A tunneling current measurement scheme to observe Majorana-zero-mode-induced crossed Andreev reflection

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Keywords: Majorana zero mode, crossed Andreev reflection, tunneling current, electron transport, resonance condition, metallic ring, Aharonov–Bohm effect

Abstract

We propose a scheme to observe the Majorana-zero-mode-induced crossed Andreev reflection by measuring tunneling current directly. In this scheme a metallic ring structure is utilized to separate electron and hole signals. Since tunneling electrons and holes have different propagating wave vectors, the conditions for them to be constructively coherent in the ring differ. We find that when the magnetic flux threading the ring varies, it is possible to observe adjacent positive and negative tunneling current peaks of about equal amplitudes.

1. Introduction

Majorana zero modes (MZMs) have been attracting the interest of the condensed matter physics community in recent years [1, 2] due to their potential application in topological quantum computation [3]. Several proposals have been made to realize MZMs in realistic systems [4–6]. Among them, one is to deposit a strong spin–orbit coupled semiconductor nanowire in the proximity of an s-wave superconductor [6]. This proposal, due to its convenience, has spurred many experimental efforts immediately [7–10]. More recently, several other different methods [11–13] have been applied for the experimental observation of MZMs.

Several ways have been proposed to probe the existence of MZMs [1]. One of them is the tunneling spectroscopy, which is based on the MZM-induced Andreev reflection (AR) [14, 15]. There are two types of AR—local AR (LAR) and crossed AR (CAR). MZM-induced LAR occurs when a normal lead is coupled to one MZM. It can result in a zero–bias peak (ZBP) of the differential conductance in the tunneling process. In theory, this ZBP is quantized to $2e^2/h$ (here $e$ is the electron charge and $h$ is the Planck constant) at zero temperature [15]. Early experiments, however, were conducted to observe the ZBP with values much less than $2e^2/h$ [7–9, 16]. Studies show that several other mechanisms may cause such a non-quantized ZBP [17–22]. Only very recently, the quantized Majorana ZBP was reported to be observed in high-quality samples of InSb nanowires covered with superconducting aluminum [23].

In contrast to LAR, MZM-induced CAR comes into play only when two independent MZMs stay close to each other and two normal leads separately couple to them [14]. The fusion process [3, 24], which is a necessary component in topological quantum computation, involves putting two or more MZMs together to measure the quantum state, and thus may require a CAR-like experiment to check if two MZMs are really close to each other. It is usually believed that MZM-induced CAR does not result in any charge tunneling signal, since the electron tunneling probability equals to that of the hole [14]. Instead of the tunneling current, it has been suggested that shot noise [25] has special characteristics in MZM-induced CAR and can, in principle, be detected [14, 15, 26]. Nonetheless, to the best of our knowledge, no experiment has ever been done till now to measure the shot noise in a CAR set-up. In this work, we propose a new scheme to directly observe tunneling current in the MZM-induced CAR process, which goes beyond the previous belief that the shot noise measurement is necessary for...
the observation of MZM-induced CAR, and expect that it may help the future work on MZM based topological quantum computation.

To demonstrate our motivation, a heuristic interpretation of the MZM-induced CAR is illustrated in figure 1. Suppose that two normal leads separately couple to two MZMs \( \gamma_1 \) and \( \gamma_2 \). When the two MZMs stay far away and have no coupling, each of them has a propagator \( G_j^{(0)}(E) = 1/(E + i\Gamma_j) \), where the subscript \( j = 1, 2 \) represents the \( j \)th MZM, \( E \) is the energy and \( \Gamma_j \) is the life time of the Majorana that originates from its coupling to the normal lead. It can be shown that \( \Gamma_j \approx 2t_j^2/v_F \), where \( t_j \) is the tunneling amplitude between Majorana and lead, and \( v_F \) is the Fermi velocity of the electron in the lead. As the two MZMs come close and begin to interact with each other, there appears a tunneling amplitude \( E_M \) between them. In the case this coupling is small in the sense that \( E_M < E \) and \( \Gamma_j \), it can be taken as a perturbation. The amplitude of CAR is determined by the propagator from \( \gamma_1 \) to \( \gamma_2 \):

\[
G_{1\to2} = G_1^{(0)}E_MG_2^{(0)} + G_1^{(0)}E_MG_2^{(0)}E_MG_1^{(0)}G_2^{(0)} + ... ,
\]

where the first term represents the process of hopping from \( \gamma_1 \) to \( \gamma_2 \), the second term represents the process of hopping from \( \gamma_1 \) to \( \gamma_2 \) and then from \( \gamma_2 \) to \( \gamma_1 \) and then from \( \gamma_1 \) to \( \gamma_2 \), and the ellipsis represents the higher order processes. Compared to it, the LAR has an amplitude that is determined by the propagator from \( \gamma_1 \) to \( \gamma_1 \) itself:

\[
G_{1\to1} = G_1^{(0)} + G_1^{(0)}E_MG_2^{(0)}E_MG_1^{(0)} + ... ,
\]

where the right hand side terms have similar meanings as that have been provided above for \( G_{1\to2} \). It follows that the two amplitudes have a relationship of \( G_{1\to2} = G_{1\to1}E_M/E + i\Gamma_j \), or

\[
G_{1\to2} = G_{1\to1}E_M/E + i\Gamma_j .
\]

Although the perturbation expansion is restricted to the situation with \( E_M \) being small, this algebraic relation between the two propagators has no such restriction. It holds also in the regime of \( E_M > E \) and \( E_M > \Gamma_j \), where it can be taken as the analytical continuation from the regime of \( E_M < E \) and \( E_M < \Gamma_j \). Thus, the process of CAR dominates over that of the LAR when \( E_M \) is much larger than \( E \) and \( \Gamma_j \), as was first shown in [14]. Furthermore, since a MZM is an equal superposition of electron and hole, it is readily seen from figure 1 that electrons and holes on the normal leads have equal status as long as only the energy is considered. Therefore, normally the tunneling current of electrons and holes have the same amplitude and cancel each other in the CAR process, as was also first shown in [14].

The key idea we pursue in this work is to separate electron and hole tunneling signals by distinguishing their wave vectors. To accomplish this goal, we employ metallic rings as wave vector filters. When a magnetic flux is threaded through a ring, an electron/hole acquires an Aharonov–Bohm (AB) phase when traveling around the ring [27]. The resonant condition for the electron/hole to be constructively coherent in the ring is

\[
\phi_D \pm \phi_M = 2n\pi \ (n \in \mathbb{Z}),
\]

where \( \phi_D = kL \) (here \( k \) is the wave vector of the electron/hole, \( L \) is the circumference of the ring) and \( \phi_M = 2\pi \Phi/\Phi_0 \) (here \( \Phi \) is the magnetic flux threading the ring, \( \Phi_0 \) is the flux quantum) are the dynamical phase and the magnetic phase (AB phase) that an electron/hole acquires when traveling one complete circle around the ring, and \( \pm \) denotes clockwise/counter-clockwise circulations. Since under a finite bias, tunneling electrons (above
the Fermi level) have larger wave vectors than tunneling holes (below the Fermi level), they need not be simultaneously at resonance in general. If we couple an external lead weakly to the ring, only when the resonant condition is fulfilled, there is a noticeable tunneling signal. Therefore, we expect to observe electron and hole tunneling peaks separately corresponding to different magnetic fluxes.

In this paper, we construct a theoretical model and perform numerical calculations to analyze the above idea in detail. The main text is organized as follows. In section 2, we elaborate on a two-ring setup and follow the spirit of the Landauer–Büttiker formalism [28, 29] to derive a set of equations determining the electron and hole tunneling probabilities. In section 3, we discuss the physical consequences that follow by solving the equations. In section 4, we provide our main conclusions.

2. Setup and model

In the early works on MZM-induced CAR [14, 15], a typical setup involves a pair of overlapping MZMs, of which each is coupled to a metallic lead. Electrons flow in along one lead and encounter the MZMs. It was shown that the electron and hole tunneling probabilities through the MZMs are equal, and no net charge current is likely to be observed on the other side. Therefore, studies have been focused on shot noise in such type of devices. In order to directly observe a tunneling current, we insert two metallic rings between the leads and MZMs, as shown in figure 2. We expect the resonance condition (4) to play an important role in the transport process, and this happens when the rings are coupled weakly to the MZMs and the external leads. The two rings in our setup have different functionalities. On the left side, a ring is connected to a source reservoir and its function is to single out the incident electrons. Only when the wave vector of an incident electron fulfills the resonance condition, it has a substantial contribution to the tunneling process. On the right side, another ring is connected to a drain reservoir and its function is to control whether the final tunneling current is dominated by electrons or by holes.

It is worth mentioning here that the structure of two directly coupled metallic rings was investigated previously [30–32], and two metallic rings coupled by a p-wave wire was studied in [33]. These studies focused on equilibrium properties, especially persistent currents inside the rings.

2.1. Setup description

We first provide a physical description of the setup as shown in figure 2. A pair of MZMs are induced at the opposite ends of the semiconductor wire when an appropriate magnetic field is applied along the wire [6]. In the

![Figure 2. A schematic setup to measure tunneling current in the MZM-induced crossed Andreev reflection. In the central part there is a strong spin–orbit-coupled semiconductor wire in proximity to an s-wave superconductor. Two coupled MZMs are induced at the opposite ends of the wire when an appropriate magnetic field is applied along it. Two metallic rings, threaded by magnetic fluxes Φ₁ and Φ₂, are coupled to the semiconductor wire and two external leads. The tunneling strengths of electrons and holes across the leading junctions and the ring–wire junctions are controlled by gate-induced barriers. The two leads are extended to a source reservoir and a drain reservoir, respectively. A voltage bias is applied on the source side and the tunneling current is measured on the drain side.](image-url)
low-energy regime, the physics of the wire is dominated by the MZMs, which can be described by a Hamiltonian of the form

$$H_M = \frac{1}{2}E_M \gamma_1 \gamma_2.$$  

Here $E_M$ is the coupling energy, $\gamma_1$ and $\gamma_2$ are Majorana operators, satisfying

$$\gamma_i \gamma_j = \delta_{ij} \delta(x - y)$$

and $\gamma_1 \gamma_2 + \gamma_2 \gamma_1 = 0$.

Two metallic rings are coupled to the semiconductor wire. In the low energy regime, this coupling is described by an effective Hamiltonian

$$H^{li} = \int \frac{dx}{\sqrt{2}} \, \frac{t_i}{\sqrt{2}} \, \gamma_i \{ \psi_i(x) - \psi_i(x) \} \delta(x) \quad (i = 1, 2),$$

where $t_i$ denotes the coupling amplitude controlled by the gate potential, $\psi_i(x)$ and $\psi_i^\dagger(x)$ represent electron annihilation and creation operators in the $i$th ring, $\delta(x)$ is a delta function indicating the coupling is local in space, and the origin $x = 0$ of the ring is selected to be its closest point to the semiconductor wire. The operators $\psi_i(x)$ and $\psi_i^\dagger(x)$ satisfy the commutation relations

$$\{ \psi_i(x), \psi_j(y) \} = \delta_{ij} \delta(x - y)$$

and

$$\{ \psi_i(x), \psi_j(y) \} = \{ \psi_i^\dagger(x), \gamma_i \} = \{ \psi_i^\dagger(x), \gamma_j \} = 0.$$

Furthermore, two metallic leads separately bridge the two reservoirs, one source and one drain, to the two rings. We can adjust the gate potential to allow the couplings between the leads and the rings to be weak in strength.

### 2.2. Model analysis

According to the Landauer–Büttiker theory [28, 29], when an electrochemical potential bias is imposed between the source and the drain, a current is triggered to flow from the source to the drain. At zero temperature, electrons in the source that have energies in the range of the bias contribute to a net current. The current flowing to the drain that we would like to measure is a sum of electron and hole currents, the magnitudes of which depend on their tunneling probabilities through the entire setup. Thus, to study the transport process, one needs to calculate the tunneling probabilities. We will proceed to give a set of equations that completely determine these probabilities in section 2.4. First, we analyze the subprocesses that constitute the entire transport process.

Electrons out of the source flow along the first lead until the end of the lead is reached. At this point, it is possible for electrons to tunnel from the lead to the ring. The junction formed by the lead and the ring has three channels: the lead and two ring arms. A general scattering process on a tri-junction is shown in figure 3, and it can be described by a 3-by-3 scattering matrix [34]:

![Figure 3. The scattering process at a lead-ring tri-junction. The arrows represent propagating electrons (or holes). The lead is labeled '1', and the two ring-arms are labeled '2' and '3'.](image-url)
where $\epsilon$ denotes the tunneling strength, $a = (\sqrt{1 - 2\epsilon} - 1)/2$ and $b = (\sqrt{1 - 2\epsilon} + 1)/2$. When the coupling between the lead and the ring is weak, $\epsilon$ is small compared to 1.

After electrons tunnel to the left ring, they propagate along the ring arms. An electron acquires a dynamic phase and a magnetic phase during this process. The dynamic phase $f_D$ equals to $kL$, where $k$ denotes the electron wave vector and $L$ denotes the distance it covers. The magnetic phase $f_M$ depends on the electromagnetic gauge chosen, however when the electron covers a complete circle, the magnetic phase it acquires does not depend on the chosen gauge but equals to $2\pi \Phi/\Phi_0$.

When electrons in the left ring come to the region where the ring couples to the MZM $\gamma_1$, they have a probability to tunnel to the right ring through MZM-induced CAR, as shown in Figure 4 (for convenience, the ring arms are drawn as straight lines). Due to the presence of superconductor, holes are generated during the process of AR. Holes, just like electrons, will propagate in both rings, and may come back to the MZM-coupled region to induce CAR again. In general, the scattering process of CAR illustrated in Figure 4 can be described by an 8-by-8 matrix:

$$
S_L = \begin{pmatrix}
-(a + b) & -\sqrt{\epsilon} & -\sqrt{\epsilon} & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{\epsilon} & a & b & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{\epsilon} & b & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

(7)

of which the eight channels are lower left electron, upper left electron, lower right electron, upper right electron, lower left hole, upper left hole, lower right hole and upper right hole. The detailed form of these matrix elements will be given and derived in the next subsection.

Lastly, electrons and holes in the right ring have a probability to tunnel to the second lead and flow along it into the drain. The scattering process at the tri-junction between the second ring and the second lead can also be described by the scattering matrix of the form of (7). Generally, the tunneling strength $\epsilon$ and the corresponding $a$ and $b$ may differ for the left tri-junction and the right tri-junction, and for electrons and holes.
2.3. Scattering matrix of the MZM-induced CAR

We now derive the MZM-induced CAR scattering matrix (8). To study the scattering process shown in figure 4, we first write a Hamiltonian that is minimal in terms of describing it

\[ H = \int dx \, \psi_1^\dagger(x) \left[ \frac{p_x^2}{2m^*} - E_F \right] \psi_1(x) + \int dy \psi_2^\dagger(y) \left[ \frac{p_y^2}{2m^*} - E_F \right] \psi_2(y) + H_{\text{int}} + H_{\text{t}}^{(1)} + H_{\text{t}}^{(2)}, \]

where the first two terms describe the electronic motion in the two metallic wires, the third term is given in (5) representing the tunneling between the two MZMs, the last two terms are given in (6) representing the tunneling between the wire and the MZM. The quantities \( m^* \) and \( E_F \) denote the electron effective mass and the Fermi energy, respectively.

Assume an operator

\[ \Gamma = \int dx \left[ D_1(x) \psi_1(x) + B_1(x) \psi_1^\dagger(x) \right] + \frac{C_1}{\sqrt{2}} \psi_1 + \int dy \left[ D_2(y) \psi_2(y) + B_2(y) \psi_2^\dagger(y) \right] + \frac{C_2}{\sqrt{2}} \psi_2 \]

that diagonalizes the Hamiltonian, in the sense that

\[ [\Gamma, H] = E \Gamma. \] (11)

Here \( E \) represents the energy of the eigenstate represented by \( \Gamma \). Substituting (9) and (10) into (11) and employing the commutation relations to simplify the formula, we obtain the Bogoliubov-de Gennes (BdG) equations

\[
\begin{align*}
&\left[ \frac{1}{2m^*} p_x^2 - E_F \right] D_1(x) - t_1 \delta(x) G_1 = E D_1(x), \\
&\left[ \frac{1}{2m^*} p_x^2 - E_F \right] B_1(x) + t_1 \delta(x) G_1 = E B_1(x), \\
&\left[ \frac{1}{2m^*} p_y^2 - E_F \right] D_2(y) - t_2 \delta(y) G_2 = E D_2(y), \\
&\left[ \frac{1}{2m^*} p_y^2 - E_F \right] B_2(y) + t_2 \delta(y) G_2 = E B_2(y), \\
&\frac{1}{\sqrt{2}} \left( -d_F(0) + B_1(0) \right) - i E_\delta G_1 = E C_1, \\
&\frac{1}{\sqrt{2}} \left( -d_F(0) + B_2(0) \right) + i E_\delta G_1 = E C_2.
\end{align*}
\]

Now suppose an electron is incident from the lower side of the left wire, as shown in figure 4. The eigenstate wave function can be written as

\[
D_1(x) = \begin{cases} \\
\epsilon^{ik_x} + r_1^{(e)} e^{-ik_x}, & x < 0, \\
t_1^{(e)} e^{ik_x}, & x > 0,
\end{cases} \]

\[
B_1(x) = \begin{cases} \\
r_1^{(h)} e^{ik_x}, & x < 0, \\
t_1^{(h)} e^{-ik_x}, & x > 0,
\end{cases} \]

\[
D_2(y) = \begin{cases} \\
w_1^{(e)} e^{-ik_y}, & y < 0, \\
w_1^{(e)} e^{ik_y}, & y > 0,
\end{cases} \]

\[
B_2(y) = \begin{cases} \\
w_1^{(h)} e^{ik_y}, & y < 0, \\
w_1^{(h)} e^{-ik_y}, & y > 0.
\end{cases} \]

Here \( k_x = \sqrt{2m^*(E_F + E)}/\hbar \) and \( k_y = \sqrt{2m^*(E_F - E)}/\hbar \) are electron and hole wave vectors, \( r_1^{(e)} \) and \( t_1^{(e)} \) represent electron reflection and transmission amplitudes, \( r_1^{(h)} \) and \( t_1^{(h)} \) represent LAR amplitudes, \( w_1^{(e)} \) and \( w_1^{(h)} \) represent electron and hole tunneling amplitudes to the right wire.

Substituting this wave function ansatz into the BdG equations, we can solve for all the unknown amplitudes. The results are

\[
t_1^{(e)} = \frac{t_1^{(e)}}{\hbar v_c} \left[ E + i \left( \frac{t_2^{(e)}}{\hbar v_h} + \frac{t_2^{(h)}}{\hbar v_h} \right) \right]/Z, \]

\[
t_1^{(e)} = 1 + r_1^{(e)}, \]

\[
r_1^{(h)} = -\frac{t_1^{(h)}}{\hbar v_h} \left[ E + i \left( \frac{t_2^{(e)}}{\hbar v_c} + \frac{t_2^{(h)}}{\hbar v_c} \right) \right]/Z, \]

\[
t_1^{(h)} = t_1^{(h)}, \]
\[ w_1^{(e)} = -E_M \left( \frac{t_1 t_2}{\hbar v_c} \right) / Z, \]  
\[ w_1^{(h)} = E_M \left( \frac{t_1 t_2}{\hbar v_h} \right) / Z, \]

and

\[ G_1 = t_1 \left[ E + i \left( \frac{t_1^2}{\hbar v_c} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z, \]
\[ C_2 = iE_M t_1 / Z, \]

where \( v_e = \hbar k_e / m^* \) and \( v_h = \hbar k_h / m^* \) are electron and hole velocities, and

\[ Z = E_M^2 - \left[ E + i \left( \frac{t_1^2}{\hbar v_c} + \frac{t_2^2}{\hbar v_h} \right) \right] \left[ E + i \left( \frac{t_1^2}{\hbar v_c} + \frac{t_2^2}{\hbar v_h} \right) \right]. \]

Next, suppose a hole is incident from the lower side of the left wire. The situation can be described by a wave function of the form

\[ \tilde{B}_1(x) = \begin{cases} e^{-i k_x x} + \tilde{z}_1^{(h)} e^{i k_x x}, & x < 0, \\ \tilde{z}_1^{(h)} e^{-i k_x x}, & x > 0, \end{cases} \]
\[ \tilde{D}_1(x) = \begin{cases} \tilde{z}_1^{(e)} e^{-i k_x x}, & x < 0, \\ \tilde{z}_1^{(e)} e^{i k_x x}, & x > 0, \end{cases} \]
\[ \tilde{B}_2(y) = \begin{cases} \tilde{w}_1^{(h)} e^{i k_y y}, & y < 0, \\ \tilde{w}_1^{(h)} e^{-i k_y y}, & y > 0, \end{cases} \]
\[ \tilde{D}_2(y) = \begin{cases} \tilde{w}_1^{(e)} e^{-i k_y y}, & y < 0, \\ \tilde{w}_1^{(e)} e^{i k_y y}, & y > 0. \end{cases} \]

Substituting these into the BdG equations and solving them, we obtain

\[ \tilde{z}_1^{(h)} = \frac{t_1^2}{\hbar v_h} \left[ E + i \left( \frac{t_1^2}{\hbar v_c} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z, \]
\[ \tilde{z}_1^{(h)} = 1 + \tilde{z}_1^{(h)}, \]
\[ \tilde{z}_1^{(e)} = -\frac{t_1^2}{\hbar v_c} \left[ E + i \left( \frac{t_1^2}{\hbar v_c} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z, \]
\[ \tilde{z}_1^{(e)} = \tilde{z}_1^{(e)}, \]
\[ \tilde{w}_1^{(h)} = -E_M \left( \frac{t_1 t_2}{\hbar v_h} \right) / Z, \]
\[ \tilde{w}_1^{(e)} = E_M \left( \frac{t_1 t_2}{\hbar v_c} \right) / Z, \]

and

\[ \tilde{C}_4 = -t_1 \left[ E + i \left( \frac{t_1^2}{\hbar v_c} + \frac{t_2^2}{\hbar v_h} \right) \right] / Z, \]
\[ \tilde{C}_2 = -iE_M t_1 / Z, \]

where \( Z \) is the same as that given in (30).

Other possible processes consist of: (1) an electron is incident from the upper side of the left wire; (2) a hole is incident from the upper side of the left wire; (3) an electron is incident from the lower side of the right wire; (4) a hole is incident from the lower side of the right wire; (5) an electron is incident from the upper side of the right wire; (6) a hole is incident from the upper side of the right wire. The solution to these transport processes is similar to the above two we have considered. The results can be summarized into the scattering matrix of the form of (8), where the (not yet prescribed) matrix elements are
\[
\begin{align*}
t_2^{(e)} &= \frac{t_2^{(e)} e^2}{2 \hbar v_F} \left[ E + i \left( \frac{t_2^{(e)} e^2}{\hbar v_F} + \frac{t_2^{(e)} e^2}{\hbar v_F} \right) \right] / Z, \\
t_2^{(e)} &= 1 + t_2^{(e)}, \\
t_2^{(b)} &= -i \frac{t_2^{(b)}}{\hbar v_F} \left[ E + i \left( \frac{t_2^{(b)}}{\hbar v_F} + \frac{t_2^{(b)}}{\hbar v_F} \right) \right] / Z, \\
t_2^{(b)} &= t_2^{(b)}, \\
w_2^{(e)} &= E_M \left( \frac{t_2^{(e)}}{\hbar v_F} \right) / Z, \\
w_2^{(b)} &= -E_M \left( \frac{t_2^{(b)}}{\hbar v_F} \right) / Z,
\end{align*}
\]

and
\[
\begin{align*}
t_2^{(b)} &= \frac{t_2^{(b)} e^2}{2 \hbar v_F} \left[ E + i \left( \frac{t_2^{(b)} e^2}{\hbar v_F} + \frac{t_2^{(b)} e^2}{\hbar v_F} \right) \right] / Z, \\
t_2^{(b)} &= 1 + t_2^{(b)}, \\
t_2^{(e)} &= -i \frac{t_2^{(e)}}{\hbar v_F} \left[ E + i \left( \frac{t_2^{(e)}}{\hbar v_F} + \frac{t_2^{(e)}}{\hbar v_F} \right) \right] / Z, \\
t_2^{(e)} &= t_2^{(e)}, \\
w_2^{(b)} &= E_M \left( \frac{t_2^{(b)}}{\hbar v_F} \right) / Z, \\
w_2^{(e)} &= -E_M \left( \frac{t_2^{(e)}}{\hbar v_F} \right) / Z.
\end{align*}
\]

In the work of Nilsson et al. [14], it is claimed that the MZM-induced CAR dominates LAR when the electron incident energy and the MZM energy width are smaller than \( E_M \). Although their conclusion is based on the calculation for a specific setup, in which the MZMs are directly coupled to metallic leads, it also holds for our setup where the MZMs are coupled to metallic ring arms. From the form of the scattering matrix elements given above, if we interpret \( t_2^{(i)} / \hbar v_F \) (\( v_F \approx v_L \approx v_h \) is the Fermi velocity) as the MZMs energy width due to their coupling to the rings, it is readily seen that the CAR probability \( |w_2^{(i)}|^2 \) is larger than the LAR probability \( |r_2^{(i)}|^2 \) when \( E_M \gg E \) and \( E_M \gg t_2^{(i)} / \hbar v_F \), as was pointed out in the introduction.

### 2.4. The entire transport process

We proceed to provide a complete study of the single electron transport process in the setup shown in figure 2. Let an electron be incident from the source. For simplicity, suppose both electron and hole tunneling processes occurring at both the left and right tri-junctions can be described uniformly by (7) with \( \epsilon \) (and the corresponding \( a, b \)) equal for all the four distinct situations. Let \( D \) and \( B \) respectively represent electron and hole propagating amplitudes, subscripts 1 and 2 respectively denote counter-clockwise and clockwise motions in the ring, superscript \((i)\) be the corresponding part in the setup as shown in figure 5 inset (b).

First, the tunneling process occurring at the tri-junction between the lower left lead and the left ring can be described as
\[
S_L \begin{pmatrix} 1 \\ D_1^{(2)} \\ D_2^{(3)} \end{pmatrix} = \begin{pmatrix} D_1^{(1)} \\ D_2^{(2)} \\ D_3^{(3)} \end{pmatrix},
\]

\[
S_L \begin{pmatrix} 0 \\ B_1^{(2)} \\ B_2^{(3)} \end{pmatrix} = \begin{pmatrix} B_1^{(1)} \\ B_2^{(2)} \\ B_3^{(3)} \end{pmatrix},
\]

where the electron and hole incoming amplitudes are 1 and 0 in the lead according to our assumption of only electrons being incident. Here \( D^{(i)} \) and \( B^{(i)} \) denote electron and hole outgoing amplitudes along the lead.

Secondly, electron and hole propagations in the left ring can be described by
\[
D_i^{(3)} = D_i^{(2)} \exp[i(-k_L L - \phi_i)],
\]

\[
B_i^{(3)} = B_i^{(2)} \exp[i(-k_L L + \phi_i)].
\]
Figure 5. A detailed analysis of the steady state transport process. Electrons are incident along the lower left lead, and the final tunneling electrons and holes flow out along the upper right lead. The inset (a) marks the geometrical length of the corresponding arcs of the rings. The inset (b) marks the different regions in the setup.

\[ D_2^{(2)} = D_2^{(2)} \exp[i(+k_c L_1^a - \phi_1^a)], \]  
\[ B_1^{(2)} = B_1^{(2)} \exp[i(+k_c L_1^a + \phi_1^a)], \]  
\[ B_2^{(2)} = B_2^{(2)} \exp[i(-k_c L_1^a + \phi_1^a)], \]  
\[ D_1^{(1)} = D_1^{(1)} \exp[i(+k_c L_1^b + \phi_1^b)], \]  
\[ D_2^{(1)} = D_2^{(1)} \exp[i(-k_c L_1^b + \phi_1^b)], \]  
\[ B_1^{(1)} = B_1^{(1)} \exp[i(-k_c L_1^b - \phi_1^b)], \]  
\[ B_2^{(1)} = B_2^{(1)} \exp[i(+k_c L_1^b - \phi_1^b)], \]  

where \(L_1^a\) (\(L_1^b\)) denotes the length of the arm \(a\) (\(b\)) of the left ring as shown in figure 5, and \(\phi_1^a\) (\(\phi_1^b\)) denotes the magnetic phase an electron acquires when circulating counter-clockwise or a hole acquires when circulating clockwise along the entire arm \(a\) (\(b\)).

Consequently, the MZM-induced CAR occurring at the central part of figure 5 can be described as

\[
\begin{bmatrix}
D_2^{(5)} \\
D_2^{(4)} \\
D_1^{(7)} \\
D_2^{(6)} \\
B_2^{(5)} \\
B_1^{(4)} \\
B_1^{(7)} \\
B_2^{(6)}
\end{bmatrix} = S_M
\begin{bmatrix}
D_2^{(5)} \\
D_2^{(4)} \\
D_1^{(7)} \\
D_2^{(6)} \\
B_2^{(5)} \\
B_1^{(4)} \\
B_1^{(7)} \\
B_2^{(6)}
\end{bmatrix}.
\]

Electron and hole propagations in the right ring can be described by

\[ D_1^{(9)} = D_1^{(6)} \exp[i(+k_c L_2^a + \phi_2^a)], \]  
\[ D_2^{(9)} = D_2^{(6)} \exp[i(-k_c L_2^a + \phi_2^a)], \]  
\[ B_1^{(9)} = B_1^{(6)} \exp[i(-k_c L_2^a - \phi_2^a)], \]  
\[ B_2^{(9)} = B_2^{(6)} \exp[i(+k_c L_2^a - \phi_2^a)]. \]
\[ D_1^{(8)} = D_1^{(7)} \exp[i(-k_L L^b - \phi_2)], \]  
\[ D_2^{(8)} = D_2^{(7)} \exp[i(+k_L L^b - \phi_2)], \]  
\[ B_1^{(8)} = B_1^{(7)} \exp[i(+k_L L^b + \phi_2)], \]  
\[ B_2^{(8)} = B_2^{(7)} \exp[i(-k_L L^b + \phi_2)], \]

where \( L^a \) (\( L^b \)) denotes the length of the arm \( a (b) \) of the right ring, and \( \phi_2^a \) (\( \phi_2^b \)) denotes the magnetic phase an electron acquires when circulating counter-clockwise or a hole acquires when circulating clockwise along the entire arm \( a (b) \).

Lastly, the tunneling process occurring at the tri-junction between the right ring and the upper right lead can be described as

\[
\begin{bmatrix}
0 \\
D_2^{(8)} \\
D_1^{(9)}
\end{bmatrix} = 
\begin{bmatrix}
D_1^{(10)} \\
D_2^{(9)}
\end{bmatrix},
\]

where on the upper right lead electron and hole incoming amplitudes are both 0. Here \( D^{(10)} \) and \( B^{(10)} \) denote the electron and hole outgoing amplitudes separately.

Combining (55) through (75), we are able to obtain all the amplitudes. Actually, this is a group of linear equations that is easy to solve numerically. What we are concerned with most in the following are \( P_e = |D^{(10)}|^2 \) and \( P_h = |B^{(10)}|^2 \), representing the electron and hole tunneling probabilities through the whole setup.

### 3. Results and discussion

#### 3.1. Tunneling probabilities and current

The electron and hole tunneling probabilities \( P_e \) and \( P_h \) are affected by the magnetic fluxes \( \Phi_M^{(1)} \) and \( \Phi_M^{(2)} \) that thread the two metallic rings. For convenience we express them as functions of the magnetic phases \( \phi_M^{(1)} (=2\pi\Phi_M^{(1)}/\Phi_0) \) and \( \phi_M^{(2)} (=2\pi\Phi_M^{(2)}/\Phi_0) \). A numerical result for \( P_e \) and \( P_h \) by solving (55) through (75) is shown in figure 6, under a group of typical settings of the relevant parameters. The magnetic fluxes are varied such that the magnetic phases \( \phi_M^{(1)} \) and \( \phi_M^{(2)} \) scan from 0 to 2\( \pi \). This includes all possibilities since any other situation is equivalent to one set of \( \phi_M^{(1)} \) and \( \phi_M^{(2)} \) in this range, due to the Byers–Yang theorem [35]. The electron incident energy \( E \) and the MZM width \( \Gamma_j = t_{ij}^2/\hbar v_F \) (resulting from their couplings to the rings) are both chosen to be much smaller than the MZM coupling energy \( E_{M} \), so that CAR dominates LAR. The circumferences of the two rings are chosen a little bit different in order to distinguish their respective resonance conditions.

We note from figure 6, that the electron/hole tunneling probability peaks when the resonance conditions \( \phi_{e/h}^{(1)} \pm \phi_{e/h}^{(1)} = 2n\pi \) and \( \phi_{e/h}^{(2)} \pm \phi_{e/h}^{(2)} = 2n\pi \) on both the rings are fulfilled simultaneously. Since under our parameter setting \( \phi_e \) and \( \phi_h \) are only slightly different from each other, the electron and hole tunneling peaks in figure 6 locate very close to each other. Nonetheless, they can be clearly distinguished. Actually, the exact locations of the peaks are a little further apart than those that the resonance conditions indicate, due to the effect of quantum mechanical level repulsion. However, the difference is small when we make the rings couple weakly to the semiconductor wire and the external leads.

The tunneling current resulting from a single incident electron is given by [36]:

\[
j = j_{\text{ino}}[P_e - (v_v / v_h)P_h],
\]

where \( j_{\text{ino}} \) represents the current carried by the incident electron, \( P_e/h \) is the electron/hole tunneling probability and \( v_v/h \) is the velocity. In our setting the incident energy \( E \) is small compared to \( E_p \), thus \( v_e \approx v_h \approx v_{0} \), and so \( j \approx j_{\text{ino}}(P_e - P_h) \).

Figure 7 shows the behavior of the tunneling current \( j \) as the magnetic fluxes are varied. From figure 7(a), we see that there exist separate regions where the tunneling current is dominated by positive and negative currents, respectively. The positive current peaks indicate the tunneling current is dominated by electrons, while the negative current peaks indicate the tunneling current is dominated by holes. The boundaries between these regions are the lines on which \( k_L L^b \pm \phi_{M}^{(2)} = 2n\pi \) holds. If we fix the magnetic flux threading the left ring, and vary the flux in the right ring, we see a negative current peak immediately followed by a positive one, or

\[^{5}\text{The simplification here seems to be in contradiction with our key prerequisite that } k_e = k_h \text{. It is worth noting that a small difference of } k_e \text{ and } k_h \text{ in the phase calculation is really the critical ingredient of our proposal, while their appearance in the amplitude can be ignored safely.}\]
vice versa, as shown in figure 7(b). The separation of adjacent opposite peaks is due to the slight difference of the electron and hole wave vectors. Furthermore, it is seen in figure 7(b) that the positive and negative tunneling current peaks have almost equal amplitudes. This implies that the process of CAR is through MZMs, since they couple in a balanced way to electrons and holes as is evident from the Hamiltonian in (6). It is worth recalling that for the same reason $P_F$ equals to $P_h$ in the case when the metallic rings are absent and two external leads are...
directly coupled to the MZMs. By introducing the rings, we are able to distinguish electrons and holes by their different wave vectors and thus break the equality of \(P_e\) and \(P_h\). However, the symmetry between electron and hole does not vanish. Its physical effect has been transformed to the equal tunneling amplitudes we have just seen.

3.2. Discussion

The above results and discussion are restricted to a single electron incident with a specific energy. In realistic situations, we connect the first lead to a source and the second lead to a drain, and apply a negative voltage bias between them. We also ground the superconductor hosting the MZMs. Electrons are incident from the source, and it depends on the flux threading the second ring whether it is an electron or hole that dominates the current to the drain. When electrons dominate, there is little current flowing through the superconductor to the ground. While if holes dominate, there is an electron current flowing through the superconductor to the ground. The external wiring of our set-up and the overall physical phenomenon are analogous to those studied in [37].

Assuming the temperature is well below the bias voltage (and also \(E_M\)), all the electron states within the range of the bias participate in the transport process, according to the Landauer–Buttiker formalism [38]. This complication, however, does not change our established picture of the tunneling current. The reason is twofold. First, the resonance has a width, originating from the coupling of the ring to the lead and MZM. This width also determines the phase resolution of the typical tunneling signal shown in figure 7(b). If the applied bias is made to be smaller than the resonance width, it is possible that all incident electrons are effectively in resonant states. Second, the positions of tunneling peaks (in the magnetic-phase versus tunneling-current diagram) may slightly vary for different incident electrons, but the overall effect is not that their tunneling currents are canceled, for the wave vectors of the holes are always smaller than those of the electrons, resulting in a separation of all the positive current peaks from all the negative ones by a boundary set by the Fermi wave vector \(k_F\). Therefore, the line shape of the tunneling current as shown in figure 7(b) can be qualitatively taken as a signature of the MZM-induced CAR.

In our model we implicitly assume that the electrons are spinless, by considering that the electron spin, in an experimental set-up, can be fully polarized in the presence of a Zeeman magnetic field. We also implicitly assume that there is only a single channel in the metallic ring. Actually, we do not intend the rings to have multiple channels, for in that case electrons in different channels can have distinct wave vectors, such that in one magnetic-phase period (from 0 to \(2\pi\)) there may exist multiple resonant states, of which if they have overlaps, the tunneling currents may cancel each other. To ensure the ring has only one channel, the geometric width of the ring should be sufficiently small.

In a typical experimental setting, we intend \(\phi_e\) and \(\phi_h\) to be only slightly different from each other, since there are always two resonance points in a \(2\pi\)-period of \(\phi_M\) corresponding to \(\phi_{e/h} + \phi_M = 2n\pi\) and \(\phi_{e/h} - \phi_M = 2n\pi\) separately. If the difference between \(\phi_e\) and \(\phi_h\) is large, there might be a cross canceling between an electron tunneling peak in one resonance point and a hole tunneling peak in another resonance point.

The parameters in our numerical calculations are chosen from an experimental verification perspective. For the MZM, we believe [7, 9] that a superconducting gap \(\Delta\) of 0.25 meV can be induced in the InSb nanowire by superconducting Nb. When the spin–orbit energy of InSb is 0.3 meV, the coherence length \(\xi\) on the wire is estimated to be 185 nm. Then, in order to let the MZM coupling energy \(E_M(=\Delta e^{-1/2})\) to be around 0.01 meV, the length of the InSb nanowire \(l\) should be about 595 nm. For the metallic rings, we want the size of the rings to be as large as possible. We follow [39] and note that the electron gas at a GaAs–GaAlAs hetero-interface can have a Fermi wavelength \(\lambda_F = 42\) nm and a Fermi velocity \(v_F = 2.6 \times 10^5\) m s\(^{-1}\), which lead to the Fermi wave vector \(k_F = 0.15\) nm\(^{-1}\) and the Fermi energy \(E_F = 12.835\) meV. The mobility of the electron gas attains a value of \(1.14 \times 10^5\) cm\(^2\) V\(^{-1}\) s\(^{-1}\), which renders the elastic mean free path to be 11 \(\mu\)m. Under this condition, we can safely stay in the phase coherent transport regime as we restrict the circumferences of the rings to be no more than \(3 \mu\)m.

We also note that the persistent current [41] inside the right ring also changes sign from the tunneling electron dominated regime to the tunneling hole dominated one when the magnetic flux is adjusted. This change in sign of the persistent current can be detected by placing a high sensitivity magnetometer near the right ring.

Finally, we would like to mention that the scheme proposed in the current paper has the potential to be applied to other normal/superconductor/normal structures to detect the CAR. For a general CAR, if on the nonlocal side the electron and hole have played a symmetric role in the coupling, which is usually the case, the two ring scheme is also helpful in distinguishing them out. However, one noteworthy point is that in the usual s-wave superconductor there is no mid-gap state, and for the tunneling to be significant there might be needed a\(^{6}\)

\(^{6}\) A more up-to-date value of the electron mobility in the GaAs–GaAlAs hetero-interface has been attained as \(10^7\) cm\(^2\) V\(^{-1}\) s\(^{-1}\), corresponding to the elastic mean free path of around 100 \(\mu\)m, see [40].
bias above the gap energy. A large bias may cause complexities like multi-channel tunneling that is detrimental to identifying a simple adjacent electron and hole tunneling signal. If the superconductor has a mid-gap state, the detection should be easier.

4. Conclusions

In conclusion, we have presented a method of using the constructive coherence condition in a metallic ring to separate the electron and hole tunneling signals of the Majorana-zero-mode-induced crossed-Andreev-reflection. When the magnetic flux in the ring close to the drain is scanned, the observation of adjacent electron and hole tunneling current peaks of about equal amplitudes can be taken as a signature of the existence of a pair of overlapping Majorana-zero-modes. It is hoped that this scheme of observing MZM-induced CAR tunneling current will play a role in the observation of the MZM fusion process in the future potential application of topological quantum computation.

Acknowledgments

LF would like to thank Javad Shabani for helpful discussions on the experimental possibilities. This work was supported by the US DOE BES Core Program (E387), and in part by the Center for Integrated Nanotechnologies, a US DOE BES user facility.

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