ON UNIQUENESS FOR TIME HARMONIC ANISOTROPIC MAXWELL’S EQUATIONS WITH PIECEWISE REGULAR COEFFICIENTS

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Abstract. We are interested in the uniqueness of solutions to Maxwell’s equations when the magnetic permeability \( \mu \) and the permittivity \( \varepsilon \) are symmetric positive definite matrix-valued functions in \( \mathbb{R}^3 \). We show that a unique continuation result for globally \( W^{1,\infty} \) coefficients in a smooth, bounded domain, allows one to prove that the solution is unique in the case of coefficients which are piecewise \( W^{1,\infty} \) with respect to a suitable countable collection of sub-domains with \( C^0 \) boundaries. Such suitable collections include any bounded finite collection. The proof relies on a general argument, not specific to Maxwell’s equations. This result is then extended to the case when within these sub-domains the permeability and permittivity are only \( L^\infty \) in sets of small measure.

1. Introduction

Suppose we are given a time-harmonic incident electric field \( E^i \) and magnetic field \( H^i \), special solutions of the time-harmonic homogeneous linear Maxwell equations of the form \( \mathcal{E}^i = \Re \left( E^i e^{-i \omega t} \right) \) and magnetic field \( \mathcal{H}^i = \Re \left( H^i e^{-i \omega t} \right) \), where \( E^i \in H^1_{\text{loc}} (\mathbb{R}^3)^3 \) and \( H^i \in H^1_{\text{loc}} (\mathbb{R}^3)^3 \) are complex-valued solutions of the homogeneous time-harmonic Maxwell equations

\[
\begin{align*}
\nabla \times E^i - i \omega \mu_0 H^i &= 0 \text{ in } \mathbb{R}^3, \\
\nabla \times H^i + i \omega \varepsilon_0 E^i &= 0 \text{ in } \mathbb{R}^3,
\end{align*}
\]

where \( \mu_0 \) and \( \varepsilon_0 \) are positive constants, representing respectively the magnetic permeability and the electric permittivity of vacuum, and \( \omega \in \mathbb{R} \setminus \{0\} \). The full time-harmonic electromagnetic field \((E, H) \in H_{\text{loc}} (\text{curl}; \mathbb{R}^3)\), where for any domain \( W \) we define

\[
H_{\text{loc}} (\text{curl}; W) := \left\{ u \in L^2_{\text{loc}} (W)^3 \text{ such that } \nabla \times u \in L^2_{\text{loc}} (W)^3 \right\},
\]

satisfies Maxwell’s equations

\[
\begin{align*}
\nabla \times E - i \omega \mu_0 \mu(x) H &= 0 \text{ in } \mathbb{R}^3, \\
\nabla \times H + i \omega \varepsilon_0 \varepsilon(x) E &= 0 \text{ in } \mathbb{R}^3,
\end{align*}
\]

where \( \varepsilon \) and \( \mu \) are real matrix-valued functions in \( L^\infty (\mathbb{R}^3)^{3 \times 3} \). Decomposing the full electromagnetic field into its incident part and its scattered part,

\[
\begin{align*}
E^s := E - E^i, \quad \text{and} \quad H^s := H - H^i,
\end{align*}
\]
we assume that the scattered field satisfies the Silver-Müller radiation condition, uniformly in all directions, that is, if \( x := r \theta \) then
\[
\lim_{r \to \infty} \sup_{\theta \in S^2} |H^r (r \theta) \wedge r \theta - r E^r (r \theta)| = 0,
\]
where \( S^2 := \{ x \in \mathbb{R}^3 \text{ such that } |x| = 1 \} \) denotes the unit sphere.

This paper is about the existence of a unique solution to (1.1) satisfying (1.2)
\[
\text{Proposition 1.}
\]
Given a bounded set \( \Omega \subset \mathbb{R}^3 \) and \( \mu \) satisfying (1.4)-(1.5) and
\[
\text{Assumption 1 holds when } I = \text{finite}.
\]

\[
\begin{align*}
\text{Assumption 1.} & \quad \text{For any } \Omega \subset \mathbb{R}^3 \text{ and } \mu \text{ satisfying (1.4)-(1.5), there exists } \delta > 0 \text{ such that } \\
\text{Proposition 1.} & \quad \text{there exists } x_j \in \partial \Omega (\Omega_j) \text{ and } \delta_j > 0 \text{ such that } B_j = B(x_j, \delta_j) \neq \emptyset \text{ for only finitely many } j \in J.
\end{align*}
\]

\[
\text{Proof.}
\]
Given \( J, x_j \in \partial \Omega (\Omega_j) \) and \( \delta_j \) as in the statement of the proposition let \( B_j = B(x_j, \delta_j) \) and let \( J' \) be the finite subset of \( J \) such that \( B_j \cap \bigcup \Omega_j = B_j \cap \bigcup \Omega_j'. \) We first show that
\[
\partial \Omega (\Omega_j) \cap B_j = \bigcup_{j \in J'} \partial \Omega (\Omega_j) \cap B_j \cap \partial \Omega_j.
\]
Indeed, let \( x \in \partial \Omega (\Omega_j) \cap B_j. \) Then \( x \notin \bigcup \Omega_j. \) We claim that there exists a sequence \( x_k \in \bigcup \Omega_j \) such that \( x_k \) tends to \( x. \) If not, for some \( \eta > 0 \) sufficiently
small, we would have \( B(x, \eta) \subset B_J \), and \( B(x, \eta) \cap \bigcup_{j \in J} \Omega_j = B(x, \eta) \cap \bigcup_{j \in J'} \Omega_j = \emptyset \).

On the other hand, there exists a sequence \( y_k \in \mathcal{U}(\Omega_J) \) such that \( y_k \) tends to \( x \). But \( B(x, \eta) \) is connected and contained in \( \overline{\Omega}_c \), thus \( B(x, \eta) \subset \mathcal{U}(\Omega_J) \). This contradiction proves the claim.

Next, we note that \( \partial \mathcal{U}(\Omega_J) \) is closed, thus complete in the subspace topology induced by \( \mathbb{R}^3 \). Its intersection with the open ball \( B_J \) is an open subspace of \( \partial \mathcal{U}(\Omega_J) \) by definition of the subspace topology. It is therefore a Baire space (see e.g. [11]). If a Baire Space is a countable union of closed sets, then one of the sets has an interior point. Using the identity (1.10), we obtain that there exists \( j_0 \) such that \( \partial \mathcal{U}(\Omega_J) \cap B_J \cap \partial \Omega_{j_0} \) admits an interior point relative to \( \partial \mathcal{U}(\Omega_J) \cap B_J \), that is, there exist \( j_0 \in J \), \( x_0 \in \partial \mathcal{U}(\Omega_J) \cap B_J \) and \( \delta > 0 \) such that \( B(x_0, \delta) \cap \partial \mathcal{U}(\Omega_J) \cap B_J \subset \partial \Omega_{j_0} \).

Since \( B_J \) is open, \( B(x_0, \delta) \cap B_J = B(x_0, \delta) \) when \( \delta \) is sufficiently small, and we have established that Assumption \( \mathbb{I} \) holds.

An example of a collection of sub-domains excluded by Assumption \( \mathbb{I} \) is a collection of concentric shells concentrating on an exterior boundary, such as

\[
(1.11) \quad \Omega_i = B \left( 0, \frac{i}{i+1} \right) \setminus \bar{B} \left( 0, \frac{i-1}{i} \right), \quad i = 1, 2, 3, \ldots
\]

In such a case, \( \partial \mathcal{U}(\Omega) \) is the unit sphere, which is not the boundary of any of the subsets. On the other hand, Assumption \( \mathbb{I} \) allows the sub-domains \( \Omega_i \) to concentrate at a point or near an interior boundary. In Figure 1, we represent on the left a non-Lipschitz non-simply connected domain \( \Omega \) which satisfies Assumption \( \mathbb{I} \). In the centre, the domain given by (1.11) excluded by Assumption \( \mathbb{I} \) is shown. On the right, we sketch a domain inspired by the one described by (1.11) which satisfies Assumption \( \mathbb{I} \) near the accumulating boundary, interior points can be found on the wedge-shaped slit in the domain.

### 2. Main result

Our main result is the following theorem.

**Theorem 2.** Assume that (1.4)(1.9) and Assumption \( \mathbb{I} \) hold. If for a given \( \omega \neq 0 \), \( E \in H_{\text{loc}}(\text{curl}; \mathbb{R}^3) \) and \( H \in H_{\text{loc}}(\text{curl}; \mathbb{R}^3) \) are solutions of (1.1)(1.3) corresponding to \( E^i = 0 \) and \( H^i = 0 \), then \( E = H = 0 \).
There is a very long history concerning this problem, under various assumptions on the coefficients, see e.g. [12, 13, 14, 15, 16] and the references therein. The improvement provided by the result in this work is that we assume that \( \varepsilon, \mu \) are matrix-valued functions and that the sub-domains \( \Omega_i \) are only of class \( C^0 \). We do not assume that the sub-domains are Lipschitz as assumed for example in [10] for the isotropic (scalar) case. The authors are not aware of the existence of a general uniqueness result for the above problem when the coefficients are just \( C^{0,\alpha} \) Hölder continuous, with \( \alpha < 1 \). For general elliptic equations, counter-examples to unique continuation, the main technique for proving uniqueness, are known in that case, see [8]. We remind the reader of the definition of a domain of class \( C^0 \).

**Definition 3.** A bounded domain \( \Omega \) of \( \mathbb{R}^3 \) is of class \( C^0 \) if for any point \( x_0 \) on the boundary \( \partial \Omega \), there exists a ball \( B(x_0, \delta) \) and an orthogonal coordinate system \( (x_1, x_2, x_3) \) with origin at \( x_0 \) such that there exists a continuous function \( f : C^0(\mathbb{R}^2; \mathbb{R}) \) that satisfies

\[
\Omega \cap B(x_0, \delta) = \{ x \in B(x_0, \delta) : x_3 > f(x_1, x_2) \}.
\]

We define \( B_0 \) as the smallest open ball containing \( \Omega \). Note that the uniqueness of the solution outside \( B_0 \) is well known, due to the so-called Rellich’s Lemma, see e.g. [2].

**Lemma 4** (Rellich’s Lemma). If for a fixed \( \omega \), \( E \in H^1_{loc}(\text{curl}; \mathbb{R}^3) \) and \( H \in H^1_{loc}(\text{curl}; \mathbb{R}^3) \) are solutions of \( (1.1)-(1.3) \) corresponding to \( E' = 0 \) and \( H' = 0 \), then \( E = H = 0 \) in \( B_0^c \).

Our proof relies on a recent unique continuation result [13] proved for globally \( W^{1,\infty} \) regular coefficients.

**Theorem 5** ([13]). Let \( V \) be a connected open set in \( \mathbb{R}^3 \). Assume that \( \varepsilon \) and \( \mu \) are two real symmetric matrix valued functions in \( V \) satisfying \((1.4)-(1.5)\), and

\[
\|\varepsilon\|_{W^{1,\infty}(V)^{3x3}} + \|\mu\|_{W^{1,\infty}(V)^{3x3}} \leq M,
\]

for some constant \( M > 0 \). Suppose \( (E, H) \in \left(L^2_{\text{loc}}(V)\right)^2 \) satisfy

\[
\nabla \wedge E - i \omega \mu_0 \mu(x) H = 0 \text{ in } V,
\]

\[
\nabla \wedge H + i \omega \varepsilon_0 \varepsilon(x) E = 0 \text{ in } V.
\]

Then, there exist \( s > 0 \) independent of \( V \), \( E \) and \( H \), such that if for some \( x_0 \in V \), and for all \( N \in \mathbb{N} \) and all \( \delta > 0 \) sufficiently small,

\[
\int_{B(x_0,\delta)} \left( |E|^2 + |H|^2 \right) \, dx \leq C_N \exp \left( -N \delta^{-s} \right)
\]

for some constant \( C_N > 0 \), then \( E = H = 0 \) in \( V \).

The proof of Theorem 5 consists of three steps. The first two steps are given by the two propositions below.

**Proposition 6.** Under the hypothesis of Theorem 5 suppose that \( A \subset \mathbb{R}^3 \) is a bounded open set and that for almost every \( x \in \U(A) \) either \( \varepsilon(x) = \mu(x) = I_3 \) or \( E(x) = H(x) = 0 \). Then \( E = H = 0 \) in \( \U(A) \).
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Proof. For any \( v, w \in L^2(U(A)) \), we have

\[
\int_{U(A)} \nabla \wedge E \cdot v \, dx - i\omega \mu_0 \int_{U(A)} \mu(x) H \cdot v \, dx = 0,
\]

\[
\int_{U(A)} \nabla \wedge H \cdot w \, dx + i\omega \varepsilon_0 \int_{U(A)} \varepsilon(x) E \cdot w \, dx = 0,
\]

where the integrals (2.1) and (2.2) are well defined by Rellich’s Lemma. Since for almost every \( x \) in \( U(A) \), either \( \varepsilon(x) = \mu(x) = I_3 \) or \( E = H = 0 \), the solutions of the system (2.1)-(2.2) can be written also in the form

\[
\int_{U(A)} \nabla \wedge E \cdot v \, dx - i\omega \mu_0 \int_{U(A)} H \cdot v \, dx = 0,
\]

\[
\int_{U(A)} \nabla \wedge H \cdot w \, dx + i\omega \varepsilon_0 \int_{U(A)} E \cdot w \, dx = 0,
\]

which is the weak formulation of

\[
\nabla \wedge E - i\omega \mu_0 H = 0 \text{ in } U(A),
\]

\[
\nabla \wedge H + i\omega \varepsilon_0 E = 0 \text{ in } U(A).
\]

Next, since \( A \) is bounded, thanks to Rellich’s Lemma, \( E = H = 0 \) in \( U(A) \cap (\mathbb{R}^3 \setminus B(R)) \), for \( R \) large enough. In particular, \( E \) and \( H \) vanish in a ball contained in \( U(A) \), which is open and connected, and the conclusion follows from Theorem applied with \( \varepsilon(x) = \mu(x) = I_3 \), which in this case reduces to a well known result concerning the Helmholtz equation. \( \square \)

Proposition 7. Let

\[ J := \{ i \in I : |E(x)|^2 + |H(x)|^2 > 0 \text{ on a set of positive measure in } \Omega_i \}. \]

Then \( J = \emptyset \).

Proof. Suppose for contradiction that \( J \) is nonempty. Then, by Assumption there exists \( x_0 \in \partial U(\Omega_J) \cap \partial \Omega_{j_0} \) such that \( B(x_0, \delta) \cap \partial U(\Omega_J) \subset \partial \Omega_{j_0} \) for some \( j_0 \in J \) and \( \delta > 0 \). To simplify notation, set \( j_0 = 1 \).

Let us show that there exist a point \( c \) on \( \partial U(\Omega_J) \cap \partial \Omega_1 \) and a radius \( \tilde{\delta} > 0 \) such that

\[
\Omega_J \cap B(c, \tilde{\delta}) = \Omega_1 \cap B(c, \tilde{\delta}).
\]

Figure sketches the configuration we have at hand around \( c \).

Since \( \Omega_1 \) has a \( C^0 \) boundary, for some (smaller) \( \delta > 0 \) there exists a continuous map \( f \) and a suitable orientation of axes such that \( B(x_0, \delta) \cap \partial \Omega_J \subset \partial \Omega_1 \) and

\[
\Omega_1 \cap B(x_0, \delta) = \{ x \in B(x_0, \delta) : x_3 > f(x_1, x_2) \}.
\]
This alone does not prove our claim, since $B(x_0, \delta)$ could still intersect $\Omega_J$ when $x_3 \leq f(x_1, x_2)$. Since $x_0 \in \partial U(\Omega_J)$, there exists a sequence $\{y_j\} \subset U(\Omega_J) \cap B(x_0, \delta)$ such that $y_j$ tends to $x_0$. Consider for a fixed and sufficiently large $j$ the line segment 
\{y_j + t e_3, t \geq 0\}, and let $\tau > 0$ be the least value of $t$ such that $y_j + t e_3 \in \partial \Omega_J$. Then, $y_j + t e_3 \notin \Omega_J$, for $t < \tau$, and $y_j + \tau e_3 \in \partial \Omega_1$. Hence $y_j + \tau e_3 \notin \bigcup_{k \in J, k > 1} \Omega_k$. Since the sets $\Omega_k$ are disjoint, the line segment does not intersect $\bigcup_{k \in J, k > 1} \Omega_k$ in $B(x_0, \delta)$. The same argument applies to any line segment $\{z + t e_3, t \geq 0\}$ for $z$ sufficiently close to $y_j$. Introducing $c = y_j + \tau e_3$ we have established that there exists a ball $B \left( c, \delta \right)$ such that $\Omega_1 \cap B \left( c, \delta \right) = \Omega_J \cap B \left( c, \delta \right)$, which is $\{23\}$.

Now, thanks to Proposition $6$ and noting that (by Fubini’s Theorem) each $\partial \Omega_i$ is of measure zero, $E = H = 0$ almost everywhere in $U(\Omega_J)$. Thus, for almost every $x \in B(c, \delta)$, either $E = H = 0$ or $e(x) = e_1(x)$, and $\mu(x) = \mu_1(x)$. Considering the weak formulation of Maxwell’s equations, and arguing as in the proof of Proposition $\ref{proposition:6}$, we note that $E$ and $H$ are weak solutions of

\[
\nabla \times E - i \omega \mu_0 \mu_1(x) H = 0 \text{ in } B(c, \delta),
\]

\[
\nabla \times H + i \omega \varepsilon_0 e_1(x) E = 0 \text{ in } B(c, \delta),
\]

and vanish on the connected non-empty open set $B(c, \delta) \cap \{x_3 < f(x_1, x_2)\}$. Since $e_1$ and $\mu_1$ satisfy $\ref{inequality:1.3}$, that is,

\[
\|e_1\|_{W^{1,\infty}(\mathbb{R}^3)} + \|\mu_1\|_{W^{1,\infty}(\mathbb{R}^3)} \leq M_1,
\]

Theorem $\ref{theorem:5}$ shows that $E = H = 0$ in $B(c, \delta)$. This in turn shows that $E$ and $H$ vanish on a ball inside $\Omega_1$, and applying Theorem $\ref{theorem:5}$ in $\Omega_1$ we obtain $E = H = 0$ almost everywhere in $\Omega_1$. This contradiction concludes the proof.  

We now turn to the final step. We have obtained that $E = H = 0$ almost everywhere in $\Omega$, and therefore either $E = H = 0$ or $e(x) = \mu(x) = I_3$ almost everywhere in $\mathbb{R}^3$. Arguing as above, we deduce that $(E, H)$ is a weak solution of $\ref{equation:1.1}$ with $e(x) = \mu(x) = I_3$ everywhere and the conclusion of Theorem $\ref{theorem:2}$ follows from Rellich’s Lemma.

3. The case of a medium with defects

We extend our result to the case when defects of small measure are allowed in the medium. One application is to liquid crystals (see $\ref{example:13}$ for more details). Namely, we assume that the permittivity and permeability are of the form

\[
\varepsilon_D = (1 - 1_D) \varepsilon + 1_D \tilde{\varepsilon},
\]

\[
\mu_D = (1 - 1_D) \mu + 1_D \tilde{\mu},
\]

where $\varepsilon$ and $\mu$ satisfy $\ref{inequality:1.4}$-$\ref{inequality:1.9}$, $1_D$ is the indicator function of a measurable bounded set $D$, such that

\[
D \subset \bigcup_{i \in I} \Omega_i, \quad D \cap \Omega_i \subset \Omega_i \text{ and } \Omega_i \setminus D \cap \Omega_i \text{ is connected for each } i \in I,
\]

and $\tilde{\varepsilon}$ and $\tilde{\mu}$ are real symmetric positive definite matrices in $L^\infty(\mathbb{R}^3)$ satisfying $\ref{inequality:1.4}$-$\ref{inequality:1.5}$. 

Theorem 8. Suppose that the electric and magnetic fields \( E_D \in H_{\text{loc}}(\text{curl};\mathbb{R}^3) \) and \( H_D \in H_{\text{loc}}(\text{curl};\mathbb{R}^3) \) are solutions of

\[
\nabla \times E_D - i \omega \mu_D(x) H_D = 0 \quad \text{in} \quad \mathbb{R}^3, \\
\nabla \times H_D + i \omega \varepsilon_D(x) E_D = 0 \quad \text{in} \quad \mathbb{R}^3,
\]

together with the Silver-Müller radiation condition (1.3), and that \( \varepsilon_D \) and \( \mu_D \) are given by (3.1), with \( D \) satisfying (3.2). Suppose Assumption [7] holds. Then, there exists a constant \( d_0 > 0 \) depending only on the measure \( |B_0| \) of \( B_0 \), \( |\omega| \) and the lower and upper bounds \( \alpha \) and \( \beta \) given in (1.4)-(1.5) such that if the measure of \( D \) satisfies \( |D| < d_0 \), then \( E_D = H_D = 0 \) almost everywhere.

To prove Theorem 8, we use the following variant of Theorem 2.

Proposition 9. Under the same assumptions as Theorem 2 and assuming that (3.2) holds,

\[
\text{supp} H_D \cup \text{supp} E_D \subset \bar{D}.
\]

Proof. The proof follows from that of Theorem 2 since by assumption for each \( i \in I, D \cap \Omega_i \subset \Omega_i \), and the boundary of \( \Omega \setminus \Omega_i \) is unaltered by the defects. \( \square \)

Proof of Theorem 8. Since (3.3) admits a weak formulation, arguing as before we see using Proposition 9 that \( E_D \in H(\text{curl};B_0) \) and \( H_D \in H(\text{curl};B_0) \) have compact support in \( B_0 \) and are also solutions of

\[
\nabla \times E_D - i \omega \mu \hat{\mu} H_D = 0 \quad \text{in} \quad B_0, \\
\nabla \times H_D + i \omega \varepsilon \hat{\varepsilon} E_D = 0 \quad \text{in} \quad B_0,
\]

where \( \hat{\varepsilon} = \mathbf{1}_3 + \mathbf{1}_D (\varepsilon - \mathbf{1}_3) \), and \( \hat{\mu} = \mathbf{1}_3 + \mathbf{1}_D (\mu - \mathbf{1}_3) \). Note that \( i \omega \mu_0 \hat{\mu} H_D \) has compact support and is divergence free. Thus the Helmholtz decomposition (see e.g., [4, 5, 9]) of \( i \omega \mu_0 \hat{\mu} H_D \) shows there exists a unique \( A_H \in H^1(B_0)^3 \) such that \( A_H \cdot \nu = 0 \), on \( \partial B_0 \), \( \text{div} (A_H) = 0 \) and such that \( i \omega \mu_0 \hat{\mu} H_D = \nabla \times A_H \). Furthermore, \( A_H \) satisfies

\[
\| \nabla A_H \|_{L^2(B_0)^3} \leq C \left( \| \nabla \cdot A_H \|_{L^2(B_0)^3} + \| A_H \|_{L^2(B_0)^3} \right)
\]

and

\[
\| A_H \|_{L^2(B_0)^3} \leq C |B_0|^{1/3} \| \nabla \times A_H \|_{L^2(B_0)^3},
\]

where \( C \) is a universal constant. Altogether this yields

\[
(3.4) \quad \| \nabla A_H \|_{L^2(B_0)^3} \leq C \beta \mu_0 |\omega| (|B_0| + 1)^{1/3} \| H_D \|_{L^2(B_0)^3}.
\]

Since \( E_D - A_H \) is curl free, we deduce that there exists \( p \in H^1(B_0) \) such that \( E_D = A_H + \nabla p \), and \( p \) is uniquely defined by setting \( \int_{B_0} p \, dx = 0 \). Noticing that \( \tilde{\varepsilon} E_D \) is divergence free, and \( \tilde{\varepsilon} - \mathbf{1}_3 \) is compactly supported in \( B_0 \) we have that \( p \) is the solution of

\[
\text{div} (\tilde{\varepsilon} \nabla p) = - \text{div} (\varepsilon A_H) \quad \text{in} \quad B_0, \\
\n\nabla p \cdot n = 0 \quad \text{on} \quad \partial B_0, \\
\n\int_{B_0} p \, dx = 0.
\]

Since \( A_H \) is divergence free, the right-hand side becomes

\[
- \text{div} (\varepsilon A_H) = - \text{div} (1_D (\tilde{\varepsilon} - \mathbf{1}_3) A_H).
\]
To proceed, we compute using the Cauchy-Schwarz inequality the following bound
\[
\alpha \|\nabla p\|_{L^2(B_0)^3}^2 \leq \int_{B_0} \hat{\varepsilon} \nabla p \cdot \nabla p \, dx = - \int_{B_0} 1_D (\hat{\varepsilon} - I_3) A_H \cdot \nabla p \, dx \\
\leq (\beta + 1) \|A_H\|_{L^2(D)} \|\nabla p\|_{L^2(B_0)^3},
\]
and we have obtained that
\[
\|\nabla p\|_{L^2(B_0)^3} \leq \frac{\beta + 1}{\alpha} \|A_H\|_{L^2(D)}.
\]
Next note using Proposition 9 that
\[
\|E_D\|_{L^2(B_0)^3} = \|E_D\|_{L^2(D)} \leq \|\nabla p\|_{L^2(B_0)^3} + \|A_H\|_{L^2(D)} \leq \frac{2\beta + 1}{\alpha} \|A_H\|_{L^2(D)}.
\]
The Sobolev-Gagliardo-Nirenberg inequality in $B_0$ shows that
\[
\|A_H\|_{L^p(B_0)^3} \leq C (|B_0| + 1)^{1/3} \|A_H\|_{H^1(B_0)^3},
\]
where $C$ is a universal constant. Therefore, using Hölder’s inequality, together with the Poincaré-Friedrichs estimate (3.4), we have
\[
\|A_H\|_{L^2(B_0)^3} \leq C \beta (|B_0| + 1)^{2/3} \mu_0 |\omega| |D|^{\frac{1}{2}} \|H_D\|_{L^2(B_0)^3}.
\]
Altogether we have obtained
\[
\|E_D\|_{L^2(B_0)^3} \leq C \frac{\beta(\beta + 1)}{\alpha} (|B_0| + 1)^{2/3} \mu_0 |\omega| |D|^{\frac{1}{2}} \|H_D\|_{L^2(B_0)^3}.
\]
Repeating the same argument, but starting with $H_D$, we obtain also
\[
\|H_D\|_{L^2(B_0)^3} \leq C \frac{\beta(\beta + 1)}{\alpha} (|B_0| + 1)^{2/3} \epsilon_0 |\omega| |D|^{\frac{1}{2}} \|E_D\|_{L^2(B_0)^3}.
\]
The inequalities (3.5) and (3.6) imply that $H_D = E_D = 0$ when
\[
|D| < d_{0} := C \frac{\alpha^3}{\beta^3(\beta + 1)^3 (|B_0| + 1)^2 (\sqrt{\epsilon_0 \mu_0} \omega)^3},
\]
where $C$ is a universal constant.

**Remark 10.** The dependence of the threshold constant $d_0$ given by (3.7) on $|\omega|$ and $|B_0|$ shows that for a permeability $\mu$ and a permittivity $\hat{\varepsilon}$ satisfying (1.4), (1.5) and (1.6) only, uniqueness for Maxwell’s equations holds provided, if $\omega$ is fixed, the domain $\Omega$ is of small measure and bounded diameter, or, for a given $\Omega$, when the absolute value of the frequency $|\omega|$ is sufficiently small. In such cases, the whole domain $\Omega$ can be taken as a defect $D$ (and a fictitious ball containing $D$ plays the role of $\Omega$). We do not claim that the dependence of $d_0$ in terms of $|\omega|$ or $|B_0|$ in (3.7) is optimal. In contrast, Theorem 2 requires additional regularity assumptions on $\mu$ and $\hat{\varepsilon}$, but does not depend on the frequency or the size of the domain.

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