DIFFERENTIABILITY OF QUASICONVEX FUNCTIONS ON SEPARABLE BANACH SPACES

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Abstract. We investigate the differentiability properties of a real-valued quasiconvex function $f$ defined on a separable Banach space $X$. Continuity is only assumed to hold at the points of a dense subset. If so, this subset is automatically residual.

Sample results that can be quoted without involving any new concept or nomenclature are as follows: (i) If $f$ is usc or strictly quasiconvex, then $f$ is Hadamard differentiable at the points of a dense subset of $X$. (ii) If $f$ is even, then $f$ is continuous and Gâteaux differentiable at the points of a dense subset of $X$. In (i) or (ii), the dense subset need not be residual but, if $X$ is also reflexive, it contains the complement of a Haar null set. Furthermore, (ii) remains true without the evenness requirement if the definition of Gâteaux differentiability is generalized in an unusual, but ultimately natural, way.

The full results are much more general and substantially stronger. In particular, they incorporate the well known theorem of Crouzeix, to the effect that every real-valued quasiconvex function on $\mathbb{R}^N$ is Fréchet differentiable a.e.

1. Introduction

According to Rockafellar [38, p. 428], it has been known since the early 20th century that a real-valued convex function on $\mathbb{R}^N$ is everywhere continuous and a.e. Fréchet differentiable, although the geometric leanings prevalent at that time made it difficult to pinpoint the origin of this result with greater accuracy.

When $\mathbb{R}^N$ is replaced by an infinite dimensional Banach space $X$, a real-valued convex function on $X$ is either continuous at every point or discontinuous at every point. This makes it obvious that the investigation of the differentiability properties of convex functions should be limited to the former class. When $X$ is separable, Mazur [31] proved the residual Gâteaux differentiability of continuous convex functions in 1933. Recall that, in Baire category terminology, a residual subset is the complement of a set of first category. Much later, in 1976, Aronszajn [1] showed that Gâteaux differentiability is also true almost everywhere, provided that a suitable generalization of null sets is used when $\dim X = \infty$ (Aronszajn null sets; see Subsection 2.1).

In the theorems of Mazur and Aronszajn, Gâteaux differentiability can be replaced by Hadamard differentiability - and hence by Fréchet differentiability when $X = \mathbb{R}^N$ - since both concepts coincide for locally Lipschitz functions (this is well known [42, p. 19] and elementary). More generally, the residual Fréchet differentiability of continuous convex functions was proved in a 1968 landmark paper by Asplund [2] when $X^*$ is separable, as well as in other cases that do not require or imply the separability of $X$.

The dichotomy everywhere/nowhere continuous is no longer true for finite quasiconvex functions, where the quasiconvexity of $f$ is understood as $f(\lambda x + (1 - \lambda)y) \leq$
max\{f(x), f(y)\} for every \(x, y\) and every \(\lambda \in [0, 1]\). Nonetheless, in 1981, Crouzeix [14] (see also [9]) proved that, just like convex functions, real-valued quasiconvex functions on \(\mathbb{R}^N\), not necessarily continuous, are a.e. Fréchet differentiable. To date, this property has not been extended in any form to infinite dimensional spaces. In fact, with only a few notable exceptions, the literature has consistently focused on differentiability under local Lipschitz continuity conditions, a topic with a long history which is still the subject of active research; see the recent text [25].

Of course, local Lipschitz continuity is grossly inadequate to handle quasiconvex functions, even continuous ones, irrespective of dim \(X\).

It is the purpose of this paper to show how Crouzeix’s theorem can be generalized when \(\mathbb{R}^N\) is replaced by a separable Banach space \(X\). We provide variants of the theorems of Mazur and Aronszajn in the more general setting of quasiconvex functions, although neither theorem has a genuine generalization. In particular, “mixed” criteria must be used to evaluate the size of the set of points of differentiability. We shall return to this and related issues further below.

As a simple first step, recall that the set of points of discontinuity of any real-valued function \(f\) on a topological space \(X\) is an \(F_\sigma\) (see e.g. [21, p.78], [39, p. 58]), so that either this set is of first category, or it has nonempty interior. Evidently, no generic differentiability result should be expected in the latter case, which dictates confining attention to quasiconvex functions that are continuous at the points of a residual subset of the (separable) Banach space \(X\). Incidentally, Mazur’s theorem breaks down for this class of functions, even when \(X = \mathbb{R}\) and continuity is assumed. Indeed, in general, a monotone continuous function on \(\mathbb{R}\) is only differentiable at the points of a set of first category ([44, Corollary 1]), even though its complement has measure 0 by Lebesgue’s theorem. As we shall see later, Aronszajn’s theorem also fails in the quasiconvex (even continuous) case when \(\dim X = \infty\).

In Banach spaces, residual sets are dense and the converse is trivially true for \(G_\delta\) sets. In particular, a real-valued function is continuous at the points of a residual subset if and only if it is densely continuous, i.e., continuous at the points of a dense subset. Common examples of densely continuous functions include the upper or lower semicontinuous functions ([32, Lemma 2.1]) and, by a theorem that goes back to Baire himself, the so-called functions of Baire first class, i.e., the pointwise limits of sequences of continuous functions ([43, p. 12]). An apparently new class (ideally quasiconvex functions) that contains all the semicontinuous quasiconvex functions -and even all the quasiconvex functions when \(X = \mathbb{R}^N\)- is described in Subsection 3.2.

As shown by the author in [37], when \(X\) is a Baire topological vector space (tvs), the densely continuous quasiconvex functions \(f\) on \(X\) have quite simple equivalent characterizations in terms of the lower level sets

\[
F_\alpha := \{ x \in X : f(x) < \alpha \} \quad \text{and} \quad F'_\alpha := \{ x \in X : f(x) \leq \alpha \},
\]

where \(\alpha \in \mathbb{R} \cup \{-\infty\}\). Of course, since \(f\) is real-valued, \(F_{-\infty} = F'_{-\infty} = \emptyset\), but the notation will occasionally be convenient. By the quasiconvexity of \(f\), all the sets \(F_\alpha\) and \(F'_\alpha\) are convex and, conversely, if all the \(F_\alpha\) (or \(F'_\alpha\)) are convex, then \(f\) is quasiconvex, which is de Finetti’s original definition [19].

In [37] and again in this paper, a special value plays a crucial role, which is the so-called topological essential infimum \(\mathcal{Tess}_X f\) of \(f\), denoted by \(m\) for

\footnote{It is only in Baire spaces (e.g., Banach spaces) that these two options are mutually exclusive.}
simplicity, defined by

\[(1.2) \quad m := \mathcal{T} \inf_X f := \sup \{ \alpha \in \mathbb{R} : F_\alpha \text{ is of first category} \} = \inf \{ \alpha \in \mathbb{R} : F_\alpha \text{ is of second category} \}.\]

The last equality in (1.2) follows from the fact that the sets $F_\alpha$ are linearly ordered by inclusion. Since $f$ is real-valued and $X$ is a Baire space, it is always true that $\mathcal{T} \inf_X f \in [-\infty, \infty)$. Note also that $\inf_X f \leq \mathcal{T} \inf_X f$ and that the set $F_m$ (but not $F_m'$) is always of first category since it is the union of countably many $F_\alpha$ with $\alpha < m$.

From now on, $X$ is a separable Banach space and $f : X \to \mathbb{R}$ is quasiconvex and densely continuous. Below, we give a synopsis of the differentiability results proved in this paper. The rough principle is that the size of the subset $F_m$ ranks the amount of differentiability of $f$ (the smaller the better). Even though $F_m$ is always of first category, various refinements will be involved. Many of the concepts and some technical results needed for the proofs are reviewed or introduced in the next two sections. In particular, several properties of $F_m'$ with direct relevance to the differentiability question are established in Section 3.

In Section 4, we focus on the differentiability of $f$ above level $m$, that is, at the points of $X \setminus F_m'$. The special features of $F_m'$ make it possible to rely on a theorem of Borwein and Wang [7] to prove that $f$ is Hadamard differentiable on $X \setminus F_m'$ except at the points of an Aronszajn null set (Theorem 4.2). If $m = -\infty$, so that $X \setminus F_m' = X$, this fully generalizes Aronszajn’s theorem and settles the differentiability issue.

Accordingly, in the remainder of the discussion, $m > -\infty$ is assumed. It turns out that $X \setminus F_m'$ is always semi-open, i.e., contained in the closure of its interior (which is not true when $F_m'$ is replaced by an arbitrary convex set), so that the differentiability result just mentioned above still shows that $f$ is Hadamard differentiable at “most” points of $X \setminus F_m'$. That $X \setminus F_m'$ may be empty does not invalidate this statement. Hence, it only remains to investigate differentiability at the points of $F_m' = F_m \cup f^{-1}(m)$.

It turns out that $f$ is never Gâteaux differentiable at any point of $F_m$ (Theorem 5.1). Thus, the differentiability of $f$ on $F_m'$ depends solely upon its differentiability at the points of $f^{-1}(m)$ and, if $f$ is Gâteaux differentiable at $x \in f^{-1}(m)$, then $Df(x) = 0$ (Theorem 5.1), as if $m$ were a genuine minimum. In summary, the problem is to evaluate the size of those points of $f^{-1}(m)$ at which the directional derivative of $f$ exists and is $0$ in all directions and to decide whether $f$ is Hadamard differentiable at such points. While this demonstrates the importance of $m$ in the differentiability question, the task is not as simple as one might first hope.

When $X = \mathbb{R}^N$, $F_m$ is not only of first category but also nowhere dense because, in finite dimension, convex subsets of first category are nowhere dense. Differentiability when $X$ is separable and $F_m$ is nowhere dense is the topic of Section 5. The technicalities depend upon whether $F_m'$ is of first or second category but, ultimately, we show that $f$ is Hadamard but not Fréchet differentiable on the complement of the union of a nowhere dense set with an Aronszajn null set (Theorem 5.4). Such unions cannot be replaced with Aronszajn null sets alone or with sets of first category alone.

Every subset which is the union of a nowhere dense set with an Aronszajn null set has empty interior (of course, this is false if “nowhere dense” is replaced by “first
category”) and the class of such subsets is an ideal, but not a σ-ideal. In particular, 
\( f \) above—or any finite collection of similar functions—is Hadamard differentiable at the points of a dense subset of \( X \). This is only a “subgeneric” differentiability property but, if \( X \) is separable and reflexive, a truly generic result holds: If \( F_m \) is nowhere dense, \( f \) is Hadamard differentiable on \( X \) except at the points of a Haar null set (Theorem 5.5). This should be related to the rather unexpected fact that, in such spaces, a quasiconvex function is densely continuous if and only if its set of points of discontinuity is Haar null (Theorem 3.3).

Examples show that Theorem 5.5 is not true for Fréchet differentiability, or if \( X \) is not reflexive, or if “Haar null” is replaced by “Aronszajn null”, even if \( f \) is continuous and \( X = \ell^2 \). Thus, Aronszajn’s theorem cannot be extended to (densely) continuous quasiconvex functions if \( \dim X = \infty \). Nonetheless, when \( X = \mathbb{R}^N \), Crouzeix’s theorem is recovered because every quasiconvex function on \( \mathbb{R}^N \) is densely continuous, the Haar null sets of \( \mathbb{R}^N \) are just the Borel subsets of Lebesgue measure 0 and Hadamard and Fréchet differentiability coincide on \( \mathbb{R}^N \).

Aside from \( X = \mathbb{R}^N \), there are several conditions ensuring that \( F_m \) is nowhere dense. In particular, if \( f \) is upper semicontinuous (usc) or, more generally, if \( \inf_X f = m \), then \( F_m = \emptyset \). Also, \( F_m \) is nowhere dense if \( f \) is strongly ideally quasiconvex (Definition 3.1, Theorem 3.4) or strictly quasiconvex and densely continuous (Corollary 5.6).

In general, \( F_m \) need not be nowhere dense if \( \dim X = \infty \), especially when \( f \) is lower semicontinuous (lsc). Thus, loosely speaking, the differentiability issue is more delicate for lsc functions than for usc ones. However, no difficulty arises as a result of passing to the lsc hull, even though doing so can only enlarge \( F_m \): In Subsection 5.2, we show that Theorems 5.4 and 5.5 are applicable to a densely continuous quasiconvex function \( f \) if and only if they are applicable to its lsc hull.

The case when \( F_m \) is not nowhere dense is discussed in Section 6. This is new territory since it never happens when \( X = \mathbb{R}^N \) or when \( f \) is usc, let alone convex and continuous. We show that a subset \( F^\perp m \) of \( X \), often much smaller than \( F_m \), is intimately related to the Gâteaux differentiability question. Specifically, \( F^\perp m \) consists of all the limits of the convergent sequences of points of \( F_m \) that remain in some finite dimensional subspace of \( X \).

In a more arcane terminology, \( F^\perp m \) is the sequential closure of \( F_m \) for the finest locally convex topology on \( X \). This type of closure has been around for quite a while in the literature, but only in connection with issues far removed from differentiability (topology, moment problem in real algebraic geometry, etc.) and often in a setting that rules out infinite dimensional Banach spaces (countable dimension).

In Theorem 6.2, we prove that if \( F^\perp m \) has empty interior, then \( f \) is both continuous and Gâteaux differentiable at the points of a dense subset of \( X \). Since Hadamard differentiability implies continuity, this is weaker than the analogous result when \( F_m \) is nowhere dense (Theorem 5.4), but applicable in greater generality.

Corollaries are given in which the assumption that \( F^\perp m \) has empty interior is replaced by a more readily verifiable condition. Most notably, we show in Corollary 6.5 that every even densely continuous quasiconvex function \( f \) on a separable Banach space is continuous and Gâteaux differentiable at the points of a dense subset of \( X \). Furthermore, if \( X \) is also reflexive, \( f \) is continuous and Gâteaux differentiable on \( X \) except at the points of a Haar null set. These properties remain true when \( f \) exhibits more general symmetries (Corollary 6.6).
Section 7 is devoted to an example showing that when $F_m$ is not nowhere dense, the differentiability properties of $f$ at the points of $F'_m$ or, equivalently, $f^{-1}(m)$, are indeed significantly weaker than when $F_m$ is nowhere dense. This confirms that an optimal outcome cannot be obtained without splitting the investigation of the two cases.

The results of Section 6, particularly those incorporating some symmetry assumption about $f$, make it legitimate to ask whether $F^+_m$ always has empty interior. If true, this would imply that Theorem 6.2 is always applicable. In Section 8 we put an early end to this speculation, by exhibiting a convex subset $C$ of $\ell^2$ of first category such that $C^\perp = \ell^2$. The construction of $C$ also shows how to produce densely continuous quasiconvex functions $f$ such that $m > -\infty$ and $F_m = C$, but we were unable to determine whether any such function fails to be Gâteaux differentiable at the points of a dense subset.

The aftermath of $F^+_m$ not always having empty interior is that, when $\dim X = \infty$, it remains an open question whether every densely continuous quasiconvex function $f$ on a separable Banach space $X$ is continuous and Gâteaux differentiable at the points of a dense subset. Nevertheless, in Section 9 this question is answered in the affirmative after the definition of Gâteaux differentiability is slightly altered.

A basic remark is that the concept of Gâteaux derivative at a point $x$ continues to make sense if it is only required that the directional derivatives at $x$ exist and are represented by a continuous linear form for some residual set of directions in the unit sphere. This suffices to define an “essential” Gâteaux derivative at $x$ in a unique way, independent of the residual set of directions (which is not true if “residual” is replaced by “dense”). With this extended definition, the problem is resolved in Theorem 9.2. In particular, if $X$ is reflexive and separable, every densely continuous quasiconvex function is continuous and essentially Gâteaux differentiable on $X$ except at the points of a Haar null set.

2. Preliminaries

We collect various definitions and related results that will be used in the sequel. Most, but not all, of them are known, some more widely than others. Whenever possible, references rather than proofs are given.

2.1. Aronszajn null sets. Let $X$ denote a separable Banach space. If $\{\xi_n\} \subset X$ is any sequence, call $\mathcal{A}(\{\xi_n\})$ the class of all Borel subsets $E \subset X$ such that $E = \bigcup_n E_n$ where $E_n$ is a Borel “null set on every line parallel to $\xi_n$”, that is, $\lambda_1((x + \mathbb{R}\xi_n) \cap E_n) = 0$ for every $x \in X$, where $\lambda_1$ is the one-dimensional Lebesgue measure. The Aronszajn null sets (11) are the (Borel) subsets $E$ of $X$ such that $E \in \mathcal{A}(\{\xi_n\})$ for every sequence $\{\xi_n\}$ such that $\text{span}(\{\xi_n\}) = X$.

The Aronszajn null sets form a $\sigma$-ring that does not contain any nonempty open subset and every Borel subset of an Aronszajn null set is Aronszajn null. When $X = \mathbb{R}^N$, they are the Borel subsets of Lebesgue measure 0. It was shown by Csörnyei [16] that the Aronszajn null sets coincide with the Gaussian null sets (Phelps [34]: these are the Borel subsets of $X$ that are null for every Gaussian probability measure on $X$) and also with the “cube null” sets of Mankiewicz [26].

\footnote{For every $\ell \in X^*$, the measure $\mu_\ell$ on $\mathbb{R}$ defined by $\mu_\ell(S) := \mu(\ell^{-1}(S))$ has a Gaussian distribution.}
Aronszajn null sets are preserved by affine diffeomorphisms of $X$, i.e., translations and bounded invertible linear transformations. As noted by Matoušková [28], if $C$ is a convex subset of $X$ with nonempty interior, the boundary $\partial C$ is Aronszajn null, but this need not be true if $\overset{\circ}{C} = \emptyset$, even if $C$ is closed and $X$ is Hilbert.

2.2. Haar null sets. Let once again $X$ denote a separable Banach space. The Borel subset $E \subset X$ is said to be Haar null if there is a Borel probability measure $\mu$ on $X$ such that $\mu(x + E) = 0$ for every $x \in X$. This definition is due to Christensen [10], [11]; see also [29]. It is obvious that every Aronszajn null set is Haar null, but the converse is false [34], except when $X = \mathbb{R}^N$. The Haar null sets also form a $\sigma$-ring (though this is not obvious from just the definition) containing no nonempty open subset, every Borel subset of a Haar null set is Haar null and, just like Aronszajn null sets, they are preserved by affine diffeomorphisms.

In practice, it is not essential that $\mu$ above be a probability measure. The definition is unchanged if $\mu$ is a Borel measure having the same null sets as a probability measure. For example, this happens if $\mu(E) := \lambda_1(\mathbb{R} \xi \cap E)$ where $\xi \in X \setminus \{0\}$ and $\lambda_1$ is the Lebesgue measure on $\mathbb{R} \xi$ (change $\lambda_1$ into $\varphi \lambda_1$ with $\varphi(t) := \pi^{-\frac{1}{2}} e^{-t^2}$).

If, in addition, $X$ is reflexive, every closed convex subset of $X$ with empty interior is Haar null (Matoušková [20]) but, as noted in the previous subsection, not necessarily Aronszajn null.

2.3. Gâteaux and Hadamard differentiability. A real-valued function $f$ on an open subset $V$ of a Banach space $X$ is Gâteaux differentiable at $x \in V$ if there is $l_x \in X^*$ such that $l_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$ for every $h \in X$. If so, $l_x$ is denoted by $Df(x)$. If, in addition, the limit is uniform for $h$ in every compact subset of $X$, then $f$ is said to be Hadamard differentiable at $x$. Equivalently, $f$ is Hadamard differentiable at $x$ if and only if $Df(x)h = \lim_{t_n \to 0} \frac{f(x + t_nh_n) - f(x)}{t_n}$ for every sequence $(t_n) \subset \mathbb{R} \setminus \{0\}$ tending to 0 and every sequence $(h_n) \subset X$ tending to $h$ in $X$. If $f$ is Hadamard differentiable at $x$, then it is continuous at $x$. This is folklore (see e.g. [8]) and elementary. Of course, Gâteaux differentiability alone does not ensure continuity. When $X = \mathbb{R}^N$, it is plain from the first definition that Hadamard differentiability and Fréchet differentiability coincide.

2.4. Cone monotone functions. Let $X$ be a Banach space and $K \subset X$ be a closed convex cone with nonempty interior. If $V \subset X$ is a nonempty open subset, a function $f : V \to \mathbb{R}$ is said to be $K$-nondecreasing if $x \in V, k \in K \setminus \{0\}$ and $x + k \in V$ imply $f(x) \leq f(x + k)$. This concept has long proved adequate to extend Lebesgue’s theorem on differentiation of monotone functions. It goes back (in $\mathbb{R}^2$) to the 1937 edition of the book by Saks [40] and resurfaced in the work of Chabriac and Crouzeix [9]. Its use in separable Banach spaces, by Borwein et al. [3] and Borwein and Wang [7], is more recent. The following result is essentially [7, Theorem 18].

**Theorem 2.1.** (Borwein-Wang). Let $X$ be a separable Banach space, $K \subset X$ be a closed convex cone with $\overset{\circ}{K} \neq \emptyset$ and $V \subset X$ be a nonempty open subset. Suppose that $f : V \to \mathbb{R}$ is $K$-nondecreasing. Then $f$ is Hadamard differentiable on $V$ except at the points of an Aronszajn null set.
In [7], Theorem 2.1 is proved only when $V = X$ but, as shown below, it is not difficult to obtain the general case as a corollary. If $x, y \in X$, we use the notation $y \leq_K x$ if $x - y \in K$ and $y <_K x$ if $x - y \in K \setminus \{0\}$. These relations are obviously transitive since $K$ is stable under addition (but $\leq_K$ is an ordering if and only if $K \cap (-K) = \{0\}$; this is unimportant).

First, $f$ in Theorem 2.1 is locally bounded (above and below) on $V$. To see this, let $x \in V$ be given and choose $k \in K$. In particular, $k \neq 0$. After replacing $k$ by $tk$ for $t > 0$ small enough, which does not affect $k \in K$, it follows from the openness of $V$ that we may assume that $x - k \in V$ and $x + k \in V$. If $z \in X$ and $|z - x| < \varepsilon$ with $\varepsilon > 0$ small enough, then $z \in V$ and $k + (z - x) \in K$, so that $x - k <_K z$, and $k - (z - x) \in \overline{0}$, so that $z <_K x + k$. Therefore, $f(x - k) \leq f(z) \leq f(x + k)$, which proves the claim.

From the above, for every $x \in V$, there is an open neighborhood $U_x \subset V$ of $x$ such that $f$ is bounded on $U_x$ and, by the separability of $X$, there is a covering of $V$ by countably many $U_x$. Thus, since a countable union of Aronszajn null sets is Aronszajn null and since $U_x \subset V$ implies that $f$ is $K$-nondecreasing on $U_x$, it is not restrictive to prove Theorem 2.1 with $V$ replaced by $U_x$ or, equivalently, to prove it under the additional assumption that $f$ is bounded above and below on $V$. This can be done by using Theorem 2.1 with $V = X$ and $f$ replaced by a finite $K$-nondecreasing extension of $f$ to $X$. Such an extension can be obtained as follows: Since $f$ is bounded on $V$, let $a, b \in \mathbb{R}$ be such that $a \leq f(x) \leq b$ for every $x \in V$ and set

$$
\tilde{f}(x) = \begin{cases} 
a & \text{if } x \notin V + K, \\
\sup \{ f(y) : y \in V, y \leq_K x \} & \text{if } x \in V + K.
\end{cases}
$$

Since $(V + K) + K = V + K$ and $\leq_K$ is transitive, it is straightforward to check that $\tilde{f}$ is $K$-nondecreasing on $X$, that $\tilde{f} = f$ on $V$ and that $a \leq \tilde{f} \leq b$, so that $\tilde{f}$ is real-valued (when $x \in V + K$, the set $\{ y \in V, y \leq_K x \}$ is not empty, so that the supremum in the definition of $\tilde{f}$ is never $-\infty$). Other extensions are described in [7], but they need not be finite when $f$ is finite.

**Remark 2.1.** The Borwein-Wang theorem was refined by Duda [18, Remark 5.2]: $f$ is Hadamard differentiable on $V$ except at the points of a subset in a class $\tilde{C}$ introduced by Preiss and Zajíček, which is strictly smaller than the class of Aronszajn null sets [36, Proposition 13]. The class $\tilde{C}$ may replace the Aronszajn null sets everywhere in this paper.

### 2.5. Convex sets and category

Convex sets have a few special properties relative to Baire category. Two useful ones are given below. Like several other results, they are stated in Banach spaces but are valid in greater generality.

**Lemma 2.2.** ([37, Lemma 3.1]) Let $X$ be a Banach space and let $U$ and $G$ be subsets of $X$ with $U$ open and $G$ convex. If $A \subset X$ is of first category and $U \setminus A \subset G$, then $U \subset G$.

The next lemma is a by-product of Lemma 2.2.

**Lemma 2.3.** ([37, Lemma 3.2]) If $X$ is a Banach space and $C \varsubsetneq X$ is convex, then $X \setminus C$ is locally of Baire second category in $X$, that is, for every open subset $U \subset X$ such that $U \cap (X \setminus C)$ is nonempty, $U \cap (X \setminus C)$ is of second category in $X$. 


Of course, the spirit of Lemma 2.3 is that even locally, the exterior of a convex subset $C \neq X$ is always a large set in some sense. While intuitively clear when $X = \mathbb{R}^N$, the existence of dense and convex proper subsets of infinite dimensional Banach spaces makes this issue less transparent in general.

3. Densely continuous quasiconvex functions

In the Introduction, we defined the densely continuous real valued functions on a tvs $X$ to be those functions that are continuous at the points of a dense subset of $X$. Recall that when $X$ is a Baire space, for instance a Banach space, this is actually equivalent to continuity at the points of a residual subset of $X$.

3.1. Some general properties. The first theorem of this section is part of the main result of [37]. It gives a characterization of the densely continuous quasiconvex functions in terms of their lower level sets. The notation (1.1) is used and will be used throughout the paper.

**Theorem 3.1.** ([37, Corollary 5.3]) Let $X$ be a Banach space and let $f : X \to \mathbb{R}$ be quasiconvex. Set $m := \text{ess inf}_X f$ (see (1.2)). Then, $f$ is densely continuous if and only if (i) $\hat{F}_\alpha \neq \emptyset$ when $\alpha > m$ and (ii) either $m = -\infty$ or $m > -\infty$ and $F_\alpha$ is nowhere dense when $\alpha < m$.

Other conditions equivalent to, or implying, dense continuity for quasiconvex functions are given in [37], but Theorem 3.1 will suffice for our purposes. Since “first category”, “nowhere dense” and “empty interior” are synonymous for convex subsets of $\mathbb{R}^N$, Theorem 3.1 shows that every real-valued quasiconvex function on $\mathbb{R}^N$ is densely continuous. This argument is independent of Crouzeix’s theorem mentioned in the Introduction.

In the next theorem, we prove a number of properties of the set $F'_m$ which will be instrumental in the discussion of the differentiability question. Part (i) was already noticed in [37]. The very short proof is given for convenience.

**Theorem 3.2.** Let $X$ be a Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. Set $m := \text{ess inf}_X f$.

(i) If $F'_m$ (i.e., $f^{-1}(m)$) is of first category, then $F'_m$ (and hence also $f^{-1}(m)$) is nowhere dense.

(ii) If $F'_m$ (i.e., $f^{-1}(m)$) is of second category, then $\hat{F}_m \neq \emptyset$. In particular, $\partial F'_m = \hat{F}_m$ is nowhere dense. If $X$ is separable, $\partial F'_m$ is Aronszajn null.

(iii) $X \setminus F'_m$ is semi-open, i.e., contained in the closure of its interior $X \setminus \overline{F'_m}$.

(iv) If $X$ is separable, the set of points of discontinuity of $f$ in $X \setminus F'_m$ is contained in an Aronszajn null set.

(v) If $X$ is separable, $F'_m$ is contained in an Aronszajn null (Haar null) set if and only if $\overline{F}'_m$ is Aronszajn null (Haar null).

**Proof.** (i) If $x \in \overline{F}'_m \setminus F'_m$, then $f$ is not continuous at $x$, so that $\overline{F}'_m \subset F'_m \cup A$ where $A$ is the set of points of discontinuity of $f$. Since $f$ is densely continuous, $A$ is of first category, whence $\overline{F}'_m$ is of first category, that is, $F'_m$ is nowhere dense.

(ii) Since $F'_m$ is of second category, $\hat{F}_m \neq \emptyset$. On the other hand, if $x \in \overline{F}'_m$ and $f$ is continuous at $x$, then $x \in F'_m$. Since the set $A$ of points of discontinuity of $f$ is
and (iv) of Theorem 3.2 show that the set of points of continuity of $x$ is dense in $X$ and has nonempty interior when $\alpha < \beta < m$. The surprising result is proved next.

In the second (i.e., $\alpha < m$) case, it is Aronszajn null. As noted in the proof of (i), $\partial F_m$ and $\overline{F}_m$ have the same boundary because $F_m$ is convex and $F_m^\circ \neq \emptyset$. Therefore, if $x \in X \setminus F_m^\circ$ and $x \notin X \setminus \overline{F}_m$, then $x \in \partial F_m = \partial (X \setminus \overline{F}_m)$ is in the closure of $X \setminus F_m^\circ$.

(iv) As before, let $A$ denote the set of points of discontinuity of $f$. The claim is that $A \setminus F_m^\circ$ is contained in an Aronszajn null set. If $x \in A \setminus F_m^\circ$, then $f(x) > m$ and there are a sequence $(x_n) \subseteq X$ with $\lim_{n \to \infty} x_n = x$ and $\alpha \in \mathbb{R}$ such that either $f(x_n) < \alpha < f(x)$ for every $n \in \mathbb{N}$ or $f(x) < \alpha < f(x_n)$ for every $n \in \mathbb{N}$. With no loss of generality, we may assume that $\alpha \in \mathbb{Q}$ and, since $f(x) > m$, that $\alpha > m$ (of course, this is redundant when $f(x) < \alpha$).

In the first case (i.e., $f(x_n) < \alpha < f(x)$), $x \in F_o$ but $x \notin F_m$, so that $x \in \partial F_o$. In the second (i.e., $f(x) < \alpha < f(x_n)$), $x \in F_o$ but $x \notin F_o$, so that once again $x \in \partial F_o$. Thus, $A \setminus F_m^\circ \subseteq \bigcup_{\alpha \in \mathbb{Q}, \alpha > m} \partial F_o$. By Theorem 3.1, $F_o \neq \emptyset$ when $\alpha > m$, so that $\partial F_o$ is Aronszajn null since $F_o$ is convex (Subsection 2.1) and so $\cup_{\alpha \in \mathbb{Q}, \alpha > m} \partial F_o$ is Aronszajn null.

(v) The sufficiency is trivial. To prove the necessity, suppose then that $F_m^\circ$ is contained in an Aronszajn null (Haar null) set. As noted in the proof of (i) above, $\overline{F}_m^\circ \subset F_m^\circ \cup A$ where $A$ is the set of points of discontinuity of $f$ and so $\overline{F}_m^\circ \subset F_m^\circ \cup (A \setminus F_m^\circ)$. By (iv), $A \setminus F_m^\circ$ is contained in an Aronszajn null set, so that $\overline{F}_m^\circ$ is Aronszajn null (Haar null) since it is Borel.

While the quasiconvexity of $f$ is actually unnecessary in part (i) of Theorem 3.2, it is important in part (ii). By Theorem 3.1 every lower level set $F_\alpha^\circ$ -not just $F_m^\circ$ -is either nowhere dense or has nonempty interior, but it is only when $\alpha = m$ that both options are possible ($F_m^\circ$ is nowhere dense when $\alpha < m$ since $F_\alpha^\circ \subset F_\beta$ with $\alpha < \beta < m$ and has nonempty interior when $\alpha > m$). In contrast, $F_m$ is always of first category but it may or may not be nowhere dense.

Remark 3.1. It is readily checked that the complement of an Aronszajn null set in a semi-open subset is dense in that subset. Thus, when $X$ is separable, parts (iii) and (iv) of Theorem 3.2 show that the set of points of continuity of $f$ in $X \setminus F_m^\circ$ is dense in $X \setminus F_m$.

In a separable and reflexive Banach space, a finite quasiconvex function is densely continuous if and only if its set of points of discontinuity is Haar null. This perhaps surprising result is proved next.
Theorem 3.3. Let $X$ be a reflexive and separable Banach space. The quasiconvex function $f : X \to \mathbb{R}$ is densely continuous if and only if the set of points of discontinuity of $f$ is Haar null.

Proof. Since the complement of a Haar null set is dense, the “if” part is clear. Conversely, since the set $A$ of points of discontinuity of $f$ is an $F_\sigma$ (hence Borel), it suffices to show that if $f$ is densely continuous, $A$ is contained in a Haar null set. This will be seen by a suitable modification of the proof of part (iv) of Theorem 3.2.

If $x \in A$, there are a sequence $(x_n) \subset X$ with $\lim_{n \to \infty} x_n = x$ and $\alpha \in \mathbb{R}$ such that either $f(x_n) < \alpha < f(x)$ for every $n \in \mathbb{N}$ or $f(x) < \alpha < f(x_n)$ for every $n \in \mathbb{N}$. Without loss of generality, we may assume that $\alpha \in \mathbb{Q}$ and $\alpha \neq m := \text{ess inf}_X f < \infty$.

In the first case, $x \in F_\alpha$ but $x \notin F_\alpha$, so that once again $x \in \partial F_\alpha$. If $(m > -\infty$ and) $\alpha < m$, it follows from Theorem 3.1 that $\partial F_\alpha = \overline{F_\alpha}$ has empty interior. Since $X$ is reflexive and separable, $\partial F_\alpha$ is Haar null (Subsection 2.2). If $\alpha > m$, then $F_\alpha \neq \emptyset$ by Theorem 3.1 so that $\partial F_\alpha$ is Aronszajn null (Subsection 2.1) and therefore Haar null. Thus, $\cup_{\alpha \in \mathbb{Q} \setminus \{m\}} \partial F_\alpha$ is Haar null and $A \subset \cup_{\alpha \in \mathbb{Q} \setminus \{m\}} \partial F_\alpha$. Since $A$ is Borel indeed an $F_\sigma$-it is Haar null.

When $X = \mathbb{R}^N$, the set of points of discontinuity of $f$ is even $\sigma$-porous; see [7, Theorem 19] (as pointed out in [37, Remark 5.2], the lsc assumption in that theorem is not needed).

3.2. Ideally quasiconvex functions. The subset $C$ of the Banach space $X$ is said to be ideally convex if $\sum_{n=1}^{\infty} \lambda_n x_n \in C$ for every bounded sequence $(x_n) \subset C$ and every sequence $(\lambda_n) \subset [0,1]$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. The boundedness of $(x_n)$ ensures that $\sum_{n=1}^{\infty} \lambda_n x_n$ is absolutely convergent. Such subsets were introduced by Lifšic [24] in 1970 but, apparently, they have not been used in connection with quasiconvex functions. Indeed, without Theorem 3.1 the purpose of doing so is not apparent.

Definition 3.1. The function $f : X \to \mathbb{R}$ is ideally quasiconvex (strongly ideally quasiconvex) if its lower level sets $F'_\alpha$ $(F_\alpha)$ are ideally convex.

It is readily seen that a strongly ideally quasiconvex function is ideally quasiconvex (use $F'_\alpha = \cap_{\beta > \alpha} F_\beta$), but the converse is false; see Remark 7.1 later. Also, if is ideally quasiconvex if and only if $f(\sum_{n=1}^{\infty} \lambda_n x_n) \leq \sup_n f(x_n)$ for every bounded sequence $(x_n) \subset X$ and every sequence $(\lambda_n) \subset [0,1]$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Strong ideal convexity amounts to $f(\sum_{n=1}^{\infty} \lambda_n x_n) < \alpha$ whenever $f(x_n) < \alpha$ for every $n$. This has nothing to do with strict quasiconvexity. For instance, every usc quasiconvex function is strongly ideally quasiconvex (see below).

If $C \subset X$ is ideally convex, it is CS-closed as defined by Jameson [22] and the two concepts coincide when $C$ is bounded. By a straightforward generalization of Carathéodory’s theorem (see for example [13]), every convex subset of $\mathbb{R}^N$ is ideally convex. Therefore, every quasiconvex function on $\mathbb{R}^N$ is ideally quasiconvex. Open convex subsets are CS-closed [20], hence ideally convex, and the same is trivially true of closed convex subsets. In particular, lsc (usc) quasiconvex functions are ideally quasiconvex (strongly ideally quasiconvex).

We shall also need the remark that an ideally convex subset $C$ can only be nowhere dense or have nonempty interior. Indeed, if $C$ is not nowhere dense, $C \cap B$
is dense in $B$ for some nonempty open ball $B$. Since $C$ and $B$ are ideally convex, $C \cap B$ is ideally convex (obvious) and bounded and therefore CS-closed. By [22 Corollary 1], this implies that $C \cap B = B$, so that $B \subseteq C$ and so $C$ has nonempty interior.

The main properties of (strongly) ideally quasiconvex functions are captured in the next theorem.

**Theorem 3.4.** Let $X$ be a Banach space.

(i) If $f : X \to \mathbb{R}$ is ideally quasiconvex, then $f$ is quasiconvex and densely continuous.

(ii) If $f : X \to \mathbb{R}$ is strongly ideally quasiconvex and $m := \text{ess inf}_X f$, then $f$ is quasiconvex and densely continuous and $F_m$ is nowhere dense. More generally, this is true if $f$ is ideally quasiconvex and $F_m$ is ideally convex.

**Proof.** (i) If $\alpha > m$, then $F'_\alpha$ is ideally convex but not nowhere dense since $F'_\alpha \supseteq F_\alpha$ is of second category by definition of $m$. Thus $F'_\alpha \neq \emptyset$ by the remark before the theorem. Now, let $\alpha < m$ and, by contradiction, suppose that $F_\alpha$ is not nowhere dense, so that the larger $F'_\alpha$ (ideally convex) is not nowhere dense either. By the remark before the theorem, $\circ F'_\alpha \neq \emptyset$ and so $\circ F_\beta \neq \emptyset$ for every $\beta > \alpha$. But $F_\beta$ is of first category by definition of $m$ if $\beta \in (\alpha, m)$ and a contradiction arises. Thus, $F_\alpha$ is nowhere dense when $\alpha < m$. That $f$ is densely continuous now follows from Theorem 3.1.

(ii) By (i), $f$ is densely continuous. Since $F_m$ is ideally convex and has empty interior (it is of first category), it is nowhere dense, once again by the remark before the theorem. □

Strongly ideally quasiconvex functions are the closest generalization of finite dimensional quasiconvex functions. As we shall see in Section 5, both have the same differentiability properties. By Theorem 3.4 (i), ideally quasiconvex functions possess only the continuity properties of the finite dimensional case.

**4. Differentiability above the essential infimum**

Throughout this section, $X$ is a separable Banach space and $f : X \to \mathbb{R}$ is quasiconvex and densely continuous. With $m := \text{ess inf}_X f$, the goal is to prove that $f$ is Hadamard differentiable at every point of $X \setminus F'_m$ except at the points of an Aronszajn null set.

The last preliminary lemma will enable us to use Theorem 2.1 to settle the differentiability question at points of $X \setminus F'_m$. The line of argument of the proof, but with other assumptions, has been used before ([15 Theorem 3.1], [24 Proposition 2]).

**Lemma 4.1.** Let $x \in X \setminus F'_m$ be a point of continuity of $f$. Then, $x \in X \setminus \overline{F}_m$ and there is an open neighborhood $U_x$ of $x$ contained in $X \setminus \overline{F}_m$ and a closed convex cone $K_x \subset X$ with nonempty interior such that $f$ is $K_x$-nondecreasing on $U_x$.

**Proof.** Since $f(x) > m$, choose $\alpha \in (m, f(x))$, so that $\circ F_\alpha \neq \emptyset$ by Theorem 3.1. Pick $x_- \in \circ F_\alpha$ and $\varepsilon > 0$ small enough that $B(x_-, 2\varepsilon) \subset F_\alpha$, that $U_x := B(x, \varepsilon) \subset X \setminus F'_\alpha$ (this is possible since $f(x) > \alpha$ and $f$ is continuous at $x$; in particular, $U_x$ is...
contained in the interior $X \setminus F'_m$ of $X \setminus F'_m$ and that $0 \notin \overline{B}(2(x - x_ -), 2\varepsilon)$. Now, set $x_+ := 2x - x_-$, so that $x$ is the midpoint of $x_-$ and $x_+$ and
\[ K_x := \cup_{\lambda \geq 0} \lambda \overline{B}(x_+ - x_-, 2\varepsilon) = \cup_{\lambda \geq 0} \lambda \overline{B}(2(x - x_-), 2\varepsilon). \]
Clearly, $K_x$ is a convex cone with nonempty interior. That it is closed easily follows from the assumption $0 \notin \overline{B}(2(x - x_-), 2\varepsilon)$. Also, if $k \in K_x \setminus \{0\}$, then $x - \frac{k}{\lambda} \in \overline{B}(x_+, \varepsilon)$ where $\lambda > 0$ is chosen such that $k \in \lambda \overline{B}(x_+, -x_-, 2\varepsilon)$. Hence, if $y \in U_x = B(x, \varepsilon)$, it follows that $y - \frac{1}{\lambda} = (y - x) + (x - \frac{k}{\lambda}) \in B(x_+, 2\varepsilon)$.
Suppose now that $y \in U_x, k \in K_x \setminus \{0\}$ and $y + k \in U_x$. From the above, $y - \frac{k}{\lambda} = z \in B(x_+, 2\varepsilon)$. By writing $\frac{1}{\lambda} = \frac{t - 1}{t}$ for some $t \in (0, 1)$, this yields $y = t(y + k) + (1 - t)z$. Since $f$ is quasiconvex, $f(y) \leq \max\{f(y + k), f(z)\} = f(y + k)$, the latter because $f(z) < \alpha < f(y + k)$ (recall $z \in F_\alpha$ and $y + k \in U_x \subset X \setminus F'_m$).

**Theorem 4.2.** Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. Set $m := \text{T ess inf}_X f$. Then, $f$ is Hadamard differentiable on $X \setminus F'_m$ except at the points of an Aronszajn null set.

**Proof.** With no loss of generality, assume that $X \setminus F'_m \neq \emptyset$. Since $X \setminus F'_m$ is dense in $X \setminus F'_m$ (Theorem 3.2(iii)), it follows that $X \setminus F'_m \neq \emptyset$. If $C_m$ denotes the set of points of continuity of $f$ in $X \setminus F'_m$, then $C_m \subset X \setminus F'_m$ by Lemma 4.1 and $C_m \neq \emptyset$ since $X \setminus F'_m$ is open and nonempty and $f$ is densely continuous.

If $x \in C_m$, it follows from Lemma 4.1 and Theorem 2.1 that there is an open neighborhood $U_x$ of $x$ contained in $X \setminus F'_m$ such that $f$ is Hadamard differentiable on $U_x$ except at the points of an Aronszajn null set. Since $X$ is separable, the open set $U := \cup_{x \in C_m} U_x \subset X \setminus F'_m$ is Lindelöf and therefore coincides with the union of countably many $U_x$. As a result, $f$ is Hadamard differentiable on $U$ except at the points of an Aronszajn null set.

Lastly, since $U$ contains all the points of continuity of $f$ in $X \setminus F'_m$, the points of $(X \setminus F'_m) \setminus U$ are the points of discontinuity of $f$ in $X \setminus F'_m$. By Theorem 3.2(iv), these points are contained in an Aronszajn null set, so that, as claimed, $f$ is Hadamard differentiable on $X \setminus F'_m$ except at the points of an Aronszajn null set.

A variant of Remark 3.1 may be repeated: Since $X \setminus F'_m$ is semi-open (Theorem 3.2(iii)), Theorem 4.2 implies that $f$ is Hadamard differentiable at the points of a dense subset of $X \setminus F'_m$. The next corollary is obvious and settles the case when $m = -\infty$.

**Corollary 4.3.** Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasi-convex and densely continuous. If $\text{T ess inf}_X f = -\infty$, then $f$ is Hadamard differentiable on $X$ except at the points of an Aronszajn null set.

5. Hadamard differentiability when $F_m$ is nowhere dense

Since the case when $m := \text{T ess inf}_X f = -\infty$ was settled in Corollary 4.3, it is henceforth assumed that $m > -\infty$. To avoid giving the impression that this is a restrictive assumption, this is not recorded in the statement of the results, which are all true when $m = -\infty$ (by Corollary 4.3 or because their assumptions implicitly rule out $m = -\infty$).

Theorem 4.2 answers the differentiability question at points of $X \setminus F'_m$, but it remains to investigate the differentiability at points of $F'_m$. To begin with, we show
that if \( x \in F'_m \), the Gâteaux derivative \( D_f(x) \) can only exist if \( x \in f^{-1}(m) \) and, if so, that \( D_f(x) = 0 \). In particular, this shows that when it comes to differentiability, \( m \) has the same property as a pointwise infimum.

**Theorem 5.1.** Let \( X \) be a Banach space and let \( f : X \to \mathbb{R} \) be quasiconvex.

(i) If \( x \in F_m \), then \( f \) is not Gâteaux differentiable at \( x \).

(ii) If \( x \in f^{-1}(m) \) and \( f \) is Gâteaux differentiable at \( x \), the Gâteaux derivative \( D_f(x) \) is 0.

**Proof.** (i) Since Gâteaux differentiability does not imply continuity, this does not follow from the fact that \( f \) is not continuous at any point of \( F_m \) (Remark 4.1).

With no loss of generality, assume that \( x = 0 \in F_m \). By the convexity of \( F_m \), the cone \( CF_m := \bigcup_{\lambda > 0} \lambda F_m \) is the countable union \( CF_m = \bigcup_{n \in \mathbb{N}} nF_m \) and hence of first category. Thus, \( X \setminus CF_m \) is nonempty and, if \( h \in X \setminus CF_m \), then \( h \neq 0 \) (since \( 0 \in F_m \subset CF_m \)) and \( th \notin F_m \) for every \( t > 0 \). As a result, \( f(th) \geq m \) when \( t > 0 \). Since \( f(0) < m \), it follows that \( \lim_{t \to 0^+} \frac{f(th) - f(0)}{t} = \infty \), so that \( f \) is not Gâteaux differentiable at 0.

(ii) Once again, by translation, we may assume that \( x = 0 \) (but now \( 0 \notin F_m \) since \( f(0) = m \)). By contradiction, assume that \( D_f(0) \neq 0 \), so that \( \Sigma := \{ h \in X : Df(0)h < 0 \} \) is an open half-space. If \( h \in \Sigma \), then \( f(th) < f(0) = m \) for \( t > 0 \) small enough, say \( 0 < t < t_h \). Thus, \( G := \{ th : h \in \Sigma, 0 < t < t_h \} \subset F_m \) and so \( G \) is of first category. On the other hand, if \( h \in \Sigma \), then \( n^{-1}h \in G \) if \( n \in \mathbb{N} \) and \( n^{-1} < t_h \). In other words, \( \Sigma \subset \bigcup_{n \in \mathbb{N}} nG \). Since \( G \) is of first category, \( \Sigma \) is of first category. Therefore, it cannot be an open half-space and a contradiction is reached.

The proof of part (ii) of Theorem 5.1 does not use the quasiconvexity of \( f \). At any rate, Theorem 5.1 makes the differentiability of \( f \) on \( F'_m \) look like a rather straightforward problem: It suffices to check whether the directional derivative of \( f \) at \( x \in f^{-1}(m) \) exists and is 0 in every direction. However, the evaluation of the size of the set of such points is not that simple. The remainder of this section deals with the case when \( F_m \) is nowhere dense.

If \( F'_m \) is of first category, then \( F'_m \) is nowhere dense (Theorem 3.2 (i)). In particular, \( F_m \) is nowhere dense and there is no need to investigate the differentiability of \( f \) at the points of \( f^{-1}(m) \) to prove the “subgeneric” Hadamard differentiability of \( f \):

**Theorem 5.2.** Let \( X \) be a separable Banach space and let \( f : X \to \mathbb{R} \) be quasiconvex and densely continuous. With \( m := \text{ess}\inf_X f \), suppose that \( F'_m \) (i.e., \( f^{-1}(m) \)) is of first category. Then, \( F'_m \) is nowhere dense and \( f \) is Hadamard differentiable on the open and dense subset \( X \setminus \overline{F'_m} \) except at the points of an Aronszajn null set.

In particular, if \( F'_m \) is contained in an Aronszajn null (Haar null) set, then \( f \) is Hadamard differentiable on \( X \) except at the points of an Aronszajn null (Haar null) set.

**Proof.** As just recalled above, \( F'_m \) is nowhere dense. Thus, \( X \setminus \overline{F'_m} \) is open and dense in \( X \) and it suffices to use Theorem 3.2.

If \( F'_m \) is contained in an Aronszajn null (Haar null) set, then \( \overline{F'_m} \) is Aronszajn null (Haar null) by Theorem 3.2 (v) and so it has empty interior. Thus, \( F'_m \) is of first category and the first part applies. That \( X \setminus F'_m \) may be replaced by \( X \) in the first part is obvious.

\( \square \)
We now prove a similar result when $F'_m$ is of second category.

**Theorem 5.3.** Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. With $m := \text{ess inf}_X f$, suppose that $F'_m$ (i.e., $f^{-1}(m)$) is of second category and that $F_m$ is nowhere dense. Then, $f$ is Hadamard differentiable on the open and dense subset $X \setminus F_m$ except at the points of an Aronszajn null set.

In particular, if $F_m$ is Aronszajn null (Haar null), then $f$ is Hadamard differentiable on $X$, except at the points of an Aronszajn null (Haar null) set.

**Proof.** By Theorem 3.2 (ii), $\partial F'_m \neq \emptyset$ and $\partial F'_m$ is Aronszajn null. On the other hand, since $\partial F'_m \neq \emptyset$ and $F_m$ is nowhere dense, $\partial F'_m \setminus F_m$ is open and dense in $\partial F'_m$.

Furthermore, since $f = m$ on $F'_m \setminus F_m$ and hence on $\partial F'_m \setminus F_m$, it follows that $f$ is Fréchet differentiable with derivative 0 on $\partial F'_m \setminus F_m$. Accordingly, the points of $F'_m \setminus F_m$ where $f$ is not Hadamard differentiable are contained in the Aronszajn null set $\partial F'_m$.

By Theorem 4.2, the points of $X \setminus F'_m$ where $f$ is not Hadamard differentiable are also contained in an Aronszajn null set and the conclusion follows from $X \setminus F_m \subseteq (X \setminus F'_m) \cup (F'_m \setminus F_m)$.

If $F_m$ is Aronszajn null (Haar null), it has empty interior, so that $F_m$ is nowhere dense and the first part applies. That $X \setminus F_m$ may be replaced by $X$ in the first part is obvious. □

By combining Theorem 5.2 and Theorem 5.3, we find

**Theorem 5.4.** Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. With that $m := \text{ess inf}_X f$, suppose that $F_m$ is nowhere dense. Then, $f$ is Hadamard differentiable on an open and dense subset of $X$ except at the points of an Aronszajn null set.

In particular, if $F'_m$ is contained in an Aronszajn null (Haar null) set, or if $F'_m$ is of second category and $F_m$ is Aronszajn null (Haar null), then $f$ is Hadamard differentiable on $X$ except at the points of an Aronszajn null (Haar null) set.

The union of two subsets of $X$, one of which is nowhere dense -not just of first category- and the other Aronszajn null, has empty interior (its complement is dense). Every finite -but not countable- union of such sets is of the same type. Thus, Theorem 5.4 not only ensures that $f$ is Hadamard differentiable on a dense subset of $X$ but also that finitely many functions satisfying its hypotheses are simultaneously Hadamard differentiable on the same dense subset of $X$. Since this need not be true for countably many functions, this property can only be called subgeneric (but see Theorem 5.5 below).

**Remark 5.1.** Phelps’ example [35, p. 80] of a Gâteaux differentiable norm on $\ell^1$ which is nowhere Fréchet differentiable shows that Hadamard differentiability cannot be replaced by Fréchet differentiability in Theorem 5.4.

If $f$ is convex and continuous, it follows from [1] that $f$ is Hadamard differentiable except at the points of an Aronszajn null set. By Theorem 5.4, this is still true when $f$ is only quasiconvex and continuous if $F'_m$ is of second category (because $F_m = \emptyset$), but false if $F'_m$ (closed) is of first category and not Aronszajn null. For instance, [6, Example 6] is a counter-example with $X$ Hilbert. However, when $X$ is reflexive, Theorem 5.4 still yields a full generalization of Crouzeix’s theorem:
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Theorem 5.5. Let $X$ be a reflexive and separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. With $m := \text{ess inf}_X f$, suppose that $F_m$ is nowhere dense. Then, $f$ is Hadamard differentiable on $X$ except at the points of a Haar null set.

Proof. Recall that in reflexive separable Banach spaces, closed convex subsets with empty interior are Haar null (Subsection 2.2). In particular, since $F_m$ is nowhere dense, $F_m$ is Haar null. If $F_m$ is of second category, the result follows directly from Theorem 5.4. On the other hand, if $F_m$ is of first category, then $F_m$ is actually nowhere dense by Theorem 3.2 (i). Thus, $F_m$ is Haar null and Theorem 5.4 is once again applicable. □

Theorem 5.5 should be put in the perspective of Theorem 3.3. When $X = \mathbb{R}^N$, every quasiconvex function is densely continuous, $F_m$ is always nowhere dense and Haar null sets have Lebesgue measure 0, so that Crouzeix’s theorem is recovered.

Theorem 5.5 is not true if $X$ is not reflexive: Every nonreflexive Banach space contains a closed convex subset $C$ with empty interior which is not Haar null ([28]). If $f := \chi_{X\setminus C}$, then $f$ is quasiconvex and continuous on $X\setminus C$ (open and dense in $X$) and hence densely continuous. Also, $m = 1$ and $F_1 = C$. Thus, $f$ is not even Gâteaux differentiable at any point of $C$ by Theorem 5.1.

Remark 5.2. Just like Theorem 5.4, Theorem 5.5 already fails in the convex case if Hadamard differentiability is replaced by Fréchet differentiability: In [27], an example is given of an equivalent norm on $\ell^2$ which is Fréchet differentiable only at the points of an Aronszajn null set. Obviously, no such set is the complement of a Haar null set.

5.1. Special cases. There are a number of special cases of Theorem 5.4 and Theorem 5.5. In particular, by Theorem 3.4, they are always applicable when $f$ is strongly ideally quasiconvex (e.g., usc) or just ideally convex with $F_m$ ideally convex. More generally, the dense continuity assumption is satisfied if $f$ is ideally convex (Theorem 3.2) or, as pointed out in the Introduction, if $f$ is the pointwise limit of continuous functions. Dense continuity can also be deduced from Theorem 3.1 or other criteria in [37].

The condition that $F_m$ is nowhere dense holds trivially if $\inf_X f = m$ (in particular, if $m = -\infty$), for then $F_m = \emptyset$. This happens when $f$ is usc, but in other cases as well; see for instance Theorem 5.4. As shown below, $F_m$ is nowhere dense -and more- when $f$ is strictly quasiconvex. Recall that this means $f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\}$ whenever $x \neq y$ and $\lambda \in (0, 1)$. It is not hard to check that $f$ is strictly quasiconvex if and only if it is quasiconvex and $f(\lambda x + (1 - \lambda)y) < f(x)$ whenever $x \neq y$, $f(x) = f(y)$ and $\lambda \in (0, 1)$. This is also equivalent to saying that $f$ is quasiconvex and nonconstant on any nontrivial line segment ([17, Theorem 9]).

Corollary 5.6. Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be strictly quasiconvex and densely continuous. Set $m := \text{ess inf}_X f$. Then, $F_m$ is nowhere dense and $f$ is Hadamard differentiable on a dense open subset of $X$ except at the points of an Aronszajn null set. Furthermore:

(i) If $F_m$ is contained in an Aronszajn null (Haar null) set, then $f$ is Hadamard differentiable on $X$ except at the points of an Aronszajn null (Haar null) set.
(ii) If \( X \) is reflexive, \( f \) is Hadamard differentiable on \( X \) except at the points of a Haar null set.

**Proof.** By Theorem 5.2 (i), \( F'_m \) is nowhere dense if (and only if) \( f^{-1}(m) \) is of first category. If so, \( F_m \subset F'_m \) is also nowhere dense, so that everything follows once again from Theorem 5.4 and Theorem 5.6.

To complete the proof, we show that \( f^{-1}(m) \) is of first category. If \( F_m = \emptyset \), the strict quasiconvexity of \( f \) shows that \( f^{-1}(m) = F'_m \) contains at most one point. If \( F_m \neq \emptyset \), assume with no loss of generality that \( 0 \in F_m \). If \( x \in f^{-1}(m) \) then \( f \left( \frac{1}{m} x \right) < \max \{ f(0), f(x) \} = m \) and so \( \frac{1}{m} x \in F_m \). As a result, \( f^{-1}(m) \subset 2F_m \). Since \( F_m \) is of first category, the same thing is true of \( 2F_m \) and hence of \( f^{-1}(m) \).

From the above proof, Corollary 5.1 is still true if \( f|_{F'_m} \) is strictly quasiconvex. Unlike ideal quasiconvexity, strict quasiconvexity does not imply dense continuity and must be checked separately. (The sum of a continuous strictly convex function and a real-valued nowhere continuous convex function is strictly convex, hence strictly quasiconvex, but nowhere continuous.)

We now show that (sub)generic Hadamard differentiability can always be obtained after a rather mild modification of the function \( f \) that does not affect the Hadamard derivative of \( f \) at any point where such a derivative exists.

**Theorem 5.7.** Let \( X \) be a separable Banach space and let \( f : X \to \mathbb{R} \) be quasiconvex and densely continuous. Suppose that \( m := \text{ess inf}_X f > -\infty \) and set \( g := \max \{ m, f \} \). The following properties hold:

(i) \( g \) is quasiconvex and densely continuous and \( f = g \) in the residual set \( X \setminus F_m \).

(ii) If \( f \) is Hadamard differentiable at \( x \), then \( g \) is Hadamard differentiable at \( x \) and \( Df(x) = Dg(x) \).

(iii) \( g \) is Hadamard differentiable on a dense open subset of \( X \) except at the points of an Aronszajn null set. Furthermore:

(iv-1) If \( F'_m \) is of first category and contained in an Aronszajn null (Haar null) set, then \( g \) is Hadamard differentiable on \( X \) except at the points of an Aronszajn null (Haar null) set.

(iv-2) If \( X \) is reflexive, \( g \) is Hadamard differentiable on \( X \) except at the points of a Haar null set.

**Proof.** (i) is obvious since every point of continuity of \( f \) is a point of continuity of \( g \). To prove (ii), assume that \( f \) is Hadamard differentiable at \( x \). If \( x \notin F'_m \), then \( f \) is continuous at \( x \) and so \( f \) and \( g \) coincide on a neighborhood of \( x \). Therefore, \( g \) is Hadamard differentiable at \( x \) and \( Dg(x) = Df(x) \). If \( x \in F'_m \), it follows from Theorem 5.1 that \( f(x) = m \) and that \( Df(x) = 0 \). Thus, \( \lim_{t \to 0} \frac{f(x+th)-f(x)}{t} = 0 \) uniformly for \( h \) in the compact subsets of \( X \). Also, \( g(x) = m \) and \( g(x+th) = f(x+th) \) if \( x + th \in F'_m \), whereas \( g(x+th) = m \) if \( x + th \in F_m \). As a result, \( \frac{|g(x+th)-g(x)|}{f(x+th)-f(x)} \leq 1 \) and so \( \lim_{t \to 0} \frac{g(x+th)-g(x)}{f(x+th)-f(x)} = 0 \) uniformly for \( h \) in the compact subsets of \( X \). This shows that \( g \) is Hadamard differentiable at \( x \) and that \( Dg(x) = 0 \).

(iii) Clearly, \( \text{ess inf}_X f = m \), \( G_m := \{ x \in X : g(x) < m \} = \emptyset \) and \( G'_m := \{ x \in X : g(x) \leq m \} = F'_m \), so that it suffices to use Theorem 5.4 and Theorem 5.5 with \( f \) replaced by \( g \). □

5.2. Differentiability of lsc hulls. Earlier, we established that every usc quasiconvex function \( f \) is Hadamard differentiable on a large subset because (a) \( f \) is
densely continuous and (b) \( F_m = \emptyset \). In contrast, if \( f \) is lsc, (a) is still true but (b) holds only if \( \inf_X f = m \). Worse, \( F_m \) need not be nowhere dense (see Remark 5.1). However, as we shall see, this cannot happen by just passing from a function \( f \) to its lsc hull \( \tilde{f} \) (largest lsc function majorized by \( f \)). More specifically, if \( f \) is quasiconvex and densely continuous, we shall see that Theorem 5.4 and Theorem 5.5 are simultaneously applicable to \( f \) and \( \tilde{f} \).

Recall that \( f(x) := \lim \inf_{x' \to x} f(x') = \inf \{ \alpha \in \mathbb{R} : x \in \overline{T}_\alpha \} \). Either characterization shows that if \( f \) is quasiconvex, then \( \tilde{f} \) is quasiconvex, but \( f \) may achieve the value \(-\infty\). Accordingly, \( x \in X \) will be called a point of continuity of \( \tilde{f} \) if \( \tilde{f}(x) \in \mathbb{R} \) and \( f \) is continuous at \( x \). Equivalently, a point of discontinuity \( x \) of \( f \) is a point where \( f(x) = -\infty \), or \( \tilde{f}(x) \in \mathbb{R} \) but \( f \) is not continuous at \( x \).

**Lemma 5.8.** Let \( f : X \to \mathbb{R} \) be quasiconvex and densely continuous and let \( \tilde{f} \) denote its lsc hull. Set \( m := \mathcal{T} \inf_X f \) and \( G_m := \{ x \in X : f(x) < m \} \). Let \( \mathcal{G}'_m := \{ x \in X : f(x) \leq m \} \). The following properties hold:

(i) \( f \) and \( \tilde{f} \) have the same points of continuity and \( f = \tilde{f} \) at such points

(ii) \( m = \mathcal{T} \inf_X f \).

(iii) \( \overline{G}_m = \overline{F}_m \).

(iv) \( F'_m \subset G'_m \subset F'_m \cup Z'_m \), where \( Z'_m \) is of first category and contained in an Aronszajn null set. In particular, \( G'_m \) is of first category (contained in an Aronszajn null set, contained in a Haar null set) if and only if the same thing is true of \( F'_m \).

**Proof.** (i) is proved in [37, Theorem 6.4].

(ii) Since \( f \) is densely continuous, its set of points of discontinuity is of first category. Thus, by (i), \( f = \tilde{f} \) except on a set of first category. This proves (ii) since it is readily checked that modifying a function on a set of first category does not affect its topological essential infimum.

(iii) is trivial when \( m = -\infty \) since \( G_{-\infty} = F_{-\infty} = \emptyset \), so assume \( m > -\infty \). First, \( F_m \subset G_m \) since \( f \leq \tilde{f} \). Next, \( G_m \subset \overline{F}_m \), for if \( x \in G_m \), there is a sequence \( (x_n) \) tending to \( x \) such that \( \lim f(x_n) = f(x) < m \). (Alternatively, notice \( G_m = \bigcup_{\alpha < m} \overline{T}_\alpha \).

(iv) That \( F'_m \subset G'_m \) follows from \( f \leq \tilde{f} \). Suppose now that \( x \in Z'_m := G'_m \setminus F'_m \). Then, \( x \) is a point of discontinuity of \( f \), for otherwise \( f(x) = \tilde{f}(x) \) by (i). This already shows that \( Z_m := G'_m \setminus F'_m \) is of first category. In addition, by Theorem 4.2, the set of points of discontinuity of \( f \) in \( X \setminus F'_m \) and therefore \( Z'_m \) is contained in an Aronszajn null set. The “in particular” part is then obvious. \( \square \)

From Lemma 5.8 if \( f \) is quasiconvex and densely continuous, the size requirements about \( F_m \) and \( F'_m \) in Theorem 5.4 or Theorem 5.5 are satisfied if and only if they are satisfied by \( G_m \) and \( G'_m \). Therefore, these theorems are applicable to \( f \) and \( \tilde{f} \) as soon as they are applicable to either of them (if \( f \) is not real-valued,
first replace \( f \) by \( \arctan f \). If so, both are Hadamard differentiable on the same dense subset of \( X \). In addition, whenever \( f \) and \( \tilde{f} \) are Hadamard differentiable at the same point \( x \), then \( Df(x) = D\tilde{f}(x) \). Indeed, by Lemma 6.1 (i), \( f(x) = f(x) \), so that, since \( f \leq \tilde{f} \),

\[
Df(x)h = \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t} \leq \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t} = D\tilde{f}(x)h,
\]

for every \( h \in X \), whence \( Df(x)h = D\tilde{f}(x)h \) upon changing \( h \) into \(-h\).

When \( f \) is replaced by the uoc hull \( \bar{f} \) of \( f \), there is no close relative of Lemma 6.1, even though \( f \) and \( \bar{f} \) coincide away from a set of first category; see [37, Lemma 5.1], where it is also shown that \( \bar{f} \) is finite. In particular, the “optimal” differentiability properties of \( \bar{f} \) do not provide any useful information about the differentiability properties of \( f \).

6. Gâteaux differentiability on a dense subset

We now investigate the differentiability properties of densely continuous quasiconvex functions without making the additional assumption that \( F_m \) is nowhere dense. Recall that \( F_m \) is always of first category and may fail to be nowhere dense only when \( m > -\infty \) and \( \dim X = \infty \). Several special cases when \( F_m \) is nowhere dense were discussed in Section 5.

If \( X \) is a vector space and \( C \subset X \), we set

\[
C^\dagger := \cup_{\dim Z < \infty} 
\]

where \( Z \) denotes a vector subspace of \( X \) and the closure \( \overline{C \cap Z} \) is relative to the natural euclidean topology of \( Z \). This means that \( C^\dagger \) is the sequential closure of \( C \) in the finest locally convex topology on \( X \) ([41, p. 56]) or in any finer topology (such as the finite topology [23]), although such characterizations will not play a role here. Nonetheless, it is informative to point out that, in general, \( (C^\dagger)\dagger \neq C^\dagger \) (see a counter example in [13]), so that \( C^\dagger \) is not the closure of \( C \) for any topology on \( X \).

In this section, the only important (though trivial) features are that \( C \subset C^\dagger \) and \( C^\dagger \subset \overline{C} \) if \( X \) is a normed space, that \( C^\dagger \) is convex if \( C \) is convex, that \( Y^\dagger = Y \) for every affine subspace \( Y \) of \( X \), plus the following alternative characterization of \( X \backslash C^\dagger \) when \( C \) is convex.

**Lemma 6.1.** Let \( X \) be a vector space and \( C \subset X \) be a convex subset. Then, \( x \in X \backslash C^\dagger \) if and only if, for every \( h \in X \backslash \{0\} \), there is \( t_h > 0 \) such that \((x - t_hh, x + t_hh) \cap C = \emptyset\).

**Proof.** Suppose first that \( x \notin C^\dagger \), so that \( x \notin C \). It suffices to prove that if \( h \in X \backslash \{0\} \), then \( x + th \notin C \) for \( t > 0 \) small enough. Indeed, since \( x \notin C \), the same property with \( h \) replaced by \(-h\) yields the existence of \( t_h \). Now, if \( x + t_hh \in C \) for some positive sequence \( (t_n) \), it is obvious that \( x \in \overline{C \cap Z} \) with \( Z := \text{span}\{x, h\} \), so that \( x \in C^\dagger \), which is a contradiction.
Suppose now that for every $h \in X \setminus \{0\}$, there is $t_h > 0$ such that $(x - t_hh, x + t_hh) \cap C = \emptyset$. In particular, $x \notin C$. By contradiction, if $x \in C^\dagger$, there is a finite dimensional subspace $Z$ such that $x \in C \cap Z$. If so, $C \cap Z \neq \emptyset$, so that the relative interior of $C \cap Z$ is also nonempty. If $z$ is any point of this relative interior, then $(x, z] \subset C \cap Z$ (see p. 45). Thus, with $h := z - x \neq 0$, it follows that $(x - th, x + th) \cap C \supset (x, x + th) \neq \emptyset$ for every $t > 0$ and a contradiction is reached.

It follows from Lemma 6.1 that if $C$ is convex, then $x \in C^\dagger$ if and only if there are $h \in X \setminus \{0\}$ and $t_h > 0$ such that $(x, x + t_hh] \subset C$. (In particular, when $C$ is convex, $C^\dagger$ is unchanged if $\dim Z \leq 2$ in 6.1.)

From Lemma 6.1 it is hardly surprising that $C^\dagger$ should have something to do with Gâteaux differentiability. The relationship is fully clarified in the proof of the next theorem which, in a weaker form, is a generalization of Theorems 5.4 and 5.5.

**Theorem 6.2.** Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. With $m := \text{ess inf}_X f$, suppose that $F^*_{m}$ has empty interior. Then, $f$ is continuous and Gâteaux differentiable on a dense subset of $X$. If $X$ is reflexive and $F^*_{m}$ is contained in a Haar null set, then $f$ is continuous and Gâteaux differentiable on $X$ except at the points of a Haar null set.

**Proof.** If $F^*_{m}$ is of first category, then $F^*_{m}$ is nowhere dense by Theorem 3.2 (i). Thus, $F^*_{m}$ is nowhere dense and a stronger result follows from Theorems 5.4 and 5.5. Thus, in the remainder of the proof, we assume that $F^*_{m}$ is of second category. In particular, $m > -\infty$.

Since $F^\circ_{m}$ is of second category, $F^\circ_{m} \neq \emptyset$ by Theorem 3.2 (ii). Let $x \in F^\circ_{m} \setminus F^\dagger_{m}$ and $h \in X \setminus \{0\}$ be given. By Lemma 6.1 there is $t_h > 0$ such that $(x - t_hh, x + t_hh) \subset X \setminus F^\circ_{m}$. After shrinking $t_h$ if necessary, we may assume that $(x - t_hh, x + t_hh) \subset F^\circ_{m} \setminus F^\dagger_{m}$. Since $F^\circ_{m} \setminus F^\dagger_{m} \subset F^\circ_{m} \setminus F^\circ_{m} \subset f^{-1}(m)$, it follows that $f = m$ on $(x - t_hh, x + t_hh)$, so that the derivative of $f$ at $x$ in the direction $h$ exists and is $0$. Since $h \in X \setminus \{0\}$ is arbitrary, $f$ is Gâteaux differentiable at $x$ with $Df(x) = 0$.

Thus, $f$ is Gâteaux differentiable at every point of $F^\circ_{m} \setminus F^\dagger_{m}$.

Now, let $x \in F^\circ_{m}$ be given and let $r > 0$ be such that $B(x, r) \subset F^\circ_{m}$. Since $F^\dagger_{m}$ has empty interior, $F^\circ_{m} \setminus F^\dagger_{m}$ is dense in $F^\circ_{m}$, so that there is $y \in B(x, r) \cap \left( F^\circ_{m} \setminus F^\dagger_{m} \right)$. Let $\varepsilon > 0$ be such that $B(y, \varepsilon) \subset B(x, r)$. Since $y \notin F^\dagger_{m}$ and $F^\dagger_{m}$ is convex, $B(y, \varepsilon) \setminus F^\dagger_{m}$ is of second category by Lemma 2.3 whereas the set of points of discontinuity of $f$ is only of first category. As a result, $B(y, \varepsilon) \setminus F^\dagger_{m}$ contains a point of continuity $z$ of $f$. Therefore, $z \in B(x, r)$ and $f$ is both continuous and Gâteaux differentiable at $z$.

From the above, $f$ is continuous and Gâteaux differentiable at the points of a dense subset of $F^\circ_{m}$. By convexity, $F^\circ_{m} \neq \emptyset$ is dense in $F^\circ_{m}$, so that the same subset is also dense in $F^\circ_{m}$. On the other hand, since Hadamard differentiability implies continuity, $f$ is continuous and Gâteaux differentiable at the points of a dense subset of $X \setminus F^\circ_{m}$ by Theorem 4.2. Altogether, this proves that $f$ is continuous and Gâteaux differentiable at the points of a dense subset of $X$.

To complete the proof, assume that $X$ is reflexive and that $F^*_{m}$ is contained in a Haar null set. In particular, $F^*_{m}$ has empty interior. Thus, from the above and Theorem 4.2 the set of points where $f$ is not Gâteaux differentiable is contained in
a Haar null set. Indeed, such points can only be in \( F'_m \cap F^\circ_m \) (contained in a Haar null set), or in \( \partial F'_m \) (Aronszajn null since \( F'_m \neq \emptyset \)) or in \( X \setminus F'_m \) and contained in an Aronszajn null set.

By Theorem 6.3, the set of points of discontinuity of \( f \) is also Haar null and so the set of points where \( f \) is not continuous or not Gâteaux differentiable is Haar null.

The counterexample to Theorem 6.3 given in the previous section also shows that the reflexivity of \( X \) cannot be omitted in the second part of Theorem 6.2.

**Remark 6.1.** Theorem 6.2 or any of its corollaries below is a statement about the differentiability properties of \( f \) on the entire space \( X \). In particular, it does not record the fact that a stronger property is true at the points of \( X \setminus F'_m \) (Theorem 4.2; recall that \( X \setminus F'_m \) is always semi-open).

The set \( F'_m \) is often much smaller than \( \overline{F}_m \). In the next four corollaries, \( F'_m \) (but not necessarily \( F_m \)) has empty interior, although this is not explicitly assumed.

**Corollary 6.3.** Let \( X \) be a separable Banach space and let \( f : X \rightarrow \mathbb{R} \) be quasiconvex and densely continuous. With \( m := \text{ess inf}_X f \), suppose that for every finite dimensional subspace \( Z \) of \( X \), there is \( \alpha < m \) such that \( F_m \cap Z \subset \overline{F}_\alpha \). Then, \( f \) is continuous and Gâteaux differentiable on a dense subset of \( X \).

If \( X \) is reflexive, \( f \) is continuous and Gâteaux differentiable on \( X \) except at the points of a Haar null set.

**Proof.** By Theorem 6.2, it suffices to show that \( F'_m \) has empty interior and that it is contained in a Haar null set if \( X \) is reflexive.

From (6.1) and the assumption \( F_m \cap Z \subset \overline{F}_\alpha \) when \( \dim Z < \infty \), it follows that \( F'_m \subset \cup_{\alpha < m} \overline{F}_\alpha \). Since the sets \( F_\alpha \) are linearly ordered by inclusion and nowhere dense when \( \alpha < m \), \( \cup_{\alpha < m} \overline{F}_\alpha = \cup_{\alpha < m, \alpha \in \mathbb{Q}} \overline{F}_\alpha \) is of first category. Thus, \( F'_m \) has empty interior. In addition, if \( X \) is reflexive, then \( \overline{F}_\alpha \) is Haar null, so that \( \cup_{\alpha < m, \alpha \in \mathbb{Q}} \overline{F}_\alpha \) is Haar null.

**Corollary 6.4.** Let \( X \) be a separable Banach space and let \( f : X \rightarrow \mathbb{R} \) be quasiconvex and densely continuous. With \( m := \text{ess inf}_X f \), suppose that \( F_m \) is contained in some proper affine subspace of \( X \). Then, \( f \) is continuous and Gâteaux differentiable on a dense subset of \( X \).

If \( X \) is reflexive and \( F_m \) is contained in some proper Borel affine subspace, \( f \) is continuous and Gâteaux differentiable on \( X \) except at the points of a Haar null set.

**Proof.** With no loss of generality, assume \( \dim X = \infty \) (if \( \dim X < \infty \), Crouzeix’s theorem gives a stronger result). Let \( Y \) denote a proper affine subspace of \( X \) containing \( F_m \). Evidently, \( F'_m \subset Y^\perp \) and, as noted at the beginning of this section, \( Y^\perp = Y \) since \( Y \) is affine. Since a proper affine subspace of \( X \) has empty interior, \( F'_m \) has empty interior, so that Theorem 6.2 applies.

Since \( Y \neq X \), there is \( \xi \in X \setminus \{0\} \) such that \( \mathbb{R}\xi \) intersects no translate of \( Y \) at more than one point. Hence, if \( E \subset X \) is Borel and \( \mu(E) := \lambda_1(\mathbb{R}\xi \cap E) \) where \( \lambda_1 \) is the Lebesgue measure on \( \mathbb{R}\xi \), then \( \mu(x + Y) = 0 \) for every \( x \in X \) provided that \( Y \) is Borel. Thus, if \( X \) is reflexive, Theorem 6.2 ensures that \( f \) is continuous and Gâteaux differentiable on \( X \) except at the points of a Haar null set.

An interesting outcome of Corollary 6.4 is:
Corollary 6.5. Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be even, quasiconvex and densely continuous. Then, $f$ is continuous and Gâteaux differentiable on a dense subset of $X$.

If $X$ is reflexive, $f$ is continuous and Gâteaux differentiable on $X$ except at the points of a Haar null set.

Proof. If $m := \text{ess inf}_X f = -\infty$, a stronger property is proved in Corollary 6.6. Accordingly, we only give a proof when $m > -\infty$ and $F_m \neq \emptyset$ since Theorems 5.3 and 5.5 yield a better result when, more generally, $F_m$ is nowhere dense.

Since $f$ is even, $F_m = -F_m$ and so $0 \in F_m$ by the convexity of $F_m \neq \emptyset$. As a result, the convex cone $CF_m := \cup_{\lambda > 0} \lambda F_m$ is the countable union $CF_m = \cup_{n \in \mathbb{N}} nF_m$ (of first category) and $CF_m = -CF_m$, so that $CF_m$ is a vector subspace of $X$ of first category. Thus, $CF_m \neq X$. Since $F_m \subset CF_m$, Corollary 6.4 ensures that $f$ is continuous and Gâteaux differentiable on a dense subset of $X$.

If $X$ is reflexive, the result also follows from Corollary 6.4 if we show that $F_m$ is contained in a Borel proper (vector) subspace of $X$. This can be seen by a modification of the above argument. Specifically, $F_\alpha = -F_\alpha$ for every $\alpha \in \mathbb{R}$, not just $\alpha = m$, whence $\overline{F_\alpha} = \overline{F_\alpha}$ and $\overline{CF_\alpha} := \cup_{\lambda > 0} \lambda \overline{F_\alpha} = \cup_{n \in \mathbb{N}} n\overline{F_\alpha}$. Thus, $\overline{CF_\alpha}$ is a Borel vector subspace of $X$ for every $\alpha \in \mathbb{R}$ such that $F_\alpha \neq \emptyset$ and $\overline{CF_\alpha} = \emptyset$ otherwise.

If $\alpha < m$, then $F_\alpha$ is nowhere dense (Theorem 3.1), whence $\overline{CF_\alpha}$ is of first category. The subspaces $\overline{CF_\alpha}$ are linearly ordered by inclusion and so $\cup_{\alpha < m} \overline{CF_\alpha} = \cup_{\alpha < m, \alpha \in \mathbb{Q}} \overline{CF_\alpha}$ is a Borel vector subspace of $X$ of first category ($F_\alpha \neq \emptyset$ ensures that $F_\alpha \neq \emptyset$ for some $\alpha < m$, so that $\overline{CF_\alpha}$ is a subspace). Obviously, $F_m = \cup_{\alpha < m} F_\alpha \subset \cup_{\alpha < m} \overline{F_\alpha} \subset \cup_{\alpha < m} \overline{CF_\alpha}$. This completes the proof.

Below, we show that differentiability on a dense subset is still true when, more generally, $f$ is invariant under the action of a linear periodic map $T$ with no fixed point. By this, we mean that $f(Tx) = f(x)$ where $T \in \mathcal{L}(X)$, $T^p = I$ for some integer $p > 1$ and 1 is not an eigenvalue of $T$, so that $Tx = x$ if and only if $x = 0$. In Corollary 6.4, $p = 2$ and $T = -I$.

Corollary 6.6. Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex, densely continuous and invariant under the action of a linear periodic map $T$ with no fixed point. Then, $f$ is continuous and Gâteaux differentiable on a dense subset of $X$.

If $X$ is reflexive, $f$ is continuous and Gâteaux differentiable on $X$ except at the points of a Haar null set.

Proof. We only show how the proof of Corollary 6.5 can be modified to yield the desired result. The key point of that proof is that, when $m > -\infty$ and $F_m \neq \emptyset$, the cone $CF_m$ is both of first category and a vector subspace of $X$. That $CF_m$ is of first category is true irrespective of any symmetry. In the proof of Corollary 6.5, it is obvious that $CF_m$ is a vector subspace of $X$ is an immediate by-product of $F_m = -F_m$, but a routine check reveals that this remains true if, for every $x \in F_m$, there is $\varepsilon > 0$ (possibly depending upon $x$) such that $-\varepsilon x \in F_m$. The proof that this holds under the hypotheses of the corollary is given below, with $F_m$ replaced by any nonempty convex subset $E \subset X$ invariant under $T$ (i.e., $T(E) \subset E$).

Since the existence of $\varepsilon$ is obvious if $x = 0$, assume $x \neq 0$. With the notation introduced before the corollary, it is clear that $x, Tx, ..., T^{p-1}x \in E$. Furthermore, since $T^p = I$, the point $y := x + Tx + ... + T^{p-1}x$ is a fixed point of $T$, (i.e., $Ty = y$) so
that \( y = 0 \) since \( T \) has no other fixed point. As a result, \( 0 = \frac{1}{p} x + \frac{1}{p} T x + \cdots + \frac{1}{p} T^{p-1} x \) and hence \( 0 \in C_x := \text{conv}\{x, ..., T^{p-1} x\} \subset E.

We claim that \( C_x \) is a neighborhood of 0 in \( Y_x := \text{span}\{x, ..., T^{p-1} x\} \). Indeed, the linear mapping \( \Lambda \) on \( \mathbb{R}^p \) defined by \( \lambda := (\lambda_0, ..., \lambda_p) \in \mathbb{R}^p \mapsto \Lambda \lambda := \sum_{j=0}^{p-1} \lambda_j T^j x \in Y_x \) is surjective and \( \ker \Lambda \) contains \( p^{-1} 1 \), where \( 1 := (1, ..., 1) \). Since the subspace \( Z := \{ \lambda \in \mathbb{R}^p : \sum_{j=0}^{p-1} \lambda_j = 0 \} \) does not contain \( p^{-1} 1 \), it follows that \( \Lambda : Z \to Y_x \) is still surjective. Let \( U \subset Z \) denote the neighborhood of 0 defined by \( U := \{ \lambda \in Z : |\lambda_j| < p^{-1} \} \). Since linear surjective maps are open, \( \Lambda(U) \) is an open neighborhood of 0 in \( Y_x \). Now, \( \Lambda(U) = \Lambda(U + p^{-1} 1) \) and, if \( \mu \in U + p^{-1} 1 \), then \( \mu = \lambda + p^{-1} 1 \) with \( \lambda \in U \), so that \( \mu_j > 0 \) and \( \sum_{j=0}^{p-1} \mu_j = \sum_{j=0}^{p-1} (\lambda_j + p^{-1}) = 1 \). Thus, every vector in the open neighborhood \( \Lambda(U) \) of 0 in \( Y_x \) is a convex combination of \( x, ..., T^{p-1} x \) and therefore in \( C_x \). This proves the claim.

Since \( -x \in Y_x \) and every neighborhood of 0 in \( Y_x \) is absorbing, it follows that \( -\varepsilon x \in C_x \subset E \) for some \( \varepsilon > 0 \), as had to be proved.

The continuity of \( T \) implies that \( \overline{F}_\alpha \) (convex) is invariant under \( T \) for every \( \alpha \in \mathbb{R} \). Thus, from the above, \( C \overline{F}_\alpha \) is a vector subspace of \( X \) whenever \( F_\alpha \neq \emptyset \). As a result, the proof of Corollary 6.5 can also be repeated when \( X \) is reflexive. \( \square \)

Corollaries 6.5 and 6.6 remain true when \( f \) is replaced by its lsc hull \( \overline{f} \) which, as is easily checked, exhibits the same symmetry as \( f \). Also, if Theorem 6.2 is applicable to \( f \), it is applicable to \( \overline{f} \) by Lemma 5.8 (ii) since \( F_m \subset G_m := \{ x \in X : f(x) < m \} \) is obvious from \( f \leq \overline{f} \). However, there seems to be no reason why the converse should be true (compare with Subsection 5.2).

7. An example

We give an example of a function \( f \) satisfying all the assumptions of Corollaries 6.5 and 6.6 (hence of Corollaries 6.3 and 6.4 as well, because Corollary 6.5 is a special case of both). This example was used in \([37]\) to produce a non trivial densely quasiconvex function. It will also show that the differentiability properties of \( f \) at the points of \( F'_m \) are generally much weaker when \( F_m \) is not nowhere dense than when \( F_m \) is nowhere dense.

Example 7.1. Let \( X \) be an infinite dimensional separable Banach space and let \( (x_n)_{n \in \mathbb{N}} \subset X \) be a dense sequence. Set \( L_n := \text{span}\{x_1, ..., x_n\} \), so that \( L := \bigcup_{n \in \mathbb{N}} L_n \) is a dense subspace of \( X \) and \( L \) is of first category since \( \dim L_n < \infty \). After passing to a subsequence, we may assume \( L \nsubseteq L_{n+1} \) without changing \( L \). Now, let \( (\alpha_n) \subset \mathbb{R} \) be a strictly increasing sequence such that \( \lim_{n \to \infty} \alpha_n = 1 \). Set \( f(x) = 1 \) if \( x \in X \setminus L \), \( f(x) = \alpha_1 \) if \( x \in L_1 \) and \( f(x) = \alpha_n \) if \( x \in L_n \setminus L_{n-1}, n \geq 2 \).

In Example 7.1 \( m := \text{ess} \inf_X f = 1, F_\alpha = X \) if \( \alpha > 1 \), \( F_1 = L \) is convex of first category but everywhere dense and \( F'_1 = X \). Also, if \( \alpha_1 < \alpha < 1 \), there is a unique \( n \in \mathbb{N} \) such that \( \alpha_n < \alpha \leq \alpha_{n+1} \) and so \( F_\alpha = L_n \) is (closed, convex and) nowhere dense. If \( \alpha \leq \alpha_1 \), then \( F_\alpha = \emptyset \). In particular, \( f \) is quasiconvex.

By Theorem 3.1 (or because \( f \) is lsc; see Remark 7.1 below), \( f \) is densely continuous. Actually, a direct verification shows that the set of points of discontinuity of \( f \) is exactly \( L \). If \( x \notin L \) and \( h \in X \setminus \{0\} \), then \( x + th \in L \) if and only if \( x + th \in L_n \) for some \( n \). This can only happen if \( t \neq 0 \) and for at most one \( t \). Indeed, if \( x + th \in L_n \)
and \( x + sh \in L_{n_2} \) with \( s \neq t \) and (say) \( n_2 \geq n_1 \), then \( x + sh - x - th = (s-t)h \in L_{n_2} \), so that \( h \in L_{n_2} \). But then, \( sh \in L_{n_2} \) and so \( x \in L_{n_2} \subseteq L \), which is a contradiction.

The above property shows that if \( x \notin L \) and \( h \in X\setminus\{0\} \), there is \( t_h > 0 \) such that \((x - t_hh, x + t_hh) \subseteq X \setminus L\). Since \( f = 1 \) on \( X \setminus L \), it follows that \( f \) is also Gâteaux differentiable at every point of \( X \setminus L \) (and \( Df(x) = 0 \)). Note that \( L \) has countable dimension, so that it is Borel and Haar null regardless of the reflexivity of \( X \).

Now, it is plain that \( f \) is even, so that it satisfies the hypotheses of Corollary \[.3\]. To see that it also satisfies the hypotheses of Corollary \[.4\] let \( Z \) be a finite dimensional subspace of \( X \) and let \( x \in F_1 \cap Z = L \cap Z \) (recall \( m = 1 \)). Obviously, \( L \cap Z = \cup_{n\in\mathbb{N}} (L_n \cap Z) \) and \( L_n \cap Z \) is a nondecreasing sequence of subspaces of \( Z \). Since \( \dim Z < \infty \), it follows that \( L_n \cap Z \) is independent of \( n \) large enough, so that \( n \geq n_0 \) and so \( L \cap Z = L_{n_0} \cap Z \). Thus, \( x \in L_{n_0} \) and so \( f(x) \leq \alpha_{n_0} < 1 = m \).

**Remark 7.1.** In Example \[7.1\], \( F_\alpha' = X \) if \( \alpha \geq 1, F_\alpha' = L_n \) if \( \alpha_1 \leq \alpha < 1 \), where \( n \in \mathbb{N} \) is the unique integer such that \( \alpha_n \leq \alpha < \alpha_{n+1} \) and \( F'_1 = \emptyset \) if \( \alpha < \alpha_1 \). Thus, \( F'_\alpha \) is always closed, so that \( f \) is lsc. Since \( X \) and all its finite dimensional subspaces are ideally convex, this also shows that \( f \) is ideally quasiconvex (Definition \[7.1\]), but \( F_1 \) is dense with empty interior, so that \( F_1 \) is not ideally quasiconvex; see the comments before Theorem \[7.4\]. Thus, \( f \) is not strongly ideally quasiconvex.

In general, when \( m > -\infty \) and \( F'_m \) is of second category, then \( F'_m \neq \emptyset \) is dense in \( F'_m \) (Theorem \[5.2\] (ii)). If also \( F_m \) is nowhere dense, then \( f = m \) on the open set \( F'_m \cap F_m \), dense in \( F'_m \), so that \( f \) is Fréchet differentiable with derivative 0 on an open and dense subset of \( F'_m \). This was used in the proof of Theorem \[5.3\].

In the above example, \( m = 1 \) and \( F'_1 = X \) is of second category. Since \( F_1 = L \) is dense in \( X \) and \( f \) is not Gâteaux differentiable at any point of \( F_1 \) (Theorem \[5.1\]), \( f \) cannot be Fréchet differentiable on an open and dense subset of \( F'_1 \) but, since it is continuous and Gâteaux differentiable on the residual subset \( f^{-1}(1) = X \setminus L \), it could conceivably be Fréchet differentiable on this set, or perhaps on a smaller but still residual subset. Below, we show not only that this is not the case, but even that the set of points of \( X \setminus L \) where \( f \) is Hadamard differentiable is of first category. This establishes that, in general, the differentiability properties at the points of \( f^{-1}(m) \) are indeed weaker when \( F_m \) is not nowhere dense than they are when \( F_m \) is nowhere dense.

The notation of Example \[7.1\] is used in the next lemma. Also, \( d(x, L_n) \) denotes the distance from \( x \in X \) to the subspace \( L_n \).

**Lemma 7.1.** Let \( (\beta_n)_{n\in\mathbb{N}} \subset (0, \infty) \) be any sequence. The set \( W := \{ x \in X : d(x, L_n) < \beta_n \} \) is residual in \( X \).

**Proof.** By the continuity of \( d(\cdot, L_n) \), the set \( W_n := \{ x \in X : d(x, L_n) < \beta_n \} \) is open in \( X \) for every \( n \in \mathbb{N} \). In addition, \( L_k \subseteq W_n \) for every \( n \geq k \), so that \( L_k \subseteq \bigcup_{n \geq j} W_n \) for every \( j, k \in \mathbb{N} \). Hence, \( L \subseteq \bigcup_{n \geq j} W_n \) for every \( j \in \mathbb{N} \) so that \( \bigcup_{n \geq j} W_n \) is open and dense in \( X \) irrespective of \( j \in \mathbb{N} \). As a result, \( \bigcap_{j \in \mathbb{N}} \bigcup_{n \geq j} W_n \) is residual and coincides with the set \( W \) of the lemma.

**Theorem 7.2.** The set of points where the function \( f \) of Example \[7.1\] is Hadamard differentiable is of first category in \( X \).

**Proof.** If \( x \in X \) and \( n \in \mathbb{N} \), let \( \xi_n = \xi_n(x) \in L_n \) be such that \( ||x - \xi_n|| = d(x, L_n) \) (recall \( \dim L_n < \infty \)). In Lemma \[7.1\], let \( \beta_n := (1 - \alpha_n)^2 \) (notation of Example \[7.1\],...
so that $W := \{ x \in X : ||x - \xi_n|| < (1 - \alpha_n)^2 \}$ for infinitely many indices $n \in \mathbb{N}$ is residual in $X$. Below, we show that $f$ is not Hadamard differentiable at any point $x \in W \setminus L$. Since $f$ is not even Gâteaux differentiable at any point of $L = F_1$ by Theorem 5.1 (i), this implies that $f$ is not Hadamard differentiable at any point of $W$, which proves the theorem.

If $x \in X \setminus L$, then $\xi_n \neq x$ for every $n$ and $f(x) = 1$. We already know that $f$ is continuous and Gâteaux differentiable at $x$ with derivative 0 (the latter also follows from Theorem 5.1 (ii)). To show that $f$ is not Hadamard differentiable at $x \in W \setminus L$, it suffices to find sequences $(h_n) \subset X$ and $(t_n) \subset (0, \infty)$, both tending to 0, such that $\lim_{n \to \infty} \frac{f(x + t_nh_n) - f(x)}{t_n} \neq 0$.

Set $h_n := \frac{\xi_n - x}{||x - \xi_n||^{1/2}}$ and $t_n := ||x - \xi_n||^{1/2}$. That $\lim_{n \to \infty} t_n = 0$ and $\lim_{n \to \infty} h_n = 0$ follows from the definition of $\xi_n$ and the denseness of $L$ in $X$. On the other hand,

$$\left| \frac{f(x + t_nh_n) - f(x)}{t_n} \right| = \frac{1 - f(\xi_n)}{||x - \xi_n||^{1/2}} \geq \frac{1 - \alpha_n}{||x - \xi_n||^{1/2}};$$

where the inequality follows from $\xi_n \in L_n$ and $f \leq \alpha_n$ on $L_n$. Since $x \in W$, there are infinitely many indices $n$ such that $||x - \xi_n||^{1/2} < 1 - \alpha_n$. Therefore, $\frac{1 - \alpha_n}{||x - \xi_n||^{1/2}} > 1$ for infinitely many indices $n$. As a result, $\frac{f(x + t_nh_n) - f(x)}{t_n}$ does not tend to 0 and the proof is complete. \hfill \Box

Although Theorem 7.2 demonstrates that the cases when $F_m$ is nowhere dense or just of first category are different, it does not rule out that $f$ is Hadamard differentiable except at the points of a Haar null set. With special choices of $X, L_n$ and $\alpha_n$, it can be shown that the set of points where $f$ is not Hadamard differentiable is not Aronszajn null, but “not Haar null” has remained elusive.

8. A peculiar convex subset of $\ell^2$

Corollaries 6.5 and 6.6 raise the question whether symmetry or any extra assumption is really needed to ensure that a densely continuous quasiconvex function is continuous and Gâteaux differentiable at the points of a dense subset of the separable Banach space $X$. By Theorem 6.2, this would be true if $C^1$ in (6.1) had empty interior whenever $C$ is a convex subset of $X$ of first category.

In this section, we show that such a property does not hold, even when $X = \ell^2$, by constructing a convex subset $C$ of $\ell^2$ of first category such that $C^1 = \ell^2$ (see (6.1)). Similar subsets can be constructed in any $\ell^p, 1 \leq p < \infty$.

Let $P := \{ \{x_j\} : x_j \geq 0 \forall j \in \mathbb{N} \} \subset \ell^2$ denote the nonnegative cone, a closed convex subset with empty interior. Set $u := (j^{-1})_{j \in \mathbb{N}} \in P$ and, for $n \in \mathbb{N}$, define $C_n := -nu + P$, an increasing sequence of closed convex subsets (not cones) with empty interior. Then, $C := \cup_{n \in \mathbb{N}} C_n$ is a convex subset (even a cone) of $\ell^2$ of first category and $P \subset C$.

**Lemma 8.1.** For every $x \in \ell^2 \setminus \{0\}$, there is $h \in P \setminus \{0\}$ such that, for every $t > 0, x_j + th_j \geq 0$ for $j$ large enough.

**Proof.** Given $x \in \ell^2$, choose a strictly increasing sequence $(n_i) \subset \mathbb{N}$ such that $\Sigma_{j=n_{i-1}+1}^{n_i} x_j^2 \leq i^{-4}$. For $j \in \mathbb{N}$, let $h_j := |x_j|$ if $1 \leq j \leq n_1$ and $h_j := i|x_j|$ if
Indeed, if $C \{f \text{ with } \alpha \text{ densely continuous}\) Indeed, with the same increasing sequence $(\alpha)$ a contradiction is reached with dimensional subspaces, if follows from (6.1) that $\Sigma$ irrespective of the choice of the sequence $(\alpha)$. But every $h \in \mathbb{P}$, so that if $t > 0$, then $x_j + th_j + nu_j > 0$ irrespective of $n \in \mathbb{N}$ for the same indices $j$ since $u_j > 0$ for every $j$. In addition, the finitely many remaining terms $x_j + th_j + nu_j$ can all be made positive if $n$ is chosen large enough. Thus, $x + th + nu \in \mathbb{P}$ if $n$ is large enough, whence $x + th \in C_n \subset C$ for every $t > 0$.

Since $t > 0$ above may be arbitrarily small, there is a sequence of span $\{x, h\} \cap C$ tending to $x$, whence $x \in C^1$. Thus, $\ell^2 \{0\} \subset C^1$. Since also $0 \in \mathbb{P} \subset C \subset C^1$, the proof that $C^1 = \ell^2$ is complete.

Anecdotaly, that $C^1 = \ell^2$ implies that $C$ has another uncommon feature. Since it is of first category, $C \neq \ell^2$ and so, by the so-called Stone-Kakutani property, $C$ is contained in a convex subset $\hat{C}$ such that $\ell^2 \hat{C}$ is also convex and nonempty ([41, p. 71]). But $\hat{C}$ cannot be a half-space $\Sigma := \{x \in \ell^2 : l(x) \leq a\}$ (let alone $\{x \in \ell^2 : l(x) < a\}$) for any $a \in \mathbb{R}$ and any linear form $l$ on $\ell^2$, continuous or not. Indeed, if $C \subset \Sigma$, then $C^1 \subset \Sigma^1$. Since every linear form is continuous on finite dimensional subspaces, if follows from [0,1] that $\Sigma^1 = \Sigma$. Since $\Sigma \neq \ell^2$ is obvious, a contradiction is reached with $C^1 = \ell^2$.

The procedure used in Example [7,4] shows how to construct quasiconvex functions $f$ with $F_m = C$ which, in addition, are continuous at every point of $X \setminus C$ and, so, densely continuous. Indeed, with the same increasing sequence $(\alpha_n)$, it suffices to define $f(x) = \alpha_1$ on $C_1$, $f(x) = \alpha_n$ on $C_n \setminus C_{n-1}$, $n \geq 2$ and $f(x) = 1$ on $X \setminus C$. If $x \notin C$, then $x \notin C_n$ for every $n$ and so there is an open ball $B(x, \varepsilon_n)$ such that $B(x, \varepsilon_n) \cap C_n = \emptyset$. Thus, if $y \in B(x, \varepsilon_n)$, then either $f(y) = 1$ (if $y \notin C$) or $f(y) \geq \alpha_{n+1}$ (if $y \in C_j$ with $j \geq n + 1$). In either case, $f(y)$ is arbitrarily close to 1 if $n$ is large enough, which proves the continuity of $f$ at $x$. Unfortunately, the Gâteaux differentiability question seems difficult to settle at the points $x \notin C$, irrespective of the choice of the sequence $(\alpha_n)$. (The characteristic function of $X \setminus C$ is nowhere Gâteaux differentiable, but since it is also nowhere continuous, it is not a counter example.)

9. Essential Gâteaux differentiability on a dense subset

In the previous section, we saw that the question whether every densely continuous quasiconvex function on a separable Banach space $X$ is continuous and Gâteaux differentiable on a dense subset of $X$ is not fully resolved by Theorem [52]. We complete this paper by showing that a positive answer can be given if the conditions for differentiability are suitably relaxed.

The classical notion of Gâteaux differentiability of $f$ at a point $x$ requires

$$
\lim_{n \to 0} \frac{f(x + \varepsilon_n h) - f(x)}{\varepsilon_n} = l_x(h)
$$

for some $l_x \in X^*$ and every $h \in X \setminus \{0\}$, that is, for every $h \in S_X$, the unit sphere of $X$. If this holds only for $h$ in a dense subset $\Omega$ of $S_X$, then $l_x$ is uniquely determined by $\Omega$, but since different subsets $\Omega$ may produce
different linear forms $l_x$, this does not provide a sound generalization of Gâteaux differentiability. On the other hand, if $\Omega$ is residual in $S_X$ and if $l'_x \in X^*$ is similarly defined when $\Omega$ is replaced by another residual set of directions $\Omega'$, then $l'_x = l_x$ since $\Omega \cap \Omega'$ is residual, and therefore dense, in $S_X$ (since $X$ is a Banach space, $S_X$ is a complete metric space and so a Baire space).

Accordingly, we shall say that $f$ is essentially Gâteaux differentiable at $x$ if $l_x(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$ for some $l_x \in X^*$ and every $h$ in some residual subset $\Omega$ of $S_X$. If so, from the above remarks, the (essential) Gâteaux derivative $Df(x) := l_x$ is well defined and independent of $\Omega$. The next theorem states that for densely continuous quasiconvex functions on a separable Banach space, continuity plus essential Gâteaux differentiability always hold on a dense subset. For clarity, we expound the simple geometric property of convex sets used in the proof.

**Lemma 9.1.** Let $X$ be a real vector space and let $C \subset X$ be a convex subset. Given $x \in X \setminus C$ and $h \in X \setminus \{0\}$, there is $t_h > 0$ such that either $(x, x + th) \subset C$ or $(x, x + th) \not\subset X \setminus C$.

**Proof.** Suppose that there is no $t_h > 0$ such that $(x, x + th) \subset C$, i.e., that $(x, x + th)$ contains a point of $X \setminus C$ for every $t > 0$. We show that there is $t_h > 0$ such that $(x, x + th) \subset X \setminus C$. If not, since $x \notin C$, there is a decreasing sequence $(t_n) \subset (0, \infty)$ such that $\lim t_n = 0$ and $x + t_n h \in C$ for every $n$. Since $C$ is convex, $x + th \in C$ for $t \in [t_n, t_1]$ and every $n$. Thus, $(x, x + t_1 h) \subset C$ contains no point of $X \setminus C$, which is a contradiction. \hfill $\square$

**Theorem 9.2.** Let $X$ be a separable Banach space and let $f : X \to \mathbb{R}$ be quasiconvex and densely continuous. Then, $f$ is continuous and essentially Gâteaux differentiable on a dense subset of $X$.

If $X$ is reflexive, $f$ is continuous and essentially Gâteaux differentiable on $X$ except at the points of a Haar null set.

**Proof.** As in the proof of Theorem 9.2, we may assume that $m := \text{ess inf}_X f > -\infty$ and that $F'_m$ is of second category, whence $F'_m \not= \emptyset$ by Theorem 3.2 (ii). We shall show that if $x \notin F'_m \setminus F_m$, the essential Gâteaux derivative of $f$ at $x$ exists and is 0. Since $F'_m \setminus F_m$ is residual in $F'_m$ and $f$ is continuous at the points of a residual subset of $F'_m$, this implies that $f$ is continuous and essentially Gâteaux differentiable at the points of a residual subset of $F'_m$ and hence at the points of a dense subset of $F'_m$. By Theorem 3.2 and Theorem 3.2 (iii), $f$ is Hadamard differentiable at the points of a dense subset of $X \setminus F'_m$, which proves the first part of the theorem.

Let then $x \in F'_m \setminus F_m$ be given. The set $H_x := \{h \in X \setminus \{0\} : (x, x + th) \subset F_m\}$ for some $t_h > 0$ is a cone (invariant by multiplication by positive scalars). Also, $H_x \subset X \setminus \{0\}$ and, if $h \in H_x$, then $h \in n(F_m - x)$ for any $n \in \mathbb{N}$ such that $n^{-1} < t_h$. Since $F_m$ is of first category, $H_x \subset \cup_{n \in \mathbb{N}} n(F_m - x)$ is of first category in $X$ and, hence, in $X \setminus \{0\}$ as well (if $E$ is a closed subset of $X$ with empty interior in $X$, then $E \setminus \{0\}$ is a closed subset of $X \setminus \{0\}$ with empty interior in $X \setminus \{0\}$).

\[3\]By the corollary of the Kuratowski-Ulam theorem used in the proof of Theorem 9.2, the definition is unchanged if it is required that $h$ belongs to a residual cone in $X \setminus \{0\}$.

\[4\]It is also convex, but this is unimportant in this proof.
That \( H_x \subset X \setminus \{0\} \) is a cone implies that it is homeomorphic to \((0, \infty) \times (H_x \cap S_X)\) through the homeomorphism \( z \mapsto (||z||, z/||z||^{-1}) \) of \( X \setminus \{0\} \) onto \((0, \infty) \times S_X \). Since homeomorphisms preserve Baire category, \((0, \infty) \times (H_x \cap S_X)\) is of first category in \((0, \infty) \times S_X \). By a classical corollary of the Kuratowski-Ulam theorem \cite{33} Theorem 15.3, it follows that \( H_x \cap S_X \) is of first category in \( S_X \). (The use of this theorem requires the topology of \( S_X \) to have a countable base, which is the case since \( X \) is separable.) Equivalently, \( S_X \setminus H_x \) is residual in \( S_X \).

Now, if \( h \in S_X \setminus H_x \), then \( [x, x + th] \subset X \setminus F_m \) for some \( t_h > 0 \) by Lemma \ref{9}. Thus, if \( h \in S_X \setminus (H_x \cup -H_x) \) (also residual in \( S_X \)), there is \( t_h > 0 \) such that \((x - th, x + th) \subset X \setminus F_m \). Since \( x \in F_m^0 \), it is not restrictive to assume, after shrinking \( t_0 \) if necessary, that \((x - th, x + th) \subset F_m^0 \setminus F_m \subset f^{-1}(m)\). This makes it obvious that the derivative of \( f \) in the direction \( h \) exists and is 0.

Suppose now that \( X \) is reflexive. From the above and Theorem \ref{4.2} the points where \( f \) does not have an essential Gâteaux derivative can only be in \( F_m \), or in \( \partial F_m^0 \) (Aronszajn null since \( F_m^0 \neq 0 \)) or in a subset of \( X \setminus F_m \) contained in an Aronszajn null set. Thus, to show that this set is contained in a Haar null set, it suffices to check that \( F_m \) is contained in a Haar null set. This follows from \( F_m \subset \bigcup_{\alpha < m, \alpha \in \mathbb{Q}} \overline{F}_\alpha \) since \( \overline{F}_\alpha \) is convex with empty interior when \( \alpha < m \) (Theorem \ref{3.1}) and \( X \) is reflexive (Subsection \ref{2.2}).

Since the set of points of discontinuity of \( f \) is Haar null (Theorem \ref{5}) and the union of two Haar null sets is Haar null, the proof is complete. \( \square \)

There is a close connection between Theorem \ref{6.2} and Theorem \ref{9.2}: If \( H_x = \emptyset \) in the above proof, then \( f \) is Gâteaux differentiable at \( x \) (and \( Df(x) = 0 \)). On the other hand, by Lemma \ref{6.1}, \( H_x = \emptyset \) if and only if \( x \notin F_m^0 \).

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