Subfactors and Planar Algebras

D. Bisch*

Abstract

An inclusion of II$_1$ factors $N \subset M$ with finite Jones index gives rise to a powerful
set of invariants that can be approached successfully in a number of different ways. We describe Jones’ pictorial description of the standard invariant of a subfactor as
a so-called planar algebra and show how this point of view leads to new structure
results for subfactors.

2000 Mathematics Subject Classification: 46L37, 46L60, 82B20, 81T05.
Keywords and Phrases: Von Neumann algebras, Subfactors, Planar algebras.

1. Introduction

Abelian von Neumann algebras are simply algebras of bounded, measurable
functions on a measure space. A general (non-abelian) von Neumann algebra can
be viewed as an algebra of “functions” (operators) on a non-commutative measure
space. The building blocks of what one might call non-commutative probability
spaces are the so-called II$_1$ factors $M$, that is those von Neumann algebras with
trivial center that are infinite dimensional and possess a distinguished tracial state
(the analogue of a non-commutative integral). The “smallest” II$_1$ factor is the
hyperfinite II$_1$ factor which is obtained as the closure in the weak operator topology
of the canonical anti-commutation relations (CAR) algebra of quantum field theory.
A II$_1$ factor comes always with a natural left representation on $L^2(M)$, the non-
commutative $L^2$-space associated to $M$. See for instance [13].

Vaughan Jones initiated in the early 80’s the theory of subfactors as a “Galois
theory” for inclusions of II$_1$ factors. A subfactor is an inclusion of II$_1$ factors $N \subset M$
such that the dimension of $M$ as left $N$-Hilbert module is finite. This dimension
is called the Jones index $[M : N]$ ([19]) and one would expect by classical results
of Murray and von Neumann that it takes on any real number $\geq 1$. One of the
early results in the theory of subfactors was Jones’ spectacular rigidity theorem
which says that this index is in fact quantized [19]: if $[M : N] \leq 4$, then it has
to be of the form $4 \cos^2 \frac{\pi}{n}$, for some $n \geq 3$. Since Jones’ early work the theory of

*Department of Mathematics, Vanderbilt University, Nashville, TN 37240 and UCSB, Department of Mathematics, Santa Barbara, CA 93106, USA. E-mail: bisch@math.vanderbilt.edu
subfactors has developed into one of the most exciting and rapidly evolving areas of operator algebras with numerous applications to different areas of mathematics (e.g. knot theory with the discovery of the Jones polynomial [20]), quantum physics and statistical mechanics. Subfactors with finite Jones index have an amazingly rich mathematical structure and an interplay of analytical, algebraic-combinatorial and topological techniques is intrinsic to the theory.

2. Subfactors

A subfactor can be viewed as a group-like object that encodes what one might call generalized symmetries of the data that went into its construction. To decode this information one needs to compute the higher relative commutants, a system of inclusions of certain finite dimensional C*-algebras naturally associated to the subfactor. This system is an invariant of the subfactor, the so-called standard invariant, which contains in many natural situations precisely the same information as the subfactor itself ([30], [32], [33]). Here is one way to construct the standard invariant: If $N \subset M$ denotes an inclusion of II$_1$ factors with finite Jones index, and $e_1$ is the orthogonal projection $L^2(M) \to L^2(N)$, then we define $M_1$ to be the von Neumann algebra generated by $M$ and $e_1$ on $L^2(M)$. $M_1$ is again a II$_1$ factor and $M \subset M_1$ has finite Jones index as well so that the previous construction can be repeated and iterated [19]. One obtains a tower of II$_1$ factors $N \subset M \subset M_1 \subset M_2 \subset \ldots$ associated to $N \subset M$, together with a remarkable sequence of projections $(e_i)_{i \geq 1}$, the so-called Jones projections, which satisfy the Temperley-Lieb relations and give rise to Jones’ braid group representation [19], [20]. The (trace preserving) isomorphism class of the system of inclusions of (automatically finite dimensional) centralizer algebras or higher relative commutants

$$\mathcal{C} = N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \subset \ldots \cup \cup \cup \mathcal{C} = M' \cap M \subset M' \cap M_1 \subset M' \cap M_2 \subset \ldots$$

is then the standard invariant $\mathcal{G}_{N,M}$ of the subfactor $N \subset M$. Each row of inclusions is given by a sequence of Bratteli diagrams, which can in fact be reconstructed from a single, possibly infinite, bipartite graph. Hence one obtains two graphs (one for each row), the so-called principal graphs of $N \subset M$, which capture the inclusion structure of the above double-tower of higher relative commutants. It turns out that if $M$ is hyperfinite and $N \subset M$ has finite depth (i.e. the principal graphs are finite graphs) [30], [32] or more generally if $N \subset M$ is amenable [33], then the standard invariant determines the subfactor. In this case the subfactor can be reconstructed from the finite dimensional data given by $\mathcal{G}_{N,M}$. In particular, subfactors of the hyperfinite II$_1$ factor $R$ with index $\leq 4$ are completely classified by their standard invariant and an explicit list can be given (see for instance [14], [16] or [33]). If the Jones index becomes $\geq 6$ such an explicit list is out of reach as the work in [6], [11] and [12] shows: there are uncountably many non-isomorphic, irreducible infinite
depth subfactors of $R$ with Jones index 6 and the same standard invariant! Partial lists of irreducible subfactors with index between 4 and 6 have been obtained by different methods (see for instance [1], [5], [6], [17], [35], [36], [37], [38]), but much work remains to be done.

There are several distinct ways to analyze the standard invariant of a subfactor (see [2], [4], [14], [22], [30], [33]). For instance, in the bimodule approach ([13], [30], see also [4], [14], [18]) $G_{N,M}$ is described as a graded tensor category of natural bimodules associated to the subfactor. $G_{N,M}$ can thus be viewed as an abstract system of (quantum) symmetries of the mathematical or physical situation from which the subfactor was constructed. It is in fact a mathematical object which generalizes for instance discrete groups and representation categories of quantum groups ([37], [38]). A variety of powerful and novel techniques have been developed over the last years that make it possible to compute and understand the standard invariant of a subfactor. A key result is Popa’s abstract characterization of the standard invariant [34]. Popa gives a set of axioms that an abstract system of inclusions of finite dimensional C*-algebras needs to satisfy in order to arise as the standard invariant of some (not necessarily hyperfinite) subfactor. This result makes it possible to analyze the structure of subfactors, which are infinite dimensional, highly non-commutative objects, by investigating the finite dimensional structures encoded in their standard invariants.

3. Planar algebras

Jones found in [22] a powerful formalism to handle complex computations with $G_{N,M}$. He showed that the standard invariant of a subfactor has an intrinsic planar structure (this will be made precise below) and that certain topological arguments can be used to manipulate the operators living in the higher relative commutants of the subfactor. The standard invariant is a so-called planar algebra. To explain this notion let us first define the planar “operad” following [22]. Elements of the planar operad are certain classes of planar $k$-tangles which determine multilinear operations on the vector spaces underlying the higher relative commutants associated to a finite index subfactor.

A planar $k$-tangle consists of the unit disk $D$ in the complex plane together with several interior disks $D_1$, $D_2$, ..., $D_n$. The boundary of $D$ is marked with $2k$ points and each $D_j$ has $2k_j$ marked points on its boundary. These marked points are connected by strings in $D$, which meet the boundary of each disk transversally. We also allow (finitely many) strings which are closed curves in the interior of $D$. The main point is that all strings are required to be disjoint (hence planarity) and to lie in the complement of the interiors of the $D_j$’s. Additional data of a planar $k$-tangle is a checkerboard shading of the connected components of $D \setminus \bigcup_{j=1}^n D_j$, and a choice of a white region at every $D_j$ (which corresponds to a choice of the first marked point on the boundary of each $D_j$). The planar operad $P$ is defined to consist of all orientation-preserving diffeomorphism classes of planar $k$-tangles.
(for all $k \geq 0$), where the diffeomorphisms leave the boundary of $D$ fixed but are allowed to move the interior disks. $\mathcal{P}$ becomes a colored operad [22] (see [28]). An example of a 4-tangle is depicted in the next figure:

Note that there are two classes of planar 0-tangles according to the shading of the tangle near the boundary of $D$.

Two planar tangles $T$ and $S$ can be composed in a natural way if the number of boundary points of $S$ matches the number of boundary points of one of the interior disks $D_j$ of $T$: To obtain the composed tangle $T \circ_j S$ shrink $S$ and paste it inside $D_j$ so that the shadings and marked white regions match up. Join the strings at the boundary of $D_j$, smooth them and erase the boundary of $D_j$. It is clear that this operation is well-defined (the checkerboard shading and choice of a white region at each disk avoid rotational ambiguity) and that it depends only on the isotopy class of each tangle. Note that there may be several different ways of composing two given tangles, each composition yielding potentially distinct planar tangles. An example of such a composition is given in the next figure (insert $S$ in the disk $D_2$ of $T$):

An abstract planar algebra is then defined to be an algebra over this planar operad ([28]). More concretely, an abstract planar algebra $\mathcal{P}$ is the disjoint union of vector spaces $\mathcal{P} = P_{\text{white}} \coprod \bigcup_{n \geq 0} P_n$ plus a morphism from the planar operad to the (colored) operad of multilinear maps between these vector spaces. In other words a planar algebra structure on $\mathcal{P}$ is a procedure that assigns to each planar $k$-tangle $T$ (with interior disks $D_j$ having $2k_j$ boundary points, $1 \leq j \leq n$) a multilinear map $Z(T) : P_{k_1} \times \cdots \times P_{k_n} \to P_k$ in such a way that composition of tangles is compatible with the usual composition of maps (naturality of composition). Note that the $P_k$'s are automatically associative algebras since the tangle
in the next figure (drawn in the case $k = 5$) defines an associative multiplication
$P_k \times P_k \rightarrow P_k$ (associativity follows from naturality of the composition).

Observe that this is a purely algebraic structure - the definition can be made
for (possibly infinite dimensional) vector spaces over an arbitrary field. The key
point is of course that this structure appears naturally in the theory of subfactors.
In order to connect with subfactors several additional conditions will be required
in the definition of a planar algebra. A planar algebra (or subfactor planar algebra
to emphasize the operator algebra context) will be an abstract planar algebra such
that $\dim P_k < \infty$ for all $k$, $\dim P_0^{\text{white}} = \dim P_0^{\text{black}} = 1$ and such that the partition
function $Z$ associated to the planar algebra is positive and non-degenerate. The partition
function is roughly obtained as follows: If $\mathcal{T}$ is a 0-tangle, then $Z(\mathcal{T})$
is a scalar since it is an element in the 1-dimensional space $P_0^{\text{white}}$ resp. $P_0^{\text{black}}$.
Note that every planar algebra comes with two parameters $\delta_1 = Z(\bigcirc)$ and $\delta_2 = Z(\circ)$,
which we require to be $\neq 0$ (the inner circles are strings, not boundaries
of disks!). In the case of a subfactor planar algebra we have $\delta \overset{\text{def}}{=} \delta_1 = \delta_2$
(which is equivalent to extremality of the subfactor [31]). In fact $\delta = [M : N]^{1/2}$ in this
case. There is an intrinsic way to define an involution on the planar algebra arising
from a subfactor which makes the partition function into a sesquilinear form on
the standard invariant. Positivity of the partition function $Z$ means then positivity
of this form. Note that $Z$ gives in particular the natural trace on the standard
invariant of the subfactor. The main result of [22] is then the following theorem.

**Theorem 3.1.** The standard invariant $G_{N,M}$ of an extremal subfactor $N \subset M$
is a subfactor planar algebra $\mathcal{P} = (P_n)_{n \geq 0}$ with $P_n = N' \cap M_{n-1}$.

This theorem says in particular that planar tangles always induce multilinear
maps ("planar operations") on the standard invariant of a subfactor. As a consequence one obtains a diagrammatic formalism that can be employed to manipulate
the operators in $N' \cap M_{n-1}$ and intricate calculations with these operators can be
Carried out using simple topological arguments. This point of view has been turned
in [9], [10] into a powerful tool to prove general structure theorems for subfactors,
and to analyze the rather complex combinatorial structure of the standard invariant
of a subfactor. It has led to a generators and relations approach to subfactors. See
also [23], [24] for more on this.

The two most fundamental examples of subfactor planar algebras are the
Temperley-Lieb systems of [19] (see also [22]) and the Fuss-Catalan systems of [7]
(see section 4).
Observe that by construction planar algebras are closely related to invariants for graphs, knots and links and to the pictorial formalism commonly used in the theory of integrable lattice models in statistical mechanics.

4. Fuss-Catalan algebras

Jones and I discovered in [7] a new hierarchy of finite dimensional algebras, which arise as the higher relative commutants of subfactors when intermediate subfactors are present. These algebras have a number of interesting combinatorial properties and they have recently been used to construct new integrable lattice models and new solutions of the Yang-Baxter equation ([15], [29]).

We show in [7] that a chain of \( k - 1 \) intermediate subfactors \( N \subset P_1 \subset P_2 \subset \ldots \subset P_{k-1} \subset M \) leads to a tower of algebras \( \left( FC_n(a_1, \ldots, a_k) \right)_{n \geq 0} \), which depend on \( k \) complex parameters \( a_1, \ldots, a_k \). The dimensions of these algebras are given by the generalized Catalan numbers or Fuss-Catalan numbers \( \frac{(k+1)n}{n+1} \) and we therefore call these algebras the Fuss-Catalan algebras. If no intermediate subfactor is present, i.e. \( P_i = N \) or \( P_i = M \) for all \( i \), then one finds the well-known Temperley-Lieb algebras (case \( k = 1 \)) [19]. The additional symmetry coming from the intermediate subfactor is captured completely by these new algebras and it is proved in [7] (see also [8]) that they constitute the minimal symmetry present whenever an intermediate subfactor occurs. See also [26].

Let us explain in more detail what happens in the case of just one intermediate subfactor. We consider \( N \subset P \subset M \), an inclusion of \( \Pi_1 \) factors with finite Jones index, and construct the associated tower of \( \Pi_1 \) factors as in section 2. One obtains an inclusion of \( \Pi_1 \) factors \( N \subset P \subset M \overset{e_1}{\subset} P_1 \overset{p_1}{\subset} M_1 \overset{e_2}{\subset} P_2 \overset{p_2}{\subset} M_2 \subset \ldots \), where the \( p_i \)'s are the orthogonal projections from \( L^2(M_{i-1}) \) onto \( L^2(P_{i-1}) \) (\( P_0 = P \), \( M_0 = M \)) and the intermediate subfactors \( P_i \) are the von Neumann algebras generated by \( M_{i-1} \) and \( p_i \). The algebra \( IA_n(\alpha, \beta) \overset{\text{def}}{=} \text{Alg}(1, e_1, \ldots, e_{n-1}, p_1, \ldots, p_{n-1}) \), generated by the \( e_i \)'s and the \( p_j \)'s, is a subalgebra of \( N' \cap M_{n-1} \). It can be shown to depend only on the two indices \( \alpha = [P : N] \) and \( \beta = [M : P] \), and not on the particular position of \( P \) in \( N \subset M \). The projections \( e_i \) and \( p_j \) satisfy again some rather nice commutation relations (see [7] for details). In order to describe the structure of these algebras let us for the moment consider the complex vector space \( FC_n(a, b) \), spanned by labelled, planar diagrams of the form

\[
\begin{array}{cc}
2n \text{ marked points} \\
\begin{array}{c}
\vspace{1cm}
\begin{array}{c}
\text{ Diagram 1 } \\
\text{ Diagram 2 }
\end{array}
\end{array}
\end{array}
\]

where \( a, b \in \mathbb{C} \setminus \{0\} \) are fixed. There is a natural multiplication of these diagrams, which makes \( FC_n(a, b) \) into an associative algebra (see [25]). To obtain \( D_1 \cdot D_2 \) put
the basis diagram $D_1$ on top of $D_2$ so that the labelling matches, remove the middle bar and all closed loops. Multiply the resulting diagram with factors of $a$ resp. $b$ according to the number of removed $a$-loops resp. $b$-loops. An example is depicted in the next figure.

Counting diagrams shows that $\dim FC_n(a, b) = \frac{1}{2n+1} \binom{3n}{n}$, the $n$-th Fuss-Catalan number [7]. Clearly $FC_n(a, b)$ embeds as a subalgebra of $FC_{n+1}(a, b)$ by adding two vertical through strings to the right of each basis diagram of $FC_n(a, b)$. A diagrammatic technique, called the middle pattern analysis in [7], can be used to compute the structure of these algebras completely in the semi-simple case. One obtains that the structure of the tower $FC_1(a, b) \subset FC_2(a, b) \subset \ldots$ of Fuss-Catalan algebras is given by the Fibonacci graph [7].

The algebras $IA_n(\alpha, \beta)$ that we are interested in can then be shown to be isomorphic to $FC_n(a, b)$, where $\alpha = a^2$, $\beta = b^2$, if the indices $\alpha$ and $\beta$ are generic, i.e. $> 4$. In the non-generic case $IA_n(\alpha, \beta)$ is a certain quotient of $FC_n(a, b)$ (see [7] for the details).

There is a natural 2-parameter Markov trace on the Fuss-Catalan algebras and the trace weights are calculated explicitly in [7]. In the special case of the Temperley-Lieb algebras this Markov trace is the one discovered by Jones in [19]. The Fuss-Catalan tower together with this Markov trace satisfies Popa’s axioms in [34] and hence, one can conclude from [34] that for every pair $(\alpha, \beta)$ of possible Jones indices, there is a subfactor whose standard invariant is given precisely by the corresponding Fuss-Catalan system $(FC_n(\sqrt{\alpha}, \sqrt{\beta}))_{n \geq 0}$. One obtains in this way uncountably many new subfactors. A complete set of generators and relations for the Fuss-Catalan algebras is also determined in [7].
It should be evident that the Fuss-Catalan algebras can be viewed as planar algebras generated by a single element in \( P_2 = N' \cap M_1 \), namely by the Jones projection \( p_1 \) onto the intermediate subfactor. This projection can be characterized abstractly \([3]\) and it satisfies a remarkable exchange relation \([9], [27]\), which plays an important role in the work described in the next section.

5. Singly generated planar algebras

Any subset \( S \) of a planar algebra \( P \) generates a planar subalgebra as the smallest graded vector space containing \( S \) and closed under planar operations. From this point of view the simplest subfactors will be those whose planar algebra is generated by the fewest elements satisfying the simplest relations, while the index may be arbitrarily large. If \( S \) is empty we obtain the Temperley-Lieb algebra. The next most complicated planar algebras after Temperley-Lieb should be those generated by a single element \( R \) which is in the \( k \)-graded subspace \( P_k \) for some \( k > 0 \). We call such an element a \( k \)-box. In \([22]\) the planar algebra generated by a single 1-box was completely analyzed so the next case is that of a planar algebra generated by a single 2-box. This means that the dimension of \( P_2 \) is at least 3 so the first case to try to understand is when \( \dim P_3 = 3 \). This dimension condition by itself imposes many relations on \( P \) but probably not enough to make a complete enumeration a realistic goal. However, if one imposes \( \dim P_3 \leq 15 \), then apart from a degenerate case, this forces enough relations to reduce the number of variables governing the planar algebra structure to be finite in number \([22] \), see also \([9]\). It seems therefore reasonable to try to find all subfactor planar algebras \( P \) generated by a single element in \( P_2 \) subject to the two restrictions \( \dim P_2 = 3 \) and \( \dim P_3 = d \) with \( d \leq 15 \).

In \([9]\) we solved this problem when \( d \leq 12 \). In fact, using planar algebra techniques we prove a much more general structure theorem for subfactors.

**Theorem 5.1.** Let \( N \subset M \) be an inclusion of \( \text{II}_1 \) factors with \( 3 < [M : N] < \infty \). Suppose that \( \dim N' \cap M_1 = 3 \) and that \( N' \cap M_2 \) is abelian modulo the basic construction ideal \( (N' \cap M_1)e_2(N' \cap M_1) \). Then there is an intermediate subfactor \( P \) of \( N \subset M, P \neq N, M \). In particular \( Jx^*J = x \) for all \( x \in N' \cap M_1 \).

The proof uses in a crucial way the abstract characterization of the intermediate subfactor projection in \([3]\) and planar algebra techniques developed in \([22]\) and \([9]\). It implies the following classification result.

**Theorem 5.2.** If \( P \) is a subfactor planar algebra generated by a 3-dimensional \( P_2 \), subject to the condition \( \dim P_3 \leq 12 \), then it must be one of the following:

a) If \( \dim P_3 = 9 \), then it is the planar algebra associated to the index 3 subfactor \( M^{23} \subset M \).

b) If \( \dim P_3 = 10 \), then it is the \( D_\infty \) planar algebra (a special FC planar algebra).

c) If \( \dim P_3 = 11 \) or 12, then it is one of the FC planar algebras.
The dimension conditions imply that a subfactor whose standard invariant is a planar algebra of the form b) or c) satisfies the hypothesis of Theorem 5.1 and hence must have an intermediate subfactor. Since the Fuss-Catalan planar algebra is the minimal symmetry associated to an intermediate subfactor it then follows easily that the planar algebra has to be one of these.

It is quite natural to expect that increasing the dimension of $P_3$ should result in a larger number of examples of planar algebras since there are more a priori undetermined structure constants in the action of planar tangles on $\mathcal{P}$. Thus the result in [10] that there is a single subfactor planar algebra satisfying the above restrictions with $d = 13$ is a complete surprise. The planar algebra which arises is that of a subfactor obtained as follows. Take an outer action of the dihedral group $D_5$ on a type II$_1$ factor $R$ and let $M$ be the crossed product $R \rtimes D_5$ and $N$ be the subfactor $R \rtimes \mathbb{Z}_2$. This particular subfactor has played a significant role in the development of subfactors and relations with knot theory and statistical mechanics. In [21] it was noted that there is a solvable statistical mechanical model associated with it and that it corresponds to an evaluation of the Kauffman polynomial invariant of a link. We prove in [10] the following

**Theorem 5.3.** Let $\mathcal{P} = (P_k)_{k \geq 0}$ be a subfactor planar algebra generated by a non-trivial element in $P_2$ (i.e. an element not contained in the Temperley-Lieb subalgebra of $P_2$) subject to the conditions $\dim P_2 = 3$ and $\dim P_3 = 13$. Then $\mathcal{P}$ is the standard invariant of the crossed product subfactor $R \rtimes \mathbb{Z}_2 \subset R \rtimes D_5$. Thus there is precisely one subfactor planar algebra $\mathcal{P}$ subject to the above conditions.

Note that this subfactor can be viewed as a Birman-Murakami-Wenzl subfactor (associated to the quantum group of $Sp(4, \mathbb{R})$ at a 5-th root of unity, see [36]). We note here that the standard invariants $\mathcal{P} = (P_k)_{k \geq 0}$ of all BMW subfactors are generated by a single non-trivial operator in $P_2$ and that they satisfy the condition $\dim P_3 \leq 15$.

The proof of this theorem uses in a crucial way theorem 5.1 and the tight restrictions imposed by compatibility of the rotation of period 3 on $P_3$ and the algebra structure.

The next phase of this enumeration project will be to tackle the case $d = 14$. Here we know that the quantum $Sp(4, \mathbb{R})$ specialization of the BMW algebra will give examples with a free parameter. We do expect however, that the general ideas of [9] and [10] will enable us to enumerate all such subfactor planar algebras.

**References**

[1] M. Asaeda & U. Haagerup, *Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2*, Comm. Math. Phys. 202 (1999), 1–63.

[2] T. Banica, *Representations of compact quantum groups and subfactors*, J. Reine Angew. Math. 509 (1999), 167–198.

[3] D. Bisch, *A note on intermediate subfactors*, Pacific Journal of Math. 163 (1994), 201-216.
[4] D. Bisch, Bimodules, higher relative commutants and the fusion algebra associated to a subfactor, The Fields Institute for Research in Math. Sciences Commun. Series, vol. 13, AMS, Providence, Rhode Island, 1997, 13-63.

[5] D. Bisch, An example of an irreducible subfactor of the hyperfinite II$_1$ factor with rational, noninteger index, J. Reine Angew. Math. 455 (1994), 21-34.

[6] D. Bisch & U. Haagerup, Composition of subfactors: new examples of infinite depth subfactors, Ann. scient. Éc. Norm. Sup. 29 (1996), 329-383.

[7] D. Bisch & V.F.R. Jones, Algebras associated to intermediate subfactors, Invent. Math. 128 (1997), 89-157.

[8] D. Bisch & V.F.R. Jones, A note on free composition of subfactors, “Geometry and Physics”, vol. 184, Marcel Dekker, Lecture Notes in Pure and Applied Mathematics, 1997, 339-361.

[9] D. Bisch & V.F.R. Jones, Singly generated planar algebras of small dimension, Duke Math. Journal 101 (2000), 41-75.

[10] D. Bisch & V.F.R. Jones, Singly generated planar algebras of small dimension, Part II, Advances in Math. (to appear).

[11] D. Bisch & S. Popa, Examples of subfactors with property T standard invariant, Geom. Funct. Anal. 9 (1999), 215-225.

[12] D. Bisch & S. Popa, A continuous family of non-isomorphic irreducible hyperfinite subfactors with the same standard invariant, in preparation.

[13] A. Connes, Noncommutative geometry, Academic Press, 1994.

[14] D. Evans & Y. Kawahigashi, Quantum symmetries on operator algebras, Oxford University Press, 1998.

[15] P. Di Francesco, New integrable lattice models from Fuss-Catalan algebras, Nuclear Phys. B 532 (1998), 609-634.

[16] F. Goodman & P. de la Harpe & V.F.R. Jones, Coxeter graphs and towers of algebras, Springer Verlag, MSRI publications, 1989.

[17] U. Haagerup, Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$, Subfactors (Kyuzeso, 1993), World Sci. Publishing, River Edge, NJ, 1994, 1–38.

[18] M. Izumi, Applications of fusion rules to classification of subfactors, Publ. RIMS, Kyoto Univ. 27 (1991), 953-994.

[19] V.F.R. Jones, Index for subfactors, Invent. Math. 72 (1983), 1-25.

[20] V.F.R. Jones, Hecke algebra representations of braid groups and link polynomials, Ann. of Math. 126, 335-388.

[21] V.F.R. Jones, On a certain value of the Kauffman polynomial, Comm. Math. Phys. 125 (1989), 459–467.

[22] V.F.R. Jones, Planar algebras I, preprint.

[23] V.F.R. Jones, The planar algebra of a bipartite graph, Knots in Hellas ’98 (Delphi), World Sci. Publishing, 2000, 94-117.

[24] V.F.R. Jones, The annular structure of subfactors, Enseign. Math. (to appear).
[25] L. Kauffman, State models and the Jones polynomial, *Topology* 26 (1987), 395-407.

[26] Z. Landau, Fuss-Catalan algebras and chains of intermediate subfactors, *Pacific J. Math.* 197 (2001), 325-36.

[27] Z. Landau, Exchange relation planar algebras, *preprint* (2000).

[28] J.P. May, Definitions: operads, algebras and modules, *Contemporary Mathematics* 202 (1997), 1-7.

[29] M. J. Martins & B. Nienhuis, Applications of Temperley-Lieb algebras to Lorentz lattice gases, *J. Phys. A* 31 (1998), L723–L729.

[30] A. Ocneanu, Quantized group string algebras and Galois theory for operator algebras, in *Operator Algebras and Applications 2*, *London Math. Soc. Lect. Notes Series* 136 (1988), 119-172.

[31] M. Pimsner & S. Popa, Entropy and index for subfactors, *Ann. scient. Ec. Norm. Sup.* 19 (1986), 57-106.

[32] S. Popa, Classification of subfactors: reduction to commuting squares, *Invent. Math.* 101 (1990), 19-43.

[33] S. Popa, Classification of amenable subfactors of type II, *Acta Math.* 172 (1994), 352-445.

[34] S. Popa, An axiomatization of the lattice of higher relative commutants, *Invent. Math.* 120 (1995), 427-445.

[35] A. Wassermann, Operator algebras and conformal field theory III, *Invent. Math.* 92 (1998), 467-538.

[36] H. Wenzl, Quantum groups and subfactors of type B, C and D, *Comm. Math. Phys.* 133, 383-432.

[37] H. Wenzl, $C^*$ tensor categories from quantum groups, *J. Amer. Math. Soc.* 11 (1998), 261-282.

[38] F. Xu, Standard $\lambda$-lattices from quantum groups, *Invent. Math.* 134 (1998), 455–487.