Scalar hairy black holes
in Einstein-Maxwell-conformally coupled scalar theory

De-Cheng Zou\textsuperscript{a,b} and Yun Soo Myung\textsuperscript{a}†

\textsuperscript{a}Institute of Basic Sciences and Department of Computer Simulation, Inje University
Gimhae 50834, Korea
\textsuperscript{b}Center for Gravitation and Cosmology and College of Physical Science and Technology,
Yangzhou University, Yangzhou 225009, China

Abstract

We obtain scalar hairy black holes from Einstein-Maxwell-conformally coupled scalar (EMCS) theory with the scalar coupling parameter $\alpha$ to the Maxwell term. In case of $\alpha = 0$, the $\alpha = 0$ EMCS theory provides constant (charged) scalar hairy black hole and charged BBMB (Bocharova-Bronnikov-Melnikov-Bekenstein) black hole where the former is stable against full perturbations, while the latter remains unstable because it belongs to an extremal black hole. It is noted that for $\alpha \neq 0$, the unstable Reissner-Nordström black holes without scalar hair imply infinite branches of $n = 0(\alpha \geq 8.019), 1(\alpha \geq 40.84), 2(\alpha \geq 99.89), \cdots$ scalarized charged black holes. In addition, for $\alpha > 0$, we develop single branch of the scalarized charged black hole based on the constant scalar hairy black hole. Finally, we obtain the numerical charged BBMB black hole solution from the $\alpha = 0$ EMCS theory.

\textsuperscript{*}e-mail address: dczou@yzu.edu.cn
\textsuperscript{†}e-mail address: ysmyung@inje.ac.kr

Typeset Using \LaTeX
1 Introduction

No-hair theorem implies that a black hole is completely described by the mass $M$, electric charge $Q$, and angular momentum $J$ \[1\]. Outside the horizon, Maxwell and gravitational fields satisfy the Gauss-law.

A minimally coupled scalar does not obey the Gauss-law and thus, a black hole cannot have a scalar hair \[2\]. The scalar-tensor theories admitting other than the Einstein gravity are rather rare. However, considering the Einstein-conformally coupled scalar theory leads to a secondary scalar hair around the BBMB black hole although the scalar hair blows up on the horizon \[3, 4\]. This was considered as the first counterexample to the no-hair conjecture for black holes. On 1991, Xanthopoulos and Zannias have pointed out that the BBMB black hole is the unique static and asymptotically flat solution to the Einstein-conformally coupled scalar theory \[5\]. It may remain an open problem that could be resolved numerically determining the exact nature of the BBMB solution. Very recently, adding the Weyl squared term to this action lead to a non-BBMB black hole solution with a primary scalar hair where we found the BBMB black hole solution numerically when turning off the Weyl squared term \[6\].

On the other hand, recently, scalarized (charged) black holes were found from the Einstein-Gauss-Bonnet-Scalar (EGBS) theory \[7, 8, 9\] (Einstein-Maxwell-Scalar (EMS) theory \[10\]) by introducing the quadratic and exponential couplings of a scalar to the Gauss-Bonnet term $f(\phi)G$ (Maxwell term $f(\phi)F^2$). In these models, we mention that the scalar is minimally coupled to the metric tensor. In this approach of spontaneous scalarization, the linearized scalar equation around the unstable black hole without scalar hair is important to determine infinite branches of the $n = 0, 1, 2, \cdots$ scalarized (charged) black holes.

In this work, we wish to introduce a combined model of Einstein-Maxwell-conformally coupled scalar (EMCS) theory to obtain various scalarized charged black holes. In case of $\alpha = 0$, the EMCS theory without scalar coupling (namely, $\alpha = 0$ EMCS theory) provides constant scalar hairy (charged) black hole and charged BBMB (Bocharova-Bronnikov-Melnikov-Bekenstein) black hole. We show that the former is stable against full perturbations. We mention that the latter remains unstable because it belongs to an extremal black hole \[11\]. For $\alpha \neq 0$, it is interesting to note that the unstable Reissner-Nordström (RN) black holes without scalar hair imply the appearance of $n = 0(\alpha \geq 8.019), 1(\alpha \geq 40.84), 2(\alpha \geq 99.89), \cdots$ scalarized charged black holes, as was shown in the EMS the-
ory \[12\]. Importantly, for \( \alpha > 0 \), we obtain single branch of scalarized charged black hole based on the constant scalar hairy black hole. Finally, we show that the allowed black hole is the charged BBMB black hole numerically in the \( \alpha = 0 \) EMCS theory when imposing the asymptotically flat condition.

## 2 EMCS theory

We start with the Einstein-Maxwell-conformally coupled scalar (EMCS) theory whose action is given by

\[
S_{\text{EMCS}} = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ R - f(\phi) F_{\mu\nu}^2 - \beta \left( \phi^2 R + 6 \partial_\mu \phi \partial^\mu \phi \right) \right],
\]

where \( f(\phi) = 1 + \alpha \phi^2 \) includes ‘\( \alpha \phi^2 \)’ (quadratic scalar coupling term with coupling parameter \( \alpha \)) and the last term corresponds to the conformally coupled scalar action with parameter \( \beta \). In the limit of \( \alpha \to 0 \), the above action recovers the \( \alpha = 0 \) EMCS theory.

It was found by Bekenstein \[4\] that a solution \((\hat{g}_{\mu\nu}, \hat{A}_\mu, \hat{\phi})\) of Einstein-Maxwell-minimally coupled scalar theory could be mapped to a solution \((g_{\mu\nu}, A_\mu, \phi)\) to the \( \alpha = 0 \) EMCS theory by the conformal transformations: \( \hat{\phi} \to \phi = \sqrt{1/\beta} \tanh(\sqrt{\beta} \phi), \hat{A}_\mu \to A_\mu = \hat{A}_\mu, \hat{g}_{\mu\nu} \to g_{\mu\nu} = (1 - \sqrt{\beta} \phi^2)^{-1} \hat{g}_{\mu\nu} \). Using this transformation, Bekenstein has discovered the charged BBMB black hole \[13\]. Also, Astorino \[14\] has recently found the constant scalar hairy black hole solution \[14\]. In this work, we choose \( \beta = 1/3 \) and \( G = 1 \) for simplicity.

We derive the Einstein equation from (1)

\[
G_{\mu\nu} = 2(1 + \alpha \phi^2) T^M_{\mu\nu} + T^\phi_{\mu\nu},
\]

where the Einstein tensor is given by \( G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu}/2 \). The energy-momentum tensors for Maxwell theory and conformally coupled scalar theory are defined by

\[
T^M_{\mu\nu} = F_{\mu\rho} F^\rho_\nu - \frac{F^2}{4} g_{\mu\nu},
\]

\[
T^\phi_{\mu\nu} = \beta \left[ \phi^2 G_{\mu\nu} + g_{\mu\nu} \nabla^2 (\phi^2) - \nabla_\mu \nabla_\nu (\phi^2) + 6 \nabla_\mu \phi \nabla_\nu \phi - 3 (\nabla \phi)^2 g_{\mu\nu} \right],
\]

where we observe the traceless condition of \( T^M_{\mu\mu} = 0 \). The Maxwell equation takes the form

\[
\nabla^\mu F_{\mu\nu} = 2 \alpha \phi \nabla^\mu (\phi) F^2.
\]

3
Finally, the scalar equation is given by

$$\nabla^2 \phi - \frac{1}{6} R \phi - \frac{\alpha}{6 \beta} F^2 \phi = 0. \tag{5}$$

Taking the trace of the Einstein equation (2) together with (5) leads to a non-vanishing Ricci scalar as

$$R = -\alpha \phi^2 F^2. \tag{6}$$

Making use of (6), one obtains the scalar equation

$$\nabla^2 \phi + \frac{\alpha}{6} \left[ \phi^2 - \frac{1}{\beta} \right] F^2 \phi = 0. \tag{7}$$

In case of $\alpha = 0$, Eq. (7) leads to a massless minimally coupled equation ($\nabla^2 \phi = 0$).

### 3 $\alpha = 0$ EMCS theory

#### 3.1 Black hole solutions

In this case, four equations of motion are obtained from the $\alpha = 0$ EMCS theory

$$G_{\mu\nu} = 2 T^M_{\mu\nu} + T^\phi_{\mu\nu}, \quad \nabla^\mu F_{\mu\nu} = 0, \quad R = 0, \quad \nabla^2 \phi = 0. \tag{8}$$

We find a constant scalar hairy (charged) black hole given by [14]

$$ds^2_{\text{csh}} = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -N(r) dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega^2, \quad \bar{\phi} = \pm \sqrt{\frac{\beta}{Q_s^2 + Q^2}}, \quad \bar{A}_t = v(r) = \frac{Q}{r} - \frac{Q}{r^+}. \tag{9}$$

which are derived from solving the background equation of $\bar{G}_{\mu\nu} = 2 \bar{T}^M_{\mu\nu}/(1 - \beta \bar{\phi}^2) = 2 \bar{T}_{\mu\nu}$ together with $\bar{T}^\mu_{\ \nu} = \frac{Q^2 + Q_s^2}{r^4} \text{diag}(-1, -1, 1, 1)$ and $\bar{\phi} = \text{const}$. Here $\beta = \kappa/6 = 4\pi G/3$.

The positions of outer and inner horizons are given by $r_\pm = m \pm \sqrt{m^2 - Q^2 - Q_s^2}$. This black hole has a primary scalar hair and its geometry is similar to a non-extremal RN black hole except that the position of the horizon is shifted by the presence of the scalar charge $Q_s$. However, we wish to point out that $\bar{\phi}$ depends on both $Q_s$ and $Q$, implying that it is not strictly a primary hair. Its thermodynamics was well established when replacing the Newtonian constant $G = 3\beta/(4\pi) = 1$ with the effective constant $\bar{G} = G(Q^2 + Q_s^2)/Q^2$.
ADM mass $M = \frac{m}{G}$ and entropy $S = \frac{A}{G^2}$. The ADM mass and entropy go to zero when the charge $Q$ approaches zero. This suggests that the constant scalar hairy black hole cannot radiate away its charge $Q$ and settle down to a constant scalar hairy black hole (uncharged). In other words, one could not find a constant scalar black hole from the Einstein-conformally coupled scalar theory and thus, the BBMB solution is the only scalar hair black hole [3, 4]. The stability of the constant scalar hair black hole will be explored in the next section.

Before we proceed, it is interesting to mention the other solution named by the charged BBMB black hole for $\alpha = 0$ [4]

$$ds^2_{\text{BBMP}} = -(1 - \frac{m}{r})^2 dt^2 + \frac{dr^2}{(1 - \frac{m}{r})^2} + r^2 d\Omega_2^2,$$

$$\tilde{\phi}(r) = \sqrt{1 - \frac{Q_s}{r - m}}, \quad \tilde{A}_t = \frac{Q}{r} - \frac{Q}{r_+}, \quad m = \sqrt{Q^2 + Q_s^2}, \quad (10)$$

where $m$ is the mass of the black hole. Here, the scalar hair $\tilde{\phi}$ is still the solution to $\tilde{\nabla}^2 \tilde{\phi} = 0$. This line element takes the same form as in the extremal RN black hole, but the scalar hair blows up at the horizon $r = m$ and it belongs to the secondary hair. It was shown forth years ago that this black hole is unstable against the scalar perturbation [11] probably since it belongs to an extremal RN black hole [15].

### 3.2 Stability of constant scalar hairy black hole

In the $\alpha = 0$ EMCS theory, all perturbations are introduced as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + a_\mu, \quad \phi = \tilde{\phi} + \varphi. \quad (11)$$

Substituting (11) into Eqs. (2), (4), and (7) together with $\alpha = 0$, their linearized equations around (9) are given by

$$\delta G_{\mu\nu}(h) = \delta T^\phi_{\mu\nu}(h, \varphi) + 2\delta T^M_{\mu\nu}(h, f), \quad (12)$$

$$\tilde{\nabla}^\mu f_{\mu\nu} = 0, \quad (13)$$

$$\tilde{\nabla}^2 \varphi = 0, \quad (14)$$
where

\[
\delta G_{\mu\nu}(h) = \delta R_{\mu\nu}(h) - \frac{1}{2} \delta R(h) g_{\mu\nu},
\]

\[
\delta T^{\phi}_{\mu\nu}(h, \varphi) = \beta \phi'^2 [\delta G_{\mu\nu}(h) + 2G_{\mu\nu}\varphi - 2\nabla_{\mu} \nabla_{\nu} \varphi],
\]

\[
\delta T^M_{\mu\nu}(h, f) = 2 \tilde{F}_{(\nu}^{\rho} f_{\mu)\rho} - \tilde{F}_{\mu\rho} \tilde{F}_{\nu\sigma} h^{\rho\sigma} + \frac{1}{2} (\tilde{F}_{\kappa\eta} f^{\kappa\eta} - \tilde{F}_{\kappa\sigma} \tilde{F}_{\rho\sigma} h^{\kappa\rho}) g_{\mu\nu} - \frac{1}{4} F^2 h_{\mu\nu}.
\]

We note that Eq. (12) becomes a coupled tensor-vector-scalar equation. Also, it is important to note that the scalar perturbation \( \delta T^\phi_{\mu\nu}(h, \varphi) \) contributes to the polar sector only. Since the odd sector is the same as that for the EMS theory [10], we do not consider the odd sector here. We may expand the metric perturbation \( h_{\mu\nu} \) in terms of tensor spherical harmonics by choosing the Regge-Wheeler gauge. For our purpose, we introduce four tensor modes \( (H_0, H_1, H_2, K) \), two vector modes \( (u_1, u_2) \), and single scalar mode \( \delta \phi_1 \) in \( \varphi = \int \omega e^{-i\omega t} \delta \phi_1(r) Y^l_m(\theta, \phi) \) for polar sector [16]. In this case, the polar sector of Eqs. (12)-(14) is given by six coupled equations as

\[
K'(r) = -\left( \frac{l(l+1) + 2N + 2rN' - 2}{2r^2} - \frac{v^2}{r^2 - 1} \right) H_1(r) + \frac{H_0(r)}{r} + \left( \frac{N' - 1}{2N} \right) K(r) + \frac{4\tilde{\phi}(rN'\delta \phi_1(r) + 2N(\delta \phi_1(r) - r\delta \phi_1'(r)))}{r^2N(\phi^2 - 1)},
\]

\[
H'_1(r) = -\frac{4iv'}{\omega(1 - \phi^2)} f_{12}(r) - \frac{H_0(r) + K(r) + N'H_1(r)}{N} + \frac{4\tilde{\phi}}{rN(1 - \phi^2)} \delta \phi_1(r),
\]

\[
H'_0(r) = \left( \frac{1}{r} - \frac{N'}{N} \right) H_0(r) - \left( \frac{1}{r} - \frac{N'}{2N} \right) K(r) + \frac{4v'}{(1 - \phi^2)N} f_{02}(r) + \frac{2\tilde{\phi} \delta \phi_1'(r)}{r(1 - \phi^2)}
\]

\[
+ \left( \frac{\omega^2}{N} - \frac{v^2}{1 - \phi^2} - \frac{l(l+1)}{2r^2} - \frac{N + rN' - 1}{r^2} \right) H_1(r) + \frac{\tilde{\phi}(rN' - 6N)}{r^2N(\phi^2 - 1)} \delta \phi_1(r),
\]

\[
f'_{02}(r) = v'K(r) + \left( \frac{l(l+1)N}{r^2\omega} - \omega \right) if_{12}(r),
\]

\[
f'_{12}(r) = -\frac{i\omega}{N^2} f_{02}(r) - \frac{N'}{N} f_{12}(r),
\]

\[
\delta \phi''_1(r) = \left[ \frac{l(l+1)}{r^2N} - \frac{\omega^2 + N'}{rN} \right] \delta \phi_1(r) - \frac{N'}{N} \delta \phi_1'(r),
\]

where \( H_2(r) = H_0(r) \) and \( f_{01}(r) = i\omega f_{12}(r) + f'_{02}(r) \) with \( f_{12} = u_2/rN \) and \( f_{01} = u_1/r \).

The full analysis of stability could be performed by obtaining quasinormal frequency \( \omega = \omega_r + i\omega_i \) for physically propagating modes when solving the above linearized equations with boundary conditions: ingoing modes at the outer horizon and purely outgoing modes at infinity. If all \( \omega_i \) are negative (no exponentially growing modes), the considering black hole
Figure 1: The real (Left) and imaginary (Right) frequencies as function of scalar charge $Q_s$ for polar $l = 2$ gravitational-led mode around the constant scalar hairy black hole. The blue line of $0.38167 - i0.08961$ starting from $Q_s = 0$ represents the quasinormal frequency for a polar $l = 2$ gravitational-led mode propagating around the RN black hole.

is stable against all physically propagating modes. Otherwise, the black hole is unstable. We will compute the lowest quasinormal modes around the constant scalar hairy black hole by making use of a direct-integration method. Interestingly, the polar $l = 2$ case includes three physically propagating modes: scalar-led, vector-led, and gravitational-led. From Fig. 1(Right), we observe that the imaginary frequencies as function of scalar charge $Q_s$ are negative for polar $l = 2$ gravitational-led mode, implying the stability of the constant scalar hairy black hole. Also, we have shown that the constant scalar hairy black hole is stable against seven physically propagating modes of one for $l = 0$ case, two for $l = 1$ case, and four for $l = 2$ case excluding polar $l = 2$ gravitational-led mode.

4 $\alpha \neq 0$ black hole solutions

4.1 Infinite scalarized black holes from RN black hole

For the EMCS theory, an allowed solution is given by the RN black hole without scalar hair

$$ds^2_{\text{RN}} = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_2,$$

$$f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad \vec{\phi} = 0, \quad \vec{A}_t = \frac{Q}{r} - \frac{Q}{r_+}, \quad \vec{R} = 0. \quad (22)$$

Here we would like to mention that the condition of zero scalar ($\vec{\phi} = 0$) is important to obtain the RN black hole solution. The new scalarized charged black holes may be found from the appearance of instability for the RN black hole without scalar hair.
The linearized equations around a non-extremal RN black hole background with \( q = Q/m = 0.7 (Q = 0.35, \ m = 1/2, \ r_+ = 0.875) \) are given by considering the perturbations of tensor \( h_{\mu \nu} \), vector \( a_\mu \), and scalar \( \varphi \) as

\[
\delta G_{\mu \nu} = 2 \delta T^M_{\mu \nu}, \quad \nabla^\mu f_{\mu \nu} = 0, \quad \left[ \nabla^2 + \frac{\alpha}{3 \beta} \frac{Q^2}{r^4} \right] \varphi = 0,
\]

where the last scalar equation is important to analyze the stability of the black hole because the first-two equations correspond to the Einstein-Maxwell linearized theory which are turned out to be stable against tensor-vector perturbations around the non-extremal RN black hole background.

For the scalar perturbation, the separation of variables is introduced around a spherically symmetric RN background \( (22) \) as

\[
\varphi(t, r, \theta, \chi) = \frac{u(r)}{r} e^{-i \omega t} Y_{lm}(\theta, \chi).
\]

(24)

Considering a tortoise coordinate \( r_* \) defined by \( r_* = \int dr/f(r) \), a radial scalar equation is given by

\[
\frac{d^2 u}{dr_*^2} + \left[ \omega^2 - V_u(r) \right] u(r) = 0.
\]

(25)

Here the scalar potential \( V_u(r) \) is given by

\[
V_u(r) = \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) \left[ \frac{2m}{r^3} + \frac{l(l+1)}{r^2} - \frac{2Q^2}{r^4} - \frac{\alpha}{3 \beta} \frac{Q^2}{r^4} \right].
\]

(26)

The \( s(l = 0) \)-mode is an allowable mode for the scalar perturbation and thus, it could be used to test the instability of the RN black hole. In this case, the sign of the last term is important to find the stability of the RN black hole. If \( \beta < 0 \), the potential is positive definite, leading to the stable black hole. If the sign is positive (\( \beta > 0 \)), it may induce a negative region outside the horizon, arriving at the unstable RN black hole. Hence, we choose \( \beta = 1/3 \) to use the previous results for the EMS theory \([12]\). A dynamical scalar equation with \( \varphi(t, r) = e^{\Omega t} \varphi(r) \) determines the instability bound as \( \alpha(q) > \alpha_{th}(q) \) where the threshold of instability is given by \( \alpha_{th}(q = 0.7) = 8.019 \). Actually, solving the static scalar equation with \( \Omega = 0 \) leads to bifurcation points for scalar clouds as \( \alpha_n(q = 0.7) = \{8.019, 40.84, 99.89, \ldots \} \) where \( n \) indicates the appearance of \( n = 0, 1, 2, \ldots \) scalarized charged black holes.
In order to find scalarized charged black holes, one assumes the metric and fields
\[
ds^2_{schb} = -\tilde{N}(r)e^{-2\phi(r)}dt^2 + \frac{dr^2}{\tilde{N}(r)} + r^2d\Omega^2,
\]
\[
\tilde{N}(r) = 1 - \frac{2m(r)}{r}, \quad \tilde{\phi}(r), \quad \tilde{A}_t = v(r).
\]
Substituting the above into Eqs. (2), (4), and (7), one finds four equations for \(m(r), \delta(r), v(r),\) and \(\tilde{\phi}(r):\)

\[
3e^{2\phi}r^2\alpha\tilde{\phi}(\tilde{\phi}^2 - 3)v'^2 - 18(m - m'r\tilde{\phi}) - r(r - 2m)(9 + \tilde{\phi}^2)\tilde{\phi}''
\]
\[-(r - 2m)[\tilde{\phi}(\tilde{\phi}^2 - 3)\delta' + (18 + r(\tilde{\phi}^2 - 9)\delta')\tilde{\phi}' - 2r\tilde{\phi}\tilde{\phi}'] = 0,
\]
\[
3e^{2\phi}r^2(1 + \alpha\tilde{\phi}^2)v'^2 + 2(r - 2m)(\tilde{\phi}^2 - 3)\delta' + 2\tilde{\phi}(3m - 2r + r(r - 2m)\delta')\tilde{\phi}'
\]
\[-3r(r - 2m)\tilde{\phi}'' + 2m'(\tilde{\phi}^2 + r\tilde{\phi}\tilde{\phi}' - 3) = 0,
\]
\[
\left(2 + r\delta' + \frac{2r\alpha\tilde{\phi}\tilde{\phi}'}{1 + \alpha\tilde{\phi}^2}\right)v' + rv'' = 0,
\]
\[
-2r\tilde{\phi}'' + \delta'(\tilde{\phi}^2 - 3 + r\tilde{\phi}\tilde{\phi}') + r\tilde{\phi}\tilde{\phi}'' = 0,
\]
where the prime (') denotes differentiation with respect to \(r\). Here, one has a relation
\[
v'(r) = \frac{2e^{-\delta}Q}{r^2(1 + \alpha\tilde{\phi}^2)}.
\]
Accepting an outer horizon located at \(r = r_+\), one may introduce an approximate solution to (28)-(31) in the near-horizon
\[
m(r) = \frac{r_+}{2} + m_1(r - r_+) + \ldots,
\]
\[
\delta(r) = \delta_0 + \delta_1(r - r_+) + \ldots,
\]
\[
\tilde{\phi}(r) = \phi_0 + \phi_1(r - r_+) + \ldots,
\]
\[
v(r) = v_1(r - r_+) + \ldots,
\]
where the coefficients are determined by
\[
m_1 = \frac{[(\alpha\phi_0^2(\phi_0^2 - 12) - 9)Q^2]}{6r_+^2(\phi_0^2 - 3)(1 + \alpha\phi_0^2)^2}, \quad v_1 = -\frac{e^{-\delta_0}Q}{r_+^2(1 + \alpha\phi_0^2)},
\]
\[
\phi_1 = \frac{\alpha\phi_0Q^2(\phi_0^2 - 3)^2}{r_+Q^2(9 - \alpha\phi_0^2(\phi_0^2 - 12)) + 3r_+^2(\phi_0^2 - 3)(1 + \alpha\phi_0^2)^2},
\]
\[
\delta_1 = \frac{2r_+(1 + \alpha\phi_0^2)}{\alpha\phi_0Q^2(\phi_0^2 - 3)} \left[Q^2(9 - \alpha\phi_0^2(\phi_0^2 - 12)) + 3r_+^2(\phi_0^2 - 3)(1 + \alpha\phi_0^2)^2\right]^{\frac{1}{2}}
\times \left[12r_+^2(\phi_0^2 - 3)(1 + \alpha\phi_0^2)^3 + Q^2\{18 + \alpha(27 + (48 + 63\alpha)\phi_0^2 + (6\alpha - 7)\phi_0^4 - 3\alpha\phi_0^6)\}\right]
\]
\[
9
\]
with the electric charge $Q$. Here we note that \( \delta_1 \) takes the complicated form because of a conformal coupling term \( \phi^2 R \). This near-horizon solution involves two parameters of \( \phi_0 = \tilde{\phi}(r_+, \alpha) \) and \( \delta_0 = \delta(r_+, \alpha) \), which can be determined by matching (32)-(35) with the asymptotic solution in the far-region

\[
\begin{align*}
    m(r) &= M - \frac{3Q^2 + Q_s^2}{6r} + \ldots, \quad \tilde{\phi}(r) = \phi_\infty + \frac{Q_s}{r} + \ldots, \\
    \delta(r) &= \frac{Q_s^2[2Q_s^2 - 6M^2 + 3Q^2(2 + \alpha)]}{108r^4} + \ldots, \quad v(r) = \Phi + \frac{Q}{r} + \ldots,
\end{align*}
\]

where the ADM mass $M$, the scalar charge $Q_s$, and the electrostatic potential $\Phi$ are included. Importantly, we set $\phi_\infty$ to be zero without any ambiguity.

We may determine the infinitely numerical solutions labeled by \( n = 0(\alpha \geq 8.019) \), \( n = 1(\alpha \geq 40.84) \), \( n = 2(\alpha \geq 99.89) \), \ldots scalarized charged black holes. Now, let us display \( n = 0, 1, 2 \) branches for black holes in terms of scalar hair $\phi_0$ on the horizon in Fig. 2.

Explicitly, we choose the horizon radius $r_+ = 0.857$ and electric charge $Q = 0.35$ to construct the \( n = 0 \) scalarized charged black hole with $\alpha = 65.25$ shown in Fig. 3. An interesting behavior is found from $\delta(r)$, compared to that in the EMS theory. It is noted that $\delta(r)$ in (37) takes a different form from $\delta(r) = 2Q_s^2/r^2$ in the EMS theory [12].

### 4.2 Scalarized black hole from constant scalar hairy black hole

Let us derive the scalarized charged black hole based on the constant scalar hairy black hole which is turned out to be stable against all perturbations. In this case, even though the expansions in the near-horizon region are the same as (32)-(35), we find a differently
Figure 3: Plot of a scalarized charged black hole with $\alpha = 65.25$ residing in the $n = 0$ branch. Here $f(r)$ represents the metric function for the RN black hole. We plot all in terms of ‘$\ln r$’ and thus, the horizon is located at $\ln r = \ln r_+ = -0.154$.

Figure 4: Plot of scalar hair $\phi_0 = \bar{\phi}(r_+, \alpha)$ on the horizon as function of the coupling $\alpha \in [0, \infty)$, comparing with $\phi_0$ in Fig. 1. The dashed line denotes $\bar{\phi} = \phi_\infty = 0.7746$. The magnification indicates that $\phi_0$ starts with $\bar{\phi}$.

approximate solution in the far-region, compared with [37] as
\[
m(r) = M - \frac{3Q_s^2(1 + \phi_\infty^2)}{2(\phi_\infty^2 - 3)^2} - \frac{Q^2(2\alpha\phi_\infty^4 - 15\alpha\phi_\infty^2 - 9)}{6(\phi_\infty^2 - 3)(1 + \alpha\phi_\infty^2)} - \frac{MQ_s\phi_\infty}{(\phi_\infty^2 - 3)r} + \ldots,
\]
\[
\delta(r) = \frac{2Q_s\phi_\infty}{(\phi_\infty^2 - 3)r} + \ldots,
\]
\[
v(r) = \Phi + \frac{Q}{(1 + \alpha\phi_\infty^2)}r + \ldots,
\]
\[
\bar{\phi}(r) = \phi_\infty + \frac{Q_s}{r} + \ldots. \tag{38}
\]

Here, we note that $\phi_\infty^2$ is taken to be $\bar{\phi}^2 = 3\frac{Q_s^2}{Q^2 + Q_s^2} < 3$ for the constant scalar hairy black hole. For the constant scalar hairy black hole, we may select electric charge $Q = 0.2$, scalar charge $Q_s = 0.1$, and mass $M = 1/2$, providing the horizon radius $r_+ = 0.9472$ and $\bar{\phi} = 0.7746$. Now, we also choose horizon radius $r_+ = 0.9472(\ln r_+ = -0.0542)$ and electric charge $Q = 0.2$ to construct single branch of the scalarized charged black hole existing from $\alpha = 0$ ($\phi_0 = \phi_\infty$) to $\alpha = \infty$ ($\phi_0 = \phi_\infty = 0.7746$) in Fig. 4.

Explicitly, we wish to display a scalarized charged black hole with $\alpha = 63.75$ in Fig. 5.
Figure 5: Plot of a scalarized black hole with $\alpha = 63.75$ based on the constant scalar hairy black hole. The horizon is located at $\ln r = \ln r_+ = -0.0542$. $\bar{N}(r)$ and $N(r)$ represents metric function for scalarized charged black hole and constant scalar hairy black hole. The magnification in the left picture shows an enlarged tendency of scalar hair $\bar{\phi}(r)$. From the right picture, we observe that $\delta(r)$ is negative.

We observe that metric function and scalar hair are similar to those of the $n = 0$ scalarized charged black hole, whereas $\delta(r)$ is negative, opposite to positive for the $n = 0$ scalarized charged black hole in Fig. 3. This arises because $\delta_0 = -7.3 \times 10^{-4}$ and $\delta(r) < 0$ in (38).

4.3 Numerical charged BBMB black hole

Let us try to find another scalarized charged black holes based on the charged BBMB solution (10). Taking into account metric and fields in (27), we consider the expansion in the near-horizon region as

\[
\bar{N}(r) = \sum_{i=2}^{\infty} N_i (r - r_+)^i, \quad \delta(r) = \sum_{i=0}^{\infty} \delta_i (r - r_+)^i, \\
\bar{\psi}(r) = \sum_{i=1}^{\infty} \psi_i (r - r_+)^i, \quad v(r) = \sum_{i=1}^{\infty} v_i (r - r_+)^i,
\]  

(39)
where we introduce $\tilde{\psi}(r) = 1/\tilde{\phi}(r)$ for a technical reason. Replacing $\tilde{\phi}(r)$ by $1/\tilde{\psi}(r)$, all equations (28)-(31) take the form

$$\alpha e^{2\delta}(1/3 - \tilde{\psi}^2)v'^2 + \tilde{\psi}^3(r\tilde{\psi}\tilde{N}' - \tilde{N}(r\delta' - 2) + 2r\tilde{\psi}') + r\tilde{N} + \tilde{\psi}\tilde{\psi}'' = 0,$$

(40)

$$(1 - 3\tilde{\psi}^2)[1 + \tilde{N}(2r\delta' - 1)] - \frac{r\tilde{N}\tilde{\psi}'}{\tilde{\psi}^2}[2\tilde{\psi}(r\delta' - 2) + 3r\tilde{\psi}']$$

$$+ 3r^2e^{2\delta}(\alpha + \tilde{\psi}^2)v'^2 + \frac{r\tilde{N}'}{\tilde{\psi}'}(r\tilde{\psi}' + 3\tilde{\psi}^3 - \tilde{\psi}) = 0,$$

(41)

$$Q\tilde{\psi}^2 + e^{\delta}r^2(\alpha + \tilde{\psi}^2)v' = 0,$$

(42)

$$\delta'(3\tilde{\psi}^3 - \tilde{\psi} + r\tilde{\psi}') + r\tilde{\psi}'' = 0.$$  

(43)

Substituting (39) into (40)-(43) leads to the lowest order equations

$$1 - \frac{r^2}{r_+}N_2 + 3\alpha e^{2\delta}r_+^2v_1^2 = 0, \quad \alpha e^{2\delta}r_+v_1^2 = 0, \quad \alpha e^{2\delta}r_+v_1 = 0, \quad \delta\psi_1 + 2\psi_2 = 0,$$

(44)

whose second and third equations imply

$$\alpha = 0.$$  

(45)

because of $r_+ \neq 0$ and $v_1 \neq 0$. This implies that the EMCS theory is not compatible with expansion (39) when deriving a scalarized extremal black hole like the charged BBMB solution. It needs to find out another expansion which is suitable for constructing scalarized black hole solution.

In this section, instead, we wish to derive the charged BBMB black hole solution (10) numerically by considering $\alpha = 0$ case because it is important to determine the exact nature of the charged BBMB solution. The first three coefficients in the near-horizon region are determined by

$$N_2 = \frac{1}{r_+^2}, \quad N_3 = -\frac{6(r_+^4 - Q^2)\psi_1^2 + 4}{3r_+^3},$$

$$N_4 = \frac{5 + 15(r_+^4 - Q^2)\psi_1^2}{3r_+^4} - \frac{3(r_+^4 - Q^2)^2\psi_1^4}{r_+^4},$$

$$\delta_0, \quad \psi_1, \quad \psi_2 = \frac{[1 - 3(r_+^2 - Q^2)\psi_1^2]}{3r_+}, \quad \psi_3 = \frac{2\psi_1[1 - 3(r_+^2 - Q^2)\psi_1^2]}{9r_+^2},$$

$$v_1 = -\frac{e^{-\delta}Q}{r_+^2}, \quad v_2 = \frac{e^{-\delta}Q}{r_+^3} \left[\frac{4}{3} - (r_+^2 - Q^2)\psi_1^2\right],$$

$$v_3 = -\frac{e^{\delta}Q}{r_+^4} \left[\frac{14}{9} - \frac{7}{3}(r_+^2 - Q^2)\psi_1^2 + 2(r_+^2 - Q^2)^2\psi_1^4\right].$$  

(46)

13
which are not appropriate near-horizon forms for the charged BBMB black hole. In this approach, two free parameters $\delta_0$ and $\psi_1$ will be determined when matching (39) with the following asymptotic solution for $r \gg r_+$:

$$\bar{N}(r) = 1 - \frac{2m}{r} + \frac{Q^2 + Q_s^2}{r^2} + \cdots, \quad \bar{\phi}(r) = \frac{\sqrt{3}Q_s}{r} + \frac{\sqrt{3}Q_sm}{r^2} + \cdots,$$

$$\delta(r) = \frac{Q_s(Q^2 + Q_s^2 - m^2)}{6r^4} + \cdots, \quad v(r) = \frac{Q}{r} - \frac{Q}{r_+},$$

(47)

which are not surely appropriate asymptotic forms for the charged BBMB black hole. In deriving a numerical black hole solution, an important requirement is to impose the asymptotic flatness. Here we may choose $\delta_0 = 0$ when considering the asymptotic boundary at $r = \infty$. Actually, we choose here $\delta_0 = 0.0001$ because the asymptotic boundary is located at $r = 100$ for numerical computation. We make Table 1 which shows numerical relations $(r_+, \psi_1, Q_s, m, Q)$ for different $r_+$ with the same $Q$. Confirming three relations from Table 1

$$r_+ \approx m, \quad m \approx \sqrt{Q^2 + Q_s^2}, \quad \psi_1 \approx \frac{1}{\sqrt{3(r_+^2 - Q^2)}} \approx \frac{1}{\sqrt{3}Q_s},$$

(48)

we read off correct coefficients

$$N_2 = \frac{1}{r_+^2}, \quad N_3 = \frac{2}{r_+^3}, \quad N_4 = \frac{3}{r_+^4} + \cdots, \quad \delta_i = 0, \quad \text{for } i = 1, \cdots,$$

$$\psi_i = 0, \quad \text{for } i = 2, \cdots,$$

$$v_1 = -\frac{Q}{r_+^2}, \quad v_2 = \frac{Q}{r_+^3}, \quad v_3 = -\frac{Q}{r_+^4} + \cdots.$$  

(49)

Making use of the above relations, we arrive at the expansion forms of charged BBMB

| $r_+$  | $\psi_1$  | $Q_s$  | $m$  | $Q$  |
|--------|-----------|--------|------|------|
| 0.75   | 1.0327    | 0.5590 | 0.7501 | 0.5 |
| 1      | 0.6666    | 0.8860 | 0.9999 | 0.5 |
| 1.25   | 0.5039    | 1.1456 | 1.2500 | 0.5 |
| 1.5    | 0.4082    | 1.4142 | 1.5000 | 0.5 |

Table 1: Table showing numerical relations among $r_+, \psi_1, Q_s, m,$ and $Q$ for different $r_+$ with the same $Q = 0.5$. These include $r_+ \approx m$, $m^2 \approx Q^2 + Q_s^2$, and $\psi_1 \approx 1/\sqrt{3}Q_s$.  

(49)
solution in the near-horizon region

\[ \tilde{N}(r) = \left(\frac{r - r_+}{r_+^2}\right)^2 - \frac{2(r - r_+)^3}{r_+^3} + \frac{3(r - r_+)^4}{r_+^4} + \ldots \rightarrow \left[\left(1 - \frac{r_+}{r}\right)^2\right]_{r = r_+}, \]

\[ v(r) = -\frac{Q(r - r_+)}{r_+^2} - \frac{Q(r - r_+)^2}{r_+^4} + \ldots \rightarrow \left[\frac{Q}{r}\right]_{r = r_+} - \frac{Q}{r_+}, \]

\[ \tilde{\psi}(r) = \frac{r - r_+}{\sqrt{3}Q_s}, \]

\[ \delta(r) \approx 0. \quad (50) \]

On the other hand, making use of \( m^2 \approx Q^2 + Q_s^2 \) leads to the asymptotic expansions of charged BBMB solution (10)

\[ \tilde{N}(r) = 1 - 2\frac{m}{r} + \frac{m^2}{r^2} = \left(1 - \frac{m}{r}\right)^2, \]

\[ \tilde{\phi}(r) = \frac{\sqrt{3}Q_s}{r} + \frac{\sqrt{3}Q_sm}{r^2} + \ldots \rightarrow \left[\frac{\sqrt{3}Q_s}{r - m}\right]_{r \gg m}, \]

\[ v(r) = \frac{Q}{r} - \frac{Q}{r_+}, \]

\[ \delta(r) \approx 0. \quad (51) \]

This shows numerically that the allowed black hole is the charged BBMB black hole only in the \( \alpha = 0 \) EMCS theory when imposing the asymptotically flat condition.

## 5 Discussions

We have investigated the EMCS theory to obtain various scalarized charged black holes. The \( \alpha = 0 \) EMCS theory provided the constant scalar hairy black hole and charged BBMB black hole. We has shown that the former is stable against full perturbations. It is interesting to note that the latter remains unstable because it belongs to an extremal black hole [11]. For \( \alpha \neq 0 \), it is reasonable to say that the unstable RN black holes imply the appearance of \( n = 0(\alpha \geq 8.019), 1(\alpha \geq 40.84), 2(\alpha \geq 99.89), \ldots \) scalarized charged black holes. We have derived infinite branches of scalarized charged black holes. Importantly, for \( \alpha > 0 \), we obtain single branch of scalarized charged black hole based on the constant scalar hairy black hole which is a stable black hole. Unfortunately, for \( \alpha > 0 \), we failed to derive another scalarized black hole based on the charged BBMB black hole. However, we have shown explicitly that the allowed black hole is given by the charged BBMB black hole
in the $\alpha = 0$ EMCS theory when requiring the asymptotic flatness. This implies that the charged BBMB solution dictates the feature of conformally coupled scalar system in the EMCS theory.

On the other hand, the EMS theory has provided infinite scalarized charged black holes based on the instability of RN black holes [10] [12].

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (No. NRF-2017R1A2B4002057).
References

[1] R. Ruffini and J. A. Wheeler, Phys. Today 24, no. 1, 30 (1971). doi:10.1063/1.3022513

[2] C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 24, no. 09, 1542014 (2015) doi:10.1142/S0218271815420146 arXiv:1504.08209 [gr-qc].

[3] N. M. Bocharova, K. A. Bronnikov and V. N. Melnikov, Vestn. Mosk. Univ. Ser. III Fiz. Astron., no. 6, 706 (1970).

[4] J. D. Bekenstein, Annals Phys. 82, 535 (1974). doi:10.1016/0003-4916(74)90124-9

[5] B. C. Xanthopoulos and T. Zannias, J. Math. Phys. 32, 1875 (1991).

[6] Y. S. Myung and D. C. Zou, Phys. Rev. D 100, no. 6, 064057 (2019) doi:10.1103/PhysRevD.100.064057 arXiv:1907.09676 [gr-qc].

[7] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120, no. 13, 131103 (2018) doi:10.1103/PhysRevLett.120.131103 arXiv:1711.01187 [gr-qc].

[8] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou and E. Berti, Phys. Rev. Lett. 120, no. 13, 131104 (2018) doi:10.1103/PhysRevLett.120.131104 arXiv:1711.02080 [gr-qc].

[9] G. Antoniou, A. Bakopoulos and P. Kanti, Phys. Rev. Lett. 120, no. 13, 131102 (2018) doi:10.1103/PhysRevLett.120.131102 arXiv:1711.03390 [hep-th].

[10] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual and J. A. Font, Phys. Rev. Lett. 121, no. 10, 101102 (2018) doi:10.1103/PhysRevLett.121.101102 arXiv:1806.05190 [gr-qc].

[11] K. A. Bronnikov and Y. N. Kireev, Phys. Lett. A 67, 95 (1978). doi:10.1016/0375-9601(78)90030-0

[12] Y. S. Myung and D. C. Zou, Eur. Phys. J. C 79, no. 3, 273 (2019) doi:10.1140/epjc/s10052-019-6792-6 arXiv:1808.02609 [gr-qc].

[13] M. Astorino, Phys. Rev. D 87, no. 8, 084029 (2013) doi:10.1103/PhysRevD.87.084029 arXiv:1301.6794 [gr-qc].

[14] M. Astorino, Phys. Rev. D 88, no. 10, 104027 (2013) doi:10.1103/PhysRevD.88.104027 arXiv:1307.4021 [gr-qc].
[15] H. Onozawa, T. Mishima, T. Okamura and H. Ishihara, Phys. Rev. D \textbf{53}, 7033 (1996) doi:10.1103/PhysRevD.53.7033 [gr-qc/9603021].

[16] Y. S. Myung and D. C. Zou, Eur. Phys. J. C \textbf{79}, no. 8, 641 (2019) doi:10.1140/epjc/s10052-019-7176-7 [arXiv:1904.09864 [gr-qc]].