Condorcet’s Jury Theorem for Consensus Clustering

Brijnesh J. Jain
Technische Universität Berlin, Germany
e-mail: brijnesh.jain@gmail.com

The goal of consensus clustering is to improve the quality of clustering by combining a sample of partitions of a dataset to a single consensus partition. This contribution extends Condorcet’s Jury Theorem to the mean partition approach of consensus clustering. As a consequence of the proposed result, we challenge and reappraise the role of diversity in consensus clustering.

1. Introduction

Ensemble learning generates multiple models and combines them to a single consensus model to solve a learning problem. The assumption is that a consensus model performs better than an individual model or at least reduces the likelihood of selecting a model with inferior performance [26]. Examples of ensemble learning are classifier ensembles [6, 22, 28, 36] and cluster ensembles (consensus clustering) [12, 29, 33, 35].

The assumptions of ensemble learning follow the idea of collective wisdom that many heads are in general better than one. The idea of group intelligence applied to societies can be traced back to Aristotle and the philosophers of antiquity (see [34]) and has been recently revived by a number of publications, including James Surowiecki’s book *The Wisdom of Crowds* [30]. In his book, Surowiecki argues that not all crowds are wise, but in order to become wise, the crowd should comply to diversity of opinion and other criteria.

Though the importance of diversity for classifier ensembles has been recognized long before Surowiecki’s book [21, 22], some authors and scholars additionally employ Surowiecki’s argument on diversity in retrospect for ensemble learning [28]. Inspired by the success of ensemble classifiers, diversity has also been suggested and adopted for consensus clustering [8, 15, 23, 29, 33].

One theoretical basis for collective wisdom can be derived from Condorcet’s Jury Theorem [4]. The theorem refers to a jury of $n$ voters that need to reach a decision
by majority vote. The assumptions of the simplest version of the theorem are:

1. There are two alternatives.
2. One of both alternatives is correct.
3. Voters decide independently.
4. The probability $p$ that a decision is correct is identical for all voters.

If the voters are competent, that is $p > 0.5$, then Condorcet’s Jury Theorem states that the probability of a correct decision by majority vote tends to one as the number $n$ of voters increases.

Condorcet’s Jury Theorem has been improved in several ways, because its assumptions are rather restrictive and partly unrealistic (see e.g. [1] and references therein). Despite its limitations, the theorem in its simplest form has been employed as a theoretical basis and underlying principle of ensemble classifiers [22, 24, 28]. In contrast, such a theoretical basis is missing for consensus clustering.

In this contribution, we extend Condorcet’s Jury Theorem to consensus clustering based on the mean partition approach [5, 7, 10, 13, 25, 29, 31, 32]. The proposed result assumes that partitions are homogeneous, which appears to be in stark contrast to the widespread opinion that partitions should be diverse. In view of the extended theorem, we reappraise the role of diversity in consensus clustering.

To derive the extended version of Condorcet’s Jury Theorem, we consider partition spaces endowed with an intrinsic metric induced by the Euclidean norm [18]. We assume existence of an unknown ground-truth partition that forms a set of correct alternatives. As additional condition to independence and competence of the voter, we demand homogeneity. Homogeneity is a concept introduced in [19] and means that all sample partitions (voters) are contained in a sufficiently small ball. We model the majority vote by mean partitions, which is justified by the Mean Partition Theorem [20]. The proof of Condorcet’s Jury Theorem for consensus clustering builds upon prior work: the Mean Partition Theorem [20], consistency of the mean partition approach [18], and homogeneity of cluster ensembles [19].

The rest of this paper is structured as follows: Section 2 introduces background material. In Section 3 we present the extended version of Condorcet’s Jury Theorem and Section 4 concludes. The proof of the extended version of Condorcet’s Jury Theorem is delegated to the appendix.

2. Background

Throughout this contribution, we assume that $\mathcal{Z} = \{z_1, \ldots, z_m\}$ is a set of $m$ data points and $\mathcal{C} = \{c_1, \ldots, c_\ell\}$ is a set of $\ell$ cluster labels.

Partitions and their Representations

Partitions usually occur in two forms, in a labelled and in an unlabelled form, where labelled partitions can be regarded as representations of unlabelled partitions.
We begin with describing labelled partitions. Consider the set
\[ \mathcal{X} = \{ X \in [0,1]^{\ell \times m} : X^T 1_\ell = 1_m \}, \]
where \( 1_\ell \in \mathbb{R}^\ell \) and \( 1_m \in \mathbb{R}^m \) are vectors of all ones. The set \( \mathcal{X} \) consists of all non-negative matrices whose rows sum to one. Any matrix \( X \in \mathcal{X} \) is a labelled partition of \( Z \). The elements \( x_{kj} \) of \( X = (x_{ij}) \) describe the degree of membership of data point \( z_j \) to the cluster with label \( c_k \). The columns \( x_j \) of \( X \) summarize the membership values of the data points \( z_j \) across all \( \ell \) clusters. The rows \( x_k \) of \( X \) represent the clusters. The position \( k \) of row \( x_k \) in \( X \) assigns label \( c_k \) to the cluster represented by \( x_k \).

Next, we describe unlabelled partitions. For this, observe that the rows of a labelled partition \( X \) describe a cluster structure. Permuting the rows of \( X \) results in a labelled partition \( X' \) with the same cluster structure but with a possibly different labelling of the clusters. In clustering, the particular labelling of the clusters is usually meaningless. What matters is the abstract cluster structure represented by a labelled partition. Since there is no natural labelling of the clusters, we define the corresponding unlabelled partition as the equivalence class of labelled partitions obtained from one another by relabelling the clusters. Formally, an unlabelled partition \( X \) corresponding to a labelled partition \( X \in \mathcal{X} \) is defined by \( X = \{ PX : P \in \Pi^\ell \} \), where \( \Pi^\ell \) is the set of all \( (\ell \times \ell) \)-permutation matrices.

The definition of an unlabelled partition as an equivalence class of labelled partitions shows that every labelled partition is a representative of a labelled one. To keep the terminology simple, we briefly call \( X \) a partition, if \( X \) is an unlabelled partition. Moreover, any labelled partition \( X' \in \mathcal{X} \) is called a representation of \( X \), henceforth. By \( \mathcal{P} \) we denote the set of all (unlabelled) partitions with \( \ell \) clusters over \( m \) data points. Since some clusters may be empty, the set \( \mathcal{P} \) also contains partitions with less than \( \ell \) clusters. Thus, we consider \( \ell \leq m \) as the maximum number of clusters we encounter. Finally, the map
\[ \pi : \mathcal{X} \rightarrow \mathcal{P}, \quad X \mapsto \pi(X) = X \]
is the natural projection that sends labelled partitions to their corresponding unlabelled partitions. In other words, \( \pi \) sends matrices to partitions they represent.

Though we are only interested in unlabelled partitions, we still need labelled partitions for two reasons: (1) Computers can not easily and efficiently cope with unlabelled partitions unless the clusters carry labels in terms of number or names. (2) Using labelled partitions considerably simplifies derivation of theoretical results.

**Intrinsic Metric**

We endow the set \( \mathcal{P} \) of partitions with an intrinsic metric \( \delta \) related to the Euclidean distance such that \( (\mathcal{P}, \delta) \) becomes a geodesic space. The Euclidean norm for matrices \( X \in \mathcal{X} \) is defined by
\[ \| X \| = \left( \sum_{k=1}^\ell \sum_{j=1}^m |x_{kj}|^2 \right)^{1/2}. \]
The Euclidean norm induces a distance function on $\mathcal{P}$ defined by

$$\delta : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}, \quad (X, Y) \mapsto \min \left\{ \|X - Y\| : X \in X, Y \in Y \right\}.$$ 

Then the pair $(\mathcal{P}, \delta)$ is a geodesic metric space \cite{18}, Theorem 2.1.

**Representations in Optimal Position**

Suppose that $X$ and $Y$ are two partitions. Then

$$\delta(X, Y) \leq \|X - Y\|$$

for all representations $X \in X$ and $Y \in Y$. For some pairs of representations $X' \in X$ and $Y' \in Y$ equality holds in Eq. (1). In this case, we say that representations $X'$ and $Y'$ are in optimal position. Note that pairs of representations in optimal position are not uniquely determined.

### 3. Condorcet’s Jury Theorem

This section derives Condorcet’s Jury Theorem for consensus clustering.

#### 3.1. Preliminaries

A hard (crisp) partition $X \in \mathcal{P}$ is a partition whose matrix representations take only binary membership values from $\{0, 1\}$. Thus, the columns of representations of hard partitions are standard basis vectors from $\mathbb{R}^\ell$. By $\mathcal{P}^+$ we denote the subset of all hard partitions. Throughout this section, we assume:

1. There is an unknown ground-truth partition $X_* \in \mathcal{P}^+$.
2. The ground-truth is a hard partition.
3. We choose an arbitrary but fixed representation $X_* \in X_*$. 

#### 3.2. Fréchet Functions

This section introduces the consensus functions of the mean partition approach and presents the Mean Partition Theorem. The Mean Partition Theorem states necessary conditions of optimality and is pivotal for deriving further interesting theoretical results \cite{20}, including the extended version of Condorcet’s Jury Theorem for consensus clustering.

To introduce consensus functions, we use the terminology of Fréchet functions \cite{11} from mathematical statistics \cite{2}. Let $(\mathcal{P}, \delta)$ be a partition space endowed with the metric $\delta$ induced by the Euclidean norm. We assume that $Q$ is a probability distribution on $\mathcal{P}$ with support $\mathcal{S}_Q$\footnote{The support of $Q$ is the smallest closed subset $\mathcal{S}_Q \subseteq \mathcal{P}$ such that $Q(\mathcal{S}_Q) = 1.$}. Suppose that $\mathcal{S}_n = (X_1, X_2, \ldots, X_n)$ is a sample of $n$
partitions $X_i \in S_Q$ drawn i.i.d. from the probability distribution $Q$. Then the Fréchet function of $S_n$ is of the form

$$F_n : \mathcal{P} \rightarrow \mathbb{R}, \quad Z \mapsto \frac{1}{n} \sum_{i=1}^{n} \delta(X_i, Z)^2.$$ 

A mean partition of sample $S_n$ is any partition $M \in \mathcal{P}$ satisfying

$$F_n(M) = \min_{X \in \mathcal{P}} F_n(X).$$

Note that a mean partition needs not to be a member of the support. In addition, a mean partition exists but is not unique, in general [18].

The Mean Partition Theorem states that any representation $M$ of a local minimum $M$ of $F_n$ is the standard mean of sample representations in optimal position with $M$.

**Theorem 3.1 (Mean Partition Theorem).** Let $S_n = (X_1, \ldots, X_n) \in \mathcal{P}^n$ be a sample of $n$ partitions. Suppose that $M \in \mathcal{P}$ is a local minimum of the Fréchet function $F_n(Z)$ of $S_n$. Then every representation $M$ of $M$ is of the form

$$M = \frac{1}{n} \sum_{i=1}^{n} X_i,$$  

(2)

where the $X_i \in X_i$ are representations in optimal position with $M$.

The Mean Partition Theorem is a special case of the same theorem on the mean of a sample of graphs [17]. Any partition can be regarded as an attributed graph without edges. Nodes represent clusters and node attributes describe the membership values of the data points. For consensus clustering, Dimitiradou et al. in [5] showed that Eq. (2) is a necessary condition of optimality. They did not explicitly stress the (perhaps obvious) property that the representations $X_i$ of the sample partitions $X_i$ are in optimal position with $M$. This property is however important for gaining further theoretical insight. We refer to [20] for examples and a proof of Theorem 3.1.

From a statistical point of view, the purpose of a mean partition is to estimate an expected partition. An expected partition of probability distribution $Q$ is any partition $M_Q \in \mathcal{P}$ that minimizes the expected Fréchet function

$$F_Q : \mathcal{P} \rightarrow \mathbb{R}, \quad Z \mapsto \int_{\mathcal{P}} \delta(X, Z)^2 dQ(X).$$

As for the sample Fréchet function $F_n$, the minimum of the expected Fréchet function $F_Q$ exists but but is not unique, in general [18].

### 3.3. Voting

We want to model the vote of a partition $X \in \mathcal{P}$ on a given data point $z \in Z$. The vote of $X$ on $z$ has two possible outcomes: The vote is correct if $X$ agrees on $z$ with the
ground-truth $X_*$, and the vote is wrong otherwise. To model the vote of a partition, we need to specify what we mean by agreeing on a data-point with the ground-truth. An agreement function of representation $X$ of $X$ is a function of the form

$$k_X : Z \to [0, 1], \quad z_j \mapsto \langle x_{j}, x^*_j \rangle$$

where $x_{j}$ and $x^*_j$ are the $j$-th columns of the representations $X$ and $X_*$, respectively. The function $k_X$ measures how strongly representation $X$ agrees with the ground-truth $X_*$ on data point $z_j$. Recall that the ground-truth $X_*$ is a hard partition by assumption. If $X$ is also a hard partition, then $k_X(z) = 1$ if $z$ occurs in the same cluster of $X$ and $X_*$, and $k_X(z) = 0$ otherwise.

The vote of representation $X$ of $X$ on data point $z$ is defined by

$$V_X(z) = \mathbb{1}\{k_X(z) > 0.5\},$$

where $\mathbb{1}\{b\}$ is the indicator function that gives 1 if the boolean expression $b$ is true, and 0 otherwise. Observe that $k_X = V_X$ for hard partitions $X \in \mathcal{P}^+$. Based on the vote of a representation we can define the vote of a partition. The vote of partition is a Bernoulli distributed random variable. We randomly select a representation $X$ of partition $X$ in optimal position with $X_*$. Then the vote $V_X(z)$ of $X$ on data point $z$ is $V_X(z)$. The probability of a correct vote on $z$ is given by

$$p_X(z) = \mathbb{P}(V_X(z) = 1).$$

3.4. Majority Vote

The setting of Condorcet’s Jury Theorem is as follows: Consider a sample of $n$ hard partitions $S_n = (X_1, \ldots, X_n)$ drawn i.i.d. from a cluster ensemble. Each of the sample partitions $X_i \in \mathcal{P}^+$ has a vote $V_i(z)$ on data point $z \in Z$ with probability $p_i(z)$ of being correct. The goal is to reach a final decision on $z$ by majority vote.

We define a majority vote $V_n(z)$ of sample $S_n$ on $z$ as follows: We randomly select a mean partition $M$ of $S_n$ and then set the majority vote $V_n(z)$ on $z$ to the vote $V_M(z)$ of the chosen mean partition $M$.

It remains to show that the vote $V_M(z)$ of any mean partition $M$ of $S_n$ is indeed a majority vote. To see this, we invoke the Mean Partition Theorem. Any representation $M$ of mean partition $M$ is of the form

$$M = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

where $X_i \in X_i$ are representations in optimal position with $M$. For a given data point $z_j \in Z$, the mean membership values are given by

$$m_j = \frac{1}{n} \sum_{i=1}^{n} x^{(i)}_{j},$$

\footnote{Recall that a mean partition is not unique in general.}
where \( \mathbf{x}^{(i)}_j \) denotes the \( j \)-th column of representation \( \mathbf{X}_i \). Since the columns of \( \mathbf{x}^{(i)}_j \) are standard basis vectors, the elements \( \mathbf{m}_{kj} \) of the \( j \)-th column \( \mathbf{m}_j \) contain the relative frequencies with which data point \( z_j \) occurs in cluster \( c_k \). Then the vote \( V_M(z_j) \) is correct if and only if the agreement function of \( M \) satisfies
\[
k_M(z_j) = \langle \mathbf{m}_j, \mathbf{x}^*_j \rangle > 0.5.
\]
This in turn implies that there is a majority \( m_{kj} > 0.5 \) for some cluster \( c_k \), because \( X_* \) is a hard partition by assumption.

### 3.5. Condorcet’s Jury Theorem

Roughly, Condorcet’s Jury Theorem states that the majority vote tends to be correct when the individual voters are independent and competent. In consensus clustering, the majority vote is based on mean partitions. Individual sample partitions \( \mathbf{X}_i \) are competent on data point \( z \in \mathcal{Z} \) if the probability of a correct vote on \( z \) is given by \( p_i(z) > 0.5 \). In the spirit of Condorcet’s Jury Theorem, we want to show that the probability \( \mathbb{P}(h_n(z) = 1) \) of the majority vote \( h_n(z) \) tends to one with increasing sample size \( n \).

In general, mean partitions are neither unique nor converge to a unique expected partition. This in turn may result in a non-convergent sequence \( (h_n(z))_{n \in \mathbb{N}} \) of majority votes for a given data points \( z \). In this case, it is not possible to establish almost sure convergence of the Condorcet’s Jury Theorem. To cope with this problem, we demand that the sample partitions are all contained in a sufficiently small ball, called asymmetry ball. The asymmetry ball \( A_Z \) of partition \( Z \in \mathcal{P} \) is the subset of the form
\[
A_Z = \{ \mathbf{X} \in \mathcal{P} : \delta(\mathbf{X}, Z) \leq \alpha_Z / 4 \},
\]
where \( \alpha_Z \) is the degree of asymmetry of \( Z \)
\[
\alpha_Z = \min \{ \| \mathbf{Z} - \mathbf{PZ} \| : \mathbf{Z} \in \mathcal{Z} \text{ and } \mathbf{P} \in \Pi \setminus \{I\} \}.
\]
A partition \( Z \) is asymmetric if \( \alpha_Z > 0 \). If \( \alpha_Z = 0 \) the partition \( Z \) is called symmetric. Any partition whose representations have mutually distinct rows is an asymmetric partition. Conversely, a partition is symmetric if it has a representation with at least two identical rows. We refer to [19] for more details on asymmetric partitions.

By \( A_Z^o \) we denote the largest open subset of \( A_Z \). We say, a probability distribution \( Q \) is homogeneous if its support \( \mathcal{S}_Q \) is contained in \( A_Z^o \) form some asymmetric partition \( Z \). A sample \( \mathcal{S}_n \) is said to be homogeneous if the sample partitions of \( \mathcal{S}_n \) are drawn from a homogeneous distribution \( Q \).

To extend Condorcet’s Jury Theorem to consensus clustering, we additionally demand that the underlying probability distribution is homogeneous. In this case, the mean partition almost surely converges to an expected partition. The assertion of Condorcet’s Jury Theorem becomes valid when the limit partition corresponds to the unknown ground-truth partition.

**Theorem 3.2 (Condorcet’s Jury Theorem).** Let
1. $X_\ast \in \mathcal{X}$ be a representation of the ground-truth $X_\ast \in \mathcal{P}^+$

2. $Q$ be a probability measure on $\mathcal{P}^+$ with support $\mathcal{S}_Q$

Suppose the following assumptions hold:

1. There is an asymmetric partition $Z \in \mathcal{P}$ such that $X_\ast \in \mathcal{A}_Z^+$ and $\mathcal{S}_Q \subseteq \mathcal{A}_Z^+$.

2. Hard partitions $X_1, \ldots, X_n \in \mathcal{P}^+$ are drawn i.i.d. according to $Q$.

3. For a given $z \in Z$, the probability $p_z = p_X(z)$ is constant for all $X \in \mathcal{S}_Q$.

Then

$$
\lim_{n \to \infty} \mathbb{P}(V_n(z) = 1) = \begin{cases} 
1 & : p_z > 0.5 \\
0 & : p_z < 0.5 \\
0.5 & : p_z = 0.5
\end{cases} \quad (3)
$$

for all $z \in Z$. If $p_z > 0.5$ for all $z \in Z$, then we have

$$
\lim_{n \to \infty} \mathbb{P}\left(\delta(M_n, X_\ast) = 0\right) = 1, \quad (4)
$$

where $(M_n)_{n \in \mathbb{N}}$ is a sequence of mean partitions.

Equation (3) corresponds to Condorcet’s Jury Theorem for majority vote on a single data point and Equation (4) shows that the sequence of mean partitions converges almost surely to the ground-truth partition. Observe that almost sure convergence in Equation (4) also holds when the probabilities $p_z$ differ for different data points $z \in Z$.

From the proof of Condorcet’s Jury Theorem follows that the ground-truth partition $X_\ast$ is an expected partition almost surely and therefore takes the form as described in the Expected Partition Theorem [20].

We conclude with a few remarks about the practical relevance of Condorcet’s Jury Theorem. As mentioned in the introduction, the assumptions of the original theorem are restrictive and unrealistic in practice. As a consequence, the simple version of Condorcet’s Jury Theorem has been generalized in several ways [1]. We claim that most generalizations carry over to Theorem 3.2 because they relax conditions such as independence that do not interfere with the assumptions specifically imposed on consensus clustering.

The assumption that the sample partitions are sufficiently close to the ground-truth partition is a strong one and usually difficult to verify. The assumption that samples are homogeneous is an additional restriction specific for consensus clustering. As shown in [19], homogeneity of a sample is neither exceptional nor pathological, but may rather occur in real-world scenarios. Moreover, homogeneity can be partly enforced by increasing the number of data points. In a practical setting, we can test whether a sample of partitions is homogeneous using a procedure proposed by [19]. The test is one-sided in the sense that a positive result confirms the assumption, whereas nothing can be said in case of a negative result.
3.6. The Role of Diversity

There is widespread acceptance to demand diversity in consensus clustering in order to improve performance [8, 15, 23, 29, 33]. We challenge this conviction in the context of Condorcet’s Jury Theorem and reappraise the potential of diversity for improving performance.

Following [23], we consider two general approaches to measure diversity of a sample $S_n = (X_1, \ldots, X_n)$. The first diversity measure is of the form

$$\nabla_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta(X_i, X_j),$$

where $\Delta$ is a dissimilarity function between partitions. The measure $\nabla_n$ quantifies diversity by the average pairwise distance. The second diversity measure is the Fréchet variation

$$\sigma_n = \frac{1}{n} \sum_{i=1}^{n} \Delta(X_i, M),$$

where $M$ is a mean partition that minimizes the Fréchet function $F_n$ corresponding to the dissimilarity function $\Delta$. Observe that the Fréchet variation $\sigma_n$ is well-defined though a mean partition that minimizes $F_n$ needs not to be unique.

We discuss the special case that the dissimilarity $\Delta$ corresponds to the squared intrinsic metric $\delta^2$. In this case, the average pairwise distance $\nabla_n$ and the Fréchet variation $\sigma_n$ are related. To show this, we introduce the notion of optimal multiple alignment. A multiple alignment of sample $S_n$ is an $n$-tuple $X = (X_1, \ldots, X_n)$ consisting of representations $X_i \in X_i$. An optimal multiple alignment of $S_n$ is any multiple alignment that minimizes the function

$$f_n(X) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \|X_i - X_j\|^2.$$

The problem of minimizing $f_n$ is equivalent to the problem of minimizing the Fréchet function $F_n$ of sample $S_n$ (see Appendix A.3, Theorem A.3). In addition, we have the following chain of inequalities

$$\sigma_n \leq \nabla_n \leq f_n^*,$$

where $f_n^*$ denotes the minimum value of $f_n$. If a sample $S_n$ is diverse with respect to $\nabla_n$, then $S_n$ is also diverse with respect to $\sigma_n$ and vice versa. To see this, assume that $\nabla_n$ is large, then $f_n^*$ is large due to the second inequality in Equation (5). By equivalence of minimizing the functions $f_n$ and $F_n$, we find that a large value $f_n^*$ yields a large value $\sigma_n$. Conversely, if $\sigma_n$ is large, then $\nabla_n$ is large due to the first inequality in Equation (5). This shows that both diversity measures $\nabla_n$ and $\sigma_n$ are related. Therefore, it is sufficient to consider one of both measures. We continue the treatment on diversity with the Fréchet variation $\sigma_n$. 

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With the Fréchet variation $\sigma_n$ as our diversity measure at hand, we present two arguments against large diversity in the context of Condorcet’s Jury Theorem.

1. The first argument asks to bound diversity. Suppose that $M$ is a mean partition of $S_n$. Then we have

$$\sigma_n = F_n(M) \leq \frac{1}{n} \sum_{i=1}^{n} \delta(X_i, X_*)^2 = F_n(X_*),$$

where $F_n(X_*)$ is the Fréchet variation of sample $S_n$ with respect to the ground truth partition $X_*$. Equation (6) implies that increasing diversity $\sigma_n$ reduces the voting competence of sample partitions $X_i$ with large squared distance from the ground-truth $X_*$. This observation suggests to bound diversity $\sigma_n$ in order to prevent incompetence of the sample partitions.

2. The second argument asks for uniqueness of the expected partition. We assume that the partitions of sample $S_n$ are diverse with respect to $\sigma_n$. Then a large Fréchet variation $\sigma_n$ suggests that the sample $S_n$ is not homogeneous. Without homogeneity, we have no guarantee that the mean and expected partition are unique. In addition, a sequence of mean partitions may not converge to a limit partition. This in turn would defy a theoretical basis for diversity in consensus clustering according to the principle of Condorcet’s Jury Theorem.

Though diversity resists a theoretical basis it can still be useful in a practical setting. To see this, we evoke the consistency results presented in [18]. Let $F_n$ denote the set of mean partitions of sample $S_n$ and let $F_Q$ be the set of expected partitions of probability distribution $Q$. Homogeneity entails that the set $F_Q$ of expected partitions consists of a singleton. Diversity means that the set $F_Q$ is a closed compact set. Then the mean partition set $F_n$ almost surely converges to the expected partition set $F_Q$. Thus, the larger the set $F_Q$, the more likely it is that $F_Q$ contains the ground-truth partition. An informed decision based on the diversity of the sample could result in a mean partition with improved quality. An open issue is how informed decisions can be attained. Figuratively speaking, homogeneity refers to the situation, where we try to catch an invisible fly with a needle. In contrast, diversity refers to the situation, where we try to catch an invisible fly with a (compact) swatter and then point to a spot on the swatter, where we assume the location of a possible hit. Our hope is that a swatter encodes more information about the possible location of the fly than a needle.

We conclude with a final note on diversity with respect to arbitrary dissimilarity functions $\Delta$. In this case, the relationship between the diversity measures $\sigma_n$ and $\nabla_n$ is unclear. In addition, conditions of uniqueness for expected partitions are unknown. The general result is that the expected partition is not unique and the mean partition set converges almost surely to an expected partition set for continuous dissimilarities [18]. Then the same argument applies as for the special case $\Delta = \delta^2$, namely a sequence of mean partitions that does not converge to a limit partition defies a theoretical basis for diversity in the context of Condorcet’s Jury Theorem.
4. Conclusion

This contribution extends the simplest version of Condorcet's Jury Theorem to the mean partition approach of consensus clustering under the additional assumptions (i) that the partition space is a geodesic metric space endowed with a metric induced by the Euclidean distance and (ii) that all sample partitions and the ground-truth are contained in some asymmetry ball. Correct alternatives are implemented by an unknown ground-truth partition and the majority vote by mean partitions. In addition, we pointed out that there is no theoretical basis for diversity in the spirit of Condorcet's Jury Theorem if diversity entails non-uniqueness of the expected partition. The proposed result serves as a first step towards a theoretical basis of consensus clustering.

A. Preliminaries

For the sake of convenience, we represent partitions as points of some geometric space, called orbit space [18]. Orbit spaces are well explored, possess a rich geometrical structure and have a natural connection to Euclidean spaces [3, 16, 27]. We introduce concepts and results necessary to prove the extended version of Condorcet’s Jury Theorem.

A.1. Partition Spaces

The group $\Pi = \Pi^\ell$ of all $(\ell \times \ell)$-permutation matrices is a discontinuous group that acts on $\mathcal{X}$ by matrix multiplication, that is

$$\cdot : \Pi \times \mathcal{X} \rightarrow \mathcal{X}, \quad (P, X) \mapsto PX.$$  

The orbit of $X \in \mathcal{X}$ is the set $[X] = \{PX : P \in \Pi\}$. The orbit space of partitions is the quotient space $\mathcal{X}/\Pi = \{[X] \in \mathcal{X} \mid X \in \mathcal{X}\}$ obtained by the action of the permutation group $\Pi$ on the set $\mathcal{X}$. We write $\mathcal{P} = \mathcal{X}/\Pi$ to denote the partition space $X \in \mathcal{X}$ to denote an orbit $[X] \in \mathcal{X}/\Pi$. The natural projection $\pi : \mathcal{X} \rightarrow \mathcal{P}$ sends matrices $X$ to the partitions $\pi(X) = [X]$ they represent. The partition space $\mathcal{P}$ is endowed with the intrinsic metric $\delta$ defined by $\delta(X, Y) = \min \{\|X - Y\| : X \in X, Y \in Y\}$.

A.2. Dirichlet Fundamental Domains

We use the following notations: By $\overline{U}$ we denote the closure of a subset $U \subseteq \mathcal{X}$, by $\partial U$ the boundary of $U$, and by $U^\circ$ the open subset $\overline{U} \setminus \partial U$. The action of permutation $P \in \Pi$ on the subset $U \subseteq \mathcal{X}$ is the set defined by $PU = \{PX : X \in U\}$. By $\Pi^* = \Pi \setminus \{I\}$ we denote the subset of $(\ell \times \ell)$-permutation matrices without identity matrix $I$.

A subset $F$ of $\mathcal{X}$ is a fundamental set for $\Pi$ if and only if $F$ contains exactly one representation $X$ from each orbit $[X] \in \mathcal{X}/\Pi$. A fundamental domain of $\Pi$ in $\mathcal{X}$ is a closed connected set $F \subseteq \mathcal{X}$ that satisfies

1. $\mathcal{X} = \bigcup_{P \in \Pi} PF$
2. $PF^\circ \cap F^\circ = \emptyset$ for all $P \in \Pi^*$.
Proposition A.1. Let $Z$ be a representation of an asymmetric partition $Z \in \mathcal{P}$. Then

$$D_Z = \{ X \in \mathcal{X} : \|X - Z\| \leq \|X - PZ\| \text{ for all } P \in \Pi \}$$

is a fundamental domain, called Dirichlet fundamental domain of $Z$.

Proof. [27], Theorem 6.6.13. ■

Lemma A.2. Let $D_Z$ be a Dirichlet fundamental domain of representation $Z$ of an asymmetric partition $Z \in \mathcal{P}$. Suppose that $X$ and $X'$ are two different representations of a partition $X$ such that $X, X' \in D_Z$. Then $X, X' \in \partial D_Z$.

Proof. [16], Prop. 3.13 and [19], Prop. A.2. ■

A.3. Multiple Alignments

Let $S_n = (X_1, \ldots, X_n)$ be a sample of $n$ partitions $X_i \in \mathcal{P}$. A multiple alignment of $S_n$ is an $n$-tuple $\mathbf{X} = (X_1, \ldots, X_n)$ consisting of representations $X_i \in \mathcal{X}$. By

$$\mathcal{A}_n = \{ \mathbf{X} = (X_1, \ldots, X_n) : X_1 \in X_1, \ldots, X_n \in X_n \}$$

we denote the set of all multiple alignments of $S_n$. A multiple alignment $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be in optimal position with representation $Z$ of a partition $Z$, if all representations $X_i$ of $\mathbf{X}$ are in optimal position with $Z$. The mean of a multiple alignment $\mathbf{X} = (X_1, \ldots, X_n)$ is denoted by

$$M_\mathbf{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$ 

An optimal multiple alignment is a multiple alignment that minimizes the function

$$f_n(\mathbf{X}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \|X_i - X_j\|^2.$$ 

The problem of finding an optimal multiple alignment is that of finding a multiple alignment with smallest average pairwise squared distances in $\mathcal{X}$. To show equivalence between mean partitions and an optimal multiple alignments, we introduce the sets of minimizers of the respective functions $F_n$ and $f_n$:

$$\mathcal{M}(F_n) = \{ M \in \mathcal{P} : F_n(M) \leq F_n(Z) \text{ for all } Z \in \mathcal{P} \}$$

$$\mathcal{M}(f_n) = \{ \mathbf{X} \in \mathcal{A}_n : f_n(\mathbf{X}) \leq f_n(\mathbf{X}') \text{ for all } \mathbf{X}' \in \mathcal{A}_n \}$$

For a given sample $S_n$, the set $\mathcal{M}(F_n)$ is the mean partition set and $\mathcal{M}(f_n)$ is the set of all optimal multiple alignments. The next result shows that any solution of $F_n$ is also a solution of $f_n$ and vice versa.

Theorem A.3. For any sample $S_n \in \mathcal{P}^n$, the map

$$\phi : \mathcal{M}(f_n) \rightarrow \mathcal{M}(F_n), \quad \mathbf{X} \mapsto \pi(M_\mathbf{X})$$

is surjective.

Proof. [20], Theorem 4.1. ■
B. Proof of Theorem 3.2

Parts 1–8 show the assertion of Equation (3) and Part 9 shows the assertion of Equation (4).

1. Let \( Z \) be a representation of \( Z \) in optimal position with the representation \( X^* \) of the ground-truth partition \( X^* \). By \( A_Z = \{ X \in X' : \| X - Z \| \leq \alpha_Z/4 \} \) we denote the asymmetry ball of representation \( Z \). By construction, we have \( X^* \in A_Z \).

2. Since \( \Pi \) acts discontinuously on \( X' \), there is a bijective isometry \( \phi : A_Z \to A_Z, \ X \mapsto \pi(X) \) according to [27], Theorem 13.1.1.

3. From [19], Theorem 3.1 follows that the mean partition \( M \) of \( S_n \) is unique. We show that \( M \in A_Z \). Suppose that \( X = (X_1, \ldots, X_n) \) is an optimal alignment in optimal position with \( Z \). Since \( \phi : A_Z \to A_Z \) is a bijective isometry, we have

\[
 f_n(X) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \| X_i - X_j \|^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \delta(X_i, X_j)^2
\]

showing that the multiple alignment \( X \) is optimal. From Theorem A.3 follows that \( M = M_X = \frac{1}{n} \sum_{i=1}^n X_i \) is a representation of a mean partition \( M \) of \( S_n \). Since \( A_Z \) is convex, we find that \( M \in A_Z \) and therefore \( M \in A_Z \).

4. From Part 1–3 of this proof follows that the multiple alignment \( X \) is in optimal position with \( X^* \). We show that there is no other multiple alignment of \( S_n \) with this property. Observe that \( A_Z \) is contained in the Dirichlet fundamental domain \( D_Z \) of representation \( Z \). Let \( S_Z = \phi(S_Q) \) be a representation of the support in \( A_Z \). Then by assumption, we have \( S_Z \subseteq A_Z \subseteq D_Z \) showing that \( S_Z \) lies in the interior of \( D_Z \). From the definition of a fundamental domain together with Lemma A.2 follows that \( X \) is the unique optimal alignment in optimal position with \( X^* \).

5. With the same argumentation as in the previous part of this proof, we find that \( M \) is the unique representation of \( M \) in optimal position with \( X^* \).

6. Let \( z \in Z \) be a data point. Since \( X_i \in X_i \) is the unique representation in optimal position with \( X^* \), the vote of \( X_i \) on data point \( z \) is of the form \( V_{X_i}(z) = V_{X^*}(z) \) for all \( i \in \{1, \ldots, n\} \). With the same argument, we have \( V_n(z) = V_M(z) = V_M(z) \).

7. By \( x^{(i)}(z) \) we denote the column of \( X_i \) that represents \( z \). By definition, we have

\[
 p_i = \mathbb{P}(V_{X_i}(z) = 1) = \mathbb{P}(\left\langle x^{(i)}(z), x^*(z) \right\rangle > 0.5)
\]

for all \( i \in \{1, \ldots, n\} \). Since \( X_i \) and \( X^* \) are both hard partitions, we find that

\[
 \left\langle x^{(i)}(z), x^*(z) \right\rangle = \mathbb{I}\{x^{(i)}(z) = x^*(z)\},
\]

13
where \( I \) denotes the indicator function.

8. From the Mean Partition Theorem follows that

\[
m(z) = \frac{1}{n} \sum_{i=1}^{n} x^{(i)}(z)
\]

is the column of \( M \) that represents \( z \). Then the agreement of \( M \) on \( z \) is given by

\[
k_M(z) = \langle m(z), x^*(z) \rangle = \frac{1}{n} \sum_{i=1}^{n} I \left\{ x^{(i)}(z) = x^*(z) \right\}.
\]

Thus, the agreement \( k_M(z) \) counts the fraction of sample partitions \( X_i \) that correctly classify \( z \).

Let

\[
p_n = P(h_n(z) = 1) = P(k_M(z) > 0.5)
\]

denote the probability that the majority of the sample partitions \( X_i \) correctly classifies \( z \).

Since the votes of the sample partitions are assumed to be independent, we can compute \( p_n \) using the binomial distribution

\[
p_n = \sum_{r=\lfloor n/2 \rfloor + 1}^{n} \binom{n}{r} p^r (1-p)^{n-r},
\]

where \( r = \lfloor n/2 \rfloor + 1 \) and \( \lfloor a \rfloor \) is the largest integer \( b \) with \( b \leq a \). Then the assertion of Equation (3) follows from [14], Theorem 1.

9. We show the assertion of Equation (4). By assumption, the support \( S_Q \) is contained in an open subset of the asymmetry ball \( A_Z \). From [19], Theorem 3.1 follows that the expected partition \( M_Q \) of \( Q \) is unique. Then the sequence \( (M_n)_{n \in \mathbb{N}} \) converges almost surely to the expected partition \( M_Q \) according to [18], Theorem 3.1 and Theorem 3.3. From the first eight parts of the proof follows that the limit partition \( M_Q \) agrees on any data point \( z \) almost surely with the ground-truth partition \( X^* \). This shows the assertion.

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