Surfaces of Albanese General type and the Severi conjecture

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Abstract

In 1932, F. Severi claimed, with an incorrect proof, that every smooth minimal projective surface $S$ of irregularity $q = q(S) > 0$ without irrational pencils of genus $q$ satisfies the topological inequality $2c_1^2(S) \geq c_2(S)$. According to the Enriques-Kodaira’s classification, the above inequality is easily verified when the Kodaira dimension of the surface is $\leq 1$, while for surfaces of general type it is still an open problem known as Severi’s conjecture. In this paper we prove Severi’s conjecture under the additional mild hypothesis that $S$ has ample canonical bundle. Moreover, under the same assumption, we prove that $2c_1^2(S) = c_2(S)$ if and only if $S$ is a double cover of an abelian surface.

Mathematics Subject Classification (2000): 14J29, 14C17, 14C20.

Introduction

Let $S$ be a complex minimal surface of general type of irregularity $q(S) > 0$ and let $\alpha: S \to \text{Alb}(S)$ be the Albanese map of $S$; it is a basic fact in the theory of surfaces that the following condition are equivalent:

1. The image of $\alpha$ has dimension 2.

2. $S$ has no irrational pencils of genus $q(S)$.

3. The image of the wedge product $\bigwedge^2 H^0(\Omega^1_S) \to H^0(\Omega^2_S)$ is non trivial.

The aim of this paper is to give numerical inequalities for the topological invariants of surfaces satisfying the above conditions. More precisely we are interested to relate Chern numbers $K_S^2 = c_1^2(S)$ and $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2$. As a mixing of the results of this paper (6.1+2.3+4.4+7.1) we get in particular the following:

Theorem 0.1. Let $S$ be a compact complex surface with ample canonical bundle such that its image under the Albanese map is 2-dimensional.

Then:

$$2c_1^2(S) - c_2(S) = 3(K_S^2 - 4\chi(\mathcal{O}_S)) \geq 0$$

and equality holds if and only if $q(S) = 2$ and the Albanese map $\alpha: S \to \text{Alb}(S)$ is a Galois double cover.

* partially supported by Italian MURST program ‘Spazi di moduli e teoria delle rappresentazioni’. Member of GNSAGA of CNR.
This result is motivated by the following classical and well known conjecture proposed by M. Reid in [18] p. 535:

Conjecture 0.2. ([18] p. 535, cf. also [3] p. 103) Let $S$ be a smooth minimal complex projective surface such that its image under the Albanese map has dimension 2, then $K_S^2 \geq 4\chi(O_S)$.

F. Severi claimed the above inequality in the paper [20] p. 305 but his proof was not correct, as Catanese pointed out in [3]; the above conjecture is usually referred as Severi’s conjecture. We note that for surfaces not of general type the Severi’s conjecture is an easy consequence of the Enriques-Kodaira’s classification (see e.g. [1, p. 188]), while if $S$ has an irrational pencil then the Severi conjecture is a consequence of the results of Xiao Gang [22]. It has been also proved by Konno that the Severi’s conjecture is true when $S$ is of general type, but $K_S$ is not ample, our arguments do not seem sufficient and the description of surfaces with $K_S^2 = 4\chi(O_S)$ given in Theorem [11] is false, although it is reasonable to conjecture that, if $K_S^2 = 4\chi(O_S)$ then the canonical model of $S$ is a flat double cover of an abelian surface.

Our proof uses elementary intersection theory and our approach is similar to the original Severi’s argument that, in modern terminology, would go as follows: first note that, by the Noether’s formula, the inequality $K^2 \geq 4\chi$ is equivalent to $2c_1^2 \geq c_2$; just to explain the idea assume that the fibres of the Albanese map are finite and let $\eta, \eta' \in H^0(\Omega_S^1)$ be generic 1-forms, then by Severi-Bogomolov’s theorem ([1, 6.6]) $\eta$ and $\eta'$ have no common integral curves. Let $T \subset S$ be the (finite by Bertini’s theorem) set of points where $\eta = 0$ and denote $R = \sum_{i=1}^{r} a_i X_i = \text{div}(\eta \wedge \eta')$ with the $X_i$’s prime divisors. It is clear that $T$ is contained in the support of $R$ and for every $i = 1, \ldots, r$ the cardinality of $T \cap X_i$ is not bigger than the number of zeroes of the pull-back of $\eta$ to the normalization of $X_i$ which is $\leq K_S \cdot X_i + X_i^2$. By summing over $r$ we get $\text{Card}(T) \leq \sum_{i=1}^{r} K_S \cdot X_i + X_i^2$ and, if the scheme $\eta = 0$ is reduced and zero dimensional, then $\text{Card}(T) = c_2(S)$ and we have $c_2(S) \leq 2R \cdot R_{red} \leq 2K_S^2 = 2c_1^2$. Note that in general it is false that $c_2(S) \leq 2R \cdot R_{red}$ (the simplest counterexamples comes from simple triple Gaio covers of abelian surfaces, cf. [10, 13]).

Our proof has been also inspired by the Fulton-Lazarsfeld’s positivity theorem [12, 12.1.7] and rests on the following simple observation: let $L$ be the tautological line bundle over the projectivised cotangent bundle $\pi: \mathbb{P}(\Omega_S^1) \to S$ (cf. [4, I.7], [13]); by a simple computation about Chern classes we have $2c_1^2(S) - c_2(S) = (L + \pi^* K_S) \cdot L^2$. If $E$ is the maximal effective divisor in $S$ such that $h^0(\Omega_S^1(-E)) = q(S)$, then, using the fact that $\Omega_S^1$ is generically generated by global sections, we can write

$$2K_S^2 - c_2(S) = (L + \pi^* K_S) \cdot L^2 = 2K_S \cdot E + (L + \pi^* K_S) \cdot C,$$

where $C$ is an effective 1-cycle in $\mathbb{P}(\Omega_S^1)$. In particular, if $\Omega_S^1(K_S)$ is nef, then Theorem 0.1 follows immediately from the above formula. In Section 2 we consider the problem, of independent interest, of characterizing for every integer $p > 0$ the surfaces $S$ of Albanese dimension 2 such that $\Omega_S^1(pK_S)$ is nef. In particular we show that $\Omega_S^1(K_S)$ is not nef if and only if there exists a rational curve $D \subset S$ with at most nodes and cusps as singularities such that $2N + T \leq 2$ and $K_S \cdot D < 2 + T$, being $N$ the number of nodes and $T$ the number of cusps of $D$.

If $\Omega_S^1(K_S)$ is not nef, then we are able to show that the term $(L + \pi^* K_S) \cdot C$ is nevertheless nonnegative by making a detailed study of the 1-cycle $C$.

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1 This was wrongly assumed to be true by Severi for every generic 1-form $\eta$ on a minimal surface without irrational pencils.
In this approach the main difficulty is to give a convenient lower bound for the multiplicity of the fibres of \( \pi \) contained in the cycle \( C \). We will see that this problem is essentially equivalent of giving a good upper bound for the Milnor number of certain singularities of curves in \( S \). Unfortunately such kind of upper bounds are easy to find only for “sufficiently nondegenerate” singularities (cf. also [11]); this forces us to make a sort of “semistable reduction” which takes a consistent part of the paper and involves degenerations and simple cyclic covers.

Most part of this paper (Sections 1,2,3,4 and 6) was done during the years 1996-97 when the author was at Scuola Normale Superiore of Pisa; the author thanks all the members of Pisa team of Algebraic Geometry and especially F. Catanese, R. Pardini and F. Zucconi for the continuous encouragement and useful discussions about the subject of this paper.

Notation and general set-up

All varieties are considered over the field of complex numbers. For every smooth projective surface \( S \) we denote by \( K_S \in \text{Pic}(S) \) its canonical line bundle and by \( q(S) = h^1(\mathcal{O}_S) = h^0(\Omega^1_S) \), \( p_g(S) = h^2(\mathcal{O}_S) = h^0(K_S) \) its irregularity and geometric genus respectively. For every effective divisor \( R \subset S \) we denote by \( R_{\text{red}} \) the support of \( R \) endowed with the reduced structure.

We denote by \( \text{Alb}(S) = \text{coker}(f : H_1(S, \mathbb{Z}) \to H^0(\Omega^1_S)) \) the Albanese variety of \( S \) and by \( \alpha : S \to \text{Alb}(S) \) the Albanese map (defined up to translations in \( \text{Alb}(S) \)). The Albanese dimension of \( S \) is the dimension of \( \alpha(S) \). We recall that, for a given point \( p \in S \), the linear map \( H^0(\Omega^1_p) \to T^\vee_p \) is canonically isomorphic to the transpose of the differential of \( \alpha \) at \( p \). According to [11] for every smooth variety \( X \) we shall denote by \( Z^i(X) \) the free abelian group of cycles of codimension \( i \), by \( A^*(X) \) its Chow ring and for every, possibly nonreduced, subvariety \( C \subset X \) of pure codimension \( i \) by \([C] \in Z^i(X) \) its associated cycle.

Let \( \mathcal{E} \) be a vector bundle of rank \( r \) on a smooth projective variety \( X \) of dimension \( n \), we denote by \( \mathbb{P}(\mathcal{E}) \to X \) the associated projective space bundle in the sense of Grothendieck (the points of \( \mathbb{P}(\mathcal{E}) \) correspond to hyperplanes of \( \mathcal{E} \), cf. [3, II.7] for a precise definition) and by \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) the tautological line bundle on \( \mathbb{P}(\mathcal{E}) \). Given a morphism of smooth varieties \( f : Y \to X \), there exists a bijection between the set of liftings of \( f \) to \( \mathbb{P}(\mathcal{E}) \) and quotient line bundles of \( f^*\mathcal{E} \) defined by taking for every lifting \( \tilde{f} : Y \to \mathbb{P}(\mathcal{E}) \) the quotient line bundle \( \tilde{f}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \). In particular the projection \( \mathbb{P}(\mathcal{E}) \to X \) gives a natural surjection \( \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) inducing isomorphisms \( H^0(\mathcal{E}) \cong H^0(\pi^*\mathcal{E}) \cong H^0(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \).

If \( X \) is a surface, \( \mathcal{E} \) a rank 2 vector bundle, \( L = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \) and \( c_1(\mathcal{E}), c_2(\mathcal{E}) \) are the Chern classes of \( \mathcal{E} \) we have the following standard numerical equalities:

- \( L^3 = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) \).
- \( L^2 \cdot \pi^*A = c_1(\mathcal{E}) \cdot A \) for every \( A \in A^1(X) \).
- \( L \cdot \pi^*A = \deg A \) for every \( A \in A^2(X) \).

1 Preliminaries

In this section \( S \) is a fixed smooth surface of general type with \( \Omega^1_S \) generically generated by global sections. It is convenient to divide the set of irreducible curves of \( S \) in 3 disjoint classes according to the behavior of the Albanese map at their generic points; our classification may appear unnatural but will be quite useful for computation.
Definition 1.1. In the above set-up, we denote by:

- $S_2 \subset S$ the (nonempty) open subset where the differential of the Albanese map $\alpha$ has rank 2.
- $S_0 \subset S$ the union of the subset of points where the differential of $\alpha$ vanishes and the (finitely many) closed curves contracted by $\alpha$.
- $S_1 = S - (S_2 \cup S_0)$.

A reduced irreducible curve $C \subset S$ is called of type $i$, $i = 0, 1, 2$, if the generic point of $C$ belongs to $S_i$.

In other words a curve is of type 0 if it is contracted by the Albanese map $\alpha$; of type 1 if it is contained in the ramification divisor of $\alpha$ but it isn’t contracted and of type 2 otherwise.

Note that every rational curve is contracted by $\alpha$ and then it is of type 0. By a well known theorem of Mumford [15], in the free abelian group generated by curves of type 0, the intersection form is negative definite; in particular if $D_1 \neq D_2$ are irreducible curves of type 0 then $D_1^2 D_2^2 > (D_1 \cdot D_2)^2$.

For every $\eta \in H^0(\Omega^1_S)$ we denote by $\Lambda_\eta$ the image of the linear map $\wedge \eta : H^0(\Omega^1_S) \to H^0(K_S)$, if $\eta \neq 0$ there exists $p \in S_2$ such that $\eta(p) \neq 0$ and then $\Lambda_\eta \neq 0$.

Define $\Lambda$ as the image of the natural map $\wedge^2 H^0(\Omega^1_S) \to H^0(K_S)$, in other words $\Lambda$ is the smallest vector subspace of $H^0(K_S)$ containing all the $\Lambda_\eta$, $\eta \in H^0(\Omega^1_S)$.

Finally let $F$ (resp.: $F_\eta$) be the fixed part of the linear system $\mathbb{P}(\Lambda)$ (resp.: $\mathbb{P}(\Lambda_\eta)$). Note that every divisor of the linear system $\mathbb{P}(\Lambda)$ is singular at the points where the differential of $\alpha$ vanishes.

Lemma 1.2. In the above notation:

1. The base locus of $\Lambda$ is $S_0 \cup S_1$, in particular the irreducible components of $F$ are exactly the curves of type 0 and 1.
2. $F = K_S$ if and only if $q(S) = 2$.
3. $F = 0$ if and only if $\Omega^1_S$ is generated by global sections outside a finite set of points (a typical case in which $F = 0$ is when $q \geq 3$ and the linear system $\mathbb{P}(\Lambda)$ contains a reduced irreducible divisor).
4. For generic $\eta \in H^0(\Omega^1_S)$, $F_\eta = F$.

In particular for generic $\eta, \mu \in H^0(\Omega^1_S)$, $\text{div}(\eta \wedge \mu) = F + D$ where every irreducible component of $D$ is of type 2 and has nonnegative selfintersection.

Proof. 1) By definition every decomposable two-form $\eta_1 \wedge \eta_2$ vanishes on $S_0 \cup S_1$ and therefore every irreducible curve of type 0,1 is contained in the base locus of the linear system $\mathbb{P}(\Lambda)$. Conversely for every $p \in S_2$ there exists $\eta_1, \eta_2$ linearly independent at $p$ and therefore $\eta_1 \wedge \eta_2(p) \neq 0$.

2) If $q(S) = 2$ then $\Lambda$ has dimension 1 and then $F = K_S$. Conversely assume $q(S) \geq 3$ and let $\eta_1 \neq 0$ be an element of the kernel of the natural map $H^0(\Omega^1_S) \to \Omega^1_{S,p}$; then $\eta_1 \wedge \eta_2(p) = 0$ for every $\eta_2$; since $p$ can be chosen arbitrarily the linear system $\mathbb{P}(\Lambda)$ contains a moving part.

3) Is an immediate consequence of 1).

4) The vector space $H^0(\Omega^1_S)$ generates the vector bundle $\Omega^1_{S_2}$ and therefore by Bertini-Sard’s theorem, for generic $\eta$, the scheme $Z = S_2 \cap \{ \eta = 0 \}$ is regular of dimension 0. According to the definition of $S_2$, the intersection of $S_2$ with the base locus of the linear system $\mathbb{P}(\Lambda_\eta)$ is contained in $Z$ and then $F_\eta$ is supported in $S_0 \cup S_1$.

On the other hand it is clear that $F \subset F_\eta$ for every $\eta$ and therefore it is sufficient to prove
that every irreducible curve \( C \subset S_0 \cup S_1 \) appear with the same multiplicity in the divisors \( F \) and \( F_\eta \) for generic \( \eta \). If \( \eta_1, \ldots, \eta_k \) is a basis of \( H^0(\Omega^1_S) \), then the multiplicity of \( C \) in \( F \) is exactly the minimum of the multiplicities of \( C \) in the divisors of \( \eta_i \wedge \eta_j \) and then there exists \( i \) such that \( \text{mult}_C(F) = \text{mult}_C(F_{\eta_i}). \)

Since \( S_0 \cup S_1 \) contains only finitely many curves, by semicontinuity of multiplicities, it follows the equality \( F = F_\eta. \)

Writing \( \mathcal{P}(\Lambda_\eta) = F + D, \) then \( D \) is a linear system on the surface \( S \) without fixed part. If \( D \) is a generic divisor of \( D \) then every irreducible component of \( D \) is of type 2 and, by general properties of linear systems, it has nonnegative self-intersection.

For every reduced irreducible curve \( C \subset S \) we denote by \( g(C) \) its geometric genus and by \( p_a(C) = 1 + \frac{1}{2}C \cdot (K_S + C) \) its arithmetic genus.

In this paper by a cusp we shall mean an irreducible double point of a curve in a surface which can be resolved by exactly one blowing-up: it’s easy to see that a singularity \( (C, p) \subset (S, p) \) is a cusp if and only if there exist local analytic coordinates \( x, y \) of \( S \) centered at \( p \) such that \( C = \{ x^2 = y^3 \}. \)

If \( \phi: B \to S \) is a nonconstant morphism from a smooth projective curve \( B \) to \( S \), the coherent sheaf \( \Omega^1_{B/S} \) is supported on a finite set of points; we shall denote by \( r(\phi) = h^0(B, \Omega^1_{B/S}) \) and we shall call \( r(\phi) \) the number of ramification points of \( \phi \). Because of the first exact sequence of differentials we also have that \( r(\phi) \) is the length of the cokernel of the morphism of \( \mathcal{O}_B \)-modules \( \phi^*\Omega^1_S \to \Omega^1_B \).

If \( C \subset S \) is a reduced irreducible curve we define \( r(C) \) as the number of ramification points of the normalization map \( \phi: B \to C \subset S \). Note that if \( C \) is a curve with at most nodes and cusps as singularities then \( r(C) \) is exactly the number of cusps of \( C \); note moreover that \( r(C) = 0 \) if \( C \) has at most ordinary singularities.

**Lemma 1.3.** For every reduced irreducible curve \( C \subset S \) we have:

\[
r(C) = \sum_{p \in C} (\text{mult}_p(C) - \text{number of branches of } C \text{ passing through } p)
\]

where for every \( p \in S \), \( \text{mult}_p(C) \) denotes the multiplicity of \( C \) at \( p \).

**Proof.** We identify the normalization of \( C \) with the set of branches of \( C \) and a branch with an equivalence class of irreducible parametrizations of \( C \) (cf. [2], IV.2); in Walker’s notation the branches are called places). It is sufficient to show that for every branch \( (B, p) \) the number of ramification points lying over \( (B, p) \) is exactly \( \text{mult}_p(B) - 1 \). Here it is convenient to think \( \text{mult}_p(B) \) as the intersection multiplicity of \( B \) with a generic smooth germ of curve passing through \( p \).

Let \( x, y \) be local coordinates on \( S \) such that \( x(p) = y(p) = 0 \) and let \( t \) be a local parameter of \( B \); then \( B \) is represented by an irreducible parametrization \( x = \alpha t^a + o(t^a), y = \beta t^b + o(t^b), \alpha \beta \neq 0 \). It is then clear that \( \text{mult}_p(B) = \min(a, b) \), while the number of ramification points over \( (B, p) \) is exactly the dimension of the vector space \( \Omega^1_{B,p}/(dx, dy) \) which is equal to \( \min(a, b) - 1 \). 

**Lemma 1.4.** For every reduced irreducible curve \( C \subset S \), \( p_a(C) - g(C) \geq r(C) \) and equality holds if and only if \( C \) has only cusps as singularities.

**Proof.** Immediate consequence of Lemma 1.3 and the formula:

\[
p_a(C) - g(C) = \frac{1}{2} \sum \text{mult}_p(C)(\text{mult}_p(C) - 1)
\]
where the sum is taken over all infinitely near points of $C$. □

**Proposition 1.5.** Let $C \subset S$ be a reduced irreducible curve such that $C^2 < 0$ and $K_S \cdot C + 2g(C) < 2 + r(C)$. Then $C$ is a rational curve with at most nodes and cusps as singularities and $C^2 \geq 2N + T - 3$ where $N$ is the number of nodes and $T$ the number of cusps of $C$.

**Proof.** Since $C^2 < 0$, by genus formula and Lemma 1.3, we have $K \cdot C > 2p_a(C) - 2 \geq 2g(C) + 2r(C) - 2$ and therefore $1 + r(C) \geq K \cdot C + 2g(C) > 4(g(C) - 1) + 2r(C) + 2$. This proves that $g(C) = 0$ and $r(C) \geq 2r(C) - 2$. By an easy computation $r(C) \leq 2$, $KC \leq r(C) + 1$ and $2p_a \leq K \cdot C + 1 \leq r(C) + 2 \leq 4$; therefore $C$ is a rational curve and either $p_a(C) \leq 1$ or $p_a(C) = r(C) = 2$.

Since every singular points of multiplicity at least 3 gives a contribution to arithmetic genus bigger or equal than 3, the curve $C$ can have at most double points as singularities. Moreover every double point which is not a node or a cusp gives a contribution to $p_a(C)$ bigger or equal to 2 while the contribution to $r(C)$ is only 1.

Thus $C$ has at most nodes and cusps as singularities and then $r(C) = 2$, $C^2 = 2N + 2T - 2 - K_S \cdot C \geq 2N + 2T - 2 - r(C) - 1 = 2N + T - 3$. □

## 2 A criterion for nefness

Let $S$ be a surface of general type with Albanese dimension 2, the goal of this section is to determine the positive integers $P$ for which the vector bundle $\Omega^1_S(pK_S)$ is nef.

We recall that a vector bundle $E$ is called nef if the line bundle $\mathcal{O}_S(E^1)$ is nef; in this section we shall use the following facts (cf. [17, 1.16]):

1. Given a line bundle $L$ on $X$, the vector bundle $E \otimes L$ is nef if and only if $\mathcal{O}_S(E^1) \otimes \pi^*L = \mathcal{O}_S(E \otimes L^1)$ is nef.

2. If $E$ is generated by global sections outside a finite set of points then $E$ is nef.

3. $E$ is nef if and only if for every smooth projective curve $B$ and every generically injective morphism $f: B \rightarrow X$ the vector bundle $f^*E$ is nef.

4. If $\mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ is an exact sequence of vector bundles then: $\mathcal{F}, \mathcal{G}$ nef $\Rightarrow \mathcal{E}$ nef $\Rightarrow \mathcal{G}$ nef.

**Lemma 2.1.** Let $X$ be a smooth algebraic variety with $\Omega^1_X$ nef; then for every smooth projective curve $B$ and every nonconstant morphism $\phi: B \rightarrow X$ we have $2g(B) \geq \h^0(\Omega^1_B/X)$. In particular $X$ does not contain rational curves.

**Proof.** This is an easy and well known result, a proof is given here for the lack of suitable reference.

Since the morphism $\phi$ is not constant the image of the pull-back morphism $\phi^*\Omega^1_X \rightarrow \Omega^1_B$ is a line bundle isomorphic to $\Omega^1_B(-R)$ for some effective divisor $R$ of degree equal to $\h^0(\Omega^1_B/X)$. According to items 3) and 4) above the line bundle $\Omega^1_B(-R)$ is nef and then of nonnegative degree.

The relation between the nefness of $\Omega^1_S(pK_S)$ and Severi inequality is given by the following easy lemma:
Lemma 2.2. Let $S$ be a smooth surface of general type with Albanese dimension 2 and $\Omega^1_S(pK_S)$ nef for some rational number $p \geq 0$. Then $K_S$ is ample and $(p+1)c^2_1(S) \geq c_2(S)$.

Proof. According to Lemma 2.1, if $S$ contains a rational curve $C$ with $K_S \cdot C \leq 0$ then $\Omega^1_S(pK_S)$ is not nef for every $p > 0$; this implies that $K_S$ is ample.

Let $\eta_1, \eta_2 \in H^0(\Omega^1_S)$ be two sections such that $\eta_1 \cap \eta_2 \neq 0$ and denote by $L_{\eta_1}, L_{\eta_2} \subset \mathbb{P}(\Omega^1_S)$ the divisors of the corresponding sections of $L = \mathcal{O}_{\mathbb{P}(\Omega^1_S)}(1)$. If $p \in S$ is a point such that $\eta_1(p) \cap \eta_2(p) \neq 0$ then $\pi^{-1}(p) \cap L_{\eta_1} \cap L_{\eta_2} = \emptyset$ and therefore we can write $L_{\eta_2} = \pi^*E + H_{\eta_2}$, where $E$ is an effective divisor of $S$ and $H_{\eta_2}$ intersects properly $L_{\eta_1}$.

By nefness $(L + p\pi^*K_S) \cdot L_{\eta_1}, H_{\eta_2} \geq 0$ and then:

$$(1 + p)c^2_1(S) - c_2(S) = (L + p\pi^*K_S) \cdot L_{\eta_1}, L_{\eta_2} = (L + p\pi^*K_S) \cdot (L_{\eta_1}, H_{\eta_2} + L_{\eta_1}, \pi^*E)$$

$$(1 + p)c^2_1(S) - c_2(S) \geq (L + p\pi^*K_S) \cdot L_{\eta_1}, \pi^*E = (1 + p)K_S \cdot E \geq 0.$$

The main result of this section is

Theorem 2.3. If $S$ is a surface of Albanese dimension 2 with ample canonical bundle $K_S$ then:

1. $\Omega^1_S(pK_S)$ is nef for every $p \geq 3$.

2. $\Omega^1_S(2K_S)$ is nef if and only if there does not exist any rational cuspidal curve $C \subset S$ with $C^2 = -1, K_S \cdot C = 1$.

3. $\Omega^1_S(K_S)$ is nef if and only if every rational curve $C \subset S$ with at most nodes and cusps as singularities satisfies the relation $C^2 \leq 2N + T - 4$, where $N$ is the number of nodes and $T$ is the number of cusps of $C$.

Remark 2.4. Since the image of the Albanese map $\alpha$ is a surface, every rational curve contained in $S$ has negative self-intersection and therefore the condition 3) can fail only if $2N + T \leq 2$.

Remark 2.5. In 2.3 we considered for simplicity only the case $p \in \mathbb{N}$ but similar results can be obtained easily for every real number $p \geq 1$ (this will be clear in the proof). For example $\Omega^1_S\left(\frac{3}{2}K_S\right)$ is nef if and only if there does not exist any rational curve $C \subset S$ with $C^2 < 0$, $K_S \cdot C \leq 1$.

Proof. Let $\phi: B \to S$ be the normalization of a reduced irreducible curve $C$ and let $R \subset B$ be the ramification divisor of $\phi$, there are three possible cases:

1) $C$ has type 2: in this case $\phi^*\Omega^1_B$ is generically generated by global sections and therefore it is nef.

2) $C$ has type 1:

Over the curve $B$ there exist a divisor $D$ and an exact sequence of vector bundle

$$0 \to \mathcal{O}_B(D + R) \to \phi^*\Omega^1_B \to \Omega^1_B(-R) \to 0$$

such that $\deg(R) = r(C), \deg(D) = \deg(\phi^*\Omega^1_B) - \deg(\Omega^1_B) = K_S \cdot C - 2g(C) + 2 \geq -C^2$.

Since the map $H^0(\Omega^1_B) \to H^0(\Omega^1_B)$ is nonzero, the degree of $\Omega^1_B(-R)$ is nonnegative and, if
$C^2 \leq 0$ then $\phi^*\Omega^1_S$ is nef.
If $C^2 > 0$ then, since $C$ is a component of the fixed part of $\mathbb{P}(\Lambda)$, there exists an effective divisor $G$ in $S$ such that $C+G = K_S$ and then $\deg(R) + \deg(D) + K_S \cdot C \geq K_S \cdot C - C^2 = CG \geq 0$.
The nefness of $\phi^*\Omega^1_S(K_S)$ follows in this case by considering the exact sequence

$$0 \to \mathcal{O}_B(D + R + \phi^*K_S) \to \phi^*\Omega^1_S(K_S) \to \Omega^1_B(-R + \phi^*K_S) \to 0$$

3) $C$ is contracted by $\alpha$, in this case $C^2 < 0$ and therefore, by the same argument used for the curves of type 1, $\phi^*\Omega^1_S(pK_S)$ is nef if and only if the degree of $\Omega^1_B(-R + p\phi^*K_S)$ is nonnegative. This condition is equivalent to $pK_S \cdot C + 2g(C) \geq 2 + r(C)$ and the conclusion now follows immediately from Proposition 1.3.

**Corollary 2.6.** Let $S$ be a surface of Albanese dimension 2 with ample canonical bundle, assume that $K_S = pH$ for some $p \geq 4$, $H \in \text{Pic}(S)$ or that $K_S = H + L$ where $H, L$ are ample line bundles such that $|H|, |L|$ are base point free linear systems, then $\Omega^1_S(K_S)$ is nef.

**Proof.** According to Theorem 2.3 we need to show that, for every rational curve $C$ with $N$ nodes, $T$ cusps and no other singularities $C^2 \leq 2N + T - 4$. Since $C^2 < 0$ it is not restrictive to assume $2N + T \leq 2$.
If $C$ is smooth, then $K_S \cdot C \geq 2$ and by genus formula $C^2 \leq -4$. If $C$ is singular then $K_S \cdot C \geq 4$: this is clear if $K_S = pH$ with $p \geq 4$; in the case $K_S = L + H$ we note that the generic pencil of $|L|$ is base point free over $C$ and therefore defines a regular morphism $C \to \mathbb{P}^1$ of degree $L \cdot C \geq 2$, similarly $H \cdot C \geq 2$ and then $K_S \cdot C \geq 4$.
By genus formula $C^2 = 2N + 2T - 2 - K_S \cdot C \leq 2N + 2T - 6 \leq 2N + T - 4$. 

**Corollary 2.7.** Let $S$ be a surface of Albanese dimension 2 with very ample canonical bundle, then $\Omega^1_S(K_S)$ is nef if and only if $S$ does not contain lines, i.e. smooth rational curves $C$ with $K_S \cdot C = 1$.

**Proof.** Let $C \subset S$ be a rational curve with $N$ nodes, $T$ cusps and $0 < K_S \cdot C \leq T + 1 \leq 3$ and let $S \to \mathbb{P}^2$ be the map induced by a generic net of $|K_S|$. The image of $C$ under this map is a plane reduced irreducible rational curve of degree $K_S \cdot C$ with $N$ nodes and $T$ cusps; it is therefore evident that the only possibility is $N = T = 0$.

### 3 Estimation of intersection products

As above let $S$ be a surface of general type with Albanese dimension 2 and Albanese map $\alpha: S \to \text{Alb}(S)$. Let $V = \mathbb{P}(\Omega^1_S)$, $\pi: V \to S$ the natural projection and $L = \mathcal{O}_V(1)$ be the tautological quotient line bundle over $V$. For every $\eta \in H^0(\Omega^1_S) = H^0(V, L)$ we denote by $L_\eta \subset V$ the divisor of the corresponding section of $L$.

**Definition 3.1.** Denote by $E \subset S$ the maximal effective divisor such that $h^0(\Omega^1_S(-E)) = q(S)$ and for every $\eta \in H^0(\Omega^1_S)$ let $H_\eta$ be the effective divisor $H_\eta = L_\eta - \pi^*E$.
Denote by $H^0(\Omega^1_S)^0 \subset H^0(\Omega^1_S)$ the subset of forms $\eta \neq 0$ such that $H_\eta$ is irreducible.

**Lemma 3.2.** In the notation of Definition 3.2 $H^0(\Omega^1_S)^0$ is a Zariski open subset of $H^0(\Omega^1_S)$. 

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Proof. For every nonzero form $\eta \in H^0(\Omega^1_2)$ the projection $L_\eta \to S$ is an isomorphism over
the open set \( \{ x \mid \eta(x) \neq 0 \} \), this implies that $L_\eta = \pi^* E_\eta + H_\eta$ where $E_\eta$ is a divisor containing $E$ and $H_\eta$ is a reduced irreducible divisor.

By Bertini’s theorem, for generic $\eta$ the divisor $E_\eta$ is supported in the proper closed subset $S_0 \cup S_1 \subset S$; by semicontinuity of multiplicities we finally get $E_\eta = E$ for generic $\eta$. \qed

Note that if $H_\eta$ is irreducible then it is also reduced and the projection $H_\eta \to S$ is birational.

Since $\Omega^1_2$ is generically generated by global sections, for generic $\eta_1, \eta_2 \in H^0(\Omega^1_2)$, $L_{\eta_1}$ intersects properly $H_{\eta_2}$ and therefore the cycle associated to the subscheme $L_{\eta_1} \cap H_{\eta_2}$ is given by the formula:

**Formula 3.3.**

$$[L_{\eta_1} \cap H_{\eta_2}] = \sum_{i=1}^r n_i C_i + \sum_{j=1}^s m_j j^{-1}(p_j), \quad n_i, m_j > 0,$$

for some points $p_j \in S$ and reduced irreducible curves $C_i \subset V$ such that the projection $C_i \to \pi(C_i)$ is generically injective. Although $[L_{\eta_1} \cap H_{\eta_2}] = [L_{\eta_2} \cap H_{\eta_1}]$ in the Chow group of $V$, in general $L_{\eta_1} \cap H_{\eta_2} \neq L_{\eta_2} \cap H_{\eta_1}$ as subschemes; this explain the asymmetry in $\eta_1, \eta_2$ in some local computations. We have moreover the following:

**Formula 3.4.**

$$\text{div}(\eta_1 \wedge \eta_2) = E + \sum_{i=1}^r n_i D_i, \quad D_i = \pi(C_i).$$

This is a consequence of the following more general fact about degeneracy loci:

Let $X$ be a smooth variety, $\mathcal{L}$ a line bundle on $X$ and $\mathcal{E}$ a rank 2 vector bundle generically generated by global sections. For every positive integer $a$, $\pi_* \mathcal{O}_{\mathcal{P}(\mathcal{E})}(a) = \mathcal{O}^a \mathcal{E}$ is the $a$-th symmetric power of $\mathcal{E}$ and therefore there exists a natural isomorphism $H^0(\mathcal{O}^a \mathcal{E} \otimes \mathcal{L}) = H^0(\mathcal{O}_{\mathcal{P}(\mathcal{E})}(a) \otimes \pi^* \mathcal{L})$. In order to simplify the notation we shall denote $\mathcal{O}(a) = \mathcal{O}_{\mathcal{P}(\mathcal{E})}(a)$ and for every $f \in H^0(\mathcal{O}^a \mathcal{E} \otimes \mathcal{L})$, by $D_f \subset \mathcal{P}(\mathcal{E})$ the divisor of the corresponding section of $\mathcal{O}_{\mathcal{P}(\mathcal{E})}(a) \otimes \pi^* \mathcal{L}$.

Given $f \in H^0(\mathcal{O}^a \mathcal{E} \otimes \mathcal{L}), g \in H^0(\mathcal{O}^b \mathcal{E})$ their resultant $r(f, g) \in H^0((\det \mathcal{E}) \otimes^a \mathcal{L} \otimes^b)$ is by definition the determinant of the morphism of vector bundles of rank $a + b$

$$\phi: (\mathcal{O}^{a-1} \mathcal{E} \otimes \mathcal{L}) \oplus (\mathcal{O}^{b-1} \mathcal{E} \otimes \mathcal{L}) \to (\mathcal{O}^{a+b-1} \mathcal{E} \otimes \mathcal{L}) \quad \phi(h, k) = hg + kf$$

By the usual properties of resultants [21, Chapt. 1], it follows that $\pi(D_f \cap D_g)$ is exactly the degeneracy locus of $\phi$. Assume $Z = D_f \cap D_g$ is a subscheme of pure codimension 2, then $\pi(Z) \neq X$ and there exists an exact sequence

$$0 \to (\mathcal{O}^{a-1} \mathcal{E} \otimes \mathcal{L}) \oplus (\mathcal{O}^{b-1} \mathcal{E} \otimes \mathcal{L}) \xrightarrow{\phi} (\mathcal{O}^{a+b-1} \mathcal{E} \otimes \mathcal{L}) \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a torsion sheaf such that $\text{Supp}(\mathcal{F}) = \pi(Z)$ and, if $Y_1, \ldots, Y_r$ are the irreducible components of $\text{Supp}(\mathcal{F})$ we have, in the notation of [7]

$$\text{div}(r(f, g)) = \sum_{i=1}^r t_{\mathcal{O}_{Y_i}, X}(\mathcal{F} \otimes \mathcal{O}_{Y_i}, X)$$

For a proof of the above equality cf. [8, A.2].

On the other hand there exists an exact sequence of sheaves on $\mathcal{P}(\mathcal{E})$

$$0 \to \mathcal{O}(-1) \to \pi^* \mathcal{L}(a - 1) \oplus \mathcal{O}(b - 1) \xrightarrow{\phi} \pi^* \mathcal{L}(a + b - 1) \to \mathcal{O}_Z(a + b - 1) \otimes \mathcal{L} \to 0$$

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In our particular case tame ramification required equality \( \text{div}(E) \) to make sense to compute the number of ramification points of the composite map \( V \). Denote cycle associated to the subscheme \( Z \). Therefore we have proved that \( \pi^*_s[Z] = \text{div}(r(f, g)) \), where \( [Z] \in Z^2(\mathbb{P}(E)) \) is the effective cycle associated to the subscheme \( Z \) (cf. [1, 1.5]). Note that if \( a = b = 1 \) then we have \( r(f, g) = f \land g \in H^0(\text{det} E \otimes \mathcal{L}) \) and therefore \( \pi^*_s[Z] = \text{div}(f \land g) \).

In our particular case \( E = \Omega^1_S, \mathcal{L} = \mathcal{O}_{S/(-E)}, \pi_s|L_n \cap H_{n_2} = \sum n_i D_i \) and then we get the required equality \( \text{div}(\eta_1 \land \eta_2) = E + \sum_{i=1}^{n_i} D_i \).

If \( C \subset S \) is a reduced irreducible curve of type 1 or 2 and \( B = \phi(C) \) is its normalization, it make sense to compute the number of ramification points of the composite map \( \alpha \circ \phi : B \rightarrow \text{Alb}(S) \). In practice this number is difficult to find; however an useful lower bound \( r(\alpha \phi) \geq t(C) \) is obtained in the following way.

We shall say first that the Albanese map has tame ramification at a pair \((p, C)\) if:

1. \( p \) is a smooth point of \( C \) and
2. there exists \( \eta \in H^0(\Omega^1_S) \) such that \( \eta(p) \neq 0 \).

We then define:

\[
t(C) = \sum_{p \in C \text{ tame}} h^0(\Omega^1_{C/\text{Alb}(S),p}).
\]

We are now interested to to give useful lower bounds for the intersection products \( L \cdot C \) where \( C \subset L_{n_1} \cap H_{n_2} \) is a reduced irreducible curve such that \( \pi : C \rightarrow \pi(C) = D \) is generically injective.

Let \( B = C \) be the normalization of \( C \) and let \( \phi = \pi \circ \nu : B \rightarrow D \subset S \) be the composite map, denote \( V_B = \mathbb{P}(\phi^* \Omega^1_S), \pi_B : V_B \rightarrow B \) the projection, \( L_B = \mathcal{O}_{V_B}(1) \). Denoting by \( C \subset V_B \) the strict transform of \( C \) under the natural map \( \nu : V_B \rightarrow V \) we have \( L \cdot C = L_B \cdot C \). Note that \( \phi : B \rightarrow D \) is the normalization map.

**Proposition 3.5.** In the above set-up:

1. if \( D \) is of type 2 then \( L \cdot C \geq t(D) \)
2. if \( D \) is of type 1 then \( L \cdot C \geq t(D) - D^2 \) and the equality holds only if \( D \) is smooth.
3. if \( D \) is of type 0 then \( L \cdot C = 2g(D) - 2 - r(D) \)

**Proof.** 1) Take a generic form \( \eta \in H^0(\Omega^1_S) \), then we have \( \phi^* L_\eta = C_\eta + \sum_i \beta_i \pi_B^{-1}(p_i) \subset V_B \) (note that \( \hat{C} = C_{\eta_1} = C_{\eta_2} \)) and, since \( D \) is of type 2, \( \hat{C} \neq C_\eta \) for generic \( \eta \). Therefore \( L \cdot C = \phi^* L_{\eta_1} \cdot \hat{C} \geq C_\eta \cdot \hat{C} \) and it is sufficient to prove that \( C_\eta \cdot \hat{C} \geq t(D) \).

This is easily proved using local parameters. Let \( p \in D \) be a tame ramification point of \( \alpha \) such that \( r = h^0(\Omega^1_{D/\text{Alb}(S),p}) > 0 \); let \( x, z \) be local coordinates at \( p \) such that \( D = \{ x = 0 \} \). By the definition of tame ramification every \( \eta \) can be written locally as \( \eta = a(z) dx + z^r b(z) dz + z \eta_1 \) with \( a(0), b(0) \neq 0 \) for \( \eta \) generic. The local contribution to \( t(D) \) at the point \( p \) is by definition \( r \). If locally \( \eta_1 = a_1(z) dz + z^r b_1(z) dz + z \eta_1 \) then in the affine subspace \( dz \neq 0 \) of \( V_B \), the local equations of \( C_\eta \) and \( \hat{C} = C_{\eta_1} \) are respectively:

\[
a(z) \frac{dx}{dz} + z^r b(z) = 0, \quad a_1(z) \frac{dx}{dz} + z^r b_1(z) = 0
\]
and, since \( a(0) \neq 0 \), the intersection multiplicity of \( C_n \) and \( \tilde{C} \) at the point \( \frac{dx}{dz} = z = 0 \) is equal or greater than \( r \).

2) Assume now \( D \) of type 1, i.e. \( D \subset V_1 \) and \( \alpha \colon D \to \text{Alb}(S) \) nonconstant. Let \( \tilde{B} \subset V_B \) be the section of the kernels of the surjective morphism of vector bundles \( \phi^*\Omega^1_S \to \Omega^0_B(-R) \), where \( R \) is the ramification divisor of the map \( \phi \). By the definition of \( \tilde{B} \), we have \( \phi^*\eta(p) = 0 \) for some \( p \in B, \eta \in H^0(\Omega^1_S) \) if and only if \( \phi^*L_n \cap \tilde{B} \cap \pi_B^{-1}(p) \neq \emptyset \). In this case \( C \) is contained in the base locus of the linear system \( |L| \) and, since for generic \( \eta \in H^0(\Omega^1_S) \), \( \phi^*\eta \neq 0 \), we have \( \tilde{B} \neq \tilde{C} \) and then \( \tilde{B} \cap \tilde{C} \geq 0 \). Therefore if \( \phi^*L_n = \tilde{C} + \sum \beta_i \pi_B^{-1}(p_i) \) we get:

\[
0 \leq \deg(\Omega^1_B(-R)) = \phi^*L_n \cdot \tilde{B} = \tilde{C} \cdot \tilde{B} + \sum \beta_i \leq 2g(B) - 2
\]

this gives \( \sum \beta_i \leq 2g(B) - 2 \) and then:

\[
L \cdot C = \phi^*L_n \cdot \tilde{C} = \phi^*L_n^2 - \sum \beta_i \geq K_S \cdot D - (2g(B) - 2) + \tilde{C} \cdot \tilde{B} \geq \tilde{C} \cdot \tilde{B} - D^2.
\]

The same argument used in the proof of item 1) shows that \( \tilde{C} \cdot \tilde{B} \geq t(D) \) and then \( L \cdot C \geq t(D) - D^2 \).

In particular \( L \cdot C \geq -D^2 \) and equality holds if and only if \( D \) is smooth and \( \tilde{B} \cap \tilde{C} = \emptyset \).

3) Let \( C \) be an irreducible component of \( L_{\eta_1} \cap H_{\eta_2} \) such that \( \pi(C) = D \) is a curve of type 0; since \( D \) is contracted by the Albanese map there exist holomorphic functions \( f_1, f_2 \), defined in a neighbourhood of \( D \) such that \( f_i|_D \equiv 0 \) and \( \eta_i = df_i \). If \( p \in D \) and \( h \) is a local equation of \( D \) at \( p \), we have \( f_2 = h^\psi \) with \( \psi|_D \neq 0 \) and then \( df_2 = h^{\psi-1}(n\psi dh + hd\psi) \).

Therefore \( D \) appears with multiplicity \( n-1 \) in the divisor \( E, \phi^*(h^{1-n}\eta_2) = 0 \) in \( \Omega^1_B(-R) \), \( \tilde{C} = \tilde{B} \) and then \( L \cdot C \) is exactly the degree of \( \Omega^1_B(-R) \).

\[
\text{Corollary 3.6.} \quad \text{In the above set-up assume } S \text{ minimal, } (L + \pi^*K_S) \cdot C = 0 \text{ and } D = \pi(C) \text{ of type 1 or 2 for some irreducible curve } C \subset L_{\eta_1} \cap H_{\eta_2}. \text{ Then } q(S) = 2, \ D = K_S \text{ is smooth of type } 1 \text{ and } \alpha \colon D \to \text{Alb}(S) \text{ is unramified.}
\]

\[
\text{Proof.} \quad \text{If } D \text{ is of type 2 then by Prop. 2.3 we get } 0 = (L+\pi^*K_S)C = LC + K_SD \geq K_SD > 0. \text{ Thus } D \text{ must be of type 1 and then } 0 = L \cdot C + K_SD \geq K_SD - D^2 + t(D). \text{ On the other hand, since the canonical divisor of a minimal surface is connected and } K_S - D \text{ is effective, we have } K_SD - D^2 = (K_S - D) \geq 0 \text{ and equality holds if and only if } D = 0, K_S; \text{ this implies that } D = K_S, t(D) = 0. \text{ By Prop. 3.5 the divisor } D \text{ is smooth and by Lemma } 1.2 \text{ } q(S) = 2. \text{ Let } \eta_1, \eta_2 \text{ be a basis of } H^0(\Omega^1_S), \text{ then } \text{div}((\eta_1 \wedge \eta_2) = D \text{ is smooth and then the differential of the Albanese map is everywhere nonzero; therefore every point of } D \text{ has tame ramification and the equality } t(D) = 0 \text{ implies that } \alpha \colon D \to \text{Alb}(S) \text{ is unramified.}
\]

4 Surfaces with \( \Omega^1(K) \) nef and \( 2c_1^2 = c_2 \)

Let’s assume now \( S \) surface of general type, of Albanese dimension 2, \( \Omega^1(S) \) nef and \( 2c_1^2(S) = c_2(S) \). In this section we prove that the Albanese map is a double cover of an abelian surface.

\[
\text{Definition 4.1. (cf. [1.8]) Let } X \text{ be an abelian variety, } V \text{ a projective variety and } f \colon V \to X \text{ a regular morphism. We shall say that } f \text{ is minimal if the following condition is satisfied: If } g \colon V' \to X \text{ is a factorization with } g' \colon X' \to X \text{ isogeny of abelian varieties, then } g \text{ is an isomorphism.}
\]
It is easy to see, cf. [1.8], that if \( f(V) \) generates \( X \) and the homomorphism \( \text{Pic}^0(X) \to \text{Pic}^0(V) \) is injective then \( f \) is minimal; in particular the Albanese map is always minimal.

**Definition 4.2.** (cf. [1.9]) Let \( X \) be an abelian variety, \( V, W \subset X \) subvarieties: we say that the pair \((V, W)\) strictly cover \( X \) if for every surjective morphism \( \pi: X \to X' \) of abelian varieties we have

\[
\dim(\pi(V)) + \dim(\pi(W)) > \dim(X').
\]

For example, in the notation of Definition 4.2, if \( W \) is irreducible then the pair \((X, W)\) strictly cover \( X \) if and only if \( W \) generates \( X \).

**Theorem 4.3 (Debarre).** Let \( X \) be an abelian variety; \( V, W \) irreducible projective varieties and \( f: V \to X \), \( g: W \to X \) morphisms. If \( V \) is smooth, \( f \) is minimal and the pair \((f(V), g(W))\) strictly cover \( X \) then the fibred product \( V \times_X W \) is connected.

**Proof.** It is a particular case of [1.4.5]. \( \square \)

It is now easy to prove the following:

**Theorem 4.4.** Let \( S \) be a surface of general type, of Albanese dimension 2 with \( \Omega_S^1(K_S) \) nef and \( 2c_1^2(S) = c_2(S) \). Then \( q(S) = 2 \) and the Albanese map \( \alpha: S \to \text{Alb}(S) \) is a ramified double cover.

**Proof.** Let \( \eta_1, \eta_2 \) be generic 1-form on \( S \); as in the proof of 2.2 we have:

\[
2c_1^2(S) - c_2(S) = (L + \pi^*K_S) \cdot L_{\eta_1} \cdot L_{\eta_2} = (L + \pi^*K_S) \cdot L_{\eta_1} \cdot H_{\eta_2} + K_S \cdot E = 0
\]

Since \( L + \pi^*K_S \) is nef and \( K_S \) ample we must have \( E = 0 \), \( L_{\eta_1} = H_{\eta_2} \) and \((L + \pi^*K_S)C = 0\) for every component of \( L_{\eta_1} \cap L_{\eta_2} \). We have seen that \( \pi_*[L_{\eta_1} \cap L_{\eta_2}] = \text{div}(\eta_1 \wedge \eta_2) \) and, since the canonical divisor contains at least one curve of type \( > 0 \), we get by 6.4 that \( q = 2 \), \( R = \text{div}(\eta_1 \wedge \eta_2) \), is smooth irreducible and the restriction \( \alpha: R \to \text{Alb}(S) \) is unramified.

Since \( g(R) = K_S^2 + 1 \geq 2 \), the image \( \alpha(R) \) is not an elliptic curve and then the pair \((\alpha(S), \alpha(R))\) strictly cover \( \text{Alb}(S) \). By 1.3 the variety \( X = S \times_{\text{Alb}(S)} R \) is connected.

We are now in position to apply the standard argument of [1.7.1], [\#. In fact the embedding \( R \subset S \) induces an open embedding \( R \to X \); therefore \( X = R \) and then \( \alpha^{-1}(\alpha(R)) = R \). As \( R \) is the ramification divisor of \( \alpha \) and \( R \) is reduced it follows that degree of \( \alpha \) must be 2. \( \square \)

**Remark 4.5.** It is proved in [\#] that, if a surface \( S \) of general type of Albanese dimension 2 and \( \Omega_S^1 \) nef satisfy the equality \( c_1^2(S) = c_2(S) \) then \( q(S) = 3 \) and the Albanese map \( \alpha \) is unramified. The Theorem 4.3 gives an improvement of this result; in fact, since \( \alpha \) is unramified and \( S \) is not elliptically fibred, the pair \((\alpha(S), \alpha(R))\) strictly cover \( \text{Alb}(S) \); therefore \( S \times_{\text{Alb}(S)} R \) is connected and \( \alpha \) is a closed embedding.

## 5 Estimation of multiplicities

We have seen that the proof of Theorem 0.1 is quite easy when \( \Omega_S^1(K_S) \) is nef. If \( \Omega_S^1(K_S) \) is not nef we need to understand the set of points where two generic 1-forms \( \eta_1, \eta_2 \) vanish together and give a lower bound of the multiplicities \( m_\eta \), appearing in the Formula 5.3.

**Lemma 5.1.** Let \( D \subset S \) be a nonempty reduced divisor whose components are curves of type 0 and \( \eta \in H^0(\Omega_S^1)^0 \) (cf. Definition 7.4). Then the set \( P_\eta \subset D \) of points \( p \) such that \( \pi^{-1}(p) \subset H_\eta \) is not empty and contains the set of singular points of \( D \).
Proof. It is not restrictive to assume $D$ connected, then $D$ is contracted by $o$ to a point in the Albanese variety. Since $\eta$ is the pull back of a closed form in the Albanese variety, there exists a neighbourhood $U$ of $D$ and a holomorphic function $f$ defined over $U$ such that $f = 0$ over $D$ and $df = \eta$. Setting $D'$ as the divisor $\{f = 0\}$, we claim that $\text{Sing}(D') \subset \text{Sing}(D') \cap D \subset D$. If $x, y$ are local holomorphic coordinates at $p \in \text{Sing}(D') \cap D$ and $h$ is the greatest common divisor of $f_x, f_y$ in the U.F.D. $\mathcal{O}_{S,p}$ then, since $p$ is singular for $D'$, we have $h^{-1}\eta(p) = 0$ and, since the equation of $H_\eta$ is $h^{-1}(f_xdx + f_ydy) = 0$, we have $\pi^{-1}(p) \subset H_\eta$. Note finally that, since $D \cdot \text{div}(f) = 0$ and $D^2 < 0$, the divisor $D'$ is always singular provided that $D \neq \emptyset$.

Let now $p \in S$ be a fixed point, for every $\eta_1, \eta_2 \in H^0(\Omega^1_S)^0$, we are interested to give a lower bound for the multiplicity $m_p(\eta_1, \eta_2)$ of $\pi^{-1}(p)$ in the cycle $[L_{\eta_1} \cap H_{\eta_2}]$. We first note that the number $m_p(\eta_1, \eta_2)$ can be easily described in terms of local coordinates.

Let $U$ be a small contractible neighbourhood of $p$ and let $x, y$ be holomorphic coordinates over $U$ such that $p = \{x = y = 0\}$. Thinking $dx, dy$ as sections of $\mathcal{O}_U(1)$, we get a trivialization $\mathcal{P}(\Omega^1_U) = U \times \mathbb{P}^1$ with $dx, dy$ homogeneous coordinates over $\mathbb{P}^1$. If, in local coordinates, $\eta = a(x, y)dx + b(x, y)dy$ then the divisor $L_\eta \cap \pi^{-1}(U)$ is defined by the equation $\eta = a(x, y)dx + b(x, y)dy = 0$.

By holomorphic Poincaré lemma, over $U$ there exist holomorphic functions $f, g$ such that $f(p) = g(p) = 0$ and $\eta_1 = dg, \eta_2 = df$. Let $h$ be the greatest common divisor of $f_x, f_y$, then, after a possible shrink of $U$, the equations of $L_{\eta_1}, H_{\eta_2}$ in the open subset $\{dx \neq 0\} \subset \pi^{-1}(U)$ are respectively $g_x + v g_y = 0, h^{-1}(f_x + v f_y) = 0$, where $v$ is the affine coordinate $\frac{dy}{dx}$.

Note that, since $\eta_2 \in H^0(\Omega^1_S)^0$, we have that the local equations of the divisors $E, R$ are respectively $\{h = 0\}$ and $h^{-1}(f_xg_y - f_yg_x) = 0$.

If $\eta_1(p) = h^{-1}\eta_2(p) = 0$, i.e. if $m_p > 0$, then the multiplicities of $f$ and $g$ at $p$ are at least $2$ and for generic $\beta \in \mathbb{C}$, the section $v = \beta$ intersects $L_{\eta_1} \cap H_{\eta_2}$ only in the component $\pi^{-1}(p)$. Therefore

$$m_p(f, g) := m_p(\eta_1, \eta_2) = \frac{1}{h}(f_x + \beta f_y, g_x + \beta g_y) \quad \text{for generic } \beta \in \mathbb{C},$$

where for any pair $f_1, f_2 \in \mathbb{C}[x, y]$ of convergent power series we shall denote by $(f_1, f_2)$ the intersection multiplicity at $x = y = 0$ of the two germs of curves of equation $f_1, f_2$.

Set-Up 5.2. We consider $x, y$ local holomorphic coordinates at a point $p \in S$, $f, g \in \mathbb{C}[x, y]$ power series such that $f(0) = g(0) = 0$, $\text{mult}(g) \geq 2, f_xg_y - f_yg_x \neq 0$ and the germ $\{f = 0\}$ is singular at $p = \{x = y = 0\}$. WE denote $h = G.C.D.(f_x, f_y), R = \text{div}(h^{-1}(f_xg_y - f_yg_x))$

Lemma 5.3. Let $f, g, h \in \mathbb{C}[x, y], R$ be as in the Set-up 5.2. Assume $\text{mult}(f) = m + 1, d = \text{mult}(h)$. Let $0 \leq \tau$ be an integer strictly smaller than the number of irreducible components of the tangent cone of $f$, then for generic $\beta \in \mathbb{C}$

$$\infty > m_p(f, g) = \frac{1}{h}(f_x + \beta f_y, g_x + \beta g_y) \geq \tau(\text{mult}_p(R) - \tau) + (\text{mult}(g) - \tau - 1)(m - \tau - d).$$

In particular if $\text{mult}(g) \geq \tau + 1$, then $m_p(f, g) \geq \tau(\text{mult}_p(R) - \tau)$.

Proof. We first prove that the above intersection product is finite for generic $\beta$. Assume that $h^{-1}(f_x + \beta f_y), g_x + \beta g_y) = \infty$ for every $\beta$, then the analytic singularity

$$(Z, 0) = \{(x, y, \beta) \in \mathbb{C}^3 \mid h^{-1}(f_x + \beta f_y) = g_x + \beta g_y = 0\}$$
has dimension 2. Since the set \( h^{-1}f_x = h^{-1}f_y = 0 \) is finite, the image of the projection onto the \( x, y \)-plane \( \pi: Z \to \mathbb{C}^2 \) is Zariski dense and this is in contradiction with the fact that \( \pi(Z) \) is contained in the set of points where \( f_xg_y - f_yg_x = 0 \).

For \( \tau = 0 \) the inequality is trivially true; assume therefore \( \tau > 0 \), in this case we have necessarily \( \mu \tau = \mu f = m \).

Let \( r + 1, r \geq 0 \), be the number of irreducible components of the tangent cone of \( f \) at 0, the pencil of tangent cones of \( f_x + \beta f_y \), contain at least \( r \) moving lines and therefore for generic \( \beta \) we have \( f_x + \beta f_y = \phi_1 \ldots \phi_r \psi \) with \( \phi_1, \ldots, \phi_r \) convergent power series of multiplicity 1 such that \( \phi_1, f_y \).

It is therefore possible to write \( f_x + \beta f_y = f' f'' h \) where \( \mu \tau = \tau, \ f' \) has no common tangent lines with \( f_y \) at 0 and then \( (f', f_y) = \tau m \). By using the relation \( g_xf_y - g_yf_x = f_y(g_x + \beta g_y) - f' f'' h g_y \), and setting \( N = \mu \tau = \mu (f_xg_y - f_yg_x) \), we get:

\[
\left( \frac{f_x + \beta f_y}{h}, g_x + \beta g_y \right) = (f', f_yg_y - f_yg_x) - (f', f_y) + (f'', g_x + \beta g_y) \geq \tau N - \tau m + (m - \tau - d)(\mu \tau - 1) = \tau (\mu \tau - 1) + (\mu \tau - 1 - \tau)(m - \tau - d).
\]

The last assertion is a consequence of the fact that \( m - \tau - d \geq 0 \).

When the tangent cone of \( f \) is a multiple line, a general and useful lower bound for the multiplicity \( m \) is at the moment unknown. For our applications we only need such a bound only in three particular cases, namely when the germ \( \{ f = g = 0 \}_\text{red} \) contains a smooth curve, when contains two smooth curves and when contains a cusp; in these cases we can obtain useful bounds (although not very sharp) by a degenerations argument together with the cyclic covering trick.

Let \( f, g \) be as in the Set-Up \ref{set-up} and let \( U \) be a small open ball centered at \( p \) with holomorphic coordinates \( x, y \). Assume that both \( f, g \) converge to holomorphic functions \( f, g \in \mathcal{O}(U) \) and consider \( \eta_1 = dg, \eta_2 = df, h = G.C.D.(f_x, f_y), R = \text{div}(\eta_1 \wedge h^{-1} \eta_2) \subset U \). After a possible shrink of \( U \) we may assume that (cf. \cite{14}):

1. the form \( h^{-1} \eta_2 \) vanishes only at \( p \).
2. \( R_\text{red} \) is smooth outside \( p \) and intersects transversally \( \partial U \simeq S^3 \).
3. for generic \( a_1, b_1 \in \mathbb{C} \) the form \( \omega_1 = a_1 dx + b_1 dy \) satisfies:
   (a) The pull-back of \( \omega_1 \) to every irreducible component of \( R \setminus \{ p \} \) is everywhere nonzero.
   (b) \( R \cap \text{div}(\omega_1 \wedge h^{-1} \eta_2) = \{ p \} \)

Put \( \omega_2 = a_2 dx + b_2 dy, \omega_3 = a_3 dx + b_3 dy \) for generic \( a_2, a_3, b_2, b_3 \in \mathbb{C} \) and let \( s(x, y) \in \mathcal{O}(U) \) be a holomorphic function such that \( P = \text{div}(s) \) is smooth and \( P \cap R \cap \partial U = \emptyset \); note that \( P \) is a germ of curve of type 2.

Given an integer \( n > 1 \) define \( X \subset U \times \mathbb{C} \) by the equation \( z^n = s(x, y) \) (\( z \) is the coordinates in the second factor \( \mathbb{C} \)) and \( \varrho: X \to U \) the associated simple cyclic cover of order \( n \); denote finally with \( \pi: V_X = \mathbb{P}(\Omega_X^1) \to X \) the natural projection.

Setting \( Q = \text{div}(z) \subset X \) we have by the Hurwitz' formula:

\[
\text{div}(\varrho^* \omega_2 \wedge \varrho^* \omega_3) = (n - 1)Q, \quad \text{div}(\varrho^* \eta_1 \wedge \varrho^* h^{-1} \eta_2) = \varrho^* R + (n - 1)Q.
\]

Let \( \tilde{Q} \subset V_X = \mathbb{P}(T_X) \) be the set of the kernels of the natural morphism of bundles \( T_X \to \varrho^* T_U \), it is obvious that \( \pi(\tilde{Q}) = Q \) and \( \tilde{Q} \subset L_{\varrho^* \omega} \) for every 1-form \( \omega \) on \( U \). Finally note that \( H_{\varrho^* \eta_2} = L_{\varrho^* h^{-1} \eta_2} \).

**Lemma 5.4.** Let \( q \in Q \) and \( \omega \) be a 1-form on \( U \), then \( \varrho^* \omega(q) = 0 \) if and only if \( ds \wedge \omega(\varrho(q)) = 0 \).
Proof. Straightforward and left to the reader. □

Lemma 5.5. In the notation above \([L_{\phi^*\omega_2} \cap L_{\phi^*\omega_3}] = (n-1)\hat{Q}\) and the 1-cycle
\[[L_{\phi^*\eta_1} \cap L_{\phi^*h^{-1}\eta_2}] - [L_{\phi^*\omega_2} \cap L_{\phi^*\omega_3}]\]
is effective and supported on \(\pi^{-1}Q^{-1}(R)\).

Proof. The first equality is an immediate consequence of Lemma 5.4 and Formula 3.4. Again by 3.4 we have \(\pi_*[L_{\phi^*\eta_1} \cap L_{\phi^*h^{-1}\eta_2}] = \phi^*R + (n-1)Q\) and the conclusion follows by observing that \(\hat{Q} \subset L_{\phi^*\eta_1} \cap L_{\phi^*h^{-1}\eta_2}\).

Therefore the cycle of \(V_X\)
\[\Delta = L_{\phi^*\omega_1} \cap ([L_{\phi^*\eta_1} \cap L_{\phi^*h^{-1}\eta_2}] - [L_{\phi^*\omega_2} \cap L_{\phi^*\omega_3}])\]
is supported in a finite set of points; by general results about conservation of numbers in intersection theory (see e.g. [5, 10.2.2]) its degree is invariant under small perturbations of \(s(x,y)\) in the Banach space \(O(U)\).

Lemma 5.6. Assume \(s(x,y)\) is a generic small perturbation of a \(s_0(x,y)\) such that \(P_0 \cap R = \{p\}\), \(P_0 = \text{div}(s_0)\). Then \(\deg \Delta = nm_p + (n-1)P_0.R\).

Proof. Write
\[[L_{\phi^*\eta_1} \cap L_{\phi^*h^{-1}\eta_2}] - [L_{\phi^*\omega_2} \cap L_{\phi^*\omega_3}] = \sum n_i C_i + \sum m_j \pi^{-1}(q_j)\]
If \(\phi(q_j) = p\) then, since \(s(p) \neq 0\), the map \(\phi\) is an isomorphism in a neighbourhood of \(q_j\) and then \(m_j = m_p\). If \(q_j \in Q\) then, since \(ds \wedge h^{-1}\eta_2 \neq 0\) over \(S \cap R\), we have by 5.4 that \(h^{-1}\eta_2(q_j) \neq 0\) and then \(m_j = 0\). This proves that \(\sum m_j = nm_p\).

Consider now a point \(q \in L_{\phi^*\omega_1} \cap C_i\); since \(R \cap \text{div}(\omega_1 \wedge h^{-1}\eta_2) = \{p\}\) and \(\omega_1\) is generic, it must be \(q \in Q\). Let \(v\) be a local equation of the irreducible component of \(R\) passing through \(\phi(q)\), then \(s, v\) are local analytic coordinates centered at \(\phi(q)\); we can write:
\[
\omega_1 = \alpha ds + \beta dv, \quad h^{-1}\eta_2 = \gamma ds + \delta dv, \quad (\alpha \delta - \beta \gamma)(\phi(q)) \neq 0
\]

Then in a neighbourhood of \(q:\)
\[
\phi^*\omega_1 = \alpha z^{n-1}dz + \beta dv, \quad \phi^*h^{-1}\eta_2 = \gamma z^{n-1}dz + \delta dv, \quad (\alpha \delta - \beta \gamma)(q) \neq 0
\]
The same local computation made in the proof of 3.3 shows that the intersection product \(L_{\phi^*\omega_1} \cdot C_i\) is obtained by setting \(v = 0\), \(\frac{dv}{dz} = t\) and computing the intersection product, in the \(z,t\)-plane, of the curves of equations \(\alpha z^{n-1} + \beta t = 0\), \(\gamma z^{n-1} + \delta t\); since \((\alpha \delta - \beta \gamma)(q) \neq 0\) the intersection product is exactly \(n - 1\). This proves that \(L_{\phi^*\omega_1} \cdot \sum n_i C_i = (n - 1)P \cdot R = (n - 1)P_0.R\).

Lemma 5.7. Assume \(s(x,y) = x - \alpha y^n\) with \(\alpha \in \mathbb{C}\); if \(\alpha\) is generic or \(n >> 0\) then the multiplicity of \(\phi^*\eta_1 \wedge \phi^*h^{-1}\eta_2\) at the point \(q = \{z = y = 0\}\) is equal to \((P \cdot R)_p + n - 1\).

Proof. The divisor of \(\phi^*\eta_1 \wedge \phi^*h^{-1}\eta_2\) is equal to \(\phi^*R + (n-1)Q\) and therefore it is sufficient to prove that for every reduced irreducible germ of curve \(p \in D \subset R\) we have \(\text{mult}_q(\phi^*D) = (P \cdot D)_q\).

Let \(\phi(x,y)\) be the equation of \(D\), if \(x\) divides \(\phi\) then the equation of \(\phi^*(D) = z^n + \alpha y^n\) and its multiplicity is \(n = (P \cdot D)_p\). If \(x\) does not divide \(\phi\) then the equations of \(\phi^*(D)\) is
\[ \psi(z, y) = \phi(z^n + \alpha y^n, y) \]

If \( n >> 0 \) then the multiplicity of \( \psi \) is equal to the multiplicity of \( \phi(0, y) \) which is equal to \((P \cdot D)_p\).

In general the multiplicity of \( \psi \) is equal to the multiplicity of \( \psi(\beta y, y) = \phi((\beta^n + \alpha)y^n, y) \) for generic \( \beta \in \mathbb{C} \); if \( \alpha \) is generic, this multiplicity is equal to the multiplicity of \( \phi(\alpha y^n, y) \) which is equal to \((P \cdot D)_p\).

**Lemma 5.8.** In the Set-up 5.2 assume that the germ \( D_0 = \{ x = 0 \} \) is contained in \( \{ f = g = 0 \} \). Let \( n_0 \) be the multiplicity of \( D_0 \) in the divisor \( R \), then \( m_p \geq n_0 - 1 \) and equality holds only if \( R = n_0 D_0 \).

**Proof.** As a first step we take an integer \( n \) cover of order \( m \) then \( s \) is a cone of \( \mathbb{C} \) curves \( D \) and Lemma 5.10. In the Set-up 5.2, assume that the germ \( \{ y = 0 \} \) is contained in \( \{ f = g = 0 \} \). Let \( n_0 \) be the multiplicity of \( D_0 \) in the divisor \( R \), then \( m_p \geq n_0 - 1 \) and equality holds only if \( R = n_0 D_0 \).

\[
\deg(\Delta) \geq m_q \geq n(P \cdot R + (n - 1) - n) = nP \cdot R - n
\]

On the other hand, considering a small generic perturbation of \( P \) we have by 5.0 \( \deg(\Delta) = n m_p + (n - 1) S \cdot R \); therefore we have \( n m_p \geq R \cdot P - n \) and then we get \( m_p \geq n_0 - 1 + \frac{P \cdot R - n}{n} \).

**Lemma 5.9.** In the Set-up 5.2, assume that \( \{ f = g = 0 \} \) contains two smooth germs of curves \( D_1, D_2 \) with contact \( D_1 \cdot D_2 = n \) and let \( n_1, n_2 \) be respectively the multiplicities of \( D_1 \) and \( D_2 \) in the divisor \( R \). Then \( m_p \geq (n - 1)(n_1 + n_2) + \text{mult}_p(R) - (2n - 1) \).

**Proof.** The proof is similar to 5.8. If \( n = 1 \) this is an immediate consequence of 5.3 (with \( \tau = 1 \)).

Assume therefore \( n > 1 \), by Weierstrass’ preparation theorem we can find local holomorphic coordinates \( x, y \) at \( p \) such that the equations of \( D_1, D_2 \) are respectively \( x = y^n \) and \( x = -y^n \).

As in the proof of 5.8 we look for a lower bound of the degree of

\[
\Delta = L_{g^* \omega_1} \cap ( [L_{g^* \eta_1} \cap L_{g^* h^{-1} \eta_2}] - [L_{g^* \omega_2} \cap L_{g^* \omega_3}]),
\]

where \( g : X \to U \) is the simple cyclic cover of order \( n \) ramified over the smooth curve of equation \( s(x, y) = x - \alpha y^n \) for a generic \( \alpha \in \mathbb{C} \).

If \( q = \{ y = z = 0 \} \in X \) then, since \( g^* \omega_1(q) \neq 0 \), the degree of \( \Delta \) is \( \geq m_q \). Since \( x^2 - y^{2n} \) divides \( f \) we have that \((z^n + \alpha y^n)^2 - y^{2n} \) divides \( g^* f \) and then by Lemma 5.3 (with \( \tau = 2n - 1 \)) and Lemma 5.2 we have:

\[
\deg(\Delta) \geq m_q \geq (2n - 1)(P \cdot R + (n - 1) - (2n - 1)) = (2n - 1)(P \cdot R - n)
\]

On the other hand, considering a small generic perturbation of \( P \), we have by 5.3 \( \deg(\Delta) = n m_p + (n - 1) P \cdot R \) and then \( m_p \geq P \cdot R - (2n - 1) \). It is now sufficient to observe that \( P \cdot R \geq (n - 1)(n_1 + n_2) + \text{mult}_p(R) \).

**Lemma 5.10.** In the Set-up 5.2, assume that the germ \( D_0 = \{ x^2 - y^3 = 0 \} \) is contained in \( \{ f = g = 0 \} \) and let \( n_0 \) be the multiplicity of \( D_0 \) in the divisor \( R \). Then \( m_p \geq \max(3, 3n_0 - 2) \geq 2n_0 \).
Proof. We prove first that \( m_\rho \geq 3 \). Let \( a \) be the multiplicity of \( D_0 \) in the divisor \( \{ f = 0 \} \). Write \( \phi = x^2 - y^3 \), \( f = \phi \tilde{f} \), \( g = \phi \tilde{g} \); we have \( h = \phi a^{-1} \tilde{h} \) with \( \tilde{h} = \text{GCD}(\tilde{f}_x, \tilde{f}_y) \).

\[
\frac{f_x + \beta f_y}{h} = a(\phi_x + \beta \phi_y) \tilde{f}_h + \phi \tilde{f}_x + \beta \tilde{f}_y
\]

\[
g_x + \beta g_y = (\phi_x + \beta \phi_y) \tilde{g} + \phi (\tilde{g}_x + \beta \tilde{g}_y)
\]

and then \( m_\rho = (h^{-1}(f_x + \beta f_y), g_x + \beta g_y) \geq (\phi_x + \beta \phi_y, \phi) = 3 \).

Consider now the double cover \( g : X \to U \) ramified over the smooth curve \( P \) of equations \( s(x, y) = y - \alpha x^2 = 0 \) for a generic \( \alpha \in \mathbb{C} \). Note that the pullback of \( D_0 \) is the union of two smooth germs \( D_1, D_2 \) with contact \( n = D_1 \cdot D_2 = 3 \) and tangent line \( q = g^{-1}(p) = \{ z = x = 0 \} \in X \).

Let’s denote by \( Q = \{ z = 0 \} \), \( R' = \text{div}(\phi^* \eta_1 \cap \phi^* h^{-1} \eta_2) = \phi^*(R) + Q \). According to 5.6 and \[5.1\] the degree of

\[
\Delta = L_{\phi^* \omega_1} \cap ([L_{\phi^* \eta_1} \cap L_{\phi^* h^{-1} \eta_2}] - [L_{\phi^* \omega_2} \cap L_{\phi^* \omega_3}])
\]

is equal to \( 2m_\rho + P \cdot R = 2m_\rho + \text{mult}(R') - 1 \).

On the other hand

\[
\Delta = L_{\phi^* \omega_1} \cap (m_q \pi^{-1}(q) + \sum n_i C_i)
\]

where, up to permutations of indices, \( n_1 = n_2 = n_0 \), \( \pi(C_1) = D_1 \), \( \pi(C_2) = D_2 \). By \[5.8\]
\[
m_q \geq 2(n_1 + n_2) + \text{mult}(R') - 5 = 4n_0 + P \cdot R - 4;
\]

Therefore:

\[
m_\rho \geq \frac{\deg \Delta - P \cdot R}{2} \geq \frac{m_q - P \cdot R}{2} + \frac{n_0}{2} (L_{\phi^* \omega_1} \cdot (C_1 + C_2)) \geq 2n_0 - 2 + \frac{n_0}{2} (L_{\phi^* \omega_1} \cdot (C_1 + C_2))
\]

And it is sufficient to prove that \( L_{\phi^* \omega_1} \cap C_i \cap \pi^{-1}(q) \neq \emptyset \) for \( i = 1, 2 \).

The local equation of \( D_i \) is \( x = \phi_i(z) \) for some convergent power series \( \phi_1, \phi_2 \) of multiplicity \( \geq 2 \) and therefore \( C_i \) is defined by \( x = \phi_i(z), dx = \phi'_i(z)dz \); since \( \phi^* \omega_1 = \gamma zdz + \delta dx \) we have that the point of coordinates \( x = z = dx = 0 \) belongs to \( C_i \cap L_{\phi^* \omega_1} \).

6 Proof of the main theorem

Using all the preparatory material of the previous section we are now able to prove the following:

**Theorem 6.1.** Let \( S \) be an algebraic surface with ample canonical bundle and let \( \alpha : S \to \text{Alb}(S) \) its Albanese map; assume that \( \alpha(S) \) is a surface, then

\[
2c_1^2(S) - c_2(S) \geq 0.
\]

and equality holds only if \( \alpha \) does not contains curves of type 0.

Note that, if \( 2c_1^2(S) - c_2(S) = 0 \), then Theorem \[6.1\] implies in particular that \( \Omega^1_S(K_S) \) is nef and then by \[1.4\] \( \alpha \) is a double cover of an abelian surface.

In the same notation of the beginning of Section 3 take \( \eta_1, \eta_2 \in H^0(\Omega^1_S) \) generic forms. We have:

\[
2c_1^2(S) - c_2(S) = (L + \pi^* K_S) \cdot L_{\eta_1} \cdot L_{\eta_2} = (L + \pi^* K_S) \cdot L_{\eta_1} \cdot H_{\eta_2} + KS \cdot E
\]
and then, since $K_S$ is ample,

$$2c_1^2(S) - c_2(S) \geq (L + \pi^*K_S) \cdot L_{\eta_1} \cdot H_{\eta_2}.$$ 

Assume $[L_{\eta_1} \cap H_{\eta_2}] = \sum_i n_i C_i + \sum_j m_j \pi^{-1}(p_j)$; we then set for every $s = 0, 1, 2$

$$R_s = \sum n_i \pi(C_i), \quad \pi(C_i) \text{ of type } s.$$

Recall that $\text{div}(\eta_1 \wedge \eta_2) = E + R_0 + R_1 + R_2$; note that, since $\text{div}(\eta_1 \wedge \eta_2) - 2E$ is effective, also $R_0 - E$ is an effective divisor.

Define also for $s = 1, 2$:

$$A_s = \sum n_i (L + \pi^*K_S) \cdot C_i, \quad \pi(C_i) \text{ of type } s$$

while for every effective subdivisor $F$ of $R_0 + E$ we define:

$$A_F = \sum n_i (L + \pi^*K_S) \cdot C_i + \sum m_j, \quad \pi(C_i) \subset \text{Supp}(F), \quad p_j \in \text{Supp}(F)$$

Let $\sigma$ be the number of connected components of $R_0 + E$; then we may write $R_0 + E = F_1 + \ldots + F_\sigma$, where the $F_j$’s are the maximal connected effective subdivisors of $R_0 + E$. It is clear that:

$$2c_1^2(S) - c_2(S) \geq (L + \pi^*K_S) \cdot L_{\eta_1} \cdot H_{\eta_2} \geq A_1 + A_2 + \sum_{j=1}^\sigma A_{F_j}.$$ 

Therefore the Theorem 6.1 follows from the following 6.2 and 6.3.

**Lemma 6.2.** In the above notation $A_1 + A_2 \geq \sigma$ and equality holds only if $\sigma = 0$.

**Lemma 6.3.** In the above notation $A_{F_j} \geq -1$ for every $j = 1, \ldots, \sigma$ and equality holds only if every component of $F_j$ is a smooth rational curve with selfintersection $-3$.

**Proof of 6.2.** Write $R_1 + R_2 = \sum_{i=1}^r n_i D_i$, with the $D_i$’s reduced and irreducible. Then by 1.2 and 1.3 we get:

$$A_1 + A_2 \geq \sum_{i=1}^r n_i D_i \cdot (K_S - D_i) \geq \sum_{i=1}^r D_i \cdot (K_S - n_i D_i) \geq \sum_{i=1}^r \sum_{j=1}^\sigma D_i \cdot F_j$$

Since every $F_j$ meets at least one $D_i$ we get $A_0 + A_1 \geq \sigma$. If equality holds then $n_i D_i \cdot (K_S - D_i) = D_i \cdot (K_S - n_i D_i)$ for every $i$ and then $R_0 + R_1$ is reduced; in this case we obtain

$$\sigma = A_1 + A_2 \geq \sum_{i=1}^r \sum_{j=1}^\sigma D_i \cdot F_j = \sum_{j=1}^\sigma (R_1 + R_2) \cdot F_j = \sum_{j=1}^\sigma (K_S - F_j) \cdot F_j \geq 2\sigma$$

which implies $\sigma = 0$.

**Proof of 6.3.** Let $F$ be a fixed connected component, with the reduced structure, of $R_0 + E$ and let $f, g$ be holomorphic functions defined in a neighbourhood of $F$ such that they vanish over $F$ and $\eta_1 = df$, $\eta_2 = dg$.

It is a straightforward consequence of 1.2 and 1.3 that, in the notation above, if $D = \pi(C)$ is of type 0 and $(L + \pi^*K_S) \cdot C < 0$ then $D$ is a rational curve with at most nodes and cusps as singularities and belongs to one of the 5 types described in the following:
Table 6.4.

| Type | $D^2$ | $K_S \cdot D$ | Singularities | $(L + \pi^*K_S) \cdot C$ |
|------|-------|---------------|---------------|-----------------|
| (i)  | -3    | 1             | $\emptyset$   | -1              |
| (ii) | -1    | 1             | 1 node        | -1              |
| (iii)| -1    | 1             | 1 cusp        | -2              |
| (iv) | -2    | 2             | 1 cusp        | -1              |
| (v)  | -1    | 3             | 2 cusps       | -1              |

We shall call for simplicity “bad curve” a curve listed in the Table 6.4. The proof follows immediately from the following Lemmas 6.5 and 6.6.

**Lemma 6.5.** Let $p \in S$ be a singular point of a bad curve $D_0 \subset F$ and let $D_0, \ldots, D_r$ be the bad curves passing through $p$. Then, if $n_i$ is the multiplicity of $D_i$ in $R_0$ and $C_i \subset V_{\mathcal{P}(\Omega^1_S)}$ is the tautological lifting of $D_i$ we have:

$$m_p + \sum_{i=1}^r n_i(L + \pi^*K_S) \cdot C_i \geq 0.$$  

**Proof.** Consider first the case $r = 0$, then, according to 5.3, $m_p \geq 2n_0 - 1 \geq n_0$ whenever $D_0$ is bad of type (ii), while according to 5.10, $m_p \geq \max(3, 3n_0 - 2) \geq 2n_0$ whenever $D_0$ is bad of type (iii),(iv) or (v). In all cases a direct computation prove the assertion.

If $r > 0$ then, by Mumford’s theorem, the curve $D_0$ must be of type (iv), the curves $D_1, \ldots, D_r$ of type (i) and $D_0 \cdot D_i = 2$ for every $i = 1, \ldots, r$. Moreover if $r \geq 2$ we would have $(D_0 + D_1 + D_2)^2 \geq 1$ which is a contradiction: therefore $r = 1$. The tangent cone of $D_0 + D_1$ at the point $p$ contains at least 2 irreducible components and then by Lemma 5.3 (with $\tau = 1$) $m_p \geq \mult_p(R_0) - 1 \geq 2n_0 + n_1 - 1 \geq n_0 + n_1$; therefore

$$m_p + \sum_{i=1}^r n_i(L + \pi^*K_S) \cdot C_i \geq m_p - n_0 - n_1 \geq 0.$$  

**Lemma 6.6.** Let $D_1, \ldots, D_r$, $r \geq 0$ be the bad curves of type (i) contained in $F$ which do not contain any singular point of a bad curve and let $F'$ be a connected component of $D_1 \cup \ldots \cup D_r$. Then $A_{F'} \geq -1$ and equality holds only if $F' = F$.

**Proof.** Denote by $n_i$ the multiplicity of $D_i$ in $R_0$. If $r = 0$ there is nothing to prove, so assume $r > 0$ and, up to permutation of indices, $F' = D_{i_1} \cup \ldots \cup D_{i_s}$, $s \leq r$. Again by Mumford’s theorem $D_i \cdot D_j \leq 2$ for every $i, j = 1, \ldots, s$; denote by $\Delta = F' \cap (F - F')$, we want to prove that $A_{F'} \geq -1$ and equality holds only if $\Delta = \emptyset$; note that $\Delta$ is contained in the singular locus of $F$ and then $m_p > 0$ for every $p \in \Delta$.

Assume first that $s > 1$. Let $p$ be a singular point of $F'$ and let $D_{i_1}, \ldots, D_{i_h}$ be the components of $F'$ passing through $p$: there are two possible cases, according to the behavior of $F'$ at the point $p$.

If the tangent cone of $F'$ at $p$ contains at least two distinct irreducible components then by
Lemma 5.3 (with $\tau = 1$) we have $m_p \geq n_{j_1} + \ldots + n_{j_k} - 1$ and equality holds only if $p \not\in \Delta$. If the tangent cone of $F'$ at $p$ contains only one irreducible component, then by Mumford’s theorem we must have $h = 2$ and $D_{j_1} \cdot D_{j_2} = 2$. According to 5.3 $m_p \geq 2(n_{j_1} + n_{j_2}) - 3 \geq n_{j_1} + n_{j_2} - 1$ and equality holds only if $p \not\in \Delta$. An easy combinatorics argument over the dual intersection graph of $F'$ proves the statement in the case $s > 1$.

Assume now $s = 1$. If $F' = F$ and $n_1 = 1$ there is nothing to prove; otherwise there exists a point $p \in D_1$ which is singular for both $\{f = 0\}_{red}$ and $\{g = 0\}$. In fact, every point of $\Delta$ satisfies this condition, while if $\Delta = \emptyset$ we argue as follows.

Consider the divisors $A = \{f = 0\}$, $B = \{g = 0\}$ in a neighbourhood $U$ of $D_1$ and let $e$ be the multiplicity of $D_1$ in $A$. By assumption $n_2$ is generic and then the divisor $B - eD_1$ is effective. Since $(A - eD_1) \cdot D_1 = 3e$, if $e > 1$ then every point of $(A - eD_1) \cap D_1$ satisfies the condition. It remains to prove that, if $e = 1$ and $(B - D_1) \cap (A - D_1) \cap D_1 = \emptyset$ then $n_1 = 1$. In fact by the theorem of Bertini-Sard we can find a point $o \in D_1$, local holomorphic coordinates $x,y$ at $o$ and a constant $\gamma \in \mathbb{C}$ such that locally we can write $g = x$ and $f + \gamma g = xy$. Therefore $dg \wedge df = (dx \wedge dy)$ proving that $n_1 = 1$.

According to 5.8 we have $m_p \geq n_1 - 1$ and equality holds only if $p \not\in \Delta$. This concludes the proof. $\square$

7 Examples, remarks and open problems

We have shown that the inequality $K^2 \geq 4\chi$ is sharp only for surfaces with irregularity $q = 2$; it is then natural to ask for a better inequality when $q > 2$. Consider first the following:

Example 7.1. Double covers: Let $X$ be a smooth algebraic surface and let $L$ be an ample line bundle on $X$; assume $K_X + L$ ample and the linear system $|2L|$ base point free. Then for every smooth divisor $D \in |2L|$ we can consider the double cover $\mathbb{S}^2 \to X$ ramified over $D$ such that $\pi_\ast O_S = O_X \oplus L^{-1}$. According to the Hurwitz formula $S$ is a surface with ample canonical bundle $K_S = \pi_\ast (K_X + L)$; by Kodaira vanishing $q(S) = q(X) + h^1(L^{-1}) = q(X)$ and a simple computation gives:

$$K_S^2 - 4\chi(O_S) = 2(K_X^2 - 4\chi(O_X)) + 2K_X \cdot L$$

We apply this construction in the following case: $X = C \times E$ with $C,E$ are smooth curves of respective genus $g(E) = 1$, $g(C) = g \geq 1$: we have $q(X) = q = g + 1$, $K_X^2 = \chi(O_X) = 0$. Let $\alpha: X \to C$, $\beta: X \to E$ be the projection, $e \in E$, $c \in C$ and $L = \alpha^\ast(ne) + \beta^\ast(e)$ with $n >> 0$. The double cover $S$ constructed as above has invariants:

$$K_S^2 - 4\chi(O_S) = 4(q - 2) \quad K_S^2 = 8(q - 2) + 4n = 4p_g(S) - 4.$$ 

Note that $K_S^2 - 4\chi(O_S)$, $K_S^2 - 4p_g(S)$ are constant, while $K_S^2$ is unbounded.

Example 7.2. Product of curves: If $C_1, C_2$ are smooth curves of respective genus $g_1, g_2 \geq 2$ and $S = C_1 \times C_2$ we have $q(S) = q = g_1 + g_2$, $K_S^2 = 8(g_1 - 1)(g_2 - 1)$ and $K_S^2 - 4\chi = 4(g_1 - 1)(g_2 - 1)$. If $g_1 = 2$ then $K_S^2 = 4\chi(O_S) + 4(q - 3) = 4p_g - 8$.

Example 7.3. If $S$ is the symmetric square of a curve of genus 3 we have $K_S^2 = 6$, $q(S) = 3, \chi(O_S) = 1$ and then $K_S^2 = 4\chi(O_S) + 2$. Conversely, according to 3.22], every surface with $p_g = q = 3$ and $K^2 = 6$ is the symmetric product of a curve of genus 3.

This examples suggest the validity of the following:
Conjecture 7.4. If $S$ is a minimal surface of general type with Albanese dimension 2 and $q(S) = 3$ then $K^2_S \geq 4\chi(O_S) + 2 = 4p_g(S) - 6$ and equality holds if and only if $S$ is the symmetric product of a curve of genus 3.

Conjecture 7.5. If $S$ is a minimal surface of general type with Albanese dimension 2 and $q(S) \geq 4$ then $K^2_S \geq 4p_g(S) - 8$. Moreover the equality holds if and only if $S$ is a product of a curve of genus 2 and a curve of genus $\geq 2$.

Conjecture 7.5 is true if one of the following condition is satisfied:

- $S$ is fibred over a curve of genus $\geq 2$.
- $K_S \cdot C \geq 2$ for every smooth rational curve $C \subset S$ and $K^2_S \geq 36(q - 3)$ (M. Manetti, unpublished).

Note that 7.4 and 7.5 are false if the surface is not of general type.

Problem 7.6. In the Set-up 5.3 let $\frac{f_xg_y - f_yg_x}{h} = J_1J_2$ be a decomposition such that every irreducible factor of $J_1$ divides both $f$ and $g$. Is it true that $m_p \geq \text{mult}(J_1) - 1$?

Since $K^2, \chi$ are topological invariants and the Albanese dimension is stable under deformations, it could be a good idea to replace $S$ with a surface $S'$ sufficiently near, in the sense of moduli, to $S$ and try to find such a $S'$ with ample canonical bundle (recall that surfaces with ample canonical bundle form a Zariski open subset in the moduli space of surfaces of general type). This argument gives additional evidences to the validity of Conjecture 7.4 but cannot be used to prove it. In fact, given a minimal surface of general type $S$, it is not always possible to deform $S$ to a surface with ample canonical bundle (see [13, 3.15] for several nice examples and recipes).

Since $K^2/\chi$ is invariant under unramified coverings, one can ask if, in the case $q(S) \geq 2$, there exists an unramified cover $Y \to X$ of the canonical model $X = S_{\text{can}}$ such that $Y$ is smoothable. As before, some of the generalized Kas' surfaces ([4, 2.5]) give examples where the above question has negative answer.

On the positive side, a surface $S$ with $K^2_S < 4\chi O_S$ has a number of moduli greater of equal to $h^1(T_S) - h^2(T_S) = 10\chi(O_S) - 2K^2_S > 2\chi(O_S)$ and then every potential counterexample to the Severi’s conjecture can be deformed with a large number of independent parameters.

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