On the realization functor of the derived category of mixed motives

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Abstract

We give an alternative construction of the Betti realization functor on the derived category of motives of complex algebraic varieties via the category of CW complexes instead of the category of complex analytic spaces. In particular, we show that the functor we define via the category of CW complexes coincide with Ayoub’s one. We deduce from this construction that Ayoub’s realization functor on geometric motives factor through Nori motives and that the image of this functor on the morphisms between the motive of a point and a Tate twist of the motive with compact support of a complex algebraic variety coincide with the classical cycle class map on higher chow groups.

Notations:

• Denote by Top the category of topological spaces. Denote by Var(k) the category of algebraic varieties over a field k, i.e. schemes of finite type over k. Let us call PVar(k) ⊂ QPVar(k) ⊂ Var(k) the full subcategories quasi-projective varieties and projective varieties respectively. Let us call PSmVar(k) ⊂ SmVar(k) ⊂ Var(k) the full subcategories of smooth varieties and smooth projective varieties respectively. Denote by CW ⊂ Top the full subcategory of CW complexes, by CS ⊂ CW the full subcategory of ∆ complexes, and by TM ⊂ CW the full subcategory of topological manifolds which admits a CW structure (a topological manifold admits a CW structure if it admits a differential structure). Denote by AnSp(k) the category of analytic spaces over k. and by AnSm(k) ⊂ AnSp(k) the full subcategory of smooth analytic spaces (i.e. complex analytic varieties).

• For V ∈ Var(C), we denote by V^{an} ∈ AnSp(C) the complex analytic space associated to V with the usual topology induced by the usual topology of C^{N}. For W ∈ AnSp(C), we denote by W^{cw} ∈ AnSp(C) the topological space given by W which is a CW complex. For simplicity, for V ∈ Var(C), we denote by V^{cw} := (V^{an})^{cw} ∈ CW. We have then
  - the analytical functor An : Var(C) → AnSp(C), An(V) = V^{an},
  - the forgetful functor Cw : AnSp(C) → CW, Cw(W) = W^{cw},
  - the composite of these two functors Cw = Cw ◦ An : Var(C) → CW, Cw(V) = V^{cw}.

• We denote by □^{n} = (P^{1}\setminus\{1\})^{n} ⊂ (P^{1})^{n}. For X ∈ Var(C) let Z^{p}(X, n) ⊂ Z^{p}(X × □^{n}) be the subgroup of p codimensional cycle in X × □^{n} meeting all faces of □^{n} properly. We denote by π_{X} : X × (P^{1})^{n} → X and π_{(P^{1})^{n}} : X × (P^{1})^{n} → (P^{1})^{n} the projections.

• For X ∈ Top a topological space, we denote by C^{sing}(X, Z) = Z\text{Hom}_{Top}(\Delta^{\bullet}, X) the complex of singular chains, ∆^{p} ⊂ R^{p} being the standard simplex. We denote by I^{n} = [0, 1]^{n} and we will consider the closed embeddings of CW complexes i_{n} : [0, 1]^{n} → □^{n} := (P^{1}\setminus\{1\})^{n} whose image is the product [0, ∞)^{n} ⊂ □^{n} of the segments R^{∞} = [0, ∞) ⊂ □^{1} (c.f. the definition of T_{z} in [11]). In particular i_{n} send 0 to 0 and 1 to ∞ and gives a morphism of complexes i : I^{n} → □^{n} in Z(CW). We denote by D^{n} ⊂ C^{n} the closed ball of radius 1, and by i'_{n} : I^{n} → D^{n} the inclusion of pro complex analytic spaces and and i'_{n}^{cw} : I^{n} → D^{n} is the corresponding inclusion of CW complexes.
\begin{itemize}
  \item For a (small) category $\mathcal{S}$, we denote by $\mathcal{PSh}(\mathcal{S}, M) := \text{Fun}(\mathcal{S}^{op}, M)$ the big category of presheaves on $\mathcal{S}$ with value in $M$. If $M$ is a model category with some extra assumptions (c.f.\cite{1}), the projective fibration (rep. the injective cofibration) and the termwise weak equivalence of $\mathcal{PSh}(\mathcal{S}, M)$ define a projective (resp. injective) model structure $\mathcal{PSh}(\mathcal{S}, M)$. In this paper, we will consider $\mathcal{M}_p(\mathcal{PSh}(\mathcal{S}, M))$ the projective model structure on $\mathcal{PSh}(\mathcal{S}, M)$.
  \item For $X \in \mathcal{S}$, we denote by $\mathbb{Z}(X) \in \mathcal{PSh}(\mathcal{S}, \text{Ab})$ the presheaf given by Yoneda embedding, that is the presheaf given by for $Z \in \mathcal{S}$, $\mathbb{Z}(X)(Z) = \mathbb{Z}(\mathcal{H}_{\mathcal{S}}(Z, X))$ and for $f : Z \to Z$ a morphism in $\mathcal{S}$, $\mathbb{Z}(X)(f) : g \in \mathcal{H}_{\mathcal{S}}(Z, X) \mapsto g \circ f \in \mathcal{H}_{\mathcal{S}}(Z', X)$
  \item For $h : X \to Y$ a morphism in $\mathcal{S}$, we denote by $\mathbb{Z}(h) : \mathbb{Z}(X) \to \mathbb{Z}(Y)$, the morphism in $\mathcal{PSh}(\mathcal{S}, \mathcal{C}(\mathbb{Z}))$ given by Yoneda embedding, that is the morphism given by for $Z \in \mathcal{S}$, $\mathbb{Z}(h)(Z) : g \in \mathcal{H}_{\mathcal{S}}(Z, X) \mapsto h \circ g \in \mathcal{H}_{\mathcal{S}}(Z, Y)$
  \item We denote by $\mathcal{C}(\mathbb{Z}) = C(\text{Ab})$ the category of abelian complexes, $C^-(\mathbb{Z}) \subset C(\mathbb{Z})$ the full subcategory consisting of bounded above complexes, $D(\mathbb{Z})$ the derived category of $C(\mathbb{Z})$ with respect to quasi-isomorphism, and $D^-(\mathbb{Z}) \subset D(\mathbb{Z})$ the image of $C^-(\mathbb{Z})$ under the localization functor $C(\mathbb{Z}) \to D(\mathbb{Z})$. For $S \in \text{Top}$, we denote by $C(S) := \mathcal{PSh}(\mathcal{S}, C(\mathbb{Z}))$ the category of complexes of presheaves on $\mathcal{S}$, $C^-(\mathbb{Z}) \subset C(\mathbb{Z})$ the full subcategory consisting of bounded above complexes, $D(S)$ the derived category of $C(S)$ with respect to the morphisms of complexes of presheaves which are quasi-isomorphisms after sheafification, and $D^-(S) \subset D(S)$ the image of $C^-(S)$ under the localization functor $C(S) \to D(S)$.
\end{itemize}

1 Introduction

Let $S \in \text{Var}(\mathbb{C})$. In \cite{1, 2} and \cite{3}, J.Ayoub has given and studied a construction of the Betti realization functor on $\text{DA}^{-}(S, \mathbb{Z})$ the derived category of mixed motives of $\text{Var}(\mathbb{C})/S$ the complex algebraic varieties together with a morphism over $S$. He mentioned also the construction of the Betti realization functor on $\text{DM}^{-}(S, \mathbb{Z})$, the derived category of mixed motives of $\text{Var}(\mathbb{C})/S$ with transfers. Let $k$ a field.

- The derived category of (effective) motives of $\text{Var}(k)/S$ is the homotopy category of the category $P^{-}(S) = \mathcal{PSh}(\text{Var}(k)^{sm}/S, C^-(\mathbb{Z}))$ of bounded above complexes of presheaves on the category of algebraic varieties over $k$ together with a smooth morphism over $S$, with respect to the projective $(\mathbb{A}^1, \text{et})$ model structure (c.f. definition \cite{4}(i)).
- Similarly, the derived category of (effective) motives of $\text{Var}(k)/S$ with transfers is the homotopy category of the category $PC^{-}(S) = \mathcal{PSh}(\text{Cor}_{k}^{\mathbb{Z}}(\text{Var}(k)^{sm}/S, C^-(\mathbb{Z}))$ of bounded above complexes of presheaves on the category of finite and surjective correspondences between algebraic varieties over $k$ together with a smooth morphism over $S$ with respect to the projective $(\mathbb{A}^1, \text{et})$ model structure (c.f. definition \cite{4}(ii)).
- For $X \in \text{Var}(k)$ and $l : D \hookrightarrow X$ a (locally closed) subvariety, the motive of the pair $(X, D)$ is $M(X, D) = D(\mathbb{A}^1, \text{et})(\mathbb{Z}_{tr}(X, D)) \in \text{DM}^{-}(k, \mathbb{Z})$, where $\mathbb{Z}_{tr}(X, D) = \text{coker}(\mathbb{Z}_{tr}(l))$ is the cokernel of the injective morphism of presheaves $\mathbb{Z}_{tr}(l) : \mathbb{Z}_{tr}(D) \hookrightarrow \mathbb{Z}_{tr}(X)$ and $D(\mathbb{A}^1, \text{et}) : \text{PC}^{-} \to \text{DM}^{-}(k, \mathbb{Z})$ is the localization functor.

The construction of J.Ayoub is defined on $\text{DA}^{-}(S)$ via $\text{AnDA}^{-}(S^{an})$, the derived category of motives of complex analytic space, which is the homotopy category of $P^{-}(\text{An}, S^{an}) = \mathcal{PSh}_{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S^{an}, C^-(\mathbb{Z}))$ of bounded above complexes of presheaves on the category of complex analytic spaces together with a smooth morphism over $S^{an}$ with respect to the projective $(\mathbb{D}^1, \text{usu})$ model structure (c.f. definition \cite{4}(i)). Similarly on $\text{DM}^{-}(S, \mathbb{Z})$ it is defined via $\text{AnDM}^{-}(S^{an}, \mathbb{Z})$, the derived category of motives of complex analytic space, which is the homotopy category of $PC^{-}(\text{An}, S^{an}) = \mathcal{PSh}_{\mathbb{Z}}(\text{Cor}_{k}^{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S^{an}), C^-(\mathbb{Z}))$ the category of bounded above complexes of presheaves on the category of finite and surjective correspondences between complex analytic spaces together with a smooth morphism over $S$ with respect to
the projective $\mathbb{D}^1$, usu) model structure (c.f. definition \[34\](ii)). For $S \in \text{AnSp}(\mathbb{C})$, we consider the commutative diagram

$$
\begin{array}{c}
\text{Cor}^{-f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S) \\
\downarrow e^{tr}_{an}(S) \\
\text{Ouv}(S)
\end{array}
\quad
\begin{array}{c}
\downarrow \text{Tr}(S) \\
\text{AnSp}(\mathbb{C})^{sm}/S
\end{array}
\quad
\begin{array}{c}
\text{Cor}^{f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S) \\
e^{an}(S)
\end{array}
$$

the morphism of sites given respectively by the inclusion functors $e^{an}(T) : \text{Ouv}(S) \to \text{AnSp}(\mathbb{C})^{sm}/S$, $e^{tr}_{an}(S) : \text{Ouv}(S) \to \text{Cor}^{f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S)$ and $\text{Tr}(S) : \text{Cor}^{-f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S) \to \text{AnSp}(\mathbb{C})^{sm}/S$. The definition of Betti realization functor by J.Ayoub is

Definition 1. \cite{73,3}

(i) The Betti realisation functor (without transfers) is the composite:

$$
\text{Bti}_0(S)^* : \text{DA}^-(S,\mathbb{Z}) \xrightarrow{\text{An}(S)^*} \text{AnDA}^-(S^{an},\mathbb{Z}) \xrightarrow{\text{Re}_{an}(S)^*} D^-(\mathbb{Z}) \quad (1)
$$

(ii) The Betti realisation functor with transfers is the composite:

$$
\text{Bti}(S)^* : \text{DM}^-(S,\mathbb{Z}) \xrightarrow{\text{An}(S)^*} \text{AnDM}^-(\mathbb{Z}) \xrightarrow{\text{Re}_{an}^\mathbb{Z}(S)^*} D^-(\mathbb{Z}) \quad (2)
$$

Since $\text{An}(S)^*$ derive trivially by proposition \[44\](ii) and and $L\text{Tr}(S^{an})^* : \text{AnDA}^-(S^{an},\mathbb{Z}) \to \text{AnDM}^-(S^{an},\mathbb{Z})$ is the inverse of $\text{Tr}(S^{an})_*$, we have $\text{Bti}_0(S)^* = \text{Bti}(S)^* \circ L\text{Tr}(S)^*$.

In \cite{2}, J.Ayoub has constructed an explicit object which gives the localization functor for the $\mathbb{D}^1$, usu model strucure. We recall this in theorem \[34\](i) and give a relative version (with and without transfers) in theorem \[34\].

Theorem 1. Let $S \in \text{AnSp}(\mathbb{C})$,

(i) For $F^* \in \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/C^-(\mathbb{Z}))$, $\text{sing}_{an} F^* \in \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}, C^-(\mathbb{Z}))$ is $\mathbb{D}^1$ local and the inclusion morphism $S(F^*) : F^* \to \text{sing}_{an} F^*$ is an $\mathbb{D}^1$, usu equivalence.

(ii) For $F^* \in \text{PSh}(\text{Cor}^{f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}), C^-(\mathbb{Z}))$, $\text{sing}_{an} F^* \in \text{PSh}(\text{Cor}^{f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}), C^-(\mathbb{Z}))$ is $\mathbb{D}^1$ local and the inclusion morphism $S(F^*) : F^* \to \text{sing}_{an} F^*$ is an $\mathbb{D}^1$, usu equivalence.

The categories $\text{AnDM}^-(S)$ and $\text{AnDA}^-(S)$, for $S \in \text{AnSp}(\mathbb{C})$ satisfy the following (see \cite{1} and \cite{3})

Theorem 2. (i) The adjunction

$$(\text{Tr}(S)^*, \text{Tr}(S)_*) : \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z})) \Rightarrow \text{PSh}(\text{Cor}^{f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(\mathbb{Z}))$$

is a Quillen equivalence for the $\mathbb{D}^1$, usu model structures. That is, the derived functor $\text{Tr}(S)_* : \text{AnDM}^-(S) \xrightarrow{\sim} \text{AnDA}^-(S, \mathbb{Z})$ is an isomorphism and $L\text{Tr}(S)^*$ is it inverse.

(ii) The adjonction $(e^{an}(S)^*, e^{an}(S)_*) : C^-(S) \Rightarrow \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z}))$ is a Quillen equivalence for the $\mathbb{D}^1$, usu model structures. That is, the derived functor $e^{an}_* : D^-(S) \xrightarrow{\sim} \text{AnDA}^-(S, \mathbb{Z})$ is an isomorphism and $\text{Re}_{an}(S)^*$ is it inverse.

(iii) The adjunction $(e^{tr}_{an}(S)^*, e^{tr}_{an}(S)_*) : C^-(S) \Rightarrow \text{PSh}(\text{Cor}^{f_\mathbb{Z}}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(\mathbb{Z}))$ is a Quillen equivalence for the $\mathbb{D}^1$, usu model structures. That is, the derived functor $e^{tr}_{an}(S)^* : D^-(S) \xrightarrow{\sim} \text{AnDM}^-(S, \mathbb{Z})$ is an isomorphism and $\text{Re}^{tr}_{an}(S)^*$ is it inverse.
In this paper we give a construction of the Betti realization functor via the category of CW complexes. The reason we do this is that the cycle class map on complex analytic spaces and the action of correspondence on homology on smooth complex analytic spaces (i.e. complex analytic manifold) are defined in a purely topological way so that we do not use the need complex structure which gives in the smooth case the Frölicher filtration on then (the Frölicher filtration gives the Hodge filtration on cohomology in the compact Kalher case).

Let $X, Y, Z \in \text{Top}$. Assume $Y$ is Hausdorff (equivalently the diagonal $\Delta_Y \subseteq Y \times Y$ is a closed subset). There is a natural composition law (c.f.33) between closed subset of $X \times Y$ which are finite and surjective over $X$ and closed subset of $Y \times Z$ which are finite and surjective over $Y$. We denote $\mathbb{I}^1 : = \{0, 1\}$. Consider now the full subcategory $\text{CW} \subseteq \text{Top}$ consisting of CW complexes. By a CW subcomplex of $X \in \text{CW}$, we mean a topological embedding $Z \hookrightarrow X$ with $Z$ a CW complex. By a closed CW subcomplex of $X \in \text{CW}$, we mean a topological closed embedding $Z \hookrightarrow X$ with $Z$ a CW complex, that is the image of the embedding is a closed subset of $X$. Let $X, S \in \text{CW}$, $S$ connected, and $h : X \to S$ a finite and surjective morphism in CW. We say that $X/S = (X, h) \in \text{CW} / S$ is reducible if $X = X_1 \cup X_2$, with $X_1, X_2$ closed CW subcomplexes finite and surjective over $S$ and $X_1, X_2 \neq X$. A pair $Y/S = (Y, h') \in \text{CW} / S$ with $Y \in \text{CW}$ and $h' : Y \to S$ a finite and surjective morphism is considered irreducible if it is not reducible. In particular $Y$ is connected.

- For $X \in \text{CW}$, $\Lambda$ a commutative ring and $p \in \mathbb{N}$, we denote by $Z_p(X, \Lambda)$ the free $\Lambda$ module generated by the closed CW subcomplex of $X$ of dimension $p$ and by $Z^p(X, \Lambda) = Z^d_p(X, \Lambda)$ the free $\Lambda$ module generated by the closed CW subcomplex of $X$ of codimension $p$.
- For $X, Y \in \text{CW}$, $X$ connected and $\Lambda$ a commutative ring, we define : $Z^{fs,i/X}(X \times Y, \Lambda) \subseteq Z^d(X \times Y, \Lambda)$ the free $\Lambda$ module generated by the closed CW subcomplexes of $X \times Y$ finite and surjective over $X$ which are irreducible.
- For $X, Y \in \text{CW}$, and $\Lambda$ a commutative ring, we define : $Z^{fs,i/X}(X \times Y, \Lambda) := \oplus_i Z^{fs,i/X_i}(X_i \times Y, \Lambda)$ where $X = \sqcup_i X_i$, with $X_i$ the connected components of $X$.

We define the category of finite surjective correspondences on CW complexes.

**Definition 2.** We define $\text{Cor}_1^{fs}(\text{CW})$ to be the category whose objects are CW complexes and whose space of morphisms between $X, Y \in \text{CW}$ is the free $\Lambda$ module $Z^{fs,i/X}(X \times Y, \Lambda)$. The composition law is the one given by $(\cite{33})$.

- Let $S \in \text{CW}$. We define $\text{CW}^{sm} / S$ to be the category whose objects are $X/S = (X, h)$ with $X \in \text{CW}$ and $h : X \to S$ a smooth morphism. Let $X/S = (X, h_1), Y/S = (Y, h_2) \in \text{CW}^{sm} / S$. A morphism $f : X/S \to Y/S$ is a morphism $f : X \to Y$ such that $f \circ h_1 = h_2$.

- Let $S \in \text{CW}$. We define $\text{Cor}_1^{fs}(\text{CW}^{sm} / S)$ to be the category whose objects are those of $\text{CW}^{sm} / S$ and whose space of morphisms between $X, Y \in \text{CW}$ is the free $\Lambda$ module $Z^{fs,i/X}(X \times_S Y, \Lambda)$. The composition law is the one given by $(\cite{32})$.

We define

- $\text{CwDA}^- (\mathbb{Z})$, the derived category of motives of CW complexes to be the homotopy category of $P^-(\text{CW}) = \text{PSh}^s_{2}(\text{CW}, C^{-}(\mathbb{Z}))$, the category of bounded above complex of presheaves on the category CW of CW complexes, with respect to the projective ($\mathbb{I}^1$, $\text{usu}$) model structure $(\cite{20})$.

- Similarly, we define $\text{CwDM}^- (\mathbb{Z})$, the derived category of motives of CW complexes, as the homotopy category of $PC^-(\text{CW}) = \text{PSh}^s_{2}(\text{Cor}_1^{fs}(\text{CW}), C^{-}(\mathbb{Z}))$, the category of bounded above complex of presheaves on the category Cor$_1^{fs}(\text{CW})$, with respect to the projective ($\mathbb{I}^1$, $\text{usu}$) model structure $(\cite{20})$.
• For $X \in \text{CW}$ and $l : D \hookrightarrow X$ a CW subcomplex, the motive of the pair $(X, D)$ is $M(X, D) = \mathcal{D}(\mathbb{I}^1, \text{usu})(Z_{tr}(X, D)) \in \text{CwDM}^- (Z)$, where $Z_{tr}(X, D) = \text{coker}(Z_{tr}(l))$ is the cokernel of the injective morphism of presheaves $Z_{tr}(l) : Z_{tr}(D) \hookrightarrow Z_{tr}(X)$ and $\mathcal{D}(\mathbb{I}^1, \text{usu}) : PC^-(\text{CW}) \to \text{CwDM}^- (Z)$ is the localization functor.

We consider the commutative diagram

$$
\begin{array}{ccc}
\text{Cor}^f_\mathbb{Z}(\text{CW}) & \xrightarrow{\text{Tr}} & \text{CW}^\text{sm} \\
\downarrow e^s_{cw} & & \downarrow e^s_{cw} \\
\{\text{pt}\} & \xrightarrow{\text{Tr}} & \{\text{pt}\}
\end{array}
$$

the morphism of sites given respectively by the inclusion functors $e^s_{cw} : \{\text{pt}\} \hookrightarrow \text{CW}$, $e^s_{cw} : \{\text{pt}\} \hookrightarrow \text{Cor}^f_\mathbb{Z}(\text{CW})$ and $\text{Tr} : \text{Cor}^f_\mathbb{Z}(\text{CW}) \hookrightarrow \text{CW}$.

For $S \in \text{CW}$, we define

• $\text{CwDA}^- (S, Z)$, the derived category of motives of CW complexes, as the homotopy category of $P^- (\text{CW}, S) = \text{PSh}_S(\text{CW}^\text{sm} / S, C^- (Z))$ the category of bounded above complex of presheaves on $\text{CW}^\text{sm} / S$ with respect to the projective $(\mathbb{I}^1, \text{usu})$ model structure (58(i)).

• Similarly, for $S \in \text{CW}$, we define $\text{CwDM}^- (S, Z)$, the derived category of motives of CW complexes, as the homotopy category of $PC^- (\text{CW}, S) = \text{PSh}_S(\text{Cor}^{f_\mathbb{Z}}(\text{CW}^\text{sm} / S), C^- (Z))$ the category of bounded above complex of presheaves on the category $\text{Cor}^{f_\mathbb{Z}}(\text{CW}^\text{sm} / S)$ with respect to the projective $(\mathbb{I}^1, \text{usu})$ model structure (58(ii)).

For $S \in \text{CW}$, we consider the commutative diagram

$$
\begin{array}{ccc}
\text{Cor}^f_\mathbb{Z}(\text{CW}^\text{sm} / S) & \xrightarrow{\text{Tr}(S)} & \text{CW}^\text{sm} / S \\
\downarrow e^{tr}_{cw}(S) & & \downarrow e^{tr}_{cw}(S) \\
\text{Ouv}(S) & \xrightarrow{\text{Tr}(S)} & \text{Ouv}(S)
\end{array}
$$

the morphism of sites given respectively by the inclusion functors $e^{tr}_{cw}(S) : \text{Ouv}(S) \hookrightarrow \text{CW}^\text{sm} / S$, $e^{tr}_{cw}(T) : \text{Ouv}(S) \hookrightarrow \text{Cor}^{f_\mathbb{Z}}(\text{CW}^\text{sm} / S)$ and $\text{Tr}(S) : \text{Cor}^{f_\mathbb{Z}}(\text{CW}^\text{sm} / S) \hookrightarrow \text{CW}^\text{sm} / S$.

We consider first the absolute case. We give an explicit object which induces the $\mathbb{I}^1$ localization functor on the category of presheaves on CW and additive presheaves on $\text{Cor}_\mathbb{Z}(\text{CW})$. More precisely by, considering $\mathbb{I}^n := [0, 1]^n$, we prove in theorem 10 the following:

**Theorem 3.**

• For $F^* \in \text{PSh}_S(\text{CW}, C^- (Z))$, $\text{sing}_S F^* \in \text{PSh}_S(\text{Cor}^{f_\mathbb{Z}}(\text{CW}), C^- (Z))$ is $\mathbb{I}^1$ local and the inclusion morphism $S(F^*) : F^* \to \text{sing}_S F^*$ is an $(\mathbb{I}^1, \text{usu})$ equivalence.

• For $F^* \in \text{PSh}_S(\text{Cor}^{f_\mathbb{Z}}(\text{CW}), C^- (Z))$, $\text{sing}_S F^* \in \text{PSh}_S(\text{Cor}^{f_\mathbb{Z}}(\text{CW}), C^- (Z))$ is $\mathbb{I}^1$ local and the inclusion morphism $S(F^*) : F^* \to \text{sing}_S F^*$ is an $(\mathbb{I}^1, \text{usu})$ equivalence.

Then, we prove that the categories $\text{CwDM}^-$ and $\text{CwDA}^-$ satisfy the following (c.f.theorem 17):

**Theorem 4.**

(i) The adjunction $(\text{Tr}^*, \text{Tr}_*) : \text{PSh}(\text{CW}, C^- (Z)) \rightleftarrows \text{PSh}_S(\text{Cor}^{f_\mathbb{Z}}(\text{CW}), C^- (Z))$ is a Quillen equivalence for the $(\mathbb{I}^1, \text{usu})$ model structures. That is, the derived functor

$$
L \text{Tr}^* : \text{CwDA}^-(Z) \rightleftarrows \text{CwDM}^-(Z)
$$

is an isomorphism and $\text{Tr}_* : \text{CwDM}^-(Z) \rightleftarrows \text{CwDA}^-(Z)$ is its inverse.
(ii) The adjunction \( (e_{cw}^*, e_{cw}) : C^-(\mathbb{Z}) \rightleftharpoons \text{PSh}_{\text{Z}}(\text{CW}, C^-(\mathbb{Z})) \) is a Quillen equivalence for the \( (\mathbb{I}^1, \text{usu}) \) model structures. That is, the derived functor \( e_{cw}^* : D^-(\mathbb{Z}) \rightrightarrow \text{CwDA}^-(\mathbb{Z}) \) is an isomorphism and \( \text{Re}_{cw} : \text{CwDA}^-(\mathbb{Z}) \rightleftarrows D^-(\mathbb{Z}) \) is its inverse.

(iii) The adjunction \( (e_{cw}^*, e_{cw}) : C^-(\mathbb{Z}) \rightleftharpoons \text{PSh}_{\text{Z}}(\text{Cor}_{\Delta}^f(\text{CW}), C^-(\mathbb{Z})) \) is a Quillen equivalence for the \( (\mathbb{I}^1, \text{usu}) \) model structures. That is, the derived functor \( e_{cw}^* : D^-(\mathbb{Z}) \rightrightarrow \text{CwDM}^-(\mathbb{Z}) \) is an isomorphism and \( \text{Re}_{cw}^* : \text{CwDM}^-(\mathbb{Z}) \rightleftarrows D^-(\mathbb{Z}) \) is its inverse.

For point (i), we use proposition [21] to prove that \( L \text{Tr}^* \) is this inverse of \( \text{Tr}_* \). In proposition [21] we prove a key result that for \( X \in \text{CW} \)

\[
\text{ad}(\text{Tr}^* \text{Tr}_*)(\text{Z}_e(X)) : \text{sing}_\mathbb{Z}(X) \rightarrow \text{Tr}_* \text{sing}_\mathbb{Z}(\text{Z}_e(X))
\]

is an equivalence usu local. To see this we use the fact that a CW complex is \( \mathbb{I}^1 \) homotopy equivalent to a \( \Delta \)-complex, and that for a \( \Delta \)-complex there exist a countable open covering such that the intersection of a finite number of members of this covering is either empty or a contractible topological space. We deduce from proposition [21] and proposition [17] the point (iii) of proposition [21] which says in particular that for \( X \in \text{CW} \), the complex \( \text{sing}_\mathbb{Z}(\text{Z}_e(X)) \), where \( \text{Z}_e(X) \) is the presheaf represented by \( X \), is quasi-isomorphic to the complex of singular chains \( C^*_{\text{sing}}(X, \mathbb{Z}) \). Indeed, considering \( \text{Z}(Y, E) = \text{coker}(\text{Z}(l)) \), the cokernel of the injective morphism of presheaves \( \text{Z}(l) : \text{Z}(E) \hookrightarrow \text{Z}(Y) \), we have

**Proposition 1.** For \( Y \in \text{CW} \) and \( l : E \rightarrow Y \) a CW subcomplex, the followings embeddings are quasi-isomorphism :

\[
C^*_{\text{sing}}(Y, E, \mathbb{Z}) \xrightarrow{\text{Z}(Y,E)(L)} \text{sing}_\mathbb{Z}(Y, E) \xrightarrow{e_{cw}} \text{ad}(\text{Tr}^* \text{Tr}_*)(\text{sing}_\mathbb{Z}(Y, E)) \xrightarrow{\text{sing}_\mathbb{Z}} \text{Z}_e(Y, E)
\]

where \( C^*_{\text{sing}}(Y, E, \mathbb{Z}) = \text{coker} l_* \) is the relative cohomology, with \( l_* : \text{Z} \text{Hom}_{\text{Cw}}(\Delta^*, E) \rightarrow \text{Z} \text{Hom}_{\text{Cw}}(\Delta^*, Y) \).

Let \( X \in \text{TM}(\mathbb{R}) \) be a differential manifold, \( Y \in \text{CW} \) and \( E \subset Y \) a subcomplex. Let \( T = \sum_i n_i T_i \in Z^f/\mathbb{T}^* \times (\mathbb{I}^n \times X \times Y) \) such that \( \partial T := \sum_{i=1}^{n} (-1)^n(T|_{\mathbb{I}^n \times X \times Y} - T|_{\mathbb{I}^n \times X \times Y}) = 0 \). Denote by \( p_X : \mathbb{I}^n \times X \times Y \rightarrow X, p_Y : \mathbb{I}^n \times X \times Y \rightarrow Y \) and \( p_{X \times Y} : \mathbb{I}^n \times X \times Y \rightarrow X \times Y \) the projections, \( m_i : T_i \rightarrow \mathbb{I}^n \times X \times Y \) the closed CW embeddings for all \( i \). Denote by \( p_X m_i \rightarrow p_X m_i : T_i \rightarrow X \) and \( p_Y m_i \rightarrow p_Y m_i : T_i \rightarrow Y \). The action of \( p_{X \times Y}(T) \in Z_{\text{dX}}(X \times Y, \mathbb{Z}) \) on homology is

\[
K_n(X, (Y, E))(p_{X \times Y}(T)) : \sum_{i} n_i (c_{Y,E}[n] \circ (p_{Y,m_i}[n]) \circ p_X m_i \in \text{Hom}_{D^-(\mathbb{Z})}(C_*(X, Z), C_*(Y, E, \mathbb{Z})[n]),
\]

where, for each \( i \) :

- \( p_X m_i \in \text{Hom}_{D^-(\mathbb{Z})}(C_*(X, Z), C_*(T_i, Z)[n]) \) is the Gysin morphism \( (p_X m_i \) is proper and \( X \in \text{TM}(\mathbb{R}) \) is a topological manifold),

- \( p_Y m_i = Z(p_{Y,m_i})(\Delta^*) : C_*(T_i, Z) \rightarrow C_*(Y, Z) \) is the classical map on singular chain,

- \( c_{Y,E} : C_*(Y, Z) \rightarrow C_*(Y, E, \mathbb{Z}) \) is the quotient map.

We identify in proposition [23] for \( X \in \text{TM}(\mathbb{R}) \) a differential manifold, \( Y \in \text{CW} \) and \( E \subset Y \) a subcomplex, the image of a morphism \( [T] \in \text{Hom}_{\text{PC}^-(_{\text{CW}})}(\text{Z}_e(X), \text{sing}_\mathbb{Z}, \text{Z}_e(Y, E)[n]) \) under the \( (\mathbb{I}^1, \text{usu}) \) localization functor with the action of \( p_{X \times Y}(T) \) on homology :

**Proposition 2.** Let \( X \in \text{TM}(\mathbb{R}) \) connected and \( Y \in \text{CW} \). Let \( l : E \rightarrow Y \) a CW subcomplex.

(i) Let \( [T] = \sum_i n_i T_i \in \text{Hom}_{\text{PC}^-(_{\text{CW}})}(\text{Z}_e(X), \text{sing}_\mathbb{Z}, (\text{Z}_e(Y, E))[n]) \) Then,

\[
\text{Re}_{cw}^* \circ D(\mathbb{I}^1, \text{usu})([T]) = K_n(X, (Y, E))(p_{X \times Y}(T)) \in \text{Hom}_{D^-(\mathbb{Z})}(\text{Z}_e(X), \text{sing}_\mathbb{Z}, (\text{Z}_e(Y, E))[n]) = \text{Hom}_{D^-(\mathbb{Z})}(C_*(X, Z), C_*(Y, E, \mathbb{Z})[n]),
\]

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(ii) If $Y$ is compact and $E \subset Y$ is closed then the factorization

$$K_n(X, (Y, E)) = K_n(X, (Y, E)) \circ [\cdot] : \mathbb{Z}_{d_X+n}(X \times Y, Z) \to \text{Hom}_{\mathbb{Z}^{-1}}(C_*(X, Z), C_*(Y, E, Z)[n])$$

where $[\cdot] : \mathbb{Z}_{d_X+n}(X \times Y, X \times E, Z) \to H_{d_X+n}^B(X \times X, X \times E, Z)$ is the fundamental class gives the classical isomorphism coming from the six functor formalism (c.f. [7])

$$\tilde{K}_n(X, (Y, E)) : H_{d_X+n}^B(X \times X, X \times E, Z) \sim \text{Hom}_{\mathbb{Z}^{-1}}(C_*(X, Z), C_*(Y, E, Z)[n]). \quad (4)$$

By definition, we have the following commutative diagram of sites:

\[
\begin{array}{ccccccccc}
\text{Cor}_{Z}^f (CW) & \xrightarrow{\text{Cor}_{Z}^f} & \text{Cor}_{Z}^f (\text{AnSm}(\mathbb{C})) & \xrightarrow{\text{An}} & \text{Cor}_{Z}^f (\text{SmVar}(\mathbb{C})) \\
\downarrow \text{Tr} & & \downarrow \text{Tr} & & \downarrow \text{Tr} \\
\text{Cor}_{Z} (CW) & \xrightarrow{\text{Cor}_{Z}} & \text{Z} \text{(AnSm}(\mathbb{C})) & \xrightarrow{\text{An}} & \text{Z} \text{(SmVar}(\mathbb{C})) \\
\downarrow {\iota_n} & & \downarrow {\iota_{\text{var}}} & & \downarrow {\iota_{\text{var}}} \\
Z \text{(AnSp}(\mathbb{C})) & \xrightarrow{\text{An}} & Z \text{(Var}(\mathbb{C})) \\
\end{array}
\]

In proposition 25(ii), we prove that $\text{Cw}^*$, and hence $\tilde{\text{Cw}}^*$, derive trivially for the $(\mathbb{B}^1, usu)$, resp. $(\mathbb{A}^1, et)$, and $(\mathbb{P}^1, usu)$ model structure. Our definition of the Betti realization functor is the following:

**Definition 3.**  
(i) The CW-Betti realization functor (without transfers) is the composite:

$$\tilde{\text{Bti}}_0 : \text{DA}^{-1}(\mathbb{C}, Z) \xrightarrow{\tilde{\text{Cw}}^*} \text{CwDA}^{-1}(\mathbb{Z}) \xrightarrow{\text{Re}_{\text{Cw}}} \text{D}^{-1}(\mathbb{Z})$$

(ii) The CW-Betti realization functor with transfers is the composite:

$$\tilde{\text{Bti}}^* : \text{DM}^{-1}(\mathbb{C}, Z) \xrightarrow{\tilde{\text{Cw}}^*} \text{CwDM}^{-1}(\mathbb{Z}) \xrightarrow{\text{Re}_{\text{Cw}}} \text{D}^{-1}(\mathbb{Z})$$

Since $\tilde{\text{Cw}}^*$ derive trivially by proposition 25(ii) and $L \text{Tr}^* : \text{CwDA}^{-1}(\mathbb{Z}) \to \text{CwDM}^{-1}(\mathbb{Z})$ is the inverse of $\text{Tr}_*$ by theorem 17(i), we have $\tilde{\text{Bti}}_0 = \tilde{\text{Bti}}^* \circ L \text{Tr}^*$. In section 3.1 we prove (c.f. theorem 18) that the absolute version of this construction coincide with Ayoub’one.

**Theorem 5.**  
(i) For $Y \in \text{Var}(\mathbb{C})$, and $E \subset Y$ a subvariety, we have $\text{Bti}^* \text{M}(Y, E) = \tilde{\text{Bti}}_0^* \text{M}(Y, E)$

(ii) For $X, Y \in \text{Var}(\mathbb{C})$, $D \subset X$, $E \subset Y$ subvarieties, and $n \in \mathbb{Z}$, $n \leq 0$, the following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^{-1}(\mathbb{C}, Z)}(\text{M}(X, D), \text{M}(Y, E)[n]) & \xrightarrow{\tilde{\text{Cw}}^*} & \text{Hom}_{\text{CwDM}^{-1}(\mathbb{Z})}(\text{M}(X, D), \text{M}(Y, E)[n]) \\
\downarrow \text{An}^* & & \downarrow \text{Re}_{\text{Cw}} \\
\text{Hom}_{\text{A}^{-1}(\mathbb{C}, Z)}(\text{M}(X, D), \text{M}(Y, E)[n]) & \xrightarrow{\text{Re}_{\text{Cw}}^{-1}} & \text{Hom}_{\text{D}^{-1}(\mathbb{Z})}(\text{M}(X, D), \text{M}(Y, E)[n]) \\
\end{array}
\]

where we denoted for simplicity $X$ for $X^{\text{an}}$ and $X^{\text{cw}}$, and similarly for $D, Y$ and $E$.

To prove this theorem, we give, for $X \in \text{AnSp}(\mathbb{C})$, an equivalence $(\mathbb{B}^1, usu)$ local $B(\text{Ztr}(X)) : \text{sing}_{\text{D}}, \text{Ztr}(X) \to \text{Cw}_*, \text{sing}_{\text{tr}}, \text{Ztr}(X^{\text{cw}})$ by considering the two canonical morphism of functors (see section 3.1),

- the morphism $\psi_{\text{Cw}}^*$, which, for $G^* \in PC^{-1}(\text{An})$, associate the following canonical morphism $\psi_{\text{Cw}}^*(G^*) : \text{Cw}^*(\text{sing}_{\text{an}}^* G^*) \to \text{sing}_{\text{tr}}^* \text{Cw}^* G^*$ in $PC^{-1}(\text{CW})$,
• the morphism \( \psi_{\text{CW}} \), which, for \( F^* \in PC^- \), associate the following canonical morphism \( \psi_{\text{CW}}(F^*) : Cw(\text{sing}_{i\text{tr}} F^*) \to \text{sing}_{i\text{tr}} Cw F^* \) in \( PC^- (CW) \).

We define the following two morphism of functors:

(i) the morphism \( W \), which, for \( G^* \in PC^- (An) \), associate the composition in \( PC^- (CW) \):

\[
W(G^*) : Cw^*(\text{sing}_{i\text{tr}} G^*) \xrightarrow{\text{Cw}^*(G^*)} \text{sing}_{i\text{tr}} Cw G^*.
\]

(ii) the morphism \( \tilde{W} \), which, for \( F^* \in PC^- \), associate the composition in \( PC^- (CW) \):

\[
\tilde{W}(F^*) : \text{Cw}^*(\text{sing}_{i\text{tr}} F^*) \xrightarrow{\text{Cw}^*(F^*)} \text{sing}_{i\text{tr}} \text{Cw}^* F^*.
\]

The morphism of functor \( B \) is defined, by associating to \( G^* \in PSh(\text{Cor}_{Z}^f \text{AnSp}(\mathbb{C}), C^- (\mathbb{Z})) \), the composite

\[
B(G^*) : \text{sing}_{i\text{tr}} G^* \xrightarrow{\text{ad}(\text{Cw}^*, Cw^*)} \text{Cw}^* G^* \xrightarrow{\text{Cw}^*(W(G^*))} \text{Cw}^* \text{sing}_{i\text{tr}} G^* \xrightarrow{\psi_{\text{CW}}(G^*)} \text{sing}_{i\text{tr}} \text{Cw}^* G^*.
\]

in \( PSh(\text{Cor}_{Z}^f \text{AnSp}(\mathbb{C}), C^- (\mathbb{Z})) \) We deduce this theorem from the proposition \( 27 \):

**Proposition 3.**

1. For \( Y \in \text{AnSp}(\mathbb{C}) \) and \( E \subset Y \) an analytic subset, \( e_{\text{an}^*}(B(Z_{\text{tr}}(Y, E))) : \text{sing}_{i\text{tr}} Z_{\text{tr}}(Y, E) \to \text{sing}_{i} Z_{\text{tr}}(Y^{\text{cw}}, E^{\text{cw}}) \)

is a quasi isomorphism in \( C^- (\mathbb{Z}) \).

2. For \( Y \in \text{AnSp}(\mathbb{C}) \), and \( E \subset Y \) an analytic subset, the morphism

\[
B(Z_{\text{tr}}(Y, E)) : \text{sing}_{i\text{tr}} Z_{\text{tr}}(Y, E) \to \text{Cw}^* \text{sing}_{i\text{tr}} Z_{\text{tr}}(Y^{\text{cw}}, E^{\text{cw}})
\]

is an equivalence \( (\mathbb{D}^1, \text{usu}) \) local in \( PSh(\text{Cor}_{Z}^f \text{AnSp}(\mathbb{C}), C^- (\mathbb{Z})) \).

We deduce the point (ii) of this proposition from point (i). To prove point (i) we reduce to the smooth case using a desingularization by \( 9 \). For \( X \in \text{AnSp}(\mathbb{C}) \) the morphism of complexes

\[
e_{\text{an}^*} B(Z_{\text{tr}}(X)) : \text{sing}_{i\text{tr}} Z_{\text{tr}}(X) \to \text{sing}_{i} Z_{\text{tr}}(X^{\text{cw}})
\]

is given by associating to \( \alpha \in Z_{\text{tr}}^{f/\mathbb{D}^n}(\mathbb{D}^n \times X) \), the restriction \( Z_{\text{tr}}(X^{\text{cw}})(i_{n})(\alpha^{\text{cw}}) = \alpha|_{\mathbb{I}^n \times X^{\text{cw}}} \) of \( \alpha^{\text{cw}} \in Z_{\text{tr}}^{f/\mathbb{D}^n}(\mathbb{D}^n \times X^{\text{cw}}) \) by the closed embedding of CW complexes \( i_{n} \times I_{X^{\text{cw}}} : \mathbb{I}^n \times X^{\text{cw}} \to \mathbb{D}^n \times X^{\text{cw}} \) induced by the closed embedding \( i_{n} : \mathbb{I}^n \to \mathbb{D}^n \). If \( X \) is smooth connected, we see using a covering by geodesically convex open subset, we see that there exist a countable open covering by open subsets isomorphic to open balls in \( \mathbb{C}^{dx} \) such that the intersection of a finite number of members are either empty or isomorphic to an open ball in \( \mathbb{C}^{dx} \).

In the section 3.2, we use our construction of the Betti realization functor via CW complexes to give, for \( X, Y \in \text{Var}(\mathbb{C}) \) and \( E \subset Y \) a closed subvariety, an explicit image of a morphism \( \alpha \in \text{Hom}_{PC^-}(Z_{\text{tr}}(X), Z_{\text{tr}}(Y, E)[n]) \) by the Betti realization functor.

**Definition 4.**

1. For \( V \in \text{Var}(\mathbb{C}) \) and \( p \in \mathbb{N} \), we consider the Bloch cycle complex \( Z^{p}(V, \ast) \), and \( Z^{p}(V^{\text{cw}}, \ast) \) the cubical complex such that for \( n \in \mathbb{N} \), \( Z^{p}(V^{\text{cw}}, n) \subset Z^{p}(\mathbb{D}^n \times X^{\text{cw}}) \) is the abelian subgroup consisting of cycle meeting the face of \( \mathbb{I}^n \) properly and whose differential is given by, for \( \gamma \in Z^{p}(V^{\text{cw}}, n) \), \( \partial \gamma = \sum_{i=1}^{n} (-1)^{i} \gamma|_{\mathbb{I}^n \times V^{\text{cw}}} - \gamma|_{\mathbb{I}_{i,0}^n \times V^{\text{cw}}} - \gamma|_{\mathbb{I}_{i,1}^n \times V^{\text{cw}}} \) where \( \mathbb{I}_{i,1}^n \subset \mathbb{I}^n \) are the faces. We then denote

\[
\hat{T}^{p} : Z^{p}(V, \ast) \to Z^{2p}(V^{\text{cw}}, \ast) , \text{ for } \alpha \in Z^{p}(V, n) \to \alpha|_{\mathbb{D}^n \times V^{\text{cw}}}
\]

the map complexes given by restriction with respect to the closed embedding of CW complexes \( i_{n} \times I_{V} : \mathbb{I}^n \times V^{\text{cw}} \to \mathbb{D}^n \times V^{\text{cw}} \) whose image is \( [0, \infty) \times V^{\text{cw}} \subset \mathbb{D}^n \times V^{\text{cw}} \) (c.f. notations).
(ii) Let $V \in \text{Var}(\mathbb{C})$ quasi-projective. Let $Y \in \text{PVar}(\mathbb{C})$ be a compactification of $V$ (e.g., the projectivisation of $V$) and $E = Y \setminus V$. The higher cycle class map \([\tilde{\alpha}]\) is the morphism of complexes of abelian groups:

$$T^n : Z^p(V, *) \to C^\text{sing}_{2dV-2p+1}(Y^{cu}, E^{cu}, Z), \quad \alpha \mapsto [p_Y(\tilde{T}_n \alpha)] = [p_Y(\tilde{\alpha}^{cu}_{\text{sing}}_{X \times Y^{cu}})],$$

where $\tilde{\alpha} \in Z^p(Y \times \square^*)$ is the closure of $\alpha$ and $p_Y : Y^{cu} \times \square^* \to Y^{cu}$ is the projection.

Let $X \in \text{SmVar}(\mathbb{C})$ and $Y \in \text{Var}(\mathbb{C})$. Let $E \subset Y$ be a closed subset and $V = Y \setminus E$. Denote by $j : V \hookrightarrow Y$ the open embeddings. We have the open embedding $I_X \times j : X \times V \hookrightarrow X \times Y$. The map of complexes

$$T^n_{X \times V} : Z^dV(X \times Y, *) \to Z^{2dV}(X^{cu} \times Y^{cu}, *)$$

induces maps denoted by the same way on the subcomplexes indicated in the following diagram in $C^-(\mathbb{Z})$:

\[
\begin{array}{ccc}
Z^dV(X \times V, *) & \xrightarrow{T^n_{X \times V}} & Z^{2dV}(X^{cu} \times V^{cu}, *) \\
\downarrow \left(1_X \times j^*\right) & & \downarrow \left(1_X \times j^{cu*}\right) \\
C_w(Z_{tr}(Y,E)(X)) & \xrightarrow{T^n_{X \times V}} & \text{sing}_wZ_{tr}(Y^{cu}, E^{cu})(X^{cu}) \\
\downarrow \left(c(Y,E) \circ Z_{tr}(j)(X)\right) & & \downarrow \left(c(Y^{cu}, E^{cu}) \circ Z_{tr}(j^{cu*})\right)(X^{cu}) \\
C_w(Z_{tr}(V)(X)) & \xrightarrow{T^n_{X \times V}} & \text{sing}_wZ_{tr}(V^{cu})(X^{cu})
\end{array}
\]

where,

- $1_X \times j^* : Z^dV(X \times Y, Z) \hookrightarrow Z^dV(X \times V, Z) ; \quad Z \mapsto Z \cap (X \times V)$
- $1_X \times j^{cu*} : Z^{2dV}(X^{cu} \times Y^{cu}, Z) \hookrightarrow Z^{2dV}(X^{cu} \times V^{cu}, Z) ; \quad Z \mapsto Z \cap (X \times V)$

Recall that $D^\text{tr}(\mathbb{A}^1, et) : PC^- \to DM^- (\mathbb{C}, \mathbb{Z})$ and $D^\text{tr}(\mathbb{A}^1, usu) : PC^- (\mathbb{C}W) \to CwDM^- (\mathbb{Z})$ are the canonical localization functors. We prove in proposition \([28]\) the following

**Proposition 4.** Let $X \in \text{SmVar}(\mathbb{C})$ and $Y \in \text{Var}(\mathbb{C})$. Let $E \subset Y$ be a closed subset and $V = Y \setminus E$. For $n \in \mathbb{Z}$, $n \leq 0$, the following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{PC^-}(Z_{tr}(X), C_wZ_{tr}(Y,E)[n]) & \xrightarrow{H^n(T^n_{X \times V})} & \text{Hom}_{PC^- (\mathbb{C}W)}(Z_{tr}(X^{cu}), \text{sing}_wZ_{tr}(Y^{cu}, E^{cu})[n]) \\
\downarrow \text{D}(\mathbb{A}^1, et) & & \downarrow \text{D}(\mathbb{A}^1, usu) \\
\text{Hom}_{DM^- (\mathbb{C}, \mathbb{Z})}(M(X), M(Y,E)[n]) & \xrightarrow{C_w^*} & \text{Hom}_{CwDM^- (\mathbb{Z})}(M(X^{cu}), M(Y^{cu}, E^{cu})[n]),
\end{array}
\]

On the other side, we identified in proposition \([28]\) in section 2.3 the image of a morphism $T \in \text{Hom}_{PC^- (\mathbb{C}W)}(Z_{tr}(X^{cu}), \text{sing}_wZ_{tr}(Y^{cu}, E^{cu})[n])$ under the $(\mathbb{A}^1, usu)$ localization functor with the action on homology : Using proposition \([28]\) and proposition \([23]\)(i), we immediately deduce from theorem \([18]\) the following (c.f.\([3]\))

**Corollary 1.** Let $X \in \text{SmVar}(\mathbb{C})$, $Y \in \text{Var}(\mathbb{C})$, $E \subset Y$ a closed subvariety and $V = Y \setminus E$ the open complementary. Let $n \in \mathbb{Z}$, $n \leq 0$. Then,

(i) the following diagram \([40]\) is commutative

\[
\begin{array}{ccc}
\text{Hom}_{PC^-}(Z_{tr}(X), C_wZ(Y,E)[n]) & \xrightarrow{H^n(T^n_{X \times V})} & \text{Hom}_{PC^- (\mathbb{C}W)}(Z_{tr}(X), \text{sing}_wZ_{tr}(Y,E)[n]) \\
\downarrow \text{An}^* \circ \text{D}(\mathbb{A}^1, et) & & \downarrow \text{R}_{\text{an}^*} \circ \text{D}(\mathbb{A}^1, usu) \\
\text{Hom}_{\text{AnDM}^- (\mathbb{Z})}(M(X), M(Y,E)[n]) & \xrightarrow{R^n_{\text{an}^*}} & \text{Hom}_{D^- (\mathbb{Z})}(\text{sing}_wZ_{tr}(X), \text{sing}_wZ_{tr}(Y,E)[n])
\end{array}
\]

where we denoted for simplicity $X$ for $X^{an}$ and $X^{cu}$, and similarly for $Y$ and $E$. 


(ii) for $\alpha \in \text{Hom}_{PC^-(\mathbb{Z}_tr(X),\mathbb{Z}(Y,E)[n])}$, we have
\[
\text{Bti}^* \circ D(\mathbb{A}^1, et)(\alpha) = K_n(X,Y)(p_{X,Y}(H^nT_{X \times V}(\alpha))),
\]
where $p_{X,Y} : \mathbb{I}^n \times X^{cw} \times Y^{cw} \to X^{cw} \times Y^{cw}$ is the projection.

We deduce from this corollary, proposition 29 and lemma 11, the following main result which say that Ayoub’s Betti realization functor factor through Nori motives. Consider the functor $\mathcal{N} : C^b(\text{Cor}_Z(\text{SmVar}(\mathbb{C}))) \to D^b(\mathcal{N})$ from the category of bounded complexes of correspondences of algebraic varieties to the derived category of Nori motives (c.f.12). This functor factor through the localization functor :
\[
C^b(\text{Cor}_Z(\text{SmVar}(\mathbb{C}))) \xrightarrow{\mathcal{N}} D^b(\mathcal{N}) \xrightarrow{\mathcal{N}} D^b(\mathbb{Z})\]
Denote by $o_N : D^b(\mathcal{N}) \to D^b(\mathbb{Z})$ the forgetful functor. In theorem 19 we prove :

**Theorem 6.** (i) For $X \in \text{SmVar}(\mathbb{C})$, $\text{Bti}^* \circ D(\mathbb{A}^1, et)(\mathbb{Z}(X)) = o_N \circ \mathcal{N}(X)$

(ii) For $X, Y \in \text{SmVar}(\mathbb{C})$, the following diagram commutes
\[
\begin{array}{ccc}
\text{Hom}_{PC^-(\mathbb{Z}(X),\mathbb{Z}(Y))} & \xrightarrow{\mathcal{N}} & \text{Hom}_{D^b(\mathcal{N})}(N(X), N(Y)) \\
\downarrow \text{D}(\mathbb{A}^1, et) & & \downarrow \text{D}(\mathbb{A}^1, et) \\
\text{Hom}_{DM^-(\mathbb{C},\mathbb{Z})}(M(X), M(Y)) & \xrightarrow{\text{Bti}^*} & \text{Hom}_{D^b(\mathbb{Z})}(C^*(X, Z), C^*(Y, Z))
\end{array}
\]

(iii) The Betti realization functor factor through Nori motives. That is $\text{Bti}^* = o_N \circ \mathcal{N}$

Let $V \in \text{Var}(\mathbb{C})$ quasi-projective. Let $Y \in \text{PVar}(\mathbb{C})$ a compactification of $V$ and $E = Y \setminus V$. Denote by $j : V \hookrightarrow Y$ the open embedding.

- For $p \leq d_V$, consider the following composition of isomorphisms of abelian groups
\[
H^{p,n}(V) : \text{CH}^p(V, n) \xrightarrow{i^*} \text{CH}^dV(\mathbb{A}^{d_V - p} \times V, n) \xrightarrow{(H^n(\mathbb{I} \times j)^*)^{-1}} \text{Hom}_{PC^-(\mathbb{Z}_tr(\mathbb{A}^{d_V - p}), \mathbb{Z}_tr(Y, E)[n])} \xrightarrow{D(\mathbb{A}^1, et)} \text{Hom}_{DM^-(\mathbb{C},\mathbb{Z})}(M(\mathbb{A}^1), M(Y, E)[n])
\]

- For $p \geq d_V$ and $E' = (Y \times \mathbb{P}^{d_V - d_V}) \setminus (V \times \mathbb{A}^{d_V - d_V})$, consider the following composition of isomorphisms of abelian groups
\[
H^{p,n}(V) : \text{CH}^p(V, n) \xrightarrow{p_V^*} \text{CH}^p(\mathbb{A}^{d_V} \times V, n) \xrightarrow{(H^n(\alpha \times j)^*)^{-1}} \text{Hom}_{PC^-(\mathbb{Z}_tr, \mathbb{Z}_tr(\mathbb{P}^{d_V - d_V} \times Y, E')[n])} \xrightarrow{D(\mathbb{A}^1, et)} \text{Hom}_{DM^-(\mathbb{C},\mathbb{Z})}(\mathbb{Z}, M(Y, E)[n])
\]

We prove that under these identifications, the image of the Betti realization functor on morphism coincide with the Bloch cycle class map (c.f. theorem 20) :

**Theorem 7.** Let $V \in \text{Var}(\mathbb{C})$. Let $Y \in \text{PVar}(\mathbb{C})$ be a compactification of $V$ and $E = Y \setminus V$. Then,
(i) for \( p \leq d_V \), the following diagram commutes:

\[
\begin{array}{ccc}
CH^n(V,n) & \xrightarrow{H^n_{TV}} & H^{n+p}(Y,E,Z) \\
H^{p,n}(V) & \sim & \sim K_n(pt,(Y,E)) \\
\end{array}
\]

\[\text{Hom}_{DM^-}(\mathbb{C},\mathbb{Z})(M(A^{\leq d_V-p}),M(Y,E)[n]) \xrightarrow{\text{Bi}^*} \text{Hom}_{DM^-}(\mathbb{Z},C_*(Y,E)[n])\]

(ii) for \( p \geq d_V \), the following diagram commutes:

\[
\begin{array}{ccc}
CH^n(V,n) & \xrightarrow{H^n_{TV}} & H^{n+p}(Y,E,Z) \\
H^{p,n}(V) & \sim & \sim K_n(pt,(Y,E)) \\
\end{array}
\]

\[\text{Hom}_{DM^-}(\mathbb{C},\mathbb{Z})(M(A^{p-d_V} \times Y,E)[n]) \xrightarrow{\text{Bi}^*} \text{Hom}_{DM^-}(\mathbb{Z},C_*(Y,E)[n])\]

In the last section we give a relative version of theorem [18]. We first give in theorem [25] a relative version of theorem [10].

**Theorem 8.** Let \( S \in \text{CW} \).

(i) For \( F^k \in \text{PSh}_2(\text{Cor}_{\mathbb{Z}}(\text{CW}^{sm}/S),C^{-}(\mathbb{Z})) \), \( \text{sing}_{\text{Z}} F^k \in \text{PSh}_2(\text{Cor}_{\mathbb{Z}}(\text{CW}^{sm}/S),C^{-}(\mathbb{Z})) \) is \( \mathbb{F} \) local and the inclusion morphism \( S(F^k) : F^k \to \text{sing}_{\text{Z}} F^k \) is an \( (\mathbb{F},\text{usu}) \) equivalence.

(ii) For \( F^k \in \text{PSh}_2(\text{Cor}_Z(\text{CW}^{sm}/S),C^{-}(\mathbb{Z})) \), \( \text{sing}_{\text{Z}} F^k \in \text{PSh}_2(\text{Cor}_Z(\text{CW}^{sm}/S),C^{-}(\mathbb{Z})) \) is \( \mathbb{F} \) local and the inclusion morphism \( S(F^k) : F^k \to \text{sing}_{\text{Z}} F^k \) is an \( (\mathbb{F},\text{usu}) \) equivalence.

We then prove in theorem [26] a relative version of point (ii) and (iii) of [17]:

**Theorem 9.** Let \( S \in \text{CW} \).

(i) The adjunction \( (e_{\text{cw}}(S)^*,e_{\text{cw}}(S)_*) : C^{-}(S) \Rightarrow \text{PSh}_2(\text{CW}^{sm}/S,C^{-}(\mathbb{Z})) \) is a Quillen equivalence for the \( (\mathbb{F},\text{usu}) \) model structures. That is, the derived functor \( e_{\text{cw}}(S)^* : D^{-}(S) \xrightarrow{\sim} \text{CwDA}^{-}(S,\mathbb{Z}) \) is an isomorphism and \( \text{Re}_{e_{\text{cw}}(S)}(S)_* : \text{CwDA}^{-}(S,\mathbb{Z}) \xrightarrow{\sim} D^{-}(S) \) is it inverse.

(ii) The adjunction \( (e_{\text{cw}}(S)^*,e_{\text{cw}}(S)_*) : C^{-}(S) \Rightarrow \text{PSh}_2(\text{Cor}_{\mathbb{Z}}(\text{CW}^{sm}/S),C^{-}(\mathbb{Z})) \) is a Quillen equivalence for the \( (\mathbb{F},\text{usu}) \) model structures. That is, the derived functor \( e_{\text{cw}}(S)^* : D^{-}(S) \xrightarrow{\sim} \text{CwDM}^{-}(S,\mathbb{Z}) \) is an isomorphism and \( \text{Re}_{e_{\text{cw}}(S)}(S)_* : \text{CwDM}^{-}(S,\mathbb{Z}) \xrightarrow{\sim} D^{-}(S) \) is it inverse.

By definition, for each \( S \in \text{Var}(\mathbb{C}) \), we have (c.f. [14]) the following commutative diagram of sites \( \text{DCat}(S) \)

\[
\begin{array}{ccc}
\text{DCat}(S) : \text{Cor}_{\mathbb{Z}}(\text{CW}^{sm}/S^{cw}) & \xrightarrow{\text{Cor}_{\mathbb{Z}}(\text{Var}(\mathbb{C})^{sm}/S^{an})} & \text{Cor}_{\mathbb{Z}}(\text{Var}(\mathbb{C})^{sm}/S) \\
\text{Tr}(S) & \xrightarrow{\text{Tr}(S)} & \text{Tr}(S) \\
\text{cw} : \text{Z}(\text{CW}^{sm}/S^{cw}) & \xrightarrow{\text{cw}(S)} & \text{Z}(\text{Var}(\mathbb{C})^{sm}/S^{an}) \\
\text{cw}(S) & \xrightarrow{\text{cw}(S)} & \text{Z}(\text{Var}(\mathbb{C})/S^{an}) \\
\text{i}_{\text{an}}(S) & \xrightarrow{\text{i}_{\text{an}}(S)} & \text{Z}(\text{Var}(\mathbb{C})/S) \\
\text{Z}(\text{AnSp}(\mathbb{C})/S^{an}) & \xrightarrow{\text{Z}(\text{Var}(\mathbb{C})^{sm}/S^{an})} & \text{Z}(\text{Var}(\mathbb{C})/S) \\
\end{array}
\]

For \( T,S \in \text{Var}(\mathbb{C}) \) and \( f : T \to S \) a morphism, the morphism of sites

- \( P(f) : \text{Var}(\mathbb{C})/T \to \text{Var}(\mathbb{C})/S, P(f^{an}) : \text{AnSp}(\mathbb{C})/T^{an} \to \text{AnSp}(\mathbb{C})/S^{an} \), and \( P(f^{cw}) : \text{CW}/T^{cw} \to \text{CW}/S^{cw} \) given by the pullback functor,
\[ P(f) : \text{DCat}(T) \to \text{DCat}(S). \] (7)

Our definition of the Betti realization functor in the relative setting is:

**Definition 5.** Let \( S \in \text{Var}(\mathbb{C}) \).

- (i) The CW-Betti realization functor (without transfers) is the composite:
  \[
  \widetilde{\text{Bti}}_0(S)^* : \text{DA}^{-}(S, \mathbb{Z}) \xrightarrow{\text{Cw}(S)^*} \text{CwDA}^{-}(S^{cw}, \mathbb{Z}) \xrightarrow{\text{Re}_{cw}(S)^*} D^{-}(S^{cw})
  \]

- (ii) The CW-Betti realisation functor with transfers is the composite:
  \[
  \widetilde{\text{Bti}}(S)^* : \text{DM}^{-}(S, \mathbb{Z}) \xrightarrow{\text{Cw}(S)^*} \text{CwDM}^{-}(S^{cw}, \mathbb{Z}) \xrightarrow{\text{Re}_{cw}(S)^*} D^{-}(S^{cw})
  \]

Similarly, in the relative case, since \( \text{Cw}(S)^* \) derive trivially by proposition 4(iii) and and \( L \text{Tr}(S^{cw})^* : \text{CwDA}^{-}(S^{cw}, \mathbb{Z}) \to \text{CwDM}^{-}(S^{cw}, \mathbb{Z}) \) is the inverse of \( \text{Tr}(S^{cw})^* \), (c.f.remark 1), we have \( \widetilde{\text{Bti}}_0(S)^* = \text{Bti}(S)^* \circ L \text{Tr}(S)^* \). As in Ayoub’s definition, it defines morphisms of homotopic 2-functors (c.f. theorem 2).

- \( S \in \text{Var}(\mathbb{C}) \mapsto \widetilde{\text{Bti}}_0(S) : \text{DA}^{-}(S, \mathbb{Z}) \to D^{-}(S) \)
- \( S \in \text{Var}(\mathbb{C}) \mapsto \widetilde{\text{Bti}}(S) : \text{DM}^{-}(S, \mathbb{Z}) \to D^{-}(S) \)

We finally prove in theorem 29 a relative version of the theorem 18 that is the construction of the relative Betti realization functor via CW complexes coincide with Ayoub’s one via analytic spaces:

**Theorem 10.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( M \in \text{DM}^{-}(S, \mathbb{Z}) \) is a constructible motive.

- (i) We have \( \text{Bti}^* M = \widetilde{\text{Bti}}^* M \)
- (ii) Let \( M_1, M_2 \in \text{DM}^{-}(S, \mathbb{Z}) \) constructible motives. Let \( F_i^* \in \text{PC}^{-}(S) \) such that \( M_i = D(\mathbb{H}_1, \text{et})(S)(F_i^*) \) for \( i = 1, 2 \). The following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^{-}(S)}(M_1, M_2) & \xrightarrow{\text{Cw}(S)^*} & \text{Hom}_{\text{CwDM}^{-}(S)}(\text{Cw}(S)^* M_1, \text{Cw}(S)^* M_2) \\
\text{An}(S)^* & \xrightarrow{\text{Re}_{\text{cw}}(S)^*} & \text{An}(S)^* M_1, \text{An}(S)^* M_2
\end{array}
\]

As in the absolute case, we define for \( S \in \text{AnSp}(\mathbb{C}) \), the morphism of functor \( B(S) \), by associating to \( G^* \in \text{PSh}(\text{Cor}_Z^f(\text{AnSp}^{sm}(\mathbb{C})/S), C^{-}(\mathbb{Z})) \), the morphism \( B(S)(G^*) \) which is the composite

\[
\begin{array}{ccc}
\text{Cw}(S)_* \text{Cw}(S)^* & \xrightarrow{\text{B}(S)(G^*)} & \text{Cw}(S)_* \text{Cw}(S)^* G^*
\end{array}
\]

in \( \text{PSh}(\text{Cor}_Z^f(\text{AnSp}(\mathbb{C})^{sm}/S), C^{-}(\mathbb{Z})) \), and deduce this theorem from the point (ii) of following proposition :
Proposition 5. Let $S \in \text{Var}(\mathbb{C})$ and $F^\bullet \in P\text{C}^- (S)$ such that $D(\mathbb{A}^1, \text{et})(S)(F^\bullet) \in \text{DM}^-(S, \mathbb{Z})$ is a constructible motive. Then,

(i) $\epsilon_{\text{an}}^*(S)_* B(S)(F^\bullet) : \text{sing}_{\mathbb{Q}} \text{An}(S)^* F^\bullet \to \text{sing}_{\mathbb{Q}} \text{Cw}(S)^* F^\bullet$ is an equivalence usu local in $C^-(S^n)$.

(ii) $B(S)(F^\bullet) : \text{sing}_{\mathbb{Q}} \text{An}(S)^* F^\bullet \to \text{Cw}(S^n)^* \text{sing}_{\mathbb{Q}} \text{Cw}(S)^* F^\bullet$ is an equivalence $(\mathbb{D}^1, \text{usu})$ local.

As in the absolute case, we deduce the point (ii) of this proposition from point (i). We prove the point (i) of this proposition using the absolute case (point (i) of the proposition 27).

2 Derived categories of motives of algebraic varieties, analytic spaces, and CW complexes

2.1 The derived category of mixed motives of algebraic varieties

For $X \in \text{Var}(k)$, and $\Lambda$ a commutative ring and $p \in \mathbb{N}$, we denote by $Z_p(X, \Lambda)$ the free $\Lambda$ module generated by the irreducible closed subspaces of $X$ of dimension $p$, by $Z^p(X, \Lambda) = Z_d X - p(X, \Lambda)$ the free $\Lambda$ module generated by the irreducible closed subspaces of $X$ of codimension $p$.

An algebraic variety $X \in \text{Var}(k)$ is aid to be proper (or complete) if the terminal map $a_X : X \to \text{Spec} k$ is universally closed, that is for all $Y \in \text{Var}(k)$, the projection $p_Y : X \times_k Y \to Y$ is closed (note that the topology on $X \times_k Y$ is finer than the product topology on the underlying topological spaces). For $X \in \text{Var}(k)$, a compactification of $X$ is a complete variety $\overline{X} \in \text{Var}(k)$ such that $X \subset \overline{X}$ is an open subset. For $X \in \text{Var}(k)$ quasi-projective, the projectivisation $X \in \text{PVar}(k)$ is a compactification of $X$.

For $X \in \text{Var}(k)$, we have the Bloch cycle complex $\mathbb{Z}(X, \ast)$, with for $n \in \mathbb{N}$, $\mathbb{Z}^p(X, n) \subset \mathbb{Z}^p(X \times \square^n)$ is the subgroup of the $p$ codimensional irreducible closed subspace of $X \times \square^n$ meeting all faces of $\square^n$ properly.

For $X, Y \in \text{Var}(k)$ and $\Lambda$ a commutative ring, we will use the following notations :

- if $X$ is smooth connected $Z^{fs/X}(X \times_k Y, \Lambda) \subset Z_d X (X \times_k Y, \Lambda)$ is the free $\Lambda$ submodule generated by the irreducible closed subspaces of $X \times_k Y$ which are finite and surjective over $X$,
- if $X$ is smooth, $Z^{fs/X}(X \times Y, \Lambda) := \bigoplus_i Z^{fs/X}(X_i \times Y, \Lambda)$ where $X = \bigcup_i X_i$, with $X_i$ the connected components of $X$,
- $Z^{ed/r}(X \times_k Y, \Lambda) \subset Z^{d + r}(X \times_k Y, \Lambda)$ the free $\Lambda$ module generated by the irreducible closed subspaces of $X \times_k Y$ dominant over $X$, and whose fibers over $X$ are either empty or equidimensional of relative dimension $r$.

By definition, for $X$ smooth, $Z^{fs/X}(X \times_k Y, \Lambda) \subset Z^{ed}/X (X \times_k Y, \Lambda)$ and $Z^{ed(p - d)/\square^n} (\square^n \times_k X, \Lambda) \subset Z_p(X, n) \otimes \mathbb{Z}$.

For $Y \in \text{Var}(k)$ irreducible and $j : V \hookrightarrow Y$ an open embedding, we denote by

$$j^* : \mathbb{Z}^p(Y, \Lambda) \to \mathbb{Z}^p(V, \Lambda); Z \mapsto Z \cap V \quad (8)$$

Definition 6. [13] We define $\text{Cor}_{\mathbb{A}}^{fs}(\text{SmVar}(k))$ to be the category whose objects are smooth algebraic varieties over $k$ and whose space of morphisms between $X, Y \in \text{SmVar}(k)$ is the free $\Lambda$ module $Z^{fs/X}(X \times_k Y, \Lambda)$. The composition of morphisms is defined in [13].

We have

- the additive embedding of categories $\text{Tr} : \mathbb{Z}(\text{SmVar}(k)) \to \text{Cor}_{\mathbb{A}}^{fs}(\text{SmVar}(k))$ which gives the corresponding morphism of sites $\text{Tr} : \text{Cor}_{\mathbb{Z}}^{fs}(\text{SmVar}(k)) \to \mathbb{Z}(\text{SmVar}(k))$.
- the inclusion functor $\epsilon_{\text{var}} : \{ pt \} \hookrightarrow \text{SmVar}(\mathbb{C})$, which gives the corresponding morphism of sites $\epsilon_{\text{var}} : \text{SmVar}(\mathbb{C}) \to \{ pt \}$.
the inclusion functor \( e_{tr}^* := \text{Tr} \circ e_{var} : \{ pt \} \hookrightarrow \text{Cor}_{Z}^{fs}(\text{SmVar}(\mathbb{C})) \) which gives the corresponding morphism of sites \( e_{tr}^* := \text{Tr} \circ e_{var} : \text{Cor}_{Z}^{fs}(\text{SmVar}(\mathbb{C})) \to \{ pt \} \).

We consider the following two big categories:

- \( \text{PSh}(\text{SmVar}(k), C^-(Z)) = \text{PSh}_{Z}(\text{SmVar}(k), C^-(Z)) \), the category of bounded above complexes of presheaves on \( \text{SmVar}(k) \), or equivalently additive presheaves on \( \text{PSh}(\text{SmVar}(k), C^-) \), sometimes, we will write for short \( P^- = \text{PSh}(\text{SmVar}(\mathbb{C}), C^-(Z)) \),
- \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)) \), the category of bounded above complexes of additive presheaves on \( \text{Cor}_{Z}^{fs}(\text{SmVar}(k)) \) sometimes, we will write for short \( PC^- = \text{PSh}(\text{Cor}_{Z}^{fs}(\text{SmVar}(\mathbb{C})), C^-(Z)) \),

and the adjunctions:

\[
\begin{align*}
(\text{Tr}^*, \text{Tr}_*) : & \text{PSh}(\text{SmVar}(k), C^-(Z)) \rightleftharpoons \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)) \\
(e_{var}^*, e_{var}) : & \text{PSh}(\text{SmVar}(\mathbb{C}), C^-(Z)) \rightleftharpoons C^-(Z), \\
(e_{tr}^*, e_{tr}) : & \text{PSh}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)) \rightleftharpoons C^-(Z),
\end{align*}
\]

given by \( \text{Tr} : \text{Cor}_{Z}^{fs}(\text{SmVar}(k)) \to \text{Z}(\text{SmVar}(k)) \), \( e_{var} : \text{SmVar}(k) \to \{ pt \} \) and \( e_{tr}^* : \text{Cor}_{Z}^{fs}(\text{SmVar}(k)) \to \{ pt \} \) respectively. We denote by \( a_{et} : \text{PSh}_{Z}(\text{SmVar}(k), Ab) \to \text{Sh}_{Z, et}(\text{SmVar}(k), Ab) \) the etale sheafification functor.

For \( X \in \text{SmVar}(k) \), we denote by
\[
Z(X) \in \text{PSh}(\text{SmVar}(k), C^-(Z)), \quad Z_{tr}(X) \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z))
\]
the presheaves represented by \( X \). They are etale sheaves.

For \( X \in \text{Var}(k) \), the have the presheaves

\[
\begin{align*}
Z_{tr}(X) & \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)), \quad Y \in \text{SmVar}(k) \mapsto Z_{tr}(X \times_{k} Y, Z), \\
Z_{eq}(X, 0) & \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)), \quad Y \in \text{SmVar}(k) \mapsto Z_{eq}(X \times_{k} Y, Z),
\end{align*}
\]

which are etale sheaves. Of course, if \( X \in \text{PVar}(k) \), then \( Z_{eq}(X, 0) = Z_{tr}(X) \).

We consider

- the usual monoidal structure on \( \text{PSh}(\text{SmVar}(k), C^-(Z)) \) and the associated internal Hom given by, for \( F^*, G^* \in \text{PSh}(\text{SmVar}(k), C^-(Z)) \) and \( Y \in \text{SmVar}(k) \),
  \[
  F^* \otimes G^*(X) : X \mapsto F^*(X) \otimes_{Z} G^*(X), \quad \text{Hom}(Z(Y), F^*) : X \mapsto F^*(X \times_{k} Y),
  \]
  \[
  \tag{10}
  \]
- the unique monoidal structure on \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)) \) such that, for \( X, Y \in \text{SmVar}(k) \),

\[
\begin{align*}
Z_{tr}(X) \otimes Z_{tr}(Y) & := Z_{tr}(X \times_{k} Y) \text{ and wich commute with colimits. It has an internal Hom which is given, for } X, Y \in \text{SmVar}(k) \text{ and } F^* \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z)), \quad \text{Hom}(Z_{tr}(Y), F^*) : X \mapsto F^*(X \times_{k} Y)
\end{align*}
\]

Together with these monoidal structure, the functor
\[
\text{Tr}^* : \text{PSh}(\text{SmVar}(k), C^-(Z)) \to \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{SmVar}(k)), C^-(Z))
\]
is monoidal.

**Definition 7.** \( \mathcal{E} \)

(i) We say that a morphism \( \phi : G_{1}^* \to G_{2}^* \) in \( \text{PSh}(\text{SmVar}(k), C^-(Z)) \) is an etale local equivalence if \( \phi_{et} : a_{et} \text{H}^{k}(G_{1}^*) \to a_{et} \text{H}^{k}(G_{2}^*) \) is an isomorphism for all \( k \in \mathbb{Z} \). The projective etale topology model structure on \( \text{PSh}(\text{SmVar}(k), C^-(Z)) \) is the left Bousfield localization of the projective model structure \( \mathcal{M}_{P}(\text{PSh}_{Z}(\text{SmVar}(k), C^-(Z))) \) with respect to the etale local equivalence.
(ii) We say that a morphism \( \phi : G^1_\bullet \to G^2_\bullet \) in \( \text{PSh}_\mathbb{Z}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z})) \) is an etale equivalence if and only if its restriction to \( \text{SmVar}(\mathbb{C}) \) \( \text{Tr}_*= G^1_\bullet \to \text{Tr}_*= G^2_\bullet \) is an etale equivalence. The projective etale topology model structure on \( \text{PSh}_\mathbb{Z}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective model structure \( \mathcal{M}_P(\text{PSh}_\mathbb{Z}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z}))) \) with respect to the etale local equivalence.

**Definition 8.**
(i) The projective \((\mathbb{A}^1, \text{et})\) model structure on \( \text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective etale topology model structure (c.f. definition 7(ii)) with respect to the class of maps \( \{ Z(X \times \mathbb{A}^1_\mathbb{C})_n [n] \to Z(X)_n, X \in \text{SmVar}(k), n \in \mathbb{Z} \} \).

(ii) The projective \((\mathbb{A}^1, \text{et})\) model structure on \( \text{PSh}_\mathbb{Z}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective etale topology model structure (c.f. definition 7(ii)) with respect to the class of maps \( \{ Z_{\text{tr}}(X \times \mathbb{A}^1_\mathbb{C})_n [n] \to Z(X)_n, X \in \text{SmVar}(k), n \in \mathbb{Z} \} \).

**Definition 9.**
(i) We define \( \text{DM}^{-}(k, \mathbb{Z})_{\text{et}} := \text{Ho}_{\mathbb{A}^1, \text{et}}(\text{PSh}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z}))) \), to be the derived category of (effective) motives, it is the homotopy category of the category \( \text{PSh}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z})) \) with respect to the projective \((\mathbb{A}^1, \text{et})\) model structure (c.f. definition 8(ii)). We denote by \( D^+(\mathbb{A}^1, \text{et}) : \text{PSh}_\mathbb{Z}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(k)), C^{-}(\mathbb{Z})) \to \text{DM}^{-}(k, \mathbb{Z})_{\text{et}} \), \( D^+(\mathbb{A}^1, \text{et})(F^\bullet) = F^\bullet \) the canonical localization functor.

(ii) By the same way, we denote \( \text{DA}^{-}(k, \mathbb{Z})_{\text{et}} := \text{Ho}_{\mathbb{A}^1, \text{et}}(\text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z}))) \) (cf. 8(ii)) and \( D(\mathbb{A}^1, \text{et}) : \text{PSh}_\mathbb{Z}(\text{Cor}^+_\mathbb{Z}(\text{SmVar}(\mathbb{C})), C^{-}(\mathbb{Z})) \to \text{DA}^{-}(k, \mathbb{Z})_{\text{et}} \), \( D(\mathbb{A}^1, \text{et})(F^\bullet) = F^\bullet \) the canonical localization functor.

For \( X \in \text{Var}(k) \),
- the (derived) motive of \( X \) is \( M(X) = D(\mathbb{A}^1, \text{et})(Z_{\text{tr}}(X)) \in \text{DM}^{-}(k) \).
- the (derived) motive with compact support of \( X \) is \( M^c(X) = D(\mathbb{A}^1, \text{et})(Z_{\text{eq}}(X, 0)) \in \text{DM}^{-}(k, \mathbb{Z}) \).

Of course, if \( X \in \text{PVar}(k) \), then \( Z_{\text{eq}}(X, 0) = Z_{\text{tr}}(X) \), so that \( M^c(X) = M(X) \).

For \( F^\bullet \in \text{PSh}(\text{SmVar}(k), \text{Ab}) \) and \( X \in \text{SmVar}(k) \), we have the complex \( F(X \times \square^1_\mathbb{k}) \) associated to the cubical object \( F(X \times \square^1_\mathbb{k}) \) in the category of abelian groups.

- If \( F^\bullet \in \text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z})) \),
  \[
  C_*F^\bullet := \text{Tot}(\text{Hom}(\mathbb{Z}(\square^1_\mathbb{k}), F^\bullet)) \in \text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z})) 
  \]
  is the total complex of presheaves associated to the bicomplex of presheaves \( X \mapsto F^\bullet(\square^1_\mathbb{k} \times_k X) \), and \( C_*F^\bullet := \varepsilon_{\text{var}}C_*F^\bullet = F^\bullet(\square^1_\mathbb{k}) \in C^{-}(\mathbb{Z}) \). We denote by \( S(F^\bullet) : F^\bullet \to C_*F^\bullet \)
  \[
  S(F^\bullet) : \cdots \to 0 \to F^\bullet \to 0 \to \cdots 
  \]
the inclusion morphism of \( \text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z})) \) : For \( f : F^\bullet_1 \to F^\bullet_2 \) a morphism \( \text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z})) \), we denote by \( S(f) : C_*F^\bullet_1 \to C_*F^\bullet_2 \), the morphism of \( \text{PSh}(\text{SmVar}(k), C^{-}(\mathbb{Z})) \) given by for \( X \in \text{SmVar}(k) \),
  \[
  S(f)(X) : \cdots \to F_1(\square^2 \times X) \to F_1(\square^1 \times X) \to F_1^\bullet(X) \to 0 \to \cdots 
  \]
  \[
  f(\square^2 \times X) \quad f(\square^1 \times X) \quad f(X) 
  \]
  \[
  \cdots \to F_2(\square^2 \times X) \to F_2(\square^1 \times X) \to F_2^\bullet(X) \to 0 \to \cdots .
  \]
• If \( F^\bullet \in \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)) \),

\[
\underline{C}_s F^\bullet := \underline{\text{Hom}}(Z_{tr}(\square^+_k), F^\bullet) \in \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)),
\]

(14)
is the complex of presheaves associated to the bicomplex of presheaves \( X \rightarrow F^\bullet(\square^+_k \times_k X) \), and \( C_s F^\bullet := e^{tr}_{cor} \underline{C}_s F^\bullet = F^\bullet(\square^+_k) \in C^{-}(Z) \). We have the inclusion morphism \( \underline{S}(\text{Tr}_s F^\bullet) : \text{Tr}_s F^\bullet \rightarrow \underline{C}_s \text{Tr}_s F^\bullet = \text{Tr}_s \underline{C}_s F^\bullet \)

which is a morphism in PSh(\( \text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z) \)) denoted the same way \( S(F^\bullet) : F^\bullet \rightarrow \underline{C}_s F^\bullet \).

For \( f : F^\bullet_1 \rightarrow F^\bullet_2 \) a morphism \( PC^- \), we have the morphism \( S(\text{Tr}_s f) : \text{Tr}_s \underline{C}_s F^\bullet_1 = \underline{C}_s \text{Tr}_s F^\bullet_1 \rightarrow \underline{C}_s \text{Tr}_s F^\bullet_2 = \text{Tr}_s \underline{C}_s F^\bullet_2 \)

which is a morphism in \( PC^- \) denoted the same way \( S(f) : \underline{C}_s F^\bullet_1 \rightarrow \underline{C}_s F^\bullet_2 \).

For \( F^\bullet \in \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)) \), we have by definition \( \text{Tr}_s \underline{C}_s F^\bullet = \underline{C}_s \text{Tr}_s F^\bullet \) and \( \text{Tr}_s S(F^\bullet) = S(\text{Tr}_s F^\bullet) \).

We now make the following definition

**Definition 10.** Let \( X \in \text{Var}(k) \) and \( D \subset X \) a subvariety. Denote by \( l : D \xhookrightarrow{} X \) the locally closed embedding. We define

- \( Z(X, D) = \text{coker}(Z(l)) \in \text{PSh}_2(\text{SmVar}(k), C^{-}(Z)) \) to be the cokernel of the injective morphism \( Z(l) : Z(D) \xhookrightarrow{} Z(X) \). By definition, we have the following exact sequence

\[
0 \rightarrow Z(D) \xrightarrow{Z(l)} Z(X) \xrightarrow{e^{c}(X,D)} Z(X,D) \rightarrow 0
\]

(15)
in \( \text{PSh}_2(\text{SmVar}(k), C^{-}(Z)) \).

- \( Z_{tr}(X, D) = \text{coker}(Z_{tr}(l)) \in \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)) \) to be the cokernel of the injective morphism \( Z_{tr}(l) : Z_{tr}(D) \xhookrightarrow{} Z_{tr}(X) \). By definition, we have the following exact sequence

\[
0 \rightarrow Z_{tr}(D) \xrightarrow{Z_{tr}(l)} Z_{tr}(X) \xrightarrow{c^{e}(X,D)} Z_{tr}(X,D) \rightarrow 0
\]

(16)
in \( \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)) \). By the exact sequences, we have \( \text{Tr}_s Z(X, D) = Z_{tr}(X, D) \)

In particular, we get the following exact sequence in \( \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)) \)

\[
0 \rightarrow \underline{C}_s Z_{tr}(D) \xrightarrow{S(Z_{tr}(l))} \underline{C}_s Z_{tr}(X) \xrightarrow{S(e^{c}(X,D))} \underline{C}_s Z_{tr}(X,D) \rightarrow 0
\]

We define

\( M(Y, E) = D(\mathbb{A}^1, et)(Z_{tr}(Y, E)) \in \text{CwDM}^- (Z) \)
to be the relative motive of the pair \((Y, E)\).

We now look at the behavior of the functors mentioned above with respect to the \((\mathbb{A}^1, et)\) model structure

**Proposition 6.**

(i) \( (\text{Tr}_s, \text{Tr}_s) : PSh(\text{SmVar}(k), C^{-}(Z)) \Rightarrow \text{PSh}_2(\text{Cor}^{fs}_Z(\text{SmVar}(k)), C^{-}(Z)) \) is a Quillen adjunction for the etale topology model structures (c.f. definition 7 (i) and (ii) respectively) and a Quillen adjunction for the \((\mathbb{A}^1, et)\) model structures (c.f. definition 3 (i) and (ii) respectively).

(ii) \( (e^{cor}, e^{cor}) : PSh(\text{SmVar}(C), C^{-}(Z)) \Rightarrow C^{-}(Z) \) is a Quillen adjunction for the etale topology model structure (c.f. definition 7 (i)) and a Quillen adjunction for the \((\mathbb{A}^1, et)\) model structure (c.f. definition 3 (i)).
Proposition 7. \([\text{[3][3]}]\)

(i) The functor \(\text{Tr}_*: \text{PSh}_Z(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}})) \to \text{PSh}(\text{SmVar}(k), C^-({\mathbb{Z}}))\) derive trivially.

(ii) For \(K^* \in C^-({\mathbb{Z}}), e_{\text{var}}K^* \in \mathbb{A}_k^1\) local.

(iii) For \(K^* \in C^-({\mathbb{Z}}), e_{\text{var}}K^* \in \mathbb{A}_k^1\) local.

Theorem 11. \([\text{[6]}]\)

(i) For \(F^* \in \text{PSh}(\text{SmVar}(k), C^-({\mathbb{Z}})), C_*F^* \in \text{PSh}(\text{SmVar}(k), C^-({\mathbb{Z}}))\) is \(\mathbb{A}_k^1\) local and the inclusion morphism \(S(F^*): F^* \to C_*F^*\) is an \((\mathbb{A}_k^1, \text{et})\) equivalence.

(ii) For \(F^* \in \text{PSh}(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}})), C_*F^* \in \text{PSh}(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}}))\) is \(\mathbb{A}_k^1\) local and the inclusion morphism \(S(F^*): F^* \to C_*F^*\) is an \((\mathbb{A}_k^1, \text{et})\) equivalence.

Theorem 12. \([\text{[2]}]\)

The adjunction \((\text{Tr}^*, \text{Tr}_*): \text{PSh}(\text{SmVar}(k), C^-({\mathbb{Z}})) \rightleftharpoons \text{PSh}(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}}))\) is a Quillen equivalence for the \((\mathbb{A}_1, \text{et})\) model structures. That is, the derived functor

\[
L \text{Tr}^*: \text{DA}^{-}(k, \mathbb{Z})_{\text{et}} \xrightarrow{\sim} \text{DM}^{-}(k, \mathbb{Z})_{\text{et}}
\]

is an isomorphism and \(\text{Tr}_*: \text{DM}^{-}(k, \mathbb{Z})_{\text{et}} \xrightarrow{\sim} \text{DA}^{-}(k, \mathbb{Z})_{\text{et}}\) is it inverse.

The following proposition identify the relative motive of a closed pair \((Y, E)\) of projective varieties to the motive of compact support of \(V = Y \setminus E\).

Proposition 8. \([\text{[6]}]\)

Let \(V \in \text{Var}(k)\) quasi projective. Let \(Y \in \text{PVar}(k)\) a compactification of \(V\) with \(E = Y \setminus V\). Denote by \(j: V \to Y\) the open embedding and \(i: E \to Y\). Then we have the following exact sequence in \(\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}}))\)

\[
0 \to C_*Z_{\text{tr}}(E) \xrightarrow{Z_{\text{tr}}(i)} C_*Z_{\text{tr}}(Y) \xrightarrow{j^*} C_*Z_{\text{eq}}(V, 0)
\]

with \(j^*: (I_X \times j)^*: Z^{fs/X}(X \times Y, \mathbb{Z}) \to Z^{ed(0)/X}(X \times V, \mathbb{Z})\). This say (c.f. definition \([10]\)) that \(j^*\) induces a quasi-isomorphism in \(\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}}))\)

\[
j^*: C_*Z_{\text{tr}}(Y, E) \to C_*Z_{\text{eq}}(V, 0)
\]

We finish this subsection by the following result of Suslin and Voevodsky:

Proposition 9. Let \(X \in \text{SmVar}(k), Y \in \text{PVar}(k)\) and \(n \in \mathbb{Z}, n < 0\). The morphism of abelian group

\[
D(\mathbb{A}_1, \text{et}) : \text{Hom}_{\text{PC}^-}(Z_{\text{tr}}(X), C_*Z_{\text{tr}}(Y)[n]) \to \text{Hom}_{\text{DM}^-}(-, (M(X), M(Y))[n])
\]

of the functor \(D(\mathbb{A}_1, \text{et}) : \text{PSh}_Z(\text{Cor}_Z^{fs}(\text{SmVar}(k)), C^-({\mathbb{Z}})) \to \text{DM}^-(k, \mathbb{Z})\) is an isomorphism.

Proof. For \(X \in \text{SmVar}(k)\) and \(Y \in \text{Var}(k)\), we have the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\text{PC}^-}(Z_{\text{tr}}(X), C_*Z_{\text{tr}}(Y)[n]) & \xrightarrow{D(\mathbb{A}_1, \text{et})} & \text{Hom}_{\text{DM}^-}(-, (M(X), M(Y))[n]) \\
\text{H}^nC_*Z_{\text{tr}}(Y)(X) & \xrightarrow{b(X)} & \text{H}^n_{\text{et}}(X, C_*Z_{\text{tr}}(Y))
\end{array}
\]

The equality on the right follow from the fact that

\[
\text{Hom}_{\text{DM}^-}(-, (Z_{\text{tr}}(X), C_*Z_{\text{tr}}(Y)[n]) = \text{Hom}_{\text{Hom}_{\text{PC}^-}}(-, (Z_{\text{tr}}(X), C_*Z_{\text{tr}}(Y)[n])
\]

since, by theorem \([11][ii])
\begin{itemize}
  \item $\mathcal{C}_eZtr(Y), \mathcal{C}_sZtr(Y)[n]$ are $\mathbb{A}^1$ local objects,
  \item $S(Ztr(X)) : Ztr(X) \rightarrow \mathcal{C}_eZtr(X)$ is an equivalence $(\mathbb{A}^1, et)$ local.
\end{itemize}

Moreover, we have (c.f. \cite{15}) $H^p_{an}(X, \mathcal{C}_eZtr(Y)) = \mathbb{H}^p_{zar}(X, \mathcal{C}_eZtr(Y))$. Now if $Y \in PVar(k)$ is projective, $\mathcal{C}_eZtr(Y) = \mathcal{C}_eZ_{eq}(Y, 0)$ satisfy Zariski descent (c.f. \cite{15}), hence $h(X)$ is an isomorphism.

\section{2.2 The derived category of mixed motives of analytic spaces}

For $X \in \text{AnSp}(\mathbb{C})$, and $\Lambda$ a commutative ring and $p \in \mathbb{N}$, we denote by $Z_p(X, \Lambda)$ the free $\Lambda$ module generated by the irreducible closed analytic subspaces of $X$ of dimension $p$.

For $X \in \text{AnSp}(\mathbb{C})$ irreducible, and $\Lambda$ a commutative ring and $p \in \mathbb{N}$, we denote by $Z_p(X, \Lambda) = Z_{d_X - p}(X, \Lambda)$ the free $\Lambda$ module generated by the irreducible closed analytic subspaces of $X$ of codimension $p$.

For $X, Y \in \text{AnSp}(\mathbb{C})$ and $\Lambda$ a commutative ring, we denote by

- if $X$ is smooth connected, $Z^{fs/X}(X \times Y, \Lambda) \subset Z_{d_X}(X \times Y, \Lambda)$ the free $\Lambda$ submodule generated by the irreducible closed analytic subspaces of $X \times Y$ which are finite and surjective over $X$,
- if $X$ is smooth, $Z^{fs/X}(X \times Y, \Lambda) := \oplus_i Z^{fs/X}(X_i \times Y, \Lambda)$ where $X = \sqcup_i X_i$, with $X_i$ the connected components of $X$.

\textbf{Definition 11.} \cite{23} Let $\text{Cor}^{fs}_\Lambda(\text{AnSm}(\mathbb{C}))$ be the category whose objects are complex analytic varieties over $\mathbb{C}$ and whose space of morphisms between $X, Y \in \text{AnSp}(\mathbb{C})$ is the free $\Lambda$ module $Z^{fs/X}(X \times Y, \Lambda)$. The composition of morphisms is defined in \cite{23}.

We have

- the additive embedding of categories $\text{Tr} : \mathbb{Z}(\text{AnSm}(\mathbb{C})) \rightarrow \text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C}))$ which gives the corresponding morphism of sites $\text{Tr} : \text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C})) \rightarrow \mathbb{Z}(\text{AnSm}(\mathbb{C}))$,
- the inclusion functor $\iota_{an} : \{pt\} \hookrightarrow \text{AnSm}(\mathbb{C})$, which gives the corresponding morphism of sites $\iota_{an} : \text{AnSm}(\mathbb{C}) \rightarrow \{pt\}$,
- the inclusion functor $\iota_{an}^r := \text{Tr} \circ \iota_{an} : \{pt\} \hookrightarrow \text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C}))$ which gives the corresponding morphism of sites $\iota_{an}^r := \text{Tr} \circ \iota_{an} : \text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C})) \rightarrow \{pt\}$.

We consider the following two big categories:

- $\text{PSh}(\text{AnSm}(\mathbb{C}), C^-(\mathbb{Z})) = \text{PSh}_{\mathbb{Z}}(\mathbb{Z}(\text{AnSm}(\mathbb{C}), C^-)(\mathbb{Z}))$, the category of bounded above complexes of presheaves on $\text{AnSm}(\mathbb{C})$, or equivalently additive presheaves on $\mathbb{Z}(\text{AnSm}(\mathbb{C}))$, sometimes, we will write for short $P^-(\text{An}) = \text{PSh}(\text{AnSm}(\mathbb{C}), C^-)(\mathbb{Z}))$,
- $\text{PSh}(\text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C})), C^-)(\mathbb{Z}))$, the category of bounded above complexes of additive presheaves on $\text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C}))$ sometimes, we will write for short $PC^-(\text{An}) = \text{PSh}(\text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C})), C^-)(\mathbb{Z}))$.

and the adjointions:

- $(\text{Tr}^*, \text{Tr}_*) : \text{PSh}(\text{AnSm}(\mathbb{C}), C^-)(\mathbb{Z})) \Rightarrow \text{PSh}(\text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C})), C^-)(\mathbb{Z}))$,
- $(\iota_{an}^*, \iota_{an^*}) : \text{PSh}(\text{AnSm}(\mathbb{C}), C^-)(\mathbb{Z})) \Rightarrow C^-(\mathbb{Z})$,
- $(\iota_{an}^r, \iota_{an^*}^r) : \text{PSh}(\text{Cor}^{fs}_\mathbb{Z}(\text{AnSm}(\mathbb{C})), C^-)(\mathbb{Z})) \Rightarrow C^-(\mathbb{Z})$,
Definition 14. (i) We define \( \phi \) and \( e^\text{tr}_{p} : \text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})) \to \{pt\} \) and \( e_{an} : \text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})) \to \{pt\} \). We denote by \( \phi_{usu} : \text{PSh}_{Z}(\text{AnSm}(\mathbb{C}), \text{Ab}) \to \text{Sh}_{Z,usu}(\text{AnSm}(\mathbb{C}), \text{Ab}) \) the sheafification functor for the usual topology.

For \( X \in \text{AnSm}(\mathbb{C}) \), we denote by

\[
Z(X) \in \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})), \ Z_{\text{tr}}(X) \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))
\]

the presheaves represented by \( X \). They are usu sheaves. For \( X \in \text{AnSp}(\mathbb{C}) \) a complex analytic space non smooth,

\[
Z_{\text{tr}}(X) \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})), Y \in \text{AnSm}(\mathbb{C}) \to Z^{fs/Y}(Y \times_{k} X, Z)
\]

is also an usu sheaf.

We consider

- the usual monoidal structure on \( \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \) and the associated internal Hom given by, for \( F^{*}, G^{*} \in \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \) and \( Y \in \text{AnSm}(\mathbb{C}) \),

\[
F^{*} \otimes G^{*}(X) : X \mapsto F^{*}(X) \otimes_{\mathbb{Z}} G^{*}(X), \ \text{Hom}(Z(Y), F^{*}) : X \mapsto F^{*}(X \times Y),
\]

- the unique monoidal structure on \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \) such that, for \( X, Y \in \text{AnSm}(\mathbb{C}) \),

\[
Z_{\text{tr}}(X) \otimes Z_{\text{tr}}(Y) := Z_{\text{tr}}(X \times Y)
\]

and which commute with colimits. It has an internal Hom which is given, for \( X, Y \in \text{AnSm}(\mathbb{C}) \) and \( F^{*} \in \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \),

\[
\text{Hom}(Z_{\text{tr}}(Y), F^{*}) : X \mapsto F^{*}(X \times Y)
\]

Together with these monoidal structure, the functor

\[
\text{Tr}^{*} : \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \to \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))
\]

is monoidal.

Definition 12. (i) We say that a morphism \( \phi : G_{1}^{*} \to G_{2}^{*} \) in \( \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \) is an usu local equivalence if \( \phi_{*} : a_{usu} H^{k}(G_{1}^{*}) \to a_{usu} H^{k}(G_{2}^{*}) \) is an isomorphism for all \( k \in \mathbb{Z} \). The projective usual topology model structure on \( \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective model structure \( \mathcal{M}_{P}(\text{PSh}_{Z}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z}))) \) with respect to the usu local equivalence.

(ii) We say that a morphism \( \phi : G_{1}^{*} \to G_{2}^{*} \) in \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \) is an usu local equivalence if and only if its restriction to \( \text{AnSm}(\mathbb{C}) \) \( \text{Tr}^{*} : \text{Tr}^{*} G_{1}^{*} \to \text{Tr}^{*} G_{2}^{*} \) is an usu local equivalence. The projective usual topology model structure on \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective model structure \( \mathcal{M}_{P}(\text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))) \) with respect to the usu local equivalence.

Definition 13. (i) The projective \( (\mathbb{D}^{1}, \text{usu}) \) model structure on \( \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective usual topology model structure (c.f. definition \( \mathcal{I} \)) with respect to the class of maps \( \{ Z(X \times \mathbb{D}^{1})[n] \to Z(X)[n], \ X \in \text{AnSm}(\mathbb{C}), n \in \mathbb{Z} \} \).

(ii) The projective \( (\mathbb{D}^{1}, \text{usu}) \) model structure on \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \) is the left Bousfield localization of the projective usual topology model structure (c.f. definition \( \mathcal{I} \)) with respect to the class of maps \( \{ Z_{\text{tr}}(X \times \mathbb{D}^{1})[n] \to Z(X)[n], \ X \in \text{AnSm}(\mathbb{C}), n \in \mathbb{Z} \} \).

Definition 14. (i) We define \( \text{AnDM}^{-}(\mathbb{Z}) := \text{Ho}_{\text{usu}}(\text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))) \), to be the derived category of motives of complex analytic space, it is the homotopy category of the category \( \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \) with respect to the projective \( (\mathbb{D}^{1}, \text{usu}) \) model structure (c.f. definition \( \mathcal{I} \)). We denote by

\[
D^{tr}(\mathbb{D}^{1}, \text{usu}) : \text{PSh}_{Z}(\text{Cor}_{Z}^{fs}(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \to \text{AnDM}^{-}(\mathbb{Z}), D^{tr}(\mathbb{D}^{1}, \text{usu})(F^{*}) = F^{*}
\]

the canonical localization functor.
(ii) We denote by the same way $\text{AnDA}^\times (Z) := \text{Ho}_{\Delta^1,usu}(\text{PSh}(\text{AnSm}(C), C^-(Z)))$ (c.f. definition [1](i)) and
\[
D(\Delta^1, usu) : \text{PSh}_{Z}(\text{AnSm}(C), C^-(Z)) \to \text{AnDA}^\times (Z), \quad D(\Delta^1, usu)(F^*) = F^*
\]
the canonical localization functor.

For $X \in \text{AnSp}(C)$, the (derived) motive of $X$ is $M(X) = D(\Delta^1, usu)(Z_{tr}(X)) \in \text{AnDM}^\times (Z)$.
Recall $\Delta$ denote the simplicial category Let

- $p_\Delta : \text{AnSm}(C) \to \Delta \times \text{AnSm}(C)$ be the morphism of site given by the projection functor $(X, i) \in \Delta \times \text{AnSm}(C) \mapsto p_\Delta((X, i)) = X \in \text{AnSm}(C)$,
- $p_\Delta : \text{Cor}_{Z}(\text{AnSm}(C)) \to \Delta \times \text{Cor}_{Z}(\text{AnSm}(C))$ be the morphism of site given by the projection functor $(X, i) \in \Delta \times \text{AnSm}(C) \mapsto p_\Delta((X, i)) = X \in \text{AnSm}(C)$

We have the adjonctions

- $(p_\Delta, p_{\Delta^+}) : \text{PSh}(\Delta \times \text{AnSm}(C), C^-(Z)) \Rightarrow \text{PSh}(\text{AnSm}(C), C^-(Z))$
- $p_\Delta : \text{PSh}(\Delta \times \text{Cor}_{Z}(\text{AnSm}(C)), C^-(Z)) \Rightarrow \text{PSh}(\text{Cor}_{Z}(\text{AnSm}(C)), C^-(Z))$.

In following lemma point (ii) is a generalization of point (i) to hypercoverings. We will only use point (i) in this paper.

**Lemma 1.**  (i) Let $X \in \text{AnSm}(C)$ and $X = \bigcup_{i \in J} U_i$ a covering by open subsets $U_i \subset X$, $J$ being a countable set. We denote by $j_i : U_i \hookrightarrow X$ the open embedding. For $I \subset J$ a finite subset, let $U_I = \bigcap_{i \in I} U_i$. Then,

- $\cdots \to \oplus_{i \in J} Z(U_i) \to \cdots \to \oplus_{i \in J} Z(U_i) \xrightarrow{\oplus_{i \in J} Z(j_i)} Z(X)$ is an usu local equivalence in $P^-(\text{An})$,
- $\cdots \to \oplus_{i \in J} Z_{tr}(U_i) \to \cdots \to \oplus_{i \in J} Z_{tr}(U_i) \xrightarrow{\oplus_{i \in J} Z_{tr}(j_i)} Z_{tr}(X)$ is an usu local equivalence in $PC^-(\text{An})$

(ii) Let $X \in \text{AnSm}(C)$ and $j_n : U_n \to X$ an hypercovering of $X$ by open subset (i.e. for all $n \in \Delta$, $U_n$ is an open subset and $j_n : U_n \to X$ is the open embedding), and $(U_\bullet, \bullet) : \cdots \to (U_n, n) \to \cdots$ the associated complex in $\Delta \times \text{AnSm}(C)$ Then,

- $j : Lp_\Delta^*(Z(U_\bullet, \bullet)) \to Z(X)$ is an isomorphism in $\text{Ho}_{usu}(\text{PSh}(\text{AnSm}(C), C^-(Z)))$
- $j : Lp_\Delta^*(Z_{tr}(U_\bullet, \bullet)) \to Z_{tr}(X)$ is an isomorphism in $\text{Ho}_{usu}(\text{PSh}_{Z}(\text{Cor}_{Z}(\text{AnSm}(C)), C^-(Z)))$

**Proof.** (i): It follows from the following two facts :

- the sequence $\cdots \to \oplus_{i \in J} Z(U_i) \to \cdots \to \oplus_{i \in J} Z(U_i)$ is clearly exact in $P^-(\text{An})$,
- the exactness of $[\oplus_{i \in J} Z(U_i) \xrightarrow{\oplus_{i \in J} Z(j_i)} Z(X) \to 0]$ in $\text{Ho}_{usu}(\text{PSh}(\text{AnSm}(C), C^-(Z)))$, follows from the fact that for $Y \in \text{AnSm}(C)$, $y \in Y$ and $f : Y \to X$ a morphism, there exists an open subset $V(y) \subset Y$ containing $y$ such that $f(V(y)) \subset U_i$, with $U_i$ containing $f(y)$.

Simaliy, the second point follows from the following two facts :

- the sequence $\cdots \to \oplus_{i \in J} Z_{tr}(U_i) \to \cdots \to \oplus_{i \in J} Z_{tr}(U_i)$ is clearly exact in $PC^-(\text{An})$,
- the exactness of $[\oplus_{i \in J} Z_{tr}(U_i) \xrightarrow{\oplus_{i \in J} Z_{tr}(j_i)} Z_{tr}(X) \to 0]$ in $\text{Ho}_{usu}(\text{PSh}_{Z}(\text{Cor}_{Z}(\text{AnSm}(C)), C^-(Z)))$, follows from the fact that for $Y \in \text{AnSm}(C)$, $y \in Y$ and $\alpha \in Z_{tr}(X)(Y)$ irreducible, there exists an open subset $V(y) \subset Y$ containing $y$ such that $\alpha_{(V(y))} \in Z_{tr}(U_i)(V(y))$, with $U_i$ containing $\alpha, y$.

(ii): see [1] Proposition 1.4 Etape 1 with our notation $p_\Delta^\times$ for $p_\Delta$. □
The following proposition is to use point (i) instead of point (ii) in the proof of the smooth case of point (i) of the proposition.

**Proposition 10.** Let \( X \in \text{AnSm}(\mathbb{C}) \) connected. Then there exist a countable open covering \( X = \bigcup_{i \in J} D_i \) such that for all finite subset \( I \subseteq J \), \( D_I := \cap_{i \in I} D_i = \emptyset \) or \( D_I \simeq \mathbb{D}^d \) is biholomorphic to an open ball \( \mathbb{D}^d \subseteq \mathbb{C}^d \).

**Proof.** Let \( x \in X \). As \( X \) is smooth, there exist an open neighborhood \( D_x \subseteq X \) of \( x \) in \( X \) such that \( D_x \) is geodesically convex and such that \( D_x \hookrightarrow \mathbb{C}^d \). As \( X \) a countable union of compact subset, we can extract a countable covering \( X = \bigcup_{i \in J} D_I \) of the open covering \( X = \cup_{x \in X} D_x \). As \( D_I \) is geodesically convex and \( D_I \) admits an open embedding \( D_I \hookrightarrow \mathbb{C}^d \) in \( \mathbb{C}^d \), \( D_I \simeq \mathbb{D}^d \).

We denote \( \mathbb{D}^n = D(0,1)^n \subseteq \mathbb{C}^n \). We see it as a pro-obj of \( \text{AnSm}(\mathbb{C}) \) \((\mathfrak{B})\). For \( F^\bullet \in \text{PSh}(\text{AnSm}(\mathbb{C}), \text{Ab}) \) and \( X \in \text{AnSm}(\mathbb{C}) \), we have the complex \( F(X \times \mathbb{D}^*) \) associated to the cubical object \( F(X \times \mathbb{D}^*) \) in the category of abelian groups.

- If \( F^\bullet \in \text{PSh}(\text{AnSm}(\mathbb{C}), C^-(\mathbb{Z})) \),

\[
\text{sing}_\mathfrak{B}, F^\bullet := \text{Hom}(\mathbb{Z}(\mathbb{D}^*), F^\bullet) \in \text{PSh}(\text{AnSm}(\mathbb{C}), C^-(\mathbb{Z})) \tag{21}
\]

is the total complex of presheaves associated to the bicomplex of presheaves \( X \mapsto F^\bullet(\mathbb{D}^* \times X) \), and \( \text{sing}_\mathfrak{D} F^\bullet := e_{\mathfrak{tr}} \text{sing}_\mathfrak{B}, F^\bullet = F^\bullet(\mathbb{D}^*) \in C^-(\mathbb{Z}) \). We denote by \( S(F^\bullet) : F^\bullet \to \text{sing}_\mathfrak{B}, F^\bullet \),

\[
\begin{align*}
\cdots & \quad \to 0 \quad \to 0 \quad \to F^\bullet \quad \to 0 \quad \to \cdots \\
\downarrow & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdots & \quad \to \text{sing}_\mathfrak{B} F^\bullet \quad \to \text{sing}_\mathfrak{B} F^\bullet \quad \to F^\bullet \quad \to 0 \quad \to \cdots
\end{align*}
\]

the inclusion morphism in \( \text{PSh}(\text{AnSm}(\mathbb{C}), C^-(\mathbb{Z})) \). For \( f : F_1^\bullet \to F_2^\bullet \) a morphism \( \text{PSh}(\text{AnSm}(\mathbb{C}), C^-(\mathbb{Z})) \),

\[
S(f)(X) : \cdots \to F_1(\mathbb{D}^2 \times X) \to F_1(\mathbb{D}^1 \times X) \to F_1^\bullet(X) \to 0 \to \cdots \tag{23}
\]

- If \( F^\bullet \in \text{PSh}_2(\text{Cor}^{fs}_\mathfrak{B}(\text{AnSm}(\mathbb{C})), C^-(\mathbb{Z})) \),

\[
\text{sing}_\mathfrak{D}, F^\bullet := \text{Hom}(\mathbb{Z}(\text{tr}(\mathbb{D}^*)), F^\bullet) \in \text{PSh}_2(\text{Cor}^{fs}_\mathfrak{B}(\text{AnSm}(\mathbb{C})), C^-(\mathbb{Z})) \tag{24}
\]

is the total complex of presheaves associated to the bicomplex of presheaves \( X \mapsto F^\bullet(\mathbb{D}^* \times X) \), and \( \text{sing}_\mathfrak{D} F^\bullet := e_{\mathfrak{tr}} \text{sing}_\mathfrak{B}, F^\bullet = F^\bullet(\mathbb{D}^*) \in C^-(\mathbb{Z}) \). We have the inclusion morphism \( S(\mathfrak{tr}, F^\bullet) : F_1^\bullet \to \text{sing}_\mathfrak{D}, F^\bullet \),

\[
S(\mathfrak{tr}, F^\bullet) : \mathfrak{tr}, F^\bullet \to \text{sing}_\mathfrak{D}, F^\bullet \quad \mathfrak{tr}, F^\bullet = \mathfrak{tr}, \text{sing}_\mathfrak{D}, F^\bullet \quad \mathfrak{tr}, F^\bullet \text{ is a morphism } PC^-(\text{An}), \quad \text{we have the morphism } \tag{22}
\]

which is a morphism in \( \text{PSh}(\text{Cor}^{fs}_\mathfrak{B}(\text{AnSm}(\mathbb{C})), C^-(\mathbb{Z})) \) denoted the same way \( S(F^\bullet) : F^\bullet \to \text{sing}_\mathfrak{D}, F^\bullet \). For \( f : F_1^\bullet \to F_2^\bullet \) a morphism \( PC^-(\text{An}) \), we have the morphism \( S(f) : \text{sing}_\mathfrak{D}, F_1^\bullet \to \text{sing}_\mathfrak{D}, F_2^\bullet \),

\[
S(f) : \text{sing}_\mathfrak{D}, F_1^\bullet \to \text{sing}_\mathfrak{D}, F_2^\bullet \quad \text{which is a morphism in } PC^-(\text{An}) \text{ denoted the same way } S(f) : \text{sing}_\mathfrak{D}, F_1^\bullet \to \text{sing}_\mathfrak{D}, F_2^\bullet.
\]
For $F^\bullet \in \text{PSh}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$, we have by definition $\text{Tr}_\ast \text{sing}_\mathcal{D}, F^\bullet = \text{sing}_\mathcal{D}, \text{Tr}_\ast F^\bullet$ and $\text{Tr}_\ast S(F^\bullet) = S(\text{Tr}_\ast F^\bullet)$.

We now make the following definition

**Definition 15.** Let $X \in \text{AnSp}(\mathbb{C})$ and $D \subset X$ an analytic subspace. Denote by $l : D \hookrightarrow X$ the locally closed embedding. We define

(i) $\mathbb{Z}(X, D) = \text{coker}(\mathbb{Z}(l)) \in \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z}))$ to be the cokernel of the injective morphism $\mathbb{Z}(l) : \mathbb{Z}_{tr}(D) \rightarrow \mathbb{Z}(X)$. By definition, we have the following exact sequence

$$0 \rightarrow \mathbb{Z}(D) \xrightarrow{\mathbb{Z}(l)} \mathbb{Z}(X) \xrightarrow{\mathcal{Z}(X,D)} \mathbb{Z}(X, D) \rightarrow 0 \quad (25)$$

in $\text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z}))$.

(ii) $\mathbb{Z}_{tr}(X, D) = \text{coker}(\mathbb{Z}_{tr}(l)) \in \text{PSh}_{\mathcal{Z}}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ to be the cokernel of the injective morphism $\mathbb{Z}_{tr}(l) : \mathbb{Z}_{tr}(D) \rightarrow \mathbb{Z}_{tr}(X)$. By definition, we have the following exact sequence

$$0 \rightarrow \mathbb{Z}_{tr}(D) \xrightarrow{\mathbb{Z}_{tr}(l)} \mathbb{Z}_{tr}(X) \xrightarrow{\mathcal{Z}(c(X,D))} \mathbb{Z}_{tr}(X, D) \rightarrow 0 \quad (26)$$

in $\text{PSh}_{\mathcal{Z}}(\text{Cor}_{\mathcal{Z}}^f(\text{SmVar}(k)), C^{-}(\mathbb{Z}))$. By the exact sequences, we have $\text{Tr}_\ast \mathbb{Z}(X, D) = \mathbb{Z}_{tr}(X, D)$.

In particular, we get the following exact sequence in $\text{PSh}_{\mathcal{Z}}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$

$$0 \rightarrow \text{sing}_\mathcal{D}, \mathbb{Z}_{tr}(D) \xrightarrow{\text{sing}_\mathcal{D}, \mathbb{Z}_{tr}(l)} \text{sing}_\mathcal{D}, \mathbb{Z}_{tr}(X) \xrightarrow{\mathcal{Z}(c(X,D))} \text{sing}_\mathcal{D}, \mathbb{Z}_{tr}(X, D) \rightarrow 0 \quad (27)$$

We define

$$M(Y, E) = D(\mathbb{D}^1, \text{usu})(\mathbb{Z}_{tr}(Y, E)) \in \text{AnDM}^{-}(\mathbb{Z})$$

to be the relative motive of the pair $(Y, E)$.

We now look at the behavior of the functors mentioned above with respect to the $(\mathbb{D}^1, \text{et})$ model structure

**Proposition 11.**

(i) $(\text{Tr}_\ast, \text{Tr}_\ast) : \text{PSh}(\text{AnSm}(k), C^{-}(\mathbb{Z})) \Rightarrow \text{PSh}_{\mathcal{Z}}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ is a Quillen adjunction for the usual topology model structures (c.f. definition §12 (i) and (ii) respectively) and a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ model structures (c.f. definition §13 (i) and (ii) respectively).

(ii) $(e_{an}^\ast, e_{an}_\ast) : \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z})) \Rightarrow C^{-}(\mathbb{Z})$ is a Quillen adjunction for the usual topology model structure (c.f. definition §12 (i)) and a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ model structure (c.f. definition §13 (ii)).

(iii) $(e_{tr}^\ast, e_{tr}_\ast) : \text{PSh}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \Rightarrow C^{-}(\mathbb{Z})$ is a Quillen adjunction for the usual topology model structure (c.f. definition §12 (ii)) and a Quillen adjunction for the $(\mathbb{D}^1, \text{usu})$ model structure (c.f. definition §13 (iii)).

**Lemma 2.**

(i) A complex of presheaves $F^\bullet \in \text{PSh}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ is $\mathbb{D}^1$ local if and only if $\text{Tr}_\ast F^\bullet \in \text{PSh}(\text{AnSm}(\mathbb{C}), C^{-}(\mathbb{Z}))$ is $\mathbb{D}^1$ local.

(ii) A morphism $\phi : F^\bullet \rightarrow G^\bullet$ in $\text{PSh}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence if and only if $\text{Tr}_\ast \phi : \text{Tr}_\ast F^\bullet \rightarrow \text{Tr}_\ast G^\bullet$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence.

**Proof.** (i): Let $g : F^\bullet \rightarrow L^\bullet$ be an usu local equivalence in $\text{PSh}(\text{Cor}_{\mathcal{Z}}^f(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ with $L^\bullet$ usu fibrant. Then,

- $F^\bullet$ is $\mathbb{D}^1$ local if and only if $L^\bullet$ is $\mathbb{D}^1$ local.
• By definition $\mathsf{Tr}_*$ preserve usu local equivalence, hence $\mathsf{Tr}_*g : \mathsf{Tr}_*F^\bullet \to \mathsf{Tr}_*L^\bullet$ is an usu local equivalence in $\mathsf{PSh}(\mathsf{AnSm}(\mathbb{C}), C^-(\mathbb{Z}))$. Thus, $\mathsf{Tr}_*F^\bullet$ is $\mathcal{D}^1$ local if and only if $\mathsf{Tr}_*L^\bullet$ is $\mathcal{D}^1$ local.

• As indicated in [3], $\mathsf{Tr}_*L^\bullet$ is also usu fibrant. By Yoneda lemma, for $X \in \mathsf{AnSm}(\mathbb{C})$, we have

$$\mathsf{Hom}_{\mathcal{P}(\mathsf{An})}(\mathbb{Z}(X)|n|, \mathsf{Tr}_*L^\bullet) = H^nL^\bullet(X) = \mathsf{Hom}_{\mathcal{P}(\mathsf{An})}(\mathbb{Z}_{tr}(X)|n|, L^\bullet)$$

Hence $L^\bullet$ is $\mathcal{D}^1$ local if and only if $\mathsf{Tr}_*L^\bullet$ is $\mathcal{D}^1$ local.

This proves (i).

(ii): Let us prove (ii).

• The only if part is proved in [3].

• Let $\phi : F^\bullet \to G^\bullet$ in $\mathsf{PSh}(\mathsf{Cor}^f_Z(\mathsf{AnSm}(\mathbb{C})), C^-(\mathbb{Z}))$ such that $\mathsf{Tr}_*\phi : \mathsf{Tr}_*F^\bullet \to \mathsf{Tr}_*G^\bullet$ is an equivalence $(\mathcal{D}^1, usu)$ local in $\mathsf{PSh}(\mathsf{AnSm}(\mathbb{C}), C^-(\mathbb{Z}))$. By definition $\mathsf{Tr}_*$ detect and preserve usu local equivalence, we can, up to replace $F^\bullet$ and $G^\bullet$ by usu equivalent presheaves, assume that $F^\bullet$ and $G^\bullet$ are usu fibrant. Let $K^\bullet \in \mathsf{PSh}(\mathsf{Cor}^f_Z(\mathsf{AnSm}(\mathbb{C})), C^-(\mathbb{Z}))$ be an $\mathcal{D}^1$ local object. By (i), $\mathsf{Tr}_*K^\bullet$ is $\mathcal{D}^1$ local. Hence, we have,

$$(\mathsf{Tr}_*F^\bullet, \mathsf{Tr}_*K^\bullet) \cong (\mathsf{Tr}_*F^\bullet, \mathsf{Tr}_*G^\bullet, \mathsf{Tr}_*K^\bullet)$$

$$(\mathsf{Tr}_*F^\bullet, \mathsf{Tr}_*K^\bullet) \cong (\mathsf{Tr}_*F^\bullet, \mathsf{Tr}_*G^\bullet, \mathsf{Tr}_*K^\bullet)$$

This proves the if part.

\[\square\]

**Proposition 12.**

(i) The functor $\mathsf{Tr}_* : \mathsf{PSh}_{\mathbb{Z}}(\mathsf{Cor}^f_Z(\mathsf{AnSm}(\mathbb{C})), C^-(\mathbb{Z})) \to \mathsf{PSh}(\mathsf{AnSm}(\mathbb{C}), C^-(\mathbb{Z}))$ derive trivially.

(ii) For $K^\bullet \in C^-(\mathbb{Z})$, $e_{an}^*K^\bullet$ is $\mathcal{D}^1$ local.

(iii) For $K^\bullet \in C^-(\mathbb{Z})$, $e_{an}^*K^\bullet$ is $\mathcal{D}^1$ local.

**Proof.**

(i): By lemma 2(ii), $\mathsf{Tr}_*$ preserve $(\mathcal{D}^1, usu)$ local equivalence.

(ii): It is proved in [1].

(iii): We have $\mathsf{Tr}_*e_{an}^*K^\bullet = e_{an}^*K^\bullet$. By (ii), $e_{an}^*K^\bullet$ is $\mathcal{D}^1$ local. By lemma 2(i), $\mathsf{Tr}_*$ detect $\mathcal{D}^1$ local object. This proves (iii).

\[\square\]

**Theorem 13.**

(i) For $F^\bullet \in \mathsf{PSh}(\mathsf{AnSm}(\mathbb{C}), C^-(\mathbb{Z}))$, $\underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet \in \mathsf{PSh}(\mathsf{AnSm}(\mathbb{C}), C^-(\mathbb{Z}))$ is $\mathcal{D}^1$ local and the inclusion morphism $S(F^\bullet) : F^\bullet \to \underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet$ is an $(\mathcal{D}^1, usu)$ equivalence.

(ii) For $F^\bullet \in \mathsf{PSh}_{\mathbb{Z}}(\mathsf{Cor}^f_Z(\mathsf{AnSm}(\mathbb{C})), C^-(\mathbb{Z}))$, $\underline{\mathsf{sing}}^\bullet_{\mathbb{Z}}F^\bullet \in \mathsf{PSh}_{\mathbb{Z}}(\mathsf{Cor}^f_Z(\mathsf{AnSm}(\mathbb{C})), C^-(\mathbb{Z}))$ is $\mathcal{D}^1$ local and the inclusion morphism $S(F^\bullet) : F^\bullet \to \underline{\mathsf{sing}}^\bullet_{\mathbb{Z}}F^\bullet$ is an $(\mathcal{D}^1, usu)$ equivalence.

**Proof.**

(i): It is proved in [2].

(ii): By (i), $\mathsf{Tr}_*\underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet = \underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet$. $\mathsf{Tr}_*F^\bullet$ is $\mathcal{D}^1$ local. By lemma 2(i), $\mathsf{Tr}_*$ detect $\mathcal{D}^1$ local object. Thus, $\underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet = \underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet$ is $\mathcal{D}^1$ local. This proves the first part of the assertion.

It follows from (i) that

$$\mathsf{Tr}_*S(F^\bullet) : \mathsf{Tr}_*F^\bullet \to \underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet$$

is an $(\mathcal{D}^1, usu)$ equivalence. Since, by lemma 2(ii), $\mathsf{Tr}_*$ detect $(\mathcal{D}^1, usu)$ equivalence, $S(F^\bullet) : F^\bullet \to \underline{\mathsf{sing}}_{\mathbb{Z}}F^\bullet$ is an $(\mathcal{D}^1, usu)$ equivalence.

\[\square\]
Theorem 14. \cite{[I]}

(i) The adjunction \((\text{Tr}_*, \text{Tr}_*) : \text{PSh}(\text{AnSm}(\mathbb{C}), C^-(\mathbb{Z})) \rightleftarrows \text{PSh}(\text{Cor}_2^{fs}(\text{AnSm}(\mathbb{C})), C^-(\mathbb{Z}))\) is a Quillen equivalence for the \((\mathcal{D}^1, \text{usu})\) model structures. That is, the derived functor

\[
L \text{Tr}_* : \text{AnDA}^- (\mathbb{Z}) \simto \text{AnDM}^- (\mathbb{Z})
\]

is an isomorphism and \(\text{Tr}_* : \text{AnDM}^- (\mathbb{Z}) \simto \text{AnDA}^- (\mathbb{Z})\) is it inverse.

(ii) The adjunction \((\iota_{an}^*, \iota_{an}^*) : C^- (\mathbb{Z}) \rightleftarrows \text{PSh}_2(\text{AnSm}(\mathbb{C}), C^- (\mathbb{Z}))\) is a Quillen equivalence for the \((\mathcal{D}^1, \text{usu})\) model structures. That is, the derived functor

\[
i_{an}^* : D^- (\mathbb{Z}) \simto \text{AnDA}^- (\mathbb{Z})
\]

is an isomorphism and \(\text{Re}_{an*} : \text{AnDA}^- (\mathbb{Z}) \simto D^- (\mathbb{Z})\) is it inverse.

(iii) The adjunction \((\iota_{tr}^*, \iota_{tr}^*) : C^- (\mathbb{Z}) \rightleftarrows \text{PSh}_2(\text{Cor}_2^{fs}(\text{AnSm}(\mathbb{C})), C^- (\mathbb{Z}))\) is a Quillen equivalence for the \((\mathcal{D}^1, \text{usu})\) model structures. That is, the derived functor

\[
i_{tr}^* : D^- (\mathbb{Z}) \simto \text{AnDM}^- (\mathbb{Z})
\]

is an isomorphism and \(\text{Re}_{tr*} : \text{AnDM}^- (\mathbb{Z}) \simto D^- (\mathbb{Z})\) is it inverse.

Definition 16. (i) We say that a morphism \(\epsilon : X' \to X\) with \(X, X' \in \text{AnSp}(\mathbb{C})\) is a proper modification (or abstract blow up) if it is proper and if there exist \(Z \subset X\) a closed analytic subspace (the discriminant) such that \(\epsilon\) induces an isomorphism \(\epsilon_{X/Z} : X' \setminus E \simto X \setminus Z\) with \(E = \epsilon^{-1}(Z)\).

(ii) The cdh topology on \(\text{AnSp}(\mathbb{C})\) is the minimal Grothendieck topology generated by open covering for the usual topology and covers \(c = \epsilon \cup l : X' \cup Z \to X\) corresponding to proper modifications \(\epsilon : X' \to X\), with \(l : Z \to X\) the closed embedding.

(iii) We say that a morphism \(\epsilon : X' \to X\) in \(\text{AnSp}(\mathbb{C})\) is a resolution of \(X\) if it is a proper modification and if \(X'\) is smooth.

We recall a version of Hironaka’s desingularisation theorem

Theorem 15. \cite{[I]} For all \(X \in \text{AnSp}(\mathbb{C})\), there exist a proper modification \(\epsilon : X' \to X\) with discriminant \(Z \subset X\) such that \(X'\) is smooth and \(\epsilon^{-1}(Z) \subset X'\) is a normal crossing divisor.

We denote by \(\iota_{an} : \text{AnSm}(\mathbb{C}) \to \text{AnSp}(\mathbb{C})\) the full embedding functor, which gives the morphism of sites \(\iota_{an} : \text{AnSp}(\mathbb{C}) \to \text{AnSm}(\mathbb{C})\). If \(F \in \text{PSh}(\text{AnSm}(\mathbb{C}), \text{Ab})\) is an usu sheaf, \(\iota_{an}^* F\) is a cdh sheaf. If \(F \in \text{PSh}(\text{AnSp}(\mathbb{C}), \text{Ab})\) is an usu sheaf, we have \(\iota_{an*}^* \iota_{an*} F = a_{cdh} F\).

Corollary 2. Let \(F \in \text{PSh}(\text{AnSm}(\mathbb{C}), \text{Ab})\) be an usu sheaf, then \(\iota_{an}^* F = 0\) if and only for all \(X \in \text{AnSm}(\mathbb{C})\) and all \(\alpha \in F(X)\), there exist a composition of blowups along smooth centers \(\epsilon : X_r \to \cdots \to X\) such that \(F(\epsilon)(\alpha) = 0 \in F(X_r)\).

Proof. It follows from theorem \cite{[I]}

Definition 17. Let \(F \in \text{PSh}(\text{AnSm}(\mathbb{C}), \text{Ab})\). Let where \(F \to I^*\) is an usu local equivalence in \(P^-(\text{An})\) with \(I^*\) a complex of injective objects (that is \(I^*\) is projectively fibrant).

(i) We say that \(F\) is homotopy invariant if \(F(p_X) : F(X) \to F(X \times \mathbb{D}^1)\) is an isomorphism, where \(p_X : X \times \mathbb{D}^1 \to X\) is the projection.

(ii) We say that \(F\) is strictly homotopy invariant if the presheaf \(H^n(I^*) : X \in \text{AnSm}(\mathbb{C}) \mapsto H^n(X, F)\), is homotopy invariant for all \(n \in \mathbb{N}\).
Proposition 13. Let $F \in \text{PSh}(\text{AnSm}(\mathbb{C}), \text{Ab})$. Then $F$ is homotopy invariant if and only if it is strictly homotopy invariant.

Lemma 3. (i) Let $e : X' \to X$ be the blow up of a smooth $X \in \text{AnSm}(\mathbb{C})$ along a smooth center $Z \subset X$. Denote by $l : Z \to X$ the closed embedding. Let $C$ (resp. $Q$) the cokernel of $Z_{tr}(e) : Z_{tr}(X') \to Z_{tr}(X)$ (resp. the cokernel of $Z_{tr}(l) \oplus Z_{tr}(e) : Z_{tr}(Z) \oplus Z_{tr}(X') \to Z_{tr}(X)$). Then for any homotopy invariant sheaf $F \in \text{PSh}(\text{Cor}(\text{AnSm}(\mathbb{C})), \text{Ab})$, $\text{Ext}^n(C, F) = \text{Ext}^n(Q, F) = 0$ for all $n \in \mathbb{N}$.

(ii) Let $F \in \text{PSh}(\text{Cor}(\text{AnSm}(\mathbb{C})), \text{Ab})$ be an usu sheaf such that $\iota_{an}^*F = 0$. Then $a_{usu}H^n\text{sing}_Z^*, F = 0$ for all $n \in \mathbb{Z}, n \leq 0$.

Proof. (i): As in the proof of [15] proposition 13.19, it follows from the fact that locally for the usual topology, since $X$ and $Z$ are smooth, $X'$ is the blow up of $Z \times \mathbb{A}^d(x - d_Z)$ along $Z \times 0$.

(ii): As in the proof of [15] theorem 13.25, (ii) follows from (i), corollary 2 and the fact that $H^n\text{sing}_Z^*$ is homotopy invariant, hence strictly homotopy invariant by proposition 13.

Proposition 14. Let $e : X' \to X$ be a proper modification, $X, X' \in \text{AnSp}(\mathbb{C})$. Then, denoting $\epsilon_Z + l'_e = S(Z_{tr}(\epsilon_Z) \oplus Z_{tr}(l'))$ and $\epsilon_e + l_e = S(Z_{tr}(\epsilon) \oplus Z_{tr}(l))$, the morphism in $\text{PC}^-\text{An}$

$$0 \to \text{sing}_Z, Z_{tr}(E) \xrightarrow{\epsilon_Z + l'_e} \text{sing}_Z, Z_{tr}(Z) \oplus \text{sing}_Z, Z_{tr}(X') \xrightarrow{\epsilon_e + l_e} \text{sing}_Z, Z_{tr}(X)$$

is an usu local equivalence.

Proof. Denote by $K \in \text{PSh}(\text{Cor}_Z(\text{AnSm}(\mathbb{C})), \text{Ab})$, the cokernel of $Z_{tr}(l) \oplus Z_{tr}(e) : Z_{tr}(Z) \oplus Z_{tr}(X') \to Z_{tr}(X)$. The sequence

$$0 \to Z_{tr}(E) \xrightarrow{Z_{tr}(\epsilon_Z) \oplus Z_{tr}(l')} Z_{tr}(Z) \oplus Z_{tr}(X') \xrightarrow{Z_{tr}(\epsilon) \oplus Z_{tr}(l)} Z_{tr}(X) \to K \to 0$$

is clearly exact in $\text{PC}^-\text{An}$. In particular, the sequence

$$0 \to \text{sing}_Z, Z_{tr}(E) \xrightarrow{\epsilon_Z + l'_e} \text{sing}_Z, Z_{tr}(Z) \oplus \text{sing}_Z, Z_{tr}(X') \xrightarrow{\epsilon_e + l_e} \text{sing}_Z, Z_{tr}(X) \to \text{sing}_Z, K \to 0$$

is exact in $\text{PC}^-\text{An}$. Hence we have to show that $a_{usu}H^n\text{sing}_Z^*, K = H^n\text{sing}_Z^*, a_{usu}K = 0$ for all $n \in \mathbb{Z}$, $n \leq 0$, $a_{usu}$ being an exact functor. By lemma 3 (ii), it suffice to show that $\iota_{an}^*a_{usu}K = 0$. But for all $U \in \text{AnSm}(\mathbb{C})$, $\alpha \in Z_{tr}(X)(U)$ and $\alpha' \in Z^{d_X'}(U \times X')$ the proper transform of $\alpha$ by $I_U \times e$, there exist, by platification and resolution of singularities, a proper modification $e : U'' \to U$, with $U'' \in \text{AnSm}(\mathbb{C})$ such that the proper transform $\alpha'' \in Z^{d_X'}(U'' \times X')$ of $\alpha'$ by $e \times I_{X'}$ is finite (and surjective) over $U''$, so that $Z_{tr}(X)(e)(\alpha) := \alpha \circ e = Z_{tr}(e')(\alpha'') \in Z_{tr}(X)(U'')$ with $\alpha'' \in Z_{tr}(X')(U'')$. It then follows from corollary 2.

We will use in the next section the following:

Proposition 15. (i) Let $G_1^\bullet, G_2^\bullet \in \text{PSh}_Z(\text{Cor}_Z^\times(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ and $f : G_1^\bullet \to G_2^\bullet$ a morphism. If $\epsilon_{an}^\text{tr}_*: S(f) : \text{sing}_Z^\bullet, G_1^\bullet \to \text{sing}_Z^\bullet, G_2^\bullet$

is a quasi-isomorphism in $C^{-}(\mathbb{Z})$, then $f : G_1^\bullet \to G_2^\bullet$ is an $(\mathbb{D}^1, \text{usu})$ local equivalence.

(ii) Let $G_1^\bullet, G_2^\bullet \in \text{PSh}_Z(\text{Cor}_Z^\times(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z}))$ and $f : G_1^\bullet \to G_2^\bullet$ a morphism. If

- $G_1^\bullet$ and $G_2^\bullet$ are $\mathbb{D}^1$ local
- $\epsilon_{an}^\text{tr}_*: \epsilon_{an}^\text{tr}_*, G_1^\bullet \to \epsilon_{an}^\text{tr}_*, G_2^\bullet$ is a quasi-isomorphism in $C^{-}(\mathbb{Z})$,
then \( f : G_1^* \rightarrow G_2^* \) is an \((\mathbb{D}^1, \text{usu})\) local equivalence.

Proof. (i): Consider the commutative diagrams

\[
\begin{array}{ccc}
G_1 & \xrightarrow{S(G_1^*)} & \text{sing}_\mathbb{D}, G_1^* \\
\downarrow f & & \downarrow S(f) \\
G_2 & \xrightarrow{S(G_2^*)} & \text{sing}_\mathbb{D}, G_2^*
\end{array}
\]

\[
\begin{array}{ccc}
G_1 & \xrightarrow{\text{ad}(e_{an}^{tr}, e_{an}^{tr})(\text{sing}_\mathbb{D}, G_1^*)} & \text{sing}_\mathbb{D}, G_1^* \\
\downarrow f & & \downarrow S(f) \\
G_2 & \xrightarrow{\text{ad}(e_{an}^{tr}, e_{an}^{tr})(\text{sing}_\mathbb{D}, G_2^*)} & \text{sing}_\mathbb{D}, G_2^*
\end{array}
\]

By theorem 13 (iii),

- \( \text{ad}(e_{an}^{tr}, e_{an}^{tr})(\text{sing}_\mathbb{D}, G_1^*) : e_{an}^{tr} e_{an}^{tr} \text{sing}_\mathbb{D}, G_1^* \rightarrow \text{sing}_\mathbb{D}, G_1^* \)
- \( \text{ad}(e_{an}^{tr}, e_{an}^{tr})(\text{sing}_\mathbb{D}, G_2^*) : e_{an}^{tr} e_{an}^{tr} \text{sing}_\mathbb{D}, G_2^* \rightarrow \text{sing}_\mathbb{D}, G_2^* \)

are \((\mathbb{D}^1, \text{usu})\) local equivalence. On the other side

\[
e_{an}^{tr} e_{an}^{tr} S(f) : e_{an}^{tr} e_{an}^{tr} \text{sing}_\mathbb{D}, G_1^* = e_{an}^{tr} \text{sing}_\mathbb{D}, G_1^* \rightarrow e_{an}^{tr} e_{an}^{tr} \text{sing}_\mathbb{D}, G_2^* = e_{an}^{tr} \text{sing}_\mathbb{D}, G_2^*
\]

is an \((\mathbb{D}^1, \text{usu})\) local equivalence (even a quasi-isomorphism), since \( e_{an}^{tr} S(f) : \text{sing}_\mathbb{D}, G_1^* \rightarrow \text{sing}_\mathbb{D}, G_2^* \) is a quasi-isomorphism in \( C^{-}(\mathbb{Z}) \) by hypothesis. Hence, by the second commutative diagram,

\[
S(f) : \text{sing}_\mathbb{D}, G_1^* \rightarrow \text{sing}_\mathbb{D}, G_2^*
\]

is an \((\mathbb{D}^1, \text{usu})\) local equivalence. Since \( S(G_1^*) \) and \( S(G_2^*) \) are \((\mathbb{D}^1, \text{usu})\) local equivalence by theorem 13 (ii), and \( S(f) \) is a \((\mathbb{D}^1, \text{usu})\) local equivalence, \( f : G_1^* \rightarrow G_2^* \) is a \((\mathbb{D}^1, \text{usu})\) local equivalence by the first commutative diagram.

(ii): Consider again the commutative diagram

\[
\begin{array}{ccc}
G_1 & \xrightarrow{S(G_1^*)} & \text{sing}_\mathbb{D}, G_1^* \\
\downarrow f & & \downarrow S(f) \\
G_2 & \xrightarrow{S(G_2^*)} & \text{sing}_\mathbb{D}, G_2^*
\end{array}
\]

Since \( S(G_1^*) \) and is a \((\mathbb{D}^1, \text{usu})\) local equivalence and \( G_1^* \) is \( \mathbb{D}^1 \) local, \( S(G_1^*) \) is an usu local equivalence. Hence \( e_{an}^{tr} S(G_1^*) \) is a quasi-isomorphism. Since \( S(G_2^*) \) and is a \((\mathbb{D}^1, \text{usu})\) local equivalence and \( G_2^* \) is \( \mathbb{D}^1 \) local, \( S(G_2^*) \) is an usu local equivalence. Hence \( e_{an}^{tr} S(G_2^*) \) is a quasi-isomorphism. On the other side, by hypothesis, \( e_{an}^{tr} f \) is a quasi-isomorphism. Hence, by the above diagram, \( e_{an}^{tr} S(f) : \text{sing}_\mathbb{D}, G_1^* \rightarrow \text{sing}_\mathbb{D}, G_2^* \) is a quasi-isomorphism. It follows then by (i) that \( f : G_1 \rightarrow G_2 \) is a \((\mathbb{D}^1, \text{usu})\) local equivalence.

\[
\square
\]

2.3 Presheaves and transfers on the category of CW complexes

Let \( X, Y \in \text{Top} \). A continuous map \( f : X \rightarrow Y \) is said to be proper if it is universally closed, that is for all \( T \in \text{Top} \), \( f \times f : X \times T \rightarrow Y \times T \) is closed. A continuous map \( f : X \rightarrow Y \) is proper if and only if it is closed and, for all \( y \in Y \), \( X_y = f^{-1}(y) \) is quasi-compact (c.f. [5]). A continuous map \( f : X \rightarrow Y \) is said to be finite if it is proper and, for all \( y \in Y \), \( X_y = f^{-1}(y) \) is a finite set. A continuous map \( f : X \rightarrow Y \) is said to be dominant if \( f(X) \subset Y \) has non empty interior. For \( X \in \text{Top} \), there exist a one point compactification \( \overline{X} \in \text{Top} \) of \( X \), that is \( \overline{X} \) is quasi-compact and there is an open embedding \( X \hookrightarrow \overline{X} \). Moreover \( \overline{X} \) is compact (i.e. quasi-compact and Hausdorff) if and only if \( X \) is locally compact and Hausdorff.

Let \( X, Y, Z \in \text{Top} \) and denote \( p_X : X \times Y \rightarrow X \) and \( p_Y : Y \times Z \rightarrow Z \), \( p_{XZ} : X \times Y \times Z \rightarrow X \times Z \) the projections. Assume that \( Y \) is Hausdorff (equivalently the diagonal \( \Delta_Y \subset Y \times Y \) is a closed subset).
Let $\Gamma_1 \subset X \times Y$, $\Gamma_2 \subset Y \times Z$, closed subsets. Hence, since $Y$ is Hausdorff, $\Gamma_1 \times_Y \Gamma_2 \subset X \times Y \times Z$ is a closed subset. Now assume that $p_{X|\Gamma_1} : \Gamma_1 \to X$ and $p_{Y|\Gamma_2} : \Gamma_2 \to Y$ are finite and surjective. Consider the commutative diagram

$$
\begin{array}{ccc}
\Gamma_1 \times_Y \Gamma_2 & \xrightarrow{p_{X|\Gamma_1} \times_Y \Gamma_2} & p_{XZ}(\Gamma_1 \times_Y \Gamma_2) \\
\downarrow p_{\Gamma_1} & & \downarrow p_{X|p_{XZ}(\Gamma_1 \times_Y \Gamma_2)} \\
\Gamma_1 & \xrightarrow{p_{\Gamma_1}} & X
\end{array}
$$

(32)

The projection $p_{\Gamma_1} : \Gamma_1 \times_Y \Gamma_2 \to \Gamma_1$ is finite and surjective (since $p_{Y|\Gamma_2} : \Gamma_2 \to Y$ is finite and surjective by hypothesis and $Y$ is Hausdorff). The projection $p_{X|\Gamma_1} : \Gamma_1 \to X$ is finite and surjective by hypothesis. Hence, by the diagram (32), the composition

$$
\Gamma_1 \times_Y \Gamma_2 \xrightarrow{p_{X|\Gamma_1} \times_Y \Gamma_2} p_{XZ}(\Gamma_1 \times_Y \Gamma_2) \xrightarrow{p_{X|p_{XZ}(\Gamma_1 \times_Y \Gamma_2)}} X
$$

is finite and surjective. Thus, $p_{X|p_{XZ}(\Gamma_1 \times_Y \Gamma_2)} : p_{XZ}(\Gamma_1 \times_Y \Gamma_2) \to X$ is finite and surjective, that is

$$
\Gamma_2 \circ \Gamma_1 := p_{XZ|\times_Y(\times_Y \Gamma_2)}(\Gamma_1 \times_Y \Gamma_2)
$$

(33)

is a closed subset of $X \times Z$ which is finite and surjective on $X$.

Let $\Gamma_1 \subset X \times Y$, $\Gamma_2 \subset Y \times Z$, closed subsets. Hence, since $Y$ is Hausdorff, $\Gamma_1 \times_Y \Gamma_2 \subset X \times Y \times Z$ is a closed subset. Now assume that $p_{X|\Gamma_1} : \Gamma_1 \to X$ and $p_{Y|\Gamma_2} : \Gamma_2 \to Y$ are finite and surjective. Consider the commutative diagram

$$
\begin{array}{ccc}
\Gamma_1 \times_Y \Gamma_2 & \xrightarrow{p_{X|\Gamma_1} \times_Y \Gamma_2} & p_{XZ}(\Gamma_1 \times_Y \Gamma_2) \\
\downarrow p_{\Gamma_1} & & \downarrow p_{X|p_{XZ}(\Gamma_1 \times_Y \Gamma_2)} \\
\Gamma_1 & \xrightarrow{p_{\Gamma_1}} & X
\end{array}
$$

(32)

The projection $p_{\Gamma_1} : \Gamma_1 \times_Y \Gamma_2 \to \Gamma_1$ is finite and surjective (since $p_{Y|\Gamma_2} : \Gamma_2 \to Y$ is finite and surjective by hypothesis and $Y$ is Hausdorff). The projection $p_{X|\Gamma_1} : \Gamma_1 \to X$ is finite and surjective by hypothesis. Hence, by the diagram (32), the composition

$$
\Gamma_1 \times_Y \Gamma_2 \xrightarrow{p_{X|\Gamma_1} \times_Y \Gamma_2} p_{XZ}(\Gamma_1 \times_Y \Gamma_2) \xrightarrow{p_{X|p_{XZ}(\Gamma_1 \times_Y \Gamma_2)}} X
$$

is finite and surjective. Thus, $p_{X|p_{XZ}(\Gamma_1 \times_Y \Gamma_2)} : p_{XZ}(\Gamma_1 \times_Y \Gamma_2) \to X$ is finite and surjective, that is

$$
\Gamma_2 \circ \Gamma_1 := p_{XZ|\times_Y(\times_Y \Gamma_2)}(\Gamma_1 \times_Y \Gamma_2)
$$

(33)

is a closed subset of $X \times Z$ which is finite and surjective on $X$.

We consider now the full subcategory $CW \subset$ Top consisting of CW complexes. Recall that CW complexes are Hausdorff, locally contractible and locally compact. By a CW subcomplex of $X \in CW$, we mean a topological embedding $Z \hookrightarrow X$ with $Z$ a CW complex. By a closed CW subcomplex of $X \in CW$, we mean a topological closed embedding $Z \hookrightarrow X$ with $Z$ a CW complex, that is the image of the embedding is a closed subset of $X$. Let $X, S \subset CW, S$ connected, and $h : X \to S$ a finite and surjective morphism in CW. We say that $X/S = (X, h) \in CW / \subset$ is reducible if $X = X_1 \cup X_2$, with $X_1, X_2$ closed CW subcomplexes finite and surjective over $S$ and $X_1, X_2 \neq X$. A pair $Y/S = (Y, h') \in CW / \subset$ with $Y \in CW$ and $h' : Y \to S$ a finite and surjective morphism is called irreducible if it is not reducible. In particular $Y$ is connected.

- For $X \in CW, \Lambda$ a commutative ring and $p \in \mathbb{N}$, we denote by $Z_p(X, \Lambda)$ the free $\Lambda$ module generated by the closed CW subcomplex of $X$ of dimension $p$ and by $Z_p(X, \Lambda) = Z_{d_X} - p(X, \Lambda)$ the free $\Lambda$ module generated by the closed CW subcomplex of $X$ of codimension $p$.

- For $X, Y \in CW, X$ connected, and $\Lambda$ a commutative ring, we define : $Z^{fs/X}(X \times Y, \Lambda) \subset Z_{d_X}(X \times Y, \Lambda)$ the free $\Lambda$ module generated by the closed CW subcomplexes of $X \times Y$ finite and surjective over $X$ which are irreducible. Note that if $Z \subset X \times Y$ is a closed CW subcomplex finite and surjective over $X$ which is irreducible, then if $Z = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset Z$ closed CW subcomplex finite and surjective over $X$, then $Z_1 = Z_2 = Z$.

- For $X, Y \in CW, \Lambda$ a commutative ring, we define : $Z^{fs/X}(X \times Y, \Lambda) := \oplus_i Z^{fs/X_i}(X_i \times Y, \Lambda)$ where $X = \sqcup_i X_i$, with $X_i$ the connected components of $X$.

For $Y \in CW$ and $j : V \hookrightarrow Y$ an open embedding, we denote by

$$
j^* : Z^p(Y, \Lambda) \to Z^p(V, \Lambda) ; Z \mapsto Z \cap V
$$

(34)

Let $X, Y, Z \in CW$ and denote $p_X : X \times Y \to X$ and $p_Y : Y \times Z \to Z$, $p_{XZ} : X \times Y \times Z \to X \times Z$ the projections. Let $\Gamma_1 \subset X \times Y$ be a closed CW subcomplex finite and surjective over $X$ which is irreducible. Let $\Gamma_2 \subset Y \times Z$ be a closed CW subcomplex finite and surjective over $Y$ which is irreducible. Hence, since $Y$ is Hausdorff, $\Gamma_1 \times_Y \Gamma_2 \subset X \times Y \times Z$ is a closed CW subcomplex finite and surjective over $X$, in particular all its irreducible components are finite and surjective over $X$. Thus, by diagram (32)

$$
\Gamma_2 \circ \Gamma_1 := p_{XZ|\times_Y(\times_Y \Gamma_2)}(\Gamma_1 \times_Y \Gamma_2)
$$

(35)

is a closed CW subcomplex of $X \times Z$ finite and surjective over $X$, in particular all its irreducible components are finite and surjective over $X$ and so define an element of $Z^{fs/X}(X \times Z)$.
Definition 18. We define $\text{Cor}^{fs}_\Lambda(CW)$ to be the category whose objects are CW complexes and whose space of morphisms between $X, Y \in CW$ is the free $\Lambda$ module $Z^{fs/X}(X \times Y, \Lambda)$. The composition is the one given by (36).

We have

- the additive embedding of categories $\text{Tr} : Z(CW) \hookrightarrow \text{Cor}^{fs}_\Lambda(CW)$ which gives the corresponding morphism of sites $\text{Tr} : \text{Cor}^{fs}_\Lambda(CW) \to Z(CW)$,
- the inclusion functor $e_{cw} : \{pt\} \hookrightarrow CW$, which gives the corresponding morphism of sites $e_{cw} : CW \to \{pt\}$,
- the inclusion functor $e^{tr}_{cw} := \text{Tr} \circ e_{cw} : \{pt\} \hookrightarrow \text{Cor}^{fs}_\Lambda(CW)$ which gives the corresponding morphism of sites $e^{tr}_{cw} := \text{Tr} \circ e_{cw} : \text{Cor}^{fs}_\Lambda(CW) \to \{pt\}$.

We consider the following two big categories:

- $\text{PSh}(CW, C^- (Z)) = \text{PSh}_{Z}(Z(CW), C^- (Z)))$, the category of bounded above complexes of presheaves on CW, or equivalently additive presheaves on $Z(CW)$, sometimes, we will write for short $P^- (CW) = \text{PSh}(CW, C^- (Z))$,
- $\text{PSh}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$, the category of bounded above complexes of additive presheaves on $\text{Cor}^{fs}_\Lambda(CW)$, sometimes, we will write for short $PC^- (CW) = \text{PSh}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$,

and the adjunctions:

- $(\text{Tr}^*, \text{Tr}_*) : \text{PSh}(CW, C^- (Z)) \leftrightarrows \text{PSh}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$,
- $(e_{cw}^*, e_{cw*}) : \text{PSh}(CW, C^- (Z)) \leftrightarrows C^- (Z)$,
- $(e^{tr}_{cw}^*, e^{tr}_{cw*}) : \text{PSh}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z)) \leftrightarrows C^- (Z)$,

given by $\text{Tr} : \text{Cor}^{fs}_\Lambda(CW) \to \Lambda(CW)$, $e_{cw} : CW \to \{pt\}$ and $e^{tr}_{cw} : \text{Cor}^{fs}_\Lambda(CW) \to \{pt\}$.

For $X \in CW$, we denote by

$$Z(X) \in \text{PSh}(CW, C^- (Z)), \quad Z_{tr}(X) \in \text{PSh}_{Z}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$$

the presheaves represented by $X$. They are usu sheaves. We denote by $a_{usu} : \text{PSh}(CW, Ab) \to \text{Sh}_{Z, usu}(CW, Ab)$ the usu sheafification functor. We consider

- the usual monoidal structure on $\text{PSh}(CW, C^- (Z))$ and the associated internal $\text{Hom}$ given by, for $F^*, G^* \in \text{PSh}(CW, C^- (Z))$ and $Y \in CW$,

$$F^* \otimes G^*(X) : X \mapsto F^*(X) \otimes_Z G^*(X), \quad \text{Hom}(Z(Y), F^*) : X \mapsto F^*(X \times Y),$$

- the unique monoidal structure on $\text{PSh}_{Z}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$ such that, for $X, Y \in CW$, $Z_{tr}(X) \otimes Z_{tr}(Y) := Z_{tr}(X \times Y)$ and wich commute with colimits. It has an internal $\text{Hom}$ which is given, for $X, Y \in CW$ and $F^* \in \text{PSh}_{Z}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$, $\text{Hom}(Z_{tr}(Y), F^*) : X \mapsto F^*(X \times Y)$

Together with these monoidal structure, the functor

$$\text{Tr}^* : \text{PSh}(CW, C^- (Z)) \to \text{PSh}_{Z}(\text{Cor}^{fs}_\Lambda(CW), C^- (Z))$$

is monoidal.

Definition 19. (i) We say that a morphism $\phi : G^*_1 \to G^*_2$ in $\text{PSh}(CW, C^- (Z))$ is an usu local equivalence if $\phi_* : a_{usu} H^k(G^*_1) \to a_{usu} H^k(G^*_2)$ is an isomorphism for all $k \in \mathbb{Z}$. The projective usual topology model structure on $\text{PSh}(CW, C^- (Z))$ is the left Bousfield localization of the projective model structure $\mathcal{M}_P(\text{PSh}_{Z}(CW, C^- (Z)))$ with respect to the usu local equivalence.
(ii) We say that a morphism $\phi : G^1_1 \to G^1_2$ in $\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z))$ is an usl local equivalence if and only if its restriction to CW $\text{Tr}_\ast, \phi : \text{Tr}_\ast G_1^1 \to \text{Tr}_\ast G_2^2$ is an usl local equivalence. The projective usual topology model structure on $\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z))$ is the left Bousfield localization of the projective model structure $\mathcal{M}_P(\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z)))$ with respect to the usl local equivalence.

**Definition 20.** 
(i) The projective $(\mathbb{I}^1, \text{usu})$ model structure on $\text{PSh}(\text{CW}, C^-(Z))$ is the left Bousfield localization of the projective usual topology model structure (c.$\mathbb{I}^1$) with respect to the class of maps $\{Z(X \times \mathbb{I}^1)[n] \to Z(X)[n], X \in \text{CW}, n \in \mathbb{Z}\}$.

(ii) The projective $(\mathbb{I}^1, \text{usu})$ model structure on $\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z))$ is the left Bousfield localization of the projective usual topology model structure (c.$\mathbb{I}^1$) with respect to the class of maps $\{Z_{tr}(X \times \mathbb{I}^1) \to Z(X), X \in \text{CW}, n \in \mathbb{Z}\}$.

**Definition 21.** 
(i) We define $\text{CwDM}^-(\mathbb{Z}) := \text{Ho}_{\mathbb{I}^1, \text{usu}}(\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{AnSm}(\mathbb{C})), C^-(\mathbb{Z})))$, the derived category of motives of CW complexes, it is the homotopy category of $\text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(\mathbb{Z}))$ with respect to the projective $(\mathbb{I}^1, \text{usu})$ model structure (20(ii)). We denote by

$$D^{\text{tr}}(\mathbb{I}^1, \text{usu}) : \text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(\mathbb{Z})) \to \text{CwDM}^-(\mathbb{Z}), D^{\text{tr}}(\mathbb{I}^1, \text{usu})(F^\bullet) = F^\bullet$$

the canonical localisation functor.

(ii) We denote by the same way $\text{CwDA}^-(\mathbb{Z}) := \text{Ho}_{\mathbb{I}^1, \text{usu}}(\text{PSh}(\text{CW}, C^-(\mathbb{Z})))$ (29(ii)) and

$$D(\mathbb{I}^1, \text{usu}) : \text{PSh}(\text{CW}, C^-(\mathbb{Z})) \to \text{CwDA}^-(\mathbb{Z}), D(\mathbb{I}^1, \text{usu})(F^\bullet) = F^\bullet$$

(38)

the canonical localisation functor.

We recall for convenience to the reader the definition of $\mathbb{I}^1$ homotopy :

- Let $X, Y \in \text{Top}$. We say that two maps $f_0 : X \to Y$, $f_1 : X \to Y$ are $\mathbb{I}^1$ homotopic, if there exist $h : X \times \mathbb{I}^1 \to Y$ such that $f_0 = h \circ (I_X \times i_0)$ and $f_1 = h \circ (I_X \times i_1)$, with $(I_X \times i_0) : X \times \{0\} \hookrightarrow X \times \mathbb{I}^1$ and $(I_X \times i_1) : X \times \{1\} \hookrightarrow X \times \mathbb{I}^1$ the inclusions.

- Let $X, Y \in \text{Top}$. We say that $X$ is $\mathbb{I}^1$ homotopy equivalent to $Y$ if there exist two maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f$ is $\mathbb{I}^1$ homotopic to $I_X$ and $f \circ g$ is $\mathbb{I}^1$ homotopic to $I_Y$.

- Let $F^\bullet, G^\bullet \in \text{PSh}(\text{CW}, C^-(\mathbb{Z}))$. We say that two maps $\phi_0 : F^\bullet \to G^\bullet$ and $\phi_1 : F^\bullet \to G^\bullet$ are $\mathbb{I}^1$ homotopic if there exist $\phi : F^\bullet \to \text{Hom}(\mathbb{I}^1, G^\bullet)$ such that $\phi_0 = \mathbb{G}^\bullet(i_0) \circ \phi$ and $\phi_1 = \mathbb{G}^\bullet(i_1) \circ \phi$, where,

  - $\mathbb{G}^\bullet(i_0) : \text{Hom}(\mathbb{I}^1, G^\bullet) \to G^\bullet$ is induced by $i_0 : \{\text{pt}\} \hookrightarrow \mathbb{I}^1$, that is, for $X \in \text{CW}$, $\mathbb{G}^\bullet(i_0)(X) = G^\bullet(I_X \times i_0) : G^\bullet(X \times \mathbb{I}^1) \to G^\bullet(X)$,

  - $\mathbb{G}^\bullet(i_1) : \text{Hom}(\mathbb{I}^1, G^\bullet) \to G^\bullet$ is induced by $i_1 : \{\text{pt}\} \hookrightarrow \mathbb{I}^1$, that is, for $X \in \text{CW}$, $\mathbb{G}^\bullet(i_1)(X) = G^\bullet(I_X \times i_1) : G^\bullet(X \times \mathbb{I}^1) \to G^\bullet(X)$.

- Let $F^\bullet, G^\bullet \in \text{PSh}(\text{CW}, C^-(\mathbb{Z}))$. We say that $F^\bullet$ is $\mathbb{I}^1$ homotopy equivalent to $G^\bullet$ if there exist two maps $\phi : F^\bullet \to G^\bullet$ and $\psi : F^\bullet \to G^\bullet$ such that $\psi \circ \phi$ is $\mathbb{I}^1$ homotopic to $I_{F^\bullet}$ and $\phi \circ \psi$ is $\mathbb{I}^1$ homotopic to $I_{G^\bullet}$.

We have the following easy lemma

**Lemma 4.** Let $X, Y \in \text{CW}$ and $f_0 : X \to Y$, $f_1 : X \to Y$ two morphisms. If $f_0$ and $f_1$ are $\mathbb{I}^1$ homotopic, then

- $Z(f_0) : Z(X) \to Z(Y)$ and $Z(f_1) : Z(X) \to Z(Y)$ are $\mathbb{I}^1$ homotopic in $\text{PSh}(\text{CW}, C^-(\mathbb{Z}))$.

- $\text{Tr}_\ast, Z_{tr}(f_0) : \text{Tr}_\ast Z_{tr}(X) \to \text{Tr}_\ast Z_{tr}(Y)$ and $\text{Tr}_\ast, Z_{tr}(f_1) : \text{Tr}_\ast Z_{tr}(X) \to \text{Tr}_\ast Z_{tr}(Y)$ are $\mathbb{I}^1$ homotopic in $\text{PSh}(\text{CW}, C^-(\mathbb{Z}))$.
Proof. Let \( h : X \times I^1 \to Y \) such that \( f_0 = h \circ (I_X \times i_0) \) and \( f_1 = h \circ (I_X \times i_1) \), with \((I_X \times i_0) : X \times \{0\} \to X \times I^1 \) and \((I_X \times i_1) : X \times \{1\} \to X \times I^1 \) the inclusions. Then

- \( \phi(h) : Z(X) \to \text{Hom}(Z(I^1), Z(Y)) \), given by for \( Z \in \text{CW} \)

\[
\phi(h)(Z) : \alpha \in \text{Hom}_{\text{CW}}(Z, X) \mapsto (\alpha \times I_{I^1}) \in \text{Hom}_{\text{CW}}(Z \times I^1, X \times I^1)
\]

\[
\Rightarrow Z(h)(Z \times I^1)(\alpha \times I_{I^1}) = h \circ (\alpha \times I_{I^1}) \in \text{Hom}_{\text{CW}}(Z \times I^1, Y)
\]

satisfy \( Z(Y)(i_0) \circ \phi(h) = Z(f_0) \) and \( Z(Y)(i_1) \circ \phi(h) = Z(f_1) \).

- \( \phi(h) : Z_{tr}(X) \to \text{Hom}(Z(I^1), Z_{tr}(Y)) \), given by for \( Z \in \text{CW} \)

\[
\phi(h)(Z) : \alpha \in \text{Hom}_{\text{Cor}^r_{\text{CW}}}(Z, X) \mapsto (\alpha \times I_{I^1}) \in \text{Hom}_{\text{Cor}^r_{\text{CW}}}(Z \times I^1, X \times I^1)
\]

\[
\Rightarrow Z_{tr}(h)(Z \times I^1)(\alpha \times I_{I^1}) = h \circ (\alpha \times I_{I^1}) \in \text{Hom}_{\text{Cor}^r_{\text{CW}}}(Z \times I^1, Y)
\]

satisfy \( Z_{tr}(Y)(i_0) \circ \phi(h) = Z_{tr}(f_0) \) and \( Z_{tr}(Y)(i_1) \circ \phi(h) = Z_{tr}(f_1) \).

Recall \( \Delta \) denote the simplicial category Let

- \( p_\Delta : \text{CW} \to \Delta \times \text{CW} \) be the morphism of site given by the projection functor \((X, i) \in \Delta \times \text{CW} \mapsto p_\Delta((X, i)) = X \in \text{CW} \)

- \( p_\Delta : \text{Cor}_{Z}(\text{CW}) \to \Delta \times \text{Cor}_{Z}(\text{CW}) \) be the morphism of site given by the projection functor \((X, i) \in \Delta \times \text{CW} \mapsto p_\Delta((X, i)) = X \in \text{CW} \)

We have the adjunctions

- \( (p_\Delta^*, p_\Delta) : \text{PSh}(\Delta \times \text{CW}, C^-(Z)) \leftrightarrows \text{PSh}(\text{CW}, C^-(Z)) \)

- \( (p_\Delta^*, p_\Delta) : \text{PSh}(\Delta \times \text{Cor}_{Z}(\text{CW}), C^-(Z)) \leftrightarrows \text{PSh}(\text{Cor}_{Z}(\text{CW}), C^-(Z)) \).

In the following lemma point (i) is a generalization of point (i) to usu hypercover. We will only use point (i) in this paper in the proof of proposition\([\mathcal{P}]\)(ii).

**Lemma 5.** (i) Let \( X \in \text{CW} \) and \( X = \cup_{i \in J} U_i \) a covering by open subsets \( U_i \subset X \), \( J \) being a countable set. We denote by \( j_i : U_i \to X \) the open embedding. For \( I \subset J \) a finite subset, let \( U_I = \cap_{i \in I} U_i \).

Then,

- \( \cdots \to +_{\text{card}} t=r Z(U_I) \to \cdots +_{i \in J} Z(U_i) \xrightarrow{\oplus_{i \in J} Z(j_I)} Z(X) \) is an usu local equivalence in \( P^-(\text{CW}) \).

- \( \cdots \to +_{\text{card}} t=r Z_{tr}(U_I) \to \cdots +_{i \in J} Z_{tr}(U_i) \xrightarrow{\oplus_{i \in J} Z_{tr}(j_I)} Z_{tr}(X) \) is an usu local equivalence in \( P^C(CW) \).

(ii) Let \( X \in \text{CW} \) and \( j_* : U_* \to X \) an hypercovering of \( X \) by open subsets (i.e. for all \( n \in \Delta, U_n \) is an open subset and \( j_n : U_n \to X \) is the open embedding), and \( (U_*, \cdot) : \cdots \to (U_n, n) \to \cdots \) the associated complex in \( \Delta \times \text{CW} \). Then,

- \( j : Lp_\Delta^* (Z(U_*, \cdot)) \to Z(X) \) is an isomorphism in \( \text{Ho}_{usu}(\text{PSh}(\text{CW}, C^-(Z))) \)

- \( j : Lp_\Delta^* (Z_{tr}(U_*, \cdot)) \to Z_{tr}(X) \) is an isomorphism in \( \text{Ho}_{usu}(\text{PSh}(\text{Cor}_{Z}(\text{CW}), C^-(Z))) \)

**Proof.** (i): The first point follows from the following two facts:

- \( \cdots \to +_{\text{card}} t=r Z(U_I) \to \cdots +_{i \in J} Z(U_i) \) is clearly exact in \( P^-(\text{CW}) \).
• the exactness of \( \bigoplus_{i \in I} Z(U_i) \xrightarrow{\bigoplus_{i \in I} \partial^1_{i, i}} Z(X) \to 0 \) in \( \text{Ho}_{\text{usu}}(\text{PSh}(CW, C^{-}(\mathbb{Z}))) \), follows from the fact that for \( Y \in CW, y \in Y \) and \( f : Y \to X \) a morphism, there exists an open subset \( V(y) \subset Y \) containing \( y \) such that \( f(V(y)) \subset U_i \), with \( U_i \) containing \( f(y) \).

Similarly, the second point follows from the following two facts:

• \( \cdots \rightarrow \bigoplus_{\text{card } i = r} Z_{tr}(U_i) \rightarrow \cdots \rightarrow \bigoplus_{i \in I} Z_{tr}(U_i) \) is clearly exact in \( \text{PSh}_{Z}(\text{Cor}^{I*}_{Z}(CW), C^{-}(\mathbb{Z})) \).

• the exactness of \( \bigoplus_{i \in I} Z_{tr}(U_i) \xrightarrow{\bigoplus_{i \in I} \partial^1_{i, i}} Z_{tr}(X) \to 0 \) in \( \text{Ho}_{\text{usu}}(\text{PSh}(\text{Cor}^{I*}_{Z}(CW), C^{-}(\mathbb{Z}))) \), follows from the fact that for \( Y \in CW, y \in Y \) and \( \alpha \in Z_{tr}(X)(Y) \) irreducible, there exists an open subset \( V(y) \subset Y \) containing \( y \) such that \( \alpha(V(y)) \in Z_{tr}(U_i)(V(y)) \), with \( U_i \) containing \( \alpha, y \).

(ii): Similar to \( \text{[1]} \) Proposition 1.4 Etape 1.

The following proposition is to use point (i) instead of point (ii) in the proof of proposition \( \text{[21]} \) ii). Recall \( CS \subset CW \) is the full subcategory of \( \Delta \) complexes.

**Proposition 16.**

(i) Let \( X \in CW \). There exist \( X' \in CS \) homotopy equivalent to \( X \), that is there exist \( g : X' \to X \) and \( h : X \to X' \) such that \( h \circ g \) is homotopic to \( I_{X'} \) and \( g \circ h \) is homotopic to \( I_X \).

(ii) Let \( X \in CS \). There exist a countable open covering \( X = \bigcup_{i \in I} U_i \) such that for all finite subset \( I \subset J, U_I := \bigcap_{i \in I} U_i = \emptyset \) or \( U_I \) is contractible.

**Proof.** (i): See \( \text{[8]} \) theorem 2C5.

(ii): Take an open star of each vertices of \( X \). As \( X \) is a countable union of compact set, we can extract a countable subcovering of this open covering.

We denote, for all \( n \in \mathbb{N}, \mathbb{I}^n := [0, 1]^n \in CW \). It gives, together with the face maps \( \partial^1_{\epsilon,i} : \mathbb{I}^{n-1} \rightarrow \mathbb{I}^n, \epsilon = 0, 1, i_1 = 1, \cdots n \) a cubical object of \( CW \), denoted \( \mathbb{I}^* \). In particular, for \( n = 1 \), we have the canonical maps \( i_0 : \{0\} \hookrightarrow \mathbb{I}^1, i_1 : \{1\} \hookrightarrow \mathbb{I}^1 \), and the terminal map \( a = a_{11} : \mathbb{I}^1 \rightarrow \{1\} \).

For \( F^* \in \text{PSh}(CW, \text{Ab}) \) and \( X \in CW \), we have the complex \( F(X \times \mathbb{I}^*) \) associated to the cubical object \( F(X \times \mathbb{I}^*) \) in the category of abelian groups, whose differential maps are

\[
\partial_1^0 = \sum_{i = 1}^n (-1)^i (F(I_X \times \partial^0_{0,i}) - F(I_X \times \partial^1_{1,i}) : F(X \times \mathbb{I}^n) \rightarrow F(X \times \mathbb{I}^{n-1})
\]

There is \( \text{[14]} \) a canonical morphism \( L : \mathbb{I}^* \rightarrow \Delta^* \) of complexes of \( \mathbb{Z}(CW) \). This gives for \( F \in \text{PSh}(CW, \text{Ab}) \) and \( X \in CW \), the morphism of complexes of abelian groups

\[
F^*(L \times I_X) : F(\Delta^* \times X) \rightarrow F(\mathbb{I}^* \times X)
\]

with \( L \times I_X : \mathbb{I}^* \times X \rightarrow \Delta^* \times X \) the corresponding morphism of complexes of \( \mathbb{Z}(CW) \). We have then following:

**Proposition 17.** \( \text{[14]} \) For \( X \in CW, \mathbb{Z}(X)(L) : \mathbb{Z}\text{Hom}(\Delta^*, X) \rightarrow \text{sing}_0, \mathbb{Z}(X) \) is a quasi-isomorphism of complexes of abelian groups.

We now introduce an explicit localization functor for the \( (\mathbb{I}^1, \text{usu}) \) model structure.

• If \( F^* \in \text{PSh}(CW, C^{-}(\mathbb{Z})) \),

\[
\text{sing}_0, F^* := \text{Hom}(\mathbb{Z}(\mathbb{I}^1), F^*) \in \text{PSh}(CW, C^{-}(\mathbb{Z}))
\]  

(39)
is the total complex of presheaves associated to the bicomplex of presheaves $X \mapsto F^\bullet(I^* \times X)$, and $\text{sing}_* F^\bullet := e_{\text{cor}, \text{sing}_*} F^\bullet = F^\bullet(I^*) \in C^-(\mathbb{Z})$. We denote by $S(F^\bullet) : F^\bullet \to \text{sing}_* F^\bullet$

\[
S(F^\bullet) : \ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow F^\bullet \longrightarrow 0 \longrightarrow \ldots
\]

\[
\cdots \longrightarrow \text{sing}_* F^\bullet \longrightarrow \text{sing}_* F^\bullet \longrightarrow F^\bullet \longrightarrow 0 \longrightarrow \ldots
\]

the inclusion morphism in $\text{PSh}(CW), C^-(\mathbb{Z})$. For $f : F^\bullet_1 \to F^\bullet_2$ a morphism of $\text{PSh}(CW, C^-(\mathbb{Z}))$, we denote by $S(f) : \text{sing}_* F^\bullet_1 \to \text{sing}_* F^\bullet_2$ the morphism of $P^-(CW)$ given by, for $X \in CW,$

\[
S(f)(X) : \ldots \longrightarrow F^\bullet_1(I^2 \times X) \longrightarrow F^\bullet_1(I^1 \times X) \longrightarrow F^\bullet_1(X) \longrightarrow 0 \longrightarrow \ldots
\]

\[
\cdots \longrightarrow F^\bullet_2(I^2 \times X) \longrightarrow F^\bullet_2(I^1 \times X) \longrightarrow F^\bullet_2(X) \longrightarrow 0 \longrightarrow \ldots
\]

- If $F^\bullet \in \text{PSh}_Z(\text{Cor}^I_Z(CW), C^-(\mathbb{Z}))$,

\[
\text{sing}_* F^\bullet := \text{Hom}(\text{Ztr}, (I^*), F^\bullet) \in \text{PSh}_Z(\text{Cor}^I_Z(CW), C^-(\mathbb{Z}))
\]

is the total complex of presheaves associated to the bicomplex of presheaves $X \mapsto F^\bullet(I^* \times X)$, and $\text{sing}_* F^\bullet := e_{\text{cor}^I_Z, \text{sing}_*} F^\bullet = F^\bullet(I^*) \in C^-(\mathbb{Z})$. We have the inclusion morphism $[40]$

\[
S(\text{Tr}_* F^\bullet) : \text{Tr}_* F^\bullet \to \text{sing}_* \text{Tr}_* F^\bullet = \text{Tr}_* \text{sing}_* F^\bullet
\]

which is a morphism in $\text{PSh}(\text{Cor}^I_Z(CW), C^-(\mathbb{Z}))$ denoted the same way $S(F^\bullet) : F^\bullet \to \text{sing}_* F^\bullet$.

For $f : F^\bullet_1 \to F^\bullet_2$ a morphism of $P^-(CW)$, we have the morphism $[41]$

\[
S(\text{Tr}_* f) : \text{Tr}_* \text{sing}_* F^\bullet_1 = \text{sing}_* \text{Tr}_* F^\bullet_1 \to \text{sing}_* \text{Tr}_* F^\bullet_2 = \text{Tr}_* \text{sing}_* F^\bullet_2
\]

which is a morphism in $PC^-(CW)$ denoted the same way $S(f) : \text{sing}_* F^\bullet_1 \to \text{sing}_* F^\bullet_2$.

For $F^\bullet \in \text{PSh}(\text{Cor}^I_Z(CW), C^-(\mathbb{Z}))$, we have by definition $\text{Tr}_* \text{sing}_* F^\bullet = \text{sing}_* \text{Tr}_* F^\bullet$ and $\text{Tr}_* S(F^\bullet) = S(\text{Tr}_* F^\bullet)$.

We now make the following definition

**Definition 22.** Let $Y \in CW$ and $E \subseteq Y$ a CW subcomplex. Denote by $l : E \hookrightarrow X$ the topological embedding. We define

- $Z(Y, E) = \text{coker}(Z(l)) \in \text{PSh}_Z(CW, C^-(\mathbb{Z}))$ to be the cokernel of the injective morphism $Z(l) : Z(E) \hookrightarrow Z(X)$.

- $Z_{tr}(Y, E) = \text{coker}(Z_{tr}(l)) \in \text{PSh}_Z(\text{Cor}^I_Z(CW), C^-(\mathbb{Z}))$ to be the cokernel of the injective morphism $Z_{tr}(l) : Z_{tr}(E) \hookrightarrow Z_{tr}(Y)$, by definition, we have the following exact sequence in $PC^-(CW)$

\[
0 \to Z_{tr}(E) \xrightarrow{Z_{tr}(l)} Z_{tr}(Y) \xrightarrow{c(Y, E)} Z_{tr}(Y, E) \to 0,
\]

- $Z_{lY, E} = \text{coker}(\text{ad}(l, l^!)(Z_X)) \in \text{Sh}(Y)$ the cokernel of the injective morphism $\text{ad}(l, l^!)(Z_X) : l_Y l^! Z_X \hookrightarrow Z_Y$ in $\text{Sh}(Y)$ By definition, there is a distinguished triangle in $D^b(Y)$

\[
l_Y l^! Z_Y \to Z_Y \to Z_{lY, E} \to l_Y l^! Z_Y[1]
\]
The relative Borel Moore homology of the pair \((Y, E)\) is \(H^{BM}_{p}(Y, E, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, a_{X}a_{X}^{l}_{e}\mathbb{Z}_{(Y, E)})\).

In particular, we get from the second point the following exact sequence in \(\text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{1}(\text{CW}), C^{-}(\mathbb{Z}))\)

\[
0 \rightarrow \text{sing}_{\ast}Z_{tr}(E) \xrightarrow{\text{sing}_{\ast}Z_{tr}(f)} \text{sing}_{\ast}Z_{tr}(Y) \xrightarrow{\text{sing}_{\ast}c_{(Y, E)}} \text{sing}_{\ast}Z_{tr}(Y, E) \rightarrow 0 \quad (43)
\]

We define

\[
M(Y, E) = D(\mathbb{I}^{1}, \text{usu})(Z_{tr}(Y, E)) \in \text{CwDM}^{-}(\mathbb{Z})
\]
to be the relative motive of the pair \((Y, E)\).

We now look at the behavior of the functors mentioned above with respect to the \((\mathbb{I}^{1}, \text{usu})\) model structures. We start with a lemma concerning the properties of \(\mathbb{I}^{1}\) homotopy maps.

**Lemma 6.** Let \(F^{\ast} \in \text{PSh}(\text{CW}, C^{-}(\mathbb{Z}))\).

(i) Let \(X, Y \in \text{CW}\) and \(f_{0}: X \rightarrow Y, f_{1}: X \rightarrow Y\) be two maps. If \(f_{0}\) and \(f_{1}\) are \(\mathbb{I}^{1}\) homotopic, then the maps of complexes

\[
\text{sing}_{\ast}F^{\ast}(f_{0}) : \text{Tot} F^{\ast}(X \times \mathbb{I}^{1}) \rightarrow \text{Tot} F^{\ast}(X \times \mathbb{I}^{1})
\]

and

\[
\text{sing}_{\ast}F^{\ast}(f_{1}) : \text{Tot} F^{\ast}(X \times \mathbb{I}^{1}) \rightarrow \text{Tot} F^{\ast}(X \times \mathbb{I}^{1})
\]

induces the same map on homology.

(ii) Let \(X, Y \in \text{CW}\), if \(f : X \rightarrow Y\) is a \(\mathbb{I}^{1}\) homotopy equivalence then

\[
\text{sing}_{\ast}F^{\ast}(f) : \text{Tot} F^{\ast}(Y \times \mathbb{I}^{1}) \rightarrow \text{Tot} F^{\ast}(X \times \mathbb{I}^{1})
\]

is a quasi-isomorphism of complexes of abelian groups.

(iii) Let \(F^{\ast}, G^{\ast} \in \text{PSh}(\text{CW}, C^{-}(\mathbb{Z}))\) and \(\phi_{0} : F^{\ast} \rightarrow G^{\ast}, \phi_{1} : F^{\ast} \rightarrow G^{\ast}\) be two maps. If \(\phi_{0}\) and \(\phi_{1}\) are \(\mathbb{I}^{1}\) homotopic, then \(\phi_{0} \equiv \phi_{1} \in \text{CwDA}^{-}\).

(iv) Let \(F^{\ast}, G^{\ast} \in \text{PSh}(\text{CW}, C^{-}(\mathbb{Z}))\), if \(\phi : F^{\ast} \rightarrow G^{\ast}\) is a \(\mathbb{I}^{1}\) homotopy equivalence then \(\phi\) is a \((\mathbb{I}^{1}, \text{usu})\) local equivalence.

**Proof.** (i): Let \(h : X \times \mathbb{I}^{1} \rightarrow Y\) an homotopy from \(f_{0} = h \circ (I_{X} \times i_{0})\) to \(f_{1} = h \circ (I_{X} \times i_{1})\). Let \(\alpha \in F^{k}(X \times \mathbb{I}^{1})\). Then, we have

\[
\partial(F^{k}(h \times I_{\mathbb{I}^{1}})(\alpha)) = \partial(F^{k}(h \times I_{\mathbb{I}^{1}})(\alpha)) + \partial(F^{k}(h \times I_{\mathbb{I}^{1}})(\alpha))
\]

\[
= F^{k}(h \times I_{\mathbb{I}^{1}})(\partial F^{k}(\alpha)) + F^{k}(f_{0})(\alpha) - F^{k}(f_{1})(\alpha) + F^{k}(h \times I_{\mathbb{I}^{1}})(\partial F^{k}(\alpha))
\]

This proves (i).

(ii): Follow immediately from (i).

(iii): Let \(\phi : F^{\ast} \rightarrow \text{Hom}(\mathbb{Z}(\mathbb{I}^{1}), G^{\ast})\) an homotopy from \(\phi_{0} = G^{\ast}(i_{0}) \circ \tilde{\phi}\) to \(\phi_{1} = G^{\ast}(i_{1}) \circ \tilde{\phi}\). Let

\[
G^{\ast}(p) : G^{\ast} \rightarrow \text{Hom}(\mathbb{Z}(\mathbb{I}^{1}), G^{\ast})
\]

be the map induced by \(a_{\mathbb{I}^{1}} : \mathbb{I}^{1} \rightarrow \{pt\}\), that is, for \(X \in \text{CW}\)

\[
G^{\ast}(p)(x) = G^{\ast}(p_{X}) : G^{\ast}(X) \rightarrow G^{\ast}(X \times \mathbb{I}^{1})
\]

with \(p_{X} = I_{X} \times a_{\mathbb{I}^{1}} : X \times \mathbb{I}^{1} \rightarrow X\) is the projection. Then, we have

\[
G^{\ast}(i_{0}) \circ G^{\ast}(p) = G^{\ast}(i_{1}) \circ G^{\ast}(p) = I_{G^{\ast}}\quad (45)
\]
Hence, it suffice to show that $G^\bullet(p) : G^\bullet \to \text{Hom}(\mathbb{Z}(\mathbb{I}^1), G^\bullet)$ is an $(\mathbb{I}^1, \text{usu})$ local equivalence, since then $G^\bullet(t_0) = G^\bullet(t_1) \circ G^\bullet(p)^{-1} \in \text{CwDA}^-$. So, consider $\psi : G^\bullet \to L^\bullet$ a morphism in $\text{PSh}(\text{CW}, C^-(\mathbb{Z}))$ with $L^\bullet$ $\mathbb{I}^1$ local. Consider the commutative diagram in $\text{Ho}_{\text{usu}} \text{PSh}(\text{CW}, C^-(\mathbb{Z}))$

\[
\begin{array}{ccc}
G^\bullet & \xrightarrow{G^\bullet(p)} & \text{Hom}(\mathbb{Z}(\mathbb{I}^1), G^\bullet) \\
\downarrow{\psi} & & \downarrow{\text{Hom}(\mathbb{Z}(\mathbb{I}^1), \psi)} \\
L^\bullet & \xrightarrow{L^\bullet(p)} & \text{Hom}(\mathbb{Z}(\mathbb{I}^1), L^\bullet)
\end{array}
\]

(46)

Since we now consider morphism in the homotopy category with respect to the usu model structure, we can assume, up to replace $L^\bullet$ by an usu local equivalent object that $L^\bullet$ is usu fibrant. Then, $L^\bullet L^\bullet(p), \text{Hom}(\mathbb{Z}(\mathbb{I}^1), L^\bullet)$ is an usu local equivalence and

\[
L^\bullet(p)^{-1} \circ \text{Hom}(\mathbb{Z}(\mathbb{I}^1), \psi) : \text{Hom}(\mathbb{Z}(\mathbb{I}^1), G^\bullet) \to L^\bullet
\]

(47)

is the only map making the diagram commute. This prove that $G^\bullet(p)$ is a $(\mathbb{I}^1, \text{usu})$ local equivalence.

(iii): Follows immediately from (iii).

\[\square\]

**Lemma 7.**

(i) A complex of presheaves $F^\bullet \in \text{PSh}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{CW}), C^-(\mathbb{Z}))$ is $\mathbb{I}^1$ local if and only if $\text{Tr}_\ast F^\bullet \in \text{PSh}(\text{CW}, C^-(\mathbb{Z}))$ is $\mathbb{I}^1$ local.

(ii) A morphism $\phi : F^\bullet \to G^\bullet$ in $\text{PSh}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{CW}), C^-(\mathbb{Z}))$ is an $(\mathbb{I}^1, \text{usu})$ local equivalence if and only if $\text{Tr}_\ast \phi : \text{Tr}_\ast F^\bullet \to \text{Tr}_\ast G^\bullet$ is an $(\mathbb{I}^1, \text{usu})$ local equivalence.

**Proof.**

(i): Let $h : F^\bullet \to L^\bullet$ be an usu local equivalence in $\text{PSh}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{CW}), C^-(\mathbb{Z}))$ with $L^\bullet$ usu fibrant. Then,

- $F^\bullet$ is $\mathbb{I}^1$ local if and only if $L^\bullet$ is $\mathbb{I}^1$ local.
- By definition $\text{Tr}_\ast$ preserve usu local equivalence, hence $\text{Tr}_\ast h : \text{Tr}_\ast F^\bullet \to \text{Tr}_\ast L^\bullet$ is an usu local equivalence in $\text{PSh}(\text{CW}, C^-(\mathbb{Z}))$. Thus, $\text{Tr}_\ast F^\bullet$ is $\mathbb{I}^1$ local if and only if $\text{Tr}_\ast L^\bullet$ is $\mathbb{I}^1$ local.
- As indicated in [3], since $\text{Tr}_\ast L^\bullet$ is also usu fibrant. By Yoneda lemma, for $X \in \text{CW}$, we have

\[
\text{Hom}_{\mathbb{P}^- (\text{CW})}(\mathbb{Z}(X)[n], \text{Tr}_\ast L^\bullet) = H^n L^\bullet(X) = \text{Hom}_{\mathbb{P}^- (\text{CW})}(\mathbb{Z}_{\text{tr}}(X)[n], L^\bullet)
\]

(48)

Hence $L^\bullet$ is $\mathbb{I}^1$ local if and only if $\text{Tr}_\ast L^\bullet$ is $\mathbb{I}^1$ local.

This prove (i).

(ii): Let us prove (ii)

- The only if part is similar to [2], lemma 2.111, we check that, for $Y \in \text{CW}$,

\[
\text{Tr}_\ast \mathbb{Z}_{\text{tr}}(p_Y) : \text{Tr}_\ast \mathbb{Z}_{\text{tr}}(Y \times \mathbb{I}^1) \to \text{Tr}_\ast \mathbb{Z}_{\text{tr}}(Y)
\]

is an equivalence $(\mathbb{I}^1, \text{usu})$ local, where $p_Y : Y \times \mathbb{I}^1 \to Y$ is the projection. By lemma [4](iv), it suffice to show that $\text{Tr}_\ast \mathbb{Z}_{\text{tr}}(p_Y)$ is an $\mathbb{I}^1$ homotopy equivalence in $\mathbb{P}^- (\text{CW})$. By lemma [3] it suffice to show that $p_Y$ is an $\mathbb{I}^1$ homotopy equivalence in $\text{CW}$. But we have the map

\[
\theta_{12}(Y) : Y \times \mathbb{I}^2 \to Y \times \mathbb{I}^1, \quad \theta_{12}(y, t_1, t_2) = (y, t_1 - t_2 t_1)
\]

(49)

which is an homotopy from $I_{Y \times \mathbb{I}^1} = \theta_{12}(Y) \circ (I_{Y \times \mathbb{I}^1} \times i_0)$ to $I_Y \times 0 = \theta_{12}(Y) \circ (I_{Y \times \mathbb{I}^1} \times i_1)$. On the other hand, $p_Y \circ (I_Y \times 0) = I_Y$. This proves the only if part.
Let \( \phi : F^* \to G^* \) in \( \text{PSh}(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z)) \) such that \( \text{Tr} \phi : \text{Tr}_* F^* \to \text{Tr}_* G^* \) is an equivalence \((\text{I}^1, \text{usu})\) local in \( \text{PSh}(\text{CW}, C^-(Z)) \). Since by definition \( \text{Tr}_* \) preserve and detect \( \text{usu} \) local equivalence, we can assume, up to replace \( F^* \) and \( G^* \) by \( \text{usu} \) equivalent presheaves, that \( F^* \) and \( G^* \) are \( \text{usu} \) fibrant. Let \( K^* \in \text{PSh}(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z)) \) an \( \text{I}^1 \) local object. By (i), \( \text{Tr}_* K^* \) is \( \text{I}^1 \) local. Hence, we have,

\[
\text{Hom}_{\text{PSh}(\text{CW})}(G^*, K^*) = \text{Hom}_{\text{PSh}(\text{CW})}(\text{Tr}_* F^*, \text{Tr}_* G^*, \text{Tr}_* K^*) \xrightarrow{\sim} \text{Hom}_{\text{PSh}(\text{CW})}(\text{Tr}_* F^*, \text{Tr}_* G^*, \text{Tr}_* K^*) \tag{50}
\]

\[
\text{Hom}_{\text{PSh}(\text{CW})}(\text{Tr}_* F^*, \text{Tr}_* G^*, \text{Tr}_* K^*) = \text{Hom}_{\text{PSh}(\text{CW})}(\text{Tr}_* F^*, \text{Tr}_* G^*, \text{Tr}_* K^*) \tag{51}
\]

This proves the if part.

**Proposition 18.**

(i) \((\text{Tr}^*, \text{Tr}_*) : \text{PSh}(\text{CW}, C^-(Z)) \xrightarrow{\sim} \text{PSh}_Z(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z))\) is a Quillen adjunction for the usual topology model structures (c.f. definition 19 (i) and (ii) respectively) and a Quillen adjunction for the \((\text{I}^1, \text{usu})\) model structures (c.f. definition 20 (i) and (ii) respectively).

(ii) \((e_{cw}^*, e_{cw*}) : \text{PSh}(\text{CW}, C^-(Z)) \xrightarrow{\sim} C^-(Z)\) is a Quillen adjunction for the usual topology model structure (c.f. definition 19 (ii)) and a Quillen adjunction for the \((\text{I}^1, \text{usu})\) model structure (c.f. definition 20 (ii)).

(iii) \((e_{cw}^{tr*}, e_{cw*}^{tr}) : \text{PSh}(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z)) \xrightarrow{\sim} C^-(Z)\) is a Quillen adjunction for the usual topology model structure (c.f. definition 19 (iii)) and a Quillen adjunction for the \((\text{I}^1, \text{usu})\) model structure (c.f. definition 20 (iii)).

**Proof.** (i): By lemma 7 (i), \( \text{Tr}_* \) preserve \( \text{I}^1 \) local objects, hence preserve the fibrations of the \((\text{I}^1, \text{usu})\) model structures. By lemma 7 (ii), \( \text{Tr}_* \) preserve \((\text{I}^1, \text{usu})\) equivalence. Thus, \( \text{Tr}_* \) preserve the trivial fibrations of the \((\text{I}^1, \text{usu})\) model structures.

(ii): It is clear that \( e_{cw}^* \) is a left Quillen functor, that is preserve cofibrations and trivial cofibrations.

(iii): It is clear that \( e_{cw}^{tr*} \) is a left Quillen functor, that is preserve cofibrations and trivial cofibrations.

**Proposition 19.**

(i) For \( F^* \in \text{PSh}(\text{CW}, C^-(Z)) \), the adjunction morphism

\[
\text{ad}(e_{cw}^*, e_{cw*}) : e_{cw}^* e_{cw*} \xrightarrow{\sim} F^* \tag{52}
\]

is an equivalence \( \text{usu} \) local.

(ii) For \( F^* \in \text{PSh}(\text{Cor}_Z^{fs}(\text{CW}), C^-(Z)) \), the adjunction morphism

\[
\text{ad}(e_{cw}^{tr*}, e_{cw*}^{tr}) : e_{cw}^{tr*} e_{cw*}^{tr} \xrightarrow{\sim} F^* \tag{53}
\]

is an equivalence \( \text{usu} \) local.

(iii) For \( K^* \in C^-(Z) \), the adjunction morphisms

\[
\text{ad}(e_{cw}^*, e_{cw*})(K^*) : K^* \to e_{cw}^* e_{cw*} K^* \quad \text{and} \quad \text{ad}(e_{cw}^{tr*}, e_{cw*}^{tr})(K^*) : K^* \to e_{cw}^{tr*} e_{cw*}^{tr} K^* \tag{54}
\]

are isomorphisms.

**Proof.** (i): The question is local. It then follows from the fact that the CW complexes are locally contractible. Indeed, let \( X \) be CW. Since the question is local and CW complexes are locally contractible, we can assume after shrinking \( X \) that \( X \) is contractible. Let \( x \in X \). Denoting \( a_X : X \to \{\text{pt}\} \) the terminal map and \( e_x : \{\text{pt}\} \to X \) the point map, there exist an homotopy between \( e_x \circ a_X \) and the identity \( I_X \) of \( X \), that is \( p_X \) is a \( \text{I}^1 \) homotopy equivalence. Hence, by lemma 9 (ii),

\[
\text{ad}(e_{cw}^*, e_{cw*})(\text{sing}_x F^*)(X) = \text{sing}_x F^*(a_X) : F^*(\text{I}^1) \to F^*(\text{I}^1 \times X) \tag{55}
\]
is an homotopy equivalence of complexes of abelian groups.

(ii): It is a particular case of point (i) since, by definition \([19]\), \(\text{Tr}_*\) detect and preserve usu local equivalence. Indeed, by (i)

\[
e_{cw}^* e_{cw} \text{sing}_{\text{tr}} \xrightarrow{\text{ad}(e_{cw}^*, e_{cw})} \text{sing}_{\text{tr}} \xrightarrow{\text{sing}_{\text{tr}} \text{Tr}_* F^*} \text{Tr}_* \text{sing}_{\text{tr}} F^* \]

is an equivalence usu local. Since \(\text{Tr}_*\) preserve usu local equivalence, this prove (ii).

(iii): Trivial.

\[
\square
\]

**Proposition 20.**  
(i) The functor \(\text{Tr}_* : \text{PSh}_\mathbb{Z}(\text{Cor}^f_\mathbb{Z}(\text{CW}), C^- (\mathbb{Z})) \to \text{PSh}(\text{CW}, C^- (\mathbb{Z}))\) derive trivially.

(ii) For \(K^* \in C^- (\mathbb{Z})\), \(e_{cw}^* K^*\) is \(\mathbb{I}^1\) local.

(iii) For \(K^* \in C^- (\mathbb{Z})\), \(e_{cw}^* K^*\) is \(\mathbb{I}^1\) local.

**Proof.** (i): By lemma \([7]\), \(\text{Tr}_*\) preserve \((\mathbb{I}^1, usu)\) local equivalences.

(ii): Let \(e_{cw}^* K^* \to L^*\) an usu local equivalence, with \(L^*\) usu fibrant. Since \(e_{cw}^* K^*\) is usu equivalent to \(L^*\) it suffices to prove that \(L^*\) is \(\mathbb{I}^1\) local. Since \(L^*\) is usu fibrant, we have to prove that

\[
\tilde{L}(p) : L^* \to \text{Hom}(\mathbb{Z}(\mathbb{I}^1), L^*)
\]

is an equivalence usu local. The proof is now similar to \([\mathbb{I}]\) proposition 1.6 etape B.

(iii): We have \(\text{Tr}_* e_{cw}^* K^* = e_{cw}^* K^*\). By (ii), \(e_{cw}^* K^*\) is \(\mathbb{I}^1\) local. By lemma \([7](i)\), \(\text{Tr}_*\) detect \(\mathbb{I}^1\) local object. This proves (iii).

\[
\square
\]

**Theorem 16.**  
(i) For \(F^* \in \text{PSh}(\text{CW}, C^- (\mathbb{Z}))\), \(\text{sing}_{\text{tr}} F^* \in \text{PSh}(\text{CW}, C^- (\mathbb{Z}))\) is \(\mathbb{I}^1\) local and the inclusion morphism \(S(F^*) : F^* \to \text{sing}_{\text{tr}} F^*\) is an \((\mathbb{I}^1, \text{usu})\) equivalence.

(ii) For \(F^* \in \text{PSh}_\mathbb{Z}(\text{Cor}^f_\mathbb{Z}(\text{CW}), C^- (\mathbb{Z}))\), \(\text{sing}_{\text{tr}} F^* \in \text{PSh}_\mathbb{Z}(\text{Cor}^f_\mathbb{Z}(\text{CW}), C^- (\mathbb{Z}))\) is \(\mathbb{I}^1\) local and the inclusion morphism \(S(F^*) : F^* \to \text{sing}_{\text{tr}} F^*\) is an \((\mathbb{I}^1, \text{usu})\) equivalence.

**Proof.** (i): Let us prove (i)

- By proposition \([19](i)\) and proposition \([20](ii)\), \(\text{sing}_{\text{tr}} F^* \in \text{PSh}(\text{CW}, C^- (\mathbb{Z}))\) is \(\mathbb{I}^1\) local. This proves the first part of the assertion.

- Consider the commutative diagram

\[
\begin{array}{ccccccc}
\cdots & 0 & \xrightarrow{I} & 0 & \xrightarrow{F^*} & 0 & \xrightarrow{I} & \cdots \\
\cdots & 0 & \xrightarrow{I} & F^* & \xrightarrow{0} & F^* & \xrightarrow{I} & \cdots \\
\cdots & 0 & \xrightarrow{\text{sing}_{\text{tr}} F^*} & \text{sing} F^* & \xrightarrow{I} & \text{sing} F^* & \xrightarrow{0} & \cdots \\
\end{array}
\]

By the diagram \([57]\), to prove that \(S(F^*) : F^* \to \text{sing}_{\text{tr}} F^*\) is an \((\mathbb{I}^1, \text{usu})\) equivalence, it suffices to show, as in \([\mathbb{I}]\), theorem 2.23 etape 2, that for all \(n \in \mathbb{Z}\), the morphism

\[
\tilde{F}^*_{(p_n)} : F^* \to \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^*), X \in \text{CW} \mapsto \tilde{F}^*_{(p_n)}(X) = F^*_{(p_n)}(X) = F^*(p_X) : F^*(X) \to F^*(\mathbb{I}^n \times X)
\]

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is an equivalence \((\mathbb{I}^1, \text{usu})\) local. For \(X \in \text{CW}\), consider the map

\[
\theta_{1,n}(X) : \mathbb{I}^n \times \mathbb{I}^1 \times X \to \mathbb{I}^n \times X \; : \; (t_1, \ldots, t_n, t_{n+1}, x) \mapsto (t_1 - t_{n+1} t_1, \ldots, t_n - t_{n+1} t_n, x)
\]

We have,

\[
\theta_{1,n}(X) \circ (I^n \times i_0 \times I_X) = I^n \times X \; , \; \theta_{1,n}(X) \circ (I^n \times i_1 \times I_X) = 0 \times I_X,
\]

that is \(\theta_{1,n}(X)\) define an \(\mathbb{I}^1\) homotopy from \(I^n \times X\) to \(0 \times I_X\); on the other side \(p_X \circ (0 \times I_X) = I_X\), with \(p_X : \mathbb{I}^n \times X \to X\) the projection. Thus,

\[
F^\ast(\theta_{1,n}) \circ F^\ast(I^n \times i_0) = F^\ast(I^n) \; , \; F^\ast(\theta_{1,n}) \circ F^\ast(I^n \times i_1) = F^\ast(0),
\]

that is \(F^\ast(\theta_{1,n})\) define an \(\mathbb{I}^1\) homotopy from \(F^\ast(I^n)\) to \(F^\ast(0)\); on the other side, \(F^\ast(0) \circ F^\ast(p_n) = I\), with

\[
\begin{align*}
- F^\ast(0) : \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast) & \to \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast), \text{ given by } X \in \text{CW} \mapsto F^\ast(0)(X) = F^\ast(0 \times I_X) : F^\ast(\mathbb{I}^n \times X) \to F^\ast(\mathbb{I}^n \times X) \\
- F^\ast(I_X) : \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast) & \to \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast), \text{ given by } X \in \text{CW} \mapsto F^\ast(I_X)(X) = F^\ast(I^n \times X) \\
- F^\ast(\theta_{1,n}) : \text{Hom}(\mathbb{Z}(\mathbb{I}^1), \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast)) & \to \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast), \text{ given by } X \in \text{CW} \mapsto F^\ast(\theta_{1,n})(X) = F^\ast(\theta_{1,n})(X) : F^\ast(\mathbb{I}^n \times X) \to F^\ast(\mathbb{I}^n \times \mathbb{I}^1 \times X).
\end{align*}
\]

Hence, \(F^\ast(p_n) : F^\ast \to \text{Hom}(\mathbb{Z}(\mathbb{I}^n), F^\ast)\) is an homotopy equivalence, hence an equivalence \((\mathbb{I}^1, \text{usu})\) local by lemma 6 (iv).

(ii): Let us prove (ii)

- By (i), \(\text{Tr}_\ast \text{sing}_\ast, F^\ast = \text{sing}_\ast, \text{Tr}_\ast F^\ast\) is \(\mathbb{I}^1\) local. By lemma 17 (i), \(\text{Tr}_\ast\) detect \(\mathbb{I}^1\) local object. Thus, \(\text{sing}_\ast F^\ast = \text{sing}_\ast F^\ast\) is \(\mathbb{I}^1\) local. This proves the first part of the assertion.

- It follows from (i) that

\[
\text{Tr}_\ast F^\ast \xrightarrow{\text{Tr}_\ast S(F^\ast)} \text{sing}_\ast, \text{Tr}_\ast F^\ast = \text{Tr}_\ast \text{sing}_\ast F^\ast \quad (58)
\]

is an \((\mathbb{I}^1, \text{usu})\) equivalence. Since, by lemma 17 (ii), \(\text{Tr}_\ast\) detect \((\mathbb{I}^1, \text{usu})\) equivalence, \(S(F^\ast) : F^\ast \to \text{sing}_\ast, F^\ast\) is an \((\mathbb{I}^1, \text{usu})\) equivalence.

\[
\square
\]

**Proposition 21.**

(i) For \(F^\ast \in \text{PSh}(\text{Cor}_\mathbb{Z}(\text{CW}), C^-(\mathbb{Z}))\),

\[
\text{ad}(\text{Tr}_\ast, \text{Tr}_\ast)(F^\ast) : F^\ast \to \text{Tr}_\ast \text{Tr}_\ast F^\ast
\]

is an isomorphism in \(\text{PSh}(\text{Cor}_\mathbb{Z}(\text{CW}), C^-(\mathbb{Z}))\).

(ii) For \(X \in \text{CW}\), the embedding

\[
\text{ad}(\text{Tr}_\ast, \text{Tr}_\ast)(\text{sing}_\ast, Z(X)) : \text{sing}_\ast, Z(X) \to \text{Tr}_\ast \text{sing}_\ast, Z_{\text{tr}}(X)
\]

in \(\text{PSh}(\text{CW}, C^-(\mathbb{Z}))\) is an equivalence usu local

*Proof.* (i): Obvious

(ii): By proposition 13 (i), there exist an homotopy equivalence \(g : X' \to X\), with \(X' \in \text{CS}\) a \(\Delta\) complex. Then,

- \(S(\mathbb{Z}(g)) : \text{sing}_\ast, Z(X') \to \text{sing}_\ast, Z(X)\) is an \(\mathbb{I}^1\) homotopy equivalence by lemma 4 thus an usu local equivalence by lemma 6 (iv) and theorem 14 (i),

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Consider the following commutative diagram in $\text{PSh}(\text{CW}, C^{-}(\mathbb{Z}))$

\[
\begin{array}{ccc}
\text{sing}_{\ast} Z(X') & \xrightarrow{\text{ad}(\text{Tr}^{\ast}, \text{Tr}_{\ast})(\text{sing}_{\ast} Z(X'))} & \text{Tr}_{\ast} \text{sing}_{\ast} Z_{tr}(X') \\
\downarrow S(Z(g)) & & \downarrow \text{Tr}_{\ast} S(Z_{tr}(g)) \\
\text{sing}_{\ast} Z(X) & \xrightarrow{\text{ad}(\text{Tr}^{\ast}, \text{Tr}_{\ast})(\text{sing}_{\ast} Z(X))} & \text{Tr}_{\ast} \text{sing}_{\ast} Z_{tr}(X)
\end{array}
\]  

(59)

By diagram (59) and the fact that $S(Z(g))$ and $\text{Tr}_{\ast} S(Z_{tr}(g))$ are usu local equivalence, it suffice to show that

\[
\text{ad}(\text{Tr}^{\ast}, \text{Tr}_{\ast})(\text{sing}_{\ast} Z(X')) : \text{sing}_{\ast} Z(X') \to \text{Tr}_{\ast} \text{sing}_{\ast} Z_{tr}(X')
\]

is an usu local equivalence. By proposition 16(ii), there exist a countable open covering $X' = \bigcup_{i \in I} U_i$ such that for all finite subset $I \subset J$, $U_I := \bigcap_{i \in I} U_i = \emptyset$ or $U_I$ is contractible. Let $(U_{\ast}, \bullet)$ be the complex

- $h : Z(U_{\ast}, \bullet) \to K^{\ast}$ be an equivalence usu local in $\text{PSh}(\Delta \times CW, C^{-}(\mathbb{Z}))$ with $K^{\ast}$ usu fibrant
- $l : Z_{tr}(U_{\ast}, \bullet) \to L^{\ast}$ be an equivalence usu local in $\text{PSh}(\Delta \times \text{Cor}_{\mathbb{Z}}(CW), C^{-}(\mathbb{Z}))$ with $L^{\ast}$ usu fibrant

We have then a unique morphism $m : K^{\ast} \to \text{Tr}_{\ast} L^{\ast}$ such that the following diagram in $\text{Ho}_{\text{usu}}(\text{PSh}(\Delta \times CW, C^{-}(\mathbb{Z})))$ commutes

\[
\begin{array}{ccc}
\text{Z}((U_{\ast}, \bullet)) & \xrightarrow{\text{ad}(\text{Tr}^{\ast}, \text{Tr}_{\ast})(\text{Z}((U_{\ast}, \bullet))))} & \text{Tr}_{\ast} \text{Z}_{tr}((U_{\ast}, \bullet)) \\
\downarrow h & & \downarrow \text{Tr}_{\ast} l \\
K^{\ast} & \xrightarrow{m} & \text{Tr}_{\ast} L^{\ast}
\end{array}
\]  

(60)

We have then the following commutative diagram in $\text{Ho}_{\text{usu}}(\text{PSh}(CW, C^{-}(\mathbb{Z})))$

\[
\begin{array}{ccc}
\text{Z}(X) & \xrightarrow{\text{ad}(\text{Tr}^{\ast}, \text{Tr}_{\ast})(\text{Z}(X'))} & \text{Tr}_{\ast} \text{Z}_{tr}(X') \\
\downarrow j & & \downarrow \text{Tr}_{\ast}(j) \\
\text{Z}((U_{\ast}, \bullet)) & \xrightarrow{\text{ad}(\text{Tr}^{\ast}, \text{Tr}_{\ast})(\text{Z}((U_{\ast}, \bullet))))} & \text{Tr}_{\ast} \text{Z}_{tr}((U_{\ast}, \bullet))
\end{array}
\]  

(61)

By lemma 5(i), and the fact that $\text{Tr}_{\ast}$ preserve usu local equivalence,

- $j : Lp^{\Delta} \text{Z}((U_{\ast}, \bullet)) = p^{\Delta} K^{\ast} \to \text{Z}(X)$, and
- $\text{Tr}_{\ast}(j) : \text{Tr}_{\ast} Lp^{\Delta} \text{Z}_{tr}((U_{\ast}, \bullet)) = p^{\Delta} \text{Tr}_{\ast} L^{\ast} \to \text{Tr}_{\ast} \text{Z}_{tr}(X)$,

are isomorphisms. On the other hand, $U_{[n]}$ is contractible for all $[n] \in \Delta$, this means that the canonical morphism $a_{U_{[n]}} : U_{[n]} \to \{pt\}$ in CW is an homotopy equivalence. Hence,

- $\text{Z}(a_{U_{[n]}}) : \text{Z}(U_{[n]}) \to \text{Z}([pt])$ is an homotopy equivalence in $\text{PSh}(CW, C^{-}(\mathbb{Z}))$, thus an $(\mathbb{I}^{1}, \text{usu})$ local equivalence by lemma 6(iv);
- $\text{Tr}_{\ast} \text{Z}_{tr}(a_{U_{[n]}}) : \text{Tr}_{\ast} \text{Z}_{tr}(U_{[n]}) \to \text{Tr}_{\ast} \text{Z}_{tr}([pt])$ is an homotopy equivalence in $\text{PSh}(CW, C^{-}(\mathbb{Z}))$, thus an $(\mathbb{I}^{1}, \text{usu})$ local equivalence by lemma 6(iv).
Thus,
\[
\text{ad}(\text{Tr}^*, \text{Tr}_*)(\mathbb{Z}((U_\bullet, \bullet))) : \mathbb{Z}((U_\bullet, \bullet)) \to \text{Tr}_* Z_{tr}(U_\bullet, \bullet)
\]
is an \((I^1, \text{usu})\) local equivalence in \(\text{PSh}(\Delta \times \text{CW}, C^-(\mathbb{Z}))\). The diagram (60) then shows that \(m : K^\bullet \to \text{Tr}_* L^\bullet\) is an isomorphism in \(\text{Hom}^1, \text{usu}(\text{PSh}(\Delta \times \text{CW}, C^-(\mathbb{Z}))\). Since \(p_{\Delta}^*\) derive trivially by definition, this implies that
\[
p_{\Delta}^*(m) : p_{\Delta}^* K^\bullet \to p_{\Delta}^* \text{Tr}_* L^\bullet
\]
is an isomorphism in \(\text{CwDM}(\mathbb{Z})^-\). The diagram (61), then shows that \(\text{ad}(\text{Tr}^*, \text{Tr}_*)(\mathbb{Z}(X')) : \mathbb{Z}(X) \to \text{Tr}_* Z_{tr}(X')\) is an isomorphism in \(\text{CwDM}(\mathbb{Z})^-\).

**Theorem 17.** (i) The adjonction \((\text{Tr}^*, \text{Tr}_*) : \text{PSh}(\text{CW}, C^-(\mathbb{Z})) \leftrightarrows \text{PSh}_2(\text{Cor}^I_{e_\text{cw}}(\text{CW}), C^-(\mathbb{Z}))\) is a Quillen equivalence for the \((I^1, \text{usu})\) model structures. That is, the derived functor
\[
L \text{Tr}^* : \text{CwDA}^- (\mathbb{Z}) \sim \to \text{CwDM}^- (\mathbb{Z})
\]
is an isomorphism and \(\text{Tr}_* : \text{CwDM}^- (\mathbb{Z}) \sim \to \text{CwDA}^- (\mathbb{Z})\) is it inverse.

(ii) The adjonction \((e_{\text{cw}}^*, e_{\text{cw}*}) : C^-(\mathbb{Z}) \leftrightarrows \text{PSh}_2(\text{Cor}^I_{e_\text{cw}}(\text{CW}), C^-(\mathbb{Z}))\) is a Quillen equivalence for the \((I^1, \text{usu})\) model structures. That is, the derived functor
\[
e_{\text{cw}}^* : D^- (\mathbb{Z}) \sim \to \text{CwDA}^- (\mathbb{Z})
\]
is an isomorphism and \(R e_{\text{cw}*} : \text{CwDA}^- (\mathbb{Z}) \sim \to D^- (\mathbb{Z})\) is it inverse.

(iii) The adjonction \((e_{\text{cw}}^{tr*}, e_{\text{cw}}^{tr*}) : C^-(\mathbb{Z}) \leftrightarrows \text{PSh}_2(\text{Cor}^I_{e_\text{cw}}(\text{CW}), C^-(\mathbb{Z}))\) is a Quillen equivalence for the \((I^1, \text{usu})\) model structures. That is, the derived functor
\[
e_{\text{cw}}^{tr*} : D^- (\mathbb{Z}) \sim \to \text{CwDM}^- (\mathbb{Z})
\]
is an isomorphism and \(R e_{\text{cw}*}^{tr} : \text{CwDM}^- (\mathbb{Z}) \sim \to D^- (\mathbb{Z})\) is it inverse.

**Proof.** (i): It follows from proposition [21](#) and theorem [16](#) 
(ii): It follows from proposition [19](#)(i) and theorem [16](#)(i).
(iii): It follows from proposition [19](#)(ii) and theorem [16](#)(ii). It also follows from (i) and (ii).

We deduce from proposition [21](#)(ii) and proposition [17](#) the following:

**Proposition 22.** For \(Y \in \text{CW}\) and \(l : E \leftrightarrow Y\) a CW subcomplex, the followings embeddings are quasi-isomorphism :
\[
C_{\ast}^{\text{sing}}(Y, E, Z) \xrightarrow{Z(Y, E)(L)} \text{sing}_t, Z(Y, E) \xrightarrow{e_{\text{cw}}^* \text{ad}(\text{Tr}^*, \text{Tr}_*)(\text{sing}_t, Z(Y, E))} \text{sing}_t, Z_{tr}(Y, E)
\]
where \(C_{\ast}^{\text{sing}}(Y, E, Z) = \text{coker} l_*\) is the relative cohomology, with \(l_* : Z \text{Hom}_{\text{CW}}(\Delta^*, E) \hookrightarrow Z \text{Hom}_{\text{CW}}(\Delta^*, Y)\).

**Proof.** Follows from proposition [21](#)(ii) and proposition [17](#)

Let \(X, Y \in \text{CW}\) and \(E \subset Y\) a subcomplex, we have

- the morphism
\[
S_n(X, (Y, E)) : \text{Hom}_{\text{PC}^-(\text{CW})}((\text{sing}_t, Z_{tr}(X), \text{sing}_t, Z_{tr}(Y, E)[n])) \to \text{Hom}_{\text{PC}^-(\text{CW})}((Z_{tr}(X), \text{sing}_t, Z_{tr}(Y, E)[n]),
\]
given by the composition \(S_n(X, (Y, E))(H) = H \circ S(Z_{tr}(X))\) with the inclusion morphism \(S(Z_{tr}(X))\),
follows from (i).

Proof. (i): Obvious
(ii): Follows from (i).

Recall $TM(\mathbb{R}) \subset CW$ is the full subcategory of topological manifolds. Let $X \in TM(\mathbb{R})$ connected topological manifold, $Y \in CW$ a CW complex, and $l : E \to Y$ a CW subcomplex. Denote by $p_X : \mathbb{I}^n \times X \times Y \to X$, $p_Y : \mathbb{I}^n \times X \times Y \to Y$ and $p_{X \times Y} : \mathbb{I}^n \times X \times Y \to X \times Y$ the projections. Let

$$T = \sum_i n_i T_i \in \text{sing}_i Z_{tr}(Y, E)(X)$$

such that $\partial_i T = 0$, where $m_i : T_i \to \mathbb{I}^n \times X \times Y$ is a closed CW subcomplex for all $i$. Denote $p_{X_i} = p_X \circ m_i : T_i \to X$ and $p_{Y_i} = p_X \circ m_i : T_i \to Y$. Consider the class of $T$:

$$[T] \in H^n \text{sing}_i Z_{tr}(Y, E)(X) = \text{Hom}_{PC-(CW)}(\mathbb{I}^n, \text{sing}_i Z_{tr}(Y, E)(X))$$

We have then

- its image

$$Re_{cw}^{tr} \circ D([\mathbb{I}^n, \text{usu}]([T]) \circ S(Z_{tr}(X)))^{-1} \in \text{Hom}_{D^{-Z}}(\text{sing}_i Z_{tr}(X), \text{sing}_i Z_{tr}(Y, E)(X))$$

by the composite functor $Re_{cw}^{tr} \circ D([\mathbb{I}^n, \text{usu}] : \text{PSh}(\text{Cor}_\mathbb{R}(CW), C^{-Z}) \to D^{-Z}(Z)$, where the last equality follows from the fact that by theorem (16): ii)

- $S(Z_{tr}(X)) : Z_{tr}(X) \to \text{sing}_i Z_{tr}(X)$ is an equivalence $([\mathbb{I}^n, \text{usu}]$ local and

- $\text{sing}_i Z_{tr}(X)$ and $\text{sing}_i Z_{tr}(Y)$ are $\mathbb{I}^n$ local objects,

- the action of $p_{X \times Y}(T) \in \mathcal{Z}_{d X + n}(X \times Y, Z)$

$$K_n(X, (Y, E))(p_{X \times Y}(T)) : \sum_i n_i(c_{Y, E}[n]) \circ (p_{Y_i}^*[n]) \circ p_{X_i}^* \in \text{Hom}_{D^{-Z}}(C_*(X, Z), C_*(Y, E, Z)[n]),$$

on homology, where, for each $i$:

- $p_{X_i}^* \in \text{Hom}_{D^{-Z}}(C_*(X, Z), C_*(T_i, Z)[n])$ is the Gysin morphism ($p_{X_i}$ is proper and $X \in TM(\mathbb{R})$ is a topological manifold),

- $p_{Y_i} = Z(p_{Y_i}^*(\Delta^*)): C_*(T_i, Z) \to C_*(Y, Z)$ is the classical map on singular chain,
Lemma 9. Let $X \in \text{TM}(\mathbb{R})$ connected, $Y \in \text{CW}$ and $E \subset Y$ a subcomplex. Let $T = \sum_{i} n_{i} T_{i} $ in $\text{sing}_{\text{tr}}(Y,E)(X)$ such that $\partial_{T} \cdot T = 0$ and let $\gamma = \sum_{i} m_{i} \gamma_{i} \in C_{p}(X,Z)$ such that $\partial \gamma = 0$. Then

$$T \circ (I_{n} \times \gamma) = K_{n}(X,(Y,E))(p_{X,Y}(T)) \circ (\gamma).$$

Proof. Denote by $T_{i,\gamma_{i}} := p_{Y}(T_{i} \circ (I_{n} \times \gamma_{i})) \subset Y$ where $p_{Y} : \mathbb{P}^{n} \times Y \rightarrow Y$ is the projection. We have then $\dim T_{i,\gamma_{i}} \leq \dim(T_{i} \circ (I_{n} \times \gamma_{i})) = p + n$ and the factorization $T_{i} \circ (I_{n} \times \gamma_{i}) = i_{\gamma_{i}} \circ T_{i,\gamma_{i}}$ where,

- $T_{i,\gamma_{i}} \subset \mathbb{P}^{n} \times T_{i,\gamma_{i}}$ is finite and surjective over $\mathbb{P}^{n}$, so that $\dim T_{i,\gamma_{i}} = p + n \geq \dim T_{i,\gamma_{i}}$,
- $i_{\gamma_{i}} : T_{i,\gamma_{i}} \rightarrow Y$ is the closed embedding.

We can assume without loss of generality that $\dim T_{i,\gamma_{i}} = p + n$. We have then,

$$p_{Y,i} p_{X,i}^{\gamma_{i}} \lbrack \text{Im}(\gamma_{i}) \rbrack \circ \partial = \lbrack \text{Im}(\gamma_{i}) \rbrack \circ \partial = \lbrack i_{\gamma_{i}} \circ \text{pt}_{\gamma_{i}}(T_{i,\gamma_{i}}) \rbrack \text{ since } p_{Y}(p_{X,i}^{-1}(\text{Im}(\gamma_{i})) \circ \partial) = T_{i,\gamma_{i}} \circ \partial = \lbrack (T_{i} \circ (I_{n} \times \gamma_{i})) \circ \partial \rbrack \text{ by definition,}$$

where $\lbrack \cdot \rbrack$ denote the fundamental class in Borel Moore homology and $\partial$ the boundary.

We deduce from lemma 8(ii) and lemma 9 the following :

Proposition 23. Let $X \in \text{TM}(\mathbb{R})$ connected and $Y \in \text{CW}$. Let $l : E \rightarrow Y$ a CW subcomplex.

(i) Let $[T] = \sum_{i} n_{i} T_{i} \in \text{Hom}_{\text{PC}(\text{CW})}(Z_{\text{tr}}(X),\text{sing}_{\text{tr}}(Z_{\text{tr}}(Y,E)))[n]$ Then,

$$\text{Re}_{\text{cw}}^{\text{tr}} \circ D(\mathbb{P}^{1}, \text{usu})([T]) = K_{n}(X,(Y,E))(p_{X,Y}(T)) \in \text{Hom}_{D_{-}(\mathbb{Z})}(\text{sing}_{\text{tr}} Z_{\text{tr}}(X),\text{sing}_{\text{tr}} Z_{\text{tr}}(Y,E))[n] = \text{Hom}_{D_{-}(\mathbb{Z})}(C_{*}(X,Z),C_{*}(Y,E,Z))[n],$$

(ii) If $Y$ is compact and $E \subset Y$ is closed then the factorization

$$K_{n}(X,(Y,E)) = K_{n}(X,(Y,E)) \circ [\cdot] : Z_{d_{\mathbb{P}^{n}X \times Y}}(X \times Y,Z) \rightarrow \text{Hom}_{D_{-}(\mathbb{Z})}(C_{*}(X,Z),C_{*}(Y,E,Z))[n]$$

where $[\cdot] : Z_{d_{\mathbb{P}^{n}X \times Y}}(X \times Y,Z) \rightarrow H_{BM}^{d_{\mathbb{P}^{n}X \times Y}}(X \times Y,Z)$ is the fundamental class gives the classical isomorphism

$$K_{n}(X,(Y,E)) : H_{BM}^{d_{\mathbb{P}^{n}X \times Y}}(X \times Y,Z) \xrightarrow{\sim} \text{Hom}_{D_{-}(\mathbb{Z})}(C_{*}(X,Z),C_{*}(Y,E,Z))[n] \quad (66)$$

Proof. (i): The equality

$$\text{Hom}_{D_{-}(\mathbb{Z})}(\text{sing}_{\text{tr}} Z_{\text{tr}}(X),\text{sing}_{\text{tr}} Z_{\text{tr}}(Y,E))[n] = \text{Hom}_{D_{-}(\mathbb{Z})}(C_{*}(X,Z),C_{*}(Y,E,Z))[n]$$

follows from proposition 22. For $p \in \mathbb{N}$ and $\gamma \in H_{p}(X,Z)$ we have

$$\text{Re}_{\text{cw}}^{\text{tr}} \circ D(\mathbb{P}^{1}, \text{usu})([T])(\gamma) = \text{Re}_{\text{cw}}^{\text{tr}}([T] \circ S(Z_{\text{tr}}(X))^{-1})(\gamma) = \text{Re}_{\text{cw}}^{\text{tr}}(K_{n}(X,Y))(T)(\gamma)$$

by lemma 8(ii)

:= \text{Re}_{\text{cw}}^{\text{tr}}(I_{n} \times \gamma) \text{ with } I_{n} \times \gamma \in Z_{\mathbb{P}^{n}X \times Y}((\mathbb{P}^{n} \times I^{p}) \times (I^{n} \times X)) = K_{n}(X,(Y,E))(p_{X,Y}(T))(\gamma) \in H_{p+n}(Y,E,Z) \text{ by lemma 9}$$

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(ii): Let \( p \in \mathbb{N} \) and \( \gamma \in H_p(X, \mathbb{Z}) \). If \( Y \) is compact, \( p_X : X \times Y \to X \) is proper and we have

\[
K_n(X, (Y, E))(p_X \times Y(T))(\gamma) = p_Y^*(p_X^*\gamma \cdot [p_X \times Y(T)]) \in H_{p+n}(Y, E, \mathbb{Z})
\]

where \( p_X^* : H_{p+n}(X, \mathbb{Z}) \to H_{p+n}(Y, \mathbb{Z}) \) is Gysin morphism, and \([p_X \times Y(T)] \in H^M_{d_X+n}(X \times Y)\) is the Borel Moore class of \( p_X \times Y(T) \in Z_{d_X+n}(X \times Y, \mathbb{Z}) \).

We finish this subsection by the following proposition:

**Proposition 24.** Let \( X, Y \in \text{CW}, E \subset Y \) a CW subcomplex and \( n \in \mathbb{Z}, n < 0 \). The morphism of abelian group

\[
D(\mathbb{I}, \text{usu}) : \text{Hom}_{PC^-(\text{CW})}(Z_{tr}(X), \text{sing}_{\mathbb{Z}} Z_{tr}(Y, E)[n]) \to \text{Hom}_{\text{DM}^-(\mathbb{Z})}(M(X), M(Y, E)[n])
\]

of the functor \( D(\mathbb{I}, \text{usu}) : \text{PSh}_\mathbb{Z}(\text{Cor}^f_{\mathbb{Z}}(\text{CW}), \text{C}^{-}(\mathbb{Z})) \to \text{DM}^-(\mathbb{Z}) \) is an isomorphism if \( X \in \text{TM}(\mathbb{R}) \) and \( Y \in \text{CW} \) is compact and \( E \subset Y \) is closed.

**Proof.** By proposition \( \ref{proposition23} \)

\[
\text{Re}^r_{\text{cwe}*} \circ D(\mathbb{I}, \text{usu}) = K_n(X, (Y, E)) : \text{Hom}_{PC^-(\text{CW})}(Z_{tr}(X), \text{sing}_{\mathbb{Z}} Z_{tr}(Y, E)[n]) \\
\rightarrow \text{Hom}_{\text{DM}^-(\mathbb{Z})}(\text{sing}_{\mathbb{Z}} Z_{tr}(X), \text{sing}_{\mathbb{Z}} Z_{tr}(Y, E)[n])
\]

Hence \( \text{Re}^r_{\text{cwe}*} \circ D(\mathbb{I}, \text{usu}) \) is an isomorphism if \( X \in \text{TM}, Y \) is compact and \( E \) is closed since \( K_n(X, (Y, E)) \) is an isomorphism by the six functor formalism as indicated in \( \ref{7} \) if \( X \in \text{TM}, Y \) is compact and \( E \) is closed. On the other hand, by theorem \( \ref{77} \) (iii)

\[
\text{Re}^r_{\text{cwe}*} : \text{Hom}_{\text{DM}^-(\mathbb{Z})}(M(X), M(Y, E)[n]) \to \text{Hom}_{\text{DM}^-(\mathbb{Z})}(\text{sing}_{\mathbb{Z}} Z_{tr}(X), \text{sing}_{\mathbb{Z}} Z_{tr}(Y, E)[n])
\]

is an isomorphism. This proves the proposition.

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**3 The Betti realization functor on the derived category of motives of complex algebraic varieties**

We will consider,

- the analytical functor \( \text{An} : \text{Var}(\mathbb{C}) \to \text{AnSp}(\mathbb{C}) \) given by \( \text{An}(V) = V^{an} \), \( \text{An}(g) = g^{an} \), and the analytical functor on transfers \( \text{An} : \text{Cor}_X^f(\text{SmVar}(\mathbb{C})) \to \text{Cor}_Z^f(\text{AnSm}(\mathbb{C})) \) given by \( \text{An}(V) = V^{an}, \text{An}(\Gamma) = \Gamma^{an} \),

- the forgetful functor \( \text{Cw} : \text{AnSp}(\mathbb{C}) \to \text{CW} \) given by \( \text{Cw}(W) = W^{cw} \), \( \text{Cw}(g) = g^{cw} \), and the forgetful functor on transfers \( \text{Cw} : \text{Cor}_X^f(\text{AnSm}(\mathbb{C})) \to \text{Cor}_Z^f(\text{CW}) \) given by \( \text{Cw}(W) = W^{cw}, \text{Cw}(\Gamma) = \Gamma^{cw} \),

- the composites \( \tilde{\text{Cw}} = \text{Cw} \circ \text{An} : \text{Var}(\mathbb{C}) \to \text{CW} \), given by \( \tilde{\text{Cw}}(V) = V^{cw}, \tilde{\text{Cw}}(g) = g^{cw} \), and \( \tilde{\text{Cw}} = \text{Cw} \circ \text{An} : \text{Cor}_X^f(\text{SmVar}(\mathbb{C})) \to \text{Cor}_Z^f(\text{CW}) \), given by \( \tilde{\text{Cw}}(V) = V^{cw}, \tilde{\text{Cw}}(\Gamma) = \Gamma^{cw} \),

- the embeddings of categories \( \iota_{an} : \text{AnSm}(\mathbb{C}) \to \text{AnSp}(\mathbb{C}) \) and \( \iota_{var} : \text{SmVar}(\mathbb{C}) \to \text{Var}(\mathbb{C}) \).

By definition, we have the following commutative diagram of sites

\[
\begin{array}{ccc}
\tilde{\text{Cw}} : \text{Cor}_Z^f(\text{CW}) & \xrightarrow{\text{Cw}} & \text{Cor}_Z^f(\text{AnSm}(\mathbb{C})) & \xrightarrow{\text{An}} & \text{Cor}_Z^f(\text{SmVar}(\mathbb{C})) \\
\text{Tr} & & \text{Tr} & & \text{Tr} \\
\tilde{\text{Cw}} : \text{Z}(\text{CW}) & \xrightarrow{\text{Cw}} & \text{Z}(\text{AnSm}(\mathbb{C})) & \xrightarrow{\text{An}} & \text{Z}(\text{SmVar}(\mathbb{C})) \\
\text{Cw} & \xrightarrow{\iota_{an}} & \text{Z}(\text{AnSp}(\mathbb{C})) & \xrightarrow{\text{An}} & \text{Z}(\text{Var}(\mathbb{C})) \\
\end{array}
\]
We note that we have the following:

**Proposition 25.**

(i) The functors

- $\text{An}^* : \text{PSh}(\text{SmVar}(\mathbb{C}), C^- (\mathbb{Z})) \to \text{PSh}(\text{AnSm}(\mathbb{C}), C^- (\mathbb{Z}))$ and
- $\text{An}^* : \text{PSh}(\text{Cor}^s_\mathbb{Z}(\text{SmVar}(\mathbb{C})), C^- (\mathbb{Z})) \to \text{PSh}(\text{Cor}^s_\mathbb{Z}(\text{AnSm}(\mathbb{C})), C^- (\mathbb{Z}))$,

derive trivially for the $(\mathbb{A}^1, \text{et})$ and $(\mathbb{D}^1, \text{usu})$ model structures.

(ii) Let $K^* \in \text{PSh}(\text{Cor}^s_\mathbb{Z}(\mathbb{C}W), C^- (\mathbb{Z}))$. If $K^*$ is $\mathbb{I}^1$ local, then $\text{Cw}_* K^*$ is $\mathbb{D}^1$ local.

- Let $K^* \in \text{PSh}(\mathbb{C}W, C^- (\mathbb{Z}))$. If $K^*$ is $\mathbb{I}^1$ local, then $\text{Cw}_* K^*$ is $\mathbb{D}^1$ local.

Moreover, the functors

- $\text{Cw}^* : \text{PSh}(\text{AnSm}(\mathbb{C}), C^- (\mathbb{Z})) \to \text{PSh}(\mathbb{C}W, C^- (\mathbb{Z}))$ and
- $\text{Cw}^* : \text{PSh}(\text{Cor}^s_\mathbb{Z}(\text{AnSm}(\mathbb{C})), C^- (\mathbb{Z})) \to \text{PSh}(\text{Cor}^s_\mathbb{Z}(\mathbb{C}W), C^- (\mathbb{Z}))$,

derive trivially for the $(\mathbb{D}^1, \text{usu})$ and $(\mathbb{I}^1, \text{usu})$ model structures.

(iii) The functors

- $\tilde{\text{Cw}}^* : \text{PSh}(\text{SmVar}(\mathbb{C}), C^- (\mathbb{Z})) \to \text{PSh}(\mathbb{C}W, C^- (\mathbb{Z}))$ and
- $\tilde{\text{Cw}}^* : \text{PSh}(\text{Cor}^s_\mathbb{Z}(\text{SmVar}(\mathbb{C})), C^- (\mathbb{Z})) \to \text{PSh}(\text{Cor}^s_\mathbb{Z}(\mathbb{C}W), C^- (\mathbb{Z}))$,

derive trivially for the $(\mathbb{A}^1, \text{et})$ and $(\mathbb{I}^1, \text{usu})$ model structures.

**Proof.** (i): It is proved in [3].

(ii): Let us prove (ii).

- Let $K^* \in \text{PSh}(\mathbb{C}W, C^- (\mathbb{Z}))$ be an $\mathbb{I}^1$ local object. Let $h : K^* \to L^*$ an equivalence usu local with $L^*$ usu fibrant. Then,
  - $L^*$ is $\mathbb{I}^1$ local, $S(L^*) : L^* \to \text{sing}_{\mathbb{I}^1} L^*$ is an equivalence usu local,
  - $\text{sing}_{\mathbb{I}^1} L^*$ is usu fibrant.

Since $\text{Cw}_*$ preserve usu local equivalence and usu fibrant object (for $X \in \text{AnSm}(\mathbb{C})$, the restriction of the functor $\text{Cw}$ to the small site of open subset of $X$ is fully faithfull),

- $\text{Cw}_* (S(L^*) \circ h) = (\text{Cw}_* S(L^*)) \circ (\text{Cw}_* h) : \text{Cw}_* K^* \to \text{Cw}_* \text{sing}_{\mathbb{I}^1} L^*$ is an equivalence usu local
- $\text{Cw}_* \text{sing}_{\mathbb{I}^1} L^*$ is usu fibrant.

Now, for $X \in \text{AnSm}(\mathbb{C})$,

$$H^n(\text{sing}_{\mathbb{I}^1} L^* (p_{X^{cw}})) : \text{Hom}_{\mathbb{Z}}(X)[n], \text{Cw}_* \text{sing}_{\mathbb{I}^1} L^*) = H^n(\text{sing}_{\mathbb{I}^1} L^* (X^{cw})) \to \text{Hom}(\mathbb{Z}(X \times \mathbb{D}^1)[n], \text{Cw}_* \text{sing}_{\mathbb{I}^1} L^*) = \text{sing}_{\mathbb{I}^1} L^* (X^{cw} \times \mathbb{D}^1)[n]$$

Moreover, the map $p_{X^{cw}} = I_{X^{cw}} \times a_{\mathbb{D}^1} : X^{cw} \times \mathbb{D}^1 \to X^{cw}$ is an homotopy equivalence. Hence, by lemma [4] i),

$$\text{sing}_{\mathbb{I}^1} L^* (p_{X^{cw}}) : \text{sing}_{\mathbb{I}^1} L^* (X^{cw}) \to \text{sing}_{\mathbb{I}^1} L^* (X^{cw} \times \mathbb{D}^1)$$

is a quasi-isomorphism. In particular $H^n(\text{sing}_{\mathbb{I}^1} L^* (p_{X^{cw}}))$ is an isomorphism. This proves that $\text{Cw}_* \text{sing}_{\mathbb{I}^1} L^*$ is $\mathbb{D}^1$ local. Now, since

- $\text{Cw}_* (S(L^*) \circ h)$ is an equivalence usu local
- $\text{Cw}_* \text{sing}_{\mathbb{I}^1} L^*$ is $\mathbb{D}^1$ local,
Cw_\cdot K^\bullet \text{ is } \mathbb{D}^1 \text{ local.}

- Let $K^\bullet \in \text{PSh}(\text{Cor}_{Z}^I(CW), C^-(Z))$ be an $\mathbb{I}^1$ local object. By lemma (7), $\text{Tr}_* K$ is $\mathbb{I}^1$ local. Hence by above, $\text{Tr}_* \text{Cw}_* K^\bullet = \text{Cw}_* K^\bullet$ is $\mathbb{I}^1$ local. Thus, by lemma (7), $\text{Tr}_* K^\bullet$ is $\mathbb{D}^1$ local.

- Let $f : G_1 \to G_2$ an equivalence ($\mathbb{D}^1, \text{usu}$) local in $\text{PSh}(\text{AnSm}(\mathbb{C}), C^-(Z))$. Let $K \in \text{PSh}(\text{CW}, C^-(Z))$ a $\mathbb{I}^1$ local object. Up to replace $K$ by an usu equivalent presheaf, we may assume that $K$ is usu fibrant. Then $\text{Cw}_* K$ is also usu fibrant. Consider the following commutative diagram:

$\begin{align*}
\text{Hom}_{P-(CW)}(\text{Cw}^* G_2, K) & \xrightarrow{Z(K)(\text{Cw}^* f)} \text{Hom}_{P-(CW)}(\text{Cw}^* G_1, K) \\
\downarrow & \text{ } \downarrow \\
\text{Hom}_{P-(\text{An})}(G_2, \text{Cw}_* K) & \xrightarrow{Z(\text{Cw}_* K)(f)} \text{Hom}_{P-(\text{An})}(G_1, \text{Cw}_* K)
\end{align*}$

By above $\text{Cw}_* K$ is $\mathbb{D}^1$ local, and $f$ is an equivalence ($\mathbb{D}^1, \text{usu}$) local. Hence, $Z(\text{Cw}_* K)(f)$ is an isomorphism since $\text{Cw}_* K$ is usu fibrant. The diagram shows then that $Z(K)(\text{Cw}^* f)$ is an isomorphism. This proves that $\text{Cw}^* f : \text{Cw}^* G_1 \to \text{Cw}^* G_2$ is an equivalence ($\mathbb{I}^1, \text{usu}$) local since $K$ is usu fibrant.

- Let $f : G_1 \to G_2$ an equivalence ($\mathbb{D}^1, \text{usu}$) local in $\text{PSh}(\text{Cor}_{Z}^I(\text{AnSm}(\mathbb{C})), C^-(Z))$. Let $K \in \text{PSh}(\text{Cor}_{Z}^I(CW), C^-(Z))$ a $\mathbb{I}^1$ local object. Up to replace $K$ by an usu equivalent presheaf, we may assume that $K$ is usu fibrant. Then $\text{Cw}_* K$ is also usu fibrant. Consider the following commutative diagram:

$\begin{align*}
\text{Hom}_{P-(CW)}(\text{Cw}^* G_2, K) & \xrightarrow{Z(K)(\text{Cw}^* f)} \text{Hom}_{P-(CW)}(\text{Cw}^* G_1, K) \\
\downarrow & \text{ } \downarrow \\
\text{Hom}_{P-(\text{An})}(G_2, \text{Cw}_* K) & \xrightarrow{Z(\text{Cw}_* K)(f)} \text{Hom}_{P-(\text{An})}(G_1, \text{Cw}_* K)
\end{align*}$

By above $\text{Cw}_* K$ is $\mathbb{D}^1$ local, and $f$ is an equivalence ($\mathbb{D}^1, \text{usu}$) local. Hence, $Z(\text{Cw}_* K)(f)$ is an isomorphism since $\text{Cw}_* K$ is usu fibrant. The diagram shows then that $Z(K)(\text{Cw}^* f)$ is an isomorphism. This proves that $\text{Cw}^* f : \text{Cw}^* G_1 \to \text{Cw}^* G_2$ is an equivalence ($\mathbb{I}^1, \text{usu}$) local since $K$ is usu fibrant.

(iii): Follows from (i) and (ii) since by definition $\widetilde{\text{Cw}} = \text{Cw} \circ \text{An}$. \hfill \square

### 3.1 Ayoub’s Betti realization functor and the Betti realisation functor via CW commplexes

We recall the definition of Ayoub’s realization functor:

**Definition 23.** \cite{Ayoub} \[\mathbb{Z}\]

(i) The Betti realisation functor (without transfers) is the composite:

$$\text{Bti}_0^*: \text{DA}^-(\mathbb{C}, \mathbb{Z}) \xrightarrow{\text{An}^*} \text{AnDA}^-(\mathbb{Z}) \xrightarrow{\text{Re}_{\text{an}^*}} \text{D}^-(\mathbb{Z}) \quad (69)$$

(ii) The Betti realisation functor with transfers is the composite:

$$\text{Bti}^*: \text{DM}^-(\mathbb{C}, \mathbb{Z}) \xrightarrow{\text{An}^*} \text{AnDM}^-(\mathbb{Z}) \xrightarrow{\text{Re}_{\text{an}^*}} \text{D}^-(\mathbb{Z}) \quad (70)$$

Since $\text{An}^*$ derive trivially by proposition \[\mathbb{Z}\] and $L \text{Tr}^*: \text{AnDA}^-(\mathbb{Z}) \to \text{AnDM}^-(\mathbb{Z})$ is the inverse of $\text{Tr}_*$ by theorem \[\mathbb{Z}\] (i), we have $\text{Bti}_0^* = \text{Bti}^* \circ L \text{Tr}^*$. 

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We now define a Betti realization functor via CW complexes. The main result of this subsection will be that this functor coincide with Ayoub’s one.

**Definition 24.**

(i) The CW-Betti realization functor (without transfers) is the composite:

$$\tilde{\text{Bti}}_0^\ast : \text{DA}^-(\mathbb{C},\mathbb{Z}) \xrightarrow{\text{Cw}} \text{CwDA}^-(\mathbb{Z}) \xrightarrow{\text{Re}_{\text{cw}}} \text{D}^-(\mathbb{Z})$$

(71)

(ii) The CW-Betti realization functor with transfers is the composite:

$$\tilde{\text{Bti}}_i^\ast : \text{DM}^-(\mathbb{C},\mathbb{Z}) \xrightarrow{\text{Cw}} \text{CwDM}^-(\mathbb{Z}) \xrightarrow{\text{Re}_{\text{cw}}} \text{D}^-(\mathbb{Z})$$

(72)

Since $\text{Cw}^\ast$ derive trivially by proposition 22(ii) and $L\text{Tr}^\ast : \text{CwDA}^-(\mathbb{Z}) \to \text{CwDM}^-(\mathbb{Z})$ is the inverse of $\text{Tr}^\ast$ by theorem 17(ii), we have $\text{Bti}_0^\ast = \text{Bti}_i^\ast \circ L\text{Tr}^\ast$.

Denote by $i_* : I_{et}^* \hookrightarrow \square_C^* = \square^* \text{ the embeddings of pro-objects in } \text{SmVar}(\mathbb{C})$

$$I_{et}^* := (U_j, a_{j,k})_{j,k \in V_\text{et}([0,\infty]^* \times \square_C), j \leq k}$$

indexed by the filtrant category of etale neighborhood of $[0, \infty]^* \times \square_C$, The morphism $i_*$ is given by the etale morphisms $i_*(l) : U_l \to \square^*_l$ associated to $l \in V_\text{et}([0,\infty]^* \times \square_C)$.

Denote by $i'_* : I_{an}^* \xrightarrow{i'} \bar{\square}^*, \bar{\square}^*_l \text{ the embeddings of pro-objects in } \text{AnSm}(\mathbb{C})$

- $I_{an}^* := (U_j, a_{j,k})_{j,k \in V_\text{an}(\square^*_l, \square^*_C), j \leq k}$ indexed by the filtrant category of etale analytic neighborhood of $\square^*_l$ in $\square^*_C$, and

- $\bar{\square}^* := (U_l, a_{l,m})_{l,m \in V_\text{an}(\bar{\square}^*_l, \bar{\square}^*_C), l \leq m}$ indexed by the filtrant category of etale analytic neighborhood of $\square^*_l$ in $\square^*_C$.

The morphism $i'_*|_{I_{an}}$ is given by the identities $i'_*(l) : U_\tau(l) = U_l \xrightarrow{I_{an}} U_l$, where $\tau : V_\text{an}(\bar{\square}^*_l, \square^*_C) \to V_\text{an}(\square^*_l, \square^*_C)$ is the natural embedding of categories.

The embedding $i_* : I_{et}^* \hookrightarrow \square^*_l \text{ of pro-objects in } \text{SmVar}(\mathbb{C})$ gives for, $F^\bullet \in \text{PSh}(\text{Cor}^{et}_Z(\text{SmVar}(\mathbb{C})), \text{C}^-(\mathbb{Z}))$, the following morphism in $\text{PSh}(\text{Cor}^{et}_Z(\text{SmVar}(\mathbb{C})), \text{C}^-(\mathbb{Z}))$,

$$F^\ast(i) : C_\ast F^\bullet \to \underline{\text{sing}_{i_*}} F^\bullet$$

(73)

given by for $X \in \text{SmVar}(\mathbb{C})$, the morphism of complexes

$$F^\ast(i)(X) : F^\bullet(X \times \square^*_l) \xrightarrow{F^\ast(i_X \times i_*)} F^\bullet(X \times I_{et}^*)$$

(74)

The morphism $i^\ast_1|_{I_{an}} : I_{an}^* \to \bar{\square}^*, \text{ of pro-object in } \text{AnSm}(\mathbb{C})$ gives for, $G^\bullet \in \text{PSh}(\text{Cor}^{et}_Z(\text{AnSm}(\mathbb{C})), \text{C}^-(\mathbb{Z}))$, the following morphism in $\text{PSh}(\text{Cor}^{et}_Z(\text{AnSm}(\mathbb{C})), \text{C}^-(\mathbb{Z}))$,

$$F^\ast(i^\ast_1) : \underline{\text{sing}_{i_1}} F^\bullet \to \underline{\text{sing}_{i^\ast_1}} F^\bullet$$

(75)

given by for $X \in \text{SmVar}(\mathbb{C})$, the morphism of complexes of abelian groups

$$F^\ast(i^\ast_1)(X) = F^\bullet(I_X \times i^\ast_1) : F^\bullet(X \times \bar{\square}^*) \to F^\bullet(X \times I_{an}^*)$$

(76)

We have two canonical morphism of functors:

- the morphism $\psi^{\text{Cw}}$, which, for $G^\bullet \in \text{PSh}(\text{Cor}^{et}_Z(\text{AnSm}(\mathbb{C})), \text{C}^-(\mathbb{Z}))$, associate the morphism

$$\psi^{\text{Cw}}(G^\bullet) : \text{Cw}^\ast(\text{sing}_{I_{an}} G^\bullet) \to \text{sing}_{i_*} \text{Cw}^\ast G^\bullet$$

in $\text{PSh}(\text{Cor}^{et}_Z(\text{CW}), \text{C}^-(\mathbb{Z}))$; the morphism $\psi^{\text{Cw}}(G^\bullet)$ is given by, for $Z \in \text{CW}$,

$$\psi^{\text{Cw}}(G^\bullet)(Z) : \lim_{X^{\text{cw}} \to Z} G^\bullet(X \times I_{an}^*) \to \lim_{Y^{\text{cw}} \to Z \times \square^*_l} G^\bullet(Y)$$

(77)

given by $(f : X^{\text{cw}} \to Z) \to (f \times I^*: (X \times I_{an}^*)^{\text{cw}} \to Z \times I^*)$ and the identity of $G^\bullet(X \times I_{an}^*)$;
• the morphism \( \psi_{CW} \), which, for \( F^* \in \text{PSh}(\text{Cor}^{fs}_Z(\text{SmVar}(\mathbb{C})), C^{-}(\mathbb{Z})) \), associate the morphism

\[
\psi_{CW}(F^*) : \widetilde{Cw}^*(\text{sing}_{et} F^*) \to \text{sing}_{et} \widetilde{Cw}^* F^*
\]

in \( \text{PSh}(\text{Cor}^{fs}_Z(CW), C^{-}(\mathbb{Z})) \); the morphism \( \psi_{CW}(F^*) \) is given by, for \( Z \in CW \),

\[
\psi_{CW}(F^*)(Z) : \lim_{X \to Z} F^*(X \times \mathbb{L}^*_{et}) \to \lim_{Y \to Z} F^*(Y) \tag{78}
\]
given by \( (f : X \to Z) \mapsto (f \times \mathbb{L}^*_{et}) : (X \times \mathbb{L}^*_{et}) \to Z \times \mathbb{L}^* \) and the identity of \( F^*(X \times \mathbb{L}^*_{et}) \).

**Definition 25.** We define the following two morphism of functors:

(i) the morphism \( W \), which, for \( G^* \in \text{PSh}(\text{Cor}^{fs}_Z(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \), associate the composition

\[
W(G^*) : Cw^*(\text{sing}_{et}, G^*) \xrightarrow{Cw^*(\text{sing}_{et}(\cdot))^c} Cw^*(\text{sing}_{et}, G^*) \xrightarrow{\psi_{CW}(G^*)} \text{sing}_{et} \text{Cw}^* G^*
\]

in \( \text{PSh}(\text{Cor}^{fs}_Z(CW), C^{-}(\mathbb{Z})) \),

(ii) the morphism \( \widetilde{W} \), which, for \( F^* \in \text{PSh}(\text{Cor}^{fs}_Z(\text{SmVar}(\mathbb{C})), C^{-}(\mathbb{Z})) \), associate the composition

\[
\widetilde{W}(F^*) : \widetilde{Cw}^* (\mathbb{C}. F^*) \xrightarrow{\widetilde{Cw}^*(\mathbb{C}. F^*)} \text{sing}_{et} \widetilde{Cw}^* F^*
\]

in \( \text{PSh}(\text{Cor}^{fs}_Z(CW), C^{-}(\mathbb{Z})) \).

**Proposition 26.** (i) For \( G^* \in \text{PSh}(\text{Cor}^{fs}_Z(\text{AnSm}(\mathbb{C})), C^{-}(\mathbb{Z})) \), \( W(G^*) : Cw^*(\text{sing}_{et}, G^*) \to \text{sing}_{et} \text{Cw}^* G^* \) is an equivalence \( (\mathbb{D}^1, usu) \) local in \( \text{PSh}(\text{Cor}^{fs}_Z(CW), C^{-}(\mathbb{Z})) \),

(ii) For \( F^* \in \text{PSh}(\text{Cor}^{fs}_Z(\text{SmVar}(\mathbb{C})), C^{-}(\mathbb{Z})) \), \( \widetilde{W}(F^*) : \text{sing}_{et} \text{Cw}^* F^* \to \text{Cw}^* F^* \) is an \( (\mathbb{D}^1, usu) \) local equivalence in \( \text{PSh}(\text{Cor}^{fs}_Z(CW), C^{-}(\mathbb{Z})) \).

**Proof.** (i): Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Cw}^* G^* & \xrightarrow{S(\text{Cw}^* G^*)} & \text{Cw}^* (\text{sing}_{et}, G^*) \\
Cw^* (S(G^*)) & \downarrow & \text{W}(G^*) \\
\text{Cw}^* (\text{sing}_{et}, G^*) & \xrightarrow{\text{sing}_{et} \text{Cw}^* G^*} & \text{sing}_{et} \text{Cw}^* G^* \\
\end{array}
\]

• By theorem \[13\] (ii) \( S(G^*) \) is a \( (\mathbb{D}^1, et) \) equivalence. Hence by proposition \[25\] (ii) \( \text{Cw}^* S(G^*) \) is a \( (\mathbb{D}^1, usu) \) equivalence.

• By theorem \[10\] (ii) \( S(\text{Cw}^* G^*) \) is a \( (\mathbb{D}^1, usu) \) equivalence.

Now, since \( \text{Cw}^* S(G^*) \) and \( S(\text{Cw}^* G^*) \) are \( (\mathbb{D}^1, usu) \) equivalence, the diagram \[79\] shows that \( W(G^*) \) is a \( (\mathbb{D}^1, usu) \) equivalence.

(ii): Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Cw}^* F^* & \xrightarrow{S(\text{Cw}^* F^*)} & \text{Cw}^* (\text{sing}_{et} F^*) \\
\text{Cw}^* (S(F^*)) & \downarrow & \text{W}(F^*) \\
\text{Cw}^* (\mathbb{C}. F^*) & \xrightarrow{\text{sing}_{et} \text{Cw}^* F^*} & \text{Cw}^* F^* \\
\end{array}
\]

\[46\]
• By theorem \[ \text{(ii)} \] \( S(F^*) \) is a \((\mathbb{I}^1, et)\) equivalence. Hence, by proposition \[ \text{25} \] (iii) \( \widetilde{\text{Cw}} S(F^*) \) is a \((\mathbb{I}^1, usu)\) equivalence.

• By theorem \[ \text{10} \] (ii) \( S(\widetilde{\text{Cw}} F^*) \) is a \((\mathbb{I}^1, usu)\) equivalence.

Now, since \( \widetilde{\text{Cw}} S(F^*) \) and \( S(\widetilde{\text{Cw}} F^*) \) are \((\mathbb{I}^1, usu)\) equivalence, the diagram \[ \text{80} \] shows that \( \widetilde{W}(F^*) \) is a \((\mathbb{I}^1, usu)\) equivalence.

\[ \square \]

Definition 26. We define the morphism of functor \( B \), by associating to \( G^* \in \text{PSh}(\text{Cor}_Z^{f^*}(\text{AnSp}(\mathbb{C})), C^- (\mathbb{Z})) \), the composite

\[
B(G^*): \text{sing}_{\text{fs}} G^* \xrightarrow{\text{ad}(\text{Cw}^*, \text{Cw}^*)(\text{sing}_{\text{fs}} G^*)} \text{Cw}^* \text{Cw}^* \text{sing}_{\text{fs}} G^* \xrightarrow{\text{Cw}^*(W(G^*))} \text{Cw}^* \text{sing}_{\text{fs}} \text{Cw}^* G^*
\]

in \( \text{PSh}(\text{Cor}_Z^{f^*}(\text{AnSp}(\mathbb{C})), C^- (\mathbb{Z})) \)

We have now the following key proposition.

Proposition 27. (i) For \( Y \in \text{AnSp}(\mathbb{C}) \) and \( E \subset Y \) an analytic subset,

\[
e^{r_{an}}(B(\text{Ztr}(Y, E))) : \text{sing}_{\text{fs}} \text{Ztr}(Y, E) \to \text{sing}_{\text{tr}} \text{Ztr}(Y, E, E^{cw})
\]

is a quasi isomorphism in \( C^- (\mathbb{Z}) \).

(ii) For \( Y \in \text{AnSp}(\mathbb{C}) \), and \( E \subset Y \) an analytic subset, the morphism

\[
B(\text{Ztr}(Y, E)) : \text{sing}_{\text{fs}} \text{Ztr}(Y, E) \to \text{Cw}^* \text{sing}_{\text{fs}} \text{Ztr}(Y, E, E^{cw})
\]

is an equivalence \( (\mathbb{I}^1, usu) \) local in \( \text{PSh}(\text{Cor}_Z^{f^*}(\text{AnSp}(\mathbb{C})), C^- (\mathbb{Z})) \).

Proof. (i):

• Consider first \( Y \in \text{AnSp}(\mathbb{C}) \). By proposition \[ \text{10} \] there exist a covering \( Y = \bigcup_{i \in J} D_i \) by a countable family of open balls \( D_i \cong \mathbb{D}^{d_Y} \) such that \( D_i = \emptyset \) or \( D_i \cong \mathbb{D}^{d_Y} \), for all \( I = \{i_1, \cdots i_l\} \subset J \). Denote by \( j_I : D_I \hookrightarrow Y \) the open embedding. We have then the following commutative diagram in \( \text{PSh}(\text{Cor}_Z^{f^*}(\text{AnSp}(\mathbb{C})), C^- (\mathbb{Z})) \):

\[
\begin{array}{ccc}
\text{sing}_{\text{fs}} \text{Ztr}(Y) & \xrightarrow{B(\text{Ztr}(Y))} & \text{Cw}^* \text{sing}_{\text{fs}} \text{Ztr}(Y, E^{cw}) \\
\text{Tot}(\text{S}(\text{Ztr}(j_I))) & & \text{Tot}(\text{Cw}^*(\text{S}(\text{Ztr}(j_I)))) \\
\text{Tot}_{\bullet, \bullet}(\oplus_{\text{card } I = 1} \text{sing}_{\text{fs}} \text{Ztr}(D_I)) & \xrightarrow{\text{Tot}(B(\text{Ztr}(D_I)))} & \text{Tot}_{\bullet, \bullet}(\oplus_{\text{card } I = 1} \text{Cw}^* \text{sing}_{\text{fs}} \text{Ztr}(D_I^{cw}))
\end{array}
\]

This gives, after applying the functor \( e^{r_{an}} \) to \[ \text{81} \], the commutative diagram in \( C^- (\mathbb{Z}) \):

\[
\begin{array}{ccc}
\text{sing}_{\text{fs}} \text{Ztr}(Y) & \xrightarrow{e^{r_{an}}(B(\text{Ztr}(Y)))} & \text{sing}_{\text{tr}} \text{Ztr}(Y, E^{cw}) \\
\text{Tot}(e^{r_{an}}(\text{S}(\text{Ztr}(j_I)))) & & \text{Tot}(e^{r_{an}}(\text{S}(\text{Ztr}(j_I)))) \\
\text{Tot}_{\bullet, \bullet}(\oplus_{\text{card } I = 1} \text{sing}_{\text{fs}} \text{Ztr}(D_I)) & \xrightarrow{\text{Tot}(e^{r_{an}}(B(\text{Ztr}(D_I))))} & \text{Tot}_{\bullet, \bullet}(\oplus_{\text{card } I = 1} \text{Cw}^* \text{sing}_{\text{fs}} \text{Ztr}(D_I^{cw})) \\
\oplus_{\text{card } I = 1} \text{H}_0(\text{sing}_{\text{fs}} \text{Ztr}(D_I)) & \xrightarrow{\text{H}_0(e^{r_{an}}(B(\text{Ztr}(D_I))))} & \oplus_{\text{card } I = 1} \text{H}_0(\text{sing}_{\text{tr}} \text{Ztr}(D_I^{cw}))
\end{array}
\]

Since \( D_I \cong \mathbb{D}^{d_Y} \),
Consider now

Hence,

\[ \text{Tot}(\epsilon_{tr}^s B(Z_{tr}(D_I))) : \text{Tot}_{*,*}(\oplus_{t=1}^{t=\text{sing}_{\mathcal{D}}} Z_{tr}(D_I)) \to \text{Tot}_{*,*}(\oplus_{t=1}^{t=\text{sing}_{\mathcal{D}}} Z_{tr}(D_I^{cw})) \]

is a quasi-isomorphism in \( C^-(\mathcal{Z}) \). On the other hand,

\[ \text{Tot}(\epsilon_{tr}^s \mathcal{S}(Z(j))), \text{Tot}_{*,*}(\oplus_{t=1}^{t=\text{sing}_{\mathcal{D}}} Z_{tr}(D_I)) \to \text{sing}_{\mathcal{D}}, Z_{tr}(Y) \]

is a quasi-isomorphism in \( C^-(\mathcal{Z}) \) since

\[ \text{Tot}(\mathcal{S}(Z(j))) : \text{Tot}_{*,*}(\oplus_{t=1}^{t=\text{sing}_{\mathcal{D}}} Z_{tr}(D_I)) \to \text{sing}_{\mathcal{D}}, Z_{tr}(Y) \]

is an equivalence \( \mathcal{D}^1, \text{usu} \) local equivalence in \( \text{PSh}(\text{Cor}^s_{\mathcal{Z}}(\text{AnSm}(\mathcal{C})), C^-(\mathcal{Z})) \) by lemma \([1]\) and theorem \([13][ii]\),

\[ \text{Tot}(\epsilon_{tr}^s \mathcal{S}(Z(j), Y)) : \text{Tot}_{*,*}(\oplus_{t=1}^{t=\text{sing}_{\mathcal{D}}} Z_{tr}(D_I)) \to \text{sing}_{\mathcal{D}}, Z_{tr}(Y^{cw}) \]

is a quasi-isomorphism in \( C^-(\mathcal{Z}) \).

Consider now \( Y \in \text{AnSp}(\mathcal{C}) \). We prove that \( \epsilon_{tr}^s B(Z_{tr}(Y)) \) is a quasi-isomorphism by induction on \( d_Y = \dim Y \). If \( d_Y = 0 \) there is nothing to prove. By theorem \([13]\) there exist a proper modification \( c : Y' \to Y \) such that \( Y' \in \text{AnSm}(\mathcal{C}) \) and \( E = \epsilon^{-1}(Z) \subset Y' \) is a normal crossing divisor. Denote by \( l : Z \hookrightarrow Y \) and \( l' : E \hookrightarrow Y' \) the closed embeddings and \( \epsilon_Z : E \to Z \) the morphism such that \( l \circ \epsilon_Z = \epsilon \circ l' \). We have then with

\[ b = \epsilon_{2*} + l'_* = \mathcal{S}(Z_{tr}(\epsilon_{Z}) \oplus Z_{tr}(l')), \quad a = \epsilon_{*} + l_* = \mathcal{S}(Z_{tr}(\epsilon) \oplus Z_{tr}(l)), \]

the following commutative diagram in \( \text{PSh}(\text{Cor}^s_{\mathcal{Z}}(\text{AnSm}(\mathcal{C})), C^-(\mathcal{Z})) \):

\[ 0 \longrightarrow \text{sing}_{\mathcal{D}}, Z_{tr}(E) \longrightarrow \text{sing}_{\mathcal{D}}, Z_{tr}(Z) \oplus \text{sing}_{\mathcal{D}}, Z_{tr}(Y') \longrightarrow \text{sing}_{\mathcal{D}}, Z_{tr}(Y) \]

\[ 0 \longrightarrow \text{Cw}_{\mathcal{D}}, Z_{tr}(E^{cw}) \longrightarrow \text{Cw}_{\mathcal{D}}, Z_{tr}(Z^{cw}) \oplus \text{sing}_{\mathcal{D}}, Z_{tr}( Y^{cw}) \longrightarrow \text{Cw}_{\mathcal{D}}, Z_{tr}(Y^{cw}) \]

But,

\[ 0 \longrightarrow \text{sing}_{\mathcal{D}}, Z_{tr}(E) \xrightarrow{b} \text{sing}_{\mathcal{D}}, Z_{tr}(Z) \oplus \text{sing}_{\mathcal{D}}, Z_{tr}(Y') \longrightarrow \text{sing}_{\mathcal{D}}, Z_{tr}(Y) \]

is a quasi-isomorphism by proposition \([14]\).
\[ 0 \to \text{sing}_{tr}^* Z_{tr}(Z) \xrightarrow{\delta} \text{sing}_{tr}^* Z_{tr}(Y) \xrightarrow{\varepsilon} \text{sing}_{tr}^* Z_{tr}(Y') \] is a quasi-isomorphism by proposition \(\text{[14]}\) and the fact that the 2-cubical diagram associated to a proper modification is of cohomological descent (see e.g. \([13]\)).

As \(Y' \in \text{SmVar}(\mathbb{C})\) is smooth and \(\dim E = d_Y - 1 < d_Y\), \(\dim Z < d_Y\), we have, by the smooth case we proved above and by the induction hypothesis, that \(e_{\text{an}*}^{tr} B(Z_{tr}(Z) \oplus Z_{tr}(Y'))\) and \(e_{\text{an}*}^{tr} B(Z_{tr}(E))\) are quasi-isomorphisms, thus, the diagram \([\text{S3}]\) shows that \(e_{\text{an}*}^{tr} B(Z_{tr}(Y))\) is a quasi-isomorphism.

- Consider now \(Y \in \text{AnSp}(\mathbb{C})\) as before and \(E \subset Y\) an analytic subspace. Denote by \(l : E \hookrightarrow Y\) the locally closed embedding. We have then the following commutative diagram in \(PC^{-1}(\text{An})\):

\[
\begin{array}{cccccc}
0 & \to & \text{sing}^*_{tr} Z_{tr}(E) & \xrightarrow{S(Z(l))} & \text{sing}^*_{tr} Z_{tr}(Y) & \xrightarrow{S_{\text{cy},E}} & \text{sing}^*_{tr} Z_{tr}(Y, E) & 0 \\
0 & \to & \text{Cw}_* \text{sing}^*_{tr} Z_{tr}(E) & \xrightarrow{\text{Cw}_* S(Z_{\text{cy}})} & \text{Cw}_* \text{sing}^*_{tr} Z_{tr}(Y) & \xrightarrow{\text{Cw}_* S_{\text{cy},E} \text{cw}} & \text{Cw}_* \text{sing}^*_{tr} Z_{tr}(Y, E) & 0 \\
\end{array}
\]

(84)

where the first row is the exact sequence \([27]\), and the second row is an exact sequence since the sequence \([23]\) is exact and \(\text{Cw}_*\) is an exact functor. Since we just showed that \(e_{\text{an}*}^{tr} B(Z_{tr}(Y))\) and \(e_{\text{an}*}^{tr} B(Z_{tr}(E))\) are quasi-isomorphisms, the diagram \([\text{S4}]\) implies that \(e_{\text{an}*}^{tr} B(Z_{tr}(Y, E))\) is a quasi isomorphism.

(ii): Follows by (i). Let us explain.

- On the one hand,
  - By theorem \([13]\)(ii), \(\text{sing}^*_{tr} Z_{tr}(Y, E)\) is \(D^1\) local.
  - By theorem \([16]\)(ii), \(\text{sing}^*_{tr} Z_{tr}(Y_{cw}, E_{cw})\) is \(l^1\) local. Hence, \(\text{Cw}_* \text{sing}^*_{tr} Z_{tr}(Y_{cw}, E_{cw})\) is \(D^1\) local, by proposition \([23]\)(ii).

- On the other hand by (i) \(e_{\text{an}*}^{tr} (B(Z_{tr}(Y, E))) : \text{sing}^*_{tr} Z_{tr}(Y, E) \to \text{sing}^*_{tr} Z_{tr}(Y_{cw}, E_{cw})\) is a quasi isomorphism in \(C^{-1}(Z)\).

Hence, by proposition \([13]\)(ii),

\[
B(Z_{tr}(Y, E)) : \text{sing}^*_{tr} Z_{tr}(Y, E) \to \text{Cw}_* \text{sing}^*_{tr} Z_{tr}(Y_{cw}, E_{cw})
\]

is an equivalence \((D^1, \text{usu})\) local.

\[\square\]

The main result of this subsection is the following:

**Theorem 18.**

(i) For \(Y \in \text{Var}(\mathbb{C})\), and \(E \subset Y\) a subvariety, we have \(\text{Bti}^* M(Y, E) = \widetilde{\text{Bti}}^* M(Y, E)\)

(ii) For \(X, Y \in \text{Var}(\mathbb{C}), D \subset X, E \subset Y\) subvarieties, and \(n \in \mathbb{Z}, n \leq 0\), the following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^{-1}(\mathbb{C}, Z)}(M(X, D), M(Y, E)[n]) & \xrightarrow{\text{Cw}^*} & \text{Hom}_{\text{DM}^{-1}(\mathbb{Z})}(M(X, D), M(Y, E)[n]) \\
\text{An}^* & & \text{Re}_{\text{cw}^*} \\
\text{Hom}_{\text{AnDM}^{-1}(\mathbb{Z})}(M(X, D), M(Y, E)[n]) & \xrightarrow{\text{Re}^*_{\text{cw}^*}} & \text{Hom}_{\text{DM}^{-1}(\mathbb{Z})}(\text{sing}_{tr}^* Z_{tr}(X, D), \text{sing}_{tr}^* Z_{tr}(Y, E)[n])
\end{array}
\]

where we denoted for simplicity \(X\) for \(X_{\text{an}}\) and \(X_{cw}\), and similarly for \(D, Y\) and \(E\).
Proof. (i): By definition, Bti* \( M(Y, E) = R^\text{tr}_{\text{an}}(Z_{tr}(Y^{an}, E^{an})) \). Since, by theorem\(^{13}(ii),\)

- \( S(Z_{tr}(Y^{an}, E^{an})) : Z_{tr}(Y^{an}, E^{an}) \to \text{sing}_D Z_{tr}(Y^{an}, E^{an}) \) is an equivalence \((D^1, usu)\) local in \( PC^-\) (An) and
- \( \text{sing}_D Z_{tr}(Y^{an}, E^{an}) \) is a \( D^1 \) local object,

we have \( Bti* M(Y, E) = e^\text{tr}_{\text{an}}(\text{sing}_D Z_{tr}(Y^{an}, E^{an})) = \text{sing}_D Z_{tr}(Y^{an}, E^{an}) \). Since

- \( B(Z_{tr}(Y^{an}, E^{an})) : \text{sing}_D Z_{tr}(Y^{an}, E^{an}) \to Cw* \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}) \) is an equivalence \((D^1, usu)\) local in \( PC^-\) (An) by proposition\(^{27}(ii),\) and
- \( Cw* \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}) \) is a \( D^1 \) local object by theorem\(^{16}(ii)\) and proposition\(^{25}(ii),\)

we have

\[
Bti* M(X) = e^\text{tr}_{\text{an}}(Cw* \text{sing}_D Z_{tr}(Y^{cw}, E^{cw})) = \text{sing}_D Z_{tr}(Y^{cw}, E^{cw})
\]

(85)

By definition, \( Bti* M(X) = R^\text{tr}_{\text{cw}}(Z_{tr}(Y^{cw}, E^{cw})) \). Since, by theorem\(^{16}(ii),\)

- \( S(Z_{tr}(Y^{cw}, E^{cw})) : Z_{tr}(Y^{cw}, E^{cw}) \to \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}) \) is an equivalence \((\Im^1, usu)\) local in \( PC^-\) (CW) and
- \( \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}) \) is an \( \Im^1 \) local object,

we have

\[
\tilde{Bti*} M(Y, E) = e^\text{tr}_{\text{cw}}(\text{sing}_D Z_{tr}(Y^{cw}, E^{cw})) = \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}) = Bti* M(Y, E) \text{ by } (85)
\]

This proves (i).

(ii): Let \( \alpha \in \text{Hom}_{\text{DM}^-((C, Z))}(M(X, D), M(Y, E)[n]) \). Consider the commutative diagram in \( \text{AnDM}(Z) \)

\[
\begin{array}{ccc}
\text{sing}_D Z_{tr}(X^{an}, D^{an}) & \xrightarrow{\text{An}^* \alpha} & \text{sing}_D Z_{tr}(Y^{an}, E^{an})[n] \\
\downarrow & & \downarrow \\
\text{Cw* sing}_D Z_{tr}(X^{cw}, D^{cw}) & \xrightarrow{\text{Cw*} \text{Cw}^* \alpha} & \text{Cw* sing}_D Z_{tr}(Y^{cw}, E^{cw})[n]
\end{array}
\]

Since \( \text{sing}_D Z_{tr}(X^{an}, D^{an}) \) and \( \text{sing}_D Z_{tr}(Y^{an}, E^{an}) \) are \( D^1 \) local objects by theorem\(^{13}(ii),\)

\[\text{An}^* \alpha \in \text{Hom}_{\text{Ho}_{\text{us}}(\text{PC}^-\text{(An))}}(\text{sing}_D Z_{tr}(X^{an}, D^{an}), \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}))\]

Thus,

\[Bti^* (\alpha) := R^\text{tr}_{\text{an}}(\text{An}^* \alpha) = e^\text{tr}_{\text{an}} \text{An}^* \alpha \]

(86)

Since \( \text{sing}_D Z_{tr}(X^{cw}, D^{cw}) \) and \( \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}) \) are \( \Im^1 \) local objects by theorem\(^{16}(ii),\)

\[\text{Cw}^* \alpha \in \text{Hom}_{\text{Ho}_{\text{us}}(\text{PC}^-\text{(CW))}}(\text{sing}_D Z_{tr}(X^{cw}, D^{cw}), \text{sing}_D Z_{tr}(Y^{cw}, E^{cw}))\]

Thus

\[\tilde{Bti}^* (\alpha) := R^\text{tr}_{\text{cw}}(\text{Cw}^* \alpha) = e^\text{tr}_{\text{cw}} \text{Cw}^* \alpha \]

(87)

Since \( \text{Cw* sing}_D Z_{tr}(X^{cw}, D^{cw}) \) and \( \text{Cw* sing}_D Z_{tr}(Y^{cw}, E^{cw}) \) are \( D^1 \) local objects by theorem\(^{16}(ii)\) and proposition\(^{25}(ii),\)

\[\text{Cw* Cw}^* \alpha \in \text{Hom}_{\text{Ho}_{\text{us}}(\text{PC}^-\text{(An))}}(\text{Cw* sing}_D Z_{tr}(X^{cw}, D^{cw}), \text{Cw* sing}_D Z_{tr}(Y^{cw}, E^{cw}))\]

(88)
Thus

\[ \mathrm{Bti}^*(\alpha) = \mathrm{Re}^{tr}_{\alpha}(\mathrm{CW}, \tilde{\mathrm{CW}} \alpha) \] since \( B(Z_{\mathrm{tr}}(X^n, D^n)) \) and \( B(Z_{\mathrm{tr}}(Y^n, E^n))[n] \) are \( (D^1, \mathrm{usu}) \) local equivalence by proposition [27(ii)]

\[ = e^*_{\alpha}(\mathrm{CW}, \tilde{\mathrm{CW}} \alpha) \] by [SS]

\[ = e^*_{\mathrm{tr}}(\mathrm{CW} \alpha) \]

\[ = \mathrm{Bti}^*(\alpha) \] by [SS].

3.2 The image of algebraic correspondences after localization by Ayoub’s Betti realization functor

**Definition 27.** Let \( V \in \mathrm{Var}(\mathbb{C}) \) quasi projective and \( p, q \in \mathbb{N} \). Recall \( Z^p(V, \_ ) \subset Z^p(\square^* \times V, \mathbb{Z}) \) is the Bloch cycle complex consisting of closed subset meeting the face of \( \square^* \) properly.

(i) Let \( Z^p(V_{\mathbb{C}w}, \_ ) \subset Z^p(\square^* \times V_{\mathbb{C}w}, \mathbb{Z}) \) be the abelian subgroup consisting of closed subset meeting the face of \( \mathbb{I}^n \) properly. We have the restriction map induced by the closed embedding of CW complexes \( i_n \times I_V : \mathbb{I}^n \times V \hookrightarrow \square^* \times V_{\mathbb{C}w} \), which gives the morphism of complexes of abelian groups:

\[ \hat{T}_p : Z^p(V, \_ ) \to Z^p(V_{\mathbb{C}w}, \_ ), \alpha \mapsto \hat{T}_\alpha = \alpha^{\mathbb{C}w}_{\mathbb{I}^n \times V_{\mathbb{C}w}}. \]

(ii) Let \( Y \in \mathrm{PVar}(\mathbb{C}) \) be a compactification of \( V \) and \( E = V \setminus Y \). The higher cycle class map \( (\mathfrak{III}) \) is the morphism of complexes of abelian groups:

\[ T^p : Z^p(V, \_ ) \to C^\text{sing}_{2dV-2p+}(Y_{\mathbb{C}w}, E_{\mathbb{C}w}, \mathbb{Z}), \alpha \mapsto [p_Y(\hat{T}_\alpha)] = [p_Y(\alpha^{\mathbb{C}w}_{\mathbb{I}^n \times V_{\mathbb{C}w}})]. \]

\( \tilde{\alpha} \in Z^p(Y \times \mathbb{I}^n) \) is the closure of \( \alpha \) and \( p_Y : Y_{\mathbb{C}w} \times \mathbb{I}^n \to Y_{\mathbb{C}w} \) is the projection

Let \( X \in \mathrm{SmVar}(\mathbb{C}) \) and \( Y \in \mathrm{Var}(\mathbb{C}) \). Let \( E \subset Y \) be a closed subset and \( V = Y \setminus E \). Denote by \( j : V \hookrightarrow Y \) the open embeddings. We have the open embedding \( I_X \times j : X \times V \hookrightarrow X \times Y \). The map of complexes \( \hat{T}^\text{dv}_{X \times Y} \) induces a map denoted by the same way on the subcomplexes indicated in the following diagram in \( C^-_{\mathcal{C}} \):

\[ \begin{array}{ccc}
Z^dV(X \times V, \_ ) & \xrightarrow{\hat{T}^{dv}_{X \times V}} & Z^{2dV}(X_{\mathbb{C}w} \times V_{\mathbb{C}w}, \_ ) \\
(\mathcal{C}_*Z_{\text{tr}}(Y, E)(X))^{\hat{T}^{dv}_{X \times V}} \downarrow & & \downarrow (\mathcal{C}_*Z_{\text{tr}}(Y_{\mathbb{C}w}, E_{\mathbb{C}w})(X_{\mathbb{C}w}))^{\hat{T}^{dv}_{X \times V}} \\
S(c(Y, E^{\mathbb{C}w})_{\mathcal{C}_*Z_{\text{tr}}(j)})^{\hat{T}^{dv}_{X \times V}} \downarrow & & \downarrow S(c(Y_{\mathbb{C}w}, E_{\mathbb{C}w}^{\mathbb{C}w})_{\mathcal{C}_*Z_{\text{tr}}(j^{\mathbb{C}w})})(X_{\mathbb{C}w})^{\hat{T}^{dv}_{X \times V}} \\
\mathcal{C}_*Z_{\text{tr}}(V)(X) & \xrightarrow{\hat{T}^{dv}_{X \times V}} & \mathcal{C}_*Z_{\text{tr}}(V_{\mathbb{C}w})(X_{\mathbb{C}w})
\end{array} \]

**Lemma 10.** Let \( Y \in \mathrm{Var}(\mathbb{C}), X \in \mathrm{SmVar}(\mathbb{C}), E \subset Y \) a closed subset and \( V = Y \setminus E \) the open complementary. Let \( n \in \mathbb{Z}, n \leq 0 \). Then, for

\[ \alpha \in \mathrm{Hom}_{PC-}(\mathcal{C}_*Z_{\text{tr}}(X), \mathcal{C}_*Z_{\text{tr}}(Y, E)[n]) = H^n\mathcal{C}_*Z_{\text{tr}}(Y, E)(X). \]

we have the following equality of morphisms of \( \mathrm{PSh}(\mathrm{Cor}_2(CW), C^-_{\mathcal{C}}(\mathbb{Z})). \)

\[ (\hat{\mathcal{W}}(Z_{\text{tr}}(Y, E))[n]) \circ (\tilde{\mathrm{CW}} \alpha) = H^n(\hat{T}^{dv}_{X \times V})(\alpha) : Z_{\text{tr}}(X_{\mathbb{C}w}) \xrightarrow{\mathcal{C}_*Z_{\text{tr}}(Y, E)[n]} \mathcal{C}_*Z_{\text{tr}}(Y_{\mathbb{C}w}, E_{\mathbb{C}w})[n] \]

where \( H^n\hat{T}^{dv}_{X \times V} \) is the map induced in cohomology of the map of complex \( \hat{T}^{dv}_{X \times V} \).
Proof. Let $\alpha_n \in C_n Ztr(Y,E)(X)$ such that $\alpha = [\alpha_n] \in H^n C_n Ztr(Y,E)(X)$. We have

$$
(W(Ztr(Y,E))[n]) \circ (\tilde{C}^w) = (\tilde{\psi}(\tilde{W}(Ztr(Y,E))[n]) \circ (\tilde{C}w^* (Ztr^*(Y,E)(i))[n]) \circ ([Cw^* \alpha_n])
$$

$$= \tilde{\psi}(\tilde{W}(Ztr(Y,E))[n]) \circ ([Cw^* (\alpha_n|_{X \times Ztr^*})])
$$

$$= [(\alpha_n|_{X \times Ztr^*})^{cw}] = [\tilde{T}_c] = H^n \tilde{T}(\alpha)
$$

We deduce from this lemma [11] and proposition [28](ii) the following

Proposition 28. Let $X \in SmVar(\mathbb{C})$, $Y \in Var(\mathbb{C})$, $E \subset Y$ a closed subset and $V = Y \setminus E$ the open complementary. Let $n \in \mathbb{Z}$, $n \leq 0$. Then, the following diagram is commutative

$$
\begin{array}{ccc}
\text{Hom}_{PC} (Ztr(X), \mathbb{C}_Z Ztr(Y,E)[n]) & \xrightarrow{H^n(\tilde{T}^d_X)} & \text{Hom}_{PC} (CW)(Ztr(X^{cw}, \text{sing}_E, Ztr(Y^{cw}, E^{cw})[n]) \\
D(A^i, et) & & D(l^i, usu)
\end{array}
$$

$\xrightarrow{(89)}$

\begin{align*}
\text{Hom}_{DM} (\mathbb{C}_Z (M(X), M(Y,E)[n]) & \xrightarrow{\tilde{C}w^*} \text{Hom}_{CwDM} (\mathbb{C}_Z (M(X^{cw}), M(Y^{cw}, E^{cw})[n]) \\
\text{Hom}_{DM} (\mathbb{C}_Z (M(X), M(Y,E)[n]) & \xrightarrow{\tilde{C}w^*} \text{Hom}_{CwDM} (\mathbb{C}_Z (M(X^{cw}), M(Y^{cw}, E^{cw})[n])
\end{align*}

Proof. Let $\alpha \in \text{Hom}_{PC} (Ztr(X), \mathbb{C}_Z Ztr(Y,E)[n])$. We have

$$
\tilde{C}w^* D(A^i, et)(\alpha) = D(l^i, usu)(\tilde{C}w^*), \text{ since } \tilde{C}w^* \text{ derive trivially by proposition [28] iii)}
$$

$$= D(l^i, usu)(W(Ztr(Y,E)) \circ \tilde{C}w^*) \text{ since } W(Ztr(Y,E)) \text{ is an } (l^i, usu)$$

$$= D(l^i, usu)(H^n \tilde{T}(\alpha)) \text{ by lemma [10]}
$$

Using proposition[28] and proposition[28] (i), we immediately deduce from theorem[18] the following:

Corollary 3. Let $X \in SmVar(\mathbb{C})$, $Y \in Var(\mathbb{C})$, $E \subset Y$ a closed subvariety and $V = Y \setminus E$ the open complementary. Let $n \in \mathbb{Z}$, $n \leq 0$. Then,

(i) the following diagram is commutative

$$
\begin{array}{ccc}
\text{Hom}_{PC} (Ztr(X), \mathbb{C}_Z Ztr(Y,E)[n]) & \xrightarrow{H^n(\tilde{T}^d_X)} & \text{Hom}_{PC} (CW)(Ztr(X), \text{sing}_E, Ztr(Y,E)[n]) \\
An^* \circ D(A^i, et) & & \text{Re}_{cw}^* \circ D(l^i, usu)
\end{array}
$$

$\xrightarrow{(90)}$

\begin{align*}
\text{Hom}_{AnDM} (\mathbb{C}_Z (M(X), M(Y,E)[n]) & \xrightarrow{\text{Re}_{cw}^*} \text{Hom}_{D-} (\mathbb{C}_Z (\text{sing}_E, Ztr(X), \text{sing}_E, Ztr(Y,E)[n])
\end{align*}

where we denoted for simplicity $X$ for $X^{an}$ and $X^{cw}$, and similarly for $Y$ and $E$.

(ii) for $\alpha \in \text{Hom}_{PC} (Ztr(X), \mathbb{C}_Z Ztr(Y,E)[n])$, we have

$$
Bti^* \circ D(A^i, et)(\alpha) = K_n(X,Y)(p_{X \times Y}(H^n \tilde{T}^d_{X \times Y}(\alpha)))
$$

where $p_{X \times Y} : X^{cw} \times Y^{cw} \times \mathbb{P}^n \to X^{cw} \times Y^{cw}$ is the projection.
Proof. (i): By theorem 18 (ii) and proposition 28 the following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{P}^{-}(Z_{tr}(X), Z(Y,E)[n]) & \xrightarrow{H^{n}(T_{X,V})} & \text{Hom}_{P}^{-}(C_{W})_{Z_{tr}(X), Z(Y,E)[n]} \\
\text{Hom}_{DM}^{-}(C_{Z}(M(X), M(Y,E)[n]) & \xrightarrow{\text{C}_{W}} & \text{Hom}_{C}^{-}(Z)(M(X), M(Y,E)[n]) \\
\text{An}^{*} & \xrightarrow{\text{Re}_{cw}^{*}} & \text{Hom}_{D}^{-}(Z)(\text{sing}_{Z_{tr}(X), Z(Y,E)[n]}) \\
\end{array}
\]

where we denote for simplicity \( X \) for \( X^{an} \) and \( X^{cw} \), and similarly for \( Y \) and \( E \). This proves (i).

(ii) : We have

\[
\text{Bt}^{*}(D(\mathbb{A}^{1}, et) (\alpha)) : = \text{Re}_{cw}^{*} \circ \text{An}^{*}(D(\mathbb{A}^{1}, et)(\alpha))
\]

\[
= \text{Re}_{cw}^{*} \circ D(\mathbb{I}^{1}, usu)(H^{n}T_{X,V}(\alpha)) \quad \text{by (i)}
\]

\[
= K_{n}(X,Y)(p_{X,Y}(H^{n}T_{X,V}(\alpha))) \quad \text{by proposition 28 (i)}
\]

This proves (ii).

\[\square\]

3.3 Ayoub's Betti realization functor and Nori motives

We denote by \( C^{b}(\text{Cor}_{\mathbb{Z}}(\text{SmVar}(\mathbb{C}))) \) the category of bounded complexes of the category \( \text{Cor}_{\mathbb{Z}}(\text{SmVar}(\mathbb{C})) \). We have the Yoneda embedding

\[
C^{b}(\text{Cor}_{\mathbb{Z}}(\text{SmVar}(\mathbb{C}))) \to \text{PSh}(\text{Cor}_{\mathbb{Z}}(\text{SmVar}(\mathbb{C})), C^{-}(\mathbb{Z}), X \in \text{SmVar}(\mathbb{C}) \mapsto Z_{tr}(X).
\]

A (small) diagram is a 1-simplicial set, that is a functor from \( \Delta^{1} \) to Set We denote by \( \text{Var}^{2}(\mathbb{C}) \) the diagram of pairs whose set of vertices are the triplet \((X, D, i)\), \(X, D \in \text{Var}(\mathbb{C})\), \(D \subset X\) a closed subvariety and \(i \in \mathbb{N}\), and whose set of edges are the morphism of pairs \((X, D, i) \to (Y, E, i)\) and if \((Z, K) \subset (X, D)\), there is an edge \((X, D, i) \to (Z, K, i - 1)\). We have a morphism of diagram

\[
H_{*} : \text{Var}^{2}(\mathbb{C}) \to \text{Ab} ; (X, D, i) \mapsto H_{i}(X, D, Z)
\]

We denote by \( \text{GV}_{\mathbb{C}}^{2}(\mathbb{C}) \subset \text{Var}^{2}(\mathbb{C}) \) the subdiagram of good pairs that is \((X, D, i)\) is a good pair if \(X, D\) are affine, \(\dim X = i\) and if \(H_{p}(X, D, Z) = 0\) is \(p \neq i\). We recall the definition of Nori motives.

**Definition-Proposition 1.** [23]

(i) We have a well defined functor \( N : \text{Var}(\mathbb{C}) \to D^{b}(\mathbb{N}) \) given by, for \(X \in \text{Var}(\mathbb{C})\), considering

- the open coverings \( X = \bigcup_{i \in J} U_{i} \) by affine varieties,
- the filtrations by closed subvarieties \( U_{1}^{0} \subset \cdots \subset U_{1} \) such that \((U_{1}^{0}, U_{1}^{p-1})\) is a good pair, where \(I \subset J\) is a finite set,

we have

\[
o_{N} \circ N(X) = \lim \text{Tot}_{p, 1}(H_{p}(U_{1}^{p,cw}, U_{1}^{p-1,cw}, p)),
\]

where the limit is indexed by open coverings and filtrations by good pairs, \(\text{Tot}\) is the total complex, and \(o_{N} : D^{b}(\mathbb{N}) \to D^{b}(\mathbb{Z})\) is the forgetful functor.

(ii) The restriction of the functor \( N : \text{Var}(\mathbb{C}) \to D^{b}(\mathbb{N}) \) to \(\text{SmVar}(\mathbb{C}) \subset \text{Var}(\mathbb{C})\) extends to a functor

\[
N : C^{b}(\text{Cor}_{\mathbb{Z}}(\text{SmVar}(\mathbb{C}))) \to D^{b}(\mathbb{N})
\]
(iii) The functor $\mathcal{N} : \text{C}^b(\text{Cor}_Z(\text{SmVar}(\mathbb{C}))) \to D^b(\mathcal{N})$ factorize trough to get the composition

$$\mathcal{N} = \tilde{\mathcal{N}} \circ D(\mathbb{A}^1, \text{et}), \quad \mathcal{N} : DM^{gm} \to D^b(\mathcal{N})$$

Using the property of good pairs, Mayer Vietoris property for open coverings for singular cohomoloy, Nori showed the following:

**Proposition 29.** [13]

(i) Let $X \in \text{Var}(\mathbb{C})$, we have the following isomorphism in $D^b(\mathbb{Z})$:

$$o_N \circ \mathcal{N}(X) \simeq C_*(X, \mathbb{Z}) \simeq \text{sing}_\text{et} Z_{tr}(X^{cw})$$

(ii) Let $X, Y \in \text{SmVar}(\mathbb{C})$, then the following diagram commutes

$$\begin{array}{ccc}
\text{Hom}_{PC^*(\mathbb{Z})}(\mathcal{N}(X), \mathcal{N}(Y)) & \xrightarrow{H^b(T_X \times Y)} & \text{Hom}_{PC^*(\mathbb{Z})}(\mathcal{N}(X), \mathcal{N}(Y)) \\
\mathcal{N} \downarrow & & \mathcal{N} \\
\text{Hom}_{DM^b(\mathcal{N})}(\mathcal{N}(X), \mathcal{N}(Y)) & \xrightarrow{o_N} & \text{Hom}_{DM^b(\mathbb{Z})}(\mathcal{N}(X), \mathcal{N}(Y))
\end{array}$$

We will use the following lemma

**Lemma 11.** Let $\mathcal{C}, \mathcal{S}$ two categories, and $D : \mathcal{C} \to \mathcal{C}'$ a localization functor. Let $F_1, F_2 : \mathcal{C}' \to \mathcal{S}$ two functors. If

$$F_1 \circ D = F_2 \circ D : \mathcal{C} \to \mathcal{S},$$

then $F_1 = F_2$.

**Proof.** Let $f' : C_1 \to C_2$ a morphism in $\mathcal{C}'$. Without loss of generality, we can assume that $f' = f \circ s^{-1}$ with $f : C_0 \to C_2$ and $s : C_0 \to C_1$ morphisms in $\mathcal{C}$, such that $s$ belongs to the class of morphisms of $\mathcal{C}$ we localize. We have then,

$$F_1(f') = F_1(f' \circ s \circ s^{-1}) = (F_1 \circ D)(f' \circ s) \circ ((F_1 \circ D)(s))^{-1} = (F_2 \circ D)(f' \circ s) \circ ((F_2 \circ D)(s))^{-1} = F_2(f' \circ s \circ s^{-1}) = F_2(f')$$

We deduce from theorem [13], corollary [3](ii), proposition [20], end lemma [11] the following main result:

**Theorem 19.** (i) For $X \in \text{SmVar}(\mathbb{C})$, $\text{Bti}^* \circ D(\mathbb{A}^1, \text{et})(Z(X)) = o_N \circ \mathcal{N}(X)$

(ii) For $X, Y \in \text{SmVar}(\mathbb{C})$, the following diagram commutes

$$\begin{array}{ccc}
\text{Hom}_{PC^*(\mathbb{Z})}(Z(X), Z(Y)) & \xrightarrow{D(\mathbb{A}^1, \text{et})} & \text{Hom}_{DM^b(\mathbb{Z})}(\mathcal{N}(X), \mathcal{N}(Y)) \\
\mathcal{N} \downarrow & & \mathcal{N} \\
\text{Hom}_{DM^b(\mathbb{Z})}(M(X), M(Y)) & \xrightarrow{\text{Bti}^*} & \text{Hom}_{DM^b(\mathbb{Z})}(\mathcal{N}(X), \mathcal{N}(Y))
\end{array}$$

(iii) The Betti realisation functor factor through Nori motives. That is $\text{Bti}^* = o_N \circ \mathcal{N}$
Proof. (i) We have
\[ B_{\text{ti}}^* \circ D(\mathbb{A}^1, \text{et})(\mathbb{Z}(X)) = \text{Re}^r_{\text{et}} D(\mathbb{I}^1, \text{usu})(\mathbb{Z}_{\text{tr}}(X^{\text{cu}})) \text{ by theorem } \text{(18) i) } \]
\[ = o_N \circ \mathcal{N}(X) \text{ by proposition } \text{(28) i). } \]

(ii) Let \( \alpha \in \text{Hom}_{\mathbb{P}C^-}(\mathbb{Z}(X), \mathbb{Z}(Y)) \), we have
\[ B_{\text{ti}}^* \circ D(\mathbb{A}^1, \text{et})(\alpha) = K_0(X, Y)(H^0_{\mathcal{T}_X \times Y}(\alpha)) \text{ by corollary } \text{(3) ii) } \]
\[ = o_N \circ \mathcal{N}(\alpha) \text{ by proposition } \text{(28) ii). } \]

(iii) By (i) and (ii), we have the equality
\[ B_{\text{ti}}^* \circ D(\mathbb{A}^1, \text{et}) = o_N \circ \mathcal{N} = o_N \circ \mathcal{N} \circ D(\mathbb{A}^1, \text{et}) \text{ (92) } \]
Lemma [11] applied to this equality [92] say that \( B_{\text{ti}}^* = o_N \circ \mathcal{N} \). This proves (iii).

3.4 The image of Ayoub’s Betti realization functor on morphism and Bloch cycle class map

Let \( V \in \text{Var}(\mathbb{C}) \) quasi-projective. Let \( Y \in \text{PVar}(\mathbb{C}) \) a compactification of \( V \) and \( E = Y \setminus V \) Denote by \( j : V \hookrightarrow Y \) the open embedding. For \( r \in \mathbb{N} \), we denote by \( i_Y^r : Y \hookrightarrow \mathbb{A}^r \times Y \) and \( i_Y^r = i_{Y|V} : V \to \mathbb{A}^r \times V \) the inclusions, by \( p_Y^r : \mathbb{A}^r \times Y \to Y \) and \( p_Y^r = p_{Y|V} : \mathbb{A}^r \times V \to V \) the projections, by \( a : \mathbb{A}^r \to \mathbb{P}^r \) the open embedding, and by \( E' = Y \times \mathbb{P}^r \setminus (V \times \mathbb{A}^r) \). We consider

- for \( p \leq d_Y \), the inclusion of complexes \( i_Y^* : \mathbb{Z}^p(V, *) \hookrightarrow \mathbb{Z}^{d_Y}(\mathbb{A}^{d_Y-p} \times V, *) \), which is a quasi-isomorphism.
- for \( p \geq d_Y \) the inclusion of complexes \( p_Y^* : \mathbb{Z}^p(\mathbb{A}^{p-d_Y}, \times V) \hookrightarrow \mathbb{Z}^p(V, *) \), which is a quasi-isomorphism.

Since \( Y \) is projective, and \( \mathbb{A}^r \) is smooth,
\[
\begin{align*}
D(\mathbb{A}^1, \text{et}) : \text{Hom}_{\mathbb{P}C^-}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^r), \mathbb{Z}(Y, E)[n]) &\to \text{Hom}_{\text{DM}^-((\mathbb{C}, \mathbb{Z}))}(M(\mathbb{A}^r), M(Y, E)[n]) \\
D(\mathbb{I}^1, \text{et}) : \text{Hom}_{\mathbb{P}C^-}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^{r, \text{cu}}), \mathbb{Z}_Y(\mathbb{Y}^{\text{cu}}, \mathbb{E}^{\text{cu}})[n]) &\to \text{Hom}_{\text{DM}^-((\mathbb{Z})}(M(\mathbb{A}^{r, \text{cu}}), M(Y^{\text{cu}}, E^{\text{cu}})[n])
\end{align*}
\]

On the other side, the inclusion of complexes of abelian groups
- for \( p \leq d_Y \), \( (I_\times j)^* : C_* \mathbb{Z}_{\text{tr}}(Y, E)(\mathbb{A}^{d_Y-p}) \hookrightarrow \mathbb{Z}^{d_Y}(\mathbb{A}^{d_Y-p} \times V, *) \)
- for \( p \geq d_Y \), \( (a \times j)^* : C_* \mathbb{Z}_{\text{tr}}(\mathbb{P}^{p-d_Y} \times Y, E')(\mathbb{Spec}(\mathbb{C})) \hookrightarrow \mathbb{Z}^p(\mathbb{A}^{p-d_Y} \times V, *) \)

is a quasi-isomorphisms.

Definition 28. Let \( V \in \text{Var}(\mathbb{C}) \) quasi-projective. Let \( Y \in \text{PVar}(\mathbb{C}) \) a compactification of \( V \) and \( E = Y \setminus V \). Recall that \( j : V \hookrightarrow Y \) is the open embedding.

(i) For \( p \leq d_Y \), we consider the following composition of isomorphisms of abelian groups
\[
\begin{align*}
\text{H}^p,n(V) : \text{CH}^p(V, n) &\xrightarrow{i_Y^*} \text{CH}^{d_Y}(\mathbb{A}^{d_Y-p} \times V, n) \xrightarrow{(H^n(I_\times j)^*)^{-1}} \text{Hom}_{\mathbb{P}C^-}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^{d_Y-p}), \mathbb{Z}_{\text{tr}}(Y, E)[n]) \\
&\xrightarrow{D(\mathbb{A}^1, \text{et})} \text{Hom}_{\text{DM}^-((\mathbb{C}, \mathbb{Z}))}(M(\mathbb{A}^r), M(Y, E)[n])
\end{align*}
\]
(ii) For $p \geq d_V$, we consider the following composition of isomorphisms of abelian groups

\[
H^{p,n}(V) : CH^p(V,n) \xrightarrow{p_1^*} CH^p(\mathbb{A}^{d_v-p} \times V,n) \xrightarrow{(H^n(\alpha \times j)^{-1})} \text{Hom}_{PC-}(Z, \mathbb{Z}, \mathbb{Z}_\text{tr}(\mathbb{P}^{d_v-p} \times V, E)[n]) \\
\xrightarrow{D(V,et)} \text{Hom}_{DM-}(C,Z)(\mathbb{Z}, M(V \times \mathbb{P}^{d_v-p}, E)[n])
\]

We now prove that under these identifications, the image of the Betti realization functor on morphism coincide with the Bloch cycle class map.

**Theorem 20.** Let $V \in \text{Var}(\mathbb{C})$. Let $Y \in \text{PVar}(\mathbb{C})$ be a compactification of $V$ and $E = Y \setminus V$. Then,

(i) for $p \leq d_V$, the following diagram commutes:

\[
\begin{array}{ccc}
CH^p(V,n) & \xrightarrow{H^nT_V} & H_{n+p}(Y,E,\mathbb{Z}) \\
H^{p,n}(V) \sim & & \sim \\
\text{Hom}_{DM-}(C,Z)(M(\mathbb{A}^{d_v-p}), M(V,E)[n]) & \xrightarrow{\text{Bi}_i^*} & \text{Hom}_{D-}(Z, C_*(Y,E)[n])
\end{array}
\]

(ii) for $p \geq d_V$, the following diagram commutes:

\[
\begin{array}{ccc}
CH^p(V,n) & \xrightarrow{H^nT_V} & H_{n+p}(Y,E,\mathbb{Z}) \\
H^{p,n}(V) \sim & & \sim \\
\text{Hom}_{DM-}(C,Z)(M(\mathbb{A}^{d_v-p} \times V), M(V,E)[n]) & \xrightarrow{\text{Bi}_i^*} & \text{Hom}_{D-}(Z, C_*(Y,E)[n])
\end{array}
\]

**Proof.** (i): By corollary 3(ii), the second square of following diagram commutes

\[
\begin{array}{ccc}
CH^d(\mathbb{A}^{d_v-p} \times V,n) & \xrightarrow{H^nT_{d_v-p}} & H_{d_V-p+n}(\mathbb{A}^{d_v-p} \times V, E, \mathbb{Z}) \\
\xrightarrow{(H^n(\alpha \times j)^{-1})} & & \xrightarrow{[p()]} \\
\text{Hom}_{PC-}(Z, \mathbb{A}^{d_v-p}, Z_\text{tr}(Y,E)[n]) & \xrightarrow{H^nT_{d_v-p}} & \text{Hom}_{PC-}(C,W)(Z, Z_\text{tr}(\mathbb{A}^{d_v-p}), Z_\text{tr}(Y,E)[n]) \\
\xrightarrow{D(V,et)} & & \xrightarrow{K_n(\mathbb{A}^{d_v-p}(Y,E))(p())} \\
\text{Hom}_{DM-}(C,Z)(M(\mathbb{A}^{d_v-p}), M(Y,E)[n]) & \xrightarrow{\text{Bi}_i^*} & \text{Hom}_{D-}(Z, C_*(Y,E)[d_V-p+n])
\end{array}
\]

where $p : \mathbb{A}^{d_v-p} \times V \times \mathbb{P}^n \to \mathbb{A}^{d_v-p} \times V$ is the projection in $CW$. Moreover $K_n(\mathbb{A}^{d_v-p}(Y,E)) \circ [\cdot] = K_n(\mathbb{A}^{d_v-p}(Y,E))$ by definition. On the other hand, we have the following commutative diagram.

\[
\begin{array}{ccc}
CH^p(V,n) & \xrightarrow{H^nT_V} & H_{d_V-p+n}(Y,E,\mathbb{Z}) \\
\xrightarrow{i_v^*} & & \xrightarrow{i_v^*} \\
CH^d(\mathbb{A}^{d_v-p} \times V,n) & \xrightarrow{H^nT_{d_v-p}} & H_{d_V-p+n}(\mathbb{A}^{d_v-p} \times Y, E, \mathbb{Z})
\end{array}
\]

This proves (i).
(ii): By corollary [H]ii, the second square of following diagram commutes

\[
\begin{array}{ccc}
C^n_p(A_{p-dV} \times V, n) & \xrightarrow{H^n f_p} & H_{dV-p+n}(A_{p-dV} \times Y, E', Z) \\
\text{Hom}_{PC_-(Z, Z)}(\mathbb{P}^{p-dV} \times Y, E'[n]) & \xrightarrow{H^n f_p} & \text{Hom}_{PC_-(CW)}(Z, Z, \mathbb{P}^{p-dV} \times Y, E'[n]) \\
\text{Hom}_{DM_-(\mathbb{C}, Z)}(Z, M(\mathbb{P}^{p-dV} \times Y, E'[n])) & \xrightarrow{\text{B}_{\text{B}^+}} & \text{Hom}_{DM_-(\mathbb{Z})}(Z, C_s(Y, E)[dV-p+n])
\end{array}
\]

where \( p : \mathbb{P}^{p-dV} \times Y \times \mathbb{S}^n \to \mathbb{P}^{p-dV} \times Y \) is the projection in CW. Moreover \( K_n(\text{pt}, \mathbb{P}^{p-dV} \times Y, E')) \circ [-] = \overline{K}_n(\text{pt}, \mathbb{P}^{p-dV} \times Y, E')) \) by definition. On the other hand, we have the following commutative diagram.

\[
\begin{array}{ccc}
C^n_p(V, n) & \xrightarrow{H^n f_p} & H_{dV-p+n}(Y, E, Z) \\
\text{CH}_p(A_{dV-p} \times V, n) & \xrightarrow{H^n f_p} & \text{CH}_p(A_{dV-p} \times Y, E, Z)
\end{array}
\]

This proves (ii).

\[\square\]

4 The relative case

4.1 The derived category of relative motives of algebraic varieties

Let \( S \in \text{Var}(k) \). The category \( \text{Var}(k)/S \) is the category whose objects are \( X/S = (X, h) \) with \( X \in \text{Var}(k) \) and \( h : X \to S \) is a morphism, and whose space of morphisms between \( X/S = (X, h_1) \) and \( Y/S = (Y, h_2) \) in \( \text{Var}(k)/S \) are the morphism \( g : X \to Y \) such that \( h_2 \circ g = h_1 \). We denote by \( \text{Var}(k)_{\text{sm}}/S \subset \text{Var}(k)/S \) the full subcategory consisting of the objects \( X/S = (X, h) \) with \( X \in \text{Var}(k) \), such that \( h : X \to S \) is a smooth morphism. For \( X/S = (X, h) \in \text{Var}(k)/S \), and \( n \in \mathbb{N} \), we denote by

- \( (X \times \mathbb{A}^1/S) := (X \times_k \mathbb{A}^1, h \circ p_X) = (X \times_S (\mathbb{A}^1 \times S)/S) \), where \( p_X : X \times_k \mathbb{A}^1 \to X \) is the projection.
- \( (X \times \mathbb{S}^n/S) := (X \times_k \mathbb{S}^n, h \circ p_X) = (X \times_S (\mathbb{S}^n \times S)/S) \), where \( p_X : X \times_k \mathbb{S}^n \to X \) is the projection.

**Definition 29.** Let \( S \in \text{Var}(k) \). We define \( \text{Cor}_{\Lambda}^{s/s}(\text{Var}(k)_{\text{sm}}/S) \) to be the category whose objects are the one of \( \text{Var}(k)_{\text{sm}}/S \) and whose space of morphisms between \( X/S \) and \( Y/S \in \text{Var}(k)_{\text{sm}}/S \) is the free \( \Lambda \) module \( \mathbb{Z}^{s/s}(X \times_S Y, \Lambda) \). The composition of morphisms is defined similarly then in the absolute case (see [20]).

We have

- the additive embedding of categories \( \text{Tr}(\mathbb{Z}) : \mathbb{Z}(\text{Var}(k)_{\text{sm}}/S) \to \text{Cor}_{\Lambda}^{s/s}(\text{Var}(k)_{\text{sm}}/S) \) which gives the corresponding morphism of sites \( \text{Tr}(\mathbb{Z}) : \text{Cor}_{\Lambda}^{s/s}(\text{Var}(k)_{\text{sm}}/S) \to \mathbb{Z}(\text{Var}(k)_{\text{sm}}/S) \).
- the inclusion functor \( e_{\text{var}}(S) : \text{Ouv}(S) \hookrightarrow \text{Var}(k)_{\text{sm}}/S \), which gives the corresponding morphism of sites \( e_{\text{var}}(S) : \text{Var}(k)_{\text{sm}}/S \to \text{Ouv}(S) \).
- the inclusion functor \( e_{\text{var}}(S) := \text{Tr} \circ e_{\text{var}} : \text{Ouv}(S) \hookrightarrow \text{Cor}_{\Lambda}^{s/s}(\text{Var}(k)_{\text{sm}}/S) \) which gives the corresponding morphism of sites \( e_{\text{var}}(S) := \text{Tr} \circ e_{\text{var}} : \text{Cor}_{\Lambda}^{s/s}(\text{Var}(k)_{\text{sm}}/S) \to \text{Ouv}(S) \).

For each morphism \( f : T \to S \) in \( \text{Var}((\mathbb{C})) \), we have
• the pullback functor $P(f) : \text{Var}(\mathbb{C})/S \to \text{Var}(\mathbb{C})/T$, $P(f)(X/S) = (X \times_T T)/T$, $P(f)(h) = h_T$, which gives the morphism of sites $P(f) : \text{Var}(\mathbb{C})/T \to \text{Var}(\mathbb{C})/S$

• the pullback functor $P(f) : \text{Cor}_{ZS}^f(\text{Var}(\mathbb{C})^{sm}/S) \to \text{Cor}_{ZS}^f(\text{Var}(\mathbb{C})^{sm}/T)$, $P(f)(X/S) = (X \times_ST/T)$, $P(f)(h) = h_T$, which gives the morphism of sites $P(f) : \text{Cor}_{ZS}^f(\text{Var}(\mathbb{C})^{sm}/T) \to \text{Cor}_{ZS}^f(\text{Var}(\mathbb{C})^{sm}/S)$.

For $S \in \text{Var}(k)$, we consider the following two big categories:

• $\text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z})) = \text{PSh}_{Z}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z}))$, the category of bounded above complexes of presheaves on $\text{Var}(k)^{sm}/S$, or equivalently additive presheaves on $Z(\text{Var}(k)^{sm}/S)$, sometimes, we will write for short $P^{-}(S) = \text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z}))$,

• $\text{PSh}_{Z}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z}))$, the category of bounded above complexes of additive presheaves on $\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S)$ sometimes, we will write for short $PC^{-}(S) = \text{PSh}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z}))$, and the adjunctions:

• $(\text{Tr}(S)^*, \text{Tr}(S)_*) : \text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z})) \Rightarrow \text{PSh}_{Z}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z}))$,

• $(e_{\text{var}}(S)^*, e_{\text{var}}(S)_*) : \text{PSh}(\text{SmVar}(\mathbb{C}), C^{-}(\mathbb{Z})) \Rightarrow C^{-}(\mathbb{Z})$,

• $(e_{\text{tr}}(S)^*, e_{\text{tr}}(S)_*) : \text{PSh}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z})) \Rightarrow C^{-}(\mathbb{Z})$,

given by $\text{Tr}(S) : \text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S) \to Z(\text{Var}(k)^{sm}/S)$, $e_{\text{var}}(S) : \text{Var}(k)^{sm}/S \to \text{Ouv}(S)$ and $e_{\text{tr}}(S) : \text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S) \to \text{Ouv}(S)$ respectively. We denote by $a_{\text{et}} : \text{PSh}_{Z}(\text{Var}(k)^{sm}/S, \text{Ab}) \to \text{Sh}_{Z, \text{et}}(\text{Var}(k)^{sm}/S, \text{Ab})$ the etale sheafification functor. For $X/S \in \text{Var}(k)^{sm}/S$, we denote by

$$Z(X/S) \in \text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z})), \quad Z_{tr}(X/S) \in \text{PSh}_{Z}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z}))$$

the presheaves represented by $X$. They are etale sheaves. For $X/S = (X, h) \in \text{Var}(k)/S$ with $h : X \to S$ non smooth,

$$Z_{tr}(X/S) \in \text{PSh}_{Z}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z})), \quad Y \in \text{Var}(k)^{sm}/S \to Z_{tr}^{f/Y}(Y \times_S X, Z)$$

is also an etale sheaf; of course if $h : X \to S$ is not dominant then $Z_{tr}(X/S) = 0$.

For a morphism $f : T \to S$ in $\text{Var}(k)$ we have the adjunctions

• $(f^*, f_*) : \text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z})) \Rightarrow \text{PSh}(\text{Var}(k)^{sm}/T, C^{-}(\mathbb{Z}))$,

• $(f^*, f_*) : \text{PSh}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z})) \Rightarrow \text{PSh}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/T), C^{-}(\mathbb{Z}))$,

given by $P(f) : \text{Var}(k)^{sm}/T \to \text{Var}(k)^{sm}/S$ and $P(f) : \text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/T) \to \text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S)$, respectively.

**Definition 30.**  
(i) The projective etale topology model structure on $\text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z}))$ is defined in the similar way of the absolute case (c.f. definition 7(i)).

(ii) The projective $(\mathbb{A}_{k}^{1}, \text{et})$ model structure on the category $\text{PSh}(\text{Var}(k)^{sm}/S, C^{-}(\mathbb{Z}))$ is the left Bousfield localization of the projective etale topology model structure with respect to the class of maps $\{Z(X \times \mathbb{A}_{k}^{1}/S)[n] \to Z(X/S)[n], X/S \in \text{Var}(k)^{sm}/S, n \in \mathbb{Z}\}$.

(iii) The projective etale topology model structure on $\text{PSh}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z}))$ is defined in the similar way of the absolute case (c.f. definition 7(ii)).

(iv) The projective $(\mathbb{A}_{k}^{1}, \text{et})$ model structure on the category $\text{PSh}_{Z}(\text{Cor}_{ZS}^f(\text{Var}(k)^{sm}/S), C^{-}(\mathbb{Z}))$ is the left Bousfield localization of the projective etale topology model structure with respect to the class of maps $\{Z_{tr}(X \times \mathbb{A}_{k}^{1}/S)[n] \to Z(X/S)[n], X/S \in \text{Var}(k)^{sm}/S, n \in \mathbb{Z}\}$. 

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**Definition 31.** Let $S \in \text{Var}(k)$

(i) We define $\text{DM}^-(S, \mathbb{Z})_{et} := \text{Ho}_{\mathbb{A}^1, et}(\text{PSh}_\mathbb{Z}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z})))))$ to be the derived category of (effective) motives, it is the homotopy category of $\text{PSh}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z}))))$ with respect to the projective ($\mathbb{A}^1, et$) model structure (c.f. definition 30(ii)). We denote by $D^{tr}(\mathbb{A}^1, et)(S) : \text{PSh}_\mathbb{Z}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z})))) \to \text{DM}^-(S, \mathbb{Z})_{et}$, $D^{tr}(\mathbb{A}^1, et)(S)(F^*) = F^*$ the canonical localization functor.

(ii) By the same way, we denote $\text{DA}^-(S, \mathbb{Z})_{et} := \text{Ho}_{\mathbb{A}^1, et}(\text{PSh}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z}))))$ (c.f. 30(i)) and $D(\mathbb{A}^1, et)(S) : \text{PSh}_\mathbb{Z}(\text{Cor}_k^{fs}(\text{SmVar}(\mathbb{C})), C^-(\mathbb{Z})))) \to \text{DA}^-(S, \mathbb{Z})_{et}$, $D(\mathbb{A}^1, et)(S)(F^*) = F^*$ the canonical localization functor.

**Proposition 30.** Let $S \in \text{Var}(k)$

(i) $(\text{Tr}(S)^*, \text{Tr}(S)*) : \text{PSh}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z}))) \Rightarrow \text{PSh}_\mathbb{Z}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z}))))$ is a Quillen adjunction for the etale topology model structures and a Quillen adjunction for the ($\mathbb{A}^1, et$) model structures (c.f. definition 30(i) and (ii) respectively).

(ii) $(e_{\text{var}}(S)^*, e_{\text{var}}(S)*) : \text{PSh}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z}))) \Rightarrow C^-(\mathbb{Z})$ is a Quillen adjunction for the etale topology model structures and a Quillen adjunction for the ($\mathbb{A}^1, et$) model structures (c.f. definition 30(i)).

(iii) $(e_{\text{var}}(S)^{tr*,}, e_{\text{var}}(S)^{tr*}) : \text{PSh}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S), C^-(\mathbb{Z}))) \Rightarrow C^-(\mathbb{Z})$ is a Quillen adjunction for the etale topology model structures and a Quillen adjunction for the ($\mathbb{A}^1, et$) model structures (c.f. definition 30(ii)).

**Proof.** (i): Follows from the fact that $\text{Tr}(S)_*$ derive trivially hence is a right Quillen functor.

(ii): Follows from the fact that $e_{\text{var}}(S)^*$ derive trivially hence is a left Quillen functor.

(iii): Follows from the fact that $e_{\text{var}}(S)^{tr*}$ derive trivially hence is a left Quillen functor.

**Proposition 31.** Let $f : T \to S$ a morphism in $\text{Var}(k)$.

(i) The functors

$- f_* : \text{PSh}(\text{Var}(k)_{sm}/T, C^-(\mathbb{Z}))) \to \text{PSh}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z})))$ and

$- f_* : \text{PSh}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/T), C^-(\mathbb{Z}))) \to \text{PSh}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S), C^-(\mathbb{Z})))$,

preserve and detect $\mathbb{A}^1$ local object, preserve etale fibrant objects.

(ii) The functors

$- f^* : \text{PSh}(\text{Var}(k)_{sm}/S, C^-(\mathbb{Z}))) \to \text{PSh}(\text{Var}(k)_{sm}/T, C^-(\mathbb{Z})))$ and

$- f^* : \text{PSh}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/S), C^-(\mathbb{Z}))) \to \text{PSh}(\text{Cor}_k^{fs}(\text{Var}(k)_{sm}/T), C^-(\mathbb{Z})))$,

preserve and detect ($\mathbb{A}^1, et$) equivalence.

**Proof.** Point (i) follows immediately from definition of $\mathbb{A}^1$ local objects. Point (ii) follows from point (i).

We immediately deduce the following

**Proposition 32.** Let $f : T \to S$ a morphism in $\text{Var}(k)$. 

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Proof. By proposition \ref{prop:quillen-equivalence}(i), \(f^*\) is a left Quillen functor.

**Theorem 21.** Let \(S \in \text{Var}(k)\). The adjunction \((\text{Tr}(S)^*, \text{Tr}(S)_*) : PSh(\text{Var}(k)^{sm}/S, C^-(\mathbb{Z})) \Rightarrow \text{PSh}_Z(\text{Cor}(k)^{sm}/S, C^-(\mathbb{Z}))\) is a Quillen equivalence, that is the derived functor

\[ L \text{Tr}(S)^* : \text{DA}^-(S, \mathbb{Z}) \xrightarrow{\sim} \text{DM}^-(S, \mathbb{Z}) \text{PSh}_Z(\text{Cor}(k)^{sm}/S, C^-(\mathbb{Z})) \]

is an isomorphism and \(\text{Tr}(S)_* : \text{DM}^-(S, \mathbb{Z}) \xrightarrow{\sim} \text{DA}^-(S, \mathbb{Z}) \text{PSh}_Z(\text{Cor}(k)^{sm}/S, C^-(\mathbb{Z}))\) is its inverse.

We have all the property of the six functor formalism on \(\text{Var}(k)\):

- \(f_1 = Rf_*\) if \(f : T \to S\) is proper,
- \(j_1\) is the left adjoint of \(j^*\) if \(j : S^o \hookrightarrow S\) is an open embedding (\(j^*\) admits a left adjoint since it is a smooth morphism).

**Proof.** See \cite{6} or \cite{4}. We just recall the definition of \(f_1\). Let \(f : T \to S\) a morphism in \(\text{Var}(k)\). Let \(\hat{T} \in P\text{Var}(k)\) a compactification of \(T\). Take \(\hat{T} \subset T \times S\) the closure of the graph of \(f\). Then, \(f = \hat{f} \circ j\) where \(j : T \hookrightarrow \hat{T}\) is the open embedding and \(\hat{f} := p_{S/T}\) is a proper morphism, with \(p_S : \hat{T} \times S \to S\) the projection. Then for \(F^* \in PC^-(T)\) a \(\mathbb{A}^1\) local and etale fibrant object \(f_1 F^* := D(\mathbb{A}^1, \text{et})(S)(\hat{f}^* j_1 F^*)\) does not depends of the compactification of \(f\) by the support property.

**Definition 32.** Let \(S \in \text{Var}(\mathbb{C})\).

- \(L S \in \text{DM}^-(S, \mathbb{Z})\) and \(p \in \mathbb{Z}\). The Tate twist of \(L S\) is \(M = L S \otimes D(\mathbb{A}^1, \text{et})(\mathbb{Z}(p)) = (\mathbb{Z}(p)_{\text{et}}(\mathbb{A}^1 \times S, \infty))^{\text{op}} \in PC^-(S)\). For \(F^* \in PC^-(S)\), we denote \(F^*(p) := F^* \otimes \mathbb{Z}(p) \in PC^-(S)\).
- A motive \(M \in \text{DM}^-(S, \mathbb{Z})\) is called constructible if it belongs to the thick subcategory generated by motives of the form \(D(\mathbb{A}^1, \text{et})(S)(\mathbb{Z}^{\text{tr}}(X/S)(p))\), with \(X/S = (X, h)\), where \(h : X \to S\) is a smooth morphism, and \(p \in \mathbb{Z}\).

**Definition-Proposition 2.** Let \(S, X \in \text{Var}(k)\) and \(h : X \to S\) a morphism.

\(i\) The motive of \(X/S = (X, h)\) is \(M(X/S) := h_! h^! \mathbb{Z}^{\text{tr}}(S/S) \in \text{DM}^-(S, \mathbb{Z})\). It is a constructible object.

\(ii\) The Borel-Moore motive of \(X/S = (X, h)\) is \(M^{BM}(X/S) := h^! h_* \mathbb{Z}^{\text{tr}}(S/S) = h_* \mathbb{Z}^{\text{tr}}(X/X) \in \text{DM}^-(S, \mathbb{Z})\). It is a constructible object.

\(iii\) The cohomological motive of \(X/S = (X, h)\) is \(M(X/S) := \text{Rh}_* h^! \mathbb{Z}^{\text{tr}}(S/S) = \text{Rh}_* \mathbb{Z}^{\text{tr}}(X/X) \in \text{DM}^-(S, \mathbb{Z})\). It is a constructible object.

\(iv\) The motive with compact support of \(X/S = (X, h)\) is \(M(X/S) := \text{Rh}_* h^! \mathbb{Z}^{\text{tr}}(S/S) \in \text{DM}^-(S, \mathbb{Z})\). It is a constructible object.

**Proof.** The fact that these four object associated to \(X/S = (X, h)\) are constructible follows from \cite{6} section 4.
4.2 The derived category of relative motives of analytic spaces

Let $S \in \text{AnSp}(C)$. The category $\text{AnSp}(C)/S$ is the category whose objects are $X/S = (X, h)$ with $X \in \text{AnSp}(C)$ and $h : X \to S$ is a morphism, and whose space of morphisms between $X/S = (X, h_1)$ and $Y/S = (Y, h_2) \in \text{SmVar}(S)$ are the morphism $g : X \to Y$ such that $h_2 \circ g = h_1$. We denote by $\text{AnSp}(C)^{sm}/S \subset \text{AnSp}(C)/S$ the full subcategory consisting of the objects $X/S = (X, h)$ with $X \in \text{AnSp}(C)$, such that $h : X \to S$ is a smooth morphism. For $X/S = (X, h) \in \text{AnSp}(C)/S$, and $n \in \mathbb{N}$, we denote by

- $(X \times \mathbb{D}^1/S) := (X \times \mathbb{D}^1, h \circ p_X) = (X \times_S (\mathbb{D}^1 \times S), p_X : X \times \mathbb{D}^1 \to X$ is the projection.
- $(X \times \mathbb{D}^n/S) := (X \times \mathbb{D}^n, h \circ p_X) = (X \times_S (\mathbb{D}^n \times S), p_X : X \times \mathbb{D}^n \to X$ is the projection.

**Definition 33.** Let $S \in \text{AnSp}(C)$. We define $\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S)$ to be the category whose objects are the one of $\text{AnSp}(C)^{sm}/S$ and whose space of morphisms between $X/S$ and $Y/S \in \text{AnSp}(C)^{sm}/S$ is the free $\Lambda$ module $Z^s(\mathbb{D}^1 X \times_SY, \Lambda)$. The composition of morphisms is defined similarly then in the absolute case.

We have

- the additive embedding of categories $\text{Tr}(S) : Z(\text{AnSp}(C)^{sm}/S) \hookrightarrow \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S)$ which gives the corresponding morphism of sites $\text{Tr}(S) : \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S) \to Z(\text{AnSp}(C)^{sm}/S)$.
- the inclusion functor $e_{an}(S) : \text{Ouv}(S) \hookrightarrow \text{AnSp}(C)^{sm}/S$, which gives the corresponding morphism of sites $e_{an}(S) : \text{AnSp}(C)^{sm}/S \to \text{Ouv}(S)$,
- the inclusion functor $e_{anr}(S) := \text{Tr}e_{an} \text{Ouv}(S) \hookrightarrow \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S)$ which gives the corresponding morphism of sites $e_{anr}(S) := \text{Tr}e_{an} : \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S) \to \text{Ouv}(S)$.

For each morphism $f : T \to S$ in $\text{AnSp}(C)$, we have

- the pullback functor $P(f) : \text{AnSp}(C)/S \to \text{AnSp}(C)/T$, $P(f)(X/S) = (X \times_S T/T, P(f)(h) = h_T$, which gives the morphism of sites $P(f) : \text{Var}(C)/T \to \text{Var}(C)/S$.
- the pullback functor $P(f) : \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S) \to \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/T)$, $P(f)(X/S) = (X \times_S T/T, P(f)(h) = h_T$, which gives the morphism of sites $P(f) : \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/T) \to \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S)$.

For $S \in \text{AnSp}(C)$, we consider the following two big categories:

- $\text{PSh}(\text{AnSp}(C)^{sm}/S, C^{-}(\mathbb{Z})) = \text{PSh}_{\mathbb{Z}}(Z(\text{AnSp}(C)^{sm}/S), C^{-}(\mathbb{Z}))$, the category of bounded above complexes of presheaves on $\text{AnSp}(C)^{sm}/S$, or equivalently additive presheaves on $Z(\text{AnSp}(C)^{sm}/S)$, sometimes, we will write for short $P^{-}(S) = \text{PSh}(\text{AnSp}(C)^{sm}/S, C^{-}(\mathbb{Z}))$,

- $\text{PSH}_{\mathbb{Z}}(\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S), C^{-}(\mathbb{Z}))$, the category of bounded above complexes of additive presheaves on $\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S)$, sometimes, we will write for short $PC^{-}(S) = \text{PSH}(\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S), C^{-}(\mathbb{Z}))$.

and the adjunctions:

- $(\text{Tr}(S)^*, \text{Tr}(S)_{!*}) : \text{PSh}(\text{AnSp}(C)^{sm}/S, C^{-}(\mathbb{Z})) \leftrightarrows \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S), C^{-}(\mathbb{Z}))$,

- $(e_{an}(S)^*, e_{an}(S)_{!*}) : \text{PSh}(\text{SmVar}(C), C^{-}(\mathbb{Z})) \leftrightarrows C^{-}(\mathbb{Z})$,

- $(e_{anr}(S)^*, e_{anr}(S)_{!*}) : \text{PSh}(\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S), C^{-}(\mathbb{Z})) \leftrightarrows C^{-}(\mathbb{Z})$,

given by

- $(\text{Tr}(S)^*, \text{Tr}(S)_{!*}) : \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S) \to Z(\text{AnSp}(C)^{sm}/S), e_{an}(S) : \text{AnSp}(C)^{sm}/S \to \text{Ouv}(S)$ and

$e_{anr}(S) : \text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S) \to \text{Ouv}(S)$ respectively. We denote by $a_{et} : \text{PSH}_{\mathbb{Z}}(\text{AnSp}(C)^{sm}/S, \text{Ab}) \to \text{Sh}_{\mathbb{Z},et}(\text{AnSp}(C)^{sm}/S, \text{Ab})$ the etale sheafification functor. For $X/S \in \text{AnSp}(C)^{sm}/S$, we denote by

$$Z(X/S) \in \text{PSh}(\text{AnSp}(C)^{sm}/S, C^{-}(\mathbb{Z})), \quad Z_{et}(X/S) \in \text{PSH}_{\mathbb{Z}}(\text{Cor}_{\text{et}}^s(\text{AnSp}(C)^{sm}/S), C^{-}(\mathbb{Z})) $$ (95)

the presheaves represented by $X$. They are ush sheaves.

For a morphism $f : T \to S$ in $\text{AnSp}(C)$ we have the adjunction

$$
$$

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• \((f^*, f_*): (P(f)^*, P(f)_*) : \mathrm{PSh}((\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z)) \rightleftharpoons \mathrm{PSh}((\mathrm{AnSp}(\mathbb{C}))^{sm}/T, C^-(Z))\),

• \((f^*, f_*): (P(f)^*, P(f)_*) : \mathrm{PSh}_Z(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z)) \rightleftharpoons \mathrm{PSh}_Z(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/T, C^-(Z))\),

given by \(P(f) : \mathrm{AnSp}(\mathbb{C})^{sm}/T \rightarrow \mathrm{AnSp}(\mathbb{C})^{sm}/S\) and \(P(f) : \mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/T \rightarrow \mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S\), respectively.

**Definition 34.** Let \(S \in \mathrm{AnSp}(\mathbb{C})\).

(i) The projective usual topology model structure on \(\mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z))\) is defined in the similar way of the absolute case (c.f. definition 34(ii)).

(ii) The projective \((\mathbb{D}^1, \mathrm{et})\) model structure on the category \(\mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z))\) is the left Bousfield localization of the projective usual topology model structure with respect to the class of maps \(\{Z(X \times \mathbb{D}^1/S)[n] \rightarrow Z(X/S)[n], X/S \in \mathrm{AnSp}(\mathbb{C})^{sm}/S, n \in \mathbb{Z}\}\).

(iii) The projective usual topology model structure on \(\mathrm{PSh}(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S), C^-(Z))\) is defined in the similar way of the absolute case (c.f. definition 34(ii)).

(iv) The projective \((\mathbb{D}^1, \mathrm{et})\) model structure on the category \(\mathrm{PSh}_Z(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S), C^-(Z))\) is the left Bousfield localization of the projective usual topology model structure with respect to the class of maps \(\{Z_{tr}(X \times \mathbb{D}^1/S)[n] \rightarrow Z(X/S)[n], X/S \in \mathrm{AnSp}(\mathbb{C})^{sm}/S, n \in \mathbb{Z}\}\).

**Definition 35.** Let \(S \in \mathrm{AnSp}(\mathbb{C})\).

(i) We define \(\mathrm{AnDM}^-(S, Z) := \mathrm{Ho}_{\mathbb{D}^1, \mathrm{et}}(\mathrm{PSh}_Z(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S), C^-(Z))\), to be the derived category of (effective) motives, it is the homotopy category of \(\mathrm{PSh}(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S), C^-(Z))\) with respect to the projective \((\mathbb{D}^1, \mathrm{usu})\) model structure (c.f. definition 34(ii)). We denote by

\[
D^r(\mathbb{D}^1, \mathrm{usu})(S) : \mathrm{PSh}_Z(\mathrm{Cor}_Z^{fs}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S), C^-(Z)) \rightarrow \mathrm{DM}^-(S, Z), D^r(\mathbb{D}^1, \mathrm{usu})(S)(F^*) = F^*
\]

the canonical localization functor.

(ii) By the same way, we denote \(\mathrm{AnDA}^-(S, Z) := \mathrm{Ho}_{\mathbb{D}^1, \mathrm{usu}}(\mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z))\) (c.f.34(i)) and

\[
D(\mathbb{D}^1, \mathrm{usu})(S) : \mathrm{PSh}_Z(\mathrm{Cor}_Z^{fs}(\mathrm{SmVar}(\mathbb{C})), C^-(Z))) \rightarrow \mathrm{AnDA}^-(S, Z), D(\mathbb{D}^1, \mathrm{usu})(S)(F^*) = F^*
\]

the canonical localization functor.

We now look at an explicit localization functor.

For \(F^* \in \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, \mathrm{Ab})\) and \(X/S \in \mathrm{AnSp}(\mathbb{C})^{sm}/S\), we have the complex \(F(X \times \mathbb{D}^*/S)\) associated to the cubical object \(F(X \times \mathbb{D}^*/S)\) in the category of abelian groups.

- If \(F^* \in \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z))\),

\[
\underline{\mathrm{sing}}_{\mathbb{D}^1}, F^* := \mathrm{Tot}(\mathrm{Hom}(\mathbb{D}^* \times S), F^*)) \in \mathrm{PSh}(\mathrm{AnSp}(\mathbb{C}))^{sm}/S, C^-(Z))
\]

is the total complex of presheaves associated to the bicomplex of presheaves \(X/S \rightarrow F^*(\mathbb{D}^* \times X/S)\), and \(\underline{\mathrm{sing}}_{\mathbb{D}^1} F^* := e_{an}(S), \underline{\mathrm{sing}}_{\mathbb{D}^1} F^* \in C^-(Z)\). We denote by \(S(F^*) : F^* \rightarrow \underline{\mathrm{sing}}_{\mathbb{D}^1} F^*, \underline{\mathrm{sing}}_{\mathbb{D}^1} F^* \rightarrow F^*, 0 \rightarrow \cdot \cdot \cdot\)
the inclusion morphism of $\text{PSh}((\text{AnSp}(\mathbb{C}))^{sm}/S, C^{-}(\mathbb{Z})) : For f : F_{1}^{\bullet} \to F_{2}^{\bullet}$ a morphism in $P^{-}(\text{An}, S)$, we denote by $S(f) : \text{sing}_{\varnothing}, F_{1}^{\bullet} \to \text{sing}_{\varnothing}, F_{2}^{\bullet}$, the morphism of $\text{PSh}((\text{AnSp}(\mathbb{C}))^{sm}/S, C^{-}(\mathbb{Z}))$ given by for $X/S \in (\text{AnSp}(\mathbb{C}))^{sm}/S$,

\[
S(f)(X/S) : \cdots \to F_{1}(\bar{\mathbb{D}}^{2} \times X/S) \to F_{1}(\bar{\mathbb{D}}^{1} \times X/S) \to F_{1}^{\bullet}(X) \to 0 \to \cdots \] 

\[
\cdots \to F_{2}(\bar{\mathbb{D}}^{2} \times X/S) \to F_{2}(\bar{\mathbb{D}}^{1} \times X/S) \to F_{2}^{\bullet}(X/S) \to 0 \to \cdots .
\]

(98)

• If $F^{\bullet} \in \text{PSh}_{2}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S, C^{-}(\mathbb{Z}))$,

\[
\text{sing}_{\varnothing}, F^{\bullet} := \text{Hom}(\mathbb{Z}_{\text{tr}}, (\bar{\mathbb{D}}^{\ast} \times S), F^{\bullet}) \in \text{PSh}_{2}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S), C^{-}(\mathbb{Z})) \tag{99}
\]

is the complex of presheaves associated to the bicomplex of presheaves $X/S \mapsto F^{\bullet}(\bar{\mathbb{D}}^{\ast} \times X/S)$, and $\text{sing}_{\varnothing}, F^{\bullet} := e_{an}^{\mathbb{C}}(S) \cdot \text{sing}_{\varnothing}, F^{\bullet} \in C^{-}(\mathbb{Z})$. We have the inclusion morphism (97)

\[
S(\text{Tr}_{\mathbb{Z}}^{\dagger}, F^{\bullet}) : \text{Tr}(S), F^{\bullet} \to \text{sing}_{\varnothing}, F^{\bullet} \] 

\[
\text{Tr}(S), F^{\bullet} = \text{Tr}(S), \text{sing}_{\varnothing}, F^{\bullet}
\]

which is a morphism in $\text{PSh}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S), C^{-}(\mathbb{Z}))$ denoted the same way $S(F^{\bullet}) : F^{\bullet} \to \text{sing}_{\varnothing}, F^{\bullet}$. For $f : F_{1}^{\bullet} \to F_{2}^{\bullet}$ a morphism $P^{-}(S)^{-}$, we have the morphism (98)

\[
S(\text{Tr}_{\mathbb{Z}}^{\dagger}, f) : \text{Tr}(S), F_{1}^{\bullet} \to \text{sing}_{\varnothing}, F_{1}^{\bullet} \] 

\[
\text{Tr}(S), F_{2}^{\bullet} = \text{Tr}(S), \text{sing}_{\varnothing}, F_{2}^{\bullet}
\]

which is a morphism in $P^{-}(\text{An}, S)$ denoted the same way $S(f) : \text{sing}_{\varnothing}, F_{1}^{\bullet} \to \text{sing}_{\varnothing}, F_{2}^{\bullet}$.

For $F^{\bullet} \in \text{PSh}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S), C^{-}(\mathbb{Z}))$, we have by definition $\text{Tr}(S), \text{sing}_{\varnothing}, F^{\bullet} = \text{sing}_{\varnothing}, \text{Tr}(S), F^{\bullet}$ and $\text{Tr}(S), S(F^{\bullet}) = S(\text{Tr}(S), F^{\bullet})$.

**Proposition 33.** Let $S \in \text{AnSp}(\mathbb{C})$

(i) $(\text{Tr}(S)^{\ast}, \text{Tr}(S)) : \text{PSh}(\text{AnSp}(\mathbb{C}))^{sm}/S, C^{-}(\mathbb{Z})) \to \text{PSh}_{2}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S), C^{-}(\mathbb{Z}))$ is a Quillen adjonction for the usual topology model structures and a Quillen adjonction for the $(\mathbb{D}^{1}, et)$ model structures (c.f. definition 30 (i) and (ii) respectively).

(ii) $(e_{an}(S)^{\ast}, e_{var}(S)) : \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S, C^{-}(\mathbb{Z})) \to C^{-}(\mathbb{Z})$ is a Quillen adjonction for the usual topology model structures and a Quillen adjonction for the $(\mathbb{D}^{1}, et)$ model structures (c.f. definition 30 (i)).

(iii) $(e_{an}(S)^{tr*}, e_{var}(S)^{tr}) : \text{PSh}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S), C^{-}(\mathbb{Z})) \to C^{-}(\mathbb{Z})$ is a Quillen adjonction for the usual topology model structures and a Quillen adjonction for the $(\mathbb{D}^{1}, et)$ model structures (c.f. definition 30 (ii)).

**Proof.** (i): Follows from the fact that $\text{Tr}(S)^{\ast}$ derive trivially hence is a right Quillen functor.

(ii): Follows from the fact that $e_{var}(S)^{\ast}$ derive trivially hence is a left Quillen functor.

(iii): Follows from the fact that $e_{var}(S)^{tr}$ derive trivially hence is a left Quillen functor. 

\[\square\]

**Proposition 34.** Let $f : T \to S$ a morphism in AnSp($\mathbb{C}$).

(i) The functors

\[- f_{s} : \text{PSh}(\text{AnSp}(\mathbb{C}))^{sm}/T, C^{-}(\mathbb{Z})) \to \text{PSh}(\text{AnSp}(\mathbb{C}))^{sm}/S, C^{-}(\mathbb{Z})) \text{ and }\]

\[- f_{s} : \text{PSh}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/T), C^{-}(\mathbb{Z})) \to \text{PSh}(\text{Cor}^{\dagger}_{Z}(\text{AnSp}(\mathbb{C}))^{sm}/S), C^{-}(\mathbb{Z})) , \]

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Proof. Point (i) follows immediately from definition of $\mathbb{D}^1$ local objects. Point (ii) follows from point (i).

We immediately deduce the following

**Proposition 35.** Let $f : T \to S$ a morphism in $\text{AnSp}(\mathbb{C})$.

(i) The adjunction $(f^*, f_*) : \text{PSh}(\text{AnSp}(\mathbb{C})^\text{sm}/S, C^{-}(\mathbb{Z})) \to \text{PSh}(\text{AnSp}(\mathbb{C})^\text{sm}/T, C^{-}(\mathbb{Z}))$ is a Quillen adjunction with respect to the usual topology model structures and the $(\mathbb{D}^1, \text{usu})$ model structures

(ii) The adjunction $(f^*, f_*) : \text{PSh}(\text{Cor}^f_{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^\text{sm}/S), C^{-}(\mathbb{Z})) \to \text{PSh}(\text{Cor}^f_{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^\text{sm}/T), C^{-}(\mathbb{Z}))$, preserve and detect $(\mathbb{D}^1, \text{usu})$ equivalence.

Proof. (i): By [H], $f^*$ derive trivially for the usual topology model structures. Now, by proposition [H](ii), $f^*$ derive trivially for the $(\mathbb{D}^1, \text{usu})$ model structures. In particular $f^*$ is a left Quillen functor.

(ii): Similar to point (i).

**Theorem 23.** Let $S \in \text{AnSp}(\mathbb{C})$

(i) For $F^* \in \text{PSh}(\text{AnSp}(\mathbb{C})^\text{sm}/S, C^{-}(\mathbb{Z}))$, $\text{sing}_S F^* \in \text{PSh}(\text{AnSp}(\mathbb{C})^\text{sm}/S, C^{-}(\mathbb{Z}))$ is $\mathbb{D}^1$ local and the inclusion morphism $S(F^*) : F^* \to \text{sing}_S F^*$ is an $(\mathbb{D}^1, \text{usu})$ equivalence.

(ii) For $F^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}^f_{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^\text{sm}/S), C^{-}(\mathbb{Z}))$, $\text{sing}_S F^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}^f_{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^\text{sm}/S), C^{-}(\mathbb{Z}))$ is $\mathbb{D}^1$ local and the inclusion morphism $S(F^*) : F^* \to \text{sing}_S F^*$ is an $(\mathbb{D}^1, \text{usu})$ equivalence.

Proof. Similar to the proof of the absolute case.

**Theorem 24.** Let $S \in \text{AnSp}(\mathbb{C})$.

(i) The adjunction $(\text{Tr}(S)^*, \text{Tr}(S)_*) : \text{PSh}(\text{AnSp}(\mathbb{C})^\text{sm}/S, C^{-}(\mathbb{Z})) \to \text{PSh}_{\mathbb{Z}}(\text{Cor}^f_{\mathbb{Z}}(\text{AnSp}(\mathbb{C})^\text{sm}/S), C^{-}(\mathbb{Z}))$ is a Quillen equivalence for the $(\mathbb{D}^1, \text{usu})$ model structures. That is, the derived functor

$$L \text{Tr}^* : \text{AnDA}^{-}(S, Z) \xrightarrow{\sim} \text{AnDM}^{-}(S, Z) \quad (100)$$

is an isomorphism and $\text{Tr}^* : \text{AnDM}^{-}(S, Z) \xrightarrow{\sim} \text{AnDA}^{-}(S, Z)$ is it inverse.

(ii) The adjunction $e_{\text{an}}(S)^*, e_{\text{an}}(S)_* : C^{-}(\mathbb{Z}) \to \text{PSh}_{\mathbb{Z}}(\text{AnSp}(\mathbb{C}), C^{-}(\mathbb{Z}))$ is a Quillen equivalence for the $(\mathbb{I}^1, \text{usu})$ model structures. That is, the derived functor

$$e_{\text{an}}^* : D^{-}(S) \xrightarrow{\sim} \text{AnDA}^{-}(S, Z) \quad (101)$$

is an isomorphism and $\text{Re}_{\text{an}}(S)_* : \text{AnDA}^{-}(S, Z) \xrightarrow{\sim} D^{-}(S)$ is it inverse.
(iii) The adjunction \((e_{an}^{tr}(S))^*, e_{an}^{tr}(S)_*): C^-(S) \cong \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}^{sm}(\mathbb{C}))/S), C^-(Z))\) is a Quillen equivalence for the \((\mathbb{D}^1, usu)\) model structures. That is, the derived functor
\[
e_{an}^{tr}(S)^*: D^-(S) \xrightarrow{\sim} \text{AnDM}^-(S, Z)
\]
is an isomorphism and \(R e_{an}^{tr}(S)_*: \text{AnDM}^-(S, Z) \xrightarrow{\sim} D^-(S)\) is it inverse.

Proof. (i): Similar to the proof of the absolute case.
(ii): It is proved in [1].
(iii): Follows from (i) and (ii).

Let \(f: T \to S\) a morphism in \(\text{AnSp}(\mathbb{C})\). There is a canonical morphism of functor \(\phi(f^*, S)\) which associate to \(F^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(Z))\) the morphism
\[
\phi(f^*, S)(F^*): f^*\text{sing}_{\mathbb{D}^1}, F^* \to \text{sing}_{\mathbb{D}^1}, f^*F^*
\]
in \(\text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}(\mathbb{C})^{sm}/T), C^-(Z))\) given by for \(Y/T \in \text{AnSp}(\mathbb{C})^{sm}/T\), the morphism
\[
\phi(f^*, S)(F^*)(Y/T): \lim_{Y/T \to T} F^*(X \times \mathbb{D}^*/S) \to \lim_{Y \times \mathbb{D}^*/T \to T} F^*(X/S)
\]
given by \((h: Y/T \to X_T/T) \mapsto (h \circ p_Y : Y \times \mathbb{D}^*/T \to X_T/T)\) and \(F^*(p_X) : (X \times \mathbb{D}^*/S) \to F^*(X/S)\).

Let \(f: T \to S\) a morphism in \(\text{AnSp}(\mathbb{C})\). There is also canonical morphism of functor \(\phi(f, e_{an})\) which associate to \(F^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(Z))\) the following morphism in \(C^-(T)\)
\[
\phi(f^*, e)(F^*) : f^*e_S, F^* \to e_T, e_T^*f^*e(S)_*, F^* = e_T, f^*e_Z, e_S, F^* \to e_T, f^*F^*
\]
given by the adjunction morphisms and denoting for simplicity \(e_S = e_{an}^{tr}(S)\) and \(e_T = e_{an}^{tr}(T)\).

Let \(f: T \to S\) a morphism in \(\text{AnSp}(\mathbb{C})\). We denote by \(\phi(f^*, e_{an}, S)\) the morphism of functor, which for \(F^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(Z))\), associate the following composition in \(C^-(T)\)
\[
\phi(f^*, e_{an}, S)(F^*) : f^*\text{sing}_{\mathbb{D}^1}, F^* \xrightarrow{\phi(f^*, e)(\text{sing}_{\mathbb{D}^1}, F^*)} e(T)_*, f^*\text{sing}_{\mathbb{D}^1}, F^* \xrightarrow{\phi(f^*, S)(F^*)} \text{sing}_{\mathbb{D}^1}, f^*F^*.
\]

By definition, we have:

**Proposition 36.** Let \(f: T \to S\) a morphism in \(\text{AnSp}(\mathbb{C})\). For \(F^* \in PC^-(\text{An}, S)\), the morphism \(\phi(f^*, e, S)(F^*) : f^*\text{sing}_{\mathbb{D}^1}, F^* \to \text{sing}_{\mathbb{D}^1}, f^*F^*\) in \(C^-(T)\) is an isomorphism.

Proof. By definition, the morphism \(\phi(f^*, e, S)(F^*)\) is given by for \(T^o \subseteq T\),
\[
\phi(f^*, e, S)(F^*)(T^o/T) : \lim_{T^o \to f^{-1}(S)} F^*(S^o \times \mathbb{D}^*/S) \to \lim_{T^o \times \mathbb{D}^*/T \to T} F^*(X/S)
\]
given by the isomorphism \((h: T^o/T \to X_T/T) \mapsto (h \circ p_T : T^o \times \mathbb{D}^*/T \to X_T/T)\) and \(F^*(p_T) : F^*(S^o \times \mathbb{D}^*/S) \to F^*(S^o/S)\).

We will use in the last subsection the following proposition:

**Proposition 37.**
(i) Let \(G_1^*, G_2^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(Z))\) and \(f: G_1^* \to G_2^*\) a morphism. If
\[
e_{an}^{tr}(G_2^*)(f) : \text{sing}_{\mathbb{D}^1}, G_1^* \to \text{sing}_{\mathbb{D}^1}, G_2^*
\]
is an equivalence usu local in \(C^-(S)\), then \(f: G_1^* \to G_2^*\) is an \((\mathbb{D}^1, usu)\) local equivalence.

(ii) Let \(G_1^*, G_2^* \in \text{PSh}_{\mathbb{Z}}(\text{Cor}_{\mathbb{Z}}^{fs}(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(Z))\) and \(f: G_1^* \to G_2^*\) a morphism. If
\[
- G_1^*, G_2^* \text{ are } \mathbb{D}^1 \text{ local},
- e_{an}^{tr}(G_2^*)(f) : e_{an}^{tr}(G_1^*) \to e_{an}^{tr}(G_2^*) \text{ is an equivalence usu local in } C^-(S),
\]
then \(f: G_1^* \to G_2^*\) is an \((\mathbb{D}^1, usu)\) local equivalence.

Proof. Similar to the proof of proposition [15] Point (i) follows from theorem [24](iii), and point (ii) follows from point (i).
4.3 Presheaves and transfers on relative CW complexes

Let $S \in \text{Top}$. The category $\text{Top}/S$ is the category whose objects are $X/S = (X,h)$ with $X \in \text{Top}$ and $h : X \to S$ is a morphism, and whose space of morphisms between $X/S = (X,h_1)$ and $Y/S = (Y,h_2) \in \text{Top}/S$ are the morphism $g : X \to Y$ such that $h_2 \circ g = h_1$.

We now restrict to the full subcategory $\text{CW}/S \subset \text{Top}$ of CW complexes.

**Definition 36.** A morphism $f : X \to Y$, $X,Y \in \text{CW}$, with $X$ connected is said to be smooth if for all $x \in X$, there exist a neighborhood $U_x \subset X$ of $x$ in $X$ and an open embedding $k : U_x \hookrightarrow U_x \times \mathbb{R}^{dx}$ such that $f|_{U_x} = p_{U_x} \circ k$.

Let $S \in \text{CW}$. The category $\text{CW}/S$ is the category whose objects are $X/S = (X,h)$ with $X \in \text{CW}$ and $h : X \to S$ is a morphism, and whose space of morphisms between $X/S = (X,h_1)$ and $Y/S = (Y,h_2) \in \text{CW}/S$ are the morphism $g : X \to Y$ such that $h_2 \circ g = h_1$. We denote by $\text{CW}^{\text{sm}}/S \subset \text{CW}/S$ the full subcategory consisting of the objects $X/S = (X,h)$ with $X \in \text{CW}$, such that $h : X \to S$ is a smooth morphism. For $X/S = (X,h) \in \text{CW}/S$, and $n \in \mathbb{N}$, we denote by

- $(X \times I^n/S) := (X \times I^n, h \circ p_X) = (X \times_S (I^n \times S)/S)$, where $p_X : X \times k I^n \to X$ is the projection.
- $(X \times \mathbb{I}^n/S) := (X \times \mathbb{I}^n, h \circ p_X) = (X \times_S (I^n \times S)/S)$, where $p_X : X \times \mathbb{I}^n \to X$ is the projection.

**Definition 37.** Let $S \in \text{CW}$. We define $\text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S)$ to be the category whose objects are the one of $\text{CW}^{\text{sm}}/S$ and whose space of morphisms between $X/S$ and $Y/S \in \text{CW}^{\text{sm}}/S$ is the free $\Lambda$ module $Z^{f/x}_{S/Y}(X \times_S Y, \Lambda)$. The composition of morphisms is defined similarly then in the absolute case.

We have

- the additive embedding of categories $\text{Tr}(S) : \mathbb{Z}^{(\text{CW}^{\text{sm}}/S)} \hookrightarrow \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S)$ which gives the corresponding morphism of sites $\text{Tr}(S) : \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S) \to \mathbb{Z}^{(\text{CW}^{\text{sm}}/S)}$.
- the inclusion functor $e_{\text{cw}}(S) : \text{Ouv}(S) \hookrightarrow \text{CW}^{\text{sm}}/S$, which gives the corresponding morphism of sites $e_{\text{cw}}(S) : \text{CW}^{\text{sm}}/S \to \text{Ouv}(S)$,
- the inclusion functor $e^f_{\text{cw}}(S) := \text{Tr} \circ e_{\text{cw}} : \text{Ouv}(S) \hookrightarrow \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S)$ which gives the corresponding morphism of sites $e^f_{\text{cw}}(S) := \text{Tr} \circ e_{\text{cw}} : \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S) \to \text{Ouv}(S)$.

For each morphism $f : T \to S$ in $\text{CW}$, we have

- the pullback functor $P(f) : \text{CW}/S \to \text{CW}/T$, $P(f)(X/S) = (X \times_S T/T)$, $P(f)(h) = h_T$, which gives the morphism of sites $P(f) : \text{CW}/T \to \text{CW}/S$.
- the pullback functor $P(f) : \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S) \to \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/T)$, $P(f)(X/S) = (X \times_S T/T)$, $P(f)(h) = h_T$, which gives the morphism of sites $P(f) : \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/T) \to \text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S)$.

For $S \in \text{CW}$, we consider the following two big categories:

- $\text{PSh}^{(\text{CW}^{\text{sm}}/S, C^-(Z))} = \text{PSh}_{\mathbb{Z}}(\mathbb{Z}^{(\text{CW}^{\text{sm}}/S), C^-(Z)})$, the category of bounded above complexes of presheaves on $\text{CW}^{\text{sm}}/S$, or equivalently additive presheaves on $\mathbb{Z}^{(\text{CW}^{\text{sm}}/S)}$, sometimes, we will write for short $P^-(\text{CW}, S) = \text{PSh}^{(\text{CW}^{\text{sm}}/S, C^-(Z))}$,
- $\text{PSh}_{\mathbb{Z}}^{(\text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S), C^-(Z))}$, the category of bounded above complexes of additive presheaves on $\text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S)$ sometimes, we will write for short $PC^-(\text{CW}, S) = \text{PSh}^{(\text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S), C^-(Z))}$

and the adjunctions:

- $(\text{Tr}(S)^*, \text{Tr}(S)_*) : \text{PSh}^{(\text{CW}^{\text{sm}}/S, C^-(Z))} \rightleftarrows \text{PSh}_{\mathbb{Z}}^{(\text{Cor}^f_{\Lambda}(\text{CW}^{\text{sm}}/S), C^-(Z))}$,
- $(e_{\text{cw}}(S)^*, e_{\text{cw}}(S)_*) : \text{PSh}(\text{SmVar}(\mathbb{C}), C^-(Z)) \rightleftarrows C^-(\mathbb{Z})$,
\begin{itemize}
\item \((e_{cw}^r(S^*), e_{cw}^r(S)) : PSh(Cor^f_Z(CW^{sm}/S), C^-(Z)) \Rightarrow C^-(Z)\),
\end{itemize}
given by \(\text{Tr}(S) : Cor^f_Z(CW^{sm}/S) \rightarrow Z(CW^{sm}/S)\), \(e_{cw}(S) : CW^{sm}/S \rightarrow \text{Ouv}(S)\) and \(e_{cw}^r(S) : Cor^f_Z(CW^{sm}/S) \rightarrow \text{Ouv}(S)\) respectively. We denote by \(d_{ct} : PSh_Z(CW^{sm}/S, Ab) \rightarrow Sh_{Z,ct}(CW^{sm}/S, Ab)\) the etale sheafification functor. For \(X/S \in CW^{sm}/S\), we denote by
\[
Z(X/S) \in PSh(CW^{sm}/S, C^-(Z)), \text{ } Z_{tr}(X/S) \in PSh_Z(Cor^f_Z(CW^{sm}/S), C^-(Z)) \quad (103)
\]
the presheaves represented by \(X\). They are usu sheaves. For \(X/S = (X, h) \in CW/S\) with \(h : X \rightarrow S\) non smooth,
\[
Z_{tr}(X/S) \in PSh_Z(Cor^f_Z(CW^{sm}/S), C^-(Z)), \text{ } Y \in CW^{sm}/S \rightarrow Z^{fs/Y}(Y \times_S X, Z) \quad (104)
\]
is also an usu sheaf; of course if \(h : X \rightarrow S\) is not dominant then \(Z_{tr}(X/S) = 0\).

For a morphism \(f : T \rightarrow S\) in CW we have the adjonctions
\begin{itemize}
\item \((f^*, f_*) : PSh(CW^{sm}/S, C^-(Z)) \Rightarrow PSh(CW^{sm}/T, C^-(Z))\),
\item \((f^*, f_*) : PSh_Z(Cor^f_Z(CW^{sm}/S), C^-(Z)) \Rightarrow PSh_Z(Cor^f_Z(CW^{sm}/T), C^-(Z))\),
\end{itemize}
given by \(P(f) : CW^{sm}/T \rightarrow CW^{sm}/S\) and \(P(f) : Cor^f_Z(CW^{sm}/T) \rightarrow Cor^f_Z(CW^{sm}/S)\), respectively.

**Definition 38.**
(i) The projective usual topology model structure on \(PSh(CW^{sm}/S, C^-(Z))\) is defined in the similar way of the absolute case (c.f. definition 12(i)).

(ii) The projective \((I^1, usu)\) model structure on the category \(PSh(CW^{sm}/S, C^-(Z))\) is the left Bousfield localization of the projective usual topology model structure with respect to the class of maps \(\{Z(X \times I^1/S)[n] \rightarrow Z(X/S)[n], X/S \in CW^{sm}/S, n \in \mathbb{Z}\}\).

(iii) The projective usual topology model structure on \(PSh(Cor^f_Z(CW^{sm}/S), C^-(Z))\) is defined in the similar way of the absolute case (c.f. definition 12(ii)).

(iv) The projective \((I^1, usu)\) model structure on the category \(PSh_Z(Cor^f_Z(CW^{sm}/S), C^-(Z))\) is the left Bousfield localization of the projective usual topology model structure with respect to the class of maps \(\{Z_{tr}(X \times I^1/S)[n] \rightarrow Z(X/S)[n], X/S \in CW^{sm}/S, n \in \mathbb{Z}\}\).

**Definition 39.** Let \(S \in CW\).

(i) We define \(\text{CwDM}^-(S, Z) := \text{Ho}_{I^1,usu}(PSh_Z(Cor^f_Z(CW^{sm}/S), C^-(Z)))\), to be the derived category of (effective) motives, it is the homotopy category of \(PSh(Cor^f_Z(CW^{sm}/S), C^-(Z))\) with respect to the projective \((I^1, usu)\) model structure (c.f. definition 13(ii)). We denote by
\[
D^{tr}(I^1, usu)(S) : PSh_Z(Cor^f_Z(CW^{sm}/S, C^-(Z))) \rightarrow \text{CwDM}^-(S, Z), \text{ } D^{tr}(I^1, usu)(S)(F^*) = F^*
\]
the canonical localization functor.

(ii) By the same way, we denote \(\text{CwDA}^-(S, Z) := \text{Ho}_{I^1,usu}(PSh(CW^{sm}/S, C^-(Z)))\) (c.f.13(i)) and
\[
D(I^1, usu)(S) : PSh_Z(Cor^f_Z(CW^{sm}/S), C^-(Z))) \rightarrow \text{CwDA}^-(S, Z), \text{ } D(I^1, usu)(S)(F^*) = F^*
\]
the canonical localization functor.

We now look at an explicit localization functor.

For \(F^* \in PSh(CW^{sm}/S, Ab)\) and \(X/S \in CW^{sm}/S\), we have the complex \(F(X \times I^*/S)\) associated to the cubical object \(F(X \times I^*/S)\) in the category of abelian groups.
If $F \in PSh(CW^{sm}/S, C^{-}(\mathbb{Z}))$, 

$$\text{sing}_{s}F := \text{Tot}(\text{Hom}(\mathbb{Z}(I^{n} \times S), F)) \in PSh(CW^{sm}/S, C^{-}(\mathbb{Z}))$$

(105)

is the total complex of presheaves associated to the bicomplex of presheaves $X/S \mapsto F^{*}(I^{*} \times X/S)$, and $\text{sing}_{t}F := \epsilon_{w}(S) \text{sing}_{s}F \in C^{-}(\mathbb{Z})$. We denote by $S(F^{*}) : F \to \text{sing}_{s}F$, 

$$S(F^{*}) : \cdots \to 0 \to 0 \to F^{*} \to 0 \to \cdots$$

(106)

the inclusion morphism of $PSh(CW^{sm}/S, C^{-}(\mathbb{Z}))$: For $f : F^{*} \to F_{2}^{*}$ a morphism $PSh(CW^{sm}/S, C^{-}(\mathbb{Z}))$, we denote by $S(f) : \text{sing}_{s}F^{*} \to \text{sing}_{s}F_{2}^{*}$, the morphism of $PSh(CW^{sm}/S, C^{-}(\mathbb{Z}))$ given by for $X/S \in CW^{sm}/S$, 

$$S(f)(X/S) : \cdots \to F_{1}(I^{2} \times X/S) \to F_{1}(I^{1} \times X/S) \to F_{1}^{*}(X/S) \to 0 \to \cdots$$

(107)

If $F^{*} \in PSh_{2}(Cor_{Z}^{fs}(CW^{sm}/S), C^{-}(\mathbb{Z}))$, 

$$\text{sing}_{s}F := \text{Hom}(\mathbb{Z}_{tr}(I^{n} \times S), F^{*}) \in PSh_{2}(Cor_{Z}^{fs}(CW^{sm}/S), C^{-}(\mathbb{Z})),$$

(108)

is the complex of presheaves associated to the bicomplex of presheaves $X/S \mapsto F^{*}(I^{*} \times X/S)$, and $\text{sing}_{s}F := \epsilon_{tr}(S) \text{sing}_{s}F \in C^{-}(\mathbb{Z})$. We have the inclusion morphism 

$$S(Tr_{s}F^{*}) : Tr(S)_{s}F^{*} \to \text{sing}_{s}, \ Tr(S)_{s}F^{*} \to \text{sing}_{s}, F^{*}$$

which is a morphism in $PSh(\text{Cor}_{Z}^{fs}(CW^{sm}/S), C^{-}(\mathbb{Z}))$ denoted the same way $S(F^{*}) : F^{*} \to \text{sing}_{s}F^{*}$. For $f : F_{1}^{*} \to F_{2}^{*}$ a morphism $PC^{-}(S)$, we have the morphism 

$$S(Tr_{s}f) : Tr(S)_{s}F_{1}^{*} = \text{sing}_{s}, Tr(S)_{s}F_{1}^{*} \to \text{sing}_{s}, Tr(S)_{s}F_{2}^{*} = \text{sing}_{s}, F_{2}^{*}$$

which is a morphism in $PC^{-}(S)$ denoted the same way $S(f) : \text{sing}_{s}F_{1}^{*} \to \text{sing}_{s}F_{2}^{*}$.

For $F^{*} \in PSh_{2}(Cor_{Z}^{fs}(CW^{sm}/S), C^{-}(\mathbb{Z}))$, we have by definition $Tr(S)_{s}F^{*} = \text{sing}_{s}, Tr(S)_{s}F^{*}$ and $Tr(S)_{s}S(F^{*}) = S(Tr(S)_{s}F^{*}).$

We recall the definition of homotopy in the relative setting:

- Let $S, X, Y \in \text{Top}$ and $h_{X} : X \to S$, $h_{Y} : Y \to S$ two maps. Let $f_{0} : X \to Y$, $f_{1} : X \to Y$ be two maps such that $h_{Y} \circ f_{0} = h_{X}$ and $h_{Y} \circ f_{1} = h_{X}$, that is such that they define morphisms from $X/S = (X, h_{X})$ to $Y/S = (Y, h_{Y})$ in Top/S. We say that $f_{0}$, $f_{1}$ are $I^{1}$ homotopic, if there exist $h : X \times I^{1} \to Y$ such that
  - $h_{Y} \circ h = h_{X} \circ p_{X}$ that is $h : X \times I^{1}/S \to Y/S$ is a morphism in Top/S,
  - $f_{0} = h \circ (I_{X} \times i_{0})$ and $f_{1} = h \circ (I_{X} \times i_{1})$,

with $(I_{X} \times i_{0}) : X \times \{0\} \to X \times I^{1}$ and $(I_{X} \times i_{1}) : X \times \{1\} \to X \times I^{1}$ the inclusions and $p_{X} : X \times I^{1} \to X$ the projection.
• Let $S \in CW$. Let $F^\bullet, G^\bullet \in \text{PSh}(CW / S, C^{-}(\mathbb{Z}))$. We say that two maps $\phi_0 : F^\bullet \to G^\bullet$ and $\phi_1 : F^\bullet \to G^\bullet$ are $I^1$ homotopic if there exist $\tilde{\phi} : F^\bullet \to \text{Hom}(\mathbb{Z}(I^1 \times S), G^\bullet)$ such that $\phi_0 = G^\bullet(i_0) \circ \tilde{\phi}$ and $\phi_1 = G^\bullet(i_1) \circ \tilde{\phi}$, where,

\[ G^\bullet(i_0) \circ \text{Hom}(\mathbb{Z}(I^1), G^\bullet) \to G^\bullet \]

is induced by $i_0 : \{\text{pt}\} \to I^1$, that is, for $X/S \in CW / S,$

\[ G^\bullet(i_0)(X/S) = G^\bullet(I_X \times i_0) : G^\bullet(X \times I^1) \to G^\bullet(X/S), \]

and

\[ G^\bullet(i_1) \circ \text{Hom}(\mathbb{Z}(I^1), G^\bullet) \to G^\bullet \]

is induced by $i_1 : \{\text{pt}\} \to I^1$, that is, for $X/S \in CW / S,$

\[ G^\bullet(i_1)(X/S) = G^\bullet(I_X \times i_1) : G^\bullet(X \times I^1) \to G^\bullet(X/S), \]

Similarly to the absolute case, we have the following lemmas:

**Lemma 12.** Let $X/S, Y/S \in CW / S$ and $f_0 : X/S \to Y/S$, $f_1 : X/S \to Y/S$ two morphisms. If $f_0$ and $f_1$ are $I^1$ homotopic, then

- $Z(f_0) : Z(X/S) \to Z(Y/S)$ and $Z(f_1) : Z(X/S) \to Z(Y)$ are $I^1$ homotopic in $\text{PSh}(CW / S, C^{-}(\mathbb{Z}))$.
- $\text{Tr}_s Z_{tr}(f_0) : \text{Tr}_s Z_{tr}(X/S) \to \text{Tr}_s Z_{tr}(Y/S)$ and $\text{Tr}_s Z_{tr}(f_1) : \text{Tr}_s Z_{tr}(X/S) \to \text{Tr}_s Z_{tr}(Y/S)$ are $I^1$ homotopic in $\text{PSh}(CW / S, C^{-}(\mathbb{Z}))$.

**Proof.** Similar to the absolute case.

**Lemma 13.** Let $S \in CW$. Let $F^\bullet \in \text{PSh}(CW / S, C^{-}(\mathbb{Z}))$.

(i) Let $X/S, Y/S \in CW / S$ and $f_0 : X/S \to Y/S$, $f_1 : X/S \to Y/S$ be two morphism. If $f_0$ and $f_1$ are $I^1$ homotopic, then the maps of complexes

\[ \text{sing}, F^\bullet(f_0) : \text{Tot} F^\bullet(Y \times I^1 / S) \to \text{Tot} F^\bullet(X \times I^1 / S) \]

and

\[ \text{sing}, F^\bullet(f_1) : \text{Tot} F^\bullet(Y \times I^1 / S) \to \text{Tot} F^\bullet(X \times I^1 / S) \]

induces the same map on homology.

(ii) Let $X/S, Y/S \in CW / S$, if $f : X \to Y$ is a $I^1$ homotopy equivalence then

\[ \text{sing}, F^\bullet(f) : \text{Tot} F^\bullet(Y \times I^1 / S) \to \text{Tot} F^\bullet(X \times I^1 / S) \]

is a quasi-isomorphism of complexes of abelian groups.

(iii) Let $F^\bullet, G^\bullet \in \text{PSh}(CW / S, C^{-}(\mathbb{Z}))$ and $\phi_0 : F^\bullet \to G^\bullet$, $\phi_1 : F^\bullet \to G^\bullet$ be two maps. If $\phi_0$ and $\phi_1$ are $I^1$ homotopic, then $\phi_0 = \phi_1 \in \text{CwDA}^{-}(S)$.

(iv) Let $F^\bullet, G^\bullet \in \text{PSh}(CW / S, C^{-}(\mathbb{Z}))$, if $\phi : F^\bullet \to G^\bullet$ is a $I^1$ homotopy equivalence then $\phi$ is a $I^1, \text{usu}$ local equivalence.

**Proof.** Similar to the absolute case.

**Lemma 14.** (i) A complex of presheaves $F^\bullet \in \text{PSh}(\text{Cor}^f_Z(CW^{-}(S), C^{-}(\mathbb{Z})))$ is $I^1$ local if and only if $\text{Tr}(S), F^\bullet \in \text{PSh}(\text{Cor}^{-}(S), C^{-}(\mathbb{Z}))$ is $I^1$ local.

(ii) A morphism $\phi : F^\bullet \to G^\bullet$ in $\text{PSh}(\text{Cor}^f_Z(CW^{-}(S), C^{-}(\mathbb{Z})))$ is an $I^1, \text{usu}$ local equivalence if and only if $\text{Tr}(S), \phi : \text{Tr}(S), F^\bullet \to \text{Tr}, G^\bullet$ is an $I^1, \text{usu}$ local equivalence.

**Proof.** (i): Similar to the absolute case.

(ii): As in the absolute case the only if part follows from lemma $\text{Loc}(ii)$ and the if part follows from (i).

**Proposition 38.** Let $S \in CW$
(i) \( (\text{Tr}(S)^*, \text{Tr}(S)_*) : \text{PSh}(\text{CW}^{sm}/S, C^- (Z)) \Rightarrow \text{PSh}_Z(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/S), C^- (Z)) \) is a Quillen adjunction for the etale topology model structures and a Quillen adjunction for the \((\mathcal{I}, \text{et})\) model structures (c.f. definition 38 (i) and (ii) respectively).

(ii) \( (e_{\text{cw}}(S)^*, e_{\text{cw}}(S)_*) : \text{PSh}(\text{CW}^{sm}/S, C^- (Z)) \Rightarrow C^- (Z) \) is a Quillen adjunction for the etale topology model structures and a Quillen adjunction for the \((\mathcal{I}, \text{et})\) model structures (c.f. definition 38 (ii)).

(iii) \( (e_{\text{cw}}(S)^{tr*}, e_{\text{cw}}(S)_*) : \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/S), C^- (Z)) \Rightarrow C^- (Z) \) is a Quillen adjunction for the etale topology model structures and a Quillen adjunction for the \((\mathcal{I}, \text{et})\) model structures (c.f. definition 38 (ii)).

**Proof.** (i): Follows from the fact that \( \text{Tr}(S)_* \) by lemma 14 derive trivially hence is a right Quillen functor.

(ii): Follows from the fact that \( e_{\text{cw}}(S)^* \) derive trivially hence is a left Quillen functor.

(iii): Follows from the fact that \( e_{\text{cw}}(S)^{tr*} \) derive trivially hence is a left Quillen functor.

\[
\text{Proposition 39. Let } f : T \rightarrow S \text{ a morphism in } \text{CW}.
\]

(i) The functors

\[
f_* : \text{PSh}(\text{CW}^{sm}/T, C^- (Z)) \rightarrow \text{PSh}(\text{CW}^{sm}/S, C^- (Z)) \text{ and }
f_* : \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/T), C^- (Z)) \rightarrow \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/S), C^- (Z)),
\]

preserve and detect \( \mathcal{I}^1 \) local object, usu fibrant objects.

(ii) The functors

\[
f^* : \text{PSh}(\text{CW}^{sm}/S, C^- (Z)) \rightarrow \text{PSh}(\text{CW}^{sm}/T, C^- (Z)) \text{ and }
f^* : \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/S), C^- (Z)) \rightarrow \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/T), C^- (Z)),
\]

preserve and detect \( \mathcal{I}^1, \text{usu} \) equivalence.

**Proof.** Point (i) follows immediately from definition of \( \mathcal{I}^1 \) local objects. Point (ii) follows from point (i).

We immediately deduce the following

**Proposition 40. Let } f : T \rightarrow S \text{ a morphism in } \text{CW}.

(i) The adjunction \( (f^*, f_*) : \text{PSh}(\text{CW}^{sm}/S, C^- (Z)) \Rightarrow \text{PSh}(\text{CW}^{sm}/T, C^- (Z)) \) is a Quillen adjunction with respect to the usu model structures and the \((\mathcal{I}, \text{usu})\) model structures

(ii) The adjunction \( (f^*, f_*) : \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/S), C^- (Z)) \Rightarrow \text{PSh}(\text{Cor}^f_{\text{Z}}(\text{CW}^{sm}/T), C^- (Z)), \) is a Quillen adjunction with respect to the usu model structures and the \((\mathcal{I}, \text{usu})\) model structures

**Proof.** (i): By 14, \( f^* \) derive trivially for the usual topology model structure. Now, by proposition 39 (ii), \( f^* \) derive trivially for the \((\mathcal{I}, \text{usu})\) model structure. In particular, \( f^* \) is a left Quillen functor.

(ii): Similar to point (i).

We deduce from lemma 13 (ii), the point (i) of the following proposition:

**Proposition 41. Let } S \in \text{CW}

(i) For \( F^* \in \text{PSh}(\text{CW}^{sm}/S, C^- (Z)) \), the adjunction morphism

\[
\text{ad}(e_{\text{cw}}(S)^*, e_{\text{cw}}(S)_*)(\text{sing}_{\text{cw}}, F^*) : e_{\text{cw}}(S)^* e_{\text{cw}}(S)_*(\text{sing}_{\text{cw}}, F^*) \rightarrow \text{sing}_{\text{cw}}, F^* \tag{110}
\]

is an equivalence usu local.
(ii) For $F^\bullet \in \text{PSh}_Z(\text{Cor}^{\text{fs}}_Z(\text{CW}^\text{sm}/S), C^- (Z))$, the adjunction morphism

$$\text{ad}(e^{tr}_{cw}(S)^*, e_{cw}(S)_*)(\text{sing}_I, F^\bullet) : e^{tr}_{cw}(S)^* e_{cw}(S)_* \text{sing}_I, F^\bullet \to \text{sing}_I, F^\bullet$$

is an equivalence usu local.

(iii) For $K^\bullet \in C^- (Z)$, the adjunction morphisms

$$- \text{ad}(e_{cw}(S)^*, e_{cw}(S)_*)(K^\bullet) : K^\bullet \to e_{cw}(S)^* e_{cw}(S)_* K^\bullet$$
$$- \text{ad}(e^{tr}_{cw}(S)^*, e^{tr}(S)_*)(K^\bullet) : K^\bullet \to e^{tr}_{cw}(S)^* e^{tr}(S)_* K^\bullet$$

are isomorphisms.

Proof. (i): Let $X/S \in \text{CW}^\text{sm}/S$. Since the question is local, we can assume, after shrinking $X/S$, that $X/S = (V \times S^*, p_S)$, with $S^* \subset S$ an open subset and $V \subset \mathbb{R}^{d_X}$ a contractible open subset and $p_S : \mathbb{R}^{d_X} \to S$ the projection. It then follows from lemma \[\square\] ii).

(ii): It is a particular point of (i), since $\text{Tr}(S)_*$ preserve usu local equivalences.

(iii): Obvious \[\square\]

Proposition 42. Let $S \in \text{CW}$

(i) The functor $\text{Tr}(S)_* : \text{PSh}_Z(\text{Cor}^{\text{fs}}_Z(\text{CW}), C^- (Z)) \to \text{PSh}(\text{CW}, C^- (Z))$ derive trivially.

(ii) For $K^\bullet \in C^- (S)$, $e_{cw}(S)^* K^\bullet$ is $I_1$ local.

(iii) For $K^\bullet \in C^- (S)$, $e^{tr}_{cw}(S)^* K^\bullet$ is $I_1^1$ local.

Proof. (i): Follows from lemma \[\square\] ii).

(ii): Let $e_{cw}(S)^* K^\bullet \to L^\bullet$ an usu local equivalence, with $L^\bullet$ usu fibrant. Since $e_{cw}(S)^* K^\bullet$ is usu equivalent to $L^\bullet$ it suffices to prove that $L^\bullet$ is $I_1$ local. Since $L^\bullet$ is usu fibrant, we have to prove that

$$L(p) : L^\bullet \to \text{Hom}(\mathbb{Z}(S \times I_1^1), L^\bullet)$$

is an equivalence usu local The proof is now similar to \[\square\] proposition 1.6 etape B.

(iii): Follows from (ii) and lemma \[\square\] i).

Theorem 25. Let $S \in \text{CW}$

(i) For $F^\bullet \in \text{PSh}(\text{CW}^\text{sm}/S, C^- (Z))$, $\text{sing}_I, F^\bullet \in \text{CW}^\text{sm}/S, C^- (Z))$ is $I_1$ local and the inclusion morphism $S(F^\bullet) : F^\bullet \to \text{sing}_I, F^\bullet$ is an $(I_1^1, \text{usu})$ equivalence.

(ii) For $F^\bullet \in \text{PSh}_Z(\text{Cor}^{\text{fs}}_Z(\text{CW}^\text{sm}/S), C^- (Z))$, $\text{sing}_I, F^\bullet \in \text{PSh}_Z(\text{Cor}^{\text{fs}}_Z(\text{CW}^\text{sm}/S), C^- (Z))$ is $I_1$ local and the inclusion morphism $S(F^\bullet) : F^\bullet \to \text{sing}_I, F^\bullet$ is an $(I_1, \text{usu})$ equivalence.

Proof. (i): As in the absolute case, the fact that $\text{sing}_I, F^\bullet$ is an $I_1$ local object follows from proposition \[\square\] i) and proposition \[\square\] ii). We now prove that $S(F^\bullet)$ is an $(I_1^1, \text{usu})$ local equivalence. As in the absolute case, it suffice to show that that for all $n \in \mathbb{Z}$, the morphism

$$F^\bullet (p_n) : F^\bullet \to \text{Hom}(\mathbb{Z}(I^n), F^\bullet), \quad X/S \in \text{CW}/S \to F^\bullet (p_n)(X/S) = F^\bullet (p_X) : F^\bullet (X/S) \to F^\bullet (I^n \times X/S)$$

is an equivalence $(I_1^1, \text{usu})$ local. For $X/S \in \text{CW}/S$, the morphism

$$\theta_{1,n}(X/S) : (I^n \times I^1 \times X)/S \to I^n \times X/S ; (t_1, \ldots, t_n, t_{n+1}, x) \mapsto (t_1 - t_n - 1, t_{1}, \ldots, t_n - t_{n+1}, x)$$

defines an $I_1$ homotopy from $I_{n+1} \times X$ to $0 \times X$ ; on the other side $p_X \circ (0 \times I_X) = I_X$, with $p_X : \mathbb{I}^n \times X \to X$ the projection. Thus, $F^\bullet (\theta_{1,n})$ define an $I_1$ homotopy from $F^\bullet (I_{n+1})$ to $F^\bullet (0)$ ; on the other side, $F^\bullet (0) \circ F^\bullet (p_n) = I$. This proves (i).

(ii): As in the absolute case, follows from (i). \[\square\]
Theorem 26. Let $S \in \text{CW}$. Then,

(i) The adjunction $(e_{cw}(S)^*, e_{cw}(S)_*): C^{-}(\mathbb{Z}) \rightleftharpoons \text{PSh}_{\mathbb{Z}}(\text{CW}^{sm} / S, C^{-}(\mathbb{Z}))$ is a Quillen equivalence for the $(I^1, \text{usu})$ model structures. That is, the derived functor

$$e_{cw}(S)^*: D^{-}(\mathbb{Z}) \rightsquigarrow \text{CwDA}^{-}(\mathbb{Z})$$

(112)

is an isomorphism and $\text{Re}_{cw}(S)_*: \text{CwDA}^{-}(\mathbb{Z}) \rightsquigarrow D^{-}(S)$ is it inverse.

(ii) The adjunction $(e^{tr}_{cw}(S)^*, e^{tr}_{cw}(S)_*): C^{-}(\mathbb{Z}) \rightleftharpoons \text{PSh}_{\mathbb{Z}}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$ is a Quillen equivalence for the $(I^1, \text{usu})$ model structures. That is, the derived functor

$$e^{tr}_{cw}(S)^*: D^{-}(\mathbb{Z}) \rightsquigarrow \text{CwDM}^{-}(S, \mathbb{Z})$$

(113)

is an isomorphism and $\text{Re}^{tr}_{cw}(S)_*: \text{CwDM}^{-}(S, \mathbb{Z}) \rightsquigarrow D^{-}(S)$ is it inverse.

(iii) The functor $\text{Tr}(S)_*: \text{PSh}(\text{CW}^{sm} / S, C^{-}(\mathbb{Z})) \rightarrow \text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$ induces an isomorphism and $\text{Tr}(S)_*: \text{CwDM}^{-}(S, \mathbb{Z}) \rightsquigarrow \text{CwDA}^{-}(S, \mathbb{Z})$.

Proof. (i): It follows from proposition (111(i) and theorem 25(i)).
(ii): It follows from proposition (111(ii) and theorem 25(ii)). It also follows from (i) and (ii).
(iii): It follows from (i) and (ii).

Remark 1. As in the absolute case, for $F^* \in \text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$, $\text{ad}(\text{Tr}^*, \text{Tr}_*)(F^*): F^* \rightarrow \text{Tr}^* \text{Tr}_* F^*$ is an isomorphism in $\text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$ and we can prove that for $X / S \in \text{CW}^{sm} / S$, the embedding

$$(\text{ad}(\text{Tr}^*, \text{Tr}_*)(\text{sing}_{*}, Z(X)) : \text{sing}_{*}, Z(X) \rightarrow \text{Tr}_* \text{sing}_{*}, Z_{tr}(X))$$

in $\text{PSh}(\text{CW}^{sm} / S, C^{-}(\mathbb{Z}))$ is an equivalence usu local. We would deduce from this that $I^1 \text{Tr}^*$ is the inverse of $\text{Tr}_*$. We will not do it since we don’t use it.

Let $f : T \rightarrow S$ a morphism in CW. There is a canonical morphism of functor $\phi(f^*, S)$ which associate to $F^* \in \text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$ the morphism

$$\phi(f^*, S)(F^*) : f^* \text{sing}_{*}, F^* \rightarrow \text{sing}_{*}, f^* F^*$$

in $\text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$ given by for $Y / T \in \text{CW}^{sm} / T$, the morphism

$$\phi(f^*, S)(F^*)(Y / T) : \lim_{Y / T \rightarrow X_T} F^*(X / Y) \rightarrow \lim_{Y / T \rightarrow X_T} F^*(X / S)$$

given by $(h : Y / T \rightarrow X_T) \mapsto (h \circ p_Y : Y \times \mathbb{I}^* / T \rightarrow X_T)$ and $F^*(p_X) : (X \times \mathbb{I}^* / S \rightarrow F^*(X / S)$.

Let $f : T \rightarrow S$ a morphism in CW. There is also canonical morphism of functor $\phi(f, e_{cw})$ which associate to $F^* \in \text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$ the morphism in $C^{-}(T)$

$$\phi(f^*, e)(F^*) : f^* e_S, F^* \rightarrow e_T, f^* e_S, F^* = e_T, f^* e_S, F^* \rightarrow e_T, f^* F^*;$$

given by the adjunction morphisms and denoting for simplicity $e_S = e_{cw}(S), e_T = e_{cw}(T)$.

Let $f : T \rightarrow S$ a morphism in CW. We denote by $\phi(f^*, e_{cw}, S)$ the morphism of functor, which for $F^* \in \text{PSh}(\text{Cor}^{fs}_{cw}(\text{CW}^{sm} / S), C^{-}(\mathbb{Z}))$, associate the following composition in $C^{-}(T)$

$$\phi(f^*, e_{cw}, S)(F^*) : f^* \text{sing}_{*}, F^* \rightarrow e(T)_* f^* \text{sing}_{*}, F^* \rightarrow e(T)_* \phi(f^*, S)(F^*) \rightarrow \text{sing}_{*}, f^* F^*.$$

By definition, we have:
Proposition 43. Let $f : T \to S$ a morphism in $\text{CW}$. For $F^* \in \text{PC}^-(\text{CW}, S)$, the morphism in $\text{C}^-(T)$ $\phi(f^*, e, S)(F^*) : f^* \text{sing}_T, F^* \to \text{sing}_T, f^*F^*$ is an isomorphism in $\text{C}^-(T)$.

Proof. By definition, the morphism $\phi(f^*, e, S)(F^*)$ is given by for $T^o \subset T$,

$$\phi(f^*, e, S)(F^*)(T^o/T) : \lim_{T^o \to F^{-1}(T^o)} F^*(S^o \times \mathbb{I}^*/S) \to \lim_{T^o \times \mathbb{I}^*/T \to X_T} F^*(X/S)$$

given by the isomorphism $(h : T^o/T \to X_T/T) \mapsto (h \circ p_T : T^o \times \mathbb{I}^*/T \to X_T/T)$ and $F^*(p_{S^o}) : F^*(S^o \times \mathbb{I}^*/S) \to F^*(S^o/S)$.

4.4 The relative Betti realisation functor

- For each $S \in \text{Var}(\mathbb{C})$ we have the analytical functor $\text{An}(S) : \text{Var}(\mathbb{C})/S \to \text{AnSp}(\mathbb{C})/S^{an}$ given by $(V/S) \mapsto V^{an}/S^{an}$ on objects and $g \mapsto g^{an}$ on morphisms, the analytical functor on transfers $\text{AnSp}(S) : \text{Cor}^f_S(\text{Var}(\mathbb{C})^{sm}/S) \to \text{Cor}_Z^f(\text{AnSp}(\mathbb{C})^{sm}/S^{an})$ given by $V/S \mapsto V^{an}/S^{an}$ on objects and $\Gamma \mapsto \Gamma^{an}$ on morphisms,

- For each $S \in \text{AnSp}(\mathbb{C})$, we have the forgetful functor $\text{Cw}(S) : \text{AnSp}(\mathbb{C})^{sm}/S \to \text{CW} / S^{cw}$ given by $W/S \mapsto W^{cw}/S^{cw}$ on objects and $g \mapsto g^{cw}$ on morphisms, and the forgetful functor on transfers $\text{Cw}(S) : \text{Cor}_Z^f(\text{AnSp}(\mathbb{C})^{sm}/S) \to \text{Cor}_Z^f(\text{CW}^{sm}/S^{cw})$ given by $W/S \mapsto W^{cw}/S^{cw}$ on objects and $\Gamma \mapsto \Gamma^{cw}$ on morphisms,

- For each $S \in \text{Var}(\mathbb{C})$, we have the composites $\widetilde{\text{Cw}}(S) = \text{Cw}(S^{an}) \circ \text{An}(S) : \text{Var}(\mathbb{C})/S \to \text{CW} / S$, given by $V/S \mapsto V^{cw}/S^{cw}$ on objects and $g \mapsto g^{cw}$ on morphisms, and $\text{Cw}(S) = \text{Cw}(S^{an}) \circ \text{An}(S) : \text{Cor}_Z^f(\text{Var}(\mathbb{C})^{sm}/S) \to \text{Cor}_Z^f(\text{CW} / S)$, given by $V/S \mapsto V^{cw}/S^{cw}$ and $\Gamma \mapsto \Gamma^{cw}$.

- For each $S \in \text{Var}(\mathbb{C})$ and $S' \in \text{AnSp}(\mathbb{C})$, we have the embeddings of categories $\iota_{an}(S') : \text{AnSp}(\mathbb{C})^{sm}/S' \to \text{AnSp}(\mathbb{C})/S'$ and $\iota_{var}(S) : \text{Var}(\mathbb{C})^{sm}/S \to \text{Var}(\mathbb{C})/S$.

By definition, for each $S \in \text{Var}(\mathbb{C})$, we have the following commutative diagram of sites $\text{DCat}(S)$

$$\text{DCat}(S) := \begin{array}{ccc}
\widetilde{\text{Cw}}(S) : \text{Cor}_Z^f(\text{CW}^{sm}/S^{cw}) & \xrightarrow{\text{Cw}(S)} & \text{Cor}_Z^f(\text{AnSp}(\mathbb{C})^{sm}/S^{an}) & \xrightarrow{\text{An}(S)} & \text{Cor}_Z^f(\text{Var}(\mathbb{C})^{sm}/S) \\
\text{Tr}(S) & & \text{Tr}(S) & & \text{Tr}(S) \\
\text{Cw}(S) : \mathbb{Z}(\text{CW}^{sm}/S^{cw}) & \xrightarrow{\text{Cw}(S)} & \mathbb{Z}(\text{AnSp}(\mathbb{C})^{sm}/S^{an}) & \xrightarrow{\text{An}(S)} & \mathbb{Z}(\text{Var}(\mathbb{C})^{sm}/S) \\
\mathbb{Z}(\text{AnSp}(\mathbb{C})^{sm}/S^{an}) & \xrightarrow{\iota_{an}(S)} & \mathbb{Z}(\text{Var}(\mathbb{C})^{sm}/S) & \xrightarrow{\iota_{var}(S)} & \mathbb{Z}(\text{Var}(\mathbb{C})/S) \\
\end{array}$$

(114)

For $T, S \in \text{Var}(\mathbb{C})$ and $f : T \to S$ a morphism, the morphism of sites

- $P(f) : \text{Var}(\mathbb{C})/T \to \text{Var}(\mathbb{C})/S, P(f^{an}) : \text{AnSp}(\mathbb{C})/T^{an} \to \text{AnSp}(\mathbb{C})/S^{an}$, and $P(f^{cw}) : \text{CW} / T^{cw} \to \text{CW} / S^{cw}$ given by the pullback functor,

- $P(f) : \text{Cor}_Z^f(\text{Var}(\mathbb{C})^{sm}/T) \to \text{Cor}_Z^f(\text{Var}(\mathbb{C})^{sm}/S), P(f^{an}) : \text{Cor}_Z^f(\text{AnSp}(\mathbb{C})/T^{an}) \to \text{Cor}_Z^f(\text{AnSp}(\mathbb{C})/S^{an})$, and $P(f^{cw}) : \text{Cor}_Z^f(\text{CW} / T^{cw}) \to \text{Cor}_Z^f(\text{CW}^{sm} / S^{cw})$ given by the pullback functor,

gives a morphism of diagram of sites

$$P(f) : \text{DCat}(T) \to \text{DCat}(S).$$

(115)

We have the relative analogue of proposition 24:

Proposition 44. (i) For $S \in \text{Var}(\mathbb{C})$, the functors
Theorem 27. The Betti realization functor define morphisms of homotopic 2-functors:

\[ \text{Bti}(S)^* : DA^-(S, \mathbb{Z}) \to \text{DA}^-(S^\text{an}, \mathbb{Z}) \] and

\[ \text{An}(S)^* : \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z})) \to \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S^{an}, C^-(\mathbb{Z})) \]

\[ \text{Bti}(S)^* : \text{DM}^-(S, \mathbb{Z}) \to \text{DM}^-(S^\text{an}, \mathbb{Z}) \]

\[ \text{An}(S)^* : \text{PSh}(\text{Cor}^S_2(\text{Var}(\mathbb{C})^{sm}/S), C^-(\mathbb{Z})) \to \text{PSh}(\text{Cor}^S_2(\text{AnSp}(\mathbb{C})^{sm}/S^{an}), C^-(\mathbb{Z})) \]

derive trivially for the \((\mathbb{A}^1, \text{et})\) and \((\mathbb{D}^1, \text{usu})\) model structures.

(ii) Let \( S \in \text{AnSp}(\mathbb{C}) \).

\[ \text{Let } K^* \in \text{PSh}(\text{Cor}^S_2(CW^{sm}/S^{cw}), C^-(\mathbb{Z})). \] If \( K^* \) is \( \mathbb{I}^1 \) local, then \( \text{Cw}(S)_* K^* \) is \( \mathbb{D}^1 \) local.
\[ \text{Let } K^* \in \text{PSh}(\text{CW}^{sm}/S^{cw}, C^-(\mathbb{Z})). \] If \( K^* \) is \( \mathbb{I}^1 \) local, then \( \text{Cw}(S)_* K^* \) is \( \mathbb{D}^1 \) local.

Moreover, the functors

\[ \text{Cw}(S)^* : \text{PSh}(\text{AnSp}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z})) \to \text{PSh}(\text{CW}^{sm}/S^{cw}, C^-(\mathbb{Z})) \] and

\[ \text{Cw}(S)^* : \text{PSh}(\text{Cor}^S_2(\text{AnSp}(\mathbb{C})^{sm}/S), C^-(\mathbb{Z})) \to \text{PSh}(\text{Cor}^S_2(\text{CW}^{sm}/S^{cw}), C^-(\mathbb{Z})) \]

derive trivially for the \((\mathbb{D}^1, \text{usu})\) and \((\mathbb{I}^1, \text{usu})\) model structures.

(iii) For \( S \in \text{Var}(\mathbb{C}) \), the functors

\[ \text{Cw}(S)^* : \text{PSh}(\text{Var}(\mathbb{C})^{sm}/S, C^-(\mathbb{Z})) \to \text{PSh}(\text{CW}^{sm}/S^{cw}, C^-(\mathbb{Z})) \] and

\[ \text{Cw}(S)^* : \text{PSh}(\text{Cor}^S_2(\text{Var}(\mathbb{C})^{sm}/S), C^-(\mathbb{Z})) \to \text{PSh}(\text{Cor}^S_2(\text{CW}^{sm}/S^{cw}), C^-(\mathbb{Z})) \]

derive trivially for the \((\mathbb{A}^1, \text{et})\) and \((\mathbb{I}^1, \text{usu})\) model structures.

Proof. The proof is completely similar to the proof of proposition 25 using lemma 13 (ii), since for \( X/S \in \text{AnSp}(\mathbb{C})/S, p_X : X^{cw} \times \mathbb{D}^1 \to X^{cw} \) is a homotopy equivalence in \( \text{CW}/S^{cw} \).

Now we have:

- The 2-functor \( \text{DM}^+ : \text{Var}(k) \to \text{TriCat} \) is an homotopic 2-functor the sense of [1] (theorem 22).
- The 2-functor \( \text{D}^+ : \text{Var}(\mathbb{C}) \to \text{TriCat}, S \in \text{Var}(\mathbb{C}) \mapsto \text{D}^-(S^{cw}) \). is an homotopic 2-functor (see [10]).

Definition 40. [1][3] Let \( S \in \text{Var}(\mathbb{C}) \),

(i) The Betti realisation functor (without transfers) is the composite:

\[ \text{Bti}_0(S)^* : \text{DA}^-(S, \mathbb{Z}) \xrightarrow{\text{An}(S)^*} \text{AnDA}^-(S^{an}, \mathbb{Z}) \xrightarrow{\text{Re}_{\text{an}}(S)^*} \text{D}^-(S^{an}) \]

(ii) The Betti realization functor with transfers is the composite:

\[ \text{Bti}(S)^* : \text{DM}^-(S, \mathbb{Z}) \xrightarrow{\text{An}(S)^*} \text{AnDM}^-(S^{an}, \mathbb{Z}) \xrightarrow{\text{Re}_{\text{an}}^T(S)^*} \text{D}^-(S^{an}) \]

Since \( \text{An}(S)^* \) derive trivially by proposition 14 (i) and and \( L \text{Tr}(S^{an})^* : \text{AnDA}^-(S^{an}, \mathbb{Z}) \to \text{CwDM}^-(S^{an}, \mathbb{Z}) \) is the inverse of \( \text{Tr}(S^{an})_* \) (c.f theorem 24 (i)), we have \( \text{Bti}_0(S)^* = \text{Bti}(S)^* \circ L \text{Tr}(S)^* \).

We have the following:

Theorem 27. [1] The Betti realization functor define morphisms of homotopic 2-functors:

- \( S \in \text{Var}(\mathbb{C}) \mapsto (\text{Bti}_0(S) : \text{DA}^-(S, \mathbb{Z}) \to \text{D}^-(S)) \)
- \( S \in \text{Var}(\mathbb{C}) \mapsto (\text{Bti}(S) : \text{DM}^-(S, \mathbb{Z}) \to \text{D}^-(S)). \)

As in the absolute case, we define:

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Definition 41. Let \( S \in \text{Var}(\mathbb{C}) \).

(i) The CW-Betti realization functor (without transfers) is the composite:

\[
\widetilde{\text{Bt}_0}(S)^* : \text{DA}^{-}(S, \mathbb{Z}) \xrightarrow{\text{Cw}(S)^*} \text{CwDA}^{-}(S^{\text{cw}}, \mathbb{Z}) \xrightarrow{\text{Rc}_{\text{cw}}(S)^*} D^{-}(S^{\text{cw}})
\]

(ii) The CW-Betti realization functor with transfers is the composite:

\[
\widetilde{\text{Bt}}(S)^* : \text{DM}^{-}(S, \mathbb{Z}) \xrightarrow{\text{Cw}(S)^*} \text{CwDM}^{-}(S^{\text{cw}}, \mathbb{Z}) \xrightarrow{\text{Rc}_{\text{cw}}^t(S)^*} D^{-}(S^{\text{cw}})
\]

Similarly, in the relative case, since \( \widetilde{\text{Cw}}(S)^* \) derive trivially by proposition 4(iii) and \( \text{LTr}(S^{\text{cw}})^* : \text{CwDA}^{-}(S^{\text{cw}}, \mathbb{Z}) \rightarrow \text{CwDM}^{-}(S^{\text{cw}}, \mathbb{Z}) \) is the inverse of \( \text{Tr}_{\text{T}} S^{\text{cw}}^{*} \) (c.f. remark 1), we have \( \text{Bt}_0(S)^* = \text{Bt}(S)^* \circ \text{LTr}(S)^* \).

We also have the following

Theorem 28. The Betti realization functor define morphisms of homotopic 2-functors:

- for \( S \in \text{AnSp}(\mathbb{C}) \), the morphism \( \psi_{\text{Cw}}(S) \) which, for \( G^* \in \text{PSh}(\text{Cor}^f_{\text{cw}}(\text{AnSp}(\mathbb{C})^{\text{sm}}/S), C^{-}(\mathbb{Z})) \), associate the morphism

\[
\psi_{\text{Cw}}(S)(G^*) : \text{Cw}(S)^{*}\text{sing}_{\text{an}} G^* \rightarrow \text{sing}_{\text{an}} \text{Cw}(S)^* G^*
\]

in \( \text{PSh}(\text{Cor}^f_{\text{cw}}(\mathbb{C}^+/S^{\text{cw}}), C^{-}(\mathbb{Z})) \); the morphism \( \psi_{\text{Cw}}(S)(G^*) \) is given by, for \( Z/S^{\text{cw}} \in \text{CW} / S^{\text{cw}} \),

\[
\psi_{\text{Cw}}(S)(Z/S^{\text{cw}}) : \lim_{Y^{\text{cw}}/S^{\text{cw}} \rightarrow Z/S^{\text{cw}}} \text{G}^* (X \times \mathbb{Y}^a_{\text{an}}/S) \rightarrow \lim_{Y^{\text{cw}}/S^{\text{cw}} \rightarrow Z/S^{\text{cw}} \times \mathbb{1}} \text{G}^* (Y/S)
\]

given by \( (f : X^{\text{cw}}/S^{\text{cw}} \rightarrow Z/S^{\text{cw}}) \rightarrow (f \times I_{\mathbb{1}}^*) : (X \times \mathbb{Y}^a_{\text{an}}/S^{\text{cw}} \rightarrow Z \times \mathbb{Y}^a_{\text{an}}/S^{\text{cw}}) \) and the identity of \( \text{G}^* (X \times \mathbb{Y}^a_{\text{an}}/S) \);

- for \( S \in \text{Var}(\mathbb{C}) \), the morphism \( \widetilde{\psi}_{\text{Cw}}(S) \) which, for \( F^* \in \text{PSh}(\text{Cor}^f_{\text{cw}}(\text{Var}(\mathbb{C})/S), C^{-}(\mathbb{Z})) \), associate the morphism

\[
\widetilde{\psi}_{\text{Cw}}(S)(F^*) : \widetilde{\text{Cw}}(S)^{*}\text{sing}_{\text{ti}} F^* \rightarrow \text{sing}_{\text{ti}} \widetilde{\text{Cw}}(S)^* F^*
\]

in \( \text{PSh}(\text{Cor}^f_{\text{cw}}(\mathbb{C}^+/S^{\text{cw}}), C^{-}(\mathbb{Z})) \); the morphism \( \widetilde{\psi}_{\text{Cw}}(S)(F^*) \) is given by, for \( Z/S^{\text{cw}} \in \text{CW} / S^{\text{cw}} \),

\[
\widetilde{\psi}_{\text{Cw}}(F^*) (Z) : \lim_{X^{\text{cw}}/S^{\text{cw}} \rightarrow Z/S^{\text{cw}}} F^* (X \times \mathbb{Y}^a_{\text{ti}}/S) \rightarrow \lim_{Y^{\text{cw}}/S^{\text{cw}} \rightarrow Z \times \mathbb{1}^+/S^{\text{cw}}} F^* (Y/S)
\]

given by \( (f : X^{\text{cw}}/S^{\text{cw}} \rightarrow Z/S^{\text{cw}}) \rightarrow (f \times I_{\mathbb{1}}^*) : (X \times \mathbb{Y}^a_{\text{ti}}/S^{\text{cw}} \rightarrow Z \times \mathbb{Y}^a_{\text{ti}}/S^{\text{cw}}) \) and the identity of \( F^* (X \times \mathbb{Y}^a_{\text{ti}}/S) \).
Definition 42. As in the absolute case, we define the following two morphisms of functors:

(i) for $S \in \text{AnSp}(\mathbb{C})$, the morphism $W(S)$, which, for $G^\bullet \in \text{PSh}(\text{Cor}_{\mathbb{Z}}^{f}(\text{AnSp}(\mathbb{C})^{sm}/S), C^{-}(Z))$, associate the composition

$$W(S)(G^\bullet) : \text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, G^\bullet) \xrightarrow{\text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, G^\bullet)} \text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, G^\bullet) \xrightarrow{\psi_{\text{Cw}(S)(G^\bullet)}} \text{sing}_{\mathbb{Z}}, \text{Cw}(S)^*G^\bullet$$

in $\text{PSh}(\text{Cor}_{\mathbb{Z}}^{f}(\text{CW}/S^{cw}), C^{-}(Z))$.

(ii) for $S \in \text{Var}(\mathbb{C})$ the morphism $\tilde{W}(S)$, which, for $F^\bullet \in \text{PSh}(\text{Cor}_{\mathbb{Z}}^{f}(\text{SmVar}(\mathbb{C})), C^{-}(Z))$, associate the composition

$$\tilde{W}(S)(F^\bullet) : \text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, F^\bullet) \xrightarrow{\text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, F^\bullet)} \text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, F^\bullet) \xrightarrow{\psi_{\text{Cw}(S)(F^\bullet)}} \text{sing}_{\mathbb{Z}}, \text{Cw}(S)^*F^\bullet$$

in $\text{PSh}(\text{Cor}_{\mathbb{Z}}^{f}(\text{CW}/S^{cw}), C^{-}(Z))$.

We have the relative analogue to proposition 20:

Proposition 45. (i) Let $S \in \text{AnSp}(\mathbb{C})$. For $G^\bullet \in PC^{-}(\text{An}, S)$, $W(G^\bullet)(S) : \text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, G^\bullet) \to \text{sing}_{\mathbb{Z}}, \text{Cw}(S)^*G^\bullet$ is an equivalence ($\mathbb{I}^1$, usu) local in $PC^{-}(\text{CW}, S^{cw})$.

(ii) Let $S \in \text{Var}(\mathbb{C})$. For $F^\bullet \in PC^{-}(\mathbb{S})$, $\tilde{W}(S)(F^\bullet) : \text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, F^\bullet) \to \text{sing}_{\mathbb{Z}}, \text{Cw}(S)^*F^\bullet$ is an ($\mathbb{I}^1$, usu) local equivalence in $PC^{-}(\text{CW}, S^{cw})$.

Proof. Similar to proposition 20.$\square$

Definition 43. For $S \in \text{AnSp}(\mathbb{C})$, we define the morphism of functor $B(S)$, by associating to $G^\bullet \in \text{PSh}(\text{Cor}_{\mathbb{Z}}^{f}(\text{AnSp}(\mathbb{C})^{sm}/S), C^{-}(Z))$, the morphism $B(S)(G^\bullet)$ which is the composite

$$\text{Cw}(S)^*\xrightarrow{\text{Cw}(S)^*(\text{sing}_{\mathbb{Z}}, G^\bullet)} \text{Cw}(S)^*\xrightarrow{\psi_{\text{Cw}(S)(G^\bullet)}} \text{sing}_{\mathbb{Z}}, \text{Cw}(S)^*G^\bullet$$

in $\text{PSh}(\text{Cor}_{\mathbb{Z}}^{f}(\text{AnSp}(\mathbb{C})^{sm}/S), C^{-}(Z))$.

We have the following key proposition.

Proposition 46. Let $S \in \text{Var}(\mathbb{C})$ and $F^\bullet \in PC^{-}(S)$ such that $D(\mathbb{A}^{1}, et)(S)(F^\bullet) \in \text{DM}^{-}(S, \mathbb{Z})$ is a constructible motive. Then,

(i) $e_{\text{an}}^{\text{tr}}(S)_{*}B(S)(F^\bullet) : \text{sing}_{\mathbb{Z}}, \text{An}(S)^*F^\bullet \to \text{sing}_{\mathbb{Z}}, \text{Cw}(S)^*F^\bullet$ is an equivalence usu local in $C^{-}(S^{an})$.

(ii) $B(S)(F^\bullet) : \text{sing}_{\mathbb{Z}}, \text{An}(S)^*F^\bullet \to \text{Cw}(S)^*F^\bullet$ is an equivalence ($\mathbb{D}^1$, usu) local.

Proof. (i): As $M = D(\mathbb{A}^{1}, et)(S)(F^\bullet)$ is a constructible motive, we may assume using induction that there exist $X/S, Y/S \in \text{Var}(\mathbb{C})^{sm}/S$, $X/S = (X, h_1), Y/S = (X, h_2)$ such that

$$M = D(\mathbb{A}^{1}, et)(S)(\text{Cone}(\alpha))$$

with $\alpha \in \text{Hom}_{PC^{-}(S)}(F(X/S), F(Y/S, p, n))$, where we have chosen $k_1 : \text{Z}_{tr}(X/S) \to F(X/S)$ and $k_2 : \text{Z}_{et}(Y/S)[p][n] \to F(Y/S, p, n)$ equivalences ($\mathbb{A}^{1}, et$) local with $F(X/S)$ and $F(X/S, p, n)$ are $\mathbb{A}^{1}$ local and etale fibrant objects. Consider the following commutative diagrams in $PC^{-}(\text{An}, S^{an})$:
(1) \( \text{sing}_D^* \mathcal{Z}_t r(X^{an}/S^{an}) \xrightarrow{\epsilon_{an}(S) \cdot B(\mathcal{Z}_t r(X^{an}/S^{an}))} \text{sing}_I^* \mathcal{Z}_t r(X^{cw}/S^{cw}) \),

\[ S(\text{An}(S) \cdot k_1) \xrightarrow{\mathcal{S}(\text{An}(S) \cdot k_2)} S(\text{Cw}(S) \cdot k_2) \]

\( \text{sing}_D^* \text{An}(S)^* F(X/S) \xrightarrow{\epsilon_{an}(S) \cdot B(\text{An}(S)^* F(X/S))} \text{sing}_I^* \text{Cw}(S)^* F(X/S) \)

(2) \( \text{sing}_D^* \mathcal{Z}_t r(Y^{an}/S^{an})(p) \xrightarrow{\epsilon_{an}(S) \cdot B(\mathcal{Z}_t r(Y^{an}/S^{an})(p))} \text{sing}_I^* \mathcal{Z}_t r(Y^{cw}/S^{cw})(p) \).

\[ \text{sing}_D^* \text{An}(S)^* F(Y/S) \xrightarrow{\epsilon_{an}(S) \cdot B(\text{An}(S)^* F(Y/S))} \text{sing}_I^* \text{Cw}(S)^* F(Y/S) \]

Then,

- \( S(\text{An}(S) \cdot k_1) : \text{sing}_D^* \mathcal{Z}_t r(X^{an}/S^{an}) \rightarrow \text{sing}_D^* \text{An}(S)^* F(X/S) \)

- \( S(\text{An}(S) \cdot k_2) : \text{sing}_D^* \mathcal{Z}_t r(Y^{an}/S^{an})(p)[n] \rightarrow \text{sing}_D^* \text{An}(S)^* F(Y/S, p, n) \)

are equivalences \((D^1, usu)\) local by proposition \(44(i)\) and theorem \(23(ii)\). Similarly,

- \( S(\text{Cw}(S) \cdot k_1) : \text{sing}_I^* \mathcal{Z}_t r(X^{cw}/S^{cw}) \rightarrow \text{sing}_I^* \text{Cw}(S)^* F(X/S) \)

- \( S(\text{Cw}(S) \cdot k_2) : \text{sing}_I^* \mathcal{Z}_t r(Y^{cw}/S^{cw})(p)[n] \rightarrow \text{sing}_I^* \text{Cw}(S)^* F(Y/S, p, n) \)

are equivalences \((I^1, usu)\) local by proposition \(44(ii)\) and theorem \(23(ii)\). On the other side,

- \( \epsilon_{an}(S) \cdot B(\mathcal{Z}_t r(X^{an}/S^{an})) : \text{sing}_D^* \mathcal{Z}_t r(X^{an}/S^{an}) \rightarrow \text{sing}_I^* \mathcal{Z}_t r(X^{cw}/S^{cw}) \) is an equivalence usu local by proposition \(27(i)\) applied to

\( \epsilon_{an}(S) \cdot B(\mathcal{Z}_t r(X^{an})) : \text{sing}_D^* \mathcal{Z}_t r(X^{an}) \rightarrow \text{sing}_D^* \mathcal{Z}_t r(X^{cw}) \)

for each \( s \in S \): since \( h_1 : X \rightarrow S \) is smooth, \( i_1^* \mathcal{Z}_t r(X^{an}/S^{an}) = \mathcal{Z}_t r(X^{an}) \) and \( i_1^* \mathcal{Z}_t r(X^{cw}/S^{cw}) = \mathcal{Z}_t r(X^{cw}) \), where \( i_1 : \{ s \} \hookrightarrow S \) is the closed embedding.

- \( \epsilon_{an}(S) \cdot B(\mathcal{Z}_t r(Y^{an}/S^{an})(p)) : \text{sing}_D^* \mathcal{Z}_t r(Y^{an}/S^{an})(p) \rightarrow \text{sing}_I^* \mathcal{Z}_t r(Y^{cw}/S^{cw})(p) \)

for each \( s \in S \): since \( h_2 : Y \rightarrow S \) is smooth \( i_2^* \mathcal{Z}_t r(Y^{an}/S^{an}) = \mathcal{Z}_t r(Y^{an}) \) and \( i_2^* \mathcal{Z}_t r(Y^{cw}/S^{cw}) = \mathcal{Z}_t r(Y^{cw}) \), where \( i_2 : \{ s \} \hookrightarrow S \) is the closed embedding.

The diagram (1) then shows that

\( \epsilon_{an}(S) \cdot B(\text{An}(S)^* F(X/S)) : \text{sing}_D^* F(X/S) \rightarrow \text{sing}_I^* \text{Cw}(S)^* F(X/S) \)

is an equivalence usu local in \( C^-(S^{an}) \), and the diagram (2) that

\( \epsilon_{an}(S) \cdot B(\text{An}(S)^* F(Y/S)) : \text{sing}_D^* F(Y/S, p, n) \rightarrow \text{sing}_I^* \text{Cw}(S)^* F(Y/S, p, n) \)

is equivalence usu local in \( C^-(S^{an}) \).

(ii): Follows from (i) Let us explain.

On the one hand,

- By theorem \(23(ii)\), \( \text{sing}_D^* \text{An}(S)^* F^* \) is \( D^1 \) local.

- By theorem \(16(ii)\), \( \text{sing}_I^* \text{Cw}(S)^* F^* \) is \( I^1 \) local. Hence, \( \text{Cw}(S^{an}) \cdot \text{sing}_I^* \text{Cw}(S)^* F^* \) is \( D^1 \) local, by proposition \(25(ii)\).
• On the other hand by (i) \( e^{tr}_{an}(S)_*(B(S)(F^*)) : \text{sing}_D, \text{An}(S)^*F^* \rightarrow \text{sing}_D, \text{Cw}(S)^*F^* \) is a quasi isomorphism in \( C^-(S) \).

Hence, by proposition [37] (ii),

\[
B(S)(F^*) : \text{sing}_D, \text{An}(S)^*F^* \rightarrow \text{Cw}(S^{an})_*, \text{sing}_D, \text{Cw}(S)^*F^*
\]

is an equivalence (\( D^1, usu \)) local.

\[\square\]

The proposition [40] gives the following relative version of theorem [18].

**Theorem 29.** Let \( S \in \text{Var}(\mathbb{C}) \). Let \( M \in \text{DM}^-(S, \mathbb{Z}) \) is a constructible motive.

(i) We have \( \text{Bti}^* M = \tilde{\text{Bti}}^* M \)

(ii) Let \( M_1, M_2 \in \text{DM}^-(S, \mathbb{Z}) \) constructible motives. Let \( F_i^*, F_2^* \in \text{PC}^-(S) \) such that \( M_i = D(\mathbb{A}^1, et)(S)(F_i^*) \in \text{DM}^-(S, \mathbb{Z}) \) for \( i = 1, 2 \). The following diagram is commutative

\[
\begin{array}{ccc}
\text{Hom}_{\text{DM}^-(S)}(M_1, M_2) & \xrightarrow{\text{An}(S)^*} & \text{Hom}_{\text{Cw}^-(S)}(\text{Cw}(S)^*M_1, \text{Cw}(S)^*M_2) \\
\text{Hom}_{\text{AnDM}^-(S)}(\text{An}(S)^*M_1, \text{An}(S)^*M_2) & \xrightarrow{\text{Re}_{\text{an}}^*(S)} & \text{Hom}_{\text{DM}^-(S)}(\text{sing}_D, \text{Cw}(S)^*F_1^*, \text{sing}_D, \text{Cw}(S)^*F^*)
\end{array}
\]

**Proof.** (i): By definition, \( \text{Bti}^* M = \text{Re}^*_{\text{an}}(S)_* \text{An}(S)^* M \). Since, by theorem [23] (ii),

- \( S(F^*) : \text{An}(S)^*F^* \rightarrow \text{sing}_D, \text{An}(S)^*F^* \) is an equivalence (\( D^1, usu \)) local in \( \text{PC}^-(\text{An}, S^{an}) \) and
- \( \text{sing}_D, \text{An}(S)^*F^* \) is a \( D^1 \) local object,

we have, since \( \text{An}(S)^* \) derive trivially by proposition [44] (i), \( \text{Bti}^* M = e^{tr}_{an}(S)_*(\text{sing}_D, \text{An}(S)^*F^*) = \text{sing}_D, \text{An}(S)^*F^* \).

Since

- \( B(\text{An}(S)^*F^*) : \text{sing}_D, \text{An}(S)^*F^* \rightarrow \text{Cw}, \text{sing}_D, \text{Cw}(S)^*F^* \) is an equivalence (\( D^1, usu \)) local in \( \text{PC}^-(\text{An}, S^{an}) \) by proposition [46] (ii), and
- \( \text{Cw}(S)^*, \text{sing}_D, \text{Cw}(S)^*F^* \) is a \( D^1 \) local object by theorem [25] (ii) and proposition [41] (ii),

we have, since \( \text{Cw}(S)^* \) derive trivially by proposition [44] (iii),

\[
\text{Bti}^* M = e^{tr}_{an}(S)_*(\text{Cw}(S)^*, \text{sing}_D, \text{Cw}(S)^*F^*) = \text{sing}_D, \text{Cw}(S)^*F^* \tag{118}
\]

By definition, \( \tilde{\text{Bti}}^* M = \text{Re}^*_{\text{an}}(\text{Cw}(S)^* M \). Since, by theorem [26] (ii),

- \( S(\text{Cw}(S)^*F^*) : \text{Cw}, \text{sing}_D, \text{Cw}(S)^*F^* \) is an equivalence (\( \mathbb{I}^1, usu \)) local in \( \text{PC}^-(\text{Cw}, S^{cw}) \) and
- \( \text{sing}_D, \text{Cw}(S)^*F^* \) is an \( \mathbb{I}^1 \) local object,

we have

\[
\tilde{\text{Bti}}^* M = e^{tr}_{an}(S)_*(\text{sing}_D, \text{Cw}(S)^*F^*) = \text{sing}_D, \text{Cw}(S)^*F^* = \text{Bti}^* M \text{ by (118)}
\]

This proves (i).
(ii): Let \( \alpha \in \text{Hom}_{\text{DM}(S, \mathbb{Z})}(M_1, M_2) \). Consider the commutative diagram in \( \text{AnDM}(S, \mathbb{Z}) \)

\[
\begin{array}{c}
\text{sing}, \text{An}(S)^* F_1^* \ar[r]^-{\text{An}(S)^* \alpha} & \text{sing}, \text{An}(S)^* F_2^* \\
\text{B}(\text{An}(S)^* F_1^*) \ar[u] & \text{Cw}(S), \text{sing}, \text{Cw}(S)^* F_1^* \ar[r]^-{\text{Cw}(S)^* \alpha} & \text{Cw}(S), \text{sing}, \text{Cw}(S)^* F_2^* \\
\end{array}
\]

Since \( \text{sing}, \text{An}(S)^* F_1^* \) and \( \text{sing}, \text{An}(S)^* F_2^* \) are \( \mathbb{D}^1 \) local objects by theorem \( 23 \)(ii) and since \( \text{An}(S) \) derive trivially by proposition \( 44 \)(i),

\[
\text{An}(S)^* \alpha \in \text{Hom}_{\text{DM}(\text{PC}^-(\text{An}(S), S^\infty))}(\text{sing}, \text{An}(S)^* F_1^*, \text{sing}, \text{An}(S)^* F_2^*)
\]

Thus,

\[
\text{Bti}(S)^* \alpha := R_{\text{an}}^\text{tr}(S)* (\text{An}(S)^* \alpha) = e_{\text{an}}^\text{tr}(S)* \text{An}(S)^* \alpha \tag{119}
\]

Since \( \text{sing}, \text{Cw}(S)^* F_1^* \) and \( \text{sing}, \text{Cw}(S)^* F_2^* \) are \( \mathbb{D}^1 \) local objects by theorem \( 25 \)(ii),

\[
\text{Cw}(S)^* \alpha \in \text{Hom}_{\text{DM}(\text{PC}^-(\text{Cw}, S^\infty))}(\text{sing}, \text{Cw}(S)^* F_1^*, \text{sing}, \text{Cw}(S)^* F_2^*)
\]

Thus, since \( \text{Cw}(S)^* \) derive trivially by proposition \( 44 \)(iii),

\[
\text{Bti}(S)^* \alpha := R_{\text{cw}}^\text{tr}(S)* (\text{Cw}(S)^* \alpha) = e_{\text{cw}}^\text{tr}(S)* \text{Cw}(S)^* \alpha \tag{120}
\]

Since \( \text{Cw}(S), \text{sing}, \text{Cw}(S)^* F_1^* \) and \( \text{Cw}(S), \text{sing}, \text{Cw}(S)^* F_2^* \) are \( \mathbb{D}^1 \) local objects by theorem \( 25 \)(ii) and proposition \( 44 \)(ii),

\[
\text{Cw}(S)* \text{Cw}(S)^* \alpha \in \text{Hom}_{\text{DM}(\text{PC}^-(\text{Cw}, S^\infty))}(\text{Cw}(S)* \text{sing}, \text{Cw}(S)^* F_1^*, \text{Cw}(S)* \text{sing}, \text{Cw}(S)^* F_2^*) \tag{121}
\]

Thus

\[
\text{Bti}^*(\alpha) = R_{\text{an}}^\text{tr}(\text{Cw}(S)* \text{Cw}(S)^* \alpha) \text{ since } B(\text{An}(S)^* F_1^*) \text{ and } B(\text{An}(S)^* F_2^*)
\]

are \( \mathbb{D}^1 \), \( \text{asu} \) local equivalence by proposition \( 46 \)(ii)

\[
= e_{\text{an}}^\text{tr}(S)* (\text{Cw}(S)* \text{Cw}(S)^* \alpha) \tag{121}
\]

\[
= e_{\text{cw}}^\text{tr}(S)* (\text{Cw}(S)^* \alpha)
\]

\[
= \text{Bti}(S)^* (\alpha) \text{ by } 120.
\]

\[\square\]

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