Color Glass Condensates in dense quark matter
and quantum Hall states of gluons

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We apply the effective theory of color glass condensate to the analysis of gluon states in dense quark matter, in which the saturation region of gluons is also present. We find that in the region two point function of gluons shows algebraic long range order. The order is completely the same as the one gluons show in the dense quark matter, which form quantum Hall states. The order leads to the vanishing of massless gluon pole. We also find that the saturation region of gluons extends from small $x$ to even large $x \lesssim 1$ in much dense quark matter. We point out a universality that the color glass condensate is a property of hadrons at high energy and of quark matter at high baryon density.

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I. INTRODUCTION

It has recently been recognized both phenomenologically and theoretically\[1\] that color glass condensate (CGC) is a realistic state of gluons in nucleons or nuclei. The state is observed in deep inelastic scattering of nucleons at high energy or high energy scattering between nuclei. Thus, it is considered to be associated with specific phenomena of QCD in such high energy scatterings. But the idea is more general and applicable to the other systems such as dense quark matter or quark gluon plasma. Here, we call the state of gluons whose number density is saturated as the state of CGC, although the use of the terminology might not be standard one\[1\]. The effective theory of CGC can clarify gluonic states, especially, their number density in both transverse $k_t$ and longitudinal $k^+$ momenta. The theory is formulated by using light cone formulation\[2\] in infinite momentum frame, $P^+ \to \infty$. ( Usually, variable of $x \equiv k^+/P^+$ is used instead of $k^+$ where $P^+$ is the longitudinal momentum of the system itself under consideration. Our notation is the standard one used in light cone formulation\[2\]. ) Then, only quantum effects of the leading order in $1/P^+$ are taken. Thus, the CGC revealed by the theory is associated with the high energy scattering. That is, gluons with much small $x$, which form the state of CGC, can be seen only by deep inelastic scattering at high energy. But, as we explain below, the region of CGC in dense quark matter extends from small $x \ll 1$ to even large $x \lesssim 1$. This implies that the CGC in the dense quark matter arises as a property of the matter in low energy physics just like color superconducting phase of quark matter.

In this paper we apply the theory\[3\] to dense quark matter, in which the state of CGC is also present. The region of the CGC in standard ($k_t, 1/x$) plane extends from saturation momentum, $k_t = Q_s(x)$ to the vanishing momentum, $k_t = 0$ because of quarks being deconfined in the dense quark matter. Since the saturation momentum, $Q_s(x)$, is proportional to the baryon number density of the quark matter, the CGC can be realized at not necessarily small $x$, e.g. $x = 0.3$ for sufficiently dense quark matter. We discuss that in the region of the CGC, two point function of gluons shows an algebraic off-diagonal long range order in the transverse directions with coordinate, $x_t$: $\langle tr(A(x_t)A(0)) \rangle \to 1/(\alpha_s|x_t|^2)$ as $|x_t| \to \infty$. It should be noted that the function is proportional to $\alpha_s^{-1}$ in the CGC region; $\alpha_s \equiv g^2/4\pi$ stands for QCD fine structure constant. This behavior of the two point function in two dimensional transverse plane indicates the existence of a certain order in the CGC. We show that exactly the same off-diagonal long range order of gluons as this one is obtained in the dense quark matter where gluons form a fractional quantum Hall state. The quantum Hall state of the gluons have recently been discussed\[4, 5, 6\] to be present in dense quark matter where a color magnetic field is generated spontaneously: under the color magnetic field gluons form quantum Hall states. It is well known that a characteristic order is present in the states. Therefore, the CGC in the dense quark matter suggests the presence of the quantum Hall state of gluons. On the other hand, in the case of nuclei, as cyclotron radius of gluons is larger than the radius of the nucleons, the quantum Hall states of gluons can not arise. Thus, the CGC in the nuclei does not represent the quantum Hall states. We have originally found the color ferromagnetic phase in the laboratory frame of the quark matter, but, obviously the quantum Hall states of gluons as well as the existence of color magnetic field still remain in the infinite momentum frame; the direction of the magnetic field is taken to point into $x_3 = x^+ - x^-$. In the next section\[7\] we evaluate the number density of gluons by using the effective theory of the CGC. In the section\[8\] we show that the number density represents a 2 point correlation function of gluons, i.e. propagator, and that the correlation in the CGC region indicates the presence of an order in the dense quark matter. In the section
we review quantum Hall states of gluons in color ferromagnetic phase of the dense quark matter. In the section VI, we show that the same correlation as the one in the CGC is obtained in the quantum Hall states of gluons. We discuss that the correlation in the CGC leads to the screening of color charge or the vanishing of gluon’s massless pole. This fact implies the presence of an order of the CGC. In the section VII, we point out that the production process of the CGC is similar to that of gluon condensation in the dense quark matter, which form quantum Hall states. We summarize our results in the section VIII.

II. NUMBER DENSITY OF GLUONS IN QUARK MATTER

A. Effective theory of CGC

In the present paper we consider $SU(N_c)$ gauge theory. First, we define a number density of gluons in the dense quark matter using light cone formulation, where in light cone gauge, $A^+ = 0$, the transverse gauge bosons, $A^i(i = 1, 2)$ is expanded in terms of annihilation operators, $a^i(k^+, k_t)$; $[a^i(p^+, p_t), a^j(\bar{k}^+, \bar{k}_t)] = \delta^{ij} \delta(k^+ - p^+) \delta^2(k_t - p_t)$,

$$A^i(x^-, x_t) = \int_{k_t > 0} \frac{dk^+ d^2k_t}{(2\pi)^3 2k^+} (a^i(k^+, k_t)\exp(ik^+x^- - ik_t x_t) + \text{h.c.})$$ (1)

with transverse (longitudinal) momenta, $k_t (k^+)$ and coordinates, $x_t (x^-)$. We have not represented explicitly time coordinate, $x^+$ and color indices. Then, the number density of gluons per unit transverse area is defined as

$$d^3N(x, k_t)/d^3k = \langle \text{matter}|a^i(\bar{k}^+, \bar{k}_t) a^i(k^+, k_t)|\text{matter}\rangle/((\pi R^2))$$ (2)

where the summation over color indices is assumed implicitly. $R$ denotes transverse radius of the matter which is supposed to be sufficiently large in order to have translational invariance in the transverse directions. We note that $R$ is independent of the baryon number density characterizing the dense quark matter, while it is proportional to $A^{1/3}$ where $A$ is the mass number of nuclei. The number density can be rewritten by using a field strength, $F^{ij}(x^-, x_t)$,

$$d^3N(\tau, k_t)/d\tau d^2k_t = k^+ d^3N(x, k_t)/d^3k$$

$$= \int d^3x d^3y \frac{4\pi^4 R^2}{\exp(ik(x - y))} \langle \text{matter}|F^{ij}(x)F^{ij}(y)|\text{matter}\rangle$$ (3)

with $\tau \equiv \log(1/x)$, where $d^3x = dx^- dx_+ d\sigma_t$ and $k(x - y) = k^+(x^- - y^-) - k_t(x_t - y_t)$ and $F^{ij}(x) = -\partial^+ A^i(x)$ in the light cone gauge.

Now, we estimate the correlation function by using effective theory of CGC. In order to do so, we will explain briefly the theory. First of all, it is assumed that there exist an effective theory including quantum effects of gluons and quarks with longitudinal momentum from infinity to $\Lambda^+$ and with all transverse momentum. It is also assumed that almost of all quantum effects of quarks are taken into account in the theory; quantum effects of quarks with smaller momenta than $\Lambda^+$ make no significant contributions. Then, the effective theory is described by standard Lagrangian of only gluons with momenta $k^+ \leq \Lambda^+$ added with a color random source term,

$$L = - \int d^4x F_{\mu\nu} F^{\mu\nu} / 4g_N c + i \int d^3x_d \left( \rho(x^-, x_t) \mathcal{P} \exp(i g \int_\infty^x dz^+ A^- (z^+, x^-, x_t)) \right),$$ (4)

where the color source, $\rho(x^-, x_t)$, represents the quantum effects of quarks and gluons with $k^+ > \Lambda^+$, which is supposed to have been integrated out in the original Lagrangian of QCD. It is static (independent of $x^+$) because it represents slowly moving components of the quarks and gluons with momenta $k^+ > \Lambda^+$. It is only coupled with a gauge field, $A^-$ and it transforms as an adjoint representation under gauge transformation. That is, the source describes modes propagating only in the positive $x^+$ direction since we take infinite momentum frame. We also note that the spatial extension in $x^-$ of the source is given by $(\Lambda^+)^{-1}$ since it involves quantum effects with momentum, $k^+$, from infinity to $\Lambda^+$; $\rho(x^-, x_t) \neq 0$ only for $0 < x^- < (\Lambda^+)^{-1}$. Using this Lagrangian, we find classical solutions, $A_{cl}(x, x_t)$ which depend on the source. Then, the effective theory of CGC dictates that quantum expectation values are obtained by averaging over the random source with a gauge invariant normalized distribution, $W_\Lambda(\rho); \int D\rho W_\Lambda(\rho) = 1$,

$$\langle \hat{O}(A(x, \rho)) \rangle_\Lambda = \int D\rho \, O(A_{cl}(x, \rho)) W_\Lambda(\rho)$$ (5)
The significant feature of the formula in eq(8) is that the number density of gluons depends only on the longitudinal momentum, \( k^+ \) such as \( \infty > k^+ > \Lambda^+ \). \( \rho \) is a gauge invariant measure. Hence, the problem of obtaining the quantum averages is to find the distribution function, \( W_\Lambda(\rho) \) and to take the average with the use of the function. All quantum averages over gluons and quarks are replaced by an average over the source with the use of \( W_\Lambda(\rho) \). This simplification is the point of the effective theory.

The classical static solution independent of \( x^+ \) is obtained easily in a transverse covariant gauge, \( \partial_\perp A^i = 0 \) by putting \( A^- = 0 \). Then, the solutions are that \( A^\mu(x^-,x_t) = \delta^{\perp +}(x^-,x_t) \) where \(-\partial_\perp^2 \alpha(x^-,x_t) = \rho_{cv}(x^-,x_t)\); \( \partial_\perp^2 \equiv \Sigma_{i=1}^2 \partial_t^2 \). ( The index \( cv \) of \( \rho_{cv} \) denotes the source in the covariant gauge. ) In order to obtain a solution in the light cone gauge, \( A^+ = 0 \), we make a gauge transformation. Hence, it follows that in the gauge, \( A^± = 0 \), \( A^i(x^-,x_t) = (i/g) V(x^-,x_t) \partial^i V(x^+,x_t) \) with \( V(x^+,x_t) = P \exp(ig \int_{-\infty}^{x^+} dz^- \alpha(z^-,x_t)) \) where \( P \) denotes path ordering.

We proceed to incorporate quantum effects of gluons with longitudinal momentum, \( b\Lambda^+ < k^+ \leq \Lambda^+ \) with \( 0 < b < 1 \). This is done by integrating the gluon fields with the momenta. As a result we obtain a new effective theory with the same Lagrangian as the one in eq(4) but with the distribution function, \( W_{b\Lambda}(\rho) \), renormalized in which the source receives corrections and its spatial extension is modified such as \( \rho(x^-,x_t) \neq 0 \) only for \( 0 < x^- < 1/(b\Lambda^+) \). The functional form of \( W_{b\Lambda}(\rho) \) is also changed. Consequently, we obtain the distribution, \( W_b(\rho) \equiv W_{b\Lambda}(\rho) \) at arbitrary longitudinal momentum scale, \( k^+ = b\Lambda^+ ; \ y = \log(\Lambda^+/k^+) = \log(1/b) \). Quantum expectation values at the scale of \( b\Lambda^+ \) can be obtained just as in eq(5) with the use of \( W_b(\rho) \).

Instead of the distribution function, \( W_b(\rho) \), of the source, it is convenient to use the distribution function, \( W_b(\rho_{cv} = -\partial_\perp^2 \alpha) \) of the gauge field, \( \alpha \). Hereafter we write it simply as \( W_b(\alpha) \). Jacobian associated with the change of variables, \( \rho \rightarrow \alpha \), is trivial.

This \( W_b(\alpha) \) obeys a renormalization group equation called as JIMWLK equation, which is a functional equation difficult to solve. Instead of solving explicitly the equation, we can derive a closed equation for a correlation function, \( S_\alpha(r_1) \equiv \frac{1}{N_c} \langle \text{matter} | \text{tr}(V(r_1) V(0)) | \text{matter} \rangle \) of a Wilson line \( V(r_1) \equiv P \exp(ig \int_{-\infty}^{\infty} dz^- \alpha_z(x_t)) \) where \( \alpha_z(x_t) \equiv z \alpha(x^-,x_t) \) with \( z = \log(x^-\Lambda) \). The equation is called as Balitsky-Kovchegov equation, which is obtained by using JIMWLK equation along with taking both large \( N_c \) limit and large baryon number density limit ( large mass number limit in the case of nuclei ),

\[
\partial_\perp S_\alpha(r_1) = -\frac{N_c \alpha_s}{\pi} \int \frac{d^2z_1}{2\pi} \frac{r_1^2}{(r_1 - z_1)^2 z_1^2} (S_\alpha(r_1) - S_\alpha(r_1 - z_1)S_\alpha(z_1)) \tag{6}
\]

With the use of \( S_\alpha(r_1) \), we can rewrite the number density of gluons,

\[
\frac{d^3 N(\tau,k_t)}{d\tau d^2 k_t} = \frac{N_c}{2\pi^3 g^2} \int d^2 r_t e^{i k_t r_t} \int_{-\infty}^{\infty} dy S_\alpha(r_1)(-\partial_\perp^2 \partial_\perp \log(S_\alpha(r_1))), \tag{7}
\]

where we have used an assumption that the correlation length in the longitudinal direction vanishes, i.e. \( \langle \alpha_x(x_t) \alpha_z(u_t) \rangle \propto \delta(y-z) \). This assumption is satisfied in the solutions of \( W_b(\alpha) \) obtained in a mean field approximation or a Gaussian approximation.

B. Saturation of gluon number density

Consequently, the number density of gluons can be obtained by solving B-K equation. Our concern is the CGC region with much large \( \tau \) or much small \( k_t \) and the asymptotic solution relevant to the region has been found,

\[
S_\alpha(r_t) \rightarrow \exp \left( -c_0 (\log(r_t^2 Q_s^2(\tau)))^2 \right) \text{ for } |r_t| \gg (Q_s(\tau))^{-1} \text{ with a numerical constant, } c_0.
\]

Therefore, it follows that

\[
\frac{d^3 N(\tau,k_t)}{d\tau d^2 k_t} \approx \frac{N_c^2 - 1}{16\pi^3 g^2 N_c} \log(Q_s^2(\tau)/k_t^2) \text{ for } k_t^2 \ll Q_s^2(\tau), \tag{8}
\]

where \( Q_s(\tau) \) is a saturation momentum. On the other hand, it behaves roughly in the large \( k_t \) such as \( \frac{d^3 N(\tau,k_t)}{d\tau d^2 k_t} \sim Q_s^2(\tau)/k_t^4 \) for \( k_t^2 \gg Q_s^2(\tau) \), where an irrelevant constant is not explicitly written. This equation implies that the number density grows very rapidly in the region of large \( k_t \) with the decrease of the transverse momentum, \( k_t \). Similarly, it grows very rapidly with the decrease of longitudinal momentum, \( x = \exp(-\tau) \) because \( Q_s^2(\tau) \sim x^{-\eta} \) with \( \eta \sim 0.3 \).

The behavior of the number density in small \( x \) or small \( k_t \) in eq(8) is quite different from the one in large \( x \) or \( k_t \). The significant feature of the formula in eq(8) is that the number density of gluons depends only on \( k_t \) through the
term, \( \log(Q_s^2(\tau)/k_t^2) \) and depends on the gauge coupling constant, \( g^2 \) inversely. That is, the number density of gluons saturates in both momenta of \( k_t \) and \( x \); it grows only in logarithmic way because \( Q_s^2(\tau) \sim x^{-\eta} \) with \( \eta \sim 0.3 \). The saturated state of gluons is called as the state of CGC. The feature is much intriguing and specific to QCD or non Abelian gauge theories. It should be stressed that the very feature gives rise to characteristic algebraic long range order of gluons in two dimensional transverse plane discussed below.

C. Saturation momentum in dense quark matter

Here we would like to mention about the saturation momentum of the dense quark matter. It may be defined in general such that \( Q_s^2(y) = \alpha_s N_c \left\langle \frac{dN}{dy} \right\rangle \). We expect that gluon number density, \( dN/dy \), is proportional to the total baryon number of the system with the radius, \( R \); \( dN/dy \propto p_B R^3 \) where \( p_B \) is the normal nuclear density of nuclei or the baryon number density of the quark matter. Thus, \( Q_s^2(y) \) is proportional to \( A^{1/3} \) in nuclei, while it is proportional to the baryon number density in the quark matter. We note that the region of the CGC is given such that \( x < x_c(k_t) \) for given \( k_t \) where \( x_c \) satisfies \( t = Q_s(y_c(k_t)) \) with \( y_c(k_t) = \log(1/x_c(k_t)) \). Then, when we consider the quark matter with \( p_B \) being sufficiently large, the region of the CGC is achieved even with large \( x \lesssim 1 \). This is because \( Q_s^2(\tau) \sim p_B x^{-\eta} \) with \( \eta \sim 0.3 \) for \( x \lesssim 1 \). For example, \( Q_s(p_n, x = 10^{-4}) = Q_s(11p_n, x = 0.3) \) where \( p_n \approx 2.8 \times 10^{14} \text{ g/cm}^3 \) stands for the normal nuclear density. Namely, the saturation momentum of nuclei at small \( x = 10^{-4} \) is equal to the saturation momentum at \( x = 0.3 \) of the quark matter with the baryon density, 11 times larger than that of the nuclei. Therefore, the CGC of such dense quark matter is not necessarily associated with high energy scattering. We can see the effects of the CGC even in low energy scattering. Therefore, the CGC of hadrons is a feature of QCD observed at high energy limit, while the CGC of quark matters is that of QCD observed at large baryon number density limit.

The result is naively suggested in the following consideration. As has been recognized, the CGC can be analyzed perturbatively because \( Q_s(x) \) is sufficiently large at sufficiently small \( x \) for \( \alpha_s(Q_s) \) to be much less than 1. But, we need to take into account infinitely large number of gluons. The analysis becomes non trivial. Similarly, the dense quark matter can be analyzed perturbatively because \( \alpha_s(p_B) \ll 1 \) with \( p_B \ll \Lambda_{QCD} \) where \( p_B \) is the chemical potential of the baryon number \( ; p_B \propto \mu_B^3 \). As we explain below, we also need to take into account infinitely large number of gluons for the analysis of dense quark matter. The analysis is non trivial and has been done in the laboratory frame of the quark matter. In this sense, the CGC in the dense quark matter may arise even at low energy scattering.

III. LONG RANGE ORDER OF GLUONS IN CGC

Now, we rewrite the number density in eq 9 in a different way. Since \( \langle \text{matter}|F^{\nu \mu}(x)F^{\mu \nu}(y)|\text{matter} \rangle = 0 \) for \( x^- \neq y^- \) in the effective theory as we have stated, we can integrate over \( x^- \) and \( y^- \) by putting \( \exp(ik^+(x^- - y^-)) = 1 \). Then, it follows that using the translational invariance in the transverse direction,

\[
d^3N(\tau, k_t)/d\tau d^2k_t = \frac{1}{4\pi^2} \int d^2r_t \exp(-ik_t x_t) \langle \text{matter}|A^\nu_+(x_t)A^\nu_+(0)|\text{matter} \rangle
\]

(9)

where we have used the fact that \( A^\nu_+(x^-; x_t) = 0 \) for \( x^- \leq 0 \) and have denoted \( A^\nu_+(x^-; x_t) \equiv A^\nu_+(x^-; x_t) \) for \( x^- > 1/k^+ \); \( \tau = \log(1/x^-) = \log(\Lambda^+ / k^+) + \log(P^+ / \Lambda^+) \). Note that since \( \alpha(x^-; x_t) = 0 \) for \( x^- > 1/k^+ \), \( A^\nu_+(x^-; x_t) \) is independent of \( x^- \) when \( x^- > 1/k^+ \).

Therefore, we find that the number density of gluons is just the propagator of the gluons in two dimensional transverse space. In other words, the number density describes the two point correlation function of gluons. Consequently, we can find an intriguing property that the correlation function shows an algebraic off-diagonal long range order in the region of CGC,

\[
\langle \text{matter}|A^\nu_+(x_t)A^\nu_+(0)|\text{matter} \rangle = c/(g^2 x_t^2) \quad \text{for} \quad |x_t| > Q_s^{-1}(\tau),
\]

(10)

where \( c \) is a numerical constant independent of that of quark number density, or \( Q_s(\tau) \). It should be noted that we can take \( |x_t| \) larger than the confinement scale \( Q_s^{-1}(\tau) \) in the dense quark matter because quarks are deconfined.

Although the behavior of the correlation at large distance is not usual one showing a long range order such as \( \langle \phi(x_t)\phi(0) \rangle \to \langle \phi(r_t) \rangle \langle \phi(0) \rangle \neq 0 \) as \( |x_t| \to \infty \), the algebraic decay of the correlation function in two dimensional spaces implies the existence of an order in the system. Actually, the algebraic decay in two dimensional spaces usually arises in the correlations of phases such as for example, \( \langle \exp(-i\theta(x_t))\exp(i\theta(0)) \rangle \propto 1/|x_t|^{\beta_k} \) in the low temperature phase of XY model; the variable of \( \exp(i\theta(x_t)) \) represents the direction of spin. When we write the spin variable
such as $S(x_l) = n(x_l) \exp(i\theta)$, then the spin shows the algebraic long range order, $\langle S(x_l)S(0) \rangle \propto 1/|x_l|^{2\nu}$, but $\langle n(x_l)n(0) \rangle \neq 0$ as $|x_l| \to \infty$ because of the magnitude, $n$, of the spin being fixed. Another example is correlations of composite bosons, $\phi$, in fractional quantum Hall states of two dimensional electrons with filling factor, $\nu = 2\pi n_e/eB : \langle \phi(x_l)\phi(0) \rangle \propto 1/|x_l|^{1/2\nu}$ as $|x_l| \to \infty$ where $n_e$ is the number density of electrons. In this case, the algebraic decay arises from the phase of the field, $\phi(x_l)$. It is well known that this algebraic decay of the correlation implies the existence of an order in the quantum Hall states of electrons. This order characterizes the quantum Hall states of two dimensional electrons.

IV. QUANTUM HALL STATES OF GLUONS

We have recently shown that the dense quark matter possesses a color ferromagnetic phase in which gluons form a quantum Hall state under a color magnetic field generated spontaneously. The phase is realized in the region of baryon number density being sufficiently large for the perturbation theory or loop expansions to be valid. But it arises in the lower baryon density than the baryon density necessary for the realization of color superconductivity. Thus, the phase is more important phenomenologically, e.g. for neutron star physics. We will show below that this quantum Hall state of gluons with $\nu = 1/4$ also exhibits the same off-diagonal long range order as the one in eq(10) as well as $1/g^2$ dependence. ( In the present section, we do use the standard formulation of gauge theories instead of the light cone formulation. Our main concern is quantum Hall states of gluons realized in two dimensional plane, which we identify as the transverse plane in the above discussion. Correlations of gluons discussed below hold even in the light cone formulation.)

The presence of color magnetic field makes lower one loop effective potential of gluons. This is perturbatively reliable result in much dense quark matter, since the gauge coupling constant is sufficiently small in the matter. ( Even if we take into account the free energy of quarks, the minimum of the effective potential is determined mainly by gluons loops. Thus, the field strength determined in this way has only the slight dependence of quark chemical potential. ) Although the presence of the field minimizes the one loop potential, there still exist more stable states, which are beyond the one loop approximation. Those are obtained by taking account of the repulsive 4 point interaction of gluons; the one loop approximation does not address with the 4 point interaction. That is, they are the quantum Hall states of gluons coupled with the field. Actually, it is easy to see in the SU(2) gauge theory that under the color magnetic field, $B \propto \sigma_3$, gluons with color $\sigma_{1,2}$ have such energy spectra as $E^2(n, k_3) = 2gB(n/2 + k_3) + k_3^2 + 2gB$ with integer $n \geq 0$ specifying Landau levels; $k_3$ is a momentum parallel to the field, $B$. Here, we take spatial color direction of $B$ be pointed into $x_3 (\sigma_3)$ direction. Thus, the gluons with $E^2(n = 0, k_3) = k_3^2 - gB < 0$ for $k_3 < \sqrt{gB}$ are unstable and their amplitudes grow unlimitedly. In other words they are produced unlimitedly if there is not the 4 point interaction. Especially, most unstable gluons ( they are modes with $k_3 = 0$ ) are two dimensional ones since there is no dependence of the coordinate $x_3$. Their amplitudes grow unlimitedly, but eventually they are saturated to form the quantum Hall state owing to the repulsive 4 point interaction of gluons. This situation is very similar to the production process of gluons with small $x$ in CGC. That is, the gluons with smaller $x$, are produced unlimitedly by the gluons with larger $x$ but they are saturated due to the nonlinear interaction of gluons. Anyway, in the dense quark matter a diagonal color magnetic field ($B \propto \sigma_j$ ) is generated spontaneously and off-diagonal gluons $\propto \sigma_{1,2}$ are produced to form quantum Hall states. Explicitly, the gluons forming the quantum Hall state are given by $\phi_a = (\Phi_1 - i\Phi_2)\sqrt{1/\ell^2}$ where $\Phi_i = (A^i + iA^i_u)/\sqrt{2}$ with spatial indices, $i$. A constant, $\ell$, appearing for the normalization of the field, $\phi_a$, denotes the spatial extension in $x_3$ direction of a region ( quantum well ) in which quantum Hall states of the gluons are fabricated; the scale is determined by the other unstable modes with the momentum, $k_3$ such as $\sqrt{gB} > k_3 > 0$. Thus, it is order of the magnetic length, $1/\sqrt{gB}$.

In order to describe the quantum Hall states of the gluons, $\phi_a$, it is convenient to use composite gluons, $\phi_a$, instead of using $\phi_a$: gluons attached with fictitious Chern-Simons flux, $a_µ [4, 5, 16, 17]$. Using the gluon composite field, $\phi_a$, Lagrangian in three space-time dimensions for describing the quantum Hall state is given by

$$L_a = [(i\partial_µ - gA_µ + a_µ)\phi_a|^2 + 2gB|\phi_a|^2 - \frac{\lambda}{2}|\phi_a|^4 + \frac{\epsilon_{\mu\nu\lambda}}{4\alpha}a_µ\partial_\nu a_\lambda],$$

where $\lambda = g^2/\ell$ and $A_\nu' = (0, -Bx_2/2, Bx_1/2)$ representing color magnetic field $B$, where $\nu$ runs from 0 to 2 and the statistical factor $\alpha$ should be taken as $\alpha = 2\pi \times$ integer for the field $\phi_a$ to represent the composite boson. Antisymmetric tensor is such that $\epsilon_{\mu\nu\lambda} = 1$ for $(\mu, \nu, \lambda) = (0, 1, 2)$, otherwise $\epsilon_{\mu\nu\lambda} = 0$. Quantization of the system leads to the following Hamiltonian,

$$H(\phi_a, \ddot{a}) = P_a^2 P_a + [(i\partial_\nu + \ddot{a} - gA_\nu)\phi_a|^2 + 2gB|\phi_a|^2 + \frac{\lambda}{2}|\phi_a|^4,$$
with commutation relations, \[ [\phi_a(x_i), P^\dagger_{a}(y_j)] = [\phi^\dagger_{a}(x_i), P_{a}(y_j)] = \delta^a(x_i - y_j), \]
where we have used a gauge condition, \( a_0 = 0 \). Additionally, we have a constraint such that \( c^{ij}\partial_a(x) = 2a_j(x) \) where \( j(x) = i(\phi^\dagger_{a}P_{a} - P^\dagger_{a}\phi_{a}) \) is color charge density. Canonical momentum of \( \phi_{a} \) is given by \( P_{a} = (\partial_0 - igA_0 + ia_0)\phi^\dagger_{a} \). Quantum Hall states are obtained by finding groundstate solutions of the Hamiltonian. Especially, in the mean field approximation we can easily obtain the spatially uniform classical solution minimizing \( H \), which represents quantum Hall states. That is, by putting \( g\vec{A} = \vec{a} \) and by introducing Lagrange multiplier, \( b_0 \) for the constraint, the classical solution is given\(^2\) by solving the following equations,

\[
2b_0|\phi_a|^2 = \frac{gB}{2\alpha} \quad \text{and} \quad b_0^2 + 2gB = \lambda|\phi_a|^2. \tag{13}
\]

The solution is that \( |\phi_a|^2 \approx \sqrt{2gB/\lambda} = \sqrt{2B/g^2} \) for small gauge coupling \( g^2 \ll 1 \). Quantum Hall states are characterized as the condensed states of the bosons, \( \langle qhs|\phi_{a}\rangle = \phi_{a}^c \) in the composite boson formulation.

Here we point out that the relation between the original gluon variable, \( \phi_{a} \) and the composite one, \( \phi_{a} \) is given by\(^{10}\)

\[
\phi_{a}(x) = \exp(i \int d^2y\theta(x - y)j(y))\phi_{a}(x), \tag{14}
\]

where \( \theta(x) \) denotes azimuthal angle of \( x \). With the use of the relation, we obtain Hamiltonian of the original gluon field, \( \phi_{a} \) with the replacement such as \( H(\phi_{a} \rightarrow \phi_{a}, a_{\mu} = 0) \) in eq\(^{13}\). Hence, Hamiltonian of the original gluon, \( \phi_{a} \), is given by

\[
H = P^\dagger_{a}P_{a} + |(i\vec{\partial} - g\vec{A})\phi_{a}|^2 - 2gB|\phi_{a}|^2 + \frac{\lambda}{2}|\phi_{a}|^4, \tag{15}
\]

which can be obtained by extracting only the mode, \( \phi_{a} \), from the original Hamiltonian of gluons. We note that the Hamiltonian looks like the one of a Higgs model. This implies that the groundstate of the Hamiltonian is formed by a condensation of the gluons, \( \phi_{a} \). Such a condensation is explicitly realized in the composite boson formulation.

**V. Long Range Order of Gluons in Dense Quark Matter**

Instead of \( \phi_{a} \) we may introduce a different form\(^{11}\) of a composite boson, \( \phi_{b} \equiv \exp(-i \int^t dt'a_{0}(t', x_i))\phi_{a} \) in the Lagrangian, then, it follows that

\[
L_{a} = |i\partial_{0}\phi_{b}|^2 - |(i\partial_{i} - gA_{i} + a_{i} - P_{i})\phi_{b}|^2 + 2gB|\phi_{b}|^2 - \frac{\lambda}{2}|\phi_{b}|^4 + \frac{\epsilon^{\mu\nu\lambda}}{4a}a_{\mu}\partial_{\nu}a_{\lambda}. \tag{16}
\]

with \( P_{i} = \partial_{i} \int^t dt'a_{0}(t', x_i) \). Using this Lagrangian, we find that longitudinal \( P_{i} \) and transverse \( a_{i} \) are canonical conjugate with each other because of the last term \( \int d^3x\frac{1}{2\alpha c_{ij}\partial_iP_{j}a_{j}} \). Actually, we can rewrite the last term such that \( \int d^3x\frac{\epsilon^{\mu\nu\lambda}}{4a}a_{\mu}\partial_{\nu}a_{\lambda} = \int d^3x\frac{1}{2\alpha}((\partial_0a - \partial_0P)\partial^2b \) where we put \( P_{i} = \partial_{i}P \) and \( a_{i} = \partial_{i}a + c^{ij}\partial_jb \). Thus, changing the variable, \( P \rightarrow P + a \), we find that only transverse component, \( b \), of \( a_i \) remains in the Lagrangian. Consequently, we may regard that \( a_i \) is transverse. Then, quantizing the canonical conjugate variables, \( \delta a_{i} = a_{i} + gA_{i} \) and \( P_{i} \) (more precisely, \( P \) and \( -\partial^2b/2\alpha \) are canonical conjugate), the term of Hamiltonian, \( |(i\partial_{i} - gA_{i} + a_{i} - P_{i})\phi_{b}|^2 \sim |\langle qhs|\phi_{b}|qhs\rangle|^2(\delta a_{i} - P_{i})^2 \), can be diagonalized where \(|\langle qhs|\phi_{b}|qhs\rangle|^2 = |\phi_{a}^c|^2 \). Then, it is straightforward to show in a similar way to the one in ref\(^{11}\) that

\[
\langle qhs|\exp(-i \int^t dt'a_{0}(t', x_i))\phi_{b}(t', x_i)\rangle \rightarrow \langle qhs|\phi_{b}(0)|qhs\rangle \rightarrow |\phi_{a}^c|^2 \left( \frac{1}{|x_t|A} \right)^{\alpha/2\pi} \text{ as } |x_t| \rightarrow \infty, \tag{17}
\]

where we have noted that \( P = \int^t dt'a_{0}(t', 0) \) and that the quantum Hall state is the groundstate of the diagonalized Hamiltonian. \( \Lambda \) denotes a typical scale in QCD, not necessarily equal to \( \Lambda_{QCD} \) although it is proportional to \( QCD \). Therefore, we find that

\[
\langle qhs|\phi_{b}^\dagger(x_t)\phi_{b}(0)|qhs\rangle \rightarrow |\phi_{a}^c|^2 \left( \frac{1}{|x_t|A} \right)^{\alpha/2\pi} = \left( \frac{2B/g^2}{|x_t|A} \right)^{\alpha/2\pi} \text{ as } |x_t| \rightarrow \infty. \tag{18}
\]
or,
\[
\langle \text{qhs}|\phi_n^i(x_i)|\phi_u(0)\rangle_{\text{qhs}} \rightarrow (2B/g\ell) \left( \frac{1}{|x_i|\Lambda} \right)^{\alpha/2\pi} \quad \text{as} \quad |x_i| \rightarrow \infty,
\]
(19)
where we have used the fact that \( \int d^2x \theta(x)\langle j \rangle = 0 \), since the color charge distribution, \( \langle j \rangle \equiv \langle \text{qhs}|j(x)\rangle_{\text{qhs}} \), in the quantum Hall state is uniform and spherical symmetric. Hence, it turns out that the correlation function in quantum Hall state of gluons shows the same algebraic off-diagonal long range order as the one in eq(10) of the color glass condensate when its filling factor, \( 2\pi \langle \text{qhs}|j(x)\rangle_{\text{qhs}}/g\mathcal{B} = \pi/\alpha \), is equal to 1/4 ( \( \alpha = 4\pi \)). We should note that since \( g\mathcal{B} \) is given by \( \Lambda_{QCD}^2 \) in the one loop approximation[14, 15] and \( \Lambda \sim \Lambda_{QCD} \), the correlation function eq(10) behaves such that \( \langle \phi_n^i(x_i)|\phi_u(0)\rangle \sim 1/(g^2|x_i|^2) \) as \( |x_i|^2 \rightarrow \infty \). The dependence of \( g^2 \) is the specific feature in eq(10) characterizing the color glass condensate.

Therefore, we find that the off-diagonal long range order, eq(10) of the gluons in CGC is completely the same as the one in eq(19) of the gluons forming the fractional quantum Hall state in dense quark matter. This similarity is stressed by rewriting eq(10) with the use of \( \phi_u \) in the case of SU(2) gauge theory,
\[
\langle \text{matter}|A^i_{3,\sigma}(r_t)A^i_{3,\sigma}(0)|\text{matter}\rangle = \langle \text{matter}|(\phi_n^i(r_t)|\phi_u(0))/l|\text{matter}\rangle + \langle \text{matter}|A^i_{3,\sigma}(r_t)A^i_{3,\sigma}(0)|\text{matter}\rangle,
\]
(20)
where \( A_{3,\sigma} \) is a gauge field in \( \sigma_3 \) direction of color space. Furthermore, we can argue that if the algebraic decay of the correlation, \( \langle \text{matter}|A^i_{3,\sigma}(r_t)A^i_{3,\sigma}(0)|\text{matter}\rangle \), comes from the gluon condensation, \( \langle \phi_u \rangle \neq 0 \), of quantum Hall states as discussed above, the color magnetic field, \( \mathcal{B} = F_3^{1,2} \) should be present.

\[
\langle F_3^{1,2} \rangle \sim \partial^3\langle A^3_3 \rangle - \partial^2\langle A^3_3 \rangle + g(\langle A^1_3 \rangle\langle A^2_3 \rangle - \langle A^1_3 \rangle\langle A^2_3 \rangle) \sim \Lambda^2/g \simeq \mathcal{B}
\]
(21)
since \( g\mathcal{B} = \Lambda_{QCD}^2 \) and \( \Lambda \sim \Lambda_{QCD} \).

On the other hand, the color magnetic field, \( F^{1,2} \), vanishes in the CGC because of the absence of a color source current pointing to the transverse direction as a leading term in \( 1/P^+ \). Thus, we may speculate based on the above discussion that non-leading terms in \( 1/P^+ \) would generate the magnetic field, \( F^{1,2} \).

As has been pointed out[11, 12], the color source, \( \rho \) is screened completely in the region of the CGC; \( \int d^2x d^2y \rho(x_i)|\rho(y_i)| = 0 \) and \( \int d^2x \rho = 0 \). This originates from the long range behavior of the two point function in eq(10) such as \( 1/|x_i|^2 \). ( Strictly speaking, the property of the screening holds only in the dense quark matter because we need to take the limit of \( |x_i| \) infinity. In the case of hadrons, \( |x_i| \) must be less than the confinement scale, \( \Lambda_{QCD}^{-1} \). On the other hand, the screening becomes complete only in the limit of \( |x_i| \) infinity because of the absence of explicit screening length.) The fact of the screening can be stated in different way. Namely, the long range behavior leads to the disappearance of the massless pole in the gluon propergater; \( k_\perp^2(d^2\mathcal{N}(\tau,k_i)/d\tau dk_i) \rightarrow 0 \) as \( k_\perp^2 \rightarrow 0 \). The screening or the disappearance of the massless pole indicates the presence of an order in the system. It apparently seems to be a Debye screening in color charged gluon gas. But it is not correct. Because the effective theory of the CGC dictates that \( F^{-i} = 0 \), that is, color electric field, \( \vec{E} \) is identical to color magnetic field, \( \vec{B} \) ( \( E^1 = B^2 \) and \( E^2 = -B^1 \) ), the electric screening implies simultaneously the magnetic screening. So the screening is not simply a Debye screening. The magnetic screening usually arises in superconducting states where Meissner mass is generated. On the other hand, there is no such mass generation in the CGC because the correlation function does not decay exponentially. Thus, the CGC in dense quark matter does not implies color superconductivity[12]. Although there is not an explicit mass generation, the screening indicates the existence of an order in this system. We speculate that the order is the one in quantum Hall states of gluons.

VI. QUANTUM HALL STATE AND CGC

Finally, we point out another similarity between the CGC and quantum Hall state of gluons. Namely, the formation process of the CGC is very similar to that of gluon condensation in quantum Hall state, which is described by a Higgs model[16]. When the population of gluons with large \( x \) is small, \( N \ll 1 \), the population grows very rapidly ( BKFL region[19] ) due to the production of gluons with smaller \( x \). But when the population becomes large enough, \( N \sim 1 \), to interact with each other as \( x \rightarrow 0 \), the nonlinear interaction becomes effective to make the population be saturated ( CGC region ). Such a situation may be described[20] by an equation of population dynamics, \( \partial_t N = N - N^2 \) with an initial condition, \( N(t = 0) \ll 1 \). Here, "time", \( t \), is qual to the rapidity, \( y = \log(1/x) \). We note that BK equation in eq(6) governing the gluon density has a similar structure to this one.
A similar equation can be derived in a Higgs model just like the Hamiltonian in eq (15) of $\phi_u$ describing quantum Hall states of unstable gluons. Although the relevant Hamiltonian in eq (15) is not a typical Higgs model which we use below, the essence in the evolution of the gluon field can be simulated in the model. That is, from the equation of motion for the spatially uniform Higgs field, $\partial^2_t \phi = \mu^2 \phi - \lambda |\phi|^2 \phi$, we can derive an equation for the square of amplitude, $N_h \equiv |\phi|^2$, of the Higgs field,

$$\partial_t N_h = 2\mu N_h \sqrt{1 - \frac{\lambda N_h}{4\mu^2}} \simeq 2\mu N_h - \frac{\lambda N_h^2}{4\mu} \quad \text{when} \quad \lambda \ll 1,$$

with the use of an condition, $\partial_t N_h = 0$ at $N_h = 0$. We have assumed the phase of Higgs field being static, for simplicity. Then, we find that when the amplitude of Higgs field is small ($N_h \ll \mu^2/\lambda$), the state is unstable and its amplitude grows exponentially in time. But when the amplitude becomes sufficiently large ($N_h \sim \mu^2/\lambda$), the field saturates and stops to grow due to the 4 point self interaction, $\lambda |\phi|^4$. This similarity suggests that the formation of the CGC in nuclei or quark matter may be understood in terms of a Higgs model in the above sense.

**VII. CONCLUSION**

We conclude that the long range behavior of the two point function of gluons in the CGC of dense quark matter is the same as that of gluons forming quantum Hall states. The behavior leads to the screening of the color source or the disappearance of the massless pole of gluons. Hence, it indicates the presence of an order in the state of the CGC of the dense quark matter, just like the order in the quantum Hall states. Furthermore, the formation process of the CGC has a similarity to that of the quantum Hall states of gluons. We may speculate that the CGC in dense quark matter might be just a condensate of gluons forming the quantum Hall states. We have also shown that the saturation region of the CGC extends from small $x \ll 1$ to large $x \lesssim 1$, e.g. $x = 0.3$. The fact implies a universality of the CGC which is realized in both hadrons at high energy scattering and quark matters at high baryon density. The universality may be applicable for quark gluon plasma at high temperature.

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