FUBINI THEOREMS FOR ANALYTIC YEH–FEYNMAN INTEGRALS ASSOCIATED WITH GAUSSIAN PROCESSES WITH APPLICATIONS

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Abstract. In this paper we study an analytic Yeh–Feynman integral associated with Gaussian processes. Fubini theorems involving the generalized analytic Yeh–Feynman integrals are established. The Fubini theorems investigated in this paper are to express the iterated generalized Yeh–Feynman integrals associated with different Gaussian processes as a single generalized Yeh–Feynman integral. As applications, we examined fundamental relationships (and with extended versions) between generalized Yeh–Fourier–Feynman transforms and convolution products (with respect to Gaussian processes) of functionals on Yeh–Wiener space.

1. Introduction

Given a positive real $T > 0$, let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions $x$ on the compact interval $[0, T]$ with $x(0) = 0$. As mentioned in [20], the usual Fubini theorem does not apply to analytic Wiener and Feynman integrals since they are not defined in terms of a countably additive nonnegative measure. Rather, they are defined in terms of a process of analytic continuation and a limiting procedure, see [4, 10]. Thus, in [20, 21], Huffman, Skoug, and Storvick investigated the structure of the Fubini theorem for analytic Feynman integrals and analytic Fourier–Feynman transforms of functionals on the classical Wiener space $C_0[0,T]$. The Fubini theorems for the analytic Feynman integral presented in [20, 21] also are effected by the concept of the scale-invariant measurability [5, 23] in $C_0[0,T]$.

In [26], Kitagawa introduced a function space which is the collection of the two variables continuous functions $x(s,t)$ on the unit square $[0,1] 	imes [0,1]$ satisfying $x(s,t) = 0$ for $(s,t) \in [0,1] \times [0,1]$ with $st = 0$, and he investigated the integration on this space. In [36], Yeh developed the measure of this space and made a logical foundation on this space. We call this space a Yeh–Wiener space and the integral a Yeh–Wiener integral.

The Fubini theorems studied in [20, 21] are related to the variance parameter defining the analytic Feynman integral. The purpose of this paper is to establish the Fubini theorem for the generalized Yeh–Feynman integral of functionals on the Yeh–Wiener space. The definition of the generalized Yeh–Feynman integral is based on the Yeh–Wiener integral of functionals in sample paths of Gaussian process $Y_h$.

1991 Mathematics Subject Classification. Primary 44A15, 46G12; Secondary 28C20, 42B10, 60G15, 60J65.

Key words and phrases. Fubini theorem, Gaussian process, generalized Yeh–Feynman integral, generalized Yeh–Fourier–Feynman transform, convolution products.
on Yeh–Wiener space \((C_0([0,S] \times [0,T]), m_y)\), see Section 2 below, as follows:

\[
\int_{C_0([0,S] \times [0,T])} F(Y_h(x; \cdot, \cdot)) dm_y(x),
\]

where \(Y_h\) is the Gaussian process on \(C_0([0,S] \times [0,T]) \times [0,S] \times [0,T]\) given by

\[
Y_h(x; s, t) = \int_0^s h(\nu, \tau) dX(\nu, \tau),
\]

and where \(h\) is a nonzero function in \(L_2([0,S] \times [0,T])\) and \(\int_0^s \int_0^\tau h(\nu, \tau) d\nu d\tau\) denotes the Paley–Wiener–Zygmund stochastic integral [17, 27, 28, 29]. The concept of the generalized Yeh–Wiener integral was introduced by Park and Skoug [31], and further developed in [32]. The Gaussian processes used in this paper, as well as in [31, 32], are generally non-stationary processes. The Fubini theorems investigated in this paper are related to the kernel functions in Gaussian processes defining the generalized Yeh–Feynman integral. We also apply our Fubini theorems to investigate fundamental relationships between the generalized Fourier–Yeh–Feynman transform and the generalized convolution product associated with Gaussian processes on the Yeh–Wiener space \(C_0(Q)\). The aesthetic value of these relations is illustrated in Section 6 below.

2. Definitions and preliminaries

Yeh–Wiener space [30] is the two parameter Wiener space \((C_0(Q), B(C_0(Q)), m_y)\) where \(Q\) is the compact rectangle \([0, S] \times [0, T]\) with nonzero area in \(\mathbb{R}^2\). \(C_0(Q)\) is the space of all real-valued continuous functions \(x\) on \(Q\) such that \(x(0, t) = x(s, 0) = 0\) for every \((s, t)\) in \(Q\). \(B(C_0(Q))\) is the Borel \(\sigma\)-field induced by the uniform norm on \(C_0(Q)\), and \(m_y\) denotes the Yeh–Wiener measure, see [26, 31, 32, 36]. The sample functions \(x\) in \(C_0(Q)\) is often called Brownian surfaces or Brownian sheets.

Let \(W(C_0(Q))\) be the class of \(m_y\)-measurable (in the sense of the Carathéodory measurability) subsets of \(C_0(Q)\). It is well known that \(W(C_0(Q))\) coincides with \(\sigma(B(C_0(Q)))\), the completion of the Borel \(\sigma\)-field \(B(C_0(Q))\). A subset \(E\) of \(C_0(Q)\) is said to be scale-invariant measurable [13, 23] provided \(\rho E\) is \(W(C_0(Q))\)-measurable for every \(\rho > 0\), and a scale-invariant measurable subset \(N\) of \(C_0(Q)\) is said to be scale-invariant null provided \(m_y(\rho N) = 0\) for every \(\rho > 0\). A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional \(F\) on \(C_0(Q)\) is said to be scale-invariant measurable provided \(F\) is defined on a scale-invariant measurable set and \(F(\rho \cdot)\) is \(W(C_0(Q))\)-measurable for every \(\rho > 0\).

The Paley–Wiener–Zygmund (PWZ) stochastic integral [17, 27, 28, 29] plays a key role throughout this paper. Let \(\{\phi_n\}\) be a complete orthonormal set in \(L_2(Q)\), each of whose elements is of bounded variation in the sense of Hardy–Krause [2] on \(Q\). Then for each \(v \in L_2(Q)\), the PWZ stochastic integral \(\langle v, x \rangle\) is defined by the formula

\[
\langle v, x \rangle = \lim_{n \to \infty} \int_Q \sum_{j=1}^n \langle v, \phi_j \rangle_2 \phi_j(s, t) dx(s, t)
\]

for all \(x \in C_0(Q)\) for which the limit exists, where \((\cdot, \cdot)_2\) denotes the \(L_2(Q)\)-inner product. We state some useful facts about the PWZ stochastic integral.

(i) For each \(v \in L_2(Q)\), the limit defining the PWZ stochastic integral \(\langle v, x \rangle\) exists for s-a.e. \(x \in C_0(Q)\) and that this limit is essentially independent of the choice of the complete orthonormal set \(\{\phi_n\}\).
(ii) If $v$ is of bounded variation on $Q$, then the PWZ stochastic integral $\langle v, x \rangle$ is equal to the Riemann–Stieltjes integral $\int_0^T \int_0^S v(s, t)dx(s, t)$ for $s$-a.e. $x \in C_0(Q)$.

(iii) The PWZ stochastic integral has the expected linearity properties. That is, for any real number $c, v \in L_2(Q)$, and $x \in C_0(Q)$, it follows that $\langle v, cx \rangle = c \langle v, x \rangle = (cv, x)$.

(iv) For each $v \in L_2(Q)$, $\langle v, x \rangle$ is a Gaussian random variable on $C_0(Q)$ with mean zero and variance $\|v\|_2^2$. From this, it follows that

\[(2.1) \quad \int_{C_0(Q)} \exp \left\{ i\alpha \langle v, x \rangle \right\} dm_\nu(x) = \exp \left\{ -\frac{\alpha^2}{2} \|v\|_2^2 \right\}.\]

for each $\alpha \in \mathbb{C}$.

(v) For all $u, v \in L_2(Q)$, it follows that

\[\int_{C_0(Q)} \langle u, x \rangle \langle v, x \rangle dm_\nu(x) = (u, v)_2.\]

Thus, if $\{v_1, \ldots, v_n\}$ is an orthogonal set in $L_2(Q)$, then the Gaussian random variables $\langle v_j, x \rangle$’s are independent.

Throughout this paper we let

\[\text{Supp}_2(Q) = \{h \in L_2(Q) : m_2^2(\text{supp}(h)) = ST \} = \{h \in L_2(Q) : h \neq 0 \text{ } m_2^2 \text{-a.e on } Q \}\]

and

\[\text{Supp}_{BV}(Q) = \{h : h \text{ is of bounded variation with } h \neq 0 \text{ } m_2^2 \text{-a.e on } Q \}\]

where $m_2^2$ denotes Lebesgue measure on $Q$. Then one can see that $\text{Supp}_{BV}(Q) \subset \text{Supp}_2(Q)$.

Given a function $h$ in $\text{Supp}_2(Q)$, we next consider the stochastic integral $\mathcal{Y}_h(x; s, t)$ given by

\[(2.2) \quad \mathcal{Y}_h(x; s, t) = \langle \chi_{[0, s] \times [0, t]}h, x \rangle,\]

for $x \in C_0(Q)$ and $(s, t) \in Q$, which it was introduced by Park and Skoug in [31].

Then the process $\mathcal{Y}_h$ on $C_0(Q) \times Q$ is a Gaussian process with mean zero and covariance function

\[\int_{C_0(Q)} \mathcal{Y}_h(x; s, t)\mathcal{Y}_h(x; s', t')dm_\nu(x) = \int_0^{\min\{t, t'\}} \int_0^{\min\{s, s'\}} h^2(\nu, \tau)d\nu d\tau.\]

Furthermore one can see that

\[(2.3) \quad \int_{C_0(Q)} \mathcal{Y}_{h_1}(x; s, t)\mathcal{Y}_{h_2}(x; s', t')dm_\nu(x) = \int_0^{\min\{t, t'\}} \int_0^{\min\{s, s'\}} h_1(\nu, \tau)h_2(\nu, \tau)d\nu d\tau.\]

Since the covariance function of $\mathcal{Y}_h(x; \cdot, \cdot)$ is stochastically continuous, we may assume that almost every sample path of $\mathcal{Y}_h(x; \cdot, \cdot)$ is in $C_0(Q)$. Also, if $h$ is a function in $\text{Supp}_{BV}(Q)$, then for all $x \in C_0(Q)$, $\mathcal{Y}_h(x; s, t)$ is continuous in $(s, t) \in Q$, and so $\mathcal{Y}_h(x; \cdot, \cdot)$ is in $C_0(Q)$. Thus, for the definition of the generalized analytic Yeh–Feynman integral of functionals on $C_0(Q)$, we require $h$ to be in $\text{Supp}_{BV}(Q)$ rather than simply in $\text{Supp}_2(Q)$. 
3. Generalized analytic Yeh–Feynman integral

In this section we introduce the generalized analytic Yeh–Feynman integral of functionals on $C_0(Q)$. We then present a class of generalized Yeh–Feynman integrable functionals.

Throughout the rest of this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively.

Given a Gaussian process $\mathcal{Y}_h$ with $h \in \text{Supp}_{BV}(Q)$, we define the (generalized) $\mathcal{Y}_h$-Yeh–Wiener integral (namely, the Yeh–Wiener integral associated with Gaussian paths $\mathcal{Y}_h(x;\cdot,\cdot)$) for functionals $F$ on $C_0(Q)$ by the formula

$$I_h[F] \equiv I_{h,x}[F(\mathcal{Y}_h(x;\cdot,\cdot))] \equiv \int_{C_0(Q)} F(\mathcal{Y}_h(x;\cdot,\cdot)) dm_\tau(x).$$

Let $F : C_0(Q) \to \mathbb{C}$ be a scale-invariant measurable functional such that

$$J_F(h;\lambda) = I_{h}[F(\lambda^{-1/2} ;)] = I_{h,x}[F(\lambda^{-1/2} \mathcal{Y}_h(x;\cdot,\cdot))]$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^q(h;\cdot)$ analytic on $\mathbb{C}_+$ such that $J_F^q(h;\lambda) = J_F(h;\lambda)$ for all $\lambda > 0$, then $J_F^q(h;\lambda)$ is defined to be the analytic $\mathcal{Y}_h$-Yeh–Wiener integral (namely, the analytic Yeh–Wiener integral associated with Gaussian paths $\mathcal{Y}_h(x;\cdot,\cdot)$) of $F$ over $C_0(Q)$ with parameter $\lambda$. For $\lambda \in \mathbb{C}_+$ we write

$$I_{h}^{an.yw^q} [F] \equiv I_{h,x}^{an.yw^q} [F(\mathcal{Y}_h(x;\cdot,\cdot))]$$

$$\equiv \int_{C_0(Q)}^{an.yw^q} F(\mathcal{Y}_h(x;\cdot,\cdot)) dm_\tau(x) = J_F^q(h;\lambda).$$

Let $q \neq 0$ be a real number, and let $F$ be a scale-invariant measurable functional whose analytic $\mathcal{Y}_h$-Yeh–Wiener integral $I_{h}^{an.yw^q} [F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic $\mathcal{Y}_h$-Yeh–Feynman integral (namely, the analytic Yeh–Feynman integral associated with Gaussian paths $\mathcal{Y}_h(x;\cdot,\cdot)$) of $F$ with parameter $q$, and we write

$$I_{h}^{an.yf^q} [F] \equiv I_{h,x}^{an.yf^q} [F(\mathcal{Y}_h(x;\cdot,\cdot))] \equiv \int_{C_0(Q)}^{an.yf^q} F(\mathcal{Y}_h(x;\cdot,\cdot)) dm_\tau(x)$$

$$= \lim_{\lambda \to -iq} I_{h,x}^{an.yw^\lambda} [F(\mathcal{Y}_h(x;\cdot,\cdot))]$$

where $\lambda$ approaches $-iq$ through values in $\mathbb{C}_+$.

Let $\mathcal{M}(L_2(Q))$ be the space of complex-valued, countably additive Borel measures on $\mathcal{B}(L_2(Q))$, the Borel class of $C_0(Q)$. Then the measure $f$ in $\mathcal{M}(L_2(Q))$ necessarily has finite total variation $\|f\|$, and $\mathcal{M}(C_0(Q))$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication, see [12, 34]. The Banach algebra $S(L_2(Q))$ consists of functionals expressible in the form

$$F(x) = \int_{L_2(Q)} \exp\{i\langle u,x\rangle\} df(u)$$

for s.a.e. $x$ in $C_0(Q)$ where $f$ is an element of $\mathcal{M}(L_2(Q))$. For a more detailed study of the Banach algebra $S(L_2(Q))$, see [11, 26, 25].

The following lemma, which follows quite easily from the definition of the PWZ stochastic integral, plays a key role in this paper.
Theorem 3.3. Let \( q \) be any nonzero real.

\[
\langle \alpha, \gamma_h(x; \cdot, \cdot) \rangle = \langle \alpha, x \rangle
\]

exists and is given by the formula

\[\Box\]
as desired.

Lemma 3.1. For each \( \alpha \in L_2(Q) \) and each \( h \in \text{Supp}_{BV}(Q) \),

\[
(3.5) \quad \langle \alpha, \gamma_h(x; \cdot, \cdot) \rangle = \langle \alpha h, x \rangle
\]

for \( s \)-a.e. \( x \in C_0(Q) \).

Lemma 3.2. Let \( F \in \mathcal{S}(L_2(Q)) \) be given by (3.4) and let \( h \) be a function in \( \text{Supp}_{BV}(Q) \). Then the functional \( G_h \) given by \( G_h(x) = F(\gamma_h(x; \cdot, \cdot)) \) belongs to the Banach algebra \( \mathcal{S}(L_2(Q)) \).

Proof. Let \( \Phi_h : L_2(Q) \to L_2(Q) \) be given by \( \Phi_h(u) = uh \), pointwise multiplication of \( u \) and \( h \) in \( L_2(Q) \). Then \( \Phi_h \) is easily seen to be continuous and so is Borel measurable. Hence \( f_{\Phi_h} \equiv f \circ \Phi_h^{-1} \) is in \( \mathcal{M}(L_2(Q)) \). In addition, for each \( \rho > 0 \), using the change of variables theorem [15, p.163] and (3.5), it follows that for \( a.e. \ x \in C_0(Q) \),

\[
\int_{L_2(Q)} \exp\{i\rho\langle u, x \rangle\}df_{\Phi_h}(u) = \int_{L_2(Q)} \exp\{i\rho(u, x)\}d(f \circ \Phi_h^{-1})(u)
\]

\[
= \int_{L_2(Q)} \exp\{i\rho(\Phi_h(u), x)\}df(u)
\]

\[
= \int_{L_2(Q)} \exp\{i\rho(uh, x)\}df(u)
\]

\[
= \int_{L_2(Q)} \exp\{i\rho(u, \gamma_h(x; \cdot, \cdot))\}df(u)
\]

\[
= F(\rho \gamma_h(x; \cdot, \cdot))
\]

\[
= G(\rho x)
\]
as desired. \( \square \)

We now state the existence theorem for the generalized analytic Yeh–Feynman integral of the functionals in \( \mathcal{S}(L_2(Q)) \).

Theorem 3.3. Let \( F \in \mathcal{S}(L_2(Q)) \) be given by (3.4). Then for all \( h \in \text{Supp}_{BV}(Q) \) and any nonzero real \( q \), the analytic \( \gamma_h \)-Yeh–Feynman integral, \( I_{h}^{\text{an},YF}[F] \) of \( F \) exists and is given by the formula

\[
(3.6) \quad I_{h}^{\text{an},YF}[F] = \int_{L_2(Q)} \exp\left\{ -\frac{i}{2q} \|uh\|_2^2 \right\} df(u).
\]

Proof. Using (3.4), the usual Fubini theorem, (3.5), and (2.1), it follows that for all \( \lambda > 0 \),

\[
J_F(h; \lambda) = \int_{C_0(Q)} F(\lambda^{-1/2} \gamma_h(x; \cdot, \cdot))dm_\gamma(x)
\]

\[
= \int_{L_2(Q)} \left[ \int_{C_0(Q)} \exp \left\{ i\lambda^{-1/2} \langle uh, x \rangle \right\} dm_\gamma \right] df(u)
\]

\[
= \int_{L_2(Q)} \exp \left\{ -\frac{1}{2\lambda} \|uh\|_2^2 \right\} df(u).
\]

Now let

\[
J_{hF}^*(h; \lambda) = \int_{L_2(Q)} \exp \left\{ -\frac{1}{2\lambda} \|uh\|_2^2 \right\} df(u)
\]
for \( \lambda \in \mathbb{C}_+ \). Then \( J_F^*(h; \lambda) = J_F(h; \lambda) \) for all \( \lambda > 0 \) and

\[
|J_F^*(h; \lambda)| \leq \int_{L^2(Q)} \exp \left\{ -\frac{\|uh\|^2}{2\lambda} \right\} |f|(u) \leq \int_{L^2(Q)} |f|(u) = \|f\| < +\infty
\]

for all \( \lambda \in \mathbb{C}_+ \), since \( \operatorname{Re}(1/\lambda) > 0 \). Thus, applying the dominated convergence theorem, we see that \( J_F^*(h; \lambda) \) is continuous on \( \tilde{C}_+ \). Also, because \( \phi(\lambda) \equiv \exp\{-\|uh\|^2/(2\lambda)\} \) is analytic on \( \mathbb{C}_+ \), applying the usual Fubini theorem and the Cauchy integration theorem it follows that
\[
\int_{\triangle} J_F^*(h; \lambda) d\lambda = \int_{L^2(Q)} \int_{\triangle} \phi(\lambda) d\lambda df(u) = 0
\]

for all rectifiable simple closed curve \( \triangle \) lying in \( \mathbb{C}_+ \). Thus by the Morera theorem, \( J_F^*(h; \lambda) \) is analytic on \( \mathbb{C}_+ \). Therefore the analytic \( Y \)-Yeh–Wiener integral

\[
I_{an} \cdot yf h[x]
\]

exists. Finally, applying the dominated convergence theorem it follows that

\[
I_{an} \cdot yf h[x] = \lim_{\lambda \to -iq} I_{an} \cdot yf h[\mathcal{Y}_h(x; \cdot, \cdot)],
\]

where

\[
(4.2) \quad \alpha_n = \left( \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} \right)^{-1}.
\]

Equation (4.1) tells us that an iterated analytic \( \mathcal{Y}_h \)-Yeh–Feynman integral can be reduced to a single analytic \( \mathcal{Y}_h \)-Yeh–Feynman integral. In this section we establish that the iterated generalized Yeh–Feynman integrals associated with different Gaussian processes also can be reduced to a single generalized Yeh–Feynman integral.

4. Fubini theorems for the generalized analytic Yeh–Feynman integral

In this section we study Fubini theorems for the iterated \( \mathcal{Y}_h \)-Yeh–Feynman integrals. In [20], Huffman, Skoug and Storvick presented a Fubini theorem involving the iterated analytic Feynman integrals for functionals on the classical Wiener space \( C_0[0, T] \). The Fubini theorem can be extended to the \( \mathcal{Y}_h \)-Yeh–Feynman integral on the Yeh–Wiener space \( C_0(Q) \) as follows:

**Theorem 4.1.** Let \( F \in \mathcal{S}(L^2(Q)) \) be given by equation (3.4) and let \( \{q_1, q_2, \ldots, q_n\} \) be a set of nonzero real numbers with

\[
\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_k} \neq 0
\]

for each \( k \in \{2, \ldots, n\} \). Then for any function \( h \) in \( \text{Supp}_{BV}(Q) \),

\[
I_{h,x_n}^{an,yf_{q_n}} \left[ I_{h,x_{n-1}}^{an,yf_{q_{n-1}}} \left[ \cdots \left[ I_{h,x_2}^{an,yf_{q_2}} \left[ I_{h,x_1}^{an,yf_{q_1}} \left[ F\left( \sum_{j=1}^{n} \mathcal{Y}_h(x_j, \cdot) \right) \right] \right] \cdots \right] \right] \right] \]

(4.1)

\[
= I_{h,x}^{an,yf_{\alpha_n}} \left[ F(\mathcal{Y}_h(x; \cdot, \cdot)) \right],
\]

where

\[
(4.2) \quad \alpha_n = \left( \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} \right)^{-1}.
\]
In order to obtain our Fubini theorems for the Yeh–Feynman integrals associated with Gaussian processes (see Theorem 4.3 below) we adopt the following conventions. Let \( h_1 \) and \( h_2 \) be nonzero functions in \( L_2(Q) \). Then there exists a nonzero function \( s \) in \( L_2(Q) \) such that

\[
s^2(s, t) = h_1^2(s, t) + h_2^2(s, t)
\]

for \( m_2^Q \)-a.e. \( (s, t) \in Q \). Note that the function ‘\( s \)’ satisfying (4.3) is not unique. We will use the symbol \( s(h_1, h_2) \) for the functions ‘\( s \)’ that satisfy (4.3) above.

Inductively, given a set \( H = \{ h_1, \ldots, h_n \} \) of nonzero functions in \( L_2(Q) \), let

\[
s(H) \equiv s(h_1, h_2, \ldots, h_n)
\]

be the set of functions \( s \) which satisfy the relation

\[
s^2(s, t) = h_1^2(s, t) + \cdots + h_n^2(s, t)
\]

for \( m_2^Q \)-a.e. \( (s, t) \in Q \). We note that if the functions \( h_1, \ldots, h_n \) are in \( \text{Supp}_{BV}(Q) \), then we can take \( s(H) \) to be in \( \text{Supp}_{BV}(Q) \). By an induction argument we see that

\[
s(s(h_1, h_2, \ldots, h_{k-1}), h_k) = s(h_1, h_2, \ldots, h_k)
\]

for all \( k \in \{2, \ldots, n\} \).

In our next lemma we obtain a Fubini theorem for the iterated Yeh–Wiener integral associated with different Gaussian paths of functionals \( F \) in \( \mathcal{S}(L_2(Q)) \).

**Lemma 4.2.** Let \( h_1 \) and \( h_2 \) be functions in \( \text{Supp}_{BV}(Q) \) and let \( F \in \mathcal{S}(L_2(Q)) \) be given by equation (3.4). Then for all \( \alpha \) and \( \beta \) in \( \mathbb{R} \),

\[
\int_{C_0(Q)} \left[ \int_{C_0(Q)} F(\alpha Y_{h_1}(x_1; \cdot, \cdot) + \beta Y_{h_2}(x_2; \cdot, \cdot)) \, dm_y(x_1) \right] \, dm_y(x_2)
\]

\[
= \int_{C_0(Q)} \left[ \int_{C_0(Q)} F(\alpha Y_{h_1}(x_1; \cdot, \cdot) + \beta Y_{h_2}(x_2; \cdot, \cdot)) \, dm_y(x_2) \right] \, dm_y(x_1).
\]

In addition, both expressions in (4.5) are given by the expression

\[
\int_{L_2(Q)} \exp \left\{ -\frac{\alpha^2}{2} \|uh_1\|_2^2 - \frac{\beta^2}{2} \|uh_2\|_2^2 \right\} df(u).
\]

**Proof.** Using (3.4) and (3.5), it follows that

\[
\int_{C_0(Q)} |F(\rho Y_h(x; \cdot, \cdot))| \, dm_y(x) \leq \int_{C_0(Q)} \| f \| \, dm_y(x) = \| f \| < +\infty
\]

for each \( \rho > 0 \). Hence by the usual Fubini theorem, we have equation (4.5) above. Furthermore, using the usual Fubini theorem, (3.5), and (2.1), it follows that for
Thus we obtain the analytic continuation

\[ I_{L_2(Q)} \left[ \int_{C_0(Q)} \left( \int_{C_0(Q)} F(\alpha h_1(x_1; \cdot, \cdot) + \beta h_2(x_2; \cdot, \cdot)) dm_2(x_2) \right) dm_2(x_1) \right] df(u) \]

\[ = \int_{L_2(Q)} \left[ \int_{C_0(Q)} \exp \left\{ i\alpha(uh_1, x_1) \right\} dm_2(x_1) \right] \times \left[ \int_{C_0(Q)} \exp \left\{ i\beta(uh_2, x_2) \right\} dm_2(x_2) \right] df(u) \]

\[ = \int_{L_2(Q)} \exp \left\{ -\frac{\alpha^2}{2} \|uh_1\|_2^2 \right\} \exp \left\{ -\frac{\beta^2}{2} \|uh_2\|_2^2 \right\} df(u) \]

as desired. \hfill \Box

**Theorem 4.3.** Let \( h_1, h_2, \) and \( F \) be as in Lemma 4.2. Then, for all nonzero real \( q \), the iterated Yeh–Feynman integral, \( I_{h_2}^{an,yf_q}[I_{h_1}^{an,yf_q}[F]] \) of \( F \) exists and is given by the formula

\[
I_{h_2,x_2}^{an,yf_q}[I_{h_1,x_1}^{an,yf_q}[F(h_1(x_1; x_2; \cdot, \cdot) + h_2(x_2; \cdot, \cdot))]]
\]

\[ = \int_{L_2(Q)} \exp \left\{ -\frac{i}{2q} \sum_{j=1}^2 \|uh_j\|_2^2 \right\} df(u). \]

Furthermore, it follows that

\[
I_{h_2,x_2}^{an,yf_q}[I_{h_1,x_1}^{an,yf_q}[F(h_1(x_1; x_2; \cdot, \cdot) + h_2(x_2; \cdot, \cdot))]] = I_{s(h_1,h_2),x}^{an,yf_q}[F(s(h_1,h_2)(x; \cdot, \cdot))]
\]

where \( s(h_1,h_2) \) is a function in \( \text{Supp}_{BV}(Q) \) satisfying relation (4.3) above.

**Proof.** Using (4.25) together with (4.40), it follows that that for all \( (\lambda_1, \lambda_2) \in (0, +\infty) \times (0, +\infty) \),

\[
I_{h_2,x_2}^{an,yf_q}[I_{h_1,x_1}^{an,yf_q}[F(h_1^{-1/2}h_1(x_1; \cdot, \cdot) + h_2^{-1/2}h_2(x_2; \cdot, \cdot))]]
\]

\[ = \int_{L_2(Q)} \exp \left\{ -\frac{1}{2\lambda_1} \|uh_1\|_2^2 - \frac{1}{2\lambda_2} \|uh_2\|_2^2 \right\} df(u). \]

For each \( \lambda_2 > 0 \) it can be analytically continued in \( \lambda_1 \) for \( \lambda_1 \in \mathbb{C}_+ \), and for each \( \lambda_1 > 0 \) it also can be analytically continued in \( \lambda_2 \) for \( \lambda_2 \in \mathbb{C}_+ \), because for any \( (\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+ \),

\[
\left| \int_{L_2(Q)} \exp \left\{ -\frac{1}{2\lambda_1} \|uh_1\|_2^2 - \frac{1}{2\lambda_2} \|uh_2\|_2^2 \right\} df(u) \right|
\]

\[ \leq \int_{L_2(Q)} \left| \exp \left\{ -\sum_{j=1}^2 \frac{\text{Re}(\lambda_j) - i\text{Im}(\lambda_j)}{2|\lambda_j|^2} \|uh_j\|_2^2 \right\} \right| df(u) \leq \|f\| < +\infty. \]

Thus we obtain the analytic continuation

\[
I_{h_2,x_2}^{an,yf_{\lambda_2}}[I_{h_1,x_1}^{an,yf_{\lambda_1}}[F(h_1(x_1; x_2; \cdot, \cdot) + h_2(x_2; \cdot, \cdot))]]
\]

\[ = \int_{L_2(Q)} \exp \left\{ -\frac{1}{2\lambda_1} \|uh_1\|_2^2 - \frac{1}{2\lambda_2} \|uh_2\|_2^2 \right\} df(u) \]
of \( I_{h_2,x_2}[H_{h_1,x_1}[F(\lambda_{1}^{-1/2}Y_{h_1}(x_1;\cdot,\cdot) + \lambda_{2}^{-1/2}Y_{h_2}(x_2;\cdot,\cdot))] | as \) a function of \((\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+\), and so it follows that

\[
\begin{align*}
I_{h_2,x_2}^{an,yf_q} & [I_{h_1,x_1}^{an,yf_q} [F(Y_{h_1}(x_1;\cdot,\cdot) + Y_{h_2}(x_2;\cdot,\cdot))] \\
& = \lim_{\lambda_2 \to -iq} \int_{C_0(Q)} \left[ \lim_{\lambda_1 \to -iq} \int_{C_0(Q)} F(\lambda_{1}^{-1/2}Y_{h_1}(x_1;\cdot,\cdot) \\
& + \lambda_{2}^{-1/2}Y_{h_2}(x_2;\cdot,\cdot)) \right] dm_{y}(x_1) \right] dm_{y}(x_2) \\
& = \int_{L_2(Q)} \exp \left\{ -\frac{i}{2q} \sum_{j=1}^{2} \|uh_j\|^2 \right\} df(u).
\end{align*}
\]

Next using (4.3), we observe that

\[
\sum_{j=1}^{2} \|uh_j\|^2 = \int_{0}^{T} \int_{0}^{S} u^2(s,t)h_1^2(s,t)dsdt + \int_{0}^{T} \int_{0}^{S} u^2(s,t)h_1^2(s,t)dsdt \\
= \int_{0}^{T} \int_{0}^{S} u^2(s,t)(h_1^2(s,t) + h_2^2(s,t))dsdt \\
= \int_{0}^{T} \int_{0}^{S} u^2(s,t)s^2(h_1, h_2)(s,t)dsdt \\
= \|us(h_1, h_2)\|^2.
\]

Using this and equation (3.3) with \( h \) replaced with \( s(h_1, h_2) \), the generalized analytic Yeh–Feynman integral \( I_{s(h_1, h_2)}^{an,yf_q}[F] \) is given by the right-hand side of equation (4.7). This completes the proof. \( \square \)

Using mathematical induction we obtain the following corollary.

**Corollary 4.4.** Let \( \mathcal{H} = \{h_1, \ldots, h_n\} \) be a set of functions in \( \text{Supp}_{BV}(Q) \) and let \( F \in \mathcal{S}(L_2(Q)) \) be given by equation (3.3). Then, for all nonzero real \( q \), the iterated analytic Yeh–Feynman integral in the following equation exist, and is given by the formula

\[
I_{h_1,x_1, \ldots, h_n,x_n}^{an,yf_q} \left[ \ldots [I_{h_2,x_2}^{an,yf_q} [I_{h_1,x_1}^{an,yf_q} [F(\sum_{j=1}^{n} \mathcal{Z}_{h_j}(x_j;\cdot))]]]] \right] \\
= \int_{L_2(Q)} \exp \left\{ -\frac{i}{2q} \sum_{j=1}^{n} \|uh_j\|^2 \right\} df(u).
\]

Moreover it follows that

\[
(4.8)
\]

\[
I_{h_1,x_1, \ldots, h_n,x_n}^{an,yf_q} \left[ \ldots [I_{h_2,x_2}^{an,yf_q} [I_{h_1,x_1}^{an,yf_q} [F(\sum_{j=1}^{n} \mathcal{Z}_{h_j}(x_j;\cdot))]]]] \right] \\
= I_{s(\mathcal{H}),x}^{an,yf_q}[F(\mathcal{Z}_{s(\mathcal{H})}(x;\cdot))]
\]

where \( s(\mathcal{H}) \equiv s(h_1, \ldots, h_n) \) is a function in \( \text{Supp}_{BV}(Q) \) satisfying relation (4.4) above.
Example 4.5. Let $\mathcal{H}_4 = \{h_1, h_2, h_3, h_4\}$ be a set of functions in $\text{Supp}_{BV}(Q)$, where
\[
\begin{aligned}
    h_1(s,t) &= \sin^2 s \cos t, \\
    h_2(s,t) &= \sin s \cos s \cos t, \\
    h_3(s,t) &= \sin s \sin t \cos t, \\
    h_4(s,t) &= \sin s \cos^2 t
\end{aligned}
\]
for $(s,t) \in Q$. In this case, we can choose the function $s(\mathcal{H}_4) \equiv s(h_1, h_2, h_3, h_4)$ to be
\[
s(\mathcal{H}_4) \equiv s(h_1, h_2, h_3, h_4) = \sqrt{2} \sin s \cos t,
\]
since
\[
\sum_{j=1}^{4} h_j^2(s,t) = 2 \sin^2 s \cos^2 t.
\]
Thus, using (4.8), it follows that
\[
I_{h_4,x_4}^{\text{an},yq} \left[ I_{h_3,x_3}^{\text{an},yq} \left[ I_{h_2,x_2}^{\text{an},yq} \left[ I_{h_1,x_1}^{\text{an},yq} \left[ F \left( \sum_{j=1}^{4} Z_{h_j}(x_j;\cdot;\cdot) \right) \right] \right] \right] \right] = I_{s(\mathcal{H}_4),x}^{\text{an},yq} \left[ F \left( Z_{s(\mathcal{H}_4)}(x;\cdot;\cdot) \right) \right].
\]

5. Generalized Fourier–Yeh–Feynman transforms

The concept of an $L_1$ analytic Fourier–Feynman transform was introduced by Brue in [3]. In [4], Cameron and Storvick introduced an $L_2$ analytic Fourier–Feynman transform. In [22], Johnson and Skoug developed an analytic Fourier–Feynman transform for $1 \leq p \leq 2$ which extended the results in [3, 4] and gave various relationships between the $L_1$ and $L_2$ theories. The transforms studied in [3, 6, 22] are defined on various classes of functionals $F$ on the classical Wiener space.

In this section we apply the Fubini theorems obtained in the previous section to study several relevant behaviors of the generalized Fourier–Yeh–Feynman transform (GFYFT) of functionals on Yeh–Wiener space $C_0(Q)$. In this paper, for simplicity, we restrict our discussion to the case $p = 1$; however most of our results hold for all $p \in [1, 2]$.

Definition 5.1. Let $\mathcal{Y}_h$ be the Gaussian process given by (2.2) with $h \in \text{Supp}_{BV}(Q)$, and let $F$ be a scale-invariant measurable functional on $C_0(Q)$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0(Q)$, let
\[
T_{\lambda,h}(F)(y) = I_{h,x}^{\text{an},yq} \left[ F(y + \mathcal{Y}_h(x;\cdot;\cdot)) \right].
\]
Then for $q \in \mathbb{R} \setminus \{0\}$, the $L_1$ analytic $\mathcal{Y}_h$-GFYFT (namely, the GFYFT associated with the Gaussian paths $\mathcal{Y}_h(x;\cdot;\cdot)$), $T_{q,h}^{(1)}(F)$ of $F$, is defined by the formula
\[
T_{q,h}^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_{\lambda,h}(F)(y)
\]
for s.a.e. $y \in C_0(Q)$ whenever this limit exists. That is to say,
\[
T_{q,h}^{(1)}(F)(y) = I_{h,x}^{\text{an},yq} \left[ F(y + \cdot) \right] \equiv I_{h,x}^{\text{an},yq} \left[ F(y + \mathcal{Y}_h(x;\cdot;\cdot)) \right]
\]
for s.a.e. $y \in C_0(Q)$. 

We note that $T^{(1)}_{q,h}(F)$ exists and if $F \approx G$, then $T^{(1)}_{q,h}(G)$ exists and $T^{(1)}_{q,h}(G) \approx T^{(1)}_{q,h}(F)$. One can see that for each $h \in L_2(Q)$, $T^{(1)}_{q,h}(F) \approx T^{(1)}_{q,-h}(F)$ since

$$
\int_{C_0(Q)} F(-x)dm_y(x) = \int_{C_0(Q)} F(x)dm_y(x).
$$

**Remark 5.2.** Note that if $h \equiv 1$ on $Q$, then the generalized analytic Yeh–Feynman integral and the analytic $Z_1$-GFYFT, $T^{(1)}_{q,1}(F)$, agree with the previous definitions of the analytic Yeh–Feynman integral and the analytic Fourier–Yeh–Feynman transform, $T^{(1)}_{q}(F)$, respectively [21, 23] because $Z_1(x, \cdot) = x$ for all $x \in C_0(Q)$.

In view of (5.11) and (3.10) with $F$ replaced with $F(y + \cdot)$, we obtain the following existence theorem.

**Theorem 5.3.** Let $F \in \mathcal{S}(L_2(Q))$ be given by equation (3.4). Then, for all $h \in \text{Supp}_{BV}(Q)$, the $L_1$ analytic $Z_h$-GFYFT, $T^{(1)}_{q,h}(F)$ of $F$ exists for all nonzero real numbers $q$, belongs to $\mathcal{S}(L_2(Q))$ and is given by the formula

$$
T^{(1)}_{q,h}(F)(y) = \int_{L_2(Q)} \exp\{i(u,y)\}df^h_i(u)
$$

for s-a.e. $y \in C_0(Q)$, where $f^h_i$ is the complex measure in $\mathcal{M}(L_2(Q))$ given by

$$
f^h_i(B) = \int_B \exp\left\{ -\frac{i}{2q}\|uh\|^2 \right\}df(u)
$$

for $B \in \mathcal{B}(L_2(Q))$.

The following corollary is a simple consequence of Theorem 5.3.

**Corollary 5.4.** Let $F$ be as in Theorem 5.3. Then, for all $h \in \text{Supp}_{BV}(Q)$ and all nonzero real $q$,

$$
T^{(1)}_{-q,h}(T^{(1)}_{q,h}(F)) \approx F.
$$

As such, the $L_1$ $Z_h$-GFYFT, $T^{(1)}_{q,h}$, has the inverse transform $\{T^{(1)}_{q,h}\}^{-1} = T^{(1)}_{-q,h}$.

**Remark 5.5.** By Theorem 5.3 and an induction argument, one can see that for any functional $F$ in $\mathcal{S}(L_2(Q))$, any nonzero real numbers $q_1, q_2, \ldots, q_n$, and any nonzero functions $h_1, \ldots, h_n$ in $\text{Supp}_{BV}(Q)$, the iterated GFYFT

$$
T^{(1)}_{q_n,h_n}(T^{(1)}_{q_{n-1},h_{n-1}}(\cdots (T^{(1)}_{q_2,h_2}(T^{(1)}_{q_1,h_1}(F)))\cdots))
$$

of $F$ exists and belongs to $\mathcal{S}(L_2(Q))$.

Next, in [21], Huffman, Skoug and Storvick studied a Fubini theorem involving ordinary Fourier–Feynman transform for functionals on classical Wiener space $C_0[0,T]$. Using (5.9) and (3.11) with $F$ replaced with $F(y + \cdot)$, we also obtain the following Fubini theorem involving the $L_1$ analytic GFYFTs of functionals in the Banach algebra $\mathcal{S}(L_2(Q))$.

**Theorem 5.6.** Let $F$ and $\{q_1, q_2, \ldots, q_n\}$ be as in Theorem 4.7. Then it follows that for each function $h \in \text{Supp}_{BV}(Q)$,

$$
T^{(1)}_{q_n,h}(\cdots (T^{(1)}_{q_2,h}(T^{(1)}_{q_1,h}(F)))\cdots)(y) = T^{(1)}_{\alpha_n,h}(F)(y)
$$

for s-a.e. $y \in C_0(Q)$, where $\alpha_n$ is a nonzero real number given by (4.12).
Equation (5.13) above tells us that iterated GFYFT with different variance parameters \( q_1, \ldots, q_n \) can be reduced to a single GFYFT. We will assert that the composition of GFYFTs associated with different Gaussian processes also can be reduced to a single GFYFT. In view of (5.9) and (4.8) with \( F \) replaced with \( F(y+) \), we obtain the following theorem, which will be very useful to prove our main theorems in next sections.

**Theorem 5.7.** Let \( \mathcal{H} = \{ h_1, \ldots, h_n \} \) and let \( F \in \mathbb{S}(L_2(Q)) \) be as in Corollary 4.4. Then it follows that for all nonzero real \( q \),

\[
T_{q,h_n}(T_{q,h_{n-1}}(\cdots(T_{q,h_2}(T_{q,h_1}(F)))\cdots))(y) = T_{q,s(\mathcal{H})}(F)(y)
\]

for \( s \text{-a.e. } y \in C_0(Q) \), where \( s(\mathcal{H}) \equiv s(h_1, \ldots, h_n) \) is a function in \( \text{Supp}_{BV}(Q) \) satisfying relation (4.4) above.

**Example 5.8.** Let \( h_1 \) and \( h_2 \) be given by

\[
h_1(s,t) = \sin \left( \frac{2\pi s}{T} \right) \sin \left( \frac{2\pi t}{T} \right) - \cos \left( \frac{2\pi s}{T} \right) \cos \left( \frac{2\pi t}{T} \right)
\]

and

\[
h_2(s,t) = \sin \left( \frac{2\pi s}{T} \right) \cos \left( \frac{2\pi t}{T} \right) + \cos \left( \frac{2\pi s}{T} \right) \sin \left( \frac{2\pi t}{T} \right)
\]

on \( Q \equiv [0,S] \times [0,T] \), respectively. Then \( h_1 \) and \( h_2 \) are in \( \text{Supp}_{BV}(Q) \) and \( s(h_1, h_2) \equiv \pm 1 \). Thus, by equation (5.14) with \( n = 2 \) and (5.10), we have

\[
T_{q,h_2}(T_{q,h_1}(F))(y) = T_{q,s(h_1, h_2)}(F)(y)
\]

for every \( F \in \mathbb{S}(L_2(Q)) \) and \( s \text{-a.e. } y \in C_0(Q) \). In fact, the right hand side of (5.15) is the ordinary FFT \( T_q^{(1)}(F) \) of \( F \), see Remark 5.2.

We give a very brief development result combining equations (5.14) and (5.13).

**Corollary 5.9.** Let \( \mathcal{H}_1 = \{ h_{11}, \ldots, h_{1n_1} \} \) and \( \mathcal{H}_2 = \{ h_{21}, \ldots, h_{2n_2} \} \) be sets of functions in \( \text{Supp}_{BV}(Q) \) which satisfy the relation

\[
s(\mathcal{H}_1) = s(\mathcal{H}_2)
\]

i.e.,

\[
s(h_{11}, \ldots, h_{1n_1}) = s(h_{21}, \ldots, h_{2n_2})
\]

for \( m_L^2 \text{-a.e. } Q \). Then it follows that for any nonzero real numbers \( q_1 \) and \( q_2 \) with \( q_1 + q_2 \neq 0 \),

\[
T_{q_2,h_{2n_2}}^{(1)}(\cdots(T_{q_2,h_1}(T_{q_1,h_{n_1}}(\cdots(T_{q_1,h_1}(F)))\cdots))(y)
\]

\[
= T_{q_1,h_{n_1}}^{(1)}(\cdots(T_{q_1,h_1}(T_{q_2,h_{n_2}}(F)))\cdots)(y)
\]

\[
= T_{q_1,s(\mathcal{H}_1)}(T_{q_2,s(\mathcal{H}_2)}(F))(y)
\]

\[
= T_{q_1,s(\mathcal{H}_1)}^{(1)}(T_{q_2,s(\mathcal{H}_2)}^{(1)}(F))(y)
\]

for \( s \text{-a.e. } y \in C_0(Q) \), where \( \mathcal{H} \) is a finite set of functions in \( \text{Supp}_{BV}(Q) \) with \( s(\mathcal{H}) = s(\mathcal{H}_1) = s(\mathcal{H}_2) \).
6. Generalized Fourier–Yeh–Feynman transform and
generalized convolution product on $C_0(Q)$

Let $E^m$ be a Euclidean space. For $f \in L_1(E^m)$, let the Fourier transform of $f$
be given by

$$\mathcal{F}(f)(\vec{u}) = \int_{E^m} e^{i\vec{u} \cdot \vec{v}} f(\vec{v}) dm_{L}^m(\vec{v})$$

and for $f, g \in L_1(E^m)$, let the convolution of $f$ and $g$ be given by

$$(f * g)(\vec{u}) = \int_{E^m} f(\vec{u} - \vec{v}) g(\vec{v}) dm_{L}^m(\vec{v})$$

where $\vec{u} \cdot \vec{v}$ denotes the dot product of vectors $\vec{u}$ and $\vec{v}$ in $E^m$, $dm_{L}^m(\vec{v})$ denotes
the normalized Lebesgue measure $(2\pi)^{-m/2} dv$ on $E^m$. As commented in [9], the
Fourier transform $\mathcal{F}$ acts like a homomorphism with convolution $*$ and ordinary
multiplication on $L_1(E^m)$ as follows: for $f, g \in L_1(E^m)$

$$(6.1) \quad \mathcal{F}(f * g) = \mathcal{F}(f) \mathcal{F}(g).$$

Also, the Fourier transform $\mathcal{F}$ and the convolution $*$ have a dual property such as

$$(6.2) \quad \mathcal{F}(f) * \mathcal{F}(g) = \mathcal{F}(fg).$$

In view of equations (6.1) and (6.2), it is worth-while to study a fundamental
relation between the GFYFT and the generalized convolution product (GCP), see
(6.3) and (6.4) below, for functionals on infinite dimensional Banach space. In
this view points, Huffman, Park, Skoug and Storvick [16, 17, 18, 33] established
fundamental relationships between the analytic Fourier–Feynman transform and
the corresponding convolution product for functionals $F$ and $G$ on the classical
Wiener space $C_0[0, T]$, as follows:

$$(6.3) \quad T_q^{(1)}((F * G)_q)(y) = T_q^{(1)}(F) \left( \frac{y}{\sqrt{2}} \right) T_q^{(1)}(G) \left( \frac{y}{\sqrt{2}} \right)$$

and

$$(6.4) \quad \left( T_q^{(1)}(F) * T_q^{(1)}(G) \right)_q(y) = T_q^{(1)} \left( \frac{F}{\sqrt{2}} \right) G \left( \frac{y}{\sqrt{2}} \right).$$

for scale-almost every $y \in C_0[0, T]$, where $(F * G)_q$ denotes the convolution product
of functionals $F$ and $G$ on $C_0[0, T]$. Equations (6.3) and (6.4) above are natural
extensions (to the case on an infinite dimensional Banach space) of the equations
(6.1) and (6.2), respectively. For an elementary introduction of the analytic Fourier–
Feynman transform and the corresponding convolution product, see [35].

Since then, in [35, 19], the authors extended the relationships (6.3) and (6.4) to
the cases between the generalized Fourier–Feynman transform and the generalized
convolution associated Gaussian processes on $C_0[0, T]$. The definition of the ordinary
Fourier–Feynman transform and the corresponding convolution are based on
the ordinary Wiener integral, see [16, 17, 18], and the definition of the generalized
Fourier–Feynman and the generalized convolution studied in [8, 19] are based on
the generalized Wiener integral [14, 30].

In this section, as applications of the Fubini theorem for the generalized Yeh–
Feynman integrals on $C_0(Q)$ we also establish more general relationships, such as
(6.3) and (6.4), between the GFYFT and the GCP on the Yeh–Wiener space $C_0(Q)$.

The following definition of the GCP on $C_0(Q)$ is due to Chang and Choi [11].
Definition 6.1. Let $F$ and $G$ be scale-invariant measurable functionals on $C_0(Q)$. For $\lambda \in \mathbb{C}_+$ and $k_1, k_2 \in \text{Supp}_{BV}(Q)$, we define their GCP with respect to $\{Z_{k_1}, Z_{k_2}\}$ (if it exists) by

$$(F \ast G)^{(k_1,k_2)}(y)$$

(6.5) $$E^\ast_{\mathbb{R},x} \left[ \begin{array}{l} F(y + Z_{k_1}(x)) \frac{1}{\sqrt{2}} G(y - Z_{k_2}(x)) \\ F(y - Z_{k_1}(x)) \frac{1}{\sqrt{2}} G(y + Z_{k_2}(x)) \end{array} \right], \quad \lambda \in \mathbb{C}_+$$

$$(\lambda = iq, \quad q \in \mathbb{R}, \quad q \neq 0).$$

When $\lambda = -iq$, we denote $(F \ast G)^{(k_1,k_2)}_\lambda$ by $(F \ast G)^{(k_1,k_2)}_q$.

Remark 6.2. Choosing $h_1 = h_2 \equiv 1$, equation (6.5) yields the convolution product studied in [24, 25]:

$$(F \ast G)^{(1,1)}_q(y) \equiv (F \ast G)_q(y) = E^\ast_{\mathbb{R},x} \left[ F\left(\frac{y + x}{\sqrt{2}}\right) G\left(\frac{y - x}{\sqrt{2}}\right) \right],$$

where $E^\ast_{\mathbb{R},x} [H(x)]$ means the (ordinary) analytic Yeh–Feynman integral of functionals $H$ on $C_0(Q)$.

Proceeding as in the proof of [19, Theorem 3.2], we can obtain the existence of the GCP of functionals in $S(L^2(Q))$.

Theorem 6.3. Let $k_1$ and $k_2$ be functions in $\text{Supp}_{BV}(Q)$ and let $F$ and $G$ be elements of $S(L^2(Q))$ with corresponding finite Borel measures $f$ and $g$ in $\mathcal{M}(L^2(Q))$. Then, the GCP $(F \ast G)^{(k_1,k_2)}_q$ of $F$ and $G$ exists for all nonzero real $q$, belongs to $S(L^2(Q))$, and is given by the formula

(6.6) $$
(F \ast G)^{(k_1,k_2)}_q(y) = \int_{L^2(Q)} \exp\{i(w,y)\} d\varphi^{k_1,k_2}_{C}(w)
$$

for $\text{s.a.e. } y \in C_0(Q)$, where

$$
\varphi^{k_1,k_2}_{C} = \varphi^{k_1,k_2} \circ \phi^{-1},
$$

$\varphi^{k_1,k_2}$ is the complex measure in $\mathcal{M}(L^2(Q))$ given by

$$
\varphi^{k_1,k_2}(B) = \int_B \exp\left\{ -\frac{i}{4q} \|uk_1 - vk_2\|^2 \right\} df(u)dg(v)
$$

for $B \in \mathcal{B}(L^2(Q))$, and $\phi : L^2(Q) \to L^2(Q)$ is the continuous function given by $\phi(u,v) = (u + v)/\sqrt{2}$.

Now we are ready to establish relationships between the GFYFT and the GCP on $C_0(Q)$.

6.1. Relationship I: GFYFT of the GCP. Our first relationship between the GFYFT and the GCP shows that the GFYFT of the GCP is a product of GFYFTs. In order to establish the first relationship we need the following lemmas.

Lemma 6.4. Let $h_1$, $h_2$, $k_1$, and $k_2$ be functions in $\text{Supp}_{BV}(Q)$, and let two stochastic processes

$$\mathcal{G}_{h_1,k_1}, \mathcal{G}_{h_2,k_2} : C_0(Q) \times C_0(Q) \times Q \to \mathbb{R}$$

be given by

$$\mathcal{G}_{h_1,k_1}(x_1,x_2,s,t) = Z_{h_1}(x_1,s,t) + Z_{k_1}(x_2,s,t)$$

for $s, t \in \mathbb{R}$.
and
\[ G_{h_1, k_1}(x_1, x_2; s, t) = Z_{h_1}(x_1; s, t) - Z_{k_1}(x_2; s, t), \]
respectively. Then the following assertions are equivalent.
(i) \( \mathcal{G}_{h, k_1} \) and \( \mathcal{G}_{h, k_2} \) are independent processes,
(ii) \( h_1 h_2 = k_1 k_2 \) in \( L_2(Q) \).

**Proof.** Since the processes \( \mathcal{G}_{h_1, k_1} \) and \( \mathcal{G}_{h_2, k_2} \) are Gaussian with mean zero, we know that \( \mathcal{G}_{h, k_1} \) and \( \mathcal{G}_{h, k_2} \) are independent processes if and only if
\[
\int_{C_2^Q(Q)} \mathcal{G}_{h_1, k_1}(x_1, x_2; s, t) \mathcal{G}_{h_2, k_2}(x_1, x_2; s', t') d(m_y \times m_y)(x_1, x_2) = 0
\]
for all \((s, t)\) and \((s', t')\) in \( Q \). But, using equation (2.3), it follows that
\[
\int_{C_2^Q(Q)} \mathcal{G}_{h_1, k_1}(x_1, x_2; s, t) \mathcal{G}_{h_2, k_2}(x_1, x_2; s', t') d(m_y \times m_y)(x_1, x_2)
\]
\[
= \int_{C_2^Q(Q)} \left\{ Z_{h_1}(x_1; s, t) Z_{h_2}(x_1; s', t') - Z_{h_1}(x_1; s, t) Z_{k_2}(x_2; s', t') + Z_{k_1}(x_2; s, t) Z_{h_2}(x_1; s', t') - Z_{k_1}(x_2; s, t) Z_{k_2}(x_2; s', t') \right\} d(m_y \times m_y)(x_1, x_2)
\]
\[
= \int_0^\min\{t, t'\} \int_0^\min\{s, s'\} h_1(\nu, \tau) h_2(\nu, \tau) d\nu d\tau
\]
\[
- \int_0^\min\{t, t'\} \int_0^\min\{s, s'\} k_1(\nu, \tau) k_2(\nu, \tau) d\nu d\tau.
\]
From this we can obtain the desired result. \( \square \)

**Lemma 6.5.** Given two functions \( h \) and \( k \) in \( \text{Supp}_{BV}(Q) \), let two stochastic processes
\[ \mathcal{E}_{h, k}^+: C_0(Q) \times C_0(Q) \times Q \to \mathbb{R} \]
be given by
\[ \mathcal{E}_{h, k}^+(x_1, x_2; s, t) = Z_h(x_1; s, t) + Z_k(x_2; s, t) \]
and
\[ \mathcal{E}_{h, k}^-(x_1, x_2; s, t) = Z_h(x_1; s, t) - Z_k(x_2; s, t), \]
respectively. Then the three processes \( \mathcal{E}_{h, k}^+, \mathcal{E}_{h, k}^- \) and \( \mathcal{Z}_{\alpha(h, k)} \) are mutually equivalent with the normal distribution \( \mathcal{N}(0, \beta_{h, k}(\cdot, \cdot)) \) where
\[
\beta_{h, k}(s, t) = \int_0^t \int_0^s \mathbf{s}^2(h, k)(\nu, \tau) d\nu d\tau.
\]

**Theorem 6.6.** Let \( k_1, k_2, F, \) and \( G \) be as in Theorem 6.3 and let \( h \) be a function in \( \text{Supp}_{BV}(Q) \). Assume that \( h^2 = k_1 k_2 \) \( m^2 \) a.e. on \( Q \). Then, for all nonzero real \( q \),
\[
T_{q, h}^{(1)}((F \ast G)_{q, (k_1, k_2)})(y)
\]
\[
= T_{q, \alpha(h, k_1)/\sqrt{2}}^{(1)}(F) \left( \frac{y}{\sqrt{2}} \right) T_{q, \alpha(h, k_2)/\sqrt{2}}^{(1)}(G) \left( \frac{y}{\sqrt{2}} \right)
\]
(6.8)
for s-a.e. \( y \in C_0(Q) \), where \( s(h, k_j) \)'s, \( j \in \{1, 2\} \), are functions in \( \text{Supp}_{BV}(Q) \) which satisfy the relation (4.1) with \( h_1 \) and \( h_2 \) replaced with \( h \) and \( k_j \), \( j \in \{1, 2\} \), respectively.

In particular, it follows that

\[
T_{q,h}^{(1)}((F * G)_q^{(h,h)})(y) = T_{q,h}^{(1)}(F)(\frac{y}{\sqrt{2}}) T_{q,h}^{(1)}(G)(\frac{y}{\sqrt{2}})
\]

for s-a.e. \( y \in C_0(Q) \).

**Proof.** By Theorems 6.3 and 5.3, \( T_{q,h}^{(1)}((F * G)_q^{(k_1,k_2)}) \) belongs to the Banach algebra \( \mathcal{S}(L^2(Q)) \) for all nonzero real \( q \). Thus, the proof given in [19, Theorem 3.3] with the current hypotheses on \( C_0(Q) \) and with Lemmas 6.4 and 6.5 also works here. □

**Remark 6.7.** Equation (6.8) above is useful in that it permits one to calculate the GFYFT of the GCP of functionals on \( C_0(Q) \) without actually calculating the GCP.

**Remark 6.8.** Under the assumptions as given in Theorem 6.6, one can prove the equation (6.8) using (6.7), (5.11) and direct calculations, but they are tedious.

Choosing \( h = k = 1 \) in equation (6.8), we have the following relationship between the ordinary FYFT and the ordinary CP on \( C_0(Q) \).

**Corollary 6.9 (Theorem 3.1 [24]).** Let \( F \) and \( G \) be as in Theorem 6.3. Then for all real \( q \in \mathbb{R} \setminus \{0\} \),

\[
T_{q}^{(1)}((F * G)_q)(y) = T_{q}^{(1)}(F)(\frac{y}{\sqrt{2}}) T_{q}^{(1)}(G)(\frac{y}{\sqrt{2}})
\]

for s-a.e. \( y \in C_0(Q) \), where \( T_{q}^{(1)}(F) \) denotes the ordinary Fourier–Yeh–Feynman transform of \( F \) (see Remark 5.2) and \( (F * G)_q \) denotes the CP of \( F \) and \( G \) given by (6.6) above.

We now present a simple example for the assumption in Theorem 6.6. Let \( k_1, k_2 \) and \( h \) be given by

\[
k_1(s, t) = 4 \sin^2 \left( \frac{2\pi s}{T} \right) \sin^2 \left( \frac{2\pi t}{T} \right),
\]

\[
k_2(s, t) = 4 \cos^2 \left( \frac{2\pi s}{T} \right) \cos^2 \left( \frac{2\pi t}{T} \right)
\]

and

\[
h(t) = \sin \left( \frac{4\pi s}{T} \right) \sin \left( \frac{4\pi t}{T} \right)
\]

on \( Q \), respectively. Then it follows that

\[
k_1(s, t)k_2(s, t) = 16 \sin^2 \left( \frac{2\pi s}{T} \right) \cos^2 \left( \frac{2\pi s}{T} \right) \sin^2 \left( \frac{2\pi t}{T} \right) \cos^2 \left( \frac{2\pi t}{T} \right)
\]

\[
= \sin^2 \left( \frac{4\pi s}{T} \right) \sin^2 \left( \frac{4\pi t}{T} \right)
\]

\[
= h^2(s, t).
\]

for all \((s, t) \in Q\).

In view of Theorems 5.7 and 6.6, we obtain the following corollary.
Corollary 6.10. Let $k_1, k_2, F$ and $G$ be as in Theorem 6.3, and let $\mathcal{H} = \{h_1, \ldots, h_n\}$ be a set of functions in $\text{Supp}_{BV}(Q)$. Assume that

$$s^2(h_1, \ldots, h_n) = k_1 k_2$$

$m_2^L$-a.e. on $Q$. Then, for all nonzero real $q$,

$$T_{q,h_n}^{(1)}(T_{q,h_{n-1}}^{(1)}(\cdots(T_{q,h_2}^{(1)}(T_{q,h_1}^{(1)}((F * G)_q^{(k_1, k_2)}))\cdots))(y)$$

$$= T_{q,*}(T_{q,h}^{(1)}((F * G)_q^{(k_1, k_2)}))(y)$$

$$= T_{q,*}(q, h_1)/\sqrt{2}(F)((y)$$

for $s$-a.e. $y \in C_0(Q)$, where $K_1 = \mathcal{H} \cup \{k_1\}$ and $K_2 = \mathcal{H} \cup \{k_2\}$.

6.2. Relationship II: GCP of GFYFTs. Our second relationship between the GFYFT and the GCP shows that the GCP of the GFYFTs can be represented as a single GFYFT.

Theorem 6.11. Let $k_1, k_2, F, G$, and $h$ be as in Theorem 6.3. Then, for all nonzero real $q$,

$$T_{q,h_1}/\sqrt{2}(F)_q^{(k_1, k_2)}(1)$$

$$= T_{q,h_1}(F(G))_q^{(k_1, k_2)}(y)$$

for $s$-a.e. $y \in C_0(Q)$, where $s(h, k_1)$'s, $j \in \{1, 2\}$, are functions in $\text{Supp}_{BV}(Q)$ which satisfy the relation 6.3 with $h_1$ and $h_2$ replaced with $h$ and $h_j$, $j \in \{1, 2\}$, respectively.

In particular, it follows that

$$T_{q,h}^{(1)}((F * G)_q^{(h, h)}(y) = T_{q,h}^{(1)}(F(G))_q^{(h, h)}(y)$$

for $s$-a.e. $y \in C_0(Q)$.

Proof. Applying 6.12, 6.8 with $F, G$, and $q$ replaced with $T_{q,h}^{(1)}(F)_q^{(h, h)}$, $T_{q,h}^{(1)}(G)_q^{(k_1, k_2)}$, and $-q$, respectively, and 6.12 again, it follows that for $s$-a.e. $y \in C_0(Q)$,

$$T_{q,h}^{(1)}((F * G)_q^{(h, h)}(y)$$

$$= T_{q,h}^{(1)}(T_{q,h}^{(1)}((F * G)_q^{(h, h)})(y)$$

$$= T_{q,h}^{(1)}\left(T_{q,h}^{(1)}((F * G)_q^{(h, h)})(y)$$

$$= T_{q,h}^{(1)}(F(G))_q^{(h, h)}(y)$$

as desired. \qed
Letting \( h = k_1 = k_2 = 1 \) in equation (6.9), one can see that equation (6.10) below holds.

**Corollary 6.12** (Theorem 3.2 [24]). Let \( F \) and \( G \) be as in Theorem 6.3. Then, for all real \( q \in \mathbb{R} \setminus \{0\} \),

\[
(T_q^{(1)}(F) * T_q^{(1)}(G))_{-q}(y) = T_q^{(1)}\left(F\left(\frac{y}{\sqrt{2}}\right)G\left(\frac{y}{\sqrt{2}}\right)\right)(y)
\]

for s-a.e. \( y \in C_0(Q) \).

We next establish two types of extension of Theorem 6.11 above.

**Theorem 6.13.** Let \( k_1, k_2, F, \) and \( G \) be as in Theorem 6.3, and let \( \mathcal{H} = \{h_1, \ldots, h_n\} \) be a finite sequence of functions in \( \text{Supp}_{BV}(Q) \). Assume that

\[
s^2(\mathcal{H}) \equiv s^2(h_1, \ldots, h_n) = k_1 k_2
\]

for \( m_1^2 \)-a.e. on \( Q \), where \( s(\mathcal{H}) \) is the function in \( \text{Supp}_{BV}(Q) \) satisfying (4.4) above. Then, for all nonzero real \( q \),

\[
\begin{align*}
(T_{q,k_1/\sqrt{2}}^{(1)}(T_{q,k_2/\sqrt{2}}^{(1)}(\cdots (T_{q,h_n/\sqrt{2}}^{(1)}(T_{q,h_1/\sqrt{2}}^{(1)}(F)) \cdots ) \cdots )))
& * T_{q,k_1/\sqrt{2}}^{(1)}(T_{q,k_2/\sqrt{2}}^{(1)}(\cdots (T_{q,h_n/\sqrt{2}}^{(1)}(T_{q,h_1/\sqrt{2}}^{(1)}(G)) \cdots ) \cdots )))\big)_{-q}(y) \\
& = (T_{q,s(\mathcal{H},k_1)/\sqrt{2}}^{(1)}(F) * T_{q,s(\mathcal{H},k_2)/\sqrt{2}}^{(1)}(G))_{-q}^{(k_1,k_2)}(y) \\
& = T_{q,s(\mathcal{H})}^{(1)}\left(F\left(\frac{y}{\sqrt{2}}\right)G\left(\frac{y}{\sqrt{2}}\right)\right)_{-q}(y)
\end{align*}
\]

for s-a.e. \( y \in C_0(Q) \), where \( s(\mathcal{H}, k_1) \) and \( s(\mathcal{H}, k_2) \) are functions in \( \text{Supp}_{BV}(Q) \) satisfying the relations

\[
s^2(\mathcal{H}, k_1) \equiv s^2(h_1, \ldots, h_n, k_1) = h_1^2 + \cdots + h_n^2 + k_1^2
\]

and

\[
s(\mathcal{H}, k_2)^2 \equiv s(h_1, \ldots, h_n, k_2)^2 = h_1^2 + \cdots + h_n^2 + k_2^2
\]

for \( m_1^2 \)-a.e. on \( Q \), respectively.

**Proof.** Applying (5.14), the first equality of (6.11) follows immediately. Next using (6.9) with \( h \) replaced with \( s(\mathcal{H}) \), the second equality of (6.11) also follows. \( \square \)

In view of equations (6.9) and (5.14), we also obtain the following assertion.

**Theorem 6.14.** Let \( F \) and \( G \) be as in Theorem 6.3. Given a function \( h \) in \( \text{Supp}_{BV}(Q) \) and finite sequences \( K_1 = \{k_{11}, k_{12}, \ldots, k_{1n}\} \) and \( K_2 = \{k_{21}, k_{22}, \ldots, k_{2m}\} \) of functions in \( \text{Supp}_{BV}(Q) \), assume that

\[
h^2 = s(K_1)s(K_2)
\]
for \( m_T^2 \)-a.e. on \( Q \). Then, for all nonzero real \( q \),

\[
\left( T_{q,h}/\sqrt{2} \left( T_{q,k_1}/\sqrt{2} \left( \cdots \left( T_{q,k_2}/\sqrt{2} \left( T_{q,k_1}/\sqrt{2} (F) \right) \cdots \right) \right) \right) \right)_{-q} (y)
\]

\[
= \left( T_{q,h}/\sqrt{2} \left( T_{q,k_1}/\sqrt{2} \left( \cdots \left( T_{q,k_2}/\sqrt{2} \left( T_{q,k_1}/\sqrt{2} (F) \right) \cdots \right) \right) \right) \right)_{-q} (y)
\]

\[
= T_{q,h} \left( F \left( \frac{\cdot}{\sqrt{2}} \right) G \left( \frac{\cdot}{\sqrt{2}} \right) \right) (y)
\]

for \( s \)-a.e. \( y \in C_0(Q) \), where \( s(h,s(K_1)) \), and \( s(h,s(K_2)) \) are functions in \( \text{Supp}_{BV}(Q) \) satisfying the relations

\[
s^2(h,s(K_1)) = h^2 + s^2(K_1) = h^2 + k_{11} + \cdots + k_{1m},
\]

and

\[
s^2(h,s(K_2)) = h^2 + s^2(K_2) = h^2 + k_{21} + \cdots + k_{2m}
\]

for \( m_T^2 \)-a.e. on \( Q \), respectively.

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