A Flat System Possessing no \((x, u)\)-Flat Output

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Abstract—In general, flat outputs of a nonlinear system may depend on the system’s state and input as well as on an arbitrary number of time derivatives of the latter. If a flat output, which also depends on time derivatives of the input, is known, one may pose the question whether there also exists a flat output which is independent of these time derivatives, i.e., an \((x, u)\)-flat output. Until now, the question whether every flat system also possesses an \((x, u)\)-flat output has been open. In this letter, this conjecture is disproved by means of a counterexample, which illustrates the difficulties one may run into when studying flatness. We present a two-input system which is differentially flat with a flat output depending on the state, the input and first-order time derivatives of the input, but which does not possess any \((x, u)\)-flat output. The proof relies on the fact that every \((x, u)\)-flat two-input system can be exactly linearized after a \(d\)-fold prolongation of one of its (new) inputs after a suitable input transformation has been applied.

Index Terms—Flatness, feedback linearization, nonlinear control.

I. INTRODUCTION

The concept of differential flatness has been introduced by Fliess, Lévine, Martin, and Rouchon in \cite{1}, \cite{2}, and has attracted a lot of interest in the control systems community. The property of a system to be differentially flat (or just “flat” for short) allows for a systematic solution of feed-forward and feedback problems, see, e.g., \cite{3}, \cite{4}, \cite{5}, \cite{6}, \cite{7}, \cite{8}. Roughly speaking, a nonlinear control system of the form

\[
\dot{x} = f(x, u)
\]

with \(\text{dim}(x) = n\) states and \(\text{dim}(u) = m\) inputs is flat if there exists an \(m\)-dimensional (fictitious) output

\[
y = \varphi(x, u, \dot{u}, \ldots, u^{(q)}),
\]

\(u^{(q)}\) denoting the \(q\)-th time derivative of the input, such that the state and the input of the system can locally be expressed as functions of this output and a finite number of its time derivatives, i.e.,

\[
x = F_x(y, \dot{y}, \ldots, y^{(r-1)})
\]

\[
u = F_u(y, \dot{y}, \ldots, y^{(r)}).
\]

Such a (fictitious) output \((2)\) is called a flat output of the system \((1)\). The computation of flat outputs for systems of the general form \((1)\) is known to be a difficult problem and an active field of research since the introduction of flatness about 30 years ago. This problem, and flatness in general, have been studied within different mathematical frameworks. In \cite{1}, \cite{2} and \cite{3}, a differential-algebraic setting is employed, whereas (infinite dimensional) differential-geometric frameworks including exterior differential systems are used, e.g., in \cite{6}, \cite{9}, \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}. Recent research in the field of flatness can be found, e.g., in \cite{15}, \cite{16}, \cite{17}, \cite{18}, \cite{19}. Despite the use of sophisticated mathematical tools, up to now, there do not exist easily verifiable necessary and sufficient conditions for flatness, though, there are results available for several classes of systems, including systems which are linearizable by static feedback \cite{20}, \cite{21}, two-input driftless systems \cite{10}, systems linearizable by a one-fold prolongation of a suitably chosen input \cite{22}, \cite{23}, two-input systems linearizable by a two-fold prolongation \cite{24}, \cite{25}, and control-affine systems with four states and two inputs \cite{26}.

One of the difficulties in checking flatness is that a flat output \((2)\) may a priori depend on time derivatives of \(u\) up to an arbitrary order \(q\).

Indeed, given a flat output \(y = (\varphi_1, \ldots, \varphi_m)\), then for an arbitrary positive integer \(\beta\),

\[
\bar{y} = (\varphi_1, \varphi_2 + \frac{\partial \varphi_1}{\partial y_1}, \varphi_3, \ldots, \varphi_m)
\]

is also a flat output. By choosing \(\beta\) large enough, we can thus easily construct a flat output which depends on time derivatives of \(u\) up to an arbitrarily high order. Thus, a general bound on \(q\) which covers all possible flat outputs of a system cannot exist. The question is whether every flat system admits a flat output \(y = \varphi(x, u, \dot{u}, \ldots, u^{(q)})\) with \(q\) bounded in terms of \(n\) and/or \(m\).

In \cite{27}, it is conjectured that there might exist a bound on \(q\) which is linear in the state dimension. This conjecture is motivated by the following example:

\[
x_1^{(q_1)} = u_1, \quad x_2^{(q_2)} = u_2, \quad \dot{x}_3 = u_1u_2,
\]
which for any integers $\alpha_1, \alpha_2 \geq 1$ admits the flat output
\[
  y_1 = x_3 + \sum_{i=1}^{\alpha_1} (-1)^i x_1^{(\alpha_1-i)} u_2^{(i-1)}
\]
\[
  y_2 = x_2.
\]

It is suspected that this system does not possess a flat output depending on derivatives of $u_1$ of order less than $\min(\alpha_1, \alpha_2) - 1$, though, a proof of that claim has not been published. In fact, the authors best knowledge, there has not even been published an example which is flat but evidently does not admit any $(x, u)$-flat output (i.e., a flat output which may depend on the state $x$ and the input $u$, but not on time derivatives of $u$).

In the following, we do exactly that. We prove that the system
\[
  \dot{x}_1 = u_1 \\
  \dot{x}_2 = u_2 \\
  \dot{x}_3 = x_1 + \frac{u_2^2}{\alpha_1} \\
\]
which admits the $(x, u, \dot{u})$-flat output (see Appendix A)
\[
  y_1 = \frac{u_2^2}{u_1} \\
  y_2 = 2x_2 - \frac{2}{u_1}(x_1 u_2 - x_3 \dot{u}_2) - 2 \frac{u_2^2}{u_1^2}(x_2 \dot{u}_2 + x_3 \dot{u}_1) \\
  + \frac{u_2^2}{u_1^3}(x_1 \ddot{u}_2 + 2x_2 \dot{u}_1) - \frac{1}{u_1} x_1 \dddot{u}_1 u_2^2
\]
cannot admit any $(x, u)$-flat output. Proofs in this letter are self contained, only some well known results about flatness are utilized without proving them.

II. Notation and Preliminaries

In this section, we introduce the notation and recall some definitions and results regarding flatness, which will be needed in the subsequent sections. Throughout this letter, all functions and vector fields are assumed to be smooth and all distributions and Jacobian matrices are assumed to have locally constant dimension and rank, resp., i.e., we consider generic points only. First order time derivatives are denoted by a dot, higher order time derivatives are denoted by superscripts in round brackets. Consider a nonlinear two-input system of the form
\[
  \dot{x}_i = f_i(x, u), \quad i = 1, \ldots, n
\]
with the state $x$ being defined on an $n$-dimensional manifold $\mathcal{X}$ and with the input $u$ taking values in a two-dimensional manifold $\mathcal{U}$. Throughout this letter, we assume that $\text{rank}(\frac{\partial f}{\partial u}) = 2$.

Definition 1: The two-input system (5) is called flat if there exist two smooth functions $y_j = \phi_j(x, u, \dot{u}, \ldots, u^{(q)})$, $j \in \{1, 2\}$ which may depend on the state $x$, the input $u$, and time derivatives of the input up to a finite order $q$, as well as smooth functions $F_{x_i}$ and $F_{u_j}$ such that locally, i.e., in some open subset,
\[
  x_i = F_{x_i}(\phi, \dot{\phi}, \ldots, \phi^{(r-1)}), \quad i = 1, \ldots, n \\
  u_j = F_{u_j}(\phi, \dot{\phi}, \ldots, \phi^{(r)}), \quad j \in \{1, 2\}.
\]
The functions $y_j = \phi_j(x, u, \dot{u}, \ldots, u^{(q)})$ are called the components of the $(x, u, \dot{u}, \ldots, u^{(q)})$-flat output $y = \phi(x, u, \dot{u}, \ldots, u^{(q)})$.

Next, let us formally define the notion of an $(x, u, \dot{u}, \ldots, u^{(q)})$-flat system.

Definition 2: A two-input system (5) is called $(x, u, \dot{u}, \ldots, u^{(q)})$-flat, if it possesses an $(x, u, \dot{u}, \ldots, u^{(q)})$-flat output.

Remark 1: In Definition 1, the notion of an $(x, u, \dot{u}, \ldots, u^{(q)})$-flat output is introduced. It should be stressed that we do not require an $(x, u, \dot{u}, \ldots, u^{(q)})$-flat output to explicitly depend on $u^{(q)}$. In particular, every $x$-flat output $y = \phi(x)$ is also an $(x, u)$-flat output.

An important implication of Definition 1 is that the time derivatives $\phi, \dot{\phi}, \ldots, \phi^{(\beta)}$ of a flat output are functionally independent for arbitrary $\beta$, which means that there does not exist any nontrivial function $\chi : \mathbb{R}^{2\beta+2} \to \mathbb{R}$ such that $\chi(\phi, \phi, \ldots, \phi^{(\beta)}) = 0$. (This property reflects the fact that there are no constraints on the time evolution of a flat output.) Furthermore, it can be shown that locally there exist unique minimal integers $r_1, r_2$ and unique maps $F_x, F_u$ such that
\[
  x = F_x(\phi_1, \dot{\phi}_1, \ldots, \phi_1^{(r_1-1)}, \phi_2, \dot{\phi}_2, \ldots, \phi_2^{(r_2-1)}) \\
  u = F_u(\phi_1, \dot{\phi}_1, \ldots, \phi_1^{(r_1)}, \phi_2, \dot{\phi}_2, \ldots, \phi_2^{(r_2)}).
\]

For a detailed proof of these properties, we refer to [14]. For $(x, u)$-flat outputs $y = \phi(x, u)$, we define the relative degree $\rho_j$ of the component $\phi_j$ as follows.

Definition 3: The relative degree $\rho_j$ of the component $\phi_j$ of an $(x, u)$-flat output $y = \phi(x, u)$ is the smallest integer $\rho_j \geq 0$ such that $\phi_j^{(\rho_j)}$ explicitly depends on $u$.

Flat systems can be exactly linearized by means of an endogenous dynamic feedback, see, e.g., [6] for the systematic construction of such a linearizing feedback. An important subclass of flat systems are those which are exactly linearizable by static feedback, i.e., which can be transformed into a linear controllable system, in particular the Brunovský normal form
\[
  \dot{x}_1 = \ddot{x}_1, \quad \dddot{x}_2 = \dddot{x}_3 = \ldots = \dddot{x}_n = 0
\]
\[
  \dot{x}_1 = \dot{x}_2, \quad \dddot{x}_2 = \dddot{x}_3 = \ldots = \dddot{x}_n = 0
\]
\[
  \dot{x}_1, \rho_1 = \dddot{x}_1, \rho_2 = \dddot{x}_2, \ldots = \dddot{x}_n, \rho_n = 0
\]
where $\rho_1 + \rho_2 + \ldots + \rho_n = n$, by means of an invertible state- and input transformation $\bar{x} = \Phi(x), \bar{u} = \Phi_u(x, u)$. A system which is static feedback linearizable is obviously flat with $y = (\bar{x}_1, \bar{x}_2)$ as a flat output. The flat parameterization (6) with respect to this flat output is a diffeomorphism and conversely, if a system possesses a flat output for which the corresponding flat parameterization is a diffeomorphism, then it is static feedback linearizable. We refer to these particular flat outputs as linearizing outputs. The relative degrees $\rho_1, \rho_2$ of the components of linearizing outputs sum to $n$ (a linearizing output is an output with a vector relative degree of $n$, see, e.g., [28]). The static feedback linearization problem

\[
  \text{Formally, } \rho_j \text{ can be defined as the smallest number such that } \frac{\partial f}{\partial \phi_j}(L^\rho_j \phi_j) = \frac{\partial f}{\partial \phi_j}(L^{\rho_j-1} \phi_j) = 0 \text{ and at least } \frac{\partial f}{\partial \phi_j}(L^\rho_j \phi_j) \neq 0 \text{ or } \frac{\partial f}{\partial \phi_j}(L^\rho_j \phi_j) \neq 0, \text{ where } L^\rho_j \text{ denotes the } \alpha \text{-th Lie derivative along the vector field } f = \sum_{i=1}^n f_i(x, u)\lambda_i.
\]
has been solved completely, see [20], [21] and [28], [29]. In the following we recall necessary and sufficient conditions for static feedback linearizability of systems of the form (5), for a proof we refer to [29]. For (5), define the distributions \( D^0 = \text{span}\{\theta_1, \theta_2\} \) and \( D^i = D^{i-1} + [f_i, D^{i-1}], i \geq 1 \) on the state and input manifold \( \Lambda \times U \), where \( f = \sum_{i=1}^n f_i(x, u)\theta_i \).

Theorem 1: The two-input system (5) is linearizable by static feedback if and only if all the distributions \( D^i \) are involutive and \( \text{dim}(D^0) = n + 2 \).

Linearizing outputs, i.e., the possible topmost variables in (7), can be computed directly from the involutive distributions \( D^i \), see, e.g., [29].

III. LINEARIZATION BY PROLONGATIONS

In this section, we prove a result regarding the linearizability of \((x, u)\)-flat two-input systems by prolongations. Based on this result, we will prove that (3) cannot be \((x, u)\)-flat.

Definition 4: The system (5) is called linearizable by a \( p \)-fold prolongation, if there exists an invertible input transformation \( u = \Phi_u(x, \hat{u}) \), such that the prolonged system

\[
x = f(x, \Phi_u(x, \hat{u})) \quad \hat{u}_1 = \hat{u}_{1,1}, \ldots, \hat{u}_{1,p-1} = \hat{u}_{1,p-1}
\]

with the state \((x, \hat{u}_1, \hat{u}_{1,1}, \ldots, \hat{u}_{1,p-1})\) and the input \((\hat{u}_{1,p}, \hat{u}_2)\) is linearizable by static feedback.

In [30] it has been shown that every \((x, u)\)-flat two-input system (5) is linearizable by an \((r_1 + r_2 - n)\)-fold prolongation, with \(r_1, r_2\) from (6) and the state dimension \(n\). According to [31], for \((x, u)\)-flat outputs, a bound on \(r_2\) is \(r_2 \leq n\). These two results imply that every \((x, u)\)-flat two-input system is linearizable by an at most \(n\)-fold prolongation. However, for the proof of our main result in Section IV, it is essential that \((x, u)\)-flat two-input systems can always be linearized by exactly \(n\) prolongations.

Proposition 1: Every \((x, u)\)-flat two-input system (5) is linearizable by an \(\text{dim}(\Lambda) = n\)-fold prolongation.

Proof: Let \(y = \varphi(x, u)\) be an \((x, u)\)-flat output of (5), let \(r_1, r_2\) be the unique minimal integers such that (6) holds and let \(p_1, p_2 \geq 0\) be the relative degrees of the components of this flat output. In the following, we only need to consider the case \(p_1 + p_2 < n\), since for \(p_1 + p_2 = n\) the system itself is already static feedback linearizable and the proposition obviously holds in this case.

The first step of the proof is to show that after applying the invertible input transformation \(\hat{u}_1 = \varphi^{(p_1)}(x, \hat{u}), \hat{u}_2 = \hat{u}_2\) (permute \(u_1\) and \(u_2\) if necessary) the derivatives of the components \(\varphi_1, \varphi_2\) of the flat output up to the orders \(r_1, r_2\) have the form

\[
\varphi_1 = \varphi_2 = \varphi_3 = \ldots = \varphi_{r_1} = \varphi_{r_2} = \varphi_{r_3} = \varphi_{(r_1 - 1)}(x)
\]

\[
\varphi^{(r_1-1)} = \varphi^{(r_2-1)} = \varphi^{(r_3-1)}(x)
\]

For the derivatives of \(\varphi_1\) this is obvious, but let us prove it in detail for the derivatives of \(\varphi_2\). By the definition of the relative degree, \(\varphi^{(p_2)}\) depends explicitly on at least one of the inputs. To show that \(\varphi^{(p_2)}(x, u)\) are independent of \(\hat{u}_2\) and its derivatives, let us recall that \(\hat{u}_2\) is by assumption the minimal integer for which (6) holds. If \(\hat{u}_2\) would occur in \(\varphi^{(i)}\) for some \(r_2 \leq s < r_2\), then \(\varphi^{(s)}\) would depend on derivatives of \(\hat{u}_2\) and hence be useless for constructing functions of \(x\) and \(u\) only. On the other hand, \(\hat{u}_2\) must explicitly occur in \(\varphi^{(r_2)}\), since otherwise \(\hat{u}_2\) could not be expressed in terms of \(\varphi_1, \varphi_2, \ldots, \varphi_{r_2}\). Furthermore, from the explicit dependence of \(\varphi^{(s)}(x, u)\) on \(\hat{u}_2\) and \(\varphi^{(r_1)}(x, u)\) the minimality of \(r_1\), it follows that \(r_1 - r_1 = r_2 - r_2\).

In the second step, let us show that (8) describes a diffeomorphism. Since the time derivatives of a flat output up to an arbitrary order are functionally independent, we only have to prove that the number \(r_1 + r_2 + 2\) of functions coincides with the number \(n + 2 + (r_1 - r_1)\) of variables \(x, \hat{u}_1, \hat{u}_{1,1}, \ldots, \hat{u}_{1,n-r_1}, \hat{u}_2\) on which they depend. Thus, we have to show that \(r_1 + r_2 = n\).

This in turn follows immediately from the fact that it must be possible to construct exactly \(n\) independent functions of \(x\) only from the functions (8), i.e., it must be possible to express the \(n\) state variables \(x\) in terms of them.

In the final step, we extend (8) by the equations

\[
\varphi^{(r_1+1)} = \varphi^{(r_1-r_1+1)}
\]

and still have a diffeomorphism. Now consider the prolonged system

\[
\dot{x} = f(x, \hat{u}_1, \hat{u}_2)
\]

\[
\hat{u}_1 = \hat{u}_{1,1}
\]

\[
\hat{u}_{1,1} = \hat{u}_{1,2}
\]

\[
\hat{u}_{1,n-1} = \hat{u}_{1,n}
\]

with the state \((x, \hat{u}_1, \hat{u}_{1,1}, \ldots, \hat{u}_{1,n-r_1}, \hat{u}_2)\) and the input \((\hat{u}_{1,n}, \hat{u}_2)\).

The parameterization of (10) by the flat output \(\varphi\) is a diffeomorphism (given by the inverse of (8) and (9), with \(\hat{u}_2\) replaced by \(\hat{u}_{1,\alpha}\)). Thus, (10) is static feedback linearizable with \(\varphi\) as a linearizing output.

---

\footnote{It should be stressed that the linearizability of a system by a \(n\)-fold prolongation of a certain input \(\hat{u}_i\) does not imply linearizability by exactly \(n\) prolongations of the same input \(\hat{u}_i\). Thus, Proposition 1 does not follow automatically from our previous results in [30] and [31]. However, the input transformation used in the proof of Proposition 1 is actually the same as in [30].}

\footnote{If \(\rho_j = 0\), the functions \(\varphi_2(x, \ldots, \varphi_{(r_1-1)}(x)\) in (8) are not present.}
IV. Example

Consider again the system (3) from the introduction. This system admits an \((x, u, \dot{u})\)-flat output (given in the introduction), and it can be shown that the system is linearizable by a 4-fold prolongation of the input \(\bar{u}_1 = \frac{\dot{u}_1}{\dot{u}_2}\) (see Appendix B). In the following, based on Proposition 1, we show that the system cannot be \((x, u)\)-flat.

Proposition 2: The system (3) does not possess any \((x, u)\)-flat output.

Proof: We prove this proposition by contradiction. Assume that (3) is \((x, u)\)-flat. Then, according to Proposition 1, the system can be rendered static feedback linearizable by \(n = 3\)-fold prolonging an input after a suitable invertible input transformation has been applied. The most general form of such an input transformation is

\[
\begin{align*}
\bar{u}_1 &= g_1(x, u_1, u_2) \\
\bar{u}_2 &= g_2(x, u_1, u_2),
\end{align*}
\]

with an inverse of the general form

\[
\begin{align*}
u_1 &= h_1(x, \bar{u}_1, \bar{u}_2) \\
u_2 &= h_2(x, \bar{u}_1, \bar{u}_2).
\end{align*}
\]

By assumption, there exists a transformation of the form (11) with inverse (12) such that the prolonged system

\[
\begin{align*}
\dot{x}_1 &= h_1(x, \bar{u}_1, \bar{u}_2) \\
\dot{x}_2 &= h_2(x, \bar{u}_1, \bar{u}_2) \\
\dot{x}_3 &= x_1 + \frac{(h_2(x, \bar{u}_1, \bar{u}_2))^2}{2h_1(x, \bar{u}_1, \bar{u}_2)} \\
\dot{\bar{u}}_1 &= \bar{u}_{1,1} \\
\dot{\bar{u}}_{1,1} &= \bar{u}_{1,2} \\
\dot{\bar{u}}_{1,2} &= \bar{u}_{1,3},
\end{align*}
\]

is static feedback linearizable. By the regularity of (12), we always have at least \(\frac{\partial h_1}{\partial \bar{u}_2} \neq 0\) or \(\frac{\partial h_2}{\partial \bar{u}_2} \neq 0\). This allows for at least one of the normalizations

\[
\begin{align*}
\dot{x}_1 &= \bar{u}_2 \\
\dot{x}_2 &= h(x, \bar{u}_1, \bar{u}_2) \\
\dot{x}_3 &= x_1 + \frac{(h_2(x, \bar{u}_1, \bar{u}_2))^2}{2h_1(x, \bar{u}_1, \bar{u}_2)} \quad \text{or} \quad \dot{x}_3 = x_1 + \frac{\bar{u}_2^2}{2h_1(x, \bar{u}_1, \bar{u}_2)} \\
\dot{\bar{u}}_1 &= \bar{u}_{1,1} \\
\dot{\bar{u}}_{1,1} &= \bar{u}_{1,2} \\
\dot{\bar{u}}_{1,2} &= \bar{u}_{1,3},
\end{align*}
\]

where by abuse of notation we denote the newly introduced input still by \(\bar{u}_2\) and renamed the composition of the remaining function \((h_2\text{ or } h_1)\) with the “inverse” of this normalization by \(h\) (without a subscript). This normalization can be done before prolonging the input \(\bar{u}_1\) since it does not involve time derivatives of \(u_1\).

In the following, we show that there cannot exist an invertible input transformation (12) such that at least one of the prolonged systems (13) is static feedback linearizable. This will allow us to conclude that the system is not linearizable by a 3-fold prolongation and consequently not \((x, u)\)-flat.

Case 1: Let us first consider the prolonged system

\[
\begin{align*}
\dot{x}_1 &= \bar{u}_2 \\
\dot{x}_2 &= h(x, \bar{u}_1, \bar{u}_2) \\
\dot{x}_3 &= x_1 + \frac{(h(x, \bar{u}_1, \bar{u}_2))^2}{2\bar{u}_2} \\
\dot{\bar{u}}_1 &= \bar{u}_{1,1} \\
\dot{\bar{u}}_{1,1} &= \bar{u}_{1,2} \\
\dot{\bar{u}}_{1,2} &= \bar{u}_{1,3},
\end{align*}
\]

where the corresponding input transformation reads as

\[
\begin{align*}
u_1 &= \bar{u}_2 \\
u_2 &= h(x, \bar{u}_1, \bar{u}_2).
\end{align*}
\]

By assumption, the prolonged system (14) is static feedback linearizable. Thus, according to Theorem 1, the distributions

\[
\begin{align*}
\mathcal{D}^0_p &= \text{span}\{\partial_{\bar{u}_1,1}, \partial_{\bar{u}_2}\} \\
\mathcal{D}^1_p &= \text{span}\{\partial_{\bar{u}_1,1}, \partial_{\bar{u}_1,2}, \partial_{\bar{u}_2}, \partial_{x_1} + \frac{\partial h}{\partial \bar{u}_2} \partial_{x_2} + \frac{2h \partial h}{\partial \bar{u}_2} \partial_{x_3} - \frac{h^2}{2\bar{u}_2^2} \partial_{x_3}\}
\end{align*}
\]

are involutive. The latter implies that

\[
\begin{align*}
\{\partial_{\bar{u}_2}, \partial_{x_1} + \frac{\partial h}{\partial \bar{u}_2} \partial_{x_2} + \frac{2h \partial h}{\partial \bar{u}_2} \partial_{x_3} - \frac{h^2}{2\bar{u}_2^2} \partial_{x_3}\} \in \mathcal{D}^1_p,
\end{align*}
\]

which can only hold if

\[
\begin{align*}
\frac{\partial^2 h}{\partial \bar{u}_2^2} &= 0 \\
\frac{\partial}{\partial \bar{u}_2} \left( \frac{2h \partial h}{\partial \bar{u}_2} \partial_{x_3} - \frac{h^2}{2\bar{u}_2^2} \partial_{x_3} \right) &= 0.
\end{align*}
\]

The first condition implies that \(h\) is affine with respect to \(\bar{u}_2\), i.e., we actually have \(h = a(x, \bar{u}_1) + b(x, \bar{u}_1)\bar{u}_2\). The second condition then yields

\[
\begin{align*}
\frac{\partial}{\partial \bar{u}_2} \left( \frac{b^2 \bar{u}_2^2 - a^2}{2\bar{u}_2^2} \right) &= 0,
\end{align*}
\]

which simplifies to \(\frac{a^2}{\bar{u}_2^2} = 0\) and thus \(a = 0\). The distribution \(\mathcal{D}^1_p\) thus simplifies to

\[
\mathcal{D}^1_p = \text{span}\{\partial_{\bar{u}_1,1}, \partial_{\bar{u}_2}, \partial_{x_2} + b \partial_{x_3} + \frac{b^2}{2} \partial_{x_3}\},
\]

and for the next distribution we obtain

\[
\begin{align*}
\mathcal{D}^2_p &= \text{span}\{\partial_{\bar{u}_1,1}, \partial_{\bar{u}_1,2}, \partial_{\bar{u}_2}, \partial_{x_1} + b \partial_{x_2} + \frac{b^2}{2} \partial_{x_3}, \\
&\quad (x_1 \frac{\partial h}{\partial x_2} + \bar{u}_{1,1} \frac{\partial h}{\partial x_3}) \partial_{x_2} + (x_1 \frac{\partial h}{\partial x_3} + \bar{u}_{1,1} \frac{\partial h}{\partial x_3} - 1) \partial_{x_3}\}
\end{align*}
\]

which of course must again be involutive. The latter in particular implies that

\[
\begin{align*}
\{\partial_{\bar{u}_1,1}, (x_1 \frac{\partial h}{\partial x_2} + \bar{u}_{1,1} \frac{\partial h}{\partial x_3}) \partial_{x_2} + (x_1 \frac{\partial h}{\partial x_3} + \bar{u}_{1,1} \frac{\partial h}{\partial x_3} - 1) \partial_{x_3}\} \in \mathcal{D}^2_p,
\end{align*}
\]

i.e., \(\frac{\partial h}{\partial x_1} \partial_{x_2} + b \frac{\partial h}{\partial x_3} \partial_{x_3} \in \mathcal{D}^2_p\), which can only hold if the vector field

\[
\frac{\partial h}{\partial x_1} \partial_{x_2} + b \frac{\partial h}{\partial x_3} \partial_{x_3}
\]

is collinear with the vector field

\[
(x_1 \frac{\partial h}{\partial x_2} + \bar{u}_{1,1} \frac{\partial h}{\partial x_3}) \partial_{x_2} + (x_1 \frac{\partial h}{\partial x_3} + \bar{u}_{1,1} \frac{\partial h}{\partial x_3} - 1) \partial_{x_3}.
\]
Therefore, we obtain the condition
\[
\frac{\partial h}{\partial u_1}(x_1 b \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1} - 1) - b \frac{\partial b}{\partial u_1}(x_1 \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1}) = 0,
\]
which simplifies to \(\frac{\partial h}{\partial u_1} = 0\). However, \(\frac{\partial h}{\partial u_2} = 0\) together with \(a = 0\) from above is a contradiction to the regularity of the transformation (15), which would then read as
\[
\begin{align*}
  u_1 &= \bar{u}_2, \\
  u_2 &= b(x)\bar{u}_2.
\end{align*}
\]
Hence, the transformation cannot be of the form (15).

**Case 2:** The second case can be handled analogously. In this case, we have to consider the prolonged system
\[
\begin{align*}
  \dot{x}_1 &= h(x, \bar{u}_1, \bar{u}_2) \\
  \dot{x}_2 &= \bar{u}_2 \\
  \dot{x}_3 &= x_1 + \frac{\bar{a}_2}{\bar{b}} + \frac{\bar{a}_1}{\bar{a}} + \frac{\bar{b}}{b} \frac{\partial b}{\partial x_3} \\
  \dot{\bar{u}}_1 &= \bar{u}_1, \\
  \dot{\bar{u}}_1 &= \bar{u}_1, \\
  \dot{\bar{u}}_1 &= \bar{u}_1,
\end{align*}
\]
and the corresponding input transformation reads as
\[
\begin{align*}
  u_1 &= h(x, \bar{u}_1, \bar{u}_2) \\
  u_2 &= \bar{u}_2.
\end{align*}
\]
By assumption, the prolonged system (16) is static feedback linearizable. Thus, the distributions
\[
\begin{align*}
  D_P^0 &= \text{span}\{\partial_{\bar{u}_1, \bar{u}_2}\} \\
  D_P^1 &= \text{span}\{\partial_{\bar{u}_1, \bar{u}_2}, \bar{u}_1, \bar{u}_2, \frac{\partial b}{\partial u_1} \partial_{x_1} + \frac{2\bar{a}_2 \bar{b} - \bar{a}_2 b}{2\bar{a}^2} \partial_{x_3}\}
\end{align*}
\]
are involutive. The involutivity of \(D_P^1\) implies that
\[\frac{\partial^2 b}{\partial u_2^2} = 0 \quad \text{and} \quad \frac{\partial}{\partial u_2}\left(\frac{2\bar{a}_1 \bar{b} + \bar{b} \bar{u}_2^2}{2(a + \bar{b} u_2)^3}\right) = 0.
\]
The first condition implies that \(h\) is affine with respect to \(\bar{u}_2\), i.e., we actually have \(h = a(x, \bar{u}_1) + b(x, \bar{u}_1)\bar{u}_2\). The second condition then yields
\[\frac{\partial}{\partial u_2}\left(\frac{2\bar{a}_1 \bar{b} + \bar{b} \bar{u}_2^2}{2(a + \bar{b} u_2)^3}\right) = 0,
\]
which simplifies to
\[\frac{a^2}{(a + \bar{b} u_2)^3} = 0.
\]
Thus, we must have \(a = 0\). As a consequence, the distribution \(D_P^1\) simplifies to
\[D_P^1 = \text{span}\{\partial_{\bar{u}_1, \bar{u}_2}, \partial_{x_1} b \partial_{x_3}, + \frac{1}{\bar{b}} \partial_{x_3}\},
\]
and for the next distribution we obtain
\[D_P^2 = \text{span}\{\partial_{\bar{u}_1, \bar{u}_2}, \bar{u}_1, \bar{u}_2, \partial_{x_1} b \partial_{x_3}, b \partial_{x_1} + \bar{b} \partial_{x_2} + \frac{1}{\bar{b}} \partial_{x_3}, x_1 \frac{\partial b}{\partial x_3} + \bar{u}_1 \frac{\partial b}{\partial u_1} + b\partial_{x_3}\},
\]
which must again be involutive. The involutivity implies that
\[\frac{\partial b}{\partial u_1}(x_1 b \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1} - 1) - b \frac{\partial b}{\partial u_1}(x_1 \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1}) = 0,
\]
which can only hold if the vector field
\[\frac{\partial b}{\partial u_1} \partial_{x_1} - \frac{1}{2\bar{b}^2} \partial_{x_3} = D_P^2
\]
is collinear with the vector field
\[\frac{\partial b}{\partial u_1}(x_1 \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1})\partial_{x_3} - (\frac{1}{2\bar{b}^2} \frac{\partial b}{\partial x_3} + \bar{u}_1 \frac{\partial b}{\partial u_1} + b)\partial_{x_3}.
\]
Therefore, we obtain the condition
\[\frac{\partial b}{\partial u_1}(x_1 \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1} + b) - \frac{1}{2\bar{b}^2} \frac{\partial b}{\partial x_3} (x_1 \frac{\partial b}{\partial x_3} + \bar{u}_1, b \frac{\partial b}{\partial u_1}) = 0,
\]
which simplifies to \(\frac{\partial h}{\partial u_1} = 0\). Thus, either \(\frac{\partial h}{\partial u_1} = 0\) or \(b = 0\). Since we already have \(a = 0\), \(b = 0\) would imply \(h = 0\), and hence the transformation would not be invertible. However, the other possibility, i.e., \(\frac{\partial h}{\partial u_1} = 0\), is in combination with \(a = 0\) also a contradiction to the regularity of the transformation (17), which would then read as
\[
\begin{align*}
  u_1 &= b(x)\bar{u}_2 \\
  u_2 &= \bar{u}_2.
\end{align*}
\]
In conclusion, there cannot exist an invertible input transformation (12) which generates an input such that a 3-fold prolongation of this input renders the system (3) static feedback linearizable. The system is thus not linearizable by a 3-fold prolongation, and as a consequence of Proposition 1, the system (3) cannot be \((x, u)\)-flat.

**V. CONCLUSION**

We have proven that there exist systems which are flat but do not admit any \((x, u)\)-flat output. It thus seems reasonable to conjecture that there also exist systems which are flat but do not admit any \((x, u, \bar{u})\)-flat output, and so on. Further research will be devoted to proving this conjecture.

**APPENDIX**

**A. Flatness of the Counterexample**

The flat parameterization of the state \(x\) and the input \(u\) of the system (3) with respect to the flat output (4) is given by
\[
\begin{align*}
  x_1 &= \frac{1}{2(1+y_1)} (2\bar{y}_1, y_1)^3 - y_1 y_2 y_3 y_2^2 y_3 + \bar{y}_1 y_2 y_3 + y_1 y_2 y_3 - y_1 y_2 y_3 \\
  x_2 &= \frac{1}{2(1+y_1)} ((-3\bar{y}_1 y_2 + y_1 y_3 y_2)^3 + (3y_1 y_2 y_3 + y_1 y_2 y_3 - y_1 y_2 y_3) y_1 \\
  x_3 &= + y_1 y_2 y_3 - y_1 y_2 y_3 + y_1 y_2 y_3) y_2 \\
  x_3 &= \frac{1}{2(1+y_1)} (2\bar{y}_1, y_1)^3 - 2\bar{y}_1, y_1)^3 \\
  x_3 &= + (3\bar{y}_1 y_2 + 3\bar{y}_1, y_2)^3 + (3\bar{y}_1 y_2 + 2\bar{y}_1, y_2)^3 \\
  x_3 &= + (3\bar{y}_1 y_2 + 2\bar{y}_1, y_2)^3 - y_1 y_2 y_3(3) y_2 \\
  x_3 &= + (3\bar{y}_1 y_2 + 2\bar{y}_1, y_2)^3 - y_1 y_2 y_3(3) y_2
\end{align*}
\]
Because of \(u_1 = \dot{x}_1\) and \(u_2 = \dot{x}_2\) (see (3)), the flat parameterization of the inputs \(u_1\) and \(u_2\) follows directly by differentiation from the parameterization of \(x_1\) and \(x_2\).
B. Linearizability of the Counterexample by Prolongations

Above we claimed that the system (3) can be rendered static feedback linearizable by a 4-fold prolongation of the input \( \tilde{u}_1 \). Let us explicitly show this. The input \( \tilde{u}_1 \) can be introduced by means of the input transformation

\[
\tilde{u}_1 = \frac{u_2}{\eta_1},
\]

(18)

(the particular choice for \( \tilde{u}_2 \) does not matter). Applying this input transformation to (3) and subsequently 4-fold prolonging \( \tilde{u}_1 \) yields the system

\[
\begin{align*}
\dot{x}_1 &= \tilde{u}_2, \\
\dot{x}_2 &= \bar{u}_2, \\
\dot{x}_3 &= x_1 + \frac{1}{4} \tilde{u}_1 \bar{u}_2, \\
\dot{\bar{u}}_1 &= \bar{u}_{1,1}, \\
\dot{\bar{u}}_1 &= \bar{u}_{1,2}, \\
\dot{\bar{u}}_1 &= \bar{u}_{1,3}, \\
\dot{\bar{u}}_1 &= \bar{u}_{1,4}
\end{align*}
\]

(19)

with the \( np = 7 \)-dimensional state \((x_1, x_2, x_3, \bar{u}_1, \tilde{u}_{1,1}, \tilde{u}_{1,2}, \tilde{u}_{1,3}, \tilde{u}_{1,4})\) and the input \((\tilde{u}_{1,4}, \tilde{u}_2)\). The distributions involved in the test for static feedback linearizability of (19) follow as

\[
\begin{align*}
D_p^0 &= \text{span}\{\partial_{\tilde{u}_{1,4}}, \partial_{\tilde{u}_2}\} \\
D_p^1 &= \text{span}\{\partial_{\tilde{u}_{1,4}}, \partial_{\tilde{u}_{1,1}}, \partial_{\tilde{u}_2}, \tilde{u}_1, \tilde{u}_2 + \frac{1}{2} \tilde{u}_1^2 \partial_{\tilde{u}_3}\} \\
D_p^2 &= \text{span}\{\partial_{\tilde{u}_{1,4}}, \partial_{\tilde{u}_{1,1}}, \partial_{\tilde{u}_{1,2}}, \partial_{\tilde{u}_2}, \tilde{u}_1, \tilde{u}_2 + \frac{1}{2} \tilde{u}_1^2 \partial_{\tilde{u}_3}, \\
&\quad \tilde{u}_{1,1} \partial_{\tilde{u}_3} + (\tilde{u}_1 \tilde{u}_{1,1} - 1) \partial_{\tilde{u}_3}\} \\
D_p^3 &= \text{span}\{\partial_{\tilde{u}_{1,4}}, \partial_{\tilde{u}_{1,1}}, \partial_{\tilde{u}_{1,2}}, \partial_{\tilde{u}_{1,3}}, \partial_{\tilde{u}_2}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3\} \\
D_p^4 &= \text{span}\{\partial_{\tilde{u}_{1,4}}, \partial_{\tilde{u}_{1,1}}, \partial_{\tilde{u}_{1,2}}, \partial_{\tilde{u}_{1,3}}, \partial_{\tilde{u}_2}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}
\end{align*}
\]

It is easy to see that all these distributions are involutive and we have \( \dim(D_p^4) = np + 2 \). Thus, according to Theorem 1, the prolonged system (19) is static feedback linearizable. A possible linearizing output of (19) follows as

\[
\begin{align*}
y_1 &= \tilde{u}_1 \\
y_2 &= (\tilde{u}_1^2 x_1 - 2 \tilde{u}_1 x_2 + 2 x_3) \tilde{u}_{1,1} - 2 \tilde{u}_1 x_1 + 2 x_2
\end{align*}
\]

which is exactly the flat output (4). Indeed, substituting \( \tilde{u}_1 = \frac{u_2}{\eta_1} \) and \( \tilde{u}_{1,1} = \frac{1}{\eta_1}(\tilde{u}_2 \tilde{u}_{1,1} - \tilde{u}_2 \tilde{u}_{1,1}) \) into the linearizing output yields (4).

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