On the connection of the generalized nonlinear sigma model with constrained stochastic dynamics

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The dynamics of a freely jointed chain in the continuous limit is described by a field theory which closely resembles the nonlinear sigma model. The generating functional $\Psi[J]$ of this field theory contains nonholonomic constraints, which are imposed by inserting in the path integral expressing $\Psi[J]$ a suitable product of delta functions. The same procedure is commonly applied in statistical mechanics in order to enforce topological conditions on a system of linked polymers. The disadvantage of this method is that the contact with the stochastic process governing the diffusion of the chain is apparently lost. The main goal of this work is to reestablish this contact. To this purpose, it is shown here that the generating functional $\Psi[J]$ coincides with the generating functional of the correlation functions of the solutions of a constrained Langevin equation. In the discrete case, this Langevin equation describes as expected the Brownian motion of beads connected together by links of fixed length.

INTRODUCTION

The subject of this work is a chain obtained by performing the continuous limit of a system of $N-1$ links of fixed length $a$ and $N$ beads of constant mass $m$. In this limit the number $N$ of beads approaches infinity, the length of the links and the mass of the beads go to zero, while the total length $L$ of the chain remains finite. The dynamics of a chain with rigid constraints of this kind has been studied in a remarkable series of papers \cite{1,2,3} using an approach based on the Langevin equation. Later on, mainly the statistical mechanics of such chains has been investigated, see e. g. \cite{4,5,6}. Dynamical models are however interesting by themselves and have also some applications, for instance in modeling the response of a chain to mechanical stresses in micromanipulations \cite{7}.

In Ref. \cite{8} the dynamics of the constrained chain has been considered using path integral
methods. The resulting model is a generalization of the nonlinear sigma model \cite{9} which will be called here the generalized nonlinear sigma model or simply GNLσM. The most striking difference from the standard nonlinear sigma model is that in the GNLσM the constraint is nonholonomic. The relation of the GNLσM with the Rouse model \cite{10,11} has been discussed in Ref. \cite{8}. It has also been shown that it gives the correct equilibrium limit in agreement with Ref. \cite{1}. Applications of the GNLσM have been developed in Refs. \cite{13,14}, computing for instance the dynamic form factor of the chain in the semiclassical approximation and the probability distribution $Z(r_{12})$ which measures the probability that in a given interval of time the average distance between two points of the chain is $r_{12}$.

One point that still needs to be clarified is if the GNLσM can be related to some stochastic process. In fact, the GNLσM has not been derived starting from a Langevin equation and applying for instance the Martin–Siggia–Rose formalism \cite{12} in order to pass the path integral formulation. The problem is that this approach becomes cumbersome if one has to deal with constraints. For this reason, in \cite{8} the constraints have been added to the path integral describing the fluctuations of the beads with the help of an insertion of Dirac delta functions. This is a widely exploited procedure in the statistical mechanics of polymers in order to impose topological conditions \cite{15,16,17,18}.

To establish a relation between the GNLσM and a stochastic process is the main goal of the present work. To this purpose, after a brief introduction to the GNLσM, we define in the next Section a two dimensional vector field $\varphi_\nu$ which satisfies a free Langevin equation and additional nonholonomic constraints. These are exactly the same constraints which appear also in the GLNσM. Our treatment is limited to two dimensions for simplicity. The generating functional $\tilde{\Psi}[J]$ of the correlation functions of the fields $\varphi_\nu$ can be constructed using the prescription of \cite{19}. The discretized version of $\tilde{\Psi}[J]$ describes the Brownian motion of a set of $N$–beads with diffusion constant $D$ which are connected together by links of fixed length. The difference between $\tilde{\Psi}[J]$ and the generating functional $\Psi[J]$ of the correlation functions of the GNLσM consists in a functional determinant. We show that this determinant is trivial by eliminating the constraints using a special set of variables, called here pseudo–polar coordinates. As a result we prove the equivalence of $\tilde{\Psi}[J]$ and $\Psi[J]$ and thus the connection of the GNLσM with a stochastic process of diffusing particles.
THE GENERALIZED NONLINEAR SIGMA MODEL AND ITS RELATION TO THE LANGEVIN EQUATION

The starting point of this Section is the generating functional of the GNLSM presented in Ref. [8]:

$$
\Psi[J] = \int \mathcal{D}R(t, s) e^{-c \int_0^{t_f} dt \int_0^N ds \mathbb{R}^2(t, s) \delta(\|\mathbb{R}'(t, s)\|^2 - 1) e^{-\int_0^{t_f} dt \int_0^L ds J(t, s) \mathbb{R}(t, s)}}
$$

(1)

with $\dot{\mathbb{R}} = \partial \mathbb{R}/\partial t$ and $\mathbb{R}' = \partial \mathbb{R}/\partial s$. The boundary conditions at $t = 0$ and $t = t_f$ of the field $\mathbb{R}(t, s)$ are respectively given by $\mathbb{R}(0, s) = \mathbb{R}_0(s)$ and $\mathbb{R}(t_f, s) = \mathbb{R}_f(s)$, where $\mathbb{R}_0(s)$ and $\mathbb{R}_f(s)$ represent given static conformations of the chain. For a ring–shaped chain the boundary conditions with respect to $s$ are periodic: $\mathbb{R}(t, s) = \mathbb{R}(t, s + L)$. An open chain with fixed ends may be described using instead the boundary conditions: $\mathbb{R}(t, 0) = r_1$, $\mathbb{R}(t, L) = r_2$, $r_1$ and $r_2$ being the locations of the fixed ends. It was shown in Refs. [8] and [13] that the above generating functional describes the dynamics of a closed chain that is the continuous version of a freely jointed chain consisting of links and beads. The constant $c$ appearing in Eq. (1) is given by:

$$
c = \frac{M}{4k_B T \tau L}
$$

(2)

Here $k_B$ denotes the Boltzmann constant, $T$ is the temperature and $\tau$ is the relaxation time which characterizes the ratio of the decay of the drift velocity of the beads. $M$ and $L$ represent the total mass and the total length of the chain respectively. The starting point to derive $\Psi[J]$ is the path integral $\Psi_N$ describing the brownian motion of a set of $N$ particles. The rigid constraints, which take into account the fact that these particles form a chain and thus are connected together by $N - 1$ massless segments of fixed length $a$, are introduced in the path integral with the help of a suitable product of Dirac delta functions. The limit of $\Psi_N$ from the discrete system to the continuous chain in which $N \rightarrow +\infty$, $a \rightarrow 0$ and $Na = L$ can be performed rigorously. The result is the generating functional of Eq. (1). This procedure is different from the usual approach to the dynamics of a chain, which is based on a Langevin equation. In this Section we are going to show that the GNL$\sigma$M can be related to a Langevin equation too. For simplicity, we restrict ourselves to the two dimensional case.

Since the GNL$\sigma$M ignores all interactions, it is natural to suppose that it should be related to a Langevin equation with no external forces:

$$
\dot{\varphi}_\nu = \nu
$$

(3)
where $\varphi_\nu$ is a two dimensional vector field and $\nu$ is a white noise source, whose components $\nu^{(i)}$, $i = 1, 2$ satisfy the basic correlation functions:

$$\langle \nu^{(i)}(t, s) \rangle = 0$$

$$\langle \nu^{(i)}(t, s) \nu^{(j)}(t', s') \rangle = \frac{\delta_{ij}}{c} \delta(t - t')\delta(s - s') \quad i, j = 1, 2$$

One may also expect that, together with Eq. (3), the field $\varphi_\nu$ must also satisfy the constraint:

$$\varphi_\nu'^2 = 1$$

The generating functional $\tilde{\Psi}[J]$ of the correlation functions of the field $\varphi_\nu$ is then given by [19]:

$$\tilde{\Psi}[J] = \int_{\varphi_\nu'^2 = 1} D\nu e^{-c \int^t_0 \int ds' ds \nu^2} \exp \left[ J \cdot \varphi_\nu \right]$$

The meaning of the statistical sum in the right hand side of the above equation becomes clear if we rewrite it as follows:

$$\tilde{\Psi}[J] = \int D\nu \int_{\mathbb{R}^2 = 1} D R e^{-c \int^t_0 \int ds' ds \nu^2} \delta(R - \varphi_\nu) e^{-\int^t_0 \int ds' ds J \cdot R}$$

The path integration over $\nu$ is now unconstrained, while that over the new field $R$ is limited to the configurations which are of the form:

$$R(t, s) = \int_0^s du (\cos \phi(t, u), \sin \phi(t, u)) + R_0(t)$$

where $R_0(t)$ is independent of $s$. The only left degree of freedom is the angle $\phi(t, s)$.

The generating functional $\Psi[J]$ of Eq. (1) differs from $\tilde{\Psi}[J]$ due to the presence of the functional Dirac delta function $\delta(R'^2 - 1)$. As a matter of fact, it is easy to show that:

$$\Psi[J] = \int D\nu \int D R e^{-c \int^t_0 \int ds' ds \nu^2} \delta(R'^2 - 1) \delta(R - \nu) e^{-\int^t_0 \int ds' ds J \cdot R}$$

The connection with the Langevin equation (3) is made by noticing that, for any solution $\varphi_\nu$ of that equation it is possible to write the formula:

$$\delta(\dot{R} - \nu) = \det^{-1} \partial_t \delta(R - \varphi_\nu)$$

Applying Eq. (11) to Eq. (10) we obtain, up to an irrelevant constant:

$$\Psi[J] = \int D R D\nu e^{-c \int^t_0 \int ds' ds \nu^2} \delta(R - \varphi_\nu) \delta(R'^2 - 1) \exp \left[ J \cdot \varphi_\nu \right]$$
As already announced, this expression of the generating functional \( \Psi[J] \) differs from \( \tilde{\Psi}[J] \) only by the fact that the condition \( R^2 = 1 \) is imposed with the help of the delta function \( \delta(R^2 - 1) \). In the next Sections the degrees of freedom which are frozen by the condition \( R^2 = 1 \) will be projected out from the path integral (12) and it will be shown that what remains is exactly the generating functional \( \tilde{\Psi}[J] \) related to the constrained stochastic process of Eqs. (3) and (6).

**THE DISCRETE GENERATING FUNCTIONAL IN PSEUDO–POLAR COORDINATES**

As a first step to show the equivalence of the generating functionals \( \Psi[J] \) and \( \tilde{\Psi}[J] \) we replace the continuous variables \( s \) and \( t \) with discrete variables \( s_m \) and \( t_n \), with \( 0 \leq m \leq M \) and \( 0 \leq n \leq N \). The spacings in the discrete \( s \) and \( t \)–lines are respectively given by:

\[
s_m - s_{m-1} = a \quad m = 2, \ldots, M
\]

\[
t_n - t_{n-1} = b \quad n = 2, \ldots, N
\]

where \( a \) and \( b \) are supposed to be very small. The continuous limit is recovered in the limit \( M, N \to +\infty, a, b \to 0 \) and \( Ma = L, Nb = t_f \). To simplify formulas, it will be used in the following the shorthand notation:

\[
R(t_n, s_m) \equiv R_{nm} \quad \nu(t_n, s_m) \equiv \nu_{nm} \quad \varphi_\nu(t_n, s_m) \equiv \varphi_{\nu, nm}
\]

In this way the discrete version of the constraint \( R^2(t,s) = 1 \) is replaced by the set of conditions:

\[
\frac{(R_{nm} - R_{n(m-1)})}{a^2} = 1 \quad n = 1, \ldots, N
\]

\[
m = 2, \ldots, M
\]

With the above settings the generating functional \( \Psi[J] \) of Eq. (12) may be rewritten as follows [20]:

\[
\Psi[J] = \lim_{N \to \infty} \lim_{M \to \infty} \int_{-\infty}^{+\infty} \left[ \prod_{n,m} d\nu_{nm} dR_{nm} \right] \exp \left\{ -abc \sum_{n,m} \nu_{nm}^2 \right\}
\]

\[
\times \prod_{n,m} \delta(R_{nm} - \varphi_{\nu, nm}) \exp \left\{ ab \sum_{n,m} J_{nm} R_{nm} \right\}
\]

\[
\times \prod_{n} \prod_{m=2}^{M} \frac{2}{a} \delta \left( \frac{|R_{nm} - R_{n(m-1)}|}{a^2} - 1 \right)
\]

(17)
Let us also note in the last line of equation (17) the normalization factor \( \prod_n \prod_{m=2}^{M} \frac{2}{a} \) in the definition of the delta function imposing the constraints. The reason of this normalization will be clear later. Without the constraints, the above equation would describe a discrete chain of \( N - 1 \) segments of length \( a \) and \( N \) beads of mass \( m \) respectively which perform a Brownian motion. The diffusion constant \( D \) is recovered from the parameter \( c \) of Eq. (2) as follows. First of all, we note that \( ca = \frac{1}{4k_B T} M a \). The ratio \( \frac{M}{L} a = m \) is nothing but the linear density of mass along the chain, so that \( \frac{M}{L} a \) is equal to the mass \( m \) of a single bead, i.e.: \( \frac{M}{L} a = m \). As a consequence, \( ca = \frac{m}{4k_B T} \). At this point we remember that the mobility of a particle \( \mu \) may be expressed in terms of the particle mass \( m \) and of the relaxation time \( \tau \) as follows: \( \frac{m}{\tau} = \frac{1}{\mu} \). Thus, \( ca = \frac{1}{4k_B T \mu} \). Due to the fact that \( D = k_B T \mu \), it is possible to write \( ca = \frac{1}{4D} \).

To eliminate the constraints (16), we pass to a new set of coordinates, which in the following will be called pseudo–polar:

\[
R_{nm} = \sum_{m'=1}^{M} l_{nm'} (\cos \phi_{nm'}, \sin \phi_{nm'}) \tag{18}
\]

The ranges of variation of the variables \( l_{mn} \) and \( \phi_{nm} \) are respectively given by:

\[
0 \leq l_{mn} < +\infty \quad 0 \leq \phi_{nm} \leq 2\pi \tag{19}
\]

The coordinate \( l_{mn} \) for \( n = 1, \ldots, N \) and \( m = 2, \ldots, M \), describes the length of the \( m \)-th segment at the instant \( t_n \). The coordinate \( l_{n1} \) is very special, because it gives the position with respect to the origin of the reference system of the first bead in the chain at the time \( t_n \). Finally, the angles \( \phi_{nm} \) tell us how the \( N - 1 \) segments are reciprocally oriented. After the transformation (18), the vector \( R_{nm} \) depends on the variables \( l_{mn} \) and \( \phi_{nm} \), i.e.:

\[
R_{nm} = R_{nm}(\{l_{nm}\}, l_{n1}; \{\phi_{nm}\}) \tag{20}
\]

where \( \{l_{nm}\} \) is the set of all \( l_{nm} \)'s for which \( m \neq 2 \) and \( \{\phi_{nm}\} \) is the set of all \( \phi_{nm} \)'s. Analogously, we denote with \( \{R_{nm}\} \) the set of all \( R_{nm} \)'s for \( m = 1, \ldots, M \) and \( n = 1, \ldots, N \). We are now able to explain the reason of the normalization factor \( \prod_n \prod_{m=2}^{M} \frac{2}{a} \) in Eq. (17). In the pseudo-polar variables the constraints (16) become: \( \frac{\mu}{a^2} = 1 \). The factor \( \frac{2}{a} \) is necessary in order to normalize the delta functions imposing these constraints. As a matter of fact, it is possible to check that \( \frac{2}{a} \int_0^{+\infty} dl_{nm} \delta\left(\frac{\mu}{a^2} - 1\right) = 1 \).
In order to perform the transformations (18) in the expression of the generating functional $\Psi[J]$ of Eq. (17), we need to compute the associated Jacobian determinant. In the rest of this Section we will prove for a general functional $f(\{R_{nm}\})$ the following formula:

$$\int \prod dR_{nm}f(\{R_{nm}\}) = \int_0^\infty \prod d\nu \int_0^{2\pi} \prod d\phi f(\{R_{nm}(\nu, \phi)\})J_{NM}$$

(21)

where the Jacobian $J_{NM}$ of the transformation (18) is given by:

$$J_{NM}(\{l_{nm}\}, l_n; \{\phi_{nm}\}) = \prod l_{nM}l_{n(M-1)} \cdots l_{n1}$$

(22)

Let’s show that $J_{NM}$ is really that given in Eq. (22). In order to proceed, it is convenient to introduce the components $x_{nm}^{(1)}$ and $x_{nm}^{(2)}$ of the vectors $R_{nm}$, i.e. $R_{nm} = (x_{nm}^{(1)}, x_{nm}^{(2)})$. Thus, Eq. (18) becomes:

$$x_{nm}^{(1)} = \sum_{m'=1}^m l_{nm'} \cos \phi_{nm'}$$
$$x_{nm}^{(2)} = \sum_{m'=1}^m l_{nm'} \sin \phi_{nm'}$$

(23)

and $J_{NM}$ may be written as follows:

$$J_{NM}(\{l_{nm}\}, l_n; \{\phi_{nm}\}) = \begin{vmatrix} \frac{\partial x_{nm}^{(1)}}{\partial l_{nm'}} & \frac{\partial x_{nm}^{(2)}}{\partial l_{nm'}} \\ \frac{\partial x_{nm}^{(1)}}{\partial \phi_{nm'}} & \frac{\partial x_{nm}^{(2)}}{\partial \phi_{nm'}} \end{vmatrix}$$

(24)

Strictly speaking, $J_{NM}$ is the determinant of a block matrix $A_{nm,n'm'}$ with composite indices $nm$ and $n'm'$. $A_{nm,n'm'}$ is composed by four $NM \times NM$ matrices, since $n,n' = 1, \ldots, N$ and $m,m' = 1, \ldots, M$. Due to the fact that $\frac{\partial x_{nm}^{(i)}}{\partial l_{nm'}} = \frac{\partial x_{nm}^{(i)}}{\partial \phi_{nm'}} = 0$ for $i = 1, 2$ if $n \neq n'$, $A_{nm,n'm'}$ is a block diagonal matrix. As a consequence, it is possible to write its determinant as follows:

$$J_{NM} = \prod_n J_{nM}$$

(25)

where

$$J_{nM} = \begin{vmatrix} \frac{\partial x_{nm}^{(1)}}{\partial l_{nm'}} & \frac{\partial x_{nm}^{(2)}}{\partial l_{nm'}} \\ \frac{\partial x_{nm}^{(1)}}{\partial \phi_{nm'}} & \frac{\partial x_{nm}^{(2)}}{\partial \phi_{nm'}} \end{vmatrix}$$

(26)

Using Eqs. (23), one finds after a few calculations that $J_{nM}$ is the determinant of the block matrix:

$$J_{nM} = \begin{vmatrix} A(n) & B(n) \\ C(n) & D(n) \end{vmatrix}$$

(27)
\(A(n), B(n), C(n), D(n)\) are lower triangular \(M \times M\) matrices with elements:

\[
\begin{align*}
A_{mm'}(n) &= \theta_{mm'} \cos \phi_{nm'} \\
B_{mm'}(n) &= \theta_{mm'} \sin \phi_{nm'} \\
C_{mm'}(n) &= -l_{nm'} \theta_{mm'} \sin \phi_{nm'} \\
D_{mm'}(n) &= l_{nm'} \theta_{mm'} \cos \phi_{nm'}
\end{align*}
\] (28)

Here the matrix \(\theta_{mm'}\) denotes the discrete equivalent of the Heaviside theta-function:

\[
\theta_{mm'} = 1 \quad \text{if} \quad m' \leq m \\
\theta_{mm'} = 0 \quad \text{if} \quad m' > m
\] (30)

If the matrices \(A(n), B(n), C(n), D(n)\) would commute, one could use a known theorem of linear algebra and write: \(J_{nM} = \det(A(n)D(n) - B(n)C(n))\). In our case these matrices do not commute, but it is still possible to compute the determinant \(J_{nM}\) by induction on \(M\).

If \(M = 1\) it is easy to show that:

\[J_{n1} = l_{n1}\] (32)

Next, we prove that

\[J_{nM} = l_{nM} J_{n(M-1)}\] (33)

To this purpose, it will be convenient to introduce new indices \(\alpha, \beta = 1, \ldots, M - 1\). At this point, we note that the \(M-\)th column of the \(2M \times 2M\) block matrix whose determinant we wish to compute in Eq. (27) has only two elements which are not zero. Thus, we expand \(J_{nM}\) with respect to the \(M-\)th column. Taking into account the necessary permutations and the fact that the two nonvanishing elements are \(A_{MM}(n) = \cos \phi_{nM}\) and \(C_{MM}(n) = -l_{nm} \sin \phi_{nM}\) we obtain:

\[
J_{nM} = \cos \phi_{nM} \det \begin{vmatrix} \theta_{\alpha\beta} \cos \phi_{n\beta} & \theta_{\alpha\beta} \sin \phi_{n\beta} & 0 \\ -l_{n\beta} \theta_{\alpha\beta} \sin \phi_{n\beta} & l_{n\beta} \theta_{\alpha\beta} \cos \phi_{n\beta} & 0 \\ -l_{n\beta} \theta_{M\beta} \sin \phi_{n\beta} & l_{n\beta} \theta_{M\beta} \cos \phi_{n\beta} & l_{nM} \cos \phi_{nM} \end{vmatrix} + (-1)^M l_{nM} \sin \phi_{nM} \det \begin{vmatrix} \theta_{\alpha\beta} \cos \phi_{n\beta} & \theta_{\alpha\beta} \sin \phi_{n\beta} & 0 \\ \theta_{M\beta} \cos \phi_{n\beta} & \theta_{M\beta} \sin \phi_{n\beta} & \sin \phi_{nM} \\ -l_{n\beta} \theta_{\alpha\beta} \sin \phi_{n\beta} & l_{n\beta} \theta_{\alpha\beta} \cos \phi_{n\beta} & 0 \end{vmatrix}
\] (34)

The determinants of the remaining two \((2M - 1) \times (2M - 1)\) matrices may be expanded according to the \((2M - 1)-\)th column, because these columns contain only one nonvanishing
After simple calculations one finds:

\[ J_{nM} = l_{nM} \det \begin{vmatrix} \theta_{\alpha\beta} \cos \phi_{n\beta} & \theta_{\alpha\beta} \sin \phi_{n\beta} \\ -l_{n\beta} \theta_{\alpha\beta} \sin \phi_{n\beta} & l_{n\beta} \theta_{\alpha\beta} \cos \phi_{n\beta} \end{vmatrix} \]  

(35)

which is exactly Eq. (33) because

\[ J_{n(M-1)} = \det \begin{vmatrix} \theta_{\alpha\beta} \cos \phi_{n\beta} & \theta_{\alpha\beta} \sin \phi_{n\beta} \\ -l_{n\beta} \theta_{\alpha\beta} \sin \phi_{n\beta} & l_{n\beta} \theta_{\alpha\beta} \cos \phi_{n\beta} \end{vmatrix} \]  

(36)

Using Eqs. (32) and (33) it is easy to show by induction that \( J_{nM} = l_{nM} l_{n(M-1)} \cdots l_{n1} \). With a straightforward application of Eq. (25) it is now possible to prove Eq. (22).

**RECOVERING THE GENERATING FUNCTIONAL \( \tilde{\Psi}[J] \) OF THE CONSTRAINED STOCHASTIC PROCESS OF EQS. (3)–(6)**

Let’s now go back to the generating functional \( \Psi[J] \) of Eq. (17). After the change of variables (18), the delta functions imposing the constraints simplify as follows:

\[
\delta \left( \frac{R_{nm} - R_{n(m-1)}}{a} - 1 \right) = \delta \left( \frac{L_{nm}}{a} - 1 \right).
\]

Further simplifications are obtained after applying the two delta function identities

\[
\delta \left( \frac{L_{nm}}{a} - a \right) = a^2 \delta (L_{nm} - a^2) \quad \text{and} \quad \delta (L_{nm} - a^2) = \frac{1}{2a} [\delta (l_{nm} - a) + \delta (l_{nm} + a)].
\]

Remembering that in our case \( l_{nm} \geq 0 \), it is possible to put:

\[
\delta (l_{nm} - a^2) = \frac{1}{2a} \delta (l_{nm} - a).
\]

As a consequence, the expression of the generating functional \( \Psi[J] \) in pseudo–polar coordinates becomes:

\[
\Psi[J] = \lim_{N \to \infty} \lim_{M \to \infty} \int_{-\infty}^{+\infty} \prod_{n,m} d\nu_{nm} \int_{0}^{+\infty} \prod_{n,m} dl_{nm} \int_{0}^{2\pi} \prod_{n,m} d\phi_{nm} \exp \left\{ -abc \sum_{n,m} \nu_{nm}^2 \right\} \\
\times \prod_{n,m} \delta (R_{nm}(\{l_{nm}\}, l_{n1}; \{\phi_{nm}\}) - \varphi_{\nu, nm}) \exp \left\{ ab \sum_{n,m} J_{nm} \cdot R_{nm}(\{l_{nm}\}, l_{n1}; \{\phi_{nm}\}) \right\} \\
\times \prod_{n} l_{n1} \left[ \prod_{n} \prod_{m=2}^{M} \delta (l_{nm} - a) \right] \prod_{m=2}^{M} l_{nM} \cdots l_{n2}
\]

(37)

In writing the above equation we have separated from the Jacobian determinant \( J_{N,M} \) the contribution coming from the \( \nu_{nm} \)'s, because these quantities denote the positions with respect to the origin of the first bead at different times \( t_n \)'s and are thus not fixed by the constraints. The integration in Eq. (37) over the \( l_{nm} \)'s, for \( n = 1, \ldots, N \) and \( m = 2, \ldots, M \), produces as
a result:

\[
\Psi[J] = \lim_{N \to \infty} \lim_{M \to \infty} a^{N(M-1)} \int_{-\infty}^{+\infty} \prod_{n,m} d\nu_{nm} \int_{0}^{+\infty} \prod_{n} dl_{n1} \int_{0}^{2\pi} \prod_{n,m} d\phi_{nm} \exp \left\{ -abc \sum_{n,m} \nu_{nm}^2 \right\} \\
\times \prod_{n,m} \delta(R_{nm}) \{a\}, l_{n1}; \{\phi_{nm}\} - \varphi_{nm} \exp \left\{ ab \sum_{n,m} J_{nm} \cdot R_{nm} \{a\}, l_{n1}; \{\phi_{nm}\} \right\}
\]

(38)

Here the symbol \(\{a\}\) denotes the set of all \(l_{nm}\)'s for \(m \neq 2\) after the imposition of the constraints \(l_{nm} = a\). We can now rewrite Eq. (38) as an integral over a restricted domain \(D\):

\[
\Psi[J] = \lim_{N \to \infty} \lim_{M \to \infty} \int_{\mathcal{D}} \prod_{n,m} dl_{nm} d\phi_{nm} \int_{-\infty}^{+\infty} \prod_{n,m} d\nu_{nm} \exp \left\{ -abc \sum_{n,m} \nu_{nm}^2 \right\} \prod_{n} l_{nM} \cdot l_{n1} \\
\times \prod_{n,m} \delta(R_{nm}) \{l_{nm}\}, l_{n1}; \{\phi_{nm}\} - \varphi_{nm} \exp \left\{ ab \sum_{n,m} J_{nm} \cdot R_{nm} \{l_{nm}\}, l_{n1}; \{\phi_{nm}\} \right\}
\]

(39)

where \(D\) is the domain of all \(l_{nm}\)'s and \(\phi_{nm}\)'s with the constraints \(l_{nm} = a\) for \(m = 2, \ldots, M\) and \(n = 1, \ldots, N\):

\[
D = \left\{ \{l_{nm}\}, \{\phi_{nm}\} \left| \begin{array}{l}
\text{l}_{nm} = a \\
0 \leq l_{n1} \leq +\infty \\
0 \leq \phi_{nm} \leq 2\pi \\
\end{array} \right. \right\} (40)
\]

At this point, using Eqs. (21) and (22) we go back to cartesian coordinates:

\[
\Psi[J] = \lim_{N \to \infty} \lim_{M \to \infty} \int_{\mathcal{D}} \prod_{n,m} dR_{nm} \int_{-\infty}^{+\infty} \prod_{n,m} d\nu_{nm} \exp \left\{ -abc \sum_{n,m} \nu_{nm}^2 \right\} \\
\times \prod_{n,m} \delta(R_{nm} - \varphi_{nm}) \exp \left\{ ab \sum_{n,m} J_{nm} \cdot R_{nm} \right\}
\]

(41)

The domain \(D\) in cartesian coordinates is given by all \(R_{nm}\)'s in the two dimensional plane subjected to the constraints (16):

\[
D = \left\{ \{R_{nm}\} \left| \begin{array}{l}
R_{nm} \in \mathbb{R}^2 \\
\frac{|R_{nm} - R_{nm(n-1)}|}{a^2} = 1 \\
\end{array} \right. \right\} (42)
\]

Finally, we rewrite the path integral in Eq. (41) in its continuous form. The result is:

\[
\Psi[J] = \int_{\mathcal{D}R} \int_{\mathcal{D}V} e^{-c \int_{t_{0}}^{t_{1}} dt \int_{V} d\nu \cdot d\nu^2 \delta(R - \varphi_{\nu}) e^{i' \int_{t_{0}}^{t_{1}} dt \int_{V} d\nu \cdot J^* \cdot dJ \cdot R}} \]

(43)

The right hand side of the above equation coincides exactly with the right hand side of Eq. (8). This proves the equivalence between the generating functional \(\Psi[J]\) of the GNLSM and the generating functional \(\bar{\Psi}[J]\) of the stochastic process of Eqs. (3)–(6).
CONCLUSIONS

In this work it has been shown that the GNLσM is related to a stochastic process which, after discretization, describes the Brownian motion of $N$ beads subjected to the constraints (16). These constraints enforce the conditions that the links connecting the beads are of fixed length. More in details, it has been proved that the generating functional $\Psi[J]$ of the GNLσM coincides with the generating functional $\tilde{\Psi}[J]$ of the solutions of the Langevin equation (3) and of the constraint (6). The fact that the two functionals are equal was not a priori obvious, because they differ by the delta function $\delta(R'^2 - 1)$ which contains quadratic powers of the fields. If $\delta(g(R))$ is a delta function imposing the condition $g(R) = 0$, then in general the following identity is valid:

$$\int \mathcal{D}R f(R) \delta(g(R)) = \int_{g(R)=0} \mathcal{D}R f(R) \det^{-1} \left| \frac{\delta g(R)}{\delta R} \right|$$

If in our case the functional determinant appearing in the right hand side of Eq. (44) would be not trivial, then there would be no chance that (8) and (12) coincide. Luckily, it turns out that, after passing to the pseudo–polar coordinates (18), the delta function $\delta(R'^2 - 1)$ produces just a functional determinant which is a trivial constant.

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[20] Unless otherwise specified, from now on it will understood that the indices \( n \) and \( m \) in sums and products will take all possible values in their respective ranges, i. e. \( 1 \leq n \leq N \) and \( 1 \leq m \leq M \).