Incidence and Combinatorial Properties of Linear Complexes

Hans Havlicek
Institut für Diskrete Mathematik und Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8–10
A-1040 Wien
Austria
havlicek@geometrie.tuwien.ac.at

Corrado Zanella
Dipartimento di Tecnica e Gestione dei Sistemi Industriali
Università di Padova
Stradella S. Nicola 3
I-36100 Vicenza
Italy
corrado.zanella@unipd.it

Dedicated to Helmut Karzel on the occasion of his 80th birthday

Abstract

In this paper a generalisation of the notion of polarity is exhibited which allows to completely describe, in an incidence-geometric way, the linear complexes of \( h \)-subspaces. A generalised polarity is defined to be a partial map which maps \((h - 1)\)-subspaces to hyperplanes, satisfying suitable linearity and reciprocity properties. Generalised polarities with the null property give rise to a linear complexes and vice versa. Given that there exists for \( h > 1 \) a linear complex of \( h \)-subspaces which contains no star –this seems to be an open problem over an arbitrary ground field--the combinatorial structure of a partition of the line set of the projective space into non-geometric spreads of its hyperplanes can be obtained. This line partition has an additional linearity property which turns out to be characteristic.

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1 Introduction

The notion of a linear complex has been investigated for longer than a century, and one could regard it as completely known. The classical approach is to define a linear complex of \( h \)-dimensional subspaces (shortly: \( h \)-subspaces) in \( \text{PG}(n, F) \),
i.e., the \( n \)-dimensional projective space over a commutative field \( F \), as the set of all \( h \)-subspaces whose Grassmann coordinates satisfy a non-trivial linear equation or, in a coordinate-free way, via a hyperplane section of the corresponding Grassmann variety. It is well known that each linear complex of lines \((h = 1)\) is the set of all lines which are contained in their polar subspace with respect to a (possibly degenerate) null polarity. The classification of these complexes reduces to the well known classification of non-zero alternating matrices over \( F \). Also, a lot was written by leading classical authors on the theory of linear complexes of \( h \)-subspaces for \( h > 1 \), but seemingly, such a theory was developed for the complex numbers only. As regards the topics which will be dealt with in this paper see, among others, \([2],[3],[5],[6],[17]\), and \([19]\).

To our knowledge, no incidence-geometric approach to linear complexes exists for \( h > 1 \). Our goal is to adopt such a point of view and coherently describe linear complexes of \( h \)-subspaces. To accomplish this task, we first collect some results about linear mappings and primes (geometric hyperplanes) of Grassmannians over arbitrary fields (commutative or skew). In a Grassmannian over a proper skew field there is only one kind of prime, whence we rule out skew fields at an early stage of our investigation. In a Grassmannian over a commutative field primes and linear complexes are the same, but this result gives no explicit information about linear complexes.

In Section 5 the notion of polarity is generalised. This leads us to an incidence-geometric construction of linear complexes of \( h \)-subspaces in \( \text{PG}(n, F) \) in terms of generalised null polarities. A crucial problem for linear complexes of \( h \)-subspaces in \( \text{PG}(n, F) \) is the existence of singular \((h - 1)\)-subspaces for \( h > 1 \). This problem is addressed in Section 6 where we also sketch a result from \([2]\) for the case of complex numbers. However, a solution over an arbitrary field \( F \) presently seems beyond reach. If there exists a linear complex of \( h \)-subspaces, \( h > 1 \), without singular \((h - 1)\)-subspaces, then there is also a linear complex of planes without singular lines. Therefore, we focus on linear complexes of planes without singular lines, where we establish an intriguing relation with linear line spreads. Any linear complex of planes without singular lines gives rise to linear line spread, which turns out to be non-geometric in several cases, e.g., for a finite ground field. Finally, Section 7 is devoted to a connection between linear complexes of planes without singular lines and linear line partitions. This connection is used to show that certain projective spaces do not admit linear line partitions.

2 The Grassmannians of a projective space

A semilinear space is a pair \( \Sigma = (\mathcal{P}, \mathcal{B}) \), where \( \mathcal{P} \) is a set of points, \( \mathcal{B} \subseteq 2^\mathcal{P} \) is a set of lines, and the following axioms hold:

\[(i) \ |\ell| \geq 2 \text{ for each } \ell \in \mathcal{B}.\]

\[(ii) \text{ For every } X \in \mathcal{P} \text{ there exists at least one } \ell \in \mathcal{B} \text{ such that } X \in \ell.\]

\[(iii) \ |\ell \cap \ell'| \leq 1 \text{ for every } \ell, \ell' \in \mathcal{B}, \ell \neq \ell'.\]

Note that some authors use slightly different axioms for a semilinear space; others speak of a partial linear space instead.
Given two points $X, Y \in P$ we write $X \sim Y$, where “$\sim$” is to be read as collinear; if there is an $\ell \in B$ such that $X, Y \in \ell$. Otherwise, $X$ and $Y$ are said to be non-collinear, $X \not\sim Y$. If $X \sim Y$ and $X \not\sim Y$, then the unique line $\ell \in B$ such that $X, Y \in \ell$ is denoted by $XY$.

Let $F$ be a (commutative or non-commutative) field. We denote by $\text{PG}(n, F)$ the $n$-dimensional projective space coordinatised by the field $F$ and collect some basic notions which will be used throughout this article.

If $X$ is a subspace of $\text{PG}(n, F)$ and $\dim X = d$, then $X$ is called a $d$-subspace. Let $U$ and $W$ be two subspaces such that $U \subseteq W$. The interval $[U, W]$ is the set of all subspaces of $\text{PG}(n, F)$ containing $U$ and contained in $W$. The set of all $d$-subspaces of $\text{PG}(n, F)$ belonging to $[U, W]$ is denoted by $[U, W]_d$. In particular, for an interval $[U, W]$ with $\dim U = h-1$ and $\dim W = h+1$ the set $[U, W]_h$ is the pencil of $h$-subspaces determined by $U$ and $W$.

Let $[U, W]$ be an interval in $\text{PG}(n, F)$, $\dim U =: k - 1$. Then $[U, W]$ carries the structure of a projective space, whose $d$-dimensional subspaces are precisely the $(k+d)$-subspaces of $\text{PG}(n, F)$ containing $U$ and contained in $W$. The projective space $[U, W]$ is isomorphic to $\text{PG}(\dim W - k, F)$.

The $h$-th Grassmannian of $\text{PG}(n, F)$, $0 \leq h \leq n - 1$, is the semilinear space $\Gamma(n, h, F) = (P, B)$, where $P$ is the set of all $h$-subspaces of $\text{PG}(n, F)$, and $B$ is the set of all pencils of $h$-subspaces. When the field is clear from the context we simply write $\Gamma(n, h)$. If $\text{PG}(n, F)$ admits a correlation (e. g., for a commutative field $F$) then $\Gamma(n, h, F)$ is isomorphic to $\Gamma(n, n - h - 1, F)$. In order to avoid confusion, the elements of $P$ and $B$ will be called $G$-points and $G$-lines, respectively, where “$G$” abbreviates “Grassmann”.

We consider the Grassmannian $\Gamma(n, h, F)$ for a fixed $h$. If a set $I$ of $G$-points is a $d$-dimensional projective space with respect to the $G$-lines contained in $I$, then $I$ is called a $d$-$G$-subspace. A $d$-$G$-subspace $I$ is thus a set of $h$-subspaces of $\text{PG}(n, F)$ pairwise meeting in $(h-1)$-subspaces. Hence there exists an $(h-1)$-subspace $U$ of $\text{PG}(n, F)$ such that $U \subseteq X$ for all $X \in I$, or there exists an $(h+1)$-subspace $V$ of $\text{PG}(n, F)$ such that $X \subseteq V$ for all $X \in I$; both conditions hold for $d \leq 1$.

Next, we describe the maximal $G$-subspaces of $\Gamma(n, h, F)$. We start by recalling two notions: If $U$ is an $(h-1)$-subspace of $\text{PG}(n, F)$, the star with centre $U$ is the set of all $h$-subspaces of $\text{PG}(n, F)$ containing $U$. Consequently, a star is an $(n-h)$-$G$-subspace. Let $S$ denote the set of stars. A dual star is the set of all $h$-subspaces of $\text{PG}(n, F)$ contained in a fixed $(h+1)$-subspace. The dual stars are $(h+1)$-$G$-subspaces, and the set of all dual stars will be denoted by $T$. For $h = 1$ a dual star is a ruled plane. If $h = 0, n-1$, then $P$ is the only maximal $G$-subspace. Otherwise, the set of all maximal $G$-subspaces partitions into the families $S$ and $T$.

3 Linear mappings

Let $P$ and $P'$ be sets. We say that $\chi$ is a partial map of $P$ into $P'$, if there is a subset $D(\chi)$ of $P$ such that $\chi : D(\chi) \rightarrow P'$ is a map in the usual sense. The set $D(\chi)$ is called the domain of $\chi$. Let $A(\chi) = P \setminus D(\chi)$ be the exceptional subset of $\chi$. The elements of $A(\chi)$ are called exceptional points of $\chi$. If $A(\chi) = \emptyset$, then $\chi$ is global. We maintain the notation $\chi : P \rightarrow P'$ even when $\chi$ is not global.

In order to have a less complicated notation we extend the definition of $\chi$ to
the power set of $\mathcal{P}$ by defining

$$\phi^\chi := (\phi \cap D(\chi)) = \{X^\chi \mid X \in \phi \cap D(\chi)\}$$

for every $\phi \in 2^\mathcal{P}$.

Thus $\chi$ assigns to every subset of $\mathcal{P}$ a subset of $\mathcal{P}'$.

Next, let $\Sigma = (\mathcal{P}, \mathcal{B})$ and $\Sigma' = (\mathcal{P}', \mathcal{B}')$ be semilinear spaces. A partial map $\chi : \mathcal{P} \to \mathcal{P}'$ is called a linear mapping, if for each $\ell \in \mathcal{B}$ one of the following holds:

(i) $\ell^\chi \in \mathcal{B}'$, and $\chi$ maps $\ell$ bijectively onto $\ell^\chi$.

(ii) $\ell^\chi = \{P'\}$, where $P' \in \mathcal{P}'$, and $|\ell \cap A(\chi)| = 1$.

(iii) $\ell \subseteq A(\chi)$.

Obviously, these three conditions are mutually exclusive. This linear mapping will also be denoted by $\chi : \Sigma \to \Sigma'$.

If $X, Y$ are distinct collinear points in $A(\chi)$, then the line $XY$ is a subset of $A(\chi)$ by (iii). In particular, for a projective space $\Sigma$ this implies that $A(\chi)$ is a subspace of $\Sigma$. In the case that $\chi$ is global, condition (i) holds for all lines, whence distinct collinear points have distinct images. However, the images of non-collinear points may coincide.

There is one type of linear mapping deserving special mention: A full projective embedding is a global and injective linear map $\chi : \Sigma \to \Sigma'$ for which $\Sigma'$ is a projective space.

The linear mappings between projective spaces allow the following explicit description:

**Theorem 1.** [4, 14] Let $\Sigma$ and $\Sigma'$ be projective spaces, and let $\chi : \Sigma \to \Sigma'$ be a linear mapping. Then the partial map $\chi$ splits into a projection from $A(\chi)$ onto a complementary subspace in $\Sigma$, say $U$, and a collineation between $U$ and a subspace of $\Sigma'$.

The linear mappings between Desarguesian projective spaces are –up to one particular case – the geometric counterpart of the semilinear maps between vector spaces; the exceptional points correspond to non-zero vectors in the kernel. The situation is different if the image of the linear mapping is a line. Here the collineation from the above theorem is just a bijection which, of course, need not be induced by a semilinear bijection between the underlying vector spaces.

We now shortly exhibit full projective embeddings of a Grassmannian $\Gamma(n, h, F)$. In doing so, the Grassmannians $\Gamma(n, 0, F)$ and $\Gamma(n, n - 1, F)$ will be excluded, since they are a priori projective spaces. Hence we have $0 < h < n - 1$ which implies $n \geq 3$.

First, assume that $F$ is a non-commutative field. By [14] Satz 3.6, no full projective embedding of $\Gamma(n, h, F)$ exists for $0 < h < n - 1$.

Next, we assume $F$ to be a commutative field. Let $N = \binom{n+1}{h+1} - 1$ and

$$\varphi_{n,h} : \Gamma(n, h, F) \to \text{PG}(N, F)$$

be the map defined by $X^{\varphi_{n,h}} = F(v_0 \land v_1 \land \ldots \land v_h)$, where $v_0, v_1, \ldots, v_h$ is a basis of $X$ meant as an $(h + 1)$-dimensional subspace of $F^{n+1}$, and $\text{PG}(N, F)$ is represented as $(h + 1)$-th exterior power of $F^{n+1}$. This definition is independent on the choice of the basis, and $\varphi_{n,h}$ is a full projective embedding, called the
Plücker embedding. Its image \( P^{\nu_{n,h}} = G_{n,h} \) is a Grassmann variety. Each Grassmann variety is intersection of quadrics \([18, pp. 184–188]\); in particular, \( G_{3,1} \) is the well-known Klein quadric.

So, the question remains of describing all full projective embeddings of \( \Gamma(n,h,F) \) when \( F \) is commutative. An answer can be given in terms of linear mappings.

**Theorem 2.** \([14]\) Let \( F \) be a commutative field and let \( \Sigma' \) be a projective space. If \( \chi : \Gamma(n,h,F) \to \Sigma' \) is a linear mapping, then there exists a unique linear mapping \( \mu : \text{PG}(N,F) \to \Sigma' \) such that \( \chi = \nu_{n,h} \mu \).

By this universal property, we may obtain all full projective embeddings of \( \Gamma(n,h,F) \) (to within collineations) as a product of \( \nu_{n,h} \) by a (possibly trivial) projection whose centre does not meet any secant of the Grassmann variety \( G_{n,h} \). Therefore, each fully embedded Grassmannian is a (possibly trivial) projection of a Grassmann variety \( G_{n,h} \). In \([22]\) and \([23]\) sufficient conditions for the (non-)existence of a non-trivial projection are given.

For other questions and literature on linear mappings the reader is referred to the book \([7]\) (different terminology), \([4]\) (including a survey of older literature), and \([15]\).

### 4 Primes and linear complexes

A *prime* of a semilinear space \((\mathcal{P}, \mathcal{B})\) is a proper subset \( L \) of \( \mathcal{P} \), such that for each \( \phi \in \mathcal{B} \) either \( \phi \subseteq L \), or \(|\phi \cap L| = 1\). Note that some authors call such a subset a geometric hyperplane.

From now on \( \Gamma(n,h,F) = (\mathcal{P}, \mathcal{B}) \) is a Grassmannian. It is easily seen that the set of all \( h \)-subspaces having non-empty intersection with a fixed \((n-h-1)\)-subspace is a prime of \( \Gamma(n,h,F) \). Somewhat surprisingly, \( \Gamma(n,h,F) \) has no other primes, if \( F \) is a non-commutative field. This beautiful result can be found in \([13]\).

Next, let \( F \) be a commutative field and \( \nu_{n,h} : \Gamma(n,h,F) \to \text{PG}(N,F) \) the related Plücker embedding. If \( H \) is a hyperplane of \( \text{PG}(N,F) \), then

\[
K = (H \cap G_{n,h})^{\nu_{n,h}}^{-1}
\]

is called a linear complex of \( h \)-subspaces in \( \text{PG}(n,F) \). Cf., e. g. \([5]\) p. 322]. If \( \varphi \) is a G-line, then either \( \varphi \subseteq K \), or \(|\varphi \cap K| = 1\). Since \( G_{n,h} \) generates \( \text{PG}(N,F) \), we have \( K \neq \mathcal{P} \). Thus each linear complex is a prime of \( \Gamma(n,h,F) \). Conversely, we have the following result; its short proof is taken from \([14]\) p. 179):

**Proposition 3.** If \( F \) is commutative, then each prime of \( \Gamma(n,h,F) \) is a linear complex.

**Proof.** Let \( L \) be a prime of \( \Gamma(n,h,F) \). Define a partial map \( \chi : \Gamma(n,h,F) \to \Sigma' \), where \( \Sigma' \) is a point, by setting \( \mathbb{A}(\chi) = L \). Obviously, this \( \chi \) is linear. By Theorem \([2]\) \( \chi = \nu_{n,h} \mu \), where \( \mu : \text{PG}(N,F) \to \Sigma' \) is a linear mapping. Then

\[
X \in L \iff X \in \mathbb{A}(\chi) \iff X^{\nu_{n,h}} \in \mathbb{A}(\mu),
\]

and the latter exceptional set is a hyperplane by Theorem \([1]\) \( \square \).

In \([20]\) a self-contained proof of Proposition \([3]\) is given.
5 Generalised polarities arising from a linear complex

Up to the end of the paper $F$ denotes a commutative field.

Our first aim is to generalise the concept of polarity. Suppose that $\chi : \Gamma(n, k) \to \text{PG}(n, F)^*$, $0 \leq k \leq n-1$, is linear. So, $\chi$ is a partial map of the set of $k$-subspaces of $\text{PG}(n, F)$ into its hyperplane set. We say that such a $\chi$ is a (generalised) polarity if for all $U_1, U_2 \in \Gamma(n, k)$ with $U_1 \sim U_2$ the following holds:

$$U_1 \subseteq U_2^\chi \implies U_2 \subseteq U_1^\chi.$$  \(1\)

In \(1\) it is understood that for $U \in \mathcal{A}(\chi)$, $U^\chi = \{U\}^\chi$ is the whole projective space. When speaking of “polarities” below, we always mean “generalised polarities”.

For $k = 0$ our definition is in accordance with the usual definition of a (possibly degenerate) polarity. Likewise, the following result is well known for polarities.

**Theorem 4.** Let $\chi : \Gamma(n, k) \to \text{PG}(n, F)^*$, $0 \leq k \leq n-1$, be a linear mapping, and assume that $\chi$ satisfies the null property

$$U \subseteq U^\chi \text{ for each } U \in \mathcal{D}(\chi).$$  \(2\)

Then $\chi$ is a polarity.

**Proof.** Assume that $U_1$ and $U_2$ are distinct $k$-subspaces such that $U_1 \sim U_2$ in $\Gamma(n, k)$. By the linearity of $\chi$, one of the following holds: (i) $U_1, U_2 \in \mathcal{D}(\chi)$ and $U_1^\chi \neq U_2^\chi$; then $(U_1U_2)^\chi$ is the pencil of hyperplanes through $U_1^\chi \cap U_2^\chi$. (ii) There is a unique $U_0 \in U_1U_2$ such that $U_0 \in \mathcal{A}(\chi)$, and a hyperplane $E$ of $\text{PG}(n, F)$ exists such that $U \in U_1U_2, U \neq U_0$ implies $U^\chi = E$. (iii) $U_1, U_2 \in \mathcal{A}(\chi)$ so that $U_1U_2 \subseteq \mathcal{A}(\chi)$.

Let $P$ be a point. In any case we have

$$P \in U_1^\chi \cap U_2^\chi \Rightarrow (P \in U^\chi \text{ for all } U \in U_1U_2).$$  \(3\)

In order to prove \(1\), assume that the above subspaces satisfy $U_1 \subseteq U_2^\chi$. Since $U_1 \subseteq U_2^\chi$, by \(3\) we have, for each $U \in U_1U_2$, that $U_1 \subseteq U^\chi$. On the other hand, $U \subseteq U^\chi$ implies $(U \cup U_1) \subseteq U^\chi$, where $U \cup U_1$ denotes the join in $\text{PG}(n, F)$. Now take $U_1', U_2' \in U_1U_2$ with $U_1' \neq U_2' \neq U_1' \neq U_2'$. For $i = 1, 2$ we obtain

$$U_1 \cup U_2 = U_1' \cup U_1 \subseteq U_1'^\chi.$$

So, \(3\) yields $U_1 \cup U_2 \subseteq U_1^\chi$, and in particular $U_2 \subseteq U_1^\chi$. \(\square\)

Taking into account the above theorem, a linear mapping $\Gamma(n, k) \to \text{PG}(n, F)^*$ with the null property \(2\) will be called a (generalised) null polarity.

Now we turn back to linear complexes:

**Proposition 5.** Let $K$ be a linear complex of $h$-subspaces of $\Sigma = \text{PG}(n, F)$, $1 \leq h \leq n-1$, and let $[U, W]$ be an interval of $\Sigma$ with $\dim U \leq h-1$ and $\dim W \geq h+1$. Then

$$K(U, W) := K \cap [U, W]_h.$$
i.e., the set consisting of all elements of $K$ containing $U$ and contained in $W$, is a linear complex of $(h - 1 - \dim U)$-subspaces in the projective space $[U, W]$, unless $[U, W]_h \subseteq K$.

Proof. A pencil $\varphi$ of $(h - 1 - \dim U)$-subspaces with respect to the projective space $[U, W]$ is a pencil of $h$-subspaces in $\Sigma$; from $\varphi \subseteq K$ or $|\varphi \cap K| = 1$ we have $\varphi \subseteq K(U, W)$ or $|\varphi \cap K(U, W)| = 1$, respectively. Therefore, if $[U, W]_h \not\subseteq K$, then $K(U, W)$ is a prime, and hence a linear complex, of the $(h - 1 - \dim U)$-th Grassmannian of $[U, W]$.

As a particular case of the above proposition, if $P$ is a point of $\Sigma$, then the set $K_P = K(P, \Sigma)$, consisting of all elements of $K$ incident with $P$, is a linear complex of $(h - 1)$-subspaces in the $(n - 1)$-dimensional projective space $[P, \Sigma]$, unless $K_P = [P, \Sigma]_h$.

As a further consequence, if $S = [U, \Sigma]_h$ is the star with centre an $(h - 1)$-subspace $U$, then either (i) $S \subseteq K$, or (ii) there is a hyperplane $E$ of $\PG(n, F)$ such that for each $X \in S$, we have $X \in K$ if, and only if, $X \subseteq E$. If (i) holds, then $U$ is called a singular $(h - 1)$-subspace of $K$; otherwise $E$ is the polar hyperplane of $U$. Similarly, if $T = [0, V]_h$, $\dim V = h + 1$, is a dual star, then either (i) $T \subseteq K$, or (ii) there is a point $P$ of $\PG(n, F)$ such that for each $X \in T$, we have $X \in K$ if and only if $P \in X$. If (i) holds, then $V$ is called a total $(h + 1)$-subspace of $K$; otherwise $P$ is the pole of $V$.

We are now in a position to introduce the following crucial notion. If $K$ is a linear complex of $h$-subspaces, $1 \leq h \leq n - 1$, we will denote by $\uparrow K$ the partial map of the set of all $(h - 1)$-subspaces of $\PG(n, F)$ into the dual projective space $\PG(n, F)^*$, defined as follows: given an $(h - 1)$-subspace, say $U$, if $U$ is singular, then $U \in \mathcal{K}(\uparrow K)$; otherwise $U^\uparrow K$ is the polar hyperplane of $U$, i.e., the union of all elements of $K$ containing $U$. In view of Theorem 6 below, this mapping $\uparrow K$ will be called the null polarity defined by $K$.

**Theorem 6.** Let $K$ be a linear complex of $h$-subspaces in $\Sigma = \PG(n, F)$, $1 \leq h \leq n - 1$. Then the following assertions hold:

(a) The partial map $\uparrow K : \Gamma(n, h - 1) \rightarrow \PG(n, F)^*$ is a null polarity with non-empty domain.

(b) The image of $\uparrow K$ generates a subspace of $\PG(n, F)^*$ with dimension at least $h$.

Proof. (a) We use induction on $h$; for $h = 1$ this is well known [5, p. 322]. So, let $h > 1$ and let $\varphi$ be a pencil of $(h - 1)$-subspaces. There is a point, say $P$, incident with every $U \in \varphi$. If $K_P = [P, \Sigma]_h$, then $\varphi \subseteq \mathcal{A}(\uparrow K)$. Otherwise, by induction assumption, $\uparrow K_P$ is a linear mapping. Thus the linearity of $\uparrow K$ follows by observing that the restrictions of $\uparrow K$ and $\uparrow K_P$ to $\varphi$ coincide. Since $U \subseteq U^\uparrow K$ holds for all $U \in \mathcal{D}(\uparrow K)$ by definition, $\uparrow K$ is a null polarity according to Theorem 4. Finally, $K \neq \Gamma(n, h)$ implies that not all $(h - 1)$-subspaces can be singular, whence the domain of $\uparrow K$ is non-empty.

(b) Assume to the contrary that the image of $\uparrow K$ generates a subspace of $\PG(n, F)^*$ with a smaller dimension. So the intersection of all hyperplanes in the image of $\uparrow K$ contains an $(n - h)$-subspace $X$, say. We show that this implies the contradiction $\mathcal{D}(\uparrow K) = \emptyset$: Given an $(h - 1)$-subspace $U$ we argue by induction on $k := \dim U \cap X$. For $k = -1$ there is no hyperplane passing
through $U \lor X$, whence $U \in \mathcal{A}(\uparrow K)$. Next, assume $k > -1$. Then there is a point $P \in \Sigma$ outside $U \lor X$. Also, there exists a point $Q \in U \cap X$. Let $V$ be a complement of $Q$ with respect to $U$. Then $U$ is an element of the pencil $[V, V \lor P \lor Q]_{h-1}$. All other elements of this pencil meet $X$ in a subspace of dimension $k - 1$, whence they belong to $\mathcal{A}(\uparrow K)$ by the induction hypothesis. The linearity of $\uparrow K$ yields $U \in \mathcal{A}(\uparrow K)$, as required.

**Corollary 7.** The set of all singular $(h-1)$-subspaces of the linear complex $K$ is equal to $(\mathcal{G}_{n,h-1} \cap W)^{\mathcal{G}_{n,h-1}}$ for some subspace $W$ of $\text{PG}((n+1)_h - 1, F)$ with

$$\left(\frac{n+1}{h}\right) - (n + 2) \leq \dim W \leq \left(\frac{n+1}{h}\right) - (h + 2).$$

**Corollary 8.** If $U_1$ and $U_2$ are singular $(h-1)$-subspaces of $K$, and $U_1 \sim U_2$ in $\Gamma(n, h - 1)$, then each element of the pencil determined by $U_1$ and $U_2$ is a singular $(h-1)$-subspace of $K$.

We have seen that $\mathcal{A}(h)$ is a null polarity. Conversely, we have:

**Theorem 9.** Let $\chi : \Gamma(n, h - 1) \to \text{PG}(n,F)^*, \ 1 \leq h \leq n - 1$, be a linear mapping with non-empty domain satisfying the null property. Then there is a unique linear complex of $h$-subspaces, say $K$, such that $\chi = \mathcal{A}(h)$.

**Proof.** We define a set $K$ of $h$-subspaces by setting $X \in K$ if, and only if, there is an $(h-1)$-subspace $U \subseteq X$ such that $X \subseteq U^\chi$. Any linear complex with the required properties necessarily has to coincide with this $K$.

The mapping $\chi$ is a polarity by Theorem 4. From 4 we infer that if $X \in K$, then $X \subseteq U^\chi$ for every $(h-1)$-subspace $W$ of $X$. If $\varphi$ is the pencil of $h$-subspaces determined by the $(h-1)$-subspace $U_\varphi$ and the $(h+1)$-subspace $V_\varphi$, $U_\varphi \subseteq V_\varphi$, then the G-points of $\varphi$ belonging to $K$ are exactly the elements $X \in \varphi$ such that $X \subseteq U_\varphi^\chi$. Therefore, either $\varphi \subseteq K$ or $|\varphi \cap K| = 1$, whence $K$ is a prime. The proof is now accomplished by applying Proposition 3.

The above theorem is a generalisation of the classical one, characterising a linear complex of lines as a set arising from a (possibly degenerate) null polarity $\text{PG}(n,F) \to \text{PG}(n,F)^*$.

By dual arguments we obtain a linear mapping $\downarrow K : \Gamma(n, h+1) \to \text{PG}(n,F)$, $0 \leq h \leq n - 2$, the dual polarity of $K$. There holds:

**Theorem 10.** Let $\chi : \Gamma(n, h + 1) \to \text{PG}(n,F)$, $0 \leq h \leq n - 2$, be a linear mapping with non-empty domain, and assume that for each $V \in \mathcal{D}(\chi)$, $V^\chi \subseteq V$. Then there is a unique linear complex of $h$-subspaces, say $K$, such that $\chi = \downarrow K$.

### 6 Existence of singular $(h-1)$-subspaces

We start with the following technical result.

**Proposition 11.** Assume that $k$, $h$, and $n$ are integers such that $1 \leq k \leq h \leq n - 1$. Let $K$ be a linear complex of $h$-subspaces of $\text{PG}(n,F)$ having no singular $(h-1)$-subspace. Then $\text{PG}(n+1-h,F)$ contains a linear complex of $k$-subspaces having no singular $(k-1)$-subspace.
Proof. Let $W$ be any $(h - k - 1)$-subspace of $\Sigma = \text{PG}(n, F)$. The intersection $K_W = K \cap [W, \Sigma]$ is a set of $k$-subspaces of $[W, \Sigma] \cong \text{PG}(n + k - h, F)$. If $L$ is a $(k - 1)$-subspace of $[W, \Sigma]$, then $L$ is an $(h - 1)$-subspace of $\Sigma$, containing $W$. By assumption an $h$-subspace $X$ of $\Sigma$ exists that contains $L$ and does not belong to $K_W$. Now Proposition 3 easily yields the assertion.

A linear complex $K$ of lines is the set of all self-conjugate lines of a null polarity with non-empty domain in $\text{PG}(n, F)$. Since each alternating matrix has even rank, $K$ always has a singular point for $n$ even; on the other hand, for each odd $n$ there are linear complexes of lines without singular points. Thus, by Proposition 11 with $k = 1$, we have that if $n - h \equiv 1 \pmod{2}$, then each $(h - 2)$-subspace in $\text{PG}(n, F)$ is contained in a singular $(h - 1)$-subspace. In our proof of Proposition 11 the subspace $W$ was chosen arbitrarily. Hence this result can be refined as follows:

**Proposition 12.** Let $K$ be a linear complex of $h$-subspaces in $\text{PG}(n, F)$. If $1 \leq h \leq n - 1$ and $n - h \equiv 1 \pmod{2}$, then each $(h - 2)$-subspace in $\text{PG}(n, F)$ is contained in a singular $(h - 1)$-subspace.

The question concerning the existence of total $(h+1)$-subspaces is dealt with in [2] over the field $\mathbb{C}$ of complex numbers as follows. The product of a linear complex of $h$-subspaces, say $K$, and a linear complex of points, i.e., a hyperplane $H$, is the set

$$K \cdot H := \{ X \in \Gamma(n, h + 1, \mathbb{C}) \mid \exists Y \in K : Y \subseteq X \cap H \}$$

If $[0, H]_h \subseteq K$ then $H$ is called a total hyperplane of $K$; in this case $K \cdot H$ is the set of all $(h + 1)$-subspaces of $\text{PG}(n, \mathbb{C})$. Otherwise, $K \cdot H$ is a linear complex of $(h + 1)$-subspaces in $\text{PG}(n, \mathbb{C})$. Let $H_0, H_1, \ldots, H_n$ be independent hyperplanes in $\text{PG}(n, \mathbb{C})$. Then the set $\Theta$ of all total $(h + 1)$-subspaces of $K$ is the intersection of $K \cdot H_0, K \cdot H_1, \ldots, K \cdot H_n$; that is, $\Theta$ is represented on the Grassmann variety $G_{n,h+1}$ as intersection with at most $n + 1$ hyperplanes. (This is dual to our Corollary and holds over an arbitrary ground field.) Since $G_{n,h+1}$ is an algebraic variety with dimension $(h + 2)(n - h - 1)$, a sufficient condition for the existence of total $(h + 1)$-subspaces is

$$(h + 2)(n - h - 1) - (n + 1) \geq 0,$$

that is

$$(h + 1)(n - h - 2) \geq 1.$$  

Since $h + 1 \geq 1$, the condition reads $n - h - 2 > 0$ or, equivalently, $h < n - 2$. Summarising, we obtain:

**Theorem 13.** [2] Each linear complex of $h$-subspaces in $\text{PG}(n, \mathbb{C})$, with $h < n - 2$, has a total $(h + 1)$-subspace.

Dually, there holds:

**Corollary 14.** Each linear complex of $h$-subspaces in $\text{PG}(n, \mathbb{C})$, with $h \geq 2$, has a singular $(h - 1)$-subspace.
A line spread of a projective space $\text{PG}(n, F)$ is a set of lines, say $\mathcal{F}$, such that each point of $\text{PG}(n, F)$ belongs to exactly one line of $\mathcal{F}$. A line spread is called geometric (or normal) if for every pair of distinct lines of $\mathcal{F}$, say $\ell$, $m$, the lines of $\mathcal{F}$ in the solid $\ell \lor m$ form a spread of $\ell \lor m$. The line spread $\mathcal{F}$ is linear if $\mathcal{F}^{\text{Pl}} \cap \mathcal{G}_{n,1}$ with a subspace of its ambient space $\text{PG}(M, F)$ with $M := (n^2 + n - 2)/2$.

**Proposition 15.** Let $K$ be a linear complex of planes in $\text{PG}(n, F)$ having no singular line. Then $n$ is even. For each hyperplane $H$ in $\text{PG}(n, F)$, let $\mathcal{F}_H$ be the set of all lines whose polar hyperplane is $H$. Then $\mathcal{F}_H$ is a line spread.

**Proof.** By Proposition 12, $n$ is even. Let $A$ and $H$ be a point and a hyperplane in $\text{PG}(n, F)$, respectively, such that $A \in H$. The set $S$ of all lines of $\text{PG}(n, F)$ containing the point $A$ is isomorphic to $\Gamma(n, 1)$. Since $\mathcal{F}^{\text{Pl}}$ is global, its restriction to $S$ is a collineation. As a consequence, $S^{\mathcal{F}}$ is the set of all hyperplanes through $A$. So, a unique line $\ell$ through $A$ exists such that $\ell^{\mathcal{F}} = H$. This proves that $\mathcal{F}_H$ is a line spread of $H$.

**Proposition 16.** Under the assumptions of Proposition 15 the line spread $\mathcal{F}_H$ is linear. More precisely, $\mathcal{F}_H^{\text{Pl}} = R \cap \mathcal{G}_{n,1}$, where $R$ is an $(M - n)$-subspace of the ambient space $\text{PG}(M, F)$ of the Grassmann variety $\mathcal{G}_{n,1}$. Furthermore, in each of the following cases the line spread $\mathcal{F}_H$ is not geometric.

(a) The field $F$ is quadratically closed.

(b) The field $F$ is finite.

**Proof.** By Theorems 2 and 6, the null polarity $\mathcal{F}$ can be written as

$$\mathcal{F}^{\text{Pl}} = \varphi_{n,1} \pi \kappa,$$

where $\varphi_{n,1}$ is the Plücker embedding, $\pi$ is a projection of $\text{PG}(M, F)$ from a subspace $C$ with dimension $(M - n - 1)$ onto a complementary $n$-subspace $D$, and $\kappa : D \to \text{PG}(n, F)^*$ is a collineation. The non-existence of singular lines yields

$$C \cap \mathcal{G}_{n,1} = \emptyset.$$

So, $\mathcal{F}_H^{\text{Pl}}$ is the intersection of $\mathcal{G}_{n,1}$ with the subspace $R := C \lor H^{\kappa - 1}$ of $\text{PG}(M, F)$.

Now suppose that (a) or (b) holds. We assume that $\mathcal{F}_H$ is a geometric line spread. By the above, there is a hyperplane, say $J$, of $\text{PG}(M, F)$, such that $J \cap R = C$, whence

$$J \cap \mathcal{F}_H^{\text{Pl}} = \emptyset.$$

As $\mathcal{F}_H$ is geometric, there is a solid $U$ of $\text{PG}(n, F)$ such that the lines of $U$ belonging to $\mathcal{F}_H$ form a line spread of $U$, say $\mathcal{F}_1$. Furthermore, $\mathcal{F}_1^{\text{Pl}}$ is the intersection of $R$ with a quadric $Q_3^+$ (the Klein quadric representing the lines of $U$). By the table in [16, pp. 29–31] (which remains true, mutatis mutandis, also for an infinite field), $\mathcal{F}_1^{\text{Pl}}$ has to be an elliptic quadric $Q_3^-$. This is impossible if (a) holds. On the other hand, for a finite field $F$ we get from $J \cap Q_3^- = \emptyset$ the contradiction that $Q_3^-$ would have an exterior plane [16, p. 17].
The proof from the above cannot be carried over to all infinite fields, since an elliptic quadric may have an exterior plane.

Remark 1. Proposition 11 for \( k = 2 \) together with the previous theorem implies the following: Any linear complex of \( h \)-subspaces, \( h > 1 \), having no singular \((h-1)\)-subspaces, yields a non-geometric linear line spread if \( F \) satisfies one of the conditions (a) or (b). The authors do not know, whether under these circumstances non-geometric linear line spreads exist or not. In case of their non-existence, Theorem 13 of Baldassarri and Corollary 14 would also hold for projective spaces over finite or quadratically closed fields.

7 Linear line partitions

A line partition \( \Omega \) of a projective space \( \text{PG}(n, F) \) is a partition of its line set into line spreads of hyperplanes such that each hyperplane contains precisely one of these spreads. Each line partition of \( \text{PG}(n, F) \) induces a surjective map \( \pi_\Omega \) of the line set onto the dual space \( \text{PG}(n, F)^* \) as follows: It assigns to each line \( \ell \) the unique hyperplane containing the equivalence class of \( \ell \). Observe that this \( \pi_\Omega \) is globally defined on the line set.

Line partitions of finite projective spaces were investigated in [9, 10, 11, 12, 21]. Here \( n \) necessarily has to be even. In particular, we quote the following result:

**Theorem 17.** [10, 11] Each finite projective space \( \text{PG}(2^i - 2, q) \) with \( i \geq 2 \) admits a line partition.

A line partition \( \Omega \) will be called linear if \( \pi_\Omega \) from the above is a linear mapping. The linear line partitions are closely related with certain linear complexes.

**Theorem 18.** Let \( \Omega \) be a line partition of \( \text{PG}(n, F) \). Denote by \( K \) the set of all planes \( \varepsilon \) such that a line \( \ell \) exists which satisfies the condition

\[ \ell \subseteq \varepsilon \subseteq \ell^{\pi_\Omega}. \]

This \( K \) is a linear complex of planes if, and only if, \( \Omega \) is linear. In this case, \( K \) has no singular line.

**Proof.** If \( \Omega \) is linear, then \( \chi := \pi_\Omega \) satisfies the assumptions of Theorem 9. This implies that \( K \) is a linear complex of planes without singular lines.

Conversely, assume that the given set \( K \) is a linear complex of planes. Let \( \ell \) be a non-singular line. Then \( \ell^{\pi_\Omega} \) and \( \ell^{\dagger K} \) are hyperplanes which obviously are identical. Next, we show that singular lines do not exist. Assume to the contrary that \( \ell'_1 \) is a singular line. Since \( \dagger K \) has a non-empty domain, also a non-singular line \( \ell'_2 \) exists. There is a line which has a point in common with \( \ell'_1 \) and \( \ell'_2 \), respectively. This line is either singular or non-singular. Hence there exists a pencil of lines containing a singular line \( \ell_1 \) and a non-singular line \( \ell_2 \), say. Since \( \dagger K \) is linear, there is another non-singular line \( \ell_3 \neq \ell_2 \) in this pencil. We infer from the above and from the linearity of \( \dagger K \) that

\[ \ell_2^{\pi_\Omega} = \ell_2^{\dagger K} = \ell_3^{\dagger K} = \ell_3^{\pi_\Omega}. \]

Thus the distinct incident lines \( \ell_2 \) and \( \ell_3 \) belong to the same line spread, a contradiction. Therefore \( \pi_\Omega = \dagger K \) which in turn implies the linearity of \( \Omega \).
By Proposition 15 and Theorem 18, there is a bijective correspondence between linear complexes of planes without singular lines and linear line partitions. The linear complex $K$ defined in the theorem will be called the linear complex of planes related to the linear line partition $\Omega$.

**Proposition 19.** No projective space $\text{PG}(4, F)$ admits linear line partitions.

**Proof.** A linear complex of planes, say $K$, in $\text{PG}(4, F)$ is dual to a linear complex of lines $K^*$ in $\text{PG}(4, F)^*$. The singular lines of $K$ are dual to ruled planes of $K^*$. Let $S$ be the set of all singular points of $K^*$ (actually, hyperplanes of $\text{PG}(4, F)$). If $S$ is a dual plane, then obviously $K^*$ contains ruled planes. Otherwise $S$ is a point and there is a line $\ell$ of $K^*$ such that $S \not\subseteq \ell$. The plane $S \cap \ell$ is a ruled plane in $K^*$. This proves that each linear complex of planes in $\text{PG}(4, F)$ has at least one singular line.

Some finite four-dimensional projective spaces admit line partitions. In particular this holds in $\text{PG}(4, q)$ for $q = 2, 3$ [10] and for $q = 5, 8, 9$ [21]. By the above all these line partitions are necessarily non-linear.

**Proposition 20.** No finite projective space $\text{PG}(6, q)$ admits linear line partitions.

**Proof.** Let $K$ be a linear complex of planes in $\text{PG}(6, q)$, without singular lines. Denote by $K_H$ the set of planes in $K$ which are contained in some hyperplane $H$. This $K_H$ is a linear complex of planes in the five-dimensional finite projective space $H$. Let $\mathcal{F}_H$ be the line spread in $H$ given according to Proposition 15. The elements of $\mathcal{F}_H$ are precisely the singular lines of $K_H$. In [8, Theorem 14] it is shown that (to within projective transformations) a unique linear complex of planes in $\text{PG}(5, q)$ exists such that its singular lines form a line spread; moreover, this spread turns out to be geometric. This contradicts Proposition 16.

**Remark 2.** It seems to be unknown whether or not linear line partitions do exist. In particular we do not know, if the line partitions of $\text{PG}(2^i - 2, q)$ by Fuji-Hara and Vanstone (cf. Theorem 17) are linear. As a matter of fact, to our knowledge, the proof of their existence is only sketched in the literature.

**Remark 3.** The authors conducted an extensive computer-based search for linear complexes of planes without singular lines in $\text{PG}(8, q)$, $q = 2, 3$, but no examples were found. Thus a proof for the (non)-existence of linear complexes of $h$-subspaces, $h \geq 2$, having no singular $(h - 1)$-subspaces remains an enticing open problem, even in the finite case.

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