Analysis and approximation of some Shape–from–Shading models for non-Lambertian surfaces

Silvia Tozza* Maurizio Falcone†

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Abstract

The reconstruction of a 3D object or a scene is a classical inverse problem in Computer Vision. In the case of a single image this is called the Shape–from–Shading (SfS) problem and is known to be ill-posed even in a simplified version like the vertical light source case. A huge number of works deals with the orthographic SfS problem based on the Lambertian reflectance model, the most common and simplest model which leads to an eikonal type equation when the light source is on the vertical axis. In this paper we want to overcome this model dealing with non-Lambertian models, more realistic and suitable whenever one has to deal with different kind of surfaces, rough or specular. We will present a unique mathematical formulation for these models, considering oblique light directions. These models lead to more complex nonlinear partial differential equations of Hamilton-Jacobi type which we are able to describe in a unified framework. The construction of approximate (weak) solutions are obtained via semi-Lagrangian schemes for the corresponding stationary Hamilton-Jacobi equations. Numerical simulations on synthetic and real images will illustrate the effectiveness of this approach.

Key words. Shape-from-Shading non-Lambertian models Stationary Hamilton-Jacobi equations semi-Lagrangian approximation

1 Introduction

The three dimensional reconstruction of an object is a topic of great interest in many different fields ranging from the biomedical application of 3D re-

* Dipartimento di Matematica, Sapienza - Università di Roma, Piazzale Aldo Moro, 5 - 00185 Roma, tozza@mat.uniroma1.it
† Dipartimento di Matematica, Sapienza - Università di Roma, Piazzale Aldo Moro, 5 - 00185 Roma, falcone@mat.uniroma1.it
construction of human tissues to the archeology for the digitization of sculptures. Other examples come from engineering for the construction of a 3D endoscope for surgical operations in real time, from astronomy for the characterization of properties of planets or other astronomical entities and from the area of security where the facial recognition of individuals has a growing importance. More recently, some robotics applications have emerged. In fact, the autonomous movement of a robot in an unknown environment requires several steps. The first step is to reconstruct the scene, locate the obstacles and define the objective. The second is to compute an optimal path to the objective from the initial position of the robot and finally the third is to implement the optimal path via the control of all the movement devices (wheels, arms, wings, ...). The first step is clearly related to the 3D reconstruction and is particularly hard because of the various lighting conditions of the objects in the scene. In fact, several light sources can appear in the environment, the object surfaces represented in the scene can have different reflection properties because they are made by different materials. It would be hard to imagine a scene which can satisfy the classical assumptions of the 3D reconstruction models, in particular the typical Lambertian assumption has to be weakened. As we have seen, the 3D reconstruction problem is ubiquitous and this is probably why has always attracted a great attention. However, despite the fact that its formulation is rather simple, there is still no global method for its resolution under realistic assumptions. The main goal of this paper is to present a unified approach to several models for non-Lambertian surfaces, to deal with light sources placed in oblique directions with respect to the surface and, finally, to develop a general numerical approximation scheme to solve the corresponding nonlinear partial differential equations for these models.

The SfS problem is a difficult inverse problem where the goal is to reconstruct the surface from a single image, that is, given a two-dimensional gray scale digital image $I(x)$, where $x := (x, y)$, the goal is to reconstruct the surface $z = u(x)$ that corresponds to it, using the information contained in the image (hence the name Shape from Shading). This problem is described in general by the irradiance equation

$$R(N(x)) = I(x)$$

where the normalized brightness of the given grey-value image $I(x)$ is put in relation with the function $R(N(x))$ that represents the reflectance map giving the value of the light reflection on the surface as a function of its orientation (i.e., of the normal $N(x)$) at each point $(x, u(x))$. Depending on how we describe the function $R$ different reflection models are determined. In the literature, the most common representation of $R$ takes into account only the angle between the outgoing normal to the surface $N(x)$ and the
light source $\omega$, that is

$$I(x) = \gamma_D(x)N(x) \cdot \omega,$$

(2)

where $\gamma_D(x)$ indicates the diffuse albedo, i.e. the diffuse reflectivity or reflecting power of a surface. It is the ratio of reflected radiation from the surface to incident radianc upon it. Its dimensionless nature is expressed as a percentage and is measured on a scale from zero for no reflection of a perfectly black surface to 1 for perfect reflection of a white surface. The data are the grey-value image $I(x)$, the direction of the light source $\omega$ and the albedo $\gamma_D(x)$. The light source $\omega$ is a unit vector, hence $|\omega| = 1$. In the simple case of a vertical light source this gives rise to an eikonal equation.

Several questions arise, even in the simple case: is a single image sufficient to determine the surface? If not, which set of additional informations is necessary to have uniqueness? How can we compute an approximate solution? Is the approximation accurate? It is well known that for Lambertian surfaces there is no uniqueness and other informations are necessary to select a unique surface (e.g. the height at each point of local maximum for $I(x)$).

However, rather accurate schemes for the classical eikonal equation are now available for the approximation. Despite its simplicity, the Lambertian assumption is very strong and does not match with many real situations.

In order to set this paper into perspective, let us remark that following the pioneering work of Horn [20, 21, 23, 24, 22] who first formulated this problem in mathematical terms, many other contributions to this area have appeared. Several approaches have been proposed, we can group them in two big classes (see the surveys [57, 14]): methods based on the resolution of partial differential equations (PDEs) and optimization methods based on a variational approximation. In the first group the unknown is directly the height of the surface $z = u(x,y)$, one can find here rather old papers based on the method of characteristics [12, 14, 21, 35, 34, 4, 29] where one typically looks for classical solutions. More recently, other contributions were developed in the framework of weak solutions in the viscosity sense starting from the seminal paper by Lions-Rouy and Tourin [31] (see e.g. [26, 28, 17, 7, 43, 3, 18, 41]). The second group contains the contribution based on minimization methods for the variational problem where the unknown are the partial derivatives of the surface, $p = u_x$ and $q = u_y$ (the so-called normal vector field) (see e.g. [23, 30, 11, 46, 19, 47, 25, 5, 54]. It is important to note that in this approach one has to couple the minimization step to compute the normal field with a local reconstruction for $u$ which is based usually on a path integration. This necessary step has also been addressed by several authors (see [13] and references therein). We should also mention that a continuous effort has been made by the scientific community to take into account more realistic scenarios including perspective deformations [33, 49, 42, 1, 32] and/or multiple images of the same object [55, 56], taken from the same point of view but with different light sources.
(photometric stereo technique, see [48, 28]) or from different points of view but with the same light source (stereoscopy, see [10]). Moreover, there are other supplementary issues that make the problem even more complex: the estimation of the albedo or of the direction of the light source (the so-called calibrated/uncalibrated problem) or the role of boundary conditions which have to be coupled with the PDE. Those are known quantities in the model but in practice the informations are not available for real images.

In this paper we will follow the PDE approach, focusing our attention on a couple of non-Lambertian reflectance models: the Oren-Nayar and the Phong models [37, 38, 36, 39, 40]. We will show that both these models lead to nonlinear stationary Hamilton-Jacobi equations of the first order. These equations have a similar structure but unfortunately it is not enough to drop the concave/convex ambiguity. However, we will be able to define a semi-Lagragian approximation scheme that can be applied to this class of Hamilton-Jacobi equations, hence to both non-Lambertian models. The paper is organized as follows. After a formulation of the general model presented in Section 2, we present the SfS models starting from the classical Lambertian model (Section 3). In Sections 4 and 5 we will give details on the construction of the nonlinear partial differential equation which corresponds respectively to the Oren-Nayar and the Phong models. Despite the differences appearing in these non-Lambertian models, we will be able to present them in a unified framework showing that the Hamilton-Jacobi equations for all the above models share a common structure which has a control theoretical interpretation. Moreover, the Hamiltonian appearing in these equations will always be convex in the gradient $\nabla u$. Then, in Section 6 we will introduce our general approximation scheme which can be applied to solve this class of problems. Finally, in the last section we will apply our approximation to a series of benchmarks based on virtual and real images. We will discuss some issue like accuracy, efficiency and the capability to obtain the maximal solution showing that the semi-Lagragian approximation is rather effective even for real images where several parameters are unknown.

2 Formulation of the general model

We fix a camera in a three-dimensional coordinate system $(Oxyz)$ in such a way that $Oxy$ coincides with the image plane and Oz with the optical axis. Let $\omega = (\omega_1, \omega_2, \omega_3) = (\tilde{\omega}, \omega_3) \in \mathbb{R}^3$ (with $\omega_3 > 0$) be the unit vector that represents the direction of the light source; let $I(x)$ be the function that measures the gray tones of the image at the point $x := (x, y)$. $I(x)$ is the datum in the model since it is measured at each pixel of the image, for example in terms of a grey level (from 0 to 255). In order to construct a continuous model, we will assume that $I(x)$ takes real values in the interval
defined in a compact domain $\Omega$ called “reconstruction domain” (with $\Omega \subset \mathbb{R}^2$ open set), $I : \Omega \to [0,1]$, where the points with a value of 0 are the dark point (blacks), while those with a value of 1 correspond to a completely reflection of the light (white dots, with a maximum reflection).

We consider the following assumptions:

A1. there is a single light source placed at infinity in the direction $\omega$ (the light rays are, therefore, parallel to each other);

A2. the observer’s eye is placed at an infinite distance from the object you are looking at (i.e. there is no perspective deformation);

A3. there are no autoreflections on the surface.

In addition to these assumptions, there are other hypothesis that depend on the different reflectance models (we will see them in the description of the individual models).

Being valid the assumption (A2) of orthographic projection, the visible part of the scene is a graph $z = u(x)$ and the unit normal to the regular surface at the point corresponding to $x$ is given by:

$$N(x) = \frac{n(x)}{|n(x)|} = \frac{-\nabla u(x), 1}{\sqrt{1 + |\nabla u(x)|^2}},$$

where $n(x)$ is the outgoing normal vector.

We assume that the height function, which is the unknown of the problem, is $u(x) \geq 0$ and the surface is standing on a flat background. We will denote by $\Omega$ the region inside the silhouette and we will assume (just for technical reasons) that $\Omega$ is an open and bounded subset of $\mathbb{R}^2$ (see Fig. 1). It is

![Figure 1: An object on a flat background: $\Omega$ indicates the region inside the silhouette, $\partial\Omega$ the boundary of it.](image)

well known that the Shape-from-Shading (SfS) problem is described by the image irradiance equation

$$I(x) = R(N(x)), \quad (4)$$
where $I(x)$ is the normalized brightness of the given grey-value image, $N(x)$ is the unit normal to the surface at the point $(x, u(x))$ and $R(N(x))$ is the reflectance map giving the value of the light reflection on the surface as a function of its orientation (i.e., of the normal) at each point. Depending on how we describe the function $R$ different reflection models are determined. We describe below some of them. To this end, it would be useful to introduce a representation of the brightness function $I(x)$ where we can distinguish different terms representing the contribution of ambient, diffuse reflected and specular reflected light. We will write then

$$I(x) = k_A I_A(x) + k_D I_D(x) + k_S I_S(x),$$  \hspace{1cm} (5)$$

where $I_A(x)$, $I_D(x)$ and $I_S(x)$ are respectively the above mentioned components and $k_A$, $k_D$ and $k_S$ indicate the percentages of these components such that their sum is equal to 1 (we do not consider absorption phenomena). Note that the diffuse or specular albedo is inside the definition of $I_D(x)$ or $I_S(x)$, respectively. In the sequel, we will always consider $I(x)$ normalized in $[0, 1]$. This will allow to switch on and off the different contributions depending on the model. Let us note that the ambient light term $I_A(x)$ represents light everywhere in a given scene. As we will see in the following sections, the intensity of diffusely reflected light in each direction is proportional to the cosine of the angle $\theta_i$ between surface normal and light source direction, without taking into account the point of view of the observer, but another diffuse model (the Oren–Nayar model) will consider it in addition. The amount of specular reflected light towards the viewer is proportional to $(\cos \theta_s)^{\alpha}$, where $\theta_s$ is the angle between the ideal (mirror) reflection direction of the incoming light and the viewer direction, $\alpha$ being a constant modelling the specularity of the material. In this way we have a more general model and, dropping the ambient and specular component, we retrieve the Lambertian reflection as a special case.

3 The Lambertian model (L–model)

For a Lambertian surface, which generates a purely diffuse model, the specular component does not exist. So, the general equation (5) becomes

$$I(x) = k_A I_A(x) + k_D I_D(x),$$  \hspace{1cm} (6)$$

whose diffuse component $I_D(x)$ is

$$I_D(x) = \gamma_D(x) N(x) \cdot \omega,$$  \hspace{1cm} (7)$$

where $\gamma_D(x)$ is the diffuse albedo. Neglecting the ambient component that can be considered as a constant (i.e. setting $k_A = 0$), recalling that the sum $k_A + k_D + k_S$ must be equal to 1, we obtain that necessarily $k_D = 1$ and we
can omit it in the following. Then, for a Lambertian surface the irradiance equation \([4]\) becomes
\[
I(x) = \gamma_D(x) N(x) \cdot \omega,
\]
where we assume to know \(\gamma_D(x)\) (in the sequel we suppose uniform albedo and we put \(\gamma_D(x) = 1\), that is all the points of the surface reflect completely the light that hits them).

Under these assumptions, the orthogonal SfS problem consists in determining the function \(u : \Omega \rightarrow \mathbb{R}\) that satisfies the equation \((8)\). The unit vector \(\omega\) and the function \(I(x)\) are the only quantity known.

In this model, we can note that the measured light in the image depends only on the scalar product between \(N(x)\) and \(\omega\) and the parameter \(\gamma_D(x)\), which describes the physical properties of the surface reflection. So in this case the surface of the object has uniform properties of light reflection (see Fig. 2). We will see in the next sections that for other models it is no longer true because other quantities are taken into account.

For Lambertian surfaces \([23, 24]\), just considering an orthographic projection of the scene, it is possible to model the SfS problem via a nonlinear PDE of the first order which describes the relation between the surface \(u(x)\) (our unknown) and the brightness function \(I(x)\). In fact, recalling the definition of the unit normal to a graph given in (3), we can write \((8)\) as
\[
I(x) \sqrt{1 + |\nabla u(x)|^2} + \tilde{\omega} \cdot \nabla u(x) - \omega_3 = 0, \quad \text{in} \ \Omega \tag{9}
\]
where \(\tilde{\omega} = (\omega_1, \omega_2)\). This is an Hamilton-Jacobi type equation which does not admit in general regular solution. It is known that the mathematical framework to describe its weak solutions is the theory of viscosity solutions.
as in [31].

**The vertical light case.**

If we choose $\omega = (0,0,1)$, the equation (9) becomes the so-called “eikonal equation”:

$$|\nabla u(x)| = f(x) \quad \text{for} \quad x \in \Omega,$$

(10)

where

$$f(x) = \sqrt{\frac{1}{I(x)^2} - 1}.$$  

(11)

The points $x \in \Omega$ where $I(x)$ assumes maximum value correspond to the case in which $\omega$ and $\mathbf{N}(x)$ have the same direction: these points are usually called “singular points”.

In order to make the problem well-posed, we need to add boundary conditions to the equations (9) or (10): they can require the value of the solution $u$ (Dirichlet boundary conditions type), or the value of its normal derivative (Neumann boundary conditions), or an equation that must be satisfied on the boundary (the so-called boundary conditions “state constraint”). In this paper we choose to consider the Dirichlet boundary conditions equal to zero, assuming the surface on a flat background

$$u(x) = 0, \quad \text{for} \quad x \in \partial \Omega,$$

(12)

but a second possibility of the same type occurs when it is known the value of $u$ on the boundary, which leads to the more general condition

$$u(x) = g(x), \quad \text{for} \quad x \in \partial \Omega.$$  

(13)

But adding a boundary condition to the PDE that describes the SfS model is not enough to obtain a unique solution because of the concave/convex ambiguity. In fact, the Dirichlet problem (9)-(13) can have several weak solutions in the viscosity sense and also several classical solutions due to this ambiguity (see [21]). As an example, all the surfaces represented in Fig. 3 are viscosity solutions of the same problem (10)-(12), which is a particular case of (9)-(13) (in fact the equation is $|u'| = -2x$ with homogenous Dirichlet boundary condition). The solution represented in Fig. 3-a is the maximal solution and is smooth. All the non-smooth a.e. solutions, which can be obtained by a reflection with respect to a horizontal axis, are still admissible weak solutions (see Fig. 3-b). In this example, the lack of uniqueness of the viscosity solution is due to the existence of a singular point where the right hand side of (10) vanishes. An additional effort is then needed to define which is the preferable solution since the lack of uniqueness is also a big drawback when trying to compute a numerical solution. In order to circumvent these difficulties, the problem is usually solved by adding some information such as height at each singular point [31].
For analytical and numerical reasons it is useful to introduce the exponential transform $\mu v(x) = 1 - e^{-\mu u(x)}$ and change the variable. Note that here $\mu$ is a free positive parameter without a physical meaning, but it is important because varying its value it is possible to modify the slope. In fact, the slope increases for increasing values of $\mu$. Assuming that the surface is standing on a flat background and following [18], we can write (9)+(12) in a fixed point form in the new variable $v$. To this end let us define $b^L : \Omega \times \partial B_3(0,1) \rightarrow \mathbb{R}^2$ and $f^L : \Omega \times \partial B_3(0,1) \times [0,1] \rightarrow \mathbb{R}$ as

$$b^L(x,a) = \frac{1}{\omega_3} \left( I(x)a_1 - \omega_1, I(x)a_2 - \omega_2 \right), \quad (14)$$

$$f^L(x,a,v(x)) = -\frac{I(x)a_3}{\omega_3} (1 - \mu v(x)) + 1 \quad (15)$$

and let $B_3$ denote the unit ball in $\mathbb{R}^3$. We obtain

$$\text{Lambertian Model}$$

$$\begin{cases} \mu v(x) = T^L(x,v(x),\nabla v) & \text{for } x \in \Omega, \\ v(x) = 0 & \text{for } x \in \partial \Omega, \end{cases} \quad (16)$$

where

$$T^L(\cdot) := \min_{a \in \partial B_1} \{ b^L(x,a) \cdot \nabla v(x) + f^L(x,a,v(x)) \}.$$ 

It is important to note for the sequel that the structure of the above first order Hamilton-Jacobi equation is similar to that related to the dynamic programming approach in control theory where $b$ is a vector field describing the dynamics of the system and $f$ is a running cost. In that framework the meaning of $v$ is that of a value function which allows to characterize the optimal trajectories (here they play the role of characteristic curves). The interested reader can find more details on this interpretation in [16].

4 The Oren-Nayar model (ON-model)

The diffuse reflectance ON–model [37, 38, 36, 39] is an extension of the previous L–model which explicitly allows to handle rough surfaces. The
idea of this model is to represent a rough surface as an aggregation of V-shaped cavities, each with Lambertian reflectance properties (see Fig. 4). In [37] and, with more details, in [39], Oren and Nayar derive a reflectance model for three types of surfaces with different slope-area distributions:

- **Uni-directional Single-Slope Distribution:** This distribution results in a non-isotropic surface where all facets have the same slope and all cavities are aligned in the same direction.

- **Isotropic Single-Slope Distribution:** Here, all facets have the same slope but they are uniformly distributed in orientation on the surface plane.

- **Gaussian Distribution:** This is the most general case examined where the slope area distribution is assumed to be normal with zero mean. The roughness of the surface is determined by the standard deviation of the normal distribution.

The reflectance model obtained for each of the above surface types is used to derive the succeeding one, called by the authors the “Qualitative Model”, a simpler version obtained by discarding the coefficient $C_3$ present in their formulation and ignoring interreflections (see Section 4.4 of [37] for more details).

![Facet model for surface patch $dA$ consisting of many V-shaped Lambertian cavities.](image)

The irradiance equation of this simpler model is then

$$E_D(x) = \gamma_D(x) \cos(\theta_i)(A + B \sin(\alpha) \tan(\beta)M(\varphi_i, \varphi_r)), $$

where $M(\varphi_i, \varphi_r) = \max[0, \cos(\varphi_r - \varphi_i)]$.

Assuming that there is a linear relation between the irradiance of the image and the image intensity, the $I_D$ *brightness equation* for the ON–model is given by

$$I_D(x) = \gamma_D(x) \cos(\theta_i)(A + B \sin(\alpha) \tan(\beta)M(\varphi_i, \varphi_r))$$

(17)

where

$$A = 1 - 0.5 \sigma^2(\sigma^2 + 0.33)^{-1}$$

(18)

$$B = 0.45\sigma^2(\sigma^2 + 0.09)^{-1}.$$  

(19)
Note that $A$ and $B$ are two non-negative constants depending on the statistics of the cavities via the roughness parameter $\sigma$. We set $\sigma \in [0, \pi/2)$, interpreting $\sigma$ as the slope of the cavities. In this model (see Fig. 5), $\theta_i$ represents the angle between the unit normal to the surface $N(x)$ and the light source direction $\omega$, $\theta_r$ stands for the angle between $N(x)$ and the observer direction $V$, $\varphi_i$ is the angle between the projection of the light source direction $\omega$ and the $x_1$ axis onto the $(x_1, x_2)$-plane tangent to the surface, $\varphi_r$ denotes the angle between the projection of the observer direction $V$ and the $x_1$ axis onto the $(x_1, x_2)$-plane tangent to the surface and the two variables $\alpha$ and $\beta$ are given by
\[
\alpha = \max[\theta_i, \theta_r] \quad \text{and} \quad \beta = \min[\theta_i, \theta_r].
\tag{20}
\]
Note that in order to make all the quantities present in this model computable for the implementation, we consider a modified version of the ON–model in which the difference $\varphi_r - \varphi_i$ is constant and depends on the data $\omega$ and $V$ given. That simplification allows for the numerical tests to compute $\max[0, \cos(\varphi_r - \varphi_i)]$ only once for a whole image.

![Figure 5: Diffuse reflectance for the ON–model.](image)

We define (see Fig. 5):
\[
\cos(\theta_i) = N \cdot \omega = \frac{-\tilde{\omega} \cdot \nabla u(x) + \omega_3}{\sqrt{1 + |\nabla u(x)|^2}} \tag{21}
\]
\[
\cos(\theta_r) = N \cdot V = \frac{-\tilde{v} \cdot \nabla u(x) + v_3}{\sqrt{1 + |\nabla u(x)|^2}} \tag{22}
\]
\[
\cos(\varphi_r - \varphi_i) = (\omega_1, \omega_2) \cdot (v_1, v_2) = \tilde{\omega} \cdot \tilde{v} \tag{23}
\]
\[
\sin(\theta_i) = \sqrt{1 - (\cos(\theta_i))^2} = \frac{g_\omega(\nabla u(x))}{\sqrt{1 + |\nabla u(x)|^2}} \tag{24}
\]
\[
\sin(\theta_r) = \sqrt{1 - (\cos(\theta_r))^2} = \frac{g_v(\nabla u(x))}{\sqrt{1 + |\nabla u(x)|^2}} \tag{25}
\]
where
\[
g_\omega(\nabla u(x)) = \sqrt{1 + |\nabla u(x)|^2} - (-\tilde{\omega} \cdot \nabla u(x) + \omega_3)^2 \tag{26}
\]
\[
g_v(\nabla u(x)) = \sqrt{1 + |\nabla u(x)|^2} - (-\tilde{v} \cdot \nabla u(x) + v_3)^2 \tag{27}
\]
For smooth surfaces, we have $\sigma = 0$ and the ON-model brings back to the L-model. In the particular case $\omega = V = (0, 0, 1)$, or, more precisely, when $\cos(\varphi_r - \varphi_i) \leq 0$ (e.g. the case when the unit vectors $\omega$ and $V$ are perpendicular we get $\cos(\varphi_r - \varphi_i) = -1$) the equation (17) simplifies and reduces to a L-model scaled by the coefficient $A$. This means that the model is more general and flexible than the L-model. This happens when only one of the two unit vectors is zero or, more in general, when the dot product between the normalized projections onto the $(x_1, x_2)$-plane of $\omega$ and $V$ is equal to zero.

Also for this diffuse model we neglect the ambient component. Then, we get $k_D = 1$ and, as a consequence, in the general equation (5) the total light intensity $I(x)$ is equal to the only diffuse component $I_D(x)$, in this case described by the equation (17). Hence, for what follow, we will write $I(x)$ instead of $I_D(x)$.

To deal with this equation one has to compute the min and max operators which appear in (17) and (20). Hence, we must consider several cases (and for simplicity we set the albedo $\gamma_D(x) = 1$):

**Case 1:** $\theta_i \geq \theta_r$ and $(\varphi_r - \varphi_i) \in [0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi]$ 

The brightness equation (17) becomes

$$I(x) = \cos(\theta_i) \left( A + B \sin(\theta_i) \frac{\sin(\theta_r)}{\cos(\theta_r)} \cos(\varphi_r - \varphi_i) \right)$$

(28)

and by using the formulas (21)-(25) we arrive to the following first order nonlinear Hamilton-Jacobi equation

$$I(x)(\sqrt{1 + |\nabla u(x)|^2}) + A(\bar{\omega} \cdot \nabla u(x) - \omega_3) - B\frac{(\bar{\omega} \cdot \bar{v}) g_{\omega}(\nabla u(x)) g_{\omega}(\nabla u(x)) - (\bar{v} \cdot \nabla u(x) + v_3)}{\sqrt{1 + |\nabla u(x)|^2}} = 0,$$

(29)

where $\bar{\omega} = (\omega_1, \omega_2)$ and $\bar{v} = (v_1, v_2)$.

**Case 2:** $\theta_i < \theta_r$ and $(\varphi_r - \varphi_i) \in [0, \frac{\pi}{2}) \cup (\frac{3}{2}\pi, 2\pi]$ 

In this case the brightness equation (17) becomes

$$I(x) = \cos(\theta_i) \left( A + B \sin(\theta_i) \frac{\sin(\theta_r)}{\cos(\theta_r)} \cos(\varphi_r - \varphi_i) \right)$$

(30)

and by using again the formulas (21)-(25) we get

$$I(x)(1 + |\nabla u(x)|^2) + A(\bar{\omega} \cdot \nabla u(x) - \omega_3) \sqrt{1 + |\nabla u(x)|^2}$$

$$- B(\bar{\omega} \cdot \bar{v}) g_{\omega}(\nabla u(x)) g_{\omega}(\nabla u(x)) = 0,$$

(31)
that is again a first order nonlinear Hamilton-Jacobi equation.

**Case 3:** \( \forall \theta_i, \theta_r \) and \((\phi_r - \phi_i) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\)

In this case we have the implication \( \max(0, \cos(\phi_r - \phi_i)) = 0 \). The brightness equation (17) simplifies in

\[
I(x) = A \cos(\theta_i)
\]

and the Hamilton-Jacobi equation associated to it becomes consequentially

\[
I(x)(\sqrt{1 + |\nabla u(x)|^2}) + A(\tilde{\omega} \cdot \nabla u(x) - \omega_3) = 0,
\]

that is equal to the L–model scaled by the coefficient \( A \).

**Case 4:** \( \theta_i = \theta_r \) and \( \phi_r = \phi_i \)

This is a particular case when the position of the light source \( \omega \) coincides with the observer direction \( V \) but there are not on the vertical axis. This choice implies \( \max[0, \cos(\phi_i - \phi_r)] = 1 \), then defining \( \theta := \theta_i = \theta_r = \alpha = \beta \), the equation (17) simplifies to

\[
I(x) = \cos(\theta) \left( A + B \sin(\theta)^2 \cos(\theta)^{-1}\right)
\]

and we arrive to a first order nonlinear Hamilton-Jacobi equation

\[
(I(x) - B)(\sqrt{1 + |\nabla u(x)|^2}) + A(\tilde{\omega} \cdot \nabla u(x) - \omega_3)

+ B \frac{(\tilde{\omega} \cdot \nabla u(x) + \omega_3)^2}{\sqrt{1 + |\nabla u(x)|^2}} = 0.
\]

**The vertical light case.**

If \( \omega = (0,0,1) \), regardless of the position of \( V \), the first three cases of the previous four cases are reduced to only one to which we can associate the following PDE to the brightness equation (17)

\[
I(x) = \frac{A}{\sqrt{1 + |\nabla u(x)|^2}}
\]

In this way we can put it in the following eikonal type equation, analogous to the Lambertian eikonal equation (10):

\[
|\nabla u(x)| = f(x) \quad \text{for} \quad x \in \Omega,
\]

where

\[
f(x) = \sqrt{\frac{A^2}{I(x)^2} - 1}.
\]
Following [52], we write the surface as
\[ S(x, z) = z - u(x) = 0, \text{ for } x \in \Omega, \]
where \( u(x) = (\nabla u(x), 1) \), so (35) becomes
\[
(I(x) - B)|\nabla S(x, z)| + A(\nabla S(x, z) \cdot \omega) \nonumber \\
+ B \left( \frac{\nabla S(x, z)}{|\nabla S(x, z)|} \cdot \omega \right)^2 |\nabla S(x, z)| = 0.
\]
Defining
\[
d(x, z) = \frac{\nabla S(x, z)}{|\nabla S(x, z)|} \nonumber \tag{40}
\]
and
\[
c(x, z) = I(x) - B + B(d(x, z) \cdot \omega)^2, \nonumber \tag{41}
\]
using the equivalence
\[
|\nabla S(x, z)| \equiv \max_{a \in \partial B_3} \{a \cdot \nabla S(x, z)\} \nonumber \tag{42}
\]
we get
\[
\max_{a \in \partial B_3} \{c(x, z) a \cdot \nabla S(x, z) - A \omega \cdot \nabla S(x, z)\} = 0. \nonumber \tag{43}
\]
Let us define the vector field for the ON-model
\[
b^{ON}(x, a) = \frac{(c(x, z) a_1 - A \omega_1, c(x, z) a_2 - A \omega_2)}{A \omega_3}, \nonumber \tag{44}
\]
and
\[
f^{ON}(x, z, a, v(x)) = -\frac{c(x, z) a_3}{A \omega_3} (1 - \mu v(x)) + 1. \nonumber \tag{45}
\]
Then, introducing the exponential transform \( \mu v(x) = 1 - e^{-\mu u(x)} \) as already done for the L-model, we can finally write the fixed point problem in the new variable \( v \) obtaining the

\[
\text{Oren-Nayar Model} \nonumber \\
\{ \begin{array}{l}
\mu v(x) = T^{ON}(x, v(x), \nabla v), \quad x \in \Omega, \\
v(x) = 0, \quad x \in \partial \Omega. 
\end{array} \nonumber \tag{46}
\]
where
\[
T^{ON}(:) := \min_{a \in \partial B_3} \{b^{ON}(x, a) \cdot \nabla v(x) + f^{ON}(x, z, a, v(x))\}
\]
Note that the simple homogeneous Dirichlet boundary condition is due to the flat background behind the object but a condition like \( u(x) = g(x) \) can also be considered if necessary. Moreover, the structure is similar to the previous Lambertian model although the definition of the vector field and cost are different.
In the particular case when $\cos(\varphi_r - \varphi_i) = 0$, the equation (17) simply reduces to

$$I(x) = A \cos(\theta)$$  \hspace{1cm} (47)

and, as a consequence, the Dirichlet problem in the variable $v$ is equal to (46) with $c(x, z) = I(x)$.

5 The Phong model (PH–model)

The PH–model introduces a specular component to the brightness function $I(x)$. As we said in Section 2, this can be described in general as the sum $I(x) = k_A I_A(x) + k_D I_D(x) + k_S I_S(x)$, where $I_A(x)$, $I_D(x)$ and $I_S(x)$ are the ambient, diffuse and specular light component, respectively. We will set for simplicity $k_A = 0$ and represent the diffuse component $I_D(x)$ as the Lambertian reflectance model.

A simple specular model is obtained putting the incidence angle equal to the reflection one and $\omega$, $N(x)$ and $R(x)$ belong to the same plane.

The PH–model is an empirical model that was developed by Phong [40] in 1975. This model describes the specular light component $I_S(x)$ as a power of the cosine of the angle between the unit vectors $V$ and $R(x)$ (it is the vector representing the reflection of the light $\omega$ on the surface), then for the Phong model

$$I_{PH}^S(x) = \gamma_S(x)(R(x) \cdot V)$$  \hspace{1cm} (48)

where $\alpha$ expresses the specular reflection characteristics of a material.

Hence, the brightness equation for the PH–model is

$$I(x) = k_D \gamma_D(x)(N(x) \cdot \omega) + k_S \gamma_S(x)(R(x) \cdot V)$$  \hspace{1cm} (49)

where $\gamma_D(x)$ and $\gamma_S(x)$ represent the diffuse and specular albedo, respectively.

Starting to see in details the PH–model in the case of oblique light source $\omega$ and oblique observer $V$.

Assuming that $N(x)$ is the bisector of the angle between $\omega$ and $R(x)$ (see Fig. 6), we obtain

$$N(x) = \frac{\omega + R(x)}{|\omega + R(x)|}$$  \hspace{1cm} (50)

which implies

$$R(x) = |\omega + R(x)||N(x) - \omega|$$.  \hspace{1cm} (51)

From the parallelogram law, taking into account that $\omega, R(x)$ and $N(x)$ are unit vectors, we can write $|\omega + R(x)| = 2(N(x) \cdot \omega)$, then we can derive
the unit vector \( \mathbf{R}(x) \) as follow:

\[
\mathbf{R}(x) = 2(\mathbf{N}(x) \cdot \omega)\mathbf{N}(x) - \omega
\]

\[
= 2 \left( \frac{-\bar{\omega} \cdot \nabla u(x) + \omega_3}{\sqrt{1 + |\nabla u(x)|^2}} \right) \mathbf{N}(x) - (\omega_1, \omega_2, \omega_3)
\]

(52)

With this definition of the unit vector \( \mathbf{R}(x) \) we have

\[
\mathbf{R}(x) \cdot \mathbf{V} = \left( \frac{-2\bar{\omega} \cdot \nabla u(x) + 2\omega_3}{1 + |\nabla u(x)|^2} \right) (-\nabla u(x), 1) - (\omega_1, \omega_2, \omega_3).
\]

(53)

Then, putting \( \alpha = 1 \), equation (49) becomes

\[
I(x)(1 + |\nabla u(x)|^2)
- k_D \gamma_D(x)(-\nabla u(x) \cdot \bar{\omega} + \omega_3)(\sqrt{1 + |\nabla u(x)|^2})
- 2k_S \gamma_S(x)(-\nabla u(x) \cdot \bar{\omega} + \omega_3)(-\bar{\omega} \cdot \nabla u(x) + \nu_3)
+ k_S \gamma_S(x)(\omega \cdot \nabla \mathbf{V})(1 + |\nabla u(x)|^2) = 0,
\]

(54)

to which we add a Dirichlet boundary condition equal to zero, assuming that the surface is standind on a flat blackground. Note that the cosine in the specular term is usually replaced by zero if \( \mathbf{R}(x) \cdot \mathbf{V} < 0 \) (and in that case we get back to the L–model).

As we have done for the previous models, we write the surface as \( S(x, z) = z - u(x) = 0 \), for \( x \in \Omega, z \in \mathbb{R} \), and \( \nabla S(x, z) = (-\nabla u(x), 1) \), so (54) will be
written as
\[
(I(x) + k_S \gamma_S(x)(\omega \cdot V))|\nabla S(x, z)|^2
- k_D \gamma_D(x)(\nabla S(x, z) \cdot \omega)(|\nabla S(x, z)|)
- 2k_S \gamma_S(x)(\nabla S(x, z) \cdot \omega)(\nabla S(x, z) \cdot V) = 0.
\] (55)

Dividing by $|\nabla S(x, z)|$, defining $d(x, z)$ as in (40) and $c(x) = I(x) + k_S \gamma_S(x)(\omega \cdot V)$, we get
\[
c(x)|\nabla S(x, z)| - k_D \gamma_D(x)(\nabla S(x, z) \cdot \omega)
- 2k_S \gamma_S(x)(\nabla S(x, z) \cdot \omega)(d(x, z) \cdot V) = 0.
\] (56)

By the equivalence $|\nabla S(x, z)| \equiv \max_{a \in \partial B_3} \{a \cdot \nabla S(x, z)\}$ we obtain
\[
\max_{a \in \partial B_3} \{c(x) a \cdot \nabla S(x, z) - k_D \gamma_D(x)(\nabla S(x, z) \cdot \omega)
- 2k_S \gamma_S(x)(\nabla S(x, z) \cdot \omega)(d(x, z) \cdot V)\} = 0.
\] (57)

Defining the vector field
\[
b^{PH}(x, a) := \frac{1}{Q^{PH}(x, z)}M^{PH}(x, z)
\] (58)
where
\[
Q^{PH}(x, z) := 2k_S \gamma_S(x)\omega_3(d(x, z) \cdot V) + k_D \gamma_D(x)\omega_3,
\] (59)
and
\[
M_i^{PH}(x, z) = (c(x)a_i - k_D \gamma_D(x)\omega_i
- 2k_S \gamma_S(x)\omega_i(d(x, z) \cdot V)), i = 1, 2.
\] (60)

Let us also define
\[
f^{PH}(x, z, a, v(x)) = - \frac{c(x) a_3}{Q^{PH}(x, z)}(1 - \mu v(x)) + 1.
\] (61)

Again, using the exponential transform $\mu v(x) = 1 - e^{-\mu u(x)}$ as done for the previous models, we can finally write the nonlinear fixed point problem

\[
\text{Phong Model}
\]
\[
\begin{cases}
\mu v(x) = T^{PH}(x, v(x), \nabla v), & x \in \Omega, \\
v(x) = 0, & x \in \partial \Omega.
\end{cases}
\] (62)

where
\[
T^{PH}(\cdot) := \min_{a \in \partial B_3} \{b^{PH}(x, a) \cdot \nabla v(x) + f^{PH}(x, z, a, v(x))\}.
\]
A. Oblique light source and vertical position of the observer.

In the case of oblique light source $\omega$ and vertical observer $V = (0, 0, 1)$, the dot product $R(x) \cdot V$ becomes

$$R(x) \cdot V = \frac{-2\tilde{\omega} \cdot \nabla u(x) + 2\omega_3}{1 + |\nabla u(x)|^2} - \omega_3$$

$$= \frac{-2\tilde{\omega} \cdot \nabla u(x) + \omega_3(1 - |\nabla u(x)|^2)}{1 + |\nabla u(x)|^2}.$$  \hspace{1cm} (63)

The fixed point problem in $v$ will be equal to (62) with the following choices

$$c(x) := I(x) + \omega_3 k_S \gamma_S(x),$$

$$Q^{PH}(x, z) := 2k_S \gamma_S(x)(d(x, z) \cdot \omega) + k_D \gamma_D(x)\omega_3,$$

$$b^{PH}(x, a) := \frac{(c(x)a_1 - k_D \gamma_D(x)\omega_1, c(x)a_2 - k_D \gamma_D(x)\omega_2)}{Q^{PH}(x, z)}.$$  \hspace{1cm} (64)

B. Vertical light source and oblique position of the observer.

When $\omega = (0, 0, 1)$ the definition of the vector $R(x)$ reported in (52) becomes

$$R(x) = \left( \frac{-2\tilde{\omega} \cdot \nabla u(x) + 2\omega_3}{1 + |\nabla u(x)|^2}, \frac{-2\tilde{\omega} \cdot \nabla u(x)}{1 + |\nabla u(x)|^2} - 1 \right)$$

and, as a consequence, the dot product $R(x) \cdot V$ with general $V$ is

$$R(x) \cdot V = \frac{-2\tilde{\omega} \cdot \nabla u(x) + \omega_3(1 - |\nabla u(x)|^2)}{1 + |\nabla u(x)|^2}.$$  \hspace{1cm} (65)

Hence, the fixed point problem in $v$ is equal to (62) with

$$c(x) := I(x) + \omega_3 k_S \gamma_S(x),$$

$$Q^{PH}(x, z) := 2k_S \gamma_S(x)(d(x, z) \cdot V) + k_D \gamma_D(x),$$

$$b^{PH}(x, a) := \frac{1}{Q^{PH}(x, z)} (c(x)a_1, c(x)a_2).$$  \hspace{1cm} (66)

C. Vertical light source and vertical position of the observer.

If we choose $\omega \equiv V = (0, 0, 1)$ the equation (54) simplifies in

$$I(x)(1 + |\nabla u(x)|^2) - k_S \gamma_S(x)(1 - |\nabla u(x)|^2) = 0.$$  \hspace{1cm} (67)

Working on this equation one can put it in the following eikonal type form, which is analogous to the Lambertian eikonal equation (10):

$$|\nabla u(x)| = f(x) \quad \text{for} \quad x \in \Omega,$$  \hspace{1cm} (68)
where now
\[
f(x) := \frac{k_D^2 \gamma_D(x)^2 - 2I_+(x)I_- (x) + k_D^2 \gamma_D(x)^2 \sqrt{Q(x)}}{2 \left( I(x) + k_S \gamma_S(x) \right)},
\]
with
\[
\begin{align*}
I_+(x) &:= I(x) + k_S \gamma_S(x), \\
I_-(x) &:= I(x) - k_S \gamma_S(x), \\
Q(x) &:= k_D^2 \gamma_D(x)^2 + 8k_S^2 \gamma_S(x)^2 + 8I(x)k_S \gamma_S(x).
\end{align*}
\]

### 6 Semi-Lagrangian Approximation

Now, let us state a general convergence theorem suitable for the class of
differential operators appearing in the models described in the previous
sections. As we noticed, the unified approach presented in this paper has
the big advantage to give a unique formulation for the three models in the form
of a fixed point problem
\[
\mu v(x) = T^M(x, v, \nabla v), \quad \text{for } x \in \Omega,
\]
where $M$ indicates the model, i.e. $M = L, ON, PH$.

We will see that the discrete operators of the ON–model and the PH–
model described in the previous sections satisfy the properties listed here.
In order to obtain the fully discrete approximation we will adopt the semi-
Lagrangian approach described in the book by Falcone and Ferretti [16],
see also [9] for a similar approach to various image processing problems
(including nonlinear diffusions models and segmentation).

Let $W_i = w(x_i)$ so that $W$ will be the vector solution giving the
approximation of the height $u$ at every node $x_i$ of the grid. Note that in one
dimension the index $i$ is an integer number, in two dimensions $i$ denotes a
multi-index, $i = (i_1, i_2)$. We consider a semi-Lagrangian scheme written in
a fixed point form, so we will write the fully discrete scheme as
\[
W_i = \hat{T}_i(W).
\]

Denoting by $G$ the global number of nodes in the grid, the operator cor-
responding to the oblique light source is $\hat{T} : \mathbb{R}^G \to \mathbb{R}^G$ that is defined
componentwise by
\[
\hat{T}_i(W) := \min_{a \in \partial B_i} \left\{ e^{-\mu h} I[W](x_i^+) - \tau F(x_i, z, a) \right\} + \tau
\]
\[
(76)
\]
where \( I[W] \) represents an interpolation operator based on the values at the grid nodes and

\[
x_i^+ := x_i + hb(x_i, a) \quad (77)
\]

\[
\tau := (1 - e^{-\mu h})/\mu \quad (78)
\]

\[
F(x_i, z, a) := P(x_i, z)a_3(1 - \mu W_i) \quad (79)
\]

\[
P: \Omega \times \mathbb{R} \to \mathbb{R} \text{ is continuous and nonnegative.} \quad (80)
\]

Since \( w(x_i + hb(x_i, a)) \) is approximated via \( I[W] \) by interpolation on \( W \) (which is defined on the grid \( G \)), it is important to use a monotone interpolation in order to preserve the properties of the continuous operator \( T \) in the discretization. To this end, the typical choice is to apply a piecewise linear (or bilinear) interpolation which gives

\[
w(x_i + hb(x_i, a)) = \sum_j \lambda_{ij}(a)W_j \quad (81)
\]

where

\[
\sum_j \lambda_{ij}(a) = 1 \quad \text{for} \quad x_i + hb(x_i, a) = \sum_j \lambda_{ij}(a)x_j. \quad (82)
\]

A simple explanation for (81)-(82) is that the coefficients \( \lambda_{ij}(a) \) represent the local coordinates of the point \( x_i + hb(x_i, a) \) with respect to the grid nodes (see [16] for more details and other choices of interpolation operators).

Note that here \( \mu \) is a free positive parameter without a specific physical meaning in the SfS problem, but it plays an important role because varying \( \mu \) it is possible to modify the slope in the change of variable \( \mu v(x) = 1 - e^{-\mu u(x)} \) (the slope increases for increasing values of \( \mu \)) so that the interval \([0, +\infty] \) is squeezed to \([0, 1] \).

Comparing (76) with its analogue for the vertical light case we can immediately note that the former has the additional term \( \tau F(x_i, z, a) \) which requires analysis.

**Theorem 6.1** Let \( \hat{T}_i(W) \) the \( i \)-th component of the operator defined as in (76). Then, the following properties hold true:

1. Let

\[
\bar{\sigma}_3 \equiv \arg \min_{a \in \partial B_3} \{ e^{-\mu h}w(x_i + hb(x_i, a)) - \tau F(x_i, z, a) \} \text{ and assume}
\]

\[
P(x_i, z)\bar{\sigma}_3 \leq 1. \quad (83)
\]

Then \( 0 \leq W \leq \frac{1}{\mu} \) implies \( 0 \leq \hat{T}(W) \leq \frac{1}{\mu} \).

2. \( \hat{T} \) is a monotone operator, i.e., \( V \leq U \) implies \( \hat{T}(V) \leq \hat{T}(U) \).
3. \( \hat{T} \) is a contraction mapping in \( L^\infty([0, 1/\mu]^G) \) if
\[ P(x_i, z)\bar{\sigma}_3 < \mu. \]

Proof.

1. To prove that \( W \leq \frac{1}{\mu} \) implies \( T(W) \leq \frac{1}{\mu} \) we just note that
\[
\hat{T}(W) \leq \frac{e^{-\mu h}}{\mu} + \tau = \frac{1}{\mu}. \quad (84)
\]
Let \( W \geq 0 \); then
\[
\hat{T}(W) \geq -\tau P(x_i, z)\bar{\sigma}_3(1 - \mu W_i) + \tau = \tau (1 - P(x_i, z)\bar{\sigma}_3(1 - \mu W_i)). \quad (85)
\]
This implies that \( \hat{T}(W) \geq 0 \) if \( P(x_i, z)\bar{\sigma}_3 \leq 1 \) since \( 0 \leq 1 - \mu W_i \leq 1 \).

2. In order to prove that \( \hat{T} \) is monotone, let us observe first that \( v(x) \leq u(x) \) for every \( x \in \Omega \) implies
\[
e^{-\mu h}[v(x + hb(x, a^*)) - u(x + hb(x, \bar{a}))] - \tau P(x, z)(a^*_3(1 - \mu v(x)) - \bar{\sigma}_3(1 - \mu u(x)))
\]
\[
\leq e^{-\mu h}[v(x + hb(x, \bar{a})) - u(x + hb(x, \bar{a}))] + \tau P(x, z)\bar{\sigma}_3(v(x) - u(x)) \quad (86)
\]
where \( a^* \) and \( \bar{a} \) are the two arguments corresponding to the minimum on \( a \) of the expression
\[
e^{-\mu h}w(x + hb(x, a)) - \tau P(x, z)a_3(1 - \mu w(x)) \quad (87)
\]
respectively for \( w = v, u \). Hence,
\[
\hat{T}_i(V) - \hat{T}_i(U) \leq e^{-\mu h}[I_1[v](x_i + hb(x_i, \bar{a})) - I_1[u](x_i + hb(x_i, \bar{a})) + \tau P(x_i, z)\bar{\sigma}_3(V_i - U_i)] \quad (88)
\]
Since the linear interpolation \( I_1 \) is monotone, we can conclude that for \( V \leq U \), \( \hat{T}(V) - \hat{T}(U) \leq 0 \). Note that this property does not require condition \([83]\) to be satisfied.

3. Let us consider now two vectors \( V \) and \( U \) dropping the condition \( V \leq U \) and assuming
\[
P(x_i, z)\bar{\sigma}_3 < \mu. \quad (89)
\]
To prove that $T$ is a contraction mapping note that following the same argument used to prove the second statement we can obtain (88). Then, by applying the definition of $I$, we get

$$\hat{T}(V) - \hat{T}(U) \leq \left( e^{-\mu h} + \tau P(x_i, z) \bar{\pi}_3 \right) \|V - U\|_\infty. \quad (90)$$

Reversing the role of $V$ and $U$, one can also obtain

$$\hat{T}(U) - \hat{T}(V) \leq \left( e^{-\mu h} + \tau P(x_i, z) \bar{\pi}_3 \right) \|V - U\|_\infty \quad (91)$$

and conclude then that $\hat{T}$ is a contraction mapping in $L^\infty$ if and only if

$$e^{-\mu h} + \tau P(x_i, z) \bar{\pi}_3 < 1 \quad (92)$$

and this holds true if the bound (89) is satisfied.

Let us consider now the algorithm based on the fixed-point iteration

$$\begin{cases}
W^n = \hat{T}(W^{n-1}), \\
W^0 \text{ given.}
\end{cases} \quad (93)$$

We can state the following convergence result

**Theorem 6.2** Let $W^k$ be the sequence generated by (93). Then the following results hold:

1. Let $W^0 \in S = \{ W \in \mathbb{R}^G : W \geq \hat{T}(W) \}$, then the $W^k$ converges monotonically decreasing to a fixed point $W^*$ of the $\hat{T}$ operator;
2. Let $\mu = 1$ and let the condition $P(x_i, z) \bar{\pi}_3 \leq \mu$ be satisfied. Then, $W^k$ converges to the unique fixed point $W^*$ of the $\hat{T}$. Moreover, if $W^0 \in S$ the convergence is monotone decreasing.

**Proof.**

1. Starting from a point in the set of super solutions $S$, the sequence is non increasing and lives in $S$ which is a closed set bounded from below (by 0). Then, $W^k$ converges and the limit is necessarily a fixed point for $\hat{T}$.
2. The assumptions guaranteed by Theorem 6.1 are satisfied and $\hat{T}$ is a contraction mapping in $[0, 1]$, so the fixed point is unique. The monotonicity of $\hat{T}$ implies that starting from $W^0 \in S$ the convergence is monotone decreasing.
Note that monotonicity can be useful to improve the speed of convergence as in [15]. Another way to improve convergence is to apply Fast Sweeping or Fast Marching methods as illustrated in [50, 51]. A crucial role is played by boundary conditions on the boundary of $\Omega$ usually we impose the homogeneous Dirichlet boundary condition, $v = 0$. This condition implies that the shadows must not cross the boundary of $\Omega$, so the choice $\omega_3 = 0$ corresponding to an infinite shadow behind the surface is not admissible. However, other choices are possible: to impose the height of the surface on $\partial\Omega$ we can set $v = g$ or to use a more neutral boundary condition we can impose $v = 1$ (state constraint boundary condition). More informations on the use of boundary conditions for these type of problems can be found in [16].

6.1 Properties of the discrete operator $\hat{T}^{ON}$ and $\hat{T}^{PH}$

We consider a semi-Lagrangian discretization of (46) written in a fixed point form, so we will write the SL fully discrete scheme for the ON–model as

$$W_i = \hat{T}^{ON}_i(W),$$

(94)

where $ON$ is the acronym identifying the ON–model. Using the same notations of the previous section, the operator corresponding to the oblique light source is $\hat{T}^{ON} : \mathbb{R}^G \rightarrow \mathbb{R}^G$ that with linear interpolation can be written as

$$\hat{T}^{ON}_i(W) = \min_{a \in \partial B_3} \{ e^{-\mu h} I_1(W)(x^+_i) - \tau F^{ON}(x_i, z, a) \} + \tau,$$

(95)

where

$$\tau := \frac{1 - e^{-\mu h}}{\mu}$$

$$b^{ON}(x_i, a) := \frac{1}{A\omega_3} (c(x_i, z)a_1 - A\omega_1, c(x_i, z)a_2 - A\omega_2)$$

$$c(x_i, z) := I(x_i) - B + B(d(x_i, z) \cdot \omega)^2$$

$$d(x_i, z) := \nabla S(x_i, z)/|\nabla S(x_i, z)|$$

$$F^{ON}(x_i, z, a) := P^{ON}(x_i, z)a_3(1 - \mu W_i)$$

$$P^{ON}(x_i, z) := \frac{c(x_i, z)}{A\omega_3}.$$  

(96)

Note that, in general, $P^{ON}$ will not be positive but that condition can be obtained tunning the parameter $\sigma$ since the coefficients $A$ and $B$ depend on $\sigma$. This explains why in some test we will not be able to get convergence for every value of $\sigma \in [0, \pi/2)$. Once the non negativity of $P^{ON}$ is guaranteed,
we can follow the same arguments of Theorem (6.1) to check that the discrete operator \( \hat{T}^{ON} \) satisfies the three properties which are necessary to guarantee convergence as in Theorem 6.2 provided we set \( P = P^{ON} \) in that statement.

For the Phong model, the semi-Lagrangian discretization of (62) written in a fixed point form gives

\[
W_i = \hat{T}_i^{PH}(W),
\]

where \( \hat{T}_i^{PH} : \mathbb{R}^G \rightarrow \mathbb{R}^G \), that is defined componentwise by

\[
\hat{T}_i^{PH}(W) = \min_{a \in \partial B_3} \left\{ e^{-\mu h} I_1(W)(x_i^+) - \tau F^{PH}(x_i, z, a) \right\} + \tau,
\]

where, in the case of oblique light source and vertical position of the observer,

\[
\tau := 1 - e^{- \mu h},
\]

\[
b^{PH}(x_i, a) := \frac{(c(x_i)b_1 - k_D \gamma_D \omega_1, c(x_i)b_2 - k_D \gamma_D \omega_2)}{Q^{PH}(x_i, z)}
\]

\[
c(x_i) := I(x_i) + \omega_3 k_S \gamma_S
\]

\[
d(x_i, z) := \nabla S(x_i, z)/|\nabla S(x_i, z)|
\]

\[
Q^{PH}(x_i, z) := 2k_S \gamma_S (d(x_i, z) \cdot \omega) + k_D \gamma_D \omega_3
\]

\[
F^{PH}(x_i, z, a) := P^{PH}(x_i, z) a_3 (1 - \mu W_i)
\]

\[
P^{PH}(x_i, z) := \frac{c(x_i)}{Q^{PH}(x_i, z)}.
\]

Here the model has less parameters and \( P^{PH} \) will always be nonnegative. Again, following the same arguments of Theorem 6.1 we can check that the discrete operator \( \hat{T}^{PH} \) satisfies the three properties which are necessary to guarantee convergence as in Theorem (6.2) provided we set \( P = P^{PH} \) in that statement.

7 Numerical Simulations

In this section we show some numerical experiments on synthetic and real images in order to analyze the behavior of the parameters involved in the ON–model and the PH–model and to compare the performances of these models with respect to the classical L–model. All the numerical tests in this section have been implemented in language C++. The computer used for the simulations is a MacBook Pro 13" Intel Core 2 Duo with speed of 2.66 GHz and 4GB of RAM (so the CPU times in the tables refer to this specific architecture).

We denote by \( G \) the discrete grid in the plane getting back to the double index notation \( x_{ij}, G := \text{card}(G) = n \times m \). We define \( G_{in} := \{ x_{ij} : x_{ij} \in \)
\[ \Omega \] as the set of grid points inside \( \Omega \); \( G_{\text{out}} := G \setminus G_{\text{in}} \). The boundary \( \partial \Omega \) will be then approximated by the nodes such that at least one of the neighboring points belongs to \( G_{\text{in}} \). For each image we define a map, called \textit{mask} representing the pixels \( x_{ij} \in G_{\text{in}} \) in white and the pixels \( x_{ij} \in G_{\text{out}} \) in black. In this way it is easy to distinguish the nodes that we have to use for the reconstruction (the nodes inside \( \Omega \)) and the nodes on the boundary \( \partial \Omega \) (see e.g. Fig. 7(b)).

### 7.1 Synthetic tests

All the synthetic images are defined on the same rectangular domain containing the support of the image, \( \Omega \equiv [-1, 1] \times [-1, 1] \). We can easily modify the number of the pixels choosing different values for the steps in space \( \Delta x \) and \( \Delta y \). All the synthetic images have 256 \( \times \) 256 pixels, unless otherwise specified. \( X \) and \( Y \) represent the real size (e.g. for \( \Omega \equiv [-1, 1] \times [-1, 1] \), \( X = 2, Y = 2 \)). Moreover, we choose the value of the tolerance \( \eta \) for the iterative process equal to \( 10^{-8} \) for the tests on synthetic images. We will see that dealing with real images we need to use a greater value of \( \eta \).

**Test 1: Sphere.** The semisphere visible in Fig. 7 is defined on \( G_{\text{in}} := \{(x, y) : x^2 + y^2 \leq r^2 \} \), as

\[
\begin{cases}
  u(x, y) = \sqrt{r^2 - x^2 - y^2} & (x, y) \in G_{\text{in}}, \\
  u(x, y) = 0 & (x, y) \in G_{\text{out}},
\end{cases}
\]

where

\[
r = \frac{\min\{X, Y\}}{2} + 2 \hat{\delta},
\]

and \( \hat{\delta} = \max\{\Delta x, \Delta y\} \).

As example, we can see in Fig. 7 the input image, the corresponding mask and the surface reconstructed by the L–model.

![Figure 7: Sphere via the L–model: (a) Input image; (b) Mask; (c) surface.](image)

The values of the parameters used in the simulations are indicated in Table 1. Note (in Table 3) that when the specular component is zero for the PH–model, we just have the contribution of the diffuse component so we have...
exactly the same error values of the L–model, as expected. By increasing the value of the coefficient \( k_S \) and, as a consequence, decreasing the value of \( k_D \), in the PH–model the \( L^2(I) \) and \( L^\infty(I) \) errors on the image grow albeit slightly and still remain of the same order of magnitude, whereas the errors on the surface decrease. For the ON–model, the same phenomenon appears when we set the roughness parameter \( \sigma \) to zero: we bring back to the L–model and, hence, we obtain the same errors on the image and the surface. Note that the errors in \( L^2(I) \) and \( L^\infty(I) \) norm for the image and the surface, decrease by increasing the value of \( \sigma \).

| Model  | \( \sigma \) | \( k_D \) | \( k_S \) | \( \alpha \) |
|--------|-------------|------------|------------|------------|
| LAM    |             |            |            |            |
| ON-00  | 0           |            |            |            |
| ON-04  | 0.4         |            |            |            |
| ON-08  | 0.8         |            |            |            |
| PH-s00 | 1           | 0          | 1          |            |
| PH-s04 | 0.6         | 0.4        | 1          |            |
| PH-s08 | 0.2         | 0.8        | 1          |            |

Table 1: Sphere: parameter values used in the models.

In Table 2 we reported the number of iterations and the CPU time (in seconds) referred to the three models with the parameter indicated in Table 1. For all the models, also varying the parameters involved, the number of iterations is always about 2000 and the CPU time slightly greater than 2 seconds, so the computation is really fast.

| SL–Schemes | Iter. | [s]    |
|------------|-------|--------|
| LAM        | 2001  | 2.14   |
| ON-00      | 2001  | 2.06   |
| ON-04      | 2020  | 2.24   |
| ON-08      | 2016  | 2.24   |
| PH-s00     | 2001  | 2.11   |
| PH-s04     | 2008  | 2.45   |
| PH-s08     | 2056  | 2.27   |

Table 2: Synthetic sphere: iterations and CPU time for the models with vertical light source \( \omega = (0,0,1) \).

Test 2: Ridge tent. In tests on synthetic images, the relevance of the choice of a model depends on which model was used to compute the images. In the previous test, the parameters that are used for the 3D reconstruction are identical to those used to compute the synthetic sphere input images, so there is a perfect match. However, for real applications, it is relevant to
Table 3: Synthetic sphere: $L^2, L^\infty$ errors with vertical light source $\omega = (0, 0, 1)$.

| SL-Schemes | $L^2(I)$ | $L^\infty(I)$ | $L^2(S)$ | $L^\infty(S)$ |
|------------|---------|---------------|---------|---------------|
| LAM        | 0.0046  | 0.0431        | 0.0529  | 0.0910        |
| ON-00      | 0.0046  | 0.0431        | 0.0529  | 0.0910        |
| ON-04      | 0.0039  | 0.0353        | 0.0513  | 0.0882        |
| ON-08      | 0.0035  | 0.0314        | 0.0506  | 0.0881        |
| PH-s00     | 0.0046  | 0.0431        | 0.0529  | 0.0910        |
| PH-s04     | 0.0064  | 0.0471        | 0.0511  | 0.0896        |
| PH-s08     | 0.0090  | 0.0706        | 0.0386  | 0.0752        |

examine the influence of an error in the parameter values. To this end we can produce an input image with the Oren-Nayar model using $\sigma = 0.1$ and then process this image with the same model using a different value of $\sigma$ to see how the results are affected by this error. This is what we are going to do for the ridge tent. Let consider the ridge tent defined by the following equation

$$
\begin{align*}
  u(x, y) &= \min \left\{ -2|x| + \frac{4}{5}X, -|y| + \frac{2}{5}Y \right\} & (x, y) \in G_{in}, \\
  u(x, y) &= 0 & (x, y) \in G_{out},
\end{align*}
$$

(102)

where

$$
G_{in} := \left\{ (x, y) : \frac{x}{X}, \frac{y}{Y} < \frac{2}{5} \right\}.
$$

In Fig. 8 we can see an example of reconstruction obtained by using the ON–model with $\sigma = 0.3$, under a vertical light source $\omega = (0, 0, 1)$. A first remark

![Tent Input](a) Tent Input ![Tent surface](b) Tent surface

Figure 8: Tent via the ON–model with $\sigma = 0.3$: (a) Input image; (b) 3D reconstruction.

is that the surface reconstruction is good even if in this case the surface is not differentiable, also note that there are no oscillations near the kinks where
there is a jump in the gradient direction. Let us examine the stability with respect to the parameters. We have produced seven input images for the ridge tent, all of size $256 \times 256$, with the following combinations of models and parameters:

LAM Lambertian model;
ON1 Oren-Nayar model with $\sigma = 0.1$;
ON3 Oren-Nayar model with $\sigma = 0.3$;
ON5 Oren-Nayar model with $\sigma = 0.5$;
PH1 Phong model with $\alpha = 1$ and $k_S = 0.1$;
PH3 Phong model with $\alpha = 1$ and $k_S = 0.3$;
PH5 Phong model with $\alpha = 1$ and $k_S = 0.5$.

Then we have computed the surfaces corresponding to all the parameter choices (i.e. matching and not matching the first choice). The results obtained in this way have been compared in terms of $L^2$ and $L^\infty$ norm errors with respect to the original surface. The errors obtained by the ON–Model are shown in Table 4 and in Table 5 for the PH–model.

Table 4: Ridge tent, ON–model: $L^2, L^\infty$ errors for the surface. In each column the model used to produce the input image, in the row the model used for the 3D reconstruction.

|       | LAM  | ON1  | ON3  | ON5  |
|-------|------|------|------|------|
| $L^2$ |      |      |      |      |
| LAM   | 0.0067 | 0.0172 | 0.0933 | 0.1920 |
| ON1   | 0.0082 | 0.0068 | 0.0821 | 0.1801 |
| ON3   | 0.0832 | 0.0700 | 0.0086 | 0.1033 |
| ON5   | 0.1946 | 0.1784 | 0.0923 | 0.0067 |
| $L^\infty$ |      |      |      |      |
| LAM   | 0.0094 | 0.0315 | 0.1942 | 0.4060 |
| ON1   | 0.0199 | 0.0093 | 0.1701 | 0.3805 |
| ON3   | 0.1784 | 0.1507 | 0.0118 | 0.2156 |
| ON5   | 0.4104 | 0.3769 | 0.1976 | 0.0094 |

Analyzing the errors in Table 4 and Table 5, we can observe that using the same model to generate the input image and to reconstruct the surface is clearly the optimal choice. The errors on the surface grow more as we consider a parameter $\sigma$ other than the one used to generate the input image.
Table 5: Ridge tent, PH-model: $L^2, L^\infty$ errors for the surface. In each column the model used to produce the input image, on the row the model used for the 3D reconstruction.

|       | $L^2$ |       |       |       |
|-------|-------|-------|-------|-------|
|       | LAM   | PH1   | PH3   | PH5   |
| LAM   | 0.0067| 0.0841| 0.2867| 0.6146|
| PH1   | 0.0586| 0.0067| 0.1996| 0.5031|
| PH3   | 0.1403| 0.0955| 0.0073| 0.2687|
| PH5   | 0.1915| 0.1664| 0.0976| 0.0060|

|       | $L^\infty$ |       |       |       |
|-------|-------------|-------|-------|-------|
|       | LAM         | PH1   | PH3   | PH5   |
| LAM   | 0.0094      | 0.1740| 0.6068| 1.3123|
| PH1   | 0.1243      | 0.0108| 0.4202| 1.0741|
| PH3   | 0.3167      | 0.2245| 0.0149| 0.5718|
| PH5   | 0.4503      | 0.4241| 0.2907| 0.0093|

as data for the 3D reconstruction. For the ON–model we lose one or two order of magnitude, depending on the “distance” of the parameter from the source model. For the PH–model we can observe that the $L^2$ and $L^\infty$ errors grow more as we consider a different $k_S$ for the generation of the image and for the reconstruction, losing one or two order of magnitude. However, the two models seem to be rather stable with respect to a variation of the parameters since the errors do not increase dramatically varying the parameters.

**Test 3: Concave/convex ambiguity for the ON–model.** We consider this test in order to show that the ON–model is not able to overcome the concave/convex ambiguity typical of the SfS problem although it is a model more realistic than the classical L–model. Let consider the following function

$$u(x, y) = \begin{cases} 
- (1 - (x^2 - y^2))^2 + 1, & \text{if } (x^2 + y^2) < 2, \\
0, & \text{otherwise.}
\end{cases} \quad (103)$$

Looking at Fig. 9 we can note that the scheme chooses the maximal viscosity solution, which does not coincide with the original surface. In order to obtain a reconstruction closer to the original surface, we fix the value in the origin at zero. In this way we forced scheme to converge to a solution different from the maximal solution (see Fig. 9(c)).

**Test 4: Concave/convex ambiguity for the PH–model.** The fourth and last synthetic numerical experiment is related to the sinusoidal function
Figure 9: Example of concave/convex ambiguity for the ON–model: (a) Original Surface; (b) Maximal Solution; (c) Approximated surface with value in the origin equal to zero.

defined as follow

\[
\begin{aligned}
  u(x, y) &= 0.5 + 0.5 \sin(\frac{\pi x}{\Delta x}) \sin(\frac{\pi y}{\Delta y}), \\
  &\quad (x, y) \in G_{in}, \\
  u(x, y) &= 0, \\
  &\quad (x, y) \in G_{out}.
\end{aligned}
\]  

(104)

With this test we want to show that also the PH–model is not able to overcome the concave/convex ambiguity typical of the Sfs problem.

---

Figure 10: Synthetic sinusoidal function: example of concave/convex ambiguity.

Fig. [1] shows the results related to the PH–model with \( k_S = 0.5, 0.8 \). We can see that even varying the parameters \( k_D \) and \( k_S \) the SL method always chooses the maximal solution.
7.2 Real tests

In this subsection we consider real input images. We start considering the golden mask of Agamemnon taken from [53] and then modified in order to get a picture in gray tones. The size of the modified image really used is $507 \times 512$. The input image is visible in Fig. 11(a), the associated mask used for the 3D reconstruction in Fig. 11(b). For the real cases, the input image is the same for all the models and we can compute only errors on the images because we do not know the height of the original surface.

![Figure 11: Agamemnon images (size 507 x 512): (a) Input image; (b) Mask.](image)

**Test 5: Agamemnon mask.** For this test we will compare the results regarding 3D reconstruction of the surface obtained with a vertical light source $\omega_{\text{vert}} = (0, 0, 1)$ and an oblique light source $\omega_{\text{obl}} = (0, 0.995, 0.995)$. The values of the parameters used in this test are reported in Table 6. For a vertical light source, we refer to Table 7 for the number of iterations and the CPU time and to Table 8 for the errors obtained with a tolerance $\eta = 10^{-8}$.

| Model | $\sigma$ | $k_D$ | $k_S$ | $\alpha$ |
|-------|---------|-------|-------|---------|
| LAM   |         |       |       |         |
| ON-00 | 0       |       |       |         |
| ON-04 | 0.4     |       |       |         |
| ON-08 | 0.8     |       |       |         |
| ON-10 | 1       |       |       |         |
| PH-s00| 1       | 0     | 1     |         |
| PH-s04| 0.6     | 0.4   | 1     |         |
| PH-s08| 0.2     | 0.8   | 1     |         |
| PH-s10| 0       | 1     | 1     |         |
for the stopping rule of the iterative process. Clearly the number of iteration and the errors of the two non-Lambertian models are the same of the classical Lambertian model when \( \sigma \) for the ON–model and \( k_S \) for the PH–model are equal to zero. In all the other cases, the non-Lambertian models are faster in terms of CPU time and need a lower number of iterations with respect to the L–model.

Table 7: Real Agamemnon mask: iterations and CPU time for the models with vertical light source \( \omega = (0, 0, 1) \).

| SL–Schemes | Iter. | [s]   |
|------------|-------|-------|
| LAM        | 3921  | 24.48 |
| ON-00      | 3921  | 15.98 |
| ON-04      | 2751  | 12.48 |
| ON-08      | 1943  | 11.41 |
| ON-10      | 1818  | 8.79  |
| PH-s00     | 3921  | 18.23 |
| PH-s04     | 2127  | 9.89  |
| PH-s08     | 1476  | 6.90  |
| PH-s10     | 1325  | 6.33  |

In Table 8 we can observe that the \( L^2 \) errors produced by the ON–model increase by increasing the value of \( \sigma \). However, the \( L^\infty \) errors are lower than the error obtained with \( \sigma = 0 \) which coincides with the Lambertian model. With respect to the PH–model, all the errors increase by increasing the value of the parameter \( k_S \), as observed for synthetic images.

Table 8: Real Agamemnon mask: \( L^2, L^\infty \) errors with vertical light source \( \omega = (0, 0, 1) \).

| SL–Schemes | \( L^2(I) \) | \( L^\infty(I) \) |
|------------|---------------|-------------------|
| LAM        | 0.0371        | 0.4745            |
| ON-00      | 0.0371        | 0.4745            |
| ON-04      | 0.0375        | 0.4627            |
| ON-08      | 0.0440        | 0.4627            |
| ON-10      | 0.0501        | 0.4627            |
| PH-s00     | 0.0371        | 0.4745            |
| PH-s04     | 0.0383        | 0.4824            |
| PH-s08     | 0.0391        | 0.4941            |
| PH-s10     | 0.0393        | 0.5098            |

In Fig. 12 we can see the output image and the 3D reconstruction in a single case for each models.

For the oblique light case, we will consider the values for the parameters reported in Table 9.
Figure 12: Agamemnon mask: results with vertical light source. On the first row the output images, on the second row the 3D reconstruction with vertical view, on the third row the 3D reconstruction with oblique view.

Table 9: Real Agamemnon mask: parameter values used in the models with an oblique light source $\omega_{obl} = (0, 0.0995, 0.9950)$.

| Model | $\sigma$ | $k_D$ | $k_S$ | $\alpha$ |
|-------|---------|-------|-------|----------|
| LAM   |         |       |       |          |
| ON-01 | 0.1     |       |       |          |
| ON-02 | 0.2     |       |       |          |
| ON-03 | 0.3     |       |       |          |
| PH-s02| 0.8     | 0.2   |       | 1        |
| PH-s03| 0.7     | 0.3   |       | 1        |
| PH-s04| 0.6     | 0.4   |       | 1        |

Looking at Table 10 we can note that the oblique cases require higher CPU time with respect to the vertical cases due to the fact that the equations are more complex because of additional terms involved. Because of these additional terms involved in the oblique case, in Table 11 we have reported the results obtained using the parameters shown in Table 9 with a value of
the tolerance $\eta$ for the stopping rule of the iterative method equal to $10^{-3}$. This is the maximum accuracy achieved by the non-Lambertian models since roundoff errors coming from several terms occur and limit the accuracy.

Table 10: Real Agamemnon mask: number of iterations and CPU time for the different models with oblique light source $\omega_{obl} = (0, 0.0995, 0.9950)$.

| SL–Schemes | Iter. | [s] |
|------------|-------|-----|
| LAM        | 321   | 117.9 |
| ON-01      | 315   | 246.0 |
| ON-02      | 361   | 281.5 |
| ON-03      | 396   | 264.6 |
| PH-s02     | 427   | 285.2 |
| PH-s03     | 564   | 373.6 |
| PH-s04     | 680   | 484.1 |

Table 11: Real Agamemnon mask: $L^2, L^\infty$ errors with oblique light source $\omega_{obl} = (0, 0.0995, 0.9950)$.

| SL–Schemes | $L^2(I)$ | $L^\infty(I)$ |
|------------|----------|---------------|
| LAM        | 0.0585   | 0.4863        |
| ON-01      | 0.0663   | 0.4588        |
| ON-02      | 0.0670   | 0.4471        |
| ON-03      | 0.0708   | 0.5451        |
| PH-s02     | 0.1141   | 0.5725        |
| PH-s03     | 0.1580   | 0.6196        |
| PH-s04     | 0.2063   | 0.6706        |

In Fig. 13 we can see the output image and the 3D reconstruction in a single case for each models.

**Test 6: Corridor.** As last example, let us consider a real image of a corridor (see Fig. 15) as a typical example of a scene which can be useful for a robot navigation problem. The size of this image is $600 \times 383$. Note that for this picture we do not know the parameters and the light direction in the scene. It seems that there is a diffused light and more than one light source. So this picture does not satisfy many assumptions we used in the theoretical part. In order to apply our numerical scheme we considered a Dirichlet boundary condition equal to zero at the wall located at the bottom of the corridor. In this way we have a better perception of the depth of the scene. In Fig. 15 we can see the output images (on the first column) and the 3D reconstructions (on the second column) obtained by L–model, ON–model with $\sigma = 0.1$ and PH–model with $k_S = 0.2$. In this example the PH–model
seems to recognize the scene better than the ON–model. In some sense this is probably due to the fact that it has less parameters so it is easier to tune to a real situation where the information on the parameters is not available.

Figure 13: Agamennon mask: results with oblique light source \( \omega_{\text{obl}} = (0, 0.0995, 0.9950) \). On the first row the output images, on the second row the 3D reconstruction with vertical view, on the third row the 3D reconstruction with oblique view.

Figure 14: Image of an real scene (size 600 × 383).
8 Conclusions

In this paper we derived nonlinear partial differential equations of first order, i.e. Hamilton-Jacobi equations, associated to the non-Lambertian reflectance models ON–model and PH–model. We derived the model equations in all the possible cases for each model, coupling vertical or oblique light source with vertical or oblique position of the observer, this gives a new and exhaustive description of these models offering a unified mathematical formulation. In this formulation we can switch on and off the different terms related to ambient, diffuse and specular reflection in a very simple way. As a result, this general model is very flexible to treat the various situations with vertical and oblique light sources. Moreover, we have observed that none of these models is able to overcome the typical concave/convex ambiguity known for the classical Lambertian model. Despite this limitations, the approach presented in this paper is able to improve the Lambertian model that is not suitable to deal with realistic images coming from medical or security applications.
application. The numerical methods presented here can be applied to solve the equations corresponding to the ON–model and the PH–model in all the situations.

As we have seen, in the complex nonlinear PDEs associated to non-Lambertian models the parameters play a crucial role to obtain accurate results. In fact, varying the value of the parameters it is possible to improve the approximation with respect to the classical L–model. We can also say that for real images the PH–model seems easier to tune.

Focusing the attention on the tests performed with an oblique light source, we have to do some comments that are common to the PH–model and the ON–model. Several terms appear in these models and each of them gives a contribution to the roundoff error. Note that the accumulation of these roundoff errors makes difficult in the oblique case to obtain a great accuracy. A possible improvement could be the use of second order schemes, that release the link between the space and the time steps which characterizes and limits the accuracy for first order schemes. This direction will be explored in a future work.

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