A new generalization of binomial coefficients

Michel Lassalle
Centre National de la Recherche Scientifique
Institut Gaspard-Monge, Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex, France
lassalle@univ-mlv.fr
http://igm.univ-mlv.fr/~lassalle

Abstract

Let \( t \) be a fixed parameter and \( x \) some indeterminate. We give some properties of the generalized binomial coefficients \( \langle x \rangle^k \) inductively defined by

\[
\frac{k}{x} \langle x \rangle^k = t \langle x-1 \rangle^{k-1} + (1-t) \langle x-2 \rangle^{k-2}.
\]

1 Definition

There are many generalizations of binomial coefficients, the most elementary of which are the Gaussian polynomials. In this note, we shall present another one-parameter extension, presumably new, encountered in the study of the symmetric groups [4]. This application will be described at the end.

Let \( t \) be a fixed parameter and \( x \) some indeterminate. For any positive integer \( k \), we define a function \( \langle x \rangle^k \) inductively by \( \langle x \rangle^0 = 0 \) for \( k < 0 \), \( \langle x \rangle^0 = 1 \) and

\[
\frac{k}{x} \langle x \rangle^k = t \langle x-1 \rangle^{k-1} + (1-t) \langle x-2 \rangle^{k-2}.
\]

Then \( \langle x \rangle^k \) is a polynomial with degree \( k \) in \( x \) and in \( t \). First values are given by

\[
\begin{align*}
\langle x \rangle^1 &= tx, \\
\langle x \rangle^2 &= t^2 \left( \frac{x}{2} \right) + (1-t) \frac{x}{2}, \\
\langle x \rangle^3 &= t^3 \left( \frac{x}{3} \right) + t(1-t) \left( \frac{x}{2} \right) - \frac{1}{3} t(1-t)x, \\
\langle x \rangle^4 &= t^4 \left( \frac{x}{4} \right) + \frac{3}{2} t^2(1-t) \left( \frac{x}{3} \right) - \frac{1}{12} (1-t)(8t^2 + 3t - 3) \left( \frac{x}{2} \right) + \frac{1}{8} (1-t^2)(2t - 1)x.
\end{align*}
\]

For \( k \) odd, it is obvious that \( \langle x \rangle^k \) is divisible by \( t \).

We have easily

\[
\langle x \rangle^k = t^k \left( \frac{x}{k} \right) + (1-t)xP(x,t),
\]

where \( P(x,t) \) is a polynomial.
with $P$ a polynomial of degree $k-2$ in $x$ and $t$. For $t = 1$ we recover the classical binomial product

$$\binom{x}{k} = \frac{1}{k!} \prod_{i=1}^{k} (x - i + 1),$$

and when $x$ is some positive integer $n$, the binomial coefficient $\binom{n}{k}$.

In this paper we shall present some notable properties of the generalized binomial coefficients $\langle \binom{x}{k} \rangle$, including a generating function, a Chu-Vandermonde identity and an explicit formula. The referee has suggested that it would be interesting to obtain a $q$-analogue of our results, using $q$-shifted factorials instead of ordinary ones, in the same way than [5] has been generalized by [3].

2 Generating function

We consider the series

$$G(u) = 1 + tu + (1 - t) \sum_{k \geq 0} \frac{(-u)^{k+2}}{(k+2)!} \prod_{i=0}^{k} (k - i + 1 + (t - 1)i),$$

$$H(u) = 1 + \sum_{k \geq 1} \frac{(-u)^{k}}{k!} \prod_{i=1}^{k} (k - i + 1 + (t - 1)i).$$

We have

$$\frac{d}{du} G(u) = t + (1 - t) H(u).$$

**Theorem 1.** The series $G(u)$ and $H(u)$ are mutually inverse.

**Proof.** Krattenthaler has pointed out that the statement is a consequence of Rothe identity [1, 2]

$$\sum_{k=0}^{n} \frac{A}{A+Bk} \binom{A+Bk}{k} \binom{C-Bk}{n-k} = \binom{A+C}{n}.$$ 

Actually if we denote

$$X_k = -\frac{k+1}{t-2}, \quad Y_k = X_{k-2} = -\frac{k-1}{t-2},$$

we have

$$G(u) = 1 + tu + \frac{1 - t}{2 - t} \sum_{k \geq 2} u^k (t - 2)^k \frac{1}{Y_k + 1} \binom{Y_k + 1}{k},$$

$$= u + \frac{1 - t}{2 - t} \sum_{k \geq 0} u^k (t - 2)^k \frac{1}{Y_k + 1} \binom{Y_k + 1}{k},$$

$$H(u) = \sum_{k \geq 0} u^k (t - 2)^k \binom{X_{k-1}}{k}. $$

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Rothe identity, written for

\[ A = 1 + \frac{1}{t-2}, \quad B = -\frac{1}{t-2}, \quad C = -1 - \frac{n+1}{t-2}, \]

yields

\[
\frac{1-t}{2-t} \sum_{k=0}^{n} \frac{1}{Y_k+1} \binom{Y_k + 1}{k} \binom{X_{n-k} - 1}{n - k} = \left( \frac{-n/(t-2)}{n} \right) = \binom{X_{n-1}}{n},
\]
since \( X_{n-1} = -n/(t-2) \). Therefore for \( n \neq 0 \) the coefficient of \( u^n \) in \( H(u)G(u) \) is

\[
(t-2)^{n-1} \binom{X_{n-1} - 1}{n-1} + (t-2)^n \binom{X_{n-1}}{n} = (t-2)^{n-1} \binom{X_{n-1} - 1}{n-1} \left( 1 + (t-2) \frac{X_{n-1}}{n} \right) = 0.
\]

\[ \square \]

**Corollary 1.** The series \( G(u) \) is the unique solution of \( G(0) = 1 \) and

\[
\frac{d}{du} G(u) = t + (1-t) \frac{u}{G(u)}.
\]

**Theorem 2.** The generating function of the numbers \( \langle x \rangle_k \) is given by

\[
\sum_{k \geq 0} \frac{x}{k} u^k = (G(u))^x.
\]

**Proof.** If we write

\[
\mathcal{F}(u; x) = \sum_{k \geq 0} \frac{x}{k} u^k,
\]

the definition of \( \langle x \rangle_k \) yields

\[
\frac{1}{x} \frac{d}{du} \mathcal{F}(u; x) = t \mathcal{F}(u; x - 1) + (1-t)u \mathcal{F}(u; x - 2).
\]

Inspired by the \( t = 1 \) case which is the classical binomial formula \( \sum_{k \geq 0} \binom{x}{k} u^k = (1+u)^x \), we may look for a generating function of the form \( \mathcal{F}(u; x) = (F(u))^x \). Then we have

\[
\frac{d}{du} F(u) = t + (1-t) \frac{u}{F(u)}.
\]

We apply Corollary 1. \[ \square \]
Corollary 2. For $k \geq 1$ we have
\[
\langle \frac{-1}{k} \rangle = \frac{(-1)^k}{k!} \prod_{i=1}^{k} (k - i + 1 + (t-1)i).
\]

Proof. Consequence of $F(u; -1) = H(u)$. □

Corollary 3. We have the generalized Chu-Vandermonde formula
\[
\sum_{i=0}^{k} \langle \binom{x}{i} \binom{y}{k-i} \rangle = \langle \frac{kx}{x+y} \rangle \langle x+y \rangle.
\]

Proof. Standard consequence of $F(u; x+y) = F(u; x)F(u; y)$. □

Corollary 4. A variant of the generalized Chu-Vandermonde formula is given by
\[
\sum_{i=0}^{k} i \langle \binom{x}{i} \binom{y}{k-i} \rangle = \langle \frac{kx}{x+y} \binom{x+y}{k} \rangle.
\]

Proof. By definition the left-hand side is given by
\[
x \sum_{i=0}^{k} \langle \binom{y}{k-i} \rangle (t \langle \binom{x-1}{i-1} \rangle + (1-t) \langle \binom{x-2}{i-2} \rangle).
\]

By the generalized Chu-Vandermonde formula, it can be written as
\[
x t \langle \binom{x+y-1}{k-1} \rangle + x(1-t) \langle \binom{x+y-2}{k-2} \rangle = \frac{kx}{x+y} \langle \binom{x+y}{k} \rangle.
\]

Remark. When $m \neq 0, 1$ we do not know any such simple expression for
\[
\sum_{i=0}^{k} \langle \binom{i}{m} \binom{x}{i} \binom{y}{k-i} \rangle.
\]

Corollary 5. For $n \geq 2$ we have
\[
\sum_{i=0}^{n} \langle \binom{n}{i-1} \rangle \langle -1 \rangle = 0.
\]

Proof. Since we have
\[
\sum_{i=0}^{n} i \langle \binom{n}{i-1} \rangle \langle -1 \rangle = \sum_{i=0}^{n-1} (i+1) \langle \binom{n}{i} \rangle \langle -1 \rangle,
\]
the property follows from Corollaries 3 and 4, written with $x = -1, y = n, k = n - 1$. Actually in that case $kx/(x+y) + 1 = 0$. □
Looking for the contributions to $u^m$ in $(G(u))^{x-1} G(u)$ we have the following generalization of Pascal’s recurrence formula

\[
\langle x\rangle - \langle x-1\rangle = t\langle x-1\rangle - (1-t) \sum_{k=0}^{m-2} \frac{(-1)^k}{(k+2)!} \langle x-1\rangle \prod_{i=0}^{k} (k-i+1+(t-1)i).
\]

Similarly with $(G(u))^{x+1} H(u)$ we get

\[
\langle x\rangle - \langle x+1\rangle = \sum_{k=1}^{m} \frac{(-1)^k}{k!} \langle x+1\rangle \prod_{i=1}^{k} (k-i+1+(t-1)i).
\]

3 New properties

Let $n$ be a positive integer. The previous properties of $\langle n \rangle$ are very similar to those of the classical binomial coefficient $\binom{n}{k}$. However some big differences must be emphasized.

Firstly $\langle n \rangle$ and $\langle n-k \rangle$ are not equal. In particular $\langle n \rangle \neq 1$. By definition we have

\[
\langle n \rangle = t\langle n-1 \rangle + (1-t)\langle n-2 \rangle.
\]

which implies by induction

\[\langle n \rangle = 1 + (t-1)\langle n-1 \rangle,\]

and

\[\langle n \rangle = \frac{1 - (t-1)^{n+1}}{2 - t}.\]

Secondly $\langle n \rangle$ is not zero for $k > n$, but divisible by $(1-t)$. Starting from the definition we have for $k \geq 2$,

\[
k\langle 1 \rangle = (1-t)\langle k-2 \rangle
\]

\[= (1-t)\frac{(-1)^k}{(k-2)!} \prod_{i=1}^{k-2} (k-i-1+(t-1)i).
\]

By induction we get

\[
\frac{k}{2} = t(1-t)\langle k-3 \rangle, \quad k \geq 3,
\]

\[
\frac{k}{3} = (1-t)(t^2 + \frac{k-1}{2}(1-t))\langle k-4 \rangle, \quad k \geq 4,
\]

\[
\frac{k}{4} = t(1-t)(t^2 + \frac{5k-8}{6}(1-t))\langle k-5 \rangle, \quad k \geq 5.
\]
More generally for \( k > n \) the definition yields

\[
\binom{k}{n} \binom{n}{k} = (1 - t) f_{n,k} \binom{n}{k - n - 1},
\]

where \( f_{n,k} \) is a monic polynomial in \( t \), inductively defined by \( f_{1,k} = 1, f_{2,k} = t \) and

\[
f_{n,k} = tf_{n-1,k-1} + (1 - t) \frac{k-1}{n-1} f_{n-2,k-2}.
\]

The coefficients \( \langle n \rangle \) with \( k > n \) may be written in terms of coefficients \( \langle k \rangle \) with \( k \leq n \). The simplest case is given below.

**Proposition 1.** We have

\[
\frac{1}{1-t} \binom{n}{n+1} = \sum_{i=0}^{n} \frac{i}{i+1} \binom{n}{n-i} \binom{n-i}{i-1}.
\]

**Proof.** Denoting the right-hand side by \( h_n \), we must prove that \( h_n = f_{n,n+1}/(n+1) \). Equivalently that \( h_n \) is inductively defined by

\[
\frac{n+1}{n} h_n = th_{n-1} + (1 - t)h_{n-2}.
\]

In other words, that we have

\[
\sum_{i=0}^{n} \frac{i}{i+1} \left( \frac{n+1}{n} \frac{n-i}{n-i} \binom{n}{n-i} - t \binom{n-1}{n-i-1} - (1 - t) \binom{n-2}{n-i-2} \right) \binom{n-i}{i-1} = 0.
\]

By the definition this may be rewritten as

\[
\sum_{i=0}^{n} \frac{i}{i+1} \left( \frac{n+1}{n-i} - 1 \right) \frac{n-i}{n} \binom{n}{n-i} \binom{n-i}{i-1} = 0.
\]

We apply Corollary 5.

\[
\square
\]

**4 Binomial expansion**

In this section we consider the expansion

\[
\binom{x}{k} = t^k \binom{x}{k} + \sum_{i=1}^{k-1} c_i(k) \binom{x}{i}.
\]

We give two methods for the evaluation of the coefficients \( c_i(k), 1 \leq i \leq k - 1 \).
4.1 First method

The definition of $\langle x \rangle_k$ may be written as

$$\sum_{i=1}^{k} \binom{k}{i} c_i(k) \binom{x-1}{i-1} = t \sum_{i=1}^{k-1} c_i(k-1) \binom{x-1}{i} + (1-t) \sum_{i=1}^{k-2} c_i(k-2) \binom{x-2}{i}.$$ 

Using the classical identity

$$\binom{x}{i} = \sum_{m=0}^{i} (-1)^m \binom{x+1}{i-m},$$

and identifying the coefficients of $\binom{x-1}{i-1}$ on both sides, we obtain

$$\frac{k}{i} c_i(k) = tc_{i-1}(k-1) + (1-t) \sum_{m=0}^{k-i-1} (-1)^m c_{i+m-1}(k-2).$$

This relation may be used to get $c_i(k)$ inductively.

For $k \geq 6$ this recurrence yields

$$\langle x \rangle_k = t^k \binom{x}{k} - \frac{k-1}{2} t^{k-2}(t-1) \binom{x}{k-1}$$

$$+ \frac{k-2}{3} t^{k-4}(t-1) \left( t^2 + \frac{3(k-3)}{8}(t-1) \right) \binom{x}{k-2}$$

$$- \frac{k-3}{4} t^{k-6}(t-1) \left( t^4 + \frac{4k-13}{6} t^2(t-1) + \frac{1}{6} \binom{k-4}{2}(t-1)^2 \right) \binom{x}{k-3}$$

$$+ \frac{k-4}{5} t^{k-8}(t-1) \left( t^6 + \frac{65k-229}{72} t^4(t-1) + \frac{5(2k-9)}{48} (k-5)t^2(t-1)^2 + \frac{5}{64} \binom{k-5}{3}(t-1)^3 \right) \binom{x}{k-4}$$

$$- \frac{k-5}{6} t^{k-10}(t-1) \left( t^8 + \frac{66k-251}{60} t^6(t-1) + \frac{85k^2-853k+2148}{240} t^4(t-1)^2 + \frac{4k-23}{48} (k-6)t^2(t-1)^3 + \frac{3}{80} \binom{k-6}{4}(t-1)^4 \right) \binom{x}{k-5}$$

$$+ \ldots .$$

It appears empirically that $c_{k-i}(k), i \neq 0$, has the form

$$c_{k-i}(k) = (-1)^i \frac{k-i}{i+1} t^{k-2i}(t-1) \sum_{m=0}^{i-1} a_m(k,i) t^{2(i-m-1)}(t-1)^m,$$

with $a_m(k,i)$ a polynomial in $k$ with degree $m$. 

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4.2 Second method

A much better expression of $c_i(k), 1 \leq i \leq k - 1,$ may be obtained by using the relation

$$\frac{1}{1-t} \binom{k}{n} \binom{n}{k} = f_{n,k} \binom{-1}{k-n-1}, \quad k \geq n + 1. \quad (4.1)$$

For instance $c_1(k)$ is given by

$$\binom{1}{k} = c_1(k),$$

hence

$$\frac{1}{1-t} k c_1(k) = \frac{1}{1-t} k \binom{1}{k} = \binom{-1}{k-2}.$$

Similarly we have

$$\binom{2}{k} = c_2(k) + 2c_1(k),$$

which yields

$$\frac{1}{1-t} \binom{k}{2} c_2(k) = t \binom{-1}{k-3} - (k-1) \binom{-1}{k-2}.$$

This property is generalized by the following explicit formula.

**Theorem 3.** Let $f_{n,k}$ be the polynomial in $t$ and $k$ inductively defined by $f_{1,k} = 1, f_{2,k} = t$ and

$$f_{n,k} = tf_{n-1,k-1} + (1-t) \frac{k-1}{n-1} f_{n-2,k-2}.$$

We have

$$\binom{x}{k} = t^k \binom{x}{k} + \sum_{i=1}^{k-1} c_i(k) \binom{x}{i},$$

with $c_i(k)$ given by

$$\frac{1}{1-t} \binom{k}{i} c_i(k) = \sum_{m=1}^{i} (-1)^{i-m} f_{m,k} \binom{k-m}{i-m} \binom{-1}{k-m-1}. $$

**Proof.** From

$$\binom{i}{k} = \sum_{m=1}^{i} c_m(k) \binom{i}{m}, \quad i \leq k - 1,$$

we deduce by inversion

$$c_i(k) = \sum_{m=1}^{i} (-1)^{i-m} \binom{i}{m} \binom{m}{k}, \quad i \leq k - 1.$$
which is a direct consequence of the classical identity
\[
\sum_{m=p}^{i} (-1)^{i-m} \binom{i}{m} \binom{m}{p} = \delta_{ip}.
\]

Therefore we have
\[
\frac{1}{1-t} \binom{k}{i} c_i(k) = \sum_{m=1}^{i} (-1)^{i-m} \frac{1}{1-t} \binom{k}{i} \binom{m}{k} = \sum_{m=1}^{i} (-1)^{i-m} \binom{k-m}{i-m} \frac{1}{1-t} \binom{k}{m} \binom{m}{k}.
\]

We apply (4.1).

The first values are given by
\[
\frac{1}{1-t} k c_{1}(k) = \left\langle -1 \right| k - 2, \\
\frac{1}{1-t} \binom{k}{2} c_{2}(k) = t \left\langle -1 \right| k - 3 - (k-1) \left\langle -1 \right| k - 2, \\
\frac{1}{1-t} \binom{k}{3} c_{3}(k) = (t^2 + \frac{k-1}{2} (1-t)) \left\langle -1 \right| k - 4 - t(k-2) \left\langle -1 \right| k - 3 + \left( k - 1 \right) \left( k - 2 \right) \left\langle -1 \right| k - 2, \\
\frac{1}{1-t} \binom{k}{4} c_{4}(k) = t(t^2 + \frac{5k-8}{6} (1-t)) \left\langle -1 \right| k - 5 - (k-3) (t^2 + \frac{k-1}{2} (1-t)) \left\langle -1 \right| k - 4 + t \left( k - 2 \right) \left\langle -1 \right| k - 3 - \left( k - 1 \right) \left( k - 3 \right) \left\langle -1 \right| k - 2.
\]

5 An application

A partition \( \rho = (\rho_1, \ldots, \rho_r) \) is a finite weakly decreasing sequence of nonnegative integers, called parts. The number \( l(\rho) \) of positive parts is called the length of \( \rho \), and \( |\rho| = \sum_{i=1}^{r} \rho_i \) the weight of \( \rho \).

For any partition \( \rho \) and any integer \( 1 \leq i \leq l(\rho) + 1 \), we denote by \( \rho^{(i)} \) the partition \( \mu \) (if it exists) such that \( \mu_j = \rho_j \) for \( j \neq i \) and \( \mu_i = \rho_i + 1 \). Similarly for any integer \( 1 \leq i \leq l(\rho) \), we denote by \( \rho^{(i)} \) the partition \( \nu \) (if it exists) such that \( \nu_j = \rho_j \) for \( j \neq i \) and \( \nu_i = \rho_i - 1 \).

In the study of the symmetric groups [4], the following differential system is encountered. To any partition \( \rho \) we associate a function \( \psi_{\rho}(u) \) with the conditions
\[
\sum_{i=1}^{l(\rho)+1} \frac{d}{du} \psi_{\rho^{(i)}}(u) = \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1) \psi_{\rho^{(i)}}(u),
\]
\[
\sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1) \frac{d}{du} \psi_{\rho^{(i)}}(u) = \sum_{i=1}^{l(\rho)+1} (\rho_i - i + 1)^2 \psi_{\rho^{(i)}}(u) + t|\rho|\psi_{\rho}(u) - (1-t) \sum_{i=1}^{l(\rho)} \psi_{\rho^{(i)}}(u).
\]

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This first order (overdetermined) differential system must be solved with the initial conditions $\psi_\rho(0) = 0$.

In this section we shall only consider the elementary case where $\rho = (r, 1^s)$ is a hook partition. In this situation the differential system becomes

$$\frac{d}{du}(\psi_{r+1,1^s}(u) + \psi_{r,2,1^{s-1}}(u) + \psi_{r,1^{s+1}}(u)) = r\psi_{r+1,1^s}(u) - (s + 1)\psi_{r,1^{s+1}}(u), \quad (5.1)$$

$$\frac{d}{du}(r\psi_{r+1,1^s}(u) - (s + 1)\psi_{r,1^{s+1}}(u)) = r^2\psi_{r+1,1^s}(u) + (s + 1)^2\psi_{r,1^{s+1}}(u) + t(r + s)\psi_{r,1^s}(u) - (1 - t)(\psi_{r-1,1^s}(u) + \psi_{r,1^{s-1}}(u)). \quad (5.2)$$

The reader may check that for $t = 1$ the solutions $\psi_{r,1^s}(u)$ are given by

$$(r + s)!\psi_{r,1^s}(u) = (e^u - 1)^{r-1}(e^{-u} - 1)^s$$

$$= \sum_{i=-s}^{r-1} (-1)^{r+s+i-1} \binom{r + s - 1}{r - i - 1} e^{iu}. \quad (5.3)$$

For $t$ arbitrary, the following partial results give an idea about the high complexity of this problem.

Let us restrict to the most elementary situation $s = 0$. By linear combination, the differential system (5.1)-(5.2) is easily transformed to

$$(r + 1)\psi'_{r+1} = r(r + 1)\psi_{r+1} + tr\psi_r - (1 - t)\psi_{r-1}, \quad (5.3)$$

$$(r + 1)\psi'_{r,1} = -(r + 1)\psi_{r,1} - tr\psi_r + (1 - t)\psi_{r-1}, \quad (5.4)$$

which must be solved with the initial conditions $\psi_r(0) = \psi_{r,1}(0) = 0$.

**Proposition 2.** We have

$$(-1)^r r! \psi_r(u) = \sum_{i=1}^{r-1} \langle \begin{array}{c} r - 1 \\ r - i - 1 \end{array} \rangle \langle \begin{array}{c} -1 \\ i - 1 \end{array} \rangle e^{iu} - \langle \begin{array}{c} r - 2 \\ r - 2 \end{array} \rangle. \quad (5.5)$$

**Proof.** The statement is easily checked for $r \leq 3$ since we have

$$\psi_1(u) = 0, \quad 2\psi_2(u) = e^u - 1, \quad -6\psi_3(u) = t(-e^{2u} + 2e^u - 1).$$

Inspired by the $t = 1$ case, we may look for a solution of (5.3) under the form

$$(-1)^r r! \psi_r(u) = \sum_{i=0}^{r-1} a_i^{(r)} e^{iu}. \quad (5.6)$$

By identification of the coefficients of exponentials, we obtain

$$\frac{r - i - 1}{r - 1} a_i^{(r)} = ta_i^{(r-1)} + (1 - t)a_i^{(r-2)}.$$
For $1 \leq i \leq r - 2$, by induction on $r$ this relation yields
\[ a_i^{(r)} = \langle \begin{array}{c} r - 1 \\ r - i - 1 \\ i - 1 \end{array} \rangle. \]

For $i = 0$ by induction on $r$ we get similarly
\[ a_0^{(r)} = -\langle r - 2 \rangle. \]

For $i = r - 1$ the value of $a_{r-1}^{(r)}$ is not defined. But the latter may be obtained from the initial condition
\[ \psi_r(0) = \sum_{i=0}^{r-1} a_i^{(r)} = 0. \]

Actually applying the generalized Chu-Vandermonde formula of Corollary 3, written with $x = -1$, $y = r - 1$ and $k = r - 2$, we have
\[ -a_{r-1}^{(r)} = \sum_{i=1}^{r-2} \langle \begin{array}{c} r - 1 \\ r - i - 1 \\ i - 1 \end{array} \rangle - \langle r - 2 \rangle 
= \sum_{i=0}^{r-3} \langle \begin{array}{c} r - 1 \\ r - i - 2 \\ i \end{array} \rangle - \sum_{i=0}^{r-2} \langle \begin{array}{c} r - 1 \\ r - i - 2 \\ i \end{array} \rangle 
= -\langle r - 2 \rangle. \]

\[ \blacksquare \]

**Proposition 3.** We have
\[ (-1)^{r+1}(r + 1)! \psi_{r,1}(u) = \sum_{i=1}^{r-1} \frac{r - i}{i + 1} \langle \begin{array}{c} r \\ r - i \\ i - 1 \end{array} \rangle e^{iu} - r \langle \begin{array}{c} r - 1 \\ r - 1 \\ i - 1 \end{array} \rangle + \frac{r + 1}{1 - t} \langle \begin{array}{c} r \\ r + 1 \\ r \end{array} \rangle e^{-u}. \]

**Proof.** It is similar to the previous one. Inspired by the $t = 1$ case, we look for a solution of (5.4) under the form
\[ (-1)^{r+1}(r + 1)! \psi_{r,1}(u) = \sum_{i=-1}^{r-1} b_i^{(r)} e^{iu}. \]

By identification of the coefficients of exponentials, we obtain
\[ (i + 1)b_i^{(r)} = r(ta_i^{(r)} + (1 - t)a_i^{(r-1)}). \]

For $1 \leq i \leq r - 1$ it yields
\[ (i + 1)b_i^{(r)} = r \left( t \langle \begin{array}{c} r - 1 \\ r - i - 1 \end{array} \rangle + (1 - t) \langle \begin{array}{c} r - 2 \\ r - i - 2 \end{array} \rangle \right) \langle \begin{array}{c} -1 \\ i - 1 \end{array} \rangle 
= (r - i) \langle \begin{array}{c} r \\ r - i \\ i - 1 \end{array} \rangle. \]
Similarly for $i = 0$ we get

$$b_0^{(r)} = -r \left( t\left\langle r - 2 \right\rangle + (1 - t)\left\langle r - 3 \right\rangle \right) = -r\left\langle r - 1 \right\rangle.$$ 

For $i = -1$ the value of $b_{-1}^{(r)}$ is not defined. But we may obtain

$$b_{-1}^{(r)} = \frac{r + 1}{1 - t} \left\langle r \right\rangle$$

from the initial condition

$$\psi_{r,1}(0) = \sum_{i=-1}^{r-1} b_i^{(r)} = 0.$$

Actually applying Corollary 3 and Proposition 1, we have

$$-b_{-1}^{(r)} = \sum_{i=1}^{r-1} \frac{r-i}{i+1} \left\langle r \right\rangle_{i-1} - r\left\langle r - 1 \right\rangle$$

$$= \sum_{i=1}^{r} \frac{r(i+1) - (r+1)i}{i+1} \left\langle r \right\rangle_{i-1} - r\left\langle r - 1 \right\rangle$$

$$= \sum_{i=0}^{r} -\frac{(r+1)i}{i+1} \left\langle r \right\rangle_{i-1} = \frac{r + 1}{1 - t} \left\langle r \right\rangle.$$

Unfortunately the situation becomes quickly very messy and a general formula for $\psi_{r,1^{s}}(u)$ is as yet unknown. The following case is obtained by putting $s = 1$ in (5.2), which leads to define $\psi_{r,1^{2}}(u)$ by

$$r\psi_{r+1,1}(u) - 2\psi_{r,1^{2}}(u) = r^2\psi_{r+1,1}(u) + 4\psi_{r,1^{2}}(u)$$

$$+ t(r + 1)\psi_{r,1}(u) - (1 - t)(\psi_{r-1,1}(u) + \psi_{r}(u)),$$

with the initial condition $\psi_{r,1^{2}}(0) = 0$. Inspired by the $t = 1$ case, we look for a solution under the form

$$(-1)^{r}(r + 2)! \psi_{r,1^{2}}(u) = \sum_{i=-2}^{r-1} c_i^{(r)} e^{iu},$$

and we obtain the recurrence relation

$$-2(i + 2)c_i^{(r)} = r(r - i)b_i^{(r+1)} - (r + 1)(r + 2)(tb_i^{(r)} + (1 - t)(b_i^{(r-1)} + a_i^{(r)})).$$

For instance

$$-4c_0^{(r)} = r^2b_0^{(r+1)} - (r + 1)(r + 2)(tb_0^{(r)} + (1 - t)(b_0^{(r-1)} + a_0^{(r)}))$$

$$= -r^2\left\langle r \right\rangle_{r} + (r + 1)(r + 2) \left( tr \left\langle r - 1 \right\rangle + (1 - t)((r - 1) + 1) \left\langle r - 2 \right\rangle \right)$$

$$= 2r(r + 1)\left\langle r \right\rangle_{r}.$$ 

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The reader may check that the solutions are given by
\[
(-1)^r(r+2)! \psi_{r,1\mathbb{Z}}(u) = \\
\sum_{i=1}^{r-1} \left( \frac{(r-i)(r-i+1)}{(i+1)(i+2)} \right) \binom{r+i+1}{r+i+1} - (1-t) \frac{i(r+1)(r+2)}{2(i+1)(i+2)} \binom{r-1}{r-i-1} \binom{-1}{i-1} e^{iu} \\
- \left( \frac{r+1}{2} \right) \binom{r}{r} + \left( \frac{r+2}{2} \right) \left( \frac{2e^{-u}/r+1}{1-t} \right) e^{-u} - \left( \frac{r-1}{r} \right) \binom{r}{r} e^{-2u} \frac{1-t}{1-t} \binom{r}{r+2} e^{iu}.
\]

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