QUANTIZATION AND INJECTIVE SUBMODULES OF DIFFERENTIAL OPERATOR MODULES

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Abstract. The Lie algebra of vector fields on $\mathbb{R}^m$ acts naturally on the spaces of differential operators between tensor field modules. Its projective subalgebra is isomorphic to $\mathfrak{sl}_{m+1}$, and its affine subalgebra is a maximal parabolic subalgebra of the projective subalgebra with Levi factor $\mathfrak{gl}_m$. We prove two results. First, we realize all injective objects of the parabolic category $\mathcal{O}_{\mathfrak{gl}_m}(\mathfrak{sl}_{m+1})$ of $\mathfrak{gl}_m$-finite $\mathfrak{sl}_{m+1}$-modules as submodules of differential operator modules. Second, we study projective quantizations of differential operator modules, i.e., $\mathfrak{sl}_{m+1}$-invariant splittings of their order filtrations. In the case of modules of differential operators from a tensor density module to an arbitrary tensor field module, we determine when there exists a unique projective quantization, when there exists no projective quantization, and when there exist multiple projective quantizations.

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1. Introduction and main results

Throughout this article we write $\mathbb{N}$ for the non-negative integers and $\mathbb{Z}^+$ for the positive integers. Let $\text{Vec} \mathbb{R}^m$ be the Lie algebra of polynomial vector fields on $\mathbb{R}^m$:

$$\text{Vec} \mathbb{R}^m := \text{Span}_{\mathbb{C}} \{ x^I \partial x_i : I = (I_1, \ldots, I_m) \in \mathbb{N}^m, 1 \leq i \leq m \}. $$

Here $x^I$ denotes the monomial $x_1^{I_1} \cdots x_m^{I_m}$ and $\partial x_i$ denotes $\partial / \partial x_i$.

Our subject is the representations of $\text{Vec} \mathbb{R}^m$ on spaces of linear differential operators between tensor field modules. In order to describe these representations, let $\text{Vec}^0 \mathbb{R}^m$ and $\text{Vec}^2_2 \mathbb{R}^m$ be the subalgebras of $\text{Vec} \mathbb{R}^m$ of vector fields vanishing to first and second order at zero, respectively: setting $|I| := \sum_{r=1}^m I_r$,

$$\text{Vec}^0 \mathbb{R}^m := \text{Span}_{\mathbb{C}} \{ x^I \partial x_i : I \in \mathbb{N}^m, |I| \geq 1, 1 \leq i \leq m \},
$$

$$\text{Vec}^2_2 \mathbb{R}^m := \text{Span}_{\mathbb{C}} \{ x^I \partial x_i : I \in \mathbb{N}^m, |I| \geq 2, 1 \leq i \leq m \}. $$

$\text{Vec}^0_2 \mathbb{R}^m$ is an ideal in $\text{Vec}^0 \mathbb{R}^m$, and the quotient $\text{Vec}^0 \mathbb{R}^m / \text{Vec}^0_2 \mathbb{R}^m$ is isomorphic to $\mathfrak{gl}_m$. Therefore any representation $V$ of $\mathfrak{gl}_m$ may be regarded as a representation of $\text{Vec}^0 \mathbb{R}^m$, as which it may be coinduced to a representation of $\text{Vec} \mathbb{R}^m$. The resulting module is denoted $\mathcal{F}(V)$, the tensor field module associated to $V$. It has a natural realization in the space $\mathbb{C}[x_1, \ldots, x_m] \otimes V$, the sections of the vector bundle over $\mathbb{R}^m$ with fiber $V$. In the case that $V$ is 1-dimensional, $\mathcal{F}(V)$ is called a tensor density module.

Given two tensor field modules $\mathcal{F}(V)$ and $\mathcal{F}(V')$, one has the space

$$\mathcal{D}(V, V') := \mathbb{C}[x_1, \ldots, x_m, \partial x_1, \ldots, \partial x_m] \otimes \text{Hom}(V, V')$$

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of linear differential operators from $\mathcal{F}(V)$ to $\mathcal{F}(V')$. The natural adjoint action of $\text{Vec} \mathbb{R}^m$ on $\mathcal{D}(V, V')$ preserves the order filtration $\mathcal{D}^k(V, V')$, $k \in \mathbb{N}$. The associated graded module $S(V, V')$ is the module of symbols, and we write $\mathcal{S}^k(V, V')$ for its homogeneous component of degree $k$. As modules of $\text{Vec} \mathbb{R}^m$, $\mathcal{D}(V, V')$ and $S(V, V')$ are not isomorphic: $S(V, V')$ is itself a tensor field module, but $\mathcal{D}(V, V')$ is not.

Two finite dimensional subalgebras of $\text{Vec} \mathbb{R}^m$ will be important for us, the *projective subalgebra* $\mathfrak{a}_m$ and the *affine subalgebra* $\mathfrak{b}_m$:

\[
\mathfrak{a}_m := \text{Span}_\mathbb{C} \{ \partial_{x_i}, x_j \partial_{x_i}, x_j \sum_{r=0}^m x_r \partial_{x_i} : 1 \leq i, j \leq m \},
\]
\[
\mathfrak{b}_m := \text{Span}_\mathbb{C} \{ \partial_{x_i}, x_j \partial_{x_i} : 1 \leq i, j \leq m \}.
\]

It is elementary that $\mathfrak{a}_m$ is isomorphic to $\mathfrak{sl}_{m+1}$ and is maximal within $\text{Vec} \mathbb{R}^m$, and $\mathfrak{b}_m$ is a maximal parabolic subalgebra of $\mathfrak{a}_m$ with Levi factor $\mathfrak{gl}_m$. Fix a system of positive roots of $\mathfrak{a}_m$ containing the roots of the nilpotent radical of $\mathfrak{b}_m$, and let $\mathcal{O}^{\mathfrak{b}_m}(\mathfrak{a}_m)$ be the corresponding parabolic category $\mathcal{O}$ of $\mathfrak{b}_m$-finite $\mathfrak{a}_m$-modules.

Suppose that $V$ and $V'$ are finite dimensional irreducible representations of $\mathfrak{gl}_m$. Then as $\mathfrak{a}_m$-modules, $\mathcal{F}(V)$, $\mathcal{D}^k(V, V')$, and $\mathcal{S}^k(V, V')$ are all elements of $\mathcal{O}^{\mathfrak{b}_m}(\mathfrak{a}_m)$. Furthermore, $\mathcal{F}(V)$ is the dual of a $\mathfrak{gl}_m$-finite parabolic Verma module, and $\mathcal{S}^k(V, V')$ is a direct sum of such Verma modules.

We now state the first of our two main results, which shows that all injective objects of $\mathcal{O}^{\mathfrak{b}_m}(\mathfrak{a}_m)$ occur in the differential operator modules $\mathcal{D}(V, V')$. The injectives of this category were first described in [RCS1], the irreducible injectives are tensor field modules, and those which are indecomposable but not irreducible have Loewy length 2 or 3 and are composed of one or two tensor field modules, respectively.

**Theorem A.** Let $I$ be an indecomposable injective object of $\mathcal{O}^{\mathfrak{b}_m}(\mathfrak{a}_m)$ of generalized infinitesimal character $\chi$. Then there exist irreducible representations $V$ and $V'$ of $\mathfrak{gl}_m$ such that $I$ is isomorphic to the $\mathfrak{a}_m$-submodule of $\mathcal{D}(V, V')$ of generalized infinitesimal character $\chi$.

We will give a constructive proof of this theorem: given any indecomposable injective, we will specify choices of $V$ and $V'$ such that $\mathcal{D}(V, V')$ contains it. There are many such choices, and it is always possible to take $V$ to be 1-dimensional. However, in any given realization of a reducible indecomposable injective the two tensor field modules composing it will be of orders $k$ and $k'$, where the difference $k - k'$ is independent of the choice of $V$ and $V'$ and can be arbitrarily large.

Differential operator realizations of injectives facilitate some types of explicit computations. For example, they can be used to observe that the Casimir element of $\mathfrak{a}_m$ acts non-semisimply on all reducible injectives.

Our second main result concerns differential operator modules $\mathcal{D}(V, V')$ whose order filtration is split under $\mathfrak{a}_m$. It is complimentary to Theorem A in the sense that reducible injectives occur precisely in those $\mathcal{D}(V, V')$ without $\mathfrak{a}_m$-splittings.

**Definition.** A projective quantization is an $\mathfrak{a}_m$-invariant splitting of the order filtration of $\mathcal{D}(V, V')$. For $V$ and $V'$ irreducible, $\mathcal{D}(V, V')$ is said to be resonant if it has either no projective quantization or more than one projective quantization.

The study of projective quantizations goes back to [GO96, CMZ97, LO99]. At first only differential operators between tensor density modules were considered: the *scalar case*. This case was completely resolved in [Le0]. More recently, projective quantizations of modules of differential operators between arbitrary tensor field
modules were examined, for example in [Ha07 MR07 CS10]. In these papers, conditions for resonance are given in terms of the spectrum of the action of the Casimir element $\Omega_{a_m}$ of $a_m$. The underlying idea is that if resonance occurs, then $\Omega_{a_m}$ has repeated eigenvalues on certain “tree-like” submodules of $\mathcal{D}(V)$; see Proposition 4.10 of [Ha07], Theorem 11 of [MR07], and Corollary 6 of [CS10].

Resonance conditions obtained from the eigenvalues of $\Omega_{a_m}$ are not sharp, as there are non-resonant modules on which $\Omega_{a_m}$ has repeated eigenvalues on some tree-like submodules. A more precise condition is given by infinitesimal characters of $a_m$, but even this is not a priori sharp. An abstract necessary and sufficient condition was given in [Mi12]:

**Theorem B.** [Mi12] For $V$ and $V'$ irreducible, $\mathcal{D}(V, V')$ is resonant if and only if there are non-trivial $a_m$-maps from $S^k(V, V')$ to $S^{k'}(V, V')$ for some $k \neq k'$.

In this article we restrict our consideration to modules of differential operators from a tensor density module $F(V)$ to an arbitrary tensor field module $F(V')$, i.e., modules $\mathcal{D}(V, V')$ such that $V$ is scalar. In this setting, Theorem B coupled with known results on homomorphisms of parabolic Verma modules gives an explicit necessary and sufficient condition for resonance. We will derive the resonance condition given by infinitesimal characters and observe a posteriori that it is sharp:

**Proposition C.** For $V$ 1-dimensional and $V'$ irreducible, $\mathcal{D}(V, V')$ is resonant if and only if $S(V, V')$ has repeated $a_m$-infinitesimal characters.

We now state the conditions under which $S(V, V')$ has repeated infinitesimal characters using the standard notation for $\mathfrak{gl}_m$-weights, which we will recall in Section 2.1. Suppose that $\delta$ is a dominant integral $\mathfrak{gl}_m$-weight:

$$\delta = (\delta_0, \delta_1, \ldots, \delta_m) \in \mathbb{C}^{m+1}, \sum_{i=0}^{m} \delta_i = 0, \text{ and } \delta_i - \delta_{i+1} \in \mathbb{N} \text{ for } 1 \leq i < m.$$  

(In the context of $\mathfrak{gl}_m$ alone, the entry $\delta_0$ is suppressed and $\delta$ is regarded simply as the element $(\delta_1, \ldots, \delta_m)$ of $\mathbb{C}^m$. We include $\delta_0$ for consistency with weights of $a_m$.)

Let $L_{\mathfrak{gl}_m}(\delta)$ be the irreducible $\mathfrak{gl}_m$-module of highest weight $\delta$. For $\gamma \in \mathbb{C}$, the scalar $\mathfrak{gl}_m$-module on which $X$ acts by $\gamma$ (trace$(X)$ is $C_{\gamma} := L_{\mathfrak{gl}_m}(-m\gamma, \gamma, \ldots, \gamma)$).

As we will see, the symbol module of $\mathcal{D}(C_{\gamma}, \mathcal{C}_{\gamma} \otimes L_{\mathfrak{gl}_m}(\delta))$ depends only on $\delta$. Therefore we use the abbreviations

$$\mathcal{D}_\gamma(\delta) := \mathcal{D}(C_{\gamma}, \mathcal{C}_{\gamma} \otimes L_{\mathfrak{gl}_m}(\delta)), \quad S(\delta) := S(C_{\gamma}, \mathcal{C}_{\gamma} \otimes L_{\mathfrak{gl}_m}(\delta)).$$

**Definition.** Fix a dominant integral $\mathfrak{gl}_m$-weight $\delta$.

(i) Let $i(\delta) \in \{1, \ldots, m\}$ be maximal such that $\delta_1 = \delta_2 = \cdots = \delta_{i(\delta)}$.

(ii) Set $\hat{\delta}_i := \delta_i - i + \sum_{j=1}^{m} \delta_j = \delta_i - i - \delta_0$. Note that $\hat{\delta}_1 > \cdots > \hat{\delta}_m$.

(iii) We say that $\delta$ is resonant if $\hat{\delta}_{i(\delta)} \in \mathbb{Z}^+.$

(iv) For $\delta$ resonant, $k \in \mathbb{Z}^+$, and $\hat{\delta}_{i(\delta)} \geq k$, let $i(\delta, k) \in \{i(\delta), \ldots, m\}$ be maximal such that $\hat{\delta}_{i(\delta, k)} \geq k$. In all other cases, $i(\delta, k)$ is not defined.

**Proposition D.** $\mathcal{D}_\gamma(\delta)$ is resonant if and only if $\delta$ is resonant. In this case, $S^k(\delta)$ and $\bigoplus_{k'<k} S^{k'}(\delta)$ share an infinitesimal character if and only if

(2) $i(\delta, k)$ is defined, $k > \frac{1}{2} \hat{\delta}_{i(\delta, k)}$, and if $i(\delta, k) \neq m$, then $k > \hat{\delta}_{i(\delta, k)+1} + 1.$
Keep in mind that if \( \delta \) is not resonant, then \( i(\delta, k) \) is not defined for any \( k \), so (2) never holds. Conversely, if \( \delta \) is resonant then (2) holds for at least one \( k \), namely, \( k = \tilde{\delta}(\delta) \). We remark that for \( i(\delta, k) < m \), (2) cannot hold unless \( \delta_i(\delta, k) > \delta_i(\delta, k) + 1 \).

Proposition D brings us to the question of which resonant modules have no projective quantizations and which have multiple projective quantizations. To date this question has been answered only in the scalar case \([Le00]\). Our second main result answers it for \( D_\gamma(\delta) \). Let us state the following condition on \( \gamma \):

\[
\text{(3) If } i(\delta, k) < m, \text{ then } k + (m + 1)\gamma = 1. \\
\text{If } i(\delta, k) = m, \text{ then } k + (m + 1)\gamma \in \{1, \ldots, \max\{1, 2k - \tilde{\delta}_1 + m - 1\}\}. 
\]

**Theorem E.**

(i) \( D^{k-1}_\gamma(\delta) \) admits a unique \( a_m \)-invariant complement in \( D^k_\gamma(\delta) \) if and only if (3) does not hold.

(ii) \( D^{k-1}_\gamma(\delta) \) admits no \( a_m \)-invariant complement in \( D^k_\gamma(\delta) \) if and only if (3) holds but (2) does not hold.

(iii) \( D^{k-1}_\gamma(\delta) \) admits more than one \( a_m \)-invariant complement in \( D^k_\gamma(\delta) \) if and only if both (2) and (3) hold.

**Corollary F.** Assume that \( \delta \) is resonant. Then \( D_\gamma(\delta) \) has more than one projective quantization if and only if for all \( k \in \mathbb{Z}^+ \) such that (2) holds, (3) also holds. Otherwise it has no projective quantization.

Note that \( i(\delta) = m \) if and only if \( \delta \) is scalar, i.e., \( \delta_1 = \delta_2 = \cdots = \delta_m \). In this case, our (2), (3), and Theorem E match (10), (11), and Theorem 8.5 of \([Le00]\), respectively.

The \( a_m \)-intertwining maps between symbol modules play a central role in the proofs of our results. We shall give these maps explicitly as powers of the divergence operator. They are a particular realization of the \( a_m \)-maps between tensor field modules, which have been thoroughly analyzed in a more general context by many authors; see \([BES88]\) and the references therein. The classification of such maps is equivalent to the classification of their duals, the \( a_m \)-maps between the \( \mathfrak{gl}_m \)-finite parabolic Verma modules. These have also been well studied; see for example \([SiSS]\) and the references therein.

The article is organized as follows. In Sections 2 and 3 we collect the necessary properties of \( a_m \) and the tensor field modules, and in Section 4 we collect those of the differential operator modules and their symbol modules. In Section 5 we study the infinitesimal characters of the symbol modules, in Section 6 we study affine and projective operators on tensor field modules, and in Section 7 we review the injectives in the parabolic category \( \mathcal{O}^{\mathfrak{b}m}(a_m) \). Section 8 presents a key feature of the action of the Casimir element \( \Omega_{a_m} \) on differential operator modules, and Section 9 contains the proofs of our main results.

2. Background

2.1. The projective subalgebra. Set \( \mathcal{E}_x := \sum_{i=1}^m x_r \partial_{x_r} \), the Euler operator. We begin with the structure of \( a_m \). Define

\[
\begin{align*}
\mathfrak{c}_m &:= \text{Span}_\mathbb{C}\{\partial_{x_i} : 1 \leq i \leq m\}, \\
\mathfrak{l}_m &:= \text{Span}_\mathbb{C}\{x_{j} \partial_{x_i} : 1 \leq i, j \leq m\}, \\
\mathfrak{b}_m &:= \text{Span}_\mathbb{C}\{x_i \partial_{x_i} : 1 \leq i \leq m\}.
\end{align*}
\]
Note that the affine subalgebra $\mathfrak{h}_m$ is $\mathfrak{c}_m \oplus \mathfrak{l}_m$, where $\mathfrak{c}_m$ is its nilradical, the constant subalgebra, and $\mathfrak{l}_m$ is its Levi factor. The center of $\mathfrak{l}_m$ is $\mathbb{C}E_x$, and $\mathfrak{h}_m$ is a Cartan subalgebra of both $\mathfrak{l}_m$ and $\mathfrak{a}_m$.

In order to give an explicit isomorphism between $\mathfrak{a}_m$ and $\mathfrak{sl}_{m+1}$, it is convenient to regard $\mathfrak{sl}_{m+1}$ as the quotient $\mathfrak{pgl}_{m+1}$ of $\mathfrak{gl}_{m+1}$ by its center. Fix coordinates $x_0, \ldots, x_m$ on $\mathbb{R}^{m+1}$, and write $e_{ij}$ for the elementary matrix with $(i, j)$th entry 1 and all other entries 0. The reader may check that $\phi : \mathfrak{pgl}_{m+1} \to \mathfrak{a}_m$,

$$\phi : e_{00} \mapsto -E_x, \quad e_{0i} \mapsto \partial_{x_i}, \quad e_{i0} \mapsto -x_iE_x, \quad e_{ij} \mapsto x_i\partial_{x_j}, \quad 1 \leq i, j \leq m,$$

is an isomorphism. We mention a conceptual proof: first check that the map $e_{ij} \mapsto x_i\partial_{x_j}$ for $0 \leq i, j \leq m$ is an injective homomorphism from $\mathfrak{gl}_{m+1}$ to $\text{Vec} \mathbb{R}^{m+1}$. Then restrict the action of its image to the hyperplane $x_0 = 1$, and identify the hyperplane with $\mathbb{R}^m$. Note that $\phi$ carries $\mathfrak{gl}_m$ isomorphically to $\mathfrak{l}_m$.

We now describe the root system of $\mathfrak{a}_m$. The set $\{e_{00}, \ldots, e_{mm}\}$ is a basis of a Cartan subalgebra of $\mathfrak{gl}_{m+1}$. Let $\{e_0, \ldots, e_m\}$ be the dual basis. Then $\{e_1 - e_0, \ldots, e_m - e_0\}$ is a basis of the dual of the corresponding Cartan subalgebra of $\mathfrak{pgl}_{m+1}$. Using the isomorphism $\phi$, we regard it as a basis of the dual $\mathfrak{h}^*_m$ of $\mathfrak{h}_m$. Thus, writing $\delta_{ij}$ for the Kronecker delta function,

$$\mathfrak{h}^*_m = \text{Span}_{\mathbb{C}} \{\epsilon_i - e_0 : 1 \leq i \leq m\}, \quad \{\epsilon_i - e_0\}(x_j\partial_{x_j}) = \delta_{ij}, \quad 1 \leq i, j \leq m.$$

The advantage of this notation is that the action of the Weyl group $W(\mathfrak{a}_m)$ of $\mathfrak{a}_m$ on $\mathfrak{h}^*_m$ is transparent: $W(\mathfrak{a}_m)$ is the symmetric group $S_{m+1}$, acting by permutation of the indices $0, 1, \ldots, m$. Note that the Weyl group $W(\mathfrak{l}_m)$ of $\mathfrak{l}_m$ is the subgroup $S_m$ of $W(\mathfrak{a}_m)$ which permutes the indices $1, \ldots, m$.

The roots of $\mathfrak{a}_m$ and $\mathfrak{l}_m$ are

$$\Delta(\mathfrak{a}_m) = \{\epsilon_i - \epsilon_j : 0 \leq i, j \leq m\} \setminus \{0\}, \quad \Delta(\mathfrak{l}_m) = \{\epsilon_i - \epsilon_j : 1 \leq i, j \leq m\} \setminus \{0\}.$$

The corresponding root vectors as follows: for $1 \leq i, j \leq m$,

$$\partial_{x_i}, \quad x_iE_x$$

has root $\epsilon_0 - \epsilon_i$, \quad $x_i\partial_{x_i}$ has root $\epsilon_i - \epsilon_0$.

The order $\epsilon_0 > \epsilon_1 > \cdots > \epsilon_m$ fixes a triangular decomposition of $\mathfrak{a}_m$ and gives the following positive and simple systems. Below them we list the corresponding positive and simple positive root vectors of $\mathfrak{a}_m$, and below those we list its negative and simple negative root vectors:

$$\Delta^+(\mathfrak{a}_m) = \{\epsilon_i - \epsilon_j : 0 \leq i < j \leq m\}, \quad \Pi^+(\mathfrak{a}_m) = \{\epsilon_{i-1} - \epsilon_i : 1 \leq i \leq m\},$$

$$\{\partial_{x_i}, x_i\partial_{x_i} : 1 \leq i < j \leq m\}, \quad \{\partial_{x_1}, x_1\partial_{x_2}, \ldots, x_{m-1}\partial_{x_m}\},$$

$$\{x_1E_x, x_2\partial_{x_1}, \ldots, x_m\partial_{x_{m-1}}\}.$$

Since $\mathfrak{l}_m$ is a standard Levi subalgebra of $\mathfrak{a}_m$, it inherits a triangular decomposition with positive and simple systems

$$\Delta^+(\mathfrak{l}_m) = \Delta^+(\mathfrak{a}_m) \cap \Delta(\mathfrak{l}_m) = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq m\},$$

$$\Pi^+(\mathfrak{l}_m) = \Pi^+(\mathfrak{a}_m) \cap \Delta(\mathfrak{l}_m) = \{\epsilon_{i-1} - \epsilon_i : 2 \leq i \leq m\}.$$

Elements of $\mathfrak{h}^*_m$ are referred to as weights. We have seen that

$$\mathfrak{h}^*_m = \{\sum_{i=0}^m \lambda_i \epsilon_i : \sum_{i=0}^m \lambda_i = 0\}.$$

A weight $\sum_{i=0}^m \lambda_i \epsilon_i$ is $\mathfrak{a}_m$-dominant if $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_m$, and $\mathfrak{a}_m$-integral if $\lambda_{i-1} - \lambda_i \in \mathbb{Z}$ for $1 \leq i \leq m$. It is $\mathfrak{l}_m$-dominant if $\lambda_1 \geq \cdots \geq \lambda_m$, and $\mathfrak{l}_m$-integral if $\lambda_{i-1} - \lambda_i \in \mathbb{Z}$ for $2 \leq i \leq m$. 


Given any weight $\lambda$ and any $\mathfrak{h}_m$-module $V$, it is conventional to write $V_\lambda$ for the $\lambda$-weight space of $V$. Given any scalar $c$, it will be convenient to write $V_{(c)}$ for the $c$-eigenspace of the Euler operator $E_x$. Thus

$$V_\lambda = \{ v \in V : Hv = \lambda(H)v \ \forall \ H \in \mathfrak{h}_m \}, \quad V_{(c)} = \{ v \in V : E_x v = cv \}.$$ 

For $\lambda \in \mathfrak{h}_m^*$, the irreducible representation of $\mathfrak{l}_m$ of highest weight $\lambda$ is denoted by $L_{\mathfrak{l}_m}(\lambda)$. This module is finite dimensional if and only if $\lambda$ is $\mathfrak{l}_m$-dominant and $\mathfrak{l}_m$-integral. All of the irreducible representations of $\mathfrak{l}_m$ we will encounter in this article will be finite dimensional.

The algebra $\mathfrak{c}_m$ of constant vector fields is $\mathfrak{l}_m$-invariant and $\mathfrak{l}_m$-irreducible: it has highest weight vector $\partial_{x_m}$ and is isomorphic to $L_{\mathfrak{l}_m}(\epsilon_0 - \epsilon_m)$. The universal enveloping algebra of $\mathfrak{c}_m$ is simply the symmetric algebra $\mathbb{C}[\partial_x]$ of polynomials in the $\partial_x$. Its homogeneous component of degree $k$ is its $(-k)$-Euler operator eigenspace, which is $\mathfrak{l}_m$-irreducible and has highest weight vector $\partial_{x_m}^k$. Thus, as $\mathfrak{l}_m$-modules,

$$\mathbb{C}[\partial_x]|_{(-k)} = \text{Span}_C \{ \partial_{x_1}^i \cdots \partial_{x_m}^m : |I| = k \} \simeq L_{\mathfrak{l}_m}(k\epsilon_0 - k\epsilon_m).$$

### 2.2. Infinitesimal characters

We now give some information pertaining to the center $\mathfrak{z}(\mathfrak{a}_m)$ of $\mathfrak{u}(\mathfrak{a}_m)$. The half-sum $\rho_{\mathfrak{a}_m}$ of the positive roots of $\mathfrak{a}_m$ is

$$\rho_{\mathfrak{a}_m} := \sum_{0 \leq i < j \leq m} \frac{1}{2} (\epsilon_i - \epsilon_j) = \frac{1}{2} (m\epsilon_0 + (m-2)\epsilon_1 + \cdots + m\epsilon_m).$$

The dot action of the Weyl group $W(\mathfrak{a}_m)$ on $\mathfrak{h}_m^*$ is $w \cdot \lambda := w(\lambda + \rho_{\mathfrak{a}_m}) - \rho_{\mathfrak{a}_m}$. Regarding elements of $\mathfrak{u}(\mathfrak{h}_m)$ as polynomials on $\mathfrak{h}_m^*$ gives a corresponding dot action on $\mathfrak{u}(\mathfrak{h}_m)$, and we write $\mathfrak{u}(\mathfrak{h}_m)^{W(\mathfrak{a}_m)}$ for the subspace of dot action invariants.

Let $\mathfrak{n}_m^+$ and $\mathfrak{n}_m^-$ be the direct sums of the positive and negative root spaces of $\mathfrak{a}_m$, respectively, and note that $\mathfrak{h}_m$ contains the Borel subalgebra $\mathfrak{n}_m^+ \oplus \mathfrak{h}_m$ of $\mathfrak{a}_m$. The zero weight space $\mathfrak{u}(\mathfrak{a}_m)_0$ of $\mathfrak{u}(\mathfrak{a}_m)$ is a subalgebra in which $(\mathfrak{n}_m \mathfrak{u}(\mathfrak{a}_m) \mathfrak{n}_m)_0$ is a two-sided ideal, and we have the decomposition

$$\mathfrak{u}(\mathfrak{a}_m)_0 = (\mathfrak{n}_m^\perp \mathfrak{u}(\mathfrak{a}_m) \mathfrak{n}_m)_0 \oplus \mathfrak{u}(\mathfrak{h}_m).$$

The Harish-Chandra homomorphism $HC : \mathfrak{u}(\mathfrak{a}_m)_0 \to \mathfrak{u}(\mathfrak{h}_m)$ is the associated projection. It is a result of Harish-Chandra that $HC$ restricts to an isomorphism from $\mathfrak{z}(\mathfrak{a}_m)$ to $\mathfrak{u}(\mathfrak{h}_m)^{W(\mathfrak{a}_m)}$. Thus any weight $\lambda \in \mathfrak{h}_m^*$ defines a character

$$\chi_\lambda : \mathfrak{z}(\mathfrak{a}_m) \to \mathbb{C}, \quad \chi_\lambda(\Omega) := HC(\Omega)(\lambda).$$

All characters of $\mathfrak{z}(\mathfrak{a}_m)$ have this form, and two characters $\chi_\lambda$ and $\chi_{\lambda'}$ are equal if and only if $\lambda' \in W(\mathfrak{a}_m) \cdot \lambda$.

Let $M$ be any $\mathfrak{a}_m$-module. We define its $N$-step $\chi_\lambda$-submodule $M^{(\lambda,N)}$, its $\chi_\lambda$-submodule $M^\lambda$, and its generalized $\chi_\lambda$-submodule $M^{(\lambda)}$ by

$$M^{(\lambda,N)} := \{ u \in M : (\Omega - \chi_\lambda(\Omega))^N u = 0 \ \forall \ \Omega \in \mathfrak{z}(\mathfrak{a}_m) \},$$

$$M^\lambda := M^{(\lambda,1)}, \quad M^{(\lambda)} := \bigcup_{N=1}^{\infty} M^{(\lambda,N)}.$$
bases \( \{e_{ij} : 1 \leq i, j \leq m \} \) and \( \{e_{ji} : 1 \leq i, j \leq m \} \) of \( \mathfrak{gl}_m \) are mutually dual via the trace form, so \( \Omega_{\mathfrak{gl}_m} = \sum_{ij=1}^m e_{ij} \otimes e_{ji} \). Applying \( \phi \) gives

\[
\Omega_{\mathfrak{gl}_m} = \sum_{i=1}^m \sum_{j=1}^m x_i \partial_{x_j} \otimes x_j \partial_{x_i}.
\]

Similarly, \( \Omega_{\mathfrak{gl}_{m+1}} = \sum_{i,j=0}^m e_{ij} \otimes e_{ji} \). Since \( \sum_{i=0}^m e_{ii} = 0 \) in \( \mathfrak{pgl}_{m+1} \), we find

\[
\Omega_{\mathfrak{pgl}_{m+1}} = \Omega_{\mathfrak{gl}_m} + \left( \sum_{i=1}^m e_{ii} \right) \otimes \left( \sum_{i=1}^m e_{ii} \right) + \sum_{i=1}^m (2e_{i,0} \otimes e_{0,i} - (m+1)e_{ii}).
\]

Applying \( \phi \) to this gives

\[
\Omega_{\mathfrak{n}} = \Omega_{\mathfrak{gl}_m} + \mathcal{E}_x \otimes (\mathcal{E}_x - m - 1) - \sum_{i=1}^m x_i \mathcal{E}_x \otimes \partial_{x_i}.
\]

For \( \lambda = \sum_{i=1}^m \lambda_i \varepsilon_i \) in \( \mathfrak{h}^* \), let \( \|\lambda\|^2 := \sum_{i=0}^m \lambda_i^2 \). Standard computations give

\[
\text{HC}(\Omega_{\mathfrak{n}}) = \mathcal{E}_x \otimes (\mathcal{E}_x - m - 1) + \sum_{i=1}^m x_i \mathcal{E}_x \otimes (x_i \partial_{x_i} + m + 1 - 2i),
\]

\[
\chi_\lambda(\Omega_{\mathfrak{n}}) = \|\lambda + \rho_{\mathfrak{n}}\|^2 - \|\rho_{\mathfrak{n}}\|^2.
\]

2.3. Tensor products. We conclude Section 2 with a lemma applicable to any reductive Lie algebra \( \mathfrak{g} \). It is not new, but we give an elementary proof. Choose a Cartan subalgebra \( \mathfrak{h} \) and a positive root system \( \Delta^+(\mathfrak{g}) \), and let \( \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \) be the associated triangular decomposition of \( \mathfrak{g} \).

Let \( \lambda \in \mathfrak{h}^* \) be any weight and let \( V \) be any \( \mathfrak{h} \)-module. As is usual, we write \( V_\lambda \) for the \( \lambda \)-weight space of \( V \) and \( \text{Supp}(V) \) for the set of weights of \( V \). Let \( L(\lambda) \) be the irreducible representation of \( \mathfrak{g} \) of highest weight \( \lambda \).

Suppose that \( \lambda \) and \( \mu \) are dominant integral weights, so that \( L(\lambda) \) and \( L(\mu) \) are finite dimensional. Observe that

\[
(L(\lambda) \otimes L(\mu))_\nu = \bigoplus_{\sigma \in \mathfrak{h}^*} L(\lambda)_\sigma \otimes L(\mu)_{\nu - \sigma}.
\]

Let \( P_{\nu,\sigma}(\lambda, \mu) \) be the projection to \( L(\lambda)_\sigma \otimes \bigoplus_{\sigma' \neq \sigma} L(\lambda)_{\sigma'} \otimes L(\mu)_{\nu - \sigma'} \) along \( \bigoplus_{\sigma' \neq \sigma} L(\lambda)_{\sigma'} \otimes L(\mu)_{\nu - \sigma'} \):

\[
P_{\nu,\sigma}(\lambda, \mu) : (L(\lambda) \otimes L(\mu))_\nu \to L(\lambda)_\sigma \otimes L(\mu)_{\nu - \sigma}.
\]

Then for all \( w \in (L(\lambda) \otimes L(\mu))_\nu \), we have \( w = \sum_{\sigma} P_{\nu,\sigma}(\lambda, \mu)(w) \).

At this point let us establish the following notation. Given any Lie algebra \( \mathfrak{t} \) and any \( \mathfrak{t} \)-module \( V \), the \( \mathfrak{t} \)-invariant subspace of \( V \) is denoted by \( V^\mathfrak{t} \):

\[
V^\mathfrak{t} := \{v \in V : Xv = 0 \ \forall \ X \in \mathfrak{t} \}.
\]

Lemma 2.1. The restriction of \( P_{\nu,\lambda}(\lambda, \mu) \) to \( (L(\lambda) \otimes L(\mu))_{\nu}^\mathfrak{t} \) is injective.

Before the proof let us make some remarks. Of course, \( (L(\lambda) \otimes L(\mu))_{\nu}^\mathfrak{t} \) is non-zero if and only if \( \nu \) is dominant integral and \( L(\nu) \) occurs as a summand of \( L(\lambda) \otimes L(\mu) \). A corollary of the lemma is the well-known fact that this can occur only if \( \nu \in \lambda + \text{Supp}(L(\mu)) \). The lemma itself may be summarized as “any highest weight vector of \( L(\lambda) \otimes L(\mu) \) contains the highest weight vector of \( L(\lambda) \) as a factor of a summand”. 
Proof. Suppose that \( w \) is a non-zero highest weight vector of weight \( \nu \) in \( L(\lambda) \otimes L(\mu) \). Let \( \sigma_0 \) be maximal with respect to the order imposed by \( \Delta^+(g) \) such that \( P_{\nu,\sigma_0}(w) \) is non-zero. (We suppress the argument \((\lambda, \mu) \) of \( P_{\nu,\sigma}(\lambda, \mu) \).) We will show that \( P_{\nu,\sigma_0}(w) \) lies in \( L(\lambda)_{\sigma_0}^+ \otimes L(\mu)_{\nu-\sigma_0} \), which implies \( \sigma_0 = \lambda \), proving the result.

For each positive root \( \alpha \) in \( \Delta^+(g) \), fix a root vector \( E_\alpha \) in the \( \alpha \)-root space \( n_\alpha^+ \). It will suffice to prove that \( P_{\nu,\sigma_0}(w) \) lies in \( L(\lambda)_{\sigma_0}^+ \otimes L(\mu)_{\nu-\sigma_0} \) for all \( \alpha \).

For each \( \sigma \in \text{Supp}(L(\lambda)) \), let \( \{ v_{\sigma,i} : 1 \leq i \leq m(\lambda) \} \) be a basis of \( L(\lambda)_{\sigma} \), where \( m(\lambda) \) is the dimension of \( L(\lambda)_{\sigma} \). Then there are unique \( w_{\sigma,i} \in L(\mu)_{\nu-\sigma} \) such that

\[
P_{\nu,\sigma}(w) = \sum_{i=1}^{m(\lambda)} v_{\sigma,i} \otimes w_{\sigma,i}, \quad w = \sum_{\sigma \in \text{Supp}(L(\lambda))} \sum_{i=1}^{m(\lambda)} v_{\sigma,i} \otimes w_{\sigma,i}.
\]

By assumption, \( E_\alpha w \) and \( w_{\sigma_0+\alpha,i} \) are zero for all \( \alpha \in \Delta^+(g) \). Apply \( E_\alpha \) to the above expression for \( w \) and use these facts to obtain

\[
0 = P_{\nu+\alpha,\sigma_0}(E_\alpha w) = \sum_{i=1}^{m(\lambda)} (E_\alpha v_{\sigma_0,i}) \otimes w_{\sigma_0,i}.
\]

Now fix some choice of \( \alpha \in \Delta^+(g) \). By elementary \( sl_2 \) theory, it is possible to choose the basis \( \{ v_{\sigma,i} \} \) of \( L(\lambda) \) so that the set of all non-zero vectors of the form \( E_\alpha v_{\sigma,i} \) is linearly independent. For such a choice, the preceding paragraph shows that \( w_{\sigma_0,i} = 0 \) for all \( i \) such that \( E_\alpha v_{\sigma_0,i} \neq 0 \). Thus \( P_{\nu,\sigma_0}(w) \) lies in \( L(\lambda)_{\sigma_0}^+ \otimes L(\mu)_{\nu-\sigma_0} \). Although the basis which gives this result depends on \( \alpha \), the result itself holds for all \( \alpha \), proving the lemma. \( \square \)

3. Tensor field modules

Henceforth we shall always use Einstein’s implied summation notation except where explicitly indicated otherwise: in products, repeated lower case indices are summed over from 1 to \( m \). We also set

\[
x = (x_1, \ldots, x_m), \quad \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_m].
\]

Definition. Given any representation \( \phi \) of \( l_m \) on a finite dimensional complex vector space \( V \), the corresponding tensor field module of \( \text{Vec} \mathbb{R}^m \) is the vector space

\[
\mathcal{F}(V) := \mathbb{C}[x] \otimes V = \{ h : \mathbb{R}^m \to V : h \text{ a polynomial function} \}.
\]

The action of \( \text{Vec} \mathbb{R}^m \) is the Lie action \( \text{Lie}_\phi \). It is defined in terms of the \( \phi \)-divergence, \( \text{Div}_\phi \). Given a vector field \( X = X_j \partial_{x_j} \) and a function \( h \) in \( \mathcal{F}(V) \),

\[
\text{Div}_\phi(X) := X + \text{Div}_\phi(X), \quad \text{Lie}_\phi(X) := X + \text{Div}_\phi(X),
\]

\[
V := X_j \partial_{x_j} h + (\partial_{x_j} X_j) \phi(x_i \partial_{x_j}).
\]

It will be useful to state explicitly the following intrinsic characterization of tensor field modules. It is obvious from the fact that \( \text{Div}_\phi \) is zero on \( \epsilon_m \). Note that if \( M \) is any \( \text{Vec} \mathbb{R}^m \)-module, then \( M^\epsilon_m \) is an \( l_m \)-module.

Lemma 3.1. (i) \( \mathcal{F}(V)^{\epsilon_m} = V \).

(ii) A \( \text{Vec} \mathbb{R}^m \)-module \( M \) is isomorphic to a tensor field module if and only if it is isomorphic to \( \mathcal{F}(M^\epsilon_m) \).

(iii) If \( V' \) is an \( l_m \)-submodule of \( V \), then \( \mathcal{F}(V') = \mathbb{C}[x] \otimes V' \) is a \( \text{Vec} \mathbb{R}^m \)-submodule of \( \mathcal{F}(V) \).
As described in the introduction, Lie_φ is coinduced from the trivial extension of φ to Vec^0 \mathbb{R}^m. To explain, note that Vec^2_0 \mathbb{R}^m is an ideal in Vec^0 \mathbb{R}^m complementary to I_m, so we may extend φ by letting Vec^2_0 \mathbb{R}^m act trivially on V. Define Hom_{Vec^0 \mathbb{R}^m}(\Omega(Vec^0 \mathbb{R}^m), V) to be the space

\[ \{ \kappa : \Omega(Vec^0 \mathbb{R}^m) \rightarrow V : \kappa(\Theta Y) + \phi(Y)\kappa(\Theta) = 0 \ \forall \ Y \in Vec^0 \mathbb{R}^m \}. \]

It is a Vec^0 \mathbb{R}^m-module under the left action \((X\kappa)(\Theta) := -\kappa(X\Theta)\). The next lemma shows that as such, it is isomorphic to \(F(V)\). The proof is elementary.

Given \(h \in F(V)\), define eval_0(h) := h(0). For any Lie algebra \(\mathfrak{g}\), let \(\Theta \mapsto \Theta^T\) be the anti-automorphism of \(\Omega(\mathfrak{g})\) that is \(-1\) on \(\mathfrak{g}\). Define

\[ \hat{h} : \Omega(Vec^0 \mathbb{R}^m) \rightarrow V, \quad \hat{h}(\Theta) := \text{eval}_0 \circ \text{Lie}_\phi(\Theta^T)h. \]

**Lemma 3.2.**

(i) \(\hat{h} \in \text{Hom}_{Vec^0 \mathbb{R}^m}(\Omega(Vec^0 \mathbb{R}^m), V)\).

(ii) \(h \mapsto \hat{h}\) is a Vec^0 \mathbb{R}^m-isomorphism from \(F(V)\) to \(\text{Hom}_{Vec^0 \mathbb{R}^m}(\Omega(Vec^0 \mathbb{R}^m), V)\).

(iii) \(\langle h, \Theta \otimes \lambda \rangle := \lambda \circ \text{eval}_0 \circ \text{Lie}_\phi(\Theta^T)h\) is a non-degenerate Vec^0 \mathbb{R}^m-invariant pairing of \(F(V)\) and \(\Omega(Vec^0 \mathbb{R}^m) \otimes_{\Omega(Vec^0 \mathbb{R}^m)} V^\ast\).

In the case that \(\lambda\) is an \(l_m\)-dominant \(l_m\)-integral weight, we abbreviate \(F(L_{l_m}(\lambda))\) by \(F(\lambda)\) and we write \(\text{Div}_\lambda\) and \(\text{Lie}_\lambda\) for the associated divergence and Lie action. **Lemma 3.2(iii)** (or an easy direct argument) gives:

**Corollary 3.3.** Under \(a_m\), \(F(\lambda)\) has highest weight \(\lambda\), highest weight space \(L_{l_m}(\lambda)\), and infinitesimal character \(\chi_\lambda\). In particular, \(\text{Lie}_\lambda(\Omega_{a_m})\) is given by \([\mathfrak{g}]\).

It is simple to check that there are at most \(m + 1\) \(l_m\)-dominant weights in any \(W(a_m)\)-dot orbit. In particular, there are at most \(m + 1\) distinct tensor field modules with any given infinitesimal character.

Let us write \(\text{Lie}_\phi \big|_{a_m}\) explicitly. If \(f\) is a scalar function and \(X\) is a vector field, then \(\text{Div}_\phi(fX) = f \text{Div}_\phi(X) + (X_i \partial_{x_i}, f)\phi(x_i \partial_{x_i})\). With this formula we find

\[ \text{Lie}_\phi(\partial_{x_i}) = \partial_{x_i}, \quad \text{Lie}_\phi(x_i \partial_{x_i}) = x_i \partial_{x_j} + \phi(x_i \partial_{x_j}), \]

\[ \text{Lie}_\phi(\mathcal{E}_x) = \mathcal{E}_x + \phi(\mathcal{E}_x), \quad \text{Lie}_\phi(x_i \mathcal{E}_x) = x_i(\mathcal{E}_x + \phi(\mathcal{E}_x)) + x_j \phi(x_i \partial_{x_j}). \]

In the case that the \(l_m\)-module \((\phi, V)\) is not necessarily irreducible, it will be illuminating to have an explicit formula for the action of \(\Omega_{a_m}\) on \(F(V)\). The proof of the following lemma is straightforward from \([9], [7], \) and \([9]\).

**Lemma 3.4.**

(i) \(\text{Lie}_\phi(\Omega_{a_m}) = \phi(\Omega_{a_m}) + \mathcal{E}_x(\mathcal{E}_x + m - 1) + 2\phi(x_i \partial_{x_j})x_j \partial_{x_i}\).

(ii) \(\text{Lie}_\phi(\Omega_{a_m}) = \phi(\Omega_{a_m}) + \phi(\mathcal{E}_x)(\phi(\mathcal{E}_x) - m - 1)\).

### 4. Linear differential operators

In this section we review the Lie action of Vec\( \mathbb{R}^m\) on spaces of differential operators between tensor field modules. The aim is to give the action of \(\Omega_{a_m}\) on these spaces. This action was previously computed by Mathonet and Radow in their investigation of conditions under which \(a_m\) acts semisimply \([MR07]\). We include a derivation here, both to establish our notation and because we give the action in a different form.

For the purposes of computation the Lie action on differential operators is transferred to the symbol spaces, where it is given in terms of the symbol calculus associated to the normal ordering. In order to clarify the exposition for readers unfamiliar with this calculus, we begin with differential operators on scalar functions.
4.1. **Scalar-valued differential operators.** For $I \in \mathbb{N}^m$, let $\partial^I_x$ denote $\partial_{x_1}^{I_1} \cdots \partial_{x_m}^{I_m}$. The algebra of linear differential operators on $\mathbb{C}[x]$ with polynomial coefficients is

$$D := \mathbb{C}[x, \partial_x] = \text{Span}_\mathbb{C}\{x^I \partial_x^J : I, J \in \mathbb{N}^m\}.$$ 

It is a non-commutative algebra filtered by order: setting $|J| := \sum^m J_i$, the subspace of operators of order $\leq k$ is

$$D^k := \text{Span}_\mathbb{C}\{x^I \partial_x^J : I, J \in \mathbb{N}^m, |J| \leq k\}.$$ 

We write $\text{Comp}$ for composition of differential operators:

$$\text{Comp} : D^{k'} \otimes D^k \to D^{k'+k}, \quad \text{Comp}(D' \otimes D) := D' \circ D.$$ 

The graded algebra $S$ associated to $D$ is the algebra of *symbols*. It is conventional to denote the symbol of $\partial_x^I$ by $\xi^I$, so that

$$S := \mathbb{C}[x, \xi] := \text{Span}_\mathbb{C}\{x^I \xi^J : I, J \in \mathbb{N}^m\}.$$ 

This algebra is commutative, with $k$th homogeneous component

$$S^k := \text{Span}_\mathbb{C}\{x^I \xi^J : I, J \in \mathbb{N}^m, |J| = k\}.$$ 

We write $\text{Mult}$ for symbol multiplication and $\text{Symb}^k$ for the canonical projection:

$$\text{Mult} : S^{k'} \otimes S^k \to S^{k'+k}, \quad \text{Mult}(\Xi' \otimes \Xi) := \Xi' \Xi$$

$$\text{Symb}^k : D^k \to S^k,$$ 

$$\text{Symb}^k(\sum_{|J| \leq k} D_J(x) \partial_x^J) := \sum_{|J| = k} D_J(x) \xi^J.$$ 

The maps $\text{Comp}$, $\text{Mult}$, and $\text{Symb}^k$ are all $\text{Vec} \mathbb{R}^m$-covariant.

The *normal order total symbol* is the linear bijection

$$\text{NS} : D \to S, \quad \text{NS}(x^I \partial_x^J) := x^I \xi^J.$$ 

It preserves order and symbols in the following sense:

$$\text{NS}^{-1}(S^k) \subseteq D^k,$$ 

$$\text{Symb}^k \circ \text{NS}^{-1}\big|_{S^k} = 1 : S^k \to S^k.$$ 

The *normal symbol calculus* is a means to give certain types of operators on $D$ expressions amenable to computation. It consists in conjugating the operators by $\text{NS}$ to obtain operators on symbol spaces, which turn out to be differential operators in the variables $x$ and $\xi$. In the literature this conjugation is often suppressed, but for precision we shall usually write it explicitly.

We will regard elements of $D$ as differential operators on $S$ which commute with constant symbols such as $\xi^J$. For example, $(x^I \partial_x^J)(x^I \xi^J) := x^I(\partial_x^J x^I)\xi^J$. Functions $f(x) \in \mathbb{C}[x]$ may be regarded as order 0 differential operators. Observe that $\exp(\partial_{\xi^I} \otimes \partial_{x^J})$ is an endomorphism of $S \otimes S$. The following lemma gives an elegant expression for $\text{Comp}$. It is well-known and simple to verify.

**Lemma 4.1.**

(i) $\text{NS} \circ \text{Comp} \circ (\text{NS}^{-1} \otimes \text{NS}^{-1}) = \text{Mult} \circ \exp(\partial_{\xi^I} \otimes \partial_{x^J})$.

(ii) In particular, $\text{NS}(\partial_x^I \circ f(x)) = \sum_{J \in \mathbb{N}^m} \prod_{r=1}^m (\frac{I_r}{J_r})(\partial_x^J)\xi^{I-J}$.

The *Lie action* $\text{Lie} \text{Vec} \mathbb{R}^m$ on $D$ is defined to be the adjoint action. Conjugation by $\text{NS}$ yields an action $\text{Lie}^{\text{NS}}$ of $\text{Vec} \mathbb{R}^m$ on $S$:

$$\text{Lie}(X)(D) := X \circ D - D \circ X, \quad \text{Lie}^{\text{NS}}(X) := \text{NS} \circ \text{Lie}(X) \circ \text{NS}^{-1}.$$ 

**Lemma 4.2.** $\text{Lie}^{\text{NS}}(X) = X - \sum_{|I| > 0} \frac{1}{|I|!} \partial_x^I (\text{NS}(X)) \partial_x^I$. 
Proof. Take $X = X_j \partial_{x_j}$ and $D = \sum_j D_j(x) \partial_{x_j}$, so that $\text{NS}(X) = X_j \xi_j$ and $\text{NS}(D) = \sum_j D_j \xi_j$. Lemma 1.1 gives the following formulas, proving the result.

$$
\text{NS}(X \circ D) = \sum (X_j D_j \xi_j + X(D_j) \xi_j), \quad \text{NS}(D \circ X) = \sum \frac{1}{m!} \partial_\xi^m (X_j \xi_j) \partial_\xi^m (D_j \xi_j). \quad \square
$$

Because composition of differential operators is commutative at the level of symbols, the Lie action on $\mathcal{D}$ preserves the filtration $(\mathcal{D}^k)_k$. Therefore it defines a graded action of $\text{Vec} \mathbb{R}^m$ on $\mathcal{S}$, the symbol action $\text{Lie}^\mathcal{S}$. It is clear that $\text{Lie}^\mathcal{S}(X)$ is the part of $\text{Lie} \text{NS}(X)$ that preserves $\xi$-degree:

$$
\text{Lie}^\mathcal{S}(X) = X - (\partial_x X_j) \xi_j \partial_{\xi_i}.
$$

Observe that $\text{Lie}^\mathcal{S}$ and $\text{Lie} \text{NS}$ are equal on $\mathfrak{b}_m$, and both are the identity on $\mathfrak{c}_m$:

$$
\text{Lie}^\mathcal{S} \big|_{\mathfrak{b}_m} = \text{Lie} \text{NS} \big|_{\mathfrak{b}_m} : \partial_{x_j} \mapsto \partial_{x_j}, \quad x_i \partial_{x_j} \mapsto x_i \partial_{x_j} - \xi_j \partial_{\xi_i}.
$$

Thus NS is a $\mathfrak{b}_m$-isomorphism. Under both $\text{Lie}^\mathcal{S}$ and $\text{Lie} \text{NS}$ we have $\mathcal{S}^\mathcal{S} = \mathbb{C}[\xi]$, and both actions restrict to the same representation $\phi_{\xi}$ of $\mathfrak{l}_m$ on $\mathbb{C}[\xi]$. Let us write $\text{Div}_\xi$ for the divergence $\text{Div}_{\phi_{\xi}}$ associated to $\phi_{\xi}$. Then

$$
\phi_{\xi}(x_i \partial_{x_j}) := -\xi_j \partial_{\xi_i}, \quad \text{Div}_\xi(X) = -(\partial_x X_j) \xi_j \partial_{\xi_i}, \quad \text{Lie}^\mathcal{S}(X) = X + \text{Div}_\xi(X).
$$

In particular, $\mathcal{S}$ is in fact the tensor field module $\mathcal{F}(\mathcal{S}^\mathcal{S})$. We remark that since the representations $(\mathcal{D}, \text{Lie})$ and $(\mathcal{S}, \text{Lie} \text{NS})$ of $\text{Vec} \mathbb{R}^m$ are isomorphic by construction, it follows from Lemma 3.1(ii) that $(\mathcal{D}, \text{Lie})$ is isomorphic to a tensor field module if and only if it is isomorphic to $(\mathcal{S}, \text{Lie}^\mathcal{S})$. It is well-known that this is false, but true (for scalar differential operators) if the modules are restricted to $\mathfrak{a}_m$. Thus $\mathcal{D}$ is not isomorphic to a tensor field module, but its restriction to $\mathfrak{a}_m$ is.

By (10), the $\mathfrak{l}_m$-submodule $(\mathcal{S}^\mathcal{S})^\mathfrak{l}_m = \mathbb{C}[\xi](-k)$ of $\mathcal{S}^\mathcal{S}_m$ is irreducible with highest weight $k(\epsilon_0 - \epsilon_m)$ and highest weight vector $\xi_k^\mathfrak{a}$. Therefore as $\text{Vec} \mathbb{R}^m$ modules,

$$
(\text{Lie}^\mathcal{S}, \mathcal{S}^\mathcal{S}) = \mathbb{C}[x] \otimes \mathbb{C}[\xi](-k) \simeq \mathcal{F}(k \epsilon_0 - k \epsilon_m).
$$

We now compute $\text{Lie}^\mathcal{S}(\Omega_{\mathfrak{a}_m})$ and $\text{Lie} \text{NS}(\Omega_{\mathfrak{a}_m})$. Set $\mathcal{E}_\xi := \xi_i \partial_{\xi_i}$, the $\xi$-Euler operator. Since $\mathcal{S}$ is a tensor field module, (6) and Lemma 3.1 give

$$
\phi_{\xi}(\mathcal{E}_\xi) = -\mathcal{E}_\xi, \quad \phi_{\xi}(\mathfrak{l}_m) = \xi_j \partial_{\xi_j} \xi_i \partial_{\xi_i} = \mathcal{E}_\xi(\mathcal{E}_\xi + m - 1), \quad \text{Lie}^\mathcal{S}(\Omega_{\mathfrak{a}_m}) = 2 \mathcal{E}_\xi(\mathcal{E}_\xi + m).
$$

Using (6) and Lemma 3.1 we find

$$
\text{Lie}^\mathcal{S}(x_i \mathcal{E}_x) = x_i(\mathcal{E}_x - \mathcal{E}_\xi) - x_j \xi_j \partial_{\xi_i}, \quad \text{Lie} \text{NS}(x_i \mathcal{E}_x) = \text{Lie}^\mathcal{S}(x_i \mathcal{E}_x) - \mathcal{E}_\xi \partial_{\xi_i}.
$$

As noted, the part of $\text{Lie} \text{NS}(\Omega_{\mathfrak{a}_m})$ which preserves $\xi$-degree is $\text{Lie}^\mathcal{S}(\Omega_{\mathfrak{a}_m})$. Since $\text{Lie} \text{NS} \big|_{\mathfrak{b}_m}$ preserves $\xi$-degree, (7) shows that

$$
\text{Lie} \text{NS}(\Omega_{\mathfrak{a}_m}) = -2(\text{Lie} \text{NS}(x_i \mathcal{E}_x) - \text{Lie}^\mathcal{S}(x_i \mathcal{E}_x)) \partial_{x_i} = 2 \mathcal{E}_\xi \partial_{\xi_i} \partial_{x_i}.
$$

The operator $\partial_{\xi_i} \partial_{x_i} : \mathcal{S} \to \mathcal{S}$ occurring here is the standard divergence operator on symbols, an operator of $\xi$-degree $-1$ usually denoted by $\text{Div}$. Thus we have

$$
\text{Lie} \text{NS}(\Omega_{\mathfrak{a}_m}) = 2 \mathcal{E}_\xi(\mathcal{E}_\xi + \text{Div} + m).
$$

Keep in mind that the terms in this product do not commute: $[\mathcal{E}_\xi, \text{Div}] = - \text{Div}$.
Remark. The notation Div appears to conflict with our notation Div\(_{\phi}\) for the \(\phi\)-
divergence of vector fields, but in fact the notations are consistent in the following
sense. The trace representation of \(l_m\) on \(\mathbb{C}\) is \(\text{tr} : x_i \partial x_j \mapsto \delta_{ij}\), and the standard
divergence \(X \mapsto \partial_x X_i\) of vector fields is both Div\(_{\text{tr}}(X)\) and Div \(\circ\) NS\((X)\). Thus the
divergence Div on symbols is Div\(_{\text{tr}}\). It is well known that it is in fact the unique
\(b_m\)-invariant operator from \(S^k\) to \(S^{k-1}\), and for \(k \geq k'\), Div\(_{k-k'}\) is the unique \(b_m\)-
invariant operator from \(S^k\) to \(S^{k'}\). We will comment on ways to regard Div\(_{\phi}\) as an
operator on symbols in Section 4.

4.2. Vector-valued differential operators. We now turn to differential opera-
tors between tensor field modules. Given any two finite dimensional represen-
tations \((\phi, V)\) and \((\phi', V')\) of \(l_m\), write \(\text{hom}(\phi, \phi')\) (or simply hom) for the adjoint action
of \(l_m\) on Hom\((V, V')\).

Let \(D(V, V')\) be the space of linear differential operators from \(F(V)\) to \(F(V')\)
with polynomial coefficients, and let \(\{D^k(V, V')\}\) be its order filtration:
\[D(V, V') := D \otimes \text{Hom}(V, V'), \quad D^k(V, V') := D^k \otimes \text{Hom}(V, V').\]
Elements of \(D(V, V')\) are maps from \(F(V)\) to \(F(V')\) as follows: for \(D \in D, \tau \in \text{Hom}(V, V'), f \in \mathbb{C}[x], \) and \(v \in V,\)
\[(D \otimes \tau)(f \otimes v) := D(f) \otimes \tau(v).\]

Let \((\phi'', V'')\) be a third representation of \(l_m\). Composition is
\[
\text{Comp} : D^k(V', V'') \otimes D^k(V, V') \rightarrow D^{k+k}(V, V''),
\]
(12)
\[
\text{Comp}((D' \otimes \tau') \otimes (D \otimes \tau)) := (D' \circ D) \otimes (\tau' \circ \tau).
\]
Note that for \(V = V'\), \(D(V, V)\) is an algebra in which \(D \otimes 1\) and \(1 \otimes \text{End}(V)\)
commute. In other words, \(D(V, V) = D \otimes \text{End}(V)\) in the category of algebras.

The graded space \(S(V, V')\) of symbols associated to \(D(V, V')\) and its \(k\)th ho-

geneous component \(S^k(V, V')\) are
\[S(V, V') := S \otimes \text{Hom}(V, V'), \quad S^k(V, V') := S^k \otimes \text{Hom}(V, V').\]
Again we have multiplication, projection to symbols, and the normal symbol:
\[
\text{Mult} : S^k(V', V'') \otimes S^k(V, V') \rightarrow S^{k+k}(V, V''),
\]
\[
\text{Mult}((\Xi' \otimes \tau') \otimes (\Xi \otimes \tau)) := (\Xi' \Xi) \otimes (\tau' \circ \tau);
\]
(13)
\[
\text{Symb}^k : D^k(V, V') \rightarrow S^k(V, V'), \quad \text{Symb}^k(D \otimes \tau) := \text{Symb}^k(D) \otimes \tau;
\]
\[
\text{NS} : D(V, V') \rightarrow S(V, V'), \quad \text{NS}(D \otimes \tau) := \text{NS}(D) \otimes \tau.
\]
As in the scalar case, Comp, Mult, and Symb are \(\text{Vec} \mathbb{R}^m\)-covariant, and
\[
\text{NS}^{-1}(S^k(V, V')) \subseteq D^k(V, V'), \quad \text{Symb}^k \circ \text{NS}^{-1} |_{S^k(V, V')} = 1.
\]

The Lie action \(\text{Lie}_{\phi, \phi'}\) of \(\text{Vec} \mathbb{R}^m\) on \(D(V, V')\) is the adjoint action
\[
\text{Lie}_{\phi, \phi'}(X)(T) := \text{Lie}_{\phi'}(X) \circ T - T \circ \text{Lie}_{\phi}(X).
\]
We transfer it via NS to the isomorphic representation \(\text{Lie}_{\phi, \phi'}^\text{NS}\) on \(S(V, V')\):
\[
\text{Lie}_{\phi, \phi'}^\text{NS}(X) := \text{NS} \circ \text{Lie}_{\phi, \phi'}(X) \circ \text{NS}^{-1}.
\]
Since \(S(V, V') = S \otimes \text{Hom}(V, V')\), we may regard elements of
\[\mathbb{C}[x, \partial_x, \xi, \partial_x] \otimes \text{End}(\text{Hom}(V, V'))\]
as operators on it. In particular, if $\pi$ is any representation of $I_m$ on $\text{Hom}(V, V')$, then $\text{Div}_\pi(X) = (O_x, X)\pi(x, \partial_x)$ acts on $S(V, V')$. We will need $\text{Div}_\pi$ for both $\pi = \text{hom}(\phi, \phi')$ and $\pi = \rho_{\phi}$, the right representation $\rho_{\phi}(Y)(\tau) := -\tau \circ \phi(Y)$.

We now generalize Lemma 4.2 to the vectorial-valued case.

**Lemma 4.3.** $\text{Lie}_{\phi, \phi'}^{NS}(X) = X + \text{Div}_{\text{hom}}(X) + \sum_{|I| > 0} \frac{1}{I!} \partial_x^I (\text{Div}_{\rho_{\phi}}(X) - \text{NS}(X)) \partial_x^I$.

**Proof.** Take $X = \sum_j X_j \partial_x^j$ and $T = \sum_j T_j \partial_x^j$, where $T_j \in \mathbb{C}[x] \otimes \text{Hom}(V, V')$. Then $\text{NS}(X) = X_j \xi_j$ and $\text{NS}(T) = \sum_j T_j \xi_j$. Note that

$$\text{Lie}_{\phi, \phi'}^{NS}(X)(T) = (X \circ T - T \circ X) + (\text{Div}_{\phi'}(X) \circ T - T \circ \text{Div}_\phi(X)).$$

By (12) and (13), Comp factors as $\text{Comp} \otimes \text{Comp}$ and NS factors as $\text{NS} \otimes 1$. It follows that Lemma 4.1 applies just as in the proof of Lemma 4.2 to give

$$\text{NS}(X(\circ T - T \circ X)) = \sum_{|I| > 0} \frac{1}{I!} \partial_x^I (\text{NS}(X))(\partial_x^I).$$

Therefore $\text{NS}(T) \mapsto \text{NS}(X(\circ T - T \circ X))$ is the operator $X - \sum_{|I| > 0} \frac{1}{I!} \partial_x^I (\text{NS}(X)) \partial_x^I$.

It remains to prove that

$$\text{NS}(T) \mapsto \text{NS}(\text{Div}_{\phi'}(X) \circ T - T \circ \text{Div}_\phi(X))$$

is the operator $\text{Div}_{\text{hom}}(X) + \sum_{|I| > 0} \frac{1}{I!} \partial_x^I (\text{Div}_{\rho_{\phi}}(X)) \partial_x^I$.

Since $\text{Div}_{\phi'}(X) = (\partial_x, X_j)\phi'(x, \partial_x)$ is an order $0$ operator on $S(V')$,

$$\text{NS}(\text{Div}_{\phi'}(X) \circ T) = (\partial_x, X_j)\phi'(x, \partial_x) \text{NS}(T).$$

By Lemma 4.1, $\text{NS}((T \circ \text{Div}_\phi(X))$ is

$$\text{NS}(\sum_j (T \circ \phi(x, \partial_x)) (\partial_x^j \circ (\partial_x, X_j))) = \sum_{I, j} -\frac{1}{I!} (\partial_x^I \partial_x, X_j) (\rho_{\phi}(x, \partial_x)(T_j)) (\partial_x^I).$$

The $I = 0$ terms combine with the $\text{Div}_{\phi'}(X)$ terms to give $\text{Div}_{\text{hom}}(X) \text{NS}(T)$, and the $|I| > 0$ terms give the $\text{Div}_{\rho_{\phi}}$ terms in the lemma. $\square$

As in the scalar case, $\text{Lie}_{\phi, \phi'}^{NS}$ preserves the order filtration, and so its $\xi$-degree $0$ component is a graded action on $S(V, V')$, the **symbol action** $\text{Lie}_{\phi, \phi'}^{S}$:

$$\text{Lie}_{\phi, \phi'}^{S}(X) = X + \text{Div}_\xi(X) + \text{Div}_{\text{hom}}(X) = X + \text{Div}_{\phi \otimes \text{hom}}(X).$$

Thus $\text{Lie}_{\phi, \phi'}^{S} = \text{Lie}_{\phi \otimes \text{hom}}$, and $S(V, V')$ is a tensor field module:

$$S(V, V') = \mathcal{F}(S(V, V') V_m) = \mathcal{F}(\mathbb{C}[x] \otimes \text{Hom}(V, V')).$$

At this point we are prepared to give explicit formulas for the restrictions of $\text{Lie}_{\phi, \phi'}^{S}$ and $\text{Lie}_{\phi, \phi'}^{NS}$ to $a_m$. As before, they are equal on $b_m$ and the identity on $c_m$:

$$\text{Lie}_{\phi, \phi'}^{S} \big|_{b_m} = \text{Lie}_{\phi, \phi'}^{NS} \big|_{b_m} : \partial_{x_j} \mapsto \partial_{x_j}, \quad x_i \partial_{x_j} \mapsto x_i \partial_{x_j} - \xi_j \partial_{\xi_i} + \text{hom}(x_i \partial_{x_j}).$$

Use (13) to obtain $\text{Lie}_{\phi, \phi'}^{S}(x_i E_x)$, and then Lemma 13 to derive $\text{Lie}_{\phi, \phi'}^{NS}(x_i E_x)$:

$$\text{Lie}_{\phi, \phi'}^{S}(x_i E_x) = x_i (E_x - E_\xi + \text{hom}(E_x)) - x_j \xi_j \partial_{\xi_i} + x_j \text{hom}(x_i \partial_{x_j}),$$

$$\text{Lie}_{\phi, \phi'}^{NS}(x_i E_x) = \text{Lie}_{\phi, \phi'}^{S}(x_i E_x) - E_\xi \partial_{\xi_i} + \rho_\phi(E_x) \partial_{\xi_i} + \rho_\phi(x_i \partial_{x_j}) \partial_{\xi_j}.$$

Finally we can deduce actions of $\Omega_{a_m}$. As we mentioned, they were obtained in [MR07] in a different form. In our applications $\phi$ will be scalar, so we give a simplified expression in that case. Recall the symbol divergence $\text{Div} = \partial_{\xi_i} \partial_{x_i}$, and for $\gamma \in \mathbb{C}$, the $1$-dimensional $I_m$-module $\mathbb{C}_\gamma$ on which $x_i \partial_{x_j}$ acts by $\gamma \delta_{ij}$.
Proof. For (i), apply Lemma 3.4 to Lie $\mathcal{D}$. The result follows from

$$ (\phi_{\xi} \otimes \text{hom})(\Omega_{\gamma}) = \text{hom}(\xi_{\gamma}) = \Omega_{\gamma} - \xi_{\gamma}, $$

for each $\xi_{\gamma}$.

For (ii), the idea leading to (11) gives

$$ (\phi_{\xi} \otimes \text{hom})(\Omega_{\gamma}) = (\text{hom}(x_{i}\partial_{x_{i}}) - \xi_{i}\partial_{x_{i}})(\text{hom}(x_{j}\partial_{x_{j}}) - \xi_{j}\partial_{x_{j}}) $$

$$ = \text{hom}(\Omega_{\gamma}) = \text{hom}(\xi_{\gamma}) - \text{hom}(x_{i}\partial_{x_{i}})\xi_{i}\partial_{x_{i}} + \xi_{\gamma}, $$

for future reference.

5. Repeated infinitesimal characters

Let $\gamma$ be any scalar and let $\delta$ be any $\ell_{m}$-dominant $\ell_{m}$-integral highest weight.

Throughout this section we work with the modules from (1):

$$ \mathcal{D}_{\gamma}(\delta) := \mathcal{D}(\mathbb{C}_{\gamma},\mathbb{C}_{\gamma} \otimes L_{\ell_{m}}(\delta)), \quad \mathcal{S}(\delta) := \mathcal{S}(\mathbb{C}_{\gamma},\mathbb{C}_{\gamma} \otimes L_{\ell_{m}}(\delta)). $$

The notation $\mathcal{S}(\delta)$ is justified by the fact that the symbol module depends only on $\delta$, because $\text{Hom}(\mathbb{C}_{\gamma},\mathbb{C}_{\gamma} \otimes L_{\ell_{m}}(\delta)) = L_{\ell_{m}}(\delta)$.

The purpose of the section is to determine when $\mathcal{D}_{\gamma}(\delta)$ has two tensor field subquotients with the same infinitesimal character. This is equivalent to determining when $\mathcal{S}(\delta)$ has two tensor field submodules with the same infinitesimal character.

Combining (10) and (13) we see that as $\text{Vec} \mathbb{R}^{\ell_{m}}$-modules, $\mathcal{S}^{k}(\delta) = \mathcal{F}(\mathbb{C}[\xi],[-k] \otimes L_{\ell_{m}}(\delta)) \simeq \mathcal{F}(L_{\ell_{m}}(k\epsilon_{0} - k\epsilon_{m}) \otimes L_{\ell_{m}}(\delta)).$

The decomposition of $L_{\ell_{m}}(k\epsilon_{0} - k\epsilon_{m}) \otimes L_{\ell_{m}}(\delta)$ into irreducible summands is a special case of the Littlewood-Richardson rule. In order to state the result, set

$$ \lambda_{K} := |K|\epsilon_{0} - \sum_{i}^{m} K_{i} \epsilon_{i} \quad \text{for} \quad K \in \mathbb{N}^{m}, $$

$$ \kappa(\delta,k) := \{ K \in \mathbb{N}^{m} : |K| = k, \quad K_{i} \leq \delta_{i} - \delta_{i+1} \quad \text{for} \quad i < m \}, $$

$$ e_{i} := \text{the} \ i^{th} \ \text{standard basis vector of} \ \mathbb{C}^{m}. $$

Proposition 5.1. As $\ell_{m}$-modules, $L_{\ell_{m}}(\lambda_{k\epsilon_{m}}) \otimes L_{\ell_{m}}(\delta) \simeq \bigoplus_{K \in \kappa(\delta,k)} L_{\ell_{m}}(\delta + \lambda_{K}).$

For future reference, at this point we specify a highest weight vector $v_{\delta}^{K}$ of weight $\delta + \lambda_{K}$ for each $K$ in $\kappa(\delta,k)$. Fix a highest weight vector $v_{\delta}$ of $L_{\ell_{m}}(\delta)$.

Lemma 5.2. For $K \in \kappa(\delta,k)$, there exists a unique highest weight vector $v_{\delta}^{K}$ in $\mathbb{C}[\xi][-k] \otimes L_{\ell_{m}}(\delta)$ of weight $\delta + \lambda_{K}$ with the following properties:

(i) $v_{\delta}^{K} = \sum_{|J|=k} \xi_{J} \otimes v_{\delta}(J)$, where $v_{\delta}(J) \in L_{\ell_{m}}(\delta)\delta + \lambda_{K} - \lambda_{J}$. 

In particular, must be a transposition (Lemma 2.1 to see that the projection \( v_\delta \) of weight \( \delta \) to its codomain is the line \( \mathbb{C}v_\delta \)). We claim that there exists a unique highest weight vector \( v^K_\delta \) of weight \( \delta + \lambda_K \) satisfying (ii). By Lemma 3.1, its domain contains a single line of highest weight vectors, and since \( L_{\lambda_{k_{\operatorname{erm}}} m} \) has 1-dimensional weight spaces, its codomain is the line \( \mathbb{C}v_\delta \otimes \xi^K \). Let \( v^K_\delta \) be the unique highest weight vector projecting to \( v_\delta \otimes \xi^K \). This proves the claim.

Finally, \( L_{\lambda_{k_{\operatorname{erm}}} m} \) is zero unless \( \lambda - \lambda_K \) is a non-negative integer combination of the simple root vectors \( \epsilon_i - \epsilon_j \), so \( v^K_\delta \) also satisfies (iii). \( \square \)

For \( K \in \kappa(\delta, k) \), write \( L_{\lambda_{k_{\operatorname{erm}}} m}(v^K_\delta) \) for the copy of \( L_{\lambda_{k_{\operatorname{erm}}} m}(\delta + \lambda_K) \) in \( \mathbb{C}[\xi](-\delta) \otimes L_{\lambda_{k_{\operatorname{erm}}} m}(\delta) \), with highest weight vector \( v^K_\delta \). For \( m \in \mathbb{C}[\xi](-\delta) \)-valued polynomials:

\[
L_{\lambda_{k_{\operatorname{erm}}} m}(v^K_\delta) := \mathcal{U}(L_{\lambda_{k_{\operatorname{erm}}} m}(\delta)) v^K_\delta, \quad \mathcal{F}(v^K_\delta) := \mathbb{C}[x] \otimes L_{\lambda_{k_{\operatorname{erm}}} m}(v^K_\delta).
\]

By Lemma 3.3, \( \mathcal{F}(v^K_\delta) \) is a \( \mathbb{C}[\xi] \)-submodule of \( S^K_\delta \) isomorphic to the tensor field module \( \mathcal{F}(\delta + \lambda_K) \). Combining this with Proposition 5.1 gives

\[
\mathbb{C}[\xi](-\delta) \otimes L_{\lambda_{k_{\operatorname{erm}}} m}(\delta) = \bigoplus_{K \in \kappa(\delta, k)} L_{\lambda_{k_{\operatorname{erm}}} m}(v^K_\delta), \quad S^K_\delta = \bigoplus_{K \in \kappa(\delta, k)} \mathcal{F}(v^K_\delta).
\]

The following proposition is the goal of the section. It sharpens the second sentence of Proposition 5.3. Recall the definitions of \( i(\delta) \), \( \tilde{\delta}_i \), resonant \( \delta \), and \( i(\delta, k) \) preceding that proposition.

**Proposition 5.3.** Fix non-negative integers \( k \geq k' \) and distinct multi-indices \( K \in \kappa(\delta, k) \) and \( K' \in \kappa(\delta, k') \). The \( a_{\operatorname{erm}} \)-infinitesimal characters of \( \mathcal{F}(v^K_\delta) \) and \( \mathcal{F}(v^{K'}_\delta) \) are equal if and only if \( i(\delta, k) \) is resonant, \( i(\delta, k) < k + K_i(\delta, k) \), and

\[
K' = K - de_{i(\delta, k)}, \quad \text{where} \quad d = k + K_i(\delta, k) - \tilde{\delta}_i(\delta, k).
\]

**Proof.** By Corollary 5.3, \( \mathcal{F}(v^K_\delta) \) has \( a_{\operatorname{erm}} \)-infinitesimal character \( \chi_{\delta + \lambda_K} \). As described in Section 2.2, \( \chi_{\delta + \lambda_K} = \chi_{\delta + \lambda_{k_{\operatorname{erm}}} m} \) if and only if \( \delta + \lambda_K + \rho_{\operatorname{erm}} \) and \( \delta + \lambda_{k_{\operatorname{erm}}} m + \rho_{\operatorname{erm}} \) are in the same \( S_{m+1} \)-orbit. Subtracting the \( S_{m+1} \)-stable vector \( (\delta_0 + \frac{1}{2} m)(1, \ldots, 1) \), the condition is that for some \( w \) in \( S_{m+1} \),

\[
(k', \tilde{\delta}_1 - K_1', \ldots, \tilde{\delta}_m - K_m') = w(k, \tilde{\delta}_1 - K_1, \ldots, \tilde{\delta}_m - K_m).
\]

From the definition of \( \kappa(\delta, k) \), we find that

\[
\tilde{\delta}_1 \geq \tilde{\delta}_1 - K_1 > \tilde{\delta}_2 \geq \tilde{\delta}_2 - K_2 > \tilde{\delta}_3 \geq \cdots > \tilde{\delta}_m \geq \tilde{\delta}_m - K_m,
\]

and similarly for \( K' \). Therefore \( \tilde{\delta}_i - K_i' \neq \tilde{\delta}_j - K_j \) for any \( i \neq j \), so if \( w \) exists, it must be a transposition \( (0 i) \). Check that (17) holds for \( w = (0 i) \) if and only if

\[
k' = \tilde{\delta}_i - K_i, \quad K'_i = \tilde{\delta}_i - k, \quad K'_j = K_j \quad \text{for} \quad j \neq i.
\]

We are assuming \( K \neq K' \) and \( k \geq k' \), so we must have \( k - k' = K_i - K_i' > 0 \). In particular, \( k, K_i, \) and \( \tilde{\delta}_i = k + K_i' \) are all in \( \mathbb{Z}^+ \). From \( K \in \kappa(\delta, k) \) we obtain \( \delta_i > \delta_i - K_i \geq \delta_{i+1} \) if \( i < m \), whence \( i(\delta) \leq i \). Since \( \delta_i(\delta) - \delta_i \in \mathbb{N}, \delta \) is resonant.
It only remains to prove that \( i(\delta, k) \) is defined and equal to \( i \), as the converse is clear. Observe that \( \delta_i = k + K'_i \geq k \), and if \( i < m \),
\[
k = \delta_i - K'_i > \delta_i - K_i = \delta_i - K_i - \delta_0 - i \geq \delta_{i+1} - \delta_0 - i = \delta_{i+1} + 1.\]
\[
\square
\]

**Corollary 6.4.** Fix \( k, k', K, \) and \( K' \) as in Proposition 6.3 and write \( w_j \) for the \((j + 1)\)-cycle \((01 \cdots j)\) in \( S_{m+1} \). If \( \chi_{\delta + \lambda_K} = \chi_{\delta + \lambda_K'} \), then there exists an \( a_m \)-dominant \( a_m \)-integral weight \( \mu \) such that
\[
\delta + \lambda_K = w_i(\delta, k) \cdot \mu, \quad \delta + \lambda_K' = w_i(\delta, k) \cdot \mu.
\]

**Proof.** Set \( i := i(\delta, k) \) and \( \sigma := \delta + \lambda_K + \rho_{a_m} \). By the proof of Proposition 6.3, \( \sigma \) is \( a_m \)-integral. To complete the proof, define \( \mu \) by \( \mu + \rho_{a_m} := (\sigma_1, \ldots, \sigma_{i-1}, \sigma_0, \sigma_{i+1}, \ldots, \sigma_m) \).

6. **Intertwining operators on tensor field modules**

In this section we describe the affine (\( \mathfrak{b}_m \)-covariant) and projective (\( \mathfrak{a}_m \)-covariant) operators on tensor field modules, and in particular, on symbol modules.

6.1. **Affine operators.** Recall the algebra \( \mathbb{C}[\partial_x] \) from [13], and write \( \mathbb{C}[\![\partial_x]\!] \) for the corresponding algebra of formal power series. Note that \( \mathbb{C}[\![\partial_x]\!] \) acts naturally on \( \mathbb{C}[x] \).

**Lemma 6.1.** For any \( \mathfrak{l}_m \)-modules \( V \) and \( V' \), there is an isomorphism
\[
\text{Hom}_{\mathfrak{l}_m}(\mathcal{F}(V), \mathcal{F}(V')) \cong \mathbb{C}[\![\partial_x]\!] \otimes \text{Hom}(V, V').
\]
It intertwines the \( \mathfrak{l}_m \)-actions, and for \( V = V' \) it is an algebra isomorphism.

**Proof.** Recall that \( \varepsilon_m \) acts solely on the first factor of \( \mathcal{F}(V) = \mathbb{C}[x] \otimes V \), so
\[
\text{Hom}_{\mathfrak{l}_m}(\mathcal{F}(V), \mathcal{F}(V')) \cong \text{End}_{\mathfrak{l}_m}(\mathbb{C}[x]) \otimes \text{Hom}(V, V').
\]
It is elementary that \( \text{End}_{\mathfrak{l}_m}(\mathbb{C}[x]) = \mathbb{C}[\![\partial_x]\!] \). For the \( \mathfrak{l}_m \)-actions, use the fact that \( \text{Div}_w(X) = \phi(X) \) for \( X \in \mathfrak{l}_m \). The rest is easy. \( \square \)

**Lemma 6.2.** Suppose that \( \mathcal{E}_x \) acts on \( V \) and \( V' \) by scalars: \( V = V_{\langle c' \rangle} \), \( V' = V'_{\langle c' \rangle} \). Then \( \text{Hom}_{\mathfrak{b}_m}(\mathcal{F}(V), \mathcal{F}(V')) \) is 0 unless \( c' - c \in \mathbb{N} \), when it is isomorphic to
\[
\left[ \mathbb{C}[\![\partial_x]\!][c'-c] \otimes \text{Hom}(V, V') \right]_{\mathfrak{l}_m}.
\]

**Proof.** Use \( \text{Hom}(V, V') = \text{Hom}(V_{\langle c' \rangle}, V'_{\langle c' \rangle}) \) together with Lemma 6.1. \( \square \)

By [13], \( \mathcal{S}^k(V, V') = \mathcal{F}(\mathbb{C}[\![\xi \!][(-k)] \otimes \text{Hom}(V, V')) \). If \( \mathcal{E}_x \) acts by scalars on \( V \) and \( V' \), then it acts by zero on \( \text{End}(\text{Hom}(V, V')) \). In this case Lemma 6.2 gives the following corollary describing the \( \mathfrak{b}_m \)-maps between symbol modules.

**Corollary 6.3.** Assume that \( \mathcal{E}_x \) acts by scalars on \( V \) and \( V' \). Then for \( k < k' \) the space \( \text{Hom}_{\mathfrak{b}_m}(\mathcal{S}^k(V, V'), \mathcal{S}^{k'}(V, V')) \) is 0, and for \( k \geq k' \) it is isomorphic to
\[
\left[ \mathbb{C}[\![\partial_x]\!][k'-k] \otimes \text{Hom}(\mathbb{C}[\![\xi \!][(-k)], \mathbb{C}[\![\xi \!][(-k')]] \otimes \text{End}(\text{Hom}(V, V')) \right]_{\mathfrak{l}_m}
\]
Remark. If $V$ and $V'$ are both 1-dimensional, elementary representation theory shows that $\delta_1$ is 1-dimensional. Therefore it is spanned by $\text{Div}^k - k'$, as stated in the remark concluding Section 4. Let us elaborate on that remark. Corresponding to any representation $\pi$ of $\mathfrak{n}_m$ on $\text{Hom}(V, V')$, there is a $\mathfrak{b}_m$-invariant operator $\text{Div}^\pi$ on $S(V, V')$, the $\pi$-symbol divergence. It is defined in the following lemma, which is simple to prove directly.

**Lemma 6.4.** $\text{Div}^\pi := \pi(x_i \partial_{x_i}) \partial_{x_j} \partial_{x_k} i$ is in $\text{Hom}_{\mathfrak{b}_m}(S^k(V, V'), S^{k-1}(V, V'))$.

The notation is chosen to reflect the fact that $\text{Div}^\pi(X) = \pi(X)$. Similarly, one finds that the operator $\text{Div}^\pi := \phi_\mathfrak{s}(x_i \partial_{x_i}) \partial_{x_j} \partial_{x_k}$ is $\mathfrak{b}_m$-invariant and simplifies to $-\mathcal{E}$ Div. Thus Proposition 4.3 (ii) may be written as

$\text{Lie}_{\mathfrak{s}}(\Omega_{\mathfrak{m}}) - \text{Lie}_{\mathfrak{s}}(\Omega_{\mathfrak{m}}) = -2(\text{Div}_{\mathfrak{s}} + \text{Div}_{\mathfrak{s}} + \rho(\mathcal{E}) \text{Div})$.

In particular, it is a $\mathfrak{b}_m$-map, as follows from the fact that $\text{Lie}_{\mathfrak{s}}(\mathfrak{b}_m)$ and $\text{Lie}_{\mathfrak{s}}(\mathfrak{b}_m)$ are equal and hence commute with both $\text{Lie}_{\mathfrak{s}}(\Omega_{\mathfrak{m}})$ and $\text{Lie}_{\mathfrak{s}}(\Omega_{\mathfrak{m}})$.

We now specialize to the setting of Section 5 and prove a key proposition about the image of $S(\delta)$ under powers of Div. We begin with two lemmas. The first is elementary:

**Lemma 6.5.** (i) $\text{Div}^d \circ (x_{i_1} \cdots x_{i_{2d}})$ acts on $C[\varepsilon] \otimes L_{\mathfrak{m}}(\delta)$ as $dl! \partial_{x_{i_1}} \cdots \partial_{x_{i_{2d}}}$.

(ii) $\text{Div}^d \circ (C[x](\delta))$ acts on $C[\varepsilon] \otimes L_{\mathfrak{m}}(\delta)$ as $C[\partial_{x}(\delta)]$.

The second lemma gives a formula for the highest weight vectors of the tensor product of the standard representation $\mathbb{C}^n$ of $\mathfrak{gl}_m$ with any finite dimensional irreducible representation of $\mathfrak{gl}_m$. It is known; we outline two proofs.

Recall from Sections 2.1 and 5 that $e_{ij}$ is the $ij$th elementary matrix in $\mathfrak{gl}_m$ and $e_i$ is the $i$th standard basis vector of $\mathbb{C}^n$. Given a dominant integral weight $\nu = \sum_{i=1}^m \nu_i e_i$ of $\mathfrak{gl}_m$ and integers $1 \leq j < i \leq m$, $r \geq 0$, and $1 \leq i_r < \cdots < i_0 \leq m$, define scalars

$c_j(i) := \frac{1}{\nu_j - \nu_i + 1 - j}, \quad b(i_0, i_1, \ldots, i_r) := (-1)^r \prod_{s=1}^r c_{i_s}(i_0) \prod_{j=1}^{i_r-1} (1 - c_j(i_0))$.

**Lemma 6.6.** Fix a highest weight vector $v_\nu$ of $L_{\mathfrak{gl}_m}(\nu)$. Suppose that $\nu_{i_0} - 1 > \nu_{i_0}$. Then $\mathbb{C}^n \otimes L_{\mathfrak{gl}_m}(\nu)$ has a highest weight vector $v_{\nu}^0$ of weight $\nu + e_{i_0}$ given by

$v_{\nu}^0 := \sum_{0 \leq r < i_0} \sum_{1 \leq i_r < \cdots < i_1 < i_0} b(i_0, \ldots, i_r) e_{i_0} \otimes \left( e_{i_0} e_{i_1} e_{i_2} \cdots e_{i_r-1} v_\nu \right)$.

**Proof.** This result can be proven using the non-commutative finite factorization of the extremal projector discovered in [AST79]. Their theorem implies that

$\left(1 - c_{i_0-1}(i_0) e_{i_0, i_0-1} e_{i_0-1, i_0}\right) \cdots \left(1 - c_{1}(i_0) e_{i_0, 0} e_{0, i_1}\right) \left(1 - c_{1}(i_0) e_{i_0, 0} e_{0, i_1}\right) (e_{i_0} \otimes v_\nu)$

is a highest weight vector. A somewhat delicate computation shows that it is $v_{\nu}^0$.

For a “low technology” proof, first use a weight argument to see that $v_{\nu}^0$ must have the given form for some scalars $b(i_0, \ldots, i_r)$. Then show that our definition of $b(i_0, \ldots, i_r)$ does give a highest weight vector by proving directly that $e_{k-1, k} v_{\nu}^0$ is zero for $1 < k \leq m$. This computation is also delicate. For example, at $k > i_r$ three terms contribute multiples of

$e_k \otimes \left( e_{i_0, i_1} \cdots e_{i_{r-1}, k} e_{k-1, i_{r+1}} \cdots e_{i_{r-1}, i_r} v_\nu \right)$.
to \(e_{k-1,k}v_{i_0}^0\). Two of them correspond to the coefficients \(b(i_0, \ldots, i_r)\) with \(i_s = k - 1\) and \(i_s = k\), and the third corresponds to

\[
b(i_0, \ldots, i_{s-1}, k, k-1, i_{s+1}, \ldots, i_r).
\]

The reader may check that the multiples they contribute sum to zero.

Similarly, at \(k = i_s\), multiples of \(e_{k-1} \otimes (e_{i_0,i_1}e_{i_1,i_2} \cdots e_{i_{r-1},k}v_{i_r})\) are contributed by three terms. Again, the three multiples sum to zero. □

Recall from Section 5 the highest weight vector \(v^K_\delta\) in \(C[\xi]_{(-K)} \otimes L_{\text{in}}(\delta)\) for \(K \in \kappa(\delta, k)\). Note that for any \(d \leq K_i\), the multi-index \(K - de_i\) is in \(\kappa(\delta, k - d)\).

**Proposition 6.7.** For \(K \in \kappa(\delta, k)\) and \(d \leq K_i\), \(L_{\text{in}}(v^K_{\delta - de_i}) \subseteq \text{Div}^d(F(v^K_\delta))\).

**Proof.** By Lemma 6.5 \(\text{Div}^d(C[x]_{(d)} \otimes L_{\text{in}}(v^K_\delta)) \subseteq C[\partial\xi]_{(d)}\) applied to \(\text{Div}^d(L_{\text{in}}(v^K_\delta))\), which is the same as \(C[\partial\xi]_{(1)}\) applied to \(L_{\text{in}}(v^K_\delta)\) \(d\) times. Thus it suffices to prove that \(C[\partial\xi]_{(1)}\) applied to \(L_{\text{in}}(v^K_\delta)\) contains \(L_{\text{in}}(v^K_{\delta - e_i})\) whenever \(K_i > 0\). In fact we need only prove that it contains \(v^K_{\delta - e_i}\), because Div is \(l_m\)-invariant.

Consider \(C[\partial\xi]_{(1)} \otimes L_{\text{in}}(v^K_\delta)\). By standard results on minuscule representations, it contains a unique line of highest weight vectors of weight \(\delta + \lambda_K - e_i\) for all \(i_0\) with \(K_{i_0} > 0\). Applying Lemma 6.6 and recalling the identifications of \(\mathfrak{g}_{l_m}\) with \(l_m\) and \(C^m\) with \(C[\partial\xi]_{(1)}\), we find that this line is spanned by

\[
v^K_{\delta, i_0} := \sum_{0 \leq r < i_0} b(i_0, \ldots, i_r) \partial x_{i_r} \otimes ((x_{i_0} \partial x_{i_1})(x_{i_1} \partial x_{i_2}) \cdots (x_{i_{r-1}} \partial x_{i_r}))v^K_\delta.
\]

Here the factors \(x_{i_r}, \partial x_{i_r}\) act on \(v^K_\delta\) via the \(l_m\)-action on \(C[\xi]_{(-K)} \otimes L_{\text{in}}(\delta)\), and the scalars \(b(i_0, \ldots, i_r)\) are defined using \(\nu = \delta + \lambda_K\). We have \(\nu_{i_0-1} > \nu_{i_0}\) because \(K_{i_0} > 0\).

We claim that the coefficient of \(v^K_{\delta, i_0}\) in \(\prod_{r=1}^{i_0} (x_{i_r-1} \partial x_{i_r})v^K_\delta\) is

\[
(-1)^r(\xi_{i_0} \partial x_{i_0})(\xi_{i_1} \partial x_{i_1}) \cdots (\xi_{i_r} \partial x_{i_r-1})\xi^K = (-1)^r K_{i_0} K_{i_1} \cdots K_{i_{r-1}} \xi^{K - e_0 - e_{i_0}}.
\]

To verify this, recall from Lemma 5.2 that the coefficient of \(v^K_\delta\) in \(v^K_{\delta, i_0}\) is \(\xi^K\). Each operator \(x_{i_r-1} \partial x_{i_r}\) is of negative weight, so the only \(\xi^K\)-multiple of \(v^K_\delta\) arising as a summand of \(\prod_{r=1}^{i_0} (x_{i_r-1} \partial x_{i_r})v^K_\delta\) occurs when every \(x_{i_r-1} \partial x_{i_r}\) acts on the factor \(\xi^K\) of the lead term \(\xi^K \otimes v^K_\delta\). Since \(\phi(x_{i_r-1} \partial x_{i_r}) = -\xi_{i_r} \partial x_{i_r-1}\), the claim follows.

Now consider the map from \(C[\partial\xi]_{(1)} \otimes C[\xi]_{(-K)} \otimes L_{\text{in}}(\delta)\) to \(C[\xi]_{1-K} \otimes L_{\text{in}}(\delta)\) given by applying the first factor to the second. This “evaluation map” is \(l_m\)-covariant. It must send \(v^K_{\delta, i_0}\) to a multiple of \(v^K_{\delta - e_0}\), because by Lemma 5.2 \(v^K_{\delta - e_0}\) is the unique highest weight vector of its weight on the right side. The proof of the proposition will be complete if we show that this multiple is non-zero.

By the last two displayed equations, the coefficient of \(v^K_\delta\) in the image of \(v^K_{\delta, i_0}\) under the evaluation map is

\[
\sum_{0 \leq r < i_0} (\xi^K)^r b(i_0, \ldots, i_r) K_{i_0} K_{i_1} \cdots K_{i_{r-1}} (K_{i_r} + 1) \xi^{K - e_0}.
\]

Now observe that \(c_i(i_0)^{-1} \in 2 + N\) for all \(j < i_0\), so \((\xi^K)^r b(i_0, \ldots, i_r)\) is always positive! The contribution at \(r = 0\) is non-zero, so the entire sum is a positive multiple of \(\xi^{K - e_0}\). □
6.2. Projective operators. The point of this section is to use Lemma 6.2 and Proposition 6.7 to show that the projective operators between tensor field modules may be viewed as powers of the divergence operator. These operators may be classified using the fact that the tensor field modules are dual to parabolic Verma modules (see Propositions 7.1 and 7.2 below). The dimension formulas for the spaces of homomorphisms between parabolic Verma modules derived in BESSSS (see also BESSSS, Section 6) give the following result.

Theorem 6.8. \( \dim \text{Hom}_{a_m}(\mathcal{F}(\lambda), \mathcal{F}(\lambda')) = 1 \) if \( \chi_\lambda = \chi_{\lambda'} \) and \( \lambda - \lambda' = \delta(\epsilon_0 - \epsilon_i) \) for some \( d \in \mathbb{N} \). Otherwise it is 0.

Recall that \( \mathcal{S}(\delta) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k(\delta) \), and by (10), \( \mathcal{S}^k(\delta) = \bigoplus_{\kappa(\delta,k)} \mathcal{F}(v^K_0) \). Define projection operators \( P_k \) and \( P_K \) compatible with these Vec \( \mathbb{R}^m \)-decompositions:

\[
P_k : \mathcal{S}(\delta) \rightarrow \mathcal{S}^k(\delta), \quad P_K : \mathcal{S}(\delta) \rightarrow \mathcal{F}(v^K_0).
\]

These projections are related by the identity \( P_k = \sum_{K \in \kappa(\delta,k)} P_K \).

Proposition 6.9. For \( K \in \kappa(\delta,k) \) and \( 0 \leq d \leq K_i \),

\[
\text{Hom}_{b_m}(\mathcal{F}(v^K_0), \mathcal{F}(v^K_0 - d\epsilon_i)) = \mathbb{C}P_{K - d\epsilon_i} \circ \text{Div}^d.
\]

This map is \( a_m \)-covariant if and only if \( \chi_\delta + \chi_K = \chi_\delta + \chi_{K - d\epsilon_i} \), i.e., \( \delta \) is resonant, \( i(\delta,k) \) is defined and equal to \( i \), and \( d = k + K - \delta_i \).

Proof. The second sentence follows from the first, Proposition 5.3, and Theorem 6.8. Observe that the first sentence is a generalization of the remark following Corollary 6.3. Since Div is a \( b_m \)-map and \( P_{K - d\epsilon_i} \big|_{\mathcal{S}^k(\delta)} \) is a Vec \( \mathbb{R}^m \)-map, \( P_{K - d\epsilon_i} \circ \text{Div}^d \) is a \( b_m \)-map, and it is non-zero by Proposition 6.7. It will suffice to prove that it is up to a scalar the unique \( b_m \)-map from \( \mathcal{F}(v^K_0) \) to \( \mathcal{F}(v^K_0 - d\epsilon_i) \).

By Lemmas 5.2 and 6.2, \( \text{Hom}_{b_m}(\mathcal{F}(v^K_0), \mathcal{F}(v^K_0 - d\epsilon_i)) \) is isomorphic to

\[
\left[ \mathbb{C}[(\delta - d) \otimes L_{l_{m}}(\delta + \chi_K)^* \otimes L_{l_{m}}(\delta + \chi_{K - d\epsilon_i})] \right]^{l_{m}}.
\]

By the Parthsarathy-Ranga Rao-Varadarajan lemma, \( L_{l_{m}}(d\epsilon_1 - d\epsilon_0) \) occurs with multiplicity 1 in \( L_{l_{m}}(\delta + \chi_K)^* \otimes L_{l_{m}}(\delta + \chi_{K - d\epsilon_i}) \), as its smallest submodule. This submodule is dual to \( \mathbb{C}[(\delta - d) \otimes L_{l_{m}}(d\epsilon_0 - d\epsilon_m)] \), so the proof is complete. \( \square \)

7. The parabolic category \( \mathcal{O}^{b_m}(a_m) \)

Here we collect some results on the parabolic category \( \mathcal{O}^{b_m}(a_m) \). For a more detailed treatment of this category we refer the reader to HUOS Chapter 9 and the references therein.

For the remainder of the paper, the symbol \( \simeq \) will indicate isomorphism as \( a_m \)-modules, and weights \( \lambda \in \mathfrak{h}_m^* \) will be called dominant, integral, or singular if they are \( a_m \)-dominant, \( a_m \)-integral, or \( a_m \)-singular, respectively.

Given a weight module \( M = \bigoplus_{\nu \in \mathfrak{h}_m^*} M_\nu \) of \( a_m \) with finite weight multiplicities, let \( M^V \) denote the module with underlying space \( \bigoplus_{\nu \in \mathfrak{h}_m^*} M_\nu^* \) and action \( (X\omega)(m) := \omega(\theta(X)m) \), where \( X \in a_m \), \( \omega \in M^V \), \( m \in M \), and \( \theta \) is the Chevalley anti-automorphism of \( a_m \). Observe that \( M \) and \( M^V \) have the same weights: \( (M^V)_\nu = M_\nu^* \) because \( \theta|_{\mathfrak{h}_m} = 1 \).

For \( \lambda \in \mathfrak{h}_m^* \), the parabolic Verma module \( M_{b_m}(\lambda) \) of \( a_m \) is \( \mathcal{U}(a_m) \otimes_{\mathcal{U}(b_m)} L_{l_{m}}(\lambda) \). Here \( L_{l_{m}}(\lambda) \) is regarded as a \( b_m \)-module via the trivial action of the nilradical.
$c_m$. Note that $M_{b_m}(\lambda)$ has a unique irreducible quotient $L(\lambda)$, the irreducible $a_m$-module of highest weight $\lambda$. Lemma 5.2(ii) implies the following.

**Proposition 7.1.** If $\lambda \in h_m^*$ is $l_m$-dominant and $l_m$-integral, then the $a_m$-modules $F(\lambda)$ and $M_{b_m}(\lambda)\gamma$ are isomorphic.

For $\mu \in h_m^*$ and $1 \leq i \leq m$, define

$$\mu[0] := \mu, \quad \mu[i] := w_i \cdot \mu,$$

where $w_i$ is the $(i + 1)$-cycle $(0 \cdots i)$ as in Corollary 5.4. Note that every regular integral weight $\lambda$ is equal to $\mu[i]$ for a unique dominant integral weight $\mu$ and a unique $0 \leq i \leq m$. The category $O_{\gamma_m}(a_m)$ has enough injectives (and projectives), and we denote by $I(\lambda)$ the injective envelope of $L(\lambda)$ in $O_{\gamma_m}(a_m)$. The following description of the injectives was first established in [RC80].

**Proposition 7.2.** Let $\lambda \in h_m^*$ be $l_m$-dominant and $l_m$-integral.

(i) If $\lambda$ is non-integral or singular integral, then $I(\lambda) = F(\lambda) = L(\lambda)$.

(ii) Let $\lambda$ be regular integral, so $\lambda = \mu[i]$ for some dominant integral $\mu$.

(a) If $i > 0$, then we have a non-split exact sequence of $a_m$-modules

$$0 \rightarrow F(\mu[i]) \rightarrow I(\lambda) \rightarrow F(\mu[i - 1]) \rightarrow 0.$$

(b) If $i = 0$, i.e., $\lambda$ is dominant integral, then $I(\lambda) \cong F(\lambda)$.

The decomposition in Proposition 7.2(ii) uniquely determines the injective $I(\lambda)$. In fact, we have the following result concerning the extensions between parabolic Verma modules. It is a particular case of the formula for the dimensions of all Ext-groups between parabolic Verma modules in the Hermitian symmetric case, deduced in [Sh88].

**Proposition 7.3.** Let $\mu$ be a dominant integral weight, $0 \leq i, j \leq m$. Then

$$\dim \text{Ext}^{i,j}_{O_{\gamma_m}(a_m)}(F(\mu[i]), F(\mu[j])) = \delta_{i-1,j}.$$

In particular, if $1 \leq i \leq m$ and

$$0 \rightarrow F(\mu[i]) \rightarrow M \rightarrow F(\mu[i - 1]) \rightarrow 0$$

is a non-split exact sequence of $a_m$-modules, then $M \cong I(\mu[i])$.

8. The Jordan form of the Casimir operator

In this section we determine those infinitesimal characters $\chi_\mu$ of $a_m$ such that the generalized $\chi_\mu$-submodule $D_\gamma(\delta)^{\mu}$ is not equal to the $\chi_\mu$-submodule $D_\gamma(\delta)^{\mu}$. En route we will see that these are precisely the infinitesimal characters for which the Casimir operator is not semisimple.

By [16] and Proposition 5.3, the graded module $S(\delta)$ of $D_\gamma(\delta)$ has at most two submodules $F(u^K)$ with any given infinitesimal character. Therefore

$$D_\gamma(\delta)^{\mu} \cong D_\gamma(\delta)^{(\mu:2)}$$

for all $\mu$. It follows from Propositions 7.2 and 7.3 that if $D_\gamma(\delta)^{\mu} \neq D_\gamma(\delta)^{\mu}$, then we may assume without loss of generality that $\mu$ is dominant integral and

$$D_\gamma(\delta)^{\mu} = I(\mu[i])$$

for some $1 \leq i \leq m$.

For convenience we will abbreviate the Vec $R^m$-actions on $D_\gamma(\delta)$ and $S(\delta)$: set

$$\text{Lie}_{\gamma,\delta} := \text{Lie}_{\gamma,C_\gamma \otimes L_{l_m}(\delta)}, \quad \text{Lie}_{\delta}^{S} := \text{Lie}_{C_\gamma,C_\gamma \otimes L_{l_m}(\delta)}.$$
In order to take advantage of Proposition 4.3 throughout we will replace Lie_{γ,δ} with the isomorphic action Lie_{γ,δ}^{NS} on S(δ).

Recall from \( f \) the projections \( P_k \) and \( P_K \) and set \( S_k(δ) := \bigoplus_{j=0}^{k} S^j(δ) \). Reviewing Section 3 we see that \( S_k(δ) \) is invariant under Lie_{γ,δ}^{NS} and the restrictions

\[
P_k|_{S_k(δ)} : S_k(δ) \rightarrow S^k(δ) \quad P_K|_{S_k(δ)} : S_k(δ) \rightarrow F(v^K_δ)
\]

are Vec \( \mathbb{R}^m \)-intertwining maps from Lie_{γ,δ}^{NS} to Lie_{γ,δ}^{NS}.

For the rest of this section, suppose that \( δ \) is resonant, \( k \in \mathbb{Z}^+ \), and \( i(δ,k) \) is defined. Assume that \( K \in κ(δ,k) \) satisfies \( K_{i(δ,k)} > \delta_{i(δ,k)} - k \). Set

\[
i : = i(δ,k), \quad d := k + K_i - \delta_i, \quad k' := k - d, \quad K' := K - de_i.
\]

Then \( K' \in κ(δ,k') \), and by Proposition 5.3 it is the only multi-index other than \( K \) such that \( χ_{δ+λK} = χ_{δ+λK'} \). In other words,

\[
(S(δ), \text{Lie}_{γ,δ}^{NS})^{δ+λK} = (S(δ), \text{Lie}_{γ,δ}^{NS})^{δ+λK'} = F(v^K_δ) \oplus F(v^{K'}_δ).
\]

We will see that in general, \( (S(δ), \text{Lie}_{γ,δ}^{NS})^{δ+λK} \neq (S(δ), \text{Lie}_{γ,δ}^{NS})^{δ+λK'} \).

Also for the rest of this section, when we write \( S(δ) \) or \( S_j(δ) \) without specifying an action it is understood that the action is \( \text{Lie}_{γ,δ}^{NS} \). For \( 0 \leq c \leq d \), define

\[
ν_c := δ + λK_{-ce_i}, \quad ω_c := χ_{ν_c}(Ω_m).
\]

**Lemma 8.1.**

(i) \( S(δ)^{(ν_0)} = S_k(δ)^{(ν_0)} = S_k(δ)^{(ν_0;2)} \).

(ii) \( S_{k-1}(δ)^{(ν_0)} = S_{k'}(δ)^{(ν_0)} = S_{k'}(δ)^{(ν_0)} \).

(iii) \( S_{k'-1}(δ)^{(ν_0)} = 0 \).

**Proof.** For all weights \( ν \), the projection \( P_j \) carries \( S_j(δ)^{(ν)} \) onto \( S_j(δ)^{(ν)} \) with kernel \( S_{j-1}(δ)^{(ν)} \). The space \( S_j(δ)^{(ν)} \) is 0 unless \( j \) is \( k \) or \( k' \), when it is \( F(v^K_δ) \) or \( F(v^{K'}_δ) \), respectively. The lemma follows. □

**Corollary 8.2.**

(i) \( P_k \) and \( P_K \) map \( S_k(δ)^{(ν_0)} \) onto \( F(v^K_δ) \) with kernel \( S_{k'}(δ)^{(ν_0)} \).

(ii) \( P_k \) and \( P_K \) map \( S_k(δ)^{(ν_0)} \) bijectively to \( F(v^{K'}_δ) \).

**Lemma 8.3.** Let \( Θ \) be any element of \( \mathfrak{S}(a_m) \) such that \( χ_{ν_0}(Θ) = 0 \). Then there exists a unique \( a_m \)-covariant map \( \mathfrak{S} : F(v^K_δ) \rightarrow F(v^{K'}_δ) \) such that

\[
\mathfrak{S} \circ P_K|_{S_k(δ)^{(ν_0)}} = P_{K'} \circ \text{Lie}_{γ,δ}^{NS}(Θ)|_{S_k(δ)^{(ν_0)}}.
\]

**Proof.** This is a commutative diagram lemma. Since \( χ_{ν_0}(Θ) = 0 \), \( \text{Lie}_{γ,δ}^{NS}(Θ) \) annihilates \( F(v^K_δ) \) and \( F(v^{K'}_δ) \). Hence by Corollary 8.2 \( \text{Lie}_{γ,δ}^{NS}(Θ) \) maps \( S_k(δ)^{(ν_0)} \) to \( S_{k'}(δ)^{(ν_0)} \) and annihilates \( S_{k'}(δ)^{(ν_0)} \). By Lemma 8.1 its restriction to \( S_k(δ)^{(ν_0)} \) factors through \( P_K \). The rest follows easily. □

In order to apply Lemma 8.3 we must choose \( Θ \) so that we can compute \( \mathfrak{S} \) explicitly. The lemma applies to any \( \mathfrak{S}(a_m) \)-multiple of \( \Omega_m - ω_0 \), but if we take for example \( Θ \) to be \( Ω_m - ω_0 \), we have no way to compute \( \mathfrak{S} \). Keeping in mind that \( ω_0 = ω_d \), define

\[
Δ := \left( \frac{1}{2} \right) d \prod_{c=0}^{d} (Ω_m - ω_c) = \left( \frac{1}{2} \right) d \prod_{c=1}^{d} (Ω_m - ω_c).
\]

**Theorem 8.4.**

(i) \( \mathfrak{S} = \left( \prod_{c}^d \left[ k - c + (m + 1)γ \right] \right) P_{K'} \circ \text{Div}^d \).
Lemma 8.7.

(i) There is a dominant integral weight $\mu$ such that $\nu_0 = \mu[i-1]$ and $\nu_d = \mu[i].$

(ii) For $k + (m + 1)\gamma \in \{1, \ldots, d\}$, $(S(\delta), \text{Lie}_{\gamma,\delta}^{\mathbb{NS}}) (\nu_0) \simeq \mathcal{F}(\nu_0) \oplus \mathcal{F}(\nu_d).$

(iii) For $k + (m + 1)\gamma \notin \{1, \ldots, d\}$, $(S(\delta), \text{Lie}_{\gamma,\delta}^{\mathbb{NS}}) (\nu_0) \simeq I(\nu_d).$

(v) $\Omega_{\alpha_m}$ acts non-semisimply on $I(\mu[i]).$

We remark that this theorem gives an alternate proof of the fact that $P_{K'} \circ \text{Div}^d$ is $\alpha_m$-covariant, independent of Proposition 8.5. To prove it we need some preliminary results. Put the standard partial order on $\mathbb{N}^m$:

$$J \leq K \ 	ext{if} \ J_i \leq K_i \ 	ext{for} \ 1 \leq i \leq m.$$

Note that if $K \in \kappa(\delta, k)$, then $J \in \kappa(\delta, |J|)$ for all $J \leq K$.

The tree-like subspace $T(v^K_\delta) \subset S(\delta)$ defined in [MR07] is

$$T(v^K_\delta) := \bigoplus_{J \leq K} \mathcal{F}(v^K_\delta).$$

Lemma 8.5. [MR07] $T(v^K_\delta)$ is invariant under $\text{Lie}_\delta^S(\text{Vec} \mathbb{R}^m)$ and $\text{Lie}_{\gamma,\delta}^{\mathbb{NS}}(\alpha_m)$.

Proof. The Lie$_\delta^S(\text{Vec} \mathbb{R}^m)$-invariance follows from that of the summands $\mathcal{F}(v^K_\delta)$. For the Lie$_{\gamma,\delta}^{\mathbb{NS}}(\alpha_m)$-invariance, use (15) to obtain

$$(\text{Lie}_\delta^S(x_i \mathcal{E}_x) - \text{Lie}_\delta^S(x_i \mathcal{E}_x) = -(\mathcal{E}_x + (m + 1)\gamma) \partial_{\xi_i}.$$  

Recall that $\mathcal{F}(v^K_\delta) = \mathbb{C}[x] \otimes L_{\alpha_m}(v^K_\delta)$ and reduce to proving

$$\partial_{\xi_i} L_{\alpha_m}(v^K_\delta) \subseteq \bigoplus_{j: J \geq 0} L_{\alpha_m}(v^K_{J} - \epsilon_j).$$

As we saw in the proof of Proposition 6.7, the highest weights of the irreducible $t_m$-submodules of $\mathbb{C}[\partial_{\xi}]_{(1)} \otimes L_{\alpha_m}(v^K_\delta)$ are all of the form $\delta + \lambda J - \epsilon_0 + \epsilon_j$. Evaluating the $\xi$-derivatives and matching highest weights with (15) completes the proof. □

For any $\mu \in \mathfrak{h}_{\gamma,\delta}$, define $P(\mu)$ to be the projection of $S(\delta)$ to its generalized $\chi_{\mu}$-submodule $S(\delta)^{(\mu)}$ along the sum $\bigoplus_{\mu' \neq \mu} S(\delta)^{(\mu')}$. couple with its other generalized $\chi_{\mu}$-submodules, note that the action of $P(\delta + \lambda J)$ on $\mathcal{F}(v^K_\delta)$ is 1 at the symbol level:

$$P_{J(J)} \circ P(\delta + \lambda J)|_{\mathcal{F}(v^K_\delta)} = P_{J(J)} \circ P(\delta + \lambda J)|_{\mathcal{F}(v^K_\delta)} = 1.$$  

It is elementary that $P(\mu) = \text{Lie}_{\gamma,\delta}^{\mathbb{NS}}(\gamma)$ for some $\gamma \in \mathcal{Z}(\alpha_m)$. Coupled with Lemma 8.5, this yields the next lemma, a sharpening of Lemma 8.4. The subsequent lemma contains the formulas necessary for the proof of Theorem 8.4(i). It will be convenient to use the convention that $P_{J} = 0$ if $J \notin \kappa(\delta, |J|)$.

Lemma 8.6. (i) $S(\delta)^{(\nu_0)} = T(v^K_\delta)^{(\nu_0)} = P^{(\nu_0)}(\mathcal{F}(v^K_\delta) \oplus \mathcal{F}(v^K_{J})).$

(ii) $\mathcal{S}_{\lambda}(\delta)^{\nu_0} = T(v^K_\delta)^{\nu_0} = P^{(\nu_0)}(\mathcal{F}(v^K_{J})).$

Lemma 8.7. Assume $0 \leq c \leq d, c < b \leq d + 1, \text{ and } \Theta \in \mathcal{Z}(\alpha_m)$. Abbreviate $\Theta^{\mathbb{NS}} := \text{Lie}_{\gamma,\delta}^{\mathbb{NS}}(\Theta),$ $\Omega_c^{\mathbb{NS}} := \text{Lie}_{\gamma,\delta}^{\mathbb{NS}}(\Omega_{\alpha_m} - \omega_c),$ $q_c := 2(k - c - 1 + (m + 1)\gamma).$

(i) $P_{\delta'} \circ \Theta^{\mathbb{NS}} \circ P_{J} = 0$ if $K' \geq J$, in particular, if $J \leq K$ and $K - J \notin \mathbb{N} e_i$.

(ii) $P_{\Omega_{\alpha_m}^{\mathbb{NS}} \circ P_{\delta'}(\mathbb{NS}) \circ P_{\delta'}(\mathbb{NS}) = q_c \text{Div} P_{\delta'}(\mathbb{NS}) \circ P_{\delta'}(\mathbb{NS}) \circ P_{\delta'}(\mathbb{NS}) \circ P_{\delta'}(\mathbb{NS}) \circ P_{\delta'}(\mathbb{NS}) \circ P_{\delta'}(\mathbb{NS}).$
(iv) $P_{K'} \circ \Theta^{NS} \circ (\prod_{a=c}^{b-1} \Omega^{a} ) \circ P_{K-c e_{i}} = (\prod_{a=c}^{b-1} \Omega^{a} ) P_{K'} \circ \Theta^{NS} \circ \text{Div}^{b-c} \circ P_{K-c e_{i}}.$

(v) $P_{K} \circ \Theta^{NS} \circ (\prod_{a=c}^{d} \Omega^{a} ) \circ P_{K-c e_{i}} = 0.$

(vi) $P_{K} \circ (\prod_{a=c}^{d} \Omega^{a} ) \circ P_{K} = (\prod_{a=c}^{d} \Omega^{a} ) P_{K'} \circ \text{Div}^{d} \circ P_{K}.$

Proof. By Lemma 8.4 the image of $\Theta^{NS} \circ P_{J}$ is in $T(v^{i}_{d})$. For $K' \not\subseteq J$, $F(v^{K'}_{d})$ is not a summand of $T(v^{i}_{d})$, proving (i).

The first equality in (ii) is immediate from Lemma 4.3(iii) and the definition of $\omega_{c}$. The argument used to prove (21) shows that $\text{Div} F(v^{K-c e_{i}}_{d})$ maps $\bigoplus_{j} F(v^{K-c e_{i}, c_{j}}_{d})$. The second equality of (ii) follows.

For (iii), apply (ii) and then use (i) to see that the terms other than $P_{K-(c+1)c_{i}}$ in the sum $\sum_{j=1}^{m} P_{K-c e_{i}-c_{j}}$ contribute nothing.

For (iv), apply (iii) repeatedly to obtain $P_{K'} \circ \Theta^{NS} \circ (\prod_{a=c}^{b-1} \Omega^{a NS} ) \circ P_{K-c e_{i}} = (\prod_{a=c}^{b-1} \Omega^{a} ) P_{K'} \circ \Theta^{NS} \circ P_{K-b e_{i}} \circ \text{Div} \circ P_{K-(b-1)c_{i}} \circ \text{Div} \circ \cdots \circ P_{K-(c+1)c_{i}} \circ \text{Div} \circ P_{K-c e_{i}}.$

Then use (i) and the second equality of (ii) to prove that this operator remains the same if the internal projections are dropped.

For (v), apply (iv) with $b = d + 1$ and note that $\Theta^{NS} \circ \text{Div}^{d-c+1}$ maps $F(v^{K-c e_{i}}_{d})$ to $S_{K-1}(\tilde{\delta})$, which is annihilated by $P_{K'}$.

Finally, (vi) is (iv) with $c = 0, b = d$, and $\Theta^{NS} = 1$. □

Proof of Theorem 8.4. We begin by using (i) to prove (iii)-(v). Part (ii) follows from Corollary 5.4.

By Proposition 8.4, $P_{K'} \circ \text{Div}^{d} F(v^{K}_{d}) \neq 0.$ Therefore if $k+(m+1)\gamma \not\in \{1, \ldots, d\}$, then $\Delta \neq 0.$ In this case $S(\delta)^{(\mu)}$ is not annihilated by $\Omega_{a_{m}}$, $\omega_{0}$, so $\Omega_{a_{m}}$ does not act semisimply on it. Hence by Proposition 7.3 and Corollary 8.2 $S(\delta)^{(\mu)} \approx I(\mu[\bar{i}]).$ This proves (iv) and (v).

In (iii), $\Delta = 0,$ so an elementary argument shows that $\Omega_{a_{m}}$ acts by $\omega_{0}$ on $S(\delta)^{(\mu)}$. Hence $S(\delta)^{(\mu)} \neq I(\mu[\bar{i}])$ by (iv), so by Proposition 7.3 it splits as desired.

To prove (i), fix $v \in F(v^{K}_{d})$ and write $\tilde{v}$ for $P^{(\mu)} v$. By (22), $P_{K} \tilde{v} = v,$ so by Lemma 8.3 $\Delta(v) = P_{K'} \circ \text{Lie}^{\gamma}_{\tilde{\delta}}(\Delta)(\tilde{v})$. By Lemma 8.6 $\tilde{v} = \sum_{j \leq K} P_{j} \tilde{v},$ and by Lemma 8.7(i) and (v) (note that $\Omega_{a_{m}}^{NS} = \Omega_{d}^{NS}$), $P_{K'} \circ \text{Lie}^{\gamma}_{\tilde{\delta}}(\Delta)(P_{j} \tilde{v}) = 0$ for $J < K$.

Therefore $\Delta(v) = P_{K'} \circ \text{Lie}^{\gamma}_{\tilde{\delta}}(\Delta)(v)$. Lemma 8.7(ii) now completes the proof. □

Remark. It follows from (20) that if $-(m+1)\gamma \in \mathbb{N}$, then $\bigoplus_{j=1}^{\infty} -(m+1)\gamma S^{j}(\delta)$ is invariant under $\text{Lie}^{\gamma}_{\tilde{\delta}}(\Delta)$ on $a_{m}$. This yields an alternate proof of Theorem 8.4(iii).

9. Proofs

In this section we prove our main results. We begin by stating a version of Theorem 8.4 explicitly for differential operators. Recall the notation of Section 5.

Theorem 9.1. Assume that $\delta$ is resonant, $i = i(\delta, k)$ is defined, $K \in \kappa(\delta, k)$, $K' \in \kappa(\delta, k'), k-k' = d \in \mathbb{Z}^{+}, \chi_{\delta}^{\lambda_{K}} = \chi_{\delta}^{\lambda_{K'}}$, $\mu \in b_{m}^{\gamma}$ is dominant integral, and $\delta + \lambda_{K} = \mu[i-1], \delta + \lambda_{K'} = \mu[i].$

(i) For $k + (m+1)\gamma \in \{1, \ldots, d\}$, the inclusion $D_{\gamma}^{k-1}(\delta) \hookrightarrow D_{\gamma}^{k}(\delta)$ does not have a unique $a_{m}$-invariant splitting. We have $D_{\gamma}^{k}(\delta)^{(\mu)} \simeq F(\mu[i-1]) \oplus F(\mu[i]), \quad D_{\gamma}^{k}(\delta)^{(\mu)} \simeq F(\mu[i]).$
(ii) For $k + (m + 1)\gamma \notin \{1, \ldots, d\}$, the inclusion $D_{\gamma}^{k-1}(\delta) \hookrightarrow D_{\gamma}^{k}(\delta)$ does not have an $a_m$-invariant splitting. We have
\[ D_{\gamma}^{k}(\delta)^{(\mu)} \simeq I(\mu[i]), \quad D_{\gamma}^{k'}(\delta)^{(\mu)} \simeq F(\mu[i]). \]

Proof. Since $\text{Lie}_{\gamma, \delta} \simeq \text{Lie}_{\gamma, \delta}^{NS}$, the two displayed equations are restatements of Corollary 8.2 and Theorem 8.4(ii)-(iii).

For (i), recall that $F(\mu[i - 1]) \simeq F(\nu_{\delta}^{k'})$ and $F(\mu[i]) \simeq F(\nu_{\delta}^{k'})$. Therefore Proposition 6.9 gives a non-zero $a_m$-map from $F(\mu[i - 1])$ to $F(\mu[i])$, so $D_{\gamma}^{k}(\delta)^{(\mu)}$ has a $1$-parameter family of $a_m$-splittings. The result follows.

For (ii), recall that any $a_m$-splitting of $D_{\gamma}^{k-1}(\delta) \hookrightarrow D_{\gamma}^{k}(\delta)$ gives a splitting of the generalized $\chi_{\mu}$-submodules $D_{\gamma}^{k-1}(\delta)^{(\mu)} \hookrightarrow D_{\gamma}^{k}(\delta)^{(\mu)}$. Thus the result follows from the fact that there is no $a_m$-splitting of $F(\mu[i]) \hookrightarrow I(\mu[i])$. □

Remark. We can now state more concretely a feature of injectives in differential operator modules mentioned in the introduction: if $I(\mu[i])$ occurs as $D_{\gamma}(\delta)^{(\mu)}$ in such a way that the quotient module $F(\mu[i - 1])$ occurs in the symbols of order $k$ and the submodule $F(\mu[i])$ occurs in the symbols of order $k'$, then
\[ k - k' = \mu_{i-1} - \mu_i + 1, \]

independent of $\gamma$ and $\delta$. This is because there must be $K$ and $K'$ as in Theorems 5.3 and 5.4 such that $\mu[i - 1] = \delta + \lambda_K$ and $\mu[i] = \delta + \lambda_{K'}$, whence (see (23) below)
\[ (\mu_{i-1} - \mu_i + 1)(\epsilon_0 - \epsilon_i) = \mu[i - 1] - \mu[i] = \lambda_K - \lambda_{K'} = (k - k')(\epsilon_0 - \epsilon_i). \]

Proof of Theorem 4. Let $a_m$-dominant $L_m$-integral weight $\lambda$, we must choose $\gamma$ and $\delta$ so that $D_{\gamma}(\delta)^{(\lambda)} \simeq I(\lambda)$. Take $\delta = \lambda$ and $(m + 1)\gamma \notin \mathbb{Z}$.

If $\delta$ either is not regular integral or is dominant integral, Proposition 5.3 and Corollary 8.4 imply that $\chi_{\delta}$ is not a repeated infinitesimal character of $D_{\gamma}(\delta)$, so
\[ F(\delta) \simeq F(\nu_{\delta}^{0}) = S^0(\delta) = S(\delta)^{\delta} = D_{\gamma}(\delta)^{(\delta)}. \]

By Section 4 in this case $I(\delta) = F(\delta)$.

Now suppose that $\delta = \mu[i]$ for some dominant integral $\mu$ and $i > 0$. We have
\[ \mu[i] = (\mu_i - 1, \mu_0 + 1, \ldots, \mu_{i-2} + 1, \mu_{i-1} - 1, \mu_{i-1} + 1, \mu_{i+1}, \ldots, \mu_m), \]
\[ \mu[i - 1] = (\mu_i - 2, \mu_0 + 1, \ldots, \mu_{i-3} + 1, \mu_{i-2} + 1, \mu_{i-2}, \ldots, \mu_m). \]

Note that $\tilde{\delta}_i = \mu_i - 1 - \mu_i + 1 \in \mathbb{Z}$ and $\mu[i - 1] - \mu[i] = \tilde{\delta}_i(\epsilon_0 - \epsilon_i)$. Take $k = \tilde{\delta}_i$, $K = k\epsilon_i$, and $K' = 0$. Then $i(\delta, k)$ is defined and equal to $i$ and Proposition 5.3 gives $\chi_{\delta + \lambda_K} = \chi_{\tilde{\delta} + \lambda_{K'}}$, so Theorem 4.1(ii) gives $I(\delta) \simeq D_{\gamma}(\delta)^{(\delta)}$. □

Proof of Proposition 6. If $S(\delta)$ has no repeated infinitesimal characters, then by Proposition 6.3 the decomposition of $D_{\gamma}(\delta)$ into its $\chi_\lambda$-subspaces gives the unique $a_m$-splitting of the order filtration: $D_{\gamma}(\delta)^{(\delta + \lambda_K)} \simeq F(\nu_{\delta}^{k'}) \simeq F(\delta + \lambda_K)$ and
\[ D_{\gamma}(\delta) = \bigoplus_{k \in \mathbb{N}} \bigoplus_{\kappa \in \kappa(\delta, k)} D_{\gamma}(\delta)^{(\delta + \lambda_K)} \simeq \bigoplus_{k \in \mathbb{N}} \bigoplus_{\kappa \in \kappa(\delta, k)} F(\delta + \lambda_K) = S(\delta). \]

If $S(\delta)$ has repeated infinitesimal characters, then by Corollary 8.4 we are in the setting of Theorem 4.1 for some $k$, $D_{\gamma}^{k-1}(\delta) \hookrightarrow D_{\gamma}^{k}(\delta)$ has either zero or more than one $a_m$-splitting. The result follows. □
Remark. Our approach to Proposition [C] is independent of [Me12] and gives an alternate proof of his Theorem [B] for $D_2(\delta)$. Indeed, by Proposition [5.3] and Theorem [6.8] there exist non-trivial $\alpha_m$-maps between distinct symbol modules $S^k(\delta)$ and $S^\delta(\delta)$ if and only if $S(\delta)$ has repeated infinitesimal characters. By Proposition [C] this occurs if and only if $D_2(\delta)$ is resonant.

Proof of Proposition [D] If $D_2(\delta)$ is resonant, then Propositions [C] and [5.3] imply that $\delta$ is resonant. For the converse of the first sentence, suppose that $\delta$ is resonant. Take $i = i(\delta, k)$, $K = e_i + (k - 1)e_m$. Check that $K \in \kappa(\delta, k)$.

If $i < m$, take $d = 1$, giving $K' = K - e_i$. If $i = m$, take $d = k$, giving $K' = 0$. In both cases, $\chi_{\delta + \lambda K} = \chi_{\delta + \lambda K'}$ by Proposition [5.3]. Let $\chi$ be such that $D_2(\delta)$ is resonant by Proposition [C].

For the second sentence, suppose that $S^k(\delta)$ and $S^\delta(\delta)$ share an infinitesimal character: $D^k = \chi_{\delta + \lambda K}$. By Proposition [5.3], $\delta$ is resonant, $i = i(\delta, k)$ is defined, and $K' = K - de_i$, where $d = k + K_i - \tilde{\delta} > 0$. Thus $\tilde{\delta}_i < k + K_i < 2k$.

If $i < m$, combining $d > 0$ with $K \in \kappa(\delta, k)$ gives $\tilde{\delta}_i < 0$. These are all integers, so $\tilde{\delta}_i + 1 < k$.

To prove the converse, suppose that $\delta$ is resonant. $i = i(\delta, k)$ is defined, and $2k > \tilde{\delta}_i$. For $i < m$, assume in addition that $k \geq \tilde{\delta}_i + 2$ and take

$$\tag{24} K = (\hat{\delta}_i - k + 1)e_i + (2k - \hat{\delta}_i - 1)e_m, \quad d = 1, \quad K' = K - e_i.$$ Then $K_i > 0$ because $i(\delta, k) = i$, so $K$ and $K'$ are in $\mathbb{N}^m$. They are in $\kappa(\delta, k)$ because $\hat{\delta}_i - \hat{\delta}_i + 1 - K_i = k - \tilde{\delta}_i - 1 - 2 \geq 0$.

For $i = m$, take $K = ke_m, d = 2k - \tilde{\delta}_m > 0$, and $K' = (\hat{\delta}_m - k)e_m$. Here $K_m' \geq 0$ because $i(\delta, m) = m$. For all values of $i$, Proposition [5.3] gives $\chi_{\delta + \lambda K} = \chi_{\delta + \lambda K'}$.

Proof of Theorem [A] The proof of (i) mirrors that of Proposition [C]. If (ii) fails, then Proposition [B] implies that $S^2(\delta)$ and $D_2^{k-1}(\delta)$ share no infinitesimal characters. Therefore the unique $\alpha_m$-splitting of $D_2^{k-1}(\delta) \hookrightarrow D_2^{k}(\delta)$ is

$$D_2^{k}(\delta) = D_2^{k-1}(\delta) \oplus \bigoplus_{K \in \kappa(\delta, k)} D_2^{k}(\delta)^{\delta + \lambda K}.$$ We first prove (ii) and (iii) for $i = i(\delta, k) < m$. Suppose that $k + (m + 1)\gamma = 1$. If $K$ is an element of $\kappa(\delta, k)$ such that $\chi_{\delta + \lambda K}$ occurs in $D_2^{k-1}(\delta)$, then Proposition [5.3] tells us that it occurs exactly once, as $\chi_{\delta + \lambda K'}$ for $K' = K - de_i$, where $d = k + K_i - \hat{\delta}_i$. For all such $K$, Theorem [0.11] shows that $D_2^{k-1}(\delta)^{\delta + \lambda K} \hookrightarrow D_2^{k}(\delta)^{\delta + \lambda K}$ is non-uniquely $\alpha_m$-split.

Conversely, if $k + (m + 1)\gamma = 1$, take $K$ and $K'$ as in (24). Then there exists a dominant integral $\mu$ such that $\mu[i] = \delta + \lambda K'$, and $\mu[i - 1] = \delta + \lambda K$. By Theorem [0.11], $D_2^{k-1}(\delta)^{\mu} \hookrightarrow D_2^{k}(\delta)^{\mu}$ is not $\alpha_m$-split. Parts (ii) and (iii) follow.

Finally we prove (ii) and (iii) for $i(\delta, k) = m$. Set $d_0 := \max\{1, 2k - \hat{\delta}_1 + m - 1\}$ and suppose that $k + (m + 1)\gamma = 1$ is in $\{1, \ldots, d_0\}$. Given any $K \in \kappa(\delta, k)$ and $K' \in \kappa(\delta, K')$ such that $k' < k$ and $\chi_{\delta + \lambda K} = \chi_{\delta + \lambda K'}$, we know that $K' = K - de_m$ where $d = k + K_m - \hat{\delta}_m$. Examining the proof in the case $i(\delta, k) < m$, we see that we must prove $d \geq d_0$. This follows from the fact that $K \in \kappa(\delta, k)$ implies $K_i \leq \delta_i - \delta_i + 1$ for $i < m$, and in particular, $K_m \geq k - \delta_1 + \delta_m$. Conversely, suppose that $k + (m + 1)\gamma = 1$ in $\{1, \ldots, d_0\}$. For $d_0 > 1$, take $K$ of minimal $K_m$ and $d = d_0$:

$$K = (\delta_1 - \delta_2)e_1 + \cdots (\delta_m - 1 - \delta_m)e_{m-1} + (k - \delta_1 + \delta_m)e_m, \quad K' = K - d_0e_m.$$
Check that $K'_m = \tilde{\delta}_m - k \geq 0$ and $d_0 = k + K_m - \tilde{\delta}_m$, so again, Theorem 9.1(ii) implies that $D_{\gamma}^{-1}(\delta) \mapsto D_{\gamma}^k(\delta)$ is not $a_m$-split.

For $d_0 = 1$ we can construct $K$ and $K' = K - e_m$ giving $\chi_{\delta + \lambda_K} = \chi_{\delta + \lambda_{K'}}$: take $K_m = \tilde{\delta}_m - k + 1$ and choose the $K_i$ with $i < m$ in any way satisfying $K_i \leq \delta_i - \delta_{i+1}$ and $|K| = k$. The theorem follows. □

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