ANISOTROPIC SINGULARITIES TO SEMI-LINEAR ELLIPTIC EQUATIONS IN A MEASURE FRAMEWORK

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Abstract. The purpose of this article is to study very weak solutions of elliptic equation
\[
\begin{cases}
-\Delta u + g(u) = 2k \frac{\partial \delta_0}{\partial x_N} + j\delta_0 & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0),
\end{cases}
\]
where \(k > 0, j \geq 0, B_1(0)\) denotes the unit ball centered at the origin in \(\mathbb{R}^N\) with \(N \geq 2, g : \mathbb{R} \to \mathbb{R}\) is an odd, nondecreasing and \(C^1\) function, \(\delta_0\) is the Dirac mass concentrated at the origin and \(\frac{\partial \delta_0}{\partial x_N}\) is defined in the distribution sense that
\[
\langle \frac{\partial \delta_0}{\partial x_N}, \zeta \rangle = \frac{\partial \zeta(0)}{\partial x_N}, \quad \forall \zeta \in C^1_c(B_1(0)).
\]
We obtain that problem (1) admits a unique very weak solution \(u_{k,j}\) under the integral subcritical assumption
\[
\int_1^\infty g(s)s^{-1-N/2-N+1}ds < +\infty.
\]
Furthermore, we prove that \(u_{k,j}\) has anisotropic singularity at the origin and we consider the odd property \(u_{k,0}\) and limit of \(\{u_{k,0}\}\) as \(k \to \infty\).

We pose the constraint on nonlinearity \(g(u)\) that we only require integrability in the principle value sense, due to the singularities only at the origin. This makes us able to search the very weak solutions in a larger scope of the nonlinearity.

1. INTRODUCTION

As early as in 1977, Lieb-Simon in [10] studied the very weak solutions to equation
\[
-\Delta u + (u - \lambda)^{3/2} = \sum_{i=1}^n m_i \delta_{a_i} \quad \text{in } \mathbb{R}^3
\]
in the description of the Thomas-Fermi theory of electric field potential determined by the nuclear charge and distribution of electrons in an atom, where \(\lambda \geq 0, t_+ = \max\{t, 0\}, m_i > 0, a_i \in \mathbb{R}^3\) and \(\delta_{a_i}\) is the Dirac mass at \(a_i\) for \(i = 1, 2, \ldots, n\). In fact, the solution of (1.1) turns out to be a classical singular solution of
\[
-\Delta u + (u - \lambda)^{3/2} = 0 \quad \text{in } \mathbb{R}^3 \setminus \{a_1, \cdots, a_n\}.
\]

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As a fundamental PDE’s model, the isolated singular problem
\[- \Delta u + |u|^{p-1}u = 0 \quad \text{in} \quad \Omega \setminus \{0\}\] (1.2)
has been studied extensively, where \(\Omega\) is a domain in \(\mathbb{R}^N\) with \(N \geq 3\). Brezis-Véron in [3] showed that problem (1.2) admits no isolated singular solution when \(p \geq \frac{N}{N-2}\). A complete classification of the isolated singularities at the origin for (1.2) was given by Véron in [18] when \(\frac{N+1}{N-1} \leq p < \frac{N}{N-2}\) as follows:

(i) either \(|x|^{2p-2}u(x)\) converges to a constant which can take only two values \(\pm [(2p - 1)(2p - 1) - N)]^{1/(p-1)}\) as \(x \to 0\),

(ii) or \(|x|^{N-2}u(x)\) converges to a constant \(c_N k\), and the couple \((u, k)\) is related to the weak solution of
\[- \Delta u + u^p = k\delta_0 \quad \text{in} \quad (C^\infty_c(\Omega))',\]
where \(c_N\) is the normalized constant of the fundamental solution \(\Gamma_N\) of \(-\Delta u = \delta_0\) in \(\mathbb{R}^N\), that is,
\[
\Gamma_N(x) = \begin{cases} 
  c_N |x|^{2-N} & \text{if } N \geq 3, \\
  c_N \log |x| & \text{if } N = 2.
\end{cases}
\] (1.3)

For \(1 < p < \frac{N+1}{N-1}\), the above classification holds under the restriction of nonnegative solutions of (1.2) and all above singular solutions are isotropic. A conjecture states that there is a rich structure of the singularities for (1.2) without the restriction of nonnegativity for \(1 < p < \frac{N+1}{N-1}\). Véron in [18] partially answered this conjecture and showed that the anisotropic singular solutions could be constructed by considering the following nonlinear eigenvalue problem on \(S^{N-1}\)
\[- \Delta_{S^{N-1}} \omega + |\omega|^{p-1} \omega = \lambda \omega,\]
where \(S^{N-1}\) is the sphere of unit ball in \(\mathbb{R}^N\) and \(\Delta_{S^{N-1}}\) is the Laplace-Beltrami operator. Later on, Chen-Matano-Véron in [5] provided the anisotropic singular solutions of (1.2) by analyzing the corresponding Laplace-Beltrami equations in the sphere. More singularities analysis see the references [12, 16, 17, 20].

In contrast with the absorption nonlinearity, the isolated singular solutions of elliptic problem with source nonlinearity
\[
\begin{align*}
- \Delta u &= u^p \quad \text{in} \quad \Omega \setminus \{0\}, \\
  u > 0 & \quad \text{in} \quad \Omega \setminus \{0\}, \\
  u &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\] (1.4)
was classified by Lions in [11], by using Schwartz’s Theorem to build that
\[- \Delta u - u^p = \sum_{|a|=0}^\infty k_a D^a \delta_0 \quad \text{in} \quad (C^\infty_c(\Omega))',\] (1.5)
and then by choosing suitable test functions in \(C^\infty_c(\Omega)\) to kill all \(D^a \delta_0\) with \(a\) multiple index and \(|a| \geq 1\), finally building the connections with the weak solutions of
\[
\begin{align*}
- \Delta u &= u^p + k \delta_0 \quad \text{in} \quad \Omega, \\
  u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\] (1.6)
Lions in [11] proved that when \(p \in (1, \frac{N}{N-2})\) with \(N \geq 3\), any solution of (1.4) is a weak solution of (1.6) for some \(k \geq 0\), and when \(p \geq \frac{N}{N-2}\), the parameter \(k = 0\). Essentially, \(D^a \delta_0\) with \(|a| \geq 1\) is killed in (1.5) because it is anisotropic singular source, that is, this
source makes the solutions anisotropic singular. For instance, the fundamental solution of 
\(-\Delta u = D^a\delta_0\) with \(a = (0, \ldots, 0, 1) \in \mathbb{R}^N\) is 
\[
P_N(x) = \hat{c}_N \frac{x_N}{|x|^N}, \quad \forall x \in \mathbb{R}^N \setminus \{0\},
\]
(1.7)
where \(\hat{c}_N\) are the normalized constants, see [4]. Obviously, \(P_N\) has anisotropic singularities.

Inspired by (1.5), we observe that \(D^a\delta_0\) with \(|a| \geq 1\) could provide anisotropic source and our motivation in this article is to make use of this kind of sources to construct anisotropic singular solutions for (1.2) and to find the criteria for more general nonlinearity. Let \(B_1(0)\) be the unit ball centered at the origin in \(\mathbb{R}^N\) with \(N \geq 2\), denote by \(\delta_0\) the Dirac mass concentrated at the origin, to be convenient, \(\frac{\partial \delta_0}{\partial x_N} = D^a\delta_0\) with \(a = (0, \ldots, 0, 1) \in \mathbb{R}^N\) and then 
\[
\langle \frac{\partial \delta_0}{\partial x_N}, \zeta \rangle = \frac{\partial \zeta(0)}{\partial x_N}, \quad \forall \zeta \in C^1_0(B_1(0)).
\]

So our concern is to study the isolated singular solutions of semilinear elliptic problem
\[
\begin{aligned}
-\Delta u + g(u) &= 2k \frac{\partial \delta_0}{\partial x_N} + j\delta_0 & \text{in } B_1(0), \\
u &= 0 & \text{on } \partial B_1(0),
\end{aligned}
\]
(1.8)
where parameters \(k > 0\) and \(j \geq 0\).

Before we state our main results, we introduce the definition of weak solution to (1.8) as follows.

**Definition 1.1.** A function \(u \in L^1(B_1(0))\) is called a very weak solution of (1.8), if \(g(u)\) be integrable in the principle value sense near the origin, \(g(u) \in L^1(B_1(0)), |x|dx\) and
\[
\int_{B_1(0)} [u(\Delta)\xi + g(u)\xi] \, dx = 2k \frac{\partial \xi(0)}{\partial x_N} + j\xi(0), \quad \forall \xi \in C^1_0(B_1(0)).
\]
(1.9)

We note that \(k \frac{\partial \delta_0}{\partial x_N}\) and \(j\delta_0\) are both visible in the distributional identity (1.9). When \(k = 0\), the definition of very weak solution of (1.8) requires \(g(u) \in L^1(B_1(0), \rho dx)\), see the references [19]. Since both sources have the support at the origin, so in this article we pose the constraint for the very weak solution that \(g(u)\) is integrable in the principle value sense near the origin means, i.e. \(\lim_{t \to 0^+} \int_{B_1(0) \setminus B_t(0)} g(u) \, dx\) exists, that provides higher possibility for searching the sign-changing singular solutions of (1.8).

Now we are ready to state our first theorem on the existence and asymptotic behavior of very weak solutions to problem (1.8).

**Theorem 1.1.** Assume that \(k > 0, j \geq 0, \Gamma_N, P_N\) is given in (1.3), (1.7) respectively, and the nonlinearity \(g: \mathbb{R} \to \mathbb{R}\) is an odd, nondecreasing and Lipschitz function satisfying
\[
\int_1^\infty g(s)s^{-\frac{N+1}{N-1}} \, ds < +\infty
\]
and
\[
g(s + t) - g(s) \leq c_1 \left[ \frac{g(s)}{1 + |s|} t + g(t) \right], \quad s, t \in \mathbb{R}, \; t \geq 0
\]
for some \(c_1 > 0\).

Then (1.8) admits a very weak solution \(u_{k,j}\) such that
\begin{enumerate}[(i)]  \item \(u_{k,j} \geq 0\) in \(B_1^+(0) = \{x = (x', x_N) \in B_1(0) : x_N > 0\}\);  \item \(u_{k,j}\) has the following singularity at the origin
\[
\lim_{t \to 0^+} \frac{u_{k,j}(te)}{P_N(te)} = 2k \quad \text{for } e = (e_1, \ldots, e_N) \in \partial B_1(0), \; e_N \neq 0
\]
\end{enumerate}
and
\[ \lim_{t \to 0^+} \frac{u_{k,j}(te)}{\Gamma_N(te)} = j \quad \text{for} \quad e \in \partial B_1(0), \quad e_N = 0; \]  
(1.13)

(iii) \( u_{k,j} \) is a classical solution of
\[ \begin{cases} -\Delta u + g(u) = 0 & \text{in} \quad B_1(0) \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial B_1(0). \end{cases} \]  
(1.14)

In the particular case that \( j = 0 \), denoting \( u_k \) the solution \( u_{k,0} \) of (1.8) with \( j = 0 \), \( u_k \) is \( x_N \)-odd, that is,
\[ u_k(x',x_N) = -u_k(x',-x_N), \quad \forall (x',x_N) \in B_1(0) \setminus \{0\}. \]

Furthermore, the \( x_N \)-odd very weak solution is unique.

We note that \( g(s) = |s|^{p-1} s \) with \( p \in (0, \frac{N}{N-1}) \) verifies (1.10) and (1.11). It follows by (1.12) that \( |u_{k,j}|^{p-1} u_{k,j} \in L^1(B_1(0)) \), but for \( p \in \left[ \frac{N}{N-1}, \frac{N+1}{N-1} \right] \) and \( k > 0 \), \( |u_{k,j}|^{p-1} u_{k,j} \notin L^1(B_1(0)) \). We can’t help to obtain the uniqueness of the very weak solution to (1.8), due to the failure of application the Kato’s inequality, which requires that the nonlinearity term \( g(u) \in L^1(B_1(0)) \).

For the existence of very weak solutions, the normal method is to approximate the Radon measure by \( C_0^1 \) functions and consider the limit of the corresponding classical solutions. When \( k > 0 \), we use a sequence of Dirac measures \( \frac{\delta_{x_N} - \delta_{-x_N}}{t} \) to approach the source \( \frac{\partial \delta_{x_N}}{\partial x_N} \) and in this approximation, the biggest challenge is to find a uniform estimate. To overcome this difficulty, our strategy is to consider \( x_N \)-odd property of solutions when \( j = 0 \) to derive the uniform bound in this approximating process.

We next state the nonexistence of very weak solution of (1.8).

**Theorem 1.2.** Assume that \( k > 0 \), \( j = 0 \), \( g(s) = |s|^{p-1} s \) with \( p \geq \frac{N+1}{N-1} \), then there is no \( x_N \)-odd weak solution for problem (1.8).

Our strategy here is to make use of \( x_N \)-odd property to deduce (1.8) into boundary data problem
\[ \begin{cases} -\Delta u + g(u) = 0 & \text{in} \quad B_1^+(0), \\ u = k\delta_0 & \text{on} \quad \partial B_1^+(0), \end{cases} \]
in the distributional sense that
\[ \int_{B_1^+(0)} u(-\Delta)\xi dx + \int_{B_1^+(0)} g(u)\xi dx = 2k \frac{\partial \xi(x_0)}{\partial x_N}, \quad \forall \xi \in C_0^1(B_1^+(0)). \]
(1.15)

It is interesting but still open to derive the nonexistence when \( j \neq 0 \).

Finally, we analyze the limit of the weak solutions \( \{u_k\}_k \) as \( k \to \infty \). From the monotonicity of \( u_k \) in \( B_1^+(0) \) and \( B_1^-(0) \) respectively, the limit of \( \{u_k\}_k \) as \( k \to \infty \) exists in \( B_1(0) \setminus \{0\} \), denote
\[ u_\infty(x) = \lim_{k \to \infty} u_k(x), \quad \forall x \in B_1(0) \setminus \{0\}. \]
(1.16)

**Theorem 1.3.** Assume that \( k > 0 \), \( j = 0 \), \( g(s) = |s|^{p-1} s \) with \( p > 1 \), \( u_k \) is the unique \( x_N \)-odd very weak solution of (1.8) and \( u_\infty \) is given by (1.16). Then \( u_\infty \) is a classical solution of
\[ \begin{cases} -\Delta u + |u|^{p-1} u = 0 & \text{in} \quad B_1(0) \setminus \{0\}, \\ u = 0 & \text{on} \quad \partial B_1(0). \end{cases} \]
(1.17)
and satisfies that
\begin{equation}
\lim_{t \to 0^+} u_\infty(te) t^{2-\sigma} = \varphi(e), \quad \forall e \in \partial B_1(0),
\end{equation}
where \( \varphi : \partial B_1(0) \to \mathbb{R} \) is a continuous \( x_N \)-odd function such that for unit vector \( e = (e_1, \ldots, e_N) \)
\[ \varphi(e) > 0 \quad \text{if} \quad e_N > 0. \]

The rest of this paper is organized as follows. In Section 2, we analyze the \( x_N \)-odd property. Section 3 is devoted to study the \( x_N \)-odd very weak solution in subcritical case when \( j = 0 \) and the nonexistence the \( x_N \)-odd very weak solution in the subcritical case. In Section 4, we consider the limit of the unique \( x_N \)-odd weak solutions \( u_k \) of (1.8) with \( j = 0 \) as \( k \to \infty \). Finally, we prove the existence of non \( x_N \)-odd very weak solution when \( j > 0 \) in Section 5.

2. Preliminary

We start this section from the \( x_N \)-odd property. Notice that an \( x_N \)-odd function \( w \) defined in a \( x_N \)-symmetric domain \( B^* \) satisfies
\[ w(x) = 0, \quad \forall x \in \{ (x', 0) \in B^* \}. \]
In what follows, we denote by \( c_i \) a generic positive constant.

**Lemma 2.1.** Assume that \( f \in C^1(B_1(0)) \) is an \( x_N \)-odd function, \( g \in C^1(\mathbb{R}) \) is an odd and nondecreasing function.

Then
\begin{equation}
\begin{cases}
\Delta u + g(u) = f & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0)
\end{cases}
\end{equation}

admits a unique classical solution \( w_f \). Moreover,

(i) \( w_f \) is \( x_N \)-odd in \( B_1(0) \);

(ii) assume more that \( f \geq 0 \) in \( B_1^+(0) = \{ x \in B_1(0) : x_N > 0 \} \) and \( f \not\equiv 0 \) in \( B_1^+(0) \), then \( w_f > 0 \) in \( B_1^+(0) \).

**Proof.** Since \( g \) is an odd and nondecreasing function, then it is standard to obtain the existence of solution by the method of super and sub solutions.

**Uniqueness.** Let \( w_f, \tilde{w}_f \) be two solutions of (2.1), \( w = w_f - \tilde{w}_f \) in \( B_1(0) \) and \( A_+ = \{ x \in B_1(0) : w(x) > 0 \} \). We claim that \( A_+ = \emptyset \). In fact, if \( A_+ \neq \emptyset \), we observe that \( w \) is a solution of
\begin{equation}
\begin{cases}
\Delta w = g(\tilde{w}_f) - g(w_f) \leq 0 & \text{in } A_+, \\
w = 0 & \text{on } \partial A_+.
\end{cases}
\end{equation}

By applying Maximum Principle, we have that
\[ w \leq 0 \quad \text{in } A_+, \]
which contradicts the definition of \( A_+ \). Then \( A_+ = \emptyset \). Similarly, \( \{ x \in B_1(0) : w(x) < 0 \} \) is empty. Therefore, \( w_f = \tilde{w}_f \) in \( B_1(0) \) and the uniqueness holds.

(i) Let \( v(x', x_N) = -w_f(x', -x_N) \), and by direct computation, we derive that
\begin{align*}
-\Delta v(x) + g(v(x)) &= -\Delta[-w_f(x', -x_N)] + g(-w_f(x', -x_N)) \\
&= \Delta w_f(x', -x_N) - g(w_f(x', -x_N)) \\
&= -f(x', -x_N) = f(x),
\end{align*}
then \( v \) is a solution of (2.1). It follows from the uniqueness of solution of (2.1) that
\[ w_f(x', x_N) = -w_f(x', -x_N), \quad \forall x = (x', x_N) \in B_1(0). \]
(ii) We observe that \( w_f = 0 \) on \( \partial B_1^+(0) \) and then \( w_f \) is a classical solution of

\[
\begin{cases}
-\Delta u + g(u) = f & \text{in } B_1^+(0), \\
u = 0 & \text{on } \partial B_1^+(0).
\end{cases}
\]  

(2.2)

We now claim that \( w_f \geq 0 \) in \( B_1^+(0) \). Indeed, if not, we have that \( \min_{B_1^+(0)} w_f < 0 \). Let \( A_- = \{ x \in B_1^+(0) : w_f(x) < \frac{1}{2} \min_{B_1^+(0)} w_f \} \), then \( \psi := w_f + \frac{1}{2} \min_{B_1^+(0)} w_f \) satisfies

\[
\begin{cases}
-\Delta \psi \geq 0 & \text{in } A_-,
\psi = 0 & \text{on } \partial A_-.
\end{cases}
\]

By Maximum Principle, we have that

\[
w_f(x) \geq \frac{1}{2} \min_{B_1^+(0)} w_f, \quad x \in A_-,
\]

which contradicts the definition of \( A_- \).

We next prove that \( w_f > 0 \) in \( B_1^+(0) \). Problem (2.2) could be seen as

\[
\begin{cases}
-\Delta u + \phi u = f & \text{in } B_1^+(0),
\phi = 0 & \text{on } \partial B_1^+(0),
\end{cases}
\]

where \( \phi(x) = \frac{g(w_f(x))}{w_f(x)} \) if \( w_f(x) \neq 0 \) and \( \phi(x) = g'(0) \) if \( w_f(x) = 0 \). It follows by \( g \in C^1(\mathbb{R}) \) that \( \phi \) is continuous and \( \phi \geq 0 \) in \( B_1^+(0) \). Since \( f \geq 0 \) in \( B_1^+(0) \), it follows by strong maximum principle that \( w_f > 0 \) or \( w_f \equiv 0 \) in \( B_1^+(0) \), then we exclude \( w_f \equiv 0 \) in \( B_1^+(0) \) by the fact that \( f \neq 0 \) in \( B_1^+(0) \). \( \square \)

**Corollary 2.1.** Assume that \( f \in C(B_1(0)) \) is an \( x_N \)-odd function such that \( f \geq 0 \) in \( B_1^+(0) \) and \( g \in C^1(\mathbb{R}) \) is an odd and nondecreasing function. Let \( w_f \) be the solution of (2.1) and \( G_{B_1(0)}[f] \) be the unique solution of

\[
\begin{cases}
-\Delta u = f & \text{in } B_1(0),
\phi = 0 & \text{on } \partial B_1(0).
\end{cases}
\]

Then \( G_{B_1(0)}[f] \) is \( x_N \)-odd and

\[
0 \leq w_f \leq G_{B_1(0)}[f] \quad \text{in } B_1^+(0).
\]

**Proof.** By applying Lemma 2.1 with \( g \equiv 0 \), we have that \( G_{B_1(0)}[f] \) is \( x_N \)-odd and

\[
G_{B_1(0)}[f] \geq 0 \quad \text{in } B_1^+(0).
\]

Denote \( v = G_{B_1(0)}[f] - w_f \), then \( v = 0 \) on \( \partial B_1^+(0) \) and \( -\Delta v = g(w_f) \geq 0 \), by Maximum Principle, we have that \( v \geq 0 \) in \( B_1^+(0) \), which ends the proof. \( \square \)

**Corollary 2.2.** Assume that \( f \in C(B_1(0)) \) is an \( x_N \)-odd function such that \( f \geq 0 \) in \( B_1^+(0) \) and \( g_1, g_2 \in C^1(\mathbb{R}) \) are odd and nondecreasing functions satisfying

\[
g_1(s) \leq g_2(s), \quad \forall s \geq 0.
\]

Let \( w_{f,i} \) be the solutions of (2.1) replaced by \( g \) by \( g_i \) with \( i = 1, 2 \) respectively.

Then

\[
|w_{f,1}(x)| \geq w_{f,2}(x), \quad \forall x \in B_1(0).
\]
**Proof.** By applying Lemma 2.1 and Corollary 2.1, we have that \(w_{f,1}, w_{f,2}\) are \(x_N\)-odd and are nonnegative in \(B_1^+(0)\). We denote \(w = w_{f,1} - w_{f,2}\), then \(w\) satisfies that

\[
\begin{cases}
-\Delta w = g_2(w_{f,2}) - g_1(w_{f,1}) & \text{in } B_1^+(0), \\
 w = 0 & \text{on } \partial B_1^+(0).
\end{cases}
\]

We first claim that \(w \geq 0\) in \(B_1^+(0)\). If not, we have that \(\min_{B_1^+(0)} w < 0\). Let us define

\[
A_- = \left\{ x \in B_1^+(0) : w(x) < \frac{1}{2} \min_{B_1^+(0)} w \right\},
\]

then \(\tilde{w} := w + \frac{1}{2} \min_{B_1^+(0)} w\) satisfies that

\[
\begin{cases}
-\Delta \tilde{w} \geq 0 & \text{in } A_-,
\\
\tilde{w} = 0 & \text{on } \partial A_-.
\end{cases}
\]

By Maximum Principle, we have that \(w(x) \geq \frac{1}{2} \min_{B_1^+(0)} w, \forall x \in A_-\), which contradicts the definition of \(A_-\). □

**Proposition 2.1.** Let \(f_1\) and \(f_2\) be \(x_N\)-odd functions in \(C^{1,1}_{\text{loc}}(\overline{B_1(0)} \setminus \{0\}) \cap L^1(B_1(0), |x|dx)\) satisfying \(f_2 \geq f_1 \geq 0\) in \(B_1^+(0)\), then the problem

\[
\begin{cases}
-\Delta u = f_i & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0)
\end{cases}
\]  

(2.3)

admits a unique \(x_N\)-odd weak solution \(u_i\) with \(i = 1, 2\) in the sense that \(u_i \in L^1(B_1(0))\),

\[
\int_{B_1(0)} u_i(-\Delta)\xi dx = \int_{B_1(0)} \xi f_i dx, \forall \xi \in C^{1,1}_0(B_1(0)), \xi(0) = 0.
\]

(2.4)

Moreover, \(u_i\) is a classical solution of

\[
\begin{cases}
-\Delta u = f_i & \text{in } B_1(0) \setminus \{0\}, \\
u = 0 & \text{on } \partial B_1(0)
\end{cases}
\]  

(2.5)

and

\[
0 \leq u_1(x) \leq u_2(x) \leq \int_{B_1^+(0)} \frac{c_2 f_2(y) |y|}{y - x (|y - x| + 2|y|)^{N-2}} dy, \forall x \in B_1^+(0).
\]

(2.6)

**Proof. Uniqueness.** Let \(w\) satisfy that

\[
\int_{B_1(0)} w(-\Delta)\xi dx = 0,
\]

(2.7)

for any \(\xi \in C^{1,1}_0(B_1(0))\) such that \(\xi(0) = 0\). Since \(0 \in L^1(B_1(0))\), then the test function could be improved into \(C^{1,1}_0(B_1(0))\) without the restriction that \(\xi(0) = 0\). Denote by \(\eta_1\) the solution of

\[
\begin{cases}
-\Delta \eta_1 = \text{sign}(w) & \text{in } B_1(0), \\
\eta_1 = 0 & \text{on } \partial B_1(0).
\end{cases}
\]

(2.8)

Then \(\eta_1 \in C^{1,1}_0(B_1(0))\) and then

\[
\int_{B_1(0)} |w| \, dx = 0.
\]
This implies $w = 0$ in $B_1(0) \setminus \{0\}$.

**Existence.** Let $f_{i,\epsilon} = f_i \chi_{B_1(0) \setminus B_0(0)}$, where $i = 1, 2$, $\chi_{B_1(0) \setminus B_0(0)} = 1$ in $B_1(0) \setminus B_0(0)$ and $\chi_{B_1(0) \setminus B_0(0)} = 0$ in $B_0(0)$, then $f_{i,\epsilon}$ is an $x_N$-odd function in $L^\infty(B_1(0))$ such that $f_{2,\epsilon} \geq f_{1,\epsilon} \geq 0$ in $B_1^+(0)$, then

\[
\begin{cases}
-\Delta u = f_{i,\epsilon} & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0)
\end{cases}
\]  

(2.9)

admits a unique solution $u_{i,\epsilon}$ satisfying

\[
\int_{B_1(0)} u_{i,\epsilon} (-\Delta) \xi dx = \int_{B_1(0)} \xi f_{i,\epsilon} dx, \quad \forall \xi \in C_0^{1,1}(B_1(0)), \xi(0) = 0.
\]  

(2.10)

Moreover, from Lemma 2.1 with $g \equiv 0$, we have that

\[
0 \leq u_{i,\epsilon} \leq u_{i,\epsilon'} \quad \text{in } B_1^+(0) \quad \text{for } 0 < \epsilon' \leq \epsilon,
\]

\[
0 \leq u_{1,\epsilon} \leq u_{2,\epsilon} \quad \text{in } B_1^+(0) \quad \text{for any } \epsilon \geq 0,
\]

and for any $x \in B_1^+(0)$,

\[
u_{2,\epsilon}(x) = \int_{B_1(0)} G_{B_1(0)}(x, y) f_{2,\epsilon}(y) dy
\]

\[
= \int_{B_1^+(0)} [G_{B_1(0)}(x, y) - G_{B_1(0)}(x, \bar{y})] f_{2,\epsilon}(y) dy,
\]

where $\bar{y} = (y', -y_N)$. We observe that $G_{B_1(0)}(x, y) = \frac{c_N}{|x-y|^{N-2}} - \hat{G}_{B_1(0)}(x, y)$, where $\hat{G}_{B_1(0)}(x, y)$ is a harmonic function in $B_1(0)$ with the boundary value $\frac{\xi_N}{|x-y|^{N-2}}$ for $y \in \partial B_1(0)$. Therefore, for $x, y \in B_1^+$, we have that $\hat{G}_{B_1(0)}(x, y) - \hat{G}_{B_1(0)}(x, \bar{y}) \geq 0$ and then

\[
G_{B_1(0)}(x, y) - G_{B_1(0)}(x, \bar{y}) \leq c_3 \left[ \frac{1}{|y-x|^{N-2}} - \frac{1}{|\bar{y} - x|^{N-2}} \right].
\]

Moreover, we see that

\[
|\bar{y} - x| \leq |y - x| + 2y_N \leq |y - x| + 2|y|
\]

and

\[
0 \leq \frac{1}{|y-x|^{N-2}} - \frac{1}{|\bar{y} - x|^{N-2}} \leq \frac{1}{|y-x|^{N-2}} - \frac{1}{(|y-x| + 2|y|)^{N-2}}
\]

\[
= \frac{(|y-x| + 2|y|)^{N-2} - |y-x|^{N-2}}{|y-x|^{N-2}(|y-x| + 2|y|)^{N-2}}
\]

\[
\leq \frac{c_4|y|}{|y-x|(|y-x| + 2|y|)^{N-2}},
\]

then

\[
u_{2}(x) \leq c_2 \int_{B_1^+(0)} \frac{|y| f_2(y)}{|y-x|(|y-x| + 2|y|)^{N-2}} dy.
\]

Therefore, we obtain a uniform bound for $u_{2,\epsilon}$, together with monotonicity, then passing to the limit as $\epsilon \to 0^+$ in (2.10), we deduces that $u_i := \lim_{\epsilon \to 0^+} u_{i,\epsilon}$ is a weak solution of (2.3) and it follows from standard stability theorem that $u_i$ is a classical solution of (2.5) for $i = 1, 2$. \qed

By direct extension, we have the following corollary.
Corollary 2.3. Let $f$ be an $x_N$-odd function in $C^1_{q,κ}(B_1(0)) \cap L^1(B_1(0), |x|^dx)$ with $i \in \mathbb{Z}$ satisfying $f \geq 0$ in $B_1^+(0)$, then the problem
\[
\begin{aligned}
-\Delta u &= f & \text{ in } B_1(0), \\
u &= 0 & \text{ on } \partial B_1(0)
\end{aligned}
\]

admits a unique $x_N$-odd weak solution $u_f$ with $i \in \mathbb{N}$, $i \geq 2$ in the sense that $u_f \in L^1(B_1(0), |x|^{i-1}dx)$,
\[
\int_{B_1(0)} u(-\Delta)\xi dx = \int_{B_1(0)} \xi fdx,
\]
for any $\xi \in C_0^{i,1}(B_1(0))$ s. t. $|\xi(x)| \leq c|x|^i$ for any $x \in B_1(0)$ and some $c > 0$. Moreover, $u_f(x) \geq 0$, $\forall x \in B_1^+(0)$.

and $u_f$ is a classical solution of
\[
\begin{aligned}
-\Delta u &= f & \text{ in } B_1(0) \setminus \{0\}, \\
u &= 0 & \text{ on } \partial B_1(0).
\end{aligned}
\]

Remark 2.1. The arguments in Proposition 2.1 and Corollary 2.3 hold when $B_1(0)$ is replaced by $\mathbb{R}^N$ and the boundary condition is done by
\[
\lim_{|x| \to +\infty} u(x) = 0.
\]

In order to study the convergence of weak solutions, we recall the definition and basic properties of the Marcinkiewicz spaces.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a domain and $\mu$ be a positive Borel measure in $\Omega$. For $\kappa > 1$, $\kappa' = \kappa/(\kappa - 1)$ and $u \in L^1_{1,loc}(\Omega, d\mu)$, we set
\[
\|u\|_{M^{\kappa}(\Omega, d\mu)} = \inf\{c \in [0, \infty] : \int_E |u|d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa}}, \forall E \subset \Omega \text{ Borel set}\}
\]
and
\[
M^{\kappa}(\Omega, d\mu) = \{u \in L^1_{1,loc}(\Omega, d\mu) : \|u\|_{M^{\kappa}(\Omega, d\mu)} < \infty\}. \tag{2.11}
\]

$M^{\kappa}(\Omega, d\mu)$ is called the Marcinkiewicz space with exponent $\kappa$ or weak $L^{\kappa}$ space and $\|\cdot\|_{M^{\kappa}(\Omega, d\mu)}$ is a quasi-norm. The following property holds.

Proposition 2.2. [1] Assume that $1 \leq q < \kappa < \infty$ and $u \in L^1_{1,loc}(\Omega, d\mu)$. Then there exists $C(q, \kappa) > 0$ such that
\[
\int_E |u|^q d\mu \leq C(q, \kappa)\|u\|_{M^{\kappa}(\Omega, d\mu)} \left( \int_E d\mu \right)^{1-q/\kappa},
\]
for any Borel set $E$ of $\Omega$.

3. $x_N$-odd very weak solution with $j = 0$

3.1. Existence of very weak solution. In this subsection, we prove the existence and uniqueness of very weak solution to problem (1.8) when $j = 0$.

Theorem 3.1. Assume that $k > 0$ $j = 0$, the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is an odd, nondecreasing and Lipchitz continuous function satisfying (1.10).

Then (1.8) admits a unique $x_N$-odd very weak solution $w_k$ such that
(i) $w_k' \geq w_k \geq 0$ in $B_1^+(0)$ for $k' \geq k > 0$;
(ii) $w_k$ satisfies (1.12);
(iii) $w_k$ is a classical solution of (1.14).
Before proving Theorem 3.1, we need following preliminaries.

**Lemma 3.1.** Assume that $k > 0$, the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is an odd, nondecreasing and Lipschitz continuous function and $u$ is a very weak solution of (1.8), locally bounded in $B_1(0) \setminus \{0\}$. Then $u$ is a classical solution of (1.14).

**Proof.** Since $\frac{\partial g}{\partial x N}$, $\delta_0$ have the support in $\{0\}$, so for any open sets $O_1, O_2$ in $B_1(0)$ such that $\overline{O_1} \subset O_2 \subset B_1(0) \setminus \{0\}$, $u$ is a very weak solution of

$$-\Delta u + g(u) = 0 \quad \text{in} \quad O_2,$$

where $u \in L^\infty(O_2)$ and $g(u) \in L^\infty(O_2)$. By standard regularity results, we have that $u$ satisfies (3.1) in $O_1$ in the classical sense. \hfill \Box

For $n \in \mathbb{N}$, we consider $\{g_n\}_n$ of $C^1$, odd, nondecreasing functions defined in $\mathbb{R}$ satisfying

$$g_n \leq g_{n+1} \leq g \quad \text{in} \quad \mathbb{R}_+, \quad \sup_{s \in \mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n \to \infty} \|g_n - g\|_{L^\infty(\mathbb{R})} = 0. \quad (3.2)$$

**Proposition 3.1.** Let $g_n$ be defined by (3.2) and

$$\mu_t = \frac{\delta_{te_N} - \delta_{-te_N}}{t}, \quad t \in (0,1).$$

Then for any $n \in \mathbb{N}$, problem

$$\begin{cases}
-\Delta u + g_n(u) = k\mu_t & \text{in} \quad B_1(0), \\
u = 0 & \text{on} \quad \partial B_1(0)
\end{cases} \quad (3.3)$$

admits a unique very weak solution $w_{k,n,t}$, which is a classical solution of

$$\begin{cases}
-\Delta u + g_n(u) = 0 & \text{in} \quad B_1(0) \setminus \{te_N, -te_N\}, \\
u = 0 & \text{on} \quad \partial B_1(0).
\end{cases}$$

Moreover, $w_{k,n,t}$ is $x_N$-odd in $B_1(0) \setminus \{te_N, -te_N\}$,

$$0 \leq w_{k,n+1,t} \leq w_{k,n,t} \leq G_{B_1(0)}[\mu_t] \quad \text{in} \quad B_1^+(0) \setminus \{te_N\}$$

and

$$w_{k+1,n,t} \geq w_{k,n,t} \quad \text{in} \quad B_1^+(0) \setminus \{te_N\}. \quad (3.4)$$

**Proof.** We observe that $\mu_t$ is a bounded Radon measure and $g_n$ is bounded, Lipschitz continuous and nondecreasing, then it follows from [19, Theorem 3.7] under the integral subcritical assumption (1.10) replaced $\frac{N}{N-2}$ by $\frac{N}{N-2}$ for $N \geq 3$ and the Kato’s inequality, problem (3.3) admits a unique weak solution $w_{k,n,t}$. Moreover, $w_{k,n,t}$ could be approximated by the classical solutions $\{w_{k,n,t,m}\}$ to problem

$$\begin{cases}
-\Delta u + g_n(u) = k\mu_{t,m} & \text{in} \quad B_1(0), \\
u = 0 & \text{on} \quad \partial B_1(0),
\end{cases} \quad (3.5)$$

where

$$\mu_{t,m}(x) = \frac{\sigma_m(x - te_N) - \sigma_m(x + te_N)}{t}$$

and $\{\sigma_m\}_m$ is a sequence of radially symmetric, nondecreasing smooth functions converging to $\delta_0$ in the distribution sense. Furthermore,

$$\int_{B_1(0)} [w_{k,n,t,m}(-\Delta) \xi + g_n(w_{k,n,t,m})\xi] dx = k \int_{B_1(0)} \mu_{t,m} \xi dx, \quad \forall \xi \in C^{1,1}_0(B_1(0)). \quad (3.6)$$
Since $\mu_{t,m}$ is $x_N$-odd and nonnegative in $B^+_1(0)$, so is $w_{k,n,t,m}$ by Lemma 2.1. We observe that $(k+1)\sigma_m \geq k\sigma_m$, it follows from Lemma 2.1 that

$$w_{k+1,n,t,m} \geq w_{k,n,t,m} \text{ in } B^+_1(0).$$

Since

$$g_n(s) \leq g_{n+1}(s), \quad \forall s \in \mathbb{R},$$

then it follows from Corollary 2.2 that

$$w_{k,n,t,m} \leq w_{k,n+1,t,m} \text{ in } B^+_1(0).$$

From the proof of Theorem 2.9 in [19], we know that

$$\mathcal{G}_{B_1(0)}[\mu_{t,m}] \rightarrow \mathcal{G}_{B_1(0)}[\mu] \text{ in } B_1(0) \setminus \{te_N,-te_N\} \text{ and in } L^q(B_1(0)) \text{ as } m \rightarrow \infty,$$

where $q \in [1,\frac{N}{N-2})$. By regularity results, any compact set $K$ and open set $O$ in $B_1(0)$ such that $K \subset O$, $O \cap \{te_N,-te_N\} = \emptyset$, there exist $c_6,c_7 > 0$ independent of $m$ such that

$$\|w_{k,n,t,m}\|_{C^2(K)} \leq c_6 \|\mathcal{G}_{B_1(0)}[\mu_{t,m}]\|_{L^\infty(O)} \leq c_7 \|G_{B_1(0)}[\mu]\|_{L^\infty(O)}.$$

Therefore, up to some subsequence, there exists a measurable function $\tilde{w}$ such that

$$w_{k,n,t,m} \rightarrow \tilde{w} \text{ in } B_1(0) \setminus \{te_N,-te_N\} \text{ and in } L^q(B_1(0)) \text{ as } m \rightarrow \infty,$$

where $q \in [1,\frac{N}{N-2})$. Then $\tilde{w}$ is $x_N$-odd, nonnegative in $B^+_1(0)$ and

$$g_n(w_{k,n,t,m}) \rightarrow g_n(\tilde{w}) \text{ in } B_1(0) \setminus \{te_N,-te_N\} \text{ and in } L^1(B_1(0)) \text{ as } m \rightarrow \infty.$$

Passing to the limit in (3.6) as $m \rightarrow \infty$, we deduce that $\tilde{w}$ is a weak solution of (3.3). By the uniqueness of weak solution of (3.3), we obtain that $w_{n,t} = \tilde{w}$. Therefore, $w_{n,t}$ is $x_N$-odd, nonnegative in $B^+_1(0)$ and it follows from (3.7) and (3.4) that

$$w_{k,n,t} \leq w_{k,n+1,t} \text{ in } B^+_1(0)$$

and

$$w_{k+1,n,t} \geq w_{k,n,t} \text{ in } B^+_1(0).$$

This ends the proof. \(\square\)

We next passing to the limit of weak solutions as $t \rightarrow 0^+$. 

**Proposition 3.2.** Let $g_n$ be defined by (3.2). Then for any $n \in \mathbb{N}$, problem

$$\begin{cases}
-\Delta u + g_n(u) = 2k \frac{\partial \delta_0}{\partial x_N} & \text{in } B_1(0), \\
\quad u = 0 & \text{on } \partial B_1(0)
\end{cases}$$

(3.8)

admits a unique very weak solution $w_{k,n}$. Moreover,

(i) $w_{k,n}$ is $x_N$-odd for any $n \in \mathbb{N}$ in $B_1(0) \setminus \{0\}$ and

$$0 \leq w_{k,n+1} \leq w_{k,n} \leq 2kG_{B_1(0)}\left[\frac{\partial \delta_0}{\partial x_N}\right] \text{ in } B^+_1(0)$$

and

$$0 \leq w_{k,n} \leq w_{k+1,n} \text{ in } B^+_1(0);$$

(ii) $w_{k,n}$ is a classical solution of

$$\begin{cases}
-\Delta u + g_n(u) = 0 & \text{in } B_1(0) \setminus \{0\}, \\
\quad u = 0 & \text{on } \partial B_1(0).
\end{cases}$$
Proof. It follows from Proposition 3.1 that problem (3.3) admits a unique very weak solution \( w_{k,n,t} \), that is,

\[
\int_{B_1(0)} [w_{k,n,t}(-\Delta)\xi + g_n(w_{k,n,t})] dx = k\frac{\xi(te_N) - \xi(-te_N)}{t}, \quad \forall \xi \in C_0^{1,1}(B_1(0)). \tag{3.9}
\]

On the one hand, we have that

\[
\lim_{t \to 0^+} \frac{\xi(te_N) - \xi(-te_N)}{t} = 2\frac{\partial \xi(0)}{\partial x_N}.
\]

On the other hand, by Proposition 3.1, we have that

\[
|w_{k,n,t}| \leq k|G_{B_1(0)}[\mu_t]| \quad \text{in} \quad B_1(0).
\]

By regularity results, for \( \sigma \in (0, 1) \) and any compact set \( K \) and open set \( O \) in \( B_1(0) \) such that \( K \subset O, O \cap \{ te_N : t \in (-\frac{1}{2}, \frac{1}{2}) \} = \emptyset \), there exist \( c_8, c_9 > 0 \) independent of \( t \) such that

\[
\|w_{k,n,t}\|_{C^{2+\sigma}(K)} \leq c_8 k G_{B_1(0)}[\mu_t]\|L^{\infty}(O) \leq c_9 k \|G_{B_1(0)}[\partial \delta_0]\|_{L^{\infty}(O)}.
\]

Moreover, by [4, Proposition 3.3], \( \{G_{B_1(0)}[\mu_t]\} \) is uniformly bounded in \( M^{\frac{N}{N-1}}(B_1(0), dx) \) if \( N \geq 3 \) and is uniformly bounded in \( M^{\frac{2}{N}}(B_1(0), dx) \) for any \( \sigma \in (0, \frac{1}{2}) \) if \( N = 2 \). Therefore, \( \{w_{k,n,t}\}_t \) is relatively compact in \( L^p(B_1(0)) \) for any \( p \in [1, \frac{N}{N-1}) \). There exists \( w_{k,n} \in L^1(B_1(0)) \) such that

\[
w_{k,n,t} \to w_{k,n} \quad \text{a.e. in} \quad B_1(0) \quad \text{and in} \quad L^1(B_1(0)),
\]

which implies that

\[
g_n(w_{k,n,t}) \to g_n(w_{k,n}) \quad \text{a.e. in} \quad B_1(0) \quad \text{and in} \quad L^1(B_1(0)) \quad \text{as} \quad t \to 0^+.
\]

Therefore, up to some subsequence, passing to the limit as \( t \to 0^+ \) in the identity (3.9), it follows that \( w_{k,n} \) is a very weak solution of (3.8). Moreover, \( w_{k,n} \) is \( x_N \)-odd and nonnegative in \( B_1^{+}(0) \).

Uniqueness. Let \( v_n \) be a weak solution of (3.8) and then \( \varphi_n := w_{k,n} - v_n \) is a very weak solution to

\[
\begin{cases}
-\Delta \varphi_n + g_n(w_{k,n}) - g_n(v_n) = 0 & \text{in} \quad B_1(0), \\
\varphi_n = 0 & \text{on} \quad \partial B_1(0).
\end{cases}
\]

By Kato’s inequality [19, Theorem 2.4] (see also [8, 9, 17]),

\[
\int_{B_1(0)} |\varphi_n|(-\Delta)\xi + \int_{B_1(0)} [g_n(w_{k,n}) - g_n(v_n)]\text{sign}(w_{k,n} - v_n)\xi dx \leq 0
\]

Taking \( \xi = G_{B_1(0)}[1] \), we have that

\[
\int_{B_1(0)} [g_n(w_{k,n}) - g_n(v_n)]\text{sign}(w_{k,n} - v_n)\xi dx \geq 0 \quad \text{and} \quad \int_{B_1(0)} |\varphi_n| dx = 0,
\]

then \( \varphi_n = 0 \) a.e. in \( B_1(0) \). Then the uniqueness is obtained. \( \square \)

The next estimate plays an important role in \( w_{k,n} \to w_k \) in \( L^p(B_1(0)) \) with \( p \in [1, \frac{N+1}{N-1}) \).

Lemma 3.2. There exists \( c_{10} > 0 \) such that

\[
\|G_{B_1(0)}[\partial \delta_0]\|_{M^{\frac{N}{N-1}}(B_1(0))} \leq c_{10} \tag{3.10}
\]

and

\[
\|G_{B_1(0)}[\partial \delta_0]\|_{M^{\frac{N+1}{N-1}}(B_1(0), |x| dx)} \leq c_{10}. \tag{3.11}
\]
Proof. We observe that
\[ G_{B_1(0)} \frac{\partial \delta_0}{\partial x_N}(x) = \frac{\partial G_{B_1(0)}(x,0)}{\partial x_N} \]
and for \( x, y \in B_1(0), x \neq y, \)
\[ G_{B_1(0)}(x,y) = \begin{cases} c_N|x-y|^{2-N} + \tilde{G}_{B_1(0)}(x,y) & \text{ if } N \geq 3, \\ -c_N \log |x-y| + \tilde{G}_{B_1(0)}(x,y) & \text{ if } N = 2, \end{cases} \]
where \( \tilde{G}_{B_1(0)} \) is a harmonic function in \( B_1(0) \times B_1(0). \) Then
\[ \frac{|\partial G_{B_1(0)}(x,0)|}{|x|^{N-1}} \leq c_{11} |x| + c_{12}. \]
Therefore, we have that
\[ |G_{B_1(0)} \frac{\partial \delta_0}{\partial x_N}| \leq \frac{c_{13}}{|x|^{N-1}}, \quad \forall \ x \in B_1(0) \setminus \{0\}. \]

Proof of (3.10). Let \( E \) be a Borel set of \( B_1(0) \) with \( |E| > 0, \) then there exists \( r_1 \in (0,1] \) such that
\[ |E| = |B_{r_1}(0)|. \]
We deduce that
\[ \int_E |G_{B_1(0)} \frac{\partial \delta_0}{\partial x_N}| \, dx = \int_{E \cap B_{r_1}(0)} \frac{c_{13}}{|x|^{N-1}} \, dx + \int_{E \setminus B_{r_1}(0)} \frac{c_{13}}{|x|^{N-1}} \, dx \]
\[ \leq \int_{B_{r_1}(0)} \frac{c_{13}}{|x|^{N-1}} \, dx \]
\[ = c_{14} r_1 = c_{15} \left( \int_E |x| \, dx \right)^{\frac{1}{N}}. \]
By the definition of Marcinkiewicz space, we have that
\[ \|G_{B_1(0)} \frac{\partial \delta_0}{\partial x_N}\|_{M^{-\frac{N}{N-1}}(B_1(0), dx)} \leq c_{15}. \]

Proof of (3.11). Let \( E \) be a Borel set of \( B_1(0) \) with \( |E| > 0, \) then there exists \( r_2 \in (0,1] \) such that
\[ \int_E |x| \, dx = \int_{B_{r_2}(0)} |x| \, dx. \]
Since
\[ \int_{B_{r_2}(0)} |x| \, dx = c_{16} r_2^{N+1}, \]
we deduce that
\[ \int_E |G_{B_1(0)} \frac{\partial \delta_0}{\partial x_N}| |x| \, dx \leq \int_E \frac{c_{13}}{|x|^{N-1}} |x| \, dx \]
\[ \leq c_{17} r_2^{2(N+1)} = c_{18} \left( \int_E |x| \, dx \right)^{\frac{2}{N+1}} \leq c_{19}. \]
This ends the proof. \( \square \)

Lemma 3.3. Assume that \( g : [0,\infty) \to [0,\infty) \) is continuous, nondecreasing and verifies (1.10). Then for \( q \geq \frac{N+1}{N-1}, \)
\[ \lim_{s \to \infty} g(s)^{s^{-q}} = 0. \]
Proof. Since
\[ \int_s^{2s} g(t)t^{-1-k_{\alpha,\beta}} dt \geq g(s)(2s)^{-1-N^{-1}} \int_s^{2s} dt = 2^{-1-N^{-1}} g(s) s^{-N^{-1}} \]
and by (1.7),
\[ \lim_{s \to \infty} \int_s^{2s} g(t)t^{-1-N^{-1}} dt = 0, \]
then
\[ \lim_{s \to \infty} g(s) s^{-N^{-1}} = 0. \]
The proof is complete.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Existence. Let \( \{g_n\} \) be a sequence of \( C^1 \) nondecreasing functions defined by (3.2). It follows that \( \{g_n\} \) is a sequence of odd, bounded and nondecreasing Lipschitz continuous functions.

By Proposition 3.2, problem (3.8) admits a unique \( x_N \)-odd weak solution \( w_{k,n} \) such that
\[ 0 \leq w_{k,n} \leq 2k G_{B_1(0)} \left[ \frac{\partial \delta_0}{\partial x_N} \right] \text{ a.e. in } B_1^+(0) \]  
(3.12)
and
\[ \int_{B_1(0)} [w_{k,n}(-\Delta)\xi + g_n(w_{k,n})\xi] dx = 2k \frac{\partial \xi(0)}{\partial x_N}, \quad \forall \xi \in C^{1,1}_0(B_1(0)). \]  
(3.13)

For \( \beta \in (0, 1) \), any compact set \( K \) and open set \( O \) in \( B_1(0) \) satisfying \( K \subset O, 0 \not\in \bar{O} \), we have that
\[ \|w_{k,n}\|_{C^{2,\beta}(K)} \leq c_{20} \|G_{B_1(0)} \left[ \frac{\partial \delta_0}{\partial x_N} \right] \|_{C^1(O)}, \]
where \( c_{20} > 0 \). Therefore, up to some subsequence, there exists \( w_k \) such that
\[ \lim_{n \to 0^+} w_{k,n} = w_k \text{ a.e. in } B_1(0). \]

Then \( \{g_n(w_{k,n})\} \) converges to \( g(w_k) \) a.e. in \( B_1(0) \). By Lemma 3.2, we have that
\[ w_{k,n} \to w_k \text{ in } L^1(B_1(0)) \text{ as } n \to +\infty, \]
and
\[ \|g_n(w_{k,n})\|_{L^1(B_1(0),|x|dx)} \leq c_{21} \|G_{B_1(0)} \left[ \frac{\partial \delta_0}{\partial x_N} \right] \|_{L^1(B_1(0),|x|dx)}, \]
by Proposition 2.2 and \( G_{B_1(0)} \left[ \frac{\partial \delta_0}{\partial x_N} \right] \in M^{N+1}_N(B_1(0),|x|dx) \), we have that
\[ m(\lambda) \leq c_{22} \lambda^{-\frac{N+1}{N-1}}, \quad \forall \lambda \geq \lambda_0, \]
where
\[ m(\lambda) = \int_{S_\lambda} |x|dx \text{ with } S_\lambda = \left\{ x \in B_1(0) : \left| G_{B_1(0)} \left[ \frac{\partial \delta_0}{\partial x_N} \right] \right| > \lambda \right\}. \]
For any Borel set $E \subset B_1(0)$, we have that
\[
\int_E |g_n(w_{k,n})||x|dx \leq \int_{E \cap S^N} g \left( 2k \left| \mathcal{G}_{B_1(0)} \left( \frac{\partial \delta_0}{\partial x_N} \right) \right| \right) |x|dx \\
+ \int_{E \cap S^N} g \left( 2k \left| \mathcal{G}_{B_1(0)} \left( \frac{\partial \delta_0}{\partial x_N} \right) \right| \right) |x|dx \\
\leq g(\lambda) \int_E |x|dx + \int_S g \left( 2k \left| \mathcal{G}_{B_1(0)} \left( \frac{\partial \delta_0}{\partial x_N} \right) \right| \right) |x|dx \\
\leq g(\lambda) \int_E |x|dx + m \left( \frac{\lambda}{2k} \right) g(\lambda) + \int_\mathbb{R}^N m(s)dg(2ks).
\]

On the other hand,
\[
\int_\frac{\lambda}{2k} g(2ks)dm(s) = \lim_{T \to \infty} \int_\frac{T}{2k} g(2ks)dm(s).
\]
Thus,
\[
m \left( \frac{\lambda}{2k} \right) g(\lambda) + \int_\frac{T}{2k} m(s)dg(2ks) \leq c_{24} g(\lambda) \left( \frac{\lambda}{2k} \right)^{-\frac{N+1}{N-1}} + c_{24} \int_\frac{T}{2k} s^{-\frac{N+1}{N-1}}dg(2ks) \\
= c_{25} T^{-\frac{N+1}{N-1}} g(T) + c_{26} \int_\frac{T}{2k} s^{-1} s^{-\frac{N+1}{N-1}}g(s)ds,
\]
where $c_{26} = \frac{c_{24}}{N-1} (2k)^{2+\frac{N+1}{N-1}}$. By assumption (1.10) and Lemma 3.3, we have that $T^{-\frac{N+1}{N-1}} g(T) \to 0$ as $T \to \infty$, therefore,
\[
m \left( \frac{\lambda}{2k} \right) g(\lambda) + \int_\frac{T}{2k} m(s) dg(2ks) \leq c_{26} \int_\frac{T}{2k} s^{-1} s^{-\frac{N+1}{N-1}}g(s)ds.
\]
Notice that the quantity on the right-hand side tends to 0 when $\lambda \to \infty$. The conclusion follows: for any $\epsilon > 0$, there exists $\lambda > 0$ such that
\[
c_{26} \int_\frac{T}{2k} s^{-1} s^{-\frac{N+1}{N-1}}g(s)ds \leq \frac{\epsilon}{2}.
\]
For $\lambda$ fixed, there exists $\delta > 0$ such that
\[
\int_E |x|dx \leq \delta \implies g(\lambda) \int_E |x|dx \leq \frac{\epsilon}{2},
\]
which implies that $\{g_n \circ w_{k,n}\}$ is uniformly integrable in $L^1(B_1(0), |x|dx)$. Then $g_n \circ w_{k,n} \to g \circ w_k$ in $L^1(B_1(0), |x|dx)$ by Vitali convergence theorem, see [7].

Furthermore, for any $\xi \in C^{1,1}_0(B_1(0))$, we know that
\[
|\xi(x) - \xi(0) - \nabla \xi(0) \cdot x| \leq c_{27} |x|^2,
\]
and it follows the odd prosperity of $w_{k,n}$, $g_n$ and $g$, that
\[
\int_{B_1(0)} g_n(w_{k,n})\xi(0) dx = 0 \quad \text{and} \quad \int_{B_1(0)} g(w_k)\xi(0) dx = 0,
\]
then

$$\left| \int_{B_1(0)} g_n(w_k,n)\xi \, dx - \int_{B_1(0)} g(w_k)\xi \, dx \right| \leq \int_{B_1(0)} |g_n(w_k,n) - g(w_k)| \nabla\xi(0) \cdot x \, dx + c_{27} \int_{B_1(0)} |g_n(w_k,n) - g(w_k)||x|^2 \, dx \leq c_{28} \|g_n(w_k,n) - g(w_k)\|_{L^1(B_1(0), |x| \, dx)} \to 0 \quad \text{as} \quad n \to +\infty.$$  

Then passing to the limit as $n \to +\infty$ in the identity (3.13), it implies that

$$\int_{B_1(0)} [w_k(-\Delta)\xi + g(w_k)\xi] \, dx = 2k \frac{\partial \xi(0)}{\partial x_N}. \quad \text{(3.16)}$$

Thus, $w_k$ is a very weak solution of (1.8). The regularity results follows by Lemma 3.1.

**Proof of (i).** Since $w_{k,n}$ is $x_N$-odd in $B_1(0) \setminus \{0\}$ and $w_k = \lim_{n \to +\infty} w_{k,n}$ in $B_1(0) \setminus \{0\}$, then it implies that $w_k$ is $x_N$-odd in $B_1(0) \setminus \{0\}$. By the fact that $w_{k+1,n} \geq w_{k,n} \geq 0$ in $B_1^+(0)$, it follows that

$$w_{k+1} \geq w_k \geq 0 \quad \text{in} \quad B_1^+(0).$$

**Proof of (ii).** We observe that

$$0 \leq w_{k,n} \leq 2kG_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}] \quad \text{in} \quad B_1^+(0),$$

then $g_n(w_{k,n}) \leq g(2kG_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}])$ for $x \in B_1^+(0)$ and let $w_g$ be the unique solution of (2.3) with $f = g(2kG_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}])$, we have that

$$2kG_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}](x) \geq w_{k,n}(x) \geq 2kG_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}](x) - w_g(x). \quad \text{(3.17)}$$

Then $w_k$ satisfies (3.17). From Proposition 2.1, it infers that for $x = te$ with $t \in (0, \frac{1}{2})$ and $e = (e_1, \cdots, e_N) \in \partial B_1(0)$ with $e_N > 0$,

$$w_y(te) \leq \int_{B_{\frac{1}{2}}(0)} \left[ \frac{|y|}{|y - te|(|y - te| + 2|y|)^{N-2}} \right] g\left(2kG_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}](y)\right) \, dy \leq \int_{B_{\frac{1}{2}}(0)} \left[ \frac{|y|}{|y - te|(|y - te| + 2|y|)^{N-2}} \right] g(2c_{13}k|y|^{1-N}) \, dy \quad := \int_{B_{\frac{1}{2}}(0)} A(t,y) \, dy.$$

For $y \in B_{\frac{1}{2}}(te)$, we have that $|y - te| \geq \frac{t}{2}$ and $\frac{t}{2} \leq |y| \leq \frac{3t}{2}$, then

$$t^{N-1} \int_{B_{\frac{1}{2}}(te)} A(t,y) \, dy \leq c_{30}r^{\frac{N+1}{N-1}} g(2c_{13}kr) \to 0 \quad \text{as} \quad r \to +\infty,$$

where $r = t^{-\frac{1}{N-1}}$. 
For $y \in B_\frac{t}{2}(0)$, we have that $|y - te| \geq \frac{t}{2}$ and
\[
\begin{aligned}
t^{N-1} \int_{B_{\frac{t}{2}}(0)} A(t, y) dy &\leq \int_{B_{\frac{t}{2}}(0)} |y||g(2c_{13}k|y|^{1-N})| dy \\
&= c_{32} \int_0^{\frac{t}{2}} g(2c_{13}ks^{1-N})s^N ds \\
&= c_{32} \int_0^{\frac{t}{2}} g(2c_{13}k) t^{-\frac{N+1}{N-1}} d\tau \\
&\to 0 \quad \text{as } t \to 0,
\end{aligned}
\]
where we have used (1.10).

For $y \in B_1^+(0) \setminus (B_\frac{t}{2}(0) \cup B_\frac{t}{2}(te))$, we have that $|y - te| \geq \frac{t}{2}$, $|y| \geq \frac{t}{2}$ and
\[
\begin{aligned}
t^{N-1} \int_{B_1^+(0) \setminus (B_\frac{t}{2}(0) \cup B_\frac{t}{2}(te))} A(t, y) dy &\leq c_{33} t \int_{B_1^+(0) \setminus (B_\frac{t}{2}(0) \cup B_\frac{t}{2}(te))} g(2c_{13}k|y|^{1-N}) dy \\
&= c_{33} t \int_1^{\frac{t}{2}} g(2c_{13}k \tau^{1-N}) \tau^{-\frac{N+1}{N-1}} d\tau \\
&= c_{34} t^{N-1} g(2c_{13}kt^{1-N}) t^{-2} \\
&\to 0 \quad \text{as } \tau \to +\infty
\end{aligned}
\]
where $\tau = 2c_{13}kt^{1-N}$. Then for any $e = (e_1, \ldots, e_N) \in \partial B_1(0)$ and $e_N \neq 0$, we have that
\[
\lim_{t \to 0^+} t^{N-1} w_y(te) = 0
\]
and
\[
\lim_{t \to 0^+} \mathcal{G}_{B_1(0)}[g((\mathcal{G}_{B_1(0)}[2k \frac{\partial \delta_0}{\partial x_N}]))(te)] t^{N-1} = 0.
\]

By (3.17),
\[
\lim_{t \to 0^+} \mathcal{G}_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}](te) t^{N-1} = e_N,
\]
and $x_N$-odd property of $w_k$, we derive that for any $e \in \partial B_1(0),
\[
\lim_{t \to 0^+} w_k(te) t^{N-1} = 2ke_N.
\]

Finally, we prove the uniqueness. Let $u_k, v_k$ be two $x_N$-odd solutions of (1.8), from Lemma 3.4, $u_k, v_k$ are two solutions of
\[
\begin{aligned}
-\Delta u + g(u) &= 0 \quad \text{in } B_1^+(0), \\
u &= k\delta_0 \quad \text{on } \partial B_1^+(0).
\end{aligned}
\tag{3.19}
\]
Form the uniqueness of the very weak solution to (3.19), see [14, Theorem 2.1], we obtain $u_k = v_k$ in $B_1^+(0)$ and combine the $x_N$-odd property we obtain the uniqueness of the very weak solution of (1.8) with $j = 0$. This ends the proof. \hfill \Box

3.2. Nonexistence. This subsection is devoted to obtain the nonexistence of very weak solutions of (1.8) in the supercritical case.

**Lemma 3.4.** Assume that $k > 0$, the nonlinearity $g: \mathbb{R} \to \mathbb{R}$ is an odd, nondecreasing and Lipschitz continuous function. Let $u$ be an $x_N$-odd very weak solution of (1.8).

Then $u$ is a very weak solution of (3.19).
**Proof.** From Lemma 3.1 and the $x_N$-odd property, we have that
\[ u = 0 \quad \text{on} \quad \partial B_1^+(0) \setminus \{0\}. \]
For any $x_N$-odd function $\xi \in C^{1,1}_0(B_1(0))$, we deduce that
\[ \xi = 0 \quad \text{on} \quad \partial B_1^+(0) \quad |\xi(x)| \leq c_{36}|x|, \quad x \in B_1(0), \]
where $c_{36} > 0$. Since $g(u) \in L^1(B_1(0), |x|dx)$, then for $x_N$-odd function $\xi \in C^{1,1}_0(B_1(0))$, we have that $g(u)\xi \in L^1(B_1(0))$ and the weak solution $u$ satisfies that
\[ \int_{B_1(0)} [u(-\Delta)\xi dx + g(u)\xi]dx = 2k\frac{\partial \xi(0)}{\partial x_N}, \quad (3.20) \]
for $x_N$-odd function $\xi \in C^{1,1}_0(B_1(0))$. By $x_N$-odd property, we have that
\[ \int_{B_1^+(0)} [u(-\Delta)\xi dx + g(u)\xi]dx = \int_{B_1^-(0)} [u(-\Delta)\xi dx + g(u)\xi]dx, \]
which, combined with (3.20), implies that
\[ \int_{B_1^+(0)} [u(-\Delta)\xi dx + g(u)\xi]dx = k\frac{\partial \xi(0)}{\partial x_N}, \quad \forall \xi \in C^{1,1}_0(B_1^+(0)). \]
So we have that $u$ is a weak solution of (3.19). □

**Proof of Theorem 1.2.** When $g(s) = |s|^{p-1}s$ with $p \geq \frac{N+1}{N-1}$ and $k > 0$, [14, Theorem 3.1] shows that the nonnegative solution of
\[ \begin{cases} -\Delta u + g(u) = 0 & \text{in} \quad B_1^+(0), \\
 u = 0 & \text{on} \quad \partial B_1^+(0) \setminus \{0\} \end{cases} \]
has removable singularity at origin, so problem (3.19) has no very weak solution, which contradicts Lemma 3.4. Then (1.8) has no $x_N$-odd weak solution when $g(s) = |s|^{p-1}s$ with $p \geq \frac{N+1}{N-1}$. □

4. **Strongly anisotropic singularity for $p \in (1, \frac{N+1}{N-1})$**

In this section, we consider the limit of weak solutions $w_k$ as $k \to \infty$ to
\[ \begin{cases} -\Delta u + |u|^{p-1}u = 2k\frac{\partial \delta_k}{\partial x_N} & \text{in} \quad B_1(0), \\
 u = 0 & \text{on} \quad \partial B_1(0), \end{cases} \quad (4.1) \]
where $p \in (1, \frac{N+1}{N-1})$. By Theorem 1.1, we observe that the mapping $k \mapsto w_k$ is nondecreasing in $B_1^+(0)$, then $\lim_{k \to +\infty} w_k(x)$ exists for any $x \in B_1(0) \setminus \{0\}$, denoting
\[ w_\infty(x) = \lim_{k \to +\infty} w_k(x) \quad \text{for} \quad x \in B_1(0) \setminus \{0\}. \quad (4.2) \]

For $w_\infty$, we have the following result.

**Proposition 4.1.** Let $p \in (1, \frac{N+1}{N-1})$, then $w_\infty$ is $x_N$-odd,
\[ 0 \leq w_\infty(x) \leq \lambda_0|x|^{-\frac{2}{p-1}}, \quad \forall \quad x \in B_1^+(0) \]
for some $\lambda_0 > 0$ and $w_\infty$ is a classical solution of
\[ \begin{cases} -\Delta u + |u|^{p-1}u = 0 & \text{in} \quad B_1(0) \setminus \{0\}, \\
 u = 0 & \text{on} \quad \partial B_1(0). \end{cases} \quad (4.3) \]
Therefore, from standard Stability Theorem, we derive that

\[ w \]

Lemma 4.1. Let \( \Box \) (4.3).

Since \( k \) by Comparison Principle, we have that \( G \)

Indeed, \( \phi \)

where \( \omega \) is arbitrary, we deduce that

\[ |w| \leq \lambda_0 v_p \quad \text{in} \quad B_{\gamma}(0) \setminus \{0\}. \]

By Comparison Principle, we have that

\[ |w| \leq \lambda_0 v_p \quad \text{in} \quad B(0) \setminus \{0\}. \]

Since \( k \) is arbitrary, we deduce that

\[ |w| \leq \lambda_0 v_p \quad \text{in} \quad B(0) \setminus \{0\}. \]

Therefore, from standard Stability Theorem, we derive that \( w \) is a classical solution of (4.3).

We next do a precise bound for \( w \) to prove (1.18).

Lemma 4.1. Let \( p \in (1, \frac{N+1}{N-1}) \) and \( w \) be defined by (4.2), then \( w \) satisfies (1.18).

Proof. We claim that

\[
\frac{1}{c_{36}} t^{\frac{2}{p-1}} P_N(e) \leq w_\infty (te) \leq c_{36} t^{\frac{2}{p-1}} P_N(e), \quad \forall t > 0, \ e = (e_1, \cdots, e_N) \in \partial B(0), e_N > 0. \tag{4.4}
\]

We have that

\[
2k G_{B_1(0)} \left[ \frac{\partial \phi}{\partial x_N} \right](x) \geq w_k(x) \geq 2k G_{B_1(0)} \left[ \frac{\partial \phi}{\partial x_N} \right](x) - \varphi_p(x), \quad \forall x \in B_1^+(0), \tag{4.5}
\]

where \( \varphi_p := k^p G_{R^N} \left[ G_{R^N} \left[ \frac{\partial \phi}{\partial x_N} \right] \right] \geq G_{B_1(0)} [w^p_k] \), and \( \varphi_p \) is a \( x_N \)-odd solution of

\[
\begin{cases}
- \Delta u = c^p_N \frac{|x_N|^{p-1} x_N}{|x|^{np}} & \text{in} \quad R^N \setminus \{0\}, \\
\lim_{|x| \to +\infty} u(x) = 0.
\end{cases} \tag{4.6}
\]

Indeed, \( G_{R^N} \left[ G_{R^N} \left[ \frac{\partial \phi}{\partial x_N} \right] \right] \) is \( x_N \)-odd and

\[
\varphi_p(x) = c_N c_N^p \int_{R^N} \frac{|y_N|^{p-1} y_N}{|x-y|^{N-2} |y|^{np}} dy
\]

\[
= c_N c_N^p |x|^{(1-N)p} \int_{R^N} \frac{|y_N|^{p-1} y_N}{|e_2 - y|^{N-2} |y|^{np}} dx,
\]

where \( e_2 = \frac{x}{|x|} \).

We observe that \( \varphi_p \) satisfies

\[
- \Delta u(x) = c^p_N \frac{|x_N|^{p-1} x_N}{|x|^{np}}, \quad \forall x \in B_1^+(0),
\]

and then it follows by Hopf’s Lemma (see [6]) that

\[
\varphi_p(x) \leq c_{37} x_N, \quad \forall x \in \partial B_1^+(0).
\]

Therefore,

\[
\varphi_p(e) \leq c_{38} P_N(e), \quad \forall e \in \partial B_1^+(0).
\]
Proof of lower bound in (4.4). It follows by (4.5) that
\[ w_k(x) \geq c_{40}k|x|^{1-N}P_N\left(\frac{x}{|x|}\right) - c_{39}k^p|x|^{(1-N)p+2}P_N\left(\frac{x}{|x|}\right), \quad \forall x \in B_\frac{1}{2}(0) \setminus \{0\}. \]
Set \( \rho_k = (2^{(N-1)(p-1)-3}c_{40}k^{p-1})^{\frac{1}{(N-1)(p-1)-2}} \), then
\[ c_{39}k^p|x|^{(1-N)p+2} \leq c_{39}k^p\left(\frac{\rho_k}{2}\right)^{(1-N)p+2} \leq \frac{c_{40}}{k}\rho_k^{1-N} \leq \frac{c_{40}}{2}k|x|^{1-N} \]
and
\[ k = c_{41}\rho_k^{1-N-\frac{2}{p-1}} \geq c_{41}|x|^{N-1-\frac{2}{p-1}}, \]
where \( c_{41} > 0 \) independent of \( k \). Thus, for \( t \in (\frac{\rho_k}{2}, \rho_k) \), \( e \in \partial B_1(0) \), \( e \cdot e_N > 0 \),
\[ w_k(te) \geq [c_{40}kt^{1-N} - c_{39}k^p(t^{1-N})^{p+2}]P_N(e) \geq \frac{c_{40}}{2}k^{1-N}P_N(e) \geq \frac{c_{40}c_{41}}{2}t^{-\frac{2}{p-1}}P_N(e). \]
Now we can choose a sequence \( \{k_n\} \subset [1, \infty) \) such that
\[ \rho_{k_{n+1}} \geq \frac{1}{2}\rho_{k_n} \]
and for any \( x \in B_\frac{1}{2}(0) \setminus \{0\} \), there exists \( k_n \) such that \( x \in B_{\frac{1}{2}} \) and
\[ w_{k_n}(x) \geq \frac{c_{40}c_{41}}{2}|x|^{-\frac{2}{p-1}}P_N\left(\frac{x}{|x|}\right). \]
Together with \( w_{k_{n+1}} \geq w_{k_n} \) in \( B_\frac{1}{2}(0) \), we have that
\[ w_\infty(x) \geq \frac{c_{40}c_{41}}{2}|x|^{-\frac{2}{p-1}}P_N\left(\frac{x}{|x|}\right), \quad \forall x \in B_\frac{1}{2}(0). \]

Proof of the upper bound in (4.4). Let \( \bar{w}_p = |x|^{-\frac{2}{p-1}}x_N \), then
\[ -\Delta \bar{w}_p = (\frac{2}{p-1} + 1)(\frac{2}{p-1} + 1 - N)|x|^{-\frac{2}{p-1}-3}x_N, \]
where \((\frac{2}{p-1} + 1)(\frac{2}{p-1} + 1 - N) > 0 \) for \( p < \frac{N+1}{N-1} \). By Comparison Principle, there exists \( t_0 > 0 \) independent of \( k \) such that
\[ u_k \leq t_0\bar{w}_p \quad \text{in} \quad B_\frac{1}{2}(0), \]
which implies that
\[ w_\infty \leq t_0\bar{w}_p \quad \text{in} \quad B_\frac{1}{2}(0). \]

Proof of (1.18). We observe there exists \( t_{00} \in (0, t_0) \) such that
\[ -\Delta t_{00}\bar{w}_p + (t_{00}\bar{w}_p)^p \leq 0 \quad \text{in} \quad \mathbb{R}^N_+. \]
Therefore,
\[ \begin{cases} -\Delta u_p + u_p^p = 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+ \setminus \{0\} \end{cases} \]
admits a unique solution \( u_p \) and by scaling property, we have that
\[ u_p(te) = t^{-\frac{2}{p-1}}u_p(e), \quad te \in \mathbb{R}^N_+. \]
By Comparison Principle, we have that
\[ u_p(te) - \max u_p(e) \leq u_\infty(te) \leq u_p(te), \quad te \in B_1^+(0), \]
which implies (1.18) with \( \varphi(e) = u_p(e) \). □

**Proof of Theorem 1.3.** From Proposition 4.1 and Lemma 4.1, one has that \( u_\infty \) is a classical solution of (1.17) satisfying (1.18). □

5. Non \( x_N \)-odd solutions

5.1. Existence. Under the assumptions on \( g \) in Theorem 1.1, it shows from [19] that the problem
\[
\begin{cases}
-\Delta u + g(u) = j\delta_0 & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0)
\end{cases}
\]
(5.1)
admits a unique weak solution, denoting by \( u_{0,j} \). In the approaching the weak solution of problem (1.8) with \( j > 0 \), a barrier will be constructed by \( u_{0,j} \) and \( u_k \), where \( u_k \) is the unique \( x_N \)-odd weak solution of (1.8) with \( j = 0 \).

**Proof of Theorem 1.1 with \( j > 0 \).** Step 1. We observe that \( g_n \) is bounded, Lipschitz continuous and nondecreasing, where \( g_n \) is defined by (3.2), then it follows from [19, Theorem 3.7] and the Kato’s inequality that
\[
\begin{cases}
-\Delta u + g_n(u) = k\mu + j\delta_0 & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0)
\end{cases}
\]
(5.2)
admits a unique weak solution \( v_{k,j,n,t} \), which is a classical solution of
\[
\begin{cases}
-\Delta u + g_n(u) = 0 & \text{in } B_1(0) \setminus \{te_N, 0, -te_N\}, \\
u = 0 & \text{on } \partial B_1(0).
\end{cases}
\]
Moreover, \( v_{k,j,n,t} \) could be approximated by the classical solutions \( \{v_{n,t,m}\} \) to
\[
\begin{cases}
-\Delta u + g_n(u) = k\mu + j\sigma_m & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0),
\end{cases}
\]
where
\[
\mu_{t,m}(x) = \frac{\sigma_m(x - te_N) - \sigma_m(x + te_N)}{t}
\]
and \( \{\sigma_m\} \) is a sequence of radially symmetric, nondecreasing smooth functions converging to \( \delta_0 \) in the distribution sense. Furthermore,
\[
\int_{B_1(0)} [v_{n,t,m}(-\Delta)\xi + g_n(v_{n,t,m})\xi] \, dx = \int_{B_1(0)} [k\mu + j\sigma_m] \xi \, dx, \quad \forall \xi \in C_0^{1,1}(B_1(0)).
\]
(5.3)
Since \( k\mu + j\sigma_m \geq k\mu, m \), it implies by Comparison Principle that
\[
w_{k,n,t,m} \leq v_{n,t,m} \leq w_{k,n,t,m} + v_{j,m} \quad \text{in } B_1(0),
\]
(5.4)
where \( w_{k,n,t,m} \) is the weak solution of (3.5) and \( v_{j,n,m} \) is the unique solution of the equation
\[
\begin{cases}
-\Delta u = j\sigma_m & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0).
\end{cases}
\]
Therefore, \( v_{k,j,n,t} \) satisfies
\[
w_{k,n,t} \leq v_{k,j,n,t} \leq w_{k,n,t} + jv_{j,m} \quad \text{in } B_1(0) \setminus \{te_N, 0, -te_N\}
\]
and
\[
v_{k',j',n,t} \geq v_{k,j,n,t} \quad \text{in } B_1^+(0) \setminus \{te_N\} \quad \text{for } k' \geq k, \ j' \geq j.
\]
(5.5)
Step 2. From Step 1, problem (5.2) admits a unique weak solution $v_{k,j,n,t}$, that is,

$$
\int_{B_1(0)} [w_{k,n,t}(-\Delta)\xi + g_n(w_{k,n,t})\xi] \, dx = \frac{\xi(t e_N) - \xi(-t e_N)}{t} + j \xi(0), \quad \forall \xi \in C_0^2(B_1(0)). \tag{5.6}
$$

On the one hand, by Lemma 2.1, we have that

$$
\lim_{t \to 0^+} \frac{\xi(t e_N) - \xi(-t e_N)}{t} = 2 \frac{\partial \xi(0)}{\partial x_N}.
$$

On the other hand, by the fact that

$$
|w_{k,n,t}| \leq k|\mathcal{G}_{B_1(0)}[\mu_t]| \quad \text{in} \ B_1(0),
$$

we have that

$$
|v_{k,j,n,t}| \leq k|\mathcal{G}_{B_1(0)}[\mu_t]| + j|\mathcal{G}_{B_1(0)}[\delta_0]| \quad \text{in} \ B_1(0).
$$

By interior regularity results, see [15], for $\sigma \in (0, 1)$ and any compact set $K$ and open set $O$ in $B_1(0)$ such that $K \subset O$, $O \cap \{te_N : t \in (-\frac{1}{2}, \frac{1}{2})\} = \emptyset$, there exist $c_{43}, c_{44} > 0$ independent of $t$ such that

$$
\|v_{k,j,n,t}\|_{C^{2+\sigma}(K)} \leq c_{43}[\|k\mathcal{G}_{B_1(0)}[\mu_t]\|_{L^\infty(O)} + \|j\mathcal{G}_{B_1(0)}[\delta_0]\|_{L^\infty(O)}]
$$

$$
\leq c_{44}[\|k\mathcal{G}_{B_1(0)}[\frac{\partial \delta_0}{\partial x_N}]\|_{L^\infty(O)} + \|j\mathcal{G}_{B_1(0)}[\delta_0]\|_{L^\infty(O)}].
$$

Moreover, by [4, Lemma 3.6] $\{\mathcal{G}_{B_1(0)}[\mu_t]\}$ is uniformly bounded in $M^{\frac{N}{N-1}}(B_1(0), dx)$ if $N \geq 3$ and in $M^{\frac{N}{N-1}}(B_1(0), dx)$ for any $\sigma \in (0, \frac{1}{2})$ if $N = 2$, $\{\mathcal{G}_{B_1(0)}[\delta_0]\}$ is uniformly bounded in $M^q(B_1(0), dx)$ if $q > 0$. Therefore, $\{v_{k,j,n,t}\}$ is relatively compact in $L^p(B_1(0))$ for any $p \in [1, \frac{N}{N-1})$.

Then there exists $v_{k,j,n} \in L^1(B_1(0))$ such that

$$
v_{k,j,n,t} \to v_{k,j,n} \quad \text{a.e. in} \ B_1(0) \quad \text{and in} \ L^1(B_1(0)),
$$

which implies that

$$
g_n(v_{k,j,n,t}) \to g_n(v_{k,j,n}) \quad \text{a.e. in} \ B_1(0) \quad \text{and in} \ L^1(B_1(0)).
$$

Therefore, up to some subsequence, passing to the limit as $t \to 0^+$ in the identity (5.3), it infers that $v_{k,j,n}$ is the unique very weak solution of

$$
\begin{cases}
-\Delta u + g_n(u) = 2k \frac{\partial \delta_0}{\partial x_N} + j \delta_0 & \text{in} \ B_1(0), \\
u = 0 & \text{on} \ \partial B_1(0).
\end{cases}
$$

Here the uniqueness follows by the Kato’s inequality. It follows by (5.4), (3.12) and $x_N$-odd property of $w_{k,n}$ that

$$
w_{k,n} \leq v_{k,j,n} \leq w_{k,n} + j\mathcal{G}_{B_1(0)}[\delta_0] \quad \text{in} \ B_1(0), \tag{5.7}
$$

where $w_{k,n}$ is the unique $x_N$-odd solution of (3.8). From the proof of Theorem 3.1, we known that for any $\xi \in C_0^{1,1}(B_1(0))$,

$$
\int_{B_1(0)} g_n(w_{k,n})\xi \, dx \to \int_{B_1(0)} g(w_k)\xi \, dx \quad \text{as} \ n \to +\infty.
$$

Step 3. It follows by (5.7) that

$$
\int_{B_1(0)} [v_{k,j,n}(-\Delta)\xi + g_n(v_{k,j,n})\xi] \, dx = 2k \frac{\partial \xi(0)}{\partial x_N} + j \xi(0), \quad \forall \xi \in C_0^{1,1}(B_1(0)). \tag{5.8}
$$
For $\beta \in (0, 1)$, any compact set $K$ and open set $O$ in $B_1(0)$ satisfying $K \subset O$, $0 \not\in \bar{O}$, we have that

$$\|v_{k,j,n}\|_{C^{2,\beta}(K)} \leq c_{45}[\|G_{B_1(0)}[\delta_0]\|_{C^1(O)} + \|G_{B_1(0)}[\delta_0]\|_{C^1(O)}].$$

Therefore, up to some subsequence, there exists $v_{k,j}$ such that

$$\lim_{n \to +\infty} v_{k,j,n} = v_{k,j} \text{ a.e. in } B_1(0).$$

Then $\{g_n(v_{k,j,n})\}$ converges to $g(v_{k,j})$ a.e. in $B_1(0)$.

We observe that $\tilde{\beta}_n := v_{k,j,n} - w_{k,n}$ is the very weak solution of

$$\left\{ \begin{array}{l l}
-\Delta u + g_n(w_{k,n} + u) - g_n(w_{k,n}) = j\delta_0 & \text{in } B_1(0), \\
u = 0 & \text{on } \partial B_1(0).
\end{array} \right. \tag{5.9}$$

Note that $0 \leq \tilde{v}_n \leq jG_{B_1(0)}[\delta_0]$ and by (1.11), it follows that

$$0 \leq g_n(w_{k,n} + \tilde{v}_n) - g_n(w_{k,n}) \leq c_1 \left[ \frac{g(w_{k,n})}{1 + |w_{k,n}|} \tilde{v}_n + g_n(\tilde{v}_n) \right].$$

Thus, $\{g_n(w_{k,n} + \tilde{v}_n) - g_n(w_{k,n})\}$ converges to $g(w_k + \tilde{v}) - g(w_k)$ a.e. in $B_1(0)$, where $\tilde{v} = v_{k,j} - w_k$.

Let $\tilde{g}_n(s) = g_n(w_{k,n} + s) - g_n(w_k)$, we see that $\tilde{g}_n(0) = 0$ and function $\tilde{g}_n$ is nondecreasing and verifies (1.10). Then it follows by Theorem 3.7 in [19] that

$$\|\tilde{g}_n(\tilde{v}_n)\|_{L^1(B_1(0))} \leq c_{46}\|G_{B_1(0)}[\delta_0]\|_{L^1(B_1(0))}$$

and

$$\tilde{v}_n \leq jG_{B_1(0)}[\delta_0].$$

Thus, it follows by (1.11) that

$$\tilde{g}_n(\tilde{v}_n(x)) \leq c_1 g(jG_{B_1(0)}[\delta_0](x)) + c_1 \frac{g(2k|G_{B_1(0)}[\delta_0]|(x))}{1 + 2k|G_{B_1(0)}[\delta_0]|(x)} jG_{B_1(0)}[\delta_0](x)$$

$$\leq c_1 g(c_{45}|x|^{2-N}) + c_{46} g(c_{45}|x|^{1-N})|x|.$$
Thus,

\[ m(\lambda)g(c_{45}\lambda^{N-2}) + \int_{\lambda}^{T} m(s)dg(c_{45}s^{N-2}) \]

\[ = c_{47}T^{-N}g(c_{45}T^{N-2}) + c_{48} \int_{\lambda}^{T} s^{-1-N}g(c_{45}s^{N-2})ds \]

\[ = c_{47}(T^{N-2})^{-\frac{N}{N-2}}g(c_{45}T^{N-2}) + c_{49} \int_{(c_{45}\lambda)^{1/(N-2)}}^{(c_{45}T)^{1/(N-2)}} t^{-1-\frac{N}{N-2}}g(t)dt \]

and

\[ m(\lambda)g(c_{45}\lambda^{N-1})\lambda^{-1} + \int_{\lambda}^{\infty} m(s)d(g(c_{45}s^{N-1})s^{-1}) \]

\[ = c_{50}T^{-N-1}g(c_{45}T^{N-1}) + c_{50} \int_{\lambda}^{T} s^{-2-N}g(c_{45}s^{N-1})ds \]

\[ = c_{50}(T^{N-1})^{-\frac{N+1}{N-1}}g(c_{45}T^{N-1}) + c_{51} \int_{(c_{45}\lambda)^{1/(N-1)}}^{(c_{45}T)^{1/(N-1)}} t^{-1-\frac{N+1}{N-1}}g(t)dt, \]

where

\[ c_{47}(T^{N-2})^{-\frac{N}{N-2}}g(c_{45}T^{N-2}) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \]

and

\[ c_{50}(T^{N-1})^{-\frac{N+1}{N-1}}g(c_{45}T^{N-1}) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \]

by the assumption (1.10) and Lemma 3.3.

Therefore,

\[ c_{1}m(\lambda)g(c_{45}\lambda^{N-2}) + c_{1} \int_{\lambda}^{\infty} m(s)dg(c_{45}s^{N-2}) \leq c_{49} \int_{(c_{45}\lambda)^{1/(N-2)}}^{(c_{45}T)^{1/(N-2)}} t^{-1-\frac{N}{N-2}}g(t)dt \]

and

\[ c_{46}m(\lambda)g(c_{45}\lambda^{N-1})\lambda^{-1} + c_{46} \int_{\lambda}^{\infty} m(s)d(g(c_{45}s^{N-1})s^{-1}) \leq c_{51} \int_{(c_{45}\lambda)^{1/(N-1)}}^{(c_{45}T)^{1/(N-1)}} t^{-1-\frac{N+1}{N-1}}g(t)dt. \]

Notice that the quantities on the right-hand side tends to 0 when \( \lambda \rightarrow \infty \). The conclusion follows: for any \( \epsilon > 0 \), there exists \( \lambda > 0 \) such that

\[ c_{49} \int_{(c_{45}\lambda)^{1/(N-2)}}^{(c_{45}T)^{1/(N-2)}} t^{-1-\frac{N}{N-2}}g(t)dt \leq \frac{\epsilon}{6} \]

and

\[ c_{51} \int_{(c_{45}\lambda)^{1/(N-1)}}^{(c_{45}T)^{1/(N-1)}} t^{-1-\frac{N+1}{N-1}}g(t)dt \leq \frac{\epsilon}{6}. \]

For \( \lambda \) fixed, there exists \( \delta > 0 \) such that

\[ \int_{E} dx \leq \delta \implies c_{1}g(c_{45}\lambda^{N-2})|E| \leq \frac{\epsilon}{6} \quad \text{and} \quad c_{46}g(c_{45}\lambda^{N-1})\lambda^{-1}|E| \leq \frac{\epsilon}{6}, \]

which implies that \( \{ \tilde{g}_{n} \circ \tilde{v}_{n} \} \) is uniformly integrable in \( L^1(B_1(0)) \). Then

\[ \tilde{g}_{n} \circ \tilde{v}_{n} \rightarrow g(w_k + \tilde{v}) - g(w_k) \quad \text{in} \quad L^1(B_1(0)) \]

by Vitali convergence theorem.
Then passing to the limit as $n \to +\infty$ in the identity (3.13), it implies that for any $\xi \in C^{1,1}_0(B_1(0))$, 
\[
\int_{B_1(0)} [w_k(-\Delta)\xi + g(w_k)\xi]dx = 2k \frac{\partial \xi(0)}{\partial x_N} + j\xi(0).
\]
Thus, $v_{k,j}$ is a very weak solution of (1.8). The regularity results follows by Lemma 3.1.

**Proof of (ii).** It follows from (1.11) and (5.7) that 
\[
v_{k,j,n}(x) \leq 2k G_{B_1(0)} \left( \frac{\partial \delta_0}{\partial x_N} \right)(x) + j G_{B_1(0)}[\delta_0](x) - \delta_0(x) - G_{B_1(0)}[g(G_{B_1(0)}[\delta_0])](x),
\]
and 
\[
v_{k,j,n}(x) \geq 2k G_{B_1(0)} \left( \frac{\partial \delta_0}{\partial x_N} \right)(x) + j G_{B_1(0)}[\delta_0](x) - \delta_0(x) - G_{B_1(0)}[g(G_{B_1(0)}[\delta_0])](x),
\]
where $w_g$ is the unique solution of (2.3) and 
\[
\lim_{|x| \to 0^+} \frac{G_{B_1(0)}[\delta_0](x)}{\Gamma_N(x)} = 1.
\]
We see that for any $e = (e_1, \ldots, e_N) \in \partial B_1(0)$ with $e_N \neq 0$, 
\[
\lim_{t \to 0^+} G_{B_1(0)}[g(G_{B_1(0)}[\delta_0])](te)t^{N-1} = 0,
\]
which, together with (3.18), implies that 
\[
\lim_{t \to 0^+} v_{k,j}(te)t^{N-1} = 0.
\]
Then 
\[
\lim_{t \to 0^+} G_{B_1(0)}[g(G_{B_1(0)}[2k \frac{\partial \delta_0}{\partial x_N}])](te)t^{N-1} = 0.
\]
Thus, 
\[
\lim_{t \to 0^+} G_{B_1(0)} \left( \frac{\partial \delta_0}{\partial x_N} \right)(te)t^{N-1} = e_N
\]
and by $x_N$-odd property of $w_k$, we derive that 
\[
\lim_{t \to 0^+} w_k(te)t^{N-1} = 2ke_N.
\]
This ends the proof. □

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