SLOW FLOCKING DYNAMICS OF THE CUCKER-SMALE ENSEMBLE WITH A CHEMOTACTIC MOVEMENT IN A TEMPERATURE FIELD

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Abstract. We study slow flocking phenomenon arising from the dynamics of Cucker-Smale (CS) ensemble with chemotactic movements in a self-consistent temperature field. For constant temperature field, our situation reduces to the previous CS model with chemotactic movements. When a large CS ensemble with chemotactic movements is placed in a self-consistent temperature field, the dynamics of the CS ensemble can be effectively described by the kinetic thermodynamic CS (TCS) equation with chemotactic movements, which corresponds to the coupled collisional transport-reaction diffusion system. For the proposed coupled model, we provide a global solvability of strong solutions and their asymptotic flocking estimates which exhibit slow algebraic relaxation toward the flocking state. Our analytical results show that asymptotic flocking is robust with respect to a small perturbation of a constant temperature.

1. Introduction. Biological complex systems often exhibit some kind of collective behaviors [17, 35, 37] which cannot be explained from the simple sum of individual motions of constituent particles (or agents). Recently, such collective dynamics has received lots of attention due to the emerging engineering applications [27, 32, 29, 30]. Among them, we are mainly interested in the “flocking” phenomenon in which a group of agents (particles) is organized into an ordered motion...
moving with the same velocity only using the limited environmental information via mutual interactions (communications) between agents, e.g., flocking of birds, swarming of fish, herding of sheep, etc. In 2007, Cucker and Smale introduced an analytically manageable particle model [10] motivated by the Vicsek model [36]. The Cucker-Smale model is a Newton-like system for the position-velocity variables, and the authors provided several sufficient conditions in terms of system parameters and initial data. Recently, Ha and Ruggeri proposed a new particle model [18] which is thermodynamically consistent and still exhibits asymptotic flocking dynamics. Their work was immediately generalized in the context of CS flocking [14, 15, 16]. In this paper, we further continue the study on the flocking dynamics of the CS flocking ensemble with chemotactic movements in a self-consistent temperature field, i.e., we are interested in the effect of dynamic interplays between temperature and chemotactic movements in the CS ensemble. To address this physical situation, we adopt the coupled collisional kinetic equation and reaction-diffusion equation for the kinetic density for the CS ensemble and local mass density of the chemotactic substance, respectively. Mathematical modeling for bacteria’s chemotactic movements has been extensively investigated in literature [1, 2, 3, 4, 9, 11, 12, 13, 20, 21, 22, 23, 24, 25, 26, 31, 33, 34]. To fix the idea, let $f = f(x, v, \theta, t)$ and $S = S(x, t)$ be the one-particle distribution of the Cucker-Smale ensemble and local mass density of the chemotactic substance, respectively. Here, $\theta$ is the temperature variable, while $v$ is the velocity variable. Then, the dynamics of $(f, S)$ is governed by the Cauchy problem to the following coupled system:

$$
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot (F_a[f]f) + \kappa_1 \partial_\theta (G_a[f]f)
= & \kappa_2 T[S](f), \quad (x, v, \theta, t) \in \mathbb{R}^{2d} \times (\mathbb{R}_+)^2, \\
F_a[f] := & \int_{\mathbb{R}^{2d} \times \mathbb{R}_+^2} \psi(|x - x_*|) \left( \frac{v}{\theta_*} - \frac{v}{\theta} \right) f(x_*, v_*, \theta_*, t) d\theta_* dv_* dx_*, \\
G_a[f] := & \int_{\mathbb{R}^{2d} \times \mathbb{R}_+^2} \zeta(|x - x_*|) \left( \frac{1}{\theta_*} - \frac{1}{\theta} \right) f(x_*, v_*, \theta_*, t) d\theta_* dv_* dx_*, \\
\partial_t S - \Delta_x S = & \kappa_3 \varphi[S, \rho], \quad \rho := \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} f(x, v, \theta, t) d\theta dv, \\
f(x, v, \theta, 0) = & f_0(x, v, \theta), \quad S(x, 0) = S_0(x), \quad (x, v, \theta) \in \mathbb{R}^{2d} \times \mathbb{R}_+, 
\end{align*}
$$

(1.1)

where $\kappa_i$, $i = 0, 1$ are nonnegative coupling strengths. The source terms $\varphi[S, \rho]$ and $T[S]$ represent interactions between the TCS particles and chemical substances, and turning operator whose properties will be discussed in Section 2.2, respectively (see Section 2.1 for details). Here the non-local operators $F_a$ and $G_a$ are velocity and temperature consensus operators, and we also assume that the kernels $\psi$ and $\zeta$ in $F_a$ and $G_a$ are bounded from above and below from zero: there exist positive constants $\psi_m, \psi_M, \psi_\infty^M, \zeta_m, \zeta_M$ and $\zeta_\infty^M$ such that

$$
0 < \psi_m \leq \psi(r) \leq \psi_M < \infty, \quad |\psi'(r)| \leq \psi_\infty^M < \infty, \quad r \in \mathbb{R}_+, \quad \|\psi\|_{C^1} < \infty, \\
0 < \zeta_m \leq \zeta(r) \leq \zeta_M < \infty, \quad |\zeta'(r)| \leq \zeta_\infty^M < \infty, \quad r \in \mathbb{R}_+, \quad \|\zeta\|_{C^1} < \infty.
$$

(1.2)
Note that for a constant temperature case with \( \theta = \theta^\infty \), system (1.1) reduces to the kinetic CS equation with chemotactic movements [5]:

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + \frac{\kappa_0}{\theta^\infty} \nabla_v \cdot (F_a[f]f) = \kappa_2 T[S](f), \quad (x,v,t) \in \mathbb{R}^{2d} \times \mathbb{R}_+,
\end{aligned}
\]

\[
F_a[f] := \int_{\mathbb{R}^{2d}} \psi(|x-x_*|)(v_* - v)f(x_*,v_*,t)dv_*dx_*,
\]

\[
\begin{aligned}
\partial_t S - \Delta_x S = \kappa_3 \varphi[S,\rho], \quad \rho := \int_{\mathbb{R}^{d} \times \mathbb{R}_+} f(x,v,t)dv,
\end{aligned}
\]

\[
f(x,v,0) = f_0(x,v), \quad S(x,0) = S_0(x), \quad (x,v) \in \mathbb{R}^{2d}.
\]

Recently, the authors in [5] studied a global well-posedness of strong solutions to (1.3) for some class of turning operator \( T[S] \) and asymptotic flocking dynamics using the Lyapunov functional approach. Hence, our focus in this paper is to figure out the additional effects on the global solvability and asymptotic properties of (1.3) by a self-consistent temperature field. In other words, we try to find feedback effects on the asymptotic flocking dynamics of (1.3) caused by the self-consistent temperature field. In the absence of chemotactic movements, the global well-posedness for (1.1) has also been studied in [7, 8, 15].

The main results of this paper are two-fold. First, we present a global well-posedness for strong solutions to system (1.1) using a local existence of strong solutions and a priori estimates. Our strong solution is a distributional weak solution to system (1.1) with the additional regularity conditions:

\[
\begin{aligned}
f &\in W^{1,\infty}(0,T; L^\infty(\mathbb{R}^{2d} \times \mathbb{R}_+)) \cap L^\infty(0,T; (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d} \times \mathbb{R}_+)),
\end{aligned}
\]

\[
S \in W^{1,\infty}(0,T; L^\infty(\mathbb{R}^{d})) \cap L^\infty(0,T; (L^1_+ \cap W^{2,\infty})(\mathbb{R}^{d})).
\]

The global solvability of (1.1) will be done by combining the local solvability and a priori estimates via continuous induction arguments (see Theorem 4.1).

Second, we derive asymptotic flocking estimates using the Lyapunov functional approach. For a sufficiently regular solution, we introduce two Lyapunov functionals measuring the concentration of temperature and velocity variables:

\[
\begin{aligned}
\mathcal{L}^\theta[f(t)] := \int_{\mathbb{R}^{d} \times (\mathbb{R}_+)^2} |\theta - \theta_*|^2 f(z_*,t)f(z,t)dz_*dz,
\end{aligned}
\]

\[
\begin{aligned}
\mathcal{L}^v[f(t)] := \int_{\mathbb{R}^{d} \times (\mathbb{R}_+)^2} |v - v_*|^2 f(z_*,t)f(z,t)dz_*dz,
\end{aligned}
\]

where we used abbreviated state variables:

\[
z = (x,v,\theta) \quad \text{and} \quad z_* = (x_*,v_*,\theta_*).
\]

Then, we can show that these functionals decay at most algebraically slow (see Proposition 1):

\[
\mathcal{L}^\theta[f(t)] \lesssim \frac{1}{(1 + t)^d} \quad \text{and} \quad \mathcal{L}^v[f(t)] \lesssim \frac{1}{(1 + t)^{d-2}}, \quad t \geq 0.
\]

Although we do not have lower bound decay estimates for \( \mathcal{L}^\theta[f] \) and \( \mathcal{L}^v[f] \), the above estimates seem to suggest faster concentration of temperature compared to the velocity concentration.

The rest of this paper is organized as follows. In Section 2, we briefly review the hierarchical TCS models, and then study basic properties of the turning operator and a priori estimates for the coupled system (1.1). In Section 3, we present a
local existence of strong solutions to system (1.1). In Section 4, we derive a priori estimates for strong solutions and then extend the local solutions to the global ones using continuous induction arguments. In Section 5, we provide asymptotic flocking estimate for a sufficiently smooth solution. Finally, Section 6 is devoted to a brief summary of our main results and discussions on the remaining issues.

Notation: 1. For $y = (y_1, \cdots, y_m) \in \mathbb{R}^m$ and $r > 0$, we set

$$|y| := \sqrt{y_1^2 + \cdots + y_m^2}, \quad B(r) := \{y \in \mathbb{R}^d : |y| \leq r\},$$

and for $f = (f_1, \cdots, f_d) \in L^\infty(\mathbb{R}^m; \mathbb{R}^d)$ and $g = g(y, t) \in L^\infty(0, T; L^\infty(\mathbb{R}^m))$, we also set

$$\|f\|_{L^\infty} := \max_{1 \leq i \leq d} \|f_i\|_{L^\infty}, \quad \|g(t)\|_{L^\infty} := \|g(\cdot, t)\|_{L^\infty(\mathbb{R}^m)}.$$

2. Let $f$ and $g$ be functions of $t$. The relation $f \lesssim g$ denotes that there exists a positive generic constant $C$ such that $f(t) \leq Cg(t)$ for all $t$.

3. For a measurable set $A \subset \mathbb{R}^d$, $m(A)$ denotes the Lebesgue measure of the set $A$.

2. Preliminaries. In this section, we briefly discuss a kinetic thermodynamic CS model, turning operator presenting discontinuous velocity jump process and present a priori estimates for the coupled system.

2.1. A kinetic TCS equation with chemotactic movements. Let $x_i, v_i$ and $\theta_i$ be the position, velocity and temperature of the $i$-th CS particle, respectively. Then, when the diffusion velocity is neglected, the dynamics of microscopic observables is governed by the continuous dynamical system [18]:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i = 1, \cdots, N,$$

$$\frac{dv_i}{dt} = \frac{\kappa_0}{N} \sum_{j=1}^{N} \psi(|x_i - x_j|) \left( \frac{v_j}{\theta_j} - \frac{v_i}{\theta_i} \right),$$

$$\frac{d\theta_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^{N} \zeta(|x_i - x_j|) \left( \frac{1}{\theta_i} - \frac{1}{\theta_j} \right),$$

where $|\cdot|$ denotes the standard $\ell^2$-norm in $\mathbb{R}^d$. Note that for a constant isothermal case $\theta_i = \theta^\infty$, equation (2.1) is automatically satisfied, and the first two equations reduce to the Cucker-Smale model:

$$\frac{dx_i}{dt} = v_i, \quad t > 0,$$

$$\frac{dv_i}{dt} = \frac{\kappa_0}{\theta^\infty N} \sum_{j=1}^{N} \psi(|x_i - x_j|)(v_j - v_i).$$

Now, we consider a large system (2.1) with $N \gg 1$ so that the numerical integration of system (2.1) is almost impossible so that we need to approximate (2.1) via the corresponding mean-field kinetic model. More precisely, let $f = f(x, v, \theta, t)$ be the one-particle distribution function of TCS particles at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^d$ and temperature $\theta \in \mathbb{R}_+$ with velocity $v \in \mathbb{R}^d$. By the standard BBGKY hierarchy
argument [28, 19], it is easy to see that the distribution function $f$ satisfies the Vlasov type equation:

$$
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla v \cdot (F_a[f]f) + \kappa_1 \partial_\theta (G_a[f]f) = 0,
$$

$$
F_a[f](z, t) = \int_{R^{2d} \times \mathbb{R}^+} \psi(|x - x_*|) \left( \frac{v_*}{\theta_*} - \frac{v}{\theta} \right) f(z_*, t) dz_*,
$$

$$
G_a[f](x, \theta, t) = \int_{R^{2d} \times \mathbb{R}^+} \zeta(|x - x_*|) \left( \frac{1}{\theta} - \frac{1}{\theta_*} \right) f(z_*, t) dz_*.
$$

(2.2)

Now we add a reaction-diffusion equation describing the evolution of the chemotactic density $S = S(x, t)$, and a nonlocal turning operator $T[S](f)$, whose kernel depends on the chemotactic density, to the right-hand-side of (2.2):

$$
\begin{align*}
\partial_t S - \Delta_x S &= \kappa_3 \varphi[S, \rho], \\
\rho &= \int_{R^d \times \mathbb{R}^+} f(x, v, \theta, t) d\theta dv, \\
f(x, v, \theta, 0) &= f_0(x, v, \theta), \\
S(x, 0) &= S_0(x), \quad (x, v, \theta) \in R^{2d} \times \mathbb{R}^+,
\end{align*}
$$

(2.3)

where $\varphi[S, \rho]$ represents chemical interactions between the TCS particles and chemical substances.

2.2. Velocity turning operator. In this subsection, we discuss the velocity jump process via the turning operator. For a given $(x, \theta, t) \in R^d \times (\mathbb{R}^+)^2$, we set $T[S](x, v, v_*, \theta, t)$ and $T^*[S](x, v, v_*, \theta, t) := T[S](x, v, v_*, \theta, t)$ to denote the rates of jumps from $v_*$ to $v$ and vice versa. Let $S = S(x, t)$ be the concentration of the chemotactic substance. Then the contribution of the rate of change in $f$ along the particle trajectory due to chemotatic movements will be registered by the turning operator $T[S](f)$:

$$
T[S](f) := T^+[S](f) - T^-[S](f),
$$

$$
T^+[S](f)(x, v, \theta, t) := \int_{R^d} T[S](x, v, v_*, \theta, t) f(x, v_*, \theta, t) dv_*,
$$

$$
T^-[S](f)(x, v, \theta, t) := \int_{R^d} T^*[S](x, v, v_*, \theta, t) f(x, v_*, \theta, t) dv_* = \lambda[S](x, v, \theta, t) f(z, t),
$$

where the quantity $\lambda[S]$ denotes the turning frequency.

Next, we provide the structural ansatz for the turning kernel $T[S]$ and the reaction term $\varphi[S, \rho]$ for the mathematical analysis of system (2.3):

- (A1). The turning kernel $T[S](x, v, v_*, \theta, t)$ is smooth, and has compact supports in $v$ and $v_*$. Moreover, $T[S]$ satisfies the following properties:

(i) $\text{supp}_{v,v_*} T[S](x, v_*, \theta, t) \subset B(R_v) \times B(R_{v_*})$, $t \geq 0$, $x \in R^d$, $\theta > 0$.

(ii) $0 \leq T[S](x, v, v_*, \theta, t) \leq S(x - v_*, t) S(x + v, t)$.

(iii) $|T[S](x, v, v_*, \theta, t) - T[S](x, v, v_*, \theta, t)| \leq |S(x - v_*, t) S(x + v, t) - S(x - v_*, t) S(x + v, t)|$.

(iv) $|\nabla_v T[S](x, v, v_*, \theta, t)| + |\nabla_{x} T[S](x, v, v_*, \theta, t)| + |\partial_\theta T[S](x, v, v_*, \theta, t)|$

$$
\leq |S(x - v_*, t) \nabla_v S(x + v, t)| + |S(x + v, t) \nabla_x S(x - v_*, t)| + S(x + v, t) S(x - v_*, t).
$$

(2.4)
(A2). The reaction term \(\varphi[S, \rho]\) takes the following ansatz:
\[
\varphi[S, \rho] = -\rho S.
\]
For a detailed discussion of the kernel, we refer to [5]. Under the above assumptions (A1) – (A2), system (2.3) becomes
\[
\begin{align*}
\partial_t f + v \cdot \nabla f + \kappa_0 \nabla_v \cdot (F_a[f]f) + \kappa_1 \partial_b(G_a[f]f) = \kappa_2 T[S](f), \\
\partial_t S - \Delta_x S = -\kappa_3 \rho S, \quad \rho := \int_{\mathbb{R}^d \times \mathbb{R}^+} f(x, v, \theta, t) d\theta dv, \quad x \in \mathbb{R}^d, \ t > 0.
\end{align*}
\]

Remark 1. We can see that the assumption (A1) can be actually met by taking the turning kernel as
\[
T[S] := C \varphi \left( \frac{v}{R_v} \right) \varphi \left( \frac{v_s}{R_v} \right) S(x - v_s, t) S(x + v, t),
\]
where \(\varphi\) denotes the standard mollifier, and \(C\) is some appropriate constant independent of \(x, v, v_s, \theta\) and \(t\).

In the sequel, we will present a priori estimates for system (2.5).

2.3. A priori estimates. In this subsection, we discuss several basic properties of a strong solution to (2.5). First, we begin with a technical lemma regarding velocity and temperature alignment terms \(F_a[f]f\) and \(G_a[f]f\).

Lemma 2.1. Let \(f = f(z, t)\) be a nonnegative global smooth solution to (2.5) which is compactly supported in \(z \in \mathbb{R}^d \times \mathbb{R}^+\) for each time \(t\). Then, we have the following estimates:

(i) \(\int_{\mathbb{R}^d \times \mathbb{R}^+} v \nabla_v \cdot (F_a[f]f) dz = 0, \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} \theta \partial_b(G_a[f]f) dz = 0.\)

(ii) \(\int_{\mathbb{R}^d \times \mathbb{R}^+} |v|^2 \nabla_v \cdot (F_a[f]f) dz = \int_{\mathbb{R}^d \times (\mathbb{R}^+)^2} \psi(|x - x_s|)(v - v_s) \cdot \left( \frac{v}{\theta} - \frac{v_s}{\theta_s} \right) f(z, t) f(z_s, t) dz_s dz.\)

(iii) \(\int_{\mathbb{R}^d \times \mathbb{R}^+} \theta^2 \partial_b(G_a[f]f) dz = \int_{\mathbb{R}^d \times (\mathbb{R}^+)^2} \zeta(|x - x_s|)^2 \left( \frac{\theta - \theta_s}{\theta_s} \right)^2 f(z, t) f(z_s, t) dz_s dz.\)

Proof. (i) We use the relation
\[
v \nabla_v \cdot (F_a[f]f) = \nabla_v \cdot (F_a[f]f \otimes v) - F_a[f]f
\]
to obtain
\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} v \nabla_v \cdot (F_a[f]f) dz = \int_{\mathbb{R}^d \times (\mathbb{R}^+)^2} \psi(|x - x_s|) \left( \frac{v}{\theta} - \frac{v_s}{\theta_s} \right) f(z, t) f(z_s, t) dz_s dz = 0.
\]
Similarly, we use the relation
\[
\partial_b(G_a[f]f) = \partial_b(\theta G_a[f]f) - G_a[f]f
\]
to obtain
\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} \theta \partial_b(G_a[f]f) dz = - \int_{\mathbb{R}^d \times (\mathbb{R}^+)^2} \zeta(|x - x_s|) \left( \frac{1}{\theta} - \frac{1}{\theta_s} \right) f(z, t) f(z_s, t) dz_s dz = 0.
\]

(ii) We use the relation
\[
|v|^2 \nabla_v \cdot (F_a[f]f) = \nabla_v \cdot \left( |v|^2 F_a[f]f \right) - 2v \cdot F_a[f]f
\]
to obtain
\[
\int_{\mathbb{R}^{2d} \times \mathbb{R}_+} |v|^2 \nabla_v \cdot (F_a[f]f) dz
\]
\[
= 2 \int_{\mathbb{R}^{2d} \times (\mathbb{R}_+)^2} \psi(|x - x_*|) v \cdot \left( \frac{v}{\theta_*} - \frac{v_s}{\theta_s} \right) f(z,t)f(z_*,t) dz_* dz
\]
\[
= \int_{\mathbb{R}^{2d} \times (\mathbb{R}_+)^2} \psi(|x - x_*|) (v - v_*) \cdot \left( \frac{v}{\theta_*} - \frac{v_s}{\theta_s} \right) f(z,t)f(z_*,t) dz_* dz.
\]

(iii) We use the relation
\[
\theta^2 \partial_\theta (G_a[f]f) = \partial_\theta (\theta^2 G_a[f]f) - 2 \theta G_a[f]f
\]
to find
\[
\int_{\mathbb{R}^{2d} \times (\mathbb{R}_+)^2} \theta^2 \partial_\theta (G_a[f]f) dz
\]
\[
= -2 \int_{\mathbb{R}^{2d} \times (\mathbb{R}_+)^2} \zeta(|x - x_*|) \theta \left( \frac{1}{\theta_*} - \frac{1}{\theta_s} \right) f(z,t)f(z_*,t) dz_* dz
\]
\[
= \int_{\mathbb{R}^{2d} \times (\mathbb{R}_+)^2} \zeta(|x - x_*|) \left( \frac{\theta - \theta_*}{\theta_*} \right)^2 f(z,t)f(z_*,t) dz_* dz.
\]

Next, we give a priori estimates for the chemotactic density $S$.

**Lemma 2.2.** Let $(f,S)$ be a nonnegative smooth solution to (2.5). Then we have
\[
\|S(t)\|_{L^1} \leq \|S_0\|_{L^1}, \quad \|S(t)\|_{L^\infty} \leq \frac{2^d}{(1 + t)^{\frac{d}{2}}} \left( \|S_0\|_{L^\infty} + \|S_0\|_{L^1} \right), \quad t \geq 0.
\]

**Proof.** (First estimate): We integrate (2.5) with respect to $x$ to obtain
\[
\frac{d}{dt} \|S(t)\|_{L^1} \leq 0.
\]
This yields the desired estimate.

• (Second estimate): By Duhamel’s principle applied to (2.5), we have
\[
S(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} S_0(y) dy
\]
\[
- \kappa_3 \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{|x-s|^2}{4(t-s)}} S(y,s) \rho(y,s) dy ds
\]
\[
\leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} S_0(y) dy \leq \|S_0\|_{L^\infty}.
\]
This yields
\[
\|S(t)\|_{L^\infty} \leq \|S_0\|_{L^\infty}.
\]
Next, we consider two cases:
\[
0 \leq t \leq 1 \quad \text{and} \quad t > 1.
\]
If $0 \leq t \leq 1$, then we have
\[
\|S(t)\|_{L^\infty} \leq \|S_0\|_{L^\infty} \leq \frac{2^\frac{d}{2}}{(1 + t)^\frac{d}{2}}\|S_0\|_{L^\infty}.
\] (2.6)

If $t > 1$, then one has
\[
\|S(t)\|_{L^\infty} \leq \frac{1}{(4\pi t)^\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4t}} S_0(y) dy \leq \frac{1}{(4\pi t)^\frac{d}{2}} \|S_0\|_{L^1}
\leq \frac{(1 + t)^\frac{d}{2}}{(4\pi t)^\frac{d}{2}} \|S_0\|_{L^1} \leq \left(\frac{2}{4\pi}\right)^\frac{d}{2} \left(\frac{2^\frac{d}{2}}{(1 + t)^\frac{d}{2}}\|S_0\|_{L^1}\right)
\leq \frac{2^\frac{d}{2}}{(1 + t)^\frac{d}{2}} \|S_0\|_{L^1}.
\] (2.7)

Finally, we combine the estimates in (2.6) and (2.7) to get
\[
\|S(t)\|_{L^\infty} \leq \frac{2^\frac{d}{2}}{(1 + t)^\frac{d}{2}} \left(\|S_0\|_{L^\infty} + \|S_0\|_{L^1}\right).
\]
where the first term in the R.H.S. of (2.8) is zero by the standard $z \leftrightarrow z_*$ trick. Then, it follows from Lemma 2.2 that

$$ |M_1(t)| \leq |M_1(0)| + 2\kappa_2 M_0 R_v m(B(R_v)) \int_0^t \|S(s)\|_{L^\infty}^2 ds $$

$$ \leq |M_1(0)| + \frac{2^{2d+1}}{d-1} \kappa_2 M_0 R_v m(B(R_v)) \|S(0)\|_{L^\infty}^2 + \|S_0\|_{L^1}^2. $$

Next, we multiply $\theta$ to the both side of (2.5) and integrate the resulting relation with respect to $x, v, \theta$ to get

$$ \frac{d\tilde{M}_1(t)}{dt} = \kappa_2 \int_{\mathbb{R}^{2d} \times \mathbb{R}^+} \theta \widetilde{T}(f)dz $$

$$ \leq \kappa_2 \int_{\mathbb{R}^{3d} \times \mathbb{R}^+} \theta \widetilde{T}(S)(x,v_*, \theta,t)f(x,v_*, \theta,t)dv_*dz $$

$$ \leq \kappa_2 \theta M_0 m(B(R_v)) M_0 \|S(t)\|_{L^\infty}. $$

By Lemma 2.2, we have

$$ \tilde{M}_1(t) \leq \tilde{M}_1(0) + \kappa_2 \theta M_0 m(B(R_v)) M_0 \int_0^t \|S(s)\|_{L^\infty}^2 ds $$

$$ \leq \tilde{M}_1(0) + \frac{2^d}{d-1} \kappa_2 \theta M_0 m(B(R_v)) M_0 \|S(0)\|_{L^\infty}^2 + \|S_0\|_{L^1}^2. $$

3. **Local-in-time existence of strong solutions.** In this section, we present a local-in-time existence of a strong solution to system (2.5). First, we recall the concept of strong solutions to (1.1) as follows.

**Definition 3.1.** Let $T \in (0, \infty)$ be given. The pair $(f, S)$ is a strong solution of system (2.5) in the time-interval $[0,T)$, if the following conditions hold:

1. (Regularity conditions): The solution $(f, S)$ has the following regularity:

   $f \in W^{1,\infty}(0,T; L^{\infty}(\mathbb{R}^{2d} \times \mathbb{R}^+)) \cap L^{\infty}(0,T; (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d} \times \mathbb{R}^+)),$

   $S \in W^{1,\infty}(0,T; L^{\infty}(\mathbb{R}^d)) \cap L^{\infty}(0,T; (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).$

2. (Distributional weak solution): For any pair of test functions $(\phi, \eta) \in C_c^\infty(\mathbb{R}^{2d} \times [0,T)) \times C_c^\infty(\mathbb{R}^d \times [0,T))$, the following integral relations hold:

   $$ - \int_0^T \int_{\mathbb{R}^{2d} \times \mathbb{R}^+} f[\partial_t \phi + v \cdot \nabla_x \phi + \kappa_0 F_0[f] \cdot \nabla_v \phi + \kappa_1 G_0[f] \partial_\theta \phi] dzdt $$

   $$ = \kappa_2 \int_0^T \int_{\mathbb{R}^{2d} \times \mathbb{R}^+} T[S](f)(z,t)\phi(z,t)dzdt + \int_{\mathbb{R}^{2d} \times \mathbb{R}^+} f_0(z)\phi(z,0)dz,$$

   $$ \int_0^T \int_{\mathbb{R}^d} S[\partial_t \eta + \Delta_x \eta] dxdt $$

   $$ = \kappa_3 \int_0^T \int_{\mathbb{R}^d} \varphi[S, \rho](x,t)\eta(x,t)dxdt + \int_{\mathbb{R}^d} S_0(x)\eta(x,0)dx.$$
Now, we state our first result on the local-in-time strong solutions as follows.

**Theorem 3.2 (Local-in-time existence of a strong solution).** For positive constants \( R_x, \theta_M \) and \( \theta_m \), suppose that the initial data \((f_0, S_0)\) satisfy

\[
\begin{align*}
&f_0 \in (L^1_x \cap W^{1,\infty})(\mathbb{R}^d \times \mathbb{R}_+), \quad S_0 \in (L^1_x \cap W^{2,\infty})(\mathbb{R}^d), \\
&M_0(0) + |M_1(0)| + M_2(0) < \infty, \\
&\text{supp}_{x,v} f_0(\cdot, \cdot) \subset B(R_x) \times B(R_v) \times [\theta_m, \theta_M] \subset \subset \mathbb{R}^d \times \mathbb{R}_+,
\end{align*}
\]

where \( R_v \) is the constant given in (A1) in (2.4). Let \( L \) be a positive constant such that

\[ L > \max \left\{ \|f_0\|_{L^1}, \sum_{0 \leq |\alpha| + |\beta| + |\gamma| \leq 1} \|\nabla_x^{\alpha} \nabla_v^{\beta} \partial_t^{\gamma} f_0\|_{L^\infty}, \sum_{0 \leq |\gamma| \leq 2} \|\nabla_x^\alpha S_0\|_{L^\infty} \right\}. \]

Then, there exists a local-in-time strong solution \((f, S)\) to (1.1) and a positive constant \( T^* \) such that

1. \( f \in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R}^d \times \mathbb{R}_+)) \cap L^\infty(0, T^*; (L^1_x \cap W^{1,\infty})(\mathbb{R}^d \times \mathbb{R}_+)). \)
2. \( S \in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R}^d)) \cap L^\infty(0, T^*; (L^1_x \cap W^{2,\infty})(\mathbb{R}^d)). \)
3. \( \sup_{0 \leq t \leq T^*} \max \{ ||f||_{W^{1,\infty}}, ||S||_{W^{2,\infty}}, |M_0|, |M_1|, |M_2| \} \leq L, \)
4. \( \text{supp}_\theta f(x, v, \cdot, t) \subset [\theta_m, \theta_M] \subset (0, \infty), \quad \forall (x, v, t) \in \mathbb{R}^d \times [0, T^*). \)
5. \( \text{supp}_\theta f(x, \cdot, \theta, t) \subset K_1^\theta := B \left( R_x + \frac{\kappa_0 \psi^\beta}{\theta_m} L t \right), \quad \forall (x, \theta, t) \in \mathbb{R}^d \times \mathbb{R}_+ \times (0, T^*]. \)

for all \( \forall (x, v, \theta, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times (0, T^*]. \)

**Proof.** Since the proof is rather lengthy, we split its proof into four steps:

- **Step A:** Use successive approximations to construct approximate solutions.
- **Step B:** Give a priori estimates for the approximate solutions.
- **Step C:** Prove the convergence of the approximate solutions.
- **Step D:** Show that the limit function is the desired strong solution.

Each step will be discussed in the following subsections. \(\square\)

3.1. **Step A (Construction of approximate solutions).** Define a sequence of approximate solutions \(\{(f^n, S^n)\}_{n \geq 0}\) for system (2.5) as follows:

- **(Initial step):** We set
  \[ f^0(z, t) = f_0(z), \quad S^0(x, t) := S_0(x), \quad t \geq 0, \quad (x, v, \theta) \in \mathbb{R}^d \times \mathbb{R}_+. \]

- **(Induction step):** Suppose the \((n-1)\)-th iterate \((f^{n-1}, S^{n-1})\) has been constructed. Then, \(n\)-th iterate \((f^n, S^n)\) is defined as a strong solution to the following system:

\[
\begin{cases}
\partial_t f^n + v \cdot \nabla_x f^n + \kappa_0 \nabla_v \cdot (F_a[f^{n-1}] f^n) + \kappa_1 \partial_v(G_a[f^{n-1}] f^n) \\
= \kappa_2 T^n[S^{n-1}] f^{n-1} - \kappa_2 \lambda [S^{n-1}] f^n, \quad t > 0, \\
\partial_t S^n - \Delta_x S^n = -\kappa_3 \rho^{n-1} S^{n-1}, \quad \rho^{n-1} := \int_{\mathbb{R}^d \times \mathbb{R}_+} f^{n-1}(x, v, \theta, t) d\theta dv, \\
f^n(z, 0) = f_0(z), \quad S^n(x, 0) = S_0(x), \quad x, v \in \mathbb{R}^d, \quad \theta \in \mathbb{R}_+. 
\end{cases}
\]

\(\square\)
First, we rewrite (3.2) as a transport form:
\[
\partial_t f^n + v \cdot \nabla_x f^n + \kappa_0 F_a[f^{n-1}] \cdot \nabla_v f^n + \kappa_1 G_a[f^{n-1}] \partial_t f^n \\
= -\left\lbrace \kappa_0 \nabla_v \cdot \left( F_a[f^{n-1}] \right) + \kappa_1 \partial_t (G_a[f^{n-1}]) + \kappa_2 \lambda [S^{n-1}] \right\rbrace f^n \\
+ \kappa_2 T^+[S^{n-1}](f^{n-1}),
\]
and consider characteristic equations for (3.3):
\[
\frac{d}{dt} x^n(t) = v^n(t), \quad t > 0, \\
\frac{d}{dt} v^n(t) = \kappa_0 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \psi(|x^n(t) - x_0|) f^{n-1} (z_0, t) \left( \frac{v}{\partial_v^0} - \frac{v^n(t)}{\partial_v^0} \right) dz_0, \\
\frac{d}{dt} \theta^n(t) = \kappa_1 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \zeta(|x^n(t) - x_0|) f^{n-1} (z_0, t) \left( \frac{1}{\partial_v^0} - \frac{1}{\partial_v^0} \right) dz_0, \\
\frac{d}{dt} f^n(t) = \kappa_2 T^+[S^{n-1}](f^{n-1})(x^n(t), v^n, \theta^n, t) \\
+ \left[ d\kappa_0 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \psi(|x^n - x_0|) f^{n-1} (z_0, t) \frac{1}{\partial_v^0} dz_0 \\
+ \kappa_1 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \zeta(|x^n - x_0|) f^{n-1} (z_0, t) \frac{1}{|\partial_v^0|^2} dz_0 \\
- \kappa_2 \lambda [S^{n-1}](x^n, v^n, \theta^n, t) \right] f^n(t) \\
=: \kappa_2 T^+[S^{n-1}](f^{n-1})(x^n(t), v^n(t), \theta^n(t), t) \\
+ h^{n-1}(x^n(t), v^n(t), \theta^n(t)) f^n(t).
\]

Equation (3.4) can be integrated along the characteristics to yield
\[
f^n(x^n(t), v^n(t), \theta^n(t), t) \\
= f_0(x^n(0), v^n(0), \theta^n(0)) e^{\int_0^t h^{n-1}(x^n(s), v^n(s), \theta^n(s), s) ds} \\
+ \kappa_2 \int_0^t \left( e^{\int_0^\tau h^{n-1}(x^n(\tau), v^n(\tau), \theta^n(\tau), \tau) d\tau} \right. \\
\times T^+[S^{n-1}](f^{n-1})(x^n(s), v^n(s), \theta^n(s), s) \bigg) ds \\
=: P^{n-1}[x^n, v^n, \theta^n](t).
\]

Now consider a backward equation of the characteristic curve: for fixed \((x, v, \theta, t) \in \mathbb{R}^{2d} \times (\mathbb{R}_+)^2\), we define the curve \((x^n(\tau; x, v, \theta, t), v^n(\tau; x, v, \theta, t), \theta^n(\tau; x, v, \theta, t))\) in variable \(\tau\) by the curve satisfying the characteristic ODE (3.4)₁, (3.4)₂ and (3.4)₃ with the following terminal condition at time \(\tau = t\):
\[
(x^n(t; x, v, \theta, t), v^n(t; x, v, \theta, t), \theta^n(t; x, v, \theta, t)) = (x, v, \theta).
\]

By using the above parametric family of curves, we define \(f^n\) as follows:
\[
f^n(x, v, \theta, t) \\
= \begin{cases} 
  P^{n-1}[x^n(\cdot; x, v, \theta, t), v^n(\cdot; x, v, \theta, t), \theta^n(\cdot; x, v, \theta, t)](t) \\
  0 & \text{otherwise.}
\end{cases}
\]
In other words, at point \((x,v,\theta, t)\), \(f^n\) is defined via the formula (3.6) if there exists a characteristic curve \((x^n, v^n, \theta^n)\) which is initiated from the boundary \(\mathbb{R}^{2d} \times [t = 0]\) and passes through the point \((x, v, \theta, t)\); otherwise, we simply set \(f^n(x,v,\theta,t) = 0\). Then \(f^n\) is a solution to (3.2) in distribution sense, i.e. for any test function \(\phi(z,t) \in C_0^\infty(0, T^*) \times \mathbb{R}^{2d} \times \mathbb{R}_+\), \(f^n\) satisfies

\[
-\int_0^T \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} f^n(\partial_t \phi + v \cdot \nabla_x \phi + \kappa_0 F_a[f^{n-1}] \cdot \nabla_v \phi + \kappa_1 G_a[f^{n-1}] \partial_\theta \phi
- \kappa_2 \lambda [S^{n-1}] \phi) dz dt = \kappa_2 \int_0^T \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} T^+[S^{n-1}](z,t) \phi(z,t) dz dt + \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} f_0(z) \phi(z,0) dz.
\]

**Remark 2.** Note that the approximation scheme (3.2) guarantees the nonnegativity of \(f^n\) and \(S^n\). More precisely, we can use the formula (3.6) with (3.5) to prove the nonnegativity of \(f^n\) inductively, and the maximum principle for parabolic equations guarantees the nonnegativity of \(S^n\) as well.

### 3.2. Step B (A priori estimates for approximate solutions)

In this subsection, we provide several a priori estimates for the approximate solution \((f^n, S^n)\). First, we study basic properties of \(S^n\) in the following lemma.

**Lemma 3.3.** Let \((f^n, S^n)\) be an approximate solution to (3.2) in the time interval \([0, T^*)\). Then, we have

\[
\sup_{0 \leq t < T^*} \|S^n(t)\|_{L^1} \leq \|S_0\|_{L^1}, \quad \sup_{0 \leq t < T^*} \|S^n(t)\|_{L^\infty} \leq \|S_0\|_{L^\infty}.
\]

**Proof.** (i) We integrate (3.2) with respect to \(x\) to obtain

\[
\frac{d}{dt} \|S^n(t)\|_{L^1} \leq 0.
\]

This yields the desired first result.

(ii) By Duhamel’s principle, we have

\[
S^n(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} S_0(y) dy
- \kappa_3 \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} S^{n-1}(y,s) \rho(y,s) dy ds
\leq \frac{1}{(4\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} S_0(y) dy = \|S_0\|_{L^\infty}.
\]

This yields the desired second estimate.

Now, we give a priori estimates for \((f^n, S^n)\) by induction on \(n\).

**Lemma 3.4.** Let \(R_M, \theta_M, \theta_m\) and \(L\) be positive constants such that

\[
\theta_M > \theta_m > 0,
L > \max \left\{ \|f_0\|_{L^1}, \sum_{0 \leq |\alpha| + |\beta| + |\gamma| \leq 1} \|\nabla_x^\alpha \nabla_v^\beta \partial_\theta^\gamma f_0\|_{L^\infty}, \sum_{0 \leq |\alpha| \leq 2} \|\nabla_x^\alpha S_0\|_{L^\infty} \right\}. \quad (3.7)
\]
Suppose that the initial data \((f_0, S_0)\) satisfy (3.1). Then, there exists a positive constant \(T^*\) such that for all \(n \geq 0\), we have

\[
\begin{align*}
(i) & \ f^n \in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R}^d \times \mathbb{R}_+)) \\
& \quad \cap L^\infty(0, T^*; (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d \times \mathbb{R}_+)).
\end{align*}
\]

\[
(ii) S^n \in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R}^d)) \cap L^\infty(0, T^*; (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).
\]

\[
(iii) \sup_{0 \leq t < T^*} \max\{|f^n|_{L^\infty}, |S^n|_{W^{2,\infty}}, |M^n_0|, |M^n_1|, |M^n_2| \leq L.
\]

\[
(iv) \sup_{\theta} f^n(x, v, t) \in [\theta_m, \theta_M] \subset (0, \infty), \quad (x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T^*).
\]

\[
(v) \sup_{\theta} f^n(x, \cdot, \theta, t) \in K^n_v := B(R_x + \frac{\kappa_0 \psi_M}{\theta_m} L \frac{t}{T^*}), \quad (x, \theta, t) \in \mathbb{R}^d \times \mathbb{R}_+ \times [0, T^*).
\]

\[
(vi) \sup_{x} f^n(\cdot, v, \theta, t) \in K^n_x := B(R_x + \frac{\kappa_0 \psi_M}{\theta_m} L \frac{T^*}{t}), \quad (v, \theta, t) \in \mathbb{R}^d \times [0, T^*).
\]

Proof. We verify (3.8) using induction on \(n\). The base case \(n = 0\) is trivial by (3.1). Now, we assume that (3.8) holds for \(n - 1\). Next we show that the estimates (3.8) hold for \(n+1\) by one.

- \((\theta\text{-support of } f^n)\): From (3.4), we have

\[
\frac{d}{dt} \theta^n(t) = \kappa_1 \int_{\mathbb{R}^{2d} \times [0, \theta_m]} \zeta(|x^n - x_s|) f^{n-1}(z_s, t) \left( \frac{1}{\theta^n(t)} - \frac{1}{\theta_s} \right) dz_s
\]

\[
\begin{cases}
\geq 0 & \text{if } 0 < \theta^n(t) \leq \theta_m, \\
\leq 0 & \text{if } \theta^n(t) \geq \theta_M.
\end{cases}
\]

Hence, for any \(t_0 \in [0, T^*), \theta^n(t_0) \in [\theta_m, \theta_M]\) implies \(\theta^n(t) \in [\theta_m, \theta_M]\) for \(t_0 \leq t < T^*\). In other words,

\[
\theta^n(t_0) \notin [\theta_m, \theta_M] \Rightarrow \theta^n(t) \notin [\theta_m, \theta_M] \text{ for } 0 \leq t \leq t_0.
\]

We combine this and the formula (3.6) to get

\[
\sup_{\theta} f^n(x, v, \cdot, t) \subset [\theta_m, \theta_M], \quad x, v \in \mathbb{R}^d, \quad 0 \leq t < T^*.
\]

- \((v\text{-support of } f^n)\): Note that

\[
\frac{d}{dt} |v^n(t)| = 2\kappa_0 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \psi(|x^n - x_s|) f^{n-1}(z_s, t) \left( \frac{v^n \cdot v_s}{\theta^n} - \frac{|v^n|^2}{\theta^n} \right) dz_s
\]

\[
\leq 2\kappa_0 |v^n| \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \psi(|x^n - x_s|) f^{n-1}(z_s, t) \frac{|v_s|}{\theta^n} dz_s
\]

\[
\leq \frac{2\kappa_0 \psi_M}{\theta_m} \sqrt{M^n_0 M^n_1} |v^n| \leq \frac{2\kappa_0 \psi_M}{\theta_m} L |v^n|, \quad 0 \leq t < T^*.
\]

This yields

\[
\frac{d}{dt} |v^n(t)| \leq \frac{\kappa_0 \psi_M}{\theta_m} L, \quad 0 < t < T^*.
\]
Hence, for any $t_0 \in [0, T^*)$, the relation $v^n(t_0) \in K^n_{t_0}$ implies

$$v^n(t) \in K^n_t \quad \text{for} \ 0 \leq t \leq T^*.$$ 

In other words,

$$v^n(t_0) \notin K^n_{t_0} \implies v^n(t) \notin K^n_t \quad \text{for} \ 0 \leq t \leq t_0.$$ 

We combine this and the formula (3.6) to get

$$\text{supp}_x f^n(t, x, \cdot, \theta) \subset K^n_t, \quad (x, \theta, t) \in \mathbb{R}^d \times \mathbb{R}_+ \times [0, T^*).$$

- (x-support of $f^n$): We integrate (3.9) to get

$$|v^n(t)| \leq R_v + \frac{\kappa_0 \psi M}{\theta_m} L T^*, \quad 0 \leq t < T^*.$$ 

We use

$$\frac{d}{dt}|x^n(t)|^2 = 2x^n \cdot v^n \leq 2|x^n||v^n| \leq 2|x^n| \left( R_v + \frac{\kappa_0 \psi M}{\theta_m} L T^* \right), \quad 0 < t < T^*$$

to get

$$\frac{d}{dt}|x^n(t)| \leq R_v + \frac{\kappa_0 \psi M}{\theta_m} L T^*, \quad 0 < t < T^*.$$ 

Hence, for any $t_0 \in [0, T^*)$ such that $x^n(t_0) \in K^n_{t_0}$, we have

$$x^n(t) \in K^n_t \quad \text{for} \ 0 \leq t < T^*.$$ 

In other words,

$$x^n(t_0) \notin K^n_{t_0} \implies x^n(t) \notin K^n_t \quad \text{for} \ 0 \leq t \leq t_0.$$ 

We combine this with the formula (3.6) to get

$$\text{supp}_x f^n(t, \cdot, \cdot, \theta) \subset K^n_t, \quad (v, \theta, t) \in \mathbb{R}^d \times \mathbb{R}_+ \times [0, T^*).$$

- (Regularity of $f^n$): Note that we can rewrite (3.4) as

$$\frac{d}{d\tau}(x^n(\tau), v^n(\tau), \theta^n(\tau)) = F(x^n(\tau), v^n(\tau), \theta^n(\tau), \tau),$$

where $F := (F_1, F_2, F_3)$ is the following vector field:

$$F_1(x^n, v^n, \theta^n, \tau) := v^n,$$

$$F_2(x^n, v^n, \theta^n, \tau) := \kappa_0 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \psi(|x^n - x_s|) f^{n-1}(z_s, \tau) \left( \frac{v^n}{\theta_s} - \frac{v^{n-1}}{\theta^n} \right) dz_s,$$

$$F_3(x^n, v^n, \theta^n, \tau) := \kappa_1 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \zeta(|x^n - x_s|) f^{n-1}(z_s, \tau) \left( \frac{1}{\theta^n} - \frac{1}{\theta^n} \right) dz_s.$$ 

Since $F$ is $C^1$ with respect to $x^n, v^n, \theta^n$ and continuous with respect to $\tau$, the solution $(x^n(\tau; x, v, \theta, t), v^n(\tau; x, v, \theta, t), \theta^n(\tau; x, v, \theta, t))$ is $C^1$ respect to $(x, v, \theta)$. By the formula (3.6) and regularity of $(f^{n-1}; S^{n-1})$, we have

$$f^n \in W^{1, \infty}(0, T^*; L^{\infty}(\mathbb{R}^{2d} \times \mathbb{R}_+)) \cap L^{\infty}(0, T^*; W^{1, \infty}(\mathbb{R}^{2d} \times \mathbb{R}_+)).$$
• (Boundedness of $M^n_0$): We integrate (3.2) with respect to $z$ to obtain

$$
\frac{d}{dt} M^n_0(t) \leq \kappa_2 \int_{\mathbb{R}^d \times \mathbb{R}_+} T[S^{n-1}](x, v, v_*, \theta, t) f^{n-1}(x, v_*, \theta, t) dv_* dz
$$

$$
= \kappa_2 \int_{\mathbb{R}^d \times \mathbb{R}_+} \left[ \int_{\mathbb{R}^d} T[S^{n-1}](x, v, v_*, \theta, t) dv \right] f^{n-1}(x, v_*, \theta, t) dv_* d\theta dx
$$

$$
\leq \kappa_2 m(B(R_0)) \| S_0 \|_{L^\infty}^2 M^n_0 \leq \kappa_2 m(B(R_0)) \| S_0 \|_{L^\infty}^2 L.
$$

Hence, for sufficiently small $T^*>0$, we have

$$
M^n_0(t) \leq M_0 + \kappa_2 m(B(R_0)) \| S_0 \|_{L^\infty}^2 LT^* \leq L, \quad 0 \leq t \leq T^*.
$$

Therefore we have $f^n \in L^\infty([0, T^*]; L^1(\mathbb{R}^d \times \mathbb{R}_+))$.

• (Boundedness of $M^n_1$ and $M^n_2$): We multiply $|v|^2$ on both side of (3.2) and integrate the resulting relation with respect to $z$ to obtain

$$
\frac{d}{dt} M^n_2(t) \leq 2\kappa_0 \int_{\mathbb{R}^d \times \mathbb{R}_+} \psi(|x - x_*|) \left( v \cdot \left( \frac{\psi''}{\theta_m} - \frac{v_*}{\theta_m} \right) \right) f^n(z, t) f^{n-1}(z_*, t) dz_* dz
$$

$$
+ \kappa_2 \int_{\mathbb{R}^d \times \mathbb{R}_+} |v|^2 T[S^{n-1}](x, v, v_*, \theta, t) f^{n-1}(x, v_*, \theta, t) dv_* dz
$$

$$
\leq \frac{\kappa_0 \psi_M}{\theta_m} \int_{\mathbb{R}^d \times \mathbb{R}_+} (3|u|^2 + |v_*|^2) f^n(z, t) f^{n-1}(z_*, t) dz_* dz
$$

$$
+ \kappa_2 \int_{\mathbb{R}^d \times \mathbb{R}_+} \left[ \int_{\mathbb{R}^d} |v|^2 T[S^{n-1}](x, v, v_*, \theta, t) dv \right] f^{n-1}(x, v_*, \theta, t) dv_* d\theta dx
$$

$$
\leq \frac{\kappa_0 \psi_M}{\theta_m} (3M^n_2 M^n_0 - 1) + \kappa_2 m(B(R_0)) \| S_0 \|_{L^\infty}^2 (R_0)^2 M^n_0
$$

$$
\leq 3\frac{\kappa_0 \psi_M \theta_m}{\theta_m} M^n_2 + \frac{\kappa_0 \psi_M \theta_m}{\theta_m} + \kappa_2 m(B(R_0)) \| S_0 \|_{L^\infty}^2 (R_0)^2 L.
$$

By Grönwall’s lemma, for sufficiently small $T^*>0$ we have

$$
M^n_2(t) \leq \| f_0 \|_{L^1} e^{\frac{3\kappa_0 \psi_M \theta_m}{\theta_m} M^n_2 T^*}
$$

$$
+ e^{\frac{3\kappa_0 \psi_M \theta_m}{\theta_m} T^*} - 1 \left( \frac{\kappa_0 \psi_M \theta_m}{\theta_m} + \kappa_2 m(B(R_0)) \| S_0 \|_{L^\infty}^2 (R_0)^2 L \right) \leq L.
$$

This yields

$$
\sup_{0 \leq t < T^*} M^n_2(t) \leq L.
$$

In the case of $M^n_1(t)$, we use the relation $M^n_1 \leq \sqrt{M^n_0 M^n_2}$ to deduce

$$
\sup_{t \in [0, T^*)} |M^n_1(t)| \leq L.
$$
• (Estimate of $\sup_{0 \leq t < T^*} \|f^n(t)\|_{L^\infty}$): For sufficiently small $T^* > 0$, it follows from (3.6) that

\[
|f^n(t)| \leq \|f_0\|_{L^\infty} e^{\left(\frac{\kappa_2 \gamma_0}{\gamma_m} M_0^{-1} + \frac{\kappa_1 \xi_0}{\xi_m} M_0^{-1}\right) T^*} + \kappa_2 T^* e^{\left(\frac{\kappa_2 \gamma_0}{\gamma_m} M_0^{-1} + \frac{\kappa_1 \xi_0}{\xi_m} M_0^{-1}\right) T^*} m(B(R_0)) \sup_{0 \leq t \leq T^*} \|f^n(t)\|_{L^\infty}
\]

Hence we have $\sup_{0 \leq t < T^*} \|f^n(t)\|_{L^\infty} \leq L$. • (Estimates of $\sup_{0 \leq t < T^*} \|\nabla_x S^n(t)\|_{L^\infty}$): We differentiate (3.2) with respect to $x_i$ to obtain

\[
|\partial_{x_i} S^n(x, t)| \leq \left| \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} \partial_{y_i} S_0(y) dy \right|
\]

\[
+ \kappa_3 \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{d/2}} \left( e^{-\frac{|x-y|^2}{4(t-s)}}, y_i \right) S^{n-1}(y, s) \rho^{n-1}(y, s) dy ds \leq \|\partial_{x_i} S_0\|_{L^\infty} + \kappa_3(\theta M - \theta_m) m(K^n_T) \|S_0\|_{L^\infty} L \int_0^t \frac{2}{\sqrt{4\pi(t-s)}} ds \leq \|\partial_{x_i} S_0\|_{L^\infty} + \kappa_3(\theta M - \theta_m) m(K^n_T) \|S_0\|_{L^\infty} L \frac{2}{\sqrt{\pi}} T^* \leq L,
\]

for sufficiently small $T^* > 0$.

• (Integral inequality for $||\nabla_x f^n(t)||_{L^\infty}$): Note that $\partial_{x_i} f^n$ satisfies

\[
\partial_{x_i} f^n + v \cdot \nabla_x f^n + \kappa_0 F_a[f^{n-1}] \cdot \nabla_v \partial_{x_i} f^n + \kappa_1 G_a[f^{n-1}] \partial_v \partial_{x_i} f^n = -\kappa_0 \partial_{x_i} f^n \nabla_v \cdot F_a[f^{n-1}] - \kappa_0 \nabla_v \cdot \left( \partial_{x_i} (F_a[f^{n-1}] f^n) \right) - \kappa_1 \partial_{x_i} f^n \partial_v (G_a[f^{n-1}] f^n) + \kappa_2 \int_{\mathbb{R}^d} \left( (\partial_{x_i} T[S^{n-1}]) f^{n-1}(x, v, \theta, t) + T[S^{n-1}] \partial_{x_i} f^{n-1}(x, v, \theta, t) \right) dv
\]

\[
- \kappa_2 \int_{\mathbb{R}^d} \left( (\partial_{x_i} T^*[S^{n-1}]) f^n(x, v, \theta, t) + T^*[S^{n-1}] \partial_{x_i} f^n(x, v, \theta, t) \right) dv.
\]

We integrate (3.10) along the particle trajectory

\[
[x^n(\tau), v^n(\tau), \theta^n(\tau)] = [x^n(\tau; x, v, \theta, t), v^n(\tau; x, v, \theta, t), \theta^n(\tau; x, v, \theta, t)]
\]

to obtain
Below, we estimate the integrands in the R.H.S of (3.11) as follows.

Finally, it follows from (3.11) and (3.12) that

\[
\|\partial_x, f^n(t)\|_{L^\infty} \\
\leq \|\partial_x, f_0\|_{L^\infty} + \int_0^t \left[ \kappa_0 \frac{d\psi_M}{\theta_m} L \|\partial_x, f^n\|_{L^\infty} + \kappa_0 \frac{\psi_M}{\theta_m} L^2 \right. \\
\left. + \frac{2\psi_M}{\theta_m} m(K_T^\infty) \sqrt{d} \|\nabla_v f^n\|_{L^\infty} + \kappa_1 \frac{\zeta_M}{\theta_m} L \|\partial_x, f^n\|_{L^\infty} + \kappa_1 \frac{\zeta_M}{\theta_m} L^2 \right. \\
\left. + \kappa_2 \frac{2\zeta^\infty}{\theta_m} L \|\partial \theta f^n\|_{L^\infty} \right] ds \\
+ 2\kappa_2 m(B(R_c)) \int_0^t \left[ 2\sqrt{d} \|S_0\|_{L^\infty} L^2 \right. \\
\left. + \kappa_2 \frac{2\zeta^\infty}{\theta_m} L \|\partial \theta f^n\|_{L^\infty} \right] ds.
\]
\[ + \|S_0\|_{L^\infty}^2 \left(\|\partial_x f^{n-1}\|_{L^\infty} + \|\partial_x f^n\|_{L^\infty} + L\right) ds \]
\[ \leq \|\nabla_x f_0\|_{L^\infty} + \left(\kappa_0 \frac{d\|\nabla_x \psi\|_{L^\infty}}{\theta_m} + \kappa_1 \frac{\|\nabla_x \zeta\|_{L^\infty}}{\theta_m^2} \right) L^2 T^* \]
\[ + 4\kappa_2 m(B(R_v)) \sqrt{d} \|S_0\|_{L^\infty} L^2 T^* + 4\kappa_2 m(B(R_v)) \|S_0\|_{L^2}^2 L T^* \]
\[ + \int_0^t \left[ \left(\kappa_0 \frac{d\psi_M}{\theta_m} L + \kappa_1 \frac{\zeta_M}{\theta_m^2} L + 2\kappa_2 m(B(R_v)) \|S_0\|_{L^\infty} \right) \|\nabla_x f^n\|_{L^\infty} \]
\[ + \kappa_0 \frac{2\psi_M^\infty m(K^n_M)}{\theta_m} L \sqrt{d} \|\nabla_v f\|_{L^\infty} + \kappa_1 \frac{2\zeta_M^\infty L}{\theta_m} \|\partial_\theta f^n\|_{L^\infty} \right] ds. \]

This yields
\[ \|\nabla_x f^n\|_{L^\infty} \leq \|\nabla_x f_0\|_{L^\infty} + CT^* \]
\[ + C \int_0^t \left(\|\nabla_x f^n(s)\|_{L^\infty} + \|\nabla_v f^n(s)\|_{L^\infty} + \|\partial_\theta f^n(s)\|_{L^\infty}\right) ds, \quad (3.13) \]

where \( C > 0 \) is a constant independent of \( n, t \) and \( T^* \).

- **(Estimates for \( \|\nabla_v f^n(t)\|_{L^\infty} \) and \( \|\partial_\theta f^n(t)\|_{L^\infty} \):** Similar to Case A, we have

\[ \|\nabla_v f^n\|_{L^\infty} \leq \|\nabla_v f_0\|_{L^\infty} + CT^* \]
\[ + C \int_0^t \left(\|\nabla_x f^n(s)\|_{L^\infty} + \|\nabla_v f^n(s)\|_{L^\infty} + \|\partial_\theta f^n(s)\|_{L^\infty}\right) ds, \]
\[ \|\partial_\theta f^n\|_{L^\infty} \leq \|\partial_\theta f_0\|_{L^\infty} + CT^* \]
\[ + C \int_0^t \left(\|\nabla_x f^n(s)\|_{L^\infty} + \|\nabla_v f^n(s)\|_{L^\infty} + \|\partial_\theta f^n(s)\|_{L^\infty}\right) ds. \quad (3.14) \]

- **(Estimates of \( \sup_{0 \leq t \leq T^*} \sum_{|\alpha| + |\beta| + |\gamma| = 1} \|\nabla_x^\alpha \nabla_v^\beta \partial_\theta^\gamma f^n(t)\|_{L^\infty} \):** Now we combine (3.13) and (3.14) to obtain

\[ \|\nabla_x f^n\|_{L^\infty} + \|\nabla_v f^n\|_{L^\infty} + \|\partial_\theta f^n\|_{L^\infty} \leq \|\nabla_x f_0\|_{L^\infty} + \|\nabla_v f_0\|_{L^\infty} + \|\partial_\theta f_0\|_{L^\infty} \]
\[ + CT^* + C \int_0^t \left(\|\nabla_x f^n(s)\|_{L^\infty} + \|\nabla_v f^n(s)\|_{L^\infty} + \|\partial_\theta f^n(s)\|_{L^\infty}\right) ds. \]

We use Grönwall’s lemma to obtain
\[ \|\nabla_x f^n\|_{L^\infty} + \|\nabla_v f^n\|_{L^\infty} + \|\partial_\theta f^n\|_{L^\infty} \]
\[ \leq \left(\|\nabla_x f_0\|_{L^\infty} + \|\nabla_v f_0\|_{L^\infty} + \|\partial_\theta f_0\|_{L^\infty} + CT^*\right) e^{CT^*} \leq L, \]

for sufficiently small \( T^* > 0 \).
(Estimates of $\sup_{0 \leq t < T} \sum_{i=1}^{n} \|\nabla_{x}^{i}S^{n}(t)\|_{L^\infty}$): We differentiate (3.2) with respect to $x_i$ and $x_j$ to obtain

$$
|\partial_{x_i}\partial_{x_j}S^{n}(x,t)|
\leq \left| \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^{d}} e^{-\frac{|x-y|^2}{4t}} \partial_{y_i}\partial_{y_j}S_0(y)dy \right|
+ \kappa_3 \int_{0}^{t} \int_{\mathbb{R}^{d}} \frac{1}{(4\pi(t-s))^{3/2}} \partial_{y_i}(S^{n-1}(y,s)\rho^{n-1}(y,s))dyds
\leq \|\partial_{x_i}\partial_{x_j}S_0\|_{L^\infty} + \kappa_3(\theta_M - \theta_m)m(K_T^f)\left(\|\partial_{x_i}S_0\|_{L^\infty}L + \|S_0\|_{L^\infty}L\right) \frac{2}{\sqrt{\pi}} T^* \leq L.
$$

for sufficiently small $T^* > 0$. Hence, we have

$$
\sup_{0 \leq t < T^*} \|\partial_{x_i}\partial_{x_j}S^{n}(t)\|_{L^\infty} \leq L.
$$

\hfill \Box

### 3.3. Step C (Convergence of approximate solutions).

Next, we show that the sequence $(f^n, S^n)$ is Cauchy. For this, we use (3.3) to derive the following equation for $f^{n+1} - f^n$:

$$
\partial_t(f^{n+1} - f^n) + v \cdot \nabla_x (f^{n+1} - f^n) + \kappa_0 \nabla_v \cdot [F_a[f^n]f^{n+1} - F_a[f^{n-1}]f^n]
+ \kappa_1 \partial_{\theta_0}[G_a[f^n]f^{n+1} - G_a[f^{n-1}]f^n] + \kappa_2(\lambda(S^n)f^{n+1} - \lambda(S^{n-1})f^n)
= \kappa_2(T+[S^n](f^n) - T+[S^{n-1}](f^{n-1})),
$$

i.e.

$$
\partial_t(f^{n+1} - f^n) + v \cdot \nabla_x (f^{n+1} - f^n) + \kappa_0 F_a[f^{n-1}] \cdot \partial_{\theta_0}(f^{n+1} - f^n)
+ \kappa_1 G_a[f^{n-1}] \partial_{\theta_0}(f^{n+1} - f^n)
= -\kappa_0 \nabla_v \cdot [(F_a[f^n] - F_a[f^{n-1}])f^{n+1}] - \kappa_1 \partial_{\theta_0}[(G_a[f^n] - G_a[f^{n-1}])f^{n+1}]
- \kappa_0 \nabla_v \cdot [F_a(f^{n-1})(f^{n+1} - f^n) - \kappa_1 \partial_{\theta_0}G_a[f^{n-1}](f^{n+1} - f^n)]
- \kappa_2(\lambda(S^n) - \lambda(S^{n-1}))f^{n+1} - \kappa_2 \lambda(S^{n-1})(f^{n+1} - f^n)
+ \kappa_2(T+[S^n] - T+[S^{n-1}](f^n)) + \kappa_2 T+[S^{n-1}](f^n - f^{n-1})
= : \sum_{i=1}^{S} B_{i},
$$

(3.15)

We also use (3.3) to derive the following equation for $S^{n+1} - S^n$:

$$
\begin{align*}
\partial_t(S^{n+1} - S^n) - \Delta_x(S^{n+1} - S^n) &= -\kappa_3(\rho^n S^n - \rho^{n-1} S^{n-1}), \quad x \in \mathbb{R}^d, \quad t > 0, \\
S^{n+1}(x,0) - S^n(x,0) &= 0, \quad x \in \mathbb{R}^d.
\end{align*}
$$

(3.16)

#### Lemma 3.5. For positive constants $R_x$, $\theta_M$ and $\theta_m$ such that $\theta_M > \theta_m > 0$, suppose that the initial data $(f_0,S_0)$ satisfy (3.1) and we choose $T^* > 0$ as in Lemma 3.4.

Then the sequence $(f^n, S^n)$ is Cauchy in the complete space $L^\infty(0,T^*;L^\infty(\mathbb{R}^d \times \mathbb{R}_+)) \times L^\infty(0,T^*;L^\infty(\mathbb{R}^d))$. 

Proof. We split its estimate into two parts. In the sequel, we denote $C$ by some positive constant independent of $n$ and $t$.

• Case A (Estimate of $\|f^{n+1} - f^n\|_{L^\infty}$): Next, we will show that

$$|\mathcal{I}_{11}| \leq C\|f^{n+1} - f^n\|_{L^\infty} \quad \text{or} \quad |\mathcal{I}_{11}| \leq C\|f^n - f^{n-1}\|_{L^\infty} \quad \text{or} \quad |\mathcal{I}_{11}| \leq C\|S^n - S^{n-1}\|_{L^\infty},$$

where $C$ is a positive constant independent of $n$.

◇ (Estimates of $\mathcal{I}_{1i}$, $i = 1, 2, 3, 4$): By Lemma 3.4, we can use the uniform bound

$$\max_{0 \leq |\beta| + |\gamma| \leq 1} \sup_{0 \leq t < T^*} \|\nabla_v \partial_x f^k(t)\|_{L^\infty} \leq L$$

and the uniform compact support

$$\text{supp}_{x,v,t} f^k(\cdot, \cdot, \cdot) \subset K^x_T \times K^v_T \times [\theta_m, \theta_M], \quad 0 \leq t < T^*, \quad k \geq 0$$

to obtain

$$|\mathcal{I}_{11}| \leq d\kappa_0 \frac{\psi_M}{\theta_m} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^n(z, t) - f^{n-1}(z, t)|dz \right) f^{n+1}(z, t)$$

$$\quad + 2d\kappa_0 \frac{\psi_M m(K^x_T)}{\theta_m} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^n(z, t) - f^{n-1}(z, t)|dz \right) |\nabla_v f^{n+1}(z, t)|$$

$$\leq C\|f^n - f^{n-1}\|_{L^\infty},$$

$$|\mathcal{I}_{12}| \leq \frac{\kappa_1 \zeta_M}{\theta_m} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^n(z, t) - f^{n-1}(z, t)|dz \right) f^{n+1}(z, t)$$

$$\quad + 2\frac{\kappa_1 \zeta_M}{\theta_m} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^n(z, t) - f^{n-1}(z, t)|dz \right) |\partial_v f^{n+1}(z, t)|$$

$$\leq C\|f^n - f^{n-1}\|_{L^\infty},$$

$$|\mathcal{I}_{13}| \leq d\kappa_0 \frac{\psi_M}{\theta_m} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^{n-1}(z, t)|dz \right) |f^{n+1}(z, t) - f^n(z, t)|$$

$$\leq C\|f^{n+1} - f^n\|_{L^\infty},$$

$$|\mathcal{I}_{14}| \leq \frac{\kappa_1 \zeta_M}{\theta_m} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^{n-1}(z, t)|dz \right) |f^{n+1}(z, t) - f^n(z, t)|$$

$$\leq C\|f^{n+1} - f^n\|_{L^\infty}.$$

◇ (Estimates of $\mathcal{I}_{1i}$, $i = 5, 6, 7, 8$): By Lemma 3.3 and Lemma 3.4, we can use the uniform bound

$$\sup_{0 \leq t < T^*} \|f^k\|_{L^\infty} \leq L, \quad \sup_{0 \leq t < T^*} \|S^k(t)\|_{L^1} \leq \|S_0\|_{L^1}, \quad \sup_{0 \leq t < T^*} \|S^k(t)\|_{L^\infty} \leq \|S_0\|_{L^\infty}$$
and the assumption (A1) to find

\[ |I_{15}| \leq \kappa_2 |f^{n+1}(z, t)| \left| \int_{\mathbb{R}^d} \left( T^* [S^n] - T^* [S^{n-1}] \right)(x, v, v_\star, \theta, t)dv_\star \right| \]
\[ \leq C \int_{B(R_\star)} \left| S^n(x - v, t)S^n(x + v_\star, t) - S^{n-1}(x - v, t)S^{n-1}(x + v_\star, t) \right| dv_\star \]
\[ \leq C \int_{B(R_\star)} \left| S^n(x - v, t)(S^n(x + v_\star, t) - S^{n-1}(x + v_\star, t)) \right| dv_\star \]
\[ + C \int_{B(R_\star)} \left| S^{n-1}(x + v_\star, t)(S^n(x - v, t) - S^{n-1}(x - v, t)) \right| dv_\star \]
\[ \leq C \| S^n - S^{n-1} \|_{L^\infty}. \]

\[ |I_{16}| = \kappa_2 \lambda [S^{n-1}](f^{n+1} - f^n)(z, t) \]
\[ \leq \kappa_2 \| f^{n+1} - f^n \|_{L^\infty} \int_{\mathbb{R}^d} S^{n-1}(x - v, t)S^{n-1}(x + v_\star, t)dv_\star \]
\[ \leq \kappa_2 \| f^{n+1} - f^n \|_{L^\infty} \| S_0 \|_{L^1} \| S_0 \|_{L^\infty} \]
\[ \leq C \| f^{n+1} - f^n \|_{L^\infty}, \]

\[ |I_{17}| \leq \kappa_2 \| f^n \|_{L^\infty} \int_{\mathbb{R}^d} |T[S^n] - T[S^{n-1}]|(x, v, v_\star, \theta, t)dv_\star \leq C \| S^n - S^{n-1} \|_{L^\infty}, \]

\[ |I_{18}| = \kappa_2 \int_{\mathbb{R}^d} T[S^n](x, v, v_\star, \theta, t)((f^n - f^{n-1})(x, v_\star, \theta, t))dv_\star \]
\[ = \kappa_2 \| f^n - f^{n-1} \|_{L^\infty} \int_{B(R_\star)} S^n(x - v_\star, t)S^n(x + v, t)dv_\star \]
\[ \leq C \| f^n - f^{n-1} \|_{L^\infty}. \]

Finally, we integrate (3.15) along the characteristic curve \((x^n(\tau), v^n(\tau), \theta(\tau)), 0 \leq \tau \leq t\) and combine the estimates of \(I_{1i}\)'s to obtain that for \(t \in [0, T^*)\).

\[ \|(f^{n+1} - f^n)(t)\|_{L^\infty} \leq C \int_0^t \|(f^{n+1} - f^n)(s)\|_{L^\infty} ds \]
\[ + C \int_0^t \left( \|(f^n - f^{n-1})(s)\|_{L^\infty} + \|(S^n - S^{n-1})(s)\|_{L^\infty} \right) ds. \]

Now, we use Grönwall’s lemma and \(f^k(x, v, 0) = f_0(x, v, \theta)\) for \(k \geq 0\) to get

\[ \|(f^{n+1} - f^n)(t)\|_{L^\infty} \]
\[ \leq C e^{CT^*} \int_0^t \left[ \|(f^n - f^{n-1})(s)\|_{L^\infty} + \|(S^n - S^{n-1})(s)\|_{L^\infty} \right] ds \]
\[ \leq C \int_0^t \left[ \|(f^n - f^{n-1})(s)\|_{L^\infty} + \|(S^n - S^{n-1})(s)\|_{L^\infty} \right] ds, \quad 0 \leq t < T^*. \]
• Case B (Estimate of $\|S^{n+1} - S^n\|_{L^\infty}$): In (3.16), we use Duhamel’s principle and take absolute values to estimate $S^{n+1} - S^n$:

$$
||(S^{n+1} - S^n)(x, t)|| \\
\leq \kappa_3 \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^\frac{d}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \rho^n(y, s) S^n(y, s) - \rho^{n-1}(y, s) S^{n-1}(y, s) |dyds
$$

In the last inequality, we used the uniform bounds (see Lemma 3.3 and Lemma 3.4):

$$
\sup_{x, v, \theta} |f^k(t)| \leq \kappa_3 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d \times \mathbb{R}^+} |f^n - f^{n-1}|(y, v, \theta, s) dv d\theta \right) S^n(y, s) |ds|
$$

We then combine (3.17) and (3.18) to obtain

$$
||(S^{n+1} - S^n)(x, t)|| \\
\leq C \int_0^t (||f^n - f^{n-1}(s)||_{L^\infty} + ||S^n - S^{n-1}(s)||_{L^\infty}) ds, \quad 0 \leq t < T^*.
$$

Finally, we combine (3.17) and (3.18) to obtain

$$
||(f^{n+1} - f^n)(t)||_{L^\infty} + ||(S^{n+1} - S^n)(t)||_{L^\infty}
$$

$$
\leq C \int_0^t (||f^n - f^{n-1}(s)||_{L^\infty} + ||S^n - S^{n-1}(s)||_{L^\infty}) ds \leq \cdots
$$

$$
\leq C^n \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} (||f^1 - f^0||_{L^\infty} + ||S^1 - S^0||_{L^\infty}) d\tau_1 \cdots d\tau_{n-1} d\tau_n
$$

$$
\leq C^n (T^*)^n \frac{1}{n!} \sup_{0 \leq t < T^*} (||f^1 - f^0||_{L^\infty} + ||S^1 - S^0||_{L^\infty}).
$$

Hence, we have

$$
\sup_{0 \leq t < T^*} \left( ||(f^{n+1} - f^n)(t)||_{L^\infty} + ||(S^{n+1} - S^n)(t)||_{L^\infty} \right)
$$

$$
\leq \frac{(CT^*)^n}{n!} \sup_{0 \leq t < T^*} \left( ||(f^1 - f^0)(t)||_{L^\infty} + ||(S^1 - S^0)(t)||_{L^\infty} \right). \tag{3.19}
$$

The right-hand side of (3.19) tends to zero as $n \to \infty$. This concludes the proof. \qed
3.4. Step D (Verification procedure). By Lemma 3.5, we can take the limit of \((f^n, S^n)\) in the product space \(L^\infty(0, T^*; L^\infty(\mathbb{R}^d \times \mathbb{R}_+)) \times L^\infty(0, T^*; L^\infty(\mathbb{R}))\) to obtain the limit function \((f, S)\). This is a solution of (2.5) in distribution sense. On the other hand, by Lemma 3.4, we have uniform bounds for \(f^n\) and \(S^n\) in the spaces:

\[
\begin{align*}
    f^n &\in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R}^d \times \mathbb{R}_+)) \cap L^\infty(0, T^*; (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d \times \mathbb{R}_+)), \\
    S^n &\in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R})) \cap L^\infty(0, T^*; (L^1_+ \cap W^{2,\infty})(\mathbb{R})).
\end{align*}
\]

Then, there exists a subsequential weak* limit \((f^\infty, S^\infty)\) such that:

\[
\begin{align*}
    f^\infty &\in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R}^d \times \mathbb{R}_+)) \cap L^\infty(0, T^*; (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d \times \mathbb{R}_+)), \\
    S^\infty &\in W^{1,\infty}(0, T^*; L^\infty(\mathbb{R})) \cap L^\infty(0, T^*; (L^1_+ \cap W^{2,\infty})(\mathbb{R})).
\end{align*}
\]

Since the sequence \((f^n, S^n)\) is Cauchy, the limit \((f, S)\) must coincide with the weak* limit \((f^\infty, S^\infty)\) as well. This completes the proof of Theorem 3.2.

4. A global existence of strong solutions. In this section, we show that a local solution in the previous section can be extended to a global-in-time strong solution of (2.5) by the continuous induction argument using a priori estimates. Next, we state our second main result as follows.

**Theorem 4.1** (Global existence of a strong solution). For positive constants \(R_x, \theta_M\) and \(\theta_m\) such that \(\theta_M > \theta_m > 0\), suppose that the initial data \(f_0, S_0\) satisfy (3.1). Then for any given \(T > 0\), there exists a strong solution \((f, S)\) to (2.5) in the sense of Definition 3.1, which has compact support in \(\mathbb{R}^d \times \mathbb{R}_+ \times [0, T]\).

4.1. A priori estimates. Next, we give a series of a priori estimates to be used in the proof of Theorem 4.1. Throughout this section, we will denote \(C > 0\) by some generic constant which may depend on the system parameters such as \(\psi_M, \zeta_M, \kappa_i\) (\(i = 0, 1, 2, 3\)), or initial data, or \(T > 0\), but \(C\) does not depend on \(t\).

**Lemma 4.2.** Let \((f, S)\) be a strong solution to (2.5) in the time interval \([0, T]\) which satisfies a priori estimate:

\[
\text{supp}_0 f(x, v, \cdot, t) \subset [\theta_m, \theta_M] \quad \text{for all } (x, v) \in \mathbb{R}^d, t > 0.
\]

Then, the first two velocity moments \(M_1\) and \(M_2\) are bounded:

\[
C_{M_1} := \sup_{t \in [0, T]} |M_1(t)| < \infty, \quad C_{M_2} := \sup_{t \in [0, T]} M_2(t) < \infty.
\]

**Proof.** First, we derive an upper bound estimate for \(M_2\). For this, we multiply \(|v|^2\) on both sides of (2.5), and integrate the resulting relation with respect to \(z\) to
Then, we use Grönwall’s lemma to derive a bound of $M_2$:

$$
\sup_{t \in [0,T]} M_2(t) \leq M_2(0) e^{CT} + e^{CT} - 1 < \infty.
$$

Now, the first estimate on the upper bound for $M_1(t)$ can be derived from the relation:

$$
|M_1(t)| \leq \sqrt{M_0(t)M_2(t)}.
$$

Next, we estimate the size of $v$-support and $x$-support of $f$. For this, consider the following characteristic equations for (2.5):

\[
\frac{d}{dt} x^\varepsilon(t) = v^\varepsilon(t), \quad t > 0,
\]

\[
\frac{d}{dt} v^\varepsilon(t) = \kappa_0 \int_{\mathbb{R}^d} \psi(|x^\varepsilon(t) - x_s|) f(z_s, t) \left( \frac{v_s}{\theta_s} - \frac{v^\varepsilon(t)}{\theta^\varepsilon(t)} \right) dz_s,
\]

\[
\frac{d}{dt} \theta^\varepsilon(t) = \kappa_1 \int_{\mathbb{R}^d} \zeta(|x^\varepsilon(t) - x_s|) f(z_s, t) \left( \frac{1}{\theta^\varepsilon(t)} - \frac{1}{\theta_s} \right) dz_s,
\]

\[
\frac{d}{dt} f(t) = \kappa_2 T^+[S](f)(x^\varepsilon(t), v^\varepsilon(t), \theta^\varepsilon(t))
\]

\[+ \left[ d\kappa_0 \int_{\mathbb{R}^d} \psi(|x^\varepsilon(t) - x_s|) f(z_s, t) \frac{1}{\theta^\varepsilon(t)} dz_s \right.
\]

\[+ \kappa_1 \int_{\mathbb{R}^d} \zeta(|x^\varepsilon(t) - x_s|) f(z_s, t) \frac{1}{|\theta^\varepsilon(t)|^2} dz_s
\]

\[- \kappa_2 \lambda[S](x^\varepsilon(t), v^\varepsilon(t), \theta^\varepsilon(t)) f(t)\right] f(t) =: \kappa_2 T^+[S](f)(x^\varepsilon(t), v^\varepsilon(t), \theta^\varepsilon(t), t) + h(x^\varepsilon(t), v^\varepsilon(t), \theta^\varepsilon(t), t) f(t).
\]

Then, it is easy to see that (4.1.4) can be rewritten as

\[
f(x^\varepsilon(t), v^\varepsilon(t), \theta^\varepsilon(t), t) = f(x^\varepsilon(t_0), v^\varepsilon(t_0), \theta^\varepsilon(t_0), t_0) e^{\int_{t_0}^t h(x^\varepsilon(s), v^\varepsilon(s), \theta^\varepsilon(s), s) ds}
\]

\[+ \kappa_2 \int_{t_0}^t e^{\int_{\tau}^t h(x^\varepsilon(\tau), v^\varepsilon(\tau), \theta^\varepsilon(\tau), \tau) d\tau} T^+[S](f)(x^\varepsilon(s), v^\varepsilon(s), \theta^\varepsilon(s), s) ds, \quad t \geq t_0.
\]
In next lemma, we show that once we can control the support of \( f \) in temperature variable, we can also control the supports of \( f \) in \( x \) and \( v \) as well.

**Lemma 4.3.** For positive constants \( R_x, \theta_M \) and \( \theta_m \) satisfying the relation \( \theta_M > \theta_m > 0 \), let \((f, S)\) be a strong solution to (2.5) in the time interval \([0, T]\) whose initial data \((f_0, S_0)\) satisfy (3.1). If \( \theta \)-support of \( f \) satisfies

\[
\text{supp}_\theta f(x, v, \cdot, t) \subset [\theta_m, \theta_M]
\]

for all \((x, v, t) \in \mathbb{R}^d \times [0, T)\), then for all \(0 \leq t < T\),

(i) \(\text{supp}_v f(x, \cdot, \theta, t) \subset \tilde{K}^v_t := B\left( R_v + \frac{\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2} t \right)\),

(ii) \(\text{supp}_x f(\cdot, v, \theta, t) \subset \tilde{K}^x_t := B\left( R_x + \left( R_v + \frac{\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2} T^* \right) t \right)\).

**Proof.** (i) Note that

\[
\frac{d}{dt} |v^c(t)|^2 = 2\kappa_0 \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x^c - y|) f(z^c, t) \left( \frac{v^c \cdot \nu}{\theta_*} - \frac{|v^c|^2}{\theta_c} \right) dz^c
\]

\[
= 2\kappa_0 |v^c| \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x^c - y|) f(z^c, t) \frac{|v^c|}{\theta_*} dz^c
\]

\[
\leq \frac{2\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2} |v^c|, \quad 0 \leq t < T.
\]

This implies

\[
\frac{d}{dt} |v^c(t)| \leq \frac{\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2}, \quad 0 \leq t < T. \tag{4.3}
\]

Hence, for any \( t \in [0, T) \), \( v^c(t) \in \tilde{K}^v_t \) implies \( v^c(s) \in \tilde{K}^v_s \) for \( t \leq s < T \). In other words,

\[
v^c(t) \notin \tilde{K}^v_t \implies v^c(s) \notin \tilde{K}^v_s \quad \text{for} \quad 0 \leq s \leq t.
\]

We combine this together with the formula (4.2) to get

\[
\text{supp}_v f(t, x, \cdot, \theta) \subset \tilde{K}^v_t, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{R}_+, \quad 0 \leq t < T.
\]

(ii) It follows from (4.3) that

\[
|v^c(t)| \leq R_v + \frac{\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2} T^*, \quad 0 \leq t < T.
\]

On the other hand, note that

\[
\frac{d}{dt} |x^c(t)|^2 = 2x^c \cdot \nu^n \leq 2|x^c||v^c| \leq 2|x^c|(R_v + \frac{\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2} T^*), \quad 0 \leq t < T.
\]

This yields

\[
\frac{d}{dt} |x^c(t)| \leq R_v + \frac{\kappa_0 \psi_M}{\theta_m} \sqrt{M_0 C M_2} T^*, \quad 0 \leq t < T.
\]

Hence, for any \( t \in [0, T) \), \( x^c(t) \in \tilde{K}^x_t \) implies \( x^c(s) \in \tilde{K}^x_s \) for \( t \leq s < T \). In other words,

\[
x^c(t) \notin \tilde{K}^x_t \implies x^c(s) \notin \tilde{K}^x_s \quad \text{for} \quad 0 \leq s \leq t.
\]

Define

\[
g(t) := \sup\{ |f(x, v, \theta, t)| : x \notin \tilde{K}^x_t, \quad v \in \mathbb{R}^d, \quad \theta \in \mathbb{R}_+, \quad 0 \leq t < T\}. \tag{4.4}
\]
We combine (4.4) together with the formula (4.2) to find
\[ g(t) \leq \kappa_2 \int_0^t e^{\left(\frac{\alpha_0 v_M}{\theta_m} + \frac{\alpha_1 v_M}{\theta_m}\right)M_0(t-s)} m(B(Re)) \|S_0\|^2_{L^\infty} g(s) ds. \]

Grönwall’s lemma yields
\[ e^{-\left(\frac{\alpha_0 v_M}{\theta_m} + \frac{\alpha_1 v_M}{\theta_m}\right)M_0 t} g(t) \leq 0, \quad \text{i.e.,} \quad g(t) \equiv 0. \]

Hence, we conclude
\[ \text{supp}_x f(\cdot, v, \theta, t) \subset \tilde{K}_T^x, \quad (v, \theta, t) \in \mathbb{R}^d \times \mathbb{R}_+ \times [0, T). \]

4.2. Proof of Theorem 4.1. In Theorem 3.2, we have already shown that system (2.5) has a local-in-time strong solution. In the sequel, we will extend these local solutions to global ones by showing the finiteness of the following quantity for a given \( T > 0 \):
\[ \sup_{0 \leq t < T} \left( \sum_{0 \leq |\alpha| \leq 2} \|\nabla_x^\alpha S(t)\|_{L^\infty} + \sum_{0 \leq |\alpha| + |\beta| + |\gamma| \leq 1} \|\nabla_x^\alpha \nabla_v^\beta \partial_\gamma f(t)\|_{L^\infty} \right) < \infty. \]

Then, by the continuous induction argument, we can easily extend the local solutions to the global ones (see the argument at the end of this subsection). Throughout this subsection, we assume that
- For positive constants \( R_x > 0 \) and \( \theta_M > \theta_m > 0 \), the initial data \((f_0, S_0)\) satisfy (3.1).
- The pair \((f, S)\) is a strong solution to (2.5) in the time interval \([0, T)\) satisfying a priori assumption:
\[ \text{supp}_y f(x, v, \cdot, t) \subset [\theta_m, \theta_M] \quad \text{for all} \quad (x, v, t) \in \mathbb{R}^{2d} \times [0, T). \]

Lemma 4.4. For any \( p \in [1, \infty] \) and \( T \in (0, \infty) \), we have
\[ \sup_{0 \leq t < T} \|S(t)\|_{L^p} < \infty, \quad C_f := \sup_{0 \leq t < T} \|f(t)\|_{L^\infty} < \infty. \]

Proof. (i) (Estimate of \( \|S(t)\|_{L^p} \)): The cases for \( p = 1 \) and \( p = \infty \) are clear from Lemma 2.2. Thus, we consider the case \( p \in (1, \infty) \). In this case, we have
\[ \|S(t)\|_{L^p} \leq \|S(t)\|_{L^\infty}^{1-\frac{1}{p}} \|S(t)\|_{L^p}^{\frac{1}{p}} \leq \|S_0\|_{L^\infty}^{1-\frac{1}{p}} \|S_0\|_{L^p}^{\frac{1}{p}}. \]

(ii) (Estimate of \( \|f\|_{L^\infty} \)): Let \( 1 < p < \infty \) be given and we set \( q := \frac{p}{p-1} \). Then, we multiply \( p|f|^{p-1} \) to both sides of (2.5), and integrate the resulting relation with
Lemma 4.5. The $L^\infty$-norm of the first-order derivative of $S$ in $x$ is bounded:

$$C_{\nabla_x S} := \sup_{0 \leq t \leq T} \| \nabla_x S(x, t) \|_{L^\infty} < \infty.$$
Proof. We differentiate (2.5) with respect to \( x_i \) to obtain

\[
|\partial_x, S(x, t)| \leq \left| \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} \partial_y, S_0(y)dy \right|
\]

\[+ \kappa_3 \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{\frac{3}{2}}} \left( e^{-\frac{|x-y|^2}{4(t-s)}} \right) \partial_y, S(y, s)\rho(y, s)dyds \]

\[\leq \|\partial_x, S_0\|_{L^\infty} + \kappa_3\|S_0\|_{L^\infty}(\theta_M - \theta_m)m(\tilde{K}_T^+)C_f \int_0^t \frac{2}{\sqrt{4\pi(t-s)}} ds \]

\[\leq \|\nabla_x S_0\|_{L^\infty} + \kappa_3\|S_0\|_{L^\infty}(\theta_M - \theta_m)m(\tilde{K}_T^+)C_f \frac{2}{\sqrt{\pi}}T < \infty.\]


Lemma 4.6. The \( L^\infty \)-norms of the first-order derivatives of \( f \) in \( x, v, \theta \) are bounded:

\[C_{\nabla f} := \sup_{0 \leq t \leq T} \sum_{|\alpha|+|\beta|+|\gamma|=1} \|\nabla_x^\alpha \nabla_v^\beta \nabla_\theta^\gamma f(t)\|_{L^\infty} < \infty.\]

Proof. We split the proof into four steps: we differentiate (2.5) with respect to \( x, v, \) and \( \theta \), and derive an integral inequality for each relation, and combine the three inequalities at the final step to get the desired result.

- Case A (Integral inequality for \( \|\nabla_x f(t)\|_{L^\infty} \)): Note that \( \partial_x f \) satisfies

\[\partial_t \partial_x f + v \cdot \nabla_x \partial_x f + \kappa_0 F_a[f] \cdot \nabla_v \partial_x f + \kappa_1 G_a[f] \partial_\theta \partial_x f \]

\[= -\kappa_0 \partial_x f \cdot F_a[f] - \kappa_0 \nabla_v \cdot (\partial_x (F_a[f])f) - \kappa_1 \partial_x f \partial_\theta (G_a[f]) \]

\[\quad + \kappa_1 \partial_\theta (\partial_x (G_a[f])f) \]

\[\quad + \kappa_2 \int_{\mathbb{R}^d} ((\partial_x, T[S])f(x, v, \theta, t) + T[S]\partial_x f(x, v, \theta, t)) dv_\ast \]

\[\quad - \kappa_2 \int_{\mathbb{R}^d} ((\partial_x, T^*[S])f(x, v, \theta, t) + T^*[S]\partial_x f(x, v, \theta, t)) dv_\ast.\]

We integrate the above relation along the particle trajectory

\[\mathcal{X}(\tau) = [x(\tau), v(\tau), \theta(\tau)] = [x(\tau; x, v, \theta, t), v(\tau; x, v, \theta, t), \theta(\tau; x, v, \theta, t)]\]

to obtain

\[|\partial_x f(t)| \leq \|\partial_x f_0\|_{L^\infty} + \int_0^t \left( \kappa_0 |\partial_x f \nabla_v \cdot (F_a[f])] + \kappa_0 |\nabla_v \cdot (\partial_x (F_a[f])f) \]

\[+ \kappa_1 |\partial_x f \partial_\theta (G_a[f]) | + \kappa_1 |\partial_\theta (\partial_x (G_a[f])f) | \right) ds \]

\[+ \kappa_2 \int_0^t \int_{\mathbb{R}^d} \left( |(\partial_x, T[S])f(x, v, \theta, s)| + |T[S]f(x, v, \theta, s)| \right) dv_\ast ds \]

\[+ \kappa_2 \int_0^t \int_{\mathbb{R}^d} \left( |(\partial_x, T^*[S])f(x, v, \theta, s)| + |T^*[S]f(x, v, \theta, s)| \right) dv_\ast ds. \]
Below, we estimate the integrands in the R.H.S. of (4.5) as follows.

- $|\partial_x f \nabla_v \cdot (F_a[f])| \leq \frac{d\psi}{\theta_m} M_0 \|\partial_x f\| L^\infty$,
- $|\nabla_v \cdot (\partial_x (F_a[f]) f)| \leq |\partial_x (\nabla_v \cdot (F_a[f]) f) + |\partial_x (F_a[f]) - \nabla_v f |
  \leq \frac{d\psi}{\theta_m} M_0 C_f + \frac{2\psi}{\theta_m} M_0 \|\nabla_v f\| L^\infty$,
- $|\partial_x f \partial_\theta (G_a[f])| \leq \frac{\zeta M}{\theta_m} M_0 \|\partial_x f\| L^\infty$,
- $|\partial_\theta (\partial_x (G_a[f]) f)| \leq |\partial_x (\partial_\theta (G_a[f]) f) + |\partial_x (G_a[f]) - \partial_\theta f |
  \leq \frac{\zeta M}{\theta_m} M_0 \|\partial_\theta f\| L^\infty$ (4.6),
- $|T|S| \leq |S(x + v, t) S(x - v_\ast, t) | \leq \|S_0\|^2 L^\infty$,
- $|\partial_x T|S| \leq |S(x + v, t) \nabla_x S(x - v_\ast, t) | + |S(x - v_\ast, t) \nabla_x S(x + v, t) |
  + |S(x + v, t) S(x - v_\ast, t) | \leq 2\sqrt{d} \|S_0\| L^\infty C \nabla_x S + \|S_0\|^2 L^\infty$.

Finally, it follows from (4.5) and (4.6) that

$$\|\partial_x f(t)\| L^\infty \leq \|\partial_x f_0\| L^\infty + \int_0^t \left[ \kappa_0 \frac{d\psi}{\theta_m} M_0 \|\partial_x f\| L^\infty + \kappa_0 \frac{d\psi}{\theta_m} M_0 C_f 
  + \kappa_0 \frac{2\psi m}{\theta_m} M_0 \sqrt{d} \|\nabla_v f\| L^\infty + \kappa_1 \frac{\zeta M}{\theta_m} M_0 \|\partial_x f\| L^\infty 
  + \kappa_1 \frac{\zeta M}{\theta_m} M_0 C_f + \kappa_1 \frac{2\zeta M}{\theta_m} M_0 \|\partial_\theta f\| L^\infty \right] ds 
  + 2\kappa_2 m(B(R_0)) \int_0^t \left( 2\sqrt{d} \|S_0\| L^\infty C \nabla_x S C_f + \|S_0\|^2 L^\infty (C_f + \|\partial_x f\| L^\infty) \right) ds 
  \leq \|\nabla_v f_0\| L^\infty + \left( \kappa_0 \frac{d\psi}{\theta_m} \|\nabla_v \| L^\infty + \kappa_1 \frac{\|\nabla_x \| L^\infty}{\theta_m} \right) M_0 C_f T 
  + 4\kappa_2 m(B(R_0)) \sqrt{d} \|S_0\| L^\infty C \nabla_x S C_f T + 2\kappa_2 m(B(R_0)) \|S_0\|^2 L^\infty C_f T 
  + \int_0^t \left[ \left( \kappa_0 \frac{d\psi}{\theta_m} M_0 + \kappa_1 \frac{\zeta M}{\theta_m} M_0 + 2\kappa_2 m(B(R_0)) \|S_0\|^2 L^\infty \right) \|\nabla_x f\| L^\infty 
  + \kappa_0 \frac{2\zeta m}{\theta_m} M_0 \sqrt{d} \|\nabla_v f\| L^\infty + \kappa_1 \frac{2\zeta M}{\theta_m} M_0 \|\partial_\theta f\| L^\infty \right] ds.$$ 

This yields

$$\|\nabla_x f\| L^\infty \leq C(1 + T) + C \int_0^t \left( \|\nabla_x f(s)\| L^\infty + \|\nabla_v f(s)\| L^\infty + \|\partial_\theta f(s)\| L^\infty \right) ds,$$ (4.7)

where $C > 0$ is a constant independent of $t$ and $T$. 


• Case B (Integral inequality for $\|\nabla_v f(t)\|_{L^\infty}$ and $\|\partial_\theta f(t)\|_{L^\infty}$): Similar to Case A, we have

$$
\|\nabla_v f\|_{L^\infty} \\
\leq C(1 + T) + C \int_0^t \left( \|\nabla_x f(s)\|_{L^\infty} + \|\nabla_v f(s)\|_{L^\infty} + \|\partial_\theta f(s)\|_{L^\infty} \right) ds,
$$

(4.8)

$$
\|\partial_\theta f\|_{L^\infty} \\
\leq C(1 + T) + C \int_0^t \left( \|\nabla_x f(s)\|_{L^\infty} + \|\nabla_v f(s)\|_{L^\infty} + \|\partial_\theta f(s)\|_{L^\infty} \right) ds.
$$

Now we combine (4.7) and (4.8) to obtain

$$
\|\nabla_x f\|_{L^\infty} + \|\nabla_v f\|_{L^\infty} + \|\partial_\theta f\|_{L^\infty} \\
\leq C(1 + T) + C \int_0^t \left( \|\nabla_x f(s)\|_{L^\infty} + \|\nabla_v f(s)\|_{L^\infty} + \|\partial_\theta f(s)\|_{L^\infty} \right) ds.
$$

We use Grönwall’s lemma to obtain

$$
\|\nabla_x f\|_{L^\infty} + \|\nabla_v f\|_{L^\infty} + \|\partial_\theta f\|_{L^\infty} \leq C(1 + T)e^{CT} < \infty.
$$

□

**Lemma 4.7.** For $i, j = 1, \cdots, d$, $\|\partial_x, \partial_x, S(t)\|_{L^\infty}$ is bounded on any finite-time interval:

$$
\sup_{0 \leq t < T} \|\partial_x, \partial_x, S(t)\|_{L^\infty} < \infty.
$$

**Proof.** We differentiate (2.5)$_2$ with respect to $x_i$ and $x_j$ and obtain

$$
|\partial_x, \partial_x, S(x,t)| \\
\leq \left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} \partial_{y_i} \partial_{y_j} S_0(y) dy \right| \\
+ \kappa_3 \left| \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi (t-s))^\frac{d}{2}} \partial_{y_i} \left( e^{-\frac{|x-y|^2}{4(t-s)}} \right) \partial_{y_j} (S(y,s)\rho(y,s)) dy ds \right| \\
\leq \|\partial_x, \partial_x, S_0\|_{L^\infty} + \kappa_3(\theta_M - \theta_m)m(\hat{K}_T)(\|\partial_x, S_0\|_{L^\infty} C_f + \|S_0\|_{L^\infty} C_{\nabla f}) \sqrt{\frac{2}{\pi}} T.
$$

□

This yields the desired estimate.

**Proof of Theorem 4.1:** We combine a priori estimate Lemma 4.4 ~ Lemma 4.7 and local existence of a strong solution to obtain a strong solution in any time interval $[0, T]$:

$$
f \in W^{1,\infty}(0, T; L^\infty(\mathbb{R}^{2d} \times \mathbb{R}_+)) \cap L^\infty(0, T; (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d} \times \mathbb{R}_+)),
$$

$$
S \in W^{1,\infty}(0, T; L^\infty(\mathbb{R})) \cap L^\infty(0, T; (L^1_+ \cap W^{2,\infty})(\mathbb{R}))
$$

using the standard continuity arguments.
5. **A priori flocking estimates.** In this section, we present a priori flocking estimates in temperature and velocity variables for the one-particle distribution function. For this, we introduce two Lyapunov functionals measuring concentration of temperature and velocity variables, respectively.

Let \((f, S)\) be a smooth solution to (2.5) in the time interval \([0, T]\) which satisfies a priori estimate:

\[
\text{supp}_t f(x, v, \cdot, t) \subset [\theta_m, \theta_M] \quad \text{for all } (x, v) \in \mathbb{R}^d, \ t > 0.
\]

Suppose that initial data \((f_0, S_0)\) satisfy (3.1). Then, it follows from Lemma 4.2 that

\[
M_0(t) + M_2(t) + \bar{M}_2(t) < \infty, \quad 0 \leq t < \infty. \quad (5.1)
\]

For \(t \geq 0\), we set

\[
\mathcal{L}^\theta[f(t)] := \int_{\mathbb{R}^d \times (\mathbb{R}_+)^2} \theta - \theta_v|^2 f(z, t) f(z, t) dz, \\
\mathcal{L}^v[f(t)] := \int_{\mathbb{R}^d \times (\mathbb{R}_+)^2} |v - v_v|^2 f(z, t) f(z, t) dz.
\]

Note that \(\mathcal{L}^\theta[f(t)]\) and \(\mathcal{L}^v[f(t)]\) are well-defined due to (5.1):

\[
\mathcal{L}^\theta[f(t)] = 2(M_0 \bar{M}_2 - |\bar{M}_1|^2) < \infty, \quad \mathcal{L}^v[f(t)] = 2(M_0 M_2 - |M_1|^2) < \infty. \quad (5.2)
\]

Next, we study time-decay estimates for the functionals \(\mathcal{L}^\theta[f]\) and \(\mathcal{L}^v[f]\). For this, we recall an elementary Grönwall type lemma as follows.

**Lemma 5.1.** [6] Let \(y : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}\) be a differential function satisfying

\[
y' \leq -\alpha y + f, \quad t > 0, \quad y(0) = y_0,
\]

where \(\alpha\) is a positive constant and \(f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}\) is a continuous function decaying to zero, as its argument goes to infinity. Then \(y\) satisfies

\[
y(t) \leq \frac{1}{\alpha} \max_{s \in [t/2, t]} |f(s)| + y_0 e^{-\alpha t} + \frac{\|f\|_{L^\infty}}{\alpha} e^{-\frac{\alpha t}{2}}, \quad t \geq 0.
\]

**Proof.** The proof can be found in Lemma A.1 [6].

Now, we are ready to provide flocking estimates.

**Proposition 1.** (Slow flocking) Suppose that the spatial dimension and communication weight functions satisfy

\[
d \geq 2, \quad \inf_{0 \leq r < \infty} \psi(r) = \psi_m > 0, \quad \inf_{0 \leq r < \infty} \zeta(r) = \zeta_m > 0.
\]

Suppose that initial data \(f_0, S_0\) satisfy (3.1), and let \((f, S)\) be a nonnegative global-in-time smooth solution to (2.5) satisfying

\[
\text{supp}_t f(x, v, \cdot, t) \subset [\theta_m, \theta_M], \quad (x, v, t) \in \mathbb{R}^d \times \mathbb{R}_+.
\]

Then, we have temperature and velocity consensus in the sense that

\[
\mathcal{L}^\theta[f(t)] \leq \frac{\mathcal{O}(1)}{(1 + t)^d} \quad \text{and} \quad \mathcal{L}^v[f(t)] \leq \frac{\mathcal{O}(1)}{(1 + t)^{d-2}}, \quad t \geq 0,
\]

where \(\mathcal{O}(1)\) is a bounded constant.
Proof. (i) First, we will derive a Grönwall’s differential inequality:

\[
\frac{d}{dt} \mathcal{L}^\theta[f(t)] \leq -2M_0 \frac{\kappa_1 \zeta_m}{\theta^2_M} \mathcal{L}^\theta[f(t)] + \mathcal{O}(1) \left(1 + t\right)^d.
\] (5.3)

Proof of (5.3): By Lemma 2.1 ∼ Lemma 2.3, one has

\[
\frac{d}{dt} \mathcal{L}^\theta[f(t)] = 2M_0 \frac{d\tilde{M}_2}{dt} - 4\tilde{M}_1 \frac{d\tilde{M}_1}{dt}
= -2M_0 \kappa_1 \int_{\mathbb{R}^{2d} \times (\mathbb{R}_+)^2} \zeta(|x - x_*|) \frac{(\theta - \theta_*)^2}{\theta \theta_*} f(z, t) f(z_*, t) dz_* dz \\
+ 2M_0 \kappa_2 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \theta^2 T[S](f)(z) dz - 4\kappa_2 \tilde{M}_1 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \theta T[S](f) dz
\leq -2M_0 \frac{\kappa_1 \zeta_m}{\theta^2_M} \mathcal{L}^\theta[f(t)] + 2M_0 \kappa_2 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \theta^2 T^+[S](f) dz \\
- 4\kappa_2 \tilde{M}_1 \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} \theta T^-[S](f) dz
\leq -2M_0 \frac{\kappa_1 \zeta_m}{\theta^2_M} \mathcal{L}^\theta[f(t)] + 2\kappa_2 \theta^2_M m(B(R_v)) M_0^2 \|S(t)\|_L^2
+ 4\kappa_2 \theta_M m(B(R_v)) M_0 \tilde{M}_1 \|S(t)\|_L^2
\leq -2M_0 \frac{\kappa_1 \zeta_m}{\theta^2_M} \mathcal{L}^\theta[f(t)] + \mathcal{O}(1) \|S(t)\|_L^2
\leq -2M_0 \frac{\kappa_1 \zeta_m}{\theta^2_M} \mathcal{L}^\theta[f(t)] + \mathcal{O}(1)(1 + t)^{-d}.
\]

Now, we apply Lemma 5.1 with

\[
y(t) = \mathcal{L}^\theta[f(t)], \quad \alpha = 2M_0 \frac{\kappa_1 \zeta_m}{\theta^2_M} \quad \text{and} \quad f(t) = \mathcal{O}(1)(1 + t)^{-d},
\]

to derive the desired decay estimate for \( \mathcal{L}^\theta[f(t)] \):

\[
\mathcal{L}^\theta[f(t)] \leq \mathcal{O}(1) \left(1 + t\right)^d, \quad t \geq 0.
\] (5.4)

(ii) Next, we derive

\[
\frac{d}{dt} \mathcal{L}^v[f(t)] \leq -\frac{M_0 \kappa_0}{\theta_M} \mathcal{L}^v[f(t)] + \frac{4P(t)^2 M_0 \kappa_0}{\theta^3_m} \mathcal{L}^\theta[f(t)] + \mathcal{O}(1)(1 + t)^{-d},
\] (5.5)

where \( P(t) \) is a positive quantity defined by

\[
P(t) := \{|v| : v \in \text{supp}_x f(x_*, \cdot, \theta, t)\}.
\]

It follows from Lemma 4.3 that

\[
P(t) \lesssim (1 + t), \quad t \geq 0.
\] (5.6)
Proof of (5.5): We use Lemma 2.1 and (5.2) to get
\[
\frac{d}{dt} \mathcal{L}^v[f(t)] = 2M_0 \frac{dM_2}{dt} - 4M_1 \cdot \frac{dM_1}{dt}
\]
\[= -2M_0 \kappa_0 \int_{\mathbb{R}^d \times (R^+)^2} \psi([x - x_*])(v - v_*) \cdot \left(\frac{1}{\theta_s} \cdot \frac{v}{\theta_s} \right) f(z, t)f(z_*, t)dz_*dz
\]
\[= 2M_0 \kappa_2 \int_{\mathbb{R}^d \times R^+} |v|^2 \mathcal{T}[S](f)dz - 4\kappa_2M_1 \cdot \int_{\mathbb{R}^d \times (R^+)^2} v \mathcal{T}[S](f)dz
\]
\[=: \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}.
\]
Below, we estimate \( \mathcal{I}_{2i} \), \( i = 1, 2, 3 \) separately.

\[\diamond \text{(Estimate of } \mathcal{I}_{21}) \] We can use Young’s inequality to obtain the following:
\[\mathcal{I}_{21} \]
\[= -2M_0 \kappa_0 \int_{\mathbb{R}^d \times (R^+)^2} \psi([x - x_*])(v - v_*) \cdot \left(\frac{1}{\theta_s} \cdot \frac{v}{\theta_s} \right) f(z, t)f(z_*, t)dz_*dz
\]
\[= 2M_0 \kappa_0 \int_{\mathbb{R}^d \times (R^+)^2} \psi([x - x_*]) |v - v_*|^2 \frac{1}{\theta_s} f(z, t)f(z_*, t)dz_*dz
\]
\[\leq 2P(t)M_0 \kappa_0 \int_{\mathbb{R}^d \times (R^+)^2} \psi([x - x_*]) |v - v_*| \theta_s f(z, t)f(z_*, t)dz_*dz
\]
\[\leq M_0 \kappa_0 \int_{\mathbb{R}^d \times (R^+)^2} \psi([x - x_*]) |v - v_*|^2 \frac{1}{\theta_s} f(z, t)f(z_*, t)dz_*dz
\]
\[\leq 4P(t)^2M_0 \kappa_0 \int_{\mathbb{R}^d \times (R^+)^2} \psi([x - x_*]) |\theta - \theta_*|^2 \theta_s f(z, t)f(z_*, t)dz_*dz
\]
\[\leq -\frac{M_0 \kappa_0 \psi m}{\theta_M} \mathcal{L}^v[f(t)] + 4P(t)^2M_0 \kappa_0 \psi m \mathcal{L}^\theta[f(t)].
\]

\[\diamond \text{(Estimate of } \mathcal{I}_{22} + \mathcal{I}_{23}) \] By Lemmas 2.2 and 2.3, we have
\[\mathcal{I}_{22} + \mathcal{I}_{23} \]
\[= 2M_0 \kappa_2 \int_{\mathbb{R}^d \times R^+} |v|^2 \mathcal{T}[S](f)dz + 4\kappa_2 |M_1| \int_{\mathbb{R}^d \times (R^+)^2} |v| \mathcal{T}[S](f)dz
\]
\[\leq 2M_0 \kappa_2 m(|B(0^+)|)R_0^2 \|S(t)\|_{L^\infty}^2 + 8\kappa_2 |M_1| m(|B(0^+)|)R_0 \|S(t)\|_{L^\infty}^2
\]
\[= \mathcal{O}(1) \|S(t)\|_{L^\infty}^2 \leq \mathcal{O}(1)(1 + t)^{-d}.
\]
In (5.7), we combine (5.8) and (5.9) to obtain the desired estimate:
\[\frac{d}{dt} \mathcal{L}^v[f(t)] \leq -\frac{M_0 \kappa_0}{\theta_M} \mathcal{L}^v[f(t)] + \frac{4P(t)^2M_0 \kappa_0}{\theta_m^3} \mathcal{L}^\theta[f(t)] + \mathcal{O}(1)(1 + t)^{-d}. \]
Note that the relations (5.3) and (5.4) yield
\[
\frac{4P(t)^2M_0\kappa_0}{\theta^3_m}L^d[f(t)] + \mathcal{O}(1)(1 + t)^{-d} \lesssim \frac{1}{(1 + t)^{d-2}}. 
\] (5.11)
Again, we apply Lemma 5.1 for (5.10) using (5.11) to derive
\[
L^d[f(t)] \lesssim \frac{1}{(1 + t)^{d-2}}, \quad t \geq 0.
\]

6. Conclusion. In this paper, we had studied a dynamic interplay due to chemotactic movements and temperature field on the flocking in the Cucker-Smale ensemble. Our proposed model consists of two coupled equations. For the evolution of the kinetic density of the Cucker-Smale particles, we employed the kinetic TCS model with a turning operator, which is a collisional Vlasov-McKean type equation, whereas for the evolution of chemical density, we used the standard reaction-diffusion equation. These equations are coupled via the non-local turning operator representing the sudden change of velocities in flocking particles. For the proposed coupled system, we have provided a global well-posedness of strong solutions and presented slow flocking estimates in temperature and velocity for the special choice of reaction term. In particular, for flocking estimate, we have employed a robust Lyapunov functional approach measuring the temperature and velocity variances. Under some a priori setting, we have shown that under suitable setting on the communication weight function and reaction term, the Lyapunov functional tends to zero algebraically fast. Of course, it would be interesting to investigate local flocking situations, e.g. formation of multi-clusters moving with different group velocities. This interesting question will be discussed in a future work.

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