TOPICAL REVIEW:
General relativistic boson stars
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Abstract. There is accumulating evidence that (fundamental) scalar fields may exist in Nature.
The gravitational collapse of such a boson cloud would lead to a boson star (BS) as a new type of a
compact object. Similarly as for white dwarfs and neutron stars, there exists a limiting mass, below
which a BS is stable against complete gravitational collapse to a black hole.

According to the form of the self-interaction of the basic constituents and the spacetime sym-
metry, we can distinguish mini-, axidilaton, soliton, charged, oscillating and rotating BSs. Their
compactness prevents a Newtonian approximation, however, modifications of general relativity, as
in the case of Jordan-Brans-Dicke theory as a low energy limit of strings, would provide them with
gravitational memory.

In general, a BS is a compact, completely regular configuration with structured layers due to the
anisotropy of scalar matter, an exponentially decreasing ‘halo’, a critical mass inversely proportional
to constituent mass, an effective radius, and a large particle number. Due to the Heisenberg principle,
there exists a completely stable branch, and as a coherent state, it allows for rotating solutions with
quantised angular momentum.

In this review, we concentrate on the fascinating possibilities of detecting the various subtypes
of (excited) BSs: Possible signals include gravitational redshift and (micro-)lensing, emission of
gravitational waves, or, in the case of a giant BS, its dark matter contribution to the rotation curves
of galactic halos.

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I. INTRODUCTION  

In this review, we will assume that fundamental scalar fields exist in Nature, that — in the early stages of the universe — they would have formed absolutely stable soliton-type configurations kept together by their self-generated gravitational field. Theoretically, such configurations are known as boson stars (BSs). In some characteristics, they resemble neutron stars; in other aspects, they are different and, thereby, astronomers may have a chance to distinguish BS signals from other compact objects, like neutron stars or black holes (BHs). BSs can be regarded as descendants of self-gravitating photonic configurations called geons (gravitational electromagnetic units) and proposed in 1955 by Wheeler [307].  

In this review, we shall mainly concentrate on the results which are not included in the first reviews from 1992 [152,184,185] as well as later in the Marcel Grossman meetings in Jerusalem and Rome [212,215]. Since then, the number of publications investigating how BSs could possibly be detected have increased; we intend to focus strongly on these more recent research results. This could provide astronomers a handling on possible signals of BSs. All of these theoretical results are only a humble beginning of the understanding where and in which scenarios a BS could be detected. We hope other researchers may find this review stimulating for gaining new ideas or for refining older research.  

A second intention of this review is to distinguish clearly the different theoretical models underlying the label BS which could lead to different observational consequences.  

The first distinction is that the matter part of a BS can be described by either a complex or by a real scalar field. Then, there can be different interactions: (a) self-interactions described by scalar field potentials, (b) minimal coupling to gauge fields, the scalar field can be carrier of a charge (electric or hypercharge, e.g.). Moreover, the BS scalar field can couple, in standard general relativity (GR), minimally, or, in scalar-tensor (ST) or Jordan-Brans-Dicke (JBD) theory, non-minimally to gravity. In the latter case, if the strength of gravitational force is influenced by the JBD scalar field, there is an interaction of the BS scalar field with the real JBD scalar field; this can lead to gravitational memory effects in BSs. JBD theory is closely related to low energy limits of superstring theories [64] which imply the primordial production of scalar fields. Then, relics like the dilaton, the axion, combined as axidilaton, or other moduli fields could remain in our present epoch as candidates.  

Already the first two papers on BSs provided the two main directions: The history of these hypothetical stars starts in 1968 with the work of Kaup [159] using a complex massive scalar field with gravitational interactions in a semi-classical manner. The energy-momentum tensor is calculated classically providing the source for gravitation; a very detailed investigation of the solution classes was done in [98]. However, one year later, Ruffini and Bonazzola [237] used field quantisation of a real scalar field and considered the ground state of N particles. The vacuum expectation value
of the field operators yield the same energy-momentum tensor and thus, not surprisingly, the same field equations. The two different physical constituents, complex versus quantised real scalar field BS, yield the same macroscopical results. It should be noted, however, that the gravitational field $g_{\mu\nu}$ is kept classical due to the non-renormalisability of standard GR, or, alternatively, treated it as principal low energy part of some renormalisable superstring model.

More recently, a BS using a field quantised complex scalar field has been constructed. BSs consisting of pure semi-classical real scalars cannot exist because their static solutions in flat spacetime are unstable due to Derrick's theorem [80], solutions may arise only if the real scalar field possesses a time-dependence leading to non-static oscillating BSs. However, if a fermion star is present as well, a real scalar component can be added and if it interacts with the fermions, then a combined boson-fermion star is the result as in the first calculation by T.D. Lee and Pang [182]; cf. Section VA.

A challenge for the BS model is that, so far, no fundamental scalar particle has been detected with certainty in the laboratory. Several are proposed by theory: The Higgs particle $h$ is a necessary ingredient of the standard model. However, the possible discovery of the Higgs boson of mass $m_h = 114.5 \text{ GeV}/c^2$ at the Large Electron Positron (LEP) collider at CERN [3,20] gives the BS strand of investigation a fresh impetus.

For the BS, we need a scalar particle which does not decay; or if it decays (like the Higgs, e.g.), one has to assume that, in the gravitational binding, the inverse process is efficient enough for an equilibrium, as is the case for the $\beta$-decay inside a neutron star. In the latter case, the direct physical predecessor of that kind of a BS is not clear; but, of course, unstable Higgs particles could not have formed a BS by themselves. In order to explore that unknown particle regime, one can play with the model parameters such as particle mass or interaction constants. Then, BSs of rather different sizes can occur: it could be just a ‘gravitational atom’; it could be as massive as the presumed BH in the central part of a galaxy; or it could be an alternative explanation for parts of the dark matter in the halo of galaxies.

In 1995, experiments [7] proved the existence of the fifth possible state of matter, the Bose-Einstein condensate (BEC); in 2001, the Nobel prize was awarded for its experimental realization in traps. BSs, if they exist, would be an astrophysical realization with a self-generated gravitational confinement, cf. [158,21].

Let us sum up some of the properties of complex scalar field BSs (following an earlier version of [296]) in Table I as we shall discuss in Section III.

### Table I. Overview of some complex scalar field BS properties.

| Property                          | BS                                                                 |
|----------------------------------|-------------------------------------------------------------------|
| Constituents                     | Scalars (Bose-Einstein-Condensation)                              |
| Pressure support                 | Heisenberg’s uncertainty relation                                 |
| Size                             | Gigantic up to very compact (few Schwarzschild radii): Table IV, Fig. 3 |
| Surface                          | Atmosphere                                                       |
| Appearance                       | Transparent (if only gravitationally interacting)                 |
| Structure                        | Einstein-Klein-Gordon equation                                   |
| Gravitational potential          | Newtonian weak up to highly relativistic                         |
| Last stable orbit                | None                                                             |
| Rotation                         | Differentially; discrete                                         |
| Avoiding a baryonic BH by        | Jet, particle dynamics                                            |
| Avoiding a scalar BH by          | Stability, evolution                                             |
| Gravitational redshift           | Comparable to neutron star values, but larger due to transparency |
| Gravitational (micro-)lensing    | Extreme deflection angles possible (Fig. 2); MACHOS              |
| Luminosity                       | Larger than luminosity of a BH                                    |
| Star disruption (tidal radius)   | Yes                                                              |
| Distinctive observational        | Broadening of emission lines                                     |
| signatures                       | Gravitational waves                                              |
|                                  | Čerenkov radiation                                               |
A. Fundamental scalar fields in Nature?

The physical nature of the spin-0-particles out of which the BS is presumed to consist, is still an open issue. Until now, no fundamental elementary scalar particle has been detected with certainty in accelerator experiments, which could serve as the main constituent of the BS. In the theory of Glashow, Weinberg, and Salam, a Higgs boson doublet \((\Phi^+, \Phi^0)\) and its anti-doublet \((\bar{\Phi}^+, \bar{\Phi}^0)\) are necessary ingredients in order to generate masses for the ‘heavy photons’, i.e. the \(W^\pm\) and \(Z^0\) gauge vector bosons \([233]\). After symmetry breaking, only one real scalar particle, the Higgs particle \(h := (\Phi^0 + \bar{\Phi}^0)/\sqrt{2}\), remains free and occurs in the state of a constant scalar field background \([229]\). As it is indicated by the rather heavy top quark \([1]\) of 176 GeV/\(c^2\), the mass of the Higgs particle is expected to be below 1000 GeV/\(c^2\). This is supported by the possible discovery of the Higgs boson of mass \(m_h = 114.5\) GeV/\(c^2\) at the Large Electron Positron (LEP) collider at CERN \([3,20]\). In order to stabilise such a light Higgs against quantum fluctuations, a supersymmetric extension of the standard model is desirable. However, then it will be accompanied by additional heavy Higgs fields \(H^0, A^0\), and a charged doublet \(H^\pm\) in the mass range of 100 GeV/\(c^2\) to 1 TeV/\(c^2\). For the unlike case of a Higgs mass \(m_h\) above 1.2 TeV/\(c^2\), the self-interaction \(U(\Phi)\) of the Higgs field is so large that any perturbative approach of the standard model becomes unreliable. Tentatively, a conformal extension of the standard model with gravity included has been analysed, cf. \([227,126]\). Future high-energy experiments at the LHC at CERN should reveal if Higgs particles really exist in Nature.

As free particles, the Higgs boson of Glashow-Weinberg-Salam theory is unstable, e.g. with respect to the decays \(h \to W^+ + W^-\) and \(h \to Z^0 + Z^0\), if it is heavier than the gauge bosons. In a compact object like the BS, these decay channels are expected to be in partial equilibrium with the inverse processes \(Z^0 \to h + \gamma\), for instance \([50]\), by utilising gravitational binding energy. This is presumably in full analogy with the neutron star \([136,273,306]\) or quark star \([160,110]\), where one finds an equilibrium of channels are expected to be in partial equilibrium with the inverse processes.

In string theories, there may exist several fundamental scalar particles having different global or local charges. For example, for complex scalar particles with global \(U(1)\) symmetry before symmetry-breaking of the group \(SU(2) \times U(1)\), this \(U(1)\) charge describes the weak hypercharge, and not an electric one. During the earliest stages of the universe, a complex scalar field with different kinds of \(U(1)\) charge (than electric or hyperweak) could have been generated.

There is also the particle/field classification with respect to the spatial reflection \(P\). Both a real and a complex scalar field can be either a scalar or a pseudo-scalar.

Thus, particle characterisation will lead to different consequences for the detection of each BS model.
B. Boson star as a self-gravitating Bose-Einstein condensate

Since Einstein and Bose it is well-known that scalar fields represent identical particles which can occupy the same ground state. Such a Bose-Einstein condensate (BEC) has been experimentally realized in 1995 for cold atoms of even number of electrons, protons, and neutrons, see Anglin and Ketterle [7] for a recent review. In the mean-field ansatz, the interaction of the atoms in a dilute gas is approximated by the effective potential

\[ U(|\Psi|^2)_{\text{eff}} = \frac{\lambda}{4} |\Psi|^4. \]  

(1.2)

This leads to a nonlinear Schrödinger equation for \( \Psi \), in this context known as Gross-Pitaevskii equation. In a microscopic approach, one introduces bosonic creation and annihilation operators \( b^\dagger \) and \( b \), respectively, satisfying

\[ [b, b^\dagger] = 1 \]  

(1.3)

and finds that every number conserving normal ordered correlation function \( \langle b_1 \cdots b_n \rangle \) splits into the sum of all possible products of contractions \( \langle b_i b_j \rangle \) as in Wick’s theorem of quantum field theory (QFT). For \( n = 1 \) one recovers the Gross-Pitaevskii equation, whereas the next order leads to the Hartree-Fock-Bogoliubov equations, see [165] for details. Therefore, it is gratifying to note that BSs with repulsive self-interaction \( U(|\Phi|^2) \) considered already in Refs. [210,58] have their counterparts in the effective potential (1.2) of BEC. Thus, some authors [158,21] advocate consideration of a cold BS as a self-gravitating BEC on an astrophysical scale.

Recently, the so-called vortices, collective excitations of BECs with angular momentum in the direction of the vortex axis \( z \), have been predicted [313] and then experimentally prepared [192]. Since, quantum-mechanically, the \( z \)-component \( J_z = aN\hbar \) of the total angular momentum is necessarily quantised by the azimuthal quantum number \( a \), it is not possible to “continuously” deform this state to the ground state, and by this circumstance, contributing to its (meta-) stability. In 1996, axisymmetric solutions of the Einstein-KG equations have been found by us [253,211,257,260] which are rotating BSs (differentially due to the frame-dragging in curved spacetime) and exhibit, as a collective state, the same relation \( J = aN \) for the total angular momentum as in the case of the vortices of an BEC and a meta-stability against the decay into the ground state as well; cf. Section III.

II. COMPLEX SCALAR FIELD BOSON STARS

In this Section, in a nutshell, BSs are localised solutions of the coupled system of Einstein and general relativistic Klein-Gordon (KG) equations of a complex scalar \( \Phi \). Depending on the self-interaction potential \( U(|\Phi|^2) \), BSs have received different labels: In 1968, Kaup [159] referred to the localised solution of the linear KG equation as a Klein-Gordon geon, whereas in 1987, the same compact object was called mini-soliton star [98] within a series of publications [180,181,98,99] where also the terms scalar soliton star or just soliton star were introduced which nowadays are coined non-topological soliton star. Before, in 1986, Colpi et al. [58] baptised a self-gravitating scalar field with \( |\Phi|^2 \) self-interaction a boson star. So, in 1989, in a paper on stability, the mini-soliton star was renamed to mini-boson star [183]. An additional \( U(1) \) charge led to the label charged BS [147]. In [47], different BS models were summarised under the label scalar stars. Let us stress that already Ruffini and Bonazzola [237] envisioned a star by calling their investigation systems of self-gravitating particles in GR.

In his perspective paper, Kaup [159] had studied for the first time the full general relativistic coupling of a linear complex KG field to gravity for a localised configuration. It is already realized that neither a Schwarzschild type event horizon nor an initial singularity occurs in the corresponding numerical solutions. Moreover, the problem of the stability of the resulting BS with respect to radial perturbations is treated. It is shown that such objects are resistant to radial gravitational collapse for a total mass \( M \) below some critical mass and low central density (related works include Refs. [73,91,285]). The resulting configuration is a macroscopic coherent state for which the KG field can be treated as a semi-classical field.

The Lagrangian density of gravitationally coupled complex scalar field \( \Phi \) reads

\[ \mathcal{L}_{\text{BS}} = \frac{\sqrt{-g}}{2\kappa} \left\{ R + \kappa \left[ g^{\mu\nu} (\partial_\mu \Phi^*)(\partial_\nu \Phi) - U(|\Phi|^2) \right] \right\}. \]  

(2.1)

Here \( \kappa = 8\pi G \) is the gravitational constant in natural units, \( g \) the determinant of the metric \( g_{\mu\nu} \), \( R := g^{\mu\omega} R_{\mu\nu} = g^{\mu\omega} \left( \partial_\nu \Gamma_{\mu\sigma}^\omega - \partial_\sigma \Gamma_{\mu\nu}^\omega + \Gamma_{\mu\sigma}^\alpha \Gamma_{\alpha\nu}^\omega - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\sigma}^\omega \right) \) the curvature scalar with Tolman’s sign convention [290]. Using the principle of variation, one finds the coupled Einstein-Klein-Gordon equations
for the radial function \( P \) the (strong) gravitational tensor field results which has to be solved numerically.

The radial Schrödinger equation in which the metric is static and the functions \( \nu \) coordinate \( r \) and its complex conjugate describe a spherically symmetric bound state of scalar fields with positive or negative frequency \( \omega \), respectively. It ensures that the BS spacetime remains static. (The case of a real scalar field can readily be accommodated in this formalism, but requires \( \omega = 0 \) due to \( \Phi = \Phi^* \).) Equations (2.3) and (2.4) give the same differential equation. BSs composed of complex scalars are symmetric under particle anti-particle conjugation \( C \), i.e., there exists a degeneracy under the sign change \( \omega \rightarrow -\omega \). In contrast to a neutron star \([122,321,273,277]\), where the ideal fluid approximation demands the isotropy of the pressure, for spherically symmetric BSs there are different stresses \( p_r \) and \( p_\perp \) in radial or tangential directions, respectively. Ruffini and Bonazzola \([237]\) display in their Fig. 3 the anisotropy of the different stresses for a quantised

\[
G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}(\Phi),
\]

\[
\left( \Box + \frac{dU}{d|\Phi|^2} \right) \Phi = 0,
\]

\[
\left( \Box + \frac{dU}{d|\Phi|^2} \right) \Phi^* = 0,
\]

where

\[
T_{\mu\nu}(\Phi) = \frac{1}{2} [ (\partial_\mu \Phi^*)(\partial_\nu \Phi) + (\partial_\mu \Phi)(\partial_\nu \Phi^* )] - g_{\mu\nu} \mathcal{L}(\Phi)/\sqrt{|g|}
\]

is the stress-energy tensor and \( \Box := \left( 1/\sqrt{|g|} \right) \partial_\mu \left( \sqrt{|g|} g^{\mu\nu} \partial_\nu \right) \) the generally covariant d’Alembertian.

The stationarity ansatz

\[
\Phi(r,t) = P(r)e^{-i\omega t}
\]

and its complex conjugate describe a spherically symmetric bound state of scalar fields with positive or negative frequency \( \omega \), respectively. It ensures that the BS spacetime remains static. (The case of a real scalar field can readily be accommodated in this formalism, but requires \( \omega = 0 \) due to \( \Phi = \Phi^* \).) Equations (2.3) and (2.4) give the same differential equation. BSs composed of complex scalars are symmetric under particle anti-particle conjugation \( C \), i.e., there exists a degeneracy under the sign change \( \omega \rightarrow -\omega \). In contrast to a neutron star \([122,321,273,277]\), where the ideal fluid approximation demands the isotropy of the pressure, for spherically symmetric BSs there are different stresses \( p_r \) and \( p_\perp \) in radial or tangential directions, respectively. Ruffini and Bonazzola \([237]\) display in their Fig. 3 the anisotropy of the different stresses for a quantised

In the case of spherical symmetry, the general line-element can be written down as follows

\[
ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right],
\]

in which the metric is static and the functions \( \nu = \nu(r) \) and \( \lambda = \lambda(r) \) depend only on the Schwarzschild type radial coordinate \( r \). From (2.2)-(2.3), a system of three coupled nonlinear equations for the radial parts of the scalar and the (strong) gravitational tensor field results which has to be solved numerically.

Similarly as in the case of a prescribed Schwarzschild background \([78]\), the self-generated spacetime curvature affects from the KG equation (2.3) the resulting radial Schrödinger equation

\[
[\partial_{r^\perp} - V_{\text{eff}}(r^\perp) + \omega^2 - m^2] P = 0
\]

for the radial function \( P(r) := \Phi e^{i\omega t} \) essentially via an effective gravitational potential \( V_{\text{eff}}(r^\perp) = e^\nu dU/(d|\Phi|^2) + e^\mu(l+1)/r^2 + (\nu' - \lambda')e^{\nu - \lambda}/2r \), when written in terms of the tortoise coordinate \( r^\perp := \int e^{(\lambda - \nu)/2} dr \), cf. \([210]\). Then, it can be easily realized that localised solutions decrease asymptotically as \( P(r) \sim (1/r) \exp (-\sqrt{m^2 - \omega^2} r) \) in a Schwarzschild-type asymptotic background. As first shown by Kaup \([159]\), cf. \([237]\) for the real scalar field case, metric and curvature associated with a BS remain completely regular.

\[\text{A. Anisotropy of scalar matter}\]

That the stress-energy tensor (2.5) of a BS, unlike a classical fluid, is in general anisotropic was already noticed by Kaup \([159]\). For a spherically symmetric configuration, it becomes diagonal, i.e., \( T_{\mu\nu}(\Phi) = \text{diag} (\rho,-p_r,-p_\perp,-p_\perp) \) with

\[
\rho = \frac{1}{2} (\omega^2 p^2 e^{-\nu} + p^2 e^{-\lambda} - U),
\]

\[
p_r = \rho - U,
\]

\[
p_\perp = p_r - p^2 e^{-\lambda}.
\]

In contrast to a neutron star \([122,321,273,277]\), where the ideal fluid approximation demands the isotropy of the pressure, for spherically symmetric BSs there are different stresses \( p_r \) and \( p_\perp \) in radial or tangential directions, respectively. Ruffini and Bonazzola \([237]\) display in their Fig. 3 the anisotropy of the different stresses for a quantised

real scalar field BS. Since it satisfies the same differential equations, we can use this figure here as well. Gleiser [107] introduced the notion of fractional anisotropy \(a_f = (p_r - p_\perp)/p_r = P^{\prime 2}e^{-\lambda}/(\rho - U)\) which depends essentially on the self-interaction. Furthermore, the contracted Bianchi identity \(\nabla^\mu T_{\mu} = 0\) is equivalent to the equation

\[
d\frac{dr}{dr}p_r = -\nu' \left( \rho + p_r - \frac{2}{r}(p_r - p_\perp) \right)
\]

(2.10)
of ‘hydrostatic’ equilibrium for an anisotropic fluid, a generalisation of the Tolman-Oppenheimer-Volkoff equation, see Ref. [211].

By defining the number density \(n := \omega e^{-\nu/2}P^2\) and the differential pressure \(p_\Delta := p_r - p_\perp = P^{\prime 2}e^{-\lambda}\), Kaup found

\[
p_\Delta = p_\Delta(\rho, p_r, n) = \rho + p_r - n^2m^2/(\rho - p_r)
\]

(2.11)
as an algebraic equation of state, valid only for the linear KG case. However, the exact form of this algebraic relation depends on the differential equations (2.2)-(2.3). Consequently, these four thermodynamical variables for the BS show a nonlocal behaviour upon perturbations depending on the boundary conditions [159]. Furthermore, Kaup derived that radial perturbations have to be nonadiabatic.

For a spherically symmetric BS in its ground state two different layers of the scalar matter are separated by \(p_\perp(R_c) = 0\), i.e. a zero of the tangential pressure. Near the centre, \(p_\perp\) is positive and, after passing through zero at the core radius \(R_c\), it stays negative until radial infinity [247]. The core radius \(R_c\) is still inside the BS and contains most of the scalar matter. Hence, all three stresses are positive inside the BS core; the boundary layer contains a matter distribution with \(p_r > 0\) and \(p_\perp < 0\).

The choice of the potential \(U\) implies a difference in the radial pressure \(p_r = \rho - U\), and, therefore, may affect the physical properties of the BS. In Table II, we shall indicate the main proposals of current literature.

### TABLE II. Complex scalar field BS models distinguished by the scalar self-interaction and year of the first publication.

| Compact Object | Self-Interaction \(U(|\Phi|^2)\) | Year, Publication |
|----------------|-----------------------------------|------------------|
| Mini-B:        | \(U_k = m^2|\Phi|^2\)            | 1968, Kaup [159] |
| Newtonian BS:  | \(U_N = m^2|\Phi|^2\)            | 1969, Ruffini-Bonazzola [237] |
| Self-interacting BS: | \(U_{\text{HKG}} = m^2|\Phi|^2 - \alpha|\Phi|^4 + \beta|\Phi|^6\) | 1981, Mielke-Scherzer [210] |
| BS:            | \(U_{\text{CSW}} = m^2|\Phi|^2 + \lambda|\Phi|^4/2\) | 1986, Colpi-Shapiro-Wasserman [58] |
| Non-topol. Soliton Star: | \(U_{\text{NTS}} = m^2|\Phi|^2(1 - |\Phi|^2/\Phi_0^2)^2\) | 1987, Friedberg-Lee-Pang [99,184] |
| General BS:    | \(U_{\text{LKi}} = U_{\text{CSW}} + \ldots + \lambda(2n+2)|\Phi|^{2n+2}\) | 1999, Ho-Kim-Lee [131] |
| Sine-Gordon BS:| \(U_{\text{SG}} = a m^2\left[\sin(\pi/2\beta \sqrt{|\Phi|^2 - 1}) + 1\right]\) | 2000, Schunck-Torres [262] |
| Cosh-Gordon BS:| \(U_{\text{CG}} = a m^2(\cosh(\beta \sqrt{|\Phi|^2} - 1))\) | 2000, Schunck-Torres [262] |
| Liouville BS:  | \(U_L = a m^2(\exp(\beta^2|\Phi|^2) - 1)\) | 2000, Schunck-Torres [262] |

The index \(U_{\text{HKG}}\) in Table II stands for Heisenberg-Klein-Gordon. As indicated, the scalar potential is a function of \(|\Phi|^2\). This means that the field \(\Phi\) can be either a scalar or a pseudo-scalar, depending whether for spatial reflections there exist an unitary operator \(P\) for which \(P\Phi(x,t)P^\dagger = \pm\Phi(-x,t)\) holds. For instance, the axion \(a\) is a pseudoscalar. The invariance of the Lagrangian under \(P\) depends on whether or not the self-interaction is even or odd. It should be noticed that the inclusion of \(|\Phi|^6\) or higher order terms into the potential implies that the scalar part of the theory is no longer renormalisable. Fig. 1 shows the Feynman diagram for a \(|\Phi|^4\) interaction.

In the Mielke-Scherzer paper [210], actually the BS is composed from several complex scalars which are in the same ground state of a t’Hooft type monopole configuration \(\Phi^I \sim R(r) P^{lI}_l(\cos \theta)\), where \(I\) indicates the \(SO(2N)\) group index. In their calculation, an averaged energy-momentum tensor is used as proposed by Power and Wheeler [231], p. 488, deriving an angular momentum term in the field equations.

Regarding the different BS labels, a historical remark is in order. In 1984, Takasugi and Yoshimura [282] calculated the gravitational collapse of a cold Bose gas by using the Tolman-Oppenheimer-Volkoff equation and called their result a cold Bose star; in the same year, the same name was given to Ruffini and Bonazzola’s BS [40]. The label boson star was first introduced by Colpi et al. in 1986 [58]. The reason for that may have been the similarity of the mass units with neutron star mass units which in both cases are the order of magnitude \(M^3_{\text{BH}}/m^2\) (for details see Section II.C).

For if the scalar field mass \(m\) is in the order of GeV/c^2, we find for a BS the same mass range as that of a neutron star. Due to this argument, Kaup’s BS with mass unit \(M^3_{\text{BH}}/m\) seems to be a small object, and it has been called
mini-BS [183]. We would like to stress that the size and total mass of a BS depends strongly on the unknown value of \( m \) and on the issue whether there are scalar self-interactions. For corresponding \( m \)'s, even a so-called mini-BS may have masses observed in active galactic nuclei; cf. Sections II M, III C, III D, III G. The existence of mini-BSs has also been proven mathematically [31].

**B. Charged boson star**

If the complex scalar field is *minimally* coupled to local \( U(1) \) gauge fields, the Lagrangian density necessarily (cf. [138]) adopts the form

\[
\mathcal{L}_{\text{CBS}} = \frac{1}{2k} \sqrt{-g} \left[ R + \frac{1}{2} \sqrt{-g} \left[ g^{\mu\nu} (D_\mu \Phi)^* (D_\nu \Phi) - U(|\Phi|^2) \right] - \frac{1}{4} \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \right].
\]  

(2.12)

In accordance with the gauge principle, the coupling between the scalar field and the \( U(1) \)-valued 1-form \( A = A_\mu dx^\mu \), is introduced via the gauge and, for scalar fields, general covariant derivative \( D_\mu \Phi = \partial_\mu \Phi + ie A_\mu \Phi \), where \( e \) denotes the \( U(1) \) coupling constant. Furthermore, the Maxwell type term for the two-form \( F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \) occurs, where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the Faraday field strength.

The expanded form

\[
(D_\mu \Phi)^* (D^\mu \Phi) = (\partial_\mu \Phi)(\partial^\mu \Phi^*) + ie A^\mu \Phi \partial_\mu \Phi^* - ie A^\mu \Phi^* \partial_\mu \Phi + e^2 A_\mu A^\mu |\Phi|^2
\]  

(2.13)

of the kinetic term in (2.12) helps us to understand the gauge interactions in flat space-time for weak couplings. In Fig. 1, the two Feynman diagrams in order \( e \) and order \( e^2 \) in the \( U(1) \) gauge coupling are illustrated for flat spacetime.

![Feynman graph for the \(|\Phi|^4\) self-interaction in Minkowski spacetime](left). Feynman graphs for the \( e A^\mu \Phi^* \partial_\mu \Phi \) (or \( e A^\mu \Phi \partial_\mu \Phi^* \), resp.) [middle] and for the \( e^2 A_\mu A^\mu |\Phi|^2 \) [right] self-interaction.

Semi-classical solutions for (2.12) have been found by Jetzer and van der Bij [147] for an electrically charged complex scalar field. These spherically symmetric configurations, called charged BSs (CBSs) [147], can only exist if the gravitational attraction is larger than the Coulomb repulsion, i.e. if \( \alpha = e^2/4\pi < \alpha_{\text{crit}} = m^2/M_{\text{Pl}}^2 \), where \( \alpha \) is the fine structure constant; cf. the stability investigation in [149]. A macroscopic CBS would accrete matter of opposite charge very rapidly or be destroyed if the accreted matter consists of anti-particles [185]. There may be vacuum instabilities induced by a CBS [154] as we discuss now.

For particle-like CBSs, a screening mechanism can occur [154]. If the CBS is more compact than the Compton wavelength of the electron and supercritically charged, then pair production of (i) electrons and positrons or of (ii) pions occurs. For fermionic pair-production, the critical CBS is about \( Z_\text{c} \sim 1/\alpha \), whereas for bosonic pairs, it is found that \( Z_\text{c} \sim 0.5/\alpha \). If a CBS has a supercritical charge, the particles with opposing charge from the pair-production screen every surplus above the critical charge whereas their partners leave the newly formed compound object. In [154] an example of a CBS is given with 66 particles very concentrated at the centre, and a cloud of 10 screening particles, e.g. pions, almost outside of the CBS (\( m^2 = 1000 m_e^2 \)); the screened CBS has a net charge of \( 56e \). There is also almost no influence on the physical properties of CBS from the condensate (apart from its electromagnetic properties at large distances). For CBSs with constituent mass \( m \) comparable to \( m_e \), the vacuum is no longer overcritical, and no pair production occurs. Furthermore, the pair production for the bosonic sector is three order of magnitudes faster than for the fermionic one, so that a pion condensate will most likely be created first. The CBS radius in this example is less than 0.1 fermi. Furthermore, it is a speculation that CHAMPs (CHArged Massive Particles) could actually form CBSs, if such charged particles like pyrgons or maximons would exist [154].

A first-order perturbative approach of a charged bosonic cloud has recently been examined [71] by which analytical formulae for mass, charge, and radius are achieved if we have \( \omega = m \), i.e., the cloud is in an energy state even above the ones of excited CBS states. This investigation is the starting point of the dynamics of CBS formation as continued in Ref. [72]; cf. Section III F.

For the detection of CBSs, we should distinguish the different meanings of a \( U(1) \) symmetry. For example, within the standard Glashow-Weinberg-Salam theory, before symmetry-breaking of the group \( SU(2) \times U(1) \), the \( U(1) \) group
describes the weak hypercharge. Hence, during the earliest stages of the universe, a complex scalar field with different kinds of $U(1)$ charge (than electric or hyperweak) could have been generated. If we allow for a spontaneous symmetry breaking, the Higgs field in the standard model becomes a real neutral scalar field afterwards, and possible mixing terms are eliminated in the unitary gauge. Since, for a bound state like a BS, the weak coupling expansion does not apply, in principle lattice field theory comes into force, cf. [50].

In a pre-CBS topic, stationary axisymmetric solutions are given in [19]. In the physically different but mathematically similar context of the Ginzburg-Landau system, cylindrically symmetric solutions for charged scalar fields without self-gravitation were found in [221,222].

C. Critical masses of boson stars

There is an essential distinction between real and complex scalar fields in the linear KG equation:

- The Lagrangian of a real scalar field is invariant only under the discrete symmetry $\Phi \rightarrow -\Phi$,
- whereas the dynamics of a complex scalar $\Phi := (\Phi_1 + i\Phi_2)/\sqrt{2}$ is invariant under the global $O(2) \simeq U(1)$ symmetry $\Phi \rightarrow \Phi e^{i\alpha}$, where $\alpha$ is a constant.

The Noether theorem associates with each continuous symmetry a locally conserved current $\partial_\mu j^\mu = 0$ and “charge” (1.1) which commutes with the Hamiltonian, i.e. $[Q, H] = 0$. For a complex scalar field, the first “constant of motion” of our coupled system of equations is given by the invariance of the Lagrangian density (2.1) under a global $U(1)$ transformation $\Phi \rightarrow \Phi e^{i\alpha}$ and so, constraining the form of the self-interaction potential. From the associated Noether current $j^\mu$, the particle number $N$ arises:

$$ j^\mu = \frac{i}{2} \sqrt{|g|} g^{\mu
u} [\Phi^* \partial_\nu \Phi - \Phi \partial_\nu \Phi^*] , \quad N := \int j^0 d^3 x ; \quad (2.14) $$

cf. [111,112]. The second “constant of motion” arises from the Abelian group of time translations, which gives rise to the concept of energy, cf. [209]. In a rest frame, this is proportional to the total gravitational mass $M$ of localised solutions. Since these are asymptotically isolated configurations we can use Tolman’s expression [289]:

$$ M := \int (2T_0^0 - T_\mu^\mu) \sqrt{|g|} d^3 x . \quad (2.15) $$

In the case of a spherically symmetric CBS, where the covector of the electromagnetic potential is given by $A_\mu = [C(r), 0, 0, 0]$, mass and particle number become

$$ M = 4\pi \int_0^\infty (\rho + p_r + 2p_\perp) e^{(\nu + \lambda)/2} \nu^2 dr $$

$$ = 4\pi \int_0^\infty \left[ 2P^2 (\omega + eC)^2 e^{-\nu} + C^2 e^{-\lambda - \nu} - U \right] e^{(\nu + \lambda)/2} \nu^2 dr , \quad (2.17) $$

$$ N = 4\pi \int_0^\infty P^2 (\omega + eC) e^{(\lambda - \nu)/2} \nu^2 dr . \quad (2.18) $$

Equivalently, the BS mass satisfies $M = 4\pi \int_0^\infty \rho r^2 dr$ which enters as a mass parameter in the metric potentials $e^{\nu(r)}$, $e^{-\lambda(r)} \rightarrow 1 - 2M/r$ of the vacuum Schwarzschild solution, or in the asymptotic regions where the BS matter tends to zero.

Since BSs are macroscopic quantum states, they are prevented from complete gravitational collapse by the Heisenberg uncertainty principle. This provides us also with a crude mass estimate: For a boson to be confined within the star of radius $R_0$, the Compton wavelength of the scalar particle has to satisfy $\lambda_\Phi = (2\pi \hbar/mc) \leq 2R_0$. On the other hand, the star’s radius should not be much less than the last stable Kepler orbit $3R_S$ around a BH of Schwarzschild radius $R_S := 2GM/c^2$ in order to avoid an instability against complete gravitational collapse.

For the mini-BS, let us therefore assume an effective radius of $R_0 \cong (\pi/2)^2 R_S = 2.47R_S$. So far, the factor $(\pi/2)^2$ has no theoretical foundation, but then the estimate
\[ M_{\text{crit}} \cong (2/\pi)M_{\text{Pl}}^2/m = 0.6366 M_{\text{Pl}}^2/m \geq M_{\text{Kaup}} = 0.633 M_{\text{Pl}}^2/m, \]  

(2.19)

cf. Ref. [152], provides a rather good upper bound on the Kaup limit \( M_{\text{Kaup}} \), which is the numerically determined maximal mass of a stable BS. Here \( M_{\text{Pl}} := \sqrt{\hbar c/G} \) is the Planck mass and \( m \) the mass of a bosonic particle. As is typical for solitonic solutions, the star becomes heavier for weaker coupling \( U(|\Phi|) \), i.e. for smaller constituent mass \( m \) in a mini-BS.

In building star-sized BSs, one needs particles of ultra-low mass \( m \) or a non-linear \([210,58,213]\) repulsive self-interaction \( U(|\Phi|)_{\text{CSW}} = m^2|\Phi|^2 \left( 1 + \Lambda |\Phi|^2/8 \right) \), where \( \Lambda = 4\lambda/m^2 \). In the latter case, for very small \( \Lambda = \lambda M_{\text{Pl}}^2/(4\pi m^2) \), the scalar field scales still as \( \Phi \simeq M_{\text{Pl}}/\sqrt{\Lambda} \) and it turns out \([213]\) that the Kaup limit (2.19) can be applied again, but for a rescaled mass \( m \to m_{\text{resc}} := m/\sqrt{1+\Lambda/8} \). Then, in this regime, the maximal mass of a stable BS is approximately

\[ M_{\text{crit}} \simeq \frac{2}{\pi} \sqrt{1/8} \frac{M_{\text{Pl}}^2}{m}. \]  

(2.20)

For large \( \Lambda \), the scalar field scales as \( \Phi \simeq M_{\text{Pl}}/\sqrt{\Lambda} \), so that \( U(|\Phi|)_{\text{CSW}} \simeq (1 + 2\pi) m^2 M_{\text{Pl}}^2/\Lambda \); the rescaled mass is now \( m \to m_{\text{resc}} := m/\sqrt{1+2\pi/\Lambda} \) and we find

\[ M_{\text{crit}} \simeq \frac{2}{\pi} \sqrt{\frac{\Lambda}{1 + 2\pi}} \frac{M_{\text{Pl}}^2}{m} = 0.236\sqrt{\Lambda} \frac{M_{\text{Pl}}^2}{m}; \]  

(2.21)

cf. Fig. 2 from Colpi et al. \([58]\) where, as well, the value for the critical mass of 0.22 is derived in Eq. (18). The complex scalar field can be redefined as a dimensionless field \( \sigma = \sqrt{\sqrt{2}}/\Phi = \sqrt{4\pi G} \Phi \) and the values of \( \sigma \) for stable BSs are in the order of unity. Then, depending on the constituent mass and the coupling constant, the critical mass of a BS can reach the Chandrasekhar limit \( M_{\text{Ch}} := M_{\text{Pl}}^2/m^2 \simeq M_{\odot} \) (if \( m \) is about the neutron mass), where \( M_{\odot} \) denotes the mass of the sun, and so can possibly even easily extend the limiting mass of 3.23 \( M_{\odot} \) for rotating neutron stars \([195]\).

**TABLE III. Critical mass and particle number for complex scalar field BS models.**

| Object          | Critical Mass \( M_{\text{crit}} \) | Particle Number \( N_{\text{crit}} \) |
|-----------------|---------------------------------------|----------------------------------------|
| Fermion Star:   | \( M_{\text{Ch}} := M_{\text{Pl}}^2/m^2 \) | \( \sim (M_{\text{Pl}}/m)^3 \) |
| Mini-BS:        | \( M_{\text{Kaup}} = 0.633 M_{\text{Pl}}^2/m \) | \( 0.653 (M_{\text{Pl}}/m)^2 \) |
| \( \beta = 1 \) | \( M = 0.638 M_{\text{Pl}}^2/m \) | \( 0.658 (M_{\text{Pl}}/m)^2 \) |
| \( M = 0.620 M_{\text{Pl}}^2/m \) | \( 0.639 (M_{\text{Pl}}/m)^2 \) |  
| Rotating mini-BS: \( (a = 1) \) | \( 1.31 M_{\text{Pl}}^2/m \) | \( 1.38 (M_{\text{Pl}}/m)^2 \) |
| Rotating mini-BS: \( (a = 2) \) | \( \geq 2.21 M_{\text{Pl}}^2/m \) | \( \geq 2.40 (M_{\text{Pl}}/m)^2 \) |
| BS: \( \lambda \gg 4\pi m^2/M_{\text{Pl}}^2 \) | \( (1/\pi \sqrt{2\pi})\sqrt{\lambda} M_{\text{Pl}}^3/m^2 \) | \( \sim \sqrt{\lambda} (M_{\text{Pl}}/m)^3 \) |
| Non-topological Soliton Star: CBS: \( \lambda = 0 \) | \( \sim 0.44/\sqrt{\varepsilon_{\text{crit}}} - \varepsilon \) \( M_{\text{Pl}}^2/m \) | \( \sim 0.44/\sqrt{\varepsilon_{\text{crit}}} - \varepsilon (M_{\text{Pl}}/m)^2 \) |
| CBS: \( \lambda \gg 4\pi m^2/M_{\text{Pl}}^2 \) | \( \sim 0.226/\sqrt{\varepsilon_{\text{crit}}} - \varepsilon \) \( M_{\text{Pl}}^3/m^2 \) | \( \sim \varepsilon \sqrt{\lambda} \sqrt{\varepsilon_{\text{crit}}} - \varepsilon (M_{\text{Pl}}/m)^3 \) |

In the case of a CBS (Table III), the charges \( \varepsilon, \varepsilon_{\text{crit}} \) are measured in units of order of magnitude \( M_{\text{Pl}}/m \) \([147]\), so that CBSs have actually a fractional mass law \([251]\). The non-topological soliton stars can easily have very large masses; but compare the discussion of fermion soliton stars in \([62,63]\) where the order of magnitude of a neutron star can be found, cf. Section VA.

In general (cf. Table III), a BS is about the huge factor \( (M_{\text{Pl}}/m) \) more massive than the mini-BS, as long as \( (M_{\text{Pl}} > m) \). However, this depends also on the value of the coupling constant \( \lambda \), which even for the real Higgs scalar of the standard model is experimentally quite unconstrained. In astrophysical terms, the critical mass of a stable BS is \( M_{\text{crit}} \cong 0.127\sqrt{\lambda} M_{\text{Ch}} = 0.1 \sqrt{\lambda} \) (GeV/mc^2)^2 \( M_{\odot} \) \([58]\). For cosmologically relevant (invisible) axions (described by a
real pseudo-scalar $a$) of $m_a \simeq 10^{-5} \text{eV}/c^2$, an axion star would have the ridiculously large mass of $M_{\text{crit}} \sim 10^{27} \sqrt{\lambda} M_\odot$ [58].

In Table IV, we show several possible sizes which such localised configurations could have. We consider two models: on the left, the BS case with an effectively large self-interaction of $[58]$ where $\lambda \gg 4\pi m^2/M_{Pl}^2$; on the right hand side, the self-interaction constant $\lambda$ does not fulfil this condition or we have a mini-BS situation, for both cases, we can use the values of $M$ and $R$ for $\lambda = 1$. The order of magnitude of the critical mass and radius are given, hence, $R$ is calculated by the Schwarzschild radius $2GM/c^2$ and so the BS configuration is effectively general relativistic. Table IV provides examples of rather different physical situations. For example, in order to obtain a BS with about a solar mass $M_\odot = 10^{30}$ kg = $10^{37}$ GeV/c$^2$, a scalar field mass of $m \sim 10^{-10}$ eV/c$^2$ for $\lambda = 0$, or $m \propto 1$ GeV/c$^2$ with $\lambda = O(1)$ is needed; for the latter situation, we find the Colpi et al. condition $\lambda \gg 10^{-37}$, hence, the self-coupling can only be neglected if it is extraordinarily tiny. In this example, the self-interacting scalar particle has a mass comparable to a neutron, leading to a BS with the dimensions of a neutron star, cf. Section III D. Furthermore, we recognise that the free scalar particle leads to a solar mass BS as well so that it is not so “mini” as its name mini-BS implies. We give also values for a mean density $\bar{\rho} = M/R^3$; if we reduce the self-interacting scalar mass further, to $m \sim 1$ MeV/c$^2$, then we find a localised object that (for $\lambda = 1$ again) has the size of the Sun but consists of $10^6$ solar masses; this is reminiscent of supermassive black holes, for example as in active galactic nuclei. A mini-BS of the same dimensions would need an extremely small scalar mass, which, for the time being, has no observational support. By reducing $m$ further, we get a BS which could play a role as a galactic halo, cf. Section II M.

Let us now increase $m$. With self-interaction and $m = 100$ GeV/c$^2$ (about the possible Higgs mass), the BS is just about $0.1$ m but with a mass of $10^{26}$ kg; for the mini-BS with the same bare mass, we find an ultra tiny atto-meter BS, clearly below one Fermi. On the left hand side of this same line, in order to derive the same object with an effectively self-interaction constant, a very heavy scalar particle is needed which is beyond experimental estimates as well. Other not mentioned values of $m$ can easily be inter- or extrapolated.

| effective SI $U_{CSW} \neq 0$ | BS configuration $\leftarrow$ needs $\rightarrow$ | uneffective SI or $U_K \neq 0$ (take $\lambda = 1$ in middle row) |
|---------------------------|--------------------------------|----------------------------------|
| $m = 10^{10.5} \text{ GeV/c}^2$ | $M \approx 10^9 \sqrt{\lambda}$ kg, $R \approx 1 \sqrt{\lambda}$ atto m, $\bar{\rho} \approx 10^{63}/\lambda$ kg/m$^3$ | $m = 100$ GeV/c$^2$ |
| $m = 10^{6.5} \text{ GeV/c}^2$ | $M \approx 10^{16} \sqrt{\lambda}$ kg, $R \approx 10 \sqrt{\lambda}$ km, $\bar{\rho} \approx 10^{47}/\lambda$ kg/m$^3$ | $m = 1$ keV/c$^2$ |
| $m = 10^5 \text{ GeV/c}^2$ | $M \approx 10^{20} \sqrt{\lambda}$ kg, $R \approx 100 \sqrt{\lambda}$ km, $\bar{\rho} \approx 10^{41}/\lambda$ kg/m$^3$ | $m = 1 \text{ eV/c}^2$ |
| $m = 100 \text{ GeV/c}^2$ | $M \approx 10^{-4} \sqrt{\lambda} M_\odot$, $R \approx 0.1 \sqrt{\lambda}$ m, $\bar{\rho} \approx 10^{29}/\lambda$ kg/m$^3$ | $m = 10^{-6} \text{ eV/c}^2$ |
| $m = 1 \text{ GeV/c}^2$ | $M \approx 10^9 \sqrt{\lambda} M_\odot$, $R \approx 10 \sqrt{\lambda}$ km, $\bar{\rho} \approx 10^{18}/\lambda$ kg/m$^3$ | $m = 10^{-10} \text{ eV/c}^2$ |
| $m = 1 \text{ MeV/c}^2$ | $M \approx 10^6 \sqrt{\lambda} M_\odot$, $R \approx 10^6 \sqrt{\lambda}$ km, $\bar{\rho} \approx 10^9/\lambda$ kg/m$^3$ | $m = 10^{-16} \text{ eV/c}^2$ |
| $m = 1 \text{ keV/c}^2$ | $M \approx 10^{12} \sqrt{\lambda} M_\odot$, $R \approx 10^{-1} \sqrt{\lambda}$ pc, $\bar{\rho} \approx 10^6/\lambda$ kg/m$^3$ | $m = 10^{-22} \text{ eV/c}^2$ |

A different approach to estimate the mass of a mini-BS is already described in [152]. There, analytic bounds are derived on the ground state of a system of $N$ self-gravitating and relativistic bosons in Newtonian approximation, without distinguishing between real or complex scalars. The Hamiltonian for an assembly of bosons is

$$H = \sum_{i=1}^{N} \sqrt{p_i^2 + m^2} - \sum_{i>j} \frac{\kappa m^2}{|r_i - r_j|}, \quad (2.22)$$

as implicitly used in Chandrasekhar’s calculation for white dwarfs [273]. The improved estimation in Ref. [234] for the lower and upper bounds on the maximal mini-BS mass is given by

$$0.8468 \ M_{Pl}^2/m < M < 1.439 \ M_{Pl}^2/m \ . \quad (2.23)$$

Since in the Hamiltonian general relativistic effects are ignored, the bounds are above the Kaup limit $M_{Kaup}$.

### D. Effective radius

The surface and thus the radius of a neutron star is characterised by a vanishing (radial) pressure [223,273]. In contradistinction to neutron stars, due to the exponentially decreasing scalar field in a BS, we can merely speak of
an *exosphere* of a BS which corresponds to the highest layer of a planet; alternatively, we can call it the *halo* of a BS. For an *effective radius* of a BS, several proposals have been delivered in the literature.

The total mass $M$ can be used to define a radius where a certain percentage of mass is reached: $R = M(95\%)$ or even $R = M(99.9\%)$, e.g. [65]. Alternatively, the energy-density $\rho$ is used in

$$R_1 := \frac{\int_0^\infty \rho^3 dr}{\int_0^\infty \rho^2 dr},$$

(2.24)

or with help of Tolman’s expression (2.15) [98]. Taken the close relation to quantum mechanical wave function into account, Mielke [204] proposed the square root of the expectation value of the space operator as BS radius

$$R_2 := \langle |x|^2 \rangle = \frac{\int_0^\infty |\Phi|^2 r^2 dr}{\int_0^\infty |\Phi|^2 dr}, \quad \text{or} \quad R_3 := \frac{\int_0^\infty |\Phi|^2 r dr}{\int_0^\infty |\Phi|^2 dr},$$

(2.25)

the latter being even simpler. All three definitions were investigated numerically in [247] for spherically symmetric BSs with the result that $R_1 < R_2 < R_3$ for one BS state; with increasing central densities the BS radius decreases, there is a minimum at $P(0) \sim 0.95\sqrt{2/\kappa}$ and a maximum around $P(0) \sim 1.5\sqrt{2/\kappa}$. However, both these values are deep in the unstable BS states.

Applying the conserved current (2.14), another definition

$$R_{\text{eff}} := \frac{1}{eN} \int_0^\infty j^0 r^3 dr$$

(2.26)

was introduced in the case of CBSs, cf. [147]. The reason is the long range nature of the electromagnetic force; in this case, the definition $R_1$ would yield an infinite radius. Let us notice that the Schwarzschild metric gives a general lower bound on the radius of any type of object, namely $R > M/M_{\text{Pl}}$.

In all these definitions, the dimensions of the normalisation and the integrated densities cancel each other, except for the radial variable $r$. Therefore, the effective radius of a mini-BS is proportional to the Compton wavelength $\lambda_B = 2\pi\hbar/mc$ of the bosonic constituent. This applies also to the case of a BS, except that we approximately can use the rescaled mass, leading to $\lambda_{\Phi_{\text{rec}}} \simeq \sqrt{\Lambda}(2\pi\hbar/mc)$, which, for large $\Lambda$, is about $(M_{\text{Pl}}/m)$ (a huge factor) times larger. Thus, the corresponding effective radius $R_{\text{eff}}$ justifies the distinction between a mini-BS and a BS.

The situation for a non-topological soliton star is different in this way that it contains an interior part where the complex scalar field is about the constant false vacuum value $\Phi_0$, a shell of width $1/m$ over which the complex scalar field decreases to zero, and the exterior vacuum. The radius is then defined by the position of the shell.

### E. Stability and Heisenberg’s uncertainty principle

In the case of a “normal” star, the equation of state is approximately that of an ideal gas, i.e. the ratio of pressure $p$ to density $\rho$ is a function of temperature. For a fermion star, as exemplified by white dwarfs or neutron stars, $p/\rho$ is independent of the temperature; for such degenerated matter, the high pressure of the electron or neutron gas, respectively, originates from Pauli’s exclusion principle, resulting in a very high Fermi energy.

Bosons are not susceptible to the exclusion principle; instead, they can all occupy the same ground state. That bosons cannot be localised within their Compton wavelength is guaranteed by Heisenberg’s uncertainty principle [107,186].

In atoms as well as in a BS, the same principle applies and is the basic warranty for its stability against collapse. From a macroscopic point of view, our mathematical model shows that the macroscopic scalar radial pressure works against gravity. In a nutshell, a cold BS is a huge Bose-Einstein condensate, as we demonstrated in more detail in Section 1B.

### F. Stability, catastrophe theory, and critical phenomena

According to Derrick’s theorem [80,5], there are no stable time-independent solutions of scalar fields in flat 4D spacetime. One ingredient in order to circumvent this no-go theorem is to allow for a time-dependent phase factor $\exp(-i\omega t)$ in the stationarity ansatz (2.6) for complex scalars. For BSs, another factor is the attraction of the gravitational field which provides a counter-balance to the effective repulsion due to kinetic energy and corresponding parts in the self-interaction $U(\Phi)$ of the scalar field.
Moreover, for such soliton-type configurations kept together by their self-generated gravitational field, the issue of stability against gravitational collapse is crucial. In the spherically symmetric case, it was shown by Gleiser [107,109], Jetzer [148–151], as well as Lee and Pang [183] that BSs with small central densities and masses below the Kaup limit are stable against small radial perturbations. (Kaup [159] investigated the stability with respect to radial perturbations as well but came to a different answer.) Shortly afterwards, we have established via catastrophe theory [170–174,248] an even simpler way to prove stability of star solutions in general. The problem of non-radial pulsations of a BS was mathematically formulated [166] and later quasinormal modes calculated [316]; cf. Section III F.

Numerical simulations [123,124] of a mini-BS together with a surrounding spherical shell consisting of a massless real scalar field have also led to qualitative stability predictions, comparable to results in [171] as we show below. In the simulation, the real scalar field implode towards the mini-BS centre; both fields are merely interacting gravitationally. The pulse then pass through the origin, explodes and continues to infinity. Meanwhile, the mini-BS is compressed into a state which ultimately forms either a BH or disperses, depending on initial data. By varying the initial amplitude of the real scalar field, a critical solution is defined as lying between initial data resulting in BH formation and initial data giving rise to dispersal; cf. the numerical work on gravitational collapse of a massless real scalar in [113].

As starting point of the numerical solution, a stable mini-BS solution is taken, and perturbed by the contribution of the real scalar field where two features are important: (i) the initial amplitude, the so-called critical parameter \( p \), and (ii) the form of initial data (gaussian or kink). Because of the choice of perturbation and a parameter value \( p \) close to a critical value \( p^\ast \), the stable mini-BS is transformed into an intermediate oscillating unstable mini-BS state, the so-called critical solution; cf. [54]. The closer \( p \) is to \( p^\ast \), the more time the unstable mini-BS oscillates; but, for \( p < p^\ast \), the mini-BS always “explodes” eventually, i.e. disperses all matter to infinity, leaving just a diffuse remnant with low mass. For \( p > p^\ast \), the perturbed mini-BS forms a BH afterwards, where a finite minimal BH mass is found. Due to this and a scaling law relating the lifetime \( \tau \) of a near-critical solution to the proximity of the solution to the critical point \( \tau \sim -\ln(p - p^\ast) \), the critical solutions belong to a so-called type I class.

The critical mini-BS solutions are a hybrid of an unstable mini-BS solution and a small halo near the outer edge of the BS; excited mini-BS states are not the explanation, they do differ significantly. The halo part seems to be just an artifact of the collision with the real scalar field.

Additionally in [123], the numerical simulation method is used to investigate and verify earlier results on the stability against radial perturbations of mini-BS states with small and large central densities. The numerical simulation starts with a stable mini-BS solution. If the critical mini-BS solution is reached, and if one ignores the matter in its halo part, the rest is described by an unstable mini-BS solution with less mass than the stable star had, but larger central density value (see Fig. 16 in Ref. [123]). In fact, the numerical method of the linear perturbation theory of [109] is compared with the simulation method. Both results coincide with each other, but, moreover, explicit values of the frequencies of the fundamental mode and the first harmonic mode are given. For every existing mode, there are either two different stable mini-BS states, or just one unstable one. Because the mode frequencies may have observational consequences, we mention them in Section III F. Here the binding energy \( M - mN \) has no meaning [123] in understanding the transitions in the BS stability or the dynamics of the simulations.

These new simulations are related to the results of an earlier paper [171] where the stability was described qualitatively by catastrophe theory. In numerical simulations, instability arises when the square of the mode frequency is becoming negative, i.e. there exist an exponentially increasing mode, and, by this, the mini-BS is destroyed. This starts to happen at extrema in the diagrams \((M, P(0))\) or \((N, P(0))\), cf. Fig. 3 in this review; these extrema belong to so-called cusps in the diagram \((M, N)\), also called bifurcation diagram. By both the numerical investigation and the catastrophe theory, one finds that, on the first cusp, one stable mode becomes unstable; on the second cusp, a second mode becomes unstable, and so on. In catastrophe theory, the appearance of each cusp is correlated to one Whitney surface (for one corresponding mode) embedded in a higher dimensional parameter space. The bifurcation diagram \((M, N)\) arises from a particular projection of this high dimensional parameter space. In Fig. 2 of [171] which is the alternative bifurcation diagram \((M - mN, N)\) using the binding energy instead of the BS mass, three cusps are shown which are labeled \( N_{C1} < N_{C2} < N_{C3} \); \( N_{C3} \) is the maximal particle number corresponding to the onset of instability; \( N_{C1} \) belongs to the second cusp (or the next minimum following \( N_{C3} \), resp.), and \( N_{C2} \) to the third cusp (or the next maximum following \( N_{C1} \), resp.). In Section VI.C of [171], the physical behaviour of non-equilibrium BSs is considered for \( N_{C2} < N < N_{C3} \). There, for constant BS mass, two BS states exist, one stable and one unstable. Perturbed BSs will exhibit oscillations around these two BS states before such a state either collapses to a BH or disperses. This depends on the strength of the perturbation, hence, on the parameter \( p \) approaching its critical value from above.

In [123], numerical simulations for exactly this particle number regime are carried out, and we see good agreements between the simulations and the general predictions from catastrophe theory. It would be interesting to see whether the numerical simulations for the other regimes could be reconfirmed: (a) for \( N < N_{C1} \), only oscillating solutions (no collapse, no dispersal), (b) for \( N_{C1} < N < N_{C2} \), a pulsation behaviour of perturbed BS states consisting of at least two different oscillation modes.
G. Newtonian approximation - Gravitational atom I

In a nutshell, a BS is a stationary solution of a (non-linear) KG equation in its own gravitational field; cf. [203,206]. We treat this problem in a semi-classical manner, because effects of the quantised gravitational field are neglected. Therefore, a Newtonian BS was also called a gravitational atom [92]; hence, the excited states of a BS is the topic of this subsection. The results of the following subsection show that a general relativistic BS (i.e. one with strong self-gravity) as well can be designated as ‘atom’. Thus, our ‘gravitational atoms’ represent coherent quantum states, which nevertheless can have macroscopic size and large masses. The gravitational field is self-generated via the energy-momentum tensor, but remains completely classical, whereas the complex scalar fields are treated to some extent as Schrödinger wave functions, which in quantum field theory are referred to as semi-classical as well.

Since a free KG equation for a complex scalar field is a relativistic generalisation of the Schrödinger equation, we consider for the ground state a generalisation of the wave function [127,207]

\[ |N, n, l, a > : \Phi_{nla}(t, r, \theta, \varphi) = R^n_l(r) Y^a_l(\theta, \varphi) e^{-i(\omega_{nl}/\hbar)t} \]

\[ = \frac{1}{\sqrt{4\pi}} R^n_l(r) P^a_l(\cos \theta) e^{-i\alpha \varphi} e^{-i(\omega_{nl}/\hbar)t}, \]  

(2.27)

where \( R^n_l(r) \) gives the radial distribution, \( Y^a_l(\theta, \varphi) \) are the spherical harmonics, \( P^a_l(\cos \theta) \) are the normalised Legendre polynomials, and \( |a| \leq l \) are the quantum numbers of azimuthal and angular momentum. Each mode has its characteristic frequency \( \omega_{nl} \). As is well-known, in the Newtonian limit Einstein’s equation reduces to the Poisson equation

\[ \Delta V = -\kappa \rho/2, \]  

(2.28)

where \( V \) is Newton’s gravitational potential, \( \rho = N m|\Phi|^2 \) is the matter density, and \( N \) the particle number. Under the condition that all bosons are in the same state, the KG equation reduces to

\[ \Delta \Phi_{nla} + 2m(\omega_{nl} + mV)\Phi_{nla} = 0, \]  

(2.29)

hence, a Schrödinger-type equation. The relation for the conserved particle number \( N = m \int |\Phi_{nla}|^2 r^2 \sin \theta dr d\theta d\varphi \) gives us a normalisation condition on the wave function \( R^n_l(r) \). No self-interaction of the scalar field is introduced.

The coupled system of (2.28) and (2.29) has to be solved numerically. A factorization as in (2.27) of the potential \( V \) and of the scalar field \( \Phi_{nla} \) yields two coupled ordinary differential equations for the radial coordinate.

Several solutions have been derived so far. The nonexistence ground state with \( (n = 1, l = a = 0) \) was calculated first by [237] and can also be found in [92] whereas in [98] even Newtonian solutions up to 5 nodes are obtained. In Fig. 3 of [92], we can see very nicely the difference in the Newtonian and GR calculation. Additionally in [92], a BS was calculated where scalar matter is distributed within the ground state plus some matter in the lowest mode with nonzero quadrupole moment (the 3d mode with \( n = 3 \) and \( l = 2 \)) so that gravitational waves can be radiated; cf. Section III.F. In [274], non-vanishing azimuthal angular momentum \( a \) is included, and so, we have rotating Newtonian BSs; explicitly, solutions for the combinations \( (l = 1, a = 0), (l = 1, a = 1), (l = 2, a = 0) \) are presented.

In cylindrical coordinates \((r, z)\), the system (2.28), (2.29) is an elliptic boundary eigenvalue problem, and a finite element method is used for solving the problem numerically in [265]. Due to the invariance of the differential equations with respect to \( z \rightarrow -z \), \( V \rightarrow V, \Phi \rightarrow \pm \Phi \), there are positive and negative parity classes where for the first the spherically symmetric solution is derived. Solutions with negative parity show two equatorial symmetric blobs of scalar matter above and below the equatorial plane, whereas the scalar field is equatorially anti-symmetric. This is comparable to a hydrogen atom in the \((n = 2, l = 1, a = 0)\) state, cf. [37]. Whereas this solution describes an axisymmetric Newtonian BS, also non-axisymmetric Newtonian BSs were obtained [319]. There, the scalar field is anti-symmetric with respect to one of the following parity transformations: (A) \((x, y, z) \rightarrow (x, y, -z), (x, -y, z), (-x, y, z)\). The type A solutions show a BS with four blobs with maximal scalar matter density in the \((x = 0)\)-plane. The type B solutions possess eight blobs of scalar matter with zero density in the \((x = 0)\)-plane; in both cases, the density vanishes also for \( y = 0 \) and \( z = 0 \).

In [317], the general relativistic extension of the axisymmetric mini-BS solutions of [265] were exhibited. These negative parity solutions are classified by the derivative \( \Phi_\gamma(r = 0, \theta = 0) \). The critical mass is given by \( M_{\text{crit}} = 1.05 M_\odot \) \( / \sqrt{\gamma} \).

An analytical solution for a BS in the limit \( A \gg 1 \) is determined in [178] for the rescaled scalar field \( \sigma_*(x_*) = \sqrt{\gamma_0 \sin(\sqrt{2} x_*)} / \sqrt{2} x_*, \) where \( \gamma_0 \) is the absolute central value of the Newtonian potential.
H. Boson stars - Gravitational atom II

In the case of the general relativistic BS, several excited BS states have been computed. The first solutions with nodes, i.e. principal quantum number \( n > 1 \), were determined by Mielke and Scherzer [210] who, motivated by Heisenberg’s non-linear spinor equation [127,207], added self-interacting terms describing the interaction between the bosonic particles in the BS configuration.

The work of Friedberg et al. [98] investigated solutions with up to 56 nodes within the mini-BS scalar field. Solutions with nodes possess a similar behaviour in the mass-particle-number function as solutions without nodes, i.e. a maximal mass or cusps in the mass-particle-number diagram were found. For the first cusp, the maximal mass depends linearly [98] on the particle number. There as well, the solutions with zero nodes are compared with solutions with ten nodes. Recently, phase-shifted BS solutions have been constructed [125]. Normally, the BS complex scalar field \( \Phi(r, t) = P(r) \cos(\omega t) + iP(r) \cos(\omega t + \delta) \) has a phase of \( \delta = \pi/2 \) between its real and its imaginary part. For \( \delta \neq \pm \pi/2 \), such solutions are investigated and show a BS like behaviour. Long-term numerical evolutions indicate that phase-shifted BSs are stable but periodic; mass-energy is exchanged between the real and imaginary fields. It seems that BSs can be more generic than previously thought. For \( \delta = 0 \), the oscillating BSs are approximated [268].

I. Rotating boson stars - Gravitational atom III

In Refs. [257,261,251,253,211], we proved numerically that regular stationary and axisymmetric solutions, which can be regarded as rapidly rotating BSs, exist in GR. In the Einstein-KG system (2.2), (2.3), we use an isotropic stationary axisymmetric line element

\[
ds^2 = f(r, \theta)dt^2 - 2k(r, \theta)dt d\varphi - l(r, \theta)d\varphi^2 - \epsilon^{\mu(r, \theta)}_a (dr^2 + r^2 d\theta^2)
\]

where \( f, k, l, \mu \) are some functions to be determined numerically. In order to find rotating BSs in ground state, the scalar field ansatz is

\[
\Phi(r, \theta, \varphi, t) := P(r, \theta) e^{-i \omega t} e^{-i a \varphi},
\]

where the uniqueness of the scalar field under a complete rotation \( \Phi(\varphi) = \Phi(\varphi + 2\pi) \) requires \( a = 0, \pm 1, \pm 2, \ldots \), as in quantum mechanics. For the resulting nonlinear elliptical system, we applied a finite difference method and found solutions for the mini-BS with \( a \) up to 500. Sequences of rotating mini-BSs with \( a = 1 \) and \( a = 2 \) were constructed in [318] where every BS is characterised by the maximal value of the scalar field which is usually not on the rotation axis. For \( a = 1 \), the energy density has a non-vanishing value on the rotation axis, whereas, for \( a \geq 2 \), it has to vanish identically there, i.e., vacuum predominates; cf. the situation of the rotating BS in (2+1)-dimensions in Section II O. The maximum mass increases above the Kaup limit: for \( a = 1 \), it is \( 1.31 M_{Pl}^2/ m \) and, for \( a = 2 \), it is at least \( 2.21 M_{Pl}^2/m \). In [199], rotating BSs were calculated with a numerically faster multigrid method.

The total angular momentum \( J \) of a BS is given by

\[
J = a N,
\]

i.e. a BS can change its rotational state only in discrete steps resembling gravitational atoms. Hence, the angular momentum \( J \) is quantised by the azimuthal quantum number \( a \). The lifting of the angular momentum degeneracy, familiar from quantum mechanics, is a gravito-magnetic effect [198] due to the rotating frame. It is, however, surprising that this still holds for macroscopic configurations. Eventually, it is now understandable why no slowly rotating BS states near the spherically symmetric ones could be found [162].

If we imagine what happens if a BS starts to rotate, we find out that the BS rearranges its structure from the sphere to a torus. The BS in this situation looks like the \((n = 2, l = 1, a = 1)\) state of a hydrogen atom, cf. [37]. Qualitatively, due to the infinite velocity of light and the infinite range of the scalar matter within the BS, it follows that the form of rotation has to be differential, not uniform. The rotating solutions of the coupled Einstein-KG equations represent the field-theoretical pendant of rotating neutron stars which have been studied numerically for various equations of state, different approximation schemes [100,60,87] as well as with quadrupole moments [195] as a model for (millisecond) pulsars.

A general insight into the physical characteristics of rotating BS matter can be obtained from the energy-momentum tensor [251].
\[ T_{\mu}{}^{\nu} = \begin{pmatrix} T_{t}{}^{t} & 0 & 0 & T_{t}{}^{\varphi} \\ 0 & T_{r}{}^{r} & T_{r}{}^{\theta} & 0 \\ 0 & T_{\theta}{}^{r} & T_{\theta}{}^{\theta} & 0 \\ T_{\varphi}{}^{t} & 0 & 0 & T_{\varphi}{}^{\varphi} \end{pmatrix}, \] (2.33)

in which the shear tensions \( T_{\theta}{}^{r} \) and \( T_{r}{}^{\theta} \), as well as all main tensions are different from each other, i.e. \( T_{r}{}^{r} \neq T_{\theta}{}^{\theta} \neq T_{\varphi}{}^{\varphi} \), and non-vanishing. Compared to a rotating ideal fluid of a neutron star with \( T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} - pg^{\mu\nu} \), there are no shear tensions at all and the main stresses generally obey \( T_{r}{}^{r} = T_{\theta}{}^{\theta} = T_{\varphi}{}^{\varphi} \).

A rotating BS, i.e. with large self-interaction constant \( \lambda \gg 4\pi m^{2}/\Lambda_{P}^{2} \), was investigated by Ryan [238]. For the construction of the solutions, the fact that the derivatives of the scalar field vanish (due to the scaling with \( \lambda \)) yields an ideal fluid of BS matter as in the non-rotating case [58]. From the solutions, multipole moments like mass, angular momentum, and mass quadrupole moment are determined. For fixed angular momentum \( J \), the maximal mass was calculated by increasing the maximal scalar field value until the numerical computation became unstable (Fig. 2 in [238]); under this condition, the maximal mass is about \( 0.58 \sqrt{\Lambda M_{P}^{2}}/m^{2} \) in our units which are different by a factor \( \sqrt{4\pi} \) to the ones in [238]; further details in Section III F.

More recently, in flat spacetime soliton rings are constructed [11] in a similar way from a complex scalar. They are metastable due to the same combination of conserved Noether charge and angular momentum.

### J. Formation

It has been demonstrated that BSs can actually form from a primordial bosonic “cloud” by Jeans instability [161,92,116,28,193,288,152,185,135,235]. (The primordial formation of non-gravitating non-topological soliton solutions was studied by Frieman et al. [101,102].) That the quantum aspects of this ground state instability of the complex scalar field acts like a classical Jeans instability is shown in [117]. The formation of a spherically symmetric mini-BS as a Bose-Einstein condensation is considered in [30]. One of the first steps in the formation process of CBSs is investigated in [71,72] by calculating the dynamics of charged bosonic clouds; cf. Section III F.

In a numerical simulation, Seidel and Suen [269] have shown for a spherically symmetric mini-BS, that there exists a dissipationless relaxation process called gravitational cooling; cf. [266,267,18]. This mechanism is similar to the violent relaxation of collisionless stellar systems which settle to a centrally denser system by sending some of their members to larger radii. Likewise, a bosonic cloud would gravitationally cool to a BS by ejecting parts of the scalar matter. It seems that the influence of the viscous term in the KG equation (2.3) is not sufficient enough, so that radiation of scalar particles is the only viable mechanism. This was demonstrated numerically by starting with a spherically symmetric configuration with \( M_{\text{initial}} \geq M_{\text{Kauf}} \), i.e. which is more massive than the Kauf limit, and then monitoring its time evolution during which the star oscillates and scalar matter is radiated. Actually, such oscillating and pulsating BS branches were predicted earlier in our stability analysis via catastrophe theory [171,248]; cf. Section IV F. Oscillating soliton stars were constructed as well by using real scalar fields which are periodic in time [268]; cf. Section IV. Without imposing spherical symmetry, the emission of gravitational waves have to be included. In Ref. [74], it is argued that the purely radial collapse considered in [269] is too artificially. A primordial bosonic cloud could have had too large angular momentum so that the minimal densities required for BS formation via gravitational cooling are not reached.

The situation of Bose gas clouds, partly even before Bose-Einstein condensation, was investigated in [137] and [77]. Details of [137] are exhibited in [152] already, so that we concentrate on the scenario of a non-isothermal Bose gas cloud [77] in Section VI in which BS formation in different perfect fluid approximations are outlined.

The influence of a cosmological phase transition on a BS with a field quantised real scalar is considered in Section IV A.

### K. Hot boson stars

The temperature dependence of the soliton star of Friedberg et al. [99] is investigated in [279]. In the calculation for the case without gravitational coupling, i.e. for a non-topological soliton (NTS), a critical temperature of \( T_{c} = 0.7875\sigma_{0} \) is derived and, in the following, it is assumed that the influence of gravitation does not change \( T_{c} \) very much. Furthermore, the zero temperature estimates of [99] can be applied as well up to a temperature of \( T = 0.7\sigma_{0} \). Without gravity, the mass of the NTS vanishes at \( T_{c} \), prohibiting their existence for \( T > T_{c} \); the same holds for the soliton star. Below the critical temperature, radius and mass of a soliton star increase with temperature as one may expect since the kinetic energy of the particles grows and the potential barrier between true and false vacuum gets
Jordan frame is the strength of whose coupling to the metric is given by a function $\tilde{\omega}$ motion inside the star as in oscillating BSs) is decisive for its existence, see Section IV. Dicke (JBD) theory [38], superstring theory [96, 46, 189]. Inflationary scenarios based upon them [175], and because a JBD model with less model-dependent than bounds from nucleosynthesis [2]. Scalar tensor theories have regained popularity through the limit of this theory agrees with current observations, which “frame” describes the true, physical metric that measures the distance between spacetime points, is a subtle and a field redefinition or “renormalisation” of the scalar field, the so-called Wagoner transformation. The question authors use in general the potential $U$ between the papers. The results lead to observable distinctions between the models. In the BS matter part, the scalar field of a JBD type theory. We point out the differences in the Lagrangians, but hope also to lighten the links between the papers. The authors use $\sqrt{|g|}$ from [303] for the real scalar BSs, temperature (or some kind of motion inside the star as in oscillating BSs) is decisive for its existence, see Section IV.

L. Boson stars in Jordan-Brans-Dicke theory

In the so-called scalar-tensor (ST) theories, a real scalar field replaces Newton’s gravitational constant $G$, the strength of whose coupling to the metric is given by a function $\omega(\phi_{\text{JBD}})$. In the simplest scenario, the Jordan-Brans-Dicke (JBD) theory [38], $\omega$ is a constant. GR is attained in the limit $1/\omega \to 0$. To ensure that the weak-field limit of this theory agrees with current observations, $\omega$ must exceed 500 at 95% confidence limit [312] from solar system timing experiments, i.e. experiments taking place in the current cosmic time. This limit is both stronger and less model-dependent than bounds from nucleosynthesis [2]. Scalar tensor theories have regained popularity through inflationary scenarios based upon them [175], and because a JBD model with $\omega = -1$ is the low-energy limit of superstring theory [96, 46, 189].

In this part, we exhibit the BS theories which contain beside the complex scalar field of the BS matter, the real scalar field of a JBD type theory. We point out the differences in the Lagrangians, but hope also to lighten the links between the papers. The results lead to observable distinctions between the models. In the BS matter part, the authors use in general the potential $U_{\text{CSW}}$ so that a scalar field self-interaction can be taken into account.

The Lagrangian for our system of ST gravity coupled to a self-interacting, complex scalar field in the (physical) Jordan frame is

$$L = \frac{\sqrt{|g|}}{2} \left[ g_{\phi\phi} \tilde{R} - \frac{\omega(\phi_{\text{JBD}})}{\tilde{\omega}(\phi_{\text{JBD}})} \tilde{g}^{\mu\nu} \partial\phi_{\text{JBD}} \partial_{\nu} \phi_{\text{JBD}} + \tilde{V}(\phi_{\text{JBD}}) + \tilde{g}^{\mu\nu} \partial_{\mu} \Phi^{*} \partial_{\nu} \Phi - U_{\text{CSW}} \right].$$  \hspace{1cm} (2.34)

The gravitational scalar is $\phi_{\text{JBD}}$ and $\omega(\phi_{\text{JBD}})$ is the Jordan frame coupling of $\phi_{\text{JBD}}$ to the matter. In more general theories, the real scalar $\phi_{\text{JBD}}$ possesses even a potential $\tilde{V}$. The complex scalar $\Phi$ has mass $m$ and is self-interacting through the potential term $U_{\text{CSW}} = U_{\text{CSW}}(|\Phi|^2)$.

There is an alternative representation of Lagrangian (2.34) in the so-called Einstein frame. The transition to this frame is effected by the conformal transformation

$$\tilde{g}_{\mu\nu} = e^{2a(\varphi_{E})} g_{\mu\nu},$$  \hspace{1cm} (2.35)

where

$$\phi_{\tilde{\text{JBD}}}^{-1} = \kappa e^{2a(\varphi_{E})}$$  \hspace{1cm} (2.36)

and $a(\varphi_{E})$ is the Wagoner transformation [303] from $\phi_{\text{JBD}}$ to the gravitational scalar $\varphi_{E}$ in the Einstein frame. The relationship between $\omega(\phi_{\text{JBD}})$ and $a(\varphi_{E})$ is obtained by requiring

$$\left( \frac{\partial a}{\partial \varphi_{E}} \right)^2 = \frac{1}{2\omega + 3}.$$  \hspace{1cm} (2.37)

Using the potential $V(\varphi_{E}) = e^{4a(\varphi_{E})} \tilde{V}[\phi_{\text{JBD}}(\varphi_{E})]$, we find the Lagrangian in the Einstein frame

$$L = \frac{\sqrt{|g|}}{2\kappa} \left[ R - 2g^{\mu\nu} \partial_{\mu} \varphi_{E} \partial_{\nu} \varphi_{E} + V(\varphi_{E}) \right] + \frac{\sqrt{|g|}}{2} e^{2a(\varphi_{E})} \left[ g^{\mu\nu} \partial_{\mu} \Phi^{*} \partial_{\nu} \Phi - e^{2a(\varphi_{E})} U_{\text{CSW}} \right].$$  \hspace{1cm} (2.38)

Mathematically, the transition from the Jordan to the Einstein frame is a conformal change of metric, cf. [202], and a field redefinition or “renormalisation” of the scalar field, the so-called Wagoner transformation. The question which “frame” describes the true, physical metric that measures the distance between spacetime points, is a subtle
one which depends on the coupling to matter, see the careful analysis of Brans [39]. However, the Einstein frame is sometimes used for calculations of BSs and in certain situations the results are close enough to GR. In the following, we will indicate which frame has been implemented by the authors.

Both the coupling functions $\varpi(\phi_{\text{JBD}})$ in the Jordan frame and the $a(\varphi_{\text{E}})$ in the Einstein frame are a priori unknown. There are, however, some theoretical reasons to motivate explicit forms. Furthermore, once a particular form for the coupling functions is taken, there are experimental constraints that can be imposed.

In Table V, we refer to publications which consider different gravitational theories and the frame in which the calculations are done; in some papers, more than one version is investigated.

| Scalar-tensor theory | Jordan frame | Einstein frame |
|----------------------|--------------|----------------|
| Non-minimal-gravity coupling | [29] | [59,16,15] |
| JBD, $\varpi = $ const, $V = 0$ | [284,119,293,294,309-311] | [315,93] |
| JBD, $\varpi = $ const, $V \neq 0$ | [292,293,310,311] | [59] |
| ST, $\varpi = \varpi(\phi_{\text{JBD}})$, $a = a(\varphi_{\text{E}})$ | [292,293,310,311] |

BS with non-minimal gravity coupling, $\phi_{\text{JBD}} = 1 + 2\kappa|\Phi|^2$, $\varpi = V = 0$: In 1987, van der Bij and Gleiser [29] considered a special JBD case, for which the gravitational scalar is actually a function of the BS complex scalar field, i.e. the BS matter changes its own gravitational field. The effective gravitational constant in the coupling constant $1/(16\pi G_{\text{eff}}) := \phi_{\text{JBD}}/(2\kappa) = 1/(16\pi G) + |\xi|\Phi^2$ can become infinite if the free parameter $\xi$ is negative. The following critical masses and particle numbers arise

$$\xi \to +\infty, \quad M_{\text{crit}} \approx 0.73\sqrt{\xi} M_{\text{Pl}}^2/m, \quad N_{\text{crit}} \approx 0.88\sqrt{\xi} M_{\text{Pl}}^2/m^2,$$

$$\xi \to -\infty, \quad M_{\text{crit}} \approx 0.66\sqrt{|\xi|} M_{\text{Pl}}^2/m, \quad N_{\text{crit}} \approx 0.72\sqrt{|\xi|} M_{\text{Pl}}^2/m^2.$$

BS in JBD theory and coupling between $\phi_{\text{JBD}}$ and $\Phi$, $\varpi = \kappa f^2/4$, $V = 0$, $m^2 \rightarrow m^2\phi_{\text{JBD}}$: In [284], Tao and Xue applied a JBD theory with positive constant $\varpi$ including a free parameter $f$. It allowed for a space-dependent boson mass, determined by the JBD gravitational scalar, the corresponding term in the complex scalar field part reads $m^2|\phi_{\text{JBD}}|^2$. After redefinition, $\varpi = G f^2/4$, calculations are done for $f^2 = 0, 1, 2 \cdot 10^3$ and a self-interaction constant $\Lambda = 0, 2, 20, 60$ (redefined in our units, see Appendix). For $f^2 = 1$, the critical values are

$$M_{\text{crit}} \approx 0.17\sqrt{\Lambda} M_{\text{Pl}}^2/m, \quad N_{\text{crit}} \approx 0.18\sqrt{\Lambda} M_{\text{Pl}}^2/m^2.$$

BS in JBD theory, $\varpi = 6$, $V = 0$: Gundersen and Jensen [119] used a pure JBD theory with a potential $U_{\text{CSW}}$ and $\varpi = 6$ is chosen for the early universe. The critical mass is found to be

$$M_{\text{crit}} \approx 0.213\sqrt{\Lambda} M_{\text{Pl}}^2/m.$$

Thus, the deviation of the critical mass is just a few percent from the GR limit value 0.218.

BS in ST theory in Jordan frame, $\varpi = \varpi(\phi_{\text{JBD}})$, $V = 0$: In 1997, Torres [292] selected three different functional forms for the JBD coupling function, namely

1. $2\varpi(\phi_{\text{JBD}}) + 3 = \frac{2B_1}{|1 - \phi_{\text{JBD}}/\phi_{\text{JBD},0}|^\alpha}$,

2. $2\varpi(\phi_{\text{JBD}}) + 3 = \varpi_0 \phi_{\text{JBD}}^n$,

3. $\varpi(x) = 0.1x, \quad 10x, \quad \ln(x), \quad \exp(0.01x)$,

where $B_1, \alpha, n$ are all positive constants, $\varpi_0$ a constant, and $\phi_{\text{JBD},0}$ is the asymptotic value of the inverse of Newton’s constant (e.g., the present value). Choice 1. had already been investigated in a cosmological context [24], for case 2. an analytical solution is known [23] and a constraint from nucleosynthesis was determined [291]. The intention for explicit functions in situation 3. was to understand how the behaviour of $\varpi$ can be managed within the radius of the star. Calculations have been done in the Jordan frame for several combinations of $B_1 = 2, 5, 8$ together with $\alpha = 0.5, 1.0, 1.5, 2.0$ and $\Lambda = 0, 10, 50, 100, 150, 200$ or $\varpi_0 = 2$ and $n = 3$, respectively. Again the order of magnitude of the BS mass varies only about a few percent when compared with GR. The influence of different external asymptotic
values of the gravitational scalar $\phi_{\text{JBD}}$ on the BS mass has been investigated as well with the result that the mass decreases with increasing time.

BS in JBD/ST theory in Einstein frame, $a = a(\varphi_E)$, $V = 0$: Comer and Shinkai [59] studied BSs in Einstein frame calculations using two forms for $a(\varphi_E)$, one being the Brans-Dicke coupling

$$a(\varphi_E) = \frac{\varphi_E - \varphi_{E\infty}}{\sqrt{2\varphi_{\text{JBD}} + 3}}, \quad (2.46)$$

with the constant $\varphi_{\text{JBD}} = 600$, and the quadratic coupling

$$a(\varphi_E) = \frac{\eta}{2} \left( \varphi_E^2 - \varphi_{E\infty}^2 \right), \quad (2.47)$$

which is a particular form considered by Damour and Nordtvedt [66] except for the additive constant; in this cosmological model there is a constraint of $\eta > 3/8$. The term $\varphi_{E\infty}$ represents the asymptotic value of the gravitational scalar field. Tests in the solar system provide the constraint $\omega_{\text{JBD}} > 500$; observations of binary pulsars require $\eta > -5$ [69]. BS calculations are performed for $\eta = 0.38$ which is close to the limit $3/8$. In the Jordan frame, the coupling function is

$$2\omega(\phi_{\text{JBD}}) + 3 = \frac{1}{\eta^2 \varphi_{E\infty}^2 - \eta \ln(G\varphi_{\text{JBD}})} \cdot (2.48)$$

In both cases, solution sequences are produced for $\Lambda = 0, 10, 100$ and for zero, one, and two nodes in the complex scalar field solution; hence, we find here the first excited states for BSs in scalar tensor theories. Again, there is no significant but still observable difference to GR in the critical mass values. Interesting is that there is a minimal value of the central $\varphi_E(0)$ for which BSs can exist. Even more, a non-uniqueness is found, i.e. two different mass values exist for one $\varphi_E(0)$, and an intersection point at which these two masses are identically. Furthermore, the stability of such BSs are discussed by using catastrophe theory following [171,172,248,173,174]. Constructions of BSs in the early universe proved that the BS mass decreases if we go back in time as Torres [292] noted before. In the early universe, only BS states with positive binding energy did exist, meaning that no stable BS could have formed. This result was later doubted by Whinnett [309] as we are going to mention below.

BS in JBD theory in Einstein frame, $\omega = 60, 600$; $V = 0$: Numerical simulations for the dynamical evolution of spherically symmetric BSs have been carried out by Balakrishna and Shinkai [16,15]. They also investigated the BH formation of a perturbed equilibrium configuration on an unstable branch. A perturbed unstable BS can also migrate to a stable solution by emitting scalar waves; it is shown how an excited BS state passes into the ground state (similar to the GR case calculations some years earlier [267,18]). Furthermore, the formation of a stable BS from a Gaussian scalar field packet is demonstrated in JBD theory, in much the same way as in GR.

BS in JBD/ST theory in the Jordan frame, $\omega = \omega(\phi_{\text{JBD}})$, $V = 0$, gravitational memory effect: The main focus in [293] was the gravitational memory effect of BSs both in JBD theory with $\omega = 400$ and in ST theory with $2\omega(\phi_{\text{JBD}}) + 3 = 10/\sqrt{1 - (\phi_{\text{JBD}}/\phi_{\text{JBD}, 0})}$ both in the Jordan frame; this ST theory is a special choice of case 1. in Eq. (2.43) [292].

When the gravitational coupling is evolving, this has important implications for astrophysical objects [22], because it means that the asymptotic boundary condition for $\phi_{\text{JBD}}$ is a function of epoch affecting the structure of the BS: (a) The star can adjust its structure in a quasi-stationary manner. Two sub cases can be distinguished: The adjustment time $t_{\text{adj}}$ is smaller than the evolution time $t_U$ of the Universe; then the BS will always be in equilibrium with the external development. If $t_{\text{adj}} > t_U$, the BS will be (quasi-)static, hence, “conserving” the value of $G$ at formation time. This effect is called gravitational memory. Actually, a BS which is not completely decoupled from the cosmological expansion will develop slowly. (b) The BS interior produces a feedback on the asymptotic gravitational constant. If a high density of BSs formed, they might be able to reduce the gravitational interaction strength in quite a significant region around themselves.

BS in JBD theory in Jordan frame, $\omega = -1$, $V = 0$, tensor mass: In [309], Whinnett analyses three different proposed mass functions for BS systems for the low-energy limit of superstring theory where $\omega = -1$ [96,46,189]. These are the Schwarzschild mass corresponding to the ADM mass in the Jordan frame, the Keplerian mass in the Jordan frame and the Keplerian mass in the Einstein frame which he calls tensor mass; cf. with the earlier investigation in [176]. In his Figure 1, the results for the mass definitions are compared with the results for the particle number in the Jordan frame. The mass definitions differ significantly leading to contrary physical interpretations for stability. The Keplerian mass in the Jordan frame yields positive binding energy for all values of central density, suggesting that every solution is generically unstable. The Schwarzschild mass in the Jordan frame instead leads to negative binding energies for every value of central density, suggesting that every solution is potentially stable. However, the tensor mass peaks at the same location as the rest mass, an important property in the application of catastrophe
theory to analyse stability properties [171]. Therefore, Whinnett adopts the tensor mass as the real mass of the star. We stress the interesting situation that whereas the (Keplerian) mass is calculated in the Einstein frame, the particle number (or rest mass) is evaluated in the Jordan frame. Finally, we remark that, in this paper, the dilaton field $\varphi$ is used, i.e. $\varphi_{\text{JBD}} = \exp(-\varphi)$.

A comparison of the critical masses shows that for the choice $\varpi = 6$ of Gunderson and Jensen, the mass values are about 7% different from the results of the tensor mass [119]. Already for $\varpi = 500$ the difference between the masses is less than $10^{-3}$, i.e. negligible.

Whinnett [309] proved also that the tensor mass and the particle number peak at the same value of the central density, which, in GR, is known due to Jetzer [151]. In the same year, Yazadjiev [315] gave a similar analytical proof for more general ST theories with a term $F(\phi_{\text{JBD}})\dot{R}$, $\varpi(\phi_{\text{JBD}}) = H(\phi_{\text{JBD}})\phi_{\text{JBD}}$, and $V(\phi_{\text{JBD}}) \neq 0$ where $F, H, V$ are general functions.

BS in JBD theory in Jordan frame, $\varpi = 400$, $V = 0$: More physical investigations for such BSs at different cosmic epochs are carried out in [294]. It results that BSs can be stable at any time of cosmic history and that equilibrium stars are denser in the past applying the tensor mass. It is shown that the radius corresponding to the critical BS mass remains roughly the same during cosmological evolution. Several new physical features are displayed: the mass-radius relation, the behaviour of the difference between the central and asymptotic $\phi_{\text{JBD}}$ value, the dependence of the binding energy on the coupling constant $\varpi$.

Charged BS in JBD/ST theory in the Jordan frame, $V = 0$: In 1999, Whinnett and Torres [310] investigated the influence of an additional $U(1)$ charge for such BSs with constant coupling function $\varpi$ and a special choice of power-law ST theory. Solutions are calculated for the following parameter combinations (always $\phi_{\text{JBD},\infty} = 1$): For $\Lambda = 0$ $\varpi = -1, 1, 500, 16\phi_{\text{JBD}}^2/3$, for $\Lambda \to \infty \varpi = 1$. It is found that there is a maximal charge per mass ratio, $q = eM_{\text{Pl}}/m$, above which weak field BS solutions are not stable. For $q = 0$, the mass evolution for four different stable BSs is shown for $\varpi = -1$ and $N = \text{constant}$ if $\phi_{\text{JBD},\infty}$ changes from 1 to 6 (Fig. 8 therein); for different constant charges $q$ the same is done in their Fig. 9. For BSs with constant central density it is demonstrated that the BS mass increases under certain conditions for the BS solutions. Additionally, the effect of spontaneous scalarization is investigated [311].

In 1994-96, two different models for fitting rotation curve data of spiral galaxies were proposed using BS-like objects [275,157,250,251,178,254-256,259]. In [275], solutions with $n = 5$ and $n = 6$ of the Newtonian BS system (2.28), (2.29)
were applied to fit the ripples of spiral galaxy NGC2998 without the contribution of visible matter. The influence of visible matter was included in [157] for NGC2998 and NGC801 by assuming some functional matter distributions for the bulge and the disk. In [178], the rotation velocities for a BS with \( \Lambda = 0 \) and \( \Lambda = 300 \) for \( n = 9 \) are shown without fitting rotation curve data. In all three papers, just the Newtonian rotation velocity formula \( v^2(r) = M(r)/r \) was applied, i.e. the non-negligible influence of the radial pressure of the anisotropic BS matter was ignored; the correct formula [255] reads

\[
v^2(r) = \frac{M(r)}{r} + \frac{\kappa}{2} p r(r) r^2 e^{\lambda(r) + \nu(r)}. \tag{2.51}
\]

In Refs. [250,251], Newtonian solutions for the massless case of the Lagrangian (2.1) were investigated. Later these solutions were designated as boson halos [255]. It was shown [250] that the boson halo mass increases linearly with radius so that the rotation velocity is constant with distance; both oscillate slightly, e.g. the rotation velocity around a constant value as

\[
v^2(r) = |\Phi(0)|^2 \left[ 1 - \frac{\sin(2\omega r)}{2\omega r} \right]. \tag{2.52}
\]

In [251], massless self-interacting complex scalar fields show a similar behaviour. However, for too strong gravitational fields, independent on the self-interaction constant, a singularity forms at the centre; cf. [199] where also the case of a massless conformally coupled complex scalar field with \( U = R|\Phi|^2/6 \) leads to a linearly increasing mass.

A more detailed investigation of the boson halos included the influence of the visible HI gas and star matter [254–256]. The spiral and dwarf galaxies fitted in [255] belong to a sample of 11 galaxies of different Hubble types and absolute magnitudes which fulfil several strong requirements [26]. All rotation curve data are measured in the 21-cm line of neutral hydrogen, so that the gas distribution extends far beyond the optical disc (at least 8 scale lengths) and the necessity of dark matter is becoming obvious. Isolation of galaxies are another constraint so that perturbative effects of nearby situated galaxies are negligible. For dwarf galaxies, the dark matter density near the centre is almost constant, i.e. dark matter has a core in these galaxies. The Newtonian solutions of [255] for a massless complex scalar field reveal a constant density near the centre as well, and so, good fits for dwarf galaxies could be obtained. In all fits for dwarf galaxies, it is recognisable that the dark matter halo dominates the luminous parts of the galaxies. Especially the maximum (and the following drop) of the DDO154 data can be matched perfectly by the boson halo; i.e. decreasing rotation curve data, near the end of observational resolution, can be explained by using a dominating dark matter component. The sample of rotation curve data of spiral galaxies are fitted as well as the universal rotation curves of [230]. Furthermore, oscillating behaviour within the optical rotation curves of some galaxies can be found, e.g. in NGC2998, which can be fitted by a dominating boson halo part. The general relativistic configurations for the boson halo reveal very large gravitational redshifts (calculated up to a value of 20). In [17], it is shown that boson halos are stable.

Recently, it has been shown that BS solutions with a soliton type potential, \( U = m^2 |\Phi|^2 (1 - \lambda_0 |\Phi|^4) \), can be used as well to fit rotation curve data of spiral and dwarf galaxies [216,217]. Further investigations on boson halos with a massive complex scalar field derive a scalar field mass of about \( 10^{-23} \text{eV}/c^2 \), if a self-interaction is included the scalar field weighs about \( 1 \text{eV}/c^2 \), cf. [8,9]. Cosmological consequences are considered in [135,240]. A prediction on core radii from a formation scenario is given in [235].

In a different approach [75,76], the dark matter part consists of a gas of massive bosonic particles, and a total of 36 galaxies were fitted including also a contribution for the baryonic matter. The best fits for all galaxies is achieved for a mass of about 60-70 eV/c^2 of the bosonic particles.

N. Classical Klein-Gordon particle

Conventionally, a BS turns out to be an object of a huge number \( N \) of particles, but there are also some attempts to construct a BS with just one particle [91,236]. Feinblum and McKinley [91], shortly before the work of Kaup, investigated the mini-BS case with two constraints, namely the normalisation of both the particle number \( N = \omega \) and of the mass \( M = \omega^2/m \). This leads to exact one ground state particle solution, possessing a vanishing binding energy by construction. The authors had numerical problems to find a normalised ground state; this may be due to the chosen mass value, \( m^2 = 0.1 \), which seems not to be the appropriate value of the single ground state, as Kaup has pointed out [159]. We may add that, in flat spacetime, the KG equation does not permit a one particle interpretation, as is well-known. In geometrodynamics, there are different approaches to describe elementary particles [308,200,201,205,207].
In 1994, Rosen [236] constructed particle-type solutions for the linear KG equation which he called *Klein-Gordon particles*. The first constraint is that the total mass is equal to the particle mass, i.e. \( M = m \). The second constraint is rather ad-hoc, namely \( m = \omega \). Furthermore, it is assumed that these particles have a clear surface where the scalar field is zero and the vacuum Schwarzschild metric starts to be present, similarly as for neutron stars. The mass of the particles are in the order of the Planck mass. In a follow-up paper [236], Rosen constructs *Proca particles* with the same conditions as above. Due to an oscillating electric potential, necessarily also an oscillating magnetic vector potential is needed for this construction. In Ref. [199], it was noted that the radial pressure for the KG and the Proca particle does not vanish at the surface, thus these solutions are unstable.

**O. D-dimensional boson stars**

Whereas in four dimensions only numerical BS solutions are known, a complex scalar with large self-interaction constant \( \Lambda \gg 1 \) allows analytical solutions in \((2+1)\) dimensions. Spherical symmetry reduces in \((2+1)\) dimensions to circular symmetry. Such circular BSs in a spacetime with negative cosmological constant exist and can be stable as long as the absolute value of the negative cosmological constant is lower than some critical one, otherwise all BSs have positive binding energy and are not energetically favourable. This result was derived for rotating BSs as well, and for non-rotating boson-fermion stars [241–243]. Furthermore, the rotating BS shows a vacuum hole at its centre, similarly as for neutron stars. The mass of the particle does not vanish at the surface, thus these solutions are unstable. Due to an oscillating electric potential, necessarily also an oscillating magnetic vector potential is needed for this construction. In Ref. [199], it was noted that the radial pressure for the KG and the Proca particle does not vanish at the surface, thus these solutions are unstable.

**P. Gauge boson solitons**

There exists a huge amount of papers on self-gravitating *SU(n)*-solutions, i.e. for Einstein-Yang-Mills theories, and combinations with other matter fields [25,41,42,302]. These globally regular solutions can be regarded as *gauge BSs*, cf. [249], except that they are unstable [278,43]. Abelian fields occur in the *gravitational-electromagnetic entities* (geons) of Wheeler [307]; cf. [6]. If the gravitational influence is weakened so much that even Newtonian gravity can be neglected, non-topological or topological solitons are obtained [97,179], cf. [4], or for cylindrical symmetry, — in a way — thread ‘BSs’ can be found [221,222].

**Q. Axidilatons from effective string models**

Commonly, in four-dimensional *effective* string theories [68], the tensor field \( g_{\mu \nu} \) of gravity is accompanied by several scalar fields. Besides the familiar dilaton \( \phi \), the 'universal' axion \( a \), a pseudo-scalar potential for the Kalb-Ramond (KR) three-form \( H := e^{\varphi/1'} da \) arises. Through spontaneous compactification from ten dimensions onto an isotropic six torus of radius \( e^\beta \), a further modulus field \( \beta \) emerges.

In the string frame \( \bar{g}_{\mu \nu} = e^{-\varphi/1'} g_{\mu \nu} \), the *effective* string Lagrangian reads

\[
L_{\text{eff}} = \sqrt{-g} e^{-\varphi/1'} \left( \frac{R}{2\kappa} + \bar{g}^{\mu \nu} \left( \partial_\mu \varphi \partial_\nu \varphi - 6 \partial_\mu \beta \partial_\nu \beta - \frac{1}{2} e^{2\varphi/1'} \partial_\mu a \partial_\nu a \right) - \frac{1}{2} e^{2\varphi/1'} U(\varphi,a) \right),
\]

**cf. Eq. (11) of Dereli et al. [79].** Via conformal change [202] of the metric \( g_{\mu \nu} \to \bar{g}_{\mu \nu} = \exp(\varphi/\sqrt{2\kappa}) g_{\mu \nu} \), we can go over to the Einstein frame. Thereby, the kinetic dilaton term changes sign and allows us to formally combine [272] the axion and the dilaton into a *single complex* scalar field

\[
\Phi := a + if(\varphi) e^{-\varphi/1'},
\]

the *axidilaton*. For this *complex* scalar field the Lagrangian reads

\[
L_{\text{eff}} = \sqrt{-g} \left( \frac{R}{2\kappa} - g^{\mu \nu} \left( \frac{1}{2} e^{2\varphi/1'} \partial_\mu \Phi \partial_\nu \Phi^* + 6 \partial_\mu \beta \partial_\nu \beta \right) - \frac{1}{2} e^{2\varphi/1'} U(\varphi,a) \right).
\]
The emerging self-interaction has, in lowest order, a mass term
\[ U(\varphi, a) := e^{2\varphi/b\epsilon} U(\varphi, a) \approx m_a^2 |\Phi|^2, \]  
(2.56)
which exhibits residual \(U(1)\) symmetry, resembling symmetry restoration. Thus, the \(U(1)\) sector of this effective string model is invariant under the global Noether symmetry \(\Phi \rightarrow e^{i\alpha} \Phi\) which allows us to establish again the global stability [171, 172] of the star.

All the occurring masses \(m\) and decay constants \(f\) of the bosonic particles are related via \(m_\varphi f_\varphi \approx m_\pi f_\pi \approx 10^{16} (eV)^2\) to those of the pion. Whereas the mass of the dilaton \(\varphi\) is conventionally related [70] to the supersymmetry breaking scale \(m_{\text{SUSY}}\) by \(m_\varphi \approx 10^{-3} (m_{\text{SUSY}}/\text{TeV}/c^2)\ eV/c^2\), this is not the only possibility. In the context of string cosmology [105], massless axions are able to seed the observed anisotropy of the Cosmic Microwave Background (CMB). The same holds for the KR axion, provided it lies in an ultra-low mass window. Besides the full dilaton interaction for the bosons as mentioned above, we could consider a very light dilaton \(\varphi\) which is stabilised [81, 82] through the axion.

Instead of combining the dilaton with the axion, one could also take recourse to the other moduli field and consider, e.g., the complex Kähler form field \(\hat{\Phi} := \varphi + i\sqrt{12}\beta\) akin to T-duality [83], without including the axion.

Are MACHOs axidilaton stars? In Ref. [213, 214], we have proposed that the weakly interacting dilatons and axions, after a partial Bose-Einstein condensation to an axidilaton star (ADS), account also for some fraction of the MACHOs (MASSive Compact Halo Objects) which are dark matter candidates, almost certainly detected by gravitational microlensing [281], though still with statistical ambiguities [118]. Since the total mass of an ADS is not well constrained, one could phenomenologically turn the argument around: By identifying the MACHOs of gravitational mass of about \(0.6 M_\odot\) (as given by microlensing observations) with the critical mass of an ADS, we are essentially “weighing”, via \(M_{\text{Kauf}}/M_{\text{crit}} \approx m\), the constituent mass to be of the order \(m_a \approx 10^{-10} \text{ eV}/c^2\). It is encouraging to note that such an ultra-low mass is perfectly compatible with the constraints on the mass range of the KR axion seeding the large-scale CMB anisotropy, cf. the recent results of Gasperini and Veneziano [105] within low-energy string cosmology.

If ADSs during their evolution would accumulate a larger mass, they can start to oscillate [171] and thereby get rid of some excess mass due to ‘gravitational cooling’ [269]. Repeated accretion leads ultimately to a BH in the upper range of the MACHO mass. These axion induced BHs [271, 104] do not carry scalar hair, which could serve to distinguish them from primordial BHs, but could have some remnant of P-violation or even “axion hair” [84], if \(a\) is interpreted as a superpotential for the KR axial torsion. In the pre-big bang scenarios, the dilaton would produce a detectable gravitational wave background [61].

Therefore, if such string-inspired scalar fields would exist in Nature, axions could not only solve the non-baryonic dark matter problem [298], but their gravitationally confined axidilaton stars would also represent the MACHOs which presumably account for about 20\% of the halo dark matter in our Galaxy.

R. Quantised complex scalar field boson star

Field quantisation of a complex scalar has been studied in [117] in the context of BS formation. In Ref. [132], a BS with a field quantised complex scalar has been introduced.

As is well-known, the complex scalar \(\Phi\) can be regarded as two real fields, namely \(\phi_1 = (\Phi + \Phi^*)/\sqrt{2}\) and \(\phi_2 = i(\Phi - \Phi^*)/\sqrt{2}\). Then, field quantisation results in the operator expansion
\[ \phi_j = \sum_{nla} b_{nl\alpha}^{(j)} R_l^{(n)}(r) Y_l^{(a)}(\theta, \varphi) e^{-i\omega_n t} + b_{nl\alpha}^{(j)*} R_l^{(n)*}(r) Y_l^{(a)*}(\theta, \varphi) e^{+i\omega_n t}, \]  
(2.57)
where \(j = 1, 2\). For a bound state, \(R_l^{(n)}(r)\) are radial distributions as generalisations of the wave function of the hydrogen atom, \(Y_l^{(a)}(\theta, \varphi) = (1/\sqrt{4\pi}) P_l^{(a)}(\cos \theta) e^{-i|\alpha|\varphi}\) the spherical harmonics given in terms of the normalised Legendre polynomials, and \(|\alpha| \leq l\) are the quantum numbers of azimuthal and angular momentum. The non-vanishing commutation relations for the bosonic creation and annihilation operators are
\[ [b_{nl\alpha}^{(j)}, b_{nn'\alpha'}^{(j)*}] = \delta_{nn'} \delta_{ll'} \delta_{\alpha\alpha'}. \]  
(2.58)
In general, there are two number operators \(N_j^{(j)} := b^{(j)} b^{(j)*}\). For the ground state of a cold BS, if \(N_1 = N_2\), we have \(|N\rangle = |N, 0, 0, \ldots\rangle := \prod_{j=1}^{\infty} b_{100}^{(j)}|0\rangle = \prod_{j=1}^{\infty} b_{12j}^{(j)}|0\rangle\), where \(|0\rangle\) is a vacuum state in the curved spacetime ‘background’; cf. Section IV A.
In Ref. [132] an attempt was made to motivate symmetry breaking in a model with $|\Phi|^4$ self-interaction and self-gravity. Three different BS models lead to three different effective coupling constants. The BS with a not quantised complex scalar [58] has simply $\Lambda/2$. The BS model with a field quantised real scalar produces $3\Lambda/4$; and the BS model with a field quantised complex scalar has $5\Lambda/8$. All three models results in the limit $\Lambda \gg 1$ to small changes in the critical mass $M_{\text{crit}} \sim K\sqrt{\Lambda}M_{\text{Pl}}/m$, where $K = 0.22, 0.225,$ and $0.223$, respectively.

III. HOW TO DETECT COMPLEX SCALAR FIELD BOSON STARS

In this section, we speculate on observational consequences for the spherically symmetric BS model given by (2.1), if not mentioned differently. In several cases, we shall recognise the important influence of the scalar field potential. Under our assumption that the complex scalar field has no interaction with baryonic matter other than gravitationally, the BS is transparent, i.e. baryonic matter can accumulate up to the centre of the BS or photons and particles can move through the BS interior, respectively.

To detect a BS depends on the physical situation; there is a corresponding wave-band where the BS might be very luminous. In the following, we shall discuss several of such scenarios.

A. Rotation curves

It might be possible to detect a BS in X-rays. Imagine a massive BS, say of $10^6 M_\odot$, for which it is likely that an accretion disk forms and since its interior solution is asymptotically Schwarzschild, nearly beyond its effective radius the BS looks similar to an Active Galactic Nucleus (AGN) where usually a black hole (BH) at the centre is assumed. In X-rays [252,258,264], one can probe close to the Schwarzschild radius or, as we propose, even inside the BS; Iwasawa et al. [139] claimed that by using data from the satellite ASCA they have probed to within 1.5 Schwarzschild radii; cf. [177,197,320]. Because this is inside the static limit for a non-rotating BH, they conclude that a Kerr geometry is required. A BS configuration could still be an alternative explanation, giving a non-singular solution where emission can occur even from the centre.

In order to differentiate between a BS and a BH, it is interesting to calculate the tangential velocities of baryonic matter that may rotate around the centre of a BS. For the static spherically symmetric metric (2.7), geodesics of a collisionless circular orbit obey

\[ v_\phi^2 = \frac{1}{2} \nu^2 e^\nu = \frac{1}{2} e^{\nu(+\lambda-1) + \frac{\kappa}{2} p_r r^2 e^{\lambda+\nu}} \simeq \frac{M(r)}{r} + \frac{\kappa}{2} p_r r^2 e^{\lambda+\nu}, \]

which reduces for a weak gravitational field into the Newtonian form $v_{\phi,\text{Newt}}^2 = M(r)/r$ if $p_r = 0$. Rotation curves for the cases $\Lambda = 0, 10$ and 300 were calculated for a critical mass BS in [252,258]. The curves increase from zero at the centre up to a maximum followed by a Keplerian decrease. The maximal rotation velocities reaches still inside the BS more than one-third of the velocity of light; similar results attain for the potentials $U_{CG}, U_{SG}, U_1$ (Fig. 8 of [262]). For a BS in Newtonian approximation, the maximal circular velocity can be just about 2000 km/s and this at more than 50 times of its Schwarzschild radius, but still inside the BS. One can conclude that large rotation velocities $v_\phi$ are not necessarily signatures of BHs. Baryonic matter rotating with the maximal velocity of about $c/3$ possesses an impressive kinetic energy of up to 6% of its rest mass. If one supposes that each year a mass of $1 M_\odot$ transfers this amount of kinetic energy into radiation, a BS would have a luminosity of $10^{44} \text{erg/s}$.

B. Gravitational redshift

If baryonic matter inside a BS emits or absorbs radiation within its gravitational potential, the spectral feature is redshifted [252,258,264]. The gravitational redshift $z_g$ within a static background is given by $1 + z_g := \sqrt{e^{\nu(R_{\text{int}})} / e^{\nu(R_{\text{int}})}}$, see e.g. Ref. [219], where emitter and receiver are located at $R_{\text{int}}$ and $R_{\text{ext}}$, respectively. The maximal possible redshift for a given configuration is obtained if the emitter is exactly at the centre $R_{\text{int}} = 0$. The receiver is always practically at infinity, hence $\exp[\nu(R_{\text{ext}})] = 1$. For all other redshifts in between, the gravitational redshift function is defined as

\[ 1 + z_g(x) = \exp\left( -\frac{\nu(x)}{2} \right). \]
Table 1 in [252] gives the results for the critical mass BS for different $\Lambda$. The redshift lies between 0.0657 for $\Lambda = -20$, 0.4565 for mini-BS with $\Lambda = 0$, and 0.6873 for the limit case $\Lambda \gg 1$. With increasing self-interaction coupling constant $\Lambda$ also the maximal redshift grows. Unstable BSs presumably may have arbitrary gravitational redshift. In Ref. [262], for the cosh-Gordon potential $U_{CG}$, the maximal redshift is found to be 0.46, for the $U(1)$-Liouville potential $U_L$, it is $z_{\text{max}} = 0.49$. For a neutron star, Markov [196] derived $z_{\text{max}} = 0.49$. In general, observed redshift values would consist of a combination of cosmological and gravitational redshifts given by $(1 + z) = (1 + z_c)(1 + z_g)$ due to the addition formula of relativistic velocities.

C. Gravitational lensing

In this section, the results of gravitational lensing of a transparent spherically symmetric mini-BS are shown [65,263]. In contrast to the last sections, it is assumed that the BS interior is empty of baryonic matter, so that deflected photons can travel freely through the BS. The deflection angle [304] is then given by

$$\hat{\alpha}(r_0) = 2 \int_{r_0}^{\infty} \frac{b e^{\lambda/2}}{r \sqrt{r^2 e^{-\nu} - b^2}} dr - \pi,$$

(3.3)

where $b$ is the impact parameter and $r_0$ denotes the closest distance between a light ray and the centre of the BS. The lens equation for small deflection angles can be expressed [218] as

$$\sin (\vartheta - \beta) = \frac{D_{ls}}{D_{os}} \sin \hat{\alpha},$$

(3.4)

where $D_{ls}$ and $D_{os}$ are the distances from the lens (deflector) to the source and from the observer to the source, respectively. The true angular position of the source is denoted by $\beta$, whereas $\vartheta$ stands for the image positions. One usually defines the reduced deflection angle to be $\alpha \equiv \vartheta - \beta = \sin^{-1} \left( \frac{D_{ls} \sin \hat{\alpha}}{D_{os}} \right)$. However, equation (3.4) relies on substitution of the distance from the source to the point of minimal approach by the distance from the lens to the source $D_{ls}$, see [218]. For large deflection angles the distance $D_{ls}$ cannot be considered a constant but it is a function of the deflection angle so that the form of the lens equation changes [65] into

$$\sin \alpha = \frac{D_{ls}}{D_{os}} \cos \vartheta \cos \left[ \arcsin \left( \frac{D_{os}}{D_{ls}} \sin (\hat{\alpha} - \alpha) \right) \right] \left[ \tan \vartheta + \tan (\hat{\alpha} - \vartheta) \right].$$

(3.5)

The reduced deflection angle can be kept defined as $\alpha \equiv \vartheta - \beta$.

Numerical computations of the reduced deflection angle for different BS potentials $U_K, U_{CG}, U_{SG}, U_L$ were performed by assuming that the BS lens is half-way between the observer and the source, i.e. $D_{ls}/D_{os} = 1/2$ (for details, see [65,262]). Observable differences are found depending on the choice of the self-interaction. In the case of a simple mass term $U_K$, the largest possible value of $\alpha$ is 23.03 degrees with an image at about $\vartheta = n \times 2.88$ arcsec where $n = n(D_{ol}, \omega)$ is the distance factor which is a function of the distance from the observer to the lens and the scalar field frequency the inverse of which is associated with the BS radius. For non-relativistic BS approximations, smaller angles will occur. The angle $\vartheta$ of the image position can have very different orders of magnitude, depending on $n$. For example, $n = 1$ fixes $\vartheta$ to be measured in arcsec. Under the assumption that the BS mass is $10^{10} M_\odot$, the distance $D_{ol}$ is about 100 pc. If the distance factor is $n = 10^{-3}$, then $\vartheta$ is measured in milli-arcsec and the BS-lens is at about 100 kpc from the observer.
FIG. 2. Reduced deflection angle for a mini-BS of critical mass. The angle $\vartheta$ for the image positions is given in units of arcsec times the distance function $n$ [65].

In Ref. [65], it is summarised that the BS has all qualitative features of a non-singular spherically symmetric transparent lens, cf. Fig. 2, the lens curve for a mini-BS of critical mass. Three images can be observed, two of them being inside the Einstein radius and one outside. An Einstein ring with infinite tangential magnification, the so-called tangential critical curve, is found, and also a radial critical curve for which two internal images merge. The appearance of the radial critical curve distinguishes BSs from other extended and non-transparent lenses [246]. For a BH or a neutron star, the radial critical curve does not exist because it is inside the event horizon or the star, respectively. Two bright images near the centre of the BS and the third image at some large distance from the centre are found. If an extended transparent source like the BS is considered, one can imagine how to detect the BS. In such a case, two images, one radially and the other tangentially elongated and both close to each other, might be observed. By looking along the line defined by these two images, the third one can be detected at a large distance.

For the Cosh-Gordon potential $U_{CG}$ [262], the maximal reduced deflection angle is $23.229$ degrees and for $U_L$ an even larger deviation, namely $24.391$ degrees, with an image at about the same place is found. Differences among all four mentioned potentials are up to 4%.

Discussions of lensing effects of different objects can also be found [239,169].

D. Gravitational microlensing and MACHOS

A statistical analysis of the galactic halo via microlensing [226,281] suggests that MACHOs (massive compact halo objects) account for a significant part (> 20%) of the total halo mass of our galaxy. Their most likely mass range seems to be in the range $0.3 - 0.8 \, M_\odot$, with an average mass of $0.5 \, M_\odot$, cf. [155,281]. If the bulge is more massive than the standard halo model assumes, the average MACHO mass [155] will be somewhat lower at $\sim 0.1 \, M_\odot$. However, there are some astrophysical difficulties with this result of estimated $\sim 0.5 \, M_\odot$ for the lenses. These cannot be hydrogen-burning stars in the halo since such objects are limited to less than 3% of the halo mass by deep star counts [114]. Modifying the halo model to slow down the lens velocities can reduce the implied lens mass somewhat, but probably not below the sub-stellar limit 0.08 $M_\odot$. Old white dwarfs have about the right mass and can evade the direct-detection constraints, but it is difficult to form them with high efficiency, and there may be problems with overproduction of metals and overproduction of light at high redshifts from the luminous stars which were the progenitors of the white dwarfs [49]. Primordial BHs are a viable possibility, though the coincidence has to be explained to have them in a stellar mass range.

Due to these difficulties of getting MACHOs in the inferred mass range without violating other constraints, there was the suggestion that BSs could be the explanation [213]. The explicit physical values for a mini-BS has been exhibited in Fig. 3 with a scalar field mass of $10^{-10} \, eV/c^2$. 

\[
\vartheta[n(D_0), \omega] \text{ arcsec}
\]

\[
\alpha(0)=0.271
\]
FIG. 3. Left: Mass $M$ and particle number $N$ (or rest mass $mN$ at infinity) of a mini-BS depending on the central density $\rho$. Right: Mass-radius dependence of a mini-BS.

At the left-hand side of Fig. 3, the dependence of the mass $M$ and the particle number $N$ (or rest mass $mN$) on the central density $\rho$ is shown. Stable non-rotating BSs exist at the lower central densities below the maximum mass. The critical values are $M_{\text{crit}} = 0.846 \, M_\odot$ and $mN_{\text{crit}} = 0.873 \, M_\odot$ for a central density of $\rho_c = 9.1 \times \rho_{\text{nucl}}$, where $\rho_{\text{nucl}} = 2.8 \times 10^{17} \, \text{kg/m}^3$ is the average density of nuclei. Since non-interacting bosons are very “soft” (due to Bose-Einstein condensation), BSs are extremely dense objects with a critical density higher than comparable ones of neutron or strange stars [110]. The figure on the right-hand side gives the mass depending on the radius. For the mass-radius diagram, 99.9% of the total mass was chosen as effective radius. This ensures that the exponentially decreasing exosphere of the BS has almost no influence on the asymptotic Schwarzschild spacetime.

Stable BSs have radii larger than the minimum at 20.5 km with a mass of 0.846 $M_\odot$. The choice for the boson mass $m$ yielded that the total mass of these relativistic BSs is just in the observed range of 0.3 to 0.8 $M_\odot$ for MACHOs.

As was the case in the discussion of axidilaton stars (Section II Q), one could turn this argument around: By identifying the most massive MACHOs with known gravitational mass of about 0.8 $M_\odot$ as BS, one is essentially ‘weighing’, via $M_{\text{Kaup}}/N_{\text{crit}} \approx m$, the boson mass to $m \sim 10^{-10} \, \text{eV}/c^2$.

### E. Čerenkov radiation from boson stars

As in Section III C, we assume that the BS interior is empty, but now electrically charged particles move fast through the gravitational BS field. In general, if a particle moves through a medium with a constant velocity greater than the velocity of light in that medium then Čerenkov radiation occurs. Because of the superluminal particle motion, a shock wave is created and this yields a loss of energy.

Accelerated charged particles emit electromagnetic radiation, but acceleration only by a gravitational field does not produce Čerenkov radiation, in agreement with the equivalence principle. There, the acceleration induced by gravity disappears in the local inertial frame (further details in the references of [47]). Despite that, the gravitational BS field can act as a medium with an effective refractive index $n_\gamma$ for light if particles move with constant velocity through it [120]. It is found

$$n_\gamma^2(k_0) = 1 - \frac{1}{k_0^2} R^i_i,$$

where $k_0$ is the frequency of the emitted photon $\gamma$ and $R^i_i$ is the sum on the spatial indices of the Ricci tensor. Čerenkov radiation is now kinematically allowed if $R^i_i < 0$, i.e. $n_\gamma^2(k_0) > 1$.

Following [120], it has been investigated in [47] whether BS models allow Čerenkov radiation. It was found that stable mini-BSs and stable BSs cannot generate Čerenkov radiation, whereas unstable BSs can. However, non-topological soliton stars do allow this effect where the position and the strength depend on the central scalar field value and the potential parameter $\Phi_0$. Then, Čerenkov radiation appears in spherical shells and only there, but this can be everywhere inside the star.

Again, we recognise how important the influence of the scalar field potential on detection can be.
F. Gravitational waves

In the early stage of BS formation, highly excited configurations are expected to exist in which the quantum numbers \( n, l \) and \( a \) of the gravitational atom, i.e. the number \( n-1 \) of nodes, the angular momentum and the azimuthal angular dependence \( e^{-ina} \) are non-zero; cf. [148]. For energetic reasons, these excited BSs are presumably only meta-stable, all initially higher modes during the BS formation have eventually to decay into the ground state \( n = 1, l = 0 \) (with \( a = 0 \) for non-rotating and \( a \neq 0 \) for rotating BSs) by a combined emission of scalar matter radiation and gravitational radiation.

If we assume a BS to be in an excited state with \( l = a = 0 \), we know that its mass and particle number depend linearly on the number of nodes \( n-1 \), cf. [98,29]. Since these states have zero quadrupole moment, the only decay channel is scalar matter radiation. For the transition of an excited BS with only \( n > 1 \) into the ground state, it is reckoned [92] that the energy released by scalar radiation is about

\[
E_{\text{rad}} \sim (n-1)M_{\text{Pl}}^2/m, \quad \Delta N \sim (n-1)(M_{\text{Pl}}/m)^2,
\]

where the loss of bosonic particles is given by \( \Delta N \).

Ferrell and Gleiser [92] considered the situation of a Newtonian BS that has the main part of particles in the 1s state and only a small number in the 3d (0.286 parts per thousand). They estimated the amount of total energy released by the transition from this small number of particles in the 3d into the 1s state. The lowest mode which has quadrupole moment and therefore can radiate gravitational waves is the 3d state with \( n = 3 \) and \( l = 2 \). For the \( \Delta n = 2, \Delta l = 2 \) transition, the BS will decay into the 1s ground state with \( n = 1 \) and \( l = 0 \), and thereby the gravitational radiation process preserves the particle number \( N \). Even though only a relative small number of particles populates the 3d state, the radiated energy is quite large [92], i.e., \( E_{\text{rad}} = 2.9 \times 10^{22} \ (\text{GeV}/\text{mc}^2) \) Ws. Highly relativistic BSs with radius near the Schwarzschild one (\( M/R \sim 1/G \)) could have a maximal power of \( 10^{52} \) W, but this is just a theoretical upper bound on the gravitational radiation of a self-interacting system. In the scenario of [92], it is concluded that the excited state will decay extremely fast to the ground state with the exponential decay time of about \( 1.1 \times 10^{-16} \ (\text{mc}^2/\text{GeV}) \) s. Thus, the final phase of the BS formation would terminate in an outburst of gravitational radiation despite the smallness of the object, cf. Table IV.

In [72], the collapse scenario of parts of a charged boson cloud with \( \omega = m \), i.e., just above the eigenfrequencies of excited CBSs, is investigated. In comparison with the results above, gravitational waves are created in a shorter time interval (\( 10^{-25} \) s) and with much less power (\( 4 \times 10^{14} \) W).

The issue of non-radial pulsations of a BS was mathematically formulated [166], whereas quasinormal modes are explicitly calculated in [316]. The interest in non-radial pulsations arises from the fact that they transport gravitational waves away from the BS. For linearised perturbations around an equilibrium BS, odd and even parity modes can be distinguished. It was shown that the odd parity modes do not couple to gravitational waves as well as for the stellar pulsation of a perfect fluid. This mode type manifests the propagation of the gravitational wave through the star, it does not change the star’s density or pressure distributions. With a different analytical approach, the even-parity quadrupole oscillations (\( l = 2 \) modes) which do couple to gravitational waves were investigated numerically in [316]. These quasinormal modes describe oscillations with emitted gravitational and scalar matter radiation (but no incoming radiation). In contradistinction to relativistic fluid stars, the imaginary parts of the frequencies are large, i.e. the \( f \) and \( p \) modes imply very short damping time scales. It is also found that the amplitude of the metric perturbation is comparable to the one of the scalar field, again different to the \( u \) modes of an ordinary star. No weakly damped modes are present within the BS. One restriction of the numerical calculations in [316] is connected with the definition of the BS surface. Due to numerical reasons, a clear surface had to be defined, but a BS has only an exponentially decreasing exosphere; this has a crucial influence on the values of the eigenfrequencies; an introduction to pulsating relativistic stars can be found in [167].

Future gravitational-wave detectors may have also different ways of searching for BSs. If a compact object of, say, a solar mass is observed to be spiralling into a central one with a much larger mass, one would be able to test the larger object. From the emitted gravitational waves, the values of the lowest few multipole moments of the central object can be extracted, like mass \( M \), angular momentum \( J \), mass quadrupole moment \( M_2 \). For a BH as central object, the no-hair theorem states that all its multipole moments are uniquely determined by the two lowest ones, e.g. \( M_2 = -J^2/M \). In [238] \( M, J, M_2 \) and the spin octopole moment \( S_3 \) are calculated for rotating BSs in distinction to a BH. For example, if one observes the combinations \( J/M^2 = 0.01, -M_2M/J^2 = 24, \) and \( S_3M^2/J^3 = 19 \), then the massive central object would be very likely a rotating BS. If instead, one finds \(-M_2M/J^2 = 1 \) and \( S_3M^2/J^3 = 1 \), then a BH is the likely source, and if one determines \(-M_2M/J^2 = 24 \) and \( S_3M^2/J^3 = 4 \), a different configuration
(BS with different kind of self-interaction, e.g.) may be detected. Further numerical combinations for BSs result from the Figures in [238].

For a rotating BS, co-rotating particles orbiting with circular geodesics in the equatorial plane were investigated in [238]. The particle energy depending on its gravitational-wave frequency for one BS solution is shown there in Figure 6. If particles could travel freely inside the BS, it would, in principle, be possible to map out the interior of the BS via the emitted gravitational waves.

If a BS experiences some kind of perturbation, we may be able to measure characteristic frequencies. In [123], the mode frequencies for the fundamental mode and the first harmonic mode of radial perturbations of mini-BSs are exhibited in Tables II and III (cf. Figs. 7 and 8 as well); cf. Fig. 6 in [267] where similar frequencies were discovered. Further details of Ref. [123] are mentioned in Section II F.

The output of gravitational waves was considered in [108] for identical BS binary systems, assuming that they are located in galactic halos. However, the gravitational radiation background contains data from ordinary binary systems as well. The estimation for critical BSs gives a contribution to the gravitational radiation background where the amplitude is $h \sim 3 \times 10^{-25} \left(\text{TeV}/mc^2\right)^{3/2}$ and the frequency $\nu \sim 25 \left(\text{TeV}/mc^2\right)^{5/4} \text{Hz}$. This result is valid if the BS binary density is about the critical density and they dominate over ordinary binaries if $m \leq 10^9 \text{TeV}/c^2$.

For mini-BS binaries with scalar field mass $m = 10^{-5} \text{eV}/c^2$, the typical amplitude is $h \sim 10^{-24}$ and the frequency $\nu \sim 4.5 \text{Hz}$, again for a mini-BS binary density of about the critical density.

G. Boson star in the galactic centre?

Several galaxy centres contain $10^6$ up to $10^9$ solar masses. The widely accepted explanation is a supermassive BH or a rich cluster of objects (stars, BHs, e.g.). Already in [252,258], a so-called mini-BS or BS has been proposed as an alternative explanation, cf. Sections III A and III B.

In [295], a supermassive mini-BS as model for the Galactic centre was compared with the observational data available for our Galaxy where, at least, the existence of a star cluster can be excluded [85,86]; cf. [106]. The central object has a mass of about $2.6 \times 10^6 M_\odot$. The more recent observations of [106] probe the gravitational potential at a radius larger than $4 \times 10^4$ Schwarzschild radii of such a BH. The mentioned data consist of the movement of stars around the Galactic centre and, in [295], they were explicitly fitted by a solution of a mini-BS with $m \sim 10^{-17} \text{eV}/c^2$ plus a stellar cluster. The same result can be received by using a BS solution with $m \sim 10^{-4} \text{GeV}/c^2 (\lambda = 1)$ and a non-topological soliton star solution ($\Phi_0 \sim m$) with $m \sim 10^4 \text{GeV}/c^2$. The data can be fitted by a central BH as well and, so far, no distinction can be obtained between a BH and a BS at the centre. The difference starts at a radius more than three orders of magnitude below the innermost data point, in case of a general relativistic mini-BS [295]. A mini-BS in the Newtonian approximation has less mass, but if we reduce the constituent mass $m$ in order to obtain the same central mass, the radius becomes about two orders of magnitude larger (cf. Fig. 5 and Table 2 in [252]). Observations are just one to two order of magnitudes away from finding differences in the BH-BS comparison. Let us also mention that an extended neutrino ball can explain the data [297] so far.

A non-rotating BS may have also the problem of accreting too much matter and so forming a BH eventually. It is obvious that inspiralling interstellar gas and stars will collide with each other, glue together within the BS, and so eventually form a BH. If instead a small BH spirals inwards and stays there, the final state of a non-rotating BS will be without doubts a BH. The investigation of the effective potentials for massless and massive particles of a spherically symmetric mini-BS can give some understanding how infalling matter behaves inside the BS [295]. All particles with orbital angular momentum and unbound orbit do not move through the BS centre, hence, leave the star; only if the orbital angular momentum is zero the particles will move exactly through the centre. In this ideal situation of non-interacting matter, every matter is removed from the BS interior and no BH can form. But, as mentioned above, presumably matter will collide and glue together so that a BH cannot be avoided (i.e., the particles have bound orbits). However, in case of a rotating BS, there is the possibility that matter could be guided along geodesics on the rotation axis forming jets; this has still to be shown. For a BS in the limit $\Lambda \gg 1$, circular orbits are stable until about $5/3$ times the radius of the doughnut hole [298].

On the basis of an effective potential, it is shown [296] that, for a spherically symmetric mini-BS, circular orbits exist for every value of the radial coordinate and are stable, even inside the BS. Hence, accretion could follow a series of stable orbits and finally end up at the centre. Further investigations are needed in order to clarify whether a BH forms.

What happens if the BS under external perturbations (gravitating scalar matter) is forced to leave the equilibrium ground state changing mass and particle number in the configuration? The numerical simulations of [267,269,18,15] show that stable mini-BSs under finite non-infinitesimal perturbations start to oscillate, emit scalar field radiation, and settle down into a new mini-BS configuration with less mass and a larger radius. If the amount of scalar accretion
matter does not exceed a critical value during some time, the BS is not jeopardised to form a BH.

The emissivity properties of a geometrically thin, optically thick, steady accretion disc with constant accretion rate around a supermassive critical mini-BS are demonstrated in \[296\] and compared with a BH of same mass. Optical thickness means that the local effective temperature must be sufficient to radiate away the local energy production. The disc calculations are only valid where the gravitational potential is \(\sim GM/r\), i.e. outside of the investigated mini-BS. One result is that, for the inner disc parts, a BS produces more power per unit area and a hotter disc than the comparable BH; for this critical mini-BS, there is a temperature difference of about two thousand degrees in the inner parts of the accretion disc. For a BS with \(\Lambda = 100\), there is about the same maximal temperature but the position of that maximum is closer to the BS centre. The calculation of the final emission spectrum reveals a difference between the mini-BS and the BH model in the far ultraviolet regime > 10^{16} \text{ Hz} (Fig. 3 in \[296\]); this disc model shows no production of X-ray or gamma-ray energy. The Eddington luminosity, at the balance of the inward force of gravity with the outward pressure of radiation, is not exceeded within the mini-BS interior. Thus, accretion could continue inwards a mini-BS. A further result \[296\] is that the disc model does not describe the Galactic centre \(\text{Sgr A}^*\) neither for a BH nor a BS due to the accretion disc properties itself. It produces several order of magnitudes too much accretion luminosity. In contrast to observations is also the peak of the standard accretion disc broad band spectrum in the infrared.

Additionally, BSs could disrupt stars \[295\]. In general, there is a distance to a central object where the extension of a normal star cannot be neglected and where tidal forces on the star become important. The corresponding so-called tidal radius is defined where \(M/r^3\) is equal to the mean internal energy of the passing star. Estimations show that BHs should have masses below \(10^8 M_\odot\) so that they provoke disruption outside the event horizon. In general, such an event takes place once in about \(10^4\) yrs. Hence, whereas a too massive BH disrupts stars just inside its event horizon, a BS produces this effect visible for observations (inside or outside of its effective radius).

Observations in the x-ray band within the central regions of active galaxies have shown an interesting feature, a Doppler and gravitationally redshifted iron Kα line \[283,89\]. It was claimed \[139\] that the data can be understood as being emitted from within 1.5 Schwarzschild radii near a rotating BH. In \[252,258\], we speculated that a BS geometry may be able to explain the data as well. Recently in \[190\], a detailed numerical analysis of a geometrically thin, optically thick accretion disc around a supermassive critical mini-BS simulated this situation for different inclination angles. It is concluded that the observed line profile of the galaxy MCG-6-30-15 cannot be fitted by a spherically symmetric mini-BS or BH for the chosen accretion disc model. For a possible identification of such a non-rotating mini-BS, the characteristic spectral feature would be the detection of an intense double peak for face-on galaxies or, for larger inclination angles, of several peaks of the long drawn-out Kα line.

H. Boson star inside a HII cloud?

BSs could exist in two more scenarios as it was pointed out in \[252,258\]:

Within host galaxies of quasars, bright HII regions are observed \[14\]. By imagining that neutral H I gas clouds could gain kinetic energy in the gravitational potentials of BSs, an excited hydrogen gas cloud, i.e. HII, results.

The second situation is that a light BS is completely contained within a nuclear burning star. In this case, both stellar structures would be interacting mutually, especially changing the model parameters of the conventional star. This resembles the so-called Thorne-Zytkow object, i.e., a neutron star core inside a supergiant \[286\]. If instead the BS matter has some kind of interaction with high energetic photons, it might be possible that the BS dissociates and the remaining scalar particles circulate freely within the conventional star. A so-called cosmion represents such a scenario and tries to explain the solar neutrino problem \[90\].

I. Gravitational memory and the evolution of boson stars

The evolution of conventional nuclear burning stars follows the Hertzsprung-Russell diagram of the luminosity as a function of the temperature. During its lifetime, such a star has different stages which are expressed by a line in this diagram. There is the possibility that a BS could go through a continually evolution as well, if the gravitational attraction changes during the evolution of the Universe.

In JBD or scalar tensor theory, a real scalar field regulates the strength of the gravitational force, cf. Section II L. Suppose that at the time of BS formation, the gravitational strength was different from the one prevailing today; following this theory, \(G\) was larger in the past. The real scalar field is actually a radial function so that the gravitational strength inside the JBD-BS is determined by a mutual gravitational interaction of the complex and the real scalar field matter. These investigations are outlined in Section II L. From the observational side, it may be important that,
inside the BS, still the strength of the gravitational constant at formation time could be effective [293]. According to
this gravitational memory effect, the BS is book-keeping the evolution of $G$. Because the JBD field is a radial function
and changes its value at infinity, there should be a repercussion on the BS. But if this change is much slower than
the cosmological evolution of $G$, the star is practically static. Due to this effect, physical properties like the radius
depend on formation time and could be different even for the same total mass.

If instead the JBD field adapts quickly inside the JBD-BS as changes occur at infinity, the BS evolves as well.
Figure 1 in [294] shows the mass against the value of the real scalar field at infinity for constant particle number. It is
derived that the JBD-BS mass increases with increasing $\phi_{\text{JBD}}(\infty) = 1/G$, i.e. with time. One can understand that an
increasing JBD field pumps energy into the BS and increases the mass. Thereby, the evolution of a BS in JBD theory
demonstrated in this Figure 1 resembles the evolution in the Hertzsprung-Russell diagram of conventional stars.
The physical properties as mass, radius, and stability is influenced by this repercussion [294], such that, for example, the
mass changes about 2%.

IV. REAL SCALAR FIELD BOSON STARS

In contrast to the complex case, real scalar fields do not possess a conserved charge or particle number. Moreover,
a static configuration $\Phi(r) = P(r)$, i.e. (2.6) for $\omega = 0$, would, in flat spacetime, be unstable due to Derrick’s theo-
rem [80], whereas a superposition of positive and negative frequency states $\phi_1(r, t) = (P(r)e^{-i\omega t} + P(r)e^{i\omega t})/\sqrt{2}$
is time-dependent and implies time-dependence of the energy-momentum tensor and the metric, as well. These configurations are called oscillating soliton stars [268,269] or ‘oscillatons’ [300,301] and are, in general, unstable. Actually, our global stability analysis via catastrophe theory [171,172] has revealed earlier the existence of oscillating BSs, as we will call them in the context of this review. Q-stars are a class of non-topological solitons which are already discussed in detail in [152]; cf. [57,191,270,12,280]. Further interesting configurations are the so-called pulsions [32,33] or oscillons [134], respectively; these are localised, time-dependent, unstable, spherically symmetric solutions to a nonlinear KG equation.

The temperature $T$ is the decisive parameter characterising the thermodynamic equilibrium of such a boson gas of
real scalar particles [187]. The particle density in momentum space is given by

$$n_\Phi = \frac{1}{\exp(E_\Phi/k_B T) - 1}, \quad (4.1)$$

where $E_\Phi = \sqrt{p^2 + m^2}$ is the energy of the particle with momentum $p$ and mass $m$. In the limit of vanishing
temperature, one obtains $n_\Phi = 0$. Hence, a BS consisting of real scalars cannot exist for $T = 0$ and a different method
has to ensure a conserved particle number.

For a massless real scalar [55], analytical solutions of the Einstein-KG system are derived first by Buchdahl [45],
cf. Wyman [314] and Baekler et al. [13]. Wyman showed that all static spherically symmetric solutions are contained
in the two-parameter family of Buchdahl. The common characteristic of all such solutions is that they are singular
or topological non-trivial [163,164]. In [244,245], the ADM mass of such solutions were calculated because the metric
is asymptotically flat and the scalar field vanishes at infinity. Since the latter has a logarithmic singularity at the
centre, for some solutions, the metric is singular as well and these are typical examples for naked singularities.

By including a mass term and a quartic self-interaction, numerical solutions of the Einstein-KG system can be
determined [153]. Again the scalar field possesses a singularity at the origin; alternatively, one finds regular solutions
at the centre, but the singularity is shifted to a finite distance. The dynamical stability investigation used a
variational perturbation method estimating upper bounds for an eigenvalue. The result showed that all such naked
singularities are unstable [153,156]. However, a direct numerical determination of the ground state eigenvalue indicates
perturbatively stable solutions [56].

In the numerical one-parameter-solutions of Ref. [153], the ‘bare’ mass $m$ is the decisive parameter, whereas $\lambda$ is not
important. In the case of complex scalars, two parameters occur: besides $m$, we have the frequency $\omega$, and globally
regular solutions exist. Thus, the parameter $\omega$ in the phase factor is actually responsible for the regularity of the
solutions. This is not surprising because this parameter is proportional to the conserved particle number $N$. This
consideration motivated the construction of the globally regular one-parameter-solutions with non-vanishing $\omega$ and
$m = 0$ in [250,251,254–256]; cf. boson halos in Section II M.
A. Quantised real scalar fields

An assembly of real scalar particles experiencing just a mass term was already considered in 1966 by Bonazzola and Pacini [34]. In this approach, the scalar field is second quantised. Later on, solutions for this model of "systems of self-gravitating" bosons were considered by Ruffini and Bonazzola [237] similarly to the Hartree-Fock atom. Because a real scalar has no anti-particle states, the corresponding KG field in a spherically symmetric spacetime metric can be merely decomposed into positive and negative frequency field operators

\[ \Phi = \Phi^+ + \Phi^-, \]

where

\[ \Phi^+ = \sum_{nla} b_{nl}^{|a|} R_{l}^{|a|} e^{-i\omega_{nl}t}, \]
\[ \Phi^- = \sum_{nla} b_{nl}^{|a|} R_{l}^{|a|} e^{+i\omega_{nl}t} \]

are again generalisations of the wave function of the hydrogen atom for a bound state. Here \( R_{l}^{|a|}(r) \) are radial distributions, \( Y_l^{|a|}(\theta, \varphi) = (1/\sqrt{4\pi}) P_l^{|a|} e^{-i|a|\varphi} \) the spherical harmonics given in terms of the normalised Legendre polynomials, and \(|a| \leq l\) are the quantum numbers of azimuthal and angular momentum. (The nodeless radial field solution \( R_0(r) \) is earlier denoted by \( P(r) \).) The non-vanishing bosonic commutation relations are

\[ [b_{nl}^{|a|}, b_{nl'}^{|a'|}] = \delta_{nn'} \delta_{ll'} \delta_{aa'}. \]

For the ground state of a cold configuration,

\[ |N\rangle = |N, 0, 0, \ldots \rangle := \prod_{i=1}^{N} b_{i00}^{|0|}\]

is chosen [237], where \(|0\rangle\) is a vacuum state in the curved spacetime 'background'. The BS energy-momentum tensor \( T_{\mu\nu}(\Phi) \) becomes now an operator. Hence, for the right-hand side of the Einstein equation (2.2), the vacuum expectation value \( \langle T_{\mu\nu} \rangle := \langle N \rangle : T_{\mu\nu} : |N\rangle \) is calculated for the ground state where \( T_{\mu\nu} : \) denotes normal ordering of the operator products.

An excited state of positive energy can be defined by \(|N, n, l, a\rangle := \Phi^+ |0\rangle\). Such a 'gravitational atom' [92] represents a coherent quantum state, which nevertheless can have macroscopic size and large mass. The gravitational field is self-generated via the mean value of the energy-momentum tensor, but remains completely classical, whereas the real scalar is treated to some extent as operator.

As we noted in the beginning of this Section, there is no conserved particle number for real scalars, and so one needs to introduce the normalisation condition

\[ 2\pi \int_0^\infty r^2 e^{(\Lambda-\nu)/2} [\omega_{nl} R_{nl}(r) + \omega_{n'l'} R_{n'l'}(r)] \, dr = \delta_{nn'} \delta_{ll'}, \]

in order to stabilise the system. Similarly as for complex scalars, the constraint is imposed that the solutions are asymptotically flat at spatial infinity.

Due to the fact that the mean value of the energy-momentum tensor is employed, the same system of differential equations (2.2), (2.3) arises as for the complex case. A later investigation [40] rediscovered these results.

An interesting issue is how a BS reacts to a second-order phase transition in the early Universe. In [132], such a spontaneous symmetry breaking has been investigated by assuming a negative quadratic term in the scalar potential. After the phase transition, the 'bare' mass \( m \) is increased and the self-interaction strength is reduced by \( \Lambda \rightarrow \Lambda/2 \), resulting in a smaller total mass \( M \) of the BS and, thus, preserving its existence at all.

In [128,129,188], the BS of a field quantised real scalar, the model of Ruffini and Bonazzola [237], has been extended by including a neutron star in perfect fluid description. The details are discussed in the 1992 reviews.

The situation of a field quantised complex scalar is exhibited in Section IIIR.
B. Dilaton star

The most general scale-invariant Lagrangian density for the real dilaton field \( \phi \) coupled to gravity adopts the form

\[
L_{\text{dil}} = \sqrt{|g|} e^{2\phi/f_\phi} \left\{ \frac{1}{2\kappa} R + \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{\lambda}{4} f_\phi^4 e^{2\phi/f_\phi} \right\},
\]

where \( \lambda \) is a coupling constant and \( f_\phi \) is the ‘decay constant’ of the dilaton. The dilaton mass itself is given by \( m_{\text{dil}} = 2\sqrt{\lambda} f_\phi \). For this non-minimal coupling, scale invariance is realized by the conformal change \( g_{\mu\nu} \rightarrow \exp(-2\alpha)g_{\mu\nu} \) and the shift \( \phi \rightarrow \phi + \alpha f_\phi \) in the dilaton field, where \( \alpha \) is some dimensionless parameter. If \( f_\phi = \sqrt{3/8\pi} m_{\text{Pl}} \), the theory is invariant under the larger symmetry group of conformal transformations, induced by a space-time dependent parameter \( \alpha = \alpha(x) \). For a scale-invariant interaction, the potential \( V(\phi) = e^{4\phi/f_\phi} \lambda f_\phi^4/4 \) is the only allowed type. For any other dilaton potential, the theory is still of JBD type in terms of \( \phi = f_\phi e^{\phi/f_\phi} \), but not anymore scale invariant. In the Einstein frame, the theory consists just of a massless real scalar plus a cosmological constant \( \Lambda = \lambda f_\phi^4/4 \) coupled to gravity.

Since the Lagrangian is scale invariant, Noether’s theorem provides the conserved dilaton charge

\[
Q_{\text{dil}} = (1 + 6/\kappa f_\phi^2) \int \sqrt{|g|} e^{2\phi/f_\phi} \partial^\mu \phi \, d^3x.
\]

It vanishes if \( \phi \) is time-independent as in the case of a static space-time of a non-rotating star. Therefore, in order to construct static dilaton stars, Gradwohl and Kälbermann [115] followed the semi-classical approach of Ruffini and Bonazzola [237] and Breit et al. [40] for real scalars. They regularise the energy-momentum operator \( P_\mu \) by means of the subtraction \( P'_\mu := P_\mu - \langle 0 | P_\mu | 0 \rangle \) of its vacuum expectation value. In curved spacetime, this Wick or normal ordering of the operator products, should be performed by the point-splitting method, in order to avoid ambiguities due to the non-uniqueness of the vacuum state, cf. [305], p. 87. Thereby, they can effectively get rid of the cosmological constant. A spontaneous breaking of the scale invariance leads to a vanishing dilaton expectation value at infinity. As a consequence, GR is recovered asymptotically and the dilaton star carries zero dilaton charge, but has a constant particle number \( N \) due to the normalisation of the dilaton field.

The following parameter combinations have been considered: \( \lambda := \lambda(f_\phi/\omega)^2 = 1, 10, 30, 100 \) and \( \alpha = (f_\phi/M_{\text{Pl}})^2 = 1/40, 3/(8\pi), 1, 10 \). The most noticeable effect on the total mass \( M \) is due to the change in the latter parameter. For a very light dilaton of mass \( m_{\text{dil}} = 10^{-11} \text{eV}/c^2 \), Gradwohl and Kälbermann [115] found as maximal values

\[
M_{\text{crit}} = 7\sqrt{\lambda} f_\phi \omega M_\odot, \quad R_{\text{crit}} = 40\sqrt{\lambda} f_\phi \omega \text{ km},
\]

where \( \omega \) is the ground state frequency.

C. Axion star

In general, axions have no effective cooling mechanism; they remain in the form of an extended gas cloud after separation from the Hubble flow. Decay processes may help in cooling down an axion cloud. In a cosmological context, there is a temperature dependence of the axion decay modes [232]. The stimulated decay process \( aa \rightarrow a\gamma\gamma \), together with \( aa \rightarrow a\gamma \) and \( aa \rightarrow \nu\bar{\nu} \) have been examined for their importance in the cooling of an axion cluster [103]. (The on-shell contribution \( aa \rightarrow aa \) followed by \( a \rightarrow \gamma\gamma \) can be ignored because this does not yield an energy loss.) As later shown in [287], the stimulated processes can only help in forming axion stars, by including the effect of large phase-space density; this process relaxes the axion clouds efficiently to form axion stars. On the other hand, free axions have a lifetime which is longer than the age of the universe so that the axion stars can be re-powered by accretion. Axion stars could be detectable as coherent cosmic masers [287]. In the same publication, an axion star for a real scalar field \( a \) with potential \( m^2a^2 + \lambda a^4 \) is investigated resulting in a critical mass in the order of magnitude of \( M_\odot \). Hence, for a real scalar field, we obtain the same result as Colpi et al. [58] in the case of complex scalars. For this potential, the axion star may radiate as a radio source. Additionally, the relaxation time of a gravitationally bounded cloud of axions as well as the luminosity of the axion star is estimated in [288].

The formation of axion stars at the QCD epoch was considered in [133]. The initial isocurvature perturbations yield small axion miniclusters of mass about \( 10^{24} \text{kg} \) with a diameter of the order of \( 10^{12} \text{m} \). In later evolution, the miniclusters undergo standard hierarchical gravitational clustering. For the minicluster density of about \( 10^{19} \text{kg/m}^3 \), axion annihilation \( aa \rightarrow \gamma\gamma \) is not important, but in axion stars with densities of about \( 10^{27} \text{kg/m}^3 \), it makes the
configuration unstable [269,74]. More recently, the formation of axion halos is discussed in [135]; cf. the constraints on scalar parameters in [235].

Numerical studies [168] including nonlinear effects in the evolution of inhomogeneities in the axion field around the QCD epoch can lead to very dense axion clusters with densities high enough so that the stimulated decay process supports to lead to Bose-Einstein condensation, and eventually, to axion stars. Femto- and picolensing by axion miniclusters is investigated in [169].

A different way how axion stars may send signals are exhibited in [140–146]. There, the oscillating soliton star model [268] is taken as example for axion stars. Since axions couple to the electromagnetic field, axion stars could dissipate their energies in magnetised conducting media surrounding white dwarfs or neutron stars. Oscillating configurations may generate monochromatic radiation with an energy equal to the axion mass. If axion stars collide with a white dwarf, the latter is heated up and a detectable amount of thermal radiation can be emitted during the collision [141]. Even stronger effects can be expected if the axion star collides with a neutron star. If the axion star dissipates its whole energy in a single outburst, this could be a possible mechanism for the observed gamma ray bursts [142]. For example, a collision with a neutron star with a strong magnetic field of about $10^9$ Tesla can produce gamma rays up to $10^{21}$ eV. More details of such collisions are given in [143]. A jet of baryons and leptons with Lorentz factors larger than 100 are generated. If the collision does not lead to the immediate destruction of the axion star and consists actually of several repeated collisions between both stars, this could be the origin of time-dependent complex properties of gamma ray bursts. A prediction for this model is the emission of radio waves with the frequency given by the axion mass as a byproduct to the gamma ray bursts. In Ref. [94,95], gamma ray bursts are explained by the relativistic detonation of electro-dilaton stars.

V. BOSON-FERMION STARS

In this part, we discuss the possibility of a BS located inside a fermion star or vice versa, depending which is the dominating component. As an example of a fermion star, a neutron star is normally considered as indicated by the equation of state. So far, there are no field quantised complex scalars involved. Since this topic was already discussed in earlier reviews we shall be rather brief here, concentrating on new publications, and show which type of scalar fields are applied within the combined star.

A. Fermion soliton star with real scalar field

A combination of a BS with a fermion star has first been examined in the context of soliton stars [182]. There, a semi-classical BS with the potential $U(\Phi) = m^2\Phi^2(1 - \Phi/\Phi_0)^2/2$ for a real scalar secures the existence of non-topological solitons, even in the absence of gravity. Hence, only a conserved fermionic particle number $N_f$ exists; that of real bosons is not well defined. Furthermore, the real scalar interacts with the fermion field $\psi$ through $-f \bar{\psi}\psi\Phi$, where $f$ is a Yukawa type coupling constant. For simplicity, one adopts for the fermion mass $m_f = f\Phi_0$ so that the fermion has zero effective mass inside the false vacuum, i.e., inside the BS. For the fermion field, a Thomas-Fermi approximation is chosen. Remarkably, $N_f$ depends as $M_{\text{Pl}}^{9/2}/m^3\Phi_0^2$ on the mass, i.e., fractional as for the CBS (cf. Section II C), whereas the total mass $M_{\text{Pl}}^4/m^2\Phi_0^2$ is similar to that of the soliton star. Fermion soliton BHs are mentioned in Section VII.

The observational properties of zero-mass fermion soliton stars are investigated in [51–53]. In Ref. [51], only the effects of baryons entering the star are considered if the universe temperature drops below some critical value. For the special choice $m_f = f\Phi_0$ of Lee and Pang, all fermions are almost massless inside the fermion soliton star. Moreover, the binding energy of constituent particles decrease to zero. Hence, if nuclei enter the star, they disintegrate into protons and neutrons, and then both into quarks. Furthermore, the fermion soliton star contents differs in the course of universe evolution. Above some critical temperature, protons cannot enter (or leave), and only quark pairs are inside the star. Below that temperature, baryons can penetrate the star’s shell and just quarks remain inside this so-called quark soliton star. The fermion soliton stars with only quark pairs would finally radiate away all its energy and cease to exist; only such stars with a net number of protons survive. These protons converts the rest energy into thermal energy so that for a short period of time (about $10^9$ years) at redshifts $z$ larger than 4, these stars could be X-ray emitters. Fermion soliton stars with dominating electron part may perturb the short wavelength character of the cosmic microwave background radiation (CMB), causing a sharp increase in the brightness temperature.

A quark soliton star at finite temperature including the same number of (disintegrated) protons and electrons, quark and $e^+e^-$ pairs, and some so far unknown fermionic particles with ponderable mass are calculated in [52,53]; the result is labelled ponderable soliton star. Whereas the quarks in the pure quark soliton star are almost massless,
the ponderable soliton star contains fermions with ponderable mass. The lifetime of the ponderable soliton star is from a few weeks to several years, they have a surface temperature of about $10^6$ K, and they radiate profusely with a luminosity as high as $10^{61}$ erg/s. Because this temperature is close to the hydrogen recombination temperature of $\sim 10^5$ K, these stars could be observed at redshift distances of $z \sim 10^5$. This short radiation time is still longer than the evolutionary time of the Universe at that epoch, and so, there may be an interaction with the CMB as well. The expected perturbations on CMB spectrum is less than 1% and it is largely around the short-wavelength at 1 mm or less, in form of point radio sources; spatial inhomogeneities of arcsec scale should be observed.

The situation of the fermion soliton star of Lee and Pang is extended in two respects in Ref. [62,63]. First, it is shown that the soliton star is connected to the Lee-Wick model with scalar potential

$$U(\Phi) = \frac{1}{2} m^2 \Phi^2 (1 - \Phi/\Phi_0)^2 + B \left[ 4 - 3(\Phi/\Phi_0) \right] (\Phi/\Phi_0)^3. \quad (5.1)$$

For a constant $B^{1/4}$ of about 100 MeV as used in hadron spectroscopy, the fermion soliton star has a critical mass of about just three solar masses. Additionally, the fermion soliton star at finite temperature is investigated and by this the formation and evolution of the stars. Above a critical temperature around 100 MeV, the universe is filled by the homogeneous real scalar $\Phi = \Phi_0$. Below the critical temperature at which first-order phase transition occurs, the real scalar adopts its true vacuum state $\Phi = 0$; the false vacuum will only survive inside some bubbles filled with fermions. The bubbles contract until they are stabilised by the fermionic pressure, i.e., the fermion soliton stars have formed. In the star’s evolution, it is assumed that the star could evaporate into hadrons either at its surface or inside via nucleated bubbles. Then, at present, fermion soliton stars would have a mass of about $10^{-6} M_\odot$.

A real scalar obeying a linear KG equation and the above mentioned Lee-Pang interaction are the ingredients of the model in [27]. The fermionic matter has zero temperature so that one should expect no real scalar matter at $T = 0$; responsible for the occurrence of the non-conserved scalar matter seems to be the Yukawa interaction. In the model, instead of the exact stress-energy tensor of a Dirac field as in [223], the perfect fluid approximation for the neutron star interior is used. Actually, Ruffini and Bonazzola [237] proved that a large number of degenerate fermions can be approximated by the perfect fluid approach. Additionally, the self-interaction factor $\bar{\psi}\psi$ is replaced by its expectation value following [237]. Then, the influence of the equation of state for an ideal Fermi gas for neutron stars is investigated, following Chandrasekhar [48], and nucleon-nucleon interactions are considered as well. It is not surprising that the hypothetical presence of a BS can change the cooling and the structure of the final state of a neutron star very effectively.

**B. Boson-fermion star with complex scalar field**

The stability of combined boson-fermion stars built from a complex scalar is investigated in [130,151]. A slowly rotating boson-neutron star based on Kaup’s linear potential and Chandrasekhar’s equation of state [48] has been analysed in [276].

In Ref. [35,36], a boson-fermion star is calculated in a generalised JBD theory with $\varepsilon = 0$ where the Brans-Dicke field has a potential and, hence, a mass; for the complex scalar, the potential $U_{\text{CSW}}$ is used and solutions are derived for $\Lambda = 0.01, 10$. This work can be seen as an extension of [93]; cf. the JBD Section II.L.

**VI. PERFECT FLUID APPROXIMATION**

Several papers on BSs followed the old idea of Oppenheimer and Volkoff [223] and applied a fluid approximation for the bosonic matter, which yields the correct order of magnitude for the total mass. In this Section, the bosonic fluid is isotropic.

Nishimura and Yamaguchi [220] constructed an equation of state for a cold Higgs fluid

$$\rho = \frac{m^4}{4\lambda} \frac{1}{(2u^2 - 1)^2} \quad (0.5 < u^2 \leq 1), \quad (6.1)$$

$$p = \frac{m^4}{3\lambda} \frac{1}{{2u^2 - 1}} \left[ \frac{2E(u^2)}{K(u^2)} - \frac{2u^2 - 5/4}{2u^2 - 1} \right], \quad (6.2)$$

where $m, \lambda$ are the parameters of the Higgs Lagrangian, $K, E$ the first and second elliptic functions, respectively, and $u$ is some parameter. For several choices of $m^4/\lambda$, it is shown that the equation of state $p(\rho)$ is an extrapolation of the equation of state of nuclear matter to higher density regions; a further characteristic is that the pressure vanishes
below some value of the central density. Substituting this equation of state into the Tolman-Oppenheimer-Volkoff equation leads to BSs with critical masses in the order of magnitude of a solar mass. In a way, by taking into account a bosonic self-interaction constant, Nishimura and Yamaguchi anticipated the result of Colpi et al. [58] who found the critical BS masses for the Lagrangian, i.e., anisotropic, description in the same mass region. Furthermore, these BSs have a finite radius, namely, the point where pressure as a function of radius vanishes.

Takasugi and Yoshimura [282] investigated a non-self-interacting cold Bose gas with the equation of state

\[ p = 3m^2n_0^2u^{8/3}/(2\rho), \quad \rho = mn_0u\sqrt{1 + \frac{9}{2}u^{2/3}}, \]

where \( m \) is the boson mass, \( n \) the number density, \( n_0 \) some constant, and \( u := n/n_0 \). We see that for small \( u \), \( p \propto \rho^{5/3} \) and for large \( u \), \( p \propto \rho/3 \). The critical mass is found to be \( 0.57 M_{Pl}^2/m \) at a radius of 5.5/\( m \) which is remarkably close to the Kaup limit 0.633 \( M_{Pl}^2/m \) of the Lagrangian description.

Three different equations of state are investigated in Ref. [299]. For massless bosons, the equation of state of photons is used \( p = \rho/3 \), leading to photon stars, or “geons”. The radius is determined where the surface pressure equals the one of the cosmological background radiation. The critical mass is about 10^{31} M_{\odot} and the radius is larger than the radius of the Universe, so that it seems that photon stars cannot be found. Alternatively, a giant MIT bag filled with gluon matter is considered, hence, the calculation stops where the pressure is equal to the bag constant \( B \) which is chosen to be 57 MeV/fm^3. The equation of state can be written as \( p = (\rho - 4B)/3 \). Then, the critical mass of the gluon star is 2 M_{\odot}. The third example describes a non-relativistic boson gas of massive particles where Bose-Einstein condensation is neglected, with an equation of state \( p = 2(\rho - mn)/3 \). Because the pressure never vanishes in this model, the surface of these BSs is defined by \( p = B \). The critical mass is about 10^{-3} M_{\odot} for \( m = 20 \text{ GeV}/c^2 \).

The MIT bag equation of state is derived in Ref. [121] for a real scalar field with self-interaction. Thereby, the effective mass of the real scalar is zero, so that the corresponding BS solutions are called precarious stars. These stars exist only for temperatures above some value which is determined by the bag constant; this coincides with the fact that the real scalar has no conserved charge, cf. Section IV.

Departing from the Newtonian spherically symmetric Poisson equation, Dehnen and Gensheimer [77] analysed a self-gravitating Bose gas cloud with temperature. A similar situation was already considered by Ingrosso and Ruffini [137] for an isothermal case, i.e., constant temperature all over the cloud; details can also be found in [152]. The difference from the Newtonian BSs of Section II G is that, here, an equation of state is derived that does not include a scalar field, but considers the Bose-Einstein statistics in its non-relativistic limit. It is found that the Bose gas obeys a polytropic relation \( p = a\rho^3 \) with some constant \( a \). The Poisson equation reduces then to the Lane-Emden equation for polytropic index 3. In general, it is found that the temperature decreases with increasing distance from the centre, hence, Bose-Einstein condensation occurs in the outer regions of the star. This result is in contrast to the isothermal case where the condensed phase is in the centre (additionally the outer regions are Boltzmannian and spatially unlimited) [137]. Furthermore, these Bose gas clouds have a finite radius where the energy density, the pressure, and the temperature vanish.

VII. SOLITON BLACK HOLE

These configurations, introduced in [99], describe a non-topological soliton star sitting completely inside the Schwarzschild horizon which is possible due to the finiteness of such a star, cf. the radius definition in Section II D. That means that the radius of the star is smaller than its Schwarzschild value. The mass \( M \) in units of \( M_{Pl}^2/(\text{m}^4\Phi^2_0) \) lies between the values 0.1256 and 16/(27\pi) = 0.1886. For this upper bound, the particle number \( N \) vanishes. It is not surprising that a soliton BH is not stable, it decays very slowly by sending scalar matter towards the horizon [181]. Outside the horizon, the Schwarzschild solution holds; cf. the more general BS result in [228]. In [225], the physical properties of a soliton BH at finite temperature was investigated following the procedure for a non-topological soliton star in [279], cf. Section II K. Fermion soliton BHs were constructed in [182] for a real scalar field; details on fermion soliton stars in Section VA.

VIII. OUTLOOK

BSs are so far hypothetical objects; what is more it is unknown whether such localised configurations really have the size of a star. Due to the unknown nature of its constituents, the dimensions of BSs could be smaller than an atom or as large as a galactic halo. Thus, all these different scenarios have to be considered and further observational consequences drawn.
Within the last three decades, a lot of models have been labelled BSs, sometimes one and the same mathematical model received different physical designations. All models agree that the constituents of a BS are scalars. They differ in regards to whether the scalars are real or complex and whether the scalars are classical or field quantised. In this way, we can distinguish four basic types of BSs. Furthermore, in each of these four BS types, sub-types can be separated due to the form of the scalar self-interaction. In this review, we tried to clarify and sum up all different boson star notions.

Further analysis is needed in order to understand these highly interesting instances of a possible fine structure in the energy levels of general relativistic BEC, the so-called BSs. In view of their rich and prospective structures, are BSs capable of explaining parts of the dark matter in the Universe?

We hope that this review can also be understood as a starting point that the topic BS leaves the pure scientific area and finds its way to popular science [224].

We think that one cannot finish a topical review on BSs any better than by quoting from [180,98,99,182] and his review [184] the last two sentences of T.-D. Lee: At present, there is no experimental evidence that soliton stars [or BSs, the authors] exist. Nevertheless, it seems reasonable that solutions of well-tested theories, such as Einstein’s GR, the Dirac equation, the KG equation, etc., should find their proper place in nature.

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APPENDIX A: CONVENTION

As usual conventions can make life easier or tougher. There are infinite possibilities to transform the BS field equations into dimensionless quantities; two are commonplace in the literature. In order to avoid recalculations, we would like to propose that future authors apply the following convention

\[ x = m r, \ \Omega = \frac{\omega}{m}, \ \phi_{\text{BD}}(x) = \frac{\phi_{\text{BD}}(x)}{M_{\text{Pl}}}, \ \sigma(x) = \sqrt{4\pi} \frac{P(x)}{M_{\text{Pl}}}, \ \Lambda = \frac{\lambda}{4\pi} \frac{M_{\text{Pl}}^2}{m^2}. \]  

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