The Electromagnetic Pion Form Factor and Instantons

Hilmar Forkel and Marina Nielsen

Department of Physics and Center for Theoretical Physics
University of Maryland, College Park, MD 20742-4111 (U.S.A.)
(February 2022)

Abstract

We calculate the electromagnetic pion form factor at intermediate spacelike momentum transfer from the QCD sum rule for the correlation function of two pseudoscalar interpolating fields and the electromagnetic current. This correlator receives essential contributions from direct (i.e., small-scale) instantons, which we evaluate under the assumption of an instanton size distribution consistent with instanton liquid and lattice simulations. The resulting form factor is in good agreement both with the sum rule based on the axial-current correlator and with experiment.

*Address after Nov. 1, 1994: European Centre for Theoretical Studies in Nuclear Physics and Related Areas, Villa Tambosi, Strada delle Tabarelle 286, I-38050 Villazzano, Italy

†Permanent address: Instituto de Física, Universidade de São Paulo, 01498 - SP- Brazil.
A central goal of strong-interaction physics is the understanding of hadrons on the basis of quantum chromodynamics (QCD). It is very unlikely that this goal can be reached without a thorough understanding of the QCD vacuum. At present, however, direct links between observed hadron properties and the vacuum structure are still rare and rely mostly on extensive numerical simulations. The aim of this letter is to study one such link – between direct instantons, *i.e.* small-scale topological vacuum fields, and electromagnetic pion properties – analytically in the framework of a QCD sum rule [1,2].

Due to the Goldstone nature of the pion, sum rule calculations of pionic properties can be based on two in principle (but not in practice!) equally suitable sets of correlation functions, corresponding to the use of pseudoscalar or axial vector interpolating fields. The pion couples strongly to both of these source currents and thus contributes to the correlators in both channels.

The pseudoscalar channel has some principal advantages for sum rule calculations. The accuracy of the standard pole-continuum parametrization for the corresponding spectral functions profits from the almost complete dominance of the pion in the low-mass region [3,4]. Furthermore, correlators involving the pseudoscalar interpolators have a simpler tensor structure. This simplifies in particular the calculation of three-point functions.

All existing sum rule applications in the pion channel (with the exception of ref. [5], see below), however, are based on axial vector current correlators [1,2,6,7]. The use of the pseudoscalar interpolating field has been avoided, since it is known to receive essential contributions from direct instantons [3]. Like instanton contributions in general, they could initially not be estimated reliably, due to insufficient knowledge of the instanton size distribution in the vacuum. The attempt of an *ab initio* description in the dilute gas approximation [8], in particular, failed for all but the smallest instanton sizes due to infrared problems with large instantons. The same problems were encountered in the attempt to supplement the conventional operator product expansion (OPE) in QCD sum rules with direct instanton contributions and led to the preference for the axial vector correlators.
In the last decade, however, mutually consistent information about the instanton size distribution has been gathered from new sources, including phenomenological estimates [9], variational studies [10] and numerical simulations of instanton liquid models [11], as well as lattice calculations [12]. The bulk features of the density $n(\rho)$ of instantons with size $\rho$ in the vacuum, which emerged from these studies, can be summarized in a simple parametrization [9],

$$n(\rho) = \bar{n} \delta(\rho - \bar{\rho}) \ ,$$

with the average (anti-) instanton density and size fixed at

$$\bar{n} = \frac{1}{2} \text{fm}^{-4} , \quad \bar{\rho} = \frac{1}{3} \text{fm} \ .$$

This form provides a reasonable approximation to the sharply peaked gaussian distribution found in ref. [11] and should be sufficiently accurate for our purpose.

Armed with the bulk features of the instanton density, we can now try to turn the former vice of the pseudoscalar interpolating field – its strong coupling to instantons – into a virtue, by using pseudoscalar correlators as a tool for “instanton diagnostics”. Linked by the corresponding sum rules to observable hadron properties, these correlators provide a theoretical laboratory both for the study of instanton effects at short and intermediate distances and for tests of the approximations used to calculate them. They also allow an independent check of the instanton distribution (1).

With this motivation in mind, we calculate in the present paper the electromagnetic pion form factor from a QCD sum rule based on pseudoscalar interpolating fields. The instanton contributions are evaluated semiclassically in the zero-mode approximation. Our investigation complements similar studies of the nucleon [13] and pion [5] mass sum rules.

The sum rule techniques for the calculation of hadron form factors at intermediate momentum transfers were developed in refs. [6,7]. Their application to the pion form factor, based on axial vector interpolators, leads to a rather good agreement with experiment (in
the momentum transfer region $0.5 < Q^2 < 3.0 \text{ GeV}^2$ \cite{6,7}. Our calculation will thus also provide a consistency check between the axial vector and pseudoscalar sum rules.

To derive the sum rule for the pion form factor, we consider the three-point correlation function of two pseudoscalar currents $j_5(x) = \overline{d}(x)i\gamma_5u(x)$ and the electromagnetic current $j_\mu^e(x) = e_u \overline{u}(x)\gamma_\mu u(x) + e_d \overline{d}(x)\gamma_\mu d(x)$,

$$\Gamma_\mu(p, p'; q) = -\int d^4x \int d^4ye^{i(p'-x-q \cdot y)} \langle 0 \mid T j_5^\dagger(x) j_\mu^e(y) j_5(0) \mid 0 \rangle,$$

with $q = p' - p$. $\Gamma_\mu$ can be decomposed into two independent Lorentz vector structures,

$$\Gamma_\mu(p, p'; q) = \Gamma_1(p^2, p'^2, q^2) \left( p' + p \right)_\mu + \Gamma_2(p^2, p'^2, q^2) q_\mu.$$

The invariant amplitudes $\Gamma_{1,2}$ have a double dispersion representation of the form

$$\Gamma_i(p^2, p'^2, q^2) = \frac{1}{\pi^2} \int_0^\infty ds \int_0^\infty ds' \frac{\rho_i(s, s', q^2)}{(s - p^2)(s' - p'^2)} + \ldots,$$

where the ellipsis represents subtraction polynomials in $p^2$ and $p'^2$, which will vanish under the Borel transform to be applied later.

Since the pseudoscalar current has a nonvanishing matrix element between the vacuum and the one-pion state,

$$\langle 0 \mid i\gamma_5u \mid \pi^+ \rangle = \sqrt{2} f_\pi K,$$

$$K = \frac{m_\pi^2}{m_u + m_d}$$

($f_\pi = 93 \text{ MeV}$ is the pion decay constant and $m_{u,d}$ are the up and down current quark masses), the pion contribution to the spectral density is of the form

$$\langle 0 \mid j_5^\dagger(\pi(p))\langle \pi(p')\mid j_\mu^e(\pi(p))\langle \pi(p)\mid j_5 \mid 0 \rangle = 2 f_\pi^2 K^2 Q_\pi F_\pi(Q^2) (p' + p)_\mu,$$

where $Q^2 = -q^2$, $Q_\pi$ is the charge of the pion and $F_\pi(Q^2)$ is the form factor to be calculated. The latter is contained in the amplitude $\Gamma_1$, on which we will concentrate in the following.

For the continuum contribution we adopt the standard form of refs. \cite{6,7}, which completes our parametrization of the spectral density $\rho_1$:

$$\rho_1(s, s', Q^2) = 2 f_\pi^2 K^2 Q_\pi F_\pi(Q^2) \delta(s) \delta(s') + \theta(s + s' - s_0) \rho_0(s, s', Q^2).$$
\((\rho_0(s, s', Q^2))\) is the free-quark spectral function and we neglected the small pion mass.) The continuum threshold \(s_0\) defines a triangular region in the \((s, s')\) plane and is related to the threshold \(s_1\) of the corresponding pseudoscalar two-point correlator as \(s_0 \simeq 1.5 s_1\) (see ref. [3] for details).

An alternative continuum ansatz with a quadratic integration region bounded by \(s_1\) was also considered in [6] for the axial-vector three-point function. Similar to ref. [3] we find almost no difference between the two parametrizations in the resulting form factor [14].

The leading instanton (and anti-instanton) contributions to \(\Gamma_\mu\) are calculated in semi-classical approximation, which amounts to evaluating the correlator (3) in the background of the (anti-) instanton fields and to averaging their collective coordinates over the corresponding vacuum distributions. We will outline only the essential features of this procedure and refer to the literature [5,9,13] for more details.

Since the sum rule probes the correlation function mainly at distances \(x, y \simeq 0.2\) fm, which are small compared to the average inter-instanton separation \(\bar{R} \simeq 1\) fm, the correlated contributions from two or more instantons should be small. We are thus led to treat only the effects of the (anti-) instanton closest to \(x\) and \(y\) explicitly and to include effects of interactions with other vacuum fields (including other instantons) at the mean-field level [15].

To obtain the explicit instanton contribution, we recall that the quark spectrum in the background of an (anti-) instanton contains one zero-mode state per flavor [16],

\[
\psi_0^\pm(x) = \frac{\rho}{\pi (r^2 + \rho^2)^{3/2}} \frac{\mp \gamma_5}{r} U, \tag{9}
\]

with \(r = x - z\), where \(z\) is the instanton position. (The spin-color matrix \(U\) satisfies \((\vec{\sigma} + \vec{\tau}) U = 0\).) Up to corrections of order \(m^* \bar{\rho}\) (see below) from higher-lying states, quarks in these two zero-modes dominate the instanton contribution to the correlator (3). Their contribution is evaluated by inserting the explicit expression (9) into (3) and by approximating quarks propagating in the higher-lying (continuum) states as non-interacting.
For quarks propagating in zero-mode states the interaction with the other vacuum fields leads at the mean-field level to the generation of an effective quark mass $m^*(\rho) = -\frac{2}{3} \pi^2 \rho^2 \langle \overline{q} q \rangle$ \cite{15}, which replaces the current mass in the zero-mode propagator. Taking this effect into account, we obtain (after a Wick-rotation to euclidean space-time) the following instanton contribution to the correlator \cite{3}:

$$
\Gamma_{\mu}(p, p'; q) = -\frac{4Q\pi}{\pi^6} \frac{\bar{n} \rho^4}{m^*(\rho)} \int d^4x \int d^4y \int d^4z \ e^{i\rho' \cdot x} e^{-i\rho \cdot y} e^{i(p' - q) \cdot z} \times
$$

$$
\left[ \frac{y^2 (y + z) \mu - (y + z)^2 y_{\mu}}{(x^2 + \bar{\rho}^2)^3 \gamma (y^2 + \bar{\rho}^2)^{3/2}} \right] + \left( \begin{array}{c}
z_{\mu} \leftrightarrow x_{\mu} \\
y_{\mu} \rightarrow -y_{\mu}
\end{array} \right),
$$

In this expression we summed over instanton and anti-instanton parts and integrated over their positions. Since we deal with a gauge-invariant correlator, the averaging over the color orientations is trivial, and the average over the remaining collective coordinate, the instanton size $\rho$, was weighted with the distribution \cite{1}.

After taking the standard double Borel transform \cite{6}, both in $p^2$ and $p'^2$, of \cite{10} (details of the rather lengthy calculation will be given in \cite{14}), we obtain

$$
\Gamma_{1}^{(\text{in})}(Q^2, M^2) = -Q\pi \frac{\bar{n} M^2}{m^*(\rho)} I_{\text{inst}}(\bar{\rho}^2 Q^2, \bar{\rho}^2 M^2),
$$

\cite{11} (11)

($M$ is the Borel mass) in terms of the dimensionless integral

$$
I_{\text{inst}}(\bar{z}^2, z^2) = \int_0^\infty d\alpha \int_0^{z^2} d\epsilon e^{-\alpha \bar{z}^2} e^{-(\alpha' + \gamma') \frac{\epsilon}{4(1 - \epsilon \bar{z}^2)}} \frac{\alpha \epsilon}{A^4(1 - \epsilon \bar{z}^2)} \times
$$

$$
\left\{ H(\alpha') H(\gamma') \left[ \frac{\alpha + \epsilon}{z^2} \left( \frac{\alpha}{16} \frac{\epsilon \bar{z}^2}{z^2} - \frac{8(1 - \epsilon \bar{z}^2)}{(1 - \epsilon \bar{z}^2)^2} - 3 \right) - \frac{2 \alpha \epsilon}{A^2}(2\alpha - A) - \frac{\alpha^3 \epsilon}{4z^2 A^2} \right] \right\},
$$

(12)

where $A = \frac{\alpha + \epsilon}{z^2} + \alpha \epsilon$, $\alpha' = \frac{\alpha}{8A}$, $\gamma' = \frac{1}{8z^2 A}$, and $H(x) = I_1(x) - I_0(x)$ is defined in terms of the modified Bessel functions $I_n(x)$.

The lowest-order perturbative contribution to $\Gamma_1$ was evaluated in ref. \cite{6}.

\footnote{Note that the corresponding expression in \cite{6} contains a misprint.}
\[ \Gamma_1^{(pert)}(Q^2, M^2) = \frac{3Q^2M^2}{16\pi^2} I_{\text{pert}}(Q^2, M^2), \]

where the continuum contributions from the parametrization (8) are subtracted in the integral

\[ I_{\text{pert}}(Q^2, M^2) = \int_{s_0/M^2}^{s_0/M^2} dx e^{-x} \int_{0}^{x} dy \frac{x^2 - y^2}{(Q^4/M^4 + 2xQ^2/M^2 + y^2)^{3/2}}. \]

As already noted in ref. [6], the power corrections to \( \Gamma_1^{(pert)} \) are small in the \( Q^2 \) range under consideration and will be neglected in this letter. A more complete analysis of the OPE can be found in [14].

Adding instanton (11) and perturbative (13) contributions and equating them to the Borel transformed dispersion representation (5) leads to the sum rule

\[ 2f_\pi^2 K^2 F_\pi(Q^2) = \left[ \frac{3Q^2M^2}{16\pi^2} I_{\text{pert}}(Q^2, M^2) - \frac{\bar{n} M^2}{m^2(\bar{\rho})} I_{\text{inst}}(\bar{\rho}^2 Q^2, \bar{\rho}^2 M^2) \right] L^{-8/9}. \]

The factor \( L^{-8/9} \), with \( L = \ln(M^2/\Lambda_{QCD}^2) / \ln(\mu^2/\Lambda_{QCD}^2) \), accounts for the scaling behavior of the three-point correlator due to the anomalous dimension of the pseudoscalar currents, and sets their renormalization point to \( \mu = 500 \) MeV. (We use the value \( \Lambda_{QCD} = 150 \) MeV for the QCD scale parameter.)

Let us now fix the remaining parameters in the sum rule. As already mentioned, the continuum threshold \( s_0 \) can be related to the corresponding threshold \( s_1 \) of the pseudoscalar two-point correlator, \( s_0 \simeq 1.5 s_1 \). The analysis of [11] (see also ref. [3]) finds \( s_1 \simeq 2.0 \) GeV\(^2\) and we will thus fix the threshold at \( s_0 = 3.0 \) GeV\(^2\). The relatively large separation of the continuum from the lowest resonance in the pseudoscalar channel (about twice of that in the vector and axial vector channels) is clearly favorable for the sum rule analysis, since it improves the accuracy of the parametrization (8).

Due to uncertainties in the determination of the light quark masses, the phenomenological value of the (scale-dependent) mass parameter \( K(\Lambda) \), defined in eq. (5), is at present not

\(^2\)See also the related discussion for the two-point correlator in [3].
accurately known. From the current bounds on the up and down quark masses [17] and with $m_{\pi} = 138$ MeV we obtain $1 \leq K(1$ GeV $) \leq 2$ GeV. In order to allow for a direct comparison with the instanton analysis of the two-point correlator [3], which takes $K = 0.7$ GeV, we fix $K$ at the lower limit of the phenomenologically acceptable range, $K = 1.0$ GeV. The effective mass $m^*$, finally, is determined as in [3,13] from the self-consistency relation [8] for the quark condensate, $\langle \bar{q}q \rangle = -2 \bar{n}/m^*(\bar{\rho})$.

Let us now proceed to the quantitative analysis of the sum rule (15). Figure 1 shows the $M^2$ dependence of both the perturbative and the instanton contributions to the pion form factor at fixed $Q^2 = 1$ GeV$^2$. Two features of the instanton part are immediately apparent: First, and as expected, it is the dominant contribution, about a factor of two larger than the perturbative part. Secondly, it improves the stability of the sum rule, i.e. it reduces the $M^2$ dependence of the form factor, and a “stability plateau” begins to develop for $M^2 > 1.2$ GeV$^2$. The same qualitative behavior is found for all values of $Q^2 \geq 0.5$ GeV$^2$.

In order to determine the $Q^2$ dependence of the form factor, we follow the procedure of ref. [6] and evaluate the sum rule for different values of $Q^2$ at a fixed value of the Borel mass, $M^2 = 1.2$ GeV$^2$, postponing a more accurate analysis to ref. [14]. The resulting form factor is shown in fig.2 and compared with the experimental data of [18] at the space-like momentum transfers accessible within our approach [4]. The agreement is clearly satisfactory and at least as good as the one from the axial vector sum rule [8] (dashed line), which we show for comparison. (A slightly better agreement between the axial vector sum rule result and the data was obtained in ref. [7] by extracting the $Q^2$ dependence of $F_\pi$ at $M^2 = 1.8$ GeV$^2$.)

The somewhat better fit of the pseudoscalar sum rule result at low $Q^2$ (where also the data become more reliable) is perhaps not accidental. The result of the axial-vector based

---

$^3$The perturbative part dominates the instanton contributions at $Q^2 \to \infty$ and shows the expected $(Q^2)^{-2}$ behavior. The so far neglected power corrections, however, will eventually blow up [14], thereby limiting the applicability of the sum rule at large $Q^2$. 

---
sum rule relies exclusively on the OPE, which breaks down at small $Q^2$, since physics of longer distance scales begins to determine the behavior of the three-point function. The instanton contribution, in contrast, which dominates our result, could optimistically remain reliable up to distances not too far below the inter-instanton separation, corresponding to $Q^2 \simeq 0.1 - 0.2 \text{ GeV}^2$. Indications supporting this expectation have been found in an analogous study of the nucleon correlation functions [19]. The applicability of the sum rule at small $Q^2$ would then be mainly limited by the unphysical singularity in the perturbative contribution (19).

To conclude, we view the above results, and in particular the stability of the pseudoscalar sum rule and its agreement with phenomenology, as further support for the importance of the instanton component in the QCD vacuum, for the bulk features of their distribution as given in eq. (2), and for the semiclassical estimate of their effects in hadron correlators at intermediate distances.

H.F. acknowledges support from the Department of Energy under Grant No. DE–FG05–93ER–40762 and M.N. acknowledges support from FAPESP Brazil.
REFERENCES

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147, 385 (1979); B147, 448 (1979); B147, 519 (1979).

[2] L. J. Reinders, H. R. Rubsteins and S. Yazaki, Nucl. Phys. B196, 125 (1982)

[3] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B191, 301 (1981).

[4] E. V. Shuryak, Rev. Mod. Phys. 65, 1 (1993).

[5] E. V. Shuryak, Nucl. Phys. B214, 237 (1983).

[6] B. L. Ioffe and A. V. Smilga, Phys. Lett. B114, 353 (1982); Nucl. Phys. B216, 373 (1983).

[7] V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. B115, 410 (1982); JETP Lett. 39, 707 (1984).

[8] C. G. Callen Jr., R. Dashen and D. J. Gross, Phys. Rev. D 17, 2717 (1978), D.G. Caldi, Phys. Rev. Lett. 39, 121 (1977)

[9] E. V. Shuryak, Nucl. Phys. B203, 93, 116, 140 (1982).

[10] D. I. Diakonov and V. Yu. Petrov, Nucl. Phys. B245, 259 (1984); Phys. Lett. B147, 351 (1984); Nucl. Phys. B272, 457 (1986).

[11] E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. B 341, 1 (1990)

[12] M.-C. Chu and S. Huang, Phys. Rev. D 45, 2446 (1992); M.-C. Chu, J. M. Grandy, S. Huang and J. W. Negele, Phys. Rev. D 49, 6039 (1994).

[13] H. Forkel and M. K. Banerjee, Phys. Rev. Lett 71, 484 (1993).

[14] M. Nielsen and H. Forkel, in preparation.

[15] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B163, 46 (1980).
[16] G. ‘t Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1976).

[17] J. Gasser and H. Leutwyler, Phys. Rep. 87, 77 (1982); H. Leutwyler, Bern University preprint BUTP-94/8 (1994) (hep-ph 9405330).

[18] C. Bebek et al., Phys. Rev. D 17, 1693 (1978).

[19] H. Forkel, University of Maryland preprint (1994).
FIG. 1. Borel mass dependence of the pion form factor from Eq. (13) at $Q^2 = 1 \text{GeV}^2$ (solid curve). The dashed and dotted curves correspond to the instanton and the perturbative contributions, respectively.

FIG. 2. The electromagnetic pion form factor from Eq. (13) (solid line) at $M^2 = 1.2 \text{GeV}^2$. The dashed line shows for comparison the result of the axial vector sum rule [3]. The experimental data are taken from ref. [18].
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9408396v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9408396v1