LI-YORKE CHAOS FOR COMPOSITION OPERATORS ON $L^p$-SPACES

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Abstract. Li-Yorke chaos is a popular and well-studied notion of chaos. Several simple and useful characterizations of this notion of chaos in the setting of linear dynamics were obtained recently. In this note we show that even simpler and more useful characterizations of Li-Yorke chaos can be given in the special setting of composition operators on $L^p$ spaces. As a consequence we obtain a simple characterization of weighted shifts which are Li-Yorke chaotic. We give numerous examples to show that our results are sharp.

1. Introduction

Throughout this note, $(X, B, \mu)$ will denote a measure space with $\mu(X) \neq 0$ and $f : X \to X$ will be a bimeasurable map (that is, $f(B) \in B$ and $f^{-1}(B) \in B$ for every $B \in B$) for which there exists a constant $c > 0$ such that

\begin{equation}
\mu(f(B)) \geq c \mu(B) \quad \text{for every } B \in B.
\end{equation}

Condition (1) ensures that the composition operator $T_f : \varphi \mapsto \varphi \circ f$ is a continuous linear operator acting on $L^p(X, B, \mu)$ ($1 \leq p < \infty$). This constitutes a natural class of operators. The topological transitivity and mixing properties of this class of operators were investigated in the recent paper [1]. Our goal here is to investigate the notion of Li-Yorke chaos and some of its variations for this class of operators. We will present several characterizations and counterexamples.

For a broad view of the area of linear dynamics, we refer the reader to the books [2, 9], to the more recent papers [3, 5, 6, 7, 8, 12], and to the references therein.

Let us recall that a continuous self-map $g$ of a metric space $(M, d)$ is said to be Li-Yorke chaotic if there exists an uncountable set $S \subset M$ (called a scrambled set for $g$) such that each pair $(x, y)$ of distinct points in $S$ is a Li-Yorke pair for $g$, in the sense that

\[ \liminf_{n \to \infty} d(g^n(x), g^n(y)) = 0 \quad \text{and} \quad \limsup_{n \to \infty} d(g^n(x), g^n(y)) > 0. \]

In the case in which $S$ can be chosen to be dense (respectively, residual) in $M$, we say that $g$ is densely (respectively, generically) Li-Yorke chaotic. This notion of chaos was introduced in [11] in the context of interval maps. It is among the most popular and well studied notions of chaos.

Li-Yorke chaotic linear operators were investigated in [3, 5]. In particular, it was shown that for any continuous linear operator $T$ on any Banach space $Y$, the following assertions are equivalent:

- $T$ is Li-Yorke chaotic;
- $T$ admits a semi-irregular vector, that is, a vector $y \in Y$ such that

\begin{equation}
\liminf_{n \to \infty} \|T^n y\| = 0 \quad \text{and} \quad \limsup_{n \to \infty} \|T^n y\| > 0;
\end{equation}

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\( T \) admits an irregular vector, that is, a vector \( z \in Y \) such that

\[
\liminf_{n \to \infty} \|T^n z\| = 0 \quad \text{and} \quad \limsup_{n \to \infty} \|T^n z\| = \infty.
\]

Moreover, characterizations for dense Li-Yorke chaos and for generic Li-Yorke chaos were also obtained in [5].

Our first result is a necessary and sufficient condition for the composition operator \( T_f \) to be Li-Yorke chaotic. It holds without any additional condition on \( \mu \) or \( f \).

**Theorem 1.1.** The composition operator \( T_f \) is Li-Yorke chaotic if and only if there is an increasing sequence \( (\alpha_i)_{i \in I} \) of positive integers and a nonempty countable family \( (B_i)_{i \in I} \) of measurable sets of finite positive \( \mu \)-measure such that:

\[
\begin{align*}
(A) & \quad \lim_{j \to \infty} \mu(f^{-\alpha_i}(B_i)) = 0 \quad \text{for all } i \in I, \\
(B) & \quad \sup \left\{ \frac{\mu(f^{-n}(B_i))}{\mu(B_i)} : i \in I, n \in \mathbb{N} \right\} = \infty.
\end{align*}
\]

Below is a consequence of Theorem 1.1.

**Corollary 1.2.** Assume \( f \) injective. The composition operator \( T_f \) is Li-Yorke chaotic if there exists a measurable set \( B \) of finite positive \( \mu \)-measure such that:

\[
\begin{align*}
(i) & \quad \liminf_{n \to \infty} \mu(f^{-n}(B)) = 0, \\
(ii) & \quad \sup \left\{ \frac{\mu(f^n(B))}{\mu(B)} : n \in \mathbb{Z}, m \in I, n < m \right\} = \infty,
\end{align*}
\]

where \( I = \{ m \in \mathbb{Z} : 0 < \mu(f^m(B)) < \infty \} \).

We will see in Example 3.4 that the injectivity hypothesis is essential in Corollary 1.2. If \( \mu \) is finite, then the converse of Corollary 1.2 holds. This follows easily from (LY3) in Theorem 1.5 below. However, for infinite measures this converse may fail (see Example 3.4). As an application of this corollary, we have the following result:

**Corollary 1.3.** Assume \( X = \mathbb{Z}, \mathcal{B} = \mathcal{P}(\mathbb{Z}) \) (the power set of \( \mathbb{Z} \)), \( f : i \in \mathbb{Z} \mapsto i + 1 \in \mathbb{Z} \) and \( 0 < \mu(\{ k \}) < \infty \) for some \( k \in \mathbb{Z} \). Then, the composition operator \( T_f \) on \( L^p(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu) \) is Li-Yorke chaotic if and only if the following conditions hold:

\[
\begin{align*}
(a) & \quad \liminf_{i \to -\infty} \mu(\{ i \}) = 0, \\
(b) & \quad \sup \left\{ \frac{\mu(\{ i \})}{\mu(\{ j \})} : i, j \in \mathbb{Z}, i < j, 0 < \mu(\{ j \}) < \infty \right\} = \infty.
\end{align*}
\]

It is well-known that the chaotic operators described in Corollary 1.3 are topologically conjugate to weighted backward shifts on \( \ell^p(\mathbb{Z}) \) with weights \( w_i = \frac{\mu(i)}{\mu(i+1)} \). For example, see Section 1.4 of [4] for more informations and relevant definitions. As a simple consequence, we get the following result. A similar characterization for one-sided backward shifts was given in [3].

**Corollary 1.4.** Let \( w = (w_n)_{n \in \mathbb{Z}} \) be a bounded sequence of positive reals. Define \( B_w : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z}) \) by \( B_w(e_n) = w_n e_{n-1} \). Then, \( B_w \) is Li-Yorke chaotic if and only if

\[
\sup \{ w_n \cdots w_m : n < m, n, m \in \mathbb{Z} \} = \infty.
\]

Now we impose some additional conditions on \( \mu \) and \( f \) in order to obtain some simpler necessary and sufficient conditions for \( T_f \) to be Li-Yorke chaotic.

**Theorem 1.5.** If \( \mu \) is finite and \( f \) is injective, then the following assertions are equivalent:

(LY1) \( T_f \) is Li-Yorke chaotic;
that \( \mu(B) < \infty \) in \((LY3)-(LY6)\), then the following implications always hold (even if \( \mu(X) = \infty \) or \( f \) is not injective):

\[
(LY6) \iff (LY7) \iff (LY1) \iff (LY2) \iff (LY3) \iff (LY5) \iff (LY4).
\]

Moreover, \((LY4) \Rightarrow (LY5)\) whenever \( \mu \) is finite and \((LY5) \Rightarrow (LY6)\) whenever \( f \) is injective. However, we will give a series of counterexamples in Section 3 showing that no other implication holds in general, even under the assumption that \( \mu \) is \( \sigma \)-finite.

Recall that \( f \) is said to be bi-Lipschitz with respect to \( \mu \) if there exist constants \( c_2 > c_1 > 0 \) such that

\[
c_1 \mu(B) \leq \mu(f(B)) \leq c_2 \mu(B) \quad \text{for every } B \in \mathcal{B}.
\]

If \( f \) is bijective and \( f^{-1} \) denotes its inverse, then this property is equivalent to saying that both composition operators \( T_f \) and \( T_{f^{-1}} \) are well-defined and continuous on \( L^p(X, \mathcal{B}, \mu) \).

In this case, note that \( T_{f^{-1}} = T_f^{-1} \).

As an immediate consequence of Theorem 1.5, we have the following result:

**Corollary 1.7.** If \( \mu \) is finite and \( f \) is bijective and bi-Lipschitz with respect to \( \mu \), then \( T_f \) is Li-Yorke chaotic if and only if so is \( T_f^{-1} \).

It is well-known that an invertible operator \( T \) on a Banach space \( Y \) can be Li-Yorke chaotic without \( T^{-1} \) being Li-Yorke chaotic. Actually, we will see in Example 3.7 that the corollary above is false without the hypothesis that \( \mu \) is finite.

Concerning the notions of dense Li-Yorke chaos and generic Li-Yorke chaos, we have the results below.

**Proposition 1.8.** If \( \mu \) is finite and \( f \) is injective, then the composition operator \( T_f \) is densely Li-Yorke chaotic if and only if it is topologically transitive.

**Proposition 1.9.** If \( \mu \) is finite, then the composition operator \( T_f \) is not generically Li-Yorke chaotic.

**Remark 1.10.** We will see in Section 3 that there exist generically Li-Yorke chaotic composition operators that are not topologically transitive. This shows that we cannot remove the hypothesis that \( \mu \) is finite in the previous propositions.

The proofs of the previous results will be given in the next section. In the case of Theorem 1.5, a key role will be played by the notions of backward weakly wandering set and forward weakly wandering set [10]. In Section 3 we will present several counterexamples.

## 2. Proofs of the main results

**Proof of Theorem 1.1**

Assume \( T_f \) Li-Yorke chaotic and let \( \varphi \in L^p(X, \mathcal{B}, \mu) \) be an irregular vector for \( T_f \). Consider the measurable sets

\[
B_i = \{ x \in X : 2^{i-1} \leq |\varphi(x)| < 2^i \} \quad (i \in \mathbb{Z})
\]
and let
\[ I = \{ i \in \mathbb{Z} : \mu(B_i) > 0 \}. \]
Because \( \sum_{i \in \mathbb{Z}} 2^{(i-1)p} \mu(B_i) \leq \int_X |\varphi|d\mu < \infty \), we have that \( 0 < \mu(B_i) < \infty \) for all \( i \in \mathbb{Z} \).
Since \( \varphi \) is an irregular vector for \( T_f \), there is an increasing sequence \( (\alpha_j)_{j \in \mathbb{N}} \) of positive integers such that \( \lim_{j \to \infty} \| T_f^{\alpha_j} \varphi \| = 0 \). This implies (A). Now, suppose that (B) is false. Then, there is a constant \( C < \infty \) such that
\[ \mu(f^{-n}(B_i)) \leq C \mu(B_i) \quad \text{whenever } i \in \mathbb{Z} \text{ and } n \in \mathbb{N}. \]
Hence, for each \( n \in \mathbb{N} \),
\[ \| T_f^n \varphi \|^p = \sum_{i \in \mathbb{Z}} \int_{f^{-n}(B_i)} |\varphi \circ f^n|^p d\mu \leq \sum_{i \in \mathbb{Z}} 2^{n p} \mu(f^{-n}(B_i)) \leq 2^p C \sum_{i \in \mathbb{Z}} 2^{(i-1)p} \mu(B_i) \leq 2^p C \| \varphi \|^p. \]
This contradicts the fact that the \( T_f \)-orbit of \( \varphi \) is unbounded.

Let us now prove the converse. Let \( Y \) be the closed linear span of \( \{ \chi_{B_i} : i \in I \} \) in \( L^p(X, \mathcal{B}, \mu) \). It follows from (A) that the set \( R_1 \) of all vectors \( \varphi \) in \( Y \) whose \( T_f \)-orbit has a subsequence converging to zero is residual in \( Y \). For each \( i \in I \), let
\[ \phi_i = \frac{1}{\mu(B_i)^{1/p}} \chi_{B_i} \in Y. \]
Then
\[ \| \phi_i \| = 1 \quad \text{and} \quad \| T_f^n \phi_i \|^p = \frac{\mu(f^{-n}(B_i))}{\mu(B_i)} \quad (n \in \mathbb{N}). \]
Therefore, (B) gives \( \sup_{n \in \mathbb{N}} \| T_f^n \|_Y = \infty \). Hence, by the Banach-Steinhaus Theorem \[\text{[14]}\] Theorem 2.5], the set \( R_2 \) of all vectors \( \varphi \) in \( Y \) whose \( T_f \)-orbit is unbounded is residual in \( Y \). Since each \( \varphi \in R_1 \cap R_2 \) is an irregular vector for \( T_f \), we conclude that \( T_f \) is Li-Yorke chaotic.

**Proof of Corollary 1.2**

Suppose that there exists such a set \( B \). If \( \lim \sup_{n \to \infty} \mu(f^{-n}(B)) \neq 0 \), then \( \varphi = \chi_B \) is a semi-irregular vector for \( T_f \) because of (i), implying that \( T_f \) is Li-Yorke chaotic. Otherwise, \( \lim_{n \to \infty} \mu(f^{-n}(B)) = 0 \). Set \( B_i = f^i(B) \) for each \( i \in \mathbb{Z} \). Since \( f \) is injective, condition (A) of Theorem \[\text{[13]}\] is satisfied. For any \( n < m \), we have that \( B_n = f^n(B) = f^{n-m}(f^m(B)) = f^{n-m}(B_m) \), which gives
\[ \sup \left\{ \frac{\mu(f^n(B))}{\mu(f^m(B))} : n \in \mathbb{Z}, m \in I, n < m \right\} = \sup \left\{ \frac{\mu(f^{n-m}(B_m))}{\mu(B_m)} : n \in \mathbb{Z}, m \in I, n < m \right\} \]
\[ = \sup \left\{ \frac{\mu(f^{-n}(B_i))}{\mu(B_i)} : i \in I, n \in \mathbb{N} \right\}. \]
In this way, (ii) implies condition (B) of Theorem \[\text{[1]}\] Thus, \( T_f \) is Li-Yorke chaotic.

**Proof of Corollary 1.3**

For the sufficiency of the conditions, it is enough to choose \( k \in \mathbb{Z} \) such that \( 0 < \mu(\{ k \}) < \infty \) and to apply Corollary 1.2 with \( B = \{ k \} \).

For the necessity of the conditions, note that (a) follows from the fact that (LY1) always implies (LY3) (Remark \[\text{[1.6]}\]). If (b) is false, then there is a constant \( C \in (0, \infty) \) with
\[ \mu(\{ i \}) \leq C \mu(\{ j \}) \quad \text{whenever } i, j \in \mathbb{Z} \text{ and } i < j. \]
Hence, for every $\varphi \in L^p(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu)$ and every $n \in \mathbb{N}$,
\[ \|T^n_f\varphi\|^p = \sum_{i \in \mathbb{Z}} |\varphi(i + n)|^p \mu(\{i\}) = \sum_{i \in \mathbb{Z}} |\varphi(i)|^p \mu(\{i - n\}) \leq C \sum_{i \in \mathbb{Z}} |\varphi(i)|^p \mu(\{i\}) = C \|\varphi\|^p. \]

Thus, all orbits under $T_f$ are bounded, contradicting the fact that $T_f$ is Li-Yorke chaotic.

**Proof of Theorem 1.5**

In order to prove this theorem, we need the concept of weakly wandering set and a couple of basic lemmas. The concept was first defined in [10]. Lemma 4 of that paper is analogous to the lemmas we prove here. However, our hypotheses on $f$ are different and hence that lemma does not apply directly to our situation. Hence, we include the proofs.

We say that a measurable set $W$ is a *backward weakly wandering set* for $f$ if there exists a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that the measurable sets \{ $W, f^{-k_1}(W), f^{-k_2}(W), f^{-k_3}(W), \ldots$ \} are pairwise disjoint. Likewise, if the measurable sets \{ $W, f^{k_1}(W), f^{k_2}(W), f^{k_3}(W), \ldots$ \} are pairwise disjoint, then we say that $W$ is a *forward weakly wandering set* for $f$.

**Lemma 2.1.** If there exists $B \in \mathcal{B}$ such that
\[ \mu(B) > 0 \quad \text{and} \quad \liminf_{k \to \infty} \mu(f^k(B)) = 0, \]
then $f$ admits a backward weakly wandering set $W \subset B$ of positive $\mu$-measure.

**Proof.** Note that, by (1) and the second condition in (5), $\mu(B)$ is necessarily finite. Let $\epsilon = \mu(B)/2$ and $\epsilon_i = \epsilon/(i \cdot I_2^i)$ for all $i \geq 1$. By the second condition in (5), we can construct a sequence $0 = k_0 < k_1 < k_2 < \cdots$ of non-negative integers such that
\[ \max \{ c^{-r} : 0 \leq r \leq k_i-1 \} \cdot \mu(f^{k_i}(B)) < \epsilon_i \quad \text{for all} \quad i \geq 1. \]

By (1), $\mu(f^{-k}(S)) \leq c^{-k} \mu(S)$ whenever $k \geq 0$ and $S \in \mathcal{B}$. This together with (6) yield, for all $0 \leq j \leq i - 1$,
\[ \mu(f^{k_i-k_j}(B)) \leq \mu(f^{-k_j}(f^{k_i}(B))) \leq c^{-k_j} \mu(f^{k_i}(B)) < \epsilon_i. \]

We claim that
\[ W = B \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} f^{k_i-k_j}(B) \]

is a backward weakly wandering set of positive $\mu$-measure. In fact, by (7), we have that
\[ \mu(W) > \mu(B) - \mu\left( \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} f^{k_i-k_j}(B) \right) \geq \mu(B) - \sum_{i=1}^{\infty} \epsilon_i = \frac{\mu(B)}{2} > 0. \]

Moreover, by the definition of $W$, for each $i \geq 1$ and $0 \leq j \leq i - 1$,
\[ W \cap f^{k_i-k_j}(W) = \emptyset, \]
and therefore
\[ \emptyset = f^{-k_i}(W \cap f^{k_i-k_j}(W)) = f^{-k_i}(W) \cap f^{-k_i}(f^{k_i-k_j}(W)) \supseteq f^{-k_i}(W) \cap f^{-k_j}(W). \]

This proves that the sets $W = f^{-k_0}(W), f^{-k_1}(W), f^{-k_2}(W), \ldots$ are pairwise disjoint, which means that $W$ is a backward weakly wandering set. \qed
Lemma 2.2. Assume $f$ injective. If there exists $B \in \mathcal{B}$ such that
\begin{equation}
0 < \mu(B) < \infty \quad \text{and} \quad \liminf_{k \to \infty} \mu(f^{-k}(B)) = 0,
\end{equation}
then $f$ admits a forward weakly wandering set $W \subset B$ of positive $\mu$-measure.

Proof. Let $\epsilon = \mu(B)/2$ and $\epsilon_i = \epsilon/(i \cdot 2^i)$ for all $i \geq 1$. By the second condition in (9), there is a sequence $0 = k_0 < k_1 < k_2 < \cdots$ of non-negative integers such that
\[
\max \{e^{-r} : 0 \leq r \leq k_{i-1}\} \cdot \mu(f^{-(k_i-k_{i-1})}(B)) < \epsilon_i \quad \text{for all } i \geq 1.
\]
Thus, for all $0 \leq j \leq i - 1$,
\[
\mu\left(f^{-(k_i-k_j)}(B)\right) = \mu\left(f^{-(k_{i-1}-k_j)}\left(f^{-(k_i-k_{i-1})}(B)\right)\right) \leq e^{-(k_{i-1}-k_j)} \mu\left(f^{-(k_i-k_{i-1})}(B)\right) < \epsilon_i.
\]
We claim that
\[
W = B \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} f^{-(k_i-k_j)}(B)
\]
is a forward weakly wandering set of positive $\mu$-measure. In fact, by (10), we have that
\[
\mu(W) \geq \mu(B) - \mu\left(\bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} f^{-(k_i-k_j)}(B)\right) \geq \mu(B) - \sum_{i=1}^{\infty} \epsilon_i = \frac{\mu(B)}{2} > 0.
\]
Moreover, by the definition of $W$, for each $i \geq 1$ and $0 \leq j \leq i - 1$, $W \cap f^{-(k_i-k_j)}(W) = \emptyset$, which is equivalent to
\begin{equation}
f^{k_i-k_j}(W) \cap W = \emptyset.
\end{equation}
By the equality $f^{k_i}(W) = f^{k_j}\left(f^{k_i-k_j}(W)\right)$, by the injectivity of $f$ and by (11), we reach
\[
f^{k_i}(W) \cap f^{k_j}(W) = f^{k_j}\left(f^{k_i-k_j}(W)\right) \cap f^{k_j}(W) = f^{k_j}\left(f^{k_i-k_j}(W) \cap W\right) = \emptyset,
\]
which proves the claim. \qed

Remark 2.3. Given any $\delta > 0$, by replacing $\epsilon = \mu(B)/2$ by $\epsilon = \mu(B)/n$ with $n$ big enough in the proofs of Lemmas 2.1 and 2.2 we see that the subset $W$ of $B$ can be chosen so that $\mu(B \setminus W) < \delta$.

Let us now prove Theorem 1.3. We begin with the implications that always hold (see Remark 1.6).

\begin{itemize}
\item (LY1) $\Rightarrow$ (LY2): Suppose that $T_f$ is Li-Yorke chaotic. Then it admits a semi-regular vector $\varphi$ in $L^p(X, \mathcal{B}, \mu)$. The second condition in (2) yields $\varphi \not= 0$.
\item (LY2) $\iff$ (LY3): Let $\varphi$ satisfy (LY2). Hence, there exists $\delta > 0$ such that the set $B = \{x \in X : |\varphi(x)| > \delta\}$ has positive $\mu$-measure. Moreover,
\begin{equation}
\left\|T_f^k \varphi\right\|^p = \int |\varphi \circ f^k|^p \, d\mu \geq \int_{f^{-k}(B)} |\varphi \circ f^k|^p \, d\mu \geq \delta^p \mu(f^{-k}(B)).
\end{equation}
By (LY2) and (12), $\liminf_{k \to \infty} \mu(f^{-k}(B)) = 0$. To prove the converse, just take $\varphi = \chi_B$.
\item (LY5) $\Rightarrow$ (LY3) and (LY4): Obvious.
\item (LY6) $\iff$ (LY7): Let $\varphi = \chi_B$, where $B \in \mathcal{B}$. Since
\[
\left\|T_f^k \varphi\right\|^p = \int |\varphi \circ f^k|^p \, d\mu = \int |\chi_B|^p \circ f^k \, d\mu = \mu\left(f^{-k}(B)\right),
\]
it is clear that (LY6) and (LY7) are equivalent properties.
(LY7) \implies (LY1): The existence of a semi-irregular vector, by itself, implies that \( T_f \) is Li-Yorke chaotic.

The next implication uses the injectivity of \( f \).

(LY5) \implies (LY6): Let \( B \) be as in (LY5). By (1) and the third condition in (LY5), \( \mu(B) \) is finite. The proof consists in constructing a measurable set \( A \) of positive \( \mu \)-measure and sequences \( n_1 < m_1 < n_2 < m_2 < \cdots \) of positive integers such that

\[
\lim_{k \to \infty} \mu(f^{-mk}(A)) = 0 \quad \text{and} \quad \limsup_{k \to \infty} \mu(f^{-mk}(A)) > 0.
\]

Set \( n_1 = 1 \) and let \( m_1 > n_1 \) be any integer such that

\[
\mu(f^{m_1-n_1}(B)) < \frac{1}{2}.
\]

Such \( m_1 \) exists because \( \liminf_{\ell \to \infty} \mu(f^{-\ell}(B)) = 0 \). Assume that \( k \geq 2 \) and that \( n_1 < m_1 < n_2 < m_2 < \cdots < n_{k-1} < m_{k-1} \) were defined. Let us define \( n_k \) and \( m_k \) as follows. By hypothesis, \( \liminf_{\ell \to \infty} \mu(f^{\ell}(B)) = 0 \). Thus, there exists \( n_k > m_{k-1} \) such that

\[
\max \{c^{-r} : 0 \leq r \leq m_{k-1}\} \cdot \mu(f^{n_k-m_k}(B)) \leq \frac{1}{2^k}.
\]

Likewise, since \( \liminf_{\ell \to \infty} \mu(f^{\ell}(B)) = 0 \), there exists \( m_k > n_k \) such that

\[
\max \{c^{-r} : 0 \leq r \leq n_k\} \cdot \mu(f^{-(m_k-n_k)}(B)) \leq \frac{1}{k \cdot 2^k}.
\]

By induction, the infinite sequences \( n_1 < m_1 < n_2 < m_2 < \cdots \) satisfy (15) and (16) for all \( k \geq 2 \). Now, let us define the set \( A \) so that (13) is satisfied. Set \( A = \bigcup_{i=1}^{\infty} f^{n_i}(B) \).

Then the second condition in (13) is automatic, because

\[
\mu(f^{-n_k}(A)) \geq \mu(B) \quad \text{for all } k \geq 1.
\]

Let us prove the first condition in (13). By the injectivity of \( f \), \( f^{-m_j}(f^{n_i}(B)) = f^{n_i-m_j}(B) \) for all \( i, j \geq 1 \). Hence,

\[
\mu(f^{-m_j}(A)) \leq \sum_{i=1}^{\infty} \mu(f^{n_i-m_j}(B)) = \sum_{i=1}^{j} \mu(f^{n_i-m_j}(B)) + \sum_{i>j} \mu(f^{n_i-m_j}(B)).
\]

To find an upper bound for the sums in (17), we proceed as follows. For all \( i \geq 2 \) and \( 1 \leq j \leq i-1 \), we have that \( 0 \leq m_j - m_i \leq m_{i-1} \). Hence, by (1), (15) and the injectivity of \( f \), for all \( i \geq 2 \) and \( 1 \leq j \leq i-1 \),

\[
\mu(f^{n_i-m_j}(B)) = \mu(f^{(m_j-m_i)}(f^{n_i-m_j}(B))) \leq c^{-(m_j-m_i)} \mu(f^{n_i-m_j}(B)) < \frac{1}{2^j}.
\]

In the same way, (15) and (16) yield, for all \( j \geq 1 \) and \( 1 \leq i \leq j \),

\[
\mu(f^{n_i-m_j}(B)) = \mu(f^{-(n_j-n_i)}(f^{-m_j}(B))) \leq c^{-(n_j-n_i)} \mu(f^{-(m_j-n_j)}(B)) < \frac{1}{j \cdot 2^j}.
\]

Hence,

\[
\sum_{i=1}^{\infty} \mu(f^{n_i-m_j}(B)) \leq \sum_{i=1}^{j} \frac{1}{j \cdot 2^j} + \sum_{i>j} \frac{1}{2^j} = \frac{1}{2^j} + \frac{1}{2^j}.
\]

By (17), \( \lim_{j \to \infty} \mu(f^{-m_j}(A)) = 0 \).

The next implication requires the additional conditions that \( \mu \) is finite and \( f \) is injective.
and Remark 2.3, there exists a measurable set $W \subset B$ and a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $\mu(W) > 0$ and the sets $W, f^{k_1}(W), f^{k_2}(W), \ldots$ are pairwise disjoint. In particular,

$$\sum_{i=1}^{\infty} \mu(f^{k_i}(W)) = \mu \left( \bigcup_{i=1}^{\infty} f^{k_i}(W) \right) \leq \mu(X) < \infty,$$

implying that $\liminf_{k \to \infty} \mu(f^k(W)) = 0$.

The next implication requires the additional condition that $\mu$ is finite.

$$(LY4) \Rightarrow (LY5):$$ By Lemma 2.2, there are a measurable set $W \subset B$ and a sequence of positive integers $k_1 < k_2 < k_3 < \cdots$ such that $\mu(W) > 0$ and the sets $W, f^{-k_1}(W), f^{-k_2}(W), \ldots$ are pairwise disjoint. In particular,

$$\sum_{i=1}^{\infty} \mu(f^{-k_i}(W)) = \mu \left( \bigcup_{i=1}^{\infty} f^{-k_i}(W) \right) \leq \mu(X) < \infty,$$

implying that $\liminf_{k \to \infty} \mu(f^{-k}(W)) = 0$. On the other hand, as $W \subset B$, we have by (LY4) that $\liminf_{k \to \infty} \mu(f^k(W)) = 0$.

**Proof of Proposition 1.8**

It was proved in [5, Theorem 10] that a continuous linear operator $T$ on a separable Banach space $Y$ is densely Li-Yorke chaotic if and only if it admits a dense set of irregular vectors (or a dense set of semi-irregular vectors). It was also observed in [5, Remark 22] that if an operator is topologically transitive, then it is densely Li-Yorke chaotic. For the converse, assume $T_f$ densely Li-Yorke chaotic and let $\varepsilon \in (0, \min\{1, \mu(X)\})$. By the above-mentioned theorem from [5], there is an irregular vector $\varphi$ for $T_f$ such that

$$\|\varphi - \chi_X\|^p < \varepsilon^{p+1}.$$ 

Set $B = \{x \in X : |\varphi(x) - 1| < \varepsilon\}$. Then $\mu(X \setminus B) < \varepsilon$. Moreover,

$$\|T_f^k \varphi\|^p \geq \int_{f^{-k}(B)} |\varphi \circ f^k| d\mu \geq (1 - \varepsilon)^p \cdot \mu(f^{-k}(B)),$$

that is,

$$\mu(f^{-k}(B)) \leq \frac{1}{(1 - \varepsilon)^p} \cdot \|T_f^k \varphi\|^p.$$ 

Since $\varphi$ is an irregular vector for $T_f$, this yields $\liminf_{n \to \infty} \mu(f^{-n}(B)) = 0$. By Lemma 2.2 and Remark 2.3, there exists a measurable set $W \subset B$ such that

$$\mu(X \setminus W) < \varepsilon \quad \text{and} \quad \liminf_{n \to \infty} \mu(f^n(W)) = 0.$$ 

By condition (C4) of [1, Remark 2.1], $T_f$ is topologically transitive.

**Proof of Proposition 1.9**

It was proved in [5, Theorem 34] that a continuous linear operator $T$ on a Banach space $Y$ is generically Li-Yorke chaotic if and only if every non-zero vector in $Y$ is semi-irregular for $T$. In our case, if $\varphi = \chi_X$ then $\|T_f^k \varphi\|^p = \mu(X)$ for all $n \geq 1$. In particular, $\varphi$ is not a semi-irregular vector for $T_f$, and so $T_f$ is not generically Li-Yorke chaotic.
3. Counterexamples

The injectivity hypothesis in Corollary [1,2]

The next example shows that we cannot omit the hypothesis that \( f \) is injective in Corollary [1,2]

Example 3.1. Consider \( X = (\mathbb{Z} \times \{0\}) \cup (\mathbb{N} \times \mathbb{N}) \) and \( \mathcal{B} = \mathcal{P}(X) \). The bimeasurable map \( f : X \to X \) is given by

\[
 f((i, 0)) = (i + 1, 0) \quad \text{and} \quad f((n, j)) = (n, j - 1) \quad (i \in \mathbb{Z}, n, j \in \mathbb{N}).
\]

The measure \( \mu : \mathcal{B} \to [0, \infty) \) is defined by

\[
 \mu(\{(i, 0)\}) = \frac{1}{2|i|} \quad \text{and} \quad \mu(\{(n, j)\}) = \begin{cases} \frac{1}{2n}, & \text{if } 1 \leq j < n \\ 1, & \text{if } j \geq n \end{cases} \quad (i \in \mathbb{Z}, n, j \in \mathbb{N}).
\]

Since \( \frac{1}{n} \mu(B) \leq \mu(f(B)) \leq 2\mu(B) \) for every \( B \in \mathcal{B}, f \) is bi-Lipschitz with respect to \( \mu \). If \( B = \{(0, 0)\} \), then conditions (i) and (ii) hold. Nevertheless, if \( B_i \in \mathcal{B} \) is nonempty and satisfies condition (A) of Theorem [1,1] then \( B_i \subset \{(k, 0) : k \leq 0\} \), and so

\[
 \sup_{n \in \mathbb{N}} \frac{\mu(f^{-n}(B_i))}{\mu(B_i)} = \frac{1}{2}
\]

Thus, by Theorem [1,1] \( T_f \) is not Li-Yorke chaotic.

The converse of Corollary [1,2] is false

In this subsection we assume that \( f \) is surjective and bi-Lipschitz with respect to \( \mu \). If \( A \in \mathcal{B} \) and \( 0 < \mu(A) < \infty \), then \( 0 < \mu(f^l(A)) < \infty \) for all \( l \in \mathbb{Z} \), and so we can define

\[
 \mathcal{Q}(f, A) = \sup \left\{ \frac{\mu(f^k(A))}{\mu(f^l(A))} : k < l \right\},
\]

\[
 \mathcal{Q}_+(f, A) = \sup \left\{ \frac{\mu(f^k(A))}{\mu(f^l(A))} : k < l, l \geq 0 \right\},
\]

\[
 \mathcal{Q}_-(f, A) = \sup \left\{ \frac{\mu(f^k(A))}{\mu(f^l(A))} : k < l < 0 \right\}.
\]

Lemma 3.2. If \( A_1, A_2 \in \mathcal{B} \) are disjoint sets of finite positive \( \mu \)-measure such that \( \mathcal{Q}(f, A_i) < \infty \) for each \( i \in \{1, 2\} \), then \( \mathcal{Q}(f, A_1 \cup A_2) < \infty \). The same holds for corresponding statements for \( \mathcal{Q}_+ \) and \( \mathcal{Q}_- \).

Proof. This follows easily from the subadditivity of \( \mu \). \hfill \Box

Lemma 3.3. If \( A \in \mathcal{B} \) is a set of finite positive \( \mu \)-measure such that \( \mu(f^i(A)) \leq \mu(f^{i+1}(A)) \) for all but finitely many \( i \in \mathbb{Z} \), then \( \mathcal{Q}(f, A) < \infty \). The same holds for corresponding statements for \( \mathcal{Q}_+ \) and \( \mathcal{Q}_- \).

Proof. Let \( N \in \mathbb{N} \) be such that \( \mu(f^i(A)) \leq \mu(f^{i+1}(A)) \) whenever \( i \in \mathbb{Z} \) and \( |i| \geq N \). Then, \( \mathcal{Q}(f, A) \leq \max \left\{ \left\{ \frac{\mu(f^k(A))}{\mu(f^l(A))} : k < l, k, l \in [-N, N] \right\} \cup \{1\} \right\} \). \hfill \Box

The next example shows that the converse of Corollary [1,2] is false in general.

Example 3.4. Let \( X = \mathbb{N} \times \mathbb{Z} \) and \( \mathcal{B} = \mathcal{P}(X) \). Let \( f : X \to X \) be the bijective bimeasurable map defined by

\[
 f(i, j) = (i, j - 1).
\]
Let

\[ X_i = \{i\} \times \mathbb{Z}, \quad D_i = \{i\} \times \{1, \ldots, i\}, \quad P_i = \{i\} \times \{2i + 1, \ldots, 4i\}, \quad G_i = \{i\} \times \{4i + 1, \ldots\}, \]

for each \( i \in \mathbb{N} \), and set

\[ D = \bigcup_{i=1}^{\infty} D_i, \quad P = \bigcup_{i=1}^{\infty} P_i, \quad G = \bigcup_{i=1}^{\infty} G_i. \]

We define \( \mu \) on \( \mathcal{B} \) by

\[
\mu(\{(i, j)\}) = \begin{cases} 
2^{-j} & \text{if } j \leq 0 \\
2^j & \text{if } 1 \leq j \leq i \\
2^{2i-j} & \text{if } i + 1 \leq j \leq 2i \\
1 & \text{if } 2i + 1 \leq j \leq 4i \\
2^{-j+4i} & \text{if } j \geq 4i + 1.
\end{cases}
\]

We note that, for all \( \mu \in \mathbb{N} \), we have that

\[
\mu(\{(i, j)\}) = \begin{cases} 
\frac{1}{2} \cdot \mu(\{(i, j)\}) & \text{if } (i, j) \in D_i \\
\mu(\{(i, j)\}) & \text{if } (i, j) \in P_i \\
2 \cdot \mu(\{(i, j)\}) & \text{otherwise}.
\end{cases}
\]

In particular, \( f \) is bi-Lipschitz with respect to \( \mu \). Moreover, if \( A \cap D = \emptyset \), then \( \mu(A) \leq \mu(f(A)) \). Now, we will establish the desired properties in a series of steps.

**Step 1.** \( T_f \) is Li-Yorke chaotic.

This follows from applying Theorem 1.1 to the sets \( B_i = \{(i, 0)\}, \, i \in \mathbb{N} \).

**Step 2.** If \( A \subset X \) is non-empty and finite, then \( Q(f, A) < \infty \).

Indeed, for every sufficiently large \( i \), \( f^{-i}(A) \cap D = \emptyset \) and \( f^{i}(A) \cap D = \emptyset \), and so

\[ \mu(f^{-i}(A)) \leq \mu(f^{i+1}(A)) \quad \text{and} \quad \mu(f^{i}(A)) \leq \mu(f^{i+1}(A)). \]

Hence, the result follows from Lemma 3.3.

**Step 3.** Fix \( i \in \mathbb{N} \). If \( A \subset G_i \) is non-empty, then \( Q(f, A) < \infty \).

Note that \( \mu(f^{j}(A)) = 1/2 \cdot \mu(f^{j+1}(A)) < \mu(f^{j+1}(A)) \) for all \( j < 0 \). Let \( k \in \mathbb{N} \) be the least such that \( (i, k) \in A \). Let \( A' = A \setminus \{(i, k)\} \). Then, for \( j > i + 1 + k \), we have that

\[
\mu(f^{j}(A)) = 2^{j-k} + \mu(f^{j}(A') \cap D_i) + \mu(f^{j}(A) \cap D_i) \\
< 2^{j-k} + \mu(f^{j}(A') \cap D_i) + 2^{i+1} \\
< 2^{j-k} + \mu(f^{j+1}(A')) + 2^{i-k} \\
= 2^{j+1-k} + \mu(f^{j+1}(A')) = \mu(f^{j+1}(A)).
\]

Now, by Lemma 3.3 we have that \( Q(f, A) < \infty \).

**Step 4.** For every \( A \subset X \) with \( 0 < \mu(A) < \infty \), we have that \( Q(f, A) < \infty \).

As \( \mu(A) < \infty \), we have that \( A \cap G \) is finite. By Step 2, \( Q(f, A \setminus G) < \infty \) if \( A \setminus G \) is non-empty. Hence, in light of Lemma 3.2 it suffices to prove the result in the case that \( A \subset G \). We will further trim \( A \). Let \( i \in \mathbb{N} \) be the least integer such that \( A \cap X_i \neq \emptyset \). Let \( k \in \mathbb{N} \) be the least integer such that \( (i, k) \in A \). Let \( E = \bigcup_{i=1}^{k} (A \cap X_i) \). By Step 3 and Lemma 3.2 \( Q(f, E) < \infty \) provided \( E \) is non-empty. Hence, we only need to prove the result for \( A \setminus E \). Therefore, we assume that \( A \subset G \) is non-empty, \( (i, k) \) is as above and, for \( i + 1 \leq l \leq k \), we have that \( A \cap X_l = \emptyset \).
Hence, assume that

\[ \text{Fix} \neq \{x \in \mathbb{R} : f(x) = x\} \]

If \( j > i \), then \( \mu(f^j(A)) = \mu(f^j(A) \cap D) \). Let \( I = \{m \in \mathbb{N} : f^j(A) \cap D_m \neq \emptyset\} \). Note that \( I \) is finite. If \( I \) is empty, then by the definition of \( \mu \) we have that \( \mu(f^j(A)) = \mu(f^j(A)) \) and we are done. Hence, assume that \( I \) is nonempty and let \( l = \max I \). Now to conclude the proof, \( \mu(f^j(A) \cap D) \leq \mu(D_i) < 2^{i+1} < 2^{j-k} \).

If \( l > k \), then

\[ \mu(f^j(A') \cap D) \leq \sum_{m \in I} \mu(D_m) < \sum_{m \in I} 2^{m+1} < 2^{i+2} < 2^{j-k}. \]

Now to conclude the proof,

\[ \mu(f^j(A)) = 2^{j-k} + \mu(f^j(A') \cap D) + \mu(f^j(A') \cap D) < 2^{j-k} + \mu(f^j(A') \cap D) + 2^{j-k} \leq 2^{j+1-k} + \mu(f^{j+1}(A')) = \mu(f^{j+1}(A)). \]

It was observed in the Introduction that the converse of Corollary 1.2 holds if \( \mu \) is finite. However, the next example shows that if we remove the injectivity hypothesis, then this converse may fail even for \( \mu \) finite.

**Example 3.5.** Let \( X = \mathbb{N} \times \mathbb{N} \) and \( B = \mathcal{P}(X) \). Let \( f : X \to X \) be the surjective bimeasurable map defined by

\[ f(i, j) = \begin{cases} (i, j-1) & \text{if } j > 1 \\ (i, 1) & \text{if } j = 1. \end{cases} \]

For each \( i \in \mathbb{N} \), let

\[ X_i = \{i\} \times \mathbb{N}, \quad F_i = \{(i, 1)\}, \quad D_i = \{i\} \times \{2, \ldots, i\}, \quad G_i = X_i \setminus (F_i \cup D_i). \]

Let

\[ F = \bigcup_{i=1}^{\infty} F_i, \quad D = \bigcup_{i=1}^{\infty} D_i, \quad G = \bigcup_{i=1}^{\infty} G_i. \]

Let \( \mu_i \) be the finite measure on \( X_i \) so that when the points of \( X_i \) are ordered in the usual fashion, their corresponding measures follow the sequence

\[ 1, 2, \ldots, 2^{i-1}, 2^i, 2^{i-1}, \ldots, 2, 1, 1/2, 1/4, 1/8, \ldots. \]

In particular, note that

\[ \mu_i(f^{-1}(\{(i, j)\})) = \begin{cases} 2 \cdot \mu_i(\{(i, j)\}) & \text{if } (i, j) \in D_i \\ 1/2 \cdot \mu_i(\{(i, j)\}) & \text{if } (i, j) \in G_i. \end{cases} \]

Let \( (\delta_i)_{i \in \mathbb{N}} \) be a sequence of positive numbers so that \( \sum_{i \in \mathbb{N}} \delta_i \mu_i(X_i) < \infty \). Define a finite measure \( \mu \) on \( B \) by \( \mu(A) = \sum_{i \in \mathbb{N}} \delta_i \mu_i(A \cap X_i) \) whenever \( A \in B \). Now, we will establish the desired properties in a series of steps.

**Step 1.** \( T_f \) is Li-Yorke chaotic.

This follows from applying Theorem 1.1 to the sets \( B_i = \{(i, 2)\}, i \in \mathbb{N} \).

**Step 2.** If \( A \subset D_i \) is non-empty, then \( \mathcal{Q}_+(f, A) < \infty \).

This simply follows from the fact that the sequence \( (\mu_i(f^{-k}(A)))_{k \in \mathbb{N}} \) is eventually decreasing and Lemma 3.3.
Step 3. If $A \subset X$ and $A \cap F \neq \emptyset$, then $Q(f, A) < \infty$.

This follows from the fact that $\mu$ is finite and $F$ is the set of fixed points of $f$.

Step 4. If $A \subset X$ is non-empty, then $Q_+(f, A) < \infty$.

This simply follows from the fact that $\mu$ is finite and $\liminf_{l \to \infty} \mu(f^l(A)) > 0$.

Step 5. If $A \subset G$ is non-empty, then $Q_-(f, A) < \infty$.

This simply follows from the fact that $\mu(f^{-1}(A)) = 1/2 \cdot \mu(A)$ for any set $A \subset G$.

Step 6. Suppose $1 \leq i < j$. Then, there exists $L_{i,j} > 1$ such that for all $A_i \subset D_i$, $A_j \subset D_j$, $A_i \neq \emptyset$ and $k > 0$, we have that

$$\mu_j(f^{-k}(A_j)) \leq L_{i,j} \cdot \mu_i(f^{-k}(A_i)).$$

As $A_i \neq \emptyset$ and $A_i \subset D_i$, we have that $2^{-k} < \mu_i(f^{-k}(A_i))$ for all $k > 0$. Now, let us consider $\mu_j(f^{-k}(A_j))$. For $0 < k \leq 2j$, we have that

$$\mu_j(f^{-k}(A_j)) \leq \mu_j(f^{-k}(D_j)) \leq (j - 1) \cdot 2^j < j \cdot 2^j \cdot 2^{j} = j \cdot 2^{3j} \cdot 2^{-k}.$$

For $k \geq 2j$, we have that

$$\mu_j(f^{-k}(A_j)) \leq \mu_j(f^{-k}(D_j)) \leq (j - 1) \cdot 2^{-k+2j} < j \cdot 2^j \cdot 2^{-k}.$$

Hence, $\mu_j(f^{-k}(A_j)) < j \cdot 2^{3j} \cdot 2^{-k}$ for all $k > 0$. Letting $L_{i,j} = j \cdot 2^{3j}$, the result follows.

Step 7. Suppose that $(\delta_i)_{i \in \mathbb{N}}$ satisfies the following additional property: $\forall j \geq 2, \delta_j < 2^{-j} \cdot \max \left\{ \frac{\delta}{L_{i,j}} : 1 \leq i \leq j - 1 \right\}$. Then, $Q(f, A) < \infty$ whenever $A \subset X$ and $\mu(A) > 0$.

Let $A \subset X$ with $\mu(A) > 0$. By Steps 3, 4 and 5, we have that $Q(f, A \cap F)$ and $Q(f, A \cap G)$ are finite, provided $A \cap F$ and $A \cap G$ are nonempty. In light of Lemma 3.2, we only need to show that $Q(f, A \cap D)$ is finite. Hence, let us assume $A \subset D$. By Step 4, we only need to show that $Q_-(f, A) < \infty$. Let $A_i = A \cap D_i$ for all $i \geq 1$. Fix $i$ to be the least positive integer for which $A_i \neq \emptyset$. Let $0 < l < k$. Then, by Step 6,

$$\frac{\mu(f^{-k}(A))}{\mu(f^{-l}(A))} = \frac{\delta_i \mu_i(f^{-k}(A_i)) + \sum_{j=l+1}^{\infty} \delta_j \mu_j(f^{-k}(A_j))}{\delta_i \mu_i(f^{-l}(A_i)) + \sum_{j=l+1}^{\infty} \delta_j \mu_j(f^{-l}(A_j))} \leq \frac{\delta_i \mu_i(f^{-k}(A_i)) + \sum_{j=l+1}^{\infty} \delta_j \cdot L_{i,j} \cdot \mu_i(f^{-k}(A_i))}{\delta_i \mu_i(f^{-l}(A_i))} \leq \frac{\delta_i \mu_i(f^{-k}(A_i)) + \sum_{j=l+1}^{\infty} 2^{-j} \cdot \delta_i \cdot \mu_i(f^{-k}(A_i))}{\delta_i \mu_i(f^{-l}(A_i))} \leq 2 \cdot \frac{\mu_i(f^{-k}(A_i))}{\mu_i(f^{-l}(A_i))}.$$

As $Q_-(f, A_i) < \infty$ (Step 2), we have that $Q_-(f, A) < \infty$, completing the proof.

The hypothesis that the measure is finite in Theorem 1.5

Our goal in this subsection is to show that the hypothesis that $\mu$ is finite is essential in Theorem 1.5 and Corollary 1.7. In all examples in this subsection we will consider $X = \mathbb{Z}$, $B = \mathcal{P}(\mathbb{Z})$ and $f : i \in \mathbb{Z} \mapsto i + 1 \in \mathbb{Z}$. Note that $f$ is a bimeasurable bijection. The measure $\mu$ will be given by its values at the points of $\mathbb{Z}$:

$$\mu_i = \mu(\{i\}) \quad (i \in \mathbb{Z}).$$

In all examples $\mu$ will be $\sigma$-finite and $f$ will be bi-Lipschitz with respect to $\mu$. 
Example 3.6. A composition operator $T_f$ satisfying only \((LY2)\) and \((LY3)\) out of the seven conditions in Theorem 1.5. Let
\[
\mu_i = \begin{cases} 
\frac{1}{2^{-i}} & \text{if } i \leq -1 \\
1 & \text{if } i \geq 0
\end{cases}
\]
Since $\mu(B) \leq \mu(f(B)) \leq 2\mu(B)$ for every $B \in \mathcal{B}$, $f$ is bi-Lipschitz with respect to $\mu$. Moreover, for any finite set $B \subset \mathbb{Z}$, $\lim_{k \to \infty} \mu(f^{-k}(B)) = 0$, proving that \((LY3)\) is true. By Remark 1.6 \((LY2)\) is also true. Now, let $\varphi \in L^p(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu)$ be arbitrary. Then
\[
\sum_{i=-\infty}^{-1} |\varphi(i)|^p \frac{1}{2^{-i}} + \sum_{i=0}^\infty |\varphi(i)|^p = \|\varphi\|^p < \infty.
\]
For each $k \geq 1$, let $j_k = k/2$ if $k$ is even and $j_k = (k-1)/2$ if $k$ is odd. Then
\[
\|T_f^k \varphi\|^p = \sum_{i \in \mathbb{Z}} |\varphi(i+k)|^p \mu(\{i\})
\]
\[
= \sum_{i=-\infty}^{-1} |\varphi(i+k)|^p \frac{1}{2^{-i}} + \sum_{i=0}^\infty |\varphi(i+k)|^p
\]
\[
= \frac{1}{2^k} \sum_{i=-\infty}^{k-1} |\varphi(i)|^p \frac{1}{2^{-i}} + \sum_{i=k}^\infty |\varphi(i)|^p
\]
\[
\leq \frac{1}{2^k} \sum_{i=-\infty}^{-1} |\varphi(i)|^p \frac{1}{2^{-i}} + \frac{1}{2^{k-j_k+1}} \sum_{i=0}^{j_k-1} |\varphi(i)|^p + \sum_{i=j_k}^\infty |\varphi(i)|^p
\]
\[
\to 0 \text{ as } k \to \infty.
\]
Thus, all orbits under $T_f$ converge to zero. In particular, \((LY1)\) is false. By Remark 1.6 \((LY6)\) and \((LY7)\) are also false. Moreover, by the definition of $\mu$, for any non-empty set $B \subset \mathbb{Z}$, $\liminf_{k \to \infty} \mu(f^k(B)) \geq 1$, proving that \((LY4)\) and \((LY5)\) are also false.

Example 3.7. A composition operator $T_f$ such that the equivalent properties \((LY6)\) and \((LY7)\) hold, but \((LY4)\) does not (in particular, $T_f$ is Li-Yorke chaotic but $(T_f)^{-1} = T_{f^{-1}}$ is not): Consider
\[
\ldots \mu_{-3} \mu_{-2} \mu_{-1} \big| \mu_0 \mu_1 \mu_2 \ldots = \ldots 1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \ldots, 1, 1, 1, \ldots
\]
where the bar indicates that $\mu_{-2} = \frac{1}{2}$, $\mu_{-1} = 1$, $\mu_0 = 1$, $\mu_1 = 1$, $\mu_2 = 1$, and so on, and in the left hand side we have successive blocks of the form
\[
1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2^{k-1}}, \frac{1}{2^{k}}, \frac{1}{2^{k-1}}, \ldots, \frac{1}{2}, \frac{1}{2}.
\]
Since $\frac{1}{2} \mu(B) \leq \mu(f(B)) \leq 2 \mu(B)$ for every $B \in \mathcal{B}$, $f$ is bi-Lipschitz with respect to $\mu$. In particular, $T_f$ and $(T_f)^{-1} = T_{f^{-1}}$ are continuous linear operators acting on $L^p(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu)$. Since $\mu(\{0\}) = 1 > 0$, $\liminf_{k \to \infty} \mu(f^{-k}(\{0\})) = 0$ and $\limsup_{k \to \infty} \mu(f^{-k}(\{0\})) > 0$, \((LY6)\) is true. By Remark 1.6 $T_f$ is Li-Yorke chaotic. As for $g = f^{-1}$, note that for any non-empty set $B$, we have that
\[
\liminf_{k \to \infty} \mu(g^{-k}(B)) = \liminf_{k \to \infty} \mu(f^k(B)) \geq 1.
\]
Hence, \((LY3)\) is false for $g$. By Remark 1.6 $(T_f)^{-1}$ is not Li-Yorke chaotic.
**Example 3.8.** A composition operator $T_f$ such that $(LY1)$ holds, but $(LY4)$ and $(LY7)$ do not: Consider

$$
\ldots \mu_{-3}\mu_{-2}, \mu_{-1} \bigg| \mu_0, \mu_1, \mu_2 \ldots = \ldots \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, 1, 2, 1, 2, 2^2, 2, 1 \ldots ,
$$

where in the right hand side we have successive blocks of the form

$$
1, 2, 2^2, \ldots, 2^{k-1}, 2^k, 2^{k-1}, \ldots, 2^2, 2, 1.
$$

Since $\frac{1}{2}\mu(B) \leq \mu(f(B)) \leq 2\mu(B)$ for every $B \in \mathcal{B}$, $f$ is bi-Lipschitz with respect to $\mu$. By Corollary 3.3, $T_f$ satisfies $(LY1)$. If a characteristic function $\chi_B$ lies in $L^p(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \mu)$, then $B \cap \mathbb{N}$ is finite, and so $T_f^n(\chi_B) \to 0$ as $n \to \infty$. This shows that $(LY7)$ is false. It is also clear that $(LY4)$ fails.

**Example 3.9.** A composition operator $T_f$ satisfying only $(LY4)$ out of the seven conditions in Theorem 1.5. It is enough to define

$$
\ldots \mu_{-3}\mu_{-2}, \mu_{-1} \bigg| \mu_0, \mu_1, \mu_2 \ldots = \ldots 1, 1, 1, 2, 2, 1 \ldots ,
$$

The injectivity hypothesis in Theorem 1.5 cannot be removed. In fact, as the next example shows, fail of the injectivity of $f$ at only 2 points may prevent $T_f$ from being Li-Yorke chaotic.

**Example 3.10.** Let $X = \{0\} \cup \left\{ \frac{1}{i} : i \geq 1 \right\}$ and $\mathcal{B} = \mathcal{P}(X)$. The finite measure $\mu$ is defined by its values at the elements of $X$ as follows:

$$
\mu(\{0\}) = 0 \quad \text{and} \quad \mu\left(\left\{ \frac{1}{i} \right\}\right) = \frac{1}{2^i} \quad \text{for } i \geq 1.
$$

The map $f : X \to X$ is given by

$$
f(0) = 0, \quad f\left(\frac{1}{i}\right) = \frac{1}{i-1} \quad \text{for } i \geq 2, \quad f(1) = 1.
$$

Clearly $f$ is surjective, but it is not injective since $f\left(\frac{1}{2}\right) = f(1)$. We claim that

$$
\frac{1}{2}\mu(B) \leq \mu(f(B)) \leq 2\mu(B) \quad \text{for every } B \in \mathcal{B}.
$$

Indeed, since the second inequality is clear, let us prove the first one. If $\left\{ \frac{1}{2}, 1 \right\} \not\subset B$, then $f$ is injective on $B$ and $\mu(f(B)) \geq \mu(B)$. Assume $\left\{ \frac{1}{2}, 1 \right\} \subset B$ and write $B$ as the union $B = \left\{ \frac{1}{2}, 1\right\} \cup B'$, where $\left\{ \frac{1}{2}, 1 \right\} \cap B' = \emptyset$. Since $\mu(f(B')) \geq \mu(B')$, we obtain

$$
\frac{\mu(f(B))}{\mu(B)} = \frac{\frac{1}{2} + \mu(B')}{\frac{1}{4} + \frac{1}{2} + \mu(B')} \geq \frac{\frac{1}{2}}{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}} = \frac{1}{2}
$$

Thus, $f$ is bi-Lipschitz with respect to $\mu$. If $B = \left\{ \frac{1}{2} \right\}$, then $\mu(B) = \frac{1}{4} > 0$ and

$$
\lim_{k \to \infty} \mu(f^{-k}(B)) = \lim_{k \to \infty} \frac{1}{2^{k+2}} = 0.
$$

Hence, $(LY3)$ is true and, by Remark 1.6, so is $(LY2)$. Now, let $\varphi \in L^p(X, \mathcal{B}, \mu)$ be arbitrary. Note that

$$
(T_f^k \varphi)\left(\frac{1}{i}\right) = \varphi\left(f^k\left(\frac{1}{i}\right)\right) = \left\{ \begin{array}{ll}
\varphi(1) & \text{if } 1 \leq i \leq k, \\
\varphi\left(\frac{1}{i-k}\right) & \text{if } i \geq k + 1.
\end{array} \right.
$$
Hence,
\[
\|T_j^k \varphi\|^p = \sum_{i=1}^{\infty} \left| (T_j^k \varphi) \left( \frac{1}{i} \right) \right|^p \mu\left( \left\{ \frac{1}{i} \right\} \right)
\]
\[
= |\varphi(1)|^p \sum_{i=1}^{k} \left| \frac{1}{2} \right|^p + \sum_{i=k+1}^{\infty} \left| \varphi\left( \frac{1}{i-k} \right) \right|^p \frac{1}{2^i}.
\]
Therefore, if $\varphi(1) \neq 0$, then $\|T_j^k \varphi\|^p \geq \frac{1}{2} |\varphi(1)|^p > 0$ for all $k \geq 1$, implying that $\varphi$ is not a semi-irregular vector for $T_j$. Suppose that $\varphi(1) = 0$. Then
\[
\|T_j^k \varphi\|^p = \frac{1}{2^k} \sum_{i=k+1}^{\infty} \left| \varphi\left( \frac{1}{i-k} \right) \right|^p \frac{1}{2^{i-k}} = \frac{1}{2^k} \|\varphi\|^p \to 0 \text{ as } k \to \infty,
\]
showing that $\varphi$ is not a semi-irregular vector of $T_j$. Therefore, $(LY 1)$ is false. By Remark 1.6, $(LY 6)$ and $(LY 7)$ are also false. Finally, note that any set $B \subset X$ of positive measure contains a point of the form $\frac{1}{i}$, for some $i \geq 1$. As $f^k \left( \frac{1}{i} \right) = 1$ for every $k$ big enough, we conclude that $\mu \left( f^k(B) \right) \geq \frac{1}{2}$ for any such $k$, proving that $(LY 4)$ (hence $(LY 5)$) is false.

A generically Li-Yorke chaotic composition operator that is not topologically transitive

It was obtained in [13, Theorem 3.13] and [5, Theorem 37] examples of unilateral weighted forward shifts

\[ T : (x_1, x_2, x_3, \ldots) \in \ell^2 \mapsto (0, w_1x_1, w_2x_2, w_3x_3, \ldots) \in \ell^2 \]

that are generically Li-Yorke chaotic. Moreover, the weights $w_j$ satisfy $1/4 \leq w_j \leq 2$ for all $j \in \mathbb{N}$. Clearly, such an operator cannot be topologically transitive. Let us see that such an operator $T$ can be regarded as a composition operator $T_j$ on $L^2(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ for suitable $\mu$ and $f$. Indeed, the measure $\mu$ is defined by

\[ \mu(\{1\}) = \infty, \quad \mu(\{2\}) = 1 \quad \text{and} \quad \mu(\{i\}) = (w_1 \cdot \ldots \cdot w_{i-2})^2 \quad \text{for } i \geq 3, \]

and the bimeasurable map $f : \mathbb{N} \to \mathbb{N}$ is given by

\[ f(1) = 1 \quad \text{and} \quad f(i) = i - 1 \quad \text{for } i \geq 2. \]

An easy computation shows that (11) holds with $c = 1/4$. Hence, the composition operator $T_j$ is well-defined on $L^2(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$. Now, let

\[ \phi : (x_n)_{n \in \mathbb{N}} \in L^2(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) \mapsto (x_2, w_1x_3, w_1w_2x_4, w_1w_2w_3x_5, \ldots) \in \ell^2. \]

Then, $\phi$ is an isometric isomorphism from $L^2(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ onto $\ell^2$ and $T \circ \phi = \phi \circ T_j$. This shows that $T$ is topologically conjugate to $T_j$. Thus, from a dynamical systems point of view, we can regard $T$ as being $T_j$.

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