A CLASS OF BANACH ALGEBRAS OF GENERALIZED MATRICES

MAYSAM MAYSAMI SADR

Abstract. We introduce a class of Banach algebras of generalized matrices and study the existence of approximate units, ideal structure, and derivations of them.

1. Introduction

Let $X$ be a compact metrizable space and $m$ be a Borel probability measure on $X$. In this note we study some aspects of the algebraic structure of a Banach algebra $M$ of generalized complex matrices whose their arrays are indexed by elements of $X^2$ and vary continuously. The multiplication of $M$ is defined similar to the ordinary matrix multiplication and uses $m$ as the weight for arrays. See Section 2 for exact definition. In the case that $m$ has full support, $M$ is isometric isomorphic to a subalgebra of compact operators acting on the Banach space of continuous functions on $X$. Indeed any element of $M$ defines an integral operator in a canonical way. Thus $M$ can be interpreted as a Banach algebra of integral operators or kernels ([2]). In Section 3 we investigate the existence of approximate units of $M$. In Section 4 we show that if $X$ is infinite then the center of $M$ is zero. In Section 5 we study ideal structure of $M$. In Section 6 we consider some classes of representations of $M$. In Section 7 we show that under some mild conditions bounded derivations on $M$ are approximately inner.

Notations. For a compact space $X$ and a Banach space $E$ we denote by $C(X; E)$ the Banach space of continuous $E$-valued functions on $X$ with supremum norm. We also let $C(X) := C(X; C)$. There is a canonical isometric isomorphism $C(X; E) \cong C(X) \hat{\otimes} E$ where $\hat{\otimes}$ denotes the completed injective tensor product. The phrase “point-wise convergence topology” is abbreviated to “pct”. By pct on $C(X; E)$ we mean the vector topology under which a net $(f_\lambda)_{\lambda} \in C(X; E)$ converges to $f$ if and only if $f_\lambda(x) \to f(x)$ in the norm of $E$ for every $x \in X$. If $f$ and $f'$ are complex functions on spaces $X$ and $X'$ then $f \otimes f'$ denotes the function on $X \times X'$ defined by $(x, x') \mapsto f(x)f'(x')$. The support of a Borel measure $m$ is denoted by $\text{Spm}$. $B_{x, \delta}$ denotes the open ball with center at $x$ and radius $\delta$.

2. The main definitions

Let $X$ be a compact metrizable space and $m$ be a Borel probability measure on $X$. By analogy with matrix multiplication we let the convolution of $f, g \in C(X^2)$ be defined by $f \star g(x, y) = \int_X f(x, z)g(z, y)dm(z)$. Also by analogy with matrix adjoint we let $f^* \in C(X^2)$ be defined by $f^*(x, y) = f(y, x)$. It is easily verified that $\star$ is an associative multiplication, $\ast$ is an involution, and also, $\|f \ast g\|_\infty \leq \|f\|_\infty \|g\|_\infty$ and $\|f^*\|_\infty = \|f\|_\infty$; thus $C(X^2)$ becomes

\textit{2010 Mathematics Subject Classification.} 46H05; 46H35; 46H10; 47B48; 46H25.

\textit{Key words and phrases.} Banach algebra; generalized matrix; approximate unit; ideal; derivation.
a Banach $*$-algebra which we denote by $\mathcal{M}_X$. If $X$ is a finite space with $n$ distinct elements $x_1, \cdots, x_n$ and $\text{Spm} = X$ then the assignment $(a_{ij}) \mapsto ((x_i, x_j) \mapsto \frac{1}{m(x_i)} a_{ij})$ defines a $*$-algebra isomorphism from the algebra of $n \times n$ matrices onto $\mathcal{M}_X$.

Beside norm and pc topologies on $\mathcal{M}_X$, we need two other topologies: Consider the canonical isometric isomorphism $f \mapsto (y \mapsto f(\cdot, y))$ from $C(X)$ onto $C(\mathbb{X}; C(X))$. We define the column-wise convergence topology (cct for short) on $\mathcal{M}_X$ to be the pull back of the pct on $C(\mathbb{X}; C(X))$ under this isomorphism. The row-wise convergence topology (rct for short) on $\mathcal{M}_X$ is defined similarly by using the other canonical isomorphism $f \mapsto (x \mapsto f(x, \cdot))$. The pct on $C(X) = \mathcal{M}_X$ is contained in the intersection of cct and rct. Column-wise and row-wise cts are adjoint to each other in the sense that the involution $\ast$ from $\mathcal{M}_X$ with cct to $\mathcal{M}_X$ with rct is a homeomorphism. If $a_\lambda \underset{\text{cct}}{\rightarrow} a$ and $b_\lambda \underset{\text{rct}}{\rightarrow} b$ in $\mathcal{M}_X$, then $c \ast a_\lambda \underset{\text{cct}}{\rightarrow} c \ast a$ and $b_\lambda \ast c \underset{\text{rct}}{\rightarrow} b \ast c$ for every $c$.

The assignment $(X, m) \mapsto \mathcal{M}_X$ can be considered as a cofunctor from the category of pairs $(X, m)$ to the category of Banach $*$-algebras: Suppose that $(X', m')$ is another pair of a compact metrizable space and a Borel probability measure on it. Let $\alpha : X' \rightarrow X$ be a measure preserving continuous map. Then $\alpha$ induces a bounded $*$-algebra morphism $\mathcal{M}\alpha$ from $\mathcal{M}_{X,m}$ into $\mathcal{M}_{X',m'}$ defined by $[(\mathcal{M}\alpha)f](x', y') = f(\alpha(x'), \alpha(y'))$. By an explicit example we show that $\mathcal{M}$ as a functor is not full: Let $\beta : X \rightarrow C$ be a continuous function with $|\beta| = 1_X$ and $\beta \neq 1_X$. Then $\hat{\beta} : \mathcal{M}_{X,m} \rightarrow \mathcal{M}_{X,m}$ defined by $\beta f(x, y) = \beta(x) f(x, y) \hat{\beta}(y)$ is an isometric $*$-algebra isomorphism. It is clear that $\hat{\beta}$ is not of the form $\mathcal{M}\alpha$ for any $\alpha : X \rightarrow X$.

Let $X_0$ be a closed subset of $X$ containing $\text{Spm}$ and let $i : X_0 \rightarrow X$ denote the embedding. Then $\mathcal{M}i : \mathcal{M}_{X,m} \rightarrow \mathcal{M}_{X_0,m}$ is surjective with kernel $I := \{ f : f|_{X_0} = 0 \}$. Thus $\mathcal{M}_{X,m}$ is an extension of $\mathcal{M}_{X_0,m}$ by the closed self-adjoint ideal $I$. Moreover, suppose that $X_0$ is a retract of $X$, i.e. there is a continuous map $\rho : X \rightarrow X_0$ with $\rho u = \text{id}_{X_0}$. It follows from functoriality of $\mathcal{M}$ that $(\mathcal{M}\rho)(\mathcal{M}\rho)$ is the identity morphism on $\mathcal{M}_{X_0,m}$. This shows that the mentioned extension splits strongly in the sense of [1] Definition 1.2. The discussion we just had, shows that by removing the null part of $m$ from $X$ we do not lose the principal part of the structure of $\mathcal{M}_{X,m}$. We will see that $\text{Spm} = X$ is a crucial condition for the study of $\mathcal{M}_{X,m}$.

For any closed subset $C$ of $X$ we have $m(C) = \inf[(1_X \otimes f) \ast 1_{X'}](x, y)$, the infimum being taken over all continuous functions $f$ on $X$ with $f(X) \subseteq [0, 1]$ and $f(C) = \{ 1 \}$. Using this and inner regularity of $m$ we can find the measure of any Borel subset. Hence we can recover $m$ from $\mathcal{M}_{X,m}$. The author does not know if the homeomorphism type of $X$ can be recovered from $\mathcal{M}_{X,m}$. Suppose that $X$ is finite with $\text{Spm} = X$. It is not so hard to see that if $\phi : \mathcal{M}_{X,m} \rightarrow \mathcal{M}_{X',m'}$ is an isometric $*$-isomorphism then there exist a measure preserving injective and surjective map $\alpha : X' \rightarrow X$ and a function $\beta : X \rightarrow C$, with $|\beta| = 1_X$, such that $\phi = (\mathcal{M}\alpha) \hat{\beta}$, where $\hat{\beta}$ is defined as above. (Note that if $\phi$ is not supposed to be isometric then this assertion is wrong.) We suggest that this conclusion is true for any arbitrary $X$ with $\text{Spm} = X$.

In Koopman’s theory, as it is well known, the operator algebras have many applications to study of dynamical systems and ergodic theory ([3]). In this direction, the study of algebraic properties of $\mathcal{M}_{X,m}$ may be useful: Let $G$ be a discrete group of measure preserving homeomorphisms of $X$. Then $G$ acts on $\mathcal{M}_{X,m}$ by isometric automorphisms and thus it is
appropriate to consider the crossed product Banach algebra \( A := G \times \mathcal{M}_X \). It is clear that any algebraic invariant of \( A \) is an invariant of the dynamical system \((\mathcal{X}, G)\). Moreover, if the suggestion stated in the preceding paragraph is true, then \((\mathcal{X}, G)\) is completely characterized by \( A \). We plan to discuss elsewhere such possible connections with ergodic theory.

### 3. Approximate units of \( \mathcal{M} \)

From now on, \( \mathcal{X} \) is a fixed compact metrizable space, \( \mathfrak{m} \) is a fixed Borel probability measure on \( \mathcal{X} \) with \( \operatorname{Spm} = \mathcal{X} \), and \( \mathcal{M} \) will denote \( \mathcal{M}_{\mathcal{X}, \mathfrak{m}} \). We also let \( \mathcal{d} \) denote a compatible metric on \( \mathcal{X} \). A right norm- (resp. pc-, cc-, rc-) approximate unit for \( \mathcal{M} \) is a net \( (u_\lambda)_{\lambda} \) in \( \mathcal{M} \) such that \( au_\lambda \to a \) in the norm topology (resp. pct, cct, rct) for every \( a \in A \). If \( \sup_\lambda \|u_\lambda\|_\infty < \infty \) then \( (u_\lambda)_{\lambda} \) is called bounded. (Bounded) left and two-sided norm- (resp. pc-, cc-, rc-) approximate units are defined similarly. It is clear that every norm-approximate unit is a pc-approximate unit. Suppose that \( \mathfrak{m} \) is an invariant of the dynamical system \((\mathcal{X}, G)\). Moreover, if the suggestion stated in the preceding paragraph is true, then \((\mathcal{X}, G)\) is completely characterized by \( A \). We plan to discuss elsewhere such possible connections with ergodic theory.

**Theorem 3.1.** There is a net in \( \mathcal{M} \) which is mutually a right cc-approximate unit and a left rc-approximate unit. Thus the same net is also a two-sided pc-approximate unit.

**Proof.** The set of all pairs \((S, \epsilon)\), in which \( S \) is a finite subset of \( \mathcal{X} \) and \( \epsilon > 0 \), with the ordering \( ((S, \epsilon) \leq (S', \epsilon')) \iff (S \subseteq S', \epsilon' \leq \epsilon) \), becomes a directed set. For any pair \((S, \epsilon)\) choose \( \delta > 0 \) such that \( \delta < \epsilon \) and \( B_{y, 2\delta} \cap B_{y', 2\delta} = \emptyset \) for \( y, y' \in S \) with \( y \neq y' \), and let \( u_{S, \epsilon} = \sum_{y \in S} \frac{1}{\mathfrak{m}(B_{y, \delta})} E_{y, \delta} \otimes E_{y, \delta} \). We show that \( (u_{S, \epsilon})_{(S, \epsilon)} \) is the desired net. Let \( f \in \mathcal{M} \) and \( r > 0 \) be arbitrary. Choose \( \epsilon > 0 \) with \( \epsilon < r \) such that for every \( z, z' \in \mathcal{X} \) if \( d(z, z') < \epsilon \) then \( |f(x, z) - f(x, z')| < r \). If \( x \) is arbitrary then for any pair \((S, \epsilon)\) with \( y \in S \) we have

\[
|f * u_{S, \epsilon} - f|(x, y) = \frac{1}{\mathfrak{m}(B_{y, \delta})} \left| \int_{B_{y, \delta}} [f(x, z) - f(x, y)] \, d\mathfrak{m}(z) + \int_{O_{x, \delta} \setminus B_{x, \delta}} f(x, z) E_{y, \delta}(z) \, d\mathfrak{m}(z) \right| \\
\leq r + r\|f\|_\infty.
\]

This shows that \( f * u_{S, \epsilon} \to f \) in cct. Similarly it is proved that \( u_{S, \epsilon} * f \to f \) in rct. \( \Box \)

**Remark 3.2.** The existence of a right (or left) pc-approximate unit for \( \mathcal{M} \) implies that \( \operatorname{Spm} = \mathcal{X} \). An easy proof is as follows. Let \( (u_\lambda)_{\lambda} \) be a right pc-approximate unit. Let \( U \) be an arbitrary nonempty open set in \( \mathcal{X} \) and let \( f \in C(\mathcal{X}) \) be such that \( f(\mathcal{X} \setminus U) = \{0\} \) and \( f(x) = 1 \) for some \( x \in U \). Then we have \( 1 = (1 \otimes f)(x, x) = \lim_\lambda [(1 \otimes f) * u_\lambda](x, x) = \lim_\lambda \int_U f(z) u_\lambda(z, x) \, d\mathfrak{m}(z) \). This implies that \( \mathfrak{m}(U) \neq 0 \). Hence \( \operatorname{Spm} = \mathcal{X} \).

**Proposition 3.3.** If \( \mathcal{M} \) has a bounded right (or left) pc-approximate unit then \( \mathcal{X} \) is finite.

**Proof.** Let \( (u_\lambda)_{\lambda} \) be a right pc-approximate unit for \( \mathcal{M} \) bounded by \( M > 0 \). First of all we show that \( \mathfrak{m}(\{x\}) \neq 0 \) for every \( x \). Assume, to get a contradiction, that \( \mathfrak{m}(\{x\}) = 0 \) for some \( x \). Let \( \epsilon > 0 \) be such that \( \epsilon M < 1/2 \). There is an open neighborhood \( U \) of \( x \) with \( \mathfrak{m}(U) < \epsilon \). Let \( f : \mathcal{X} \to [0, 1] \) be a continuous function with \( f(x) = 1 \) and \( f(\mathcal{X} \setminus U) = \{0\} \). For every \( \lambda \) we have \( |(1 \otimes f) * u_\lambda|(x, x) \leq \int_U |f(z)| u_\lambda(z, x) \, d\mathfrak{m}(z) \leq \epsilon M < 1/2 \). But this is impossible because \( [(1 \otimes f) * u_\lambda](x, x) \to 1 \).
Now, since $m(\mathcal{X}) = 1$, it is concluded that $\mathcal{X}$ must be a countable space. Suppose that $\mathcal{X}$ is not finite. Then there is an infinite discrete subset $\{x_1, x_2, \cdots\}$ of $\mathcal{X}$. For every $n$ let $f_n \in \mathcal{M}$ be defined by $f_n(z, z') = 1$ if $z = z' = x_n$ and otherwise $f_n(z, z') = 0$. Then we have $1 = f_n(x_n, x_n) = \lim_{\lambda}(f_n \ast u_{\lambda})(x_n, x_n) = \lim_{\lambda} m\{x_n\} u_{\lambda}(x_n, x_n)$. It follows that $m\{x_n\} \geq 1/M$. But this contradicts $\lim_{n \to \infty} m\{x_n\} = 0$. Hence, $\mathcal{X}$ is finite. □

**Theorem 3.4.** The following statements are equivalent.

(a) $\mathcal{X}$ is finite.
(b) $\mathcal{M}$ has a bounded right (or left) pc-approximate unit.
(c) $\mathcal{M}$ has a unit.

**Proof.** (b)$\Rightarrow$(a) is the statement of Proposition [3.3] (c)$\Rightarrow$(b) is trivial. (a)$\Rightarrow$(c) is easily verified by analogy with ordinary matrix algebras. □

**Lemma 3.5.** Let $x \in X$. The function $r \mapsto m(B_{x,r})$ is continuous at $r_0 \in [0, \infty)$ if and only if $m\{y : \varnothing(x, y) = r_0\} = 0$. (Note that $B_{x,0} = \emptyset$.)

**Proof.** Straightforward. □

**Lemma 3.6.** The function $x \mapsto m(B_{x,r})$ is continuous at $x_0$ if $m\{y : \varnothing(x_0, y) = r\} = 0$.

**Proof.** For $\epsilon > 0$ by Lemma [3.5] there is $\delta > 0$ such that $m(B_{x_0,r+\delta} \setminus B_{x_0,r-\delta}) < \epsilon$. Suppose that $y \in B_{x_0,\delta}$. Then $m(B_{x_0,r-\delta}) = m(B_{y,r}) \leq m(B_{x_0,r+\delta})$. So $m(B_{x_0,r}) - m(B_{y,r}) < \epsilon$. □

**Lemma 3.7.** Let $\delta > 0$ be such that $m\{y : \varnothing(x, y) = \delta\} = 0$ for every $x \in \mathcal{X}$. Then there exists $\delta' < 2\delta$ such that $m(B_{x,\delta'} \setminus B_{x,\delta}) < \delta m(B_{x,\delta})$ for every $x \in \mathcal{X}$.

**Proof.** Assume, to reach a contradiction, that there is no $\delta'$ with the desired properties. For sufficiently large $n$ we have $\delta + n^{-1} < 2\delta$ and hence there is a $x_n$ such that $m(B_{x_n,\delta+n^{-1}}) - m(B_{x_n,\delta}) \geq \delta m(B_{x_n,\delta})$. Without lost of generality we can suppose that the sequence $(x_n)_n$ converges to an element $x$. Let $r > 0$ be arbitrary. For sufficiently large $n$ we have $m(B_{x_n,\delta+n^{-1}}) \leq m(B_{x_n,\delta+r})$ and hence $\delta m(B_{x_n,\delta}) \leq m(B_{x_n,\delta+r}) - m(B_{x_n,\delta})$. It follows from Lemma [3.6] that $\delta m(B_{x,\delta}) \leq m(B_{x,\delta+r} \setminus B_{x,\delta})$. Letting $r \to 0$ and using Lemma [3.5] we conclude that $m(B_{x,\delta}) = 0$, a contradiction. □

**Theorem 3.8.** Suppose that the following condition is satisfied. (C1) $\mathcal{X}$ has a compatible metric $\varnothing$ under which there is a decreasing sequence $(\delta_n)_n$ of strictly positive numbers such that $\inf_n \delta_n = 0$ and $m\{y : \varnothing(x, y) = \delta_n\} = 0$ for every $n$ and every $x \in \mathcal{X}$. Then $\mathcal{M}$ has a right (resp. left) norm-approximate unit. Moreover, that approximate unit can be chosen so as to be a sequence.

**Proof.** For every $n$ let $\delta'_n$ be such that the statement of Lemma [3.7] is satisfied with $\delta, \delta'$ replaced by $\delta_n, \delta'_n$. Let $K_n = \{(x, y) : \varnothing(x, y) \leq \delta_n\}$ and $U_n = \{(x, y) : \varnothing(x, y) < \delta'_n\}$. Choose a continuous function $E_n : \mathcal{X}^2 \to [0, 1]$ such that $E_n(K_n) = \{1\}$ and $E_n(\mathcal{X}^2 \setminus U_n) = \{0\}$ and let $E_n$ (resp. $E'_n$) be defined by $(x, y) \mapsto E_n(x, y)/m(B_{y,\delta_n})$ (resp. $(x, y) \mapsto E_n(x, y)/m(B_{x,\delta_n})$). (Note that by Lemma [3.6] $E_n, E'_n \in \mathcal{M}$.) Using Lemma [3.7] it is easily verified that $(E_n)_n$ (resp. $(E'_n)_n$) is a right (resp. left) norm-approximate unite for $\mathcal{M}$. □
Theorem 3.9. Suppose that the following condition is satisfied. (C2) \( X \) has a compatible metric \( d \) under which there exists a sequence \( (\delta_n)_n \) satisfying all properties stated in (C1) and, in addition, \( m(B_{x,\delta_n}) = m(B_{y,\delta_n}) \) for every \( n \) and every \( x, y \in X \). Then \( M \) has a two-sided norm-approximate unit.

Proof. It is concluded from \( E_n = E'_n \) where \( E_n, E'_n \) are as in the proof of Theorem 3.8.

Example 3.10. If \( X \) is the closure of a nonempty bounded open subset of \( \mathbb{R}^n \) with the normalized \( n \)-dimensional Lebesgue measure and with the Euclidean metric, then \( X \) satisfies conditions of Theorem 3.8. More generally, if an open subset of a Riemannian manifold has compact closure \( X \) then \( X \), with the geodesic distance \( d \) and normalized Riemannian volume \( m \), satisfies conditions of Theorem 3.8. Indeed, \( m\{y : d(x, y) = r\} = 0 \) for every \( r \) and \( x \).

Example 3.11. Any closed Riemannian manifold \( X \) which has constant (positive) sectional curvature (e.g. standard spheres and tori, compact Lie groups with invariant Riemannian metrics), with geodesic distance \( d \) and normalized Riemannian volume \( m \), satisfies conditions of Theorem 3.8. Indeed, in addition to the property mentioned in Example 3.10 we have \( m(B_{x,r}) = m(B_{y,r}) \) for every \( r, x, y \).

Example 3.12. Let \( X \) be a second countable compact Hausdorff group. It is well-known that \( X \) has a compatible bi-invariant metric \( d \) i.e. \( d(xx', zyz') = d(x, y) \) for every \( x, y, z, z' \in X \) (see [8] or [6] Corollary A4.19). We show that \( d \) with the normalized Haar measure \( m \) satisfies (C1) and hence (because of invariant property of \( m \)) satisfies (C2): Suppose, on the contrary, that there is no sequence \( (\delta_n)_n \) satisfying (C1) for \( d \). So there must be \( \epsilon > 0 \) such that \( m\{y : d(e, y) = r\} \neq 0 \) for every nonzero \( r < \epsilon \); thus \( m(B_{e,\epsilon}) = \infty \), a contradiction.

4. The center of \( M \)

It is clear that if \( X \) is finite then the center of \( M \) is the one-dimensional subalgebra of scalar multiples of the unit of \( M \). But in the infinite case the situation is different:

Theorem 4.1. If \( X \) is infinite then the center of \( M \) is zero.

Proof. Suppose that \( f \) is in the center of \( M \). Let \( x, y \) be arbitrary in \( X \) with \( x \neq y \), and \( \delta > 0 \) be such that \( d(x, y) > 4\delta \). Let \( g := \frac{1}{m(B_{x,\delta})} E_{x,\delta} \otimes E_{x,\delta} \) and \( h_{\delta} := \frac{1}{m(B_{x,\delta})} E_{x,\delta} \otimes E_{y,\delta} \). Then we have \( f \ast g(x, y) = 0 \) and hence \( g \ast f(x, y) = 0 \). We have,

\[
|f|(x, y) = |g \ast f - f|(x, y) \leq \frac{1}{m(B_{x,\delta})} \int_{B_{x,\delta}} |f(z, y) - f(x, y)|dm(z) + \delta \|f\|_{\infty}.
\]

By this inequality and continuity of \( f \) we conclude that \( f(x, y) = 0 \). Also, a simple computation shows that \( \lim_{\delta \to 0} f \ast h_{\delta}(x, y) = f(x, x) \) and \( \lim_{\delta \to 0} h_{\delta} \ast f(x, y) = f(y, y) \). Thus we have \( f(x, x) = f(y, y) \). Now, suppose that \( X \) is infinite. Then there is a sequence \( (x_n)_{n \geq 0} \) such that \( x_n \to x_0 \) and \( x_0 \neq x_n \) for every \( n \geq 1 \) Thus \( f(x_0, x_0) = \lim_{n \to \infty} f(x_0, x_n) = 0 \) and hence \( f(x, x) = f(x_0, x_0) = 0 \). This completes the proof. □
5. The Ideal Structure of $\mathcal{M}$

It is clear that the involution $*$ induces a one-to-one correspondence between norm- (resp. rc-, cc-, pc-) closed right ideals and norm- (resp. cc-, rc-, pc-) closed left ideals of $\mathcal{M}$. Also any self-adjoint right or left ideal is a two-sided ideal. The rc-closure of any right ideal is a right ideal and the cc-closure of any left ideal is a left ideal. For any norm-closed linear subspace $V$ of $\mathbb{C}(\mathfrak{X})$ we let $\mathcal{R}_V := \{ f \in \mathcal{M} : f(\cdot, y) \in V \}$ and $\mathcal{L}_V := \{ f \in \mathcal{M} : f(x, \cdot) \in V \}$.

It is clear that $\mathcal{R}_V^* = \mathcal{L}_V$ and $\mathcal{L}_V^* = \mathcal{R}_V$ where $V := \{ \bar{f} : f \in V \}$.

**Theorem 5.1.** $\mathcal{R}_V$ (resp. $\mathcal{L}_V$) is a cc-closed right (resp. rc-closed left) ideal in $\mathcal{M}$. Moreover, if $V$ is pc-closed then $\mathcal{R}_V$ (resp. $\mathcal{L}_V$) is pc-closed.

**Proof.** It is clear that $\mathcal{R}_V$ is a cc-closed linear subspace of $\mathcal{M}$. Let $f \in \mathcal{R}_V$ and $g \in \mathcal{M}$. For every $y$ let $h_y : \mathfrak{X} \to V$ be defined by $h_y(z) = f(\cdot, z)g(z, y)$. Then the Bochner integral $\int_{\mathfrak{X}} h_y \text{d}m$ exists and belongs to $V$ ([7 Proposition 1.31]). Since $f \ast g(\cdot, y) = \int_{\mathfrak{X}} h_y \text{d}m$, we have $f \ast g \in \mathcal{R}_V$. Thus $\mathcal{R}_V$ is a right ideal. Also, $\mathcal{L}_V = \mathcal{R}_V^*$ is a rc-closed left ideal. The second part of the theorem is trivial. □

**Theorem 5.2.** Let $R$ be a norm-closed right ideal of $\mathcal{M}$ and let $V = \{ f(\cdot, y) : f \in R, y \in \mathfrak{X} \}$. Then $V$ is a norm-closed linear subspace of $\mathcal{C}(\mathfrak{X})$ and the cc-closure of $R$ is equal to $\mathcal{R}_V$.

Moreover, if $R$ is pc-closed then $V$ is pc-closed.

**Proof.** Suppose that $f \in R$ and $y \in \mathfrak{X}$. Let $\epsilon > 0$ be arbitrary and $\delta > 0$ with $\delta < \epsilon$ be such that if $d(z, z') < \delta$ then $|f(x, z) - f(x, z')| < \epsilon$ for every $x$. Then for every $x, y'$ we have $|\frac{1}{m(B_y, \delta)} \mathbb{E}_{g, \delta}(x, \cdot) - f(x, y')| < \epsilon$ and $|f(x, y') - f(x, y)| \leq \epsilon + \epsilon \|f\|_{\infty}$. This implies that there exists $F_{f, y} \in R$ with $F_{f, y}(x, y) = f(x, y)$ for every $x, z$. Let $h, h' \in V$. Let $f, f' \in R$ and $y, y' \in \mathfrak{X}$ be such that $h = f(\cdot, y)$ and $h' = f'(\cdot, y')$. We have $h + h' = [F_{f, y} + F_{f, y}'](\cdot, z)$ for any arbitrary $z$ and thus $h + h' \in V$. This shows that $V$ is a linear subspace. Suppose that $g \in \mathcal{C}(\mathfrak{X})$ is a limit point of $V$. There are sequences $(f_n)_n \in R$ and $(y_n)_n \in \mathfrak{X}$ such that $f_n(\cdot, y_n) \to g$. It is clear that the sequence $(F_{f_n, y_n})_n \in R$ converges to an element $G$ of $R$ with $G(\cdot, z) = g$ for every $z$. This shows that $V$ is norm-closed. (A similar argument shows that if $R$ is pc-closed then $V$ is pc-closed.) To complete the proof, it is enough to show that if $K \in \mathcal{R}_V$ then there exists a net in $R$ converging to $K$ in cc. Let $K \in \mathcal{R}_V$ be fixed. For every $y$ there are $k_y \in R$ and $\alpha(y) \in \mathfrak{X}$ such that $K(\cdot, y) = k_y(\cdot, \alpha(y))$. For every $\epsilon > 0$ and every finite subset $S$ of $\mathfrak{X}$ there exists $\delta > 0$ with the following three properties.

- $\delta \|k_y\|_{\infty} < \epsilon/2$ for every $y \in S$.
- $B_{y, 2\delta} \cap B_{y', 2\delta} = \emptyset$ for $y, y' \in S$ with $y \neq y'$.
- If $d(z, z') < 2\delta$ then $|k_y(x, z) - k_y(x, z')| < \epsilon/2$ for every $y \in S$.

Let $K_{S, \epsilon} := \sum_{y \in S} \frac{1}{m(B_{\alpha(y), \delta})} h_y \ast (\mathbb{E}_{\alpha(y), \delta} \otimes \mathbb{E}_{\alpha(y), \delta}) \in R$. Then $\|K_{S, \epsilon}(\cdot, y) - G(\cdot, y)\|_{\infty} < \epsilon$ for every $y \in S$. Considering the set of all pairs $(S, \epsilon)$ as a directed set in the obvious way, shows that $K_{S, \epsilon} \cto K$. □

Passing through the involution and using Theorem 5.2 we conclude that for any norm-closed left ideal $L$ of $\mathcal{M}$, $V := \{ f(x, \cdot) : f \in L, x \in \mathfrak{X} \}$ is a norm-closed linear subspace and rc-closure of $L$ is equal to $\mathcal{L}_V$. Moreover, if $L$ is pc-closed then $V$ is pc-closed.
Corollary 5.3. The mapping \( V \mapsto \mathcal{R}_V \) (resp. \( V \mapsto \mathcal{L}_V \)) establishes a 1-1 correspondence between norm-closed linear subspaces of \( C(\mathcal{X}) \) and cc-closed right (resp. rc-closed left) ideals of \( \mathcal{M} \), and also between pc-closed linear subspaces of \( C(\mathcal{X}) \) and pc-closed right (resp. left) ideals of \( \mathcal{M} \). In particular, 1-dimensional and norm-closed 1-codimensional subspaces of \( C(\mathcal{X}) \) correspond respectively to minimal and maximal cc-closed right (resp. rc-closed left) ideals of \( \mathcal{M} \).

Corollary 5.4. There is no nontrivial ideal in \( \mathcal{M} \) mutually closed under both cct and rct. In particular, there is no nontrivial pc-closed ideal in \( \mathcal{M} \).

Proof. Let \( I \) be a nonzero cc-closed and rc-closed ideal. There are closed linear subspaces \( V, W \subseteq C(\mathcal{X}) \) such that \( I = \mathcal{R}_V = \mathcal{L}_W \). Since \( V \neq 0 \) there are \( f_0 \in V \) and \( x_0 \in \mathcal{X} \) with \( f_0(x_0) = 1 \). For every \( g \in C(\mathcal{X}) \) we have \( f_0 \otimes g \in \mathcal{R}_V \). Thus \( g = (f_0 \otimes g)(x_0, -) \in W \) and \( W = C(\mathcal{X}) \). So, \( I = \mathcal{M} \). \( \square \)

6. Canonical representations of \( \mathcal{M} \)

For a Banach algebra \( \mathcal{A} \) a Banach space \( E \) is called Banach left \( \mathcal{A} \)-module if \( E \) is a left \( \mathcal{A} \)-module in the algebraic sense and such that the action of \( \mathcal{A} \) on \( E \) is a bounded bilinear operator. Banach right \( \mathcal{A} \)-modules and Banach \( \mathcal{A} \)-bimodules are defined similarly. Let \( B(E) \) denote the Banach algebra of bounded linear operators on \( E \) and \( K(E) \subseteq B(E) \) be the closed ideal of compact operators. Any Banach left \( \mathcal{A} \)-module structure on \( E \) gives rise to a bounded representation \( \mathcal{A} \to B(E) \), \( a \mapsto (\omega \mapsto a\omega) \), and vice versa. The statements of the following theorem are standard results and can be found for instance in [5].

Theorem 6.1. Let \( E \) denote any of the Banach spaces \( L^p(m) \) \((1 \leq p \leq \infty)\) or \( C(\mathcal{X}) \). Then \( \rho : \mathcal{M} \to K(E) \), defined by \( [\rho(f)g](x) = \int_X f(x,y)g(y)dm(y) \) \((g \in E)\), is a well-defined faithful bounded representation. Moreover, the following statements hold.

(i) In the case that \( E = L^2(m) \), \( \rho \) is a *-representation.
(ii) In the case that \( E = L^\infty(m) \) or \( E = C(\mathcal{X}) \), \( \rho \) is isometric.

It is clear that for any Banach space \( E \), \( C(\mathcal{X}; E) \) is a Banach right (resp. left) \( \mathcal{M} \)-module in the canonical way. Its module action is denoted by the same symbol \( * \) and is given by \((g * f)(y) = \int_X g(z)f(z,y)dm(z) \) (resp. \((f * g)(x) = \int_X f(x,z)g(z)dm(z)) \) for \( f \in \mathcal{M} \) and \( g \in C(\mathcal{X}; E) \). Similarly, \( C(\mathcal{X}^2; E) \) becomes a Banach \( \mathcal{M} \)-bimodule.

7. Derivations on \( \mathcal{M} \)

Let \( \mathcal{A} \) be a Banach algebra and \( E \) be a Banach \( \mathcal{A} \)-bimodule. A (bounded) derivation from \( \mathcal{A} \) to \( E \) is a (bounded) linear map \( D : \mathcal{A} \to E \) satisfying \( D(ab) = aD(b) + D(a)b \) \((a, b \in \mathcal{A})\). \( D \) is called inner if there exists \( \omega \in E \) such that \( D(a) = a\omega - \omega a \) for every \( a \). \( D \) is called approximately inner [4] if there is a net \( (\omega_\lambda)_\lambda \) in \( E \) such that \( D(a) = \lim_\lambda a\omega_\lambda - \omega_\lambda a \). If \( (\omega_\lambda)_\lambda \) can be chosen so as to be a sequence then \( D \) is called sequentially approximate inner.

Theorem 7.1. Suppose that the condition (C2) of Theorem 7.9 is satisfied, and let \( E \) be a Banach \( \mathcal{M} \)-bimodule such that its module operation \( \circ : \mathcal{M} \otimes E \otimes \mathcal{M} \to E \) is continuous w.r.t. injective tensor norm, and such that for every norm approximate unit \((E_n)_n \) of \( \mathcal{M} \) we
have $E_n \odot \omega \to \omega$ for every $\omega \in E$. Then any bounded derivation from $\mathcal{M}$ to $E$ is sequentially approximate inner.

**Proof.** Let $D : \mathcal{M} \to E$ be a bounded derivation. Let $\Gamma : \mathcal{M} \bar{\otimes} \mathcal{M} \to E$ be the bounded linear map defined by $f \otimes g \mapsto f \circ D(g)$. Also let $\Lambda : \mathcal{M} \bar{\otimes} \mathcal{M} \to \mathcal{M}$ denote the convolution product. It is not hard to verify the following two identities for $h \in \mathcal{M}$ and $F \in \mathcal{M} \bar{\otimes} \mathcal{M}$.

$$\Gamma(h \star F) = h \circ \Gamma(F), \quad \Gamma(F \star h) = \Lambda(F) \circ D(h) + \Gamma(F) \circ h.$$ 

Let the sequence $(\delta_n)_n$ be as in the statement of Theorem 3.3 and let $\alpha_n = m(B_{x, \delta_n})$ for every $x \in \mathcal{X}$. By Lemma 3.7 there is $r_n$ such that $\delta_n < r_n < 2\delta_n$ and $m(B_{x, r_n} - B_{x, \delta_n}) < \delta_n\alpha_n$. Choose a continuous function $G_n : \mathcal{X}^2 \to [0, 1]$ such that $G_n$ has constant values 1 and 0 respectively on $\{(x, y) : \vartheta(x, y) \leq \delta_n\}$ and $\{(x, y) : \vartheta(x, y) \geq r_n\}$, and let $G_n \in C(\mathcal{X}^4)$ be defined by $G_n(x, z, z', y) = \frac{1}{\alpha_n} G_\delta(x, y)$. Note that we have $\Lambda(G_n) = \frac{1}{\alpha_n} G_n$. It is not hard to verify that $(\Lambda(G_n))_n$ is a two-sided norm-approximate unit for $\mathcal{M}$ and $\lim_{n \to \infty} f \star G_n - G_n \star f = 0$ for every $f \in \mathcal{M}$. Let $K_n = \Gamma(G_n) \in E$. For the sequence $(K_n)_n$ and $h \in \mathcal{M}$ we have,

$$\lim_{n \to \infty} h \circ K_n - K_n \circ h = \lim_{n \to \infty} h \circ \Gamma(G_n) - \Gamma(G_n) \circ h = \lim_{n \to \infty} \Gamma(h \star G_n) - \Gamma(G_n \star h) + \Lambda(G_n) \circ D(h) = \Gamma(\lim_{n \to \infty} h \star G_n - G_n \star h) + D(h) = D(h).$$

This completes the proof. \qed

For any Banach space $E$, the Banach $\mathcal{M}$-bimodule $C(\mathcal{X}^2; E)$, mentioned in the preceding section, satisfies the conditions of Theorem 7.1.

**References**

1. W.G. Bade, H.G. Dales, Z.A. Lykova, *Algebraic and strong splittings of extensions of Banach algebras*, No. 656. American Mathematical Soc., 1999.
2. S. Beaver, *Banach algebras of integral operators, off-diagonal decay, and applications in wireless communications*, Ph.D. thesis, University of California, arXiv:math/0406198 [math.OA] (2004).
3. T. Eisner, B. Farkas, M. Haase, R. Nagel, *Operator theoretic aspects of ergodic theory*, Springer International Publishing AG, 2015.
4. F. Ghahramani, R.J. Loy, *Generalized notions of amenability*, Journal of Functional Analysis 208, no. 1 (2004): 229–260.
5. P.R. Halmos, V.S. Sunder, *Bounded integral operators on $L^2$ spaces*, Vol. 96. Springer Science & Business Media, 2012.
6. K.H. Hofmann, S.A. Morris, *The structure of compact groups: a primer for students-a handbook for the expert*, Vol. 25. Walter de Gruyter, 2006.
7. T. Hytonen, J. van Neerven, M. Veraar, L. Weis, *Analysis in Banach spaces*, in preparation (2015).
8. V.L. Klee, *Invariant metrics in groups (solution of a problem of Banach)*, Proceedings of the American Mathematical Society 3, no. 3 (1952): 484–487.

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), P.O. Box 45195-1159, Zanjan 45137-66731, Iran

E-mail address: sadr@iasbs.ac.ir