Topological charge and spin pumping in a semiconductor nanowire

Zhi-Hai Liu$^{1,}$\textsuperscript{,}$\dagger$ and H. Q. Xu$^{1,}\ddagger$

$^{1}$Beijing Key Laboratory of Quantum Devices, Key Laboratory for the Physics and Chemistry of Nanodevices, and Department of Electronics, Peking University, Beijing 100871, China
$^{2}$Beijing Academy of Quantum Information Sciences, Beijing 100193, China

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The implementation of adiabatic topological pumping in a semiconductor nanowire is proposed by modulating the arranged gate potentials periodically. The topological charge pumping is demonstrated by simulating the Rice-Mele model on a periodic double-quantum-dot chain, and manifested by the gapless edge states in a finite chain with open boundary conditions. When exposed to an external staggered magnetic field, we show that the semiconductor nanowire can be used to implement the topological spin pumping and, meanwhile, serves as a dynamic version of the topological insulator in the presence of Rashba spin-orbit interaction. Explicitly, all the adiabatic pumps are confirmed by the nontrivial topological invariants of the Bloch functions under different modulations and, when the double-quantum-dot chain is coupled to the external leads, the topological transports can also be verified by exploiting the scattering matrix formulation.

I. INTRODUCTION

Recently, adiabatic topological pumping has attracted intensity attentions, as it is important to implement the quantized dissipationless transports\textsuperscript{1,2} and simulate higher dimensional topological physics\textsuperscript{3–5}. The topological pumping was first introduced by Thouless\textsuperscript{6,7}, by which the quantization of particle transport can be realized by slowly modulating a one-dimensional periodic potential. Heuristically, some topological phases of matter can be illustrated by an adiabatic quantized pump. As such, the integer quantum Hall effect is explained by Laughlin’s pumping argument\textsuperscript{8} and the topological insulator is associated with the changing of time-reversal polarization in a half cycle\textsuperscript{9}. However, up to now, the experimental realizations of the topological pumping are almost limited to the dynamic wave systems, such as the optical superlattices\textsuperscript{10–12} and the photonic quasicrystals\textsuperscript{13,14}, and it is therefore of interest to explore the topological transport in conventional semiconductors.

Both experimentally and theoretically, the adiabatic quantum pumping of electron charge and spin in semiconductor nanostructures are always one of the hotspot issue due to the state-of-the-art nanotechnology\textsuperscript{15–24}. Especially, the parametric pumping facilitates the adiabetic quantum transfer of non-interacting electrons in unbiased quantum dots\textsuperscript{16–18}, but the transferred number in a cycle is not necessarily quantized under this pump scheme\textsuperscript{16,17}. Undoubtedly, the pumped number is quantized through the adiabatic topological pumping, and that is deeply rooted in the nontrivial topology of the periodic wave functions\textsuperscript{6,7,25,26}. However, so far a systematic analysis of the topological pumping in a semiconductor nanowire is still scarce, and we hope our discussions will promote the study of the adiabatic transport in the semiconductor nanostructures.

In this paper, we propose the implementation of adiabatic topological pumping in a semiconductor nanowire by periodically modulating the gate potentials. To be more specific, here we assume that a periodic double-quantum-dot (DQD) chain is fabricated by the arranged barrier potentials on a semiconductor nanowire, as shown in Fig. 1(a). In the language of the second quantization, we show that the discrete chain is equivalent to the Rice-Mele model\textsuperscript{27}, and then the topological charge pumping can be realized by a nontrivial modulation of the gate potentials. Moreover, it is found that the synthetic lattice can be used to implement the topological spin pumping when exposed to an external staggered magnetic field, and simulate the topological insulator in the presence of Rashba spin-orbit coupling. In order to ensure the adiabaticity throughout the pumping process, the cyclic time is assumed to be much larger than the characteristic time involving in the interdot tunnelings, and can be denoted as the unit time for the convenience of our representation. By analogy with the quantum Hall and quantum spin Hall effect\textsuperscript{28,29}, the nontrivial topological pumping can be manifested by the gapless edge states in a finite chain with open boundary conditions. Moreover, based on the bulk-boundary correspondence\textsuperscript{30}, the exotic edge states are correlated to the nontrivial topology of the Bloch functions, and that results in the pumping of quantized polarizations in a cycle\textsuperscript{9}. While when the DQD chain is open to the external electron reservoirs, we also show that the topological pumping can be verified by the quantized charge and spin transports based...
on the scattering matrix method.

Undoubtedly, the electronic correlations play an vital role in quantum dots, and generates the inter-dot spin exchange interaction in the Coulomb regime\textsuperscript{31,32}. Nevertheless, the topological spin pumps have been demonstrated by periodically modulating the alternating exchange coupling strengths under an external staggered or tilted magnetic field\textsuperscript{12,33}. Meanwhile, because the electron occupancies of the quantum dots can be regulated by the gate potentials on a semiconductor nanowire\textsuperscript{34,35}, it is possible to keep the DQD chain at half filling, on which the electronic interaction is negligible and the non-interacting pumping scheme is suitable.

This paper is organized as followings. In Sec. II the continuous Hamiltonian of the non-interacting double-quantum-dot chain is introduced, and the implementation of the topological charge pumping is demonstrated therein. When exposed to an external staggered magnetic field, the adiabatic spin pumping in the semiconductor nanowire is analyzed in Sec. III. Under the effect of Rashba spin-orbit interaction, the pumping of time-reversal polarization in a half cycle is addressed in Sec. IV. For an open semiconductor nanowire coupled to the external leads, the topological charge and spin transports are investigated in Sec. V by exploiting the scattering matrix formalism. Finally, we present a brief conclusion in Sec. VI.

II. TOPOLOGICAL CHARGE PUMPING

For a semiconductor nanowire subjected to the periodic potential $U(t,x)$ applied in the wire-axial direction, as shown in Fig. 1(a), the effective non-interacting Hamiltonian of the nanowire is

\begin{equation}
H_0 = \frac{p^2}{2m_e} - V_c(t) + U(t,x),
\end{equation}

with $p = -i \hbar \partial / \partial x$ denoting the momentum operator, $m_e$ being the effective electron mass, and $V_c(t)$ corresponding to the off-set chemical potential. Structurally, it is suggested to separate $U(t,x)$ into two terms $U(t,x) = U_A(t,x) + U_V(t,x)$, with $U_A(t,x)$ comprised two barrier gate potentials $V(t)$ and $V_2(t)$ to define two quantum dots, QD1 and QD2, in a unit cell and $U_V(t,x)$ consisted of plunger gate potentials $\pm V_0(t)/2$ to modulate their on-site energies. To be more specific, let $2L$ represent the periodic length of the chain, $2d (2d \ll 2L)$ is the width of each barrier potential, and the spatial distributions of the gate potentials are square to facilitate the determination of the Bloch spectrum of the lattice.\textsuperscript{36} Therefore, within a unit cell of $-L \leq x < L$, the analytical forms of the two separated components take the form of

\begin{equation}
U_A(t,x) = \frac{1}{2} V_0(t) [f_1^+(x) - f_2^+(x)] + V_2(t) [1 + f_2^-(x)],
\end{equation}

with $f_1^+(x) = \Theta(x + d) \pm \Theta(x - d)$, $f_2^+(x) = \Theta(x + d - L) \pm \Theta(x + L - d)$, and $\Theta(x)$ the Heaviside function. To facilitate the study of the pumping, we assume the sum of the two barrier potentials $\bar{V} = V_1(t) + V_2(t)$ is fixed as a constant during the following calculations. Therefore, the energy spectrum of the periodic semiconductor nanowire can be regulated by the potentials $V_0$ and $\delta V = V_2 - V_1$ in the practice. For the case of $V_0(t) = 6 \sin(2\pi t) \text{meV}$ and $\delta V(t) = 8 + 6 \cos(2\pi t) \text{meV}$, the Bloch spectrum of the lattice on the $k - t$ plane is displayed in Fig. 1(b), with $k$ indicating the Bloch vector [in a unit of 1/(2L)] along the $x$ direction. Here, we assume that the synthetic lattice is defined on an InAs nanowire with $m_j = 0.023m_0$ (m$_0$ being the mass of a free electron), $2L = 120 \text{nm}$, $2d = 20 \text{nm}$, and $\bar{V} = 36 \text{meV}$ throughout this paper.

Meanwhile, based on the lowest orbital on each dot the discrete Hamiltonian of the double-quantum-dot chain can be derived by taking the second quantization method. Interestingly, it is found that the effective Hamiltonian is identical to that of the Rice-Mele model\textsuperscript{27}

\begin{equation}
H_{0,T} = \sum_n \left( t_{in,0} a_n^d b_n^+ + t_{ex,0} a_{n+1}^d b_n^+ + \text{h.c.} \right) - \frac{\Delta_0}{2} \sum_n \left( a_n^d a_n^d - b_n^+ b_n^+ \right),
\end{equation}

with $a_n^d$ and $b_n^+$ representing the creation operators of the lowest localized states on the two alternating QDs in the $n$-th unit cell, $t_{in/ex,0}$ corresponding to the intra/inter-cell hopping amplitude, and $\pm \Delta_0/2$ denoting the effective on-site energies. By

![FIG. 2. (color online) (a) The energy spectrum of a finite DQD chain comprising 30 sites versus $t$ under a nontrivial modulation mode with $V_0(t) = 6 \sin(2\pi t) \text{meV}$ and $\delta V(t) = 6 \cos(2\pi t) \text{meV}$, and the changing contour of $\bar{V} = (V_0, \delta V)$ shown in the inset. (b) The same as (a), but under a trivial modulation mode with $V_0 = 6 \sin(2\pi t) \text{meV}$ and $\delta V(t) = 8 + 6 \cos(2\pi t) \text{meV}$. (c) The spatial density distributions of the two gapless states, as indicated by the circle and square symbols in (a). (d) The numerical distribution of the Berry curvature $F(k,t)$ on the $k - t$ plane under the nontrivial modulation mode, by which the time-evolution of charge polarization $Q(t)$ is also depicted in the inset of (c).]
analog with the topological charge pumping in optical superlattices\textsuperscript{10,11}, the quantized charge transport can be realized when the changing trajectory of the parameter vector $\langle \Delta_n, \delta V \rangle$, with $\delta V = t_{\text{in}}, \delta t_{\text{ex}}$, obeys a nonzero winding number in a period. Here, because $\Delta_n$ and $\delta t_{\text{ex}}$ can be regulated by the potentials $V_0$ and $\delta V$ respectively, then we can deduce that the topological pumping can be implemented if the changing contour of the modulation vector $\vec{V} = (V_0, \delta V)$ possesses a nonzero winding number in a cycle.

By analogy to the quantum Hall effect\textsuperscript{29}, and regarding $t$ as an virtual momentum perpendicular to the axial direction, the nontrivial charge pumping can also be reflected by the gapless edge states in a finite DQD chain with open boundary conditions. To illustrate this point, the energy spectra of a finite chain versus $t$ are displayed in Figs. 2(a) and 2(b) under two modulation modes. As expected, Fig. 2(a) indicates that there exists two gapless edge states crossing the bandgap of the chain when the modulation vector $\vec{V}$ is shown to have a nonzero winding number in a cycle, see the inset. Besides, the probability density distributions of the two gapless states, denoted as $P_n$, at a fixed energy value within the bandgap ($E = -0.17$ meV) are also displayed in Fig. 2(c). For the sake of comparison, Fig 2(b) shows the energy spectrum under a trivial modulation mode, with the changing contour of $\vec{V}$ displayed in the inset, and it does confirm the vanishing of the gapless edge states at this case.

In fact, the presence of the gapless edge states is correlated to the nontrivial topology of a periodic chain based on the bulk-boundary correspondence\textsuperscript{30}. For this end, the Berry curvature of the occupied Bloch band is introduced\textsuperscript{25}

\begin{equation}
F(k, t) = i \langle \partial_t u | \partial_k u \rangle - \langle \partial_k u | \partial_t u \rangle \tag{4}
\end{equation}

with $|u(k, t)\rangle$ representing the occupied Bloch function, and the nontrivial topology is quantified by a nonzero Chern number\textsuperscript{6}

\begin{equation}
C = \frac{1}{2\pi} \int_0^T \int_{-\pi}^{\pi} F(k, t)dk dt . \tag{5}
\end{equation}

For instance, when subjected to a nontrivial modulation mode as illustrated in Fig. 2(a), the numerical distribution of the Berry curvature on the $k - t$ plane is displayed on Fig. 2(d), by which the nonzero Chern number is verified, i.e., $C = 1$ accordingly. Specifically, the numerical values of $F(k, t)$ are derived by taking the discretization method introduced by Fukuiet al. in Ref. 38. Meanwhile, based on the modern theory of polarization\textsuperscript{39,40}, we can interpret $Q(t) = \frac{1}{2\pi} \int_0^T \int_{-\pi}^{\pi} F(k, t)dk dt$ as the time-dependent charge polarization (in a unit of $-e$) pumped in this process, and then the effective pumping can be visualized by the time-evolution of $Q(t)$ in a cycle, just as shown in the inset of Fig. 2(c) with $Q(T) = 1$ under this nontrivial modulation pattern.

\section*{III. TOPOLOGICAL SPIN PUMPING}

Instead of the alternating plunger gate potentials, the DQD chain can be used to implement the topological spin pumping when exposed to an external staggered magnetic field applied in a one transverse direction, such as the $z$ direction. In this case, the continuous Hamiltonian of the spinfull DQD chain can be written as

\begin{equation}
H_Z = \frac{P^2}{2m_e} - V_c(t) + U_A(t, x) + U_Z(t, x)\sigma_z , \tag{6}
\end{equation}

with $U_Z(t, x)$ representing the periodic spin-splitting induced by the staggered magnetic field $B(t)$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ being the three Pauli matrices. Notably, the effect of the vector potential in the Landau gauge is ignored in Eq. (6) due to the strong confinement along the other transverse direction. For a unit cell with $-L \leq x < L$, $U_Z(t, x)$ takes the form of

\begin{equation}
U_Z(t, x) = \frac{\Delta_z(t)}{2} \left[ f^+_z(x) - f^-_z(x) \right] , \tag{7}
\end{equation}

with $\Delta_z(t) = g\mu_B B(t)$ denoting the Zeeman splitting, $g$ being the electron Landé factor, and $\mu_B$ the magneton constant. Moreover, based on the two lowest Zeeman splitting levels on each dot, the discrete Hamiltonian of the DQD chain is

\begin{equation}
H_{Z,T} = \sum_n \left( t_{\text{in}} \sigma^0_n b_n^\dagger + t_{\text{ex}} \sigma^{n+1}_n b_n + \text{h.c.} \right) \tag{8}
\end{equation}

\begin{equation}
\quad - \frac{\Delta_z}{2} \sum_n \left( \alpha_n \sigma_z a_n - b^\dagger_n \sigma^+_z b_n \right) , \quad \text{where} \quad \alpha_n = \Delta_n \sigma^+_n + \Delta_n^* \sigma^-_n , \tag{9}
\end{equation}

\begin{equation}
\quad \text{with} \quad a_n = a_{n+1} + \Delta_n \sigma^+_n a_n , \quad b_n = b^\dagger_{n+1} + \Delta_n^* \sigma^-_n b_n . \tag{10}
\end{equation}
with \( a_n = \{ a_{n^x, a_{n^y}} \}, b_n = \{ b_{n^x, b_{n^y}} \} \), and the subscripts \( \chi = \uparrow, \downarrow \) indexing the two eigenstates of \( \sigma_\chi \), i.e., \( \sigma_\uparrow = \uparrow \) and \( \sigma_\downarrow = \downarrow \).

Meanwhile, the Bloch spectrum of the spinfull chain can be ascertained by exploiting the periodic boundary conditions of the Bloch function of \( H_Z \). For the case of \( \Lambda_0(t) = 0.87 \sin(2\pi t) \) meV, which corresponds to the oscillating magnetic field \( B(t) = \cos(2\pi t) \) T for InAs with the Landé factor \( g = 15 \), and \( \delta V(t) = 6 \cos(2\pi t) \) meV, the two-dimensional Bloch spectrum on the \( k - t \) plane is presented in Fig. 3(a). Notably, because \( [H_Z, \sigma_\chi] = 0 \), each Bloch band shown in Fig. 3(a) is degenerate and comprised by the two different spin states. At the same time, it is also allowed to separate \( H_Z \) into two independent spin-polarized terms \( H_{Z\uparrow} + H_{Z\downarrow} \), with \( H_{Z\chi} = \sum_n \left( t_{in,\chi} a_{n,\chi}^\dagger b_{n,\chi} + t_{ex,\chi} b_{n+1,\chi}^\dagger a_{n,\chi} + \text{h.c.} \right) + (\Delta_0/2) \sum_n \left( a_{n,\chi}^2 - b_{n,\chi}^2 \right) \). Then, by analogy with the discrete Hamiltonian in Eq. (3), we can deduce that the spin-up electron can be pumped if the changing contour of the modulation vector \( M = (\Delta_0, \delta V) \) obeys a nonzero winding number in a cycle. The conclusion can also be applied to the spin-down electron but with the modulation vector replaced by \( M^* = (-\Delta_0, \delta V) \). Therefore, both the spin-up and spin-down electrons are pumped for a nontrivial modulation mode illustrated in Fig. 3(a), and can be manifested by the two doubly-degenerate spin edge states crossing the bandgap of a finite chain, just as shown in Fig. 3(b). In this case, all the bulk states are also degenerate due to the spin degeneracy. However, because the changing trajectories of the modulation vectors \( \vec{M} \) and \( \vec{M}^* \) are shown to have different winding directions in a cycle [seen the inset of Fig. 3(d)], the spin-up and spin-down electrons will be propagated in the different directions \( N \), and then results in the topological spin pumping.

To illustrate this point, the Berry curvatures of the two spin-dependent occupied Bloch bands are introduced \( F_{\chi=}\dagger, (k, t) = i(\partial_{\chi} u_{\chi}(k) \partial_{\chi} u_{\chi}(k)) - (\partial_{\chi} u_{\chi}(k) \partial_{\chi} u_{\chi}(k)) \), with \( |u_{\chi}(k, t) \rangle \) denoting the occupied Bloch functions. Specifically, the numerical distributions of the two Berry curvatures on the \( k - t \) plane are displayed in Fig. 3(c) under this nontrivial modulation and besides, on each spin subset the time-evolution of the pumped charge \( Q_\chi(t) = \frac{1}{2} \sum_{k} \int_{0}^{t} F_{\chi}(k, t) dk dt \) are also presented in Fig. 3(d). Numerically, the different signs of \( Q_\uparrow(t) \) and \( Q_\downarrow(t) \) implies the distinct propagating directions of the two different spin states \( 12 \), and the identities \( Q_\uparrow(t) = \pm 1 \) show the effective topological pumping on each spin subspace in a cycle. However, the equality \( Q_\uparrow(T) + Q_\downarrow(T) = 0 \) demonstrates that there is no net charge transport in the whole process, but it does verify the effective topological spin pumping (in a unit of \(-h\)) because \( \frac{1}{2} \{ Q_\uparrow(T) - Q_\downarrow(T) \} = 1 \).

IV. TIME-REVERSAL POLARIZATION PUMPING

It is well known that the spin-orbit interaction plays an important role in the development of the topological insulator \( 11,12 \). In the presence of Rashba spin-orbit interaction (SOI), we will demonstrate that the DQD chain can be served as a dynamic version of the topological insulator when the parameter vector \( \vec{M} = (\Delta_0, \delta V) \) is driven by a nontrivial modulation pattern. At this point, the continuous Hamiltonian of the DQD chain can be written as

\[
H = H_Z + \alpha \sigma_\chi, \tag{9}
\]

with the Hamiltonian \( H_Z \) given in Eq. (6) and \( \alpha \) representing the strength of the SOI, which can also be quantified by the spin-orbit length \( x_{so} \approx h/(m \alpha) \).

Actually, the lowest two Zeeman splitting levels on each dot are the quasi-spin states due to the SOI and the second quantized Hamiltonian of the DQD chain with \( x_{so} \gg d \) can be approximated as

\[
H_T = \sum_n \left( a_n^\dagger t_{in} b_n + a_{n^+1}^\dagger t_{ex} b_n + \text{h.c.} \right) - \Delta_0 \sum_{n} \left( a_n^\dagger \sigma_z a_n - b_n^\dagger \sigma_z b_n \right). \tag{10}
\]

It is important to note that \( a_n = \{ a_{n^x, a_{n^y}} \} \) and \( b_n = \{ b_{n^x, b_{n^y}} \} \) represent the spinors comprised by the two quasi-spin states on the dots and the intra/inter-cell tunneling amplitude can be written as a matrix operating in the quasi-spin space

\[
t_{in/ex} = t_{in/ex,0} \exp[i \phi_{in/ex} \sigma_z], \tag{11}
\]
with the phase factors $\varphi_{\text{in/ex}} \approx L/x_{ao}$. Actually, the exponential factor in Eq. (11) can be illustrated as the Aharonov-Casher effect in the interdot spin tunneling with spin-orbit coupling.$^{43,44}$

Similarly, the Bloch spectrum of the DQD chain with SOI can also be derived based on the original Hamiltonian in Eq. (9). For the case of $x_{ao} = 180$ nm,$^{45}$ Fig. 4(a) shows the Bloch spectrum of the lowest four Bloch bands on the $k-t$ plane when the modulation vector $\hat{M}$ is driven by the mode illustrated in Fig. 3(a). Under this nontrivial modulation model, it indicates that the spin degeneracy is generally lifted under the effect of SOI, except at the moments with $t = 0, 0.5$ and 1. In fact, because of the zero magnetic field at these times, the spin degeneracy manifests as the Kramers’ degeneracy at the time-reversal momenta $k = 0, \pm \pi$. Meanwhile, based on the effective discrete Hamiltonian in Eq. (10), the energy spectrum of a finite DQD chain versus $t$ is displayed in Fig. 4(b) under this nontrivial modulation pattern. As expected, there are four quasi-spin edge states crossing the band-gap of the finite chain and the double degeneracy in the energy levels (including the bulk and edge states) can only occur when $t = 0.5$.

Moreover, the energy spectrum of the spin-orbit DQD chain under a trivial modulation mode with $\Delta_0 = 0.87 \sin(2\pi t)$ meV and $\delta V = 8 + 6 \cos(2\pi t)$ meV is displayed in Fig. 4(c) for comparison and obviously, the gapless quasi-spin edge states vanish at this case. Actually, by analogy with the topological insulator, the presence of the quasi-spin gapless edge states is associated with the pumping of the time-reversal polarization in a half period.$^6$. To be more specific, the polarization can be derived as the difference between the centers of 1D maximally localized Wannier functions constructed by the occupied Bloch wave functions$^{46,47}$. For this end, the periodic wave vector is first discretized $k_j = 2(j/G)\pi - \pi$ with $j = 1, 2, ..., G$ and, at a fixed time, the overlap matrix between two adjacent points is also introduced $\Lambda_{\gamma \nu}^{(j + 1)j} = \langle u_\gamma(k_j, t)|u_\nu(k_{j+1}, t)\rangle$, with $\nu, \gamma = 1, 2$ indexing the two occupied Bloch bands. Then, based on the Wilson loop

$$W(t) = \Lambda^{(1,2)}\Lambda^{(2,3)}...\Lambda^{(G-1,G)}\Lambda^{(G,1)}$$

with $\lambda_j(t)$ and $\lambda_j(t)$ denoting its two time-dependent eigenvalues, the Wannier centers can be calculated as $\bar{x}_\gamma(t) = \text{arg}(|\lambda_\gamma(t)|/(2\pi)) \in [-0.5, 0.5]$ with $\bar{x}_1 \geq \bar{x}_2$ and therefore time-reversal polarization is given by$^{46,47}$

$$P(t) = \bar{x}_1(t) - \bar{x}_2(t).$$

Under these two different modulation patterns, the time-evolutions of $\bar{x}_\gamma(t)$ are displayed in Fig. 4(d) and besides, the corresponding changing contours of the modulation vector $\hat{M} = (\Delta_0, \delta V)$ are also displayed in the inset. For a nontrivial modulation mode, as indicated by the solid (blue) lines shown in Fig. 4(d), the time-reversal polarization is increased by one in the half cycle, i.e., $P(T/2) - P(0) = 1$, and that is in contrast to a trivial pumping with $P(T/2) - P(0) = 0$. Practically, the variation of $P(t)$ in a half cycle can be identified as the $\Sigma_2$ invariant, which is used to distinguish the topological insulator theoretically$^{41,42}$. Therefore, the DQD chain can be served as a dynamical version of the topological insulator, with the topology can be manipulated by the parameter vector $\hat{M}$.

![FIG. 5](image)

**FIG. 5.** (color online) (a) The schematic diagram of the one-dimensional tight-binding version of an open double-quantum-dot chain comprising $N$ sites and coupled to the external leads by the tunneling coupling $J'$. (b) The transferred charge $\Delta_\nu$ in a unit of $-e$, as a function of $t$ when the modulation vector $\hat{V} = (V_0, \delta V)$ is driven by the two different modes illustrated in Fig. 2. Here, the trivial and nontrivial modulation modes are denoted by “T” and “NT”, respectively. (c) The pumped spin $\Delta_\nu$, in a unit of $-\hbar$, versus $t$ with the vector $\hat{M} = (\Delta_0, \delta V)$ driven by the two different modulation patterns addressed in Fig. 4(d).

**V. TOPOLOGICAL PUMPING IN AN OPEN DQD CHAIN**

Experimentally, a chain of quantum dots fabricated on a semiconductor nanowire is often open to the electron reservoirs$^{34,35}$ and generally, the electrons in the external leads are crucial to sustain the balance during the adiabatic pumping. Therefore, it is important to analysis the response of the asymptotic electron states to the adiabatic modulation as discussed above.

For this end, we simplify all the system by a one-dimensional tight-binding model, as shown in Fig. 5(a). In the case of a finite DQD chain comprised by $N$ sites, the left and right external leads are described by the two lattice Hamiltonians $H_l = J_0 \sum_{n=1}^{N-1} \left( \hat{c}_n^\dagger \hat{c}_{n+1} + \text{h.c.} \right)$ and $H_r = J_0 \sum_{n=N-1}^{+1} \left( \hat{c}_n^\dagger \hat{c}_{n+1} + \text{h.c.} \right)$, with $\hat{c}_n^\dagger$ representing the electron creation operator on the $n$-site of the leads and $J_0$ denoting the uniform inter-site tunneling amplitude. Meanwhile, the coupling to the DQD chain is depicted by the tunneling coupling interaction $H_c = J' \left( a_0 a_0^\dagger + b_0 b_0^\dagger + \text{h.c.} \right)$, with $J'$ being the spin-independent coupling strength. Therefore, the total Hamiltonian of the open DQD chain can be written as

$$H_{\text{tot}} = H_{\text{DQD}} + H_l + H_r + H_c$$

where $H_{\text{DQD}}$ represents the discrete Hamiltonian of the DQD chain and is shown to have different explicit forms under different conditions.

Apparently, for the two semi-infinite external leads, the energy eigenvalue in the momentum space can be derived as $E = 2J_0 \cos(k)$, with the degenerate eigenstates given by $\exp(\pm ikn)$. Generally, when the DQD chain is subjected to a right-moving input wave, the scattered wave in the leads
takes the form of

$$|\psi_t\rangle = \begin{cases} \frac{\exp(ikn) + R(t)\exp(-ikn)}{\mathcal{T}(t): \exp(ikn)} & n \leq 0 \\ \frac{\mathcal{T}(t)\exp(ikn)}{n \geq N + 1} & \end{cases}, \quad (15)$$

with $\mathcal{T}(t)$ and $R(t)$ representing the time-dependent transmission and reflection amplitudes, respectively. Practically, all the amplitudes can be ascertained by the transfer method as illustrated in Appendix A for the spinless case, with $H_{\text{DQD}}$ given by $H_{0,T}$ in Eq. (3). Then, based on the scattering matrix approach the transferred charge in a cycle can be calculated as

$$\Delta q = \frac{e}{2\pi i} \int_0^T dt R^* (t) \frac{dR(t)}{dt} \cdot \quad (16)$$

For a spinfull DQD chain, $R(t)$ corresponds to a $2 \times 2$ matrix in spin space, and it reduces to a diagonal matrix in the absence of SOI. In this case, the discrete Hamiltonian $H_{\text{DQD}}$ becomes $H_{\text{z,1}}$ in Eq. (8) and the transferred spin for each period is given by

$$\Delta \xi = \frac{\hbar}{4\pi i} \int_0^T dt \left( R^* (t) \frac{dR(t)}{dt} - R(t) \frac{dR^* (t)}{dt} \right), \quad (17)$$

with $R^* (t)$ and $R(t)$ representing the time-dependent spin-up and spin-down reflection amplitudes, respectively. While under the effect of SOI, there exists spin flipping involving in the reflection and the optimal topological spin pumping in a spin-orbit discrete chain has been demonstrated in Ref. 2. Therefore, we will limit ourselves to investigate the case without SOI and verify the quantized charge and spin transport in a semiconductors of topological phases of matter on the one-dimensional nanostructures, experimentally.

VI. CONCLUSION

In this paper, we have demonstrated the implementation of topological charge and spin pumping in a semiconductor double-quantum-dot chain by periodically modulating the arranged gate potentials and an external staggered magnetic field. Experimentally, the fabrication of a periodic array of quantum dots on a semiconductor nanowire can be achieved by exploiting the nanoimprint lithography\cite{34,35,49}, and there is no specific requirement of the gate potentials. Although the spatial distribution of the gate potentials is assumed to be square in our model, the prerequisite for implementing the topological pumping is the tunability of the discrete parameters of the double-quantum-dot chain, such as the on-site energies and the staggered interdot tunneling amplitudes, during the modulation. Especially, the fabrication of high controllable quantum dots on the nanomaterials with long coherence times, such as the carbon nanotube or Ge/Si heterostructure nanowire\cite{30,51}, will promote the implementation of adiabatic topological pumping experimentally. Moreover, the micromagnet, which are widely used in the spin-based quantum dot quantum computing\cite{32,53}, offers a platform to generate the staggered magnetic field and the tunable Rashba spin-orbit interaction in III-V semiconductors\cite{45} facilitates the realization of a dynamic version of the topological insulator. Overall, our theoretical study should promote the implementation of adiabatic topological pumping and the simulation of higher dimensional topological phases of matter on the one-dimensional nanostructures, experimentally.

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Appendix A: Derivation of the reflection amplitude

In this appendix, the detailed process to derive the reflection coefficient of an open spinless double-quantum-dot chain is given. Under this circumstance, the explicit form of the total Hamiltonian in Eq. (14) can be written as

$$H_{\text{tot}} = \sum_{n=1}^N (t_{n,0}a_n^\dagger b_n + t_{n,0}^\dagger a_n b_n + \text{h.c.}) - \Delta_0$$

$$\times \sum_{n=1}^N (a_n^\dagger a_n - b_n^\dagger b_n) + J_0 \sum_{n} \gamma_n (c_n^\dagger c_{n+1} + \text{h.c.})$$

$$+ J' (a_1^\dagger c_0 + b_N^\dagger c_{N+1} + \text{h.c.}),$$

(A1)

with the coefficients

$$\gamma_n = \begin{cases} 1 & n < -1 \text{ or } n > N + 1 \\ 0 & -1 \leq n \leq N + 1 \end{cases}$$

and

$$\xi_n = 1 - \delta_{n,N}.$$

When the DQD chain is subjected to a right-moving electron wave, the scattering solution in the leads is given by Eq. (15). By exploiting the scattering matrix formalism, the
amplitudes of the electron wave on the left side are connected to the amplitudes on the right side by the transfer equation:

\[ T(t) e^{ik(N+2)} = \mathcal{V} \left[ e^{-ikt} + \mathcal{R}(t) e^{ikt} \right], \]  \hspace{1cm} (A2)

with \( T(t) \) and \( \mathcal{R}(t) \) representing the instantaneous transmission and reflection coefficients and, explicitly, the transfer matrix \( \mathcal{V} \) can be expanded as

\[ \mathcal{V} = IP^{-N}O. \]  \hspace{1cm} (A3)

Here, \( I \) represents the matrix responsible for the input transition \( e^{-ikt} + \mathcal{R}(t) e^{ikt} \) with

\[ I = \left[ \begin{array}{cc} \frac{E+\Delta}{t_{\alpha 0}} & -\frac{J}{t_{\alpha 0}} \\ \frac{J}{t_{\alpha 0}} & \frac{E-\Delta}{t_{\alpha 0}} \end{array} \right] \]  \hspace{1cm} (A4)

\( P \) corresponds to the transfer matrix within the DQD chain \[ \left[ \begin{array}{c} B_n \\ A_n \end{array} \right] \rightarrow \left[ \begin{array}{c} B_{n+1} \\ A_{n+1} \end{array} \right] \) and takes the form of

\[ P = \left[ \begin{array}{cc} E+\Delta & -\frac{J}{t_{\alpha 0}} \\ \frac{J}{t_{\alpha 0}} & E-\Delta \end{array} \right] \left[ \begin{array}{cc} 1 & t_{\alpha 0} \\ 1 & 0 \end{array} \right]. \]  \hspace{1cm} (A5)

\( Q \) denotes the output matrix and, based on the system Hamiltonian in Eq. (A1), can be written as

\[ O = \left[ \begin{array}{cc} E & -\frac{J}{t_{\alpha 0}} \\ \frac{J}{t_{\alpha 0}} & E \end{array} \right] \left[ \begin{array}{cc} 1 & t_{\alpha 0} \\ 1 & 0 \end{array} \right]. \]  \hspace{1cm} (A6)

Then, by substituting Eqs. (A4-A6) into Eq. (A3), we can derive the explicit form of the transfer matrix \( \mathcal{V} \) and then, based on Eq. (A2), the analytic expression of the reflection amplitude

\[ \mathcal{R}(t) = \frac{[\mathcal{V}]_{2,1} e^{2ikt} + [\mathcal{V}]_{1,2} e^{ikt} - [\mathcal{V}]_{1,1}}{[\mathcal{V}]_{1,1} - [\mathcal{V}]_{2,2} + [\mathcal{V}]_{1,2} e^{ikt} - [\mathcal{V}]_{2,1} e^{ikt}}. \]  \hspace{1cm} (A7)

with \( [\mathcal{V}]_{n,m} \) representing the matrix elements of the transfer matrix.
29 X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057–1110 (2011).
30 Y. Hatsugai and T. Fukui, Bulk-edge correspondence in topological pumping, Phys. Rev. B 94, 041102(R) (2016).
31 G. Burkard, D. Loss, and D. P. Divincenzo, Coupled quantum dots as quantum gates, Phys. Rev. B 59, 2070 (1999).
32 Z.-H. Liu, O. Entin-Wohlman, A. Aharony, and J. Q. You, Control of the two-electron exchange interaction in a nanowire double quantum dot, Phys. Rev. B 98, 241303(R) (2018).
33 R. Shindou, Quantum spin pump in $S = 1/2$ antiferromagnetic chains - Holonomy of phase operators in sine-Gordon theory, J. Phys. Soc. Jpn. 74, 1214 (2005).
34 J.-Y. Wang, S. Huang, G.-Y. Huang, D. Pan, J. Zhao, and H. Q. Xu, Coherent transport in a linear triple quantum dot made from a pure-phase InAs nanowire, Nano Lett. 17(7), 4158-4164 (2017).
35 J. Mu, S. Huang, Z.-H. Liu, W. Li, J.-Y. Wang, D. Pan, G.-Y. Huang, Y. Chen, J. Zhao, and H. Q. Xu, A highly tunable quadruple quantum dot in a narrow bandgap semiconductor InAs nanowire, Nanoscale 13, 3983 (2021).
36 Z.-H. Liu, O. Entin-Wohlman, A. Aharony, J. Q. You, and H. Q. Xu, Su-Schrieffer-Heeger model: Interplay of spin-orbit and Zeeman interactions, arXiv:2104.07291.
37 R. Winkler, Spin-orbit Coupling Effects in Two-dimensional Electron and Hole Systems, (Springer, Berlin, 2003).
38 T. Fukui, Y. Hatsugai and H. Suzuki, Chern Numbers in Discretized Brillouin Zone: Efficient Method of Computing (Spin) Hall Conductances, J. Phys. Soc. Jpn. 74,1674 (2005).
39 R. D. King-Smith and D. Vanderbilt, Theory of polarization of crystalline solids, Phys. Rev. B 47, 1651(R) (1993).
40 J. K. Asbóth, L. Orozslyán, and A. Pályi, Current Operator and Particle Pumping, A Short Course on Topological Insulators. Lecture Notes in Physics Vol. 919 (Springer, Cham, 2016).
41 C. L. Kane and E. J. Mele, $Z_2$ Topological Order and the Quantum Spin Hall Effect, Phys. Rev. Lett. 95, 146802 (2005).
42 B. A. Bernevig, T. L. Hughes, and S.C. Zhang, Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells, Science 314, 1757–1761 (2006).
43 Y. Aharonov and A. Casher, Topological Quantum Effects for Neutral Particles, Phys. Rev. Lett. 53, 319 (1984).
44 T. V. Shahbazyan and M. E. Raikh, Low-Field Anomaly in 2D Hopping Magnetoresistance Caused by Spin-Orbit Term in the Energy Spectrum, Phys. Rev. Lett. 73, 1408 (1994); O. Entin-Wohlman and A. Aharony, DC Spin geometric phases in hopping magnetoconductance, Phys. Rev. Research 1, 033112 (2019).
45 Z. Scherübl, G. Fülöp, M. H. Madsen, J. Nygård, and S. Csonka, Electrical tuning of Rashba spin-orbit interaction in multigated InAs nanowires, Phys. Rev. B 94, 035444 (2016).
46 R. Yu, X. L. Qi, A. Bernevig, Z. Fang and Xi Dai, Equivalent expression of $Z_2$ topological invariant for band insulators using the non-Abelian Berry connection, Phys. Rev. B 84, 075119 (2011).
47 A. A. Soluyanov and D. Vanderbilt, Wannier representation of $Z_2$ topological insulators, Phys. Rev. B 83, 035108 (2011); Computing topological invariants without inversion symmetry, Phys. Rev. B 83, 235401 (2011).
48 C. Q. Zhou, Y. F. Zhang, L. Sheng, R. Shen, D. N. Sheng, and D. Y. Xing, Proposal for a topological spin Chern pump, Phys. Rev. B 90,085133 (2014).
49 I. Martini, M. Kamp, F. Fischer, L. Worschech, J. Koeth, A. Forchel, Fabrication of quantum point contacts and quantum dots by imprint lithography, Microelectron. Eng. 57, 397-403 (2001).
50 M. J. Biercuk, S. Garaj, N. Mason, J. M. Chow, and C. M. Marcus, Gate-defined quantum dots on carbon nanotubes, Nano Lett. 5(7), 1267-1271 (2005).
51 Y. Hu, H. Churchill, D. Reilly, J. Xiang, C. M. Lieber, and C. M. Marcus, A Ge/Si heterostructure nanowire-based double quantum dot with integrated charge sensor, Nat. Nanotechnol. 2, 622-625 (2007).
52 D. Kim, Z. Shi, C. B. Simmons, D. R. Ward, J. R. Prance, T. S. Koh, J. K. Gamble, D. E. Savage, M. G. Lagally, M. Friesen, S. N. Coppersmith, and M. A. Eriksson, Quantum control and process tomography of a semiconductor quantum dot hybrid qubit, Nature 511, 70 (2018).
53 T. Ito, T. Otsuka, T. Nakajima, M. R. Delbecq, S. Amaha, J. Yoneda, K. Takeda, A. Noiri, G. Allison, A. Ludwig, A. D. Wieck, and S. Tarucha, Four single-spin Rabi oscillations in a quadruple quantum dot, Appl. Phys. Lett.113, 093102 (2018).