On Bohr–Sommerfeld bases

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Abstract

This paper combines algebraic and Lagrangian geometry to construct a special basis in every space of conformal blocks, the Bohr–Sommerfeld (BS) basis. We use the method of Borthwick–Paul–Uribe [BPU], whereby every vector of a BS basis is defined by some half-weighted Legendrian distribution coming from a Bohr–Sommerfeld fibre of a real polarization of the underlying symplectic manifold. The advantage of BS bases (compared to bases of theta functions in [T1]) is that we can use information from the skillful analysis of the asymptotics of quantum states. This gives that Bohr–Sommerfeld bases are unitary quasi-classically. Thus we can apply these bases to compare the Hitchin connection [H] with the KZ connection defined by the monodromy of the Knizhnik–Zamolodchikov equation in combinatorial theory (see, for example, Kohno [K1] and [K2]).

1 Degree 0 cycles

Let $X$ be a Kähler manifold with Kähler form $\omega$ and $L$ an algebraic geometric polarization with $c_1(L) = [\omega] \in H^2(X, \mathbb{Z})$. Suppose in addition that the canonical class is even:

$$K_X = 0 \mod 2,$$

and fix a metaplectic structure on $X$, that is, a line bundle $L_{K/2}$ such that $L_{K/2}^{\otimes 2} = L_{K_X}$. Then $L^k \otimes L_{K/2}$ is a holomorphic line bundle on $X$ for any $k \in \mathbb{Z}^+$, here called the level. We get spaces

$$\mathcal{H}^k = H^0(X, L^k \otimes L_{K/2}) \quad \text{for } k \geq 0$$

(1.1)
of holomorphic sections of $L^k \otimes L_{K/2}$; here $I$ denotes the complex structure of $X$. Thus if $\mathcal{M}$ is a family of polarized complex structures on the underlying smooth compact manifold $X$, the spaces (1.1) define holomorphic vector bundles

$$\mathcal{H}^k \to \mathcal{M},$$

with fibres (1.1) for $k \gg 0$.

For the applications we have in mind, it is sufficient to work under the following restriction: there exists an integer $d$ such that

$$L_{K/2} = L^d.$$ (1.3)

Thus twisting by $L_{K/2}$ just shifts the level $k$ to $k + d$. We use the hyperplane section class

$$c_1(L) = [\omega] \in H^2(X, \mathbb{Z}),$$

to define the degree of a cycle $C \subset X$: it is the integer

$$\deg C = [C] \cdot [\omega]^{(\dim C)/2} = \int_C (\omega|_C)^{(\dim C)/2}.$$

For any holomorphic (effective algebraic) subcycle $C$

$$\dim C > 0 \implies \deg C > 0.$$  

More precisely, there is an exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

and restricting global sections (constants) to $C$ defines a distinguished line in the space of sections:

$$\mathcal{C} = H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_C).$$

Our line bundle $L^{k+d}$ defines a rational (that is, meromorphic) map $X \to \mathbb{P}(\mathcal{H}_I^{k+d})^\vee$, sending $x \in X$ to the hyperplane $H^0(\mathcal{I}_x \otimes L^{k+d}) \subset H^0(L^{k+d})$ of sections vanishing at $x$. For an open set $U$, a trivialization of $L|_U$ is given by a section $s_U$ that is everywhere nonvanishing on $U$, and $x \in U$ defines a
covector of $H^0(L^k)$ of evaluation at $x$, using $s_U$ to identify the fibre at $x$ with $\mathbb{C}$. More technically, there is an exact sequence

$$0 \to \mathcal{I}_x \otimes L^{k+d} \to L^{k+d} \to \mathcal{O}_x \to 0 \quad (1.4)$$

where the epimorphism is the restriction homomorphism. Part of its cohomology sequence

$$H^0(\mathcal{I}_x \otimes L^{k+d}) \to H^0(L^{k+d}) \to \mathbb{C} \to H^1(\mathcal{I}_x \otimes L^{k+d}) \quad (1.5)$$

shows that $H^0(\mathcal{I}_x \otimes L^{k+d})$ is indeed a hyperplane if $H^1(\mathcal{I}_x \otimes L^{k+d}) = 0$. Now by Serre’s classical Theorems A and B, $H^1(\mathcal{I}_x \otimes L^{k+d}) = 0$ for $k \gg 0$, and we have a map

$$\mathbb{P}\varphi_k: X \to \mathbb{P}H^0(L^{k+d}),$$

(1.6)

which is an embedding for $k \gg 0$. This standard construction of algebraic geometry reduces the study of $X$ to projective geometry.

The holomorphic structure of $L$ admits a Hermitian connection $a_L$, defined by the complex structure, with curvature form $2\pi i\omega$. On the other hand, our Kähler metric defines a Levi-Civita Hermitian connection $a_{LC}$ on $L_{K/2}$. We suppose also that

$$a_{L,d} = a_{LC} \quad (1.7)$$

(see (1.3)).

We can now define a Hermitian form on $H^0(L^{k+d})$ in two steps: first, every section $\tilde{s}$ of $L^{k+d}$ is locally of the form $s \cdot hF$, where $s$ is a section of $L^k$ and $hF$ a section of $L_{K/2}$ (a half-form). For two such sections $\tilde{s}_1 = s_1 \cdot hF_1$ and $\tilde{s}_2 = s_2 \cdot hF_2$, set

$$\langle \tilde{s}_1, \tilde{s}_2 \rangle = \int_X (s_1, s_2) \cdot (hF_1, hF_2). \quad (1.8)$$

We get an identification of vectors and covectors:

$$H^0(L^{k+d}) = H^0(L^{k+d})^*.$$  

In particular, a trivialization over an open $U$ sends

$$\varphi_U: U \to H^0(L^{k+d})$$
and the projectivization of this is just the complex conjugate of (1.6):

\[ \mathbb{P}\varphi_U: U \to \mathbb{P}H^0(L^{k+d}). \]  

(1.9)

In the set-up of complex quantization, vectors in \( H^0(L^{k+d}) \) are called states, and vectors \( \varphi(x) \) for \( x \in X \) are called coherent states (recall that states are distributions, not functions).

From now on we can forget about what \( K/2 \) means geometrically, and consider a twisting by the metaplectic structure as a shift of level.

Now inverting the usual way of thinking, we ask whether there are submanifolds of \( X \) of degree 0 that define hyperplanes in the spaces of sections \( H^0(X, L^k) \), and what kind of submanifolds these are. We strengthen the condition \( \text{deg} \ C = 0 \) to \( \omega|_C = 0 \); in other words, every such submanifold \( \mathcal{L} \) must be isotropic with respect to the Kähler form \( \omega \). Thus a maximal dimensional submanifold must be Lagrangian. Just as an algebraic subvariety may be singular, we do not need to restrict ourselves to Lagrangian submanifolds: in what follows we consider Lagrangian cycles a priori admitting singularities. The main property of any such cycle is

*it can’t be contained in any proper algebraic subvariety*

(in particular, in a divisor). Thus any holomorphic object is uniquely determined by its restriction to a Lagrangian cycle \( \mathcal{L} \). Thus restrictions to \( \mathcal{L} \) can serve as boundary conditions for holomorphic sections of line bundles with curvature proportional to \( \omega \).

**Remark** Geometrically, if we consider Lagrangian cycles as supports of boundary conditions for holomorphic objects, they have the minimal possible dimension. Usual boundary conditions deal with boundaries of complex domains of real codimension 1. Thus it is only for Riemann surfaces that Lagrangian boundary conditions coincide with the usual boundary conditions. In this case, in the modern theory of integrable systems the restriction of holomorphic objects to a small circle around a point reduces many analytical problems to algebraic geometry of curves (see for example the survey [DKN]). Thus we should add the role of Lagrangian submanifolds as boundary conditions for holomorphic objects to Alan Weinstein’s proclamation [Wei1], p. 5. It seems reasonable to expect that restrictions to Lagrangian submanifolds give a higher dimensional generalization of the modern version of the theory of integrable systems.
If we forget for a minute the complex structure $I$ on $X$, the polarization $L$ gives us a quadruple

$$(X, \omega, L, a_L),$$

where $\omega$ is the Kähler form and $a_L$ a Hermitian connection on $L$ with curvature form

$$F_a = 2\pi i \cdot \omega,$$

of Hodge type $(1, 1)$ for the given holomorphic structure on $L$. Thus the pair $(X, \omega)$ is a symplectic manifold, the phase space of a mechanical system. There are no invariants of an embedding of a Lagrangian submanifold $L$ in a symplectic manifold. There are two ways of getting invariants:

1. considering families of Lagrangian manifolds admitting invariants (in particular limit singular subcycles); or
2. giving submanifolds an additional structure (such as a section of some bundle or an Hermitian connection on the trivial line bundle).

The restriction of the pair $(L^k, a_{L^k})$ to a Lagrangian cycle $\mathcal{L}$ gives this type of additional structure. It defines the space of covariant constant sections:

$$H^0_\alpha((L^k, a_{L^k})|_{\mathcal{L}})$$

which can be nonzero, as for points. Indeed, restricting to any Lagrangian submanifold $\mathcal{L}$ gives a topologically trivial line bundle on $\mathcal{L}$ with flat connection. A connection of this type is defined by its monodromy character

$$\chi: \pi_1(\mathcal{L}) \rightarrow U(1),$$

and admits a covariant constant section (as for restriction to a point) if and only if this character is trivial.

**Definition 1.1** A Lagrangian cycle $\mathcal{L}$ is a level $k$ Bohr–Sommerfeld ($\text{BS}_k$) cycle if the character $[\mathcal{L}]$ for $(L^k, a_{L^k})$ is trivial.

In particular, just as for points,

$$\mathcal{L} \text{ is BS}_k \implies H^0_\alpha((L, a_L)|_{\mathcal{L}}) = \mathbb{C}.$$
Moreover such a section defines a *trivialization* of the restriction $L_k^{k+d}|_{\mathcal{L}}$, which identifies $C^\infty$ sections with complex valued functions on $\mathcal{L}$:

$$\Gamma(L_k^{k+d}|_{\mathcal{L}}) = C^\infty_C(\mathcal{L}).$$

Thus the restriction to $\mathcal{L}$ defines an embedding

$$\text{res}: H^0(L^{k+d}) \hookrightarrow C^\infty_C(\mathcal{L}). \quad (1.12)$$

**Definition 1.2** The image

$$\text{res}(H^0(L^k)) = \mathcal{H}_L \subset C^\infty(\mathcal{L}) \quad (1.13)$$

is called the *analog of the Hardy space*.

Now recall that our space $\mathcal{H}_I^k$ (1.1) is the space of twisted holomorphic half-forms:

$$\mathcal{H}_I^k = H^0(L^k \otimes L_{K/2}).$$

To preserve the geometric meaning, fix a half-form $hF$ on $\mathcal{L}$; we call a pair $(\mathcal{L}, hF)$ a *half-weighted Lagrangian cycle* or a Lagrangian cycle marked with a half-form. Now we can identify the space of functions with the space of half-forms

$$\Gamma(L^{k+d})|_{\mathcal{L}} = C^\infty_C(\mathcal{L}) \cdot hF = \Gamma(\Delta^{1/2}), \quad (1.14)$$

where $\Delta$ is the complex volumes bundle on $\mathcal{L}$. This space is selfadjoint with respect to a Hermitian form like (1.8).

Following Borthwick, Paul and Uribe [BPU], we can construct a distribution in some completion of $C^\infty(\mathcal{L}) \cdot hF$. Its restriction to the image of $\mathcal{H}_I^k$ gives a covector or a state. The BPU method uses usual codimension 1 boundary conditions rather than Lagrangian boundary conditions, and the original Hardy spaces of strictly pseudoconvex domains rather than the analog of Hardy space (1.13). We refer the reader to the beautiful paper [BPU] for the details, which we cannot reproduce here; this paper realizes a very large program. The construction is following:

1. Our Hermitian connection on $L^*$ defines a contact structure on the unit circle bundle $P$ of $L^*$. 


(2) The disc bundle in $L^*$ is a strictly pseudoconvex domain, and there is the Szegö orthogonal projector $\Pi: L^2(P) \to \mathcal{H}$ to the Hardy space of boundary values of holomorphic functions on the disc bundle.

(3) The contact manifold $P$ is a principal $U(1)$-bundle, and the natural $U(1)$-action on $P$ commutes with $\Pi$ and gives a decomposition $\mathcal{H} = \bigoplus_k H^0(L^{k+d})$ of the Hardy space.

(4) If we fix a metaplectic structure on $P$, we can lift every BS$^{k+d}$ submanifold to a Legendrian submanifold $\Lambda \subset P$ over it, marked with the lifted half-form $hF$.

(5) $\Lambda$ has an associated space of Legendrian distributions of order $m$, which is the Szegö projection of space of conormal distributions to $\Lambda$ of order $m + \frac{1}{2} \dim X$ (see [BPU], 2.1).

(6) A half-form on $\Lambda$ is identified with the symbol of a Legendrian distribution of order $m$ (see [BPU], 2.2); thus at the level of symbols, all Legendrian distributions look like delta functions or their derivatives.

(7) For a Legendrian submanifold $\Lambda$ with a half-form we fix the Legendrian distribution of order $\frac{1}{2}$ with symbol $hF$ which is the Szegö projection of the delta function $\delta_\Lambda$.

In summary, we have:

(1) For every lift $\Lambda \subset P$ of a BS$^{k+d}$ submanifold $\mathcal{L}$ marked with a half-form $hF$ we have a vector

$$BPU_{k+d}(\Lambda, hF) = \Pi^k_{hF}(\delta_\Lambda) \in H^0(L^{k+d}), \quad (1.15)$$

where $\Pi^k_{hF}$ is the Szegö projection to the $(k + d)$th component of the Hardy space of the distribution with symbol $hF$.

(2) Every such lifting is defined up to $U(1)$-action on $P$; thus a pair $(\mathcal{L}, hF)$ defines a point of the projectivization

$$BPU_{k+d}(\mathcal{L}, hF) = \mathbb{P}(\Pi^k_{hF}(\delta_\Lambda)) \in \mathbb{P}H^0(L^{k+d}). \quad (1.16)$$

Our observations are the following:
(1) This construction holds literally in the case that $\mathcal{L}$ has the structure of a smooth orbifold.

(2) If (1.3) holds, there exists a canonical geodesic lifting (see Section 2). Thus $(\mathcal{L}, hF)$ defines a section of $L^{k+d}$.

The next step is the Analog of Serre’s Theorems A and B proved in [BPU], Section 3:

**Theorem 1.1** If $k$ is large enough then $\text{BPU}_k(\Lambda, hF) \neq 0$.

There are two or three canonical ways to give any Lagrangian submanifold $\mathcal{L}$ a half-form:

(1) If $X$ is a Kähler manifold with a metaplectic structure. Then this metaplectic structure defines a metalinear structure on $\mathcal{L}$ (see for example Guillemin [Gu]), and the Kähler metric $g$ defines a half-form $hF_g$ on $\mathcal{L}$. (This method is of course the most important for our applications.)

(2) The graph of a metasymplectomorphism with symplectic volume as square of the half-form.

(3) If $\mathcal{L}$ admits a free torus action.

We have seen that half-weighted Bohr–Sommerfeld orbifolds look geometrically like points. Let us denote by $\mathcal{L}M$ the family of all cycles that are Lagrangian with respect to $\omega$. A polarization $(\mathcal{L}, a_{\mathcal{L}})$ defines a subspace

$$\text{BS}^{k+d}_l(\mathcal{L}) \subset \mathcal{L}M$$

of Bohr–Sommerfeld Lagrangians; we decorate it by the index $l = [\mathcal{L}] \in H^{\dim_C X}(X, \mathbb{Z})$ (the cohomology class of the cycles), and by the level $k$ (which we sometimes omit). This space breaks up into connected components according to the topological type of generic Lagrangian cycles. Recall that by the Darboux–Weinstein theorem we can identify a small tubular neighborhood of $\mathcal{L}$ with a neighborhood of the zero section of the cotangent bundle of $\mathcal{L}$, and any Lagrangian cycle in this neighborhood can be identified with a closed 1-form on $\mathcal{L}$. We get a system of “charts” for $\mathcal{L}M$, and the tangent space

$$T\mathcal{L}M_{\mathcal{L}} = \{\alpha \in \Omega_{\mathcal{L}} \mid d\alpha = 0\}$$
is the space of closed 1-forms on $\mathcal{L}$. On the other hand, the periods of these forms give infinitesimal deformations of the character (1.11). Thus if $\mathcal{L} \in \text{BS}(L)$, the tangent space is the space of exact forms on $\mathcal{L}$:

$$T_{\text{BS}(X,L)} = \{ \alpha \in \Omega_1 | \alpha = \partial f \} = C^\infty(\mathcal{L})/\mathbb{R}.$$  

(1.19)

Thus we get the following result.

**Proposition 1.1** The normal space of $\text{BS}(L)$ in $\mathcal{LM}$ at $L$ is

$$\text{NBS}(L)_{\mathcal{L}} = H^1(\mathcal{L}, \mathbb{R}).$$  

(1.20)

**Remark** The subspace BS($L$) of the space of all Lagrangian cycles is a partial case of isodrastic deformations of Lagrangian cycles (see [Wei2]).

Now for every family of Lagrangian cycles with base $B$, for a smooth element $\mathcal{L}$ we have the “Kodaira–Spencer” map

$$\text{KS}: TB_{\mathcal{L}} \to H^1(\mathcal{L}, \mathbb{R}).$$

The base $B$ of any family of Lagrangian cycles contains the subspace of BS cycles

$$B \cap \text{BS}^k(L) \subset B.$$

**Corollary 1.1** The codimension of this subset at a smooth cycle $\mathcal{L}$ is

$$\text{codim}(B \cap \text{BS}(L)) = b_1(\mathcal{L}) - \text{corank KS}.$$

We write

$$h\text{WBS}^{k+d}_L$$

(1.21)

for the family of half-weighted BS cycles marked with half-forms, that is, the set of pairs $\{(\mathcal{L}, hF)\}$. Then the BPU construction gives a “rational” map

$$\mathbb{P}\varphi_k : h\text{WBS}^{k+d}_L \to \mathbb{P}H^0(L^{k+d})_*$$

(1.22)

which is regular if $k \gg 0$ (just as the map (1.6) for points).

There are three types of finite dimensional families of Lagrangian cycles where we can expect the existence of a finite set of BS cycles.
Example 1. Real polarization  A real polarization of \((S, \omega, L, a)\) is a fibration
\[ \pi: S \to B, \]
(1.23)
such that \(\omega|_{\pi^{-1}(b)} = 0\) for every point \(b \in B\) and for generic \(b\) the fibre \(\pi^{-1}(b)\) is a smooth Lagrangian.

Thus if we consider the pair \((S, \omega)\) as the phase space of a mechanical system, it admits a real polarization if and only if it is completely integrable. In the compact case a generic fibre is a \(n\)-torus \(T^n\) (where \(2n = \dim \mathbb{R} X\)), and \(\dim B = n\); thus
\[ \dim B = \text{rank} H^1(T^n, \mathbb{R}) \implies \dim B \cap \text{BS}(L) = 0. \]
(1.24)

Remark  A priori, there is no consistent way to introduce a preferred orientation in the space of fibres of a real polarization. A metaplectic structure provides it in some cases.

Example 2. Moduli spaces of spLag cycles  Let \(\mathcal{L}\) be a special Lagrangian cycle and \(\mathcal{M}^{[\mathcal{L}]\text{c}}\) the “moduli space” of all deformation of \(\mathcal{L}\) as a special Lagrangian cycle in \(X\) (see \([13]\)). Then by McLean’s theorem the tangent space \(T \mathcal{M}^{[\mathcal{L}]\text{c}} = H^1(\mathcal{L}, \mathbb{R})\) is the space of harmonic 1-forms and the Kodaira–Spencer map has corank \(\text{KS} = 0\). Therefore, by definition, every smooth BS cycle must be infinitesimally rigid, so that
\[ \dim(\mathcal{M}^{[\mathcal{L}]\text{c}} \cap \text{BS}_k)(L) = 0. \]
(1.25)

In particular, if \(X\) is a Calabi–Yau threefold polarized by a Ricci flat metric with fixed complex orientation we have the system of functions of any level \(k\)
\[ H^3(X, \mathbb{Z}) \to \mathbb{Z} \]
(1.26)
sending a cohomology class \(l \in H^3(X, \mathbb{Z})\) to the number
\[ \#(\mathcal{M}^d \cap \text{BS}_k(L)) \]
(see \([14]\) and \([12]\) for the relation of this function with the Casson–Donaldson invariant).
Now fixing a half-form on all the BS$^{k+d}$ cycles of these families, we get a finite set of points in $\mathbb{P}H^0(L^{k+d})$.

Moreover in many cases this collection of points in $\mathbb{P}H^0(L^{k+d})$ can be lifted up to finite ambiguity to a basis of the vector space $H^0(L^{k+d})$ (as predicted in [BPU], Remark on p. 400).

It was realized by Poincaré for the case $X = C$ is an algebraic curve of genus $g > 1$.

**Example 3. Relative Poincaré series** Let $C$ be an algebraic curve of genus $g > 1$ with a fixed Spin$^C$ structure, that is, with a fixed theta structure $L$ such that

$$L^2 = L_{K_C}.$$

Then $L$ defines a metaplectic structure and a polarization.

This line bundle has a Hermitian connection with square the Levi-Civita Hermitian connection on $T^*C$. Then every 1-cycle $\mathcal{L}$ is Lagrangian.

$$\mathcal{L} \in \text{BS}^2(L) \text{ if and only if it is geodesic.}$$

If $k > 1$ then it is BS$^k$ if and only if it is $k$-geodesic, that is, its holonomy is a $k$th root of unit.

Parametrizing such a cycle by arclength gives a half-form $hF$ on it. Thus it defines the Poincaré series of $H^0(L^{2k})$ as an automorphic form given by the relative Poincaré series (see [BPU], Section 4).

In the same vein, a BS$^{k+d}$ cycle $\mathcal{L}$ defines a section of $L^{k+d}$ if the canonical class $[K_X] = dc_1(L)$ with $d \in \mathbb{Z}$ and the Hermitian connection $a$ is “proportional” to the connection on $\det T^*X$ induced by the Levi-Civita connection (see below).

**Remark** As in the original proof of Serre’s Theorems A and B, we can’t avoid some technical work in functional analysis. Our aim is to localize these techniques in one place, the proof of Theorem 1.1. After this the theory becomes a combination of projective algebraic geometry and Lagrangian geometry. We call this hybrid aLag geometry.
2 BPU construction, geodesic lifting and geometric quantization

For our applications, we extend slightly the BPU construction described in (1.15–16) of the previous section. We must repeat some of the details. We stay in the situation of the starting point of Section 1: let $X$ be a Kähler manifold with a polarization $L$ having a Hermitian connection with the Kähler form as curvature form.

Consider the principal $U(1)$-bundle $P$ of the dual line bundle $L^*$, the unit circle bundle in $L^*$. Let

$$D \subset L^*, \quad \partial D = P$$

be the unit disc subfibration with boundary. Our $L$ is positive, hence $D$ is a strictly pseudoconvex domain. The Hermitian connection on $L^*$ is given by 1-form $\alpha$ on $P$ which defines a contact structure on $P$ with volume form

$$\frac{1}{2\pi} \alpha \wedge d\alpha^n,$$

where $n = \dim \mathbb{C} X$.

The null space of $\alpha$ at a point $p \in P$ is the maximal complex subspace of the tangent space.

The Hardy subspace

$$\mathcal{H}_I \subset L^2(P)$$

consists of boundary values of holomorphic functions on $D$. We have the Szegö orthogonal projector

$$\Pi_I : L^2(P) \to \mathcal{H}_I. \quad (2.2)$$

The natural action of $U(1) = \text{Hom } P$ on $P$ as a principal bundle commutes with $\Pi_I$ and decomposes the space $\mathcal{H}_I$ as a Hilbert direct sum of isotypes:

$$\mathcal{H}_I = \bigoplus_{k=0}^{\infty} (\mathcal{H}_I^k = H^0(L^{k+d})), \quad (2.3)$$

with only positive characters. Thus

$$\Pi_I = \bigoplus_{k=0}^{\infty} \Pi_k. \quad (2.4)$$
Our first addition to [BPU] is the following: suppose that the group \( G \) acts on \( X \) preserving \( \omega \) and \((L, a_L)\). Then there exists a central extension \( \tilde{G} \):
\[
1 \to U(1) \to \tilde{G} \to G \to 1,
\]
(2.5)
where the centre \( U(1) = \text{Hom} P \) acts on \( P \) as a group of a principal bundle. This action induces a natural representation
\[
\rho: \tilde{G} \to \text{Op}(L^2(P))
\]
(2.6)
on the operator algebra of the space of functions.

If the transformations of \( G \) preserve our complex structure \( I \) then the projector \( \Pi_I \) (2.4) defines a representation
\[
\Pi_I \circ \rho \circ \Pi_I: \tilde{G} \to \text{Op}(\mathcal{H}_I)
\]
(2.7)
which commutes with the action of \( U(1) \). So this action decomposes this representation as a Hilbert direct sum of isotypes
\[
\rho^k_I = \Pi_k \circ \rho \circ \Pi_k: \tilde{G} \to \text{End} H^0(L^k).
\]
(2.8)
These representations can be projectivized
\[
\mathbb{P}\rho^k_I: G \to \text{Hom}\mathbb{P}H^0(L^k).
\]
Recall that our \( X \) and the principal bundle \( P \) have given metaplectic structures.

Now if \( \mathcal{L} \) is a half-weighted BS orbifold, the complex conjugate of a covariant constant section gives a lift of it to a Legendrian cycle \( \Lambda \) on \( P \) marked with half-forms \((\Lambda, hF)\), and the construction (1.15) defines a section
\[
\text{BPU}_{1-d}(\Lambda, hF) \in H^0(X, L).
\]
In the same vein, we get lifts \( \Lambda_{k+d} \) of \( \mathcal{L} \) from \( \text{BS}^{k+d}(L) \) to the Legendrian cycle on \( P \) which is a cyclic \((k + d)\)-cover of it and the system of sections
\[
\text{BPU}_{k+d}(\Lambda_{k+d}, hF) \in H^0(X, L^{k+d}).
\]
(2.9)
The Lagrangian cycle \( \mathcal{L} \) can be reconstructed from the set of its BPU images as the quasi-classic limit as \( 1/k = \text{Planck’s constant} \to 0 \): the wave
fronts of distributions concentrate on $\mathcal{L}$ (see [BPU] and the references given there).

Now for two Lagrangian cycles ($\mathcal{L}_1, hF_1$) and ($\mathcal{L}_2, hF_2$), the asymptotic behaviour of the scalar product $\langle \text{BPU}_k(\mathcal{L}_1, hF_1), \text{BPU}_k(\mathcal{L}_2, hF_2) \rangle$ (see (1.8)) can be computed in terms of the intersection $\mathcal{L}_1 \cap \mathcal{L}_2$ (see [BPU]). In particular,

$$\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset \implies \text{BPU}_k(\mathcal{L}_1, hF_1) \perp \text{BPU}_k(\mathcal{L}_2, hF_2)$$

asymptotically as $k \to \infty$. (For the orbifold case these asymptotics are somewhat weaker, but are still quite expressive for geometric corollaries). This asymptotic technique comes from the physical interpretation of this setup as the “classical” (= pre-BRST) geometric quantization (GQ for short).

More precisely the asymptotic analysis of quantum states gives

$$\langle \text{BPU}_k(\mathcal{L}, hF), \text{BPU}_k(\mathcal{L}, hF) \rangle^k \sim \left(\frac{k}{\pi}\right)^{\frac{1}{2}\dim X} \int_{\mathcal{L}} \|hF\|^2 + O(k^{\frac{1}{4}(\dim X-1)}), \quad (2.11)$$

and if $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$ then

$$\langle \text{BPU}_k(\mathcal{L}_1, hF_1), \text{BPU}_k(\mathcal{L}_2, hF_2) \rangle^k \sim \left(\frac{k}{\pi}\right)^{\frac{1}{2}(\dim X-1)} + O(k^{\frac{1}{4}(\dim X-2)}). \quad (2.12)$$

From this it easy to see that

**Proposition 2.1** Let $\pi: X \to B$ be any real polarization (see Section 1, Example 1). Then for $k \gg 0$, the BPU vectors in $H^0(L^{k+d})$ span a subspace of dimension

$$\text{rank} \, \langle \text{BPU}_{k+d}(BS^{k+d}(L) \cap B) \rangle = \#(BS^k(L) \cap B).$$

**Remark** A more sophisticated analysis of the asymptotics of quantum states extends this observation as follows: let

$$\mathcal{L}_1, \ldots, \mathcal{L}_{N_{\text{max}}} \subset BS^{k+d}(L)$$

be a maximal collection of disjoint $BS^{k+d}$ cycles. Then

$$N_{\text{max}} \leq \text{rank} \, H^0(L^{k+d}),$$

and the right-hand side is given by the Riemann–Roch theorem.
Geodesic lifting

The lifting of a BS Lagrangian cycle on $X$ to a Legendrian cycle on $R$ we have described is defined up to the natural $U(1)$-action on $R$ and the states $\text{BPU}_k(\mathcal{L}, \hbar F)$ are defined up to scaling. But in our applications we can do this almost canonically (up to a finite ambiguity) and get an actual basis of $H^0(L^{k+d})$.

To describe this almost canonical lifting we must consider the Lagrangian Grassmannian of the tangent bundle of $X$ as described in [T3]. Pointwise, the tangent space $(TX)_x$ at a point $x \to X$ is $\mathbb{C}^n$ with the constant symplectic form $\langle \ , \ \rangle = \omega_x$ and the constant Euclidean metric $g_x$, giving the Hermitian triple $(\omega_x, I_x, g_x)$. Define the Lagrangian Grassmannian $(\Lambda\uparrow)_x = \Lambda\uparrow(TX)_x$ to be the Grassmannian of oriented Lagrangian subspaces in $(TX)_x$. Taking this space over every point of $X$ gives the oriented Lagrangian Grassmannization of $TX$

$$\pi : \Lambda\uparrow(TX) \to X \text{ with } \pi^{-1}(x) = (\Lambda\uparrow)_x.$$  \hfill (2.13)

A complex structure $I_x$ on $(TX)_x$ gives the standard identification

$$(\Lambda\uparrow)_x = \text{U}(n)/\text{SO}(n).$$  \hfill (2.14)

This space admits a canonical map

$$\det : (\Lambda\uparrow)_x \to \text{U}(1) = S^1_x \text{ sending } u \in \text{U}(n) \to \det u \in \text{U}(1) = S^1.$$  \hfill (2.15)

Taking this map over every point of $X$ gives the map

$$\det : \Lambda\uparrow(TX) \to S^1(L-K),$$  \hfill (2.16)

where $S^1(L-K)$ is the unit circle bundle of the line bundle $\bigwedge^n TX = \det TX$, with first Chern class

$$c_1(\det TX) = -K_X,$$

where $K_X$ is the canonical class of $X$ (see for example [12] and [13]).

We have already noted that our Lagrangian cycles does not usually have an orientation defined a priori. Thus we must consider the Lagrangian Grassmannian $\Lambda(TX)$ forgetting orientations. Then we get a map

$$\det : \Lambda(TX) \to S^1(L-K/2)$$
in place of (2.16).

Now for every oriented Lagrangian cycle \( \mathcal{L} \subset X \), we have the Gaussian lift of the embedding \( i: \mathcal{L} \to X \) to a section

\[
G(i): \mathcal{L} \to \Lambda(TX)|_{\mathcal{L}},
\]

(2.17)

sending \( x \in \mathcal{L} \) to the subspace \( T_{\mathcal{L}}x \subset (TX)_x \). The composite of this Gauss map with the projection \( \det \) gives the map

\[
\det \circ G(i): \mathcal{L} \to S^1(L_{-K/2})|_{\mathcal{L}}.
\]

(2.18)

Thus every Lagrangian cycle \( \mathcal{L} \) defines a Legendrian subcycle

\[
\Lambda = \det \circ G(i)(\mathcal{L}) \subset S^1(L_{-K/2}).
\]

(2.19)

The Levi-Civita connection of the Kähler metric defines a Hermitian connection \( a_{LC} \) on \( L_{-K/2} \).

**Definition 2.1** A Lagrangian cycle \( \mathcal{L} \) is *almost geodesic* if the Legendrian cycle \( \Lambda = \det \circ G(i)(\mathcal{L}) \) is horizontal with respect to the Levi-Civita connection \( a_{LC} \) on \( L_{-K/2} \).

We now use property (1.3). The line bundle \( L_{-K/2} \) is \( L^{-d} \), where \( L \) is the line bundle of the polarization and we suppose that the Levi-Civita connection is induced by the connection \( a_L \) on \( L \). Then we have

**Proposition 2.2** A Lagrangian cycle \( \mathcal{L} \) is BS\(^0\) if and only if it is almost geodesic.

The Hermitian structures of our line bundles define a map

\[
\mu_d: S^1(L^*) \to S^1(L_{-K/2})
\]

(2.20)

of the principal U(1)-bundles of these line bundles, which fibrewise is minus the isogeny of degree \( d \). Thus every Lagrangian cycle \( \mathcal{L} \) defines an oriented Legendrian subcycle (see (2.1))

\[
\Lambda = \mu_d^{-1}(\det \circ G(i)(\mathcal{L})) \subset S^1(L^*) = P.
\]

(2.21)

Now consider the pair of isogenies

\[
\mu_{d+k}: S^1(L^d) \to S^1(L^{d(k+d)})
\]

(2.22)
and
\[ \mu_d : S^1(L^{k+d}) \to S^1(L^{d(k+d)}) \]
and the lift
\[ l : \mathcal{L} \to S^1(L^{k+d}) \]
given by a covariant constant section over a BS\(^{k+d}\) cycle \(\mathcal{L}\).

**Definition 2.2** The lift \(l\) is *almost geodesic* if
\[ \mu_{d+k} \circ \det \circ G(i)(\mathcal{L}) = \mu_d \circ l(\mathcal{L}). \]

The number of geodesic lifts is obviously \( \leq |d(k + d)| \).

In summary, let \(\Lambda^k \mathcal{M}\) be the space of Legendrian subcycles of \(P\) the images of whose projection to \(X\) is the \(k\)th root of unity cover of a BS\(^{k+d}\) Lagrangian cycle on \(X\) (such Legendrian cycles are sometimes called *Planckian cycles*). Then the natural projection
\[ p : \Lambda^{k+d} \mathcal{M} \to \text{BS}\(^{k+d}\) (L) \]
which sends a Legendrian cycle to Lagrangian cycle is a principal U(1)-bundle.

The geodesic lifting
\[ l : \text{BS}\(^{k+d}\) (L) \to \Lambda^{k+d} \mathcal{M} \]
we have described is a multisection of this principal bundle and
\[ p : \text{BS}\(^{k+d}\) (L) \to \text{BS}\(^{k+d}\) (L) \]
is a finite cyclic cover.

Consider a real polarization \(\pi : X \to B\) (1.23). Then we have a finite set of Bohr–Sommerfeld fibres
\[ B \cap \text{BS}\(^{k+d}\) (L) = \{\mathcal{L}_i\}, \text{ for } i = 1, \ldots, N_\pi^{k+d}. \]

**Definition 2.3** A choice of geodesic lifts
\[ \{\tilde{\mathcal{L}}_i\} \subset \Lambda^{k+d} \mathcal{M} \]
is called a choice of *theta structure* of the real polarization \(\pi\).
Marking these Lagrangian cycles with the half-forms given by our Kähler metric $g$ (see (1) below Theorem 1.1), we get a finite set

$$\{\tilde{L}_i, hF_g\}$$

of half-weighted Legendrian cycles.

We know that

$$\text{BPU}_k(\{\tilde{L}_i, hF_g\}) \subset \mathbb{P}H^0(L^{k+d})$$

is a linear independent system of vectors (states) if $k \gg 0$. In particular, if

$$\#(\text{BS}^{k+d}(L) \cap B) = \text{rank} H^0(L^{k+d}), \quad (2.24)$$

we get a Bohr–Sommerfeld basis.

**Remark** It is easy to see that we are imitating the geometric situation of Section 1, Example 3. For other descriptions and applications of the geodesic lift from Lagrangian to Legendrian cycles see [T1], [T2] and [T3].

**Geometric quantization**

There is a deep reason for coincidences such as (2.24) for Bohr–Sommerfeld fibres of a real polarization of the phase space of a classical mechanical system: we can view any symplectic manifold $(S, \omega)$ as the phase space of some classical mechanical system, and the pair $(L, a_L)$, where $a_L$ is an Hermitian connection on line bundle $L$ with curvature form $F_a = 2\pi i \cdot \omega$ as a prequantization data of this system.

Bohr–Sommerfeld bases identify two approaches to the geometric quantization of $(S, \omega, L, a)$ (see [A], [S1] or [W]). The first approach is a choice of a complex polarization, which is nothing other than a choice of a complex structure $I$ on $S$ such that $S_I = X$ is a Kähler manifold with Kähler form $\omega$. Then the curvature form of the Hermitian connection $a$ is of type $(1, 1)$, hence for any level $k \in \mathbb{Z}^+$, the line bundle $L^k$ is a holomorphic line bundle on $S_I$. Complex quantization provides the space of wave functions of level $k$ (1.1) (see Kirillov’s survey [K]).

The second approach to geometric quantization is the choice of a real polarization of $(S, \omega, L, a)$ (see Section 1, Example 1) which is a fibration
\( \pi : S \to B \) (1.23). We have already seen that restricting \((L, a)\) to a Lagrangian fibre gives a flat connection or equivalently, a character of the fundamental group \( \chi : \pi_1(\text{fibre}) \to \text{U}(1) \).

Let \( \mathcal{L}_\pi \) be the sheaf of sections of \( L \) that are covariant constant along fibres. Then we get the space \( \mathcal{H}_\pi = \bigoplus_i H^i(S, \mathcal{L}_\pi) \) and in the regular case, Śniatycki proved that \( H^i(S, \mathcal{L}_\pi) = 0 \) for \( i \neq n \). To compute the last component \( H^n(S, \mathcal{L}_\pi) \) we need to involve Bohr–Sommerfeld fibres: we expect to get a finite number of Bohr–Sommerfeld fibres, and in the regular case,

\[
H^n(S, \mathcal{L}_\pi) = \bigoplus_{\text{BS} \cap B} \mathbb{C} \cdot s_i,
\]

where \( s_i \) is the covariant constant section of the restriction of \((L, a)\) to a Bohr–Sommerfeld fibre of the real polarization \( \pi \) (see [S2]).

In the general case, we can use this to define a new collection of spaces of wave functions (of level \( k \)):

\[
\mathcal{H}^k_\pi = \bigoplus_{\text{BS}^{k+d} \cap B} \mathbb{C} \cdot s_i,
\]

(2.25)

and use the Borthwick–Paul–Uribe construction to compare (1.1) with (2.25).

There is a canonical way of describing the subset \( \text{BS}^{k+d}(L) \cap B \) using special coordinates on \( B \), the so-called action coordinates, which are part of the action angle coordinates (see [A], [GS1], ...).

An important observation, proved mathematically in some cases, is that the projectivization of the spaces (1.1) does not depend on the choice of complex structure:

\[
\frac{\partial \mathbb{P}\mathcal{H}^k_\pi}{\partial I} = 0.
\]

(2.26)

In other words, the vector bundle (1.2) admits a projective flat connection. Thus spaces of wave functions are given purely by the symplectic prequantization data (see for example [H]). The same is true for the projectivization of the spaces (2.25). Moreover, these spaces do not depend on the real polarization \( \pi \) (1.21), provided that we extend our prequantization data \((S, \omega, L, a, )\) by adding some half-density or half-form on every BS fibre (see [GS1]) to define the half-form pairing of Blattner, Kostant and Sternberg (for the difference between half-density and half-form quantizations see [M]).
To compare the spaces $\mathcal{H}_I^k$ and $\mathcal{H}_\pi^k$ by the BPU method, we have to arrange for them to have the same rank. This arithmetical problem can be solved directly in many interesting cases; see, for example, [K1] and [JW1]. For the geometry behind these coincidences see [T1].

3 Application: theory of non-Abelian theta functions

The classical theory of theta functions serves as a beautiful model for our theory. Although this theory is realized by many approaches to geometric quantization (see [T1]), we must demonstrate that all classical bases of theta functions can be described as Bohr–Sommerfeld bases given by the BPU method, that the geodesic lifting is a choice of theta structure and so on. This is a beautiful but quite serious job, and will be done in a special paper (or book). Here we will discuss this theory as a model for the theory of non-Abelian theta functions.

Let $A$ be a principal polarized Abelian variety (ppAv) of complex dimension $g$ with zero element $o \in A$ and with flat metric $g$. Then the tangent bundle $TA$ has the standard constant Hermitian structure (that is, the Euclidean metric, symplectic form and complex structure $I$). The Kähler form $\omega$ gives a polarization of degree 1. In the equality (1.3) we have $d = 0$. We fix a smooth Lagrangian decomposition of $A$

$$A = T^g_+ \times T^g_-.$$  \hspace{1cm} (3.1)

with both tori Lagrangian with respect to $\omega$ (recall that, smoothly, $A$ is the standard torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ with the standard constant integer form $\omega$ and this decomposition is nothing other than reducing the integer form $\omega$ to normal form). Let $L$ be a holomorphic line bundle with holomorphic structure given by a Hermitian connection $a$ with curvature form $F_a = 2\pi i \cdot \omega$, and $L = \mathcal{O}_A(\Theta)$, where $\Theta$ is the classical symmetric theta divisor. The decomposition (3.1) induces a decomposition $H^1(A, \mathbb{Z}) = \mathbb{Z}^g_+ \times \mathbb{Z}^g_-$, and a Lagrangian decomposition

$$A_k = (T^g_+)_k \times (T^g_-)_k.$$  \hspace{1cm} (3.2)
of the $k$-torsion subgroup.

In this case, complex quantization is nothing other than the classical theory of theta functions. Indeed, the Lagrangian decomposition (3.2) of the $k$-torsion subgroup defines a collection of compatible theta structures of every level $k$ and a decomposition of the spaces of wave functions

$$\mathcal{H}_I^k = H^0(A, L^k) = \bigoplus_{w \in (\mathbb{Z}g)^k} \mathbb{C} \cdot \theta_w \quad \text{with} \quad \text{rank} \mathcal{H}_I^k = k^g, \quad (3.3)$$

where $\theta_w$ is the theta function with characteristic $w$ (see [Mum]).

On the other hand, the direct product (3.1) gives us a real polarization

$$\pi: A \rightarrow T^g = B. \quad (3.4)$$

Remark that in this case the action coordinates are just flat coordinates on $T^g = B$, and under this identification

$$B \cap \text{BS}^k(L) = (T^g)_k \quad (3.5)$$

is the $k$-torsion subgroup.

Thus applying geometric quantization to the real polarization (3.4) of the phase space $(A, \omega, L^k, a_L)$, where $a_k$ is the Hermitian connection defining the holomorphic structure on $L^k$, we get the decomposition

$$\mathcal{H}_\pi^k = \bigoplus_{\rho \in \text{U}(1)^g_k} \mathbb{C} \cdot s_\rho. \quad (3.6)$$

**Corollary 3.1** (1) $\text{rank} \mathcal{H}_L^k = \text{rank} \mathcal{H}_\pi^k = k^g$.

(2) Moreover, there exists a isomorphism

$$\mathcal{H}_I^k = \mathcal{H}_\pi^k,$$

and this is canonical up to a scaling factor.

Indeed, the identification of the BS$^k$ fibres of $\pi$ given by the projection to $T^g$ gives us at the same time a lift of the BS$^k$ fibres to Legendrian submanifolds of $P$. Moreover the canonical class $K_A = 0$. So there exists a canonical
metaplectic structure on $A$, and a canonical collection of half-forms on Bohr–Sommerfeld fibres invariant with respect to translations defined up to a common phase factor. We can use the Borthwick–Paul–Uribe homomorphism (2.23) which is an inclusion

$$ \text{BPU}_k: \mathcal{H}_k \hookrightarrow \mathcal{H}_i^k $$

because sections are orthogonal and which is an isomorphism (because the ranks are equal). Moreover it easy to prove

**Proposition 3.1** The homomorphism $\text{BPU}_k$ extends to an inclusion of $H_k$-modules, where $H_k$ is the Heisenberg group of level $k$.

**Corollary 3.2** For every BS$_k$ torus $T^{2g}_j$ of (3.3) the BPU quantum mechanical state $\text{BPU}_k(T^{2g}_j)$ coincides with the corresponding theta function.

The functions making up the special bases of these spaces are called classical theta functions of level $k$ with characteristic.

**Remark** This coincidence should of course be proved directly using the form of the Schwartz kernel of $\Pi_k$, Fourier images of delta functions and the interpretation of theta functions as solutions of the heat equation.

Thus combining the constructions of complex and real polarizations gives us some orthogonal bases in complete linear systems. However, if we start with any polarized Kähler manifold $X$, the main question is about a real polarization of the form (1.23) on $X$ (possibly with degenerate fibres).

Finally, let $\Sigma$ be a Riemann surface of genus $g$ and $A = J_{\Sigma}$ its Jacobian. Then as a real manifold

$$ J_{\Sigma} = T^{2g} = \text{Hom}(\pi_1(\Sigma), U(1)),$$

with the symplectic form $\omega$ and the line bundle $L$ with a Hermitian connection $a$ with curvature $F_a = 2\pi i \omega$. Thus we can apply these constructions to the quadruple

$$ (\text{Hom}(\pi_1(\Sigma), U(1)), \omega, L_{\Theta}, a). $$

Then the Lagrangian decomposition (3.1) gives a real polarization with BS$_k$ fibres (3.5).
Now giving $\Sigma$ a complex structure $I$ defines a complex polarization of $\text{Hom}(\pi_1(\Sigma), U(1)) = J_\Sigma$. So the collection of spaces (1.1)

$$\mathcal{H}^k_I = H^0(J_\Sigma, L^k_\Theta)$$

are fibres of the holomorphic vector bundles (1.2)

$$\mathcal{H}^k \rightarrow \mathcal{M}_g$$

over the moduli space of Riemann surfaces of genus $g$.

Then the identification (3.7) shows that these spaces are actually independent of the complex structure $I$; that is, there exists a projective flat connection on every vector bundle (3.8). These connections may be described by a heat equation as in [H] and [W2].

Here we apply this method to the following noncommutative generalization of this situation: consider the $(6g - 6)$-manifold

$$R_g = \text{Hom}(\pi_1(\Sigma_2), SU(2))/PU(2),$$

the space of classes of $SU(2)$-representations of the fundamental group of this Riemann surface (it only depends on $g$) and apply the BPU construction to its GQ.

The new feature of this situation is the fact that $R_g$ is singular. So before this case, we must consider as a model the following singular Abelian case. Let

$$K_\Sigma = J_\Sigma/\{\pm \text{id}\}$$

be the Kummer variety. All geometric objects (3.8) are invariant with respect to the involution $-\text{id}$ and we get the prequantized mechanical system

$$(K_\Sigma, \omega, L_\Theta, a)$$

with singular phase space

$$\text{Sing } K_\Sigma = (J_\Sigma)_2,$$

that is, the 2-torsion points of the Jacobian.

A complex structure $I$ on $\Sigma$ defines a complex polarization of $K_\Sigma$; but now we only consider spaces of wave functions of even level

$$\mathcal{H}^k_I = H^0(K_\Sigma, L^k_\Theta) = H^0_{ev}(J_\Sigma, L^k_\Theta),$$

(3.14)
the space of even (symmetric) theta functions.

To describe a real polarization of $K_{\Sigma}$, represent $\Sigma$ as a connected sum of $g$ 2-tori:

$$\Sigma = T^2_1 \# T^2_2 \# \cdots \# T^2_g,$$

and fix a standard pair of generators of the fundamental group $(a_i, b_i)$ of each 2-torus $T^2_i$. Then we get the standard presentation of the fundamental group of $\Sigma$

$$\pi_1(\Sigma) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = \text{id} \rangle. \quad (3.16)$$

Killing the generators $a_i$ defines the handlebody $\tilde{\Sigma}_a$ with boundary

$$\partial \tilde{\Sigma}_a = \Sigma, \quad (3.17)$$

and fundamental group

$$\pi_1(\tilde{\Sigma}_a) = \langle b_1, \ldots, b_g \rangle, \quad (3.18)$$

the free group on the $b_i$.

Now we can define the Jacobian of a handlebody

$$J_{\tilde{\Sigma}_a} = H_1(\tilde{\Sigma}_a, \mathbb{R})/H_1(\tilde{\Sigma}_a, \mathbb{Z}), \quad (3.19)$$

and its Kummer variety:

$$K_{\tilde{\Sigma}_a} = J_{\tilde{\Sigma}_a}/\{\pm \text{id} \}.$$

Our real polarization (3.4) can be described as the natural map

$$\pi: J_{\Sigma} \rightarrow J_{\tilde{\Sigma}_a} \quad (3.21)$$

providing a real polarization of the Kummer variety

$$\pi: K_{\Sigma} \rightarrow K_{\tilde{\Sigma}_a}. \quad (3.22)$$

The fibres of this polarization are Lagrangian, and the $2^g$ Kummer varieties of $g$-dimensional tori over $(K_{\tilde{\Sigma}_a})_2$ which are singular, and

$$\text{Sing} \pi^{-1}(w) = (\pi^{-1}(w))_2, \quad \text{for } w \in (J_{\tilde{\Sigma}_a})_2. \quad (3.23)$$
Obviously the involution $-\text{id}$ on $J_{\tilde{\Sigma}_{a}}$ preserves the BS fibres of (3.4) and acts freely on

$$ (J_{\tilde{\Sigma}_{a}})_{2k} \setminus (J_{\tilde{\Sigma}_{a}})_{2} $$

preserving pointwise the subset

$$ (J_{\tilde{\Sigma}_{a}})_{2}. $$

So the number of BS$_{2k}$ fibres of the real polarization (3.21)

$$ \#(K_{\tilde{\Sigma}_{a}} \cap \text{BS}^{2k}(K_{\Sigma}, L_{\Theta})) = 2^{g-1}(k^{g} + 1) $$

is equal to rank of the space of even theta functions of level $2k$ (3.14).

Now the BS fibres over

$$ \left( (J_{\tilde{\Sigma}_{a}})_{2k} \setminus (J_{\tilde{\Sigma}_{a}})_{2} \right) \setminus \{ \pm \text{id} \} $$

are nonsingular, and the fibres over $(J_{\tilde{\Sigma}_{a}})_{2}$ are singular and simply connected. The following statement also holds in the non-Abelian case:

**Proposition 3.2** For a singular BS$_{k}$ fibre $\pi^{-1}(w)$,

$$ H_{1}(\pi^{-1}(w)) = H_{1}(\pi^{-1}(w) \cap \text{Sing} K_{\Sigma}). $$

As usual we have the space of wave functions (like (3.6)) for the real polarization (3.22)

$$ H^{2k}_{ev} = \bigoplus_{w \in ((J_{\tilde{\Sigma}_{a}})_{2k} \setminus (J_{\tilde{\Sigma}_{a}})_{2}) \setminus \{ \pm \text{id} \}} \mathbb{C} \cdot s_{w} + \bigoplus_{w \in (J_{\tilde{\Sigma}_{a}})_{2}} \mathbb{C} \cdot s_{w}. $$

(3.27)

Summarizing, for even level we get the following orthogonal decomposition of the space (3.3) of theta functions

$$ H_{I}^{2k} = H_{ev}^{0}(J_{\Sigma}, L^{2k}) \oplus H_{odd}^{0}(J_{\Sigma}, L^{2k}) $$

(3.28)

into even and odd theta functions, and the even component is the space of wave functions (3.14) of a complex polarization of the Kummer variety. Then the direct BPU construction gives a linear embedding

$$ \text{BPU}: \bigoplus_{w \in ((J_{\tilde{\Sigma}_{a}})_{2k} \setminus (J_{\tilde{\Sigma}_{a}})_{2}) \setminus \{ \pm \text{id} \}} \mathbb{C} \cdot s_{w} \rightarrow H_{ev}^{0}(J_{\Sigma}, L^{2k}) $$

(3.29)
for nonsingular BS fibres and a slight modification of it for the orbifold case gives us the full identification

$$H^0(K_{\Sigma}, L_{O}^{2k}) = \mathcal{H}_r^{2k}. \quad (3.30)$$

Under this identification, (3.27) corresponds to the decomposition

$$H^0(L_{O}^{2k}) = H^0(O(J_{\Sigma})^2) \oplus H^0(J_{\Sigma} \otimes L^k).$$

Unfortunately in the non-Abelian case the singularities are much worse than in the case of orbifolds and we apply the following strategy: for the smooth and orbifold cases we use the BPU method directly and its orbifold modification, but for heavy singularities we use the special features of our situation avoiding analysis.

**Complex quantization of $R_g$**

The space (3.10) is stratified by the subspace of reducible representations

$$R^{triv} \subset R^{red}_g \subset R_g, \quad R^{irr}_g = R_g - R^{red}_g. \quad (3.31)$$

Using symplectic reduction arguments, we get a nondegenerate closed symplectic form $\Omega$ on this space. This form $\Omega$ defines a symplectic structure on $R^{irr}_g$. There exists a Hermitian line bundle $L$ with the U(1)-connection $A_{CS}$ on $R_g$ (the Chern–Simons connection, see [RSW] or [T1], §3). By definition, the curvature form of this connection is

$$F_{A_{CS}} = 2\pi i \cdot \Omega. \quad (3.32)$$

Thus the quadruple

$$(R_g, \Omega, L, A_{CS}) \quad (3.33)$$

is a prequantum system.

The standard way of getting a complex polarization is to give the Riemann surface $\Sigma$ of genus $g$ a conformal structure $I$. We get a complex structure on the space of classes of representations $R_g$ such that $R_{\Sigma} = R_g = \mathcal{M}^{ss}$ is the moduli space of semistable holomorphic vector bundles on $\Sigma$ (see [H] for references).
On Bohr–Sommerfeld bases

The form $F_{\text{ACS}} (3.32)$ is a $(1,1)$-form and the line bundle $L$ admits a unique holomorphic structure compatible with the Hermitian connection $A_{\text{CS}}$. Moreover, a complex structure $I$ on $\Sigma$ defines a Kähler–Weil–Petersson metric on $\mathcal{M}^{\text{ss}}$ with Kähler form $\omega_{\text{WP}} = \Omega$. This metric defines the Levi-Civita connection on the complex tangent bundle $T\mathcal{M}^{\text{ss}}$, and hence a Hermitian connection $A_{\text{LC}}$ on the line bundle

$$\det T\mathcal{M}^{\text{ss}} = L^\otimes 4,$$

(3.34)

and a Hermitian connection $A_{I/\text{LC}}$ on $L$ compatible with the holomorphic structure on $L$. Thus $A_{\text{LC}} = A_{4\text{CS}}$ and the equality (1.3) holds with $d = -2$. We can use the geodesic lifting (2.23).

The result of complex quantization of the prequantum system (3.33) can be viewed as the space of wave functions of level $k$, that is, the space of $I$-holomorphic sections

$$\mathcal{H}_I^k = H^0(R_I, L^{k-2})$$

(3.35)

One knows that this system of spaces and monomorphisms is related to the system of representations of $\mathfrak{sl}(2, \mathbb{C})$ in the Weiss–Zumino–Novikov–Witten model of CQFT. The ranks of these spaces are given by the Verlinde formula (see [B]):

$$\text{rank } \mathcal{H}_I^k = \frac{k^{g-1}}{2^{g-1}} \sum_{n=1}^{k-1} \frac{1}{(\sin(\frac{n\pi}{k}))^{2g-2}}$$

(3.36)

(please note the shift $k \mapsto k - 2$). Many beautiful features of the geometry of embeddings (1.7)

$$\mathbb{P}_{\varphi_k}: R_I = \mathcal{M}^{\text{ss}} \rightarrow \mathbb{P}^0H^0(L^k)^*,$$

(3.37)

observed by Beauville, Laszlo, Pauly, Sorger and many others, make it reasonable to call this area of mathematics the theory of non-Abelian theta functions. But we would like to mention specially the observation of Oxbury and Ramanan about the spaces of level 4 [O]. In this case the space (3.35) is the natural direct sum of spaces of Abelian theta functions of Jacobian and Pryms of a Riemann surface and the union of classical theta bases (3.3) is the Bohr–Sommerfeld basis for this case.
Moreover the vector bundle (1.2)
\[ \mathcal{H}^k \to M_g \] (3.38)
over the moduli space of Riemann surfaces of genus \( g \) admits the projectively flat Hitchin connection [H].

Our space \( R_g \), with complex structure induced by a complex structure \( I \) on \( \Sigma \), is a singular algebraic variety \( R_I \) and
\[ \text{Sing} R_I = K_\Sigma \] (3.39)
is the Kummer variety of \( \Sigma \). The restriction
\[ L^k|_{K_\Sigma} = L^{2k}_{\mathcal{O}}. \] (3.40)
It is easy to see that if \( k \gg 0 \), the restriction gives the epimorphism
\[ \text{res}: H^0(L^k) \to H^0(L^{2k}_{\mathcal{O}}) \to 0 \] (3.41)
and using our Hermitian structure (1.8) we get the orthogonal decomposition
\[ H^0(L^k) = H^0(L^{2k}_{\mathcal{O}}) \oplus H^0(J_{K_\Sigma} \otimes L^k). \] (3.42)
For the first component of this decomposition we still have the special basis (3.29), (3.30). Moreover this component decomposes as
\[ H^0(L^{2k}_{\mathcal{O}}) = H^0(\mathcal{O}(J_{\mathcal{O}})_2) \oplus H^0(J_{(J_{\mathcal{O}})_2} \otimes L^k). \] (3.43)
Summarizing, we have a filtration of the vector bundle (3.38):
\[ \mathcal{H}_{\text{triv}} \subset \mathcal{H}_{\text{red}}^{2k} \subset \mathcal{H}^{k+2} \] (3.44)
corresponding to the flag (3.31) and the decompositions (3.42) and (3.43) (see also (3.27) and (3.30)). Every bundle of this flag has a projective flat connection, these connections are hereditary. The monodromies of the first pair of bundles
\[ \text{Mon } \mathcal{H}_{\text{triv}} = \text{Sp}(2g, \mathbb{Z}_2) \quad \text{and} \quad \text{Mon } \mathcal{H}_{\text{red}}^{2k} = \text{Sp}(2g, \mathbb{Z}_{2k}) \] (3.45)
are finite. Our main result (see 4.18) reduces the question of the monodromy of \( \mathcal{H}^{k+2} \) to purely combinatorial question about the representation (4.15). This representation is the subject of an absolutely different and very beautiful theory providing many 3-manifold invariants.
4 Combinatorial theory and identifications

Let \( \Gamma \) be any 3-valent graph having vertices \( V(\Gamma) \) and edges \( E(\Gamma) \), with \( |V(\Gamma)| = 2g - 2 \) and \( |E(\Gamma)| = 3g - 3 \); consider functions

\[
w : E(\Gamma) \to \left\{ 0, \frac{1}{2k}, \ldots, \frac{1}{2} \right\}
\]

(4.1)
on the edges of \( \Gamma \) to rational numbers with denominator \( \frac{1}{2k} \) in \([0, \frac{1}{2}]\) satisfying:

(0) \( w(C_i) \in \mathbb{Z} \cdot \frac{1}{k} \) if \( C_i \) disconnects \( \Gamma \);

and for any three edges \( C_l, C_m, C_n \) meeting at a vertex \( P_i \):

(1) \( w(C_l) + w(C_m) + w(C_n) \in \frac{1}{k} \cdot \mathbb{Z} \);

(2) \( w(C_l) + w(C_m) + w(C_n) \leq 1 \);

(3) for any ordering of \( C_l, C_m, C_n \),

\[
|w(C_l) - w(C_m)| \leq w(C_n) \leq w(C_l) + w(C_m).
\]

(4.2)

A function \( w \) satisfying these conditions is called an **admissible integer weight** of level \( k \) on \( \Gamma \). Let \( W^k_g(\Gamma) \) be the set of admissible integer weights. This set is canonically embedded in the set \( W^k(\Gamma) \) of all (unrestricted) functions

\[
W^k_g(\Gamma) \subset W^k(\Gamma),
\]

(4.3)

which are obviously \( |W^k_g| = (k + 1)^{3g-3} \) in number.

**Proposition 4.1 (see for example [K1])** The number \( |W^k_g(\Gamma)| \) of admissible weights of level \( k \) is independent of \( \Gamma \).

The restrictions (4.2) are called the **Clebsch–Gordan conditions**. We can consider the space of all real functions with values in \([0, 1]\) subject to these conditions to get a complex \( \Delta_\Gamma \) (see [JW1]). We thus have a space

\[
\mathcal{H}^k_\Gamma = \bigoplus_{w \in W^k_g(\Gamma)} \mathbb{C} \cdot w.
\]

(4.4)
with a natural Hermitian pairing $\langle \cdot, \cdot \rangle_c$ such that the $\{w\}$ form a unitary orthonormal basis. All these spaces are of course canonically contained in the common space

$$\mathcal{H}_k^\Gamma \subset \mathcal{H}_g^k = \bigoplus_{w \in W_g^k} \mathbb{C} \cdot w. \quad (4.5)$$

The geometry of the projective configuration

$$\bigcup_{\text{all graphs}} \mathbb{P} \mathcal{H}_k^\Gamma \subset \mathbb{P} \mathcal{H}_g^k$$

reflects properties of the Moore–Seiberg complex ([MS]) for 3-valent graphs.

This combinatorial description has the following geometric meaning: consider $\mathbb{R}^{3g-3}$ with coordinates $c_i$ corresponding to $\{C_i\} = E(\Gamma)$. It contains the complex $\Delta_\Gamma$ and the integer sublattice $\mathbb{Z}^{3g-3} \subset \mathbb{R}^{3g-3}$, and we can consider the \textit{“action”} torus:

$$T^A = \mathbb{R}^{3g-3}/\mathbb{Z}^{3g-3}. \quad (4.6)$$

Then $T^A$ contains a topological complex $\overline{\Delta}_\Gamma$ obtained by glueing together the boundary points of the polytope $\Delta_\Gamma$.

Of course every unrestricted function $w \in W_g^k$ (4.1) defines a $2k$-torsion point $w \in T^A_{2k}$ on the action torus. Thus we have an identification:

$$W_g^k = T^A_k \quad (4.7)$$

In particular, the admissible integer weights $W_g^k$ can be viewed as a subset of the $2k$-torsion points of the action torus:

$$W_g^k(\Gamma) \subset T^A_k. \quad (4.8)$$

Now if we pump up the edges of a trivalent graph $\Gamma$ to tubes, and the vertices to small 2-spheres we get a Riemann surface $\Sigma_\Gamma$ of genus $g$ marked with $3g - 3$ disjoint, noncontractible, pairwise nonisotopic smooth circles $\{C_i\}$ on $\Sigma$, the meridian circles of the tubes. The isotopy class of such a collection of circles is called a \textit{marking} of the Riemann surface. It is easy to see that the complement is the union

$$\Sigma_g - \{C_1, \ldots, C_{3g-3}\} = \prod_{i=1}^{2g-2} P_i \quad (4.9)$$
of $2g - 2$ trinions $P_i$, where every trinion is a 2-sphere with 3 disjoint discs deleted:

$$P_i = S^2 \setminus (D_1 \cup D_2 \cup D_3) \quad \text{with} \quad \overline{D}_i \cap \overline{D}_j = \emptyset \quad \text{for} \quad i \neq j. \quad (4.10)$$

On the other hand any trinion decomposition of our Riemann surface $\Sigma$, given by a choice of a maximal collection of disjoint, noncontractible, pairwise nonisotopic smooth circles on $\Sigma$. It is easy to see that any such system contains $3g - 3$ simple closed circles

$$C_1, \ldots, C_{3g-3} \subset \Sigma_g,$$  

with complement the union of $2g - 2$ trinions $P_j$. The type of such a decomposition is given by its 3-valent dual graph $\Gamma(\{C_i\})$, associating a vertex to each trinion $P_i$, and an edge linking $P_i$ and $P_j$ to a circle $C_i$ such that

$$C_i \subset \partial P_i \cap \partial P_j.$$  

Thus the isotopy class of a trinion decomposition is given by a 3-valent graph $\Gamma$.

Now the modular group $\text{Mod}_g$ which acts on $R_g$ by symplectomorphisms preserving the prequantum data. Every element $\gamma \in \text{Mod}_g$ changes the system of loops $\{[C_i]\} \to \{\gamma([C_i])\}$ but the graph of the trinion decomposition is precisely the same:

$$\Gamma(\{[C_i]\}) = \Gamma(\{\gamma([C_i])\}). \quad (4.12)$$

Thus the set of admissible integer weights $W_k^g(\Gamma(\{C_i\}))$ is the same, and defines the basis (4.4) in the space $H_k^\Gamma$.

Moreover, using the fusion matrices that describe the monodromy of the Knizhnik–Zamolodchikov equation, Kohno [K1] constructed a canonical isomorphism

$$H^k_{\Gamma_1} = H^k_{\Gamma_2} \quad \text{for any two graphs } \Gamma_1 \text{ and } \Gamma_2, \quad (4.13)$$

and as a consequence, he obtained unitary linear representations of the central extensions

$$1 \to Z(k) \to \widetilde{\text{Mod}}_g^k \to \text{Mod}_g \to 1,$$  

(4.14)
where \( Z(k) \) is the cyclic group generated by \( \exp(2\pi i \frac{k}{8(k+2)}) \); Kohno’s representations are

\[
\rho^k_c : \widetilde{\text{Mod}}_g \to \text{End} \mathcal{H}^k_\Gamma.
\]

The decomposition with these components is parallel to the decomposition of the highest weight representation of the affine Lie algebra of \( \mathfrak{sl}(2, \mathbb{C}) \) by eigenspaces of the operator \( L_0 \) from Sugawara’s construction of the representation of the Virasoro Lie algebra (see [K2]).

Using this representation, we construct the vector bundles

\[
\pi : \mathcal{H}^k_c \to M_g
\]

(4.16)

(the subscript \( c \) stands for “combinatorial”) over the moduli space \( M_g \) of curves of genus \( g \) having fibres

\[
\pi^{-1}(\Sigma_\Gamma) = \mathcal{H}^k_\Gamma,
\]

(4.17)

with the projective unitary connection \( a_c \) with the monodromy representation (4.15). Indeed, \( \text{Mod}_g \) acts transitively both on all markings \( \{[C_i]\} \), and on all trinion decompositions.

The main result

We want to identify the projectivizations of the spaces (3.35) and (4.4):

**Theorem 4.1** For \( k \gg 0 \) (depending on the genus \( g \)) then there exists a canonical identification

\[
\mathcal{H}^{k+2} = \mathcal{H}^k_{\Gamma_0}
\]

(4.18)

up to finite ambiguity for the special 3-valent graph \( \Gamma_0 \) described below.

This identification gives a chain of identifications of objects and construction of two theories: the complex quantization of (3.33) (the WZNW model of CQFT) and the combinatorial theory of Witten, Reshetikhin–Turaev, Tsuchiya–Kanie, Drinfeld, Moore–Seiberg and Kohno. Thus we can use results of the theory of non-Abelian theta functions in algebraic geometric as an effective means of computing topological invariants of 3-manifolds. On the other hand, the standard bases in the spaces \( \mathcal{H}^k_\Gamma \) (4.4) define non-Abelian
theta functions with characteristic of level \( k \) and, following Mumford, we should write down special equations in these bases defining the images of \( \mathcal{M}^a \) in spaces of conformal blocks.

We realize this program using the BPU method of Sections 1–2. (Thus we only get the identification (4.18) for \( k \gg 0 \).

We must construct \( BS^{k+2} \) cycles on \( R_g \) indexed by the set \( W^k_g(\Gamma) \) (4.3). This was done by Jeffrey and Weitsman [JW1]: for a marked Riemann surface \( \Sigma_\Gamma \), the map

\[
\pi_{\{C_i\}} : R_g \to \mathbb{R}^{3g-3}
\]

(4.19)

with fixed coordinates \((c_1, \ldots, c_{3g-3})\) such that

\[
c_i(\pi_{\{C_i\}}(\rho)) = \frac{1}{\pi} \cos^{-1}\left(\frac{1}{2} \text{tr} \rho([C_i])\right) \in [0, 1],
\]

(4.20)

where \( \{C_i\} = E(\Gamma) \). It is well known that

(1) The map \( \pi_{\{C_i\}} \) is a real polarization of the system \((R_g, k \cdot \omega, L^k, k \cdot A_{CS})\).

(2) The coordinates \( c_i \) are action coordinates for this Hamiltonian system.

(3) The map \( \pi_{\{C_i\}} \) is a moment map for the action of \( T^{3g-3} \) on \( R_g \)

\[
R_g \times T^{3g-3} \to R_g
\]

constructed by Goldman [G].

(4) The image of \( R_g \) under \( \pi_{\{C_i\}} \) is a convex polyhedron

\[
\Delta_{\{C_i\}} \subset [0, 1]^{3g-3}.
\]

(4.22)

(5) The symplectic volume of \( R_g \) equals the Euclidean volume of \( \Delta_{\{C_i\}} \):

\[
\int_{R_g} \omega^{3g-3} = \text{Vol} \Delta_{\{C_i\}} = \frac{2 \cdot \zeta(2g - 2)}{(2\pi)^{g-1}}.
\]

(6) The expected number of Bohr–Sommerfeld orbits of the real polarization \( \{C_i\} \)

\[
N_{BS}(\pi_{\{C_i\}}, R_g, \omega, L, A_{CS})
\]

(4.23)
is equal to the number of half-integer points in the polyhedron $\Delta_{\{C_i\}}$, and
\[
\lim_{k \to \infty} \frac{N_{k,\text{BS}}}{k^{3g-3}} = \int_{R_g} \omega^{3g-3} = \text{Vol} \Delta_{\{C_i\}}. \tag{4.24}
\]

These functions $c_i$ are continuous on all $R_g$ and smooth over $(0, 1)$.

Every $w \in W^k_g(\Gamma)$ defines a point of $\Delta_{\{C_i\}}$ (4.22) with coordinates

\[ c_i = 2w(C_i). \]

**Proposition 4.2 ([JW1])**

1. The map $x \mapsto 2x$ sends the complex $\Delta_\Gamma$ (4.7) to $\Delta_{\{C_i\}}$ (4.22)

\[ 2\Delta_\Gamma = \Delta_{\{C_i\}}; \]

2. This transformation sends $W^k_g(\Gamma)$ to the set of BS$_k$ fibres of the real polarization $\pi_\Gamma = \pi_{\{C_i\}}$ (4.19):

\[ 2W^k_g(\Gamma) = \Delta_{\{C_i\}} \cap \text{BS}^{k+2}(L); \tag{4.25} \]

3. In particular

\[ |W^k_g(\Gamma)| = \#(\Delta_{\{C_i\}} \cap \text{BS}^{k+2}(L)). \]

In summary:

1. Fixing the graph $\Gamma(\{[C_i]\})$ of a trinion decomposition we enumerate canonically the set of $k$-Bohr–Sommerfeld fibres of all polarizations with the same graph as the set $W^k_g(\Gamma(\{[C_i]\}))$ of integer weights on this graph;

2. Fixing a collection of loops $\{[C_i]\}$ we get a finite set of disjoint $k$-Bohr–Sommerfeld oriented cycles $L_w$ for $w \in W^k_g(\Gamma(\{[C_i]\}))$ in $R_g$;

3. For any level $k$, any complex Riemann surface $\Sigma$, and any trinion decomposition $\{C_i\}$, we have

\[ \text{rank } \mathcal{H}^{k+2}_I = \text{rank } \mathcal{H}^k_\Gamma = \text{rank } \mathcal{H}^k_\pi = \text{Verlinde number } (3.36). \tag{4.26} \]
We complete the description in [JW1] of the BS\textsuperscript{k+2} fibres of a real polarization \( \pi_{\Gamma} \) by describing the fibres \( \pi_{\Gamma}^{-1}(w) \) for which
\[
\pi_{\Gamma}^{-1}(w) \cap K_{\Sigma} \neq \emptyset. \tag{4.27}
\]

**Remark** We will see below that BS\textsuperscript{k+2} fibres disjoint from \( K_{\Sigma} = \text{Sing} R_{I} \) can only have orbifold singularities. So we can apply the BPU construction to it, and get a partial basis of “theta functions with characteristic”.

Return to the geometric procedure described after formula (4.8). Pumping up our graph \( \Gamma \) we get a handlebody \( \tilde{\Gamma} \) with boundary
\[
\partial \tilde{\Gamma} = \Sigma, \tag{4.28}
\]
giving an exact sequence of fundamental groups
\[
1 \to \ker \to \pi_{1}(\Sigma) \to \pi_{1}(\tilde{\Gamma}) \to 1,
\]
where the kernel is the subgroup of the fundamental group of Riemann surface of cycles homotopic to a point in \( \tilde{\Gamma} \). To recognize our previous handlebody \( \tilde{\Sigma}_{a} \) (3.17), recall that the standard presentation (3.16) of the fundamental group of \( \Sigma_{g} \) defines another “dual” presentation given by the following GNW construction (Guruprasad–Nilakantan–Weil, see [JW] for references): set
\[
\alpha_{i} = r_{i-1}b_{i}^{-1}r_{i}^{-1}, \quad \beta_{i} = r_{i}a_{i}^{-1}r_{i}^{-1}, \quad \text{where} \quad r_{i} = \prod_{j=1}^{i}[a_{j}, b_{j}].
\]
Then
\[
\pi_{1}(\Sigma_{g}) = \left\langle \alpha_{1}, \ldots, \alpha_{g}, \beta_{1}, \ldots, \beta_{g} \mid \prod_{j=1}^{g}[\alpha_{j}, \beta_{j}] = 1 \right\rangle
\]
is another presentation of \( \pi_{1}(\Sigma_{g}) \). Sending the generators \( a_{i}, b_{j} \) to \( \alpha_{i}, \beta_{j} \) gives an automorphism \( W \) of \( \pi_{1}(\Sigma_{g}) \), that is, \( W \in \text{Mod}_{g} \), and it is an involution: \( W^{2} = \text{id} \).

To show that
\[
\tilde{\Gamma} = \tilde{W}(\Sigma)_{a} = \tilde{\Sigma}_{W(a)}, \tag{4.29}
\]
consider the special 3-valent graph \( \Gamma_{0} \) corresponding to the presentation \( \Sigma \) as a connected sum of \( g \) 2-tori (see [K], Figs. 12a and 13b). We get a basis (3.16). To get \( \Gamma_{0} \), we fix the following system of cycles \( \{C_{i}\} \) on \( \Sigma \): they consist of three groups:
(1) \(a_1, \ldots, a_g\), the cycles \(a_i\) of \([K]\), Fig. 13b;

(2) \(a'_2, \ldots, a'_{g-1}\), the cycles \(c_i\) of \([K]\), Fig. 13b;

(3) \(c_1 = [a_1, b_1], \ldots, c_{g-1} = [a_{g-1}, b_{g-1}]\), the cycles \(b_i\) of \([K]\), Fig. 13b.

Then

\[
\{c_i\} \subset [\pi_g, \pi_g] \tag{4.30}
\]

is the commutator subgroup of the fundamental group. Since \(E(\Gamma_0)\) contains the subset

\[
E(\Gamma_0)^a = \{a_1, \ldots, a_g\} \tag{4.31}
\]

Our handlebody (3.17) transformed by \(W\) is equal to \(\tilde{\Gamma}_0\).

We label the coordinates \(\{c_i\}\) (4.19) and (4.20) by the same symbols \(a_i, a'_j, c_k\). Then

\[
\rho \in K\Sigma \implies c_i(\rho) = 0 \text{ for } i = 1, 2, \ldots, g - 1; \tag{4.32}
\]

\[
\rho \in K\Sigma \implies a_i = a'_i \text{ for } i = 2, \ldots, g - 1. \tag{4.33}
\]

Thus

\[
\dim \pi_{\Gamma_0}(K\Sigma) = g;
\]

more precisely, we have:

**Proposition 4.3**

\[
\pi_{\Gamma_0}(K\Sigma) = K_{\tilde{\Gamma}_0};
\]

and \(\pi_{\Gamma_0}\) is the real polarization (3.22).

We must now check the following:

**Proposition 4.4**

\[
K_{\tilde{\Gamma}_0} \cap \text{BS}_k(R_g, L) = K_{\tilde{\Gamma}_0} \cap \text{BS}^{2k}(K\Sigma, L\Theta)
\]

The proof follows immediately from the description of BS\(_k\) fibres in Proposition 5.2 and Corollary 5.1 of the next section.
Corollary 4.1  (1) $W^k_g(\Gamma_0)$ contains the subset

$$W^k_g(\Gamma_0)_{\text{Ab}} = K_{\Gamma_0} \cap \text{BS}^{k+2}(L) \subset W^k_g(\Gamma_0)$$

of weights that we call Abelian weights.

(2) Weights of the set

$$W^k_g(\Gamma_0)_{\text{non-Ab}} = W^k_g(\Gamma_0) \setminus W^k_g(\Gamma_0)_{\text{Ab}}$$

are called non-Abelian weights.

(3) The space (4.4) can be decomposed into Abelian and non-Abelian parts

$$H^k_{\Gamma_0} = \left( \bigoplus_{w \in W^k_g(\Gamma_0)_{\text{Ab}}} \mathbb{C} \cdot w \right) \oplus \left( \bigoplus_{w \in W^k_g(\Gamma_0)_{\text{non-Ab}}} \mathbb{C} \cdot w \right).$$

(4) (3.30) identifies the Abelian component:

$$\left( \bigoplus_{w \in W^k_g(\Gamma_0)_{\text{Ab}}} \mathbb{C} \cdot w \right) = H^0(K_{\Sigma}, L^{2k}).$$

To get a basis of “theta functions with characteristic” in all $H^0(L^k)$ related to a trinion decomposition of $\Sigma$ and a linear isomorphism $BPU_k: H^k_{\pi_{\Gamma_0}} \to H^0(L^k)$ (2.24) we must construct on every fibre $L_w$ for $w \in W^k_g$ an almost canonical half-form $hF_w$ in such a way that the Szegö projector (2.2) extends to a class of distributions including $(\Lambda_w, hF_w)$ for every $w \in W^k_g(\Gamma)$.

5 Covariant constant half-forms and singularities

To put covariant constant half-forms on BS fibres, recall some facts about its structure. Let $\Gamma$ be a 3-valent graph with vertices $V(\Gamma)$ and edges $E(\Gamma)$, and suppose that

$$w: E(\Gamma) \to \left\{ 0, \frac{1}{2k}, \ldots, \frac{1}{2} \right\}$$
is an integer admissible weight. For $\alpha \in \{0, \frac{1}{2k}, \ldots, 1\}$ let

$$w^{-1}(\alpha) = \Gamma_1(\alpha) \cup \cdots \cup \Gamma_n(\alpha) \subset \Gamma$$ (5.1)

be the decomposition into connected components. Then every component $\Gamma_i(\alpha)$ is a 3-valent graph with $n_i$ univalent vertices $a_1, \ldots, a_{n_i}$ (see [K1]).

Every gauge class of connections contains a connection $a_0$ adapted to a trinion decomposition (see [JW1], Definition 2.2). Fix the filtration

$$Z(\text{SU}(2)) = \mathbb{Z}_2 \subset \text{U}(1) \subset \text{SU}(2),$$ (5.2)

and view it as the triple

$$G = \{\mathbb{Z}_2, \text{U}(1), \text{SU}(2)\}$$ (5.3)

For $[a] \in \pi^{-1}(w)$, we have the function

$$e_w: E(\Gamma) \rightarrow G$$ (5.4)

sending every loop $C_j$ to the element of $G$ conjugate to the stabilizer of the monodromy of $[a]$ around this loop, and the function

$$v_w: V(\Gamma) \rightarrow G$$ (5.5)

sending a trinion $P_i$ to the stabilizer of the flat connection $a|_{P_n}$. Of course,

$$\begin{align*}
C_j \subset \partial P_n &\implies v_w(P_n) \subset e_w(C_j); \\
C_1 \cup C_2 \cup C_3 = \partial P_n \quad \text{and} \quad e_w(C_1) = e_w(C_2) = \text{SU}(2) &\implies e_w(C_3) = \text{SU}(2) \implies v_w(P_n) = \text{SU}(2),
\end{align*}$$ (5.6)

and so on.

Obviously

$$e_w(C_j) = \text{U}(1) \text{ or } \text{SU}(2).$$ (5.7)

**Proposition 5.1** The functions $e_w$ and $v_w$ depend on only $w$ and not on the choice of $[a] \in \pi^{-1}(w)$. 

More precisely, they depend on the combinatorics of the decomposition (5.1).

Thus \( w \) defines direct products

\[
\prod_{C \in E(\Gamma)} e_w(C) \quad \text{and} \quad \prod_{P \in V(\Gamma)} v_w(P),
\]

and \( \prod_{P \in V(\Gamma)} v_w(P) \) acts on \( \prod_{C \in E(\Gamma)} e_w(C) \) as follows: for

\[
g = (g_1, \ldots, g_{2g-2}) \in \prod_{P \in V(\Gamma)} v_w(P) \quad \text{with} \quad g_i \in v_w(P_i), \quad \text{and}
\]

\[
(t_1, \ldots, t_{3g-3}) \in \prod_{C \in E(\Gamma)} e_w(C) \quad \text{with} \quad t_n \in e_w(C_n),
\]

if \( C_n \subset \partial P_i \cap \partial P_j \) then

\[
g(t_n) = g_i \circ t_n \circ g_j^{-1}. \tag{5.8}
\]

**Proposition 5.2 ([JW1], Theorem 2.5)** The fibre \( \pi^{-1}(w) \) is given by

\[
\pi^{-1}(w) = \prod_{C \in E(\Gamma)} e_w(C) / \prod_{P \in V(\Gamma)} v_w(P). \tag{5.9}
\]

Applying this description to

\[
w \in K_{\Gamma_0} \cap BS^k(R_g, L)
\]

proves Proposition 4.4: in this case, for every \( P \in V(\Gamma_0) \)

\[
U(1) \subset v_w(P) \quad \text{and} \quad e_w(c_i) = SU(2)
\]

for \( c_i \) from (4.30).

**Corollary 5.1** The fibre \( \pi^{-1}(w) \) is isomorphic to

\[
\pi^{-1}(w) \cong T^t \times [(S^3)^p \times (S^2)^s] / G_w, \tag{5.10}
\]

where \( t, p \) and \( s \) are nonnegative integers and \( G_w \) is the finite Abelian group defined by \( w \), or more precisely by the combinatoric data (4.1); moreover,

\[
H_1(\pi^{-1}(w)) = \mathbb{Z}^t \oplus \mathbb{Z}^p. \tag{5.11}
\]
Translations along the torus $T^t$ in (4.10) are induced by Hamiltonians lifted from the target space $\mathbb{R}^{3g-3}$ of $\pi$. We consider below half-forms invariant under such translations.

Jeffrey and Weitsman [JW1] use the normalization of the action coordinates via branched covers to construct a covariant constant section $s_w$ of the restrictions of $(L^k, A_{k-CS})$ to $\pi^{-1}(w)$.

Our groups (5.2–3) admit bi-invariant half-forms $hF_1$ on $U(1)$ and $hF_3$ on $SU(2)$. For every $w$ we can normalize these form $hF_1(w)$ and $hF_2(w)$ so that the half-form

$$hF_w = (hF_1(w))^{t-s} \cdot (hF_2(w))^{p+s}$$

is homogeneous of degree 1 on $\pi^{-1}(w)$ (see (5.10)) with respect to scaling $hF_i(w) \to t \cdot hF_i(w)$. We say that such half-form is homogeneous normalized.

It’s easy to see ([JW2], 4.7) that a normalized half-forms for a nonsingular BS$^{k+2}$ fibre is given by a Hamiltonian vector field with Hamiltonian in $\mathbb{R}^{3g-3}$ of volume 1.

Thus every BS$^{k+2}$ fibre is given the covariant constant half-form (5.12), and we can proceed to construct the corresponding Legendrian distributions in $P$. Recall that $R_g$ is almost homogeneous with respect to the Goldman torus action (4.21). Thus the Schwartz kernel of coherent states does not depend on points and, outside singular points of fibres, they behave as in the homogeneous case (see [BPU], (10–13)). By lifting to $P$ every BS$^{k+2}$ fibre $\pi^{-1}(w)$ defines Legendrian subcycle $\Lambda_w \subset P$ marked with the half-form

$$\overline{hF}_w = (\varphi_4 \circ \det \circ G(i))^* hF_w,$$

and having monodromy a $k$th root of 1.

To apply the BPU construction the principal bundle $P = S^1(L^*)$ must be given a metaplectic structure. This can be done at once using (3.34).

For the identification (4.18), consider the decompositions (3.42)

$$H^0(L^k) = H^0(K_{\Sigma}, L_{\Theta}^{2k}) \oplus H^0(J_{K_{\Sigma}} \otimes L^k)$$

and

$$\mathcal{H}^k_{\Gamma_0} = \left( \bigoplus_{w \in W_{\delta}(\Gamma_0)_{Ab}} \mathbb{C} \cdot w \right) \oplus \left( \bigoplus_{w \in W_{\delta}(\Gamma_0)_{non-Ab}} \mathbb{C} \cdot w \right)$$
Of Corollary 4.1. Then (3.30) identifies the first (Abelian) components and
the BPU construction identifies the second (non-Abelian) components. This
method (and its verbatim modification for orbifolds) is applicable because
the non-Abelian BS\(_k\) fibres are contained in the smooth part of \(R_g\).

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References

[A] V.I. Arnold, Mathematical methods of classical mechanics, 2nd edi-
tion, Springer Verlag, 1989

[AG] V.I. Arnold and A. B. Givental, Symplectic geometry, “Itogi Nauki”
4 (1985), Moscow, 5–140

[BPU] D. Borthwick, T. Paul and A. Uribe, Legendrian distributions with
applications to the non-vanishing of Poincaré series of large weight,
Invent. math, 122 (1995), 359–402 or hep-th/9406036

[B] A. Beauville, Vector bundles on Riemann surfaces and conformal field
theory, in “Algebraic geometric methods in math. physics”, Cacively
(1993), 145–166

[DKN] B. A. Dubrovin, I. M. Krichever and S. P. Novikov, Integrable systems
I, “Itogi Nauki” 4 (1985), Moscow, 179–288

[G] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows
of surface group representations, Invent. Math. 85 (1986), 263–302.

[GS1] V. Guillemin and S. Sternberg, Symplectic techniques in physics, CUP
(1983)

[Gu] V. Guillemin, Symplectic spinors and partial differential equations,
Coll. Inter. C.N.R.S., Aix-en-Provence, 1974) (1975), 217–252
On Bohr–Sommerfeld bases 42

[H] N. J. Hitchin, Flat connections and geometric quantization, Commun. Math. Phys., 131 (1990), 347–380

[JW1] L. C. Jeffrey and J. Weitsman, Bohr–Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula, Commun. Math. Phys. 150 (1992), 593–630

[JW2] L. C. Jeffrey and J. Weitsman, Half density quantization of the moduli space of flat connections and Witten’s semiclassical invariants, Topology 32 (1993), 509–529

[K1] T. Kohno, Topological invariants for 3-manifolds using representations of mapping class group I, Topology 31 (1992), 203–230

[K2] T. Kohno, Topological invariants for 3-manifolds using representations of mapping class group II; Estimating tunnel number of knots, Contemporary mathematics 175 (1994), 193–217

[K] A. A. Kirillov, Geometric quantization, “Itogi Nauki”, vol. 4, Moscow, (1985), 141–178

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, Comm. Math. Phys. 123 (1989), 177–254

[Mum] D. Mumford, Tata lectures on theta. I: Progr. Math, 28, Birkhäuser (1983). II. Jacobian theta functions and differential equations: Progr. Math, 43, Birkhäuser (1984). III. Progr. Math, 97. Birkhäuser (1991)

[O] W. M. Oxbury, Prym varieties and the moduli of spin bundles, Algebraic geometry, ed. P.E. Newstead, Lect. Notes in Pure App. Math. vol. 200 (Marcel Dekker 1998), 351–376

[RSW] T.R. Ramadas, L.M. Singer, J. Weitsman, Some comments on Chern–Simmons gauge theory, Commun. Math. Phys. 126 (1989), 409–420

[S1] J. Śniatycki, Geometric quantization and quantum mechanics, Applied Math Sciences 30, Springer (1980)

[S2] J. Śniatycki, Bohr–Sommerfeld conditions in geometric quantization, Reports in Math. Phys. 7 (1974), 127–135
On Bohr–Sommerfeld bases

[T1] Andrei Tyurin, Quantization and “theta functions”, Jussieu preprint Avril 1999/Publication 216, e-print math.AG/9904046, 32 pp.

[T2] Andrei Tyurin, Geometric quantization and mirror symmetry, Warwick preprint 22/1999, alg-geom 9902027, 53 pp.

[T3] Andrei Tyurin, Special Lagrangian geometry and slightly deformed algebraic geometry (spLag and sdAG), Warwick preprint 8/1998, alg-geom 9806006, 45 pp.

[T4] Andrei Tyurin, Non-Abelian analogue of Abel’s theorem, ICTP Preprint 157 (1997), 55 pp.

[Wei1] A. Weinstein, Symplectic geometry, BAMS, 5 (1981), 1–13

[Wei2] A. Weinstein, Connections of Berry and Hannay type for moving Lagrangian submanifolds, Advances in Math. 82 (1990), 133-159

[W] N. Woodhouse, Geometric quantization, Oxford Math Monographs, OUP (1980)

[We] G. Welters, Polarized Abelian varieties and the heat equations, Comp. Math. 49 (1983), 173–194

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