Effects of anisotropy on optimal dense coding

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Abstract

Optimal dense coding with thermal entangled states of a two-qubit anisotropic XXZ model and a Heisenberg model with Dzyaloshinskii–Moriya (DM) interactions is studied in this paper. The DM interaction is another kind of anisotropic antisymmetric exchange interaction. The effects of these two kinds of anisotropies on dense coding are studied in detail for both the antiferromagnetic and ferromagnetic cases. For the two models, I give the conditions that the parameters of the models have to satisfy for a valid dense coding. I also found that even though there is entanglement, it is unavailable for our optimal dense coding, which is the same as entanglement teleportation.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Entanglement is one of the most fascinating features of quantum mechanics and plays a central role in quantum information processing, such as quantum key distribution [1], quantum teleportation [2], dense coding [3], and so on. In the initial dense coding protocol [4], the sender can transmit two bits of classical information to the receiver by sending a single qubit if they share a two-qubit maximally entangled state (an Einstein–Podolsky–Rosen (EPR) state). Since then, many works on dense coding have been presented experimentally [5] or theoretically [6–8]. We know that, in a general dense coding, the sender performs one of the local unitary transformations $U_i \in U(d)$ on the $d$-dimensional quantum system to put the initially shared entangled state $\rho$ in $\rho_i = (U_i \otimes I_d)\rho(U_i^\dagger \otimes I_d)$ with a priori probability $p_i (i = 0, 1, \ldots, i_{\text{max}})$, and then the sender sends off his quantum system to the receiver. Upon receiving this quantum system, the receiver performs a suitable measurement on $\rho_i$ to extract the signal. The optimal amount of information that can be conveyed is known to be bounded by the Holevo quantity [9]: $\chi = S(\bar{\rho}) - \sum_{i=0}^{i_{\text{max}}} p_i S(\rho_i)$, where $S(\rho)$ denotes the von Neumann entropy and $S(\bar{\rho}) = \sum_{i=0}^{i_{\text{max}}} p_i \rho_i$ is the average density matrix of the signal ensemble. Since the Holevo quantity is asymptotically achievable [10], one can use $\chi = S(\bar{\rho}) - \sum_{i=0}^{i_{\text{max}}} p_i S(\rho_i)$ as the definition of the capacity of dense coding. Moreover, the von Neumann entropy is invariant under unitary transformations, $S(\rho_i) = S(\rho)$. Therefore, the dense coding capacity can be rewritten as $\chi = S(\bar{\rho}) - S(\rho)$. The following problem is to find the optimal signal ensemble $\{\rho_i; p_i\}_{i=0}^{i_{\text{max}}}$ that maximizes $\chi$. The author of [11] showed that the $d^2$ signal states ($i_{\text{max}} = d^2 - 1$) generated by mutually orthogonal unitary transformations with equal probabilities yield the maximum $\chi$, which is called optimal dense coding, and considered the optimal dense coding when the shared entangled state was a general mixed one.

The quantum entanglement in solid-state systems such as spin chains has been an important emerging field since the founding of thermal entanglement [12]. Spin chains are natural candidates for the realization of the entanglement compared with other physical systems. As the thermal fluctuation is introduced into the system, the state of a typical solid-state system at thermal equilibrium (temperature $T$) is $\rho(T) = e^{-\beta H}/Z$, where $H$ is the Hamiltonian, $Z = \text{tr}e^{-\beta H}$ is the partition function and $\beta = 1/(kT)$, where $k$ is the Boltzmann constant. For simplicity, I write $k = 1$. As $\rho(T)$ represents a thermal state, the entanglement in the
state is called the thermal entanglement. The study of the thermal entanglement properties in Heisenberg systems has received a great deal of attention [13–18]. Some authors have considered the availability of thermal entanglement. Quantum teleportation that uses the thermal entangled state as a channel has been proposed in [19–21].

In this paper, I study the optimal dense coding [11] using the thermal entangled states of a two-qubit anisotropic XXZ model and a Heisenberg model with Dzyaloshinskii–Moriya (DM) interactions. I investigate in detail the effects of two kinds of anisotropies on dense coding for both the antiferromagnetic (AFM) and ferromagnetic (FM) cases and give the conditions that the parameters of the model have to satisfy for dense coding. The paper is organized as follows: in section 2, optimal dense coding using the thermal entangled states of a two-qubit anisotropic XXZ model is investigated; optimal dense coding using the thermal entangled state of a Heisenberg model with DM interactions is studied in section 3 and conclusions are given in section 4.

2. Optimal dense coding using thermal states of a two-qubit anisotropic XXZ chain

Let us consider a two-qubit anisotropic XXZ Heisenberg model

\[ H = \frac{J}{2}(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + \Delta\sigma_1^z\sigma_2^z) = J(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y) + \frac{\Delta}{2}\sigma_1^z\sigma_2^z, \]

where \(\sigma_{j\alpha}(j = 1, 2, \alpha = x, y, z)\) are the Pauli matrices. \(J\) is the real coupling constant, \(J > 0\) corresponding to the AFM case and \(J < 0\) to the FM case. The operators \(\sigma_{j\pm} = (1/2)(\sigma_{jx} \pm i\sigma_{jy})\). Without loss of generality, I define \(|0\rangle\) (|1\rangle) as the ground (excited) state of a two-level particle. The eigensystem of \(H\) is \(H|00\rangle = (J/\Delta|00\rangle, H|\Psi^+\rangle = (-J/\Delta|\pm\rangle)\) and \(H|11\rangle = (J/\Delta|11\rangle\). The thermal state of the system at equilibrium (temperature \(T\)) is

\[ \rho = \frac{1}{Z_1}[e^{-\beta(J/\Delta)|00\rangle\langle00|} + e^{-\beta(-J/\Delta)|\Psi^+\rangle\langle\Psi^+|} + e^{-\beta(J/\Delta)|11\rangle\langle11|}], \]

where \(Z_1 = 2\lambda e^{-J/2T}\) is the partition function and \(\lambda = 1 + e^{J/\Delta/T}\) \(\cosh[J/T]\). In [22], the concurrence [23] of the model is considered as a measure of thermal entanglement.

Now I carry out the optimal dense coding using the thermal entangled states of the two-qubit system as a channel. The set of mutually orthogonal unitary transformations [11] of the optimal dense coding for two-qubit is

\[ U_{00}|x\rangle = |x\rangle, \quad U_{01}|x\rangle = e^{\sqrt{\chi}(2\pi/2)|x\rangle}, \]

\[ U_{10}|x\rangle = |x + 1(\text{mod}2),\]

\[ U_{11}|x\rangle = e^{\sqrt{\chi}(2\pi/2)|x + 1(\text{mod}2)},\]

\[ \rho = \frac{1}{4}\sum_{i=0}^3(U_i \otimes I_2)\rho(U_i^\dagger \otimes I_2), \]

where we have assumed \(0 \rightarrow 00; 1 \rightarrow 01; 2 \rightarrow 10; 3 \rightarrow 11\), and \(\rho\) is the thermal state of equation (2). Through straightforward algebra, we have

\[ \rho = \frac{1}{4}|\langle00|00\rangle + |01\rangle\langle01| + |10\rangle\langle10| + |11\rangle\langle11|\]. \]

After completing the set of mutually orthogonal unitary transformations, the maximal dense coding capacity \(\chi\) can be written as

\[ \chi = S(\rho) - 2S(\rho). \]

Here, \(S(\rho)\) is the von Neumann entropy of the quantum state \(\rho\). Thus, the value of \(\chi\) is

\[ \chi = \frac{T\lambda\ln[4] - 2\ln[\lambda] + 2J\xi e^{(J/\Delta)/T}}{T\lambda\ln[4]}, \]

where \(\xi = \Delta \cosh[J/T] + \sinh[J/T].\)

It is found that \(\chi(\Delta, J) = \chi(-J, -\Delta),\) which indicates that \(\chi\) satisfies \(\chi_{\text{AFM}}(\Delta) = \chi_{\text{FM}}(-\Delta).\) The result is the same as the concurrence [22]. I now give the numerical analysis of \(\chi\). In figure 1, the optimal dense coding capacity \(\chi\) as a function of the coupling constant \(J\) and anisotropy \(\Delta\) is plotted at a definite temperature. From the analytical point of view, in order to carry out the optimal dense coding successfully, the parameters of the model must satisfy

\[ \chi > \log_2[2] = 1 \Leftrightarrow \Delta \xi e^{(J/\Delta)/T} > T\lambda\ln[\lambda]. \]

For different values of \(\Delta\) and \(J\), there must be a critical temperature \(T_{\text{critical}}\), beyond which we cannot give an optimal dense coding with this two-qubit Heisenberg XXZ chain. In the following, I will investigate explicitly the effects of \(\Delta\) and \(T\) on \(\chi\).

Case 1. Anisotropy \(\Delta = 0\). We find that \(\chi = 2\) when \(T \rightarrow 0\) is always true regardless of the AFM or FM case. The result is the same when the channel is a two-qubit EPR state. This is because in this case, the thermal state is \(|\Psi^+\rangle\) (for the AFM case) or \(|\Psi^-\rangle\) (for the FM case), which are the EPR states, so the sender can transmit 2 bits of classical information by sending 1 qubit.
Figure 2. Plot of $\chi$ versus the anisotropy $\Delta$. The left panel corresponds to the AFM case ($J = 1$) and the right panel corresponds to the FM case ($J = -1$). From top to bottom, the temperature is 0.005, 0.5 and 1, respectively.

Case 2. Anisotropy $|\Delta| \gg 0$. We have

$$\chi = \begin{cases} \frac{1 + T \ln[4] - T \ln[1 + e^{2\Delta/T}]}{T \ln[2]} + \tanh[1/T], & \text{if } \Delta \to +\infty; \\ 1, & \text{if } \Delta \to -\infty, \end{cases}$$

(9)

for $J = 1$, and

$$\chi = \begin{cases} 1, & \text{if } \Delta \to +\infty; \\ \frac{1 + T \ln[4] - T \ln[1 + e^{2\Delta/T}]}{T \ln[2]} + \tanh[1/T], & \text{if } \Delta \to -\infty, \end{cases}$$

(10)

for $J = -1$. These features can be seen in figure 2. For $J = 1$ and $\Delta \to -\infty$ or for $J = -1$ and $\Delta \to +\infty$, we have $\chi = 1$, which means that the quantum channel is not valid for optimal dense coding, because now the thermal state is $\| \langle 00 \rangle \| + \| \langle 11 \rangle \|$, which is a superposition of two product states. But for $J = 1$ and $\Delta \to +\infty$ or for $J = -1$ and $\Delta \to -\infty$, the value of $\chi$ is the same and depends on the temperature. In these two cases, the thermal state is

$$\rho = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm \tanh[1/T] & 0 \\ 0 & \mp \tanh[1/T] & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(11)

where ‘+’ corresponds to $J = -1$ and $\Delta \to -\infty$, and ‘−’ corresponds to $J = 1$ and $\Delta \to +\infty$. The concurrence [23] of equation (11) is $C = \tanh[1/T]$, which means that equation (11) is an entangled state for a finite temperature. In order to get a valid optimal dense coding when anisotropy $|\Delta| \gg 0$, here we must have

$$\frac{1 + T \ln[4] - T \ln[1 + e^{2\Delta/T}]}{T \ln[2]} + \tanh[1/T] > 1$$

$$\Leftrightarrow \frac{1 + \tanh[1/T]}{\ln[(1 + e^{2\Delta/T})/2]} > T.$$ 

(12)

This can be true for any temperature. We can see this also in figure 2. Moreover, since $\chi(J = 1, \Delta = -1) = \chi(J = -1, \Delta = 1) = \ln[4]/\ln[2] \approx 0.69314$ when $T \to 0$, the value will decrease as the temperature increases. This can be easily understood since the concurrence of this model is zero when $\Delta = -1$ for the AFM case and $\Delta = 1$ for the FM case. Therefore, $\chi$ has an abrupt transition at the two points.

Case 3. For a finite anisotropy and temperature. In figure 3, $\chi$ as a function of $T$ is plotted for four different values of anisotropy. From this figure, when $J = 1$, if $\Delta < -1$, regardless of the temperature, $\chi$ is always less than 1 and the thermal entangled states are not valid for optimal dense coding. This can be easily explained since the concurrence of this model $C_{\text{AFM}} = 0$ for $\Delta < -1$. Moreover, we can see that with increasing the $\Delta$, the range of $T$ for which the optimal dense coding is feasible becomes wider. Accordingly, for $J = -1$, we must have $\Delta < 1$ in order to make $\chi > 1$ at some temperature. This is easily understood because $C_{\text{FM}} = 0$ for $\Delta > 1$. But the range of $T$ for which the optimal dense coding is feasible becomes narrower.

3. The effects of DM interaction on optimal dense coding

Another kind of anisotropy that we investigate is the DM anisotropic antisymmetric interaction, which arises from spin–orbit coupling [24, 25]. Now we consider the Heisenberg model with DM interactions

$$H_{\text{DM}} = \frac{J}{2} \left[ (\sigma_1 \sigma_2 + \sigma_1 \sigma_2 - \sigma_1 \sigma_2) + \vec{D} \cdot (\vec{\sigma}_1 \times \vec{\sigma}_2) \right].$$

(13)

where $\vec{D}$ is the DM vector coupling. For simplicity, I choose $\vec{D} = D \vec{z}$, and the Hamiltonian $H_{\text{DM}}$ becomes

$$H_{\text{DM}} = \frac{J}{2} \left[ (\sigma_1 \sigma_2 + \sigma_1 \sigma_2 - \sigma_1 \sigma_2) + D(\sigma_1 \sigma_2 - \sigma_1 \sigma_2) \right]$$

$$= J \left[ (1 + iD)\sigma_1 \sigma_2 + (1 - iD)\sigma_1 \sigma_2 + \frac{J}{2} \sigma_1 \sigma_2 \right].$$

(14)

We notice that $H_{\text{DM}}(D = 0) = H(\Delta = 1)$. The eigenvalues and eigenvectors of $H_{\text{DM}}$ are $H_{\text{DM}}(00) = \frac{1}{2} (00)$, $H_{\text{DM}}(11) = \frac{\sqrt{2}}{2} (11)$, $H_{\text{DM}}(1\pm) = (\pm 1/\sqrt{\sqrt{2} + D^2 - \frac{\Delta^2}{2}})$, with $(\pm) = (1/\sqrt{2}) ((01) \pm e^{i\theta}|10\rangle)$ and $\theta = \arctan D$. 

Figure 3. Plot of $\chi$ versus $T$. The left panel corresponds to the AFM case ($J = 1$) and the right panel corresponds to the FM case ($J = -1$). Left panel: solid (black) curve for $\Delta = -2$, dashed (blue) curve for $\Delta = -0.9$, dotted (red) curve for $\Delta = 0$ and dash-dotted (green) curve for $\Delta = 1$. Right panel: solid (black) curve for $\Delta = -1$, dashed (blue) curve for $\Delta = 0$, dotted (red) curve for $\Delta = 0.9$ and dash-dotted (green) curve for $\Delta = 2$.
As the thermal fluctuation is introduced into the system, in the standard basis \{\{11\}, \{10\}, \{01\}, \{00\}\}, the state can be expressed as
\[
\rho_{\text{DM}} = \frac{1}{Z_2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} e^{\frac{\delta}{2}(J - \beta)} (1 + e^{\theta i}) & \frac{1}{2} e^{i \theta} e^{\frac{\delta}{2}(J - \beta)} (1 - e^{\theta i}) & 0 \\
0 & \frac{1}{2} e^{i \theta} e^{\frac{\delta}{2}(J - \beta)} (1 - e^{\theta i}) & \frac{1}{2} e^{\frac{\delta}{2}(J - \beta)} (1 + e^{\theta i}) & 0 \\
0 & 0 & 0 & e^{-\frac{T}{2}}
\end{pmatrix},
\]
where \(Z_2 = 2 \eta e^{-J/2T}, \ \eta = 1 + e^{J/T} \cosh[\delta/(2T)]\) and \(\delta = 2J \sqrt{1 + D^2}\). The entanglement of this model has been studied in [21] by means of concurrence. Through equations (3) and (4), we have also
\[
\bar{\rho}_{\text{DM}} = \frac{1}{2} ([|00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11|). \tag{16}
\]
Similarly, after a straight calculation, the value of \(\chi\) is given by
\[
\chi = \frac{T \eta \ln[4] - 2 \ln[\eta] + \zeta e^{J/T}}{T \eta \ln[4]}, \tag{17}
\]
where \(\zeta = 2J \cosh[\delta/(2T)] + \delta \sinh[\delta/(2T)]\). In order to carry out the optimal dense coding successfully, the parameters of the model must satisfy \(\chi > \log_2[2] = 1 \Leftrightarrow \zeta e^{J/T} > 2T \eta \ln[\eta]\), which can be returned to equation (8) where \(\Delta = 1\) for a vanishing DM interaction.

In figure 4, we give a plot of \(\chi\) as a function of DM interaction and the spin coupling constant at a definite temperature. The variation of \(\chi\) with \(D\) is very similar to that of concurrence for the AFM case. But for the FM case, the behavior of \(\chi\) is different from that of concurrence. These results can be found by comparing figure 4 with figure 1 in [21]. Next, we consider the effects of DM interaction on dense coding capacity.

Case 1. DM interaction \(D \gg 0\). The \(\chi\) is always equal to \(\log_2[4] = 2\). For the AFM and FM cases since the thermal state is \(\frac{1}{2} \{01\} \{01\} + i \{01\} \{10\} + i \{01\} \{10\} + i \{10\} \{01\}\), which is an EPR-type state and its concurrence is 1.

Case 2. For a finite DM interaction and temperature. The variation of \(\chi\) with \(D\) for \(J = 1\) and \(J = -1\) is plotted in figure 5. As the temperature increases, the range of \(D\) useful for optimal dense coding becomes narrow whether \(J > 0\) or \(J < 0\). Although there exists some \(D\) for the thermal state that is not valid for optimal dense coding, for these \(D\) values the concurrence of the model is not zero. Therefore, even though there is entanglement, it is unavailable for our optimal dense coding, which is the same as entanglement teleportation. Comparing the left panel with the right panel, DM interaction must be stronger for the FM case in order to make \(\chi > 1\) at the same temperature for the AFM case. Moreover, for \(D = 0\), \(\chi\) is always less than 2 for the FM case no matter what the temperature is, which can be easily understood since the entanglement is zero. In figure 6, \(\chi\) as a function of \(T\) is plotted for different DM interactions. The critical value of \(T\) when the thermal entangled state is valid for optimal dense coding (\(\chi > 1\)) for \(D = 5\) is larger than that for \(D = 1\). At zero temperature, regardless of the DM interaction, \(\chi = 2\) for the AFM case, because the state is \(\{+\} = \frac{1}{\sqrt{2}} (|01\rangle - e^{-\theta i} |10\rangle\), which is an EPR-type state. However, the thermal state is uncertain for the FM case at zero temperature. For nonzero \(D\), the state is \(\{|+\} = \frac{1}{\sqrt{2}} (|01\rangle + e^{-\theta i} |10\rangle\), so \(\chi(D \neq 0) = 2\). At zero temperature, \(\chi(D = 0) < 1\), because the state is \(\frac{1}{2} \{00\} \{00\} + i \{01\} \{01\} + i \{10\} \{10\} - i(01) |01\rangle + |10\rangle + 2 |11\rangle |11\rangle\), which is not an entangled state.

4. Conclusions

In conclusion, we studied analytically the effects of two kinds of anisotropy on the optimal dense coding in an anisotropic XXZ model and a Heisenberg model with DM interactions. We have demonstrated that whether the optimal dense coding is valid or not depends on both the anisotropic parameters and the sign of exchange constants \(J\). The conditions for a valid optimal dense coding have been given. For the AFM XXZ model, anisotropy must be larger than \(-1\), and the critical temperature above which \(\chi\) is less than 1 will increase as anisotropy increases. But anisotropy must be less than 1 and
the critical temperature will decrease with the increasing of anisotropy for the FM case. The dependence trends of $\chi$ on DM interaction are the same for both the AFM and FM cases of the Heisenberg model with DM interactions, and the relatively stronger DM interaction will be helpful for optimal dense coding. We also found that even though there is entanglement, it is unavailable for our optimal dense coding, which is the same as entanglement teleportation.

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