BRST Structures and Symplectic Geometry on a Class of Supermanifolds

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Abstract

By investigating the symplectic geometry and geometric quantization on a class of supermanifolds, we exhibit BRST structures for a certain kind of algebras. We discuss the undeformed and $q$-deformed cases in the classical as well as in the quantum cases.

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Quantum groups and quantum algebras \cite{1} play an important role in many physical problems such as exactly soluble models in statistical mechanics, conformal field theory, integrable model field theories \cite{2}. This is due to their rich mathematical structures and geometric properties. Efforts have also been made to explore possible applications to gauge theory in terms of non-commutative geometry \cite{3}. As the form of a q-gauge invariant theory should be essentially determined by the constraint structure, the generalization of the geometric structure of BRST invariance might provide insight into possible applications of quantum groups to gauge fields. The construction of the Balatin-Frandkin-Vilkovisky BRST charge related to the $SU_q(2)$ algebra has also been discussed from this point of view \cite{4}.

In fact it is also worthwhile to study the BRST constructions for q-deformed algebras for general systems of non-first class constraints. In this letter by investigating the symplectic geometry and geometric quantization on a class of special supermanifolds we exhibit BRST structures for a class of algebras including $SU_q(2)$, either undeformed or q-deformed with three algebraic elements. We discuss the classical as well as the quantum cases, as the q-deformation and $\hbar$-quantization are different in principle \cite{5,6}.

It is well known that for a finite dimensional first class constrained Hamiltonian system, the constraints $K_a$, $a = 1, \ldots, n$, satisfy Poisson algebra relations

$$[K_a, K_b]_{P.B.} = f^c_{ab} K_c ,$$

with $f^c_{ab}$ real coefficients and $[, ,]$ standing for the Poisson bracket. The BRST charge $Q$ is given by \cite{7}

$$Q = C^a K_a + \frac{1}{2} f^c_{ab} C^b C^c ,$$

where $C^a$ and $\tilde{C}^b$ are anticommuting grassmann quantities with grassmann parities 1 and $-1$ respectively and such that $(C^a)^2 = 0$, $(\tilde{C}^b)^2 = 0$, $C^a \tilde{C}^b = -\tilde{C}^b C^a$ and $[C_a, \tilde{C}^b]_{P.B.} = \delta^b_a$ ($\delta^b_a$ being the Kronecker’s symbol). It is easy to prove that the BRST charge $Q$ is nilpotent $[Q, Q]_{P.B.} = 0$ and satisfies

$$[Q, \tilde{C}^a]_{P.B.} \overset{def.}{=} \tilde{K}_a = K_a + f^c_{ab} \tilde{C}^b C^c , \quad [\tilde{K}_a, K_b]_{P.B.} = f^c_{ab} \tilde{K}_c .$$

$$[\tilde{K}_a, \tilde{C}^b]_{P.B.} = f^c_{ab} \tilde{C}^c , \quad [Q, \tilde{K}_a]_{P.B.} = 0 .$$

$$[Q, C^a]_{P.B.} + \frac{1}{2} f^c_{bc} C^b C^c = 0 .$$

Equation (3) is the Maurer-Cartan equation related to the BRST cohomology.
To give a systematic description of the BRST structures for general constrained Hamiltonian systems with three constraints, we consider the supermanifold $M = M_+ \times M_0 \times M_-$, where $M_0$ is an usual commutative differentiable manifold with even dimensions, $M_+$ and $M_-$ are the anticommuting parts of $M$ with grassmann parity $+1$ resp. $-1$ (these parts correspond to “ghost” and “antighost” respectively in the BRST formalism). We will use freely notions and notations from supermanifold theory as described in [8], here we limit ourselves to a brief summary. The tangent space $T_p(M)$ of the supermanifold $M$ at point $p$ is spanned by the local supervector basis $i e = (\frac{\partial}{\partial x^i})_p$ or $e_i = (\frac{\partial}{\partial x^i})_p$, satisfying the relations $i e = (-1)^i e_i$, where $x^i$ is the local coordinate and the exponent of $(-1)$ is the grassmann parity of $x^i$. The arrow represents the acting direction of the partial derivatives. The grassmann parity takes values 0 or $\pm 1$ with respect to commuting (“c-type”) variables and anticommuting (“a-type”) variables respectively. A tangent vector $X$ on $M$ may be expressed as $X = X^i e_i = (-1)^i X_i e_i = (-1)^i X^i$ is the $i$-th coordinate of the tangent space and $X$ in the exponent of $(-1)$ is the grassmann parity of $X$.

The local basis for the dual space $T^*_p(M)$ is $e^i = i e = dx^i$ satisfying $< i e, e^j > = i \delta^j_i$ and $i \delta^j_i = (-1)^i \delta^j_i = (-1)^{i+j} \delta^j_i$. Similarly a local dual vector $V$ takes the form $V = e^i V = (-1)^i V^i e^i = V^i e_i$, where $V_i \equiv (-1)^i V^i$.

As $M_0$ is even dimensional, a real, c-type, super-antisymmetric and closed two form $\omega$ can be defined. We have thus $d\omega = 0$, $\omega$ is a symplectic form on the supermanifold $M$, and is non-degenerate, i.e., $X = 0$ if $X | \omega = 0$, where $X$ is the supervector field on $M$ and $| \omega$ denotes the left inner product defined by $(X | \omega)(Y) = \omega(X, Y)$ for any two vectors $X$ and $Y$ on $M$. Locally,

$$\omega = \frac{1}{2} dx^i i \omega_j dx^j,$$

where $i \omega_j$ has the supersymmetric property, $i \omega_j = (-1)^{i+j} \omega_j$ and the non degeneracy property, $sdet(i \omega_j) \neq 0$. Here $sdet$ stands for superdeterminant.

The canonical transformations are $\omega$-preserving diffeomorphisms of $M$ onto itself. A vector $X$ on $M$ corresponds to an infinitesimal canonical transformation if and only if the Lie derivative of $\omega$ with respect to $X$ vanishes,

$$\mathcal{L}_X \omega = X | d \omega + d (X | \omega) = 0.$$

A vector field $X$ satisfying (7) is said to be a Hamiltonian vector field on the supermanifold.
Let $\mathcal{F}(M)$ denotes the set of differentiable functions on the supermanifold $M$. It can be proved following [10] that for any function $f \in \mathcal{F}(M)$, there exits a unique Hamiltonian vector field $X_f$ satisfying

$$X_f \omega = -df . \quad (8)$$

For $f, g \in \mathcal{F}(M)$ the super Poisson bracket $[f, g]_{P.B.}$ is defined by

$$[f, g]_{P.B.} = -\omega(X_f, X_g) = \omega(X_g, X_f) = -X_f g = (-1)^{fg}X_g f . \quad (9)$$

It has the following properties

$$[f, g]_{P.B.} = -(-1)^{fg}[f, g]_{P.B.} ,$$

$$[f, [g, h]_{P.B.}]_{P.B.} + (-1)^{f(g+h)}[g, [h, f]_{P.B.}]_{P.B.} + (-1)^{h(f+g)}[h, [f, g]_{P.B.}]_{P.B.} = 0 .$$

For $f \in \mathcal{F}(M)$ the related Hamiltonian vector field $X_f$ has locally the form

$$X_f = f \sum_{i,j} \frac{\partial}{\partial x^i} i^i \omega^j \frac{\partial}{\partial x^j} . \quad (10)$$

Hence from (8) we have

$$[f, g]_{P.B.} = -f \sum_{i,j} \frac{\partial}{\partial x^i} i^i \omega^j \frac{\partial}{\partial x^j} g , \quad (11)$$

where $(i^i \omega^j)$ is the inverse of $(i \omega_j)$ (which exists since by assumption $\omega$ is non degenerate).

Now we assume the manifold $M_0$ to be a 2-dimensional smooth manifold defined by

$$F(S_1, S_2, S_3) = 0 \quad (12)$$

for some continuously differentiable function $F: \mathbb{R}^3 \rightarrow \mathbb{C}$, where $S_i$, $i = 1, 2, 3$ are the dynamical variables on phase space, i.e., the constraints of the Hamiltonian system (i.e. we have the case considered in [11] with $n = 3$). We recall that the a-type coordinates $C^a$ resp. $\bar{C}_b$ of the manifolds $M_+$ resp. $M_-$ with grassmann parity +1 resp. −1 satisfy (as in [11]):

$$(C^a)^2 = 0 , \quad (\bar{C}_b)^2 = 0 , \quad C^a \bar{C}_b = -\bar{C}_b C^a , \quad a, b = 1, 2, 3 . \quad (13)$$

**Proposition 1.** The symplectic form $\omega$ on the supermanifold $M$ is given by

$$\omega = \frac{1}{2} \sum_{ijk} \epsilon_{ijk} B_i dS_j \wedge dS_k + \sum_a d\bar{C}_a \wedge dC^a , \quad (14)$$

where $B_i$ are differentiable functions of $S_i$ satisfying

$$\sum_{i=1}^3 B_i \frac{\partial F}{\partial S_i} = -\frac{1}{\alpha} , \quad (15)$$


for some real constant \( \alpha \). And for \( f \in \mathcal{F}(M) \), the Hamiltonian vector field \( X_f \) is given by

\[
X_f = \alpha \sum_{ijk} \epsilon_{ijk} \frac{\partial f}{\partial S_i} \frac{\partial F}{\partial S_j} \frac{\partial}{\partial S_k} + f \frac{\partial}{\partial C^a} \frac{\partial}{\partial C^a} + f \frac{\partial}{\partial \bar{C}^a} \frac{\partial}{\partial \bar{C}^a}.
\] (16)

**Proof.** The proof is straightforward by showing \( X_f \omega = -df \), \( \forall f \in \mathcal{F}(M) \) in terms of relations (12) and (13). It is also obvious that the right hand side of (14) gives a closed form since \( M_0 \) is here a 2-dimensional manifold. \( \blacksquare \)

From the definition of the Poisson bracket (9) and formula (16), the Poisson bracket for \( f, g \in \mathcal{F}(M) \) is then given by

\[
[f, g]_{P.B.} = - \left( \alpha \sum_{ijk} \epsilon_{ijk} \frac{\partial f}{\partial S_i} \frac{\partial F}{\partial S_j} \frac{\partial g}{\partial S_k} + f \frac{\partial}{\partial C^a} \frac{\partial g}{\partial C^a} + f \frac{\partial}{\partial \bar{C}^a} \frac{\partial g}{\partial \bar{C}^a} \right).
\] (17)

We remark that this Poisson bracket is uniquely given by the manifold up to an algebraic isomorphism. It does not depend on the form of the symplectic form \( \omega \).

Formula (17) gives rise to the classical Poisson bracket relations of \( \bar{C}^a \) and \( C^b \),

\[
[C^a, \bar{C}^b]_{P.B.} = \delta^b_a
\] (18)

with all other Poisson brackets being zero.

From formulas (16) and (17) we easily have

\[
X_{S_i} = \alpha \sum_{ijk} \epsilon_{ijk} \frac{\partial F}{\partial S_j} \frac{\partial}{\partial S_k}
\] (19)

and

\[
[S_i, S_j]_{P.B.} = \alpha \sum_k \epsilon_{ijk} \frac{\partial F}{\partial S_k}.
\] (20)

Therefore \( S_i, i = 1, 2, 3 \) constitute a Poisson algebra. For different values of the constant \( \alpha \) the Poisson algebras are algebraic isomorphic. Henceforth for simplicity we will take \( \alpha \) to be \( \frac{1}{2} \). Formula (21) expresses the fact that the 2-dimensional manifold given by (12) is related to a certain algebra. When \( F(S_1, S_2, S_3) \) is a quadratic function of \( S_i \), then it gives rise to a linear algebra, the first class constraint of the Hamiltonian system.

To investigate the BRST cohomology for both undeformed and q-deformed cases in terms of symplectic geometry and geometric quantization, we take the manifold \( M_0 \) defined by \( F \) to be of the form

\[
F = S_1^2 + S_2^2 + G(S_3) - const. = 0,
\] (21)
where $G$ is a continuously differentiable function. \textit{Const} represents the Casimir invariant after geometric quantization. From formula (20) we deduce the Poisson algebraic relations with respect to the manifold (21) (“classical constraint algebra”):

\begin{equation}
[S_1, S_2]_{P.B.} = \frac{\partial G(S_3)}{2\partial S_3}, \quad [S_2, S_3]_{P.B.} = S_1, \quad [S_3, S_1]_{P.B.} = S_2. \tag{22}
\end{equation}

This can be rewritten in the form

\begin{equation}
[S_+, S_-]_{P.B.} = -i\frac{\partial G(S_3)}{\partial S_3}, \quad [S_3, S_\pm]_{P.B.} = \mp iS_\pm, \tag{23}
\end{equation}

where $S_\pm = S_1 \pm iS_2$. We set $C_\pm = \frac{1}{2}(C^1 \mp iC^2)$. From (18) we have then

\begin{equation}
[C_- C_+, C_\pm]_{P.B.} = \frac{1}{2}, \quad [C_3 C_\pm]_{P.B.} = 1. \tag{24}
\end{equation}

\textbf{Proposition 2.} The BRST charge associated with the algebra (22) has the following general form

\begin{equation}
Q = S_+ C^+ + S_- C^- + A_1 C^3 + A_2 C_- C^3 + A_3 C_+ C^3 + A_4 C_3 C^+ C^- \\
+ A_5 \bar{C}_+ \bar{C}_- C^+ C^3, \tag{25}
\end{equation}

where $A_l \equiv A_l(S_3), l = 1, \ldots, 5$ are continuously differentiable functions. $Q$ is nilpotent (in the sense that $[Q, Q]_{P.B.} = 0$) if

\begin{equation}
A_4 A_1 = i \frac{\partial G(S_3)}{\partial S_3}, \quad A_2 = 2i \frac{\partial A_1}{\partial S_3}, \quad A_3 = -A_2, \quad A_5 = 2i \frac{\partial A_2}{\partial S_3}. \tag{26}
\end{equation}

\textbf{[Proof].} From formula (17) we have

\begin{equation}
[S_-, A_l]_{P.B.} = -iS_- \frac{\partial A_l}{\partial S_3}, \quad [S_+, A_l]_{P.B.} = iS_+ \frac{\partial A_l}{\partial S_3}. \tag{27}
\end{equation}

Let $Q$ be given by (25). Then we have

\begin{equation}
[Q, Q]_{P.B.} = 2(-i \frac{\partial G(S_3)}{\partial S_3} C^+ C^- + iS_+ \frac{\partial A_1}{\partial S_3} C^+ C^3 + iS_- \frac{\partial A_2}{\partial S_3} C^+ \bar{C}_- C^3 \\
+ \frac{1}{2} S_+ A_3 C^3 + \frac{1}{2} S_+ A_5 \bar{C}_+ C^3 - iS_- \frac{\partial A_1}{\partial S_3} C^- C^3 \\
+ \frac{1}{2} S_- A_2 C^- C^3 - iS_- \frac{\partial A_2}{\partial S_3} C_- \bar{C}_+ C^3 - \frac{1}{2} S_- A_5 \bar{C}_+ C^3 C^3 \\
+ A_1 A_4 C^+ C^- + \frac{1}{2} A_2 A_4 C^+ C^3 \bar{C}_3 C^+ + \frac{1}{2} A_3 A_4 C^+ C^3 \bar{C}_3 C^-). \tag{28}
\end{equation}

Hence we have $[Q, Q]_{P.B.} = 0$ under condition (26).
In addition, from the definition of Hamiltonian vector field $X_Q$ associated with the BRST charge $Q$ we have $\mathcal{L}_{X_Q}\omega = 0$. Hence the symplectic form (14) and the corresponding phase space are BRST invariant.

In this way we get a general BRST construction for all the constraint algebras of the form (23). Each $(A_l)_{l=1,\ldots,5}$ gives rise to a solution of (26) and hence to an expression of BRST charge. Here are two simple kinds of solutions:

**i.** $A_1 = \frac{\partial G(S_3)}{\partial S_3}$, $A_2 = -A_3 = 2i \frac{\partial^2 G(S_3)}{\partial^2 S_3}$, $A_4 = i$, $A_5 = -4 \frac{\partial^3 G(S_3)}{\partial^3 S_3}$ (29)

which inserted in (23) gives rise to the following expression of $Q$ (denoted by $Q_1$):

$$Q_1 = S_+ C^+ + S_- C^- + \frac{\partial G(S_3)}{\partial S_3} C^3 + 2i \frac{\partial^2 G(S_3)}{\partial^2 S_3} (\bar{C}_- C^- C^3 - \bar{C}_+ C^+ C^3)$$

$$+ i \bar{C}_3 C^+ C^- - 4 \frac{\partial^3 G(S_3)}{\partial^3 S_3} \bar{C}_+ C^- C^+ C^3 \, .$$ (30)

**ii.** When $\frac{\partial G(S_3)}{\partial S_3}$ can be written as $2H(S_3)S_3$ for some function $H(S_3)$, another simple solution of (26) is

$$A_1 = S_3 \, , \quad A_2 = -A_3 = 2i \, , \quad A_4 = 2iH(S_3) \, , \quad A_5 = 0 \, .$$ (31)

The corresponding $Q$ from (25) is:

$$Q_2 = S_+ C^+ + S_- C^- + S_3 C^3 + 2i \bar{C}_- C^- C^3 - 2i \bar{C}_+ C^+ C^3 + 2iH(S_3) \bar{C}_3 C^+ C^- \, .$$ (32)

Let us now give two particular examples of functions $F$ entering (21). We take $G$ resp. $\text{const}$ in (21) as given by $G(S_3) = -\frac{\sinh(2\gamma S_3)}{2\gamma \cosh \gamma}$ resp. $\text{const} = \frac{\sinh \gamma}{2\gamma \cosh \gamma}$ so that (21) reaches

$$S_1^2 + S_2^2 = \frac{\sinh(2\gamma S_3)}{2\gamma \cosh \gamma} = \frac{\sinh \gamma}{2\gamma \cosh \gamma} \, .$$ (33)

$\gamma = \ln q$ is a deformation parameter, $q > 0$.

**Proposition 3.** For the manifold $M_0$ given by (33) the symplectic form (14) on the supermanifold $M$ is given by

$$\omega = -\frac{\gamma \cosh \gamma}{\sinh \gamma} (S_1 dS_2 \wedge dS_3 + S_2 dS_3 \wedge dS_1) + \frac{\sinh(2\gamma S_3) \cosh \gamma}{\cosh(2\gamma S_3) \sinh \gamma} dS_1 \wedge dS_2$$

$$+ \sum_{i=1}^3 d\bar{C}_i \wedge dC^i \, .$$ (34)
From proposition 1, we see that what we have to verify is condition (15). Using (33) we have \[ 2B_1S_1 + 2B_2S_2 - \frac{\cosh(2\gamma S_3)}{\cosh \gamma} B_3 = -2. \] Substituting the functions \( B_i \) taken from (34) we see that (15) is satisfied. \( \Box \)

**Proposition 4.** The Poisson algebra related to the manifold (33) is just the q-deformed simple harmonic oscillator algebra \( \mathcal{H}_q(4), \)

\[
[S_+, S_-]_{P.B.} = i \frac{\cosh(2\gamma S_3)}{\cosh \gamma}, \quad [S_3, S_{\pm}]_{P.B.} = \mp i S_{\pm} .
\]

**[Proof].** The proof is straightforward by using formula (17). \( \Box \)

Corresponding to the case (30) we have the following expression for the BRST charge

\[
Q = S_+ C^+ + S_- C^- - \frac{\cosh(2\gamma S_3)}{\cosh \gamma} C^3 - i \frac{4\gamma \sinh(2\gamma S_3)}{\cosh \gamma} (C_- C^- C^3 - C_+ C^+ C^3) + i C_3 C^+ C^- + 16 \frac{\gamma^2 \cosh(2\gamma S_3)}{\cosh \gamma} C_+ C_- C^+ C^3 .
\]

We remark that in the limit \( \gamma \to 0 \) the manifold (33) becomes the elliptic paraboloid \( S_1^2 + S_2^2 - S_3 = \frac{1}{2} \) and the algebra (33) becomes the \( \mathcal{H}(4) \) algebra (of the simple classical harmonic oscillator).

For another application we consider the symmetry in the Kepler problem on a two dimensional sphere. It is described \([9]\) by the algebra generated by \( R_1, R_2 \) and \( L \), defined by

\[
[R_1, R_2]_{P.B.} = \frac{1}{4} (\lambda - 8E) L + 8\lambda L^3), \quad [R_2, L]_{P.B.} = R_1 , \quad [L, R_1]_{P.B.} = R_2 , \]

where \( \lambda \) is the curvature of the sphere, \( E \) is the energy eigenvalue, \( R_1, R_2 \) are the Runge-Lenz vectors and \( L \) is the 3-d component of the angular momentum.

**Proposition 5.** The symmetry algebra (37) of the 2-dimensional Kepler problem can be described by the symplectic geometry on the supermanifold \( M \) with \( M_0 \) given by

\[
R_1^2 + R_2^2 + \frac{1}{4} (\lambda - 8E) L^2 + \lambda L^4 = C_K
\]

with supersymplectic form

\[
\omega = -\frac{1}{C_K} \left( R_1 dR_2 \wedge dL + R_2 dL \wedge dR_1 + \frac{(\lambda - 8E)L + 4\lambda L^3}{\lambda - 8E + 8\lambda L^2} dR_1 \wedge dR_2 \right) + \sum_{i=1}^3 d\bar{C}_i \wedge dC^i ,
\]

where \( C_K \) is a constant.
[Proof]. Comparing formulas (20) and (37) we have \( \frac{1}{2} \frac{\partial F}{\partial R_2} = R_2 \), \( \frac{1}{2} \frac{\partial F}{\partial R_1} = R_1 \). Hence \( dF = d(R_1^2 + R_2^2 + \frac{1}{4} (\lambda - 8E)L^2 + \lambda L^4) \), which gives rise to the manifold given by (38). The expression for \( \omega \) is then deduced using (14). \( \Box \)

The classical BRST charge given by the formula (30) is then

\[
Q_1 = R_+ C^+ + R_- C^- + \left( \frac{1}{2} (\lambda - 8E)L + 4\lambda L^3 \right) C^3 + i\bar{C}_3 C^+ C^- - 96\lambda L \bar{C}_+ \bar{C}_- C^+ C^- C^3 \\
+ 2i \left( \frac{1}{2} (\lambda - 8E) + 12\lambda L^2 \right) (\bar{C}_- C^+ C^- - \bar{C}_+ C^+ C^3).
\]

(40)

As here \( \frac{\partial G(L)}{\partial L} = \left( \frac{1}{2} (\lambda - 8E) + 4\lambda L^2 \right) \cdot L \), we have also another solution of type (31) for \( Q \), namely

\[
Q_2 = R_+ C^+ + R_- C^- + L_2 C^3 + 2i\bar{C}_- C^+ C^- - 2iC_+ C^+ C^3 \\
+ i \left( \frac{1}{2} (\lambda - 8E) + 4\lambda L^2 \right) \bar{C}_3 C^+ C^-,
\]

(41)

where \( R_\pm = R_1 \pm iR_2 \).

Remark. By similar methods we can give the BRST construction for other algebras given by the supersymplectic geometry on \( M \), with \( M_0 \) of the form (21).

Now we discuss relations similar to the case of the first class constrained systems (1). Corresponding to the relations (3) we define \( \tilde{S}_\pm = \tilde{S}_1 \pm i\tilde{S}_2 = [Q, \bar{C}_1 \pm i\bar{C}_2]_{P.B.} \). Using the general form of BRST charge (25) we have

\[
\tilde{S}_+ = [Q, 2\bar{C}_+]_{P.B.} = S_- - A_3 \bar{C}_+ C^3 - A_4 \bar{C}_3 C^- + A_5 \bar{C}_+ \bar{C}_- C^- C^3 \\
\tilde{S}_- = [Q, 2\bar{C}_-]_{P.B.} = S_- - A_2 \bar{C}_- C^3 + A_4 \bar{C}_3 C^+ - A_5 \bar{C}_+ \bar{C}_- C^+ C^3 \\
\tilde{S}_3 = [Q, \bar{C}_3]_{P.B.} = A_1 + A_2 \bar{C}_- C^- + A_3 \bar{C}_+ C^+ + A_5 \bar{C}_+ \bar{C}_- C^+ C^- \\
\]

(42)

Proposition 6. The algebraic relations of \( \tilde{S}_{\pm,3} \) are in general no longer similar to the ones in (3). Nevertheless, in the case of the solution (31) one has

\[
\left[ \tilde{S}_+, \tilde{S}_- \right]_{P.B.} = -2iH(S_3)\tilde{S}_3 + iS_+ \frac{\partial A_4}{\partial S_3} \bar{C}_3 C^+ - iS_- \frac{\partial A_4}{\partial S_3} \bar{C}_3 C^-,
\]

(43)
[Proof]. A direct calculation of relations among \( \tilde{S}_{\pm, 3} \) gives

\[
\begin{align*}
[\tilde{S}_3, \tilde{S}_+]_{P.B.} & = -iS_+ \frac{\partial A_1}{\partial S_3} - \frac{1}{2} A_3^2 \tilde{C}_+ C^3 + \frac{1}{2} A_2 A_4 \tilde{C}_3 C^- + A_3 A_5 \tilde{C}_+ \tilde{C}_- C^- C^3 \\
& - iS_+ \left( \frac{\partial A_2}{\partial S_3} \tilde{C}_- C^- + \frac{\partial A_3}{\partial S_3} \tilde{C}_+ C^+ \right) - iS_+ \frac{\partial A_5}{\partial S_3} \tilde{C}_+ \tilde{C}_- C^+ C^- \\
& - \frac{1}{2} A_4 A_5 \tilde{C}_3 \tilde{C}_+ C^+ C^- \\
[\tilde{S}_3, \tilde{S}_-]_{P.B.} & = iS_- \frac{\partial A_1}{\partial S_3} - \frac{1}{2} A_3^2 \tilde{C}_- C^- - \frac{1}{2} A_5 A_2 \tilde{C}_3 C^- - A_3 A_4 \tilde{C}_+ C^- C^3 \\
& + iS_- \left( \frac{\partial A_2}{\partial S_3} \tilde{C}_- C^- + \frac{\partial A_3}{\partial S_3} \tilde{C}_+ C^+ \right) + iS_- \frac{\partial A_5}{\partial S_3} \tilde{C}_+ \tilde{C}_- C^+ C^- \\
& - \frac{1}{2} A_4 A_5 \tilde{C}_3 \tilde{C}_- C^+ C^- \\
[\tilde{S}_+, \tilde{S}_-]_{P.B.} & = -i \frac{\partial G}{\partial S_3} - A_2 A_4 \tilde{C}_- C^- - A_3 A_4 \tilde{C}_+ C^+ - 2 A_4 A_5 \tilde{C}_+ \tilde{C}_- C^- C^3 \\
& - iS_+ \frac{\partial A_2}{\partial S_3} \tilde{C}_- C^- - iS_- \frac{\partial A_3}{\partial S_3} \tilde{C}_+ C^+ + iS_+ \frac{\partial A_4}{\partial S_3} \tilde{C}_3 C^- - iS_- \frac{\partial A_5}{\partial S_3} \tilde{C}_3 C^- \\
& - iS_+ \frac{\partial A_5}{\partial S_3} \tilde{C}_+ \tilde{C}_- C^- C^3 + iS_- \frac{\partial A_5}{\partial S_3} \tilde{C}_+ \tilde{C}_- C^- C^3 \\
& - \frac{1}{2} A_4 A_5 (\tilde{C}_3 \tilde{C}_+ C^+ C^3 + \tilde{C}_3 \tilde{C}_- C^- C^3)
\end{align*}
\tag{44}
\]

Substituting (31) into (44) one gets (43). \( \blacksquare \)

By using the algebraic relations (23) and (24) it is also easy to verify relations similar to (1) and (3) when the solution (31) is taken into account.

Proposition 7. In the case described by (31) we have

\[
\begin{align*}
[\tilde{S}_a, \tilde{C}_b]_{P.B.} & = f_{ab} \tilde{C}_c, \quad [Q, \tilde{S}_a]_{P.B.} = 0, \quad \tag{45}
\end{align*}
\]

\[
\begin{align*}
[Q, C^a]_{P.B.} + \frac{1}{2} f^a_{bc} C^b C^c & = 0, \quad \tag{46}
\end{align*}
\]

where \( a, b, c = 1, 2, 3 \), \( f_{bc} = -f_{cb} \), \( f_{12}^2 = H(S_3), f_{23}^1 = f_{31}^2 = 1 \), \( \tilde{S}_1 = \frac{1}{2} (\tilde{S}_+ + \tilde{S}_-) \), \( \tilde{S}_2 = \frac{1}{2} (\tilde{S}_+ - \tilde{S}_-) \). \( \blacksquare \)

Therefore when \( \frac{\partial G(S_3)}{\partial S_3} \) can be analytically written as \( 2H(S_3)S_3 \), for the algebra (23) there exits a kind of BRST construction with Maurer-Cartan equations given by (16) and relations (13), similar to the formulas (3) and (1) in the first class constrained system.

Now we consider the quantization of above BRST systems. In terms of geometric quantization, the physical system is quantized by constructing a prequantization line
bundle on the symplectic manifold \((M, \omega)\) such that its connection one form is a symplectic potential and the section curvature is \(\omega\), and introducing a polarization \([[[1]]]\). The quantum Hilbert space is defined to be the subspace of the product bundle’s section space which is covariantly constant along the chosen polarization. For BRST systems, the additional condition is that \(\hat{Q}\) annihilates the physical Hilbert space, where \(\hat{Q}\) is the quantum operator associated with the classical BRST charge \(Q\). The quantization gives rise to a map from classical observables \(f \in \mathcal{F}(M)\) to quantum operators \(\hat{f}\). The expression of the quantum operator \(\hat{Q}\) depends on its relation to the classical phase space as expressed by \(Q\), and the selection of polarization. Here rather than discussing in details a physical constrained system, we will only discuss the algebraic construction of a quantum BRST system.

The role of the manifold \([[[2]]]\) is taken up in the quantum case by the corresponding Casimir operator (for \(SU_q(2)\) see \([[[3]]]\)). The algebraic relations \([[[2]]]\) and \([[[3]]]\) become quantum ones,

\[
[\hat{S}_+, \hat{S}_-] = \frac{\partial G(S_3)}{\partial S_3} \text{evaluated at } [\hat{S}_3], \quad [\hat{S}_3, \hat{S}_\pm] = \pm \hat{S}_\pm,
\]

\[
[\hat{C}^\pm, \hat{C}^\pm] = \frac{i}{2}, \quad [\hat{C}^3, \hat{C}^\pm] = i,
\]

where \([,]\) represents the supercommutator defined by \([A, B] = AB - (-1)^{AB} BA\). For the Schrödinger polarization of the anticommuting part there are explicit expressions, and \(\hat{C}^a = C^a, \hat{C}^a = i\frac{\partial G}{\partial C^a}, a = 1, 2, 3\), see \([[[1]]]\).

**Proposition 8.** The quantum BRST charge given by

\[
\hat{Q} = S_+ \hat{C}^+ + S_- \hat{C}^- + A_1(\hat{S}_3) \hat{C}_3 + A_2(\hat{S}_3) \hat{C}_- \hat{C}^- \hat{C}_3 + A_3(\hat{S}_3) \hat{C}_+ \hat{C}^+ \hat{C}_3
\]

\[+ A_4(\hat{S}_3) \hat{C}_3 \hat{C}^+ \hat{C}^- + A_5(\hat{S}_3) \hat{C}_+ \hat{C}^- \hat{C}^+ \hat{C}^- \hat{C}_3,\]

is nilpotent in the sense that \([\hat{Q}, \hat{Q}] = 2\hat{Q}^2 = 0\). Here we have

\[
A_4(\hat{S}_3)A_1(\hat{S}_3) = i[[[\hat{S}_3]]], \quad A_2(\hat{S}_3) = 2i(A_1(\hat{S}_3) - A_1(\hat{S}_3 - 1))
\]

\[
A_3(\hat{S}_3) = 2i(A_1(\hat{S}_3) - A_1(\hat{S}_3 + 1)), \quad A_5 = 4(2A_1(\hat{S}_3) - A_1(\hat{S}_3 - 1) - A_1(\hat{S}_3 + 1))\).
\]

**[Proof].** The proof is straightforward by using relations \([[[17]]]\) and \([[[18]]]\).

Corresponding to the classical cases \([[[2]]]\) resp. \([[[32]]]\), we have two simple solutions of \([[[50]]]\) and two quantum expressions of \(\hat{Q}\),

\[
i. \quad A_1(\hat{S}_3) = [[[\hat{S}_3]]], \quad A_2(\hat{S}_3) = 2i([[\hat{S}_3]] - [[[\hat{S}_3] - 1]]), \quad A_4(\hat{S}_3) = i,
\]

\[
A_3(\hat{S}_3) = 2i([[\hat{S}_3]] - [[[\hat{S}_3] + 1]]), \quad A_5 = 4(2[[\hat{S}_3]] - [[[\hat{S}_3] - 1]] - [[[\hat{S}_3] + 1]])\).
\]
Using the solution (51) we have the quantum BRST charge to the commutation relations

\[ \hat{Q}_1 = \hat{S}_+ \hat{C}^+ + \hat{S}_- \hat{C}^- + [[\hat{S}_3]] \hat{C}^3 + 2i([[\hat{S}_3]] - [[\hat{S}_3] - 1])\hat{C}_- \hat{C}^- \hat{C}^3 \\
+ 2i([[[\hat{S}_3]] - [[\hat{S}_3] + 1]])\hat{C}_+ \hat{C}^+ \hat{C}^3 + i\hat{C}_3 \hat{C}^+ \hat{C}^-
+ 4(2[[\hat{S}_3]] - [[\hat{S}_3] - 1] - [[\hat{S}_3] + 1])\hat{C}_+ \hat{C}_- \hat{C}^+ \hat{C}^- \hat{C}^3. \]

(52)

ii. When \([\hat{S}_3]\) can be analytically written as \(2H(\hat{S}_3)\hat{S}_3\), we have

\[ A_1(\hat{S}_3) = \hat{S}_3, \quad A_2(\hat{S}_3) = -A_3(\hat{S}_3) = 2i, \quad A_4(\hat{S}_3) = 2iH(\hat{S}_3), \quad A_5(\hat{S}_3) = 0, \]

(53)

\[ \hat{Q}_2 = \hat{S}_+ \hat{C}^+ + \hat{S}_- \hat{C}^- + \hat{S}_3 \hat{C}^3 + 2i\hat{C}_- \hat{C}^- \hat{C}^3 - 2i\hat{C}_+ \hat{C}^+ \hat{C}^3 + 2iH(\hat{S}_3)\hat{C}_3 \hat{C}^+ \hat{C}^- . \]

(54)

For the more particular case of the \(H_q(4)\) algebra (occurring in Prop. 4), (55) becomes

\[ [[\hat{S}_+ , \hat{S}_- ]] = -\frac{\cosh(2\gamma \hat{S}_3)}{\cosh \gamma}, \quad [[\hat{S}_3 , \hat{S}_\pm ]] = \pm \hat{S}_\pm . \]

Using the solution (51) we have the quantum BRST charge

\[ \hat{Q} = \hat{S}_+ \hat{C}^+ + \hat{S}_- \hat{C}^- - \frac{\cosh(2\gamma \hat{S}_3)}{\cosh \gamma} \hat{C}^3 + \frac{2i}{\cosh \gamma} (\cosh 2\gamma (\hat{S}_3 - 1) - \cosh 2\gamma \hat{S}_3)\hat{C}_- \hat{C}^- \hat{C}^3 \\
+ \frac{2i}{\cosh \gamma} (\cosh 2\gamma (\hat{S}_3 + 1) - \cosh 2\gamma \hat{S}_3)\hat{C}_+ \hat{C}^+ \hat{C}^3 - i\hat{C}_3 \hat{C}^+ \hat{C}^- \\
+ \frac{8}{\cosh \gamma} \cosh(2\gamma \hat{S}_3)(\cosh 2\gamma - 1)\hat{C}_+ \hat{C}_- \hat{C}^+ \hat{C}^- \hat{C}^3 . \]

For the algebra (37) of the Kepler problem on the 2-sphere, the quantization gives rise to the commutation relations

\[ [[\hat{R}_1 , \hat{R}_2 ]] = \frac{i}{4}(\lambda - 8E)\hat{L} + 8\lambda \hat{L}^3) ,\]

\[ [[\hat{R}_2 , \hat{L} ]] = i\hat{R}_1, \quad [[\hat{L} , \hat{R}_1 ]] = i\hat{R}_2 . \]

Corresponding to (52) and (54) we have

\[ \hat{Q}_1 = \hat{R}_+ \hat{C}^+ + \hat{R}_- \hat{C}^- + \frac{1}{2}(\lambda - 8E)\hat{L} + 8\lambda \hat{L}^3)\hat{C}^3 \\
+ i(\lambda - 8E + 8\lambda(3\hat{L}^2 - 3\hat{L} + 1))\hat{C}_- \hat{C}^- \hat{C}^3 \\
+ i(8E - \lambda - 8\lambda(3\hat{L}^2 + 3\hat{L} + 1))\hat{C}_+ \hat{C}^+ \hat{C}^3 \\
+ i\hat{C}_3 \hat{C}^+ \hat{C}^- + 96\lambda \hat{L} \hat{C}_+ \hat{C}_- \hat{C}^+ \hat{C}^- \hat{C}^3 . \]

\[ \hat{Q}_2 = \hat{R}_+ \hat{C}^+ + \hat{R}_- \hat{C}^- + \hat{S}_3 \hat{C}^3 + 2i\hat{C}_- \hat{C}^- \hat{C}^3 - 2i\hat{C}_+ \hat{C}^+ \hat{C}^3 \\
+ \frac{i}{2}(\lambda - 8E + 8\lambda \hat{L}^2)\hat{C}_3 \hat{C}^+ \hat{C}^- . \]
where $\hat{R}_\pm = \hat{R}_1 \pm i\hat{R}_2$.

By using the solution (53), we have

**Proposition 9.**

$$[\hat{S}_a, \hat{C}_b] = f_{ab}^c \hat{C}_c, \quad [\hat{Q}, \hat{S}_a] = 0,$$

$$[\hat{Q}, \hat{C}^a] + \frac{1}{2} f_{bc}^a \hat{C}^b \hat{C}^c = 0,$$

where $a, b, c = 1, 2, 3$, $f_{bc}^a = -f_{cb}^a$, $f_{12}^3 = iH(\hat{S}_3)$, $f_{23}^1 = f_{31}^2 = i$, $i\hat{S}_a = [\hat{Q}, \hat{C}_a]$.

Summarizing, we have investigated the symplectic geometry on a class of supermanifolds $M$ with $M_0$ defined by (21). The related BRST structures with classical constraint algebra (23) have been discussed in detail, as well as the quantum BRST structure with respect to the quantum algebra (47). Two examples were given as applications. The BRST systems we discussed here apply to a class of constrained systems with both undeformed and q-deformed algebras, for instance, $SU_q(2)$ and $SU_q(1,1)$, their related 2-dimensional manifolds are just special examples of the manifold described by (21), see \[6\].

In addition, we found that the expression of the BRST charge is not unique. Other constructions from equations (24) (classical case) resp. (50) (quantum case) can also be investigated. It would also be interesting to consider the case where $q = e^{\gamma}$ is a root of unit.

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