Common Decomposition of Correlated Brownian Motions and its Financial Applications

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Abstract

In this paper, we develop a theory of common decomposition for two correlated Brownian motions, in which, by using change of time method, the correlated Brownian motions are represented by a triple of processes, \((X, Y, T)\), where \(X\) and \(Y\) are independent Brownian motions. We show the equivalent conditions for the triple being independent. We discuss the connection and difference of the common decomposition with the local correlation model. Indicated by the discussion, we propose a new method for constructing correlated Brownian motions which performs very well in simulation. For applications, we use these very general results for pricing of two-factor financial derivatives whose payoffs rely very much on the correlations of underlyings. And in addition, with the help of numerical method, we also make a discussion of the pricing deviation when substituting a constant correlation model for a general one.

1 Introduction

The correlation between assets plays an important role in finance. Whenever we meet a problem involving two stochastic factors, the correlation risk is unavoidable. The problem may be from areas of asset allocation, pairs trading, risk management and typically, multi-assets derivative’s pricing. Generally speaking, there are two methods in financial modelling to induce dependence between assets, one is by copula, the other is in SDE models by assuming a correlation structure for processes driving the model. Since Brownian motion is the most commonly used driving process stemming from Bachelier, correlation between Brownian motions is crucially important in the latter.

To formulate correlated Brownian motions, many models adopt the constant local correlation assumption, i.e., \(d[B, W]_t = \rho dt\) or conventionally, \(dB_t dW_t = \rho dt\), for Brownian motions \(B\) and \(W\) and a constant \(\rho \in \mathbb{R}\). However, more and more empirical works proved that the dependence between financial factors varies over time and depending on the economic status. There is a short review of literatures rejecting constant correlation before the 2008 crisis in Buraschi et al. [2010] and the authors themselves studied the joint correlation of stock index around the crisis. Other empirical evidences include Chiang et al. [2007] finding a significant increasing for correlations between Asian market after the crisis, Syllignakis and Kouretas [2011] and Junior and Franca [2012] getting similar results for the European and global markets, Xiong et al. [2018] for time-varying correlation between policy index and stock return in China and Balcilar et al. [2018] for dynamic correlation between oil price and inflation in South Africa.

Probable for this reason, there is a growing literature in recent years applying dynamic local correlation for financial problems. Since the value of local correlation, i.e., \(\rho\) introduced above, must be in \([-1, 1]\), these literatures
adopted various techniques to assure this. Osajima 2007 and Fernández et al. 2013 modeled \( \rho \) as a bounded deterministic function of time \( t \) for SABR model while Teng et al. 2015 adopted the same idea in geometric-Brownian-motion model and applied it to pricing Quanto. Note that in these models, \( \rho \) is dynamic but nonstochastic. For stochastic \( \rho \), Van Emmerich 2006, Langnau 2010, Teng et al. 2016 and Carr 2017 expressed \( \rho \) as a bounded function of some stochastic state processes and applied it in derivatives pricing problems. And some literatures modeled \( \rho \) directly by a bounded stochastic process. For example, bounded Jacobi process is a kind of bounded diffusion process driven by Brownian motion and was introduced to model \( \rho \) with applications in option pricing, including vanilla option (Teng et al. 2016), correlation swap (Meissner 2016), Quanto (Ma 2009) and multi-asset option (Ma 2009), and in portfolio selection and risk management (Buraschi et al. 2010). Hull et al. 2010 modeled the local correlation as a step process where each step is a beta-distributional random variable; Märkus and Kumar 2019 made a comparison of several stochastic local correlation models. Moreover, regime switching model is a well used model in finance where all the parameters, including \( \rho \), could be driven by a common continuous-time finite-state stationary Markov process, and thus provide another way to model stochastic local correlation, e.g. Zhou and Yin 2003.

The main focus of this paper is on proposing a new method for formulation and analysis of the dependence structure for general correlated Brownian motions. Dynamic correlation is widely considered in discrete time model. Engle 2002 proposed DCC-GARCH model to generate conditional correlation and Christodoulakis and Satchell 2002 used the correlated ARCH process. Tsay 2005 investigated a large number of multivariate time series and multivariate volatility models, and apply them into financial markets. Copula is a well developed method for characterizing dependency structure between random variables. Jaworski and Krzywda 2013 and Bosc 2012 apply Copula to correlated Brownian motions. Given the Copula of correlated Brownian motions, they solved local correlation from PDEs and found that the Copula of Brownian motions can not be Gumbel and Clayton Copula. Wishart process can establish stochastic covariance directly, and the local correlation obtained from covariance matrix is stochastic as well. Da Fonseca et al. 2007 discussed the Wishart process for multi-asset option pricing and found that there is a correlation leverage affect in call on max style option. Double Heston model also allows a special kind of local correlation between asset and stochastic volatility, see Costabile et al. 2012 and Christoffersen et al. 2009 for more details.

Except correlated Brownian motions, there are also other ways to construct correlated stochastic processes. Wang 2009 obtained correlated variance gamma processes by Brownian motions with constant correlation compound with time changes. Mendoza-Arriaga and Linetsky 2016 and Barndorff-Nielsen et al. 2001 describe correlated stochastic processes by independent background stochastic processes with dependent Lévy subordinators. Ballotta and Bonfiglioli 2016 proposed factor model for Lévy process, each asset is governed by a systematic component and a specific component.

In this article, we construct correlated Brownian motions by change-of-time. Change-of-time is a developed technique to construct stochastic processes (see Barndorff-Nielsen and Shiryaev 2015), and is widely applied to mathematical finance (e.g. Carr et al. 2003 and Geman et al. 2001). However, as far as we know, there are few works apply change-of-time technique into modeling correlated Brownian motions.

After constructing correlated Brownian motions, we apply our method into multi-asset option pricing by Fourier transform. Fourier transform method in option pricing is developed by Carr and Madan 1999, more recent papers studied Fourier transform method to price multi-asset options, e.g. Hurd and Zhou 2010 for spread option, Wang 2009 for rainbow options and Leentvaar and Oosterlee 2008 gave a numerical method for multi-asset options without explicit expression. Recall the main result in Chen et al. 2018 if \( \{B_n\}_{n \geq 0} \) and \( \{W_n\}_{n \geq 0} \) are two random walks with filtration
\( \{F_n\}_{n \geq 0} \) satisfy
\[
P(B_n - B_{n-1} = 1 | F_{n-1}) = P(B_n - B_{n-1} = -1 | F_{n-1}) = P(W_n - W_{n-1} = 1 | F_{n-1}) = P(W_n - W_{n-1} = -1 | F_{n-1}) = \frac{1}{2}.
\]

Then \( \{B_n\}_{n \geq 0} \) and \( \{W_n\}_{n \geq 0} \) can be decomposed as
\[
(B_n, W_n) = (X_{T_n} + Y_{S_n}, X_{T_n} - Y_{S_n}),
\]
where \( \{X_n\}_{n \geq 0}, \{Y_n\}_{n \geq 0} \) are two independent random walks, and \( T_n + S_n = n \). Moreover, if \( P(B_n - B_{n-1} = 1, W_n - W_{n-1} = 1 | F_{n-1}) = \vartheta \) is constant, then \( \{X_n\}_{n \geq 0}, \{Y_n\}_{n \geq 0} \) and \( \{T_n\}_{n \geq 0} \) are mutually independent, and \( \{T_n\}_{n \geq 0} \) is increment independent with
\[
P(T_n - T_{n-1} = 1) = 2\vartheta, P(T_n - T_{n-1} = 0) = 1 - 2\vartheta.
\]

As we all know, by Donsker’s theorem,
\[
\left( \frac{B_{|n|}}{\sqrt{n}}, \frac{W_{|n|}}{\sqrt{n}} \right) \xrightarrow{d} (\tilde{B}_t, \tilde{W}_t),
\]
\[
\left( \frac{X_{|n|}}{\sqrt{n}}, \frac{Y_{|n|}}{\sqrt{n}} \right) \xrightarrow{d} (\tilde{X}_t, \tilde{Y}_t),
\]
where \( (\tilde{B}_t, \tilde{W}_t) \) are two Brownian motions with constant correlation coefficient \( 4\vartheta - 1 \), \( (\tilde{X}_t, \tilde{Y}_t) \) are two independent Brownian motions. According to strong law of large numbers, \( \frac{T_{|n|}}{n} \xrightarrow{a.s.} 2\vartheta, \frac{S_{|n|}}{n} \xrightarrow{a.s.} t - 2\vartheta \). Hence, by Billingsley [1968] Section 14, Lemma
\[
\left( \frac{X_{|n|}}{\sqrt{n}}, \frac{Y_{|n|}}{\sqrt{n}} \right) \xrightarrow{d} (\tilde{X}_{2\vartheta t}, \tilde{Y}_{(1-2\vartheta)t}).
\]

From
\[
\lim_{n \to \infty} \left( \frac{B_{|n|}}{\sqrt{n}}, \frac{W_{|n|}}{\sqrt{n}} \right) = \lim_{n \to \infty} \left( \frac{X_{|n|}}{\sqrt{n}}, \frac{Y_{|n|}}{\sqrt{n}} \right) = \left( \frac{X_{2\vartheta t}}{\sqrt{2\vartheta t}}, \frac{Y_{(1-2\vartheta)t}}{\sqrt{2\vartheta t}} \right),
\]
we know that
\[
(\tilde{B}_t, \tilde{W}_t) \overset{d}{=} (\tilde{X}_{2\vartheta t} + \tilde{Y}_{(1-2\vartheta)t}, \tilde{X}_{2\vartheta t} - \tilde{Y}_{(1-2\vartheta)t}).
\]

Since we have proved two Brownian motions with constant correlation coefficient have the decomposition similar with Chen et al. [2018] in the sense of distribution, in the rest of this article, we consider the general two correlated Brownian motions. Furthermore, we decompose Brownian motions in the sense of almost surely.

From another view, every step have the same size in random walk, so we can say common or counter movements and put the common (counter) movements together. However, size of two Brownian motions movements may very different in a very short time period. Thus, we need to develop a new kind of common and counter movements.

### 2 Common Decomposition of Two Correlated Brownian Motions

In this section, we consider the dependency structure of two correlated Brownian motions. In Section 2.1 we propose the definition of *common decomposition* of two correlated Brownian motions and give some notations. In Section 2.2 we investigate the independency property of stochastic processes obtained from the common decomposition. In Section 2.3 we study the connection of the common decomposition and *local correlation* of two correlated Brownian motions.
2.1 Model Setup

Throughout this section we consider, on a complete probability space \((\Omega, \mathcal{F}, P)\), two correlated Brownian motions, \(\{B_t\}_{t \geq 0}\) and \(\{W_t\}_{t \geq 0}\), with respect to the same filtration \(\mathcal{F}_t = \{\mathcal{F}_t\}_{t \geq 0}\) which is assumed to satisfy the usual conditions.

Define
\[
T_t \triangleq \frac{t + [B, W]_t}{2}, \quad S_t \triangleq \frac{t - [B, W]_t}{2},
\]
By immediate calculation, if \(s < t\) then we have
\[
-t + s = \frac{[B]_t - [W]_t + [B]_s + [W]_s}{2} \leq [B, W]_t - [B, W]_s \leq \frac{[B]_t + [W]_t - [B]_s - [W]_s}{2} = t - s,
\]
therefore
\[
0 \leq T_t - T_s \leq t - s, \quad 0 \leq S_t - S_s \leq t - s.
\]
That is to say, \(T_t\) and \(S_t\) are increasing processes with \(T_t + S_t = t\). And they are both absolutely continuous with respect to \(t\). \(T\) and \(S\) could be regarded as special “timers” that record the time with special correlation information.

Example 2.1. For example, if the correlation coefficient of \(B\) and \(W\) is a constant \(\rho\), i.e., \([B, W]_t = \rho t\), then \(T_t = \frac{1 + \rho}{2} t\) and \(S_t = \frac{1 - \rho}{2} t\). Specifically,

- when \(B\) and \(W\) are completely positive correlated, i.e., \([B, W]_t = t\), \(T_t = t\) and \(S_t = 0\);
- when \(B\) and \(W\) are completely negative correlated, i.e., \([B, W]_t = -t\), \(T_t = 0\) and \(S_t = t\);
- when \(B\) and \(W\) are independent with each other, \(T_t = S_t = \frac{t}{2}\).

Let
\[
\tau_t = \inf\{u : T_u > t\}, \quad \varsigma_t = \inf\{u : S_u > t\},
\]
by definition, \(\{\tau_t\}_{t \geq 0}\) and \(\{\varsigma_t\}_{t \geq 0}\) are time changes of filtration \(\mathcal{F}_t\). By contrast, \(T\) is a time change of \(\{\mathcal{F}_t\}_{t \geq 0}\) and \(S\) is a time change of \(\{\mathcal{F}_t\}_{t \geq 0}\).

Loosely speaking, the so-called common decomposition in this article could be given through time-changed processes \(X_t \triangleq \frac{B_t + W_t}{2}, Y_t \triangleq \frac{B_t - W_t}{2}\). By definitions of \(T, S, \tau\) and \(\varsigma\), we observe that, in most cases, \(X_{T_t} = \frac{B_t + W_t}{2}\) and \(Y_{S_t} = \frac{B_t - W_t}{2}\).

However, given \(\omega \in \Omega\), when the limit of \(T_u(\omega)\) exists and is finite, i.e., whenever
\[
T_{\infty}(\omega) \triangleq \lim_{u \to \infty} T_u(\omega)
\]
is finite, \(X_t(\omega)\) is not well-defined for \(t \geq T_{\infty}(\omega)\). For example, if \(B\) and \(W\) are completely negative correlated, then \([B, W]_t = -t\), \(T_t = 0\) for any \(t \geq 0\), and \(\tau_t = \infty\). The same happens to \(S\) and \(Y\). (Here let \(S_{\infty}(\omega) \triangleq \lim_{u \to \infty} S_u(\omega)\).)

In order to overcome this limitation, we apply the similar method as in [Revuz and Yor 2013][chapter V] to modify the definition of \(X\) and \(Y\). We assume the probability space \((\Omega, \mathcal{F}, P)\) are rich enough to support Brownian motions that are independent of known Brownian motions and \(\mathcal{F}_\infty\).

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1A time change \(C_t\) is a family \(C_t, \mathbb{C} \geq 0\), of stopping times such that the map \(s \to C_t\) are a.s. increasing and right-continuous [Revuz and Yor 2013][chapter V, Definition 1.2].
Suppose \( \{ \tilde{X}_t, \tilde{Y}_t \}_{t \geq 0} \) is a 2-dimensional Brownian motion independent from \( \mathcal{F}_\infty \). We modify the definition of \( \{ X_t \}_{t \geq 0} \) and \( \{ Y_t \}_{t \geq 0} \) as follows\(^2\):

\[
X_t \triangleq \begin{cases} \frac{B_t + W_{T_t}}{2}, & \text{if } t < T_\infty, \\ X_\infty + \tilde{X}_{T_t - T_\infty}, & \text{if } t \geq T_\infty, \end{cases} \quad Y_t \triangleq \begin{cases} \frac{B_t - W_{\tilde{T}_t}}{2}, & \text{if } t < S_\infty, \\ Y_\infty + \tilde{Y}_{\tilde{T}_t - S_\infty}, & \text{if } t \geq S_\infty. \end{cases}
\tag{3}
\]

By the definitions of \( X, Y, T \) and \( S \),

\[
B_t = X_{T_t} + Y_{S_t}, \quad W_t = X_{T_t} - Y_{S_t}.
\tag{4}
\]

Thus we obtained a representation of \((B, W)\) through the three new-defined processes \( X, Y \), and \( T \) (it always holds that \( S_t = t - T_t \)). We call this the common decomposition of \((B, W)\) and denote it by \((X, Y, T)\).

**Remark 2.2.** The choice of \((\tilde{X}, \tilde{Y})\) can only affect definition of \((X, Y)\), but has no influence on the decomposition of \( B \) and \( W \). To be more specific, for \( \forall t \geq 0 \), if \( T_t < T_\infty \), by definition, \( X_{T_t} \) does not depend on \( \tilde{X} \). If \( T_t = T_\infty \), \( X_{T_t} = X_{T_\infty} = X_\infty \), does not depend on \( \tilde{X} \), either. The same is true for \( Y \).

For the readers’ convenience, we introduce some notations here:

- \( \mathcal{F}_t^X \): natural filtration of stochastic process \( \{ X_t \}_{t \geq 0} \). It is remarkable that \( \mathcal{F}_t^T = \sigma(T_u : u \leq t) = \sigma(S_u : u \leq t) = \mathcal{F}_t^S \).
- \( A \perp B|C \): \( A \) and \( B \) are conditional independent under \( C \).

### 2.2 Properties of the Common Decomposition

In the previous section, we introduced the so called common decomposition \((X, Y, T)\) of Brownian motions \( B \) and \( W \). In this part we give some basic properties of this decomposition. Proofs can be found in the Section 6.

Our first result is to illustrate the essence of and the relationship between \( X \) and \( Y \).

**Theorem 2.3.** Given Brownian motions \( \{ B_t \}_{t \geq 0} \) and \( \{ W_t \}_{t \geq 0} \) and their decomposition \((X, Y, T)\) as in Section 2.1, we have that \( \{ X_t \}_{t \geq 0} \) (resp., \( \{ Y_t \}_{t \geq 0} \)) is a Brownian motion of the filtration \( \{ \mathcal{F}_t^X \}_{t \geq 0} \) (resp., \( \{ \mathcal{F}_t^Y \}_{t \geq 0} \)), and that \( X \) and \( Y \) are independent.

From Theorem 2.3, the common decomposition represents \( B \) (resp. \( W \)) as the sum (resp. difference) of two time-changed Brownian motions. The dependency structure of \( B \) and \( W \) is embodied in \( T \) as well as in the dependencies between it and the two new-defined Brownian motions. Hence for clarity and convenience, the independency of \( X, Y \) and \( T \) is worth studying. In the following theorem, a sufficient and necessary condition is given for mutual independency of them.

**Theorem 2.4.** Under the conditions and notations as in Theorem 2.3, the processes of decomposition triple, i.e., \( X, Y \) and \( T \), are mutually independent if and only if:

\[(C1) \quad \mathcal{F}_\infty^B \perp \mathcal{F}_\infty^T \mid \mathcal{F}_\infty^{B,W} \quad \text{and} \quad \mathcal{F}_\infty^W \perp \mathcal{F}_\infty^T \mid \mathcal{F}_\infty^{B,W}.\]

As an example to understand the condition, when \( B \) and \( W \) has a constant correlation say, \( \rho \), \( (C1) \) is satisfied since \( T_t = \frac{1+\rho}{2} t \) and \( \mathcal{F}_\infty^T \) is a trivial \( \sigma \)-algebra. More general cases will be discussed in Section 2.3 later.

\(^2\)Please note that according to Revuz and Yor \[2013\] Chapter V, Proposition 1.8], \( X_\infty \triangleq \lim_{t \to \infty} \frac{B_t + W_t}{2} \) and \( Y_\infty \triangleq \lim_{t \to \infty} \frac{B_t - W_t}{2} \) exist.
The above two theorems give a more visual interpretation of the common decomposition. During the two Brownian motions’ movings, sometimes they move as if with positive correlation and sometimes quite the contrary. These “common” or “opposite” moving times are picked out to form new “clocks” \( T_i \) or \( S_i \). And their revolutions are decomposed thereupon according to the new clocks. By Theorem 2.3 under the new clocks, they keep their Brownian-motion features and these features are independent under the two clocks. Thus dependency structures and Brownian features are decomposed either. By Theorem 2.4 if they satisfy condition (C1), their dependency information is only contained in \( T \). The decomposition is quite complete and clear. In this case, if one studied the dependency of two correlated Brownian motions, all he need is to consider precess \( T \) in common decomposition.

The following proposition gives an equivalent condition of (C1) from another aspect.

**Proposition 2.5.** Suppose the assumptions in Theorem 2.3 hold. Then condition (C1) is equivalent with the following statement.

(C2) Given two processes \( \{\phi_1^i\}_{i \geq 0} \) and \( \{\phi_2^i\}_{i \geq 0} \), which are progressively measurable with \( \{\mathcal{F}_i^T\}_{i \geq 0} \) that

\[
E \left[ \exp \left( \frac{1}{2} \int_0^t \left( \phi_1^i \right)^2 \, dT_u + \frac{1}{2} \int_0^t \left( \phi_2^i \right)^2 \, dS_u \right) \right] = E \left[ \exp \left( \frac{1}{2} \int_0^t \left( \left( \phi_1^i \right)^2 - \left( \phi_2^i \right)^2 \right) \, dT_u + \frac{1}{2} \int_0^t \left( \phi_2^i \right)^2 \, du \right) \right] < \infty, \forall t,
\]

let

\[
D_t^\phi \triangleq \exp \left( \int_0^t \phi_1^i \, dX_u + \int_0^t \phi_2^i \, dS_u - \frac{1}{2} \int_0^t \left( \phi_1^i \right)^2 \, dT_u - \frac{1}{2} \int_0^t \left( \phi_2^i \right)^2 \, dS_u \right),
\]

then \( D_t^\phi \) is a martingale and \( \frac{dQ}{dP} |_{\mathcal{F}_t} = D_t^\phi \) defines a probability measure such that

\[
(X_t^\phi, Y_t^\phi)_Q \overset{d}{=} (X_t, Y_t)_P,
\]

where \( X_t^\phi = X_{T_t} - \int_0^t \phi_1^i \, dT_u, Y_t^\phi = Y_{S_t} - \int_0^t \phi_2^i \, dS_u \).

This proposition link the independency of the decomposition triple with conditions similar to Girsanov theorem. Undoubtedly it may attract our attention to consider its connection with financial modelling.

**Example 2.6.** In financial models, the Girsanov transform is typically used to change the drift parts of diffusions that modeling the prices. Consider two drifted Brownian motions,

\[
\int_0^t \theta_1^i \, du + B_t, \int_0^t \theta_2^i \, du + W_t,
\]

where \( \theta^i, i = 1, 2 \) are bounded, progressively measurable with \( \{\mathcal{F}_i^T\}_{i \geq 0} \). According to Theorem 2.3 these two processes can be represented as

\[
\int_0^t \theta_1^i \, du + X_{T_t} + Y_{S_t}, \int_0^t \theta_2^i \, du + X_{T_t} + Y_{S_t}.
\]

Let \( \lambda \) and \( \mu \) denote the densities of \( T \) and \( S \),

\[
\lambda_t \triangleq \frac{dT_t}{dt}, \mu_t \triangleq \frac{dS_t}{dt},
\]

and suppose that \( \inf \{ t \geq 0 : \lambda_t, \mu_t \} > 0 \).

If \( (B, W) \) satisfies condition (C1), then from Proposition 2.5 the two drifted Brownian motions can be transformed to

\[
\int_0^t \theta_1^i \, du + B_t = X_t^\phi_{\lambda} + Y_t^\phi_{\mu}, \int_0^t \theta_2^i \, du + W_t = X_t^\phi_{\lambda} - Y_t^\phi_{\mu} := W_t^\phi,
\]

(6)
where \( \phi = (\phi^1, \phi^2) \) is defined as

\[
\phi^1_t = \frac{\theta^1_t + \theta^2_t}{2\lambda_t}, \quad \phi^2_t = \frac{\theta^1_t - \theta^2_t}{2\mu_t}.
\]

Under the probability \( Q \) as defined in Proposition 2.5, it is notable that by (5),

\[
(B, W)_P \overset{d}{=} (B^\phi, W^\phi)_Q,
\]

thus the drift parts vanish after change of probability measure.

Consider the common decomposition of \((B^\phi, W^\phi)\), denoted by \((X^\phi, Y^\phi, T^\phi)\). From (6),

\[
T_t = T^\phi_t.
\]

Moreover, from (5) we have

\[
(\{T_t\}_{t \geq 0})_P \overset{d}{=} (\{T_t\}_{t \geq 0})_Q.
\]

\[\text{Remark 2.7.} \] (8) and (9) reveal the invariance property of \( T \) under change of measure. From the application point of view, this implies that in financial modelling after change of numeraire, the common decomposition method is still valid. And from empirical view, we can estimate parameters from real probability measure and apply to risk neutral measure directly. For example, Ballotta and Bonfiglioli 2016 bring correlation matrix estimated from observed asset prices in option pricing model, and we give a theoretical foundation. This is quite convenient for derivatives pricing which are lack of public data.

This also shows that we can simplify two correlated Brownian motions with drifts by changing of measure, and keep the dependency structure of them.

\[\text{2.3 Connection with Local Correlation Models}\]

Let \( \rho_t = \frac{d[B, W]}{dt} \), \( \rho \) is called the local correlation process of \( B \) and \( W \).

When \( \rho \) is a constant, i.e., \( d[B, W] = \rho dt \), this is the most commonly used model for correlated Brownian motions, so common that it sometimes looks like a routine operation to add a \( \rho \) to include dependency structure in a Brownian-motion-driven model. Without doubt, when dependency is a critical element for the issues under study, the constant local correlation assumption seems neither realistic (e.g., Chiang et al. 2007) nor sufficiently precise (e.g., Driessen et al. 2013). Some researchers have noticed this problem and brought in more realistic assumptions for \( \rho \). For example, among others, Osajima 2007 and Teng et al. 2015 supposed \( \rho \) to be deterministic but time variant. Langnau 2010 modeled \( \rho \) as a function of the asset price thus is dynamic and stochastic; and some introduced a SDE to directly model a dynamic and stochastic \( \rho \), such as Ankirchner and Heyne 2012, Jaworski and Krzywda 2013, Bosc 2012, Langnau 2010, etc.

In this section, we take a new look at the common decomposition via the local correlation process. We consider the differences and connections of the common decomposition method and the local correlation model. Still, the proofs can be found in Section 6.

\[\text{2.3.1 Different Decomposing Methods for Correlated Brownian Motions}\]

Let us first recall a well used \( \rho \)-based decomposition method representing correlated Brownian motions as linear combinations of independent Brownian motions. Suppose \( \tilde{Z} \) is a Brownian motion independent of \( \mathcal{F}_\infty \) and \((\tilde{X}, \tilde{Y})\) in (3), define
\[
Z_t \triangleq \int_0^t \frac{1_{\{\rho_u \neq \pm 1\}}}{\sqrt{1 - \rho_u^2}} (dW_u - \rho_u dB_u) + \int_0^t 1_{\{\rho_u = \pm 1\}} dZ_u \tag{10}
\]

Particularly, if \(\rho_t \neq \pm 1\), a.s. \(\forall t\), then

\[
Z_t = \int_0^t \frac{1}{\sqrt{1 - \rho_u^2}} dW_u - \int_0^t \frac{\rho_u}{\sqrt{1 - \rho_u^2}} dB_u.
\]

It is not difficult to verify that \(\{Z_t\}_{t \geq 0}\) is a Brownian motion independent of \(\{B_t\}_{t \geq 0}\), and that the local correlation of \(Z\) and \(W\) is \(\sqrt{1 - \rho_t^2}\).

By definition of \(Z_t\), we have the local-correlation based decomposition of \((B, W)\),

\[
(B_t, W_t) = (B_t, \int_0^t \rho_s dB_s + \int_0^t \sqrt{1 - \rho_s^2} dZ_s). \tag{11}
\]

If we start from the right side of the equation, i.e., starting from independent Brownian motions \(B, Z\) and local correlation process \(\rho\), we have got a commonly used model for construction correlated \((B, W)\) are Brownian motions.

As a comparison, recall that the common decomposition in the current paper of \((B, W)\) is

\[
(B_t, W_t) = (X_{Ti} + Y_{Si}, X_{Ti} - Y_{Si}).
\]

Similarly, if we can start from the right side, i.e., from independent Brownian motions \(X, Y\) and time-change process \(T\), and make the construction, under some conditions, to be correlated Brownian motions, then we have got a new construction method of \((B, W)\). We will make further discussions of this in the next section.

**Remark 2.8.** The different ideas behind the two methods look clear from the above comparison: the local-correlation method model dependency of the Brownian motions from a spatial perspective while the common-decomposition method from a temporal perspective.

The next proposition give a connection between local-correlation based decomposition and common decomposition. The two method would share the same equivalent conditions when considering completely-independent decomposition.

**Proposition 2.9.** Under the conditions stated in Theorem 2.3 \(X, Y\) and \(T\) are mutually independent if and only if the following condition holds:

1. \((C3)\) \(\rho, \) \(B\) and \(Z\) in local-correlation model are mutually independent.

2.3.2 Some Discussions on \(T\)

From the setup, we can see the important role \(T\) played in the common decomposition. Since \(X\) and \(Y\) are independent, \(T\) is the one relevant to the dependency structure of \((B, W)\) in the decomposition triple (in the case of...
complete decomposition where \( X, Y \) and \( T \) are independent, \( T \) contains all the dependency information. On the other hand, if we treat \( T \) as a special timer, a “clock”, it is obvious that this clock’s movings are affected by the correlation degree of \((B, W)\). In this section, we make some more specific discussions of \( T \) via \( \rho \) to get a better understanding of the common decomposition.

First, by definitions, there is a connection between \( T \) and \( \rho \):

\[
T_t = \int_0^t \frac{1 + \rho_u}{2} du, \quad S_t = \int_0^t \frac{1 - \rho_u}{2} du,
\]

in which, \( 1 + \rho_t \) is in fact the distance between local correlation \( \rho_t \) and \(-1\), \( 1 - \rho_t \) is the distance between \( \rho_t \) and \(+1\), and the denominator 2 is the distance between \(-1\) and \(+1\). Thus the integrands could be regarded as normalizations of the deviation of \((B, W)\)'s correlation from complete correlation. Think of the case when \( \rho \) is always close to 1 and far away from \(-1\), then the “clock \( T \)” runs faster than \( S \), and it is the clock focusing on positive correlation.

Consider the readings of the two clocks, at any time \( t \), they satisfy

\[
T_t + S_t = t,
\]

\[
T_t - S_t = \int_0^t \rho_u du.
\]

That is to say, the sum of the readings represents the calendar time, while the difference of them shows the cumulated correlation of \((B, W)\) till time \( t \).

And the average correlation coefficient process which is defined as

\[
\bar{\rho}_t = \frac{1}{t} \int_0^t \rho_u du,
\]

could also be represented by \( T \) and \( S \),

\[
\bar{\rho}_t = \frac{T_t - S_t}{t} = \frac{T_t - S_t}{T_t + S_t}.
\]

The change-of-time technique did be used in stochastic correlations, Barndorff-Nielsen et al. [2001] and Mendoza-Arriaga and Linetsky [2016] describe dependent stochastic processes by independent stochastic processes with dependent Lévy subordinators. However as far as we know, there are few works model correlated Brownian motions by time-changed process.

In two-factor derivative’s pricing, when local correlation of the two factors varies stochastically over time, it is always difficult to obtain the option prices. The average correlation coefficient process, \( \bar{\rho} \), usually plays an important role under this circumstances. For example, in Ma [2009] the price of foreign equity option was approximated by the moments of \( \bar{\rho} \), in Van Emmerich [2006] and Teng et al. [2016], the price of a Quanto is determined by the Laplace transform of \( \bar{\rho} \). Note that \( \bar{\rho} \) is an integral in local correlation model, but in our method, \( \bar{\rho}_t = \frac{2T_t}{t} - 1 \). This is one reason indicating the advantage of using common-decomposition method in financial modelling. We will discuss this further in Section 4.

In the next part we use a simple example to reveal the concepts mentioned above.

**Example 2.10.** Suppose \( B \) and \( W \) are two Brownian motions with constant correlation \( \rho \in (-1, 1) \). then by local-correlation method,

\[
(B_{t}, W_{t}) = (B_{t_i}, \rho B_t + \sqrt{1 - \rho^2} Z_t).
\]
In this case, the condition in Proposition 2.9 is satisfied, thus the processes of common-decomposition triple, X, Y and T, are mutually independent. And they can be calculated accurately:

\[ T_t = \frac{1 + \rho_t}{2} t, \quad S_t = \frac{1 - \rho_t}{2} t, \]

\[ X_t = \frac{1}{2} B_{\frac{t}{\Delta t}} + \frac{1}{2} W_{\frac{t}{\Delta t}}, \quad Y_t = \frac{1}{2} B_{\frac{t}{\Delta t}} - \frac{1}{2} W_{\frac{t}{\Delta t}} \]

and the decomposition of (B, W),

\[ (B_t, W_t) = (X_{\frac{t}{\Delta t}} + Y_{\frac{t}{\Delta t}}, X_{\frac{t}{\Delta t}} - Y_{\frac{t}{\Delta t}}), \]

In this example, it shows clearly that,

1) T and S conform a decomposition of the “calendar time” in any time period. They are composed by special “time points” picked out according to the correlation structure of (B, W). They can be considered as special clocks that moves only at special time.

2) If \( \rho > 0 \), clock T runs faster than clock S, vice versa.

3) Consider the family of convex combinations of B and W, \( C = \{ \alpha B + (1 - \alpha) W | 0 \leq \alpha \leq 1 \} \). The local correlation of every two processes in \( C \) with parameters \( \alpha \) and \( \beta \) is

\[ \rho_{\alpha, \beta} = (1 - \rho)[(2\alpha - 1)\beta - \alpha] + 1. \]

If \( \alpha = \frac{1}{2}, \rho_{\alpha, \beta} = \frac{\rho + 1}{2} > 0, \forall \beta \in [0, 1] \). Otherwise, there always exists a \( \beta \in [0, 1] \) such that \( \rho_{\alpha, \beta} \leq 0 \). That is to say, \( \frac{B + W}{2} \) is the only process in \( C \) that is strictly positive correlated with any other process in \( C \). Note that this process is in fact X under clock T, thus X represents the common structures in B and W.

Remark 2.11. Actually, if the local correlation of B and W is not constant, the three descriptions for Example 2.10 remain valid. For 1) and 2), the results remain the same. For 3), we can prove

\[ \frac{1}{\Delta t} \text{Cov}(\alpha B_t + (1 - \alpha) W_t, \beta B_t + (1 - \beta) W_t) = (1 - \text{Corr}(B_t, W_t))[2\alpha - 1 \beta - \alpha] + 1, \]

where Cov and Corr denote covariance and correlation respectively. With the similar discussion, \( \frac{B + W}{2} \) is the only process in \( C \) that is strictly positive correlated with any convex combination of B, W.

2.4 Composition of T and S

The example in previous section demonstrated what the processes in common decomposition look like and how to construct the clock T when \( \rho_t \equiv \rho \in (0, 1) \). In this section, similar analysis is carried from a distributional aspect for general cases by discretizing \( \rho \). In this part, we also start with two correlated Brownian motion B and W with local correlation process \( \rho \), and all the other notations defined in previous sections are followed.

Given \( t \geq 0 \), let \( \Pi \) be a partition of \( [0, t] \):

\[ 0 = t_0 < t_1 < t_2 < \cdots < t_n = t, \]

and write \( ||\Pi|| = \max\{t_i - t_{i-1} : i = 1, \ldots, n\} \). For \( \forall \omega \in \Omega \), define

\[ A^{\Pi}(\omega) = \bigcup_{i=0}^{n-1} (t_i, t_i + \frac{1 + \rho_i(\omega)}{2} \Delta t_i). \]
Note that by the construction of $A^\Pi$, the stochastic processes $\{1_{\{u \in A^\Pi\}}\}_{0 \leq u \leq t}$ and $\{1_{\{u \notin A^\Pi\}}\}_{0 \leq u \leq t}$ are predictable. Set
\[
\tilde{X}^\Pi_s \triangleq \int_0^s 1_{\{u \in A^\Pi\}} dB_u, \quad \tilde{Y}^\Pi_s \triangleq \int_0^s 1_{\{u \notin A^\Pi\}} dB_u,
\]
i.e., $\tilde{X}^\Pi$ keeps in step with $B$ in $A^\Pi$ and stays still at other time while $Y^\Pi$ does the opposite. Let
\[
\tilde{W}^\Pi_s \triangleq \tilde{X}^\Pi_s - \tilde{Y}^\Pi_s = \int_0^s 1_{\{u \in A^\Pi\}} dB_u - \int_0^s 1_{\{u \notin A^\Pi\}} dB_u.
\]
(13)

Then $\tilde{W}^\Pi$ is a Brownian motion moving commonly with $B$ in $A^\Pi$ and oppositely in $(A^\Pi)^c$. And $X^\Pi$ and $Y^\Pi$ represent the common movements and counter movements of $B$ and $W^\Pi$.

At any time $s \leq t$, the time period $[0, s]$ is divided into two parts: the commonly-moving period $A^\Pi \cap [0, s]$ and the oppositely-moving period $(A^\Pi)^c \cap [0, s]$, whose total lengths could be calculated respectively as (suppose $t_i < s \leq t_{i+1}$)
\[
\tilde{T}^\Pi_s(\omega) \triangleq m\left([0, s] \cap A^\Pi(\omega)\right) = t_i + \sum_{k=0}^i \rho_k(\omega) \Delta t_k + m\left((t_i, s] \cap A^\Pi(\omega)\right),
\]
\[
\tilde{S}^\Pi_s(\omega) \triangleq m\left([0, s] \cap E^\Pi(\omega)\right) = t_i - \sum_{k=0}^i \rho_k(\omega) \Delta t_k + m\left((t_i, s] \cap E^\Pi(\omega)\right),
\]
where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$. Obviously,
\[
\lim_{||\Pi|| \to 0} \tilde{T}^\Pi_s = \frac{s + \int_0^s \rho_u du}{2} = T_s, \quad \lim_{||\Pi|| \to 0} \tilde{S}^\Pi_s = \frac{s - \int_0^s \rho_u du}{2} = S_s, \forall s \in [0, t].
\]
(14)

The following proposition considers the limitation of $X^\Pi$, $Y^\Pi$ and $W^\Pi$ in distribution.

**Proposition 2.12.** Suppose the assumptions in model setup and the conditions in Proposition 2.9 hold, under the former notations, for any given $0 \leq u_1 < u_2 < \cdots < u_K < \infty, 0 \leq v_1 < v_2 < \cdots < v_L < \infty$, as $||\Pi|| \to 0$, we have
\[
(B_{u_1}, B_{u_2}, \ldots, B_{u_K}, W_{v_1}, \tilde{W}_{v_1}^\Pi, \tilde{W}_{v_2}^\Pi, \ldots, \tilde{W}_{v_L}^\Pi) \overset{d}{\to} (B_{u_1}, B_{u_2}, \ldots, B_{u_K}, W_{v_1}, W_{v_2}, \ldots, W_{v_L}).
\]

Proposition 2.12 guarantees that $(B, \tilde{W}^\Pi)$ converge to $(B, W)$ in the sense of distribution as $||\Pi|| \to 0$. For simplicity, we still denote this distributional convergence of processes by “$\overset{d}{\to}$”. Thus,
\[
(B, \tilde{W}^\Pi) \overset{d}{\to} (B, W).
\]
as a consequence
\[
(\tilde{X}^\Pi, \tilde{Y}^\Pi) = \left(\frac{B + \tilde{W}^\Pi}{2}, \frac{B - \tilde{W}^\Pi}{2}\right) \overset{d}{\to} (X_T, Y_S).
\]
(15)

The convergence properties (15) and (14) reveal the connections of $X$ and $Y$ with common and counter movements of $(B, W)$ in some sense, and give an intuitive explanation for $T$ and $S$ to be considered as clocks recording positive correlation and negative correlation of $(B, W)$. 

11
3 A New Method for Construction of Correlated Brownian Motions

In the previous section, we demonstrated the common decomposition representation of two correlated Brownian motions. For any two Brownian motions $B$ and $W$, we can find a decomposition triple $(X, Y, T)$ to represent them by change of time method. While in practice, a converse problem may be also worthy of concern and research. That is, is it possible to construct two Brownian motions with desired dependency structure from two independent Brownian motions by common decomposition method? In this section we will focus on this problem.

In tradition, in order to construct correlated Brownian motions, local correlation model and copula method are both considered. Ma 2009, Ma 2009 and Teng et al. 2016 modelled local correlation by bounded Jacobi process. Teng et al. 2016 studied stochastic local correlation by an O-U process compound with hyperbolic function tanh since local correlation is bounded. Bosc 2012 and Jaworski and Krzywda 2013 discussed correlated Brownian motions by copula. In the following, we construct correlated Brownian motions by common decomposition method.

**Theorem 3.1.** Let $(X, Y)$ be a 2-dimensional standard Brownian motion and $(T_t)_{t \geq 0}, (S_t)_{t \geq 0}$ be time changes with respect to $\mathbb{F}$. If $\mathcal{F}^X_t \perp \mathcal{F}^Y_s \mid \mathcal{F}_t$, then $B_t \triangleq X_{T_t} + Y_{S_t}$ and $W_t \triangleq X_{T_t} - Y_{S_t}$ are two correlated Brownian motions with respect to $\mathcal{F}^{B,W}$ with $[B, W]_t = T_t - S_t$.

Immediately, we have a convenient way to construct correlated Brownian motions from Theorem 3.1.

**Corollary 3.2.** Suppose that $T, S$ are increasing processes satisfying $T_t + S_t = t, \forall t \geq 0$, and $X, Y$ are independent Brownian motions. If $X, Y, T$ are mutually independent, then $B_t \triangleq X_{T_t} + Y_{S_t}$ and $W_t \triangleq X_{T_t} - Y_{S_t}$ are two correlated Brownian motions with respect to $\mathcal{F}^{B,W}$ with $[B, W]_t = T_t - S_t$.

Regime switching is a commonly used model in finance, it can capture features in financial data well, for example see Schaller and Norden 1997. Also, Casarin et al. 2018 and Pelletier 2006 have considered regime switching model for correlations in discrete time. In the next example, we consider regime switching model to construct correlated Brownian motions by common decomposition method.

**Example 3.3.** (Regime switching model) Suppose $Q_t$ is a continuous time stationary Markov Chain taking value in a finite state space $\{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the unit vector. The Markov chain $Q_t$ has a stationary transition probability matrix $P(t) = (p_{ij}(t))_{n \times n}$, where

$$p_{ij}(t) = P(Q_{t+s} = e_j \mid Q_s = e_i).$$

And the homogeneous generator $A = (a_{ij})_{n \times n}$ is defined as

$$A \triangleq \lim_{t \to 0} \frac{P(t) - I}{t},$$

$I$ denotes the identity matrix. Then we have

$$\frac{dP(t)}{dt} = AP(t) = P(t)A,$$
solve the ODE we obtain

\[ P(t) = e^{At}. \]  

(16)

Let \( \omega = [\omega_1, \omega_2, \ldots, \omega_n]^T, \omega_i \in [0, 1], \forall i \) and

\[ T_t = \int_0^t \omega^T Q s ds, S_t = t - T_t = \int_0^t (1 - \omega)^T Q_s ds. \]

Obviously, \( \{ T_t \}_{t \geq 0}, \{ S_t \}_{t \geq 0} \) are increasing processes. Let \( \{ X_t \}_{t \geq 0}, \{ Y_t \}_{t \geq 0} \) be 2-dimensional standard Brownian motion independent with \( Q_t \). Then from Corollary 3.2 we have \( \{ X_{T_t} + Y_{S_t} \}_{t \geq 0} \) and \( \{ X_{T_t} - Y_{S_t} \}_{t \geq 0} \) are two correlated Brownian motions.

The common decomposition method also gives a new way to simulate correlated Brownian motions. One can simulate \( T_t \) at first, then simulate \( X_{T_t} \) and \( Y_{S_t} \) under the condition of \( T_t \). Particularly, if \( X, Y \) and \( T \) are mutually independent, the distribution of \( X_{T_t} \) and \( Y_{S_t} \) under the condition of \( T_t \) is normal distribution, this may bring advantages of our construction compared with local correlation model.

One of the most common simulation method for local correlation model is Euler-Maruyama scheme (see Kloeden and Platen 2013). Firstly, given a partition \( \Pi \) of \([0, t]\), let

\[ W_{t i}^\Pi = \int_0^{t} \rho_u^\Pi dB_u + \int_0^{t} \sqrt{1 - (\rho_u^\Pi)^2} dZ_u = \sum_{k=0}^{n-1} (\rho_{i k} \Delta B_{i k} + \sqrt{1 - \rho_{i k}^2} \Delta Z_{i k}), \]

(17)

where \( \Delta B_{i k} = B_{i k+1} - B_{i k}, \Delta Z_{i k} = Z_{i k+1} - Z_{i k} \) and \( \{ \rho_{i u}^\Pi \}_{0 \leq u \leq t} \) is defined as

\[ \rho_u^\Pi = \rho_{i u}, \ i \leq u < i + 1. \]

Secondly, we simulate \((B, W)\) by (17). Thus, the simulation result is \((B, W^\Pi)\) eventually, and there will be a simulation error.

Under the condition \( X, Y \) and \( T \) are mutually independent, Table 1 and Table 2 show the specific steps of simulation of common decomposition method when we do not have the explicit expression of \( T \)’s distribution and comparing with the Euler-Maruyama scheme of local correlation model. The common decomposition of \((B, W^\Pi)\) is denoted as \((X^\Pi, Y^\Pi, T^\Pi)\). From Table 1 the advantages of common decomposition method in simulation include:

- If we only need \( B_t \) and \( W_t \) at time \( t \), and the trajectory is not necessary, common decomposition method can reduce the time of simulations. If \( \rho_{i} \) is a stochastic process, we need to simulate \( 3n \) random numbers, i.e. \( \Delta B_{i u}, \Delta Z_{i u}, \rho_{i u}, \ i = 0, 1, \ldots, n - 1. \) However, in common decomposition method we only need to simulate \( n + 2 \) random numbers, i.e. \( T^\Pi, T^\Pi_{1/2}, \ldots, T^\Pi_n, X^\Pi_{T^\Pi} \) and \( Y^\Pi_{S^\Pi} \). Moreover, if we have the explicit expression of \( T \)’s distribution, we can simulate \( T_t \) directly, then we only need to simulate \( X_{T_t} \) and \( Y_{S_t} \), hence simulation can be reduced to 3 times.

Remark 3.4. The simulation error can be controlled as long as the simulation error of \( T^\Pi_{T^\Pi} \) can be controlled, since

\[ E|X_{T_t} - X_{T^\Pi_t}|^2 = E|T_t - T^\Pi_t| \leq (E|T_t - T^\Pi_t|^2)^{1/2}. \]

Therefore, if the explicit expression of \( T \)’s distribution is obtained, one can simulate \( T \) directly, and then simulate \( X_T \) and \( Y_S \) with the similar steps in Table 3 and Table 4. There is no simulation error for \( T \), hence we can simulate \((B, W)\) accurately while this is impossible for local correlation model. If the explicit expression of \( T \)’s distribution is unobtained, the simulation error of two methods is same, because both the simulation result of two methods is \((B, W^\Pi)\).
From Table 2 if the trajectory is needed, and we do not have the explicit expression of $T$’s distribution, there is no much difference in simulation for two methods.

| Common decomposition method | Local correlation model (Euler-Maruyama scheme) |
|----------------------------|-----------------------------------------------|
| Simulate $T^{ii}_1, T^{ii}_2, \ldots, T^{ii}_{tn}$ one by one | Simulate $\rho_{i0}, \rho_{i1}, \ldots, \rho_{it-1}$ one by one |
| Simulate $X^{ii}_1$ and $Y^{ii}_1, Y^{ii}_2, \ldots, Y^{ii}_{tn}$ | Simulate $\Delta B_t, \Delta B_{t1}, \ldots, \Delta B_{tn-1}$ and $\Delta Z_{t0}, \Delta Z_{t1}, \ldots, \Delta Z_{tn-1}$ |
| Calculate ($B_t, W^{ii}$) | Calculate ($B_t, W^{ii}$) |

Table 1: Simulate ($B_t, W_t$) (Explicit expression of $T$’s distribution is unobtained)

| Common decomposition method | Local correlation model (Euler-Maruyama scheme) |
|----------------------------|-----------------------------------------------|
| Simulate $T^{ii}_1, T^{ii}_2, \ldots, T^{ii}_{tn}$ one by one | Simulate $\rho_{i0}, \rho_{i1}, \ldots, \rho_{it-1}$ one by one |
| Simulate $\Delta X^{ii}_1, \Delta X^{ii}_2, \ldots, \Delta X^{ii}_{tn-1}$ and $\Delta Y^{ii}_1, \Delta Y^{ii}_2, \ldots, \Delta Y^{ii}_{tn-1}$ | Simulate $\Delta B_t, \Delta B_{t1}, \ldots, \Delta B_{tn-1}$ and $\Delta Z_{t0}, \Delta Z_{t1}, \ldots, \Delta Z_{tn-1}$ |
| Calculate $B_{t1}, B_{t2}, \ldots, B_{tn}$ and $W^{ii}_1, W^{ii}_2, \ldots, W^{ii}_{tn}$ | Calculate $B_{t1}, B_{t2}, \ldots, B_{tn}$ and $W^{ii}_1, W^{ii}_2, \ldots, W^{ii}_{tn}$ |

1 According to $T^{ii}_1 = \frac{t+\sum_{i=2}^{n} \rho_i \Delta t_i}{2}$, simulate $T^{ii}_0, T^{ii}_1, \ldots, T^{ii}_{tn}$ is equivalent with simulate $\rho_{i0}, \rho_{i1}, \ldots, \rho_{it-1}$.

2 Under the condition of $T^{ii}_1, T^{ii}_2, \ldots, T^{ii}_{tn-1}$, $\Delta X^{ii}_0, \Delta X^{ii}_1, \ldots, \Delta X^{ii}_{tn-1}$ and $\Delta Y^{ii}_0, \Delta Y^{ii}_1, \ldots, \Delta Y^{ii}_{tn-1}$ are independent normal distributions with mean zero and variance $\Delta T^{ii}_0, \Delta T^{ii}_1, \ldots, \Delta T^{ii}_{tn-1}, \Delta S^{ii}_0, \Delta S^{ii}_1, \ldots, \Delta S^{ii}_{tn-1}$, respectively.

Figure 1(a) Figure 1(b) Figure 1(c) display how we simulate the trajectory of ($B, W$) in $[0, t]$ through common decomposition method (explicit expression of $T$’s distribution is unobtained) step by step, parameters are taken as follow:

\[ Q_0 = [1, 0, 0]^T, \omega = [0.3, 0.6, 0.9]^T, A = \begin{bmatrix} -1 & 0.8 & 0.2 \\ 0.4 & -1 & 0.6 \\ 0.3 & 0.7 & -1 \end{bmatrix}, t = 1, \Delta t = 0.01, \forall i. \] (18)

Figure 1: Simulate ($B_t, W_t$) by Common Decomposition Method (Explicit expression of $T$’s distribution is unobtained)
In order to compare two simulation methods in practical, we consider the regime switching model in Example 3.3. Thanks to (16), simulation for regime switching model is feasible. We calculate the mean of $B_t + W_t$ by simulate $(B_t, W_t)$ with $N = 5000$ replications and take parameters same as in (18). We implement Monte Carlo methods by MATLAB with a Core i7 2.8GHZ CPU.

Obviously the mean of $B_t + W_t$ is 0 in theoretical, Table 3 shows that the simulation error of two methods are very close, because their standard deviation are truly close. And common decomposition method runs much faster than local correlation model with Euler-Maruyama scheme.

| Table 3: Comparing two simulation methods                  | $E(B_t + W_t)$ | Std Dev    | Running time |
|-----------------------------------------------------------|----------------|------------|--------------|
| Common decomposition method (explicit expression of $T$'s distribution is unobtained) | -0.0034        | $1.8593 \times 10^{-2}$ | 3.1868 seconds |
| Local correlation model (Euler-Maruyama scheme)           | 0.0265         | $1.8561 \times 10^{-2}$ | 11.4762 seconds |

4 Applications of Common Decomposition Method in Financial Derivative Pricing

In financial derivatives’ pricing, there are quite a few chances to meet with the situation of handling two stochastic factors. For example, in stochastic volatility models, the risky price and the stochastic volatility are two factors; in cross-currency derivatives, the evolution of two currencies are driven by different stochastic factors; in two-asset or multi-asset derivatives, the price movements may be modeled by two stochastic processes, etc. In these problems, modeling the stochastic factors by two Brownian motions has been a common-used method, see, among others, Heston 1993, Dai et al. 2004 and Hurd and Zhou 2010. In most situations, from a practical aspect, the two stochastic factors (hence the two Brownian motions) should be correlated to each other. And empirical researches has indicated that their dependence changed over time and depending on the market conditions, e.g., [2] for cross-currency derivatives, Engle and Sheppard 2001 for multi-asset and Benhamou et al. 2010 for stochastic volatility models.

In the previous two sections, we considered the common decomposition of two Brownian motions, whose dependence structure could be very general. In Section 2, we showed how to decompose Brownian motions $(B, W)$ to a triplet $(X, Y, T)$, and in Section 3 we answered how to construct two correlated Brownian motions from a given triplet $(X, Y, T)$. In this section, we will apply the common decomposition method to study the pricing of some typical two-factor derivatives that modeled by two correlated Brownian motions. We first give two examples showing direct usage of the decomposing triplet $(X, Y, T)$ in deriving pricing formula. And then we will focus on the pricing of two-color rainbow options, because a wide variety of contingent claims have a payoff function which includes two risky assets. There are several examples for two-color rainbow options, one is given by option-bonds, see Stulz 1982 for details; besides, a special kind of two-color rainbow options, spread options, are ubiquitous in financial markets, including equity, fixed income, foreign exchange, commodities and energy markets, Carmona and Durrleman 2003 present a overview of examples and common features of spread options.

For simplicity, we assume that $X$, $Y$ and $T$ are mutually independent in this section, i.e., $\rho$ is independent from $(B, Z)$ in the local correlation model by Proposition 2.9. This assumption is not so rigorous as to go against the

\footnote{Note that we do not need to simulate the trajectory here.}

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reality. For example, in [Ma 2009], when considering the pricing problem of foreign equity options with stochastic correlations, the author illustrated independency of $\rho$, $B$ and $Z$ from an empirical view.

4.1 Pricing of Covariance Swap and Covariance Option

Options dependent on exchange rate movements, such as those paying in a currency different from the underlying currency, have an exposure to movements of the correlation between the asset and the exchange rate. This risk may be eliminated by two ways, a straightforward approach is Quanto option and will be discussed in Section 4.2; the other approach that we focus on this section is Covariance Options or Correlation Options, see Swishchuk 2016 for more details. By combining variance and covariance options, the realised variance of return on a portfolio can be locked in. Carr and Madan 1999 illustrated that the covariance swaps can be constructed by options and futures, in other words, options can be perfectly hedged by covariance swaps and futures. In the following part, we consider the so called covariance options which is designed to cope with the covariance risks of two underlying assets.

Suppose that the prices of the two assets, $(S^1, S^2)$, can be characterized as

$$\begin{align*}
\frac{dS^1_t}{S^1_t} &= \mu_1 dt + \sigma_1 dB_t, \\
\frac{dS^2_t}{S^2_t} &= \mu_2 dt + \sigma_2 dW_t,
\end{align*}$$

where the drifts and volatilities of underlying prices are assumed to be constant.

Example 4.1 (Swap and Option on Realized Covariance of Returns). Consider two risky assets whose prices evolve as in (19). Then according to Example 2.6, $(S^1, S^2)$ could be transformed to, under proper conditions,

$$\begin{align*}
\frac{dS^1_t}{S^1_t} &= r dt + \sigma_1 d\tilde{B}_t, \\
\frac{dS^2_t}{S^2_t} &= r dt + \sigma_2 d\tilde{W}_t,
\end{align*}$$

where $\tilde{B}$ and $\tilde{W}$ are Brownian motions under the risk neutral measure $Q$, and $r$ denotes the constant risk free interest rate.

Continuously compounded rate returns of two assets are $\ln S^1_t / S^1_0$ and $\ln S^2_t / S^2_0$. Accordingly, define the realized covariance of returns of two underlying assets as the quadratic covariation of $\ln S^1_t / S^1_0$ and $\ln S^2_t / S^2_0$

$$\text{Cov}_R(S^1_t, S^2_t) = [\ln \frac{S^1_t}{S^1_0}, \ln \frac{S^2_t}{S^2_0}]_t,$$

then the payoff of covariance swap and covariance option of the underlying equity $S^1$ and $S^2$ at expiration is

$$\text{Cov}_R(S^1_t, S^2_t) - K,$$

and

$$\max\{\text{Cov}_R(S^1_t, S^2_t) - K, 0\},$$

where $K$ represents the strike price. So

$$\text{Cov}_R(S^1_t, S^2_t) = [\ln \frac{S^1_t}{S^1_0}, \ln \frac{S^2_t}{S^2_0}]_t = \int_0^T \sigma_1 \sigma_2 d[B, W]_t = \sigma_1 \sigma_2 (T_t - S_t) = \sigma_1 \sigma_2 (2T_t - t),$$

the price of covariance swap and covariance option only depend on the expectation and distribution of $T_t$. The financial derivatives correlation swap and correlation option is similar.
4.2 Pricing of Quanto Option

Quanto option is a famous cross-currency financial product trading in organized exchanges as well as in OTC. Its payoff is calculated in one currency but is settled in another currency at a fixed exchange rate. It is designed to hedge the risks of delivering foreign investments to domestic currency. Hence the correlation between the underlying price and the exchange rate plays an ultimate role in pricing. Usually, this correlation structure is modeled by two correlated Brownian motions. In Section 2, we have showed that part of the dependency of two Brownian motions could be described by $T$ in common decomposition. In the following example, we will show the essential role of $T$ in the pricing of an European-style Quanto.

**Example 4.2.** Let us consider an European-style Quanto. Suppose the price of underlying equity $S$ in foreign currency and the exchange rate $R$ are modeled, under the risk neutral probability in the domestic currency, as follows:

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dB_t, \quad dR_t = \mu_2 R_t dt + \sigma_2 R_t dW_t,$$

the payoff of a Quanto put option is

$$R_0 \max(K - S_t).$$

Under the arbitrage free assumption in domestic currency world, let $r_1, r_2$ represent the risk free interest under domestic currency and foreign currency respectively, one can get

$$R_0 = \exp(-r_1 t) E[\exp(r_2 t) R_t], \quad S_0 R_0 = \exp(-r_1 t) E[S_t R_t].$$

(20) represent the bank account in foreign currency. Note that

$$E[R_t] = R_0 \exp(\mu_2 t), \quad E[S_t R_t] = S_0 R_0 \exp\left((\mu_1 + \mu_2 - \frac{1}{2} \sigma_1^2 - \frac{1}{2} \sigma_2^2) t\right) E[\exp(\sigma_1 B_t + \sigma_2 W_t)],$$

and under the condition (C3), we have

$$E[\exp(\sigma_1 B_t + \sigma_2 W_t)] = \exp\left((\sigma_1^2 + \sigma_2^2) \frac{t}{2}\right) E\left[\exp(\sigma_1 \sigma_2 \int_0^t \rho_u du)\right].$$

After simple calculations,

$$\mu_2 = r_1 - r_2, \quad \mu_1 = r_1 - \mu_2 - \frac{1}{t} \ln E\left[\exp(\sigma_1 \sigma_2 \int_0^t \rho_u du)\right].$$

According to Van Emmerich 2006 and Teng et al. 2016, Quantos’ price is a function of $\ln E\left[\exp(\sigma_1 \sigma_2 \int_0^t \rho_u du)\right]$.

$$P_{Quanto} = R_0 \left(Ke^{-r_1 t} N(-d_2) - S_0 e^{-(r_1 t - r_2 t + \ln E(\exp(\sigma_1 \sigma_2 \int_0^t \rho_u du)))} N(-d_1)\right),$$

where

$$d_1 = \frac{\log(S_0/K) + (r_2 + \sigma_2^2/2)t - \ln E(\exp(\sigma_1 \sigma_2 \int_0^t \rho_u du))}{\sigma_1 \sqrt{t}}, \quad d_2 = d_1 - \sigma_1 \sqrt{t}.$$

Note that

$$\ln E\left[\exp(\sigma_1 \sigma_2 \int_0^t \rho_u du)\right] = \ln E\left[\exp(\sigma_1 \sigma_2 (2T_t - t))\right],$$

then Quantos’ price is actually a function of $T_t$ and determined by Laplace transform of $T_t$. 

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4.3 Pricing of 2-Color Rainbow Options

In this section, we focus on a class of multi-asset options, the 2-color rainbow option which is written on the maximum or minimum of two risky assets. This kind of option was first studied in [Margrabe 1978] and the most well known [Stulz 1982] where the author showed its extensive applications in valuing many financial instruments such as foreign currency bonds, option-bonds, risk-sharing contracts in corporate finance, secured debt, etc.

In this part we use the same asset-price models as in Section 4.1.

The payoff of a rainbow option with maturity $\tau$ may have the forms listed in Table 4 (see [Ouwehand and West 2006]). We will demonstrate that all these types of rainbow options could be valuated through a unified approach.

| Option Style       | Payoff                      |
|--------------------|-----------------------------|
| Best of assets or cash | $\max(S^1_t, S^2_t, K)$   |
| Put 2 and Call 1   | $\max(S^1_t - S^2_t, 0)$   |
| Call on max        | $\max(\max(S^1_t, S^2_t) - K, 0)$ |
| Call on min        | $\max(\min(S^1_t, S^2_t) - K, 0)$ |
| Put on max         | $\max(K - \max(S^1_t, S^2_t), 0)$ |
| Put on min         | $\max(K - \min(S^1_t, S^2_t), 0)$ |

Define a 2-dimensional process $M_t = (X_{T_t}, Y_{S_t})^\top$. Similar to the cases studied in [Carr and Wu 2004], the payoffs in Table 4 could be reformulated as

$$
(a_1 + b_1 e^{\theta_1^T M_t}) 1_{\{e_1^T M_t \leq k_1\}} 1_{\{e^T M_t \leq k\}} + (a_2 + b_2 e^{\theta_2^T M_t}) 1_{\{e_2^T M_t \leq k_2\}} 1_{\{e^T M_t \geq k\}},
$$

with some proper parameters.

For example, consider the Call-on-max option, whose payoff is $\max(\max(S^1_t, S^2_t) - K, 0)$, the parameters are (for $i = 1, 2$)

$$
a_i = -K, b_i = S^i_0 e^{(r - \frac{1}{2} \sigma_i^2) \tau}, \theta_1 = \left(\frac{\gamma_1}{\sigma_1}, \frac{\gamma_2}{\sigma_2}\right), \theta_2 = \left(\frac{\gamma_1}{\sigma_1}, \frac{\gamma_2}{\sigma_2}\right),
$$

$$
c_i = -\theta_i, k_i = -\ln \frac{K}{b_i}, c = \theta_2 - \theta_1, k = \ln \frac{b_1}{b_2}.
$$

It is easy to check that

$$
\{e_1^T M_t \leq k_1\} = \{S_1^t \geq K\}, \{e^T M_t \leq k\} = \{S_1^t \geq S_2^t\}.
$$

Now we can present a unified valuation approach for options with payoffs in Table 4 through process $M$. First, for given parameters $\gamma_1, \gamma_2 \in \mathbb{R}, \gamma_3, \gamma_4, \gamma_5 \in \mathbb{R}^2$, define an intermediate valuation function $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$
G(x_1, x_2; \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) \triangleq E^Q \left[ (\gamma_1 + \gamma_2 e^{\gamma_3^T M_t}) 1_{\{e_1^T M_t \leq x_1\}} 1_{\{e^T M_t \leq x\}} \right], 
$$

where $E^Q$ indicates the expectation under the risk-neutral measure $Q$. It is obvious that the initial price of a rainbow option could be given by $G$ as

$$
e^{-r \tau} \left[ G(k_1, k_1; a_1, b_1, \theta_1, c_1, c) + G(k_2, -k_2; a_2, b_2, \theta_2, c_2, -c) \right].
$$

The following proposition gives a general rule to calculate function $G$.
Proposition 4.3. Let $G(x_1, x)$, $(x_1, x) \in \mathbb{R}^2$ be given as in (22) and $L_t$ represents the Fourier transform of $T_t$. Define
\[ \Phi_{M_t}(z_1, z_2) = e^{-\frac{1}{2}z_1^2}L_t(\frac{1}{2}(z_1^2 - z_2^2)), \]
then the generalized Fourier transform of $G(x_1, x)$, denoted by $\hat{G}(\lambda_1, \lambda)$, is given as
\[ \hat{G}(\lambda_1, \lambda) = -\frac{\gamma_1}{\lambda_1}\Phi_{M_t}(\lambda_1 \gamma_4 + \lambda \gamma_5) - \frac{\gamma_2}{\lambda \lambda_1}\Phi_{M_t}(\lambda \gamma_4 + \lambda_1 \gamma_5 - i \gamma_3), \]
where $\text{Im} \lambda, \text{Im} \lambda_1 > 0$. In particular, if $\rho_t = \rho$ is a constant, then $L_t(z) = \exp(\frac{1+i\rho t}{2}z)$ and $\hat{G}(\lambda_1, \lambda)$ can be obtained from (24) and (25).

Given Proposition 4.3, the function $G(x_1, x; \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ could be calculated by the inversion formula and then the prices of rainbow options are obtained from (23).

Remark 4.4. For general cases where the payoffs cannot be represented as before, Proposition 4.3 no longer applies. But we can still apply the Fourier-transform method directly to pricing functionals. For given parameters $(S_0, \tau, r, \sigma_1, \sigma_2)$, rewrite the option payoffs as $V(y_1 + B_\tau, y_2 + W_\tau)$, where $y_i = (\frac{S_i}{S_0} - \frac{1}{2}) \tau, i = 1, 2$. Denote by $f(b, w)$ the joint probability density of $B_\tau$ and $W_\tau$ under $Q$, then the price of $V(y_1 + B_\tau, y_2 + W_\tau)$ is
\[ C(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(y_1 + b, y_2 + w) f(b, w) db dw. \]

According to Leentvaar and Oosterlee 2008, the Fourier transform of $C(y_1, y_2)$ is
\[ \hat{C}(\lambda_1, \lambda_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda_1 y_1 + i\lambda_2 y_2} V(y_1 + b, y_2 + w) f(b, w) db dw dy_1 dy_2 \]
\[ = F^V(\lambda_1, \lambda_2) E^Q \left[ e^{-i\lambda_1 B_\tau - i\lambda_2 W_\tau} \right] \]
where $F^V$ denotes the Fourier transform of $V$. In general, $F^V$ has no explicit expression and thus is usually be calculated numerically.

Leentvaar and Oosterlee 2008 have put forward a numerical method for the cases when the correlation coefficient of $B$ and $W$ is constant. In fact, if this is the case, $E^Q \left[ e^{-i\lambda_1 B_t - i\lambda_2 W_t} \right]$ could be calculated explicitly.
\[ E^Q \left[ e^{-i\lambda_1 B_t - i\lambda_2 W_t} \right] = \exp \left( - (\lambda_1^2 + \lambda_2^2 + 2\rho_1 \lambda_1 \lambda_2) \tau \right). \]

When the correlation coefficient of $B$ and $W$ is not constant, we can still use similar approaches as in Leentvaar and Oosterlee 2008 by means of common decomposition. Continuing to use the notions as before, we have
\[ E^Q \left[ e^{-i\lambda_1 B_t - i\lambda_2 W_t} \right] = \Phi_{M_t}(-\lambda_1 - \lambda_2, -\lambda_1 + \lambda_2) = e^{-(\lambda_1 - \lambda_2)^2 \tau L_t(-2\lambda_1 \lambda_2)}. \]
Consequently,
\[ \hat{C}(\lambda_1, \lambda_2) = F^V(\lambda_1, \lambda_2) e^{-(\lambda_1 - \lambda_2)^2 \tau L_t(-2\lambda_1 \lambda_2)}. \]
Hence when the Fourier transform of $T_t$, $L_t$, is known, the price will be obtained by inverse Fourier transform formula.

\footnote{For simplicity, we omit the parameters $\gamma_i$, $i = 1, \ldots, 5$, in the function expressions when there is no confusion.}
In the last few pages, we considered how to calculate the price of a rainbow option. Actually, following similar approach outlined in Proposition 4.3, we could give a Fourier-transform method for calculating Greeks. The next corollary set forth an example of this.

**Corollary 4.5.** Consider the Delta of $S^1$ for a Call-on-Max option listed in Table 4

\[
\Delta(S^1) = \frac{1}{S_0^2} \left( \frac{\partial G}{\partial x_1} (k_1, k; a_1, b_1, \theta_1, c_1, c) + \frac{\partial G}{\partial x} (k_1, k; a_1, b_1, \theta_1, c_1, c) - \frac{\partial G}{\partial x} (k_2, -k; a_2, b_2, \theta_2, c_2, -c) \right) \\
+ e^{r \left( -\frac{1}{2} \sigma_1^2 \right) t} \frac{\partial G}{\partial \gamma} (k_1, k; a_1, b_1, \theta_1, c_1, c) \\
:= g_1(k_1, k) + g_2(k_1, k),
\]

where

\[
g_1(k_1, k) = \left( \frac{1}{S_0^2} \left( \frac{\partial G}{\partial x_1} + \frac{\partial G}{\partial x} \right) + e^{r \left( -\frac{1}{2} \sigma_1^2 \right) t} \frac{\partial G}{\partial \gamma} \right) (k_1, k; a_1, b_1, \theta_1, c_1, c),
\]

\[
g_2(k_1, k) = \left( -\frac{1}{S_0^2} \frac{\partial G}{\partial x} \right) (k_1, k; a_1, b_1, \theta_1, c_1, c).
\]

The Fourier transform of $g_1$ has an explicit expression as

\[
\frac{ia_1}{S_0^2} \frac{1}{\lambda + 1} \Phi_{M_1}(\lambda_1 c_1 + \lambda c) + \left( \frac{ib_1}{S_0^2} \frac{1}{\lambda_1} - \frac{e^{r \left( -\frac{1}{2} \sigma_1^2 \right) t}}{\lambda_1} \right) \Phi_{M_1}(\lambda_1 c_1 + \lambda c - i \theta_1),
\]

and the expression of Fourier transform of $g_2$ is

\[-\frac{ia_2}{S_0^2} \Phi_{M_1}(\lambda_2 c_2 - \lambda c) - \frac{ib_2}{S_0^2} \Phi_{M_1}(\lambda_2 c_2 - \lambda c - i \theta_2).
\]

And Delta will be obtained by the inverse Fourier transform formula.

**Remark 4.6.** Other Greeks can be derived along the same procedures.

From the foregoing content of this section, we know that, thanks to the common decomposition method, to calculate the price and Greeks of a rainbow option, we only need to find out the Fourier transform of $T_i$. We consider some specific models of $T_i$ in the following examples to give the readers more intuitive insights.

The first example is under the regime switching model, which is widely used in financial modelling.

**Example 4.7.** Consider the regime switching model given in Example 3.3 by Lemma A.1 in Buffington and Elliott 2002, the Fourier trans form of $T_i$ is

\[
L_i(z) = e^{z T_i} = 1^\top e^{(A + z \text{diag} \omega) t} Q_0,
\]

where $\text{diag} \omega = \begin{bmatrix} \omega_1 & 0 & \ldots & 0 \\ 0 & \omega_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \omega_n \end{bmatrix}$, $A = (a_{ij})_{n \times n}$ is the generator of $Q_t$. Then by Proposition 4.3, we can get the option price from $L_i(z)$. For example, if the option style is Call-on-max, then

\[
\hat{\mathcal{G}}(\lambda_1, \lambda; a_1, b_1, \theta_1, c_1, c) = \frac{K}{\lambda_1} e^{-\frac{1}{2} \left( \lambda_1 c_1 - \lambda_2 \right)^2} 1^\top e^{(A - 2 \lambda_1 \lambda_2 c_1 c_2 \text{diag} \omega) t} Q_0 \\
- \frac{S_0^2}{\lambda_1} e^{r \left( -\frac{1}{2} \sigma_1^2 \right) t} \left( \lambda_1 c_1 + i \theta_1 \right) 1^\top e^{(A - 2 \lambda_1 \lambda_2 c_1 c_2 \text{diag} \omega) t} Q_0.
\]
In the next example, \( \{T_t\}_{t \geq 0} \) has a specific modelling through a bounded function of some stochastic processes and the Fourier transform of \( T_t \) is given by a PDE.

**Example 4.8.** Suppose that \( h \) is a bounded function with values in \([0, 1]\) and \( \nu \) is a diffusion process satisfying the following SDE

\[
d \nu_t = \mu(t, \nu_t) dt + \sigma(t, \nu_t) dZ_t,
\]

where \( \{Z_t\}_{t \geq 0} \) is a Brownian motion and \( \mu(t, x), \sigma(t, x) \) are determined functions such that the SDE have an unique solution.

Let \( T_t = \int_0^t h(\nu_s) ds \). By Feynman-Kac formula, the Laplace transform of \( T_t - T_s \) for fixed \( t \) under the condition \( \nu_s \), which is denoted by \( L(s, \nu_s; t, z) \), satisfies the following PDE:

\[
\frac{\partial L}{\partial s} + \mu(t, \nu) \frac{\partial L}{\partial \nu} + \frac{1}{2} \sigma^2(t, \nu) \frac{\partial^2 L}{\partial \nu^2} + zf(\nu)L = 0,
\]

with terminal condition \( L(t, \nu_t; t, z) = 1 \). The solution of \( (27) \) are related with Sturm-Liouville problem, see Polyanin 2002 1.8.6.5 and 1.8.9 for more details.

Particularly, Teng et al. 2016 considered \( f(x) = \frac{1 + \tanh(x)}{2} \) for modelling stochastic correlation. Ma 2009 discussed \( f(x) = \frac{1+x^2}{2} \) and \( \nu_t \) as a bounded Jacobi process

\[
d \nu_t = \kappa(\theta - \nu_t) dt + \sigma \sqrt{(h - \nu_t)(\nu_t - l)} dZ_t,
\]

the boundary for bounded Jacobi process is \([l, h]\) when

\[
\kappa(\theta - l) > \frac{1}{2} \sigma^2 (h - l), \kappa(h - \theta) > \frac{1}{2} \sigma^2 (h - l).
\]

Sometimes, there is no closed-form solution of financial derivatives, so Monte Carlo method is needed. And we have illustrated simulation steps in Section 3.

### 5 Numerical Results

In literatures that study the pricing of two-asset derivatives with models driven by two Brownian motions, \( B \) and \( W \), it is a commonly used assumption that the local correlation of \( B \) and \( W \) is a constant, i.e., \( d[B, W]_t = \rho dt \) for some \( \rho \in \mathbb{R} \). However, as we have mentioned before, this assumption is inconsistent with empirical studies. For example, based on data from different markets around the world, Chiang et al. 2007, Syllignakis and Kouretras 2011 and Junior and Franca 2012 all found that the correlation coefficients changed as time and economic situations changed. Then it is natural to ask, when the actual correlation coefficient is dynamic and stochastic, how much it would influent the pricing error if we still applied the constant-correlation model?

In this part, we consider as an example the pricing of two-color rainbow options. We investigate the difference of option prices under constant and dynamic stochastic correlations by numerical experiments and try to summarize when this difference is negligible or nonnegligible.

Regime switching model is widely considered in financial modelling, thus we apply the regime switching model in this section which has been introduced in Example 3.3 and 4.7. Suppose that the market has three different states described by a finite-state-space Markov process \( \{Q_t\}_{t \geq 0} \) with an initial value \( Q_0 \) and a transition rate matrix \( A \) that

\[
A = \begin{bmatrix}
-1 & 0.8 & 0.2 \\
0.4 & -1 & 0.6 \\
0.3 & 0.7 & -1
\end{bmatrix}.
\]
Since our concern is in correlations, we assume for simplicity that all coefficients of the underlying assets, except for the local correlation, are constants. Thus the underlying prices are assumed to satisfy (under the risk neutral probability)

\[
\frac{dS^1_t}{S^1_t} = r dt + \sigma_1 dB_t, \quad \frac{dS^2_t}{S^2_t} = r dt + \sigma_2 dW_t,
\]

where \( r = 0.05, S^1_0 = 100, S^2_0 = 120, \sigma_1 = 0.2, \sigma_2 = 0.3 \). And the local correlation process of \( B \) and \( W \),

\[ \rho_t = 2\omega^\top Q_t - 1, \]

where \( \omega \in R_+^3 \) which indicates the switching states for local correlation coefficient of log prices, \( \frac{d[\log S^1_t, \log S^2_t]}{\sigma_1^2 \sigma_2^2} := \rho_t \). For example, if \( \omega = [0.3, 0.6, 0.9]^\top \), at any time \( t \), \( \rho_t \) switches among \(-0.4, 0.2 \) and 0.8 according to the market conditions.

Consider the two-color rainbow options as in Section 4.3, note that, under the above model, if \( \rho \) is considered as a constant, the option prices can be given in closed form as in Stulz [1982]. While for the actual case with a regime-switching \( \rho \), we can apply Proposition 4.3 to derive the true prices. Following the notations in Proposition 4.3, by the inversion fourier formula, we have

\[
G(x_1, x) = \int_{-\infty}^{\infty} e^{-i\lambda_1 x} \hat{G}(\lambda_1, \lambda) d\lambda_1 d\lambda,
\]

where \( \lambda_1, \lambda \), denote the imaginary part of \( \lambda \) and \( \lambda_1 \). In the subsequent numerical experiment, we choose \( \lambda_{11} = \lambda_1 = 1 \), since \( \hat{G}(\lambda_1, \lambda) \) is well defined only for \( \lambda_1, \lambda \) with strictly positive imaginary. And we approximate (29) by

\[
G(x_1, x) \approx \sum_{j=-N_1}^{N_1} \sum_{k=-N}^{N} e^{\lambda_1 x_1 + \lambda_1 x j (j \eta_1 + k \eta)} \hat{G}(j \eta_1 + i \lambda_{11}, k \eta + i \lambda_1 \eta_1 \eta),
\]

where we set \( N_1 = N = 1000 \) and \( \eta_1 = \eta = 0.1 \).

Suppose that the contract life of the option is \( \tau = 0.25 \) and the strike is \( K = 90 \). Let \( \omega = [0.6, 0.6, 0.6]^\top \), then the regime switching model degenerates to the constant correlation model. We verified the group of parameters are accurate enough and the difference of option price obtained from Stulz [1982] and Proposition 4.3 is smaller than \( 10^{-13} \).

In the following two sections, 5.1 and 5.2, we compare the option prices induced by the constant-\( \rho \) models in Stulz [1982] to the prices given by the regime-switching-\( \rho \) models (through (23)). Since we have assumed the regime-switching case to be actual, the latter could be regarded as the “true” prices. And thus the comparison results will indicate how large the pricing error would be when we substituted a constant for the original nonconstant \( \rho \). For clarity, we make comparison in an ideal situation that the investor knows exactly the other coefficients except for \( \rho \).

In Section 5.1 we compare the two cases in a more theoretical way. We assume that the investor estimates \( \rho \) historically from the observed stock prices. The numerical results in this section show that there may be big differences between the prices. In Section 5.2, we adopt an approach more close to the practical procedure. We suppose the investor calibrate the constant correlation model to option prices he observed (which were calculated from the regime-switching model). And then the calibrated model is used for pricing. And it shows that there will be a big pricing error by using constant correlation model, especially for those options deep out of the money. This is in line with the results given in Costin et al. [2016] for CDS options.

\[ \text{In empirical, the risk free interest } r \text{ can be observed and } \sigma_1, \sigma_2 \text{ can be calibrated precisely from vanilla options.} \]
5.1 Constant and Nonconstant Correlation in Pricing Rainbow Options

In this section, we estimate a constant correlation coefficient \( \hat{\rho} \) from the historical data which are given by the regime switching model, and then calculate the option prices derived from this \( \hat{\rho} \). By comparing these option prices with those deriving directly from the regime switching model, we can get a general idea of the error we would make when applying constant correlation model in the situations where the actual correlation coefficients are dynamic and stochastic. For the robustness of the results, we consider the comparisons in 5 different cases with different vector \( \omega \)s.

Since we have assumed that all the other parameters can be obtained precisely, the investor actually could get the data of \((B, W)\) by observing prices of the underlying assets. Suppose that he has got these historical data of a long term and with a relatively high frequency as \((B_{t_i}, W_{t_i}), i = 0, 1, \ldots, n\), where \(0 = t_0 < t_1 < \ldots < t_n = t\). According to definition, the estimated constant correlation based on data till time \(t\) is

\[
\hat{\rho} \triangleq \frac{\sum_{i=0}^{n-1} \Delta B_{t_i} \Delta W_{t_i}}{t}.
\]

Note that, setting \(\Delta t = \max\{t_{i+1} - t_i|i = 0, \ldots, n\}\), we have

\[
\frac{\sum_{i=0}^{n-1} \Delta B_{t_i} \Delta W_{t_i}}{t} \xrightarrow{\Delta t \to 0} \frac{[B, W]_t}{t} = \frac{T_t - S_t}{t} = \frac{1}{t} \int_0^t (2\omega - 1)^\top Q_s ds,
\]

and according to the Ergodic Theorem of Markov processes,

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t Q_s ds = \pi,
\]

where \(\pi\) denotes the stationary distribution of Markov process \(Q_t\).

Thus, as long as we assume these data to be long-term and with a relatively high frequency, we always have

\[
\hat{\rho} \approx \frac{1}{t} \int_0^t (2\omega - 1)^\top Q_s ds \approx 2\omega^\top \pi - \boxed{1}\]  

(30)

In this case, no matter how violently the correlation coefficient switches over time, the investors may have similar estimates from long-term historical data. And thus the option prices calculated along these estimates may deviate a lot from the “true” prices. We will show these price deviations by the relative error defined as

\[
\text{Relative error} = \frac{\text{Price with constant } \hat{\rho} - \text{Price with regime switching } \rho}{\text{Price with regime switching } \rho}.
\]

(31)

In the numerical experiments, for each case, we simulate a path of \((B, W)\) to present the historical data, where we choose \(t = 20\) and \(\Delta t_i = 0.05, \forall i\). In order to make consistent comparison, we randomly choose 5 different \(\omega\), which all satisfy the condition \(2\omega^\top \pi - 1 = 0.2\). That is to say, by (30), the option prices calculated from the \(8\)We have illustrated in Remark \(27\) that it is feasible to apply directly the estimated \(\hat{\rho}\) from historical data into option pricing. \(9\)Note that the stationary distribution \(\pi\) satisfies the following equations

\[
A^\top \pi = 0, \pi 1^\top = 1,
\]

where \(A\) denotes the generator of \(Q\). In our numerical experiments, \(A = \begin{bmatrix}-1 & 0.8 & 0.2 \\ 0.4 & -1 & 0.6 \\ 0.3 & 0.7 & -1\end{bmatrix}\), and then \(\pi = [0.2636, 0.4273, 0.3091]^\top\).
estimated coefficients are similar since in all cases \( \hat{\rho} \approx 0.2 \). While on the contrary, we shall see that the prices calculated from original model are quite different from each other.

We list the numerical results in Table 5 in which the second column shows the “true” prices calculated from the original regime switching model, the third column shows the \( \hat{\rho} \) estimated from the “historical data”, the forth column shows the prices obtained by constant correlation model with \( \hat{\rho} \), while the last column shows the relative errors defined as in (31).

| \( \omega \)          | True Prices | \( \hat{\rho} \) | Prices with \( \hat{\rho} \) | Relative errors |
|----------------------|-------------|------------------|------------------------------|-----------------|
| \([0.7665, 0.7551, 0.2436]^\top\) | 37.2642     | 0.2377           | 35.2623                      | -5.37%          |
| \([0.8068, 0.8772, 0.0404]^\top\) | 38.2361     | 0.2103           | 35.3671                      | -7.50%          |
| \([0.6824, 0.6178, 0.5051]^\top\) | 35.9230     | 0.2436           | 35.2398                      | -1.90%          |
| \([0.5559, 0.4063, 0.9054]^\top\) | 33.8134     | 0.1911           | 35.4403                      | 4.81%           |
| \([0.6, 0.6, 0.6]^\top\)     | 35.4064     | 0.2177           | 35.3388                      | 0.19%           |

It is obviously from Table 5 that there may be big pricing errors when using constant correlation coefficient estimated from historical data. In this numerical example, although all the other coefficients were assumed to induce zero error, the relative errors for pricing can mount to unacceptable levels. It is almost certain that these high errors come from the substitution of \( \hat{\rho} \) for the real dynamic stochastic \( \rho \). As a verification, we considered the case of \( \omega = [0.6, 0.6, 0.6]^\top \), where the regime switching model degenerates to the constant correlation model. The results were shown in the last row of the table. We can see that there is only a small relative error, 0.19%, which presents the technical error other than substitution of constant correlations to dynamic ones.

More specifically, we can see that in all cases the estimated \( \hat{\rho} \)s are around 0.2, and thus the resulting option prices are around 35.3, while the true prices deviate from as high as 38.2 to as low as 33.8. There would be a big unexpected loss if the investor applied the constant correlation model to value these options and used these prices as a guidance of his investments.

### 5.2 Calibrating a Constant Correlation Model from Data Given by the Dynamic Correlation Model

In this section, we investigate the difference between option prices under constant correlation model and dynamic stochastic correlation model through a more practical way. First, in practice, when considering derivatives’ pricing, investors do not use coefficients estimated from historical data commonly. More often, they observe the market prices of a class of derivatives, and calibrate the theoretical model to the observed prices. In our case, the “market prices” are supposed to be given by the regime switching model, and the “theoretical model” held by investors is supposed to be the constant correlation model. And “calibration of the theoretical model” reduces to “finding the optimum \( \rho \) to fit the market prices” since this is assumed to be the only unknown parameter for the theoretical model. On the other hand, just like the idea of “implied volatility”, each observed option price can deduce an “implied correlation”, \( \rho_{\text{imp}} \). The change of \( \rho_{\text{imp}} \) with strikes can also indicate the deviation of option prices given by constant correlation model from actual prices based on dynamic correlation.

The numerical simulations are carried out along the following procedure.

First, we give the prices for options with a maturity \( \tau = 0.25 \) and strikes \( K = 80, 90, \ldots, 140 \) under regime switching model by the Fourier transform method. These will play the part of “initial market data” in our numerical experiment.
Then based on these data, we calibrate the constant correlation model to a proper $\rho$. This is done by minimizing the following cumulative square error function by Gradient Descent method:

$$L(\rho) = \sum_n \left( \text{Price}_n^{\text{constant}}(\rho) - \text{Price}_n^{\text{dynamic}} \right)^2.$$

And then, the calibrated correlation coefficients are applied to the constant correlation model for pricing options with strikes $K = 82, 84, \ldots, 88, 92, 94, \ldots, 98, \ldots, 132, 134, \ldots, 138$. The resulting prices will be compared with the prices under regime switching model.

To see the variations of implied correlation, we apply the definition of $\rho_{imp}$ given by Da Fonseca et al. 2007 which satisfies

$$\text{Price} = \text{Price}^{\text{constant}}(\rho_{imp}),$$

to the prices given by regime switching models with more strikes $K = 80, 82, 84, \ldots, 140$.

In the following, We run through the calibrating-pricing procedure for Call on Min, Call on Max, Put on Max and Put on Min options, consider their relative errors defined as in (31), and calculate the implied correlations respectively. We show the results in Figures 2-5. In each figure, the dotted line separates the curve into two parts, the out-of-the-money cases (in figures, the left part for puts or the right for calls) and the in-the-money cases. The intersection is at-the-money case.

![Graph 1](image1.png)

(a) Relative error (Calibrated $\rho = -0.3190$)

![Graph 2](image2.png)

(b) Implied correlation

Figure 2: Call on Min option with $Q_0 = [1, 0, 0]^\top$, $\omega = [0.3, 0.6, 0.9]^\top$

On the first try, we choose parameters $Q_0 = [1, 0, 0]^\top$ and $\omega = [0.3, 0.6, 0.9]^\top$ to generate the regime switching model. The immediate observation is the huge pricing error for deep-out-of-the-money options of Put on Max and Call on Min. The relative error reaches more than 70%, which are shown in Figure 2(a) and 4(a). While for Call on Max option, the relative error is no more than 0.1%, as shown in Figure 3(a). And it is also small for Put on Min option whose figure is omitted here since the relative error always lies below the level 0.5%.

---

10] Just as before, all the other coefficients are supposed to be known exactly.

11] The initial value is taken as $\rho = 0$. The step size is set as $|0.01/L'(0)|$ where $L'$ denotes the first derivative of $L$. The gradient descent method terminates when $|L'(\rho)|$ is smaller than $10^{-4}$. 

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Figure 3: Call on Max option with $Q_0 = [1, 0, 0]^T$, $\omega = [0.3, 0.6, 0.9]^T$

Figure 4: Put on Max option with $Q_0 = [1, 0, 0]^T$, $\omega = [0.3, 0.6, 0.9]^T$
To see whether this is a common property or not, we change the initial regime switching model to a new one with parameters $Q_0 = [0.2, 0, 0.8]^\top$ and $\omega = [0.3, 0.6, 0.95]^\top$, and repeat the calibrating-pricing procedure. For Call on Max, Call on Min and Put on Max, the results are really similar with the previous group of parameters and we omit the figures. But for Put on Min, the result is different from former one, relative error could be more than 10% for out-of-the-money options as shown in Figure 5 which is also nonnegligible. We will try to give a reasonable explanation for this in the next section, Section 5.3. And also there in addition, we will explain why the calibrated option prices perform well for in-the-money and at-the-money options but terribly bad for deep-out-of-the-money options and why the Call on Max seems different from the other options.

On the other side, for implied correlation, we can see in Figure 2(b)-5(b), the implied correlation always changes sharply for out-of-the money cases and mildly for in-the-money cases, which is similar with the calibrated $\rho$. For Call on Max options, though there are only tiny pricing errors, the implied correlations change a lot with different strikes.

Figure 6 investigate Put on Max options again and the maturity considered as $\tau = 0.5$. Comparing with Figure 4, the main features of them are similar, but we can find in Figure 6, the calibrated error is a little smaller and the implied correlation changes a little milder. This implies the maturity has little effect on our discoveries.

### 5.3 Error Analysis

The pricing errors coming from setting the dynamic stochastic correlation of underlying log prices to be constant, as shown in the last section, are further analyzed in this part. This analysis is from a theoretical view but with the help of numerical simulations. Through this analysis, we try to explain the phenomenon discovered in Section 5.2.

Now we consider options with payoffs $V(S_1^T, S_2^T, \tau, K)$, then the price of the option is

$$E_Q \left[ e^{-r\tau} V(S_1^T, S_2^T, \tau, K) \right] = E_Q \left[ e^{-r\tau} V \left( S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)\tau + \sigma_1 B_\tau}, S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)\tau + \sigma_2 W_\tau}, \tau, K \right) \right],$$

where $Q$ denotes the risk neutral probability measure.

Note that all the payoffs considered in previous numerical simulations are in this way.
When the local correlation process is a constant $\rho$, since $(B_\tau, W_\tau) \sim N \left( (0, 0), \tau \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$, the option price is a function of $\rho$ which will be denoted as $\text{Price}^c(\rho)$ in the following.

For more general case where $\rho$ is a stochastic process, we first recall the term of average correlation coefficient

$$\bar{\rho}_t = \frac{1}{t} \int_0^t \rho_u du,$$

which by common decomposition, can be rewritten as

$$\bar{\rho}_t = T_t - S_t. \quad (32)$$

Since under the condition of $\mathcal{F}_\tau^T$, $(B_\tau, W_\tau) = (X_\tau + Y_\tau, X_\tau - Y_\tau) \sim N \left( (0, 0), \tau \begin{pmatrix} 1 & \bar{\rho}_\tau \\ \bar{\rho}_\tau & 1 \end{pmatrix} \right)$, following the discussions in the constant-$\rho$ case, the option price (denoted by $\text{Price}^d$) equals

$$\text{Price}^d = E^{\mathbb{Q}} \left[ E^{\mathbb{Q}} \left[ e^{-r\tau} V(S_1^T, S_2^T, K) | \mathcal{F}_\tau^T ] \right] = E^{\mathbb{Q}} \left[ \text{Price}^c(\bar{\rho}_\tau) \right]. \right.$$ 

If $\text{Price}^c$ is an affine function of $\rho$, i.e., $\exists a, b \in \mathbb{R}$, $\text{Price}^c(\rho) = a\rho + b$, we have

$$\text{Price}^d = E^{\mathbb{Q}} \left[ \text{Price}^c(\bar{\rho}_\tau) \right] = E^{\mathbb{Q}}[a\bar{\rho}_\tau + b] = \text{Price}^c(E^{\mathbb{Q}}[\bar{\rho}_\tau]). \quad (33)$$

In other words, when the option price under constant-$\rho$ model is linear in $\rho$, the price under a general dynamic correlation model is exactly the same as that with a constant correlation coefficient $E^{\mathbb{Q}}[\bar{\rho}_\tau]$.

Otherwise, for general $\text{Price}^c$, by Taylor’s expansion, we can get the following approximation formula,

$$\text{Price}^d = E^{\mathbb{Q}} \left[ \text{Price}^c(\bar{\rho}_\tau) \right] \approx \text{Price}^c(E^{\mathbb{Q}}[\bar{\rho}_\tau]) + \frac{1}{2} \text{Var}^{\mathbb{Q}}(\bar{\rho}_\tau) \frac{\partial^2 \text{Price}^c}{\partial \rho^2}(E^{\mathbb{Q}}[\bar{\rho}_\tau]). \quad (34)$$

In the following, based on the above analysis, we try to explore causes for the two phenomena found in the former section: 1) the pricing errors seem more remarkable for out-the-money options when applying constant correlation model; 2) the pricing errors for Call-on-Max options seem relatively small than other kind of options.
We first consider relations between $Price^c(\rho)$ and $\rho$ in the cases of in-the-money, at-the-money and out-of-the-money for Put-on-Max options. Choosing parameters as $r = 0.05, \tau = 0.5, S_0^1 = 100, S_0^2 = 120, \sigma_1 = 0.2, \sigma_2 = 0.3$, we draw diagrams for $Price^c(\rho)$ when Strike = 150 (in the money), Strike = 120 (at the money) and Strike = 90 (out of the money) and list them in Figure 7. The results show that, for in-the-money and at-the-money cases, $Price^c(\rho)$ reveals a strong linearity on $\rho$ except for when $\rho$ is near to 1. But it is quite nonlinear for out the money case. We conduct similar diagraming with different parameters for Put-on-Max option as well as Put-on-Min, Call-on-Min and Call-on-Max options, and get similar results. Recall the approximations (33) and (34), the above results give an explanation for why constant correlation model performs well on the whole for in-the-money and at-the-money options but poorly for out-of-the-money options.

![Figure 7: Put on max option price in constant correlation model](image)

| $\omega$         | $Q_0$    | $\tau = 0.25$ | $\tau = 0.5$ |
|------------------|----------|---------------|---------------|
| $[0.3, 0.6, 0.9]$| $[1, 0, 0]$ | -0.3177       | -0.2488       |
| $[0.3, 0.6, 0.95]$| $[0.2, 0, 0.8]$ | 0.5784        | 0.5298        |

We can find in Figure 2(b), 3(b), 4(b), 5(b), 6(b) and Table 6 when strike is in-the-money and at-the-money, the implied correlation of each option is very close to $E\tilde{\rho}_\tau$; on the contrary, when strike is out-of-the-money, the implied correlation changes sharply and far away from $E\tilde{\rho}_\tau$. This is coincident with the conclusion in previous and (33), that is why calibrating constant correlation model into dynamic correlation model perform well when option is in-the-money and at-the-money.

We now turn to the Call-on-Max option whose performance in calibration in Section 5.2 seemed quite different from the others that the calibrated constant correlation model always performs well, even for out-the-money case. Note that as mentioned before, we have already got diagrams for this kind of option which have similar linear or nonlinear shapes like other options and we did not include them in the main text. A interesting question is, now that the shape of $Price^c(\rho)$ for out-the-money case looks apparently nonlinear, why does it still approximate the true price well? We choose the same parameters as before except for $\tau = 0.25$ and draw the diagram of $Price^c(\rho)$ for Call-on-Max option for the case Strike = 130 (out-of-the-money) in Figure 8. The diagram looks still quite nonlinear, but it is worth noting that in the fifure $Price^c(\rho)$ just changes from 3.97 to 3.995. In other words, when $\rho$ changes in its full range, the price changes only about 0.6% which implies that, for Call-on-max option, the correlation between underlying assets has only a small, almost negligible, impact on the option price. While on the contrary, think about calibrating $\rho$ from option prices, a small deviation in the price may cause great changes.
in the implied $\rho$. This result on one hand explains why the implied correlation of Call-on-Max option is volatile but the calibrated constant correlation model always performs well and on the other indicates that when the data are from out-of-the-money Call-on-Max options, correlation-coefficient calibrating may be unsuitable since the implied correlation is too sensitive with the price.

### 6 Some Proofs

**Proof of Theorem 2.3** First note that $\frac{B+W}{2}$ and $\frac{B-W}{2}$ are continuous martingales and

$$\left[ \frac{B+W}{2}, \frac{B-W}{2} \right]_t = \frac{1}{16} (|2B|_t - |2W|_t) = 0.$$  

By the definitions of $\tau$ and $\varsigma$,

$$\tau_t = \inf \left\{ u : \left[ \frac{B+W}{2} \right]_u > t \right\}, \quad \varsigma_t = \inf \left\{ u : \left[ \frac{B-W}{2} \right]_u > t \right\}.$$  

Then according to [Revuz and Yor 2013](Chapter V, Theorem 1.10), $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ are two independent Brownian motions.

Before going further, we first prove three lemmas as preparations.

**Lemma 6.1.** Suppose the conditions in Theorem 2.3 and the following condition (E) hold, then $\{X_t\}_{t \geq 0}$, $\{Y_t\}_{t \geq 0}$ and $\{T_t\}_{t \geq 0}$ in the common decomposition are mutually independent.

(E) For any $F^T$-progressively measurable processes $\{\phi_1^1\}_{t \geq 0}$ and $\{\phi_1^2\}_{t \geq 0}$ that guarantee

$$E \left[ \exp \left( \frac{1}{2} \int_0^T (\phi_1^1)^2 dT_u + \frac{1}{2} \int_0^T (\phi_1^2)^2 dS_u \right) \right] < \infty.$$  

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we have

\[ E \left[ \exp \left( \int_0^\infty \Phi_u^1 dX_u + \int_0^\infty \Phi_u^2 dY_u \right) \right] \bigg| \mathcal{F}_\infty^T = \exp \left( \frac{1}{2} \int_0^\infty (\Phi_u^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\Phi_u^2)^2 dS_u \right). \]

Proof. For \( \forall n, m \in \mathbb{N} \), and \( 0 = t_0 < t_1 < \cdots < t_n \), \( 0 = s_0 < s_1 < \cdots < s_m \), we consider the joint distribution of \( \{X_{t_1}, \ldots, X_{t_n}, Y_{s_1}, \ldots, Y_{s_m}\} \) conditional on \( \mathcal{F}_\infty^T \) by calculating

\[ E \left[ \exp \left( \sum_{i=1}^n \theta_i^1 (X_{t_i} - X_{t_{i-1}}) + \sum_{j=1}^m \theta_j^2 (Y_{s_j} - Y_{s_{j-1}}) \right) \right] \bigg| \mathcal{F}_\infty^T \]

(35)

where \( \theta_i^1, \theta_j^2 \in \mathbb{R}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, m. \)

Define

\[ \Phi_u^1 = \sum_{i=1}^n \theta_i^1 1_{\{t_{i-1} \leq u < t_i\}}, \quad \Phi_u^2 = \sum_{j=1}^m \theta_j^2 1_{\{s_{j-1} \leq u < s_j\}}, \]

it is easy to verify \( \int_0^\infty \Phi_u^1 dX_u = \sum_{i=1}^n \theta_i^1 (X_{t_i} - X_{t_{i-1}}) \), \( \int_0^\infty \Phi_u^2 dY_u = \sum_{j=1}^m \theta_j^2 (Y_{s_j} - Y_{s_{j-1}}) \) and

\[ E \left[ \exp \left( \int_0^\infty (\Phi_u^1)^2 dT_u + \int_0^\infty (\Phi_u^2)^2 dS_u \right) \right] < \infty. \]

By definitions of \( X \) and \( Y \) (for simplicity, we set \( \int_{t_\infty}^\infty \Phi_u^1 dX_u = 0 \) when \( T_\infty = \infty \)),

\[ \int_0^\infty \Phi_u^1 dX_u = \int_0^{t_\infty} \Phi_u^1 dX_u + \int_{t_\infty}^\infty \Phi_u^1 dX_u = \int_0^{t_\infty} \Phi_u^1 dX_u + \int_0^{t_\infty} \Phi_u^1 dX_u, \]

\[ \int_0^\infty \Phi_u^2 dY_u = \int_0^{s_\infty} \Phi_u^2 dY_u + \int_{s_\infty}^\infty \Phi_u^2 dY_u = \int_0^{s_\infty} \Phi_u^2 dY_u + \int_0^{s_\infty} \Phi_u^2 dY_u. \]

Thus

\[ E \left[ \exp \left( \int_0^\infty \Phi_u^1 dX_u + \int_0^\infty \Phi_u^2 dY_u \right) \right] \bigg| \mathcal{F}_\infty^T \]

(36)

Note that \( \mathcal{F}_\infty^T \subset \mathcal{F}_{t_\infty}^{X_t} \vee \mathcal{F}_{s_\infty}^{Y_s}, \ X \) and \( Y \) are martingales with respect to \( \mathcal{F}_t^X \vee \mathcal{F}_t^Y, \mathcal{F}_\infty^X \) and \( \mathcal{F}_\infty^Y \) are independent with respect to \( \mathcal{F}_{t_\infty}^{X_t}, \mathcal{F}_{s_\infty}^{Y_s} \) and \( \mathcal{F}_\infty^T \), so exp \( \left( \int_0^T \Phi_u^1 dX_u + \int_0^T \Phi_u^2 dY_u - \frac{1}{2} \int_0^T (\Phi_u^1)^2 du - \frac{1}{2} \int_0^T (\Phi_u^2)^2 du \right) \) is a local martingale with \( \mathcal{F}_{t_\infty}^{X_t} \vee \mathcal{F}_{s_\infty}^{Y_s} \vee \mathcal{F}_t^X \vee \mathcal{F}_t^Y \) by Itô’s lemma. Since any positive local martingale is a supermartingale, and according to Karatzas and Shreve [2012, Chapter 3, Proposition 5.12], we have

\[ E \left[ \exp \left( \int_0^T \Phi_u^1 dX_u + \int_0^T \Phi_u^2 dY_u - \frac{1}{2} \int_0^T (\Phi_u^1)^2 du - \frac{1}{2} \int_0^T (\Phi_u^2)^2 du \right) \right] = 1, \quad \forall t. \]
which implies \( \{ \exp \left( \int_0^t \Phi^1_{u+T_\infty} d\bar{X}_u + \int_0^t \Phi^2_{u+S_\infty} d\bar{Y}_u - \frac{1}{2} \int_0^t (\Phi^1_{u+T_\infty})^2 du - \frac{1}{2} \int_0^t (\Phi^2_{u+S_\infty})^2 du \right) \}_{t \geq 0} \) is a martingale. Hence,

\[
E \left[ \exp \left( \int_0^\infty \Phi^1_{u+T_\infty} d\bar{X}_u + \int_0^\infty \Phi^2_{u+S_\infty} d\bar{Y}_u \right) \bigg| \mathcal{F}_{\infty}^T \right] = \exp \left( \frac{1}{2} \int_0^\infty (\Phi^1_{u+T_\infty})^2 du + \frac{1}{2} \int_0^\infty (\Phi^2_{u+S_\infty})^2 du \right).
\]

Substitute (37) into (36),

\[
E \left[ \exp \left( \int_0^\infty \Phi^1_{u+T_\infty} d\bar{X}_u + \int_0^\infty \Phi^2_{u+S_\infty} d\bar{Y}_u \right) \bigg| \mathcal{F}_{\infty}^T \right] = \exp \left( \frac{1}{2} \int_0^\infty (\Phi^1_{u+T_\infty})^2 du + \frac{1}{2} \int_0^\infty (\Phi^2_{u+S_\infty})^2 du \right).
\]

i.e.

\[
E \left[ \exp \left( \sum_{i=1}^m \theta_i^1 (X_{t_i} - X_{t_{i-1}}) + \sum_{j=1}^n \theta_j^2 (Y_{s_j} - Y_{s_{j-1}}) \right) \bigg| \mathcal{F}_{\infty}^T \right] = \exp \left( \sum_{i=1}^m \frac{1}{2} (\theta_i^1)^2 (t_k - t_{k-1}) + \sum_{j=1}^n \frac{1}{2} (\theta_j^2)^2 (s_j - s_{j-1}) \right),
\]

which implies \( X \) and \( Y \) are independent and \( \mathcal{F}_{\infty}^T \) does not affect the distribution of \( \{X_t, Y_t\}_{t \geq 0} \). Hence, \( \{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0} \) and \( \{T_t\}_{t \geq 0} \) are mutually independent.

**Lemma 6.2.** Suppose \( (X_t, Y_t)_{t \geq 0} \) is a 2-dimensional standard Brownian motion and \( \{T_t\}_{t \geq 0}, \{S_t\}_{t \geq 0} \) are two increasing processes with \( T_t + S_t = t, \forall t \). If \( X \), \( Y \) and \( T \) are mutually independent, then we have the following consequences:

1. the condition (E) holds;
2. \( \mathcal{F}_{\infty}^{X_T} \perp \mathcal{F}_{\infty}^{Y_S} \mid \mathcal{F}_t^{X_T} \) and \( \mathcal{F}_t^{X_T} \perp \mathcal{F}_t^{Y_S} \mid \mathcal{F}_t^{T}; \)
3. \( \{X_t\}_{t \geq 0} \) and \( \{Y_t\}_{t \geq 0} \) are martingales with respect to \( \{\mathcal{F}_t^{X_T} \vee \mathcal{F}_{\infty}^T\}_{t \geq 0} \).

**Proof.** (1) Set \( S_t = t - T_t \), let \( \tau \) and \( \zeta \) be the inverse of \( T \) and \( S \) as defined in [2]. Define

\[
\Phi^1_u = \Phi^1_{t_{\tau(u)}} 1_{(u \leq T_\infty)}, \Phi^2_u = \Phi^2_{t_{\zeta(u)}} 1_{(u \leq S_\infty)},
\]

then

\[
\int_0^\infty (\Phi^1_u)^2 du = \int_0^{T_\infty} (\Phi^1_u)^2 du = \int_0^\infty (\Phi^1_u)^2 dU_u, \quad \int_0^\infty (\Phi^2_u)^2 du = \int_0^{S_\infty} (\Phi^2_u)^2 du = \int_0^\infty (\Phi^2_u)^2 dS_u,
\]

and

\[
E \left[ \frac{1}{2} \int_0^\infty (\Phi^1_u)^2 du + \frac{1}{2} \int_0^\infty (\Phi^2_u)^2 du \right] = E \left[ \frac{1}{2} \int_0^\infty (\Phi^1_u)^2 dU_u + \frac{1}{2} \int_0^\infty (\Phi^2_u)^2 dS_u \right] \leq E \exp \left( \frac{1}{2} \int_0^\infty (\Phi^1_u)^2 dU_u + \frac{1}{2} \int_0^\infty (\Phi^2_u)^2 dS_u \right) < \infty.
\]
Hence \( \int_0^\infty \Phi_u^1 dX_u \) and \( \int_0^\infty \Phi_u^2 dY_u \) are well defined and

\[
\int_0^\infty \Phi_u^1 dX_u = \int_0^T \Phi_{\tau_u}^1 dX_u = \int_0^\infty \phi_u^1 dX_{\tau_u}, \quad \int_0^\infty \Phi_u^2 dY_u = \int_0^\infty \phi_u^2 dY_u = \int_0^\infty \phi^2 dY_u. \tag{40}
\]

Since \(X, Y, T\) are independent, \(X\) and \(Y\) are martingales with respect to \(\{\mathcal{F}_T^X \cup \mathcal{F}_T^Y \cup \mathcal{F}_\infty^T\}_{t \geq 0}\). Observe that \(E \exp \left( \frac{1}{2} \int_0^\infty (\Phi_u^1)^2 du + \frac{1}{2} \int_0^\infty (\Phi_u^2)^2 du \right) < \infty\), thus by \textit{Karatzas and Shreve 2012 [Chapter 3, Proposition 5.12]}, \(E \exp \left( \int_0^t \Phi_u^1 dX_u + \int_0^t \Phi_u^2 dY_u - \frac{1}{2} \int_0^t (\Phi_u^1)^2 du - \frac{1}{2} \int_0^t (\Phi_u^2)^2 du \right) \) is a martingale. Consequently

\[
E \left[ \exp \left( \int_0^\infty \Phi_u^1 dX_u + \int_0^\infty \Phi_u^2 dY_u - \frac{1}{2} \int_0^\infty (\Phi_u^1)^2 du - \frac{1}{2} \int_0^\infty (\Phi_u^2)^2 du \right) \middle| \mathcal{F}_\infty^T \right] = 1.
\]

Recall (39), (40), and substituting to the former equation, we have

\[
E \left[ \exp \left( \int_0^\infty \Phi_u^1 dX_u + \int_0^\infty \Phi_u^2 dY_u - \frac{1}{2} \int_0^\infty (\Phi_u^1)^2 du - \frac{1}{2} \int_0^\infty (\Phi_u^2)^2 du \right) \middle| \mathcal{F}_T^X \right] = 1.
\]

Note that \(\exp \left( \frac{1}{2} \int_0^\infty (\Phi_u^1)^2 du + \frac{1}{2} \int_0^\infty (\Phi_u^2)^2 du \right)\) is measurable with \(\mathcal{F}_\infty^T\), and the desired result holds immediately.

(2) First note that, when \(X, Y, T\) are independent, by the former result, (E) is true. As a direct consequence of (E), \(\mathcal{F}_t^{XY}\) and \(\mathcal{F}_t^{YS}\) is conditional independent given \(\mathcal{F}_t^X, \forall t \in [0, +\infty]\).

Thus for every \(\mathcal{F}_T^{YS}\)-measurable random variable \(\eta, E[\eta|\mathcal{F}_t^{XY} \cup \mathcal{F}_T^X] = E[\eta|\mathcal{F}_t^X]\). Furthermore, since \(\mathcal{F}_t^T \subset \mathcal{F}_t^{XY}, \forall t \in [0, +\infty]\), we have

\[
E[\eta|\mathcal{F}_t^T] = E[\eta|\mathcal{F}_t^{XY}], \quad \forall t \in [0, +\infty]. \tag{41}
\]

To prove the result of this part, i.e., \(\mathcal{F}_t^{YS}\) and \(\mathcal{F}_\infty^{XT}\) are conditional independent given \(\mathcal{F}_t^{XT}\), it is sufficient to prove that for any \(\mathcal{F}^T\)-progressively measurable process \(\phi\) satisfying conditions in (E),

\[
E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_\infty^{XT} \right] = E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_t^{XT} \right].
\]

By (41),

\[
E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_\infty^{XT} \right] = E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_t^{XT} \right] = E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_\infty^X \right] = \exp \left( \frac{1}{2} \int_0^t (\phi_u)^2 dS_u \right).
\]

where the second equality comes from (E) immediately.

Since \(E \exp \left( \frac{1}{2} \int_0^t (\phi_u)^2 dS_u \right) \in \mathcal{F}_t^T\), and applying (41) again, we have

\[
E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_\infty^{XT} \right] = E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_t^{XT} \right] = E \left[ \exp \left( \int_0^t \phi_u dY_u \right) \middle| \mathcal{F}_t^{XY} \right],
\]

which is the desired conclusion. By similar proofs, we have \(\mathcal{F}_t^{XY} \perp \mathcal{F}_\infty^{YS} | \mathcal{F}_t^{XY}\).
(3) Given $\mathcal{F}_{s \wedge t}$, we can see $X_{T_{t \wedge s}} - X_{T_{t \wedge s}}$ and $\mathcal{F}_{t}^{Y_{s}}$ are mutually independent according to (E). Hence,

$$E \left[ X_{T_{t \wedge s}} - X_{T_{t \wedge s}} \mid \mathcal{F}_{t}^{X_{t}} \vee \mathcal{F}_{t}^{Y_{s}} \vee \mathcal{F}_{t}^{T_{s \wedge t}} \right] = E \left[ X_{T_{t \wedge s}} - X_{T_{t \wedge s}} \mid \mathcal{F}_{t}^{T_{s \wedge t}} \right] = 0.$$

Observe that, $\mathcal{F}_{t}^{X_{t}} \vee \mathcal{F}_{t}^{Y_{s}} = \mathcal{F}_{t}^{S_{s \wedge t}}$, $X_{t}$ is a martingale with $\mathcal{F}_{t}^{S_{s \wedge t}} \vee \mathcal{F}_{t}^{T_{s \wedge t}}$. The same argument for $Y_{s}$.

\[\square\]

**Lemma 6.3.** Suppose $\{X_{t}\}_{t \geq 0}$ is a Brownian motion and $\{T_{t}\}_{t \geq 0}$ is a nondecreasing stochastic process independent with $\{X_{t}\}_{t \geq 0}$. Given $\{\phi_{t}\}_{t \geq 0}$ and $\{\theta_{t}\}_{t \geq 0}$, which are progressively measurable with $\{\mathcal{F}_{t}^{X_{t}}\}_{t \geq 0}$ and

$$E \left[ \exp \left( \frac{1}{2} \int_{0}^{t} (\phi_{u})^{2} dT_{u} \right) \right] < \infty, E \left[ \exp \left( \frac{1}{2} \int_{0}^{t} (\theta_{u})^{2} dT_{u} \right) \right] < \infty, \forall t,$$

let

$$X_{t}^{\phi} = X_{t} - \int_{0}^{t} \phi_{u} d\tau_{u}, \tau_{u} = \inf \{ u : T_{u} \geq u \},$$

then we have

$$E \left[ \exp \left( \int_{0}^{T_{t \wedge s}} \theta_{s} dX_{u}^{\phi} \right) \exp \left( \int_{0}^{T_{t \wedge s}} \phi_{s} dX_{u} - \frac{1}{2} \int_{0}^{T_{t \wedge s}} (\phi_{s})^{2} d\tau_{s} \right) \mid \mathcal{F}_{s \wedge t}^{X_{t}} \vee \mathcal{F}_{s \wedge t}^{T_{s \wedge t}} \right] = \exp \left( \int_{0}^{T_{t \wedge s}} (\theta_{s} \phi_{s})^{2} d\tau_{s} \right).$$

\[\text{Proof.}\] Fix $t$, from

$$E \exp \left( \frac{1}{2} \int_{0}^{s \wedge t} (\phi_{u})^{2} d\tau_{u} \right) \leq E \exp \left( \frac{1}{2} \int_{0}^{t} (\phi_{u})^{2} d\tau_{u} \right) = E \exp \left( \frac{1}{2} \int_{0}^{t} (\phi_{u})^{2} dT_{u} \right) < \infty,$$

and [Karatzas and Shreve 2012, Chapter 3, Proposition 5.12] we have $\left\{ \exp \left( \int_{0}^{s \wedge t} \phi_{u} dX_{u} - \frac{1}{2} \int_{0}^{s \wedge t} (\phi_{u})^{2} d\tau_{u} \right) \right\}_{s \geq 0}$ is a martingale with respect to $\{\mathcal{F}_{s}^{X_{t}} \vee \mathcal{F}_{s \wedge t}^{T_{s \wedge t}}\}_{s \geq 0}$. Let

$$\frac{dQ}{dP} \mid \mathcal{F}_{s \wedge t}^{X_{t}} \vee \mathcal{F}_{s \wedge t}^{T_{s \wedge t}} = \exp \left( \int_{0}^{s \wedge t} \theta_{s} dX_{u} - \frac{1}{2} \int_{0}^{s \wedge t} (\phi_{s})^{2} d\tau_{s} \right),$$

note that $X$ is a Brownian motion with respect to $\{\mathcal{F}_{s}^{X_{t}} \vee \mathcal{F}_{s \wedge t}^{T_{s \wedge t}}\}_{s \geq 0}$, then by Girsanov theorem,

$$\tilde{X}_{s}^{\phi} \equiv X_{s} - \int_{0}^{s \wedge t} \phi_{u} d\tau_{u}$$

is a Brownian motion with $\{\mathcal{F}_{s}^{X_{t}} \vee \mathcal{F}_{s \wedge t}^{T_{s \wedge t}}\}_{s \geq 0}$ under probability measure $Q$. Hence $\left\{ \exp \left( \int_{0}^{s \wedge t} \phi_{u} d\tilde{X}_{u}^{\phi} - \frac{1}{2} \int_{0}^{s \wedge t} (\theta_{u})^{2} d\tau_{u} \right) \right\}_{s \geq 0}$ is a martingale under $Q$, by optional stopping theorem\footnote{In Girsanov theorem, we need to determine an upper bound $t$ in advance, then $\tilde{X}^{\phi}$ is a Brownian motion with $\{\mathcal{F}_{s}^{X_{t}} \vee \mathcal{F}_{s \wedge t}^{T_{s \wedge t}}\}_{0 \leq s \leq t}$ in $[0, t]$. Thanks to $0 \leq t \wedge T_{t} \leq t$, optional stopping theorem for $t \wedge T_{t}$ remains valid.} we obtain

$$E^{P} \left[ \exp \left( \int_{0}^{T_{t \wedge s}} \theta_{s} d\tilde{X}_{u}^{\phi} - \frac{1}{2} \int_{0}^{T_{t \wedge s}} (\theta_{s})^{2} d\tau_{s} \right) \mid \mathcal{F}_{s \wedge t}^{T_{s \wedge t}} \right] = 1,$$

i.e.,

$$E^{P} \left[ \exp \left( \int_{0}^{T_{t \wedge s}} \theta_{s} d\tilde{X}_{u}^{\phi} - \frac{1}{2} \int_{0}^{T_{t \wedge s}} (\theta_{s})^{2} d\tau_{s} \right) \mid \mathcal{F}_{s \wedge t}^{T_{s \wedge t}} \right] = \exp \left( \frac{1}{2} \int_{0}^{T_{t \wedge s}} (\theta_{s})^{2} d\tau_{s} \right).$$

Note that $\tilde{X}_{s}^{\phi} = X_{s}, \forall s \in [0, t \wedge T_{t}]$, we get desired result immediately. \[\square\]
Proof of Theorem 2.4. For the “if” part: since $\mathcal{F}_t^S \perp \mathcal{F}_t^W$ and $\mathcal{F}_t^{B,W} \subset \mathcal{F}_t$, we have

$$E \left[ B_t - B_s | \mathcal{F}_\infty^T \vee \mathcal{F}_s^{B,W} \right] = E \left[ B_t - B_s | \mathcal{F}_s^{B,W} \right] = 0,$$

(42)

therefore the process $B$ is a martingale with respect to $\{\mathcal{F}_t^{B,W} \vee \mathcal{F}_\infty^T \}_{t \geq 0}$, so is the process $W$ by similar analysis.

As a consequence, $X_t = B_t + Y_t$, $Y_t = B_t - W_t$, are martingales with respect to the same filtration. So for any $\mathcal{F}_t$-progressively measurable process $\phi^1$, $\phi^2$ satisfying the conditions in (E),

$$D^\phi_t \triangleq \exp \left( \int_0^t \phi_u^1 dX_u + \int_0^t \phi_u^2 dY_u - \frac{1}{2} \int_0^t (\phi_u^1)^2 dT_u - \frac{1}{2} \int_0^t (\phi_u^2)^2 dS_u \right), \quad t \in [0, +\infty)$$
is a local martingale with respect to $\{\mathcal{F}_t^{B,W} \vee \mathcal{F}_\infty^T \}_{t \geq 0}$ by Itô’s lemma. From [Karatzas and Shreve 2012 Chapter 3, Proposition 5.12], and

$$E \left[ \exp \left( \frac{1}{2} \int_0^\infty (\phi_u^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\phi_u^2)^2 dS_u \right) \right] < \infty,$$

we have $ED^\phi_t = 1, \forall t$, consequently, $D^\phi_t$ is a martingale. Moreover,

$$E \left[ \left( \int_0^\infty (\phi_u^1)^2 dT_u + \int_0^\infty (\phi_u^2)^2 dS_u \right) \right] < \infty,$

implies $D^\phi_\infty$ exists. And thus

$$E[D^\phi_0 | \mathcal{F}_0^{B,W} \vee \mathcal{F}_\infty^T] = D^\phi_0 = 1,$$
i.e.

$$E \left[ \exp \left( \int_0^\infty \phi_u^1 dX_u + \int_0^\infty \phi_u^2 dY_u \right) | \mathcal{F}_\infty^T \right] = \exp \left( \frac{1}{2} \int_0^\infty (\phi_u^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\phi_u^2)^2 dS_u \right).$$

According to Lemma 6.1, the desired result is obtained.

For the “only if” part: if $\{X_t\}_{t \geq 0}$, $\{Y_t\}_{t \geq 0}$ and $\{T_t\}_{t \geq 0}$ are independent, by Lemma 6.2, $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ are martingales with respect to $\mathcal{F}_t^{B,W} \vee \mathcal{F}_\infty^T$.

Consequently, $B_t = X_t + Y_t$, $W_t = X_t - Y_t$, are martingales with respect to $\mathcal{F}_t^{B,W} \vee \mathcal{F}_\infty^T$. Since $|B_t| = |W_t| = t$, $B_t$ and $W_t$ are Brownian motions with respect to $\mathcal{F}_t^{B,W} \vee \mathcal{F}_\infty^T$ according to Lévy characterisation.

On the other hand, $B_t$ and $W_t$ are Brownian motions with respect to $\mathcal{F}_t^{B,W}$ as well. That is to say, for any $t \geq 0$, the conditional distribution of process $B$ given $\mathcal{F}_\infty^T \vee \mathcal{F}_\infty^T$ is coincident with its conditional distribution given $\mathcal{F}_t^{B,W}$. Then we can conclude that $\mathcal{F}_\infty^S \perp \mathcal{F}_\infty^T | \mathcal{F}_t^{B,W}$. Similarly, $\mathcal{F}_\infty^W \perp \mathcal{F}_\infty^T | \mathcal{F}_t^{B,W}$. □

Proof of Proposition 2.5.

We prove that the independency of $X$, $Y$ and $T$ is equivalent with condition (C2), then from Theorem 2.4, we have condition (C1) is equivalent with condition (C2).

For the “⇒” part: It is obvious that $D^\phi_t$ is a positive local martingale according to Itô’s lemma, hence $D^\phi_t$ is a supermartingale. From [Karatzas and Shreve 2012 Chapter 3, Proposition 5.12], and

$$E \left[ \exp \left( \frac{1}{2} \int_0^t (\phi_u^1)^2 dT_u + \frac{1}{2} \int_0^t (\phi_u^2)^2 dS_u \right) \right] < \infty,$$

we have $ED^\phi_t = 1, \forall t$, consequently, $D^\phi_t$ is a martingale.
Suppose $\theta_i, i = 1, 2$ are bounded determined processes, then
\[
E^Q \left[ \exp \left( \int_0^t \theta_1^a dX_t^a + \int_0^t \theta_2^a dY_t^a \right) \right] = E^P \left[ \exp \left( \int_0^t \phi_1^a dX_t^a + \int_0^t \phi_2^a dY_t^a \right) D_t^f \right]
= E^P \left[ \exp \left( \int_0^t \phi_1^a dX_t^a \right) \exp \left( \int_0^t \phi_2^a dY_t^a - \frac{1}{2} \int_0^t (\phi_2^a)^2 dT_u \right) \right]
= E^P \left[ \exp \left( \int_0^t \phi_2^a dY_t^a \right) \exp \left( \int_0^t \phi_2^a dY_t^a - \frac{1}{2} \int_0^t (\phi_2^a)^2 dT_u \right) \right].
\]

According to the independency of $X, Y, T$, we have
\[
E^P \left[ \exp \left( \int_0^t \varphi_1^a dY_t^a \right) \exp \left( \int_0^t \varphi_2^a dY_t^a - \frac{1}{2} \int_0^t (\varphi_2^a)^2 dS_u \right) \right] = E^P \left[ \exp \left( \int_0^t \varphi_2^a dY_t^a \right) \right].
\]

where $Y_t^a = Y_t - \int_0^t \phi_2^a dS_u = Y_t - \int_0^t \phi_2^a dT_u$. Observe that $t \wedge S_t = S_t$, then from Lemma 6.3 we have
\[
E^P \left[ \exp \left( \int_0^{S_t} \varphi_2^a dY_t^a \right) \right] = E^P \left[ \exp \left( \int_0^{S_t} \varphi_2^a dY_t^a \right) \right] = E^P \left[ \exp \left( \int_0^{S_t} \varphi_2^a dY_t^a \right) \right].
\]

Substitute (44) and (45) into (43),
\[
E^Q \left[ \exp \left( \int_0^t \theta_1^a dX_t^a + \int_0^t \theta_2^a dY_t^a \right) \right] = E^P \left[ \exp \left( \int_0^t \phi_1^a dX_t^a \right) \exp \left( \int_0^t \phi_1^a dX_t^a - \frac{1}{2} \int_0^t (\phi_1^a)^2 dT_u \right) \exp \left( \int_0^t \phi_2^a dY_t^a \right) \exp \left( \int_0^t \phi_2^a dY_t^a - \frac{1}{2} \int_0^t (\phi_2^a)^2 dT_u \right) \right]
= E^P \left[ \exp \left( \int_0^t \phi_2^a dY_t^a \right) \exp \left( \int_0^t \phi_2^a dY_t^a - \frac{1}{2} \int_0^t (\phi_2^a)^2 dT_u \right) \right].
\]

apply Lemma 6.3 to the former equation again, we obtain
\[
E^Q \left[ \exp \left( \int_0^t \theta_1^a dX_t^a + \int_0^t \theta_2^a dY_t^a \right) \right] = E^P \left[ \exp \left( \int_0^t \theta_2^a dY_t^a \right) \exp \left( \int_0^t \theta_2^a dY_t^a - \frac{1}{2} \int_0^t (\theta_2^a)^2 dT_u \right) \right] = E^P \left[ \exp \left( \int_0^t \theta_2^a dY_t^a + \int_0^t \theta_2^a dY_t^a \right) \right].
\]
If \( \theta_i, i = 1, 2 \) are complex, the proof remains valid, hence we have \((\hat{X}_\theta, \hat{Y}_\theta)_Q \overset{d}{=} (X_T, Y_S)_F\) immediately.

For the "\(\Rightarrow\)" part: Suppose \( \{\phi_i^1\}_{i \geq 0} \) and \( \{\phi_i^2\}_{i \geq 0} \) satisfy the conditions in (E) (note that the range of \( \{\phi_i^1\}_{i \geq 0} \) and \( \{\phi_i^2\}_{i \geq 0} \) in (E) is smaller than \( \{(C2)\} \)).

\[
E \left[ \left( \frac{1}{2} \int_0^\infty (\phi_u^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\phi_u^2)^2 dS_u \right) \right] \leq E \left[ \exp \left( \frac{1}{2} \int_0^\infty (\phi_u^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\phi_u^2)^2 dS_u \right) \right] < \infty,
\]

accordingly \( D_\phi^\infty \) exists. We first claim that

\[
E^P[D_\phi^\infty|\mathcal{F}_\infty^T] = 1 \quad \text{a.s.}
\]

To see this, we only need to prove for any \( A \in \mathcal{F}_\infty^T \)

\[
E^P[D_\phi^\infty 1_A] = P(A).
\]

Let

\[
D \triangleq \{ A \in \mathcal{F}|E^P[D_\phi^\infty 1_A] = P(A) \}, \quad \mathcal{P} \triangleq \{ \bigcap_{i=1}^n A_{t_i} | A_{t_i} \in \sigma(T_{t_i}), n \geq 1, t_1 < t_2 < \cdots < t_n \},
\]

note that \( E^P[D_\phi^\infty] = 1 \), so \( D \) is a \( \lambda \)-system and obviously \( \mathcal{P} \) is a \( \pi \)-system, moreover, \( \sigma(\mathcal{P}) = \mathcal{F}_\infty^T \). Then for any \( A = \bigcap_{i=1}^n A_{t_i} \in \mathcal{P} \), let \( A_{t_i} = \{ T_{t_i} \in B_i \} \), where \( B_i \) is a Borel set, we have

\[
E^P[D_\phi^\infty 1_A] = E^P[1_A E^P[D_\phi^\infty|\mathcal{F}_\infty]|\mathcal{F}_\infty] = E^P[D_\phi^\infty 1_A]. \tag{46}
\]

Since \((\hat{X}_\theta, \hat{Y}_\theta)_Q \overset{d}{=} (X_T, Y_S)_F\), and

\[
[X_T]_t = X_T^2 - \int_0^t X_T dX_T,
\]

so we have \((X_T, Y_S)_Q \overset{d}{=} ([X_T], [Y_S])_F\), i.e. \((T, S)_Q \overset{d}{=} (T, S)_F\). Consequently,

\[
P(A) = P(T_{t_i} \in B_i, i = 1, 2, \ldots, n) = Q(T_{t_i} \in B_i, i = 1, 2, \ldots, n) = E^Q[1_A] = E^P[D_\phi^\infty 1_A], \tag{47}
\]

combine (46) and (47) we know that \( \mathcal{P} \subset D \). According to \( \pi - \lambda \) theorem we can conclude

\[
\mathcal{F}_\infty^T = \sigma(\mathcal{P}) \subset D,
\]

hence, we have proved our claim. \( E^P[D_\phi^\infty|\mathcal{F}_\infty] = 1 \) implies

\[
E \left[ \exp \left( \int_0^\infty \phi_u^1 dX_T + \int_0^\infty \phi_u^2 dY_S \right) | \mathcal{F}_\infty \right] = \exp \left( \frac{1}{2} \int_0^\infty (\phi_u^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\phi_u^2)^2 dS_u \right),
\]

we complete proof by Lemma 6.1.

\[\Box\]

**Proof of Proposition 2.9** We prove it by the equivalence of condition (C3) and (C1).

\(\Rightarrow\)(C1)\(\Rightarrow\)(C3): According to \( \mathcal{F}_\infty^T \perp \mathcal{F}_\infty^T|\mathcal{F}_\infty^{B,W} \) and (42), we have \( \{B_i\}_{i \geq 0} \) is a martingale with respect to \( \mathcal{F}_\infty^{B,W} \). Because \( \mathcal{F}_\infty^T \perp \mathcal{F}_\infty^T \) \( \perp \mathcal{F}_\infty^T \) (actually, \( \mathcal{F}_\infty^T \subset \mathcal{F}_\infty^{B,W} \)), then for any \( \xi \in \mathcal{F}_\infty^T \),

\[
E \left[ \xi \bigg| \mathcal{F}_\infty^{B,W} \right] = E[\xi] = E \left[ \xi \bigg| \mathcal{F}_\infty^T \right],
\]

37
which is equivalent with $F^Z_t \perp F^S_{\infty} | F^B_{\infty} \cup F_T^T$. Hence,
\[
E\left[ B_t - B_s \bigg| F^S_{s} \cup F^T_T \cup F^Z_T \right] = E\left[ B_t - B_s \bigg| F^B_{s} \cup F_T^T \right] = 0,
\]
and equivalently, $\{B_t\}_{t \geq 0}$ is a martingale with respect to $\{F^B_{t} \cup F^T_T \cup F^Z_T\}_{t \geq 0}$. With the same arguments, $\{W_t\}_{t \geq 0}$ is a martingale with respect to $\{F^B_{t} \cup F^T_T \cup F^Z_T\}$ as well. Obviously, $\{Z_t\}_{t \geq 0}$ is a martingale with respect to $\{F^B_{t} \cup F^T_T \cup F^Z_T\}$, so from the definition of $Z_t$, we know $\{Z_t\}_{t \geq 0}$ is a martingale with respect to $\{F^B_{t} \cup F^T_T \cup F^Z_T\}$ and $[Z]_t = t[E[B_tZ_t]] = 0$. According to Lévy characterisation (see Shreve 2004), $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are also two independent Brownian motions with respect to $\{F^B_{t} \cup F^T_T \cup F^Z_T\}$. Since $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are adapted with $F^B_{t} \cup F^Z_T \subset F^B_{t} \cup F^T_T \cup F^Z_T$ and $F^B_{t} \cup F^Z_T \cup F^S_{t} \cup F^T_T$, so $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are also two independent Brownian motions with respect to $\{F^B_{t} \cup F^Z_T\}$. Consequently, the joint distribution of $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ is same under the condition of $\{F^B_{t} \cup F^T_T \cup F^Z_T\}$ and $\{F^B_{t} \cup F^Z_T\}$ which implies
\[
F^Z_T \cup F^S_{\infty} \perp F^T_T | F^B_{t} \cup F^Z_T.
\]
Let $t = 0$ in (48) we obtain $F^Z_T \cup F^S_{\infty} \perp F^T_T$ and note that $\{B_t\}_{t \geq 0}$ is independent with $\{Z_t\}_{t \geq 0}$ we can conclude $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are mutually independent.

"(C3)⇒(C1)". Note that $F^B_{t} \cup F^Z_T \subset F^B_{t} \cup F^T_T$, and obviously $F^B_{0}, F^T_T$ and $F^Z_T$ are mutually independent under the condition of $F^B_{T}$ by (C3). Then for any $\xi \in F^B_{0},$ \[
E\left[\xi \bigg| F^B_{t} \cup F^Z_T \right] = E\left[ E[\xi | F^B_{0} \cup F^Z_T] | F^T_T \right] = E\left[ E[\xi | F^B_{0} \cup F^Z_T] | F^B_{t} \cup F^Z_T \right] = E\left[ \xi | F^B_{t} \cup F^Z_T \right]
\]
with similar approach we can prove $E\left[\xi | F^B_{t} \cup F^Z_T \right] = E\left[ \xi | F^B_{0} \right]$ as well, immediately
\[
E\left[\xi \bigg| F^B_{t} \cup F^Z_T \right] = E\left[ \xi | F^B_{0} \right], \forall \xi \in F^B_{0},
\]
which is equivalent to $F^B_{t} \cup F^Z_T \perp F^T_T | F^B_{t} \cup F^Z_T$.

As for $F^Z_T \perp F^T_T | F^B_{t} \cup F^Z_T$, we first observe that $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are martingales with respect to $F^B_{t} \cup F^Z_T \cup F^T_T$ by the independency of $\{B_t\}_{t \geq 0}$, $\{Z_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$. So according to
\[
W_t = \int_0^t \rho_s dB_s + \int_0^t \sqrt{1 - \rho^2_s} dZ_s,
\]
$\{W_t\}_{t \geq 0}$ is a martingale with respect to $F^B_{t} \cup F^Z_T \cup F^T_T$. Since $\{W_t\}_{t \geq 0}$ is adapted to $F^B_{t} \cup F^Z_T \cup F^T_T$ and $F^B_{t} \cup F^Z_T \cup F^T_T$, $\{W_t\}_{t \geq 0}$ is a martingale with respect to $F^B_{t} \cup F^Z_T \cup F^T_T$ respectively, so $\{W_t\}_{t \geq 0}$ is a martingale with respect to $F^B_{t} \cup F^Z_T \cup F^T_T$ respectively. Hence, by Lévy characterisation, $\{W_t\}_{t \geq 0}$ is a Brownian motion with respect to $F^B_{t} \cup F^Z_T \cup F^T_T$, respectively. Thus, the distribution of $\{W_t\}_{t \geq 0}$ is same under the condition of $F^B_{t} \cup F^Z_T \cup F^T_T$ and $F^B_{t} \cup F^Z_T \cup F^T_T$, which result in $F^Z_T \perp F^T_T | F^B_{t} \cup F^Z_T$.

Proof of Proposition 2.12 Let
\[
W_t^\Pi = \int_0^s \rho_u^\Pi dB_u + \int_0^s \sqrt{1 - (\rho^\Pi_u)^2} dZ_u = \sum_{k=0}^i \left( \rho_k \Delta B_k + \sqrt{1 - \rho_k^2} \Delta Z_k \right) + \rho_i (B_s - B_t) + \sqrt{1 - \rho_i^2} (Z_s - Z_t), t_i \leq s < t_{i+1}
\]
where $\Delta B_{i_k} = B_{i_k+1} - B_{i_k}, \Delta Z_{i_k} = Z_{i_k+1} - Z_{i_k}$.

Observe that given $F^T_{\infty}$, the conditional distribution of $(\Delta B_{i_k}, \Delta W_{i_k}^{\Pi})$ is

$$(\Delta B_{i_k}, \Delta W_{i_k}^{\Pi}) \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Delta t_i & \rho_{i_k} \Delta t_i \\ \rho_{i_k} \Delta t_i & \Delta t_i \end{pmatrix} \right),$$

which is just the same as the conditional distribution of $(\Delta B_{i_k}, \Delta W_{i_k}^{\Pi})$. If condition (C3) holds, $\{B_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are independent Brownian motions with respect to $F^B_1 \vee F^T_\infty$. Hence, by the independent property of increments, given $F^T_{\infty}$, we have

$$(\Delta B_{i_0}, \Delta B_{i_1}, \ldots, \Delta B_{i_{n-1}}, \Delta W_{i_0}^{\Pi}, \Delta W_{i_1}^{\Pi}, \ldots, \Delta W_{i_{n-1}}^{\Pi}) \overset{d}{=} (\Delta B_{i_0}, \Delta B_{i_1}, \ldots, \Delta B_{i_{n-1}}, \Delta W_{i_0}^{\Pi}, \Delta W_{i_1}^{\Pi}, \ldots, \Delta W_{i_{n-1}}^{\Pi}).$$

Consequently,

$$(B_{i_0}, B_{i_1}, B_{i_2}, \ldots, B_{i_{n-1}}, W_{i_0}^{\Pi}, W_{i_1}^{\Pi}, \ldots, W_{i_{n-1}}^{\Pi}) \overset{d}{=} (B_{i_0}, B_{i_1}, B_{i_2}, \ldots, W_{i_0}^{\Pi}, W_{i_1}^{\Pi}, \ldots, W_{i_{n-1}}^{\Pi}). \tag{49}$$

Next, for any given $u_k, v_l, k = 1, 2, \ldots, K, l = 1, 2, \ldots, L$, we consider the difference between the distribution of $(B_{i_1}, B_{i_2}, \ldots, B_{i_K}, W_{i_1}^{\Pi}, W_{i_2}^{\Pi}, \ldots, W_{i_K}^{\Pi})$ and $(B_{i_1}, B_{i_2}, \ldots, B_{i_K}, W_{i_1}^{\Pi}, W_{i_2}^{\Pi}, \ldots, W_{i_K}^{\Pi})$. Let

$$i_k = \sup \{z \in \mathbb{Z} : t_z < u_k \}, j_l = \sup \{z \in \mathbb{Z} : t_z < v_l \}, k = 1, 2, \ldots, K, l = 1, 2, \ldots, L.$$ 

Then, for all $\delta > 0$,

$$P(B_{i_k} \leq a_1 - \delta, \ldots, B_{i_k} \leq a_K - \delta, W_{i_k}^{\Pi} \leq b_1 - \delta, \ldots, W_{i_k}^{\Pi} \leq b_L - \delta) \leq P\left( B_{a_k} \leq a_1, \ldots, B_{a_k} \leq a_K, W_{a_k}^{\Pi} \leq b_1, \ldots, W_{a_k}^{\Pi} \leq b_L \right) \leq P\left( B_{a_k} \leq a_1 + \delta, \ldots, B_{a_k} \leq a_K + \delta, W_{a_k}^{\Pi} \leq b_1 + \delta, \ldots, W_{a_k}^{\Pi} \leq b_L + \delta \right) + \sum_{k=1}^{K} P(B_{a_k} - B_{i_k} \leq -\delta) + \sum_{l=1}^{L} P(W_{a_k}^{\Pi} - W_{i_k}^{\Pi} \leq -\delta),$$

similar inequality holds for $P(B_{a_k} \leq a_1, \ldots, B_{a_k} \leq a_K, W_{a_k}^{\Pi} \leq b_1, \ldots, W_{a_k}^{\Pi} \leq b_L, W_{a_k}^{\Pi} \leq b_1, \ldots, W_{a_k}^{\Pi} \leq b_L)$. Let $H = \{ B_{a_k} - B_{i_k} \leq \delta, W_{a_k}^{\Pi} - W_{i_k}^{\Pi} \leq \delta, k = 1, \ldots, K, l = 1, \ldots, L \}$, then

$$P(B_{a_k} \leq a_1, \ldots, B_{a_k} \leq a_K, W_{a_k}^{\Pi} \leq b_1, \ldots, W_{a_k}^{\Pi} \leq b_L) - P(B_{a_k} \leq a_1, \ldots, B_{a_k} \leq a_K, W_{a_k}^{\Pi} \leq b_1, \ldots, W_{a_k}^{\Pi} \leq b_L) \leq P(B_{i_k} \leq a_1 + \delta, \ldots, B_{i_k} \leq a_K + \delta, W_{i_k}^{\Pi} \leq b_1 + \delta, \ldots, W_{i_k}^{\Pi} \leq b_L + \delta) + \sum_{k=1}^{K} P(B_{a_k} - B_{i_k} \leq -\delta) + \sum_{l=1}^{L} P(W_{a_k}^{\Pi} - W_{i_k}^{\Pi} \leq -\delta) - P(B_{i_k} \leq a_1 - \delta, \ldots, B_{i_k} \leq a_K - \delta, W_{i_k}^{\Pi} \leq b_1 - \delta, \ldots, W_{i_k}^{\Pi} \leq b_L - \delta, H). \tag{50}$$

Note that (49) implies $P(B_{i_k} \leq a_1 - \delta, \ldots, B_{i_k} \leq a_K - \delta, W_{i_k}^{\Pi} \leq b_1 - \delta, \ldots, W_{i_k}^{\Pi} \leq b_L - \delta) = P(B_{i_k} \leq a_1 - \delta, \ldots, B_{i_k} \leq a_K - \delta, W_{i_k}^{\Pi} \leq b_1 - \delta, \ldots, W_{i_k}^{\Pi} \leq b_L - \delta)$, and compared the first term and last term in the right hand
of (50), we have

\[ P(B_{t_1} \leq a_1 + \delta, \ldots, B_{t_k} \leq a_K + \delta, W_{t_1}^{II} \leq b_1 + \delta, \ldots, W_{t_k}^{II} \leq b_L + \delta) \]

\[ - \quad P(B_{t_1} \leq a_1 - \delta, \ldots, B_{t_k} \leq a_K - \delta, W_{t_1}^{II} \leq b_1 - \delta, \ldots, W_{t_k}^{II} \leq b_L - \delta, H) \]

\[ = P(B_{t_1} \leq a_1 + \delta, \ldots, B_{t_k} \leq a_K + \delta, W_{t_1}^{II} \leq b_1 + \delta, \ldots, W_{t_k}^{II} \leq b_L + \delta) \]

\[ - \quad P(B_{t_1} \leq a_1 - \delta, \ldots, B_{t_k} \leq a_K - \delta, W_{t_1}^{II} \leq b_1 - \delta, \ldots, W_{t_k}^{II} \leq b_L - \delta) \]

\[ + \quad P(B_{t_1} \leq a_1 - \delta, \ldots, B_{t_k} \leq a_K - \delta, W_{t_1}^{II} \leq b_1 - \delta, \ldots, W_{t_k}^{II} \leq b_L - \delta, H^c) \]

\[ \leq P(B_{t_1} \leq a_1 + \delta, \ldots, B_{t_k} \leq a_K + \delta, W_{t_1}^{II} \leq b_1 + \delta, \ldots, W_{t_k}^{II} \leq b_L + \delta) \]

\[ - \quad P(B_{t_1} \leq a_1 - \delta, \ldots, B_{t_k} \leq a_K - \delta, W_{t_1}^{II} \leq b_1 - \delta, \ldots, W_{t_k}^{II} \leq b_L - \delta) + P(H^c) \]

\[ \leq \sum_{k=1}^{K} P(|B_{t_k} - a_k| \leq \delta) + \sum_{l=1}^{L} P(|W_{t_l}^{II} - b_l| \leq \delta) + P(H^c). \]  

(51)

Substitute (51) into (50), we obtain

\[ P(B_{t_1} \leq a_1, \ldots, B_{t_k} \leq a_K, W_{v_1}^{II} \leq b_1, \ldots, W_{v_k}^{II} \leq b_L) - P(B_{t_1} \leq a_1, \ldots, B_{t_k} \leq a_K, W_{v_1}^{II} \leq b_1, \ldots, W_{v_k}^{II} \leq b_L) \]

\[ \leq \sum_{k=1}^{K} P(B_{t_k} - B_{t_{k-1}} \leq -\delta) + \sum_{l=1}^{L} P(W_{v_l}^{II} - W_{v_{l-1}}^{II} \leq -\delta) + \sum_{k=1}^{K} P(|B_{t_k} - a_k| \leq \delta) + \sum_{l=1}^{L} P(|W_{v_l}^{II} - b_l| \leq \delta) + P(H^c) \]

\[ \leq \sum_{k=1}^{K} P(B_{t_k} - B_{t_{k-1}} \leq -\delta) + \sum_{l=1}^{L} P(W_{v_l}^{II} - W_{v_{l-1}}^{II} \leq -\delta) + \sum_{k=1}^{K} P(|B_{t_k} - a_k| \leq \delta) + \sum_{l=1}^{L} P(|W_{v_l}^{II} - b_l| \leq \delta) \]

\[ + \quad \sum_{k=1}^{K} P(B_{t_k} - B_{t_{k-1}} \geq \delta) + \sum_{l=1}^{L} P(W_{v_l}^{II} - W_{v_{l-1}}^{II} \geq \delta) \]

\[ = 2 \sum_{k=1}^{K} \Phi\left(\frac{\delta}{\sqrt{u_k - t_{t_k}}}\right) - 2 \sum_{l=1}^{L} \Phi\left(\frac{\delta}{\sqrt{v_l - t_{v_l}}}\right) + \sum_{k=1}^{K} P(|B_{t_k} - a_k| \leq \delta) + \sum_{l=1}^{L} P(|W_{v_l}^{II} - b_l| \leq \delta), \]

where \( \Phi \) denotes the standard normal distribution. For any \( \epsilon > 0 \), we first give a \( \delta \) small enough such that for any \( a_k, k = 1, 2, \ldots, K \) and \( b_l, l = 1, 2, \ldots, L, \)

\[ \sum_{k=1}^{K} P(|B_{t_k} - a_k| \leq \delta) + \sum_{l=1}^{L} P(|W_{v_l}^{II} - b_l| \leq \delta) < \frac{\epsilon}{2}, \]

then fix \( \delta \) and let \( ||II|| \) small enough such that

\[ 2 \sum_{k=1}^{K} \Phi\left(\frac{\delta}{\sqrt{u_k - t_{t_k}}}\right) + 2 \sum_{l=1}^{L} \Phi\left(\frac{\delta}{\sqrt{v_l - t_{v_l}}}\right) \leq 2(K + L) \Phi\left(\frac{\delta}{\sqrt{||II||}}\right) < \frac{\epsilon}{2}. \]

As a consequence

\[ P(B_{t_1} \leq a_1, \ldots, B_{t_k} \leq a_K, W_{v_1}^{II} \leq b_1, \ldots, W_{v_k}^{II} \leq b_L) - P(B_{t_1} \leq a_1, \ldots, B_{t_k} \leq a_K, W_{v_1}^{II} \leq b_1, \ldots, W_{v_k}^{II} \leq b_L) \leq \epsilon, \]

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similarly,

\[ P(B_{t_1} \leq a_1, \ldots, B_{t_K} \leq a_K, \bar{W}_{t_1} \leq b_1, \ldots, \bar{W}_{t_L} \leq b_L) - P(B_{t_1} \leq a_1, \ldots, B_{t_K} \leq a_K, W_{t_1} \leq b_1, \ldots, W_{t_L} \leq b_L) \leq \epsilon, \]

i.e.

\[ |P(B_{t_1} \leq a_1, \ldots, B_{t_K} \leq a_K, W_{t_1} \leq b_1, \ldots, W_{t_L} \leq b_L) - P(B_{t_1} \leq a_1, \ldots, B_{t_K} \leq a_K, W_{t_1} \leq b_1, \ldots, W_{t_L} \leq b_L)| \leq \epsilon, \]

From the definition of Itô’s integral, we have

\[ (B_{u_1}, B_{u_2}, \ldots, B_{u_K}, W_{v_1}, W_{v_2}, \ldots, W_{v_L}) \xrightarrow{d} (B_{u_1}, B_{u_2}, \ldots, B_{u_K}, W_{v_1}, W_{v_2}, \ldots, W_{v_L}). \]  

(52)

Combine (52) and (53), as \(|II| \to 0,\)

\[ (B_{u_1}, B_{u_2}, \ldots, B_{u_K}, W_{v_1}, W_{v_2}, \ldots, W_{v_L}) \xrightarrow{d} (B_{u_1}, B_{u_2}, \ldots, B_{u_K}, W_{v_1}, W_{v_2}, \ldots, W_{v_L}). \]

Remark 6.4. Suppose \(\{M_t\}_{t \geq 0}, \{N_t\}_{t \geq 0}\) are two local martingales with respect to \(\mathcal{F}_t\) and \(|M|_t = |N|_t, \forall t,\) then Theorem 2.3, Lemma 6.1, Proposition 2.5 and Proposition 2.9 can be generalized directly. Set

\[ T_t = \frac{|M|_t + |M, N|_t}{2}, S_t = \frac{|M|_t - |M, N|_t}{2}, \]

and define \(X_t, Y_t\) similarly with Section 2.1. Then we have

\[ (M_t, N_t) = (X_{T_t} + Y_{S_t}, X_{T_t} - Y_{S_t}), \]

and \(\{X_t\}_{t \geq 0}\) and \(\{Y_t\}_{t \geq 0}\) are two independent Brownian motions. \(\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}\) and \(\{T_t\}_{t \geq 0}\) are mutually independent is equivalent with following statements:

(E) for any \(\mathcal{F}_\infty^{T,S}\) measurable processes \(\{\phi_1^i\}_{t \geq 0}\) and \(\{\phi_2^i\}_{t \geq 0}\) which satisfy \(\int_0^\infty (\phi_1^1)^2 dT_u + \int_0^\infty (\phi_2^1)^2 dS_u\) are bounded, we have

\[ E \left[ \exp \left( \int_0^\infty \phi_1^1 dX_{T_u} + \int_0^\infty \phi_2^1 dY_{S_u} \right) | \mathcal{F}_\infty^T \right] = \exp \left( \frac{1}{2} \int_0^\infty (\phi_1^1)^2 dT_u + \frac{1}{2} \int_0^\infty (\phi_2^1)^2 dS_u \right). \]

(D1) \(\mathcal{F}_\infty^M \perp \mathcal{F}_\infty^{T,S} | \mathcal{F}_\infty^{M,N}\) and \(\mathcal{F}_\infty^N \perp \mathcal{F}_\infty^{T,S} | \mathcal{F}_\infty^{M,N}\).

(D2) suppose \(\{\phi_1^1\}_{t \geq 0}\) and \(\{\phi_2^1\}_{t \geq 0}\) are two bounded predictable processes which is measurable with \(\mathcal{F}_\infty^{T,S}\), then

\[ D_t^\phi = \exp \left( \int_0^t \phi_1^1 dX_{T_u} + \int_0^t \phi_2^1 dY_{S_u} - \frac{1}{2} \int_0^t (\phi_1^1)^2 dT_u - \frac{1}{2} \int_0^t (\phi_2^1)^2 dS_u \right) \]

is a martingale and \(dQ | \mathcal{F}_t \) \(= D_t^\phi\) defines a new probability satisfy

\[ (X_t^\phi, Y_{\tilde{S}}^\phi)_{Q} \overset{d}{=} (X_{T_t}, Y_{S_t})_{P}, \]

where \(X_t^\phi = X_{T_t} - \int_0^t \phi_1^1 dT_u, Y_{\tilde{S}}^\phi = Y_{S_t} - \int_0^t \phi_2^1 dS_u.\)
Moreover, if \([M]_t = [N]_t\) is absolute continuous with \(t\), then according to martingale representation theorem, we can rewrite \((M_t, N_t)\) as

\[
(M_t, N_t) = \left( \int_0^t \theta_u dB_u, \int_0^t \zeta_u dB_u + \int_0^t \eta_u dZ_u \right),
\]

where \(B\) and \(Z\) are two independent Brownian motions. It is evident that \(\mathcal{F}^T, S \subset \mathcal{F}^B, Z, T\). Then \(\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}\) and \(\{T_t\}_{t \geq 0}\) are mutually independent is equivalent with

(D3) \(\mathcal{F}^0, Z, \{B_t\}_{t \geq 0}\) and \(\{Z_t\}_{t \geq 0}\) are mutually independent.

Particularly, the equivalence condition (D2), (D3) of \(\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}\) and \(\{T_t\}_{t \geq 0}\) are mutually independent in 1-dimension situation, i.e. Ocone martingale, illustrated in Kallsen 2006 and Vostrikova and Yor 2000 is a special case of \(M_t = N_t\). Ocone martingale has been widely used in financial mathematics, such as Carr et al. 2005 and Geman et al. 2001.

**Proof of Theorem 3.1** The martingale properties of \(X_T, Y_S\) are easy to check, since immediately by optional stopping theorem, \(X_T, Y_S\) are martingales under \(\mathcal{F}^{X_T, Y_S}\) respectively and \(\mathcal{F}^X S \subset \mathcal{F}^{X_T, Y_S}\), \(\mathcal{F}^{Y_S} \subset \mathcal{F}^{X_T, Y_S}\) guarantee that

\[
E \left[ X_{T_u} | \mathcal{F}^{X_T, Y_S}_t \right] = E \left[ X_{T_u} | \mathcal{F}^{X_T}_t \right], \quad E \left[ Y_{S_u} | \mathcal{F}^{X_T, Y_S}_t \right] = E \left[ Y_{S_u} | \mathcal{F}^{Y_S}_t \right], \quad u \geq t.
\]

When \(T_t + S_t = t\), \(X_T\) and \(Y_S\) are continuous martingales under \(\mathcal{F}^{X_T, Y_S}\). By direct calculation, for \(u \geq t \geq 0\)

\[
E \left[ X_{T_u} Y_{S_u} | \mathcal{F}^{X_T, Y_S}_t \right] = E \left[ X_{T_u} | \mathcal{F}^{X_T, Y_S}_t \right] E \left[ Y_{S_u} | \mathcal{F}^{X_T, Y_S}_t \right]
\]

Accordingly, \([X_T, Y_S]_t = < X_T, Y_S >_t = 0\). And thus,

\[
[B]_t = [X_T + Y_S]_t = [X_T]_t + [Y_S]_t + 2[X_T, Y_S]_t = T_t + S_t = t.
\]

Similarly,

\[
[W]_t = [X_T - Y_S]_t = t.
\]

Hence \(B\) and \(W\) are Brownian motions with respect to \(\mathcal{F}^{B, W}\) (which is equal to \(\mathcal{F}^{X_T, Y_S}\)). And \([B, W]_t = [X_T + Y_S, X_T - Y_S]_t = T_t - S_t, t \geq 0\). \(\square\)

**Proof of Corollary 3.2** Let

\[
\mathcal{F}_t \triangleq \sigma \{ X_u, Y_u, \{ T_v \leq u \}, \{ S_v \leq u \} : u \leq t, \forall v \},
\]

then \(\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}\) are two standard Brownian motions with respect to \(\mathcal{F}_t\) result from the independency of \(\{X_t\}_{t \geq 0}, \{Y_t\}_{t \geq 0}, \{T_t\}_{t \geq 0}\). By definition of \(\mathcal{F}_t\), \(\{ T_u \leq t \}, \{ S_u \leq t \} \in \mathcal{F}_t\) for any \(u > 0\), hence \(T_u, S_u\) are stopping times, and \(\{T_t\}_{t \geq 0}, \{S_t\}_{t \geq 0}\) are time changes of \(\mathcal{F}^t\).

Then by Lemma 6.2 the conditions in Theorem 3.1 are satisfied, and we get the desired result. \(\square\)
\textbf{Proof of Proposition 4.3} By definition,
\[
\hat{G}(\lambda_1, \lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda_1 x_1 + i\lambda x} G(x_1, x) \, dx_1 \, dx.
\]

According to Fubini theorem,
\[
\int_{-\infty}^{\infty} e^{i\lambda_1 x_1} G(x_1, x) \, dx_1 = E \left[ \int_{-\infty}^{\infty} e^{i\lambda_1 x_1} (\gamma_1 + \gamma_2 e^{i\tau_3 M_T}) 1_{\{\gamma_4 M_T \leq x\}} 1_{\{\gamma_5 M_T \leq x\}} \, dx_1 \right]
= E \left[ (\gamma_1 + \gamma_2 e^{i\tau_3 M_T}) 1_{\{\gamma_4 M_T \leq x\}} \int_{-\infty}^{\infty} e^{i\lambda_1 x_1} 1_{\{\gamma_5 M_T \leq x_1\}} \, dx_1 \right]
= E \left[ (\gamma_1 + \gamma_2 e^{i\tau_3 M_T}) \frac{1}{i\lambda_1} e^{i\lambda_1 x_1} \int_{\gamma_4}^{\gamma_5} e^{i\lambda_1 x_1} \, dx_1 \right]
= \frac{1}{i\lambda_1} E \left[ e^{i\lambda_1 \gamma_4 M_T} (\gamma_1 + \gamma_2 e^{i\tau_3 M_T}) 1_{\{\gamma_5 M_T \leq x\}} \right],
\]
where the last equality comes from the fact that the imaginary part of $\lambda_1$ is positive, which deduces that $e^{i\lambda_1 x_1} |_{x=\infty} = 0$. With similar calculation for $x$, we have
\[
\hat{G}(\lambda_1, \lambda) = \int_{-\infty}^{\infty} \frac{1}{i\lambda_1} e^{i\lambda x} E \left[ e^{i\lambda_1 \gamma_4 M_T} (\gamma_1 + \gamma_2 e^{i\tau_3 M_T}) 1_{\{\gamma_5 M_T \leq x\}} \right] \, dx
= -\frac{1}{i\lambda_1} E \left[ e^{i\lambda_1 \gamma_4 M_T + i\lambda \tau_3 M_T} (\gamma_1 + \gamma_2 e^{i\tau_3 M_T}) \right]
= -\frac{\gamma_1}{\lambda_1} \Phi_{M_T}(\lambda_1 \gamma_4 + \lambda \gamma_5) - \frac{\gamma_2}{\lambda_1} \Phi_{M_T}(\lambda_1 \gamma_4 + \lambda \gamma_5 - i\tau_3),
\]
$\Phi_{M_T}$ denotes the characteristic function of $M_T$. In our model, it can be calculated by conditional expectation
\[
\Phi_{M_T}(z_1, z_2) = E e^{i z_1 Y_T + i z_2 Y_T} = E \left[ E[e^{i z_1 X_T + i z_2 Y_T} \mid T_t, S_t] \right] = e^{-\frac{1}{2} T_t \sigma_1^2 - \frac{1}{2} S_t \sigma_2^2}
= e^{-\frac{1}{2} T_t z_1^2} e^{-\frac{1}{2} (z_1 - z_2)^2 \sigma_1^2} = e^{-\frac{1}{2} T_t z_1^2} L_t \left(-\frac{1}{2} (z_1 - z_2)^2 \right),
\]
where $L_t$ means the generalized fourier transform of $T_t$ at time $t$. \hfill \qed

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