The alternating PBW basis for the positive part of $U_q(\hat{sl}_2)$

Paul Terwilliger

Abstract

The positive part $U^+_q$ of $U_q(\hat{sl}_2)$ has a presentation with two generators $A, B$ that satisfy the cubic $q$-Serre relations. We introduce a PBW basis for $U^+_q$, said to be alternating. Each element of this PBW basis commutes with exactly one of $A, B, qAB - q^{-1}BA$. This gives three types of PBW basis elements; the elements of each type mutually commute. We interpret the alternating PBW basis in terms of a $q$-shuffle algebra associated with affine $sl_2$. We show how the alternating PBW basis is related to the PBW basis for $U^+_q$ found by Damiani in 1993.

Keywords. $q$-Onsager algebra, $q$-shuffle algebra, PBW basis.

2010 Mathematics Subject Classification. Primary: 17B37. Secondary 05E15.

1 Introduction

This paper is motivated by a recent development in statistical mechanics, concerning the $q$-Onsager algebra $O_q$ [11, 12]. In [3] Baseilhac and Koizumi introduced a current algebra $A_q$ for $O_q$, in order to solve boundary integrable systems with hidden symmetries. In [5, Definition 3.1] Baseilhac and Shigechi give a presentation of $A_q$ by generators and relations. The generators are denoted $\{W_{-k}\}_{k=0}^{\infty}$, $\{W_{k+1}\}_{k=0}^{\infty}$, $\{G_{k+1}\}_{k=0}^{\infty}$, $\{\tilde{G}_{k+1}\}_{k=0}^{\infty}$. The relations involve $q$ and a nonzero scalar parameter $\rho$. In an attempt to understand $A_q$ we considered the limiting case $\rho = 0$. For this value of $\rho$ the algebra $O_q$ gets replaced by an algebra $U^+_q$ called the positive part of the quantum group $U_q(\hat{sl}_2)$. The algebra $U^+_q$ has a presentation with two generators $A, B$ that satisfy the cubic $q$-Serre relations; see Definition 2.2 below. In this paper we display some elements in $U^+_q$, denoted

$$\{W_{-k}\}_{k=0}^{\infty}, \ {W_{k+1}\}_{k=0}^{\infty}, \ {G_{k+1}\}_{k=0}^{\infty}, \ {\tilde{G}_{k+1}\}_{k=0}^{\infty},$$

(1)

that satisfy the $\rho = 0$ analog of the relations in [5, Definition 3.1]. These relations are given in Propositions 5.7, 5.10 below; see Propositions 5.11, 6.3, 8.1 for some additional relations.

We defined the elements (1) and obtained the above relations in the following way. Start with the free algebra $V$ on two generators $x, y$. The standard (linear) basis for $V$ consists of the words in $x, y$. In [10,11] M. Rosso introduced an algebra structure on $V$, called a $q$-shuffle
For $u, v \in \{x, y\}$ their $q$-shuffle product is $u \star v = uv + q^{\langle u, v \rangle}vu$, where $\langle u, v \rangle = 2$ (resp. $\langle u, v \rangle = -2$) if $u = v$ (resp. $u \neq v$). Rosso gave an injective algebra homomorphism $\natural$ from $U_q^+$ into the $q$-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$. Let $U$ denote the image of $U_q^+$ under $\natural$. A word $v_1v_2\cdots v_n$ in $\mathbb{V}$ is said to be alternating whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. We name the alternating words as follows:

$$W_0 = x, \quad W_{-1} = xyx, \quad W_{-2} = xyxyx, \quad \ldots$$

$$W_1 = y, \quad W_2 = yxy, \quad W_3 = yxyxy, \quad \ldots$$

$$G_1 = yx, \quad G_2 = yxyx, \quad G_3 = yxyxyx, \quad \ldots$$

$$\tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \ldots$$

We describe the $q$-shuffle product of every pair of alternating words. Using this description we show that the alternating words satisfy the relations mentioned below (1). Using these relations we show that $U$ contains the alternating words.

We use the alternating words to obtain some PBW bases for $U$. For instance, we show that the elements $\{W_{-k}\}_{k=0}^{\infty}$, $\{W_{k+1}\}_{k=0}^{\infty}$, $\{\tilde{G}_{k+1}\}_{k=0}^{\infty}$ (in appropriate linear order) give a PBW basis for $U$, said to be alternating. The elements $\{W_{-k}\}_{k=0}^{\infty}$, $\{W_{k+1}\}_{k=0}^{\infty}$, $\{G_{k+1}\}_{k=0}^{\infty}$ give a similar PBW basis for $U$. We describe how the alternating PBW basis is related to the PBW basis for $U_q^+$ found by Damiani in [6].

This paper is organized as follows. In Section 2, we recall the notion of a PBW basis, and describe the one for $U_q^+$ found by Damiani. In Section 3 we obtain some slightly technical facts about $U_q^+$ that will be used later in the paper. In Section 4 we describe the algebra homomorphism $\natural$ from $U_q^+$ into the $q$-shuffle algebra $\mathbb{V}$. In Section 5 we introduce the alternating words in $\mathbb{V}$, and obtain some relations involving these words. In Sections 6, 7 we use these relations to obtain a commutator relation for every pair of alternating words. In Section 8 we obtain some additional relations for the alternating words, which get used to show that the alternating words are contained in $U$. In Section 9 the alternating words are related using generating functions. In Section 10 we use the alternating words to obtain some PBW bases for $U$, including the alternating PBW basis. In Section 11 we show how the alternating PBW basis is related to the Damiani PBW basis. In Section 12 we give some slightly technical comments about some relations in Sections 6, 8. In Section 13 we give some open problems. In Appendices A, B we present the commutator relations in an alternative way. In Appendix C we give some examples of the commutator relations.

## 2 The algebra $U_q^+$

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote a field. We will be discussing vector spaces, tensor products, and algebras. Each vector space and tensor product discussed is over $\mathbb{F}$. Each algebra discussed is associative, over $\mathbb{F}$, and has a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra.
Definition 2.1. (See [6, p. 299].) Let $\mathcal{A}$ denote an algebra. A Poincaré-Birkhoff-Witt (or PBW) basis for $\mathcal{A}$ consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on $\Omega$, such that the following is a basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \leq a_2 \leq \cdots \leq a_n.$$ 

We interpret the empty product as the multiplicative identity in $\mathcal{A}$.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$ 

For elements $X, Y$ in any algebra, define their commutator and $q$-commutator by

$$[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1}YX.$$ 

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3.$$ 

Definition 2.2. (See [9, Corollary 3.2.6].) Define the algebra $U_q^+$ by generators $A, B$ and relations

$$[A, [A, [A, B]_q]_{q^{-1}}] = 0, \quad [B, [B, [B, A]_q]_{q^{-1}}] = 0. \quad (2)$$ 

We call $U_q^+$ the positive part of $U_q(\widehat{\mathfrak{sl}_2})$. The relations (2) are called the $q$-Serre relations.

For an algebra $\mathcal{A}$, by an automorphism of $\mathcal{A}$ we mean an algebra isomorphism $\mathcal{A} \to \mathcal{A}$. By an antiautomorphism of $\mathcal{A}$, we mean an $F$-linear bijection $\gamma: \mathcal{A} \to \mathcal{A}$ such that $(ab)\gamma = b\gamma a\gamma$ for all $a, b \in \mathcal{A}$.

Lemma 2.3. There exists a unique automorphism $\sigma$ of $U_q^+$ that swaps $A, B$. There exists a unique antiautomorphism $S$ of $U_q^+$ that fixes each of $A, B$.

In [6, p. 299] Damiani obtained a PBW basis for $U_q^+$, involving some elements

$$\{E_{n\delta + \alpha_0}\}_{n=0}^\infty, \quad \{E_{n\delta + \alpha_1}\}_{n=0}^\infty, \quad \{E_{n\delta}\}_{n=1}^\infty. \quad (3)$$ 

These elements are recursively defined as follows:

$$E_{\alpha_0} = A, \quad E_{\alpha_1} = B, \quad E_\delta = q^{-2}BA - AB, \quad (4)$$ 

and for $n \geq 1$,

$$E_{n\delta + \alpha_0} = \frac{[E_\delta, E_{(n-1)\delta + \alpha_0}]}{q + q^{-1}}, \quad E_{n\delta + \alpha_1} = \frac{[E_{(n-1)\delta + \alpha_1}, E_\delta]}{q + q^{-1}}, \quad (5)$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta + \alpha_1}A - AE_{(n-1)\delta + \alpha_1}. \quad (6)$$
Proposition 2.4. (See [6, p. 308].) The elements (3) in the linear order

\[ E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1} \]

form a PBW basis for \( U_q^+ \).

The PBW basis elements (3) are known to satisfy certain relations [6, Section 4]. For instance the elements \( \{E_{n\delta}\}_{n=1}^{\infty} \) mutually commute [6, p. 307].

Next we describe a grading for the algebra \( U_q^+ \). Note that the \( q \)-Serre relations are homogeneous in both \( A \) and \( B \). Therefore the algebra \( U_q^+ \) has a \( \mathbb{N}^2 \)-grading for which \( A \) and \( B \) are homogeneous, with degrees (1, 0) and (0, 1) respectively. For this grading the PBW basis elements (3) are homogeneous with degrees shown below:

| PBW basis element | degree          |
|-------------------|-----------------|
| \( E_{n\delta+\alpha_0} \) | \((n+1, n)\)    |
| \( E_{n\delta+\alpha_1} \) | \((n, n+1)\)   |
| \( E_{n\delta} \)     | \((n, n)\)     |

Using this data and Proposition 2.4, one can obtain the dimension of each homogeneous component for the \( \mathbb{N}^2 \)-grading of \( U_q^+ \). This calculation is elementary but useful later in the paper, so we will go through it in detail. This will be done in the next section.

3 The dimensions of the \( U_q^+ \) homogeneous components

In this section we do two things. First we compute the dimension of each homogeneous component for the \( \mathbb{N}^2 \)-grading of \( U_q^+ \). Then we use this data to characterize a certain type of PBW basis for \( U_q^+ \); this characterization will be invoked later in the paper to obtain the alternating PBW basis.

Definition 3.1. Let the set \( \mathcal{R} \) consist of the ordered pairs \((r, s)\) \( \in \mathbb{N}^2 \) such that \(|r - s| \leq 1\) and \((r, s) \neq (0, 0)\).

Definition 3.2. Define a generating function in two commuting indeterminates \( \lambda, \mu \):

\[
\Phi(\lambda, \mu) = \prod_{(r, s) \in \mathcal{R}} \frac{1}{1 - \lambda^r \mu^s}
\]

In more detail,

\[
\Phi(\lambda, \mu) = \prod_{\ell=1}^{\infty} \frac{1}{1 - \lambda^\ell \mu^{\ell-1}} \frac{1}{1 - \lambda^\ell \mu^\ell} \frac{1}{1 - \lambda^{\ell-1} \mu^\ell}.
\]

Definition 3.3. For \((i, j)\) \( \in \mathbb{N}^2 \) let \( d_{i,j} \) denote the coefficient of \( \lambda^i \mu^j \) in \( \Phi(\lambda, \mu) \). Thus

\[
\Phi(\lambda, \mu) = \prod_{(r, s) \in \mathcal{R}} (1 + \lambda^r \mu^s + \lambda^{2r} \mu^{2s} + \cdots) = \sum_{(i, j) \in \mathbb{N}^2} d_{i,j} \lambda^i \mu^j.
\]

Note that \( d_{i,j} \in \mathbb{N} \) for \((i, j)\) \( \in \mathbb{N}^2 \).
Example 3.4. For $0 \leq i, j \leq 6$ the number $d_{i,j}$ is given in the $(i, j)$-entry of the matrix below:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 3 & 3 & 3 & 3 \\
1 & 3 & 6 & 8 & 9 & 9 & 9 \\
1 & 3 & 8 & 14 & 19 & 21 & 22 \\
1 & 3 & 9 & 19 & 32 & 42 & 48 \\
1 & 3 & 9 & 21 & 42 & 66 & 87 \\
1 & 3 & 9 & 22 & 48 & 87 & 134
\end{pmatrix}
$$

Definition 3.5. A subset $\Omega \subseteq U^+_q$ is called feasible whenever

(i) each element of $\Omega$ is homogeneous with respect to the $\mathbb{N}^2$-grading of $U^+_q$;

(ii) there is a bijection $\Omega \rightarrow \mathcal{R}$ that sends each element of $\Omega$ to its degree.

Let $\Omega$ denote a feasible subset of $U^+_q$ and let $<$ denote a linear order on $\Omega$. Consider the following vectors in $U^+_q$:

$$a_1 a_2 \cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \leq a_2 \leq \cdots \leq a_n. \quad (7)$$

Note that each vector in (7) is homogeneous with respect to the $\mathbb{N}^2$-grading of $U^+_q$.

Lemma 3.6. With the above notation, for $(i, j) \in \mathbb{N}^2$ the number of vectors in (7) that have degree $(i, j)$ is equal to the integer $d_{i,j}$ from Definition 3.3.

Proof. For $a \in \Omega$ with degree $(r, s)$ the contribution of $a$ to $\Phi(\lambda, \mu)$ is

$$\frac{1}{1 - \lambda^r \mu^s} = 1 + \lambda^r \mu^s + \lambda^{2r} \mu^{2s} + \cdots. \quad \square$$

Corollary 3.7. For $(i, j) \in \mathbb{N}^2$ the $(i, j)$-homogeneous component of $U^+_q$ has dimension $d_{i,j}$.

Proof. Let the set $\Omega$ consist of the elements (3). The set $\Omega$ is feasible by Definition 3.5 and the table below Proposition 2.4. Endow $\Omega$ with the linear order $<$ from Proposition 2.4. By Definition 2.1 and Proposition 2.4 the vectors (7) form a basis for the vector space $U^+_q$. We mentioned earlier that every vector in (7) is homogeneous with respect to the $\mathbb{N}^2$-grading of $U^+_q$. So for $(i, j) \in \mathbb{N}^2$ the set of vectors in (7) that have degree $(i, j)$ is a basis for the $(i, j)$-homogeneous component of $U^+_q$. The result follows in view of Lemma 3.6. \quad \square

Lemma 3.8. Let $\Omega$ denote a feasible subset of $U^+_q$ and let $<$ denote a linear order on $\Omega$. Then for $(i, j) \in \mathbb{N}^2$ the following (i)–(iii) are equivalent:

(i) the vectors in (7) that have degree $(i, j)$ span the $(i, j)$-homogeneous component of $U^+_q$;

(ii) the vectors in (7) that have degree $(i, j)$ are linearly independent;
Proposition 3.9. Let $\Omega$ denote a feasible subset of $U_q^+$ and let $<$ denote a linear order on $\Omega$. Then the following (i)–(v) are equivalent:

(i) the equivalent conditions of Lemma 3.8 hold for all $(i, j) \in \mathbb{N}^2$;
(ii) the vectors (7) span $U_q^+$;
(iii) the vectors (7) are linearly independent;
(iv) the vectors (7) form a basis for $U_q^+$;
(v) $\Omega$ in order $<$ forms a PBW basis for $U_q^+$.

Proof. Condition (i) is equivalent to each of (ii), (iii), (iv) since $U_q^+$ is a direct sum of its homogeneous components. Conditions (iv), (v) are equivalent by Definition 2.4. In Section 10 we will use Proposition 3.9 to obtain the alternating PBW basis for $U_q^+$.

4 Embedding $U_q^+$ into a $q$-shuffle algebra

In this section we recall an embedding, due to Rosso [10, 11], of $U_q^+$ into a $q$-shuffle algebra. For this $q$-shuffle algebra the underlying vector space is a free algebra on two generators. We begin by describing this free algebra.

Let $x, y$ denote noncommuting indeterminates, and let $V$ denote the free algebra with generators $x, y$. By a letter in $V$ we mean $x$ or $y$. For $n \in \mathbb{N}$, a word of length $n$ in $V$ is a product of letters $v_1 v_2 \cdots v_n$. We interpret the word of length zero to be the multiplicative identity in $V$; this word is called trivial and denoted by 1. The vector space $V$ has a basis consisting of its words; this basis is called standard.

We mention some symmetries of the free algebra $V$.

Lemma 4.1. There exists a unique automorphism $\sigma$ of the free algebra $V$ that swaps $x, y$. There exists a unique antiautomorphism $S$ of the free algebra $V$ that fixes each of $x, y$.

The free algebra $V$ has a $\mathbb{N}^2$-grading for which $x$ and $y$ are homogeneous, with degrees $(1, 0)$ and $(0, 1)$ respectively. For $(i, j) \in \mathbb{N}^2$ let $V_{i,j}$ denote the $(i, j)$-homogeneous component. These homogeneous components are described as follows. Let $w = v_1 v_2 \cdots v_n$ denote a word in $V$. The $x$-degree of $w$ is the cardinality of the set $\{i | 1 \leq i \leq n, v_i = x\}$. The $y$-degree of $w$ is similarly defined. For $(i, j) \in \mathbb{N}^2$ the subspace $V_{i,j}$ has a basis consisting of the words in $V$ that have $x$-degree $i$ and $y$-degree $j$. The dimension of $V_{i,j}$ is equal to the binomial coefficient $\binom{i+j}{i}$. 

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We have been discussing the free algebra $V$. There is another algebra structure on $V$, called the $q$-shuffle algebra. This algebra was introduced by Rosso [10, 11] and described further by Green [7]. We will adopt the approach of [7], which is well suited to our purpose. The $q$-shuffle product is denoted by $\star$. To describe this product, we start with some special cases. We have $1 \star v = v \star 1 = v$ for $v \in V$. For letters $u, v$ we have

$$u \star v = uv + vuq^{\langle u, v \rangle}$$

where

$$\begin{array}{c|cc}
\langle, \rangle & x & y \\
\hline
x & 2 & -2 \\
y & -2 & 2 \\
\end{array}$$

So

$$x \star y = xy + q^{-2}yx, \quad y \star x = yx + q^{-2}xy,$$

$$x \star x = (1 + q^2)xx, \quad y \star y = (1 + q^2)yy.$$  \hfill (8)

For a letter $u$ and a nontrivial word $v = v_1v_2 \cdots v_n$,

$$u \star v = \sum_{i=0}^{n} v_1 \cdots v_iuv_{i+1} \cdots v_nq^{\langle v_1, u \rangle + \langle v_2, u \rangle + \cdots + \langle v_n, u \rangle},$$  \hfill (10)

$$v \star u = \sum_{i=0}^{n} v_1 \cdots v_iuv_{i+1} \cdots v_nq^{\langle v_n, u \rangle + \langle v_{n-1}, u \rangle + \cdots + \langle v_1, u \rangle}.$$  \hfill (11)

For example

$$x \star (yyy) = xyy + q^{-2}yxy + q^{-4}yyxy + q^{-6}yyyx + q^{-8}yxyy + q^{-10}xyyx + q^{-12}yxxy,$$

$$(xy) \star y = xyxy + (1 + q^2)xyyx + q^{-2}yxyx.$$  \hfill (9)

For nontrivial words $u = u_1u_2 \cdots u_r$ and $v = v_1v_2 \cdots v_s$ in $V$,

$$u \star v = u_1((u_2 \cdots u_r) \star v) + v_1(u \star (v_2 \cdots v_s))q^{\langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle + \cdots + \langle u_r, v_1 \rangle},$$  \hfill (12)

$$u \star v = (u \star (v_1 \cdots v_{s-1}))v_s + ((u_1 \cdots u_{r-1}) \star v)u_rq^{\langle u_r, v_1 \rangle + \langle u_r, v_2 \rangle + \cdots + \langle u_r, v_s \rangle}.$$  \hfill (13)

For example

$$(xx) \star (yyy) = xxyy + q^{-2}xxyy + q^{-4}xxyy + q^{-6}xxxy + q^{-8}xxxy + q^{-10}xxx + q^{-12}xxxyy,$$

$$(xy) \star (xxy) = xxyy + xxyy + [2]q^2 xxyy + [3]q^2 xxyy.$$  \hfill (14)

The map $\sigma$ from Lemma 4.1 is an automorphism of the $q$-shuffle algebra $V$. The map $S$ from Lemma 4.1 is an antiautomorphism of the $q$-shuffle algebra $V$. Below Lemma 4.1 we mentioned an $\mathbb{N}^2$-grading for the free algebra $V$. This is also an $\mathbb{N}^2$-grading for the $q$-shuffle algebra $V$. 7
Definition 4.2. Let $U$ denote the subalgebra of the $q$-shuffle algebra $V$ that is generated by $x, y$.

Shortly we will see that $U \neq V$.

The algebra $U$ is described as follows. With some work (or by [10, Theorem 13], [7, p. 10]) one obtains

\[
x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0, \tag{14}
\]

\[
y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0. \tag{15}
\]

So in the $q$-shuffle algebra $V$ the elements $x, y$ satisfy the $q$-Serre relations. Consequently there exists an algebra homomorphism $\natural$ from $U^+_q$ to the $q$-shuffle algebra $V$, that sends $A \mapsto x$ and $B \mapsto y$. The map $\natural$ has image $U$ by Definition 4.2 and is injective by [11, Theorem 15]. Therefore $\natural : U^+_q \to U$ is an algebra isomorphism. By construction the following diagrams commute:

\[
\begin{array}{ccc}
U^+_q & \xrightarrow{\natural} & V \\
\sigma \downarrow & & \downarrow \sigma \\
V & \xrightarrow{\natural} & V
\end{array}
\]

\[
\begin{array}{ccc}
U^+_q & \xrightarrow{\natural} & V \\
& \downarrow & \\
U^+_q & \xrightarrow{\natural} & V
\end{array}
\]

Earlier we mentioned an $\mathbb{N}^2$-grading for both the algebra $U^+_q$ and the $q$-shuffle algebra $V$. These gradings are related as follows. The algebra $U$ has an $\mathbb{N}^2$-grading inherited from $U^+_q$ via $\natural$. With respect to this grading, for $(i, j) \in \mathbb{N}^2$ the $(i, j)$-homogeneous component of $U$ is the $\natural$-image of the $(i, j)$-homogeneous component of $U^+_q$. This homogeneous component is equal to $\mathbb{V}_{i,j} \cap U$.

Next we show that $U \neq V$. For $(i, j) \in \mathbb{N}^2$, the dimension of $\mathbb{V}_{i,j} \cap U$ is $d_{i,j}$ and the dimension of $\mathbb{V}_{i,j}$ is $\binom{i+j}{i}$. There exists $(i, j) \in \mathbb{N}^2$ such that $d_{i,j} < \binom{i+j}{i}$. Therefore $U \neq V$.

Next we describe how the map $\natural$ acts on the PBW basis elements (3). Define $\overline{x} = 1$ and $\overline{y} = -1$. A word $v_1v_2 \cdots v_n$ in $V$ is Catalan whenever $\overline{v}_1 + \overline{v}_2 + \cdots + \overline{v}_i$ is nonnegative for $1 \leq i \leq n-1$ and zero for $i = n$. In this case $n$ is even. For $n \geq 0$ define

\[
C_n = \sum v_1v_2 \cdots v_{2n}[1][1 + \overline{v}_1][1 + \overline{v}_1 + \overline{v}_2] \cdots [1 + \overline{v}_1 + \overline{v}_2 + \cdots + \overline{v}_{2n}],
\]

where the sum is over all the Catalan words $v_1v_2 \cdots v_{2n}$ in $V$ that have length $2n$. For example

\[
C_0 = 1, \quad C_1 = [2]_q xy, \quad C_2 = [2]_q^2 xyx + [3]_q [2]_q^2 xxyy.
\]

By [17, Theorem 1.7] the map $\natural$ sends

\[
E_{n\delta + \alpha_0} \mapsto q^{-2n}(q - q^{-1})^{2n}xC_n, \quad E_{n\delta + \alpha_1} \mapsto q^{-2n}(q - q^{-1})^{2n}C_ny
\]

for $n \geq 0$, and

\[
E_{n\delta} \mapsto -q^{-2n}(q - q^{-1})^{2n-1}C_n
\]

for $n \geq 1$. See [8, p. 696] for more information about $\natural$. 

8
The alternating words

In this section we introduce a type of word in \( V \), said to be alternating. We display some relations in the \( q \)-shuffle algebra \( V \), that are satisfied by the alternating words. Later in the paper we will use these relations to show that the alternating words are contained in \( U \).

Definition 5.1. A word \( v_1v_2\cdots v_n \) in \( V \) is called alternating whenever \( n \geq 1 \) and \( v_{i-1} \neq v_i \) for \( 2 \leq i \leq n \). Thus an alternating word has the form \( \cdots xyxy\cdots \).

Definition 5.2. We name the alternating words as follows:

\[
\begin{align*}
W_0 &= x, & W_{-1} &= xy, & W_{-2} &= xyxy, & \cdots \tag{16} \\
W_1 &= y, & W_2 &= yx, & W_3 &= xyxy, & \cdots \tag{17} \\
G_1 &= yx, & G_2 &= yxy, & G_3 &= xyxyx, & \cdots \tag{18} \\
\tilde{G}_1 &= xy, & \tilde{G}_2 &= xyx, & \tilde{G}_3 &= xyxyx, & \cdots \tag{19}
\end{align*}
\]

For notational convenience define \( G_0 = 1 \) and \( \tilde{G}_0 = 1 \). So for \( k \in \mathbb{N} \),

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{name} & \text{description} & \text{x-degree} & \text{y-degree} & \text{length} \\
\hline
W_{-k} & xyxy\cdots x & k+1 & k & 2k+1 \\
W_{k+1} & yxyx\cdots y & k & k+1 & 2k+1 \\
G_k & yxyx\cdots x & k & k & 2k \\
\tilde{G}_k & xyxy\cdots y & k & k & 2k \\
\hline
\end{array}
\]

Lemma 5.3. The maps \( \sigma, S \) from Lemma 4.1 act on the alternating words as follows. For \( k \in \mathbb{N} \),

(i) the map \( \sigma \) sends

\[
W_{-k} \mapsto W_{k+1}, \quad W_{k+1} \mapsto W_{-k}, \quad G_k \mapsto \tilde{G}_k, \quad \tilde{G}_k \mapsto G_k;
\]

(ii) the map \( S \) sends

\[
W_{-k} \mapsto W_{-k}, \quad W_{k+1} \mapsto W_{k+1}, \quad G_k \mapsto \tilde{G}_k, \quad \tilde{G}_k \mapsto G_k.
\]

Proof. By Lemma 4.1 and Definition 5.2.

Lemma 5.4. For \( k \in \mathbb{N} \) the following holds in the free algebra \( V \):

\[
\begin{align*}
W_{-k} &= xG_k = \tilde{G}_kx, & G_{k+1} &= yW_{-k} = W_{k+1}x, \\
W_{k+1} &= y\tilde{G}_k = G_ky, & \tilde{G}_{k+1} &= xW_{k+1} = W_{-k}y.
\end{align*}
\]

Proof. Use Definition 5.2.
We are going to show that $U$ contains every alternating word. As a warmup, consider the alternating words $xy$ and $yx$. Using (8),

$$xy = q \frac{q x \ast y - q^{-1} y \ast x}{q^2 - q^{-2}}, \quad yx = q \frac{q y \ast x - q^{-1} x \ast y}{q^2 - q^{-2}}. \tag{20}$$

Therefore $U$ contains $xy$ and $yx$. In order to handle longer alternating words, we will develop some relations involving the $q$-shuffle product. Next we describe the $q$-shuffle product of a letter and an alternating word.

**Lemma 5.5.** For $k \in \mathbb{N}$,

$$x \ast W_{-k} = (1 + q^2) \sum_{i=0}^{k} \tilde{G}_i x^2 G_{k-i}, \tag{21}$$

$$x \ast W_{k+1} = q^{-2} G_{k+1} + \tilde{G}_{k+1} + (1 + q^{-2}) \sum_{i=0}^{k-1} W_{i+1} x^2 W_{k-i}, \tag{22}$$

$$x \ast G_k = W_{-k} + (1 + q^{-2}) \sum_{i=0}^{k-1} W_{i+1} x^2 G_{k-i-1}, \tag{23}$$

$$x \ast \tilde{G}_k = W_{-k} + (1 + q^2) \sum_{i=0}^{k-1} \tilde{G}_{k-i-1} x^2 W_{i+1} \tag{24}$$

and

$$y \ast W_{-k} = G_{k+1} + q^{-2} \tilde{G}_{k+1} + (1 + q^{-2}) \sum_{i=0}^{k-1} W_{-i} y^2 W_{i-k+1}, \tag{25}$$

$$y \ast W_{k+1} = (1 + q^2) \sum_{i=0}^{k} G_i y^2 \tilde{G}_{k-i}, \tag{26}$$

$$y \ast G_k = W_{k+1} + (1 + q^2) \sum_{i=0}^{k-1} G_{k-i-1} y^2 W_{-i}, \tag{27}$$

$$y \ast \tilde{G}_k = W_{k+1} + (1 + q^{-2}) \sum_{i=0}^{k-1} W_{-i} y^2 \tilde{G}_{k-i-1} \tag{28}$$

**Proof.** Use (10) and Definition 5.2. □

Next we describe the $q$-shuffle product of an alternating word and a letter.
Lemma 5.6. For $k \in \mathbb{N}$,

\begin{equation}
W_{-k} \star x = (1 + q^2) \sum_{i=0}^{k} \tilde{G}_{k-i} x^2 G_i, \tag{29}
\end{equation}

\begin{equation}
W_{k+1} \star x = G_{k+1} + q^{-2} \tilde{G}_{k+1} + (1 + q^{-2}) \sum_{i=0}^{k-1} W_{k-i} x^2 W_{i+1}, \tag{30}
\end{equation}

\begin{equation}
G_{k} \star x = W_{-k} + (1 + q^2) \sum_{i=0}^{k-1} W_{i+1} x^2 G_{k-i-1}, \tag{31}
\end{equation}

\begin{equation}
\tilde{G}_{k} \star x = W_{-k} + (1 + q^{-2}) \sum_{i=0}^{k-1} \tilde{G}_{k-i-1} x^2 W_{i+1} \tag{32}
\end{equation}

and

\begin{equation}
W_{-k} \star y = q^{-2} G_{k+1} + \tilde{G}_{k+1} + (1 + q^{-2}) \sum_{i=0}^{k-1} W_{k-i-1} y^2 W_i, \tag{33}
\end{equation}

\begin{equation}
W_{k+1} \star y = (1 + q^2) \sum_{i=0}^{k} G_{k-i} y^2 \tilde{G}_i, \tag{34}
\end{equation}

\begin{equation}
G_{k} \star y = W_{k+1} + (1 + q^{-2}) \sum_{i=0}^{k-1} G_{k-i-1} y^2 W_i, \tag{35}
\end{equation}

\begin{equation}
\tilde{G}_{k} \star y = W_{k+1} + (1 + q^2) \sum_{i=0}^{k-1} W_{i} y^2 \tilde{G}_{k-i-1}. \tag{36}
\end{equation}

Proof. Use (11) and Definition 5.2. \qed

Proposition 5.7. For $k \in \mathbb{N}$ the following holds in the $q$-shuffle algebra $\mathbb{V}$:

\begin{equation}
[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \tag{37}
\end{equation}

\begin{equation}
[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1}) W_{-k-1}. \tag{38}
\end{equation}

\begin{equation}
[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1}) W_{k+2}. \tag{39}
\end{equation}

Proof. These relations are routinely checked using Lemmas 5.5, 5.6. \qed

Note 5.8. We have a comment about notation. In Proposition 5.7 we used the commutator and $q$-commutator notation. Throughout the paper, for any equation in the $q$-shuffle algebra $\mathbb{V}$ that involves a commutator or $q$-commutator, it is understood that these commutators are computed using the $q$-shuffle product $\star$.

We just displayed some relations for the alternating words in $\mathbb{V}$. Shortly we will display some more general relations for the alternating words. To obtain these relations we use the following identities.
Lemma 5.9. For \( k, \ell \in \mathbb{N} \),
\[
W_{-k} \ast W_{-\ell} = x(G_k \ast W_{-\ell}) + x(W_{-k} \ast G_{\ell}) q^2 = q^2 (\tilde{G}_k \ast W_{-\ell}) x + (W_{-k} \ast \tilde{G}_{\ell}) x,
\]
\[
W_{-k} \ast W_{\ell+1} = x(G_k \ast W_{\ell+1}) + y(W_{-k} \ast G_{\ell}) q^2 = q^{-2} (\tilde{G}_k \ast W_{\ell+1}) x + (W_{-k} \ast G_{\ell}) y,
\]
\[
W_{k+1} \ast W_{-\ell} = y(\tilde{G}_k \ast W_{-\ell}) + x(W_{k+1} \ast G_{\ell}) q^2 = q^{-2} (G_k \ast W_{-\ell}) y + (W_{k+1} \ast \tilde{G}_{\ell}) x,
\]
\[
W_{k+1} \ast W_{\ell+1} = y(\tilde{G}_k \ast W_{\ell+1}) + x(W_{k+1} \ast \tilde{G}_{\ell}) q^2 = q^2 (G_k \ast W_{\ell+1}) y + (W_{k+1} \ast G_{\ell}) y
\]
and
\[
W_{-k} \ast G_{\ell+1} = x(G_k \ast G_{\ell+1}) + y(W_{-k} \ast W_{-\ell}) q^2 = (\tilde{G}_k \ast G_{\ell+1}) x + (W_{-k} \ast W_{\ell+1}) x,
\]
\[
W_{-k} \ast \tilde{G}_{\ell+1} = x(G_k \ast \tilde{G}_{\ell+1}) + x(W_{-k} \ast W_{\ell+1}) q^2 = (\tilde{G}_k \ast \tilde{G}_{\ell+1}) x + (W_{-k} \ast W_{-\ell}) y,
\]
\[
W_{k+1} \ast G_{\ell+1} = y(\tilde{G}_k \ast G_{\ell+1}) + y(W_{k+1} \ast W_{-\ell}) q^2 = (G_k \ast G_{\ell+1}) y + (W_{k+1} \ast W_{\ell+1}) x,
\]
\[
W_{k+1} \ast \tilde{G}_{\ell+1} = y(\tilde{G}_k \ast \tilde{G}_{\ell+1}) + x(W_{k+1} \ast \tilde{G}_{\ell+1}) q^2 = (G_k \ast \tilde{G}_{\ell+1}) y + (W_{k+1} \ast G_{\ell}) y
\]
and
\[
G_{k+1} \ast W_{-\ell} = y(W_{-k} \ast W_{-\ell}) + x(G_{k+1} \ast G_{\ell}) q^2 = (W_{k+1} \ast W_{-\ell}) x + (G_{k+1} \ast \tilde{G}_{\ell}) x,
\]
\[
G_{k+1} \ast W_{\ell+1} = y(W_{-k} \ast W_{\ell+1}) + y(G_{k+1} \ast \tilde{G}_{\ell}) q^2 = q^{-2} (W_{k+1} \ast W_{\ell+1}) x + (G_{k+1} \ast G_{\ell}) y,
\]
\[
G_{k+1} \ast W_{\ell} = x(W_{k+1} \ast W_{-\ell}) + x(G_{k+1} \ast G_{\ell}) q^2 = q^{-2} (W_{-k} \ast W_{-\ell}) y + (\tilde{G}_{k+1} \ast G_{\ell}) x,
\]
\[
\tilde{G}_{k+1} \ast W_{\ell+1} = x(W_{k+1} \ast W_{\ell+1}) + y(\tilde{G}_{k+1} \ast \tilde{G}_{\ell}) q^2 = q^2 (W_{-k} \ast W_{\ell+1}) y + (\tilde{G}_{k+1} \ast G_{\ell}) y
\]
and
\[
G_{k+1} \ast G_{\ell+1} = y(W_{-k} \ast G_{\ell+1}) + y(G_{k+1} \ast W_{-\ell}) = (W_{k+1} \ast G_{\ell+1}) x + (G_{k+1} \ast W_{\ell+1}) x,
\]
\[
G_{k+1} \ast \tilde{G}_{\ell+1} = y(W_{-k} \ast \tilde{G}_{\ell+1}) + x(G_{k+1} \ast \tilde{G}_{\ell+1}) = (W_{k+1} \ast \tilde{G}_{\ell+1}) x + (G_{k+1} \ast W_{-\ell}) y,
\]
\[
\tilde{G}_{k+1} \ast G_{\ell+1} = x(W_{k+1} \ast G_{\ell+1}) + y(\tilde{G}_{k+1} \ast W_{-\ell}) = (W_{-k} \ast G_{\ell+1}) y + (G_{k+1} \ast W_{\ell+1}) x,
\]
\[
\tilde{G}_{k+1} \ast \tilde{G}_{\ell+1} = x(W_{k+1} \ast \tilde{G}_{\ell+1}) + x(G_{k+1} \ast \tilde{G}_{\ell+1}) = (W_{-k} \ast \tilde{G}_{\ell+1}) y + (\tilde{G}_{k+1} \ast G_{\ell}) y.
\]

Proof. Use (12), (13).

Proposition 5.10. For \( k, \ell \in \mathbb{N} \) the following relations hold in the \( q \)-shuffle algebra \( \mathbb{V} \):
\[
[W_{-k}, W_{-\ell}] = 0, \quad [W_{k+1}, W_{\ell+1}] = 0,
\]
\[
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0,
\]
\[
[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0,
\]
\[
[W_{-k}, \tilde{G}_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,
\]
\[
[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0,
\]
\[
[G_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, G_{\ell+1}] = 0,
\]
\[
[\tilde{G}_{k+1}, G_{\ell+1}] = 0, \quad [\tilde{G}_{k+1}, G_{\ell+1}] = 0.
\]

Proof. Use Lemma 5.9 and induction on \( k + \ell \).
Proposition 5.11. For \( k, \ell \in \mathbb{N} \) the following relations hold in the \( q \)-shuffle algebra \( V \):

\[
\begin{align*}
[W_{-k}, G_{\ell}]_q &= [W_{-\ell}, G_k]_q, & [G_k, W_{\ell+1}]_q &= [G_{\ell}, W_{k+1}]_q, \\
[\tilde{G}_k, W_{-\ell}]_q &= [\tilde{G}_{\ell}, W_{-k}]_q, & [W_{\ell+1}, \tilde{G}_k]_q &= [W_{k+1}, \tilde{G}_{\ell}]_q, \\
[G_k, \tilde{G}_{\ell+1}] - [G_{\ell}, \tilde{G}_{k+1}] &= q[W_{-\ell}, W_{k+1}]_q - q[W_{-k}, W_{\ell+1}]_q, \\
[\tilde{G}_k, G_{\ell+1}] - [\tilde{G}_{\ell}, G_{k+1}] &= q[W_{\ell+1}, W_{-k}]_q - q[W_{k+1}, W_{-\ell}]_q, \\
[G_{k+1}, \tilde{G}_{\ell+1}]_q - [G_{\ell+1}, \tilde{G}_{k+1}]_q &= q[W_{-\ell}, W_{k+2}] - q[W_{-k}, W_{\ell+2}], \\
[\tilde{G}_{k+1}, G_{\ell+1}]_q - [\tilde{G}_{\ell+1}, G_{k+1}]_q &= q[W_{\ell+1}, W_{-k-1}] - q[W_{k+1}, W_{-\ell-1}].
\end{align*}
\]

Proof. Each equation is verified by evaluating both sides using Lemma 5.9 and simplifying the result using Proposition 5.10.

We emphasize a few points about Definition 5.2.

Lemma 5.12. Referring to Definition 5.2, the following (i)–(v) hold.

(i) For each of the lines (16)–(19) the listed elements mutually commute.

(ii) An alternating word is listed in (16) if and only if it commutes with \( x \).

(iii) An alternating word is listed in (17) if and only if it commutes with \( y \).

(iv) An alternating word is listed in (18) if and only if it commutes with \( qy \star x - q^{-1}x \star y \).

(v) An alternating word is listed in (19) if and only if it commutes with \( qx \star y - q^{-1}y \star x \).

Proof. (i) By (40), (46).

(ii) Use (23), (24) and (37), (38).

(iii) Use (ii) above and the \( \sigma \) action from Lemma 5.3.

(iv) Use (20) together with (42), (44), (51) at \( \ell = 0 \).

(v) Use (iv) above and the \( \sigma \) action from Lemma 5.3.

6 Commutator relations for alternating words, part I

Our next general goal is to obtain a commutator relation for every pair of alternating words. As we pursue this goal, it is convenient to split the argument into two cases. In the present (resp. next) section we treat the case in which the pair of alternating words have length of opposite (resp. same) parity. Throughout this section fix \( n \in \mathbb{N} \).
In Table 1 below, we see four cases. For each case we have a set $L$ and a set $R$:

| case | set $L$ | set $R$ |
|------|---------|---------|
| 1    | $\{G_i \ast W_{-j} | i, j \in \mathbb{N}, \ i + j = n\}$  | $\{W_{-j} \ast G_i | i, j \in \mathbb{N}, \ i + j = n\}$ |
| 2    | $\{G_i \ast W_{j+1} | i, j \in \mathbb{N}, \ i + j = n\}$  | $\{W_{j+1} \ast G_i | i, j \in \mathbb{N}, \ i + j = n\}$ |
| 3    | $\{\tilde{G}_i \ast W_{-j} | i, j \in \mathbb{N}, \ i + j = n\}$  | $\{W_{-j} \ast \tilde{G}_i | i, j \in \mathbb{N}, \ i + j = n\}$ |
| 4    | $\{G_i \ast W_{j+1} | i, j \in \mathbb{N}, \ i + j = n\}$  | $\{W_{j+1} \ast G_i | i, j \in \mathbb{N}, \ i + j = n\}$ |

Table 1

In each case, we will show that the sets $L$ and $R$ have the same span. To this end, order the set $L$ as follows:

| case | ordering of $L$ |
|------|----------------|
| 1    | $G_0 \ast W_{-n}, \ G_n \ast W_0, \ G_1 \ast W_{1-n}, \ G_{n-1} \ast W_{-1}, \ G_2 \ast W_{2-n}, \ G_{n-2} \ast W_{-2}, \ldots$ |
| 2    | $G_0 \ast W_{n+1}, \ G_n \ast W_1, \ G_1 \ast W_n, \ G_{n-1} \ast W_2, \ G_2 \ast W_{n-1}, \ G_{n-2} \ast W_3, \ldots$ |
| 3    | $\tilde{G}_0 \ast W_{-n}, \ \tilde{G}_n \ast W_0, \ \tilde{G}_1 \ast W_{1-n}, \ \tilde{G}_{n-1} \ast W_{-1}, \ \tilde{G}_2 \ast W_{2-n}, \ \tilde{G}_{n-2} \ast W_{-2}, \ldots$ |
| 4    | $\tilde{G}_0 \ast W_{n+1}, \ \tilde{G}_n \ast W_1, \ \tilde{G}_1 \ast W_n, \ \tilde{G}_{n-1} \ast W_2, \ \tilde{G}_2 \ast W_{n-1}, \ \tilde{G}_{n-2} \ast W_3, \ldots$ |

Order the set $R$ as follows:

| case | ordering of $R$ |
|------|----------------|
| 1    | $W_{-n} \ast G_0, \ W_0 \ast G_n, \ W_{1-n} \ast G_1, \ W_{-1} \ast G_{n-1}, \ W_{2-n} \ast G_2, \ W_{-2} \ast G_{n-2}, \ldots$ |
| 2    | $W_{n+1} \ast G_0, \ W_1 \ast G_n, \ W_n \ast G_1, \ W_2 \ast G_{n-1}, \ W_{n-1} \ast G_2, \ W_3 \ast G_{n-2}, \ldots$ |
| 3    | $W_{-n} \ast \tilde{G}_0, \ W_0 \ast \tilde{G}_n, \ W_{1-n} \ast \tilde{G}_1, \ W_{-1} \ast \tilde{G}_{n-1}, \ W_{2-n} \ast \tilde{G}_2, \ W_{-2} \ast \tilde{G}_{n-2}, \ldots$ |
| 4    | $W_{n+1} \ast \tilde{G}_0, \ W_1 \ast \tilde{G}_n, \ W_n \ast \tilde{G}_1, \ W_2 \ast \tilde{G}_{n-1}, \ W_{n-1} \ast \tilde{G}_2, \ W_3 \ast \tilde{G}_{n-2}, \ldots$ |

Define sequences $\{u_i\}_{i=0}^n$, $\{v_i\}_{i=0}^n$ as follows. In cases 1 and 4 let $\{u_i\}_{i=0}^n$ (resp. $\{v_i\}_{i=0}^n$) denote the given ordering of $L$ (resp. $R$). In cases 2 and 3 let $\{u_i\}_{i=0}^n$ (resp. $\{v_i\}_{i=0}^n$) denote the given ordering of $R$ (resp. $L$).

**Lemma 6.1.** In each case 1–4 above, the following holds for $0 \leq j \leq n$.

(i) For $j$ even,

$$u_j = v_j + (1 - q^2) \sum_{i=0}^{j-1} (-1)^i v_i, \quad v_j = u_j + (1 - q^{-2}) \sum_{i=0}^{j-1} (-1)^i u_i.$$ 

(ii) For $j$ odd,

$$u_j = q^2 v_j + (1 - q^2) \sum_{i=0}^{j-1} (-1)^i v_i, \quad v_j = q^{-2} u_j + (1 - q^{-2}) \sum_{i=0}^{j-1} (-1)^i u_i.$$ 

**Proof.** To obtain the result for case 1, use Propositions 5.7, 5.10, 5.11 and induction on $j$. To obtain the result for case 2 (resp. case 3) (resp. case 4), apply $\sigma S$ (resp. $S$) (resp. $\sigma$) to everything from case 1. \qed
In Appendix A we present Lemma 6.1 in an alternative form.

**Proposition 6.2.** For each case in Table 1, the sets $L$ and $R$ have the same span.

**Proof.** By Lemma 6.1.

We mention some relations for later use.

**Proposition 6.3.** For $n \in \mathbb{N}$,

\[
\sum_{k=0}^{n} G_{n-k} \star W_{-k}q^{2k-n} = \sum_{k=0}^{n} W_{-k} \star G_{n-k}q^{n-2k}, \tag{54}
\]

\[
\sum_{k=0}^{n} G_{n-k} \star W_{k+1}q^{n-2k} = \sum_{k=0}^{n} W_{k+1} \star G_{n-k}q^{2k-n}, \tag{55}
\]

\[
\sum_{k=0}^{n} \tilde{G}_{n-k} \star W_{-k}q^{n-2k} = \sum_{k=0}^{n} W_{-k} \star \tilde{G}_{n-k}q^{2k-n}. \tag{56}
\]

\[
\sum_{k=0}^{n} \tilde{G}_{n-k} \star W_{k+1}q^{2k-n} = \sum_{k=0}^{n} W_{k+1} \star \tilde{G}_{n-k}q^{n-2k}, \tag{57}
\]

**Proof.** To verify (54), evaluate each summand on the left using case 1 of Lemma 6.1 and simplify the result. To obtain (55) (resp. (56)) (resp. (57)) apply $S$ (resp. $\tilde{S}$) (resp. $\sigma$) to everything in (54).

\[ \square \]

## 7 Commutator relations for alternating words, part II

In this section we obtain a commutator relation for every pair of alternating words that have length of the same parity. Throughout this section fix an integer $n \geq 1$.

Consider the following sets:

\[ N = \{ \tilde{G}_i \star G_j | i, j \in \mathbb{N}, \ i + j = n \}, \]

\[ S = \{ G_i \star \tilde{G}_j | i, j \in \mathbb{N}, \ i + j = n \}, \]

\[ E = \{ W_{i+1} \star W_{-j} | i, j \in \mathbb{N}, \ i + j = n - 1 \}, \]

\[ W = \{ W_{-i} \star W_{j+1} | i, j \in \mathbb{N}, \ i + j = n - 1 \}. \]

We are going to show that the sets

\[ N \cup E, \quad N \cup W, \quad S \cup E, \quad S \cup W \]

all have the same span. To this end, for each of $N, S, E, W$ we order the elements in two ways:
ordering of $N$

| I  | $G_0 \ast G_n, G_n \ast G_0, G_1 \ast G_{n-1}, G_{n-1} \ast G_1, G_2 \ast G_{n-2}, G_{n-2} \ast G_2, \ldots$ |
|----|--------------------------------------------------------------------------------------------------|
| II | $\tilde{G}_n \ast G_0, \tilde{G}_0 \ast G_n, \tilde{G}_{n-1} \ast G_1, \tilde{G}_1 \ast G_{n-1}, \tilde{G}_{n-2} \ast G_2, \tilde{G}_2 \ast G_{n-2}, \ldots$ |

ordering of $S$

| I  | $G_n \ast G_0, G_0 \ast G_n, G_{n-1} \ast G_1, G_1 \ast G_{n-1}, G_{n-2} \ast G_2, G_2 \ast G_{n-2}, \ldots$ |
|----|--------------------------------------------------------------------------------------------------|
| II | $G_0 \ast G_n, G_n \ast \tilde{G}_0, G_{n-1} \ast \tilde{G}_1, G_1 \ast G_{n-1}, G_{n-2} \ast \tilde{G}_2, G_2 \ast G_{n-2}, \ldots$ |

ordering of $E$

| I  | $W_n \ast W_0, W_1 \ast W_{n-1}, W_{n-1} \ast W_1, W_2 \ast W_{2-n}, W_{2-n} \ast W_2, W_3 \ast W_{3-n}, \ldots$ |
|----|--------------------------------------------------------------------------------------------------|
| II | $W_1 \ast W_{1-n}, W_n \ast W_0, W_2 \ast W_{2-n}, W_{n-1} \ast W_1, W_{3-n} \ast W_3, W_{2-n} \ast W_2, \ldots$ |

ordering of $W$

| I  | $W_0 \ast W_n, W_{1-n} \ast W_1, W_1 \ast W_{n-1}, W_{2-n} \ast W_2, W_{n-1} \ast W_{2-n}, W_{3-n} \ast W_3, \ldots$ |
|----|--------------------------------------------------------------------------------------------------|
| II | $W_{1-n} \ast W_1, W_0 \ast W_n, W_2-n \ast W_2, W_1 \ast W_{n-1}, W_{3-n} \ast W_3, W_{2-n} \ast W_{2-n}, \ldots$ |

Next we define sequences $\{u_i\}_{i=0}^n$, $\{v_i\}_{i=0}^n$, $\{U_i\}_{i=0}^{n-1}$, $\{V_i\}_{i=0}^{n-1}$ as follows. There are four cases:

| case | $\{u_i\}_{i=0}^n$ | $\{v_i\}_{i=0}^n$ | $\{U_i\}_{i=0}^{n-1}$ | $\{V_i\}_{i=0}^{n-1}$ |
|------|-----------------|-----------------|-----------------|-----------------|
| 1    | ordering I of $N$ | ordering I of $S$ | ordering I of $E$ | ordering I of $W$ |
| 2    | ordering I of $S$ | ordering I of $N$ | ordering II of $E$ | ordering I of $W$ |
| 3    | ordering II of $N$ | ordering II of $S$ | ordering I of $W$ | ordering II of $E$ |
| 4    | ordering II of $S$ | ordering II of $N$ | ordering II of $W$ | ordering II of $E$ |

**Lemma 7.1.** For each case 1–4 above we have the following.

(i) For $0 \leq j \leq n$ and $j$ even,

$$u_j = v_j + (1 - q^2) \sum_{i=0}^{j-1} (-1)^i v_i, \quad v_j = u_j - (1 - q^2) \sum_{i=0}^{j-1} (-1)^i U_i.$$

(ii) For $0 \leq j \leq n$ and $j$ odd,

$$u_j = v_j + (1 - q^2) \sum_{i=0}^{j-2} (-1)^i v_i, \quad v_j = u_j - (1 - q^2) \sum_{i=0}^{j-2} (-1)^i U_i.$$

(iii) For $0 \leq j \leq n - 1$ and $j$ even,

$$U_j = V_j + (1 - q^{-2}) \sum_{i=0}^{j+1} (-1)^i v_i, \quad V_j = U_j - (1 - q^{-2}) \sum_{i=0}^{j+1} (-1)^i u_i.$$

(iv) For $0 \leq j \leq n - 1$ and $j$ odd,

$$U_j = V_j + (1 - q^{-2}) \sum_{i=0}^{j} (-1)^i v_i, \quad V_j = U_j - (1 - q^{-2}) \sum_{i=0}^{j} (-1)^i u_i.$$
Proof. To obtain the result for case 1, use Propositions 5.7, 5.10, 5.11 and induction on $j$. To obtain the result for case 2 (resp. case 3) (resp. case 4), apply $\sigma S$ (resp. $S$) (resp. $\sigma$) to everything from case 1.

In Appendix B we present Lemma 7.1 in an alternative form.

**Proposition 7.2.** The sets

\[ N \cup E, \quad N \cup W, \quad S \cup E, \quad S \cup W \]

all have the same span.

**Proof.** Let $V$ denote the span of $N \cup S \cup E \cup W$. By case 1 or case 4 of Lemma 7.1 we find that $N \cup E$ and $S \cup W$ have the same span, which must be $V$. By case 2 or case 3 of Lemma 7.1 we find that $N \cup W$ and $S \cup E$ have the same span, which must be $V$. The result follows.

8 Each alternating word is contained in $U$

In Propositions 5.7, 5.10, 5.11 we obtained some relations that involve the alternating words. In this section we obtain some additional relations for the alternating words; these resemble the relations in Proposition 6.3. As we will see, these additional relations together with Propositions 5.7, 5.10, 5.11 imply that each alternating word is contained in $U$.

**Proposition 8.1.** For $n \geq 1$,

\[ \sum_{k=0}^{n} G_k \star \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{n-1-2k}, \]  
\[ (58) \]

\[ \sum_{k=0}^{n} G_k \star \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} \star W_{-k} q^{n-1-2k}, \]  
\[ (59) \]

\[ \sum_{k=0}^{n} \tilde{G}_k \star G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} \star W_{-k} q^{2k+1-n}, \]  
\[ (60) \]

\[ \sum_{k=0}^{n} \tilde{G}_k \star G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} \star W_{n-k} q^{2k+1-n}. \]  
\[ (61) \]

**Proof.** We first verify (58) by evaluating each side. In (58) the $k$-summand on the left is $\tilde{G}_n q^n$ (resp. $G_n q^n$) for $k = 0$ (resp. $k = n$). By Lemma 5.4 we have $\tilde{G}_n = x W_n$ and $G_n = y W_{1-n}$. By Lemma 5.9

\[ G_k \star \tilde{G}_{n-k} = y(W_{1-k} \star \tilde{G}_{n-k}) + x(G_k \star W_{n-k}) \]

for $1 \leq k \leq n - 1$ and

\[ W_{-k} \star W_{n-k} = x(G_k \star W_{n-k}) + y(W_{-k} \star \tilde{G}_{n-k-1}) q^{-2} \]

for $0 \leq k \leq n - 1$. Using these comments, one checks that the two sides of (58) are equal. To obtain (59) (resp. (60)) (resp. (61)), apply $S$ (resp. $\sigma$) (resp. $\sigma S$) to everything in (58).
Lemma 8.2. Using the equations below, the alternating words in $\mathcal{V}$ are recursively obtained from $x, y$ in the following order:

$$W_0, W_1, G_1, W_{-1}, W_2, G_2, W_{-2}, W_3, \ldots$$

We have $W_0 = x$ and $W_1 = y$. For $n \geq 1$,

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \ast \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n \ast W_0 - W_0 \ast W_n}{(1 + q^{-2n})(1 - q^{-2})}, \quad (62)$$

$$\tilde{G}_n = G_n + \frac{W_0 \ast W_n - W_n \ast W_0}{1 - q^{-2}}, \quad (63)$$

$$W_{-n} = \frac{q W_0 \ast G_n - q^{-1} G_n \ast W_0}{q - q^{-1}}, \quad (64)$$

$$W_{n+1} = \frac{q G_n \ast W_1 - q^{-1} W_1 \ast G_n}{q - q^{-1}}. \quad (65)$$

Proof. Equation (63) is from (37). To obtain (62), subtract $q^n$ times (63) from (58), and simplify the result. Equations (64), (65) are from (38), (39). \qed

Theorem 8.3. Each alternating word of $\mathcal{V}$ is contained in $U$.

Proof. By Lemma 8.2. \qed

9 Some generating functions

We continue to discuss the alternating words in $\mathcal{V}$. In previous sections we found many relations involving these words. In this section we express some of these relations using generating functions. We use these generating functions to solve for the alternating words (18) in terms of the alternating words (16), (17), (19).

Definition 9.1. We define some generating functions in an indeterminate $t$:

$$G(t) = \sum_{n \in \mathbb{N}} t^n G_n, \quad \tilde{G}(t) = \sum_{n \in \mathbb{N}} t^n \tilde{G}_n,$$

$$W^+(t) = \sum_{n \in \mathbb{N}} t^n W_{n+1}, \quad W^-(t) = \sum_{n \in \mathbb{N}} t^n W_{-n}.$$

Lemma 9.2. We have

$$G(q^{-1}t) \ast W^-(qt) = W^-(q^{-1}t) \ast G(qt),$$

$$\tilde{G}(q^{-1}t) \ast W^+(qt) = W^+(q^{-1}t) \ast \tilde{G}(qt),$$

$$G(qt) \ast W^+(q^{-1}t) = W^+(qt) \ast G(q^{-1}t),$$

$$\tilde{G}(qt) \ast W^-(q^{-1}t) = W^-(qt) \ast \tilde{G}(q^{-1}t).$$

Proof. Routine consequence of Proposition 6.3. \qed
Lemma 9.3. We have
\[ G(q^{-1}t) \ast \tilde{G}(qt) - qtW^-(q^{-1}t) \ast W^+(qt) = 1, \]
\[ G(qt) \ast \tilde{G}(q^{-1}t) - qtW^+(qt) \ast W^-(q^{-1}t) = 1, \]
\[ \tilde{G}(q^{-1}t) \ast G(qt) - qtW^+(q^{-1}t) \ast W^-(qt) = 1, \]
\[ \tilde{G}(qt) \ast G(q^{-1}t) - qtW^-(qt) \ast W^+(q^{-1}t) = 1. \]

Proof. Routine consequence of Proposition 8.1. \qed

Remark 9.4. By Lemmas 9.2, 9.3 the following matrices are inverses with respect to the \( q \)-shuffle product:
\[
\begin{pmatrix}
G(q^{-1}t) & qtW^-(q^{-1}t) \\
W^+(q^{-1}t) & \tilde{G}(q^{-1}t)
\end{pmatrix}, \quad \begin{pmatrix}
\tilde{G}(qt) & -qtW^-(qt) \\
-W^+(qt) & G(qt)
\end{pmatrix}.
\]

Our next goal is to solve for \( G(t) \). To this end, we introduce some elements \( \{D_n\}_{n \in \mathbb{N}} \) in \( \mathcal{V} \). These elements will be defined recursively.

Definition 9.5. Define \( \{D_n\}_{n \in \mathbb{N}} \) in \( \mathcal{V} \) such that \( D_0 = 1 \) and for \( n \geq 1 \),
\[ D_0 \ast \tilde{G}_n + D_1 \ast \tilde{G}_{n-1} + \cdots + D_n \ast \tilde{G}_0 = 0. \] (66)

Example 9.6. We have
\begin{align*}
D_1 &= -\tilde{G}_1, \\
D_2 &= \tilde{G}_1 \ast \tilde{G}_1 - \tilde{G}_2, \\
D_3 &= 2\tilde{G}_1 \ast \tilde{G}_2 - \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 - \tilde{G}_3, \\
D_4 &= \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 + 2\tilde{G}_1 \ast \tilde{G}_3 + \tilde{G}_2 \ast \tilde{G}_2 - 3\tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_2 - \tilde{G}_4
\end{align*}
and
\begin{align*}
\tilde{G}_1 &= -D_1, \\
\tilde{G}_2 &= D_1 \ast D_1 - D_2, \\
\tilde{G}_3 &= 2D_1 \ast D_2 - D_1 \ast D_1 \ast D_1 - D_3, \\
\tilde{G}_4 &= D_1 \ast D_1 \ast D_1 \ast D_1 + 2D_1 \ast D_3 + D_2 \ast D_2 - 3D_1 \ast D_1 \ast D_2 - D_4.
\end{align*}

Lemma 9.7. For \( n \geq 1 \) the following hold in the \( q \)-shuffle algebra \( \mathcal{V} \).

(i) \( D_n \) is a homogeneous polynomial in \( \tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n \) that has total degree \( n \), where we view each \( \tilde{G}_i \) as having degree \( i \). In this polynomial the coefficient of \( \tilde{G}_n \) is \( -1 \).

(ii) \( \tilde{G}_n \) is a homogeneous polynomial in \( D_1, D_2, \ldots, D_n \) that has total degree \( n \), where we view each \( D_i \) as having degree \( i \). In this polynomial the coefficient of \( D_n \) is \( -1 \).

Proof. By Definition 9.5 and induction on \( n \). \qed
Lemma 9.8. The following coincide:

(i) the subalgebra of the $q$-shuffle algebra $\mathbb{V}$ generated by $\{D_n\}_{n=1}^{\infty}$;
(ii) the subalgebra of the $q$-shuffle algebra $\mathbb{V}$ generated by $\{\tilde{G}_n\}_{n=1}^{\infty}$.

Proof. Each subalgebra contains the other by Lemma 9.7.

We emphasize a few points.

Lemma 9.9. For $n \in \mathbb{N}$ we have $D_n \in U$.

Proof. By Theorem 8.3 and Lemma 9.8.

Lemma 9.10. For $i, j \in \mathbb{N}$ the following holds in the $q$-shuffle algebra $\mathbb{V}$:

\[[D_i, D_j] = 0, \quad [D_i, \tilde{G}_j] = 0.\]

Proof. By Lemma 9.8 and since the $\{\tilde{G}_j\}_{j \in \mathbb{N}}$ mutually commute.

Definition 9.11. We define a generating function in the indeterminate $t$:

\[D(t) = \sum_{n \in \mathbb{N}} t^n D_n.\]

Lemma 9.12. We have

\[D(t) \ast \tilde{G}(t) = 1 = \tilde{G}(t) \ast D(t).\]

Proof. Use Definition 9.5 and Lemma 9.10.

Lemma 9.13. We have

\[
W^-(q^{-1}t) \ast D(q^{-1}t) = D(qt) \ast W^-(qt), \\
D(q^{-1}t) \ast W^+(q^{-1}t) = W^+(qt) \ast D(qt).
\]

Proof. By the two equations in Lemma 9.2 that involve $\tilde{G}$, together with Lemma 9.12.

Proposition 9.14. We have

\[
G(t) = D(q^2t) + q^2 t W^-(t) \ast D(t) \ast W^+(t), \quad (67) \\
G(t) = D(q^{-2}t) + t W^+(t) \ast D(t) \ast W^-(t). \quad (68)
\]

Proof. In the equations of Lemma 9.3 eliminate the $\tilde{G}$ term using Lemma 9.12 evaluate the result using Lemma 9.13 and then make a change of variables $t \mapsto q^{\pm 1}t$ as needed.

Theorem 9.15. For $n \in \mathbb{N}$ we have

\[
G_n = q^{2n} D_n + q^2 \sum_{i+j+k+1=n \atop i,j,k \geq 0} W_{-i} \ast D_j \ast W_{k+1}, \quad (69) \\
G_n = q^{-2n} D_n + \sum_{i+j+k+1=n \atop i,j,k \geq 0} W_{k+1} \ast D_j \ast W_{-i}. \quad (70)
\]
Proof. For the equations (67) and (68), compare the coefficient of $t^n$ on either side. ☐

Example 9.16. For $1 \leq n \leq 4$ the equations (69), (70) are given below:

\[
\begin{align*}
G_1 &= q^2D_1 + q^2W_0 \ast W_1, \\
G_2 &= q^4D_2 + q^2W_0 \ast W_2 + q^2W_{-1} \ast W_1 + q^2W_0 \ast D_1 \ast W_1, \\
G_3 &= q^6D_3 + q^2W_0 \ast W_3 + q^2W_{-1} \ast W_2 + q^2W_{-2} \ast W_1 + q^2W_0 \ast D_1 \ast W_2 \\
&\quad + q^2W_{-1} \ast D_1 \ast W_1 + q^2W_0 \ast D_2 \ast W_1, \\
G_4 &= q^8D_4 + q^2W_0 \ast W_4 + q^2W_{-1} \ast W_3 + q^2W_{-2} \ast W_2 + q^2W_{-3} \ast W_1 \\
&\quad + q^2W_0 \ast D_1 \ast W_3 + q^2W_{-1} \ast D_1 \ast W_2 + q^2W_{-2} \ast D_1 \ast W_1 + q^2W_0 \ast D_2 \ast W_2 \\
&\quad + q^2W_{-1} \ast D_2 \ast W_1 + q^2W_0 \ast D_3 \ast W_1,
\end{align*}
\]

and

\[
\begin{align*}
G_1 &= q^{-2}D_1 + W_1 \ast W_0, \\
G_2 &= q^{-4}D_2 + W_1 \ast W_{-1} + W_2 \ast W_0 + W_1 \ast D_1 \ast W_0, \\
G_3 &= q^{-6}D_3 + W_1 \ast W_{-2} + W_2 \ast W_{-1} + W_3 \ast W_0 + W_1 \ast D_1 \ast W_{-1} \\
&\quad + W_2 \ast D_1 \ast W_0 + W_1 \ast D_2 \ast W_0, \\
G_4 &= q^{-8}D_4 + W_1 \ast W_{-3} + W_2 \ast W_{-2} + W_3 \ast W_{-1} + W_4 \ast W_0 \\
&\quad + W_1 \ast D_1 \ast W_{-2} + W_2 \ast D_1 \ast W_{-1} + W_3 \ast D_1 \ast W_0 + W_1 \ast D_2 \ast W_{-1} \\
&\quad + W_2 \ast D_2 \ast W_0 + W_1 \ast D_3 \ast W_0.
\end{align*}
\]

Note that $D_1, D_2, D_3, D_4$ are given in Example 9.6.

10 The alternating PBW basis for $U$

In this section we obtain some PBW bases for $U$, including the alternating PBW basis.

Theorem 10.1. A PBW basis for $U$ is obtained by the elements

\[
\{W_{-i}\}_{i \in \mathbb{N}}, \quad \{G_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}} \tag{71}
\]

in any linear order $<$ that satisfies one of (i)--(vi) below:

(i) $W_{-i} < \tilde{G}_{j+1} < W_{k+1}$ for $i, j, k \in \mathbb{N};$

(ii) $W_{k+1} < \tilde{G}_{j+1} < W_{-i}$ for $i, j, k \in \mathbb{N};$

(iii) $W_{k+1} < W_{-i} < \tilde{G}_{j+1}$ for $i, j, k \in \mathbb{N};$

(iv) $W_{-i} < W_{k+1} < \tilde{G}_{j+1}$ for $i, j, k \in \mathbb{N};$

(v) $\tilde{G}_{j+1} < W_{k+1} < W_{-i}$ for $i, j, k \in \mathbb{N};$

(vi) $\tilde{G}_{j+1} < W_{-i} < W_{k+1}$ for $i, j, k \in \mathbb{N}.$
Proof. (i) We invoke Proposition \ref{prop:3.9}. For notational convenience we identify the algebras $U_q^+$ and $U$, via the isomorphism $\zeta$ from below \eqref{e:15}. Let $\Omega$ denote the set of alternating words listed in \eqref{e:71}. We have $\Omega \subseteq U$ by Theorem \ref{thm:8.3}. We show that $\Omega$ is feasible in the sense of Definition \ref{def:3.5} Definition \ref{def:3.5}(i) holds since each element of $\Omega$ is a word in $V$. Definition \ref{def:3.5}(ii) holds by the table above Lemma \ref{lem:5.3}. We have shown that $\Omega$ is feasible. The set $\Omega$ has a linear order $<$ from the theorem statement. We show that $\Omega$ in order $<$ satisfies Proposition \ref{prop:3.9}(ii). In the present notation, we must show that the vector space $U$ is spanned by

$$a_1 \ast a_2 \ast \cdots \ast a_n, \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \leq a_2 \leq \cdots \leq a_n. \quad \tag{72}$$

Let $\mathbb{A}$ denote the set of alternating words in $V$. So $\mathbb{A}$ is the union of $\Omega$ and $\{G_{\ell+1}\}_{\ell \in \mathbb{N}}$. The generators $x, y$ of $U$ are contained in $\mathbb{A}$, so the algebra $U$ is generated by $\mathbb{A}$. Therefore the vector space $U$ is spanned by

$$a_1 \ast a_2 \ast \cdots \ast a_n, \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \mathbb{A}. \quad \tag{73}$$

We have a linear order $<$ on $\Omega$; extend $<$ to $\mathbb{A}$ such that

$$W_{-i} < G_{\ell+1} < \tilde{G}_{j+1} < W_{k+1} \quad \text{for } i, j, k, \ell \in \mathbb{N}. \quad \tag{74}$$

For every product $a_1 \ast a_2 \ast \cdots \ast a_n$ in \eqref{e:73} and for every pair $a_{s-1}, a_s$ ($2 \leq s \leq n$) of adjacent terms such that $a_{s-1} > a_s$, we can use the commutator relations from Sections 6, 7 to express $a_{s-1} \ast a_s$ as a linear combination of products $a'_{s-1} \ast a'_s$ such that $a'_{s-1}, a'_s \in \mathbb{A}$ and $a'_{s-1} \leq a'_s$. Specifically, use Lemma \ref{lem:6.1} (resp. Lemma \ref{lem:7.1}) if the lengths of $a_{s-1}$ and $a_s$ have opposite (resp. same) parity. This argument shows that the vector space $U$ is spanned by the products

$$a_1 \ast a_2 \ast \cdots \ast a_n, \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \mathbb{A}, \quad a_1 \leq a_2 \leq \cdots \leq a_n.$$

Let $a_1 \ast a_2 \ast \cdots \ast a_n$ denote one of the above products. We show that $a_1 \ast a_2 \ast \cdots \ast a_n$ is contained in the span of \eqref{e:72}. Our proof is by induction on the number $\zeta$ of terms among $a_1, a_2, \ldots, a_n$ that are contained in $\{G_{\ell+1}\}_{\ell \in \mathbb{N}}$. First assume that $\zeta = 0$. Then $a_i \in \Omega$ for $1 \leq i \leq n$, so $a_1 \ast a_2 \ast \cdots \ast a_n$ is listed in \eqref{e:72}. Next assume that $\zeta \geq 1$. At least one of $a_1, a_2, \ldots, a_n$ is contained in $\{G_{\ell+1}\}_{\ell \in \mathbb{N}}$. Pick the maximal integer $s$ such that $1 \leq s \leq n$ and $a_s$ is contained in $\{G_{\ell+1}\}_{\ell \in \mathbb{N}}$. Eliminate $a_s$ using \eqref{e:69} and straighten the result using the commutator relations in Lemma \ref{lem:6.1}. These moves and induction show that $a_1 \ast a_2 \ast \cdots \ast a_n$ is a linear combination of vectors, each contained in the span of \eqref{e:72}. So $a_1 \ast a_2 \ast \cdots \ast a_n$ is contained in the span of \eqref{e:72}, as desired. By the above comments the vector space $U$ is spanned by \eqref{e:72}. Proposition \ref{prop:3.9}(ii) is now satisfied. By that proposition we find that $\Omega$ in order $<$ is a PBW basis for $U$.

(ii) Similar to the proof of (i) above, with line \eqref{e:74} replaced by

$$W_{k+1} < G_{\ell+1} < \tilde{G}_{j+1} < W_{-i} \quad \text{for } i, j, k, \ell \in \mathbb{N},$$

and use \eqref{e:70} instead of \eqref{e:69}.

(iii) By (ii) above and case 3 of Proposition \ref{prop:6.2}.

(iv) By (i) above and case 4 of Proposition \ref{prop:6.2}.

(v) By (ii) above and case 4 of Proposition \ref{prop:6.2}.

(vi) By (i) above and case 3 of Proposition \ref{prop:6.2}.

$\square$
Theorem 10.2. A PBW basis for $U$ is obtained by the elements

\[ \{W_{-i}\}_{i \in \mathbb{N}}, \quad \{G_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}} \]

in any linear order $<$ that satisfies one of (i)–(vi) below:

(i) $W_{-i} < G_{j+1} < W_{k+1}$ for $i, j, k \in \mathbb{N}$;
(ii) $W_{k+1} < G_{j+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;
(iii) $W_{k+1} < W_{-i} < G_{j+1}$ for $i, j, k \in \mathbb{N}$;
(iv) $W_{-i} < W_{k+1} < G_{j+1}$ for $i, j, k \in \mathbb{N}$;
(v) $G_{j+1} < W_{k+1} < W_{-i}$ for $i, j, k \in \mathbb{N}$;
(vi) $G_{j+1} < W_{-i} < W_{k+1}$ for $i, j, k \in \mathbb{N}$.

Proof. Apply the automorphism $\sigma$ to everything in Theorem 10.2 and use Lemma 5.3(i). \qed

In our view, the above twelve PBW bases for $U$ are not substantially different. So we focus on the most convenient one, which is from Theorem 10.1(i).

Definition 10.3. The alternating PBW basis for $U$ consists of the elements

\[ \{W_{-i}\}_{i \in \mathbb{N}}, \quad \{\tilde{G}_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}} \]

in a linear order $<$ that satisfies

\[ W_{-i} < \tilde{G}_{j+1} < W_{k+1} \quad i, j, k \in \mathbb{N}. \]

11 Comparing the Damiani PBW basis and the alternating PBW basis

In Section 2 we described the Damiani PBW basis. In the previous section we defined the alternating PBW basis. In this section we show how these two PBW bases are related.

We will adopt the following point of view. Instead of working directly with the Damiani PBW basis (3), we will work with the closely related elements $\{xC_n\}_{n=0}^{\infty}$, $\{C_ny\}_{n=0}^{\infty}$, $\{C_n\}_{n=1}^{\infty}$ from the end of Section 4. We begin with $\{xC_n\}_{n=0}^{\infty}$ and $\{C_ny\}_{n=0}^{\infty}$.

Definition 11.1. We define some generating functions in the indeterminate $t$:

\[ C^-(t) = \sum_{n \in \mathbb{N}} t^n (xC_n), \quad C^+(t) = \sum_{n \in \mathbb{N}} t^n (C_ny). \]

Proposition 11.2. We have

\[ C^-(\omega t) = W^-(\omega^{-1}t) \star D(q^{-1}t) = D(qt) \star W^-(qt), \]
\[ C^+(\omega t) = D(q^{-1}t) \star W^+(q^{-1}t) = W^+(qt) \star D(qt). \]
Proof. We first verify $C^-(t) = W^-(q^{-1}t) \ast D(q^{-1}t)$. To do this, it suffices to show that for $n \in \mathbb{N}$,

$$xC_n = (-1)^n q^{-n} \sum_{i=0}^{n} W_{-i} \ast D_{n-i}.$$  \hfill (75)

Let $\hat{x}C_n$ denote the expression on the right in (75). We show that $xC_n = \hat{x}C_n$. We will use induction on $n$. The result holds for $n = 0$, since $xC_0 = x = \hat{x}C_0$. Next assume that $n \geq 1$. By the equation on the left in (5) together with the discussion at the end of Section 4,

$$(q - q^{-1})(xC_n) = (xC_{n-1}) \ast (xy) - (xy) \ast (xC_{n-1}).$$  \hfill (76)

By induction, the right-hand side of (76) is equal to

$$\hat{x}C_{n-1} \ast (xy) - (xy) \ast \hat{x}C_{n-1},$$

which by construction and $xy = \tilde{G}_1$ is equal to

$$(-1)^{n-1} q^{1-n} \sum_{i=0}^{n-1} (W_{-i} \ast D_{n-i-1} \ast \tilde{G}_1 - \tilde{G}_1 \ast W_{-i} \ast D_{n-i-1}),$$

which by Lemma 9.10 is equal to

$$(-1)^{n-1} q^{1-n} \sum_{i=0}^{n-1} (W_{-i} \ast \tilde{G}_1 - \tilde{G}_1 \ast W_{-i}) \ast D_{n-i-1},$$

which by (13) is equal to

$$(-1)^{n-1} q^{1-n} \sum_{i=0}^{n-1} (W_0 \ast \tilde{G}_{i+1} - \tilde{G}_{i+1} \ast W_0) \ast D_{n-i-1},$$

which after a change of variables $i \mapsto i - 1$ is equal to

$$(-1)^{n-1} q^{1-n} \sum_{i=0}^{n} (W_0 \ast \tilde{G}_i - \tilde{G}_i \ast W_0) \ast D_{n-i},$$

which by $\tilde{G}_0 = 1$ and (38) is equal to

$$(-1)^{n-1} q^{1-n} (1 - q^{-2}) \sum_{i=0}^{n} (W_0 \ast \tilde{G}_i - W_{-i} \ast D_{n-i}),$$

which by (66) and algebra is equal to

$$(-1)^{n} q^{-n} (q - q^{-1}) \sum_{i=0}^{n} W_{-i} \ast D_{n-i},$$

which is equal to $(q - q^{-1})\hat{x}C_n$. We have shown that $xC_n = \hat{x}C_n$, so (75) holds. We have verified $C^-(t) = W^-(q^{-1}t) \ast D(q^{-1}t)$. In this equation apply $\sigma S$ to each side, to get $C^+(t) = D(q^{-1}t) \ast W^+(q^{-1}t)$. The remaining equations in the proposition statement are from Lemma 9.13. \hfill \square
Next we restate Proposition 11.2 using the style of (15).

**Theorem 11.3.** For \( n \in \mathbb{N} \),

\[
xC_n = (-1)^n q^{-n} \sum_{i=0}^{n} W_{-i} \star D_{n-i} = (-1)^n q^n \sum_{i=0}^{n} D_{n-i} \star W_{-i},
\]

\[C_n y = (-1)^n q^{-n} \sum_{i=0}^{n} D_{n-i} \star W_{i+1} = (-1)^n q^n \sum_{i=0}^{n} W_{i+1} \star D_{n-i}.\]

**Proof.** For each equation in Proposition 11.2 compare the coefficient of \( t^n \) on either side. \( \square \)

**Proposition 11.4.** We have

\[
W^{-}(t) = C^{-}(-qt) \star \tilde{G}(t) = \tilde{G}(t) \star C^{-}(-q^{-1}t),
\]

\[
W^{+}(t) = \tilde{G}(t) \star C^{+}(-qt) = C^{+}(-q^{-1}t) \star \tilde{G}(t).
\]

**Proof.** Evaluate the equations in Proposition 11.2 using Lemma 9.12. \( \square \)

**Theorem 11.5.** For \( n \in \mathbb{N} \),

\[
W_{-n} = \sum_{i=0}^{n} (-1)^i q^i (xC_i) \star \tilde{G}_{n-i} = \sum_{i=0}^{n} (-1)^i q^{-i} \tilde{G}_{n-i} \star (xC_i),
\]

\[
W_{n+1} = \sum_{i=0}^{n} (-1)^i q^i \tilde{G}_{n-i} \star (C_i y) = \sum_{i=0}^{n} (-1)^i q^{-i} (C_i y) \star \tilde{G}_{n-i}.
\]

**Proof.** For each equation in Proposition 11.4 compare the coefficients of \( t^n \) on either side. \( \square \)

Now we bring in \( \{C_n\}_{n=1}^{\infty} \). Before getting into detail we emphasize one point.

**Lemma 11.6.** In the \( q \)-shuffle algebra \( \mathbb{V} \) the elements \( \{C_n\}_{n \in \mathbb{N}} \) mutually commute.

**Proof.** By the comment below Proposition 2.4 and the discussion at the end of Section 4. \( \square \)

**Definition 11.7.** We define a generating function in the indeterminate \( t \):

\[
C(t) = \sum_{n \in \mathbb{N}} t^n C_n.
\]

**Proposition 11.8.** We have

\[
C(-t) = D(qt) \star D(q^{-1}t).
\]

(77)

**Proof.** We first show that for \( n \in \mathbb{N} \),

\[
G_n = \sum_{i=0}^{n} (-1)^i q^i C_i \star \tilde{G}_{n-i} + q^2 \sum_{i=0}^{n-1} (-1)^i q^i (xC_i) \star W_{n-i}.
\]

(78)
For $n = 0$ the equation (78) holds, since each side is equal to 1. Next assume that $n \geq 1$. By the first equation in Theorem 11.5,

$$W_n = (-1)^n q^n x C_n + \sum_{i=0}^{n-1} (-1)^i q^i (xC_i) \star \tilde{G}_{n-i}. \quad (79)$$

We will evaluate (79) after some comments. By Lemma 5.4 we have $W_n = x G_n$. Also by Lemma 5.4 we have $\tilde{G}_{n-i} = W_{n-i}$ for $0 \leq i \leq n - 1$. By this and (12),

$$(xC_i) \star \tilde{G}_{n-i} = x (C_i \star \tilde{G}_{n-i} + q^2 (xC_i) \star W_{n-i}) \quad (0 \leq i \leq n - 1). \quad (80)$$

Evaluating (79) using the above comments, we obtain

$$x G_n = x \sum_{i=0}^{n} (-1)^i q^i C_i \star \tilde{G}_{n-i} + q^2 x \sum_{i=0}^{n-1} (-1)^i q^i (xC_i) \star W_{n-i}. \quad (78)$$

In the above equation, each term has an $x$ on the left; removing such $x$ we obtain (78). In terms of generating functions, (78) becomes

$$G(t) = C(-qt) \star \tilde{G}(t) + q^2 t C(-qt) \star W^+(t).$$

Adjusting this equation using the first equation in Proposition 11.2 we obtain

$$G(t) = C(-qt) \star \tilde{G}(t) + q^2 t W^-(t) \star D(t) \star W^+(t). \quad (81)$$

Comparing (67), (81) we obtain $D(q^2 t) = C(-qt) \star \tilde{G}(t)$. In this equation replace $t$ by $q^{-1} t$ and use Lemma 9.12 to obtain (77).

**Proposition 11.9.** For $n \in \mathbb{N}$,

$$C_n = (-1)^n \sum_{i=0}^{n} q^{2i-n} D_i \star D_{n-i}$$

**Proof.** In the equation (77) compare the coefficient of $t^n$ on either side.

**Corollary 11.10.** For $n \geq 1$ the following hold in the $q$-shuffle algebra $\mathbb{V}$.

(i) $C_n$ is a homogeneous polynomial in $D_1, D_2, \ldots, D_n$ that has total degree $n$, where we view each $D_i$ as having degree $i$. In this polynomial the coefficient of $D_n$ is $(-1)^n (q^n + q^{-n})$.

(ii) $D_n$ is a homogeneous polynomial in $C_1, C_2, \ldots, C_n$ that has total degree $n$, where we view each $C_i$ as having degree $i$. In this polynomial the coefficient of $C_n$ is $(-1)^n (q^n + q^{-n})^{-1}$.

**Proof.** (i) By Proposition 11.9

(ii) By (i) above and induction on $n$. 

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Corollary 11.11. For \( n \geq 1 \) the following hold in the \( q \)-shuffle algebra \( \mathbb{V} \).

(i) \( C_n \) is a homogeneous polynomial in \( \tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n \) that has total degree \( n \), where we view each \( \tilde{G}_i \) as having degree \( i \). In this polynomial the coefficient of \( \tilde{G}_n \) is \((-1)^{n+1}(q^n + q^{-n})\).

(ii) \( \tilde{G}_n \) is a homogeneous polynomial in \( C_1, C_2, \ldots, C_n \) that has total degree \( n \), where we view each \( C_i \) as having degree \( i \). In this polynomial the coefficient of \( C_n \) is \((-1)^{n+1}(q^n + q^{-n})^{-1}\).

Proof. Combine Lemma 9.7 and and Corollary 11.10.

Corollary 11.12. The following (i)–(iii) coincide:

(i) the subalgebra of the \( q \)-shuffle algebra \( \mathbb{V} \) generated by \( \{C_n\}_{n=1}^{\infty} \);

(ii) the subalgebra of the \( q \)-shuffle algebra \( \mathbb{V} \) generated by \( \{D_n\}_{n=1}^{\infty} \);

(iii) the subalgebra of the \( q \)-shuffle algebra \( \mathbb{V} \) generated by \( \{\tilde{G}_n\}_{n=1}^{\infty} \).

Proof. By Lemma 9.8 and Corollary 11.11.

In Corollary 11.12 we saw that \( \{C_n\}_{n=1}^{\infty} \) and \( \{\tilde{G}_n\}_{n=1}^{\infty} \) generate the same subalgebra of the \( q \)-shuffle algebra \( \mathbb{V} \). Next we discuss in more detail how \( \{C_n\}_{n=1}^{\infty} \) and \( \{\tilde{G}_n\}_{n=1}^{\infty} \) are related.

Lemma 11.13. We have

\[
C(-qt) \star \tilde{G}(q^2t) = D(t) = C(-q^{-1}t) \star \tilde{G}(q^{-2}t).
\]

Proof. To verify the above equations, eliminate \( C(-qt) \) and \( C(-q^{-1}t) \) using Proposition 11.8 and evaluate the result using Lemma 9.12.

Theorem 11.14. For \( n \in \mathbb{N} \),

\[
0 = \sum_{i=0}^{n} (-1)^i [2n - i]_q C_i \star \tilde{G}_{n-i}.
\] \hspace{1cm} (82)

Proof. By Lemma 11.13

\[
C(-qt) \star \tilde{G}(q^2t) = C(-q^{-1}t) \star \tilde{G}(q^{-2}t).
\]

In this equation, compare the coefficient of \( t^n \) on either side.

Remark 11.15. Using (82) we can recursively solve for \( \{C_n\}_{n=1}^{\infty} \) in terms of \( \{\tilde{G}_n\}_{n=1}^{\infty} \), and also \( \{\tilde{G}_n\}_{n=1}^{\infty} \) in terms of \( \{C_n\}_{n=1}^{\infty} \).
12 Some comments about Propositions 6.3 and 8.1

Consider the equations in Propositions 6.3 and 8.1. For each equation, it is natural to ask what is the common value of each side, in terms of the standard basis for $V$. In this section we compute this common value, and give some additional results of a similar nature.

Near the end of Section 4 we defined the Catalan words in $V$, using the notation $x = 1$ and $y = -1$. We now use this notation to define another kind of word.

**Definition 12.1.** A word $v_1v_2 \cdots v_n$ in $V$ is *constrained* whenever $v_1 + v_2 + \cdots + v_i \in \{0, \pm 1\}$ for $1 \leq i \leq n - 1$ and $v_1 + v_2 + \cdots + v_n = 0$. In this case $n$ is even.

**Example 12.2.** For $0 \leq n \leq 3$ we give the constrained words of length $2n$.

| $n$  | constrained words of length $2n$               |
|------|----------------------------------------------|
| 0    | 1                                            |
| 1    | $xy, yx$                                    |
| 2    | $xyxy, xyyx, yxyx, yxx$                    |
| 3    | $xyxyxy, xyyxy, yxxxy, xyyxy, yxxxy, yxyxy$ |

**Lemma 12.3.** For $n \in \mathbb{N}$ there are $2^n$ constrained words of length $2n$. These words have the form $b_1 b_2 \cdots b_n$ with $b_i \in \{xy, yx\}$ for $1 \leq i \leq n$.

For $n \in \mathbb{N}$ consider the sum of the constrained words that have length $2n$.

**Lemma 12.4.** For $n \in \mathbb{N}$ the above sum is equal to $(xy + yx)^n$, where the exponent is with respect to the free product.

The following results can be obtained using Lemma 5.9 and induction on $n$. The proofs are straightforward and omitted.

**Proposition 12.5.** For $n \in \mathbb{N}$,

\[
\sum_{k=0}^{n} G_{n-k} \ast W_{-k} q^{2k-n} = \sum_{k=0}^{n} W_{-k} \ast G_{n-k} q^{n-2k} = [2]_q^n (xy + yx)^n x,
\]

\[
\sum_{k=0}^{n} G_{n-k} \ast W_{k+1} q^{n-2k} = \sum_{k=0}^{n} W_{k+1} \ast G_{n-k} q^{2k-n} = [2]_q^n y (xy + yx)^n,
\]

\[
\sum_{k=0}^{n} \tilde{G}_{n-k} \ast W_{-k} q^{2k-n} = \sum_{k=0}^{n} W_{-k} \ast \tilde{G}_{n-k} q^{n-2k} = [2]_q^n x (xy + yx)^n,
\]

\[
\sum_{k=0}^{n} \tilde{G}_{n-k} \ast W_{k+1} q^{n-2k} = \sum_{k=0}^{n} W_{k+1} \ast \tilde{G}_{n-k} q^{2k-n} = [2]_q^n y (xy + yx)^n.
\]
Proposition 12.6. For $n \geq 1$,
\[
\sum_{k=0}^{n} G_{k} \ast \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k} = [2]^{n-1}_{q} (x y + y x)^{n-1} (q x y + q^{-1} y x),
\]
\[
\sum_{k=0}^{n} G_{k} \ast \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k} = [2]^{n-1}_{q} (q^{-1} x y + q y x)(x y + y x)^{n-1},
\]
\[
\sum_{k=0}^{n} \tilde{G}_{k} \ast G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k} = [2]^{n-1}_{q} (x y + y x)^{n-1} (q^{-1} x y + q y x),
\]
\[
\sum_{k=0}^{n} \tilde{G}_{k} \ast G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k} = [2]^{n-1}_{q} (q x y + q^{-1} y x)(x y + y x)^{n-1}.
\]

Proposition 12.7. For $n \in \mathbb{N}$,
\[
\sum_{k=0}^{n} W_{-k} \ast W_{k-n} q^{2k-n} = \sum_{k=0}^{n} W_{-k} \ast W_{n-k} q^{n-2k} = q [2]^{n+1}_{q} x (x y + y x)^{n} x,
\]
\[
\sum_{k=0}^{n} W_{k+1} \ast W_{n-k+1} q^{2k-n} = \sum_{k=0}^{n} W_{k+1} \ast W_{n-k+1} q^{n-2k} = q [2]^{n+1}_{q} y (x y + y x)^{n} y.
\]

Proposition 12.8. For $n \geq 1$,
\[
\sum_{k=0}^{n} G_{k} \ast G_{n-k} q^{2k-n} = \sum_{k=0}^{n} G_{k} \ast G_{n-k} q^{n-2k} = [2]^{n}_{q} y (x y + y x)^{n-1} x,
\]
\[
\sum_{k=0}^{n} \tilde{G}_{k} \ast \tilde{G}_{n-k} q^{2k-n} = \sum_{k=0}^{n} \tilde{G}_{k} \ast \tilde{G}_{n-k} q^{n-2k} = [2]^{n}_{q} x (x y + y x)^{n-1} y.
\]
Proposition 12.9. For $n \geq 1$,
\[ \sum_{k=0}^{n} G_{n-k} \star W_{-k} q^{n-2k} = q[2]_q^n (q^{-1}xy + qyx)(xy + yx)^{n-1}x, \]
\[ \sum_{k=0}^{n} W_{-k} \star G_{n-k} q^{2k-n} = q^{-1}[2]_q^n (qxy + q^{-1}yx)(xy + yx)^{n-1}x, \]
\[ \sum_{k=0}^{n} G_{n-k} \star W_{k+1} q^{2k-n} = q^{-1}[2]_q^n (qxy + q^{-1}yx)^{n-1}, \]
\[ \sum_{k=0}^{n} W_{k+1} \star G_{n-k} q^{n-2k} = q[2]_q^n (q^{-1}xy + qyx)^{n-1}x. \]

13 Directions for future research

Problem 13.1. Determine if the relations in Proposition 5.11 can be obtained directly from the relations in Propositions 5.7, 5.10.

Problem 13.2. Consider the algebra $U_q^+$ defined by generators (1) subject to the relations in Propositions 5.7, 5.10, 5.11. By those propositions the algebra $U_q^+$ is a homomorphic preimage of $U_q^+$. Determine the center of $U_q^+$ and the kernel of the homomorphism.

Problem 13.3. Find some generators \( \{W_{-k}\}_{k=0}^{\infty}, \{W_{k+1}\}_{k=0}^{\infty}, \{G_{k+1}\}_{k=0}^{\infty}, \{\tilde{G}_{k+1}\}_{k=0}^{\infty} \) for $O_q$ that satisfy the relations in [5, Definition 3.1]. The papers [2], [13], [14], [15], [16], [18] might be helpful in this direction.

Problem 13.4. In [4] Baseilhac and Kolb obtained a PBW basis for $O_q$ that involves some elements
\[ \{B_{n+\alpha_0}\}_{n=0}^{\infty}, \{B_{n+\alpha_1}\}_{n=0}^{\infty}, \{B_{n}\}_{n=1}^{\infty}, \] (83)
This PBW basis is roughly analogous to the PBW basis for $U_q^+$ given in (3). Find the relationship between the elements (83) and the $O_q$ generators in Problem 13.3. We expect that the relationship generalizes the results in Section 11 of the present paper.
14 Acknowledgment

The author thanks Pascal Baseilhac, Samuel Belliard, Jonas Hartwig, Jae-ho Lee, Kazumasa Nomura, and Travis Scrimshaw for valuable discussions about this paper and related topics.

15 Appendix A: Commutator relations for alternating words, part I

In this appendix we give a reformulation of Lemma 6.1.

Lemma 15.1. For \( i, j \in \mathbb{N} \) the following holds in the \( q \)-shuffle algebra \( V \).

(i) For \( i \leq j \),

\[
G_i \star W_{-j} = W_{-j} \star G_i + (1 - q^2) \sum_{\ell=1}^{i} W_{-\ell-j} \star G_{i-\ell} + (q^2 - 1) \sum_{\ell=1}^{i} W_{\ell-i} \star G_{j+\ell},
\]

\[
W_{-j} \star G_i = G_i \star W_{-j} + (1 - q^2) \sum_{\ell=1}^{i} G_{i-\ell} \star W_{-\ell-j} + (q^2 - 1) \sum_{\ell=1}^{i} G_{j+\ell} \star W_{\ell-i}
\]

and

\[
G_i \star W_{j+1} = W_{j+1} \star G_i + (1 - q^{-2}) \sum_{\ell=1}^{i} W_{\ell+j+1} \star G_{i-\ell} + (q^{-2} - 1) \sum_{\ell=1}^{i} W_{i-\ell+1} \star G_{j+\ell},
\]

\[
W_{j+1} \star G_i = G_i \star W_{j+1} + (1 - q^{-2}) \sum_{\ell=1}^{i} G_{i-\ell} \star W_{\ell+j+1} + (q^{-2} - 1) \sum_{\ell=1}^{i} G_{j+\ell} \star W_{i-\ell+1}
\]

and

\[
\tilde{G}_i \star W_{-j} = W_{-j} \star \tilde{G}_i + (1 - q^{-2}) \sum_{\ell=1}^{i} W_{-\ell-j} \star \tilde{G}_{i-\ell} + (q^{-2} - 1) \sum_{\ell=1}^{i} W_{\ell-i} \star \tilde{G}_{j+\ell},
\]

\[
W_{-j} \star \tilde{G}_i = \tilde{G}_i \star W_{-j} + (1 - q^{-2}) \sum_{\ell=1}^{i} \tilde{G}_{i-\ell} \star W_{-\ell-j} + (q^{-2} - 1) \sum_{\ell=1}^{i} \tilde{G}_{j+\ell} \star W_{\ell-i}
\]

and

\[
\tilde{G}_i \star W_{j+1} = W_{j+1} \star \tilde{G}_i + (1 - q^{-2}) \sum_{\ell=1}^{i} W_{\ell+j+1} \star \tilde{G}_{i-\ell} + (q^{-2} - 1) \sum_{\ell=1}^{i} W_{i-\ell+1} \star \tilde{G}_{j+\ell},
\]

\[
W_{j+1} \star \tilde{G}_i = \tilde{G}_i \star W_{j+1} + (1 - q^{-2}) \sum_{\ell=1}^{i} \tilde{G}_{i-\ell} \star W_{\ell+j+1} + (q^{-2} - 1) \sum_{\ell=1}^{i} \tilde{G}_{j+\ell} \star W_{i-\ell+1}.
\]
(ii) For $i > j$,

$$G_i \star W_{-j} = q^2 W_{-j} \star G_i + (1 - q^2) \sum_{\ell=0}^{j} W_{-i-\ell} \star G_{j-\ell} + (q^2 - 1) \sum_{\ell=1}^{j} W_{-j} \star G_{i+\ell},$$

$$W_{-j} \star G_i = q^{-2} G_i \star W_{-j} + (1 - q^{-2}) \sum_{\ell=0}^{j} G_{j-\ell} \star W_{-i-\ell} + (q^{-2} - 1) \sum_{\ell=1}^{j} G_{i+\ell} \star W_{-j}$$

and

$$G_i \star W_{j+1} = q^{-2} W_{j+1} \star G_i + (1 - q^{-2}) \sum_{\ell=0}^{j} W_{i+\ell+1} \star G_{j-\ell} + (q^{-2} - 1) \sum_{\ell=1}^{j} W_{j} \star G_{i+\ell},$$

$$W_{j+1} \star G_i = q^2 G_i \star W_{j+1} + (1 - q^2) \sum_{\ell=0}^{j} G_{j-\ell} \star W_{i+\ell+1} + (q^2 - 1) \sum_{\ell=1}^{j} G_{i+\ell} \star W_{j-\ell+1}$$

and

$$\tilde{G}_i \star W_{-j} = q^{-2} W_{-j} \star \tilde{G}_i + (1 - q^{-2}) \sum_{\ell=0}^{j} W_{-i-\ell} \star \tilde{G}_{j-\ell} + (q^{-2} - 1) \sum_{\ell=1}^{j} W_{-j} \star \tilde{G}_{i+\ell},$$

$$W_{-j} \star \tilde{G}_i = q^2 \tilde{G}_i \star W_{-j} + (1 - q^2) \sum_{\ell=0}^{j} \tilde{G}_{j-\ell} \star W_{-i-\ell} + (q^2 - 1) \sum_{\ell=1}^{j} \tilde{G}_{i+\ell} \star W_{-j}$$

and

$$\tilde{G}_i \star W_{j+1} = q^2 W_{j+1} \star \tilde{G}_i + (1 - q^2) \sum_{\ell=0}^{j} W_{i+\ell+1} \star \tilde{G}_{j-\ell} + (q^2 - 1) \sum_{\ell=1}^{j} W_{j} \star \tilde{G}_{i+\ell},$$

$$W_{j+1} \star \tilde{G}_i = q^{-2} \tilde{G}_i \star W_{j+1} + (1 - q^{-2}) \sum_{\ell=0}^{j} \tilde{G}_{j-\ell} \star W_{i+\ell+1} + (q^{-2} - 1) \sum_{\ell=1}^{j} \tilde{G}_{i+\ell} \star W_{j-\ell+1}.$$
16 Appendix B: Commutator relations for alternating words, part II

In this appendix we give a reformulation of Lemma 7.1.

Lemma 16.1. For $i, j \in \mathbb{N}$ the following holds in the $q$-shuffle algebra $\mathcal{V}$.

(i) For $i \leq j$,

\[
\tilde{G}_i \star G_j = G_j \star \tilde{G}_i + (1 - q^2) \sum_{\ell=1}^{i} W_{\ell-i} \star W_{j+\ell} - (1 - q^2) \sum_{\ell=0}^{i-1} W_{-j-\ell} \star W_{i-\ell},
\]

\[
G_i \star \tilde{G}_j = \tilde{G}_j \star G_i + (1 - q^2) \sum_{\ell=1}^{i} W_{i-\ell+1} \star W_{-j-\ell+1} - (1 - q^2) \sum_{\ell=0}^{i-1} W_{j+\ell+1} \star W_{i-\ell+1},
\]

\[
\tilde{G}_j \star G_i = G_i \star \tilde{G}_j + (1 - q^2) \sum_{\ell=1}^{i} W_{j+\ell} \star W_{-\ell-i} - (1 - q^2) \sum_{\ell=0}^{i-1} W_{i-\ell} \star W_{-j-\ell},
\]

\[
G_j \star \tilde{G}_i = \tilde{G}_i \star G_j + (1 - q^2) \sum_{\ell=1}^{i} W_{-j-\ell+1} \star W_{i-\ell+1} - (1 - q^2) \sum_{\ell=0}^{i-1} W_{i-\ell+1} \star W_{j+\ell+1},
\]

\[
W_{i+1} \star W_{-j} = W_{-j} \star W_{i+1} + (1 - q^{-2}) \sum_{\ell=0}^{i} G_{j+1+\ell} \star \tilde{G}_{i-\ell} - (1 - q^{-2}) \sum_{\ell=0}^{i} G_{i+1+\ell} \star \tilde{G}_{j+1+\ell},
\]

\[
W_{-i} \star W_{j+1} = W_{j+1} \star W_{-i} + (1 - q^{-2}) \sum_{\ell=0}^{i} \tilde{G}_{j+1+\ell} \star G_{i-\ell} - (1 - q^{-2}) \sum_{\ell=0}^{i} \tilde{G}_{i-\ell} \star G_{j+1+\ell},
\]

\[
W_{-j} \star W_{i+1} = W_{i+1} \star W_{-j} + (1 - q^{-2}) \sum_{\ell=0}^{i} G_{i-\ell} \star \tilde{G}_{j+1+\ell} - (1 - q^{-2}) \sum_{\ell=0}^{i} G_{j+1+\ell} \star \tilde{G}_{i-\ell},
\]

\[
W_{j+1} \star W_{-i} = W_{-i} \star W_{j+1} + (1 - q^{-2}) \sum_{\ell=0}^{i} \tilde{G}_{i-\ell} \star G_{j+1+\ell} - (1 - q^{-2}) \sum_{\ell=0}^{i} \tilde{G}_{j+1+\ell} \star G_{i-\ell}.
\]
(ii) For $i > j$,

\[
\tilde{G}_i \star G_j = G_j \star \tilde{G}_i + (1 - q^2) \sum_{\ell=1}^{j} W_{\ell-j} \star W_{i+\ell} - (1 - q^2) \sum_{\ell=0}^{j-1} W_{-i-\ell} \star W_{j-\ell},
\]

\[
G_i \star \tilde{G}_j = \tilde{G}_j \star G_i + (1 - q^2) \sum_{\ell=1}^{j} W_{j-\ell+1} \star W_{-i-\ell+1} - (1 - q^2) \sum_{\ell=0}^{j-1} W_{i+\ell+1} \star W_{\ell-j+1},
\]

\[
\tilde{G}_j \star G_i = G_i \star \tilde{G}_j + (1 - q^2) \sum_{\ell=1}^{j} W_{i+\ell} \star W_{\ell-j} - (1 - q^2) \sum_{\ell=0}^{j-1} W_{j-\ell} \star W_{-i-\ell},
\]

\[
G_j \star \tilde{G}_i = \tilde{G}_i \star G_j + (1 - q^2) \sum_{\ell=1}^{j} W_{-i-\ell+1} \star W_{j-\ell+1} - (1 - q^2) \sum_{\ell=0}^{j-1} W_{\ell-j+1} \star W_{i+\ell+1},
\]

\[
W_{i+1} \star W_{-j} = W_{-j} \star W_{i+1} + (1 - q^{-2}) \sum_{\ell=0}^{j} G_{i+1+\ell} \star \tilde{G}_{j-\ell} - (1 - q^{-2}) \sum_{\ell=0}^{j} G_{j-\ell} \star \tilde{G}_{i+1+\ell},
\]

\[
W_{-i} \star W_{j+1} = W_{j+1} \star W_{-i} + (1 - q^{-2}) \sum_{\ell=0}^{j} \tilde{G}_{i+1+\ell} \star G_{j-\ell} - (1 - q^{-2}) \sum_{\ell=0}^{j} \tilde{G}_{j-\ell} \star G_{i+1+\ell},
\]

\[
W_{-j} \star W_{i+1} = W_{i+1} \star W_{-j} + (1 - q^{-2}) \sum_{\ell=0}^{j} G_{j-\ell} \star \tilde{G}_{i+1+\ell} - (1 - q^{-2}) \sum_{\ell=0}^{j} G_{i+1+\ell} \star \tilde{G}_{j-\ell},
\]

\[
W_{j+1} \star W_{-i} = W_{-i} \star W_{j+1} + (1 - q^{-2}) \sum_{\ell=0}^{j} \tilde{G}_{j-\ell} \star G_{i+1+\ell} - (1 - q^{-2}) \sum_{\ell=0}^{j} \tilde{G}_{i+1+\ell} \star G_{j-\ell}.
\]
Appendix C: Examples of commutator relations

In this appendix we give some examples of commutator relations from Lemmas 6.1 and 7.1.

Length 2:

\[ W_1 \ast W_0 = W_0 \ast W_1 + (1 - q^{-2})(G_1 - \tilde{G}_1). \]

Length 3:

\[
\begin{align*}
G_1 \ast W_0 &= q^2 W_0 \ast G_1 + (1 - q^2)W_{-1}, \\
W_1 \ast G_1 &= q^2 G_1 \ast W_1 + (1 - q^2)W_2, \\
\tilde{G}_1 \ast W_0 &= q^{-2} W_0 \ast \tilde{G}_1 + (1 - q^{-2})W_{-1}, \\
W_1 \ast \tilde{G}_1 &= q^{-2} \tilde{G}_1 \ast W_1 + (1 - q^{-2})W_2.
\end{align*}
\]

Length 4:

\[
\begin{align*}
W_2 \ast W_0 &= W_0 \ast W_2 + (1 - q^{-2})(G_2 - \tilde{G}_2), \\
W_1 \ast W_{-1} &= W_{-1} \ast W_1 + (1 - q^{-2})(G_2 - \tilde{G}_2), \\
\tilde{G}_1 \ast G_1 &= G_1 \ast \tilde{G}_1 + (1 - q^2)(W_0 \ast W_2 - W_{-1} \ast W_1).
\end{align*}
\]

Length 5:

\[
\begin{align*}
G_2 \ast W_0 &= q^2 W_0 \ast G_2 + (1 - q^2)W_{-2}, \\
G_1 \ast W_{-1} &= (q^2 - 1)W_0 \ast G_2 + W_{-1} \ast G_1 + (1 - q^2)W_{-2}, \\
W_1 \ast G_2 &= q^2 G_2 \ast W_1 + (1 - q^2)W_3, \\
W_2 \ast G_1 &= (q^2 - 1)G_2 \ast W_1 + G_1 \ast W_2 + (1 - q^2)W_3, \\
\tilde{G}_2 \ast W_0 &= q^{-2} W_0 \ast \tilde{G}_2 + (1 - q^{-2})W_{-2}, \\
\tilde{G}_1 \ast W_{-1} &= (q^{-2} - 1)W_0 \ast \tilde{G}_2 + W_{-1} \ast \tilde{G}_1 + (1 - q^{-2})W_{-2}, \\
W_1 \ast \tilde{G}_2 &= q^{-2} \tilde{G}_2 \ast W_1 + (1 - q^{-2})W_3, \\
W_2 \ast \tilde{G}_1 &= (q^{-2} - 1)\tilde{G}_2 \ast W_1 + \tilde{G}_1 \ast W_2 + (1 - q^{-2})W_3.
\end{align*}
\]

Length 6:

\[
\begin{align*}
W_3 \ast W_0 &= W_0 \ast W_3 + (1 - q^{-2})(G_3 - \tilde{G}_3), \\
W_2 \ast W_{-1} &= W_{-1} \ast W_2 + (1 - q^{-2})(G_2 \ast \tilde{G}_1 + G_3 - G_1 \ast \tilde{G}_2 - \tilde{G}_3), \\
W_1 \ast W_{-2} &= W_{-2} \ast W_1 + (1 - q^{-2})(G_3 - \tilde{G}_3), \\
\tilde{G}_1 \ast G_2 &= G_2 \ast \tilde{G}_1 + (1 - q^2)(W_0 \ast W_3 - W_{-2} \ast W_1), \\
\tilde{G}_2 \ast G_1 &= G_1 \ast \tilde{G}_2 + (1 - q^2)(W_0 \ast W_3 - W_{-2} \ast W_1).
\end{align*}
\]
\[ G_3 \star W_0 = q^2 W_0 \star G_3 + (1 - q^2) W_{-3}, \]
\[ G_2 \star W_{-1} = q^2 W_{-1} \star G_2 + (1 - q^2) W_{-2} \star G_1 + (q^2 - 1) W_0 \star G_3 + (1 - q^2) W_{-3}, \]
\[ G_1 \star W_{-2} = W_{-2} \star G_1 + (q^2 - 1) W_0 \star G_3 + (1 - q^2) W_{-3}, \]
\[ W_1 \star G_3 = q^2 G_3 \star W_1 + (1 - q^2) W_4, \]
\[ W_2 \star G_2 = q^2 G_2 \star W_2 + (1 - q^2) G_1 \star W_3 + (q^2 - 1) G_3 \star W_1 + (1 - q^2) W_4, \]
\[ W_3 \star G_1 = G_1 \star W_3 + (q^2 - 1) G_3 \star W_1 + (1 - q^2) W_4, \]
\[ \tilde{G}_3 \star W_0 = q^{-2} W_0 \star \tilde{G}_3 + (1 - q^{-2}) W_{-3}, \]
\[ \tilde{G}_2 \star W_{-1} = q^{-2} W_{-1} \star \tilde{G}_2 + (1 - q^{-2}) W_{-2} \star \tilde{G}_1 + (q^{-2} - 1) W_0 \star \tilde{G}_3 + (1 - q^{-2}) W_{-3}, \]
\[ \tilde{G}_1 \star W_{-2} = W_{-2} \star \tilde{G}_1 + (q^{-2} - 1) W_0 \star \tilde{G}_3 + (1 - q^{-2}) W_{-3}, \]
\[ W_1 \star \tilde{G}_3 = q^{-2} \tilde{G}_3 \star W_1 + (1 - q^{-2}) W_4, \]
\[ W_2 \star \tilde{G}_2 = q^{-2} \tilde{G}_2 \star W_2 + (1 - q^{-2}) \tilde{G}_1 \star W_3 + (q^{-2} - 1) \tilde{G}_3 \star W_1 + (1 - q^{-2}) W_4, \]
\[ W_3 \star \tilde{G}_1 = \tilde{G}_1 \star W_3 + (q^{-2} - 1) \tilde{G}_3 \star W_1 + (1 - q^{-2}) W_4. \]

Length 8:

\[ W_4 \star W_0 = W_0 \star W_4 + (1 - q^{-2})(G_4 - \tilde{G}_4), \]
\[ W_3 \star W_{-1} = W_{-1} \star W_3 + (1 - q^{-2})(G_3 \star \tilde{G}_1 + G_4 - G_1 \star \tilde{G}_3 - \tilde{G}_4), \]
\[ W_2 \star W_{-2} = W_{-2} \star W_2 + (1 - q^{-2})(G_3 \star \tilde{G}_1 + G_4 - G_1 \star \tilde{G}_3 - \tilde{G}_4), \]
\[ W_1 \star W_{-3} = W_{-3} \star W_1 + (1 - q^{-2})(G_4 - \tilde{G}_4), \]
\[ \tilde{G}_1 \star G_3 = G_3 \star \tilde{G}_1 + (1 - q^2)(W_0 \star W_1 - W_{-3} \star W_1), \]
\[ \tilde{G}_2 \star G_2 = G_2 \star \tilde{G}_2 + (1 - q^2)(W_0 \star W_4 + W_{-1} \star W_3 - W_{-2} \star W_2 - W_{-3} \star W_1), \]
\[ \tilde{G}_3 \star G_1 = G_1 \star \tilde{G}_3 + (1 - q^2)(W_0 \star W_4 - W_{-3} \star W_1). \]

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Paul Terwilliger
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, WI 53706-1388 USA
email: terwilli@math.wisc.edu

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