Classical limits, quantum duality and Lie-Poisson structures

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Abstract
Quantum duality principle is applied to study classical limits of quantum algebras and groups. For a certain type of Hopf algebras the explicit procedure to construct both classical limits is presented. The canonical forms of quantized Lie-bialgebras are proved to be two-parametric varieties with two classical limits called dual. When considered from the point of view of quantized symmetries such varieties can have boundaries that are noncommutative and noncocommutative. In this case the quantum duality and dual limits still exist while instead of Lie bialgebra one has a pair of tangent vector fields. The properties of these constructions called quantizations of Hopf pairs are studied and illustrated on examples.

1 Introduction
Quantum duality principle [1, 2] asserts that quantization of a Lie bialgebra \((A, A^*)\) gives rise to a dual pair of Hopf algebras \((U_p(A), U_p(A^*))\) or in dual terms \((\text{Fun}_p(G), \text{Fun}_p(G^*))\). In the standard form quantum algebras and groups do not exhibit this duality explicitly. This is clearly seen when the classical limit is concerned. The quantum algebra \(U_p(A)\) is meant as a member of 1-dimensional family of Hopf algebras – the deformation curve. Its

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classical limit $U(A)$ is a fixed point of the orbit $\text{Orb}(U(A))$ where the deformation curve starts. Due to quantum duality formulated in terms of quantum formal series Hopf algebras quantum algebra can be interpreted as a quantum group

\[ U_p(A) \approx (\text{Fun}(G^*))_p \]

for the universal covering group $G^*$ with Lie algebra $A^*$. So there must be another classical limit, i.e. another deformation curve that starts at a fixed point of $\text{Orb}(\text{Fun}(G^*))$ and contains the Hopf algebra $U_p(A)$.

Thus the natural form of deformation quantization of Lie bialgebra $(A, A^*)$ must be a 2-parametric family of Hopf algebras with two dual classical limits. Within certain assumptions this family forms an analytic variety $Q$ and the classical limits – its boundary. The existence of a variety $Q$ with such properties is equivalent to attributing its member the quantum duality. The Lie bialgebra appears here in the form of two vector fields tangent to $Q$. From the point of view of Lie-Poisson structures and their symmetries it can be shown natural to consider the varieties of this type and their boundaries entirely placed in the domain of noncommutative and noncocommutative Hopf algebras. Preserving the main property of quantum duality we find that in this case some other characteristics are not conserved. In particular the lifted tangent fields may not form a Lie-bialgebra any more. But dual parameters and dual limits are still present there and can play important role in applications.

The paper is organised as follows. In the subsection 2.1 we describe how to construct the dual classical limits and under what conditions it can be done. The explicit example is considered in subsection 2.2 where it is also shown how inverting the dual variety $Q^*$ one can change the role of dual parameters. In the subsection 3.1 lifted varieties $Q_\varepsilon$ (that are called the quantized Hopf pairs) are studied. Their Lie-Poisson properties are considered in 3.2 and in 3.3 the nontrivial example of such $Q_\varepsilon$ variety is presented. In Appendix the two-parametric form for the standard quantization of $sl(n, \mathbb{C})$ is given explicitly.

## 2 Dual classical limits

### 2.1 General scheme

Let us construct the second classical limit for a quantum Lie algebra $U_p(A)$. Consider the variety $\mathcal{H}$ of Hopf algebras with fixed number of generators. Its points $H \in \mathcal{H}$ are parametrized by the corresponding structure constants. We must find a (smooth) curve in $\mathcal{H}$ containing $U_p(A)$ and intersecting with the orbit $\text{Orb}(\text{Fun}(G^*))$. In the limit to be obtained the multiplication $m$ in $U_p(A)$ must become Abelian. For the universal enveloping algebra

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A such a procedure is trivially described by a linear contraction. The corresponding transformation of basis \( \{a_i\} \),
\[
B(t) : a_i \rightarrow a_i/t,
\]
leads to new structure constants \( C'_{ij} \)
\[
C'_{ij}(t) = tC_{ij}.
\]
The costructure of \( U(A) \) being primitive is insensitive to this transformation. Algebras \( U(A_t) \) form a line in \( \text{Orb} U(A) \) with the limit point \( U(A_0) \equiv \text{Abelian} \). Let us apply operators \( B(t) \) to \( U_p(A) \). Such a sequence of equivalence transformations (for each value of \( p \) fixed) generates the smooth one-parametric curve
\[
B(t)U_p(A) \equiv U_p(A_t)
\]
belonging to the orbit \( \text{Orb}(U_p(A)) \). Note that in general case Hopf algebras \( U_p(A) \) are not equivalent for different \( p \). We thus obtain in \( \mathcal{H} \) the 2-parametric subset
\[
\{U_p(A_t)\}_{p>0,t>0} \equiv Q(A, A^*)
\]
formed by the dense set of smooth curves. Being the deformation quantization \( \{U_p(A)\}_{p>0} \) is an analytic family. So \( Q(A, A^*) \) is a smooth variety with the global co-ordinates \( p \) and \( t \). Nevertheless these co-ordinates are not appropriate for our task because the limit
\[
\lim_{t \rightarrow 0} U_{p>0}(A_t)
\]
does not exist. This is clearly seen from the properties of the coproduct in \( U_p(A) \),
\[
\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i + p(D^{kl}_i a_k \otimes a_l + S^{kl}_i a_k \otimes a_l + D^{(k,m)}_i a_k a_m \otimes a_l + \cdots).
\]
Here the first deforming function is divided into symmetric and antisymmetric parts and \( D^{kl}_i \) are the structure constants of \( A^* \). The transformation (3) leads to the expression divergent for \( t \rightarrow 0 \):
\[
\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i + p/t(D^{kl}_i a_k \otimes a_l + \cdots).
\]
Here the unwritten terms in (4) may contain the higher negative powers of \( t \).

It is easy to define a class of deformation quantizations for which one can overcome this difficulty. For each \( U_p(A) \) consider the family of groups parametrized by \( p \) and defined on the space \( \text{Mor}(U_p(A), K) \) by the convolution multiplication
\[
\phi_1 \ast \phi_2 = (\cdot K) (\phi_1 \otimes \phi_2) \Delta, \quad \phi_1, \phi_2 \in \text{Mor}(U_p(A), K)
\]
(\( K \) here is the main field). Let \( P \) be the subspace of functionals dual to the space of algebra \( A \). On this space (when certain conditions are fulfilled) the convolution also generates
a family of groups parametrised by \( p \). This analytic family describes the contraction of the obtained group \( \mathcal{P} \) to the additive Abelian vector group on the space dual to that of Lie algebra \( A \).

Suppose now that

(a) the system of equations
\[
(\cdot U_p)(\text{id} \otimes S)\Delta = (\cdot U_p)(\text{id} \otimes S)\Delta = \eta \varepsilon
\]
on the basic elements \( a_i \) fixes the antipode \( S \) of \( U_p(A) \) (see [3]).

(b) for \( U_p(A) \) the contraction \( \mathcal{P}_p \rightarrow \mathcal{AB} \) is equivalent to the trivial, that is induced by such a contraction of its algebra \( A^* \) where the structure constants in the lowest order are proportional to \( p \).

It is easy to verify that for such deformation quantizations in every monome of the coproduct (5) the power of \( p \) is less than the total degree of the basic elements \( a_i \) by one.

If the Hopf algebra \( U_p(A) \) belongs to the class described above the second classical limit can be obtained as follows. Let us change the co-ordinates:

\[
(p, t) \Rightarrow (h, t), \quad h = p/t.
\]

Then the coproduct (6) becomes well defined in the limit \( t \to 0 \) (with \( h \) fixed) because now its structure constants can bare only positive powers of \( t \) (and in the two lowest orders can depend only on \( h \)).

The multiplication structure constants in these new co-ordinates also have the finite limit values. To see this let us go to the dual picture. Consider the space of linear functionals on the space of Hopf algebra \( U_p(A_t) \) and the canonical dual basis \( \{ f^i \} \) such that the elements \( f^i \) dual to the basic elements of \( A \),

\[
< a_i, f^j > = \delta^j_i,
\]
form the basis of \( A^* \). Construct the dual Hopf algebras for the elements of \( Q(A, A^*) \)
\[
\{(U_p(A_t))^*\}_{p>0,t>0} = \{\text{Fun}_p(G_t)\}_{p>0,t>0} \equiv Q^*(A, A^*)
\]
and also for the border line
\[
\{(U(A_t))^*\}_{t \geq 0} = \{\text{Fun}(G_t)\}_{t \geq 0}.
\]

We shall consider these Hopf algebras as quantum formal series groups [4]. The basis transformation \( B^*(t) \) dual to (1)

\[
B^*(t) : f^i \to tf^i
\]
leads to the following compositions:

$$[f^i, f^k] = p/t(D^i_t f^l + tE^i_{su}f^s f^u + \cdots),$$  \hspace{0.5cm} (13)$$

and

$$\Delta f^i = f^i \otimes 1 + 1 \otimes f^i + t(C^i_{kl} f^k \otimes f^l + T^i_{kl}f^k \otimes f^l + tC^i_{(k,l)(u)}f^k f^l \otimes f^u + \cdots)$$  \hspace{0.5cm} (14)$$

For simplicity the renormalization factors for high power monomials are incorporated in the corresponding structure constants. These coproduct structure constants (in general) depend on $p$ but they all have well defined limits when $p \to 0$. These limits describe the power series expansion of the multiplication law in $G_t$ in terms of exponential co-ordinate functions. So the corresponding Taylor series can be written for them in the neighbourhood of $(\cdot)p = 0$. Substituting $p = \hbar t$ in (14) and going to the limit $t \to 0$ we see that all coefficients are finite. (Note that in the framework of formal series Hopf algebras this conclusion is true for all $p \geq 0$.)

We have shown that for algebras $U_h(A_t) \in Q(A, A^*)$ of the described class the limit $U_h(A_0) \equiv \lim_{t \to 0} U_h(A_t)$ exists. According to the quantum duality $U_h(A_t) \approx \text{Fun}_t(G^*_h)$, so

$$\lim_{t \to 0} U_h(A_t) \equiv U_h(A_0) \approx \text{Fun}(G^*_h)$$  \hspace{0.5cm} (15)$$

Thus every such deformation quantization can be written in the form $U_h(A_t)$ (respectively $\text{Fun}_t(G^*_h)$) that reveals two canonical dual classical limits:

$$\begin{array}{ccc}
U_h(A_t) & \overset{h \to 0}{\Rightarrow} & U(A_t) \\
\text{Fun}_h(G_t) & \overset{t \to 0}{\Rightarrow} & \text{Fun}(G^*_h) & \approx & U_h(A_b) \\
\text{Fun}(G_t) & \overset{t \to 0}{\Rightarrow} & \text{Fun}_h(\mathcal{A}B) & \approx & U(A^*_h) \\
\end{array}$$  \hspace{0.5cm} (16)$$

All the reasoning is invariant with respect to interchange $A \rightleftharpoons A^*$. So the set $\{\text{Fun}(G^*_h)\}$ can be also considered as a straight line in the orbit $\text{Orb}(\text{Fun}(G^*))$ – the trivial contraction of the group $G^*$ into the abelian additive vector group $\mathcal{A}B$. The lines $\{U(A_t)\}$ and $\{\text{Fun}(G^*_h)\}$ intersect in the point $U(A_b) \approx \text{Fun}(\mathcal{A}B)$.

The conclusion can be formulated as follows.

**Proposition 1** If $U_p(A)$ is a deformation quantization of a Lie bialgebra $(A, A^*)$ with the properties (a)-(b) then in $\mathcal{H}$ there exists an analytic submanifold $Q(A, A^*)$ (respectively $Q^*(A, A^*)$). It can be globally parametrised by co-ordinates $(h, t)$. The trivial contraction lines $U(A_t)$ and $\text{Fun}(G^*_h)$ (respectively $U(A^*_h)$ and $\text{Fun}(G_t)$) together with their intersection
point \( U(\mathcal{A}) = \text{Fun}(\mathcal{A}B) \) form a boundary of \( Q(\mathcal{A}, \mathcal{A}^*) \) (respectively \( Q^*(\mathcal{A}, \mathcal{A}^*) \)) and supply its elements with the dual classical limits:

\[
\begin{align*}
\lim_{h \to 0} H_{h,t} &= U(\mathcal{A}_t) \quad \text{for } Q, \\
\lim_{t \to 0} H_{h,t} &= \text{Fun}(G_{h}^*)
\end{align*}
\]

(17)

and

\[
\begin{align*}
\lim_{h \to 0} H_{h,t}^* &= \text{Fun}(G_{t}^*) \quad \text{for } Q^*, \\
\lim_{t \to 0} H_{h,t}^* &= U(\mathcal{A}_t^*)
\end{align*}
\]

(18)

Suppose now that

1. \( t \) and \( h \) parametrize the intersecting trivial contraction lines \( U(\mathcal{A}_t) \) and \( \text{Fun}(G_{h}^*) \) in the orbits \( \text{Orb}(U(\mathcal{A})) \) and \( \text{Orb}(\text{Fun}(G')) \) (respectively \( \text{Orb}(\text{Fun}(G)), \text{Orb}(U(\mathcal{A}')) \)) for a certain pair of inequivalent algebras \( (\mathcal{A}, \mathcal{A}') \) of equal dimension and corresponding universal covering Lie groups \( G \) and \( G' \). The intersection point coincides with the contraction limit.

2. In the variety \( \mathcal{H} \) of Hopf algebras with the generators \( \{a_i, 1\} \) (\( \{a_i\} \) is the basic of \( \mathcal{A} \)) there exists the analytical 2-dimensional subvariety \( Q(\mathcal{A}, \mathcal{A}') \) (respectively \( Q^*(\mathcal{A}, \mathcal{A}') \)) of Hopf algebras \( H \) and the disunqued union \( U(\mathcal{A}_t) \cup \text{Fun}(G_{h}^*) \) is the boundary of \( Q(\mathcal{A}, \mathcal{A}') \).

It is easy to check the validity of the following statement

**Proposition 2** If for algebras \( \mathcal{A} \) and \( \mathcal{A}' \) the conditions (1)-(2) are fulfilled – they are dual, Hopf algebras \( H \in Q(\mathcal{A}, \mathcal{A}') \) are the deformation quantizations of the Lie bialgebra \( (\mathcal{A}, \mathcal{A}' \approx \mathcal{A}^*) \) and the contraction curves \( U(\mathcal{A}_t) \) and \( \text{Fun}(G_{h}^*) \) supply the dual classical limits to the points of \( Q(\mathcal{A}, \mathcal{A}') \).

It is possible to invert the dual list \( Q^* \) with respect to \( Q \) so that the dual limits will refer to dual algebras (respectively groups). This is due to the fact that in the \((h, t)\) - co-ordinates the Hopf algebras \( H_{h,t}^* \) and \( H_{t,h}^* \) are equivalent, they are characterised by the same parameter \( p = ht \) (see (9)) and are connected by the transformation \( B^*(h/t) \). Thus one can always introduce the transformed nondegenerate bilinear form \( <, >_{h/t} \) such that the parameters on the list \( Q^* \) will be interchanged with respect to \( Q \),

\[
H_{h,t} \overset{<, >_{h/t}}{\leftrightarrow} H_{t,h}^*.
\]

(19)

Now the correlation between the canonical classical limits for \( Q \) and \( Q^* \) will differ from that
described by the diagram (16):

![Diagram](image)

Note that here in contrast to (16) the duality holds only for points of \( Q \) and \( Q^* \).

2.2 Example: \( U_h(sl_t(2)) \approx \text{Fun}_t(\widehat{E}_h(2)) \)

Consider the standard quantum algebra \( U_v(sl(2, C)) \):

\[
\begin{align*}
\Delta L &= L \otimes 1 + 1 \otimes L, & [L, M] &= M, \\
\Delta N &= N \otimes 1 + e^{-2pL} \otimes N, & [L, N] &= -N, \\
\Delta M &= M \otimes e^{2pL} + 1 \otimes M, & [M, N] &= 2 \frac{\sinh 2pL}{1 - e^{-2p}}.
\end{align*}
\]  

(21)

Applying the transformation \( B(t) \) and introducing the new co-ordinates \( (h = p/t, t) \) (see (1)) we get the canonical form for Hopf algebras of the variety \( Q(sl(2), e(2)) \):

\[
\begin{align*}
\Delta L &= L \otimes 1 + 1 \otimes L, & [L, M] &= tM, \\
\Delta N &= N \otimes 1 + e^{-2hL} \otimes N, & [L, N] &= -tN, \\
\Delta M &= M \otimes e^{2hL} + 1 \otimes M, & [M, N] &= 2t^2 \frac{\sinh 2hL}{1 - e^{-2h}}.
\end{align*}
\]  

(22)

The dual classical limits are

\[
\begin{align*}
\lim_{h \to 0} U_h(sl_t(2)) &= U(sl_t(2)), \\
\lim_{t \to 0} U_h(sl_t(2)) &= \text{Fun}(\widehat{E}_h(2)).
\end{align*}
\]  

(23)

Here \( \widehat{E}(2) \) is the group of flat motions with the quasiorthogonal rotation.

The canonical dualization produces the variety \( Q^*(sl(2), e(2)) \) whose points can be interpreted as \( \text{Fun}_h(SL_t(2)) \) and the parametrization is as follows:

\[
\begin{align*}
[\lambda, \mu] &= -2h\mu, \\
[\lambda, \nu] &= -2h\nu, \\
[\mu, \nu] &= 0, \\
\Delta \mu &= 1 \otimes \mu + (\mu \otimes e^{-t\lambda})(1 \otimes 1 + t^2 \mu \otimes \nu)^{-1}, \\
\Delta \nu &= \nu \otimes 1 + (e^{-t\lambda} \otimes \nu)(1 \otimes 1 + t^2 \mu \otimes \nu)^{-1}, \\
\Delta \lambda &= \lambda \otimes 1 + 1 \otimes \lambda - 4ht \sum_{n=1}^{\infty} t^{2n-1} \frac{(-\mu \otimes \nu)^n}{1 - e^{-2nh}}.
\end{align*}
\]  

(24)
We have used here the realization of $\text{Fun}_h(SL(2))$ obtained in \[5\]. On the dual list $Q^*$ the classical limits are

\[
\lim_{h \to 0} \text{Fun}_h(SL_t(2)) = \text{Fun}(SL_t(2)), \quad \lim_{t \to 0} \text{Fun}_h(SL_t(2)) = U(\overline{e_h(2)}). \tag{25}
\]

The first of these two limits is treated here as the formal series group whose generic elements are the basic exponential co-ordinate functions on $SL_t(2)$ in the neighbourhood of unit.

Using the equivalence transformation $B^*(h/t)$ on $Q^*$ the parametric dualization can be defined:

\[
< L, \lambda >_{h/t} = < M, \mu >_{h/t} = < N, \nu >_{h/t} = h/t. \tag{26}
\]

It inverts the dual list $Q^*$ with respect to $Q$ and shows explicitly that quantum algebras of $A$ and $A^*$ as well as quantum groups of $G$ and $G^*$ form the dual pairs of Hopf algebras (see (20)):

\[
\begin{array}{ccc}
U_h(slt(2)) & \xrightarrow{h \to 0} & \overline{U_h(sl_t(2))} \\
U(sl_t(2)) & \xleftarrow{t \to 0} & \text{Fun}(SL_{h(2)}) \\
U(e_t(2)) & \xrightarrow{h \to 0} & \text{Fun}(SL_{h(2)})
\end{array}
\]

(27)

In the Appendix the 2-parametric variety $Q$ for quantum algebras $U_h(sl(n, C))$ is explicitly described.

3 Quantization of Hopf pairs

3.1

The geometric description of the quantum duality properties of the deformation quantization given by the criteria (1)-(2) leads to the natural generalisation of this notion.

Let us call the deformation quantization of the Hopf pair $(H(\theta,0), H(0,\tau))$ the following construction in the variety $\mathcal{H}$ of Hopf algebras with the generators $\{a_i, 1\}$:

(1’) $\theta$ and $\tau$ parametrize the intersecting contraction curves $H(\theta,0)$ and $H(0,\tau)$ such that in the first order the coproducts of generators remain undeformed in $H(\theta,0)$ and the products of generators – in $H(0,\tau)$; the intersection point $H(0,0)$ coincides with the contraction limit for both $H(\theta,0)$ and $H(0,\tau)$.

(2’) In the variety $\mathcal{H}$ there exists the analytic 2-dimensional manifold $Q(H(\theta,0), H(0,\tau))$ (respectively $Q^*(H(\theta,0), H(0,\tau))$) of (noncommutative and noncocommutative) Hopf algebras $H(\theta,\tau)$. The disjoint union $H(\theta,0) \cup H(0,\tau) \cup H(0,0)$ is the boundary of $Q(H(\theta,0), H(0,\tau))$. (Respectively $H^*(\theta,0) \cup H^*(0,\tau) \cup H^*(0,0)$ – the boundary of $Q^*$.)
Then it is easy to check that on the curves $H_{(\theta,0)}$ and $H_{(0,\tau)}$ the smooth vector fields of the first deforming functions $V(\theta)$ and $W(\tau)$ exist such that the limit vectors

$$\lim_{\theta \to 0} V(\theta) \equiv V(0),$$  \hspace{1cm} (28)

$$\lim_{\tau \to 0} W(\tau) \equiv W(0),$$  \hspace{1cm} (29)

are the first deformation functions for $H_{(0,0)}$ in the direction of $H_{(0,\tau)}$ and $H_{(\theta,0)}$ respectively.

To connect this purely geometric construction with the usual deformation quantization picture it is natural to formulate the additional condition.

(3') In $\mathcal{H}$ there exists a smooth 1-parametric family $Q_\varepsilon$ of varieties $Q(H_{(\theta,0)}, H_{(0,\tau)})$ such that its limit

$$\lim_{\varepsilon \to 0} Q_\varepsilon \equiv Q_0(H_{0(\theta,0)}, H_{0(0,\tau)})$$

satisfies the conditions (1)-(2).

The main difference with the canonical case is that the intersection point algebra $H_\varepsilon(0,0)$ may be noncommutative and noncocommutative for $\varepsilon \neq 0$. From this point of view the canonical dualization is the dualization with respect to the Abelian and coAbelian Hopf algebra. While in the quantization of a Hopf pair no such restrictions on $H_\varepsilon(0,0)$ are imposed. The parameters $\theta$ and $\tau$ are dual with respect to the common limit $H_{0,0}$ of the boundary curves $H_{(\theta,0)}$ and $H_{(0,\tau)}$. The pair of vector fields $(V_\varepsilon(\theta), W_\varepsilon(\tau))$ plays here the role of a Lie-bialgebra. In the limit $\varepsilon \to 0$ the pair $(V_0(0), W_0(0))$ becomes a Lie-bialgebra.

The existence of the deformation quantization of a Hopf pair is tightly connected with the contraction properties of Hopf algebras. In [3] it was demonstrated how the deformation parameters of the quantum group can be dualized with the quantization parameters of the corresponding quantum algebra. The algebra $U_\phi G_{0,k_2}$ was constructed to illustrate this effect:

$$[P_3, P_1] = \frac{e^{i\phi P_3} - e^{-i\phi P_3}}{2i\phi},$$

$$[P_3, P_2] = -k_2 P_1,$$

$$[P_1, P_2] = 0.$$  \hspace{1cm} (30)

$$\Delta(P_{1,3}) = e^{-i\phi P_3/2} \otimes P_{1,3} + P_{1,3} \otimes e^{i\phi P_3/2},$$

$$\Delta(P_2) = P_2 \otimes 1 + 1 \otimes P_2.$$  \hspace{1cm} (31)

The authors emphasise that in the dual Hopf algebra $(U_\phi G_{0,k_2})^*$ the contraction parameter $k_2$ measures the deformation of coproduct and thus obtain the features of the quantization parameter.

From our point of view it must be also stressed that in the limit

$$\lim_{k_2 \to 0} U_\phi G_{(0,k_2)} = U_\phi G_{(0,0)}$$
we do not get the classical algebra of functions on the dual group. The co-ordinate functions remain noncommutative. So the parameters $k_2$ and $\phi$ are not canonically dual.

The situation becomes clear in terms of quantization of a Hopf pair. The contraction curves $\text{Fun}_\phi(G^*_\phi) \equiv H_{(0,\phi)}$ and $UG_{(0,k_2)} \equiv H_{(k_2,0)}$ have the common limit $UG_{(0,0)} \equiv H_{(0,0)}$. Here $H_{(0,0)}$ is the Hiesenberg algebra. The union $\text{Fun}_\phi(G^*_\phi) \cup UG_{(0,k_2)}$ form the boundary of the 2-dimensional variety $Q(H_{(k_2,0)},H_{(0,\phi)})$. The vector fields $V$ and $W$ are trivial in the sense that their projections $V_{\downarrow G\wedge G\to G}$ and $W_{\downarrow G\to G\wedge G}$ do not depend on $k_2$ and $\phi$ respectively. The first deforming function 

$$[P_3, P_2]_{(1)} = -P_1$$

with the only nonzero co-ordinate $V_{[3,2]}^1 = -1$ generates 

$$V(\phi) = V(0)$$

and the first deforming function 

$$\Delta_{(1)} P_{1,3} = -i/2 (P_2 \otimes P_{1,3} - P_{1,3} \otimes P_2)$$

with nontrivial co-ordinates $W_{1}^{[2,1]} = -i/2$, $W_{3}^{[2,3]} = -i/2$ generates the field 

$$W(k_2) = W(0).$$

Notice that here the pair $(V_{\downarrow G\wedge G\to G}, W_{\downarrow G\to G\wedge G})$ is just a Lie-bialgebra. This fact is tightly connected with the possibility to fulfil the condition $(3')$. The family $Q_\varepsilon$ is obtained by a simple transformation of generators 

$$P_{1,3} = p_{1,3}/\varepsilon, \quad P_2 = p_2$$

that doesn’t touch the tangent vectors $(V, W)$. In terms of new generators $p_i$ the left-hand side of the first commutator in (31) acquires the multiplier $\varepsilon^2$ and in the limit $\varepsilon \to 0$ the 2-dimensional subvariety $Q_0$ will have all the properties characteristic to the canonical deformation quantization scheme (see the conditions (1)-(3)):

$$
\begin{align*}
[p_3, p_1] &= 0, \\
\Delta(p_{1,3}) &= e^{-i\phi p_2/2} \otimes p_{1,3} + p_{1,3} \otimes e^{i\phi p_2/2}, \\
[p_3, p_2] &= -k_2 p_1, \quad \Delta(p_2) = p_2 \otimes 1 + 1 \otimes p_2, \\
[p_1, p_2] &= 0.
\end{align*}
$$

We come to the conclusion that the variety described by (30) and (31) is the deformation of a Hopf pair $(H_{(k_2,0)}, H_{(0,\phi)})$. The parameters $\phi$ and $k_2$ are dual with respect to $H_{\varepsilon(0,0)}$ the universal enveloping algebra of the Hiesenberg algebra. In the limit $\varepsilon \to 0$ they become canonically dual.
It is obviously clear that considering $U_{\phi} G_{(0,k_2)}$ with fixed $k_2$ just as the canonical deformation quantization of $U G_{(0,k_2)}$, one can easily find the parameter $\psi$ canonically dual to $\phi$. The standard procedure described in section 1 leads to the following variety $Q(G_{(0,k_2)}, e(2))$,

$$\begin{align*}
[P_3, P_1] &= \frac{\psi}{2i\phi}(e^{i\phi}P_2 - e^{-i\phi}P_2), \\
[P_3, P_2] &= -\psi k_2 P_1, \\
[P_1, P_2] &= 0, \\
\Delta P_{1,3} &= e^{-i/2\phi}P_2 \otimes P_{1,3} + P_{1,3} \otimes e^{i/2\phi}P_2, \\
\Delta P_2 &= P_2 \otimes 1 + 1 \otimes P_2.
\end{align*}$$

In the considered example of the quantization of a Hopf pair one of the contraction curves $-H_{\varepsilon(k_2,0)}$ remains canonical (with a primitive coproduct) even for $\varepsilon \neq 0$. The reason is that the contraction described by the parameter $k_2$ does not touch the coproducts of basic coordinate functions. In terms of the quantization of a Hopf pair this means that only one of the contractions, namely the Fun($G^*_\phi$) is "lifted" by the $\varepsilon$-deformation nontrivially. In the subsection 3.3 we demonstrate an example where both contraction lines are nontrivially lifted.

### 3.2 Lie-Poisson structures in a case of quantized pair

In the deformation quantization of a Lie bialgebra the quantum algebra $U_h(A_t)$ refers to the initial Lie-Poisson structure as to the Poisson-Hopf algebra $(U(A_t), A^*)$, where the Poisson comultiplication is described by a Lie structure of $A^*$. In construction of a Hopf algebra $U_h(A_t)$ the comultiplication is given a preference – it is quantized first. The deformations of commutation relations play an auxiliary role. It starts when the first order deformation with respect to parameter $h$ is already performed in $\Delta_{A'}$’s by $A^*$.

Considering $U_h(A_t)$ as a quantum group Fun$_t(G^*_h)$ one obtains the dual scenario. Now one has a Lie-Poisson group (Fun($G^*_h$), $A$). The multiplication is quantized first and the deformation of the costructure is auxiliary. For example, when the group $G^*_h$ is Abelian the noncommutative co-ordinates can always be introduced without any deformation of $G^*_h$.

Having the applications in mind one can consider such a Lie-Poisson structure $A^*$ on Fun($G_t$) that do not form a Poisson-Hopf algebra with $U(A)$, but nevertheless the simultaneous Hopf deformations of both multiplication and comultiplication exist. In this case the first order deforming function of $U(A_t)$ will have the nontrivial multiplication constituents, that naturally may not form a Lie algebra themselves. They depend on $t$ and tend to zero when $t \to 0$ if $t$ is still dual to $h$ with respect to $U(\text{Ab})$. This means that the desired Poisson properties can be imposed on the group $G$ only together with the symmetry deformation. In
other words the Poisson structure on Fun($G_t$) defined by $A'$ does not perform the group $G_t$ into a Lie-Poisson group but does it for a certain infinitesimal deformation $(G_t)_h$. From the Hopf pair quantization point of view this case is based on such a deformation quantization where the field $V(t)$ (or both $V(t)$ and $W(h)$) has not only comultiplicative but also multiplicative nontrivial parts. The Lie bialgebra is reobtained in the limit $t, h \to 0$ (see (28), (29)).

It must be noticed that the canonical scheme is stable (in the sense of generalisation described above) for semisimple algebras $A$ (or $A^*$). For them a first deforming function can be always set to zero by a similarity transformation of $A$. But for physical applications the nonsemisimple algebras play an important role and for them a deformation with a nontrivial first order may exist.

The other possibility is to consider deformations of quantum algebras induced by deformations of classical ones [3]. The example considered in the 3.1 illustrates this case. There the Hopf algebra $H_{(0,0)}$ is nontrivial but classical Lie universal enveloping. In general the pair of deforming functions $(V(0), W(0))$ must not form a Lie bialgebra in such a case. The Lie bialgebra structure must be obtained as a limit for $\varepsilon \to 0$ if the condition (3') is fulfilled. This is the case when the Poisson structure that form a Poisson-Hopf algebra with $A$ and gives rise to a global deformation is applicable also for a classical deformation $A_t$ with the analogous results.

The general case described by the conditions (1')-(3') can be considered as a combination of these two possibilities. Here instead of Poisson structures on Lie group $G$ a first deforming function of some initial Poisson structure on $G$ appears. And instead of constructing a Lie structure for the initial multiplication described by $A$ and supposed to become Poisson we see a deformation problem for some already existing Lie structure, compatible with $A$. The details of this situation are explicitly demonstrated in the following subsection.

### 3.3 Example of nontrivial lifting

To obtain a nontrivial lifting for both multiplications and comultiplications one must chose quantum algebras that have nontrivial deformations. A favourable situation may be found in case of Hopf algebras obtained as Drinfeld doubles [1].

Let us construct the double $D$ for the Hopf algebras $U_h(sl_t(2, \mathbb{C}))$ and $\text{Fun}_h(\tilde{E}_t(2))$ (see (22), (24)). To make the compositions more transparent The modified notations for the basic elements,

\[
\{L, M, N, \lambda, \mu, \nu \}
\]

\[
\downarrow
\]

\[
\{L, X_+, X_-, H, Y_+, Y_- \}
\]
make the compositions of the double more transparent:

\[ [L, X_{\pm}] = \pm hX_{\pm}, \]
\[ [X_{+}, X_{-}] = h^2 e^{tL} - e^{-tL}, \]
\[ [H, Y_{\pm}] = -tY_{\pm}, \]
\[ [Y_{+}, Y_{-}] = 0, \]
\[ [H, L] = 0, \]
\[ [H, X_{\pm}] = tX_{\pm} \pm \frac{2th^2}{1-e^{-th}} Y_{\pm}, \]
\[ [Y_{\pm}, L] = \pm hY_{\pm}, \]
\[ [Y_{\pm}, X_{\pm}] = (\mp e^{\pm tL} \pm e^{-hH}) \]
\[ [Y_{\pm}, X_{\mp}] = \pm h^2 Y_{\pm}^2 + (e^{-th} - 1)X_{\mp}Y_{\pm}, \]
\[ \Delta L = L \otimes 1 + 1 \otimes L, \]
\[ \Delta X_{+} = X_{+} \otimes e^{tL} + 1 \otimes X_{+}, \]
\[ \Delta X_{-} = X_{-} \otimes 1 + e^{-tL} \otimes X_{-}, \]
\[ \Delta H = H \otimes 1 + 1 \otimes H - 2t \sum_{n=1}^{\infty} \frac{(-h^2(Y_{-} \otimes Y_{+}))^n}{1-e^{-nth}}, \]
\[ \Delta Y_{+} = Y_{+} \otimes 1 + (e^{-hH} \otimes Y_{+})(1 \otimes 1 + h^2 Y_{-} \otimes Y_{+})^{-1}, \]
\[ \Delta Y_{-} = 1 \otimes Y_{-} + (Y_{-} \otimes e^{-hH})(1 \otimes 1 + h^2 Y_{-} \otimes Y_{+})^{-1}. \]

The costructure of this quantum algebra describes the direct product \( \tilde{E}_t(2) \times SL_h(2, \mathbb{C}) \).

Consider a pair of new parameters \((\tau, \theta)\) that describe the contractions

\[
\begin{array}{c}
SL_h(2, \mathbb{C}) \overset{\text{contract}}{\tau \rightarrow 0} E_h(2, \mathbb{C}) \\
sl_h(2, \mathbb{C}) \overset{\text{contract}}{\theta \rightarrow 0} e_h(2, \mathbb{C})
\end{array}
\]

lifted to the variety \( \mathcal{H} \) of quantum algebras with six generators. After the necessary reparametrization of the type described in section 1 we get the two-dimensional family \( D_{\tau,\theta} \) (on this stage of construction the parameters \( t \) and \( h \) may be fixed). The elements of this variety differ from the initial Hopf algebra \( D \) in the following compositions:

\[ [X_{+}, X_{-}] = \tau \theta^2 h^2 e^{tL} - e^{-tL}, \]
\[ [H, X_{\pm}] = tX_{\pm} \pm \tau \theta \frac{2th^2}{1-e^{-th}} Y_{\pm}, \]
\[ [Y_{\pm}, X_{\pm}] = \theta(\mp e^{\pm tL} \pm e^{-hH}) \]
\[ [Y_{\pm}, X_{\mp}] = \pm \tau \theta h^2 Y_{\pm}^2 + (e^{-th} - 1)X_{\mp}Y_{\pm}, \]
\[ \Delta H = H \otimes 1 + 1 \otimes H - 2t \sum_{n=1}^{\infty} \frac{(-h^2(Y_{-} \otimes Y_{+}))^n}{1-e^{-nth}}, \]
\[ \Delta Y_{+} = Y_{+} \otimes 1 + (e^{-hH} \otimes Y_{+})(1 \otimes 1 + \tau h^2 Y_{-} \otimes Y_{+})^{-1}, \]
\[ \Delta Y_{-} = 1 \otimes Y_{-} + (Y_{-} \otimes e^{-hH})(1 \otimes 1 + \tau h^2 Y_{-} \otimes Y_{+})^{-1}. \]
The smooth subvariety $D_{\theta,\tau}$ contains two nontrivial contraction curves $D_{\theta,0}$ and $D_{0,\tau}$ having the noncommutative and noncocommutative Hopf algebra $D_{0,0}$ as the common limit:

\[
\begin{align*}
[\mathcal{L}, X_\pm] &= \pm h X_\pm, \\
[H, Y_\pm] &= -i Y_\pm, \\
[H, X_\pm] &= t X_\pm, \\
[Y_\pm, X_\mp] &= (e^{-th} - 1) X_\mp Y_\pm,
\end{align*}
\]

(All other pairs of generators commute.)

\[
\begin{align*}
\Delta \mathcal{L} &= \mathcal{L} \otimes 1 + 1 \otimes \mathcal{L}, \\
\Delta X_+ &= X_+ \otimes e^{tL} + 1 \otimes X_+, \\
\Delta X_- &= X_- \otimes 1 + e^{-tL} \otimes X_-, \\
\Delta H &= H \otimes 1 + 1 \otimes H, \\
\Delta Y_+ &= Y_+ \otimes 1 + e^{-hH} \otimes Y_+, \\
\Delta Y_- &= 1 \otimes Y_- + Y_- \otimes e^{-hH}.
\end{align*}
\]

The picture of the smooth subvariety $D_{\theta,\tau}$ and its boundary looks like follows:

The points of the curve $D_{(\theta,0)}$ differ from that of $D_{(0,0)}$ by a single commutator

\[
[Y_\pm, X_\mp] = \theta(\mp e^{\pm tL} \pm e^{-hH})).
\]

For the contraction curve $D_{(0,\tau)}$ the coproducts $\Delta H, \Delta Y_\pm$ have the initial form (36) while the multiplication coincides with that of $D_{(0,0)}$ (see (11)). The vector field $V(\theta)$ is described by the deformation function:

\[
\begin{align*}
[X_+, X_-]_{(1)} &= \theta^2 h^2 \frac{e^{tL} - e^{-tL}}{1 - e^{-th}}, \\
[H, X_\pm]_{(1)} &= \pm \theta \frac{2th^2}{1 - e^{-th}} Y_\mp, \\
[Y_\pm, X_\mp]_{(1)} &= \pm \theta h^2 Y_\mp^2 + (e^{-th} - 1) X_\mp Y_\pm,
\end{align*}
\]

\[
\begin{align*}
(\Delta - \Delta^{opp})_{(1)} H &= 2th^2 \frac{Y_- \otimes Y_+ - Y_+ \otimes Y_-}{1 - e^{-th}}, \\
(\Delta - \Delta^{opp})_{(1)} Y_\pm &= \mp h^2[(e^{-hH} \otimes Y_\mp)(Y_\mp \otimes Y_\pm) - (Y_\pm \otimes e^{-hH})(Y_\mp \otimes Y_\pm)].
\end{align*}
\]
In the vector field $W(\tau)$ only some antisymmetric multiplication structure constants on $A \wedge A$ are different from zero,

\begin{align}
[H, X_{\pm}] &= \pm \tau \frac{2iL^2}{1-e^{-i\tau}} Y_{\pm}, \\
[Y_{\pm}, X_{\mp}] &= \pm \tau h^2 Y_{\pm}^2, \\
[Y_{\pm}, X_{\mp}] &= (\mp e^{\pm tL} \pm e^{-hH}).
\end{align}

(48)

Both $V(\theta)$ and $W(\tau)$ nontrivially depend on the co-ordinates of contraction curves. The limit values $V(0)$ and $W(0)$ are defined by the relations (47) and (49) respectively.

To complete the construction we must find such an $\varepsilon$-dependent family $D_\varepsilon$ of 2-dimensional varieties that in the limit $\varepsilon \to 0$ satisfies the canonical conditions (1)-(2). This may be done using the parameters $h$ and $t$. In the structure relations for $D_{\theta,\tau}$ we perform the substitution:

\begin{align}
h &\Rightarrow \varepsilon^2, \\
t &\Rightarrow \varepsilon^2, \\
X_{\pm} &\Rightarrow \varepsilon X_{\pm}, \\
Y_{\pm} &\Rightarrow \varepsilon Y_{\pm}
\end{align}

(50)

and obtain in the limit $\varepsilon \to 0$ the following compositions. For the internal points of $D_{\varepsilon(\theta,\tau)}$:

\begin{align*}
D_0(\theta, \tau) \begin{cases}
[X_+, X_-] &= 2\tau \theta^2 L, \\
[Y_{\pm}, X_{\mp}] &= -\theta(L \mp H), \\
\Delta &\leftarrow \text{primitive,}
\end{cases}
\end{align*}

(51)

and for the points of the boundary:

\begin{align*}
D_0(\theta, 0) \begin{cases}
[Y_\pm, X_\pm] &= -\theta(L \mp H), \\
\Delta &\leftarrow \text{primitive,}
\end{cases}
\end{align*}

(52)

\begin{align*}
D_0(0, \tau) \begin{cases}
&\text{Abelian,} \\
&\Delta \leftarrow \text{primitive.}
\end{cases}
\end{align*}

(53)

\begin{align*}
D_0(0, 0) \begin{cases}
&\text{Abelian,} \\
&\Delta \leftarrow \text{primitive.}
\end{cases}
\end{align*}

(54)

All the Hopf algebras here are the classical universal enveloping and the points of the curve $D_{\varepsilon(0,\tau)}$ are found to be trivialised. The deformation diagram obtains the form

\begin{align*}
D_{\theta', \tau'} \quad | \\
\quad \theta \to 0 \\
\quad \tau \to 0 \\
\quad D_{\theta', 0} \\
\quad D_{0, 0} \approx U(Ab)
\end{align*}

(55)

It reflects the fact that the algebra $D_{0(\theta,\tau)}$ – a two-step classical first order deformation (of Abelian algebra) – can also be treated as the second order deformation for each $\tau \neq 0$.

So the limit $\varepsilon \to 0$ leads us to the degenerated case. The quantization removes this degeneracy.
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5 Appendix

The transformation $B(t)$ and the reparametrization lead to the following 2-parametric variety $Q(sl(n, C), sl^*(n, C))$ for quantum $sl(n, C)$ algebras. (We use here the basis introduced in [7].)

\[
[H_i, H_j] = 0, \quad i, j = 1, \ldots, n - 1;
\]

\[
[H_i, X_{\pm(j,j+1)}] = \pm t\alpha_{ij} X_{\pm(j,j+1)},
\]

\[
[X_{\pm(i,i+1)}, X_{\pm(j,j+1)}] = t^2 \delta_{ij} e^{(ht/2)\delta_{i+1,n} - e^{-ht/2}} \delta_{i+1,n-1} - e^{-ht/2} - e^{ht/2},
\]

\[
[X_{\pm(i,i+1)}, X_{\pm(j,j+1)}] = 0, \quad \text{for } |i - j| > 1;
\]

\[
\Delta H_i = H_i \otimes 1 + 1 \otimes H_i,
\]

\[
\Delta X_{\pm(i,i+1)} = X_{\pm(i,i+1)} \otimes e^{-(ht/2)H_{i+1,n-1}} + e^{-(ht/2)H_{i,n-1}} \otimes X_{\pm(i,i+1)},
\]

\[
S(H_i) = -H_i,
\]

\[
S(X_{\pm(i,i+1)}) = -e^{H_{i,n-1}/2} X_{\pm(i,i+1)} e^{H_{i,n-1}/2},
\]

\[
\varepsilon(X_{\pm(i,i+1)}) = \varepsilon(H_i) = 0
\]

Here $\alpha_{ij}$ is the Cartan matrix and

\[
H_{i,n-1} = H_{i,i+1} + H_{i+1,i+2} + \cdots + H_{n-2,n-1}.
\]

The modified ($\pm$)-co-ordinate functions were applied so that the relation

\[
t e^{\pm h t/2(1+\delta_{i,j})} X_{\pm(i,j)} e^{-h H_{i+1,n-1}/2} = [X_{\pm(i,i+1)}, X_{\pm(i+1,j)}]_{\exp(\pm(ht/2)\delta_{i,j})}.
\]

holds. In these terms the dual group $SL^*(n, C)$ has the simplest form. The coproduct for an arbitrary basic element $X_{\pm(i,j)}$ looks like

\[
\Delta X_{\pm(i,j)} = X_{\pm(i,j)} \otimes e^{-h H_{i,n-1}/2} + e^{-h H_{i,n-1}/2} \otimes X_{\pm(i,j)}
\]

\[
+ \frac{(1 - e^{ht})}{t} \sum_{k=i+1}^{j-1} X_{\pm(i,k)} \otimes X_{\pm(k,j)}.
\]
In the limit $t \to 0$ it describes the compositions of the solvable group $SL^*(n, \mathbb{C})$:

\[
\begin{align*}
\Delta H_i &= H_i \otimes 1 + 1 \otimes H_i, \\
\Delta X_{\pm(i,j)} &= X_{\pm(i,j)} \otimes e^{-\hbar H_{j,n-1}/2} + e^{-\hbar H_{i,n-1}/2} \otimes X_{\pm(i,j)} \\
&\pm \hbar \sum_{k=i+1}^{j-1} X_{\pm(i,k)} \otimes X_{\pm(k,j)}.
\end{align*}
\]

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