On global classical solutions to one-dimensional compressible Navier–Stokes/Allen–Cahn system with density-dependent viscosity and vacuum

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Abstract
In this paper, by using the energy estimates, the structure of the equations, and the properties of one dimension, we establish the global existence and uniqueness of strong and classical solutions to the initial boundary value problem of compressible Navier–Stokes/Allen–Cahn system in one-dimensional bounded domain with the viscosity depending on density. Here, we emphasize that the time does not need to be bounded and the initial vacuum is still permitted. Furthermore, we also show the large time behavior of the velocity.

MSC: 35D35; 76D03

1 Introduction
The Navier–Stokes/Allen–Cahn system, which is a combination of the compressible Navier–Stokes equations with an Allen–Cahn phase field description, is considered in this paper. Mathematically, in one dimension, this model reads as follows [5] (cf. [1]):

\[ \rho_t + (\rho u)_x = 0, \]
\[ \rho u_t + \rho uu_x + (\rho^\gamma)_x = (v(\rho)u)_x - \frac{\delta}{2} (\chi^2_x)_x, \]
\[ \rho \chi_t + \rho u \chi_x = -\mu, \]
\[ \rho \mu = -\delta \chi_{xx} + \frac{\rho}{\delta} (\chi^3 - \chi) \]

for \((t, x) \in (0, +\infty) \times [0, 1]\). Here, \(\rho, u, \) and \(\chi\) represent the density of the fluid, the mean velocity of the fluid mixture, and the concentration of one selected constituent, respectively; \(\mu\) is the chemical potential, \(\sqrt{\delta}\) represents the thickness of the interfacial region. The viscous coefficient \(v(\rho) > 0\) satisfies

\[ 0 < \tilde{v} \leq v(\rho). \]
We supplement (1)–(4) with the initial value conditions

\[(\rho, u, \chi)(0, x) = (\rho_0, u_0, \chi_0)(x), \quad x \in [0, 1], \quad (6)\]

and the no-slip boundary conditions for viscous fluids and the concentration difference

\[(u, \chi)(t, 0) = (u, \chi)(t, 1) = (0, 0), \quad t \geq 0. \quad (7)\]

Before stating our main results, we review some previous works on this topic. For 1-dimensional compressible Navier–Stokes/Allen–Cahn system, Ding et al. [5] established the existence and uniqueness of local and global classical solutions for initial data \(\rho_0\) without vacuum states. Besides, Ding et al. [6] proved the existence and uniqueness of global strong solutions to (1)–(4) with free boundary conditions and with the lower bound of the initial density. Yin et al. [18] investigated the large time behavior of the solutions to the inflow problem in the half space, and they obtained that the nonlinear wave is asymptotically stable if the initial data has a small perturbation. Recently, Luo et al. [15] (see also [14]) proved that the system tends to the rarefaction wave time-asymptotically, where the strength of the rarefaction wave is not required to be small. Chen et al. [2] established the global strong and classical solutions with initial vacuum in bounded domains. After that, Chen et al. [4] established the blowup criterion of the strong solutions with the viscosity depending on the density and the concentration of one selected constituent. Very recently, Yan et al. [17] considered the global existence of strong solutions with the phase variable dependent viscosity and the temperature dependent heat-conductivity without vacuum.

For the multi-dimensional compressible Navier–Stokes/Allen–Cahn system, Kotschote [11] established the local existence of a unique strong solution without initial vacuum. Later on, Feireisl et al. [8] proved the existence of weak solutions in 3D, where the density \(\rho\) is a measurable function, and they [9] obtained the global weak solutions in the bounded domain of \(\mathbb{R}^3\) without any restriction on the initial data for \(\gamma > 6\), which was extended to \(\gamma > 2\) by Chen et al. [3]. Hošek et al. [10] considered the weak-strong uniqueness result in a bounded domain of \(\mathbb{R}^3\) under the incompressibility assumption, which is relying on the relative entropy method. Very recently, Feireisl et al. [7] proved that the model is thermodynamically consistent, particularly, a variant of the relative energy inequality holds. At the same time, they obtained the weak-strong uniqueness principle and showed the low Mach number limit to the standard incompressible model. For more related results, we refer the readers to Zheng et al. [19], Liu et al. [12], and Ma et al. [16].

Although considerable progress has been made to the compressible Navier–Stokes/Allen–Cahn system, one of the natural questions is whether one could obtain the global classical solutions without any small assumption on the initial data or perturbations, where the time \(t\) could tend to \(+\infty\)? Motivated by [13], we give a partial answer to this question.

Our first main result in the paper is the following.

**Theorem 1** Assume that \(0 \leq \rho_0 \in H^1\), \(\mu_0 \in L^2\), \(u_0 \in H^1_0\), and \(\chi_0 \in H^1_0 \cap H^2\). Then there exists a global strong solution \((\rho, u, \chi)\) to the initial boundary value problem (1)–(7) such
that, for all $T \in (0, +\infty)$,

\[
\begin{aligned}
\rho &\in C([0, T]; H^1), \\
\rho_t &\in L^\infty(0, T; L^2), \\
u &\in L^\infty(0, T; H_0^1) \cap L^2(0, T; H^2), \\
\sqrt{\rho} u_t &\in L^2(0, T; L^2), \\
\chi &\in L^\infty(0, T; H_0^1 \cap H^2), \\
(\sqrt{\rho} u, \chi) &\in L^\infty(0, T; H^2), \\
(\sqrt{\rho} u_t, \chi_t) &\in L^2(0, T; H_0^1).
\end{aligned}
\] (8)

Especially, the density can remain uniformly bounded for all time, that is,

\[
\begin{aligned}
\sup_{0 \leq t < +\infty} \| \rho(t, \cdot) \|_{L^\infty} < +\infty, \\
\sup_{0 \leq t < +\infty} \| \chi(t, \cdot) \|_{H_0^1 \cap H^2} < +\infty,
\end{aligned}
\] (9) (10)

and

\[
\lim_{t \to +\infty} \| u(t, \cdot) \|_{W^{1,p}} = 0, \quad \forall p \in [1, \infty).
\] (11)

The following result means that the strong solution obtained by Theorem 1 is a classical solution provided that the initial data $(\rho_0, u_0, \chi_0)$ satisfies some additional conditions.

**Theorem 2** Assume $v(\rho) \in C^2[0, \infty)$, and the initial data $(\rho_0, u_0, \chi_0)$ satisfies

\[
(\rho_0, \rho_0^\gamma) \in H^2, \quad (u_0, \chi_0) \in H_0^1 \cap H^2
\]

and the following compatibility condition

\[
\begin{aligned}
[v(\rho_0) u_0]_x - a(\rho_0^\gamma)_x &= \sqrt{\rho_0} g, \\
\mu_0 &= \sqrt{\rho_0} h,
\end{aligned}
\] (12)

where $g \in L^2$ and $h \in H^1$. Then the strong solution $(\rho, u, \chi)$ obtained in Theorem 1 becomes a classical solution and satisfies, for any $0 < T < +\infty$,

\[
\begin{aligned}
\rho, \rho^\gamma &\in C([0, T]; H^2), \\
\rho_t, \rho^\gamma_t &\in C([0, T]; H^1), \\
\rho_u, \rho^\gamma_u &\in L^2(0, T; L^2), \\
u &\in C([0, T]; H^2) \cap L^2(0, T; H^3), \\
\chi &\in C([0, T]; H^3), \\
(u_t, \chi_t) &\in L^2(0, T; H_0^1), \\
(\sqrt{\rho} u, \chi) &\in L^\infty(0, T; H^2), \\
(\sqrt{\rho} u_t, \sqrt{\rho} \chi_t) &\in L^\infty(0, T; H^2) \cap L^2(0, T; H^3), \\
(\sqrt{\rho} u_t, \sqrt{\rho} \chi_t) &\in L^2(0, T; L^2).
\end{aligned}
\] (13)
A few remarks are listed in order:

**Remark 1** Compared with the previous results [2], ours are more general. First, the viscosity $\nu(\rho)$ depends on the density; Second, we remove the compatibility condition in obtaining the strong solution in Theorem 1, and Theorem 2 is established under (12), which has improved Theorem 2 in [2]; Third, the density $\rho$ is uniformly bounded for all time and the large time behavior of $u$ is also obtained, see (11) for details.

**Remark 2** Similar to [2], we have to use the no-slip boundary condition on $\chi$ to deal with the term $\int \rho^2 u_x \chi^2 \, dx$ in (29) because $\|u_\tau\|_{L^\infty}$ is not time-integrable when we establish the time-independent lower order estimates, see (26) below.

**Remark 3** The concentration $\chi$ is uniformly (in time) bounded with higher order estimates in (10), without any decay as $t \to +\infty$, perhaps because it appears in the hyperbolicity in (3) rather than in the parabolicity in (24).

We now make some comments on the analysis of this paper. To obtain the results stated in Theorems 1 and 2, which mainly establish the time-independent lower order estimates and the time-dependent higher order ones, the method used in [2] is not suitable here, due to the all time-dependent a priori estimates. Moreover, it is difficult to obtain the large time behavior of solutions (11). Here, it is noted that we borrow some ideas from [13], where they discussed the global large classical solutions to the compressible Navier–Stokes equations. The key uniform upper bound of the density is obtained by Zlotnik's inequality, which is also successfully used to system (1)–(7) (see Lemma 4). Furthermore, the key time-independent $L^2$-norm of $u_x$ is bounded by the material derivative $u_\tau + uu_x$ (see Lemma 5). With the lower order estimates obtained in Lemmas 3–6, the time-dependent higher order estimates on $(\rho, u, \chi)$ are obtained by standard energy estimates and the properties of one dimension.

The paper is organized as follows. In the next section, we deduce the desired estimates globally in time. By using the a priori estimates obtained in Sect. 2, we complete the proofs of Theorems 1 and 2 in Sect. 3.

2 A priori estimates

In this section, we establish some necessary a priori estimates of the solutions to (1)–(7) to extend the local solution to a global one, which is guaranteed by the following Lemma 1, whose proof can be obtained by similar arguments as those in [5].

**Lemma 1** Assume that $\rho_0 \in C^{1,\alpha}$ satisfies $0 < C_0^{-1} \leq \rho_0 \leq C_0$ for some constant $\alpha \in (0, 1)$ and $C_0 > 0, u_0, \chi_0 \in C^{2,\alpha}$. Then there exists a small time $T_0 > 0$ depending only on $(\rho_0, u_0, \chi_0)$ such that the initial boundary value problem (1)–(7) admits a unique classical solution $(\rho, u, \chi)$ satisfying that

$$(\rho_\tau, \rho_x) \in C^{0, \frac{\alpha}{2}}, \quad 0 < C^{-1} \leq \rho \leq C, \quad (u, \chi) \in C^{2+\alpha, \frac{2+\alpha}{2}},$$

where the $C^{a,b}$ is the usual Hölder space.

Before starting the a priori estimates, we list Zlotnik's inequality which could be found in [20] and will be used to establish the uniform upper bound of the density.
Lemma 2  Let the function \( y \) satisfy
\[
y'(t) = g(y) + b'(t), \quad \text{on } [0, T], \quad y(0) = y^0,
\]
with \( g \in C(\mathbb{R}) \) and \( y, b \in W^{1,1}(0, T) \). If \( g(\infty) = -\infty \) and
\[
b(t_2) - b(t_1) \leq N_0 + N_1 (t_2 - t_1),
\]
for all \( 0 \leq t_1 < t_2 \leq T \) with some nonnegative constants \( N_0 \) and \( N_1 \), then
\[
y(t) \leq \max \{ y^0, \bar{\zeta} \} + N_0 < \infty \quad \text{on } [0, T],
\]
where \( \bar{\zeta} \) is a constant such that
\[
g(\zeta) \leq -N_1 \quad \text{for } \zeta \geq \bar{\zeta}.
\]

2.1 A priori estimates (I): Lower order estimates
We emphasize that, in this subsection, \( C \) denotes some positive constant, which may be changed line by line and depends only on \( \bar{\nu}, \delta, \gamma \) and the initial data \( (\rho_0, u_0, \chi_0) \), but without the lower bound of the initial density \( \rho_0 \) and the length of \( T \). First of all, we have the following basic energy estimates.

Lemma 3  Let \( (\rho, u, \chi) \) be a smooth solution of (1)–(7) on \( (0, T) \times [0,1] \). Then one has
\[
\sup_{t \in [0,T]} \int \left( \frac{\rho u^2}{2} + \frac{\rho \gamma}{\gamma - 1} + \frac{\rho (\chi^2 - 1)^2}{4} + \frac{\chi_x^2}{2} \right) dx + \int_0^T \int (u_x^2 + \mu^2) dx dt \leq C. \tag{14}
\]

Proof  This lemma can be obtained by standard energy estimates. Multiplying (2), (3) by \( u \) and \( \mu \), respectively, by integrating by parts and by using (1) and (4), we obtain (14), the details can be found in [2]. \( \square \)

Due to the basic energy inequality (14), we first consider the uniform upper bound of the density \( \rho \), which does not depend on the length of time \( T \).

Lemma 4  Let \( (\rho, u, \chi) \) be a smooth solution of (1)–(7) on \( (0, T) \times [0,1] \). Then one has
\[
0 \leq \rho(t,x) \leq C. \tag{15}
\]

Proof  To prove this lemma, we borrow some some ideas of [13]. First of all, integrating (2) over \( (0, x) \), we obtain
\[
\frac{\partial}{\partial t} \int_0^x \rho u \, dy + \rho u_x^2 + \rho \gamma - v(\rho) u_x + \frac{\delta}{2} \chi_x^2 \tag{16}
\]
\[
= \left[ \rho u_x^2 + \rho \gamma - v(\rho) u_x + \frac{\delta}{2} \chi_x^2 \right](t,0),
\]

which implies that
\[
\left[ \rho u^2 + \rho \gamma - v(\rho)u_x + \frac{\delta}{2} \chi^2 \right](t,0) \tag{17}
\]
\[
= \frac{\partial}{\partial t} \int_0^1 \int_0^x \rho u dy \, dx + \int_0^1 \rho u^2 \, dx + \int_0^1 \rho \gamma \, dx - \int_0^1 v(\rho)u_x \, dx + \frac{\delta}{2} \int_0^1 \chi^2 \, dx.
\]
Combining (16) with (17), it follows from (14) that
\[
\rho \gamma - v(\rho)u_x + \frac{\delta}{2} \chi^2 \tag{18}
\]
\[
= \int_0^1 \rho u^2 \, dx + \int_0^1 \rho \gamma \, dx - \int_0^1 \rho (\rho)u_x \, dx + \frac{\delta}{2} \int_0^1 \chi^2 \, dx
+ \frac{\partial}{\partial t} \int_0^1 \rho \int_0^\rho v(s)s^{-2} \, ds \, dx + \int_0^1 \int_0^x \rho u dy \, dx - \int_0^1 \rho u dy
\]
\[
\leq C + \frac{\partial}{\partial t}(B_1(t) + B_2(t) + B_3(t)),
\]
where we have used the following fact (due to (1))
\[
-\int_0^1 v(\rho)u_x \, dx = D_t \int_0^1 \rho \int_0^\rho v(s)s^{-2} \, ds \, dx.
\]
The notion \(D_t f(t,x)\) denotes the derivation operator \(D_t f(t,x) = \partial_t f(t,x) + u \partial_x f(t,x)\).

Next, direct calculations show that
\[
- v(\rho)u_x = D_t \int_1^\rho v(s)s^{-1} \, ds,
\]
which together with (18) yields
\[
D_t \int_1^\rho v(s)s^{-1} \, ds \leq -\rho \gamma - \frac{\delta}{2} \chi^2 + C + D_t (B_1(t) + B_2(t) + B_3(t)) \tag{19}
\]
\[
\leq -\rho \gamma + C + D_t (B_1(t) + B_2(t) + B_3(t)).
\]
Now, we focus on the estimates of the last term on the right-hand side of (19). First, by (14) and Hölder’s inequality, we easily obtain
\[
|B_2(t)| + |B_3(t)| \leq C \int_0^1 \int_0^x \rho dy \int_0^x \rho u^2 \, dx + \int_0^1 \rho dy \int_0^1 \rho u^2 \, dx \leq C. \tag{20}
\]
Next, we also have
\[
B_1(t) \leq \frac{1}{2} \int_0^\max(\sup_{[0,T] \times [0,1]} \rho)^2 \int_0^\rho v(s)s^{-1} \, ds.
\]
Finally, it follows from Lemma 2, Zlotnik’s inequality, and (19) that
\[
\int_1^\rho v(s)^{-1} \, ds \leq C + \frac{1}{2} \int_1^\rho v(s)^{-1} \, ds,
\]
which together with (5) shows (15). This completes the proof. \(\square\)

Next, we focus on \(L^2\)-estimates about \(\rho \chi_t, u_s,\) and \(\chi_{xx},\) which are the key estimates for the proofs of the main theorems.

**Lemma 5** Let \((\rho, u, \chi)\) be a smooth solution of (1)–(7) on \((0, T) \times [0, 1].\) Then one has
\[
\sup_{t \in [0, T]} \left( \|\rho \chi_t\|_{L^2}^2 + \|u_s\|_{L^2}^2 + \|\chi_{xx}\|_{L^2}^2 \right)
\]
\[
+ \int_0^T \left( \|\chi_{xx}\|_{L^2}^2 + \sqrt{\rho} \|u_t + uu_s\|_{L^2}^2 \right) \, dt \leq C.
\]

**Proof** First, multiplying (2) by \(u_t + uu_s\) and integrating the resultant equality by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 v(\rho) u_s^2 \, dx + \int_0^1 \rho (u_t + uu_s)^2 \, dx
\]
\[
= \frac{d}{dt} \left( \int_0^1 \rho^\gamma u_s \, dx + \frac{\delta}{2} \int_0^1 \chi_s^2 u_s \, dx \right) - \frac{1}{2} \int_0^1 \left[ v(\rho) - \rho^\gamma(\rho) \right] u_s^2 \, dx
\]
\[
+ \gamma \int_0^1 \rho^\gamma u_s^2 \, dx - \delta \int_0^1 \chi_s \chi_{xx} u_s \, dx - \delta \int_0^1 \chi_s \chi_{xx} uu_s \, dx
\]
\[
\leq \frac{d}{dt} \left( \int_0^1 \rho^\gamma u_s \, dx + \frac{\delta}{2} \int_0^1 \chi_s^2 u_s \, dx \right) + C \|u_s\|_{L^2}^3 + C \|u_t\|_{L^2}^2
\]
\[
+ C \|\chi_t\|_{L^\infty} \|\chi_{xx}\|_{L^2} \|u_s\|_{L^2} + C \|\chi_x\|_{L^\infty} \|\chi_{xx}\|_{L^2} \|u_s\|_{L^\infty} \|u_s\|_{L^2}
\]
\[
\leq \frac{d}{dt} \left( \int_0^1 \rho^\gamma u_s \, dx + \frac{\delta}{2} \int_0^1 \chi_s^2 u_s \, dx \right) + C \|u_s\|_{L^\infty} \|u_s\|_{L^2}^2 + C \|u_s\|_{L^2}^2
\]
\[
+ C \|\chi_{xx}\|_{L^2} \|\chi_{xx}\|_{L^2} \|u_s\|_{L^2} + C \|\chi_{xx}\|_{L^2} \|\chi_{xx}\|_{L^2} \|u_s\|_{L^2}^2.
\]

Next, let us rewrite (3) and (4) as
\[
\rho^2 \chi_t + \rho^2 u \chi_x = \chi_{xx} - \frac{\rho}{\delta} (\chi^3 - \chi),
\]
from which, due to (14), (15), and Young’s inequality, we obtain
\[
\|\chi_{xx}\|_{L^2} \leq C \|\rho \chi_t\|_{L^2} + C \|\sqrt{\rho} u\|_{L^2} \|\chi_s\|_{L^\infty} + C
\]
\[
\leq \|\rho \chi_t\|_{L^2} + \frac{1}{2} \|\chi_{xx}\|_{L^2} + C.
\]

Next, due to (2), (14), (15), and (25), we obtain
\[
\|u_s\|_{L^\infty} \leq C \|v(\rho) u_x - \rho \gamma\|_{L^1} + C \|\rho^\gamma\|_{L^\infty}
\]
\[
\leq C \|v(\rho) u_x - \rho \gamma\|_{L^1} + C \|v(\rho) u_x - \rho \gamma\|_{L^1} + C.
\]
\[
\leq C \|v(\rho)u_x\|_{L^1} + C \|\rho\\|_{L^1} + C \|\rho(u_t + uu_x)\|_{L^1} + C \|\chi_x\|_{L^2} \|\chi_{xx}\|_{L^2} + C
\]
\[
\leq C \int_0^1 v(\rho)u_x^2 \, dx + C \|\sqrt{\rho}(u_t + uu_x)\|_{L^2} + C \|\chi_{xx}\|_{L^2} + C.
\]

Then, substituting (25) and (26) into (23), and integrating the resultant inequality over \((0, t)\), one has

\[
\int_0^1 v(\rho)u_x^2 \, dx + \int_0^t \int_0^1 \rho(u_t + uu_x)^2 \, dx \, dt
\]
\[
\leq C + C \int_0^1 \left( \int_0^1 v(\rho)u_x^2 \, dx \right)^2 \, ds + C_1 \|\chi_t\|_{L^2}^2 + \varepsilon \int_0^t \|\chi_{xx}\|_{L^2}^2 \, ds
\]
\[
+ C \int_0^t \left( \int_0^1 v(\rho)u_x^2 \, dx \right) \|\rho\chi_t\|_{L^2}^2 \, ds,
\]

where we have used the following fact:

\[
\int_0^1 \rho^\gamma u_x \, dx + \int_0^1 \chi^2 u_x \, dx \leq \frac{1}{4} \int v(\rho)u_x^2 \, dx + C_1 \|\rho\chi_t\|_{L^2}^2 + C
\]
due to (15) and (25).

Next, differentiating (24) with respect to \(t\), we deduce that

\[
\rho^2 \chi_{tt} + (\rho^2) \chi_t + \rho^2 u \chi_{xt}
\]
\[
= \chi_{xt} - (\rho^2)u \chi_x - \rho^2 u_t \chi_x - \rho (\chi^3 - \chi) - \rho (3\chi^2 - 1) \chi_t.
\]

Then, multiplying (28) by \(\chi_t\) and integrating the resultant equality by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int \rho^2 \chi_t^2 \, dx + \int \chi_{tt}^2 \, dx
\]
\[
= \frac{1}{2} \int \rho^2 u_x \chi_t^2 \, dx - 2 \int \rho^2 u \chi_t \chi_{xt} \, dx - \int \rho^2 u_u \chi_x \chi_{xt} \, dx - \int \rho^2 u^2 \chi_{xx} \chi_t \, dx
\]
\[
- \int \rho^2 u^2 \chi_{xx} \chi_{xt} \, dx - \int \rho^2 u_x \chi_x \chi_{xt} \, dx - \int \rho u (3\chi^2 - 1) \chi_{xt} \, dx
\]
\[
- \int \rho u (\chi^3 - \chi) \chi_{xt} \, dx - \int \rho (3\chi^2 - 1) \chi_t^2 \, dx = \sum_{i=1}^9 I_i.
\]

Now, we estimate each term on the right-hand side of (29). First, due to Sobolev’s inequality, Hölder’s inequality, and Young’s inequality, one has

\[
I_1 \leq C \|\rho\|_{L^\infty} \|\chi_t\|_{L^\infty} \|u_x\|_{L^2} \|\rho\chi_t\|_{L^2}
\]
\[
\leq \varepsilon \|\chi_{xx}\|_{L^2}^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) \|\rho\chi_t\|_{L^2}^2.
\]
Similarly, we obtain
\[ I_2 \leq \varepsilon \||x_0||_{L^2}^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) \||x_1||_{L^2}^2, \]
\[ I_3 + I_5 + I_8 \leq \varepsilon \||x_0||_{L^2}^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right)^2. \]

Due to (25), we obtain
\[ I_4 \leq C \||u||_{L^\infty}^2 \||\rho x_1||_{L^2}^2 \|x_0\|_{L^2}^2 \|
\leq C \||u||_{L^2}^2 \||\rho x_1||_{L^2}^2 + C \||u||_{L^2}^3 + C \||u||_{L^2}^2
\leq C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) \||x_1||_{L^2}^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right)^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right).

From (3), (14), and (25), we have
\[ I_6 = -\int_0^1 \rho^2(u_x + uu_x)x_1x_1 \, dx + \int_0^1 \rho^2 uu_x x_1x_1 \, dx
\leq \int_0^1 \rho^2(u_x + uu_x)x_1x_1 \, dx + \int_0^1 \rho(u_x + uu_x)x_1 \mu \, dx + \int_0^1 \rho^2 uu_x x_1x_1 \, dx
\leq C \sqrt{\rho(u_x + uu_x)} \||x_1||_{L^2} \||x_2||_{L^2}^2 + C \||x_1||_{L^\infty} \sqrt{\rho(u_x + uu_x)} \||x_1||_{L^2} \||\mu||_{L^2}
+ C \||x_1||_{L^\infty} \||x_2||_{L^2} \|ho x_1||_{L^2}
\leq C \sqrt{\rho(u_x + uu_x)} \||x_1||_{L^2}^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) \||x_1||_{L^2}^2
+ C \||\mu||_{L^2}^2 \left( \||x_1||_{L^2}^2 + \int_0^1 v(\rho)u_x^2 \, dx \right)
+ C \left( \int_0^1 v(\rho)u_x^2 \, dx \right)^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) + C \||\mu||_{L^2}^3.

By (3), (14), (15), and (25), \( I_7 \) could be rewritten and estimated as
\[ I_7 = \int_0^1 \rho u_x^2 (3 \chi^2 - 1) \chi_x^2 \, dx + \int_0^1 u(3 \chi^2 - 1) \chi_x \mu \, dx
\leq C \||u||_{L^\infty}^2 \||x_2||_{L^2}^2 + C \||u||_{L^\infty} \||x_2||_{L^2} \||\mu||_{L^2}
\leq C \||u||_{L^2}^2 \||x_2||_{L^2}^2 \||x_2||_{L^2} + C \||u||_{L^2} \||x_2||_{L^2} \||\mu||_{L^2}
\leq C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) \||x_1||_{L^2}^2 + C \left( \int_0^1 v(\rho)u_x^2 \, dx \right)^2
+ C \left( \int_0^1 v(\rho)u_x^2 \, dx \right) + C \||\mu||_{L^2}^2.
Similarly, the last term $I_9$ is

$$I_9 = \int_0^1 \rho u (3 \chi^2 - 1) \chi_s \chi_t \, dx + \int_0^1 (3 \chi^2 - 1) \mu \chi_t \, dx$$

$$\leq C \|u\|_{L^\infty} \|\chi_t\|_{L^2} \|\chi_s\|_{L^2} + C \|\chi_t\|_{L^\infty} \|\mu\|_{L^2}$$

$$\leq C \|u\|_{L^\infty} \|\chi_t\|_{L^2} \|\chi_s\|_{L^2} + C \|\chi_{xt}\|_{L^2} \|\mu\|_{L^2}$$

$$\leq \varepsilon \|\chi_{st}\|_{L^2}^2 + C \left( \int_0^1 v(\rho) u^2_t \, dx \right) + C \|\mu\|_{L^2}^2.$$ 

Then, substituting all the above estimates into (29), and then integrating it over $(\tau, t)$, where $\tau \in (0, t)$, we obtain

$$\int_0^1 \rho \chi^2(t) \, dx + \int_\tau^t \int_0^1 \chi_s^2 \, dx \, ds \leq C \int_\tau^t \left( \int_0^1 v(\rho) u^2_t \, dx \right) \|\rho \chi_t\|_{L^2}^2 \, ds + C \int_\tau^t \left( \int_0^1 v(\rho) u^2_t \, dx \right) \, ds$$

$$+ \varepsilon \int_\tau^t \|\sqrt{\rho}(u_t + uu_s)\|_{L^2}^2 \, ds + \|\rho \chi_t(\tau)\|_{L^2}^2 + C.$$ 

On the other hand, multiplying (3) by $\rho \chi_t$, we have

$$\int_0^1 \rho \chi^2(t) \, dx = -\int_0^1 \rho^2 u \chi_x \chi_s(t) \, dx - \int_0^1 \rho \chi_t \mu(t) \, dx$$

$$\leq C \|\rho \chi_t(\tau)\|_{L^2} \|u(\tau)\|_{L^\infty} \|\chi_s(\tau)\|_{L^2} + C \|\rho \chi_t(\tau)\|_{L^2} \|\mu(\tau)\|_{L^2}$$

$$\leq \frac{1}{2} \|\rho \chi_t(\tau)\|_{L^2}^2 + C \|u\|_{L^\infty} \|\chi_s\|_{L^2}^2 + C \|\mu\|_{L^2}^2.$$ 

Substituting the above inequality into (30), then letting $\tau \to 0^+$, we conclude that

$$\int_0^1 \rho \chi^2(t) \, dx + \int_0^t \int_0^1 \chi_s^2 \, dx \, ds \leq C \int_0^1 \left( \int_0^1 v(\rho) u^2_t \, dx \right) \|\rho \chi_t\|_{L^2}^2 \, ds + C \int_0^t \left( \int_0^1 v(\rho) u^2_t \, dx \right) \, ds$$

$$+ \varepsilon \int_0^t \|\sqrt{\rho}(u_t + uu_s)\|_{L^2}^2 \, ds + C.$$ 

At last, multiplying (31) by $1 + C_1$, then adding the resultant inequality into (27), and choosing $\varepsilon$ sufficiently small, we have

$$\|\rho \chi_t\|_{L^2}^2 + \int_0^1 v(\rho) u^2_t \, dx + \int_0^t \left( \|\chi_{st}\|_{L^2}^2 + \|\sqrt{\rho}(u_t + uu_s)\|_{L^2}^2 \right) \, ds$$

$$\leq C \int_0^1 \left( \int_0^1 v(\rho) u^2_t \, dx \right) \, ds + C \int_0^t \left( \int_0^1 v(\rho) u^2_t \, dx \right) \|\sqrt{\rho} \chi_t\|_{L^2}^2 \, ds + C,$$ 

which together with Gronwall’s inequality, (4), (14), and (25) shows (22). This completes the proof. \(\square\)
Lemma 6 Let \((\rho, u, \chi)\) be a smooth solution of (1)–(7) on \((0, T) \times [0, 1]\). Then one has

\[
\sup_{t \in [0, T]} \sigma(t) \left( \|u\|_{W^{1,\infty}}^2 + \|\sqrt{\rho}(u_t + uu_x)\|_{L^2}^2 + \int_0^T \|u_t + uu_x\|_{L^2}^2 \, dt \right) \leq C, \tag{32}
\]

where \(\sigma(t) \equiv \min\{1, t\}\).

Proof Taking the operator \(\partial_t + (u\cdot \nabla)\) to (2), and multiplying it by \(u_t + uu_x\), then integrating the resultant equality by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \rho |u_t + uu_x|^2 \, dx + \int_0^1 \rho \int_0^1 v(\rho) \left( |u_t + uu_x_x| \right)^2 \, dx \tag{33}
\]

\[
= \gamma \int_0^1 \rho \rho v(u_t + uu_x_x) \, dx + \int_0^1 \left[ v(\rho) + v'(\rho) \rho \right] u_t^2 (u_t + uu_x)_x \, dx
\]

\[
+ \delta \int_0^1 \left( \chi_x x (u_t + uu_x)_x + u \chi_x x(u_t + uu_x)_x \right) \, dx
\]

\[
\leq C \left( 1 + \|u_x\|_{L^\infty} \|u_x\|_{L^2} \|u_t + uu_x\|_{L^2} + C \|u_x\|_{L^2} \|\chi_x\|_{L^2} \|u_t + uu_x\|_{L^2} \right)
\]

\[
+ C \|\chi_x\|_{L^\infty} \|\chi_x\|_{L^2} \|\chi_x\|_{L^2} \|u_t + uu_x\|_{L^2}
\]

\[
\leq C \left( 1 + \|\sqrt{\rho}(u_t + uu_x)\|_{L^2} \|u_x\|_{L^2} \|u_t + uu_x\|_{L^2} + C \|\chi_x\|_{L^2} \right)
\]

\[
\leq \frac{1}{2} \|u_t + uu_x\|_{L^2}^2 + C \|u_x\|_{L^2}^2 + C \|u_x\|_{L^2}^2 \left( \|u_t + uu_x\|_{L^2} \right)^2 + C \|\chi_x\|_{L^2}^2,
\]

due to (22), (25), and (26). Then, multiplying (33) by \(\sigma(t)\) and integrating the resultant inequality over \((0, t)\), one has

\[
\sup_{t \in [0, T]} \sigma(t) \int_0^1 \rho |u_t + uu_x|^2 \, dx + \int_0^1 \rho \int_0^1 v(\rho) \left( |u_t + uu_x_x| \right)^2 \, dx \, dt \leq C, \tag{34}
\]

due to (14), (22), and Gronwall’s inequality. Furthermore, (34) together with (22) and (26) yields

\[
\sup_{t \in [0, T]} |u_x|_{L^\infty} \leq C. \tag{35}
\]

Combining (34) and (35) leads to (32). This completes the proof. \(\square\)

2.2 A priori estimates (II): higher order estimates

In this subsection, we derive the higher-order estimates of the smooth solution \((\rho, u, \chi)\) to system (1)–(7). Particularly, in this subsection the constant \(C\) may depend on the initial data \((\rho_0, u_0, \chi_0), \gamma, \delta, \text{and } \bar{\nu}\). Almost the a priori estimates obtained in this subsection could be obtained by similar arguments as in [2], we estimate them here to make the paper self-contained and satisfy the new assumptions on the initial data and compatibility condition (12).
Lemma 7 Let \((\rho, u, \chi)\) be a smooth solution of (1)–(7) on \((0, T) \times [0,1]\). Then one has

\[
\sup_{t \in [0,T]} \left( \|\rho_t\|_{L^2}^2 + \|\rho_x\|_{L^2}^2 \right) \leq C \tag{36}
\]

and

\[
\sup_{t \in [0,T]} \left( \sigma(t) \|\sqrt{\rho}u_t\|_{L^2}^2 + \sigma(t) \|u_{xx}\|_{L^2}^2 \right) + \int_0^T \left( \|u_{xx}\|_{L^2}^2 + \sigma(t) \|u_{xx}\|_{L^2}^2 \right) dt \leq C. \tag{37}
\]

Proof Differentiating (1) with respect to \(x\) leads to

\[
\rho_{xt} + \rho_{xx} u + 2 \rho_x u_x + \rho u_{xx} = 0. \tag{38}
\]

Then, multiplying (38) by \(\rho_x\) and integrating the resultant equality by parts, by (15), one shows that

\[
\frac{d}{dt} \|\rho_x\|_{L^2} \leq C\|u_x\|_{L^\infty} \|\rho_x\|_{L^2} + C\|u_{xx}\|_{L^2}. \tag{39}
\]

To prove the last term on the right-hand side of (39), let us rewrite (2) as

\[
v(\rho)u_{xx} = \rho(u_t + uu_x) + (\rho')_x - v'(\rho)\rho_x u_x - \delta \chi_x \chi_{xx}, \tag{40}
\]

which together with (5), (15), (22), and (32) leads to

\[
\|u_{xx}\|_{L^2} \leq C \|\sqrt{\rho}(u_t + uu_x)\|_{L^2} + C(1 + \|u_x\|_{L^\infty}) \|\rho_x\|_{L^2} + C\|\chi_x\|_{L^\infty} \|\chi_{xx}\|_{L^2} \tag{41}
\]

\[
\leq C(\sigma(t))^{-1/2}(1 + \|\rho_x\|_{L^2}).
\]

Therefore, substituting (41) into (39), then Gronwall’s inequality gives

\[
\sup_{t \in [0,T]} \|\rho_x\|_{L^2} \leq C, \tag{42}
\]

which together with (1), (15), and (22) yields

\[
\sup_{t \in [0,T]} \|\rho_t\|_{L^2} \leq C\|u\|_{L^\infty} \|\rho_x\|_{L^2} + C\|u_x\|_{L^2} \leq C. \tag{43}
\]

Next, it follows from (15), (22), and (41) that

\[
\|\sqrt{\rho}u_t\|_{L^2} \leq C\|\sqrt{\rho}(u_t + uu_x)\|_{L^2} + C\|\sqrt{\rho}u_{xx}\|_{L^2} \tag{44}
\]

\[
\leq C\|\sqrt{\rho}(u_t + uu_x)\|_{L^2} + C,
\]

and

\[
\|u_{xx}\|_{L^2} \leq C\|u_t + uu_x\|_{L^2} + C\|u_x\|_{L^2} + C\|u_{xx}\|_{L^2} \tag{45}
\]

\[
\leq C\|u_t + uu_x\|_{L^2} + C\|u_x\|_{L^4} \|u_{xx}\|_{L^2}^{1/4} + C\|u_{xx}\|_{L^2} \leq C\|u_t + uu_x\|_{L^2} + C.
\]
then (44) and (45) together with (32) and (41) yield (37). This completes the proof of Lemma 7.

From now on, suppose that \( (\rho, u, \chi) \) is a smooth solution of problem (1)–(7) with the smooth initial data satisfying the conditions in Theorem 2 and \( v(\cdot) \in C^2[0, \infty) \).

**Lemma 8** Let \( (\rho, u, \chi) \) be a smooth solution of (1)–(7) on \( (0, T) \times [0,1] \). Then one has

\[
\sup_{t \in [0, T]} \left( \|u_{xx}\|_{L^2}^2 + \|u_t\|_{W^{1,\infty}} + \|\sqrt{\rho} u_t\|_{L^2}^2 \right) + \int_0^T \|u_{xx}\|_{L^2}^2 \leq C. \tag{46}
\]

**Proof** Based on (33) and the compatibility condition (12), the initial data \( \|\sqrt{\rho_0} (u_0 + u_0u_0)\|_{L^2}^2 \leq C \). Therefore integrating (33) over \( (0, t) \), one has

\[
\sup_{t \in [0, T]} \int_0^1 \rho |u_t + uu_x|^2 \, dx + \int_0^T \int_0^1 \nu(\rho) |(u_t + uu_x)_x|^2 \, dx \, dt \leq C,
\]

which together with (22), (26), (41), (44), and (45) yields (46). This completes the proof. \( \square \)

**Lemma 9** Let \( (\rho, u, \chi) \) be a smooth solution of (1)–(7) on \( (0, T) \times [0,1] \). Then one has

\[
\sup_{t \in [0, T]} \left( \|\rho_{xx}\|_{L^2}^2 + \|\rho_{tt}\|_{L^2}^2 + \left\| \left( \rho^{\gamma'} \right)^2 \right\|_{L^2}^2 + \left\| \left( \rho^{\gamma'} \right)^2 \right\|_{L^2}^2 + \left\| v_{xt}(\rho) \right\|_{L^2}^2 \right) \tag{47}
\]

\[
+ \int_0^T \left( \|u_{xxx}\|_{L^2}^2 + \|\chi_{xxx}\|_{L^2}^2 + \|\rho_{xx}\|_{L^2}^2 + \left\| \left( \rho^{\gamma'} \right)^2 \right\|_{L^2}^2 + \left\| v_{tt}(\rho) \right\|_{L^2}^2 \right) \leq C.
\]

**Proof** Differentiating (1) twice with respect to \( x \), and multiplying the resultant equality by \( \rho_{xx} \), then integrating it by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int \rho_{xx}^2 \, dx \leq C \|u_x\|_{L^\infty} \|\rho_{xx}\|_{L^2}^2 + C \|\rho_x\|_{L^2} \|\rho_{xx}\|_{L^2} \|u_{xx}\|_{L^2}
\]

\[
+ C \|\rho_{xx}\|_{L^2} \|u_{xxx}\|_{L^2}
\]

\[
\leq C \|\rho_{xx}\|_{L^2}^2 + C \left( \|\rho_x\|_{L^2} + \|\rho_{xx}\|_{L^2} \right) \|\rho_{xx}\|_{L^2} + C \|u_{xxx}\|_{L^2}^2
\]

\[
\leq C \|\rho_{xx}\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C
\]

due to (15), (36), and (46). Furthermore, it follows from (1) that

\[
(\rho^{\gamma'})_x + (\rho^{\gamma'} u)_x + (\gamma - 1) \rho^{\gamma'} u_x = 0,
\]

from which, by the same arguments as in (48), we deduce

\[
\frac{1}{2} \frac{d}{dt} \int (\rho^{\gamma'})_{xx}^2 \, dx \leq C \left\| \left( \rho^{\gamma'} \right)_{xx} \right\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C.
\]

Combining the above inequality with (48) leads to

\[
\frac{1}{2} \frac{d}{dt} \left( \|\rho_{xx}\|_{L^2}^2 + \left\| \left( \rho^{\gamma'} \right)_{xx} \right\|_{L^2}^2 \right) \tag{49}
\]

\[
\leq C \|\rho_{xx}\|_{L^2}^2 + \left\| \left( \rho^{\gamma'} \right)_{xx} \right\|_{L^2}^2 + C \|u_{xxx}\|_{L^2}^2 + C.
\]
To obtain the second term on the right-hand side of (49), it follows from (2) that

\[
v'(\rho) u_{xxx} = \rho_{xx} u + \rho u_{xt} + \rho_{x} u_{xx} + \rho u_{x}^2 + \rho u u_{xx} + (\rho')_{xx} + \delta \chi_{xxx} + \delta \chi \chi_{xxx} - 2v'(\rho) \rho_{xx} u - v'(\rho) \rho_{xx} u_{x},
\]

which gives the following estimates:

\[
\|u_{xxx}\|_{L^2} \leq C \|u_t\|_{L^\infty} \|\rho_{xx}\|_{L^2} + C \|u_{xx}\|_{L^2} \|u_{x}\|_{L^\infty} \|\rho_{x}\|_{L^2} + C \left( \|\rho_{xx}\|_{L^4} + \|\rho_{xx}\|_{L^2} \right) L^2
\]

\[
+ C \|\chi_{x}\|_{L^\infty} \|\chi_{xxx}\|_{L^2} + C \|\rho_{x}\|_{L^\infty} \|u_{xx}\|_{L^2} + C \|\rho_{x}\|_{L^2} \|u_{x}\|_{L^\infty} + C \|u_{x}\|_{L^\infty} \|\rho_{xx}\|_{L^2}
\]

\[
\leq C \left( \|\rho'\|_{xx} \right)_{L^2} + C \|\rho_{xx}\|_{L^2} + \|u_{xx}\|_{L^2} + C \|\chi_{xxx}\|_{L^2} + C,
\]

where we have used (15), (22), (36), and (46). Furthermore, it follows from (24) that

\[
\chi_{xxx} = 2\rho \rho_{xx} \chi_{x} + \rho^2 \chi_{xx} + 2\rho \rho_{xx} \chi_{x} + \rho^2 \chi_{xxx} + \rho^3 \chi_{xxx}
\]

\[
+ \rho_{x} (\chi^3 - \chi) + \rho (3\chi^2 - 1) \chi_{x},
\]

which, due to (14), (22), (36), and (46), implies that

\[
\|\chi_{xxx}\|_{L^2} \leq C \left( 1 + \|\chi_{x}\|_{L^\infty} + \|\chi_{xx}\|_{L^2} \right) \left( \|\rho_{xx}\|_{L^2} + \|u_{xx}\|_{L^2} + 1 \right) + C
\]

\[
\leq C \|\rho_{xx}\|_{L^2} + C.
\]

Then, substituting (50) and (51), by using Gronwall's inequality, (22), and (46), we obtain

\[
\sup_{t \in [0, T]} \left( \|\rho_{xx}\|_{L^2}^2 + \|\rho'\|_{xx}^2 \right) \leq C.
\]

Next, it follows from (38) that

\[
\|\rho_{xt}\|_{L^2} \leq C \|\rho_{xx}\|_{L^2} \|u_{x}\|_{L^2} + C \|\rho_{x}\|_{L^2} \|u_{x}\|_{L^\infty} + C \|u_{xx}\|_{L^2} \leq C
\]

due to (22), (36), (46), and (52). Furthermore, it follows from (1) that

\[
\rho_{tt} = -\rho_{xt} u + \rho_{x} u_{x} + \rho_{t} u_{x} - \rho u_{xt},
\]

from which we have

\[
\int_0^T \|\rho_{xt}\|_{L^2}^2 \, dt
\]

\[
\leq C \int_0^T \left( \|u_{x}\|_{L^2}^2 \|\rho_{xt}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 \|\rho_{x}\|_{L^2}^2 + C \|u_{x}\|_{L^\infty}^2 \|\rho_{t}\|_{L^2}^2 + C \|u_{xt}\|_{L^2}^2 \right) \, dt
\]

\[
\leq C \int_0^T \|u_{xt}\|_{L^2}^2 \, dt + C.
\]
due to (22), (36), (46), and (53). Similarly, we obtain

\[
\sup_{t \in [0,T]} \left( \| \rho \|_{L^2}^2 + \| v_{xt}(\rho) \|_{L^2}^2 \right) + \int_0^T \left( \| \rho \|_{H^1}^2 + \| v_{tt}(\rho) \|_{L^2}^2 \right) dt \leq C,
\]

which together with (22) and (50)–(54) shows (47). Therefore, we complete the proof. \(\square\)

**Lemma 10** Let \((\rho, u, \chi)\) be a smooth solution of (1)–(7) on \((0, T) \times [0, 1]\). Then we have

\[
\sup_{t \in [0,T]} \left( t \| u_{xt} \|_{L^2}^2 + t \| \rho \chi_{tt} \|_{L^2}^2 + \| X_{xt} \|_{L^2}^2 + t \| u_{xxx} \|_{L^2}^2 + \| \chi_{xxx} \|_{L^2}^2 \right)
\]

\[
+ \int_0^T \left( t \| \sqrt{\rho} u_{xt} \|_{L^2}^2 + t \| u_{xx} \|_{L^2}^2 + t \| \chi_{x} \|_{L^2}^2 + t \| \chi_{xx} \|_{L^2}^2 + \| \rho \chi_{x} \|_{L^2}^2 \right) dt \leq C.
\]

**Proof** Multiplying (28) by \(\chi_{tt}\), integrating the resultant equality by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \chi_{tt}^2 \, dx + \int_0^1 \rho^2 \chi_{tt}^2 \, dx
\]

\[
= -\frac{d}{dt} \int_0^1 \rho_t (\chi^3 - \chi) \chi_t \, dx - \int_0^1 \rho^2 u_{xx} \chi_t \, dx - 2 \int_0^1 \rho u_{tx} \chi_{tt} \, dx
\]

\[
- \int_0^1 \rho^2 u_t \chi_{tt} \, dx - 2 \int_0^1 \rho \chi_{tx} \chi_{tt} \, dx - \int_0^1 \rho (3 \chi^2 - 1) \chi_t \chi_{tt} \, dx
\]

\[
+ \int_0^1 \rho \chi_t (\chi^3 - \chi) \chi_t \, dx + \int_0^1 \rho (3 \chi^2 - 1) \chi_t^2 \, dx
\]

\[
\leq -\frac{d}{dt} \int_0^1 \rho_t (\chi^3 - \chi) \chi_t \, dx + C \| u \|_{L^\infty} \| \rho \chi_{tt} \|_{L^2} \| X_{xt} \|_{L^2}
\]

\[
+ C \| u \|_{L^\infty} \| \rho \chi_{tt} \|_{L^2} \| X_{x} \|_{L^\infty} \| \rho_t \|_{L^2} + C \| \rho \chi_{tt} \|_{L^2} \| \sqrt{\rho} u_{tx} \|_{L^2} \| X_{t} \|_{L^\infty}
\]

\[
+ C \| \rho \chi_{tt} \|_{L^2} \left( \| \rho_t \|_{L^2} \| X_{t} \|_{L^\infty} + \| X_{t} \|_{L^\infty} \right) + C \| \rho_t \|_{L^2} \| X_{t} \|_{L^\infty}
\]

\[
+ C \| \rho_t \|_{L^2} \| X_{tt} \|_{L^\infty}
\]

\[
\leq -\frac{d}{dt} \int_0^1 \rho_t (\chi^3 - \chi) \chi_t \, dx + \frac{1}{2} \| \rho \chi_{tt} \|_{L^2}^2 + C \| \chi_{xt} \|_{L^2}^2 + C,
\]

which together with the compatibility condition (12) yields

\[
\sup_{t \in [0,T]} \| \chi_{xt} \|_{L^2}^2 + \int_0^T \| \rho \chi_{tt} \|_{L^2}^2 \, dt \leq C,
\]

by using the fact that

\[
\int_0^1 \rho_t (\chi^3 - \chi) \chi_t \, dx \leq \frac{1}{4} \| \chi_{xt} \|_{L^2}^2 + C \| \rho_t \|_{L^2}^2.
\]

Furthermore, (56) together with (51) leads to

\[
\sup_{t \in [0,T]} \| \chi_{xxx} \|_{L^2}^2 \leq C.
\]
To proceed, differentiating (28) with respect to $t$, multiplying the resultant equation by $\chi_{tt}$, and integrating by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_0^1 \rho^2 \chi_{tt}^2 \, dx + \int_0^1 \chi_{tt}^2 \, dx
$$

$$
= -\frac{1}{2} \int_0^1 \rho^2 \chi_{tt}^2 \, dx - 4 \int_0^1 \rho \rho_t \chi_{tt}^2 \, dx - 2 \int_0^1 \rho_t^2 \chi_{tt} \, dx
$$

$$
- 2 \int_0^1 \rho \chi_{tt} \chi_{tt} \, dx - 4 \int_0^1 \rho_t \chi_{tt} \chi_{tt} \, dx - 4 \int_0^1 \rho_t \chi_{tt} \, dx
$$

$$
- 2 \int_0^1 \rho_t^2 \chi_{tt} \, dx - 2 \int_0^1 \rho_t \chi_{tt} \, dx - 4 \int_0^1 \rho_t (\chi^2 - \chi) \chi_{tt} \, dx
$$

$$
- 2 \int_0^1 \rho (3 \chi^2 - 1) \chi_{tt} \, dx - 4 \int_0^1 \rho \chi_{tt} \chi_{tt} \, dx - 4 \int_0^1 \rho (3 \chi^2 - 1) \chi_{tt} \, dx
$$

$$
\leq C \|u_t\|_{L^\infty} \|\chi_{tt}\|_{L^2}^2 + C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

$$
+ C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

$$
+ C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

$$
+ C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

$$
+ C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

$$
\leq \frac{1}{2} \|\chi_{tt}\|_{L^2}^2 + C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

$$
+ C \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2} \|\rho \chi_{tt}\|_{L^2} + C \|\chi_{tt}\|_{L^\infty} \|\chi_{tt}\|_{L^\infty} \|\rho_t\|_{L^2}^2
$$

Multiplying the above inequality by $\tau$ and using Gronwall’s inequality, we obtain

$$
\sup_{t \in [0,T]} \|\chi_{tt}\|_{L^2}^2 + \int_0^T \|\rho \chi_{tt}\|_{L^2}^2 \, dt \leq \varepsilon \int_0^T \|\sqrt{\rho} \chi_{tt}\|_{L^2}^2 \, dt + C,
$$

(58)

where we have used (46), (47), and (56).

Then, differentiating (2) with respect to $t$ leads to

$$
\rho u_{tt} + \rho uu_{tt} - \left[ v(\rho) u_t \right]_t = -\rho_t u_t + uu_{tt} - \frac{\delta}{2} \left( \chi_e^2 \right)_{tt}.
$$

(59)

Multiplying the above equation by $\rho u_{tt}$ and integrating the resultant equality by parts, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_0^1 v(\rho) u_{tt}^2 \, dx + \|\sqrt{\rho} u_{tt}\|_{L^2}^2
$$

$$
= \frac{1}{2} \int_0^1 v_t(\rho) u_{tt}^2 \, dx - \int_0^1 v_t(\rho) u_t u_{tt} \, dx - \int_0^1 \rho_t (u_t + uu_x) u_{tt} \, dx
$$

$$
- \int_0^1 \rho u_t u_{tt} \, dx - \int_0^1 \left( \rho \right)_{tt} u_{tt} \, dx - \frac{\delta}{2} \int_0^1 \left( \chi_e^2 \right)_{tt} u_{tt} \, dx = \sum_{i=1}^6 l_i.
$$

(60)
Now, we focus on the estimates of the terms on the right-hand side of (60). First, due to (36) and (47), we obtain

\[ J_1 \leq C \| v_t(\rho) \|_{L^\infty} \| u_{xt} \|_{L^2}^2 \leq C \| \rho_t \|_{L^2}^{1/2} \| v_{st}(\rho) \|_{L^2}^{1/2} \| u_{xt} \|_{L^2}^2 \leq C \| u_{xt} \|_{L^2}^2. \]

It follows from (46) and (47) that

\[ J_2 = -\frac{d}{dt} \int_0^1 v_t(\rho) u_t u_{xt} \, dx + \frac{1}{2} \int_0^1 v_{tt}(\rho) u_{xt} \, dx + \frac{1}{2} \int_0^1 v_t(\rho) u_{xt}^2 \, dx \\
\leq -\frac{d}{dt} \int_0^1 v_t(\rho) u_t u_{xt} \, dx + C \| u_t \|_{L^\infty} \| v_{st}(\rho) \|_{L^2} \| u_{xt} \|_{L^2} \\
+ C \| v_t(\rho) \|_{L^\infty} \| u_{xt} \|_{L^2}^2 \leq -\frac{d}{dt} \int_0^1 v_t(\rho) u_t u_{xt} \, dx + C \| u_{xt} \|_{L^2}^2 + C \| v_t(\rho) \|_{L^2}^2. \]

Next, it follows from (1) and integration by parts that

\[ J_3 = \int_0^1 (\rho u)(u_t + uu_x)u_{xt} \, dx = -\int_0^1 \rho u(u_t + uu_x)u_{xt} \, dx \\
- \int_0^1 \rho u(u_t + uu_x)u_{xt} \, dx \\
\leq -\frac{d}{dt} \int_0^1 \rho u(u_t + uu_x)u_{xt} \, dx + C \| \sqrt{\rho} u_{xt} \|_{L^2} \| (u_t + uu_x)_x \|_{L^2} \\
+ \int_0^1 \rho_t u(u_t + uu_x)u_{xt} \, dx + \int_0^1 \rho_{tt} u(u_t + uu_x)u_{xt} \, dx \\
+ \int_0^1 \rho u(u_t + uu_x)u_{xt} \, dx \\
\leq -\frac{d}{dt} \int_0^1 \rho u(u_t + uu_x)u_{xt} \, dx + C \| \sqrt{\rho} u_{xt} \|_{L^2} \| (u_t + uu_x)_x \|_{L^2} \\
+ C \| u_t + uu_x \|_{L^\infty} \| \rho_t \|_{L^2} \| u_{xt} \|_{L^2} \\
+ C \| u_t + uu_x \|_{L^\infty} \| \sqrt{\rho} u_t \|_{L^2} \| u_{xt} \|_{L^2} + C \| u_{xt} \|_{L^2}^2 \leq -\frac{d}{dt} \int_0^1 \rho u(u_t + uu_x)u_{xt} \, dx + \frac{1}{2} \| \sqrt{\rho} u_{xt} \|_{L^2}^2 + C \| u_{xt} \|_{L^2}^2 \\
+ C \| (u_t + uu_x)_x \|_{L^2}^2, \]

where we have used (15), (36), (46). Moreover, by (46), one easily shows that

\[ J_4 \leq C \| u_x \|_{L^\infty} \| \sqrt{\rho} u_t \|_{L^2} \| \sqrt{\rho} u_{xt} \|_{L^2} \leq \epsilon \| \sqrt{\rho} u_{xt} \|_{L^2}^2 + C. \]

Furthermore, \( J_5 \) is estimated as

\[ J_5 = \frac{d}{dt} \int_0^1 (\rho^\gamma)_{u_{xt}} \, dx - \int_0^1 (\rho^\gamma)_{u_{xt}} \, dx \]

\[ \leq \frac{d}{dt} \int_0^1 (\rho^\gamma)_{u_{xt}} \, dx + C \| (\rho^\gamma)_{u_{xt}} \|_{L^2}^2 + C \| u_{xt} \|_{L^2}^2. \]
Similarly, we deduce that
\[
J_0 = \frac{\delta}{2} \frac{d}{dt} \int_0^1 (\chi_x^2) u_{xt} dx - \frac{\delta}{2} \int_0^1 \chi_x^2 u_{xt} dx - \frac{\delta}{2} \int_0^1 \chi_x \chi_{xt} u_{xt} dx
\]
\[
\leq \frac{\delta}{2} \frac{d}{dt} \int_0^1 (\chi_x^2) u_{xt} dx + C \|\chi_{xt}\|_{L^2}^2 \|u_{xt}\|_{L^2} + C \|\chi_x\|_{L^\infty} \|\chi_{xt}\|_{L^2} \|u_{xt}\|_{L^2}
\]
\[
\leq \frac{\delta}{2} \frac{d}{dt} \int_0^1 (\chi_x^2) u_{xt} dx + C \|u_{xt}\|_{L^2}^2 + C \|\chi_{xt}\|_{L^2}^2 + C \|\chi_{xtt}\|_{L^2}^2.
\]
Substituting all the above estimates into (60), we obtain
\[
\frac{d}{dt} F(t) + \|\sqrt{\rho} u_{tt}\|_{L^2}^2 \leq C \|u_{xt}\|_{L^2}^2 + C \|\chi_{xt}\|_{L^2}^2 + C \|\rho^\gamma\|_{L^2}^2 + C \|\sqrt{\rho} u_{tt}\|_{L^2}^2
\]
\[
+ C \|u_t + uu_s\|_{L^2}^2 + C \|v_t\|_{L^2}^2 + C.
\]
Multiplying the above inequality by \(\gamma\) and adding (58) to the resultant inequality, then integrating it over \((0, t)\), after choosing \(\epsilon\) small enough, and using (46), (47), (56), Gronwall’s inequality, we show that
\[
\sup_{t \in [0, T]} t (\|\rho \chi_{tt}\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2) + \int_0^T t (\|\chi_{xt}\|_{L^2}^2 + \|\sqrt{\rho} u_{tt}\|_{L^2}^2) dt \leq C,
\]
due to
\[
F(t) = \frac{1}{2} \int_0^1 v(\rho) u_{xt}^2 dx + \int_0^1 v_t(\rho) u_{xt} dx + \int_0^1 \rho u_t(u_t + uu_s) u_{xt} dx
\]
\[- \int_0^1 (\rho^\gamma) u_{xt} dx - \frac{\delta}{2} \int_0^1 (\chi_x^2) u_{xt} dx,
\]
which together with (62) leads to
\[
\int_0^T t \|u_{xxt}\|_{L^2}^2 dt \leq C.
\]
Similarly, due to (28), one has
\[
\int_0^T t \| \chi_{xxt} \|_{L^2}^2 \, dt \leq C,
\]
which together with (57), (62), (63), (64), and (56) shows (55). This completes the proof. □

3 Proofs of the main theorems

In this section, based on the a priori estimates derived in Sect. 2, we extend the local classical solution obtained in Lemma 1 to a global one.

Proof of Theorem 1 To prove Theorem 1, we first construct a sequence of approximate solutions by giving the density without initial vacuum. Let \( j_\delta(x) \) be a standard mollifier with width \( \delta \) and define the initial density
\[
\rho_0^\delta = j_\delta * \rho_0 + \delta, \quad u_0^\delta = u_0 * j_\delta, \quad \chi_0^\delta = \chi_0 * j_\delta,
\]
where \( \rho_0^\delta \to \rho_0, \quad u_0^\delta \to u_0, \quad \chi_0^\delta \to \chi_0 \) in \( H^1 \), as \( \delta \to 0 \).

Due to Lemma 1, the initial boundary value problem (1)–(7) with initial data \( (\rho_0^\delta, u_0^\delta, \chi_0^\delta) \) has a classical solution \( (\rho_0^\delta, u_0^\delta, \chi_0^\delta) \) on \([0, T_0]\times[0,1]\). Moreover, the estimates obtained in Lemmas 3–7 show that the solution \( (\rho_0^\delta, u_0^\delta, \chi_0^\delta) \) satisfies, for any \( 0 < T < +\infty \),
\[
\sup_{t \in [0,T]} \left( \| \rho_0^\delta \|_{H^1} + \| (\rho_0^\delta, u_0^\delta, \chi_0^\delta) \|_{H^2} \right) + \int_0^T \left( \| u_0^\delta \|_{H^2}^2 + t \| u_{xxt} \|_{L^2}^2 + \| \chi_{xxt} \|_{L^2}^2 \right) \leq C,
\]
where \( C \) is independent of \( \delta \). With all the estimates at hand, one easily extracts subsequences to some limit \( (\rho, u, \chi) \) in the weak sense. Then letting \( \delta \to 0 \), we deduce that \( (\rho, u, \chi) \) is a strong solution to (1)–(7).

Furthermore, the uniqueness of the strong solution \( (\rho, u, \chi) \) could be obtained by the similar argument as in [2]. For simplicity, we omit the details here.

Next, we will extend the local existence time \( T_0 \) of the strong solution to be infinity and therefore prove the global existence result. Let \( T^* \) be the maximal time of existence for the strong solution, thus, \( T^* \geq T_0 \). For any \( 0 < \tau < T \leq T^* \) with \( T \) finite, we obtain
\[
(u, \chi) \in L^\infty(0,T; H^1_0) \cap L^2(0,T; H^2), \quad (u_t, \chi_t) \in L^2(\tau,T; H^1),
\]
then
\[
(u, \chi) \in C([\tau, T]; H^1_0). \quad (65)
\]

Moreover, it follows from
\[
\rho \in L^\infty(0,T; H^1), \quad \rho \in L^\infty(0,T; L^2)
\]
that
\[ \rho \in C([0, T]; H^1). \]  \hspace{1cm} (66)

Let
\[ (\rho^*, u^*, \chi^*) \triangleq (\rho, u, \chi)(T^*, x) = \lim_{t \to T^*} (\rho, u, \chi)(t, x), \]

it follows from (65) and (66) that \((\rho^*, u^*, \chi^*)\) satisfies the initial condition stated in Theorem 1.

Therefore, we take \((\rho^*, u^*, \chi^*)\) as the initial data at \(T^*\) and then use the local result, Lemma 1, to extend the strong solution beyond the maximum existence time \(T^*\). This contradicts the assumption on \(T^*\). We finally show that \(T^*\) could be infinity and complete the proof of the global existence of the strong solution.

It remains to prove (11), process of which is similar to that in in [13], we sketch it here for completeness. Due to integration by parts, we have
\[
\frac{d}{dt} \| u_x \|_{L^2}^2 = 2 \int_0^1 u_x u_{xt} \, dx
\]
\[
= 2 \int_0^1 u_x (u_t + uu_x)_x \, dx - (uu_x)_x \, dx
\]
\[
= 2 \int_0^1 u_x (u_t + uu_x)_x \, dx - \int_0^1 u_x^2 \, dx
\]
\[
\leq C \| (u_t + uu_x)_x \|_{L^2}^2 + C (1 + \| u_x \|_{L^\infty}) \| u_x \|_{L^2}^2,
\]
which together with (14) and (32) leads to
\[
\int_1^{+\infty} \left( \| u_x \|_{L^2}^2 + \left| \frac{d}{dt} \| u_x \|_{L^2}^2 \right| \right) \, dt \leq C.
\]

Therefore, we obtain
\[
\lim_{t \to +\infty} \| u_x \|_{L^2}^2 = 0,
\]
which together with (32) yields
\[
\lim_{t \to +\infty} \| u \|_{W^{1,p}} = 0, \quad \forall p \in [1, \infty).
\]

This completes the proof. \(\Box\)

Proof of Theorem 2 With the higher-order estimates in Lemmas 8–10 at hand, the proof of Theorem 2 is similar to that of Theorem 1, and so it is omitted here for simplicity. \(\Box\)

Acknowledgements
The author is indebted to the editor’s kind help and the anonymous reviewers’ invaluable suggestions which improved the manuscript greatly.
This work was supported by the National Natural Science Foundation of China (no. 11671188).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors’ contributions
The main idea of this paper was proposed by MLS. MLS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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