AN EXPLICIT SELF-DUALITY

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Abstract. We provide an exposition of the canonical self-duality associated to a presentation of a finite, flat, complete intersection over a Noetherian ring, following work of Scheja and Storch.

1. Introduction

Consider a finite ring map \( A \to B \) and assume that \( A \) is Noetherian. Coherent duality for proper morphisms provides a functor \( f^! : D(\text{Spec } A) \to D(\text{Spec } B) \) on derived categories. The finiteness assumption on \( f \) implies that \( f^! A \) is isomorphic to the sheaf on \( B \) associated to \( \text{Hom}_A(B, A) \). See for example [Har66, Ideal Theorem and III 6]. If we assume moreover that \( f : \text{Spec } B \to \text{Spec } A \) is a local complete intersection morphism, then \( f^! A \) is locally free [Sta18, 0B6V]. We thus obtain an isomorphism

\[
\text{Hom}_A(B, A) \cong B
\]

of \( B \)-modules under additional hypotheses, for example if we assume that \( B \) local.

An explicit presentation of \( B \) as

\[
B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)
\]

provides a canonical choice for the isomorphism (1.0.1). In this expository paper, we follow the approach of [SS75] to construct this canonical isomorphism for \( B \) a finite, flat \( A \)-algebra equipped with a presentation (1.0.2).

The approach is as follows: Consider the ideals

\[
(f_1 \otimes 1 - 1 \otimes f_1, \ldots, f_n \otimes 1 - 1 \otimes f_n) \subset (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)
\]

of \( A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n] \). One writes

\[
f_j \otimes 1 - 1 \otimes f_j = \sum a_{ij}(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n).
\]

and defines the element \( \Delta \in B \otimes \Omega B \) as the image of \( \text{det}(a_{ij}) \) under the morphism \( A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n] \to B \otimes B \). This is shown to be independent of the choice of \( a_{ij} \). There is a canonical \( A \)-module morphism

\[
\chi : B \otimes A B \to \text{Hom}_A(\text{Hom}_A(B, A), B).
\]

Let \( I \) denote the kernel of multiplication \( B \otimes_A B \to B \), or in other words the image of \( (x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n) \). One checks that \( \chi \) restricts to an isomorphism

\[
\chi : \text{Ann}_B B \otimes_A I \to \text{Hom}_B(\text{Hom}_A(B, A), B)
\]

of \( B \)-modules and identifies the annihilator as \( \text{Ann}_B B \otimes_A I \cong \Delta \). Finally, one shows that

\[
\chi(\Delta) =: \Theta \in \text{Hom}_B(\text{Hom}_A(B, A), B)
\]

provides the desired isomorphism of \( B \)-modules \( \Theta : \text{Hom}_A(B, A) \to B \) guaranteed by the general theory of coherent duality.

Our arguments largely follow the outline of [SS75], although we make more use of Koszul homology in some proofs than the original did, and provide a self-contained proof of Lemma 2.4; the goal in large part is to provide an English reference for this material. See also [Kun05, Appendices H and I].
Remark 1.1. One motivation for providing an explicit description of this isomorphism is to describe the resulting $A$-valued bilinear form on $B$. This form is defined via
\[ \langle b, c \rangle \mapsto \Theta^{-1}(b)(c) = \eta(bc) \in A, \]
where $\eta = \Theta^{-1}(1)$. The form $\langle -,- \rangle$ has been used to give a notion of degree [EL77] [Eis78, some remaining questions (3)]. For example, it computes the local $A^1$-Brouwer degree of Morel [KW19].

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2. Commutative Algebra Preliminaries

Lemma 2.1. [SS75, 1.2] Let $A$ be a noetherian ring and suppose that $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are sequences satisfying the following hypotheses:

(i) $b = (g_1, \ldots, g_n) \subset a = (f_1, \ldots, f_n)$

(ii) If $p$ is a prime such that $a \subset p$, then the sequence $f_1, \ldots, f_n$ is a regular sequence in $A_p$, as is $g_1, \ldots, g_n$.

Write $g_i = \sum_{i=1}^{n} a_{ij} f_j$, and let $(a_{ij})$ be the resulting matrix of coefficients.

\[ \Delta := \det(a_{ij}). \]

Define $\Sigma$ to be the image of $\Delta$ under the map $A \to A/b$. Then:

(a) The element $\Sigma$ is independent of the choices of $a_{ij}$.

(b) We have an equality (of $A/b$-ideals):

\[ (\Delta) = \text{Fit}_{A/b}(a/b), \]

where Fit denotes the 0-th Fitting ideal.

(c) We have an equality of ideals:

\[ (\Delta) = \text{Ann}_{A/b}(a/b), \]

and

\[ a/b = \text{Ann}_{A/b}(\Delta). \]

Remark 2.2. We comment on condition (ii). If $(A, p)$ is a local ring and $a \subset p$, then condition (ii) is equivalent to asking that $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are regular sequences. In general, condition (ii) asks only that they are regular sequences after localizing at primes containing $a$ (e.g., they may not be regular sequences on $A$).

Proof. First, we may assume that $A$ is a local ring and each of the $f_i$'s and $g_i$'s are in the maximal ideal $m$.

(a): Write $g_i = \sum_{i=1}^{n} b_{ij} f_j$. We want to show that $\det(a_{ij}) - \det(b_{ij})$ is in $b$. It suffices to consider the case where $a_{ij} = b_{ij}$ for all $j$ and for $i = 1, \ldots, n - 1$, as this allows us to change the presentation of one $g_i$ at a time, and thus all of them. Define

\[ c_{ij} = \begin{cases} a_{ij} = b_{ij} & i = 1, \ldots, n - 1 \\ a_{ij} - b_{ij} & i = n, \end{cases} \]

By cofactor expansion along the $j$-th row, we have that

\[ \det(a_{ij}) - \det(b_{ij}) = \det(c_{ij}). \]

But now

\[ (c_{ij}) \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \\ 0 \end{pmatrix} \]
We will abbreviate Tor which means that det(c_{ij}) = det(g_{ij}) \in (g_1, \ldots, g_n),
which means det(c_{ij}) \cdot a \in (g_1, \ldots, g_n).
But g_n \in a and hence det(c_{ij}) \cdot g_n \in (g_1, \ldots, g_n),
which means that det(c_{ij}) \in (g_1, \ldots, g_n) = b since g_1, \ldots, g_n is a regular sequence.

(b): First observe that
Fit_A(a/b) \mod b = Fit_A/(a/b).
Therefore, to prove the claim, it suffices to prove that
Fit_A(a/b) = \Delta + I,
where I \subset b.
To prove this claim, note that the Fitting ideal of the A-module a/b is computed by a presentation:
\[ A^\oplus n \oplus A^\oplus (\mathbb{Z}) \xrightarrow{T} A^\oplus n \to a/b \to 0, \]
where T is given by:
\[ (a_{ij}) \times d_{Kosz}^2. \]
In other words, the matrix of T has the first n-columns are just given by a_{ij} and the last \( \binom{n}{2} \) columns are composed of the usual Koszul relations among the f_i. (Note that the sequence f_1, \ldots, f_n is regular in our local ring, so the corresponding Koszul complex produces a resolution of a [Sta18, 062F].)

Now, the Fitting ideal is given by the n \times n-minors of the matrix of T. The first minor is \( \Delta \).
If \( \Delta' \) is another n \times n minor, then it is the determinant of a matrix T', which is composed of some r columns of (a_{ij}) and n - r columns of d_{Kosz}^1; without loss of generality we may assume T' contains the first r columns of (a_{ij}) (if not, simply reorder the g_i, using that the ring A is local and thus regularity of the sequence of g_i preserved). Applying T' to \( (f_k) \) we get
\[
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_n
\end{pmatrix}
= \begin{pmatrix}
g_1 \\
\vdots \\
g_r \\
0
\end{pmatrix}
\]
We again conclude that \( \Delta' f_i = \det(T') f_i \in b \) for each \( i = 1, \ldots, n \). Thus,
\[ \Delta' \cdot a \in (g_1, \ldots, g_n), \]
and in particular
\[ \Delta' \cdot g_n \in (g_1, \ldots, g_n), \]
which by regularity of the g_i, means that \( \Delta' \in b \) and thus Fit_A(a/b) = \Delta + I with I \subset b.

(c): First, we claim that we have an isomorphism:
Ann_{A/b}(a/b) \cong Tor^A_n(A/b, A/a).
We will abbreviate Tor^A by \( Tor_j \) and \( \otimes_A \) by \( \otimes \) in what follows. To prove this, we deploy the Koszul complex. (As noted above, a regular sequence is Koszul-regular by [Sta18, 062F].) We thus have a quasi-isomorphism:
\[ K_\bullet(f_1, \ldots, f_n) \simeq A/a \]
Therefore the Tor group above is computed as the kernel of \( 1 \otimes d_{Kosz}^n \) in the complex \( A/b \otimes K_\bullet(f_1, \ldots, f_n): \)
\[ 0 \to A/b \xrightarrow{(f_1, \ldots, f_n)} (A/b)^\oplus n. \]
Indeed, the cohomology of this small complex is the desired annihilator and thus we obtain the desired isomorphism.

On the other hand, we claim that $\text{Tor}_n(A/a, A/b) \cong \Delta \cdot A/b$. To see this note that we have a short exact sequence of $A$-modules:

$$0 \to a/b \to A/b \to A/a \to 0.$$  

We claim that the induced long exact sequence splits into short exact sequences for $j \geq 1$

$$0 \to \text{Tor}_j(A/b, a/b) \to \text{Tor}_j(A/b, A/b) \to \text{Tor}_j(A/b, A/a) \to 0$$

Indeed, via the Koszul complex for $A$, we see that for $j \geq 1$:

$$(2.0.1) \quad \text{Tor}_j(A/b, a/b) \cong (a/b)^{(7)}; \quad \text{Tor}_j(A/b, A/b) \cong (A/b)^{(7)},$$

and the map $\text{Tor}_j(A/b, a/b) \to \text{Tor}_j(A/b, A/b)$ is identified with the direct sum of copies of the injection $a/b \to A/b$. To conclude, the functoriality of the Koszul complex [Sta18, 0624] yields a morphism of complexes

$$A/b \otimes K_\bullet(g_1, \ldots, g_n) \to A/b \otimes K_\bullet(f_1, \ldots, f_n);$$

where the left end is as follows:

$$(2.0.2) \quad A/b \xrightarrow{0} (A/b)^{\oplus n} \xrightarrow{\varphi} A/b^{(f_1, \ldots, f_n)}(A/b)^{\oplus n}.$$  

Since the map $\text{Tor}_j(A/b, A/b) \to \text{Tor}_j(A/b, A/a)$ is a surjection, we conclude that

$$\text{Tor}_n(A/b, A/a) \cong \text{Im}(\varphi) \cong \Delta \cdot A/b$$

as desired.

For the second claim, note that the ideal $\text{Ann}_{A/b}(\varphi)$ is obtained as the kernel of the left vertical map in (2.0.2), and is thus isomorphic to $\text{Tor}_n(A/b, a/b)$, which we already know is isomorphic to $a/b$ by (2.0.1).

\[\square\]

A module $M$ over a ring $R$ is said to be reflexive if the natural map $R \to \text{Hom}_R(\text{Hom}_R(M, R), R)$ is an isomorphism [Sta18, 0AUY]. A form of the following lemma is in the stacks project [Sta18, 0AVA], but assumes that $A$ is integral and that $A = B$. The following is [SS75, 1.3].

**Lemma 2.3.** Let $A$ be a Noetherian ring and $B$ a finite flat $A$-algebra. A finite $B$-module $M$ is reflexive if and only if the following conditions hold:

(i) If $p \subseteq A$ is a prime ideal with depth $A_p \leq 1$, then $M_p$ is a reflexive $B_p$-module.

(ii) If $p \subseteq A$ is a prime ideal with depth $A_p \geq 2$, then depth$_{A_p}(M_p) \geq 2$.

**Proof.** The property of being reflexive is preserved under any localization of $B$ [Sta18, 0EB9], and can be checked locally on $B$ [Sta18, 0AV1]. Therefore reflexivity of $M$ implies (i). Reflexivity implies (ii): Any regular sequence in $A_p$ is a regular sequence on $B_p$ by flatness. Let $a_1, a_2$ be a length 2 regular sequence on $A_p$. Let $N$ be any $B_p$-module. Then $a_1$ is a nonzerodivisor on $\text{Hom}_B(N, B_p)$. The cokernel of multiplication by $a_1$ is a submodule of $\text{Hom}_B(N, B_p/a_1B_p)$, on which $a_2$ is a nonzerodivisor. This shows the claim. (Note: This is almost [Sta18, 0AV5], except that we take $\text{Hom}$ but want the A-depth.)

Conversely, suppose $M$ is not reflexive. We assume for the sake of contradiction that properties (i) and (ii) hold. Since reflexivity can be checked locally, there is some minimal $p \subseteq A$ among all prime ideals of $A$ for which $M_p$ is a not a reflexive $B_p$-module. Without loss of generality, we may assume that $A$ is local with maximal ideal $p$. Since $M_p$ is not reflexive, we must have that depth $A_p \geq 2$ and therefore depth$_{A_p}(M_p) \geq 2$. We consider the exact sequence

$$0 \to \text{Ker}\varphi \to M \to \text{Hom}_B(\text{Hom}_B(M, B), B) \to \text{Coker}\varphi \to 0,$$
where \( \varphi \) is the canonical map to the double-dual. By assumption, \( \varphi \) becomes an isomorphism after localizing at any prime of \( A \) different from \( p \). It follows that \( \text{Ker} \varphi \) and \( \text{Coker} \varphi \) have finite length. Since \( \text{depth}_A M > 1 \), there exists some \( x \in A \) which is a nonzerodivisor on \( M \). But then \( x \) is a nonzerodivisor on the finite-length module \( \text{Ker} \varphi \), which therefore must vanish. Since \( \text{Hom}_B(\text{Hom}_B(M, B), B) \) is reflexive (as a \( B \)-module), it has \( A \)-depth \( \geq 2 \) by the forward implication of the lemma. The exact sequence

\[
0 \to M \to \text{Hom}_B(\text{Hom}_B(M, B), B) \to \text{Coker} \varphi \to 0,
\]

then shows that \( \text{depth}_A \text{Coker} \varphi \geq 1 \) by the standard behavior of depth in short exact sequences [Sta18, 00LX]. Therefore the cokernel must vanish, which shows that \( M \) is reflexive. \( \square \)

**Lemma 2.4.** [SS75, 1.4] Let \( A \) be a Noetherian ring and let \( B \) be a finite flat \( A \)-algebra. Let \( M \) be a finite \( B \)-module, which is projective as an \( A \)-module. If \( \text{Hom}_B(M, B) \) is projective as a \( B \)-module, then \( M \) is projective as a \( B \)-module. In particular, if \( \text{Hom}_B(M, B) \) is free, then \( M \) is free.

**Proof.** It is enough to show that \( M \) is reflexive. We are therefore reduced to checking the conditions (i) and (ii) of Lemma 2.3. Clearly, (ii) holds, since \( M \) is projective over \( A \). It remains to check (i). We may therefore assume that \( A \) is a Noetherian local ring with \( \text{depth} A \leq 1 \), and we want to show that \( M \) is projective as a \( B \)-module. Since \( B \) is finite flat over \( A \), we have \( \text{depth}_B \mathfrak{m} = \text{depth} A \) for every maximal ideal \( \mathfrak{m} \) of \( B \) [Sta18, 0337].

Throughout, we will write \( N^* := \text{Hom}_B(N, B) \) for a \( B \)-module \( N \). Consider the map

\[
\varphi : M \to M^{**}.
\]

Let \( C := \text{Coker} \varphi \). Taking a presentation of \( M \), we obtain an exact sequence

\[
0 \to U \to F \to M \to 0
\]

with \( F \) free. Consider the dual sequence

\[
0 \to M^* \to F^* \to U^*,
\]

and let \( Q := \text{Im}(F^* \to U^*) \). Since \( M^* \) is projective by assumption, \( Q \) has projective dimension 0 or 1 as a \( B \)-module.

We have the commutative diagram

\[
\begin{array}{ccc}
F & \longrightarrow & M \\
\downarrow & & \downarrow \\
F^{**} & \longrightarrow & M^{**} \longrightarrow \text{Ext}_B^1(Q, B) \longrightarrow 0
\end{array}
\]

with exact lower row. Since \( F \to M \) is a surjection, we see that \( C = \text{Ext}_B^1(Q, B) \). Suppose \( \text{depth} A = 0 \). Apply the Auslander–Buchsbaum formula [Sta18, 090V] to the \( B_\mathfrak{m} \)-module \( Q_\mathfrak{m} \) for each maximal ideal \( \mathfrak{m} \) of \( B \). We find that \( Q_\mathfrak{m} \) has projective dimension zero, i.e., is projective. Therefore \( C_\mathfrak{m} = 0 \) and \( C = 0 \).

Now suppose that \( \text{depth} A = 1 \). Then \( \text{depth}_{B_\mathfrak{m}} U^*_\mathfrak{m} \geq 1 \) by [Sta18, 0AV5], whence

\[
\text{depth}_{B_\mathfrak{m}} Q_\mathfrak{m} \geq 1
\]

by [Sta18, 00LX]. Again by Auslander–Buchsbaum, we find that \( Q_\mathfrak{m} \) is projective, and that \( C = 0 \).

We have shown that in any case \( M \to M^{**} \) is surjective. Since \( M^{**} \) is projective, this implies \( M \cong M^{**} \otimes N \) for some \( B \)-module \( N \). It follows that \( N^* = 0 \) and that \( N \) is again free as an \( A \)-module.

By assumption both \( M \) and \( M^{**} \) are free over the local ring \( A \). A surjection of finite free \( A \)-modules is an isomorphism if they have the same rank. To show two finite free modules have the same rank, we may localize at a minimal prime ideal \( \mathfrak{q} \) of \( A \), so that also \( B_\mathfrak{q} \) is a zero-dimensional ring. Over the Artinian ring \( B_\mathfrak{q} \), \( \text{Hom}_{B_\mathfrak{q}}(N_\mathfrak{q}, B_\mathfrak{q}) = 0 \) implies \( N_\mathfrak{q} = 0 \). (To see this, note that we may assume that \( B \) is local, with maximal ideal \( \mathfrak{m} \). Then \( N_\mathfrak{q} \to \mathfrak{m} N_\mathfrak{q} \) is nonzero.
by Nakayama’s lemma. Since $B_q$ has finite length, there is a nonzero element annihilated by $m$ whence a $B$-homomorphism $B/m \to B_q$.) Thus $M_q$ and $M_q^{**}$ have the same rank, and therefore $M \to M^{**}$ is an isomorphism. □

3. The explicit isomorphism

Recall that a ring map $A \to B$ is a relative global complete intersection if there exists a presentation $A[x_1, \ldots, x_n]/(f_1, \ldots, f_c) \cong B$, and every nonempty fiber of $\text{Spec} B \to \text{Spec} A$ has dimension $n - c$ [Sta18, 00SP]. Note that in this case the $f_i$ form a regular sequence [Sta18, 00SV].

We note that a global complete intersection is flat [Sta18, 00SW], and thus syntomic. We will be interested in the situation where $A \to B$ is furthermore assumed to be a finite flat global complete intersection.

Construction 3.1. Suppose that $A \to B$ is a finite flat global complete intersection. Choose a presentation

$$A[x_1, \ldots, x_n] \xrightarrow{\pi} B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$

Consider the commutative diagram

$$
\begin{array}{ccc}
A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n] & \xrightarrow{m_1} & A[x_1, \ldots, x_n] \\
\downarrow \pi \otimes \pi & & \downarrow \pi \\
B \otimes_A B & \xrightarrow{m} & B,
\end{array}
$$

with $m_1, m$ the obvious multiplication maps. We note that the elements

$$\{f_j \otimes 1 - 1 \otimes f_j\}_{j=1,\ldots,n}$$

are all in $\ker(m_1)$, which is generated by the $x_i \otimes 1 - 1 \otimes x_i$ for $i = 1, \ldots, n$, whence we have a relation

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^n a_{ij} (x_i \otimes 1 - 1 \otimes x_i).$$

Define $\Delta := (\pi \otimes \pi)(\det(a_{ij})) \in B \otimes_A B$. Define also $I := \ker m$.

Proposition 3.2. The following properties of $\Delta$ hold:

(a) The element $\Delta$ is independent of the choice of $a_{ij}$.

(b) We have an equality of $B \otimes_A B$-ideals:

$$\langle \Delta \rangle = \text{Fit}_{B \otimes_A B} I,$$

(c) we have an equality of ideals

$$\langle \Delta \rangle = \text{Ann}_{B \otimes_A B} I \quad \text{Ann}_{B \otimes_A B}(\Delta) = I.$$

Proof. Consider the ring map

$$\pi \otimes 1 : A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n] \to B \otimes_A A[x_1, \ldots, x_n] \cong B[x_1, \ldots, x_n].$$

Since

$$f_i \otimes 1 - 1 \otimes f_i = \sum_{i=1}^n a_{ij} (x_i \otimes 1 - 1 \otimes x_i)$$

in $A[x_1, \ldots, x_n] \otimes A[x_1, \ldots, x_n]$, we have that

$$-1 \otimes f_i = \sum_{i=1}^n a_{ij} (\pi(x_i) \otimes 1 - 1 \otimes x_i)$$

in $B \otimes_A A[x_1, \ldots, x_n]$.

Note that $\Delta$ is the image of $\det(a_{ij})$ under the obvious morphism $B \otimes_A A[x_1, \ldots, x_n] \to B \otimes_A B$, and that if $a$ is the ideal generated by the $\pi(x_i) \otimes 1 - 1 \otimes x_i$ and $b$ the ideal generated by the $(-1 \otimes f_i)$, then $I$ is $a/b$. The desired properties will then follow immediately from
applying Lemma 2.1 to $b = (−1 \otimes f_j) \subset (π(x_i) \otimes 1 − 1 \otimes x_i) = a$, once we show that the conditions of the Lemma are satisfied. It suffices to show that each is a regular sequence.

We claim that $\{−1 \otimes f_j\} \subset B \otimes_A A[x_1, \ldots, x_n]$ is a regular sequence. Indeed, since relative global complete intersections are flat [Sta18, 00SW] and regular sequences are preserved under flat morphisms, this follows by regularity of the $f_j$ in $A[x_1, \ldots, x_n]$ and flatness of $A \rightarrow B$. It is immediate also that $(π(x_i) − x_i)$ forms a regular sequence in $B[x_1, \ldots, x_n]$ as well (the $π(x_i)$ are just elements $b_i$ of $B$, and $(x_i − b_i)$ is always a regular sequence in $B[x_1, \ldots, x_n]$).

Thus, the proposition follows by Lemma 2.1. □

Now, retain our setup from Construction 3.1. There is a canonical map of $A$-modules

$$χ : B \otimes_A B \rightarrow \text{Hom}_A(\text{Hom}_A(B, A), B) \quad χ(b \otimes c) = (φ \mapsto ϕ(b)c).$$

Both $B \otimes_A B$ and $\text{Hom}_A(\text{Hom}_A(B, A), B)$ carry two natural $B$-module structures:

1. $B$ acts on $B \otimes_A B$ as multiplication on either the left or right factor (i.e., either $a(b \otimes c) = ab \otimes c$ or $a(b \otimes c) = b \otimes ac$).
2. $B$ acts on $\text{Hom}_A(\text{Hom}_A(B, A), B)$ as either pre- or post-composing a homomorphism by multiplication (i.e., either $ϕc : ψ \mapsto ϕ(ψc)$ or $ϕc : ψ \mapsto ϕc(ψ)$).

**Lemma 3.3.** $χ$ induces a $B$-module isomorphism $\text{Ann}_{B \otimes_A B} I \cong \text{Hom}_B(\text{Hom}_A(B, A), B)$.

**Proof.** We note first that this map is an isomorphism of $A$-modules, for which it suffices to check that it’s bijective: Since $B$ is a projective $A$-module we have that $B$ is canonically isomorphic to $B^\vee$ (where we denote by $\vee$ the $A$-module dual), so that we have isomorphisms of $A$-modules

$$B \otimes_A B \cong (B^\vee)^\vee \cong \text{Hom}_A(B^\vee, B) \cong \text{Hom}_A(\text{Hom}_A(B, A), B);$$

one can check that $χ$ is simply the composition of these canonical isomorphisms.

It’s immediately checked that the morphism $χ$ is in fact a $B$-bimodule homomorphism for the $B$-module structures of $B \otimes_A B$ and $\text{Hom}_A(\text{Hom}_A(B, A), B)$ given by right multiplication and post-composition.

Now, we note the following:

1. The largest submodule of $B \otimes_A B$ where the two $B$-module structures agree is $\text{Ann}_{B \otimes_A B} I$: this follows since an element $r \in B \otimes_A B$ is annihilated by all $a \otimes 1 − 1 \otimes a$ exactly when $(a \otimes 1)r = (1 \otimes a)r$ for all $a$, which occurs exactly when the action of every $a$ on $r$ is the same under the two $B$-module structures.
2. The largest submodule of $\text{Hom}_A(\text{Hom}_A(B, A), B)$ where the two $B$-module structures agree is

$$\text{Hom}_B(\text{Hom}_A(B, A), B) \subset \text{Hom}_A(\text{Hom}_A(B, A), B);$$

this is clear since the condition of pre- and post-multiplying by elements of $B$ being the same is exactly $B$-linearity.

Putting this together, we have that $χ$ induces an isomorphism of $B$-modules

$$χ : \text{Ann}_{B \otimes_A B} I \rightarrow \text{Hom}_B(\text{Hom}_A(B, A), B),$$

which was our desired claim. □

**Theorem 3.4.** The map $χ(∆) : \text{Hom}_A(B, A) \rightarrow B$ is an isomorphism of $B$-modules.

**Proof.** Applying Lemma 3.2(c) we have that $\text{Ann}_{B \otimes_A B} I = ∆(B \otimes_A B)$, and further that $\text{Ann}_{B \otimes_A B} ∆(B \otimes_A B) = I$. Thus, we have that

$$\text{Ann}_{B \otimes_A B} I = ∆(B \otimes_A B) \cong ∆(B \otimes_A B)/\text{Ann}_{B \otimes_A B} ∆ = ∆(B \otimes_A B)/I \cong m(∆)B.$$ 

Applying Lemma 3.3, we have then that $\text{Hom}_B(\text{Hom}_A(B, A), B)$ is a free $B$-module with basis $χ(∆)$. Applying Lemma 2.4, this implies that $\text{Hom}_A(B, A)$ is a free $B$-module of rank 1. We must then have that the $B$-module homomorphism $χ(∆) : \text{Hom}_A(B, A) \rightarrow B$ is an isomorphism, as desired. □
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