Open Membranes in Matrix Theory

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We discuss how to construct open membranes in the recently proposed matrix model of M theory. In order to sustain an open membrane, two boundary terms are needed in the construction. These boundary terms are available in the system of the longitudinal five-branes and D0-branes.

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M theory is recently conjectured to be described by a large N quantum mechanical system based on D-particle dynamics [1]. One of the major clues for this conjecture is the realization of the light-cone membrane Hamiltonian as a large N gauged quantum mechanics [2]. The conjecture is further supported by an explicit calculation of the membrane tension [1] and by the Dirac quantization of the membrane charge in the background of a longitudinal five-brane [3]. Further supporting evidence has been collected in [4] [5] [6].

A closed, transverse membrane with toroidal topology can be constructed by utilizing the basis for large N matrices spanned by \( U, V \) with \( UV = \exp(2\pi i/N)VU \). \( U \) and \( V \) in turn can be realized with \( U = \exp(ip), V = \exp(iq), [q,p] = 2\pi i/N \). Now \( p \) and \( q \) parametrize a torus of unit radii, it is then natural that only closed membranes are realized using this basis. We wish to point out in this note that with a slight abuse, this basis can be used to construct open, transverse membranes with cylindric topology. Before proceeding to our construction, it is useful to briefly review the light-cone Hamiltonian of [1] and the construction of closed membranes there.

Compactifying M theory on a circle of radius \( R \), D0-branes emerge as supergravitons carrying positive unit momentum \( p_{11} = 1/R \). These particles, regarded as massive particles in 10 dimensions, has a mass \( m = p_{11} \). The light-cone Hamiltonian of multiple particles is given by the dimensionally reduced 10D super Yang-Mills theory. In the temporal gauge \( A_0 = 0 \), the bosonic part of the Hamiltonian reads

\[
H = P_- = \frac{m}{2} \Tr \left( \sum_i (\dot{X}^i)^2 - C \sum_{i<j} [X^i, X^j]^2 \right),
\]

where constant \( C = (2\pi\alpha')^{-2} \). A number of properties of M theory have been checked using this Hamiltonian, as we mentioned before. In particular, a transverse membrane is constructed by \( X^8 = R_8 p, X^9 = R_9 q \) and \( X^i = 0, i = 1, \ldots, 7 \). Here \( p \) and \( q \) both have a period 2\( \pi \), so \( X^8 \) and \( X^9 \) have periods \( 2\pi R_8 \) and \( 2\pi R_9 \) respectively. Note that the configuration satisfies the stationary equations of motion

\[
\sum_i [X^i, [X^j, X^i]] = 0
\]

and \([X^8, X^9] \neq 0\). This is possible only in the large N limit, since for a finite N, the equations of motion always imply \([X^i, X^j] = 0\). The light-cone Hamiltonian is not vanishing, and the mass squared of the membrane is given by the relation \( M^2 = 2P_{11} H = 2NmH \).

This is shown to agree with the membrane tension formula [1].
It appears that it is impossible to discuss open membranes with the basis \((p,q)\), as it has toroidal topology. One way to get around this problem is to introduce a pair of noncommutative variables parametrizing a finite cylinder. We don’t know how to do this at this moment. Instead we propose to study open membranes using the basis \((p,q)\) with a slight modification for the ansatz \(X^8, X^9\). Since it is generally believed that only wrapped membrane is stable, we shall still compactify \(X^9\) with radius \(R_9\), and leave \(X^8\) uncompactified. The open membrane we are interested in will wrap around \(X^9\) once, and stretched from \(X^8 = x_1\) to \(X^8 = x_2\). The ansatz we propose is the following

\[
X^9 = R_9 p, \quad X^8 = \begin{cases} 
  x_1, & q < q_1 \\
  x_2, & q > q_2 \\
  \frac{x_2 - x_1}{q_2 - q_1} q + \frac{x_1 q_2 - x_2 q_1}{q_2 - q_1}, & q_1 < q < q_2
\end{cases}
\] (2)

where \(q_1 \geq 0\) and \(q_2 \leq 2\pi\). The strategy here is to break the circle parametrized by \(q\) by assuming that \(X^8\) collapses to a constant point at both ends. It is important to keep in mind that \(X^8\) is a function of only \(q\), and is diagonal in the diagonal basis of \(q\). Let \(X^8 = f(q)\). Taking derivative twice,

\[
f^{(2)}(q) = \frac{\Delta x}{\Delta q} \left( \delta(q - q_1) - \delta(q - q_2) \right).
\] (3)

Before we proceed to justify the above equation, we compute the membrane tension first. The commutator \([X^8, X^9] = R_9 (2\pi i/N) f'(q)\), and the first derivative of \(f(q)\) is a step function. It is straightforward to evaluate the light-cone Hamiltonian, using the method presented in [1]. The mass squared of the open membrane is

\[
M^2 = \frac{m^2 (2\pi R_9 \Delta x)^2}{2\pi \Delta q},
\]

where we take \(2\pi \alpha' = 1\). This gives rise to a membrane tension

\[
T^2 = \frac{m^2}{2\pi \Delta q} \geq \left( \frac{m}{2\pi} \right)^2,
\] (4)

since \(\Delta q \leq 2\pi\). We see that the membrane tension is bounded from below by the true membrane tension. With \(\Delta q < 2\pi\), the open membrane must be interpreted as meta-stable, since its spectrum is continuous, and nothing prevents it from decaying to its lower bound. Actually, it will be shown later that only in the limit \(\Delta q = 2\pi\), it is possible to maintain some unbroken supersymmetry.
Next, we show that (3) can be interpreted as forces needed to sustain the open membrane at its boundaries. The two relevant equations of motion resulting from ansatz (2) are

\[
\begin{align*}
X^{8}, [X^{9}, X^{8}] & = 0, \\ [X^{9}, [X^{8}, X^{9}]] & = (2\pi R_{9} N)^{2} \frac{\Delta x}{\Delta q} (\delta(q - q_{1}) - \delta(q - q_{2})).
\end{align*}
\] (5)

Roughly speaking, the second equation implies that a force is needed at boundary \(X^{8} = x_{1}\) and an opposite force is needed at boundary \(X^{8} = x_{2}\), in order to sustain the open membrane.

The physical origin of such force must be found in the system consisting of five-branes and D0-branes. Such a system is discussed in [3]. A transverse open membrane can end on a longitudinal five-brane, and the latter is interpreted as a D4-brane in 10 dimensions. Typically, what has been considered before in the M theory context is a static open membrane stretched between two parallel five-branes [7]. Due to 11 dimensional Lorentz invariance, one can always boost the open membrane in one of the longitudinal directions of the five-branes. Here in the matrix model context, a transverse membrane always carries an infinite longitudinal momentum, that is, it moves in the speed of light along \(X^{11}\). To see that the boundary terms in (5) are actually available in the system studied in [3], we first find out the corresponding term in the Hamiltonian phenomenologically. With a finite \(N\), \(q\) has eigen-values \(q = 2\pi (i - 1)/N\), \(i = 1, \ldots, N\). Let the \(i\)'s corresponding to \(q_{1}\) and \(q_{2}\) be \(i_{1}\) and \(i_{2}\). The delta function \(\delta(q - q_{1})\) can be replaced by a diagonal matrix (in the basis in which \(q\) is diagonal) with only one nonvanishing entry at the \(i_{1}\)-th row and the \(i_{1}\)-th column with a value \(N/(2\pi)\). Let \(v_{i}\) be the unit vector with only one nonvanishing entry at the \(i\)-th row, then \(\delta(q - q_{1}) = N/(2\pi) v_{i_{1}} v_{i_{1}}^{+}\). A term in the Hamiltonian such as

\[
\Delta H = mR_{9}^{2} \frac{2\pi}{N} (\frac{\Delta x}{\Delta q}) \text{Tr} X^{8} (v_{i_{1}}^{+} v_{i_{1}} - v_{i_{2}}^{+} v_{i_{2}}^{+})
\] (6)

would reproduce the desired boundary terms in (5).

Now, consider two parallel longitudinal five-branes, with 5 longitudinal spatial coordinates \(X^{m}\), and one of them is \(X^{11}\), and 5 transverse coordinates \(X^{a}\). Let one of the longitudinal coordinates be \(X^{9}\), the same as the circle around which the open membrane is wrapped. And let one of the transverse coordinates be \(X^{8}\), along which the open membrane is stretched. Further, let the location of the two five-branes be \(X^{8} = x_{1}, x_{2}\). In the background of these five-branes, there are additional modes in the dynamics of D0-branes, these correspond to open strings stretched between a five-brane and a D0-brane.
In particular, there are two sets of bosons transforming in the fundamental representation of $U(N)$, call them $V^\rho_I$, where $\rho$ index the spinor representation of the positive chirality of $SO(4)_L$, the rotation group of the 4 longitudinal directions excluding $X^{11}$, and $I = 1, 2$. We refer to ref.[3] for further notation. The bosonic part of the additional Hamiltonian is

$$\Delta H = |\partial_t V^\rho_I|^2 + V^\rho_I (X^a - x^a_I)^2 V^\rho_I - V^\rho_I [X^m, X^n] \sigma^\rho_{mn} V^I_\sigma + |V^I|^4.$$  

Since there is only one nontrivial $X^m$, that is $X^9 = R_9 p$, the third term in the above Hamiltonian drops out, hence the equation of motion for $X^9$ is just $[X_8, [X^9, X^8]] = 0$ which is satisfied by our ansatz. So the only nontrivial terms relevant to our problem are

$$\Delta H = |\partial_t V^\rho_I|^2 + V^\rho_I (X^8 - x^8_I)^2 V^\rho_I + |V^I|^4, \tag{7}$$

where $x^8_1 = x_1$, $x^8_2 = x_2$. This additional part is very similar to the required term in (6). Indeed, the above Hamiltonian will reproduce the desired boundary terms provided

$$2(X^8 - x_1)V^I_1 = mR_9^2 \left( \frac{2\pi}{N} \right) \frac{\Delta x}{\Delta q} v_i v^+_i,$$

$$2(X^8 - x_2)V^I_2 = -mR_9^2 \left( \frac{2\pi}{N} \right) \frac{\Delta x}{\Delta q} v_i v^+_i. \tag{8}$$

So in order to have the right boundary terms, the vector fields $V_I$ must be excited at the boundary, or the vector fields must have nonvanishing quantum fluctuations close to the boundary. Before exploring this possibility, let us first notice that the above equations can be satisfied in principle. Examine the first equation. The R.H.S. tells us that $V_1$ must have only one nonvanishing entry at the $i_1$-th row. Since all eigenvalues of $X^8$ must be equal to or greater than $x_1$, so $X^8 - x_1 \geq 0$, and the signs on the both sides agree. In the second equation, $X^8 - x_2 \leq 0$, again the signs on the both sides of the second equation agree.

Actually, the problem is a little more involved than represented by (8). The fermions transforming as vector of $U(N)$ will likely to contribute to the boundary forces. We believe that the physics is however captured by (8). The difficulty to satisfy (8) at the classical level is that the potential for the vectors $V_I$ is always positive, so it is impossible to give $V_I$ a vacuum expectation value, even only at boundaries of the open membrane. Quantum mechanically, the possibility for the existence of nonvanishing $V_I$ at boundaries exists. Consider a single D0-brane close to one of the five-brane. It is known there is an attractive force between the two objects, and a bound state can form [8]. The typical transverse
radius for such a bound state is just the 11 dimensional Planck length. Such a force is not known if there are many D0-branes and they are not arranged in a commutative fashion.

Eqs. (8) imply that $V_1 \sim v_i$ and $V_2 \sim v_i$, so $X^8 - x_I$ vanishes when applied to $V_I V_I^\dagger$. This is due to the fact that we have identified $x_I$ in (3) with $x_I$ in (5). Physically one may introduce a cut-off for $X^8 - x_I$. This cut-off distance between a boundary of the open membrane and a five-brane is constrained by the condition that the additional Hamiltonian (5) should not contribute to the membrane tension in the large N limit. Let, say $(X^8 - x_1)$ in (8) scales as $1/N^\alpha$ in the large N limit, then $|V_1|^2 \sim 1/N^{1-\alpha}$ according to (8). The second term in (7) will contribute an amount $1/N^{1+\alpha}$ to the Hamiltonian. The condition that this term can be neglected in computing the membrane tension is $\alpha > 0$. (This implies that the cut-off $1/N^\alpha \rightarrow 0$ in the large N limit.) On the other hand, the last term in (7) scales as $1/N^{2(1-\alpha)}$ and can be neglected if $\alpha < 1/2$.

We need to examine the supersymmetry transformation in order to see which value of the parameter $\Delta q$ is allowed. It can be seen that supersymmetry vanishes acting on all fields except $\theta$, the superpartner of $X^\mu$. The SUSY transformation of this field, using the notation of [3], is

$$\delta \theta^\rho = 2[X^8, X^9](\gamma_{89}\eta)^\rho_\alpha + \eta'_\alpha + V I V I^\dagger \eta_\sigma \alpha,$$

where the new index $\alpha$ is the spinor index of the transverse rotation group $SO(5)$. For a closed membrane considered in [3], there is no a third term. The first term is proportional to the identity matrix, so can be cancelled by the second term. For the open membrane, the first term is not proportional to the identity matrix, a step function sets in. The commutator is

$$[X^8, X^9] = \frac{2\pi i}{N} R_9 \frac{\Delta x}{\Delta q} (\theta(q - q_1) - \theta(q - q_2)),$$

and vanishes when $q < q_1$ or $q > q_2$. As in the closed membrane case, one can always choose $\eta'$ to cancel the first term for $q_1 < q < q_2$. For $q$ outside this range, the first term vanishes, then the constant term $\eta'$ must be cancelled by the third term. However, in the large N limit, $\eta'$ has infinitely many nonvanishing diagonal entries to be cancelled by the third term if, say, $q_1 > 0$. This can not be constructed from a finite number vectors $V_I^\rho \bar{\rho}$, since the third term will be close to a “pure state” density matrix, while the part of $\eta'$ to be cancelled is rather a “mixed state” density matrix. A resolution of this problem again comes from the fact that $|V|_2$ is nonvanishing only at the quantum level. If one interprets
$V_1V_1^+$ as the quantum average $\langle V_1V_1^+ \rangle$, then this matrix can be a “mixed state” density matrix. From cancelation of SUSY (9), we deduce that

$$\langle V_1V_1^+ \rangle \sim \frac{1}{N} \sum_{i=1}^{i_1} v_i v_i^+,$$

and a similar result for $\langle V_2V_2^+ \rangle$.

The above ansatz contradicts (8) however, since there only a single $v_{i_1}$ appears. To resolve this problem, recall that there is certain arbitrariness in regularizing the delta functions in (5). If $q_1 \sim i_1/N \to 0$ in the large $N$ limit (we shall show this is the case indeed), then it is equally good to use the following

$$\delta(q - q_1) = \frac{N}{2\pi i_1} \sum_{i=1}^{i_1} v_i v_i^+,$$

and consequently the R.H.S. of (9) is modified. Use (10) in this modified condition, we find $i_1 \sim N^\alpha$. The condition that the second term of (4) does not contribute to the membrane tension is still $\alpha > 0$. The condition that the third term does not contribute to the membrane tension is $|V_1|^4 \sim i_1^2/N^2 << 1/N$, so $\alpha < 1/2$, also the same as the condition we derived using a single $v_{i_1}$. Finally, we see that

$$q_1 = i_1/N \sim 1/N^{1-\alpha} \to 0.$$

Similarly, in the large $N$ limit $q_2 \to 2\pi$, and $\Delta q \to 2\pi$. It must be emphasized that although $q_1 \to 0$, but $i_1 \sim N^\alpha \to \infty$, and this justifies our ansatz (2) and equations of motion (4).

It remains to determine the exponent $\alpha$. A closer examination of the large $N$ dynamics is necessary in order to do this and to work out more details of our construction. Here we shall make only a plausible guess for the boundary dynamics. In general, one may replace our ansatz with a general boundary function such that equations of motion (4) are still valid at the boundary. With SUSY (3) unbroken for some choice of $\eta$ and $\eta'$, generally one expects that the one-loop correction is vanishing. In particular, the large one-loop kinetic energy in (7) must be canceled by that of fermions. It is plausible that the major contribution to the second and third terms in (7) is from higher loops. If so, in the perturbative calculation, the third term becomes important. If one assumes that these two terms are comparable, then one deduces $i_1 \sim N^\alpha = N^{1/3}$, that is, the exponent $\alpha = 1/3$. 

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Finally, a remark on central charges. A closed membrane carries a rank 2 tensor central charge. In an appropriate formulation of super algebra of the matrix model, this central charge should be of form $\text{tr}[X^i, X^j]$. The closed membrane solution indeed has a nonvanishing commutator. For an open membrane, its ends appear as a closed string in the five-branes. A closed string in a five-brane carries a central charge corresponding to the anti-self-dual tensor field in the tensor multiplet. Again, in an appropriate formulation of super algebra, one expects a vector central charge. In the matrix model, the natural candidate for this is just $\sum_j [X^j, [X^i, X^j]]$. This quantity must have nonvanishing value only at a boundary of the open membrane. This further justifies our equations of motion (5).

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