Tunable optical activity due to the chiral anomaly in Weyl semimetals

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Weyl semimetals are a three dimensional gapless topological phase in which bands intersect at arbitrary points – the Weyl nodes – in the Brillouin zone. These points carry a topological quantum number known as the chirality and always appear in pairs of opposite chiralities. The notion of chirality leads to anomalous non-conservation of chiral charge, known as the chiral anomaly, according to which charge can be pumped between Weyl nodes of opposite chiralities by an electromagnetic field \( \mathbf{E} \cdot \mathbf{B} \). Here, we propose probing the chiral anomaly by measuring of optical activity of Weyl semimetals. In particular, we observe that applying an \( \mathbf{E} \cdot \mathbf{B} \) field on this state gives it a non-zero gyrotropic coefficient or a Hall-like conductivity, which can then be seen in routine Kerr and Faraday effect experiments. More generally, any experiment that can probe a correlation function that has the same symmetries as the gyrotropic coefficient can, in principle, detect the chiral anomaly as well. We estimate typical sizes of the optical effects and find the Faraday effect to be well within experimental reach.

WSMs present a condensed matter realization of the phenomenon. In this context, the anomaly implies that the charge in the momentum states near left-handed Weyl nodes is not conserved, with the compensation being provided by right-handed Weyl nodes to ensure total charge conservation. In the simplest case of a WSM with just two Weyl nodes, the anomaly can be written as

\[
\partial_\mu j^\mu_{\text{ch}} = \frac{e^2}{4\pi^2\hbar^2} \mathbf{E} \cdot \mathbf{B}
\]  

where \( j^\mu_{\text{ch}} = (j^\mu_+ - j^\mu_-)/2 \) is the four dimensional chiral current and the subscripts \( \pm \) on the currents denote the chirality of the Weyl node contributing to them, and the right hand side states that the pumping is driven by parallel electric and magnetic fields. The electromagnetic fields are space and time-dependent in general. In the absence of any spatial variations, (2) reduces to

\[
\partial_\mu \rho_{\text{ch}} = \frac{e^2}{4\pi^2\hbar^2} \mathbf{E}(t) \cdot \mathbf{B}(t)
\]

For time-independent \( \mathbf{E} \) and \( \mathbf{B} \), \( \rho_{\text{ch}} \) grows linearly with time until a scattering process cuts off the growth by relaxing the charge imbalance between the Weyl nodes. (2) is a two-dimensions higher version of the chiral anomaly present at the edge of an integer quantum Hall state \[16\], where charge can be pumped from one edge to the other by a longitudinal electric field: \( \partial_\mu j^\mu_{\text{ch}} = \frac{e^2}{2\pi^2\hbar} \cdot \mathbf{E} \). An important difference, however, is that the chiral currents \( j^\mu_{\pm} \) in the integer quantum Hall state reside on spatially separated edges and can be observed individually with local probes, whereas the chiral currents in WSMs are separated in momentum space and cannot be distinguished by spatially local probes. The question is, what kind of probe can qualitatively see the chiral anomaly?

A transport phenomenon intimately tied to the chiral anomaly that was noticed early on was a large longitudinal magnetoconductivity. This occurs because relaxation of chiral charge involves large momentum scattering and hence takes a long time in clean systems.  

where \( k \) is the momentum relative to the Weyl node and \( \sigma_i \) are Pauli matrices in the local band basis. Hence, the name WSMs.

WSMs are bestowed with a physical property known as the Adler-Bell-Jackiw anomaly or the chiral anomaly[6, 7]. This is a well-known phenomenon in high energy physics. It represents an anomalous non-conservation of chiral charge in the presence of appropriate external electromagnetic fields, even though the Hamiltonian enjoys the continuous symmetry – the chiral gauge symmetry – corresponding to chiral charge conservation via Noether’s theorem. The resolution to this paradox lies in the fact that the chiral gauge transformation modifies the integration measure in the path integral, and hence the path integral itself. Thus, the chiral anomaly is a purely quantum process that the classical Hamiltonian is oblivious to. Viewed differently, it is an artifact of the low energy Weyl theory, and appropriate regularization at high energies that smoothly interpolates between Weyl Hamiltonians of opposite chiralities would destroy the chiral gauge symmetry[2].
However, this effect is quantitative and is difficult to identify unambiguously in magnetotransport data\cite{17}. Recently, another transport experiment was proposed in which chiral charge pumping according to\cite{3} resulted in a large enhancement of the length scale over which an applied local voltage decayed\cite{18}. There exist various other predictions for transport phenomena whose origins can be traced to the chiral anomaly, most famously, the \textit{chiral magnetic effect}\cite{10,19,21}, in which a current flows along an applied magnetic field, and a non-quantized anomalous Hall effect\cite{4,12,14,21,24}. All these experiments are purely transport-based, however, and it would be nice to find signatures of the anomaly in other kinds of experiments as well. More importantly, one might ask: what material properties are sensitive to the chirality of a given system? Experiments that measure these properties can, in principle, be designed to measure the chiral anomaly in WSMs as well.

\textbf{Gyrotropy:} In this work, we propose an optical signature of the chiral anomaly, and show that an appropriate material property to measure is its gyrotropic coefficient $\gamma$, which is defined in terms of the dielectric tensor $\varepsilon_{ij}(q, \omega)$ as\cite{25}

$$\varepsilon_{ij}(q, \omega) = \varepsilon_0(\omega)\delta_{ij} + i\gamma(\omega)\varepsilon_{ijk}q_k + \mathcal{O}(q^2). \quad (4)$$

where $\delta_{ij}$ and $\varepsilon_{ijk}$ are the Kronecker delta and the antisymmetric tensor, respectively. More generally, any three-point correlation function that has the same symmetries as $\gamma$ (and correspondingly, experiments measuring that function) will be sensitive to the anomaly.

$\gamma$ is even under time-reversal, but vanishes in systems that have a mirror symmetry. Systems that break all mirror symmetries and hence, break inversion symmetry, have a non-vanishing $\gamma$ in general and can be assigned a handedness equal to $\text{sgn}(\gamma)$. In particular, a single Weyl node is chiral and can have a non-zero $\gamma \propto \chi$\cite{20}. However, any symmetries relating Weyl nodes of opposite chiralities will make the total $\gamma$ of the WSM vanish. Since spatially local probes only detect the total response of all the Weyl nodes, they will then see a null result for $\gamma$. To get a non-zero result, one must find a way to subtract, rather than add, contributions to $\gamma$ from Weyl nodes of opposite chirality.

Such a subtraction can be done by observing that $\gamma$ is related to the Hall conductivity through the relation $\sigma_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_0}$, where $\varepsilon_0$ is the permittivity of free space, so it must have contributions that are odd in the charge of the quasiparticles. If one can somehow arrange for the doping to be different at the $\chi = +1$ and $\chi = -1$ Weyl nodes, their contributions to $\gamma$ will no longer cancel. The anomaly induced by $\mathbf{E} \cdot \mathbf{B}$ (\cite{2} and \cite{49}) precisely ensures such a charge imbalance between the two chiralities. In other words, once a finite amount of charge has been pumped from $\chi = -1$ to $\chi = +1$, the $\chi = -1$ ($\chi = +1$) Weyl node is surrounded by a hole (electron) Fermi surface assuming they were undoped initially. If they were already doped, their local Fermi levels will become different. In the language of symmetries, chiral charge pumping ensures that all symmetries relating Weyl nodes of opposite chiralities are broken because they have different Fermi levels relative to the Weyl points.

Physically, $\gamma$ is a measure of the optical activity of the system. In particular, systems with non-zero $\gamma$ exhibit the Faraday effect – rotation of the plane of polarization of linearly polarized light as it propagates through the system. This can be seen as follows. The eigenmodes of the dielectric tensor (4) correspond to circularly polarized light; the associated eigenvalues determine the refractive indices for the two polarizations, $n_{L,R}^2(\omega) = \varepsilon_0(\omega) \pm \gamma q$. Since $n_L \neq n_R$, the two circular polarizations making up the linear polarization pick up a relative phase as the light propagates through the system, resulting in the Faraday effect. It is straightforward to show that the rotation per unit length is given by

$$\frac{d\theta_F}{dl} = \frac{\omega^2}{2c^2} \mathcal{R}e[\gamma(\omega)] \quad (5)$$

where $c$ is the speed of light. If the system is dissipative, then it has been argued that it also exhibits a Kerr effect – rotation of the polarization plane during reflection. Solving Maxwell’s equations at the boundary between vacuum and a gyrotropic medium and carefully applying boundary conditions at the interface gives the Kerr angle\cite{27}:

$$\tan \theta_K = -\frac{\omega}{c}n_{\mathcal{R}e} \left[ \frac{\gamma(\omega)}{\varepsilon_{xx}(\omega) - 1} \right] \quad (6)$$

Thus, we propose detecting the chiral anomaly \textit{optically}. Later in the paper, we estimate the sizes of these effects for typical WSMs and find the Faraday effect to be well within experimental limits.

While the Faraday effect is a purely bulk phenomenon, the Kerr effect involves reflection off a surface. An arbitrary surface of a WSM carries Fermi arc surface states, which can pollute the results of a Kerr measurement focusing on gyrotropic effects. At the moment, there is no theory that allows computation of the Kerr angle because of the Fermi arcs, and developing one is outside the scope of this paper. Rather, we propose avoiding the effect of the Fermi arcs by performing the Kerr experiment on a surface that has no Fermi arcs, if possible. On the other hand, the Faraday effect is not expected to be affected by surface phenomenon such as Fermi arcs. Thus, we favor measuring the Faraday effect over the Kerr effect, although we present results for both based on\cite{5} and\cite{9}.

A caveat, though, is that WSMs can at most preserve only one out of $\mathcal{I}$ and $\mathcal{T}$ symmetries. Consequently, a $\mathcal{T}$-symmetric WSM already has a non-zero $\gamma$ in general, without external fields, while an $\mathcal{I}$-symmetric WSM has
vanishing \( \gamma \) but can be optically active due to possible ferromagnetic moments. Later we will describe how the background contributions to optical activity can be separated from the anomaly-based ones by a clever separation of their frequencies.

**Single Weyl node results:** In the following, we will first calculate the gyrotropic coefficient of a single Weyl node and then apply the results to WSMs with multiple Weyl nodes. The calculation is straightforward. For simplicity, we first determine the contribution to \( \gamma \) due to a constant \( E \cdot B \) field. Later we argue that time-dependent fields greatly facilitate separating the anomalous contribution to \( \gamma \) from possible background terms, and modify the results accordingly.

We start with the Hamiltonian \( H^X_k = \chi \hbar v_F k \cdot \sigma - \mu_\chi \) for a single Weyl node \( W^X \) of chirality \( \chi \) and chemical potential \( \mu_\chi \) above the Weyl point. \( \mu_\chi \) consists of two parts in general – the background chemical potential due to doping already present in the system, and the change because of charge pumping. The latter grows linearly with time in the absence of large momentum scattering; in practice, such scattering is present and the system reaches a non-equilibrium steady state characterized by a relaxation time \( \tau \). \( \chi \) dictates that the density of electrons pumped into the neighborhood of \( W_\chi \) in this time is \( \Delta \rho_\chi = \chi \frac{e^2}{2 \hbar c} E \cdot B \tau \). We then calculate the dielectric tensor \( \epsilon_{ij}(q, \omega) \) using the Kubo formula:

\[
\epsilon_{ij}(q, \omega) = \delta_{ij} + \frac{\Pi_{ij}(q, i\omega \rightarrow \omega + i\delta)}{\varepsilon_0 \omega^2} \tag{7}
\]

\[\Pi_{ij}(q, i\omega) = \int_{k,\Omega} \text{Tr} \left[ \sigma_i \frac{1}{i(\Omega + \omega) - H^X_k - q} \sigma_j \frac{1}{i\Omega - H^X_{k+q}} \right] \tag{8}\]

Assuming \( q = q \hat{z} \) without loss of generality, we get, to first order in \( q \) (See Appendix for details of the calculation and more general expressions)

\[
\epsilon_{xy}(q, \omega) = -\chi \frac{\alpha}{3\pi} \frac{q}{\omega} \left( \frac{\pi}{8} - i \right) \left[ \frac{\Lambda}{\hbar \omega} + 4 \left( \frac{\mu_\chi}{\hbar \omega} \right)^3 \right] \tag{9}
\]

\[
\epsilon_{xx}(q, \omega) = 1 - \frac{\alpha}{6\pi} \left( \frac{2\mu_\chi}{\hbar \omega} \right)^2 \left\{ -\log \left| \frac{4\Lambda^2}{4\mu_\chi - \omega^2} - i \pi \text{sgn}(\omega) \right| \right\} \tag{10}
\]

Kerr effects in a real WSM. For simplicity, we assume an \( \mathcal{I} \)-symmetric system with two Weyl nodes, one of each chirality. The results can be trivially scaled to \( N \) nodes. If the Fermi level is tuned to the Weyl point in the absence of external electromagnetic fields – the iridate (candidate) WSMs are expected to be in this limit because of their stoichiometry – then, there is no background chemical potential and \( \mu_\chi \) only depends on the charge pumped:

\[\mu_\chi^{\text{undoped}} = \chi \left( \frac{3e^2 \hbar v_F^3}{2} E \cdot B \tau \right)^{1/3} \tag{11}\]

On the other hand, if there is already a background
Moreover, the potentially large background and is thus error-prone. The procedure can work in principle, it involves subtracting \( \omega \) terms that break effect measurements are commonly used to study systems that break time-dependent. Thus, apply polarization or magnetization.

\[
\gamma(\omega) = \frac{4\alpha e^2 v_F^2 \tau}{\hbar^2 \omega^4} E \cdot B
\]  

(12)

\( \gamma(\omega) \) is identical for both doped and undoped (without external fields) WSMs because of the assumption \( |\mu_\gamma| \ll \hbar \omega \), a physically relevant limit for the Faraday effect. The Faraday and Kerr angles according to (\( \gamma \)) and (\( \theta \)) are

\[
\frac{d\theta_F}{d\ell} = -\frac{e^2 v_F^2 \tau}{\hbar^2 \omega^2} \frac{12\pi}{4 \log^2 \frac{|E|}{\hbar \omega} + \pi^2} E \cdot B
\]  

(13)

where we have assumed \( |\mu_+^{doped} - \mu_-^{doped}| \ll \epsilon_F \) in writing the Kerr angle expression (14). Putting in realistic values of parameters, \( v_F = 10^8 \text{m/s}^{-1}, |E| = 1 \text{V/mm}, \) \( |B| = 1 \text{T}, \tau = 10 \text{ps}, \hbar \omega = 0.1 \text{eV}, \Lambda \sim 1 \text{eV} \) gives \( |d\theta_F/d\ell| \sim 1 \text{mrad/mm} \) and \( |\theta_K| \sim 0.1 \text{prad} \). With these values, the Faraday effect should be easily measurable while the Kerr effect is rather small. Moreover, the effects will be enhanced in a sufficiently clean system, in which internode scattering will be suppressed since it involves a large momentum transfer and the relaxation time \( \tau \) will consequently be amplified.

**Subtracting the background:** Faraday and Kerr effect measurements are commonly used to study systems that break \( T \) or \( \mathcal{I} \) symmetry. WSMs break at least one of \( T \) and \( \mathcal{I} \) symmetries and thus, exhibit intrinsic, i.e., \( T\times B \) independent optical activity in general. The final piece of the puzzle of probing the anomaly via gyrotropy is being able to subtract this background.

One way to do so is to simply do an experiment without \( E \) and \( B \) fields and subtract the results from the results in the presence of an \( E \times B \) field. While this procedure can work in principle, it involves subtracting a potentially large background and is thus error-prone. Moreover, the \( E \) and \( B \) fields can change the optical activity independently of the chiral anomaly as well, for instance, by inducing polarization or magnetization.

A cleaner procedure would be to make the fields time-dependent. Thus, apply \( E(t) = E_0 \cos \Omega_1 t \) and \( B(t) = B_0 \cos \Omega_2 t \), such that \( \Omega_1, \Omega_2 \ll \tau^{-1} \), and measure the optical activity at a higher frequency \( \omega \gg \tau^{-1} \). Then, \( E(t) \) and \( B(t) \) can be treated quasistatically and the preceding analysis can be applied with minor modifications. In particular, the gyroscopic coefficient picks up a slow time-dependence:

\[
\gamma(\omega; t) = \frac{4\alpha e^2 v_F^2 \tau}{\hbar^2 \omega^4} E_0 \cdot B_0 \cos \Omega_1 t \cos \Omega_2 t
\]  

(15)

for \( \tau \ll t \ll \Omega_1^{-1} \), and thus has components at frequencies \( \Omega_1 \pm \Omega_2 \) which should be easily separable from other frequency components. In addition to the time-dependence of \( \gamma \), its dependence on the relative angle between \( E \) and \( B \) should be easy to observe as well on top of the constant background.

Before concluding, we compare the Faraday and Kerr effects described here with those in another system in which an \( E \cdot B \) electromagnetic field plays a vital role — the strong topological insulator WSMs as well. There, an \( E \cdot B \) term appears directly in the action, and has been shown to lead to measurable Faraday and Kerr effects. In WSMs, the effects depend on the \( \omega, v_F, \tau \) and \( E \cdot B \) besides fundamental constants, and the Faraday rotation angle grows with distance. In contrast, topological insulators with gapped surface states — either due to magnetization or due to finite size — exhibit quantized, constant rotation angles in both Faraday and Kerr experiments that only depend on fundamental constants and the permittivity and permeability of the system.

In summary, we have described method to probe the chiral anomaly in WSMs via optical measurements of their dielectric properties. Unlike the anomaly at a one dimensional quantum Hall edge, the anomaly in WSMs cannot be probed by spatially local probes. Instead, we observe that their optical properties can be controlled by an \( E \cdot B \) electromagnetic field — a dependence characteristic of the chiral anomaly. The precise material parameter involved is the gyrotropy \( \gamma \), a Hall-like contribution to the dielectric tensor. \( \gamma \) determines the optical activity of the system and is non-vanishing only when all mirror symmetries and hence, \( \mathcal{I} \) symmetry, are broken. A single Weyl node has a non-zero \( \gamma \) proportional to its chirality. However, the total \( \gamma \) vanishes in an \( \mathcal{I} \)-symmetric WSM because it has degenerate Weyl nodes of opposite chirality. An \( E \cdot B \) field breaks \( \mathcal{I} \) by pumping charge between Weyl nodes of opposite chirality due to the chiral anomaly, resulting in a non-vanishing total \( \gamma \) proportional to the amount of charge pumped. A routine Faraday rotation experiment can then detect the effect. Applying a time-dependent \( E \cdot B \) will help separate the anomalous contributions from non-anomalous ones. We estimate typical sizes of the effect and find it to be accessible by current experiments. Finally, other experiments that can measure the gyroscopic coefficient can, in principle, be designed to probe the chiral anomaly in WSMs as well.

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[1] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011)

[2] P. Hosur and X. Qi. ArXiv e-prints (2013), arXiv:1309.4464 [cond-mat.str-el]
where we have used the definition of the current operator: $j_i^X = \frac{\partial H_k^X}{\partial k_i} = \chi \sigma_i$. The sum over Matsubara frequencies $i\Omega$ is carried out by the standard contour integration technique, which introduces Fermi functions $f(k) = [e^{\beta E_k} + 1]^{-1}$.
into the expression for $\Pi_{ij}(q, \omega)$:

$$\Pi_{ij}(q, \omega) = - \int_k \frac{f(-|k+q|-\mu)}{-2|k+q|} \text{Tr} \left[ \sigma_i (-|k+q| + \chi(k+q) \cdot \sigma) \sigma_j \frac{-i\omega - |k+q| + \chi \cdot \sigma}{-i\omega - |k+q|^2 - k^2} \right]$$

$$- \int_k \frac{f(|k+q|-\mu)}{2|k+q|} \text{Tr} \left[ \sigma_i (|k+q| + \chi(k+q) \cdot \sigma) \sigma_j \frac{-i\omega + |k+q| + \chi \cdot \sigma}{-i\omega + |k+q|^2 - k^2} \right]$$

$$- \int_k \frac{f(-k-\mu)}{-2k} \text{Tr} \left[ \sigma_i i\omega - k + \chi(k+q) \cdot \sigma_i \frac{-i\omega - k + \chi \cdot \sigma}{i\omega - k^2 - |k+q|^2} \right]$$

$$- \int_k \frac{f(k-\mu)}{2k} \text{Tr} \left[ \sigma_i i\omega + k + \chi(k+q) \cdot \sigma_j \frac{i\omega + k + \chi \cdot \sigma}{i\omega + k^2 - |k+q|^2} \right]$$

(A4)

Shifting $k$ by $q$ in the first two terms and grouping terms with the same argument in the Fermi functions together gives

$$\Pi_{ij}(q, \omega) = - \int_k \frac{f(-k-\mu)}{-2k} \text{Tr} \left[ \sigma_i (-k + \chi \cdot \sigma) \sigma_j \frac{-i\omega - k + \chi(k-q) \cdot \sigma}{-i\omega - k^2 - |k-q|^2} + \sigma_i i\omega - k + \chi \cdot \sigma \right]$$

$$- \int_k \frac{f(k-\mu)}{2k} \text{Tr} \left[ \sigma_i (k + \chi \cdot \sigma) \sigma_j \frac{-i\omega + k + \chi(k-q) \cdot \sigma}{i\omega + k^2 - |k-q|^2} + \sigma_i i\omega - k + \chi \cdot \sigma \right]$$

(A5)

The traces of products of Pauli matrices are given by

$$\text{Tr}[\sigma_i \sigma_j] = 2\delta_{ij}$$

$$\text{Tr}[\sigma_i \sigma_j \sigma_k] = 2i\varepsilon_{ijk}$$

$$\text{Tr}[\sigma_i \sigma_j \sigma_k \sigma_l] = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

(A6)\ (A7)\ (A8)

Using these

$$\Pi_{ij}(q, \omega) = - \int_k \frac{f(-k-\mu)}{-k} \left\{ \delta_{ij} \frac{k(i\omega + k) + i\varepsilon_{ijk} \chi(kq_l + k_i \omega) + \delta_{il} \delta_{jm} - \delta_{ij} \delta_{km} + \delta_{im} \delta_{jl}}{[i\omega - k^2 - |k+q|^2]} k_l(k_m - q_m) \right\} + (k \rightarrow -k)$$

$$- \int_k \frac{f(-k-\mu)}{-k} \left\{ -\delta_{ij} \frac{i\varepsilon_{ijk} \chi(i\omega k_l + q_j k_l)}{[i\omega - k^2 - |k+q|^2]} \right\} + (k \rightarrow -k)$$

(A9)

Doing the $m$-sum

$$\Pi_{ij}(q, \omega) = - \int_k \frac{f(-k-\mu)}{-k} \left\{ \delta_{ij} \frac{k(i\omega + k) + i\varepsilon_{ijk} \chi(kq_l + k_i \omega) + \delta_{il} k_l k_m - \delta_{ij} k_m - \delta_{im} k_l} {[i\omega - k^2 - |k+q|^2]} \right\} + (k \rightarrow -k)$$

$$- \int_k \frac{f(-k-\mu)}{-k} \left\{ -\delta_{ij} \frac{i\varepsilon_{ijk} \chi(i\omega k_l + q_j k_l)}{[i\omega - k^2 - |k+q|^2]} \right\} + (k \rightarrow -k)$$

(A10)

The integrals have azimuthal symmetry about $q$. We therefore choose $q = q\hat{z}$ without loss of generality and calculate various longitudinal and Hall components of $\Pi_{ij}$.

1. Longitudinal susceptibility

The longitudinal susceptibility is

$$\Pi_{xx}(q, \omega) = \int_k \frac{f(-k-\mu)}{k} \left[ \frac{k(i\omega + k) - k_z (k_z - q)}{[i\omega + k]^2 - |k-q|^2} - \frac{(i\omega - k) + k_z (k_z + q)}{[i\omega - k]^2 - |k+q|^2} \right]$$

$$+ \int_k \frac{f(-k-\mu)}{-k} \left[ -\frac{k(i\omega + k) - k_z (k_z - q)}{[i\omega - k]^2 - |k-q|^2} - \frac{(i\omega - k) + k_z (k_z + q)}{[i\omega + k]^2 - |k+q|^2} \right]$$

(A11)
This is clearly even in $q$. For small $q$, we can Taylor expand in powers of $q$. The leading order ($q = 0$) term is

$$
\Pi_{xx}(0, i\omega) = \frac{1}{i\omega} \int_k [f(-k - \mu) - f(k - \mu)] \left[ \frac{i\omega + 2k/3}{i\omega + 2k} - \frac{i\omega - 2k/3}{i\omega - 2k} \right]
$$

(A12)

where we have replaced $k_2 \rightarrow k^2/3$ in the spherical integral. We now analytically continue to real time by making the substitution $i\omega \rightarrow \omega + i\delta$ and evaluate the real and imaginary parts separately. The imaginary part is

$$
\mathcal{Im}[\Pi_{xx}(\omega)] = -\frac{4}{3\pi^2} \int k^3 dk \Im \left[ \frac{f(-k - \mu) - f(k - \mu)}{(\omega + i\delta)^2 - 4k^2} \right]
$$

(A13)

at zero temperature. In the above equations, the $\delta$-functions in the second line indicate the fact that longitudinal conduction occurs via resonant absorption of photons driving transitions between electronic states related by particle-hole symmetry. This is why the third line above contains $\Theta(\omega/2 - |\mu|)$, since a transition can only occur from an occupied to an unoccupied state. Clearly, such processes are only possible in systems with particle-hole symmetry; an ordinary metal only has a Drude peak at $\omega = 0$ with a temperature dependent width and thus has a vanishing $\mathcal{Im}[\Pi_{xx}(\omega)]$ at $\omega \neq 0$ at zero temperature.

The real part of $\Pi_{xx}(\omega)$ at zero temperature is

$$
\mathcal{Re}[\Pi_{xx}(\omega)] = -\frac{4}{3\pi^2} \int k^3 dk \frac{\Theta(\mu + k) - \Theta(\mu - k)}{\omega^2 - 4k^2}
$$

(A14)

where we have introduced a cutoff $\Lambda$. $\Lambda$ physically corresponds to the energy scale at which non-linearities in the dispersion become important and is typically an $\mathcal{O}(1)$ or $\mathcal{O}(1/10)$ fraction of the bandwidth. In the last line, we have subtracted an unobservable $\omega$- and $\mu$-independent part of the response $\propto \Lambda^2$, which arises from the filled valence band. The plasma frequency $\omega_p$, obtained by solving $1 + \mathcal{Re}[\Pi_{xx}(\omega_p)]/\varepsilon_0 \omega_p^2$, is equal to $\mu$ up to proportionality factors of $\mathcal{O}(1)$ and logarithmic corrections. This means Faraday experiments can only be performed at frequencies $\omega \gtrsim |\mu|$, simply because the EM wave will not be able to penetrate into the system at lower frequencies.

2. Hall susceptibility

We now calculate the Hall susceptibilities $\Pi_{ij}$ for $i \neq j$. By azimuthal symmetry, $\Pi_{xz}(q\hat{z}, i\omega) = \Pi_{yz}(q\hat{z}, i\omega) = 0$. The remaining component is

$$
\Pi_{xy}(q, i\omega) = -\int_k \frac{f(-k - \mu)}{k} \{i\chi(kq + kz i\omega) \left\{ \frac{1}{-i\omega - k^2} - \frac{1}{|k - q|^2} \right\} + \frac{1}{i\omega - k^2} - \frac{1}{|k + q|^2} \}
$$

(A15)

$$
-\int_k \frac{f(k - \mu)}{k} \{i\chi(-kq + kz i\omega) \left\{ \frac{1}{-i\omega + k^2} - \frac{1}{|k - q|^2} \right\} + \frac{1}{i\omega + k^2} - \frac{1}{|k + q|^2} \}
$$

(A16)
Taylor expanding to $\mathcal{O}(q)$ gives

\[
\Pi_{xy}(q, i\omega) = -\int \frac{f(-k - \mu)}{-k} \{i\chi(kq + kz\omega)\} \left\{ \frac{1}{(i\omega)^2 + 2i\omega k} - \frac{2kzq}{[(i\omega)^2 + 2i\omega k]^2} + \frac{1}{(i\omega)^2 - 2i\omega k} + \frac{2kzq}{[(i\omega)^2 - 2i\omega k]^2} \right\}
\]

\[
-\int \frac{f(k - \mu)}{k} \{i\chi(-kq + kz\omega)\} \left\{ \frac{1}{(i\omega)^2 - 2i\omega k} - \frac{2kzq}{[(i\omega)^2 - 2i\omega k]^2} + \frac{1}{(i\omega)^2 + 2i\omega k} + \frac{2kzq}{[(i\omega)^2 + 2i\omega k]^2} \right\}
\]

(A17)

The 0th order term in $q$ vanishes because the integrand becomes rotationally symmetric at $q = 0$. To order $q$,

\[
\Pi_{xy}(q, i\omega) = -\int \frac{f(-k - \mu)}{-k} \{i\chi\} \left\{ \frac{kq}{(i\omega)^2 + 2i\omega k} - \frac{2k^2 q i\omega/3}{[(i\omega)^2 + 2i\omega k]^2} + \frac{kq}{(i\omega)^2 - 2i\omega k} + \frac{2k^2 q i\omega/3}{[(i\omega)^2 - 2i\omega k]^2} \right\}
\]

\[
-\int \frac{f(k - \mu)}{k} \{i\chi\} \left\{ \frac{-kq}{(i\omega)^2 - 2i\omega k} - \frac{2k^2 q i\omega/3}{[(i\omega)^2 - 2i\omega k]^2} + \frac{-kq}{(i\omega)^2 + 2i\omega k} + \frac{2k^2 q i\omega/3}{[(i\omega)^2 + 2i\omega k]^2} \right\}
\]

(A18)

where we have replaced $k^2$ by $k^2/3$ because the rest of the integrals in those terms is rotationally invariant. This simplifies to

\[
\Pi_{xy}(q, i\omega) = \frac{i\chi q}{\pi^2} \int k^2 dk \left[ f(-k - \mu) + f(k - \mu) \right] \frac{(i\omega)^2 - 4k^2/3}{[(i\omega)^2 - (2k)^2]^2}
\]

(A19)

At zero temperature, this becomes

\[
\Pi_{xy}(q, i\omega) = \frac{i\chi q}{\pi^2} \int k^2 dk [\Theta(\mu + k) + \Theta(\mu - k)] \frac{(i\omega)^2 - 4k^2/3}{[(i\omega)^2 - (2k)^2]^2}
\]

(A20)

Analytically continuing to real time

\[
\Pi_{xy}(q, i\omega \rightarrow \omega + i\delta) = \frac{i\chi q}{\pi^2} \int k^2 dk [\Theta(\mu + k) + \Theta(\mu - k)] \frac{(\omega + i\delta)^2 - 4k^2/3}{[\omega + i\delta - 2k]^2[\omega + i\delta + 2k]^2}
\]

(A21)

\[
= \frac{i\chi q}{\pi^2} \int k^2 dk [\Theta(\mu + k) + \Theta(\mu - k)] (\omega^2 - 4k^2/3) \left[ \frac{1}{(\omega^2 - 4k^2)^2} + i\pi \left( \frac{\delta'(\omega - 2k)}{(\omega + 2k)^2} + \frac{\delta'(\omega + 2k)}{(\omega - 2k)^2} \right) \right]
\]

The imaginary part is

\[
\mathcal{I}m[\Pi_{xy}(q, \omega)] = \frac{\chi q}{\pi^2} \int k^2 dk [\Theta(\mu + k) + \Theta(\mu - k)] \frac{\omega^2 - 4k^2/3}{(\omega^2 - 4k^2)^2}
\]

\[
= \frac{\chi q}{\pi^2} \left[ \frac{-k^3}{3(4k^2 - \omega^2)} \right]_0^\Lambda + \frac{\chi q}{\pi^2} \text{sgn}(\mu) \left[ \frac{-k^3}{3(4k^2 - \omega^2)} \right]_0^{|\mu|}
\]

\[
= -\frac{\chi q}{3\pi^2} \left[ \frac{\Lambda^3}{4\Lambda^2 - \omega^2} + \frac{\mu^3}{4\mu^2 - \omega^2} \right]
\]

(A22)

For $|\mu| \ll |\omega| \ll \Lambda$, this becomes

\[
\mathcal{I}m[\Pi_{xy}(q, \omega)] \approx -\frac{\chi q}{3\pi^2} \left[ \frac{\Lambda^3}{4} + \frac{\mu^3}{\omega^2} \right]
\]

(A23)

The real part is

\[
\mathcal{R}e[\Pi_{xy}(q, \omega)] = -\frac{\chi q}{\pi} \int k^2 dk [\Theta(\mu + k) + \Theta(\mu - k)] (\omega^2 - 4k^2/3) \left( \frac{\delta'(\omega - 2k)}{(\omega - 2k)^2} + \frac{\delta'(\omega + 2k)}{(\omega + 2k)^2} \right)
\]

\[
= -\frac{\chi q \omega}{90\pi} \{1 + \text{sgn}(\mu) [\Theta(|\mu| - |\omega|/2) - |\omega|\delta(|\mu| - |\omega|/2)] \}
\]

\[
= \frac{\chi q \omega}{90\pi}
\]

(A24)

for $|\mu| \ll |\omega|$. 