1. Definitions

In this note we are only concerned with simple undirected graphs $G = (V, E)$ where $V$ is a set and $E \subseteq \mathcal{P}_2(V)$ where

$$\mathcal{P}_2(V) = \{\{x, y\} : x, y \in V \text{ and } x \neq y\}.$$  

We denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. Moreover, for any cardinal $\alpha$ we denote the complete graph on $\alpha$ points by $K_\alpha$.

For any graph $G$, disjoint subsets $S, T \subseteq V(G)$ are said to be connected to each other if there are $s \in S, t \in T$ with $\{s, t\} \in E(G)$.

Given a collection $\mathcal{D}$ of pairwise disjoint, nonempty, connected subsets of $V$, we associated with $\mathcal{D}$ a graph $G(\mathcal{D})$ with vertex set $\mathcal{D}$ and

$$E(G(\mathcal{D})) = \{\{d, e\} : d \neq e \in \mathcal{D} \text{ and } d, e \text{ are connected to each other}\}.$$  

We say that a graph $M$ is a minor of a graph $G$ if there is a collection $\mathcal{D}$ of pairwise disjoint, nonempty, connected subsets of $V$ and an injective graph homomorphism $f : M \to G(\mathcal{D})$.

This implies that $K_\alpha$ is a minor of a graph $G$ if and only if there is a collection $\{S_\beta : \beta \in \alpha\}$ of nonempty, connected and pairwise disjoint subsets of $V(G)$ such that for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ the sets $S_\beta$ and $S_\gamma$ are connected to each other.

2. Different statements of Hadwiger’s conjecture

The statement of Hadwiger’s conjecture that is usually found in the literature is the following:

**\(H\)**: If $G$ is a simple undirected graph and $\lambda = \chi(G)$ then the complete graph $K_\lambda$ is a minor of $G$.

---

[2010 Mathematics Subject Classification. 05C15, 05C83.]
The next version of Hadwiger’s statement has a bit of a different flavor, and we will compare it to (H) in the finite and infinite contexts in the following sections.

\textbf{(ModH)}: For every graph $G$ there is a minor $M$ of $G$ such that

(1) $M \not\cong G$, and
(2) $\chi(M) = \chi(G)$.

There is a version of (ModH) that appears to be similar, but we will see later that it is worthwhile to look at the statement separately.

\textbf{(HomH)}: For every graph $G$ there is a minor $M$ of $G$ such that

(1) $M \not\cong G$, and
(2) there is a graph homomorphism $f : G \to M$.

Last, the following weaker version of this was studied in [4]:

\textbf{(WeakH)}: Whenever $\lambda$ is a cardinal such that there is no graph homomorphism $c : G \to K_\lambda$ then $K_\lambda$ is a minor of $G$.

3. The finite case

Overview:

- (H) is a long-standing open problem.
- (ModH) is equivalent to (H) for finite graphs (see proposition 3.1).
- (HomH) is also equivalent to (H) for finite graphs.
- (WeakH) is implied by (H).

\textbf{Proposition 3.1.} For finite graphs $G$, the statements (H) and (ModH) are equivalent.

\textit{Proof.} Given a finite non-complete graph $G = (V, E)$, the statement (H) implies that $K = K_{\chi(G)}$ is a minor of $G$. Since $K$ is complete, but not $G$, they are not isomorphic, so (ModH) holds.

For the other implication, take any finite graph $G$ and let $n = \chi(G)$. Use (ModH) to get a proper minor $M_1$ such that $\chi(M_1) = n$. If $M_1$ is complete, we have proved (H), otherwise use (ModH) again to find a proper minor $M_2$ of $M_1$ with $\chi(M_2) = n$, and so on. Since $G$ is finite, this procedure is bound to end at some $M_k$ for some $k \in \mathbb{N}$, which implies that $M_k$ is complete and has $n$ points.

It is easy to modify Proposition 3.1 to see that in the finite case, (H) and (HomH) are equivalent.
In the finite setting, the statement (WeakH) amounts to saying that if \( \chi(G) = t > 0 \) then \( K_{t-1} \) is a minor of \( G \). This is weaker than (H); whether it is strictly weaker is an open question (see section 5).

### 4. The infinite case

#### 4.1. Infinite chromatic number. Overview:

- (H) is \textbf{false}: Let \( G \) be the disjoint union of all \( K_n, n \in \mathbb{N} \). Then \( \chi(G) = \omega \), but \( K_\omega \) is not a minor of \( G \).
- (ModH) is \textbf{true}, see proposition 4.1.
- (HomH) is open.
- (WeakH) is \textbf{true}, see [4].

So that is why we separately introduced (HomH) in addition to (ModH): they might be different for graphs with infinite chromatic number.

**Proposition 4.1.** For graphs with infinite chromatic number, (ModH) is true.

**Proof.** Let \( I \) be the set of isolated vertices of \( G \).

\textbf{Case 1.} \( I \neq \emptyset \). We set \( M = G \setminus I \). It is easy to see that \( M \not\cong G \) as \( M \) contains no isolated points. Since \( \chi(G) \geq \aleph_0 \) we have \( \chi(M) = \chi(G) \).

\textbf{Case 2.} \( I = \emptyset \). Fix \( v_0 \in V(G) \). Let \( M = (V(G), E) \) where

\[ E = \{ e \in E(G) : v_0 \notin e \}, \]

that is we remove all edges connecting \( v_0 \) to some other vertex in \( V(G) \). Since \( M \) has \( v_0 \) as an isolated point, but \( G \) has no isolated points, we have \( M \not\cong G \), and it is easy to verify that \( \chi(M) = \chi(G) \).

#### 4.2. Finite chromatic number. For infinite graphs with finite chromatic number we get the following results:

- It is not known whether (H) and (WeakH) are true;
- (ModH) is \textbf{true}: the theorem of De Bruijn and Erdős [1] implies that if \( G \) is infinite with finite chromatic number, there is a finite subgraph \( M \) of \( G \) with \( \chi(M) = \chi(G) \).
- (HomH) is true for the same reason (note that a coloring is always a graph homomorphism to a complete graph).

### 5. Open questions

**Question 1.** Does the weak Hadwiger conjecture (WeakH) hold for finite graphs?
(WeakH) might be as elusive has (H) has been so far; so here is a different problem:

**Question 2.** When we restrict ourselves to finite graphs, does the weak Hadwiger conjecture (WeakH) imply the statement of the Hadwiger conjecture?

The next question is a stronger version of (ModH) and focuses on finite graphs.

**Question 3.** Suppose that $G$ is a finite, connected graph such that whenever you contract 1 edge or 2 edges, the chromatic number decreases. Does this imply $G$ is complete?

Finally we turn to infinite graphs:

**Question 4.** Does (WeakH) hold for infinite graphs with finite chromatic number?

**Question 5.** Does (HomH) hold for graphs with infinite chromatic number?

6. **Acknowledgement**

I would like to thank user @bof from mathoverflow.net for his argument used in proposition [4][5].

**References**

[1] de Bruijn Nicolaas, Erdős Paul, *A colour problem for infinite graphs and a problem in the theory of relations*, Nederl. Akad. Wetensch. Proc. Ser. A, **53** (1951), 371–373.

[2] Hadwiger Hugo, *Über eine Klassifikation der Streckenkomplexe*, Vierteljschr. Naturforsch. Ges. Zürich, **88** (1943), 133–143.

[3] Robertson Neil, Seymour Paul, Thomas Robin, *Excluding subdivisions of infinite cliques*, Trans. Amer. Math. Soc. (**332**) (1992), no. 1, 211–223.

[4] van der Zypen Dominic, *A weak form of Hadwiger’s conjecture*, SOP Trans. on Applied Mathematics **1** (2014), no. 2, 84–87.

[5] [http://mathoverflow.net/a/221663/8628](http://mathoverflow.net/a/221663/8628)

Federal office of social insurance, Effingerstrasse 20, CH-3003 Bern, Switzerland

E-mail address: dominic.zypen@gmail.com