Efficient Reduced-Rank DOA Estimation Algorithms Using Alternating Low-Rank Decompositions

Yunlong Cai, Linzheng Qiu, Rodrigo C. de Lamare, and Minjian Zhao

Abstract

In this work, we propose an alternating low-rank decomposition (ALRD) approach and novel subspace algorithms for direction-of-arrival (DOA) estimation. In the ALRD scheme, the decomposition matrix for rank reduction is composed of a set of basis vectors. A low-rank auxiliary parameter vector is then employed to compute the output power spectrum. Alternating optimization strategies based on recursive least squares (RLS), denoted as ALRD-RLS and modified ALRD-RLS (MARLD-RLS), are devised to compute the basis vectors and the auxiliary parameter vector. Simulations for large sensor arrays with both uncorrelated and correlated sources are presented, showing that the proposed algorithms are superior to existing techniques.

Index Terms

DOA estimation, low-rank decomposition, parameter estimation.

I. INTRODUCTION

Array signal processing has been widely used in areas such as radar, sonar and wireless communications. Many applications related to array signal processing require the estimation of the direction-of-arrival (DOA) of the sources impinging on a sensor array [1]. Among the well-known DOA estimation schemes are the Capon method and subspace-based algorithms [2] such as Multiple-Signal Classification (MUSIC) [3] and Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) [4]. The Capon method calculates the output power spectrum for each scanning angle according to the constrained minimum variance (CMV) criterion. Then the estimated DOAs can be obtained by finding the peaks of the output power spectrum [5]. MUSIC, ESPRIT and their improved versions [6], [7], [8], [9], [10], [11] estimate the DOAs by exploiting the signal and the noise subspaces of the signal correlation matrix. Due to the eigenvalue decomposition (EVD) and/or the singular-value decomposition (SVD), MUSIC and ESPRIT require a high computational cost, especially for large sensor arrays. The recently proposed subspace-based auxiliary vector (AV) [12], the conjugate gradient (CG) [13] and the joint iterative optimization (JIO) algorithms [14] employ basis vectors to build the signal subspace instead of the EVD or the SVD. However, the iterative construction of the basis vectors yields a complexity comparable to the EVD. Moreover, the AV and CG algorithms cannot provide a satisfactory performance for large sensor arrays with many sources.

In recent years, large sensor arrays have gained importance for applications such as radar and future communication systems. The performance of direction finding algorithms depends on the data record and the array size. Resorting to large arrays or more snapshots leads to higher resolution [2]. However, direction finding for large arrays also requires large data records and are associated with high computational costs. Beamspace DOA estimation [15], [16], [17], [18] is an effective method to reduce the computational burden. Nevertheless, the beamspace-based algorithms are sensitive to the presence of sources located outside the angular sectors-of-interest [19].

In this paper, we present an alternating low-rank decomposition (ALRD) approach for DOA estimation in large sensor arrays with a large number of sources. In the ALRD scheme, a subspace decomposition matrix which consists of a set of basis vectors and an auxiliary parameter vector are employed to compute the output power spectrum for each scanning angle. In order to

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This work was supported in part by the National Natural Science Foundation of China under Grants 61471319, Zhejiang Provincial Natural Science Foundation of China under Grant LY14F010013, the Fundamental Research Funds for the Central Universities, and the National High Technology Research and Development Program (863 Program) of China under Grant 2014AA01A707.
avoid matrix inversions, we develop recursive least squares (RLS) type algorithms \cite{20} to compute the basis vectors and the auxiliary parameter vector, which reduces the computational complexity. The proposed DOA estimation algorithms are referred to as ALRD-RLS and modified ALRD-RLS (MALRD-RLS), which employs a single basis vector.

The paper is organized as follows. In Section III we outline the system model and the problem of DOA estimation. The proposed ALRD scheme and algorithms are presented in Section III. In Section IV, we illustrate and discuss the simulation results. Finally, Section V concludes this work.

II. SYSTEM MODEL AND PROBLEM Formulation

We consider a uniform linear array (ULA) with \( M \) omnidirectional sensor elements and suppose that \( K \) narrowband source signals impinge on the ULA from directions \( \theta_1, \theta_2, \ldots, \theta_K \), respectively, where \( M \) is a large number with \( K \ll M \). The \( i \)th snapshot of the received signal can be expressed by an \( M \times 1 \) vector as

\[
\mathbf{r}(i) = \sum_{k=1}^{K} \mathbf{a}(\theta_k)s_k(i) + \mathbf{n}(i)
\]

where \( s_k(i) \) is the \( k \)th source signal with power \( \sigma_n^2 \). \( \mathbf{n}(i) \) denotes the noise vector which is assumed to be temporally and spatially white Gaussian with zero mean and variance \( \sigma_n^2 \). The array steering vector \( \mathbf{a}(\theta_k) \) is defined as

\[
\mathbf{a}(\theta_k) = [1, e^{-2\pi j \frac{\lambda}{d} \cos \theta_k}, \ldots, e^{-2\pi j (M-1) \frac{\lambda}{d} \cos \theta_k}]^T
\]

where \(( \cdot )^T\) denotes the transpose operation and \( \lambda \) is the signal wavelength. The parameter \( d_s = \frac{\lambda}{\pi} \) represents the array inter-element spacing.

Direction finding algorithms aim to estimate the DOAs \( \Theta = [\theta_1, \ldots, \theta_K]^T \) by processing \( \mathbf{r}(i) \). The correlation matrix of \( \mathbf{r}(i) \) is given by

\[
\mathbf{R} = \mathbb{E}\{\mathbf{r}(i)\mathbf{r}^H(i)\} = \sum_{k=1}^{K} \mathbf{a}(\theta_k)\mathbf{R}_{s,k}\mathbf{a}^H(\theta_k) + \sigma_n^2\mathbf{I}_M
\]

where \( \mathbb{E}\{\cdot\} \) denotes expectation, \(( \cdot )^H \) is the Hermitian operator and \( \mathbf{I}_M \) is the identity matrix with dimension \( M \). \( \mathbf{R}_{s,k} \) is the correlation matrix of the \( k \)th signal with \( \mathbf{R}_{s,k} = \mathbb{E}\{s_k(i)s_k^H(i)\} \). \( \mathbf{R}_n = \mathbb{E}\{\mathbf{n}(i)\mathbf{n}^H(i)\} = \sigma_n^2\mathbf{I}_M \) is the correlation matrix of the noise vector. Note that the exact knowledge of \( \mathbf{R} \) is difficult to obtain, thus estimation by sample averages is employed in practice, which is \( \hat{\mathbf{R}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}(i)\mathbf{r}^H(i) \), with \( N \) being the number of available snapshots.

III. PROPOSED ARLD SCHEME ALRD-RLS AND MALRD-RLS ALGORITHMS

In this section, we detail the proposed ALRD scheme and the ALRD-RLS and MALRD-RLS DOA estimation algorithms. The ALRD scheme divides the received vector into several segments and processes each segment with an individual basis vector. The basis vectors constitute the columns of the decomposition matrix, which performs dimensionality reduction \cite{21, 22, 23, 24, 25, 26, 27, 28}. Then, a lower dimensional data vector is processed by the auxiliary parameter vector to construct the output power spectrum. The ALRD-RLS and the MALRD-RLS algorithms are based on an alternating optimization procedure of the basis vectors and the reduced-rank auxiliary parameter vector.

![Fig. 1. Block diagram of the ALRD scheme](image-url)
A. Proposed ALRD Scheme

The block diagram of the ALRD scheme is depicted in Fig. 1. The received vector \( r(i) = [r_0(i) \ldots r_{M-1}(i)]^T \) is processed by an \( M \times D \) decomposition matrix \( T(i) \) which consists of a set of \( I \times 1 \) basis vectors \( s_d(i) \), where \( d \in \{1, \ldots D\} \). The resulting data vector can be expressed by

\[
\mathbf{r}_D(i) = T^H(i)r(i) = \sum_{d=1}^{D} q_d d_d^H \mathbf{R}(i) s_d^T(i)
\]

where \( q_d \) is a \( D \times 1 \) vector with a one in the \( d \)th position and zeros elsewhere. \( d_d \) is the \( D \times 1 \) selection vector to divide \( r(i) \) into \( D \) segments, which are defined as:

\[
d_d = [0 \ldots 0 \ 1 \ 0 \ldots 0]^T
\]

where \( \mu_d \) is the selection pattern which is chosen as \( \mu_d = (d-1) \left\lfloor \frac{M}{D} \right\rfloor \). The \( M \times I \) matrix \( \mathbf{R}(i) \) corresponding to \( r(i) \) has a Hankel structure [29], which is described by

\[
\mathbf{R}(i) = \begin{pmatrix}
  r_{0}(i) & r_{1}(i) & \cdots & r_{I-1}(i) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{M-I}(i) & r_{M-I+1}(i) & \cdots & r_{M-1}(i) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{M-2}(i) & r_{M-1}(i) & 0 & 0 \\
  r_{M-1}(i) & 0 & 0 & 0
\end{pmatrix}
\]

The \( I \times 1 \) basis vectors \( s_1(i) \ldots s_D(i) \) form the \( M \times D \) decomposition matrix \( T(i) \). After dimensionality reduction, the \( D \times 1 \) data vector \( \mathbf{r}_D(i) \) is processed by an auxiliary parameter vector \( \tilde{\omega}(i) \) to compute the output power spectrum. As seen from Fig. 1, the basis vectors \( s_d(i) \) and the auxiliary parameter vector \( \tilde{\omega}(i) \) are alternately optimized according to some prescribed criterion, which is introduced in what follows.

B. Proposed ALRD-RLS DOA Estimation Algorithm

The ALRD-RLS algorithm solves the optimization problem:

\[
\min_{\tilde{\omega}(i), s_d(i)} \sum_{l=1}^{i} \alpha^{i-l} |\tilde{\omega}(i)^H \sum_{d=1}^{D} q_d d_d^H \mathbf{R}(l) s_d^T(i)|^2
\]

subject to \( \tilde{\omega}(i)^H \sum_{d=1}^{D} q_d d_d^H \mathbf{A}_n s_d^T(i) = 1 \)

where \( \alpha \) is a forgetting factor close to but smaller than 1. \( \mathbf{A}_n \) is the \( M \times I \) Hankel matrix of the scanning steering vector \( \mathbf{a}(\theta_n) = [a_0(\theta_n) \ldots a_{M-1}(\theta_n)]^T \) given by

\[
\mathbf{A}_n(i) = \begin{pmatrix}
  a_0(\theta_n) & a_1(\theta_n) & \cdots & a_{I-1}(\theta_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{M-I}(\theta_n) & a_{M-I+1}(\theta_n) & \cdots & a_{M-1}(\theta_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{M-2}(\theta_n) & a_{M-1}(\theta_n) & 0 & 0 \\
  a_{M-1}(\theta_n) & 0 & 0 & 0
\end{pmatrix}
\]

The optimization problem in (7) can be solved by the method of Lagrange multipliers whose Lagrangian is described by

\[
\mathcal{L}(i) = \sum_{l=1}^{i} \alpha^{i-l} |\tilde{\omega}(i)^H \sum_{d=1}^{D} q_d d_d^H \mathbf{R}(l) s_d^T(i)|^2 + 2\Re\{\lambda \tilde{\omega}(i)^H \sum_{d=1}^{D} q_d d_d^H \mathbf{A}_n s_d^T(i) - 1\}
\]
where $\Re\{\cdot\}$ selects the real part of the argument. By taking the gradient of (9) with respect to $s_d(i)$, we obtain
\[
\frac{\partial \mathcal{L}(i)}{\partial s_d(i)} = \sum_{l=1}^{i} \alpha^{i-l} \mathcal{R}^H(l) d_d q_d^H \bar{\omega}(i) \bar{\omega}^H(i) q_d d_d^H \mathcal{R}(l) s_d(i) \\
+ \sum_{l=1}^{i} \alpha^{i-l} \mathcal{R}^H(l) d_d q_d^H \bar{\omega}(i) \bar{\omega}^H(i) \sum_{j \neq d} q_j d_j^H \mathcal{R}(l) s^*_j(i) \\
+ \lambda^* \mathcal{A}_n^H d_d q_d^H \bar{\omega}(i).
\]
By equating (10) to zero and solving for $s_d(i)$, we have
\[
R_{s,d}(i) s_d(i) = -\sum_{j \neq d}^D P_{s,j}(i) - \lambda^* \mathcal{A}_n^T d_d q_d^H \bar{\omega}^*(i)
\]
where
\[
R_{s,d}(i) = \sum_{l=1}^{i} \alpha^{i-l} \mathcal{R}^T(l) q_d d_d^H \bar{\omega}^*(i) \bar{\omega}^T(i) q_d d_d^H \mathcal{R}^*(l)
\]
\[
P_{s,j}(i) = \sum_{l=1}^{i} \alpha^{i-l} \mathcal{R}^T(l) d_d q_d^H \bar{\omega}^*(i) \bar{\omega}^T(i) q_j d_j^H \mathcal{R}^*(l) s_j(i).
\]
Then $I \times 1$ basis $s_d(i)$ is described as:
\[
s_d(i) = -R_{s,d}^{-1}(i) \sum_{j \neq d} P_{s,j}(i) - \lambda R_{s,d}^{-1}(i) \mathcal{A}_n^T d_d q_d^H \bar{\omega}^*(i).
\]
Substituting (14) into (7), we obtain the Lagrange multiplier:
\[
\lambda = \frac{\sum_{j \neq d} q_d^H \bar{\omega}(i) d_d^H \mathcal{A}_n^* s_j(i-1) - 1 - \prod(i) R_{s,d}^{-1}(i) \sum_{j \neq d} P_{s,j}(i)}{\prod(i) R_{s,d}(i) \prod(i)}
\]
where $\prod(i) = q_d^H \bar{\omega}(i) d_d^H \mathcal{A}_n^*$. Based on (14) and (15), we obtain the $d$th basis vector $s_d(i)$.

Next we consider the update of $R_{s,d}^{-1}(i)$. By applying the matrix inversion lemma (20) to (12), we obtain
\[
g_{s,d}(i) = \frac{R_{s,d}(i-1) \mathcal{R}(l) d_d}{\alpha + d_d^H \mathcal{R}^*(l) R_{s,d}(i-1) \mathcal{R}(l) d_d}
\]
\[
R_{s,d}^{-1}(i) = \alpha^{-1} R_{s,d}^{-1}(i-1) - \alpha^{-1} g_{s,d}(i) d_d^H \mathcal{R}^*(l) R_{s,d}(i-1) - 1
\]
where $\beta = (q_d^H \bar{\omega}^*(i) \bar{\omega}^T(i) q_d)^{-1}$. As with $P_{s,j}(i)$, we obtain it through iterations:
\[
P_{s,j}(i) = \mathcal{R}^T(l) d_d q_d^H \bar{\omega}^*(i) \bar{\omega}^T(i) q_j d_j^H \mathcal{R}^*(l) s_j(i)
+ \alpha P_{s,j}(i-1)
\]
By employing (14)-(18), we can update $s_d(i)$ for $d \in \{1, \ldots, D\}$. Given the values of $s_d(i)$, we can compute $\bar{\omega}(i)$. Defining $\bar{a}(i) = \sum_{d=1}^{D} q_d d_d^H \mathcal{A}_n s_d^*(i)$, (7) can be modified as
\[
\min_{\bar{\omega}(i)} \sum_{l=1}^{i} \alpha^{i-l} |\bar{\omega}^H(i) \bar{\omega}_D(l)|^2 \\
\text{subject to} \quad \bar{\omega}^H(i) \bar{a}(i) = 1
\]
Solving for $\bar{\omega}(i)$, we have
\[
g_D(i) = \frac{R_D^{-1}(i-1) \bar{\omega}_D(i)}{\alpha + \bar{\omega}_D^H(i) R_D^{-1}(i-1) \bar{\omega}_D(i)}
\]
\[
R_D^{-1}(i) = \alpha^{-1} R_D^{-1}(i-1) - \alpha^{-1} g_D(i) \bar{\omega}_D^H(i) R_D^{-1}(i-1)
\]
\[
\bar{\omega}(i) = \frac{R_D^{-1}(i) \bar{a}(i)}{\bar{a}^H(i) R_D^{-1}(i) \bar{a}(i)}
\]
where $R_D^{-1}(i) = \sum_{l=1}^{i} \alpha^{i-l} \bar{\omega}_D(i) \bar{\omega}_D^H(i)$. Based on the previous derivations, we calculate the output power for each scanning angle $\theta_n$:
\[
P(\theta_n) = \sum_{l=1}^{i} \alpha^{i-l} |\bar{\omega}^H(i) \bar{\omega}_D(i)|^2 = \frac{1}{\bar{a}^H(i) R_D^{-1}(i) \bar{a}(i)}.
\]
The ALRD-RLS algorithm is summarized in Table III.

### Table I

| Step | Description |
|------|-------------|
| 1    | Set $N$ and $\alpha$ |
| 2    | for each scanning angle $\theta_i$, do |
| 3    | Initialize $R_{s,j}^{(0)}(0)$, $P_{s,j}(0)$, $s_d(0)$, $\bar{\omega}(0)$ |
| 4    | for each snapshot $i$ ($i = 1, \ldots, N$) do |
| 5    | for each basis $d$ ($d = 1, \ldots, D$) do |
| 6    | Update $s_d(i)$ based on (14)-(15) |
| 7    | Update $\bar{\omega}(i)$ based on (20)-(22) |
| 8    | Calculate the output power $P(\theta_n) = (\bar{\alpha}^H(N)R_D^{-1}(N)\bar{\alpha}(N))^{-1}$ |
| 9    | Determine the estimated DOA $\hat{\theta} = \arg \max_{\theta_n} P(\theta_n)$ |

### C. Proposed MALRD-RLS DOA Estimation Algorithm

By examining the structure of the ALRD scheme, we can reduce its computational cost by using a single basis vector in the decomposition matrix. From this observation, we come up with a modified version of the ALRD-RLS algorithm, i.e., the MALRD-RLS algorithm. Specifically, the columns of the decomposition matrix in the MALRD-RLS algorithm are formed by shifted versions of the same basis vector $s(i)$, which results in $\bar{r}_D(i) = Q \mathbf{R}(i)s^*(i)$, where $Q = \sum_{d=1}^{D} q_d d_d^T$. Therefore, the optimization problem solved by the MALRD-RLS algorithm is:

$$\min_{\alpha} \sum_{i=1}^{N} |\alpha|^{i-1} |\bar{\omega}^H(i)Q \mathbf{R}(l)s^*(i)|^2,$$

subject to $\bar{\omega}^H(i)Q \mathbf{A}_n s^*(i) = 1$. This problem can be solved by following the same procedure as in the ALRD-RLS algorithm. Firstly, we construct the Lagrangian function as

$$\mathcal{L}(i) = \sum_{l=1}^{i} \alpha^{i-l} |\bar{\omega}^H(i)Q \mathbf{R}(l)s^*(i)|^2 + 2\Re\{\lambda[\bar{\omega}^H(i)Q \mathbf{A}_n s^*(i) - 1]\}$$

(24)

(25)

Secondly, we take the gradient of (25) with respect to $s(i)$, set the result to zero and solve for $s(i)$. The update equation of $s(i)$ is given by

$$s(i) = \frac{R_s^{-1}(i)\mathbf{A}_n^TQ^T\bar{\omega}^*(i)}{\bar{\omega}^T(i)Q \mathbf{A}_n R_s^{-1}(i)\mathbf{A}_n^TQ^T\bar{\omega}^*(i)}$$

(26)

where $R_s(i) = \sum_{l=1}^{i} \alpha^{i-l} \mathbf{R}(l)Q^T\bar{\omega}^*(i)\bar{\omega}^T(i)Q \mathbf{R}(l)$. The matrix $R_s^{-1}(i)$ can be computed as:

$$g_s(i) = \frac{R_s^{-1}(i-1)\mathbf{R}(i)Q^T\bar{\omega}^*(i)}{\alpha + \bar{\omega}^T(i)Q \mathbf{R}(i)R_s^{-1}(i-1)\mathbf{R}(i)Q^T\bar{\omega}^*(i)}$$

$$R_s^{-1}(i) = \alpha^{-1}R_s^{-1}(i-1) - \alpha^{-1}g_s(i)\bar{\omega}^T(i)Q \mathbf{R}(i)R_s^{-1}(i-1).$$

(27)

(28)

Next, we discuss the update of $\bar{\omega}(i)$. By redefining $\bar{\alpha}(i) = Q \mathbf{A}_n s^*(i)$, the cost function for the update of $\bar{\omega}(i)$ is the same as that in (19). Hence $\bar{\omega}(i)$ can also be constructed by (20)-(22) in the MALRD-RLS algorithm.

A brief summary of the MALRD-RLS algorithm is illustrated in Table III. After the update of $s(i)$ and $\bar{\omega}(i)$, we calculate the output power spectrum based on $P(\theta_n) = \sum_{l=1}^{N} |\alpha|^{i-1} |\bar{\omega}^H(i)\bar{r}_D(i)|^2 = (\bar{\alpha}^H(i)R_D^{-1}(i)\bar{\alpha}(i))^{-1}$. The peaks of the power spectrum are the estimated DOAs.

### D. Computational Complexity

Here we detail the computational complexity of the proposed ALRD-RLS and MALRD-RLS algorithms and several existing DOA estimation algorithms. ESPRIT uses an EVD of $\mathbf{R}$, which has complexity of $O(M^3)$. MUSIC employs both the EVD and grid search, resulting in a cost of $O(M^3 + (180/\Delta)M^2)$, with $\Delta$ being the search step. Matrix inversions and grid searches
are essential for Capon, whose complexity is $O((180/\Delta)M^3)$. For the AV and CG algorithms, the construction of the basis vectors leads to a complexity which is higher than that of the ESPRIT algorithm [12] [13]. The JIO-RLS algorithm has a cost of $O((180/\Delta)N(M^2 + D^2))$, with $D$ being the length of the reduced-rank received vector. ALRD-RLS avoids the EVD, the matrix inversion and the construction of the transformation matrix, and the update of $D$ basis vectors and an auxiliary parameter vector requires $O((180/\Delta)N(DI^2 + D^2))$. MALRD-RLS only uses one basis vector and costs $O((180/\Delta)N(I^2 + D^2))$. The computational complexity of the analyzed algorithms is depicted in Table III. Even in a large sensor array, $I$ and $D$ are small numbers, with $I \ll M$ and $D \ll M$, the cost of ALRD-RLS and MALRD-RLS can be less than those of the existing algorithms.

### IV. Simulations

In this section, we evaluate the ALRD-RLS and MALRD-RLS algorithms through simulations. We compare ALRD-RLS and MALRD-RLS with MUSIC, ESPRIT, Capon, CG, AV and the JIO-RLS algorithms. A ULA with $M = 60$ elements is adopted in the experiments. $K = 15$ narrowband source signals impinge on the ULA from directions $\theta_1, \ldots, \theta_K$, with 2 of them being correlated and the others uncorrelated. The correlated source samples are generated from a first-order autoregressive process:

$$s_1 \sim \mathcal{N}(0, \sigma_z^2) \quad \text{and} \quad s_2(i) = \rho s_1(i) + \sqrt{1 - \rho^2} e(i)$$

where $e \sim \mathcal{N}(0, \sigma_e^2)$. $\rho$ is the correlation coefficient fixed as 0.7 in this work. We assume that a small number of snapshots are available for DOA estimation and fix $N = 20$ in the simulations. The source signals with powers $\sigma_z^2 = 1$ are modulated by the binary phase shift keying (BPSK) scheme. The search step is chosen as $0.3\lambda$ for the algorithms based on grid search. We assume that the source DOAs are resolved if $|\hat{\theta}_k - \theta_k| < |\theta_k - \theta_{k-1}|/2$. In each experiment, 100 independent Monte Carlo runs are conducted to obtain the curves.

In the first example shown in Fig. 3 we plot the resolution probability versus the input signal-to-noise ratio (SNR) of the analyzed algorithms. We set the parameters for JIO-RLS to $D = 5$ and $\lambda = 0.998$. For MALRD-RLS and ALRD-RLS, we choose $I = 12$, $D = 5$, $\lambda = 0.998$. Note that higher $I$ and $D$ yield higher probability of resolution, yet they lead to higher cost as well. We have examined the values of $I, D \in [3, 15]$ and observed that $I = 12$, $D = 5$ provides a satisfactory performance with acceptable complexity. MALRD-RLS achieves the best performance in the large sensor arrays for different SNR values, followed by ALRD-RLS, MUSIC, JIO-RLS, Capon and ESPRIT. The AV and CG algorithms fail to resolve the DOAs for

**TABLE II**

**THE MALRD-RLS DOA ESTIMATION ALGORITHM.**

| Step | Algorithm | Complexity |
|------|-----------|------------|
| 1 Set | $N$ and $\alpha$ | $O(M^2)$ |
| 2 for | each | angle $\theta_n$ | $O((180/\Delta)N)$ |
| 3 Initialize | $\hat{\omega}(0), \omega(0)$ | $O(M^3)$ |
| 4 for | each | snapshot $i (i = 1, \ldots, N)$ | $O((180/\Delta)N)$ |
| 5 Update | $s(i)$ | $O((180/\Delta)N)$ |
| 6 Redefine | $\hat{a}(i) = Q(I)\hat{a}^*(i)$ | $O((180/\Delta)N)$ |
| 7 Calculate | the output power $P(\theta_n) = (\hat{a}^H(N)\hat{R}_{\omega}^H(N)\hat{a}(N))^{-1}$ | $O((180/\Delta)N)$ |
| 8 Determine | the estimated DOA $\hat{\theta} = \arg \max_{\theta_n} P(\theta_n)$ | $O((180/\Delta)N)$ |

**TABLE III**

**COMPARISON OF COMPUTATIONAL COMPLEXITY.**

| Algorithm | Complexity |
|-----------|------------|
| ESPRIT [4] | $O(M^2)$ |
| MUSIC [3] | $O(M^3 + (180/\Delta)M^2)$ |
| Capon [5] | $O((180/\Delta)N(M^2 + D^2))$ |
| AV [12] | $O((180/\Delta)N(M^2 + D^2))$ |
| CG [13] | $O((180/\Delta)N(M^2 + D^2))$ |
| JIO-RLS [14] | $O((180/\Delta)N(M^2 + D^2))$ |
| ALRD-RLS | $O((180/\Delta)N(M^2 + D^2))$ |
| MALRD-RLS | $O((180/\Delta)N(I^2 + D^2))$ |
most of the SNR values when many sources are present. Note that MUSIC, ESPRIT, Capon, AV and CG require forward backward averaging (FBA) \[30\], \[31\] to ensure satisfactory performance for correlated signals.

\[
\text{RMSE} = \sqrt{\frac{1}{NK} \sum_{n=1}^{N} \sum_{k=1}^{K} (\hat{\theta}_k - \theta_k)^2}
\]

Fig. 2. Probability of resolution versus input SNR.

We then evaluate the root mean square error (RMSE) performance of the analyzed algorithms, which is calculated as \(\text{RMSE} = \sqrt{\frac{1}{NK} \sum_{n=1}^{N} \sum_{k=1}^{K} (\hat{\theta}_k - \theta_k)^2}\). From Fig. 3 MALRD-RLS provides a superior RMSE performance with the lowest threshold SNR and the lowest RMSE level in high SNRs. The gap between the RMSE of MALRD-RLS and the Cramer-Rao bound (CRB) is due to the small number of available snapshots and the fact that the correlated sources degrade the performance.

\[
\begin{align*}
\text{RMSE} & = \sqrt{\frac{1}{NK} \sum_{n=1}^{N} \sum_{k=1}^{K} (\hat{\theta}_k - \theta_k)^2} \\
\end{align*}
\]

Fig. 3. RMSE versus input SNR.

V. CONCLUSION

In this paper, we have proposed the ALRD scheme and the ALRD-RLS and MALRD-RLS subspace DOA estimation algorithms based on alternating optimization. The proposed algorithms are suitable for large sensor arrays and have a lower computational cost than existing techniques. Simulation results show that MALRD-RLS and ALRD-RLS outperform previously reported algorithms.

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