Three-Loop $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ Corrections to Hadronic Higgs Decays

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Abstract

We calculate the top-quark-induced three-loop corrections of $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ to the Yukawa couplings of the first five quark flavours in the framework of the minimal standard model with an intermediate-mass Higgs boson, with mass $M_H \ll 2M_t$. The calculation is performed using an effective-Lagrangian approach implemented with the hard-mass procedure. As an application, we derive the $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ corrections to the $H \rightarrow q\bar{q}$ partial decay widths, including the case $q = b$. The couplings of the Higgs boson to pairs of leptons and intermediate bosons being known to $\mathcal{O}(\alpha_s^2 G_F M_t^2)$, this completes the knowledge of such corrections in the Higgs sector. We express the results both in the $\overline{\text{MS}}$ and on-shell schemes of mass renormalization. We recover the notion that the QCD perturbation expansions exhibit a worse convergence behaviour in the on-shell scheme than they do in the $\overline{\text{MS}}$ scheme.

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1 Introduction

One of the longstanding questions of elementary particle physics is whether nature makes use of the Higgs mechanism of spontaneous symmetry breaking to endow the particles with their masses. The Higgs boson, $H$, is the missing link sought to verify this theoretical conjecture in the standard model (SM). The possible range of the Higgs-boson mass, $M_H$, is constrained from below both experimentally and theoretically. The failure of experiments at the CERN Large Electron-Positron Collider (LEP 1) to observe the decay $Z \rightarrow f\bar{f}H$ has ruled out the mass range $M_H \leq 65.6$ GeV at the 95% confidence level [1]. Depending on the precise value of the top-quark mass, $M_t$, the requirement that the vacuum be the true ground state provides an even more stringent theoretical lower bound [2]. However, this bound may be somewhat relaxed by taking into account the possibility that the physical minimum of the effective potential might be metastable [3].

Other theoretical arguments bound $M_H$ from above. The requirements that partial-wave unitarity in intermediate-boson scattering at high energies be satisfied [4] or that perturbation theory in the Higgs sector be meaningful [5] establish an upper bound on $M_H$ at about $(8\pi\sqrt{2}/3G_F)^{1/2} \approx 1$ TeV, where $G_F$ denotes Fermi’s constant, in a weakly interacting SM. The triviality bounds, i.e., $M_H$ upper bounds derived through perturbative [6] or lattice [7] computations by requiring that the running Higgs self-coupling, $\lambda(\mu)$, stay finite for renormalization scales $\mu < \Lambda$, where $\Lambda$ is the cutoff beyond which new physics operates, are somewhat stronger. In the following, we shall focus our attention on the lower end of the allowed $M_H$ range, where $M_H \ll 2M_t$, which will be accessed by colliding-beam experiments in the near future.

A Higgs boson with $M_H \approx 135$ GeV decays dominantly to $b\bar{b}$ pairs [8]. This decay mode will be of prime importance for Higgs-boson searches at LEP 2 [9], the Fermilab Tevatron [10] or a possible 4-TeV upgrade thereof [11], a next-generation $e^+e^-$ linear collider [12], and a future $\mu^+\mu^-$ collider [13]. Techniques for the measurement of the $H \rightarrow bb$ branching fraction at a $\sqrt{s} = 500$ GeV $e^+e^-$ linear collider have been elaborated in Ref. [14]. The branching ratio of $H \rightarrow c\bar{c}$ is roughly 30 times smaller than that of $H \rightarrow bb$ [8].

Once a novel scalar particle is discovered, it will be crucial to decide if it is the very Higgs boson of the SM or if it lives in some more extended Higgs sector. To that end, precise knowledge of the SM predictions will be mandatory, i.e., quantum corrections must be taken into account. The present knowledge of quantum corrections to the $H \rightarrow q\bar{q}$ partial decay widths has recently been reviewed in Ref. [15]. At one loop, the electroweak [16, 17] and quantum-chromodynamical (QCD) [18] corrections are known for arbitrary masses. The leading high-$M_H$ term, of $O(G_F M_H^2)$, was first derived by Veltman [19]. In the limit $M_H \ll 2M_t$, which we are interested in here, the terms of $O(X_t)$, where $X_t = (G_F M_t^2/8\pi^2\sqrt{2})$, tend to be dominant. They arise in part from the renormalizations of the Higgs wave function and vacuum expectation value, which are independent of the quark flavour $q$ [20]. In the case of $q = b$, there is an additional non-universal $O(X_t)$ contribution [10, 17, 21], which partly cancels the flavour-independent one. At two loops, the universal [22] and bottom-specific [23, 24] $O(\alpha_s X_t)$ terms are available. Furthermore,
the first [23] and second [20] terms of the expansion in $M_q^2/M_H^2$ of the $O(\alpha_s^2)$ five-flavour QCD correction have been found. As for the top-quark-induced correction in $O(\alpha_s^2)$, the full $M_t$ dependence of the non-singlet (double-bubble) contribution [27] as well as the first four terms of the $M_H^2/M_t^2$ expansion of the singlet (double-triangle) contribution [28] have been computed. At three loops, the $O(\alpha_s^3)$ non-singlet correction is known in the massless approximation [29].

In this paper, we shall take the next step, to three loops including virtual top-quark effects. Specifically, we shall evaluate the $O(\alpha_s^2 X_t)$ corrections to the decay widths of $H \rightarrow q\bar{q}$, and in particular of $H \rightarrow b\bar{b}$. These corrections may be divided into three classes, which are separately finite and gauge independent: (i) The universal correction originates in the renormalizations of the Higgs wave function and vacuum expectation value and occurs as a building block in the computation of quantum corrections to any Higgs-boson production or decay process. It is related to the Higgs- and $W$-boson self-energies. In the case of the lepton decay, $H \rightarrow l^+l^-$, this is the only source of $O(\alpha_s^2 X_t)$ corrections [31]. (ii) The quark-specific correction arises from Feynman diagrams where the top quark only appears in a closed loop which is connected with the external quark line by two gluons (and, in some cases, one additional weak neutral boson). It just depends on the third component of weak isospin of the considered quark. (iii) The bottom-specific correction emerges from the one-loop seed diagram where one charged boson is emitted and re-absorbed from the external bottom-quark line, by appropriately adding gluon lines (and, in some cases, one quark or ghost loop). It is the natural extension of the bottom-specific $O(\alpha_s X_t)$ correction to the $H \rightarrow b\bar{b}$ decay width [23, 24] by one order of $\alpha_s$.

The next-to-leading-order QCD corrections to the top-quark-induced shifts in the $l^+l^- H$, $W^+W^- H$, and $ZZH$ couplings have been found in Ref. [30]. The present paper completes the knowledge of the $O(\alpha_s^2 X_t)$ corrections to the SM Yukawa couplings, still excluding that of the top quark. The Yukawa couplings of the first four quark flavours receive corrections from the same class of diagrams, while, in the case of the bottom Yukawa coupling, an additional class of diagrams must be included.

In Ref. [31], it was noticed that the QCD perturbation expansions of the $l^+l^- H$, $W^+W^- H$, and $ZZH$ couplings and the electroweak $\rho$ parameter, for which the $O(\alpha_s^2 X_t)$ correction is also known [31], exhibit striking similarities. If the top-quark mass is renormalized according to the on-shell scheme, the coefficients of $(\alpha_s/\pi)X_t$ and $(\alpha_s/\pi)^2 X_t$ are in each case negative and of increasing magnitude. On the other hand, the corresponding coefficients in the $\overline{\text{MS}}$ scheme are much smaller and of variant sign. This gave support to the notion that the use of the pole mass deteriorates the convergence properties of the QCD perturbation series, which may also be motivated from the study of renormalons [32]. It is interesting to find out whether this observation is substantiated by the analysis of the light-quark and bottom Yukawa couplings. We shall return to this issue in Sect. 5.

Our key result for the $H \rightarrow b\bar{b}$ decay width to $O(\alpha_s^2 X_t)$ has recently been presented without derivation in a brief note [33]. This paper provides the full details of our analysis and also deals with the $H \rightarrow q\bar{q}$ decays, where $q = u, d, s, c$. It is organized as follows. After introducing our notations in Sect. 2, we shall derive, in Sect. 3, a heavy-top-quark effective Lagrangian with two coefficient functions to be determined by diagrammatic
calculation. From this Lagrangian, we shall obtain a generic formula for the \( H \rightarrow q\bar{q} \) decay widths valid through \( \mathcal{O}(\alpha_s^2 X_t) \). In Sect. 4, we shall calculate, at three loops, the coefficient functions relevant for the first four quark flavours. In Sect. 5, we shall extend this analysis to also include the case \( q = b \). Our conclusions will be summarized in Sect. 6.

2 Notations

In this section, the notation is fixed and useful formulae, which will be necessary to numerically evaluate our results, are provided. The calculation is performed in the framework of dimensional regularization with space-time dimension \( n = 4 - 2\varepsilon \). The QCD gauge group is taken to be \( \text{SU}(N_c) \), with \( N_c \) arbitrary. The colour factors corresponding to the Casimir operators of the fundamental and adjoint representations are \( C_F = (N_c^2 - 1)/(2N_c) \) and \( C_A = N_c \), respectively. For the numerical evaluation we set \( N_c = 3 \). The trace normalization of the fundamental representation is \( T = 1/2 \). The number of active quark flavours is denoted by \( n_f \). Unless otherwise stated, we work in the \( \overline{\text{MS}} \) scheme, with \( \mu \) being the renormalization scale.

The \( \mu \) dependence of the strong coupling constant, \( \alpha_s(\mu) \), and any quark mass, \( m(\mu) \), is governed by the renormalization-group (RG) equations,

\[
\frac{\mu^2 d \alpha_s(\mu)}{d\mu^2} = \beta(\alpha_s) = -\left( \frac{\alpha_s}{\pi} \right)^2 \sum_{k \geq 0} \beta_k \left( \frac{\alpha_s}{\pi} \right)^k ,
\]

\[
\frac{\mu^2 d \ln m(\mu)}{d\mu^2} = \gamma_m(\alpha_s) = -\frac{\alpha_s}{\pi} \sum_{k \geq 0} \gamma_m^k \left( \frac{\alpha_s}{\pi} \right)^k ,
\]

where

\[
\beta_0 = \frac{1}{4} \left[ \frac{11}{3} C_A - \frac{4}{3} T n_f \right] , \quad \beta_1 = \frac{1}{16} \left[ \frac{34}{3} C_A^2 - 4 C_F T n_f - \frac{20}{3} C_A T n_f \right] ,
\]

\[
\gamma_0 = \frac{1}{4} \left[ 3C_F \right] , \quad \gamma_1 = \frac{1}{16} \left[ \frac{3}{2} C_F^2 + \frac{97}{6} C_F C_A - \frac{10}{3} C_F T n_f \right] .
\]

are the first few coefficients of the QCD \( \beta \) and \( \gamma \) functions. The next-to-leading-order (two-loop) solution of Eq. (1) reads

\[
\frac{\alpha_s(\mu)}{\pi} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} \left[ 1 - \frac{\beta_1}{\beta_0^2} \ln(\ln(\mu^2/\Lambda^2)) \right] ,
\]

where \( \Lambda \) is the asymptotic scale parameter.

The relation between the \( \overline{\text{MS}} \) mass \( m(\mu) \) and the on-shell mass \( M_t \) of the top quark is given by

\[
\frac{m_t(\mu)}{M_t} = 1 + X_t \left( 4 + \frac{3}{2} \ln \frac{\mu^2}{M_t^2} \right) + \frac{\alpha_s(n_f)(\mu)}{\pi} C_F \left( -1 - \frac{3}{4} \ln \frac{\mu^2}{M_t^2} \right) .
\]
+ \left( \frac{\alpha_s(n_f)(\mu)}{\pi} \right)^2 \left\{ C_F T \left( \frac{3}{4} - \frac{3}{2} \zeta(2) \right) \right.
+ C_F^2 \left[ \frac{7}{128} + 3 \zeta(2) \left( \frac{5}{8} + \ln 2 \right) - \frac{3}{4} \zeta(3) + \frac{21}{32} \ln \frac{\mu^2}{M_t^2} + \frac{9}{32} \ln^2 \frac{\mu^2}{M_t^2} \right]
+ C_A C_F \left[ -\frac{1111}{384} + \frac{1}{2} \zeta(2) (1 - 3 \ln 2) + \frac{3}{8} \zeta(3) - \frac{185}{96} \ln \frac{\mu^2}{M_t^2} - \frac{11}{32} \ln^2 \frac{\mu^2}{M_t^2} \right]
+ C_F T n_f \left( \frac{71}{96} + \frac{1}{2} \zeta(2) + \frac{13}{24} \ln \frac{\mu^2}{M_t^2} + \frac{1}{8} \ln^2 \frac{\mu^2}{M_t^2} \right) \right\} + O(\alpha_s^3, \alpha_s X_t, X_t^2),
(6)

where \( n_f = 6, X_t = \left( G_F M_t^2 / 8 \pi^2 \sqrt{3} \right) \), and \( \zeta \) is Riemann’s zeta function, with values \( \zeta(2) = \pi^2 / 6 \) and \( \zeta(3) \approx 1.202057 \). The \( O(\alpha_s^2) \) and \( O(X_t) \) corrections were calculated in Refs. [34] and [35, 36], respectively. Iterating Eq. (6), one obtains the scale-independent \( \overline{\text{MS}} \) mass \( \mu_t = m_t(\mu_t) \) as

\[
\frac{\mu_t}{M_t} = 1 + 4 X_t - \frac{\alpha_s(n_f)(M_t)}{\pi} C_F + \left( \frac{\alpha_s(n_f)(M_t)}{\pi} \right)^2 \left\{ C_F T \left( \frac{3}{4} - \frac{3}{2} \zeta(2) \right) \right.
+ C_F^2 \left[ \frac{199}{128} + 3 \zeta(2) \left( \frac{5}{8} + \ln 2 \right) - \frac{3}{4} \zeta(3) \right] + C_A C_F \left[ -\frac{1111}{384} 
+ \frac{1}{2} \zeta(2) (1 - 3 \ln 2) + \frac{3}{8} \zeta(3) \right]
+ C_F T n_f \left( \frac{71}{96} + \frac{1}{2} \zeta(2) \right) \right\} + O(\alpha_s^3, \alpha_s X_t, X_t^2),
(7)

In the \( \overline{\text{MS}} \) scheme, the relation between the values of \( \alpha_s(\mu) \) for \( n_f = 5 \) and \( n_f = 6 \) is given by

\[
\alpha_s^{(5)}(\mu) = \alpha_s^{(6)}(\mu) \left\{ 1 + \frac{\alpha_s^{(6)}(\mu)}{\pi} T \left[ \frac{1}{3} \ln \frac{\mu^2}{m_t^2} + x_t \left( \frac{2}{3} + \ln \frac{\mu^2}{m_t^2} \right) \right] + O(\alpha_s^2) \right\},
(8)
\]

where \( x_t(\mu) = \left[ G_F m_t^2(\mu) / 8 \pi^2 \sqrt{2} \right] \). The constant term of \( O(\alpha_s^2 x_t) \) in Eq. (8) represents a new result, which follows from the analysis of the next section. Inserting the \( O(X_t) \) term of Eq. (8) into Eq. (8), one obtains

\[
\alpha_s^{(5)}(\mu) = \alpha_s^{(6)}(\mu) \left\{ 1 + \frac{\alpha_s^{(6)}(\mu)}{\pi} T \left[ \frac{1}{3} \ln \frac{\mu^2}{M_t^2} + X_t \left( \frac{2}{3} + \ln \frac{\mu^2}{M_t^2} \right) \right] + O(\alpha_s^2) \right\}.
(9)
\]

3 Effective Lagrangian

In this section, we construct an effective Lagrangian for the interaction between an intermediate-mass Higgs boson and a quark-antiquark pair. Therefore, in addition to the pure QCD Lagrangian, the couplings of the quarks to the Higgs boson, \( H \), the neutral Goldstone boson, \( \chi \), and the charged Goldstone bosons, \( \phi^\pm \), must be taken into account. This will produce corrections proportional to \( X_t \). In this paper, we are not interested in corrections of \( O(X_t^2) \), and our formulae will not in general be valid in this order.
As a starting point, we consider the bare Yukawa Lagrangian,

$$\mathcal{L}_Y = -\frac{H^0 v^0}{v^0} J,$$

(10)

where $v$ is the Higgs vacuum-expectation value, the superscript 0 labels bare quantities, and the operator $J$ is defined as

$$J = \sum_q m_q^0 q^0 \bar{q}^0 + m_t^0 \bar{t}^0 t^0.$$ 

(11)

Here, $q$ runs over $u, d, s, c, \text{ and } b$. It is easy to see that $J$ is a finite operator, in the sense that no additional renormalization constant is needed. Our aim is to construct the equivalent expression for $J$ in the effective theory where the top quark is integrated out. Because only the leading terms in $M_t$ are considered, $\mathcal{L}_Y$ may be written as a linear combination of three physical operators with mass dimension four, namely,

$$\mathcal{L}_Y \xrightarrow{m_t^0 \to \infty} \mathcal{L}^\text{eff}_Y = -\frac{H^0}{v^0} \left[ C_1^0 O_1' + \sum_q \left( C_{2q}^0 O_{2q}' + C_{3q}^0 O_{3q}' \right) \right],$$

(12)

where

$$O_1' = \left( G_{a\mu\nu}^{0\nu} \right)^2,$$

$$O_{2q}' = m_q^0 \bar{q}^0 q^0,$$

$$O_{3q}' = \bar{q}^0 \left[ \frac{i}{2} \left( \gamma^{0\nu} - \gamma^{\nu \bar{q}^0 q} - m_q^0 \right) q^0 \right]$$

are bare operators in the $n_f = 5$ effective theory and $C_i^0 = C_i^0(\alpha_s, m_t^0, \mu) \ (i = 1, 2q, 3q)$ are their bare coefficient functions. Here, $G_{a\mu\nu}$ is the colour field strength, $D_\mu$ is the covariant derivative, and the parameters and fields of the effective theory are marked by a prime. Notice that $O_{3q}'$ vanishes by the fermionic equation of motion and may be omitted once $C_1^0$ and $C_{2q}^0$ are determined.

The relations between the parameters and fields in the full and effective theories read

$$q^{0\nu} = \left( \zeta_{2q}^0 \right)^{1/2} q^0,$$

$$C_{a\mu}^{0\nu} = \left( \zeta_3^0 \right)^{1/2} G_{a\mu}^0,$$

$$m_q^{0\nu} = \zeta_{m,q}^0 m_q^0,$$

$$g_s^{0\nu} = \zeta_{g_s}^0 g_s,$$

(14)

where $G_{a\mu}$ denotes the gluon field. The renormalization constants $\zeta_{2q}^0, \zeta_{m,q}^0, \text{ and } \zeta_3^0$ play an important rôles in the determination of the coefficient functions $C_i^0$. As may be seen using the method of projectors \cite{37}, they may be calculated from the vector and scalar
parts of the quark self-energy, $\Sigma_V(p^2)$ and $\Sigma_S(p^2)$, and the transverse part of the gluon self-energy, $\Pi(p^2)$, via

\[
\zeta_{2q}^0 = 1 + \Sigma_V^0(0), \quad (15) \\
\zeta_{m,q}^0 = \frac{1 - \Sigma_S^0(0)}{1 + \Sigma_V^0(0)}, \quad (16) \\
\zeta_3^0 = 1 + \Pi^0t(0), \quad (17)
\]

where the superscript $t$ indicates that only diagrams containing the top quark have to be considered and the superscript 0 reminds us of that the parameters are still in their bare forms. In our convention, the bare quark and gluon propagators are proportional to

\[
\begin{align*}
\zeta_{0q} &= 1 + \Sigma_V^0t(0), \\
\zeta_{0m,q} &= 1 - \Sigma_S^0t(0) \quad (15) \\
\zeta_0^3 &= 1 + \Pi^0t(0), \quad (16)
\end{align*}
\]

Notice that the axial-vector part of the quark self-energy, $\Sigma_A(p^2)$, does not enter our analysis because it leads to corrections of $O(x^2t)$. The renormalization constant $\zeta_0^g$ only enters the stage via Eqs. (8) and (9). Through $O(\alpha_s^2x^2t)$, we have $\zeta_0^g = (\zeta_0^3)_{-1/2}$.

By means of the higher-order formulation [38] of a well-known low-energy theorem (LET) [39] in connection with the method of projectors [37], one may derive the following relations which allow one to calculate the coefficient functions:

\[
\begin{align*}
\zeta_0^3 C_1^0 &= -m_0^0 \frac{\partial}{\partial m_0^0} \Pi^0t(0), \quad (18) \\
\zeta_{2q}^0 \zeta_{m,q}^0 \left( C_{2q}^0 - C_{3q}^0 \right) &= 1 - \Sigma_S^0(0) - \frac{m_0^0}{m_0} \frac{\partial}{\partial m_0} \Sigma_S^0(0), \quad (19) \\
\zeta_{2q}^0 C_{3q}^0 &= -m_0^0 \frac{\partial}{\partial m_0} \Sigma_V^0(0). \quad (20)
\end{align*}
\]

Here it is understood that the operator $(m_0^0 \partial/\partial m_0^0)$ only acts on those appearances of $m_0^0$ which remain after every $1/v^0$ is saturated by one power of $m_0^0$ to give $m_0^0/v^0$. As may be seen from Eqs. (18)–(20), the calculation of the coefficient functions involves the quark- and gluon-propagator diagrams containing the top quark, with nullified external momenta. By means of the LET, the corresponding vertex diagrams may be generated by attaching an external Higgs-boson line with zero momentum to the quark lines. To the order of interest here, this requires the evaluation of one-, two-, and three-loop tadpole diagrams. They are calculated with the help of the program package MATAD, written in FORM [40], which makes use of integration-by-parts identities developed in Ref. [41].

So far, we have constructed an expression for $J$ in terms of bare operators and coefficient functions, which both still contain poles in $\varepsilon$. As is well known, different operators of the same dimension and quantum numbers in general mix under renormalization. Specifically, the renormalized operators, which will be denoted by square brackets, are related to the unrenormalized ones according to [12]

\[
\begin{align*}
\left[ O_1' \right] &= \left[ 1 + 2 \left( \frac{\alpha_s \partial}{\partial \alpha_s} \ln Z_g \right) \right] O_1' - 4 \left( \frac{\alpha_s \partial}{\partial \alpha_s} \ln Z_m \right) \sum_q O_{2q}', \\
\left[ O_{2q}' \right] &= O_{2q}'. \quad (21)
\end{align*}
\]
where $Z_g$ and $Z_m$ are the coupling and mass renormalization constants, respectively, in pure QCD with $n_f = 5$ active flavours. It hence follows that

$$\frac{\mu^2 d}{d\mu^2} [O'_1] = -\left(\frac{\alpha_s \partial}{\partial \alpha_s} \frac{\pi \beta}{\alpha_s}\right) [O'_1] + 4 \left(\frac{\alpha_s \partial}{\partial \alpha_s} \gamma_m\right) \sum_q [O'_{2q}],$$

$$\frac{\mu^2 d}{d\mu^2} [O'_{2q}] = 0. \quad (22)$$

Note that $O'_{3q}$ does not mix with $O'_1$ and $O'_{2q}$. On the other hand, the coefficient functions are renormalized according to

$$C_1 = \frac{1}{1 + 2(\alpha_s \partial/\partial \alpha_s) \ln Z_g} C^0_1,$$

$$C_{2q} = \frac{4(\alpha_s \partial/\partial \alpha_s) \ln Z_m}{1 + 2(\alpha_s \partial/\partial \alpha_s) \ln Z_g} C^0_1 + C^0_{2q}. \quad (23)$$

Consequently, the $n_f = 5$ effective Lagrangian takes the form

$$\mathcal{L}_Y^{\text{eff}} = -\frac{H^0}{v^0} \left[ C_1 [O'_1] + \sum_q \left( C_{2q} [O'_{2q}] + C_{3q} [O'_{3q}] \right) \right]. \quad (24)$$

The new coefficient functions and the operators are individually finite, but, with the exception of $[O'_{2q}]$, they are not separately RG invariant. From now on, $[O'_{3q}]$ will be omitted because it vanishes on mass shell and thus does not contribute to the $H \to q\bar{q}$ decay widths.

It is desirable to arrange for the renormalized coefficient functions and operators to be separately $\mu$ independent up to higher orders. This may be achieved by reshuffling the terms in Eq. (24). Exploiting the RG invariance of $[O'_{2q}]$ and of the trace of the energy-momentum tensor,

$$[\Theta^\mu] = \frac{\pi \beta^{(5)}}{2\alpha_s^{(5)}} [O'_1] + \left(1 - 2\gamma_m^{(5)}\right) \sum_q [O'_{2q}],$$

$$[O'_{3q}] = [O'_{2q}]. \quad (25)$$

where all quantities are defined in the $n_f = 5$ effective theory, we may construct two new operators which are indeed RG invariant, e.g.,

$$[O'_g] = -\frac{2\pi}{\beta_0^{(5)}} \left( \frac{\pi \beta^{(5)}}{2\alpha_s^{(5)}} [O'_1] - 2\gamma_m^{(5)} \sum_q [O'_{2q}] \right),$$

$$[O'_q] = [O'_{2q}]. \quad (26)$$

Thus, the relevant part of the $n_f = 5$ effective Lagrangian may be written as

$$\mathcal{L}_Y^{\text{eff}} = -\frac{H^0}{v^0} \left( C_g [O'_g] + \sum_q C_q [O'_q] \right). \quad (27)$$
where

\[ C_g = -\frac{\alpha_s^{(5)}(5)}{\pi^2 \beta_0^{(5)}} C_1, \]
\[ C_q = \frac{4\alpha_s^{(5)}(5)}{\pi \beta_0^{(5)}} C_1 + C_{2q}. \tag{28} \]

As the new coefficient functions and operators are separately RG invariant, we may choose \( \mu = \mu_t \) for \( C_g \) and \( C_q \) and \( \mu = M_H \) for \( [O'_g] \) and \( [O'_{2q}] \).

In order to calculate the \( H \rightarrow q\bar{q} \) decay width, it is more convenient to re-express \( \mathcal{L}_{\text{eff}}^Y \) in terms of the operators \( [O'_1] \) and \( [O'_{2q}] \). At the same time, the separation of the scales \( M_H \) and \( \mu_t \) is kept, so that Eq. (27) becomes

\[ \mathcal{L}_{\text{eff}}^Y = -\frac{H_0^0}{\nu^0} \left( C_1 [O'_1] + \sum_q C_{2q} [O'_{2q}] \right), \tag{29} \]

where

\[ C_1(\mu_t, M_H) = \frac{\alpha_s^{(5)}(\mu_t) \beta_0^{(5)}(M_H)}{\alpha_s^{(5)}(M_H) \beta_0^{(5)}(\mu_t)} C_1(\mu_t), \]
\[ C_{2q}(\mu_t, M_H) = \frac{4\alpha_s^{(5)}(\mu_t)}{\pi \beta_0^{(5)}(\mu_t)} \left( \gamma_m^{(5)}(\mu_t) - \gamma_m^{(5)}(M_H) \right) C_1(\mu_t) + C_{2q}(\mu_t). \tag{30} \]

Notice that, for our purposes, \( C_1 \) and \( C_{2q} \) are needed up to \( \mathcal{O}(\alpha_s x_t) \) and \( \mathcal{O}(\alpha_s^2 x_t) \), respectively. It is instructive to expand Eq. (31) in terms of \( \alpha_s^{(6)}(\mu_t) \), keeping only terms relevant for our \( \mathcal{O}(\alpha_s^2 x_t) \) calculation. Observing that \( C_1 \) starts at \( \mathcal{O}(\alpha_s) \), while \( C_{2q} = 1 \) to lowest order, this yields

\[ C_1(\mu_t, M_H) = C_1(\mu_t), \]
\[ C_{2q}(\mu_t, M_H) = -4 \gamma_0^0 \frac{\alpha_s^{(6)}(\mu_t)}{\pi} C_1(\mu_t) \ln \frac{\mu_t^2}{M_H^2} + C_{2q}(\mu_t). \tag{31} \]

The coefficients \( C_1 \) and \( C_{2q} \) must be computed diagrammatically. This will be done for the light-quark flavours in Sect. [4] and for \( q = b \) in Sect. [5]. In the remainder of this section, we present a generic formula, derived from Eq. (27), for the \( H \rightarrow q\bar{q} \) decay width, appropriate for \( M_H \ll 2M_t \), which accommodates all presently known corrections. It reads

\[ \Gamma(H \rightarrow q\bar{q}) = \Gamma_{\text{Born}}^{qq} \left( 1 + \delta_{\text{Born}}^q \right)^2 \left[ \left( 1 + \Delta_q^{\text{QED}} \right) \left( 1 + \Delta_q^{\text{weak}} \right) \left( 1 + \Delta_q^{\text{QCD}} \right) C_{2q} \right]^2 \]
\[ + \Xi_q^{\text{QCD}} C_1 C_{2q}. \tag{32} \]

Here,

\[ \Gamma_{qq}^{\text{Born}} = \frac{N_c G_F M_H m_q^2}{4\pi \sqrt{2}} \left( 1 - \frac{4m_q^2}{M_H^2} \right)^{3/2}, \tag{33} \]
is the Born result including the full mass dependence. As is well known \[18\], we may avoid the appearance of large logarithms of the type $\ln(M_H^2/m_q^2)$ in the QCD correction, $\Delta_q^{\text{QCD}}$, by taking $m_q$ in Eq. (33) to be the $\overline{\text{MS}}$ mass for $n_f = 5$, $m_q^{(5)}(\mu)$, with $\mu$ of order $M_H$. Consequently, we may put $m_q = 0$ in $\Delta_q^{\text{QCD}}$. We may proceed similarly with the quantum-electrodynamical (QED) correction, $\Delta_q^{\text{QED}}$, which then takes the form

$$\Delta_q^{\text{QED}} = \frac{\alpha(\mu)}{\pi} Q_q^2 \left( \frac{17}{4} + \frac{3}{2} \ln \frac{\mu^2}{M_H^2} \right),$$

where $\alpha(\mu)$ is the $\overline{\text{MS}}$ fine-structure constant and $Q_q$ is the fractional quark charge. In turn, $m_q^{(5)}(\mu)$ is then also shifted by a QED correction \[33\] from the pole mass, $M_q$. $\Delta_q^{\text{weak}|_{x_t=0}}$ denotes the weak correction with the leading $O(\alpha)$ term stripped off. If we put $m_q = 0$ and consider the limit $M_H \ll 2M_W$, $\Delta_q^{\text{weak}|_{x_t=0}}$ simplifies to \[17\]

$$\Delta_q^{\text{weak}|_{x_t=0}} = \frac{G_F M_Z^2}{8\pi^2 \sqrt{2}} \left[ \frac{1}{2} - 3 \left( 1 - 4 s_w^2 |Q_q| \right)^2 + c_w^2 \left( \frac{3}{s_w^2} \ln c_w^2 - 5 \right) \right],$$

where $c_w^2 = 1 - s_w^2 = M_W^2/M_Z^2$, $M_Z$ is the $Z$-boson mass, and $M_W$ is the $W$-boson mass. $\Delta_q^{\text{QCD}}$ is the well-known QCD correction in the $n_f = 5$ effective theory \[25\].

$$\Delta_q^{\text{QCD}} = \frac{\alpha_s^{(n_f)}(\mu)}{\pi} C_F \left( \frac{17}{4} + \frac{3}{2} \ln \frac{\mu^2}{M_H^2} \right)$$

$$+ \left( \frac{\alpha_s^{(n_f)}(\mu)}{\pi} \right)^2 \left[ C_F^2 \left( \frac{691}{64} - \frac{9}{4} \zeta(2) - \frac{9}{4} \zeta(3) + \frac{105}{16} \ln \frac{\mu^2}{M_H^2} + \frac{9}{8} \ln^2 \frac{\mu^2}{M_H^2} \right) \right]$$

$$+ C_F C_A \left( \frac{893}{64} - \frac{11}{8} \zeta(2) - \frac{31}{8} \zeta(3) + \frac{71}{12} \ln \frac{\mu^2}{M_H^2} + \frac{11}{16} \ln^2 \frac{\mu^2}{M_H^2} \right)$$

$$+ C_F T n_f \left( - \frac{65}{16} + \frac{1}{2} \zeta(2) + \zeta(3) - \frac{11}{6} \ln \frac{\mu^2}{M_H^2} - \frac{1}{4} \ln^2 \frac{\mu^2}{M_H^2} \right)$$

$$= \frac{\alpha_s^{(5)}(\mu)}{\pi} \left( \frac{17}{3} + 2 \ln \frac{\mu^2}{M_H^2} \right)$$

$$+ \left( \frac{\alpha_s^{(5)}(\mu)}{\pi} \right)^2 \left[ \frac{8851}{144} - \frac{47}{6} \zeta(2) - \frac{97}{6} \zeta(3) + \frac{263}{9} \ln \frac{\mu^2}{M_H^2} + \frac{47}{12} \ln^2 \frac{\mu^2}{M_H^2} \right].$$

This correction originates from the class of diagrams where the Higgs boson directly couples to the final-state $q\bar{q}$ pair, i.e., it does not comprise the double-triangle topologies of Ref. \[28\]. The latter will be discussed below. The scale $\mu$ in $\Delta_q^{\text{QCD}}$ must be identified with that of $m_q^{(5)}(\mu)$ in $\Gamma_{q\bar{q}}^\text{Born}$. It is natural to choose $\mu = M_H$ in order to suppress the logarithms of RG origin.

The corrections of $O(\alpha_s^n x_t)$, with $n = 0, 1, 2$, as well as the $n_f = 6$ QCD corrections of $O(\alpha_s)$ and $O(\alpha_s^2)$ are all contained in $\delta_u$, $C_1$, and $C_2q$. $\delta_u$ contains the universal $O(\alpha_s^n x_t)$
corrections, which originate from the renormalizations of the Higgs wave function and vacuum expectation value. Specifically, we have

\[ \frac{H^0}{v^0} = 2^{1/4}G_F^{1/2}H \left( 1 + \bar{\delta}_u \right), \] (37)

with \[30\]

\[ \bar{\delta}_u = x_t \left\{ \frac{7}{2} + \frac{\alpha_s^{(6)}(\mu)}{\pi} \left( \frac{19}{3} - 2\zeta(2) + 7 \ln \frac{\mu^2}{m_t^2} \right) + \left( \frac{\alpha_s^{(6)}(\mu)}{\pi} \right)^2 \left[ \frac{13307}{864} - \frac{2377}{108} \zeta(2) \right. \right. \]

\[ \left. - \frac{178}{9} \zeta(3) + \frac{143}{12} \zeta(4) - \frac{1}{6} B_4 - \frac{1}{12} D_3 + \frac{1323}{16} S_2 + \left( \frac{267}{8} - \frac{15}{2} \zeta(2) \right) \ln \frac{\mu^2}{m_t^2} \right. \]

\[ + \frac{105}{8} \ln^2 \frac{\mu^2}{m_t^2} \right\}, \] (38)

where \( S_2 \approx 0.260\,434, D_3 \approx -3.027\,009, \) and \( B_4 \approx -1.762\,800 \) are mathematical constants related to certain three-loop tadpole diagrams. The formula for \( N_c \) arbitrary may be found in Ref. \[30\]. The choice \( \mu = \mu_t \) eliminates the logarithms in Eq. (38) and may thus be considered natural.

As is well known \[28\], starting at \( O(\alpha_s^2) \), \( \Gamma (H \to q\bar{q}) \) also receives leading contributions from the \( q\bar{q} \) and \( qg\bar{q} \) cuts of the double-triangle diagrams where the top quark circulates in one of the triangles. The additional exchange of a virtual Higgs or Goldstone boson within the top-quark triangle of the double-triangle seed diagram gives rise to a \( O(\alpha_s^2 x_t) \) correction, which we must include in our analysis. In the framework of the \( n_f = 5 \) effective theory, where the top quark only appears in the coefficient functions, this class of contributions is generated by the interference diagram of the operators \[ O'_2 \] and \[ O'_3 \] depicted in Fig. 1. The absorptive part of this diagram also includes a contribution from the \( gg \) cut, which is well known and must be subtracted in order to obtain the desired \( O(\alpha_s^2 x_t) \) correction to \( \Gamma (H \to q\bar{q}) \). We so obtain

\[ \Xi^{QCD}_q = \frac{\alpha_s^{(5)}(\mu)}{\pi} C_F \left( -19 + 6\zeta(2) - \ln^2 \frac{m_q^2}{M_H^2} - 6 \ln \frac{\mu^2}{M_H^2} \right). \] (39)

While \( \Xi^{QCD}_q \) is obviously not RG invariant, the physical observable \( \Gamma (H \to q\bar{q}) \) is because, in Eq. (32), the \( \mu \) dependence of \( \Xi^{QCD}_q \) is compensated by that of \( C_{2q} \), as will become apparent in the next section.

As a by-product, we may derive from the \( n_f = 5 \) effective Lagrangian \[24\] a formula for the \( H \to gg \) decay width which includes the \( O(x_t) \) correction. Our result is

\[ \Gamma (H \to gg) = A_{gg} \left( 1 + \bar{\delta}_u \right)^2 (C_1)^2, \] (40)

where

\[ A_{gg} = \frac{N_c C_F G_F M_H^3}{\pi \sqrt{2}}. \] (41)
Notice that, to the order of our calculation, \( C_1 \) is independent of the quark flavour.

At this point, we should mention that the QCD correction \([44]\) to \( \Gamma(H \to gg) \) also includes contributions due to \( q\bar{q}g \) final states. These may also be interpreted as \( \mathcal{O}(\alpha_s^3 M_H^2/m_q^2) \) corrections to the respective \( H \to q\bar{q} \) decay widths \([8, 45]\). Specifically, these corrections would appear in form of a term proportional to \( \left( \frac{\alpha_s^{(5)}}{\pi} \right) (M_H^2/m_q^2)(C_1)^2 \) within the square brackets of Eq. \((32)\). In the following, this term will not be taken into account, since it is formally one order of \( \alpha_s \) beyond our considerations.

In the next section, we shall derive \( C_1 \) as well as \( C_2q \) for \( q \neq b \). This will enable us to calculate \( \Gamma(H \to q\bar{q}) \) through \( \mathcal{O}(\alpha_s^2 x_t) \) from Eq. \((32)\) and \( \Gamma(H \to gg) \) through \( \mathcal{O}(x_t) \) from Eq. \((40)\). The case \( q = b \) is more complicated and will be treated in Sect. \( 5 \).

### 4 \( H \to q\bar{q} \) decay to \( \mathcal{O}(\alpha_s^2 x_t) \)

In this section, we calculate the \( \mathcal{O}(\alpha_s^2 x_t) \) corrections to the \( H \to q\bar{q} \) decay widths, where \( q \neq b \). The relevant types of diagrams are depicted in Figs. \(2(a)\)–\(c\). The diagrams of types (b) and (c) emerge from the pure QCD diagram in Fig. \(2(a)\) by allowing for the exchange of one virtual Higgs or Goldstone boson. Although the pure QCD diagram only contributes in \( \mathcal{O}(\alpha_s^2) \), it is needed for the renormalization. There, the top-quark mass appears logarithmically and has to be replaced according to \([35, 36]\)

\[
m_t^0 = m_t \left( 1 + \frac{3}{2\varepsilon} x_t \right).
\]

(42)

The counterterm thus obtained cancels the ultraviolet subdivergences of the type-(b) diagrams; the remaining overall divergences are removed if the LET is applied. We checked that this is true separately for the contributions due to the \( H, \chi, \) and \( \phi^\pm \) bosons.

The diagrams of type (c), where the scalar-boson lines are attached to both the top-quark loop and the light-quark line, are also separately finite upon application of the LET. Due to the appearance of one power of the light-quark Yukawa coupling, in our approximation, they only contribute to the scalar part of light-quark self-energy, \( \Sigma_S(0) \).
Figure 2: Typical diagrams generating universal $\mathcal{O}(\alpha_s^2 x_t)$ corrections to $\Gamma (H \to q\bar{q})$. The dashed line in diagram (b) represents a $H$, $\chi$, or $\phi^\pm$ boson, while in diagram (c) only the $H$ and $\chi$ bosons appear. In total, there are 12 diagrams of type (b) and 12 diagrams of type (c).

A potential problem arises in the diagrams of type (c) if the scalar particle is the neutral Goldstone boson $\chi$, since then only one $\gamma^5$ matrix appears in each quark line. However, the arbitrariness of the $\gamma^5$ definition only affects the finite terms of the original diagrams, which are removed by taking the derivative according to the LET.

In the MS scheme, the coefficient functions are found to be

$$C_1 = \frac{\alpha_s^{(6)}(\mu)}{\pi} T \left( -\frac{1}{6} + \frac{1}{2} x_t \right),$$

$$C_{2q} = 1 + \left( \frac{\alpha_s^{(6)}(\mu)}{\pi} \right)^2 C_F T \left\{ \frac{5}{12} - \frac{1}{2} \ln \frac{\mu^2}{m_t^2} + x_t \left[ \frac{7}{2} - 18 \zeta(3) + I_q (6 + 36 \zeta(3)) \right] \right\},$$

where $I_q$ is the third component of weak isospin, i.e., $I_q = 1/2$ for up-type quark flavours and $I_q = -1/2$ for down-type quark flavours. The dependence on $I_q$ stems from the $q\bar{q}\chi$ coupling, which appears linearly in the $\chi$-exchange diagrams of type (c). There are several checks for Eqs. (43) and (44). If we ignore the RG improvement of Eq. (30) and insert the $\mathcal{O}(\alpha_s)$ term of $C_1$ and the $\mathcal{O}(\alpha_s^2)$ term of $C_{2q}$ into Eq. (32), then we recover the $\mathcal{O}(\alpha_s^2)$ double-triangle contribution to $\Gamma (H \to q\bar{q})$ found in Ref. [28]. The $\mathcal{O}(\alpha_s x_t)$ term of $C_1$ may be deduced from Ref. [46]. The $\mathcal{O}(\alpha_s^2 x_t)$ term of $C_2$ is new. In compliance with RG invariance, it does not contain a logarithm of the type $\ln(\mu^2/m_t^2)$.

Since the measured top-quark mass corresponds to the pole mass $M_t$, it is convenient to directly express the perturbative expansions in terms of $M_t$. In Eqs. (32) and (40), the radiative corrections are arranged in such a way that, in the limit $M_H \ll 2 M_t$, all dependence on the top-quark mass is concentrated in $\delta_u, C_1,$ and $C_{2q}$. Consequently, we just need to substitute Eq. (6) into Eqs. (38), (43), and (44). In the case of $C_{2q}$, this leads to

$$C_{2q}^{\text{OS}} = 1 + \left( \frac{\alpha_s^{(6)}(\mu)}{\pi} \right)^2 C_F T \left\{ \frac{5}{12} - \frac{1}{2} \ln \frac{\mu^2}{M_t^2} + X_t \left[ \frac{15}{2} - 18 \zeta(3) + I_q (6 + 36 \zeta(3)) \right] \right\}.$$
\[ \Gamma(H \rightarrow gg) = \frac{A_{gg}}{144} \left( \frac{\alpha_s^{(0)}(\mu)}{\pi} \right)^2 (1 + X_t), \tag{46} \]

which is in agreement with Ref. [46].

5 \textbf{\textit{H → b b decay to } } \mathcal{O}(\alpha_s^2 x_t)

In this section, we upgrade the \( \mathcal{O}(\alpha_s^2 x_t) \) calculation of \( \Gamma(H \rightarrow q\bar{q}) \) to include the case \( q = b \). In addition to the diagrams in Fig. 2, we must now consider diagrams where the primary bottom quark branches into a top quark and a charged Goldstone boson \( \phi^\pm \). Typical examples are depicted in Fig. 3. At the three-loop level, there is a total of 54 such diagrams. These diagrams only affect \( C_{2q} \), while \( C_1 \) remains unchanged. Because the top-quark-induced corrections already start at \( \mathcal{O}(x_t) \), the renormalization constants \( \zeta^0_{2q} \) and \( \zeta^0_{m,q} \) of Eqs. (13) and (16), respectively, do contribute to Eqs. (19) and (20), from which we extract \( C_{2b}^0 \). Thereby, it is important to observe that the interference of \( \mathcal{O}(x_t) \) and \( \mathcal{O}(\alpha_s^2) \) terms contribute in \( \mathcal{O}(\alpha_s^2 x_t) \).

Figure 3: Typical diagrams generating non-universal \( \mathcal{O}(\alpha_s^n x_t) \) \((n = 0, 1, 2) \) corrections to \( \Gamma(H \rightarrow b\bar{b}) \). The dashed lines represent the charged Goldstone boson \( \phi^\pm \). In total, there are 4 (54) irreducible two-loop (three-loop) diagrams.

In the \( \overline{\text{MS}} \) scheme, \( C_{2b} \) is found to be

\[ C_{2b} = C_{2d} + x_t \left\{ -3 + \frac{\alpha_s^{(n_f)}}{\pi} C_F \left( -\frac{21}{4} - \frac{9}{2} \ln \frac{\mu^2}{m_t^2} \right) + \left( \frac{\alpha_s^{(n_f)}}{\pi} \right)^2 \right\} \left[ C_F \left( -\frac{441}{64} \right) \right] \]
+ 9\zeta(2) + \frac{99}{4}\zeta(3) - \frac{27}{16}\ln\frac{\mu^2}{m_t^2} - \frac{27}{8}\ln^2\frac{\mu^2}{m_t^2} + CFCA\left(-\frac{3629}{192} + \frac{11}{2}\zeta(2) - \frac{45}{8}\zeta(3)\right) \\
- \frac{87}{8}\ln\frac{\mu^2}{m_t^2} - \frac{33}{16}\ln^2\frac{\mu^2}{m_t^2} + CF\tau\left(\frac{241}{48} - 2\zeta(2) + 3\ln\frac{\mu^2}{m_t^2} + \frac{3}{4}\ln^2\frac{\mu^2}{m_t^2}\right)\right}\}
= C_{2d} + x_t\left\{-3 + \frac{\alpha_s^{(n)}(\mu)}{\pi}\left(-7 - 6\ln\frac{\mu^2}{m_t^2} + \left(\frac{\alpha_s^{(n)}(\mu)}{\pi}\right)^2\right)\right\}
\times \left[-\frac{3253}{48} + 30\zeta(2) + \frac{43}{2}\zeta(3) - \frac{69}{2}\ln\frac{\mu^2}{m_t^2} - \frac{45}{4}\ln^2\frac{\mu^2}{m_t^2}\right], \tag{47}

where \(n_f = 6\). Replacing \(m_t\) by \(M_t\) via Eq. (13) leads to

\[ C_{2b}^{OS} = C_{2d}^{OS} + X_t\left\{-3 + \frac{\alpha_s^{(n)}(\mu)}{\pi}(\frac{\alpha_s^{(n)}(\mu)}{\pi})^2\left(C_F\left(-\frac{9}{2} + 9\zeta(2)\right) + \frac{2}{3}\left(-\frac{27}{32} + 9\zeta(2)\left(\frac{9}{4} - 2\ln 2\right) + \frac{117}{4}\zeta(3)\right) + C_A\left(-\frac{37}{24} + \zeta(2)\left(\frac{5}{2} + 9\ln 2\right)\right)\right)\right\}
\times \left[-\frac{67}{3} + 4\zeta(2)(8 + 2\ln 2) + \frac{41}{2}\zeta(3) + \frac{7}{4}\ln\frac{\mu^2}{M_t^2}\right], \tag{48}

To simplify Eqs. (47) and (48), we set \(N_F = 3\), insert the numerical values of the mathematical constants, and eliminate the RG logarithms by choosing \(\mu = \mu_t\) and \(\mu = M_t\), respectively. We so obtain

\[ C_{2b} = 1 + 18\left(\frac{\alpha_s^{(n)}(\mu_t)}{\pi}\right)^2 + x_t(\mu_t)\left[-3 - 7\frac{\alpha_s^{(n)}(\mu_t)}{\pi} + \left(\frac{\alpha_s^{(n)}(\mu_t)}{\pi}\right)^2\right]\times (-28.018 + 1.154 n_f), \tag{49}
\]

\[ C_{2b}^{OS} = 1 + 18\left(\frac{\alpha_s^{(n)}(M_t)}{\pi}\right)^2 + X_t\left[-3 + \frac{\alpha_s^{(n)}(M_t)}{\pi} + \left(\frac{\alpha_s^{(n)}(M_t)}{\pi}\right)^2\right]\times (64.223 - 5.094 n_f), \tag{50}

where we have displayed the coefficient of \(n_f = 6\), for reasons which will become clear in a moment.

Equation (50) extends the non-universal correction \(\delta_{nt}\) defined by Eq. (12) of Ref. [23] to \(O(\alpha_s^2X_t)\). From Eq. (50) we read off that the leading \(O(X_t)\) term receives the QCD
correction factor $\left[1 - 0.333 \alpha_s^{(0)}(M_t)/\pi - 11.219 \left(\alpha_s^{(6)}(M_t)/\pi\right)^2\right]$. We thus recover a pattern similar to the electroweak parameter $\Delta \rho$ \cite{31} and the corrections $\delta_u$, $\delta_{WW}$, and $\delta_{ZZH}$ \cite{30} to the $l^+l^−H$, $W^+W^−H$, and $ZZH$ vertices, respectively. In fact, the corresponding QCD expansions in $\alpha_s^{(6)}(M_t)/\pi$ of these four observables all have negative coefficients which dramatically increase in magnitude as one passes from two to three loops \cite{30}. On the other hand, if the top-quark mass is renormalized in the \MS scheme at scale $\mu = M_t$, then the respective QCD expansions in $\alpha_s^{(6)}(\mu_t)/\pi$ are found to have coefficients which have variant signs and nicely group themselves around zero \cite{30}. In the case of $C_{2b}$, the QCD correction factor reads $\left[1 + 2.333 \alpha_s^{(6)}(\mu_t)/\pi + 7.032 \left(\alpha_s^{(6)}(\mu_t)/\pi\right)^2\right]$. We conclude that, also in the case of the $bbH$ interaction, the QCD expansion in the on-shell scheme exhibits a worse convergence behaviour than the one in the \MS scheme. However, the difference is less striking than in the previous four cases.

Finally, we would like to test Broadhurst's rule concerning the naïve non-abelianization of QCD \cite{47}. Guided by the observation that the $n_f$-independent term of $\beta_0$ in Eq. (3) emerges from the coefficient of $n_f$ by multiplication with $−33/2$, Broadhurst conjectured that this very relation between the $n_f$-independent term and the coefficient of $n_f$ approximately holds for any observable at next-to-leading order in QCD. In Ref. \cite{30}, this rule was applied to $\Delta \rho$, $\delta_u$, $\delta_{WW}$, and $\delta_{ZZH}$, and it was found that, in all four cases, the signs and orders of magnitude of the $n_f$-independent terms are correctly predicted. Except for $\delta_{ZZH}$, these predictions come, in fact, very close to the true values. If we multiply the coefficients of $n_f$ in Eqs. (13) and (50) with $−33/2$, we obtain $−19.041$ and $84.055$, which has to be compared with the respective $n_f$-independent terms, $−28.018$ and $64.223$. Once again, the signs and orders of magnitude of the $n_f$-independent terms, which are usually much harder to compute than the coefficients of $n_f$, come out correctly.

6 Discussion and summary

In this paper, we calculated the three-loop $\mathcal{O}(\alpha_s^2 X_t)$ corrections to the $H \rightarrow q\bar{q}$ decay widths of the SM Higgs boson with mass $M_H \ll 2M_t$, including the case $q = b$. To this end, we constructed a $n_f = 5$ effective Yukawa Lagrangian by integrating out the top quark. This Lagrangian is a linear combination of dimension-four operators acting in QCD with $n_f = 5$ quark flavours, while all $M_t$ dependence is contained in the coefficient functions. We renormalized this Lagrangian and, by exploiting the RG invariance of the energy-momentum tensor, rearranged it in such a way that the renormalized operators and coefficient functions are separately independent of the renormalization scale, $\mu$, to the order considered. The RG-improved formulation thus obtained provides a natural separation of the $n_f = 5$ QCD corrections at scale $\mu = M_H$ and the top-quark-induced $n_f = 6$ corrections at scale $\mu = M_t$, in the sense that the final result does not contain logarithms of the type $\ln(M_t^2/M_H^2)$ if the $n_f = 5$ and $n_f = 6$ corrections are expanded in $\alpha_s^{(5)}(M_H)$ and $\alpha_s^{(6)}(M_t)$, respectively.

In contrast to the two-loop $\mathcal{O}(\alpha_s X_t)$ case \cite{23}, where it was sufficient to consider just
one term in the Lagrangian, we needed to take into account three types of operators and to allow for them to mix under renormalization. The mixing terms are related to
the $O(\alpha^2)$ double-triangle contribution considered in Ref. [28] and extend the latter to $O(\alpha_s^2 X_t)$.

Similarly to Ref. [23], we could take advantage of a low-energy theorem to simplify the calculation of the coefficient functions. This allowed us to relate a huge number of three-loop three-point diagrams to a manageable number of three-loop two-point diagrams. Specifically, we had to compute 24 irreducible three-loop two-point diagrams for $q \neq b$ and, in addition, 54 ones for $q = b$. Such a theorem is not available for the gauge interactions, which might explain why three-loop $O(\alpha^2 s X_t)$ corrections have not yet been calculated for the $Z \to q\bar{q}$ decay widths, including the important case of $Z \to b\bar{b}$, which has recently attracted much attention in connection with the so-called $R_b$ anomaly.

To illustrate the effect of the $O(\alpha_s^2 X_t)$ corrections to the $H \to q\bar{q}$ decay widths, we rewrite Eq. (32) as

$$
\Gamma (H \to q\bar{q}) = \Gamma_{q\bar{q}}^{\text{Born}} \left[ (1 + \Delta_q^{\text{QED}}) \left( 1 + \Delta_q^{\text{weak}} \big|_{x_t=0} \right) \left( 1 + \Delta_q^{\text{QCD}} \right) \left( 1 + \Delta_q^t \right) + \Xi_q^{\text{QCD}} \Xi_q^t \right],
$$

where

$$
\Delta_q^t = \left( 1 + \bar{\delta}_u^q \right)^2 (C_{2q})^2 - 1,
$$

$$
\Xi_q^t = \left( 1 + \bar{\delta}_u^q \right)^2 C_1 C_{2q}
$$

contain the leading $M_t$ dependence and refer to the $n_f = 6$ theory, while $\Delta_q^{\text{QCD}}$ and $\Xi_q^{\text{QCD}}$ are confined to pure QCD with $n_f = 5$. We now collect our final results for $\Delta_q^t$ and $\Xi_q^t$. For simplicity, we undo the RG improvement of Eq. (30) and employ the expanded versions of $C_1$ and $C_{2q}$ given in Eq. (31) instead. In the MS scheme with $\mu = \mu_t$, we then have

$$
\Delta_u^t = \left( \frac{\alpha_s^{(6)}(\mu_t)}{\pi} \right)^2 \left[ \left( \frac{5}{9} + \frac{2}{3} \ln \frac{\mu_t^2}{M_H^2} \right) + x_t(\mu_t) \right] \left[ 7 + 6.087 \frac{\alpha_s^{(6)}(\mu_t)}{\pi} + \left( \frac{\alpha_s^{(6)}(\mu_t)}{\pi} \right)^2 \right] \times \left( -6.640 + \frac{8}{3} \ln \frac{\mu_t^2}{M_H^2} \right),
$$

$$
\Delta_d^t = \left( \frac{\alpha_s^{(6)}(\mu_t)}{\pi} \right)^2 \left[ \left( \frac{5}{9} + \frac{2}{3} \ln \frac{\mu_t^2}{M_H^2} \right) + x_t(\mu_t) \right] \left[ 7 + 6.087 \frac{\alpha_s^{(6)}(\mu_t)}{\pi} + \left( \frac{\alpha_s^{(6)}(\mu_t)}{\pi} \right)^2 \right] \times \left( -72.339 + \frac{8}{3} \ln \frac{\mu_t^2}{M_H^2} \right),
$$

$$
\Delta_b^t = \left( \frac{\alpha_s^{(6)}(\mu_t)}{\pi} \right)^2 \left[ \left( \frac{5}{9} + \frac{2}{3} \ln \frac{\mu_t^2}{M_H^2} \right) + x_t(\mu_t) \right] \left[ 1 - 7.913 \frac{\alpha_s^{(6)}(\mu_t)}{\pi} + \left( \frac{\alpha_s^{(6)}(\mu_t)}{\pi} \right)^2 \right] \times \left( -59.163 + \frac{2}{3} \ln \frac{\mu_t^2}{M_H^2} \right).
$$
where \( q = u, d, s, c \). Clearly, \( \Delta_t^q = \Delta_t^u \) and \( \Delta_t^s = \Delta_t^d \). Introducing the top-quark pole mass \( M_t \) and choosing \( \mu = M_t \), we find the equivalent expressions

\[
\Delta_{t, \text{OS}}^u = \left( \frac{\alpha_s(M_t)}{\pi} \right) 2 \left( \frac{5}{9} + 2 \ln \frac{M_t^2}{M_H^2} \right) + X_t \left[ 7 - 12.580 \frac{\alpha_s(M_t)}{\pi} + \left( \frac{\alpha_s(M_t)}{\pi} \right)^2 \right] \\
\Delta_{t, \text{OS}}^d = \left( \frac{\alpha_s(M_t)}{\pi} \right) 2 \left( \frac{5}{9} + 2 \ln \frac{M_t^2}{M_H^2} \right) + X_t \left[ 7 - 12.580 \frac{\alpha_s(M_t)}{\pi} + \left( \frac{\alpha_s(M_t)}{\pi} \right)^2 \right] \\
\Delta_{t, \text{OS}}^b = \left( \frac{\alpha_s(M_t)}{\pi} \right) 2 \left( \frac{5}{9} + 2 \ln \frac{M_t^2}{M_H^2} \right) + X_t \left[ 1 - 10.580 \frac{\alpha_s(M_t)}{\pi} + \left( \frac{\alpha_s(M_t)}{\pi} \right)^2 \right] \\
\times \left( -43.868 + \frac{2}{3} \ln \frac{M_t^2}{M_H^2} \right). \tag{54}
\]
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