EXACT SOLUTION OF A GENERALIZED ANNNI MODEL ON A
CAYLEY TREE

U. A. ROZIKOV, H. AKIN, AND S. UĞUZ

Abstract. We consider the Ising model on a Cayley tree of order two with nearest
neighbor interactions and competing next nearest neighbor interactions restricted to
spins belonging to the same branch of the tree. This model was studied by Vannimenus
and found a new modulated phase, in addition to the paramagnetic, ferromagnetic, an-
tiferromagnetic phases and a (+ + - -) periodic phase. Vannimenus’s results based
on the recurrence equations (relating the partition function of an n− generation tree to the
partition function of its subsystems containing (n − 1) generations) and most results are
obtained numerically. In this paper we analytically study the recurrence equations and
obtain some exact results: critical temperatures and curves, number of several phases,
partition function.

Mathematical Subject Classification: 82B20, 82B26.

Keywords: Cayley tree, configuration, Ising model, phase, Gibbs measure.

1. Introduction

The model considered by Vannimenus [20] consists of Ising spins (σ = ±1) on a Cayley
tree of branching ratio 2, so that every spin has three nearest-neighbor (NN). Two kinds
of bonds are present: NN interactions of strength $J_1$ and next-nearest-neighbor (NNN)
interactions $J_2$, these being restricted to spins belonging to the same branch of the tree.
The phase diagram described by Vannimenus contains a modulated phase, as found for
similar models on periodic lattices, but the multicritical Lifshitz point is at zero tempe-
arture. The variation of the wavevector with temperature in the modulated phase is studied
in detail, it is shown narrow commensurate steps between incommensurate regions. The
behavior of the coherence length near the different transitions is also analyzed.

The Vannimenus’s model was then generalized in many directions:

In [8] it was considered a model with the competing NN and NNN interactions Ising
model on a Cayley tree but in their case it is allowed for all interbranch NNN interactions
on the coordination number three which was discussed earlier in [9] and it were obtained
in addition to the expected paramagnetic, ferromagnetic and antiferromagnetic phases,
an intermediate range of $J_2/J_1 < 0$ values where the local magnetization has chaotic
oscillatory glass-like behavior.

Another generalization is due to Mariz et al [10] these authors studied the phase dia-
gram for the Ising model on a Cayley tree with competing NN interactions $J_1$ and NNN
interactions $J_2$ and $J_3$ in the presence of an external magnetic field. At vanishing te-
mperature, the phase diagram is fully determined, for all values and signs of $J_2/J_1$ and $J_3/J_2$; in
particular, it was verified that values of $J_3/J_2$ high enough favor the paramagnetic phase.
At finite temperatures, several interesting features (evolution of reentrances, separation of
the modulated region into two disconnected pieces, etc.) are exhibited for typical values
of $J_2/J_1$ and $J_3/J_2$. 
The next generalization is considered in [11], where the lattice spin model with $Q$-component discrete spin variables restricted to having orientations orthogonal to the faces of $Q$-dimensional hypercube is considered on the Cayley tree (Bethe lattice). The partition function of the model with dipole-dipole and quadrupole-quadrupole interaction is presented in terms of double graph expansions. By analyzing the regions of stability of different types of fixed points of the system of recurrent relations (which is generalization of the Vannimenus’s equations), the phase diagrams of the model are plotted. For $Q ≤ 2$ the phase diagram of the model is found to have three tricritical points.

The next generalizations are considered in [2], [3] and [4]. These authors have studied the phase diagram for Potts model on a Cayley tree with competing NN interactions $J_1$, prolonged NNN interactions $J_p$ and one level NNN interactions $J_o$. In [4] the Potts model with $J_o ≠ 0$ is considered and it is shown that for some values of $J_o$ the multicritical Lifshitz point be at non-zero temperature and proven that as soon as the same-level interaction $J_o$ is nonzero, the paramagnetic phase found at high temperatures for $J_o = 0$ disappears, while Ising model does not obtain such property.

But most results of the above mentioned works are obtained numerically. Thus it is natural to try to get some these results by an analytical way.

In this paper we consider the same model which was considered by Vannimenus (not its generalization) and study its phases by an analytical argument. Here we shall combine analytical arguments of papers [5], [14], [15], [19].

2. Preliminaries

The Cayley tree (Bethe lattice [1]) $Γ^k$ of order $k ≥ 1$ is an infinite tree, i.e., a graph without cycles, such that from each vertex of which issues exactly $k + 1$ edges. Let $Γ^k = (V, L)$, where $V$ is the set of vertices of $Γ^k$, $L$ is the set of edges of $Γ^k$. Two vertices $x$ and $y$ are called nearest neighbors (NN) if there exists an edge $l = ⟨x, y⟩$. A collection of the pairs ⟨$x_1$, $x_2$⟩, ..., ⟨$x_d$, $y$⟩ is called a path from $x$ to $y$. Then the distance $d(x, y)$, $x, y ∈ V$, on the Cayley tree, is the number of edges in the shortest path from $x$ to $y$.

For a fixed $x^0 ∈ V$ we set

$$W_n = \{x ∈ V | d(x, x^0) = n\}, \quad V_n = \bigcup_{k=1}^n W_k.$$

Denote

$$S(x) = \{y ∈ W_{n+1} : d(x, y) = 1\}, \quad x ∈ W_n,$$

this set is called a set of direct successors of $x$.

The vertices $x$ and $y$ are called next-nearest-neighbor (NNN) which is denoted by ⟨$x$, $y$⟩, if there exists a vertex $z ∈ V$ such that $x$, $z$ and $y$, $z$ are NN. We will consider only prolonged NNN ⟨$x$, $y$⟩, for which there exist $n$ such that $x ∈ W_n$ and $y ∈ W_{n+2}$, this kind of NNN is considered in [20] and [3].

We consider Ising model with competing NN and prolonged NNN interactions on a Cayley tree where the spin takes values in the set $Φ := \{-1, 1\}$, and is assigned to the vertices of the tree [20]. A configuration $σ$ on $V$ is then defined as a function $x ∈ V ↦ σ(x) ∈ Φ$; the set of all configurations is $Φ^V$. The (formal) Hamiltonian of the model is

$$H(σ) = -J_1 \sum_{⟨x, y⟩ ∈ L} σ(x)σ(y) - J_2 \sum_{⟨x, y⟩} σ(x)σ(y),$$

where $J_1, J_2 ∈ R$ are coupling constants and ⟨$x$, $y$⟩ stands for NN vertices and ⟨$x$, $y$⟩ stands for prolonged NNN.
As usual, one can introduce the notions of Gibbs measure (phase) of the Ising model with a competing interactions on the Cayley tree [7], [18], [13].

The standard approach consists in writing down recurrence equation relating the partition function

\[ Z_n = \sum_{\sigma_n \in \Phi_{V_n}} \exp\{-\beta H(\sigma_n)\}, \]

of \( n \)-generation tree to the partition function \( Z_{n-1} \) of its subsystems containing \((n - 1)\) generations. In [20] the partition function \( Z_n \) of the Hamiltonian (2.1) is given by

\[ Z_n = \left( u_1^{(n)} + u_2^{(n)} \right)^2 + \left( u_3^{(n)} + u_4^{(n)} \right)^2, \quad n \geq 1. \]  

(2.2)

Here \( u^{(n)} = (u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, u_4^{(n)}) \) satisfies the following recurrent equation

\[
\begin{align*}
  u_1^{(n+1)} &= a \left( bu_1^{(n)} + b^{-1} u_2^{(n)} \right)^2 \\
  u_2^{(n+1)} &= a^{-1} \left( bu_3^{(n)} + b^{-1} u_4^{(n)} \right)^2 \\
  u_3^{(n+1)} &= a^{-1} \left( b^{-1} u_1^{(n)} + bu_2^{(n)} \right)^2 \\
  u_4^{(n+1)} &= a \left( b^{-1} u_3^{(n)} + bu_4^{(n)} \right)^2,
\end{align*}
\]

(2.3)

where \( a = \exp(J_1\beta), \ b = \exp(J_2\beta) \).

Consider mapping \( F : u = (u_1, u_2, u_3, u_4) \in R_+^4 \rightarrow F(u) = (u_1', u_2', u_3', u_4') \in R_+^4 \) defined by

\[
\begin{align*}
  u_1' &= a \left( bu_1 + b^{-1} u_2 \right)^2 \\
  u_2' &= a^{-1} \left( bu_3 + b^{-1} u_4 \right)^2 \\
  u_3' &= a^{-1} \left( b^{-1} u_1 + bu_2 \right)^2 \\
  u_4' &= a \left( b^{-1} u_3 + bu_4 \right)^2.
\end{align*}
\]

(2.4)

Then the recurrent equation (2.3) can be written as \( u^{(n+1)} = F(u^{(n)}), \ n \geq 0 \) which in the theory of dynamical systems is called trajectory of the initial point \( u^{(0)} \) under action of the mapping \( F \). Thus asymptotic behavior of \( Z_n \) for \( n \rightarrow \infty \) can be determined by values of \( \lim u^{(n)} \) i.e., trajectory of \( u^{(0)} \) under action of \( F \). In this paper we study the trajectory (dynamical system) for a given initial point \( u^{(0)} \in R_+^4 \).

3. Dynamics of \( F \)

3.1. Fixed points. In this subsection we are going to define fixed points, i.e., solutions to \( F(u) = u \).

Denote \( \text{Fix}(F) = \{ u : F(u) = u \} \).

We introduce the new variables \( \alpha = \sqrt{a}, \ \nu_i = \sqrt{u_i}, \ i = 1, 2, 3, 4 \). Then the equation \( F(u) = u \) becomes as

\[
\begin{align*}
  v_1 &= \alpha \left( bv_1^2 + b^{-1} v_2^2 \right) \\
  v_2 &= \alpha^{-1} \left( bv_3^2 + b^{-1} v_4^2 \right) \\
  v_3 &= \alpha^{-1} \left( b^{-1} v_1^2 + bv_2^2 \right) \\
  v_4 &= \alpha \left( b^{-1} v_3^2 + bv_4^2 \right),
\end{align*}
\]

(3.1)
Lemma 3.2. If a vector \( u \) is a fixed point of \( F \) then \( u \in M_1 = \{ u = (u_1, u_2, u_3, u_4) \in R_+^4 : u_1 = u_4, u_2 = u_3 \} \) or \( u \in M_2 = \{ u = (u_1, u_2, u_3, u_4) \in R_+^4 : \sqrt{u_1} + \sqrt{u_4} = \varphi(\sqrt{u_2} + \sqrt{u_3}) \} \), where \( \varphi(x) = \frac{1 + a^{-1}b x}{ab + (b^2 - b^2)x} \).

Proof. From (3.1) we get

\[
\begin{align*}
(v_1 - v_4)[ab(v_1 + v_4) - 1] + (v_2 - v_3)[ab^{-1}(v_2 + v_3)] &= 0, \\
(v_1 - v_4)[(ab)^{-1}(v_1 + v_4)] + (v_2 - v_3)[1 + \alpha^{-1}b(v_2 + v_3)] &= 0.
\end{align*}
\]

If \( v_1 = v_4 \) (resp. \( v_2 = v_3 \)) from the second equation of (3.2) we get \( v_2 = v_3 \) (resp. \( v_1 = v_4 \)). Thus \( v_1 = v_4 \) if and only if \( v_2 = v_3 \). Assume now \( v_1 \neq v_4 \) and \( v_2 \neq v_3 \) then system (3.2) can be reduced to the equation

\[
(b^2 - b^{-2})(v_1 + v_4)(v_2 + v_3) + ab(v_1 + v_4) - \alpha^{-1}b(v_2 + v_3) - 1 = 0.
\]

The equation (3.3) gives \( v_1 + v_4 = \varphi(v_2 + v_3) \). \( \Box \)

Let us first study fixed points of \( F \) which belong in \( M_1 \): the condition \( u_1 = u_4, u_2 = u_3 \) reduces the equation \( F(u) = u \) to the following equation

\[
x = f(x) \equiv a^2 \left( \frac{1 + b^2x}{b^2 + x} \right)^2,
\]

where \( x = \frac{u_1}{u_2} \). Denote

\[
\bar{a} = a^{-2}b^{-6}, \quad \bar{b} = b^4, \quad y = b^2x.
\]

The following lemma gives full description of solutions to (3.4).

Lemma 3.2. (Cf. Lemma 10.7 in [13]). Equation (3.4) has a unique positive, stable solution if \( \bar{b} \leq 9 \). If \( \bar{b} > 9 \), then there exist \( v_1(\bar{b}) \) and \( v_2(\bar{b}) \) such that the conditions 0 < \( v_1(\bar{b}) < v_2(\bar{b}) \) are satisfied and equation (3.4) has three solutions, \( x_1^* < x_2^* < x_3^* \) and \( x_3^* \) is unstable, if \( v_1(\bar{b}) < \bar{a} < v_2(\bar{b}) \) and has two solutions, \( x_1^*, x_2^* \), \( x_3^* \) is unstable (saddle) and \( x_2^* \) is stable, if \( \bar{a} = v_1(\bar{b}) \) or \( \bar{a} = v_2(\bar{b}) \). In this case, we have

\[
v_1(\bar{b}) = \frac{1}{y_1} \left( \frac{1 + y_1}{\bar{b} + y_1} \right)^2,
\]

where \( y_1 \) and \( y_2 \) are the solutions of the equation \( y^2 + (3 - \bar{b})y + \bar{b} = 0 \).

Now we shall give some argument to find fixed points of \( F \) which belong in \( M_2 \). Again use variables \( v_i, i = 1, 2, 3, 4 \), assume \( v_2 + v_3 = C, \) with \( C > \max \{ 0, \frac{ab}{b^2 - b^2} \} \). Using Lemma 3.1 we get \( v_2 = C - v_2 \) and \( v_4 = \varphi(C) - v_1 \). Then from the first equation of (3.1) we obtain \( v_2 = \sqrt{b(\alpha^{-1}v_1 - \frac{v_1^2}{\bar{b}})} \) and from the second equation of (3.1) we have \( P_1(v_1) = 0 \) with a polynomial \( P_1 \) of degree 4, coefficients of which depend on parameters \( \alpha, b \) and \( C \). Thus a quartic equation can be obtained. Such an equations can be solved using known formulas (see [http://mathworld.wolfram.com/QuarticEquation.html]), since we will have some complicated formulas for the coefficients and the solutions, we do not present the solution here.

3.2. Periodic points. A point \( u \) in \( R_+^4 \) is called periodic point of \( F \) if there exists \( p \) so that \( F^p(u) = u \) where \( F^p \) is the \( p \)th iterate of \( F \). The smallest positive integer \( p \) satisfying the above is called the prime period or least period of the point \( u \). Denote by \( \text{Per}_p(F) \) the set of periodic points with prime period \( p \).

Note that the set \( M_1 \) is invariant with respect to \( F \) i.e., \( F(M_1) \subset M_1 \). In this subsection we shall describe some periodic points of \( F \) which belong in \( M_1 \).
Let us first describe periodic points with \( p = 2 \) on \( M_1 \), in this case the equation 
\[
F(F(u)) = u
\]
can be reduced to description of 2-periodic points of the function \( f \) defined in (3.4) i.e., to solution of the equation
\[
f(f(x)) = x. \tag{3.5}
\]

Note that the fixed points of \( f \) are solutions to (3.5), to find other solutions we consider the equation
\[
\frac{f(f(x)) - x}{f(x) - x} = 0,
\]
simple calculations show that the last equation is equivalent to the following
\[
b^4(1 + a^2b^2)^2x^2 + a^2\{b^8 + 2(a^{-2} + a^2)b^6 + 4b^4 - 1\}x + b^4(a^2 + b^2)^2 = 0. \tag{3.6}
\]
The equation has two positive solutions iff
\[
\frac{b^4(1 + a^2b^2)^2}{a^2} + \frac{a^2\{b^8 + 2(a^{-2} + a^2)b^6 + 4b^4 - 1\}}{a^2}x + \frac{b^4(a^2 + b^2)^2}{a^2} = 0.
\]
The solutions to (3.5) which are different from fixed points of \( f \) are vary as
\[
B = \begin{cases} 
> 0, & \text{if } b \geq \sqrt{2} - 1 \text{ or } b \leq \sqrt{2} - 1, a^2 \in (0, b^-) \cup (b^+, +\infty) \\
0, & \text{if } b \leq \sqrt{2} - 1, a^2 = b^-, b^+ \\
< 0, & \text{if } b \leq \sqrt{2} - 1, a^2 \in (b^-, b^+)
\end{cases}
\]
where
\[
b^\pm = \frac{1 - 4b^4 - b^8 \pm (1 - b^4)\sqrt{(b^4 - 1)^2 - 4b^4}}{4b^6};
\]
\[
D = -a^2(b^4 - 1)^2(4b^6a^4 + (3b^8 + 6b^4 - 1)a^2 + 4b^6) = \begin{cases} 
< 0, & \text{if } \sqrt{3^{-1}} < b, b \neq 1 \text{ or } b \leq \sqrt{3^{-1}}, a^2 \in (0, b^-) \cup (b^+, +\infty) \\
0, & \text{if } b = 1 \text{ or } b \leq \sqrt{3^{-1}}, a^2 = b^-, b^+ \\
> 0, & \text{if } b \leq \sqrt{3^{-1}}, a^2 \in (b^-, b^+)
\end{cases}
\]
where
\[
b^\pm_+ = \frac{1 - 3b^8 - 6b^4 \pm \sqrt{(b^4 - 1)^2(9b^4 - 1)}}{8b^6}.
\]
One can check that \( \sqrt{3^{-1}} < \sqrt{2} - 1 \), and for \( b \leq \sqrt{3^{-1}} \) one has \( b^- \leq b^-_* \) and \( b^+ \leq b^+ \).
Thus we have proved the following

**Lemma 3.3.** The solutions to (3.5) which are different from fixed points of \( f \) are vary as follows:
1) If \( \sqrt{3^{-1}} < b, b \neq 1 \) or \( b \leq \sqrt{3^{-1}}, a^2 \in (0, b^-) \cup (b^+, +\infty) \) then the equation (3.6) has no positive solution.
2) If \( b = 1 \) or \( b \leq \sqrt{3^{-1}} \) and \( a^2 = b^-, b^+ \) then the equation (3.6) has unique positive solution \( x_1 = \frac{B - B^0}{2B(1 + a^2b^2)^2} \).
3) If \( b \leq \sqrt{3^{-1}}, a^2 \in (b^-, b^+) \) then there are two positive solutions \( x^\pm = \frac{-B^\pm \sqrt{D}}{2B(1 + a^2b^2)^2} \) to (3.6).

The following lemma gives useful properties of the function \( f \).
Lemma 3.4. 1) If $b > 1$ then the sequence $x_n = f(x_{n-1})$, $n = 1, 2, \ldots$ converges for any initial point $x_0 > 0$, where $f$ is defined in (5.4).

2) If $b < 1$ then the sequence $y_n = g(y_{n-1})$, $n = 1, 2, \ldots$ converges for any initial point $y_0 > 0$, where $g(x) = f(f(x))$.

Proof. 1) For $b > 1$ we have $f'(x) = 2a^2(b^4 - 1) \frac{1 + b^2 x}{(b^2 - x)^3} > 0$ i.e., $f$ is an increasing function. Here we consider the case when the function $f$ has three fixed points $x_i^*$, $i = 1, 2, 3$ (see Lemma 3.2). This proof is more simple for cases when $f$ has one or two fixed points). We have that the point $x_2^*$ is repeller i.e., $f'(x_2^*) > 1$ and the points $x_1^*, x_3^*$ are attractive i.e., $f'(x_1^*) < 1$, $f'(x_3^*) < 1$. Now we shall take arbitrary $x_0 > 0$ and prove that $x_n = f(x_{n-1})$, $n \geq 1$ converges as $n \to \infty$. Consider the following partition $(0, +\infty) = (0, x_1^*) \cup (x_1^*, x_2^*) \cup (x_2^*, x_3^*) \cup (x_3^*, +\infty)$. For any $x \in (0, x_1^*)$ we have $x < f(x) < x_1^*$, since $f$ is an increasing function, from the last inequalities we get $x < f(x) < f^2(x) < f^3(x) = x_1^*$ iterating this argument we obtain $f^{n-1}(x) < f^n(x) < x_1^*$, which for any $x_0 \in (0, x_1^*)$ gives $x_{n-1} < x_n < x_1^*$ i.e., $x_n$ converges and its limit is a fixed point of $f$, since $f$ has unique fixed point $x_1^*$ in $(0, x_1^*)$ we conclude that the limit is $x_1^*$.

For $x \in (x_1^*, x_2^*)$ we have $x_2^* > x > f(x) > x_1^*$, consequently $x_n > x_{n+1}$ i.e., $x_n$ converges and its limit is again $x_1^*$. Similarly, one can show that if $x_0 > x_2^*$ then $x_n \to x_3^*$ as $n \to \infty$.

2) For $b < 1$ we have $f$ is decreasing and has unique fixed point $x_1$ which is repelling, but $g$ is increasing since $g'(x) = f'(f(x))f'(x) > 0$. By Lemma 3.3 we have that $g$ has at most three fixed points (including $x_1$). The point $x_1$ is repelling for $g$ too, since $g'(x_1) = f'(f(x_1))f'(x_1) = (f'(x_1))^2 > 1$. But fixed points $x_-, x_+$ (see Lemma 3.3) of $g$ are attractive. Hence one can repeat the same argument of the proof of the part 1) for the increasing function $g$ and complete the proof.

Lemma 3.3 shows that if $b > 1$ i.e., $J_2 > 0$ then there is no any 2-periodic trajectory of $F$ on $M_1$. Moreover, the following lemma says that if $J_2 > 0$ then there is no any periodic trajectory on $M_1$.

Lemma 3.5. 1) If $J_2 > 0$ then for any $p \geq 2$ the equation $F^p(u) = u$ has no solution $u \in M_1 \backslash \text{Fix}(F)$.

2) If $J_2 < 0$ then for any $p \geq 3$ the equation $F^p(u) = u$ has no solution $u \in M_1 \backslash (\text{Fix}(F) \cup \text{Per}_2(F))$.

Proof. 1) Assume there is a solution $u^{(0)} \in M_1 \backslash \text{Fix}(F)$ then we get $p$–periodic trajectories $u^{(n+p)}_i = u^{(n)}_i$, $i = 1, 2; n = 0, 1, 2, \ldots$. Since the set $M_1$ is invariant with respect to $F$, we obtain

$$x_{n+p} = \frac{u^{(n+p)}_1}{u^{(n+p)}_2} = \frac{u^{(n)}_1}{u^{(n)}_2} = x_n = f^n(x_0).$$

This is a contradiction, since by Lemma 3.3 we have $x_n$ is not periodic.

2) Assume there is a solution $u^{(0)} \in M_1 \backslash (\text{Fix}(F) \cup \text{Per}_2(F))$ then we have

$$y_{n+p} = x_{2n+2p} = \frac{u^{(2n+2p)}_1}{u^{(2n+2p)}_2} = \frac{u^{(2n)}_1}{u^{(2n)}_2} = y_n = f^{2n}(y_0).$$

This is a contradiction, since by Lemma 3.3 we have $y_n$ is not periodic. □

4. Exact Values

Starting from random initial condition (with $u^{(0)}_1 \neq u^{(0)}_4$ and $u^{(0)}_2 \neq u^{(0)}_3$), one iterates the recurrence equations (2.3) and observes their behavior after large number of iterations.
Lemma 3.5 gives

In the simplest situation a fixed point \( u^* = (u^*_1, u^*_2, u^*_3, u^*_4) \in \mathbb{R}_+^4 \) is reached. It corresponds to (see \([20]\)):

- a paramagnetic phase if \( u^* \in M_1 \);
- a ferromagnetic phase if \( u^* \in M_2 \).

If the iterations give a cyclic (periodic), say with period \( p \), sequence then the corresponding phase is called \((p-)commensurate phase\). Finally, the system may remain aperiodic, which corresponds to an \( incommensurate phase\).

The condition \( b > 9 \) of Lemma 3.2 requires that \( J_2 > 0 \). Denote

\[
T_c = \frac{2J_2}{\ln 3}, \quad J_2 > 0.
\]

Lemma 3.2 gives the following

**Theorem 4.1.** If \( T \geq T_c \) then the model (2.1) has unique paramagnetic phase; If \( T < T_c \) then there are exactly three (resp. two) paramagnetic phases if \((J_1, J_2)\) is such that \( b^3 \sqrt{T_1} < a^{-1} < b^3 \sqrt{T_2} \) (resp. \( a^{-1} = b^3 \sqrt{T_1} \) or \( a^{-1} = b^3 \sqrt{T_2} \)).

For the condition \( b < \sqrt{3^{-1}} \) of Lemma 3.3 we need to condition \( J_2 < 0 \). In this case we have \( T_c = \frac{-2J_2}{\ln 3} \), \( J_2 < 0 \).

From Lemma 3.3 we get the following

**Theorem 4.2.** If \( T \geq T_c \) then the model (2.1) (on \( M_1 \)) has unique 2-commensurate phase; If \( T < T_c \) then there are exactly two (resp. one) 2-commensurate phases if \( a^2 \in (b^+_v, b^+_s) \) (resp. \( a^2 = b^-_v \) or \( a^2 = b^+_s \)).

For a fixed temperature \( T = \beta^{-1} < T_c \) we have two critical curves \( a^2 = b^+_v \) i.e., on terms of \( J_1 \) and \( J_2 \) are given by the following explicit relations

\[
J_1 = \frac{1}{2\beta} \ln \left( 8^{-1} \{1 - 3e^{8J_2 \beta} - 6e^{4J_2 \beta} \pm \sqrt{(e^{4J_2 \beta} - 1)^3(9e^{4J_2 \beta} - 1)}\} \right) - 3J_2.
\]

Using Lemma 3.3 and formula (2.2) we can get explicit formulas for the sequence of periodic partition functions:

\[
Z_n = Z_n(y) = 2a^{-2/3} \times
\]

\[
\left\{
\begin{array}{ll}
\left( ab(b + \frac{1}{by})^2 + \frac{1}{ab}(\frac{a}{b} + b)\right)^{-\frac{4}{3}} + a^{\frac{2}{3}} \left( \frac{b}{a} + \frac{b}{y} + b - \frac{2}{a(\frac{b}{y} + b)} \right)^{-\frac{4}{3}} \right)^2, & n = 2m \\
\left( ab(b + \frac{1}{fy(y)})^2 + \frac{1}{ab}(\frac{a}{b} + \frac{f(y)}{y})\right)^{-\frac{4}{3}} + a^{\frac{2}{3}} \left( \frac{b}{a} + \frac{b}{f(y)} + \frac{2}{a(\frac{b}{f(y)} + b)} \right)^{-\frac{4}{3}} \right)^2, & n = 2m + 1
\end{array}
\right.
\]

where \( m = 0, 1, 2, \ldots \); \( y \) is one of \( x_1, x_2, x_3 \) defined in Lemma 3.3 and function \( f \) is given in (3.4).

It is easy to see that if \( x \) is a fixed point of \( f \) then corresponding fixed point of \( F \) has the form \( u^*(x) = (u^*_1(x), u^*_2(x), u^*_3(x), u^*_4(x)) \) with \( u^*_1(x) = a^{-1}(b + (bx)^{-1})^{-2} \) and \( u^*_2(x) = a(b + b^{-1}x)^{-2} \). If \( y \) is a fixed point of \( g \) then corresponding 2-periodic point of \( F \) has the form \( u_{1,2}^{\text{per}}(y) = (u_{1,2}^{\text{per}}(y), u_{2,1}^{\text{per}}(y), u_{1}^{\text{per}}(y)) \) with

\[
\begin{align*}
& u_{1}^{\text{per}}(y) = a^{-1/3} \left( ab(b + (by)^{-1})^{-2} + (ab)^{-1}(by^{-1} + b^{-1})^{-2} \right)^{-2/3}, \\
& u_{2}^{\text{per}}(y) = a^{1/3} \left( a^{-1}b(b + b^{-1}y)^{-2} + ab^{-1}(by + b^{-1})^{-2} \right)^{-2/3}.
\end{align*}
\]

Lemma 3.5 gives
Theorem 4.3. The model (2.1) (on $M_1$) has uncountable set $S$ of incommensurate phases $\mu_u$, where $u \in M_1 \setminus (\text{Fix}(F) \cup \text{Per}_2(F))$. Moreover the set of incommensurate phases can be classified to (uncountable) subsets

$$S_x = \{\mu_u : u \in M_1 \setminus (\text{Fix}(F) \cup \text{Per}_2(F)) \text{ with } \lim_{n \to \infty} F^n(u) = u^*(x)\},$$

where $x$ is an attractive fixed point of $f$ and

$$S^\text{per}_y = \{\mu_u : u \in M_1 \setminus (\text{Fix}(F) \cup \text{Per}_2(F)) \text{ with } \lim_{n \to \infty} F^{2n}(u) = u^\text{per}(y)\},$$

where $y$ is an attractive fixed point of $g$.

5. Concluding remarks

Usually, to describe phases (Gibbs measures) of a given Hamiltonian on a Cayley tree one has correspondence between Gibbs measures and a collection of vectors (real numbers in some particular cases) $\{h_x, x \in V\}$, which satisfy a non-linear equation (see for example, [5], [6], [12]–[16], [19]). The recurrent equation (2.3) considered in this paper (which was obtained in [20]) describes a vector function $\{u^{(n)}, n \in N\}$ which is a particular case of the above mentioned function $h_x$ obtained as $h_x = u^{(n)}$ if $x \in W_n$ i.e., depends only on number of the generation set where belongs $x$ but not on $x$ itself. Thus the solutions to the recurrent equation (2.3) do not fully describe phases of the model (2.1). But deriving of the functional equation for $h_x$ corresponding to the Hamiltonian (2.1) is also difficult, since there is prolonged NNN interaction. Such model can be also studied by a contour argument (see [17] and the references therein), but this argument does not give exact solutions, in general.

By a process of iteration, for the model (2.1) Vannimenus found a new modulated phase, in addition to the expected paramagnetic and ferromagnetic (fixed point) phases and a ($+ + --$) periodic (four cycle antiferromagnetic phase, which consisted of commensurate (periodic) and incommensurate (aperiodic) regions corresponding to so called "devil’s staircase". In this paper, using theory of dynamical systems we have analytically proved many above mentioned results, i.e., the following exact results are obtained:

**Paramagnetic phase**: The exact critical temperature and exact critical curves are found. It is proven that the number of the paramagnetic phases can be at most three. (Theorem 4.1).

**Ferromagnetic phase**: We reduced description of such phases to a quartic equation (i.e., solution of the equations on $M_2$). But we were not able to study the periodic solutions on $M_2$.

**Commensurate phase**: The exact critical temperature (which is obtained from the critical temperature of the paramagnetic phase by replacing $J_2$ with $-J_2$) and exact critical curves are found. On the set $M_1$ it is proven that the number of the 2-commensurate phases can be at most two and there is not $p$-commensurate phases if $p \geq 3$ (Lemma 3.5, Theorem 4.2). We also described exact values of periodic partition functions.

**Incommensurate phase**: We proved that the model has uncountably many such phases. Moreover we classified them in two classes: the first class contains the phases which are "asymptotically fixed" (set $S_x$); the second class contains the phases which are "asymptotically periodic" (set $S^\text{per}_y$). Note that for the usual Ising model with external field on Cayley trees such infinitely many phases are known (see [7], p.250).
ACKNOWLEDGEMENTS

UAR thanks the Scientific and Technological Research Council of Turkey (TUBITAK) for support and Zirve and Harran Universities for kind hospitality. This work was completed in the Abdus Salam International Center for Theoretical Physics (ICTP), Trieste, Italy and UAR thanks ICTP for providing financial support and all facilities and IMU/CDE-program for travel support. He also thanks the TWAS Research Grant: 09-009 RG/Maths/As-I; UNESCO FR: 3240230333.

REFERENCES

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, (Academic Press, London/New York, 1982).
[2] N.N. Ganikhodjaev, H. Akin, S. Temir, Potts model with two competing binary interactions. *Turk. J. Math.* **31**: 229–238 (2007).
[3] N.N. Ganikhodjaev, F.M. Mukhamedov, C.H. Pah, Phase diagram of the three states Potts model with next nearest neighbour interactions on the Bethe lattice. *Phys. Lett. A* **373**: 33–38 (2008).
[4] N.N. Ganikhodjaev, S. Temir, H. Akin, Modulated phase of a Potts model with competing binary interactions on a Cayley tree. *J. Stat. Phys.* **137**: 701–715 (2009).
[5] N.N. Ganikhodjaev, U.A. Rozikov, A description of periodic extremal Gibbs measures of some lattice models on the Cayley tree, *Theor. Math. Phys.* **111**: 480-486 (1997).
[6] N.N. Ganikhodjaev, U.A. Rozikov, On Ising model with four competing interactions on Cayley tree. *Math. Phys. Anal. Geom.* **12**: 141–156 (2009).
[7] H.O. Georgii, Gibbs measures and phase transitions (Walter de Gruyter, Berlin, 1988).
[8] S. Inawashiro, C.J. Thompson, Competing Ising interactions and chaotic glass-like behaviour on a Cayley tree. *Phys. Lett. B* **97A**: 245–248 (1983).
[9] S. Katsura, M. Takizawa, Bethe lattice and the Bethe approximation, *Prog. Theor.Phys. B* **51**: 82–98 (1974).
[10] A. Mariz, C. Tsallis, E.L. Albuquerque, Phase diagram of the Ising model on a Cayley tree in the presence of competing interactions and magnetic field. *J. Stat. Phys.* **40**: 577–592 (1985).
[11] V.R. Ohanyan, L.N.Ananikyan, N.S.Ananikian, An exact solution on the ferromagnetic face-cubic spin model on a Bethe lattice. *Physica A.* **377**: 501-513 (2007).
[12] R. Pemantle, Y. Peres, The critical ising model on trees, concave recursions and nonlinear capacity, *Ann. Probab.* **38**: 184-206 (2010).
[13] C. Preston, *Gibbs states on countable sets* (Cambridge University Press, London 1974).
[14] U.A. Rozikov, Partition structures of the Cayley tree and applications for describing periodic Gibbs distributions. *Theor. Math. Phys.* **112**: 929–933 (1997).
[15] U.A. Rozikov, Sh.A. Shoyusupov, Fertile three state HC models on Cayley tree. *Theor. Math. Phys.* **156**: 1319-1330 (2008).
[16] U.A. Rozikov, Y.M. Suhov, Gibbs measures for SOS model on a Cayley tree. *Inf. Dim. Anal. Quant. Prob. Rel. Fields* 9: 471-488 (2006).
[17] U.A. Rozikov, A contour method on Cayley trees. *J. Stat. Phys.* **130**: 801–813 (2008).
[18] Ya.G. Sinai, *Theory of phase transitions: Rigorous Results* (Pergamon, Oxford, 1982).
[19] Y.M. Suhov, U.A. Rozikov, A hard-core model on a Cayley tree: an example of a loss network, *Queueing Systems* **46**: 197–212 (2004).
[20] J. Vannimenus, Modulated phase of an Ising system with competing interactions on a Cayley tree. *Z. Phys. B* **43**: 141–148 (1981).

Utkir Rozikov, Institute of Mathematics and Information Technologies, 29, Do’rmon Yo’li str., 100125, Tashkent, Uzbekistan, email: rozikovu@yandex.ru

Hasan Akin, Zirve University, Faculty of Education, Department of Mathematics, Kızılhısar Campus, 27260, Gaziantep, Turkey, email: hasanakin69@gmail.com

Selman Uğuz, Department of Mathematics, Arts and Science Faculty, Harran University, Sanliurfa, 63120, Turkey, email: selmanuguz@gmail.com