FORMULAE OF SOME GLOBAL CR INVARIANTS FOR SasakiAN $\eta$-EINSTEIN MANIFOLDS

YUYA TAKEUCHI

Abstract. In this paper, we give explicit formulae of the Burns-Epstein invariant and global CR invariants via renormalized characteristic forms introduced by Marugame for Sasakian $\eta$-Einstein manifolds. As an application, we show that the latter invariants are algebraically independent.

1. Introduction

The biholomorphic equivalence problem is one of the major problems in several complex variables. An approach to this problem is to study the CR structures on the boundaries of domains. In this direction, Fefferman [Feff74] has shown that two bounded strictly pseudoconvex domains in $\mathbb{C}^{n+1}$ are biholomorphic if and only if their boundaries are isomorphic as CR manifolds. Since then, it has been of great importance to construct and compute CR invariants.

Recently, some researchers have introduced and studied invariants for strictly pseudoconvex CR manifolds admitting a pseudo-Einstein contact form: the total $Q'$-curvature [CY13, Hir14], the Burns-Epstein invariant [BE90, Mar16], the total $T'$-curvatures [CG17, Mar19, CT20], and global CR invariants via renormalized characteristic forms [Mar19]. Note that the first and third ones are special cases of the last one. In general, it is hard to compute these invariants of a given CR manifold. However, we have already obtained explicit formulae of the first and third ones for Sasakian $\eta$-Einstein manifolds, which are pseudo-Hermitian manifolds satisfying a strong Einstein condition. This has been carried out for the total $Q'$-curvature by Case and Gover [CG17] and the author [Tak18] independently, and for the total $T'$-curvatures by Marugame [Mar19] implicitly. The purpose of this paper is to give formulae of the remaining cases, the Burns-Epstein invariant and global CR invariants via renormalized characteristic forms, for Sasakian $\eta$-Einstein manifolds.

We first consider global CR invariants via renormalized characteristic forms. Let $\Omega$ be an $(n+1)$-dimensional strictly pseudoconvex domain with boundary $M$. Assume that $M$ admits a pseudo-Einstein contact form. Take a Fefferman defining function $\rho$ of $\Omega$; see Section 3.2 for the definition. The $(1,1)$-form

$$\omega_+ = -dd^c \log(-\rho)$$

defines a Kähler metric $g_+$ near the boundary, where $d^c = (\sqrt{-1}/2)(\bar{\partial} - \partial)$. The Chern connection with respect to $g_+$ diverges on the boundary since so does $g_+$. However, we obtain Burns-Epstein’s renormalized connection, which is smooth up
to the boundary, via a c-projective compactification [ČG19]. The corresponding curvature form is denoted by Θ. Let \( \Phi \) be a (not necessarily homogeneous) \( GL(n+1, \mathbb{C}) \)-invariant polynomial of degree at most \( n \). Marugame [Mar19] has proved that

\[
\mathcal{I}_\Phi(M) = -\sum_{m=0}^{n} \frac{1}{l^p} \int_{\rho<\epsilon} \frac{d\log(-\rho) \wedge d^c \log(-\rho)}{2\pi} \wedge \left( \frac{\omega_+}{2\pi} \right)^{n-m} \wedge \Phi \left( \frac{\sqrt{-1}}{2\pi} \Theta \right)
\]

is independent of the choice of \( \rho \), and gives a global CR invariant of \( M \). More precisely, he has shown that \( \mathcal{I}_\Phi \) is written as the integral of a linear combination of the complete contractions of polynomials in the Tanaka-Webster torsion, curvature, and their covariant derivatives. (Our definition is a little bit different from Marugame’s one. Our modification is to guarantee \( \mathcal{I}_\Phi \) is written as the integral of a linear combination of \( GL(n+1, \mathbb{C}) \)-invariant polynomials \( \Phi_1 \) and \( \Phi_2 \) of degree at most \( n \.)

Our first result is to give an explicit expression of \( \mathcal{I}_\Phi \) in terms of \( \Phi \), the Einstein constant, and the Chern tensor \( S^\alpha_\beta \) for Sasakian \( \eta \)-Einstein manifolds. A Sasakian \( \eta \)-Einstein manifold is a pseudo-Hermitian manifold \( (S, T^{1,0}S, \eta) \) of dimension \( 2n + 1 \) such that the Tanaka-Webster torsion \( A^\alpha_\beta \) and Ric curvature \( \text{Ric} \) satisfy

\[
A^\alpha_\beta = 0, \quad \text{Ric}_{\alpha\beta} = (n+1)\lambda \sigma_{\alpha\beta},
\]

where \( \lambda \in \mathbb{R} \) and \( \sigma_{\alpha\beta} \) is the Levi form. We call the constant \( (n+1)\lambda \) the Einstein constant of \( (S, T^{1,0}S, \eta) \).

**Theorem 1.1.** Let \( (S, T^{1,0}S, \eta) \) be a closed \((2n + 1)\)-dimensional Sasakian \( \eta \)-Einstein manifold with Einstein constant \( (n + 1)\lambda \). For a \( GL(n+1, \mathbb{C}) \)-invariant polynomial \( \Phi \) of degree at most \( n \), define a \( GL(n, \mathbb{C}) \)-invariant polynomial \( \Phi' \) by

\[
\Phi'(A) = \Phi \left( \begin{array}{cc} A & 0 \\ 0 & 0 \end{array} \right).
\]

Then

\[
\mathcal{I}_\Phi(S) = \sum_{m=0}^{n} \int_S \left( -\frac{\lambda}{2\pi} \eta \right) \wedge \left( -\frac{\lambda}{2\pi} \eta \right)^{n-m} \wedge \Phi' \left( \frac{\sqrt{-1}}{2\pi} \Xi \right),
\]

where

\[
\Xi^\alpha_\beta = S^\alpha_\beta \theta^\rho \wedge \theta^\rho.
\]

A typical example of Sasakian \( \eta \)-Einstein manifolds is a circle bundle over a Kähler-Einstein manifold. Let \( Y \) be an \( n \)-dimensional complex manifold and \( (L, h) \) a Hermitian holomorphic line bundle over \( Y \) such that

\[
\omega = -\sqrt{-1} \Theta_h = dd^c \log h
\]

defines a Kähler-Einstein metric on \( Y \) with Einstein constant \( (n + 1)\lambda \). Consider the circle bundle

\[
S = \{ v \in L | h(v, v) = 1 \}
\]

over \( Y \), which is a real hypersurface in the total space of \( L \). The triple

\[
(S, T^{1,0}S = T^{1,0}L|_S \cap (TS \otimes \mathbb{C}), \eta = d^c \log h|_S)
\]

is a \((2n + 1)\)-dimensional Sasakian \( \eta \)-Einstein manifold with Einstein constant \((n + 1)\lambda \) and called the circle bundle associated with \((Y, L, h)\); see Section 2.3 for details. In this case, we can calculate \( \mathcal{I}_\Phi \) in terms of the Einstein constant and characteristic numbers of \( Y \). In order to simplify computation, we introduce some.
invariant polynomials. Let $m$ be a positive integer. Define a $GL(m, \mathbb{C})$-invariant homogeneous polynomial $\text{ch}_k$ of degree $k$ by

$$\text{ch}_k(A) = \frac{1}{k!} \text{tr} A^k.$$ 

A partition of $n$ is an $n$-tuple $\varsigma = (\varsigma_1, \ldots, \varsigma_n) \in \mathbb{N}^n$ with $\sum_{k=1}^n k\varsigma_k = n$. The space of partitions of $n$ is written as $\text{Part}(n)$. For a partition $\varsigma$ of $n$, set

$$(1.1) \quad \Phi_{\varsigma}(A) = \prod_{k=2}^{n} \text{ch}_k(A)^{\varsigma_k},$$

which is a $GL(m, \mathbb{C})$-invariant homogeneous polynomial of degree $n - \varsigma_1$. We will write $\mathcal{I}_\varsigma$ as $\mathcal{I}$ for simplicity. It can be seen that any $\mathcal{I}$ is written as a linear combination of $(\mathcal{I}_{\varsigma})_{\varsigma \in \text{Part}(n)}$ (Lemma 4.3). Hence it suffices to compute $\mathcal{I}$.

**Theorem 1.2.** Let $(L, h)$ be a Hermitian holomorphic line bundle over a closed $n$-dimensional complex manifold $Y$ such that $\omega = -\sqrt{-1} \Theta_h$ defines a Kähler-Einstein metric on $Y$ with Einstein constant $(n + 1)\lambda$. Denote by $(S, T^{1,0}S, \eta)$ the circle bundle associated with $(Y, L, h)$. For a partition $\varsigma$ of $n$,

$$\mathcal{I}(S) = -\lambda \int_Y (\lambda \varsigma_1(L))^n \prod_{k=2}^{n} \left[ \sum_{j=0}^{k} \frac{1}{(k-j)!} (\lambda \varsigma_1(L))^{k-j} \text{ch}_j(T^{1,0}Y \oplus C) \right]^{\varsigma_k}.$$ 

As an application of the above theorem, we will show that $(\mathcal{I}_{\varsigma})_{\varsigma \in \text{Part}(n)}$ are essentially different invariants.

**Theorem 1.3.** The invariants $(\mathcal{I}_{\varsigma})_{\varsigma \in \text{Part}(n)}$ are algebraically independent over $\mathbb{C}$—that is, there exist no non-trivial polynomial relations between $(\mathcal{I}_{\varsigma})_{\varsigma \in \text{Part}(n)}$ with coefficients in $\mathbb{C}$.

As a corollary, we have a criterion for the triviality of $\mathcal{I}$.

**Corollary 1.4.** For a $GL(n + 1, \mathbb{C})$-invariant polynomial $\Phi$ of degree at most $n$, the invariant $\mathcal{I}$ is trivial if and only if $\Phi \equiv 0$ modulo $\text{ch}_1$.

This is a generalization of the latter statement of [Mar19, Proposition 6.7]. We will compute the Burns-Epstein invariant $\mu$ for Sasakian $\eta$-Einstein manifolds also. Burns and Epstein [BE90] have introduced this invariant for the boundaries of bounded strictly pseudoconvex domains in $\mathbb{C}^{n+1}$. Marugame [Mar16] has generalized this invariant for closed strictly pseudoconvex CR manifolds admitting a pseudo-Einstein contact form. He has defined this invariant as the boundary term of the renormalized Gauss-Bonnet-Chern formula

$$\int_{\Omega} c_{n+1}(\Theta) = \chi(\Omega) + \mu(M),$$

where $\Omega$ is a strictly pseudoconvex domain bounded by $M$.

**Theorem 1.5.** Let $(S, T^{1,0}S, \eta)$ be a closed $(2n + 1)$-dimensional Sasakian $\eta$-Einstein manifold with Einstein constant $(n + 1)\lambda$. The Burns-Epstein invariant $\mu(S)$ of $S$ is given by

$$\mu(S) = \sum_{m=0}^{n} \int_{S} \left( -\frac{\lambda}{2\pi} \eta \right) \wedge \left( \frac{\lambda}{2\pi} d\eta \right)^{n-m} \wedge c_{m} \left( \sqrt{-1} \Omega \right),$$

where $\Omega$ is the Tanaka-Webster curvature form.
Similar to Theorem 1.2, we obtain an expression of $\mu$ for the circle bundle case in terms of the Einstein constant and characteristic number $s$.

**Corollary 1.6.** Let $(L, h)$ be a Hermitian holomorphic line bundle over a closed $n$-dimensional complex manifold $Y$ such that $\omega = -\sqrt{-1}\Theta h$ defines a Kähler-Einstein metric on $Y$ with Einstein constant $(n + 1)\lambda$. Denote by $(S, T^{1,0}S, \eta)$ the circle bundle associated with $(Y, L, h)$. The Burns-Epstein invariant $\mu(S)$ of $S$ is given by

$$
\mu(S) = -\lambda \sum_{m=0}^{n} \int_{Y} (\lambda c_1(L))^{n-m} c_m(T^{1,0}Y).
$$

This paper is organized as follows. In Section 2 (resp. Section 3), we recall basic facts on CR and Sasakian manifolds (resp. strictly pseudoconvex domains). The renormalized connection and the invariants $I_\varphi$ are introduced in Section 4. Section 5 is devoted to proofs of Theorems 1.1 to 1.3. Section 6 deals with the Burns-Epstein invariant. In Section 7, we pose a conjecture on global CR invariants and give a proof of it in low dimensions.

**Notation.** We use Einstein’s summation convention and assume that

- lowercase Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1, ..., $n$;
- lowercase Latin indices $a, b, c, \ldots$ run from 1, ..., $n, \infty$.

Suppose that a function $I(\epsilon)$ admits an asymptotic expansion, as $\epsilon \to +0$,

$$
I(\epsilon) = \sum_{m=1}^{k} a_m \epsilon^{-m} + b \log \epsilon + O(1).
$$

Then the logarithmic part $\log I(\epsilon)$ of $I(\epsilon)$ is the constant $b$.

## 2. CR GEOMETRY

### 2.1. CR structures.

Let $M$ be a smooth $(2n + 1)$-dimensional manifold without boundary. A **CR structure** is a rank $n$ complex subbundle $T^{1,0}M$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ such that

$$
T^{1,0}M \cap T^{0,1}M = 0, \quad [\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M),
$$

where $T^{0,1}M$ is the complex conjugate of $T^{1,0}M$ in $TM \otimes \mathbb{C}$. Set $HM = \text{Re} T^{1,0}M$ and let $J: HM \to HM$ be the unique complex structure on $HM$ such that

$$
T^{1,0}M = \ker(J - \sqrt{-1} : HM \otimes \mathbb{C} \to HM \otimes \mathbb{C}).
$$

A typical example of CR manifolds is a real hypersurface $M$ in an $(n+1)$-dimensional complex manifold $X$; this $M$ has the canonical CR structure

$$
T^{1,0}M = T^{1,0}X|_M \cap (TM \otimes \mathbb{C}).
$$

A CR structure $T^{1,0}M$ is said to be **strictly pseudoconvex** if there exists a nowhere-vanishing real one-form $\theta$ on $M$ such that $\theta$ annihilates $T^{1,0}M$ and

$$
-\sqrt{-1}d\theta(Z, \overline{Z}) > 0, \quad 0 \neq Z \in T^{1,0}M.
$$

We call such a one-form a **contact form**. The triple $(M, T^{1,0}M, \theta)$ is called a pseudo-Hermitian manifold. Denote by $T$ the Reeb vector field with respect to $\theta$; that is, the unique vector field satisfying

$$
\theta(T) = 1, \quad T \cdot d\theta = 0.
$$
Let \((Z_\alpha)\) be a local frame of \(T^{1,0}M\), and set \(Z_\overline{\alpha} = \overline{Z_\alpha}\). Then \((T, Z_\alpha, Z_\overline{\alpha})\) gives a local frame of \(TM \otimes \mathbb{C}\), called an admissible frame. Its dual frame \(\theta, \theta^\alpha, \theta^\overline{\alpha}\) is called an admissible coframe. The two-form \(d\theta\) is written as
\[
d\theta = \sqrt{-1} l_\alpha \omega^\alpha \wedge \theta^\overline{\alpha},
\]
where \((l_\alpha)\) is a positive definite Hermitian matrix. We use \(l_\alpha\) and its inverse \(l_\alpha^{\overline{\alpha}}\) to raise and lower indices of tensors.

2.2. Tanaka-Webster connection and pseudo-Einstein condition. A contact form \(\theta\) induces a canonical connection \(\nabla\), called the Tanaka-Webster connection with respect to \(\theta\). It is defined by
\[
\nabla T = 0, \quad \nabla Z_\alpha = \omega_\alpha^\beta Z_\beta, \quad \nabla Z_\overline{\alpha} = \omega_\alpha^{\overline{\beta}} Z_\overline{\beta} \quad \left(\omega_\alpha^\beta = \omega_\alpha^{\overline{\beta}}\right)
\]
with the following structure equations:
\[
\begin{align}
\label{e:2.1}
d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha^\beta + A_\alpha^\beta \theta \wedge \theta^\overline{\alpha}, \\
\label{e:2.2}dl_\alpha &= \omega_\alpha^\gamma l_\gamma + l_\alpha \omega^{\overline{\beta}}.
\end{align}
\]
The tensor \(A_{\alpha\beta} = A_\alpha^{\overline{\beta}}\) is shown to be symmetric and is called the Tanaka-Webster torsion. We denote the components of a successive covariant derivative of a tensor by subscripts preceded by a comma, for example, \(K_{\rho\sigma\gamma}\); we omit the comma if the derivatives are applied to a function.

The curvature form \(\Omega^\beta_\alpha = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta\) of the Tanaka-Webster connection satisfies
\[
\Omega^\beta_\alpha = R^\beta_{\rho\sigma\alpha} \theta^\rho \wedge \theta^\sigma - A_\alpha^\gamma \theta \wedge \theta^\overline{\alpha} + A_\rho^{\overline{\gamma}} \alpha \theta \wedge \theta^\overline{\rho} - \sqrt{-1} A_\overline{\alpha}^{\overline{\rho}} \alpha \theta \wedge \theta^\rho.
\]
We call the tensor \(R^\beta_{\alpha\rho\sigma}\) the Tanaka-Webster curvature. This tensor has the symmetry
\[
R^\beta_{\alpha\rho\sigma} = R^\beta_{\rho\alpha\sigma} = R^\beta_{\alpha\sigma\rho}.
\]
Contraction of indices gives the Tanaka-Webster Ricci curvature \(\text{Ric}_{\rho\sigma} = R^\alpha_{\alpha\rho\sigma}\) and the Tanaka-Webster scalar curvature \(\text{Scal} = \text{Ric}_\rho^\rho\). The Chern tensor \(S_{\alpha\beta\rho\sigma}\) is defined by
\[
S_{\alpha\beta\rho\sigma} = R_{\alpha\beta\rho\sigma} - P_{\alpha\beta} l_{\rho\sigma} - P_{\rho\sigma} l_{\alpha\beta} - P_{\alpha\rho} l_{\beta\sigma} - P_{\beta\sigma} l_{\alpha\rho},
\]
where
\[
P_{\alpha\beta} = \frac{1}{n+2} \left( R_{\alpha\beta} - \frac{\text{Scal}}{2(n+1)} l_{\alpha\beta} \right).
\]
A contact form \(\theta\) is said to be pseudo-Einstein if the following two equalities hold:
\[
\begin{align}
\label{e:2.4}
\text{Ric}_{\alpha\beta} &= \frac{1}{n} \text{Scal} \cdot l_{\alpha\beta}, \quad \text{Scal}_{\alpha} = \sqrt{-1} \text{In} A_{\alpha\beta, \overline{\beta}},
\end{align}
\]
From Bianchi identities for the Tanaka-Webster connection, we obtain
\[
\left( R_{\alpha\beta} - \frac{1}{n} \text{Scal} \cdot l_{\alpha\beta} \right) = \frac{n-1}{n} \left( \text{Scal}_{\alpha} - \sqrt{-1} \text{In} A_{\alpha\beta, \overline{\beta}} \right);
\]
see [Hir14, Lemma 5.7(iii)] for example. Hence the latter equality of (2.4) follows from the former one if \(n \geq 2\). On the other hand, the former equality of (2.4) automatically holds if \(n = 1\), and the latter one is a non-trivial condition.
2.3. Sasakian manifolds. Sasakian manifolds are an important class of pseudo-Hermitian manifolds. See [BG08] for a comprehensive introduction to Sasakian manifolds.

A Sasakian manifold is a pseudo-Hermitian manifold $(S, T^{1,0}S, \eta)$ with vanishing Tanaka-Webster torsion. This condition is equivalent to that the Reeb vector field $\xi$ with respect to $\eta$ preserves the CR structure $T^{1,0}S$. An almost complex structure $I$ on the cone $C(S) = \mathbb{R}_+ \times S$ of $S$ is defined by

$$I(a(r\partial/\partial r) + b\xi + V) = -b(r\partial/\partial r) + a\xi + JV,$$

where $r$ is the coordinate of $\mathbb{R}_+$, $a, b \in \mathbb{R}$, and $V \in HS$. The bundle $T^{1,0}C(S)$ of $(1,0)$-vectors with respect to $I$ is given by

$$T^{1,0}C(S) = C(r\partial/\partial r - \sqrt{-1}\xi) \oplus T^{1,0}S.$$

The vanishing of the Tanaka-Webster torsion implies that $I$ is integrable; that is, $(C(S), I)$ is a complex manifold. Moreover, the one-form $\eta$ is equal to $d\omega \log r^2$. In what follows, we identify $S$ with the level set $\{1\} \times S \subset C(S)$.

A $(2n + 1)$-dimensional Sasakian manifold $(S, T^{1,0}S, \eta)$ is said to be Sasakian $\eta$-Einstein with Einstein constant $(n + 1)\lambda$ if the Tanaka-Webster Ricci curvature satisfies

$$\text{Ric}_{\alpha\beta} = (n + 1)\lambda g_{\alpha\beta}.$$

Note that $\eta$ is a pseudo-Einstein contact form on $S$.

A typical example of Sasakian manifolds is the circle bundle associated with a negative Hermitian line bundle. Let $Y$ be an $n$-dimensional complex manifold and $(L, h)$ a Hermitian holomorphic line bundle over $Y$ such that

$$\omega = -\sqrt{-1} \Theta_h = d\omega \log h$$

is a Kähler form on $Y$. Note that $c_1(L) = -[\omega/2\pi]$. Consider the circle bundle

$$S = \{ v \in L \mid h(v, v) = 1 \}$$

over $Y$, which is a real hypersurface in the total space of $L$. The one-form $\eta = d\omega \log h|_S$ is a connection one-form of the principal $S^1$-bundle $p: S \to Y$ and satisfies $d\eta = p^*\omega$. Moreover, the natural CR structure $T^{1,0}Y$ coincides with the horizontal lift of $T^{1,0}Y$ with respect to $\eta$. Since $\omega$ is a Kähler form, we have

$$-\sqrt{-1}d\eta(Z, \overline{Z}) = -\sqrt{-1}\omega(p(Z, \overline{Z}) > 0$$

for all non-zero $Z \in T^{1,0}S$. This implies that $(S, T^{1,0}S)$ is a strictly pseudoconvex CR manifold and $\eta$ is a contact form on $S$. Note that the Reeb vector field $\xi$ with respect to $\eta$ is the generator of the $S^1$-action on $S$.

Consider the Tanaka-Webster connection with respect to $\eta$. Take a local coordinate $(z^1, \ldots, z^n)$ of $Y$. The Kähler form $\omega$ is written as

$$\omega = \sqrt{-1}g_{\alpha\overline{\beta}}dz^\alpha \wedge d\overline{z}^\beta,$$

where $(g_{\alpha\overline{\beta}})$ is a positive definite Hermitian matrix. Let $Z_\alpha$ be the horizontal lift of $\partial/\partial z^\alpha$. Then $(\xi, Z_\alpha, Z_{\overline{\alpha}} = \overline{Z_\alpha})$ is an admissible frame on $S$. The corresponding admissible coframe is given by $(\eta, \theta^\alpha = p^*(dz^\alpha), \overline{\theta}^\overline{\beta} = p^*(d\overline{z}^{\overline{\beta}}))$. Since $d\eta = p^*\omega$, we have

$$d\eta = \sqrt{-1}(p^*g_{\alpha\overline{\beta}})\theta^\alpha \wedge \overline{\theta}^{\overline{\beta}},$$
which implies \( l^\alpha_\beta = p^*g^\alpha_\beta \). The connection form \( \pi^\alpha_\beta \) of the Kähler metric with respect to the frame \( (\partial/\partial z^\alpha) \) satisfies

\[
0 = d(dz^\beta) = dz^\alpha \wedge \pi^\alpha_\beta, \quad d g^\alpha_\beta = \pi^\alpha_\gamma g^\gamma_\beta + g^\alpha_\gamma \pi^\gamma_\beta \quad \left( \pi^\beta_\beta = \pi^\alpha_\beta \right).
\]

We write as \( \Pi^\alpha_\beta \) the curvature form of the Kähler metric. Pulling back (2.5) by \( p \) gives

\[
d\theta^\beta = \theta^\alpha \wedge (p^*\pi^\alpha_\beta), \quad dl^\alpha_\beta = (p^*\pi^\beta_\alpha)l^\gamma_\beta + l^\gamma_\alpha(p^*\pi^\beta_\gamma) \]

This yields that \( \omega^\alpha_\beta = p^*\pi^\alpha_\beta \), and the Tanaka-Webster torsion vanishes identically; that is, \((S, T^{1,0}S, \eta)\) is a Sasaki manifold. Moreover, the curvature form \( \Omega^\alpha_\beta \) of the Tanaka-Webster connection is given by \( \Omega^\alpha_\beta = p^*\Pi^\alpha_\beta \). In particular, \((S, T^{1,0}S, \eta)\) is a Sasaki \( \eta \)-Einstein manifold with Einstein constant \((n + 1)\lambda\) if and only if \( \omega \) defines a Kähler-Einstein metric with Einstein constant \((n + 1)\lambda\).

3. Strictly pseudoconvex domains

Let \( \Omega \) be a relatively compact domain in an \((n+1)\)-dimensional complex manifold \( X \) with smooth boundary \( M = \partial \Omega \). There exists a smooth function \( \rho \) on \( X \) such that

\[
\Omega = \rho^{-1}((-\infty, 0)), \quad M = \rho^{-1}(0), \quad dp \neq 0 \text{ on } M;
\]

such a \( \rho \) is called a defining function of \( \Omega \). A domain \( \Omega \) is said to be strictly pseudoconvex if we can take a defining function \( \rho \) of \( \Omega \) that is strictly plurisubharmonic near \( M \). The boundary of a strictly pseudoconvex domain is a closed strictly pseudoconvex real hypersurface. Conversely, it is known that any closed connected strictly pseudoconvex CR manifold of dimension at least five can be realized as the boundary of a strictly pseudoconvex domain in a complex projective manifold [BDM75, HL75, Lem95].

3.1. Graham-Lee connection. Let \( \rho \) be a defining function of a strictly pseudoconvex domain \( \Omega \) in an \((n+1)\)-dimensional complex manifold \( X \). There exists the unique \((1,0)\)-vector field \( \tilde{Z}_\infty \) near the boundary such that

\[
\tilde{Z}_\infty \rho = 1, \quad \tilde{Z}_\infty \partial \bar{\rho} = \kappa \bar{\rho}
\]

for a smooth function \( \kappa \), which is called the transverse curvature. Take a local frame \( (\tilde{Z}_\alpha) \) of \( \text{Ker} \partial \rho \), and let \( (\bar{\rho}, \bar{\kappa} = \partial \rho) \) be the dual frame of \( (\tilde{Z}_\alpha, \tilde{Z}_\infty) \). By the strict pseudoconvexity, there exists a positive Hermitian matrix \( (\tilde{l}^\alpha_\beta) \) such that

\[
\bar{d} \bar{\rho} = \sqrt{-1} \tilde{l}^\alpha_\beta \bar{\theta}^\alpha \wedge \bar{\theta}^\beta + \kappa \bar{\rho} \wedge d^c \rho.
\]

We use \( \tilde{l}^\alpha_\beta \) and its inverse \( \tilde{n}^\alpha_\beta \) to raise and lower indices of tensors. The Graham-Lee connection \( \nabla \) is the unique connection on \( TX \) defined by

\[
\nabla \tilde{Z}_\alpha = \tilde{\omega}^\alpha_\beta \tilde{Z}_\beta, \quad \nabla \tilde{Z}_\infty = \tilde{\omega}^\infty_\beta \tilde{Z}_\beta \quad \left( \tilde{\omega}^\alpha_\beta = \tilde{\omega}^\alpha_\beta \right), \quad \nabla \tilde{Z}_\infty = \nabla \tilde{Z}_\infty = 0,
\]

with the following structure equations:

\[
d\tilde{\theta}^\beta = \tilde{\theta}^\alpha \wedge \tilde{\omega}^\alpha_\beta - \sqrt{-1} \tilde{A}^\alpha_\beta \partial \rho \wedge \bar{\theta}^\beta - \kappa \bar{\rho} \wedge \bar{\theta}^\beta + \frac{1}{2} \kappa \bar{\rho} \wedge \bar{\theta}^\beta, \]

\[
d \tilde{l}^\alpha_\beta = \tilde{\omega}^\gamma_\alpha \tilde{l}^\gamma_\beta + \tilde{l}^\gamma_\alpha \tilde{\omega}^\gamma_\beta;
\]
see [GL88, Proposition 1.1] for a proof of the existence and uniqueness. Note that the restriction of $\tilde{\nabla}$ to $M$ coincides with the Tanaka-Webster connection with respect to $\theta = d^c\rho|_M$.

### 3.2. Fefferman defining functions

Let $\Omega$ be a strictly pseudoconvex domain in a complex manifold $X$ of dimension $n + 1$. Fix a local coordinate $z$ on $X$ near a point of the boundary $M = \partial \Omega$. A differential operator $J_z$ is defined by

$$J_z[\phi] = -\det \left( \frac{\partial \phi}{\partial z^i}, \frac{\partial \phi}{\partial \bar{z}^i}, \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} \right).$$

A Fefferman defining function is a defining function of $\Omega$ such that

$$dd^c \log J_z[\rho] = dd^c O(\rho^{n+2}).$$

This condition is independent of the choice of a local coordinate $z$. It is known that $\Omega$ admits a Fefferman defining function if and only if the boundary $M$ has a pseudo-Einstein contact form; see [Lee88, Theorem 4.2], [Hir93, Lemma 7.2], and [HPT08, Proposition 2.10].

### 4. Global CR invariants via renormalized characteristic forms

#### 4.1. Burns-Epstein’s renormalized connection

Let $\Omega$ be a strictly pseudoconvex domain of dimension $n + 1$. Assume that its boundary $M$ admits a pseudo-Einstein contact form. For a Fefferman defining function $\rho$ of $\Omega$, the $(1,1)$-form

$$\omega_+ = -dd^c \log(-\rho)$$

defines a Kähler metric $g_+$ near the boundary. We extend $g_+$ to a Hermitian metric on $\Omega$. Let $\psi_{a\overline{b}}$ be the Chern connection with respect to $g_+$. This connection diverges on the boundary since so does $g_+$. The renormalized connection form $\theta_{a\overline{b}}$ is defined by

$$\theta_{a\overline{b}} = \psi_{a\overline{b}} + \frac{1}{\rho}(\delta_{a\overline{b}} \rho_c + \delta_{b\overline{b}} \rho_a)\tilde{\theta}.$$ 

This connection form can be extended smoothly up to the boundary; see [Mar19, Proposition 4.1] for example. The corresponding curvature form is denoted by $\Theta_{a\overline{b}}$, which satisfies

$$\text{tr} \Theta = -dd^c \log J_z[\rho] = dd^c O(\rho^{n+2});$$

see [Mar16, Equation (4.7)]. Near the boundary, $\theta_{a\overline{b}}$ is written in terms of the Graham-Lee connection.

**Lemma 4.1** ([Mar16, Proposition 3.5]). For the frame $(\tilde{\theta}^\alpha, \tilde{\theta}^\infty = \partial \rho)$,

\[
\begin{align*}
\theta_{a}^{\overline{b}} &= \tilde{\theta}_{a}^{\overline{b}} + \frac{1}{2} \kappa (\partial \rho - \overline{\partial} \rho) \delta_{a}^{\overline{b}}, \\
\theta_{\infty}^{\overline{a}} &= \kappa \tilde{\theta}^{\overline{a}} - \sqrt{-1} A^{\beta}_{a} \tilde{\theta}^{\overline{\beta}} - \kappa \overline{\partial} \rho, \\
\theta_{a}^{\infty} &= -\tilde{A}_{a\beta} \tilde{\theta}^{\overline{\beta}} - \rho (1 - \kappa \rho)^{-1} \kappa_{a} \partial \rho + \sqrt{-1} \rho (1 - \kappa \rho)^{-1} \tilde{A}_{a\beta} \tilde{\theta}^{\overline{\beta}}, \\
\theta_{\infty}^{\infty} &= -\kappa \overline{\partial} \rho - \rho \kappa (1 - \kappa \rho)^{-1} \partial \rho - \rho (1 - \kappa \rho)^{-1} \partial \kappa.
\end{align*}
\]
4.2. Global CR invariants via renormalized characteristic forms. Let $\Omega$ be an $(n+1)$-dimensional strictly pseudoconvex domain with boundary $M$. Assume that $M$ admits a pseudo-Einstein contact form, and take a Fefferman defining function $\rho$ of $\Omega$.

**Definition 4.2** ([Mar19, Theorem 1.1 and Proposition 4.9]). Let $\Phi$ be a $GL(n+1, \mathbb{C})$-invariant polynomial of degree at most $n$. Then

$$\mathcal{F}_\Phi(M) = -\sum_{m=0}^{n} \left[ \int_{\rho < -\epsilon} \frac{d \log(-\rho) \wedge d^c \log(-\rho)}{2\pi} \wedge \left( \frac{\omega_+}{2\pi} \right)^{n-m} \wedge \Phi \left( \frac{\sqrt{-1}}{2\pi} \Theta \right) \right]$$

is independent of the choice of $\rho$, and gives a global CR invariant of $M$.

In the remainder of this section, we show that any $\mathcal{F}_\Phi$ is written as a linear combination of $\mathcal{F}_\varsigma$, which appears in the introduction. It is known that $(\text{ch}_k)_{k=1}^{n+1}$ generate the algebra of $GL(n+1, \mathbb{C})$-invariant polynomials and are algebraically independent over $\mathbb{C}$. For a $GL(n+1, \mathbb{C})$-invariant polynomial $\Phi$ of degree at most $n$, there exists a $GL(n+1, \mathbb{C})$-invariant polynomial $\tilde{\Phi}$ of degree at most $n-1$ and a family $(C_\varsigma^\Phi)_{\varsigma \in \text{Part}(n)}$ of complex numbers such that

$$\Phi = \text{ch}_1 \tilde{\Phi} + \sum_{\varsigma \in \text{Part}(n)} C_\varsigma^\Phi \Phi_\varsigma,$$

where $\Phi_\varsigma$ is defined by (1.1). Note that $\tilde{\Phi}$ and $C_\varsigma^\Phi$ are unique.

**Lemma 4.3.** For a $GL(n+1, \mathbb{C})$-invariant polynomial $\Phi$ of degree at most $n$,

$$\mathcal{F}_\Phi = \sum_{\varsigma \in \text{Part}(n)} C_\varsigma^\Phi \mathcal{F}_\varsigma.$$

In particular, $\mathcal{F}_\Phi$ is trivial if $\Phi \equiv 0$ modulo $\text{ch}_1$.

**Proof.** It suffices to show that $\mathcal{F}_\Phi$ is trivial if $\Phi$ is of the form $\text{ch}_1 \tilde{\Phi}$. (4.1) implies that

$$\text{ch}_1(\Theta) = O(\rho^{n+1}) \mod dp.$$

Hence, for $1 \leq m \leq n$,

$$\int_{\rho < -\epsilon} \frac{d \log(-\rho) \wedge d^c \log(-\rho)}{2\pi} \wedge \left( \frac{\omega_+}{2\pi} \right)^{n-m} \wedge \Phi \left( \frac{\sqrt{-1}}{2\pi} \Theta \right) = O(1).$$

Therefore the integral

$$\int_{\rho < -\epsilon} \frac{d \log(-\rho) \wedge d^c \log(-\rho)}{2\pi} \wedge \left( \frac{\omega_+}{2\pi} \right)^{n-m} \wedge \Phi \left( \frac{\sqrt{-1}}{2\pi} \Theta \right)$$

has a finite limit as $\epsilon \to +0$. On the other hand, since $\Omega$ is of complex dimension $n+1$, the differential form

$$\frac{d \log(-\rho) \wedge d^c \log(-\rho)}{2\pi} \wedge \left( \frac{\omega_+}{2\pi} \right)^{n} \wedge \Phi \left( \frac{\sqrt{-1}}{2\pi} \Theta \right)$$

is identically zero. Therefore $\mathcal{F}_\Phi = 0$. □
5. Global CR invariants for Sasakian $\eta$-Einstein manifolds

Let $(S, T^{1,0}S, \eta)$ be a $(2n + 1)$-dimensional Sasakian $\eta$-Einstein manifold with Einstein constant $(n + 1)\lambda$. Then

$$\rho = \begin{cases} \lambda^{-1}(\nu^2 - 1) & \lambda \neq 0, \\ \log r^2 & \lambda = 0, \end{cases}$$

is a Fefferman defining function of $\{ r < 1 \}$ in $C(S)$ [Tak18, Proposition 3.1]. Note that

$$dp = (1 + \lambda p)d\log r^2, \quad d^c p = (1 + \lambda p)\eta.$$

Let $(\eta, \theta^\alpha, \theta^\beta)$ be an admissible coframe on $S$. Then

$$dd^c p = \sqrt{-1}(1 + \lambda p)\omega_{\alpha\beta}(\theta^\alpha \wedge \theta^\beta) + \lambda(1 + \lambda p)^{-1} dp \wedge d^c p.$$

In particular,

$$\tilde{\omega}_{\alpha\beta} = (1 + \lambda p)\omega_{\alpha\beta}, \quad \kappa = \lambda(1 + \lambda p)^{-1}.$$

We compute the Graham-Lee connection with respect to $\rho$. (2.1) and (2.2) yield that

$$d\theta^\beta = \theta^\alpha \wedge \omega_{\alpha\beta} = \theta^\alpha \wedge \left(\omega_{\alpha\beta} + \frac{1}{2}\lambda(d\log r^2)\delta_{\alpha\beta}\right) + \frac{1}{2}\lambda(1 + \lambda p)^{-1} dp \wedge \theta^\beta,$$

$$d\bar{\rho} = \lambda(1 + \lambda p)d\log r^2 \cdot \omega_{\alpha\beta} + (1 + \lambda p)d\rho_{\alpha\beta}$$

$$= \left(\omega_{\alpha\beta} + \frac{1}{2}\lambda(d\log r^2)\delta_{\alpha\beta}\right)\bar{\omega}_{\beta\gamma} + \bar{\omega}_{\alpha\beta}\left(\omega_{\alpha\gamma} + \frac{1}{2}\lambda(d\log r^2)\delta_{\alpha\gamma}\right).$$

Hence the uniqueness of the Graham-Lee connection implies

$$\tilde{\omega}_{\alpha\beta} = \omega_{\alpha\beta} + \frac{1}{2}\lambda(d\log r^2)\delta_{\alpha\beta}, \quad \bar{\omega}_{\alpha\beta} = 0.$$

**Lemma 5.1.** Let $(S, T^{1,0}S, \eta)$ be a $(2n + 1)$-dimensional Sasakian $\eta$-Einstein manifold with Einstein constant $(n + 1)\lambda$. For a Fefferman defining function $\rho$ given by (5.1), the renormalized connection and curvature satisfy

$$\theta_{\alpha\beta} = \omega_{\alpha\beta} + \lambda(\partial\log r^2)\delta_{\alpha\beta}, \quad \theta_{\infty\beta} = \lambda(1 + \lambda p)^{-1}\theta^\beta,$$

$$\theta_{\alpha\infty} = -(1 + \lambda p)\omega_{\alpha\gamma}\theta^\gamma, \quad \theta_{\infty\infty} = -\lambda\theta^\gamma \log r^2,$$

$$\Theta_{\alpha\beta} = \Omega_{\alpha\beta}^\beta + \sqrt{-1}\Delta d\eta \cdot \delta_{\alpha\beta} - \lambda\omega_{\alpha\gamma}\theta^\gamma \wedge \theta^\beta = S_{\alpha\beta}^{} \theta^\rho \wedge \theta^\sigma,$$

$$\Theta_{\alpha\infty} = 0, \quad \Theta_{\alpha\infty} = 0, \quad \Theta_{\infty\infty} = 0.$$

**Proof.** The equalities for $\theta_{\alpha\beta}^\rho$ are consequences of Lemma 4.1 and (5.2) and (5.3). The curvature form $\Theta$ is defined by

$$\Theta_{\alpha\beta} = d\theta_{\alpha\beta}^\rho - \theta_{\alpha\gamma} \wedge \theta_{\beta\gamma}^\rho.$$
Similarly, (2.1) yields
\[
\Theta_\infty = -\lambda^2(1 + \lambda \rho)^{-1}(d \log r^2) \wedge \theta^\beta + \lambda(1 + \lambda \rho)^{-1}d \theta^\beta
- \lambda(1 + \lambda \rho)^{-1} \theta^\beta \wedge (\omega_{\gamma}^\beta + \lambda(\partial \log r^2)\delta_{\gamma}^\beta)
+ \lambda^2(1 + \lambda \rho)^{-1}(\partial \log r^2) \wedge \theta^\beta
= 0.
\]
By using both (2.1) and (2.2), we have
\[
\Theta_\lambda = -\lambda(1 + \lambda \rho)\lambda_{\alpha\sigma}(d \log r^2) \wedge \theta^\beta - (1 + \lambda \rho)\lambda_{\alpha\sigma} d \theta^\beta
+ (1 + \lambda \rho)\lambda_{\alpha\sigma}(\omega_{\alpha}^\beta + \lambda(\partial \log r^2)\delta_{\alpha}^\beta) \wedge \theta^\beta
- \lambda(1 + \lambda \rho)\lambda_{\alpha\sigma}\theta^\beta \wedge (\partial \log r^2)
= 0.
\]
Finally,
\[
\Theta_\infty = -\lambda \delta \partial \log r^2 + \lambda \lambda_{\rho\sigma}\theta^\beta \wedge \theta^\gamma = 0.
\]
These complete the proof. \(\square\)

**Proof of Theorem 1.1.** From the definition of \(\mathcal{J}_B\) and Lemma 5.1, it follows that
\[
\mathcal{J}_B(S) = -\sum_{m=0}^n \int_{\mathbb{C}} \frac{(1 + \lambda \rho)^{n-m-1}}{(\lambda \rho-n)^{n-m+2}} dp \wedge \left(\frac{d \lambda}{2 \pi}\right)^{n-m} \wedge \Phi\left(\frac{\sqrt{\lambda}}{2 \pi} \Theta\right)
= -\sum_{m=0}^n \left(\int_{\mathbb{C}} \frac{(1 + \lambda \rho)^{n-m-1}}{(\lambda \rho-n)^{n-m+2}} dp \right) \int_{S} \left(\frac{d \lambda}{2 \pi}\right)^{n-m} \wedge \Phi\left(\frac{\sqrt{\lambda}}{2 \pi} \Xi\right)
= \sum_{m=0}^n \int_{S} \left(\frac{\lambda}{2 \pi}\eta\right)^{n-m} \wedge \left(-\frac{\lambda}{2 \pi} d \eta\right)^{n-m} \wedge \Phi\left(\frac{\sqrt{\lambda}}{2 \pi} \Xi\right).
\]
This gives the desired conclusion. \(\square\)

Let \((L, h)\) be a Hermitian line bundle over an \(n\)-dimensional complex manifold \(Y\) such that \(\omega = -\sqrt{-1} \Theta h\) defines a Kähler-Einstein metric on \(Y\) with Einstein constant \((n+1)\lambda\). Consider the circle bundle \((S, T^{1,0}S, \eta)\) associated with \((Y, L, h)\). The Chern tensor \(S_\alpha^\beta_\sigma\) coincides with the Bochner tensor \(B_\alpha^\beta_\sigma\) of \((Y, \omega)\). Define \(\operatorname{End}(T^{1,0}Y)\)-valued two-forms \(B\) and \(K\) by
\[
B_\alpha^\beta_\sigma = B_\alpha^\beta_\sigma dz^\sigma \wedge d \bar{z}^\sigma, \quad K_\alpha^\beta = K_\alpha^\beta = B_\alpha^\beta_\sigma - B_\alpha^\beta = \lambda(\delta_{\alpha}^\beta g_{\sigma\rho} + g_{\alpha\sigma} \delta_{\rho}^\beta) dz^\sigma \wedge d \bar{z}^\sigma.
\]

**Lemma 5.2.**
\[
\left[\chi_h \left(\frac{\sqrt{1}}{2 \pi} B\right)\right] = \sum_{j=0}^k \frac{1}{(k-j)!} (\lambda c_1(L))^{k-j} \chi_j(T^{1,0}Y \oplus \mathbb{C}).
\]

**Proof.** The two-forms \(\Pi\) and \(K\) satisfy the following equations:

\[
\text{tr} \Pi = -\sqrt{-1}(n+1)\lambda \omega, \quad \text{tr} K = -\sqrt{-1}(n+1)\lambda \omega,
\]
\[
\Pi \wedge K = K \wedge \Pi = -\sqrt{-1}\lambda \omega \wedge \Pi, \quad K \wedge K = -\sqrt{-1}\lambda \omega \wedge K.
\]
Hence

\[ \left( \frac{-1}{2\pi} B \right)^k = \sum_{j=1}^{k} \binom{k}{j} \left( \frac{-1}{2\pi} k \right)^{k-j} \wedge \left( \frac{-1}{2\pi} \Pi \right)^j + \left( \frac{-1}{2\pi} K \right)^k \]

and

\[ \text{ch}_k \left( \frac{-1}{2\pi} B \right) = \sum_{j=1}^{k} \frac{1}{(k-j)!} \left( -\frac{\lambda}{2\pi} \omega \right)^{k-j} \wedge \frac{1}{j!} \text{tr} \left( \frac{-1}{2\pi} \Pi \right)^j + \frac{n+1}{k!} \left( -\frac{\lambda}{2\pi} \omega \right)^k. \]

Therefore we have

\[ \left[ \text{ch}_k \left( \frac{-1}{2\pi} B \right) \right] = \sum_{j=0}^{k} \frac{1}{(k-j)!} \left( \lambda c_1(L) \right)^{k-j} \text{ch}_j(T^{1,0} Y) + \frac{n+1}{k!} \left( \lambda c_1(L) \right)^k \]

which completes the proof. \( \square \)

**Remark 5.3.** Formally,

\[ \left[ \text{ch}_k \left( \frac{-1}{2\pi} B \right) \right] = \text{ch}_k((T^{1,0} Y \oplus \mathbb{C}) \otimes L^\lambda). \]

**Proof of Theorem 1.2.** In our setting, \( d\eta = p^* \omega \) and \( \Xi = p^* B \). From Theorem 1.1, we obtain

\[ I_\varsigma(S) = \int_S \left( -\frac{\lambda}{2\pi} \eta \right) \wedge \left( -\frac{\lambda}{2\pi} d\eta \right) \wedge \Phi_c \left( \frac{-1}{2\pi} \Xi \right) \]

\[ = \int_S \left( -\frac{\lambda}{2\pi} \eta \right) \wedge p^* \left[ \left( \frac{\lambda}{2\pi} \omega \right) \wedge \Phi_c \left( \frac{-1}{2\pi} B \right) \right] \]

\[ = -\lambda \int_Y \left( -\frac{\lambda}{2\pi} \omega \right) \wedge \Phi_c \left( \frac{-1}{2\pi} B \right) \]

\[ = -\lambda \int_Y (\lambda c_1(L))^{\varsigma_1} \prod_{k=2}^{n} \left[ \sum_{j=0}^{k} \frac{1}{(k-j)!} (\lambda c_1(L))^{k-j} \text{ch}_j(T^{1,0} Y \oplus \mathbb{C}) \right]^{\varsigma_0} ; \]

here, the third equality follows from integration along fibers. \( \square \)

As an application, we show the algebraic independence of \((\mathcal{F}_\varsigma)_{\varsigma \in \text{Part}(n)}\). To this end, we consider a smooth complete intersection variety \( Y_d \) of multi-degree \( d = (d_1, \ldots, d_n) \) in \( \mathbb{C}P^{2n} \). Take as \( L \) the restriction of \( \mathcal{O}(-1) \) to \( Y_d \), and denote by \( \tau \) the first Chern class of \( L^{-1} \) for simplicity. The normal bundle of \( Y_d \) is isomorphic
to $\bigoplus_{i=1}^{n} \mathcal{O}(d_i)_{|Y_d}$, and so
\[
\text{ch}_k(T^{1,0}Y_d \oplus \mathbb{C}) = -\text{ch}_k \left( \bigoplus_{i=1}^{n} \mathcal{O}(d_i)_{|Y_d} \right) + \text{ch}_k(T^{1,0}\mathbb{C}P^{2n}_{|Y_d} \oplus \mathbb{C})
\]
\[
= -\text{ch}_k \left( \bigoplus_{i=1}^{n} \mathcal{O}(d_i)_{|Y_d} \right) + \text{ch}_k(\mathcal{O}(1)^{\oplus(2n+1)}_{|Y_d})
\]
\[
= -\sum_{i=1}^{n} \text{ch}_k(\mathcal{O}(d_i)_{|Y_d}) + (2n+1) \text{ch}_k(\mathcal{O}(1)_{|Y_d})
\]
\[
= (-s_k(d) + C_k) \tau^k,
\]
where
\[
s_k(d) = \frac{1}{k!}(d_1^k + \cdots + d_n^k), \quad C_k = \frac{2n+1}{k!}.
\]
In particular,
\[
c_1(T^{1,0}Y_d) = \text{ch}_1(T^{1,0}Y_d \oplus \mathbb{C}) = (-s_1(d) + 2n+1) \tau.
\]
If $s_1(d) > 2n + 1$, then there exists a Hermitian metric $h$ on $L$ such that $\omega = -\sqrt{-1}\Theta_h$ defines a Kähler-Einstein manifold with Einstein constant $(n+1)\lambda_d = -s_1(d) + 2n+1$ by the Aubin-Yau theorem. Set
\[
\nu_k(d) = -\sum_{j=1}^{k} \frac{s_1(d)^{k-j} s_j(d)}{(n+1)^{k-j}(k-j)!} + \frac{s_1(d)^k}{(n+1)^{k-1}k!},
\]
which is a homogeneous symmetric polynomial in $d$ of degree $k$. The cohomology class
\[
\left[ \text{ch}_k \left( \frac{\sqrt{-1}}{2\pi} B \right) \right] = \sum_{j=0}^{k} \frac{1}{(k-j)!} (\lambda_d c_1(L))^{k-j} \text{ch}_j(T^{1,0}Y_d \oplus \mathbb{C})
\]
is equal to
\[
[\nu_k(d) + (\text{a symmetric polynomial in } d \text{ of degree at most } k-1)] \tau^k;
\]
this follows from (5.4). For $\zeta \in \text{Part}(n)$, define a homogeneous symmetric polynomial $p_{\zeta}(d)$ in $d$ of degree $n$ by
\[
p_{\zeta}(d) = \left( \frac{s_1(d)}{n+1} \right)^{\zeta_1} \prod_{k=2}^{n} \nu_k(d)^{\zeta_k}.
\]
Similar to the above, the cohomology class
\[
(\lambda_d c_1(L))^{\zeta_1} \prod_{k=2}^{n} \left[ \sum_{j=0}^{k} \frac{1}{(k-j)!} (\lambda_d c_1(L))^{k-j} \text{ch}_j(T^{1,0}Y_d \oplus \mathbb{C}) \right]^{\zeta_k}
\]
is of the form
\[
[p_{\zeta}(d) + (\text{a symmetric polynomial in } d \text{ of degree at most } n-1)] \tau^n.
\]
Denote by $S_d$ the circle bundle associated with $(Y_d, L, h)$. From Theorem 1.2 and $\int_{Y_d} \tau^n = d_1 \cdots d_n$, it follows that the invariant $\mathcal{K}(S_d)$ is a symmetric polynomial in $d$, and its leading term is given by
\[
\frac{d_1 \cdots d_n s_1(d)}{n+1} p_{\zeta}(d).
\]
Proof of Theorem 1.3. Suppose that there exists a non-constant polynomial \( f(x) \) such that \( f(\mathscr{I}) \) is identically zero. Let \( m(\geq 1) \) be the degree of \( f \), and \( f_m \) the leading part of \( f \), which is a non-trivial homogeneous polynomial of degree \( m \). From \( f(\mathscr{I}(S_d)) = 0 \), it follows that
\[
\left( \frac{d_1 \cdots d_n s_1(d)}{n + 1} \right)^m f_m(p_\varsigma(d)) = 0,
\]
or equivalently, \( f_m(p_\varsigma(d)) = 0 \) as a polynomial in \( d \). The polynomial \( p_\varsigma \) is written as a linear combination of \( (s_1^{\varsigma_1} \cdots s_n^{\varsigma_n})_{\varsigma \in \text{Part}(n)} \). Conversely, for any \( \varsigma' \in \text{Part}(n) \), the polynomial \( s_1^{\varsigma_1} \cdots s_n^{\varsigma_n} \) is a linear combination of \( (p_\varsigma)_{\varsigma \in \text{Part}(n)} \). Thus we obtain a non-trivial homogeneous polynomial \( g_m(x) \) of degree \( m \) such that \( g_m(s_1^{\varsigma_1} \cdots s_n^{\varsigma_n}) = 0 \) as a polynomial in \( d \). This contradicts the fact that \( s_1, \ldots, s_n \) are algebraically independent over \( \mathbb{C} \). Therefore the invariants \( (\mathscr{I})_{\varsigma \in \text{Part}(n)} \) do not satisfy any non-trivial polynomial relation with coefficient in \( \mathbb{C} \).

Theorem 1.3 gives a criterion for the triviality of \( \mathscr{I}_\Phi \).

Proof of Corollary 1.4. We have already shown that \( \mathscr{I}_\Phi \) is trivial if \( \Phi \equiv 0 \) modulo \( \text{ch}_1 \) in Lemma 4.3. Conversely, assume that \( \mathscr{I}_\Phi \) is trivial. It follows from Lemma 4.3 that
\[
\sum_{\varsigma \in \text{Part}(n)} C^\Phi_\varsigma \mathscr{I}_\varsigma = 0.
\]

Theorem 1.3 implies that the coefficients \( C^\Phi_\varsigma \) are zero, and so \( \Phi = \text{ch}_1 \tilde{\Phi} \).

6. Burns-Epstein invariant for Sasakian \( \eta \)-Einstein manifolds

Let \( \Omega \) be an \((n+1)\)-dimensional strictly pseudoconvex domain with boundary \( M \), which admits a pseudo-Einstein contact form. Fix a Fefferman defining function \( \rho \) of \( \Omega \) and consider the renormalized connection. The Burns-Epstein invariant \( \mu(M) \) is defined by
\[
\mu(M) = \frac{1}{n!} \left( \frac{\sqrt{-1}}{2\pi} \right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \int_M (\Phi_0^{(0)} - \Phi_1^{(1)}),
\]
where \( \mathfrak{S}_n \) denotes the symmetric group of order \( n \) and
\[
\begin{align*}
\Phi_0^{(0)} &= \sum_{\sigma, \tau \in \mathfrak{S}_n} \text{sgn}(\sigma \tau) \theta_\infty \wedge \theta_{\sigma(1)} \wedge \cdots \wedge \theta_{\sigma(n)} \wedge \theta_\infty \tau(n), \\
\Phi_k^{(0)} &= \sum_{\sigma, \tau \in \mathfrak{S}_n} \text{sgn}(\sigma \tau) \theta_\infty \wedge \Theta_{\sigma(1)} \tau(1) \wedge \cdots \wedge \Theta_{\sigma(k)} \tau(k) \\
&\wedge \theta_{\sigma(k+1)} \wedge \theta_\infty \tau(k+1) \wedge \cdots \wedge \theta_{\sigma(n)} \wedge \theta_\infty \tau(n) \quad (1 \leq k \leq n), \\
\Phi_0^{(1)} &= \sum_{\sigma, \tau \in \mathfrak{S}_n} \text{sgn}(\sigma \tau) \Theta_{\sigma(1)} \wedge \theta_\infty \tau(1) \wedge \theta_{\sigma(2)} \wedge \theta_\infty \tau(2) \wedge \cdots \wedge \theta_{\sigma(n)} \wedge \theta_\infty \tau(n), \\
\Phi_k^{(1)} &= \sum_{\sigma, \tau \in \mathfrak{S}_n} \text{sgn}(\sigma \tau) \Theta_{\sigma(1)} \wedge \theta_\infty \tau(1) \wedge \Theta_{\sigma(2)} \tau(2) \wedge \cdots \wedge \Theta_{\sigma(k+1)} \tau(k+1) \\
&\wedge \theta_{\sigma(k+2)} \wedge \theta_\infty \tau(k+2) \cdots \wedge \theta_{\sigma(n)} \wedge \theta_\infty \tau(n) \quad (1 \leq k \leq n - 1), \\
\Phi_n^{(1)} &= 0.
\end{align*}
\]
This \( \mu(M) \) is independent of the choice of a Fefferman defining function, and gives a global CR invariant of \( M \) [Mar16, Theorem 4.6].
In the previous section, we have given formulae of the renormalized connection and curvature for Sasakian $\eta$-Einstein manifolds. Combining those with some algebraic facts yields a proof of Theorem 1.5.

**Proof of Theorem 1.5.** By Lemma 5.1, the renormalized connection and curvature on $S$ are given by

$$\theta_\alpha^\beta = \omega_\alpha^\beta + \sqrt{-1} \lambda \eta \delta_\alpha^\beta, \quad \theta_\infty^\beta = \lambda \theta^\beta,$$

$$\Theta_\alpha^\beta = \Omega_\alpha^\beta + \sqrt{-1} \lambda d\eta \cdot \delta_\alpha^\beta - \lambda l_\alpha^\beta \theta^\beta \wedge \theta^\gamma,$$

where $\eta = \sqrt{-1} \lambda d\eta$, $\Theta_\infty^\beta = 0$, $\Theta_\alpha^\infty = 0$, $\Theta_\infty^\infty = 0$.

Since $\Theta_\infty^\alpha = 0$, the $(2n+1)$-form $\Phi_1^{(1)}$ vanishes identically for any $0 \leq k \leq n$. On the other hand,

$$\frac{1}{n!} \left( \frac{\sqrt{-1}}{2\pi} \right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \Phi_1^{(0)} = \left( \frac{\sqrt{-1}}{2\pi} \right)^{n+1} \theta_\infty^\alpha \wedge \det (\Theta_\alpha^\beta + \theta_\alpha^\infty \wedge \theta_\infty^\beta)$$

$$= \left( \frac{-\lambda}{2\pi} \eta \right) \wedge \det \left( -\frac{\lambda}{2\pi} d\eta \cdot \delta_\alpha^\beta + \frac{\sqrt{-1}}{2\pi} \Omega_\alpha^\beta \right)$$

$$= \sum_{m=0}^{n} \left( -\frac{\lambda}{2\pi} \eta \right) \wedge \left( -\frac{\lambda}{2\pi} d\eta \right)^{n-m} \wedge c_m \left( \frac{\sqrt{-1}}{2\pi} \Omega \right).$$

Hence

$$\mu(S) = \sum_{m=0}^{n} \int_S \left( -\frac{\lambda}{2\pi} \eta \right) \wedge \left( -\frac{\lambda}{2\pi} d\eta \right)^{n-m} \wedge c_m \left( \frac{\sqrt{-1}}{2\pi} \Omega \right),$$

which completes the proof. \hfill $\square$

Similar to the proof of Theorem 1.2, we obtain a formula of the Burns-Epstein invariant for the circle bundle case in terms of the Einstein constant and characteristic numbers.

**Proof of Corollary 1.6.** In this setting, integration along fibers gives that

$$\mu(S) = -\lambda \sum_{m=0}^{n} \int_Y (\frac{-\lambda}{2\pi} \omega)^{n-m} \wedge c_m (\frac{\sqrt{-1}}{2\pi} \Omega)$$

$$= -\lambda \sum_{m=0}^{n} \int_Y (\lambda c_1(L))^{n-m} c_m (T^{1,0}Y).$$

This gives the desired formula. \hfill $\square$

### 7. Concluding remarks

In this section, we propose a conjecture on the Burns-Epstein invariant and global CR invariants via renormalized characteristic forms.

In Theorems 1.1 and 1.5, we computed global CR invariants via renormalized characteristic forms and the Burns-Epstein invariant for Sasakian $\eta$-Einstein manifolds. Similar to the proof in Lemma 5.2, we have the following equality for any closed Sasakian $\eta$-Einstein manifold $(S, T^{1,0}S, \eta)$ of dimension $2n+1$:

$$\mu(S) = \sum_{\kappa \in \text{Part}(n)} C_{\kappa} \mathcal{K}_{\kappa}(S),$$
where $C_\varsigma$ is a real constant depending only on $\varsigma$. In general dimensions, it is hard to compute the coefficient $C_\varsigma$ explicitly. However, we can do it in low dimensions. When $n \leq 3$, we have

$$
\mu(S) = \begin{cases} 
-\mathcal{I}_{(1)}(S) & n = 1, \\
-\mathcal{I}_{(2,0)}(S) - \mathcal{I}_{(0,1)}(S) & n = 2, \\
-\mathcal{I}_{(3,0,0)}(S) + \mathcal{I}_{(1,1,0)}(S) + 2\mathcal{I}_{(0,0,1)}(S) & n = 3.
\end{cases}
$$

When $n = 4$, a computation shows that

$$
\mu(S) = \mathcal{I}_{(4,0,0,0)}(S) - \mathcal{I}_{(2,1,0,0)}(S) - 2\mathcal{I}_{(1,0,1,0)}(S) + \frac{1}{2}\mathcal{I}_{(0,2,0,0)}(S) - 6\mathcal{I}_{(0,0,0,1)}(S).
$$

It is natural to expect that the same equality as above holds for general closed strictly pseudoconvex CR manifolds.

**Conjecture 7.1.** Fix a positive integer $n$. There exists a family $(C_\varsigma)_{\varsigma \in \text{Part}(n)}$ of real numbers such that

$$
\mu(M) = \sum_{\varsigma \in \text{Part}(n)} C_\varsigma \mathcal{I}_\varsigma(M)
$$

for any closed strictly pseudoconvex CR manifold $(M, T^{1,0}M)$ of dimension $2n + 1$ admitting a pseudo-Einstein contact form.

Here we show that this conjecture holds in low dimensions.

**Proposition 7.2.** Conjecture 7.1 is true for $n = 1$ and 2.

**Proof.** We first note that

$$
\mathcal{I}_{(n,0,\ldots,0)}(M) = \frac{(-1)^{n+1}}{2(2\pi)^{n+1}(n!)^2} \overline{Q}(M),
$$

where $\overline{Q}(M)$ is the total $Q'$-curvature; see [Hir14, Theorem 5.6] for a proof. In dimension three, $\overline{Q}(M) = -8\pi^2 \mu(M)$ [Hir14, Theorem 6.6], and so

$$
\mu(M) = -\mathcal{I}_{(1)}(M).
$$

In dimension five, $\mathcal{I}_{(0,1)}(M)$ is equal to

$$
\mathcal{I}_{(0,1)}(M) = \frac{1}{16\pi^3} \overline{I}(M),
$$

where $\overline{I}(M)$ is the total $\overline{I}$-curvature introduced by Case and Gover [CG17, Proposition 8.8]; this follows from [Mar19, Section 5.6 and Theorem 6.6]. On the other hand, it is known that

$$
\overline{Q}(M) + 64\pi^3 \mu(M) = -4\overline{I}(M);
$$

see [HMM17, (1.11)]. Thus we have

$$
\mu(M) = \mathcal{I}_{(2,0)}(M) - \mathcal{I}_{(0,1)}(M),
$$

which completes the proof.

**Acknowledgements**

The author would like to thank Jeffrey Case and Taiji Marugame for helpful comments.
REFERENCES

[BdM75] L. Boutet de Monvel, *Intégration des équations de Cauchy-Riemann induites formelles*, Séminaire Goulaouic-Lions-Schwartz 1974–1975; équations aux dérivées partielles linéaires et non linéaires, 1975, pp. Exp. No. 9, 14.

[BE90] D. Burns and C. L. Epstein, *Characteristic numbers of bounded domains*, Acta Math. 164 (1990), no. 1-2, 29–71.

[BG08] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[CG17] J. S. Case and A. R. Gover, *The P′-operator, the Q′-curvature, and the CR tractor calculus*, 2017. arXiv:1709.08057, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[CT20] J. S. Case and Y. Takeuchi, *T′-curvatures in higher dimensions and the Hirachi conjecture*, 2020. arXiv:2003.08201.

[CY13] J. S. Case and P. Yang, *A Paneitz-type operator for CR pluriharmonic functions*, Bull. Inst. Math. Acad. Sin. (N.S.) 8 (2013), no. 3, 285–322.

[Fe74] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. 26 (1974), 1–65.

[GLe88] C. R. Graham and J. M. Lee, *Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains*, Duke Math. J. 57 (1988), no. 3, 697–720.

[Hir14] K. Hirachi, *Q-prime curvature on CR manifolds*, Differential Geom. Appl. 33 (2014), no. suppl., 213–245.

[Hir93] K. Hirachi, *Scalar pseudo-Hermitian invariants and the Szegő kernel on three-dimensional CR manifolds*, Complex geometry (Osaka, 1990), 1993, pp. 67–76.

[HL75] F. R. Harvey and H. B. Lawson Jr., *On boundaries of complex analytic varieties. I*, Ann. of Math. (2) 102 (1975), no. 2, 223–290.

[HMM17] K. Hirachi, T. Marugame, and Y. Matsumoto, *Variation of total Q-prime curvature on CR manifolds*, Adv. Math. 306 (2017), 1333–1376.

[HPT08] P. D. Hislop, P. A. Perry, and S.-H. Tang, *CR-invariants and the scattering operator for complex manifolds with boundary*, Anal. PDE 1 (2008), no. 2, 197–227.

[Lee88] J. M. Lee, *Pseudo-Einstein structures on CR manifolds*, Amer. J. Math. 110 (1988), no. 1, 157–178.

[Lem95] L. Lempert, *Algebraic approximations in analytic geometry*, Invent. Math. 121 (1995), no. 2, 335–353.

[Mar16] T. Marugame, *Renormalized Chern-Gauss-Bonnet formula for complete Kähler-Einstein metrics*, Amer. J. Math. 138 (2016), no. 4, 1067–1094.

[Mar19] T. Marugame, *Renormalized characteristic forms of the Cheng–Yau metric and global CR invariants*, 2019. arXiv:1912.10684.

[Tak18] Y. Takeuchi, *Ambient constructions for Sasakian η-Einstein manifolds*, Adv. Math. 328 (2018), 82–111.

[ČG19] A. Cap and A. R. Gover, *c-projective compactification; (quasi-)Kähler metrics and CR boundaries*, Amer. J. Math. 141 (2019), no. 3, 813–856.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, 1-1 MACHIKANEYAMA-CHO, TOYONAKA, OSAKA 560-0043, JAPAN

E-mail address: yu-takeuchi@cr.math.sci.osaka-u.ac.jp, yuya.takeuchi.math@gmail.com