a positive constant \( D \) such that

\[
\|T_1T_2\|_{A_1} \leq D\|T_1\|_{A_1}\|T_2\|_{A_2} \left( \left( \frac{\|T_1\|_{A_2}}{\|T_1\|_{A_1}} \right)^\theta + \left( \frac{\|T_2\|_{A_2}}{\|T_2\|_{A_1}} \right)^\theta \right)
\]

for all nonzero \( T_1, T_2 \in A_1 \) \( [6, 35, 43, 48] \). We remark that differential subalgebras include many families of Banach algebras of infinite matrices with certain off-diagonal decay and localized integral operators \( [31, 33, 44, 45, 47, 49, 50, 48] \), and they have been widely used in operator theory, non-commutative geometry, frame theory, algebra of pseudodifferential operators, numerical analysis, signal processing, control and optimization etc, see \( [6, 19, 32, 35, 40, 41, 43, 46, 48] \), the survey papers \( [29, 36] \) and references therein.

Next we define the conjugate \( T^* \) of a localized linear operator \( T \) between (weak-) Hilbert-dense Banach spaces. Given Banach spaces \( B_i \) and their dense Hilbert subspaces \( H_i, i = 1, 2 \), we assume that linear operators \( T \) reside in a Banach subspace \( B \) of \( B(H_1, H_2) \) and also of \( B(B_1, B_2) \). For a linear operator \( T \in B \), its restriction \( T|_{H_1} \) to \( H_1 \) is a bounded operator from \( H_1 \) to \( H_2 \), hence its conjugate \( (T|_{H_1})^* \) is well-defined on \( H_2 \), and the “conjugate” of \( T \) is well-defined if the conjugate \( (T|_{H_1})^* \) can be extended to a bounded operator from \( B_2 \) to \( B_1 \). The above approach to define the conjugate requires certain localization for linear operators in \( B \), which will be stated precisely in the next theorem, cf. (3.4).
**Theorem 3.1.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with the property that for $i = 1, 2$, $\mathcal{H}_i \subset \mathcal{B}_i$, $\mathcal{H}_i$ is dense in $\mathcal{B}_i$, and

$$\sup_{0 \neq x \in \mathcal{H}_i} \frac{\|x\|_{\mathcal{B}_i}}{\|x\|_{\mathcal{H}_i}} < \infty. \tag{3.2}$$

Assume that Banach algebra $\mathcal{A}$ with norm $\| \cdot \|_\mathcal{A}$ is a unital Banach subalgebra of $\mathcal{B}(\mathcal{B}_1)$ and a differential subalgebra of $\mathcal{B}(\mathcal{H}_1)$ of order $\theta \in (0, 1]$. Let $\mathcal{B}$ be a Banach subspace of both $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$, and $\mathcal{B}^*$ be a Banach subspace of both $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $\mathcal{B}(\mathcal{B}_2, \mathcal{B}_1)$ such that

(i) $ST \in \mathcal{A}$ for all $S \in \mathcal{B}^*$ and $T \in \mathcal{B}$. Moreover,

$$\sup_{0 \neq T \in \mathcal{B}, 0 \neq S \in \mathcal{B}^*} \frac{\|ST\|_{\mathcal{A}}}{\|S\|_{\mathcal{B}^*} \|T\|_{\mathcal{B}}} < \infty. \tag{3.3}$$

(ii) For any $T \in \mathcal{B}$ and $S \in \mathcal{B}^*$ there exist unique $T^* \in \mathcal{B}^*$ and $S^* \in \mathcal{B}$ with the property that $\|T^*\|_{\mathcal{B}^*} = \|T\|_{\mathcal{B}}$, $\|S^*\|_{\mathcal{B}} = \|S\|_{\mathcal{B}^*}$, and

$$\langle Tu, w \rangle = \langle u, T^* w \rangle \text{ and } \langle Sw, u \rangle = \langle w, S^* u \rangle \text{ for all } u \in \mathcal{H}_1 \text{ and } w \in \mathcal{H}_2. \tag{3.4}$$

Assume that $F$ is a differentiable map from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that its derivative $F'$ is continuous and bounded from $\mathcal{B}_1$ into $\mathcal{B}$, and

$$\beta_{F,T} = \sup_{x \in \mathcal{B}_1} \sup_{u \in \mathcal{H}_1} \left\| \frac{F'(x)u}{\|F'(x)u\|_{\mathcal{H}_2}} - \frac{Tu}{\|Tu\|_{\mathcal{H}_2}} \right\|_{\mathcal{H}_2} < \sqrt{2} \tag{3.5}$$

for some linear operator $T \in \mathcal{B}$. Take an initial $x_0 \in \mathcal{B}_1$ and a noisy observation data $z_\epsilon = F(x^0) + \epsilon$ for some $x^0 \in \mathcal{B}_1$ and $\epsilon \in \mathcal{B}_2$, define $x_n, n \geq 1$, iteratively by

$$x_{n+1} = x_n - \mu T^*(F(x_n) - z_\epsilon), \quad n \geq 0, \tag{3.6}$$

where the relaxation factor $\mu$ satisfies

$$0 < \mu < (2 - \beta_{F,T}^2) \left( \inf_{\|u\|_{\mathcal{H}_1} = 1} \|Tu\|_{\mathcal{H}_2} \right)^2 \left( \sup_{\|u\|_{\mathcal{H}_1} = 1} \|Tu\|_{\mathcal{H}_2} \right)^2 \left( \inf_{\|u\|_{\mathcal{H}_1} = 1} \|F'(v)u\|_{\mathcal{H}_2} \right)^2 \left( \sup_{\|u\|_{\mathcal{H}_1} = 1} \|F'(v)u\|_{\mathcal{H}_2} \right)^2, \tag{3.7}$$

and $T^* \in \mathcal{B}^*$ is the conjugate operator defined by (3.4). Then $x_n, n \geq 0$, converges exponentially to some $x_\infty \in \mathcal{B}_1$ with

$$\|x_\infty - x^0\|_{\mathcal{B}_1} \leq C \|\epsilon\|_{\mathcal{B}_2}, \tag{3.8}$$

where $C$ is an absolute positive constant.

Given a Banach space $\mathcal{B}$, we say that a map $G : \mathcal{B} \to \mathcal{B}$ is a contraction if there exists $r \in [0, 1)$ such that

$$\|G(x) - G(y)\|_{\mathcal{B}} \leq r \|x - y\|_{\mathcal{B}} \text{ for all } x, y \in \mathcal{B}. \tag{3.1}$$

For a contraction $G$ on a Banach space $\mathcal{B}$, the Banach fixed point theorem states that there is a unique fixed point $x^*$ for the contraction $G$ (i.e., $G(x^*) = x^*$), and for any initial $x_0 \in \mathcal{B}$, the sequence $x_{n+1} = G(x_n), n \geq 0$, converges exponentially to the fixed point $x^*$ \cite{20}. To prove Theorem 3.1, we need a fixed point theorem for differentiable maps on a Banach space with its derivative being continuous and bounded in a differential Banach subalgebra and its restriction on a dense Hilbert subspace being a contraction.
Theorem 3.2. Let $\mathbf{B}$ be a Banach space, $\mathbf{H}$ be a Hilbert space such that $\mathbf{H} \subset \mathbf{B}$ is dense in $\mathbf{B}$ and

\[(3.9) \sup_{0 \neq x \in \mathbf{H}} \frac{\|x\|_{\mathbf{B}}}{\|x\|_{\mathbf{H}}} < \infty,\]

and let $\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}(\mathbf{B})$ and also a differential subalgebra of $\mathcal{B}(\mathbf{H})$ of order $\theta \in (0, 1]$. If $G$ is a differentiable map on $\mathbf{B}$ whose derivative $G'$ is continuous and bounded from $\mathbf{B}$ into $\mathcal{A}$ and there exists $r \in [0, 1)$ such that

\[(3.10) \quad \|G'(x)\|_{\mathcal{B}(\mathbf{H})} \leq r \quad \text{for all} \quad x \in \mathbf{B},\]

then there exists a unique fixed point $x^*$ for the map $G$. Furthermore given any initial $x_0 \in \mathbf{B}$, the sequence $x_n, n \geq 0$, defined by

\[(3.11) \quad x_{n+1} = G(x_n), \quad n \geq 0,
\]

converges exponentially to the fixed point $x^*$.

Proof. Let $x_n, n \geq 0$, be as in (3.11). It follows from the continuity of $G'$ in the Banach subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathbf{B})$ that

\[(3.12) \quad x_{n+1} - x_n = G(x_n) - G(x_{n-1})
\]

\[= \left( \int_0^1 G'(x_{n-1} + t(x_n - x_{n-1})) dt \right) (x_n - x_{n-1})
\]

\[= T_n(x_n - x_{n-1}), \quad n \geq 1.\]

Observe that

\[(3.13) \quad \|T_n\|_{\mathcal{B}(\mathbf{H})} \leq \int_0^1 \|G'(x_{n-1} + t(x_n - x_{n-1}))\|_{\mathcal{B}(\mathbf{H})} dt \leq r
\]

and

\[(3.14) \quad \|T_n\|_{\mathcal{A}} \leq \int_0^1 \|G'(x_{n-1} + t(x_n - x_{n-1}))\|_{\mathcal{A}} dt \leq M
\]

where $M = \sup_{x \in \mathbf{B}} \|G'(x)\|_{\mathcal{A}} < \infty$ by the assumption on the map $G$. Set

\[b_n = \sup_{l \geq 1} \|T_{l+n-1}T_{l+n-2} \cdots T_l\|_{\mathcal{A}}, \quad n \geq 1.
\]

Then we obtain from (3.1), (3.13) and (3.14) that

\[b_{2n+1} \leq \left( \sup_{m \geq 1} \|T_m\|_{\mathcal{A}} \right) b_{2n} \leq Mb_{2n}
\]

and

\[b_{2n} \leq 2D\left( \sup_{l \geq 1} \|T_{l+n-1} \cdots T_l\|_{\mathcal{B}(\mathbf{H})} \right)^{\theta} (b_n)^{2-\theta} \leq 2Dr^{n\theta}(b_n)^{2-\theta}
\]

for all $n \geq 1$. Thus

\[b_n \leq M^{\eta_0} b_{n-\eta_0} \leq M^{\eta_0}(2D)^{r^{\theta(\eta_0-\eta_0)/2}}(b_{(n-\eta_0)/2})^{2-\theta}
\]

\[\leq M^{\eta_0+(2-\theta)\eta_1}(2D)^{1+(2-\theta)r^{\theta}} (b_{(n-\eta_0)-(n-\eta_0-2\epsilon)}^{2-\theta}) (b_{(n-\eta_0-2\epsilon_1)/4})^{2-\theta}
\]

\[\leq \cdots
\]

\[\leq M^{\sum_{i=0}^{l} \epsilon_i (2-\theta)^i} (2D)^{\sum_{i=0}^{l-1} (2-\theta)^i} r^{\theta} r^{\sum_{i=0}^{l-1} \sum_{j=i+1}^{l} r^j} (2-\theta)^i,
\]
where \( n = \sum_{i=0}^t \epsilon_i 2^i \) with \( \epsilon_i \in \{0, 1\} \) and \( \epsilon_t = 1 \). Therefore
\[
(3.15) \quad b_n \leq \left( (2D)^{1/(1-\theta)} (M/r_0) (2-\theta)/(1-\theta) \right)^{n \log_2 (2-\theta)} r^n, \quad n \geq 1
\]
if \( \theta \in (0, 1) \), and
\[
(3.16) \quad b_n \leq \frac{M}{r} \left( 2DM/r \right)^{\log_2 r} r^n, \quad n \geq 1
\]
if \( \theta = 1 \). By (3.15) and (3.16), for any \( r_1 \in (r, 1) \) there exists a positive constant \( C \) such that
\[
(3.17) \quad \|T_n T_{n_1} \cdots T_1\|_A \leq C r_1^n, \quad n \geq 1.
\]
Recall that \( A \) is a Banach subalgebra of \( B(B) \). We then obtain from (3.12) and (3.17) that
\[
\|x_{n+1} - x_n\|_B \leq C r_1^n \|x_1 - x_0\|_B, \quad n \geq 1,
\]
which proves the exponential convergence of the sequence \( x_n, n \geq 0 \).

Finally we prove the uniqueness of the fixed point for the map \( G \). Let \( x^* \) and \( \tilde{x}^* \) be fixed points of the map \( G \). Then \( x^* - \tilde{x}^* \) is a fixed point of the linear operator \( T := \int_0^1 G'(\tilde{x}^* + t(x^* - \tilde{x}^*)) \, dt \in A \), because
\[
(3.18) \quad x^* - \tilde{x}^* = G(x^*) - G(\tilde{x}^*) = T(x^* - \tilde{x}^*).
\]
Following the argument to prove (3.17), we obtain that \( \lim_{n \to \infty} \|T^n\|_{B(B)} = 0 \). This together with (3.18) implies that \( x^* = \tilde{x}^* \), the uniqueness of fixed points for the map \( G \).

Now we apply Theorem 3.2 to prove Theorem 3.1.

**Proof of Theorem 3.1.** Define \( G : B_1 \to B_1 \) by
\[
(3.19) \quad G(x) = x - \mu T^* (F(x) - z_\epsilon), \quad x \in B_1.
\]
Then \( G \) is differentiable on \( B_1 \) and its derivative \( G'(x) = I - \mu T^* F'(x), \quad x \in B_1 \), is continuous and bounded in \( A \) by the assumption on \( F \) and the Banach spaces \( B \) and \( B^* \). Set
\[
m_0 = \frac{2 - \beta^2_{F,T}}{2} \left( \inf_{0 \neq u \in H_1} \frac{\|Tu\|_{H_2}}{\|u\|_{H_1}} \right) \left( \inf_{x \in B_1} \frac{\|F'(x)u\|_{H_2}}{\|u\|_{H_1}} \right)
\]
and
\[
M_0 = \left( \sup_{0 \neq u \in H_1} \frac{\|Tu\|_{H_2}}{\|u\|_{H_1}} \right) \left( \sup_{x \in B_1} \frac{\|F'(x)u\|_{H_2}}{\|u\|_{H_1}} \right).
\]
Observe that
\[
\|G'(x)\|_A \leq \|I\|_A + \mu \|T\|_B \left( \sup_{x \in B_1} \|F'(x)\|_B \right)
\]
\[
\times \left( \sup_{0 \neq S_1, S_2 \in B} \frac{\|S_1 S_2\|_A}{\|S_1\|_B \|S_2\|_B} \right) < \infty,
\]
\[
(3.20)
\]
and
\[
\|G'(x)\|_{\mathcal{B}(\mathcal{H}_1)} \leq \|(I + \mu T^* F'(x))^{-1}\|_{\mathcal{B}(\mathcal{H}_1)} \|1 - \mu^2 (T^* F'(x))^2\|_{\mathcal{B}(\mathcal{H}_1)}
\]
\[
\leq (1 + M_0^2 \mu^2) \sup_{0 \neq u \in \mathcal{H}_1} \frac{\|u\|^2_{\mathcal{H}_1}}{(u, (1 + \mu T^* F'(x)))_{\mathcal{H}_1}}
\]
(3.21)
\[
\leq \frac{1 + M_0^2 \mu^2}{1 + m_0 \mu} < 1, \ x \in \mathcal{B}_1,
\]
where the second inequality holds as
\[
\|(I + \mu T^* F'(x))^{-1}\|_{\mathcal{B}(\mathcal{H}_1)} = \sup_{0 \neq u \in \mathcal{H}_1} \frac{\|u\|_{\mathcal{H}_1}}{\|(I + \mu T^* F'(x))u\|_{\mathcal{H}_1}}
\]
and the third inequality follows from (2.8). Combining the above two estimates about \(G'(x), x \in \mathcal{B}_1\), with Theorem 3.2 proves the exponential convergence of \(x_n, n \geq 0, \text{ in } \mathcal{B}_1\).

Denote by \(x_\infty\) the limit of \(x_n, n \geq 0, \text{ in } \mathcal{B}_1\). Then taking limit in the iterative algorithm (3.6) yields
\[
T^* F(x_\infty) - T^* F(x^0) = T^* \epsilon.
\]
Thus
(3.22)
\[
A_\infty(x_\infty - x^0) = T^* \epsilon,
\]
where \(A_\infty = \int_0^1 T^* F'(x_0 + t(x_\infty - x^0))dt\). Following the argument to prove Theorem 3.2 and applying (3.20) and (3.21), there exists a positive constant \(C_r\) for any \(r \in ((1 + M_0^2 \mu^2)/(1 + m_0 \mu), 1)\) such that
\[
\|(I - \mu A_\infty)^n\|_A \leq C_r r^n, \ n \geq 1.
\]
Thus \(A_\infty\) is invertible in \(A\) and
(3.23)
\[
\|(A_\infty)^{-1}\|_A \leq \mu \sum_{n=0}^\infty \|(I - \mu A_\infty)^n\|_A \leq \mu (\|I\|_A + C_r/(1 - r)).
\]
Combining (3.22) and (3.23) leads to
\[
\|x_\infty - x^0\|_{\mathcal{B}_1} \leq \|(A_\infty)^{-1}\|_{\mathcal{B}(\mathcal{B}_1)} \|T^* \epsilon\|_{\mathcal{B}_1}
\]
\[
\leq \|(A_\infty)^{-1}\|_A \|T\|_\mathcal{B} \left( \sup_{0 \neq S \in \mathcal{A}} \|S\|_{\mathcal{B}(\mathcal{B}_1)} \right) \left( \sup_{0 \neq U \in \mathcal{B}^*} \frac{\|U\|_{\mathcal{B}(\mathcal{B}_2, \mathcal{B}_1)}}{\|U\|_{\mathcal{B}^*}} \right) \|\epsilon\|_{\mathcal{B}_2}.
\]
This proves the error estimate (3.8). □

Remark 3.3. Our model of Hilbert-dense Banach spaces in Theorem 3.1 is \(\ell^p(\Lambda)\), the space of \(p\)-summable sequences \(\ell^p(\Lambda)\), with \(2 \leq p < \infty\). For that case, exponential convergence of the Van-Cittert algorithm, which is similar to the iterative algorithm (3.6) in Theorem 3.1, is established in [48] under slightly different restriction on the relaxation factor \(\mu\). For weak-Hilbert-dense Banach spaces, the iterative algorithm (3.6) in Theorem 3.1 still has exponential convergence if operators in \(\mathcal{B}\) and \(\mathcal{B}^*\) are assumed additionally to be uniformly continuous in the weak topologies of Banach spaces, that is, \(\sup_{\|T\|_\mathcal{B} \leq 1} |f(Tx_n) - f(Tx_\infty)| \to 0\) for any bounded linear functional \(f\) on \(\mathcal{B}_2\) if \(x_n\) tends to \(x_\infty\) in the weak topology of \(\mathcal{B}_1\); and \(\sup_{\|S\|_{\mathcal{B}^*} \leq 1} |g(Sy_n) - g(Sy_\infty)| \to 0\) for any bounded linear functional \(g\) on \(\mathcal{B}_1\).
if \( y_n \) tends to \( y_\infty \) in the weak topology of \( B_2 \). We leave the detailed arguments to interested readers.

4. Sparse reconstruction and optimization

In this section, we show that sparse signals \( x \in A \) could be reconstructed from their nonlinear measurements \( F(x) \) via the optimization approach \([1.21]\).

**Theorem 4.1.** Let \( H_1 \) and \( H_2 \) be Hilbert spaces, \( M \) be a Banach space, \( A = \cup_{i \in I} A_i \) be union of closed linear subspaces of \( H_1 \), \( s_A \) and \( a_A \) be in \([1.18]\) and \([1.19]\) respectively, and let \( F \) be a continuous map from \( H_1 \) to \( H_2 \) normalized so that \( F(0) = 0 \). If \((A, M, H_1)\) forms a sparse approximation triple, and if \( F \) has the sparse Riesz property \([1.22]\) and the almost linear property \([1.23]\) with \( D, \beta, \gamma_1, \gamma_2 \geq 0 \) satisfying

\[
\gamma_3 := 1 - 2D\gamma_1 - (D\gamma_1 + D\gamma_2 + \beta)\sqrt{a_A s_A} > 0,
\]

then given \( x^0 \in M \) and \( \varepsilon > 0 \), the solution \( x^0_M \) of the optimization problem \([1.21]\) provides a suboptimal approximation to \( x^0 \),

\[
\|x^0_M - x^0\|_{H_1} \leq \left( \frac{2 + 8D\gamma_2 + 4\beta}{\gamma_3} \right) \sqrt{a_A s_A} \sigma_{A,M}(x^0) + \frac{2 + \sqrt{a_A s_A}}{\gamma_3} D \varepsilon
\]

and

\[
\|x^0_M - x^0\|_M \leq \left( \frac{2 - 4D\gamma_1 + 2(D\gamma_1 + 2D\gamma_2 + \beta)\sqrt{a_A s_A}}{\gamma_3} \right) \sigma_{A,M}(x^0) + \frac{2D}{\gamma_3} \sqrt{s_A} \varepsilon.
\]

To prove Theorem 4.1, we need the following approximation property for sparse approximation triples.

**Proposition 4.2.** Let \((A, M, H_1)\) be a sparse approximation triple and \( a_A \) be as in \([1.19]\). Then

\[
\|x - x_{A,M}\|_{H_1} \leq a_A \|x\|_M, \quad x \in M,
\]

where \( x_{A,M} \) is a best approximator of \( x \in M \).

We postpone the proof of the above proposition to Appendix B and start the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Let \( x^0_{A,M} := \arg\min_{\hat{x} \in A} \|x^0 - \hat{x}\|_M \) be a best approximator in \( A \) to \( x^0 \), where the existence follows from the proximinality property of the triple \((A, M, H_1)\). Denote by \( A(x^0_{A,M}) \) the linear space in \( A \) containing \( x^0_{A,M} \). Then

\[
x^0_{A,M} = \arg\min_{\hat{x} \in A(x^0_{A,M})} \|x^0 - \hat{x}\|_M = \arg\min_{\hat{x} \in A(x^0_{A,M})} \|x^0 - \hat{x}\|_{H_1},
\]

by the common best approximator property \([1.14]\); and

\[
\|x^0\|_M = \|x^0_{A,M}\|_M + \|x^0 - x^0_{A,M}\|_M = \|x^0_{A,M}\|_M + \sigma_{A,M}(x^0)
\]

by the norm splitting properties \([1.15]\) and \([1.16]\).
Let \( x_0^{A,M} + h_0 := \arg\min_{\hat{x} \in A(x_0^{A,M})} \| x_0^M - \hat{x} \|_M \in A(x_0^{A,M}) \) be a best approximator to \( x_0^M \) in \( A(x_0^{A,M}) \). Then
\[
(4.7) \quad x_0^{A,M} + h_0 = \arg\min_{\hat{x} \in A(x_0^{A,M})} \| x_0^M - \hat{x} \|_{H_1}
\]
and
\[
(4.8) \quad \| x_0^M \|_M = \| x_0^{A,M} + h_0 \|_M + \| x_0^M - x_0^{A,M} - h_0 \|_M
\]
by the common best approximator property and norm splitting property of the triple \((A, M, H_1)\).

Set \( h := x_0^M - x^0 \). We then obtain from (1.21), (4.6) and (4.8) that
\[
\| h - h_0 \|_M \leq \| (x^0 - x_0^{A,M}) + (h - h_0) \|_M + \sigma_{A,M}(x^0)
\]
\[
= \| x_0^M \|_M - \| x_0^{A,M} + h_0 \|_M + \sigma_{A,M}(x^0)
\]
\[
\leq \| x_0^M \|_M - \| x_0^{A,M} + h_0 \|_M + \sigma_{A,M}(x^0)
\]
\[
\leq \| h_0 \|_M + 2\sigma_{A,M}(x^0).
\]
(4.9)

Let \( h_1 := \arg\min_{h \in A} \| h - h_0 - \hat{h} \|_M \) be a best approximator of \( h - h_0 \). Then
\[
(4.10) \quad \| h - h_0 \|_{H_1}^2 = \| h_1 \|_{H_1}^2 + \| h - h_0 - h_1 \|_{H_1}^2,
\]
(4.11)
\[
\| h - h_0 \|_M = \| h_1 \|_M + \| h - h_0 - h_1 \|_M,
\]
and
\[
(4.12) \quad \| h - h_0 - h_1 \|_{H_1} \leq \sqrt{a_A} \| h - h_0 \|_M
\]
by (1.14), (1.15) and (4.4).

From (1.21), (1.23), (4.4) and (4.9), it follows that
\[
\| F(h) \| \leq \| F(x_0^M) - F(x^0) + F(h) \| + \varepsilon
\]
\[
\leq \gamma_1 \| h \|_{H_1} + \gamma_2 \sqrt{a_A}(\sigma_{A,M}(x^0) + \sigma_{A,M}(x_0^M)) + \varepsilon
\]
\[
\leq \gamma_1 \| h_0 \|_{H_1} + \gamma_1 \| h_1 \|_{H_1} + \gamma_1 \| h - h_0 - h_1 \|_{H_1}
\]
\[
+ \gamma_2 \sqrt{a_A}(\sigma_{A,M}(x^0) + \| (x^0 - x_0^{A,M}) + (h - h_0) \|_M) + \varepsilon
\]
\[
\leq \gamma_1 \| h_0 \|_{H_1} + \gamma_1 \| h_1 \|_{H_1} + (\gamma_1 + \gamma_2) \sqrt{a_A} \| h - h_0 \|_M
\]
\[
+ 2\gamma_2 \sqrt{a_A} \sigma_{A,M}(x^0) + \varepsilon
\]
\[
\leq \gamma_1 \| h_0 \|_{H_1} + \gamma_1 \| h_1 \|_{H_1} + (\gamma_1 + \gamma_2) \sqrt{a_A} \| h_0 \|_M
\]
\[
+ 2(\gamma_1 + 2\gamma_2) \sqrt{a_A} \sigma_{A,M}(x^0) + \varepsilon.
\]
(4.13)

By the definition of \( x_0^{A,M} \), we have that
\[
x_0^{A,M} = P_{A(x_0^{A,M})}(x^0) \quad \text{and} \quad x_0^{A,M} + h_0 = P_{A(x_0^{A,M})}(x_M^0),
\]
where \( P_V \) is the projection operator from \( H_1 \) to its closed subspace \( V \). Therefore
\[
(4.14) \quad h_0 = P_{A(x_0^{A,M})}(h).
\]
By (1.22), (4.7), (4.10) and (4.14) we get
\[ \|h_0\|_{H_1} = \|P_A(x_{A,M}^0)(h)\|_{H_1} \leq \|h\|_{H_1} \]
\[ \leq D\|F(h)\| + \beta \sqrt{a_A} \sigma_{A,M}(h) \]
(4.15)
\[ \leq D\|F(h)\| + \beta \sqrt{a_A} \|h - h_0\|_M \]
and
\[ \|h_1\|_{H_1} = \|P_A(h_1)(I - P_A(x_{A,M}^0))(h)\|_{H_1} \leq \|h\|_{H_1} \]
(4.16)
\[ \leq D\|F(h)\| + \beta \sqrt{a_A} \|h - h_0\|_M. \]
Hence by (1.18), (4.9), (4.12), (4.15) and (4.16), we have
\[ \|h_0\|_{H_1} \leq D\gamma_1\|h_0\|_{H_1} + (D\gamma_1 + D\gamma_2 + \beta)\sqrt{a_A}\|h_0\|_M + D\gamma_1\|h_1\|_{H_1} \]
(4.17)
\[ + 2(D\gamma_1 + 2D\gamma_2 + \beta)\sqrt{a_A} \sigma_{A,M}(x^0) + D\varepsilon; \]
and
\[ \|h_1\|_{H_1} \leq D\gamma_1\|h_0\|_{H_1} + (D\gamma_1 + D\gamma_2 + \beta)\sqrt{a_A}\|h_0\|_M + D\gamma_1\|h_1\|_{H_1} \]
(4.18)
\[ + 2(D\gamma_1 + 2D\gamma_2 + \beta)\sqrt{a_A} \sigma_{A,M}(x^0) + D\varepsilon. \]
Combining (4.17) and (4.18) and using \( \|h_0\|_M \leq \sqrt{a_A}\|h_0\|_{H_1} \) lead to
\[ \|h_0\|_{H_1} \leq \frac{2(D\gamma_1 + 2D\gamma_2 + \beta)\sqrt{a_A} \sigma_{A,M}(x^0) + D\varepsilon}{1 - 2D\gamma_1 - (D\gamma_1 + D\gamma_2 + \beta)\sqrt{a_A}s_A} \]
(4.19)
and
\[ \|h_1\|_{H_1} \leq \frac{2(D\gamma_1 + 2D\gamma_2 + \beta)\sqrt{a_A} \sigma_{A,M}(x^0) + D\varepsilon}{1 - 2D\gamma_1 - (D\gamma_1 + D\gamma_2 + \beta)\sqrt{a_A}s_A}. \]
On the other hand,
\[ \|h - h_0 - h_1\|_{H_1} \leq \sqrt{a_A}\|h - h_0\|_M \leq \sqrt{a_A}\|h_0\|_M + 2\sqrt{a_A} \sigma_{A,M}(x^0) \]
(4.21)
\[ \leq \sqrt{a_A}s_A\|h_0\|_{H_1} + 2\sqrt{a_A} \sigma_{A,M}(x^0) \]
by (1.18), (4.9) and (4.12). Therefore the error estimates (4.2) and (4.3) follow from (1.18), (4.9), (4.19), (4.20) and (4.21). \( \Box \)

As a corollary, we have the following result for linear mapping \( F \), cf. [51, Theorem 1.1] in the classical sparse recovery setting.

**Corollary 4.3.** Let \( M, A, H_1, H_2 \) be as in Theorem 4.1, and let \( F : H_1 \rightarrow H_2 \) be linear and have the sparse Riesz property (1.22) with \( D > 0 \) and \( \beta \in (0, 1/\sqrt{a_A}s_A) \). Given \( x^0 \in M \) and \( \varepsilon > 0 \), the optimization solution of (1.21) satisfies
\[ \|x_M^0 - x^0\|_{H_1} \leq \left( \frac{2 + 4\beta}{1 - \beta\sqrt{a_A}s_A} \right) \sqrt{a_A} \sigma_{A,M}(x^0) + \frac{(2 + \sqrt{a_A}s_A)D}{1 - \beta\sqrt{a_A}s_A} \varepsilon \]
and
\[ \|x_M^0 - x^0\|_M \leq \left( \frac{2 + 2\beta\sqrt{a_A}s_A}{1 - \beta\sqrt{a_A}s_A} \right) \sigma_{A,M}(x^0) + \frac{2D}{1 - \beta\sqrt{a_A}s_A} \sqrt{s_A} \varepsilon, \]
where \( \sigma_{A,M}(x^0) = \inf_{\hat{x} \in A} \|\hat{x} - x^0\|_M \).
5. Sparse Riesz property and almost linear property

In this section, we consider the sparse Riesz property \( (1.22) \) and almost linear property \( (1.23) \) for nonlinear maps not far from a linear operator with the restricted isometry property \( (1.26) \).

**Theorem 5.1.** Let \( H_1 \) and \( H_2 \) be Hilbert spaces, \( M \) be a Banach space, and \( A = \bigcup_{i \in I} A_i \) be a union of closed linear subspaces of \( H_1 \). Assume that \( (A, M, H_1) \) is a sparse approximation triple, and \( T \in \mathcal{B}(H_1, H_2) \) has the restricted isometry property \( (1.26) \) on \( 2A \) with \( \delta_2(A)(T) < \sqrt{2}/2 \). If \( F \) is a continuous map from \( H_1 \) to \( H_2 \) with \( F(0) = 0 \) and

\[
\gamma_{F,T}(2A) < \frac{\sqrt{2}}{2} - \sqrt{\delta_2(A)(T)},
\]

then \( F \) has the sparse Riesz property \( (1.22) \),

\[
\|F(x)\|_{H_2} \geq (1 - 2(\sqrt{\delta_2(A)(T)} + \gamma_{F,T}(2A)))\|x\|_{H_1} - (\sqrt{\delta_2(A)(T)} + \gamma_{F,T}(2A))\sqrt{\sigma_A}\sigma_{M}(x), \quad x \in M.
\]

For any \( x \in M \), define

\[
x_{A,M}^{k+1} = x_{A,M}^k + \arg\min_{\tilde{x} \in A} \|x - x_{A,M}^k - \tilde{x}\|_{M}, \quad k \geq 0,
\]

with initial \( x_{A,M}^0 = 0 \). To prove Theorem 5.1, we need convergence of the above greedy algorithm.

**Proposition 5.2.** Let \( (A, M, H_1) \) be a sparse approximation triple. Then \( x_{A,M}^k, k \geq 0, \) in the greedy algorithm \( (5.3) \) converges to \( x \in M \),

\[
\lim_{k \to \infty} \|x_{A,M}^k - x\|_{M} = 0.
\]

We postpone the proof of the above proposition to Appendix B and start the proof of Theorem 5.1.

**Proof.** Take \( x \in M \), let \( x_{A,M}^k, k \geq 0, \) be as in the greedy algorithm \( (5.3) \). Then from Proposition 5.2, the continuity of \( F \) on \( H_1 \), and the continuous imbedding of \( M \) into \( H_1 \) it follows that

\[
\lim_{k \to \infty} \|F(x_{A,M}^k) - F(x)\|_{H_2} = 0.
\]

Write \( u_k = x_{A,M}^{k+1} - x_{A,M}^k, k \geq 0 \). Then \( u_k \in A \), and

\[
\|F(x) - Tx\|_{H_2} \leq \sum_{k=0}^{\infty} \|F(x_{A,M}^{k+1}) - F(x_{A,M}^k) - Tu_k\|_{H_2} \leq \gamma_{F,T}(2A) \sum_{k=0}^{\infty} \|u_k\|_{H_1}
\]

by \( (1.26), (1.27), (5.5) \) and the assumption \( F(0) = 0 \).

Observe that \( 4(Tu_k, Tu_{k'}) = \|T(\tilde{u}_k + \tilde{u}_{k'})\|_{H_2}^2 - \|T(\tilde{u}_k - \tilde{u}_{k'})\|_{H_2}^2, \) where \( \tilde{u}_k = u_k/\|u_k\|_{H_1}, k \geq 0 \). Then

\[
\|Tu_k, Tu_{k'}\| \leq \delta_2(A)(T)\|u_k\|_{H_1}\|u_{k'}\|_{H_1}, \quad k, k' \geq 0,
\]

so

\[
\|u_k\|_{H_1} \leq \frac{\sqrt{2}}{2} - \delta_2(A)(T)\|u_k\|_{H_1}, \quad k \geq 0.
\]
by the restricted isometry property (1.26). We remark that in the classical sparse recovery setting, the inner product $\langle u_k, u_{k'} \rangle$ between different $u_k$ and $u_{k'}$ is always zero, but it may be nonzero in our setting. Hence for $K \geq 1$,

$$
\|Tx_{A,M}\|_{2}^{2} = \|T \left( \sum_{k=0}^{K} u_k \right) \|_{2}^{2} = \sum_{k=0}^{K} \|Tu_k\|_{2}^{2} + \sum_{0 \leq k \neq k' \leq K} \langle Tu_k, Tu_{k'} \rangle 
\leq (1 + \delta_{2A}(T)) \sum_{k=0}^{K} \|u_k\|_{H_1}^{2} + \sum_{0 \leq k \neq k' \leq K} \langle u_k, u_{k'} \rangle 
+ \delta_{2A}(T) \sum_{0 \leq k \neq k' \leq K} \|u_k\|_{H_1} \|u_{k'}\|_{H_1} 
= \|x_{A,M}^{K+1}\|_{H_1}^{2} + \delta_{2A}(T) \left( \sum_{k=0}^{K} \|u_k\|_{H_1} \right)^{2},
$$

and similarly

$$
\|Tx_{A,M}\|_{2}^{2} \geq \|x_{A,M}^{K+1}\|_{H_1}^{2} - \delta_{2A}(T) \left( \sum_{k=0}^{K} \|u_k\|_{H_1} \right)^{2}.
$$

Therefore combining (5.8) and (5.9), and then applying (1.17) and (5.4) when taking limit as $K \to \infty$, we obtain

$$
-\delta_{2A}(T) \left( \sum_{k \geq 0} \|u_k\|_{H_1} \right)^{2} \leq \|Tx\|_{H_1} - \|x\|_{H_1} \leq \delta_{2A}(T) \left( \sum_{k \geq 0} \|u_k\|_{H_1} \right)^{2},
$$

which implies that

$$
\|Tx\|_{H_2} - \sqrt{\delta_{2A}(T)} \sum_{k \geq 0} \|u_k\|_{H_1} \leq \|x\|_{H_1} \leq \|Tx\|_{H_2} + \sqrt{\delta_{2A}(T)} \sum_{k \geq 0} \|u_k\|_{H_1}.
$$

By (1.19),

$$
\|u_k\|_{H_1} \leq \sqrt{a_A} \|u_{k-1}\|_{M}, \ k \geq 1.
$$

This together with (1.14) and (1.15) implies that

$$
\sum_{k \geq 0} \|u_k\|_{H_1} \leq \|u_0\|_{H_1} + \|u_1\|_{H_1} + \sqrt{a_A} \sum_{k \geq 2} \|u_{k-1}\|_{M} \leq \sqrt{2} \|x\|_{H_1} + \sqrt{a_A} \sigma_{A,M}(x).
$$

Combining (5.6), (5.10) and (5.12) gives

$$
\|F(x)\|_{H_2} - \|x\|_{H_1} \leq (\sqrt{\delta_{2A}(T)} + \gamma_{F,T}(2A)) \left( \sqrt{2} \|x\|_{H_1} + \sqrt{a_A} \sigma_{A,M}(x) \right).
$$

Reformulating the above estimates completes the proof of the estimate (5.2) for the sparse Riesz property of $F$.

**Theorem 5.3.** Let $H_1, H_2, M, A, T, F$ be as in Theorem 5.1 with additional assumption that $\gamma_{F,T}(4A) < \infty$. Then $F$ has the almost linear property on $A$,

$$
\|F(x) - F(y) - F(x - y)\|_{H_2} \leq 2\gamma_{F,T}(4A) \|x - y\|_{H_1} + 2(\gamma_{F,T}(2A) + \gamma_{F,T}(4A)) \sqrt{a_A} \sigma_{A,M}(x + \sigma_{A,M}(y)).
$$
Proof. Take \( x, y \in \mathbf{M} \), and let \( x_{A,M}^k \) and \( y_{A,M}^k \), \( k \geq 0 \), be as in the greedy algorithm (5.3) to approximate \( x \) and \( y \) in \( \mathbf{M} \) respectively. Write

\[
\|F(x) - F(y) - T(x - y)\|_{\mathbf{H}_2} \leq \|F(x) - F(x_{A,M}^k) - T(x - x_{A,M}^k)\|_{\mathbf{H}_2} + \|F(y) - F(y_{A,M}^k) - T(y - y_{A,M}^k)\|_{\mathbf{H}_2} + \|F(x_{A,M}^k) - F(y_{A,M}^k) - T(x_{A,M}^k - y_{A,M}^k)\|_{\mathbf{H}_2}
\]

(5.14)

\[
=: I_1 + I_2 + I_3.
\]

By (1.19), (1.26), (1.27), (5.4) and the continuity of \( F \),

\[
\|x - y\| = \|F(x) - F(x_{A,M}^k) - T(x - x_{A,M}^k)\|_{\mathbf{H}_2} + \|F(y) - F(y_{A,M}^k) - T(y - y_{A,M}^k)\|_{\mathbf{H}_2} + \|F(x_{A,M}^k) - F(y_{A,M}^k) - T(x_{A,M}^k - y_{A,M}^k)\|_{\mathbf{H}_2}
\]

(5.14)

By (1.19), (1.26), (1.27), (5.4) and the continuity of \( F \) and \( T \) on \( H_1 \), we get

\[
I_1 \leq \sum_{k \geq 2} \|F(x_{A,M}^{k+1}) - F(x_{A,M}^k) - T(x_{A,M}^{k+1} - x_{A,M}^k)\|_{\mathbf{H}_2}
\]

\[
\leq \gamma_{F,T}(A) \sum_{k \geq 2} \|x_{A,M}^{k+1} - x_{A,M}^k\|_{\mathbf{H}_1}
\]

(5.15)

\[
\leq \gamma_{F,T}(A) \sqrt{a_A} \sigma_{A,M}(x) \leq \gamma_{F,T}(2A) \sqrt{a_A} \sigma_{A,M}(x)
\]

and similarly

\[
I_2 \leq \gamma_{F,T}(2A) \sqrt{a_A} \sigma_{A,M}(y).
\]

For the term \( I_3 \), we obtain from (1.19) and (1.27) that

\[
I_3 \leq \gamma_{F,T}(4A) \|x_{A,M}^2 - y_{A,M}^2\|_{\mathbf{H}_1}
\]

\[
\leq \gamma_{F,T}(4A) (\|x - y\|_{\mathbf{H}_1} + \|x - x_{A,M}^2\|_{\mathbf{H}_1} + \|y - y_{A,M}^2\|_{\mathbf{H}_1})
\]

(5.17)

\[
\leq \gamma_{F,T}(4A) \left( \|x - y\|_{\mathbf{H}_1} + \sqrt{a_A} (\sigma_{A,M}(x) + \sigma_{A,M}(y)) \right).
\]

Combining estimates in (5.14)-(5.17) gives

\[
\|F(x) - F(y) - T(x - y)\| \leq \gamma_{F,T}(4A) \|x - y\|_{\mathbf{H}_1} + \left( \gamma_{F,T}(2A) + \gamma_{F,T}(4A) \right) \sqrt{a_A} (\sigma_{A,M}(x) + \sigma_{A,M}(y)).
\]

(5.18)

Write

\[
\|F(x - y) - T(x - y)\|
\]

\[
\leq \|F(x - y) - F(x_{A,M}^2 - y_{A,M}^2 - T(x - x_{A,M}^2)\|
\]

\[
+ \|F(x_{A,M}^2 - y_{A,M}^2) - F(x_{A,M}^2 - y_{A,M}^2) - T(y_{A,M}^2 - y_{A,M}^2)\|
\]

\[
+ \|F(x_{A,M}^2 - y_{A,M}^2 - T(x_{A,M}^2 - y_{A,M}^2)\|
\]

Following the arguments used to establish (5.18), we have

\[
\|F(x - y) - T(x - y)\| \leq \gamma_{F,T}(4A) \|x - y\|_{\mathbf{H}_1} + \left( \gamma_{F,T}(2A) + \gamma_{F,T}(4A) \right) \sqrt{a_A} (\sigma_{A,M}(x) + \sigma_{A,M}(y)).
\]

(5.19)

Combining (5.18) and (5.19) proves the estimate (5.13) for the almost linear property of the map \( F \).

Combining Theorems 4.1, 5.1 and 5.3 leads to the following result on the stable reconstruction of sparse signals \( x \) from their nonlinear measurements \( F(x) \) when \( F \) is not far away from a measurement matrix \( T \) with the restricted isometry property (1.26).
**Theorem 5.4.** Let $H_1, M, A, T, F$ be as in Theorem 5.1 with

$$\sqrt{2(\sqrt{\delta_2 A(T)} + \gamma_{F,T}(2A))} + 4\gamma_{F,T}(4A) + (\sqrt{\delta_2 A(T)} + 3\gamma_{F,T}(2A) + 4\sqrt{\delta_1 A(T)})\sqrt{a_A a_M} < 1.$$ 

Then for any given $x^0 \in M$ and $\varepsilon > 0$, the solution $x^0_M$ of the minimization problem (1.21) has the following error estimates:

$$\|x^0_M - x^0\|_{H_1} \leq C_1 \sqrt{a_A} \sigma_{A,M}(x^0) + C_2 \varepsilon$$

and

$$\|x^0_M - x^0\|_M \leq C_1 \sigma_{A,M}(x^0) + C_2 \sqrt{a_M} \varepsilon$$

where $C_1$ and $C_2$ are absolute constants independent on $x^0 \in M$ and $\varepsilon \geq 0$.

Applying Theorem 5.4 to linear maps, we have the following corollary.

**Corollary 5.5.** Let $H_1, M, A, H_2$ and $T$ be as in Theorem 5.4. If

$$\delta_2 A(T) < (\sqrt{2} + \sqrt{a_A a_M})^{-2},$$

then for any given $x^0 \in M$ and $\varepsilon > 0$, the solution $x^0_M$ of the minimization problem (1.21) with $F = T$ has the error estimates (5.20) and (5.21).

For classical sparse recovery problems, the conclusions in Corollary 5.5 have been established under weaker assumptions on the restricted isometry constant $\delta_2 A(T)$, see [11] and references therein.

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**Appendix A. Bi-Lipschitz Map and Uniform Stability**

In this appendix, we provide some sufficient conditions, mostly optimal, for a differentiable map to have the bi-Lipschitz property (1.1), see Theorems A.3 and A.5 in Banach space setting, and Theorems A.7 and A.9 in Hilbert space setting.

For a differentiable map $F$ from one Banach space $B_1$ to another Banach space $B_2$ that has the bi-Lipschitz property (1.1), we have

$$A\|y\| \leq \frac{\|F(x + ty) - F(x)\|}{t} \leq B\|y\| \quad \text{for all} \quad x, y \in B_1 \text{ and } t > 0,$$

where $A, B$ are the constants in the bi-Lipschitz property (1.1). Then taking limit as $t \to 0$ leads to a necessary condition for a differentiable bi-Lipschitz map.

**Theorem A.1.** Let $B_1$ and $B_2$ be Banach spaces. If $F : B_1 \to B_2$ is a differentiable map that has the bi-Lipschitz property (1.1), then its derivative $F'(x), x \in B_1$, has the uniform stability property (1.2).

For $B_1 = B_2 = \mathbb{R}$, a differentiable map $F$ with the uniform stability property (1.2) for its derivative has the bi-Lipschitz property (1.1), but it is not true in general Banach space setting. Maps $E_{p,\varepsilon}, 1 \leq p \leq \infty, \varepsilon \in (0, \pi/4)$, from $\mathbb{R}$ to $\mathbb{R}^2$ in the example below are such examples.
Example A.2. For $1 \leq p \leq \infty$ and $\epsilon \in [0, \pi/4)$, define $E_{p,\epsilon} : \mathbb{R} \rightarrow \mathbb{R}^2$ by
\begin{equation}
E_{p,\epsilon}(t) = \begin{cases}
(-\cos \epsilon, \sin \epsilon) - (\sin \epsilon, \cos \epsilon)(t + \pi/2 + \epsilon) & \text{if } t \in (-\infty, -\pi/2 - \epsilon), \\
(\sin t, -\cos t) & \text{if } t \in [-\pi/2 - \epsilon, \pi/2 + \epsilon], \\
(\cos \epsilon, \sin \epsilon) + (-\sin \epsilon, \cos \epsilon)(t - \pi/2 - \epsilon) & \text{if } t \in (\pi/2 + \epsilon, \infty),
\end{cases}
\end{equation}
see Figure 1. The maps $E_{p,\epsilon}$ just defined do not have the bi-Lipschitz property (1.1), but their derivatives $E'_{p,\epsilon}$ have the uniform stability property (1.2),
\begin{align*}
\frac{\sqrt{2}}{2} |\hat{t}| \leq \|E'_{p,\epsilon}(t)\hat{t}\|_p &= \begin{cases}
\| (\tilde{t} \sin \epsilon, \tilde{t} \cos \epsilon) \|_p & \text{if } t < -\pi/2 - \epsilon, \\
\| (\tilde{t} \cos t, \tilde{t} \sin t) \|_p & \text{if } |t| \leq \pi/2 + \epsilon, \\
\| (-\tilde{t} \sin \epsilon, \tilde{t} \cos \epsilon) \|_p & \text{if } t > \pi/2 + \epsilon
\end{cases} \\
&\leq 2|\hat{t}| \quad \text{for all } t, \hat{t} \in \mathbb{R},
\end{align*}
where $\| \cdot \|_p, 1 \leq p \leq \infty$, is the $p$-norm on the Euclidean space $\mathbb{R}^2$.

Given a differentiable bi-Lipschitz map $F$ from one Banach space $B_1$ to another Banach space $B_2$ such that its derivative $F'(x)$ is uniformly stable, define
\begin{equation}
\alpha_F := \sup_{\|y\|=1} \inf_{\|z\|=1} \sup_{x \in B_1} \frac{\|F'(x)y - z\|}{\|F'(x)y\|}.
\end{equation}
The quantity $\alpha_F$ is the minimal radius such that for any $0 \neq y \in B_1$, the set $\mathbb{B}(y)$ of unit vectors $F'(x)y/\|F'(x)y\|, x \in B_1$, is contained in a ball of radius $\alpha_F < 1$ centered at a unit vector. Our next theorem shows that a differentiable bi-Lipschitz map $F$ with its derivative $F'(x)$ being uniformly stable and continuous and with $\alpha_F$ in (A.2) satisfying $\alpha_F < 1$ has the bi-Lipschitz property (1.1).

**Theorem A.3.** Let $B_1$ and $B_2$ be Banach spaces, and $F$ be a continuously differentiable map from $B_1$ to $B_2$ with the property that its derivative has the uniform stability property (1.2). If $\alpha_F$ in (A.2) satisfies
\begin{equation}
\alpha_F < 1,
\end{equation}
then $F$ is bi-Lipschitz.
then $F$ is a bi-Lipschitz map.

Proof. Given $x, y \in B_1$ with $y \neq 0$,

$$F(x + y) - F(x) = \int_0^1 F'(x + ty)y dt = \left( \int_0^1 \|F'(x + ty)y\| dt \right) z$$

$$+ \int_0^1 \|F'(x + ty)y\| \left( \frac{F'(x + ty)y}{\|F'(x + ty)y\|} - z \right) dt,$$

where $z \in B_2$ with $\|z\| = 1$. Thus

$$\|F(x + y) - F(x)\| \geq \left( \int_0^1 \|F'(x + ty)y\| dt \right)$$

$$\times \left( 1 - \inf_{\|z\| = 1} \sup_{0 \leq t \leq 1} \left\| \frac{F'(x + ty)y}{\|F'(x + ty)y\|} - z \right\| \right)$$

$$\geq (1 - \alpha_F) \left( \int_0^1 \|F'(x + ty)y\| dt \right) \geq (1 - \alpha_F) A \|y\|,$$

and

$$\|F(x + y) - F(x)\| \leq \int_0^1 \|F'(x + ty)y\| dt \leq B \|y\|,$$

where $A, B$ are lower and upper stability bounds in the uniform stability property (1.2). Combining the above two estimates completes the proof.

Remark A.4. The U-shaped map $E_{p,\epsilon}$ in Example A.2 with $p = \infty$ and $\epsilon = 0$ is not a bi-Lipschitz map and

$$\alpha_{E_{\infty,0}} = \inf_{\|z\| = 1} \sup_{|t| \leq \pi/2} \left\| \frac{(\cos t, \sin t)}{\max(|\cos t|, |\sin t|)} - z \right\|_{\infty}$$

$$= \sup_{|t| \leq \pi/2} \left\| \frac{(\cos t, \sin t)}{\max(|\cos t|, |\sin t|)} - (1, 0) \right\|_{\infty} = 1.$$

This indicates that the geometric condition [A.3] about $\alpha_F$ is optimal.

For a differentiable map $F$ not far away from a bounded below linear operator $T$, we suggest using $Ty/\|Ty\|$ as the center of the ball containing the set of unit vectors $F'(x)y/\|F'(x)y\|, x \in B_1$, and define the minimal radius of that ball by $\beta_{F,T}$ in (1.5). Then obviously

$$\alpha_F \leq \beta_{F,T}. \tag{A.4}$$

This together with Theorem A.3 implies that a differentiable map $F$ satisfying $\beta_{F,T} < 1$ is a bi-Lipschitz map.

Theorem A.5. Let $B_1$ and $B_2$ be Banach spaces, and $F$ be a continuously differentiable map from $B_1$ to $B_2$ with its derivative having the uniform stability property (1.2). If $T \in B(B_1, B_2)$ is bounded below and satisfies (1.4), then $F$ is a bi-Lipschitz map.
We may use the following quantity to measure the distance between differentiable map $F$ and bounded below linear operator $T$,

$$\delta_{F,T} := \sup_{0 \neq y \in B_1} \sup_{x \in B_1} \frac{\|F(x + y) - F(x) - Ty\|}{\|Ty\|} = \sup_{0 \neq y \in B_1} \sup_{z \in B_1} \frac{\|F'(z)y - Ty\|}{\|Ty\|}.$$  

By direct computation,

$$\beta_{F,T} \leq \sup_{\|y\| = 1} \sup_{x \in B_1} \left( \frac{\|F'(x)y - Ty\|}{\|F'(x)y\|} + \frac{\|F'(x)y\| - \|Ty\|}{\|F'(x)y\|} \right) \leq \frac{2\delta_{F,T}}{1 - \delta_{F,T}}.$$  

Thus the geometric condition (1.4) in Theorem A.5 can be replaced by the condition $\delta_{F,T} < 1/3$.

**Corollary A.6.** Let $B_1, B_2, F$ and $T$ be as in Theorem A.5. If $\delta_{F,T} < 1/3$, then $F$ is a bi-Lipschitz map.

The geometric condition (1.4) to guarantee the bi-Lipschitz property for the map $F$ is optimal in general Banach space setting, as $\beta_{E_{\infty,0}, T_1} = 1$ for the U-shaped map $E_{\infty,0}$ in Example A.2 and the linear operator $T_1 t := (t, 0), t \in \mathbb{R}$. But in Hilbert space setting, as shown in the next theorem, the geometric condition (1.4) could be relaxed to $\beta_{F,T} < \sqrt{2}$.

**Theorem A.7.** Let $H_1$ and $H_2$ be Hilbert spaces, and let $F : H_1 \rightarrow H_2$ be a continuously differentiable map with its derivative having the uniform stability property (1.2). If there exists a linear operator $T \in \mathcal{B}(H_1, H_2)$ satisfying (1.3) and (1.6), then $F$ is a bi-Lipschitz map.

**Proof.** Take $u, v \in H_1$ with $v \neq 0$. Then

$$\|F(u + v) - F(u)\| \leq \int_0^1 \|F'(u + tv)v\| dt \leq B\|v\|,$$

where $B$ is the upper stability bound in (1.2). Observe that

$$\langle F'(u)v, Tv \rangle = \|F'(u)v\| \|Tv\| \left( 1 - \frac{1}{2} \frac{\|F'(u)v\|}{\|Tv\|} - \frac{Tu}{\|Tv\|} \right).$$  

Then

$$\langle F'(u)v, Tv \rangle \geq \frac{2 - (\beta_{F,T})^2}{2} \|F'(u)v\| \|Tv\|,$$

which implies that

$$\langle F(u + v) - F(u), Tv \rangle = \int_0^1 \langle F'(u + tv)v, Tv \rangle dt \geq \frac{2 - (\beta_{F,T})^2}{2} \left( \int_0^1 \|F'(u + tv)v\| dt \right) \|Tv\| \geq \frac{2 - (\beta_{F,T})^2}{2} A \|Tv\| \|v\|,$$

where $A$ is the lower stability bound in (1.2). Hence

$$\|F(u + v) - F(u)\| \geq \frac{\langle F(u + v) - F(u), Tv \rangle}{\|Tv\|} \geq \frac{2 - (\beta_{F,T})^2}{2} A \|v\|.$$
Combining (A.6) and (A.8) proves the bi-Lipschitz property for $F$.

**Remark A.8.** The geometric condition (1.6) is optimal as for the U-shaped map $E_{p, \epsilon}$ in Example A.2 with $p = 2$ and $\epsilon = 0$,

\[
\beta_{E_{2,0,T_1}} = \sup_{\tilde{t} \neq 0} \sup_{|t| \leq \pi/2} \left\| \frac{(\tilde{t} \cos t, \tilde{t} \sin t)}{(\cos^2 t + \sin^2 t)^{1/2}|\tilde{t}|} - (\tilde{t}, 0) \right\|_2 = \sqrt{2}
\]

where $T_1 \tilde{t} = (\tilde{t}, 0), \tilde{t} \in \mathbb{R}$.

Define

\[
\theta_{F,T} = \sup_{u \in H_1, v \neq 0} \arccos \left( \frac{\langle F'(u)v, Tv \rangle}{\|F'(u)v\|\|Tv\|} \right),
\]

the maximal angle between vectors $F'(u)v$ and $Tv$ in the Hilbert space $H_2$. Then

\[
\beta_{F,T} = 2 \sin \frac{\theta_{F,T}}{2}.
\]

So the geometric condition (1.6) can be interpreted as that the angles between $F'(u)v$ and $Tv$ are less than or equal to $\theta_{F,T} \in [0, \pi/2)$ for all $u, v \in H_1$. The above equivalence between the geometric condition (1.6) and the angle condition $\theta_{F,T} < \pi/2$, together with (1.2) and (1.3), implies the existence of positive constants $A_1, B_1$ such that

\[
(A.10) \quad A_1 \|Tv\|^2 \leq \langle F'(u)v, Tv \rangle \leq B_1 \|Tv\|^2, \quad u, v \in H_1.
\]

The converse can be proved to be true too. Thus $\beta_{F,T} < \sqrt{2}$ if and only if $S := T^*F$ is strictly monotonic [55] if there exist positive constants $m$ and $M$ such that

\[
m\|u - v\|^2 \leq \langle u - v, S(u) - S(v) \rangle \leq M\|u - v\|^2 \quad \text{for all } u, v \in H.
\]

As an application of the above equivalence, Theorem A.7 can be reformulated as follows.

**Theorem A.9.** Let $H_1$ and $H_2$ be Hilbert spaces, and let $F : H_1 \to H_2$ be a continuously differentiable map with its derivative having the uniform stability property (1.2). If there exists a linear operator $T \in \mathcal{B}(H_1, H_2)$ satisfying (1.3) and (A.10), then $F$ is a bi-Lipschitz map.

From Theorem A.9 we obtain the following result similar to the one in Corollary A.6.

**Corollary A.10.** Let $H_1, H_2$ and $F$ be as in Theorem A.9. If there exists a bounded below linear operator $T \in \mathcal{B}(H_1, H_2)$ with $\delta_{F,T} < \sqrt{2} - 1$, then $F$ is a bi-Lipschitz map.

Given a differentiable map $F$, it is quite technical in general to construct linear operator $T$ satisfying (1.3) and (1.4) in Banach space setting (respectively (1.3) and (A.10) in Hilbert space setting). A conventional selection is that $T = F'(x_0)$ for some $x_0 \in B_1$, but such a selection is not always favorable. Let $\Phi = (\phi_\lambda)_{\lambda \in \Lambda}$ be impulse response vector with its entry $\phi_\lambda$ being the impulse response of the signal generating device at the innovation position $\lambda \in \Lambda$, and $\Psi = (\psi_\gamma)_{\gamma \in \Gamma}$ be sampling functional vector with entry $\psi_\gamma$ reflecting the characteristics of the acquisition
device at the sampling position $\gamma \in \Gamma$. In order to consider bi-Lipschitz property of the nonlinear sampling map

$$S_{f, \Phi, \Psi} : \ell^2(\Lambda) \ni x \mapsto x^T \Phi \text{ companding} \mapsto f(x^T \Phi) \text{ sampling} \mapsto \langle f(x^T \Phi), \Psi \rangle \in \ell^2(\Gamma)$$

related to instantaneous companding $h(t) \mapsto f(h(t))$, a linear operator

$$T := A_{\Phi, \Psi}(A_{\Phi, \Psi}^{-1}A_{\Phi, \Psi})^{-1}A_{\Phi, \Psi}(A_{\Phi, \Psi})^{-1}$$

satisfying (1.3) and (A.10) is implicitly introduced in [48], where

$$A_{\Phi, \Psi} = ((\phi_\lambda, \psi_\gamma))_{\lambda \in \Lambda, \gamma \in \Gamma}$$

is the inter-correction matrix between $\Phi$ and $\Psi$.

**APPENDIX B. SPARSE APPROXIMATION TRIPLE**

In this appendix, we prove Propositions 5.2 and 4.2, and conclude it with a remark on the greedy algorithm (5.3).

**Proof of Proposition 5.2.** The convergence of $x_{A,M}^k$, $k \geq 0$, follows from

$$\sum_{k=0}^{K} \|x_{A,M}^{k+1} - x_{A,M}^k\|_M = \|x\|_M - \|x - x_{A,M}^{K+1}\|_M \leq \|x\|_M, \quad K \geq 0,$$

by the norm splitting property (1.15). Denote by $x_{A,M}^\infty \in M$ the limit of $x_{A,M}^k$, $k \geq 0$. Then the limit $x_{A,M}^\infty$ satisfies the following consistency condition:

(B.1) $$\langle x_{A,M}^\infty, y \rangle = \langle x, y \rangle$$

for all $y \in A$. The above consistency condition holds as $0 = \arg\min_{x \in A} \|x - x_{A,M}^\infty - \hat{x}\|_M$, which together with the norm-splitting property (1.14) in $H_1$ implies that the projection of $x - x_{A,M}^\infty$ onto $A_i$ are zero for all $i \in I$. From the consistency condition (B.1), we conclude that (B.1) hold for all $y \in kA$, $k \geq 0$, and hence for all $y$ in the closure of $\cup_{k \geq 0} kA$. This together with the sparse density property of the sparse approximation triple $(A, M, H_1)$ proves the convergence of $x_{A,M}^k$, $k \geq 0$, to $x \in M$. \qed

**Proof of Proposition 4.2.** Take $0 \neq x \in M$ and let $x_{A,M}^k$, $k \geq 0$, be as in the greedy algorithm (5.3). Write $u_k = x_{A,M}^{k+1} - x_{A,M}^k$, $k \geq 0$. Thus

$$u_k = \arg\min_{x \in A} \|x - x_{A,M}^k - \hat{x}\|_M = \arg\min_{x \in A} \|(x - x_{A,M}^{k-1}) - u_{k-1} - \hat{x}\|_M \in A,$$

and $x - x_{A,M} = \sum_{k \geq 1} u_k$ by Proposition 5.2. This together with (1.15) and (1.19) implies that

$$\|x - x_{A,M}\|_{H_1} \leq \sum_{k \geq 1} \|u_k\|_{H_1} \leq \sqrt{a_A} \sum_{k \geq 1} \|u_{k-1}\|_M = \sqrt{a_A} \|x\|_M.$$

This completes the proof. \qed
Given a sparse approximation triple \((A, M, H_1)\), we say that \(x \in M\) is compressible \([8, 25, 26, 42]\) if \(\{\sigma_{kA, M}(x)\}_{k=1}^{\infty}\) having rapid decay, such as
\[
\sigma_{kA, M}(x) \leq C k^{-\alpha} \quad \text{for some } C, \alpha > 0,
\]
where \(\sigma_{kA, M}(x)\) is the best approximation error of \(x\) from \(kA\),
\[
(B.2) \quad \sigma_{kA, M}(x) := \inf_{\hat{x} \in kA} \|\hat{x} - x\|_M, \quad k \geq 1.
\]

For the sequence \(x_{A, M}^k, k \geq 0\), in the greedy algorithm \((5.3)\), we have
\[
(B.3) \quad \|x_{A, M}^k - x\|_M \geq \sigma_{kA, M}(x),
\]
as \(x_{A, M}^k \in kA, k \geq 1\). The above inequality becomes an equality in the classical sparse recovery setting. We do not know whether and when the greedy algorithm \((5.3)\) is suboptimal, i.e., there exists a positive constant \(C\) such that
\[
(B.4) \quad \|x_{A, M}^k - x\|_M \leq C \sigma_{kA, M}(x), \quad x \in M,
\]
even for compressible signals. The reader may refer to \([54]\) for the study of various greedy algorithms.

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Qiyu Sun: Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA. Email: qiyu.sun@ucf.edu

Wai-shing Tang, Department of Mathematics, National University of Singapore, Singapore 119076 Republic of Singapore. Email: mattws@nus.edu.sg