A Classical Explanation of the Bohm-Aharonov Effect

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Abstract

The motion of a system of particles under electromagnetic interaction is considered. Under the assumption that the force acting on an electric charge is given by the sum of the electromagnetic fields produced by any other charged particles in its neighborhood, we prove that the vector potential of the electromagnetic field has to be considered for the balance of kinetic momentum. The theory cannot be quantized in the usual form—because it involves a mass matrix that depends on spatial variables—and the Hamilton’s function becomes singular at a distance equal to the geometric mean of the electrodynamic radiiuses of electrons and protons.

1 Introduction

In previous works [1] [2], we have shown that, according to classical mechanics and electrodynamics: a neutral system of electric charges that passes through a region where there is an inhomogeneous magnetic field, experiences a force, even if its internal kinetic angular momentum is equal to zero. Given that this challenges the common interpretation of the Stern-Gerlach experiment—as evidence that there are intrinsic angular momenta—we have considered necessary to study the motion of systems of electric charges where the internal magnetic force is not neglected, as it is usual in common classical treatments.
In this paper we study the motion of electric charges under electromagnetic interaction. Neglecting radiative effects, we assume that the force acting on an electric charge is given by the sum of the electromagnetic fields produced by any other particles in its neighborhood. From the invariance of the Lagrange’s function, we find that the vector potential of the electromagnetic field must be considered for the balance of linear and angular momentum, thus predicting a classical 

Bohm-Aharonov Effect.

A Hamilton’s function is obtained also for the system of two particles. The result is a theory that cannot be quantized but approximately in the usual form, since it involves a mass matrix that depends on spatial variables. Also, the Hamilton’s function becomes singular where the distance between the particles satisfies the relation:

\[ r = \frac{e^2}{c^2 m_e m_p^{1/2}}. \]

For the sake of completeness, we include a section where the Lagrange’s function and the equations of motion for the center of mass and the vector of relative position are obtained.

2 The General Law of Motion

We study the classical motion of an electron and a proton, under electromagnetic interaction.

Neglecting any retardation and/or radiative effects, we use the formulas

\[ \phi(\vec{x}, t) = \frac{q}{\|\vec{x} - \vec{r}(t)\|} \quad \text{and} \quad \vec{A}(\vec{x}, t) = \frac{q}{c} \frac{\vec{v}(t)}{\|\vec{x} - \vec{r}(t)\|}, \]

(1)

to find the electrodynamic potentials associated to a punctual charge \( q \) moving along the path \( \vec{r}(t) \). (Where \( \vec{v} = \frac{d\vec{r}}{dt} \).) The corresponding electromagnetic field is

\[ \vec{E}(\vec{x}, t) = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = \]

\[ \frac{q(\vec{x} - \vec{r})}{\|\vec{x} - \vec{r}\|^3} - \frac{q}{c^2} \left( \frac{\vec{v}}{\|\vec{x} - \vec{r}\|} + \frac{(\vec{x} - \vec{r}) \cdot \vec{v}}{\|\vec{x} - \vec{r}\|^3} \right) \]

\[ \vec{H}(\vec{x}, t) = \nabla \times \vec{A} = \frac{q \vec{v} \times (\vec{x} - \vec{r})}{c \|\vec{x} - \vec{r}\|^3}. \]

(2)
Further, we suppose—as it’s done when only Coulomb’s field is considered—that the force that acts on the electron is that due to the proton’s electromagnetic field, and *vice versa*. The equations of motion are:

\[
m_e \ddot{\vec{v}}_e = - \frac{e^2 (\vec{r}_e - \vec{r}_p)}{r^3} + \frac{e^2}{c^2} \left( \frac{\dot{\vec{v}}_p}{r} + \frac{(\vec{r}_e - \vec{r}_p) \cdot \vec{v}_p \vec{v}_p}{r^3} \right)
\]

and

\[
m_p \ddot{\vec{v}}_p = - \frac{e^2 (\vec{r}_p - \vec{r}_e)}{r^3} + \frac{e^2}{c^2} \left( \frac{\dot{\vec{v}}_e}{r} + \frac{(\vec{r}_p - \vec{r}_e) \cdot \vec{v}_e \vec{v}_e}{r^3} \right)
\]

(Here we have introduced the notation
\[
r = \|\vec{r}_p - \vec{r}_e\|
\]
that simplifies the equations.)

These are the Euler-Lagrange’s Equations for the Lagrange’s Function:

\[
L(\vec{r}_e, \vec{r}_p, \vec{v}_e, \vec{v}_p) = \frac{1}{2} m_e \vec{v}_e^2 + \frac{1}{2} m_p \vec{v}_p^2 + \frac{e^2}{c^2} \vec{v}_e \cdot \vec{v}_p - \frac{e^2}{c^2} \vec{r}_e \cdot \vec{v}_p,
\]
as we’ll prove for the equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \vec{v}_p} - \frac{\partial L}{\partial \vec{r}_p} = 0.
\]

From (7) we get

\[
\frac{d}{dt} \frac{\partial L}{\partial \vec{v}_p} = m_p \ddot{\vec{v}}_p - \frac{e^2}{c^2} \frac{d}{dt} \left( \frac{\vec{v}_e}{r} \right).
\]

Further:

\[
\frac{d}{dt} \left( \frac{\vec{v}_e}{r} \right) = \frac{\vec{v}_e}{r} - \frac{(\vec{r}_p - \vec{r}_e) \cdot \vec{v}_p \vec{v}_p}{r^3} + \frac{((\vec{r}_p - \vec{r}_e) \cdot \vec{v}_e \vec{v}_e)}{r^3},
\]

and

\[
\frac{\partial L}{\partial \vec{r}_p} = - \frac{e^2 (\vec{r}_p - \vec{r}_e)}{r^3} + \frac{e^2}{c^2} \left( \vec{v}_e \cdot \vec{v}_p \vec{r}_p \cdot \vec{r}_p - \vec{r}_e \right)
\]

(3)
From equations 8, 10, and 11—and the identity $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$—we can easily prove that equation 8 is equivalent to equation 5.

The generalized momenta are

$$\vec{p}_e = m_e \vec{v}_e - \frac{e^2}{c^2} \frac{\vec{v}_p}{r} = m_e \vec{v}_e - \frac{e}{c} \vec{A}_p(\vec{r}_e)$$  \hspace{1cm} (12)$$

and

$$\vec{p}_p = m_p \vec{v}_p - \frac{e^2}{c^2} \frac{\vec{v}_e}{r} = m_p \vec{v}_p + \frac{e}{c} \vec{A}_e(\vec{r}_p)$$  \hspace{1cm} (13)$$

where $\vec{A}_e(\vec{r})$ and $\vec{A}_p(\vec{r})$ are the vector potentials for the field of the electron and the field of the proton, respectively.

The Lagrange’s function 7 is invariant under translations and rotations of the reference system; therefore, the sum of the generalized momenta

$$\vec{p}_e + \vec{p}_p = m_e \vec{v}_e + m_p \vec{v}_p - \frac{e}{c} \vec{A}_p(\vec{r}_e) + \frac{e}{c} \vec{A}_e(\vec{r}_p),$$  \hspace{1cm} (14)$$

and the total angular momentum (which is not equal to the kinetic momentum and, therefore, the dipolar field is not enough to describe the magnetic properties of the system)

$$\vec{L} = \vec{r}_e \times \vec{p}_e + \vec{r}_p \times \vec{p}_p, \hspace{1cm} (15)$$

are constants of motion. In consequence, the center of mass of the system,

$$\vec{R} = \frac{m_e \vec{r}_e + m_p \vec{r}_p}{m_e + m_p}, \hspace{1cm} (16)$$

does not move according to Newton’s First Law. This was expected given that equations 4 & 5 are not in compliance with Newton’s Third Law either.

In the case of a system of $n$ particles with masses $m_1, \ldots, m_n$ and charges $q_1, \ldots, q_n$, the Lagrange’s Function assumes the form

$$L = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_i^2 - \frac{1}{2} \sum_{q_i \neq q_j} \frac{q_i q_j}{r_{ij}} \left( 1 - \frac{\vec{v}_i \cdot \vec{v}_j}{c^2} \right).$$  \hspace{1cm} (17)$$

Therefore, the generalized momentum of the $i$th particle is

$$\vec{p}_i = m_i \vec{v}_i + \frac{q_i}{c} \sum_{j \neq i} \frac{q_j \vec{v}_j}{r_{ij}} = m_i \vec{v}_i + \frac{q_i}{c} \vec{A}_i(\vec{r}_i), \hspace{1cm} (18)$$
where $\vec{A}_i$ is the vector potential of the magnetic field produced by the other particles.

Given that the function $17$ is also invariant under arbitrary translations and/or rotations, we come again to the conclusion that the sum of the generalized momenta and the angular momentum, are constants of motion. Therefore, as it has been confirmed by Bohm and Aharonov[3], the vector potential of the electromagnetic field acting on each particle must be considered for the balance of linear momentum.

The problem of gauge invariance is not an issue for us. Under the gauge transformation

$$L' = L + \frac{\partial \lambda}{\partial t} + \sum_{i=1}^{n} \vec{v}_i \cdot \frac{\partial \lambda}{\partial \vec{r}_i}$$

(19)

the momenta are transformed as:

$$\vec{p}_i' = \vec{p}_i + \frac{\partial \lambda}{\partial \vec{r}_i};$$

(20)

If the sum of the momenta is going to be a constant of motion, $\lambda$ must be invariant under arbitrary translations. In other words

$$\sum_{i=1}^{n} \frac{\partial \lambda}{\partial \vec{r}_i} = \vec{0},$$

(21)

and, in those circumstances:

$$\sum_{i=1}^{n} \vec{p}_i' = \sum_{i=1}^{n} \vec{p}_i.$$

Going back to the electron-proton system, its energy

$$E = \frac{1}{2} m_e \vec{v}_e^2 + \frac{1}{2} m_p \vec{v}_p^2 - \frac{e^2 \vec{v}_e \cdot \vec{v}_p}{c^2} - \frac{e^2}{r}$$

(22)

$$= \frac{1}{2}(\vec{p}_e \cdot \vec{v}_e + \vec{p}_p \cdot \vec{v}_p) - \frac{e^2}{r},$$

is also a constant of motion.

Solving $12$ and $13$ for the velocities, we find

$$\vec{v}_e = \frac{\vec{p}_e - \frac{e^2 \vec{p}_p}{m_e m_p c^2 r}}{1 - \frac{e^2}{m_e m_p c^2 r}}$$

(23)
\[ \vec{v}_p = \frac{\vec{p}_p}{m_p} - \frac{e^2 \vec{p}_p}{e^4 m_p c^2 r^2} \] (24)

From this and (22) we get the Hamilton’s Function

\[ H = \frac{\vec{p}_e^2}{2m_e} + \frac{\vec{p}_p^2}{2m_p} - \frac{e^2 \vec{p}_p \vec{p}_e}{m_e m_p c^2 r} - \frac{e^2}{r} \] (25)

Where

\[ r >> \frac{e^2}{c^2 (m_e m_p)^{1/2}} \] (26)

this function coincides with the function used in [4] to approximate the eigenvalues of the corresponding quantum system.

Further, we notice that (25) is singular where

\[ r = \frac{e^2}{c^2 (m_e m_p)^{1/2}} \] (27)

At this distance the equations of motion (4) and (5) cannot be solved for the accelerations, which is fundamental for the applicability of the theorem of existence and uniqueness of solutions. Therefore, even if the Principle of Least Action is valid, to determine a particular solution additional conditions have to be imposed. There are another two possibilities which we shall not investigate further in this paper:

1. That inequality (26) defines the limits of validity of electrodynamics.
2. That a fully relativistic approach is required. In this case the effects of retardation have to be considered; the equations of motion are difference-differential equations; and there is not room for a variational approach. (At least not for a variational approach that does not explicitly accounts for the action of the entire electromagnetic field.)

3 Separation of the Internal Motion

As it was shown before, the center of mass of the system does not move according to Newton’s First Law. Notwithstanding, and for the sake of completeness, we’ll carry out the decomposition of the motion.
into the motion of the center of mass and an internal motion. Let’s
consider the substitutions:
\[ \vec{R} = \frac{m_p \vec{r}_p + m_e \vec{r}_e}{M}; \quad \vec{r} = \vec{r}_e - \vec{r}_p \] (28)

(where \( M = m_p + m_e \)), in such way that:
\[ \vec{r}_p = \vec{R} - \frac{m_e}{M} \vec{r}; \quad \vec{r}_e = \vec{R} + \frac{m_p}{M} \vec{r}. \] (29)

and
\[ \vec{v}_p \cdot \vec{v}_e = \dot{\vec{R}}^2 + K_L \dot{\vec{R}} \cdot \dot{\vec{r}} - \frac{m_em_p}{M^2} \dot{\vec{r}}^2, \]

where
\[ K_L = \frac{m_p - m_e}{M}. \]

The Lagrange’s function takes the form:
\[ L(\vec{R}, \vec{r}, \dot{\vec{R}}, \dot{\vec{r}}) = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 + \frac{e^2}{r} \frac{\dot{\vec{R}}^2}{c^2} + K_L \dot{\vec{R}} \cdot \dot{\vec{r}} - \frac{m_em_p}{M^2} \dot{\vec{r}}^2. \] (30)

The momenta are:
\[ \vec{P}_R = M \left( 1 - \frac{2e^2}{Mc^2r} \right) \dot{\vec{R}} - \frac{K_L e^2}{c^2r} \dot{\vec{r}}, \] (31)

and
\[ \vec{p}_r = -\frac{K_L e^2}{c^2r} \dot{\vec{R}} + \mu \left( 1 + \frac{2e^2}{Mc^2r} \right) \dot{\vec{r}}. \] (32)

The energy is:
\[ E(\vec{R}, \vec{r}, \dot{\vec{R}}, \dot{\vec{r}}) = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{e^2}{r} \frac{\dot{\vec{R}}^2}{c^2} + K_L \dot{\vec{R}} \cdot \dot{\vec{r}} - \frac{m_em_p}{M^2} \dot{\vec{r}}^2 \]
\[ = \frac{1}{2} (\vec{P}_R \cdot \dot{\vec{R}} + \vec{p}_r \cdot \dot{\vec{r}}) - \frac{e^2}{r}. \] (33)

Solving equations 31 and 32 for the velocities we find
\[ \dot{\vec{R}} = \frac{\mu \left( 1 + \frac{2e^2}{Mc^2r} \right) \vec{P}_R + K_L e^2 \vec{p}_r}{\Delta}, \] (34)

and
\[ \dot{\vec{r}} = \frac{M \left( 1 - \frac{2e^2}{Mc^2r} \right) \vec{p}_r + K_L e^2 \vec{P}_R}{\Delta}, \] (35)
where

$$\Delta = M\mu \left( 1 - \frac{4e^4}{M^2e^4r^2} \right) - \frac{K^2 e^4}{c^4r^2} = m_pm_e - \frac{e^4}{c^4r^2}.$$  

Now we are ready to write the Hamilton’s Function

$$H = \frac{1}{2} \left( 1 + \frac{2e^2}{Mce^2r} \right) \vec{p}^2 M + \left( 1 - \frac{2e^2}{Mce^2r} \right) \vec{p}^2 \mu + \frac{2K_L e^2}{m_pm_e c^2r} \vec{R} \cdot \vec{p} - \frac{e^2}{r}.$$  

(36)

References

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