Decomposition of planar graphs with forbidden configurations

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Abstract

A \((d, h)\)-decomposition of a graph \(G\) is an ordered pair \((D, H)\) such that \(H\) is a subgraph of \(G\) of maximum degree at most \(h\) and \(D\) is an acyclic orientation of \(G - E(H)\) with maximum out-degree at most \(d\). In this paper, we prove that for \(l \in \{5, 6, 7, 8, 9\}\), every planar graph without 4- and \(l\)-cycles is \((2, 1)\)-decomposable. As a consequence, for every planar graph \(G\) without 4- and \(l\)-cycles, there exists a matching \(M\), such that \(G - M\) is 3-DP-colorable and has Alon-Tarsi number at most 3. In particular, \(G\) is 1-defective 3-DP-colorable, 1-defective 3-paintable and 1-defective 3-choosable. These strengthen the results in [Discrete Appl. Math. 157 (2) (2009) 433–436] and [Discrete Math. 343 (2020) 111797].

Keywords: decomposition; list coloring; defective coloring; Alon-Tarsi number; DP-coloring

1 Introduction

A proper \(k\)-coloring of a graph \(G\) is a mapping \(\phi : V(G) \to [k]\) such that \(\phi(u) \neq \phi(v)\), whenever \(uv \in E(G)\), where and herein after, \([k] = \{1, 2, \ldots, k\}\). The least integer \(k\) such that \(G\) admits a proper \(k\)-coloring is the chromatic number \(\chi(G)\) of \(G\). Let \(h\) be a non-negative integer. An \(h\)-defective \(k\)-coloring of \(G\) is a mapping \(\phi : V(G) \to [k]\) such that each color class induces a subgraph of maximum degree at most \(h\). In particular, a 0-defective coloring is a proper coloring of \(G\).

A \(k\)-list assignment of \(G\) is a mapping \(L\) that assigns a list \(L(v)\) of \(k\) colors to each vertex \(v\) in \(G\). An \(h\)-defective \(L\)-coloring of \(G\) is an \(h\)-defective coloring \(\psi\) of \(G\) such that \(\psi(v) \in L(v)\) for all \(v \in V(G)\). A graph \(G\) is \(h\)-defective \(k\)-choosable if \(G\) admits an \(h\)-defective \(L\)-coloring for each \(k\)-list assignment \(L\). In particular, if \(G\) is 0-defective \(k\)-choosable, then we call it \(k\)-choosable. The choice number \(ch(G)\) is the smallest integer \(k\) such that \(G\) is \(k\)-choosable.

Cowen, Cowen, and Woodall [2] proved that every outerplanar graph is 2-defective 2-colorable, and every planar graph is 2-defective 3-colorable. Eaton and Hull [6], and independently, Škreklovski [12] proved that every outerplanar graph is 2-defective 2-choosable, and every planar graph is 2-defective 3-choosable. Cushing and Kierstead [3] proved that every planar graph is 1-defective 4-choosable. Let \(\mathcal{G}_{4, l}\) be the family of planar graphs which contain no 4-cycles and no \(l\)-cycles. Lih et al. [10] proved that for each \(l \in \{5, 6, 7\}\), every graph \(G \in \mathcal{G}_{4, l}\) is 1-defective 3-choosable. Dong and Xu [4] proved that for each \(l \in \{8, 9\}\), every graph \(G \in \mathcal{G}_{4, l}\) is 1-defective 3-choosable.

Note that a graph being \(h\)-defective \(k\)-choosable means that for every \(k\)-list assignment \(L\) of \(G\), there exists a subgraph \(H\) (depending on \(L\)) of \(G\) with \(\Delta(H) \leq h\) such that \(G - E(H)\) is \(L\)-colorable. The subgraph \(H\) may be different for different \(L\). As a strengthening of the above results, the following problem is studied in the literature: For \((h, k) \in \{(2, 3), (1, 4)\}\), is it true that every planar graph \(G\) has a subgraph of maximum

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neighbors. It is well-known and easy to see that
matching \( M \) of a graph \( G \) of maximum
degree \( h \) creates the above mentioned linear ordering and the corresponding coloring is easily obtained by using a
We just note here that for any graph \( G \) larger than each of
et al. [1] proved that every planar graph is
AT
such that
AT
Tarsi number
Fig. 1 and Fig. 2, and if \( G \) has no subgraph isomorphic to any configuration in
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Indeed, stronger results were proved in [7, 11]. The results concern two other graph parameters: The Alon-
Tarsi number \( AT(G) \) of \( G \) and the paint number \( \chi_P(G) \) of \( G \). The reader is referred to [7] for the definitions.
We just note here that for any graph \( G \), \( ch(G) \) ≤ \( \chi_P(G) \) ≤ \( AT(G) \), and the differences \( \chi_P(G) - ch(G) \) and \( AT(G) - \chi_P(G) \) can be arbitrarily large. It was proved in [7] that every planar graph \( G \) has a matching \( M \) such that \( AT(G - M) \) ≤ 4, and proved in [11] that for \( l \) ∈ \{5, 6, 7\}, every graph \( G \) ∈ \( G_{4,l} \) has a matching \( M \) such that \( AT(G - M) \) ≤ 3.
In this paper, we consider further strengthening of the results concerning graphs in \( G_{4,l} \) for \( l \) ∈ \{5, 6, 7, 8, 9\}. (Note that the result in [11] does not cover the cases for \( l = 8 \) and 9). We strengthen the above results in two aspects: a larger class of graphs with a stronger property.
Given two non-negative integers \( d, h \) and a graph \( G \), a \((d, h)\)-decomposition of \( G \) is a pair \((D, H)\) such that \( H \) is a subgraph of \( G \) of maximum degree at most \( h \) and \( D \) is an acyclic orientation of \( G - E(H) \) with maximum out-degree at most \( d \). We say \( G \) is \((d, h)\)-decomposable if \( G \) has a \((d, h)\)-decomposition. Cho et al. [1] proved that every planar graph is \((4, 1)\)-decomposable, \((3, 2)\)-decomposable and \((2, 6)\)-decomposable.
Note that a graph \( H \) which has an acyclic orientation of maximum out-degree at most \( d \) if and only if \( H \) is \( d\)-degenerate, i.e., the vertices of \( H \) can be linearly ordered so that each vertex has at most \( d \) backward neighbors. It is well-known and easy to see that \( d\)-degenerate graphs not only have choice number, paint number, Alon-Tarsi number and DP-chromatic number at most \( d + 1 \), there is a linear time algorithm that creates the above mentioned linear ordering and the corresponding coloring is easily obtained by using a greedy coloring algorithm. The reader is referred to [5] for the definition of DP-chromatic number \( \chi_{DP}(G) \) of a graph \( G \). We just mention here that \( ch(G) \) ≤ \( \chi_{DP}(G) \), and there are graphs \( G \) for which \( \chi_{DP}(G) \) are larger than each of \( AT(G) \) and \( \chi_P(G) \), there are also graphs \( G \) for which \( \chi_{DP}(G) \) are smaller than each of \( AT(G) \) and \( \chi_P(G) \) [9]. This paper proves the following result:

**Theorem 1.1.** Assume \( G \) is a plane graph. Then \( G \) is \((2, 1)\)-decomposable if one of the following holds:

1. \( G \) has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2.
2. \( G \) has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3.
3. \( G \) ∈ \( G_{4,9} \).

Note that if \( G \) ∈ \( G_{4,l} \) for some \( l \) ∈ \{5, 6, 7\}, then \( G \) has no subgraph isomorphic to any configuration in
Fig. 1 and Fig. 2, and if \( G \) ∈ \( G_{4,8} \), then \( G \) has no subgraph isomorphic to any configuration in Fig. 1 and
Fig. 2: Forbidden configurations in (1) of Theorem 1.1.

Fig. 3: Forbidden configurations in (2) of Theorem 1.1.

All graphs in this paper are finite and simple. For a plane graph $G$, we use $V(G)$, $E(G)$ and $F(G)$ to denote the vertex set, edge set and face set of $G$, respectively. For any element $x \in V(G) \cup F(G)$, the degree of $x$ is denoted by $d(x)$. A vertex $v$ in $G$ is called a $k$-vertex, or $k^+$-vertex, or $k^-$-vertex, if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. Analogously, one can define $k$-face, $k^+$-face, and $k^-$-face. An $n$-face $[x_1x_2\ldots x_n]$ is a $(d_1, d_2, \ldots, d_n)$-face if $d(x_i) = d_i$ for $1 \leq i \leq n$. Let $D$ be an orientation of a graph $G$, we use $d^+_D(v)$ and $d^-_D(v)$ to denote the out-degree and in-degree of a vertex $v$ in $D$, respectively. Let $\Delta^+(D)$ denote the maximum out-degree of vertices in $D$. Two cycles (or faces) are adjacent if they have at least one common edge. Two cycles (or faces) are normally adjacent if they intersect in exactly two vertices. Let $G$ be a plane graph and $xy$ be a given boundary edge of $G$. A vertex $v \neq x, y$ is called a normal vertex. A vertex $v$ is special if $v$ is a $5^+$-vertex or $v \in \{x, y\}$. A face is internal if it is not the outer face $f_0$. A face is special if it is an internal $7^+$-face or the outer face $f_0$. A normal vertex $v$ is minor if $d(v) = 3$ and it is incident with an internal $4^-$-face. A good 5-face is an internal 5-face adjacent to at least one internal 3-face. An edge contained in a triangle is a triangular edge. Note that in all three cases, there are no adjacent triangles. So every triangular edge is contained in a unique triangle.

2 Proof of Theorem 1.1

For the purpose of using induction, we prove the following result. Assume $G$ is a plane graph and $e = xy$ is a boundary edge of $G$. A nice decomposition of $(G, e)$ is a pair $(D, M)$ such that $M$ is a matching and $D$ is an acyclic orientation of $G - M$ with $d^+_D(x) = d^-_D(y) = 0$ and $\Delta^+(D) \leq 2$. Note that in a nice decomposition $(D, M)$ of $(G, e)$, since $d^+_D(x) = d^-_D(y) = 0$, we conclude that $e = xy \in M$.

**Theorem 2.1.** If $G$ is a plane graph satisfying the condition of Theorem 1.1 and $e$ is a boundary edge of $G$, then $(G, e)$ has a nice decomposition.

Assume Theorem 2.1 is not true and $G$ is a counterexample with minimum number of vertices. We shall
derive a sequence of properties of $G$ that lead to a contradiction. It is obvious that $G$ is connected, for otherwise we can consider each component of $G$ separately.

**Lemma 2.2.** $G$ is 2-connected.

**Proof.** Assume to the contrary that $G$ has a cut-vertex $x'$. Let $G = H_1 \cup H_2$, $V(H_1 \cap H_2) = \{x'\}$ and $e = xy \in E(H_1)$. Let $e' = x'y'$ be a boundary edge of $H_2$. By the minimality of $G$, there is a nice decomposition $(D_1, M_1)$ of $(H_1, e)$ and a nice decomposition $(D_2, M_2)$ of $(H_2, e')$. Let $M = (M_1 \cup M_2) \setminus \{x'y'\}$ and $D = D_1 \cup D_2 \cup \{x'y'\}$. It is straightforward to verify that $(D, M)$ is a nice decomposition of $(G, e)$. \hfill \Box

**Lemma 2.3.** For any $v \in V(G) \setminus \{x, y\}$, $d(v) \geq 3$.

**Proof.** Assume $v \in V(G) \setminus \{x, y\}$ and $d(v) \leq 2$. By the minimality of $G$, there exists a nice decomposition $(D, M)$ of $(G - v, e)$. Let $D'$ be obtained from $D$ by orienting edges incident with $v$ as out-going edges from $v$. Then $(D', M)$ is a nice decomposition of $(G, e)$. \hfill \Box

**Lemma 2.4.** If $u$ and $v$ are two adjacent 3-vertices, then $\{u, v\} \cap \{x, y\} \neq \emptyset$.

**Proof.** Suppose that $u$ and $v$ are two adjacent 3-vertices with $\{u, v\} \cap \{x, y\} = \emptyset$. By the minimality of $G$, there is a nice decomposition $(D, M)$ of $(G - \{u, v\}, e)$. Let $M' = M \cup \{uv\}$, and $D'$ be obtained from $D$ by orienting the other edges incident with $u, v$ as out-going edges from $u, v$. Then $(D', M')$ is a nice decomposition of $(G, e)$. \hfill \Box

For an internal face $f$, let $t_f$ be the number of incident normal 3-vertices and let $s_f$ be the number of adjacent internal 3-faces. Note that each 3-vertex of $f$ is incident with at most one 3-face adjacent to $f$. Thus we have the following corollary.

**Corollary 2.5.** For any internal face $f$, $t_f \leq d(f)/2$ and $t_f + s_f \leq d(f)$.

The following four lemmas first appeared in [11], although the hypotheses and some definitions are slightly different. For the completeness of this paper, we include the short proofs with illustration figures.

![Fig. 4: (a) A bad 5-cycle and an adjacent triangle. (b) For the proof of Lemma 2.6. Here and in figures below, a solid triangle represents a 3-vertex, a solid square represents a 4-vertex, a thick line represents an edge in the matching $M$.](image)

A 5-cycle $[u_1 u_2 u_3 u_4 u_5]$ is a bad 5-cycle if it is adjacent to a triangle $[u_1 u_5 u_6]$ with $u_i \notin \{x, y\}$, where $1 \leq i \leq 6$, and $d(u_1) = d(u_3) = 3$, and $d(u_2) = d(u_4) = d(u_5) = d(u_6) = 4$, as depicted in Fig. 4(a).

**Lemma 2.6** (Lemma 5.2 in [11]). There are no bad 5-cycles in $G$. -

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Proof of Lemma 2.6. Assume $C = [u_1u_2u_3u_4u_5]$ is a bad 5-cycle and $T = [u_1u_5u_6]$ is a triangle adjacent to $C$, where $d(u_1) = d(u_3) = 3$ and $d(u_i) = 4$ for $i \in \{2, 4, 5\}$, as depicted in Fig. 4(a). A nice decomposition of $G - \{u_1, u_2, \ldots, u_6\}$ is extended to a nice decomposition as in Fig. 4(b).

A triangle $T$ is minor if $T$ is a $(3, 4, 4)$-triangle and $T \cap \{x, y\} = \emptyset$. A triangle chain in $G$ is a subgraph of $G - \{x, y\}$ consisting of vertices $w_1, w_2, \ldots, w_{k+1}, u_1, u_2, \ldots, u_k$ in which $[w_iw_{i+1}u_i]$ is a $(4, 4, 4)$-cycle for $1 \leq i \leq k$, as depicted in Fig. 5. We denote $T_i$ the triangle $[w_iw_{i+1}u_i]$ and denote such a triangle chain by $T_1T_2\ldots T_k$. If a triangle $T$ has exactly one common vertex with a triangle chain $T_1T_2\ldots T_k$ and the common vertex is in $T_1$, then we say $T$ intersects the triangle chain $T_1T_2\ldots T_k$.

![Fig. 5: A triangle chain.](image)

Lemma 2.7 (Lemma 2.10 in [11]). If a minor triangle $T_0 = [w_0w_1u_0]$ intersects a triangle chain $T_1T_2\ldots T_k$, then every 3-vertex adjacent to a vertex in $T_k$ belongs to $\{x, y\} \cup V(T_0)$.

The $k = 0$ case of the above lemma asserts that every 3-vertex adjacent to a vertex in $T_0$ belongs to $\{x, y\}$.

Proof of Lemma 2.7. Assume $G$ has a minor triangle $T_0 = [w_0w_1u_0]$ intersecting a triangle chain $T_1T_2\ldots T_k$, and $z \notin \{x, y\} \cup V(T_0)$ is a 3-vertex adjacent to a vertex in $T_k$, as depicted in Fig. 6(a). A nice decomposition of $G - (\bigcup_{i=0}^k V(T_i) \cup \{z\})$ is extended to a nice decomposition of $G$ as in Fig. 6(b).

Lemma 2.8 (Lemma 2.11 in [11]). If a minor triangle $T_0$ intersects a triangle chain $T_1T_2\ldots T_k$, then the distance between $T_k$ and another minor triangle is at least two.
Proof of Lemma 2.8. Assume to the contrary that $T_1 T_2 \ldots T_k$ with $T_i = [w_i w_{i+1} u_i], 1 \leq i \leq k$, is a triangle chain that intersects a minor triangle $T_0 = [w_0 w_1 u_0]$, and the distance between $T_k$ and another minor triangle $T_0' = [z_1 z_2]$ with $d(z_1) = 3$ is less than 2. By Lemma 2.7, we may assume $w_{k+1} z$ is a $(4, 4)$-edge connecting $T_k$ and $T_0'$, as depicted in Fig. 7(a). A nice decomposition of $G - (\bigcup_{i=0}^{k} V(T_i) \cup V(T_0'))$ is extended to a nice decomposition of $G$ as in Fig. 7(b).\hfill\square

Lemma 2.9 (Lemma 3.1 in [11]). Assume that $f$ is a 6-face adjacent to five 3-faces, and none of the vertices on these 3-faces is in \{x, y\}. If $f$ is incident with a 3-vertex, then there is at least one 5$^+$-vertex on these five 3-faces.

Proof of Lemma 2.9. Let $f = [v_1 v_2 v_3 v_4 v_5 v_6]$ be a 6-face, $v_1$ be a 3-vertex and $T_i = [v_i v_{i+1} u_i], 1 \leq i \leq 5$, be the five 3-faces. Assume to the contrary that there is no 5$^+$-vertex on $T_i$. By Lemma 2.7, we may assume all $v_{i+1}$ and $u_i$ are 4-vertices for $1 \leq i \leq 5$, as depicted in Fig. 8(a). A nice decomposition of $G - (\bigcup_{i=1}^{5} V(T_i))$ is extended to a nice decomposition of $G$ as in Fig. 8(b).\hfill\square
The above lemmas present some reducible configurations. We use standard discharging method to prove that there must be some reducible configurations in a minimum counterexample, which leads to a contradiction.

First, we define an initial charge function by \( \mu(x) = d(x) - 4 \), \( \mu(y) = d(y) - 4 \), \( \mu(f_0) = d(f_0) + 4 \), and \( \mu(v) = d(v) - 4 \) for each vertex \( v \in V(G) \setminus \{x, y\} \), \( \mu(f) = d(f) - 4 \) for each face \( f \) other than \( f_0 \). By Euler’s formula and handshaking theorem, we obtain that the sum of all the initial charges is zero, i.e.,

\[
(d(x) - 4) + (d(y) - 4) + (d(f_0) + 4) + \sum_{v \neq x, y} (d(v) - 4) + \sum_{f \neq f_0} (d(f) - 4) = 0.
\]

Next, we design some discharging rules to redistribute the charges, such that the sum of the final charges is not zero, which leads to a contradiction.

### Discharging Rules

**R1.** Every internal 3-face \( f \) receives \( \frac{1}{3} \) from each adjacent face.

**R2.** Assume \( v \) is a normal 3-vertex. If \( v \) is incident with an internal 4\(^{-}\)-face, then it receives \( \frac{1}{2} \) from each of the other two incident faces. Otherwise it receives \( \frac{1}{3} \) from each incident face.

**R3.** Let \( v \) be a normal 5-vertex. Then \( v \) sends \( \frac{1}{3} \) to each incident 4\(^{+}\)-face. If \( v \) is incident with a 3-face \( g = [uvw] \), then \( v \) sends \( \frac{1}{6} \) to the other face \( g' \) incident with uw. Moreover, if \( v \) is incident with three consecutive faces \( f_1, f_2, f_3 \) and \( f_1, f_3 \) are 3-faces, then \( v \) sends an extra \( \frac{1}{6} \) to \( f_2 \).

**R4.** Let \( v \) be a normal 6\(^{+}\)-vertex. Then \( v \) sends \( \frac{1}{3} \) to each incident 4\(^{+}\)-face. If \( v \) is incident with a 3-face \( g = [uvw] \), then \( v \) sends \( \frac{1}{4} \) to the other face \( g' \) incident with uw.

**R5.** Let \( v \) be a vertex in \( \{x, y\} \). Then it sends \( \frac{1}{3} \) to every incident internal 4\(^{+}\)-face. If \( v \) is incident with a 3-face \( g = [uvw] \), then \( v \) sends \( \frac{1}{3} \) to the other face \( g' \) incident with uw.

**R6.** \( f_0 \) sends \( \frac{1}{3} \) to each adjacent 4\(^{+}\)-face.

**R7.** In Case 2 (i.e., \( G \) has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3), every internal 5-face receives \( \frac{1}{8} \) from adjacent internal 6\(^{+}\)-faces via each common edge.

**R8.** In Case 3 (i.e., \( G \in \mathcal{G}_{4,9} \)), every good 5-face receives \( \frac{1}{3} \) from adjacent internal 7\(^{+}\)-faces via each common edge.

For \( z \in V(G) \cup F(G) \), let \( \mu'(z) \) be the final charge of \( z \). In the remainder of this paper, we prove that

\[
\sum_{z \in V(G) \cup F(G)} \mu'(z) > 0,
\]

which contradicts the fact that \( \sum_{z \in V(G) \cup F(G)} \mu'(z) = \sum_{z \in V(G) \cup F(G)} \mu(z) = 0 \).

Note that R7 only applies to Case 2 and R8 only applies to Case 3. Moreover, R7 and R8 only involve 5\(^{+}\)-faces.

It follows from R5 that for \( v \in \{x, y\} \)

\[
\mu'(v) \geq \mu(v) - (d(v) - 1) \times \frac{1}{3} = \frac{2d(v) - 11}{3} \geq -\frac{7}{3}.
\]

Note that \( f_0 \) sends \( \frac{1}{2} \) to each adjacent internal face by R1 and R6, and sends at most \( \frac{1}{7} \) to each incident normal 3-vertex by R2. It follows from Lemma 2.4 that \( f_0 \) is incident with at most \( \frac{d(f_0)}{12} \) normal 3-vertices. Then

\[
\mu'(f_0) \geq \mu(f_0) - \frac{d(f_0)}{2} \times \frac{1}{2} - d(f_0) \times \frac{1}{3} \geq \frac{5d(f_0)}{12} + 4 \geq \frac{21}{4}.
\]

Hence, \( \mu'(x) + \mu'(y) + \mu'(f_0) > 0 \). 

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Assume \( v \) is a normal 3-vertex. If \( v \) is incident with an internal 4\(^{-}\)-face, then the other two incident faces are 5\(^{+}\)-faces or the outer face \( f_0 \). Hence \( \mu'(v) = \mu(v) + 2 \times \frac{1}{2} = 0 \). Otherwise each face incident with \( v \) is a 5\(^{+}\)-face or \( f_0 \), and \( \mu'(v) = \mu(v) + 3 \times \frac{1}{3} = 0 \) by R2.

If \( v \) is a normal 4-vertex, then \( \mu'(v) = \mu(v) = 0 \). If \( v \) is a normal 5-vertex, then it is incident with at most two 3-faces, and then \( \mu'(v) \geq \mu(v) - 5 \times \frac{1}{5} - \frac{1}{6} = 0 \) by R3. If \( v \) is a normal 6\(^{+}\)-vertex, then \( \mu'(v) = \mu(v) - d(v) \times \frac{1}{3} = \frac{2(d(v) - 6)}{3} \geq 0 \) by R4.

If \( f \) is an internal 3-face, then it receives \( \frac{1}{3} \) via each incident edge, and \( \mu'(f) = \mu(f) + 3 \times \frac{1}{3} = 0 \) by R1. If \( f \) is an internal 4-face, then \( \mu'(f) \geq \mu(f) = 0 \).

It remains to show that \( \mu'(f) \geq 0 \) for internal 5\(^{+}\)-faces \( f \).

In the remainder of the paper, we consider the three cases separately in three subsections.

### 2.1 \( G \) has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 2

Assume that \( f = [v_1v_2v_3v_4v_5] \) is an internal 5-face. By Corollary 2.5, \( t_f \leq 2 \). If \( f \) is not adjacent to any internal 3-face, then \( \mu'(f) \geq \mu(f) - 2 \times \frac{1}{2} = 0 \) by R2. So we may assume that \( f \) is adjacent to at least one internal 3-face. Since the configurations Fig. 2(a)–2(d) are forbidden, \( f \) is adjacent to exactly one internal 3-face \( f^* \) and no 4-faces. If \( t_f \leq 1 \), then \( \mu'(f) \geq \mu(f) - \frac{1}{3} - \frac{1}{2} = 0 \) by R1 and R2. Assume \( t_f = 2 \) and \( f^* = [w_1w_2v_3] \) is an internal 3-face. If there are some special vertices in \( \{u, v_1, v_2, \ldots, v_5\} \), then \( f \) receives at least \( \frac{1}{6} \) from special vertices, and then \( \mu'(f) \geq \mu(f) - \frac{1}{3} - \left(\frac{1}{3} + \frac{1}{2}\right) + \frac{1}{6} = 0 \) by R1, R2, R3, R4 and R5. So we may assume that none of \( \{u, v_1, v_2, \ldots, v_5\} \) is a special vertex. It follows that \( f \) is incident with two 3-vertices and three 4-vertices. If neither \( v_1 \) nor \( v_2 \) is a 3-vertex, then \( \mu'(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{4} = 0 \) by R1 and R2.

Without loss of generality, assume that \( d(v_2) = 3 \) and \( d(v_1) = d(v_3) = d(u) = 4 \). If \( d(v_4) = 3 \) and \( d(v_5) = 4 \), then it contradicts Lemma 2.6. If \( d(v_4) = 4 \) and \( d(v_5) = 3 \), then it contradicts Lemma 2.7.

Assume that \( f = [v_1v_2v_3v_4v_5] \) is an internal 6-face. By Corollary 2.5, \( t_f \leq 3 \).

- \( t_f = 3 \). Without loss of generality, assume that \( v_1, v_3 \) and \( v_5 \) are normal 3-vertices.

By Corollary 2.5, \( s_f \leq 3 \). If \( s_f \leq 1 \), then \( \mu'(f) \geq \mu(f) - \frac{1}{3} - 3 \times \frac{1}{4} = 0 \) by R1 and R2.

Assume that \( s_f = 2 \). By symmetry, assume that one of the adjacent internal 3-face is \( [v_1v_2u] \). By Lemma 2.7, one vertex in \( \{u, v_2\} \) is a special vertex. Thus, \( \mu'(f) \geq \mu(f) - 2 \times \frac{1}{3} - 3 \times \frac{1}{4} + \frac{1}{6} = 0 \) by R1, R2, R3, R4 and R5.

Assume that \( s_f = 3 \).

(i) \( v_i v_{i+1} \) is incident with an internal 3-face \( [v_{i+1}u_i] \) for \( i \in \{1, 3, 5\} \). For each \( i \in \{1, 3, 5\} \), by Lemma 2.7, there is a special vertex in \( \{u_i, v_{i+1}\} \). Thus \( f \) receives at least \( \frac{1}{3} \) from \( \{u_i, v_{i+1}\} \) by R3, R4 and R5. Hence, \( \mu'(f) \geq \mu(f) - 3 \times \frac{1}{3} - 3 \times \frac{1}{4} + \frac{1}{6} = 0 \) by R1, R2, R3, R4 and R5.

(ii) \( v_i v_{i+1} \) is incident with an internal 3-face \( [v_{i+1}u_i] \) for \( i \in \{1, 2, 5\} \). If \( v_2 \) is a special vertex, then \( f \) receives \( \frac{1}{4} \) from \( v_2 \). Otherwise, \( v_2 \) is a normal 4-vertex. By Lemma 2.7, both \( u_1 \) and \( u_2 \) are special vertices. Then \( f \) receives at least \( \frac{1}{3} \) from \( u_1 \) and \( u_2 \) by R3, R4 and R5. In any way, \( f \) receives at least \( \frac{1}{6} \) from \( \{u_1, u_2, v_2\} \). On the other hand, one of \( u_3 \) and \( v_6 \) is also a special vertex, and \( f \) receives at least \( \frac{1}{3} \) from \( \{u_3, v_6\} \) by R3, R4 and R5. Thus, \( \mu'(f) \geq \mu(f) - 3 \times \frac{1}{3} - 3 \times \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = 0 \).

- \( t_f = 2 \). By Corollary 2.5, \( s_f \leq 4 \). If \( s_f \leq 3 \), then \( \mu'(f) \geq \mu(f) - 3 \times \frac{1}{3} - 2 \times \frac{1}{2} = 0 \) by R1 and R2.

Assume \( s_f = 4 \). We claim that \( f \) will receive at least \( \frac{1}{3} \) from vertices. If \( f \) is incident with a 2-vertex, then the 2-vertex must be in \( \{x, y\} \), and \( f \) receives at least \( \frac{1}{3} \) from incident 2-vertices by R5. So we may assume that \( f \) is not incident with any 2-vertex. By symmetry, it suffices to consider five cases.

(1) The four adjacent internal 3-faces are \( [v_i v_{i+1} u_i] \) for \( 1 \leq i \leq 4 \). Thus, the two normal 3-vertices must be \( v_1 \) and \( v_5 \). If one of \( v_2, v_3 \) and \( v_4 \) is a special vertex, then \( f \) receives \( \frac{1}{3} \) from it by R3, R4 and R5. So we may assume that \( v_2, v_3 \) and \( v_4 \) are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, there are at least two special vertices in \( \{u_1, u_2, u_3, u_4\} \), thus \( f \) receives at least \( 2 \times \frac{1}{6} = \frac{1}{3} \) from these vertices by R3, R4 and R5.
(2) The four adjacent internal 3-faces are \([v_i v_{i+1} u_i] \) for \( i \in \{1, 2, 3, 5\} \), while \( v_1 \) and \( v_4 \) are normal 3-vertices. Similarly, if \( v_2 \) or \( v_3 \) is a special vertex, then \( f \) receives at least \( \frac{1}{6} \) from it. So we may assume that \( v_2 \) and \( v_3 \) are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, there are at least two special vertices in \( \{u_1, u_2, u_3\} \), thus \( f \) receives at least \( 2 \times \frac{1}{6} = \frac{1}{3} \) from these vertices by R3, R4 and R5.

(3) The four adjacent internal 3-faces are \([v_i v_{i+1} u_i] \) for \( i \in \{1, 2, 3, 5\} \), while \( v_1 \) and \( v_4 \) are normal 3-vertices. By Lemma 2.7, \( u_2 \) and \( u_5 \) is a special vertex; one of \( \{v_2, v_3, u_1, u_2, u_3\} \) is a special vertex. Thus, \( f \) receives at least \( 2 \times \frac{1}{6} = \frac{1}{3} \) from \( \{u_1, u_2\} \) by R3, R4 and R5.

(4) The four adjacent internal 3-faces are \([v_i v_{i+1} u_i] \) for \( i \in \{1, 2, 4, 5\} \), while \( v_1 \) and \( v_3 \) are normal 3-vertices. If \( v_2 \) is a special vertex, then \( f \) receives \( \frac{1}{6} \) from it by R3, R4 and R5. Otherwise, \( v_2 \) is a normal 4-vertex. By Lemma 2.7, each of \( u_1 \) and \( u_2 \) is a special vertex, thus \( f \) receives at least \( 2 \times \frac{1}{6} = \frac{1}{3} \) from \( \{u_1, u_2\} \) by R3, R4 and R5.

(5) The four adjacent internal 3-faces are \([v_i v_{i+1} u_i] \) for \( i \in \{1, 2, 4, 5\} \), while \( v_1 \) and \( v_4 \) are normal 3-vertices. By Lemma 2.7, there is at least one special vertex in \( \{u_1, u_2, v_2, v_3\} \), and there is at least one special vertex in \( \{u_4, u_5, v_5, v_6\} \). Thus, \( f \) receives at least \( 2 \times \frac{1}{6} = \frac{1}{3} \) from these vertices by R3, R4 and R5.

To sum up, \( f \) always receives at least \( \frac{1}{3} \) from some vertices in the above five cases. Therefore, \( \mu'(f) \geq \mu(f) - 4 \times \frac{1}{3} - 2 \times \frac{1}{2} + \frac{1}{3} = 0 \) by R1 and R2.

- \( t_f = 1 \). By Corollary 2.5, \( s_f \leq 5 \). If \( s_f \leq 4 \), then \( \mu'(f) \geq \mu(f) - \frac{1}{2} - 4 \times \frac{1}{3} > 0 \). Assume that \( s_f = 5 \) and for \( 1 \leq i \leq 5 \), \( v_i v_{i+1} u_i \) is an internal 3-face. Let \( X = \{v_1, \ldots, v_6, u_1, \ldots, u_5\} \). By Lemma 2.9, there is a special vertex in \( X \). Therefore, \( f \) receives at least \( \frac{1}{6} \) from the special vertices in \( X \), and \( \mu'(f) \geq \mu(f) - \frac{1}{2} - 5 \times \frac{1}{3} + \frac{1}{6} = 0 \) by R3, R4 and R5.

- \( t_f = 0 \). Then \( f \) sends nothing to incident vertices, and \( \mu'(f) \geq \mu(f) - 6 \times \frac{1}{3} = 0 \).

If \( f \) is an internal 7+ face, then \( f \) sends out charges by R1 and R2. As \( t_f + s_f \leq d(f) \), we have

\[
\mu'(f) \geq \mu(f) - \frac{s_f}{3} - \frac{t_f}{2} \geq \frac{2}{3} d(f) - 4 - \frac{t_f}{6} \geq \frac{7}{12} d(f) - 4 > 0.
\]

This completes the proof of Case 1 of Theorem 1.1.

### 2.2 \( G \) has no subgraph isomorphic to any configuration in Fig. 1 and Fig. 3

Lemma 2.10 below follows easily from the fact that configurations in Fig. 1 and Fig. 3 are forbidden.

**Lemma 2.10.** If two 5-faces have two consecutive common edges on their boundaries, then one of the 5-face is the outer face \( f_0 \) (see Fig. 9).

Now we calculate the final charge of internal 5+ faces.

Assume \( f \) is an internal 6-face. If \( f \) is incident with a 2-vertex, then the 2-vertex belongs to \( \{x, y\} \), and \( f \) is adjacent to at most \( d - 2 \) internal faces. By R5, \( f \) receives \( \frac{1}{3} \) from each of \( x \) and \( y \). By R6, \( f \) receives \( \frac{1}{3} \) via each common edge with the outer face \( f_0 \). By R1 and R7, \( f \) sends at most \( \frac{1}{4} \) to each adjacent internal face. By R2,
By Lemma 2.4, $\mu'(f) \geq d - 4 + 2 \times \frac{1}{9} + 2 \times \frac{1}{9} - (d - 2) \times \frac{1}{3} - \lfloor \frac{d}{2} \rfloor \times \frac{1}{3} \geq \frac{5d - 24}{12} > 0$.

Assume that $f$ is not incident with any 2-vertex. By Lemma 2.10, there are no adjacent internal 5-faces. By Lemma 2.4, $f$ is adjacent to at most $d - t_f$ internal 5-faces.

**d = 5.** Assume that $f = [v_1v_2v_3v_4v_5]$. Since adjacent triangles and a triangle normally adjacent to a 7-cycle are forbidden, $s_f \leq 2$. By Corollary 2.5, $t_f \leq 2$. It follows that $f$ is incident with at most two minor 3-vertices.

If $s_f = 0$, then $\mu'(f) \geq \mu(f) - 2 \times \frac{1}{3} = 0$ by R2.

Assume $s_f \geq 1$. Since Fig. 1 and Fig. 3(c) are forbidden, $f$ is not adjacent to any 4-face. It follows that every face adjacent to $f$ is a 3-face or a $6^+$-face. Thus, $f$ is adjacent to at least three $6^+$-faces (the number of adjacent $6^+$-faces is counted by the number of common edges). If $f$ is incident with at most one minor 3-vertex, then $\mu'(f) \geq 5 - 4 - 2 \times \frac{1}{9} - (\frac{1}{3} + \frac{1}{3}) + 3 \times \frac{1}{6} - 2 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0$ by R1, R2 and R7. Assume $f$ is incident with exactly two minor 3-vertices. That is $t_f = 2$ and $s_f = 2$. By symmetry, we have three subcases to consider:

- $f$ is adjacent to two internal 3-faces $[v_1v_2u_1]$, $[v_3v_4u_3]$, and $v_1$, $v_3$ are minor 3-vertices.
- $f$ is adjacent to two internal 3-faces $[v_1v_2u_1]$, $[v_3v_4u_3]$, and $v_1$, $v_4$ are minor 3-vertices.
- $f$ is adjacent to two internal 3-faces $[v_1v_2u_1]$, $[v_3v_4u_3]$, and $v_1$, $v_3$ are minor 3-vertices.

By Lemma 2.7 and Lemma 2.8, the two 3-faces are incident with at least one special vertex. By R3, R4 and R5, $f$ receives at least $\frac{1}{6}$ from these special vertices. Hence, $\mu'(f) \geq 5 - 4 + \frac{1}{6} + 3 \times \frac{1}{6} - 2 \times \frac{1}{3} - 2 \times \frac{1}{3} = 0$.

**d = 6.** Assume that $f = [v_1v_2v_3v_4v_5]$. If $s_f = 0$, then it sends at most $\frac{1}{6}$ to each incident normal 3-vertex, and sends $\frac{1}{6}$ to each adjacent 5-face, thus $\mu'(f) \geq 6 - 4 - t_f \times \frac{1}{3} - (6 - t_f) \times \frac{1}{6} = 1 - \frac{t_f}{3} \geq 0$ by R2 and R7.

Suppose that $f$ is adjacent to an internal 3-face. Then they are normally adjacent. Since the configurations in Fig. 1 and Fig. 3(c) are forbidden, $s_f = 1$. By Corollary 2.5, $t_f \leq 3$. If $t_f \leq 2$, then $\mu'(f) \geq 6 - 4 - \frac{1}{3} - t_f \times \frac{1}{6} - (6 - t_f) \times \frac{1}{6} = 2 \times \frac{t_f}{3} \geq 0$ by R1, R2 and R7.

Assume $t_f = 3$ and the 3-face is $[uv_1v_2]$. By Lemma 2.4, we may assume $v_1$, $v_3$ and $v_5$ are the three normal 3-vertices. By Lemma 2.7, there is a special vertex in $\{u, v_2\}$, thus $f$ receives at least $\frac{1}{6}$ from $\{u, v_2\}$. Since the configurations in Fig. 1 and Fig. 3 are all forbidden, $v_3$ cannot be incident with an internal 4$^-$-face. Thus, $f$ is incident with at most two minor 3-vertices, which implies that $\mu'(f) \geq 6 - 4 - (2 \times \frac{1}{3} + \frac{1}{6}) - \frac{1}{3} - (6 - 3) \times \frac{1}{6} + \frac{1}{3} = 0$.

**d = 7.** Let $f$ be a 7-face. As Fig. 3(c) is forbidden, $s_f = 0$. By Corollary 2.5, $t_f \leq 3$. By R2, $f$ sends at most $\frac{1}{6}$ to each incident normal 3-vertex. By R7, $f$ sends $\frac{1}{6}$ to each adjacent internal 5-face. Hence, $\mu'(f) \geq 7 - 4 - t_f \times \frac{1}{6} - (7 - t_f) \times \frac{1}{6} = \frac{11 - 2t_f}{6} > 0$.

**d \geq 8.** Let $f$ be a $8^+$-face. Then $f$ sends at most $\frac{1}{6}$ to each incident normal 3-vertex, and $\frac{1}{6}$ to each adjacent internal 3-face, and $\frac{1}{6}$ to each adjacent internal 5-face. Combining with Corollary 2.5, we have that

$$\mu'(f) \geq d - 4 - t_f \times \frac{1}{2} - s_f \times \frac{1}{3} - (d - s_f) \times \frac{1}{6} = \frac{5}{6}d - \frac{1}{2}t_f - \frac{1}{6}s_f - 4 \geq \frac{d}{2} - 4 \geq 0.$$

This completes the proof of Case 2.

### 2.3 $G \in \mathcal{G}_{4,9}$

**Lemma 2.11.** A 5-cycle contains at most three triangular edges.

**Proof.** Assume $[x_1x_2x_3x_4x_5]$ is a 5-cycle, and $[x_1x_2x_6],[x_2x_3x_7],[x_3x_4x_8]$ and $[x_4x_5x_9]$ are four triangles. Since there is no 4-cycle in $G$, $x_1,x_2,\ldots,x_9$ are nine distinct vertices. Thus, $[x_1x_6x_2x_7x_3x_8x_4x_9x_5]$ is a 9-cycle, a contradiction.

□
Lemma 2.12. Let $f = [x_1 x_2 x_3 x_4 x_5]$ and $g = [x_5 x_1 u v w]$ be two adjacent 5-faces. If $d(x_1) \geq 3$ and $d(x_5) \geq 3$, then $f$ and $g$ are normally adjacent, and neither $x_2 x_3$ nor $x_3 x_4$ is adjacent to a 3-face. Moreover, if $x_1 x_2$ is incident with a 3-face, then $x_1$ is a 3-vertex and the 3-face is $[x_1 x_2 u]$.

Proof. Since $d(x_1) \geq 3$ and $d(x_5) \geq 3$, we have that $x_2 \neq u$ and $x_4 \neq w$. Since $G$ has no 4-cycle, $x_1, x_2, \ldots, x_5, u, v, w$ are distinct. Therefore, $f$ and $g$ are normally adjacent.

By the symmetry of $x_2 x_3$ and $x_3 x_4$, suppose that $x_2 x_3$ is incident with a 3-face $[x_2 x_3 x_7]$. Since there are no 4-cycles in $G$, $x_7$ is not incident with $f$ or $g$. Thus, $[x_5 x_4 x_3 x_7 x_2 x_1 u v w]$ is a 9-cycle, a contradiction. Hence, neither $x_2 x_3$ nor $x_3 x_4$ is incident with a 3-face.

Let $x_1 x_2$ be incident with a 3-face $[x_1 x_2 x_6]$. Since $f$ has no chord, $x_6 \notin \{x_3, x_4, x_5, v, w\}$. If $x_6 \neq u$, then $[x_5 x_4 x_3 x_2 x_6 x_1 u v w]$ is a 9-cycle, a contradiction. Thus $x_6 = u$ and $x_1$ is a 3-vertex.

Lemma 2.13. Let $f = [x_1 x_2 x_3 x_4 x_5]$ and $g = [x_5 x_1 u p q w]$ be two adjacent faces. If $d(x_1) \geq 3$ and $d(x_5) \geq 3$, then $\{u, w\} \cap \{x_1, \ldots, x_5\} = \emptyset$, while $\{p, q\} \cap \{x_2, x_3, x_4\} = \{p\}$ or $\{p, q\} \cap \{x_2, x_3, x_4\} = \{q\} = \{x_4\}$.

Proof. Since $G$ has no 9-cycle, $\{x_2, x_3, x_4\} \cap \{u, p, q, w\} \neq \emptyset$. For $d(x_1) \geq 3$ and $d(x_5) \geq 3$, we have that $x_2 \neq u$ and $x_4 \neq w$. Note that there are no 4-cycles, it follows that $\{x_2, x_3, x_4\} \cap \{u, w\} = \emptyset$, $x_3 \notin \{p, q\}$, $x_4 \neq p$ and $x_2 \neq q$. Therefore, $\{p, q\} \cap \{x_2, x_4\} = \{p\} = \{x_2\}$ or $\{p, q\} \cap \{x_2, x_4\} = \{q\} = \{x_4\}$.

Lemma 2.14. Let $f = [x_1 x_2 x_3 x_4 x_5]$ be a 5-face adjacent to two 3-faces, that are either $[x_1 x_2 x_6]$ and $[x_2 x_3 x_7]$, or $[x_1 x_2 x_6]$ and $[x_3 x_4 x_8]$ (see Fig. 10(a) and Fig. 10(b)). If $d(x_1) = 3$, $d(x_5) \geq 3$ and $d(x_6) \geq 3$, and $x_3 x_1 x_6$ is incident with a 6-face $g$, then $g$ is a 6-face $[x_5 x_1 x_6 u v w]$, where $\{u, w\} \cap \{x_1, x_2, \ldots, x_8\} = \emptyset$, $v = x_4$ and $d(x_4) \geq 4$ ($d(x_4) \geq 5$ for the case of Fig. 10(b)).

Proof. We only consider the case of Fig. 10(a) here, the case of Fig. 10(b) is quite similar. Suppose that $g = [x_5 x_1 x_6 u v w]$. Since $d(x_5) \geq 3$ and $d(x_6) \geq 3$, $x_1, x_2, x_5, u$ are four distinct vertices, and $x_1, x_4, x_5, w$ are four distinct vertices. As there is no 4-cycle in $G$, $x_1, x_2, \ldots, x_7, u, w$ are distinct. It follows that $g$ must be a 5- or 6-face. If $g$ is a 5-face, then $g = [x_5 x_1 x_6 u v w]$ and $[x_5 x_4 x_3 x_7 x_2 x_1 x_6 u v w]$ is a 9-cycle, a contradiction. Let $g = [x_5 x_1 x_6 u v w]$ be a 6-face. If $v \notin \{x_2, x_3, x_4\}$, then $[u v w x_5 x_4 x_3 x_2 x_1 x_6]$ is a 9-cycle, a contradiction. If $v = x_2$, then $[u x_6 x_1 x_2]$ is a 4-cycle, a contradiction. If $v = x_3$, then $[u x_6 x_2 x_3]$ is a 4-cycle, a contradiction. Hence, $v = x_4$ and $[x_3 x_5 x_6]$ is a triangle.

Lemma 2.15. Let $f = [x_1 x_2 x_3 x_4 x_5]$ be a 5-face adjacent to two 3-faces $[x_1 x_2 x_6]$ and $[x_3 x_4 x_8]$. If $d(x_2) = 3$, $d(x_3) \geq 4$ and $d(x_6) \geq 3$, then $x_3 x_2 x_6$ is incident with a 7-face.
Proof. Suppose that \( x_3x_2x_6 \) is incident with a face \( g = [x_3x_2x_6w \ldots w] \). Since \( d(x_3) \geq 4 \) and \( d(x_6) \geq 3 \), we have that \( x_2, x_3, x_4, x_8, w \) are five distinct vertices, and \( x_1, x_2, x_6, u \) are four distinct vertices. Since there are no 4-cycles, we have that \( x_1, x_2, \ldots, x_6, x_8, u, w \) are distinct. It follows that \( g \) must be a 5\(^{+}\)face. If \( g \) is a 5-face, then \( g = [x_3x_2x_6wu] \) and \( [x_3x_8x_4x_5x_1x_2x_6uw] \) is a 9-cycle, a contradiction. Let \( g \) be a 6-face \([x_3x_2x_6uvw]\). If \( v \notin \{x_1, x_4, x_5\} \), then \([uwux_3x_4x_5x_1x_2x_6]\) is a 9-cycle, a contradiction. If \( v = x_1 \), then \([ux_6x_2x_1]\) is a 4-cycle, a contradiction. If \( v = x_4 \), then \([ux_3x_8x_4]\) is a 4-cycle, a contradiction. If \( v = x_5 \), then \([ux_6x_1x_5]\) is a 4-cycle, a contradiction. Therefore, \( x_3x_2x_6 \) is incident with a 7\(^{+}\)-face.

Lemma 2.16. Let \( f = [x_1x_2x_3 \ldots] \) be a 7\(^{+}\)-face. If \( x_2 \) is a normal 3-vertex, then at most one of \( x_1x_2 \) and \( x_2x_3 \) is incident with a good 5-face.

Proof. Suppose to the contrary that \( x_1x_2 \) is incident with a good 5-face \( g_1 = [x_1x_2v_3v_4v_5] \) and \( x_2x_3 \) is incident with a good 5-face \( g_2 = [x_3x_2v_4u_4u_5] \). Note that \( g_1 \) and \( g_2 \) are all internal faces. By Lemma 2.3, \( v_3 \) cannot be a 2-vertex. By Lemma 2.12, \( g_1 \) and \( g_2 \) are normally adjacent. Moreover, \( v_3 \) is a 3-vertex, and \( g_3 = [v_3v_4u_4] \) is an internal 3-face. It is observed that \( g_1, g_2, \) and \( g_3 \) are all internal faces. It follows that \( v_3 \) does not belong to \( \{x, y\} \), but this contradicts Lemma 2.4.

Let \( \tau(\rightarrow f) \) be the number of charges that \( f \) receives from other elements.

Claim 1. If \( f \) is an internal 5-face and \( s_f = 1 \), then \( \tau(\rightarrow f) \geq \frac{1}{3} \).

Proof. Let \( f = [v_1v_2v_3v_4v_5] \) be an internal 5-face, and let \([v_1v_2v_6]\) be an internal 3-face. Since \( f \) has no chord, \( v_1, v_2, \ldots, v_6 \) are six distinct vertices. If \( v_i \in \{x, y\} \) for any \( 1 \leq i \leq 6 \), then \( v_i \) sends \( \frac{1}{3} \) to \( f \) by R5, we are done. Assume \([v_1, v_2, \ldots, v_6] \cap \{x, y\} = \emptyset \). By Lemma 2.3, \( d(v_i) \geq 3 \) for \( 1 \leq i \leq 6 \).

Next, we show that \( f \) is adjacent to a special face. By the hypothesis, neither \( v_3v_4 \) nor \( v_4v_5 \) is incident with an internal 4\(^{-}\)-face. By Lemma 2.12, neither \( v_3v_4 \) nor \( v_4v_5 \) is incident with a 5-face. If \( v_3v_4 \) or \( v_4v_5 \) is incident with an internal 7\(^{+}\)-face or \( f_0 \), we are done. So we may assume that each of \( v_3v_4 \) and \( v_4v_5 \) is incident with an internal 6-face. By Lemma 2.13, \( v_3v_4 \) is incident with a 6-face \([v_3v_4uwv_2]\). If \([v_2v_3w]\) bounds a 3-face, then \( d(w) = 2 \) and \( v_2v_3 \) is incident with the outer face \([v_2v_3w]\), we are done. Hence, we can assume that \( v_2v_3 \) is not incident with a 3-face. By Lemma 2.12, \( v_2v_3 \) cannot be incident with a 5-face. Since there are no 9-cycles, \( v_2v_3 \) cannot be incident with a 6-face. Hence, \( v_2v_3 \) is incident with a 7\(^{+}\)-face. Therefore, \( f \) is adjacent to at least one specific face in any case. By R6 and R8, \( f \) receives \( \frac{1}{3} \) from each adjacent special face, thus \( \tau(\rightarrow f) \geq \frac{1}{3} \).

Claim 2. Let \( f \) be an internal 5-face and \( s_f = 2 \). If \( f \) is incident with one minor 3-vertex, then \( \tau(\rightarrow f) \geq \frac{1}{3} \).

Proof. Assume that \( f = [x_1x_2x_3x_4x_5] \). If \( x \) or \( y \) is incident with \( f \) or one of the adjacent 3-faces, then it sends at least \( \frac{1}{3} \) to \( f \) by R5. So we may assume that neither \( x \) nor \( y \) is incident with \( f \) or the adjacent 3-faces. Now we show that \( f \) is adjacent to at least one 7\(^{+}\)-face sending \( \frac{1}{3} \) to \( f \) by R6 and R8.

Case 1. Let \([x_1x_2x_6]\) and \([x_2x_3x_7]\) be internal 3-faces, and let \( x_1 \) be a minor 3-vertex. By Lemma 2.3 and Lemma 2.4, \( d(x_5) \geq 4 \) and \( d(x_6) \geq 4 \). By Lemma 2.14, if \( x_5x_1x_6 \) is incident with a 6\(^{-}\)-face, then \([x_4x_5w]\) is a triangle but it does not bound a 3-face, thus \( x_4x_5 \) is incident with a 7\(^{+}\)-face. Hence, either \( x_3x_1x_6 \) or \( x_4x_5 \) is incident with a 7\(^{+}\)-face.

Case 2. Let \([x_1x_2x_6]\) and \([x_3x_4x_8]\) be internal 3-faces, and let \( x_1 \) be a minor 3-vertex. By Lemma 2.3, Lemma 2.4 and Lemma 2.14, we also get that either \( x_3x_1x_6 \) or \( x_4x_5 \) is incident with a 7\(^{+}\)-face.

Case 3. Let \([x_1x_2x_6]\) and \([x_3x_4x_8]\) be internal 3-faces, and let \( x_2 \) be a minor 3-vertex. By Lemma 2.3 and Lemma 2.4, \( d(x_3) \geq 4 \) and \( d(x_8) \geq 4 \). By Lemma 2.15, \( x_2x_3 \) is incident with a 7\(^{+}\)-face.

Claim 3. Let \( f \) be an internal 5-face and \( s_f \geq 2 \). If \( f \) is incident with two minor 3-vertices, then \( \tau(\rightarrow f) \geq 1 \).
Lemma 2.12, Lemma 2.14, neither vertex in
assume that $g_i$ for $i \in \{1, 2, 3, 4, 5\}$.

Proof. Assume $f = [x_1x_2x_3x_4x_5]$. If $x_i$ is a 2-vertex, then $x_i \in \{x, y\}$ and $x_{i-1}, x_{i+1}$ is incident with the outer face $f_0$. By R5, $f$ receives $\frac{1}{3}$ from each of $x$ and $y$. By R6, $f$ receives $\frac{1}{3}$ via each of $x_i-1, x_{i+1}$. Thus, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3} + 2 \times \frac{1}{3} > 1$. So we may assume that $d(x_i) \geq 3$ for any $1 \leq i \leq 5$. Denote the adjacent face incident with $x_ix_{i+1}$ by $g_i$.

Case 1. Let $[x_1x_2x_6]$ and $[x_2x_3x_7]$ be internal 3-faces, and let $x_1$ and $x_3$ be minor 3-vertices. Suppose that $x_6$ is a 2-vertex. It follows that $\{x_2, x_6\} = \{x, y\}$ and $g_5 = f_0$. By R5, $f$ receives $\frac{1}{3}$ from each of $x_2$ and $x_6$. By R6, $f$ receives at least $\frac{1}{3}$ from the outer face $f_0$. Thus, $\tau(\rightarrow f) \geq 3 \times \frac{1}{3} = 1$.

So we may assume that $d(x_6) \geq 3$, and by symmetry, $d(x_7) \geq 3$. Firstly, we claim that $f$ receives at least $\frac{1}{3}$ from $\{x_2, x_6, x_7\}$. If $x_2$ is a special vertex, then $f$ receives $\frac{1}{3}$ from $x_2$ by R3, R4 and R5. So we may assume that $x_2$ is a normal 4-vertex. It follows from Lemma 2.7 that both $x_6$ and $x_7$ are special vertices. By R3, R4 and R5, $f$ receives at least $\frac{1}{6} \times 2 = \frac{1}{3}$ from $x_6$ and $x_7$.

Next, we show that $f$ is adjacent to at least two special faces. Since $f$ receives at least $2 \times \frac{1}{3} = \frac{2}{3}$ from adjacent special faces by R6 and R8, we are done. By Lemma 2.14, we get that both $g_3$ and $g_4$ are 6+-faces, and $g_3, g_5$ cannot be 6-face simultaneously. If both $g_3$ and $g_5$ are 7+-faces, then we are done. By symmetry, assume that $g_5$ is a 6-face and $g_3$ is a 7+-face. It follows that $g_4$ is the outer 3-face or a 7+-face. That is, $g_3$ and $g_4$ are the special faces, we are done.

Case 2. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let $x_1$ and $x_4$ be minor 3-vertices. Similar to Case 1, we may assume that $d(x_6) \geq 3$ and $d(x_8) \geq 3$. Note that $x_1$ and $x_4$ are 3-vertices. Since there are no 4-cycles, neither $g_1$ nor $g_5$ is a 4-vertex. By Lemma 2.12, neither $g_4$ nor $g_5$ is a 5-face. By Lemma 2.13 and Lemma 2.14, neither $g_1$ nor $g_5$ is a 6-face. So both $g_4$ and $g_5$ are 7+-faces. Thus, $f$ receives at least $\frac{1}{3} \times 2 = \frac{2}{3}$ from these 7+-faces. Next we show that $f$ will receive at least $\frac{1}{3}$ from others.

If $g_2$ is a 7+-face, then we are done. By Lemma 2.13, $g_2$ cannot be a 6-face. Assume $g_2$ is a 5-face. By Lemma 2.12, $d(x_2) = d(x_3) = 3$. By Lemma 2.4, we have that $\{x_2, x_3\} = \{x, y\}$. By R5, $f$ receives $\frac{1}{3}$ from each of $x_2$ and $x_3$, we are done. It is clear that $g_2$ cannot be a 4-face. Suppose that $g_2$ is a 3-face $[x_2x_3x_7]$. If there is one special vertex in $\{x_2, x_3\}$, then we are done by R3, R4 and R5. So we may assume that both $x_2$ and $x_3$ are normal 4-vertices. By Lemma 2.7 and Lemma 2.8, at least two of $x_6, x_7$ and $x_8$ are special vertices, thus $f$ receives at least $2 \times \frac{1}{6} = \frac{1}{3}$ from these special vertices, we are done.

Case 3. Let $[x_1x_2x_6]$ and $[x_3x_4x_8]$ be internal 3-faces, and let $x_1$ and $x_3$ be minor 3-vertices. Similar to Case 1, assume $d(x_6) \geq 3$ and $d(x_8) \geq 3$. By Lemma 2.7, one of $\{x_2, x_6\}$ is a special vertex. By R3, R4 and R5, $f$ receives at least $\frac{1}{3}$ from $\{x_2, x_6\}$.

Since there are no 4-cycles, we have that $g_2$ cannot be a 4-vertex. Suppose that $g_2$ is a 5-face. By Lemma 2.12, we have that $d(x_2) = d(x_3) = 3$. By Lemma 2.4, $x_2$ belongs to $\{x, y\}$. As a consequence, $\{x, y\} \subseteq \{x, y\}$ and $g_2$ is the outer face $f_0$. By R5 and R6, $\tau(\rightarrow f) \geq 2 \times \frac{1}{3} + \frac{1}{3} = 1$, we are done. By Lemma 2.13, $g_2$ cannot be a 6-face. Thus, we may assume that $g_2$ is a 7+-face. By R8, $f$ receives $\frac{1}{3}$ from $g_2$.

Next we show that $f$ receives at least $\frac{1}{3}$ from others. By Lemma 2.12 and Lemma 2.13, $g_4$ cannot be a 5- or 6-face. Thus, $g_4$ is a 3- or 7+-face. Suppose that $g_4$ is a 3-face $[x_4x_5x_9]$. If $x_9$ is a 2-vertex, then $\{x, y\} \subseteq \{x, y\}$, and then $f$ receives at least $2 \times \frac{1}{3} = \frac{1}{3}$ from $x$ and $y$ by R5. So we may assume that $d(x_9) \geq 3$. By Lemma 2.12 and Lemma 2.13, $g_5$ is a 7+-face sending $\frac{1}{3}$ to $f$. By Lemma 2.7, there is a special vertex in $\{x_4, x_5, x_8, x_9\}$ sending at least $\frac{1}{3}$ to $f$. Thus, $f$ receives at least $\frac{1}{3} + \frac{1}{3} = \frac{1}{3}$ from $g_5$ and the special vertex. Suppose that $g_4$ is a 7+-face. If $g_5$ is also a 7+-face, then $f$ receives at least $2 \times \frac{1}{3} = \frac{1}{3}$ from $g_4$ and $g_5$, we are done. So we may assume that $g_5$ is a 6-vertex. By Lemma 2.14, $d(x_4) \geq 5$. By R3, R4 and R5, $f$ receives at least $\frac{1}{6}$ from $x_4$. Therefore, $f$ still receives at least $\frac{1}{6} + \frac{1}{3} = \frac{1}{3}$ from $g_4$ and $x_4$.

Claim 4. Let $f$ be an internal 5-face and $s_f = 2$. If $t_f = 2$, and exactly one of the two normal 3-vertices is minor, then $\tau(\rightarrow f) \geq \frac{1}{2}$. 

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Proof. Assume \( f = [x_1x_2x_3x_4x_5] \). By the definition of normal 3-vertex and minor 3-vertex, we only need to consider two cases.

**Case 1.** Let \([x_1x_2x_6]\) and \([x_2x_3x_7]\) be internal 3-faces, and let \( x_1 \) and \( x_4 \) be normal 3-vertices. If \( x_5 \) or \( x_6 \) is a 2-vertex, then \( x_5 \) or \( x_6 \) belongs to \( \{x, y\} \). It follows that \( x_1x_5 \) is incident with the outer face \( f_0 \). By R5, \( f \) receives at least \( \frac{1}{3} \) from \( \{x, y\} \). By R6, \( f \) receives \( \frac{1}{3} \) from the outer face \( f_0 \). Thus, \( \tau(\rightarrow f) \geq 2 \times \frac{1}{3} = \frac{2}{3} \).

So we may assume that \( d(x_5) \geq 3 \) and \( d(x_6) \geq 3 \). Note that \( x_4 \) is a 3-vertex. By Lemma 2.14, \( x_1x_5 \) cannot be incident with a 6-face. That is, \( x_1x_5 \) is incident with a 7+ face which sends \( \frac{1}{3} \) to \( f \). On the other hand, by Lemma 2.7, one vertex in \( \{x_2, x_3, x_6, x_7\} \) is a special vertex which sends at least \( \frac{1}{6} \) to \( f \). Thus, \( \tau(\rightarrow f) \geq \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \).

**Case 2.** Let \([x_1x_2x_6]\) and \([x_3x_4x_8]\) be internal 3-faces, and let \( x_2 \) and \( x_5 \) be normal 3-vertices. If \( x_6 \) is a 2-vertex, then \( x_6 \in \{x, y\} \). Since \( x_2 \) is a normal vertex, \( \{x, y\} = \{x_1, x_6\} \). Thus, \( f \) receives \( \frac{1}{3} \) from each of \( x_1 \) and \( x_6 \) by R5, and thus \( \tau(\rightarrow f) \geq \frac{1}{4} + \frac{1}{4} \geq \frac{1}{2} \). Assume \( d(x_6) \geq 3 \). By Lemma 2.7, at least one of \( x_1 \) and \( x_6 \) is a special vertex. By R3, R4 and R5, \( f \) receives at least \( \frac{1}{3} \) from these special vertices. If \( x_3 \) is a 3-vertex, then \( x_3 \in \{x, y\} \) by Lemma 2.4. By R5, \( f \) receives \( \frac{1}{3} \) from \( x_3 \). Thus, \( \tau(\rightarrow f) \geq \frac{1}{3} + \frac{1}{3} = \frac{1}{2} \).

So we may assume that \( d(x_3) \geq 4 \). By Lemma 2.15, \( x_2x_3 \) is incident with a 7+ face. By R8, \( f \) receives \( \frac{1}{3} \) from each adjacent 7+ face. Thus, \( \tau(\rightarrow f) \geq \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \). \( \Box \)

**Claim 5.** If \( f \) is an internal 5-face and \( s_f = 3 \), then \( \tau(\rightarrow f) \geq \frac{4}{5} \).

Proof. Assume \( f = [x_1x_2x_3x_4x_5] \). According to symmetry, we only need to consider two cases.

**Case 1.** Let \([x_1x_2x_6], [x_2x_3x_7] \) and \([x_4x_5x_9] \) be internal 3-faces. Assume \( d(x_6) = 2 \). By Lemma 2.4, \( \{x, y\} = \{x_1, x_6\} \) or \( \{x, y\} = \{x_2, x_6\} \). By R5, \( f \) receives \( \frac{1}{4} \) from each of \( x \) and \( y \), thus \( \tau(\rightarrow f) \geq 2 \times \frac{1}{4} = \frac{1}{2} \).

So we may assume that \( d(x_6) \geq 3 \). Similarly, we can assume that \( d(x_7) \geq 3 \) and \( d(x_9) \geq 3 \). It is clear that neither \( x_1x_5 \) nor \( x_3x_4 \) is incident with a 4-face. By Lemma 2.12, neither \( x_1x_5 \) nor \( x_3x_4 \) is incident with a 5-face. By Lemma 2.13, neither \( x_1x_5 \) nor \( x_3x_4 \) is incident with a 6-face. Hence, \( f \) is adjacent to two 7+ faces. By R6 and R8, \( \tau(\rightarrow f) \geq 2 \times \frac{1}{4} = \frac{2}{3} \).

**Case 2.** Let \([x_1x_2x_6], [x_2x_3x_7] \) and \([x_3x_4x_8] \) be internal 3-faces. If \( d(x_i) = 2 \) for \( i \in \{5, 6, 7, 8, \} \), then \( x_i \in \{x, y\} \) by Lemma 2.3. Since \( x \) and \( y \) are adjacent, we have that \( \{x, y\} \subset \{x_1, x_2, \ldots, x_8\} \). By R5, \( f \) receives \( \frac{1}{4} \) from each of \( x \) and \( y \), thus \( \tau(\rightarrow f) \geq 2 \times \frac{1}{4} = \frac{1}{2} \).

So we may assume that \( x_3, x_6, x_7 \) and \( x_8 \) are all 3+ vertices. It is clear that neither \( x_4x_5 \) nor \( x_1x_5 \) is contained in a 4-face. By Lemma 2.12, neither \( x_1x_5 \) nor \( x_3x_4 \) is incident with a 5-face. Recall that \( x_6 \) is a 3+ vertex and \( x_4x_5 \) is not contained in a triangle. By Lemma 2.13, \( x_1x_5 \) cannot be incident with a 6-face. Hence, \( x_1x_5 \) is incident with a 7+ face. By symmetry, \( x_4x_5 \) is also incident with a 7+ face. By R6 and R8, \( \tau(\rightarrow f) \geq 2 \times \frac{1}{3} = \frac{2}{3} \). \( \Box \)

Now we calculate the final charge of internal 5+ faces. Let \( f = [v_1v_2 \ldots v_d] \) be an internal d-face for \( d \geq 5 \). By Lemma 2.2, every face in \( \mathcal{G} \) is bounded by a cycle. Since there are no 9-cycles, \( d \neq 9 \).

If \( v_i \) is a 2-vertex, then \( v_i \in \{x, y\} \) and \( v_{i-1}v_iv_{i+1} \) is incident with the outer face \( f_0 \). Thus, \( f \) is adjacent to at most \( d-2 \) internal faces. By Corollary 2.5, \( t_f \leq \frac{d}{2} \). By R1 and R8, \( f \) sends at most \( \frac{1}{2} \) to each adjacent internal face. By R2, \( f \) sends at most \( \frac{1}{3} \) to each incident normal 3-vertex. By R5, \( f \) receives \( \frac{1}{3} \) from each of \( x \) and \( y \). By R6, \( f \) receives \( \frac{1}{3} \) via each of \( v_{i-1}v_i \) and \( v_iv_{i+1} \). Hence, \( \mu'(f) \geq d-4+4 \times \frac{1}{3}-(d-2) \times \frac{1}{3} \approx \frac{d}{2} \times \frac{1}{2} > 0 \).

So we may assume that there is no 2-vertex incident with \( f \).

• \( d = 5 \).

By Corollary 2.5 and Lemma 2.11, \( t_f \leq 2 \) and \( s_f \leq 3 \). If \( s_f = 0 \), then \( \mu'(f) \geq 5-4-2 \times \frac{1}{3} = \frac{2}{3} > 0 \) by R2.

If \( s_f = 1 \), then \( f \) is incident with at most one minor 3-vertex. By Claim 1, R1 and R2, \( \mu'(f) \geq 5-4+\frac{1}{3} - \frac{1}{3} (\frac{1}{2} + \frac{1}{2}) = 0 \).

Assume \( s_f = 2 \). If \( t_f = 0 \), then \( \mu'(f) \geq 5-4-2 \times \frac{1}{3} = \frac{1}{3} > 0 \) by R1. Let \( t_f = 1 \). If the normal 3-vertex is not minor, then \( \mu'(f) \geq 5-4-2 \times \frac{1}{3} = \frac{1}{3} > 0 \) by R1 and R2. If the normal 3-vertex is
minor, then $\mu'(f) \geq 5 - 4 + \frac{1}{3} - 2 \times \frac{1}{3} - \frac{1}{2} > 0$ by Claim 2, R1 and R2. Let $t_f = 2$. It is observed that $f$ is incident with at least one minor 3-vertex. If $f$ is incident with exactly one minor 3-vertex, then $\mu'(f) \geq 5 - 4 + \frac{1}{2} - 2 \times \frac{1}{2} - (\frac{1}{2} + \frac{1}{3}) = 0$ by Claim 4, R1 and R2. The other situation, $f$ is incident with exactly two minor 3-vertices. Thus, $\mu'(f) \geq 5 - 4 + 1 - 2 \times \frac{1}{3} - 2 \times \frac{1}{2} - \frac{1}{2} > 0$ by Claim 3, R1 and R2.

Assume $s_f = 3$. If $t_f = 0$, then $\mu'(f) \geq 5 - 4 - 3 \times \frac{1}{3} = 0$ by R1. If $t_f = 1$, then $\mu'(f) \geq 5 - 4 + \frac{2}{3} - 3 \times \frac{1}{3} - \frac{1}{2} > 0$ by Claim 5, R1 and R2. If $t_f = 2$, then it is incident with two minor 3-vertices, and then $\mu'(f) \geq 5 - 4 + 1 - 3 \times \frac{1}{3} - 2 \times \frac{1}{2} = 0$ by Claim 3, R1 and R2.

• $d = 6$.

Note that there are no 4-cycle in $G$. If $f$ is adjacent to a 3-face, then it must be normally adjacent to the 3-face. Since there are no 9-cycles in $G$, $f$ is adjacent to at most two 3-faces. It follows that $f$ is incident with at most two minor 3-vertices. By R1 and R2, $\mu'(f) \geq 6 - 4 - 2 \times \frac{1}{3} - (2 \times \frac{1}{2} + \frac{1}{3}) = 0$.

• $d = 7$.

If $f$ is adjacent to a 3-face, then it must be normally adjacent to the 3-face. Otherwise, there is a 4-cycle in $G$. Since there are no 9-cycles in $G$, $f$ is adjacent to at most 3-face. It follows that $f$ is incident with at most one minor 3-vertex. By Corollary 2.5, $t_f \leq 3$. If $t_f = 3$, then $f$ is adjacent to at most four good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu'(f) \geq 7 - 4 - (1 + 4) \times \frac{1}{3} - (\frac{1}{2} + 2 \times \frac{1}{3}) > 0$ by R1, R2 and R8. If $t_f = 2$, then $f$ is adjacent to at most five good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu'(f) \geq 7 - 4 - (1 + 5) \times \frac{1}{3} - (\frac{1}{2} + \frac{1}{3}) > 0$ by R1, R2 and R8. If $t_f = 1$, then $f$ is adjacent to at most six good 5-faces by Lemma 2.4 and Lemma 2.16, and then $\mu'(f) \geq 7 - 4 - (1 + 6) \times \frac{1}{3} - \frac{1}{2} > 0$ by R1, R2 and R8. If $t_f = 0$, then $\mu'(f) \geq 7 - 4 - 7 \times \frac{1}{3} > 0$ by R1 and R8.

• $d = 8$.

Similar to the above cases, if $f$ is adjacent to a 3-face, then it must be normally adjacent to the 3-face. Since there are no 9-cycles, $f$ is not adjacent to any 3-face. Thus, $f$ is not incident with any minor 3-vertex. By R2 and R8, $\mu'(f) \geq 8 - 4 - 4 \times \frac{1}{3} - 4 \times \frac{1}{3} = 0$.

• $d \geq 10$.

By R1 and R8, $f$ sends at most $\frac{1}{3}$ via each incident edge. It follows that $\mu'(f) \geq d - 4 - d \times \frac{1}{3} - \frac{1}{2} \times \frac{1}{2} > 0$.

This completes the proof of Theorem 2.1.

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