Phase retrapping in a point-like $\varphi$ Josephson junction: the butterfly effect

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We consider a $\varphi$ Josephson junction, which has a bistable zero-voltage state with the stationary phases $\psi = \pm \varphi$. In the non-zero voltage state the phase “moves” viscously along a tilted periodic double-well potential. When the tilting is reduced quasistatically, the phase is retrapped in one of the potential wells. We study the viscous phase dynamics to determine in which well ($-\varphi$ or $+\varphi$) the phase is retrapped for a given damping, when the junction returns from the finite-voltage state back to zero-voltage state. In the limit of low damping the $\varphi$ Josephson junction exhibits a butterfly effect — extreme sensitivity of the destination well on damping. This leads to an impossibility to predict the destination well.

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The term butterfly effect is widely used to denote the extreme sensitivity of complex dynamical systems to initial conditions[1, 2]. The effect puts a clear distinction between determinism and predictability. For example, due to the butterfly effect it is not possible to predict the weather reliably for more than 3–5 days in advance. Although the original work[3] was related to simulation of atmospheric processes, it was discovered later on that in quite a few problems of nonlinear physics a tiny perturbations of initial conditions might lead to completely different final states. In particular, the butterfly effect was observed in simulations of long Josephson junction subjected to an oscillating magnetic field[4].

Consider now a point-like $\varphi$ Josephson junction ($\varphi$-JJ) proposed theoretically[5–7] and recently demonstrated experimentally[8]. This $\varphi$ JJ has a doubly degenerate ground state phase $\psi = \pm \varphi$, which is a result of the unusual Josephson energy profile

$$U_J(\psi) = 1 - \cos(\psi) + \frac{\Gamma_0}{4} [1 - \cos(2\psi)].$$

(1)

The energy $U_J(\psi)$ has a form of a $2\pi$-periodic double-well potential with wells at $\psi = \pm \varphi$, see Fig. 1. The ground state phase $\varphi = \arccos(-1/\Gamma_0)$ . The parameter $\Gamma_0 < 0$ defines the depth of the wells [7, 9]. The potential has two wells per period for $\Gamma_0 < -1$ . Application of a bias current $\gamma$ tilts the potential as shown in Fig. 1.

Since in the zero-voltage state the $\varphi$-JJ is bistable, it is interesting to understand, in which of these two states the phase is retrapped when the $\varphi$ JJ returns from the finite-voltage state to the zero-voltage state upon quasistatic decrease of the tilt (bias current density) $\gamma$. Note, that for conventional 0 or for $\pi$ JJs with only a single energy minimum per period of Josephson energy, such a question does not arise. Earlier[10] we have naively suggested that upon returning from the positive-voltage state to the zero-voltage state the phase is retrapped in the $+\varphi$ state.

This is indeed true for rather large damping[8]. However, at lower damping the behavior is non-trivial and often experimentally seems to be non-deterministic[8].

Here we study the retrapping process of the Josephson phase in a point-like $\varphi$ JJ and demonstrate that at low damping the system exhibits the butterfly effect.

The dynamics of a $\varphi$ JJ can be described by the equation of motion for the phase $\psi(t)$ (see the supplement)

$$\ddot{\psi} + \frac{\partial U_J}{\partial \psi} = \gamma - \alpha \dot{\psi},$$

(2)

where $\Gamma_0 < -1$ is a parameter of the potential defining its depth , $\gamma$ is the bias current density normalized to the average critical current density $\langle j_c(x) \rangle$ , $\alpha$ is the dimensionless damping coefficient (also normalized using

FIG. 1. Tilted periodic double well (Josephson) potential $U(\psi)$ for $\Gamma_0 = -4$ and different values of bias current (tilt) $\gamma$. 

Since in the zero-voltage state the $\varphi$-JJ is bistable, it is interesting to understand, in which of these two states the phase is retrapped when the $\varphi$ JJ returns from the finite-voltage state to the zero-voltage state upon quasistatic decrease of the tilt (bias current density) $\gamma$. Note, that for conventional 0 or for $\pi$ JJs with only a single energy minimum per period of Josephson energy, such a question does not arise. Earlier[10] we have naively suggested that upon returning from the positive-voltage state to the zero-voltage state the phase is retrapped in the $+\varphi$ state.
This model describes well a $\varphi$ JJ made out of a $0-\pi$ JJ with facet lengths $L_0 \leq \lambda_{J,0}$ and $L_\pi \leq \lambda_{J,\pi}$ [5–7, 9], where $\lambda_{J,0}$ and $\lambda_{J,\pi}$ are the Josephson lengths in the 0 and $\pi$ parts, accordingly. For $0-\pi$ JJs with somewhat longer facets, like in experiment[8], it holds qualitatively. For experimental parameters the double well potential $U(\psi)$ calculated numerically is not so deep as in the model (1) as can be seen in Fig. 1 of Ref. 8. This leads only to quantitative rather than qualitative difference in the results obtained.

Equation (2) describes a phase (point-like particle of a unit mass with the coordinate $\psi$) moving viscously (term $-\alpha \dot{\psi}$) in a tilted $2\pi$-periodic double-well potential

$$U(\psi) = U_J(\psi) - \gamma \psi,$$

see Fig. 1. The bias current density $\gamma$ sets the tilt.

The main process that we are interested in here is the dynamics of switching from the finite-voltage state to the zero-voltage state. At $\gamma = 0$ the phase is trapped in one of the wells of the potential $U(\psi)$, i.e., at $\psi = -\varphi$ or at $\psi = +\varphi$. Upon increase of the bias current $\gamma$, the potential $U(\psi)$ tilts and, at some value of the bias current $\gamma$, the phase escapes because the corresponding well disappears. For $\gamma > 0$ the phase escapes from the $-\varphi$ well at $\gamma = \gamma_c^-$ and from the $+\varphi$ well at $\gamma = \gamma_c^+ > \gamma_c^-$ as found earlier[7, 8], see Fig. 1. After escape, the phase slides viscously along the periodic potential. The voltage across the junction is proportional to the velocity of the phase motion $\dot{\psi}(t)$. Further, we start decreasing the bias current density (tilt) $\gamma$ quasistatically. At some $\gamma = \gamma_R$, which depends on the damping $\alpha$, the phase is retrapped in one of the wells. It is this retrapping process, which is the main subject of this study.

Note that in general the damping $\alpha$ is a function of temperature $T$. However, the temperature is also responsible for thermal fluctuations that can be added as an additional stochastic current to the r.h.s. of Eq. (2). In the following we assume that such fluctuations are negligible (zero) and the only effect of temperature is the change in $\alpha$. At the end we discuss shortly the effect of these fluctuations on our results.

To analyze the retrapping process, first, we search the value of the tilt $\gamma_R$ (retrapping current) at a given damping $\alpha$. The retrapping situation corresponds to the trajectory, on which the phase starts with zero velocity at the main maximum of the potential $U(\psi)$ situated at $\psi = \psi_L$, see Fig. 1, slides down viscously, passes two minima and one maximum and arrives to $\psi_R = \psi_L + 2\pi$ with zero velocity. The value of $\psi_L$ is one of the roots of the equation $\partial U/\partial \psi = 0$, i.e., from Eq. (3),

$$\left.\frac{\partial U_J}{\partial \psi}\right|_{\psi=\psi_L} = \gamma,$$

corresponding to the maximum of $U(\psi)$.

Since $\psi_L$ depends on $\gamma$, it is more convenient to fix the tilt $\gamma$ and look for the critical/retrapping value of $\alpha_R(\gamma)$, rather than looking for $\gamma_R(\alpha)$. To find $\psi(t)$ the Eq. (2) was solved for fixed $\gamma$ and $\alpha$ with initial conditions $\psi(0) = \psi_L + \epsilon$ and $\dot{\psi}(0) = 0$ (typically we use $\epsilon \sim 10^{-6}$). The solution was calculated up to the point where either $\psi(t) > \psi_L + 2\pi$ or where $\psi < 0$. In the first case the particle is not trapped for given $\gamma$ and $\alpha$ and moves to the next period of the potential. In the second case the particle is trapped. By varying $\alpha$ we repeat the simulation to find the boundary values $\alpha_R(\gamma)$ between the above two cases with a given accuracy of $10^{-6}$. The resulting plots of already inverted $\gamma_R(\alpha)$ dependences for different values of $\Gamma_0$ are shown in Fig. 2. One can see that the dependences $\gamma_R(\alpha)$ are almost linear.

![FIG. 2. The dependence $\gamma_R(\alpha)$ for different values of $\Gamma_0$. Symbols represent the results of direct numerical simulation. Lines show PT results, given by Eq. (10) for $\alpha \to 0$. The horizontal dashed lines show the values of the depinning current $\gamma_{c-}$ for given $\Gamma_0$, i.e., the current, at which the $-\varphi$ well disappears. For $\gamma > \gamma_{c-}$ the potential has only one $+\varphi$ well and the phase is retrapped there. The vertical dashed line shows the corresponding value of $\alpha_R(\gamma_{c-})$. For JJ with $\alpha > \alpha_R(\gamma_{c-})$ the retrapping current $\gamma_R(\alpha) > \gamma_{c-}$ and potential has only one $+\varphi$ well where the phase is retrapped.](https://example.com/image.png)
where $C = 2 - \frac{\Gamma_0}{\alpha}$ is determined from the initial condition for the retrapping trajectory: $\psi(-\infty) = 0, \psi(-\infty) = \psi_L$. Now, if we turn on the small damping $\alpha$ and the bias $\gamma$, they will lead to dissipation and driving, correspondingly. The dissipated energy $Q$ along the “critical” path from $\psi_L = -\pi$ to $\psi_L + 2\pi = +\pi$ (for $\gamma = 0$) is

$$Q = \alpha \int_{-\pi}^{+\pi} \psi d\psi = \alpha I(\Gamma_0),$$

where, using Eq. (6), we define

$$I(\Gamma_0) = \int_{-\pi}^{+\pi} \sqrt{2[1 + \cos(\psi)] - \Gamma_0 \sin^2(2\psi)} d\psi,$$

which can be calculated numerically for any $\Gamma_0$.

The energy input due to the tilt $\gamma$ is

$$E_\gamma = \gamma [\psi_L + 2\pi - \psi_L] = 2\pi \gamma.$$

In the case of retrapping trajectory, $E_\gamma$ compensates $Q$ and brings the particle exactly to the position $\psi_L + 2\pi$ with zero velocity. Thus, from $E_\gamma = Q$ we get

$$\gamma_R(\alpha) = \frac{I(\Gamma_0)}{2\pi\alpha}.$$

We note that in the limit $\Gamma_0 \to 0$ we have $I(\Gamma_0) \to 8$ and obtain a well known result[11] valid for conventional JJ with sinusoidal CPR, namely $\gamma_R = \frac{4}{\pi} \alpha$. The lines corresponding to $\gamma_R(\alpha)$ dependences (10) are shown in Fig. 2 and agree well with numerical data for $\alpha \to 0$.

Knowing the dependence $\gamma_R(\gamma)$ we now take various values of $\gamma$, take the corresponding $\gamma_R(\gamma)$, put the phase at the turning point $\psi = \psi_R(\gamma) - \epsilon$, see Fig. 1, and follow its time-evolution. The ultimate goal is to see in which well ($-\varphi$ or $+\varphi$) the phase is trapped. The decision about trapping is taken when the velocity $\dot{\psi}$ changes the sign two times in a row on the same side relative to the energy barrier separating the two wells. Examples of the trajectories on the phase plane ($\psi, \dot{\psi}$) are shown in Fig. 3. Thus we get destination well vs. $\gamma_R(\gamma)$ dependence.

Figure 4 shows the destination well ($-\varphi$ or $+\varphi$) as a function of $\alpha$. One can see that, indeed, for large $\alpha$ the phase is trapped in the $+\varphi$ well, as predicted[7] and demonstrated experimentally[8]. However, as $\alpha$ decreases, the destination well changes from $+\varphi$ to $-\varphi$ then back to $+\varphi$ and so on. The intervals of $\alpha$, corresponding to the retrapping in a particular well, become smaller and small even on a logarithmic scale, see Fig. 4. Thus, in the limit of small $\alpha$ the destination well is extremely sensitive to the initial conditions — tiny variation (or fluctuation) of $\alpha$ or $\gamma$ (thermal or electronic noise) results in a global effect — retrapping in a different well. Thus our $\varphi$ JJ exhibits the butterfly effect.

In our case, the butterfly effect prevents one to forecast, in which well the phase will be retracted in experiment in the limit of small $\alpha$. In fact, already in the first

![FIG. 3. The retrapping trajectories in the phase plane ($\psi, \dot{\psi}$) for $\Gamma_0 = -3$ and different tilt $\gamma$. The phase starts at $\psi_L$, where $U(\psi)$ has a maximum, with $\dot{\psi} = 0$ and arrives to $\psi_R = \psi_L + 2\pi$ (next $U(\psi)$ maximum) with $\dot{\psi} = 0$. Then the phase falls back and is trapped in one of the minima of the $U(\psi)$.](image-url)
rigorous treatment of noise will be presented elsewhere.

At the end we would like to mention an interesting detail. When simulating a retrapping dynamics, we also have counted how many times \( N \) the phase crossed the barrier separating the wells before being trapped in one of the wells. In fact the well (+\( \varphi \) or −\( \varphi \)) plotted in Fig. 4 is just \( \varphi[1−2(N \bmod 2)] \). Fig. 5 shows \( N(\alpha) \) plots for different values of \( \Gamma_0 \). Note that \( N \) is an integer so it changes step-wise, as it is well visible for large \( \alpha \). For small \( \alpha \) (large \( N \)) the dependence looks almost continuous and can be well approximated by \( N \approx C_\alpha/\alpha \). The coefficient \( C_\alpha \) depends on \( \Gamma_0 \). In supplement it is proven analytically that \( NC_\alpha/\alpha \) for any potential \( U(\psi) \) in the PT limit \( \alpha \ll 1 \). In our case, using \( \psi_{\text{dec}} = \arccos[(2 + \Gamma_0)/\Gamma_0] \), see Fig. 1, and \( \psi(E_{\text{max}}) = \pi \) we get

\[
C_\alpha^{\text{PT}} = \int_{\psi_{\text{dec}}}^{\pi} \frac{\partial U(\psi_{\text{st}})}{\partial \psi_{\text{st}}} \frac{d\psi_{\text{st}}}{W(\psi_{\text{st}})},
\]

where

\[
W(\psi_{\text{st}}) = \int_{-\psi_{\text{st}}}^{+\psi_{\text{st}}} \sqrt{2[U(\psi_{\text{st}}) - U(\psi)]} d\psi.
\]

The values of \( C_\alpha \) obtained from direct simulations as well as \( C_\alpha^{\text{PT}} \) calculated using Eq. (11) are summarized in Tab. I together with other key numbers.

| \( \Gamma_0 \) | \( \gamma_{\text{c}} \) | \( \alpha_R(\gamma_{\text{c}}) \) | \( C_\alpha(\Gamma_0) \) | \( C_\alpha^{\text{PT}}(\Gamma_0) \) |
|---|---|---|---|---|
| −1.5 | 0.153 | 0.100 | 0.521 | 0.522 |
| −2.0 | 0.369 | 0.229 | 0.412 | 0.412 |
| −3.0 | 0.840 | 0.492 | 0.313 | 0.314 |
| −4.0 | 1.327 | 1.327 | 0.263 | 0.264 |

**TABLE I.** The values of key quantities.

In conclusion, we have studied the retrapping of the phase in a point-like \( \varphi \) JJ upon transition to zero-voltage state. For given damping \( \alpha \), we have calculated the retrapping current \( \gamma_R \) and the destination well, where the phase is trapped. For large \( \alpha \) it is always a deeper well (\( +\varphi \) for \( \gamma > 0 \)). However, as \( \alpha \) decreases, the dependence of the destination well of \( \alpha \) is an oscillating function, with oscillations (switching of the destination well) happening faster and faster even on the logarithmic scale, see Fig. 4. Thus, at \( \alpha \to 0 \) a tiny variation of \( \alpha \) or \( \gamma \) (noise) leads to a different destination well, i.e., to a butterfly effect. Detailed treatment of the noise will be presented elsewhere.

The butterfly effect at small damping does not allow to manipulate the \( \varphi \) JJ by means of the bias current as described earlier[8, 10]. Simultaneously, in this regime one can use a \( \varphi \) JJ as a random number generator (coin/dice) giving the output of −\( \varphi \) or +\( \varphi \) randomly. The extreme sensitivity may also be exploited in amplifiers or detectors as well as for the investigation of the fine details of the JJ dynamics itself. In the quantum regime the dynamics described here may lead to extremely strong mixing/entanglement of the states \(|−\varphi\rangle \) and \(|+\varphi\rangle \).

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