Implementing 64-bit Maximally Equidistributed Mersenne Twisters

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CPUs and operating systems are moving from 32 to 64 bits, and hence it is important to have good pseudorandom number generators designed to fully exploit these word lengths. However, existing 64-bit very long period generators based on linear recurrences modulo 2 are not completely optimized in terms of the equidistribution properties. Here we develop 64-bit maximally equidistributed pseudorandom number generators that are optimal in this respect and have speeds equivalent to 64-bit Mersenne Twisters. We provide a table of specific parameters with period lengths from $2^{607} - 1$ to $2^{44497} - 1$.

Categories and Subject Descriptors: G.4 [Mathematical Software]: Algorithm design and analysis; I.6 [Computing Methodologies]: Simulation and Modeling

1. INTRODUCTION

Monte Carlo simulations are a basic tool in financial engineering, computational physics, statistics, and other fields. To obtain precise simulation results, the quality of pseudorandom number generators is important. At present, the 32-bit Mersenne Twister (MT) generator MT19937 (with period $2^{19937} - 1$) [Matsumoto and Nishimura 1998] is one of the most widely used pseudorandom number generators. However, modern CPUs and operating systems are moving from 32 to 64 bits, and hence it is important to have high-quality generators designed to fully exploit 64-bit words.

Many pseudorandom number generators, including Mersenne Twisters, are based on linear recurrences modulo 2; these are called $F_2$-linear generators. One advantage of these generators is that they can be assessed by means of the dimension of equidistribution with $v$-bit accuracy, which is a most informative criterion for high dimensional uniformity of the output sequences. In fact, MT19937 is not completely optimized in this respect. Panneton et al. [2006] developed the Well Equidistributed
Long-period Linear (WELL) generators with periods from $2^{512} - 1$ to $2^{44497} - 1$, which are completely optimized for this criterion (called maximally equidistributed), but the parameter sets were only searched for the case of 32-bit generators. Conversely, there exist several 64-bit $\mathbb{F}_2$-linear generators. Nishimura [2000] developed 64-bit Mersenne Twisters, and the SIMD-oriented Fast Mersenne Twister (SFMT) generator [Saito and Matsumoto 2009] has a function to generate 64-bit unsigned integers.

For graphics processing units, MTGPDC [Saito and Matsumoto 2013] is also a good candidate. However, these generators are not maximally equidistributed. In earlier work, L’Ecuyer [1999] searched for 64-bit maximally equidistributed combined Tausworthe generators (with some additional properties). At present, though, to the best of our knowledge, there exists no 64-bit maximally equidistributed MT-type $\mathbb{F}_2$-linear generator with very long period $2^{19937} - 1$, such as a 64-bit variant of the WELL generators.

The aim of this article is to develop 64-bit maximally equidistributed $\mathbb{F}_2$-linear generators without loss of speed as compared with 64-bit Mersenne Twisters [Nishimura 2000]. The key techniques are (i) state transitions with double feedbacks [Panneton et al. 2006; Saito and Matsumoto 2009] and (ii) linear output transformations with several memory references [Harase 2009]. We provide a table of specific parameters with periods from $2^{607} - 1$ to $2^{44497} - 1$. The design of our generators is based on a combination of existing techniques, such as the WELL and dSFMT generators [Panneton et al. 2006; Saito and Matsumoto 2009], but we select state transitions different from those of the original WELL generators to maintain the generation speed. We denote these generators 64-bit maximally equidistributed Mersenne Twisters (MEMTs).

In practice, we often convert unsigned integers into 53-bit double-precision floating-point numbers in $[0, 1)$ in IEEE 754 format. Our 64-bit generators are useful for this. To generate 64-bit output values, one can either use a pseudorandom number generator whose linear recurrence is implemented with 32-bit integers and then take two successive 32-bit blocks or, instead, use a pseudorandom number generator whose recurrence is implemented directly over 64-bit integers. As described below, the former method may degrade the dimension of equidistribution with $v$-bit accuracy, compared with simply using 32-bit output values. We consider the case of the 32-bit MT19937 generator in Section 4. For this reason, we develop 64-bit maximally equidistributed Mersenne Twisters to directly generate 64-bit unsigned integers.

The article is organized as follows. In the next section, we review $\mathbb{F}_2$-linear generators and their theoretical criteria. We also summarize the framework of Mersenne Twisters. Section 3 is devoted to our main result: 64-bit MEMTs. In Section 4, we compare our generators with others in terms of speeds, theoretical criteria, and empirical statistical tests. Section 5 concludes.

2. PRELIMINARIES

2.1. $\mathbb{F}_2$-Linear Generators

Let $\mathbb{F}_2 := \{0, 1\}$ be the two-element field, i.e., addition and multiplication are performed modulo 2. We consider the following class of generators.

**Definition 2.1 ($\mathbb{F}_2$-linear generator).** Let $S := \mathbb{F}_2^p$ be a $p$-dimensional state space (of the possible states of the memory assigned for generators). Let $f : S \rightarrow S$ be an $\mathbb{F}_2$-linear state transition function. Let $O := \mathbb{F}_2^w$ be the set of outputs, where $w$ is the word size of the intended machine, and let $o : S \rightarrow O$ be an $\mathbb{F}_2$-linear output function. For an initial state $s_0 \in S$, at every time step, the state is changed by the recursion

$$s_{i+1} = f(s_i) \quad (i = 0, 1, 2, \ldots),$$

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and the output sequence is given by
\[ o(s_0), o(s_1), o(s_2), \ldots \in O. \] (2)

We identify \( O \) as a set of unsigned \( w \)-bit binary integers. A generator with these properties is called \( \mathbb{F}_2 \)-linear generator.

Let \( P(z) \) be the characteristic polynomial of \( f \). The recurrence (1) has a period of length \( 2^w - 1 \) (its maximal possible value) if and only if \( P(z) \) is a primitive polynomial modulo 2 [Niederreiter 1992; Knuth 1997]. When this value is reached, we say that the \( \mathbb{F}_2 \)-linear generator has maximal period. Unless otherwise noted, we assume throughout that this condition holds. We refer the reader to [Matsumoto et al. 2006; L’Ecuyer and Panneton 2009] for details.

### 2.2. Quality Criteria

A most informative criterion for high dimensional uniformity is the dimension of equidistribution with \( v \)-bit accuracy. Assume that an \( \mathbb{F}_2 \)-linear generator has the maximal period \( 2^w - 1 \). We identify the output set \( O := \mathbb{F}_2^w \) as a set of unsigned \( w \)-bit binary integers. We focus on the \( v \) most significant bits of the output, and regard these bits as the output with \( v \)-bit accuracy. This amounts to considering the composition \( o_v : S \rightarrow \mathbb{F}_2^w \rightarrow \mathbb{F}_2^w \), where the latter mapping denotes taking the \( v \) most significant bits. We define the \( k \)-tuple output function as
\[ o_v^{(k)} : S \rightarrow (\mathbb{F}_2^w)^k, \quad s_0 \mapsto (o_v(s_0), o_v(f(s_0)), \ldots, o_v(f^{k-1}(s_0))). \]

Thus, \( o_v^{(k)}(s_0) \) is the vector formed by the \( v \) most significant bits of \( k \) consecutive output values of the pseudorandom number generators from a state \( s_0 \).

**Definition 2.2 (Dimension of equidistribution with \( v \)-bit accuracy).** The generator is said to be \( k \)-dimensionally equidistributed with \( v \)-bit accuracy if and only if \( o_v^{(k)} : S \rightarrow (\mathbb{F}_2^w)^k \) is surjective. The largest value of \( k \) with this property is called the dimension of equidistribution with \( v \)-bit accuracy, denoted by \( k(v) \).

Because \( o_v^{(k)} \) is \( \mathbb{F}_2 \)-linear, \( k \)-dimensional equidistribution with \( v \)-bit accuracy means that every element in \( (\mathbb{F}_2^w)^k \) occurs with the same probability, when the initial state \( s_0 \) is uniformly distributed over the state space \( S \). As a criterion of uniformity, larger values of \( k(v) \) for each \( 1 \leq v \leq w \) are desirable [Tootill et al. 1973]. We have a trivial upper bound \( k(v) \leq \lfloor p/v \rfloor \). The gap \( d(v) := \lfloor p/v \rfloor - k(v) \) is called the dimension defect at \( v \), and the sum of the gaps \( \Delta := \sum_{v=1}^w (\lfloor p/v \rfloor - k(v)) \) is the total dimension defect. If \( \Delta = 0 \), the generator is said to be maximally equidistributed. For \( \mathbb{F}_2 \)-linear generators, one can quickly compute \( k(v) \) for \( v = 1, \ldots, w \) by using lattice reduction algorithms over formal power series fields [Harase et al. 2011; Harase 2011]. These are closely related to the lattice reduction algorithm originally proposed by Couture and L’Ecuyer [2000].

As another criterion, the number \( N_1 \) of nonzero coefficients for \( P(z) \) should be close to \( p/2 \) [Compagner 1991; Wang and Compagner 1993]. For example, generators for which \( P(z) \) is a trinomial or pentanomial fail in statistical tests [Lindholm 1968; Matsumoto and Kurita 1996; Matsumoto and Nishimura 2002]. If \( N_1 \) is not large enough, the generator suffers a long-lasting impact for poor initialization known as a 0-excess state \( s_0 \in S \), which contains only a few bits set to 1 [Panneton et al. 2006]. Thus, \( N_1 \) should be in the vicinity of \( p/2 \). See [Matsumoto et al. 2006; L’Ecuyer and Panneton 2009] for the general theory of such criteria.
2.3. Mersenne Twisters

In general, to ensure a maximal period, testing the primitivity of $P(z)$ is the bottleneck in searching for long-period generators (because the factorization of $2^p - 1$ is required). If $p$ is a Mersenne exponent (i.e., $2^p - 1$ is a Mersenne prime), one can instead use an irreducibility test that is easier and equivalent. Matsumoto and Nishimura [1998] developed a pseudorandom number generator with a Mersenne prime period $2^p - 1$ known as the Mersenne Twister. We briefly review their method. The state space $S$ and state transition $f : S \to S$ are expressed as

$$\begin{align*}
(w^u_1, w_{i+1}, w_{i+2}, \ldots, w_{i+N-1}) &\mapsto (w^u_{i+1}, w_{i+2}, w_{i+3}, \ldots, w_{i+N}),
\end{align*}$$

where $w_i \in \mathbb{F}_2^w$ is a $w$-bit word vector, $w^u_{i+1} \in \mathbb{F}_2^u$ denotes the $w - r$ most significant bits of $w_i$, $N := \lceil p/w \rceil$, and $r$ is a non-negative integer such that $p = Nw - r$, so that the elements of $S$ are split into arrays whose $r$ bits are unused and padded with zeros. The state transition (3) is implemented as

$$w_{i+N} := w_{i+N} \oplus (w^u_{i+1} \mid w^r_{i+1})A,$$

where $w^r_{i+1} \in \mathbb{F}_2^r$ represents the $r$ least significant bits of $w_{i+1}$, $\oplus$ is the bitwise exclusive-OR (i.e., addition in $\mathbb{F}_2^r$), and $(w^u_{i+1} \mid w^r_{i+1})$, a $w$-bit vector, is the concatenation of the $(w - r)$-bit vector $w^u_{i+1}$ and the $r$-bit vector $w^r_{i+1}$ in that order. $M$ is an integer such that $0 < M < N - 1$, and $A \in \mathbb{F}_2^{w \times w}$ is a $(w \times w)$-regular matrix (with the format in (9) below). Furthermore, to improve $k(v)$, for the right-hand side of (3), the output transformation $o : S \to O$ is implemented as

$$\begin{align*}
(w^u_{i+1}, w_{i+2}, \ldots, w_{i+N}) &\in S \mapsto w_{i+N}T \in O,
\end{align*}$$

where $T \in \mathbb{F}_2^{w \times w}$ is a suitable $(w \times w)$-regular matrix. This technique is called tempering. From this, we obtain an output sequence $w_1T, w_{N+1}T, w_{N+2}T, \ldots \in O$ by multiplying a matrix $T$ by the sequence from (4). Matsumoto and Nishimura [1998] and [Nishimura 2000] searched for 32- and 64-bit Mersenne Twisters with period length $2^{19937} - 1$, respectively. In Appendix B of Matsumoto and Nishimura [1998], it is proved that these generators cannot attain the maximal equidistribution (e.g., $\Delta = 6750$ for MT19937 in [Matsumoto and Nishimura 1998]). In fact, the state transition in (3)–(4) is very simple, but the linear output transformation $T$ in (5) is rather complicated (see Matsumoto and Nishimura [1998]; Nishimura [2000] for details).

3. MAIN RESULT: 64-BIT MAXIMALLY EQUIDISTRIBUTED MERSSENTE TWISTERS

3.1. Design

To obtain maximally equidistributed generators without of loss of speed, we try to shift the balance of costs in $f$ and $o$. The key technique is a suitable choice of (i) state transitions with double feedbacks [Panneton et al. 2006], [Saito and Matsumoto 2009], and (ii) linear output transformations with several memory references [Harase 2009]. First, we adopt recursion formulas with double feedbacks as used by Panneton et al. [2006] and Saito and Matsumoto [2009]. Let $N := \lceil p/w \rceil$. We divide $S := \mathbb{F}_2^p$ into two parts $S := \mathbb{F}_2^u \times \mathbb{F}_2^w$ and consider the state transition $f : S \to S$ with

$$\begin{align*}
(w^u_{i+1}, w_{i+1}, w_{i+2}, \ldots, w_{i+N-1}, u_i) &\mapsto (w^u_{i+1}, w_{i+2}, \ldots, w_{i+N-1}, u_{i+1}).
\end{align*}$$

The first $p - w$ bits are stored in an array whose remaining $r$ bits are unused and padded with zeros, as are the original MTs. Note that the number of words is $N - 1$, not $N$. The remaining word $u_i$ is expected to be stored in a register of the CPU and updated as $u_{i+1}$ at the next step, so that the implementation requires only a single word (see a formal algorithm in this subsection below).
$u_i$ is called the lung, which was originally proposed by [Panneton et al. 2006] and refined by [Saito and Matsumoto 2009]. (Note that $v_{i,0}$ in Fig. 1 of [Panneton et al. 2006] corresponds to the lung.) This approach proves to represent a key technique for drastically improving $N_1$ and $\Delta$. By refining the recursion formulas proposed in [Saito and Matsumoto 2009], we implement the state transition $f$ with the following recursions:

$$u_{i+1} := (w_{i-1}^r \mid w_{i+1}^r)A \oplus w_{i+2} \oplus u_i B,$$

(7)

$$w_{i+N-1} := (w_{i-1}^r \mid w_{i+1}^r) \oplus u_{i+1} C.$$

(8)

$A$, $B$, and $C$ are $(w \times w)$-matrices as follows:

$$wA := \begin{cases} (w \gg 1) & \text{if } w_{w-1} = 0, \\ (w \gg 1) \oplus a & \text{if } w_{w-1} = 1, \end{cases}$$

(9)

$$wB := w \oplus (w \ll sh_1),$$

(10)

$$wC := w \oplus (w \gg sh_2),$$

(11)

where $w = (w_0, \ldots, w_{w-1}) \in \mathbb{F}_2^w$ and $a = (a_0, \ldots, a_{w-1}) \in \mathbb{F}_2^w$ are $w$-bit vectors, $sh_1$ and $sh_2$ are integers with $0 < sh_1, sh_2 < w$, and “$w \gg l$” and “$w \ll l$” denote left and right logical (i.e., zero-padded) shifts by $l$ bits, respectively. Note that $u_i$ is one component in a state $s_i \in S$ and is not output.

To attain the maximal equidistribution exactly, we design a linear output transformation using another word in the state array, which comes from [Harase 2009]. More precisely, for the right-hand side of (6), we consider the following linear output transformation $o : S \rightarrow O$ with one more memory reference:

$$(w_{i+2}^r, w_{i+3}, \ldots, w_{i+N-1}, u_{i+1}) \in S \mapsto w_{i+N-1}T_1 \oplus w_{i+L}T_2 \in O.$$

(12)

Here $T_1$ and $T_2$ are $(w \times w)$-matrices given by

$$wT_1 := w \oplus (w \ll sh_3),$$

(13)

$$wT_2 := (w \& b),$$

(14)

where $L$ is an integer with $0 < L < N - 2$, $sh_3$ is an integer with $0 < sh_3 < w$, and $\&$ denotes a bitwise AND with a suitable bit mask $b = (b_0, \ldots, b_{w-1}) \in \mathbb{F}_2^w$. A circuit-like description of the proposed generators is shown in Figure 1.
Set $m^{w-r} \leftarrow (1, \ldots, 1, 0, \ldots, 0)$ and $\tilde{m}^r \leftarrow (0, \ldots, 0, 1, \ldots, 1)$. Here $m^{w-r}$ is a bit mask that retains the first $w - r$ bits and sets the other $r$ bits to zero, whereas $\tilde{m}^r$ is its bitwise complement. The symbol $\wedge$ denotes bitwise exclusive-OR.

**Step 0.** Store $w[0], w[1], \ldots, w[N-2]$, $u \leftarrow$ initial values except for all zero.

**Step 1.** Set an integer variable $i \leftarrow 0$.

**Step 2.** Set $x \leftarrow (w[i] \& m^{w-r}) \wedge (w[(i+1) \mod N-1] \& \tilde{m}^r)$. // compute $(w_i^{w-r} \mid w_{i+1}^{w-r})$.

**Step 3.** Set $u \leftarrow xA \wedge w[(i+M) \mod N-1] \wedge uB$. // compute Eq. (7).

**Step 4.** Set $w[i] \leftarrow x \wedge uC$. // compute Eq. (8).

**Step 5.** Set $y \leftarrow w[i]T_1 \wedge w[(i + L) \mod N-1]T_2$. // compute Eq. (12).

**Step 6.** Output $y$.

**Step 7.** Increment $i \leftarrow i + 1$. If $i \geq N - 1$, then $i \leftarrow i \mod N - 1$.

**Step 8.** Return to Step 2.

### 3.2. Specific Parameters

We search for specific parameters in the following way. First, we look for $M, sh_1, sh_2$, and $a$ in (7)–(11) at random such that $f$ attains the maximal period $2^p - 1$. In general, because we can obtain several parameters, we choose a parameter set whose $N_i$ is large enough and whose output has large $k(v)$ for $v = 1, \ldots, w$ as far as possible in the case where the outputs are identical (i.e., $T_1$ and $T_2$ are the identity and zero matrices, respectively). In the next step, we search for $L, sh_3$, and $b$ in (12)–(14) at random such that the generator is “almost” maximally equidistributed (i.e., $\Delta$ is almost 0). Finally, to obtain $\Delta = 0$ strictly, we apply a slight modification to the bit mask $b$ by using the backtracking algorithm in [Harase 2009] (with some trial and error). Table I lists specific parameters for 64-bit maximally equidistributed generators with periods ranging from $2^{207} - 1$ to $2^{44497} - 1$. We call these the 64-bit maximally equidistributed Mersenne Twisters (MEMTs). The code in C is available at [https://github.com/sharase/memt-64](https://github.com/sharase/memt-64).

**Remark 3.1.** In parallel computing, an important requirement is the availability of pseudorandom number generators with disjoint streams. These are usually implemented by partitioning the output sequences of a long-period generator into long disjoint subsequences whose starting points are found by making large jumps in the original sequences. For this purpose, [Haramoto et al. 2008] proposed a fast jumping-ahead algorithm for $F_2$-linear generators. We also implemented this algorithm for our MEMT generators. The code is also available at the above website. The default skip size is $2^{256}$.

### 4. PERFORMANCE

We compare the following $F_2$-linear generators corresponding to 64-bit integer output sequences:

- MEMT19937-64: our new generator;
- MT19937-64: the 64-bit Mersenne Twister (downloaded from [http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt64.html](http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt64.html));
- MEMT19937-64 (ID3): a 64-bit Mersenne Twister based on a five-term recursion (ID3) [Nishimura 2000];
- SFMT19937-64 (without SIMD): the 64-bit integer output of the SIMD-oriented Fast Mersenne Twister SFMT19937 without SIMD [Saito and Matsumoto 2008];
- SFMT19937-64 (with SIMD): the 64-bit integer output of the foregoing with SIMD [Saito and Matsumoto 2008].
### Table I. Specific Parameters of 64-bit Maximally Equidistributed Mersenne Twisters (MEMTs)

| $p$   | $w$ | $r$ | $N_1$ | $\Delta$ | $M$ | $L$ | $sh_1$ | $sh_2$ | $a$ | $b$ | $N_1$ |
|-------|-----|-----|-------|---------|-----|-----|--------|--------|-----|-----|-------|
| 607   | 64  | 33  | 10    | 313     | 5   | 13  | 35     | 811fde8012348bc | 325 | 66edc62a6bf8c826 | 0 |
| 1279  | 64  | 1   | 20    | 641     | 7   | 22  | 37     | 1afe2d52ed3952b  | 35  | 3a23d78e8fb5e349 | 0 |
| 2281  | 64  | 23  | 36    | 1145    | 17  | 36  | 21     | 7cbe23eca8ade36 | 546 | 6aede6fd97b338ec | 0 |
| 4253  | 64  | 35  | 67    | 9603    | 29  | 30  | 20     | fac168c6f4d1722 | 2129 | e4e22d26e15abe | 0 |
| 11213 | 64  | 51  | 176   | 7820    | 9   | 5   | 5      | db6f6e652e1c757 | 5455 | bd2d125e89593f | 0 |
| 19937 | 64  | 31  | 312   | 9603    | 45  | 33  | 13     | 5c32e06f730fc42 | 9603 | 6aede6fd97b338ec | 0 |
| 44497 | 64  | 64  | 696   | 19475   | 6711 | 64  | 14     | 4fa9ca362f39c9a9 | 19475 | 6fbb29aee9f91 | 0 |

The first three generators have the periods of $2^{19937} - 1$. SFMT19937-64 has the period of a multiple of $2^{19937} - 1$ (see Proposition 1 in [Saito and Matsumoto 2008] for details).

Table II summarizes the figures of merit $\Delta$, $N_1$ and timings. In this table, we report the CPU time (in seconds) taken to generate $10^9$ 64-bit unsigned integers for each generator. The timings were obtained using two 64-bit CPUs: (i) a 3.40 GHz Intel Core i7-3770 and (ii) a 2.70 GHz Phenom II X6 1045T. The code was written in C and compiled with GCC using the -O3 optimization flag on 64-bit Linux operating systems. For SFMT19937-64, we measured the CPU time for the case of sequential generation (see [Saito and Matsumoto 2008] for details).

MEMT19937-64 has the maximally equidistribution (i.e., $\Delta = 0$), and also has $N_1 \approx p/2$. These values are the best in this table. In terms of generation speed, MEMT19937-64 is comparable to or even slightly faster than the MT19937-64 generators on the above two platforms. SFMT19937-64 without SIMD is comparable to or faster than MEMT19937-64, and SFMT19937-64 with SIMD is more than twice as fast as MEMT19937-64. However, $\Delta$ for SFMT19937-64 is rather large. In fact, the SFMT generators are optimized under the assumption that one will mainly be using 32-bit output sequences, so that the dimensions of equidistribution with $v$-bit accuracy for 64-bit output sequences are worse than those for 32-bit cases ($\Delta = 4188$). For this, we analyze the structure of SFMT19937 in the online Appendix.
Finally, we **convert** 64-bit integers into double-precision floating-point numbers \( u_0, u_1, u_2, \ldots \) in \([0, 1]\), and apply them to the statistical tests included in the SmallCrush, Crush, and BigCrush batteries of TestU01 [LEcuyer and Simard 2007]. Note that these batteries have 32-bit resolution and have not yet been tailored to 64-bit integers. In our C-implementation, for MEMT19937-64 and MT19937-64, we generate the above uniform real numbers by \( (x \gg 11) \times (1.0/9007199254740992.0) \), where \( x \) is a 64-bit unsigned integer output (see also Remark 4.3). For SFMT19937-64, we use a function \( \text{sfmt\_genrand\_res53()} \), which is obtained by dividing 64-bit unsigned integers \( x \) by \( 2^{32} \), i.e., \( x \times (1.0/18446744073709551616.0) \); see the online Appendix for details. In any case, we investigate the 32 most significant bits of 64-bit outputs in TestU01. The generators in Table I passed all the tests; except for the linear complexity tests (unconditional failure) and matrix-rank tests (failure only for small \( p \)), which measure the \( \mathbb{F}_2 \)-linear dependency of the outputs and reject \( \mathbb{F}_2 \)-linear generators. This is a limitation of \( \mathbb{F}_2 \)-linear generators. However, for some matrix-rank tests, we can observe differences between MEMT and SFMT generators. Table III summarizes the \( p \)-values on the matrix-rank test of No. 60 of Crush for five initial states. (SFMT1279-64, SFMT2281-64, and SFMT19937-64 denote the results for double-precision floating-point numbers in \([0, 1]\) converted from the 64-bit integer outputs of SFMT1279, SFMT2281, and SFMT19937 in [Saito and Matsumoto 2008], respectively.)

**Remark 4.1.** We occasionally see the use of the least significant bits of pseudorandom numbers in applications. An example is the case in which one generates uniform integers from 0 to 15 by taking the bit mask of the 4 least significant bits or modulo 16. For this, we invert the order of the bits (i.e., the \( i \)-th bit is exchanged with the \((w-i)\)-th bit) in each integer and compute the dimension of equidistribution with \( v \)-bit accuracy, dimension defect at \( v \), and total dimension defect, which are denoted by \( k(v), d'(v), \) and \( \Delta' \), respectively. In this case, MEMT19937-64 is not maximally equidistributed, but \( \Delta' = 4047 \) and \( d'(v) \) is 0 or 1 for each \( v \leq 11 \). Note that \( \Delta' \) takes values 9022, 8984, 21341 for MT19937-64, MT19937-64 (ID3), SFMT19937-64, respectively. \( \Delta' \) of MEMT19937-64 is still smaller than \( \Delta \) of the other generators in Table III. However, as far as possible, we recommend using the most significant bits (e.g., by taking the right-shift in the above example), because our generators are optimized preferentially from the most significant bits.

**Remark 4.2.** For 32-bit generators, there have been some implementations that produce 64-bit unsigned integers or 53-bit double-precision floating-point numbers (in IEEE 754 format) by concatenating two consecutive 32-bit unsigned integers. We note that such conversions might not be preferable from the viewpoint of \( k(v) \). As an example, consider the 32-bit MT19937 generator in the header \(<\text{random}>\) of the C++11 STL in GCC (see [ISO 2012, Chapter 26.5]). Let \( z_0, z_1, z_2, \ldots \in \mathbb{F}_2^{32} \) be a 32-bit unsigned integer sequence from 32-bit MT19937. To obtain 64-bit unsigned integers, the GCC implements a random engine adaptor \texttt{independent\_bit\_engine} to produce 64-bit unsigned integers.

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**Table III. \( p \)-Values on the Matrix-Rank Test of No. 60 of Crush in TestU01**

| Generator      | 1st | 2nd | 3rd | 4th | 5th |
|----------------|-----|-----|-----|-----|-----|
| MEMT1279-64   | 0.89| 0.41| 0.11| 0.70| 0.22|
| MEMT2281-64   | 0.70| 0.02| 0.62| 0.49| 0.98|
| MEMT19937-64  | 0.13| 0.32| 0.14| 0.85| 0.60|
| SFMT1279-64   | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) |
| SFMT2281-64   | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) | \(< 1.0 \times 10^{-300}\) |
| SFMT19937-64  | 0.29| 0.02| 0.06| 0.49| 0.83|
unsigned integers from the concatenations as
\[(z_0, z_1), (z_2, z_3), (z_4, z_5), (z_6, z_7), \ldots \in F_2^{64}.
\]
To generate 53-bit double-precision floating-point numbers in \([0, 1)\) (i.e., uniform real distribution \((0,1)\) for MT19937), the GCC implementation generates 64-bit unsigned integers
\[(z_1, z_0), (z_3, z_2), (z_5, z_4), (z_7, z_6), \ldots \in F_2^{64}
\]
by concatenating two consecutive 32-bit integer outputs and dividing them by the maximum value \(2^{64}\). The sequences (15) and (16) can be viewed as \(F_2\)-linear generators with the state transition \(f^2\) in (3), so that we can compute \(k(v)\). As a result, we have \(\Delta = 13543\) for \(v = 1, \ldots, 64\) in (15) and \(\Delta = 13161\) for \(v = 1, \ldots, 52\) in (16), which are worse than \(\Delta\) of MT19937-64. In particular, \(k(12) = 623 < \lfloor 19937/12 \rfloor = 1661\) for each case. For this reason, we feel that there is a need to design 64-bit high-quality pseudorandom number generators.

Remark 4.3. For our generators, we have also implemented a function to produce double-precision floating-point numbers \(u_0, u_1, u_2, \ldots\) in \([0, 1)\) in IEEE 754 format by using the method of Section 2 of [Saito and Matsumoto 2009] (i.e., the 12 bits for sign and exponent are kept constant, and the 52 bits of the significand are taken from the generator output). We note that this function is preferable from the viewpoint of \(k(v)\) because it does not appear approximation errors by division.

5. CONCLUSIONS
In this article, we have designed 64-bit maximally equidistributed Mersenne Twisters and searched for specific parameters with period lengths from \(2^{607} - 1\) to \(2^{44497} - 1\). The key techniques are (i) state transitions with double feedbacks and (ii) linear output transformations with several memory references. As a result, the generation speed is still competitive with 64-bit Mersenne Twisters on some platforms. The code in C is available at [https://github.com/sharase/memt-64](https://github.com/sharase/memt-64). Pseudorandom number generation is a trade-off between speed and quality. Our generators offer both high performance and computational efficiency.

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REFERENCES
A. Compagner. 1991. The hierarchy of correlations in random binary sequences. *Journal of Statistical Physics* 63 (1991), 883–896. Issue 5. [http://dx.doi.org/10.1007/BF01029989](http://dx.doi.org/10.1007/BF01029989).
R. Couture and P. L’Ecuyer. 2000. Lattice computations for random numbers. *Math. Comput.* 69, 230 (2000), 757–765. [http://dx.doi.org/10.1090/S0025-5718-99-01112-6](http://dx.doi.org/10.1090/S0025-5718-99-01112-6).
H. Haramoto, M. Matsumoto, T. Nishimura, F. Panneton, and P. L’Ecuyer. 2008. Efficient jump ahead for \(F_2\)-linear random number generators. *INFORMS J. Comput.* 20, 3 (2008), 385–390.
S. Harase. 2009. Maximally equidistributed pseudorandom number generators via linear output transformations. *Math. Comput. Simul.* 79, 5 (2009), 1512–1519. [http://dx.doi.org/10.1016/j.matcom.2008.06.006](http://dx.doi.org/10.1016/j.matcom.2008.06.006).
S. Harase. 2011. An efficient lattice reduction method for \(F_2\)-linear pseudorandom number generators using Mulders and Storjohann algorithm. *J. Comput. Appl. Math.* 236, 2 (2011), 141 – 149. [http://dx.doi.org/10.1016/j.cam.2011.06.005](http://dx.doi.org/10.1016/j.cam.2011.06.005).

ACM Transactions on Mathematical Software, Vol. V, No. N, Article A, Publication date: January YYYYY.
S. Harase and T. Kimoto

S. Harase, M. Matsumoto, and M. Saito. 2011. Fast lattice reduction for $F_2$-linear pseudorandom number generators. *Math. Comput.* 80, 273 (2011), 395–407.

ISO. 2012. *ISO/IEC 14882:2011 Information technology — Programming languages — C++*. International Organization for Standardization, Geneva, Switzerland. 1338 (est.) pages.

D. E. Knuth. 1997. *The Art of Computer Programming, Volume 2 (3rd ed.): Seminumerical Algorithms*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.

P. L’Ecuyer. 1999. Tables of maximally equidistributed combined LFSR generators. *Math. Comp.* 68, 225 (1999), 261–269. DOI: http://dx.doi.org/10.1090/S0025-5718-99-01039-X

P. L’Ecuyer and F. Panneton. 2009. $F_2$-Linear Random Number Generators. In *Advancing the Frontiers of Simulation: A Festschrift in Honor of George Samuel Fishman*, C. Alexopoulos, D. Goldman, and J. R. Wilson (Eds.). Springer-Verlag, New York, 169–193.

P. L’Ecuyer and R. Simard. 2007. TestU01: a C library for empirical testing of random number generators. *ACM Trans. Math. Software* 33, 4 (2007), Art. 22, 40. DOI: http://dx.doi.org/10.1145/1268776.1268777

H. Niederreiter. 1992. *Random Number Generation and Quasi-Monte Carlo Methods*. SIAM CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 63. SIAM, Philadelphia.

M. Saito and M. Matsumoto. 2008. SIMD-oriented Fast Mersenne Twister: a 128-bit Pseudorandom Number Generator. In *Monte Carlo and Quasi-Monte Carlo Methods 2006*, A. Keller, S. Heinrich, and H. Niederreiter (Eds.). Springer-Verlag, Berlin, 607–622. [http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/SFMT/index.html](http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/SFMT/index.html)

M. Saito and M. Matsumoto. 2009. A PRNG specialized in double precision floating point numbers using an affine transition. In *Monte Carlo and quasi-Monte Carlo methods 2008*. Springer, Berlin, 589–602. DOI: [http://dx.doi.org/10.1007/978-3-642-04107-5_38](http://dx.doi.org/10.1007/978-3-642-04107-5_38)

M. Saito and M. Matsumoto. 2013. Variants of Mersenne twister suitable for graphic processors. *ACM Trans. Math. Software* 39, 2 (2013), Art. 12, 20. DOI: [http://dx.doi.org/10.1145/2427023.2427029](http://dx.doi.org/10.1145/2427023.2427029)

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