SHARP EQUALITY CONDITION ABOUT RESTRICTION FORMULA OF JUMPING NUMBERS

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Abstract. In the present article, we establish the equality condition about the restriction formula of jumping numbers by giving a sharp lower bound of the dimension of the support of a related coherent sheaf. As an application, we obtain a sharp equality condition about a generalized version of the fundamental subadditivity property of complex singularity exponents. We also obtain a sharp upper bound and a semicontinuity property of jumping numbers, the slicing result of complex singularity exponents, and a relationship between the fibrewise Bergman kernels and integrability.

1. Introduction

1.1. Restriction formula of jumping numbers and some consequences of Demailly’s strong openness conjecture. Let Ω be a domain in \( \mathbb{C}^n \) and \( o \in \Omega \). Let \( u \) be a plurisubharmonic function on \( \Omega \). Nadel [30] introduced the multiplier ideal sheaf \( I(u) \) which can be defined as the sheaf of germs of holomorphic functions \( f \) such that \( |f|^2 e^{-2u} \) is locally integrable. Here \( u \) is regarded as the weight of \( I(u) \).

The complex singularity exponent is defined as follows (see [40], see also [7, 9]) (or log canonical threshold in algebraic geometry see [33, 28]):

**Definition 1.1.** Let \( o \in \Omega \). The complex singularity exponent is defined to be

\[
c_o(u) := \sup\{c \geq 0 : I(cu)_o = O_o\}.
\]

Since \( I(c_o(u))_o \) is a coherent ideal sheaf [30], then it follows that \( \{z | c_z(u) \leq c_o(u)\} = \text{Supp}(O/I(c_o(u))_o) \) is an analytic set by the openness conjecture, i.e., \( I(c_z(u))_z \neq O_z \), which was posed by Demailly and Kollár (see [12]) and proved by Berndtsson [3].

Let \( I \subseteq O_o \) be an coherent ideal. As a generalization of the complex singularity exponents, the jumping number is defined as follows (see e.g. [25, 26]):

**Definition 1.2.** The jumping number is defined to be

\[
c'_I(u) := \sup\{c \geq 0 : I(cu)_o \supseteq I\}.
\]

Especially, when \( I = O_o \), the jumping number degenerates to the complex singularity exponent.

In [11] (see also (14.1) in [8]), the following restriction formula of multiplier ideal sheaves has been established by using Ohsawa-Takegoshi \( L^2 \) extension theorem:
Restriction formula (multiplier ideal). For any regular complex submanifold \((H, o) \subset (\mathbb{C}^n, o)\), one can obtain
\[
\mathcal{I}(u|_H) \subseteq \mathcal{I}(u)|_H. \tag{1.1}
\]

Using our solution of Demailly’s strong openness conjecture, i.e., \(\mathcal{I}(\tilde{c}_o^I(u)) \supseteq \mathcal{I}\), which was posed in \([7, 8]\) and proved in \([20, 21, 22]\), it follows that inequality \((1.1)\) is equivalent to

Restriction formula (jumping number). For any regular complex submanifold \((H, o) \subset (\mathbb{C}^n, o)\), one can obtain
\[
c_o(u|_H) \leq \sup\{c_o(\tilde{u}^I|_H) \subseteq \mathcal{O}_o^\mathcal{I} \}
\]

By our solution of Demailly’s strong openness conjecture, one can also obtain the analogue of inequality \((1.2)\)
\[
c_o(u \circ f) \leq \sup\{c_o(\tilde{u}^I|_H) \subseteq \mathcal{O}_o^\mathcal{I} \}
\]

which is equivalent to the comparison of the multiplier ideals \(\mathcal{I}(u \circ f) \subseteq \mathcal{I}(u)\) (see \([11]\), see also (14.3) in \([8]\)), where \(f\) is an holomorphic map.

When \(\mathcal{I}(u|_H)_o = \mathcal{O}_o\), the restriction formula about jumping numbers degenerates to the following restriction formula ("important monotonicity result") about complex singularity exponents in \([12]\):

Proposition 1.3. \([12]\) For any regular complex submanifold \((H, o) \subset (\mathbb{C}^n, o)\),
\[
c_o(u|_H) \leq c_o(u) \tag{1.3}
\]

holds, where \(u|_H \neq -\infty\).

In \([11]\) (see also Theorem (14.2) in \([8]\)), the following subadditive theorem of jumping numbers has been presented

Theorem 1.4. Subadditivity Theorem
(a) \(\pi_i := \Omega_1 \times \Omega_2 \rightarrow \Omega_i\) \(i = 1, 2\) the projections, and let \(u_i\) be a plurisubharmonic function on \(\Omega_i\). Then
\[
\mathcal{I}(u_1 \circ \pi_1 + u_2 \circ \pi_2) = \pi_1^* \mathcal{I}(u_1) + \pi_2^* \mathcal{I}(u_2).
\]

(b) Let \(\Omega\) be a domain and let \(u\) and \(v\) be two plurisubharmonic functions on \(\Omega\). Then
\[
\mathcal{I}(u + v) \subseteq \mathcal{I}(u) \cdot \mathcal{I}(v).
\]

By our solution Demailly’s strong openness conjecture, it follows that Theorem \(1.4\) is equivalent to the following result:

Theorem 1.5.
(a) Let \(\Omega_1 \supset o_1\) and \(\Omega_2 \supset o_2\) be two domains, \(\pi_i := \Omega_1 \times \Omega_2 \rightarrow \Omega_i\) \(i = 1, 2\) the projections, and let \(u_i\) be a plurisubharmonic function on \(\Omega_i\). Then
\[
c_o^I(u_1 \circ \pi_1 + u_2 \circ \pi_2) = \sup\{\min\{c_{o_1}(u_1), c_{o_2}(u_2)\}|J_1 \cdot J_2 \supset \hat{I}\}, \tag{1.4}
\]

where \(\hat{I}\) is a coherent ideal in \(\mathcal{O}_{o_1 \times o_2}\), \(J_1\) and \(J_2\) are coherent ideals in \(\mathcal{O}_{o_1}\) and \(\mathcal{O}_{o_2}\) respectively.
Let $\Omega$ be a domain and let $u$ and $v$ be two plurisubharmonic functions on $\Omega \ni o$. Then
\begin{equation}
\label{eq:1.5}
c_{l}^{1}(u + v) \leq \sup \{\min \{c_{o}^{1}(u), c_{o}^{1}(v)\}|I_{1} \cdot I_{2} \supseteq I\}
\end{equation}
where $I_{1}$ and $I_{2}$ are coherent ideals in $\mathcal{O}_{o}$.

1.2. Main theorem: sharp equality condition about restriction formula of jumping numbers. Let $n \geq 2$, $H = \{z_{k+1} = \cdots = z_{n} = 0\}$. Let $u = \log (\sum_{1 \leq j \leq l} |z_{j}|^{2})^{1/2}$ Note that
\begin{enumerate}
\item when $l > k$, $c_{o}(u) = l > k = c_{o}(u|_{H})$;
\item when $l \leq k$, $c_{o}(u) = l = c_{o}(u|_{H})$.
\end{enumerate}
Then it is natural to consider the following problem for the sharp equality condition about the above restriction formula of jumping numbers:

**Problem 1.6.** Suppose that there exists a germ of regular complex submanifold $(H, o) \subset (\mathbb{C}^{n}, o)$ with dimension $k$ such that
\begin{equation}
\label{eq:1.6}
c_{o}^{1}(u|_{H}) = \sup \{c_{o}(u)\} \subseteq O_{o}\&\tilde{I}|_{H} = I\} =: c.
\end{equation}
Can one obtain that
\begin{equation}
\label{eq:1.7}
dim_{o}(\text{Supp}(\mathcal{O}/I(cu))) \geq n - k?
\end{equation}

For the case $I = \mathcal{O}_{o}$ and $(k, n) = (1, 2)$, Problem \[1.6\] was solved by Blel-Mimouni \[4\] and Favre-Jonsson \[16\].

For the case $I = \mathcal{O}_{o}$ and $(k, n) = (1, n)$, Problem \[1.6\] was solved in \[21\]. Recently, combining with the recent result in \[13\] and Proposition 2.4 in \[24\], Rashkovskii reproved the case $I = \mathcal{O}_{o}$ and $(k, n) = (1, n)$ in \[32\].

In the present article, we solve Problem \[1.6\] as follows:

**Theorem 1.7.** (main theorem) Suppose that there exists a germ of regular complex submanifold $(H, o) \subset (\mathbb{C}^{n}, o)$ with dimension $k$ such that equality \[1.6\] holds. Then inequality \[1.7\] holds.

**Remark 1.8.** Note that the points in $\text{Supp}(\mathcal{O}/I(cu)) \cap H$ are not considered in the proof of Theorem \[1.7\], then we obtain a more subtle conclusion:
\begin{equation}
\label{eq:1.8}
dim_{o}(\text{Supp}(\mathcal{O}/I(cu)) \setminus H) \geq n - k.
\end{equation}

Let $I = \mathcal{O}_{o}$, then we obtain the following corollary of Theorem \[1.7\].

**Corollary 1.9.** Suppose that there exists a germ of regular complex submanifold $(H, o) \subset (\mathbb{C}^{n}, o)$ with dimension $k$ such that
\begin{equation}
\label{eq:1.9}
c_{o}(u) = c_{o}(u|_{H}) =: c.
\end{equation}
Then we have
\begin{equation}
\label{eq:1.10}
dim_{o}(\text{Supp}(\mathcal{O}/I(cu))) \geq n - k.
\end{equation}

**Remark 1.10.** When $k = 1$, one can obtain that
\begin{equation}
\label{eq:1.11}\{z|c_{z}(u) \leq c_{o}(u)\} = \text{Supp}(\mathcal{O}/I(cu))
\end{equation}
is regular at $o$ by Siu’s decomposition of positive closed currents \[35\], see also \[9\]. Details are referred to \[21\].

However, when $k > 1$ and $n > 2$, $\{z|c_{z}(u) \leq c_{o}(u)\}$ may not be regular at $o$, e.g. $u := \log |z_{1}| + \log |z_{2}|$ and $H = \{z_{3} = 0\}$, then $\{z|c_{z}(u) \leq c_{o}(u)\} = (\{z_{1} = 0\} \cup \{z_{2} = 0\})$, and $c_{o}(u) = c_{o}(u|_{H}) = 1$. 

(b) Let $\Omega$ be a domain and let $u$ and $v$ be two plurisubharmonic functions on $\Omega \ni o$. Then
\begin{equation}
\label{eq:1.12}
c_{l}^{1}(u + v) \leq \sup \{\min \{c_{o}^{1}(u), c_{o}^{1}(v)\}|I_{1} \cdot I_{2} \supseteq I\}
\end{equation}
where $I_{1}$ and $I_{2}$ are coherent ideals in $\mathcal{O}_{o}$. 

**Remark 1.8.**
Let $C(V_k) := c_o(u|V_k)$ be a function on the Grassmannian $G(k, n)$ of $k$-dimensional subspaces $V_k$ in $\mathbb{C}^n$.

Simulated by Siu’s slicing theorem of Lelong numbers [35], one can reformulate a slicing result of complex singularity exponents, which is implied by the combination of Berndtsson’s log-subharmonicity of Bergman kernels [2] and the openness conjecture:

**Remark 1.11.** There exists $c \in \mathbb{R}^+ \cup \{+\infty\}$ such that $C(V_k) = c$ almost everywhere in the sense of the unique $U(n)$-invariant measure of mass 1 on the Grassmannian $G(p, n)$. Moreover $c$ is the upper bound of $c_o(u|V_k)$ for any $V_k$ (details see Lemma 1.12).

When $k = 1$, it follows that $c_o(V_k) = \frac{1}{\nu(u|V_k, o)}$, where $\nu(u|V_k, o)$ is the Lelong number of $u|V_k$ at $o$. Then Remark 1.11 degenerates to Siu’s slicing theorem on Lelong numbers [35] (see also [9]) when $k = 1$.

1.3. **An application:** Sharp equality condition about a generalized version of the fundamental subadditivity property of the complex singularity exponents. In this subsection, by using Theorem 1.12 we obtain the sharp equality condition about a generalized version of the fundamental subadditivity property of complex singularity exponents.

In [12] (see also (13.17) in [8]), the fundamental subadditivity property of complex singularity exponents has been presented as follows

**Theorem 1.12.** [12] Let $I$ and $J$ be coherent ideals on $O_o$. Let $u = \log |I|$ and $v = \log |J|$. $c_o(\max\{u, v\}) \leq c_o(u) + c_o(v)$.

Motivated by the proof of Theorem 1.12 in [12] (see also (13.17) in [8]), using Theorem 2.9 [21, 23], our solution of a conjecture posed by Demailly and Kollar in [12], we obtain

**Proposition 1.13.** Let $I_1$ and $I_2$ be coherent ideals in $O_{o_1}$ and $O_{o_2}$ respectively. Let $u$ and $v$ be plurisubharmonic functions on $\Omega_1$ and $\Omega_2$ respectively as in Theorem 1.12

$$c_o^{I_1} \times c_o^{I_2}(\max\{u \circ \pi_1 + v \circ \pi_2\}) = c_o^{J_1}(u) + c_o^{J_2}(v).$$

Details of the proof of Proposition 1.13 is in subsection 2.5.

When $I_1 = O_o$ and $I_2 = O_o$, by using Proposition 1.3 we generalize Theorem 1.12 as follows

**Theorem 1.14.** Let $u$ and $v$ be plurisubharmonic functions on $\Delta^n$. Then

$$c_o(\max\{u, v\}) \leq c_o(u) + c_o(v).$$

Note that

(a) if $u = v = \log |z|$, then $c_o(\max\{u, v\}) = \frac{1}{n} = \frac{2}{n} < c_o(u) + c_o(v)$;

(b) if $n \geq 2$, $u = \log |z'|$ and $v = \log |z''|$, then $c_o(\max\{u, v\}) = n = k + (n - k) = c_o(u) + c_o(v)$, where $z' = (z_1, \ldots, z_k)$ and $z'' = (z_{k+1}, \ldots, z_n)$,

then it is natural to consider the following problem for the sharp equality condition about the generalized version of the fundamental subadditivity property of complex singularity exponents:

**Problem 1.15.** Let $u$ and $v$ be plurisubharmonic functions on $\Delta^n$. If

$$c_o(\max\{u, v\}) = c_o(u) + c_o(v) =: c,$$

(1.8)
can one obtain that

\[ \dim_o \text{Supp}(\mathcal{O}/\mathcal{I}(cu)) + \dim_o \text{Supp}(\mathcal{O}/\mathcal{I}(cv)) \geq n? \] (1.9)

Let \( H \) be the diagonal of \( \Delta^n \times \Delta^n \). Using Proposition 1.13 (\( \Omega_1 \sim \Delta^n, \Omega_2 \sim \Delta^n, I_1 \sim O_o, I_2 \sim O_o \)) and Corollary 1.18 (\( u \sim \max\{u \circ \pi_1, v \circ \pi_2\}, n \sim 2n, k \sim n \)), we obtain that

\[ \dim_o \text{Supp}(\mathcal{O}/\mathcal{I}(c \max\{u \circ \pi_1, v \circ \pi_2\})) \geq n. \]

Note that

\[ \text{Supp}(\mathcal{O}/\mathcal{I}(c \max\{u \circ \pi_1, v \circ \pi_2\})) \subseteq \text{Supp}(\mathcal{O}/\mathcal{I}(cu \circ \pi_1)) \cap \text{Supp}(\mathcal{O}/\mathcal{I}(cv \circ \pi_2)), \]

then we give an affirmative answer to the above problem:

**Theorem 1.16.** If equality (1.8) holds, then inequality (1.9) holds.

1.4. **A sharp upper bound and semicontinuity property of jumping numbers.** Let \( I \subseteq O_o \) and \( J \subseteq I(c_o(u))_o = I(c_o^I(u)) \) be coherent ideals. Using our solution of Demailly’s strong openness conjecture, i.e., \( I(c_o(u))_o \nsubseteq I \), which was posed in \([7, 8]\) and proved in \([20, 21, 22]\). Related effectiveness result is referred to \([23]\), we obtain the following inequality of jumping numbers:

**Theorem 1.17.** \[ \frac{c_o^I(u)}{c_o(u) - c_o^I(u)} \geq c_o^I(\log |J|). \]

Given coherent ideal \( J \subseteq O_o \), and letting \( I = O_o \), we obtain the following sharp upper bound of the jumping numbers represented by the complex singularity exponents:

**Corollary 1.18.** \[ c_o^I(u) \leq \frac{c_o(u)}{c_o(\log |J|)} + c_o(u). \]

The following remark illustrates the sharpness of Corollary 1.18.

**Remark 1.19.** Let \( (z_1, \ldots, z_n) \) be the coordinates of \( \mathbb{C}^n \). Suppose \( u := c \log |z_n| \) and \( J = (z_n^k) \). Then we have \( c_o^I(u) = \frac{k+1}{c}, c_o(\log |J|) = \frac{1}{k} \) and \( c_o(u) = \frac{1}{c} \). This gives the sharpness of Corollary 1.18.

More general, when \( J \) is principal ideal (i.e., \( J = (f)_o \)), and \( u = c \log |f| \), then we have \( c_o^I(u) = \frac{1+c_o(\log |f|)}{c} \) and \( c_o(u) = \frac{c_o(\log |f|)}{c} \). This implies the sharpness of Corollary 1.18.

Let coherent sheaf \( \mathcal{F} \subseteq O \) be an ideal sheaf.

By the definition of \( c_{\mathcal{F}}^z(u) \), it follows that

\[ c_{\mathcal{F}}^z(u) > p \implies \mathcal{F} \subseteq \mathcal{I}(pu)_z. \] (1.10)

By the solution of Demailly’s strong openness conjecture, it follows that

\[ c_{\mathcal{F}}^z(u) \leq p \implies \mathcal{F} \nsubseteq \mathcal{I}(pu)_z. \] (1.11)

Using inequalities (1.10) and (1.11) and the fact that the supports of coherent analytic sheaves are analytic subsets, one can obtain the following

**Remark 1.20.** The lowerlevel set of jumping numbers

\[ \{ z | c_{\mathcal{F}}^z(u) \leq p \} = \text{Supp}(\mathcal{F}/(\mathcal{F} \cap \mathcal{I}(pu))) \]

is an analytic subset.
1.5. **Organization of the paper.** The paper is organized as follows. In the present section, we recall the statement of the restriction formula of jumping numbers and present the main results of the present paper: the sharp equality condition about the restriction formula of jumping numbers (Theorem 1.7, main theorem); sharp equality conditions about the fundamental subadditivity property of complex singularity exponents (Theorem 1.16, an application of the main theorem); a sharp upper bound of jumping numbers (Corollary 1.18 of Theorem 1.17); a semicontinuity property of jumping numbers (Remark 1.20); the slicing result of complex singularity exponents (Remark 1.11). In Section 2, we recall or give some preliminary results used in the proof of the main theorem and applications. In Section 3, we prove the main theorem (Theorem 1.7). In Section 4, we prove the a sharp upper bound of jumping numbers (Corollary 1.18 of Theorem 1.17). In section 5, we present a relationship between the fibrewise Bergman kernels and integrability.

2. **Some preparatory results for the proof of the main theorem and applications**

In this section, we recall and present some preparatory results for the proof of the main theorem and applications.

2.1. **Ohsawa-Takegoshi $L^2$ extension theorem of Manivel-Demailly type.**

We recall the Ohsawa-Takegoshi $L^2$ extension theorem of Manivel-Demailly type as follows:

**Theorem 2.1.** ([29, 6], see also [9, 8]) Let $D$ be a bounded pseudo-convex domain in $\mathbb{C}^{k+1}$. Let $u$ be a plurisubharmonic function on $D$. Let $H = \{z_k+1 = 0\}$ be a complex hyperplane in $\mathbb{C}^n$. Then for any holomorphic function on $H \cap D$ satisfying

$$\int_{H \cap D} |f|^2 e^{-2u} d\lambda_H < +\infty,$$

there exists a holomorphic function $F$ on $D$ satisfying $F|_{H \cap D} = f$, and

$$\int_D |F|^2 e^{-2u-2a\log|z_{k+1}|} d\lambda_n \leq C_{D,a} \int_{H \cap D} |f|^2 e^{-2u} d\lambda_H,$$

where $a \in [0,1)$, $C_{D,a}$ only depends on the diameter of $D$ and $a$, and $d\lambda_H$ is the Lebesgue measure on $H$.

The optimal estimates version of Theorem 2.1 could be referred to [18, 19].

2.2. **Strict inequality about jumping numbers.** Let $\Omega \ni o$ be a domain in $\mathbb{C}^{k+1}$. Let $I \subseteq O_o$ be a coherent ideal.

Let $v$ be a plurisubharmonic function on $\Delta^{k+1}$ with coordinates $(z_1, \cdots, z_k, z_{k+1})$. Using Theorem 2.1 we obtain the following

**Lemma 2.2.** If $c_o^I(v|_{\{z_{k+1}=0\}}) = 1$, then for any $N > 0$,

$$\sup_{\tilde{I} \subseteq O_o, i(z_{k+1}=0)=I} \{c_o^\tilde{I}(\frac{1}{2} \log(e^{2u} + |z_{k+1}|^{2N}))\} > 1$$

holds, where $\tilde{I} \subseteq O_o$ is coherent ideal, $o$ is the origin in $\mathbb{C}^{k+1}$. 


Proof. By Hölder inequality, it follows that $e^{2(1-\varepsilon)v}|z_{k+1}|^{2}\varepsilon N \leq (1-\varepsilon)e^{2v+\varepsilon}|z_{k+1}|^{2N}$, which implies

$$
\frac{1}{e^{2v}+|z_{k+1}|^{2N}} \leq e^{-(1-\varepsilon)v}|z_{k+1}|^{-2\varepsilon N},
$$

(2.1)

where $\varepsilon \in (0, 1)$. As $c_{l}'(v|_{z_{k+1}=0}) = 1$, then $|I|e^{-2(1-\varepsilon)v|_{z_{k+1}=0}}$ is integrable near $o$.

By Theorem 2.4 (u \sim (1-\varepsilon)v, a \sim \varepsilon N, f \sim I) and choosing $\varepsilon \in (0, \frac{1}{2})$, it follows that there exists $\tilde{l}$ such that $|\tilde{l}|e^{-2(1-\varepsilon)v|_{z_{k+1}=0}}$ is locally integrable near $o$. Using inequality (2.1) we obtain that $|\tilde{l}|e^{-2(1-\varepsilon)v|_{z_{k+1}=0}}$ is locally integrable near $o$. Using our solution of Demailly’s strong openness conjecture, we obtain the present lemma. \qed

Note that $c_{o}(\frac{1}{2}\log(e^{2v}+|z_{k+1}|^{2N})) = c_{o}(\max\{v, N \log |z_{k+1}|\})$ for any $N > 0$ ($\Leftarrow e^{2\max\{v, N \log |z_{k+1}|\}} \leq e^{2v}+|z_{k+1}|^{2N}$), then it follows that

Corollary 2.3. If $c_{l}'(v|_{z_{k+1}=0}) = 1$, then

$$
\sup_{\tilde{l} \in (z_{k+1}=0)=I} \{c_{l}'(\max\{2v, \log |z_{k+1}|^{N}\}) > 1
$$

for any $N > 0$.

2.3. A useful proposition in [24]. In [24], using Demailly’s idea of equisingular approximations of quasiplusharmonic functions (see [8], see also [10]) and our solution of Demailly’s strong openness conjecture, we have obtained the following proposition:

Proposition 2.4. [24] Let $D$ be a bounded domain in $\mathbb{C}^{n}$, and the origin $o \in D$. Let $u$ be a plurisubharmonic function on $D$. Let $(g_{j})$ be a (finite) local basis of $\mathcal{I}(u)_{o}$. Then there exists $l > 1$ such that $e^{-2u} - e^{-2\max\{u, -\sum_{j=1}^{l} \log |g_{j}|\}}$ is integrable on a small enough neighborhood $V_{o}$ of $o$.

Let coherent ideal $l \subset \mathcal{I}(u)_{o}$ and $(h_{j})$ be the basis of $l$. Using $(h_{j})$ instead of $(g_{j})$ in the proof of Proposition 2.4 in [24], one can obtain

Remark 2.5. For any $l \subset \mathcal{I}(u)_{o}$, we have $e^{-2u} - e^{-2\max\{u, -\sum_{j=1}^{l} \log |h_{j}|\}} < +\infty$, where $|l| = \sum_{j} |h_{j}|$, and $l \in (1, c_{l}'(u))$ is the positive number in Proposition 2.4.

Let $n = k + 1$. It is well-known that if $\{z|\mathcal{I}(u)_{z} \neq \mathcal{O}_{z}\} \subset \{z_{k+1} = 0\}$, then there exists $N_{0} > 0$ large enough such that $(z_{k+1}^{N_{0}})_{o} \subset \mathcal{I}(u)_{o}$.

Corollary 2.6. If $c_{l}'(u) \leq 1$ ($\Rightarrow |J|e^{-2u}$ is not integrable near $o$) by using our solution of Demailly’s strong openness conjecture, then

$$
c_{l}'(\max\{u, N \log |z_{k+1}|\}) \leq 1
$$

for any $N \geq \frac{1}{2}N_{0}$ (independent of $J$), where $J \subset \mathcal{O}_{o}$ is a coherent ideal, $l \in (1, c_{l}'(u))$ and $I := (z_{k+1}^{N_{0}})_{o}$. Especially, if $c_{l}'(u) = 1$, then

$$
c_{l}'(\max\{u, N \log |z_{k+1}|\}) = 1.
$$

Proof. Using Remark 2.5 one can obtain that $e^{-2u} - e^{-2\max\{u, -\sum_{j=1}^{l} \log |z_{k+1}|\}}$ is locally integrable near $o$ by letting $l = (z_{k+1}^{N_{0}})_{o}$. As $|J|e^{-2u}$ is not locally integrable.
near $o$ ($\Leftrightarrow c_o'(u) = 1$), then it follows that $e^{-2\max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}}$ is not locally integrable near $o$, which implies $c_o'(\max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}) \leq 1$.

Since $c_o'(\max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}) \geq c_o'(u) = 1$ ($\Leftrightarrow \max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\} \geq u$), then it follows that $c_o'(\max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}) = 1$.

As $N \log |z_{k+1}| \leq \frac{1}{T} N_0 \log |z_{k+1}|$, then it follows that $u \leq \max\{u, N \log |z_{k+1}|\} \leq \max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}$, which implies

$$c_o'(u) \leq c_o'(\max\{u, N \log |z_{k+1}|\}) \leq c_o'(\max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}).$$

Note that $c_o'(u) = c_o'(\max\{u, \frac{1}{T} N_0 \log |z_{k+1}|\}) = 1$, then we obtain the present corollary.

Let $I \subseteq O_o$. Let $J$ be an coherent ideal satisfying $J \subseteq I(c_o'(u)u)$. Let $\{f_j\}_{j=1,2,\ldots,s}$ be a local basis of $J_o$. Denoted by $|J| := \sum_{i=1}^s |f_i|$ and $\tilde{u}_l := \max\{c_o'(u)u, \frac{1}{T} \log |J|\}$, where $l \in (1, \frac{c_o'(u)}{c_o'(u)})$. The proof of Proposition 2.4 in [24] can also imply the following

**Remark 2.7.** For any $l \in (1, \frac{c_o'(u)}{c_o'(u)})$, we have $e^{-2c_o'(u)}u - e^{-2\tilde{u}_l}$ is locally integrable near $o$, which implies $c_o'(\tilde{u}_l) = 1$.

### 2.4. Measure along the fibres

Let $X := \{z_{k+1} = \cdots = z_n = 0\}$. Consider a map $g$ from $\mathbb{C}^n \setminus X \to \mathbb{C}^{n-k-1}$: $q(z_1, \ldots, z_n) = (z_{k+1}, \ldots, z_n)$.

Let $Y$ be an analytic set on $\mathbb{B}^n$ whose complex dimension is smaller than $n-k$. By the same proof as Lemma 2.8 in [24] (methods can be referred to [17] and [5]), one can obtain that

**Lemma 2.8.** For almost all $(z_{k+1}, \ldots, z_n)$, the complex dimension of $q^{-1}(z_{k+1}, \ldots, z_n) \cap Y$ is zero, i.e., $(q^{-1}(z_{k+1}, \ldots, z_n) \cap Y, o) = (X \cap Y, o)$.

**Proof.** Note that the $2(n-k-2)$ dimensional Hausdorff measure on $\mathbb{C}^{n-k-1}$ is positive, and $2(n-k)$ dimensional Hausdorff measure of $Y$ is zero, then it follows that for almost all $(z_{k+1}, \ldots, z_n)$, the 2 dimensional Hausdorff measure of $q^{-1}(z_{k+1}, \ldots, z_n) \cap Y$ is zero, i.e., the complex dimension of $q^{-1}(z_{k+1}, \ldots, z_n) \cap Y$ is zero. Then we obtain the present lemma.

### 2.5. Proof of Proposition 1.13

By the definition of jumping numbers, it follows that for any $\varepsilon > 0$, there exists a neighborhood $U_\varepsilon$ of $o$ and $C_\varepsilon > 0$ such that

$$r^{-2(c_o'(u) - \varepsilon)} \int_{\Delta^n} I_{\{u < \log r\} \cap U_\varepsilon} |I|^2 d\lambda_n < C_\varepsilon$$

holds for any $r > 0$, which implies

$$\liminf_{r \to 0^+} \frac{\log(\int_{\Delta^n} I_{\{u < \log r\} \cap U_\varepsilon} |I|^2 d\lambda_n)}{2 \log r} \geq c_o'(u) - \varepsilon. \quad (2.2)$$

We recall our solution of the conjecture posed by Demailly-Kollar [12],

$$\liminf_{r \to 0^+} (-r^{2c_o'(u)} \int_{\Delta^n} I_{\{u < \log r\} \cap U_\varepsilon} d\lambda_n) > 0,$$

as follows
Theorem 2.9. [21, 23] Let $u$ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$ and $I$ be a coherent ideal in $\mathcal{O}_o$. Then for any neighborhood $U$, there exists $C_\varepsilon > 0$ such that

$$(-r^{2c_o^I(u)}) \int_{\Delta^n} I_{(u < \log r) \cap U} |I|^2 d\lambda_n > C_\varepsilon.$$  

By Theorem [23], it follows that for any neighborhood $U$ of $o$,

$$\limsup_{r \to 0^+} \frac{\log \int_{\Delta^n} I_{(u < \log r) \cap U} |I|^2 d\lambda_n}{2 \log r} \leq c_o^I(u) \tag{2.3}$$

holds.

As $\{\max\{u \circ \pi_1, v \circ \pi_2\} < \log r\} \cap (\pi_1^{-1}(U) \cap \pi_2^{-1}(V)) = \{(u < \log r) \cap U\} \times \{(v < \log r) \cap V\}$, then it follows that

$$\int_{\Delta^n \times \Delta^n} I_{\max\{u \circ \pi_1, v \circ \pi_2\} < \log r} \cap (\pi_1^{-1}(U) \cap \pi_2^{-1}(V)) |(\pi_1^* I) \times |(\pi_2^* J)|^2 d\lambda_{2n}$$

$$= \int_{\Delta^n} I_{(u < \log r) \cap U} |I|^2 d\lambda_n \times \int_{\Delta^n} I_{(v < \log r) \cap V} |J|^2 d\lambda_n \tag{2.4}$$

where $U$ and $V$ are neighborhoods of $o \in \mathbb{C}^n$, $I$ and $J$ are coherent ideals in $\mathcal{O}_o$.

By inequality [22], it follows that for any $\varepsilon > 0$, there exist neighborhoods $U_\varepsilon$ and $V_\varepsilon$ of $o$ such that

$$\liminf_{r \to 0^+} \frac{\log \int_{\Delta^n} I_{(u < \log r) \cap U_\varepsilon} |I|^2 d\lambda_n}{2 \log r} \geq c_o^I(u) - \varepsilon$$

and

$$\liminf_{r \to 0^+} \frac{\log \int_{\Delta^n} I_{(v < \log r) \cap V_\varepsilon} |J|^2 d\lambda_n}{2 \log r} \geq c_o^J(v) - \varepsilon. \tag{2.5}$$

By inequality [23], it follows that

$$c_o^{I \times J}(\max\{u \circ \pi_1, v \circ \pi_2\})$$

$$= \limsup_{r \to 0^+} \frac{\log \int_{\Delta^n \times \Delta^n} I_{\max\{u \circ \pi_1, v \circ \pi_2\} < \log r} \cap (\pi_1^{-1}(U_\varepsilon) \cap \pi_2^{-1}(V_\varepsilon)) |(\pi_1^* I) \times |(\pi_2^* J)|^2 d\lambda_{2n}}{2 \log r}$$

$$\geq \liminf_{r \to 0^+} \frac{\log \int_{\Delta^n} I_{(u < \log r) \cap U_\varepsilon} |I|^2 d\lambda_n}{2 \log r} + \liminf_{r \to 0^+} \frac{\log \int_{\Delta^n} I_{(v < \log r) \cap V_\varepsilon} |J|^2 d\lambda_n}{2 \log r}$$

$$\geq (c_o^I(u) - \varepsilon) + (c_o^J(v) - \varepsilon).$$

By inequality [23], it follows that for any $\varepsilon > 0$, there exist neighborhoods $U_\varepsilon'$ and $V_\varepsilon'$ of $o$ such that

$$\liminf_{r \to 0^+} \frac{\log \int_{\Delta^n \times \Delta^n} I_{\max\{u \circ \pi_1, v \circ \pi_2\} < \log r} \cap (\pi_1^{-1}(U_\varepsilon') \cap \pi_2^{-1}(V_\varepsilon')) |(\pi_1^* I) \times |(\pi_2^* J)|^2 d\lambda_{2n}}{2 \log r}$$

$$\geq c_o^{I \times J}(\max\{u \circ \pi_1, v \circ \pi_2\}) - \varepsilon. \tag{2.6}$$
By inequality \(2.3\), it follows that
\[
c'_o(u) + c'_0(v) \geq \lim_{r \to 0^+} \sup \frac{\log \int_{\Delta_n} \mathbb{1}_{(u < \log r) \cap V_2} |I|^2 d\lambda_n}{2 \log r} + \lim_{r \to 0^+} \sup \frac{\log \int_{\Delta_n} \mathbb{1}_{(v < \log r) \cap V_2} |J|^2 d\lambda_n}{2 \log r}.
\]

By the arbitrariness of \(r\), we obtain Proposition \(2.3\).

3. PROOF OF THEOREM \(1.7\) (MAIN THEOREM)

By changing of the local coordinates, \(H\) can be chosen as the complex linear space \(\{z_{k+1} = \cdots = z_n = 0\}\). It suffices to consider the case \(c'_o(u|H) = 1\) (consider \(c'_o(u)|H\) instead of \(u\)).

By \(\sup\{c'_o(u)\} I \subseteq \mathcal{O} \& I|_H = I\) = 1 \((\Rightarrow |I|^2 e^{-2u}\) is not locally integrable near \(o\)), and Proposition \(2.3\), it follows that
\[
\sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O} \& I|_H = I\} \leq 1,
\]
where \(\tilde{u} := \max\{u, \frac{1}{\log} \sum_j |y_j|\}\). Since \(\tilde{u} \geq u\), which implies
\[
\sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O} \& I|_H = I\} \geq \sup\{c'_o(u)|I \subseteq \mathcal{O} \& I|_H = I\} \geq 1,
\]
then it follows that \(\sup\{c'_o(u)|I \subseteq \mathcal{O} \& I|_H = I\} = 1\).

By the restriction formula for jumping numbers, it follows that \(\sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O} \& I|_H = I\} \geq c'_o(\tilde{u}|H) \geq c'_o(u|H) = 1\) \((\Leftarrow \tilde{u}|_H \geq u|_H\), which implies \(c_o(\tilde{u}|_H) = \sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O} \& I|_H = I\} = 1\).

Let \(Y := \text{Supp}\{z|c_o(\tilde{u}) \leq 1\} = \text{Supp}(\mathcal{O}/I(c'_o(u|H)u))\). We prove Theorem \(1.4\) by contradiction. If not, then \(\dim Y < n-k\). By Lemma \(2.8\), there exists a \(k+1\) dimensional plane \(H_1 \supset H\) such that \(H_1 \cap Y = H \cap Y\) (without loss of generality, one can retract the \(\Delta^n\)). By changing of the coordinates, we set \(H_1 := \{z_{k+2} = \cdots = z_n\}\).

By the restriction formula for jumping numbers, it follows that
\[
\sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O} \& I|_H = I\} \geq \sup\{c'_o(u)|I \subseteq \mathcal{O}|_{\{z_{k+1} = 0\}} \& I|_H = I\} \geq c'_o(\tilde{u}|H).
\]
As
\[
\sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O} \& I|_H = I\} = c'_o(\tilde{u}|H) = 1,
\]
then it follows that
\[
\sup\{c'_o(\tilde{u})|I \subseteq \mathcal{O}|_{\{z_{k+1} = 0\}} \& I|_H = I\} = c_o(\tilde{u}|_H) = 1.
\]

As \(H_1 \cap Y = H \cap Y\), then \(\{z|\mathcal{I}(\tilde{u}|_{H_1})_z \neq \mathcal{O}_z, z \in H_1\} \subseteq H \cap Y\). By Corollary \(2.6\) on \(H_1 (u \sim \tilde{u}|_{H_1}, J \sim I)\), it follows that there exists \(N > 0\) (independent of \(I \subseteq \mathcal{O}|_{\{z_{k+1} = 0\}}\)) such that
\[
\sup_{\mathcal{I}(\tilde{u}|_{H_1}, N \log |z_{k+1}|)} \{c'_o(\tilde{u}|H, N \log |z_{k+1}|)\} \leq 1. \tag{3.1}
\]
By Corollary 2.3, it follows that

$$\sup_{t, |t| = 1} \{ c^{(f)}_t(\max\{ \bar{u}|_{H_1}, N \log |z_{k+1}| \}) \} > 1,$$

which contradicts inequality 3.1.

Then the present theorem has thus been proved.

4. PROOF OF THEOREM 1.17

In the present section, we prove Theorem 1.17 and Corollary 1.18.

4.1. Proof of Remark 2.7 For the convenience of the readers, we recall the proof in [24] as follows:

It is clear that there exists a small enough neighborhood $V_2 \ni o$ such that

1. $\int_{V_1} |J|^2 e^{-2\bar{c}_t(u)} < \infty$;
2. $\{ f_j \}_{j=1,2,\ldots} \text{ generates } J|_{V_1}$.

Given any real number $l \in (1, \frac{2\bar{c}_t(u)}{c^{(f)}(o)})$, by the proof of Demailly's strong openness conjecture ([20, 21], see also [23]), there exists a small neighborhood $V_2$ of $o$ such that

$$\int_{V_2} |J|^2 e^{-2\bar{c}_t(u)} < \infty.$$ (4.1)

Then

$$\int_{V_2} (e^{-2\bar{c}_t(u)} - e^{-2\bar{u}_t}) \leq \int_{\{ u < \frac{1}{(l-1)c^{(f)}_t(o)} \log |J| \cap V_2 \}} e^{-2\bar{c}_t(u)}$$
$$= \int_{\{ u < \frac{1}{(l-1)c^{(f)}_t(o)} \log |J| \cap V_2 \}} e^{2(l-1)c^{(f)}_t(u) - 2\bar{c}_t(u)}$$
$$\leq \int_{\{ u < \frac{1}{(l-1)c^{(f)}_t(o)} \log |J| \cap V_2 \}} e^{2\log |J| - 2\bar{c}_t(u)}$$
$$\leq \int_{V_2} |J|^2 e^{-2\bar{c}_t(u)} < +\infty,$$ (4.2)

where the last inequality follows from inequality 4.1.

As $e^{-2\bar{c}_t(u)} - e^{-2\bar{u}_t}$ is integrable near $o$, it follows that $|J|^2 e^{-2\bar{c}_t(u)} - e^{-2\bar{u}_t}$ is integrable near $o$. Since $|J|^2 e^{-2\bar{c}_t(u)}$ is not integrable near $o$, it follows that $|J|^2 e^{-2\bar{u}_t}$ is not integrable near $o$, which implies $c^{(f)}_t(\bar{u}_t) \leq 1$.

As $c^{(f)}_t(u) \leq \bar{u}_t$, it follows that

$$|J|^2 e^{-2\bar{c}_t(u)} \geq |J|^2 e^{-2\bar{c}_t}$$

for any $c > 0$. When $c \in (0, 1)$, by the definition of jumping numbers, it follows that $|J|^2 e^{-2\bar{c}_t(u)}$ is locally integrable near $o$, which implies $|J|^2 e^{-2\bar{c}_t}$ is locally integrable near $o$, i.e., $c^{(f)}_t(\bar{u}_t) \geq 1$. Then we have $c^{(f)}_t(\bar{u}_t) = 1$.

4.2. Proof of Theorem 1.17 By the proof of Demailly’s strong openness conjecture ([20, 21], see also [23]), and $c^{(f)}_t(\bar{u}_t) = 1$ in Remark 2.7, it follows that $c^{(f)}_t(\max\{ c^{(f)}_t(u), \frac{1}{c^{(f)}_t(u)} \log |J| \}) = 1$. By the comparison of complex singularity
Lemma 5.1. \( c \) (for any \( r > 0 \) kernel associated with \( H \) on \( \Delta \) where \( c \) where 

\[
\frac{c_o(u)}{c_v(u) - c_o(u)} \geq c_o(\log |J|). \tag{4.3}
\]

Then Theorem 1.17 has thus been proved.

For the sake of completeness, we give a proof of the following equivalence

\[
J \subseteq I(c_o(u) \epsilon) \iff c_o(u) > c_o(u).
\]

Firstly, we prove "\( \Rightarrow \)". Since \( J \subseteq I(c_o(u) \epsilon) \) implies that \( |J|^2 e^{-2c_o(u)} u \) is locally integrable near \( o \), then it follows that \( c_o(u) > c_o(u) \) by our solution of Demailly’s strong openness conjecture.

Secondly, we prove "\( \Leftarrow \)". Since \( J \not\subseteq I(c_o(u) \epsilon) \) implies that \( |J|^2 e^{-2c_o(u)} u \) is not locally integrable near \( o \), then it follows that \( c_o(u) \leq c_o(u) \) by the definition of \( c_o(u) \).

4.3. Proof of Corollary 1.18. When \( I = O_o \), inequality 4.3 degenerates to

\[
\frac{c_o(u)}{c_o(u) - c_o(u)} \geq c_o(\log |J|), \tag{4.4}
\]

i.e.

\[
c_o(u) \leq \frac{c_o(u)}{c_o(\log |J|)} + c_o(u). \tag{4.5}
\]

Then Corollary 1.18 has been proved for \( J \subseteq I(c_o(u) \epsilon) \).

If \( J \) does not satisfy \( J \subseteq I(c_o(u) \epsilon) \), then \( |J| e^{-2c_o(u)} u \) is not integrable near \( o \), which implies \( c_o(u) \leq c_o(u) \). Note that \( |J|^2 \) is locally bounded near \( o \), then it follows that \( |J| e^{-2c_o(u)} \) is locally integrable near \( o \) for any \( c < c_o(u) \), which implies \( c_o(u) \geq c_o(u) \). Then it is clear that \( c_o(u) = c_o(u) \).

Then Corollary 1.18 has been proved.

5. Some properties of complex singularity exponents and fiberwise Bergman kernels

In this section, we recall and present some properties complex singularity exponents and fiberwise Bergman kernels.

5.1. Berndtsson’s log-subharmonicity, the openness conjecture and slicing result of complex singularity exponent. Let \( v \) be a plurisubharmonic function on \( \Delta^a \). Let \( H_{2v}(\Delta^a) \) be the Hilbert space of the holomorphic function \( f \) on \( \Delta^a \) satisfying (the norm) \( (\int_{\Delta^a} |f|^2 e^{-2v})^{1/2} < +\infty \). Let \( K_{\Delta^a, 2v} \) be the Bergman kernel associated with \( H_{2v}(\Delta^a) \).

It is easy to see that \( \int_{\Delta^a} e^{-2v} = +\infty \) (for any \( r > 0 \)) if and only if \( K_{\Delta^a, 2v}(o) = 0 \), where \( o \) is the origin of \( \mathbb{C}^n \). By definition of \( c_o(v) \), it follows that \( \int_{\Delta^a} e^{-2v} = +\infty \) (for any \( r > 0 \)) implies \( c_o(v) \leq 1 \); by Berndtsson’s solution of openness conjecture, \( c_o(v) \leq 1 \) implies \( \int_{\Delta^a} e^{-2v} = +\infty \) (for any \( r > 0 \)). Then one can obtain

\textbf{Lemma 5.1.} \( c_o(v) \leq 1 \) if and only if \( K_{\Delta^a, 2v}(o) = 0 \).

Let \( p : \Delta^a \times \Delta^m \to \Delta^m \) satisfying \( p(z_1, \cdots, z_n, w_1, \cdots, w_m) = (w_1, \cdots, w_m) \), where \( (z_1, \cdots, z_n) \) and \( (w_1, \cdots, w_m) \) are coordinates on \( \mathbb{C}^n \) and \( \mathbb{C}^m \).
Let $u$ be a plurisubharmonic function on $\Delta^n \times \Delta^m$. Let $K_{2Cu}$ be the fiberwise Bergman kernel on $\Delta^n \times \Delta^m$ such that $K_{2Cu}|_{p^{-1}(w)}$ is the Bergman kernel associated with the Hilbert space $H_{2Cu}|_{p^{-1}(w)}$ ($p^{-1}(w)$) (see [2]).

Berndtsson’s important result of log-subharmonicity of the Bergman kernels [2] tells us that log $K_{2Cu}$ is plurisubharmonic. Combining with Lemma 5.1 one can obtain

**Lemma 5.2.** For any $a > 0$, \{w | c_{(o,w)}(u|_{p^{-1}(w)}) \leq a\} is a complete pluripolar set on $\Delta^m$, which implies that $c_{(o,w)}(u|_{p^{-1}(w)})$ are the same (denoted by $C$) for almost every $w$ in the sense of Lebesgue measure on $\mathbb{C}^m$. Moreover $C = \sup_{w \in \Delta^m} \{c_{(o,w)}(u|_{p^{-1}(w)})\}$.

**Proof.** By Lemma 5.1 ($v = au|_{p^{-1}(w)}$), it follows that
\[
\{w | c_{(o,w)}(u|_{p^{-1}(w)}) \leq a\} = \{w | \log K_{2Cu}(o, w) = -\infty\},
\]
which implies \{w | c_{(o,w)}(u|_{p^{-1}(w)}) \leq a\} is a complete pluripolar set on $\Delta^m$.

Note that the Lebesgue measure of pluripolar set is 0 or $\pi$. It follows that $c_{(o,w)}(u|_{p^{-1}(w)})$ are the same (denoted by $C$) for almost every $w$.

We prove "moreover" part by contradiction: if not, then it follows that there exists $w$ satisfying $c_{(o,w)}(u|_{p^{-1}(w)}) > C$, which implies
\[
\log K_{2Cu}(o, w) > -\infty.
\]
As log $K_{2Cu}(o, w)$ is plurisubharmonic, then it follows that there exists a neighborhood $U$ of $w$ such that
\[
\log K_{2Cu}(o, \cdot) > -\infty
\]
for almost all point in $U$. Using the openness conjecture, one can obtain
\[
c_{(o, \cdot)}(u|_{p^{-1}(\cdot)}) > C
\]
holds for almost all point in $U$, which contradicts "$c_{(o,w)}(u|_{p^{-1}(w)}) = C$ for almost all $w \in \Delta^m$. □

### 5.2. Berndtsson’s log subharmonicity and integrability

In this subsection, we present a relationship between Berndtsson’s log subharmonicity and integrability.

We recall the original form of Ohsawa-Takegoshi $L^2$ extension theorem as follows:

**Theorem 5.3.** ([31], see also [30, 21, 22], etc.) Let $D$ be a bounded pseudo-convex domain in $\mathbb{C}^n$. Let $u$ be a plurisubharmonic function on $D$. Let $H$ be an $m$-dimensional complex plane in $\mathbb{C}^n$. Then for any holomorphic function on $H \cap D$ satisfying
\[
\int_{H \cap D} |f|^2 e^{-2u} d\lambda_H < +\infty,
\]
there exists a holomorphic function $F$ on $D$ satisfying $F|_{H \cap D} = f$, and
\[
\int_D |F|^2 e^{-2u} d\lambda_n \leq C_D \int_{H \cap D} |f|^2 e^{-2u} d\lambda_H,
\]
where $C_D$ only depends on the diameter of $D$ and $m$, and $d\lambda_H$ is the Lebesgue measure on $H$.

We recall a lemma which was used in [20, 21, 22] to prove Demailly’s strong openness conjecture:
Lemma 5.4. (see [20, 21]) Let $h_a$ be a holomorphic function on unit disc $\Delta \subset \mathbb{C}$ which satisfies $h_a(o) = 0$ and $h_a(a) = 1$ for any $a$, then we have
\[
\int_{\Delta} |h_a|^2 dA_1 > C_1 |a|^{-2},
\]
where $a \in \Delta$ whose norm is smaller than $\frac{1}{6}$, $C_1$ is a positive constant independent of $a$ and $h_a$.

Let $u$ be a plurisubharmonic function on $\Delta^n \times \Delta^m$ ($n = k, m = 1$) with coordinates $(z_1, \cdots, z_k, w)$, and $p$ be the projection with $p(z_1, \cdots, z_k, w) = w$ and $K_{2u}$ be the fiberwise Bergman kernel as in the above subsection.

Lemma 5.5. If $u > 0$, then $e^{-2u}$ is integrable near the origin $(o, o_w) \in \mathbb{C}^{k+1}$ if and only if $K_{2u}^{-1}(o, w)$ is integrable near the origin $o_w$ with respect to $w$, i.e.,
\[
\nu(\frac{1}{2} \log K_{2u}(o, \cdot), o_w) \geq 1.
\]

Proof. It is clear that if $e^{-2u}$ is integrable near origin $(o, o_w)$, then $K_{2u}^{-1}(o, w)$ is integrable near $o_w$. Then it suffices to prove "only if" part, i.e., if $e^{-2u}$ is not integrable near $(o, o_w)$, then $K_{2u}^{-1}(o, w)$ is not integrable near $o_w$.

We use our idea of movably using Ohsawa-Takegoshi $L^2$ extension theorem ([20, 21, 22]) to prove "only if" part:

As $e^{-2u}$ is not integrable near $o$, then it follows from Theorem 5.3 ($H = p^{-1}(a)$) that for any $a \in \Delta$, there exists holomorphic function $F_a$ on $\Delta^{k+1}$ such that

1. $F_a(o, a) = 1$;
2. $\int_{\Delta^{k+1}} |F_a|^{-2u} \leq C_D K_{2u}^{-1}(o, a)$;
3. $F_a(o, o_w) = 0$.

(Using the definition of $K_{2u}$, one can choose holomorphic $f_a$ on $\Delta^k \times \Delta$ satisfying $f_a(o, a) = 1$ and
\[
\int_{p^{-1}(a)} |f_a|^2 e^{-2u} = K_{2u}^{-1}(o, a),
\]
and $F_a$ is the Ohsawa-Takegoshi $L^2$ extension of $f_a$.)

By Lemma 5.3 ($F_a(z_1, \cdots, z_k, \cdot) = h_a(\cdot)$) and the submean inequality of $|F_a|^2$, it follows that
\[
\int_{\Delta^{k+1}} |F_a|^2 > C_2 \frac{1}{|a|^2},
\]
where $C_2 > 0$ is independent of $a$.

As $u > 0$, then it follows from (2) that
\[
K_{2u}^{-1}(o, a) \geq \frac{1}{C_D} \int_{\Delta^{k+1}} |F_a|^2 e^{-2u} > C_2 \frac{1}{C_D |a|^2}.
\]

Then the present lemma has been done.

Lemma 5.6. If $e^{-2u}$ is integrable near $o$, then $e^{-2u-2c \log |w|}$ is also integrable near $o$, where $c \in (0, 1)$.

Proof. As $e^{-2u}$ is integrable near $o$, it follows that
\[
\nu(\log K_{2u}(o, \cdot), o_w) = 0.
\]
Since
\[
K_{2u+2c \log |\cdot|}(o, \cdot) = |\cdot|^{2c} K_{2u}(o, \cdot),
\]

then

\[ \nu \left( \frac{1}{2} \log K_{2u + 2c \log |z|} \right) = c < 1. \]

By Lemma 5.8, the present lemma has thus been done. \qed

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