PONCELET TRIANGLES:
A THEORY FOR LOCUS ELLIPTICITY

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ABSTRACT. We present a theory which predicts if the locus of a triangle center over certain
Poncelet triangle families is a conic or not. We consider families interscribed in (i) the
confocal pair and (ii) an outer ellipse and an inner concentric circular caustic. Previously,
determining if a locus was a conic was done on a case-by-case basis. In the confocal case,
we also derive conditions under which a locus degenerates to a segment or a circle. We
show the locus’ turning number is either \( \pm 3 \), while predicting its monotonicity with respect
to the motion of a vertex of the triangle family.

Keywords Poncelet, ellipse, triangle center.

MSC 51M04 and 51N20 and 51N35 and 68T20

1. INTRODUCTION

Paraphrasing [7, Thm 2.14, p.22], Poncelet’s closure theorem says that given two real
conics \( \mathcal{C}, \mathcal{C}' \), if an \( N \)-gon can be drawn with all vertices on \( \mathcal{C} \) and with all sides tangent to
\( \mathcal{C}' \), then a 1d family of such \( N \)-gons exists, where any point on \( \mathcal{C} \) can be a vertex of the
family.

Here we consider \( N = 3 \) families, termed “Poncelet triangles”. A first natural question
is: what are curves swept by some of its notable points such as the incenter, barycenter,
circumcenter, etc., or more generally by a vast array of its triangle centers, catalogued in
[15]? A curious phenomenon, readily observed by simulation, is that some centers will
sweep conics while others will not.

As an example, Figure 1 shows Poncelet triangles embedded in a pair of confocal ellipses
(also known as the “elliptic billiard”). While in [10, 22] it is shown that the incenter sweeps
an ellipse, in [13] the locus of the symmedian point is a quartic.

\[ \]
FIGURE 2. Six “famous” concentric, axis-parallel (CAP) families, and their stationary triangle centers at the center. Video

Though locus ellipticity can be painstakingly proved on a case-by-case basis, see for example [9, 23], we hereby describe a theory to explain the phenomenon over a wider range of cases. We employ a technique known as “Blaschke’s Parametrization” which allows us to wield Poncelet triangles as roots of symmetric polynomials over the complex numbers.

**Main results** We show that if a triangle center $X$ can be expressed as a fixed affine combination of barycenter, circumcenter, and $Y$, where $Y$ is stationary over a given family, then the locus of $X$ will be an ellipse as well.

Referring to 6 Poncelet triangle families shown in Figure 2, let us henceforth refer to them as: confocal (elliptic billiard), with incircle, with circumcircle, homothetic, dual, and excentral, respectively.

Henceforth we shall refer to triangle centers as $X_k$, after Kimberling [15]. In [11, 21] it was shown that the mittenpunkt $X_9$ (resp. symmedian point $X_6$) is stationary over the confocal (resp. excentral) family.

For the confocal case we (i) predict which triangle centers sweep ellipses, and (ii) derive conditions under which they degenerate to a circle or a segment. We then apply the same method for the incircle family, showing how to extend it to the excentral family. We still lack a strategy for locus-type prediction for the circumcircle, homothetic, and dual families, given that their stationary centers are already fixed linear combinations of barycenter $X_2$ and circumcenter $X_3$.

We also show that for a generic family, a locus’ turning number is always $\pm 3$ and derive conditions for monotonicity with respect to the motion of family vertices.

To the best of our knowledge, the above results are new, e.g., they are not found in classical treatments such as [7] nor in well-known surveys of Poncelet’s theorem such as [4, 8, 17].

**Related Work** In [18] loci of triangle centers are studied over the so-called “poristic” family (Poncelet triangles embedded between two circles). Conic, circular, and pointwise loci are identified. In [23] the loci of vertex, perimeter, and area centroids are studied over a generic Poncelet family indicating that the first and last are always ellipses while in general

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1 Also known as Chapple’s porism.
the perimeter one is not a conic. The locus of the “circumcenter-of-mass” (a generalization of the circumcenter for N-gons), studied in [24], is shown to be a conic over Poncelet N-gon families, see [5].

Over the confocal family, the elliptic locus of (i) the incenter was proved in [9, 10, 22]; (ii) of the barycenter in [10, 23]; and (iii) of the circumcenter in [9, 10]. The elliptic locus of the Spieker center $X_{10}$ (which is the perimeter centroid of a triangle) was proved in [10]. Some properties and invariants of the confocal family are described in [21]; $N = 3$ subcases are proved in [13]. Some invariants have been proved for all $N \geq 3$ in [2, 3, 5].

**Article Structure** in Section 2 reviews preliminaries, and Blaschke’s parametrization. Section 3 proves locus phenomena for triangles in a generic pair of ellipses. Section 4 analyzes loci of triangle centers in the confocal pair. In Section 5 this analysis is extended to the incircle family. We include links to animations in the caption of most figures, all of which are collected in Table 2.

Appendix A lists triangle centers whose loci are (numerically) ellipses (and/or circles) over other concentric, axis-parallel families.

## 2. Preliminaries

Referring to Figure 3, we regard Poncelet triangles in a generic ellipse pair as the image of a fixed affine transformation applied to the solutions of $B(z) = \lambda$ for each $\lambda$ in the unit circle of $C$, where $B(\cdot)$ is a degree-3 Blaschke product [6]. In [14], such a parametrization is used to derive the following fact (see Figure 4):

**Theorem 1.** Over the family of Poncelet triangles interscribed in a generic nested pair of ellipses (non-concentric, non-axis-aligned), if $X_{\alpha,\beta}$ is a fixed linear combination of $X_2$ and $X_3$, i.e., $X_{\alpha,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then its locus is an ellipse.

Recall the following observation reproduced from [14]:

**Observation 1.** Amongst the 40k+ centers listed on [15], about 4.9k triangle centers lie on the Euler line [16]. Out of these, only 226 are always fixed affine combinations of $X_2$ and $X_3$. For $k < 1000$, these amount to $X_{k, k =2, 3, 4, 5, 20, 140, 376, 381, 382, 546, 547, 548, 549, 550, 631, 632}$. Consider the affine image of Poncelet triangles in a generic pair of ellipses such that the external ellipse is sent to the unit circle, Figure 3. Let $f, g$ be the foci of the caustic thus obtained.
Let $p = (a + b)/2$ and $q = (a - b)/2$. If, as in Theorem 1, $\mathcal{X}_{a,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then the latter can be parametrized as $\mathcal{X}_{a,\beta} = u\lambda + v\lambda^1 + w$ where [14, Eqn. 3]:

\[
\begin{align*}
u := & \frac{\beta pq(q - fg)}{(q - p)(p + q)} + \frac{1}{3} \alpha fgq \\
w := & \frac{q (\overline{f} + \overline{g}) (p^2(\alpha + 3\beta) - \alpha q^2)}{3(p - q)(p + q)} + p(f + g)(\alpha p^2 - q^2(\alpha + 3\beta))
\end{align*}
\]

**Remark 1.** In [6, Lemma 3.4, p. 28] it is shown that (i) for each $\lambda$ on the complex unit circle there are 3 solutions for the equation $B(z) = \lambda$, where $B(z)$ is a degree-3 Blaschke product, and (ii) these 3 solutions move monotonically and in the same direction as $\lambda$. This means that as $\lambda$ sweeps the unit circle monotonically, Poncelet triangles sweep the outer ellipse monotonically and in the same direction as $\lambda$. Moreover, as $\lambda$ sweeps the unit circle monotonically each vertex of a Poncelet triangle sweeps the outer ellipse exactly once.

The following Lemma, proved in [14], was used to prove Theorem 1 and gives us a tool to explicitly find the semiaxis and rotation angle of said elliptic loci:

**Lemma 1.** If $u, v, w \in \mathbb{C}$ and $\lambda$ is a parameter that varies over the unit circle $\mathbb{T} \subset \mathbb{C}$, then the curve parametrized by

\[ F(\lambda) = u\lambda + v\frac{1}{\lambda} + w \]

is an ellipse centered at $w$, with semiaxis $|u| + |v|$ and \(||u| - |v|||$, rotated with respect to the canonical axis of $\mathbb{C}$ by an angle of $(\arg u + \arg v)/2$.

**Definition 1** (Degenerate Locus). When the elliptic locus of a triangle center is either segment or a fixed point, i.e., either one or both of its axes have shrunk to zero, we will call it “degenerate”.

### 3. Generic Ellipse Pair

In this section we consider Poncelet triangles in a generic pair of ellipses (non-concentric, and non-axis-parallel). We (i) specify a method to determine locus ellipticity if the family has a stationary center, (ii) we derive conditions for locus monotonicity, and (iii) compute the winding number of loci.

**Corollary 1.** If a triangle center $X_k$ is stationary over a Poncelet triangle family, then the locus of any triangle center $X$ which is a fixed linear combination of $X_2, X_3, X_k$ will be an ellipse.

**Proof.** The triangle center $X = \alpha X_2 + \beta X_3 + \gamma X_k$ is the linear combination $\mathcal{X}_{a,\beta} := \alpha X_2 + \beta X_3$ under a fixed translation by $\gamma X_k$, because both $\gamma$ and $X_k$ are fixed over the family. The result then follows from Theorem 1. □

**Proposition 1.** Let $\mathcal{X}$ be a fixed linear combination of $X_2$, $X_3$, and $X_k$, where $X_k$ is some stationary center over the family of Poncelet triangles. As the vertices of said triangles sweep the outer ellipse monotonically, the path of $\mathcal{X}$ in its elliptical locus is monotonic as well, except for when this locus is degenerate.
which only happens when the locus of \( X \) we derive

By Theorem 1, the locus of \( \{X_i\}_{i=2,3,4,5,20} \) is elliptic. Thus, we can now assume that

Denoting \( u = u_0 + u_1 \) and \( v = v_0 + v_1 \) with \( u_0, u_1, v_0, v_1 \in \mathbb{R} \), we can directly compute

\[
\frac{d}{dt} X(t) = |u|^2 + |v|^2 + 2|u| \frac{v}{|v|} + 2 \sin(2t)(u_2 v_1 - u_1 v_2) - 2 \cos(2t)(u_1 v_1 + u_2 v_2)
\]

Since \( (u_2 v_1 - u_1 v_2)^2 + (u_1 v_1 + u_2 v_2)^2 = (u_1^2 + u_2^2)(v_1^2 + v_2^2) = |u|^2 |v|^2 \), there is some angle \( \phi \in [0, 2\pi] \) (the angle between the vectors \( (u_1, u_2) \) and \( (v_1, v_2) \)) such that \( u_1 v_1 + u_2 v_2 = |u| |v| \cos \phi \) and \( u_2 v_1 - u_1 v_2 = |u| |v| \sin \phi \). Substituting this back in the previous equation, we derive

\[
\frac{d}{dt} X(t) = |u||v| \left( \frac{|u|}{|v|} + \frac{|v|}{|u|} + 2 \sin(2t) \sin \phi - 2 \cos(2t) \cos \phi \right)
\]

By AM-GM inequality, this last quantity is always strictly greater than 0 unless \( |u| = |v| \), which only happens when the locus of \( X \) is degenerate (see Lemma 1). If \( |u| \neq |v| \), we will have \( \frac{d}{dt} X(t) > 0 \), and hence the velocity vector never vanishes, meaning that the \( X \) sweeps its smooth locus monotonically, as desired.

\[\square\]

**Proposition 2.** Let \( X \) be a fixed linear combination of \( X_2, X_3, \) and \( X_4 \), where \( X_k \) is some stationary center over the family of Poncelet triangles. Over a full cycle of said triangles,
the winding number of $\mathcal{X}$ over its elliptical locus is $\pm 3$, except for when this locus is degenerate.

Proof. By Theorem 1, the locus of $\mathcal{X}$ can be parametrized by $u\lambda + v\frac{1}{\lambda} + w$ for some $u, v, w \in \mathbb{C}$. From Remark 1, one can see that the winding number of $\lambda$ associated to Poncelet triangles is $+3$ for each full counterclockwise cycle of said triangles over the outer Poncelet ellipse. Thus, it is sufficient to prove that the winding number of $\mathcal{X}$ over its elliptical locus is $\pm 1$ as $\lambda$ goes around the complex unit circle just once in the counterclockwise direction.

Since $w$ is the center of the elliptic locus of $\mathcal{X}$ (see Lemma 1), we compute the winding number of $\mathcal{X}$ around $w$. Parametrizing $\mathcal{X}$ as $\mathcal{X}(t) = u\lambda + v\lambda^{-1} + w$ where $\lambda = e^{it}$, one can directly compute the winding number as [1, Lemma 1, p. 114]:

$$\frac{1}{2\pi i} \oint_{\mathcal{X}} \frac{d\zeta}{\zeta - w} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\mathcal{X}'(t)}{\mathcal{X}(t) - w} dt = \text{sign}(|u|^2 - |v|^2)$$

By Lemma 1, the only way we can have $|u| = |v|$ is if the locus of $\mathcal{X}$ is degenerate. Thus, whenever this locus is not degenerate, the winding number of $\mathcal{X}$ around its locus as $\lambda$ sweeps the unit circle once is equal to 1 if $|u| > |v|$ and $-1$ when $|u| < |v|$, as desired. □

4. CONFOCAL PAIR

Consider a confocal ellipse which admits a family of Poncelet triangles. If $a, b$ denote the outer ellipse’s semi-axes, the caustic semi-axes are given by [10]:

$$a_c = \frac{a(\delta - b^2)}{c^2}, \quad b_c = \frac{b(a^2 - \delta)}{c^2},$$

where $c^2 = a^2 - b^2$ and $\delta = \sqrt{a^4 - a^2b^2 + b^4}$.

The affine transformation $x \rightarrow x/a, y \rightarrow y/b$ sends the confocal pair to a pair where the outer conic is the unit circle and the caustic $\delta_c'$ is a concentric ellipse. The foci $f, g$ of the $\delta_c'$ are given by:

$$f = (-c', 0), \quad g = (c', 0)$$

where $c' = (1/c') \sqrt{2\delta - a^2 - b^2}$.

We are now in a position to offer a compact alternate proof to the ellipticity of $X_1$ previously done in [10, 22]:

Corollary 2. In the confocal pair, the locus of $X_1$ is an ellipse.

Proof. For any triangle, $X_1$ can be expressed as the linear combination $X_1 = \alpha X_2 + \beta X_3 + \gamma X_0$ of $X_2, X_3$ and $X_0$ with:

$$\alpha = \frac{6}{\rho + 2}, \quad \beta = \frac{2\rho}{\rho + 2}, \quad \gamma = \frac{-\rho - 4}{\rho + 2}$$

where $\rho = r/R$, is the ratio of inradius to circumradius. Since in the confocal family $X_0$ is stationary and $\rho$ is invariant [21], the claim follows. □

Interestingly, and as originally stated in [14], ample experimental evidence suggests that:

Conjecture 1. The locus of $X_1$ is a non-degenerate conic iff the pair is confocal.

We are now in a position to expand this last result to many other triangle centers in the confocal pair (elliptic billiard), as many of these are fixed linear combinations of $X_2, X_3$, and $X_0$. In order to do so, we compute several of such linear combinations in Table 1, most
of which were derived from the “combos” in [15]. We use the existence of these linear combinations to prove the following very comprehensive result:

**Corollary 3.** In the confocal pair, from $X_1$ to $X_{200}$, the locus of $X_k$ is an ellipse for $k = 1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 20, 21, 35, 36, 40, 46, 55, 56, 57, 63, 65, 72, 78, 79, 80, 88^1, 84, 90, 100, 104, 119, 140, 142, 144, 145, 149, 153, 162^1, 165, 190^1, 191, 200.$

**Proof.** As in Corollary 2, one can write $X_1$ as a fixed linear combination of $X_2, X_3,$ and $X_9$, given that the ratio $p = r/R$ is constant in the confocal pair. Using the formulas on Table 1, all these triangle centers (except for $X_{88}, X_{162},$ and $X_{190}$) are fixed linear combinations of $X_1, X_2, X_3,$ and $X_9$, and therefore they are fixed linear combinations of $X_2, X_3,$ and $X_9$ as well. By Corollary 1, given that $X_9$ is stationary over the confocal family, this implies the loci of all these triangle centers are ellipses. □

Note: the loci of $X_{88}, X_{162},$ and $X_{190}$ are also ellipses because by definition they lie on the circumsnconic centered on $X_9$ [15, X(9)]. Centers railed to said circumsnconic were called “swans” in [20], as over the confocal family they execute an intricate dance, which can be seen live [here].

Referring to Figure 5:

**Proposition 3.** In the confocal pair, the locus of $X_k = \alpha X_2 + \beta X_3$ for $\alpha, \beta \in \mathbb{R}$ is degenerate when:

$$\frac{\alpha}{\beta} = \frac{2a^2 - b^2 + \delta}{2b^2}, \text{ or } \frac{\alpha}{\beta} = \frac{2b^2 - a^2 + \delta}{2a^2}$$

**Proof.** By Lemma 1, this will happen when $|u| = |v|$ in Theorem 1. In the confocal pair, when $\alpha, \beta \in \mathbb{R}$, both $u$ and $v$ are real numbers as well. Thus, the ratios $\alpha/\beta$ that yield degenerate loci can be computed directly by solving $u = \pm v$. □
**Observation 2.** These ratios $\alpha/\beta$ that make the locus of $\mathcal{X}$ be degenerate can also be expressed in terms of $\rho = r/R$ as:

$$\frac{\alpha}{\beta} = \frac{3}{2} \left( \frac{1 \pm \sqrt{1 - 2\rho}}{\rho + 1 \pm \sqrt{1 - 2\rho}} \right)$$

**Proposition 4.** In the confocal pair, the locus of $\mathcal{X} = \alpha X_2 + \beta X_3$ for $\alpha, \beta \in \mathbb{R}$ is a circle when:

$$\frac{\alpha}{\beta} = \frac{\delta - 3ab \pm 2(a^2 + b^2)}{2ab}$$

**Proof.** By Lemma 1, this will happen when $|u| + |v| = ||u| - |v||$ with $u, v$ from Theorem 1. In the confocal pair, when $\alpha, \beta \in \mathbb{R}$, both $u$ and $v$ are real numbers as well. Thus, this condition holds if and only if either $u = 0$ or $v = 0$. The ratios $\alpha/\beta$ that yield circular loci can then be computed directly. \( \square \)

**Observation 3.** It follows that $(\alpha/\beta)_+ + (\alpha/\beta)_- = -3$.

### 5. Incircle Family and Beyond

Triangle centers whose loci are (numerically) ellipses (and/or circles) over other concentric, axis-parallel (CAP) families, appear in Appendix A. Here we adapt the method in the previous section to these families.

Let the “incircle family”, shown in Figure 2, is a Poncelet triangle family interscribed in an external ellipse and a concentric internal circle.

**Proposition 5.** In the incircle family, from $X_1$ to $X_{200}$, the locus of $X_k$ is an ellipse for $k = 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 20, 21, 35, 36, 40, 46, 55, 56, 57, 63, 72, 78, 79, 80, 84, 90, 100, 104, 119, 140, 142, 144, 145, 149, 153, 165, 191, 200.

**Proof.** It [11, Thm. 1] is is shown that the circumradius $R$ is constant over the incircle family. Since the inner ellipse is a circle with constant radius $r$, the ratio $\rho = r/R$ is constant.
over the incircle family. Using the formulas from Table 1, all these triangle centers are fixed linear combinations of $X_1$, $X_2$, and $X_3$. By Corollary 1, given that $X_1$ is stationary over the incircle family, this implies the loci of all these triangle centers are ellipses. □

Note: $X_1$, a fixed point over the incircle family, can be regarded as a degenerate ellipse.

Referring to Figure 2(bottom right), since in the excentral family $X_6$ is stationary, we could use a similar strategy to prove the ellipticity of certain centers. Nevertheless, we still lack a theory for this case.

The following is suggested by experimental results:

**Conjecture 2.** If a triangle center’s barycentric coordinates are rational on the squares of a triangle’s sidelengths, its locus will be an ellipse.

The following are open questions:
- Why over the incircle family are the loci of certain centers circles?
- How can we predict locus ellipticity in the homothetic, circumcircle, and dual families? Note $X_2$, $X_3$, and $X_4$ are stationary, so we don’t have a third stationary center, independent of $X_2$ and $X_3$ which would allow us to apply our method.
- Extend Appendix A to well-known non-concentric pairs such as the Brocard porism [19], the bicentric family (Chapple’s porism), the MacBeath family (excentral triangles to bicentrics family), etc.
- Prove Conjectures 1 and 2.

**Videos** Animations illustrating some phenomena herein are listed on Table 2.

**ACKNOWLEDGEMENTS**

We would like to thank A. Akopyan for valuable insights. The first author is fellow of CNPq and coordinator of Project PRONEX/ CNPq/ FAPEG 2017 10 26 7000 508.

### A. TRIANGLE CENTERS WHOSE LOCUS IS AN ELLIPSE

Referring to Figure 2, here we report lists of Kimberling centers $X_k$ [15] whose locus under various classic Poncelet triangle families are ellipses (or circles).

- **Confocal pair (stationary $X_0$):** Ellipses: 1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 20, 21, 35, 36, 40, 46, 55, 56, 57, 63, 65, 72, 78, 79, 80, 84, 88, 90, 100, 104, 119, 140, 142, 144, 145, 149, 153, 162, 165, 190, 191, 200. Note: the first 29 in the list were proved in [12]. Note: the following centers lie on the $X_0$-centered circumellipse: 88, 100, 162, 190 [15].
- **Incircle (stationary $X_1$):** Ellipses: 2, 4, 7, 8, 9, 10, 20, 21, 63, 72, 78, 79, 84, 90, 100, 104, 140, 142, 144, 145, 153, 191, 200. Circles: 3, 5, 11, 12, 35, 36, 40, 46, 55, 56, 57, 65, 80, 119, 165.
- **Circumcircle (stationary $X_3$):** Ellipses: 6, 49, 51, 52, 54, 64, 66, 67, 68, 69, 70, 71, 113, 125, 141, 143, 146, 154, 155, 159, 161, 182, 184, 185, 193, 195. Circles: 2, 4, 5, 20, 22, 23, 24,
Homothetic (stationary $X_2$): Ellipses: 3, 4, 5, 6, 17, 20, 32, 39, 62, 69, 76, 83, 98, 99, 114, 115, 140, 141, 147, 148, 182, 183, 187, 190, 193, 194. Circles: 13, 14, 15, 16.

Dual (stationary $X_4$) Ellipses: 2, 3, 5, 20, 64, 107, 122, 133, 140, 154.

Excentral (stationary $X_6$): Ellipses: 2, 3, 4, 5, 20, 22, 23, 24, 25, 26, 49, 51, 52, 54, 64, 66, 67, 68, 69, 70, 74, 110, 113, 125, 140, 141, 143, 146, 154, 155, 156, 159, 161, 182, 184, 185, 186, 193, 195.

REFERENCES

[1] Ahlfors, L. V. (1979). Complex Analysis: an Introduction to Theory of Analytic Functions of One Complex Variable. McGraw Hill.

[2] Akopyan, A., Schwartz, R., Tabachnikov, S. (2020). Billiards in ellipses revisited. Eur. J. Math. doi: 10.1007/s40879-020-00426-9.

[3] Bialy, M., Tabachnikov, S. (2020). Dan Reznik’s identities and more. Eur. J. Math. doi:10.1007/s40879-020-00428-7.

[4] Bos, H. J. M., Kers, C., Oort, F., Raven, D. W. (1987). Poncelet’s closure theorem. Exposition. Math., 5(3): 289–364.

[5] Chavez-Caliz, A. (2020). More about areas and centers of Poncelet polygons. Arnold Math J. doi:10.1007/s40598-020-00154-8.

[6] Daepp, U., Gorkin, P., Shaffer, A., Voss, K. (2018). Finding Ellipses: What Blaschke Products, Poncelet’s Theorem, and the Numerical Range Know about Each Other. Carus Mathematical Monographs, 34. Providence, RI: MAA Press.

[7] Dragović, V., Radnović, M. (2011). Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics. Frontiers in Mathematics. Basel: Springer.

[8] Dragović, V., Radnović, M. (2014). Bicentennial of the great Poncelet theorem (1813–2013): current advances. Bull. Amer. Math. Soc. (N.S.), 51(3): 373–445.

[9] Fierobe, C. (2021). On the circumcenters of triangular orbits in elliptic billiard. Journal of Dynamical and Control Systems.

[10] Garcia, R. (2019). Elliptic billiards and ellipses associated to the 3-periodic orbits. American Mathematical Monthly, 126(06): 491–504.

[11] Garcia, R., Reznik, D. (2022). Family ties: Relating Poncelet 3-periodics by their properties. J. Croatian Soc. for Geom. Gr. (KoG). To appear.

[12] Garcia, R., Reznik, D., Koiller, J. (2020). Loci of 3-periodics in an elliptic billiard: why so many ellipses? arXiv:2001.08041.

[13] Garcia, R., Reznik, D., Koiller, J. (2021). New properties of triangular orbits in elliptic billiards. Amer. Math. Monthly, 128(10): 898–910.

[14] Helman, M., Laurain, D., Garcia, R., Reznik, D. (2022). Invariant center power and elliptic loci of poncelet triangles. J. of Dyn. & Control Systems. To appear.

[15] Kimberling, C. (2019). Encyclopedia of triangle centers. bit.ly/3a0Qj7r.

[16] Kimberling, C. (2020). Central lines of triangle centers. bit.ly/34vVoJ8.

[17] Martini, H. (1995). Recent results in elementary geometry, II. In: Proceedings of the 2nd Gauss Symposium. Conference A: Mathematics and Theoretical Physics (Munich, 1993), Sympos. Gaussiana. de Gruyter, Berlin, pp. 419–443.

[18] Odehnal, B. (2011). Poristic loci of triangle centers. J. Geom. Graph., 15(1): 45–67.

[19] Reznik, D., Garcia, R. (2021). Related by similarity II: Poncelet 3-periodics in the homothetic pair and the brocard porism. Intl. J. of Geom., 10(4): 18–31.

[20] Reznik, D., Garcia, R., Koiller, J. (2020). The ballet of triangle centers on the elliptic billiard. Journal for Geometry and Graphics, 24(1): 079–101.

[21] Reznik, D., Garcia, R., Koiller, J. (2020). Can the elliptic billiard still surprise us? Math Intelligencer, 42: 6–17.

[22] Romaskevich, O. (2014). On the incenters of triangular orbits on elliptic billiards. Enseign. Math., 60: 247–255.

[23] Schwartz, R., Tabachnikov, S. (2016). Centers of mass of Poncelet polygons, 200 years after. Math. Intelligencer, 38(2): 29–34.

[24] Tabachnikov, S., Tsukerman, E. (2014). Circumcenter of mass and generalized euler line. Discrete Comput. Geom., 51: 815–836.