COHEN-MACAULAY-NESS IN CODIMENSION FOR SIMPLICIAL COMPLEXES AND EXPANSION FUNCTOR

RAHIM RAHMATI-ASGHAR

Abstract. In this paper we show that expansion of a Buchsbaum simplicial complex is CM, for an optimal integer \( t \geq 1 \). Also, by imposing extra assumptions on a CM\(_t\) simplicial complex, we prove that it can be obtained from a Buchsbaum complex.

Introduction

Set \([n] := \{x_1, \ldots, x_n\}\). Let \( K \) be a field and \( S = K[x_1, \ldots, x_n] \), a polynomial ring over \( K \). Let \( \Delta \) be a simplicial complex over \([n]\). For an integer \( t \geq 0 \), Haghighi, Yassemi and Zaare-Nahandi introduced the concept of CM\(_t\)-ness which is the pure version of simplicial complexes Cohen-Macaulay in codimension \( t \) studied in [7]. A reason for the importance of CM\(_t\) simplicial complexes is that they generalizes two notions for simplicial complexes: being Cohen-Macaulay and Buchsbaum. In particular, by the results from [9, 11], CM\(_0\) is the same as Cohen-Macaulay and CM\(_1\) is identical with Buchsbaum.

In [3], the authors described some combinatorial properties of CM\(_t\) simplicial complexes and gave some characterizations of them and generalized some results of [6, 8]. Then, in [4], they generalized a characterization of Cohen-Macaulay bipartite graphs from [5] and [2] on unmixed Buchsbaum graphs.

Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded \( S \)-modules and studied some homological behaviors of this functor (see [1]). The expansion functor helps us to present other multigraded \( S \)-modules from a given finitely generated multigraded \( S \)-module which may have some of algebraic properties of the primary module. This allows to introduce new structures of a given multigraded \( S \)-module with the same properties and especially to extend some homological or algebraic results for larger classes (see for example [1, Theorem 4.2]). There are some combinatorial versions of expansion functor which we will recall in this paper.

The purpose of this paper is the study of behaviors of expansion functor on CM\(_t\) complexes. We first recall some notations and definitions of CM\(_t\) simplicial complexes in Section 1. In the next section we describe the expansion functor in three contexts, the expansion of a simplicial complex, the expansion of a simple graph and the expansion of a monomial ideal. We show that there is a close relationship between these three contexts. In Section 3 we prove that the expansion of a CM\(_t\) complex \( \Delta \) with respect to \( \alpha \) is CM\(_{t+e-k+1}\) but it is not CM\(_{t+e-k}\) where \( e = \dim(\Delta^\alpha) + 1 \) and \( k \) is the minimum of the components of \( \alpha \) (see Theorem 3.3). In Section 4, we introduce a new functor, called contraction, which acts in
contrast to expansion functor. As a main result of this section we show that if the
contraction of a CM\(_t\) complex is pure and all components of the vector obtained
from contraction are greater than or equal to \(t\) then it is Buchsbaum (see Theorem
4.6). The section is finished with a view towards the contraction of simple graphs.

1. Preliminaries

Let \(t\) be a non-negative integer. We recall from [3] that a simplicial complex \(\Delta\)
is called CM\(_t\) or Cohen-Macaulay in codimension \(t\) if it is pure and for every face
\(F \in \Delta\) with \(\#(F) \geq t\), \(\text{link}_\Delta(F)\) is Cohen-Macaulay. Every CM\(_t\) complex is also
CM\(_r\) for all \(r \geq t\). For \(t < 0\), CM\(_t\) means CM\(_0\). The properties CM\(_0\) and CM\(_1\) are
the same as Cohen-Macaulay-ness and Buchsbaum-ness, respectively.

The link of a face \(F\) in a simplicial complex \(\Delta\) is denoted by \(\text{link}_\Delta(F)\) and is
\[
\text{link}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}.
\]
The following lemma is useful for checking the CM\(_t\) property of simplicial com-
plexes:

Lemma 1.1. ([3, Lemma 2.3]) Let \(t \geq 1\) and let \(\Delta\) be a nonempty complex. Then
\(\Delta\) is CM\(_t\) if and only if \(\Delta\) is pure and \(\text{link}_\Delta(v)\) is CM\(_{t-1}\) for every vertex \(v \in \Delta\).

Let \(\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))\) be a simple graph with vertex set \(V\) and edge set \(E\). The
independence complex of \(\mathcal{G}\) is the complex \(\Delta_{\mathcal{G}}\) with vertex set \(V\) and with faces
consisting of independent sets of vertices of \(\mathcal{G}\). Thus \(F\) is a face of \(\Delta_{\mathcal{G}}\) if and only
if there is no edge of \(\mathcal{G}\) joining any two vertices of \(F\).

The edge ideal of a simple graph \(\mathcal{G}\), denoted by \(I(\mathcal{G})\), is an ideal of \(S\) generated
by all squarefree monomials \(x_ix_j\) with \(x_ix_j \in E(\mathcal{G})\).

A simple graph \(\mathcal{G}\) is called CM\(_t\) if \(\Delta_{\mathcal{G}}\) is CM\(_t\) and it is called unmixed if \(\Delta_{\mathcal{G}}\) is
pure.

For a monomial ideal \(I \subset S\), We denote by \(G(I)\) the unique minimal set of
monomial generators of \(I\).

2. The Expansion Functor in Combinatorial and Algebraic Concepts

In this section we define the expansion of a simplicial complex and recall the
expansion of a simple graph from [10] and the expansion of a monomial ideal from
[1]. We show that these concepts are intimately related to each other.

(1) Let \(\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n\). For \(F = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq \{x_1, \ldots, x_n\}\) define
\[
F^\alpha = \{x_{i_1k_1}, \ldots, x_{i_rk_1}, \ldots, x_{i_1k_r}, \ldots, x_{i_rk_r}\}
\]
as a subset of \([n]^n := \{x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}\}\). \(F^\alpha\) is called the expansion
of \(F\) with respect to \(\alpha\).

For a simplicial complex \(\Delta = \langle F_1, \ldots, F_r \rangle\) on \([n]\), we define the expansion of \(\Delta\)
with respect to \(\alpha\) as the simplicial complex
\[
\Delta^\alpha = \langle F_1^\alpha, \ldots, F_r^\alpha \rangle.
\]

(2) The duplication of a vertex \(x_i\) of a simple graph \(\mathcal{G}\) was first introduced by
Schrijver [10] and it means extending its vertex set \(V(\mathcal{G})\) by a new vertex \(x'_i\) and
replacing \(E(\mathcal{G})\) by
\[
E(\mathcal{G}) \cup \{(e \setminus \{x_i\}) \cup \{x'_i\} : x_i \in E(\mathcal{G})\}.
\]
For the $n$-tuple $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$, with positive integer entries, the expansion of the simple graph $G$ is denoted by $G^\alpha$ and it is obtained from $G$ by successively duplicating $k_i - 1$ times every vertex $x_i$.

(3) In [1] Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded $S$-modules and studied some homological behaviors of this functor. We recall the expansion functor defined by them only in the category of monomial ideals and refer the reader to [1] for more general case in the category of finitely generated multigraded $S$-modules.

Let $S^\alpha$ be a polynomial ring over $K$ in the variables

$$x_{11}, \ldots, x_{1k_1}, \ldots, x_{nk_n}.$$ Whenever $I \subset S$ is a monomial ideal minimally generated by $u_1, \ldots, u_r$, the expansion of $I$ with respect to $\alpha$ is defined by

$$I^\alpha = \sum_{i=1}^r P_{i1}^{u_{i1}(u_i)} \cdots P_{in}^{u_{in}(u_i)} \subset S^\alpha$$

where $P_j = (x_{j1}, \ldots, x_{jk_j})$ is a prime ideal of $S^\alpha$ and $\nu_j(u_i)$ is the exponent of $x_j$ in $u_i$.

It was shown in [1] that the expansion functor is exact and so $(S/I)^\alpha = S^\alpha / I^\alpha$.

In the following lemmas we describe the relations between the above three concepts of expansion functor.

**Lemma 2.1.** For a simplicial complex $\Delta$ we have $I_\Delta^\alpha = I_{\Delta^\alpha}$. In particular, $K[\Delta]^\alpha = K[\Delta^\alpha]$.

**Proof.** Let $\Delta = \langle F_1, \ldots, F_r \rangle$. Since $I_\Delta = \bigcap_{i=1}^r P_{F_i}$, it follows from Lemma 1.1 in [1] that $I_\Delta^\alpha = \bigcap_{i=1}^r P_{F_i}^\alpha$. The result is obtained by the fact that $P_{F_i}^\alpha = P_{(F_i^\alpha)_c}$. \hfill \Box

Let $u = x_{i_1} \cdots x_{i_n} \in S$ be a monomial and $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$. We set $u^\alpha = G((u)^\alpha)$ and for a set $A$ of monomials in $S$, $A^\alpha$ is defined

$$A^\alpha = \bigcup_{u \in A} u^\alpha.$$ One can easily obtain the following lemma.

**Lemma 2.2.** Let $I \subset S$ be a monomial ideal and $\alpha \in \mathbb{N}^n$. Then $G(I^\alpha) = G(I)^\alpha$.

**Lemma 2.3.** For a simple graph $G$ on the vertex set $[n]$ and $\alpha \in \mathbb{N}^n$ we have $I(G^\alpha) = I(G)^\alpha$.

**Proof.** Let $\alpha = (k_1, \ldots, k_n)$ and $P_j = (x_{j1}, \ldots, x_{jk_j})$. Then it follows from Lemma 11(ii,iii) of [1] that

$$I(G^\alpha) = \langle x_{ir}x_{js} : x_i x_j \in E(G), 1 \leq r \leq k_i, 1 \leq s \leq k_j \rangle = \sum_{x_i \in E(G)} P_i P_j$$

$$= \sum_{x_i \in E(G)} (x_i)^\alpha (x_j)^\alpha = (\sum_{x_i \in E(G)} (x_i)(x_j))^\alpha = I(G)^\alpha.$$ \hfill \Box
3. The expansion of a CM$_t$ complex

The following proposition gives us some information about the expansion of a simplicial complex which are useful in the proof of the next results.

**Proposition 3.1.** Let $\Delta$ be a simplicial complex and let $\alpha \in \mathbb{N}^n$.

(i) For all $i \leq \dim(\Delta)$, there exists an epimorphism $\theta : \tilde{H}_i(\Delta^\alpha; K) \to \tilde{H}_i(\Delta; K)$.

In particular in this case

$$\tilde{H}_i(\Delta^\alpha; K)/\ker(\theta) \cong \tilde{H}_i(\Delta; K);$$

(ii) For $F \in \Delta^\alpha$ such that $F = G^\alpha$ for some $G \in \Delta$, we have

$$\link_{\Delta^\alpha}(F) = (\link_{\Delta}(G))^\alpha;$$

(iii) For $F \in \Delta^\alpha$ such that $F \neq G^\alpha$ for every $G \in \Delta$, we have

$$\link_{\Delta^\alpha}F = \langle U^\alpha \setminus F \rangle \ast \link_{\Delta^\alpha}U^\alpha$$

for some $U \in \Delta$ with $F \subseteq U^\alpha$. Here $*$ means the join of two simplicial complexes.

In the third case, $\link_{\Delta^\alpha}F$ is a cone and so acyclic, i.e., $\tilde{H}_i(\link_{\Delta^\alpha}F; K) = 0$ for all $i > 0$.

**Proof.** (i) Consider the map $\pi : [n]^\alpha \to [n]$ by $\pi(x_{ij}) = x_i$ for all $i, j$. Let the simplicial map $\varphi : \Delta^\alpha \to \Delta$ be defined by $\varphi([x_{i_1j_1}, \ldots, x_{i_kj_k}]) = \{\pi(x_{i_1j_1}), \ldots, \pi(x_{i_kj_k})\} = \{x_{i_1}, \ldots, x_{i_k}\}$. Actually, $\varphi$ is an extension of $\pi$ to $\Delta^\alpha$ by linearity. Define $\varphi_\# : \tilde{C}_q(\Delta^\alpha; K) \to \tilde{C}_q(\Delta; K)$, for each $q$, by

$$\varphi_\#([x_{i_0j_0}, \ldots, x_{i_kj_k}]) = \left\{ \begin{array}{ll} 0 & \text{if for some indices } i_r = i_t \\ [\varphi([x_{i_0j_0}]), \ldots, \varphi([x_{i_kj_k}])] & \text{otherwise.} \end{array} \right.$$ 

It is clear from the definitions of $\tilde{C}_q(\Delta^\alpha; K)$ and $\tilde{C}_q(\Delta; K)$ that $\varphi_\#$ is well-defined. Also, define $\varphi_\alpha : \tilde{H}_i(\Delta^\alpha; K) \to \tilde{H}_i(\Delta; K)$ by

$$\varphi_\alpha : z + B_i(\Delta^\alpha) \to \varphi_\#(z) + B_i(\Delta).$$

It is trivial that $\varphi_\alpha$ is onto.

(ii) The inclusion $\link_{\Delta^\alpha}(F) \supseteq (\link_{\Delta}(G))^\alpha$ is trivial. So we show the reverse inclusion. Let $\sigma \in \link_{\Delta^\alpha}(G^\alpha)$. Then $\sigma \cap G^\alpha = \emptyset$ and $\sigma \cup G^\alpha \in \Delta^\alpha$. We want to show $\pi(\sigma) \in \link_{\Delta}(G)$. Because in this case, $\pi(\sigma)^\alpha \in (\link_{\Delta}(G))^\alpha$ and since that $\sigma \subseteq \pi(\sigma)^\alpha$, we can conclude that $\sigma \in (\link_{\Delta}(G))^\alpha$.

Clearly, $\pi(\sigma) \cup G \in \Delta$. To show that $\pi(\sigma) \cap G = \emptyset$, suppose, on the contrary, that $x_i \in \pi(\sigma) \cap G$. Then $x_{ij} \in \sigma$ for some $j$. Especially, $x_{ij} \in G^\alpha$. Therefore $\sigma \cap G^\alpha \neq \emptyset$, a contradiction.

(iii) Let $\tau \in \link_{\Delta^\alpha}F$. Let $\tau \cap \pi(F)^\alpha = \emptyset$. It follows from $\tau \cup F \in \Delta^\alpha$ that $\pi(\tau)^\alpha \cup \pi(F)^\alpha \in \Delta^\alpha$. Now by $\tau \subseteq \pi(\tau)^\alpha$ it follows that $\tau \cup \pi(F)^\alpha \in \Delta^\alpha$. Hence $\tau \in \link_{\Delta^\alpha}(\pi(F)^\alpha)$. So we suppose that $\tau \cap \pi(F)^\alpha = \emptyset$. We write $\tau = (\tau \cap \pi(F)^\alpha) \cup (\tau \setminus \pi(F)^\alpha)$. It is clear that $\tau \cap \pi(F)^\alpha \subseteq \pi(F)^\alpha \setminus F$ and $\tau \setminus \pi(F)^\alpha \in \link_{\Delta^\alpha}(\pi(F)^\alpha)$.

The reverse inclusion is trivial. 

**Remark 3.2.** Let $\Delta = \langle x_1, x_2, x_2x_3 \rangle$ be a complex on $[3]$ and $\alpha = (2, 1, 1) \in \mathbb{N}^3$. Then $\Delta^\alpha = \langle x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31} \rangle$ is a complex on $\{x_{11}, x_{12}, x_{21}, x_{23}, x_{31}\}$. Notice that $\Delta$ is pure but $\Delta^\alpha$ is not. Therefore, the expansion of a pure simplicial complex is not necessarily pure.
Theorem 3.3. Let $\Delta$ be a simplicial complex on $[n]$ of dimension $d-1$ and let $t \geq 0$ be the least integer that $\Delta$ is CM$_t$. Suppose that $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$ such that $k_i > 1$ for some $i$ and $\Delta^\alpha$ is pure. Then $\Delta^\alpha$ is CM$_{t+e-k+1}$ but it is not CM$_{t+e-k}$, where $e = \dim(\Delta^\alpha) + 1$ and $k = \min\{k_i : k_i > 1\}$.

Proof. We use induction on $e \geq 2$. If $e = 2$, then $\dim(\Delta^\alpha) = 1$ and $\Delta$ should be only in form $\Delta = \langle x_1, \ldots, x_n \rangle$. In particular, $\Delta^\alpha$ is of the form

$$\Delta^\alpha = \langle \{x_{i_1}, x_{i_2} \}, \{x_{i_2}, x_{i_3} \}, \ldots, \{x_{i_s}, x_{i_s+1} \} \rangle.$$ 

It is clear that $\Delta^\alpha$ is CM$_1$ but it is not Cohen-Macaulay.

Assume that $e > 2$. Let $\{x_{ij}\} \in \Delta^\alpha$. We want to show that $\text{link}_{\Delta^\alpha}(x_{ij})$ is CM$_{t-k}$. Consider the following cases:

Case 1: $k_i > 1$. Then

$$\text{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \setminus x_{ij} \rangle \ast (\text{link}_{\Delta}(x_i))^\alpha.$$ 

$(\text{link}_{\Delta}(x_i))^\alpha$ is of dimension $e-k_i-1$ and, by induction hypothesis, it is CM$_{t+e-k_i-k+1}$. On the other hand, $\langle \{x_i\}^\alpha \setminus x_{ij} \rangle$ is Cohen-Macaulay of dimension $k_i-2$. Therefore, it follows from Theorem 1.1(i) of [4] that $\text{link}_{\Delta^\alpha}(x_{ij})$ is CM$_{t+e-k}$.

Case 2: $k_i = 1$. Then

$$\text{link}_{\Delta^\alpha}(x_{ij}) = (\text{link}_{\Delta}(x_i))^\alpha$$

which is of dimension $e-2$ and, by induction, it is CM$_{t+e-k}$.

Now suppose that $e > 2$ and $k_s = k$ for some $s \in [n]$. Let $F$ be a facet of $\Delta$ such that $x_s$ belongs to $F$.

If $\dim(\Delta) = 0$, then $k_l = k$ for all $l \in [n]$. In particular, $e = k$. It is clear that $\Delta^\alpha$ is not CM$_{t+e-k}$ (or Cohen-Macaulay). So suppose that $\dim(\Delta) > 0$. Choose $x_i \in F \setminus x_s$. Then

$$\text{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \setminus x_{ij} \rangle \ast (\text{link}_{\Delta}(x_i))^\alpha.$$ 

By induction hypothesis, $(\text{link}_{\Delta}(x_i))^\alpha$ is not CM$_{t+e-k_i-k}$. It follows from Theorem 3.1(ii) of [4] that $\text{link}_{\Delta^\alpha}(x_{ij})$ is not CM$_{t+e-k-1}$. Therefore $\Delta^\alpha$ is not CM$_{t+e-k}$.

Corollary 3.4. Let $\Delta$ be a non-empty Cohen-Macaulay simplicial complex on $[n]$. Then for any $\alpha \in \mathbb{N}^n$, with $\alpha \neq \mathbf{1}$, $\Delta^\alpha$ can never be Cohen-Macaulay.

4. The contraction functor

Let $\Delta = \langle F_1, \ldots, F_r \rangle$ be a simplicial complex on $[n]$. Consider the equivalence relation $\sim$ on the vertices of $\Delta$ given by

$$x_i \sim x_j \iff \langle x_i \rangle \ast \text{link}_{\Delta}(x_i) = \langle x_j \rangle \ast \text{link}_{\Delta}(x_j).$$

In fact $\langle x_i \rangle \ast \text{link}_{\Delta}(x_i)$ is the cone over $\text{link}_{\Delta}(x_i)$, and the elements of $\langle x_i \rangle \ast \text{link}_{\Delta}(x_i)$ are those faces of $\Delta$, which contain $x_i$. Hence $\langle x_i \rangle \ast \text{link}_{\Delta}(x_i) = \langle x_j \rangle \ast \text{link}_{\Delta}(x_j)$, means the cone with vertex $x_i$ is equal to the cone with vertex $x_j$. In other words, $x_i \sim x_j$ is equivalent to saying that for a facet $F \in \Delta$, $F$ contains $x_i$ if and only if it contains $x_j$.

Let $[\bar{m}] = \{\bar{y}_1, \ldots, \bar{y}_m\}$ be the set of equivalence classes under $\sim$. Let $\bar{y}_i = \{x_{i1}, \ldots, x_{ia_i}\}$. Set $\alpha = (a_1, \ldots, a_m)$. For $F_i \in \Delta$, define $G_i = \{\bar{y}_i : \bar{y} \in F_i\}$ and let $\Gamma$ be a simplicial complex on the vertex set $[\bar{m}]$ with facets $G_1, \ldots, G_r$. We call $\Gamma$ the contraction of $\Delta$ by $\alpha$ and $\alpha$ is called the vector obtained from contraction.

For example, consider the simplicial complex $\Delta = \langle x_1x_2x_3, x_2x_3x_4, x_1x_4x_5, x_2x_3x_5 \rangle$ on the vertex set $[5] = \{x_1, \ldots, x_5\}$. Then $\bar{y}_1 = \{x_1\}$, $\bar{y}_2 = \{x_2, x_3\}$, $\bar{y}_3 = \{x_4\}$.
\[ \bar{y}_4 = \{x_5\} \text{ and } \alpha = (1, 2, 1, 1). \] Therefore, the contraction of \( \Delta \) by \( \alpha \) is \( \Gamma = \langle \bar{y}_1\bar{y}_2; \bar{y}_2\bar{y}_3; \bar{y}_1\bar{y}_3\bar{y}_4; \bar{y}_2\bar{y}_4 \rangle \) a complex on the vertex set \( \bar{A} = \{\bar{y}_1, \ldots, \bar{y}_4\}. \)

**Remark 4.1.** Note that if \( \Delta \) is a pure simplicial complex then the contraction of \( \Delta \) is not necessarily pure (see the above example). In the special case where the vector \( \alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n \) and \( k_i = k_j \) for all \( i, j \), it is easy to check that in this case \( \Delta \) is pure if and only if \( \Delta^\alpha \) is pure. Another case is introduced in the following proposition.

**Proposition 4.2.** Let \( \Delta \) be a simplicial complex on \([n]\) and assume that \( \alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n \) satisfies the following condition:

\( \langle \dagger \rangle \) for all facets \( F, G \in \Delta \), if \( x_i \in F \setminus G \) and \( x_j \in G \setminus F \) then \( k_i = k_j \).

Then \( \Delta \) is pure if and only if \( \Delta^\alpha \) is pure.

**Proof.** Let \( \Delta \) be a pure simplicial complex and let \( F, G \in \Delta \) be two facets of \( \Delta \). Then

\[
|F^\alpha| - |G^\alpha| = \sum_{x_i \in F} k_i - \sum_{x_i \in G} k_i = \sum_{x_i \in F \setminus G} k_i - \sum_{x_i \in G \setminus F} k_i.
\]

Now the condition \( \langle \dagger \rangle \) implies that \( |F^\alpha| = |G^\alpha| \). This means that all facets of \( \Delta^\alpha \) have the same cardinality.

Let \( \Delta^\alpha \) be pure. Suppose that \( F, G \) are two facets in \( \Delta \). If \( |F| > |G| \) then \( |F^\alpha| > |G^\alpha| \). Therefore \( \sum_{x_i \in F \setminus G} k_i > \sum_{x_i \in G \setminus F} k_i \). This concludes that \( |F^\alpha| = \sum_{x_i \in F} k_i > \sum_{x_i \in G} k_i = |G^\alpha| \), a contradiction. \( \square \)

There is a close relationship between a simplicial complex and its contraction. In fact, the expansion of the contraction of a simplicial complex is the same complex. The precise statement is the following.

**Lemma 4.3.** Let \( \Gamma \) be the contraction of \( \Delta \) by \( \alpha \). Then \( \Gamma^\alpha \cong \Delta \).

**Proof.** Suppose that \( \Delta \) and \( \Gamma \) are on the vertex sets \([n] = \{x_1, \ldots, x_n\}\) and \([\bar{m}] = \{\bar{y}_1, \ldots, \bar{y}_m\}\), respectively. Let \( \alpha = (a_1, \ldots, a_m) \). For \( \bar{y}_i \in \Gamma \), suppose that \( \{\bar{y}_i\}^\alpha = \{\bar{y}_{i_1}, \ldots, \bar{y}_{i_a}\} \). So \( \Gamma^\alpha \) is a simplicial complex on the vertex set \([\bar{m}]^\alpha = \{\bar{y}_{ij} : i = 1, \ldots, m, j = 1, \ldots, a_i\}\). Now define \( \varphi : [\bar{m}]^\alpha \to [n] \) by \( \varphi(\bar{y}_{ij}) = x_{ij} \). Extending \( \varphi \), we obtain the isomorphism \( \varphi : \Gamma^\alpha \to \Delta \). \( \square \)

**Proposition 4.4.** Let \( \Delta \) be a simplicial complex and assume that \( \Delta^\alpha \) is Cohen-Macaulay for some \( \alpha \in \mathbb{N}^n \). Then \( \Delta \) is Cohen-Macaulay.

**Proof.** By Lemma 3.1(i), for all \( i \leq \dim(\text{link}_\Delta F) \) and all \( F \in \Delta \) there exists an epimorphism \( \theta : \text{link}_\Delta^\alpha F^\alpha \to \text{link}_\Delta F \) such that

\[
\tilde{H}_i(\text{link}_\Delta^\alpha F^\alpha; K)/\ker(\theta) \cong \tilde{H}_i(\text{link}_\Delta F; K).
\]

Now suppose that \( i < \dim(\text{link}_\Delta F) \). Then \( i < \dim(\text{link}_\Delta^\alpha F^\alpha) \) and by Cohen-Macaulayness of \( \Delta^\alpha \), \( \tilde{H}_i(\text{link}_\Delta^\alpha F^\alpha; K) = 0 \). Therefore \( \tilde{H}_i(\text{link}_\Delta F; K) = 0 \). This means that \( \Delta \) is Cohen-Macaulay. \( \square \)

It follows from Proposition 4.4 that:

**Corollary 4.5.** The contraction of a Cohen-Macaulay simplicial complex \( \Delta \) is Cohen-Macaulay.

This can be generalized in the following theorem.
Theorem 4.6. Let $\Gamma$ be the contraction of a CM$_{t}$ simplicial complex $\Delta$, for some $t \geq 0$, by $\alpha = (k_1, \ldots, k_n)$. If $k_i \geq t$ for all $i$ and $\Gamma$ is pure, then $\Gamma$ is Buchsbaum.

Proof. If $t = 0$, then we saw in Corollary 4.5 that $\Gamma$ is Cohen-Macaulay and so it is CM$_{t}$. Hence assume that $t > 0$. Let $\Delta = \langle F_1, \ldots, F_r \rangle$. We have to show that $\tilde{H}_i(\text{link}_{\Gamma} G; K) = 0$, for all faces $G \in \Gamma$ with $|G| \geq 1$ and all $i < \text{dim(link}_{\Gamma} G)$.

Let $G \in \Gamma$ with $|G| \geq 1$. Then $|G^\alpha| \geq t$. It follows from Lemma 1.1 and CM$_{t}$-ness of $\Delta$ that $\tilde{H}_i(\text{link}_{\Gamma} G; K) \cong \tilde{H}_i(\text{link}_{\Delta} G^\alpha; K) = 0$ for $i < \text{dim(link}_{\Delta} G^\alpha)$ and, particularly, for $i < \text{dim(link}_{\Gamma} G)$. Therefore $\Gamma$ is Buchsbaum.

Corollary 4.7. Let $\Gamma$ be the contraction of a Buchsbaum simplicial complex $\Delta$. If $\Gamma$ is pure, then $\Gamma$ is also Buchsbaum.

Let $G$ be a simple graph on the vertex set $[n]$ and let $\Delta_G$ be its independence complex on $[n]$, i.e., a simplicial complex whose faces are the independent vertex sets of $G$. Let $\Gamma$ be the contraction of $\Delta_G$. In the following we show that $\Gamma$ is the independence complex of a simple graph $H$. We call $H$ the contraction of $G$.

Lemma 4.8. Let $G$ be a simple graph. The contraction of $\Delta_G$ is the independence complex of a simple graph $H$.

Proof. It suffices to show that $I_{\Gamma}$ is a squarefree monomial ideal generated in degree 2. Let $\Gamma$ be the contraction of $\Delta_G$ and let $\alpha = (k_1, \ldots, k_n)$ be the vector obtained from the contraction. Let $[n] = \{x_1, \ldots, x_n\}$ be the vertex set of $\Gamma$. Suppose that $u = x_{i_1} \ldots x_{i_t} \in G(I_{\Gamma})$. Then $u^\alpha \subset G(I_{\Gamma})^\alpha = G(I_{\Delta_G}) = G(I_G)$. Since $u^\alpha = \{x_{i_1j_1} \ldots x_{i_tj_t} : 1 \leq j_l \leq k_i, 1 \leq l \leq t\}$ we have $t = 2$ and the proof is completed.

Example 4.9. Let $G_1$ and $G_2$ be, respectively, from left to right the following graphs:

The contraction of $G_1$ and $G_2$ are

The contraction of $G_1$ is equal to itself but $G_2$ is contracted to an edge and the vector obtained from contraction is $\alpha = (2, 3)$.

We recall that a simple graph is CM$_{t}$ for some $t \geq 0$, if the associated independence complex is CM$_{t}$.
Remark 4.10. The simple graph $G'$ obtained from $G$ in Lemma 4.3 and Theorem 4.4 of [4] is the expansion of $G$. Actually, suppose that $G$ is a bipartite graph on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \ldots, x_d\}$ and $W = \{x_{d+1}, \ldots, x_{2d}\}$. Then for $\alpha = (n_1, \ldots, n_d, n_1, \ldots, n_d)$ we have $G' = G^\alpha$. It follows from Theorem 3.3 that if $G$ is CM$_t$ for some $t \geq 0$ then $G'$ is CM$_{t+n-n_0+1}$ where $n = \sum_{i=1}^d n_i$ and $n_0 = \min\{n_i > 1 : i = 1, \ldots, d\}$. This implies that the first part of Theorem 4.4 of [4] is an obvious consequence of Theorem 3.3 for $t = 0$.

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Rahim Rahmati-Asghar,
Department of Mathematics, Faculty of Basic Sciences,
University of Maragheh, P. O. Box 55181-83111, Maragheh, Iran.
E-mail: rahmatiasghar.r@gmail.ac.ir