Feasible platform to study negative temperatures

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Abstract

We afford an experimentally feasible platform to study Boltzmann negative temperatures. Our proposal takes advantage of well-known techniques of engineering Hamiltonian to achieve steady states with highly controllable population inversion. Our model is completely general and can be applied in a number of contexts, such as trapped ions, cavity-QED, quantum dots coupled to optical cavities, circuit-QED, and so on. To exemplify, we use Hamiltonian models currently used in optical cavities and trapped ions domain, where the level of precision achieved the control of the freedom degrees of a single atom inside a cavity as well as a single trapped ion. We show several interesting effects such as absence of thermalization between systems with inverted population and cooling by heating in these unconventional systems.

Keywords: negative temperatures, Boltzmann entropy, population inversion, trapped ions

(Some figures may appear in colour only in the online journal)

1. Introduction

Negative temperature $T$ is associated with systems with a bounded energy spectrum for which $1/T = (\partial S/\partial E)_V < 0$, where $S$ is the Boltzmann entropy. In general, negative temperatures can be observed in population-inverted states, which can be achieved by injecting energy into these systems in the right way. The concept of negative temperature was proposed in the early 1950s [1, 2], and has since been discussed in a number of papers [3–12]. Very recently, negative temperature was experimentally observed using ultra cold quantum gases [13]. The experiment in [13] triggered a vivid debate regarding the existence of negative absolute temperatures, with the opponents claiming that negative absolute temperature should not exist if the ‘right’ (Gibbs) entropy is used [14–17], while those in favor of Boltzmann entropy claiming that negative temperatures not only exist as they are necessary to explain systems presenting population inversion [18–21]. Although this debate is far from a consensus, it seems to indicate that while Boltzmann’s entropy is adequate to describe the canonical ensemble in the thermodynamical limit, Gibbs’s entropy should be used to describe the microcanonical ensemble [16, 17] and, at least tentatively, systems with finite dimension [15]. Anyway, it is not our aim to enter in the debate on what is the right absolute temperature, or, equivalently, the right entropy. Rather, here we take a pragmatic approach to consider Boltzmann entropy, which leads to temperatures that may be negative for systems presenting inverted population. This can be useful for future applications, as for example thermal machines with negative temperatures [22].

Taking advantage of well-known techniques of Hamiltonian engineering [23, 24], in this paper we afford an experimentally feasible platform in several contexts, as for example quantum circuits, cavity QED and trapped ions. Particularly, to engineer systems displaying negative Boltzmann temperatures, we explore a model in the context of optical cavities and trapped ions in which the extraordinary level of precision reaches the control of individual atoms. To our aim, we work with an ion trapped in a harmonic potential pumped by a laser field to build steady states of the excited ionic levels, therefore with inverted population. Similar inverted population can also be achieved with single trapped atom inside an optical cavity, by properly engineering the atom-cavity mode interaction with the help of external laser fields as was done in [25]. Within our model, we are able to...
and the trap frequency is \( \omega_t \) such that \( \delta = \omega - \omega_t = k \nu \) on resonance with the TL ion, can be set in the following time-independent form:

\[
H_I = (g_k \sigma_x a^k + g^*_k \sigma_x a^k),
\]

In this manner, we are left with interactions (i) carrier \((k = 0, g_0 = \Omega/2)\), with \( \eta_0 \ll 1 \), (ii) the first \((k = 1, g_1 = \Omega \eta_1 L/2)\), with \( \eta_1 L \ll 1 \), (iii) the second \((k = 2, g_2 = -\eta_2^L/4)\), with \( \eta_2^L \ll 1 \), and (iv) the third blue sideband \((k = 3, g_3 = -i \Omega \eta_3^L/6)\), with \( \eta_3^L \ll 1 \) [27, 28]. This Hamiltonian experimentally model a number of systems, as for example trapped ions [26, 29], a TL atom pumped by a classical electromagnetic field (EM) and interacting with a quantized mode of a EM [30], a coupling between spin and nano mechanical resonator [31] or either a TL neutral atom in a dipole trap or a TL ion in a harmonic trap [32–34]. The dynamics of this model for weak system-reservoir coupling and \( g_k \ll \omega_0, \nu \) can be described by the master equation formalism, which, for the Hamiltonian (equation (2.2)) reads [35–37]

\[
\frac{\partial \rho}{\partial t} = -i[H_I, \rho] + \kappa (n_f + 1) D(\sigma_+ \rho + \kappa n_f D[\sigma^+] \rho + \gamma (n_a + 1) D[\sigma_+ \rho + \gamma n_a D[\sigma_+ \rho]),
\]

where \( \kappa \) and \( \gamma \) are the spontaneous emission rates for the vibrational mode and atom, respectively, \( n_f(n_a) \) is the average photon number for the vibrational mode (atom) reservoir, and \( D[A] \rho \equiv 2 \lambda A \rho A - A \rho A - \rho A A \).

Next, we solve numerically equation (2.3) to obtain the steady state of the system at \( t \to \infty \) by imposing \( \partial \rho/\partial t = 0 \) and thus to calculate the corresponding thermodynamic properties. We note that for solving numerically this system we must truncate the infinite Fock basis of the bosonic field somewhere. This truncation, i.e. how many states we shall take into account in our numerical simulation, depends on the mean number of excitations in the bosonic field (vibrational mode). Also, as we are particularly interested on the TL system, whose asymptotic behavior can present inverted population, we trace over the bosonic field variables. We then use the master equation (2.3) to investigate negative temperatures for a range of parameters represented by the cooperativity \( C_I = g_2^2/\gamma \kappa \). The cooperativity is the experimentally relevant parameter and depends on the physical context. For instance, in the context of optical cavities, the experiment reported in [32] used \( g = 2 \pi \times 16 \text{ MHz} \), \( \gamma = 2 \pi \times 3 \text{ MHz} \) and \( \kappa = 2 \pi \times 1.4 \text{ MHz} \), which corresponds to \( C \sim 61 \).

It is important to mention some differences and similarities between our model and the system used in [13]. As in [13], we use external fields to engineer the effective Hamiltonian that drives the system to a stationary regime with population inversion. In the experiment with cold atoms in optical lattices described in [13], via Feshbach resonance, the

provide the range of values that negative temperatures can be observed. Next, we study thermalization of two systems when one of them, displaying negative temperatures, is put in contact with a second one. We show that steady states of systems involving negative temperatures may present very surprising behaviors, as for example absence of thermalization with other systems.

2. Model

To our purpose, we make use of the so-called generalized anti-Jaynes–Cummings model (AJCM) [26], which can be derived as follows. Consider a trapped two-level (TL) ion whose transition frequency between its excited and ground states is \( \omega_0 \) and the trap frequency is \( \nu \). The quantum of vibrational energy of the center of mass of the ion is described by \( a^\dagger a \). In the Schrödinger picture, the Hamiltonian that describes such a system is \( H = H_f + H_a + H_{int}(t) \), with \( H_f = \nu a^\dagger a, H_a = \omega_0 \sigma_z/2 \). In the interaction picture, after adjusting the frequency \( \nu \), we trace over the bosonic field variables. We then use the master equation (2.3) to investigate negative temperatures for a range of parameters represented by the cooperativity \( C_I = g_2^2/\gamma \kappa \). The cooperativity is the experimentally relevant parameter and depends on the physical context. For instance, in the context of optical cavities, the experiment reported in [32] used \( g = 2 \pi \times 16 \text{ MHz} \), \( \gamma = 2 \pi \times 3 \text{ MHz} \) and \( \kappa = 2 \pi \times 1.4 \text{ MHz} \), which corresponds to \( C \sim 61 \).

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effective interaction is engineered to prepare a steady state with negative temperature, that is, starting from a state without population inversion, the system reaches a state with population inversion for sufficiently long times. However, it is important to note that the relevant dissipative mechanics considered in [13] is sufficiently weak such that it can be disregarded in that experiment, allowing to treat the atomic ensemble as an isolated system. In our model, on the other hand, the dissipation of the atom is taken into account and is responsible for destroying the state with population inversion. However, the engineered effective interaction with the bosonic mode plus its dissipation result in an effective engineered reservoir leading the system to a state with inverted population. Thus, as we take into account the ion dissipation in our model, here we have a competition between the action of the natural dissipation of the ion (spontaneous emission) and the action of the engineered interaction.

3. Producing steady states with negative temperatures

Let us begin by investigating how the energy and the corresponding effective temperature of the TL system varies with the cooperativity. As said before, we are interested in the steady state, such that we consider the asymptotic limit $t \to \infty$ by imposing $\partial \rho / \partial t = 0$ for $k = 1, 2, 3$. In figures 1(a)–(f), where the cooperativity parameter is displayed in logarithmic scale for clarity, we have considered the environments with average thermal photons $n = 0, 0.5, 2.0$. From now on, we assume $\nu = \omega_0$, such that $n_f = n_e = n$. We note that if we assume $\nu \ll \omega_0$, as it happens in trapped ion experiments, then $n_f$ will be different from $n_e$ for the same temperature $T$ for both systems. However, as the whole environment is cooled down to $T \approx 0$, the thermal photons at optical frequency is zero, while the effective thermal phonon occupation can be set as $n_f = 0.05$ [26]. To this case, according to our numerical calculations, the results are qualitatively the same. In fact, according to our numerical calculation, the quantitative difference is very small for $n_f = 0.05$–2.0, and exactly the same for $n_f = 0$, since this case is valid for $T = 0$ K.

From figures 1(a), (c), (e) we see that all steady states end with inverted population $\langle \sigma_i \rangle > 0$, thus leading to negative temperatures. Figures 1(b), (d), (f) show the rescaled temperature $k_B T / \omega_0$ versus cooperativity. Here $T$ was obtained through $T = 1 / \kappa_\theta (d S / d (H_\theta))$, where $S_\theta$ is the von Neumann entropy to the ion. Note that the role of nonlinearity is to populate the excited state, thus enhancing the inverted population: the greater $l$, the greater the positive average $\langle \sigma_i \rangle$, which is also reflected in the negative temperature. Note that inverted population can be obtained with small values of cooperativity simply increasing the nonlinearity of the AJCM. Also, note that the environments to the bosonic and atomic systems slightly suppress the population inversion, figures 1(a), (c), (e), thus avoiding to reach hotter negative temperatures for the TL system. From figures 1(b), (d), (f) we can see that for sufficiently high values of the cooperativity, negative temperatures approach to 0 K to all $k = 1, 2, 3$. A numerical inspection shows us that, for the set of parameters we have used, negative temperatures only occur for $C \gtrsim 0.65$.

4. Negative temperatures and coupled systems

Here we show how our platform provides an excellent tool for studying thermalization of two coupled systems and how to control the heating or cooling of both systems through cooperativity. Since systems presenting inverted population is hotter than other systems with positive temperatures, it is interesting to understand what happens when these hotter than hot systems are coupled to each other. Let us consider a TL atom (A) coupled to another one (B) through a simple exchange interaction $H_0 = \lambda (\sigma^A_+ \sigma^B_- + \sigma^A_- \sigma^B_+)$, which is the usual interaction resulting, for example, of collision processes [38]. The atom A is coupled to a quantum bosonic mode through the effective AJCM equation (2.2) and interacts via $H_N$ with atom B. For simplicity, we assume the vibrational mode, the atom A, and the atom B decaying at the same rate $(\gamma_A = \gamma_B = \kappa_\gamma)$. In figure 2 we show the average energy versus cooperativity, figures 2(a), (c), (e), and the scaled effective temperature $k_B T / \omega_0$ versus cooperativity, figures (a), (c), (e), and the scaled effective

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4 Boltzmann temperatures scale from cold to hot according to $+0K \ldots +300K$, $-\infty K \ldots -300K \ldots -0K$.

5 Actually, hotter than hot is a criticism against negative temperatures, see [14].
temperature versus cooperativity, figures 2(b), (d), (f), for 
k = 1 considering the environment thermal photon averages 
n = 0, 0.5, and 2.0. The behavior of these curves show that 
atom A presents population inversion from C ≥ 3.5, different 
from atom B which always has positive temperatures (for 
λ = 3γ). Also, for a certain value of C the energies of atoms 
A and B are not the same, thus indicating that there will be no 
thermalization. We note that the effect of increasing the 
environment temperature is to produce steady states with 
lower negative temperatures. In other words, the positive 
(natural) reservoir tends to decrease the negative temperature 
of atom A. Another surprising effect is that temperature does 
not increase monotonically with cooperativity. This effect can 
be better appreciated looking at the atom B curve in 
figure 2(b): by increasing the cooperativity, the temperature of 
atom B, as given by the dashed blue line, first increase until 
C ≈ 4.5 attaining its maximum, and then decays to zero for 
large values of the cooperativity.

The models k = 2, 3 present a similar pattern an will not 
be shown here: atom B, which is coupled only to atom A, 
ever inverts its population (for λ = 3γ), but cannot be 
heated arbitrarily, since its temperature reaches a maximum. 
Otherwise, atom A, which is coupled both to atom B and to a 
bosonic mode, is the one that has its population inverted, 
being this population (and so its negative temperature) 
diminished when the temperature of the environment reser-
voir is increased (from n = 0 to 2.0). Also, the atomic steady 
states do not thermalize neither with each other nor with the 
environment.

Now, let us recall that the cooperativity parameter com-
prehends the coupling between the ion A and its vibrational 
mode as well as the ion A decay rate and the vibrational mode 
damping through \( C_k = \frac{g_k^2}{\gamma A \kappa} \). Thus, it is also interesting to 
study how the scaled average energy and the effective 
temperature behave when varying the strength coupling \( \lambda \) 
between atoms A and B and also to ask if it is possible to have 
thermalization between them. In figures 3(a)–(f) we show the 
scaled average internal atom energy \( \langle \sigma \rangle \) and the temperature 
versus the scaled strength coupling \( \lambda/\gamma \) for k = 1 
considering environment thermal photon averages \( n = 0, 0.5, \) 
and 2.0. Now, when the thermal photon averages is either 
n = 0 or 0.5, both atoms can present negative temperatures. 
Note that atom B energy increases until crossing the zero 
energy line, thus inverting its population and acquiring 
negative temperature. After that, atom B energy starts to 
decrease, until its energy crosses back the zero energy line, 
acquiring positive temperature. On the contrary, atom A, 
which starts with inverted population, diminishes its internal 
energy until crossing the zero energy line, acquiring a positive 
temperature for sufficiently high value of the rate \( \lambda/\gamma \). Also, 
for sufficiently high values of the rate \( \lambda/\gamma \) both atoms A and 
B thermalize with each other, although they do not thermalize 
with their environments. It is interesting to note that atom A 
and B may thermalize even at negative temperatures, as seen 
in figures 3(a)–(d). The role of the atoms A and B environ-
ments is to diminish population inversion, see the blue-dash 
line in figures 3(a), (c), (e), and thus the negative temperature 
effect. For the parameters used here, the negative temperature

\[
\lambda = \frac{\gamma_A}{\gamma} = \frac{\gamma_B}{\gamma} = \kappa,
\]

Figure 3. Scaled average energy \( \langle \sigma \rangle \) versus scaled atom–atom 
coupling \( \lambda/\gamma \), figures (a), (c), (e), and scaled effective temperature 
\( k_B T/\omega_0 \) versus \( \lambda/\gamma \), figures (b), (d), (f), for k = 1. The environment 
thermal photon averages are \( n = 0, 0.5, \) and 2.0, respectively. 
Curves for atom A (B) is indicated by solid black (blue dashed) line. 
Here we used \( g_k = \sqrt{10} \gamma \) and \( \gamma = \gamma_A = \gamma_B = \kappa \).

5. Unconventional cooling by heating (CBH)

Now let us analyze the case where both the bosonic mode and 
atoms A and B are surrounded by environments at the same 
temperature \( T \) characterized by an average number \( n \) of 
thermal photons. We again assume here that both atoms decay 
with the same rate \( \gamma_A = \gamma_B = \kappa = \gamma \) and interact with each 
other through the natural coupling \( H_N = \lambda (\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+). \) 
The cooperativity is fixed at \( C = 10 \). Meanwhile, atom A is 
pumped with a laser leading to the effective Hamiltonian 
equation (2.2) between atom A and the quantized vibrational 
mode. Since in this case the interaction considered between 
the two atoms is the usual (or natural) one, one could expect 
thermalization between the atoms, which would eventually

\[
H_N = \lambda (\sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+) + \kappa \left( \sigma_A^+ + \sigma_A^- \right) + \kappa \left( \sigma_B^+ + \sigma_B^- \right) + g_k \left( \sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+ \right) + \kappa \left( \sigma_A^+ + \sigma_A^- \right) + \kappa \left( \sigma_B^+ + \sigma_B^- \right) + g_k \left( \sigma_A^+ \sigma_B^- + \sigma_A^- \sigma_B^+ \right).
\]
prevent CBH [27], at least in conventional systems with positive temperatures. Actually, as discussed, e.g. in [27], to achieve CBH in conventional systems (positive temperatures) it is necessary to engineer a Hamiltonian whose major contribution is due to counter-rotating terms This is because two systems, modeled by the usual Hamiltonian as given by matter-radiation interaction, generally thermalize with their environment and, when in contact, thermalize with each other. However, as we saw in figure 2, the steady state of atom A can display negative temperature, and thus an unconventional CBH with systems presenting negative temperatures can indeed occur, despite the coupling being a natural (not engineered) one. To see this, in figures 4(a)–(f) we show the scaled average internal atom energy and the scaled effective temperature versus the environment thermal photon average \( n \) for the models \( k = 1, 2, 3 \). Figures 4(a), (b), for \( k = 1 \), show that both the atom B, with positive temperature, and the atom A, with negative temperature (see footnote 3), cool down by decreasing their internal energy when their environments are heated up. This effect is saturated near \( n \sim 0.5 \) for the atom B, which has positive temperature, see figures 4(a), (b). This effect can be better appreciated by thinking in the opposite manner, i.e. in the heating by cooling: if the whole environment is cooled down, no matter if the atoms are with negative or positive temperatures, both atoms always heat up. This is a remarkable result, since, as emphasized above, it is usually expected that the energy flux between two systems with temperatures of opposite signs is from the one with negative to that with positive temperature, no matter as high the positive temperature is [18–21]. On the other hand, figures 4(a), (b) show that the internal energy of atom A, and hence its temperature (solid black line), always decreases, as expected for systems with negative temperatures.

It is interesting to note that for \( k = 2 \) and \( k = 3 \), figures 4(c), (d) and (e), (f), respectively, the CBH occurs only for atom A: when the environment temperature increases, atom B (with positive temperature) is heated while atom A (with negative temperature) cools.

### 6. Conclusion

We have proposed an experimentally feasible platform to study systems capable to display population inversion in their steady state, therefore presenting Boltzmann negative temperatures. Our platform includes (i) one qubit and a bosonic mode coupled through an effective Hamiltonian, the so-called AJCM and (ii) the previous system in (i) plus another qubit, which is coupled to the first TL system through a natural Hamiltonian model, such as that stemming from collisions. Using our platform as a theoretical tool, we were able to study a variety of phenomena, such as the thermalization for two qubits when one or both of them presents negative temperature, the control of negative temperatures through the cooperativity parameter, and also what we have called unconventional CBH. Unconventional CBH occurs when decreasing (increasing) the temperature of the entire environment, the temperature of one or both the two qubits increases (decreases), even when the temperature of one qubit is negative. This is a striking result if we remember that systems with negative temperatures are expected to decrease their negative temperatures as in contact with another system having positive temperature. Our proposal can be nowadays engineered in several contexts, such as trapped ions [26, 29], cavity QED [30], nano mechanical resonator [31], among others [32–34].

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