ENDPOINT STRICHARTZ ESTIMATES FOR THE MAGNETIC
SCHRÖDINGER EQUATION

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Abstract. We prove Strichartz estimates for the Schrödinger equation with an electromagnetic potential, in dimension $n \geq 3$. The decay and regularity assumptions on the potentials are almost critical, i.e., close to the Coulomb case. In addition, we require repulsivity and a non trapping condition, which are expressed as smallness of suitable components of the potentials. However, the potentials themselves can be large, and we avoid completely any a priori spectral assumption on the operator. The proof is based on smoothing estimates and new Sobolev embeddings for spaces associated to magnetic potentials.

1. Introduction

Recent research on linear and nonlinear dispersive equations is largely focused on measuring precisely the rate of decay of solutions. Indeed, decay and Strichartz estimates are one of the central tools of the theory, with immediate applications to local and global well posedness, existence of low regularity solutions, and scattering. This point of view includes most fundamental equations of physics like the Schrödinger, Klein-Gordon, wave and Dirac equations. Strichartz estimates appeared in [32]; the basic framework for this study was laid out in the two papers [14], [20], which examined in an exhaustive way the case of constant coefficient, unperturbed equations. This leads naturally to the possible extensions to equations perturbed with electromagnetic potentials or with variable coefficients; a general theory of dispersive properties for such equations is still under construction and very actively researched.

In the present paper we shall focus on the time dependent Schrödinger equation

\begin{equation}
\begin{aligned}
    i\partial_t u(t,x) &= H u(t,x), \\
    u(0,x) &= \varphi(x), \\
    x &\in \mathbb{R}^n, \quad n \geq 3
\end{aligned}
\end{equation}

associated with the electromagnetic Schrödinger operator

\begin{equation}
H := -\nabla^2_A + V(x), \quad \nabla_A := \nabla - iA(x)
\end{equation}

where $A = (A^1, \ldots, A^n) : \mathbb{R}^n \to \mathbb{R}^n$, $V : \mathbb{R}^n \to \mathbb{R}$. We recall that in the unperturbed case $A \equiv 0$, $V \equiv 0$, dispersive properties are best expressed in terms of the mixed norms on $\mathbb{R}^{1+n}$

\[ L^p L^q := L^p(\mathbb{R}^1; L^q(\mathbb{R}^n)) \]

as follows: for every $n \geq 3$,

\[ \| e^{itA} \varphi \|_{L^p L^q} \leq c_n \| \varphi \|_{L^2}, \]

provided the couple $(p, q)$ satisfies the admissibility condition

\begin{equation}
\frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \quad 2 \leq p \leq \infty.
\end{equation}
These estimates are usually referred to as Strichartz estimates. Our main goal is to find sufficient conditions on the potentials $A, V$ such that Strichartz estimates are true for the perturbed equation (1.1).

In the purely electric case $A \equiv 0$ the literature is extensive and almost complete; we may cite among many others the papers [3], [16], [25]. It is now clear that in this case the decay $V(x) \sim 1/|x|^2$ is critical for the validity of Strichartz estimates; suitable counterexamples were constructed in [17]. In the magnetic case $A \neq 0$, the Coulomb decay $|A| \sim 1/|x|$ is likely to be critical, however no explicit counterexamples are available at the time. An intense research is ongoing concerning Strichartz estimates for the magnetic Schrödinger equation, see e.g. [9], [7], [8], [13]; see also [24] for a more general class of first order perturbations.

Due to the perturbative techniques used in the above mentioned papers, an assumption concerning absence of zero-energy resonances for the perturbed operator $H$ is typically required in order to preserve the dispersion. In the case $A \equiv 0$ it was shown in [3] how this abstract condition can be dispensed with, by directly proving some weak dispersive estimates (also called Morawetz or smoothing estimates) via multipliers methods. Here we shall give a very short proof of Strichartz estimates for the magnetic Schrödinger equation with potentials of almost Coulomb decay, based uniquely on the weak dispersive estimates proved in [12]. The leading theme is that direct multiplier techniques allow to avoid, under suitable repulsivity conditions on $V$ and non-trapping conditions on $A$ (see also [11]), the presence of non-dispersive components, and to preserve Strichartz estimates.

We begin by introducing some notations. Regarding as usual the potential $A$ as a 1-form, we define the corresponding magnetic field as the 2-form $B = dA$, which can be identified with the anti-symmetric gradient of $A$:

$$B \in \mathcal{M}_{n \times n}, \quad B = DA - (DA)^t,$$

where $(DA)_{ij} = \partial_i A^j$, $(DA)^t_{ij} = (DA)_{ji}$. In dimension 3, $B$ is uniquely determined by the vector field curl $A$ via the vector product

$$Bv = \text{curl} A \times v, \quad \forall v \in \mathbb{R}^3.$$

We define the trapping component of $B$ as

$$B_\tau(x) = \frac{x}{|x|} B(x);$$

when $n = 3$ this reduces to

$$B_\tau(x) = \frac{x}{|x|} \times \text{curl} A(x), \quad n = 3,$$

thus we see that $B_\tau$ is a tangential vector. The trapping component may be interpreted as an obstruction to the dispersion of solutions; some explicit examples of potentials $A$ with $B_\tau = 0$ in dimension 3 are given in [11], [12].

Moreover, by

$$\partial_\tau V = \nabla V \cdot \frac{x}{|x|},$$

we denote the radial derivative of $V$, and we decompose it into its positive and negative part

$$\partial_\tau V = (\partial_\tau V)_+ - (\partial_\tau V)_-.$$

The positive part $(\partial_\tau V)_+$ also represents an obstruction to dispersion, and indeed we shall require it to be small in a suitable sense. To ensure good spectral properties of the operator we shall also assume that the negative part $V_-$ is not too large in the sense of the Kato norm:
Definition 1.1. Let $n \geq 3$. A measurable function $V(x)$ is said to be in the Kato class $K_n$ provided

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y| \leq r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0.$$ 

We shall usually omit the reference to the space dimension and write simply $K$ instead of $K_n$. The Kato norm is defined as

$$\|V\|_K = \sup_{x \in \mathbb{R}^n} \int \frac{|V(y)|}{|x-y|^{n-2}} dy.$$ 

A last notation we shall need is the radial-tangential norm

$$\|f\|^p_{L^2 L^\infty(S_r)} := \int_0^\infty \sup_{|x| = r} |f|^p dr.$$ 

In our results we always assume that the operators $H$ and $-\Delta_A := -(\nabla - iA)^2$ are self-adjoint and positive on $L^2$, in order to ensure the existence of the propagator $e^{itH}$ and of the powers $H^s$ via the spectral theorem. There are several sufficient conditions for selfadjointness and positivity, which can be expressed in terms of the local integrability properties of the coefficients (see the standard references [6], [21]); here we prefer to leave this as an abstract assumption. Our main result is the following:

**Theorem 1.1.** Let $n \geq 3$. Given $A, V \in C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, assume the operators $\Delta_A = -(\nabla - iA)^2$ and $H = -\Delta_A + V$ are selfadjoint and positive on $L^2$. Moreover assume that

$$\|V^{-}\|_K < \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} - 1)}$$

for a sufficiently small $\epsilon > 0$ depending on $A$ and

$$\sum_{j \in \mathbb{Z}} 2^j \sup_{|x| \sim 2^j} |A| + \sum_{j \in \mathbb{Z}} 2^j \sup_{|x| \sim 2^j} |V| < \infty,$$

and the Coulomb gauge condition

$$\text{div } A = 0.$$ 

Finally, when $n = 3$, we assume that for some $M > 0$

$$\left(\frac{M + 1}{M}\right)^2 \| |x| B_r \|_{L^2 L^\infty(S_r)}^2 + (2M + 1) \| |x|^2 (\partial_r V)_+ \|_{L^1 L^\infty(S_r)} < \frac{1}{2},$$

while for $n \geq 4$ we assume that

$$\| |x|^2 B_r (x) \|_{L^\infty}^2 + 2 \| |x|^3 (\partial_r V)_+ (x) \|_{L^{\infty}} \leq \frac{2}{3} (n - 1) (n - 3).$$

Then, for any Schrödinger admissible couple $(p, q)$, the following Strichartz estimates hold:

$$\| e^{itH} \varphi \|_{L^p L^q} \leq C \| \varphi \|_{L^2}, \quad \frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \quad p \geq 2, \quad p \neq 2 \text{ if } n = 3.$$ 

In dimension $n = 3$, we have the endpoint estimate

$$\| |D\frac{r}{2} e^{itH} \varphi \|_{L^2 L^6} \leq \| H \frac{1}{2} \varphi \|_{L^2},$$

**Remark 1.1.** Let us remark that the regularity assumption $A, V \in C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ is actually stronger than what we really require. For the validity of the Theorem, we just need to give meaning to inequalities (1.11), (1.12).
Remark 1.2. Assumptions (1.10), (1.11), and (1.12) imply the weak dispersion of the propagator $e^{itH}$ (see Theorems 1.9, 1.10, assumptions (1.24), and (1.27) in [12]). Actually assumption (1.24) in [12] seems to be stronger than (1.11), but reading carefully the proof of Theorem 1.9 in [12] it is clear that the real assumption is our (1.11) (see inequality (3.14) in [12]). The strict inequality in (1.11), (1.12) is essential, in order to dispose of the weighted $L^2$-estimate in the above mentioned Theorems by [12] (see also inequality (3.5) below).

Remark 1.3. We emphasize that in Theorem 1.1 we do not require absence of resonances at energy zero, in contrast with [7], [8]. Indeed, this is possible thanks to the non-trapping and repulsivity conditions (1.11), (1.12); notice however that these conditions can be checked easily in concrete examples, which is not the case for the abstract assumption on resonances.

The derivation of Strichartz estimates from the weak dispersive ones turns out to be remarkably simple if working on the half derivative $|D|^{1/2}u$, see Section 3 for details. As a drawback, the final estimates are expressed in terms of fractional Sobolev spaces generated by the perturbed magnetic operator $-\Delta_A$. Thus, in order to revert to standard Strichartz norms as in (1.13), we need suitable bounds for the perturbed Sobolev norms in terms of the standard ones. This is provided by the following theorem, which we think is of independent interest.

**Theorem 1.2.** Let $n \geq 3$. Given $A \in L^2_{\text{loc}}(\mathbb{R}^n), V : \mathbb{R}^n \to \mathbb{R}$, assume the operators $\Delta_A = -(\nabla - iA)^2$ and $H = -\Delta_A + V$ are selfadjoint and positive on $L^2$. Moreover, assume that $V_-$ is of Kato class, $V_-$ satisfies

\begin{equation}
\|V_+\|_K < \frac{\pi^{\frac{n}{2}}}{\Gamma (\frac{n}{2} - 1)},
\end{equation}

and

\begin{equation}
|A|^2 + V \in L^{n/2, \infty}, \quad A \in L^{n, \infty}.
\end{equation}

Then the following estimate holds:

\begin{equation}
\|H^{1/4}f\|_{L^q} \leq C_q \||D|^{1/2}f\|_{L^q}, \quad 1 < q < 2n, \quad n \geq 3.
\end{equation}

In addition we have the reverse estimate

\begin{equation}
\|H^{1/4}f\|_{L^q} \geq c_q \||D|^{1/2}f\|_{L^q}, \quad 4/3 < q < 4, \quad n \geq 3.
\end{equation}

2. Proof of Theorem 1.2

We start with the proof of Theorem 1.2, divided into several steps. First we need to prove that the heat kernel associated with the operator $H$ is well behaved under quite general assumptions:

**Proposition 2.1.** Consider the selfadjoint operator $H = -(\nabla - iA(x))^2 + V(x)$ on $L^2(\mathbb{R}^n), n \geq 3$. Assume that $A \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, moreover the positive and negative parts $V_+$ of $V$ satisfy

\begin{equation}
V_+ \text{ is of Kato class},
\end{equation}

\begin{equation}
\|V_+\|_K < c_n = \frac{\pi^{n/2}}{\Gamma (n/2 - 1)}.
\end{equation}

Then $e^{-tH}$ is an integral operator and its heat kernel $p_t(x, y)$ satisfies the pointwise estimate

\begin{equation}
|p_t(x, y)| \leq \frac{(2\pi t)^{-n/2}}{1 - \|V_+\|_K/c_n} e^{-|x - y|^2/(8t)}.
\end{equation}
Proof. We recall Simon’s diamagnetic pointwise inequality (see e.g. Theorem B.13.2 in [29]), which holds under weaker assumptions than ours: for any test function \( g(x) \),
\[
|e^{i((\nabla - iA(x))^2 - V)}| \leq e^{i(\Delta - V)}|g|.
\]
Notice that by choosing a delta sequence \( g_\varepsilon \) of (positive) test functions, this implies an analogous pointwise inequality for the corresponding heat kernels. Now we can apply the second part of Proposition 5.1 in [10] which gives precisely estimate (2.3) for the heat kernel of \( e^{-t(\Delta - V)} \) under (2.1), (2.2).

The second tool we shall use is a weak type estimate for imaginary powers of selfadjoint operators, defined in the sense of spectral theory. This follows easily from the previous heat kernel bound and the techniques of Sikora and Wright (see [28]):

**Proposition 2.2.** Let \( H \) be as in Proposition 2.1, and assume in addition that \( H \geq 0 \). Then for all \( y \in \mathbb{R} \) the imaginary powers \( H^{iy} \) satisfy the \((1,1)\) weak type estimate
\[
\|H^{iy}\|_{L^1 \to L^1} \leq C(1 + |y|)^{n/2}.
\]

*Proof.* By Theorem 3 in [27] we obtain immediately that our heat kernel bound (2.3) implies the finite speed of propagation for the wave kernel \( \cos(t\sqrt{H}) \), in the sense of [27], [28], i.e.,
\[
(\cos(t\sqrt{H})\phi, \psi)_{L^2} = 0
\]
for all \( \phi, \psi \in L^2 \) with support in \( B(\xi_1, x_1), B(\xi_2, x_2) \) respectively, provided \( |t| < 2^{-1/2}(|x_1 - x_2| - \xi_1 - \xi_2) \). Then we are in position to apply Theorem 2 from [28] which gives the required bound.

We are ready to prove the first part of Theorem 1.2.

**Proof of (1.17).** We shall use the Stein-Weiss interpolation theorem applied to the analytic family of operators
\[
T_z = H^z \cdot (-\Delta)^{-z}.
\]
Here \( H^z \) is defined by spectral theory while \( (-\Delta)^{-z} \) e.g. by the Fourier transform. Writing \( z = x + iy \), we can decompose
\[
T_z = H^{iy}H^{x}(-\Delta)^{-x}(-\Delta)^{-iy}, \quad y \in \mathbb{R}, \quad x \in [0, 1].
\]
The operators \( H^{iy} \) and \( (-\Delta)^{iy} \) are obviously bounded on \( L^2 \). On the side \( \Re z = 0 \) the operator reduces to the composition of pure imaginary powers
\[
T_{iy} = H^{iy}(-\Delta)^{-iy}
\]
and by the weak type estimate (2.4) we obtain immediately by interpolation that \( H^{iy} \), and hence \( T_{iy} \) is bounded on \( L^p \) for all \( 1 < p < \infty \):
\[
\|T_z f\|_{L^p} \leq C(1 + |y|)^{n/2}\|f\|_{L^p} \quad \text{for} \quad \Re z = 0, \quad 1 < p < \infty.
\]
Next we consider the case \( \Re z = 1 \). We start by proving the estimate
\[
\|H f\|_{L^r} \leq C\|\Delta f\|_{L^r}, \quad 1 < r < \frac{n}{2}.
\]
For \( f \in C_c^\infty(\mathbb{R}^n) \) we can write
\[
H f = -\Delta f - 2iA \cdot \nabla f + (|A|^2 - i\nabla \cdot A + V)f.
\]
We have then by Hölder’s inequality in Lorentz spaces and assumption (1.16)
\[
\|A \cdot \nabla f\|_{L^r} \leq C\|A\|_{L^{n, \infty}}\|\nabla f\|_{L^{\frac{n}{r}, r}}, \quad 1 \leq r < n,
\]
and using the precise Sobolev embedding
\[ \|g\|_{L^{n/2},\infty} \leq C \|\nabla g\|_{L^r} \]
(and the boundedness of Riesz operators, which rules out the case \( r = 1 \)) we obtain
\[ \|A \cdot \nabla f\|_{L^r} \leq C \|\Delta f\|_{L^r}, \quad 1 < r < n. \]
In a similar way,
\[ \|(\|A\|^2 - i\nabla \cdot A + V)f\|_{L^r} \leq C \|\|A\|^2 - i\nabla \cdot A + V\|_{L^{n/2},\infty} \|f\|_{L^{n/2},\infty} \quad 1 < r < \frac{n}{2} \]
and again by the Sobolev embedding
\[ \|f\|_{L^{n/2},\infty} \leq C \|\Delta f\|_{L^r} \]
we conclude that
\[ \|(\|A\|^2 - i\nabla \cdot A + V)f\|_{L^r} \leq C \|\Delta f\|_{L^r}, \quad 1 < r < \frac{n}{2} \]
Summing up we obtain (2.6). Combining (2.6) with the \( L^r \)-boundedness of the purely imaginary powers \( H^{iy} \) and \(-\Delta^{iy}\), we get
\[ (2.7) \quad T_{1+iy} : L^r \rightarrow L^r, \quad 1 < r < \frac{n}{2}. \]
Interpolating (2.7) with (2.5) we obtain
\[ \|T_{1/4}f\|_{L^p} \leq C \|f\|_{L^p} \]
for
\[ \frac{1}{q} = \frac{3}{4p} + \frac{1}{4r} \quad 1 < p < \infty, \quad 1 < r < \frac{n}{2} \implies 1 < q < 2n \]
which concludes the proof. \( \square \)

We pass now to the proof of the reverse estimate (1.18). We shall need the following lemma:

**Lemma 2.3.** Assume that
\[ (2.8) \quad A \in L^2_{\text{loc}}(\mathbb{R}^n), \quad \|V_-\|_K < 4\pi^{n/2}/\Gamma(n/2 - 1). \]
Then for some constnt \( a < 1 \) the following inequality holds:
\[ (2.9) \quad \int V_- |f|^2 dx \leq a \|\nabla A f\|_{L^2}^2. \]

**Proof.** The proof follows a standard argument. We begin by showing that
\[ (2.10) \quad \int V_- |f|^2 dx \leq a \|\nabla f\|_{L^2}^2, \]
for some \( a < 1 \). This can be restated as
\[ (V_{-}^{1/2}(-\Delta)^{-1/2} f, V_{-}^{1/2}(-\Delta)^{-1/2} f) \leq a \|f\|^2, \quad a < 1 \]
i.e. we must prove that the operator \( T = V^{1/2}(-\Delta)^{-1/2} \) is bounded on \( L^2 \) with norm smaller than one. Equivalently, we must prove that the operator
\[ TT^* = V^{1/2}(-\Delta)^{-1} V^{1/2} \]
satisfies
\[ \|V^{1/2}(-\Delta)^{-1} V^{1/2} f\|^2 \leq b \|f\|^2, \quad b < 1. \]
Writing explicitly the kernel of \((-\Delta)^{-1}\), we are reduced to prove
\[ I = \int |V_- (x)| \left( \int \frac{|V_- (y)|^{1/2}}{|x-y|^{n-2}} f(y) dy \right)^2 dx \leq k_n^2 b \|f\|^2 \]
where $k_n = 4\pi^{n/2}/\Gamma(n/2 - 1)$, $b < 1$. Now by Cauchy-Schwartz
\[
I \leq \int |V_+(x)| \left( \int \frac{|V_+(y)|}{|x-y|^{n-2}} dy \right) \left( \int \frac{|f(y)|^2}{|x-y|^{n-2}} dy \right) dx
\]
which gives
\[
I \leq \|V_+\|_K \int \frac{|V_+(x)|}{|x-y|^{n-2}} |f(y)|^2 dy dx = \|V_+\|_K^2 \|f\|^2
\]
and this proves (2.10) under the smallness assumption (2.8). Applying the same computation to the function $|f|$ instead of $f$, we deduce from (2.10) that
\[
(2.11) \quad \int V_- |f|^2 dx \leq a \|\nabla f\|_{L^2}^2.
\]
Since $A \in L^2_{loc}$, we can apply the diamagnetic inequality
\[
|\nabla f| \leq |\nabla A f|, \quad \text{a.e. in } \mathbb{R}^n
\]
(see e.g. [22]) to obtain (2.9). \hfill \Box

**Proof of (1.18).** We begin by proving the $L^2$ inequality
\[
(2.12) \quad \|(-\Delta)^{1/2} f\|_{L^2} \simeq \|\nabla f\|_{L^2} \leq C \|H^{1/2} f\|_{L^2}.
\]
We can write, with the notation $\nabla A = \nabla - iA$,
\[
\|H^{1/2} f\|^2 = \langle H f, f \rangle = (-\nabla_A^2 f, f) + \int V|f|^2 = \|\nabla A f\|_{L^2}^2 + \int V|f|^2
\]
and this implies
\[
\|H^{1/2} f\|^2 \geq \|\nabla A f\|_{L^2}^2 - \int V_- |f|^2.
\]
Thus by Lemma 2.3 we have for some $a < 1$
\[
\|H^{1/2} f\|^2 \geq (1 - a) \|\nabla A f\|_{L^2}^2
\]
so that, in order to prove (2.12), it is sufficient to prove the inequality
\[
(2.13) \quad \|\nabla f\|_{L^2} \leq C \|\nabla A f\|_{L^2}.
\]
Now, using as in the first half of the proof the Hölder inequality and the Sobolev embedding in Lorentz spaces, we can write
\[
\|A f\|_{L^2}^2 + \int V|f|^2 \leq C \|A f\|_{L^{n/2}} + V_{L^\infty} \|f\|_{L^{2n/2}} \leq C \|\nabla f\|_{L^2}
\]
by assumption (1.16). Then, by the diamagnetic inequality
\[
|\nabla f| \leq |\nabla A f|
\]
we obtain
\[
(2.14) \quad \|A f\|_{L^2}^2 + \int V|f|^2 \leq C \|\nabla A f\|_{L^2}^2.
\]
Moreover we have
\[
\|(-\nabla - iA)^2 f\|_{L^2} \geq \|\nabla f - |A f|\|^2 \Rightarrow |\nabla f|^2 \leq 2|\nabla A f|^2 + 2|A f|^2
\]
which implies
\[
\|\nabla f\|_{L^2}^2 \leq 2\|\nabla A f\|_{L^2}^2 + 2\|A f\|_{L^2}^2
\]
and combining this with (2.14) we get
\[
\|\nabla f\|_{L^2}^2 \leq C \|\nabla A f\|_{L^2}^2 - 2 \int V|f|^2 \leq C \|\nabla A f\|_{L^2}^2 - 2 \int V_- |f|^2.
\]
Using again Lemma 2.3 we finally arrive at (2.13), so that the claimed estimate (2.12) is proved.
Now we can use again the Stein-Weiss interpolation theorem, applied to the analytic family of operators

$$T_z = (-\Delta)^z H^{-z}$$

with $z$ in the range $0 \leq \Re z \leq 1/2$. Writing $z = x + iy$ we have

$$T_z = (-\Delta)^{iy}(-\Delta)^x H^{-iy}$$,

and arguing as in the proof of (2.5) we get that $T_{iy}$ is bounded on $L^2$. This shows that $(-\Delta)^{\frac{1}{2}H^{-\frac{1}{2}}}$ is also bounded on $L^2$. Then by the Stein-Weiss interpolation theorem we obtain as above

$$\|T_z f\|_{L^p} \leq C\|f\|_{L^2} \quad \text{for} \quad \Re z = 1/2.$$  \hspace{1cm} (2.15)

On the side $\Re z = 0$ the operator reduces to the composition of pure imaginary powers

$$T_{iy} = (-\Delta)^{iy}H^{-iy}$$

and arguing as in the proof of (2.5) we get that $T_{iy}$ is bounded on $L^p$ for all $1 < p < \infty$:

$$\|T_z f\|_{L^p} \leq C(1 + |y|)^{n/2}\|f\|_{L^p} \quad \text{for} \quad \Re z = 0, \quad 1 < p < \infty.$$  \hspace{1cm} (2.16)

Then by the Stein-Weiss interpolation theorem we obtain as above

$$\|T_{1/4} f\|_{L^q} \leq C\|f\|_{L^q}$$

with

$$\frac{1}{q} = \frac{1}{4} + \frac{1}{2p}, \quad 1 < p < \infty \quad \implies \quad \frac{2}{3} < q < 4.$$  \hspace{1cm} \Box

## 3. Proof of Strichartz estimates

Let us first recall some well known facts about the free propagator. First of all, the free Strichartz estimates for $T(t) = e^{it\Delta}$, its dual operator and the operator $TT^*$ are

$$\|e^{it\Delta} \varphi\|_{L^p L^q} \leq C\|\varphi\|_{L^2}, \hspace{1cm} (3.1)$$

$$\left\| \int e^{-is\Delta} F(s, \cdot) ds \right\|_{L^2} \leq C\|F\|_{L^{p'} L^{q'}}, \hspace{1cm} (3.2)$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds \right\|_{L^p L^q} \leq C\|F\|_{L^{p'} L^{q'}}, \hspace{1cm} (3.3)$$

for all Schrödinger admissible couples $(p, q), (\tilde{p}, \tilde{q})$ satisfying

$$\frac{2}{p} = \frac{n}{2} - \frac{n}{q}, \quad p \geq 2,$$

with $p \neq 2$ if $n = 2$ (see [14], [20]). Moreover, we recall the following estimate:

$$\left\| D^{\frac{1}{2}} \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds \right\|_{L^p L^q} \leq \sum_{j \in \mathbb{Z}} 2^{j/2}\|F_j\|_{L^2 L^2}, \hspace{1cm} (3.4)$$

for any admissible couple $(p, q)$ as above, where

$$F = \sum_{j \in \mathbb{Z}} F_j, \quad \text{supp} F_j \subset \{2^j \leq |x| \leq 2^{j+1}\} \times \mathbb{R}$$

Estimate (3.4) was proved in [26] first; actually it follows by mixing the free Strichartz estimates for $T(t)$ with the dual of the local smoothing estimates which were proved independently by [5], [30] and [33]. In the paper [26] the endpoint estimate for $p = 2$ is not proved (and indeed it predates the Keel-Tao paper [20]). The endpoint case $p = 2$ in dimension $n \geq 3$ is a consequence of Lemma 3 in [18].
Finally, we need to recall the local smoothing estimates for the magnetic propagator $e^{itH}$ proved in [12] under assumptions less restrictive than the ones of the present paper: we have
\begin{equation}
\sup_{R>0} \frac{1}{R} \int_{|x| \leq R} |\nabla_A e^{itH} \varphi| \, dx \, dt + \sup_{R>0} \frac{1}{R^2} \int_{|x| = R} |e^{itH} \varphi|^2 \, d\sigma_R \, dt \lesssim \|(-\Delta_A)^{\frac{1}{2}} \varphi\|^2_{L^2},
\end{equation}
where the constant in the inequality only depends on $B_r$ and $(\partial_r V)_+.$

We are now ready to prove Theorem 1.1. Since $\text{div} A = 0$, we can expand $H$ as follows:
\begin{equation}
H = -\Delta + 2i A \cdot \nabla_A - |A|^2 + V.
\end{equation}
As a consequence, by the Duhamel formula we can write
\begin{equation}
e^{itH} \varphi = e^{it\Delta} \varphi + \int_0^t e^{i(t-s)\Delta} R(x, D) e^{itH} \varphi \, ds,
\end{equation}
where the perturbative operator $R(x, D)$ is given by
\begin{equation}
R(x, D) = 2i A \cdot \nabla_A - |A|^2 + V.
\end{equation}
By (3.1) and (3.4) we have
\begin{equation}
\sum_{j \in \mathbb{Z}} 2^j \|\chi_j R(x, D) e^{itH} \varphi\|_{L^2 L^2}
\end{equation}
\begin{equation}
\lesssim \sum_{j \in \mathbb{Z}} 2^j \left(\|\chi_j A \cdot \nabla_A e^{itH} \varphi\|_{L^2 L^2} + \|\chi_j (V - |A|^2) e^{itH} \varphi\|_{L^2 L^2}\right) = I + II.
\end{equation}

By Hölder inequality, assumption (1.9) and the smoothing estimates (3.5) we have
\begin{equation}
I \leq \left( \sum_{j \in \mathbb{Z}} 2^j \sup_{|x| \sim 2^j} |A| \right) \cdot \left( \sup_{R>0} \frac{1}{R} \int_{|x| \leq R} |\nabla_A e^{itH} \varphi|^2 \, dx \, dt \right)^{\frac{1}{2}} \lesssim \|(-\Delta_A)^{\frac{1}{2}} \varphi\|_{L^2}
\end{equation}
\begin{equation}
II \leq \left( \sum_{j \in \mathbb{Z}} 2^{2j} \sup_{|x| \sim 2^j} (|V| + |A|^2) \right) \cdot \left( \sup_{R>0} \frac{1}{R^2} \int_{|x| = R} |e^{itH} \varphi|^2 \, d\sigma_R \, dt \right)^{\frac{1}{2}}
\end{equation}
\begin{equation}
\lesssim \|(-\Delta_A)^{\frac{1}{2}} \varphi\|_{L^2}.
\end{equation}

Now we remark that all the assumptions of Theorem 1.2 are satisfied. Indeed, we know that $A \in L^2_{\text{loc}}$; moreover, assumption (1.9) implies that $|A| \lesssim 1/|x|$ and $|V| \lesssim 1/|x|^2$, hence (1.16) is satisfied. Thus by Theorem 1.2 (which holds also in the special case $V \equiv 0$) we get
\begin{equation}
\|(-\Delta_A)^{\frac{1}{2}} \varphi\|_{L^2} \lesssim C \|\varphi\|_{H^{\frac{3}{2}}} \lesssim \|H^{\frac{1}{2}} \varphi\|_{L^2}.
\end{equation}
Collecting (3.11), (3.12) and (3.13) we obtain
\begin{equation}
\sum_{j \in \mathbb{Z}} 2^j \|\chi_j R(x, D) e^{itH} \varphi\|_{L^2 L^2} \lesssim \|H^{\frac{1}{2}} \varphi\|_{L^2}
\end{equation}
and by (3.9), (3.10) and (3.14) we deduce

\[(3.15) \quad \left\| D^\alpha e^{-itH}\varphi \right\|_{L^pL^q} \lesssim \| H^{\frac{1}{2}}\varphi \|_{L^2}, \]

for any admissible couple \((p, q)\); notice that this includes also the 3D endpoint estimate (1.14). In order to conclude the proof, it is now sufficient to use estimate (1.17) which gives

\[(3.16) \quad \left\| H^{\frac{1}{2}}e^{-itH}\varphi \right\|_{L^pL^q} \lesssim \| H^{\frac{1}{2}}\varphi \|_{L^2}, \]

and commuting \(H^{1/4}\) with the flow \(e^{itH}\) we obtain (1.13). However, in dimension 3 (1.17) does not cover the endpoint \(q = 6\) and we are left with (3.15).

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