Internal Space for the Noncommutative Geometry Standard Model and Strings

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Abstract

In this paper I discuss connections between the noncommutative geometry approach to the standard model on one side, and the internal space coming from strings on the other. The standard model in noncommutative geometry is described via the spectral action. I argue that an internal noncommutative manifold compactified at the renormalization scale, could give rise to the almost commutative geometry required by the spectral action. I then speculate how this could arise from the noncommutative geometry given by the vertex operators of a string theory.
1 Introduction

In this paper I present some speculative conjectures linking the programme started in [1], where the ordinary standard model of particle physics is seen as particular noncommutative geometry [2, 3], with some features of string theory [4, 5]. The standard model in noncommutative geometry is described by the spectral action [6, 7] of a noncommutative geometry. All topological information of a Hausdorff space is contained in the commutative algebra of continuous complex-valued functions defined over it, and a noncommutative space is the generalization to noncommutative $C^*$-algebras. Although the most familiar examples are deformation of ordinary manifolds (such as the Moyal plane or the noncommutative torus), noncommutative geometry has a much wider scope, and there are examples of noncommutative geometries (such as the space of Penrose tilings [2]) which are not deformations of a ordinary commutative manifolds.

While the topology is encoded in a $C^*$-algebras, much of the geometry is encoded in the (generalized) Dirac operator, a self adjoint operator which acts on the Hilbert space on which the algebra is represented as bounded operators. With the Dirac operator is possible to define a differential calculus representing also forms as operators, and it is possible to build the action, which is the measure of the fluctuations of the geometry. The initial great success of this programme has been the fact that, by considering as geometry an almost commutative geometry composed of ordinary spacetime and an internal zero-dimensional space, it is possible to reproduce the action of the standard model, with the interpretation of the Higgs field as a an intermediate “vector” boson, on a a par with the photon, W and Z particles. The “vectoriality” of Higgs is in the internal space. The gravitational part of the action also emerges, and all parameters of the model (masses, Yukawa couplings etc.) are encoded in the zero-dimensional part of the Dirac operator.

This approach is not without problems, and recently there have been attempts [8, 9] to solve some of them by considering the internal space to have some characteristics of a six dimensional space. The problem is that on the one side the internal space should be zero dimensional, because it is represented by a finite algebra, but on the other its chirality and charge conjugations should behave like those of a space of six (modulo 8) dimensions. Connes [9] has solved this by noting that the concept of deimenion is not uniquely determined in noncommutative geometry, and considering the
KO-dimensional, different from the metric dimension. The two coincide for ordinary (commutative) manifolds and Dirac operators, but are not necessarily the same for noncommutative spaces.

In this paper I conjecture that the metric dimension of the internal (compactified and noncommutative) space is indeed six, but that all but a finite number of the degrees of freedom are “frozen” at low energy, smaller than the renormalization scale. I then go on to further conjecture that this internal space could be coming from compactified strings. The conjectures here are based on circumstantial evidence, and no hard calculations are performed. In particular I do not exhibit the noncommutative geometry which could give rise to the standard model, but only indicate some of its features. There are other issues relevant to the spectral action and the standard model that I do not tackle. Nevertheless I feel that the considerations presented here, speculative and unmoulded as they are, can be of interest and sparkle some interest. A much more ambitious programme would be the investigation of the possible landscapes of string theory giving realistic models of particle physics.

The paper is organized as follows. In the first two section I describe the noncommutative geometry and the standard action respectively. I then describe in section 4 how a continuous noncommutative geometry can be frozen to appear finite dimensional in the context of the standard action. In section 5 I speculate how such a mechanism could be present in a string theory. I conclude with some final remarks.

2 The standard model as an almost commutative geometry

In this section I give a brief sketch of how the standard model arises in noncommutative geometry, more details can be found in the reviews [10, 11]. In the work of Connes and collaborators [2, 12] the standard model is a particular “almost commutative” geometry. The functions on this space are the tensor product of $C(M)$, the algebra of continuous functions on a four dimensional manifold $M$, times a finite dimensional algebra which represents an internal, noncommutative, space. To reproduce the standard model the internal space is taken to be

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) ,$$

(2.1)
where the factors are complex numbers, quaternions and complex valued three by three matrices. Thus the continuous functions which describe this noncommutative geometry (and its topology) form the algebra

$$\mathcal{A} = C(M) \otimes \mathcal{A}_F.$$  \hspace{1cm} (2.2)

The gauge group is given by the unimodular (unitary elements of the algebra of unit determinant) part of the algebra, and in this case it correctly reproduces the standard model gauge group, i.e. functions valued in $U(1) \times SU(2) \times SU(3)$.

The algebra is represented as operator on the Hilbert space of spaces, which again is a tensor products of a continuous part composed of spinors on the four dimensional space, times and internal space.

$$\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_F$$ \hspace{1cm} (2.3)

where $C$ is for continuum and $F$ for finite, then it is possible to split both the spacetime and the internal space spinors into left and right parts. The internal Hilbert space can be split into the different chiralities and charge as

$$\mathcal{H}_F = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c$$ \hspace{1cm} (2.4)

with $L, R$ referring to chirality, and the the superscript $c$ referring to the charge conjugate (antiparticle) sector. It comprises all fermionic degrees of freedom of the standard model, i.e. four degrees of freedom for electrons and positron of both helicities, two for neutrinos, 24 for quarks (including colour), which makes 30 degrees of freedom per generation, for a total of 90 degrees of freedom. Since spinors already contain the degrees of freedom of left and right particles and antiparticles there is an overcounting. Note also that at this stage I am not considering right-handed neutrinos. I will come back to these aspects. The representation of $\mathcal{A}_F$ on $\mathcal{H}_F$ acts in a different way in the particle and antiparticle sectors, and this is crucial for the bimodule structure introduced below. I do not give the explicit finite dimensional representation, and refer to the literature for details.

The geometry of (ordinary or noncommutative) spaces is encoded in the Dirac operator. For example, the (metric) dimension of the space is given by the exponent in the growth of the number of eigenvalues of it, i.e., if call the list of $d_k$ these eigenvalues in ascending order (and repeated according to the degeneracy), then if there exist two real numbers $c$ and $n$ such that

$$d_k \leq C k^{-n}$$ \hspace{1cm} (2.5)
then \( n \) is called the metric dimension of the space. If the operator is finite
the metric dimension is of course zero.

For the case at hand also the Dirac operator again is split in the continuous four dimensional operator and internal, finite matrix, representing a zero dimensional internal space:

\[
D = \partial \otimes \mathbb{I} + \gamma_5 \otimes D_F .
\]  

(2.6)

The presence of \( \gamma_5 \) is conventional, its insertion ensures that the Laplacian has a similar splitting into an internal and an external part. The four dimensional part it is the usual \( \partial = \gamma^\mu \partial_\mu \) while \( D_F \), in the \( L, R \) basis is

\[
D_F = \begin{pmatrix}
0 & M & 0 & 0 \\
M^\dagger & 0 & 0 & 0 \\
0 & 0 & 0 & M^* \\
0 & 0 & M^T & 0
\end{pmatrix} .
\]  

(2.7)

The \( 24 \times 21 \) mass matrix \( M \) contain information on the masses of all fermions, and the Cabibbo-Kobayashi-Maskawa mixing angles.

The algebra \( \mathcal{A} \), the Hilbert space \( \mathcal{H} \) and the Dirac operator \( D \) form the spectral triple of the standard model. There are however two other main ingredients to compose a real spectral triple. The first is the chirality operator\(^1\)

\[
\gamma = \gamma_5 \otimes \gamma_F .
\]  

(2.8)

The second is appropriate generalization of the usual charge conjugation operator \( \mathcal{C} \) to comprise an operator acting on the finite part.

\[
J = \mathcal{C} \otimes J_F .
\]  

(2.9)

The construction of a real spectral triple is very sensitive to the number of dimensions modulo 8, and the operators \( D, J \) and \( \gamma \) have to satisfy some consistency relations:

\[
\begin{align*}
J^2 &= \varepsilon \mathbb{I} , \\
JD &= \varepsilon' DJ , \\
J\gamma &= (-1)^{n/2} \gamma J , \quad \text{if } n \text{ is even},
\end{align*}
\]  

(2.10)

\(^1\)This operator exists only for even spectral triple. Otherwise the triple is called odd.
and the mod 8 periodic functions $\varepsilon$ and $\varepsilon'$ given by

$$
\varepsilon = (1, 1, -1, -1, -1, 1, 1),
$$
$$
\varepsilon' = (1, -1, 1, 1, 1, -1, 1, 1), \quad n = 0, 1, \ldots, 7. \quad (2.11)
$$

Furthermore, the map

$$
b \mapsto b^o = Jb^*J^{-1}, \quad (2.12)
$$
determines a representation of the opposite algebra $A^o$ on $\mathcal{H}$ which commutes with $A$ and its commutators with $D$

$$
[a, Jb^*J^{-1}] = 0, \quad (2.13)
$$
$$
[[D, a], Jb^*J^{-1}] = 0, \quad \forall \ a, b \in A, \quad (2.14)
$$

thus giving $\mathcal{H}$ a bimodule structure, whereby $b^o$ effectively “acts on the right” on the fermionic states.

In this case

$$
\gamma_F = \begin{pmatrix}
\sigma_3 & 0 \\
0 & \sigma_3
\end{pmatrix},
$$
$$
J_F = \begin{pmatrix}
0 & \mathbb{I} \\
\mathbb{I} & 0
\end{pmatrix}. \quad (2.15)
$$

One can verify that relations (2.10) hold with the correct zero dimensions for a space described by a finite dimensional algebra.

The geometry of the standard model is therefore that of an almost commutative geometry, in the sense that it is the product of the ordinary spacetime, times a matrix algebra. So in the end the structure of spacetime remains unchanged.

### 3 The spectral action principle and the standard model

I need to write down the action of the model, to be able to build the standard model. With the Dirac operator one can construct differential one-forms, which are self-adjoint operators of the form

$$
A = \sum_i a_i[D, b_i], \quad a_i, b_i \in A \quad (3.1)
$$
and a covariant Dirac operator

\[ D_A = D + A + JAJ^\dagger \]  (3.2)

Note that the presence of the conjugation by \( J \) is crucial for \( D_A \) to act in a symmetric way on particles and antiparticles, since the representation of \( \mathcal{A} \) threatens them in a different way.

I have now sketched all ingredients for the construction of the action. This is the sum of bosonic and a fermionic part [13, 6, 7]:

\[ S = \text{tr} \chi \left( \frac{D_A}{\Lambda} \right) + \langle \Psi | D_A | \Psi \rangle \]  (3.3)

with \( \Psi \in \mathcal{H} \). Here \( \text{tr} \) is the usual trace in the Hilbert space \( \mathcal{H} \), \( \Lambda \) is a cutoff parameter and \( \chi \) is a suitable function which removes all eigenvalues of \( D_A \) larger than \( \Lambda \), a smoothened version of a the characteristic function of the interval \([0,1]\). The spectral action has to be read in the Wilson renormalization scheme sense. All physical quantities are bare and they are subject to renormalization thus getting an energy dependent meaning. The spectral action is invariant under the gauge action of the inner automorphisms given by

\[ A \rightarrow UAU^\dagger + U[D, U^\dagger] \]  (3.4)

with \( U \) an unitary element of \( \mathcal{A} \).

The computation of the action (3.3) is conceptually simple, but quite involved [7, 14]. One computes the square of the Dirac operator with Lichnerowicz’ formula [15] and the trace with a suitable heat kernel expansion [16], so as to get an expansion in powers of the parameter \( \Lambda \). The key points are general identities valid for differential elliptic operators of degree \( d \) on an \( m \)-dimensional manifold \( M \) with metric \( g \):

\[ \text{tr} \left( O^{-s} \right) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{-s} \, \text{tr} \, e^{-tO} \quad \text{Re}(s) \geq 0 \quad , \]  (3.5)

and (heat kernel expansion)

\[ \text{tr} \, e^{-tO} = \sum_{n \geq 0} t^{\frac{n-m}{d}} \int_M d\mu_g(x) a_n(x, O) \quad . \]  (3.6)

The \( a_n \)'s are called Seeley-de Witt coefficients and their expression (as well as the method to calculate them) can be found in [16]. The trace (3.3) will
then depend on the $a_n$’s and on the “momenta” of $\chi$:

$$\text{tr}\,\chi(O) = \sum_{n \geq 0} f_n a_n(O) ,$$

(3.7)

with

$$f_0 = \int_0^\infty dt \, t \, \chi(t) ,$$
$$f_2 = \int_0^\infty dt \, \chi(t) ,$$
$$f_{2n+4} = (-1)^n \partial^n_t \chi(t)|_{t=0} .$$

(3.8)

If $\chi$ is not too different from the characteristic function of the interval $[0,1]$ but just a smoothened version of it, it is safe to assume $f_2 = f_4 = 2 f_0 = 1$. In this case the spectral action just counts the eigenvalues of $D^2_A$ which are less than the cutoff $\Lambda^2$.

In (3.3) appears the covariant Dirac operator, square root the Laplacian, implicitly it contains information about the metric and other relevant geometric aspects of the manifold. It should therefore come as no surprise that they contain Riemann curvature, Christoffel symbols etc. What is more surprising is that one obtains an action which contains all of the ingredients of the standard model plus gravity. The calculation is quite long and involved, for details refer to [7, 14, 11]. The final result is:

$$S_B = \int_M d\mu_g \left( I_1 \Lambda^2 + I_2 + I_3 \frac{1}{\Lambda^2} + O\left(\frac{1}{\Lambda^4}\right) \right) .$$

(3.9)

The first coefficient is constant:

$$I_1 = \frac{45}{8\pi^2} ,$$

(3.10)

and gives as leading term a cosmological constant. This coefficient is subject to the renormalization group flow, and to agree with the observed absence (or at least smallness) of the cosmological constant, a high degree of fine tuning is necessary. The second coefficient is

$$I_2 = \frac{1}{16\pi^2} \left( -15 R - 8 K_1 |\Phi|^2 \right) ,$$

(3.11)

with $R$ the Ricci scalar, $K_1 = \text{tr} (3 M_u^\dagger M_u + 3 M_d^\dagger M_d + M_e^\dagger M_e)$, the $M$’s are mass matrices for up and down quark-types, and leptons. This term contains
the Einstein-Hilbert Lagrangian as well as the quadratic term of the Higgs potential. The remaining terms of the potential, as well as the kinetic terms for the gauge fields are in the term of order $\Lambda^{-2}$:

$$I_3 = \frac{1}{16\pi^2} \left( 240 F^{\mu\nu} F_{\mu\nu} + 12 G^{\mu\nu} G_{\mu\nu} + 4 K_1 |D_{\alpha} \Phi|^2 
- \frac{2}{3} K_1 R |\Phi|^2 + K_2 |\Phi|^4 - \frac{9}{4} C^2 \right) + \text{surface terms} \quad (3.12)$$

with $F$ and $G$ the gauge field strength for the abelian and nonabelian part respectively, $K_2 = \text{tr} (3(M_1^+ M_u)^2 + 3(M_4^+ M_d)^2 + (M_1^+ M_e)^2)$, and $C$ the Weyl tensor defined as

$$C_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} - g_{\nu[\rho} R_{\mu]\sigma} + \frac{1}{6} (G_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) R \quad (3.13)$$

Hence the full action reproduces all terms of the standard model and gravity, plus some unavoidable terms quadratic in the curvature. As I said the action must be read in a Wilsonian sense, and to make predictions it is necessary to study the renormalization of the coupling constants. The model sketched here has relations among coupling constants which are $g_3 = g_2 = (5/3)^{1/2} g_1$, a large renormalization is needed and one gets [11, 17], for $\Lambda \simeq 10^{13-17}$ GeV the mass $m_H = 175.1^{+5.8}_{-9.8}$ GeV. The value for $m_H$ is (at present) still realistic, while the relation among couplings is not.

The spectral action principle gives an interpretation of the standard model which shows a path to a deeper comprehension of it. In this sense the mass relations, or other possible phenomenological predictions are not its nicest feature. What is most important is the fact that Connes (and collaborators) succeeded in putting a generic Yang-Mills theory in a larger geometrical context, so that the model is at the same time more rigid, and more flexible.

It is more rigid in the sense that not all gauge groups, and all representations, can be a candidate. In fact the noncommutative geometry model building is severely restricted. It is important that the Standard Model can be seen in this framework, while the (non supersymmetric) $SU(5)$ unified theory, ruled out by experiments, cannot. It is more flexible in that it sees the standard model as coming from an internal (noncommutative) space. This space does not have to be an ordinary manifold (as in Kaluza-Klein), and so this opens the way for several generalizations. If this approach is correct phenomenological and experimental predictions will certainly follow.

There are however several problems with this approach. A partial list goes as follows:
1. **The model is Euclidean**

The whole construction of spectral triples is mathematically consistent only for compact Euclidean manifold. The non compact case can be dealt without excessive problems (see for example [15]) but putting a Lorentzian signature is less natural. The wave functions of the fields are not anymore a Hilbert space, just to mention one problem. There have been however constructions of Lorentzian spectral triples [10, 19, 20].

2. **Fermion doubling**

I have already noted this. The problem [21] is that the four dimensional part of the spectral triple requires the presence of fields which are ordinary Dirac fermions, but the constructions put in the finite internal Hilbert space all degrees of freedom, so that the four degrees of freedom corresponding which comprise a Dirac spinor field (say electron and positron of right and left chiralities) are multiplied tensorially by the same degrees of freedom appearing in the internal Hilbert space.

3. **Neutrino masses**

At the time of construction of the original model neutrinos where believed to be massless, so the model was built this way. There is now experimental evidence to the contrary, so the model has to accomodate these masses, either as Dirac or Majorana masses [22, 23]. Majorana masses create problems with the *first order condition* (2.14) which imposes that forms have to commute with functions. The presence of a Majorana mass introduces in the matrix of the finite part of the Dirac operator which connect the left with the right chirality, and this spoils the commutation relation $[JaJ^{-1}, [D, b]]$. Dirac masses are instead spoil Poincaré duality, the noncommutative geometry version of the duality between forms of degree $n$ and of degree $d - n$ [12]. While a violation of the first order condition creates unsurmountable problems with gauge invariance, it is not clear which problems a violation of Poincaré duality would create. Phenomenologist in general favour strongly the see-saw mechanism to give mass to the neutrinos, and this requires necessarily Majorana masses.

\footnote{In principle it is possible that the first order condition is satisfied up to an infinitesimal, this happens in [21].}
4. **The model is *ad hoc***

The construction is very complicated, especially the representation of the Hilbert space. Everything is put there by hand, just to fit the standard model. While it is true that the aim of the game (at this stage) is to understand the standard model, not predict it, nevertheless it would be desirable to have some of the features come out in a more natural way.

5. **Classical vs. Quantum**

The original Connes-Lott is classical, and its mass relations do not survive quantization [25], although some relations can still be inferred [26]. The spectral action on the other side is quantum by nature, as it has to read in the Wilson renormalization scheme, but suffers from the fact that there are relations among coupling constants, which are the same as in SU(5), and are not phenomenologically favoured. In general one would like some quantization process more on a par with the rest of the mathematical construction. In this respect more work on the roots of the spectral action is necessary.

6. **Fine Tuning**

The spectral action requires several instances of fine tuning, to start with a huge cosmological constant. An analysis of this problem in the light of the recent indications of a non-vanishing cosmological constant would be of great interest.

7. **The model is almost commutative**

In other words this is an effective model, and one would like to see more of the full power of noncommutative geometry at work, rather than just some discrete Kaluza-Klein. For instance, where is the internal (matrix) space coming from?

Some of these drawbacks have been examined in two recent articles. Let us examine them in turn. In [8] Barret deals with points [1] and [2]. Let us recall that the problem with fermion doubling pointed out in [21] is not so much the presence of more states than one would expect. This can be fixed projecting out the unwanted states, and identifying the remaining ones. The problem is in that the projection spoils the action. If the Hilbert space is the product of the continuum part by the discrete part indicated in [2,3].
The troublesome terms in $\mathcal{H}$ are the ones coming from the crossed products of states with different chiralities. These do not properly transform under CP. The idea in [21] was to project out those states. This is possible, and the fermionic action can be correctly written to contain only the proper degrees of freedom. But the problem is that if one uses a reduced Hilbert space, then the trace which gives the bosonic action will be over a reduced space, and the final result is not the correction for the standard model, both for the Connes-Lott model or the spectral action. We tried to fix this by giving a physical meaning to the extra “mirror” fermions [27]. This is not a satisfactory solution which somehow nullify the very interesting connections between the quantum number assignments of the standard model and cancellation of anomalies. Moreover there is a fine tuning problem. In [28] a slightly different form of the bosonic part of the Connes-Lott action, reinstating the missing pieces of the action. The problem here is that in the commutative limit this action vanishes, and moreover no solution for the spectral action is offered.

Barrett notices that if one uses the chirality $\gamma$ and $J$ operators coming from the Minkowski, and not Euclidean, assignments for the four-dimensional part, and the operators for a six-dimensional space for the internal part, then the doubling disappears from the Fermionic part of the action if one considers as Hilbert space the subspace of $\mathcal{H}$ which is eigenspace of $\gamma$ and $J$, with eigenvalue $i$ and $1$ respectively. Barret does not consider the bosonic part.

This is similar to the projection in [21], in fact the proper projection operator is $P = \frac{1 + \gamma}{2}$ on the particle and antiparticle sectors. The problem is that with this assignment there is no way to have the bosonic action come out correct. In fact the calculation is similar the one done in [27], without the mirror fermions and with a slight modification of the Dirac operator.

At the same time as the paper by Barrett, a paper by Connes [9] appeared. Like Barrett, Connes notices the mismatch between the dimension of the internal space (zero), and the values of $\varepsilon$ and $\varepsilon'$ required, which would be the ones indicated in (2.11) by dimension 6 (mod. 8). This is interpreted as a mismatch between the metric dimension, and the $KO$-dimension. The former is the one encoded in the growth of the eigenvalues of the Dirac operator. The latter is the one encoded in (2.11). For ordinary, commutative, manifolds these definitions coincide, but this may not be so for noncommutative geometries. It is known that (even ordinary) spaces can have different definitions of dimension which do not necessarily coincide. This is common
for fractals. For noncommutative geometries this happens commonly for
spaces such as the Podles sphere \[29\] which have metric dimension zero, but
KO-dimension 2 \[24\].

There also in Connes’ paper the assignments for \(J_F\) and \(\gamma_f\) are changed
to make them like those a six-dimensional space. It turns out that this is
equivalent (in the Euclidean case) to just reversing the sign of \(J\) and \(\gamma\) in
the antiparticle sector. Since he is in the Euclidean framework the total
KO-dimension is now \(10 = 2 = 26 \pmod 8\), and the connection with strings
immediately comes to mind. There is another subtle, but not minor, change
in this version of the spectral action, namely the presence of \(J\) in the fermionic
action, so that now the action reads

\[
S' = \text{tr} \chi \left( \frac{D_A}{\Lambda} \right) + \langle J\Psi | D_A | \Psi \rangle
\]

with \(\Psi\) belonging to the projected Hilbert space of physical particles. The
presence of \(J\) is crucial for the correct pairings of particles and antiparticles.
What is left is then to crank again the machine, which in Connes’ word
is an “excruciating calculation, rewarded by the result of an action which
reproduces the gauge sector of the standard model. The novel choice of \(J\)
and \(\gamma\), and the modification of the fermionic action eliminate the doubling
and also allows for the presence of right-handed neutrinos and the see-saw
mechanism.

Therefore the internal space of noncommutative geometry has metric di-
mension zero, and KO-dimension six. In the next section we will see how
a (noncommutative) six dimensional internal space can give rise, with the
spectral action, to a noncommutative geometry which is well approximated
by an almost commutative one.

4 Nonabelian Yang-Mills from internal non-
commutative spaces

The spectral action is a natural evolution of the central philosophy on non-
commutative geometry, for which the geometry is given by the Dirac oper-
or, it is however an effective action, and for example there is a nonpertur-
bative versions of the action \[30\] based on superconnection \[31\]. Being an
effective action, valid only at certain scales, it is possible that the geometry
we are probing with the standard model is only a low energy geometry. In this section I see how an effective geometry, with the characteristics of the standard model, might emerge.

The bosonic part of the spectral action is sensitive only to the eigenvalues of the Dirac operator which are less than the energy scale given by $\Lambda$. This is the renormalization scale, a quantity that is safe to assume of an order of magnitude between the grand unification scale ($\sim 10^{16}$ GeV) and Planck energy ($\sim 10^{19}$ GeV). As for the spacetime part, even considering (for technical or physical reasons) spacetime compact, there are obviously many eigenvalues much smaller than $\Lambda$, and this requires the use of heath kernel techniques.

If however the internal space is compactified on a scale of the order of $\Lambda$ things are different. Consider again, as in (2.2) a manifold which is the product of spacetime by an internal space, which is taken to be a noncommutative space described by a noncommutative algebra $A_I$, represented on an Hilbert space $H_I$. One may think of the noncommutative space as a deformation of a compact ordinary manifold. So the full algebra and Hilbert spaces of the triple will be

$$\mathcal{A} = C(M) \otimes A_I,$$
$$\mathcal{H} = H_C \otimes H_I$$

(4.1)

Consider now the case in which the eigenvalues of the Dirac operator, call them $d_i$, have the characteristic that not only few of them are small (with respect to $\Lambda$), but that also their difference is small only for a limited range of indices

$$|d_k - d_l| \gg \Lambda \text{ if } k, l > N$$

(4.2)

i.e. only a finite number of pairs of eigenvalues are much smaller than the renormalization scale. The details of the noncommutative space are not important at this stage. What is important is that, since $A_I$ is a $C^*$-algebra, it can always be represented as bounded operator on $H_I$. Therefore it is possible to consider $A_I$ to be made of infinite matrices, and consider them in the basis which diagonalizes $D_{I0}$. An element of $\mathcal{A}$ will be denoted by $a_{kl}(x)$. A generic one-form will be of the kind:

$$A_{kl}(x) = \sum_i a^{(i)}[D_0, b^{(i)}] = \sum_i \sum_{m \in \mathbb{Z}} a^{(i)}_{km}(x)(d_l - d_m)\partial b^{(i)}_{ml}(x)$$

(4.3)
I am considering low energy fluctuations on the vacuum, and therefore $a$ and $b$ are much smaller than $\lambda$, so that $D_A = (\partial + d_k)\delta_{kl} + A_{kl}$ has all elements, except the first $N$ rows and columns, much larger than $\Lambda$. Because of the function $\chi$, then effectively only the $N \times N$ minor contributes to the action, which therefore becomes effectively the spectral action for an almost commutative geometry with gauge group $U(N)$.

With the same mechanism, and different scales, it is possible to have scenarios for which the internal space is a product of matrix algebras. It will suffice to modify (4.2) into

$$|d_k - d_l| \gg \Lambda \text{ if } k, l > N_1, \text{ or } N_1 < k, l < N_2, \ldots$$

(4.4)

The important point is that the spectral action measures the fluctuations of the space, as long as they do not exceed the scale $\Lambda$, and this may happen only if there are no quantities which exceed this scale, if the theory can be considered perturbatively.

Let us try to be slightly more precise (but still highly speculative). Consider the internal space to be a deformation of a two-dimensional space. I assume the deformation to be so “mild” that the relevant structures of the undeformed case survive. The internal Hilbert space in this case is made of two dimensional spinors, whose components are functions on the noncommutative space. The other elements of internal part of the spectral triple can be

$$D_{I0} = \begin{pmatrix} 0 & \nabla_2 + i \nabla_i \\ \nabla_2 - i \nabla_i & 0 \end{pmatrix}, \quad \gamma_I = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad J_I = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix},$$

(4.5)

With $C$ complex conjugation, $\nabla_i$ the two covariant derivatives which take into account the curvature of the space. We are making lots of assumptions here, and there is no warranty that these operators satisfy relations (2.10) for a two-dimensional space, nor that they exist for that matter.

The eigenvalues of $D_{I0}$ come in pairs differentiated only by the sign. The process of diagonalization of $D_{I0}$ can be done in two steps, first diagonalize $\nabla_2 + i \nabla_1$, or rather its deformed equivalent, with the infinite dimensional matrix $u$, hence

$$D'_{I0} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} D_{I0} \begin{pmatrix} u^\dagger & 0 \\ 0 & u^\dagger \end{pmatrix} = \begin{pmatrix} 0 & d \\ d^\dagger & 0 \end{pmatrix}$$

(4.6)

\(^3\)I am not considering $J$ for the moment, to keep things simple.
where $d$ is diagonal. Then block diagonalize with the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ \tilde{d}^* & -\tilde{d}^* \end{pmatrix}$$

(4.7)

where $\tilde{d}$ is a diagonal matrix of elements $d_k/|d_k|$ if $d_k \neq 0$, 1 otherwise, hence all of its elements are phases. In the new basis

$$U\gamma_I U^{-1} = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$$

$$UJ_I U^{-1} = \begin{pmatrix} \tilde{d}C & 0 \\ 0 & -\tilde{d}C \end{pmatrix}$$

(4.8)

If now the we operate the reduction to the minor of small matrix elements of $D_A$, we pick up a minor composed of the first few rows and columns of both blocks and we have a duplication of the internal structure, inherited from the two-dimensional spinors. The chirality $\gamma_I$ maintains the same structure in the minor, and the phases appearing in $J_I$ can be eliminated with a further conjugation by a diagonal unitary matrix.

Effectively only a matrix part of the noncommutative algebra $A_I$ contributes to the spectral action. We end up with a $U(n) \times U(n)$ gauge theory, of the kind very familiar in noncommutative geometry [11, 12]. With a judicious choice of the Dirac operator the gauge group can be broken to a smaller diagonal group. A crucial difference is that however the commutation relation of the reduced $D_I$, $J_I$ and $\gamma_I$ have remained those of the original two-dimensional theory, and are block reduction of those of a zero-dimensional space. The point is that in reality the internal space is two-dimensional, but the particular choice of eigenvalues effectively freezes out the higher modes.

5 Noncommutative internal space and strings

In this section I will attempt a connection between the spectral action and string theory, arguing on the possibility of a way to extract the standard model from a string theory. I will discuss how the internal space of strings could give rise, with a freezing process like the one discussed earlier, to a model with the characteristics of the standard model in the framework of the spectral action.
The first step is the construction of the noncommutative geometry of strings. Connections between noncommutative geometry and strings have become common in the literature since the work of Seiberg and Witten [32], but attempts to construct the noncommutative geometry of strings, in the framework of Connes’ spectral geometry, go back to Frohlich and Gawedzky [33] and has been further developed in [34, 35, 36, 37].

I now give the essential elements of the construction of the (closed) strings spectral triple, the Frohlich-Gawedzky triple, and refer to the above papers, especially [36, 37], for details. Consider bosonic strings compactified on a $d$ dimensional torus $\mathbb{T}_d$ with radii $L_i$. This torus (or a deformation of it) will be the internal space. The basic operators of the theory are the Fubini-Veneziano fields

$$X^i_\pm(z_\pm) = x^i_\pm + i g^{ij} p^j_\pm \log z_\pm + \sum_{n\neq 0}^{\infty} \frac{1}{in} \alpha^{(\pm)i}_n z_\pm^{-n} \tag{5.1}$$

and the chiral Heisenberg fields

$$\alpha^i_\pm(z_\pm) = -i \partial_\pm X^i_\pm(z_\pm) = \sum_{n=-\infty}^{\infty} \alpha^{(\pm)i}_n z_\pm^{-n-1} \quad ; \quad \alpha^{(\pm)i}_0 \equiv g^{ij} p^j_\pm \tag{5.2}$$

where $z_\pm \in \mathbb{C} \cup \{\infty\}$. The $z_\pm$'s are the world sheet coordinates and $g^{ij}$ is the target space metric. The center of mass of the string is $x = x_+ + x_-$, its momentum $p = p_+ + p_-$, the winding is $p = p_+ - p_-$. The $\alpha_n$ for $n$ negative (resp. positive) are the creation (resp. annihilation) modes for the oscillatory modes of the string.

The states of this string are described by an Hilbert space which takes into account the momentum, the winding number and the left and right modes of the string:

$$\mathcal{H}_I = L^2_S(\mathbb{T}_d \times \mathbb{T}_d^*) \otimes \mathcal{F}^+ \otimes \mathcal{F}^- \tag{5.3}$$

where by $L^2_S$ I mean the square integrable spinors, $\mathcal{F}^\pm$ are the Fock spaces generated by the left and right moving oscillatory creation operators, $\mathbb{T}_d^*$ is the dual torus whose presence is due to the states in which the strings winds around the compact dimension. The radii of the dual torus are $L_{\ell}^{-1}$. The states of the string are created by the action of vertex operators acting on a vacuum (operator states):

$$|\Psi\rangle = V_\Psi(z_+, z_-) |0\rangle \tag{5.4}$$
The vertex operator algebra is specific of a particular background and specific string theory, it is natural to take as second element of the spectral triple the (properly regularized) vertex operator algebra of the theory. For a toroidal background the fundamental vertex operators for the low lying states (tachyonic for the bosonic string) are of the kind

\[ V_{q\pm} = : e^{i\theta^\pm x_\pm} : \]  

(5.5)

where \( : \cdot : \) means normal ordering with respect to the creation and annihilation operators and the \( q \)'s are the momenta of the states.

The left and right movers of the theory split, so that each copy has its set of reciprocally anticommuting gamma matrices

\[ \{ \gamma_i^\pm, \gamma_j^\pm \} = \pm 2g_{ij} \]  

(5.6)

each sector has his own \( \gamma^\pm = \gamma_1^\pm \ldots \gamma_d^\pm \), and I define

\[ \gamma_I = (-1)^d \gamma^+ \gamma^- \]  

(5.7)

Each sector has his own Dirac operator as well:

\[ D^\pm = \gamma^\pm \alpha^\mu \partial_\mu (z_\pm) \]  

(5.8)

with which I can build two Dirac operators

\[ D_I = D^+ + D^- \quad \bar{D}_I = D^+ - D^- \]  

(5.9)

The two operators are unitarily related by \textit{T-duality transformation}. This transformation exchanges winding and momentum modes. Ignoring the oscillatory modes, and with the usual identification of the momentum operator with the derivative, the operator \( D_I \) becomes the usual Dirac operator on the compactified space, while \( \bar{D}_I \) is the operator on the dual torus. This can be identified with the set of “position” for the dual (in the sense of position-momentum duality) of the winding number.

The last element is \( J_I \), which can be taken to act on the Hilbert space as:

\[ J_I |\Psi\rangle = V^\dagger_\Psi |\Psi\rangle \]  

(5.10)

The bimodule structure is preserved, in fact:

\[ V_1 J_I V_2 J_I |\Psi\rangle = V_1 J_I V_2 J_I V_\Psi |0\rangle = V_1 J_I V_2 V_\Psi^\dagger |0\rangle = V_1 V_\Psi V_2^\dagger |0\rangle \]
The relations between $D_I$, $\gamma_I$ and $J_I$ satisfy those of a 2d-dimensional space. The reasons for this doubling of the dimensions is that both momentum and winding modes contribute. In fact we have shown in [37] that there is a range of energies for which the internal space is effectively composed of two copies of a noncommutative torus. We have also shown [35] that on the contrary, if the compactification is very large then the degrees of freedom are those of commutative “subtriples” which reproduces spacetime, with the winding modes playing, as it should, no role.

Let us now, very heuristically, discuss the features of a string internal landscape which could give rise to the standard model. Consider the space made of four uncompactified dimensions, for which the usual geometry will apply, and some dimensions compactified on a small scale. I now speculate how some internal space of strings could give rise to an almost commutative geometry with some features of the standard model. Assume that the scale of compactification of the internal space, and of the lowest eigenvalues of the Dirac operator) will be of order $\Lambda$, while the scale of the oscillatory modes is higher, at the Planck scale. This is not a new scenario in string theory [38]. This means however that we can ignore the oscillatory modes in a spectral action, but that we would have to take into account both the momentum and winding modes.

A torus does not give the desired hierarchy of eigenvalues, so we suppose that the compactified geometry is deformed to give the desired set of eigenvalues, but still keeps the main features of the toroidal case. In particular the presence of momentum and winding number modes. One expects that, as in the torus case, the noncommutativity in the internal space comes form the interplay of these two modes, but if the space is not a torus the two spaces are not the same.

Let us look at the features that the noncommutative geometry of compactified strings has in common with the standard model. There are two sectors, the left and right movers. They are the eigenspaces of the chirality operator $\gamma$. The two sectors are connected by T-duality, which exchanges momentum and winding modes. Again the torus is not apt to reproduce the standard model, since it has a perfect symmetry between these two sector, in contrast with the chiral characteristics of the standard model, but is should be possible to envisage a (deformed) geometry for which the momenta have
a different symmetry from the windings. The charge conjugation operator acts transforming the string into the hermitian conjugate, exchanging left and right sectors, and giving the bimodule structure. The Dirac operator feels the geometrical structure of the internal space, which is not necessarily a deformation of a commutative space, but just an algebra generated by a vertex operator algebra. The value of its eigenvalues can therefore be quite general.

The low lying states which contribute to the action are a vacuum state, and then the $d$ states created by the operators (5.5) acting on it. For $d = 3$, in view of the argument described in section 4, one would have that the relevant states of the vertex algebra freezes to a $\mathbb{C} \oplus M_3(\mathbb{C})$. The left-right structure we conjecture is given by T-duality. A chiral theory would be a consequence of a non T-dual state. At present I have do not see how to obtain a quaternionic structure, nor the presence of more than one generation.

6 Final Remarks

The speculative character of this paper is such that it poses many questions and gives no answers. The main task is to build a geometry with at least some of the desired features of the theory, in terms of the spectrum of the Dirac operator. On one side this is a difficult task because there are not so many noncommutative manifolds which are studied sufficiently. On the other side the possibility to have noncommutative space gives added degrees of freedom which effectively enlarge the scope of the investigation. What I envisage is a programme “top-down”, which starting form the internal geometries sees which sort of gauge/gravity model they can give. This is dual to the conventional approaches in noncommutative geometry which start from the standard model (with all of its parameters) and try to fit the geometry which fits it.

String theory provides us with an extreme wealth of possible internal geometries, and structure which I do not have considered here, supersymmetry and branes the most important ones, they will definitely have consequences for the matter discussed here. In general more work is needed (independently of the conjectures of this paper) to understand string theory in the framework of the spectral action.
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