A GEOMETRIC HAMILTON-JACOBI THEORY FOR CLASSICAL FIELD THEORIES

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Abstract. In this paper we extend the geometric formalism of the Hamilton-Jacobi theory for hamiltonian mechanics to the case of classical field theories in the framework of multisymplectic geometry and Ehresmann connections.

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1. Introduction

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the principal function) such that

$$\frac{\partial S}{\partial t} + H(q^A, \frac{\partial S}{\partial q^A}) = 0.$$  

(1.1)
If we put $S(t, q^A) = W(q^A) - tE$, where $E$ is a constant, then $W$ satisfies
\[ H(q^A, \partial W/\partial q^A) = E; \] (1.2)

$W$ is called the \textbf{characteristic function}.

Equations (1.1) and (1.2) are indistinctly referred as the \textbf{Hamilton-Jacobi equation}.

There are some recent attempts to extend this theory for classical field theories in the framework of the so-called multisymplectic formalism \cite{15,16}. For a classical field theory the hamiltonian is a function $H = H(x^\mu, y^i, p^\mu_i)$, where $(x^\mu)$ are coordinates in the space-time, $(y^i)$ represent the field coordinates, and $(p^\mu_i)$ are the conjugate momenta.

In this context, the Hamilton-Jacobi equation is \cite{17}
\[ \frac{\partial S^\mu}{\partial x^\nu} + H(x^\nu, y^i, \partial S^\mu/\partial y^i) = 0 \] (1.3)

where $S^\mu = S^\mu(x^\nu, y^j)$.

In this paper we introduce a geometric version for the Hamilton-Jacobi theory based in two facts: (1) the recent geometric description for Hamiltonian mechanics developed in \cite{6} (see \cite{8} for the case of nonholonomic mechanics); (2) the multisymplectic formalism for classical field theories \cite{3,4,5,7} in terms of Ehresmann connections \cite{9,10,11,12}.

We shall also adopt the convention that a repeated index implies summation over the range of the index.

\section{A geometric Hamilton-Jacobi theory for Hamiltonian mechanics}

First of all, we give a geometric version of the standard Hamilton-Jacobi theory which will be useful in the sequel.

Let $Q$ be the configuration manifold, and $T^*Q$ its cotangent bundle equipped with the canonical symplectic form
\[ \omega_Q = dq^A \wedge dp_A \]
where $(q^A)$ are coordinates in $Q$ and $(q^A, p_A)$ are the induced ones in $T^*Q$.

Let $H : T^*Q \to \mathbb{R}$ a hamiltonian function and $X_H$ the corresponding hamiltonian vector field:
\[ i_{X_H} \omega_Q = dH \]

The integral curves of $X_H, (q^A(t), p_A(t))$, satisfy the Hamilton equations:
\[ \frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}, \quad \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A} \]
Theorem 2.1 (Hamilton-Jacobi Theorem). Let \( \lambda \) be a closed 1-form on \( Q \) (that is, \( d\lambda = 0 \) and, locally \( \lambda = dW \)). Then, the following conditions are equivalent:

(i) If \( \sigma : I \to Q \) satisfies the equation
\[
\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}
\]
then \( \lambda \circ \sigma \) is a solution of the Hamilton equations;
(ii) \( d(H \circ \lambda) = 0 \).

To go further in this analysis, define a vector field on \( Q \):
\[
X^\lambda_H = T\pi_Q \circ X_H \circ \lambda
\]
as we can see in the following diagram:

\[
\begin{array}{ccc}
T^*Q & \xrightarrow{X_H} & T(T^*Q) \\
\downarrow \pi_Q & & \downarrow T\pi_Q \\
Q & \xrightarrow{X^\lambda_H} & TQ
\end{array}
\]

Notice that the following conditions are equivalent:

(i) If \( \sigma : I \to Q \) satisfies the equation
\[
\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A}
\]
then \( \lambda \circ \sigma \) is a solution of the Hamilton equations;
(i)' If \( \sigma : I \to Q \) is an integral curve of \( X^\lambda_H \), then \( \lambda \circ \sigma \) is an integral curve of \( X_H \);
(i)" \( X_H \) and \( X^\lambda_H \) are \( \lambda \)-related, i.e.
\[
T\lambda(X^\lambda_H) = X_H \circ \lambda
\]
so that the above theorem can be stated as follows:

Theorem 2.2 (Hamilton-Jacobi Theorem). Let \( \lambda \) be a closed 1-form on \( Q \). Then, the following conditions are equivalent:

(i) \( X^\lambda_H \) and \( X_H \) are \( \lambda \)-related;
(ii) \( d(H \circ \lambda) = 0 \).

3. The multisymplectic formalism

3.1. Multisymplectic bundles. The configuration manifold in Mechanics is substituted by a fibred manifold
\[
\pi : E \to M
\]
such that

(i) \( \dim M = n, \dim E = n + m \)

(ii) \( M \) is endowed with a volume form \( \eta \).

We can choose fibred coordinates \( (x^\mu, y^i) \) such that

\[
\eta = dx^1 \wedge \cdots \wedge dx^n.
\]

We will use the following useful notations:

\[
d^n x = dx^1 \wedge \cdots \wedge dx^n
\]

\[
d^{n-1} x^\mu = i_{\partial/\partial x^\mu} d^n x.
\]

Denote by \( V\pi = \ker T\pi \) the vertical bundle of \( \pi \), that is, their elements are the tangent vectors to \( E \) which are \( \pi \)-vertical.

Denote by \( \Pi : \Lambda^n E \to E \)

the vector bundle of \( n \)-forms on \( E \).

The total space \( \Lambda^n E \) is equipped with a canonical \( n \)-form \( \Theta \):

\[
\Theta(\alpha)(X_1, \ldots, X_n) = \alpha(e)(T\Pi(X_1), \ldots, T\Pi(X_n))
\]

where \( X_1, \ldots, X_n \in T_e(\Lambda^n E) \) and \( \alpha \) is an \( n \)-form at \( e \in E \).

The \((n + 1)\)-form

\[
\Omega = -d\Theta,
\]

is called the canonical multisymplectic form on \( \Lambda^n E \).

Denote by \( \Lambda^r_1 E \) the bundle of \( r \)-semibasic \( n \)-forms on \( E \), say

\[
\Lambda^r_1 E = \{ \alpha \in \Lambda^n E \mid i_{v_1 \wedge \cdots \wedge v_r} \alpha = 0, \text{ whenever } v_1, \ldots, v_r \text{ are } \pi \text{-vertical} \}
\]

Since \( \Lambda^r_1 E \) is a submanifold of \( \Lambda^n E \) it is equipped with a multisymplectic form \( \Omega_r \), which is just the restriction of \( \Omega \).

Two bundles of semibasic forms play an special role: \( \Lambda^1_1 E \) and \( \Lambda^2_1 E \). The elements of these spaces have the following local expressions:

\[
\Lambda^1_1 E : p_0 d^n x
\]

\[
\Lambda^2_1 E : p_0 d^n x + p_0^\mu dy^i \wedge d^{n-1} x^\mu.
\]

which permits to introduce local coordinates \( (x^\mu, y^i, p_0) \) and \( (x^\mu, y^i, p_0, p_0^\mu) \) in \( \Lambda^1_1 E \) and \( \Lambda^2_1 E \), respectively.

Since \( \Lambda^1_1 E \) is a vector subbundle of \( \Lambda^2_1 E \) over \( E \), we can obtain the quotient vector space denoted by \( J^1 \pi^* \) which completes the following exact sequence of vector bundles:

\[
0 \to \Lambda^1_1 E \to \Lambda^2_1 E \to J^1 \pi^* \to 0.
\]

We denote by \( \pi^*_{1,0} : J^1 \pi^* \to E \) and \( \pi_1 : J^1 \pi^* \to M \) the induced fibrations.
3.2. Ehresmann Connections in the fibration $\pi_1 : J^1 \pi^* \rightarrow M$.

A **connection** (in the sense of Ehresmann) in $\pi_1$ is a horizontal subbundle $H$ which is complementary to $V \pi_1$; namely,

$$T(J^1 \pi^*) = H \oplus V \pi_1$$

where $V \pi_1 = \ker T \pi_1$ is the vertical bundle of $\pi_1$. Thus, we have:

(i) there exists a (unique) horizontal lift of every tangent vector to $M$;

(ii) in fibred coordinates $(x^\mu, y^i, p^\mu_i)$ on $J^1 \pi^*$, then

$$V \pi_1 = \text{span} \left\{ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p^\mu_i} \right\}, \quad H = \text{span} \left\{ H_\mu \right\},$$

where $H_\mu$ is the horizontal lift of $\frac{\partial}{\partial x^\mu}$.

(iii) there is a horizontal projector $h : T(J^1 \pi^*) \rightarrow H$.

3.3. Hamiltonian sections. Consider a hamiltonian section

$$h : J^1 \pi^* \rightarrow \Lambda^n_2 E$$

of the canonical projection $\mu : \Lambda^n_2 E \rightarrow J^1 \pi^*$ which in local coordinates read as

$$h(x^\mu, y^i, p^\mu_i) = (x^\mu, y^i, -H(x, y, p), p^\mu_i).$$

Denote by $\Omega_h = h^* \Omega_2$, where $\Omega_2$ is the multisymplectic form on $\Lambda^n_2 E$.

The field equations can be written as follows:

$$i_h \Omega_h = (n - 1) \Omega_h,$$  \hspace{1cm} (3.1)

where $h$ denotes the horizontal projection of an Ehresmann connection in the fibred manifold $\pi_1 : J^1 \pi^* \rightarrow M$.

The local expressions of $\Omega_2$ and $\Omega_h$ are:

$$\Omega_2 = -d(p_0 d^nx + p^\mu_i dy^i \wedge d^{n-1}x^\mu),$$

$$\Omega_h = -d(-H d^nx + p^\mu_i dy^i \wedge d^{n-1}x^\mu).$$

3.4. The field equations. Next, we go back to the Equation (3.1).

The horizontal subspaces are locally spanned by the local vector fields

$$H_\mu = h\left( \frac{\partial}{\partial x^\mu} \right) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + (\Gamma^\nu_\mu)^j \frac{\partial}{\partial p^\nu_j},$$

where $\Gamma^i_\mu$ and $(\Gamma^\nu_\mu)^j$ are the Christoffel components of the connection.

Assume that $\tau$ is an integral section of $h$; this means that $\tau : M \rightarrow J^1 \pi^*$ is a local section of the canonical projection $\pi_1 : J^1 \pi^* \rightarrow M$ such that $T \tau(x)(T_x M) = H_{\tau(x)}$, for all $x \in M$.

If $\tau(x^\mu) = (x^\mu, \tau^i(x), \tau^\mu_i(x))$ then the above conditions becomes

$$\frac{\partial \tau^i}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_i}, \quad \frac{\partial \tau^\mu_i}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}.$$
which are the Hamilton equations.

4. The Hamilton-Jacobi Theory

Let $\lambda$ be a 2-semibasic $n$-form on $E$; in local coordinates we have

$$\lambda = \lambda_0(x, y) \, d^n x + \lambda^\mu_i(x, y) \, dy^i \wedge d^{n-1} x^\mu.$$ 

Alternatively, we can see it as a section $\lambda : E \rightarrow \Lambda^2_n E$, and then we have

$$\lambda(x^\mu, y^i) = (x^\mu, y^i, \lambda_0(x, y), \lambda^\mu_i(x, y)).$$

A direct computation shows that

$$d\lambda = \left( \frac{\partial \lambda_0}{\partial y^i} - \frac{\partial \lambda^\mu_i}{\partial x^\mu} \right) \, dy^i \wedge d^n x + \frac{\partial \lambda^\mu_i}{\partial y^j} \, dy^j \wedge dy^i \wedge d^{n-1} x^\mu.$$ 

Therefore, $d\lambda = 0$ if and only if

$$\frac{\partial \lambda_0}{\partial y^i} = \frac{\partial \lambda^\mu_i}{\partial x^\mu} \quad (4.1)$$ 

$$\frac{\partial \lambda^\mu_i}{\partial y^j} = \frac{\partial \lambda^\mu_j}{\partial y^i}. \quad (4.2)$$

Using $\lambda$ and $h$ we construct an induced connection in the fibred manifold $\pi : E \rightarrow M$ by defining its horizontal projector as follows:

$$\tilde{h}_e : T_e E \rightarrow T_e E$$

$$\tilde{h}_e(X) = T_{\pi_1,0} \circ h_{(\mu \circ \lambda)(e)} \circ \epsilon(X)$$

where $\epsilon(X) \in T_{(\mu \circ \lambda)(e)}(J^1 \pi^*)$ is an arbitrary tangent vector which projects onto $X$.

From the above definition we immediately proves that

(i) $\tilde{h}$ is a well-defined connection in the fibration $\pi : E \rightarrow M$.

(ii) The corresponding horizontal subspaces are locally spanned by

$$\tilde{H}_\mu = \tilde{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu((\mu \circ \lambda)(x, y)) \frac{\partial}{\partial y^i}.$$ 

The following theorem is the main result of this paper.

**Theorem 4.1.** Assume that $\lambda$ is a closed 2-semibasic form on $E$ and that $\tilde{h}$ is a flat connection on $\pi : E \rightarrow M$. Then the following conditions are equivalent:

(i) If $\sigma$ is an integral section of $\tilde{h}$ then $\mu \circ \lambda \circ \sigma$ is a solution of the Hamilton equations.

(ii) The n-form $h \circ \mu \circ \lambda$ is closed.
Before to begin with the proof, let us consider some preliminary results.

We have

\[(h \circ \mu \circ \lambda)(x, y) = (x^\mu, y^i, -H(x^\mu, y^i, \lambda^\mu_i(x, y)), \lambda^i_j(x, y)),\]

that is

\[h \circ \mu \circ \lambda = -H(x^\mu, y^i, \lambda^\mu_i(x, y)) d^n x + \lambda^\mu_i dy^i \wedge d^{n-1} x^\mu.\]

Notice that \(h \circ \mu \circ \lambda\) is again a 2-semibasic \(n\)-form on \(E\).

A direct computation shows that

\[d(h \circ \mu \circ \lambda) = -\left( \frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^j_i} \frac{\partial \lambda^\mu_i}{\partial y^j} + \frac{\partial \lambda^\mu_i}{\partial x^\mu} \right) dy^i \wedge d^n x\]

\[+ \frac{\partial \lambda^\mu_i}{\partial y^i} dy^i \wedge dy^i \wedge d^{n-1} x^\mu.\]

Therefore, we have the following result.

**Lemma 4.2.** Assume \(d\lambda = 0\); then

\[d(h \circ \mu \circ \lambda) = 0\]

if and only if

\[\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^j_i} \frac{\partial \lambda^\mu_i}{\partial y^j} + \frac{\partial \lambda^\mu_i}{\partial x^\mu} = 0.\]

**Proof of the Theorem**

\((i) \Rightarrow (ii)\)

It should be remarked the meaning of \((i)\).

Assume that

\[\sigma(x^\mu) = (x^\mu, \sigma^i(x))\]

is an integral section of \(\tilde{h}\); then

\[\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial p^i_i}.\]

\((i)\) states that in the above conditions,

\[(\mu \circ \lambda \circ \sigma)(x^\mu) = (x^\mu, \sigma^i(x), \tilde{\sigma}^\nu_j = \lambda^\nu_j(\sigma(x)))\]

is a solution of the Hamilton equations, that is,

\[\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial \lambda^\mu_i}{\partial x^\mu} \frac{\partial \sigma^i}{\partial y^j} \frac{\partial \lambda^\mu_i}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}.\]
Assume (i). Then
\[
\frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^\nu} \frac{\partial \lambda_i^\nu}{\partial y^j} + \frac{\partial \lambda_i^\mu}{\partial x^\mu} = \frac{\partial H}{\partial y^i} + \frac{\partial H}{\partial p^\nu} \frac{\partial \lambda_i^\nu}{\partial y^j} + \frac{\partial \lambda_i^\mu}{\partial x^\mu},
\]
(since \(d\lambda = 0\))
\[
\frac{\partial H}{\partial y^i} + \frac{\partial \sigma^j}{\partial x^\nu} \frac{\partial \lambda_i^\nu}{\partial y^j} + \frac{\partial \lambda_i^\mu}{\partial x^\mu},
\]
(since the first Hamilton equation)
\[
= 0 \quad (\text{since } (i))
\]
which implies (ii) by Lemma 4.2

(ii) \(\Rightarrow\) (i)

Assume that \(d(h \circ \mu \circ \lambda) = 0\).

Since \(\tilde{h}\) is a flat connection, we may consider an integral section \(\sigma\) of \(\tilde{h}\). Suppose that
\[
\sigma(x^\mu) = (x^\mu, \sigma^i(x)).
\]
Then, we have that
\[
\frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial p^\mu_i}.
\]
Thus,
\[
\frac{\partial \sigma^\mu_j}{\partial x^\mu} = \frac{\partial \lambda_i^\mu}{\partial x^\mu} + \frac{\partial \lambda_i^\nu}{\partial y^j} \frac{\partial \sigma^i}{\partial x^\mu},
\]
(since \(d\lambda = 0\))
\[
= \frac{\partial \lambda_i^\mu}{\partial x^\mu} + \frac{\partial \lambda_i^\nu}{\partial y^j} \frac{\partial \sigma^i}{\partial x^\mu},
\]
(since the first Hamilton equation)
\[
= -\frac{\partial H}{\partial y^j}, \quad (\text{since } (ii)). \quad \square
\]

Assume that \(\lambda = dS\), where \(S\) is a 1-semibasic \((n - 1)\)-form, say
\[
S = S^\mu d^{n-1}x^\mu
\]
Therefore, we have
\[
\lambda_0 = \frac{\partial S^\mu}{\partial x^\mu}, \quad \lambda_i^\mu = \frac{\partial S^\mu}{\partial y^i}
\]
and the Hamilton-Jacobi equation has the form
\[
\frac{\partial}{\partial y^i} \left( \frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^j}) \right) = 0.
\]
The above equations mean that
\[
\frac{\partial S^\mu}{\partial x^\mu} + H(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^j}) = f(x^\mu)
\]
so that if we put $\tilde{H} = H - f$ we deduce the standard form of the Hamilton-Jacobi equation (since $H$ and $\tilde{H}$ give the same Hamilton equations):

$$\frac{\partial S^\mu}{\partial x^\mu} + \tilde{H}(x^\nu, y^i, \frac{\partial S^\mu}{\partial y^i}) = 0 .$$

An alternative geometric approach of the Hamilton-Jacobi theory for Classical Field Theories in a multisymplectic setting was discussed in [15, 16].

5. Time-dependent mechanics

A Hamiltonian time-dependent mechanical system corresponds to a classical field theory when the base is $M = \mathbb{R}$.

We have the following identification $\Lambda^1_1E = T^*E$ and we have local coordinates $(t, y^i, p_0, p_i)$ and $(t, y^i, p_i)$ on $T^*E$ and $J^1\pi^*$, respectively. The Hamiltonian section is given by

$$h(t, y^i, p_i) = (t, y^i, -H(t, y, p), p_i) ,$$

and therefore we obtain

$$\Omega_h = dH \wedge dt - dp_i \wedge dy^i .$$

If we denote by $\eta = dt$ the different pull-backs of $dt$ to the fibred manifolds over $M$, we have the following result.

The pair $(\Omega_h, dt)$ is a cosymplectic structure on $E$, that is, $\Omega_h$ and $dt$ are closed forms and $dt \wedge \Omega^n_h = dt \wedge \Omega_h \wedge \cdots \wedge \Omega_h$ is a volume form, where $dimE = 2n + 1$. The Reeb vector field $R_h$ of the structure $(\Omega_h, dt)$ satisfies

$$i_{R_h} \Omega_h = 0 , \ i_{R_h} dt = 1 .$$

The integral curves of $R_h$ are just the solutions of the Hamilton equations for $H$.

The relation with the multisymplectic approach is the following:

$$h = R_h \otimes dt ,$$

or, equivalently,

$$h(\frac{\partial}{\partial t}) = R_h .$$

A closed 1-form $\lambda$ on $E$ is locally represented by

$$\lambda = \lambda_0 dt + \lambda_i dy^i .$$

Using $\lambda$ we obtain a vector field on $E$:

$$(R_h)_\lambda = T\pi_{1,0} \circ R_h \circ \mu \circ \lambda$$

such that the induced connection is

$$\tilde{h} = (R_h)_\lambda \otimes dt$$
Therefore, we have the following result.

**Theorem 5.1.** The following conditions are equivalent:

(i) $(R_h)_\lambda$ and $R_h$ are $(\mu \circ \lambda)$-related.

(ii) The 1-form $h \circ \mu \circ \lambda$ is closed.

**Remark 5.2.** An equivalent result to Theorem 5 was proved in [14] (see Corollary 5 in [14]).

Now, if

\[ \lambda = dS = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial y^i} dy^i, \]

then we obtain the Hamilton-Jacobi equation

\[ \frac{\partial}{\partial y^i} \left( \frac{\partial S}{\partial t} + H(t, y^i, \frac{\partial S}{\partial y^i}) \right) = 0. \]

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