SWITCHED GRAPHS OF SOME STRONGLY REGULAR GRAPHS RELATED TO THE SYMPLECTIC GRAPH

ALICE M.W. HUI, BERNARDO RODRIGUES

huimanwa@gmail.com, Rodrigues@ukzn.ac.za, May 25, 2016

Abstract. Applying a method of Godsil and McKay [6] to some graphs related to the symplectic graph, a series of new infinite families of strongly regular graphs with parameters $(2^n \pm 2^{(n-1)/2}, 2^{(n-1)/2} \pm 2, 2^{n-2} \pm 2^{(n-3)/2}, 2^{n-2} \pm 2^{(n-1)/2})$ are constructed for any odd $n \geq 5$. The construction is described in terms of geometry of quadric in projective space. The binary linear codes of the switched graphs are $[2^n \mp 2^{n-1}, n + 3, 2^{t+1}]_2$-code or $[2^n \mp 2^{n-1}, n + 3, 2^{t+2}]_2$-code.

Keywords: strongly regular graph, cospectral graphs, linear code of a graph

MSC 2010: 05E30 05C50 94B25 51A50

1. Introduction

Consider the $n$-dimensional projective space PG$(n, 2)$ over the finite field $\mathbb{F}_2$. That is, PG$(n, 2) = \mathbb{F}_2^{n+1} \setminus \{0\}$. When $n$ is odd, there are two non-equivalent non-singular quadrics in PG$(n, 2)$, namely elliptic and hyperbolic. For general references, see [9, Ch. 5] and [10, Ch. 22]. Both quadrics define a symplectic polarity (null polarity) in PG$(n, 2)$ [9 Theorem 5.28].

Let $n \geq 5$ be an odd number. Let $\mathcal{Q}$ be a non-singular quadric in PG$(n, 2)$. Define the graph $\Gamma_\mathcal{Q} = (V_\mathcal{Q}, E_\mathcal{Q})$ as follows. The vertex set $V_\mathcal{Q}$ is the set of points of PG$(n, 2)$ not in $\mathcal{Q}$. Two vertices $x$ and $y$ are adjacent in $\Gamma_\mathcal{Q}$ if and only if the line $xy$ joining them is an external line of $\mathcal{Q}$. $\Gamma_\mathcal{Q}$ is the complement of a subgraph of the symplectic graph $Sp(n+1, 2)$, which is the graph of the perpendicular relation induced by a non-degenerate symplectic form of $\mathbb{F}_2^{n+1}$ on the non-zero vectors of $\mathbb{F}_2^{n+1}$. In [8] [7] [11], $\Gamma_\mathcal{Q}$ is denoted by $\mathcal{N}_{n+1}^\epsilon$, where $\epsilon$ is $+$ (plus) if $\mathcal{Q}$ is hyperbolic, and $-$ (minus) if $\mathcal{Q}$ is elliptic.

A strongly regular graph with parameters $(v, k, \lambda, \mu)$ is a graph with $v$ vertices such that each vertex lies on exactly $k$ edges; any two adjacent vertices have exactly $\lambda$ neighbours in common; and any two non-adjacent vertices have exactly $\mu$ neighbours in common. The adjacency matrix of a strongly regular graph has exactly three eigenvalues. One is $k$ with multiplicity 1, and the remaining two are usually denoted by $r$ and $s$, $r > s$ with multiplicities $f$ and $g$ respectively. For general references, see
It is well-known that $\Gamma_Q$ defined above is a strongly regular graph. Table 1 shows the parameters of $\Gamma_Q$ for the different quadrics in $\text{PG}(n, 2)$ (see [4]).

| $\mathcal{Q}$ | graph $\Gamma_Q$ | $v$ | $k$ | $\lambda$ | $\mu$ |
|--------------|-----------------|-----|-----|--------|-----|
| elliptic     | $\Gamma_Q = \mathcal{N}_{n+1}$ | $2^n + 2 \frac{n-3}{2}$ | $2^n - 2 \frac{n-1}{2}$ | $2^n - 2 \frac{n-5}{4}$ | $2^n - 2 \frac{n-3}{2}$ |
| hyperbolic   | $\Gamma_Q = \mathcal{N}_{n+1}^\perp$ | $2^n - 2 \frac{n-1}{2}$ | $2^n - 2 \frac{n-3}{2}$ | $2^n - 2 \frac{n-1}{2}$ | $2^n - 2 \frac{n-3}{2}$ |
| elliptic     | $\Gamma_Q = \mathcal{N}_{n+1}$ | $2^n - 2 \frac{n-3}{2}$ | $-2^n - \frac{n-3}{2}$ | $\frac{1}{3}(2^n + 4)$ | $2^n + 2 \frac{n-3}{2}$ |
| hyperbolic   | $\Gamma_Q = \mathcal{N}_{n+1}^\perp$ | $2^n - 2 \frac{n-3}{2}$ | $-2^n - \frac{n-3}{2}$ | $\frac{1}{3}(2^n + 4)$ | $2^n - 2 \frac{n-3}{2}$ |

**Table 1. Parameters of $\Gamma_Q$**

Godsil and McKay (1982) introduced a method to generate graphs with the same adjacency spectrum [6] i.e. the adjacency matrices of the graphs have equal multisets of eigenvalues. The method is described as follows. Let $\Gamma$ be a graph. Let $S$ be a subset of the vertex set such that the subgraph of $\Gamma$ with vertex set $S$ is regular. Suppose any vertex outside $S$ has $0$, $|S|$ or $\frac{1}{2}|S|$ neighbours in $S$. Consider the graph $\Gamma'$ obtained by switching $\Gamma$ as follows: for any vertex $x$ of $\Gamma$ outside $S$, if $x$ has $\frac{1}{2}|S|$ neighbours in $S$, then delete those $\frac{1}{2}|S|$ edges and join $x$ to the other $\frac{1}{2}|S|$ vertices. We call $S$ a Godsil and McKay switching set of $\Gamma$. By Godsil and McKay [6], $\Gamma'$ has the same adjacency spectrum as $\Gamma$. In the case where $\Gamma$ is a strongly regular graph, $\Gamma'$ has the same adjacency spectrum as $\Gamma$ and thus is also a strongly regular graph with the same parameters (see [4]). Recently, there has been interest in constructing new strongly regular graphs from known ones using the method of Godsil-McKay described above, see for example [1] and [3].

In this article, we apply the method of Godsil-McKay to $\Gamma_Q$ as described above. The paper is organized as follows: After a brief description of our terminology in Section 2, we give two constructions of Godsil-McKay switching sets for $\Gamma_Q$ in Section 3. In Sections 4 and 5, we study the binary code spanned by the rows of the adjacency matrix $\Gamma_Q$ and that of its switched graphs. In Section 6, we give a number of switched graphs found and find the parameters of the codes of the switched graphs.

### 2. Terminology and notation

For any $m = 0, 1, 2, \cdots, n-1$, a subspace of dimension $m$, or $m$-space, of $\text{PG}(n, 2)$ is a set of points all of whose representing vectors form, together with the zero, a subspace of dimension $m + 1$ of $\mathbb{F}_2^{n+1}$. The number of points of an $m$-space in $\text{PG}(n, 2)$ is $2^{m+1} - 1$ [9, Theorem 3.1].

A quadric $Q_n$ in $\text{PG}(n, 2)$ is the set of points $[X_0, X_1, \cdots, X_n]$ satisfying a non-zero homogeneous equation of degree two, i.e. $\sum_{i \leq j, i,j=0}^n a_{ij}X_iX_j = 0$ for some $a_{ij} \in \mathbb{F}_q$. 

[1] Ch.9 and [5] Ch.2.
not all zero. If the equation can be reduced to fewer than \( n + 1 \) variables by a change of basis, \( Q_n \) is called singular. Otherwise, it is non-singular.

Depending on the parity of \( n \), there is one or there are two quadrics under the action of the automorphism group of \( \text{PG}(n, 2) \). For \( n \) odd, there are two distinct non-singular quadrics, respectively the elliptic quadric with canonical equation \( f(X_0, X_1) + X_2X_3 + \cdots + X_{n-1}X_n = 0 \) where \( f \) is an irreducible binary quadratic form, and the hyperbolic quadric with canonical equation \( X_0X_1 + X_2X_3 + \cdots + X_{n-1}X_n = 0 \). For \( n \) even, there is the parabolic quadric with canonical equation \( X_0^2 + X_1X_2 + \cdots + X_{n-1}X_n = 0 \). For a parabolic quadric \( Q_n \), there is an unique point in \( \text{PG}(n, 2) \setminus Q_n \), called the nucleus of \( Q_n \), such that all line through the nucleus is tangent to \( Q_n \) (see [10, page 10]). Table 2 shows the number of points of different non-singular quadrics.

| quadric \( Q_n \) | Elliptic | Hyperbolic | Parabolic |
|-------------------|----------|------------|-----------|
| number of points  | \( 2^n - 2^{\frac{n}{2}} - 1 \) | \( 2^n + 2^{\frac{n}{2}} - 1 \) | \( 2^n - 1 \) |

Table 2. Number of points in non-singular quadrics

A singular quadric in \( \text{PG}(n, 2) \) is either an \( m \)-space, \( m < n \), or a cone \( \Pi_{n-t-1}Q_t \) which is the set of points on the lines joining an \( (n-t-1) \)-space \( \Pi_{n-t-1} \) to a non-singular quadric \( Q_t \) in a \( t \)-space \( \Pi_t \) with \( \Pi_{n-t-1} \cap \Pi_t = \emptyset \). The number of points of such a cone is

\[
|\Pi_{n-t-1}Q_t| = (2^{n-t} - 1) + 2^{n-t}|Q_t|.
\]

A polarity \( \rho \) of \( \text{PG}(n, 2) \) is an order-two bijection on its subspaces that reverses containment. That is, for an \( m \)-space \( \Pi_m \) and \( m' \)-space \( \Pi_{m'} \) of \( \text{PG}(n, 2) \), if \( \Pi_m \subset \Pi_{m'} \), then \( \Pi_{m'}^\rho \subset \Pi_m^\rho \). In particular, a polarity interchanges \( m \)-spaces and \( (n-1-m) \)-spaces. For a general reference on polarities, see [9, Section 2.1].

The (binary linear) code \( C(\Gamma) \) of a graph \( \Gamma = (V, E) \) is the subspace in the vector space \( \mathbb{F}_2^{|V|} \) generated by the rows of the adjacency matrix of \( \Gamma \) modulo 2. The length \( n \) of \( C(\Gamma) \) is \( |V| \), and the dimension \( k \) of \( C(\Gamma) \) is the dimension of \( C(\Gamma) \) as a subspace in \( \mathbb{F}_2^{|V|} \). For any vector \( w = (w_x)_{x \in V} \in \mathbb{F}_2^{|V|} \), the weight \( \text{wt}(w) \) of \( w \) is

\[
\text{wt}(w) = |\{x \in V|w_x \neq 0\}|.
\]

The minimum weight \( d \) of a code is the minimum of the weight of its non-zero codewords. A binary linear code of length \( n \), dimension \( k \) and minimum weight \( d \) will be referred to as an \([n, k, d]_2\). For any subset \( U \subset V \), the characteristic vector of \( U \), denoted by \( v^U \), is the vector \((w_x)_{x \in V} \) where \( w_x = 1 \) if \( x \in U \), and \( w_x = 0 \) if \( x \notin U \). For a general reference on codes, see [2].

For the graph \( \Gamma_Q = (V_Q, E_Q) \) defined in Section [11] \( C(\Gamma_Q) \) is a \([2^n + 2^{\frac{n-1}{2}}, n + 1, 2^{n-1}]_2 \) code if \( Q \) is elliptic, and is a \([2^n + 2^{\frac{n-1}{2}}, n + 1, 2^{n-1} - 2^{\frac{n-1}{2}}]_2 \) code if \( Q \) is hyperbolic. A vector \( w \in \mathbb{F}_2^{|V_Q|} \) is a codeword of \( C(\Gamma_Q) \) if and only if it is the
characteristic vector of \((\text{PG}(n,2) \setminus \mathcal{Q}) \setminus \Sigma\) for some \((n-1)\)-space \(\Sigma\) in \(\text{PG}(n,2)\). The weight distribution of \(C(\Gamma_{\mathcal{Q}})\) is shown in Tables 3 and 4 (see for example [7]).

| weight | 0 | \(2^{n-1}\) | \(2^{n-1} + 2^{\frac{n-1}{2}}\) |
|--------|---|-------------|------------------|
| number of codewords | 1 | \(2^n - 2^{\frac{n+1}{2}} - 1\) | \(2^n + 2^{\frac{n-1}{2}}\) |

**Table 3.** Weight distribution of \(C(\Gamma_{\mathcal{Q}})\) if \(\mathcal{Q}\) is elliptic

| weight | 0 | \(2^{n-1} - 2^{\frac{n-1}{2}}\) | \(2^{n-1}\) |
|--------|---|-------------|-------|
| number of codewords | 1 | \(2^n - 2^{\frac{n-1}{2}}\) | \(2^n + 2^{\frac{n-1}{2}} - 1\) |

**Table 4.** Weight distribution of \(C(\Gamma_{\mathcal{Q}})\) if \(\mathcal{Q}\) is hyperbolic

3. TWO CONSTRUCTIONS OF GODSIL-MCKAY SWITCHING SETS OF \(\Gamma_{\mathcal{Q}}\)

In this section, we will prove Theorems 3.A and 3.B which give constructions of Godsil-McKay switching sets of the graph \(\Gamma_{\mathcal{Q}}\) defined in Section 1 for quadrics \(\mathcal{Q}\) in \(\text{PG}(n,2)\).

Theorems 3.A and 3.B are as follows.

**Theorem 3.A.** Let \(\mathcal{Q}\) be a non-singular quadric in \(\text{PG}(n,2)\) where \(n \geq 5\) is odd. Let \(t\) be an integer such that \(0 < t \leq \frac{n-3}{2}\), \(\alpha\) be a \(t\)-space in \(\mathcal{Q}\), and \(\Pi\) be a \((t+1)\)-space meeting \(\mathcal{Q}\) in exactly \(\alpha\). Let \(\Gamma_{\mathcal{Q}}\) be as defined in Section 1. Then

\[
(3.1) \quad S_t := \Pi \setminus \alpha
\]

is a Godsil-McKay switching set of \(\Gamma_{\mathcal{Q}}\) of size \(2^{t+1}\). Let \(\Gamma_{\mathcal{Q},t}\) be the graph obtained by Godsil-McKay switching with switching set \(S_t\). Then \(\Gamma_{\mathcal{Q},t}\) is a strongly regular graph with the same parameters as \(\Gamma_{\mathcal{Q}}\) (which are listed as in Table 1). Furthermore, if \(\perp\) is the polarity of \(\text{PG}(n,2)\) induced by \(\mathcal{Q}\), then

\[
(3.2) \quad T_t := (\text{PG}(n,2) \setminus \mathcal{Q}) \setminus \alpha_{\perp}
\]

is the set of vertices in \(\Gamma_{\mathcal{Q}}\) outside \(S_t\) which have exactly \(\frac{1}{2}|S_t|\) neighbours in \(S_t\).

**Theorem 3.B.** Let \(\mathcal{Q}\) be a non-singular quadric in \(\text{PG}(n,2)\) where \(n \geq 5\) is odd. If \(\mathcal{Q}\) is elliptic, then let \(t\) be an integer such that \(0 < t \leq \frac{n-3}{2}\). If \(\mathcal{Q}\) is hyperbolic, then let \(t\) be an integer such that \(0 < t \leq \frac{n-5}{2}\). In \(\text{PG}(n,2)\) where \(n \geq 5\) is odd, let \(\mathcal{Q}\) be a non-singular quadric. Let \(\alpha\) be a \(t\)-space in \(\mathcal{Q}\). Let \(\Pi, \Pi'\) be distinct \((t+1)\)-spaces meeting \(\mathcal{Q}\) in exactly \(\alpha\) such that the space spanned by \(\Pi\) and \(\Pi'\) meet \(\mathcal{Q}\) in exactly \(\alpha\). Let \(\Gamma_{\mathcal{Q}}\) be as defined in Section 1. Then

\[
(3.3) \quad S_{t,t} := (\Pi \cup \Pi') \setminus \alpha
\]
is a Godsil-McKay switching set of $\Gamma_Q$. Let $\Gamma_{Q,t,t}$ be the graph obtained by Godsil-McKay switching with switching set $S_{t,t}$. Then $\Gamma_{Q,t,t}$ is a strongly regular graph with the same parameters as $\Gamma_Q$ (these are listed as in Table 1). Furthermore, if $\perp$ is the polarity of $\text{PG}(n, 2)$ induced by $Q$, then

$$T_{t,t} = T_t \cup \left[\left(\left(\Pi^{\perp} \Delta \Pi^{\perp} \right) \setminus S_{t,t}\right) \setminus Q\right]$$

is the set of vertices in $\Gamma_Q$ outside $S_{t,t}$ which have exactly $|\frac{1}{2}S_{t,t}|$ neighbours in $S_{t,t}$, where $\Delta$ is the symmetric difference.

**Remark.** In both Theorems 3.A and 3.B, $t \leq \frac{n-3}{2}$ or $t \leq \frac{n-5}{2}$. This is a necessary and sufficient condition for the existence of $\alpha$, $\Pi$ and $\Pi'$ by [10, Theorem 22.8.3].

**Remark.** In Theorem 3.B, by the dimension theorem for subspaces, the space $\langle \Pi, \Pi' \rangle$ spanned by $\Pi$ and $\Pi'$ is an $(t+2)$-subspace. By [9, Theorem 3.1], there are exactly three planes through $\alpha$ in $\langle \Pi, \Pi' \rangle$. Let $\Pi''$ be the plane through $\alpha$ other than $\Pi$ and $\Pi'$. Since $Q$ is a quadric, either $\langle \Pi, \Pi' \rangle \cap Q = \Pi''$ or $\langle \Pi, \Pi' \rangle \cap Q = \alpha$ holds by [10, Theorem 22.8.3]. In the former case, since $\langle \Pi, \Pi' \rangle$ is one dimension higher than that of $\Pi''$, $(\Pi \cup \Pi') \setminus \alpha = \langle \Pi, \Pi' \rangle \setminus \Pi''$ is a Godsil-McKay switching set of $\Gamma_Q$ by Theorem 3.A. The latter case is treated in Theorem 3.B.

Throughout this article, we will work under the assumptions of Theorems 3.A or 3.B. In particular, the symbols $n,Q, \perp, t, \alpha, \Pi, \Pi', \Gamma_Q, \Gamma_{Q,t}, \Gamma_{Q,t,t}, S_t, S_{t,t}, T_t$ and $T_{t,t}$ are preserved as defined in Theorems 3.A or 3.B.

We first check the size of switching sets described in Theorems 3.A or 3.B.

**Lemma 3.1.** The size $|S_t|$ of $S_t$ and the size $|S_{t,t}|$ of $S_{t,t}$ are respectively $2^{t+1}$ and $2^{t+2}$.

**Proof.** Since $\Pi$ is a $(t+1)$-space and $\alpha$ is a $t$-space,

$$|S_t| = |\Pi \setminus \alpha| = (2^{t+2} - 1) - (2^{t+1} - 1) = 2^{t+1}.$$ 

For $S_{t,t}$, because of $\Pi \cap \Pi' = \alpha$, we have

$$|S_{t,t}| = |(\Pi \cup \Pi') \setminus \alpha| = 2(2^{t+2} - 1) - (2^{t+1} - 1) - (2^{t+1} - 1) = 2^{t+2}.$$ 

We now determine the structure of the subgraphs of $\Gamma_Q$ with vertex sets $S_t$ and $S_{t,t}$ respectively.

**Lemma 3.2.** The subgraph of $\Gamma_Q$ with vertex set $S_t$ is null.

**Proof.** Since $\alpha$ is one dimension less than that of $\Pi$, a line in $\Pi$ either lies in $\alpha$ or is tangent to $\alpha$, and thus to $Q$. In other words, no two vertices are joined in $\Gamma_Q$. □

**Lemma 3.3.** The subgraph of $\Gamma_Q$ with vertex set $S_{t,t}$ is a regular subgraph of degree $2^{t+1}$.
Proof. Let $x$ be a point in $S_{t,t}$. Without loss of generality, assume $x \in \Pi$. We count the number of neighbours of $x$. By the same argument used in Lemma 3.2, $x$ is not adjacent to any vertex in $\Pi \setminus \alpha$.

Since the span of $\Pi$ and $\Pi'$ meets $Q$ in exactly $\alpha$ by assumption, any line through $x$ and a point in $\Pi' \setminus \alpha$ is an external line of $Q$. In other words, the vertex is adjacent to any vertex in $\Pi' \setminus \alpha$. Since the size of $\Pi' \setminus \alpha$ is $2^{t+1}$, the result follows. $\square$

Lemma 3.4. $S_t$ is a Godsil-McKay switching set of $\Gamma_Q$.

Proof. By Lemma 3.2, the subgraph of $\Gamma_Q$ with vertex set $S_t$ is null. By Lemma 3.1, $|S_t| = 2^{t+1}$. Let $x$ be a point in $(\text{PG}(n,2) \setminus Q) \setminus S_t$. It suffices to show that either any line joining $x$ and a point of $S_t$ meets $Q$, or there are exactly $2^t$ or $2^{t+1}$ points $y$ in $S_t$ such that the line $xy$ is an external line to $Q$. Since any line in $\text{PG}(n,2)$ has exactly three points, any line through two external points of $Q$ is either a tangent or an external line. Thus, it also suffices to show that either any line joining $x$ and a point of $S_t$ is an external line, or there are exactly $2^t$ or $2^{t+1}$ points $y$ in $S_t$ such that the line $xy$ is tangent to $Q$.

Suppose the line $xy$ joining $x$ and a point $y$ of $S_t$ is tangent to $Q$. Since every line has only three points, the unique point $z$ on $xy$ other than $x$ and $y$ is a point of $Q$. Let $\Sigma$ be the $(t+1)$-space spanned by $z$ and $\alpha$. Then $\Sigma$ meets $Q$ in the $t$-space $\alpha$ and at least one point not in $\alpha$, namely $z$. By [10, Theorem 22.8.3], $\Sigma$ either lies in $Q$ or meets $Q$ in exactly two $t$-spaces.

If $\Sigma$ lies in $Q$, then every line through $x$ and a point of $S_t$ is a tangent to $Q$, and we are done.

If $\Sigma$ meets $Q$ in exactly two $t$-spaces, say $\alpha$ and $\alpha'$, then $\alpha$ and $\alpha'$ meet in a $(t-1)$-space. Then a line $xy'$ through $x$ and a point $y'$ of $S_t$ is tangent to $Q$ if and only if $y'$ is in $S_t \cap \langle x, \alpha' \rangle$. Since
\[
S_t \cap \langle x, \alpha' \rangle = (\Pi \setminus \alpha) \cap \langle x, \alpha' \rangle = (\Pi \cap \langle x, \alpha' \rangle) \setminus (\alpha \cap \langle x, \alpha' \rangle) = \langle y, \alpha \cap \alpha' \rangle \setminus (\alpha \cap \alpha')
\]
has $(2^{t+1} - 1) - (2^t - 1) = 2^t$ points, the result follows. $\square$

We need to make use of a property of $\perp$ to prove the following two lemmas. Recall from [10, Lemma 22.3.3] that, for any point $y \in Q$, $y^\perp$ comprises the points on the tangents to $Q$ at $y$ and the lines in $Q$ through $y$; for any point $y \notin Q$, $y^\perp$ consists of the points on the tangents to $Q$ through $y$.

Lemma 3.5. The following inclusions hold:

1) $S_t \subset \Pi^\perp \subset \alpha^\perp$.
2) $S_{t,t} \subset \Pi^\perp \Delta \Pi'^\perp \subset \alpha^\perp$. 
Proof. Since $\alpha$ is a subset of $\Pi$ and $\Pi'$, by the definition of a polarity, $\Pi^\perp$ and $\Pi'^\perp$ are subsets of $\alpha^\perp$.

Since the line through any two points of $\Pi$ is either a tangent of $Q$ or a line of $Q$. By [10] Lemma 22.3.3, $\Pi$ is a subset of $\Pi^\perp$. Similarly, $\Pi'$ is a subset $\Pi'^\perp$. Hence $S_t \subset \Pi^\perp$ and $S_{t,t} \subset \Pi^\perp \cup \Pi'^\perp$.

As stated in Theorem 3.4, the space spanned by $\Pi$ and $\Pi'$ meets $Q$ in exactly $\alpha$. Thus, any line through joining a point of $\Pi$ and a point of $\Pi'$ is a tangent of $Q$. By [10] Lemma 22.3.3, $\Pi \cap \Pi'^\perp = \emptyset$ and $\Pi^\perp \cap \Pi' = \emptyset$, and so $S_{t,t} \cap \Pi^\perp \cap \Pi'^\perp = \emptyset$. The result follows.

To determine $T_t$, we prepare a lemma about polarities in $PG(n,2)$.

**Lemma 3.6.** Let $\rho$ be a polarity of $PG(n,2)$. Let $\Sigma$ be an $(m + 1)$-space of $PG(n,2)$ where $0 \leq m < n - 1$. Let $x$ be a point in $PG(n,2)$. Then exactly one of the following cases occurs.

1. $x$ is in $\Sigma^\rho$.
2. $x$ is in $\pi^\rho \setminus \Sigma^\rho$ for exactly one $m$-space $\pi$ in $\Sigma$.

**Proof.** By [9] Theorem 3.1], there are exactly $N = 2^{m+2} - 1$ $m$-spaces in $\Sigma$. Let $\pi_1, \pi_2, \ldots, \pi_N$ be the $m$-spaces contained in $\Sigma$. Since $\rho$ is a polarity, $\pi_i^\rho$, $i = 1, 2, \ldots, N$, are $(n - 1 - m)$-spaces containing the $(n - 2 - m)$-space $\Sigma^\rho$. For distinct $i, j \in \{1, 2, \ldots, N\}$, $\pi_i^\rho \cap \pi_j^\rho = (\pi_i, \pi_j)^\rho = \Sigma^\rho$. Thus, the number of points in $\bigcup_{i=1}^{N} \pi_i^\rho$ is $|\Sigma^\rho| + \sum_{i=1}^{N} |\pi_i^\rho \setminus \Sigma^\rho| = (2^{n-1-m} - 1) + N(2^{n-m} - 1) = 2^{n+1} - 1$, which is the number of points in $PG(n,2)$. Now, the result follows.

**Lemma 3.7.** Let $x$ be a point not in $Q$. Then exactly one of the following cases occurs.

1. $x$ is in $\Pi^\perp$; any line joining $x$ and a point in $S_t$ is not an external line of $Q$.
2. $x$ is in $\pi^\perp \setminus \Pi^\perp$ for exactly one $t$-space $\pi \neq \alpha$ in $\Pi$; the line $xy$ through $x$ and a point $y \in S_t$ is an external line of $Q$ if and only if $y \notin \pi$. Furthermore, there are $2^t$ such points $y$.
3. $x$ is in $\alpha^\perp \setminus \Pi^\perp$; any line through a point of $S_t$ and $x$ is an external line of $Q$.

**Proof.** By [9] Theorem 3.1], there are exactly $N = 2^{t+2} - 1$ $t$-spaces in $\Pi$. Let $\pi_0, \pi_1, \ldots, \pi_{N-1}$ be the $t$-spaces contained in $\Pi$. Without loss of generality, assume $\pi_0 = \alpha$.

Let $x$ be a point not in $Q$. By Lemma 3.6 $x$ is either in $\Pi^\perp$ or in $\pi_i^\perp \setminus \Pi^\perp$ for exactly one $i \in \{0, 1, \ldots, N-1\}$.

1. Suppose $x \in \Pi^\perp$. Then $x \in y^\perp$ for all $y \in \Pi$. Thus the line through $x$ and a point $y \in \Pi \setminus Q$ is a tangent to $Q$. In other words, no point $y$ in $\Pi \setminus Q = \Pi \setminus \alpha = S_t$ satisfies the condition that the line $xy$ is an external line of $Q$.  

7
(2) Suppose \( x \in \pi_i^\perp \setminus \Pi^\perp \) for exactly one \( i \neq 0 \). By a similar argument, it follows that no point \( y \) in \( \pi_i \setminus Q = \pi_i \setminus \alpha \) satisfies the condition that the line \( xy \) is an external line of \( Q \). Suppose there exists \( z \in S_t \setminus \pi_i \) such that the line through \( xz \) is not an external line of \( Q \). Since every line contains exactly three points, that line is tangent to \( Q \) and thus \( x \) is in \( z^\perp \). Then \( x \in z^\perp \cap \pi_i = \langle z, \pi_i \rangle = \Pi^\perp \). This gives a contradiction, and thus the line through \( x \) and a point \( y \in S_t \) is an external line of \( Q \) if and only if \( y \notin \pi_i \). Since \( \alpha \cap \pi_i \) is a \((t - 1)\)-space, there are exactly

\[
|S_t \setminus \pi_i| = |(\Pi \setminus \pi_i) \setminus (\alpha \setminus \pi_i)| = [(2^{t+2} - 1) - (2^{t+1} - 1)] - [(2^{t+1} - 1) - (2^{t-1})] = 2^t
\]

points \( y \) in \( S_t \) such that the line \( xy \) is an external line of \( Q \).

(3) Suppose \( x \in \alpha^\perp \setminus \Pi^\perp \). Suppose there exists \( y \in S_t \) such that the line \( xy \) is not an external line of \( Q \). Then that line is a tangent to \( Q \) and thus \( x \in y^\perp \). Then \( x \in y^\perp \cap \alpha^\perp = \langle y, \alpha \rangle = \Pi^\perp \). This gives a contradiction and the result follows.

We are ready to give a proof of Theorem 3.A.

Proof of Theorem 3.A. By Lemma 3.4, \( S_t \) is a Godsil-McKay switching set for \( \Gamma_Q \).

By Godsil and McKay [6], \( \Gamma_Q,t \) has a same adjacency spectrum as \( \Gamma_Q \). Since \( \Gamma_Q \) is a strongly regular graph, \( \Gamma_Q,t \) is also a strongly regular graph with the same parameters (see the first three paragraphs on [4, Subsection 14.5.1]), where the parameters are listed as in Table 1 on 2.

By the definition of \( T_t \) and Lemma 3.7,

\[
T_t = (\text{PG}(n, 2) \setminus Q) \cap \left[ \bigcup_{\pi \neq \alpha} \pi^\perp \setminus \Pi^\perp \right] \setminus S_t
\]

where \( \pi \) runs over all \( t \)-space of \( \Pi \) except \( \alpha \). By Lemma 3.6, \( \bigcup_{\pi \neq \alpha} \pi^\perp \setminus \Pi^\perp \) = \( \text{PG}(n, 2) \setminus \alpha^\perp \). Since \( S_t \) is in \( \alpha^\perp \), the result follows.

With Lemma 3.7, we prove Theorem 3.B.

Proof of Theorem 3.B. By Lemma 3.8 the subgraph of \( \Gamma_Q \) with vertex set \( S_{t,t} \) is a regular subgraph of degree \( 2^{t+1} \).

Let \( x \) be a point in \( (\text{PG}(n, 2) \setminus Q) \setminus S_{t,t} \). By Lemma 3.9 one of the following cases occurs.

1. \( x \in \Pi^\perp \) and \( x \in \Pi'^\perp \).
2. \( x \in \Pi^\perp \) and \( x \in \alpha^\perp \setminus \Pi'^\perp \).
3. \( x \in \Pi^\perp \) and \( x \in \pi' \setminus \Pi'^\perp \) for some \( t \)-space \( \pi' \neq \alpha \) of \( \Pi' \).
4. \( x \in \alpha^\perp \setminus \Pi^\perp \) and \( x \in \Pi'^\perp \).
(5) $x \in \alpha^\perp \setminus \Pi^\perp$ and $x \in \alpha^\perp \setminus \Pi'^\perp$.  
(6) $x \in \pi^\perp \setminus \Pi^\perp$ for some $t$-space $\pi \neq \alpha$ of $\Pi$, and $x \in \Pi'^\perp$.  
(7) $x \in \pi^\perp \setminus \Pi^\perp$ for some $t$-space $\pi \neq \alpha$ of $\Pi$, and $x \in \pi'^\perp \setminus \Pi'^\perp$ for some $t$-space $\pi' \neq \alpha$ of $\Pi'$.  

Note that case (3) never occurs. Indeed, since $\alpha$ is a subset of $\Pi$, we have $\Pi^\perp \subseteq \alpha^\perp$. Indeed, if $x$ is in $\Pi^\perp$, then $x$ is in $\alpha^\perp$ by Lemma 3.3. By Lemma 3.6, $\alpha^\perp = (\alpha^\perp \setminus \Pi'^\perp) \cup \Pi'^\perp$ is disjoint from $\pi^\perp \setminus \Pi'^\perp$. Similarly, case (6) never occurs.

For the remaining cases, by Lemma 3.7 there are respectively $0+0 = 0, 0+2t+1 = 2t+1, 2t+1 + 0 = 2t+1, 2t+1 + 2t+1 = 2t+2, 2t + 2t = 2t+1$ points $y$ in $(\Pi \cup \Pi') \setminus \alpha$ such that the line $xy$ is an external line of $Q$. Therefore, $S_{t,t}$ is a Godsil-McKay switching set of $\Gamma_Q$ because we have $|S_{t,t}| = 2t+2$ by Lemma 3.4.

Similarly, by Godsil and McKay [6], $\Gamma_{Q,t,t}$ has a same adjacency spectrum as $\Gamma_Q$. By [4], $\Gamma_{Q,t,t}$ is also a strongly regular graph with the same parameters, where the parameters are listed as in Table 1.

The vertex $x$ is adjacent to none or all vertices in $S_{t,t}$, if and only if case (1) or (5) holds, if and only if $x \in \alpha^\perp \setminus (\Pi^\perp \Delta \Pi'^\perp)$. The result for $T_{t,t}$ now follows. 

4. Some codewords of the switched graphs

We shall use the same notation $n, Q, t, \tau, \alpha, \Pi, \Pi', \Gamma_Q, \Gamma_{Q,t,t}, S_t, S_{t,t}, T_t, T_{t,t}$, as described in Theorems 3.1 or 3.2. Recall from Section 2 that $C(\Gamma_{Q,t})$ and $C(\Gamma_{Q,t,t})$ are respectively the code of $C(\Gamma_{Q,t})$ and $C(\Gamma_{Q,t,t})$. In this section, we aim to prove $v^{S_t}, v^{T_t} \in C(\Gamma_{Q,t})$ and $v^{S_{t,t}}, v^{T_{t,t}} \in C(\Gamma_{Q,t,t})$.

Since we will need frequently the number of external lines of a non-singular quadric through a point, we give these numbers in the following lemma for ease of reference.

**Lemma 4.1.** Let $Q_m$ be a non-singular quadric in $PG(m,2)$. Let $x$ be a point not in $Q_m$. If $m$ is odd, there are $2^{m-2} \pm 2^{\frac{m-1}{2}}$ external lines through $x$, where the upper sign of $\pm$ is taken when if $Q_m$ is elliptic, and otherwise if $Q_m$ is hyperbolic. If $m$ is even, there are $0$ or $2^{m-2} - 1$ external lines through $x$, depending on whether $x$ is the nucleus of $Q_m$ or not.

**Proof.** When $m$ is odd, $Q_m$ has $2^m \mp 2^{\frac{m-1}{2}} - 1$ points (see Table 1). Thus, there are $|PG(m,2)| - |Q_m| = 2^m \pm 2^{(m-1)/2}$ non-quadric points. By [10] Theorem 22.6.6, these non-quadric points are in the same orbit under the subgroup $Aut(Q_m)$ of the automorphism group of $PG(m,2)$ which fixes $Q_m$. Thus, through each point, there are a same number of external lines. The result follows because there are $\frac{1}{3}(2^{m-2})(2^{\frac{m-1}{2}} \mp 1)(2^{\frac{m-1}{2}} \pm 1)$ external lines in $PG(m,2)$ [10] Lemma 22.8.1.

Similarly, when $m$ is even, there are $2^m$ non-quadric points. Recall from Section 2 that all line through the nucleus of $Q_m$ is tangent to $Q_m$. By [10] Theorem 22.6.6, any
non-quadric points, other than the nucleus, are in the same orbit under Aut($Q_m$). The result follows similarly because there are $\frac{1}{3}(2^{m-2})(2^m - 1)$ external lines in PG$(m, 2)$ [10, Lemma 22.8.1].

In the following lemma, whenever we use the signs $\pm$ or $\mp$, the upper sign is always taken when $Q$ is elliptic, and lower sign is always taken when $Q$ is hyperbolic.

**Lemma 4.2.** There is an external line $l$ of $Q$ such that $l$ and $\alpha^\perp$ are disjoint.

*Proof.* Let $x$ be a non-quadric point not in $\alpha^\perp$. Let $\Sigma$ be the $(n-t)$-space spanned by $\{x, \alpha^\perp\}$. If there is an external line of $Q$ through $x$ but not in $\Sigma$, then such a line will be disjoint from $\alpha^\perp$ and we are done.

We first consider the case for $t = 1$. By Lemma 3.6, $x \in \pi^\perp$ for a unique 1-space $\pi$ of $\Pi$. Since $x$ is not in $\alpha^\perp$, we have $\pi \neq \alpha$ and so $\pi \cap \alpha$ is a point of $Q$. By Theorem [10, Theorem 22.7.2], $\Sigma \cap Q$ is a parabolic quadric. If $x$ is the nucleus of $\Sigma \cap Q$, then there is no external line (of both $Q$ and $\Sigma \cap Q$) in $\Sigma$ and through $x$, as desired. If $x$ is not the nucleus of $\Sigma \cap Q$, then there are $2n^3 - 1$ external lines in $\Sigma$ and through $x$ by Lemma 4.1. Since $n$ is not less than 5, this number is less than the number of external lines in PG$(n, 2)$ through $x$ found in Lemma 4.1 and thus there is an external line of $Q$ through $x$ but not in $\Sigma$, as desired.

Similarly, in case $t = 2$, $\Sigma$ is an $(n-2)$-space meeting $Q$ in a line cone $\Pi_1 Q^-_{n-4}$ over an elliptic quadric $Q^-_{n-4}$ if $Q$ is elliptic, and a line cone $\Pi_1 Q^+_{n-4}$ over a hyperbolic quadric $Q^+_{n-4}$ if $Q$ is hyperbolic. Since $\Pi_1 Q^-_{n-4}$ has

$$|\Pi_1 Q^-_{n-4}| = 3 + 4(2^{n-4} - 2^{\frac{n-5}{2}} - 1) = 2^{n-2} - 2^{(n-1)/2} - 1$$

points and $\Pi_1 Q^+_{n-4}$ has

$$|\Pi_1 Q^+_{n-4}| = 3 + 4(2^{n-4} + 2^{\frac{n-5}{2}} - 1) = 2^{n-2} + 2^{(n-1)/2} - 1$$

points, there are

$$|\Sigma| - |\Pi_1 Q^-_{n-4}| = 2^{n-2} \pm 2^{\frac{n-1}{2}}, \epsilon \in \{-, +\}$$

non-quadric points in the $(n-2)$-space $\Sigma$. Thus, there are at most

$$\frac{2^{n-2} \pm 2^{\frac{n-1}{2}}}{2} = 2^{n-3} \pm 2^{\frac{n-3}{2}}$$

external lines in $\Sigma$ through $x$. Since this number is less than the number of external lines in PG$(n, 2)$ through $x$ found in Lemma 4.1, there is an external line of $Q$ through $x$ but not in $\Sigma$, as desired.

We now consider the case for $t > 2$. By [9] Theorem 3.1, through $x$, there are $2^{n-t} - 1$
Lemma 4.3. The vector \( v^{S_t} \) is in \( C(\Gamma_{Q,t}) \). The vector \( v^{S_{t,t}} \) is in \( C(\Gamma_{Q,t,t}) \).

**Proof.** Let \( l = \{x_1, x_2, x_3\} \) be an external line of \( Q \) such that \( l \) and \( \alpha^\perp \) are disjoint. This exists by Lemma 4.2.

For each \( i = 1, 2, 3 \), let \( r_i, \dot{r}_i \) and \( \ddot{r}_i \) respectively be the row of the adjacency matrices of \( \Gamma_Q, \Gamma_{Q,t}, \Gamma_{Q,t,t} \) corresponding to \( x_i \). Then \( r_i \) is the characteristic vector of \( (\textrm{PG}(n,2) \setminus Q) \setminus x_i^\perp \). By Lemma 3.6, \( \textrm{PG}(n,2) \setminus Q \) is the disjoint union of \( l^\perp \setminus Q \), \( (x_i^\perp \setminus l^\perp) \setminus Q \), \( (x_i^\perp \setminus l^\perp) \setminus Q \). Since \( l^\perp \) is a subset of \( x_i^\perp \) for \( i = 1, 2, 3 \), we have

\[
(4.3) \quad r_1 + r_2 + r_3 = 0
\]
in \( F_2^{|V_0|} \).

Since \( l \) is disjoint from \( \alpha^\perp \), we have \( l \subset T_t \) and \( l \subset T_{t,t} \). By the definitions of \( \Gamma_{Q,t} \) and \( \Gamma_{Q,t,t} \), for each \( i = 1, 2, 3 \), we have

\[
(4.4) \quad \dot{r}_i = r_i + v^{S_t}
\]
and

\[
(4.5) \quad \ddot{r}_i = r_i + v^{S_{t,t}}.
\]

By (4.3) and (4.4), \( \dot{r}_1 + \dot{r}_2 + \dot{r}_3 = v^{S_t} \) and so \( v^{S_t} \) is a codeword of \( C(\Gamma_{Q,t}) \). Similarly, \( v^{S_{t,t}} \) is a codeword of \( C(\Gamma_{Q,t,t}) \) because \( \dot{r}_1 + \ddot{r}_2 + \ddot{r}_3 = v^{S_{t,t}} \) by (4.3) and (4.5). □

The purpose and proof of following lemma are similar to those of Lemma 4.2 and we apply this lemma to prove \( v^{T_t} \in C(\Gamma_{Q,t}) \) and \( v^{T_{t,t}} \in C(\Gamma_{Q,t,t}) \).

**Lemma 4.4.** Let \( x \) be a non-quadratic point in \( \alpha^\perp \). Then there is an external line \( l \) of \( Q \) through \( x \) such that \( l \) is tangent to \( \alpha^\perp \) at \( x \).

**Proof.** To prove the lemma, it suffices to show some of external line through \( x \) does not lie in \( \alpha^\perp \).

We first consider the case for \( t = 1 \). Then \( \alpha^\perp \) is an \((n-2)\)-space. By [10, Theorem 22.7.2], \( \alpha^\perp \cap Q \) is a line cone \( \Pi_1 Q_{n-4}^- \) over an elliptic quadric \( Q_{n-4}^- \) if \( Q \) is elliptic, and a line cone \( \Pi_1 Q_{n-4}^+ \) over a hyperbolic quadric \( Q_{n-4}^+ \) if \( Q \) is hyperbolic. For either \( Q \) elliptic or hyperbolic, the set of points \( y \)'s in \( \alpha^\perp \cap Q \) such that the line \( xy \) is tangent to \( \alpha^\perp \cap Q \) forms a line cone \( \Pi_1 Q_{n-5} \) over a parabolic quadric \( Q_{n-5} \). Since \( Q_{n-5} \) has \( 2^{n-5} - 1 \) points [9, Theorem 5.21], there are

\[
|\Pi_1 Q_{n-5}| = [3 + 4(2^{n-5} - 1)] = 2^{n-3} - 1
\]
tangents in $\alpha^\perp$ through $x$. Using (4.1) and (4.2), there are
\[\frac{|\alpha^\perp \cap Q| - |\Pi_1 Q_{n-5}|}{2} = 2^{n-4} \mp 2^{(n-3)/2}\]
secants in $\alpha^\perp$ through $x$. Since there are $2^{n-2} - 1$ lines in $\alpha^\perp$ through $x$ [9, Theorem 3.1], there are
\[2^{n-2} - 1 = 2^{n-4} \mp 2^{(n-3)/2}\]
external lines of $Q$ in $\alpha^\perp$ through $x$, where the upper signs of $\pm$ and $\mp$ are taken if $Q$ is elliptic and the lower sign if $Q$ is hyperbolic. Since the number in (4.6) is less than the number of external lines through $x$ found in Lemma 4.1, there is an external line of $Q$ through $x$ but not in $\alpha^\perp$, as desired.

We now consider the case for $t > 1$. By [9, Theorem 3.1], through $x$, there are only
\[2^{n-1-t} - 1\]
lines of the $(n-1-t)$-space $\alpha^\perp$. Since this number is less than the number of external lines through $x$ found in Lemma 4.1, there is an external line of $Q$ through $x$ but not in $\alpha^\perp$, as desired. □

**Lemma 4.5.** The vector $v^{T_t}$ is in $C(\Gamma_{Q,t})$. The vector $v^{T_{t,t}}$ is in $C(\Gamma_{Q,t,t})$.

**Proof.** Let $x_1 \in S_t$. Note that $x_1 \in \alpha^\perp$. Take an external line $l = \{x_1, x_2, x_3\}$ of $Q$ through $x$ such that $l$ is tangent to $\alpha^\perp$ at $x_1$. It exists by Lemma 4.4.

For each $i = 1, 2, 3$, let $r_i$, $\dot{r}_i$ and $\ddot{r}_i$ respectively be the row of the adjacency matrices of $\Gamma_Q$, $\Gamma_{Q,t}$, $\Gamma_{Q,t,t}$ corresponding to $x_i$. By the same argument used in the proof of Lemma 4.3, we have
\[(4.7) \quad r_1 + r_2 + r_3 = 0.\]
Because of $x_1 \in S_t \subset S_{t,t}$, by the definitions of $\Gamma_{Q,t}$ and $\Gamma_{Q,t,t}$, we have
\[(4.8) \quad \dot{r}_1 = r_1 + v^{T_t}\]
and
\[(4.9) \quad \ddot{r}_1 = r_1 + v^{T_{t,t}}.\]
Since $x_2, x_3$ are not in $\alpha^\perp$, they are in $T_t$ and $T_{t,t}$ by (3.2) and (3.4). So, for $i = 2, 3$, we have
\[(4.10) \quad \dot{r}_i = r_i + v^{S_t}\]
and
\[(4.11) \quad \ddot{r}_i = r_i + v^{S_{t,t}}.\]
By (4.7), (4.8) and (4.10), \( r_1 + r_2 + r_3 = v^{T_t} \) and so \( v^{T_t} \) is a codeword of \( C(\Gamma_{Q,t}) \). Similarly, \( v^{T_t} \) is a codeword of \( C(\Gamma_{Q,t,t}) \) because \( r_1 + r_2 + r_3 = v^{T_t} \) by (4.7), (4.9) and (4.11).

\[ \square \]

5. THE MINIMUM WORD OF \( C(\Gamma_{Q,t}) \) AND \( C(\Gamma_{Q,t,t}) \)

In this section, we use the same notation \( n, Q, T, t, \alpha, \Pi, \Pi', \Gamma_Q, \Gamma_{Q,t}, \Gamma_{Q,t,t}, S_t, S_{t,t}, T_t \) and \( T_{t,t} \) as in Section 4, except requiring \( n \geq 7 \).

Let

\begin{equation}
C_t = \langle C(\Gamma_{Q,t}), v^{S_t}, v^{T_t} \rangle
\end{equation}

and

\begin{equation}
C_{t,t} = \langle C(\Gamma_{Q,t,t}), v^{S_{t,t}}, v^{T_{t,t}} \rangle
\end{equation}

In this section, we aim to prove the minimum word of \( C_t \) and \( C_{t,t} \) are respectively \( v^{S_t} \) and \( v^{S_{t,t}} \). This will give the minimum word of \( C(\Gamma_{Q,t}) \) and \( C(\Gamma_{Q,t,t}) \) once we prove that \( C_t = C(\Gamma_{Q,t}) \) and \( C_{t,t} = C(\Gamma_{Q,t,t}) \) in the next section.

Lemma 5.1. Let \( w \in C(\Gamma_Q) \). Then \( \text{wt}(w + v^{S_t}) > 2^{t+1} \) and \( \text{wt}(w + v^{S_{t,t}}) > 2^{t+2} \).

Proof. From Table 3 if \( Q \) is elliptic, the weight \( \text{wt}(w) \) of \( w \) satisfies \( \text{wt}(w) \geq 2^{n-1} \). By Lemma 3.1 \( \text{wt}(v^{S_t}) = 2^{t+1} \) and \( \text{wt}(v^{S_{t,t}}) = 2^{t+2} \). So,

\[ \text{wt}(w + v^{S_t}) \geq \text{wt}(w) - \text{wt}(v^{S_t}) = 2^{n-1} - 2^{t+1}, \]

\[ \text{wt}(w + v^{S_{t,t}}) \geq \text{wt}(w) - \text{wt}(v^{S_{t,t}}) = 2^{n-1} - 2^{t+2}. \]

Since we have assumed \( n \geq 7 \) in this section and we have \( t \leq \frac{n-3}{2} \) under the assumption in Theorems 3.4 and 3.5, it is straightforward to verify that \( \text{wt}(w + v^{S_t}) > 2^{t+1} \) and \( \text{wt}(w + v^{S_{t,t}}) > 2^{t+2} \).

From Table 4 if \( Q \) is hyperbolic, then \( \text{wt}(w) \geq 2^{n-1} - 2^{n-t} \). Similarly, since \( n \geq 7 \), it is straightforward to verify that \( \text{wt}(w + v^{S_t}) > 2^{t+1} \) and \( \text{wt}(w + v^{S_{t,t}}) > 2^{t+2} \) with \( t \) in the range stated in Theorems 3.3 and 3.4. \( \square \)

For any subset \( U \) of points of \( \text{PG}(n, 2) \), denoted by \( \hat{U} \) the set \( U \setminus Q \). Recall that whenever we use the signs \( \pm \) or \( \mp \), the upper sign is always taken when \( Q \) is elliptic, and lower sign is always taken when \( Q \) is hyperbolic.

Lemma 5.2.

1. \( |\hat{\alpha}| = 2^{n-t-1} \pm 2^{n-t-2}. \)
2. \( \hat{A} = (\Pi^\perp \Delta \Pi'^\perp) \setminus S_{t,t} \). Then \( |\hat{A}| = 2^{n-t-2} \pm 2^{n-t-3} - 2^{t+2}. \)
3. \( \Sigma \) be an \( (n-1) \)-space. Then exactly one of the following holds:
   a. \( \Sigma \cap Q = Q_{n-1} \); \( |\hat{\Sigma}| = 2^{n-1}. \)
   b. \( \Sigma \cap Q = \Pi_0 Q_{n-2} \) where \( Q_{n-2} \) and \( Q \) are both elliptic or both hyperbolic; \( |\hat{\Sigma}| = 2^{n-1} \pm 2^{n-t-2}. \)
Lemma 5.3. The size of $T_t$ and $T_{t,t}$ are respectively $|T_t| = 2^n - 2^{n-t-1}$ and $|T_{t,t}| = 2^n + 2^{n-t-1} - 2^{n-t-2} - 2^{t+2}$. Furthermore, the following holds:

1. $|T_t| > 2^{t+1}$.
2. $|T_t \triangle S_t| > 2^{t+1}$.
3. $|T_{t,t}| > 2^{t+2}$.
4. $|T_{t,t} \triangle S_{t,t}| > 2^{t+2}$.

Proof. Using (3.2) and Lemma 5.2[1], we obtain

$$|T_t| = |\text{PG}(n,2)| - |Q| - |\tilde{\alpha}^\perp| = (2^{n+1} - 1) - (2^n + 2^{n-t-1} - 1) - (2^{n-t-1} + 2^{n-t-1} - 1) = 2^n - 2^{n-t-1}.$$

Proof. (1) Since $\alpha$ is in $Q$, by [10] Theorem 22.8.3, $\alpha^\perp \cap Q$ is a cone $\Pi_tQ_{n-2t-2}$ where $Q_{n-2t-2}$ is elliptic if $Q$ is elliptic, and is hyperbolic otherwise. By (2.1) and Table I, we have

$$|\alpha^\perp \cap Q| = (2^{t+1} - 1) + 2^{t+1}(2^{n-2t-2} + 2^{n-2t-3} - 1) = 2^n - 1.$$

Since $\alpha^\perp$ is an $(n - t - 1)$-space, it follows that

$$|\tilde{\alpha}^\perp| = |\alpha^\perp| - |\alpha^\perp \cap Q| = (2^{t+1} - 1) - [2^{n-t-1} + 2^{n-t-1} - 1] = 2^n - 1 + 2^{n-t}.$$

(2) Similar to (1), we have

$$|\tilde{\Pi}^\perp| = |\Pi^\perp| - |\Pi^\perp \cap Q| = |\Pi_tQ_{n-2t-3}| = (2^{n-t-1} - 1) - [(2^{t+1} - 1) + 2^{t+1}(2^{n-2t-3} - 1) - 1] = 2^n - 2.$$

and

$$|\langle \langle \Pi, \Pi' \rangle, \Pi^\perp \rangle| = |\langle \langle \Pi, \Pi' \rangle^\perp \rangle \cap Q| = |\langle \Pi, \Pi' \rangle^\perp| = |\Pi_tQ_{n-2t-4}|$$

$$= (2^{n-t-2} - 1) - [(2^{t+1} - 1) + 2^{t+1}(2^{n-2t-4} - 1)] = 2^{n-t-3} + 2^{n-2}.$$

where $Q_{n-2t-4}$ is hyperbolic if $Q$ is elliptic; $Q_{n-2t-4}$ is elliptic if $Q$ is hyperbolic. Recall from Lemma 5.5, $S_{t,t} \subset \Pi^\perp \cap \Pi^\perp$. Now using Lemma 3.1, we deduce

$$|\hat{A}| = |\tilde{\Pi}^\perp| + |\tilde{\Pi}^\perp| - 2|\langle \langle \Pi, \Pi' \rangle, \Pi^\perp \rangle| - |S_{t,t}| = 2^n - 2^{n-t-1} + 2^{n-2} - 2^{t+2}.$$

(3) By [10] Theorem 22.8.5, $\Sigma \cap Q$ is either (a) $Q_{n-1}$ or (b) $\Pi_0Q_{n-2}$ where $Q_{n-2}$ and $Q$ are both elliptic or both hyperbolic. The result follows by (2.1) and Table I.

□
Since \(0 < t \leq \frac{n-3}{2}\), we have
\[
|T_t| - 2^{t+1} = 2^{n-t-1}(2^{t+1} - 1) - 2^{t+1} > 3 \cdot 2^{n-t-1} - 2^{t+1} > 0.
\]
So, \(|T_t| > 2^{t+1} \).

Using (3.4) and Lemma 5.2(2), we have
\[
|T_{t,t}| = |T_t| + |\hat{A}| = 2^n - 2^{n-t-2} + 2^{\frac{n-1}{2}} - 2^{t+2}
\]
where \(A = (\Pi^\perp \triangle \Pi'^\perp) \setminus S_{t,t}\). Because of \(t > 0\), we have
\[
|T_{t,t}| - 2^{t+2} = 2^{n-t-2}(2^{t+2} - 1) + 2^{\frac{n-1}{2}} - 2^{t+3} \geq 7 \cdot 2^{n-t-2} + 2^{\frac{n-1}{2}} - 2^{t+3}.
\]
When \(Q\) is elliptic, \(t \leq \frac{n-3}{2}\) and so
\[
7 \cdot 2^{n-t-2} + 2^{\frac{n-1}{2}} - 2^{t+3} > 0.
\]
When \(Q\) is hyperbolic, \(t \leq \frac{n-5}{2}\) and so
\[
7 \cdot 2^{n-t-2} - 2^{\frac{n-1}{2}} - 2^{t+3} \geq 7 \cdot 2^{\frac{n-1}{2}} - 2^{\frac{n-1}{2}} - 2^{t+3} > 0.
\]
In both cases, \(|T_{t,t}| > 2^{t+2}\). The results of \(T_t \triangle S_t\) and \(T_{t,t} \triangle S_{t,t}\) follow because of \(T_t \cap S_t = \emptyset\) and \(T_{t,t} \cap S_{t,t} = \emptyset\). □

**Lemma 5.4.** Let \(R = (\text{PG}(n,2) \setminus Q) \setminus \Sigma\) for some \((n-1)\)-space \(\Sigma\) of PG\((n,2)\). Then the following holds:

1. \(|R \triangle T_t| > 2^{t+1}\).
2. \(|R \triangle T_t \triangle S_t| > 2^{t+1}\).
3. \(|R \triangle T_{t,t}| > 2^{t+2}\).
4. \(|R \triangle T_{t,t} \triangle S_{t,t}| > 2^{t+2}\).

**Proof.** The complement \(R^c\) of \(R\) in PG\((n,2) \setminus Q\) is \(R^c = \hat{\Sigma}\).

Let \(A := ((\Pi^\perp \triangle \Pi'^\perp) \setminus S_{t,t}) \setminus Q\). By (3.2) and (3.4), we have

\[
T_t^c = \hat{\alpha}^\perp,
T_{t,t} = T_t \cup A.
\]

Recall for any subsets \(U_1, U_2, U_3\) of PG\((n,2) \setminus Q\), we have \(U_1 \triangle U_2 = U_1^c \triangle U_2^c\); \((U_1 \cup U_2)^c = U_1^c \cap U_2^c\); \((U_1 \triangle U_2) \triangle U_3 = U_1 \triangle (U_2 \triangle U_3)\); \(U_1 \triangle U_2 \supset U_1 \setminus U_2\), and equality holds if and only if \(U_1 \subset U_2\). Further because of \(S_t, S_{t,t} \subset \alpha^\perp\) by Lemma 3.3 and
$S_{t,t} \cap A = \emptyset$, we have

$$R \triangle T_t = \hat{\Sigma} \triangle \hat{\alpha} \supset \hat{\Sigma} \setminus \hat{\alpha};$$

$$R \triangle T_t \triangle S_t = (\hat{\Sigma} \triangle \hat{\alpha}) \triangle S_t = \hat{\Sigma} \triangle (\hat{\alpha} \setminus S_t) \supset \hat{\Sigma} \setminus (\hat{\alpha} \setminus S_t) \supset \hat{\Sigma} \setminus \hat{\alpha}^t;$$

$$R \triangle T_{t,t} \triangle S_{t,t} = \hat{\Sigma} \triangle [(\hat{\alpha} \setminus \hat{A}) \setminus S_{t,t}] \supset \hat{\Sigma} \setminus [(\hat{\alpha} \setminus \hat{A}) \setminus S_{t,t}] \supset \hat{\Sigma} \setminus (\hat{\alpha} \setminus \hat{A}).$$

Thus, it suffices to show (i) $|\hat{\Sigma} \setminus \hat{\alpha}| > 2^{t+1}$ and (ii) $|\hat{\Sigma} \setminus (\hat{\alpha} \setminus \hat{A})| > 2^{t+2}$ for $t$ within the range mentioned in Theorems 3.A and 3.B By [10, Theorem 22.8.3], $\Sigma \cap Q$ is either (a) a parabolic quadric $Q_{n-1}$, or (b) a point cone $\Pi_0 Q_{n-2}$ where $Q_{n-2}$ and $Q$ are both elliptic or both hyperbolic.

(a) If $\Sigma \cap Q = Q_{n-1}$, then by [10, Theorem 22.7.2], we have $\Sigma^\bot \notin Q$ and so $\Sigma^\bot \notin \alpha$. By the definition of a polarity, we have $\alpha^\bot \notin \Sigma$. Since $\Sigma$ is a hyperplane and $\alpha^\bot$ is an $(n-1-t)$-space, $\Sigma \cap \alpha^\bot$ is a $(n-2-t)$-space.

(i) By Lemma 5.2(3a) and $0 < t \leq \frac{n-3}{2}$, we have

$$|\hat{\Sigma} \setminus \hat{\alpha^\bot}| - 2^{t+1} \geq |\hat{\Sigma}| - |\Sigma \cap \alpha^\bot| - 2^{t+1}$$

$$= 2^{n-1} - (2^{n-t-1} - 1) - 2^{t+1} = 2^{n-t}(2^t - 1) + 1 - 2^{t+1}$$

$$\geq 2^{n-t-1} - 2^{t+1} + 1 > 0.$$

(ii) Similarly, since $\hat{\Sigma} \setminus (\hat{\alpha} \setminus \hat{A}) \subset \hat{\Sigma} \setminus \hat{\alpha}$, we have

$$|\hat{\Sigma} \setminus (\hat{\alpha} \setminus \hat{A})| - 2^{t+2} \geq |\hat{\Sigma}| - |\Sigma \cap \alpha^\bot| - 2^{t+2}$$

$$\geq 2^{n-t-1} - 2^{t+2} + 1 > 0.$$

(b) (i) If $\Sigma \cap Q = \Pi_0 Q_{n-2}$, then by Lemma 5.2(3b), and because of $t > 0$, we have

$$|\hat{\Sigma} \setminus \hat{\alpha}| - 2^{t+1} \geq |\hat{\Sigma}| - |\alpha| - 2^{t+1}$$

$$= (2^{n-1} + 2^{n-1}) - (2^{n-t-1} + 2^{n-t}) - 2^{t+1}$$

$$= 2^{n-t-1} - 2^{t+1} \geq 2^{n-1-t} - 2^{t+1} > 0.$$
Proposition 5.5. Let $\Sigma \cap Q = \Pi_0 Q_{n-2}$, then by Lemma 5.2 and because of $t > 0$, we have

\[
|\hat{\Sigma} \setminus (\alpha^\perp \setminus \hat{A})| - 2^{t+2} \\
\geq |\hat{\Sigma}| - |\alpha^\perp| + |\hat{A}| - 2^{t+2} \\
= (2^{n-1} + 2^{\frac{n-1}{2}}) - (2^{n-t-1} + 2^{\frac{n-1}{2}}) + (2^{n-t-2} + 2^{\frac{n-1}{2}} - 2^{t+2}) - 2^{t+2} \\
= 2^{n-2-t}(2^{t+1} - 1) + 2^{\frac{n-1}{2}} - 2^{t+3} \\
\geq 3 \cdot 2^{n-2-t} + 2^{\frac{n-1}{2}} - 2^{t+3}
\]

where the last equality holds if and only if $t = 1$. If $Q$ is elliptic, then because of $t \leq \frac{n-3}{2}$, we have

\[
3 \cdot 2^{n-2-t} + 2^{\frac{n-1}{2}} - 2^{t+3} \geq 0
\]

where the equality holds if and only if $t = \frac{n-3}{2}$. Because of $n \geq 7$, it is impossible to have $1 = t = \frac{n-3}{2}$. Combining (5.5) and (5.6), we have $|\hat{\Sigma} \setminus (\alpha^\perp \setminus \hat{A})| > 2^{t+2}$.

If $Q$ is hyperbolic, then because of $t \leq \frac{n-5}{2}$, we have

\[
3 \cdot 2^{n-2-t} - 2^{\frac{n-1}{2}} - 2^{t+3} > 0.
\]

Combining (5.5) and (5.7), we have $|\hat{\Sigma} \setminus (\alpha^\perp \setminus \hat{A})| > 2^{t+2}$.

\[\square\]

Proposition 5.5. Let $u$ be a non-zero vector in $C_t$. Then $\text{wt}(u) \geq 2^{t+1}$, and equality holds if and only if $u = v^S_t$.

Proof. Let $u$ be a non-zero vector in $C_t$. Then $u$ is one of the following: $w, w + v^S_t, w + v^{T_1}, w + v^{T_1} + v^S_t, v^{T_1}, w^{T_1} + v^S_t$ or $v^S_t$ for some $w \in C(\Gamma_Q)$. By Tables 3 and 4, $\text{wt}(w) > 2^{t+1}$, and by Lemma 5.1, $\text{wt}(w + v^S_t) > 2^{t+1}$. Note that for any subsets $U_1, U_2$ of $\text{PG}(n, 2) \setminus Q, v^{U_1} + v^{U_2} = v^{U_1 \Delta U_2}$. The result follows from Lemmas 3.1 and 5.3 because $w = v^R$ where $R = \text{PG}(n, 2) \setminus Q \setminus \Sigma$ for some $(n-1)$-space $\Sigma$. \[\square\]

Proposition 5.6. Let $u$ be a non-zero vector in $C_{t,t}$. Then $\text{wt}(u) \geq 2^{t+2}$, and equality holds if and only if $u = v^{S_{t,t}}$.

Proof. It follows using arguments that are similar to those in the proof of Proposition 5.5. \[\square\]

6. Numbers of Switched Graphs Found

With the notation as given in Section 5 for $n, Q, \perp, t, \alpha, \Pi, \Pi', \Gamma_Q, \Gamma_{Q,t}, \Gamma_{Q,t,t}, S_t, S_{t,t}, T_t$ and $T_{t,t}$, we assume $n \geq 7$. In this section, we will prove $C(\Gamma_{Q,t}) = C_t$ and
$C(\Gamma_{Q,t,t}) = C_{t,t}$ as claimed in Section 4 and then count the number of non-isomorphic graphs constructed through Theorems 3.A and 3.B.

Let $A, A_t, A_{t,t}$ be the adjacency matrices of $\Gamma_Q$, $\Gamma_{Q,t}$ and $\Gamma_{Q,t,t}$.

Since $S_t \subset S_{t,t}$ and $T_t \subset T_{t,t}$, we may assume that the first $|S_t|$ rows and columns of $A, A_t, A_{t,t}$ correspond to points of $\text{PG}(n, 2) \setminus Q$ in $S_t$; the next $|S_{t,t} \setminus S_t|$ rows and columns correspond to those in $S_{t,t} \setminus S_t$; the last $|T_{t,t}|$ rows and columns correspond to points in $T_{t,t}$ such that the last $|T_t|$ rows and columns correspond to points in $T_t$.

By the definition of $\Gamma_{Q,t}$,

$$A_t = A + M_t, \text{ where } M_t = \begin{pmatrix} O & O & J_t \\ O & O & O \\ J_t' & O & O \end{pmatrix}$$

where $J_t$ is the $|S_t|$-by-$|T_t|$ all-ones matrix. Similarly, by the definition of $\Gamma_{Q,t,t}$,

$$A_{t,t} = A + M_{t,t}, \text{ where } M_{t,t} = \begin{pmatrix} O & O & J_{t,t} \\ O & O & O \\ J_{t,t}' & O & O \end{pmatrix}$$

where $J_{t,t}$ is the $|S_{t,t}|$-by-$|T_{t,t}|$ all-ones matrix.

**Lemma 6.1.** None of $v^{T_t}$ or $v^{T_{t,t}}$ is in $C(\Gamma_Q)$.

**Proof.** Suppose $v^{T_t}$ is in $C(\Gamma_Q)$. Recall any codeword in $C(\Gamma_Q)$ is $v^R$ where $R = (\text{PG}(n, 2) \setminus Q) \setminus \Sigma$ for some $(n - 1)$-space $\Sigma$. By (3.2),

$$\text{PG}(n, 2) \setminus Q \setminus \alpha^\perp = (\text{PG}(n, 2) \setminus Q) \setminus \Sigma.$$ 

This implies $\Sigma \setminus Q = \alpha^\perp \setminus Q$. Considering the size of $\Sigma \setminus Q$ and $\alpha^\perp \setminus Q$ given in Lemma 5.2, we have $n = 3$ or $t = 0$, which contradicts the range of $n$ and $t$ stated in Theorem 3.A or Theorem 3.B. \qed

We now prove $C(\Gamma_{Q,t}) = C_t$ and $C(\Gamma_{Q,t,t}) = C_{t,t}$ as announced in Section 5.

**Lemma 6.2.** $C(\Gamma_{Q,t}) = \langle C(\Gamma_{Q,t}), v^{S_t}, v^{T_t} \rangle$ and the 2-rank of $C(\Gamma_{Q,t})$ is $n + 3$.

**Proof.** By Lemmas 4.3 and 4.5, $v^{S_t}$ and $v^{T_t}$ are codewords of $C(\Gamma_{Q,t})$. By (6.1), a row of the adjacency matrix of $\Gamma_{Q,t}$ either is a row of the adjacency matrix of $\Gamma_Q$ or differs from such a row by $v^{S_t}$ or $v^{T_t}$. Thus, any row of the adjacency matrix of $\Gamma_Q$ is a codeword of $C(\Gamma_{Q,t})$.

By Lemma 6.1, $v^{T_t} \notin C(\Gamma_Q)$ and by Proposition 5.3, for any $w \in C(\Gamma_Q)$, we have that none of $w$ and $w + v^{T_t}$ is the vector $v^{S_t}$. Thus, $v^{S_t}, v^{T_t}$ and a basis of $C(\Gamma_Q)$ form a linearly independent set of size $2 + (n + 1) = n + 3$.

In (6.1), since the 2-rank of $M_t$ is 2, the 2-rank of $C(\Gamma_Q)$ differs from that of $C(\Gamma_{Q,t})$ by at most 2. Since $v^{S_t}$ and $v^{T_t}$ and a basis of $C(\Gamma_Q)$ form a linearly independent set in $C(\Gamma_{Q,t})$ with size two more than the 2-rank of $C(\Gamma_Q)$, they form a basis of $C(\Gamma_{Q,t})$. \qed

18
Lemma 6.3. \( C(\Gamma_{Q,t,t}) = \langle C(\Gamma_{Q,t,t}), v^{S_{t,t}}, v^{T_{t,t}} \rangle \) and the 2-rank of \( C(\Gamma_{Q,t,t}) \) is \( n + 3 \).

Proof. The proof is similar to that of Lemma 6.2.

We now give the parameters of \( C(\Gamma_{Q,t}) \) and \( C(\Gamma_{Q,t,t}) \). Recall the upper sign of \( \mp \) is taken when if \( Q \) is elliptic, and otherwise if \( Q \) is hyperbolic.

Theorem 6.4. \( C(\Gamma_{Q,t}) \) is a \([2^{n \mp 2^{\frac{n+1}{2}}}, n + 3, 2^{t+1}]_2\)-code. \( C(\Gamma_{Q,t,t}) \) is a \([2^{n \mp 2^{\frac{n-1}{2}}}, n + 3, 2^{t+2}]_2\)-code

Proof. The length of \( C(\Gamma_{Q,t}) \) and \( C(\Gamma_{Q,t,t}) \) are the number of vertices of their respective graphs, which is \( 2^{n \mp 2^{\frac{n+1}{2}}} \). Other parameters of the codes follow from Lemmas 6.2, 6.3, and Proportions 5.5, 5.6.

Theorem 6.5. The graphs \( \Gamma_Q, \Gamma_{Q,1}, \Gamma_{Q,2}, \cdots, \Gamma_{Q,\frac{n-1}{2}}, \Gamma_{Q,1,1}, \Gamma_{Q,2,2}, \cdots, \Gamma_{Q,m,m} \) are distinct up to isomorphism, where \( m = \frac{n-3}{2} \) if \( Q \) is elliptic and \( m = \frac{n-5}{2} \) if \( Q \) is hyperbolic.

Proof. \( \Gamma_Q \) is distinct from other graphs in the list because it has a 2-rank \( n + 1 \) \([11, \text{Theorem 5.3}] \) but others do not by Lemmas 6.2 and 6.3. Let \( \Gamma, \Gamma' \) be two graphs listed above other than \( \Gamma_Q \). Let \( S \) and \( S' \) be switching sets of \( \Gamma_Q \) such that \( \Gamma, \Gamma' \) are obtained from \( \Gamma_Q \) with switching sets respectively \( S \) and \( S' \).

Suppose there is an isomorphism \( \phi \) between \( \Gamma \) and \( \Gamma' \). Then \( \phi \) induces a code isomorphism \( \Phi \) between \( C(\Gamma) \) and \( C(\Gamma') \). Since \( \Phi \) maps minimum word(s) of \( C(\Gamma) \) to those of \( C(\Gamma') \), we have \( \Phi(S) = S' \) by Propositions 5.5 and 5.6. Considering the size of the switching sets given in Lemma 3.1, we may assume without loss of generality that \( \Gamma = \Gamma_{Q,t+1} \) and \( \Gamma' = \Gamma_{Q,t,t} \) for some \( t \). By Lemma 6.2, the subgraph of \( \Gamma_Q \), and hence of \( \Gamma \), with vertex set \( S = S_{t,t+1} \) is null. But by Lemma 5.5, the subgraph of \( \Gamma_Q \), and hence of \( \Gamma' \), with vertex set \( S' = S_{t,t} \) is not null. This contradicts \( \Phi(S) = S' \), and so \( \Gamma \) and \( \Gamma' \) are non-isomorphic.

Since we work under the assumption that \( n \geq 7 \) in Sections 5 and 6, Theorem 6.5 is valid under the same assumption. However, it can be checked directly that in case \( Q \) is elliptic and \( n = 5 \), \( \Gamma_Q, \Gamma_{Q,1} \) and \( \Gamma_{Q,1,1} \) are non-isomorphic; in case \( Q \) is hyperbolic and \( n = 5 \), \( \Gamma_Q \) and \( \Gamma_{Q,1} \) are non-isomorphic. In conclusion, for \( n \geq 5 \), if \( Q \) is an elliptic quadric in PG\((n, 2)\), then Theorems 3.A and 3.B give \( n - 3 \) non-isomorphic graphs, other than \( \Gamma_Q \), with the same parameters as \( \Gamma_Q \), where the parameters are shown in Table I. For \( n \geq 5 \), if \( Q \) is a hyperbolic quadric in PG\((n, 2)\), then Theorems 3.A and 3.B give \( n - 2 \) non-isomorphic graphs, other than \( \Gamma_Q \), with the same parameters as \( \Gamma_Q \).

References

[1] Abiad, A. & Haemers, W.H. Switched symplectic graphs and their 2-ranks. Appeared online in Des. Codes Cryptogr.

19
[2] Assmus, E.F. & Key, J.D. (1992). Designs and their codes. Cambridge University Press, Cambridge.
[3] Barwick, S.G., Jackson, W-A. & Penttila, T. New families of strongly regular graphs. Manuscript.
[4] Brouwer, A.E. & Haemers, W.H. (2011). Spectra of graphs. Springer, New York.
[5] Cameron, P.J. & van Lint, J.H. (1991). Designs, graphs, codes and their links, Cambridge University Press, Cambridge.
[6] Godsil, C.D. & McKay, B.D. (1982). Constructing cospectral graphs. Aequationes Math, 25, 257–268.
[7] Haemers, W.H., Peeters, M.J.P. & van Rijckevorsel, J.M. (1999). Binary codes of strongly regular graphs. Des. Codes Cryptogr., 17, 187–209.
[8] Hall, J.I. & Shult, E.E. (1985). Locally cotriangular graphs. Geometriae Dedicata, 18, 113–159.
[9] Hirschfeld, J.W.P. (1979). Projective geometries over finite fields. Oxford Univ. Press.
[10] Hirschfeld, J.W.P. & Thas, J.A. (1991). General Galois geometries. Oxford Sci. Publ., Clarendon Press.
[11] Peeters, M.J.P. (1995). Uniqueness of strongly regular graphs having minimal $p$-rank. Linear Algebra Appl. 226–228, 9–31.