1. Introduction

An easy corollary of Simpson’s nonabelian Hodge theorem is the following.

**Theorem 1.1.** [Sim92, Corollary 4.3] Let $X$ and $Y$ be smooth projective complex varieties, and let $f: Y \to X$ be a morphism that induces a surjection:

$$f_*: \pi_1(Y) \to \pi_1(X).$$

Let $\mathbb{L}$ be a $\mathbb{C}$-local system on $X$ such that $f^*\mathbb{L}$ underlies a complex polarized variation of Hodge structures on $Y$. Then $\mathbb{L}$ underlies a complex polarized variation of Hodge structures on $X$.

As Simpson notes, this is especially useful when $Y$ is the smooth complete intersection of smooth hyperplane sections of $X$. This article is concerned with a logarithmic and arithmetic analog of Theorem 1.1. Here the arithmetic analog of a complex polarized variation of Hodge structures is a (logarithmic) crystalline representation. To state this precisely, we need the following notation.

**Setup 1.2.** Let $k \cong \mathbb{F}_q$ be a finite field of odd characteristic, let $W := W(k)$ be the ring of Witt vectors, and let $K := \text{Frac}(W)$ be the field of fractions. Let $X/W$ be a smooth projective scheme of relative dimension at least 2. Let $S \subset X$ be a strict normal crossings divisor, flat over $W$. Let $j: D \hookrightarrow X$ be a relative smooth ample divisor, flat over $W$ that intersects $S$ transversely, so that $S \cap D \subset D$ is a strict normal crossings divisor. Let $X_K = X_K - S_K$ and let $D_K = D_K - (D_K \cap S_K)$. There is a natural continuous surjective homomorphism

$$j_{K*}: \pi_1(D_K^o) \to \pi_1(X_K^o).$$

**Question 1.3.** In the context of **Setup 1.2**, let $\rho_X: \pi_1(X_K^o) \to \text{GL}_N(\mathbb{Z}_{p^f})$ be a continuous $p$-adic representation. Restricting $\rho_X$ via $j_K$, one gets a representation $\rho_D: \pi_1(D_K^o) \to \text{GL}_N(\mathbb{Z}_{p^f})$. Suppose $\rho_D$ is logarithmic crystalline. Is $\rho_X$ also logarithmic crystalline?

In this article, we answer **Question 1.3** under several additional assumptions.

**Theorem 1.4.** In the context of **Question 1.3**, suppose further that:

1. $N^2 < p - \dim X$;
2. $\rho_D$ is geometrically absolutely residually irreducible, i.e., the composite
   $$\pi_1(D_K^o) \to \pi_1(D_K^o) \to \text{GL}_N(\mathbb{Z}_{p^f}) \to \text{GL}_N(\mathbb{F}_p)$$
   is an irreducible representation;
3. the line bundle $\mathcal{O}_X(D - S)$ is ample on $X$; and
(4) if \( S \neq \emptyset \), then \( \text{Conjecture 4.6} \) holds for \( \rho_D \). (This is known for instance when \( \rho_D \) strictly comes from geometry, in the sense of \( \text{Definition 4.8} \).)

Then \( \rho_X \) is (logarithmic) crystalline.

Remark 1.5. In (4), when \( S \neq \emptyset \), we have an extra condition, namely, that \( \text{Conjecture 4.6} \) holds for \( \rho_D \). This is a natural conjecture in relative \( p \)-adic Hodge theory and will be explicated in Section 4, but let us briefly explain what it means for \( \text{Conjecture 4.6} \) to hold for \( \rho_D \). (We maintain notation from Setup 1.2.) Let \( \rho_D \) be a logarithmic crystalline representation, associated to a logarithmic Fontaine-Faltings module \((M, \nabla, \text{Fil}, \varphi)_D\). Then the conjecture says the following two filtered de Rham bundles over the scheme \((U \cap D)_K\) are isomorphic:

- \((M, \nabla, \text{Fil})|_{D_K}\)
- \(\mathbb{D}^{\text{alg}}(\rho_D|_{D_K})\),

where \(\mathbb{D}^{\text{alg}}\) is the algebraic \( p \)-adic Riemann-Hilbert functor of \([\text{DLLZ18}, \text{Theorem 1.1}]\). (For this to make sense in the context of \([\text{DLLZ18}]\), it suffices to note that the local system \(\rho_D|_{D_K}\) is crystalline and hence de Rham by \([\text{TT19}]\).)

To prove this conjecture, one would need to develop a logarithmic variant of \([\text{TT19}]\). In particular, one needs to construct the logarithmic version crystalline period sheaf as in \([\text{TT19}]\) such that it naturally embeds into the logarithmic de Rham period sheaf constructed in \([\text{DLLZ18}, \text{Definition 2.2.10}]\).

We emphasize that for \(\rho_D\) strictly coming from geometry (as in \( \text{Definition 4.8} \)), \( \text{Conjecture 4.6} \) holds. This is explained in Section 4.

We do not yet know how to relax the other assumptions. As a corollary of \( \text{Theorem 1.4} \), we have the following.

Corollary 1.6. Setup as in \( \text{Setup 1.2} \) and suppose that \( S = \emptyset \). Let \( A_D \to D \) be an abelian scheme. Let \( \rho: \pi_1(D_K) \to \text{GL}_{2g}(\mathbb{Z}_p) \) be the representation induced from the \( p \)-adic Tate module of \( A_D \to D \) over the generic fiber \( D_K \). Suppose that:

1. \( 4g^2 < p - \text{dim } X \);
2. \( \rho_D \) is geometrically absolutely residually irreducible, as in \( \text{Theorem 1.4} \).

Then the following are equivalent.

- \( A_D \to D \) extends to an abelian scheme \( A_X \to X \).
- the local system \( \rho_D: \pi_1(D_K) \to \text{GL}_{2g}(\mathbb{Z}_p) \) extends to a local system \( \rho_X: \pi_1(X_K) \to \text{GL}_{2g}(\mathbb{Z}_p) \).

Note that if \( \text{dim}(X) \geq 3 \), then \( j_*: \pi_1(D_K) \to \pi_1(X_K) \) is an isomorphism; hence every local system on \( D_K \) extends to a local system on \( X_K \). Using our proof technique and a standard spreading out argument, one recovers a very special case of a corollary of Simpson’s \( \text{Theorem 1.1} \).

Corollary 1.7. Let \( X/\mathbb{C} \) be a smooth projective variety of dimension at least 2, let \( D \subset X \) be a smooth ample divisor, and let \( f_D: A_D \to D \) be an abelian scheme such that the associated graded Higgs bundle attached to de Rham cohomology

\[ (E_D, \theta_D) := \text{Gr}_{\text{Fil}_{\text{Hodge}}} (\mathcal{H}^1_{\text{dR}}(A_D/D), \nabla) \]

is stable\(^1\). Then the following are equivalent.

- \( A_D \to D \) extends to an abelian scheme \( A_X \to X \).

\(^1\)This stability is equivalent to the local system \( R^1f_D*_\mathbb{C} \) being irreducible, but we wished to give a purely coherent algebraic formulation of the result.
the graded Higgs bundle \((E_D, \theta_D)\) extends to a graded Higgs bundle \((E, \theta)\) on \(X\).
Moreover, if \(\dim(X) \geq 3\), then \(A_D \to D\) always extends to an abelian scheme \(A_X \to X\).

While Corollary 1.7 is indeed true without any stability assumption on \((E_D, \theta_D)\) by Simpson’s Theorem 1.1, the proof in [Sim92] is highly complex analytic. On the other hand, our method is arithmetic/algebro-geometric.

Question 1.3 was inspired by a question of Simpson. To state this question, we first need a definition.

**Definition 1.8.** Let \(X/K\) be a smooth variety and let \(L\) be a \(K\)-local system on \(X\). We say that \(L\) is motivic if there exists a dense open \(U \subset X\), a smooth projective morphism \(f: Y \to U\), and an integer \(i \geq 0\), such that \(L|_U\) is a subquotient of \(R^i f_* \mathcal{C}\).

**Question 1.9 (Simpson).** Let \(X/K\) be a smooth, projective variety of dimension at least 2, and let \(D \subset X\) be a smooth ample divisor. Let \(L\) be a \(K\)-local system on \(X\) such that \(X|_D\) is motivic. Then is \(L\) motivic?

Question 1.9 is compatible with other conjectural characterizations of motivic local systems, e.g., Simpson’s Standard conjecture [Sim90, p. 372]. We show that Theorem 1.4 has the following sample application, which provides some evidence for Question 1.9.

**Corollary 1.10.** Let \(X/O_K[1/N]\) be a smooth projective scheme over a ring of integers in a number field of relative dimension at least 3 and pick an embedding \(\iota: K \to \mathbb{C}\). Let \(D \subset X\) be a relative smooth ample divisor. Let \(f_D: Y_D \to D\) be a smooth projective morphism and \(i \geq 0\). Suppose the following equivalent conditions hold.

- The associated graded Higgs bundle \(\text{GrFil}_{\text{Hodge}}(H^i_{\text{dR}}(Y_D/D), \nabla)\) is a stable Higgs bundle over \(D_K\).
- For any embedding \(\iota: K \to \mathbb{C}\), the complex local system \(R^i f_{D,C}^{\iota}(\mathbb{C})\) on \(D_C := D \times_{K,\iota} \mathbb{C}\) is irreducible.

Then, after potentially replacing \(K\) by a finite extension and \(N\) by a larger integer, we have the following:

1. the Gauss-Manin connection together with its Hodge filtration \((H^i_{\text{dR}}(Y_D/D), \nabla_{\text{GM}}, \text{Fil}_{\text{Hodge}})\) canonically extends to a filtered flat connection \((\mathcal{H}, \nabla, \text{Fil})\) on \(X\).
2. For all primes \(p \gg 0\) of \(K\), the \(p\)-adic completion of \((\mathcal{H}, \nabla, \text{Fil})\) underlies a filtered Frobenius crystal.
3. For all primes \(p \gg 0\) of \(K\), the \(p\)-adic completion of the associated graded Higgs bundle:

\[
(E, \theta) := \text{GrFil}(\mathcal{H}, \nabla)
\]

is \(p\)-adically 1-periodic.
4. For any embedding \(\iota: O_K \to \mathbb{C}\), the filtered flat connection \((\mathcal{H}, \nabla, \text{Fil}) \times_{K,\iota} \mathbb{C}\) on \(X \times_{K,\iota} \mathbb{C}\) underlies a \(\mathbb{Z}\) polarized variation of Hodge structures.

We find it very likely that each of conditions (2) and (3) in fact characterizes those flat connections which are motivic. While (2) formally implies (3), we first prove (3) and use it to deduce (2). Note that condition (2) in particular guarantees that the flat connection \((\mathcal{H}_X, \nabla)\) is globally nilpotent, i.e., that for all \(p \gg 0\), the \(p\)-curvature is nilpotent on the mod \(p\) reduction of \(X\).

\(^2\)equivalently, by a theorem of Deligne, a summand
Remark 1.11. Note that the assumptions of Theorem 1.4 imply that we may Tate twist $\rho_D$ so that it has Hodge-Tate weights in the interval $[0, \sqrt{p - \dim(X)}]$. As we have assumed that $p \geq 3$ and $\dim(X) \geq 2$, this means that $\sqrt{p - \dim(X)} \leq p - 2$. Tate twists do not change the property of “being crystalline”, so we may assume without loss of generality that $\rho_D$ has Hodge-Tate weights in the interval $[0, p - 2]$. This is necessary to apply the theory of Lan-Sheng-Zuo.

We briefly explain the structure of the proof of Theorem 1.4.

Step 1 In Section 3, we transform the question into a problem about extending a periodic Higgs-de Rham flow. The theory of Higgs-de Rham flows has its origins in the seminal work of Ogus-Vologodsky on nonabelian Hodge theory in characteristic $p$ [OV07]. This theory has recently been enhanced to a $p$-adic theory by Lan-Sheng-Zuo. According to the theory of Lan-Sheng-Zuo, there is an equivalence between the category of certain crystalline representations (with bounds on the Hodge-Tate weights) and periodic Higgs-de Rham flows. Let $HDF_D$ be the logarithmic Higgs-de Rham flow over $(D, S \cap D)$ associated to the representation $\rho_D$.

\begin{align*}
(E_D, \theta_D)_0 & \quad (V_D, \nabla_D, \Fil_D)_0 \\
(E_D, \theta_D)_1 & \quad (V_D, \nabla_D, \Fil_D)_1 \\
(E_D, \theta_D)_2 & \quad \cdots
\end{align*}

Then we need to extend $HDF_D$ to some periodic Higgs-de Rham flow over $(X, S)$.

Step 2 In Section 4.2, we extend $(E_D, \theta_D)$, to a graded logarithmic semistable Higgs bundles $(E_X, \theta_X)$ over $(X, S)$. Using Scholze’s notion of de Rham local systems together with a rigidity theorem due to Liu-Zhu (and Diao-Lan-Liu-Zhu), we construct a graded logarithmic Higgs bundle $(E_{X_K}, \theta_{X_K})$ over $(X_K, S_K)$ such that

$$(E_{X_K}, \theta_{X_K})|_{D_K} = (E_D, \theta_D)|_{D_K}.$$

One gets graded Higgs bundles $(E_{X_K}, \theta_{X_K})$, extending $(E_D, \theta_D)|_{D_K}$. In Section 5, using a result of Langer (extending work of Langton), we extend $(E_{X_K}, \theta_{X_K}) = (E_{X_K}, \theta_{X_K})_0$ to a semistable Higgs torsion free sheaf $(E_{X_K}, \theta_{X_K})$ on $(X, S)$. We show that this extension $(E_X, \theta_X)$ is unique up to an isomorphism and has trivial Chern classes in Section 6, which implies that $(E_X, \theta_X)$ is locally free using work of Langer.

Step 3 We show that $(E_X, \theta_X)$ constructed in Step 2 has stable reduction modulo $p$ and is graded. To do this, we prove a Lefschetz theorem for semistable Higgs bundles with vanishing Chern classes using a vanishing theorem of Arapura in Section 7. It is here that our assumptions transform from $N < p$ to $N^2 < p - \dim(X)$. The argument that $(E_X, \theta_X)$ is graded is contained in Section 8.

Step 4 In Section 9, we extend $HDF_D|_{D_1}$ to a Higgs-de Rham flow $HDF_{X_1}$ over $X_1$ (here the subscript $n$ denotes reduction modulo $p^n$). By the stability of the Higgs bundle, this flow extends $HDF_D|_{D_1}$.

Step 5 In Section 10, we deform $HDF_{X_1}$ to a $(p$-adic, periodic) Higgs-de Rham flow $HDF_X$ over $X$, which extends $HDF_D$. To do this, we use results of Krishnamoorthy-Yang-Zuo [KYZ20] that explicitly calculate the obstruction class of deforming each piece of the Higgs-de Rham flow together with a Lefschetz theorem relating this obstruction class to the obstruction class over $D$.

Finally, in Section 11, we explain the proofs of Corollary 1.6, Corollary 1.7, and Corollary 1.10.
Acknowledgement. We thank Adrian Langer for several useful emails about his work. We thank Ruochuan Liu for explanations and clarifications on his joint work with Xinwen Zhu. We thank Carlos Simpson for the suggestion of Question 1.9 as well as comments on this article and his interest in this work. We also thank an anonymous referee for detailed commentary and thoughtful suggestions. R.K. thanks Ambrus Pál and Carlos Simpson for early conversations which suggested that Question 1.3 could possibly have an affirmative answer. R.K. also thanks Philip Engel and Daniel Litt for several helpful discussions. R.K. thanks the Universität Mainz, where much of this work was conducted, for pleasant working conditions. R.K gratefully acknowledges support from NSF Grant No. DMS-1344994 (RTG in Algebra, Algebraic Geometry and Number Theory at the University of Georgia).

2. Notation

The following notation is in full force for Section 3-Section 10.

- $p$ is an odd prime number.
- $k \cong \mathbb{F}_q$ be a finite field of characteristic $p$.
  - $W := W(k)$,
  - $K := \text{Frac} W$.
- $(X, S)$: $X$ is a smooth projective scheme over $\text{Spec}(W)$ and $S \subset X$ is a relative (strict) normal crossings divisor, flat over $W$.
  - $(X_n, S_n)$: the reduction of $(X, S)$ modulo $p^n$ for any $n \geq 1$;
  - $(X_K, S_K)$: the generic fiber of $(X, S)$;
  - $X^o = X - S$
  - $X$: the $p$-adic formal completion of $X$ along the special fiber $X_1$;
  - $X_K$: the rigid-analytic space associated to $X$.
- $D \subset X$: a relative smooth ample divisor, flat over $W$, that intersects $S$ transversely.
  - Same notation for $D_1, D_n, D_K, D, D_K, D^o$, etc.

3. The theory of Lan-Sheng-Zuo

The following fundamental theorem is a combination of work of Lan-Sheng-Zuo ([LSZ19, Theorem 1.4] for non-logarithmic case and joint with Y. Yang in [LSYZ19, Theorem 1.1] for the logarithmic setting) together with the work of Faltings [Fal89, Theorem 2.6(*)], p. 43] relating logarithmic Fontaine-Faltings modules to crystalline (lisse) $p$-adic sheaves.

Theorem 3.1. (Lan-Sheng-Zuo) Let $X/W$ be a smooth projective scheme and let $S \subset X$ be a relative simple normal crossings divisor, and let $X_K = X \setminus S_K$. Then for each natural number $f \in \mathbb{N}$, there is an equivalence between the category of logarithmic crystalline representations $\pi_1(X_K^o) \to \text{GL}_N(\mathbb{Z}_{p^f})$ with Hodge-Tate weights in the interval $[0, p - 2]$ and the category of $f$-periodic Higgs-de Rham flow over $(X, S)$ where the exponents of nilpotency are less than or equal to $p - 2$.

Let us use this theorem to investigate Question 1.3. Suppose that the Hodge-Tate weights of $\rho_D$ are in $[0, p - 2]$. Then there is a periodic Higgs-de Rham flow $\text{HDF}_D$ on $\mathcal{D}$ associated to this logarithmic crystalline representation under Theorem 3.1. On the other hand, if $\rho_D$ is indecomposable and $N \leq p - 2$, then there exists an integral Tate twist $\rho_D(n)$ with Hodge-Tate weights in $[0, p - 2]$.

Suppose further that

$$(\rho_D)_Q : \pi_1(D_K^o) \to \text{GL}_N(\mathbb{Q}_{p^f})$$
is semi-simple. Let $\tau_Q: \pi_1(D_K^0) \to GL_M(\mathbb{Q}_p f)$ be an irreducible summand of $(\rho_D)_Q$. As $\pi_1(D_K^0)$ is a compact topological group, it follows that we may conjugate $\tau_Q$ to obtain a representation $\tau$ as follows:

$$\tau_Q: \pi_1(D_K^0) \xrightarrow{\tau} GL_M(\mathbb{Q}_p f) \hookrightarrow GL_M(\mathbb{Q}_p f).$$

(If the residual representation is not absolutely irreducible, then there is no guarantee that this choice of lattice is unique up to homothety.) It then follows from the last sentence of [Fal89, Theorem 2.6(*), p. 43] that $\tau$ is again logarithmic crystalline, with Hodge-Tate weights in $[0, p - 2]$. (Note that this holds for any choice of lattice $\tau \subset \tau_Q$.)

Lemma 3.2. Setup as in Question 1.3 and suppose further that $\rho_D$ is indecomposable and $N \leq p - 2$. Then Question 1.3 has an affirmative answer if and only if there exists a periodic Higgs-de Rham flow over $(\mathcal{X}, \mathcal{S})$ extending the Higgs-de Rham flow $\text{HDF}_D$ on $(\mathcal{D}, \mathcal{D} \cap \mathcal{S})$.

Proof. If $\rho_X$ is logarithmic crystalline, we choose the Higgs-de Rham flow $\text{HDF}_X$ associated to $\rho_X$.

Conversely, suppose there exists a Higgs-de Rham flow $\text{HDF}'_X$ on $X$ extending the Higgs-de Rham flow $\text{HDF}_D$ on $(\mathcal{D}, \mathcal{D} \cap \mathcal{S})$. As $N \leq p - 2$, the exponents of nilpotency are all $\leq p - 2$. Then by [Theorem 3.1] one obtains a logarithmic crystalline representation $\rho'_X$ extending $\rho_D$. As the map

$$\pi_1(D_K^0) \to \pi_1(X_K^0)$$

is surjective (by e.g. [EK16, Theorem 1.1(a)]), we see that $\rho_X$ is isomorphic to $\rho'_X$; hence $\rho_X$ is logarithmic crystalline as desired.

Remark 3.3. One may formulate a version of [Lemma 3.2] with the assumption that $\rho_D$ is semi-simple instead of indecomposable, but the precise formulation is a bit laborious due to the different Hodge-Tate ranges on the summands. Nonetheless, we indicate the key point: we claim that if $\rho_D$ has Hodge-Tate weights in the range $[0, p - 2]$, any lattice $\tau$ in any summand of $(\rho_D)_Q: \pi_1(D_K^0) \to GL_M(\mathbb{Q}_p f)$ is logarithmic crystalline with Hodge-Tate weights in $[0, p - 2]$.

4. Local systems and Higgs bundles

4.1. Logarithmic de Rham and crystalline local systems

For the reader’s convenience, we recall Faltings’ crystalline local systems and Scholze’s de Rham local systems.

A filtered de Rham bundle is defined to be vector bundle together with a separated and exhaustive decreasing filtration by locally direct summands, and an integrable connection satisfying Griffiths transversality.

Setup 4.1. Let $Y$ be a smooth scheme over $W$ (not necessary projective). Denote by $\mathcal{Y}$ the $p$-adic formal completion of $Y$ along the special fiber $Y_1$ and by $\mathcal{Y}_K$ the rigid-analytic space associated to $\mathcal{Y}$, which is an open subset of $Y_K^{an}$.

A $\mathbb{Z}_p$-local system $\mathcal{L}$ over the rigid-analytic space $\mathcal{Y}_K$ is called crystalline, if it has an associated Fontaine-Faltings module $(\mathcal{M}, \nabla, \text{Fil}, \varphi)_\mathcal{Y}$ over $\mathcal{Y}$. Using Faltings’ $\mathcal{D}$-functor

[3] In the non-logarithmic case, this follows from the last sentence of [Fal89, Theorem 2.6(*), p. 43].
[Fal89] p.36], locally on a small affine open set \( U = \text{Spec}(R) \), one can reconstruct the local system \( L \) as follows

\[
L(U_K) = \mathbb{D}(M_Y)(U_K) = \lim_{\rightarrow n} \text{Hom}\left(M_Y(U) \otimes B^+(\hat{R})/p^nB^+(\hat{R}), B^+(\hat{R})[1/p]/B^+(\hat{R})\right),
\]

where homomorphisms on the right hand side are \( B^+(\hat{R}) \)-linear and respect the filtrations and the \( \varphi \)'s. In particular, one has

\[
(4.1) \quad M'_Y(U) \otimes_R B^+(\hat{R}) = L(U_K) \otimes_{\mathbb{Z}_p} B^+(\hat{R}).
\]

A \( \mathbb{Z}_p \)-local system over the \( K \)-scheme \( Y_K \) is called \textit{crystalline}, if its restriction on \( Y_K \) is crystalline. If \( Y/W \) is proper, then every finite étale cover of \( Y_K \) extends to a finite étale cover on \( Y_K \). Therefore, every local system over \( Y_K \) can be extended to a unique local system on \( Y_K \). \( \square \)

Following [Sch13] Definition 7.5 and Definition 8.3, a \( \mathbb{Z}_p \)-local system \( L \) on \( \mathcal{Y}_K \) is said to be \textit{de Rham} if there exists a filtered de Rham bundle \((\mathcal{E}, \nabla, \text{Fil})_{\mathcal{Y}_K} \) over \( \mathcal{Y}_K \) such that

\[
\mathcal{E}_{\mathcal{Y}_K} \otimes_{\mathcal{O}_{\mathcal{Y}_K}} \mathcal{O}_{\mathcal{B}_{\text{dR}}} \simeq L \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{B}_{\text{dR}}}. \]

A \( \mathbb{Z}_p \)-local system over \( Y_K \) is called de Rham, if the restriction of the local system on \( \mathcal{Y}_K \) is de Rham.

Following [TT19] Definition 3.10, one can also use the crystalline period sheaf \( \mathcal{O}_{\mathcal{B}_{\text{cris}}} \) to redefine Faltings’ notion of a crystalline representation. A local system \( L \) on \( \mathcal{Y}_K \) is said to be \textit{crystalline} if there exists a filtered \( F \)-isocrystal on \( Y_1 \) with realization \((\mathcal{E}, \nabla, \text{Fil})_{\mathcal{Y}_K} \) over \( \mathcal{Y}_K \) such that there is an isomorphism

\[
\mathcal{E}_{\mathcal{Y}_K} \otimes_{\mathcal{O}_{\mathcal{Y}_K}} \mathcal{O}_{\mathcal{B}_{\text{cris}}} \simeq L \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{B}_{\text{cris}}}
\]

preserving the connection, filtration, and Frobenius. Using the natural inclusion \( \mathcal{O}_{\mathcal{B}_{\text{cris}}} \hookrightarrow \mathcal{O}_{\mathcal{B}_{\text{dR}}} \) [TT19] Corollary 2.25(1)], Tan-Tong showed that any crystalline local system over \( \mathcal{Y}_K \) is also de Rham. \( \square \)

\textbf{Remark 4.2}. Assume that \( L \) is a crystalline local system over \( \mathcal{Y}_K \). Then there is an associated Fontaine-Faltings module \((M, \nabla, \text{Fil}, \varphi)_{\mathcal{Y}} \) over \( \mathcal{Y} \). Since crystalline local systems are always de Rham, there is also a filtered de Rham bundle \((\mathcal{E}, \nabla, \text{Fil})_{\mathcal{Y}_K} \) over \( \mathcal{Y}_K \) associated to \( L \). One then has

\[
(M, \nabla, \text{Fil})_{\mathcal{D}} \big|_{D_K} = (\mathcal{E}, \nabla, \text{Fil})_{\mathcal{D}_K}^{\varphi}\text{.}
\]

The appearance of a dual is simply because Faltings’ original functor is contravariant.

Now, we consider the logarithmic case.

\textbf{Setup 4.3}. Let \( Y, Y_K, \mathcal{Y} \) and \( \mathcal{Y}_K \) be given as in [Setup 4.1]. Let \( Z/S \) be a relative smooth normal crossing divisor in \( Y \) and denote \( U \) the complement of \( Z \) in \( Y \). Then we also construct spaces \( Z_K, \mathcal{Z}_K, U_K, \mathcal{U} \) and \( \mathcal{U}_K \) similar as that for \( Y \) in [Setup 4.1].

We note that \( \mathcal{U}_K \) is a \( p \)-adic rigid analytic open subset of \( \mathcal{Y}_K^o := \mathcal{Y}_K - Z_K \), and in general it is strictly smaller. For instance, consider the following example: \( Y = \text{Spec}W[T] \) and \( Z \) is defined by equation \( T = 0 \). Then \( \mathcal{U}_K \) is the annulus in the analytification of the affine line \( \mathbb{A}^1_K \) given by \( \lvert T \rvert_p = 1 \), but \( \mathcal{Y}_K^o \) is the annulus given by \( 0 < \lvert T \rvert_p \leq 1 \).

\textsuperscript{4}This argument occurs on p. 42 of [Fal89] Theorem 2.6\textsuperscript{4}.

\textsuperscript{5}Here, this isomorphism is of sheaves on the pro-étale site of \( \mathcal{Y}_K \), the rigid-analytic generic fiber.

\textsuperscript{6}In the first line of [Niz01] Proof of 5.1, Niziol also mentions this result.
In [Fal89], Faltings also defined the category Fontaine-Faltings modules in the logarithmic case. Roughly speaking a logarithmic Fontaine-Faltings modules over \((\mathcal{Y}, \mathcal{Z})\) is a filtered logarithmic de Rham bundle \((\mathcal{Y}, \mathcal{Z})\) together with a \(\varphi\)-structure satisfying some axioms with respect to the filtration. Let \(M\) be a logarithmic Fontaine-Faltings module over \((\mathcal{Y}, \mathcal{Z})\). By restricting onto \(\mathcal{U}\) and forgetting the logarithmic structure, one gets a Fontaine-Faltings module over \(\mathcal{U}\). Then Faltings’ \(\mathbb{D}\)-functor sends this module to a crystalline local system \(\mathbb{L}_{\mathcal{U}}\) over \(\mathcal{U}_K\). In [Fal89, page 43 i]], Faltings claims that everything there extends to logarithmic context. In particular, one has following result.

**Proposition 4.4.** Let \(M\) be a logarithmic Fontaine-Faltings module over \((\mathcal{Y}, \mathcal{Z})\). Denote \(\mathbb{L}_{\mathcal{U}}\) extends uniquely to a local system \(\mathbb{L}_{\mathcal{Y}_K^p}\) over \(\mathcal{Y}_K^p\).

(1) Suppose \(Y\) is proper over \(W\). Then the local system \(\mathbb{L}_{\mathcal{Y}_K^p}\) is algebraic. That is, there exists a local system \(\mathbb{L}_{\mathcal{Y}_K^p}\) over \(\mathcal{Y}_K^p = Y_K - Z_K\) such that \(\mathbb{L}_{\mathcal{Y}_K^p} = \mathbb{L}_{\mathcal{Y}_K^p}|_{\mathcal{Y}_K^p}\).

By abusinot notation, we denote by \(\mathbb{D}(M)\) for both local systems \(\mathbb{L}_{\mathcal{Y}_K^p}\) and \(\mathbb{L}_{\mathcal{Y}_K^p}\).

A \(\mathbb{Z}_p\)-local system over \(\mathcal{Y}_K^p\) is called logarithmic crystalline over \((\mathcal{Y}_K, \mathcal{Z}_K)\), if it comes from a logarithmic Fontaine-Faltings over \((\mathcal{Y}, \mathcal{Z})\). A \(\mathbb{Z}_p\)-local system \(\mathcal{Y}_K^p\) is called logarithmic crystalline over \((\mathcal{Y}_K, \mathcal{Z}_K)\), if its restriction on \(\mathcal{Y}_K^p\) is logarithmic crystalline over \((\mathcal{Y}_K, \mathcal{Z}_K)\).

**Remark 4.5.** In [Fal89, page 43 i]], Faltings wrote a very rough idea about logarithmic context. Here we write down a proof of Proposition 4.4 explicitly. We crucially use the non-logarithmic case [Fal89, Theorem 2.6].

**Proof of Proposition 4.4.** For the first statement, since it is a local property, so we may assume \(\mathcal{Y} = \text{Spec}(R)\) is affine with a toric chart \(W[T_1, \ldots, T_d] \to R\) such that the relative normal crossing divisor \(Z\) on \(X\) is defined by the equation \(T_1 \cdots T_m = 0\).

Let’s recall a Faltings’ observation in [Fal89, Remark in page 43]. Denote by \(M_n := M (\text{mod } p^n)\), which is sent to the \(\mathbb{Z}/p^n\mathbb{Z}\)-local system \(\mathbb{L}_n = \mathbb{L} \text{ (mod } p^n)\) over \(\mathcal{U}_K\) under Faltings’ \(\mathbb{D}\)-functor. Consider the base extension

\[(4.2) \quad \pi_n : R \to R_n = R[T_1^{p^{-n}}, \ldots, T_m^{p^{-n}}].\]

Denote by \(Y_n = \text{Spec}(R_n)\). It is clear that the map \(\pi_n : Y_n \to Y\) is ramified along \(Z\) of order \(p^n\). Because

\[
d \log T_i = p^n \cdot d \log T_i^{p^{-n}}
\]

and \(M_n\) is killed by \(p^n\), the pulled back logarithmic Fontaine-Faltings module \(\pi_n^* M_n\) over \((\mathcal{Y}_n, \mathcal{Z}_n)\) no longer has logarithmic poles along \(\mathcal{Z}_n := \pi_n^{-1}(Z)!\) Hence \(\pi_n^* M_n\) is a Fontaine-Faltings module over \(\mathcal{Y}_n\). By taking the Faltings’ \(\mathbb{D}\)-functor for this Fontaine-Faltings module over \(\mathcal{Y}_n\), one gets a local system, denoted by \(\mathbb{L}_{\mathcal{Y}_n,K}\), over \(\mathcal{Y}_n,K\). We note that this local system extends the pulled back local system \(\pi_n^* \mathbb{L}_n\) over \(\mathcal{U}_n,K\) by the compatibility of \(\mathbb{D}\)-functor over \(\mathcal{Y}_n\) and that over \(\mathcal{U}_n\).

Since \(\pi_n^* M_n\) is a pulled-back Fontaine-Faltings module, the covering group \(\text{Gal}(\pi_n)\) naturally acts on \(\pi_n^* M_n\). It induces an action of the covering group \(\text{Gal}(\pi_n)\) on \(\mathbb{L}_{\mathcal{Y}_n,K}\) under \(\mathbb{D}\)-functor. In general this action does not always permit descent, because the stabilizer of some point does not act trivially on the fiber at this point. Once we restrict the local system \(\mathbb{L}_{\mathcal{Y}_n,K}\) on the étale locus \(\mathcal{Y}_n,K = \mathcal{Z}_n,K\), then the obstruction to descending this local system vanishes. By descending, one gets the wanted local system, denoted by \(\mathbb{L}_{\mathcal{Y}_K,K}\), over \(\mathcal{Y}_K = \mathcal{Y}_K - \mathcal{Z}_K\).

For the second statement, we need to use the global condition of properness. Here is a high-level summary of the proof. We first find a suitable global ramified cover of \(Y\) such
that the pullback local system can be extended globally. It will be algebraic by formal GAGA (as in p. 42 of [Fal89, Theorem 2.6*]). Then by descending, we get the local system we want.

In order to illustrate the idea, we first consider the projective line $Y = \mathbb{P}_W^1$ together with $m$ distinct $W$-points $Z = \{x_1, \ldots, x_m\}$ such that $x_i \neq x_j \pmod{p}$, where $m$ is an integer greater than 1. Without loss generality, we assume that $x_i \neq 0 \pmod{p}$. By adding a $p^n$-root of $\frac{z-x_i}{z-x_1}$ for each $1 < i \leq m$ into the rational function field of $Y$ one gets an extension field

$$L = K(z) \left[ \left( \frac{z-x_2}{z-x_1} \right)^{p^n}, \ldots, \left( \frac{z-x_m}{z-x_1} \right)^{p^n} \right]$$

where $z$ is a parameter of the projective line. We denote by $Y_n$ the normalization of $Y$ in $L$. Then one has the normalization morphism

$$\tau_n : Y_n \rightarrow Y.$$ 

Denote by $Z_n = \pi_n^* Z$. In general, the normalization $Y_n$ is not smooth over $W$, but its generic fiber $Y_{n,K}$ is smooth over $\text{Spec}(K)$. This is because the degree of $p^n$ is not coprime with $\text{char}(k) = p$ and it is coprime with $\text{char}(K) = 0$. By the construction of $L$, the morphism $\tau_n$ is étale over the generic fiber $Y_{n,K}^0$ of $Y^0 = Y - Z$ and along $Z$. The morphism $\tau_n$ locally factors through the Kummer cover constructed in local situation above. For the reader’s convenience, we write down a local factorization in following. Firstly, we choose an affine covering $\{V^{(0)}, V^{(\infty)}, V^{(x_1)}, \ldots, V^{(x_m)}\}$ of $\mathbb{P}_W^1$ with toric charts:

$$V^{(0)} = \text{Spec}(R^{(0)}) \quad R^{(0)} = W[z, \frac{1}{z-x_1}, \ldots, \frac{1}{z-x_m}] \rightarrow W[z],$$

$$V^{(\infty)} = \text{Spec}(R^{(\infty)}) \quad R^{(\infty)} = W[\frac{1}{z}, \frac{1}{z-x_1}, \ldots, \frac{1}{z-x_m}] \rightarrow W[\frac{1}{z}],$$

$$V^{(x_1)} = \text{Spec}(R^{(x_1)}) \quad R^{(x_1)} = W[z, \frac{1}{z-x_1}, \ldots, \frac{1}{z-x_{m-1}}, \frac{z-x_i}{z-x_1}] \rightarrow W[\frac{z-x_1}{z-x_i}],$$

$$V^{(x_i)} = \text{Spec}(R^{(x_i)}) \quad R^{(x_i)} = W[z, \frac{1}{z-x_1}, \ldots, \frac{1}{z-x_{i-2}}, \frac{z-x_i}{z-x_{i-1}}, \frac{z-x_i}{z-x_{i-1}}, \ldots, \frac{z-x_i}{z-x_{i+1}}] \rightarrow W[\frac{z-x_i}{z-x_1}] \quad \text{for } i = 2, \ldots, m$$

For any $V \in \{V^{(0)}, V^{(\infty)}, V^{(x_1)}, \ldots, V^{(x_m)}\}$, denote $R = \mathcal{O}_{\mathbb{P}_W^1}(V)$. Recall the Kummer covering $V'_n = \text{Spec}(R_n)$ constructed in local situation (4.2), one has

$$R_n = \begin{cases} R^{(0)} & \text{if } V = V^{(0)}; \\
R^{(\infty)} & \text{if } V = V^{(\infty)}; \\
R^{(x_1)} \left[ \left( \frac{z-x_1}{z-x_2} \right)^{p^n} \right] & \text{if } V = V^{(x_1)}; \\
R^{(x_i)} \left[ \left( \frac{z-x_i}{z-x_1} \right)^{p^n} \right] & \text{if } V = V^{(x_i)} \text{ and } i \in \{2, \ldots, m\}; \end{cases}$$

Denote by $V_n = V \times_{\mathbb{P}_W^1} Y_n$ the base-change open subset of $Y_n$. Then locally over $V$, the map $\tau_n$ factors through $\pi_n$. Hence one has morphisms

$$Y_n \supset V_n \xrightarrow{\pi'_n} V' \xrightarrow{\pi_n} V \subset Y.$$
Considering the Raynaud generic fibers, one has following diagram

\[
\begin{array}{ccc}
\mathcal{V}^{\sigma}_{n,K} & \xrightarrow{\pi_n} & \mathcal{V}^{\sigma}_K \\
\cap & \cap & \cap \\
\mathcal{Y}_{n,K} & \xrightarrow{\pi_n'} & \mathcal{Y}_{K} \\
\tau_n & \xrightarrow{} & \\
\end{array}
\]

Consider the local system \( L = L_{Y_K} \) (mod \( p^n \)) over \( \mathcal{Y}^{\sigma}_K \) obtained from the Fontaine-Faltings module \( M \) (mod \( p^n \)). By the first statement of this proposition, we know that the local system \( \pi_n^* (L_{n}|_{\mathcal{V}^{\sigma}_K}) \) over \( \mathcal{V}^{\sigma}_K \) extends to a local system over \( \mathcal{V}_K \) and the extended local system endows with an action of the covering group \( \text{Gal}(\pi_n) \). Taking \( \pi_n^{**} \), then one gets a local system over \( \mathcal{V}_{n,K} \) with an action of the covering group \( \text{Gal}(\tau_n) \). By gluing, one gets a \( p^n \)-torsion local system \( \mathbb{L}_{\tau_n} \) with an action of \( \text{Gal}(\tau_n) \) over \( \mathcal{Y}_{n,K} \). Since our space \( Y \) is proper over \( W \), so is \( Y_{n,K} \) over \( K \). Now, by formal GAGA, one gets a \( p^n \)-torsion local system \( \mathbb{L}_{\tau_n} \) over \( Y_{n,K} \) with an action of \( \text{Gal}(\tau_n) \) over \( Y_{n,K} \). By descending, one gets a local system \( L_{n,Y_K} \) over \( Y^{\sigma}_K \). By its construction, it is just the algebraization of \( \mathbb{L}_{n} \).

Taking the projective limits, one obtains the desired \( \mathbb{Z}_p \)-local system

\[
\mathbb{L}_{Y_{n,K}} = \lim_{\leftarrow n} \mathbb{L}_{n,Y_{n,K}}.
\]

Let’s consider the general situation of higher dimensional \( Y \). As in the case of projective line, we only need find a covering of \( Y_n \) of \( Y \), which locally factors through the Kummer cover \( \pi_n \) constructed in (4.2). Here we use the Kawamata cover trick [Kaw81] to find such a covering \( \tau_n: Y_n \to Y \).

Write \( Z = \sum_{i=1}^{n} Z(i) \) as the sum of irreducible components. Since \( Y \) is projective over \( W \), there exists a relative ample line bundle \( A \) over \( Y \). Choose a sufficiently large power \( B = A^n \) of \( A \) such that the following conditions hold.

- \( B^{p^n}(-Z(i)) \) is generated by global sections for each \( i = 1, \cdots, m \);
- For each \( i \in \{1, \cdots, m\} \), there are \( d = \dim_W Y \) \( W \)-relative smooth divisors \( H(i,1), \cdots, H(i,d) \) for each such that \( H(i,j) \notin Z \),

\[
\mathcal{O}_Y(H(i,j)) = B^{p^n}(-Z(i)),
\]

and moreover, if we set \( H = \sum_{i,j} H(i,j) \), the divisor \( Z + H \subset Y \) is relative simple normal crossing over \( W \).

The existence of such an \( n \) and the \( H(i,j) \) follows from Poonen’s Bertini theorems over finite fields [Poo04, Theorem 1.3]. Indeed, for \( n \) sufficiently divisible, one can satisfy the two conditions on the special fiber of \( Y \), and, at the cost of increasing \( n \), one can also assure that for each \( i \), all global sections of \( B^{p^n}(-Z(i)) \) on the special fiber of \( Y \) lift to global sections of \( B^{p^n}(-Z(i)) \) on all of \( Y \).

Denote by \( Z_{ij} \) be the cyclic cover obtained by taking the \( p^n \)-th root out of \( H(i,j) + Z(i) \). In other word, the local coordinate ring of the cyclic covering is obtained by adding the \( p^n \)-th root of the global section, denote by \( T_{ij} \), of the line bundle \( B^{p^n} \) corresponding to the divisor \( H(i,j) + Z(i) \). Let \( Y_n \) be the normalization of

\[
Z_{11} \times_Y \cdots \times_Y Z_{1d} \times_Y Z_{21} \times_Y \cdots \times_Y Z_{2d} \times_Y \cdots \times_Y Z_{md}.
\]

Denote by \( \tau_n \) the morphism

\[
\tau_n: Y_n \to Y,
\]

\[
\tau_n^*: \mathcal{Y}_{n,K} \to \mathcal{Y}_K.
\]
which is étale over $Y_K - (Z + H)_K$. For the details of the cyclic cover trick we invite
readers to look at Kawamata original paper [Kaw81], or the book by Esnault-Viehweg
([EV92], Lemma 3.19).

Now, we show that the covering morphism $\pi_n$ constructed above factors through the
Kummer covering given in (4.2). Without loss of generality, we choose a point

$$x \in \bigcap_{i=1}^r Z_i \setminus \bigcup_{i=r+1}^m Z_i,$$

and choose a sufficiently small affine neighborhood $V$ of $x$. By the construction, for
each $i$ in $\{1, \cdots, r\}$, there exists $j_i \in \{1, \cdots, d\}$ such that $H(i, j_i)$ does not vanish at
$x$. This is because $Z + H$ is relatively normal crossing over $W$, there are no more than
d irreducible components of $Z + H$ pass through any given point. Consider the global
section $T_i := T_{i, j_i}$ determined by the $H(i, j_i) + Z(i)$ for each $i = 1, \cdots, r$. Since $Z$ is normal
crossing and $H(i, j_i)$ does not vanish at $x$, the system $\{T_1, \cdots, T_r\}$ can be extended to a
local parameter system $\{T_1, \cdots, T_d\}$ at $x$. Similar as the projective line case, we denote
$R = \mathcal{O}_Y(V)$, $R_n = R[T_1^{p^{-n}}, \cdots, T_r^{p^{-n}}]$, $V'_n = \text{Spec}(R_n)$ and $V_n = V \times_Y Y_n$. Then we have morphisms

$$V_n \xrightarrow{\pi_n} V'_n \xrightarrow{\pi_n} V$$

Following exactly the same argument as for projective line, one gets the desired local
system. This completes the proof.

Finally, we pose a basic conjecture in logarithmic $p$-adic Hodge theory. The conjecture roughly says the following: two natural filtered de Rham bundles associated to a
logarithmic crystalline representation coincide.

**Conjecture 4.6.** Let $Y/W$ be a smooth proper scheme and let $Z \subset Y$ be a relative simple
normal crossings divisor. Let $\mathbb{L}$ be a logarithmic crystalline local system on $(Y, Z)$, with
associated logarithmic Fontaine-Faltings module $(M, \nabla, \text{Fil}, \phi)_{Y, K}$. As $Y/W$ is proper,
the triple $(M, \nabla, \text{Fil})$ is indeed algebraic, and hence defined over the scheme $Y_K$. By
**Remark 4.2** and the rigidity of de Rham local systems, the local system satisfies the con-
dition in [DLLZ18, Theorem 1.1]. Then there is an isomorphism of logarithmic filtered
de Rham bundles on $Y_K$

$$(M, \nabla, \text{Fil})_Y |_{Y, K} = D_{\text{alg}}^{\text{dR}}(\mathbb{L} \otimes \mathbb{Q}_p)^\vee,$$

where the latter is as in [DLLZ18, Theorem 1.1].

**Remark 4.7.** Suppose one had a logarithmic version crystalline periodic sheaf $\mathcal{O}_{\mathbb{B}_{\text{cris, log}}}$
satisfying a logarithmic version of the Tan-Tong theorem. This means that

(1) a local system $\mathbb{L}$ is logarithmic crystalline if and only if there exist a filtered
logarithmic $F$-isocrystal $\mathcal{E}$ satisfying

$$\mathcal{E} \otimes \mathcal{O}_{\mathbb{B}_{\text{cris, log}}} \simeq \mathcal{L} \otimes \mathcal{O}_{\mathbb{B}_{\text{cris, log}}};$$

(2) there is a natural injective map $\mathcal{O}_{\mathbb{B}_{\text{cris, log}}} \hookrightarrow \mathcal{O}_{\mathbb{B}_{\text{dR, log}}}$, where the latter is the
logarithmic de Rham period sheaf defined in [DLLZ18].

Then we claim that the conjecture holds. Indeed, by forgetting the Frobenius structure
on both sides in (4.3) and extending the coefficient to $\mathcal{O}_{\mathbb{B}_{\text{dR, log}}}$ one gets the required
equation to ensure the $\mathbb{L}$ being de Rham.
4.2. De Rham local systems and graded Higgs bundles.

Let $\mathbb{L}_{Y, \kappa}$ (respectively $\mathbb{L}_{D, \kappa}$) be the $\mathbb{Z}_p$-local system on $Y_{\kappa}$ (respectively $D_{\kappa}$) associated to the representation $\rho_{X}$ (respectively $\rho_{D}$) given in \textbf{Theorem 1.4}. Denote by $\mathbb{L}_{Y, \kappa}$ and $\mathbb{L}_{D, \kappa}$ their analytifications. By Faltings’ definition of logarithmic crystalline representation, there is an associated logarithmic Fontaine-Faltings module $(M, \nabla, \Fil, \varphi)_{D}$ over $(\mathcal{D}, \mathcal{D} \cap S)$ associated to $\rho_{D}$. By taking the associated graded of the underlying filtered logarithmic de Rham bundle $(M, \nabla, \Fil)_{D}$, we obtain a logarithmic Higgs bundle $(E_{\kappa}, \theta_{K})$ over $(\mathcal{D}, \mathcal{D} \cap S)$. Under the condition that $\rho_{D}$ comes from geometry, we show in this section that the restriction of $(E_{\kappa}, \theta_{K})$ onto $(\mathcal{D}_{\kappa}, \mathcal{D}_{\kappa} \cap S_{\kappa})$ extends to a logarithmic Higgs bundle $(E_{X_{\kappa}}, \theta_{X_{\kappa}})$ over $(X_{\kappa}, S_{\kappa})$. We will now explain that under certain circumstances, Conjecture 4.6 holds. Maintain notation as in \textbf{Setup 4.3}. First, we define what it means for a logarithmic crystalline local system on $(Y, Z)$ to strictly come from geometry. Let $Y'/W$ be a smooth scheme and let $Z' \subset Y'$ be a $W$-flat relative simple normal crossings divisor. Suppose we are equipped with a map $f : (Y', Z') \rightarrow (Y, Z)$ that is log smooth, i.e., such that the differential map $df : f^{*}(\Omega^{1}_{Y'/W}(dZ)) \rightarrow \Omega^{1}_{Y'/W}(d \log Z')$ is locally a direct summand. Suppose $\mathcal{E}$ is a logarithmic Fontaine-Faltings module over $(Y', Z')$ (the associated $p$-adic formal completions of $Y'$ and $Z'$ respectively) with Hodge-Tate weights in $[0, a]$. Let $\mathbb{L} = \mathbb{D}(\mathcal{E})$ be the associated logarithmic local system over $(Y'_{\kappa}, Z'_{\kappa})$. Denote by $d$ the relative dimension of $Y'$ over $Y$. In the case $a + \min\{b, d\} \leq p - 2$, according to \textbf{FalK9} Theorem 6.2, the local system $R^{b}f^{\circ*}_{et}(\mathbb{L})_{Y'}$ is still logarithmic crystalline and it is the associated logarithmic local system of the Fontaine-Faltings module $R^{b}f^{\circ}_{cryst}(\mathcal{E})$. Namely, one has $R^{b}f^{\circ*}_{et}(\mathbb{L})_{Y'} = \mathbb{D}(R^{b}f^{\circ}_{cryst}(\mathcal{E}))$.

In particular, for trivial local system $\mathbb{L}_{\kappa}$ over $Y'_{\kappa}$ and its associated Fontaine-Faltings module $(\mathcal{O}_{Y'}, \nabla, \Fil_{\text{triv}}, \varphi_{\text{triv}})$, one has $R^{b}f^{\circ*}_{et}(\mathbb{L}_{\kappa}) = \mathbb{D}(R^{b}f^{\circ}_{cryst}(\mathcal{O}_{Y'}, d, \Fil_{\text{triv}}, \varphi_{\text{triv}}))^{\vee}$.

\textbf{Definition 4.8}. Let $Y/W$ be a smooth proper scheme of relative dimension $d$ and let $Z \subset Y$ be a $W$-flat relative simple normal crossings divisor. A logarithmic crystalline local system $\mathbb{L}$ on $(Y_{\kappa}, Z_{\kappa})$ is said to be \textit{strictly coming from geometry} if there exists a pair $(Y', Z')/W$ together with a log smooth morphism $(Y', Z') \rightarrow (Y, Z)$ of relative dimension $d$ and an integer $\min\{b, d\} \leq p - 2$ such that such that $\mathbb{L}$ is a subquotient of $R^{b}f^{\circ*}_{et}(\mathbb{L}_{\kappa})$.

\textbf{Lemma 4.9}. Let $Y/W$ be a smooth proper scheme of relative dimension $d$ and let $Z \subset Y$ be a $W$-flat relative simple normal crossings divisor. Let $\mathbb{L}_{Y_{\kappa}}$ be a logarithmic crystalline representation on $(Y, Z)$ strictly coming from geometry in the sense of \textbf{Definition 4.8}. By Faltings’ definition of logarithmic crystalline representation, there is an associated logarithmic Fontaine-Faltings module $(M, \nabla, \Fil, \varphi)_{Y}$ over $(Y, Y \cap Z)$ associated to $\mathbb{L}_{Y_{\kappa}}$. Taking the functor $D_{\text{dR}, \text{log}}$ in \textbf{DLLZ18} Theorem 1.7, one has a logarithmic de Rham bundle $D_{\text{dR}, \text{log}}(\mathbb{L}_{Y_{\kappa}} \otimes \mathbb{Q}_{p})$ over $(Y_{\kappa}, Z_{\kappa})$. Then $(M, \nabla, \Fil)_{Y, \kappa} = D_{\text{dR}, \text{log}}(\mathbb{L}_{Y_{\kappa}} \otimes \mathbb{Q}_{p})^{\vee}$, i.e., \textbf{Conjecture 4.6} holds for $\mathbb{L}_{D_{\kappa}}$. 


Proof. We first consider the case \( \mathbb{L}_{D^\log_k} = R^if_{\text{et}}\mathbb{Z}_p \), where \( f: (Y, Z) \to (D, D \cap S) \) is a logarithmic smooth family. Then by the functorial property of Faltings’ \( \mathbb{D} \)-functor with respect to logarithmic smooth family \cite[Theorem 6.3]{Fal89}, one has

\[
(M, \nabla, \text{Fil}, \varphi)^\vee_D = R^if_{\text{et}}^{\text{crys}}((\mathcal{O}_Y, d, \text{Fil}_{\text{triv}}, \varphi_{\text{triv}}).
\]

Recall the functorial property of the functor \( D_{\text{dR, log}} \) \cite[Theorem 3.5.8 on p. 38]{DLLZ18},

\[
D_{\text{dR, log}}(\mathbb{L}_{D^\log_k} \otimes \mathbb{Q}_p) = R^if_{\text{et}}^{\text{dR}}((\mathcal{O}_Y, d, \text{Fil}_{\text{triv}}) |_{y_K}).
\]

Since the crystalline direct image coincides with de Rham direct image in characteristic 0, one has

\[
(M, \nabla, \text{Fil})_D |_{D_K} = D_{\text{dR, log}}(\mathbb{L}_{D^\log_k} \otimes \mathbb{Q}_p)^\vee.
\]

Next, we consider the general case. Assume \( \mathbb{L}_{D^\log_k} \) is a subquotient of \( R^if_{\text{et}}\mathbb{Z}_p \). Since crystalline representations and de Rham representations are preserved under taking sub and quotient. And the functor \( D \) is contravariant and the \( D_{\text{dR, log}} \) is covariant. Then taking the corresponding subquotients, one will gets

\[
(M, \nabla, \text{Fil})_D |_{D_K} = D_{\text{dR, log}}(\mathbb{L}_{D^\log_k} \otimes \mathbb{Q}_p)^\vee.
\]

With this result in hand, we have the following.

**Proposition 4.10.** Maintain notation as in Setup 1.3 and Question 1.3. Suppose further that if \( S \neq \emptyset \), then Conjecture 4.6 holds for \( \rho_D \) (this is the case, e.g., if \( \rho_D \) strictly comes from geometry in the sense of Definition 4.8). By Faltings’ definition of logarithmic crystalline representation, there is an associated logarithmic Fontaine-Faltings module \((M, \nabla, \text{Fil}, \varphi)_D \) over \((D, D \cap S) \) associated to \( \rho_D \). By taking the associated graded of the underlying logarithmic de Rham bundle \((M, \nabla, \text{Fil})_D \), we obtain a logarithmic Higgs bundle \((E, \theta)_D \) over \((D, D \cap S) \). Then we have the following.

1. There exists a logarithmic filtered de Rham bundle \((M, \nabla, \text{Fil})_{X_K} \) over \((X_K, S_K) \) such that

\[
(M, \nabla, \text{Fil})_D |_{D_K} = (M, \nabla, \text{Fil})_{X_K} |_{D_K}.
\]

Furthermore, the connection has nilpotent residues around \( S_K \).

2. There exists a logarithmic graded Higgs bundle \((E, \theta)_{X_K} \) over \((X_K, S_K) \) extending \((E, \theta)_D \) \( D_K \).

**Proof.** Taking the associated graded, one derives (2) from (1) trivially. To prove (1), we must construct a logarithmic filtered de Rham bundle.

If \( S = \emptyset \), then the representation \( \rho_X \) is de Rham by \cite{LZ17}. Then the existence of \((M, \nabla, \text{Fil})_{X_K} \) follows from by Remark 4.2. In the logarithmic setting, we will use the functor \( D_{\text{dR}} \) in \cite[Theorem 1.1]{DLLZ18} and then show its dual satisfies the requirements in (1).

Denote by \( \mathbb{L}_{D^\log_k} \) and \( \mathbb{L}_{X^\log_k} \) the corresponding local systems associated to \( \rho_D \) and \( \rho_X \). We first prove that the condition in \cite[Theorem 1.1]{DLLZ18} is satisfied for the local system \( \mathbb{L}_{X^\log_k} \otimes \mathbb{Q}_p \) over \( X^\circ_k \), i.e. \( \mathbb{L}_{X^\log_k} |_{(X^\circ_k)^{an}} \) is de Rham. Denote by \( \mathcal{U} \) the \( p \)-adic completion of \( D - D \cap S \) along its special fiber, and denote by \( \mathcal{U}_K \) the analytic generic fiber of \( \mathcal{U} \) which is an open subset of \((D^\log_K)^an\). By forgetting the logarithmic structure, the local system \( \mathbb{L}_{D^\log_k} |_{\mathcal{U}_K} \) is crystalline (the local version in the sense of \cite[II(g)]{Fal89}) with associated Fontaine-Faltings module \((M, \nabla, \text{Fil}, \varphi)_D |_{\mathcal{U}} \). Then by \cite[Corollary 2.25(1)]{TT19}, \( \mathbb{L}_{D^\log_k} |_{\mathcal{U}_K} \) is also de Rham with attached filtered de Rham bundle

\[
D_{\text{dR}}(\mathbb{L}_{D^\log_k} |_{\mathcal{U}_K}) = (M, \nabla, \text{Fil})_D |_{\mathcal{U}_K},
\]
where $D_{dR}$ is Liu-Zhu’s functor in [LZ17, Theorem 3.8]. Since $\mathbb{L}_{\mathbb{X}_K^\circ}$ extends $\mathbb{L}_{D_{\circ}}$, by rigidity of de Rham local system [LZ17, Theorem 1.5](iii)], $\mathbb{L}_{\mathbb{X}_K^\circ}|_{(X_K^\circ)^{\text{an}}}$ is also de Rham.

Now, the functor $D_{dR}^{\text{alg}}$ in [DLLZ18, Theorem 1.1] gives us a logarithmic filtered de Rham bundle $D_{dR}^{\text{alg}}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p)$ over $(X_K, S_K)$. We denote $(M, \nabla, \text{Fil})_{X_K}$ to be the dual of $D_{dR}^{\text{alg}}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p)$. That is

$$(M, \nabla, \text{Fil})_{X_K} = D_{dR}^{\text{alg}}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p)^\vee.$$ 

Finally, we show that $(M, \nabla, \text{Fil})_{X_K}$ satisfies our requirement in (1). From the construction of $D_{dR}^{\text{alg}}$, it is the algebraization of the functor $D_{dR, \log}$ in [DLLZ18, Theorem 1.7], i.e.

$$D_{dR, \log}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p) = D_{dR}^{\text{alg}}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p) |_{X_K^\circ}.$$ 

By then functorial property of $D_{dR, \log}$, one has

$$(4.5) \quad D_{dR, \log}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p) = D_{dR, \log}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p) |_{D_K}.$$ 

By Lemma 4.9, we have

$$(M, \nabla, \text{Fil})_{D_K} = D_{dR, \log}(\mathbb{L}_{\mathbb{X}_K^\circ} \otimes \mathbb{Q}_p)^\vee.$$ 

Summing up the equations above, one has

$$(M, \nabla, \text{Fil})_{X_K} |_{D_K} = (M, \nabla, \text{Fil})_{D_K} |_{D_K}.$$ 

Note that $D_K$ meets every component of $S_K$ as $D_K \subset X_K$ is ample. As $\rho_D$ is logarithmic crystalline, the pair $(E_{D_K}, \nabla)$ has nilpotent residues around $S_K \cap D_K$. On the other hand, the residues are constant along each component of $S_K$; therefore $\nabla$ has nilpotent residues around $S_K$.$^7$

The fact that $M_{X_K}$ admits an integrable connection with logarithmic poles and nilpotent residues implies that its de Rham Chern classes all vanish [EV86, Appendix B]. By comparison between de Rham and $l$-adic cohomology, this implies that the $\mathbb{Q}_l$ Chern classes also vanish.

Consider the associated graded Higgs bundle $(E, \theta)_{X_K} = \text{Gr}((M, \nabla, \text{Fil})_{X_K})$ over $X_K$, which extends the Higgs bundle $(E, \theta)_D |_{D_K}$. Since $(E, \theta)_{X_K} |_{D_K}$ is semistable, it follows that $(E, \theta)_{X_K}$ is also semistable. As the Chern classes of $M_X$ all vanish, so do the Chern classes of $E$.

In summary, we have constructed a logarithmic Higgs bundle $(E, \theta)_{X_K}$ extending the Higgs bundle $(E_D, \theta_D) |_{D_K}$ that is semistable, and has (rationally) trivial Chern classes. In the next sections, we will extend $(E, \theta)_{X_K}$ to a Higgs bundle on $X$ whose special fiber is also semistable.

## 5. A theorem in the style of Langton

In this section, we apply results due to Langer [Lan14, Lan19] in the vein of Langton to graded semistable logarithmic Higgs bundles. We fix an auxiliary prime $\ell \neq p$. (All subsequent results are independent of the choice of $\ell$.)

**Theorem 5.1.** Let $(E_{X_K}, \theta_{X_K})$ be a graded semistable logarithmic Higgs bundle over $X_K$ such that the underlying vector bundle has rank $r \leq p$ and trivial $\mathbb{Q}_\ell$ Chern classes. Then there exists a semi-stable logarithmic Higgs bundle $(E_X, \theta_X)$ over $X$, which satisfies

$^7$Another argument for the nilpotence of the residues; the local monodromy of $\rho_D$ along $D_K \cap S_K$ is unipotent. This implies that the local monodromy of $\rho_X$ along $S_K$ is also unipotent because $D_K$ intersects each component of $S_K$ non-trivially and transversally. Then simply apply [DLLZ18, Theorem 3.2.12].
Step 2. We now claim that there exists a logarithmic Higgs sheaf \((M \to M)\) of Langer \([Lan19, \text{Theorem 1.2}]\). Consider the constructible map \(M\) inverse Cartier \(C\), whose field of definition is the same as the field of definition of the flat connection Simpson filtration semistable logarithmic flat connection has a distinguished gr-semistable Griffiths transverse filtration, the logarithmic Higgs bundle of rank \(r\).

There exists a finite field \(F\) where \(Gr\) is the associated graded of the Simpson filtration. (Compare with \([Lan19, \text{Theorem 1.6}]\).

normal crossings divisor defined over \(F\) for all \(i > 1\).

Proof. The proof works in several steps.

Step 1. We first construct an auxiliary logarithmic coherent Higgs sheaf \((F_X, \Theta_X)\) on \(X\) such that

(a) \(F_X\) is reflexive (and hence torsion-free);
(b) there is an isomorphism \((F_X, \Theta_X)|_{X_K} \cong (E_{X_K}, \theta_{X_K})\).

First of all, \(E_K\) admits a coherent extension \(F_X\) to \(X\). Replacing \(F_X\) with \(F_X^*\), this is reflexive.

We must construct a logarithmic Higgs field. We know that \(F_X \subset F_{X_K} \cong E_{X_K}\). Work locally, over open \(W\)-affines \(U\), and pick a finite set of generators \(f_i\) for \(F_{U\alpha}\). Then, for each \(\alpha\), there exists an \(r_\alpha \geq 0\) with

\[
\theta_{X_K}(f_i) \subset p^{-r_\alpha}F_{U\alpha} \otimes_{\mathcal{O}_{U\alpha}} \Omega^1_{U\alpha/W}(\log S \cap U\alpha)
\]

for each \(i\). As \(X\) is noetherian, there are only finitely many \(\alpha\) and hence there is a uniform \(r\) such that

\[
p^r \theta_{X_K} : F_X \to F_X \otimes_{\mathcal{O}_X} \Omega^1_{X/W}(\log S).
\]

Set \(\Theta_X := p^r \theta_{X_K}\); this makes sense by the above formula \((5.1)\). Then \((F_X, \Theta_X)\) is a Higgs sheaf on \(X\). We claim that \((F_X, \Theta_X)|_{X_K} \cong (E_{X_K}, \theta_{X_K})\). Indeed, as \((E_{X_K}, \theta_{X_K})\) is a graded Higgs bundle, there exists an isomorphism \((E_{X_K}, p^r \theta_{X_K}) \cong (E_{X_K}, \theta_{X_K})\).

Finally, as \(F_X\) is torsion-free, it is automatically \(W\)-flat.

Step 2. We now claim that there exists a logarithmic Higgs sheaf \((E_X, \theta_X)\) extending \((E_{X_K}, \theta_{X_K})\) such that the logarithmic Higgs sheaf \((E_{X_1}, \theta_{X_1})\) on the special fiber \(X_1\) is semistable. This is a direct consequence of [Lan14, \text{Theorem 5.1}]\footnote{To apply Langer’s theorem directly, set \(L\) to be the smooth Lie algebroid whose underlying coherent sheaf is the logarithmic tangent sheaf, and whose bracket and anchor maps are trivial.} We emphasize that \((E_X, \theta_X)\) is merely a Higgs sheaf; we don’t yet know it is a vector bundle.

The \((E_X, \theta_X)\) constructed by Langer’s theorem is torsion-free by design, hence also \(W\)-flat. We claim that the Chern classes of \((E_{X_1}, \theta_{X_1})\) vanish because the Chern classes of \((E_{X_K}, \theta_{X_K})\) do; this is what we do in Section 6.

Step 3. The special fiber \((E_{X_1}, \theta_{X_1})\) is a semistable logarithmic Higgs sheaf with rank \(r \leq p\). Moreover, the Chern classes all vanish. Then it directly follows from [Lan19, \text{Theorem 2.2}] that \((E_{X_1}, \theta_{X_1})\) is locally free.\footnote{See Section 1.6 and Lemma 1.7 of loc. cit. for why all Chern classes vanishing implies that \(\Delta_i = 0\) for all \(i > 1\) and hence why the hypotheses Lemma 2.5 of loc. cit. are met.}
Step 4. Finally, it follows from the easy argument below that $E_X$ is a vector bundle. 

**Lemma 5.2.** Let $F_X$ be a torsion free and coherent sheaf over $X$. If $F_X |_{X_i}$ is locally free, then $F_X$ is locally free.

**Proof.** Since $F_X$ is torsion free and coherent sheaf, its singular locus $Z$ is a closed sub scheme of $X$. And $F_X$ is locally free if and only if $Z$ is empty. Since $F_X |_{X_i}$ is locally free, its singular locus $Z \cap X_i$ is empty. On the other hand $Z$ is empty if and only if $Z \cap X_1$ is empty, because $X$ is proper over $W$. Thus $Z$ is also empty. □

We emphasize that we do not know the Higgs sheaf $(E_{X_1}, \theta_{X_1})$ is graded in general. We will deduce this in our situation using a Lefschetz-style theorem.

### 6. Local constancy of Chern classes

To explain the local constancy of this section, we first need some preliminaries. Let $S$ be a separated scheme and let $f: X \to S$ be smooth projective morphism. Let $\ell$ be invertible on $S$. Then the $\ell$-adic sheaf $R^i f_* \mathcal{Z}_\ell(j)$ is a constructible, locally constant sheaf, i.e., a lisse $\ell$-adic sheaf, by the smooth base change theorem. Moreover, for any point $s \in S$, the proper base change theorem implies that the natural morphism $R^i f_* \mathcal{Z}_\ell(j)_s \to R^i(f_*)_s \mathcal{Z}_\ell(j)$ is an isomorphism. In particular, if $\xi \in H^0(S, R^i f_* \mathcal{Z}_\ell(j))$ then it is automatically “locally constant”.

**Proposition 6.1.** Let $S$ be a separated irreducible scheme. Let $f: X \to S$ be a smooth proper morphism. Let $\mathcal{E}$ be a vector bundle on $X$. If $s$ is a geometric point such that the $\ell$-adic Chern classes of $\mathcal{E}_s$ vanish, then the Chern classes of $\mathcal{E}$ vanish at every geometric point.

**Proof.** As discussed above, the result directly follows if we show that the Chern classes live in $H^0(S, R^i f_* \mathcal{Z}_\ell(j))$. This amounts to defining Chern classes in this level of generality. We indicate how to do this. The key is the splitting principle; see, e.g., [Sta20, 02UK] for a reference (in the context of Chow groups). Let $f: \mathbb{P}\mathcal{E} \to X$ be the associated full flag scheme; then the following two properties hold:

- The induced map $f^*: H^1_{\text{ét}}(X, \mathbb{Q}_\ell) \to H^1_{\text{ét}}(\mathbb{P}\mathcal{E}, \mathbb{Q}_\ell)$ is injective.
- The vector bundle $f^*\mathcal{E}$ has a filtration whose subquotients are line bundles.

By the splitting principle, it suffices to construct the first Chern class for line bundles in the appropriate cohomology group. Let $L$ be a line bundle on $X$. Then the isomorphism class of $L$ gives a class in $H^1_{\text{ét}}(X, \mathcal{O}_X^*)$. Consider the Leray spectral sequence:

$$E_2^{pq} := H^p_{\text{ét}}(S, R^q f_* \mathcal{O}_X^*) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathcal{O}_X^*)$$

By the low-degree terms of the Leray spectral sequence, there is a natural exact sequence

$$0 \to H^1_{\text{ét}}(S, f_* \mathcal{O}_X^*) \to H^1_{\text{ét}}(X, \mathcal{O}_X^*) \to H^0(S, L) \to 0.$$  

(See e.g. [FGIT05, Eqns. 9.2.11.3 and 9.2.11.4 on p. 256-257].) For a line bundle, we define the $\ell$-adic Chern class via the Kummer sequence, which induces a map

$$R^1 f_*(\mathcal{O}_X^*) \to R^2 f_* \mathcal{Z}_\ell(-1).$$

This map is compatible with base change and hence agrees with the notion of $\ell$-adic first Chern class for smooth projective varieties over fields. In particular, we obtain a class $c_1(L)$ in $H^0_{\text{ét}}(X, R^2 f_* \mathcal{Z}_\ell(-1))$. The result follows. □

The set of all $\mathbb{F}_p$-points of $\mathcal{M}$ is finite and preserved by this self map. Thus every $\mathbb{F}_p$-point in this moduli space is preperiodic. This preperiodicity result is also obtained as [Ara19, Theorem 8(2)].
**Corollary 6.2.** Let $S$ be a separated irreducible scheme. Let $f : X \to S$ be a smooth proper morphism. Let $E$ be a coherent sheaf over $X$ with a finite locally free resolution

$$0 \to E^m \to E^{m-1} \to \cdots \to E^0 \to E \to 0.$$ 

Suppose $E$ is flat over $S$, i.e. for all $x \in X$, the stalk of $E$ at $x$ is flat over the local ring $O_{S,s}$ where $s = f(x) \in S$. Suppose there is a point $s \in S$ such that the $\mathbb{Q}_\ell$ Chern classes of $E_s$ vanish. Then for every point $s \in S$, the $\mathbb{Q}_\ell$ Chern classes of $E_s$ vanish.

**Proof.** We only need to show that the restriction on the fiber $X_s$ for any point $s \in S$ induces the following exact sequence

$$(6.1) \quad 0 \to E^m |_{X_s} \to E^{m-1} |_{X_s} \to \cdots \to E^0 |_{X_s} \to E |_{X_s} \to 0,$$

where $E^i |_{X_s}$ is the restriction of $E^i$ on $X_s$. The Whitney sum formula implies that

$$c(E |_{X_s}) = \prod_{i=0}^m c(E^i |_{X_s})^{(-1)^i},$$

where $c(\cdot)$ is the total Chern class. Then Proposition 6.1 implies the result.

In the following we prove the exactness of (6.1). Recall that a complex of sheaves is exact if and only if the corresponding complexes on all stalks are exact. Thus we only need to show the exactness of the following complex of $O_{X_s,x}$-modules

$$(6.2) \quad 0 \to (E^m |_{X_s})_x \to (E^{m-1} |_{X_s})_x \to \cdots \to (E^0 |_{X_s})_x \to 0,$$

for all $x \in X_s$, where $(F |_{X_s})_x$ is the stalk of $F |_{X_s}$ at $x$. By assumption one has an exact sequence of $O_{X,x}$-modules

$$(6.3) \quad 0 \to E^m_x \to E^{m-1}_x \to \cdots \to E^0_x \to E_x \to 0,$$

for all $x \in X_s$, where $E^i_x$ (resp. $E_x$) is the stalk of $E^i$ (resp. $E$) at $x$. Since $E_x$ is flat over $O_{S,s}$ by assumption and $E^i_x$ is flat over $O_{S,s}$ by the local freeness of $E^i$, the exactness of (6.3) is preserved by tensoring any $O_{S,s}$-module. In particular, the functor $- \otimes_{O_{S,s}} k(s)$ preserves the exactness of (6.3), i.e. one has the following exact sequence of $k(s)$-modules

$$(6.4) \quad 0 \to E^m_x \otimes_{O_{S,s}} k(s) \to \cdots \to E^0_x \otimes_{O_{S,s}} k(s) \to E_x \otimes_{O_{S,s}} k(s) \to 0,$$

According the following Cartesian squares

$$\begin{array}{ccc}
\text{Spec}(O_{X_s,x}) & \to & \text{Spec}(O_{X,x}) \\
\downarrow & & \downarrow \\
X_s & \to & X \\
f_s & & f \\
\downarrow & & \downarrow \\
s & \to & S
\end{array}$$

one has $O_{X_s,x} = O_{X,x} \otimes_{O_{S,s}} k(s)$. Hence $- \otimes_{O_{S,s}} k(s) = - \otimes_{O_{X,s}} O_{X_s,x}$ for all $O_{X,x}$-modules. In particular for any $O_X$-module $F$, one has canonical isomorphisms of $k(s)$-modules

$$F_x \otimes_{O_{S,s}} k(s) \xrightarrow{\cong} (F \otimes_{O_X} O_{X,x}) \otimes_{O_{S,s}} O_{X_s,x} = F \otimes_{O_X} O_{X_s,x} = F |_{X_s} \otimes_{O_{X,s}} O_{X_s,x} = (F |_{X_s})_x.$$

Thus the exactness of (6.4) implies the exactness of (6.2). \qed
7. A Lefschetz-style theorem for morphisms of Higgs bundle, after Arapura

In this section, we temporarily change the notation. Let $Y/k$ be a $d$-dimensional smooth projective variety defined over an algebraically closed field. Let $S$ be a normal crossing divisor. We first review a vanishing theorem of Arapura.

Recall that a logarithmic Higgs bundle over $(Y, S)$ is a vector bundle $E$ over $Y$ together with an $O_Y$-linear map $\theta: E \to E \otimes \Omega^1_Y(\log S)$ such that $\theta \wedge \theta = 0$. This integrability condition induces a Higgs complex

$$DR(E, \theta) = (0 \to E \to E \otimes \Omega^1_Y(\log S) \to E \otimes \Omega^2_Y(\log S) \to \cdots).$$

We set the Higgs cohomology to be:

$$H^*_{Higgs}(Y, (E, \theta)) := H(Y, DR(E, \theta))$$

The following is a fundamental vanishing theorem due to Arapura.

**Theorem 7.1.** [Ara19, Theorem 1 on p.297] Let $(E, \theta)$ be a nilpotent semistable Higgs bundle on $(Y, S)$ with vanishing Chern classes in $H^*(Y_{et}, \mathbb{Q}_\ell)$. Let $L$ be an ample line bundle on $Y$. Suppose that either

(a) $\text{char}(k) = 0$, or

(b) $\text{char}(k) = p$, $(Y, S)$ is liftable modulo $p^2$, $d + \text{rank } E < p$.

Then

$$H^i(Y, DR(E, \theta) \otimes L) = 0$$

for $i > d$.

As Arapura notes, all one really needs to assume is that $c_1(E) = 0$ and $c_2(E).L^{d-2} = 0$ in $H^*(Y_{et}, \mathbb{Q}_\ell)$.

Note that if $(E, \theta)$ is semistable with vanishing Chern classes, then so is the dual. By Grothendieck-Serre duality, one immediately deduces [Ara19, Lemma 4.3]

$$H^i\left(Y, DR(E, \theta) \otimes L^\vee(-S)\right) = 0$$

for any $i < d$.

We use Theorem 7.1 to prove a Lefschetz theorem.

**Setup 7.2.** Let $Y/k$ be a smooth projective variety over a perfect field of characteristic $p$ and of dimension $d \geq 2$. Let $S \subset Y$ be a simple normal crossings divisor (possibly empty). Let $D \subset Y$ be a smooth ample divisor that meets $S$ transversely and such that $O(D - S)$ is also ample. Let $j: D \to Y$ denote the closed embedding.

We suppose that $(Y, S)$ has a lifting $(\tilde{Y}, \tilde{S})$ over $W_2(k)$. We may define $\tilde{D} \subset \tilde{Y}$ to have the same topological space as $D$ and the structure sheaf induced from $O_{\tilde{Y}}$.

Let $(E, \theta)$ be a logarithmic Higgs bundle over $Y$. The Higgs field $\theta|_D$ on $E|_D$ is defined as the composite map as in the following diagram:

$$\begin{array}{ccc}
E|_D \xrightarrow{j^*} E|_D \otimes j^* \Omega^1_Y & \xrightarrow{\theta|_D} & E|_D \otimes \Omega^1_D \\
\downarrow_{id \otimes dj} & & \downarrow_{id \otimes dj} \\
E|_D \otimes \Omega^1_D. & & \end{array}$$

Then one has the following result.
Lemma 7.3. Setup as in Setup 7.2. Let \((E, \theta)\) be a semistable logarithmic Higgs bundle on \(Y\) of rank \(r\) with trivial Chern classes and semistable restriction on \(D\). Suppose further that \(d + r \leq p\). Then the restriction functor induces isomorphisms

\[
res: H^i_{\text{Hig}}(Y, (E, \theta)) \xrightarrow{\sim} H^i_{\text{Hig}}(D, (E, \theta)|_D)
\]

for all \(i \leq d - 2\) and an injection

\[
res: H^{d-1}_{\text{Hig}}(Y, (E, \theta)) \hookrightarrow H^{d-1}_{\text{Hig}}(D, (E, \theta)|_D)
\]

Corollary 7.4. Setup as in Setup 7.2. Let \((E, \theta)\) and \((E, \theta)'\) be two semistable logarithmic Higgs bundles over \(Y\) of rank \(r\) and \(r'\) respectively, where \(d + rr' \leq p\). Suppose further that both Higgs bundles have trivial \(\mathbb{Q}_\ell\) Chern classes and semistable restrictions to \(D\). Then one has

1. an isomorphism

\[
\text{Hom}((E, \theta), (E, \theta)') \cong \text{Hom}((E, \theta)|_D, (E, \theta)'|_D)
\]

2. an injection

\[
H^i_{\text{Hig}}(Y, \text{Hom}((E, \theta), (E, \theta)')) \hookrightarrow H^i_{\text{Hig}}(D, \text{Hom}((E, \theta)|_D, (E, \theta)'|_D)).
\]

Proof. Denote \((E, \Theta) := \text{Hom}((E, \theta), (E, \theta)')\). Then

\[
(E, \Theta)|_D \cong \text{Hom}((E, \theta)|_D, (E, \theta)'|_D).
\]

Note that \((E, \theta)\) and \((E, \theta)\)' are both strongly semistable as \(r, r' \leq p\). It follows that \((E, \Theta)\) is also strongly semistable and hence semistable. Similarly, \((E, \Theta)|_D\) is also semistable. Then the result follows Lemma 7.3. □

Proof of Lemma 7.3. In the following, we need to show the restriction functor induces an isomorphism between these two Higgs cohomology groups. According Arapura’s Theorem 7.1, one has

\[
\mathbb{H}^i(Y, DR(E, \theta) \otimes \mathcal{O}_Y(-D)) = 0,
\]

for all \(i < d\). The following exact sequence of complexes

\[
\begin{array}{ccccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
DR(E, \theta) \otimes \mathcal{O}_Y(-D) & \rightarrow & E(-D) & \rightarrow & E(-D) \otimes \Omega^1(\log S) & \rightarrow & E(-D) \otimes \Omega^2(\log S) & \rightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
DR(E, \theta) & \rightarrow & E & \rightarrow & E \otimes \Omega^1(\log S) & \rightarrow & E \otimes \Omega^2(\log S) & \rightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
DR(E, \theta) \otimes j_! \mathcal{O}_D & \rightarrow & j_!(E|_D) & \rightarrow & j_!(E|_D) \otimes \Omega^1(\log S) & \rightarrow & j_!(E|_D) \otimes \Omega^2(\log S) & \rightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

induces

\[
H^i_{\text{Hig}}(Y, (E, \theta)) \xrightarrow{\sim} \mathbb{H}^i(Y, DR(E, \theta) \otimes j_! \mathcal{O}_D)
\]

for all \(i \leq d - 2\) and an injection

\[
H^{d-1}_{\text{Hig}}(Y, (E, \theta)) \hookrightarrow \mathbb{H}^{d-1}(Y, DR(E, \theta) \otimes j_! \mathcal{O}_D).
\]

\footnote{Following [LSZ19], we call a Higgs bundle strongly semistable if it initiates a Higgs-de Rham flow where all terms are Higgs semistable and defined over a common finite field. Then the strong semistability of \((E, \theta)\) and \((E, \theta)\)' follows from \footnote{10} by preperiodicity.}
On the other hand, we have
\[ \mathbb{H}^i(D, DR((E, \theta) |_D) \otimes \mathcal{I}/\mathcal{I}^2[-1]) = \mathbb{H}^{i-1}(D, DR((E, \theta) |_D) \otimes \mathcal{I}/\mathcal{I}^2) = 0, \]
for all \( i < d \). This is because that \( \mathcal{I}/\mathcal{I}^2 = \mathcal{O}(-D) |_D \) is more negative than \( \mathcal{O}_D(-D \cap S) \).

The long exact sequence of the following exact sequence of complexes
\[
\begin{array}{c}
0 & \to & 0 & \to & 0 & \to & 0 & \to \\
DR((E, \theta) |_D) \otimes \mathcal{I}/\mathcal{I}^2[-1] & = & 0 & \to & E |_D \otimes \mathcal{I}/\mathcal{I}^2 & \to & \theta |_D & \to \\
& & 0 & \to & E |_D & \to & E |_D \otimes \Omega^1_{\mathcal{I}}(\log S) & \to \\
& & 0 & \to & E |_D & \to & E |_D \otimes \Omega^2_{\mathcal{I}}(\log S) & \to \\
& & 0 & \to & E |_D & \to & E |_D & \to \\
& & 0 & \to & 0 & \to & 0 & \\
\end{array}
\]
induces
\[ \mathbb{H}^i(D, DR(E, \theta) |_D) \xrightarrow{\sim} H^1_{\text{Hig}}(D, (E, \theta) |_D) \]
for all \( i \leq d - 2 \) and an injection
\[ \mathbb{H}^{d-1}(D, DR(E, \theta) |_D) \hookrightarrow H^{d-1}_{\text{Hig}}(D, (E, \theta) |_D). \]
Then the lemma follows from the fact that
\[ \mathbb{H}^i(Y, DR(E, \theta) \otimes j_* \mathcal{O}_D) \xrightarrow{\sim} \mathbb{H}^i(D, DR(E, \theta) |_D). \]
\[ \square \]

8. The Higgs bundle \((E_X, \theta_X)\) is graded.

**Proposition 8.1.** Setup as in [Theorem 1.4] Let \((E_{X,k}, \theta_{X,k})\) be the logarithmic Higgs bundle attached to \( \rho_X \) in [Section 4.2] Let \((E_X, \theta_X)\) be the Higgs bundle constructed in [Theorem 5.1] Then the Higgs bundle \((E_X, \theta_X)\) is graded.

To prove this, recall that a Higgs bundle \((E, \theta)\) is graded if and only if it is invariant under the \( \mathbb{G}_m \) action, i.e., if
\[ (E, \theta) \simeq (E, t\theta) \quad \text{for all } t \in \mathbb{G}_m. \]

**Proof.** Recall that \((E_D, \theta_D)\) is graded. For any \( t \in \mathbb{G}_m \), one has
\[ f_D: (E_D, \theta_D) \simeq (E_D, t\theta_D). \]
By [Corollary 7.4] one gets \( f_{X_1}: (E_{X_1}, \theta_{X_1}) \simeq (E_{X_1}, t\theta_{X_1}) \). Consider the obstruction \( c \) to lift \( f_{X_1} \) to \( W_2 \), which is located in
\[ c \in H^1_{\text{Hig}}(\mathcal{H}\text{om}((E_{X_1}, \theta_{X_1}), (E_{X_1}, t\theta_{X_1}))). \]
By [Corollary 7.4] one has an injective map
\[ \text{res}: H^1_{\text{Hig}}(\mathcal{H}\text{om}((E_{X_1}, \theta_{X_1}), (E_{X_1}, t\theta_{X_1}))) \hookrightarrow H^1_{\text{Hig}}(\mathcal{H}\text{om}((E_D, \theta_D), (E_D, t\theta_D))). \]
Since \( f_{D_1} \) is liftable, the image of \( c \) under \( \text{res} \) vanishes. Thus \( c = 0 \) and there is lifting of \( f_{X_1} \)
\[ f'_{X_2}: (E_{X_2}, \theta_{X_2}) \to (E_{X_2}, t\theta_{X_2}). \]
In general $f_{X_2}'|_{D_2} \neq f_D|_{D_2}$. We consider the difference $c' \in H^0_{rig}(D_1, \text{End}(E_{D_1}, \theta_{D_1}))$ between $f_{X_2}'|_{D_2}$ and $f_D|_{D_2}$. Since the restriction map in 1) of Corollary 7.4 is a bijection, we have a unique preimage $\text{res}^{-1}(c')$ of $c'$. Then we get a new isomorphism

$$f_{X_2}: (E_{X_2}, \theta_{X_2}) \to (E_{X_2}, t\theta_{X_2}).$$

by modifying the lifting $f_{X_2}'$ via $\text{res}^{-1}(c')$. Inductively on the truncated level, we can lift $f_{X_1}$ to an unique isomorphism $f_X: (E_X, \theta_X) \simeq (E_X, t\theta_X)$ such that $f_X|_D D = f_D$. Thus $(E_X, \theta_X)$ is graded.

\[\square\]

9. Higgs-de Rham flow $HDF_{X_1}$ with initial term $(E_{X_1}, \theta_{X_1})$

Setup as in Question 1.3 and assume that

(1) $N^2 < p - \dim(X)$ and

(2) the representation $\rho_X$ is geometrically absolutely residually irreducible.

By the work in Section 8, there is a graded logarithmic Higgs bundle $(E_X, \theta_X)$ such that $(E_{X_1}, \theta_{X_1})$ and $(E_{X_{\kappa}}, \theta_{X_{\kappa}})$ are semistable. By the setup of Question 1.3 we assume that the $p$-adic Higgs bundle $(E_D, \theta_D)$ is periodic. As explained above, our goal is to prove that $(E_X, \theta_X)$ is $p$-adic periodical and that the nilpotence of each term in the attached Higgs-de Rham flow is $\leq p - 2$. We will first address the periodicity of $(E_{X_1}, \theta_{X_1})$.

Since $(E_{X_1}, \theta_{X_1}) = (E_X, \theta_X)|_{X_1}$ is a graded semistable (logarithmic) Higgs bundle with trivial Chern classes and $N \leq p$, it is preperiodic by Lan-Sheng-Zuo [LSZ19] Theorem 1.5 (and footnote 10), i.e., there exists a preperiodic Higgs-de Rham flow $HDF_{X_1}$ with initial term $(E_{X_1}, \theta_{X_1})$. Restricting this preperiodic flow to $D_1$, one obtains a preperiodic Higgs-de Rham flow $HDF_{X_1}|_{D_1}$ with initial term $(E_{D_1}, \theta_{D_1})_0 := (E_{X_1}, \theta_{X_1})|_{D_1}$ (9.1)

\begin{align*}
(E_{D_1}, \theta_{D_1})_0 & \quad \quad (V_{D_1}, \nabla_{D_1}, \text{Fil}_{D_1})_0 \\
(E_{D_1}, \theta_{D_1})_1 & \quad \quad (V_{D_1}, \nabla_{D_1}, \text{Fil}_{D_1})_1 \\
& \quad \quad \vdots
\end{align*}

On the other hand, $(E_{D_1}, \theta_{D_1})_0 := (E_{D_1}, \theta_{D_1})$ is periodic by assumption; therefore, it initiates a periodic Higgs-de Rham flow:

(9.2)

\begin{align*}
(E_{D_1}, \theta_{D_1})_0 & \quad \quad (V_{D_1}, \nabla_{D_1}, \text{Fil}_{D_1})_0 \\
(E_{D_1}, \theta_{D_1})_1 & \quad \quad (V_{D_1}, \nabla_{D_1}, \text{Fil}_{D_1})_1 \\
& \quad \quad \vdots
\end{align*}

By assumption, the first Higgs terms of the two flows are isomorphic. We will prove that under our hypotheses, these two flows over $D_1$ are isomorphic. By Lan-Sheng-Zuo [LSZ19] Theorem 1.5, $(E_{D_1}, \theta_{D_1})_0$ is semistable\(^{12}\). There are two extensions $(E_D, \theta_D)_0$ and $(E_X, \theta_X)|_D$ of $(E_{D_{\kappa}}, \theta_{D_{\kappa}})$. Moreover, the hypothesis that $\rho_D$ is geometrically absolutely residually irreducible implies that $(E_{D_1}, \theta_{D_1})$ is stable. Since one extension has stable reduction over $D_1$ and the other one has semistable reduction over $D_1$, it follows that

$$(E_D, \theta_D)_0 \simeq (E_X, \theta_X)|_D$$

\(^{12}\)Strictly speaking, [LSZ19] Theorem 1.5 only deals with the non-logarithmic case. However the argument immediately carries to the logarithmic case, as follows. As $(E_{X_1}, \theta_{X_1})_0$ is preperiodic, so is $(E_{D_1}, \theta_{D_1})_0$. Preperiodic logarithmic Higgs bundles with trivial Chern classes are automatically semistable, see also footnote 10.
by the following Langton-style lemma.

Lemma 9.1. (Lan-Sheng-Zuo) Let $R$ be a discrete valuation ring with fraction field $K$ and residue field $k$. Let $X/R$ be a smooth projective scheme and let $S \subset R$ be a relative (strict) normal crossings divisor. Let $(E_1, \theta_1)_X$ and $(E_2, \theta_2)_X$ be logarithmic Higgs bundle on $X$ that are isomorphic over $K$. Suppose that $(E_1, \theta_1)_X$ is semistable and $(E_2, \theta_2)_X$ is stable. Then $(E_1, \theta_1)_X$ and $(E_2, \theta_2)_X$ are isomorphic.

Proof. The non-logarithmic version may be found as Lemma 6.4 in arXiv 1311.6424v1. The argument in the logarithmic version setting requires no changes. \hfill \square

Now, we identify $(E_D, \theta_D)_0$ with $(E_X, \theta_X)_0|_D$ with this isomorphism. In particular, one has

$$(E_D, \theta_D)_0 = (E_X, \theta_X)_0|_D = (E_D, \theta_D)_0|_{D_1} =: (E_D, \theta_D)_0.$$  

(9.3)

On the other hand, one has periodic Higgs-de Rham flow $\text{HDF}_D|_{D_1}$ with initial term $(E_D, \theta_D)_0$.

Lemma 9.2. $\text{HDF}_{X_1} |_{D_1} = \text{HDF}_D |_{D_1}$.

Proof. Since $(E_D, \theta_D)_0$ is stable and $\text{HDF}_D |_{D_1}$ is periodic, all terms appeared in $\text{HDF}_D |_{D_1}$ are stable. This is because

- $(V_{D_i}, \nabla_{D_i}), i$ is stable if and only if $(E_D, \theta_D), i$ is stable;
- If $(E_D, \theta_D), i+1$ is stable, then $(V_{D_i}, \nabla_{D_i}), i$ is stable.

Since Cartier inverse functor is compatible with restriction, one has $(V_{D_i}, \nabla_{D_i})_0 = (V_{D_i}, \nabla_{D_i})_0$. By [LSZ19] Lemma 7.1\footnote{The argument in loc. cit is written for flat connections, but the proof extends more generally to logarithmic flat connections.} $\text{Fil}_{D_i, 0} = \text{Fil}_{D_i, 0}$. Taking the associated graded, one gets $(E_D, \theta_D)_1 = (E_D, \theta_D)_1$. Inductively, one shows that $\text{HDF}_{X_1} |_{D_1} = \text{HDF}_D |_{D_1}$. \hfill \square

By the fully faithfulness of the restriction functor as in [Lemma 7.3] the map $\varphi_{D_1} : (E_D, \theta_D)_f \rightarrow (E_D, \theta_D)_0$ that witnesses the periodicity of $\text{HDF}_{D_1}$ can be lifted canonically to a map $\varphi_{X_1} : (E_X, \theta_X)_f \rightarrow (E_X, \theta_X)_0$. This implies that the Higgs-de Rham flow $\text{HDF}_{X_1}$ is also $f$-periodic.

10. Higgs-de Rham flow $\text{HDF}_{X}$ with initial term $(E_X, \theta_X)_0$

In this section, we use two results of Krishnamoorthy-Yang-Zuo [KYZ20] to lift $\text{HDF}_{X_1}$ onto $X$.

Proposition 10.1. There is an unique $f$-periodic Higgs-de Rham flow $\text{HDF}_{X}$ over $X$, which

- lifts $\text{HDF}_{X_1}$,
- with initial term $(E_X, \theta_X)$ and satisfying $\text{HDF}_{X} |_{D} \simeq \text{HDF}_D$. 

In this section, we use two results of Krishnamoorthy-Yang-Zuo [KYZ20] to lift $\text{HDF}_{X_1}$ onto $X$.

Proposition 10.1. There is an unique $f$-periodic Higgs-de Rham flow $\text{HDF}_{X}$ over $X$, which

- lifts $\text{HDF}_{X_1}$,
- with initial term $(E_X, \theta_X)$ and satisfying $\text{HDF}_{X} |_{D} \simeq \text{HDF}_D$. 

In this section, we use two results of Krishnamoorthy-Yang-Zuo [KYZ20] to lift $\text{HDF}_{X_1}$ onto $X$.
We prove this result inductively on the truncated level; in particular, we may assume we have already lifted $HDF_{X_1}$ to an $f$-periodic Higgs-de Rham flow $HDF_{X_n}$ over $X_n$, where $n \geq 1$ is a positive integer:

(10.1) \[ (E_n, \theta_n) \rightarrow (V_n, \nabla_n, \Fil_{n+1}) \rightarrow \cdots \rightarrow (E_n, \theta_n) \]

satisfying $HDF_{X_n} \mid_{D_n} \simeq HDF \mid_{D_n}$ and with initial term $(E_n, \theta_n)$. We only need to lift $HDF_{X_n}$ to an $f$-periodic Higgs-de Rham flow $HDF_{X_{n+1}}$ over $X_{n+1}$ satisfying $HDF_{X_{n+1}} \mid_{D_{n+1}} \simeq HDF \mid_{D_{n+1}}$ and with initial term $(E_n, \theta_n)$ as following.

First, taking the $(n+1)$-truncated level Cartier inverse functor, one gets

\[ (V_{n+1}, \nabla_{n+1})_0 := C_n^{-1}((E_{n+1}, \theta_{n+1})_0, (V_n, \nabla_n, \Fil_{n+1})_f-1, \varphi_n) \]

which is a vector bundle over $X_{n+1}$ together with an integrable connection. This filtration lifts $(V_n, \nabla_n)_0$ and satisfies $(V_{n+1}, \nabla_{n+1})_0 \mid_{D_{n+1}} = (V_D, \nabla_D)_0 \mid_{D_{n+1}}$.

**Proposition 10.2. Notation as above.**

1. There is a unique Hodge (Griffith transversal) filtration $\Fil_{X+1}$ on $(V_{n+1}, \nabla_{n+1})$, which lifts $\Fil_{X_n}$ and satisfying $\Fil_{X+1} \mid_{D_{n+1}} = \Fil_D \mid_{D_{n+1}}$.

2. Taking the associated graded, we obtain a Higgs bundle

\[ (E_{n+1}, \theta_{n+1}) = \Gr(V_{n+1}, \nabla_{n+1}, \Fil_{n+1}) \]

which lifts $(E_n, \theta_n)$ and satisfies

\[ (E_{n+1}, \theta_{n+1}) \mid_{D_{n+1}} = (E_{D+1}, \theta_{D+1}) \]

To prove this result, we need a result of Krishnamoorthy-Yang-Zuo [KYZ20] about the obstruction to lifting the Hodge filtration. We recall this result.

Let $(V, \nabla, F^*)$ be a filtered de Rham bundle over $X_n$, i.e. a vector bundle bundle together with a flat connection with logarithmic singularities and a filtration by sub-bundles that satisfies Griffiths transversality over $X_n$. Denote its modulo $p$ reduction by $(\overline{V}, \overline{\nabla}, \overline{F}^*)$. Let $(\tilde{V}, \overline{\nabla})$ be a lifting of the flat bundle $(V, \nabla)$ on $X_n$. The $\overline{F}$ induces a Hodge filtration $\Fil^\End_{\overline{V}}$ on $(\End(\overline{V}), \overline{\End})$ defined by

\[ \Fil^\End_{\overline{V}} := \sum_{\ell_i} (\overline{V}/\overline{F}^1)^{\vee} \otimes \overline{F}^{1+\ell-1}. \]

As $(\End(\overline{V}), \overline{\End}, \Fil^\End)$ satisfies Griffith transversality, the de Rham complex $(\End(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\End})$ induces the following complex denoted by $\mathcal{C}$

\[ 0 \rightarrow \End(\overline{V})/\Fil^0 \End(\overline{V}) \rightarrow \End(\overline{V})/\Fil^1 \End(\overline{V}) \otimes \Omega^1_{X_1} \End(\overline{V})/\Fil^2 \End(\overline{V}) \otimes \Omega^2_{X_1} \rightarrow \cdots \]

which is the quotient of $(\End(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\End})$ by the sub complex $(\Fil^\End \End(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\End})$, i.e., we have the following exact sequence of complexes of sheaves over $X_1$:

\[ 0 \rightarrow (\Fil^\End \End(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\End}) \rightarrow (\End(\overline{V}) \otimes \Omega^*_{X_1}, \overline{\End}) \rightarrow \mathcal{C} \rightarrow 0. \]

Denote $(\overline{E}, \overline{\theta}) = \Gr(\overline{V}, \overline{\nabla}, \overline{F}^*)$. Then $(\End(\overline{E}), \overline{\theta}^\End) = \End((\overline{E}, \overline{\theta}))$ is also a graded Higgs bundle. Here is the key observation about the complex $\mathcal{C}$: it is a successive extension of
The restriction induces Proposition 10.4.

\[ \left\{ (\text{End}(\mathcal{E}), \tilde{\theta})^{\text{End}} \right\} \]

**Theorem 10.3.** [KYZ20] Theorem 3.9 
Notation as above.

1. The obstruction of lifting the filtration $F^*$ onto $(\tilde{V}, \tilde{\nabla})$ with Griffiths transversality is located in $\mathbb{H}^1(X_1, \mathcal{C})$.

2. If the above the obstruction vanishes, then the lifting space is an $\mathbb{H}^0(X_1, \mathcal{C})$-torsor.

Associated to the filtered de Rham bundle $(V_{X_1}, \nabla_{X_1}, \text{Fil}_{X_1})$ and $(V_{D_1}, \nabla_{D_1}, \text{Fil}_{D_1}) = (V_{X_1}, \nabla_{X_1}, \text{Fil}_{X_1})|_{D_1}$, one has complex of sheaves over $X_1$

\[
0 \to \text{End}(V_{X_1})/\text{Fil}^0 \text{End}(V_{X_1}) \xrightarrow{\nabla_{X_1}} \text{End}(V_{X_1})/\text{Fil}^{-1} \text{End}(V_{X_1}) \otimes \Omega^1_{X_1} \xrightarrow{\nabla_{X_1}} \cdots
\]

and complex $\mathcal{C}_{D_1}$ of sheaves over $D_1$

\[
0 \to \text{End}(V_{D_1})/\text{Fil}^0 \text{End}(V_{D_1}) \xrightarrow{\nabla_{D_1}} \text{End}(V_{D_1})/\text{Fil}^{-1} \text{End}(V_{D_1}) \otimes \Omega^1_{D_1} \xrightarrow{\nabla_{D_1}} \cdots
\]

satisfying

\[ \mathcal{C}_{D_1} = \mathcal{C}_{X_1} |_{D_1}. \]

**Proposition 10.4.** The restriction induces

1. an isomorphism $\mathbb{H}^0(X_1, \mathcal{C}_{X_1}) \cong \mathbb{H}^0(D_1, \mathcal{C}_{D_1})$, and
2. an injection $\mathbb{H}^1(X_1, \mathcal{C}_{X_1}) \hookrightarrow \mathbb{H}^1(D_1, \mathcal{C}_{D_1})$.

**Proof.** Since the complex $\mathcal{C}$ is a successive extension of direct summands of the Higgs complex, one gets the results by Lemma 7.3 and the five lemma. \( \square \)

**Proof of Proposition 10.2.** The uniqueness of the Hodge filtration follows Theorem 10.3 and Proposition 10.4. For the existence, we denote by $c \in \mathbb{H}^1(X_1, \mathcal{C}_{X_1})$ the obstruction to lift Fil$_{X_1}$ onto $X_{n+1}$. Since Fil$_{D_1}$ is liftable, the image res$(c) \in \mathbb{H}^1(D_1, \mathcal{C}_{D_1})$ of $c$ under res vanishes. Since res is an injection, $c = 0$ and Fil$_D|_{D_{n+1}}$ is also liftable. We choose a lifting Fil’$_{X_{n+1}}$ and denote by $c' \in \mathbb{H}^0(D_1, \mathcal{C}_{D_1})$ the difference between Fil’$_{X_{n+1}}|_{D_{n+1}}$ and Fil$_D|_{D_{n+1}}$. Since res is an isomorphism, one uses res$^{-1}(c')$ to modify the original filtration Fil’$_{X_{n+1}}$. Then one gets a new filtration Fil$_{X_{n+1}}$ such that Fil$_{X_{n+1}}|_{D_{n+1}} = \text{Fil}_D|_{D_{n+1}}$. \( \square \)

Run the Higgs-de Rham flow with initial term $((E_{n+1}, \theta_{X_{n+1}}), (V_{X_n}, \nabla_{X_n}, \text{Fil}_{X_n})_{f^{-1}}, \varphi_n)$ together with the Hodge filtrations constructed as in Proposition 10.2. Then one constructs a Higgs-de Rham flow $HDF_{X_{n+1}}$ over $X_{n+1}$

\[
\begin{align*}
(E_{n+1}, \theta_{X_{n+1}}) & \quad \nabla_{X_{n+1}}, \text{Fil}_{X_{n+1}}) \\
(V_{X_{n+1}}, \nabla_{X_{n+1}}, \text{Fil}_{X_{n+1}})_{f^{-1}} & \quad \cdots \\
(E_{n+1}, \theta_{X_{n+1}}) & \quad \nabla_{X_{n+1}}, \text{Fil}_{X_{n+1}})
\end{align*}
\]

satisfying

- with initial term $((E_{n+1}, \theta_{X_{n+1}}), (V_{X_n}, \nabla_{X_n}, \text{Fil}_{X_n})_{f^{-1}}, \varphi_n)$;
- $HDF_{X_{n+1}}|_{D_{n+1}} \simeq HDF_{D}|_{D_{n+1}}$.

For the last step, we need to show this lifting flow is $f$-periodic. In other words, we need the following result.
Lemma 10.5. There exists an isomorphism
\[ \varphi_{X,n+1} : (E_{X,n+1}, \theta_{X,n+1})_f \xrightarrow{\sim} (E_{X,n+1}, \theta_{X,n+1})_0 \]
which lifts \( \varphi_X \) and satisfying \( \varphi_{X,n+1} |_{D_{n+1}} = \varphi_{D_{n+1}} \).

To prove this result, we need another result on the obstruction class to lifting a Higgs bundle.

Theorem 10.6. [KYZ20 Proposition 4.2] Let \((E, \theta)\) be a logarithmic Higgs bundle over \(X_n\). Denote \((\bar{E}, \bar{\theta}) = (E, \theta) |_{X_1}\). Then
1. if \((E, \theta)\) has lifting \((\bar{E}, \bar{\theta})\), then the lifting set is an \(H^1_{Hig}(X_1, \text{End}((\bar{E}, \bar{\theta})))\)-torsor;
2. the infinitesimal automorphism group of \((\bar{E}, \bar{\theta})\) over \((E, \theta)\) is \(H^0_{Hig}(X_1, \text{End}((\bar{E}, \bar{\theta})))\).

Directly from Corollary 7.4, one gets results.

Proposition 10.7. The restriction induces
1. an isomorphism \(\text{res} : H^0_{Hig}(X_1, \text{End}(E_{X_1}, \theta_{X_1})) \xrightarrow{\sim} H^0_{Hig}(D_1, \text{End}(E_{D_1}, \theta_{D_1}))\), and
2. an injection \(\text{res} : H^1_{Hig}(X_1, \text{End}(E_{X_1}, \theta_{X_1})) \hookrightarrow H^1_{Hig}(D_1, \text{End}(E_{D_1}, \theta_{D_1}))\).

Proof of Lemma 10.5. We identify \((E_{X_1}, \theta_{X_1})_0\) and \((E_{X_1}, \theta_{X_1})_f\) via \(\varphi_{X,n}\). Since both \((E_{X,n+1}, \theta_{X,n+1})_0\) and \((E_{X,n+1}, \theta_{X,n+1})_f\) lift \((E_{X,n}, \theta_{X,n})_0\), they differ by an element \(c \in H^1_{Hig}(X_1, \text{End}(E_{X_1}, \theta_{X_1}))\).

Since \(HDF_{X,n+1} |_{D_{n+1}}\) is \(f\)-periodic, one has \(\text{res}(c) = 0 \in H^1_{Hig}(D_1, \text{End}(E_{D_1}, \theta_{D_1}))\). By the injection of the restriction map in 2) of Proposition 10.7, \(c = 0\) and there is an isomorphism
\[ \varphi_{X,n+1} : (E_{X,n+1}, \theta_{X,n+1})_f \longrightarrow (E_{X,n+1}, \theta_{X,n+1})_0. \]
In general \(\varphi_{X,n+1} |_{D_{n+1}} \neq \varphi_{D_{n+1}}\). We consider the difference
\[ c' \in H^0_{Hig}(D_1, \text{End}(E_{D_1}, \theta_{D_1})) \]
between \(\varphi_{X,n+1}' |_{D_{n+1}}\) and \(\varphi_{D_{n+1}}\). Since the restriction map in 1) of Proposition 10.7 is a bijection, we have a unique preimage \(\text{res}^{-1}(c')\) of \(c'\). We obtain a new isomorphism
\[ \varphi_{X,n+1} : (E_{X,n+1}, \theta_{X,n+1})_f \longrightarrow (E_{X,n+1}, \theta_{X,n+1})_0. \]
by modifying the lifting \(\varphi_{X,n+1}'\) via \(\text{res}^{-1}(c')\). Then \(\varphi_{X,n+1}\) satisfies our required property, i.e. \(\varphi_{X,n+1} |_{D_{n+1}} = \varphi_{D_{n+1}}\). \(\square\)

11. Applications

In this section, we provide the aforementioned applications. The notation in this section differs somewhat from Section 3-Section 10. We have done this because we want the applications to be self-contained. Therefore, the notation for each application is contained within the statement in Section 1.

Proof of Corollary 1.6. It is clear that if the abelian scheme extends, then the local system extends. We must prove that if the local system extends, then the abelian scheme extends. By Theorem 1.4 \(\rho_X\) is crystalline. As crystalline representations have (locally) constant Hodge-Tate weights, the Hodge-Tate weights are in particular in \([0, 1]\), i.e., \(\rho_X\) is a crystalline representation whose associated Fontaine-Faltings module lives in \(\mathcal{MF}^\nabla_{[0, 1]}(X)\).

There is a \(p\)-divisible group \(G_X\) over \(X\) whose \(p\)-adic Tate module over \(X_K\) is isomorphic to \(\rho_X\) by Faltings Theorem 7.1]. We claim that a polarization on \(A_D \to D\) naturally induces
a quasi-polarization on $G_X$. To see this, first of all: if $Y/W$ is a smooth scheme, then the functor $\mathcal{M}_Y \to \text{Rep}_{\pi_1(YW)}(\mathbb{Z}_p)$ is fully faithful by \cite[Theorem 2.6]{Fal}. In our case, we have a surjection $\pi_1(DK) \to \pi_1(XK)$. It follows that a skew-symmetric isogeny $\rho_0 \to \rho_D^*$ extends to an skew-symmetric isogeny $\rho_X \to \rho_X^*$, and therefore any skew-symmetric isogeny $G_D \to G'_D$ extends to a skew-symmetric isogeny $G_X \to G'_X$. It follows that any polarization $\lambda_D$ on $A_D \to D$ naturally yields a quasi-polarisation on $G_X$.

It follows from \cite[Corollary 8.6]{KP} that there exists an abelian scheme $A_{X_1} \to X_1$ with $A_{X_1}[p^{\infty}] \cong G_{X_1}$. Moreover, any polarization $\lambda_{D_1}$ on $A_{D_1}$ extends to a polarization on $A_{X_1}$ after possibly multiplying by a power of $p$. We may therefore pick polarization $\lambda_D$ such that the polarization $\lambda_{D_1}$ on $A_{D_1} \to D_1$ extends to a polarization $\lambda_X$ on $A_{X_1} \to X_1$.

By our choice of polarization, it follows that $G_X$ is a deformation of $A_{X_1}[p^{\infty}]$ as a quasi-polarized $p$-divisible group. Then Serre-Tate theory \cite[Theorem 1.2.1]{Kat} implies that there is a formal abelian scheme $A_X \to \mathcal{X}$ that is polarizable. Grothendieck’s algebraization theorem then implies that $A_X$ uniquely algebraizes to an abelian scheme $A_X \to X$, as desired.

\textbf{Proof of Corollary 1.7.} We first show the equivalence of the first two items. Let $X, D, f_D: A_D \to D$, and $(E_X, \theta_X)$ be given as in the statement of the theorem. We use the following fact: for any finitely generated $\mathbb{Z}$ algebra $R$ that is an integral domain, there exist infinitely many primes $p$ such that $R$ embeds in $\mathbb{Z}_p$. This follows from Cassels’ embedding theorem \cite{Cas}. Indeed, pick a set of generators $t_i$ of $R$ over $\mathbb{Z}$. Then there exists infinitely many $p$ such that $\text{Frac}(R)$ embeds in $\mathbb{Q}_p$ and moreover such that the image of each of the $t_i$ is in $\mathbb{Z}_p^\times$. This implies the result as the $t_i$ generated $R$ as a $\mathbb{Z}$ algebra.

By spreading-out and the above observation, it follows that there exists infinitely many primes $p$ such that the following holds.

- $X, D, f_D: A_D \to D$, and $(E_X, \theta_X)$ may all be defined over a copy of $\mathbb{Z}_p \subset \mathbb{C}$, and moreover $X, D$, and $f_D: A_D \to D$ are all smooth and projective over $\mathbb{Z}_p$.
- The Higgs bundle $(E_D, \theta_D)$ is stable modulo $p$. (Stability is an open condition.)
- The Higgs bundle $(E_D, \theta_D)$ is 1-periodic over $\mathbb{Z}_p$. (The Higgs bundle is the associated graded of de Rham cohomology of a smooth projective morphism over $D$.)
- If $A_D \to D$ has relative dimension $g$, then $p > 4g^2 + \text{dim}(X)$.

Now, the arguments starting in \textbf{Section 6} apply and we deduce that $(E_X, \theta_X)$ is 1-periodic. It follows from \cite[Theorem 1.4]{LSZ} that the hypotheses of \textbf{Corollary 1.6} are all satisfied. We deduce that $A_D \to D$ (over $\mathbb{Z}_p$) extends to an abelian scheme $A_X \to X$. The result follows.

To prove the second part, we simply repeat the spreading out argument above and note that $\pi_1(D) \to \pi_1(X)$ is an isomorphism by the classical Lefschetz theorem. Then \textbf{Corollary 1.6} implies that $A_D \to D$ extends to an abelian scheme $A_X \to X$. \hfill \Box

To prove our final application, we first several preliminaries.

\textbf{Definition 11.1.} Let $X/k$ be a smooth variety over a field of characteristic 0. The category $\text{MIC}(X)$ is the category of vector bundles equipped with an integrable connection $(V, \nabla)$.

In fact, the category $\text{MIC}(X)$ is $k$-linear Tannakian category, which is neutralized by the fiber at any $k$-point $x$ of $X$, written $\omega_x$. We set the algebraic fundamental group to
be the Tannakian group
\[ \text{Aut}^\oplus(\omega_x) =: \pi_1^{\text{alg}}(X). \]

We emphasize that this group does not satisfy base change, see e.g. [Esn12] p. 3-4.

**Lemma 11.2.** Let \( X/k \) be a smooth projective variety of dimension at least 2 over a field of characteristic 0. Let \( D \subset X \) be a smooth ample divisor. Then the natural functor
\[ \text{MIC}(X) \to \text{MIC}(D) \]
is fully faithful, and when \( \dim(X) \geq 3 \) and \( k = \overline{k} \), it is an equivalence of categories.

**Proof.** We first prove that the above when \( k = \overline{k} \). This argument is essentially contained in [Esn12 Proposition 2.1]. More precisely, when \( \dim(X) \geq 3 \), one may directly apply [Esn12 Proposition 2.1], which shows that \( \pi_1^{\text{alg}}(X_{\overline{k}}) \to \pi_1^{\text{alg}}(D_{\overline{k}}) \) is an isomorphism. When \( \dim(X) = 2 \), then one simply repeats the first part of the argument, replacing the use of the isomorphism furnished by [Gro70 Théorème 1.2(b)] with the surjection furnished by [Gro70 Théorème 1.2(a)] to prove that \( \pi_1^{\text{alg}}(X_{\overline{k}}) \to \pi_1^{\text{alg}}(D_{\overline{k}}) \) is surjective.14

When \( k \) is not algebraically closed, the above implies that \( (V, \nabla) \) and \( (V', \nabla') \) are two flat connections on \( X \) such that \( (V_D, \nabla) \cong (V'_D, \nabla') \) in \( \text{MIC}(D) \), then the above implies that \( (V, \nabla_{\overline{k}}) \cong (V', \nabla'_{\overline{k}}) \). This implies that \( (V, \nabla) \cong (V', \nabla') \), and therefore the above functor is always fully faithful.

\[ \square \]

**Proof of Corollary 1.10.** We remind the reader that in this corollary, the notation is somewhat different from the rest of the paper. In particular, \( X/\mathcal{O}_K[1/N] \) is a smooth projective scheme over a ring of integers in a number field, and \( D \subset X \) is a relative smooth ample divisor.

(1) immediately follows from Lemma 11.2

We now prove (4). Pick an embedding \( \iota : K \to \mathbb{C} \). The flat connection \((\mathcal{H}, \nabla)_C\) corresponds under Riemann-Hilbert, by construction (i.e., the proof of [Esn12 Proposition 2.1]), to the topological local system \( \pi_1(X_C) \cong \pi_1(D_C) \to \text{GL}_N(\mathbb{Z}) \subset \text{GL}_N(\mathbb{C}) \). By Simpson’s [Theorem 1.1] the representation \( \pi_1(X_C) \to \text{GL}_N(\mathbb{C}) \) underlies a \( \mathbb{Z} \) PVHS. In particular, there is an induced Griffiths transverse filtration \( \text{Fil}'_C \) on \((\mathcal{H}, \nabla)_C\).

Consider the associated graded Higgs bundle
\[ (E', \theta')_C := \text{Gr}_{\text{Fil}'_C}(\mathcal{H}, \nabla)_C \]
on \( X_C \). This is a stable Higgs bundle.

Consider the Simpson filtration \( \text{Fil} \) on \((\mathcal{H}, \nabla)_K\), constructed in [LSZ19 A.4]. The associated graded Higgs bundle \( (E, \theta)_K \) is semistable by construction. On the other hand, \( \text{Gr}_{\text{Fil}}(\mathcal{H}, \nabla)_C \) is stable. This implies that the same is true of the Simpson filtration on \((\mathcal{H}, \nabla)_K\) by [LSZ19 Lemma 7.1], and moreover, that \( \text{Fil}_C = \text{Fil}'_C \) up to a shift. It follows immediately that \( (E', \theta')_C \cong (E, \theta)_C \) descends to \( K \) and hence to \( \mathcal{O}_K[1/N] \) (again, after potentially increasing \( N \)).

We come to (2) and (3). Let \( p \) be an unramified prime ideal with sufficiently large residue characteristic. Denote by \( \mathcal{O}_p \) the ring of integers in \( p \)-adic field \( K_p \). Then the \( p \)-adic completion \( (\mathcal{H}_{\mathcal{O}_p}, \nabla_{\mathcal{O}_p})|_{D_{\mathcal{O}_p}} \) of \((\mathcal{H}_{dR}(Y_D/D), \nabla_{GM})\) underlies a Fontaine-Faltings module. (This is because it comes from the cohomology of a smooth, proper family and

\[ \text{14} \]
Both of these go under the name “Grothendieck-Malcev” theorem, and they use the fact that the topological fundamental group of a smooth, quasi-projective variety is finitely generated.

\[ \text{15} \]
It follows from [Sim91 p. 331] that under the complex Simpson correspondence, \((\mathcal{H}, \nabla)_C\) is simply sent to the associated graded Higgs bundle under the filtration described above.
exists a 1-periodic Higgs-de Rham flow over the $p$-adic completion $X_{O_\rho}$ in the proof of our main theorem, there exists a 1-periodic Higgs-de Rham flow over the $p$-adic completion $X_{O_\rho}$ such that its restriction on $D_{O_\rho}$ is equal to that in (11.1). In particular, one has $(E_{O_\rho}, \theta_{O_\rho}) |_{D_{O_\rho}} \cong (E_{O_\rho}, \theta_{O_\rho}) |_{D_{O_\rho}}$.

We have constructed two Higgs bundles over the $p$-adic formal scheme $X_{O_\rho}$: $(E_{O_\rho}, \theta_{O_\rho})$ and $(E'_{O_\rho}, \theta'_{O_\rho})$. (The former was constructed to live over a ring of integers in $K$, the latter was constructed via the main theorem of this article and a priori only lives on the $p$-adic formal scheme.) To prove (3), we only need to show that these two Higgs bundles are isomorphic. Since both of $(E, \theta)_{K_p} |_{X_{k_p}}$ and $(E'_{O_\rho}, \theta'_{O_\rho}) |_{X_{k_p}}$ extend the same Higgs bundle $(E_{O_\rho}, \theta_{O_\rho}) |_{D_{k_p}}$ over the special fiber $D_{k_p}$ of $D$, Corollary 7.4 implies that

$$(E, \theta)_{O_\rho} |_{X_{k_p}} \cong (E'_{O_\rho}, \theta'_{O_\rho}) |_{X_{k_p}}.$$  

By Theorem 10.6 and Proposition 10.7, an extension of $(E_{O_\rho}, \theta_{O_\rho})$ over $X_{O_\rho}$ is uniquely determined by its modulo $p$ reduction. Thus $(E, \theta)_{O_\rho} \cong (E'_{O_\rho}, \theta'_{O_\rho})$. This verifies (3).

Finally, to prove (2), we only need to show the filtered de Rham bundle $(\mathcal{H}_{O_\rho}, \nabla_{O_\rho}, \text{Fil}_{O_\rho})$ in the flow is isomorphic to the $p$-adic completion $(\mathcal{H}_{O_\rho}, \nabla_{O_\rho}, \text{Fil}_{O_\rho})$ of $(\mathcal{H}, \nabla, \text{Fil})$. By (3), we may identify the two Higgs bundles $(E_{O_\rho}, \theta_{O_\rho}) |_{D_{O_\rho}}$ and $(E'_{O_\rho}, \theta'_{O_\rho}) |_{D_{O_\rho}}$. This Higgs bundle is 1-periodic and stable modulo $p$. Thus up to an isomorphism, it initiates a unique 1-periodic Higgs-de Rham flow. So the restriction of the flow in (11.2) onto $D_{k_p}$ is isomorphic the flow in (11.1). In particular, one has an isomorphism $	au_{D_{O_\rho}} : (\mathcal{H}_{O_\rho}, \nabla_{O_\rho}, \text{Fil}_{O_\rho}) |_{D_{O_\rho}} \cong (\mathcal{H}_{O_\rho}, \nabla_{O_\rho}, \text{Fil}_{O_\rho}) |_{D_{O_\rho}}$ as filtered flat connections over $D_{O_\rho}$.

The reductions modulo $p$ of both $(\mathcal{H}_{O_\rho}, \nabla_{O_\rho})$ and $(\mathcal{H}_{O_\rho}, \nabla_{O_\rho})$ are stable flat connections. By Lemma 11.2 we may extend $\tau_{D_{O_\rho}}$ to an isomorphism

$$\tau_{X_{k_p}} : (\mathcal{H}_{O_\rho}, \nabla'_{O_\rho}) |_{X_{k_p}} \cong (\mathcal{H}_{O_\rho}, \nabla_{O_\rho}) |_{X_{k_p}}.$$  

Again using [LSZ19, Lemma 7.1], over $X_{k_p}$, the filtrations Fil and Fil' only differ by a shift under $\tau_{X_{k_p}}$ because their associated graded Higgs bundles are stable. As $\tau_{D_{O_\rho}}$ preserves the filtration, so does $\tau_{X_{k_p}}$. 

28
Now, the modulo $p$ reductions of both $(\mathcal{H}_p', \nabla_{\mathcal{O}_p'})$ and $(\mathcal{H}_p, \nabla_{\mathcal{O}_p})$ are stable and the two flat connections are isomorphic over $X_{K_p}$. It then follows from Langer’s Langton-style theorem \cite[Theorem 5.2]{Lan14} that $\tau_{X_{K_p}}$ extends to an isomorphism $\tau_{X_{\mathcal{O}_p}} : (\mathcal{H}_{\mathcal{O}_p}, \nabla_{\mathcal{O}_p}) \cong p^r(\mathcal{H}_{\mathcal{O}_p}, \nabla_{\mathcal{O}_p})$ for some $r \in \mathbb{Z}$. As $\tau_p$ is an isomorphism over $D$, it follows that $r = 0$, i.e., that $\tau_{X_{\mathcal{O}_p}} : (\mathcal{H}', \nabla')_{\mathcal{O}_p} \cong (\mathcal{H}_{\mathcal{O}_p}, \nabla_{\mathcal{O}_p})$. We may once again apply \cite[Proposition 7.5]{LSZ19} to deduce that $\tau_{X_{\mathcal{O}_p}}(\text{Fil}_{\mathcal{O}_p}) = \text{Fil}_{\mathcal{O}_p}$. Thus $\tau_{X_{\mathcal{O}_p}}$ is the desired isomorphism. 

\begin{thebibliography}{99}
\bibitem[Ara19]{Ara19} Donu Arapura. Kodaira-Saito vanishing via Higgs bundles in positive characteristic. \textit{J. Reine Angew. Math.}, 755:293–312, 2019.
\bibitem[Cas76]{Cas76} J.W.S. Cassels. An embedding theorem for fields. \textit{Bull. Austral. Math. Soc.}, 14, 1976.
\bibitem[DLLZ18]{DLLZ18} Hansheng Dao, Kai-Wen Lan, Ruochuan Liu, and Xinwen Zhu. Logarithmic Riemann-Hilbert correspondences for rigid varieties. \textit{arXiv preprint arXiv:1803.05786}, 2018.
\bibitem[EK16]{EK16} Hélène Esnault and Lars Kindler. Lefschetz theorems for tamely ramified coverings., 2016.
\bibitem[Esn12]{Esn12} Hélène Esnault. Flat bundles in characteristic 0 and $p > 0$.
\bibitem[EV86]{EV86} Hélène Esnault and Eckart Viehweg. Logarithmic de Rham complexes and vanishing theorems. \textit{Inventiones mathematicae}, 86(1):161–194, 1986.
\bibitem[EV92]{EV92} Hélène Esnault and Eckart Viehweg. \textit{Lectures on vanishing theorems}, volume 20 of \textit{DMV Seminar}. Birkhäuser Verlag, Basel, 1992.
\bibitem[Fal89]{Fal89} Gerd Faltings. Crystalline cohomology and $p$-adic Galois-representations. In \textit{Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)}, pages 25–80. Johns Hopkins Univ. Press, Baltimore, MD, 1989.
\bibitem[FGI+05]{FGI+05} Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. \textit{Fundamental algebraic geometry: Grothendieck’s FGA explained.}, volume 123. Providence, RI: American Mathematical Society (AMS), 2005.
\bibitem[Gro70]{Gro70} Alexander Grothendieck. Representations lineaires et compactification profinie des groupes discrets. \textit{Manuscr. Math.}, 2:375–396, 1970.
\bibitem[Kat81]{Kat81} Nick Katz. Serre-Tate local moduli. In \textit{Algebraic surfaces (Orsay, 1976–78)}, volume 868 of \textit{Lecture Notes in Math.}, pages 138–202. Springer, Berlin-New York, 1981.
\bibitem[Kaw81]{Kaw81} Yujiro Kawamata. Characterization of abelian varieties. \textit{Compositio Math.}, 43(2):253–276, 1981.
\bibitem[KP21]{KP21} Raju Krishnamoorthy and Ambrus Pál. Rank 2 local systems and abelian varieties. \textit{Sel. Math., New Ser.}, 27(4):40, 2021.
\bibitem[KYZ20]{KYZ20} Raju Krishnamoorthy, Jinbang Yang, and Kang Zuo. Deformations of periodic Higgs-de Rham flows. \textit{arXiv preprint arXiv:2005.00570}, 2020.
\bibitem[Lan14]{Lan14} Adrian Langer. Semistable modules over Lie algebroids in positive characteristic. \textit{Doc. Math.}, 19:509–540, 2014.
\bibitem[Lan19]{Lan19} Adrian Langer. Nearby cycles and semipositivity in positive characteristic. \textit{arXiv preprint arXiv:1902.05745v3}, 2019.
\bibitem[LSY19]{LSY19} Guitang Lan, Mao Sheng, Yanhong Yang, and Kang Zuo. Uniformization of $p$-adic curves via Higgs-de Rham flows. \textit{J. Reine Angew. Math.}, 747:63–108, 2019.
\bibitem[LSZ19]{LSZ19} Guitang Lan, Mao Sheng, and Kang Zuo. Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups. \textit{J. Eur. Math. Soc. (JEMS)}, 21(10):3053–3112, 2019.
\bibitem[LZ17]{LZ17} Ruochuan Liu and Xinwen Zhu. Rigidity and a Riemann-Hilbert correspondence for $p$-adic local systems. \textit{Invent. Math.}, 207(1):291–343, 2017.
\bibitem[Niz01]{Niz01} Wiesława Nizioł. Cohomology of crystalline smooth sheaves. \textit{Compos. Math.}, 129(2):123–147, 2001.
\bibitem[OV07]{OV07} A. Ogus and V. Vologodsky. Nonabelian Hodge theory in characteristic $p$. \textit{Publ. Math. Inst. Hautes Études Sci.}, (106):1–138, 2007.
\bibitem[Poo04]{Poo04} Bjorn Poonen. Bertini theorems over finite fields. \textit{Ann. of Math. (2)}, 160(3):1099–1127, 2004.
\bibitem[Sch13]{Sch13} Peter Scholze. $p$-adic Hodge theory for rigid-analytic varieties. \textit{Forum Math. Pi}, 1:e1, 77, 2013.
\bibitem[Sim90]{Sim90} Carlos T. Simpson. Transcendental aspects of the Riemann-Hilbert correspondence. \textit{Ill. J. Math.}, 34(2):368–391, 1990.
\end{thebibliography}
[Sim91] Carlos T. Simpson. The ubiquity of variations of Hodge structure. In Complex geometry and Lie theory. Proceedings of a symposium, held at Sundance, UT, USA, May 26-30, 1989, pages 329–348. Providence, RI: American Mathematical Society, 1991.

[Sim92] Carlos T. Simpson. Higgs bundles and local systems. Publ. Math., Inst. Hautes Étud. Sci., 75:5–95, 1992.

[Sta20] The Stacks Project Authors. Stacks Project. https://stacks.math.columbia.edu 2020.

[TT19] Fucheng Tan and Jilong Tong. Crystalline comparison isomorphisms in $p$-adic Hodge theory: the absolutely unramified case. Algebra Number Theory, 13(7):1509–1581, 2019.

Email address: krishnamoorthy@alum.mit.edu

Arbeitsgruppe Algebra und Zahlentheorie, Bergische Universität Wuppertal, F 13.05 Gaussstrasse 20, Wuppertal 42119, Germany

Email address: yjb@mail.ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China

Email address: zuok@uni-mainz.de

Institut für Mathematik, Universität Mainz, Mainz 55099, Germany