Geometry of branched minimal surfaces of finite index

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Abstract

Given \( I, B \in \mathbb{N} \cup \{0\} \), we investigate the existence and geometry of complete finitely branched minimal surfaces \( M \) in \( \mathbb{R}^3 \) with Morse index at most \( I \) and total branching order at most \( B \). Previous works of Fischer-Colbrie [3] and Ros [13] explain that such surfaces are precisely the complete minimal surfaces in \( \mathbb{R}^3 \) of finite total curvature and finite total branching order. Among other things, we derive scale-invariant weak chord-arc type results for such an \( M \) with estimates that are given in terms of \( I \) and \( B \). In order to obtain some of our main results for these special surfaces, we obtain general intrinsic monotonicity of area formulas for \( m \)-dimensional submanifolds \( \Sigma \) of an \( n \)-dimensional Riemannian manifold \( X \), where these area estimates depend on the geometry of \( X \) and upper bounds on the lengths of the mean curvature vectors of \( \Sigma \). We also describe a family of complete, finitely branched minimal surfaces in \( \mathbb{R}^3 \) that are stable and non-orientable; these examples generalize the classical Henneberg minimal surface.

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1 Introduction

Let \( X \) be a complete Riemannian 3-manifold with positive injectivity radius \( \text{Inj}(X) \). Let \( M \) be a complete immersed surface in \( X \) of constant mean curvature (CMC). The Jacobi operator of \( M \) is the Schrödinger operator

\[
L = \Delta + |A_M|^2 + \text{Ric}(N),
\]

where \( \Delta \) is the Laplace-Beltrami operator on \( M \), \( |A_M|^2 \) is the square of the norm of its second fundamental form and \( \text{Ric}(N) \) denotes the Ricci curvature of \( X \) in the direction of the unit normal vector \( N \) to \( M \); the index of \( M \) is the index of \( L \),

\[
\text{Index}(M) = \lim_{R \to \infty} \text{Index}(B_M(p, R)),
\]

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where $B_M(p, R)$ is the intrinsic metric ball in $M$ of radius $R > 0$ centered at a point $p \in M$, and $\text{Index}(B_M(p, R))$ is the number of negative eigenvalues of $L$ on $B_M(p, R)$ with Dirichlet boundary conditions. Here, we have assumed that the immersion is two-sided (which is the case when the constant value $H$ of the mean curvature of $M$ is not zero). In the case, the immersion is one-sided, then the index is defined in a similar manner using compactly supported variations in the normal bundle; see Definition 3.2 for details.

Given $I, B \in \mathbb{N} \cup \{0\}$, we investigate the existence and geometry of complete finitely branched minimal surfaces $M$ in $\mathbb{R}^3$ with index at most $I$ and total branching order at most $B$; let $\mathcal{M}(I, B)$ be the space of such examples. Works of Fischer-Colbrie [3] and Ros [13] ensure that the surfaces $M \in \bigcup_{I, B \in \mathbb{N} \cup \{0\}} \mathcal{M}(I, B)$ are precisely the complete minimal surfaces in $\mathbb{R}^3$ of finite total curvature and finite total branching order. One goal of this paper is to derive certain scale-invariant weak chord-arc type results for surfaces in $\mathcal{M}(I, B)$ with explicit estimates given in terms of $I$ and $B$; see Proposition 4.1 for these estimates. We also describe some interesting new examples of non-orientable surfaces in $\mathcal{M}(0, B)$, $B \geq 2$. These new examples of complete stable branched minimal surfaces generalize the classical Henneberg surface of finite total curvature $-2\pi$ that has two simple branch points; these surfaces are described analytically and geometrically at the end of Section 3. In Section 3 we also explain how to extend to $\mathcal{M}(I, B)$ the geometric and topological lower bound estimates for the index of complete unbranched minimal surfaces with finite total curvature due to Chodosh and Maximo [2].

In Section 2 we study the area of intrinsic balls $B_M(x, R)$ of an $n$-dimensional submanifold of a Riemannian $m$-manifold $M$, where $x \in M$ and $0 < R \leq \text{Inj}(X)$. In particular, we derive explicit upper bounds for the area growth of $B_M(x, R)$ as a function of $R \in (0, \text{Inj}(X)]$, that depend on upper bounds for the sectional curvature of the extrinsic geodesic ball $B_X(x, R)$ and for the length of the mean curvature vector of $M$ restricted to $B_M(x, R)$. In Section 4 we will apply this intrinsic area estimate to obtain certain scale-invariant weak chord-arc bounds for any surface $M \in \mathcal{M}(I, B)$; see Proposition 4.1.

The intrinsic area estimates in Section 2 will also be applied in our papers [8] and [9] to study CMC surfaces of bounded index in spaces $X$ of dimension three. The monotonicity-of-area type formulae in Proposition 2.4, the weak chord-arc results given in Proposition 4.1 and other theoretical results in Section 3, such as the aforementioned extension of the Chodosh and Maximo lower bound estimates for the index of surfaces in $\mathcal{M}(I, B)$, have important applications to the proof to the Hierarchy Structure Theorem 1.1 in [9]; this theorem is a fundamental result that describes the structure of complete CMC surfaces of finite index in a 3-dimensional $X$ with $\text{Inj}(X) > \delta > 0$ and having a fixed an upper bound on its absolute sectional curvature function, and it was our main motivation for developing the results in the present paper.

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2 Volume growth of intrinsic balls in submanifolds of bounded mean curvature vector

Let $M$ be an immersed $n$-dimensional submanifold in a geodesic ball $B_X(x_0, R_1)$ of an $m$-dimensional manifold $(X, g)$, with $x_0 \in M$ and $R_1$ less or than equal to the injectivity radius function $\text{Inj}_X(x_0)$ of $X$ at $x_0$. In this section we will find lower bounds for the $n$-dimensional volume $A(r)$ of $B_M(x_0, r)$, as a function of $r \in (0, \text{Inj}_X(x_0)]$; see Proposition 2.4 below for a precise description.

Let us denote by $\Delta$, $\Delta$ the Laplacians in $X$ and $M$, respectively. Analogously, $\nabla, \nabla$ will stand for the Levi-Civita connections and gradient operators. Let $N_{n+1}, \ldots, N_m$ be a local orthonormal basis of the normal bundle to $M$, and let $\vec{H}$ be the mean curvature vector of $M$. We start with a well-known formula.

**Lemma 2.1.** Given $f \in C^\infty(X)$, $(\Delta f)|_M = \Delta(f|_M) - n \vec{H}(f) + \sum_{j=n+1}^m g(\nabla_{N_j} \nabla f, N_j)$.

**Proof.** Let $\{v_1, \ldots, v_n\}$ be a local orthonormal basis for $TM$.

\[
(\Delta f)|_M = \sum_{i=1}^n g(\nabla_{v_i} \nabla f, v_i) + \sum_{j=n+1}^m g(\nabla_{N_j} \nabla f, N_j)
\]

\[
= \sum_{i=1}^n g(\nabla_{v_i} \nabla f + \sum_{j=n+1}^m N_j(f) N_j, v_i) + \sum_{j=n+1}^m g(\nabla_{N_j} \nabla f, N_j)
\]

\[
= \sum_{i=1}^n g(\nabla_{v_i} \nabla f, v_i) + \sum_{j=n+1}^m N_j(f) \sum_{i=1}^n g(\nabla_{v_i} N_j, v_i) + \sum_{j=n+1}^m g(\nabla_{N_j} \nabla f, N_j)
\]

\[
= \Delta(f|_M) - n \vec{H}(f) + \sum_{j=n+1}^m g(\nabla_{N_j} \nabla f, N_j).
\]

Given $a \in \mathbb{R}$, let $s_a(t)$ be the unique solution of $x''(t) + ax(t) = 0$, $x(0) = 0$, $x'(0) = 1$. We will denote by $I_a$, the interval $[0, \pi/\sqrt{a}]$ when $a > 0$, and $I_0 = [0, \infty)$ if $a \leq 0$. Thus, $s_a(t) > 0$ for all $t \in I_a \setminus \{0\}$. Let $f_a : I_a \to \mathbb{R}$ be the smooth function given by

\[
f_a(t) = \frac{1}{t^2} \left(1 - t \frac{s_a'(t)}{s_a(t)}\right), \quad t \in I_a.
\]

A direct computation gives that

\[
f_a(t) = \begin{cases} \frac{1}{t^2} (1 - t \sqrt{a} \cot(\sqrt{at})) & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ \frac{1}{t^2} (1 - t \sqrt{-a} \coth(\sqrt{-at})) & \text{if } a < 0. \end{cases}
\]

The last equality implies that $f_a(t)$ is smooth at $t = 0$, with value $f_a(0) = a/3$.

**Lemma 2.2.** Let $R : B_X(x_0, R_1) \to [0, R_1)$ denote the extrinsic Riemannian distance function in $X$ to $x_0$. 
1. The intrinsic Laplacian of the restriction of $R^2$ to $M$ is

$$
\Delta((R^2)|_M) = 2(m-1)RH^{S(R)} + 2nR\bar{H}(R) + 2|\nabla(R)|_M|^2 - 2R \sum_{j=n+1}^{m} II^{S(R)}(N_j^T, N_j^T),
$$

where $H^{S(R)}$ denotes the mean curvature of the geodesic sphere $S(R) = \partial B(x_0, R)$ with respect to the unit normal $-\nabla R$, $N_j^T = N_j - N_j(R)\nabla R$ is the projection of $N_j$ tangent to $S(R)$, and $II^{S(R)}$ is the second fundamental form of $S(R)$ with respect to $-\nabla R$.

2. If the sectional curvature of $X$ satisfies $K_{sec} \leq a$ for some $a \in \mathbb{R}$, then

$$
\Delta((R^2)|_M) \geq 2n + 2nR\bar{H}(R) - 2R^2 \int_{a}^{\infty} \left( n - |\nabla(R)|_M|^2 \right),
$$

and equality holds in (2.3) if $K_{sec} = a$. In particular if $X$ is flat, then

$$
\Delta((R^2)|_M) = 2n + 2nR\bar{H}(R).
$$

Remark 2.3. For $a = 0$, equation (2.4) generalizes the well-known formula $\Delta((R^2)|_M) = 2n$ for minimal submanifolds of Euclidean space. Similarly, if we assume $K_{sec} \leq 0$, inequality (2.3) generalizes the inequality $\Delta((R^2)|_M) \geq 2n$ for minimal submanifolds given by Yau in [15, equation (7.1)].

Proof. Lemma 2.1 applied to $R^2$ gives

$$
\Delta((R^2)|_M) = (\Delta(R^2))|_M + 2nR\bar{H}(R) - \sum_{j=n+1}^{m} g(\nabla N_j \nabla(R^2), N_j). \tag{2.5}
$$

We now compute the first and third terms of the last RHS. On the one hand, since $|\nabla R| = 1$,

$$
\Delta(R^2) = 2 + 2R\Delta R. \tag{2.6}
$$

As $\nabla R$ is unitary and orthogonal to the geodesic spheres centered at $x_0$, we can take an orthonormal basis of $TX$ of the form $\{E_1, \ldots, E_{m-1}, \nabla R\}$ where $E_1, \ldots, E_{m-1}$ is an orthonormal basis of the tangent space to $S(R)$. Thus,

$$
\Delta R = \sum_{i=1}^{m-1} g(\nabla_{E_i} \nabla R, E_i) + g(\nabla_{\nabla R} \nabla R, \nabla R).
$$

The first term in the last RHS equals $(m-1)H^{S(R)}$, and the second term clearly vanishes. Thus,

$$
\Delta R = (m-1)H^{S(R)}, \tag{2.7}
$$

and

$$
\Delta(R^2) = 2 + 2(m-1)R H^{S(R)}. \tag{2.8}
$$
On the other hand,
\[ g(\nabla_{N_j} \nabla (R^2), N_j) = 2g(\nabla_{N_j} (R \nabla R), N_j) = 2N_j(R) + 2R g(\nabla_{N_j} \nabla R, N_j). \] (2.9)

Decomposing \( N_j = N_j^T + N_j(R) \nabla R \) where \( N_j^T \) is tangent to \( S(R) \), the bilinearity of the second term of the last RHS with respect to \( N_j \) allows us to write
\[ g(\nabla_{N_j} \nabla R, N_j) = g(\nabla_{N_j^T} \nabla R, N_j^T) = \Pi^{S(R)}(N_j^T, N_j^T), \] (2.10)

where we have used that \( g(\nabla \nabla R, \nabla R) = 0 \) and that \( g(\nabla_{N_j^T} \nabla R, \nabla R) = 0 \) because \( \nabla R \) has constant length. From (2.5), (2.8), (2.9) and (2.10) we have
\[ \Delta((R^2)|_M) = 2 + 2(m - 1)R H^{S(R)} + 2n R \bar{H}(R) - 2 \sum_{j=n+1}^{m} N_j(R)^2 - 2R \sum_{j=n+1}^{m} \Pi^{S(R)}(N_j^T, N_j^T). \] (2.11)

Since \( \nabla R = \nabla (R|_M) + \sum_j N_j(R) N_j \), then
\[ 1 = |\nabla R|^2 = |\nabla (R|_M)|^2 + \sum_{j=n+1}^{m} N_j(R)^2. \] (2.12)

Plugging (2.12) into (2.11) we obtain item 1 of the lemma.

As for item 2, we will assume that \( K_{\sec} \leq a \) for some \( a \in \mathbb{R} \). Let \( e_1, \ldots, e_{m-1} \) be an orthonormal basis of principal directions of \( T_x S(R) \), with respective principal curvatures \( \lambda_1, \ldots, \lambda_{m-1} \) with respect to the unit normal \( -\nabla R \) to \( S(R) \). For each \( j = n + 1, \ldots, m \) we can write \( N_j^T = \sum_{i=1}^{m-1} a_{ij} e_i \) where \( a_{ij} = g(e_i, N_j^T) = g(e_i, N_j) \in \mathbb{R} \). Thus,
\[ (m - 1)H^{S(R)} = \sum_{i=1}^{m-1} \lambda_i \quad \text{and} \quad \Pi^{S(R)}(N_j^T, N_j^T) = \sum_{i=1}^{m-1} \lambda_i a_{ij}^2. \]

Hence, we can write the formula in item 1 of the lemma as
\[ \Delta((R^2)|_M) = 2R \sum_{i=1}^{m-1} \lambda_i \left( 1 - \sum_{j=n+1}^{m} a_{ij}^2 \right) + 2n R \bar{H}(R) + 2|\nabla (R|_M)|^2. \] (2.13)

Observe that given any tangent vector \( v \) to \( S(R) \),
\[ \Pi^{S(R)}(v, v) = g(\nabla_v \nabla R, v) = (\nabla^2 R)(v, v), \] (2.14)
where \( \nabla^2 R \) denotes the hessian of \( R \). Since \( K_{\sec} \leq a \), standard comparison results (see e.g. [12, Theorem 27]) give
\[ \frac{s'_a(R)}{s_a(R)} g_R \leq \nabla^2 R, \] (2.15)
where $g_R$ is the induced metric by $g$ on $S(R)$. Evaluating (2.15) at the principal directions $e_i,$ we have
\[
\frac{s_i'(R)}{s_a(R)} \leq \lambda_i, \quad \text{for all } i = 1, \ldots, m - 1.
\] (2.16)

Given $i = 1, \ldots, m - 1,$ we decompose $e_i$ in its tangent and normal components to $M$
\[
e_i = e_i^{T, M} + \sum_{j=n+1}^{m} g(e_i, N_j)N_j = e_i^{T, M} + \sum_{j=n+1}^{m} a_{ij}N_j,
\]
from where
\[
1 = |e_i|^2 \geq \left| \sum_{j=n+1}^{m} a_{ij}N_j \right|^2 = \sum_{j=n+1}^{m} a_{ij}^2.
\]

This last inequality together with (2.13) and (2.16), give
\[
\Delta((R^2)|_M) \geq 2R\frac{s_i'(R)}{s_a(R)} \sum_{i=1}^{m-1} \left( 1 - \sum_{j=n+1}^{m} a_{ij}^2 \right) + 2n R \bar{H}(R) + 2|\nabla (R|_M)|^2
\]
\[
= 2R \frac{s_i'(R)}{s_a(R)} \left( m - 1 - \sum_{j=n+1}^{m} |N_j|^2 \right) + 2n R \bar{H}(R) + 2|\nabla (R|_M)|^2
\]
\[
(2.1) \quad 2 \left( 1 - R^2 f_a(R) \right) \left( m - 1 - \sum_{j=n+1}^{m} |N_j|^2 \right) + 2n R \bar{H}(R) + 2|\nabla (R|_M)|^2
\]
\[
= 2(m - 1) - 2 \sum_{j=n+1}^{m} |N_j|^2 - 2R^2 f_a(R) \left( m - 1 - \sum_{j=n+1}^{m} |N_j|^2 \right)
\]
\[
+ 2n R \bar{H}(R) + 2|\nabla (R|_M)|^2
\]
\[
(\ast) \quad 2n - 2R^2 f_a(R) \left( n - |\nabla (R|_M)|^2 \right) + 2n R \bar{H}(R),
\]
where in (\ast) we have used that
\[
1 - |\nabla (R|_M)|^2 + \sum_{j=n+1}^{m} |N_j|^2 \overset{(2.12)}{=} \sum_{j=n+1}^{m} N_j(R)^2 + \sum_{j=n+1}^{m} |N_j|^2 = \sum_{j=n+1}^{m} |N_j|^2 = m - n.
\]

Now inequality (2.3) is proved. If $K_{sec} = a,$ then both (2.15) and (2.16) are equalities, and the above argument shows that (2.3) is also an equality. In the case $X$ is flat, then $a = 0$ and $f_0(t) = 0,$ which gives (2.4).

The next result generalizes the classical monotonicity of area formula of Allard [1, Section 5.1] for hypersurfaces of bounded mean curvature, in part since it does not require the hypersurface to be proper in the ambient space. Proposition 2.4 is motivated by the calculations in the last two pages of Yau [15], where he derived the lower bound area estimate given in (2.18) when $a \leq 0,$ $H_0 = 0.$
Proposition 2.4 (Intrinsic monotonicity of area formula). Let $\overline{B}_X(x_0, R_1)$ denote a closed geodesic ball in an $m$-dimensional manifold $(X, g)$, where $0 < R_1 \leq \text{Inj}_X(x_0)$, and suppose that $K_{\text{sec}} \leq a$ on $B_X(x_0, R_1)$ for some $a \in \mathbb{R}$. Given $H_0 \geq 0$, define

$$R_0(a, H_0) = \begin{cases} \frac{1}{\sqrt{a}} \arccot \left( \frac{H_0}{\sqrt{a}} \right) & \text{if } a > 0, \\ 1/H_0 & \text{if } a = 0 \quad (\text{if } H_0 = 0 \text{ we take } R_0(0, 0) = \infty) \\ \frac{1}{\sqrt{-a}} \arccosh \left( \frac{H_0}{\sqrt{-a}} \right) & \text{if } a < 0 \quad (\text{if } \frac{H_0}{\sqrt{-a}} \geq 1 \text{ we take } R_0(a, H_0) = \infty), \end{cases}$$

and let $r_1 = r_1(R_1, a, H_0) = \min \{R_1, R_0(a, H_0)\}$.

Suppose $M$ is a complete, immersed, connected $n$-dimensional submanifold of $X$ and $x_0 \in M$ is a point such that when $\partial M \neq \emptyset$, $d_M(x_0, \partial M) \geq R_1$ and the length of the mean curvature vector $\vec{H}$ of $M$ restricted to $\overline{B}_X(x_0, R_1)$ is bounded from above by $H_0$. Then:

1. If $M$ is compact without boundary, then there exists $y \in M$ such that the extrinsic distance from $x_0$ to $y$ is greater than or equal to $r_1$.

2. The $n$-dimensional volume $A(r)$ of $B_M(x_0, r)$ is a strictly increasing function of $r \in (0, r_1]$.

3. For all $r \in (0, r_1)$ when $r_1 \neq \infty$ or otherwise, for all $r \in (0, \infty)$:

$$A(r) \geq \begin{cases} \omega_n r^n e^{-nH_0r} & \text{if } a \leq 0, \\ \omega_n r^n e^{-n(aH_0 + \frac{1}{2}f_a(r_1)r)} & \text{if } a > 0, \end{cases}$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ and the function $f_a$ is defined in (2.2).

Proof. Let $\mathbb{M}^m(a)$ denote the $m$-dimensional, simply-connected space form of constant sectional curvature $a \in \mathbb{R}$. Recall that the number $R_0(a, H_0)$ represents the radius of a geodesic sphere in $\mathbb{M}^m(a)$ with constant mean curvature $H_0$, and that geodesic spheres in $\mathbb{M}^m(a)$ of radii less than $R_0(a, H_0)$ have mean curvature greater than $H_0$.

We first prove item 1 of the lemma. Fix a point $x_0 \in M$ and let $r \in (0, r_1)$. Suppose $M$ is compact with empty boundary and suppose that there does not exist a point $y \in M$ such that the extrinsic distance from $x_0$ to $y$ is greater than $r_1$; in this case we have $M \subset \overline{B}_X(x_0, r_1)$. Since $r_1 \leq R_1 \leq \text{Inj}_X(x_0)$, all the distance spheres $\partial B_X(x_0, r)$ with $r \in (0, r_1)$ are geodesic spheres. Since the absolute sectional curvature of $X$ is bounded by $a$, comparison results imply that $\partial B_X(x_0, r)$ has normal curvatures greater than $H_0$ because $r < R_0(a, H_0)$ in this case. Assume for the moment that $M$ is contained in $\overline{B}_X(x_0, r)$. As $M$ is closed, there exists a largest $r_2 \in (0, r]$ such that $M \subset \overline{B}_X(x_0, r_2)$, and by compactness of $M$ there exists a point $x \in M \cap \partial B_X(x_0, r_2)$. Therefore, all the normal curvatures of $M$ at $x$ are greater than $H_0$, which implies that the length of the mean curvature vector of $M$ is greater than $H_0$, thereby contradicting one of the hypotheses on $M$. This contradiction proves that $M(x_0)$ cannot be contained in $\overline{B}_X(x_0, r)$. Since this non-containment equation holds for every $r \in (0, r_1)$ and $M$ is compact, we conclude that $M$ cannot be contained in $B_X(x_0, r_1)$. Item 1 is now proved.

To see that item 2 holds, consider two values $r_2 < r_3$ in $[0, r_1]$. By item 1 and the hypotheses on $M$, then $B_M(x_0, r_3) \setminus \overline{B}_M(x_0, r_3)$ is a non-empty open subset of $M$, hence $A(r_2) < A(r_3)$. 

7
It remains to prove the lower bound estimates for $A(r)$ given in item 3. In what follows, we only consider values $r \in (0, r_1]$. By Stokes’ Theorem,

$$\int_{B_M(x_0,r)} \Delta((R^2)|_M) \leq \int_{\partial B_M(x_0,r)} |\nabla (R^2)| = 2 \int_{\partial B_M(x_0,r)} R|\nabla R| \leq 2r l(r), \quad (2.19)$$

where $l(r) = \text{Volume}(\partial B_M(x_0, r)) = A'(r)$ is the $(n - 1)$-dimensional volume of $\partial B_M(x_0, r)$.

Since $K_{\text{sec}} \leq a$, inequality (2.3) implies that

$$\int_{B_M(x_0,r)} \Delta((R^2)|_M) \geq 2n A(r) + 2n \int_{B_M(x_0,r)} R \bar{H}(R) - 2 \int_{B_M(x_0,r)} R^2 f_a(R) \left(n - |\nabla (R|_M)|^2\right). \quad (2.20)$$

Since $R \leq r$ and $|\bar{H}(R)| = |\langle \bar{H}, \nabla R \rangle| \leq |\bar{H}| \leq H_0$, we have $R \bar{H}(R) \geq -H_0 r$, and thus,

$$\int_{B_M(x_0,r)} R \bar{H}(R) \geq -H_0 r A(r). \quad (2.21)$$

Next we analyze the second integral in the RHS of (2.20).

If $a = 0$, then $f_a \equiv 0$ and the second integral in the RHS of (2.20) vanishes. In this case, (2.19), (2.20) and (2.21) give

$$2r A'(r) = 2r l(r) \geq \int_{B_M(x_0,r)} \Delta((R^2)|_M) \geq 2n A(r) - 2n H_0 r A(r), \quad (2.22)$$

that is,

$$\frac{A'(r)}{A(r)} \geq \frac{n}{r} - n H_0 \quad \forall r \in (0, r_1],$$

which implies that

$$\frac{d}{dr} \left( \frac{A(r)}{r^ne^{-nH_0r}} \right) \geq 0,$$

hence the function

$$r \mapsto \frac{A(r)}{r^ne^{-nH_0r}}$$

is non-decreasing for $r \in (0, r_1]$. Since the limit as $r \to 0^+$ of this last function is $\omega_n$, we deduce:

If $a = 0$, then $A(r) \geq \omega_n r^n e^{-nH_0r}$ for all $r \in (0, r_1]$.

In fact, the last estimate holds if $a \leq 0$, because $f_a \leq 0$ in $(0, \infty)$ in this case, and hence,

$$-2 \int_{B_M(x_0,r)} R^2 f_a(R) \left(n - |\nabla (R|_M)|^2\right) \geq 0 \quad (2.23)$$

so the same computations of case $a = 0$ are valid for $a \leq 0$. 

8
We next study the case $a > 0$. Now $f_a(t)$ is strictly positive, increasing in the interval $I_a = [0, \pi/\sqrt{a})$ and limits to $a/3 > 0$ as $t \to 0^+$ and to $+\infty$ as $t \to (\pi/\sqrt{a})^-$. Whenever $r \in (0, r_1]$,

$$2r A'(r) \geq \int_{B_M(x_0, r)} \Delta((R^2)|_M)$$

\begin{align*}
&\geq 2n A(r) - 2nH_0 r A(r) - 2 \int_{B_M(x_0, r)} R^2 f_a(R) \left(n - |\nabla(R|_M)|^2\right) \\
&\geq 2n \left[1 - r^2 f_a(r_1')\right] A(r) - 2nrH_0 A(r) \tag{2.24}
\end{align*}

where in $(A)$ we have bounded $R \leq r$, $f_a(R) \leq f_a(r_1)$ and $n - |\nabla(R|_M)|^2 \leq n$.

Finally, (2.24) implies

$$\frac{d}{dr} \left(\frac{A(r)}{r^n e^{-nrH_0 + \frac{1}{2} f_a(r_1)r}}\right) \geq 0;$$

hence the function

$$r \mapsto \frac{A(r)}{r^n e^{-nrH_0 + \frac{1}{2} f_a(r_1)r}}$$

is non-decreasing for $r \in [0, r_1]$. Since the limit as $r \to 0^+$ of this last function is $\omega_n$, we deduce the inequality (2.18) in the case where $a > 0$; thus, the proposition is proved. \qed

**Remark 2.5.**

1. Proposition 2.4 holds regardless whether or not the normal bundle of the submanifold $M$ is trivial, since item 2 of Lemma 2.2 does not depend on whether or not the normal bundle of $M$ admits a global trivialization.

2. The proof of Proposition 2.4 shows that if $M$ has local density $k \in \mathbb{N}$ at $x_0$, then the RHS in (2.18) cab replaced by $k$ times the same expression.

3. In the case $a > 0$, it holds that $A(r) \geq \omega_n r^n e^{-nrH_0 + \frac{1}{2} f_a(r_1)r}$ for every $r \in (0, r_1]$. This can be proved by following the same proof for values $r \leq r'_1$ where $r'_1$ is any number less than or equal to $r_1$.

4. If $H_0 \neq 0$ or $a \neq 0$, the inequality (2.18) is strict.

**Corollary 2.6.** Let $R_1 > 0$, $a \in \mathbb{R}$ and $H_0 \geq 0$, and suppose that $X$ is a complete Riemannian $m$-dimensional manifold with injectivity radius at least $R_1 > 0$ and $K_{sec} \leq a$. If $M \hookrightarrow X$ is a complete, non-compact immersed $n$-dimensional submanifold with empty boundary and the mean curvature vector $\vec{H}$ of $M$ satisfies $|\vec{H}| \leq H_0$, then $M$ has infinite volume.

**Proof.** Let $r_1 > 0$ be the number given by Proposition 2.4. Observe that by Proposition 2.4, the $n$-dimensional volume of each component of $M$ is at least $A(r_1) > 0$. Therefore, if $M$ has infinitely many components, then $M$ has infinite volume. So assume that $M$ has a finite number of components. Since $M$ is non-compact, then we can replace $M$ by a non-compact component. Take a point $x_0 \in M$ and let $\gamma : [0, \infty) \to M$ a length-minimizing ray starting at $x_0$ and parameterized by arc length. Consider the pairwise disjoint intrinsic balls $B_M(\gamma(2kr_1), r_1), k \in \mathbb{N}$. Since each of these balls has volume at least $A(r_1)$ by Proposition 2.4, then we conclude that $M$ has infinite volume. \qed
Proposition 2.7. Given $R_1 > 0$, $a \in \mathbb{R}$ and $H_0 \geq 0$, there exists $r_2 = r_2(R_1, a, H_0) \in (0, r_1]$ (here $r_1$ is given by Proposition 2.4) such that if $X$ is a complete Riemannian 3-manifold with injectivity radius at least $R_1 > 0$ and $K_{sec} \leq a$, and if $M \hookrightarrow X$ is a complete, connected immersed surface with boundary, whose mean curvature vector $\vec{H}$ satisfies $|\vec{H}| \leq H_0$, then for all $p \in \text{Int}(M)$ we have

$$\text{Area}[B_M(p, r)] \geq 3r^2, \quad \text{whenever } 0 < r \leq \min\{r_2, d_M(p, \partial M)\}. \quad (2.25)$$

Furthermore, given $\varepsilon_0 > 0$ define $C_A = \min\{\varepsilon_0, \frac{r_2}{\varepsilon_0}\}$. If $p \in M$ satisfies $d_M(p, \partial M) \geq \varepsilon_0$, then

$$\text{Area}[B_M(p, d_M(p, \partial M))] \geq C_A d_M(p, \partial M) \quad (2.26)$$
and

$$\text{Area}[B_M(p, \varepsilon_0)] \geq C_A \varepsilon_0, \quad (2.27)$$

Proof. First suppose that $a > 0$. By (2.18), we have that whenever $0 < r \leq \min\{r_1, d_M(p, \partial M)\}$,

$$\text{Area}[B_M(p, r)] \geq \pi r^2 e^{-2r(H_0 + \frac{1}{2}f_a(r_1)r)} = \phi(r)r^2, \quad (2.28)$$

where $\phi(r) = \pi e^{-2r(H_0 + \frac{1}{2}f_a(r_1)r)}$ for all $r > 0$. Choose $r_2 = r_2(R_1, a, H_0) \in (0, r_1]$ such that $\phi(r_2) \geq 3$, which can be done since $\phi$ is continuous and $\phi(0) = \pi$. As $r > 0 \mapsto \phi(r)$ is decreasing, we have that if $0 < r \leq (0, \min\{r_2, d_M(p, \partial M)\})$, then

$$\text{Area}[B_M(p, r)] \geq \phi(r)r^2 \geq \phi(r_2)r^2 \geq 3r^2; \quad (2.28)$$

which proves (2.25) assuming $a > 0$. The proof of (2.25) when $a \leq 0$ is similar and we leave it for the reader.

Next assume that $p \in M$ satisfies $d_M(p, \partial M) \geq \varepsilon_0$, and we will show that (2.26) and (2.27) hold. Let $\gamma: [0, d_M(p, \partial M)) \rightarrow M$ be a minimizing geodesic from $p$ to $\partial M$, parameterized by arc length. Choose the largest $k \in \mathbb{N}$ such that

$$(2k - 1)\varepsilon_0 \leq d_M(p, \partial M) < (2k + 1)\varepsilon_0 \leq 3k\varepsilon_0. \quad (2.29)$$

By the triangle inequality, the collection $\mathcal{B} = \{B_M(\gamma(2(i - 1)\varepsilon_0), \varepsilon_0)\}_{i=1}^k$ is pairwise disjoint and $\cup \mathcal{B}$ is contained in $B_M(p, d_M(p, \partial M))$; hence,

$$\text{Area}[B_M(p, d_M(p, \partial M))] \geq \sum_{i=1}^k \text{Area}(B_M(\gamma(2(i - 1)\varepsilon_0), \varepsilon_0)). \quad (2.30)$$

Also observe that given $i \in \{1, \ldots, k\}$, (2.29) implies

$$\varepsilon_0 \leq d_M(\gamma(2(i - 1)\varepsilon_0), \partial M). \quad (2.31)$$

We next prove (2.26) and (2.27) by consideration of two cases.
• Suppose \( \varepsilon_0 \leq r_2 \). By (2.31), for each \( i \in \{1, \ldots, k\} \) we have

\[
\varepsilon_0 \leq \min \{r_2, d_M(\gamma(2(i-1)\varepsilon_0), \partial M)\}.
\]

The last inequality allows us to use (2.25) to conclude that

\[
\text{Area}[B_M(\gamma(2(i-1)\varepsilon_0), \varepsilon_0)] \geq 3\varepsilon_0^2. \tag{2.32}
\]

Note that \( C_A = \varepsilon_0 \) in this case. Taking \( i = 1 \) in (2.32), we have

\[
\text{Area}[B_M(p, \varepsilon_0)] \geq 3\varepsilon_0^2 > \varepsilon_0^2 = C_A\varepsilon_0, \quad \text{hence (2.27) holds.}
\]

As the collection \( B \) is pairwise disjoint, (2.30) and (2.32) imply

\[
\text{Area}[B_M(p, d_M(p, \partial M))] \geq 3k\varepsilon_0^2 = 3kC_A\varepsilon_0 \geq C_A d_M(p, \partial M),
\]

hence (2.26) also holds in this case.

• Suppose \( \varepsilon_0 > r_2 \). By (2.31), for each \( i \in \{1, \ldots, k\} \) we have \( r_2 < d_M(\gamma(2(i-1)\varepsilon_0), \partial M) \); hence (2.25) implies that

\[
\text{Area}[B_M(\gamma(2(i-1)\varepsilon_0), \varepsilon_0)] \geq 3r_2^2. \tag{2.33}
\]

Since \( d_M(p, \partial M) < 3k\varepsilon_0 \) and \( C_A = \frac{r_2^2}{\varepsilon_0} \) in this case,

\[
\text{Area}[B_M(p, d_M(p, \partial M))] \geq 3kr_2^2 = 3kC_A\varepsilon_0 \geq C_A d_M(p, \partial M),
\]

which proves that (2.26). The inequality (2.27) follows from (2.26) after replacing \( M \) by the closure of \( B_M(p, \varepsilon_0) \).

\( \square \)

Remark 2.8. A straightforward adaptation of the proof of Proposition 2.7 gives a related statement and proof for any \( n \)-dimensional submanifold \( M \), with a fixed upper bound on the length of its mean curvature vector field, in a Riemannian \( m \)-manifold \( X \) which has injectivity radius at least \( R_1 > 0 \) and sectional curvature bounded from above by some \( a \in \mathbb{R} \); in this setting, \( 3r_2^2 \) in (2.25) is replaced by \( c_n r_2^n \), where \( c_n \) is any positive number less than \( \omega_n \).

3 Index of finitely branched minimal surfaces in \( \mathbb{R}^3 \)

Definition 3.1. Let \( \Sigma \) be a smooth surface endowed with a conformal class of metrics. We say that a harmonic map \( f : \Sigma \to \mathbb{R}^3 \) is a (possibly non-orientable) branched minimal surface if it is a conformal immersion outside of a locally finite set of points \( B_\Sigma \subset \Sigma \), where \( f \) fails to be an immersion. Points in \( B_\Sigma \) are called branch points of \( f \). It is well-known (see e.g. Micallef and White [10, Theorem 1.4]) that given \( p \in B_\Sigma \), there exist a conformal coordinate \( (\overline{D}, z) \) for \( \Sigma \) centered at \( p \) (here \( \overline{D} \) is the closed unit disk in the plane), a diffeomorphism \( u \) of \( \overline{D} \) and a rotation \( \phi \) of \( \mathbb{R}^3 \) such that \( \phi \circ f \circ u \) has the form

\[
z \mapsto (z^q, x(z)) \in \mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3.
\]
for $z$ near 0, where $q \in \mathbb{N}$, $q \geq 2$, $x$ is of class $C^2$, and $x(z) = o(|z|^q)$. The branching order $B(p) \in \mathbb{N}$ is defined to be $q - 1$. The total branching order of $f$ is

$$B(\Sigma) := \sum_{p \in B_0} B(p).$$

**Definition 3.2.** Given a 1-sided minimal immersion $F : M \looparrowright X$, let $\tilde{M} \rightarrow M$ be the two-sided cover of $M$ and let $\tau : \tilde{M} \rightarrow \tilde{M}$ be the associated deck transformation of order 2. Denote by $\tilde{\Delta}$, $|\tilde{A}|^2$ the Laplacian and squared norm of the second fundamental form of $\tilde{M}$, and let $N : \tilde{M} \rightarrow TX$ be a unitary normal vector field. The index of $F$ is defined as the number of negative eigenvalues of the elliptic, self-adjoint operator $\tilde{\Delta} + |\tilde{A}|^2 + \text{Ric}(N, N)$ defined over the space of compactly supported smooth functions $\phi : \tilde{M} \rightarrow \mathbb{R}$ such that $\phi \circ \tau = -\phi$.

We next recall a fundamental lower bound for the index $I(f)$ of a connected, complete, possibly finitely branched minimal surface $f : \Sigma \looparrowright \mathbb{R}^3$ with finite total curvature, which is due to Chodosh and Maximo [2], and to Karpukhin [6]:

$$3I(f) \geq \begin{cases} 2g(\Sigma) + 2 \sum_{j=1}^{e} (d_j + 1) - 2B - 5 & \text{if } \Sigma \text{ is orientable,} \\ g(\tilde{\Sigma}) + 2 \sum_{j=1}^{e} (d_j + 1) - 2B - 4 & \text{if } \Sigma \text{ is non-orientable,} \end{cases} \quad (3.1)$$

where $g(\Sigma)$ is the genus of $\Sigma$ if $\Sigma$ is orientable (resp. $g(\tilde{\Sigma})$ is the genus of the orientable cover $\tilde{\Sigma}$ of $\Sigma$ if $\Sigma$ is not orientable$^1$), $e$ and $B$ are respectively the number of ends and the total branching order of $\Sigma$, and for each end $E_j$ of $\Sigma$, $d_j$ is the multiplicity of $E_j$ as a multi-graph over the limiting tangent plane of $E_j$.

Inequality (3.1) has not been explicitly stated in the literature, so an explanation is in order. Ros [13] proved that $3I(f) \geq 2g(\Sigma)$ using harmonic square integrable 1-forms on $\Sigma$ for a minimal immersion $f : \Sigma \looparrowright \mathbb{R}^3$ with finite total curvature, in order to produce test functions for the index operator of $f$. Chodosh and Maximo [2, Theorem 1] improved Ros’ technique with an enlarged space of harmonic 1-forms which admit certain singularities at the ends of $\Sigma$ that take care of the spinning (multiplicity) of each end of such an immersion $f$, obtaining a simplified version of (3.1) without the term $-2B$. Finally, Karpukhin [6, Proposition 2.3 and Remark 2.4] included the study of branch points although he made use of the original space of $L^2(\Sigma)$ harmonic 1-forms considered by Ros. Formula (3.1) is the combined inequality that one can deduce from [2] and [6].

The class of complete, non-flat, finitely branched, stable minimal surfaces in $\mathbb{R}^3$ contains an interesting non-trivial family of surfaces, as we explain next.

1. Any non-orientable, complete, finitely branched minimal surface $f : \Sigma \looparrowright \mathbb{R}^3$ with finite total curvature, whose extended unoriented Gauss map $G : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a diffeomorphism, is

$^1$If $\Sigma$ is a compact non-orientable surface and $\tilde{\Sigma} := \Sigma$ denotes the oriented cover of $\Sigma$, then the genus of $\tilde{\Sigma}$ plus 1 equals the number of cross-caps in $\Sigma$. 

12
stable (observe that the conformal compactification of $\Sigma$ must be $\mathbb{P}^2$). We prove this property by contradiction: if $f$ is not stable, then the first eigenvalue $\lambda_1$ of the Jacobi operator on $\Sigma$ is negative, which implies that there exists an eigenfunction $\phi: S^2 \to \mathbb{R}$ of the lifted Jacobi operator on the orientable cover $\pi: \widetilde{\Sigma} \to \Sigma$ of $\Sigma$ so that $\phi \circ \tau = -\phi$ and $L \phi + \lambda \phi = 0$ on $\widetilde{\Sigma}$, where $\lambda < 0$ and $\tau: S^2 \to S^2$ is the antipodal map. Let $\Omega$ be a component of $\phi^{-1}(0, \infty)$. As $\phi$ is odd, $\tau(\Omega) \subset \phi^{-1}(-\infty, 0)$ and so, $\pi|_{\Omega}: \Omega \to \pi(\Omega)$ is a diffeomorphism. In particular, $\pi(\Omega)$ is an orientable domain in $\Sigma$. Since $G$ is also a diffeomorphism, $G(\pi(\Omega))$ is an orientable domain in $\mathbb{P}^2$. Thus, $G(\pi(\Omega))$ lifts to two disjoint diffeomorphic domains in $S^2$ of the form $g(\Omega)$, $(g \circ \tau)(\Omega)$ (here $g: \widetilde{\Sigma} \to S^2$ is the Gauss map of $\widetilde{\Sigma}$). In particular, $\text{Area}(g \circ \tau)(\Omega)) = \text{Area}(g(\Omega)) \leq 2\pi$, which implies that the first eigenvalue of the Jacobi operator $L$ on $\Omega$ is non-negative. This is a contradiction, as the first Dirichlet eigenvalue of $L$ on $\Omega$ (defined as the supremum of the first Dirichlet eigenvalues of $L$ on a increasing sequence of compact smooth domains $\Omega_i \nearrow \Omega$) is $\lambda < 0$. This contradiction proves that $\Sigma$ is stable.

2. Using the Weierstrass representation for non-orientable minimal surfaces in [7], the classical Henneberg minimal surface given by the Weierstrass data$^2$ on its oriented covering $\mathbb{C} \setminus \{0\}$

$$g(z) = z, \quad \omega = z^{-4}(z^4 - 1)\, dz,$$

is a non-orientable, complete branched minimal surface $f: \mathbb{P}^2 \setminus \{0, \infty\} \leftrightarrow \mathbb{R}^3$ with two branch points of order 1 at $\{1, -1\}$, $\{i, -i\} \in \mathbb{P}^2$ and a single end of spinning $3$ at $\{0, \infty\}$. Since its extended Gauss map is a diffeomorphism from $\mathbb{P}^2$ to $\mathbb{P}^2$, then the Henneberg minimal surface $H_1 = f(\mathbb{P}^2 \setminus \{0, \infty\})$ is stable. After translating the surface in $\mathbb{R}^3$ so that $f(e^{i\pi/4}) = \widetilde{0}$, the branch points are mapped by $f$ into $\pm(0, 0, 1)$.

Henneberg’s surface can be generalized as follows. Given an odd integer $m \in \mathbb{N}$, consider the following Weierstrass data on $\mathbb{C} \setminus \{0\}$,

$$g(z) = z, \quad \omega = z^{-(3+m)}(z^{2m+2} - 1)\, dz$$

which produces a two-sheeted cover of a complete minimal Mobius strip $f: \mathbb{P}^2 \setminus \{0, \infty\} \leftrightarrow \mathbb{R}^3$ which is stable with $m + 1$ branch points of order 1 at the $(m + 1)$ pairs of antipodal $(2m + 2)$-roots of unity and a single end of spinning $m + 2$ at $\{0, \infty\}$. Henneberg’s minimal surface corresponds to the case $m = 1$. After translating the surface $H_m = f(\mathbb{P}^2 \setminus \{0, \infty\})$ in $\mathbb{R}^3$ so that $f(e^{i\pi/(m+1)}) = \widetilde{0}$, the branch points of $H_m$ are located at $(0, 0, \pm \sqrt{2}/(m+1))$, and a parameterization of $H_m$ in polar coordinates is

$$f(re^{i\theta}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \left( \frac{r^m \cos(m\theta)}{m} + \frac{\cos((m+2)\theta)}{(m+2)r^{m+2}} \right) - \frac{1}{2} \left( \frac{r^{m+2} \cos((m+2)\theta)}{m+2} + \frac{\cos(m\theta)}{mr^m} \right) \\ \frac{1}{2} \left( \frac{\sin((m+2)\theta)}{(m+2)r^{m+2}} - \frac{r^m \sin(m\theta)}{m} \right) - \frac{1}{2} \left( \frac{r^{m+2} \sin((m+2)\theta)}{m+2} - \frac{\sin(m\theta)}{mr^m} \right) \\ \frac{1}{m+1} \left( r^{m+1} + \frac{1}{r^{m+1}} \right) \cos((m+1)\theta) \end{pmatrix}. $$

$^2$This means that $f(z) = \text{Re} \left( \frac{1}{2}(1 - g^2)\omega, \frac{1}{2}(1 + g^2)\omega, g\omega \right)$ parameterizes the surface.
We note that \( f \) maps each of the \( m + 1 \) pairs of opposite half-lines

\[
\left\{ l_j = \left\{ re^{i\frac{j\pi}{2(m+1)}} \mid r > 0 \right\}, -l_j \right\}
\]

(for each \( j \) odd) into a horizontal line of \( \mathbb{R}^3 \) that passes through \( \vec{0} \), and the union \( L \) of these \( m + 1 \) horizontal lines forms an equiangular system contained in \( H_m \). Therefore, the reflection in \( \mathbb{C} \setminus \{0\} \) about \( l_j \cup (-l_j) \) induces a symmetry of \( H_m \). Reflections in the \( m + 1 \) vertical planes that bisect each of the angles between the lines in \( L \) are planes of symmetry of \( H_m \). Rotations of angle \( \pi \) about each of the lines in \( L \) together with these \( m + 1 \) planar reflections form the group of isometries of \( H_m \) (all of which extend to ambient isometries), which, when considered to be a subgroup of \( O(3) \), is the antiprismatic group \( A_{2(m+1)} \). In fact, every intrinsic isometry of \( H_m \) extends to an extrinsic isometry, since such an intrinsic isometry produces a conformal diffeomorphism of \( \mathbb{C} \setminus \{0\} \) into itself that preserves the set of \((2m + 2)\)-roots of unity.

For odd \( m \geq 3 \), these generalized Henneberg surfaces \( H_m \) can be deformed to less symmetric examples of non-orientable, complete finitely branched stable minimal surfaces in \( \mathbb{R}^3 \) whose branch locus consists of \( m + 1 \) pairs of antipodal points in \( \mathbb{C} \setminus \{0, \infty\} \) (\( H_1 \) can be proven to be the unique such surface for \( m = 1 \)); see [11] for a description and special properties of these deformed Henneberg-type examples.

### 4 Scale invariant weak chord-arc type estimates for branched minimal surfaces of finite index in \( \mathbb{R}^3 \)

**Proposition 4.1.** Given \( I, B \in \mathbb{N} \cup \{0\} \), let \( f : (\Sigma, p_0) \looparrowright (\mathbb{R}^3, \vec{0}) \) be a complete, connected, pointed branched minimal surface with index at most \( I \) and total branching order at most \( B \). Given \( R > 0 \), let \( \Omega_R \) denote the component of \( f^{-1}(\mathbb{B}(R)) \) that contains \( p_0 \). Then, the following scale-invariant estimates hold and depend only on \( I, B \):

1. For any \( p \in \Omega_R \),

\[
d_{\Omega_R}(p, \partial \Omega_R) < \hat{L} R,
\]

where \( \hat{L} = \sqrt{\frac{1}{2}(3I + 2B + 3)} \).

2. If \( f \) is injective with image a plane, then the distance between any two points of \( \Omega_R \) is less than or equal to \( 2R \). Otherwise, given points \( p, q \) in \( \Omega_R \),

\[
d_{\Omega_{2R}}(p, q) < \hat{C} R,
\]

where \( \hat{C} = \hat{C}(I, B) = 8\hat{L}^3 + 2\pi\hat{L}^2 - 20\hat{L} - \frac{\pi}{2} \). In particular, \( \Omega_R \subset B_{\Sigma}(p, \hat{C} R) \) for every \( p \in \Omega_R \).
Proof. Since (4.1) and (4.2) are invariant under re-scaling, we do not lose generality by assuming $R = 1$. Let $f : (\Sigma, p_0) \leftrightarrow (\mathbb{R}^3, \tilde{0})$ be a complete, connected, pointed branched minimal surface in $\mathbb{R}^3$ with index $I(f) \leq I$ and total branching order $B(\Sigma) \leq B$. Observe that such an $f$ has finite total curvature [3, 4, 13]. Thus, $f$ is proper and $\Omega_1$ is compact with non-empty boundary $\partial \Omega_1$. Given a point $p \in \text{Int}(\Omega_1)$, let $L = d_{\Omega_1}(p, \partial \Omega_1)$ and consider a length minimizing geodesic arc parameterized by arc length $\gamma : [0, L] \to \Sigma$ joining $\gamma(0) = p$ to $\partial \Omega_1$. Observe that the intrinsic ball of center $p$ and radius $L$ satisfies $B_{\Sigma}(p, L) \subset \Omega_1$. The intrinsic version of the monotonicity formula for minimal surfaces described in Proposition 2.4 applied to the particular case $m = 3$, $a = 0$, $R_1 = \infty$, $n = 2$ and $H_0 = 0$, gives that $\text{Area}[B_{\Sigma}(p, L)] \geq \pi L^2$ (with the notation of Proposition 2.4, the number $r_1 = r_1(R_1, a, H_0)$ equals $\infty$ in this case; observe that the proof of Proposition 2.4 works for branched minimal surfaces; also see the last page of Yau [15] for the special case $a \leq 0$, $H_0 = 0$ in inequality (2.18)). Hence,

$$\text{Area}(\Omega_1) \geq \text{Area}[B_{\Sigma}(p, L)] \geq \pi L^2.$$

(4.3)

Next we deduce an upper bound for $\text{Area}(\Omega_1)$. Inequality (3.1) implies that regardless of the orientability character of $\Sigma$, we have $3I \geq 3I(f) \geq 2S + 2e - 2B(\Sigma) - 5$, where $e$ is the number of ends of $\Sigma$, and $S$ is the total spinning of the ends. Hence,

$$2S \leq 3I - 2e + 2B(\Sigma) + 5 \leq 3I - 2e + 2B + 5.$$

(4.4)

As $e \geq 1$, we have

$$\text{Area}(\Omega_1) \leq \text{Area}[f^{-1}(\mathbb{R})(1))] \leq \pi S \overset{(4.4)}{\leq} \frac{\pi}{2} (3I + 2B + 3) = \pi \widehat{L}^2,$$

(4.5)

where in (*) we have used that the asymptotic area growth of $\Sigma$ in balls of large radius $R$ is $\pi SR^2$ (see e.g., [5]) and the classical (extrinsic) monotonicity formula. Now, (4.3) and (4.5) give

$$d_{\Omega_1}(p, \partial \Omega_1) = L \leq \widehat{L}$$

(4.6)

for any $p \in \text{Int}(\Omega_1)$, which implies that $d_{\Omega_R}(p, \partial \Omega_R) \leq \widehat{L}R$; notice that this last inequality is strict (otherwise $f(\Sigma)$ is a possibly branched plane passing through the origin by the extrinsic monotonicity formula, in which case the first inequality in (4.4) is strict). This implies that the inequality (4.1) is strict, and item 1 of Proposition 4.1 is proven.

In order to obtain item 2, we will need the following auxiliary property: If $f$ is not an embedded plane, then

$$d_{\Omega_R}(p, q) \leq 2\widehat{L}(3I + 2B - 1)R + \frac{1}{2}\text{Length}(\partial \Omega_R).$$

(4.7)

Observe that (4.7) is invariant under re-scaling. We will divide the proof of (4.7) into four claims.

Claim 4.2. For any $p, q \in \Omega_1$,

$$d_{\Omega_1}(p, q) \leq \sup_{p', q' \in \Omega_1} d_{\Omega_1}(p', q') = \lim_{r \to 1^{+}} \sup_{p'' \in \Omega_1} d_{\Omega_R}(p'', q'').$$

(4.8)

15
Proof. The first inequality in (4.8) holds by definition of supremum and so Claim 4.2 reduces to checking that the equality part of (4.8) holds. For each \( r \in (1, 2] \), let \( p_r, q_r \) be points of \( \Omega_r \) such that
\[
L_r := d_{\Omega_r}(p_r, q_r) = \sup_{p'', q'' \in \Omega_r} d_{\Omega_r}(p'', q'')
\]
and let \( \alpha_r : [0, L_r] \to \Omega_r \subset \Omega_2 \) be a Lipschitz curve contained in \( \Omega_r \) with Lipschitz constant 1 that realizes the minimum distance \( L_r \) in \( \Omega_r \) between \( p_r, q_r \). Taking a sequence \( r_j \downarrow 1 \), after passing to a subsequence we obtain a limit Lipschitz curve \( \alpha_1 \) of the \( \alpha_{r_j} \) with Lipschitz constant 1 joining points \( p_1, q_1 \in \Omega_1 \) of positive length \( L_1 := \lim_{r_j \downarrow 1} L_{r_j} \). It straightforward to check that
\[
L_1 = d_{\Omega_1}(p_1, q_1) = \sup_{p', q' \in \Omega_1} d_{\Omega_1}(p', q'),
\]
which shows the equality part on the RHS of (4.8). □

Claim 4.3. If (4.7) holds whenever \( \Omega_R \) is transverse to \( S^2(R) \) along its boundary, then (4.7) holds for \( R = 1 \) (and thus, it also holds for any \( R > 0 \)).

Proof. This is a direct consequence of Claim 4.2 since almost all spheres centered at \( \bar{0} \) are transverse to \( f \) by Sard’s theorem. □

By Claim 4.3, we can reduce the proof of (4.7) to the case that \( R = 1 \) and \( f \) is transverse to \( S^2(1) \) along \( \partial \Omega_1 \). This transversality assumption implies that \( \Omega_1 \) is a smooth, connected, compact surface with a finite set \( \{ \partial_1, \ldots, \partial_b \} \) of boundary components, \( b \in \mathbb{N} \).

Claim 4.4. For any \( p, q \in \Omega_1 \),
\[
d_{\Omega_1}(p, q) \leq 2\widetilde{L} + \frac{1}{2}\text{Length}(\partial \Omega_1).
\]

Proof. Assuming \( b > 1 \), there is a geodesic arc \( \alpha_1 \subset \Omega_1 \) that minimizes the distance from \( \partial_1 \) to the set \( \bigcup_{i=2}^b \partial_i \), and, possibly after re-indexing, we may assume that \( \alpha_1 \) joins \( \partial_1 \) to \( \partial_2 \). Notice that the distance from the midpoint of \( \alpha_1 \) to \( \partial \Omega_1 \) is half the length of \( \alpha_1 \), and so (4.1) implies that the length of \( \alpha_1 \) is less than \( 2\widetilde{L} \). Assuming that \( b > 2 \), let \( \alpha_2 \) be a minimizing geodesic in \( \Omega_1 \) from \( \partial_1 \cup \partial_2 \) to the set \( \bigcup_{i=3}^b \partial_i \), which also has length less than \( 2\widetilde{L} \) by similar reasoning as in the case of \( \alpha_1 \); again after possibly re-indexing, we can assume that the end point of \( \alpha_2 \) which does not lie in \( \partial_1 \cup \partial_2 \) lies in \( \partial_3 \). Continuing inductively, we obtain a collection of arcs \( \{ \alpha_1, \alpha_2, \ldots, \alpha_{b-1} \} \) in \( \Omega_1 \), each with length less than \( 2\widetilde{L} \) and the set
\[
C := \partial \Omega_1 \cup \alpha_1 \cup \ldots \cup \alpha_{b-1}
\]
is path connected. Note that if \( b = 1 \), then \( C = \partial \Omega_1 = \partial_1 \).

For any pair of points \( p', q' \in C \), the intrinsic distance \( d_C(p', q') \) measured in \( C \) can be realized as the length of an embedded piecewise smooth arc in \( C \) consisting of arcs alternating between arcs in components of \( \partial \Omega_1 \) and arcs in \( \alpha_1 \cup \ldots \cup \alpha_{b-1} \). In particular,
\[
d_{\Omega_1}(p', q') \leq d_C(p', q') \leq 2(b - 1)\widetilde{L} + \frac{1}{2}\text{Length}(\partial \Omega_1).
\]
(4.9)

Let \( p, q \) be points in \( \Omega_1 \). Let \( p', q' \in \partial \Omega_1 \) be the end points of respective length-minimizing geodesics in \( \Omega_1 \) joining \( p \) and \( q \) to \( \partial \Omega_1 \). Applying (4.1) to \( p \) and \( q \) together with the estimate in (4.9), we have
\[
d_{\Omega_1}(p, q) \leq d_{\Omega_1}(p, \partial \Omega_1) + d_{\Omega_1}(q, \partial \Omega_1) + d_C(p', q') \leq 2b\widetilde{L} + \frac{1}{2}\text{Length}(\partial \Omega_1),
\]
which proves Claim 4.4. □
Claim 4.5. Inequality (4.7) holds.

Proof. Since (4.7) is invariant under re-scaling, it suffices to prove it for $R = 1$. By Claim 4.4, we have that (4.7) will follow by proving that

\[ b \leq 3I(f) + 2B(\Sigma) - 1. \tag{4.10} \]

Recall that $f|_{\Omega_1}$ is transverse to $\partial B(1)$ and that $\partial \Omega_1 = \{ \partial_1, \ldots, \partial_b \}$. Each $\partial_i$ is a simple closed curve in $\Sigma$, and $\partial_i$ admits a small tubular neighborhood $U_i$ in $\Sigma$ which is topologically an annulus.

Assume for the moment that $\Sigma$ is orientable, and we will prove that $b \leq g(\Sigma) + e$. Let $\Delta = \{ \Delta_1, \ldots, \Delta_k \}$ denote the set of components of $\Sigma \setminus \text{Int}(\Omega_1)$ and since each of these components has at least one end, then $k \leq e$. Given $i \in \{ 1, \ldots, k \}$, let $A_i$ denote the set of components of $\partial \Delta_i$ with one of the components arbitrarily removed; in particular the number of components in $A := \bigcup_{i=1}^k A_i$ is $b - k$. Note that for each component $\beta \in A$, there is a simple closed curve $\gamma_\beta$ in $\Sigma$ that intersects $A$ transversely in a single point of $\beta$, where $\gamma_\beta$ consists of an arc in $\Omega_1$ together with an arc in the component $\Delta_j \in \Delta$ that has $\beta$ in its boundary. It follows that the collection of simple closed curves $A$ does not separate $\Sigma$ and so, by the definition of genus, the number of elements in $A$, which is $b - k$, is less than or equal to $g(\Sigma)$. Since $k \leq e$, then $b \leq g(\Sigma) + e$, which proves the desired inequality when $\Sigma$ is orientable.

When $\Sigma$ is non-orientable, then $\Sigma$ is the connected sum of $g(\Sigma) + 1$ projective planes punctured in $e$ points, where $\widetilde{\Sigma}$ is the oriented cover of $\Sigma$, and a similar argument just carried out in the orientable case shows that $b \leq g(\widetilde{\Sigma}) + e + 1$.

According to the hypothesis stated for inequality (4.7), $f$ is assumed not to be an embedded plane. Thus the total spinning $S$ of $f$ satisfies $S \geq 2$. If $S = 2$, then the extrinsic monotonicity formula for minimal surfaces implies that either $f$ has one end with multiplicity 2 (in this case $f(\Sigma)$ is a plane, $B(\Sigma) = b = 1$ and $I(f) = 0$, so (4.10) is an equality in this case), or $f$ is injective and has two ends. In this last case, $f(\Sigma)$ is a catenoid by Schoen [14], $b \leq 2$, $B(\Sigma) = 0$ and $I(f) = 1$, which implies that (4.10) holds in this case.

If $S \geq 3$ and $\Sigma$ is orientable, then

\[
\begin{align*}
 b & \leq g(\Sigma) + e \\
 & \leq 2g(\Sigma) + e \quad \text{(because } g(\Sigma) \geq 0) \\
 & \leq 3I(f) - 2S - e + 2B(\Sigma) + 5 \quad \text{(by (3.1))} \\
 & \leq 3I(f) + 2B(\Sigma) - 2 \quad \text{(because } S \geq 3 \text{ and } e \geq 1),
\end{align*}
\]

hence (4.10) holds. Finally, if $S \geq 3$ and $\Sigma$ is non-orientable, then

\[
\begin{align*}
 b & \leq g(\widetilde{\Sigma}) + e + 1 \\
 & \leq 3I(f) - 2S - e + 2B(\Sigma) + 5 \quad \text{(by (3.1))} \\
 & \leq 3I(f) + 2B(\Sigma) - 2 \quad \text{(because } S \geq 3 \text{ and } e \geq 1),
\end{align*}
\]

hence (4.10) again holds. Therefore, inequality (4.10) holds in every case, and as observed above, this suffices to finish the proof of Claim 4.5.
With the auxiliary property (4.7) at hand, we next prove item 2 of Proposition 4.1. The first statement for \( f \) injective with image a plane is obvious. Assume \( f \) is not in this case and we will prove (4.2) for \( R = 1 \).

First suppose that \( f : (\Sigma, p_0) \to (\mathbb{R}^3, \vec{0}) \) is injective with image a catenoid \( C \). After a possible rotation of \( C \) fixing the origin, we can assume that the \((x_1, x_3)\)-plane \( P \) is a plane of symmetry of \( C \) and the axis of \( C \) is parallel to the \( x_3 \)-axis. As we observed previously, for estimating distances between pairs of points in \( \Omega_1 \), we may assume that the boundary sphere \( \partial \mathbb{B}(1) \) is transverse to \( C \). Then \( C \cap P \cap \mathbb{B}(1) \) contains a component arc \( \Gamma \) with non-vanishing curvature passing through the origin. By convexity, \( \Gamma \) has length less than the length of the boundary circle of the disk \( P \cap \mathbb{B}(1) \), and so, \( \text{length}(\Gamma) < 2\pi \). As the axis of \( C \) is parallel to the \( x_3 \)-axis, we deduce that \( \Gamma \) can be parameterized by its third coordinate as \( \Gamma = \{(x_1(t), 0, t) \mid t \in [a, b]\} \) for some \(-1 \leq a < 0 < b \leq 1\). Let \( C(1) = C \cap \{(x_1, x_2, x_3) \mid a \leq x_3 \leq b\} \); clearly \( \Omega_1 \subset C(1) \) and \( \Omega_1 \cap \partial C(1) = \{(x_1(a), 0, a), (x_1(b), 0, b)\} \). Similar comparison estimates also prove that each horizontal disk \( \{x_3 = t\} \cap \mathbb{B}(1) \) with \( t \in [a, b] \) intersects \( \Omega_1 \) in a connected component \( \Lambda(t) \) passing through \((x(t), 0, t) \in \Gamma \), and \( \Lambda(t) \) is invariant under reflection across \( P \). \( \Lambda(t) \) is either a horizontal circle of radius less than 1, a circular arc of length less than \( 2\pi \) or just the point \((x(t), 0, t) \) when \( t = a \) or \( t = b \). In particular, for any pair of points \( p, q \in \Omega_1 \) there exists a piecewise smooth path in \( \Omega_1 \) joining \( p \) and \( q \), which consists of a pair of horizontal circular arcs that join \( p \) and \( q \) to \( \Gamma \) together with an arc in \( \Gamma \) joining the end points of these two horizontal arcs.

It follows that the distance \( d_{\Omega_1}(p, q) < 4\pi \). Direct substitution of \( I = 1 \) and \( B = 0 \) in the RHS of (4.2) shows that the inequality (4.2) holds in this case that \( f : (\Sigma, p_0) \to (\mathbb{R}^3, \vec{0}) \) is injective with \( f(\Sigma) = C \).

If \( S = 2 \), then the arguments in the fifth paragraph of the proof of Claim 4.5 show that either \( f \) is injective with \( f(\Sigma) \) being a catenoid (hence (4.2) holds by the last paragraph), or else \( f(\Sigma) \) is a plane passing through the origin with \( B(\Sigma) = 1 \); in this last case the intrinsic distance between any two points of \( \Omega_1 \) is less than or equal to 4, and so, (4.2) is also seen to hold.

It remains to show that (4.2) holds if \( S \geq 3 \). Assume now that \( S \geq 3 \). We proved in the sixth paragraph of the proof of Claim 4.5 that if \( S \geq 3 \), then \( b \leq 3I(f) + 2B(\Sigma) - 2 \). Plugging this estimate of \( b \) into the inequality in Claim 4.4 and using the scale invariance of this inequality, we get the following estimate for all points \( p, q \in \Omega_R \) and for all \( R > 0 \):

\[
d_{\Omega_R}(p, q) \leq 2(3I(f) + 2B(\Sigma) - 2)LR + \frac{1}{3}\text{Length}(\partial \Omega_R). \tag{4.11}
\]

By the extrinsic monotonicity formula, \( \pi R^2 < \text{Area}[f^{-1}(\mathbb{B}(R))] \leq \pi SR^2 \) for each \( R > 0 \), where the strict inequality holds since \( f(\Sigma) \) is assumed not to be injective with image a plane passing through the origin. Taking \( R = 1 \) in the first of these inequalities and \( R = 2 \) in the second one, we deduce that

\[
\text{Area}[f^{-1}(\mathbb{B}(2) - \mathbb{B}(1))] < 4\pi S - \pi. \tag{4.12}
\]

By the co-area formula,

\[
\min_{r \in [1, 2]} \text{Length}[f^{-1}(\partial \mathbb{B}(r))] \leq \text{Area}[f^{-1}(\mathbb{B}(2) - \mathbb{B}(1))]. \tag{4.13}
\]
Let $\rho \in [1, 2]$ be such that \(\text{Length}[f^{-1}(\partial\mathbb{B}(\rho))]\) equals the minimum in the LHS of (4.13). Given $p, q \in \Omega_1$,
\[
\begin{align*}
d_{\Omega_\rho}(p, q) & \leq 2(3I + 2B - 2)\tilde{L}\rho + \frac{1}{2}\text{Length}(\partial\Omega_\rho) \quad \text{(by (4.11))} \\
& \leq 2(3I + 2B - 2)\tilde{L}\rho + \frac{1}{2}\text{Length}[f^{-1}(\partial\mathbb{B}(\rho))] \quad \text{(because $\partial\Omega_\rho \subset f^{-1}(\partial\mathbb{B}(\rho))$)} \\
& < 2(3I + 2B - 2)\tilde{L}\rho + 2\pi S - \frac{\pi}{2} \quad \text{(by (4.12) and (4.13))} \\
& \leq 4(3I + 2B - 2)\tilde{L} + \pi(3I - 2e + 2B + 5) - \frac{\pi}{2} \quad \text{(by (4.4) and $\rho \leq 2$)} \\
& \leq 4(3I + 2B - 2)\tilde{L} + \pi(3I + 2B + 3) - \frac{\pi}{2} \quad \text{(because $e \geq 1$)}
\end{align*}
\]

Since $\rho \leq 2$ and $3I + 2B = 2\tilde{L}^2 - 3$, then $d_{\Omega_\rho}(p, q) \leq d_{\Omega_\rho}(p, q) < 8\tilde{L}^3 + 2\pi\tilde{L}^2 - 20\tilde{L} - \frac{\pi}{2}$, which proves (4.2) holds. This completes the proof of Proposition 4.1.}

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