Dynamical generation and dynamical reconstruction

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A definition of ‘dynamical generation’, a hotly debated topic at present, is proposed and its implications are discussed. This definition, in turn, leads to a method allowing to distinguish in principle tetraquark and molecular states. The different concept of ‘dynamical reconstruction’ is also introduced and applies to the generation of preexisting mesons (quark-antiquark, glueballs, ...) via unitarization methods applied to low-energy effective Lagrangians. Large $N_c$ arguments play an important role in all these investigations. A simple toy model with two scalar fields is introduced to elucidate these concepts. The large $N_c$ behavior of the parameters is chosen in order that the two scalar fields behave as quark-antiquark mesons. When the heavier field is integrated out, one is left with an effective Lagrangian with the lighter field only. A unitarization method applied to the latter allows to ‘reconstruct’ the heavier ‘quarkonium-like’ field, which was previously integrated out. It is shown that a Bethe-Salpeter (BS) analysis is capable to reproduce the preformed quark-antiquark state, and that the corresponding large-$N_c$ behavior can be brought in agreement with the expected large $N_c$ limit; this is a subtle and interesting issue on its own. However, when only the lowest term of the effective Lagrangian is retained, the large $N_c$ limit of the reconstructed state is not reproduced: instead of the correct large $N_c$ quarkonium limit, it fades out as a molecular state would do. Implications of these results are presented: it is proposed that axial-vector, tensor and (some) scalar mesons just above 1 GeV obtained via the BS approach from the corresponding low-energy, effective Lagrangian in which only the lowest term is kept, are quarkonia states, in agreement with the constituent quark model, although they might fade away as molecular states in the large $N_c$ limit.

I. INTRODUCTION

A central topic of past and modern hadron physics is the determination of the wave function of resonances in terms of quark and gluon degrees of freedom, both in the baryon and meson sectors and both for light and heavy quarks (for reviews see refs. [1,2]). In the mesonic sector, beyond conventional quark-antiquark ($\bar{q}q$) mesons, one has glueball states, multiquark states such as tetraquarks and ‘dynamically generated resonances’, most notably molecular states. Indeed, different authors used the term ‘dynamical generation’ in rather different contexts. In the works of Refs. [3,4,5,6] a dynamical generated state is regarded as a resonance obtained via unitarization methods from a low-energy Lagrangian; in Refs. [7,8,9] it is considered as a state which does not follow the quark-antiquark pattern in large-$N_c$ expansion. In Refs. [10,11,12,13] the concept of ‘additional, companion poles’ as dynamically generated states is introduced, while in Refs. [14,15,16] a dynamically generated resonance is regarded as loosely bound molecular state. These various definitions are not mutually exclusive and describe different points of view of the problem.

In this article (Sec. II and III) a definition for a dynamically generated state is proposed and its implications, also in connection with the aforementioned works, are presented. This definition, in turn, leads to a method allowing to distinguish in principle tetraquark and molecular states, although they are both four-quark states. The concept of ‘dynamical reconstruction’ is then introduced and discussed: it applies to resonances which are obtained from low-energy effective Lagrangians via unitarization methods, but still correspond to ‘fundamental’ (not dynamically generated) $\bar{q}q$, glueball or multiquark states. In this context, the study of the large $N_c$ behavior of these resonance constitutes a useful tool to discuss their nature. In the end of Sec. III some general thoughts about the form of an effective theory of hadrons valid up to 2 GeV are also presented.

In Sec. IV the attention is focused on a simple toy model, in which only two scalar fields are considered. The parameters of the toy model are chosen in a way that both fields behave as quarkonium-like state in the large $N_c$ limit. The heavier state is first integrated out in order to obtain an effective low-energy Lagrangian in the toy world and then is reobtained via a Bethe-Salpeter (BS) study applied to the low-energy Lagrangian. It is shown that this is (at least in some cases) possible and that the corresponding large $N_c$ behavior can be brought in agreement with the expected large $N_c$ limit; this is a subtle and interesting issue on its own. However, when only the lowest term of the effective, low-energy Lagrangian is retained, the large $N_c$ limit of the reconstructed state is not reproduced: instead of the correct large $N_c$ quarkonium limit (which must hold by construction), it fades out as a molecular state would do. Implications of these results are presented: it is proposed that axial-vector, tensor and (some) scalar mesons just above 1 GeV obtained via the BS approach from the corresponding low-energy, effective Lagrangian in which only the lowest term is kept, are (reconstructed) quark-antiquark fields, in agreement with the constituent quark model.
with the constituent quark model, although they might fade away as molecular states in the large $N_c$ limit. Finally, in Sec. V the conclusions are presented.

II. DYNAMICAL GENERATION

Consider a physical system which is correctly and completely described in the energy range $0 \leq E \lesssim E_{\text{max}}$ by a quantum field theory in which $N$ fields $\phi_1, \phi_2, \ldots, \phi_N$, their masses and interactions are encoded in a Lagrangian $\mathcal{L} = \mathcal{L}(\phi_1, E_{\text{max}})$. Each Lagrangian has such an $E_{\text{max}}$ beyond which it cannot be trusted. In particular, we shall refer to $E_{\text{max}}$ in the following sense: All the masses $M_i$ of the states $\phi_i$ and the energy transfer in a two-body scattering $\phi_i + \phi_j \rightarrow \phi_i + \phi_j$ should be smaller than $E_{\text{max}}$ ($M_i < E_{\text{max}}$ and, in the s-channel, $\sqrt{s} \lesssim E_{\text{max}}$).

Moreover, we also assume that: (i) The theory is not confining, thus, to each field $\phi_i$ there is a corresponding, measurable resonance (and at least one with zero width). (ii) If not renormalizable, an appropriate regularization shall be specified.

A resonance $R$, emerging in the system described by $\mathcal{L}$, is said to be dynamically generated if it does not correspond to any of the original fields $\phi_1, \phi_2, \ldots, \phi_N \subset \mathcal{L}$ in the Lagrangian and if its mass $M_R$ lies below $E_{\text{max}}$ ($M_R \lesssim E_{\text{max}}$).

The last requirement $M_R \lesssim E_{\text{max}}$ is natural because the state $R$ can be regarded as an additional, dynamically generated resonance only if it belongs to the energy range in which the theory is valid. This simple consideration plays an important role in the following discussion. Clearly, the dynamical generated state $R$ emerges via interactions of the original resonances $\phi_i$. When switching them off, $R$ must disappear. For this reason a dynamically generated mesonic resonance in QCD fades out in the large-$N_c$ limit, which corresponds to a decreasing interaction strength of mesons, see later for more details.

Some examples and comments are in order:

(a) $\mathcal{L} = \mathcal{L}_{\text{QED}}$, in which the electron and the photon fields are the basic fields. This theory is valid up to a very large $E_{\text{max}}$ (GUT scale). Positronium states are molecular, electron-positron bound states. They appear as poles close to the real axis just below the threshold $2m_e$, but slightly shifted due to their nonzero decay widths into photons. Clearly, positronium states are ‘dynamically generated’ according to the given definition and should not be included in the original QED Lagrangian, otherwise they would be double-counted. Note, the number of positronium states is infinite.

(b) In Lagrangians describing nucleon-nucleon interaction via meson exchange ($\omega, \rho, \pi$ and $\sigma$) a bound state close to threshold, called the deuteron, emerges via Yukawa-interactions, see for instance ref. [17] and refs. therein. The deuteron is a dynamically generated molecular state. In this case, when lowering the interaction strength below a critical value (by reducing the coupling and/or increasing the mass of the exchanged particle(s)), the bound state disappears. In fact, the number of molecular states which can be obtained via a Yukawa potential is finite [18], and eventually zero if the attraction is too weak. In such models the deuteron should not be included in the original Lagrangian in order to avoid double-counting [18].

(c) $\mathcal{L} = \mathcal{L}_{\text{F}}$ the Fermi theory of the weak interaction, in which the neutrino and electron fields interact via a local, quartic interaction. This theory is valid up to $E_{\text{max}} \ll M_W$, where $W$ is the boson mediator of the weak force. As already mentioned in ref. [2], the linear rise of the $\pi e^-e^-$ cross section -as calculated from $\mathcal{L}_{\text{F}}$- shows a loss of unitarity at high energy. Unitarization applied to $\mathcal{L}_{\text{F}}$ implies that a resonance well above $E_{\text{max}}$ exists, and this resonance is exactly the $W$ meson. However, being $M_W > E_{\text{max}}$ one cannot state in the framework of the Fermi theory if the $W$ meson is dynamically generated or not. A straightforward way to answer this question is the knowledge of the corresponding theory valid up to an energy $E_{\text{max}} > M_W$. Of course, this theory is known: it is the electroweak theory described by the Lagrangian $\mathcal{L}_{\text{EW}}$, which is part of the Standard Model [20] and is valid up to a very high energy (GUT scale). In the framework of $\mathcal{L}_{\text{EW}}$ the neutrino, the electron and the $W$ meson are all elementary fields. One can then conclude that the $W$ meson is not a dynamically generated state.

Indeed, $\mathcal{L}_{\text{F}}$ can be seen as the result of integrating out the $W$ field from the electroweak Lagrangian $\mathcal{L}_{\text{EW}}$. Unitarization arguments applied to the Fermi Lagrangian $\mathcal{L}_{\text{F}}$ allow in a sense to ‘dynamically reconstruct’ the $W$, which is already present as a fundamental field in $\mathcal{L}_{\text{EW}}$.

(d) It is important to discuss in more depth and to formalize the issue raised in the previous example. To this end let us consider the Lagrangian $\mathcal{L} = \mathcal{L}(\phi_i, E_{\text{max}})$ as the low-energy limit of a Lagrangian $\mathcal{L}' = \mathcal{L}'(\phi_i, \varphi_k, E_{\text{max}}')$ valid up to an energy $E_{\text{max}}' > E_{\text{max}}$. Beyond the fields $\phi_i$, $\mathcal{L}'$ depends also on the fields $\varphi_k$, which are heavier than $E_{\text{max}}$. Formally, when integrating out the fields $\varphi_k$ from $\mathcal{L}'$ one obtains $\mathcal{L}$.

In general, a unitarization scheme uses the information encoded in a low-energy effective Lagrangian and the principle of unitarity in quantum field theories, in order to deduce the existence and some properties of resonances beyond the limit of validity of the theory itself. When applying a unitarization scheme to the Lagrangian $\mathcal{L}$, an energy window between $E_{\text{max}}$ and a new energy scale $E_U > E_{\text{max}}$ - which depends on the detail of the unitarization - becomes (partially) accessible. For our purposes, we assume that $E_U \lesssim E_{\text{max}}$.

Let $R$ be a resonance with mass $E_{\text{max}} < M_R < E_U$ obtained from $\mathcal{L}$ via a unitarization approach. Is this resonance $R$ dynamically generated or not? A straightforward way to answer this question would be the knowledge of $\mathcal{L}'$. If $R$ corresponds to one of the fields $\varphi_k$ is not dynamically generated and vice-versa. However, if $\mathcal{L}'$ is not known it is not possible to answer this question at the level of the unitarized version of $\mathcal{L}$ only.
In conclusion, although the unitarization approach opens a window between $E_{\text{max}}$ and $E_L$ and the existence of resonances in this range can be inferred from the unitarized Lagrangian $\mathcal{L}$ only, still the knowledge of the latter is not complete \[21\]. If $\mathcal{L}'$ is unknown, some other kind of additional information is required to deduce the nature of $R$. In the framework of low-energy QCD this additional information can be provided by large-$N_c$ arguments, see Sec. III.C.

(e) Let us consider a scalar field $\varphi = \varphi(t, x)$ in a 1+1 dimensional world $(t, x)$ subject to the potential $V(\varphi) = \frac{1}{4}(\varphi^2 - F^2)^2$. We assume that this theory is valid up to high energies. When expanding around one of the two minima $\varphi = \pm F$, the mass of $\varphi$ is found to be $m = \lambda F^2/3$. In addition, this theory admits also a soliton with mass $M = \frac{2m}{\lambda}$, which is large if $\lambda$ is small \[22\]. In this example the solitonic state with mass $M$ can be regarded as a ‘dynamically generated state’.

(f) Mixing can take place among two ‘fundamental fields’ $\phi_i$ and $\phi_k$: Two physical resonances arise as an admixture of these two fields. One is predominantly $\phi_i$ and the other predominantly $\phi_k$. Also, mixing can take place among a dynamically generated resonance $R$ and one (or more) of the $\phi_i$. It decreases when the interaction is lowered (large-$N_c$ limit in the mesonic world): one state reduces to the original, preexisting resonance and the other disappears. In conclusion, mixing surely represents a source of technical complication which renders the identification of states (extremely) more difficult, but it does not change the number of states and the meaning of the previous discussion.

III. APPLICATION TO MESONS

A. Effective hadron theory up to 2 GeV

Let us turn to the hadronic world below 2 GeV. The basic ingredients of each low-energy hadronic Lagrangian are quark-antiquark mesons and three-quark baryons. In the framework of our formalism, we shall consider each quark-antiquark (3-quark) state as a fundamental state, which is described by a corresponding field in the hadronic Lagrangian (as long as its mass is below an upper energy $E_{\text{max}}$).

Let us formalize this point in the mesonic sector as it follows. Consider the correct, effective theory describing mesons up to $E_{\text{max}} \simeq 2$ GeV given by

$$\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}}, N_c),$$

where $N_c$ is the number of colors. Its precise form is, unfortunately, unknown. In fact, because confinement has not yet been analytically solved, it is not possible to derive $\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}}, N_c)$ from the QCD Lagrangian. In the limit $N_c \to \infty$ the effective Lagrangian $\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}}, N_c)$ is expected to be more simple. Although even in this limit a mathematical derivation is not possible, it is known that it must primarily consists of non-interacting quark-antiquark states. In fact, their masses scale as $N_c^0$ and their decay widths as $N_c^{-1}$ respectively \[23, 24, 25, 26\]. Considering that the $\overline{c}q$ mass scales as $N_c^0$ these states shall be clearly present also when going back from the large $N_c$ limit to the physical world $N_c = 3$. The next-expected states which are present in the large $N_c$ limit are glueballs, i.e. bound states of pure gluonic nature. Their masses also scale as $N_c^0$ and the decay widths as $N_c^{-2}$. They thus are also expected to be present in the real world for $N_c = 3$. In particular, the lightest glueball is a scalar field which is strongly related to the trace anomaly, i.e. the breaking of the classical dilatation invariance of the QCD Lagrangian (see also Sec. III.D).

In addition to quark-antiquark and glueball states, also hybrid states survive in the large-$N_c$ limit \[26\]. They constitute an interesting subject of meson spectroscopy (see ref. \[27\] and refs. therein), but will not be considered in the following discussion. All these states are therefore ‘preexisting’ and not-dynamically generated states of the mesonic Lagrangian under consideration.

An intermediate state is devoted to baryon states: they have an linearly increasing mass with $N_c (M \sim N_c)$ which exceeds $E_{\text{max}}$ for a large enough $N_c$ and are therefore not present in the large-$N_c$ limit of $\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}}, N_c)$. Thus, although they appear in the $N_c = 3$ world, they are not present in the effective Lagrangian because of the way in which the limit is constructed. If, instead, we construct the large $N_c$ limit as $\mathcal{L}^{\text{had}}_{\text{eff}}(N_c, N_{\text{max}} \to \infty)$ (in such a way that at $N_c = 3$ it coincides with Eq. (1)), baryons are well defined states as proved originally in ref. \[24\]. For simplicity baryons will not be discussed in this paper, but together with hybrid states, should be included in a more complete treatment.

The reason why the value $E_{\text{max}} \simeq 2$ GeV is chosen is that all the resonances under study in this work are lighter than 2 GeV. Thus, they either correspond to a field in $\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}}, N_c = 3)$ or arise as additional resonances (i.e. dynamically generated) via interaction of preexisting states of $\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}}, N_c = 3)$. The full knowledge of the Lagrangian $\mathcal{L}^{\text{had}}_{\text{eff}}(E_{\text{max}} \simeq 2$ GeV, $N_c = 3$) would allow to answer if a resonance lighter than 2 GeV is dynamically generated or not by a simple look at it. Clearly, if one would be interested in a resonance whose mass is heavier than 2 GeV, $E_{\text{max}}$ should be increased. Moreover, one expects to find below 2 GeV all the relevant ground-state mesons in the channel $J^{PC} = 0^{+}, 0^{++}, 1^{-}, 1^{++}, 2^{++}$. Thus, $\mathcal{L}^{\text{had}}_{\text{eff}}(2$ GeV, 3) may be described by an effective Lagrangian which exhibits linear realization of chiral symmetry and its spontaneous breakdown, see section III.D for a closer discussion.

Most of the mesonic resonances listed in the Particle Data Group \[28\] can be immediately associated to a corresponding, underlying quark-antiquark state. Yet, the question if some resonances of ref. \[28\] are not $\overline{c}q$ is interesting and is the basis of many studies. In the mesonic sector, two alternative possibilities are well-known:
• Molecular states: they are bound state of two distinct quark-antiquark mesons. They correspond to the example of the positronium (example (a) Sec. II). Just as the positronium states are not included in the QED Lagrangian, hadronic molecular states should not be included directly in $\mathcal{L}_{\text{eff}}(E_{\text{max}}, N_c)$. They arise upon meson-meson interactions in the $N_c = 3$ physical world, see the general discussion above. However, they inevitably fade out in the large-$N_c$ limit because the interaction of a n-leg meson vertex decreases as $N_c^{-(n-2)/2}$. This is therefore a clear example of dynamically generated states within a mesonic system (see also the next subsection for a closer description of physical candidates below 1 GeV).

• Tetraquark states: they consist of two distinct, colored ‘bumps’, in contrast to a molecular state, which is made of two colorless, quark-antiquark bumps [29]. Loops of $\bar{q}q$ mesons, corresponding to the interaction of two colorless states, cannot generate the color distribution of a tetraquark. If present at $N_c = 3$, they shall be included directly in the effective Lagrangian $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c = 3)$ and, in view of the given definition, should not be regarded as dynamically generated states.

A second, slightly different way to see it is the following: let us imagine to construct the Lagrangian $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$. We first put in quark-antiquark and glueball states, that is those configurations which surely survive in the large-$N_c$ limit and corresponds to non-dynamically generates states: $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c) = \mathcal{L}_{\text{eff}}^{\text{qq}} + \mathcal{L}_{\text{eff}}^{\text{g}}(E_{\text{max}}, N_c)$, Then the question is: ‘does this Lagrangian describe the physical world for $N_c = 3$’? (Note, loops shall be taken into account and dynamically generated states can eventually emerge out of this Lagrangian.) If the answer is positive, no multiquark states are needed. If the answer is negative, the basic Lagrangian shall be extended to include from the very beginning multiquark states, most notably tetraquark states: $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, 3) = \mathcal{L}_{\text{eff}}^{\text{qq}} + \mathcal{L}_{\text{eff}}^{\text{g}}(E_{\text{max}}, 3) + \mathcal{L}_{\text{eff}}^{\text{multiquark}}(E_{\text{max}}, 3)$.

A third approach to the problem goes via large-$N_c$ arguments. In refs. [24, 25] it has been shown that a tetraquark state also vanishes in the large-$N_c$ limit. However, for $N_c = 3$ the most prominent and potentially relevant for spectroscopy is the ‘good’ diquark, which is antisymmetric in color space: $d_a = \varepsilon_{abc}q^b q^c$, (with $a, b, c = 1, 2, 3$) [30]. The tetraquark is the composition of a good diquark and a good antiquark: $d_a \bar{d}_a$. The extension to $N_c$ of a good diquark is the antisymmetric configuration $d_{a_1} = \varepsilon_{a_1a_2...a_{N_c}}q^{a_1}q^{a_2}...q^{a_{N_c}}$ with $a_1,...,a_{N_c} = 1,...,N_c$, which constitutes of $(N_c - 1)$ quarks. Thus, the generalization of the tetraquark to the $N_c$ world is not a diquark-antidiquark object, but the state $\chi = \sum_{a_1=1}^{N_c} d_{a_1} \bar{d}_{a_1}$ which is made of $(N_c - 1)$ quarks and $(N_c - 1)$ antiquarks, see also the discussion in ref. [31]. It is the dibaryon already described in ref. [24] which has a well defined large-$N_c$ limit: its mass scale as $M_\chi \propto 2(N_c - 1)$ and decays into a baryon and an antibaryon. The state $\chi$, while not present in $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c \to \infty)$ because its mass overshoots $E_{\text{max}}$, appears in $\mathcal{L}_{\text{eff}}^{\text{had}}(N_c, E_{\text{max}}, N_c \to \infty)$ in which also the baryons survive: this is contrary to a dynamically generated state, which disappears also in this case.

As a result of our discussion, tetraquark states and molecular states, although both formally four-quark states, are crucially different: The former are ‘elementary’ and should be directly included in the effective Lagrangian $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c = 3)$, the latter can emerge as dynamically generated resonances. We now turn to the particular case of the light scalar mesons, where all these concepts play an important role.

### B. Light scalar mesons

One of the fundamental questions of low energy QCD concerns the nature of the lightest scalar states $\sigma \equiv f_0(600), k \equiv k(800), f_0 \equiv f_0(980)$ and $a_0 \equiv a_0(980)$. Shall these states be included from the very beginning in $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$? If yes, they correspond to quark-antiquark or tetraquark nonets (one of them can be also related to a light scalar glueball). If not, they shall be regarded as dynamically generated states. The main point of the following subsection is to discuss previous works about light scalar mesons in connection with the proposed definition of dynamical generation. In fact, it is easy to classify previous works into two classes (not dynamically generated and dynamically generated), thus allowing to order different works of the last three decades in a clear way. We first review works in which scalar states are not dynamically generated and then works in which they are dynamically generated.

The light scalar states are not dynamically generated and should be directly included in $\mathcal{L}_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$:

(i) In the quark-antiquark picture these light scalar states form the nonet of chiral partners of pseudoscalar mesons. Their flavor wave functions reads $\sigma \equiv \sqrt{\frac{\sqrt{2}}{2}}(u \bar{u} + \bar{d}d), f_0 \equiv 3s, a_0 \equiv u \bar{d}, k^+ \equiv u \bar{s}$. At the microscopic level this is the prediction of the NJL model [32], where a $\sigma$ mass of about $2m^*$ is obtained and where $m^* \sim 300$ MeV is the constituent quark mass. This is usually the picture adopted in linear sigma models at zero [33] and at nonzero density and temperature [34]. However, this assignment encounters a series of problems: it can hardly explain the mass degeneracy of $a_0$ and $f_0$, the strong coupling of $a_0$ to kaon-kaon, the large mass difference with the other p-wave nonets of tensor and axial-vector mesons [35], it is at odd with large-$N_c$ studies (see later on) and with recent lattice works [36].

(ii) Tetraquark picture, first proposed by Jaffe [37] and revisited in Refs. [38, 39, 40]: $\sigma \equiv \frac{1}{\sqrt{2}} \mathcal{F}(f_{12}, d)[u, d], f_0 \equiv \frac{1}{\sqrt{2}} \mathcal{F}(f_{12}, S)[u, s] + \mathcal{F}(d, S)[d, s], a_0 \equiv \frac{1}{\sqrt{2}} \mathcal{F}(d, d)[u, d], k^+ \equiv \frac{1}{\sqrt{2}} \mathcal{F}(d, s)[u, d]$ where $\mathcal{F}[, , ]$ stands for antisymmetric configuration in flavor space (which, together with the
already mentioned antisymmetric configuration in color space, implies also a s-wave and spinless structure of the diquarks and of the tetraquarks). Degeneracy of $a_0$ and $f_0$ is a natural consequence. A good phenomenology of decays can be obtained if also the next-to-leading order contribution in the large-$N_c$ expansion is taken into account and/or if instanton induced terms are included. Linear sigma models with an additional nonet of scalar states can be constructed. The quark-antiquark states lie above 1 GeV and mix with the scalar glueball whose mass is placed at $\sim 1.7$ GeV by lattice QCD calculations. This reversed scenario directly affects the physics of chiral restoration at non-zero temperature.

(iii) Different assignments, in which also the glueball state shows up below 1 GeV have been proposed, see refs. and refs. therein.

(iv) In these assignments the very existence of the scalar mesons is due to some preformed compact bare fields entering in $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c = 3)$. By removing the corresponding bare resonances from $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c = 3)$, they disappear. Dressing via meson-meson loops, such as $\pi\pi$ for $\sigma$, $K\pi$ for $f_0$ and $K\bar{K}$ for $a_0$ and $f_0$, surely takes place. In particular, due to the intensity in these channels and the closeness to thresholds, they can cause a strong distortion and affect the properties of the scalar states. However, the important point is that in all these scenarios mesonic loops represent a further complication of light scalars, but are not the reason of their existence.

(v) As discussed in Section III.A, non-dynamically generated scalar states survive in the large-$N_c$ limit, although in a different way according to quarkonium, glueball or tetraquark interpretations.

The light scalars states are dynamically generated and should not be included in $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$:

(i) In Ref. the $\sigma$ pole arises as a broad enhancement due to the inclusion of $\rho$ mesons in the t-channel isoscalar $\pi\pi$ scattering. In this case the $\sigma$ is ‘dynamically generated’ and arises because of a Yukawa-like interaction due to $\rho$ meson exchange (pretty much as the deuteron described above, but above threshold). When reducing the $\rho\pi\pi$ coupling $g_{\rho\pi\pi}$ (which, in the large-$N_c$ limit scales as $1/\sqrt{N_c}$), the $\sigma$ fades out. Alternatively, the limit $M_\rho \to \infty$ also implies a disappearance of the $\sigma$-enhancement.

(ii) Similar conclusions for the $f_0(980)$ and $a_0(980)$ meson, described as molecular $KK$ bound states just below threshold, have been obtained in refs. In particular, in ref. the origin of these states is directly related to a one-meson-exchange potential. Within all these approaches the $a_0(980)$ and the $f_0(980)$ are ‘dynamically generated’.

(iii) In the model of ref. the $a_0$ state also arises as an additional, dynamically generated state, but in a different way. Scalar and pseudoscalar quark-antiquark mesons are the original states. A bare scalar state with a mass of 1.6 GeV is the original, quark-antiquark ‘seed’. When loops of pseudoscalar mesons are switched on, the mass is slightly lowered and the state is identified with $a_0(1450)$. In addition, a second state, arising in this model as a further zero of the real part of the denominator of the propagator, is identified with the $a_0(980)$ meson: it is dynamically generated and disappears in the large-$N_c$ limit, where only the original quark-antiquark seed survives. More in general, we refer to for the emergence of additional, companion poles not originally present as preexisting states in the starting Lagrangian. In particular, in ref. the conventional scalar quark-antiquark states, calculated within an harmonic oscillator confining potential, lie above 1 GeV. When meson loops are switched on, a complete second nonet of dynamically generated states below 1 GeV emerge. Note, in all these studies the validity of the employed theories lies well above 1 GeV, so that the definition of dynamical generation given in Section II holds for the light scalars.

(iv) Note, in (i) and (ii) the emergence of states is due to t-channel forces. This is not the case in (iii). However, a common point of them is that the light scalar states disappear in the large-$N_c$ limit.

It is clear that the situation concerning light scalars is by far not understood. We wish, however, to stress once more that there is a crucial difference among the two outlined options in relation to the Lagrangian $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$. Note also that in this subsection we only discussed works for which it is possible to immediately conclude if the scalar mesons are dynamically generated or not according to the definition given in Sec. II. Unitarization methods were not discussed here; in fact, when the latter are applied, care is needed. This is the subject of the next subsection.

C. Low-energy Lagrangians, unitarization and dynamical reconstruction

The Lagrangian $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$ with $E_{\text{max}} \simeq 2$ GeV induces breakdown of chiral symmetry $SU_A(N_f)$, where $N_f$ is the number of light flavors. There are therefore $N_f^2 - 1$ Goldstone bosons: the pion triplet for $N_f = 2$, in addition four kaonic states and the $\eta$ meson for $N_f = 3$.

If we integrate out all the fields in $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$ besides the three light pions, we obtain the Lagrangian of chiral perturbation theory (see ref. and refs. therein) for $N_f = 2$:

$$L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c) \to L_{\chi PT}(E_{\chi PT}, N_c),$$

where $E_{\chi PT}$ should be lower than the first resonance heavier than the pions ($\sim 400$ MeV). $L_{\chi PT}(E_{\chi PT}, N_c)$ is recasted in an expansion of the pion momentum $O(p^{2n})$ and for each $n$ there is a certain number of low-energy coupling constants, which in principle could be calculated from $L_{\text{eff}}^{\text{had}}(E_{\text{max}}, N_c)$, if it were known. Being this not the case, they are directly determined by experimental data. (Similar properties hold when the kaons and the $\eta$ are retained in $L_{\chi PT}(E_{\chi PT}, N_c)$).
For instance, the vector isotriplet \( \rho \) meson is predicted by a large variety of approaches (such as quark models) to be a preexisting 1\(^-\) quark-antiquark field. In this sense it is a fundamental field appearing in \( \mathcal{L}_{\text{eff}} \) (2 GeV, 3), which is integrated out (together with other fields) to obtain \( \mathcal{L}_{\chi PT} (E_{\chi PT}, 3) \). However, the \( \rho \) meson spectral function cannot be obtained from chiral perturbation theory unless a unitarization scheme is employed. As an example, via the IAM (Inverse Amplitude Method) unitarization scheme applied to \( \mathcal{L}_{\chi PT} (E_{\chi PT}, N_c = 3) \), a window between the original energy \( E_{\chi PT} \) and \( 4\pi f_\rho \sim 1 \text{ GeV} \) is opened: resonances with masses in this window, such as the \( \rho \) meson, can be described within unitarized \( \chi PT \). As discussed in the point (d) of Sec. II, the very last question if the \( \rho \) meson is dynamically generated or not cannot be answered at the level of unitarized \( \chi PT \). One still does not know if \( \rho \) corresponds to a basic, preexisting field entering in \( \mathcal{L}_{\text{eff}} \) (2 GeV, \( N_c \)) or not.

Some additional information is needed. In the interesting and important case of large-\( N_c \) studies of unitarized \( \chi PT \), the required additional knowledge is provided by the large-\( N_c \) scaling of the low-energy constants: It has been shown in ref. \[7\] that the mass scales as \( \propto N_c^{\frac{2}{3}} \) and the width as \( N_c^{-\frac{1}{3}} \) and thus the \( \rho \) meson should be considered as a fundamental (not dynamically generated) quark-antiquark field, which shall be directly included in \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c = 3 \)). We also refer to the analytic results of \[8\] where the large-\( N_c \) limit is evident.

Let us turn to the lightest scalar-isoscalar resonance \( \sigma \equiv f_0(600) \) as obtained from (unitarized) \( \chi PT \). In refs. \[52, 53\] precise determinations of the \( \sigma \) pole are obtained, but as stated in ref. \[52\] it is difficult to understand its properties in terms of quarks and gluons. In ref. \[7\] a study of the \( \sigma \) pole within the IAM-scheme in the large-\( N_c \) limit has been performed: a result which is at odd with a predominantly quarkonium, or a glueball, interpretation of the \( \sigma \) meson has been obtained. The mass is not constant and the width does not decrease. However, even at this stage one still cannot say if the \( \sigma \) is dynamically generated or ‘reconstructed’ in relation to \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c = 3 \)) because it is hard to distinguish the molecular and the tetraquark assignments in the large-\( N_c \) limit (see discussion above).

Recently, the Bethe-Salpeter unitarization approach has been used to generate various axial-vector \[2, 3\], tensor and scalar mesons above 1 GeV \[6\]. The starting point are low-energy Lagrangians for the vector-pseudoscalar (such as \( \rho \pi \)) and vector-vector (such as \( \rho \rho \)) interactions. These Lagrangians are also in principle derivable by integrating out heavier fields from the complete \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c = 3 \)). For instance, in the \( \rho \pi \) axial-vector channel the \( a_1(1260) \) meson is obtained and in the \( \rho \rho \) tensor and scalar channels the states \( f_2(1270) \) and \( f_0(1370) \) are found. Are these dynamically generated states of molecular type? The answer is: not necessarily. In fact, the masses of the obtained states lie above the energy limit of the low-energy effective theories out of which they are derived. Even for these states the possibility of ‘dynamical reconstruction’ - just as for the \( \rho \) meson described above- is not excluded: In this scenario, these resonances above 1 GeV are intrinsic, preexisting quark-antiquark or glueball (multiquark states are improbable here) fields of \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c = 3 \)). While first integrated out to obtain the low-energy Lagrangians, unitarization methods applied to the latter allow to reconstruct them. In the next section a toy model is presented, in which this mechanism is explicitly shown: although a state obtained via BS-equation looks like a molecule, it still can represent a fundamental, preexisting quark-antiquark (or glueball) state.

As discussed in the summary of the PDG compilation \[54\] and in refs. \[1, 2\] (and refs. therein) the tensor resonances \( f_2(1275), f_2(1525), a_2(1320) \) and \( K_2(1430) \) represent a nonet of quark-antiquark states. The ideal mixing, the very well measured strong and electromagnetic decay rates \[53\], the masses and the mass splitting are all in excellent agreement with the quark-antiquark assignment. In this case they are fundamental (intrinsic) fields of \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c = 3 \)), which can be dynamically reconstructed (rather than generated) via unitarization scheme(s) applied to a low-energy Lagrangians.

Although experimentally and theoretically more involved, the same can hold in the axial-vector channel: the resonances \( f_1(1285), f_1(1510), a_1(1260) \) and \( K_1(1270) \) are in good agreement with the low-lying 1\(^-\) quark-antiquark assignment. Even more complicated is the situation in the scalar channel: the low-lying quark-antiquark states mix with the scalar glueball \[15\]. Also in this case, however, the possibility of ‘dynamical reconstruction’ rather then ‘generation’ is upheld.

If dynamical reconstruction takes place, there is no conflict between the quark model assignment of ref. \[54\] and the above mentioned recent studies. Note, also, that dynamical reconstruction is in agreement with the discussion of ref. \[56\].

D. A simplification of the Lagrangian \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c \))

The Lagrangian \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c \)) with \( E_{\max} \sim 2 \text{ GeV} \) has been a key ingredient allover the present discussion but it has not been made explicit because unknown. An improved knowledge of (at least parts of) \( \mathcal{L}_{\text{eff}} \) (\( E_{\max}, N_c \)) would surely represent a progress in understanding the low-energy hadron system. As a last step we discuss which properties it might have. Clearly, it must reflect the symmetries of QCD, most notably spontaneous breakdown of chiral symmetry. The (pseudo-) scalar meson matrix \( \Phi \), the (axial-)vector, tensor mesons and the scalar glueball are its basic building blocks. Moreover, if additional non-dynamically generated scalar states such as tetraquarks exist, they shall be also included. We thus have a complicated, general \( \sigma \)-model Lagrangian with many terms, in which operators of all orders can
enter, because renormalization is not a property that an effective hadronic Lagrangian necessarily have.
The question is if it is possible to obtain a (relatively) simple form out of this complicated picture, see also \([42, 43, 44, 57]\).

A possibility to substantially simplify the situation is via dilution invariance: let us consider the (pseudo-)scalar meson matrix \(\Phi\), which transforms as \(\Phi \to R\Phi L^\dagger\) \((R, L \subset SU(3))\) under chiral transformation and the dilaton field \(G\), subject to the potential \(V_G = \propto G^4 (\log G/\Lambda_G + 1/4)\), where \(\Lambda_G\) is a dimensional parameter of the order of \(\Lambda_{QCD}\) (the glueball emerges upon shifting \(G\) around the minimum of its potential \(G_0 \sim \Lambda_G\)). Consider \(\mathcal{L}_{eff}^\text{had}(E_{\text{max}}, N_c) = T - V\), where \(T\) is the dynamical part and \(V = V[G, \Phi, ...]\) is the potential describing masses and interactions of the fields (dots refer to other degrees of freedom, such as (axial-)vector ones). We assume that: (i) In the chiral limit the only term in \(V\) which breaks dilution invariance -and thus mimics the trace anomaly of QCD- is encoded in \(V_G\) (via the dimensional parameter \(\Lambda_G\)). (ii) The potential \(V\) is finite for any finite value of the fields. As a consequence of (i) only operators of order (exactly) four can be included. They have the form \(G^4 Tr[\Phi^4], Tr[\Phi^3 \Phi^2], Tr[\Phi^2 \Phi]\), ... As a consequence of (ii) a huge set of operators are not admitted. In fact, an operator of the kind \(G^{-2} Tr[\partial^2 \Phi \partial^2 \Phi]^2\), is excluded because, although of dimension 4, it blows up for \(G \to 0\). In this way we are left with a sizably smaller number of terms, even smaller than what renormalizability alone would impose \(59\). Work alone this direction, including (axial-)vector degrees of freedom is ongoing \(60\) and can constitute an important source of informations for spectroscopy and for future developments at nonzero temperature and densities, where in the chirally restored phase a degeneration of chiral partners is manifest.

In conclusion, a way to implement these ideas and use the definition of dynamical generation can be sketched as follows: after writing a general chirally symmetric Lagrangian up to fourth order including the glueball and the quark-antiquark fields as basic states, one should attempt without further inclusion of any other state to describe physical processes up to \(\sim 2\) GeV, as pion-pion scattering, decay widths, etc. In doing this one should of course include loops. If, for instance, we start with a basic scalar-isoscalar quark-antiquark field above 1 GeV, do we correctly reproduce the resonance \(f_0(600)\) when solving the Bethe-Salpeter channel in the \(\pi\pi\) sector below 1 GeV? If the answer is positive, the latter resonance is dynamically generated and there is no need of any other additional state. If, albeit including loops, the attraction among pions turns out to be too weak to generate the resonance \(f_0(600)\), we conclude that it is necessary to enlarge our model by explicitly introducing a field which describes it. As argued previously, this field can be identified as a tetraquark state. If this ambitious program will lead to a successful result is matter of future research.

IV. A TOY MODEL FOR DYNAMICAL RECONSTRUCTION

A. Definitions and general discussion

In this section we start from a toy Lagrangian, in which two mesons \(\varphi\) and \(S\) interact. A large-\(N_c\) dependence is introduced in such a way that both fields behave as quarkonium states. Then, the field \(S\), which is taken to be heavier than \(\varphi\), is integrated out and a low-energy Lagrangian with the field \(\varphi\) only is obtained. A Bethe-Salpeter study is applied to the latter Lagrangian: the question is if the original state \(S\), which was previously integrated out, can be reobtained in this way. The answer is generally positive, however care is needed concerning the large-\(N_c\) limit. In a straight BS-approach the quarkonium-like large-\(N_c\) limit of the state \(S\) cannot be reproduced. However, as it shall be shown, within a modified BS approach the large-\(N_c\) limit can be correctly obtained.

The toy Lagrangian \(61, 62\) consisting of the two fields \(\varphi\) (with mass \(m\)) and \(S\) (with bare mass \(M_0 > 2m\)) reads

\[
\mathcal{L}_{\text{toy}}(E_{\text{max}}, N_c) = -\frac{1}{2}g(\varphi + m^2)\varphi - \frac{1}{2}S(\varphi^2 + M_0^2)S + gS \varphi^2,
\]

which we assume to be valid up to \(E_{\text{max}} >> M_0\). The \(N_c\) dependence of the effective Lagrangian is encoded in \(g\) only: \(g = g(N_c) = g_0/\sqrt{3N_c}\). In this way both masses behave like \(N_c^{1/2}\) and the decay amplitude for \(S \to 2\varphi\) scales as \(1/\sqrt{N_c}\), just as if \(\varphi\) and \(S\) were quarkonia states. \(\mathcal{L}_{\text{toy}}(E_{\text{max}}, N_c = 3)\) is the analogous of \(\mathcal{L}_{\text{eff}}^\text{had}(E_{\text{max}}, N_c = 3)\) in a simplified toy world. For definiteness we refer to values in GeV: \(m = 0.3\), \(M_0 = 1\), \(g_0\) will be varied within 1.5 and 5.

The propagator of the field \(S\) is modified via \(\varphi\)-meson loops and takes the form (at the resummed 1-loop level, see Fig. 1):

\[
\Delta(i) = i \left[ p^2 - M_0^2 + (\sqrt{2}g)\Sigma_{\Lambda}(p^2) \right]^{-1}
\]

where \(\Sigma_{\Lambda}(p^2)\) is the 1-loop contribution, which is regularized via a 3-d sharp cutoff \(\Lambda\) \(63\). The dressed mass can be defined via the zero of the real part of \(\Delta^{-1}\), i.e.

\[
M^2 - M_0^2 + (\sqrt{2}g)\Re \Sigma_{\Lambda}(M^2) = 0.
\]

In the large-\(N_c\) limit \(M \to M_0\). This is true whatever definition of the mass of the resonance is chosen. For finite \(N_c\) one has in general \(M < M_0\) due to the loop corrections (see ref. \(61\) for details). The \(T\) matrix for \(\varphi\varphi\) scattering in the \(s\)-channel upon 1-loop resummation is depicted in Fig. 1 and reads \(64\):

\[
T(p^2) = i(\sqrt{2}g)^2 \Delta = \frac{1}{K^{-1} + \Sigma_{\Lambda}(p^2)}, \quad K = \frac{(\sqrt{2}g)^2}{M_0^2 - p^2}.
\]

Note, the present interest is focused on the 1-particle pole of the \(S\) resonance and its corresponding enhancement in
The Lagrangian $\mathcal{L}_{\text{le}}(E_{\text{le}}, N_c)$ contains only quartic term of the kind $\varphi^4$, $\varphi^2 \square \varphi^2$, ... $\mathcal{L}_{\text{le}}(E_{\text{le}}, N_c)$ is the analogue of chiral perturbation theory or, more in general, of a low-energy Lagrangian in this simplified system. The fact that we know explicitly the form of $\mathcal{L}_{\text{toy}}(E_{\text{max}}, N_c)$ allows us to calculate the ‘low-energy constants’ $L^{(k)}$ of eq. (8). If the form of $\mathcal{L}_{\text{toy}}(E_{\text{max}}, N_c)$ were unknown, then also the $L^{(k)}$ would be such. Note, each $L^{(k)}$ scales as $N_c^{-1}$.

**BS-inspired unitarization, way 1:** As a first exercise let us consider the low-energy Lagrangian $\mathcal{L}_{\text{le}}$ up to a certain order $n$ by approximating the potential to $V(n) = \sum_{k=0}^{n} V^{(k)}$. By performing a Bethe-Salpeter study with this approximate potential, see Fig. 1, we obtain the following $T$ matrix:

$$
T(p^2, n) = -K(n) + K(n) \Sigma_\Lambda(p^2) T(p^2, n)
$$

$$
T(p^2, n) = \frac{1}{-K(n) - \Sigma_\Lambda(p^2)},
$$

$$
K(n) = \frac{(\sqrt{2g})^2 \sum_{k=0}^{n} \left( \frac{g^2}{M_0^2} \right)^k}{1 - \Sigma_\Lambda(p^2)},
$$

where $K(n)$ is the bare tree-level amplitude corresponding to the sum of all the quartic terms up to order $n$.

Clearly, $T(p^2, n)$ is an approximate form of $T(p^2)$ of eq. (8). The larger $n$, the better the approximation. Formally one has $\lim_{n \to \infty} T(p^2, n) = T(p^2)$. What we are doing is a ‘dynamical reconstruction’ of the state $S$ via a Bethe-Salpeter analysis applied to the low-energy Lagrangian $\mathcal{L}_{\text{le}}$: We reobtain the state $S$ which has been previously integrated out.

Let us keep $n$ fixed and perform a large-$N_c$ study of $T(p^2, n)$. Do we obtain the correct result, that is $M = M_0$? The answer is no. In fact, in the large-$N_c$ limit $K(n)$ scales as $1/N_c$ due to dependence encoded in $g$, while $\Sigma_\Lambda(p^2)$ scales as $N_c^0$ (we assume that the cutoff does not scale with $N_c$ [67]). In the large-$N_c$ limit we obtain $T(p^2, n) \approx -K(n)$. But $K(n)$ is a polynomial in $p^2$ and, for any finite $n$, does not admit poles for finite $p^2$, but only for $p^2 \to \infty$. Thus, we find the incorrect result that in the large-$N_c$ limit the mass of the dynamically reconstructed state is infinity. This is shown in Fig. 2 for a particular numerical choice.

Although our analysis has been applied to a simple toy model, the form of eq. (10) is general. One has a polynomial form for $K(n)$ as function of $p^2$ and a mesonic loop $\Sigma_\Lambda(p^2)$ which is independent on $N_c$. Complications due to different quantum numbers do not alter the conclusion. We also note that these results are in agreement with the discussion of ref. [68], where the scalar $\sigma$ meson is first integrated out and then ‘reconstructed’ in the framework of the linear sigma model.

**IAM inspired unitarization** If we would, instead, apply the IAM unitarization scheme to the $n = 1$ approximate form we would obtain the correct result in the large-$N_c$ expansion. In fact, in this case one schematically has (neglecting $t$ and $u$ channels):

$$
T_{1\text{AM}} \simeq T_2 \left( T_2 - T_4 - iT_2 \sigma T_2 \right)^{-1} T_2
$$

(11)
in agreement with the large-N_{c} inspired unitarization described above (way 1), it is possible to follow a different BS-inspired approach which is in agreement with the large-N_{c} limit. For simplicity we discuss it in the explicit case n = 1 [60]. One has:

\[ K(1) = \left( \frac{\sqrt{2}g}{M_{0}} \right)^{2} \left( 1 + \frac{p^{2}}{M_{0}} \right) \left( 1 - \frac{p^{2}}{M_{0}} \right) \]  

Now, instead of plugging K(1)^{-1} directly into eq. (10), we first invert it obtaining the approximate form K(1)_{way2}^{-1} valid up to order O(p^4/M^4_0):

\[ K(1)^{-1}_{way2} = \frac{M_{0}^{2}}{(\sqrt{2}g)^{2}} \left( 1 - \frac{p^{2}}{M_{0}^{2}} + ... \right) \]  

The next step is to write the T matrix in terms of K(1)_{way2}^{-1}:

\[ T(p^{2},1)_{way2} = \frac{1}{-K(1)_{way2}^{-1} + \Sigma_{\Lambda}(p^{2})} \approx \frac{(\sqrt{2}g)^{2}}{p^{2} - M_{0}^{2} + (\sqrt{2}g)^{2}\Sigma_{\Lambda}(p^{2})}. \]  

Thus, this new approximate form derived from the BS equation is now in agreement with the large-N_{c} limit and is equivalent to the IAM-inspired unitarization approach described above. This shows an important fact in this discussion: it is not the BS method which fails in BS-way 1, but rather the adopted perturbative expansion. We could as well develop a second IAM-inspired unitarization which fails to reproduce the correct large-N_{c} results and is equivalent to BS-way1. From this perspective we can rearrange the unitarizations as ‘large N_{c} correct’ (BS-way2 and IAM) and ‘large N_{c} violating’ (BS-way1 and IAM-way2). The reason why we associate to the different unitarizations the names BS or IAM is simply due to the way the equations are settled down in the different cases. It offers a simple mnemonic to their development.

**B. BS equation with the lowest term only**

In most studies employing the BS analysis only the lowest term of the effective low-energy Lagrangian is kept. Within the present toy model it is not possible to reconstruct a resonance with mass M > 2m with only the lowest term (n = 0) [71]. However, a simple modification of the model which allows for such a study is possible:

\[ \mathcal{L}_{new}(E_{max}, N_{c}) = \mathcal{L}_{toy}(E_{max}, N_{c}) + \frac{g}{2M_{0}^{2}} \varphi^{4} \]  

In this way an extra-repulsion (whose quartic form is assumed to be valid up to E_{max}) has been introduced.
The $T$ matrix takes the form:

$$T(p^2) = \frac{1}{-K^{-1} + \Sigma_{\Lambda}(p^2)}, \quad K = \frac{(\sqrt{2g})^2}{M_0^2 - p^2} - \frac{(\sqrt{2g})^2}{M_0^2}$$

(18)

When deriving the low-energy Lagrangian everything goes as before, but the $k = 0$ term is now absent:

$$V(n) = \sum_{k=1}^{n} V^{(k)}, \quad V^{(k)} = L^{(1)} \phi^2 (-\Box)^{k} \phi^2$$

Note, in this case the $\phi^2$ scattering vanish at low momenta and in the chiral limit $m \to 0$ (just as the $\pi\pi$ scattering in reality).

A study of the case $n = 1$ (corresponding to the first term only in the expansion) is now possible. We consider the following approximate form for the $\phi^2$ potential at the lowest energy at the approximate form $\phi^2$ is unknown. Moreover, from low-energy informations only one does not know the value of the cutoff $\Lambda$ to be employed in mesonic loops: a new cutoff $\Lambda$, not necessarily equal to the original $\Lambda$, is also introduced as a free parameter. From the perspective of low-energy phenomenology, one writes down the following approximate form for the $T$ matrix, which depend on two ‘free parameters’ $L^{(1)}$ and $\tilde{\Lambda}$:

$$\tilde{T}(p^2) = T(p^2, 1) = \frac{1}{-K^{-1} + \Sigma_{\tilde{\Lambda}}(p^2)}, \quad \tilde{K} = K(1) = 4L^{(1)} p^2.$$  

(19)

The question is if it is possible to vary $L^{(1)}$ and $\tilde{\Lambda}$ such a way that the approximate $T$-matrix $\tilde{T}(p^2)$ reproduces the ‘full’ result $T(p^2)$ of eq. (18) between, say, $2m = 0.6$ GeV and $1.3$ GeV for $N_c = 3$.

The answer is that this is generally possible, but the results for $L^{(1)}$ and $\tilde{\Lambda}$ vary drastically with the coupling constant $g_0$ in the original Lagrangian. In particular, if $g_0$ is small a good fit implies a very large and unnatural value of $\tilde{\Lambda}$ (Fig. 3.a). For instance, for $g_0 = g(N_c = 3) = 1.5$ GeV the mass $M = 0.95$ GeV is only slightly shifted from the bare mass $M_0 = 1$ GeV. In this case the approximate form $|\tilde{T}(p^2)|$ reproduces $|T(p^2)|$ only if $\tilde{\Lambda} \sim 10^4 \Lambda$ (astronomically high and seemingly unnatural from the perspective of the low-energy theory).

The situation changes completely if $g_0$ is large: it is possible to find a satisfactory description in which $\Lambda \sim \tilde{\Lambda}$. For instance, for $g_0 = 5$ GeV one has $M = 0.65$ GeV and the approximate $|\tilde{T}(p^2)|$ reproduces well $|T(p^2)|$ for $\tilde{\Lambda} = \Lambda$ (Fig. 3.d).

In the first case the failure of the dynamical reconstruction with a meaningful value of the cutoff $\tilde{\Lambda}$ is due to the quantitative inappropriate behavior of the Bethe-Salpeter approach when only the first term is kept. In the second case a rather satisfactory description is possible for a meaningful value of the cutoff. On the light of the results of the low-energy Lagrangian only, one could also propose the interpretation that the obtained state $S$ is dynamically generated, and shall be regarded as a $\phi\phi$ molecular state. This is, however, not the correct interpretation in the present example. We know, in fact, that this state corresponds -by construction- to the original, preexisting, quarkonium-like state $S$.

In both cases, as soon as we increase the number of colors, the approximate $T$-matrix $|\tilde{T}(p^2)|$ and the full $|T(p^2)|$ show a completely different behavior (Fig. 3, panels (b,c) and (e,f)). While the peak of $|T(p^2)|$ approaches $M_0 = 1$ GeV and becomes narrower according to the correct large-$N_c$ limit of the $S$ meson, the dynamically reconstructed state fades out, because of the incorrect behavior of BS unitarization with large-$N_c$. This is clearly visible from the interaction term $V^{(1)} = L^{(1)} \varphi^2 (-\Box) \varphi^2$, because $L^{(1)}$ scales as $N_c^{-1}$. However, although the interaction term disappears with large-$N_c$, the state $S$ is still the original quark-antiquark state. This example shows that the reconstruction of the state $S$ is not possible in the large-$N_c$ limit, but does not mean that $S$ is a dynamically generate state of molecular type. Note, this is just a sub-case of the previous general discussion on large-$N_c$ dependence: the fact that only one term is kept generates a much faster ‘fading out’ of the reconstructed state, compare Fig. 2 and Fig. 3.
the large-\(N_c\) limit- a second BS-unitarization allows for a correct description of the large-\(N_c\) limit. The second BS-unitarization is however not applicable in the present case. In fact, at least two terms in the expansion of \(K(n)\) are needed to follow it. If only the lowest term is kept, as done here with the term \(n = 1\) in Eq. (19), this is no longer feasible. This is similar to the fact that also the IAM method needs at least two terms in the expansion of the amplitude \(K\) in order to be applicable [71].

C. Analogy with the real world

The original toy-Lagrangian \(\mathcal{L}_{\text{toy}}(E_{\text{max}}, N_c)\) of eq. (9) is assumed to be valid up to an energy \(E_{\text{max}} >> M_0\). The corresponding low-energy Lagrangian \(\mathcal{L}_{\text{le}}(E_{\text{le}}, N_c)\) of eq. (7) -obtained by integrating out the \(S\) field- is valid up to an energy \(E_{\text{le}} << M_0\). When unitarizing \(\mathcal{L}_{\text{le}}(E_{\text{le}}, N_c)\), one can enlarge the validity of the low-energy theory up to \(M_0\) and then infer the existence of the resonance \(S\) with mass \(M < M_0\). However, if no other input is known, the nature of the state \(S\) cannot be further studied, see the general discussion of the point (d) in Sec. II.

This situation is similar to the example (c) in Sec. II: the Fermi Lagrangian \(\mathcal{L}_V\) alone does not allow to deduce the nature of the \(W\) meson, even if the existence of the latter is inferred by unitarization arguments applied to \(\mathcal{L}_V\). It is also similar to the cases studied in Sec. III.C: when a resonance is obtained by unitarizing a low-energy mesonic Lagrangian, (at first) no statement about its nature can be done.

Further information is needed: in the case of the \(W\) meson the full electroweak Lagrangian \(\mathcal{L}_{\text{EW}}\) is known and leads to the straightforward conclusion that the \(W\) meson is not a dynamically generated state, but a fundamental field of the standard model. In the framework of the toy model, this corresponds to the knowledge of the ‘full Lagrangian’ \(\mathcal{L}_{\text{toy}}(E_{\text{max}}, N_c)\) of eq. (3). The ‘quarkonium’ nature of \(S\) can then be easily deduced.

In the case of low-energy mesonic theories discussed in Sec. III.C, the full hadronic Lagrangian is not known. The only additional knowledge is the large-\(N_c\) scaling of the low-energy constants of the low-energy Lagrangian(s). In the framework of the toy model, this corresponds to the knowledge of low-energy Lagrangian \(\mathcal{L}_{\text{le}}(E_{\text{le}}, N_c)\) of eq. (7) (up to a certain \(n\)) together with the scaling of the quantities \(L(k)\) in eq. (8). The latter additional knowledge can lead to the correct conclusions about the nature of the \(S\) meson, although -as discussed in Sec. IV.A- care is needed when the BS method is chosen.

Moreover, as further studied in Section. IV.B, when only the lowest term of the low-energy Lagrangian is kept, it is not possible to reproduce the correct large-\(N_c\) behavior of the resonance \(S\). Although the ‘dynamical reconstruction’ of the state \(S\) is possible, the state \(S\) ‘looks like’ a molecular state which fades out in the large-\(N_c\) limit. This however is not true: In fact, we know from the very beginning that the state \(S\) corresponds ‘by construction’ to a quark-antiquark state.

Although this discussion is based on toy models and the real world is much more complicated than this, the same qualitative picture can hold in low-energy QCD. In fact, the use of the BS equation in the Literature is often limited to the lowest term of a low-energy Lagrangian for \(\pi\pi, \rho\rho, \ldots\) interactions. In our view, such low-energy Lagrangians emerge upon integrating out all the heavier fields in \(\mathcal{L}_{\text{eff}}(E_{\text{max}}, N_c = 3)\), in which tensor, axial-vector and scalar quark-antiquark fields must exist below 2 GeV. Then, the use of the BS equation, similarly to the dynamical reconstruction of \(S\) in this simple example, leads to the dynamical reconstruction of the axial-vector, tensor and scalar mesons above 1 GeV: they are preexisting, quark-antiquark states, which are reobtained from low-energy Lagrangians via unitarization methods. Future unitarization studies, involving the leading and the next-to-leading-terms in the effective Lagrangians, may shed light on this point.

V. CONCLUSIONS AND OUTLOOK

In this work we studied the issue of dynamical generation both in a general context and in low-energy QCD. A dynamically generated resonance has been defined as a state which does not correspond to any of the fields of the original Lagrangian describing the system up to a certain maximal energy \(E_{\text{max}}\), provided that its mass lies below this maximal energy. This discussion also offered us the possibility to distinguish in principle tetraquark from mesonic molecular states in low-energy QCD: while the former are fundamental and shall be included as bare fields in the (yet-unknown) hadronic Lagrangian \(\mathcal{L}_{\text{had}}(E_{\text{max}}, N_c = 3)\), this is not the case for the latter. Note, \(\mathcal{L}_{\text{had}}(E_{\text{max}}, N_c = 3)\) represents the complete hadron theory valid up to \(E_{\text{max}} \simeq 2\) GeV.

In the application to the hadronic world we also discussed dynamical reconstruction of resonances: these are resonances which are obtained via unitarization methods from low-energy effective Lagrangians, but still represent fundamental fields (such as quark-antiquark states) in \(\mathcal{L}_{\text{eff}}(E_{\text{max}}, N_c = 3)\). Note, the low-energy effective Lagrangians can be seen as the result of integrating out heavier (quarkonia, glueballs, ...) fields representing intrinsic, fundamental states in \(\mathcal{L}_{\text{eff}}(E_{\text{max}}, N_c = 3)\). In the scenario of dynamical reconstruction, one reconstructs these heavier resonances by unitarizing the appropriate low-energy Lagrangian.

Within a simple toy model these issues have been examined. This model consists of two fields, \(\varphi\) and \(S\), with the latter being heavier and with a nonzero decay width into \(\varphi\varphi\). We introduced a large-\(N_c\) dependence which mimics that of quarkonium states in QCD. The field \(S\) has been first integrated out and the emerging low-energy interaction Lagrangian involving only the field \(\varphi\) has been derived. Out of it the state \(S\) has been ‘dynamically re-
constructed’.

In order to do it we have used a unitarization inspired by the Bethe-Salpeter equation and we have shown that the original, quarkonium-like large-$N_c$ behavior of $S$ cannot be reproduced if only the lowest term in the effective Lagrangian is kept. (Note, when more terms are kept this problem can be easily solved and the large $N_c$ result is correct also within the BS approach. The problem is not the latter but the adopted perturbative expansion, see Sec IV. A). We then proposed that a similar, although more complicated, dynamical reconstruction mechanism takes place for tensor, axial-vector and scalar mesons above 1 GeV; these resonances, studied in recent works, can be interpreted as fundamental quark-antiquark states, which are reobtained when unitarizing low-energy effective Lagrangians. In this scenario there is no conflict between the ‘old’ quark model assignments and recent developments, because they would both represent a dual description of the same, preexisting quark-antiquark resonances. This interpretation, although not yet conclusive, represents a possibility which deserves further study.

Dynamical reconstruction can also hold for light scalar mesons below 1 GeV, if the latter form a quarkonium (quite improbable) or a tetraquark nonet. The situation in this case is, as discussed in the text, still unclear. In this work we limited the study to the light mesonic sector, but the present discussion about dynamical generation/reconstruction can also hold, with due changes, in the baryon and in the heavy quark sectors.

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For instance, in the positronium case the interaction Lagrangian (neglecting nonlocality) has the form $L = gP\gamma^{\mu}\gamma^{\nu}\psi$. Surely, this Lagrangian is useful to study properties of the positronium state, but can describe the $e^-e^+$ interaction only in a very small energy range close to $2m_e$, where the positronium resonance dominates. Moreover, as soon as the photon is added to the Lagrangian (a obviously necessary step), one shall not include $P$ from the very beginning, otherwise it is double counted. Indeed, the compositness condition can be directly related to the Bethe-Salpeter approach applied to $Q_{GPD}$, where the composite nature of the positronium is manifest. Similar arguments hold for the deu-teron.

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