Spikes and diffusion waves in a one-dimensional model of chemotaxis

Grzegorz Karch\textsuperscript{1} and Kanako Suzuki\textsuperscript{2}

\textsuperscript{1} Instytut Matematyczny, Uniwersytet Wrocławski, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
\textsuperscript{2} Institute for International Advanced Interdisciplinary Research, Tohoku University, 6-3 Aramaki-aza-Aoba, Aoba-ku, Sendai 980-8578, Japan

E-mail: grzegorz.karch@math.uni.wroc.pl and kasuzu-is@m.tains.tohoku.ac.jp

Received 10 May 2010, in final form 14 October 2010
Published 4 November 2010
Online at stacks.iop.org/Non/23/3119

Recommended by A L Bertozzi

Abstract

We consider the one-dimensional initial value problem for the viscous transport equation with nonlocal velocity
\[ u_t = u_{xx} - (u(K' * u))_x \]
and a given kernel \( K' \in L^1(\mathbb{R}) \). We show the existence of global-in-time nonnegative solutions and we study their large time asymptotics. Depending on \( K' \), we obtain either linear diffusion waves (i.e. the fundamental solution of the heat equation) or nonlinear diffusion waves (the fundamental solution of the viscous Burgers equation) in asymptotic expansions of solutions as \( t \to \infty \). Moreover, for certain aggregation kernels, we show a concentration of solution on an initial time interval, which resemble a phenomenon of the spike creation, typical in chemotaxis models.

Mathematics Subject Classification: 35Q, 35K55, 35B40

1. Introduction

In this work, we study the large time behaviour of solutions to the one-dimensional initial value problem

\[ u_t = u_{xx} - (u(K' * u))_x \quad \text{for } x \in \mathbb{R}, \quad t > 0, \quad (1.1) \]
\[ u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \quad (1.2) \]

where the aggregation kernel \( K' \in L^1(\mathbb{R}) \) is a given function (the symbol ‘*’ denotes the convolution with respect to the variable \( x \)) and the initial datum \( u_0 \in L^1(\mathbb{R}) \) is nonnegative. Such models have been used to describe a collective motion and aggregation phenomena in biology and mechanics of continuous media. In this case, the unknown function \( u = u(x, t) \geq 0 \) is either the population density of a species or the density of particles in a granular
The kernel $K'$ in (1.1) can be understood as the derivative of a certain function $K$, that is, $K'$ stands for $dK/dx$. We use this notation to emphasize that cell interaction described by equation (1.1) takes place by means of a potential $K$ (see [7] for derivation of such equations).

Equation (1.1) and its generalizations, considered either in the whole space or in a bounded domain, have been studied in several recent works. First, note that in the particular case of $K(x) = e^{-|x|}/2$, equation (1.1) corresponds to the one-dimensional parabolic–elliptic system of chemotaxis

$$u_t = u_{xx} - (uv_x)_x, \quad -v_{xx} + v = u, \quad x \in \mathbb{R}, \ t > 0. \quad (1.3)$$

Indeed, since $K(x) = e^{-|x|}/2$ is the fundamental solution of the operator $-\partial_x^2 + \text{Id}$, one can rewrite the second equation in (1.3) as $v = K * u$. Now, it suffices to substitute this formula into the first equation in (1.3) to obtain (1.1). We refer the reader to the works [3, 4, 6, 14, 19, 24] (this list is far from being complete) for mathematical results and for additional references on systems modelling chemotaxis.

Next, one should mention the inviscid aggregation equation $u_t + \nabla \cdot (u \nabla K * u) = 0$, describing the evolution of a cell density, which was derived as a macroscopic equation from the so-called ‘individual cell-based model’ [7, 31]. Here, the reader is referred to [1, 2, 7, 8, 20] for recent results on the existence and the blowup of solutions to the initial value problem for the inviscid aggregation equation.

To handle diffusion phenomena, equations describing aggregation are supplemented with additional terms. One possible approach is to add a nonlinear term modelling a degenerate diffusion as in the porous medium equation, see e.g. [34]. Results and other references on the chemotaxis model with a degenerate diffusion can be found in [5, 32] and on more general aggregation equations in [17, 22, 33].

In several cases, the mechanism of spreading out of organisms resembles a Lévy flight, hence, the anomalous diffusion is better modelled by nonlocal pseudodifferential operators. Recent works [3, 4, 12, 21] contain several mathematical results on a chemotaxis system and on an aggregation equation with either the fractional Laplacian or a more general Lévy operator.

In our recent work [18], we have studied the multidimensional version of problem (1.1)–(1.2) and we have answered questions how singularities of the gradient of the aggregation kernel $K$ influence on the existence and the nonexistence of global-in-time solutions. In this paper, we complete those results in the one-dimensional case by proving the existence of global-in-time solutions for every $K' \in L^1(\mathbb{R})$, and by studying their large time asymptotics. We show that asymptotic profiles, as $t \to \infty$, of solutions to (1.1)–(1.2) with general $K' \in L^1(\mathbb{R})$ are given either by the fundamental solution of the linear heat equation or by self-similar solutions of the viscous Burgers equation. Moreover, under another set of assumptions (which are satisfied, e.g., by the kernel $K(x) = e^{-|x|}$), we prove a certain concentration property solutions to (1.1)–(1.2) which can be observed on an initial time interval.

Notation. In this work, the usual norm of the Lebesgue space $L^p(\mathbb{R})$ is denoted by $\| \cdot \|_p$ for any $p \in [1, \infty]$ and $W^{k,p}(\mathbb{R})$ is the corresponding Sobolev space. $C_c^\infty(\mathbb{R})$ denotes the set of smooth and compactly supported functions. The constants (always independent of $x$ and $t$) will be denoted by the same letter $C$, even if they may vary from line to line. Sometimes, we write, e.g., $C = C(\alpha, \beta, \gamma, ...)$ when we want to emphasize the dependence of $C$ on parameters $\alpha, \beta, \gamma, ...$. 
2. Results and comments

We begin our study of properties of solutions to the initial value problem (1.1)–(1.2) by showing that it has a unique and global-in-time solution for a large class of initial conditions and aggregation kernels. The results on the global-in-time existence and regularity of solutions from the following theorem are more-or-less standard and we state them for the completeness of the exposition. They are also not surprising because it is well known that solutions to the one-dimensional Patlak–Keller–Segel model of chemotaxis do not blow up in finite time (see, e.g., [12, 15, 28]). On the other hand, the fact that all solutions to problem (1.1)–(1.2) are uniformly bounded in time as the $L^p$-valued functions (see proposition 3.2) seem to be new and are a first step towards the understanding of the large time behaviour of solutions.

**Theorem 2.1 (Existence of global-in-time solution).** Assume that

\[ K' \in L^1(\mathbb{R}), \quad (2.1) \]

\[ u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R}) \quad \text{for some } q \in [2, \infty). \quad (2.2) \]

Suppose also that $u_0 \geq 0$. Then the initial value problem (1.1)–(1.2) has a unique, nonnegative, global-in-time solution $u \in C([0, \infty), L^1(\mathbb{R}) \cap L^q(\mathbb{R}))$. This solution has the following regularity property $u \in C^1((0, \infty), L^p(\mathbb{R})) \cap C((0, \infty), W^{1,p}(\mathbb{R}))$ for each $p \in [1, \infty]$. Moreover, the solution conserves the integral (‘mass’)

\[ M \equiv \|u(t)\|_1 = \int_\mathbb{R} u(x, t) \, dx = \int_\mathbb{R} u_0(x) \, dx = \|u_0\|_1 \quad \text{for all } t \geq 0, \quad (2.3) \]

and for each $t_0 > 0$ and all $p \in (1, \infty)$ we have $\sup_{t \geq t_0} \|u(t)\|_p < \infty$.

In our work [18], we have proved that nonnegative solutions to (1.1)–(1.2) exist globally in time, in the case of ‘mildly singular’ kernels satisfying $K' \in L^q(\mathbb{R})$ for some $q \in (1, \infty)$, see [18, theorem 2.5]. Theorem 2.1 improves those results in the one-dimensional case by showing the global-in-time existence of nonnegative solutions to (1.1)–(1.2) for every kernel $K' \in L^1(\mathbb{R})$. This result holds true, in particular, for every ‘strongly singular’ kernel [18] satisfying $K' \in L^1(\mathbb{R}) \setminus L^q(\mathbb{R})$ for any $q > 1$.

**Remark 2.2.** Under the only assumption $u_0 \in L^1(\mathbb{R})$, one can adapt the classical two-norm approach by Weissler [35] to obtain a mild solution of problem (1.1)–(1.2) satisfying $u \in C([0, T], L^1(\mathbb{R})) \cap C((0, T], L^q(\mathbb{R}))$ for certain $q \geq 2$ and $T > 0$. Hence, without loss of generality, we assume in this work that $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ for some $q \in [2, \infty)$.

Next, we state conditions on $K'$ under which solutions to (1.1)–(1.2) decay as $t \to \infty$.

**Theorem 2.3 (L^p-decay of solutions).** Assume that $u = u(x, t)$ is a nonnegative solution to problem (1.1)–(1.2) with $K'$ and $u_0$ satisfying (2.1) and (2.2), respectively. There exists $D > 0$ and $C = C(p, \|K'\|_1, \|u_0\|_1) > 0$ independent of $t$ such that if $\|K'\|_1 \|u_0\|_1 \leq D$, then for each $p \in [1, \infty]$ we have

\[ \|u(t)\|_p \leq CT^{-(1-1/p)/2} \quad \text{for all } t > 0. \quad (2.4) \]

**Remark 2.4.** In the proof of theorem 2.3, we choose $D = 1/C_{\text{GNS}}$, where $C_{\text{GNS}}$ is the optimal constant in the Gagliardo–Nirenberg–Sobolev inequality (4.2).

We do not know if the decay estimate (2.4) holds true for every nonnegative solution which does not necessarily satisfy the condition $\|K'\|_1 \|u_0\|_1 \leq D$. Reasons that such decay estimates may fail for certain kernels $K'$ and for initial conditions with large mass can be found in theorem 2.7 and in the discussion following it. Here, however, we show that estimates (2.4) hold true for each solution of problem (1.1)–(1.2) which tends to zero as $t \to \infty$ without a priori assumed decay rate.
Theorem 2.5. Let the assumptions of theorem 2.1 hold true. Assume, moreover, that there exists \( p_0 \in (1, \infty) \) such that \( \| u(t) \|_{p_0} \to 0 \) as \( t \to \infty \). Then, for each \( p \in [1, \infty] \) there is \( C = C(p, u) > 0 \) independent of \( t \) such that \( \| u(t) \|_p \leq C t^{-(1-1/p)/2} \) for all \( t > 0 \).

Theorem 2.5 is proved at the end of section 5.

The main goal of this work is to derive an asymptotic profile as \( t \to \infty \) of those solutions from theorem 2.1 which satisfy the \( L^p \)-decay estimates (2.4).

Theorem 2.6 (Self-similar asymptotics). Under the assumptions of theorem 2.1, every solution \( u = u(x, t) \) of problem (1.1)–(1.2) satisfying estimate (2.4) has a self-similar asymptotic profile as \( t \to \infty \). More precisely,

(i) if \( \int_{\mathbb{R}} K'(x) \, dx = 0 \), we have

\[
 t^{(1-1/p)/2} \| u(t) - MG(t) \|_p \to 0 \quad \text{as} \quad t \to \infty
\]

for every \( p \in [1, \infty] \), where \( M = \int_{\mathbb{R}} u_0(x) \, dx \) and \( G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp(-|x|^2/4t) \) is the heat kernel.

(ii) On the other hand, if \( A \equiv \int_{\mathbb{R}} K'(x) \, dx \neq 0 \), we have

\[
 t^{(1-1/p)/2} \| u(t) - U_{M,A}(x, t) \|_p \to 0 \quad \text{as} \quad t \to \infty
\]

for every \( p \in [1, \infty] \), where \( U_{M,A}(x, t) = \frac{1}{\sqrt{t}} U_{M,A}(x/\sqrt{t}, 1) \) is the so-called nonlinear diffusion wave and is defined as the unique self-similar solution of the initial value problem for the viscous Burgers equation

\[
 U_t = U_{xx} - A(U^2)_x, \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0, \quad (2.7)
\]

\[
 U(x, 0) = M \delta_0, \quad (2.8)
\]

where \( \delta_0 \) is the Dirac measure.

Let us recall properties of solutions to (2.7)–(2.8) which will be useful in the proof of theorem 2.6. It is well known that the Hopf–Cole transformation allows us to solve this initial value problem to obtain the following explicit solution

\[
 U_{M,A}(x, t) = \frac{At^{-1/2} \exp(-|x|^2/(4t))}{C_{M,A} + \frac{1}{2} \int_0^{\sqrt{t}} \exp(-\xi^2/4) \, d\xi}, \quad (2.9)
\]

where \( C_{M,A} \) is a constant which is determined uniquely as a function of \( M \) and \( A \) by the condition \( \int_{\mathbb{R}} U_{M,A}(\eta, 1) \, d\eta = M \). The important point to note here is that for every \( M \in \mathbb{R} \) the function \( U_{M,A} \) is a unique solution to equation (2.7) in the space \( C((0, \infty); L^1(\mathbb{R})) \) having the properties

\[
 \int_{\mathbb{R}} U_{M,A}(x, t) \, dx = M \quad \text{for all} \quad t > 0
\]

and

\[
 \int_{\mathbb{R}} U_{M,A}(x, t) \varphi(x) \, dx \to M \varphi(0) \quad \text{as} \quad t \to 0
\]

for all \( \varphi \in C^\infty_c(\mathbb{R}) \) [11, section 4]. Such a solution is called a fundamental solution in the linear theory and a source solution in the nonlinear case (cf [9, 11, 23]).

In the proof of theorem 2.6, we study the behaviour, as \( \lambda \to \infty \), of the rescaled family of functions

\[
 u_{\lambda}(x, t) = \lambda u(\lambda x, \lambda^2 t) \quad \text{and} \quad K_{\lambda}'(x) = \lambda K'(\lambda x), \quad \text{for every} \quad \lambda > 0, \quad (2.10)
\]
which satisfy the initial value problems
\[
\begin{align*}
\partial_t u_\lambda &= \partial_x^2 u_\lambda - \partial_x(u_\lambda(K_\lambda' * u_\lambda)), \\
u_\lambda(x,0) &= u_{0,\lambda}(x) = \lambda u_0(\lambda x).
\end{align*}
\] (2.11) (2.12)

Note that if \( u = u(x,t) \) is a nonnegative solution of problem (1.1)–(1.2) obtained in theorem 2.1, then by (2.3) and by a simple change of variables, the following identities
\[
\|u_\lambda(t)\|_1 = \|u_0\|_1 \quad \text{and} \quad \|K_\lambda'\|_1 = \|K'\|_1
\] (2.13)
hold true for all \( t > 0 \) and all \( \lambda > 0 \).

Let us now emphasize that we obtain an asymptotic profile of all solutions to problem (1.1)–(1.2) which tend to zero as \( t \to \infty \). Additional assumptions on the kernel \( K' \) and on initial conditions (as those in theorem 2.3) seem to be necessary to prove the decay of solutions because, in our next theorem, we show a concentration phenomenon for some solutions to (1.1)–(1.2). Here, in order to understand our result, one should keep in mind that if a solution \( u = u(x,t) \) behaves for large \( t \) either as the heat kernel or the nonlinear diffusion wave, one should expect that \( \|u(t)\|_\infty \) decreases and the first moment \( \int_{\mathbb{R}} u(x,t) \, |x| \, dx \) increases in \( t \).

Here, we find a large class of kernels and initial conditions such that corresponding solutions have different behaviour, at least, on a certain initial time interval.

**Theorem 2.7 (Concentration phenomenon).** Assume that the kernel \( K' \) satisfies
\[
\begin{itemize}
\item \( K'(x) = K'(|x|) \, \text{sgn} \, x \) for all \( x \in \mathbb{R} \setminus \{0\} \),
\item \( K'(x) \leq 0 \) for all \( x > 0 \),
\item there exist \( \delta > 0 \) and \( \gamma > 0 \) such that \( \sup_{0 \leq x \leq \delta} K'(x) \leq -\gamma \).
\end{itemize}

Let \( u_0 \in C_c^\infty(\mathbb{R}) \) be nonnegative, nontrivial and even. For every \( P > 0 \), set \( u_{0,P}(x) = P^3 u_0(Px) \) and denote by \( u_P = u_P(x,t) \) the corresponding solution of problem (1.1)–(1.2) with the kernel \( K' \) and \( u_{0,P} \) as the initial datum. If \( P > 0 \) is sufficiently large, then the first moment \( I_P(t) = \int_{\mathbb{R}} u_P(x,t) \, |x| \, dx \) is a strictly decreasing function of \( t \in [0, T] \) for some \( T = T(P) > 0 \).

Some remarks on theorem 2.7 are in order.

**Remark 2.8.** Assumptions on the kernel \( K \), as those stated in theorem 2.7, were imposed in [4, 18] to show the finite-time blowup of solutions to aggregation equations either in the dimension \( n \geq 2 \) or with an anomalous diffusion modelled by the fractional Laplacian. In this work, however, solutions are global-in-time by theorem 2.1 (the bound of the \( L^p \)-norm prevents the solution from collapsing to the delta measure), but they have a tendency to form spiky-like structures as those discussed, e.g., in [14].

**Remark 2.9.** By the uniqueness, the solution \( u_P(x,t) \) considered in theorem 2.7 is an even function of \( x \) for every \( t > 0 \). Hence, it is expected that, for suitable initial conditions, we have \( u_P(0,t) = \max_{x \in \mathbb{R}} u_P(x,t) \). It follows from the proof of theorem 2.7 that the quantity \( u_P(0,t) \) has to increase on an interval \([0, T]\); however, we do not know if it increases monotonically. This phenomenon is in perfect agreement with numerical simulations of spikes in the one-dimensional Keller–Segel model, which are reported in [15].

**Remark 2.10.** Note that \( M_P = \int_{\mathbb{R}} u_{0,P}(x) \, dx = P^3 \int_{\mathbb{R}} u_0(x) \, dx \). Hence, the concentration phenomenon described in theorem 2.7 appears only if mass of a solution is sufficiently large. This result should be compared with theorem 2.3, where solutions are shown to decay for sufficiently small masses. Moreover, in that case, by the inspection of the proof of theorem 2.3, one can show that the \( L^2 \)-norm of solutions decays monotonically for all \( t > 0 \).
Remark 2.11. It is shown in theorem 2.7 that the first moment \( I_P(t) \) is strictly decreasing on a certain initial time interval, however, it cannot converge to zero when \( t \to \infty \). Indeed, such a decay of \( I_P(t) \) cannot be true because of the inequality (see. e.g. [3, remark 2.6])

\[
\left( \int_{\mathbb{R}} u_P(x, t) \, dx \right)^{2-1/p} \leq C \| u_P(t) \|_p \, I_P(t)^{1-1/p},
\]

the conservation of mass (2.3), and the boundedness of the \( L^p \)-norm of solutions to (1.1)–(1.2), shown in proposition 3.2.

Remark 2.12. It is not clear for us what is the large time behaviour of solutions to problem (1.1)–(1.2) which concentrate initially in the sense described in theorem 2.7. Our numerical simulations of solutions to equation (1.1) on a finite interval and with the Neumann boundary conditions show their convergence towards nonconstant stationary solutions (these results will be published in our subsequent paper). The large time behaviour of solutions to problem (1.1)–(1.2) on the whole line \( x \in \mathbb{R} \) seems to be more complicated, because one can easily shown that equation (1.1) has no non-zero stationary solutions which decay at infinity sufficiently fast. Indeed, the equation \( w_{xx} - (wK' + w)_x = 0 \) implies \( w_x - wK' + w = C \), and the constant \( C \) has to vanish. Hence, for \( v(x) = \int_{-\infty}^{x} K'(y) \, dy \), we have \( (ve^{-v})_x = 0 \) and, consequently, \( w(x) = Ce^{v(x)} \). It is easy to check that \( w \in L^1(\mathbb{R}) \) and \( K' \in L^1(\mathbb{R}) \) implies \( v \in L^\infty(\mathbb{R}) \), hence, relation \( w = Ce^v \) can be true for \( C = 0 \) and \( v \equiv 0 \), only.

Remark 2.13. In [26, 27], the authors study the large time behaviour of solutions to the Cauchy problem for the so-called parabolic–parabolic model of chemotaxis

\[
u_t = \Delta u - \nabla \cdot (u \nabla v), \quad v_t = \Delta v - v + u, \quad x \in \mathbb{R}^n, \quad t > 0.
\]

Their results can be summarized as follows. If the solution \((u(x, t), v(x, t))\) of (2.14) satisfies \( \sup_{t \to 0} (\| u(t) \|_p + \| v(t) \|_p) < \infty \) for every \( p \in [1, \infty) \), then \( u(x, t) \) decays as a solution to the linear heat equation and its large time behaviour is described by the heat kernel. Moreover, a higher order term of the asymptotic expansion of \( u \) is calculated. Here, however, because of a technical obstacle, we cannot apply methods from [26, 27] to show a decay of solutions to (1.1)–(1.2).

3. Existence of solutions—proof of theorem 2.1

The proof of the existence of local-in-time solutions to (1.1)–(1.2) is standard, hence, we only sketch that reasoning.

Step 1. Local-in-time solutions. We construct local-in-time mild solutions of (1.1)–(1.2) which are solutions of the following integral equation

\[
u(t) = G(\cdot, t) * u_0 - \int_0^t \partial_x G(\cdot, t-s) * (u(K' * u))(s) \, ds
\]

with the heat kernel \( G(x, t) = (4\pi t)^{-1/2} \exp(-|x|^2/(4t)) \). In our reasoning, we use the following estimates which result immediately from the Young inequality for the convolution:

\[
\| G(\cdot, t) * f \|_{L^p} \leq Ct^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p})} \| f \|_{L^q},
\]

\[
\| \partial_x G(\cdot, t) * f \|_{L^p} \leq Ct^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} \| f \|_{L^q}
\]

for every \( 1 \leq q \leq p \leq \infty \), each \( f \in L^q(\mathbb{R}) \), and \( C = C(p, q) \) independent of \( t, f \). Note that \( C = 1 \) in inequality (3.2) for \( p = q \) because \( \| G(\cdot, t) \|_{L^1} = 1 \) for all \( t > 0 \).
Lemma 3.1 (Local existence). Assume that $K' \in L^1(\mathbb{R})$ and $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ for some $q \in [2, \infty)$. Then there exists $T = T(\|u_0\|_1, \|u_0\|_{L^q}, \|K'\|_1) > 0$ such that the integral equation (3.1) has the unique solution in the space $\mathcal{Y}_T = C([0, T], L^1(\mathbb{R})) \cap C([0, T], L^q(\mathbb{R}))$. Moreover, this solution satisfies $u \in C((0, T], L^p(\mathbb{R}))$ for all $1 \leq p < \infty$.

Proof. Here, it suffices to follow the reasoning from [18, theorem 2.3], where local-in-time existence of solutions to the equation (3.1), written as $u(t) = \mathcal{G}(\cdot, t) * u_0 + B(u, u)(t)$ with the bilinear form

$$B(u, v)(t) = - \int_0^t \partial_s \mathcal{G}(\cdot, t-s) * (u(K' * v))(s) \, ds,$$  \hspace{1cm} (3.4)

are constructed in the space $\mathcal{Y}_T$ supplemented with the norm $\|u\|_{\mathcal{Y}_T} = \sup_{0 \leq t \leq T} \|u\|_1 + \sup_{0 \leq t \leq T} \|u\|_{L^q}$. Note that $\mathcal{G}(\cdot, t) * u_0 \in \mathcal{Y}_T$ by (3.2). To apply ideas from [18, theorem 2.3], one should prove the following estimates of the bilinear form (3.4).

First, for every $u, v \in \mathcal{Y}_T$, using (3.2)–(3.3) and the Young and the Hölder inequalities, we obtain

$$\|B(u, v)(t)\|_1 \leq C \int_0^t (t-s)^{-1/2} \|u(K' * v)(s)\|_1 \, ds \leq C \int_0^t (t-s)^{-1/2} \|u(s)\|_q \|v(s)\|_{L^q} \|K'\|_1 \, ds,$$

where $1/q + 1/q' = 1$. Since $1 < q' \leq q$ for $q \geq 2$, by a standard interpolation, we have $\|v(s)\|_{L^q} \leq C(\|v(s)\|_1 + \|v(s)\|_{L^q})$. Therefore, using the definitions of the norm in $\mathcal{Y}_T$ we obtain

$$\|B(u, v)(t)\|_1 \leq C T^{1/2} \|K'\|_1 \|u\|_{\mathcal{Y}_T} \|v\|_{\mathcal{Y}_T}.$$

In a similar way, we prove the following $L^q$-estimate

$$\|B(u, v)(t)\|_q \leq C \int_0^t (t-s)^{-1-(1/q)/2 - 1/2} \|u(K' * v)(s)\|_1 \, ds \leq C T^{1/2} \|K'\|_1 \|u\|_{\mathcal{Y}_T} \|v\|_{\mathcal{Y}_T}.$$  \hspace{1cm} (3.5)

Summing up these inequalities, we obtain the following estimate of the bilinear form

$$\|B(u, v)\|_{\mathcal{Y}_T} \leq C(T^{1/2} + T^{1/2q}) \|K'\|_1 \|u\|_{\mathcal{Y}_T} \|v\|_{\mathcal{Y}_T}.$$ 

Hence, choosing $T > 0$ such that $4C(T^{1/2} + T^{1/2q}) \|K'\|_1 (\|u_0\|_1 + \|u_0\|_q) < 1$, we obtain the solution in $\mathcal{Y}_T$ by [18, lemma 3.1].

Obviously, by an interpolation inequality, the solution of the integral equation (3.1) in the space $\mathcal{Y}_T$ belongs also to $C((0, T], L^p(\mathbb{R}))$ for every $p \in [1, q]$. To show that $u \in C((0, T], L^p(\mathbb{R}))$ for all $q < p < \infty$, it is sufficient to note that $\mathcal{G}(t) * u_0$ belongs to this space by (3.2). Moreover, for every $u, v \in \mathcal{Y}_T$,

$$\int_0^t \partial_s \mathcal{G}(\cdot, t-s) * (u(K' * v))(s) \, ds \in C((0, T], L^p(\mathbb{R}))$$

for each $p \in (q, \infty]$ by estimates similar to those in (3.5). \hfill \Box

Step 2. Regularity of mild solutions. Results on regularity of mild solutions to (1.1)–(1.2) are standard and well known, see, e.g., the monograph by Pazy [29]. In particular, by a bootstrap argument, one can show that any mild solution $u \in C((0, T], L^p(\mathbb{R}))$ obtained in lemma 3.1 satisfies $u \in C^1((0, T], L^p(\mathbb{R})) \cap C((0, T], W^{2,p}(\mathbb{R}))$. Note that a solution is, in fact, much more regular that can be show by adapting the reasoning form recent works [10,13].
Step 3. Positivity and mass conservation. If the initial condition is nonnegative, the same property is shared by the corresponding solution. Obviously, this fact holds true for the viscous transport equation \( u_t - u_{xx} + (b(x, t)u)_x = 0 \). Now, it suffices to substitute \( b = \mathcal{K}' * u \) to show that solutions to (1.1) are nonnegative if initial conditions are so.

Next, integrating equation (3.1) with respect to \( x \), using the Fubini theorem, and the identities \( \int_R \mathcal{G}(x, t) \, dx = 1 \) and \( \int_R \partial_x \mathcal{G}(x, t) \, dx = 0 \) for all \( t > 0 \), we obtain the conservation of the integral and the \( L^1 \)-norm of nonnegative solutions stated in (2.3).

Step 4. Uniform \( L^p \)-estimate. In order to show that the solution constructed in lemma 3.1 exists for all \( T > 0 \), it is sufficient to prove that its \( L^p \)-norm does not blow up in finite time. Here, we prove that, in fact, the quantity \( \| u(t) \|_p \) is uniformly bounded for large \( t > 0 \) for every \( p \in [1, \infty) \).

**Proposition 3.2.** Assume that \( u \in C([0, T], L^1(R)) \cap C([0, T], L^4(R)) \) is a nonnegative local-in-time solution of problem (1.1)–(1.2) for some \( T > 0 \), with the kernel \( \mathcal{K}' \), and the initial datum \( u_0 \) satisfying (2.1) and (2.2), respectively. For every \( p \in [1, \infty) \) and for every \( t_0 > 0 \), there exists \( C = C(p, t_0, \| u_0 \|_1, \| u_0 \|_q, \| u_0 \|_1) \) independent of \( t \) and of \( T > 0 \) such that \( \| u(t) \|_p \leq C \) for all \( t > 0 \).

**Proof.** It follows from lemma 3.1 that for every \( t_0 \in (0, T] \) we have \( u(t_0) \in L^p(R) \) for all \( p \in [1, \infty) \). Hence, without loss of generality, we can assume that \( u_0 \in L^p(R) \) and \( t_0 = 0 \). Moreover, the regularity of solutions discussed in step 2 allows us to justify our calculations below.

We proceed by induction to show the uniform bound for the \( L^p \)-norm with \( p = 2^n \), \( n \in \mathbb{N} \cap \{0\} \). The estimates for other \( p \in [1, \infty) \) are a simple consequence of the Hölder inequality.

For \( p = 1 \), we have \( \| u(t) \|_1 = \| u_0 \|_1 \) because \( u \) is nonnegative and because the integral of \( u \) is conserved in time, see (2.3).

Now, let \( \varepsilon > 0 \) be small and we fix it at the end of this proof. For \( p = 2^n \) with \( n \geq 1 \), we multiply equation (1.1) by \( u^{p-1} \) and integrate over \( R \) to obtain

\[
\frac{1}{p} \frac{1}{\partial_t} \int_R u^p \, dx = - \frac{4(p-1)}{p^2} \int_R \left| (u^{p/2})_x \right|^2 \, dx + (p-1) \int_R u^{p-1} u_x (\mathcal{K}' * u) \, dx
\]

\[
+ (p-1) \int_R u^{p-1} u_x ((\mathcal{K}' - \mathcal{K}''') * u) \, dx,
\]  

(3.6)  

where the auxiliary kernel \( \mathcal{K}' \in C^\infty_c(R) \) satisfies \( \| \mathcal{K}' - \mathcal{K}''' \|_1 \leq \varepsilon \).

The second term on the right-hand side of (3.6) is estimated by the \( \varepsilon \)-Young inequality \( ab \leq \varepsilon a^2 + C(\varepsilon) b^2 \) and by (2.3) as follows:

\[
(p-1) \left\| \int_R u^{p-1} u_x (\mathcal{K}' * u) \, dx \right\|
\leq (p-1) \varepsilon \int_R u^{p-2} |u_x|^2 \, dx + C(\varepsilon) \| (\mathcal{K}' * u)(t) \|_\infty^2 \int_R u^p \, dx
\leq \frac{4(p-1)}{p^2} \varepsilon \int_R \left| (u^{p/2})_x \right|^2 \, dx + C(\varepsilon) \| \mathcal{K}''' \|_\infty^2 \| u_0 \|_1^2 \int_R u^p \, dx.
\]  

(3.7)
Concerning the third term on the right-hand side of (3.6), it follows from the identity $u^{p-1}u_s = (2/p)(u/p^2), u/p^2$, from the Hölder inequality, and from the Young inequality that

$$(p - 1) \left| \int_{\mathbb{R}} u^{p-1}u_s (K' - \tilde{K}') * u \, dx \right| \leq \frac{2(p - 1)}{p} \|u\|_p u/p^2 \|u/p^2\|_{2} \|K' - \tilde{K}'\|_1 \|u(t)\|_{p+2}$$

where $\|K' - \tilde{K}'\|_1 \leq \varepsilon$. Now, note that the Gagliardo–Nirenberg–Sobolev inequality leads to

$$\|u/p^2\|_{2} \leq C \|u\|_{p+2}^{2(p+1)/(3(p+2))} \|u\|_{p+2}^{2(p+1)/(3(p+2))}.$$ 

Moreover, applying the induction hypothesis for $p/2 = 2^{n-1}$, we have $\|u/p^2(t)\|_{1} \leq C$ with $C > 0$ independent of $t$ and $T > 0$. Hence, coming back to inequality (3.8) we obtain

$$(p - 1) \left| \int_{\mathbb{R}} u^{p-1}u_s ((K' - \tilde{K}') * u) \, dx \right| \leq \varepsilon C \|u/p^2\|_{2} \|u/p^2\|_{2} \|u\|_{p+2}^{1+2/p} \|u\|_{p+2}^{1+2/p},$$

(3.8)

with $C = C(p, \|K\|_{\infty}, \|u_0\|_{1})$ independent of $t$ and $T$.

Since $\kappa(p) = 4(p + 1)/3p \leq 2$ for all $p \geq 2$, using the elementary inequality $s^{\kappa(p)} \leq C(s^2 + 1)$ for all $s \geq 0$ and fixed $C > 0$ independent of $s$, we deduce from (3.9) that

$$(p - 1) \left| \int_{\mathbb{R}} u^{p-1}u_s ((K' - \tilde{K}') * u) \, dx \right| \leq C \varepsilon \|u/p^2\|_{2} \|u/p^2\|_{2} + C \varepsilon C.$$ 

(3.10)

Now, by estimates (3.7) and (3.10), we obtain from (3.6) the following inequality

$$\frac{d}{dt} \int_{\mathbb{R}} u^p \, dx \leq -(1 - \varepsilon C) \|u/p^2\|_{2} \|u\|_p^p + C \|u\|_p^p + \varepsilon C,$$

(3.11)

where $C = C(p, \|K\|_{1}, \|\tilde{K}\|_{\infty}, \|u_0\|_{1})$ denotes various constants independent of $\varepsilon, u, t$ and $T$.

Now, we require $\varepsilon < C^{-1}$ in the first term on the right-hand side of (3.11). Using the Nash inequality

$$\|u\|_2 \leq C \|u\|_{1/3}^{1/3} \|u\|_{1/2}^{1/2},$$

(3.12)

which is valid for all $v \in L^1(\mathbb{R})$ such that $v \in L^2(\mathbb{R})$, and applying the inductive hypothesis (note that $p/2 = 2^{n-1}$), we have

$$\|u(t)\|_{p/2}^p = \|u/p^2(t)\|_{2} \leq C \|u/p^2(t)\|_{2} \|u/p^2(t)\|_{2} \|u/p^2(t)\|_{2} \leq C \|u/p^2(t)\|_{2} \|u/p^2(t)\|_{2} \|u/p^2(t)\|_{2},$$

(3.13)

where $C$ is independent of $u, t$ and $T$. Hence, applying estimate (3.13) in (3.11) we obtain the following differential inequality for $\|u(t)\|_{p}$:

$$\frac{d}{dt} \|u(t)\|_{p}^p \leq -C(1 - \varepsilon C) (\|u(t)\|_{p}^p)^3 + C \|u(t)\|_{p}^p + \varepsilon C,$$

where $C = C(p, K', \|u_0\|_{1}) > 0$ is independent of $\varepsilon, u, t$ and $T$.

We leave for the reader the proof that any nonnegative solution of the differential inequality $\dot{f} \leq -(1 - \varepsilon C) f^3 + Cf + \varepsilon C$ is bounded, provided $\varepsilon > 0$ is sufficiently small. Hence, by the recurrence argument, $\|u(t)\|_{p}$ is bounded for any $p = 2^n, n \in \mathbb{N} \cup \{0\}$ and this completes the proof of proposition 3.2.
Remark 3.3. Note that we do not control the growth in $p$ of the constants $C$ in proposition 3.2, hence we are not able to pass to the limit $p \to \infty$ to obtain the global-in-time bound of the $L^\infty$-norm of the solution. However, in theorem 2.3, we show a decay estimate of $\|u(t)\|_\infty$ under additional assumptions on the kernel $K'$.

Proof of theorem 2.1. Steps 1–4, described above, contain all details of the proof. □

4. Optimal $L^p$-decay of solutions

We are in a position to prove the decay estimates from theorem 2.3 and we do it in two steps. First, we obtain the optimal decay estimate of $L^2$-norm using Gagliardo–Nirenberg–Sobolev inequalities. Next, estimates of other $L^p$-norms are shown by applying the integral formulation (3.1) of the initial value problem (1.1)–(1.2).

Proof of theorem 2.3. Let $p = 2$. Multiplying equation (1.1) by $u$ and integrating the resulting equation over $\mathbb{R}$ we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 \, dx = - \int_{\mathbb{R}} |u_x|^2 \, dx + \int_{\mathbb{R}} uu_x(K' * u) \, dx. \quad (4.1)$$

By the Hölder inequality, the Young inequality, the following Gagliardo–Nirenberg–Sobolev inequality

$$\|u(t)\|_4 \leq C_{GNS} \|u_x(t)\|_2^{1/2} \|u(t)\|_1^{1/2}, \quad (4.2)$$

and identity (2.3), the second term on the right-hand side of (4.1) is estimated as follows:

$$\int_{\mathbb{R}} uu_x(K' * u) \, dx \leq \|uu_x(t)\|_4 \|(K' * u)(t)\|_4 \leq \|u(t)\|_2^2 \|u_x(t)\|_2 \|K'\|_1 \leq C_{GNS} \|u_x(t)\|_2^2 \|K'\|_1 \|u_0\|_1. \quad (4.3)$$

Coming back to (4.1) we see that

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq -(1 - C_{GNS} \|K'\|_1 \|u_0\|_1) \|u_x(t)\|_2^2,$$

hence, if $\|K'\|_1 \|u_0\|_1 < 1/C_{GNS}$, we obtain

$$\frac{d}{dt} \|u(t)\|_2^2 + C \|u_x(t)\|_2^2 \leq 0, \quad (4.4)$$

where $C = C(\|K'\|_1, \|u_0\|_1) > 0$. Now, by the Nash inequality (3.12), since the $L^4$-norm of the solution is constant in time by (2.3), we obtain the differential inequality

$$\frac{d}{dt} \|u(t)\|_2^2 + C \|u_0\|_1^{-2} \|u(t)\|_2^3 \leq 0, \quad (4.5)$$

which implies $\|u(t)\|_2 \leq Ct^{-1/4}$ for all $t > 0$ and $C > 0$ independent of $t$.

Now, we are going to use systematically the above $L^2$-decay estimate to show the decay of other $L^p$-norms. First, we consider $p \in [1, \infty)$ and we compute the $L^p$-norm of both sides of the integral equation (3.1) to obtain

$$\|u(t)\|_p \leq \|\mathcal{G}(\cdot, t) * u_0\|_p + \int_0^t \|\partial_s \mathcal{G}(\cdot, t - s) * (u(K' * u))(s)\|_p \, ds \, ds \leq C t^{-1/2} \|\mathcal{G}(\cdot, t) * u_0\|_1 + C \int_0^t \|u_0\|_1^{1-p} \|u_k\|_1^{p-1} \|K'\|_1 \|u(s)\|_2^2 \, ds$$

$$\leq C t^{-1/2} \|\mathcal{G}(\cdot, t) * u_0\|_1 + C - \frac{1}{2} \|u_0\|_1^{1-p} \|u_k\|_1^{p-1} \|K'\|_1 \|u(s)\|_2^2 \, ds \quad (4.6)$$
after applying inequalities (3.2)–(3.3). Hence, by the $L^2$-decay estimate, we have
\[
\|u(t)\|_p \leq C t^{-\frac{1}{2}(1-\frac{1}{p})} \|u_0\|_1 + C(\|K'\|_1, \|u_0\|_1) \int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})} s^{-\frac{1}{2}} \, ds \\
= C t^{-\frac{1}{2}(1-\frac{1}{p})}, \quad \text{for all } t > 0,
\]
because \(\int_0^t (t-s)^{-\frac{1}{2}(1-\frac{1}{p})} s^{-\frac{1}{2}} \, ds = t^{-\frac{1}{2}(1-\frac{1}{p})} \int_0^1 (1-\rho)^{-\frac{1}{2}(1-\frac{1}{p})} \rho^{-\frac{1}{2}} \, d\rho\).

To deal with the case \(p = \infty\), we use the already proved decay estimate for any \(p > 2\). Proceeding in a way similar to that in (4.6), we obtain
\[
\|u(t)\|_{\infty} \leq \|\mathcal{G}(\cdot, t) * u_0\|_{\infty} + \int_0^t \|\partial_t \mathcal{G}(\cdot, t-s) * (u(K' * u))(s)\|_{\infty} \, ds \\
\leq C t^{-\frac{1}{2}} \|u_0\|_1 + \int_0^t C (t-s)^{-\frac{1}{2}} \|u(K' * u)(s)\|_{p/2} \, ds \\
\leq C t^{-\frac{1}{2}} \|u_0\|_1 + \int_0^t C (t-s)^{-\frac{1}{2}} \|K'\|_1 \|u(s)\|_p^2 \, ds \\
\leq C t^{-\frac{1}{2}} \|u_0\|_1 + C(p, \|K'\|_1, \|u_0\|_1) \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \, ds \\
= C(p, \|K'\|_1, \|u_0\|_1) t^{-\frac{1}{2}}
\]
for all \(t > 0\). This completes the proof of theorem 2.3.

In the proof of theorem 2.5, we need the following auxiliary result.

**Lemma 4.1.** Let \(v_t \in L^2(\mathbb{R})\). For \(\tilde{K}' \in C_c(\mathbb{R})\), we define \(\tilde{K}(\cdot) = \int_{-\infty}^\cdot \tilde{K}'(y) \, dy\). Denote \(A = \int_{\mathbb{R}} \tilde{K}'(y) \, dy = \lim_{y \to +\infty} \tilde{K}(y)\). Then
\[
\|\tilde{K}' * v - Av\|_{\infty} \leq (\|\tilde{K}\|_{L^2(-\infty,0)} + \|\tilde{K} - A\|_{L^2(0, +\infty)}) \|v_t\|_{2}.
\]

**Proof.** First, note that \(\tilde{K} \in L^2((-\infty, 0])\) and \(\tilde{K} - A \in L^2([0, +\infty))\), because \(\tilde{K}' \in C_c(\mathbb{R})\). Hence, the proof is the immediate consequence of the integration by parts and of the Schwartz inequality in view of the following identities
\[
\tilde{K}' * v(x) - Av(x) = \int_{\mathbb{R}} \tilde{K}(y) v_t(x-y) \, dy - A \int_{-\infty}^{+\infty} v_t(x-y) \, dy \\
= \int_{-\infty}^0 \tilde{K}(y) v_t(x-y) \, dy + \int_0^{+\infty} [\tilde{K}'(y) - A] v_t(x-y) \, dy.
\]

**Proof of theorem 2.5.** Without loss of generality, we can assume that \(\|u(t)\|_2 \to 0\) as \(t \to \infty\), because this is the immediate consequence of the Hölder inequality, proposition 3.2, and the assumption on the decay of the \(L^p\)-norm.

To show the optimal decay of the \(L^2\)-norm, we use the following equality (cf (3.6))
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 = -\|u_t(t)\|_2^2 + \int_{\mathbb{R}} u(t)u_t(t)(K' - \tilde{K}') * u(t) \, dx \\
+ \int_{\mathbb{R}} u(t)u_t(t)\tilde{K}' * u(t) \, dx,
\]
where \(\tilde{K} \in C_c(\mathbb{R})\) satisfies \(\|K' - \tilde{K}'\|_1 \leq \varepsilon\) with \(\varepsilon > 0\) to be chosen later on.
It follows from the Hölder inequality, the Young inequality and from the Gagliardo–Nirenberg–Sobolev inequality (4.2) (see the calculations which lead to (4.3)) that
\[
\left| \int_{\mathbb{R}} u(t)u_x(t)(K^\prime - \tilde{K}^\prime) * u(t) \, dx \right| \leq C \|u_x(t)\|_2^2 \|K^\prime - \tilde{K}^\prime\|_1 \|u(t)\|_1
\]
\[
\leq C \varepsilon \|u_0\|_1 \|u_x(t)\|_2^2. \tag{4.8}
\]

Next, note that
\[
\int_{\mathbb{R}} u^2 u_x \, dx = 0 \text{ for all } u \in W^{1,2}(\mathbb{R}),
\]

hence, for \(A = \int_{\mathbb{R}} \tilde{K}^\prime(y) \, dy\), the last term on the right-hand side of (4.7) is estimated as follows
\[
\left| \int_{\mathbb{R}} u(t)u_x(t)\tilde{K}^\prime * u(t) \, dx \right| = \left| \int_{\mathbb{R}} u(t)u_x(t)(\tilde{K}^\prime * u(t) - Au(t)) \, dx \right|
\]
\[
\leq \|\tilde{K}^\prime * u(t) - Au(t)\|_\infty \|u(t)\|_2 \|u_x(t)\|_2. \tag{4.9}
\]

Now, we apply both estimates (4.8) and (4.9) in equality (4.7). Using, moreover, lemma 4.1 to deal with the last term on the right-hand side of (4.9), we deduce the following inequality:
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq (-1 + C \varepsilon \|u_0\|_1 + C \|u(t)\|_2)(\|\tilde{K}\|_{L^1(-\infty,0)} + \|\tilde{K} - A\|_{L^1(0,\infty)})\|u_x(t)\|_2^2. \tag{4.10}
\]

Hence, for sufficiently small \(\varepsilon > 0\) and for sufficiently large \(T_0 > 0\), since \(\|u(t)\|_2 \to 0\) if \(t \to \infty\), we obtain the estimate
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 \leq -C \|u_x(t)\|_2^2 \tag{4.11}
\]
for all \(t \geq T_0\) and \(C > 0\) independent of \(t\) and \(u\). Now, it remains to repeat the reasoning from the proof of theorem 2.3 for \(p = 2\) (see inequalities (4.4)–(4.5)) to obtain the required decay of the \(L^2\)-norm. To show the decay estimate for other \(L^p\)-norms, one should copy the corresponding arguments from the proof of theorem 2.3. \(\square\)

5. Self-similar large time behaviour

Our goal in this section is to prove theorem 2.6. Here, we always assume that \(u = u(x, t)\) is the nonnegative global-in-time solution of the initial value problem (1.1)–(1.2) with \(K\) and \(u_0\) satisfying (2.1) and (2.2), respectively. Moreover, we assume that this solution satisfies the following decay estimates
\[
\|u(t)\|_p \leq C t^{-\frac{2}{p} \left(1 - \frac{1}{p}\right)} \tag{5.1}
\]
for each \(p \in [1, \infty]\), all \(t > 0\), and \(C\) independent of \(t\).

The proof that the large time behaviour of the solution \(u = u(x, t)\) is described either by the fundamental solution of the heat equation or by the self-similar solution of the viscous Burgers equation is based on the so-called scaling method which is often used in the study of asymptotic properties of solutions to nonlinear evolution equation (see, e.g. the review paper [34] for some applications of this method to the porous media equation). Here, for every \(\lambda > 0\), we denote by \(u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)\) the solution of the initial value problem (2.11)–(2.12). In the following, we systematically use identities (2.13) as well as the decay estimate (5.1).

Now, we prove a series of technical lemmas which usually should be obtained to apply the scaling method.
Lemma 5.1. For each $p \in [1, \infty]$ there exists $C = C(\|K\|_1, \|u_0\|_1) > 0$, independent of $t$ and of $\lambda$, such that
\[
\|u_\lambda(t)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}
\]
for all $t > 0$ and all $\lambda > 0$.

Proof. By the change of variables and estimate (5.1) we obtain
\[
\|u_\lambda(t)\|_p = \lambda^{\frac{1}{2}} \|u(\cdot, \lambda^2 t)\|_p \leq C\lambda^{\frac{1}{2}} (\lambda^2 t)^{-\frac{1}{2}(1-\frac{1}{p})} = Ct^{-\frac{1}{2}(1-\frac{1}{p})}. \quad \square
\]

Lemma 5.2. For each $p \in [1, \infty)$ there exists $C = C(p, \|K\|_1, \|u_0\|_1) > 0$, independent of $t$ and of $\lambda$, such that $\|\partial_x u_\lambda(t)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}$ for all $t > 0$ and all $\lambda > 0$.

Remark 5.3. Note that applying the approach by Giga and Sawada [13] and Dong and Du [10] one can improve lemma 5.2 by showing that $\|\partial^k x u_\lambda(t)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}$ for each $k \in \mathbb{N}$. Here, we do not reproduce that reasoning, because the particular case $k \in [0, 1]$ is sufficient in our study of the large time behaviour of solutions to problem (1.1)–(1.2).

Proof of lemma 5.2. Here, we use the following counterpart of the integral equation (3.1)
\[
\partial_t u_\lambda(t + 1) = \partial_t G(t) * u_\lambda(1) + \int_0^t \partial_t G(t - s) * ((u_\lambda)_x(K_\lambda^* u_\lambda) + u_\lambda(K_\lambda^* (u_\lambda)_x))(s + 1) \, ds
\]
for all $t > 0$. Hence, computing the $L^p$-norm and using (3.2), (3.3), (2.13) we obtain
\[
\|\partial_t u_\lambda(t + 1)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}\|u_\lambda(1)\|_1 + \int_0^t C(t - s)^{-\frac{1}{2}} (\|u_\lambda(K_\lambda^* u_\lambda)(s + 1)\|_p + \|u_\lambda(K_\lambda^* (u_\lambda)_x)(s + 1)\|_p) \, ds
\]
\[
\leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}\|u_\lambda(1)\|_1 + C\|K\|_1 \int_0^t (t - s)^{-\frac{1}{2}} \|\partial_x u_\lambda(s + 1)\|_p \|u_\lambda(s + 1)\|_\infty \, ds.
\]
Next, we use inequality (5.2) with $p = \infty$:
\[
\|u_\lambda(s + 1)\|_\infty \leq C(\|K\|_1, \|u_0\|_1)(s + 1)^{-\frac{1}{2}} \leq C(\|K\|_1, \|u_0\|_1)
\]
for all $s > 0$, consequently,
\[
\|\partial_t u_\lambda(t + 1)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}\|u_\lambda(1)\|_1 + C\int_0^t (t - s)^{-\frac{1}{2}} \|\partial_x u_\lambda(s + 1)\|_p \, ds.
\]
Applying the singular Gronwall lemma (see, e.g., [16, chapter 7]), we conclude that
\[
\|\partial_t u_\lambda(t + 1)\|_p \leq C(p, t, \|K\|_1, \|u_0\|_1),
\]
for all $t > 0$, where the right-hand side of this inequality is independent of $\lambda$. In particular, for $t = 1$, there exists $C = C(\|K\|_1, \|u_0\|_1)$ independent of $\lambda$ such that $\|\partial_t u_\lambda(2)\|_p \leq C$ for all $\lambda > 0$. Next, using the definition of $u_\lambda$, we obtain $\|\partial_t u_\lambda(2)\|_p = \lambda^{\frac{1}{2}} \|\partial_x u(\cdot, 2\lambda^2)\|_p$. Hence, after substituting $\lambda = \sqrt{t/2}$, we arrive at $\|\partial_t u(\cdot, t)\|_p \leq Ct^{-\frac{1}{2}(1-\frac{1}{p})}$ for all $t > 0$. \quad \square

The proof of our next lemma relies on a form of Aubin–Simon’s compactness result that we recall below.
Theorem 5.4 ([30, theorem 5]). Let $X$, $B$ and $Y$ be Banach spaces satisfying $X \subset B \subset Y$ with compact embedding $X \subset B$. Assume, for $1 \leq p \leq \infty$ and $T > 0$, that

- $F$ is bounded in $L^p(0, T; X)$,
- $\{\partial_t f : f \in F\}$ is bounded in $L^p(0, T; Y)$.

Then $F$ is relatively compact in $L^p(0, T; B)$ (and in $C(0, T; B)$ if $p = \infty$).

Lemma 5.5 (Compactness in $L^1_{\text{loc}}(\mathbb{R})$). For every $0 < t_1 < t_2 < \infty$ and every $R > 0$, the set $\{u_\lambda\}_{\lambda > 0} \subseteq C([-R, R])$ is relatively compact.

Proof. We apply theorem 5.4 with $p = \infty$, $F = \{u_\lambda\}_{\lambda > 0}$, and

$$X = W^{1,1}([-R, R]), \quad B = L^{1}([-R, R]), \quad Y = W^{-1,1}([-R, R])$$

where $R > 0$ is fixed and arbitrary, and $Y$ is the dual space of $W^{1,1}_0([-R, R])$. Obviously, the embedding $X \subset B$ is compact by the Rellich–Kondrashov theorem.

By lemmas 5.1 and 5.2 with $p = 1$, the sets $\{u_\lambda\}_{\lambda > 0} \subseteq L^\infty([-t_1, t_2], L^1([-R, R]))$ and $\{\partial_t u_\lambda\}_{\lambda > 0} \subseteq L^\infty([-t_1, t_2], L^1([-R, T]))$ are bounded.

To check the second condition of Aubin–Simon’s compactness criterion, it suffices to show that there is a positive constant $C$ which independent of $\lambda > 0$ such that $\sup_{t \in [t_1, t_2]} \|\partial_t u_\lambda\|_Y \leq C$. Let us show this estimate by a duality argument. For every $\phi \in C_c^\infty((−R, R))$ and $t \in [t_1, t_2]$ we have

$$\left| \int_\mathbb{R} \partial_t u_\lambda(x, t) \phi(x) \, dx \right| = -\int_\mathbb{R} \partial_t u_\lambda \partial_x \phi \, dx + \int_\mathbb{R} (\partial_t \phi) u_\lambda(K_\lambda' * u_\lambda) \, dx \leq \|\phi\|_\infty (\|u_\lambda(t)\|_1 + \|K_\lambda'\|_1 \|u_\lambda(t)\|_2^2) \leq \|\phi\|_\infty C(t_1, t_2, \|K'\|_1, \|u_0\|_1)$$

by virtue of lemma 5.1. Hence, lemma 5.5 is proved. $\square$

Lemma 5.6 (Compactness in $L^1(\mathbb{R})$). For every $0 < t_1 < t_2 < \infty$, the set $\{u_\lambda\}_{\lambda > 0} \subseteq C([-t_1, t_2], L^1(\mathbb{R}))$ is relatively compact.

Proof. Let $\psi \in C_c^\infty(\mathbb{R})$ be nonnegative and satisfy $\psi(x) = 0$ for $|x| < 1$ and $\psi(x) = 1$ for $|x| > 2$. Put $\psi_R(x) = \psi(x/R)$ for every $R > 0$. Since $u$ is nonnegative, in view of lemma 5.5, using a standard diagonal argument, it suffices to show that

$$\sup_{t \in [t_1, t_2]} \|u_\lambda(t)\|_{\psi_R} \to 0 \quad \text{as} \quad R \to \infty, \quad \text{uniformly in} \quad \lambda \geq 1. \quad (5.3)$$

Multiplying both sides of equation (2.11) by $\psi_R$ and integrating over $\mathbb{R}$ and from 0 to $t$, we obtain

$$\int_\mathbb{R} u_\lambda(x, t) \psi_R(x) \, dx - \int_\mathbb{R} u_\lambda(x, 0) \psi_R(x) \, dx = \int_0^t \int_\mathbb{R} \partial_t \psi_R(x) u_\lambda(x, s) \, dx \, ds + \int_0^t \int_\mathbb{R} \psi(x) (u_\lambda(x, s)(K_\lambda' * u_\lambda)(x, s)) \, dx \, ds.$$
Since $\partial_x \psi (x) = \psi' (x/R)/R^2$ and $\partial_x \psi_R (x) = \psi' (x/R)/R$, we have
\[
\int_{\mathbb{R}} u_{\lambda} (x, t) \psi (x) \, dx \leq \int_{|x| > R} u_{\lambda, 0} (x) \psi (x) \, dx + \frac{\| \psi' \|_{C^1}}{R^2} \int_{0}^{t} \| u_{\lambda} (s) \|_1 \, ds
\]
\[
+ \frac{\| \psi' \|}{R} \left( K' \| u_{\lambda} (s) \|_1 \right) s \int_{0}^{t} \| u_{\lambda} (s) \|_1 \, ds
\]
\[
\leq \int_{\mathbb{R}} u_{\lambda, 0} (x) \psi (x) \, dx + \frac{f}{R} \| \psi_{xx} \|_\infty \| u_{\lambda} \|_1 + \frac{\| \psi' \|_{C^1}}{R} \int_{0}^{t} \| u_{\lambda} (s) \|_1 \, ds
\]
(5.4)
because $\| u_{\lambda} (s) \|_1 = \| u_0 \|_1$.

Now, note that, by the change of variables, we have
\[
\int_{|x| > R} u_{\lambda, 0} (x) \, dx = \int_{|x| > R} u_0 (x) \, dx \leq \int_{|x| > R} u_0 (x) \, dx
\]
for all $\lambda > 1$. Moreover, it follows from lemma 5.1 that $\| u_{\lambda} (x) \|_\infty \leq C s^{-1/2}$ with $C$ independent of $\lambda$. Hence, we see for all $t \in [t_1, t_2]$ that
\[
\int_{\mathbb{R}} u_{\lambda} (x, t) \psi (x) \, dx \leq \int_{|x| > R} u_0 (x) \, dx + C_2 \left( \frac{t_2}{R^2} + \frac{t_1^{1/2}}{R} \right)
\]
where $C_2 = C (\| \psi_{xx} \|, \| \psi_x \|, \| K' \|_1, \| u_0 \|_1)$ is independent of $\lambda > 1$. This proves our claim (5.3) because $u_0 \in L^1 (\mathbb{R})$. \[\square\]

Lemma 5.7 (Initial condition). For every test function $\phi \in C^\infty_c (\mathbb{R})$, there exists $C = C (\phi, \| K' \|_1, \| u_0 \|_1)$ independent of $\lambda$ such that
\[
\left| \int_{\mathbb{R}} u_{\lambda} (x, t) \phi (x) \, dx - \int_{\mathbb{R}} u_{0, \lambda} (x) \phi (x) \, dx \right| \leq C (1 + t^{1/2}).
\]
(5.5)

Proof. Following the calculations which lead to estimates (5.4) with $\psi_R$ replaced by $\phi \in C^\infty_c (\mathbb{R})$ we obtain
\[
\left| \int_{\mathbb{R}} u_{\lambda} (x, t) \phi (x) \, dx - \int_{\mathbb{R}} u_{0, \lambda} (x) \phi (x) \, dx \right|
\]
\[
\leq \int_{0}^{t} \| \phi_{xx} \|_\infty \| u_{\lambda} (s) \|_1 \, ds + \int_{0}^{t} \| \phi' \|_{C^1} \| u_{\lambda} (s) \|_1 \| u_{\lambda} (s) \|_\infty \, ds
\]
\[
\leq \| \phi_{xx} \|_\infty \| u_0 \|_1 t + C \| \phi_x \|_{C^1} \| K' \|_1 \| u_0 \|_1 \int_{0}^{t} s^{-1/2} \, ds
\]
\[
\leq C (1 + t^{1/2}),
\]
where $C > 0$ is independent of $\lambda$. \[\square\]

Now, we are in a position to prove our main result on the large time behaviour of solutions to problem (1.1)–(1.2).

Proof of theorem 2.6. By lemma 5.6, for every $0 < t_1 < t_2 < \infty$, the family $\{ u_{\lambda} \}_{\lambda > 0}$ is relatively compact in $C ([t_1, t_2], L^1 (\mathbb{R}))$ for any $0 < t_1 < t_2 < \infty$. Consequently, there exists a subsequence of $\{ u_{\lambda} \}_{\lambda > 0}$ (not relabelled) and a function $\bar{u} \in C ((0, \infty), L^1 (\mathbb{R}))$ such that
\[
u_{\lambda} \to \bar{u} \quad \text{in} \quad C ([t_1, t_2], L^1 (\mathbb{R})) \quad \text{as} \quad \lambda \to \infty.
\]
(5.6)
Passing to a subsequence, we can assume that
\[ u_{\lambda}(x, t) \to \bar{u}(x, t) \quad \text{as} \quad \lambda \to \infty \] (5.7)
almost everywhere in \((x, t) \in \mathbb{R} \times (0, \infty)\).

Now, multiplying equation (2.11) by a test function \(\phi \in C_c^\infty(\mathbb{R} \times (0, \infty))\) and integrating the resulting equation over \(\mathbb{R} \times (0, \infty)\), we obtain the identity
\[ -\int_0^\infty \int_\mathbb{R} u_{\lambda} \phi_t \, dx \, dt = \int_\mathbb{R} \int_0^\infty u_{\lambda} \phi_{xx} \, dx \, dt + \int_\mathbb{R} \int_0^\infty u_{\lambda}(K_{\lambda}' \ast u_{\lambda}) \phi_t \, dx \, dt. \] (5.8)

Recall that \(K_{\lambda}'(x) = \lambda K'(\lambda x)\) and \(A = \int_\mathbb{R} K'(y) \, dy\) for all \(\lambda > 0\), hence, by the well-known property of an approximation of the identity, we have \(K_{\lambda}' \ast \bar{u}(t) \to A\bar{u}(t)\) in \(L^1(\mathbb{R})\) as \(\lambda \to \infty\). Consequently, by the Young inequality, (2.13), and (5.6), we have \(\|K_{\lambda}' \ast u_{\lambda}(t) - A\bar{u}(t)\|_1 \leq \|K'\|_1 \|u_{\lambda}(t) - \bar{u}(t)\|_1 + \|K_{\lambda}' \ast \bar{u}(t) - A\bar{u}(t)\|_1 \to 0\) as \(\lambda \to \infty\) for every \(t > 0\). Therefore, passing to the limit \(\lambda \to \infty\) in equality (5.8) and using the properties of the sequence \(\{u_{\lambda}\}_{\lambda>0}\) stated in (5.6) and (5.7), we obtain that \(\bar{u}(x, t)\) is a weak solution of the equation
\[ \bar{u}_t = \bar{u}_{xx} - A(\bar{u}^2)_x, \] (5.9)
with \(A = \int_\mathbb{R} K'(x) \, dx\). Now, note that, by the change of variables and the dominated convergence theorem, we obtain \(\int_\mathbb{R} u_{\lambda} \phi(x) \, dx = \int_\mathbb{R} u_0(x) \phi(x/\lambda) \, dx \to M\phi(0)\) as \(\lambda \to \infty\). Hence, it follows from lemma 5.7 that \(\bar{u}(x, 0) = M\delta_0\) in the sense of bounded measures. Thus, \(\bar{u}\) is a weak solution of the initial value problem
\[ \bar{u}_t = \bar{u}_{xx} - A(\bar{u}^2)_x, \] (5.10)
\[ \bar{u}(x, 0) = M\delta_0. \]

Since problem (5.9)–(5.10) has a unique solution (see, e.g., [11, section 4]), the whole family \(\{u_{\lambda}\}_{\lambda>0}\) converge to \(\bar{u}\) in \(C(0, \infty), L^1(\mathbb{R})\).

Obviously, if \(A = 0\), this limit function is the multiple of Gauss–Weierstrass kernel
\[ \bar{u}(x, t) = MG(x, t) = M \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{|x|^2}{4t}\right). \] (5.11)
For \(A \neq 0\), we obtain the self-similar solution \(\bar{u} = U_{M, A}\) of the viscous Burgers equation, given by the explicit formula (2.9).

Hence, by (5.6), we have
\[ \lim_{\lambda \to \infty} \|u_{\lambda}(1) - \bar{u}(1)\| = 0 \]
and, after setting \(\lambda = \sqrt{t}\) and using the self-similar form of \(\bar{u}(x, t) = t^{-1/2}\bar{u}(xt^{-1/2}, 1)\), we obtain
\[ \lim_{t \to \infty} \|u(t) - \bar{u}(t)\| = 0. \] (5.12)

The convergence of \(u(\cdot, t)\) towards the self-similar profile in the \(L^p\)-norms for \(p \in (1, \infty)\) is the immediate consequence of the H"older inequality, the decay estimate (5.1) with \(p = \infty\), and (5.12). Indeed, we have
\[ \|u(t) - \bar{u}(t)\|_p \leq \left(\|u(t)\|_\infty + \|\bar{u}(t)\|_\infty\right)^{1-p/2} \|u(t) - \bar{u}(t)\|_1^{1/p} \to 0 \quad \text{as} \quad t \to \infty. \] (5.13)

To complete the proof of theorem 2.6, it remains to show the convergence in the \(L^\infty\)-norm. Here, however, it suffices note the decay estimate \(\|u_{\lambda}(t)\|_2 \leq Ct^{-3/4}\), provided by lemma 5.2 with \(\lambda = 1\), and the identity \(\|\bar{u}_{\lambda}(t)\|_2 = t^{-3/4}\|\bar{u}_{\lambda}(1)\|_2\) resulting from the explicit
formulae (5.11) and (2.9). Hence, by the Gagliardo–Nirenberg–Sobolev inequality and by (5.13) with \( p = 2 \), we obtain
\[
\|u(t) - \bar{u}(t)\|_\infty \leq C(\|u_+(t)\|_2 + \|\bar{u}_+(t)\|_2)^{1/2} \|u(t) - \bar{u}(t)\|_2^{1/2} = o(r^{-1/2})
\]
as \( t \to \infty \).

6. Concentration phenomenon

**Proof of theorem 2.7.** First, note that equation (1.1) is invariant under the transformation \( x \mapsto -x \) because, by the assumptions, the kernel \( K' \) is an odd function. Hence, by the uniqueness of solutions to problem (1.1)–(1.2), the solution \( u_p \) corresponding to the even and nonnegative initial datum \( u_0, p \) satisfies \( u_p(x, t) = u_p(-x, t) \geq 0 \) for all \( x \in \mathbb{R} \) and \( t > 0 \).

We study the evolution of the first moment \( I_P(t) = \int_{\mathbb{R}} u_p(x, t) |x| \, dx \) following ideas from the recent papers [4, 18]. First, note that integrating by parts we have
\[
\int_{\mathbb{R}} |\partial_x^2 u_p(x, t)| \, dx = 2 \int_0^\infty x \partial_x^2 u_p(x, t) \, dx = -2 \int_0^\infty \partial_t u_p(x, t) \, dx = 2u_p(0, t),
\]
because \( \partial_t u_p(0, t) = 0 \) for all \( t > 0 \) in the case of the even function \( u_p(\cdot, t) \). Next, by the assumptions, we have \( K'(x) = \frac{1}{|x|} K'(|x|) \) where \( \frac{1}{|x|} = \text{sign } x \). Hence, using equation (1.1), we obtain
\[
\frac{d}{dt} I_P(t) = \int_{\mathbb{R}} \partial_t u_p(x, t) |x| \, dx = \int_{\mathbb{R}} (\partial_x^2 u_p(x, t) - \partial_x(u_p(x, t)K' \ast u_p(x, t))) |x| \, dx
\]
\[
= 2u_p(0, t) + \int_{\mathbb{R}} \int_{\mathbb{R}} u_p(x, t)u_p(y, t) \frac{x - y}{|x|} \frac{x - y}{|y|} K'(|x - y|) \, dx \, dy
\]
\[
= 2u_p(0, t) + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} u_p(x, t)u_p(y, t) \left( \frac{x}{|x|} - \frac{y}{|y|} \right) \frac{x - y}{|x - y|}
\]
\[
\times K'(|x - y|) \, dx \, dy
\]
(6.1)
by the symmetrization of the double integral on the right-hand side. Now, we use the elementary identity
\[
\left( \frac{x}{|x|} - \frac{y}{|y|} \right) \frac{x - y}{|x - y|} = \left( \frac{|x| + |y|}{|x - y|} \right) \left( 1 - \frac{x}{|x|} \cdot \frac{y}{|y|} \right)
\]
and the inequalities
\[
1 \leq \frac{|x| + |y|}{|x - y|} \quad \text{and} \quad 1 - \frac{x}{|x|} \cdot \frac{y}{|y|} \geq 0
\]
which are valid for all \( x, y \in \mathbb{R} \setminus \{0\} \). Moreover, using the properties of \( K' \) stated in theorem 2.7, we deduce from (6.1) that
\[
\frac{d}{dt} I_P(t) \leq 2u_p(0, t) + \frac{1}{2} \int_{\mathbb{R}} \int_{|y| < \delta/2} u_p(x, t)u_p(y, t) \left( 1 - \frac{x}{|x|} \cdot \frac{y}{|y|} \right) K'(|x - y|) \, dx \, dy
\]
\[
\leq 2u_p(0, t) - \frac{\nu}{2} \int_{\mathbb{R}} \int_{|y| < \delta/2} u_p(x, t)u_p(y, t) \left( 1 - \frac{x}{|x|} \cdot \frac{y}{|y|} \right) \, dx \, dy
\]
\[
\leq 2u_p(0, t) - \frac{\nu}{2} \int_{\mathbb{R}} \int_{|y| < \delta/2} u_p(x, t)u_p(y, t) \left( 1 - \frac{x}{|x|} \cdot \frac{y}{|y|} \right) \, dx \, dy
\]
\[
+ \gamma \int_{\mathbb{R}} \int_{|y| > \delta/2} u_p(x, t)u_p(y, t) \left( 1 - \frac{x}{|x|} \cdot \frac{y}{|y|} \right) \, dx \, dy.
\]
(6.2)
Note now that
\[ \int_{\mathbb{R}} u_P(x, t) \frac{x}{|x|} \, dx = 0 \quad \text{and} \quad \int_{|x| \geq \delta/2} u_P(x, t) \frac{x}{|x|} \, dx = 0 \]
because \( u_P(x, t) = u_P(-x, t) \). Hence, denoting \( M_P = \int_{\mathbb{R}} u_P(x, t) \, dx \) it follows from (6.2) that
\[
\frac{d}{dt} I_P(t) \leq 2u_P(0, t) - \frac{\gamma}{2} M_P^2 + \gamma M_P \int_{|y| > \delta/2} u_P(y, t) \, dy
\leq 2u_P(0, t) - \frac{\gamma}{2} M_P^2 + \frac{2\gamma}{\delta} M_P I_P(t). \tag{6.3}
\]
Now, we use the dependence of the initial datum on \( P \). Putting
\[
M = \int_{\mathbb{R}} u_0(x) \, dx, \quad I(0) = \int_{\mathbb{R}} u_0(x)|x| \, dx,
\]
and changing the variables we have
\[
2u_0(0) - \frac{\gamma}{2} M_P^2 + \frac{2\gamma}{\delta} M_P I_P(0) = 2u_0(0)P^3 - \frac{\gamma}{2} M^2 P^4 + \frac{2\gamma}{\delta} M I(0) P^3 < 0
\]
if \( P > 0 \) is sufficiently large. Hence, for such \( P \), the right-hand side of inequality (6.3) is negative for \( t = 0 \). Consequently by the continuity of the functions \( u_P \) and \( I_P \), the right-hand side of inequality (6.3) is negative for every \( t \in [0, T] \) with some \( T = T(P) > 0 \). This completes the proof of theorem 2.7. \( \square \)

Acknowledgments

This work was partially supported by the MNiSzW grant No N N201 418839, the Japan–Poland Research Cooperative Program (2008–2009), MEXT the Grant-in-Aid for Young Scientists (B) 20740887 and the Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007–2013 funded by European Regional Development Fund (PhD Programme: Mathematical Methods in Natural Sciences).

References

[1] Bertozzi A L, Carrillo J A and Laurent T 2009 Blowup in multidimensional aggregation equations with mildly singular interaction kernels Nonlinearity 22 683–710
[2] Bertozzi A L and Laurent T 2007 Finite-time blow-up of solutions of an aggregation equation in \( \mathbb{R}^d \) Commun. Math. Phys. 274 717–35
[3] Biler P and Karch G 2010 Blowup of solutions to generalized Keller–Segel model J. Evol. Eqns 10 247–62
[4] Biler P, Karch G and Laurençot Ph 2009 Blowup of solutions to a diffusive aggregation model Nonlinearity 22 1559–68
[5] Blanchet A, Carrillo J A and Laurençot Ph 2009 Critical mass for a Patlak–Keller–Segel model with degenerate diffusion in higher dimensions Calc. Var. Partial Diff. Eqns 35 133–68
[6] Blanchet A, Dolbeault J and Perthame B 2006 Two dimensional Keller–Segel model: optimal critical mass and qualitative properties of the solutions Electron. J. Diff. Eqns 44 1–33
[7] Bodnar M and Velázquez J J L 2005 Derivation of macroscopic equations for individual cell-based models: a formal approach Math. Methods Appl. Sci. 28 1757–79
[8] Bodnar M and Velázquez J J L 2006 An integro-differential equation arising as a limit of individual cell-based models J. Diff. Eqns 222 341–80
[9] Chern I L and Liu T P 1987 Convergence to diffusion waves of solutions for viscous conservation laws Commun. Math. Phys. 110 1103–33
[10] Dong H and Du D 2007 On the smoothness of solutions of the Navier–Stokes equations J. Math. Fluid Mech. 9 139–52
One-dimensional model of chemotaxis

[11] Escobedo M, Vázquez J L and Zuazua E 1993 Asymptotic behaviour and source-type solutions for a diffusion–convection equation Arch. Ration. Mech. Anal. 124 43–65
[12] Escudero C 2006 The fractional Keller–Segel model Nonlinearity 19 2009–18
[13] Giga Y and Sawada O 2003 On regularizing-decay rate estimates for solutions to the Navier–Stokes initial value problem Nonlinear Analysis and Applications vol 1, 2 (Dordrecht: Kluwer Academic Publishing) pp 549–62 (V Lakshmikantham on his 80th Birthday)
[14] Hillen T 2007 A classification of spikes and plateaus SIAM Rev. 49 35–51
[15] Hillen T and Potapov A 2004 The one-dimensional chemotaxis model: global existence and asymptotic profile Math. Methods Appl. Sci. 27 1783–801
[16] Henry D 1981 Geometric Theory of Semilinear Parabolic Equations (Lecture Note in Mathematics vol 840) (Berlin: Springer)
[17] Ikeda T and Nagai T 1987 Stability of localized stationary solutions Japan. J. Appl. Math. 4 73–97
[18] Karch G and Suzuki K 2009 Blow-up versus global existence of solutions to aggregation equation with diffusion 1–14, arXiv:1004.4021
[19] Kozono H and Sugiyama Y 2008 Local existence and finite time blow-up of solutions in the 2-D Keller–Segel system J. Evol. Eqns 8 353–78
[20] Laurent T 2007 Local and global existence for an aggregation equation Commun. Partial Diff. Eqns 32 1941–64
[21] Li D and Rodrigo J 2009 Finite-time singularities of an aggregation equation in $\mathbb{R}^n$ with fractional dissipation Commun. Math. Phys. 287 687–703
[22] Li D and Zhang X 2010 On a nonlocal aggregation model with nonlinear diffusion Discrete Contin. Dyn. Syst. 27 301–23
[23] Liu T P and Pierre M 1984 Source solutions and asymptotic behavior in conservation laws J. Diff. Eqns 51 419–41
[24] Murray J D 2003 Mathematical Biology. II, Spatial Models and Biomedical Applications (Interdisciplinary Applied Mathematics vol 18) 3rd edn (New York: Springer)
[25] Nagai T 2000 Behavior of solutions to a parabolic–elliptic system modelling chemotaxis J. Korean Math. Soc. 37 721–32
[26] Nagai T, Syukinii R and Umesako M 2003 Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in $\mathbb{R}^N$ Funkcial. Ekvac. 46 383–407
[27] Nagai T and Yamada T 2007 Large time behavior of bounded solutions to a parabolic system of chemotaxis in the whole space J. Math. Anal. Appl. 336 704–26
[28] Osaki K and Yagi A 2001 Finite dimensional attractor for one-dimensional Keller–Segel equations Funkcial. Ekvac. 44 441–69
[29] Pazy A 1983 Semigroups of Linear Operators and Applications to Partial Differential Equations (Berlin: Springer)
[30] Simon J 1987 Compact sets in the space $L^p(0, T; B)$ Ann. Math. Pura Appl. 146 65–96
[31] Stevens A 2000 A stochastic cellular automaton modeling gliding and aggregation of myxobacteria SIAM J. Appl. Math. 61 172–28
[32] Sugiyama Y 2009 On $\epsilon$-regularity theorem and asymptotic behaviors of solutions for Keller–Segel systems SIAM J. Math. Anal. 41 1664–92
[33] Topaz C M, Bertozzi A L and Lewis M A 2006 A nonlocal continuum model for biological aggregation Bull. Math. Biol. 68 1601–23
[34] Vázquez J L 2003 Asymptotic behaviour for the porous medium equation posed in the whole space J. Evol. Eqns 3 67–118 (Dedicated to Philippe Bénilan)
[35] Weissler F B 1980 The Navier–Stokes initial value problem in $L^p$ Arch. Ration. Mech. Anal. 74 219–30