AMPLITUDE INEQUALITIES FOR DIFFERENTIAL GRADED MODULES

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Abstract. Differential Graded Algebras can be studied through their Differential Graded modules. Among these, the compact ones attract particular attention.

This paper proves that over a suitable chain Differential Graded Algebra $R$, each compact Differential Graded module $M$ satisfies $\text{amp } M \geq \text{amp } R$, where amp denotes amplitude which is defined in a straightforward way in terms of the homology of a DG module.

In other words, the homology of each compact DG module $M$ is at least as long as the homology of $R$ itself. Conversely, DG modules with shorter homology than $R$ are not compact, and so in general, there exist DG modules with finitely generated homology which are not compact.

Hence, in contrast to ring theory, it makes no sense to define finite global dimension of DGAs by the condition that each DG module with finitely generated homology must be compact.

0. Introduction

Differential Graded Algebras (DGAs) play an important role in both ring theory and algebraic topology. For instance, if $M$ is a complex of modules, then the endomorphism complex $\text{Hom}(M, M)$ is a DGA with multiplication given by composition of endomorphisms, and this can be used to prove ring theoretical results, see [12] and [13]. Another example is that over a commutative ring, the Koszul complex on a series of elements is a DGA, see [17, sec. 4.5], and again, ring theoretical results ensue, see [10].

Likewise, DGAs occur naturally in algebraic topology, where the canonical example is the singular cochain complex $C^*(X)$ of a topological space $X$. Other constructions also give DGAs; for instance, if

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$G$ is a topological monoid, then the singular chain complex $C_\ast(G)$ is a DGA whose multiplication is induced by the composition of $G$; see [3].

Just as rings can be studied through their modules, DGAs can be studied through their Differential Graded modules (DG modules), and this is the subject of the present paper.

The main results are a number of “amplitude inequalities” which give bounds on the amplitudes of various types of DG modules. Such results have been known for complexes of modules over rings since Iversen’s paper [9], and it is natural to seek to extend them to DG modules.

Another main point, implied by one of the amplitude inequalities, is that, in contrast to ring theory, it appears to make no sense to define finite global dimension of DGAs by the condition that each DG module with finitely generated homology must be compact. This is of interest since several people have been asking how one might define finite global dimension for DGAs.

**First main Theorem.** To get to the first main Theorem of the paper, recall from [13] that if $R$ is a DGA then a good setting for DG modules over $R$ is the derived category of DG left-$R$-modules $D(R)$.

A DG left-$R$-module is called compact if it is in the smallest triangulated subcategory of $D(R)$ containing $R$, or, to use the language of topologists, if it can be finitely built from $R$. The compact DG left-$R$-modules form a triangulated subcategory $D_c(R)$ of $D(R)$, and play the same important role as finitely presented modules of finite projective dimension do in ring theory.

The amplitude of a DG module $M$ is defined in terms of the homology $H(M)$ by

$$
amp M = \sup \{ i \mid H_i(M) \neq 0 \} - \inf \{ i \mid H_i(M) \neq 0 \}.
$$

**Theorem A.** Assume $\amp R < \infty$. Let $L$ be in $D_c(R)$ and suppose $L \neq 0$. Then

$$
\amp L \geq \amp R.
$$

Expressed in words, this says that among the compact DG modules, none can be shorter than $R$ itself. The Theorem will be proved in the situation specified in Setup [13] below; the main point is that $R$ is a local DGA, that is, a chain DGA for which $H_0(R)$ is a local commutative noetherian ring. The multiplication in $H_0(R)$ is induced by the multiplication in $R$.
Of equal significance to Theorem A is perhaps the following consequence: If \( \text{amp } R \geq 1 \), that is, if \( R \) is a true DGA in the sense that it is not quasi-isomorphic to a ring, then a DG module with amplitude zero cannot be compact. There are many such DG modules and they can even be chosen so that their homology \( H(M) \) is finitely generated as a module over the ring \( H_0(R) \). The scalar multiplication of \( H_0(R) \) on \( H(M) \) is induced by the scalar multiplication of \( R \) on \( M \). For a concrete example, note that \( H_0(R) \) itself can be viewed as a DG module via the canonical surjection \( R \twoheadrightarrow H_0(R) \) which exists because \( R \) is a chain DGA.

So if \( \text{amp } R \geq 1 \) then there are DG modules with finitely generated homology over \( H_0(R) \) which are not compact. Hence, as mentioned above, it appears to make no sense to define finite global dimension of DGAs by the condition that each DG module with finitely generated homology must be compact. This contrasts sharply with ring theory where this precise definition works for several classes of rings, such as the local commutative noetherian ones.

**Second main Theorem.** To explain the second main Theorem of the paper, let me first give an alternative, equivalent formulation of Theorem A.

Let \( D^\text{fg}_+(R) \) denote the DG left-\( R \)-modules \( M \) for which each \( H_i(M) \) is finitely generated over \( H_0(R) \), and for which \( H_i(M) = 0 \) for \( i \ll 0 \). It turns out that the compact DG left-\( R \)-modules are exactly the DG modules in \( D^\text{fg}_+(R) \) for which \( \text{k.pd}_R M < \infty \), where \( \text{k.pd} \) denotes \( k \)-projective dimension, see Notation 1.5 and Lemma 2.2. Hence the following is an equivalent formulation of Theorem A.

**Theorem A’.** Assume \( \text{amp } R < \infty \). Let \( L \) be in \( D^\text{fg}_+(R) \) and suppose \( \text{k.pd}_R L < \infty \) and \( L \not\equiv 0 \). Then

\[
\text{amp } L \geq \text{amp } R.
\]

The dual of Theorem A’ is now the second main Theorem of the paper, which will also be proved in the situation specified in Setup 1.1. To state it, some more notation is necessary.

Let \( A \) denote a local commutative noetherian ground ring over which \( R \) is a DGA. Let \( D^\text{fg}_-(R) \) be the DG left-\( R \)-modules \( M \) for which each \( H_i(M) \) is finitely generated over \( H_0(R) \), and for which \( H_i(M) = 0 \) for \( i \gg 0 \). Let \( \text{k.id} \) denote \( k \)-injective dimension, see Notation 1.5 and let
cmd$_A R$ denote the so-called Cohen-Macaulay defect of $R$ over $A$, see [2, (1.1)] or Notation 1.6.

**Theorem B.** Assume amp $R < \infty$. Let $I$ be in $D_{fg}^+(R)$ and suppose k.id$_R I < \infty$ and $I \neq 0$. Then

$$\text{amp } I \geq \text{cmd}_A R.$$ 

**Comments and connections.** Theorems A, A’, and B will be obtained as corollaries of a more general amplitude inequality, Theorem 4.1 which is a DGA generalization of the first of Iversen’s amplitude inequalities from [9].

Theorem B can be written in a more evocative form for certain DGAs. Suppose that the ground ring $A$ has a dualizing complex $C$ and consider $D = \text{RHom}_A(R, C)$ which is sometimes a so-called dualizing DG module for $R$, see [6]. Since amp $R < \infty$ implies cmd$_A R = \text{amp RHom}_A(R, C) = \text{amp } D$ by [2, (1.3.2)], Theorem B takes the form

$$\text{amp } I \geq \text{amp } D.$$ 

So if $D$ is indeed a dualizing DG module for $R$, then, expressed in words, Theorem B says that among the DG modules in $D_{fg}^+(R)$ with k.id$_R I < \infty$, none can be shorter than the dualizing DG module $D$.

Theorems A, A’, and B are specific to the finite amplitude case, and fail completely if the amplitude of $R$ is permitted to be infinite: Let $K$ be a field and consider the polynomial algebra $K[X]$ as a DGA where $X$ is placed in homological degree 1 and where the differential is zero. There is a distinguished triangle

$$\Sigma K[X] \to K[X] \to K \to$$

in $D(K[X])$, involving $K[X]$, the suspension $\Sigma K[X]$ and the trivial DG module $K$. Applying the functor $\text{RHom}_{K[X]}(-, K)$ gives a distinguished triangle

$$\text{RHom}_{K[X]}(K, K) \to K \to \Sigma^{-1} K \to$$

which shows that $\text{RHom}_{K[X]}(K, K)$ has bounded homology whence k.pd$_{K[X]} K < \infty$ and k.id$_{K[X]} K < \infty$, see Notation 1.5.

However, it is clear that amp $K = 0$ and amp $K[X] = \infty$, and not hard to show cmd$_K K[X] = \infty$. Hence

$$\text{amp } K < \text{amp } K[X],$$
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showing that Theorems A′ and A fail, and

\[ \amp K < \cmd K[X], \]

showing that Theorem B fails.

Note that while Theorems A and A′ are uninteresting if \( R \) is a ring concentrated in degree zero (for which \( \amp R = 0 \)), Theorem B is already interesting in this case. For instance, if \( R \) is equal to the ground ring \( A \) placed in degree zero, then Theorem B says that \( \amp I \geq \cmd A \) when \( I \neq 0 \) is a complex in \( \text{D}^b_f(A) \) with \( \k.id_A I < \infty \).

This implies the classical conjecture by Bass that if \( A \) has a finitely generated module \( M \) with finite injective dimension, then \( A \) is a Cohen-Macaulay ring. To see so, apply Theorem B to the injective resolution \( I \) of \( M \). This gives \( 0 = \amp I \geq \cmd A \) whence \( \cmd A = 0 \), that is, \( A \) is Cohen-Macaulay. The ring case of Theorem B and the fact that it implies the conjecture by Bass has been known for a good while to commutative ring theorists, but a published source seems hard to find.

The paper is organized as follows. Section 1 explains some of the notation and terminology. Sections 2 and 3 are preparatory; they prove a number of homological estimates and set up some base change machinery. Finally, Section 4 proves Theorems A and A′, and Section 5 proves Theorem B.

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1. Background

This Section explains some of the notation and terminology of the paper. The usage will be standard and corresponds largely to such references as [1], [6], [7], and [13].

All proofs will be given under the following Setup. Note, however, that the results also hold in some other situations, see Remark 1.7

Setup 1.1. By \( A \) is denoted a local commutative noetherian ring, and by \( R \) a chain DGA (that is, \( R_i = 0 \) for \( i \leq -1 \)) over \( A \) for which each \( H_i(R) \) is a finitely generated \( A \)-module.

It will be assumed that the canonical ring homomorphism \( A \to H_0(R) \) is surjective.
Remark 1.2. Since $A$ is the ground ring for $R$, everything in sight will have an $A$-structure. □

Remark 1.3. Since $A \to H_0(R)$ is surjective, $H_0(R)$ is a quotient of $A$ and so $H_0(R)$ is a local commutative noetherian ring.

In fact, $A \to H_0(R)$ is equal to the composition $A \to R \to H_0(R)$, where $R \to H_0(R)$ is the canonical surjection which exists because $R$ is a chain DGA. Through these morphisms, the residue class field $k$ of the local ring $H_0(R)$ can be viewed as a DG bi-module over $R$, and as a module over $A$, and these will be denoted simply by $k$.

Note that $k$ viewed as a $A$-module is the residue class field of $A$, because $A \to H_0(R)$ is surjective. □

Remark 1.4. Any ring can be viewed as a trivial DGA concentrated in degree zero. A DG module over such a DGA is just a complex of modules over the ring in question.

Also, an ordinary module over a ring can be viewed as a complex concentrated in degree zero, and hence as a DG module over the ring viewed as a DGA.

When there is more than one ring or DGA action on the same object, I will sometimes use subscripts to indicate the actions. For instance, a DG left-$R$-right-$R$-module might be denoted $_R D_R$. □

Notation 1.5. By $R^o$ is denoted the opposite DGA of $R$ with product defined in terms of the product of $R$ by $r \cdot s = (-1)^{|r||s|}sr$. DG left-$R^o$-modules will be identified with DG right-$R$-modules, and $D(R^o)$, the derived category of DG left-$R^o$-modules, will be identified with the derived category of DG right-$R$-modules.

The supremum and the infimum of the DG module $M$ are defined by

$$
\sup M = \sup \{ i \mid H_i(M) \neq 0 \}, \\
\inf M = \inf \{ i \mid H_i(M) \neq 0 \};
$$

in these terms the amplitude of $M$ is

$$
\amp M = \sup M - \inf M.
$$

The $k$-projective dimension, the $k$-injective dimension, and the depth are defined by

$$
kPD R_M = -\inf \text{RHom}_R(M,k), \\
kID R_M = -\inf \text{RHom}_R(k,M), \\
\depth R_M = -\sup \text{RHom}_R(k,M),
$$
see [2, def. (1.1)]. Here $\text{RHom}$ is the right derived functor of $\text{Hom}$ which will be used along with $\mathop{\mathcal{L}}\limits^\ast\otimes$, the left derived functor of $\otimes$.

Derived functors are defined on derived categories. Some notation for these was already given above, but let me collect it here. The derived category of DG left-$R$-modules is denoted by $\mathcal{D}(R)$. The full subcategory of compact objects is denoted by $\mathcal{D}^c(\mathcal{R})$. The full subcategory of $\mathcal{D}(\mathcal{R})$ consisting of DG modules with each $H^i(M)$ finitely generated over $H^0(\mathcal{R})$ and $H^i(M) = 0$ for $i \ll 0$ is denoted by $\mathcal{D}_{fg}^+(\mathcal{R})$. Finally, $\mathcal{D}_{fg}^-(\mathcal{R})$ is denoted by $\mathcal{D}_{b}^{fg}(\mathcal{R})$.

If DG modules are viewed as having the differentials pointing to the right, then $\mathcal{D}_{fg}^+(\mathcal{R})$ consists of DG modules with homology extending to the left, $\mathcal{D}_{fg}^-(\mathcal{R})$ consists of DG modules with homology extending to the right, and $\mathcal{D}_{b}^{fg}(\mathcal{R})$ consists of DG modules with bounded homology.

Observe that $R$ could just be $A$ or $H^0(\mathcal{R})$ concentrated in degree zero. Hence the notations introduced so far can also be applied to $A$ and $H^0(\mathcal{R})$, and define triangulated subcategories $\mathcal{D}^c$, $\mathcal{D}_{fg}^+ \cap \mathcal{D}_{fg}^-(\mathcal{R})$ is denoted by $\mathcal{D}_{b}^{fg}(\mathcal{R})$.

Notation 1.6. It is well known that homological invariants such as projective dimension (often denoted $\text{pd}$) and depth can be extended from modules to complexes of modules, see for instance [5] and [9].

I will need two other extended invariants which are less well known, those of Krull dimension and Cohen-Macaulay defect. The Krull dimension can be found in both [5] and [9], and the Cohen-Macaulay defect in [2]. Let $M$ be in $\mathcal{D}_{fg}^+(A)$. The Krull dimension of $M$ may be defined as

$$\text{Kdim}_A M = \sup \{ \text{Kdim}_A H^i(M) - i \}, \quad (1)$$

see [5, prop. 3.5]. (Note the sign change induced by the difference between the present homological notation and the cohomological notation of [5].) The Cohen-Macaulay defect of $M$ is then

$$\text{cmd}_A M = \text{Kdim}_A M - \text{depth}_A M,$$

see [2, (1.1)].

Remark 1.7. The final condition of Setup 1.1 is that the canonical ring homomorphism $A \to H^0(\mathcal{R})$ is surjective.
This implies that when localizing the Setup at a prime ideal of the ground ring $A$, the ring $H_0(R)$ remains local; a fact needed in some of the proofs.

However, the results of the paper sometimes apply even if $A \to H_0(R)$ is not surjective.

Namely, suppose that the conditions of Setup 1.1 are satisfied except that $A \to H_0(R)$ is not surjective. Suppose moreover that $R_0$ is central in $R$ and that $H_0(R)$ is finitely generated as an $A$-algebra by $\xi_1, \ldots, \xi_n$. Then the results of this paper still apply to $R$.

To see this, pick cycles $\Xi_1, \ldots, \Xi_n$ in $R_0$ representing $\xi_1, \ldots, \xi_n$, set $A' = A[X_1, \ldots, X_n]$, and consider the $A$-linear ring homomorphism $A' \to R_0$ given by $X_i \mapsto \Xi_i$. Then $R$ is a DGA over $A'$ and the canonical ring homomorphism $A' \to H_0(R)$ is surjective.

To achieve the situation of Setup 1.1 it remains to make the ground ring local. For this, let $m$ be the maximal ideal of $H_0(R)$ and let $p$ be the contraction to $A'$. Replace $A'$ and $R$ by the base changed versions

$$\tilde{A} = A'_p \otimes_{A'} A' \cong A'_p \quad \text{and} \quad \tilde{R} = A'_p \otimes_{A'} R.$$

Then $\tilde{A}$ is local and the canonical ring homomorphism $\tilde{A} \to H_0(\tilde{R})$ is surjective. Hence the pair $\tilde{A}$ and $\tilde{R}$ fall under Setup 1.1 and so the results of the paper apply to $\tilde{R}$.

Now, the localization at $p$ inverts the elements of $A'$ outside $p$. Such elements are mapped to elements of $H_0(R)$ outside $m$, and these are already invertible. Hence the homology of the canonical morphism

$$R \to A'_p \otimes_{A'} R$$

is an isomorphism; that is, the canonical morphism

$$R \to \tilde{R}$$

is a quasi-isomorphism. This implies that $R$ and $\tilde{R}$ have equivalent derived categories, see [14, III.4.2], and so, since the results of this paper apply to $\tilde{R}$, they also apply to $R$. $\Box$

\section{2. Homological estimates}

This Section provides some estimates which will be used as input for the proofs of the main Theorems.
The following Lemma is well known. It holds because \( H_0(R) \) is local. The proof is a simple application of the Eilenberg-Moore spectral sequence, see [4, exam. 1, p. 280].

**Lemma 2.1.** Let \( X \) be in \( \mathcal{D}^\ell_+ (R) \) and let \( Y \) be in \( \mathcal{D}^\ell_+(R) \). Then

\[
\inf (X \otimes_R Y) = \inf X + \inf Y.
\]

Consequently, if \( X \not\cong 0 \) and \( Y \not\cong 0 \) then \( X \otimes_R Y \not\cong 0 \). \( \square \)

For the following Lemma, note that \( H_0(R) \) can be viewed as a DG left-\( H_0(R) \)-right-\( R \)-module; in subscript notation, \( H_0(R) H_0(R) R \). If \( L = RL \) is a DG left-\( R \)-module, then

\[
H_0(R) \otimes_R L = H_0(R) R \otimes_R R L
\]

inherits a DG left-\( H_0(R) \)-module structure. Since a DG left-\( H_0(R) \)-module is just a complex of left-\( H_0(R) \)-modules, \( H_0(R) \otimes_R L \) is hence a complex of left-\( H_0(R) \)-modules.

**Lemma 2.2.** Let \( L \) be in \( \mathcal{D}(R) \). Then

\( L \) is in \( \mathcal{D}^c(R) \) \( \iff \) \( L \) is in \( \mathcal{D}^\ell_+(R) \) and \( k.pd_R L < \infty \).

If these equivalent statements hold, then \( H_0(R) \otimes_R L \) is in \( \mathcal{D}^c(H_0(R)) \), and

\[
pd_{H_0(R)}(H_0(R) \otimes_R L) = k.pd_R L.
\]

**Proof.** \( \Rightarrow \) Let \( L \) be in \( \mathcal{D}^c(R) \); that is, \( L \) is finitely built from \( R \) in \( \mathcal{D}(R) \). Setup [4.1] implies that \( R \) is in \( \mathcal{D}^\ell_+(R) \). Moreover, \( \sup k \otimes_R R = \sup k = 0 < \infty \). But then \( L \), being finitely built from \( R \), is also in \( \mathcal{D}^\ell_+(R) \) and has \( \sup k \otimes_R L < \infty \). And \( \sup k \otimes_R L < \infty \) implies \( k.pd_R L < \infty \) by [7, rmk. (1.2)].

\( \Leftarrow \) When \( L \) is in \( \mathcal{D}^\ell_+(R) \), there is a minimal semi-free resolution \( F \rightarrow L \) by [4 (0.5)]. When \( k.pd_R L < \infty \), it is not hard to see from [4 (0.5) and lem. (1.7)] that there is a semi-free filtration of \( F \) which only contains finitely many quotients of the form \( \Sigma^i R^\alpha \) where \( \Sigma^i \) denotes the \( i \)'th suspension and where \( \alpha \) is finite. This means that \( F \) and hence \( L \) is finitely built from \( R \).

Now suppose that the equivalent statements hold. It is clear that \( H_0(R) \otimes_R R \cong H_0(R) \) is in \( \mathcal{D}_b^\ell(H_0(R)) \). As \( L \) is finitely built from \( R \),
it follows that $H_0(R) \overset{L}{\otimes}_R L$ is also in $D^f_h(H_0(R))$. Therefore the first $=$ in the following computation holds by [3 (A.5.7.3)],
\[
\text{pd}_{H_0(R)}(H_0(R) \overset{L}{\otimes}_R L) = -\inf \text{RHom}_{H_0(R)}(H_0(R) \overset{L}{\otimes}_R L, k) = -\inf \text{RHom}_R(L, k) = k.pd_R L,
\]
where the second $=$ is by adjunction and the last $=$ is by definition. □

The following Lemmas use that, as noted in Remark 1.2 all objects in sight have an $A$-structure.

**Lemma 2.3.** Let $X$ be in $D^f_c(R)$ and let $L$ be in $D_c(R)$. Then
\[
\text{depth}_A(X \overset{L}{\otimes}_R L) = \text{depth}_A X - k.pd_R L.
\]

**Proof.** The Lemma can be proved by a small variation of a well known proof of the Auslander-Buchsbaum theorem, as given for instance in [11, thm. 3.2]. Let me give a summary for the benefit of the reader.

Since $L$ is finitely built from $R$ in $D(R)$, there is an isomorphism
\[
\text{RHom}_A(k, X \overset{L}{\otimes}_R L) \cong \text{RHom}_A(k, X) \overset{L}{\otimes}_R L = (*).
\]
Replace $\text{RHom}_A(k, X)$ with a quasi-isomorphic truncation $T$ concentrated in homological degrees $\leq \sup \text{RHom}_A(k, X)$; see [7 (0.4)]. Replace $L$ with a minimal semi-free resolution $F$; see [7 (0.5)]. Then
\[
(*) \cong T \overset{R}{\otimes}_R F,
\]
and hence
\[
\sup \text{RHom}_A(k, X \overset{L}{\otimes}_R L) = \sup T \overset{R}{\otimes}_R F.
\]
The claim of the Lemma is that
\[
\sup \text{RHom}_A(k, X \overset{L}{\otimes}_R L) = \sup \text{RHom}_A(k, X) + k.pd_R L,
\]
and by the above this amounts to
\[
\sup T \overset{R}{\otimes}_R F = \sup \text{RHom}_R(k, X) + k.pd_R L. \tag{2}
\]

Forgetting the differentials of $R$ and $F$ gives the underlying graded algebra $R^\natural$ and the underlying graded module $F^\natural$, and [7 (0.5)] says that
\[
F^\natural \cong \prod_{i \leq k.pd_R L} \Sigma^i (R^\natural)^{\beta_i}.
\]
Hence
\[(T \otimes_R F)_* \cong T^* \otimes_R F_* \cong T^* \otimes_R \coprod_{i \leq k.pd_R L} \Sigma^i(T^*)^{\beta_i} \cong \prod_{i \leq k.pd_R L} \Sigma^i(T^*)^{\beta_i}.\]

The right hand side is just a collection of copies of \(T^*\) moved around by \(\Sigma^i\), so since \(T^*\) and hence \(T^*\) is concentrated in homological degrees \(\leq \sup \text{RHom}_A(k, X)\), the right hand side and therefore the left hand side is concentrated in homological degrees \(\leq \sup \text{RHom}_A(k, X) + k.pd_R L\). This implies
\[\sup T \otimes_R F \leq \sup \text{RHom}_A(k, X) + k.pd_R L.\]

Using that \(\beta_{k.pd_R L} \neq 0\) by \([7, \text{lem. } (1.7)]\), it is possible also to see
\[\sup T \otimes_R F \geq \sup \text{RHom}_A(k, X) + k.pd_R L.\]

This proves Equation (2) and hence the Lemma. \(\square\)

Through the canonical morphism \(R \to H_0(R)\), an \(H_0(R)\)-module \(M\) can be viewed as a DG right-\(R\)-module. If \(M\) is finitely generated over \(H_0(R)\), then as a DG right-\(R\)-module it is in \(D^{fg}_c(R)\).

**Lemma 2.4.** Let \(M\) be a finitely generated \(H_0(R)\)-module and let \(L\) be in \(D^{fg}_c(R)\). Suppose \(M \neq 0\) and \(L \neq 0\). View \(M\) as a DG right-\(R\)-module in \(D^{fg}_b(R)\), and suppose
\[\text{Kdim}_A H_i(M \otimes_R L) \leq 0\]
for each \(i\). Then
\[k.pd_R L \geq \text{Kdim}_A M + \text{inf} L.\]

**Proof.** If \(k.pd_R L = \infty\) then the Lemma holds trivially, so suppose \(k.pd_R L < \infty\). Then Lemma 2.2 says that \(H_0(R) \otimes_R L\) is in \(D^c(H_0(R))\). That is, \(H_0(R) \otimes_R L\) is finitely built from \(H_0(R)\), so \(H_0(R) \otimes_R L\) is isomorphic to a bounded complex of finitely generated free \(H_0(R)\)-modules. Also, Lemma 2.1 implies \(H_0(R) \otimes_R L \neq 0\), and hence \([9, \text{thm. } 4.1]\) says
\[\text{pd}_H_0(R)(H_0(R) \otimes_R L) \geq \text{Kdim}_{H_0(R)} M - \text{Kdim}_{H_0(R)}(M \otimes_{H_0(R)} (H_0(R) \otimes_R L)).\]

Note that the assumption in \([9]\) that the ring is equicharacteristic is unnecessary: The assumption is only used to ensure that the so-called
new intersection theorem is valid, and this was later proved for all local noetherian commutative rings in [16, thm. 1].

Moving the parentheses in the last term gets rid of tensoring with $H_0(R)$, and Krull dimensions over $H_0(R)$ can be replaced with Krull dimensions over $A$ because $A \rightarrow H_0(R)$ is surjective, so the inequality is

$$\text{pd}_{H_0(R)}(H_0(R) \otimes_R L) \geq \text{Kdim}_A M - \text{Kdim}_A(M \otimes_R L).$$

(3)

The first term here is

$$\text{pd}_{H_0(R)}(H_0(R) \otimes_R L) = k \cdot \text{pd}_R L$$

(4)

by Lemma 2.2. For the third term, note that the assumption

$$\text{Kdim}_A H_i(M \otimes_R L) \leq 0$$

for each $i$ implies

$$\text{Kdim}_A(M \otimes_R L) = - \inf(M \otimes_R L) = (*)$$

(5)

see Notation 1.6. But Lemma 2.1 implies

$$(*) = - \inf L.$$  

(6)

Substituting Equations (4) to (6) into the inequality (3) gives the inequality claimed in the Lemma. □

3. Flat base change

This Section sets up a theory of flat base change which will be used in the proofs of the main Theorems.

Let $\tilde{A}$ be a local noetherian commutative ring and let $A \rightarrow \tilde{A}$ be a flat ring homomorphism.

It is clear that

$$\tilde{R} = \tilde{A} \otimes_A R$$

is a chain DGA over $\tilde{A}$. The homology is

$$H_i(\tilde{R}) = H_i(\tilde{A} \otimes_A R) \cong \tilde{A} \otimes_A H_i(R)$$

and this is finitely generated over $\tilde{A}$ for each $i$. The canonical ring homomorphism $A \rightarrow H_0(R)$ is surjective, so $\tilde{A} \otimes_A A \rightarrow \tilde{A} \otimes_A H_0(R)$ is also surjective, but this map is isomorphic to the canonical ring homomorphism

$$\tilde{A} \rightarrow H_0(\tilde{R})$$
which is hence surjective. So Setup 1.1 applies to the DGA $\tilde{R}$ over the ring $\tilde{A}$.

There is a morphism of DGAs

$$R \to \tilde{R}$$

given by $r \mapsto 1 \otimes r$, and this defines a base change functor of DG left modules

$$\tilde{R}^L \otimes_R : \text{D}(R) \to \text{D}(\tilde{R})$$

which in fact is just given by

$$\tilde{R}^L \otimes_R = (\tilde{A} \otimes_A R)^L \otimes_R \cong \tilde{A} \otimes_A (R^L \otimes_R \cong \tilde{A} \otimes_A -). \quad (7)$$

There is also a base change functor of DG right modules.

It is easy to see that the base change functors preserve membership of the subcategories $D^c$, $D^b_+$, $D^b_-$, and $D^b_f$.

If $X$ is in $D(\mathcal{R})$ and $Y$ is in $D(R)$, then it is an exercise to compute the derived tensor product of the base changed DG modules $\tilde{X} = X^L \otimes_R \tilde{R}$ and $\tilde{Y} = \tilde{R}^L \otimes_R Y$ as

$$\tilde{X}^L \otimes_R \tilde{Y} \cong \tilde{A} \otimes_A (X^L \otimes_R Y). \quad (8)$$

4. AMPLITUDE INEQUALITIES FOR COMPACT OBJECTS

This Section proves Theorem 4.1 which is a DGA generalization of the first of Iversen’s amplitude inequalities from [2]. Theorems A and A’ from the Introduction follow easily.

**Theorem 4.1.** Let $X$ be in $D^b_+(\mathcal{R})$ and let $L$ be in $D^c(R)$. Suppose $X \not\cong 0$ and $L \not\cong 0$. Then

$$\text{amp}(X^L \otimes_R L) \geq \text{amp}(X).$$

**Proof.** The inequality says

$$\sup(X^L \otimes_R L) - \inf(X^L \otimes_R L) \geq \sup X - \inf X,$$

which by Lemma 2.1 is the same as

$$\sup(X^L \otimes_R L) \geq \sup X + \inf L. \quad (9)$$

Write

$$M = H_{\sup X}(X)$$
for the top homology of $X$. With this notation, [3] prop. 3.17] says
\[
dept_A X \leq \Kdim_A M - \sup X. \tag{10}
\]
(Note again the difference between the homological notation of this paper and the cohomological notation of [3].)

To prove the Theorem, consider first the special case where
\[
\Kdim_A H_i(M \otimes_R L) \leq 0
\]
for each $i$. Then
\[
\sup (X \otimes_R L) \geq -\depth_A (X \otimes_R L)
\]
\[
\geq -\depth_A X + k.pd_R L
\]
\[
\geq -\depth_A X + \Kdim_A M + \inf L
\]
\[
\geq -\Kdim_A M + \sup X + \Kdim_A M + \inf L
\]
\[
= \sup X + \inf L
\]
proving (10). Here (a) is by [3 eq. (3.3)], (b) is by Lemma 2.3 (c) is by Lemma 2.4 and (d) is by Equation (10).

Next the general case which will be reduced to the above special case by localization. Observe that $M \otimes_R L \not\cong 0$ by Lemma 2.1. Pick a prime ideal $p$ of $A$ which is minimal in
\[
\bigcup_i \Supp_A H_i(M \otimes_R L)
\]
and consider the flat ring homomorphism $A \to A_p$. Set
\[
\tilde{R} = A_p \otimes_A R, \quad \tilde{X} = A_p \otimes_A X, \quad \tilde{L} = A_p \otimes_A L
\]
so that $\tilde{X}$ and $\tilde{L}$ are the base changes of $X$ and $L$ to $\tilde{R}$, see Section 3.

Let me check that the above special case of the Theorem applies to $\tilde{X}$ and $\tilde{L}$. The theory of Section 3 says that Setup 1.1 applies to $\tilde{R}$, $\tilde{X}$ is in $D^{fg}_b(\tilde{R}^e)$, and $\tilde{L}$ is in $D^c(\tilde{R})$. Moreover, $p$ is in the support of some $H_i(M \otimes_R L)$ in $A$ so must be in the support of $M = H_{\sup X}(X)$ and in the support of some $H_i(L)$. It follows that $\tilde{X} \not\cong 0$ and $\tilde{L} \not\cong 0$.

Since $p$ is in the support of $M = H_{\sup X}(X)$, it even follows that
\[
\sup \tilde{X} = \sup X \tag{12}
\]
and 
\[ \widetilde{M} = H_{\text{sup}, \widetilde{X}}(\widetilde{X}) = H_{\text{sup}, X}(A_p \otimes_A X) \cong A_p \otimes_A H_{\text{sup}, X}(X) = A_p \otimes_A M. \]

Finally, Equation (8) from Section 3 implies
\[ H_i(\widetilde{M} \otimes_R \widetilde{L}) \cong A_p \otimes_A H_i(M \otimes_R L). \]

The support of each of these modules in \( A_p \) is either empty or equal to the maximal ideal \( \mathfrak{p}_p \) since \( \mathfrak{p}_p \) was chosen minimal in the set (11), and each of the modules is finitely generated over \( A_p \) because each \( H_i(M \otimes_R L) \) is finitely generated over \( H_0(R) \) and hence over \( A \). So
\[ \text{Kdim}_{A_p} H_i(\widetilde{M} \otimes_R \widetilde{L}) \leq 0 \]
for each \( i \).

Hence the above special case of the Theorem does apply and gives
\[ \text{sup}(\widetilde{X} \otimes_R \widetilde{L}) \geq \text{sup} \widetilde{X} + \text{inf} \widetilde{L}, \]
which by Equation (8) again is
\[ \text{sup}(A_p \otimes_A (X \otimes_R L)) \geq \text{sup} \widetilde{X} + \text{inf} \widetilde{L}. \]

So
\[ \text{sup}(X \otimes_R L) \geq \text{sup}(A_p \otimes_A (X \otimes_R L)) \]
\[ \geq \text{sup} \widetilde{X} + \text{inf} \widetilde{L} \]
\[ = \text{sup} X + \text{inf} \widetilde{L} \]
\[ \geq \text{sup} X + \text{inf} L \]
proving (12). Here (e) is by (13) and (f) is by (12). \( \square \)

**Proof (of Theorems A and A').** Theorem A follows by setting \( X = R \) in Theorem 4.1, and Theorem A' is equivalent to Theorem A by Lemma 2.2. \( \square \)

### 5. Amplitude Inequality for Objects with Finite \( k \)-Injective Dimension

This Section proves Theorem B from the Introduction. The proof uses dualizing complexes; see [8, chp. V]. Since, on one hand, not all rings have dualizing complexes, while, on the other, complete local
noetherian commutative rings do, it is also necessary to include some material on completions.

The following Proposition assumes that the ground ring \( A \) has a dualizing complex \( C \), and considers the DG left-\( R \)-right-\( R \)-module
\[
RD_R = \text{RHom}_A(RR_R, C)
\]
whose left-structure comes from the right-structure of the \( R \) in the first argument of \( \text{RHom} \), and vice versa. By forgetting the right-structure, I can get a DG left-\( R \)-module \( R \).

**Proposition 5.1.** Suppose that \( A \) has a dualizing complex \( C \) and set \( R \) \( D \) \( R \) \( \text{RHom}_A(RR_R, C) \). Let \( I \) be in \( D_{fg}^\leq(R) \). Then the following conditions are equivalent.

(i) \( k \cdot \text{id}_R I < \infty \).

(ii) \( R \) \( I \) is finitely built from \( R \) \( D \) in \( D(R) \).

(iii) \( R \) \( I \cong R \) \( D \) \( R \) \( L \) \( \otimes_R \) \( R \) \( L \) for an \( R \) \( L \) in \( D_c(R) \).

**Proof.** (i) \( \Rightarrow \) (iii). Let
\[
M_R = \text{RHom}_A(RI, C)
\]
be the dual of \( I \). Since \( I \) is in \( D_{fg}^\leq(R) \) and hence in \( D_{fg}^\leq(A) \), it follows that \( M \) is \( D_{fg}^\leq(A) \) and hence in \( D_{fg}^\leq(R^o) \). Moreover,
\[
M \overset{L}{\otimes}_R k = \text{RHom}_A(I, C) \overset{L}{\otimes}_R k \overset{(a)}{=} \text{RHom}_A(\text{RHom}_R(k, I), C) = (*),
\]
where \( (a) \) holds because \( C \) is isomorphic in \( D(A) \) to a bounded complex of injective modules, cf. [3, (A.4.24)]. The assumption \( k \cdot \text{id}_R I < \infty \) implies that \( \text{RHom}_R(k, I) \) has bounded homology, so the same is true for \( (*) \) whence \( \sup M \overset{L}{\otimes}_R k < \infty \). This implies \( k \cdot \text{pd}_{R^o} M < \infty \) by [7, rmk. (1.2)].

By Lemma 2.2 this means that \( M \) is in \( D^c(R^o) \), that is, \( M_R \) is finitely built from \( R_R \). But then \( R \) \( L = \text{RHom}_{R^o}(M_R, R R_R) \) is finitely built from \( R R \) and satisfies
\[
M_R \cong \text{RHom}_R(RL, R R_R),
\]
and hence
\[ R I \cong R \text{Hom}_A(\text{Hom}_A(RI, C), C) \]
\[ = R \text{Hom}_A(M_R, C) \]
\[ \cong R \text{Hom}_A(\text{Hom}_R(RL, RR), C) \]
\[ \cong R \text{Hom}_A(RR, C) \otimes_R RL \]
\[ = R D_R \otimes_R RL, \]
proving (iii). Here (b) is by [3, thm. (A.8.5)] and (c) is because \( RL \) is finitely built from \( RR \).

(iii) \( \Rightarrow \) (ii). For \( RL \) to be in \( D(R) \) means that \( RL \) is finitely built from \( RR \) in \( D(R) \). But then
\[ R I \cong R D_R \otimes_R RL \]
is finitely built from
\[ R D_R \otimes_R RR \cong R D \]
in \( D(R) \).

(ii) \( \Rightarrow \) (i). Without loss of generality, I can assume that \( C \) is normalized, that is, \( R \text{Hom}_A(k, C) \cong k \). Then
\[ R \text{Hom}_R(k, RD) = R \text{Hom}_R(k, R \text{Hom}_A(RR, C)) \]
\[ \cong R \text{Hom}_A(RR \otimes_R k, C) \]
\[ \cong R \text{Hom}_A(k, C) \]
\[ \cong k \]
has bounded homology, where (d) is by adjunction. When \( RI \) is finitely built from \( RD \), then the homology of \( R \text{Hom}_R(k, RI) \) is also bounded, and then
\[ k \text{id}_R I = - \inf R \text{Hom}_R(k, RI) < \infty. \]

For the remaining part of the paper, let \( \mathfrak{m} \) be the maximal ideal of \( A \) and consider \( \hat{A} \), the completion of \( A \) in the \( \mathfrak{m} \)-adic topology, which is a local noetherian commutative ring by [15, p. 63, (4)].

The canonical ring homomorphism \( A \to \hat{A} \) is flat by [15, p. 63, (3)], and the theory of Section 3 gives a new chain DGA
\[ \hat{R} = \hat{A} \otimes_A R \]
over \( \hat{A} \), and base change functors for DG modules from \( R \) to \( \hat{R} \).
Lemma 5.2. Let $I$ be in $D(R)$ and consider the base changed DG module $\hat{I} = \hat{R} \otimes_R I$ in $D(\hat{R})$. Then

(i) $\text{amp}\, \hat{I} = \text{amp}\, I$.
(ii) $k.\text{id}_R \hat{I} = k.\text{id}_R I$.

Proof. (i). Equation (7) from Section 3 says

$$\hat{I} = \hat{R} \otimes_R I \cong \hat{A} \otimes_A I.$$ 

Since $\hat{A}$ is faithfully flat over $A$ by [15, p. 63, (3)], part (i) is clear.

(ii). The residue class field of $H_0(R)$ is $k$, so the residue class field of $H_0(\hat{R}) \cong \hat{A} \otimes_A H_0(R)$ is $\hat{A} \otimes_A k$ which by Equation (7) is $\hat{A} \otimes_A k \cong \hat{R} \otimes_R k$.

Hence

$$k.\text{id}_R \hat{I} = - \inf R\text{Hom}_R(\hat{R} \otimes_R k, \hat{I}) = (\ast).$$

But

$$R\text{Hom}_R(\hat{R} \otimes_R k, \hat{I}) \overset{(a)}{=} R\text{Hom}_R(k, R\text{Hom}_R(\hat{R}, \hat{I}))$$

$$\cong R\text{Hom}_R(k, \hat{I})$$

$$\cong R\text{Hom}_R(k, \hat{A} \otimes_A I)$$

$$\overset{(b)}{=} \hat{A} \otimes_A R\text{Hom}_R(k, I),$$

where (a) is by adjunction and (b) is because $\hat{A}$ is flat over $A$ while $k$ has finitely generated homology, cf. [3] (A.4.23)]. Hence

$$(\ast) = - \inf \hat{A} \otimes_A R\text{Hom}_R(k, I) \overset{(c)}{=} - \inf \text{Hom}_R(k, I) = k.\text{id}_R I,$$

proving part (ii). Here (c) is again because $\hat{A}$ is faithfully flat over $A$. $\Box$

Proof (of Theorem B). The base change $A \to \hat{A}$ induces the change from $R$ and $I$ to $\hat{R}$ and $\hat{I}$.

$\hat{I}$ is in $D_{fg}(\hat{R})$ by the theory of Section 3. Lemma 5.2 implies that $k.\text{id}_R \hat{I} < \infty$ and that $\text{amp}\, \hat{I} = \text{amp}\, I$. Moreover, $\text{cmd}_A \hat{R} = \text{cmd}_A R$ by [2] prop. (1.2)]. So it is enough to prove Theorem B for $\hat{I}$ over $\hat{R}$.
Setup 1.1 applies to $\hat{R}$ over $\hat{A}$ by Section 3, so the results proved so far apply to DG modules over $\hat{R}$. Since $\hat{A}$ is complete, it has a dualizing complex $C$ by [8, sec. V.10.4]. Hence Proposition 5.1 gives

$$\hat{I} \cong \text{RHom}_{\hat{A}}(\hat{R}, C) \overset{L}{\otimes}_{\hat{R}} L$$

for an $L$ in $D^c(\hat{R})$. But then Theorem 4.1 gives

$$\text{amp } \hat{I} \geq \text{amp } \text{RHom}_{\hat{A}}(\hat{R}, C) = \text{cmd } \hat{A} \hat{R}$$

as desired, where the $=$ is by [2] (1.3.2). □

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