One-loop quantum cosmological correction to the gravitational constant using the kink solution in de Sitter universe

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Abstract

In this paper, we show the equivalence between a classical static scalar field theory and the (closed) de Sitter cosmological model whose potential represents shape invariance property. Based on this equivalence, we calculate the one-loop quantum cosmological correction to the ground state energy of the kink-like solution in the (closed) de Sitter cosmological model in which the fluctuation potential $V''$ has a shape invariance property. It is shown that this type of correction, which yields a renormalized mass in the case of scalar field theory, may be interpreted as a renormalized gravitational constant in the case of (closed) de Sitter cosmological model.

Keywords: One-loop correction; kink energy; shape invariance; zeta function regularization; de Sitter universe.

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1 Introduction

The quantum corrections to the mass of classical topological defects plays an important role in the semi-classical approach to quantum field theory [1, 2]. The computation of quantum energies around classical configurations in (1 + 1)-dimensional kinks has been developed in [3-6] by using topological boundary conditions, the derivative expansion method [7-11], the scattering phase shift technique [12], the zeta-function regularization technique [13-14], and also the dimensional regularization method [15]. In a previous paper by one of the authors, the one-loop renormalized kink quantum mass correction in a (1 + 1)-dimensional scalar field theory model was derived using the generalized zeta function method for those potentials where the fluctuation potential $V''$ has the shape invariance property [16]. These potentials are very important since they possess a shape invariant operator in their prefactor which makes the corrections of this kind of potential exact by the heat kernel method. This kind of potential occurs in different fields of physics particularly in quantum gravity and cosmology. An empty closed Friedmann-Robertson-Walker (FRW) universe with a decaying cosmological term $\Lambda \sim a^{-m}$ is an example ($a$ is the scale factor and $m$ is a parameter $0 \leq m \leq 2$) which is equivalent to a cosmology with the exotic matter equation of state $p = (m/3 - 1)\rho$ [17]. The special case $m = 0$ leads to de Sitter spacetime which is of particular interest in the present work.

One of the basic motivations to study the quantum gravity on de Sitter background was the fact that it might be a very reasonable candidate for vacuum state when the classical action contains a positive cosmological term [18-30]. Moreover, such investigations could provide the framework to solve the cosmological constant problem [18-30]. A quite interesting model in this regard was the Coleman-Weinberg type suppression for the effective cosmological constant using the large-distance limit of the quantum gravity one-loop effective action on the de Sitter background [30]. In order to slightly improve the situation such background was replaced by an hyperbolic background (i.e. a gravitational theory with negative cosmological constant) [31]. The one-loop effective action for 4-dimensional gauged supergravity with negative cosmological constant, was also investigated in space-times with compact hyperbolic spatial section. The explicit expansion of the effective action as a power series of the curvature on hyperbolic background was derived, making use of heat-kernel and zeta-regularization techniques [13], and the induced cosmological and Newton constants were computed [32]. It is also worth noticing to numbers of interesting works where quantum vacuum energy in quantum gravity on de Sitter background has been calculated [33-35].

On the other hand, we know that quantum cosmology is a rather simplified and approximate version of quantum gravity where the key role is played by the quantum Wheeler-DeWitt equation. In fact, quantum cosmology is just the quantum mechanics of the whole universe, not a quantum field theory of gravity. Therefore, for the same reason that we are interested in calculation of the vac-
uum energy and its corrections in quantum gravity on de Sitter background, we may study the vacuum energy and its corrections in quantum cosmology in de Sitter universe. However, we have to keep in mind that these approximate corrections are just valuable in the context of quantum cosmology, not quantum gravity, and any nontrivial result obtained in this way should be interpreted quantum mechanically and not quantum field theoretically. For example, one may study the impact of these quantum cosmological corrections on the quantum tunneling rate from *nothing* to de Sitter universe \[36\]. The type of these corrections, however, depends on the way we implement our quantum mechanical approximation.

In this paper, we aim to make maximum use of the shape invariance property of the quantum mechanical potential which appears in the quantum cosmology (Wheeler-DeWitt equation) of de Sitter universe. In this regard, we first show the equivalence between the configuration space of (closed) de Sitter cosmological model and a *classical* static scalar field theory. As mentioned above, we know how to compute the quantum corrections to the vacuum energy of the classical scalar field, which is interpreted as the quantum corrections to the mass of the classical kink solution. Based on this equivalence, we implement the above mentioned techniques to calculate the one-loop quantum cosmological correction to the ground state energy of the kink-like solution in the (closed) de Sitter cosmological model\[1\]. To this end, we first study the quantum cosmology of de Sitter spacetime and construct the Euclidean action in the path integral. Then, by a technic known as Duru-Kleinert equivalency we change this action into the standard quadratic one with shape invariant potential. Using the shape invariance property of the potential, the heat kernel method, and the generalized zeta function regularization method to implement our setup for describing semi-classical kink-like states, we obtain the one-loop quantum correction to the corresponding mass-like term in the kink-like solution of the cosmological model at hand. It is shown that this type of one-loop correction, which yields quantum correction to the mass of the kink solution in the case of classical scalar field theory, may be *interpreted* as a renormalized gravitational constant in the case of (closed) de Sitter cosmological model.

2 Quantum cosmology of de Sitter universe

We shall consider an empty closed \((k = 1)\) FRW universe with a non vanishing cosmological constant \(\Lambda\). The line element is given by

\[
ds^2 = -N(t)^2 c^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \right],
\]

\[\text{(1)}\]

\[\text{It is worth noticing that usually it is not necessary in quantum gravity to use the kink solution for calculation of vacuum energy, but here in quantum cosmology we are making use of the shape invariance property and the equivalence between the configuration space of (closed) de Sitter cosmological model and a classical static scalar field theory, so we shall use the technique where kink solution is involved to compute the quantum energies around the classical kink-like configurations in de Sitter universe.}\]
where \( a(t) \) as the scale factor is the only dynamical degree of freedom and the laps function \( N(t) \) is a pure gauge variable. The pure gravitational action corresponding to (1) is

\[
S = \frac{c^4}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} (R - 2\Lambda) + \frac{c^2}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{g(3)} K
\]

\[
= \frac{3c^2\pi}{4G} \int \left( -\frac{a\dot{a}^2}{N} + c^2 Na - \frac{\Lambda}{3} Na^3 \right) dt := \int dtL.
\]

In equation (2) \( \Lambda \) is the cosmological constant, \( \mathcal{M} = \mathcal{R} \times S^3 \) is the spacetime manifold, \( K \) is the trace of the extrinsic curvature of the spacelike boundary \( \partial\mathcal{M} = S^3 \) and the lagrangian is defined as

\[
L = \frac{3c^2\pi}{4G} \left( -\frac{a\dot{a}^2}{N} + c^2 Na - \frac{\Lambda}{3} Na^3 \right).
\]

The Hamiltonian corresponding to (3) is obtained

\[
H = \Pi_a \dot{a} - L = -N \frac{G}{3\pi c^2 a} \Pi_a^2 - \frac{3c^2\pi}{4G} (c^2 Na - \frac{\Lambda}{3} Na^3),
\]

where \( \Pi_a = -2a\dot{a}/N \) is the canonical momentum conjugate to \( a \). Also it is assumed in (4) that \( \Lambda > 0 \). Gauge invariance of action (2) yields the Hamiltonian constraint

\[
- \frac{\partial H}{\partial N} = \frac{1}{2ma} \Pi_a^2 + \frac{1}{2} mc^2 a - \frac{ma}{6} a^3 = 0,
\]

where \( m = \frac{3\pi c^2}{2G} \) is interpreted as the mass equivalent of de Sitter universe. The Hamiltonian constraint requires a gauge fixing condition. Choosing \( N = 1 \), the time variable \( t \) becomes essentially the proper time. In this gauge the solution of the classical equations of motion is given by

\[
a(t) = a_0 e \cosh \left( \frac{t}{a_0} \right),
\]

where \( a_0^2 = \frac{3}{\Lambda} \) is interpreted as the minimum radius of the universe after tunneling from nothing and the initial conditions \( a(0) = a_0 \) and \( \dot{a}(0) = 0 \) are used. This solution corresponds to a usual de Sitter spacetime where a phase of contraction from infinitely past time is followed by an expansion phase where the scale factor has reached its minimum \( a_0 \). However, a different cosmological scenario can give rise to an identical de Sitter expansion if we consider analytic continuation \( \tau = it + (\pi/2)a_0 \). Then one can obtain the instanton solution as

\[
a_E = ca_0 \sin \left( \frac{\tau}{a_0} \right),
\]

so that the instanton is a 4-sphere of radius \( a_0 \) if \( \tau/a_0 \in \{-\pi/2, \pi/2\} \), with spherical three-dimensional sections labelled by the latitude angle \( \theta = a_0\tau \).
Both metrics are related by the analytic continuation into the complex plane of the Euclidean “time” $\tau$
\[ \tau = \frac{\pi}{2} a_0 + it, \quad a = a_E \left( \frac{\pi}{2} a_0 + it \right), \quad (8) \]
which is a Wick rotation with respect to the point $\tau = (\pi/2)a_0$ in this plane. This analytic continuation can be interpreted as a quantum nucleation of the Lorentzian de Sitter spacetime from the Euclidean hemisphere as a matching of the two manifolds across the equatorial section $\tau = (\pi/2)a_0$ ($t = 0$), the bounce surface of zero extrinsic curvature. Canonical quantization of this simple cosmological model in the coordinate representation is accomplished by the operator realizations
\[ a = a, \quad \Pi_a = -i\hbar \frac{\partial}{\partial a}. \quad (9) \]
The Hamiltonian constraint becomes the Wheeler-DeWitt equation for the wave function of the de Sitter spacetime
\[ \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial a^2} + \frac{1}{2} mc^2 a^2 - \frac{m a^4}{2a_0^2} \right) \Psi(a) = 0, \quad (10) \]
where the operator ordering in the kinetic term is neglected. The corresponding one-dimensional potential
\[ V(a) = \frac{1}{2} mc^2 a^2 - \frac{m a^4}{2a_0^2}, \quad (11) \]
is unbounded from below, and consequently the time of flight of a classical particle from the largest turning point to infinity is finite, namely
\[ \int_{x_o > a_0}^\infty \frac{dx}{\sqrt{|V(x)|}} < \infty. \quad (12) \]
This may be in contraction with (6) at first sight, because this equation implies $t \to \infty$ as $a \to \infty$. However, this is only an apparent contradiction. If we use “conformal time” gauge $N = a(t)$, then the kinetic terms have the standard form. The relation between the conformal time and proper time is
\[ d\eta = \frac{1}{\sqrt{ma(t)}} dt, \quad (13) \]
and consequently the classical solution in the conformal gauge is
\[ a(\eta) = \frac{a_0 c}{\cos(c\sqrt{m\eta})}. \quad (14) \]
It is then clear that the “particle” reaches infinity indeed after finite conformal time, $\eta = \pi/(2c\sqrt{m})$. The corresponding Euclidean solution will be
\[ a(\tau) = \frac{a_0 c}{\cosh(c\sqrt{m\tau})}. \quad (15) \]
The action (2) is invariant under the transformations
\[ \delta a = \epsilon(t)\{a, H\}, \]
\[ \delta \Pi_a = \epsilon(t)\{\Pi_a, H\}, \]
\[ \delta N = \dot{\epsilon}(t), \]
provided the parameters vanish at the end points. This transformations defines a “gauge” equivalence class of histories. The path integral in the Euclidian region for the propagator amplitude between fixed initial and final configurations can be written as
\[ (a_f|a_i) = \int_0^\infty dN <a_f, N|a_i, 0> \int D\mu[a]e^{-S_E(a,N)}, \]
where \(<a_f, N|a_i, 0>\) is a Green function for the WDW equation and the Euclidian action \(S_E\) is defined in the gauge \(\dot{N} = 0\) as
\[ S_E = \frac{m^2}{2} \int_0^\infty d\tau N^{-1} a^{-1} \left( \frac{\dot{a}^2}{N^2a^{-2}} + a^2(c^2 - a^2) \right), \]
and integration measure in one-loop approximation is given by
\[ D\mu[a] = \prod_t da(t)\left(\frac{2a(t)}{N(t)}\right) + O(\hbar). \]
The Euclidean action (18) is not suitable to be used in instanton calculation techniques. The reason is that the kinetic term is not in its standard quadratic form. It has been shown that in such quantum cosmological model one may use the Duru-Kleinert equivalence to work with the standard form of the action [38, 39]. Using this procedure, we find the Duru-Kleinert equivalent path integral in the present quantum cosmological model as follows
\[ (a_f|a_i) = \int_0^\infty dN \int D[a]e^{-S_0}, \]
where \(S_0\) has the standard quadratic form
\[ S_0 = \frac{m^2}{2} \int_{\tau_0}^{\tau_f} d\tau \left( a^2 + a^2c^2 - a^4 \right). \]

3 Semi-Classical soliton States

In this section, we quote briefly how one can calculate the zeta function of an operator through the heat kernel method, in the case of scalar field \(\Phi(x)\). Classical configuration space is found by static configuration \(\Phi(x)\), so that the energy functional
\[ E[\Phi] = \int dx \left[ \frac{1}{2} \partial_\mu \Phi^\mu + V(\Phi) \right], \]
is finite. One can describe quantum evolution in Schrödinger picture by the following functional equation

\[ i\hbar \frac{\partial}{\partial t} \Phi[\phi(x), t] = H \Phi[\phi(x), t], \quad (23) \]

so that quantum Hamiltonian operator is given by

\[ H = \int dx \left[ -\frac{\hbar^2}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(x)} + E[\phi] \right]. \quad (24) \]

In the field representation the matrix elements of evolution operator are given by

\[ G(\phi^{(f)}(x), \phi^{(i)}(x), T) = \langle \phi^{(f)} | e^{-i\bar{\hbar} H} | \phi^{(i)} \rangle = \int D[\phi(x, t)] \exp \left( \frac{-i}{\bar{\hbar}} S[\phi] \right), \quad (25) \]

where the initial conditions are those of static kink solutions of classical equations where \( \phi^{(i)}(x, 0) = \phi_k(x) \), \( \phi^{(f)}(x, T) = \phi_k(x) \). In semi-classical picture, we are interested in loop expansion for evolution operator up to the first quantum correction

\[ G(\phi^{(f)}(x), \phi^{(i)}(x), \beta) = \exp \left( -\frac{\beta}{\bar{\hbar}} E[\phi_k] \right) \text{Det}^{-\frac{1}{2}} \left[ -\partial^2 + P\Delta \right] (1 + O(\hbar)), \quad (26) \]

where we use analytic continuation to Euclidean time, \( t = -i\tau, T = -i\beta \), and \( \Delta \) is the differential operator

\[ \Delta = -\frac{d^2}{dx^2} + \frac{d^2 V}{d\phi^2} \big|_{\phi = \phi_k}, \quad (27) \]

\( P \) is the projector over the strictly positive part of spectrum of \( \Delta \)

\[ \Delta \xi_n(x) = \omega_n^2 \xi_n(x), \quad \omega_n^2 \in \text{Spec}(\Delta) = \text{Spec}(P\Delta) + \{0\}. \quad (28) \]

We write functional determinant in the form

\[ \text{Det} \left[ -\frac{\partial^2}{\partial \tau^2} + \Delta \right] = \prod_n \text{det} \left[ -\frac{\partial^2}{\partial \tau^2} + \omega_n^2 \right]. \quad (29) \]

All determinants in infinite product correspond to harmonic oscillators of frequency \( \omega_n \). On the other hand, it is well known that

\[ \text{det} \left( -\frac{\partial^2}{\partial \tau^2} + \omega_n^2 \right)^{-\frac{1}{2}} = \prod_j N \left( \frac{\omega_n^2}{\beta} + \omega_n^2 \right)^{-\frac{1}{2}} \]

\[ = \prod_j \left( \frac{\omega_n^2}{\beta} \right)^{-\frac{1}{2}} \prod_j \left( 1 + \frac{\omega_n^2}{\beta} \right)^{-\frac{1}{2}}. \quad (30) \]
The first product does not depend on $\omega_n$ and combines with the Jacobian and other factors we have collected into a single constant. The second factor has the limit
\[
\left[ \frac{\sinh(\omega_n \beta)}{\omega_n \beta} \right]^{-\frac{1}{2}},
\]
and thus, with an appropriate normalization, we obtain for large $\beta$
\[
G(\phi^{(f)}(x), \phi^{(i)}(x), \beta) \cong \exp \left( -\frac{\beta}{\hbar} E[\phi_k] \right) \prod_n \left( \frac{\omega_n}{\pi \hbar} \right)^{\frac{1}{2}} \exp \left( -\frac{\beta}{2} \sum_n \omega_n (1 + \mathcal{O}(\hbar)) \right)
\]
where eigenvalues in the kernel of $\Delta$ have been excluded. Interesting eigenenergy wave functionals
\[
H \Phi_j[\phi_k(x)] = \varepsilon_j \Phi_j[\phi_k(x)]
\]
we have an alternative expression for $G_E$ for $\beta \to \infty$.
\[
G(\phi^{(f)}(x), \phi^{(i)}(x), \beta) \cong \Phi_0^*[\phi_k(x)] \Phi_0[\phi_k(x)] \exp \left( -\frac{\beta}{2} \frac{\omega_n}{\hbar} \right),
\]
and, therefore, from (31) and (33) we obtain
\[
\varepsilon_0^k = E[\phi_k] + \frac{\hbar}{2} \sum_{\omega_n > 0} \omega_n + \mathcal{O}(\hbar^2),
\]
\[
|\Phi_0[\phi_k(x)]|^2 = \text{Det} \left[ \frac{P\Delta}{\pi^2 \hbar^2} \right],
\]
as the Kink ground state energy and wave functional up to One-Loop order. If we define the generalized zeta function
\[
\zeta_{P\Delta}(s) = Tr(P\Delta)^{-s} = \sum_{\omega_n^2 > 0} \frac{1}{(\omega_n^2)^s},
\]
associated to differential operator $P\Delta$, then
\[
\varepsilon_0^k = E[\phi_k] + \frac{\hbar}{2} Tr(P\Delta)^{\frac{1}{2}} + \mathcal{O}(\hbar^2) = \left. E[\phi_k] + \frac{\hbar}{2} \zeta_{P\Delta}(\frac{1}{2}) + \mathcal{O}(\hbar^2) \right|_{s = \frac{1}{2}}.
\]
The eigenfunction of $\Delta$ is a basis for quantum fluctuations around kink background, therefore sum of the associated zero-point energies encoded in $\zeta_{P\Delta}(\frac{1}{2})$ in (36) is infinite. According to zeta function regularization procedure, energy and mass renormalization prescription, the renormalized kink energy in semiclassical limit becomes [37]
\[
\varepsilon^k(s) = E[\phi_k] + \Delta M_k + \mathcal{O}(\hbar^2) = \left. E[\phi_k] + \lim_{s \to \frac{1}{2}} \frac{\delta_1 \varepsilon^k(s) + \delta_2 \varepsilon^k(s)}{s} \right| \mathcal{O}(\hbar^2),
\]
where

\[ \delta_1 \epsilon^k(s) = \frac{\hbar}{2\mu^{2s+1}}[\zeta_P \Delta(s) - \zeta_\nu(s)], \]

\[ \delta_2 \epsilon^k(s) = \lim_{L \to \infty} \frac{\hbar}{2\mu^{2s+1}} \frac{\Gamma(s+1)}{\Gamma(s)} \zeta_\nu(s+1) \times \]

\[ \int_{L/2}^{L/2} dx \left( \frac{d^2V}{d\phi^2} \phi_k - \frac{d^2V}{d\phi^2} \phi_\nu \right). \]

Here \( \phi_\nu \) is a constant minimum of potential \( V(\phi) \), \( E[\phi_k] \) is the corresponding classical energy where \( \mu \) has the unit \( \text{length}^{-1} \) dimension, introduced to make the terms in \( \delta_1 \epsilon^k(s) \) and \( \delta_2 \epsilon^k(s) \) homogeneous from a dimensional point of view and \( \zeta_\nu \) denoted zeta function associated with vacuum \( \phi_\nu \).

Now we explain very briefly how one can calculate zeta function of an operator though heat kernel method. We introduce generalized Riemann zeta function of operator \( A \) by

\[ \zeta_A(s) = \sum_n \frac{1}{|\lambda_n|^s}, \]

where \( \lambda_n \) are eigenvalues of operator \( A \). On the other hand, \( \zeta_A(s) \) is the Mellin transformation of heat kernel \( G(x,y,t) \) which satisfies the following heat diffusion equation

\[ AG(x,y,t) = -\frac{\partial}{\partial t} G(x,y,t), \]

with an initial condition \( G(x,y,0) = \delta(x - y) \). Note that \( G(x,y,t) \) can be written in terms of its spectrum

\[ G(x,y,t) = \sum_n e^{-\lambda_n t} \psi_n^*(x) \psi_n(y), \]

and as usual, if the spectrum is continues, one should integrate it. From relation (40), (41) and (42) it is clear that

\[ \zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \int_{-\infty}^{\infty} G(x,x,\tau) dx. \]

Hence, if we know the associated Green function of an operator, we can calculate the generalized zeta function corresponding to that operator. In the next section, we use the shape invariance property of the potential and the heat kernel method to obtain the quantum corrections to the kink masses.

4 Renormalized ground state energy of the Kink solution in de Sitter universe

Comparison of the two previous sections reveals that the system of closed de Sitter universe may be equivalent to the classical configuration space of the static
field $\Phi(x)$. Therefore, the one-loop quantum corrections to the mass quantity of closed de Sitter universe, namely $m = 3\pi c^2/(2G)$, is equivalent to the one-loop quantum corrections to the kink mass or ground state energy of the scalar field $\Phi(x)$. We call this procedure as the one-loop quantum corrections to the ground state energy of the Kink-like solution of closed de Sitter universe. In other words, what is physically meant by the one-loop quantum corrections to the ground state energy in de Sitter universe is nothing but the one-loop quantum corrections to the mass quantity $m = 3\pi c^2/(2G)$ in de Sitter universe which resembles the kink’s mass or energy in the equivalent system of the classical static field $\Phi(x)$ configuration.

Using the techniques of the previous section implemented on the scalar field theory, we now compute the one-loop quantum correction to the ground state energy of the Kink-like solution of de Sitter universe. To this end, we need the spectrum of differential operator (27) and the corresponding vacuum. According to the previous section, the operator (27) which acts on the eigenfunctions becomes

$$\Delta_{l+h} = \frac{mc^2}{\hbar} \left( -\frac{d^2}{dx^2} + l^2 - \frac{l(l+1)}{\cosh^2 x} + h \right), \quad (44)$$

where $x = \sqrt{mc}\tau$, $h = 1 - 2l$ and $l = 2$. The combination $mc^2/\hbar$ is a necessary result of the well known fact that in every quantum mechanical problem of gravity where $\hbar$ appears, $m$ is also expected to appear [41].

Also the operator acting on the vacuum has the following form

$$\Delta_{l+h}(0) = \frac{mc^2}{\hbar} \left( -\frac{d^2}{dx^2} + l^2 + h \right). \quad (45)$$

Note that we have the constant shift $h$ in the spectrum that we add it in our calculations latter (see Eq.(69)), also since we have $\zeta\Delta(s) = |\sigma|^{-s}\zeta(s)$ so we ignore $\sigma = \frac{mc^2}{\hbar}$ here in our calculations and we add it at last steps. In the reminding of this section, to obtain the spectrum of (44) we will use the shape invariance property. First we review briefly concepts that we will use.

Consider the following one-dimensional bound-state Hamiltonian

$$H = -\frac{d^2}{dx^2} + U(x), \quad x \in I \subset \mathcal{R} \quad (46)$$

where $I$ is the domain of $x$ and $U(x)$ is a real function of $x$, which can be singular only in the boundary points of the domain. Let us denote by $E_n$ and $\psi_n(x)$ the eigenvalues and eigenfunctions of $H$ respectively. We use factorization method which consists of writing Hamiltonian as the product of two first order mutually adjoint differential operators $A$ and $A^\dagger$. If the ground state eigenvalue and eigenfunctions are known, then one can factorize Hamiltonian (46) as

$$H = A^\dagger A + E_0, \quad (47)$$

10
where $E_0$ denotes the ground-state eigenvalue,
\begin{equation}
A = \frac{d}{dx} + W(x),
\end{equation}
\begin{equation}
A^\dagger = -\frac{d}{dx} + W(x),
\end{equation}
and
\begin{equation}
W(x) = -\frac{d}{dx} \ln(\psi_0).
\end{equation}

Supersymmetric quantum mechanics (SUSY QM) begins with a set of two matrix operators, known as supercharges
\begin{equation}
Q^+ = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}.
\end{equation}

This operators form the following superalgebra \cite{42}
\begin{equation}
\{Q^+, Q^-\} = H_{SS}, \quad [H_{SS}, Q^\pm] = (Q^\pm)^2 = 0,
\end{equation}
where SUSY Hamiltonian $H_{SS}$ is defined as
\begin{equation}
H_{SS} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}.
\end{equation}

In terms of the Hamiltonian supercharges
\begin{equation}
Q_1 = \frac{1}{\sqrt{2}} (Q^+ + Q^-), \quad Q_2 = \frac{1}{\sqrt{2}i} (Q^+ - Q^-),
\end{equation}
the superalgebra takes the form
\begin{equation}
\{Q_i, Q_j\} = H_{SS} \delta_{ij}, \quad [H_{SS}, Q_i] = 0, \quad i, j = 1, 2.
\end{equation}

The operators $H_1$ and $H_2$
\begin{equation}
H_1 = A^\dagger A = -\frac{d^2}{dx^2} + U_1 = -\frac{d^2}{dx^2} + W^2 - \frac{dW}{dx},
\end{equation}
\begin{equation}
H_2 = AA^\dagger = -\frac{d^2}{dx^2} + U_2 = -\frac{d^2}{dx^2} + W^2 + \frac{dW}{dx},
\end{equation}
are called SUSY partner Hamiltonians and the function $W$ is called the superpotential. Now, let us denote by $\psi^{(1)}_l$ and $\psi^{(2)}_l$ the eigenfunctions of $H_1$ and $H_2$ with eigenvalues $E^{(1)}_l$ and $E^{(2)}_l$, respectively. It is easy to see that the eigenvalues of the above Hamiltonians are positive and isospectral, i.e., they have
almost the same energy eigenvalues, except for the ground state energy of $H_1$. According to the [12], their energy spectra are related as

$$E_l = E_l^{(1)} + E_0, \quad E_0^{(1)} = 0, \quad \psi_l = \psi_l^{(1)}, \quad l = 0, 1, 2, ..., \quad (56)$$

$$E_l^{(2)} = E_{l+1}^{(1)}, \quad \psi_l^{(2)} = [E_{l+1}^{(1)}]^\dagger A \psi_{l+1}^{(1)}, \quad \psi_{l+1}^{(1)} = [E_l^{(1)}]^\dagger A^\dagger \psi_l^{(2)}. \quad (60)$$

Therefore if the eigenvalues and eigenfunctions of $H_1$ were known, one could immediately derive the spectrum of $H_2$. However the above relations only give the relationship between the eigenvalues and eigenfunctions of the two partner Hamiltonians. A condition of an exactly solvability is known as the shape invariance condition. This condition means the pair of SUSY partner potentials $U_{1,2}(x)$ are similar in shape and differ only in the parameters that appears in them [43]

$$U_2(x; a_1) = U_2(x; a_2) + R(a_1), \quad (57)$$

where $a_1$ is a set of parameters and $a_2$ is a function of $a_1$. Then the eigenvalues of $H_1$ are given by

$$E_l^{(1)} = R(a_1) + R(a_2) + ... + R(a_l), \quad (58)$$

and the corresponding eigenfunctions are

$$\psi_l = \prod_{m=1}^{l} \frac{A_l(x; a_m)}{\sqrt{E_m}} \psi_0(x; a_{l+1}). \quad (59)$$

The shape invariance condition [57] can be rewritten in terms of the factorization operators defined in equation (48)

$$A(x; a_1) A^\dagger(x; a_1) = A^\dagger(x; a_2) A(x; a_2) + R(a_1), \quad (60)$$

where $a_2 = f(a_1)$. Now we are ready to obtain spectra of $\Delta_l$ operator defined in [29]. For a given eigenspectrum of $E_l$, we introduce the following factorization operators

$$A_l = \frac{d}{dx} + l \tanh(x), \quad (61)$$

$$A_l^\dagger = -\frac{d}{dx} + l \tanh(x),$$

the operator $\Delta_l$ can be factorized as

$$A_l^\dagger(x) A_l(x) \psi_n^{(1)}(x) = E_n^{(1)} \psi_n^{(1)}(x), \quad (62)$$

$$A_l(x) A_l^\dagger(x) \psi_n^{(2)}(x) = E_n^{(2)} \psi_n^{(2)}(x).$$
Therefor for a given \( l \), its first bounded excited state can be obtained from the ground state of \( l - 1 \) and consequently the excited state \( m \) of a given \( l \), \( \psi_{l,m}(x) \), using (59) can be written as

\[
\psi_{l,m}(x) = \sqrt{\frac{2(2m-1)!}{\prod_{j=1}^{m} j(2l-j)!}} \frac{1}{2^m(m-1)!} A_{l}^{1}(x)A_{l-1}^{1}(x)\ldots A_{m+1}^{1}(x) \frac{1}{\cosh^m(x)},
\]

with eigenvalue \( E_{l,m} = m(2l - m) \). Obviously its ground state with \( E_{l,0} = 0 \) is given by \( \psi_{l,0} \propto \cosh^{-1}(x) \). Also its continuous spectrum consists of

\[
\psi_{l,k}(x) = \frac{A_{l}^{1}(x)}{\sqrt{k^2 + l^2}} \frac{A_{l-1}^{1}(x)}{\sqrt{k^2 + (l-1)^2}} \ldots \frac{A_{1}^{1}(x)}{\sqrt{k^2 + 1}} e^{ikx},
\]

with eigenvalues \( E_{l,k} = l^2 + k^2 \) with following normalization condition

\[
\int_{-\infty}^{\infty} \psi_{l,k}^{\ast}(x)\psi_{l,k'}(x)dx = \delta(k - k').
\]

Therefor, using equations (41), (42), (63) and (64) we find

\[
G_{\Delta_{i}(0)}(x,y,\tau) = \frac{e^{-\frac{r^2}{4}}}{2\sqrt{\pi}} e^{-\frac{(x-y)^2}{4r}},
\]

and

\[
G_{\Delta_{i}(x,y,\tau)} = \sum_{m=1}^{l-1} \psi_{l,m}(x)\psi_{l,m}(y)e^{-m(2l-m)r}
\]

\[
+ \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-\frac{(k^2 + l^2)r}{4}} \left( \prod_{m=1}^{l} A_{m}^{1}(x)e^{ikx}\right)^{\ast} \left( \prod_{m=1}^{l} A_{m}^{1}(y)e^{iky}\right).
\]

In the case of de Sitter spacetime we left with \( l = 2 \) and then using (63) we have

\[
\zeta \delta_{2}(s) - \zeta \delta_{2}(0)(s) = 3^{-s} - \frac{3}{2} \int_{-\infty}^{\infty} \frac{dk}{(k^2 + 4)^{s+1}} =
\]

\[
3^{-s} - \frac{3}{2} \frac{2^{-2s+1} \Gamma(s+\frac{1}{2})}{\Gamma(s+1)}.
\]

Consequently we have

\[
\delta_{1} \varepsilon^{k}(s) = \frac{h}{2} \mu^{2s+1} \left( \frac{mc^2}{h}\right)^{-2s} \left[ \zeta_{P\Delta_{i}h}(s) - \zeta_{\Delta_{i}h}(s) \right]_{t=2} =
\]

\[
\frac{h}{2} \mu^{2s+1} \left( \frac{mc^2}{h}\right)^{-2s} \left( 3^{-s} - \frac{3}{2} \frac{2^{-2s+1} \Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \right).
\]

Also we obtain

\[
\delta_{2} \varepsilon^{k}(s) = \lim_{L \rightarrow \infty} \frac{h}{2\pi} \left( \frac{mc^2}{h}\right)^{-2s} \mu^{2s+1} \frac{\Gamma(s+1)}{\Gamma(s)} \left[ \zeta_{\Delta_{i}h}\right]_{s+1}(s + 1)_{l=2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx (-6 \cosh^{-2}(x))
\]

\[
= - \frac{3h}{\sqrt{\pi}} \mu^{2s+1} \left( \frac{mc^2}{h}\right)^{-2s} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}.
\]
Finally we have

\[ \lim_{s \to -\frac{1}{2}} \left( \delta_1 \varepsilon^k(s) + \delta_2 \varepsilon^k(s) \right) = \frac{h}{2} \mu^{2s+1} \left( \frac{mc^2}{\bar{h}} \right)^{-2s} \left( 3^{-s} - \frac{3}{\sqrt{\pi}} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \right) \]

\[ = \frac{h}{2} \frac{mc^2}{\bar{h}} \left( \sqrt{3} - \frac{3}{\sqrt{\pi}} \right). \]  

(71)

At last, we find the one-loop correction to the kink energy (mass) as

\[ \varepsilon^k = mc^2 + \frac{mc^2}{2} \left( \sqrt{3} - \frac{3}{\sqrt{\pi}} \right). \]  

(72)

Now, the renormalized gravitational constant \( G_{one-loop} \) is obtained through

\[ G = \frac{3\pi c^2}{mc^2} \]  

as

\[ G_{one-loop} = \frac{G}{1 + \frac{3}{2} \left( \sqrt{3} - \frac{3}{\sqrt{\pi}} \right)}, \]  

(73)

which indicates that the one-loop renormalized gravitational constant is smaller than the original gravitational constant.

5 Conclusion

The shape invariance property of the fluctuation operator is of particular importance in order to find the exact one-loop quantum correction to the mass of kink solutions, namely the instanton solutions of classical field equations. The (1+1)-dimensional Sine-Gordon and \( \phi^4 \) field theories are some of these examples. This kind of potential occurs in different fields of physics particularly in quantum gravity and cosmology. The de Sitter spacetime is one of those examples whose quantum cosmology reveals the shape invariance property in the action.

In this paper, we have shown that the system of closed de Sitter universe is equivalent to the classical configuration space of the static field \( \Phi(x) \). Therefore, we expect kink-like solutions in the system of closed de Sitter universe. The one-loop quantum corrections to the mass-like quantity of closed de Sitter universe, namely \( m = \frac{3\pi c^2}{2G} \), is therefore equivalent to the computation of the one-loop quantum corrections to the kink mass or ground state energy of the scalar field \( \Phi(x) \). In fact, what is physically meant by the one-loop quantum corrections to the ground state energy in de Sitter universe, in the present paper, is nothing but the one-loop quantum corrections to the mass quantity \( m = \frac{3\pi c^2}{2G} \) in de Sitter universe which plays the role of kink’s mass in the equivalent system of the classical static field \( \Phi(x) \) configuration. We have therefore computed the quantum corrections to the ground state energy of the Kink-like solution in the closed de Sitter universe. From mass-energy equivalence principle we know that any corrections to the energy is equivalent to the corresponding corrections to an equivalent mass quantity. In de Sitter cosmology this mass quantity
coincides with the gravitational constant as \( m = \frac{3\pi c^2}{2G} \). Therefore, it is shown that the one-loop quantum corrections to the ground state energy of the Kink-like solution in the closed de Sitter universe may be interpreted as the renormalization of the gravitational constant which turns out to be smaller than the original gravitational constant. The obtained correction may become more viable and important whenever we study the other interesting aspects of quantum cosmology of closed de Sitter universe. For example, this correction may have considerable impact in evaluation of the tunneling rate from nothing to a closed de Sitter universe. In fact, since the tunneling rate depends on the classical action which itself is dependent on the gravitational constant, then any renormalization on the gravitational constant will alter the tunneling rate.

We feel it necessary to compare and discuss on the present semi classical one-loop quantum cosmological corrections and the well known one-loop quantum gravitational corrections. In the latter approach, for instance in de Sitter spacetime, one usually uses of zeta-function regularization technique and heat-kernel methods to compute the leading part of the one-loop contribution to the effective action. As a result, addition of this contribution to the classical action leads to the one-loop effective action in the large-distance limit as

\[
\Gamma_{\text{eff}} = S + \Gamma(1) \\
= \int d^4x \sqrt{|g|} [(8\pi G)^{-1} \Lambda + \beta_\Lambda |\Lambda|^2 \log(|\Lambda|\mu^{-2})] \\
- \int d^4x \sqrt{|g|} R [(16\pi G)^{-1} + \beta_G |\Lambda| \log(|\Lambda|\mu^{-2})],
\]

where the effective or induced Newton and cosmological constants are given by

\[
\Lambda_{\text{eff}} = \Lambda \frac{1 + \kappa_\Lambda 8\pi G |\Lambda| \log(|\Lambda|\mu^{-2})}{1 + \beta_G 16\pi G |\Lambda| \log(|\Lambda|\mu^{-2})},
\]

\[
(GA)_{\text{eff}} = (GA) \frac{1 + \kappa_\Lambda 8\pi G |\Lambda| \log(|\Lambda|\mu^{-2})}{[1 + \beta_G 16\pi G |\Lambda| \log(|\Lambda|\mu^{-2})]^2}.
\]

In fact, for any background like de Sitter, one should calculate vacuum energy (effective) action which yields effective gravitational and cosmological constants. It is then impossible to calculate only gravitational constant unless we prove that the induced cosmological constant is zero by some (yet unknown) mechanisms.

In the present paper, however, rather than the effective action approach we are just dealing with the semi classical correction on the kink-like solution in de Sitter universe and no such corrections are imposed by effective action on the gravitational and cosmological constants. Thus, we just obtain a semi classical correction of mass which may be interpreted as a correction to the gravitational constant due to the relation \( m = \frac{3\pi c^2}{2G} \). The cosmological constant here appears just as the inverse squared of the minimum radius of the universe after tunneling from nothing, namely \( \Lambda = \frac{3}{a_0^2} \), which indicates the bouncing point or the width of the potential barrier in the Wheeler-DeWitt equation, while the gravitational constant plays the role of the mass of kink-like solution. Therefore,
it is reasonable to expect that the mass term, namely gravitational constant, as a dynamical quantity bears quantum correction but the cosmological constant as a parameter in the potential bears no such a correction.

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References

[1] R. Rajaraman, Kinks and Instantons (North Holland, Amsterdam, 1987).

[2] C. Rebbi and G. Soliani, Solitons and Particles (World Scientific, Singapore, 1984).

[3] H. Nastase, M. Stephanov, P. Van Nieuwenhuizen, and A. Rebhan, Nucl. Phys. B 542, 471 (1999).

[4] N. Graham and R. L. Jaffe, Phys. Lett. B 435, 145 (1998).

[5] N. Graham and R. L. Jaffe, Nucl. Phys. B 544, 432 (1999).

[6] N. Graham and R. L. Jaffe, Nucl. Phys. B 549, 516 (1999).

[7] G. Dunne and K. Rao, JHEP 0001, 019 (2000).

[8] I. J. R. Aitchison and C. M. Fraser, Phys. Rev. D 31, 2605 (1985).

[9] L-H. Chan, Phys. Rev. Lett. 54, 1222 (1985).

[10] L-H. Chan, Phys. Rev. D 55, 6223 (1997).

[11] G. V. Dunne, Phys. Lett. B 467, 238 (1999).

[12] A. S. Goldhaber, A. Litvintsev, and P. Van Nieuwenhuizen, Phys. Rev. D 64, 045013 (2001).

[13] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, Zeta Regularization Techniques with Applications, (World Scientific, Singapore, 1994).

[14] M. Bordag, A. S. Goldhaber, P. Van Nieuwenhuizen, and D. Vassilevich, Phys. Rev. D 66, 125014 (2002).

[15] A. Rebhan, P. Van Nieuwenhuizen, and R. Wimmer, New J. Phys. 4, 31 (2002).

[16] S. Rafei, S. Jalalzadeh, and K. Ghafoori Tabrizi, Chinese. J. Physics, 46, 401 (2008).

[17] M. A. Jafarizadeh, F. Darabi, A. Rezaei-Aghdam, A. R. Rastegar, Phys. Rev. D 60, 063514 (1999).

[18] G. W. Gibbons, S. W. Hawking and M. J. Perry. Nucl. Phys. B138, 141, (1978).

[19] G. W. Gibbons and M. J. Perry. Nucl. Phys. B146, 90, (1978).

[20] S. W. Hawking and W. Israel, An Einstein Centenary Survey, Cambridge University Press, Cambridge, (1979).
[21] S. M. Christensen and M. J. Duff. Nucl. Phys. B170, 480, (1980).
[22] S. M. Christensen, M. J. Duff, G.W. Gibbons and M. Rocek. Phys. Rev. Lett. 45, 161, (1980).
[23] I. Antoniadis, J. Iliopoulos and T. N. Tomaras. Phys. Rev. Lett. 56, 1319, (1986).
[24] L. Ford. Phys. Rev. D31, 710 (1985).
[25] B. Allen and M. Turyn. Nucl. Phys. B292, 813, (1987).
[26] S. D. Odintso, Europhys. Lett. 10, 287, (1989).
[27] S. D. Odintso, Theor. Math. Phys. 82, 61, (1990).
[28] I. Antoniadis and E. Mottola. J. Math. Phys. 32, 1037, (1991).
[29] E. S. Fradkin and A. A. Tseytlin. Nucl. Phys. B234, 472, (1984).
[30] T. R. Taylor and G. Veneziano. Nucl. Phys. B345, 210, (1990).
[31] A. A. Bysenko, S. D. Odintso and S. Zerbini, Class. Quant. Grav. 12, 1, (1995); Erratum-ibid.12, 2355, (1995).
[32] A. A. Bysenko, S. D. Odintso and S. Zerbini, Phys. Lett. B336, 355, (1994).
[33] I. L. Buchbinder, E. N. Kirillova, and S. D. Odintso, Mod. Phys. Lett. A4, 633, (1989).
[34] S. D. Odintso, Europhys. Lett. 10, 287, (1989).
[35] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintso, and S. Zerbini, JCAP 0502, 010, (2005).
[36] M. A. Jafarizadeh, F. Darabi, A. Rezaei-Aghdam, and A. R. Rastegar, Phys. Rev. D60, 063514 (1999).
[37] A. A. Izquierdo, J. M. Guilarte, M. A. G. Leon and W. G. Fuertes, Nucl. Phys. B 635, 525 (2002).
[38] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics* (World Scientific, Singapore, 1991).
[39] M. A. Jafarizadeh, F. Darabi, A. R. Rastegar, Phys.Lett. A 248, 19 (1998).
[40] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw Hill, New York, 1965).
[41] J. J. Sakurai, *Modern Quantum Mechanics*, (Addison-Wesley, 1985).
[42] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).
[43] L. Gendenshtein, JETP Lett. 38, 356 (1983).