THE z-CLASSES OF ISOMETRIES

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ABSTRACT. Let $G$ be a group. Two elements $x, y$ are said to be in the same $z$-class if their centralizers are conjugate in $G$. The conjugacy classes give a partition of $G$. Further decomposition of the conjugacy classes into $z$-classes provides an important information about the internal structure of the group.

Let $V$ be a vector space of dimension $n$ over a field $F$ of characteristic different from 2. Let $B$ be a non-degenerate symmetric, or skew-symmetric, bilinear form on $V$. Let $I(V, B)$ denote the group of isometries of $(V, B)$. We parametrize the $z$-classes in $I(V, B)$. We show that the number of $z$-classes in $I(V, B)$ is finite when $F$ is perfect and has the property that it has only finitely many field extensions of degree $\leq n$. Along the way we also determine the conjugacy classes in $I(V, B)$.

1. Introduction

Let $F$ be a field of characteristic $\neq 2$. Let $V$ be a vector space of dimension $n$ over $F$. Let $V$ be equipped with a non-degenerate symmetric or symplectic (i.e. skew-symmetric) bilinear form $B$. We call $(V, B)$ a non-degenerate space. The group of isometries of $(V, B)$ is denoted by $I(V, B)$. It is a linear algebraic group. When $B$ is symmetric, resp. symplectic, $(V, B)$ is called a quadratic, resp. symplectic space, and the group of isometries is denoted by $O(V, B)$, resp. $Sp(V, B)$. They are called the orthogonal and the symplectic group respectively. For an isometry $f$, let $Z(f)$ denote its centralizer in $I(V, B)$. Let $W$ be a subspace of $V$. The restriction of $B$ on $W$, viz. the form $B : W \times W \to F$, is denoted by $B|_W$.

Let $G$ be a group. For $x, y$ in $G$ we say $x \sim y$ if their centralizers $Z_G(x)$ and $Z_G(y)$ are conjugate in $G$. We call the equivalence class of $x$ defined by this relation, the $z$-class of $x$. The $z$-classes are pairwise disjoint and give a stratification of the group $G$. This provides an important information about the internal structure of the group, cf. [8] for the elaboration of this theme. The structure of each $z$-class can be expressed as a certain set theoretic fibration, cf. Theorem 2.1 [8]. In general, a group may be infinite and it may have infinitely many conjugacy classes, but the number of $z$-classes is often finite. For example, if $G$ is a compact Lie group, then it is implicit in Weyl’s structure theory cf. [14], Borel-De Siebenthal [1], that the number of $z$-classes is finite. Analogously, Steinberg cf. [12] p. 107 has remarked on the finiteness of $z$-classes in reductive algebraic groups.
over a field of good characteristic. In [8], Kulkarni proposed to interpret the $z$-classes as an internal ingredient in a group $G$ which determine the “dynamics” of $G$ when acting on any set $X$. For example, in classical plane geometries over $\mathbb{R}$ or $\mathbb{C}$ it is observed that the “dynamical types” that our mind can perceive are just finite in number. Kulkarni [8] proposed that this finiteness of “dynamical types” can be interpreted as a phenomenon related to the finiteness of the $z$-classes in the corresponding group of the geometry. The work [4] strengthened this proposal. It was proved in [4] that the number of $z$-classes in the group of isometries of the hyperbolic $n$-space is finite, and this fact was interpreted as accounting for the finiteness of “dynamical types” in the hyperbolic geometries.

Apart from geometric motivations, it is an interesting problem in its own right to parametrize the conjugacy and the $z$-classes of a group. These are useful informations in order to understand the internal structure of a group. The conjugacy classes, $z$-classes and the set of operators themselves of the general linear groups and the affine groups have been parametrized by Kulkarni [7]. This has been extended to linear operators over division ring by Rony Gouraige [6]. There have been attempts to classify the $z$-classes in exceptional groups also. Recently Anupam Singh [10] has proved finiteness result for the $z$-classes in the compact real form $G_2$.

In this paper we have given a complete account of the conjugacy classification, and also obtained the classification of the $z$-classes in $I(V, B)$. The classification of the $z$-classes appears to be new. Our parametrization of the conjugacy classes may be compared with the previous attempts cf. [2], [9], [11], [13], [15].

An element $g$ in a group $G$ is called real if it is conjugate to its inverse in $G$. It is an interesting problem to classify the real elements in a given group. As a consequence of the conjugacy classification, we observe the following result. Previously a stronger version of it was proved by Wonenburger [17], see also [3].

**Theorem 1.1.** Every element in the orthogonal group $O(V, B)$ is real.

The classification of the conjugacy and the $z$-classes are given in section 3 to section 5 of this paper. In these sections we have related the classification problems with the problems of classifying the equivalence classes of hermitian forms over cyclic $\mathbb{F}$-algebras. To each isometry we associate certain data. The data arise from the internal structure of the isometry, and from the action of the isometry on the space. We prove that these data determine the respective classes, i.e. two isometries are in the same class if and only if they have the same data. It is easier to classify quadratic forms over fields than the hermitian forms over cyclic algebras. In section 8 we consider $\mathbb{F}$ to be a perfect field. In this case, by a remarkable property of linear algebraic groups, every element in $I(V, B)$ has a unique Jordan decomposition. A systematic use of Jordan decomposition leads to a neater classification of the conjugacy and the $z$-classes. We prove that
**Theorem 1.2.** If $\mathbb{F}$ is perfect and has the property that it has only finitely many field extensions of degree $\leq n$, then the number of $z$-classes in $I(\mathbb{V}, B)$ is finite.

In particular we have the following important consequence.

**Theorem 1.3.** Let $\mathbb{F}$ be the field of real numbers. Then the number of $z$-classes in $O(\mathbb{V}, B)$, resp. $Sp(\mathbb{V}, B)$ is finite.

The non-degenerate quadratic forms over the reals are classified by their rank $n$ and signature $(p, n - p)$. The orthogonal group in this case is often denoted by $O(p, q)$, $q = n - p$. The orthogonal groups over the reals are precisely the isometries of the pseudo-riemannian geometries of constant curvature [16]. A remarkable feature of the $O(p, q)$ action on a pseudo-riemannian space of constant curvature is that the group $O(p, q)$ is infinite. But the “dynamical types” that our find can perceive are just finite in number. Can we account for this fact in terms of the internal group structure alone?

The number of conjugacy classes in $O(p, q)$, as we shall see, is infinite. This infinity roughly arises from the eigenvalues of the minimal polynomials. These are the “numerical invariants” (cf. [8]) of a transformation. However, as we mention above, the number of $z$-classes is finite. The “spatial invariants” associated to each transformation are the orthogonal decomposition of the space, and the signatures of the associated form on the summands. Roughly speaking the spatial invariants determine the $z$-classes. The same is true for the symplectic group over the reals. The centralizers are described in terms of group-structure itself. Thus the finiteness of the number of $z$-classes is interpreted as accounting for the finiteness of “dynamical types” in pseudo-riemannian and symplectic geometries. This extends the previous work in [4]. Thus the orthogonal and the symplectic groups over the reals provide significant examples of the philosophy that was suggested in [8].

2. Preliminaries

2.1. **Self-dual polynomial.** Let $\mathbb{F}[x]$ be the ring of polynomials over $\mathbb{F}$. For a polynomial $g(x)$ let $c_k(g)$ denote the coefficient of $x^k$ in $g(x)$. Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$.

Let $f(x)$ be a monic polynomial of degree $n$ over $\mathbb{F}$ such that $0, 1$ and $-1$ are not its roots. Over $\overline{\mathbb{F}}$ let

$$f(x) = (x - c_1)(x - c_2)\ldots(x - c_n).$$

Then the polynomial

$$f^*(x) = (x - c_1^{-1})(x - c_2^{-1})\ldots(x - c_n^{-1})$$

is said to be the dual to $f(x)$. It is easy to see that

$$f^*(x) = f(0)^{-1}x^nf(x^{-1}).$$
Clearly, \( c_k(f^*) = f(0)^{-1}c_{n-k} \).

**Definition 2.1.** Let \( f(x) \) be a monic polynomial over \( \mathbb{F} \) such that \(-1, 0, 1\) are not its roots. The polynomial \( f(x) \) is called **reciprocal** or **self-dual** if \( f(x) = f^*(x) \).

Thus for \( f(x) \) self-dual, degree of \( f(x) \) is even, and \( c_k(f) = c_{n-k}(f) \).

**2.2. Elementary divisors of a linear map.** Let \( V \) be a vector space of finite dimension over a field \( \mathbb{F} \). Let \( T : V \rightarrow V \) be a linear map. We can consider \( V \) as a finite-dimensional \( \mathbb{F}[x] \) module, where for \( v \) in \( V \) the module multiplication is defined by

\[
(2.2.1) \quad x.v = Tv.
\]

By the structure theorem of finitely generated modules over a principal ideal domain, there exists monic polynomials \( f_1(x), ..., f_k(x) \) in \( \mathbb{F}[x] \) such that

\[
V = V_1 + V_2 + ... + V_k,
\]

where \(+\) denote the (usual) direct sum, \( V_i = \mathbb{F}[x]/(f_i(x)) \), and \( f_1(x)|f_2(x)|...|f_k(x) \). For each \( i \), \( 1 \leq i \leq k \), \( f_i(x) \) is the minimal polynomial of \( T|_{V_i} : V_i \rightarrow V_i \). Moreover \( f_k(x) \) is the minimal polynomial of \( T : V \rightarrow V \), and the characteristic polynomial of \( T \) is given by

\[
\chi_T(x) = f_1(x)f_2(x)...f_k(x).
\]

The polynomials \( f_1(x), ..., f_k(x) \) are called the **invariant factors** of \( T \). Let \( p(x) \) be a prime divisor of \( f_k(x) \), and suppose that the exact powers of \( p(x) \) in \( f_1(x), ..., f_k(x) \) are \( p(x)^{e_1}, ..., p(x)^{e_k}, e_i \geq 0, e_k > 0 \). The prime powers other than 1 thus obtained, as \( p(x) \) runs through the prime divisors of \( f_k(x) \), are called the **elementary divisors** of \( T \). If a prime power \( p(x)^k \) occurs \( d \) times as an elementary divisor, We call \( d \) the multiplicity of \( p(x)^k \).

**2.3. The Standard Form.** Let \( W \) be a vector space over a field \( \mathbb{F} \). Let \( W^* \) be the dual space to \( W \). There is a canonical pairing \( \beta : W^* \times W \rightarrow \mathbb{F} \) given by

\[
\text{for } w^* \in W^*, v \in W, \quad \beta(w^*, v) = w^*(v).
\]

Moreover \( \beta \) is non-degenerate, i.e. for each \( w^* \) in \( W^* \), there is a \( v \) in \( W \) such that \( \beta(w^*, v) \neq 0 \).

Now, given any linear map \( T : W \rightarrow W \) there is a corresponding linear map

\[
T^* : W^* \rightarrow W^* \text{ defined by}
\]

\[
\text{for } v^* \in W^*, u \in W, \quad T^*(v^*)(u) = v^*(Tu).
\]

We call \( T^* \) the **dual** of \( T \). Note that \((ST)^* = T^*S^*\). Moreover \( GL(W, \mathbb{F}) \) acts on the left on \( W^* \) by \((T \circ w^*)(v) = w^*(T^{-1}v)\). For simplicity of notation we shall often write
$T \circ w^* = Tw^*$. With this we have

$$\beta(Tw^*, Tv) = Tw^*(Tv) = w^*(T^{-1}Tv) = w^*(v) = \beta(w^*, v).$$

In this sense $T$ preserves the pairing $\beta$.

Now consider the vector space $V = W^* + W$. The pairing $\beta : W^* \times W \to \mathbb{F}$ can be extended canonically to a quadratic (resp. symplectic) form $b$ on $V$ defined as follows.

(i) For $w \in W$, $b(w, w) = 0$,
(ii) For $w^* \in W^*$, $b(w^*, w^*) = 0$,
(iii) For $w \in W$ and $v^* \in W^*$, $b(w, v^*) = v^*(w) = b(v^*, w)$, resp. $-b(v^*, w)$. Since $\beta$ is non-degenerate, we see that $b$ is a non-degenerate symmetric (resp. symplectic) bilinear form. Moreover every invertible linear transformation $T : W \to W$ gives rise to an isometry as follows.

**Proposition 2.2.** There is a canonical embedding of $GL(W, \mathbb{F})$ into $I(V, b)$.

**Proof.** Let $T : W \to W$ be an invertible linear map and let $T^* : W^* \to W^*$ be its dual. Define the linear map $h_T : V \to V$ as follows

$$h_T(v) = \begin{cases} T(v) & \text{if } v \in W \\ T^*(v) & \text{if } v \in W^* \end{cases}$$

Now observe that for $u \in W$, $w^* \in W^*$,

$$b(h_T u, h_T w^*) = h_T w^*(h_T u) = (T w^*)(Tu) = w^*(T^{-1}Tu) = w^*(u) = b(u, w^*).$$

This shows that $h_T$ is an isometry. The correspondence $T \mapsto h_T$ gives us the desired embedding.

**Definition 2.3.** Let $V$ be a vector space equipped with a non-degenerate symmetric (resp. symplectic) bilinear form $b$. $(V, b)$ is said to be a standard quadratic (resp. symplectic) space if there exists a subspace $W$ of $V$ such that $V = W^* + W$, and $b|_W = 0 = b|_{W^*}$.

3. **Decomposition of the space relative to an isometry**

Let $(V, B)$ be a non-degenerate space over $\mathbb{F}$. Let $T : V \to V$ be an isometry. Let $m_T(x)$ denote the minimal polynomial of $T$. Suppose $p_1(x), ..., p_t(x)$ are pairwise distinct irreducible polynomials over $\mathbb{F}$, and $m_T(x) = p_1(x)^{d_1} \cdots p_t(x)^{d_t}$. Suppose degree of $m_T(x)$ is $m$. The integer $d_i$ is called the exponent or the multiplicity of the prime factor $p_i(x)$.

Let $E = \mathbb{F}[x]/(m_T(x))$. The image of the indeterminate $x$ in $E$ is denoted by $t$. There is a canonical algebra structure on $E$ defined by $tv = Tv$. The $\mathbb{F}$-algebra $E = \mathbb{F}[t]$ is spanned by $\{1, t, t^2, ..., t^{m-1}\}$. In particular, if the minimal polynomial is irreducible, then $E$ is an extension field of $\mathbb{F}$.
Lemma 3.1. (i) The minimal polynomial of an isometry of a non-degenerate space is self-dual.

(ii) There is a unique automorphism \( e \mapsto \bar{e} \) of \( \mathbb{E} \) over \( \mathbb{F} \) which carries \( t \) to \( t^{-1} \).

Proof. For any \( f(x) \) in \( \mathbb{F}[x] \), and for any \( u, v \) in \( \mathbb{V} \) we have the identity

\[
B(f(t)u, v) = B(u, f(t^{-1})v).
\]

Applying this to the minimal polynomial we see that \( m_T(t^{-1}) = 0 \). This shows that the minimal polynomial is self-dual. Further it follows that \( \mathbb{E} = \mathbb{F}[t^{-1}] \). Hence there is a unique automorphism \( e \mapsto \bar{e} \) which carries \( t \) to \( t^{-1} \). In particular we have for all \( u, v \) in \( \mathbb{V} \),

\[
B(eu, v) = B(u, \bar{e}v).
\]

Suppose \( p(x) \) is an irreducible factor of \( m_T(x) \). From (i) of the above lemma it follows that \( p(x) \) can be one of the following three types.

(i) \( p(x) \) is self-dual.

(ii) \( p(x) = x \pm 1 \).

(iii) \( p(x) \) is not self-dual. In this case there is an irreducible factor \( p^*(x)^d \) of the minimal polynomial such that \( p^*(x) \) is dual to \( p(x) \).

Among the irreducible factors of \( m_T(x) \), suppose \( p_i(x) \) is self-dual for \( i = 1, 2, ..., k_1 \). Let the other irreducible factors are \( p_j(x), p^*_j(x) \) for \( j = 1, 2, ..., k_2 \). For a prime-power polynomial \( p(x)^d \), let \( \mathbb{V}_p = \ker p(T)^d \). Let \( \oplus \) denote the orthogonal sum, and \( + \) denote the usual sum of subspaces. It is easy to see that if \( p_i(x) \) is not dual to \( p_j(x) \), then \( \mathbb{V}_{p_i} \) is orthogonal to \( \mathbb{V}_{p_j} \). In particular, when \( p_j(x) \) is not self-dual, \( B|_{\mathbb{V}_{p_j}} = 0 = B|_{\mathbb{V}_{p^*_j}} \). This gives us the primary decomposition of \( \mathbb{V} \) (with respect to \( T \)) into \( T \)-invariant non-degenerate subspaces:

\[
\mathbb{V} = \bigoplus_{i=1}^{k_1} \mathbb{V}_i \bigoplus_{j=1}^{k_2} \mathbb{V}_j,
\]

where for \( i = 1, 2, ..., k_1, \mathbb{V}_i = \mathbb{V}_{p_i} \); for \( j = 1, 2, ..., k_2, \mathbb{V}_j = \mathbb{V}_{p_j} + \mathbb{V}_{p^*_j}, B|_{\mathbb{V}_{p_j}} = 0 = B|_{\mathbb{V}_{p^*_j}} \). In particular, from the non-degeneracy of \( B \) it follows that \( \dim \mathbb{V}_{p_j} = \dim \mathbb{V}_{p^*_j} \). The non-degenerate \( T \)-invariant components in this orthogonal decomposition are called the primary components of \( \mathbb{V} \) with respect to \( T \), or simply the primary subspaces of \( T \). Let \( T_i \) denote the restriction of \( T \) to \( \mathbb{V}_i \). Then \( m_{T_i}(x) = p_i(x)^{d_i} \) for \( i = 1, 2, ..., k_1 \), and \( m_{T_j}(x) = p_j(x)^{d_j}p^*_j(x)^{d_j} \) for \( j = 1, 2, ..., k_2 \). We observe that this decomposition is in fact invariant under \( Z(T) \). Moreover we have a canonical decomposition

\[
Z(T) = \Pi_{i=1}^{k_1} Z(T_i) \times \Pi_{j=1}^{k_2} Z(T_j).
\]

Thus the conjugacy and the \( z \)-classes of \( T \) are determined by the restriction of \( T \) to each of the primary subspaces. So to determine the conjugacy and the \( z \)-classes of \( T \) we have
reduced to the case when the minimal polynomial of the isometry is one of the following three types:

(i) \( m_T(x) = p(x)^d \), where \( p(x) \) is monic, irreducible over \( \mathbb{F} \), and is self-dual,

(ii) \( m_T(x) = (x - 1)^d \), or \( (x + 1)^d \).

(iii) \( m_T(x) = q(x)^d q^*(x)^d \), where \( q(x) \) is monic, irreducible over \( \mathbb{F} \) and is not self-dual, and, \( q^*(x) \) is dual to \( q(x) \).

**Lemma 3.2.** Suppose \( T : V \rightarrow V \) is such that the minimal polynomial is one of the types (i), (ii) above. Suppose \( m_T(x) = q(x)^d \), and degree of \( q(x) \) is \( m \). Then there is an orthogonal decomposition, unique up to isomorphism, \( V = \oplus_{i=1}^{k} V_{d_i} \), where \( 1 \leq d_1 \leq \ldots \leq d_k = d \), and for each \( i = 1, \ldots, k \), \( V_{d_i} \) is free over the algebra \( \mathbb{F}[x]/(q(x)^{d_i}) \). The summand \( V_{d_i} \) corresponds to the elementary divisor \( q(x)^{d_i} \) of \( T \). Let the multiplicity of the elementary divisor \( q(x)^{d_i} \) is \( k_i \). Then the dimension of \( V_{d_i} \) is \( md_i k_i \).

**Proof.** From the theory of elementary divisors in linear algebra, we have a direct sum decomposition: \( V = \sum_{i=1}^{k} V_{d_i} \), where \( 1 \leq d_1 \leq \ldots \leq d_k = d \), and for each \( i = 1, \ldots, k \), \( V_{d_i} \) is \( T \)-invariant and is free over the algebra \( \mathbb{F}[x]/(q(x)^{d_i}) \). To prove the lemma it is sufficient to show that \( V_{d_i} \) is non-degenerate. Then the lemma follows by applying similar decomposition successively to the orthogonal complement of \( V_{d_i} \). If possible, suppose \( V_{d_i} \) is degenerate. Let \( R(V_{d_i}) \) be the radical of \( V_{d_i} \), that is,

\[
R(V_{d_i}) = \{ v \in V_{d_i} \mid B(v, V_{d_i}) = 0 \}.
\]

Let \( v \) be a non-zero element in \( R(V_{d_i}) \). Since \( R(V_{d_i}) \) is \( T \)-invariant, let \( q(T)v = 0 \). Then there exist a \( u \) in \( V_{d_i} \) such that \( q(T)^{d-1}u = v \). Then for all \( i < d \), and \( w \) in \( V_{d_i} \),

\[
B(q(T)^{d-1}u, w) = B(u, q(T^{-1})^{d-1}w) = 0.
\]

Hence \( v \) is orthogonal to \( V \), a contradiction to the non-degeneracy of \( B \). Thus \( V_{d_i} \) must be non-degenerate. This completes the proof of the lemma. \( \square \)

4. **The Induced Form by an Isometry**

4.1. **The case when the minimal polynomial is prime-power.**

**Theorem 4.1.** Let \( T : V \rightarrow V \) be an isometry such that \( m_T(x) = p(x)^d \), where \( p(x) \) is an irreducible polynomial over \( \mathbb{F} \). Assume that \( p(x) \) is either self-dual, or \( x - 1 \). If \( p(x) = x - 1 \), then \( d > 1 \). Consider the cyclic \( \mathbb{F} \)-algebra \( \mathbb{E}_T^T = \mathbb{F}[x]/(p(x)^d) \). We simply denote it by \( \mathbb{E}_T^T \) when there is no confusion about \( d \). The \( \mathbb{E}_T^T \)-module \( V \) is denoted by \( V_T \).

Then we have the following.

(i) There is a unique automorphism \( e \rightarrow e \) of \( \mathbb{E}_T^T \) over \( \mathbb{F} \) which carries \( t \) to \( t^{-1} \).

(ii) There exists an \( \mathbb{F} \)-linear function \( h_T : \mathbb{E}_T^T \rightarrow \mathbb{F} \) such that the symmetric bilinear map \( h_T : (a, b) \mapsto h_T(ab) \) on \( \mathbb{E}_T^T \times \mathbb{E}_T^T \) is non-degenerate. Also there exists \( c \in \mathbb{E}_T^T \) such
that for all \( e \), \( h^T(\epsilon) = h^T(ce) \). Moreover, when \( T \) is not unipotent, we can take \( c = 1 \). For \( T \) unipotent, \( c = (-1)^{d-1} \).

(iii) The module \( \mathbb{V} \) over \( \mathbb{E}^T \) admits a unique \( c \)-hermitian form \( H^T(u,v) = cH^T(v,u) \), \( \mathbb{E}^T \)-linear in the first variable, and is related to the original \( \mathbb{F} \)-valued inner product by the identity

\[
B(u, v) = h^T(H^T(u, v)).
\]

**Proof.** (i) is clear from Lemma 3.1

(ii) When \( p(x) \neq x - 1 \), let \( 2k \) be the degree of \( p(x) \). In this case the degree of the minimal polynomial is \( 2kd \). Define an \( \mathbb{F} \)-linear function \( h^T : \mathbb{E}^T \rightarrow \mathbb{F} \) such that for all \( s(x) \) relatively prime to \( p(x) \), \( h^T([p(x)]^{d-1}[s(x)]) \neq 0 \). Now we claim that \( \tilde{h}^T \) is non-degenerate. For suppose, \( [f(x)] \neq 0 \) in \( \mathbb{E}^T \) be such that for all \( b \) in \( \mathbb{E}^T \), \( \tilde{h}^T([f(x)]b) = 0 \). Suppose \( f(x) = p(x)^i s(x) \), where \( s(x) \) is relatively prime to \( p(x) \) and \( i \geq 0 \). Then we have \( h^T([s(x)]^{d}) = 0 \), this is a contradiction. Hence \( \tilde{h}^T \) must be non-degenerate. Now note that \( h' : e \rightarrow h^T(\epsilon) \) is a linear map, hence by the non-degeneracy of \( \tilde{h}^T \) it follows that there exists \( c \in \mathbb{E}^T \) such that for all \( e \) in \( \mathbb{E}^T \), \( h^T(\epsilon) = h^T(ce) \). Moreover,

\[
ch^T([s(x)]^{d}) = h^T([s(x)]^{d}) = h^T([s(x)]^{d}) = c h^T([s(x)]^{d})
\]

where \( s(x) = \frac{p(x)}{x^r} = s(x^{-1}) \), and \( \epsilon = 1 \) or \( (-1)^{d-1} \), according to \( p(x) \neq x - 1 \), or, \( p(x) = \epsilon 1 \). Hence for \( T \) non-unipotent, we can take \( c = 1 \), and for \( T \) unipotent, \( c = (-1)^{d-1} \).

(iii) For \( u, v \) in \( \mathbb{V} \), consider the linear map \( L : \mathbb{E}^T \rightarrow \mathbb{F} \) given by \( L(e) = B(eu, v) \). There exists a unique \( e' \) in \( \mathbb{E}^T \) such that \( h^T(ce') = L(e) \). We define \( H^T(u, v) \) to be this element \( e' \). That is, \( H^T(u, v) \) is defined as follows:

\[
\text{for all } e \text{ in } \mathbb{E}^T, \text{ and for } u, v \text{ in } \mathbb{V}, \quad h^T(eH^T(u, v)) = B(eu, v).
\]

In particular taking \( e = 1 \) we have

\[
(4.1.1) \quad h^T(H^T(u, v)) = B(u, v)
\]

Now we see that for \( u_1, u_2, v \) in \( \mathbb{V} \),

\[
h^T(e(H^T(u_1, v) + H^T(u_2, v))) = h^T(eH^T(u_1, v)) + h^T(eH^T(u_2, v)) = B(eu_1, v) + B(eu_2, v) = B(eu_1 + eu_2, v) = B(e(u_1 + u_2), v) = h^T(H^T(u_1 + u_2, v))
\]

\[
(4.1.2) \quad \Rightarrow H^T(u_1, v) + H^T(u_2, v) = H^T(u_1 + u_2, v)
\]
Now for all $e'$ in $\mathbb{E}_T$ we have
\[
 h^T(e' e H^T(u, v)) = B(e' eu, v) \\
= B(e'(eu), v) = h^T(e'H^T(eu, v))
\]
(4.1.3) \[\Rightarrow e H^T(u, v) = H^T(eu, v) \]

This shows that $h^T$ is $\mathbb{E}_T$-linear in the first variable.

Given any hermitian form $H(u, v)$ satisfying (4.1.1), (4.1.3) we see that
\[
 h^T(eH(u, v)) = h^T(H(eu, v)) = B(eu, v).
\]

So such $H^T(u, v)$ is unique.

Further, for all $e$ in $\mathbb{E}_T$,
\[
 h^T(e(H^T(u, v))) = ch^T(\bar{e} H^T(u, v)) \\
= cB(\bar{e} u, v) \\
= cB(ev, u) = h^T(ec H^T(v, u))
\]
(4.1.4) \[\Rightarrow H^T(u, v) = c H^T(v, u) \]

This proves the theorem. \[\square\]

**Corollary 4.2.** Let $T : V \to V$ be an isometry such that $m_T(x) = p(x)^d$, where $p(x)$ is irreducible over $\mathbb{F}$ and is either self-dual or, $x - 1$. Then $T$ and $T^{-1}$ induce the same hermitian form.

**Proof.** When $p(x) \neq x - 1$, we have $\mathbb{E}_T = \mathbb{E}_T^{-1}$. Hence we can take $h^T = h^{-1}$, and thus $H^T = H^{-1}$.

For $p(x) = x - 1$, $h^T = (-1)^{d-1}h^{-1}$. But in this case the hermitian forms are equivalent, hence $T$ is conjugate to $T^{-1}$. \[\square\]

### 4.2. The case when the minimal polynomial is a product of polynomials dual to each other.

Let $T : V \to V$ be an isometry such that $m_T(x) = q(x)^d q^*(x)^d$, where $q(x)$, $q^*(x)$ are irreducible polynomials over $\mathbb{F}$ such that they are dual to each other. Consider the $\mathbb{F}$-algebra $\mathcal{E}_d^T = \mathbb{F}[x]/(q(x)^d) + \mathbb{F}[x]/(q^*(x)^d)$. For simplicity we denote it by $\mathcal{E}^T$. The $\mathcal{E}^T$-module $V$ is denoted by $V^T$. Replacing $\mathbb{E}_T$ by $\mathcal{E}^T$ in the statement, it can be seen by similar methods as above that Theorem 4.1 holds in this case also, and we can take $c = 1$ in part (ii) of the theorem.
5. Classification of Isometries with Prime-Power Minimal Polynomial

Let $S : V \to V$ and $T : V \to V$ be two isometries such that $m_S(x) = p(x)^d = m_T(x)$, where $p(x)$ is either $x - 1$, or, is irreducible and self-dual. Then $E^S$ and $E^T$ are both $F$-isomorphic to $F[x]/(p(x)^d)$. Let $f : E^S \to E^T$ be an $F$-isomorphism such that $f(s) = t$. Let $h^S : E^S \to F$ be the linear map of Theorem 4.1. Then $h^T = h^S \circ f^{-1}$ is such a linear map on $E^T$, and this map induces a hermitian form $H'$ on $V'$. Since such a hermitian form is unique, hence we must have $H' = H^T$. Thus for $u, v$ in $V'$, $h^S(H^S(u, v)) = H(u, v)$, and for $u', v'$ in $V'$, $h^T(H^T(u', v')) = h^S \circ f^{-1}(H^T(u', v'))$.

**Definition 5.1.** Suppose $E$ and $E'$ are isomorphic modules over $F$, and let $f : E \to E'$ be an isomorphism. Let $H$ be an $E$-valued hermitian form on $V$ and $H'$ be an $E'$-valued hermitian form on $V'$. Then $(V, H)$ and $(V', H')$ are equivalent if there exists an $F$-isomorphism $T : V \to V'$ such that for all $u, v$ in $V$ and for all $e$ in $E$ the following conditions are satisfied.

(i) $T(ev) = f(e)T(v)$, and
(ii) $H'(T(u), T(v)) = f(H(u, v))$.

When $E = E'$, we take $f$ to be the identity in the definition.

**Lemma 5.2.** Suppose $S$ and $T$ are isometries of $(V, B)$. Let the minimal polynomial of both $S$ and $T$ be $p(x)^d$, resp. $(x - 1)^d$, where $p(x)$ is monic, irreducible over $F$, and self-dual. Let $H^S$ and $H^T$ be the induced hermitian, resp. $(-1)^{d-1}$-hermitian, form by $S$ and $T$ respectively.

1. Then $S$ and $T$ are conjugate in $I(V, B)$ if and only if $H^S$ and $H^T$ are equivalent.
2. Let $Z(T)$ be the centralizer of $T$ in $I(V, B)$. Then an isometry $C$ is in $Z(T)$ if and only if $C$ preserves $H^T$, i.e. $Z(T) = U(V', H^T)$.

**Proof.** Suppose $S$ is conjugate to $T$ in $I(V, B)$. Let $C$ in $I(V, B)$ be such that $T = CSC^{-1}$. Then $C : V \to V'$ is an $F$-isomorphism. For $l \geq 1$, and $v$ in $V'$,

$$C(s^l v) = C \circ s^l(v) = T^l \circ C(v) = t^l C(v) = f(s^l) C(v).$$

It follows that, for all $e$ in $E^S$, and $v$ in $V'$, $C(ev) = f(e)C(v)$. For $u, v$ in $V'$, note that

$$h^S(f^{-1}(H^T(C(u), C(v)))) = h^S \circ f^{-1}(H^T(C(u), C(v))) = h^T(H^T(C(u), C(v))) = B(C(u), C(v)) = B(u, v) = h^S(H^S(u, v)).$$

Hence, by the uniqueness of $H^S$ we have, $f^{-1}(H^T(C(u), C(v))) = H^S(u, v)$, i.e. $H^T(C(u), C(v)) = f(H^S(u, v))$. This shows that $H^S$ and $H^T$ are equivalent.
Conversely, suppose $H^S$ and $H^T$ are equivalent. Let $C : V^S \to V^T$ be an $\mathbb{F}$-isomorphism such that (i) and (ii) in Definition 5.1 hold. We have for $v$ in $V$,

\[
CS(v) = C(sv) = f(s)C(v) = tC(v) = TC(v).
\]

that is, $CSC^{-1} = T$. Further, for $x$, $y$ in $V$,

\[
B(C(x), C(y)) = h^T(H^T(C(x), C(y)) = h^T(f(H^S(x, y)) = h^S(H^S(x, y)) = B(x, y).
\]

Hence $C : V \to V$ is an isometry. This completes the proof.

(ii) Note that an invertible linear transformation $C : V \to V$ is $E^T$-linear if and only if $CT = TC$. Now replacing $S$ by $T$, and $f$ by identity in the proof of (i) the lemma follows. \hfill \Box

5.1. Conjugacy class.

**Theorem 5.3.** Let $T$ be an isometry of a non-degenerate space $V$. Let the minimal polynomial of $T$ be $p(x)^d$, resp. $(x - 1)^d$, where $p(x)$ is monic, irreducible over $\mathbb{F}$, and self-dual.

1. The conjugacy class of $T$ is determined by the following data.

   (i) The elementary divisors of $T$.

   (ii) The finite sequence of equivalence classes of hermitian, resp. $(-1)^{d_i-1}$-hermitian, spaces

\[
\{(V^T_{d_1}, H^T_{d_1}), \ldots, (V^T_{d_k}, H^T_{d_k})\},
\]

where $1 \leq d_1 \leq d_2 \leq \ldots \leq d_k = d$, and for each $i$, $H^T_{d_i}$ takes values in the cyclic algebra $E_{d_i}$ which is isomorphic to $\mathbb{F}[x]/(p(x)^{d_i})$, resp. $\mathbb{F}[x]/((x - 1)^{d_i})$.

2. The centralizer of $T$ is the direct product

\[
U(V^T_{d_1}, H^T_{d_1}) \times \ldots \times U(V^T_{d_k}, H^T_{d_k}).
\]

**Proof.** Let $S : V \to V$ and $T : V \to V$ are two isometries. If $S$ and $T$ are conjugate in $I(V, B)$, then by the previous theorem it is clear that they have the same data.

Conversely, suppose $S$ and $T$ have the same data. The elementary divisors of $S$ and $T$ determine orthogonal decompositions of $V$ as

\[
V = V^S_{d_1} \oplus \ldots \oplus V^S_{d_k},
\]

\[
V = V^T_{d_1} \oplus \ldots \oplus V^T_{d_k},
\]

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where $1 \leq d_1 \leq \ldots \leq d_k = d$, and for each $i$, $\mathbb{V}_{d_i}^S$ (resp. $\mathbb{V}_{d_i}^T$) is free when considered as a module over $\mathbb{E}_{d_i} = \mathbb{F}[x]/(p(x)^{d_i})$. Since $S$ and $T$ have the same set of elementary divisors, $\mathbb{V}_{d_i}^S$ is isomorphic to $\mathbb{V}_{d_i}^T$ as a free module over $\mathbb{E}_{d_i}$, for $i = 1, 2, \ldots, k$. Since $(\mathbb{V}_{d_i}^S, H_{d_i}^S)$ is equivalent to $(\mathbb{V}_{d_i}^T, H_{d_i}^T)$, by the previous lemma, $S|_{\mathbb{V}^S_{d_i}}$ is conjugate to $T|_{\mathbb{V}^T_{d_i}}$ for $i = 1, 2, \ldots, k$. Hence $S$ is conjugate to $T$.

The description of $Z(T)$ is clear from the orthogonal decomposition of $V$ and (2) of Lemma 5.2. \hfill $\Box$

The following corollary is immediate from Corollary 4.2.

**Corollary 5.4.** Let $T$ be an isometry which is either unipotent, or its minimal polynomial is self-dual and prime-power. Then $T$ is real.

5.2. The $z$-Class. Let $T$ be an isometry of $(\mathbb{V}, B)$, and $m_T(x) = p(x)^d$, where $p(x)$ as in Theorem 5.3. Let degree of $p(x)$ is $m$. Let the $T$-invariant orthogonal decomposition of $V$ relative to the elementary divisors of $T$ is given by:

$$(5.2.1) \quad \mathbb{V} = \mathbb{V}_{d_1}^T \oplus \ldots \oplus \mathbb{V}_{d_k}^T,$$

$1 \leq d_1 \leq \ldots \leq d_k = d$, and for each $i = 1, \ldots, k$, $\mathbb{V}_{d_i}^T$ is free when considered as a module over $\mathbb{E}_{d_i}^T = \mathbb{F}[x]/((p(x)^{d_i})$.

Let $S$ be an isometry such that $m_S(x) = q(x)^d$. Let the degree of $q(x)$ is also $m$. Suppose the elementary divisors of $S$ and $T$ have the same degrees, and the elementary divisors with same degree appear with equal multiplicities. Then the $S$-invariant orthogonal decomposition of $V$ is isomorphic to the above decomposition, i.e. for each $i = 1, 2, \ldots, k$, $\mathbb{E}_{d_i}^S$ is isomorphic to $\mathbb{E}_{d_i}^T$, and $\mathbb{V}_{d_i}^S$ is isomorphic to $\mathbb{V}_{d_i}^T$. If $l_i$ is the multiplicity of the elementary divisor $p(x)^{d_i}$, then $n = m\Sigma_{i=1}^k d_i l_i$.

Thus the decomposition (5.2.1) corresponding to an isometry is determined up to isomorphism by the data:

(i) The degree of the irreducible divisor $p(x)$ of the isometry, 
(ii) a sequence of integers $(d_1, \ldots, d_k)$, where for $i = 1, \ldots, k$, $d_i$ correspond to the powers to the prime in the elementary divisors. The $d_i$’s, and the multiplicities of the elementary divisors give the partition $\pi : \frac{n}{m} = \Sigma_{i=1}^k d_i l_i$.

(iii) The finite sequence of isomorphism classes of cyclic $\mathbb{F}$-algebras $(\mathbb{E}_{d_1}, \ldots, \mathbb{E}_{d_k})$.

In addition to the above three, there is another data associated to each isometry, viz.

(iv) A finite sequence $(H_{d_1}, \ldots, H_{d_k})$ of equivalence classes of hermitian, resp. $(-1)^{d_i-1}$-hermitian, forms. For $i = 1, 2, \ldots, k$, $H_{d_i}$ is the $\mathbb{E}_{d_i}$-valued hermitian form induced by the isometry.

The data (i)-(iv) completely characterize the $z$-class of an isometry.
Theorem 5.5. The $z$-class of an isometry with minimal polynomial $p(x)^d$, where $p(x)$ is self-dual and irreducible over $\mathbb{F}$, resp. $x - 1$, is determined by the following data.

(i) The degree $m$ of the prime divisor of the minimal polynomial.

(ii) A non-decreasing sequence of integers $(d_1, ..., d_k)$ such that

$$\frac{n}{m} = \sum_{i=1}^{k} d_i l_i.$$ 

The integers $d_i$'s correspond to the powers of the prime in the elementary divisors.

(iii) A sequence $(E_{d_1}, ..., E_{d_k})$ of isomorphism classes of cyclic algebras over $\mathbb{F}$, where for each $i = 1, 2, ..., k$, $E_{d_i}$ is isomorphic to $\mathbb{F}[x]/(p(x)^{d_i})$.

(iv) A finite sequence of equivalence classes of hermitian (resp. $(−1)^{d_i−1}$-hermitian) forms $(H_{d_1}, ..., H_{d_k})$, where each $H_{d_i}$ takes values in $E_{d_i}$.

Proof. Let $S$ and $T$ be two isometries with same data $(i) - (iv)$. Let $m_S(x) = p(x)^d$, and $m_T(x) = q(x)^d$, degree of $p(x) = degree of q(x) = m$. For $i = 1, 2, ..., k$, we identify $E_{d_i}^S$ and $E_{d_i}^T$ with $E_{d_i}$. Also the $E_{d_i}$-valued hermitian forms $H_{d_i}^S$ and $H_{d_i}^T$ are equivalent, and we identify them with $H_{d_i}$. From Theorem 5.3(2), we see that both $Z(S)$ and $Z(T)$ are equal to

$$U(\mathbb{V}_{d_1}, H_{d_1}) \times U(\mathbb{V}_{d_2}, H_{d_2}) \times ....U(\mathbb{V}_{d_k}, H_{d_k}).$$

Hence $S$ and $T$ belong to the same $z$-class.

Conversely suppose $S$ and $T$ are in the same $z$-class. Replacing $S$ by its conjugate, we may assume $Z(S) = Z(T)$. Hence, after renaming the indices, if necessary, we may assume, for $i = 1, 2, ..., k$, $(\mathbb{V}_{d_i}^S, H_{d_i}^S)$ and $(\mathbb{V}_{d_i}^T, H_{d_i}^T)$ are equivalent. In particular, $E_{d_i}^S$ and $E_{d_i}^T$ are isomorphic, and their common dimension over $\mathbb{F}$ is $md_i$. This gives us the orthogonal decomposition: $\mathbb{V} = \bigoplus_{i=1}^{k} \mathbb{V}_{d_i}$, where each $\mathbb{V}_{d_i}$ is free over $E_{d_i} \cong E_{d_i}^S \cong E_{d_i}^T$, and consequently rise to the partition $\pi : \frac{n}{m} = \sum_{i=1}^{k} d_i l_i$. Thus $S$ and $T$ have the same data $(i) - (iv)$.

This completes the proof. \qed

6. The case when the minimal polynomial is a product of polynomials dual to each other

Let $T : \mathbb{V} \to \mathbb{V}$ be an isometry such that

$$m_T(x) = q(x)^d q^*(x)^d,$$

where $q(x), q^*(x)$ are distinct irreducible polynomials over $\mathbb{F}$ and are dual to each other. We have

$$\mathbb{V} = \mathbb{V}_q + \mathbb{V}_{q^*},$$
and $B|_{\mathbb{V}_q} = 0 = B|_{\mathbb{V}_q^*}$, $\dim \mathbb{V}_q = \dim \mathbb{V}_q^*$. Since $B$ is non-degenerate, we can choose a basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ such that for each $i$, $e_i \in \mathbb{V}_p$, $f_i \in \mathbb{V}_p^*$, and

$$B(e_i, e_i) = 0 = B(f_i, f_i), \quad B(e_i, f_j) = \delta_{ij} \text{ or } -\delta_{ij}.$$  

For each $w^* \in \mathbb{V}_q^*$, define the linear map $w^* : v \to B(v, w)$. These maps enable us to identify $\mathbb{V}_q^*$ with the dual of $\mathbb{V}_q$. Thus $(\mathbb{V}, B)$ is a standard space, and $T = T_L + T_L^T$, where $T_L$, the restriction of $T$ to $\mathbb{V}_q$, is an element of $\mathfrak{GL}(\mathbb{V})$. Conversely given an element in $\mathfrak{GL}(\mathbb{V}_q)$ we have seen in section 2.3 that it can be extended to an isometry of $(\mathbb{V}, B)$. Hence the conjugacy classes of $T$ are parametrized by the usual theory of linear maps. For a modern viewpoint of this theory cf. [7].

However, the above description does not give the $z$-class. To compute the centralizers and the $z$-classes we need to appeal to Theorem 4.1. Let $\mathcal{E}_{d_i}$ denote the cyclic algebra isomorphic to $\mathbb{F}[x]/(q(x)^{d_i})$, and let $\mathcal{E}_{d_i}^*$ be its dual space which is isomorphic to $\mathbb{F}[x]/(q^*(x)^{d_i})$. Then we have a $T$-invariant decomposition of $\mathbb{V}$ corresponding to the elementary divisors of the minimal polynomial as $\mathbb{V} = \bigoplus_{i=1}^k \mathbb{V}_d^T$, where for each $i = 1, 2, \ldots, k$, $\mathbb{V}_d^T$ is free when considered as a module over $\mathcal{E}_{d_i} = \mathcal{E}_{d_i} + \mathcal{E}_{d_i}^*$. Then by the version of Theorem 4.1 let $H_d^T$ be the induced $\mathcal{E}_{d_i}$-valued hermitian form on $\mathbb{V}_d$. Using similar methods as in the previous section it follows that, replacing $\mathcal{E}_{d_i}$ by $\mathcal{E}_{d_i}$, versions of Theorem 5.3 and Theorem 5.5 also hold in this case. The $z$-classes are characterized by the analogous version of Theorem 5.5 which is given below.

6.1. The $z$-Class.

**Theorem 6.1.** The $z$-class of an isometry whose minimal polynomial is $q(x)^d q^*(x)^d$ where $q(x), q^*(x)$ are of degree $m$ and irreducible over $\mathbb{F}$ and are dual to each other, is determined by the following data.

(i) The degree $m$ of one of the prime divisors of the minimal polynomial.

(ii) A non-decreasing sequence of integers $(d_1, \ldots, d_k)$ such that

$$\frac{n}{2m} = \Sigma_{i=1}^k d_i l_i.$$  

(iii) A sequence $(\mathcal{E}_{d_1}, \ldots, \mathcal{E}_{d_k})$ of isomorphism classes of algebras over $\mathbb{F}$, where for each $i = 1, 2, \ldots, k$, $\mathcal{E}_{d_i}$ is isomorphic to the algebra $\mathbb{F}[x]/(q(x)^{d_i}) + \mathbb{F}[x]/(q^*(x)^{d_i})$.

(iv) A finite sequence of equivalence classes of hermitian forms $(H_{d_1}, \ldots, H_{d_k})$, where for each $i = 1, 2, \ldots, k$, $H_{d_i}$ takes values in $\mathcal{E}_{d_i}$.

6.2. The case when the minimal polynomial is $(x + 1)^d$.

Note that, two isometries $S$ and $T$ are conjugate if and only if $-S$ and $-T$ are conjugate. Now, suppose $T$ is an isometry with minimal polynomial $(x - 1)^d$. Then $-T : \mathbb{V} \to \mathbb{V}$ is also an isometry, and $m_{-T}(x) = (x + 1)^d$. Conversely, if $T$ is unipotent, then $-T$ has
minimal polynomial \((x + 1)^d\). Thus this case is reduced to the unipotent case, and the parametrization of the conjugacy and the \(z\)-classes of \(T\) are similar to those of \(-T\).

7. Proof of Theorem 1.1

Let \(T\) be an element in \(O(V, B)\). It is enough to prove the reality of \(T\) on each of the primary components. When the minimal polynomial of \(T\) is \((x \pm 1)^d\), or \(p(x)^d\) for \(p(x)\) irreducible over \(F\) and self-dual, it follows from Corollary 5.4 that \(T\) is real. Suppose the minimal polynomial of \(T\) is \(m_T(x) = q(x)^d q^*(x)^d\), where \(q(x)\) is irreducible over \(F\) and is not self-dual, \(q^*(x)\) is dual to \(q(x)\). Suppose degree of \(q(x)^d\) is \(m\). In this case \(T = T_L + T_L^*\), where \(T_L\) is in \(GL(m, F)\). Since \(T\) is self-dual, \(T\) and \(T^{-1}\) have the same elementary divisors. Then \(T^{-1} = T_L^{-1} + T_L^{-1*}\), where we may take \(T_L^{-1*}\) to be in \(GL(m, F)\). Clearly \(T_L\) and \(T_L^{-1*}\) have the same set of elementary divisors. Hence they are conjugate in \(GL(m, F)\). Let \(S\) be the conjugating element. Then \(T_L^*\) and \(T_L^{-1}\) are conjugate by \(S^*\). Hence \(T\) is conjugate to \(T^{-1}\) in \(O(V, B)\) by the isometry \(S + S^*\).

This completes the proof of Theorem 1.1

8. Classifications over a perfect field

Let \(F\) be a perfect field of characteristic different from two. The group of isometries \(I(V, B)\) is a linear algebraic group. A significant property of a linear algebraic group over a perfect field of characteristic different from two is that each \(T\) in \(I(V, B)\) has the Jordan decomposition \(T = T_s T_u\), where \(T_s\) is semisimple, (that is every \(T_s\) invariant subspace has a \(T_s\)-invariant complement, and \(T_u\) is unipotent. Moreover \(T_s, T_u\) are also elements of \(I(V, B)\), they commute with each other, and are polynomials in \(T\) (cf. [5]). Moreover we have \(Z(T) = Z(T_s) \cap Z(T_u)\). To some extent, the Jordan decomposition reduces the study of conjugacy and \(z\)-classes in \(I(V, B)\) to the study of conjugacy and \(z\)-classes of semisimple, resp. unipotent elements. Also the use of Jordan decomposition leads to a more neat and simpler classification of the conjugacy, and the \(z\)-classes.

Suppose \(T : V \rightarrow V\) is a semisimple isometry with prime and self-dual minimal polynomial. Suppose \(E = F[x]/(p(x))\). Then \(E\) is a finite simple field extension of \(F\), \([E : F] = \text{degree of } p(x)\). Thus the cyclic algebras in Theorem 5.3 and Theorem 5.5 are isomorphic to the field \(E\), and the hermitian forms \(H_d\) are \(E\)-valued.

Now suppose \(T\) is an arbitrary semisimple isometry, and let its minimal polynomial be a product of pairwise distinct prime polynomials over \(F\). Suppose

\[m_T(x) = (x - 1)^e (x + 1)^{e'} \Pi_{i=1}^k p_i(x) \Pi_{j=1}^l q_j(x) q_j^*(x),\]

where \(e, e' = 0\) or \(1\), \(p_1(x), ..., p_k(x)\) are self-dual, and for \(j = 1, 2, ..., l\), \(q_j(x)\) is dual to \(q_j^*(x)\). Suppose for each \(i\), degree of \(p_i(x)\) is \(2m_i\), and for each \(j\), degree of \(q_j(x)\) is \(m_j^l\).
Let the characteristic polynomial of $T$ be

$$
\chi_T(x) = (x - 1)^t(x + 1)^m \prod_{i=1}^k p_i(x)^{d_i} \prod_{j=1}^l q_j(x)^{e_j} q_j^*(x)^{e_j}.
$$

Then we have the following orthogonal decomposition of $V$ into $T$-invariant subspaces:

$$
V = V_1 \oplus V_{-1} \oplus \bigoplus_{i=1}^k V_i \bigoplus \bigoplus_{j=1}^l (W_j + W_j^*),
$$

where $V_1 = \ker (T - I)^t$, $V_{-1} = \ker (T + I)^m$, for each $i = 1, \ldots, k$, $V_i = \ker p_i(T)$, and for each $j = 1, 2, \ldots, l$, $W_j = \ker q_j(T)$, $W_j^* = \ker q_j^*(T)$. We have, $\dim V_i = 2m_i d_i$, and $\dim W_j = m_j e_j$. Let $E_i$ be the field isomorphic to $F[x]/((p_i(x)))$, and $K_j$, resp. $K_j^*$ be the field isomorphic to $F[x]/((q_j(x)))$, resp. $F[x]/((q_j^*(x)))$. As a vector space over $E_i$, $V_i$ is the direct sum of $d_i$ copies of $E_i$. The vector space $W_j$, resp. $W_j^*$, is the orthogonal sum of $e_j$ copies of $K_j$, resp. $K_j^*$. Thus the characteristic, and the minimal polynomial of $T$ determine the primary decomposition of $V$ with respect to $T$ in this case.

Since restriction of $T$ to $W_j$ can be considered as an element of $GL(W_j)$, the conjugacy classes on the primary component $(W_j + W_j^*)$ are determined by the theory described in [7]. Hence we may assume $m_T(x) = \prod_{i=1}^k p_i(x)$. In this case, the decomposition of $V$ gives a partition $\pi : n = \sum_{i=1}^k 2m_i d_i$. The following description of the conjugacy classes follows from Theorem 6.1.

**Theorem 8.1.** Let $T : V \to V$ be an isometry whose minimal polynomial is a product of pairwise distinct prime, and self-dual polynomials. Suppose the minimal polynomial is $m_T(x) = \prod_{i=1}^k p_i(x)$.

Then the conjugacy class of $T$ is determined by the following data.

(i) The minimal and the characteristic polynomial.

(ii) The finite sequence of equivalence classes of hermitian spaces

$$
\{(V_1, H_1), \ldots, (V_k, H_k)\},
$$

where for each $i = 1, 2, \ldots, k$, $V_i = \ker p_i(T)$, and $H_i$ is the induced hermitian form by $T$, and takes values in the extension field $E_i = F[x]/((p_i(x)))$ of $F$.

8.1. **Classification of z-classes for semisimple isometries.** Suppose $T : V \to V$ is a semisimple isometry. Without loss of generality we may assume that the minimal polynomial in this case has no root 1 or $-1$. Then the classification of the z-classes follows from Theorem 5.5 and Theorem 6.1.

**Theorem 8.2.** Let $T : V \to V$ be a semisimple isometry whose minimal polynomial is self-dual. Then the z-class is characterized by

(i) The finite sequence of integers $(m_1, \ldots, m_{k_1}, l_1, \ldots, l_{k_2})$, where $2m_i \leq n$, $1 \leq i \leq k_1$, are the degrees of the distinct prime and self-dual factors of the minimal polynomial and, $l_j$, $1 \leq j \leq k_2$, are the degrees of those prime factors to which there is a dual factor.
(ii) A partition of $n$, $\pi : n = 2\sum_{i=1}^{k_1} m_i d_i + 2\sum_{j=1}^{k_2} l_j e_j$.

(iii) Simple field extensions $E_i$, $1 \leq i \leq k_1$ of $F$, $[E_i : F] = 2m_i$, and isomorphism class of simple field extensions $E_j$, $1 \leq j \leq k_2$, $[E_j : F] = l_j$.

(iv) Equivalence classes of $E_i$-valued hermitian forms $H_i$, $1 \leq i \leq k_1$, and $E_j^2$-valued hermitian forms $H_j'$, $1 \leq j \leq k_2$.

8.2. Finiteness of the $z$-classes. Suppose $F$ has only finitely many field extensions of degree $\leq n$. Then the number of distinct equivalence classes of quadratic forms of rank $\leq n$ is finite. Hence the number of equivalence classes of hermitian forms of rank $\leq n$ over an extension field of $F$ is finite. Combining this fact with Theorem 8.2 and the fact that there are only finitely many partitions of $n$, we obtain the following.

**Theorem 8.3.** If $F$ is perfect and has the property that it has only finitely many field extensions of degree $\leq n$, then the number of $z$-classes in the group $I(V, B)$ is finite. This holds for example when the field $F$ is algebraically closed, the field of real numbers, or a local field.

In fact for unipotent isometries it follows for similar reasons that

**Proposition 8.4.** If $F$ has the property that it has only finitely many field extensions of degree $\leq n$, then the number of unipotent conjugacy classes in the group $I(V, B)$ is finite.

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