Distribution Function for Shot Noise in Disordered Multichannel Quantum Conductors

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Abstract

We obtain the generating function for shot noise through a disordered multichannel conductor in the zero temperature quantum regime. The distribution of charge transmitted over a fixed time interval is found to be approximately Gaussian.
The nature of shot noise in mesoscopic systems at low temperatures has attracted considerable interest in recent years. Lesovik [1] showed that the intensity of the shot noise in a single channel two terminal conductor is suppressed in the quantum regime compared to its classical value. This result has been generalized and applied to various other cases [2, 3, 4]. Levitov and Lesovik [5] later was able to obtain the entire distribution function for the charge transport over a fixed time interval for a single channel conductor and showed that the distribution function becomes binomial in the quantum regime as opposed to Poissonian in the classical regime. This result has been generalized to include shot noise in a superconductor-normal metal point contact [6] as well.

In the present work we generalize the work of ref. [5] and evaluate the shot noise distribution function for a multichannel conductor. We obtain the generating function for the distribution of charge transmitted over a time interval $t_0$ and show that the charge distribution is approximately Gaussian. We find that in some limit the shot noise is reduced by a factor close to $1/3$ compared to the Poisson value as obtained in ref. [3], but we show that there is a correction term independent of the value of conductance that can be important in the large $t_0$ limit. Our results disagree with a recent calculation by Lee et al [7], and we comment on the differences.

When a dc voltage $V$ is applied across the system in a two terminal geometry, the shot noise generating function for a multichannel conductor may be written as a product over generating function for independent channels, each of which has a binomial distribution [3]:

$$\chi(\lambda) = \prod_a (T_a z + 1 - T_a)^M; \quad z = e^{i\lambda},$$

(1)

where $M = (e/h)VT_0$ can be interpreted as the number of attempts in time $t_0$, and $T_a$ is
the transmission probability corresponding to channel $a$. The probability distribution for $Q$ charges [each of charge $e$] transferred in a time $t_0$ is

$$P(Q, t_0) = \int_{-\pi}^\pi d\lambda e^{iQ\lambda} < \chi(\lambda, t_0) >$$

(2)

where the angular bracket represents an average over the distribution of the $T_a$'s in disordered mesoscopic conductors. One can also read off the moments or the cumulants of the charge distribution directly by expanding $< \chi(\lambda, t_0) >$ or $\ln < \chi(\lambda, t_0) >$ in power series of $\lambda$.

It has been shown [8] that the distribution of the transmission probability in the diffusive regime for an $N$-channel disordered conductor can be obtained from an appropriate random matrix theory constructed for the matrix $X = [TT^\dagger + (TT^\dagger)^{-1} - 2I]/4$ where $T$ is the $2N \times 2N$ transfer matrix describing the conductor and $I$ is the identity matrix. The $N$ doubly degenerate real non-negative eigenvalues $x_a$ of the matrix $X$ are related to the transmission probability $T_a$ by the relation

$$T_a = \frac{1}{1 + x_a}; \quad 0 \leq x_a \leq \infty.$$  

(3)

According to the random matrix theory [9], the joint probability distribution for the $N$ eigenvalues $x_a$ can then be written in the form

$$p(x_1 \cdots x_N) = \prod_{a<b} |x_a - x_b|^\alpha \prod_c e^{-V(x_c)},$$

(4)

where $V(x)$ is a Lagrange multiplier function [10] that incorporates physical constraints like a given value of the first moment, and $\alpha$ is a symmetry parameter. The value of $\alpha$ is 1, 2 or 4 depending on whether the symmetry of the matrix ensemble is orthogonal, unitary or symplectic. A reasonably good description of a disordered conductor in the
diffusive regime \([8, 11, 12]\), is obtained from the choice \(V(x) = tx\), where the parameter \(t\) is related to the dimensionless ohmic conductance \(g_0\) of the system by \(t = \alpha g_0^2 / 2N\). The conductance \(g\) is given by the relation \(g = \sum_a \frac{1}{1 + x_a}\), and is therefore a linear statistic of the eigenvalues \(x_a\); the variance of such linear statistics have well known properties \([9, 13, 14]\).

The generating function for shot noise is, however, not a linear statistic; it is a product statistic as given by eq. (1). To our knowledge, fluctuation properties of such product statistics within a random matrix framework have not been calculated before. [One might attempt to avoid the problem by evaluating the average \(<\ln \chi>\), which is a linear statistic \([7]\), instead of the appropriate \(\ln <\chi>\); this is a reasonable approximation only if the distribution is very sharply peaked. Our results differ from those of \([7]\).] We will evaluate the average of the product statistic within the random matrix framework using a continuum approach \([15]\), which is known to work very well in the diffusive regime. In this approach the probability distribution (4) is written as the exponential of a fictitious Hamiltonian where the eigenvalues are considered as particles repelling each other logarithmically as in a Coulomb fluid, within a confining potential \(V(x)\), at a temperature \(1/\alpha\).

In terms of the variable \(x_a\), the generating function can be rewritten as

\[
\chi(\lambda, t_0) = e^{M \sum_a \ln[(x_a + 2)/(x_a + 1)]}.
\]

The average of \(\chi\) over the distribution of \(x_a\) can then be written as

\[
\langle \chi(\lambda) \rangle = Z_M / Z_0 = e^{-(F_M - F_0)}
\]
where
\[
Z_M = \int_0^\infty \prod_a dx_a \exp \left[ \alpha \sum_{a<b} \ln |x_a - x_b| - t \sum_a x_a + M \sum_a \ln \frac{x_a + z}{x_a + 1} \right].
\] (7)

Here \(Z_0\), defined as \(Z_{M=0}\) is the partition function and \(F_0\), defined as \(F_{M=0}\) is the free energy for the coulomb fluid characterized by the logarithmic repulsion and a linear confining potential as shown in eq. (7) for \(M = 0\). It is well known that for a linear confining potential the density of eigenvalues at the origin scales with the number of eigenvalues \(N\). Therefore the large \(N\) continuum approximation is valid [15], and we can use the coulomb fluid results
\[
U'_M(x) - \alpha \int_I \frac{dy}{x-y} \sigma_M(y) = 0,
\] (8)

where \(\sigma_M(x)\) is the ‘effective’ density of eigenvalues contained in an interval \(I\), the prime denotes a derivative with respect to \(x\), and
\[
U_M(x) = U_0(x) + U_c(x) = V(x) - M \ln \left[ \frac{x + z}{x + 1} \right]
\] (9)
is the ‘effective’ confining potential, which includes the confining potential \(U_0(x) = U_{M=0}(x) = V(x)\), as well as the ‘correction’ \(U_c(x) = M \ln \left[ \frac{x + z}{x + 1} \right]\). [We use the word ‘effective’ within a quotation mark because the parameter \(z\) is complex, and the density or the confining potential has only a mathematical meaning.] The range of integration \(I\) is determined by the allowed range of eigenvalues as well as the normalization requirement.

In terms of the density, the free energy \(F_M\) can be obtained [16] from the relation
\[
F_M = (1/2) A_M N + \int_I dx \sigma_M(x) U_M(x),
\] (10)

where \(A_M\) is the chemical potential determined from the integration constant,
\[
U_M(x) - \alpha \int_I dy \sigma_M(y) \ln |x - y| = A_M.
\] (11)
The general solution to equation (8) for the density is given by \[ I = (0, b) \]

\[
\sigma_M(x) = \frac{1}{\pi^2 \alpha} \sqrt{\frac{b-x}{x}} \int_0^b \frac{dy}{y-x} \sqrt{\frac{y}{b-y}} U'_M(y),
\] (12)

where \( b \) is the upper limit for the density set by the symmetry and normalization requirements.

To obtain explicit expressions, we write

\[
\sigma_M(x) = \sigma_0(x) + \sigma_c(x),
\] (13)

where \( \sigma_0(x) = \sigma_{M=0}(x) \) is the density for the potential \( U_0(x) \) alone, and \( \sigma_c(x) \) is the correction to the density due to \( U_c(x) \). Then for \( U_0(x) = tx \), and for non-negative eigenvalues, we obtain

\[
\sigma_0(x) = \frac{t}{\pi \alpha} \sqrt{\frac{b-x}{x}},
\] (14)

and

\[
\sigma_c(x) = \frac{M(1 - \sqrt{z})}{\pi \alpha \sqrt{x}} \frac{x - \sqrt{z}}{(x+1)(x+z)}; \quad |argz| < \pi.
\] (15)

The normalization condition \( \int_0^b dx \sigma_M(x) = N \) gives us the value for \( b \), \( N = tb/2\alpha \). Note that the contribution from \( \sigma_c \) to the normalization integral is zero \[17\],

\[
\int_0^\infty dx \sigma_c(x) = 0.
\] (16)

Therefore we may conclude that the chemical potential \( A_M \) differs from \( A_0 \) by a negligible amount as \( N \to \infty \). This provides a major simplification, and the change in free energy is then given by

\[
F_M - F_0 = (1/2) \int_0^b dx \sigma_0 U_c + (1/2) \int_0^\infty dx \sigma_c U_0 + (1/2) \int_0^\infty dx \sigma_c U_c,
\] (17)
where $\sigma_0$, $U_0$, $\sigma_c$ and $U_c$ are as defined before. In the limit of large system size, $b = 4N^2/g_0^2 \to \infty$ ($g_0 = Nl/L$, where $l$ is the mean free path and $L$ is the system size), all integrals can be explicitly evaluated, and the generating function is given by
\[
\langle \chi(\lambda) \rangle = \exp \left[ -\frac{1 + \pi}{2\pi} M g_0 (1 - \sqrt{z}) + \frac{M g_0^2}{4N} (1 - z) + \frac{4M^2}{\alpha} \ln \frac{1 + \sqrt{z}}{2\sqrt{1/4}} \right], \quad z = e^{i\lambda}, \ |\lambda| < \pi
\] (18)
where we have used the conductance $g_0$ instead of the parameter $t$. This is our principal result. Again for large system size, $\frac{M g_0^2}{N} << M g_0$, and the term proportional to $(1 - z)$ can be neglected compared to the term proportional to $(1 - \sqrt{z})$. However, for long enough measurement time $t_0$ (a single energy generating function can be considered only in this limit), the term proportional to $M^2$ may not be negligible ($M = eV t_0/\hbar$). Note that this term is independent of the conductance $g_0$, but depends on the symmetry parameter $\alpha$.

We can use eq. (2) to evaluate an approximate probability distribution $P(Q, t_0)$. We use the fact that major contributions to the integral in (2) come from regions around $\lambda = 0$ where expansions in powers of $\lambda$ can be used. If we keep terms up to order $\lambda^2$, the limits of the integral can then be extended to $\pm \infty$. The result, valid around the mean $Q_0$, is a Gaussian distribution:
\[
P(Q, t_0) \sim \exp \left[ -\frac{(Q - Q_0)^2}{2 \left( \frac{Q_0}{2} + \frac{M^2}{8\alpha} \right)} \right]
\] (19)
where the mean value
\[
< Q >= Q_0 = \frac{1 + \pi}{2\pi} M g_0.
\] (20)
We can also expand $\ln \langle \chi \rangle$ in power series of $\lambda$ to obtain directly the cumulants of the charge distribution. For $\frac{M g_0^2}{N} \to 0$, we get
\[
\ln \langle \chi \rangle = Q_0 x + \left( \frac{Q_0}{2} + \frac{M^2}{8\alpha} \right) \frac{x^2}{2} + \left( \frac{Q_0}{4} \right) \frac{x^3}{3!} + \left( \frac{Q_0}{8} - \frac{M^2}{64\alpha} \right) \frac{x^4}{4!} + \ldots
\] (21)
where the cumulant $\langle\langle Q^k \rangle\rangle$ can simply be read off as the coefficient of $x^k/k!$ in the expansion (21). The first two cumulants agree with the approximate Gaussian distribution obtained above. Note that $Q_0$ defines the average charge transferred in time $t_0$, while the intensity of the shot noise is given by $\langle\langle Q^2 \rangle\rangle$. A completely uncorrelated classical system would have a Poisson distribution $\langle\langle Q \rangle\rangle$ with $\langle\langle Q \rangle\rangle$ as well as $\langle\langle Q^2 \rangle\rangle$ given by $Q_P = Mg_0$. If the term proportional to $M^2$ is neglected, the shot noise is reduced compared to $Q_P$ by a factor $\frac{1+\pi^2}{4\pi}$, which is close to $1/3$ obtained in ref. [3]. However the correction term, which is independent of $g_0$, may become important for long enough observation time and/or stronger disorder. On the other hand, for stronger disorder, the appropriate random matrix model [11] has a confining potential $V(x) \sim [\ln x]^2$, for very large $x$, and the density near the origin no longer scales with $N$. In this regime the usual large $N$ continuum approximation breaks down. The shot noise distribution in this regime is under investigation.

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