Finite-size scaling of pseudo-critical point distributions in the random transverse-field Ising chain

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We study the distribution of finite size pseudo-critical points in a one-dimensional random quantum magnet with a quantum phase transition described by an infinite randomness fixed point. Pseudo-critical points are defined in three different ways: the position of the maximum of the average entanglement entropy, the scaling behavior of the surface magnetization, and the energy of a soft mode. All three lead to a log-normal distribution of the pseudo-critical transverse fields, where the width scales as $L^{-1/\nu}$ with $\nu = 2$ and the shift of the average value scales as $L^{-1/\nu_{\text{typ}}}$ with $\nu_{\text{typ}} = 1$, which we related to the scaling of average and typical quantities in the critical region.

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Quenched disorder has a profound effect on the physical characteristics of phase transitions in classical and quantum mechanical systems. A theoretically and experimentally important issue is the measurement of physical observables in disordered systems at or near critical points. These measurements are always performed on finite samples and on one particular realization of disorder. Finite size scaling (FSS) is the systematic way to extract informations on the thermodynamic limit by studying finite systems, and the objects to be analyzed by FSS of disordered systems are the distributions of physical properties in the ensemble of the disorder realizations. This also sheds light on the question about whether a single experimental measurement on a rather large system is representative for the whole ensemble of random systems to which it belongs. This is very much connected to the important issue of self-averaging of thermodynamic quantities like the expectation values for order parameter, specific heat or susceptibilities.

In an infinite system these observables display a characteristic singularity at a critical point, where for instance the susceptibility diverges. This divergence is suppressed in a finite system and replaced by a finite maximum, the location of which is called pseudo-critical point and is slightly shifted against the infinite system’s critical point. In pure systems this shift depends in a systematic way on the lateral systems size $L$, usually proportional to $L^{-1/\nu_P}$, where $\nu_P$ is the correlation length exponent of the pure system. In finite disordered systems the susceptibility usually has several maxima in the critical region and each one is slightly shifted against the critical point of an infinite system. One identifies the location of the largest maximum with the pseudo-critical point of the corresponding sample and an intriguing question therefore concerns the distributions of these pseudo-critical points.

If the disorder is irrelevant according to the Harris criterion $\nu_P > d/2$, $d$ being the dimension of the variation of the disorder (usually identical with the system’s spatial dimension), the width of the distribution scales as $L^{-d/2}$. In this case the shift of the average finite-size transition point is proportional to $L^{-1/\nu_P}$ as the shift of the finite size pseudo-critical point in the pure case.

For relevant disorder $\nu_P < d/2$, there is a new random (R) fixed point at which the exponent, $\nu_R \neq \nu_P$, satisfies the relation $\nu_R \geq d/2$. In many random systems (diluted magnets, random field problems, spin glasses, etc.) the random fixed point is of conventional form, which means that thermal and disorder fluctuations remain of the same order at large scales, i.e. during renormalization. According to the finite-size scaling theory of conventional random critical points, both the shift and the width of the distribution of pseudo-critical points are characterized by the same random exponent and there is a lack of self-averaging. These predictions were checked for various models.

In this paper we intend to examine the distribution of pseudo-critical points and its finite size scaling for a quantum phase transition in a paradigmatic model for a random magnet with an infinite randomness fixed point. This new type of random fixed points has been observed in various systems (disordered quantum magnets at $T = 0$, random stochastic models, etc.) in which the disorder plays a dominant role over quantum, thermal, or stochastic fluctuations: during renormalization the strength of disorder grows without limits. Many asymptotically exact results have been obtained for one dimensional systems, partially by the use of a strong dis-
order renormalization group (SDRG) method\textsuperscript{10}. We study in detail the random transverse-field Ising spin chain\textsuperscript{11} (RTFIC) defined by the Hamiltonian:

\[ H = - \sum_l J_l \sigma_l^x \sigma_{l+1}^x - \sum_l h_l \sigma_l^z , \]

in terms of the \( \sigma_l^{x,z} \) Pauli matrices at site \( l \). Here the \( J_l \) exchange couplings and the \( h_l \) transverse-fields are independent random variables. We are interested in the properties of the system in its ground state, i.e., at \( T = 0 \). In the thermodynamic limit the control-parameter is given by\textsuperscript{12}

\[ \delta = [\ln h]_{\text{av}} - [\ln J]_{\text{av}} , \]

where \([ \ldots ]_{\text{av}}\) denotes averaging over quenched disorder, so that \( \delta = 0 \) at the critical point\textsuperscript{12}. In the following, we consider the case of random couplings \( J_l \) and homogeneous transverse fields \( h_l = h \)\textsuperscript{13}, so \( h \) is the analog of the temperature in thermal transitions. Its critical value in the thermodynamic limit is given by \( h_c(\infty) = [\ln h]_{\text{av}} \). In the vicinity of the critical point the average correlation length involves the exponent \( \nu = 2 \), whereas the typical correlation length diverges with a different exponent \( \nu_{\text{gap}} = 1 \)\textsuperscript{2}. The characteristic time-scale, \( \tau \), which is related to the smallest gap as \( \tau \sim \epsilon^{-1} \), scales logarithmically at the critical point, \( \ln \tau \sim \sqrt{\tau} \). It remains divergent also in an extended region of the off-critical region, in the so-called Griffiths phase, where \( \tau \sim L^{z} \) with a dynamical exponent, \( z \), which depends on the distance from the critical point.

Finite-size scaling of the RTFIC has been studied in\textsuperscript{14, 15, 16}. The distribution of the surface magnetization could be computed analytically\textsuperscript{12, 16}, and the distribution of the gap and the end-to-end correlation function by the SDRG method\textsuperscript{10}. They turned out to be different in the micro-canonical and the canonical ensembles. Here we define sample dependent critical parameters \( h_c(\alpha, L) \) (where \( \alpha \) indicates a particular disorder realization) and study their distribution. The standard approach, defining \( h_c(\alpha, L) \) through the rounding of the singularity of the susceptibility, does not work here, since the susceptibility is divergent also in the Griffiths phase. A similar problem arises for the rounding of the specific heat, since it has only a very weak essential singularity.

a. Pseudo-critical points through the maximum of the average entropy Here we suggest that for random quantum systems the pseudo-critical transition points can be conveniently defined through the rounding of the average entanglement entropy. For the RTFIC we consider a periodic sample (\( \alpha \)) of length \( L \) and calculate the entanglement entropy between the two halves of the chain, which is then averaged over all possible starting points of the block. In the limit \( L \rightarrow \infty \) the average entropy is divergent at the critical point\textsuperscript{17}, whereas in a finite sample we use the position of its maximum to define \( h_c(\alpha, L) \).

In the numerical calculations we have used efficient free-fermion techniques\textsuperscript{18, 19} with which we could calculated the entanglement entropy up to sizes \( L = 512 \).

The couplings in Eq.\textsuperscript{11} are taken from a uniform box-like distribution, which is centered at \( J = 1 \) and has a width \( \Delta J \). For different strengths of disorder (\( \Delta J = 0.2, 0.4, 0.6 \) and \( 1.0 \)) and for each system size \( L \) 10000 disorder realizations were generated. Additionally the entropy of each sample is averaged over \( L/2 \) starting position of the block. As an illustration we show in Fig.\textsuperscript{1} the probability distributions of \( \ln h_c(\alpha, L) \) at a disorder \( \Delta J = 0.4 \) for different sizes. The distribution functions for different \( L \) are symmetric and in terms of rescaled variables: \( [\ln h_c(\alpha, L) - [\ln h_c(\alpha, L)]_{\text{av}}]/\Delta \ln h_c(\alpha, L) \) the transformed distributions are well fitted by the same Gaussian form, as shown in the inset of Fig.\textsuperscript{1}. For different strength of disorder we have analyzed the shift of the average value, \([\ln h_c(\alpha, L)]_{\text{av}}\), as well as the standard deviation, \( \Delta \ln h_c(\alpha, L) \), which are shown in Fig.\textsuperscript{2}. Interestingly, the average transition point for a given \( L \) is practically independent of the strength of disorder and corresponds to the value in the pure system.

Our numerical data are compatible with a FSS form for the shift that is given by \( h_c(\infty) - h_c(\alpha, L) \sim L^{-2} \ln L \). The scaling of the width of the distributions is found to follow \( \Delta h_c(L) \sim L^{-1/\nu} \), where the exponent is given by \( 1/\nu = 0.50(1) \) independently of the strength of disorder (Fig.\textsuperscript{1}). Thus, our numerical estimate for \( \nu \) agrees well with the exponent of the average correlation length of the RTFIC. We can thus conclude that the distribution of \( \ln h_c(\alpha, L) \) in Fig.\textsuperscript{1} is well described by a Gaussian with a variance of \( O(1/L) \) and by a shift of \( O(L^{-2} \ln L) \).

b. Pseudo-critical points through the surface magnetization The surface magnetization is perhaps the simplest physical quantity of the RTFIC. Fixing the spin at one end of the chain say at \( l = L + 1 \), which amounts to have \( h_{L+1} = 0 \), the magnetization at the other end of the chain at \( l = 1 \) is given by the exact formula\textsuperscript{20}:

\[ m_s = [1 + \sum_{l=1}^{L} \prod_{j=1}^{l} (h_j/J_j)^2]^{-1/2} . \]

\[ \text{FIG. 1: (Color online) Distribution of } \ln h_c(\alpha, L) \text{ for a disorder with } \Delta J = 0.4 \text{ for different sizes. The inset the distributions of scaled variables is well described by a Gaussian.} \]
In the following we use a doubling procedure. For a given random sample \( \alpha \) of length \( L \), we construct a replicated sample \( 2\alpha \) of length \( 2L \) by gluing two copies of \( \alpha \) together, and study the ratio of the surface magnetizations: \( r(\alpha, L) = m_s(2\alpha, 2L)/m_s(\alpha, L) \). We rewrite the exact expression in Eq. (3) as

\[
m_s(\alpha, L) = [Z_1(L) + 1]^{-1/2}, \quad Z_1(L) = \sum_{i=1}^{L} e^{-U(i)},
\]

in terms of a random walk variable: \( U(i) = 2 \sum_{j=1}^{i} \ln J_i \). Similarly, we obtain for the replicated sample

\[
m_s(2\alpha, 2L) = [Z_2(2L) + 1]^{-1/2}, \quad Z_2(L) = \sum_{i=1}^{2L} e^{-U_2(i)}
\]

where \( U_2(i) = U(i) \) for \( 1 \leq i \leq L \) and it is \( U_2(i) = U(L) + U(i - L) \) for \( L + 1 \leq i \leq 2L \). The expression in Eq. (4) simplifies into \( Z_2(L) = (1 + e^{-U(L)}) Z_1(L) \). Since in the critical region \( \ln Z_1(L) \sim L^{1/2} \) for large \( L \) the ratio of the two surface magnetizations is given by:

\[
r(\alpha, L) \simeq \left(1 + e^{-U(L)}\right)^{-1/2}
\]

For the pure chain in the ordered phase with \( \delta_P = h - h_c(\infty) = h - 1 < 0 \) we have \( U(L) \to \infty \), and in a large finite system the ratio behaves as \( r(L) = 1 - e^{-2\delta_P L}/4 \), whereas in the disordered phase with \( \delta_P > 0 \) we have \( U(L) \to -\infty \) and the finite-size correction reads as \( r(L) = e^{-\delta_P L} \). Then at the critical point \( U(L) = 0 \) and the ratio has a non-trivial value, \( r(L) = 1/\sqrt{2} \). For the random chain we have the same type of trivial fixed points corresponding to the disordered \( U(L) \to \infty \) and the ordered \( U(L) \to -\infty \) phases, respectively, and it is natural to use the condition, \( U(L) = 0 \), to define finite-size critical transverse fields. This leads to the microcanonical condition:

\[
\frac{1}{L} \sum_{l=1}^{L} \ln J_l = \frac{1}{L} \sum_{l=1}^{L} \ln h_l = \ln h_c(\alpha, L).
\]

Using this definition we obtain for the finite-size behavior in the disordered phase for a given sample \( \ln m_s(\alpha, L) \sim -(\ln h_c(\alpha, L) - \ln h) L \) and for its average: \( \ln m_s(\alpha, L)_{av} \sim -(\ln h_c(\infty) - \ln h) L \sim -|\delta| L \), which involves the typical exponent, \( \nu_{typ} = 1 \).

c. Pseudo-critical points through a soft mode

Criticality of the RTFIC in the free fermion representation is related to the vanishing of the excitation energy of a special fermionic mode. We recall that for closed chains, i.e. with \( J_L \neq 0 \) the even and odd number of excitations are taken from two different sectors and the ground state is the vacuum of the even sector, whereas the first excited state is in the odd sector and contains one fermion with energy, \( \Lambda_1 \). For the pure system its energy is given by \( \Lambda_1^f = h - J = \delta_P \), thus it changes sign at the critical point. Here we use the same condition, \( \Lambda_1 = 0 \), to define criticality a in finite random system, too. This leads to the matrix equation [18]: \( (A + B) \Phi_1 = 0 \) with

\[
(A + B) = \begin{pmatrix}
h_1 & J_1 & J_2 & J_3 & \cdots \h_{L-1} & J_{L-1} \\
J_L & h_2 & J_3 & J_4 & \cdots \h_L & J_{L-1} \\
J_{L-1} & J_L & h_3 & J_4 & \cdots \h_{L-1} & J_{L-2} \\
\vdots & \vdots & \vdots & \ddots & \ddots \ddots \ddots \vdots \\
J_2 & J_3 & J_4 & \cdots & \cdots & \cdots & J_L \\
J_1 & J_2 & J_3 & \cdots & \cdots & \cdots & J_{L-1} \\
h_3 & J_4 & J_5 & \cdots & \cdots & \cdots & h_{L-1} \\
h_2 & J_3 & J_4 & \cdots & \cdots & \cdots & h_L
\end{pmatrix}
\]

the solution of which is just the micro-canonical condition in Eq. (7).

In the following we study the scaling behavior of the smallest gap \( \epsilon(\alpha, L) \) in the ordered phase. For this it is more convenient to use open chains with \( L \) bonds where we have the asymptotic expression [14]:

\[
\epsilon(\alpha, L) \sim m_s(\alpha, L) |m_s(\alpha, L)| \prod_{i=1}^{L} \frac{h_i}{J_i} h_i.
\]

provided the scaled gap, \( \epsilon(\alpha, L)L \), goes to zero. Here we take \( h_{L+1} = h_1 \), \( m_s(\alpha, L) \) and \( m_s(\alpha, L) \) denote the surface magnetization at the two ends of the chain, which are both \( O(1/\sqrt{L}) \) in the ordered phase. Consequently for a given sample, \( \ln \epsilon(\alpha, L) \sim -(\ln h - \ln h_c(\alpha, L))L \) and
for its average: \( \ln c(\alpha, L)_{av} \sim -(\ln h - \ln h_c(\infty))L \sim -\delta L \). Thus the finite-size correction involves the typical exponent, \( \nu_{typ} = 1 \).

d. Log-normal distribution We have used three different methods to determine pseudo-critical points, all of which gave coherent results for the pseudo-critical point distributions. We found that the distribution of \( x = \ln h_c(\alpha, L) \) is Gaussian around \( \overline{\tau} = \ln h_c(\alpha, L)_{av} \)

\[
P_L(x = \ln h_c(\alpha, L)) = \sqrt{\frac{L}{2\pi\sigma^2}} e^{-\frac{(x - \overline{\tau})^2}{2\sigma^2}}
\] (10)

Thus \( \Delta \ln h_c(\alpha, L) = \sigma/\sqrt{L} \) and the fluctuations are governed by the exponent \( \nu = 2 \). The shift of the average, \( \ln h_c(\infty) - \overline{\tau} \), is of \( O(L^{-2} \ln L) \) from the average entropy and zero by the other two methods. The average of the critical transverse fields is given by:

\[
[h_c(\alpha, L)]_{av} = e^{\overline{\tau}} e^{\frac{\sigma^2}{2L}} = h_c(\infty)(1 + \frac{2\sigma^2}{L} + \ldots),
\]

thus \( [h_c(\alpha, L)]_{av} - h_c(\infty) = \lambda(L) / L \), which is consistent with \( \nu_{typ} = 1 \). We can thus conclude that \( \Delta h_c(\alpha, L)/\lambda(L) \) tends to infinity, which can be taken as a definition of an infinite randomness fixed point.

We now use these results to explain the role of the averaged correlation length \( \delta^{-2} \). In the disordered phase \( (h > h_c(\infty)) \), the average surface magnetization \( m_s(h, L)_{av} \) is dominated by the rare ordered samples having \( m_s(\alpha, L) = O(1) \), for which \( h_c(\alpha, L) > h > h_c(\infty) \). From their probability we obtain:

\[
m_s(h, L)_{av} \sim \text{Prob}(\ln h_c(\alpha, L) > \ln h) \sim e^{-\frac{h_h}{\delta^2}}
\] (11)

Its exponential decay thus involves the critical exponent \( \nu = 2 \). In the ordered phase \( (h < h_c(\infty)) \) we consider the average gap, \( [\epsilon(h, L)]_{av} \), which is dominated by those rare realizations, for which \( h_c(\alpha, L) < h < h_c(\infty) \) and \( \epsilon(\alpha, L) = O(1) \). From their probability we obtain:

\[
[\epsilon(h, L)]_{av} \sim \text{Prob}(\ln h_c(\alpha, L) < \ln h) \sim e^{-\frac{h_h}{\delta^2}}
\] (12)

which also involves the critical exponent \( \nu = 2 \). We note that previously we have shown that the typical quantities, both the surface magnetization and the gap, have an exponential decay: \( e^{-L[\delta]} \), which involve the typical exponent, \( \nu_{typ} = 1 \).

Our results for the RTFIC are also relevant for other systems displaying an infinite randomness fixed point, like the random antiferromagnetic XX chain and various stochastic models with quenched disorder, such as the Sinai-walk and the partially asymmetric exclusion process. Also in higher dimensional realizations of infinite randomness fixed points, like the two-dimensional random transverse Ising model \([21, 22]\), we expect a similar scenario as we described here, which can be checked for instance by calculating the entropy numerically by the SDRG method \([22]\).

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