WEIGHTED COMPOSITION OPERATORS ON SPACES OF ANALYTIC VECTOR-VALUED LIPSCHITZ FUNCTIONS

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ABSTRACT. Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( \psi \) be an analytic operator-valued function on \( \mathbb{D} \), where \( \mathbb{D} \) is the unit disk. We provide necessary and sufficient conditions for the boundedness and compactness of weighted composition operators \( W_{\Psi, \varphi} : f \mapsto \Psi(f \circ \varphi) \) on \( \operatorname{Lip}_A(\mathbb{D}, X, \alpha) \) and \( \operatorname{lip}_A(\mathbb{D}, X, \alpha) \), the spaces of analytic \( X \)-valued Lipschitz functions \( f \), where \( X \) is a complex Banach space and \( \alpha \in (0, 1] \).

1. INTRODUCTION AND PRELIMINARIES

Let \((S, d)\) be a metric space, \( X \) be a Banach space and \( \alpha \in (0, 1] \). The space of all functions \( f : S \to X \) for which
\[
p_\alpha(f) = \sup \left\{ \frac{\|f(s_1) - f(s_2)\|}{d^\alpha(s_1, s_2)} : s_1, s_2 \in S, s_1 \neq s_2 \right\} < \infty,
\]
is denoted by \( \operatorname{Lip}_\alpha(S, X) \). The subspace of functions \( f \) for which
\[
\lim_{d(s_1, s_2) \to 0} \frac{\|f(s_1) - f(s_2)\|}{d^\alpha(s_1, s_2)} = 0,
\]
is denoted by \( \operatorname{lip}_\alpha(S, X) \). The spaces \( \operatorname{Lip}_\alpha(S, X) \) and \( \operatorname{lip}_\alpha(S, X) \) equipped with norm \( \|f\|_\alpha = \|f\|_S + p_\alpha(f) \) are Banach spaces, where,
\[
\|f\|_S = \sup_{s \in S} \{\|f(s)\| : s \in S\}.
\]
These spaces are called \( X \)-valued Lipschitz spaces. The spaces \( \operatorname{Lip}_\alpha(S, X) \) and \( \operatorname{lip}_\alpha(S, X) \) were studied by Johnson for the first time [6].

For a Banach space \( X \), let \( H(\mathbb{D}, X) \) be the space of all analytic \( X \)-valued functions on the open unit disc \( \mathbb{D} \) and \( A(\overline{\mathbb{D}}, X) \) be the Banach space of all continuous functions \( f : \overline{\mathbb{D}} \to X \) which are analytic on \( \mathbb{D} \). For \( \alpha \in (0, 1] \) we define the spaces
\[
\Lambda_\alpha(X) = \operatorname{Lip}_\alpha(\mathbb{D}, X) \cap H(\mathbb{D}, X)
\]
and
\[
\operatorname{Lip}_A(\overline{\mathbb{D}}, X, \alpha) = \operatorname{Lip}_\alpha(\overline{\mathbb{D}}, X) \cap A(\overline{\mathbb{D}}, X).
\]
Clearly, \( \Lambda_\alpha(X) \) and \( \operatorname{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \), equipped with Lipschitz norm \( \| \cdot \|_\alpha \), are Banach spaces and the spaces
\[
\Lambda_\alpha^0(X) = \operatorname{lip}_\alpha(\mathbb{D}, X) \cap H(\mathbb{D}, X)
\]

2010 Mathematics Subject Classification. Primary 46E40; Secondary 47A56, 47B33.

Key words and phrases. Analytic vector-valued Lipschitz functions; vector-valued Bloch spaces; weighted composition operators; compact operators.
and
\[ \text{lip}_A(\overline{D}, X, \alpha) = \text{lip}_\alpha(\overline{D}, X) \cap A(\overline{D}, X) \]
are Banach subspaces of \( \Lambda_\alpha(X) \) and \( \text{Lip}_A(\overline{D}, X, \alpha) \), respectively.

We consider the weighted Banach space
\[ H_{\nu}^\infty(X) = \{ f \in H(\overline{D}, X) : \| f \|_\nu = \sup_{z \in \overline{D}} \nu(z) \| f(z) \| < \infty \} \]
endowed with norm \( \| \cdot \|_\nu \), where \( \nu : \overline{D} \to (0, +\infty) \) is a bounded continuous weight function.

For a positive real number \( \alpha \), \( B_\alpha(X) \) denotes the \( X \)-valued Bloch type space of all analytic functions \( f : \overline{D} \to X \) satisfying
\[ \sup_{z \in \overline{D}} (1 - |z|^2)^\alpha \| f'(z) \| < \infty. \]
The space \( B_\alpha(X) \) endowed with norm
\[ \| f \|_{B_\alpha} = \| f(0) \| + \sup_{z \in \overline{D}} (1 - |z|^2)^\alpha \| f'(z) \|, \quad (f \in B_\alpha(X)), \]
is a Banach space.

In the case \( X = C \), we omit \( X \) in the notation.

For Banach spaces \( X \) and \( Y \), by \( L(X, Y) \) (\( K(X, Y) \)), we mean the Banach space of all bounded (compact) linear operators from \( X \) to \( Y \). Let \( S(\overline{D}, X) \) and \( S(\overline{D}, Y) \) be subspaces of \( A(\overline{D}, X) \) and \( A(\overline{D}, Y) \), respectively. Let \( \phi \in A(\overline{D}) \) be a nonconstant self map of \( \overline{D} \) and \( \Psi : \overline{D} \to L(X, Y) \) be a continuous operator-valued function analytic on \( \overline{D} \). Then the weighted composition operator \( W_{\Psi, \phi} \) from \( S(\overline{D}, X) \) to \( S(\overline{D}, Y) \) is defined to be the linear operator of the form
\[ W_{\Psi, \phi}(f)(z) = \psi(z)(f(\phi(z)), \quad (f \in S(\overline{D}, X), \quad z \in \overline{D}). \]
For simplicity of notation, we write \( \Psi_z \) instead of \( \Psi(z) \).

Note that if \( \Psi_z \) is the identity map on \( X \) for every \( z \in \overline{D} \), then \( W_{\Psi, \phi} \) is the composition operator on \( S(\overline{D}, X) \). In the scalar case, a weighted composition operator is a composition operator followed by a multiplier.

There is recent interest into properties of the composition operators and weighted composition operators between Banach spaces of vector-valued functions. For instance, the weakly compact composition operators on Hardy spaces, weighted Bergman spaces, Bloch spaces and BMOA, in the vector-valued case, have been characterized in [2, 8, 9, 11, 12]. Weighted composition operators between vector-valued Lipschitz spaces and weighted Banach spaces of vector-valued analytic functions have been studied in [5, 10]. Also, composition operators on analytic Lipschitz spaces in the scalar-valued case have been investigated in [1, 13]. The present study is aimed at finding some necessary and sufficient conditions for boundedness and compactness of weighted composition operators on the spaces of analytic \( X \)-valued Lipschitz functions.

The rest of this paper is designed as follows. In section 2 we shall characterize bounded and compact weighted composition operators between analytic vector-valued Lipschitz spaces. Section 3 is devoted to discussing this kinds of operators between analytic vector-valued little Lipschitz spaces.
2. Weighted Composition Operators on \( \text{Lip}_A(\mathbb{D}, X, \alpha) \) and \( \Lambda_\alpha(X) \)

In this section, we provide necessary and sufficient conditions for weighted composition operators between analytic vector-valued Lipschitz spaces to be bounded and compact. In what follows, we will assume that \( X \) and \( Y \) are Banach spaces.

We begin with some elementary properties of analytic vector-valued Lipschitz spaces. The best reference here is [2]. Let \( E \) be a Banach subspace of \( H(\mathbb{D}) \) which contains constant functions and its closed unit ball \( U(E) \) is compact for the compact open topology. Then the space

\[ *E := \{ u \in E^* : u|_{U(E)} \text{ is } \text{co-continuous} \}, \]

endowed with the norm induced by \( E^* \), is a Banach space and the evaluation map \( f \mapsto [u \mapsto u(f)] \) from \( E \) into \((*E)^*\), is an isometric isomorphism. In particular, \( *E \) is a predual of \( E \). Furthermore, the vector-valued space

\[ E[X] := \{ f \in H(\mathbb{D}, X) : x^* \circ f \in E, \ x^* \in X^* \}, \]

by the norm \( \|f\|_{E[X]} = \sup_{\|x^*\|\leq 1} \|x^* \circ f\| \) is a Banach space.

Bonet, et al. in [2, Lemma 10] argued that the map \( \Delta : \mathbb{D} \rightarrow *E, \Delta(z) = \delta_z \), where \( \delta_z \) is the evaluation map on \( E \), is analytic and the linear operator \( \chi : L(*E, X) \rightarrow E[X], \chi(T) = T \circ \Delta \) is bounded. Defining \( \psi(g)(u) : X^* \rightarrow \mathbb{C} \) by \( \psi(g)(u)(x^*) = u(x^* \circ g) \) for \( g \in E[X] \) and \( u \in *E \), they showed that \( \psi(g) \in L(*E, X^*) \) and \( \psi(g)(\delta_z) \in L(*E, X) \). Besides, using operators \( \chi \) and \( \psi \), they deduced that the space \( E[X] \) is isomorphic to \( L(*E, X) \).

In the following proposition we use the above mentioned result for the spaces \( \Lambda_\alpha[X] \) and \( B_{1-\alpha}[X] \) to show that the norms \( \| \cdot \|_\alpha \) and \( \| \cdot \|_{B_{1-\alpha}} \) are equivalent, whenever \( \alpha \in (0, 1) \).

**Proposition 2.1.** Let \( \alpha \in (0, 1) \). Then \( \Lambda_\alpha(X) = B_{1-\alpha}(X) \) and the norms \( \| \cdot \|_\alpha \) and \( \| \cdot \|_{B_{1-\alpha}} \) are equivalent.

**Proof.** Using Hardy-Littlewood theorem for \( \alpha \in (0, 1) \), we can see that \( \Lambda_\alpha = B_{1-\alpha} \) and \( \| \cdot \|_\alpha \asymp \| \cdot \|_{B_{1-\alpha}} \) (i.e. for some positive constants \( a \) and \( b \), \( a \| \cdot \|_\alpha \leq \| \cdot \|_{B_{1-\alpha}} \leq b \| \cdot \|_\alpha \)). Hence, \( \Lambda_\alpha = *B_{1-\alpha} \), where \( * \Lambda_\alpha \) and \( *B_{1-\alpha} \) are the preduals of \( \Lambda_\alpha \) and \( B_{1-\alpha} \) mentioned above, respectively. Thus \( \Lambda_\alpha[X] = B_{1-\alpha}[X] \) and the linear operators

\[ \text{id} : \Lambda_\alpha[X] \xrightarrow{\psi} L(*\Lambda_\alpha, X) = L(*B_{1-\alpha}, X) \xrightarrow{\chi} B_{1-\alpha}[X] \]

and

\[ \text{id} : B_{1-\alpha}[X] \xrightarrow{\psi} L(*B_{1-\alpha}, X) = L(*\Lambda_\alpha, X) \xrightarrow{\chi} \Lambda_\alpha[X] \]

are bounded. This shows that \( \| \cdot \|_{\Lambda_\alpha[X]} \asymp \| \cdot \|_{B_{1-\alpha}[X]} \). Since \( \Lambda_\alpha(X) = \Lambda_\alpha[X] \) and \( B_{1-\alpha}(X) = B_{1-\alpha}[X] \), we conclude that

\[ \| \cdot \|_\alpha \asymp \| \cdot \|_{\Lambda_\alpha[X]} \asymp \| \cdot \|_{B_{1-\alpha}[X]} = \| \cdot \|_{B_{1-\alpha}}. \]

\[ \square \]

From Proposition 2.1 we deduce that for \( \alpha \in (0, 1) \) the norm

\[ \|f\|_{\Lambda_\alpha(X)} = \|f(0)\| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{(1-\alpha)} \|f'(z)\|, \quad (f \in \Lambda_\alpha(X)) \]
defines an equivalent norm on $\Lambda_\alpha(X)$. Hereafter we use this norm for $\Lambda_\alpha(X)$ and $\Lambda^0_\alpha(X)$, whenever $\alpha \in (0,1)$.

In [13], Mahyar and Sanatpour proved that every function $f \in \text{Lip}_\alpha(\mathbb{D})$ has a unique continuous extension $E(f)$ to $\overline{\mathbb{D}}$, such that $E(f) \in \text{Lip}_\alpha(\mathbb{D})$. In fact, for every $w \in \partial \mathbb{D}$ they defined $E(f)(w) = \lim_{n \to \infty} f(z_n)$, where $\{z_n\}$ is any sequence in $\mathbb{D}$ converging to $w$. By the same method, one can see that every function $f \in \text{Lip}_\alpha(\mathbb{D}, X)$ has a unique continuous extension $E(f)$ to $\overline{\mathbb{D}}$ such that $E(f) \in \text{Lip}_\alpha(\overline{\mathbb{D}}, X)$.

Furthermore, as in the proof of [13, Proposition 2.1], it can be seen that the mapping $f \mapsto E(f)$ is a homeomorphism from $(\Lambda_\alpha(X), \| \cdot \|_{\Lambda_\alpha(X)})$ onto $(\text{Lip}_A(\overline{\mathbb{D}}), X, \alpha, \| \cdot \|_\alpha)$. Thus we can modify the problem of boundedness and compactness of the weighted composition operators $W_{\psi,\varphi} : \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \to \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta)$, to the problem of boundedness and compactness of the weighted composition operators $W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)$, where $\psi = \Psi_{|\mathbb{D}}$ and $\varphi = \phi_{|\mathbb{D}} : \mathbb{D} \to \mathbb{D}$.

**Proposition 2.2.** Let $\Psi \in \text{Lip}_A(\overline{\mathbb{D}}, L(X,Y), \beta)$ and $\Phi \in A(\mathbb{D})$ be a self map of $\mathbb{D}$. Then $W_{\psi,\varphi} : \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) \to \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta)$ is bounded (compact) if and only if $W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)$ is bounded (compact), where $\psi = \Psi_{|\mathbb{D}}$ and $\varphi = \phi_{|\mathbb{D}} : \mathbb{D} \to \mathbb{D}$.

**Proof.** Let $R$ be the restriction map from $\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$ into $\Lambda_\alpha(X)$. Clearly, $R$ is a bounded linear operator. Let $z \in \partial \mathbb{D}$ and $\{z_n\}$ be any sequence in $\mathbb{D}$ converging to $z$. An easy computation shows that for every $f \in \text{Lip}_A(\overline{\mathbb{D}}, X, \alpha)$,

$$W_{\psi,\varphi}(f)(z) = \lim_{n \to \infty} \psi(z_n) f(\varphi(z_n)).$$

On the other hand,

$$(E \circ W_{\psi,\varphi} \circ R)(f)(z) = \lim_{n \to \infty} W_{\psi,\varphi}(R(f))(z_n) = \lim_{n \to \infty} \psi(z_n) f(\varphi(z_n))$$

holds for every $f \in \Lambda_\alpha(X)$. Thus $E \circ W_{\psi,\varphi} \circ R = W_{\psi,\varphi}$ and the diagram

$$\begin{array}{ccc}
\Lambda_\alpha(X) & \xrightarrow{W_{\psi,\varphi}} & \Lambda_\beta(Y) \\
\uparrow R & & \downarrow E \\
\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) & \xrightarrow{W_{\psi,\varphi}} & \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta)
\end{array}$$

is commutative.

Likewise, $W_{\psi,\varphi} = R \circ W_{\psi,\varphi} \circ E$ and the diagram

$$\begin{array}{ccc}
\Lambda_\alpha(X) & \xrightarrow{W_{\psi,\varphi}} & \Lambda_\beta(Y) \\
\uparrow E & & \downarrow R \\
\text{Lip}_A(\overline{\mathbb{D}}, X, \alpha) & \xrightarrow{W_{\psi,\varphi}} & \text{Lip}_A(\overline{\mathbb{D}}, Y, \beta)
\end{array}$$

is commutative and the proof is complete. \qed

For every $x \in X$ and every scalar-valued function $f \in \Lambda_\alpha$, the function $f_x(z) = f(z)x$, $z \in \mathbb{D}$ belongs to $\Lambda_\alpha(X)$ and $\|f_x\|_{\Lambda_\alpha(X)} = \|f\|_{\Lambda_\alpha} \|x\|$. Moreover,

$$(W_{\psi,\varphi}(f_x))'(z) = \varphi'(z) f'(\varphi(z)) \psi_z(x) + f(\varphi(z)) \psi'_z(x).$$
In particular, for each \( x \in X \) the constant function \( 1_x \) exists in \( \Lambda_\alpha(X) \) and the Banach space \( X \) can be considered as a subspace of \( \Lambda_\alpha(X) \).

For every vector-valued function \( f \in H(\mathbb{D}, X) \) and every \( z \in \mathbb{D} \) we have
\[
(\Psi_{\varphi, \psi}(f))'(z) = \varphi'(z)\psi_z(f'(\varphi(z))) + \psi_z'(f(\varphi(z))).
\]
Hence, \( DW_{\varphi, \psi} = \Psi_{\varphi, \psi} \circ D + \Psi_{\varphi, \psi} \), where \( D \) is the derivation operator. By [10, Theorem 2.1], for \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \) we have
\[
\|W_{\varphi, \psi} : H^{\infty}_\nu(X) \to H^{\infty}_\nu(Y)\| \asymp \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta}}{(1 - |\varphi(z)|^2)^{\alpha}} \|\psi_z\|, \tag{2.1}
\]
where \( \nu_\alpha \) is the standard weight \( \nu_\alpha(z) = (1 - |z|^2)^{\alpha} \).

By the same method, one can see that \( \Psi_{\varphi, \psi} : H^{\infty}(X) \to H^{\infty}_\nu(Y) \) is bounded if and only if \( \psi \in H^{\infty}(L(X, Y)) \). Moreover, \( \|W_{\varphi, \psi} : H^{\infty}(X) \to H^{\infty}_\nu(Y)\| \asymp \|\psi\|_{H^{\infty}_\nu} \).

The following theorem characterizes the bounded weighted composition operators between analytic vector-valued Lipschitz spaces.

**Theorem 2.3.** For \( 0 < \alpha < 1 \) the operator \( W_{\varphi, \psi} \) maps \( \Lambda_\alpha(X) \) boundedly into \( \Lambda_\beta(Y) \) if and only if \( \psi \in \Lambda_\beta(L(X, Y)) \) and
\[
q_{\alpha, \beta} = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)||\psi_z| < \infty.
\]
Moreover,
\[
\|W_{\varphi, \psi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)\| \asymp \max\{q_{\alpha, \beta}, \|\psi\|_{\Lambda_\beta(L(X, Y))}\}. \tag{2.2}
\]

**Proof.** The proof of the first part is a straightforward modification of that of [7, Theorem 2.1]. We prove that in the case \( W_{\varphi, \psi} \) is bounded, the relation (2.2) holds. Let \( W_{\varphi, \psi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y) \) be bounded. Using \( DW_{\varphi, \psi} = \Psi_{\varphi, \psi} \circ D + \Psi_{\varphi, \psi} \), one can easily show that the operators \( W_{\varphi, \psi} : H^{\infty}_\nu(X) \to H^{\infty}_\nu(Y) \) and \( W_{\varphi, \psi} : \Lambda_\alpha(X) \to H^{\infty}_\nu(Y) \) are bounded and
\[
\|W_{\varphi, \psi} : \Lambda_\alpha(X) \to H^{\infty}_\nu(Y)\| \leq \frac{2}{\alpha} \|\psi\|_{\Lambda_\beta(L(X, Y))}.
\]
Therefore,
\[
\|W_{\varphi, \psi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)\| \leq \|W_{\varphi, \psi} : H^{\infty}_\nu(X) \to H^{\infty}_\nu(Y)\|
+ \|W_{\varphi, \psi} : \Lambda_\alpha(X) \to H^{\infty}_\nu(Y)\|.
\]
From relation (2.1), for some positive constant \( C \) we have
\[
\|W_{\varphi, \psi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)\| \leq C \max\{q_{\alpha, \beta}, \|\psi\|_{\Lambda_\beta(L(X, Y))}\}.
\]

For the converse, since
\[
(1 - |z|^2)^{1-\beta} \|\psi_z'\| = \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|\psi_z'(x)\|
= \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|(W_{\varphi, \psi}(1_x))'(z)\|
\]
holds for every \( z \in \mathbb{D} \) and since \( W_{\varphi, \psi} \) is bounded, we have
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|\psi_z'\| \leq \|W_{\varphi, \psi}\|.
\]
For every nonzero $a \in \mathbb{D}$ and every $x \in X$ define
\[ f_{a,x}(z) = \frac{1}{\alpha} \left( \frac{1 - |a|^2}{(1 - \overline{a}z)^{1-\alpha}} - (1 - \overline{a}z)^\alpha \right) x. \]

It could be shown that $\{f_{a,x} : a \neq 0, \|x\| \leq 1\}$ is a bounded subset of $\Lambda_\alpha(X)$. Moreover, $f_{a,x}(a) = 0$ and $f'_a(a) = \frac{x}{(1 - |a|^2)(1 - \alpha)}$. Then for some positive constant $C$ we have
\[ q_{\alpha,\beta} = \sup_{\|x\| \leq 1} \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f_{\varphi(z),x}))'(z)\| \leq C\|W_{\psi,\varphi}\|, \]
which completes the proof.

**Theorem 2.4.** The operator $W_{\psi,\varphi} : \Lambda_1(X) \to \Lambda_\beta(Y)$ is well defined and bounded if and only if $\psi \in \Lambda_\beta(L(X,Y))$ and $\varphi'\psi \in H^\infty_{\nu_1 - \beta}(L(X,Y))$. Furthermore,
\[ \|W_{\psi,\varphi} : \Lambda_1(X) \to \Lambda_\beta(Y)\| \asymp \max\{\|\varphi'\psi\|_{H^\infty_{\nu_1 - \beta}}, \|\psi\|_{\Lambda_\beta}\}. \]

**Proof.** A simple computation gives that $f \in \Lambda_1(X)$ if and only if $f' \in H^\infty(X)$ and $\|f\| \asymp \|f\|_B + \|f'\|_B$.

Let $W_{\psi,\varphi}$ be bounded. Defining $f_{a,x}(z) = (z - a)x$, for any $a, z \in \mathbb{D}$ and $x \in X$, one can see that $\{f_{a,x} : a \in \mathbb{D}, \|x\| \leq 1\}$ is a bounded sequence in $\Lambda_1(X)$. Since $W_{\psi,\varphi}$ is bounded, for every $z \in \mathbb{D}$,
\[ (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_x\| = \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \|\psi_x\| \]
\[ = \sup_{\|x\| \leq 1} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f_{\varphi(z),x}))'(z)\| \]
\[ \leq \sup_{\|x\| \leq 1} \|W_{\psi,\varphi}\| \|f_{\varphi(z),x}\|_{\Lambda_1(X)} < 3\|W_{\psi,\varphi}\|. \]

Thus $\varphi'\psi \in H^\infty_{\nu_1 - \beta}(L(X,Y))$. The rest of the proof runs as the proof of Theorem 2.3.

The next lemma shows that the compact open topology on $\Lambda_\alpha_0(X)$ is stronger than the weak topology.

**Lemma 2.5.** Let $\alpha \in (0,1)$ and $\{f_n\}$ be a bounded sequence in $\Lambda_\alpha_0(X)$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Then $\{f_n\}$ converges weakly to zero.

**Proof.** For every $f \in \Lambda_\alpha_0(X)$, consider the function $\tilde{f}(z) = (1 - |z|^2)^{1-\alpha} f'(z)$ and set $\tilde{\Lambda} = \{\tilde{f} : f \in \Lambda_\alpha_0(X)\}$. Clearly, $\tilde{\Lambda}$ is a subspace of $C_0(\mathbb{D}, \mathbb{X})$. Let $T$ be a bounded linear functional on $\Lambda_\alpha_0(X)$. By Hahn- Banach theorem, for some measure $\mu \in M(\mathbb{D}, \mathbb{X}^*)$ we have $Tf = \int_{\mathbb{D}} \tilde{f} d\mu$, for every $f \in \Lambda_\alpha_0(X)$ (see [4, Corollary 2, p. 387]). Without loss of generality we assume that $\|f_n\|_{\Lambda_\alpha_0(X)} \leq 1$.

Fixing $\epsilon > 0$, let $\{r_m\}$ be an increasing sequence in $(0,1)$ converging to 1 and $D_m = \{z \in \mathbb{D} : |z| \leq r_m\}$. Then $\mathbb{D} = \bigcup_{m=1}^\infty D_m$ and $|\mu|(\mathbb{D} \setminus D_m) < \frac{\epsilon}{2}$ for some $m$. Since $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, we deduce that
\[ \sup_{z \in D_m} \|\tilde{f}_n(z)\| < \frac{\epsilon}{2\|\mu\|} \] for \( n \) sufficiently large. Therefore

\[
|T(f_n)| \leq |\int_{D \setminus D_m} \tilde{f}_n d\mu| + |\int_{D_m} \tilde{f}_n d\mu| \leq \int_{D \setminus D_m} \|\tilde{f}_n(z)\| d|\mu|(z) + \int_{D_m} \|\tilde{f}_n(z)\| d|\mu|(z) \leq |\mu|(D \setminus D_m) + |\mu|(D_m) \frac{\epsilon}{2\|\mu\|} < \epsilon,
\]

which shows that \( \lim_{n \to \infty} T(f_n) = 0 \). Thus \( \{f_n\} \) converges weakly to zero as desired. \( \square \)

In the next theorem we provide a necessary and sufficient condition for weighted composition operator \( W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y) \) to be compact, whenever \( \alpha \in (0, 1) \). We use the idea of [10] and define \( T_\psi : X \to \Lambda_\beta(Y) \), by \( T_\psi(x)(z) = \psi_z(x) \). In the case \( W_{\psi,\varphi} \) is bounded, \( T_\psi \) is a bounded linear operator and

\[ \|T_\psi : X \to \Lambda_\beta(Y)\| \leq \|W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)\|. \]

For \( n \in \mathbb{N} \), we define \( L_n : \Lambda_\alpha(X) \to \Lambda_\beta(Y) \) by \( L_n(f) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k \) for every \( f \in \Lambda_\alpha(X) \), where \( \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \) is the Taylor expansion of \( f \). One can easily show that for every \( f \in \Lambda^0_\alpha(X) \), \( \|L_n f - f\|_{\Lambda_\alpha(X)} \to 0 \) as \( n \to \infty \).

For \( r \in (0, 1) \) we define the linear operator \( K_r : \Lambda_\alpha(X) \to \Lambda_\alpha(X) \), \( K_r(f)(z) = f(rz) \) for every \( f \in \Lambda_\alpha(X) \) and \( z \in D \). Clearly, \( K_r \) is a bounded linear operator from \( \Lambda_\alpha(X) \) into \( \Lambda^0_\alpha(X) \) and for every \( f \in \Lambda_\alpha(X) \), \( \|K_r f - f\|_{\Lambda_\alpha(X)} \to 0 \) as \( r \to 1^- \).

**Theorem 2.6.** Let \( 0 < \alpha < 1 \) and \( W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y) \) be a bounded weighted composition operator. Then \( W_{\psi,\varphi} \) is compact if and only if \( T_\psi \) is compact and

\[
\limsup_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| |\psi_z| = 0. \tag{2.3}
\]

**Proof.** Let \( W_{\psi,\varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y) \) be compact. Considering the bounded operator \( T : X \to \Lambda_\alpha(X), Tx = 1_x \), we get \( T_\psi = W_{\psi,\varphi} \circ T \) is compact. If (2.3) does not hold, one can find a sequence \( \{z_n\} \) of \( D \) such that \( |\varphi(z_n)| > \frac{1}{2}, |\varphi(z_n)| \to 1 \) and

\[
\lim_{n \to \infty} \frac{(1 - |z_n|^2)^{1-\beta}}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} |\varphi'(z_n)| |\psi_{z_n}| > 0.
\]

For every \( n \), define

\[
f_n(z) = \frac{1}{\varphi(z_n)} \left( \frac{(1 - |\varphi(z_n)|^2)^2}{(1 - |\varphi(z_n)|^2)^{2-\alpha}} - \frac{1 - |\varphi(z_n)|^2}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} \right) x_n
\]

where \( \{x_n\} \) is a sequence in \( X \), for which \( \|x_n\| \leq 1 \) and \( \frac{n}{n+1} \|\psi_{z_n}\| < \|\psi_{z_n}(x_n)\| \). Clearly, \( \{f_n\} \) is a bounded sequence in \( \Lambda^0_\alpha(X) \) converging to zero on compact subsets of \( D \). Moreover, \( f_n(\varphi(z_n)) = 0 \) and \( f'_n(\varphi(z_n)) = \frac{1}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} x_n \). Since
$W_{\psi, \varphi}$ is compact, from lemma 2.5 we deduce that $W_{\psi, \varphi}(f_n) \to 0$, as $n \to \infty$. But

$$\|W_{\psi, \varphi}(f_n)\|_{L_\beta(Y)} \geq (1 - |z_n|^2)^{1-\beta} \|\varphi'(z_n) f'_n(\varphi(z_n)) \psi z_n(x_n) + f_n(\varphi(z_n)) \psi z_n(x_n)\|$$

$$= \frac{(1 - |z_n|^2)^{1-\beta}}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} |\varphi'(z_n)||\psi z_n(x_n)|$$

$$> \frac{(1 - |z_n|^2)^{1-\beta}}{(1 - |\varphi(z_n)|^2)^{1-\alpha}} |\varphi'(z_n)||\psi z_n(x_n)| \frac{n}{n+1},$$

implies that $\lim_{n \to \infty} \|W_{\psi, \varphi}(f_n)\|_{L_\beta(Y)} > 0$, which is impossible.

Conversely, let $T_\psi$ be compact and (2.3) holds. Consider the operator $q_k : \Lambda_\alpha(X) \to X$, $f \mapsto \frac{f^{(k)}(0)}{k!}$. By Cauchy’s integral theorem, for every $f \in \Lambda_\alpha(X)$ we have

$$\|q_k(f)\| \leq \frac{1}{2\pi} \oint_{|z|=\frac{1}{2}} \frac{|f'(z)|}{|z|^k} |dz| \leq \frac{2^k}{C} \|f\|_{L_\alpha(X)},$$

where $C$ is a positive constant. Therefore $q_k$ is a bounded linear operator. We show that $M_{\varphi^k} : \Lambda_\beta(Y) \to \Lambda_\beta(Y), f \mapsto \varphi^k f$ is bounded. Since $W_{\psi, \varphi} : \Lambda_\alpha(X) \to \Lambda_\beta(Y)$ is bounded, from Theorem 2.3 we conclude that

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| < \infty.$$

Thus for every $f \in \Lambda_\beta(Y)$ we have

$$\|M_{\varphi^k}(f)\|_{L_\beta(Y)} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} |(\varphi^k f)'(z)|$$

$$\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} \|f'(z)\| + k \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1-\beta} |\varphi'(z)||f(z)||$$

$$\leq \|f\|_{L_\beta(Y)} + k \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} \frac{(1 - |\varphi(z)|^2)^{1-\alpha} |\varphi'(z)||f(z)||}{(1 - |\varphi(z)|^2)^{1-\alpha}}$$

$$\leq (2 + kC) \|f\|_{L_\alpha(Y)},$$

where $C$ is a positive constant. This shows that $M_{\varphi^k}$ is a bounded operator on $\Lambda_\beta(Y)$ and since $T_\psi$ is compact, we deduce that $W_{\psi, \varphi} \circ L_n = \sum_{k=0}^{n} M_{\varphi^k} \circ T_\psi \circ q_k$ is a compact operator. Since $K_r f \in \Lambda^0_\alpha(X)$, we see that $\|K_r f - L_n(K_r f)\|_{L_\alpha(X)} \to 0$ as $n \to \infty$. Hence $\|W_{\psi, \varphi} \circ K_r - W_{\psi, \varphi} \circ L_n \circ K_r\| \to 0$ as $n \to \infty$, which shows that $W_{\psi, \varphi} \circ K_r$ is a compact operator. For completing the proof we show that $\limsup_{r \to 1^-} \|W_{\psi, \varphi} - W_{\psi, \varphi} \circ K_r\| = 0$. For this, fix $0 < \delta < 1$. For every $f \in \Lambda_\alpha(X)$
with \( \|f\|_{\Lambda_\alpha(X)} \leq 1 \) we have
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^{1 - \beta} \|(W_{\psi, \varphi}(f - K_r f))'(z)\|
\leq \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1 - \beta} |\varphi'(z)||\psi_z||f'(\varphi(z)) - r f'(r \varphi(z))||
+ \sup_{z \in \mathbb{D}} (1 - |z|^2)^{1 - \beta} ||\varphi'_z||f(\varphi(z)) - f(r \varphi(z))||
\leq \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^{1 - \beta} |\varphi'(z)||\psi_z||f'(\varphi(z)) - f'(r \varphi(z))| \quad (2.4)
+ \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^{1 - \beta} |\varphi'(z)||\psi_z||(f - f_r)'
(1 - r)
+ \sup_{|\varphi(z)| > \delta} (1 - |z|^2)^{1 - \beta} |\varphi'(z)||\psi_z||(f - f_r)'
(1 - r)
+ \sup_{\delta \rightarrow 1} \sup_{|\varphi(z)| > \delta} (1 - |z|^2)^{1 - \beta} |\varphi'(z)||\psi_z||f(\varphi(z)) - f(r \varphi(z))||. \quad (2.7)
\]
By Cauchy's integral theorem, (2.4) is not bigger than \( \frac{(1 - r)}{(1 - \delta)^{1 - \alpha}} \) and converges to zero whenever \( r \rightarrow 1^- \). Also,
\[
(2.5) \leq \sup_{|\varphi(z)| \leq \delta} \frac{(1 - |z|^2)^{1 - \beta}}{(1 - |\varphi(z)|^2)^{1 - \alpha}} |\varphi'(z)||\psi_z||(1 - r) \rightarrow 0,
\]
as \( r \rightarrow 1^- \) and
\[
(2.6) \leq 2 \sup_{|\varphi(z)| > \delta} \frac{(1 - |z|^2)^{1 - \beta}}{(1 - |\varphi(z)|^2)^{1 - \alpha}} |\varphi'(z)||\psi_z|.
\]
Moreover,
\[
(2.6) \leq \sup_{z \in \mathbb{D}} ||\psi_z|| ||f||_\alpha |\varphi(z)|(1 - r)^\alpha \rightarrow 0,
\]
as \( r \rightarrow 1^- \). Thus
\[
\lim_{r \rightarrow 1^-} \|W_{\psi, \varphi} - W_{\psi, \varphi} \circ K_r\| = \lim_{r \rightarrow 1^-} \sup_{||f|| \leq 1} ||W_{\psi, \varphi}(f) - W_{\psi, \varphi} \circ K_r(f)||
\leq 2 \sup_{|\varphi(z)| > \delta} \frac{(1 - |z|^2)^{1 - \beta}}{(1 - |\varphi(z)|^2)^{1 - \alpha}} |\varphi'(z)||\psi_z|.
\]
Letting \( \delta \rightarrow 1 \), we have
\[
\lim_{r \rightarrow 1^-} \|W_{\psi, \varphi} - W_{\psi, \varphi} \circ K_r\| \leq 2 \lim_{\delta \rightarrow 1} \sup_{|\varphi(z)| > \delta} \frac{(1 - |z|^2)^{1 - \beta}}{(1 - |\varphi(z)|^2)^{1 - \alpha}} |\varphi'(z)||\psi_z| = 0,
\]
which ensures that \( W_{\psi, \varphi} \) is compact. \( \square \)

3. Weighted Composition Operators on \( \text{lip}_A(\overline{\mathbb{D}}, X, \alpha) \) and \( \Lambda_\alpha^0(X) \)

In this section we characterize bounded and compact weighted composition operators on the spaces of analytic vector-valued little Lipschitz functions.

Ohno, et al. in [7, Theorem 4.1] showed that for \( \alpha \in (0, 1) \), the operator \( W_{\psi, \varphi} : \Lambda_\alpha^0 \rightarrow \Lambda_\beta^0 \) is bounded if and only if \( \psi \in \Lambda_\beta^0, W_{\psi, \varphi} : \Lambda_\alpha \rightarrow \Lambda_\beta \) is bounded and \( \lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{1 - \beta} \varphi'(z) \psi_z = 0 \). For the vector-valued case, we have weaker
results. That is, if \( W_{\psi,\varphi} : \Lambda_0^0(X) \to \Lambda_0^0(Y) \) is well defined and bounded, then \( \psi \in \Lambda_\beta(\mathbb{D}, L(X,Y)) \) and the point wise limit of \( (1 - |z|^2)^{1-\beta} \varphi'(z) \psi_z \) is zero, whenever \( |z| \to 1^- \).

By the next theorem, we provide some sufficient conditions for the weighted composition operator \( W_{\psi,\varphi} : \Lambda_0^0(X) \to \Lambda_0^0(Y) \) to be well defined and bounded.

**Theorem 3.1.** Let \( W_{\psi,\varphi} : \Lambda_0(X) \to \Lambda_\beta(Y) \) be bounded, \( \psi \in \Lambda_\beta(\mathbb{D}, L(X,Y)) \) and \( \lim_{|z| \to 1^-} (1 - |z|^2)^{1-\beta} \varphi'(z) \psi_z = 0. \) Then \( W_{\psi,\varphi} : \Lambda_0^0(X) \to \Lambda_0^0(Y) \) is well defined and bounded.

**Proof.** We just show that \( W_{\psi,\varphi} : \Lambda_0^0(X) \to \Lambda_0^0(Y) \) is well defined. The boundedness can be shown by means of the closed graph theorem. Let \( f \in \Lambda_0^0(X) \). Given \( \epsilon > 0 \), for some \( r < |z| < 1 \), we have

\[
(1 - |z|^2)^{1-\beta} \| \psi_z' \| < \frac{\alpha \epsilon}{4 \| f \|_{\Lambda_0(X)}},
\]

\[
(1 - |z|^2)^{1-\alpha} \| f'(z) \| < \frac{\epsilon}{4M},
\]

\[
(1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \| < \frac{\epsilon}{4L},
\]

where

\[
M = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \|}{(1 - |\varphi(z)|^2)^{1-\alpha}} < \infty,
\]

and

\[
L = \max \{ \sup_{|z| \leq r} \| f(z) \|, \sup_{|z| \leq r} \| f'(z) \| \} < \infty.
\]

Fix \( z \in \mathbb{D} \) with \( r < |z| < 1 \). If \( r < |\varphi(z)| < 1 \), then

\[
(1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \| \| f'(\varphi(z)) \| \leq \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} (1 - |\varphi(z)|^2)^{1-\alpha} |\varphi'(z)| \| \psi_z \| \| f'(\varphi(z)) \| < \frac{\epsilon}{4},
\]

and for \( |\varphi(z)| \leq r \) we have

\[
(1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \| \| f'(\varphi(z)) \| \leq L(1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \| < \frac{\epsilon}{4}.
\]

This shows that

\[
\sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \| \| f'(\varphi(z)) \| < \frac{\epsilon}{2}.
\]

It is easy to check that \( \| f((\varphi(z))) \| \leq \frac{1}{\alpha} \| f \|_{\Lambda_0(X)} \). Thus

\[
\sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \| (W_{\psi,\varphi}(f))'(z) \| \leq \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} |\varphi'(z)| \| \psi_z \| \| f'(\varphi(z)) \| + \sup_{r < |z| < 1} (1 - |z|^2)^{1-\beta} \| \psi_z' \| \| f(\varphi(z)) \| < \epsilon,
\]

which ensures that \( W_{\psi,\varphi}(f) \in \Lambda_0^0(Y). \)
For characterizing the compact weighted composition operators between analytic little Lipschitz spaces we need the next lemma.

**Lemma 3.2.** A subset $K$ of $\text{lip}_A(\overline{D}, X, \alpha)$ is relatively compact if and only if

(i) $K$ is bounded,

(ii) $K(z) = \{f(z) : f \in K\}$ is relatively compact for every $z \in \overline{D},$

(iii) $\lim_{|z| \to 1^-} \sup_{f \in K} (1 - |z|^2)^{1-\alpha} \|f'(z)\| = 0.$

**Proof.** We begin by proving that (iii) is necessary. Suppose that $K$ is relatively compact and let $C$ be a positive constant such that $\|\cdot\|_{\Lambda_\alpha(X)} \leq C\|\cdot\|_\alpha.$ Given $\epsilon > 0,$ there are functions $f_1, f_2, \ldots, f_n \in K$ such that for every $f \in K$ and for some $1 \leq j \leq n,$ we have $\|f - f_j\|_\alpha < \frac{\epsilon}{2C}$ which ensures that

$$(1 - |z|^2)^{1-\alpha} \|f'(z)\| \leq (1 - |z|^2)^{1-\alpha} \|f_j'(z)\| + \frac{\epsilon}{2}, \quad (z \in \overline{D}).$$

For each $1 \leq j \leq n,$ there exists $r_j \in (0, 1)$ such that

$$\sup_{r_j < |z| < 1} (1 - |z|^2)^{1-\alpha} \|f_j'(z)\| < \frac{\epsilon}{2}.$$

Setting $r = \max\{r_1, \ldots, r_n\}$ yields the assertion

$$\sup_{r < |z| < 1} (1 - |z|^2)^{1-\alpha} \|f'(z)\| \leq \epsilon, \quad (f \in K)$$

and (iii) holds.

For the converse, let $\{f_n\}$ be a bounded sequence in $\text{lip}_A(\overline{D}, X, \alpha).$ Hence, $\{f_n\}$ is an equicontinuous sequence in $C(\overline{D}, X).$ From (ii) and by generalized Arzela-Ascoli theorem, [3, Theorem A], $\{f_n\}$ is relatively compact in $C(\overline{D}, X).$ Thus there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is Cauchy in $C(\overline{D}, X).$ Let $g_n = f_{n_k}|_D.$ We show that $\{g_{n_k}\}$ is a Cauchy sequence in $\Lambda_\alpha^0(X).$ Fix $\epsilon > 0.$ By (iii), there exists $0 < r < 1$ such that for every $j,$

$$(1 - |z|^2)^{1-\alpha} \|g_{n_j}'(z)\| < \frac{\epsilon}{4}, \quad (r < |z| < 1).$$

It is easy to check that $\{g_{n_k}'\}$ is Cauchy with respect to compact open topology on $C(\overline{D}, X).$ Thus for $k$ and $l$ sufficiently large, we have

$$\sup_{|z| \leq r} \|g_{n_k}'(z) - g_{n_l}'(z)\| < \frac{\epsilon}{2}.$$  

Hence,

$$\sup_{z \in \overline{D}} (1 - |z|^2)^{1-\alpha} \|g_{n_k}'(z) - g_{n_l}'(z)\| \leq \sup_{|z| \leq r} (1 - |z|^2)^{1-\alpha} \|g_{n_k}'(z) - g_{n_l}'(z)\|$$

$$+ \sup_{r < |z| < 1} (1 - |z|^2)^{1-\alpha} \|g_{n_k}'(z) - g_{n_l}'(z)\|$$

$$< \epsilon,$$

which implies that $\{g_{n_k}\}$ is a Cauchy sequence in $\Lambda_\alpha^0(X).$ Thus $\{f_n\}$ is Cauchy in $\text{lip}_A(\overline{D}, X, \alpha),$ as desired. \qed
Regarding the arguments following the proof of Proposition 2.1, for every \( \psi \in \Lambda_{\beta}(K(X,Y)) \) there exists an operator-valued function \( \Psi \in \text{Lip}_A(\overline{D}, L(X,Y), \beta) \) such that \( \Psi|_D = \psi \). More precisely, for every \( z \in \partial D \), \( \Psi_z = \lim_{n \to \infty} \psi_{z_n} \), where \( \{z_n\} \) is any sequence in \( D \) converging to \( z \). Since for every \( n, \psi_{z_n} : X \to Y \) is compact, we deduce that \( \Psi \) is a compact linear operator from \( X \) in to \( Y \) and \( \Psi \in \text{Lip}_A(\overline{D}, K(X,Y), \beta) \).

Now we can characterize the compact weighted composition operators from \( \Lambda^0_{\alpha}(X) \) into \( \Lambda^0_{\beta}(Y) \).

**Theorem 3.3.** Let \( W_{\psi,\varphi} : \Lambda^0_{\alpha}(X) \to \Lambda^0_{\beta}(Y) \) be a bounded weighted composition operator.

(i) If \( W_{\psi,\varphi} \) is compact, then \( \psi \in \Lambda_{\beta}(K(X,Y)) \) and

\[
\lim_{|z| \to 1^-} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \psi_z = 0. \tag{3.1}
\]

(ii) If \( \psi \in \Lambda^0_{\beta}(K(X,Y)) \) and (3.1) holds, then \( W_{\psi,\varphi} \) is compact.

**Proof.** (i) Let \( U_\alpha \) be the closed unit ball of \( \Lambda^0_{\alpha}(X) \). Since \( W_{\psi,\varphi} \) is compact, \( W_{\psi,\varphi}(U_\alpha) \) is relatively compact in \( \Lambda^0_{\beta}(Y) \) and hence, \( E(W_{\psi,\varphi}(U_\alpha)) \) is relatively compact in \( \text{Lip}_A(\overline{D}, Y, \beta) \). By Lemma 3.2, we have

\[
\lim_{|z| \to 1^-} \sup_{f \in U_\alpha} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f))'(z)\| = 0.
\]

Given \( \epsilon > 0 \), there exists \( r \in (0,1) \) such that

\[
\sup_{r < |z| < 1} \sup_{f \in U_\alpha} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f))'(z)\| < \frac{\epsilon}{3}. \tag{3.2}
\]

For every \( x \in X \) and every nonzero \( a \in D \), define

\[
f_{a,x}(z) = \frac{1}{r} \left( \frac{1 - |a|^2}{(1 - \overline{a}z)^{1-\alpha}} - (1 - \overline{a}z)^\alpha \right) x.
\]

One can see that \( \{f_{a,x} : 0 \neq a, ||x|| \leq 1\} \) is a bounded subset of \( \Lambda^0_{\alpha}(X) \). Moreover, \( f_{a,x}(a) = 0 \), \( f_{a,x}'(a) = \frac{x}{(1 - |a|^2)^{1-\alpha}} \) and \( \sup_{||x|| \leq 1} ||f_{a,x}||_{\Lambda_{\alpha}(X)} \leq 3 \). Since \( (W_{\psi,\varphi}(f_{\varphi(z),x}))'(z) = \frac{\varphi'(z)}{(1 - |\varphi(z)|^2)^{1-\alpha}} \psi_z(x) \), by relation (3.2) we have

\[
\sup_{r < |z| < 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| \psi_z = \sup_{r < |z| < 1, ||x|| \leq 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| ||\psi_z||
\]

\[
= \sup_{r < |z| < 1, ||x|| \leq 1} (1 - |z|^2)^{1-\beta} \|(W_{\psi,\varphi}(f_{\varphi(z),x}))'(z)\|
\]

\[
< \epsilon.
\]

(ii) Let \( \psi \in \Lambda^0_{\beta}(K(X,Y)) \) and (3.1) holds. Fixing \( \epsilon > 0 \), for some \( r \in (0,1) \) we have

\[
\sup_{r < |z| < 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\varphi(z)|^2)^{1-\alpha}} |\varphi'(z)| ||\psi_z|| < \frac{\epsilon}{8}.
\]
and
\[
\sup_{r<|z|<1} (1 - |z|^2)^{1-\beta} \|\psi'_z\| < \frac{\alpha\epsilon}{8}.
\]

Let \(\{f_n\}\) be a bounded sequence in \(A_0^\alpha(X)\) such that \(\|f_n\|_{A_\alpha(X)} \leq 1\). Then \(\{E(f_n)\}\) is a bounded sequence in \(\text{lip}_A(D, X, \alpha)\) and \(\{W_{\Psi,\phi}(E(f_n))\}\) is a bounded sequence in \(\text{lip}_A(D, Y, \beta)\), which implies that \(\{W_{\Psi,\phi}(E(f_n))\}\) is equicontinuous. For every \(z \in D\), \(\{E(f_n)(\phi(z))\}\) is a bounded sequence in \(X\) and \(\Psi_z : X \to Y\) is a compact operator. Hence, \(\{W_{\Psi,\phi}(E(f_n))(z)\}\) is relatively compact in \(Y\). From Arzela-Ascoli theorem we deduce that \(\{W_{\Psi,\phi}(E(f_n))\}\) is relatively compact in \(C(D, Y)\). Therefore, there exists a subsequence \(\{f_{n_k}\}\) of \(\{f_n\}\) such that \(\{W_{\Psi,\phi}(f_{n_k})\}\) is uniformly convergent on compact subsets of \(D\). Using Cauchy’s integral theorem, one can show that \(\{(W_{\Psi,\phi}(f_{n_k}))'\}\) is convergent with respect to the compact open topology. Thus for \(k\) and \(l\) sufficiently large,
\[
\sup_{|z| \leq r} \| (W_{\Psi,\phi}(f_{n_k}))'(z) - (W_{\Psi,\phi}(f_{n_l}))'(z) \| < \frac{\epsilon}{4}.
\]

Therefore,
\[
\begin{align*}
\sup_{z \in D} (1 - |z|^2)^{1-\beta} \| & (W_{\Psi,\phi}(f_{n_k}))'(z) - (W_{\Psi,\phi}(f_{n_l}))'(z) \| \\
& \leq \sup_{|z| \leq r} (1 - |z|^2)^{1-\beta} \| (W_{\Psi,\phi}(f_{n_k}))'(z) - (W_{\Psi,\phi}(f_{n_l}))'(z) \| \\
& + \sup_{r<|z|<1} (1 - |z|^2)^{1-\beta} \| \phi'(z) \| \| \psi_z(f_{n_k}'(\phi(z)) - f_{n_l}'(\phi(z))) \| \\
& + \sup_{r<|z|<1} (1 - |z|^2)^{1-\beta} \| \psi'_z(f_{n_k}(\phi(z)) - f_{n_l}(\phi(z))) \| \\
& < \frac{\epsilon}{4} + 2 \sup_{r<|z|<1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\phi(z)|^2)^{1-\alpha}} |\phi'(z)| \| \psi_z \| \\
& + \frac{2}{\alpha} \sup_{r<|z|<1} (1 - |z|^2)^{1-\beta} \| \psi'_z \| < \epsilon.
\end{align*}
\]

We conclude that \(\{W_{\Psi,\phi}(f_{n_k})\}\) is a Cauchy sequence in \(A_0^\beta(Y)\) and hence \(W_{\Psi,\phi}\) is a compact operator. \(\square\)

The next corollary is an immediate consequence of Theorem 3.3.

**Corollary 3.4.** Let \(W_{\Psi,\phi} : \text{lip}_A(D, X, \alpha) \to \text{lip}_A(D, Y, \beta)\) be a bounded weighted composition operator.

(i) If \(W_{\Psi,\phi}\) is compact, then \(\Psi \in \text{Lip}_A(D, K(X, Y), \beta)\) and
\[
\lim_{|z| \to 1} \frac{(1 - |z|^2)^{1-\beta}}{(1 - |\phi(z)|^2)^{1-\alpha}} |\phi'(z)| \| \psi_z \| = 0. \tag{3.3}
\]

(ii) If \(\Psi \in \text{lip}_A(D, K(X, Y), \beta)\) and (3.3) holds, then \(W_{\Psi,\phi}\) is compact.
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