Nodal domain distribution of rectangular drums *

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We consider the sequence of nodal counts for eigenfunctions of the Laplace-Beltrami operator in two dimensional domains. It was conjectured recently that this sequence stores some information pertaining to the geometry of the domain, and we show explicitly that this is the case for the family of rectangular domains with Dirichlet boundary conditions.

I. INTRODUCTION

One of the major problems in mathematical physics is concerned with the geometrical information stored in the spectrum of the Laplace Beltrami operator

$$- \Delta \psi_j (r) = E_j \psi_j (r); \quad r \in \Omega (\alpha) .$$

(1)

The spectrum is ordered such that $E_{j-1} \leq E_j \leq E_{j+1}$ and $\Omega (\alpha)$ is a connected compact region, parameterized by $\alpha$, on a 2D Riemannian manifold. If $\Omega (\alpha)$ has a boundary, Dirichlet boundary conditions are assumed. The corresponding physical system could be a vibrating drum. In 1911 H. Weyl showed that the number of eigenvalues up to energy $E$ is

$$N (E) \sim \frac{AE}{4\pi}, \quad \text{as } E \to \infty$$

(2)

where $A$ is the area of $\Omega$. Subsequent research have shown (see e.g., [1]) that each of the terms in the asymptotic series of $N(E)$ provides further geometrical information on the boundary. This prompted M. Kac to ask, 'can one hear the shape of a drum?' [2]. That is, 'is it possible to uniquely define the shape of the drum from the spectrum?' It is known by now that for certain classes of domains the answer to Kac's question is positive, whereas there exists a large set of isospectral domains which are not isometric. (Ref. [3] gives an updated review of this subject.)

In the present note we would like to investigate the geometrical information stored in yet another sequence of numbers which are derived from the eigenfunctions $\psi_j$. Considering real eigenfunctions $\psi_j$, we count the number $\nu_j$ of nodal domains which are the connected domain where $\psi_j$ has a constant sign. The nodal domains are separated by the nodal lines where $\psi_j = 0$. The sequence $\{\nu_j\}_{j=1}^{\infty}$ is the sequence of nodal counts. According to Courant's Nodal theorem $\nu_j \leq j$. This fundamental theorem reveals the deep connection between the spectrum and the nodal count. It is convenient to define the normalized nodal domain numbers $\xi_j = \nu_j/j$. Because of Courant's theorem $0 \leq \xi_j \leq 1$. This estimate has been further refined (for domains in $\mathbb{R}^2$)

$$\limsup_{j \to \infty} \xi_j = 0.691 \ldots$$

(3)

Following [4], we study the distribution of the normalized nodal numbers in the spectral interval $I = [E_0, E^1]$

$$P(\xi, I) = \frac{1}{N_I} \sum_{E_j \in I} \delta(\xi - \xi_j)$$

(4)

where $N_I$ is the number of levels in the interval $I$.

In Ref. [3] the above distribution has been introduced as a tool to distinguish between systems which are integrable (separable) or classically chaotic. For the class of separable domains, it was shown that the limit distribution

$$P(\xi) = \lim_{E \to \infty} P(\xi, I)$$

(5)

exists. This has universal features: (a) there exists a system dependent parameter $\xi '$, maximum value of the nodal domain number, such that $P(\xi) = 0$ for $\xi > \xi '$ and (b) for $\xi \approx \xi '$,

$$P(\xi) = \frac{C}{\sqrt{1 - \xi/\xi '}}$$

(6)

The constant $C$ is system dependent, but the order of the singularity is universal and depends only on the dimensionality. (It was recently shown that the exponent for domains in $d$ dimensions is $(d - 3)/2$.)

The dependence on the geometry of the domain can come only through the parameters $\xi ', C$ or the details of the function $P(\xi)$ away from the universal domain. Indeed, the limiting distributions for the rectangular and circular boundaries were computed in [4] and found to be different as expected. However, as will be shown below, the function $P(\xi)$ does not distinguish between different rectangles. That is $\xi ' = 2/\pi$ and

$$P(\xi) = \left[1 - \left(\frac{\pi \xi}{2}\right)^2\right]^{-1/2}$$

(7)

for all rectangles! Note that for $\xi \approx 2/\pi$, this result coincides with the universal expression [4] with $C = 1/\sqrt{2}$.

The new result of the present note is that the dependence of $P(\xi, I)$ on the finite spectral interval $I$ contains

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sufficient information to resolve between different rectangles. Thus, by counting nodal domains one can deduce the shape of the (rectangular) drum. It should be emphasized at the outset that the nodal count sequence involves dimensionless integers, and therefore it cannot provide any scale information. Hence, when we say “resolve” we mean “resolve up to a scale”.

II. RECTANGLES

We consider the Dirichlet spectrum of a domain bounded in a rectangle with sides $L_x$ and $L_y$. Denoting $\alpha = L_x/L_y$ and choosing $L_x = \pi$, the spectrum is given by

$$E = n^2 + \alpha^2 m^2,$$

where $n, m = 1, 2, 3 \ldots$ and $0 < \alpha < 1$. Since the system is separable in rectangular co-ordinates the nodal domain number is simply $\nu_j = nm$, and $j = N(n^2 + \alpha^2 m^2)$ where $N(E)$ is the spectral counting function. The leading terms in the asymptotic expansion of $N(E)$ are

$$N(E) \simeq \frac{1}{4\pi} \left[ AE - L\sqrt{E} \right],$$

where $A, L$ are the area and perimeter of the boundary respectively. In terms of $\alpha$,

$$N(E) \simeq \frac{\pi E}{4\alpha} \left( 1 - \frac{2}{\pi} \frac{1 + \alpha}{\sqrt{E}} \right).$$

Introducing the transformation

$$n(E, \theta) = \sqrt{E} \cos \theta; \quad m(E, \theta) = \sqrt{E} \sin \theta/\alpha,$$

the normalized nodal-domain number can be approximated by

$$\xi_j(E, \theta) = \frac{2}{\pi} \sin 2\theta \left[ 1 - \frac{2}{\pi} \frac{1 + \alpha}{\sqrt{E}} \right]^{-1}. (12)$$

Converting the summation in eq. (11) into an integral, we obtain the leading terms in the asymptotic expansion of $P(\xi, I)$ in the large $E$ limit

$$P(\xi, I) \simeq \frac{1}{2\alpha N_I} \int_{E^0}^{E^1} \int_0^{\pi/2} \delta \left[ \xi - \xi_j(E, \theta) \right] dE d\theta,$$

where

$$N_I \simeq \frac{\pi}{4\alpha} \left\{ E^1 - E^0 \right\} - \frac{2}{\pi} (1 + \alpha) \left( \sqrt{E^1} - \sqrt{E^0} \right).$$

Introducing the variable $x = \sqrt{E/E^0}$

$$P(\xi, I) = \frac{E^0}{2\alpha N_I} \int_1^\infty \int_0^{\pi/2} x \delta \left[ \xi - \frac{2}{\pi} \frac{\sin 2\theta}{(1 - \epsilon/x)} \right] dx d\theta,$$

where

$$g = \sqrt{\frac{E^1}{E^0}}, \quad \epsilon(\alpha) = \frac{2}{\pi} \frac{(1 + \alpha)}{\sqrt{E^0}}. \quad (16)$$

The integral reduces to

$$P(\xi, I) = \frac{E^0}{\alpha N_I} \int_1^\infty x \left[ \frac{2}{\pi} \frac{\cos 2\theta_0}{(1 - \epsilon/x)} \right]^{-1} dx,$$

where $\sin 2\theta_0 = \frac{\pi \xi}{2} \left( 1 - \frac{1}{\epsilon} \right)$ and

$$l = \left\{ \begin{array}{ll}
\frac{g}{\epsilon}, & \text{if } \xi < \frac{\pi}{2} \\
\min \left\{ \frac{g}{\epsilon}, \frac{\pi \xi}{2} \left( 1 - \epsilon \right)^{-1} \right\}, & \text{if } \frac{\pi}{2} < \xi \leq \frac{3\pi}{4}.
\end{array} \right.$$ (18)

Note that $P(\xi, I) = 0$ for $\xi > 2\pi/1 - \epsilon$. The above integral can be rewritten as

$$P(\xi, I) = \frac{\pi E^0}{2\alpha N_I} \int_1^\infty \frac{x(x - \epsilon)}{\sqrt{a + bx + cx^2}} dx,$$

where

$$a = -\epsilon^2 \left( \frac{\pi \xi}{2} \right)^2, \quad b = 2\epsilon \left( \frac{\pi \xi}{2} \right)^2, \quad c = 1 - \left( \frac{\pi \xi}{2} \right)^2.$$ (20)

This integral can be computed for any given value of the parameters.

III. RESULTS

Using the above expression, it is possible to show that the derivative

$$P' = \frac{\partial P}{\partial \alpha} \bigg|_{\alpha = \alpha_0} (21)$$

FIG. 1: Typical behavior of the derivative $P'$ for $\xi < 2/\pi$. For $\xi > 2/\pi$, $P'$ is not defined as the function $P$ is not smooth.
In Figure 2, the nodal domain distribution given by the eq. (19) is shown for different $\alpha$, along with the corresponding numerical data. The limiting distribution is obtained by taking the spectral interval to infinity. In this limit, $\epsilon = 0$ and

$$P(\xi) = \begin{cases} 
\left[1 - \left(\frac{\pi \xi}{2}\right)^2\right]^{-1/2}, & \xi < 2/\pi \\
0, & \xi > 2/\pi 
\end{cases}$$

which is independent of $\alpha$. Thus the parameter dependence is arising from the leading finite energy correction to $P(\xi, I)$.

The problem studied above shows clearly that the nodal sequence stores geometrical information, which, in the present case suffices to determine unambiguously the rectangle for which the nodal sequence is given. Attempts to generalize these ideas to other separable systems such as e.g., smooth surfaces of revolutions or flat tori are under way.

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[1] C. Clark, “The asymptotic distribution of eigenvalues and eigenfunctions for elliptic boundary value problems,” *SIAM Review*, vol. 9 (4), pp. 627, 1967.

[2] M. Kac, “Can one hear the shape of a drum?,” *Amer. Math. Monthly*, vol. 73, pp. 1, 1966.

[3] S. Zeldich, “Survey of the inverse spectral problem,” *math.SP/0402356*.

[4] A. Pleijel, “Remarks on Courant’s nodal line theorem,” *Commun. Pure Appl. Math.*, vol. IX, pp. 543, 1956.

[5] G. Blum, S. Gnutzmann and U. Smilansky, “Nodal domain statistics: a criterion for quantum chaos,” *Phys. Rev. Lett.*, vol. 88, pp. 114101, 2002.

[6] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics: Part I*, McGraw-Hill, New York, pp. 761, 1953.

[7] I.S. Gradshteyn and I.M. Ryzik, *Table of Integrals, Series and Products*, 6th edn., Academic Press, USA, pp. 94, 2000.