Static spherically symmetric solutions of the Einstein–Kalb–Ramond (KR) field equations are obtained. Besides an earlier known exact solution, we also find an approximate, asymptotically flat solution for which the metric coefficients are obtained as an infinite series in $\frac{1}{r}$. Subsequently, we study gravitational lensing and perihelion precession in these spacetimes and obtain explicit formulae which include corrections to these effects in the presence of the KR field.

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I. INTRODUCTION

Gravitational theories in a curved background spacetime with torsion has been an area of investigation for a long time. Torsion, which appears as an antisymmetric tensorial piece in the connection, is arguably an inescapable consequence when the matter fields giving rise to spacetime curvature are possessed with spin. Therefore, the simplest level of torsion theory can provide a classical background for quantum matter fields. Starting from the original Einstein-Cartan (EC) theory, several papers have appeared which explore the various consequences of the torsion field, its impact on gravitational and cosmological solutions of the general relativistic field equations and on the nature of torsion couplings to arbitrary spin fields. In recent years, a lot of inspiration in this regard is provided by superstring theory. In fact, it has been shown that spacetime torsion can be identified with the strength of the massless antisymmetric second-rank tensor field, viz., the Kalb-Ramond (KR) field $B_{\mu\nu}$, appearing in the heterotic string spectrum. Thus, torsion is, in some sense, an inherent feature in the low-energy effective string action.

Extensive studies have already been carried out regarding the coupling of torsion with other spin fields, especially the electromagnetic field, where the well-known problem of violation of $U(1)$ gauge-invariance is explored. In the recent work of Majumdar and SenGupta, this problem is sorted out using the Chern-Simons (CS) extension which is augmented with the KR field strength $\partial_\mu B_{\nu\lambda}$ on account of anomaly-cancellation of the corresponding quantum theory. Such a formalism can have some important implications in explaining certain astrophysically/cosmologically observable phenomena such as cosmic optical activity, fermion helicity flip, etc. which may provide direct tests for the viability of string theory. However, a complete understanding of such phenomena do require appropriate knowledge of the various possible solutions of the gravitational field equations in spacetimes with torsion (or, equivalently the KR field). In particular, in this regard, we have explored earlier the possible existence of solutions for the static spherical symmetric gravitational field equations in vacuum as well as the cosmological solutions in presence of spacetime torsion. Identifying the KR field strength three-tensor as the Hodge-dual of the derivative of the pseudoscalar axion $H$ of string spectrum, it has been shown that spherical symmetric solutions can actually exist for a suitable form of the torsion (i.e., KR) field and can have interesting features like that of the spacetime
admitting a ‘wormhole’ or a ‘naked singularity’. However, apart from these exact solutions there may exist other solutions as well for spherical symmetric spacetimes with torsion.

In this paper, we try to carry out the most general study of the existence of possible static spherical symmetric solutions of the vacuum field equations, especially to check the uniqueness of those obtained in the earlier work [1]. Our aim is to establish a physically meaningful general solution which match the limiting requirements and provide a proper understanding about how the standard Schwarzschild solution gets modified in the presence of different forms of the torsion field. Implications thereof, are obtained through the study of geodesic motion in such spacetimes. We also investigate the corrections inflicted by torsion on some of the standard tests of general relativity theory. As has been done in the earlier paper [3], here also we seek solutions satisfying the boundary requirement of asymptotic flatness.

Although the full low energy effective action of string theory includes the graviton, dilaton, axion as well as other fields which may arise out of different type of compactifications, here we attempt to focus on the effects of the axion only. We investigate whether the presence of axion ( appearing as a dual field to the Kalb-Ramond induced torsion ) is perceptible at all through the various observational and solar system tests. This is in contrast with the volume of works on the dilaton-graviton system[13] and our work here aims to fill a gap in the literature.

II. STATIC SPHERICALLY SYMMETRIC SOLUTIONS IN A KALB-RAMOND BACKGROUND

Following the identification [4] of the totally skew-symmetric torsion tensor with the modified KR field strength $H_{\mu \nu \lambda}$, the action for gauge-invariant EC-KR coupling is given by

$$S = \int d^4x \sqrt{-g} \left[ R(g) - \frac{1}{12} H_{\mu \nu \lambda} H^{\mu \nu \lambda} \right]$$

(1)

which has exact correspondence with that in the low-energy effective string theory. $R(g)$ is the Ricci scalar curvature and $\kappa \sim (\text{Planck mass})^{-2}$ is the gravitational coupling constant. The modified KR field strength three-form $H$ is defined by the KR field strength plus the $U(1)$ electromagnetic CS three-form: $H = dB + \sqrt{\kappa} A \wedge F$. Due to the Planck mass suppression we neglect the CS term in the present analysis. The field equations that can be obtained from the above action are given as

$$R_{\mu \nu} - \frac{1}{2}g_{\mu \nu}R = \kappa T_{\mu \nu}$$

(2)

$$D_{\mu}H^{\mu \nu \lambda} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} H^{\mu \nu \lambda}) = 0$$

(3)

where $R_{\mu \nu}$ is the Ricci tensor of Riemannian geometry; and $T_{\mu \nu}$ is a symmetric 2-tensor, analogous to the energy-momentum tensor, and is given by

$$T_{\mu \nu} = \frac{1}{4} \left( 3g_{\nu \rho}H_{\alpha \beta \mu}H^{\alpha \beta \rho} - \frac{1}{2} g_{\mu \nu}H_{\alpha \beta \gamma}H^{\alpha \beta \gamma} \right).$$

(4)

Taking the line element in its most general spherically symmetric form

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(5)

and expressing the three-form $H_{\mu \nu \lambda}$ in terms of its Hodge-dual one-form — a pseudovector — with independent components $H_{012}, H_{013}, H_{023}$ and $H_{123}$; it has been shown in [11] that static spherical symmetric solutions, consistent with the basic requirement of asymptotic flatness, are possible only when $H_{023} \neq 0$ and all other components vanish. This corresponds to the situation that the dual pseudoscalar ‘axion’ $h$ defined through the relation

$$H_{\mu \nu \lambda} = \epsilon_{\mu \nu \lambda} \partial_\rho H.$$

(6)

depends on the radial coordinate $r$ only. Denoting $H_{023} = h_{023}$ by $[h(r)]^2$ the field equations can be expressed as

$$e^{-\lambda} \left( \frac{1}{r^2} - \frac{\nu'}{r} \right) - \frac{1}{r^2} = \tilde{h}^2$$

(7)

$$e^{-\lambda} \left( \frac{r^2}{2} + \frac{\nu'}{2} \right) - \frac{1}{r^2} = - \tilde{h}^2$$

(8)

$$e^{-\lambda} \left( \nu'' + \frac{\nu'^2}{2} - \frac{\nu' \nu''}{2} + \frac{\nu' - \nu''}{r} \right) = 2\tilde{h}^2$$

(9)

$$\partial_1 \left( r^2 e^{\frac{\nu'}{r}} H' \right) = \partial_1 \left( r^2 h \ e^{\nu/2} \right) = 0.$$  

(10)
where a prime stands for derivative with respect to \( r \); and the constant \( \bar{\kappa} = 3/4\kappa \). The last equation in the above set is obtained by using both Eq.(3) and the Bianchi identity for the KR field, viz., \( \epsilon^{\mu\nu\lambda\sigma} \partial_\sigma H_{\mu\nu\lambda} = 0 \). The above equations can readily be solved to obtain

\[
h(r) = H'(r) e^{-\lambda/2} = \frac{b_0}{r^2} e^{-\nu/2}
\]  

(11)

and

\[
e^{-\lambda} = 1 + \frac{c_1}{r^2} + \frac{\tau(r)}{r}
\]  

(12)

\[
e^\nu = \frac{c_2}{r(r + \tau(r) + c_1)} e^{\int^r \frac{2dr}{r + \tau(r) + c_1}}
\]  

(13)

where \( b_0, c_1 \) and \( c_2 \) are the constants of the integrations and

\[
\tau(r) = \bar{\kappa} \int^r r^2 h^2(r) dr.
\]

(14)

The above solutions are consistent only when they satisfy the asymptotic flatness requirement, viz., \( e^{\pm \nu}, e^{\pm \lambda} \to 1 \) as \( r \to \infty \), and a consistency condition derived from the field equations:

\[
\tau'' + \frac{\tau'}{r} = \frac{\tau'(\tau' - 1)}{r + c_1 + \tau}.
\]

(15)

The asymptotic flatness condition on the solutions demands that we must have \( c_2 = 1 \). This can readily be verified in the limit where torsion vanishes, i.e., \( \tau(r) = 0 \). For non-zero torsion, if we further put \( c_1 = 0 \), then as is shown in [11], a typical exact solution satisfying the above requirement (15) can be obtained for a specific form of \( \tau(r) \), viz., \( \tau(r) = -b/r; \ b = \bar{\kappa}b_0^2 = \) constant (i.e., \( h(r) \sim \frac{1}{r^2} \)), whence

\[
e^{-\lambda} = 1 - \frac{b}{r^2}
\]  

(16)

\[
e^\nu = 1.
\]

(17)

and we have a wormhole for a real KR field, i.e., a positive \( b \). Note that this geometry has been discussed many times in the literature beginning with the work of Ellis [14], though its appearance in the context of the Kalb–Ramond field coupled to gravity had not been noticed till recently.

Now, to investigate whether the above solution is unique, or whether there exist other physically meaningful solution(s) consistent with the boundary requirements, we take a general functional form of \( \tau(r) \), which depends on the KR field strength \( h(r) \), in the following section. Moreover, throughout our subsequent analysis we take the KR field to be real.

Observing that \( \tau(r) \) does not involve any additive constant, we can generally express it in the form

\[
\tau(r) = \sum_{m=1}^{\infty} a_m r^m + \sum_{n=1}^{\infty} b_n r^n.
\]

(18)

But from Eqs.(11) and (14) we find

\[
\tau'(r) = \frac{b}{r^2} e^{-\nu}, \ b = \bar{\kappa}b_0^2.
\]

(19)

Since \( e^{-\nu} \equiv g^{00} \to 1 \) as \( r \to \infty \), Eqs.(18) and (19) are consistent only when all the \( a_m \)'s in \( \tau(r) \) vanish, i.e., \( \tau(r) = \sum_{n=1}^{\infty} b_n r^n \). Plugging this in Eq.(15) and matching the coefficients of equal powers of \( r \) from both sides, we obtain

\[
\tau(r) = b_1 \left[ \frac{1}{r} - \frac{c_1}{2r^2} + \frac{c_2}{3r^3} - \left( 1 - \frac{b_1}{6c_1^2} \right) \frac{c_3}{4r^4} \right.
\]

\[
+ \left( 1 - \frac{b_1}{2c_1^2} \right) \frac{c_4}{5r^5} \left. - \left( 1 - \frac{59b_1}{60c_1^2} + \frac{3b_1^2}{80c_1^3} \right) \frac{c_5}{6r^6} + \cdots \right] \]

(20)
Computing $\tau'(r)$ and comparing with Eq.(19) we find $b_1 = -b$ and the solutions can, in general, be expressed as

$$e^\nu \equiv g_{00}(r) = 1 + \frac{c_1}{r} + \frac{bc_1}{6r^3} - \frac{bc_1^2}{6r^4} + \frac{6bc_1^3}{40r^5} + 3b^2c_1 + \cdots \quad (21)$$

$$e^{-\lambda} \equiv -g^{11}(r) = 1 + \frac{c_1}{r} - \frac{b}{r^2} + \frac{bc_1}{2r^3} - \frac{bc_1^2}{3r^4} + \left(\frac{bc_1^3 + b^2c_1}{6}\right)\frac{1}{4r^5} + \cdots \quad (22)$$

and the solution for the KR field is given by

$$h(r) = \sqrt{\frac{b}{c}} \frac{1}{r^2} \left[1 - \frac{c_1}{r} + \frac{c_1^2}{r^2} - \left(\frac{c_1^3 + bc_1}{6}\right)\frac{1}{r^3} + \left(\frac{c_1^4 + bc_1^2}{2}\right)\frac{1}{r^4} + \cdots\right]. \quad (23)$$

The above solutions are, by construction, asymptotically flat and reproduce the exact solution found in [11] for $c_1 = 0$. For $c_1 \neq 0$, we obtain the standard Schwarzschild solution, viz., $e^\nu = e^{-\lambda} = 1 - r_s/r$ in the zero-torsion limit, i.e., $b = 0$, provided $-c_1 = r_s = 2GM$ (the Schwarzschild radius). In fact, whenever $c_1$ is non-vanishing, being a constant we can always identify it with $-r_s$, thereby obtaining the Schwarzschild solution in the limit $b \to 0$.

### III. GEODESICS, LENSING AND PERIHELION PRECESSION IN KR BACKGROUND

The equations of geodesics for the general static spherically symmetric metric is given by [15, 16]:

$$\dot{r}^2 = \left(\frac{dr}{d\tau}\right)^2 = e^{-\lambda(r)} \left[ e^{-\nu(r)} E^2 - \frac{J^2}{r^2} - L \right] \quad (24)$$

$$\dot{\phi} = \frac{d\phi}{d\tau} = \frac{J}{r^2} \quad (25)$$

$$\dot{t} = \frac{dt}{d\tau} = E e^{-\nu(r)} \quad (26)$$

where the motion is as usual considered to be taking place in the $\theta = \pi/2$ plane and the constants $E$ and $J$ having the respective interpretations of energy per unit mass and angular momentum about an axis perpendicular to the invariant plane ($\theta = \pi/2$). Here, $\tau$ is an affine parameter and $L$ is the Lagrangian having the values 0 and 1 respectively for null and time-like particles. We are not concerned about the space-like particles herein. Now, let us consider separately the following two cases:

#### A. Case : $c_1 = 0$

In this case, the metric coefficients $e^\nu$ and $e^\lambda$ given by Eqs.(16) and (17) yield the equations for the radial geodesics ($J = 0$) :

$$\left(\frac{dr}{dt}\right)^2 = (1 - L/E^2) \left(1 - \frac{b}{r^2}\right); \quad \frac{dt}{d\tau} = E. \quad (27)$$

with solution

$$t = \pm \sqrt{\frac{r^2 - b}{1 - L/E^2}} + \text{constant} \quad (28)$$

and the affine parameter $\tau \propto t$. The above equation represents a hyperbola and shows that to an external observer a radially infalling particle (time-like or null) approaches the radius $r = \sqrt{b}$ asymptotically but can never reach it. As $\tau$ is linear in $t$ we find that the $\tau - r$ relationship also represents a hyperbola. Now, for time-like geodesics ($L = 1$), $\tau$ is the proper time and hence an observer falling with a time-like particle also skirts the physical singularity at $r = 0$ by asymptotically grazing the critical radius at $r = \sqrt{b}$. This feature is in sharp contrast with what happens in a Schwarzschild spacetime and is the characteristic of a wormhole spacetime.

For the general motion of geodesics in the present case, we obtain from Eqs.(24) and (25) the equation of orbit

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{E^2 - L}{J^2} - u^2\right)(1 - bu^2). \quad (29)$$
where \( u = 1/r \). Now, in order to have bound orbits \((E^2 < 1)\) the equation \( du/d\phi = 0 \) must have at least two real, positive roots not coinciding with any physical or coordinate singularity. Such roots exclusively imply the two turning points of the closed orbit. In the present case, the real positive values of \( u \) for which \( du/d\phi \) vanishes are \( 1/\sqrt{b} \) and \( E/J \) for null geodesics \((L = 0)\). However, as the metric diverges at \( u \equiv 1/r = 1/\sqrt{b} \) we infer that the null geodesics cannot follow closed orbits in a wormhole spacetime. For time-like geodesics \((L = 1)\) in such a spacetime, there is no positive real value of \( u \) for which the metric is non-singular and \( du/d\phi = 0 \). Therefore, bound orbits are not permissible for time-like geodesics also, i.e., for all kinds of particles we can only have unbound orbits \((E^2 > 1)\).

A close inspection of Eq.(29) raised to the second order, viz.,

\[
\frac{d^2u}{d\phi^2} + \left[ 1 + \frac{b}{D^2} \right] u = 2bu^3; \quad D = \frac{J}{\sqrt{E^2 - L}}
\]  

(30)

shows that the KR field, represented by the parameter \( b \), inflicts a two-fold change in the \( u - \phi \) straight line \( u \sim \sin(\phi - \phi_\infty) \) that corresponds to the limiting Minkowski spacetime \((b = 0)\). The KR field not only alters the intercept on the \( \phi \)-axis but also produces a departure from the straight line motion due to the term \( bu^3 \) on the right of the above equation. Although the impact parameter \( D \) and hence the changes in the intercept are different for massive and massless particles, the amount of bending near the origin of force is same for both kinds of particles. At \( r = r_0 \), the distance of closest approach towards the origin of force \( du/d\phi = 0 \), whence Eq.(29) gives \( r_0 = D \). Replacing back \( u \) by \( 1/r \) and solving Eq.(29) we obtain

\[
\phi(r) - \phi_\infty = \sin^{-1}(r_0/r) + \frac{b}{4r_0^2} \left[ \sin^{-1}(r_0/r) - (r_0/r)\sqrt{1 - (r_0/r)^2} \right] + O \left( \frac{b}{r_0^2} \right)^2.
\]

(31)

The angle of bending for all types of particles is given by

\[
\Delta\phi = 2 | \phi(r_0) - \phi_\infty | - \pi = \frac{b}{r_0^2} \frac{\pi}{4} + O \left( \frac{b^2}{r_0^4} \right).
\]

(32)

Now, instead of finding an expression for \( \Delta\phi \) as an expansion in powers of \( \frac{b}{r_0^2} \equiv x \)(say), one can also perform an exact integration and obtain the following expression for the amount of bending:

\[
\Delta\phi = 2 \left( K \left[ \frac{b}{r_0^2} \right] \right) - \pi
\]

(33)

where \( K[x] \) is the complete elliptic integral of the first kind. A plot of \( \Delta\phi \) as a function of \( x \) shows a linear region for small values of \( b << r_0^2 \) with a slope reasonably close to \( \pi/4 \) – a fact which is demonstrated in the approximate calculation of the bending angle discussed above.

\begin{center}
\bf{Deflection}
\end{center}

![Graph](image-url)
Let us now turn towards the radial equation for $u(\phi)$ quoted above. In order to obtain information about the trajectories of photons in the geometrical optics limit we can, alternatively, solve for $u(\phi)$, in terms of elliptic functions by directly integrating the equation given below.

\[
\left(\frac{du}{d\phi}\right)^2 = b \left(\frac{1}{b} - u^2\right) \left(\frac{1}{r_0^2} - u^2\right)
\] (34)

For $r_0^2 = b$ the integration is straightforward and yields :

\[r(\phi) = \frac{1}{u(\phi)} = \sqrt{b} \coth(\phi - \phi_0)\] (35)

In terms of the proper radial distance \(l = \pm \sqrt{r^2 - b}\) we have:

\[l = \pm \sqrt{b} \text{cosech}(\phi - \phi_0)\] (36)

The general solution for $u(\phi)$ is given as :

\[(\phi - \phi_0) = \frac{\sqrt{b}}{r_0} F\left[\arcsin r_0 u, \frac{b}{r_0^2}\right]\] (37)

where $F$ denotes the incomplete elliptic integral of the first kind. On inverting we obtain :

\[u(\phi) = \frac{1}{r_0} \text{sn} \left[\frac{r_0}{\sqrt{b}} (\phi - \phi_0), \frac{b}{r_0^2}\right]\] (38)

where $\text{sn}$ denotes the Jacobian elliptic function. A plot of $r_0 u$ versus $\phi$ with $\phi_0 = 0$ and $\frac{r_0}{\sqrt{b}} = 3$ is shown below.

For $r_0 = \sqrt{b}$ it can be seen that the $\text{sn}$ reduces to the hyperbolic tangent which yields the expression discussed above for this case. As $r_0 \rightarrow \sqrt{b}$ the $\Delta \phi$ shoots up rapidly. The trajectory exhibits multiple winding in the vicinity of the throat. Why does this happen? Does the negative energy density present in the vicinity of the throat result in such peculiar behaviour? It is also clear that for other values of the ratio $x = b/r_0^2$ ($x < 1$) we find that the negative energy source (here the Kalb–Ramond field) deflects the light ray by an amount which is crucially dependent on the KR parameter $\sqrt{b}$. The explicit behaviour of the null geodesics including the case of multiple windings for $r_0 = \sqrt{b}$ is also exhibited in the functional forms for the trajectories derived above.
B. Case : \(c_1 \neq 0\)

Here we have the complete series solution given by Eqs.(21) and (22) with \(c_1\) identified as \(-r_s\), the Schwarzschild radius. Since both \(e^\nu\) and \(e^\lambda\) are convergent in the region \(r >> r_s\), we study the motion of geodesics in such a region where for simplicity, we consider the torsion to be small, i.e., \(b/r^2 << 1\). This implies that we are assuming that the small torsion (or, equivalently the KR field) does not completely change the nature of the trajectory of particles as in the case \(c_1 = 0\). It rather inflicts a correction over the general relativistic phenomena like bending of light and the perihelic precession of planetary orbits. Dropping terms of order cubic or more in \(r_s/r\) and \(b/r^2\) we can approximately write the solutions (21) and (22) as

\[
\begin{align*}
e^\nu &= 1 - \frac{r_s}{r} \\
e^{-\lambda} &= 1 - \frac{r_s}{r} - \frac{b}{r^2}.
\end{align*}
\]

Indeed, an analysis similar to that in the case \(c_1 = 0\) shows that for time-like particles closed orbits are really possible for the above truncated series form of the metric coefficients.

I. Bending of light rays (lensing)

For photons the trajectory equations (24) and (25) yield

\[
\left(\frac{dr}{d\phi}\right)^2 = r^4 e^{-\lambda(r)} \left(\frac{\nu^{-\nu(r)}}{D^2} - \frac{1}{r^2}\right); \quad D = \frac{J}{E}
\]

with solution in the form of a quadrature:

\[
\phi(r) - \phi_\infty = \int_r^\infty \frac{e^{\lambda(r)/2} \, dr/r}{\sqrt{e^{-\nu(r)} \frac{1}{r^2} - 1}}
\]

At the distance of closest approach \((r_0)\) to the center of force, \(\frac{dr}{d\phi}\big|_{r=r_0} = 0\), whence Eq.(41) gives \(D^2 = r_0^2 e^{-\nu(r_0)}\). Plugging this in Eq.(42) and using the specific expressions for the metric components given in Eqs.(39) and (40), we obtain

\[
\begin{align*}
\phi(r) &= \phi_\infty + \sin^{-1}(r_0/r) + \frac{r_s}{2r_0} \left(2 - \sqrt{1 - (r_0/r)^2} - \sqrt{\frac{r-r_0}{r+r_0}}\right) \\
&\quad + \frac{3r_s^2}{8r_0} \left[\frac{1}{2} \sin^{-1}\left(\frac{r_0}{r}\right) - 2 \cos^{-1}\left(\frac{r_0}{r}\right) - \frac{r_0}{2r} \sqrt{1 - \left(\frac{r_0}{r}\right)^2} + \frac{3}{2} \sqrt{\frac{r-r_0}{r+r_0} - \frac{1}{6} \left(\frac{r-r_0}{r+r_0}\right)^2}\right] \\
&\quad + \frac{b}{4r_0^2} \left[\sin^{-1}(r_0/r) - (r_0/r) \sqrt{1 - (r_0/r)^2}\right] + \text{higher order terms.}
\end{align*}
\]

The angle of bending is given by

\[
\Delta \phi = 2 |\phi(r_0) - \phi_\infty| - \pi = \frac{2r_s}{r_0} + \frac{3\pi}{16} \left(\frac{r_s}{r_0}\right)^2 + \frac{b}{r_0^2} \frac{\pi}{4} + \text{higher order corrections.}
\]

Looking at the above expression we note that the first and third terms are relevant as long as we are interested in results valid up to first order in the mass \(M (r_s \sim GM)\) and the KR parameter \(b\). The first term is of course the usual Schwarzschild bending, whereas the third term comes from the KR field. Using these two terms we may rewrite the total bending as follows:

\[
\Delta \phi = (\Delta \phi)_{Schw} \left[1 + \frac{(\Delta \phi)_{KR}}{(\Delta \phi)_{Schw}}\right]
\]

(45)
Now the maximum KR energy density (note that this is negative) is obtained by using the minimum value of \( r \) (which is \( r_0 \), the impact parameter). This energy density can be written as:

\[
|\rho_{KR}^{\text{max}}| = \frac{c^4}{8\pi G r_0^3} \frac{b}{r_0^3} = \frac{8}{3\pi} \frac{M c^2}{V_0} \frac{(\Delta \phi)_{KR}}{(\Delta \phi)_{Schw}}
\]

where \( V_0 \) is the volume \( \frac{4}{3}\pi r_0^3 \).

We have calculated this energy density using the error bars for the current deflection of light measurements for Sun [17]. The amount of energy per unit volume is enormous resulting in an unacceptable ambient KR temperature much larger than that for CMBR. This suggests that \( (\Delta \phi)_{KR} \) has to be far less than the value of these error bars in order to give a reasonable KR field energy density and therefore remains undetectable within the present day experimental precision.

2. Precession of perihelion of planetary orbits

As has been mentioned earlier bound orbits are really possible for time-like particles when \( c_1 \neq 0 \). In the case of elliptic planetary orbits, we obtain from Eqs.(24) and (25)

\[
\left( \frac{dr}{d\phi} \right)^2 = r^4 e^{-\lambda(r)} \left( e^{-\nu(r)} E^2 - 1 - \frac{1}{r^2} \right).
\]

At perihelia and aphelion, \( r = r_\pm \) and \( \frac{dr}{d\phi} \bigg|_{r=r_\pm} = 0 \), whence the above equation yields

\[
\frac{1}{r_\pm^2} - \frac{e^{-\nu(r_\pm)} E^2}{J^2} = - \frac{1}{J^2}
\]

Solving for \( E \) and \( J \) we obtain the trajectory equation in the form of the quadrature [14]:

\[
\phi(r) - \phi(r_-) = \int_{r_-}^{r} \frac{e^{\lambda(r)/2} Y^{-1/2}(r)}{r^2} \frac{dr}{r^2}
\]

where

\[
Y(r) = \frac{e^{\nu(r_+)} r_+^2 [e^{\nu(r)} - e^{\nu(r_-)}] - e^{\nu(r_-)} r_-^2 [e^{\nu(r)} - e^{\nu(r_+)}] + \frac{1}{r^2}}{r_+^2 r_-^2 e^{\nu(r_+)} [e^{\nu(r_+)} - e^{\nu(r_-)}] - e^{\nu(r_-)} r_-^2 [e^{\nu(r)} - e^{\nu(r_+)}] + \frac{1}{r^2}}
\]

Simplifying the above expressions and carrying out the integration, we obtain for the metric in Eqs.(39) and (40) the amount of rotation of the orbit per revolution:

\[
(\Delta \phi) = 2 |\phi(r_\pm) - \phi(r_-)| = 2 \pi \left[ \frac{3r_s}{l} + \frac{3r_s^2}{8l^2} (18 + e^2) + \frac{b}{l^2} \left( 1 + \frac{e^2}{2} \right) \right] \pi
\]

\[+ \text{ higher order corrections}.\]

Here \( L \) and \( e \) are respectively the semilatus rectum and eccentricity of the elliptic orbit and are defined by \( 2/l = (1/r_+ + 1/r_-) \), \( r_\pm = (1 \pm e)a \), \( a \) being the semimajor axis.

An analysis similar to the preceding section shows that the maximum KR energy density that can be obtained by using the minimum value of \( r \) (which in this case is the perihelion distance \( r_- \)) is given by

\[
|\rho_{KR}^{\text{max}}| = \frac{c^4}{8\pi G r_0^3} \frac{b}{r_0^3} \sim \frac{M c^2}{V} \frac{(\Delta \phi)_{KR}}{(\Delta \phi)_{Schw}}
\]

where we have considered the eccentricity \( e \) to be fairly small so that \( r_- \sim l \), the semilatus rectum, and \( V \) is the volume \( \frac{4}{3}\pi l^3 \) of the two-body system. Using the standard observational data for the perihelion precessions of Mercury, Earth and Icarus [17] we have calculated \( |\rho_{KR}^{\text{max}}| \) with \( (\Delta \phi)_{KR} \) of the order of error bars. Once again the energy density is found to be extremely high, thus suggesting that \( (\Delta \phi)_{KR} \) should be much less than the present day experimental error bars.
IV. CONCLUSIONS

In this paper we have extended the previous work by to two of us (SSG and SS) on spherically symmetric static solutions of the Kalb–Ramond field coupled to gravity. We have arrived at an approximate asymptotically flat solution to the field equations. The ‘approximation’ is necessitated by the fact that, as far as we could see, the resulting equations are not exactly solvable. This approximate solution has the nice feature that it has a Schwarzschild piece and another term related to the KR field. In a sense, therefore our approximate solution represents the exterior gravitational field of a massive spherically symmetric body in the presence of a Kalb–Ramond field. In order to arrive at observable signatures, or measure the extent of the KR field we have subsequently studied gravitational lensing and perihelion precession. For the exact background (Ellis geometry) we have been able to find exact expressions for the amount of bending. For the approximate solutions the amount of bending and perihelion precession is approximate but the formulae look reasonable enough to justify our approximation. We also make a suggestion about how the bending/perihelion precession due to the KR field may be tested in observations. Our findings seem to imply that if the amount of bending/perihelion precession has to lie within the error bars of the observational results for the Sun, we would end up with an abnormally large KR energy density and, subsequently a huge ambient background temperature (a cosmic KR background). This being absurd we conclude that the bending/perihelion precession amount is far below the known error bars and, therefore, cannot be detected by the present level of precision in the solar system tests employed to test GR. Therefore, alternative tests (besides those that are related to cosmology) need to be thought out in order to prove the existence of the KR field. Such possibilities have already been explored in the context of optical activity of the electromagnetic radiation from distant galactic sources, helicity flip of solar neutrino and in other areas. Future work along these lines is in progress and will be communicated in due course.

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