An optimal transport approach to data compression in distributionally robust control

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Abstract—We consider the problem of controlling a stochastic linear time-invariant system using a behavioural approach based on the direct optimization of controllers over input-output pairs drawn from a large dataset. In order to improve the computational efficiency of controllers implemented online, we propose a method for compressing this large data set to a smaller synthetic set of representative behaviours using techniques based on optimal transport. Specifically, we choose our synthetic data by minimizing the Wasserstein distance between atomic distributions supported on both the original data set and our synthetic one. We show that a distributionally robust optimal control computed using our synthetic dataset enjoys the same performance guarantees onto an arbitrarily larger ambiguity set relative to the original one. Finally, we illustrate the robustness and control performances over the original and compressed datasets through numerical simulations.

I. INTRODUCTION

Strongly influenced by the current trend in nearby and complementary scientific areas, learning from data promises to be one of the near future quintessential problems also for the system-and-control community. The growing complexity of today’s systems, which drastically limits the traditional model-based control design, along with the increasing data storage capacity and data availability, make data-driven control approaches timely and more attractive.

Specifically, recent works were built upon a well-known result in subspace identification, the so called Willems’ lemma [1]. Essentially, under a specific technical condition, such result enables to recover a nonparametric minimal linear time-invariant (LTI) realization of an unknown system from a matrix of input-output measures, namely the column space of the data matrix span all the possible trajectories of a corresponding LTI system. In this context, the interpretation of the Willems’ lemma generated two main research directions: data-dependent parametrization analysis and control [2], [3], [4], and data-enabled (robust) optimal control [5], [6], [7].

In this work, we deal with a specific issue raised within this latter approach. Specifically, since the crucial result in [1] establishes equivalence between a (possibly noisy) data matrix and a nonparametric LTI model under the persistent excitation of the system input, which directly translates on the input-output observations length, one might be induce to collect disproportionately large datasets. As a consequence, the real-time implementation of optimization-based (robust) control law may lead to quite challenging online computations. This limits the field of possible applications, e.g., ruling out systems that require high frequency rates, or obtaining too conservative performances, e.g., by choosing short control horizons to reduce the computational time.

In system-and-control, data compression and dimensionality reduction have been widely used to design synthetic sets of samples that “best” capture the informative content of original, noise-corrupted datasets. Tracing back over the past decades, statistical and manifold learning [8], [9], principal component analysis and related variants [10], [11], [12], or, more generally, subspace identification approaches [13], [14], [15], represent only few examples amongst the most diffused techniques. Conceptually, these methods tacitly neglect the stochastic nature of the noise itself by comparing sets of data on topological or Hilbert spaces. Successively, they eliminate all those components considered less significant, according to some criterion, and potentially associated with noise.

Breaking away from the literature, we propose an optimal transport approach [16], [17] for the synthesis of a compressed set of samples. Defining a metric on a probability space, the Wasserstein distance is key to setup a linear program (LP) to minimize the distance between the discrete measure associated to the synthetic dataset and the empirical distribution of the noisy data. As a particular case of the family of variational Wasserstein problems [17, §9], each atom of the synthetic dataset identifies a specific barycentre for a cluster of the original samples, then seeking to maximize its informative content. In the context of data-enabled robust control of stochastic LTI systems [6], we show that the distributionally robust optimization problem, built upon a Wasserstein ball as ambiguity set (and hence admitting a convex reformulation), enjoys the same performance guarantees of the original dataset, at a price of considering an arbitrarily bigger radius that depends on the number of synthetic atoms adopted. Remarkably, the term accounting for extra robustness vanishes as the number of atoms approaches the cardinality of the original dataset.

After briefly recalling some notions of optimal transport (§II), in §III we introduce the distributionally robust control problem addressed. The variational Wasserstein problem is then formalized and discussed in §IV, while in §V we compare the control performances obtained over the original and compressed datasets through a numerical case study.

Notation: The set of real and non-negative numbers is represented by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$, respectively, while the set of natural numbers is denoted by $\mathbb{N}$. For vectors $x_1, \ldots, x_N \in \mathbb{R}^n$ and $I := \{1, \ldots, N\}$, we denote $\text{col}(x_i)_{i \in I} := (x_1^\top, \ldots, x_N^\top)^\top$. The authors are with the Department of Engineering Science, University of Oxford, OX1 3PJ, United Kingdom {filippo.fabiani, paul.goulart}@eng.ox.ac.uk. This work was partially supported by Innovate UK, part of UK Research and Innovation, under Project LEO (Ref. 104781).
Given a matrix $X \in \mathbb{R}^{n \times m}$, its $(i, j)$ entry is denoted by $[X]_{i,j}$, while its transpose as $X^\top$. The symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in the appropriate space, i.e., $\langle x, y \rangle = x^\top y$ for $x, y \in \mathbb{R}^n$ and $(X, Y) = \text{trace}(X^\top Y)$ for $X, Y \in \mathbb{R}^{n \times n}$.

We define the Wasserstein ball of radius $Q$, $\mathcal{W}_Q$, as the set of all probability measures $\pi$ on $\Omega$, where $\Omega$ is a metric space. This set is nonempty, we stress that the solution to the LP in (5), attained on the vertices of $\mathcal{T}$, may not be unique. Finally, since $\mathcal{W}_Q$ defines a metric, in the discrete setting $\mathcal{W}_{Q\nu}(\mathcal{P}, \mathcal{Q}) = 0$ if and only if $\alpha = \beta$, and therefore the triangle inequality holds as $\mathcal{W}_{Q\nu}(\mathcal{P}, \mathcal{Q}) \leq \mathcal{W}_{Q\nu}(\mathcal{P}, \mathcal{G}) + \mathcal{W}_{Q\nu}(\mathcal{G}, \mathcal{Q})$, for some discrete measures $\mathcal{P}, \mathcal{Q}, \mathcal{G} \in \mathcal{P}(\Omega)$.

Throughout the paper, we will consider the so called Kantorovich-Rubinstein distance, a special case of the Wasserstein one obtained by setting in (2) $p = 1$ (and hence we will omit the subscript). Moreover, we assume the metric $d$ be induced by an arbitrary norm $\| \cdot \|$ on $\mathbb{R}^n$.

### III. Data-enabled optimal control

First, we formulate the predictive control problem for stochastic LTI systems addressed. Then, after a brief digression on nonparametric models for deterministic LTI systems, we finally recall from [6] a distributionally robust reformulation of the optimal control problem, along with the main reasons that motivate us to consider a synthetic dataset.

#### A. Constrained control of stochastic LTI systems

Let us consider the following discrete time, stochastic LTI system:

\[
\begin{align*}
x(k + 1) & = Ax(k) + Bu(k) + En(k), \\
y(k) & = Cx(k) + Du(k) + Fv(k),
\end{align*}
\]

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $E \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{l \times n}$, $D \in \mathbb{R}^{l \times m}$ and $F \in \mathbb{R}^{l \times q}$. Here, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^l$ and $v(k) \in \mathbb{R}^q$ are the state, control input, output and stochastic disturbance of the system at time $k \in \mathbb{Z}_+$ respectively. Throughout the paper, we assume the following.

**Standing Assumption 1:** The pair $(A, B)$ is controllable, while the pair $(A, C)$ is observable.

Then, we assume that the system matrices defining (3) are unknown, and that we have access to input-output measurements, $(\hat{u}(k), \hat{y}(k)), k \in \mathbb{Z}$. Any $\hat{y}(k)$ is therefore affected by the realization of the uncertainty $\nu(k)$, drawn from an unknown probability distribution $\mathcal{P}_{\nu}$, supported on $\mathcal{T} \subseteq \mathbb{R}^q$.

Along the line of stochastic model predictive control [18], we consider a finite horizon control problem over the time horizon $K \in \mathbb{N}$ for (4), where the main goal is to design a (possibly constrained) sequence of inputs, namely $u := \text{col}(u(k), \ldots, u(k + K - 1)) \in \mathcal{U} \subseteq \mathbb{R}^{mK}$, while minimizing a given cost function $J : \mathbb{R}^{nK} \times \mathbb{R}^{lK} \to \mathbb{R}$. Specifically, due the stochastic nature of the problem in question, we aim at solving the following optimization problem:

\[
\begin{align*}
\inf_{w \in \mathcal{W}} & \mathbb{E}_{\nu,w}[J(u, y)],
\end{align*}
\]

where $\mathcal{P}_\nu^K := \mathcal{P}_\nu \times \cdots \times \mathcal{P}_\nu$ is the $K$-fold product distribution characterizing $\nu$ over the whole horizon $K$. For the remainder, we adopt the same assumptions on the cost function postulated in [6], as formalized next.
Standing Assumption 2: For all \((u,y) \in \mathbb{R}^{(m+1)K}\), \(J(u,y)\) is a separable function, namely \(J(u,y) := J_1(u) + J_2(y)\), where \(J_1 : \mathbb{R}^{mK} \rightarrow \mathbb{R}\), \(J_2 : \mathbb{R}^{K} \rightarrow \mathbb{R}\) are convex and continuous. In addition, \(J_2(\cdot)\) is such that \(\Xi := \{\xi \in \mathbb{R}^{IK} \mid J_2^* (\xi) < \infty\} \subseteq \mathbb{R}^{IK}\) is a bounded set. □

B. Learning models in deterministic LTI systems

Recently, the Willems’ fundamental lemma [1, Th. 1] has been exploited (or provided inspiration) to develop data-consistent, nonparametric model for deterministic, LTI systems [6], [2], [4], [3]. Specifically, let us consider the system in (4) with \(\nu(k) = 0\), for all \(k \in \mathbb{Z}\), and let us assume to conduct several experiments of length \(N\) to collect input-output measurements for different time shifts. The data can be organized within a matrix \(\mathcal{H}_K := \text{col}(\mathcal{U}_{0,K,N-K+1}, \mathcal{Y}_{0,N-K+1}) \in \mathbb{R}^{(m+1)K \times N-K+1}\), where

\[
\mathcal{U}_{0,K,N-K+1} := \begin{bmatrix}
\hat{u}(0) & \hat{u}(1) & \cdots & \hat{u}(N-K) \\
\hat{u}(1) & \hat{u}(2) & \cdots & \hat{u}(N-K+1) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{u}(K-1) & \hat{u}(K) & \cdots & \hat{u}(N-1)
\end{bmatrix}
\]

and \(\mathcal{Y}_{0,K,N-K+1} \in \mathbb{R}^{IK \times N-K+1}\) is defined similarly. Here, the first subscript refers to the time index of the top-left entry of a given matrix, the second refers to the number of block-rows, while the third one to the number of columns. Note that both \(\mathcal{U}_{0,K,N-K+1} \in \mathbb{R}^{mK \times N-K+1}\) and \(\mathcal{Y}_{0,K,N-K+1}\) have constant vector entries along the block anti-diagonals, and therefore \(\mathcal{H}_K\) belongs to the class of Hankel matrices.

As crucial assumption, the Willems’ fundamental lemma restricts the class of input sequences over the horizon \(K\) to the persistently exciting signals, defined as follow.

Definition 2: [1] A signal \(x \in \mathbb{R}^w\), observed over \(N\) samples, is persistently exciting of order \(K\) if the corresponding Hankel matrix \(\mathcal{H}_{K,N-K+1}\) has full rank \(wK\). □

Therefore, it follows immediately that a signal can be persistently exciting of order \(K\) shall be sufficiently long, i.e., \(N \geq (w+1)K - 1\). Then, by relying on Definition 2, the Willems’ fundamental lemma is stated next.

Lemma 1: ([1, Th. 1]) If the control signal \(\hat{u}(0), \ldots, \hat{u}(N-1)\) is persistently exciting of order \(n + K\), then for any \(K\)-long input-output trajectory of the deterministic version of the system in (4), \(\text{col}(u,y)\), there exists \(g \in \mathbb{R}^{N-K+1}\) such that

\[
\text{col}(u,y) = \text{col}(\mathcal{U}_{0,K,N-K+1}, \mathcal{Y}_{0,N-K+1})g.
\]

In other words, any linear combination of the columns of the data matrix \(\mathcal{H}_K\) is an input-output trajectory of length \(K\) for the system or, equivalently, the column space of \(\mathcal{H}_K\) span all the possible trajectories of the system. Remarkably, the solution to (6) is in general not unique, i.e., given a certain data set \(\mathcal{H}_K\), there might exist several vectors \(g\) that associate the data to a predefined input-output trajectory, \(\text{col}(u,y)\). Thus, given the current time \(k \in \mathbb{N}\), we rearrange the available input-output data set to restate the deterministic version of (5). Specifically, we split \(\mathcal{H}_K\) into \(\mathcal{U}_f, \mathcal{Y}_f, \mathcal{U}_g\), and \(\mathcal{Y}_g\). While the former \((\mathcal{U}_f\) and \(\mathcal{Y}_f\)) are some data matrices for the “forward” propagation (from \(k\) onward) of control sequence and output prediction, respectively, the latter \((\mathcal{U}_g\) and \(\mathcal{Y}_g\)) define the consistency constraints associated with less recent measurements, together with the sequences \(\hat{u} := \text{col}(\hat{u}(0), \ldots, \hat{u}(k-1))\) and \(\hat{y} := \text{col}(\hat{y}(0), \ldots, \hat{y}(k-1))\).

Thus, Lemma 1 enables to define a data-consistent, nonparametric model for a deterministic LTI system, which leads to the deterministic counterpart of (5):

\[
\begin{align*}
\min_{g} & \ J(\mathcal{U}_f g, \mathcal{Y}_f g) \\
\text{s.t.} & \ \begin{bmatrix} \mathcal{U}_g \\ \mathcal{Y}_g \end{bmatrix} g = \begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix}, \quad (7)
\end{align*}
\]

C. A distributionally robust data-enabled control problem

Although not explicitly used, Lemma 1 provided the insight in [6] to design a robust data-enabled predictive control (DeePC) algorithm to solve the predictive control problem in (5). The realization of the uncertain parameter \(\nu\), indeed, poses several challenges around the consistency constraints in (7), since it takes values over the unknown distribution \(P_\nu\), directly affecting the output trajectories. Therefore, after some manipulations, a distributionally robust, semi-infinite reformulation of (7) was proposed in [6], which reads as

\[
\inf_{v \in V} \sup_{Q \in \Pi_v(\mathcal{P}_K)} E_Q[f(\kappa, v)], \quad (8)
\]

where \(v\) corresponds to the vector \(g\) stacked with a constant term, and therefore the set \(V \subseteq \mathbb{R}^{N-K+2}\) coincides with the feasible set of (7) extended to \(v\), accordingly. Then, \(\nu\) stacks all the objects in (7) affected by the realization of \(\nu\), i.e., \(\mathcal{H}_K, \mathcal{U}_f\) and \(\hat{y}\), whose (unknown) probability distribution is denoted by \(\mathcal{P}_\kappa\), supported on \(\Theta \subseteq V \subseteq \mathbb{R}^{(K)(N-K+2)}\). As in [19], [6], we next postulate a light-tailed assumption for the probability distribution \(\mathcal{P}_\kappa\), which is however automatically satisfied in case \(\Theta\) identifies a compact set.

Assumption 1: There exists some \(a > 0\) such that

\[
E_{\mathcal{P}_\kappa}[e^{||s||^a}] := \int_{\Theta} e^{||s||^a} \mathcal{P}_\kappa(ds) < \infty \quad \square
\]

Moreover, with \(\kappa\) we identify the effective measurements, i.e., realizations of the uncertain objects \(\kappa\) that also affect the constraints set \(V\), and \(\mathcal{P}_\kappa\) is the empirical distribution of such measurements. Finally, \(f\) is obtained by massaging \(J\) and substituting vectors \(\kappa\) and \(v\).

By leveraging on the results in [19], under Assumption 1 [6, Th. 4.2] shows that considering the Wasserstein ball as ambiguity set in (8) allows for a convex reformulation, whose solution upper bound the out-of-sample performance, \(E_{\mathcal{P}_\kappa}[f(\kappa, v)]\), with high confidence. Specifically, for some \(\lambda > 0\), the semi-infinite problem in (8) is upper-bounded by

\[
\min_{v \in V} f(\kappa, v) + \varepsilon \cdot \max \left\{ \sup_{\xi \in S} ||\xi||_\infty ||\text{col}(g, 0)||_\infty, \lambda ||v||_\infty \right\}. \quad (9)
\]

Thus, by denoting \(v^*\) as an optimal solution to (9), with \(g^*\) subvector of \(v^*\), the control action \(u^* = \mathcal{U}_f g^*\) provides...
probabilistic guarantees to obtain good control performance with respect to possible realizations of the stochastic trajectory $y$ associated with the ambiguity set $B_l(P_s)$.

However, due to the persistent excitation assumption, along with possibly large historical dataset, the Hankel matrix gathering all the measurements might be disproportionately large, leading to a quite challenging online computation. This fairly limits the field of possible applications, ruling out all those critical systems that require high frequency samplings. Moreover, since $N \geq (m+1)(n+K)-1$, one may also be induced to choose short control horizons $K$, and therefore obtaining exceedingly conservative control performances. These reasons motivate us to design a suitable procedure to compress the intrinsic information concealed within the whole dataset $\mathcal{H}_K$ into a synthetic one.

IV. DATA COMPRESSION AS A VARIATIONAL WASSERSTEIN PROBLEM

In this section, we design an offline, optimal transport-based procedure which allows to compress the original dataset, i.e., to select a limited set of samples that “best” summarize the information content of $\mathcal{H}_K$. This clearly entails a lower computational burden in solving (9), and hence making such a distributionally robust control synthesis more appealing for an online implementation. Moreover, we show that the optimal solution obtained by means of the synthetic dataset enjoys the same probabilistic guarantees onto a Wasserstein ball with slightly bigger radius. This robust overestimate, which depends on the number of samples adopted, vanishes as the set of synthetic data increases.

Thus, let the dimensions of $\mathcal{H}_K \in \mathbb{R}^{r \times R}$ be fixed, i.e., $N \geq (m+1)(n+K)-1$ be chosen so that the control input $u$ is persistently exciting of order $n+K$ as in Lemma 1 where $r := (m+l)K$, $R := N-K+1$. The empirical distribution of such measurements is defined as $P_\kappa = \frac{1}{R} \sum_{i \in R} \delta_{h_i}$, where $R := \{1, \ldots, R\}$ and $h_i$ is the $i$-th column of $\mathcal{H}_K$. In this context, our goal is to find a set of locations $\mathcal{J}_K \in \mathbb{R}^{p \times S}$, with $S \leq R$, whose empirical probability distribution $P_s = \frac{1}{S} \sum_{i \in S} \delta_{s_i}$, $S := \{1, \ldots, S\}$, is close to the one of the original dataset. Hence, our problem can be formulated in an optimal transport fashion as

$$\min_{\mathcal{J}_K} W(\hat{P}_\kappa, \hat{P}_s) = \min_{\mathcal{J}_K} \min_{T \in \mathcal{T}(1_R \times 1_S / S)} \langle T, D(\mathcal{H}_K, \mathcal{J}_K) \rangle. \tag{10}$$

Such an optimization problem has a strong practical interpretation: an optimal solution to (10), $\mathcal{J}_K^{*}$, is generated by a synthetic set of samples (or atoms) defining the closest (in term of Wasserstein distance, and hence transportation cost) empirical probability distribution, $P_s$, that approximate the original one, $P_\kappa$. The nested optimization in (10) is a variational Wasserstein problem, representing a particular case of the Wasserstein barycentres problem [20]. Specifically, it directly lies into the set of $k$-means problems [21]. In addition to allowing for a semi-discrete setting, the advantages of adopting the Wasserstein distance as metric to compare probability measures have been proved in many practical and theoretical problems, spanning from dictionary and statistical learning [22], [23], to vision and image processing [24], [25].

A. On the optimal transport problem (10)

Despite its appealing structure and strong practical interpretation, the variational Wasserstein problem in (10) is convex in each single variable, i.e., $\mathcal{J}_K$ and $T$, but not jointly. In practice, this might lead to compute local optimal solutions, where every atom defining each optimal $\mathcal{J}_K^{*}$ identifies a specific barycentre for a subset of samples in $\mathcal{H}_K$. Specifically, (10) corresponds to

$$\min_{\mathcal{J}_K} \min_{T \in \mathcal{T}(1_R / 1_S / S)} \sum_{j \in S} \sum_{i \in R} t_{i,j} \|h_i - s_j\|,$$

where $t_{i,j} := [T]_{i,j}$. Thus, fixed any $j \in S$, every non null $t_{i,j}$ defines the quantity of a predefined sample $h_i$ of $\mathcal{H}_K$ that is associated to the barycentre $s_j$. Then, it follows immediately that every atom defining $\mathcal{J}_K^{*}$ belongs to conv($\mathcal{H}_K$). In general, since each $t_{i,j} \geq 0$ assumes continuous values, we note that every sample $h_i$ may be split among several barycentres $s_j$, and hence generating overlapping clusters.

Since it represents the minimum of affine functions, we note that the Wasserstein distance itself is not smooth in its arguments. To circumvent this problem, the cost function in (10) can be regularized by means of a strictly convex, weighted entropic term, i.e., $\gamma (T, \log(T))$, for some strictly positive coefficient $\gamma$, whose benefits are twofold [26]: i) the inner optimization problem in (10) admits a closed form, which translates in a matrix balancing problem, and ii) the Wasserstein distance is differentiable for any $\gamma > 0$.

However, by exploiting the convexity in each single variable, a possible solution algorithm may be represented by a typical alternate block-coordinate descent (ABCD) method, whose main steps are summarized in Algorithm 1:

\begin{algorithm}
\textbf{Initialization:} Select $\mathcal{J}_K(0) \in \mathbb{R}^{p \times S}$

\textbf{Iteration} ($k \in \mathbb{N}$):

1. Compute

$$T(k) \in \arg\min_{T \in \mathcal{T}(1_R / 1_S / S)} \langle T, D(\mathcal{H}_K, \mathcal{J}_K(k - 1)) \rangle$$

2. Compute $\mathcal{J}_K(k) \in \arg\min_{\mathcal{J}_K} \langle T(k), D(\mathcal{H}_K, \mathcal{J}_K) \rangle$

\end{algorithm}

B. Robust performance guarantees

In this context, the strategy is to solve the optimal transport problem in (10) to reformulate the robust optimization problem in (8) with a Wasserstein ball centred in $\hat{P}_s$ and (possibly) different radius as ambiguity set, which reads as

$$\inf_{v \in V} \sup_{Q \in B_E(P_s)} E_Q[f(\kappa, v)]. \tag{11}$$

\textbf{Remark 1:} As long as $S < E$, (11) is defined on a lower dimensional space compared to (8). However, for simplicity’s sake, we will keep the same notation. \hfill $\Box$
therefore paves the way to possibly longer control horizons can be solved online with a lower computational time. This reformulation equivalent to the one in (9), which however in (11) can be manipulated to obtain a tractable convex ambiguity set in [6, Th. 4.1]. Finally, although defined on a performance between \( \hat{\kappa}(0) \) and \( \bar{\kappa} \).

Algorithm 2: Receding horizon robust synDeePC

**Offline:** Given \( \mathcal{R}_K \), set \( S \leq R \), compute

\[
\mathcal{R}_K^* \in \arg\min_{\mathcal{R}_K} W(\tilde{P}_n, \tilde{P}_s)
\]

**Initialization:** Set \( \hat{\mathcal{V}}(0) \), \( \hat{\kappa}(0) \) and \( \hat{\varepsilon} \)

**Iteration** \((k \in \mathbb{N})\):

1. Compute

\[
\nu^*(k) := \arg\min_{v \in \mathcal{V}(k)} f(\hat{\kappa}(k), v) + \bar{\varepsilon} \cdot \max \left\{ \sup_{\xi \in \Xi} \| \xi \|_{\infty} \| \text{col}(g(\xi, 0)) \|, \lambda \| v \|_{\star} \right\}
\]

2. Set \( u^*(k) = \mathcal{R}_K \nu^*(k) \)

3. Retrieve current measurements, update \( \mathcal{V}(k+1) \), \( \hat{\kappa}(k+1) \)

Next, we show that an optimal solution to the robust optimization problem in (11) upper bounds the out-of-sample performance \( \mathbb{E}_{P_n}[f(\kappa, v)] \) with high confidence.

**Proposition 1:** Let \( \beta \in (0, 1) \) and \( S \leq R \) be given. Under Assumption 1, there exists some \( \bar{\varepsilon} = \varepsilon(\beta, S) > 0 \) such that, for all \( v \in \mathcal{V} \),

\[
P^S_n \left\{ \mathbb{E}_{P_n}[f(\kappa, v)] \leq \sup_{Q \in B_{\bar{\varepsilon}}(P_n)} \mathbb{E}_Q[f(\kappa, v)] \right\} \geq 1 - \beta.
\]

**Proof:** First, given any \( S \leq R \), let us define \( \eta(S) := \min_{\mathcal{R}_K} W(\tilde{P}_n, \tilde{P}_s) \). Then, the triangle inequality for the Wasserstein metric ensures that \( W(\tilde{P}_n, \tilde{P}_s) \leq W(\tilde{P}_n, \tilde{P}_n) + W(\tilde{P}_n, \tilde{P}_s) \leq W(\tilde{P}_n, \tilde{P}_s) + \eta(S) \). Furthermore, in view of Assumption 1 it follows from [19, Th. 3.4] that \( P^S_n \{ W(\tilde{P}_n, \tilde{P}_s) \leq \varepsilon(\beta) \} \geq 1 - \beta \), and therefore, since \( \tilde{P}_n \) is an empirical distribution, we obtain

\[
P^S_n \left\{ W(\tilde{P}_n, \tilde{P}_s) \leq \varepsilon(\beta) + \eta(S) \right\} \geq 1 - \beta.
\]

This latter relation, which can be equivalently restated as \( P^S_n \{ \tilde{P}_n \in B_{\varepsilon}(\tilde{P}_s) \} \geq 1 - \beta \), where \( \varepsilon := \varepsilon(\beta) + \eta(S) \), directly implies \( \mathbb{E}_{P_n}[f(\kappa, v)] \leq \sup_{Q \in B_{\varepsilon}(P_n)} \mathbb{E}_Q[f(\kappa, v)] \) with probability \( 1 - \beta \), thus concluding the proof.

As a remark, we note that the Wasserstein distance between \( \tilde{P}_n \) and \( \tilde{P}_s \) can be made arbitrarily small while increasing the number of synthetic data \( S \), since \( \eta(S) \to 0 \) as \( S \to R \). In this case, we precisely recover the radius of the ambiguity set in [6, Th. 4.1]. Finally, although defined on a lower dimensional space, the robust optimization problem in (11) can be manipulated to obtain a tractable convex reformulation equivalent to the one in (9), which however can be solved online with a lower computational time. This therefore paves the way to possibly longer control horizons \( K \) and higher sampling frequencies, as shown next.

V. NUMERICAL SIMULATION

We apply here the (syn)DeePC method presented in [6] and summarized in Algorithm 2 on a linearized model of a quadcopter in a receding horizon fashion, comparing control and computational performances when considering the original dataset and the compressed one. The simulations are run in Matlab environment on a laptop with a Quad-Core Intel Core i5 2.4 GHz CPU and 8 Gb RAM, with main parameters summarized in Table I.

The linear model adopted is valid around a hover position, where the state vector is \( \text{col}(x, y, z, \dot{x}, \dot{y}, \dot{z}, \phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) \in \mathbb{R}^{12} \). Specifically, \( x, y \) and \( z \) are the 3 spatial coordinates and relative velocities, while \( \phi, \theta, \psi \) are the angular ones, with relative rates. The control inputs are represented by four rotors, constrained to the set \( \mathcal{U} = [-0.7007, 0.2993]^4 \).

By assuming full state measurement, we use the same state-space matrices adopted in [6], as well as same original cost function, \( J(u, y) = \| u \|_1 + c \| y - r \|_1 \), where \( r \) denotes the parametrized, 8-figure trajectory. In this context, \( T_s \) in Table I represents the temporal resolution with which the reference trajectory is sampled, and hence ideally corresponding to the sampling time. Moreover, the columns of the matrix \( \mathcal{R}_K \) are filled by means of random inputs drawn from a uniform distribution on \( \mathcal{U} \) to guarantee the persistency of excitation, according to Definition 2.

Some a-priori considerations on the choice of the parameter \( S \) in Algorithm 2 can be made, e.g., in a data-driven fashion by evaluating the behaviour of the Wasserstein distance \( W(\tilde{P}_n, \tilde{P}_s) \) when \( S \) varies. Thus, according to Fig. 1 the function \( \eta(S) \) seems assuming reasonable values for \( S \leq 92 \), leading to an offline step in Algorithm 2 taking less than six minutes. Interestingly, we note that the effect of the local minima seems preventing the Wasserstein distance from being monotonically decreasing for values of \( S > 92 \),
In Fig. 2 and 3 we compare the trajectory tracking performances of the quadrotor controlled by means of the DeePC with full matrix $H_K$ (solid lines) and synthetic dataset, $\mathcal{S}_K$, obtained first by reducing to the 50\% the total number of samples, i.e., $S = 92$ (solid-dashed lines), which also correspond to a reduction of the 70\% when considering a longer control horizon, i.e., $K = 50$ (dotted lines). As shown in Fig. 3 where the position errors of the spatial coordinates are illustrated, the performances of the robust controller computed by means of the compressed dataset do not exceedingly deteriorate compared with the one computed by means of $\mathcal{H}_K$, also exhibiting an almost overlapping behaviour when the control horizon $K$ increases. Moreover, from our numerical experience, the step (51) in Algorithm 2 with compressed dataset takes approximately 0.64[s] on average, in contrast with the 1.73[s] required by the original dataset, see Fig. 4. On the other hand, a longer control horizon does not lead to a much higher computational time, i.e., 0.85[s], while considering the whole dataset would take around 3[s] to solve the DeePC optimization problem.

Fig. 2: Dynamical evolution of the controlled quadcopter while following a figure-8 trajectory.

Fig. 3: Spatial coordinate tracking errors.

Fig. 4: Computational time over the whole trajectory tracking control problem.

VI. CONCLUSION AND OUTLOOK

In data-driven robust control problems, optimal transport theory promises to be the key tool for the design of synthetic datasets able to provide both robust performance guarantees and reasonable computational time for real-time implementation. Specifically, we have investigated the benefits of adopting the Wasserstein metric to compress the informative content of a large dataset into a smaller one, also illustrating the performance of the robust controller obtained by means of the synthetic dataset compared to the original one. Future research directions will focus on the impact that the discrete measure adopted to compare the empirical distribution associated with the original data, i.e., the vector $\beta$ in (2), has on the robustness. Given the input-output structure of the gathered data, we will investigate also the possibility to use different distances to define the Wasserstein metric, as well as a jointly convex reformulation of (10).

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