Weak Identification with Bounds
in a Class of Minimum Distance Models*

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Abstract

When parameters are weakly identified, bounds on the parameters may provide a valuable source of information. Existing weak identification estimation and inference results are unable to combine weak identification with bounds. Within a class of minimum distance models, this paper proposes identification-robust inference that incorporates information from bounds when parameters are weakly identified. The inference is based on limit theory that combines weak identification theory with parameter-on-the-boundary theory. This paper demonstrates the role of the bounds and identification-robust inference in two example factor models. This paper also demonstrates the identification-robust inference in an empirical application, a factor model for parental investments in children.

Keywords: Identification-Robust Inference, Partial Identification, Boundary, Subvector Inference, Uniform Inference, Factor Model, Weak Factors

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1 Introduction

Minimum distance models are commonly used to estimate and test hypotheses on structural parameters of interest. The link function, which maps the structural parameters to reduced-form parameters, is usually nonlinear, and may not be one-to-one. If the link function is not one-to-one, the structural parameters are not identified. Without identification, the minimum distance estimator is inconsistent, and standard hypothesis tests are size-distorted. Furthermore, weak identification arises in a neighborhood of parameter values that are not identified.

When identification is lost, information about the structural parameters may still be available from bounds on the identified set. This paper combines weak identification with bounds by (1) formulating weak identification limit theory that accommodates an identified set that is defined by an intersection of bounds, (2) proposing an identification- and boundary-robust test that has power against hypotheses that violate the bounds when identification is weak, and (3) demonstrating this test in low-dimensional factor models and an empirical application. Existing weak-identification-robust tests are either invalid in the presence of bounds or do not have power against hypotheses that violate the bounds.

In this paper, the identified set is defined by the intersection of bounds. The bounds are informative for the parameters when identification is weak. Other papers in the weak identification literature assume the parameter space is a product space between the strongly identified parameters and the weakly identified parameters, excluding the possibility of informative bounds.\(^1\) Without any bounds, Dufour (1997) shows that any confidence interval for a non-identified parameter must have infinite length with probability at least \(1 - \alpha\) asymptotically. Using bounds sidesteps this result. Valid confidence intervals with finite length exist and can be quite informative.

We derive limit theory for an estimator along sequences of parameters. Following Andrews and Cheng (2012), we define three classes of sequences of parameters, strong, semi-strong, and weak, indicating the strength of identification. These sequences may converge to the boundary of the parameter space. The strong and semi-strong sequences lead to a parameter-on-the-boundary limiting distribution (see Andrews (1999)), which reduces to the normal limiting distribution when the se-

\(^1\)For example, see page 1060 in Stock and Wright (2000), Assumption A in Guggenberger and Smith (2005), or Assumption B1(ii) in Andrews and Cheng (2012).
sequence converges to a point in the interior. Weak sequences lead to a nonstandard limiting distribution for the estimator.

The limit theory is used to define an identification- and boundary-robust test. When testing a hypothesis on the structural parameters, we recommend calculating the quasi-likelihood ratio (QLR) statistic and comparing it to a critical value that is calculated from the quantiles of the limiting distribution under weak identification. This robust quasi-likelihood ratio (RQLR) test is very flexible, applying to full-vector inference and subvector inference and allowing strongly identified or weakly identified nuisance parameters. It has many desirable features. First, the test is uniformly valid, which follows from the fact that the three classes of sequences, strong, semi-strong, and weak, are sufficient for uniform size control. Second, simulations show the test has power against hypotheses that violate the bounds when identification is weak. Third, under strong identification the critical value converges to a quantile of the chi-squared distribution when the sequence converges to a point in the interior of the parameter space, and thus asymptotically coincides with the standard QLR test.

In contrast, other available identification-robust (IR) tests do not share these desirable features. Surprisingly, this includes tests based on the Anderson-Rubin (AR) statistic from Stock and Wright (2000) and the K and CLR statistics from Kleibergen (2005). (In Stock and Wright (2000), the AR statistic is denoted “S” and in Kleibergen (2005), the CLR statistic is denoted “GMM-M.”) Specifically, the versions of these tests that plug in estimates of strongly identified nuisance parameters either do not have power against hypotheses that violate the bounds or do not control size for hypotheses near the boundary of the identified set.\(^2\) It is well-known that, in general, one cannot plug in weakly identified nuisance parameters for these tests. The fact that plugging in strongly identified nuisance parameters is problematic near the boundary of the identified set is novel and is documented in the simulations found in Section 5.

There are many possible applications of weak identification with bounds. We focus on the low-dimensional factor models considered in Cox (2022b). In factor models, weak identification arises when factor loadings are close to zero, or when factor loadings for two factors are too close to each other, in some sense. This paper

\(^2\)Controlling size near the boundary of the identified set is especially important because, otherwise, a confidence set formed by inverting such a test is too small and under-covers asymptotically. Likewise, having consistent power against hypotheses that violate the bounds is also important because then a confidence set formed by inverting such a test is not unnecessarily large.
considers two example factor models, one with one factor and one with two factors. In both cases, informative bounds on the weakly identified parameters can be derived from the nonnegativity of the variances of the latent factors and errors. We provide the implementation details for the RQLR test in these two examples, verifying the conditions for uniform size control. Simulations show good performance of the RQLR test relative to other IR tests.

Weak identification may be relevant in many empirical applications of factor models, especially when the number of factors is unknown. If a researcher does not know whether a given dataset contains one or two factors, for example, then the possibility the dataset contains a weakly identified factor should not be ruled out. We consider one such application, estimating a model of parental investments in children. Attanasio et al. (2020) use a factor model with one factor to analyze the effects of a randomized intervention designed to improve the quality of maternal-child interactions on parental investments in children. Following Cox (2022b), we note indicators that the model contains a second factor that may be weakly identified. We demonstrate the relevance of the bounds and the value of inference based on the RQLR test in the weakly identified specification with two factors.

Other related papers, including Andrews (1999), Moon and Schorfheide (2009), Ketz (2018), and Ketz and McCloskey (2021), allow the parameter to be on the boundary of the parameter space. Those papers require strongly identified parameters. Andrews (2001) allows some parameters to be on the boundary and others to be weakly identified. However, Andrews (2001) requires the parameter space to be a product space between the strongly identified and weakly identified parameters; see expression (3.1) in Andrews (2001). As before, this precludes the bounds from being informative for the weakly identified parameters.

The rest of the paper is organized as follows. Section 2 defines the class of minimum distance models and presents the examples. Section 3 states the limit theory for an estimator along sequences of parameters. Section 4 states the limit theory for the QLR statistic and proposes the RQLR test. Section 5 provides simulations of IR tests in the two example factor models. Section 6 presents the empirical application. Section 7 concludes. Proofs and details for the examples and application are available in supplemental materials.

For clarity, we introduce some notation used throughout the paper. We use $'$ to denote the transpose of a vector or matrix. We write $(a, b)$ for $(a', b')'$ for stacking two
column vectors, $a$ and $b$. We write $d_a$ to denote the dimension of an arbitrary vector, $a$. Let $e_1, \ldots, e_d$ denote the standard normal basis vectors in $\mathbb{R}^d$. Given a function $f$, we write $f(a, b)$ for $f((a, b))$ for simplicity. Given a real-valued function $f(x)$, we write $\partial_x f(x)$ to denote the row vector of derivatives of $f(x)$. If $f(x)$ is a vector-valued function, $\partial_x f(x)$ denotes the matrix of derivatives where each row denotes the derivatives of the corresponding component of $f(x)$. If $f$ is real-valued, we use $\partial_{xy} f(x, y)$ and $\partial_{xx} f(x)$ to denote $\partial_y(\partial_x f(x, y))'$ and $\partial_x(\partial_x f(x))'$, respectively, which are matrices of second derivatives. Given any two sets, $A$ and $B$, let \( \vec{d}(A, B) = \max(\sup_{a \in A} \inf_{b \in B} d(a, b), 0) \) denote a directed distance between $A$ and $B$. We write the Hausdorff distance between two sets, $A$ and $B$, as \( d_H(A, B) = \max \left( \vec{d}(A, B), \vec{d}(B, A) \right) \). For a sequence of stochastic processes, $Q_n(\theta)$ and $Q(\theta)$ over $\theta \in K$, let $Q_n(\theta) \Rightarrow K Q(\theta)$ denote weak convergence in the space of bounded functions on $K$ with the uniform norm.

2 A Class of Minimum Distance Models

Minimum distance models are commonly used to estimate and test hypotheses on structural parameters of interest. This section introduces a class of minimum distance models that satisfy a key assumption and describes two example factor models that satisfy the assumption.

In minimum distance models, the link function, $\delta(\cdot)$, maps structural parameters, $\theta \in \Theta$, to reduced-form parameters, $\delta$. We assume the reduced-form parameters, $\delta$, are identified, while identification of the structural parameters, $\theta$, is determined by the mapping, $\delta(\cdot)$. That is, a given parameter value $\theta^*$ is identified if $\delta(\theta^*) = \delta(\theta)$ implies $\theta^* = \theta$. In order to analyze the identification of $\theta$, we place more structure on the mapping, $\delta(\cdot)$. The structural parameters, $\theta$, are composed of two subvectors, $\theta = (\pi, \beta)$. The following assumption says that $\pi$ is also a subvector of $\delta$. That is, $\delta$ is composed of two subvectors, $\delta = (\pi, \tau)$, where $\pi$ is common to both $\theta$ and $\delta$. This focuses the identification analysis on $\beta$ and $\tau$.

**Assumption 1.** There exists a twice continuously differentiable function $\tau : \Theta \rightarrow \mathbb{R}^d$, such that $\delta(\pi, \beta) = (\pi, \tau(\pi, \beta))$ for all $(\pi, \beta) \in \Theta$.

**Remarks:** 2.1. Weak identification is often defined by some structure that isolates the source of the identification loss. For example, see Assumption C in Stock

\footnote{We let $\delta$ denote both the mapping and the reduced-form parameters themselves.}
and Wright (2000) or Assumption A in Andrews and Cheng (2012). Assumption 1 is a new type of identification structure that isolates $\tau(\pi, \beta)$ as a mapping that determines identification of $\beta$. For a fixed value of $\pi$, $\tau(\pi, \beta)$ may not be invertible for $\beta$, indicating that $\beta$ is not identified.

2.2. Verifying Assumption 1 often requires a reparameterization. In this paper we provide closed-form reparameterizations for the examples. However, solving numerically for a reparameterization satisfying Assumption 1 is also possible. Han and McCloskey (2019) provide a general strategy for finding a reparameterization based on solving a sequence of differential equations.

In the remainder of this section, we describe two example factor models and demonstrate how to reparameterize them to satisfy Assumption 1.

Example 1: One Latent Factor. A factor model states that a $p$-vector of observed variables, $X_i$, is related to an $m$-vector of unobserved factors, $f_i$, by the equation,

$$X_i = \Lambda f_i + \epsilon_i,$$

where $\epsilon_i$ is a $p$-vector of unobserved errors and $\Lambda$ is a $p \times m$ matrix of coefficients called factor loadings.\(^4\) Let $\Sigma$ be an $m \times m$ symmetric and positive definite matrix that denotes the covariance matrix of the factors and let $\Phi$ be a $p \times p$ diagonal matrix that denotes the covariance matrix of the errors. We assume that $\epsilon_i$ is uncorrelated with $f_i$.

When there is only one factor and three observed variables, the factor loadings matrix is given by

$$\Lambda = \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix},$$

where $\lambda_1 = 1$ normalizes the scale of the factor in terms of $X_{1i}$. The other parameters in the model are the (scalar) variance of the factor, $\sigma^2$, and the variances of the errors, $\Phi = diag(\phi_1, \phi_2, \phi_3)$.

Identification is determined by the covariance matrix of $X_i$:

$$Cov(X_i) = \Lambda \Sigma \Lambda' + \Phi.$$  \(^{(2.2)}\)

\(^4\)For a classical treatment of factor models, see Lawley and Maxwell (1971) or Anderson (1984).
The covariance matrix of $X_i$ is identified and can be written as a nonlinear function of the parameters. The parameters are identified if and only if this nonlinear function can be inverted. We reparameterize the model so the elements of $\text{Cov}(X_i)$ are the reduced-form parameters, $\delta$. Writing out equation (2.2),

$$
\begin{bmatrix}
\omega_1 & \rho_2 & \rho_3 \\
\rho_2 & \omega_2 & \tau \\
\rho_3 & \tau & \omega_3
\end{bmatrix} =
\begin{bmatrix}
\sigma^2 + \phi_1 & \lambda_2 \sigma^2 & \lambda_3 \sigma^2 \\
\lambda_2 \sigma^2 & \lambda_2^2 \sigma^2 + \phi_2 & \lambda_2 \lambda_3 \sigma^2 \\
\lambda_3 \sigma^2 & \lambda_2 \lambda_3 \sigma^2 & \lambda_3^2 \sigma^2 + \phi_3
\end{bmatrix},
$$

where $\omega_j = \text{Var}(X_{ji})$, $\rho_j = \text{Cov}(X_{1i}, X_{ji})$, and $\tau = \text{Cov}(X_{2i}, X_{3i})$.

We use equation (2.3) to define the reparameterization: $(\lambda_2, \lambda_3, \sigma^2, \phi_1, \phi_2, \phi_3) \mapsto (\rho_2, \rho_3, \omega_1, \omega_2, \omega_3, \beta)$, where $\beta = \sigma^2$. Cox (2022b) proposes this reparameterization to analyze weak identification without bounds. The reparameterized model satisfies Assumption 1 by setting $\pi = (\rho_2, \rho_3, \omega_1, \omega_2, \omega_3)$). The function $\tau(\pi, \beta)$ is found by looking at the unused equation in (2.3), associated with $\tau$, written in terms of the new parameters:

$$
\tau(\pi, \beta) = \rho_2 \rho_3 \beta^{-1}.
$$

$\beta$ is not identified when either $\rho_2 = 0$ or $\rho_3 = 0$. This corresponds to $\lambda_2 = 0$ or $\lambda_3 = 0$. Thus, the variance of the factor is identified if and only if all three observed variables have nonzero factor loadings. In this paper, we assume $\rho_2 \neq 0$ in order to focus on the effect of $\rho_3$.

When $\rho_3 = 0$, $\beta$ is still partially identified by bounds. These bounds come from the nonnegativity of the variance parameters. The inequality $\phi_1 \geq 0$, rewritten in the new parameters, is $\beta \leq \omega_1$, which is an upper bound on $\beta$. Similarly, $\phi_2 \geq 0$, rewritten in the new parameters, is $\beta \geq \rho_2^2/\omega_2$. Intuitively, the variance of the factor cannot be larger than the variance of the normalizing variable, and the variance of the factor must be large enough to explain the covariance between $X_{1i}$ and $X_{2i}$. Thus, the identified set for $\beta$, when $\rho_3 = 0$, is the interval,

$$
\left[ \rho_2^2/\omega_2, \omega_1 \right].
$$

As the variances of the errors, $\phi_j$, get smaller, so that the factors are measured with less error, the interval shrinks. Under weak identification, these bounds provide a

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5Anderson and Rubin (1956) originally derived this condition for identification in factor models.
valuable source of information about $\beta$. \hfill \square

**Example 2: Two Latent Factors.** Consider equation (2.1) when there are two factors ($m = 2$) and five observed variables ($p = 5$). In this case, the matrix of factor loadings is given by

$$
\Lambda = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\lambda_{31} & \lambda_{32} \\
\lambda_{41} & \lambda_{42} \\
\lambda_{51} & \lambda_{52}
\end{bmatrix},
$$

where the first two rows impose a standard normalization in the factor model literature. The other parameters in the model are the variance of the first factor, $\sigma_1^2$, the variance of the second factor, $\sigma_2^2$, the covariance between the two factors, $\sigma_{12}$, and the variances of the error terms, $\phi_j$ for $j \in \{1, 2, \ldots, 5\}$. In this paper, we assume $\lambda_{31} \neq 0$, $\lambda_{41} \neq 0$, $\lambda_{32} \neq 0$, and $\lambda_{12} \neq 0$. Intuitively, this says that the third and fourth observed variables each provide some information about both factors. This assumption allows us to focus on the main source of identification loss.

As before, identification is determined by the covariance matrix of $X_i$ using equation (2.2). The following equations expand equation (2.2) element by element:

- $\omega_1 = \text{Var}(X_{1i}) = \sigma_1^2 + \phi_1$
- $\omega_2 = \text{Var}(X_{2i}) = \sigma_2^2 + \phi_2$
- $\omega_3 = \text{Var}(X_{3i}) = \lambda_{31}^2 \sigma_1^2 + 2\lambda_{31} \lambda_{32} \sigma_{12} + \lambda_{32}^2 \sigma_2^2 + \phi_3$
- $\omega_4 = \text{Var}(X_{4i}) = \lambda_{41}^2 \sigma_1^2 + 2\lambda_{41} \lambda_{42} \sigma_{12} + \lambda_{42}^2 \sigma_2^2 + \phi_4$
- $\omega_5 = \text{Var}(X_{5i}) = \lambda_{51}^2 \sigma_1^2 + 2\lambda_{51} \lambda_{52} \sigma_{12} + \lambda_{52}^2 \sigma_2^2 + \phi_5$
- $\rho_{31} = \text{Cov}(X_{1i}, X_{3i}) = \lambda_{31} \sigma_1^2 + \lambda_{32} \sigma_{12}$
- $\rho_{32} = \text{Cov}(X_{2i}, X_{3i}) = \lambda_{31} \sigma_{12} + \lambda_{32} \sigma_2^2$
- $\rho_{41} = \text{Cov}(X_{1i}, X_{4i}) = \lambda_{41} \sigma_1^2 + \lambda_{42} \sigma_{12}$
- $\rho_{42} = \text{Cov}(X_{2i}, X_{4i}) = \lambda_{41} \sigma_{12} + \lambda_{42} \sigma_2^2$
- $\rho_{51} = \text{Cov}(X_{1i}, X_{5i}) = \lambda_{51} \sigma_1^2 + \lambda_{52} \sigma_{12}$
- $\rho_{52} = \text{Cov}(X_{2i}, X_{5i}) = \lambda_{51} \sigma_{12} + \lambda_{52} \sigma_2^2$
- $\chi = \text{Cov}(X_{3i}, X_{4i}) = \lambda_{31} \lambda_{41} \sigma_1^2 + (\lambda_{31} \lambda_{42} + \lambda_{32} \lambda_{41}) \sigma_{12} + \lambda_{32} \lambda_{42} \sigma_2^2$. 


Cox (2022b) uses the above equations, together with setting $\beta = \sigma^2_2$, to define a reparameterization to analyze weak identification without bounds. Cox (2022b) shows that this reparameterization is well-defined and invertible.

This reparameterization allows us to write the covariance of $X_i$ as

$$
\text{Cov}(X_i) = \begin{bmatrix}
\omega_1 & \sigma_{12} & \rho_{31} & \rho_{41} & \rho_{51} \\
\sigma_{12} & \omega_2 & \rho_{32} & \rho_{42} & \rho_{52} \\
\rho_{31} & \rho_{32} & \omega_3 & \chi & \tau_1(\pi, \beta) \\
\rho_{41} & \rho_{42} & \chi & \omega_4 & \tau_2(\pi, \beta) \\
\rho_{51} & \rho_{52} & \tau_1(\pi, \beta) & \tau_2(\pi, \beta) & \omega_5
\end{bmatrix},
$$

where $\rho = (\rho_{31}, \rho_{32}, \rho_{41}, \rho_{42}, \rho_{51}, \rho_{52}), \omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5), \pi = (\rho, \omega, \sigma_{12}, \chi)$, and

$$
\text{Cov}(X_{3i}, X_{5i}) = \tau_1(\pi, \beta) = \frac{\rho_{32}(\rho_{52}\rho_{41} - \rho_{51}\rho_{42}) + \chi(\beta \rho_{51} - \sigma_{12}\rho_{52})}{\beta \rho_{41} - \sigma_{12}\rho_{42}},
$$

$$
\text{Cov}(X_{4i}, X_{5i}) = \tau_2(\pi, \beta) = \frac{\rho_{42}(\rho_{52}\rho_{31} - \rho_{51}\rho_{32}) + \chi(\beta \rho_{51} - \sigma_{12}\rho_{52})}{\beta \rho_{31} - \sigma_{12}\rho_{32}}.
$$

The parameter $\beta$ is identified as long as either $\tau_1(\pi, \beta)$ or $\tau_2(\pi, \beta)$ can be inverted for $\beta$. Proposition 1 in Cox (2022b) implies that this occurs if and only if $\rho_{52}\rho_{41} - \rho_{51}\rho_{42} \neq 0$ or $\rho_{52}\rho_{31} - \rho_{51}\rho_{32} \neq 0$.\footnote{Relative to Section 4 in Cox (2022b), we have added the assumptions $\lambda_3 \neq 0$ and $\lambda_4 \neq 0$. These imply that $\rho_{31}\rho_{42} - \chi \sigma_{12} = \lambda_{31}\lambda_{42}(\sigma_1^2\sigma_2^2 - \sigma_{12}^2) \neq 0$ and $\rho_{32}\rho_{41} - \chi \sigma_{12} = \lambda_{32}\lambda_{41}(\sigma_1^2\sigma_2^2 - \sigma_{12}^2) \neq 0$.} In terms of the original parameters, identification is lost when $\lambda_{52}\lambda_{41} - \lambda_{51}\lambda_{42} = 0$ and $\lambda_{52}\lambda_{31} - \lambda_{51}\lambda_{32} = 0$. This says that the rows of the factor loadings for variables three through five are collinear with each other, and there is therefore not enough separate variation to distinguish the two factors.

When $\rho_{52}\rho_{41} - \rho_{51}\rho_{42} = 0$ and $\rho_{52}\rho_{31} - \rho_{51}\rho_{32} = 0$, $\beta$ is still partially identified by bounds. These bounds come from nonnegativity of the variances of the errors. In this example, the bounds are more complicated than with one factor. They are given
by $\ell(\pi, \beta) \leq 0$, where

$$
\ell_j(\pi, \beta) = \begin{cases} 
\chi \sigma_{12}^2 - \sigma_{12}(\rho_{31}\rho_{42} + \rho_{32}\rho_{41}) + \beta \rho_{41}\rho_{31} - \omega_1 & \text{if } j = 1 \\
\beta - \omega_2 & \text{if } j = 2 \\
\rho_{31}(\chi \beta - \rho_{32}\rho_{42}) + \rho_{32}(\rho_{32}\rho_{41} - \chi \sigma_{12}) - \omega_3 & \text{if } j = 3 \\
\rho_{31}(\chi \beta - \rho_{32}\rho_{42}) + \rho_{42}(\rho_{31}\rho_{42} - \chi \sigma_{12}) - \omega_4 & \text{if } j = 4.
\end{cases} \quad (2.5)
$$

One advantage of the results in Section 3 is that they hold for any identified set with a Lipschitz boundary. Thus, a tractable characterization of the identified set is not needed.\footnote{Even a closed-form characterization of the bounds, as presented here, is not needed. As long as the bounds can be written as $\ell(\theta) \leq 0$, where the function $\ell(\theta)$ can be calculated numerically, the following theory can be used.}

Any model that combines weak identification with bounds is a potential application. An earlier version of this paper, Cox (2020b), shows that Assumption 1 can be satisfied in other kinds of models, including a linear instrumental variables model, a structural vector autoregression, and a model of household expenditures. For brevity, we focus on factor models as the running examples in this paper.

### 3 Estimation Limit Theory

This section states conditions for limit theory of an estimator under weak identification with bounds within the class of minimum distance models introduced in Section 2. Weak identification is defined by the limit of $\tau(\pi, \beta)$ evaluated at a sequence of parameter values that is indexed by the sample size. The conditions are carefully formulated to characterize the influence of the bounds on the limit theory. Section 4 translates this limit theory to the QLR statistic for the purpose of inference.

Let $P$ denote the distribution of a sample, $W_1, ..., W_n$, indexed by the sample size, $n$. Let $P = P_{\theta, \xi}$ depend on the structural parameters, $\theta$, and an additional, possibly infinite dimensional parameter, $\xi$. We fix a sequence of true parameter values, $\theta_n = (\pi_n, \beta_n) \to \theta_* = (\pi_*, \beta_*)$ and $\xi_n \to \xi_*$, indexed by the sample size. Throughout this section, limits are taken as $n \to \infty$ and probabilities are calculated using $P = P_{\theta_n, \xi_n}$. 
3.1 Consistency

This subsection defines the minimum distance estimator and states sufficient conditions for a type of consistency result. Let \( \Theta \) be an open set containing \( \Theta \). Suppose \( \tau(\cdot) \) and \( \delta(\cdot) \) can be extended to functions on \( \tilde{\Theta} \). Let \( Q_n(\theta) \) denote a real-valued data-dependent objective function on \( \tilde{\Theta} \). We take the estimator, \( \hat{\theta}_n \), to be a random vector in \( \Theta \) satisfying

\[
Q_n(\hat{\theta}_n) \leq \inf_{\theta \in \Theta} Q_n(\theta) + o_p(1).
\]

(3.1)

We write \( \hat{\theta}_n = (\hat{\pi}_n, \hat{\beta}_n) \).

This definition of \( \hat{\theta}_n \) imposes the bounds through the boundary of \( \Theta \). Typically, we have \( \Theta = \{ \theta \in \mathbb{R}^d | \ell(\theta) \leq 0 \} \), where a vector-valued function \( \ell(\theta) \) defines the bounds. This is a very general way to incorporate bounds. In particular, it allows the identified set to be an intersection of bounds, and it does not require a characterization of the extreme points of the identified set for \( \beta \) as a function of \( \pi \).

The following assumption says that \( Q_n(\theta) \) approximately depends on \( \theta \) only through the reduced-form parameters, \( \delta \).

**Assumption 2.** \( T_n(\delta) \) is a real-valued data-dependent function on \( \delta(\tilde{\Theta}) \) that is almost surely twice differentiable and satisfies

- \( a \) \( \sup_{\theta \in \tilde{\Theta}} |Q_n(\theta) - T_n(\delta(\theta))| = o_p(1) \), and
- \( b \) for every sequence of positive constants, \( \eta_n \to 0 \),

\[
\sup_{\theta \in \tilde{\Theta} : \|\delta(\theta) - \delta(\theta_n)\| \leq \eta_n} \frac{Q_n(\theta) - T_n(\delta(\theta))}{(1 + \sqrt{n}\|\delta(\theta) - \delta(\theta_n)\|)^2} = o_p(n^{-1}).
\]

**Remark: 3.1.** Assumption 2 is satisfied without approximation \( (Q_n(\theta) \equiv T_n(\delta(\theta))) \) in Examples 1 and 2 and any minimum distance objective function. Furthermore, Assumption 2 can also be satisfied with other types of objective functions, including GMM and maximum likelihood, as long as appropriate reduced-form parameters can be defined.

The following assumption ensures the reduced-form parameters are identified and that \( T_n(\delta) \) can consistently estimate them.

**Assumption 3.** There exists a nonstochastic real-valued function, \( T(\delta) \), continuous on \( \delta(\tilde{\Theta}) \), such that
(a) \( \sup_{\delta \in \delta(\hat{\Theta})} |T_n(\delta) - T(\delta)| \to_p 0 \), and

(b) for every neighborhood, \( U \), of \( \delta_* = \delta(\theta_*) \),

\[
\inf_{\delta \in \delta(\hat{\Theta})/U} T(\delta) - T(\delta_*) > 0.
\]

Assumption 3 is sufficient for consistent estimation of the reduced form parameters, including \( \pi \). However, a stronger consistency result is desired—one that guarantees concentration of the support of the distribution of \( \hat{\beta}_n \) on the identified set. For this purpose, the next assumption places a restriction on the shape of the parameter space. For any \( \Pi \subseteq \mathbb{R}^{d_\pi} \), let

\[
\mathcal{B}(\Pi) = \{ \beta \in \mathbb{R}^{d_\beta} | (\pi, \beta) \in \Theta \text{ for some } \pi \in \Pi \},
\]

which defines the cross-section of the parameter space. Let \( \mathcal{B}_* = \mathcal{B}(\pi_*) \) denote the identified set for \( \beta \) when \( \tau(\pi_*, \beta) \) is not invertible for \( \beta \).\(^8\) Recall \( d_H(A, B) \) denotes the Hausdorff distance between the sets \( A \) and \( B \).

**Assumption 4.** (a) \( \Theta \) is closed, \( \mathcal{B}_* \) is compact, and

(b) there exists an \( M > 0 \) such that for all \( \pi_1 \) and \( \pi_2 \) in a neighborhood of \( \pi_* \),

\[
d_H(\mathcal{B}(\pi_1), \mathcal{B}(\pi_2)) \leq M \| \pi_1 - \pi_2 \|.
\]

**Remark: 3.2.** Part (b) assumes the cross-section for \( \beta \) is locally Lipschitz continuous in the Hausdorff distance as a function of \( \pi \). Importantly, this does not require differentiability of the bounds, thus permitting kink points associated with the intersection of multiple bounds. When \( \mathcal{B}(\pi) \) is an interval, as in Examples 1 and 2, Assumption 4(b) only requires the endpoints of the interval to be locally Lipschitz continuous in \( \pi \).

The following lemma shows that \( \hat{\pi}_n \to_p \pi_* \) and that the support of the distribution of \( \hat{\beta}_n \) concentrates on \( \mathcal{B}_* \).

**Lemma 1.** Under Assumptions 1, 2(a), 3, and 4, there exists a sequence of neighborhoods of \( \pi_* \), \( \Pi_n \), such that \( \mathcal{B}_n := \mathcal{B}(\Pi_n) \) is compact and satisfies

\(^8\)For simplicity, we write \( \mathcal{B}(\pi) \) for \( \mathcal{B}(\{\pi\}) \).
Example 1 and Example 2, Continued. We estimate the factor models using a minimum distance objective function that is quadratic in the reduced-form parameters. Let $\hat{\Omega}$ denote an estimator of the covariance matrix of $X_i$, and let

$$Q_n(\theta) = (\delta(\theta) - vech(\hat{\Omega}))' \hat{W}_n(\delta(\theta) - vech(\hat{\Omega}))/2,$$  \hspace{1cm} (3.2)

denote the objective function, where $\hat{W}_n$ converges in probability to a positive definite weight matrix.\(^9\)

The limit theory is driven by the limit theory of $\hat{\Omega}$. We assume $\hat{\Omega}$ is consistent and asymptotically normal along the sequence $P_{\theta_n, \xi_n}$. That is, there exists a $d_\delta \times d_\delta$ positive definite matrix, $V$, such that $\sqrt{n}(vech(\hat{\Omega}) - \delta(\theta_n)) \rightarrow_d N(0, V)$. This can be verified by a triangular-array central limit theorem, for example if $X_i$ is independent across $i$ with $4 + \epsilon$ finite moments. Alternative central limit theorems that allow dependence are also possible. Let $\hat{V}$ be a consistent estimator of $V$. If $\hat{W}_n = \hat{V}^{-1}$, then the objective function is optimally weighted, but this is not required.

3.2 Types of Sequences

The limit theory depends on what type of sequence we have. There are three types of sequences, strong, semi-strong, and weak, referring to the strength of identification. The relevant quantity for determining identification is $\tau(\pi_n, \beta) - \tau(\pi_n, \beta_n)$, viewed as a function of $\beta$. The next assumption defines the different types of sequences based on the rate of convergence of $\tau(\pi_n, \beta) - \tau(\pi_n, \beta_n)$. Let $\partial_\pi \tau(\pi, \beta)$ and $\partial_\beta \tau(\pi, \beta)$ denote the $d_\pi \times d_\pi$ and $d_\pi \times d_\beta$ matrices of derivatives of $\tau(\pi, \beta)$ with respect to $\pi$ and $\beta$, respectively.

Assumption 5(S). \hspace{1cm} (a) $\tau(\pi, \beta) = \tau(\pi, \beta_*)$ implies $\beta = \beta_*$. \hspace{1cm} (b) $C := \partial_\beta \tau(\pi, \beta_*)$ has full rank $d_\beta$.

Assumption 5(SS). There exists a sequence of positive constants, $a_n$, such that $a_n \rightarrow \infty$ and $n^{-1/2}a_n \rightarrow 0$ as $n \rightarrow \infty$.

\(^9\)The $vech(\cdot)$ function vectorizes symmetric matrices. We take the ordering of the elements in $vech$ to coincide with $\delta$.  

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(a) There exists a function, \(c(\beta)\), such that
\[
a_n(\tau(\pi_n, \beta) - \tau(\pi_n, \beta_n)) \to c(\beta),
\]
uniformly over \(\beta \in B_1\), and \(c(\beta) = 0\) implies \(\beta = \beta^*\).\(^{10}\)

(b) There exists a matrix-valued function, \(C(\beta)\), such that
\[
a_n \partial_\beta \tau(\pi_n, \beta) \to C(\beta),
\]
uniformly over \(\beta\) in a neighborhood of \(\beta^*\), and \(C := C(\beta^*)\) has full rank \(d_\beta\).

**Assumption 5(W).** There exists a function, \(c(\beta)\), such that
\[
\sqrt{n}(\tau(\pi_n, \beta) - \tau(\pi_n, \beta_n)) \to c(\beta),
\]
uniformly over \(\beta \in B_1\).

**Remarks: 3.3.** Assumption 5 characterizes different types of sequences according to the rate \(\tau(\pi_n, \beta) - \tau(\pi_n, \beta_n)\) converges to zero as a function of \(\beta \in B_1\). “S” stands for strongly identified, “SS” stands for semi-strongly identified, and “W” stands for weakly identified. Assumption 5(S) is satisfied when \(\beta\) is identified in the limit. Assumption 5(SS) is satisfied when \(\tau(\pi_n, \beta) - \tau(\pi_n, \beta_n)\) converges to zero slower than a \(\sqrt{n}\) rate, while Assumption 5(W) is satisfied when this quantity converges to zero at a \(\sqrt{n}\) rate or faster. Assumption 5(W) covers the important case that \(\beta\) is not identified for any \(n\). This corresponds to \(\pi_n = \pi^*\) for all \(n\). In this case, \(c(\beta) \equiv 0\).

3.4. In Assumption 5(SS), \(a_n\) indexes the weakness of the sequence. By extension, we can take \(a_n = 1\) for all \(n\) under Assumption 5(S) and \(a_n = \sqrt{n}\) for all \(n\) under Assumption 5(W).

**Example 1, Continued.** Recall \(\tau(\pi, \beta) = \rho_2 \rho_3/\beta\). Invertibility of this mapping for \(\beta\) depends on the value of \(\rho_3\). Accordingly, Assumption 5 characterizes sequences of \(\theta_n = (\rho_{2n}, \rho_{3n}, \omega_{1n}, \omega_{2n}, \omega_{3n}, \beta_n) \to \theta_* = (\rho_{2*}, \rho_{3*}, \omega_{1*}, \omega_{2*}, \omega_{3*}, \beta_*)\) based on the value of \(\rho_{3n}\). Strong sequences have \(\rho_{3*} \neq 0\). Weak sequences satisfy \(\sqrt{n} \rho_{3n} \to s_* \in \mathbb{R}\). These sequences satisfy Assumption 5(W) with \(c(\beta) = s_* \rho_{2*} (\beta^{-1} - \beta_*^{-1})\). Semi-strong sequences have \(\rho_{3*} = 0\) and satisfy \(\sqrt{n} \rho_{3n} \to \pm \infty\). These sequences

\(^{10}\)Here and throughout, \(B_1\) denotes the first element of the sequence \(B_n\).
satisfy Assumption 5(SS) with \(c(\beta) = \pm \rho_{2*}(\beta^{-1} - \beta_*^{-1})\), \(C(\beta) = \mp \rho_{2*}\beta^{-2}\), and \(a_n = |\rho_{3n}|^{-1}\).

Example 2, Continued. Recall invertibility of \(\tau_1(\pi, \beta)\) or \(\tau_2(\pi, \beta)\) for \(\beta\) depends on the values of \(\rho_{52}\rho_{41} - \rho_{51}\rho_{42}\) and \(\rho_{52}\rho_{31} - \rho_{51}\rho_{32}\). Accordingly, let \(s_1(\pi) = \rho_{52}\rho_{41} - \rho_{51}\rho_{42}\) and \(s_2(\pi) = \rho_{52}\rho_{31} - \rho_{51}\rho_{32}\). Assumption 5 characterizes sequences of \(\theta_n = (\pi_n, \beta_n) \rightarrow \theta_* = (\pi_*, \beta_*)\), where \(\pi_* = (\rho_*, \omega_*, \chi_*, \sigma_{12*})\) and \(\rho_* = (\rho_{31*}, \rho_{32*}, \rho_{41*}, \rho_{42*}, \rho_{51*}, \rho_{52*})\), based on the values of \(s_1(\pi_n)\) and \(s_2(\pi_n)\). Strong sequences have \(s_1(\pi_*) \neq 0\) or \(s_2(\pi_*) \neq 0\). Weak sequences satisfy \(\sqrt{n}s_1(\pi_n) \rightarrow s_{1*} \in \mathbb{R}\) and \(\sqrt{n}s_2(\pi_n) \rightarrow s_{2*} \in \mathbb{R}\). These sequences satisfy Assumption 5(W) with \(c(\beta) = (c_1(\beta), c_2(\beta))'\) and

\[
c_1(\beta) = s_1(\beta - \beta_*)(\chi_*\sigma_{12*} - \rho_{32*}\rho_{41*})(\beta\rho_{41*} - \sigma_{12*}\rho_{42*})^{-1}(\beta\rho_{41*} - \sigma_{12*}\rho_{42*})^{-1} (3.3) \\
c_2(\beta) = s_2(\beta - \beta_*)(\chi_*\sigma_{12*} - \rho_{31*}\rho_{42*})(\beta\rho_{31*} - \sigma_{12*}\rho_{32*})^{-1}(\beta\rho_{31*} - \sigma_{12*}\rho_{32*})^{-1}.
\]

Semi-strong sequences have \(s_1(\pi_*) = s_2(\pi_*) = 0\) and satisfy \(\sqrt{n}s_1(\pi_n) \rightarrow \pm \infty\) or \(\sqrt{n}s_2(\pi_n) \rightarrow \pm \infty\). If we let \(a_n = (s_1(\pi_n)^2 + s_2(\pi_n)^2)^{-1/2}\), these sequences can also be taken to satisfy \(a_n(s_1(\pi_n), s_2(\pi_n)) \rightarrow (s_{1*}, s_{2*})\). These sequences satisfy Assumption 5(SS) with

\[
C(\beta) = \begin{bmatrix}
s_1(\chi_*\sigma_{12*} - \rho_{32*}\rho_{41*})(\beta\rho_{41*} - \sigma_{12*}\rho_{42*})^{-2} \\
s_2(\chi_*\sigma_{12*} - \rho_{31*}\rho_{42*})(\beta\rho_{31*} - \sigma_{12*}\rho_{32*})^{-2}
\end{bmatrix},
\]

and \(c(\beta)\) given by (3.3).

\[\square\]

3.3 Convergence Rate

This section derives the convergence rate of the components of \(\hat{\theta}_n\) along the different types of sequences defined in Section 3.2. We first assume the reduced-form objective function is regular in the sense that it satisfies a quadratic expansion. Let \(\delta_n = \delta(\theta_n)\). For every \(\delta \in \delta(\hat{\theta})\), let

\[
R_n(\delta) = T_n(\delta) - T_n(\delta_n) - \partial_\delta T_n(\delta_n)'(\delta - \delta_n) - \frac{1}{2}(\delta - \delta_n)'\partial_{\delta\delta} T_n(\delta_n)(\delta - \delta_n), (3.4)
\]

where \(\partial_\delta T_n(\delta)\) denotes the first derivative vector and \(\partial_{\delta\delta} T_n(\delta)\) denotes the second derivative matrix of \(T_n(\delta)\). \(R_n(\delta)\) is the remainder term in a quadratic expansion of \(T_n(\delta)\) around \(\delta = \delta_n\).
**Assumption 6.** For every sequence of positive constants, \( \eta_n \to 0 \),

\[
\sup_{\delta \in \delta(\Theta); \|\delta - \delta_n\| \leq \eta_n} \frac{|n R_n(\delta)|}{(1 + \sqrt{n}\|\delta - \delta_n\|)^2} = o_p(1).
\]

In addition, we assume the derivatives of \( T_n(\delta) \) satisfy regular limits so the reduced-form model is locally asymptotically normal around \( \delta_n \).

**Assumption 7.**

(a) There exists a random vector, \( Y \), and a positive definite and symmetric \( d_\delta \times d_\delta \) matrix, \( V \), such that

\[
\sqrt{n} \partial_\delta T_n(\delta_n) \rightarrow_d Y \sim N(0, V).
\]

(b) There exists a positive definite and symmetric \( d_\delta \times d_\delta \) matrix, \( H \), such that

\[
\partial_{\delta\delta} T_n(\delta_n) \rightarrow_p H.
\]

**Remarks: 3.5.** Assumption 6 ensures the reduced-form objective function satisfies a quadratic expansion around \( \delta_n \). In Examples 1 and 2, Assumption 6 is satisfied without remainder (\( R_n(\delta) \equiv 0 \)) because \( T_n(\delta) \) is quadratic in \( \delta \). More generally, sufficient conditions for Assumption 6 are that \( T_n(\delta) \) is twice continuously differentiable almost surely and the second derivative is stochastically equicontinuous.  

**3.6.** Assumption 7(a) can be verified by a central limit theorem, and Assumption 7(b) can be verified by a law of large numbers. In Examples 1 and 2, if \( W \) denotes the probability limit of \( \hat{W}_n \), then Assumption 7 is satisfied with \( V = \hat{W} \hat{V} \hat{W} \) and \( H = W \). When \( W = \hat{V}^{-1} \), so the objective function is optimally weighted, \( H = V = \hat{V}^{-1} \).

**3.7.** These assumptions are reasonable for the reduced-form parameters because they are strongly identified. Analogous conditions on the structural parameters would need to be modified to account for the weak identification. See Assumption C1 in Andrews and Cheng (2012) for the type of quadratic expansion that would be needed on the structural parameters.

Under these assumptions, we can derive the rate of convergence of \( \hat{\theta}_n \) for each type of sequence.

**Lemma 2.** Under Assumptions 1, 2, 3, 4, 6, and 7, the following hold.

(a) Under Assumption 5(S), \( \sqrt{n}(\hat{\theta}_n - \theta_n) = O_p(1) \).

\[\text{See Andrews (1994) for conditions for stochastic equicontinuity in strongly identified models.}\]
Under Assumption 5(SS), \( \sqrt{n}(\hat{\pi}_n - \pi_n) = O_p(1) \) and \( \sqrt{n}(\hat{\beta}_n - \beta_n)/a_n = O_p(1) \).

Under Assumption 5(W), \( \sqrt{n}(\hat{\pi}_n - \pi_n) = O_p(1) \) and \( \hat{\beta}_n = O_p(1) \).

Remark: 3.8. In part (a), the rate is what we expect under strong identification. In part (c), under weak identification, \( \hat{\beta}_n \) is no longer consistent. In part (b), under Assumption 5(SS), we see that the rate depends on how weak the sequence is. As the sequence gets weaker \( (a_n \text{ diverges faster}) \) the rate of convergence gets slower until \( \hat{\beta}_n \) is not consistent.

The limit theory is determined by the limit of the objective function locally at the rate determined by Lemma 2. Let \( \psi = (\psi_1, \psi_2) \in \mathbb{R}^{d_\psi} \) denote local parameters, where \( \psi_1 \in \mathbb{R}^{d_\pi} \) and \( \psi_2 \in \mathbb{R}^{d_\beta} \). We define the local objective functions and their limits next. Let

\[
q_n^S(\psi) = Q_n(\theta_n + n^{-1/2}\psi) - Q_n(\theta_n)
\]

\[
q_n^{SS}(\psi_1, \psi_2) = Q_n(\pi_n + n^{-1/2}\psi_1, \beta_n + a_n n^{-1/2}\psi_2) - Q_n(\theta_n)
\]

\[
q_n^W(\psi_1, \beta) = Q_n(\pi_n + n^{-1/2}\psi_1, \beta) - Q_n(\theta_n)
\]

denote the local objective functions. We use the following notation to define the limiting objective functions. Let

\[
D(\theta) = \partial_\theta \delta(\theta) = \begin{bmatrix}
I_{d_\pi} & 0_{d_\pi \times d_\beta} \\
\partial_\pi \tau(\theta) & \partial_\beta \tau(\theta)
\end{bmatrix}
\]  

(3.5)

denote the derivative of the link function. Let \( D_* = D(\theta_*) \). Also let

\[
D_1(\theta) = \partial_\pi \delta(\theta) = \begin{bmatrix}
I_{d_\pi} \\
\partial_\pi \tau(\theta)
\end{bmatrix}
\]

denote the derivative only with respect to \( \pi \). The limits of the local objective functions under strong and semi-strong sequences depend on \( J = D'_* H D_* \) and \( Z = -J^{-1} D'_* Y \).\(^{12}\)

\(^{12}\)The limit under semi-strong sequences uses these formulas with

\[
D(\theta_*) = \begin{bmatrix}
I_{d_\pi} & 0_{d_\pi \times d_\beta} \\
\partial_\pi \tau(\theta_*) & C
\end{bmatrix},
\]  

(3.6)

where \( C \) is defined in Assumption 5(SS).
The limit of the local objective function under weak sequences depends on

\[ J_{11}(\beta) = D_1(\pi_*, \beta)'HD_1(\pi_*, \beta) \]
\[ g(\beta) = \begin{pmatrix} 0_{d_r \times 1} \\ c(\beta) \end{pmatrix} \]
\[ Z_1(\beta) = -J_{11}^{-1}(\beta)D_1(\pi_*, \beta)'(Y + Hg(\beta)) \]
\[ \tilde{q}^W(\beta) = g(\beta)'Hg(\beta)/2 + Y'g(\beta) - Z_1(\beta)'J_{11}(\beta)Z_1(\beta)/2, \hspace{1cm} (3.7) \]

viewed as functions of \( \beta \in \mathcal{B}_1 \). Let

\[ q^S(\psi) = (Z - \psi)'J(Z - \psi)/2 - Z'JZ/2 \]
\[ q^W(\psi_1, \beta) = (Z_1(\beta) - \psi_1)'J_{11}(\beta)(Z_1(\beta) - \psi_1)/2 + \tilde{q}^W(\beta) \]

denote the limits of the local objective functions. The limit is the same for strong and semi-strong sequences, as the following lemma shows. Recall that \( \Rightarrow_K \) denotes weak convergence in the space of bounded functions on \( K \) with the uniform norm. The following lemma shows weak convergence of the local objective functions uniformly over all compact subsets of their domain.

**Lemma 3.** Under Assumptions 1, 2, 3, 4, 6, and 7, the following hold.

(a) Under Assumption 5(S), for all compact \( K \subseteq \mathbb{R}^{d_\theta} \), \( nq^S_n(\psi) \Rightarrow_K q^S(\psi) \).

(b) Under Assumption 5(SS), for all compact \( K \subseteq \mathbb{R}^{d_\theta} \), \( nq^{SS}_n(\psi) \Rightarrow_K q^S(\psi) \).

(c) Under Assumption 5(W), for all compact \( K \subseteq \mathbb{R}^{d_\pi \times \mathcal{B}_1} \), \( nq^W_n(\psi_1, \beta) \Rightarrow_K q^W(\psi_1, \beta) \).

**Remark:** 3.9. The limit under strong and semi-strong sequences is the same, and is a quadratic function of \( \psi \). Under weak sequences, the limit is a quadratic function of \( \psi_1 \), but the coefficients in the quadratic function depend on \( \beta \) in a non-quadratic way. This is the source of the nonstandard distribution under weak identification. \( \square \)

### 3.4 The Influence of the Boundary

We allow the sequence of parameters to be on or approach the boundary of the parameter space. When this happens, the limit theory depends on the setwise limit of the local parameter space.
The type of setwise limit that we use is called Kuratowski convergence. We say that a sequence of sets, $\Psi_n$, satisfies $\Psi_n \rightarrow_K \Psi$ for some set, $\Psi$, if for every compact set, $K$, $\max\left(\tilde{d}(A \cap K, B), \tilde{d}(B \cap K, A)\right) \rightarrow 0$. Kuratowski convergence is a very weak way to define setwise convergence, and in particular is weaker than Hausdorff convergence.

**Assumption 8.**

(a) For a closed set, $\Psi \subseteq \mathbb{R}^d$, $\{(\sqrt{n}(\pi - \pi_n), \sqrt{n}(\beta - \beta_n)/a_n) | (\pi, \beta) \in \Theta\} \rightarrow_K \Psi$.

(b) $\Psi$ is convex.

**Remarks:** 3.10. When $\theta_n$ is converging to the interior of $\Theta$, Assumption 8 is satisfied with $\Psi = \mathbb{R}^d$. When $\theta_n$ is on the boundary, then Assumption 8 is satisfied with $\Psi$ being a cone. When $\theta_n$ is local to the boundary at the $\sqrt{n}$ rate, Assumption 8 is satisfied with $\Psi$ being the translation of a cone.

3.11. Andrews (1999) uses a related definition of setwise convergence that applies when the true parameter is a fixed point on the boundary. The type of convergence used in Assumption 8 generalizes the convergence in Andrews (1999) to allow the sequence to drift toward the boundary.

3.12. Assumption 8 is not needed for the results under Assumption 5(W). Furthermore, Assumption 8(b) is only needed for the estimation results in Theorem 1, and not for the inference results in Section 4.14

Assumption 8 is not needed for weak sequences. The corresponding result for weak sequences follows from Assumption 4.

**Lemma 4.** If $\Theta$ satisfies Assumption 4, then $\{(\sqrt{n}(\pi - \pi_n), \beta) | (\pi, \beta) \in \Theta\} \rightarrow_K \mathbb{R}^{d_\pi} \times \mathcal{B}_\pi$.

**Remark:** 3.13. Lemma 4 characterizes the influence of the bounds on the limit theory. For weak sequences, we know from Lemma 2 that the local parameters are derived by rescaling only the $\pi$ directions and not the $\beta$ directions. This stretches out the local parameter space so that the limit is a product between $\mathbb{R}^{d_\pi}$ and $\mathcal{B}_\pi$. This

---

13 Recall $\tilde{d}(A, B) = \max\left(\sup_{a \in A} \inf_{b \in B} d(a, b), 0\right)$. Lemma A.1 in the supplemental materials shows that this definition of Kuratowski convergence is the same as the usual definition given in, for example, Definition 1.1.1 in Aubin and Frankowska (1990).

14 Remarkably, Assumption 8(a) is not needed for Theorem 3, below, either.
effectively redirects the bounds so that they only enforce the \( \beta \) directions, under weak sequences.

**Example 1 and Example 2, Continued.** In Example 1, the boundary is given by \( \ell(\theta) \leq 0 \), where \( \ell(\theta) = (\beta - \omega_1, \rho_2^2 - \omega_2 \beta)' \). In Example 2, the boundary is given by \( \ell(\theta) \leq 0 \), where \( \ell(\theta) \) is defined in (2.5). Suppose \( \sqrt{n \ell(\theta_n)} / a_n \to \bar{\ell} \), where \( \bar{\ell} \) may have some elements equal to \(-\infty\). In both examples, Assumption 8 is satisfied with \( \Psi = \{ \psi \in \mathbb{R}^{d_\theta} | \bar{\ell} + \partial_\theta \ell(\theta^*_n) \psi \leq 0 \} \) under strong sequences and \( \Psi = \{ (\psi_1, \psi_2) \in \mathbb{R}^{d_\theta} | \bar{\ell} + \partial_\beta \ell(\theta_n) \psi_2 \leq 0 \} \) under semi-strong sequences by linearizing \( \ell(\theta) \) around \( \theta^*_n \). 

### 3.5 Limit Theory

To state the limit theory, one additional assumption is needed.

**Assumption 9.** Almost surely, the sample path of \( \hat{q}^W(\beta) \) is uniquely minimized over \( B_\ast \). Denote the unique minimizing point by \( \beta_{MIN} \).

**Remarks:** 3.14. As we know from Lemma 3, \( \hat{q}^W(\beta) \) is the limit of the local objective function under weak sequences after concentrating out the \( \psi_1 \) parameters: \( \hat{q}^W(\beta) = \inf_{\psi_1, \beta'} q^W(\psi_1, \beta) \). As stated below, \( \beta_{MIN} \) is the limit of \( \hat{\beta}_n \) under weak sequences. Assumption 9 ensures the limit is well-defined. This assumption is only used for the estimation results. It is not needed for the inference results in Section 4.

3.15. Theorem 4 in Cox (2020a) provides sufficient conditions for verifying Assumption 9 based on the derivative of \( \hat{q}^W(\beta) \) with respect to \( Y \). We use a modification of that theorem to verify Assumption 9 in Examples 1 and 2.

The following theorem states the limit theory for \( \hat{\theta}_n \) under all three types of sequences.

**Theorem 1.** (a) Under Assumptions 1, 2, 3, 4, 5(S), 6, 7, and 8,

\[
\sqrt{n}(\hat{\theta}_n - \theta_n) \to_d \arg\min_{\psi \in \Psi} q^S(\psi).
\]

(b) Under Assumptions 1, 2, 3, 4, 5(SS), 6, 7, and 8,

\[
\left( \sqrt{n}(\hat{\pi}_n - \pi_n), \sqrt{n}(\hat{\beta}_n - \beta_n) / a_n \right) \to_d \arg\min_{\psi \in \Psi} q^S(\psi).
\]
(c) Under Assumptions 1, 2, 3, 4, 5(W), 6, 7, and 9,

\[
\left( \frac{\sqrt{n}(\hat{\pi}_n - \pi_n)}{\hat{\beta}_n} \right) \rightarrow_d \text{argmin}_{(\psi_1, \beta) \in \mathbb{R}^{d_\psi} \times \mathcal{B}_x} q^W(\psi_1, \beta) = \left( Z_{1}(\beta_{MIN}) \right).
\]

Remarks: 3.16. When \( \theta_n \) converges to a point on the interior of \( \Theta \), Assumption 8 is satisfied with \( \Psi = \mathbb{R}^{d_\theta} \), and the limits in parts (a) and (b) simplify to \( Z = -J^{-1}D'Y \sim N(0, J^{-1}D'VD_*J^{-1}) \), which coincides with the usual asymptotic normal distribution. Part (b) shows that the limit theory is the same under Assumption 5(SS) as under Assumption 5(S) with one difference: the rate of convergence of \( \hat{\beta}_n \) is diminished by \( a_n \), the weakness of the sequence.

3.17. Part (c) shows that nonstandard limit theory applies under Assumption 5(W). \( \hat{\beta}_n \) is inconsistent and converges to the argmin of a stochastic process over the identified set, \( \beta_{MIN} \). \( \hat{\pi}_n \) is \( \sqrt{n} \)-consistent, but with a nonstandard limiting distribution that can be characterized as a Gaussian stochastic process function of \( \beta_{MIN} \).

3.18. The proof of Theorem 1 relies on a new argmax theorem from Cox (2022a). The new argmax theorem allows the domain (the local parameter space) to change with the sample size. Importantly, it only requires Kuratowski convergence of the domain. This new technical tool is the key for proving Theorem 1.

4 Inference Recommendation

This section provides a recommendation for testing hypotheses of the form

\[ H_0 : r_1(\pi) = 0 \text{ and } r_2(\beta) = 0. \] (4.1)

This type of hypothesis is very flexible and includes subvector inference as a special case. The simulations of Examples 1 and 2 test \( H_0 : \beta = \beta_0 \), for some hypothesized value, \( \beta_0 \). This amounts to testing the variance of the factor in Example 1 and the variance of the second factor in Example 2. Let \( r(\pi, \beta) = (r_1(\pi), r_2(\beta)) \), and let \( d_r = d_{r_1} + d_{r_2} \) denote the number of restrictions.

We test \( H_0 \) using a quasi-likelihood ratio (QLR) statistic. Under weak identification, this statistic has nonstandard limit theory. Let \( \Theta^r = \{ \theta \in \Theta | r(\theta) = 0 \} \) denote
the restricted parameter space. Define the QLR statistic to be

\[ QLR_n = 2n \left( \inf_{\theta \in \Theta^r} Q_n(\theta) - \inf_{\theta \in \Theta} Q_n(\theta) \right). \]

We test \( H_0 \) by comparing \( QLR_n \) to a critical value based on the asymptotic distribution of \( QLR_n \) under weak identification.

Let \( \theta_n = (\pi_n, \beta_n) \to \theta_* = (\pi_*, \beta_*) \) and \( \xi_n \to \xi_* \) be sequences of parameters, as in the previous section. Let \( \mathcal{B}^r(\pi) = \{ \beta \in \mathcal{B}(\pi) | r_2(\beta) = 0 \} \) denote the restricted cross section for \( \beta \). Then, \( \mathcal{B}_*^r = \mathcal{B}^r(\pi_*) \) is the subset of the identified set that satisfies the null hypothesis. The limit theory depends on whether \( \mathcal{B}_*^r \) is a singleton. When \( \mathcal{B}_*^r \) is a singleton, all the nuisance parameters are strongly identified under the null. Otherwise, there exist weakly identified nuisance parameters under the null. The following assumption makes this distinction and imposes regularity conditions on \( r(\theta) \).

**Assumption 10.**

(a) \( r(\theta) \) is continuously differentiable, and \( R_1 := \partial_x r_1(\pi_*) \) has full rank, \( d_{r_1} \).

(b) \( r_2(\beta) = 0 \) implies \( \beta = \beta_* \).

(c) There exists an \( M > 0 \) such that for all \( \pi_1 \) and \( \pi_2 \) in a neighborhood of \( \pi_* \) satisfying \( r_1(\pi_1) = r_1(\pi_2) = 0 \), \( d_H(\mathcal{B}^r(\pi_1), \mathcal{B}^r(\pi_2)) \leq M \| \pi_1 - \pi_2 \| \).

(d) For a closed set, \( \Psi^r \subseteq \mathbb{R}^{d_\theta} \), \( \{(\sqrt{n}(\pi - \pi_n), \sqrt{n}(\beta - \beta_n)/a_n) | (\pi, \beta) \in \Theta^r \} \to K \Psi^r \).

**Remark:** 4.1. Part (b) states that there are no weakly identified nuisance parameters under the null hypothesis. This is the first case, denoted by “W1” in the following. Otherwise, when there are weakly identified parameters under the null hypothesis, we assume part (c), which guarantees the null-imposed cross-sections are Lipschitz in the Hausdorff distance, similar to Assumption 4(b). This second case is denoted by “W2” in the following. Part (d) is the restricted version of Assumption 8(a).

Assumption 10(d) gives the limit of the localized parameter space under Assumptions 5(S) or (SS). It will also be used under Assumption 5(W) for case W1. In case W2, the corresponding result follows from Assumption 10(c), as the following lemma shows. Let \( R_1^\perp = \{ \psi_1 \in \mathbb{R}^{d_\pi} | R_1 \psi_1 = 0_{d_{r_1}} \} \). 22
Lemma 5. If $\Theta^r$ satisfies Assumption 10(a,c), then \[\{(\sqrt{n}(\pi-\pi_n), \beta)| (\pi, \beta) \in \Theta^r\} \rightarrow_K R^\perp_1 \times \mathcal{B}^r_\ast.\]

The following definition is used to state the limit of $QLR_n$ under $W_2$. For every $\beta \in \mathcal{B}_1$, let

$$\tilde{q}^{W,r}(\beta) = \tilde{q}^{W}(\beta) + Z_1(\beta)'R_1' (R_1J_{11}(\beta)^{-1}R_1')^{-1} R_1Z_1(\beta)/2. \quad (4.2)$$

This formula characterizes the contribution of the $r_1(\pi)$ restrictions to the concentrated limit objective function from Remark 3.14. The following theorem characterizes the asymptotic distribution of $QLR_n$.

Theorem 2. Suppose Assumptions 1, 2, 3, 4, 6, and 7 hold, and suppose that $r(\theta_n) = 0$ for all $n$.

(a) Under Assumptions 5(S), 8(a), and 10(d),

$$QLR_n \rightarrow_d QLRS := \inf_{\psi \in \Psi^r} (Z - \psi)'J(Z - \psi) - \inf_{\psi \in \Psi} (Z - \psi)'J(Z - \psi).$$

(b) Under Assumptions 5(SS), 8(a), and 10(a,(b or c),d),

$$QLR_n \rightarrow_d QLRS = \inf_{\psi \in \Psi^r} (Z - \psi)'J(Z - \psi) - \inf_{\psi \in \Psi} (Z - \psi)'J(Z - \psi).$$

(c) Under Assumptions 5(W) and 10(b,d),

$$QLR_n \rightarrow_d QLR_{1*}^{W1} := \inf_{(\psi_1, \psi_2) \in \Psi^r} (Z_1(\beta_*) - \psi_1)'J_{11}(\beta_*) (Z_1(\beta_*) - \psi_1)$$

$$- Z_1(\beta_*)'J_{11}(\beta_*) Z_1(\beta_*) \inf_{\beta \in \mathcal{B}_*} 2\tilde{q}^{W}(\beta).$$

(d) Under Assumptions 5(W) and 10(a,c),

$$QLR_n \rightarrow_d QLR_{*2}^{W2} := \inf_{\beta \in \mathcal{B}_*^r} 2\tilde{q}^{W,r}(\beta) - \inf_{\beta \in \mathcal{B}_*} 2\tilde{q}^{W}(\beta).$$

Remarks: 4.2. The limit theory for parts (a) and (b) agrees with the limit theory derived in Andrews (2001). The limit reduces to $\chi^2_{d_\theta}$ when $\Psi = \mathbb{R}^{d_\theta}$, $\Psi^r$ is a subspace of dimension $d_\theta - d_r$, and $H = V$. 

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4.3. Parts (c) and (d) state limit theory for the QLR statistic under weak identification. For both, the limit theory of the unrestricted part is the same. Under Assumption 10(b,d), the limit theory for the null-imposed part resembles the strong identification case, while under Assumption 10(a,c), the limit theory for the null-imposed part resembles the weak identification case.

We use the quantiles of the distribution of \( QLR^L \) to define an IR critical value, where \( L \in \{S,W1,W2\} \). At the end of this section, we give instructions for simulating the quantiles of \( QLR^L \). Notice that the distribution of \( QLR^L \) depends on the objects: \( \theta_*, H, V, c(\cdot), \Psi \), and \( \Psi^r \). Recall \( \theta_* \) is the limit of the sequence of parameter values, \( \theta_n \), \( H \) and \( V \) are defined in Assumptions 6 and 7, \( c(\cdot) \) is defined in Assumption 5(W), \( \Psi \) is defined in Assumption 8(a), and \( \Psi^r \) is defined in Assumption 10(d). The distribution also depends on \( B_* \) and \( B'_* \), but we write these as \( B(\pi_*) \) and \( B'(\pi_*) \), respectively, making the dependence on \( \pi_* \) explicit. Let \( F_L(\cdot; \theta_*, H, V, c(\cdot), \Psi, \Psi^r) \) denote the distribution function of \( QLR^L_L \), for \( L \in \{S,W1,W2\} \).

Let \( \alpha \in (0,1) \) denote the nominal size of the test. Let \( q\alpha(1-\alpha; \theta_*, H, V, c(\cdot), \Psi, \Psi^r) \) denote the \( 1-\alpha \) quantile of the \( F_L(\cdot; \theta_*, H, V, c(\cdot), \Psi, \Psi^r) \) distribution. The following assumption requires the quantile to be a continuity point of the distribution.

**Assumption 11(\( \alpha,L \)).** For a given \( \theta_* \), \( H, V, c(\cdot), \Psi \), and \( \Psi^r \), \( q\alpha(1-\alpha; \theta_*, H, V, c(\cdot), \Psi, \Psi^r) \) is a continuity point of \( F_L(\cdot; \theta_*, H, V, c(\cdot), \Psi, \Psi^r) \).

**Remark:** 4.4. Assumption 11(\( \alpha,L \)) ensures the probability the test statistic is equal to the critical value is zero in the limit. We use Assumption 11(\( \alpha,L \)) for different values of \( \alpha \) and \( L \in \{S,W1,W2\} \) in Theorem 3 below.

In order to estimate the quantiles of the distribution of \( QLR^L \), we need estimators of the parameters indexing \( q\alpha(1-\alpha; \theta_*, H, V, c(\cdot), \Psi, \Psi^r) \) for \( L \in \{S,W1,W2\} \). In what follows, we describe estimators for these parameters and state high-level conditions for asymptotic uniform validity of the resulting critical values. The high-level conditions are verified in Examples 1 and 2 in the supplemental materials.

We assume there exist consistent estimators of \( \pi \), \( H \), and \( V \), denoted by \( \hat{\pi} \), \( \hat{H} \), and \( \hat{V} \). For \( \beta \), a consistent estimator usually does not exist under weak identification.

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\(^{15}\) \( F_L(\cdot) \) does not depend on all of the objects for particular values of \( L \). Specifically, \( F_S(\cdot) \) does not depend on \( c(\cdot) \), \( F_{W1}(\cdot) \) does not depend on \( \Psi \), and \( F_{W2}(\cdot) \) does not depend on \( \Psi \) or \( \Psi^r \). We abuse notation by dropping the irrelevant arguments when a particular value of \( L \) is considered.

\(^{16}\) The \( 1-\alpha \) quantile is defined as \( q(1-\alpha) := \inf\{u \in \mathbb{R} | F(u) \geq 1-\alpha\} \).
For that reason, we assume $\hat{\beta}$ is a consistent estimator under strong and semi-strong identification only.

For $c(\cdot)$, a consistent estimator usually does not exist under weak identification. Thus, we use a type of confidence set for $c(\cdot)$ with level $\alpha_c \in [0, \alpha)$. Let $\tilde{\Pi}$ denote a set of values of $\pi$. For every $\pi^*_\epsilon \in \tilde{\Pi}$, let

$$\hat{c}(\beta; \pi^*_\epsilon, \beta^*_\epsilon) = \sqrt{n}(\tau(\pi^*_\epsilon, \beta^*_\epsilon) - \tau(\pi^*_\epsilon, \beta^*_\epsilon)). \quad (4.3)$$

We assume the set of all $\hat{c}(\beta; \pi^*_\epsilon, \beta^*_\epsilon)$ for $\pi^*_\epsilon \in \tilde{\Pi}$ contains an approximation to the true $c(\cdot)$ with prespecified probability $1 - \alpha_c$. Below we give formulas for $\tilde{\Pi}$ in Examples 1 and 2 that satisfy this condition.

We also estimate $\Psi$ and $\Psi^r$ by sets $\hat{\Psi}$ and $\hat{\Psi}^r$. These objects depend on how close $\theta_n$ is to the boundary of the parameter space. We assume $\sqrt{n}(\Theta^r - \theta_n)$ “covers” $\hat{\Psi}^r$ and $\hat{\Psi}$ “covers” $\sqrt{n}(\Theta - \theta_n)$ with probability at least $1 - \alpha_\Psi$, where “covers” is precisely defined in the following assumption, and where $\alpha_\Psi \in [0, \alpha)$. Sets, $\hat{\Psi}$ and $\hat{\Psi}^r$, satisfying this condition can be defined by adapting the approach in Romano et al. (2014) for constructing a confidence set for the slackness parameters of a collection of inequalities.

The following assumption states high-level conditions to be placed on the estimators. Not all the conditions are assumed for all identification strengths. That is, some of the conditions only hold for some of the identification strengths (weak, strong, or semi-strong), and are not required to hold for others.

**Assumption 12.** Under a given sequence, $\theta_n \to \theta_*$ and $\xi_n \to \xi_*$, the objects $\hat{\pi}$, $\hat{\beta}$, $\hat{H}$, $\hat{V}$, $\tilde{\Pi}$, $\hat{\Psi}$, and $\hat{\Psi}^r$ satisfy the following conditions.

(a) $\sqrt{n}(\hat{\pi} - \pi_n) = O_p(1)$, $r_1(\hat{\pi}) = 0$, $\hat{H} \to_p H$, and $\hat{V} \to_p V$, where $H$ and $V$ are defined in Assumption 7. $\hat{H}$ and $\hat{V}$ are positive definite and symmetric almost surely.

(b) $\hat{\beta} \to_p \beta^*_\epsilon$.

(c) For every $\epsilon > 0$,

$$\liminf_{n \to \infty} P_{\theta_n, \xi_n}\left(\inf_{\hat{\pi}^*_\epsilon \in \tilde{\Pi}} \inf_{\hat{\beta}^*_\epsilon \in \mathcal{B}^r(\hat{\beta})} \left(\|\hat{\beta}^*_\epsilon - \beta^*_\epsilon\| + \sup_{\beta^*_\epsilon \in \mathcal{B}_1} \|\hat{c}(\beta; \hat{\pi}^*_\epsilon, \hat{\beta}^*_\epsilon) - c(\beta)\|\right) \leq \epsilon\right) \geq 1 - \alpha_c,$$

where $c(\beta)$ is defined in Assumption 5(W), and $\mathcal{B}_1$ is defined in Lemma 1.
(d) There exists a neighborhood, \(U\), of \(\beta^*\) such that

\[
\inf_{\hat{\pi}^* \in \Pi} \inf_{\beta^* \in \mathcal{B}^r(\hat{\pi}^*)} \left( \frac{\sqrt{n}||\hat{\beta}_* - \beta_n||}{a_n} + \sup_{\beta \in B_1} \left| \frac{\sqrt{n}}{a_n} \hat{c}(\beta; \hat{\pi}_*, \hat{\beta}_*) - c(\beta) \right| \right)
+ \sup_{\beta \in \mathcal{U}} \left| \frac{\sqrt{n}}{a_n} \partial_{\beta} \hat{c}(\beta; \hat{\pi}_*, \hat{\beta}_*) - C(\beta) \right| \to_p 0.
\]

where \(c(\beta), C(\beta),\) and \(a_n\) are defined in Assumption 5(SS) and \(B_1\) is defined in Lemma 1.

(e) For every \(K,\) compact, and for every \(\epsilon > 0,\)

\[
\lim \inf_{n \to \infty} P_{\theta_n, \xi_n} \left( \bar{d} \left( \sqrt{n}(\Theta - \theta_n) \cap K, \hat{\Psi} \right) \leq \epsilon \right) \geq 1 - \alpha_{\Psi}.
\]

(f) For every \(K,\) compact, and for every \(\epsilon > 0,\)

\[
\lim \inf_{n \to \infty} P_{\theta_n, \xi_n} \left( \bar{d} \left( \hat{\Psi}^r \cap K, \sqrt{n}(\Theta^r - \theta_n) \right) \leq \epsilon \text{ and } 0 \in \hat{\Psi}^r \right) \geq 1 - \alpha_{\Psi}.
\]

**Remarks: 4.5.** Part (a) is used for consistent estimation of \(H, V,\) and \(\pi.\) It is assumed for all types of sequences, strong, semi-strong, and weak. Part (b) is used for consistent estimation of \(\beta.\) It is only assumed for strong and semi-strong sequences. Part (c) is only used for weak sequences. It ensures the set of all \(\hat{c}(\beta; \pi_*, \beta_*)\) for \(\pi_* \in \hat{\Pi}\) contains an approximation to the true \(c(\beta)\) with probability at least \(1 - \alpha_c\) asymptotically. Part (d) is only used for semi-strong sequences. It says that \(\hat{c}(\beta; \pi_*, \beta_*)\) for \(\pi_* \in \hat{\Pi}\) is capable of estimating both \(c(\beta)\) and \(C(\beta)\) (by its derivative) simultaneously with probability approaching 1. Parts (e) and (f) formalize the sense in which \(\hat{\Psi}\) and \(\hat{\Psi}^r\) approximate the limit of \(\sqrt{n}(\Theta - \theta_n)\) and \(\sqrt{n}(\Theta^r - \theta_n)\), respectively. Part (e) says that \(\hat{\Psi}\) must contain points that are arbitrarily close to any point of \(\sqrt{n}(\Theta - \theta_n)\) with probability at least \(1 - \alpha_{\Psi}\) asymptotically. Part (f) says that \(\hat{\Psi}^r\) only contains points that are arbitrarily close to some point of \(\sqrt{n}(\Theta^r - \theta_n)\) with probability at least \(1 - \alpha_{\Psi}\) asymptotically. Part (e) is used for strong and semi-strong sequences, while part (f) is used for strong, semi-strong, and \(W_1\) sequences.

4.6. Below we give formulas for \(\hat{\pi}, \hat{\beta}, \hat{H}, \hat{V}, \hat{\Pi}, \hat{\Psi},\) and \(\hat{\Psi}^r\) in Examples 1 and 2, and verify Assumption 12 in the supplemental materials. The arguments used to verify Assumption 12 are not specific to these examples and can be easily generalized.
4.7. The role of $\alpha_c$ and $\alpha_\Psi$ is to capture the finite-sample probability that the sets $\hat{\Pi}$, $\hat{\Psi}$, or $\hat{\Psi}^r$ do not contain/cover $\pi_*$, $\Psi$, or $\Psi^r$, respectively. It is possible to make these values zero by growing the sets as the sample size increases. This leads to less conservative inference asymptotically, but may not accurately represent the uncertainty from the estimation of these parameters in finite sample. We cover this possibility in the theoretical results of this paper (by not requiring $\alpha_c$ or $\alpha_\Psi$ to be positive), but in the simulations and the empirical application we use $\alpha_c = \alpha_\Psi = \alpha/10$ in case W1 and $\alpha_c = 2\alpha_\Psi = \alpha/5$ in case W2.

We follow Andrews and Cheng (2012) and use an identification-category-selection (ICS) statistic that distinguishes weak identification from strong identification. Let $\hat{\kappa}$ be a statistic taking values in $[0, 1]$. We use the following conditions on $\hat{\kappa}$.

**Assumption 13.** Under the given sequence, $\theta_n \to \theta_*$ and $\xi_n \to \xi_*$,

(a) $\hat{\kappa} \to_p 1$, or

(b) $\hat{\kappa} \to_p 0$.

**Remarks:** 4.8. Under weak identification, we assume $\hat{\kappa}$ satisfies Assumption 13(a), while under strong identification, we assume $\hat{\kappa}$ satisfies Assumption 13(b). In this way, $\hat{\kappa}$ is an indicator for weak identification.

4.9. ICS statistics are common in weakly identified models. For examples, see Andrews and Cheng (2012), Andrews (2017, 2018), and Andrews and Guggenberger (2017, 2019), among others. If one wants to avoid using an ICS statistic, below we describe a simple modification that avoids an ICS statistic without being too conservative.

**Example 1, Continued.** In Example 1, we recommend the following objects to satisfy Assumptions 12 and 13. Let $\tilde{\Theta}^r = \{(\pi, \beta) \in \tilde{\Theta}|\beta = \beta_0\}$. $\tilde{\Theta}^r$ imposes the null hypothesis, but does not impose the bounds. Let $\tilde{\theta}_n$ be an element of $\tilde{\Theta}^r$ that satisfies $Q_n(\tilde{\theta}_n) \leq \inf_{\theta \in \tilde{\Theta}^r} Q_n(\theta) + o_p(n^{-1})$. This estimator is useful because it is asymptotically normal irrespective of the boundary. We take $\hat{\pi}$ to be the first five components of $\tilde{\theta}_n = (\hat{\rho}_2, \hat{\rho}_3, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \beta_0)$. We can also take $\hat{\beta} = \beta_0$. In Example 1, we weight the objective with $\hat{W}_n = \hat{V}^{-1}$, given by the empirical fourth moments of $X_i$, and take $\hat{H}$ and $\hat{V}$ to be $\hat{V}^{-1}$. 

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For $\tilde{\Pi}$, we use the asymptotic normality of $\tilde{\rho}_3$ to construct a confidence set for $\rho_3$ with coverage probability at least $1 - \alpha_c$. One can show that $\sqrt{n}(\tilde{\rho}_3 - \rho_{3n}) \to_d N(0, e_2' J_{11}(\beta_0)^{-1} e_2)$, where $\rho_{3n}$ is the second element of $\pi_n$ and $e_2$ is the second standard normal basis vector. Let

$$\tilde{\Pi} = \left\{ (\tilde{\rho}_2, \rho_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) \mid \sqrt{n}|\rho_3 - \tilde{\rho}_3| \leq z_{1-\alpha_c/2} \sqrt{e_2' J_{11}^{-1} e_2} \right\},$$

where $J_{11} = D_1(\tilde{\theta}_n)' \tilde{V}^{-1} D_1(\tilde{\theta}_n)$. This definition of $\tilde{\Pi}$ satisfies Assumption 12(c) under weak sequences and Assumption 12(d) under semi-strong sequences. Note that a $1 - \alpha_c$ confidence interval is used for $\rho_3$ (since $\rho_3$ is the key parameter that determines identification), while only consistent estimators are needed for the other parameters.

For $\hat{\Psi}$ and $\hat{\Psi}^r$, we linearize $\ell(\theta)$ around $\theta_*$. Let

$$\bar{\ell} = \left[ \frac{\sqrt{n} \ell_1(\tilde{\theta}_n) \pm \left( \partial_x \ell_1(\tilde{\theta}_n)' J_{11}^{-1} \partial_x \ell_1(\tilde{\theta}_n) \right)^{1/2} z_{1-\alpha_c}} {\sqrt{n} \ell_2(\tilde{\theta}_n) \pm \left( \partial_x \ell_2(\tilde{\theta}_n)' J_{11}^{-1} \partial_x \ell_2(\tilde{\theta}_n) \right)^{1/2} z_{1-\alpha_c}} \right]$$

be worst case values of $\bar{\ell}$ in one-sided confidence sets. We use the upper bounds for $\hat{\Psi}$ and the lower bounds for $\hat{\Psi}^r$: $\hat{\Psi} = \{ \psi \in \mathbb{R}^d \mid \min(\bar{\ell}^-, 0) + \partial_x \ell(\tilde{\theta}_n)\psi \leq 0 \}$ and $\hat{\Psi}^r = \{ (\psi_1, 0) \in \mathbb{R}^d \mid \min(\bar{\ell}^+, 0) + \partial_x \ell(\tilde{\theta}_n)\psi_1 \leq 0 \}$.

Also, $\check{\kappa} = 1 \{ \inf_1 n \rho_3^2 \leq \log(n) e_2' J_{11}^{-1} e_2 \}$ is an ICS statistic that satisfies Assumption 13(a) under weak sequences and Assumption 13(b) under strong sequences.

**Example 2, Continued.** We recommend the following objects to satisfy Assumptions 12 and 13 in Example 2. As above, let $\check{\Theta}^r = \{ (\pi, \beta) \in \check{\Theta} \mid \beta = \beta_0 \}$, and let $\check{\theta}_n$ be an element of $\check{\Theta}^r$ that satisfies $Q_n(\check{\theta}_n) \leq \inf_{\theta \in \check{\Theta}^r} Q_n(\theta) + o_p(n^{-1})$. We write $\hat{\theta}_n = (\check{\pi}_n, \beta_0)$ and take $\check{\pi}$ to be $\check{\pi}_n$. We can also take $\check{\beta} = \beta_0$. In Example 2, we weight the objective with $\hat{W}_n = \hat{V}^{-1}$, given by the empirical fourth moments of $X_i$, and take $\check{H}$ and $\check{V}$ to be $\check{V}^{-1}$.

For $\check{\Pi}$, we use the asymptotic normality of $\check{\pi}_n$ to construct a confidence set for $s(\pi) = (s_1(\pi), s_2(\pi)) = (\rho_{52} \rho_{41} - \rho_{51} \rho_{42}, \rho_{52} \rho_{32} - \rho_{51} \rho_{31})$. Let $\check{s}_n = (s_1(\check{\pi}_n), s_2(\check{\pi}_n))$, and notice that $\sqrt{n}(\check{s}_n - s(\pi_n)) \to_d N(0, S J_{11}(\beta_0)^{-1} S')$, where $S = \partial_x s(\pi_*)$. Let

$$\check{\Pi} = \{ (\check{\rho}_{31}, \check{\rho}_{32}, \check{\rho}_{41}, \check{\rho}_{42}, \rho_{51}, \rho_{52}, \check{\omega}, \check{\sigma}, \check{\sigma}_1) : \sqrt{n}|\rho_{52} \check{\rho}_{41} - \rho_{51} \check{\rho}_{42} - s_1(\check{\pi}_n)| \leq \check{V}_{s_1}^{1/2} z_{1-\alpha_c/4}$$

and $\sqrt{n}|\rho_{52} \check{\rho}_{31} - \rho_{51} \check{\rho}_{32} - s_2(\check{\pi}_n)| \leq \check{V}_{s_2}^{1/2} z_{1-\alpha_c/4} \}$.  

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where \( \hat{\pi}_n = ((\hat{\rho}_{31}, \hat{\rho}_{32}, \hat{\rho}_{41}, \hat{\rho}_{42}, \hat{\rho}_{51}, \hat{\rho}_{52}), \hat{\omega}, \hat{\xi}, \hat{\sigma}_{12}), \hat{V}_{sq} = \partial_\pi s_j(\hat{\pi}_n)\hat{J}_{11}^{-1}\partial_\pi s_j(\hat{\pi}_n)', \) and \( \hat{J}_{11} = D_1(\hat{\theta}_n)\hat{V}^{-1}D_1(\hat{\theta}_n). \) This definition of \( \hat{\Pi} \) satisfies Assumption 12(c) under weak sequences and Assumption 12(d) under semi-strong sequences.

For \( \hat{\Psi} \) and \( \hat{\Psi}^r \), we use the same objects as Example 1 with the following changes. We extend the definition of \( \hat{\ell}^\pm \) in (4.4) to four bounds, and we replace \( \alpha_{\Psi} \) by \( \alpha_{\Psi}/2 \) because with two factors, at most two error variances can be zero at the same time.\(^{17}\)

Also, \( \hat{\kappa} = 1\{ns_1(\hat{\pi}_n)^2\hat{V}^{-1}_{s_1} + ns_2(\hat{\pi}_n)^2\hat{V}^{-1}_{s_2} \leq 2\log(n) \} \) is an ICS statistic that satisfies Assumption 13(a) under weak sequences and Assumption 13(b) under strong sequences. \( \Box \)

We next define the IR critical value. Let \( \alpha_{W_1} = \alpha - \alpha_c - \alpha_{\Psi}, \alpha_{W_2} = \alpha - \alpha_c, \) and \( \alpha_S = \alpha - 2\alpha_{\Psi}. \) For \( L \in \{W_1, W_2\} \), we assume \( 0 < \alpha_L \leq \alpha_S. \) Let

\[
\overline{CV}_L = \hat{\kappa} \sup_{\hat{\pi} \in \Pi} \sup_{\hat{\beta} \in \mathcal{B}^r(\hat{\pi})} q_L(1 - \alpha_L; \hat{\pi}, \hat{\beta}, \hat{\Pi}, \hat{V}, \hat{c}(\cdot; \hat{\pi}, \hat{\beta}), \hat{\Psi}^r) + (1 - \hat{\kappa})q_S(1 - \alpha_S; \hat{\pi}, \hat{\beta}, \hat{\Pi}, \hat{V}, \hat{\Psi}, \hat{\Psi}^r).
\]

This critical value uses \( q_S \) when identification is strong and \( q_{W_1} \) or \( q_{W_2} \) when identification is weak. When identification is semi-strong, all the categories, \( q_S, q_{W_1}, \) and \( q_{W_2}, \) naturally adjust to approximate the semi-strong limit of \( QLR_n. \) The definition of \( \hat{\kappa} \) often involves a tuning parameter. One way to avoid this, and achieve tuning-parameter-free inference, is to take the maximum between the \( q_S \) and \( q_L \) quantiles. This may be conservative, but not as conservative as one might think.\(^{18}\) In the examples and the empirical application, we use \( \overline{CV}_L \) defined above, but note that some practitioners may prefer taking the max to achieve tuning-parameter-free inference.

Up to this point, all results and assumptions have been stated for a fixed sequence, \( \{\theta_n\}_{n=1}^\infty. \) Now, we consider classes of sequences of parameters in order to show that the RQLR test is uniformly valid. We now state the final assumption, a subsequencing condition that guarantees that the three types of sequences, strong, semi-strong, and weak, are sufficient for uniform size control. Let \( \Gamma \) denote the joint parameter space for \( \theta \) and \( \xi. \) Let \( \Theta_1 \subseteq \Theta \) and \( \Gamma_1 = \{ (\theta, \xi) \in \Gamma | \theta \in \Theta_1, \xi \in \Xi_1(\theta), r(\theta) = 0 \}, \) where

\(^{17}\)This is a general result in factor models. No more than \( m, \) the number of factors, error variances can be zero, or else the covariance matrix of observables is singular. This corresponds to the number of bounds that can bind since the bounds are derived from the nonnegativity of the error variances.

\(^{18}\)Under strong identification with \( H = V \) and when \( \Psi \) and \( \Psi^r \) are subspaces of dimensions \( d_\theta \) and \( d_\theta - d_x, \) respectively, both quantiles converge to the \( \chi^2_{d_\theta} \) quantile, so the conservativeness only occurs at the boundary or under weak identification.
\( \Xi^\dagger(\theta) \) is a set of possible values for \( \xi \) for every \( \theta \in \Theta^\dagger \). We seek to control size uniformly over \( \Gamma^\dagger \). Let \( \Upsilon_L \) for \( L \in \{S, SS, W\} \) denote sets of converging sequences, \( \{\gamma_n\}_{n=1}^\infty \subseteq \Gamma \), such that the following assumption holds.

**Assumption 14.** For every sequence \( \{\gamma_n^\dagger\}_{n=1}^\infty \subseteq \Gamma^\dagger \), and for every subsequence, \( n_m \), there exists a sequence, \( \{\gamma_n\}_{n=1}^\infty \in \Upsilon_S \cup \Upsilon_{SS} \cup \Upsilon_W \), and a further subsequence, \( n_q \), such that for all \( q \), \( \gamma_n^\dagger = \gamma_{n_q}^\dagger \).

**Remark: 4.10.** Assumption 14 guarantees that sequences contained in \( \Upsilon_S \cup \Upsilon_{SS} \cup \Upsilon_W \) are sufficiently “dense” in the set of all sequences that they characterize uniformity over \( \Gamma^\dagger \). Relative to Andrews et al. (2020), this approach to uniformity separates out the subsequencing condition from the condition that controls rejection probabilities, which are conceptually distinct.

The following theorem states that the IR critical values are able to control size in a uniform sense.

**Theorem 3.** Suppose Assumption 14 holds. Let \( L \in \{W_1, W_2\} \):

- Suppose for every \( \{\gamma_n\}_{n=1}^\infty \in \Upsilon_S \), Assumptions 1, 2, 3, 4, 5(S), 6, 7, 11(\( \alpha_S, S \)), 12(a,b,e,f), and 13(b) hold.

- Suppose for every \( \{\gamma_n\}_{n=1}^\infty \in \Upsilon_{SS} \), Assumptions 1, 2, 3, 4, 5(SS), 6, 7, 10(a), 11(\( \alpha_S, S \)), and 12(a,b,d,e,f) hold. In addition, when \( L = W_1 \), suppose Assumption 10(b) holds, and when \( L = W_2 \), suppose Assumption 10(c) holds.

- Suppose for every \( \{\gamma_n\}_{n=1}^\infty \in \Upsilon_W \), Assumptions 1, 2, 3, 4, 5(W), 6, 7, 11(\( \alpha_L, L \)), 12(a,c), and 13(a) hold. In addition, when \( L = W_1 \), suppose Assumptions 10(b) and 12(f) hold, and when \( L = W_2 \), suppose Assumption 10(a,c) holds.

Then, for \( L \in \{W_1, W_2\} \),

\[
\lim_{n \to \infty} \sup_{\gamma^\dagger \in \Gamma^\dagger} \sup_{\gamma^\dagger \in \Gamma^\dagger} P_{\gamma^\dagger} \left( QLR_n > \bar{CV}_L \right) \leq \alpha.
\]

**Remarks: 4.11.** Theorem 3 shows that the robust critical values control size uniformly over \( \gamma \in \Gamma^\dagger \). This shows that the RQLR test is identification- and boundary-robust because \( \Gamma^\dagger \) includes parameters that may be weakly identified and/or on the boundary.
4.12. The assumptions for Theorem 3 are high-level, but quite weak. Most of the assumptions are on the reduced form model, and thus are analogous to conditions in strongly identified models. Only the relevant assumptions for each type of sequence need to be verified in a given model. Also notice that Assumptions 8, 9, and 10(d) are not needed at all. This is quite surprising. Furthermore, if one takes \( \hat{CV}_L \) to be the maximum between the \( q_S \) and \( q_L \) quantiles, then Assumption 13 and an ICS statistic are not needed.

4.13. Under some conditions, the robust critical value reduces to a chi-squared critical value asymptotically. If \( H = V, \theta^* \) is in the interior of \( \Theta \), and identification is strong or semi-strong, then \( \hat{CV}_L \) converges in probability to a chi-squared critical value. Furthermore, if \( \alpha_c = \alpha_\Psi = 0 \), then the RQLR test is asymptotically equivalent to the usual QLR test.

4.14. The proof of Theorem 3 is intuitive, but contains a challenging step. The basic outline is to show that the quantiles, \( q_S \) and \( q_L \), are continuous in their parameters, so the estimators from Assumption 12 can be plugged in. The challenging step is showing that \( q_L \) adapts to approximate the \( q_S \) limit under semi-strong sequences; see Lemma F.3 in the supplemental materials.

4.15. While Theorem 3 focuses on hypothesis testing, the RQLR test can also be used to construct confidence sets for \( r(\theta) \) by inverting a family of tests. By incorporating sequences of null hypotheses as in Andrews et al. (2020), such confidence sets can be shown to have asymptotically uniformly valid coverage probability.

To finish this section, we give instructions for simulating the quantiles of \( QLR^L_\ast \) for \( L \in \{S,W1,W2\} \). For simplicity, we state the instructions for \( L = W2 \) because the instructions for the other values of \( L \) are similar. Suppose we have estimators \( \hat{\pi}, \hat{H}, \hat{V}, \hat{\pi}^* \in \hat{\Pi}, \) and \( \hat{\beta}^* \in B^r(\hat{\pi}) \).

Step 1: Draw a large number, \( B \), draws of \( Y_b \sim N(0, \hat{V}) \) for \( b = 1, \ldots, B \).

Step 2: For each \( b \), calculate \( q^W(\beta) \) and \( q^{W..}(\beta) \) as a function of \( \beta \in B(\hat{\pi}) \), using the formulas in (3.7) and (4.2), where \( \pi^* \) is replaced by \( \hat{\pi} \), \( H \) is replaced by \( \hat{H} \), and \( c(\beta) \) is replaced by \( \hat{c}(\beta; \hat{\pi}^*, \hat{\beta}^*) \), defined in (4.3).

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Kuratowski convergence is so weak that for every sequence of sets, \( \sqrt{n}(\Theta - \theta_n) \), there is always a subsequence, \( n_m \), and a closed set \( \Psi \) such that \( \sqrt{n_m}(\Theta - \theta_{n_m}) \rightarrow K \Psi \). This implies that Assumptions 8(a) and 10(d) are automatically satisfied along a subsequence.
Step 3: For each $b$, calculate $QLR_{W}^{*2}$ by minimizing $\tilde{q}^{W}(\beta)$ over $\beta \in \mathcal{B}(\hat{\pi})$ and by minimizing $\tilde{q}^{W,r}(\beta)$ over $\beta \in \mathcal{B}^r(\hat{\pi})$.

Step 4: Let the simulated $1 - \alpha$ quantile of $QLR_{W}^{*2}$ be the $(1 - \alpha)B$ order statistic of the values of $QLR_{W}^{*2}$.

5 Simulations

This section reports simulated rejection probabilities for hypothesis tests in Examples 1 and 2. We find that the RQLR test is the only test that can control size up to and including the boundary of the identified set and still have power against hypotheses that violate the bounds.

In Example 1, we test hypotheses on $\sigma^2$, the variance of the factor. Suppose the true values of the parameters are $\sigma^2 = 1$, $\lambda_2 = 1$, $\lambda_3 = 0$, and $\phi_j = 1$ for $j \in \{1, 2, 3\}$. The fact that $\lambda_3 = 0$ implies that $\sigma^2$ is not identified. It is only partially identified by bounds, which characterize the identified set to be $[0.5, 2]$. In this case, by observational equivalence, the rejection probability should be less than or equal to $\alpha$ for every hypothesized value in the identified set. We simulate 1000 samples of size $n = 500$ with iid normally distributed factors and errors and calculate rejection probabilities for tests of the hypothesis $H_0: \sigma^2 = \sigma_0^2$, where $\sigma_0^2$ takes values in $[0.4, 2.15]$.

Figure 1 reports rejection probabilities for the Anderson-Rubin (AR) test from Stock and Wright (2000), the GELR test from Guggenberger and Smith (2005), the likelihood ratio test (AM) from Andrews and Mikusheva (2016a), the SR-AR test in Andrews and Guggenberger (2019), and the RQLR test. For all these tests, the strongly identified nuisance parameters are plugged in.\textsuperscript{20}

Remarks: 5.1. Several IR tests are not included in Figure 1. The K and CLR tests defined in Kleibergen (2005) reduce to the AR test when the model is just identified, as Example 1 is. Also, the versions of the AR, K, and CLR tests defined for minimum distance models in Magnusson (2010) are equivalent to the GMM versions when the link function is additively separable between the reduced-form parameters and the structural parameters, as it is in Examples 1 and 2. We also omit projected

\textsuperscript{20}We employ the reparameterization in Section 2, so all the nuisance parameters are strongly identified. This allows us to use plug-in versions of tests that are designed for full-vector inference. Otherwise, those tests would have to be projected.
versions of the IR tests defined for full-vector inference, including the AR, K, and CLR tests because we expect them to be very conservative. Tests that explicitly allow for weakly identified nuisance parameters, including the tests in Chaudhuri and Zivot (2011), Andrews and Mikusheva (2016b), Andrews (2017), and Andrews (2018), are compared in Cox (2022b). Cox (2022b) found that those tests can be very conservative under weak identification relative to tests that reparameterize the model so the nuisance parameters are strongly identified.

5.2. The solid black line denotes the 5% threshold over the identified set. The RQLR test approximately controls size when \( \sigma_0^2 \) is in the identified set and has rejection probability above 5% when \( \sigma_0^2 \) is outside the identified set. The AR, GELR, and AM tests have rejection probabilities above 5% when \( \sigma_0^2 \) is in the identified set, and therefore do not control size. Indeed, when \( \sigma_0^2 = 2 \), the maximal rejection probabilities over the identified set are 11.1%, 10.3%, and 28.5%, respectively. The SR-AR test rejects with probability approximately 5% for any value of \( \sigma_0^2 \), so it does not have consistent power against hypotheses that violate the bounds.

5.3. The problem with plugging in strongly identified nuisance parameters is that the estimators for the nuisance parameters are not asymptotically normal when the bounds are imposed. Instead, they have a parameter-on-the-boundary asymptotic
Figure 2: Simulated Rejection Probabilities of IR Tests in Example 2

![Graph showing rejection probabilities for different tests]

(a) Lower Bound

(b) Upper Bound

Note: Finite sample \((n = 500)\) rejection probabilities of IR tests for \(H_0 : \sigma_2^2 = \sigma_{2,0}^2\) in Example 2, calculated with 1000 simulations for each value. The true data generating process simulates jointly normally distributed errors and factors with \(\sigma_1^2 = \sigma_2^2 = 1\), \(\sigma_{12} = 0\), \(\lambda_{jk} = 1\) for \(j \in \{3, 4, 5\}\) and \(k \in \{1, 2\}\), and \(\phi_j = 1\) for \(j \in \{1, 2, 3, 4, 5\}\).

distribution; see Andrews (1999). The argument used in Stock and Wright (2000), Kleibergen (2005), and others to show that strongly identified nuisance parameters can be plugged in relies essentially on asymptotic normality.

5.4. On average, the AR, GELR, AM, SR-AR, and RQLR tests take 0.01, 0.3, 269, 0.03, and 35 seconds per simulation to compute, respectively. The RQLR test is more computationally expensive than AR, GELR, and SR-AR, but is less computationally expensive than AM. See Section B.3 in the Supplemental Materials for details on the computation of these tests.

Similar results hold for Example 2. In Example 2, we test hypotheses on \(\sigma_2^2\), the variance of the second factor. Suppose the true values of the parameters are \(\sigma_1^2 = \sigma_2^2 = 1\), \(\sigma_{12} = 0\), \(\phi_j = 1\) for \(j \in \{1, 2, 3, 4, 5\}\), and \(\lambda_{jk} = 1\) for \(j \in \{3, 4, 5\}\) and \(k \in \{1, 2\}\). The fact that \(\lambda_{jk} = 1\) for all \(j \in \{3, 4, 5\}\) and \(k \in \{1, 2\}\) implies that we cannot separately identify the variances of the two factors. The variance of the second factor is only partially identified by bounds, which characterize the identified set to be \([2/3, 2]\).

As before, we simulate 1000 samples of size \(n = 500\) with iid normally distributed factors and errors and calculate rejection probabilities for tests of the hypothesis \(H_0 : \sigma_2^2 = \sigma_{2,0}^2\), for various values of \(\sigma_{2,0}^2\). Figure 2 reports the rejection probabilities...
for the Anderson-Rubin (AR) test from Stock and Wright (2000), the K and CLR tests from Kleibergen (2005), the SR-AR test from Andrews and Guggenberger (2019), and the RQLR test.

**Remarks: 5.5.** The same conclusions as Figure 1 hold. The AR, K, and CLR tests reject with probability above 5% when \( \sigma^2_{2,0} \) is in the identified set. Indeed, when \( \sigma^2_{2,0} = 2 \), the maximal rejection probabilities over the identified set are 9.7%, 12.0%, and 10.2%, respectively. The SR-AR test rejects with probability approximately 5% for any value of \( \sigma^2_{2,0} \), so it does not have consistent power against hypotheses that violate the bounds. The RQLR test is the only test that can control size up to and including the boundary of the identified set and still have power against hypotheses that violate the bounds.

**5.6.** The supplemental materials reports the rejection probabilities for other IR tests, including empirical likelihood tests from Guggenberger and Smith (2005) and Guggenberger et al. (2012) and the SR-CQLR test from Andrews and Guggenberger (2019). The simulations in the supplemental materials show that all of these tests fit into one of two categories: either the rejection probability is above \( \alpha \) near the boundary of the identified set, or there is trivial power against hypotheses that violate the bounds.

## 6 Empirical Application

The literature on childhood development commonly uses factor models to deal with measurement error in observed measures of parental investments and skills. We demonstrate our IR inference in the factor model used by Attanasio et al. (2020) to analyze the effects of a randomized intervention in Colombia.

Attanasio et al. (2014) document a significant improvement in cognition and language development of young children following a randomized controlled trial of weekly home visits by trained local women designed to improve the quality of maternal-child interactions. Attanasio et al. (2020) analyze this intervention using a factor model for measures of parental time and material investments. They find that an increase in parental investments is sufficient to explain the effects of the intervention.

The factor model used by Attanasio et al. (2020) specifies one factor for the control group and one factor for the treatment group.\(^{21} \) Cox (2022b) calculates a specification

\(^{21}\)Attanasio et al. (2020) ran a variety of exploratory factor analyses on the measurement system
Table 1: Parental Investment Factors

|                  | Control          | Treatment        |
|------------------|------------------|------------------|
|                  | $\sigma^2_{c,1}$ | $\sigma^2_{c,2}$ | $\sigma^2_{t,1}$ | $\sigma^2_{t,2}$ |
| Point Estimate   | 0.93             | 0.32             | 1.00             | 0.08             |
| Standard CI      | [0.80,1.06]      | [0.20,0.44]      | [0.83,1.18]      | [0.04,0.12]      |
| CLR CI           | [0.90,1.34]      | [0.17,0.34]      | [0.98,1.0]       | [0.03,0.08]      |
| RQLR CI          | [0.84,0.95]      | [0.23,0.40]      | [0.88,1.12]      | [0.08,0.14]      |

Note: The point estimate minimizes the minimum distance objective function. The standard CI is calculated using a t-statistic. The CLR CI is calculated by inverting the CLR-Plug test without imposing the bounds. The RQLR CI is calculated by inverting the RQLR test.

test for the one-factor models for material investments and notes that they are likely misspecified with p-values below $10^{-8}$. To account for this, Cox (2022b) estimates factor models with two factors, allowing the second factor to be weakly identified.

To be specific, suppose the dataset contains $J$ measures of material investments in children. Individual $i$ with treatment status $s \in \{c, t\}$ has observed material investment measure $j \in \{1, \ldots, J\}$ given by

$$X_{ij}^s = \mu_j^s + \lambda_j^s f_i^s + \epsilon_{ij}^s,$$

where $\mu_j^s$ is the mean of measure $j$, $\lambda_j^s$ are the factor loadings for measure $j$, $f_i^s$ are the parental investment factors, and $\epsilon_{ij}^s$ are error terms assumed uncorrelated with each other and with the factors. Let $\sigma^2_{s,1}$ and $\sigma^2_{s,2}$ denote the variances of the two factors for $s \in \{c, t\}$.

Cox (2022b) compares IR hypothesis tests in low-dimensional factor models and recommends a test that uses a reparameterization so the nuisance parameters are strongly identified, such as the CLR test. Table 1 reports point estimates and confidence intervals (CIs) for the variances of the factors, including a standard CI calculated with plug-in standard errors, the CLR CI that inverts the CLR test, and the RQLR CI that inverts the RQLR test. We make the following remarks on Table 1.

Remarks. 6.1. The Standard CI is only valid under strong or semi-strong

for parental investments that includes both time and material investments and found anywhere between 1 and 4 factors; see Table C.1 in Attanasio et al. (2020).

These are just variables indicating the number and type of toys the children have. See the supplemental materials for more details on the dataset and factor model specification.
identification. The CLR CI is calculated by inverting a plug-in version of the CLR test from Kleibergen (2005) using a reparameterization to make the nuisance parameters strongly identified. The CLR statistic is calculated using a version of the AR and K statistics that does not impose the bounds in (2.5). This is because the simulations in Section 5 indicate that the version of the CLR test that does impose the bounds is invalid near the ends of the identified set. The RQLR CI is calculated by inverting the RQLR test using the same formulas for the critical value as Example 2.

6.2. The CLR CI for $\sigma^2_{t,1}$ is very wide, with right endpoint at 10. This is because 10 was used as an arbitrary upper bound on the values of these parameters for practical/numerical reasons. If this number were increased, the right endpoint would likely be larger and possibly unbounded. This is a common problem with IR CIs.\textsuperscript{23} In factor models, this is especially problematic because the variance of a factor cannot be larger than the variance of its normalizing variable. For $\sigma^2_{t,1}$, the point estimate of the upper bound is 1.01. Imposing the bounds and using the RQLR test is a remedy to this problem. The RQLR CI is reasonable in length and does not extend much beyond the point estimate of the upper bound.

7 Conclusion

This paper combines weak identification with bounds in a class of minimum distance models. Limit theory for the minimum distance estimator and the QLR statistic is derived for strong, semi-strong, and weak sequences of parameters. An identification-and boundary-robust test (the RQLR test) is recommended for testing hypotheses on the structural parameters. The RQLR test is implemented in two example factor models with weak factors and an empirical application, demonstrating the value of bounds as a source of information when parameters are weakly identified.

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