Inverse problems for Schrödinger equations with Yang-Mills potentials in domains with obstacles and the Aharonov-Bohm effect.

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Abstract

We study the inverse boundary value problems for the Schrödinger equations with Yang-Mills potentials in a bounded domain $\Omega_0 \subset \mathbb{R}^n$ containing finite number of smooth obstacles $\Omega_j$, $1 \leq j \leq r$. We prove that the Dirichlet-to-Neumann operator on $\partial \Omega_0$ determines the gauge equivalence class of the Yang-Mills potentials. We also prove that the metric tensor can be recovered up to a diffeomorphism that is identity on $\partial \Omega_0$.

1 Introduction.

Let $\Omega_0$ be a smooth bounded domain in $\mathbb{R}^n$, diffeomorphic to a ball, $n \geq 2$, containing $r$ smooth nonintersecting obstacles $\Omega_j$, $1 \leq j \leq r$. Consider the Schrödinger equation in $\Omega = \Omega_0 \setminus (\cup_{j=1}^r \Omega_j)$ with Yang-Mills potentials

$$\sum_{j=1}^n \left(-i \frac{\partial}{\partial x_j} I_m + A_j(x)\right)^2 u + V(x) u - k^2 u = 0$$

with the boundary conditions

$$u \big|_{\partial \Omega_j} = 0, \quad 1 \leq j \leq r,$$
where $A_j(x), V(x), u(x)$ are $m \times m$ matrices, $I_m$ is the identity matrix in $\mathbb{C}^m$. Let $G(\Omega)$ be the gauge group of all smooth nonsingular matrices in $\overline{\Omega}$. Potentials $A(x) = (A_1, \ldots, A_n), V$ and $A'(x) = (A'_1, \ldots, A'_n), V'(x)$ are called gauge equivalent if there exists $g(x) \in G(\Omega)$ such that

\begin{equation}
A'(x) = g^{-1}Ag - ig^{-1}(x)\frac{\partial g}{\partial x}, \quad V' = g^{-1}Vg.
\end{equation}

Let $\Lambda$ be the Dirichlet-to-Neumann (D-to-N) operator on $\partial\Omega_0$, i.e.

$$\Lambda f = \left(\frac{\partial u}{\partial \nu} + i(A \cdot \nu)u\right) |_{\partial\Omega_0},$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal to $\partial\Omega_0$ and $u(x)$ is the solution of (1.1), (1.2), (1.3). We assume that the Dirichlet problem (1.1), (1.2), (1.3) has a unique solution. We shall say that the D-to-N operators $\Lambda$ and $\Lambda'$ are gauge equivalent if there exists $g_0 \in G(\Omega)$ such that

$$\Lambda' = g_{0,\partial\Omega_0}^{-1}A_{0,\partial\Omega_0},$$

where $g_{0,\partial\Omega_0}$ is the restriction of $g_0$ to $\partial\Omega_0$. We shall prove the following theorem:

**Theorem 1.1.** Suppose that D-to-N operators $\Lambda'$ and $\Lambda$ corresponding to potentials $(A', V')$ and $(A, V)$ respectively are gauge equivalent for all $k \in (k_0 - \delta_0, k_0 + \delta_0)$, where $k_0 > 0$, $\delta_0 > 0$. Then potentials $(A', V')$ and $(A, V)$ are gauge equivalent too.

If we replace $A', V'$ by $A^{(1)} = g_0^{-1}A'g_0 - ig_0^{-1}\frac{\partial g_0}{\partial x}, \quad V^{(1)} = g_0^{-1}Vg_0$ then $\Lambda = \Lambda_1$ where $\Lambda_1$ is the D-to-N operator corresponding to $(A^{(1)}, V^{(1)})$. The proof of Theorem 1.1 gives that if $\Lambda = \Lambda_1$ then $(A, V)$ and $(A^{(1)}, V^{(1)})$ are gauge equivalent with a gauge $g \in G(\Omega)$ such that $g|_{\partial\Omega_0} = I_m$. We shall denote the subgroup of $G(\Omega)$ consisting of $g$ such that $g(x)|_{\partial\Omega_0} = I_m$ by $G_0(\Omega)$. In the case when $\Omega_0$ contains no obstacles Theorem 1.1 was proven in [E] for $n \geq 3$ and in [E3] for $n = 2$. Note that the result of [E] is stronger since it requires that $\Lambda = \Lambda^{(1)}$ for one value of $k$ only. In the case $n = 2$ the proof of Theorem 1.1 is simpler than that in [E3] since it does not rely on the uniqueness of the inversion of the non-abelian Radon transform.
We shall prove Theorem 1.1 in two steps. In §2 we shall prove that \((A, V)\) and \((A^{(1)}, V^{(2)})\) are locally gauge equivalent using the reduction to the inverse problem for the hyperbolic equations as in [B], [B1], [KKL], [KL], [E1], and in §3 we shall prove the global gauge equivalence using the results of §2 and of [E2]. Following Yang and Wu (see [WY]) one can describe the gauge equivalence class of \(A = (A_1, \ldots, A_n)\). Fix a point \(x^{(0)} \in \partial \Omega_0\) and consider all closed paths \(\gamma\) in \(\Omega\) starting and ending at \(x^{(0)}\). Let \(x = \gamma(\tau), \ 0 \leq \tau \leq \tau_0\), be a parametric equation of \(\gamma\), \(\gamma(0) = \gamma(\tau_0) = x^{(0)}\). Consider the Cauchy problem for the system

\[
\frac{\partial}{\partial \tau} c(\tau, \gamma) = \frac{d\gamma(\tau)}{d\tau} \cdot A(\gamma(\tau))c(\tau, \gamma), \quad c(0, \gamma) = I_m.
\]

By the definition the gauge phase factor \(c(\gamma, A)\) is \(c(\tau_0, \gamma)\). Therefore \(A\) defines a map of the group of paths to \(GL(m, \mathbb{C})\). The image of this map is a subgroup of \(GL(m, \mathbb{C})\) which is called the holonomy group of \(A\) (see [Va]).

It is easy to show (c.f. §3) that \(c(\gamma, A^{(1)}) = c(\gamma, A^{(2)})\) for all closed paths \(\gamma\) iff \(A^{(1)}\) and \(A^{(2)}\) are gauge equivalent in \(\Omega\). As it was shown by Aharonov and Bohm [AB] the presence of distinct gauge equivalent classes of potentials can be detected in an experiment and this phenomenon is called the Aharonov-Bohm effect. In §4 we consider the recovery of the Riemannian metrics from the D-to-N operator in domains with obstacles.

## 2 Inverse problem for the hyperbolic system.

Consider two hyperbolic system:

(2.1) \[ L^{(p)} u = \frac{\partial^2}{\partial t^2} u^{(p)} + \sum_{j=1}^{n} (-i \frac{\partial}{\partial x_j} I_m + A_j^{(p)}(x)^2 u^{(p)} + V^{(p)}(x) u^{(p)} = 0, \quad p = 1, 2, \]

in \(\Omega \times (0, T_0)\) with zero initial conditions

(2.2) \[ u^{(p)}(x, 0) = u_t^{(p)}(x, 0) = 0 \]

and the Dirichlet boundary conditions

(2.3) \[ u^{(p)} \big|_{\partial \Omega_j \times (0, T_0)} = 0, \quad 1 \leq j \leq r, \quad u^{(p)} \big|_{\partial \Omega_0 \times (0, T_0)} = f(x', t), \quad p = 1, 2. \]
Here $\Omega = \Omega_0 \setminus (\cup_{j=1}^r \Omega_j)$ is the same as in §1, $A_j^{(p)}(x), 1 \leq j \leq n, V^{(p)}(x), u^{(p)}(x,t), p = 1, 2,$ are smooth $m \times m$ matrices. As in §1 introduce D-to-N operators $\Lambda^{(p)} f = \left( \frac{\partial}{\partial t} + i \sum_{j=1}^n A_j^{(p)} \cdot \nu_j \right) u |_{\partial \Omega_0 \times (0,T_0)}, p = 1, 2.$

Making the Fourier transform in $t$ one can show that the D-to-N operator for (2.1) when $T_0 = \infty$ determines the D-to-N operator for (1.1) for all $k$ except a discrete set, and vice versa.

We shall prove the following theorem:

**Theorem 2.1.** Suppose $\Lambda^{(1)} = \Lambda^{(2)}$ and $T_0 > \max_{x \in M} d(x, \partial \Omega_0)$ where $d(x, \partial \Omega_0)$ is the distance in $\Omega$ from $x \in \Omega$ to $\partial \Omega_0$. Then potentials $A_j^{(1)}(x), 1 \leq j \leq n, V^1(x)$ and $A_j^{(2)}(x), 1 \leq j \leq n, V^2(x)$ are gauge equivalent in $\Omega$, i.e. (1.4) holds with $g \in G_0(\Omega)$.

Note that Theorem 2.1 implies Theorem 1.1. We can consider a more general than (2.1) equation when the Euclidean metric is replaced by an arbitrary Riemannian metric:

\[
\frac{\partial^2 u^{(p)}}{\partial t^2} + \sum_{j,k=1}^n \frac{1}{\sqrt{g_p(x)}} (-i \frac{\partial}{\partial x_j}) I_m \left( A_j^{(p)}(x) \right) \sqrt{g_p(x)} g_p^{jk}(x) \left( -i \frac{\partial}{\partial x_j} \right) I_m A_k^{(p)}(x) u^{(p)} + V^{(p)}(x) u^{(p)}(x,t) = 0,
\]

(2.4)

where $\|g_p^{jk}(x)\|^{-1}$ are metric tensors in $\Omega_p^{(p)}$, $g_p(x) = \det \|g_p^{jk}\|^{-1}, A_j^{(p)}(x), V^{(p)}(x)$ are the same as in (2.1), $\Omega_p^{(p)} = \Omega_0 \setminus \overline{\Omega_p}, \Omega_p = \cup_{j=1}^r \Omega_{jp}$. Let $\Gamma$ be an open subset of $\partial \Omega_0$ and let $0 < T < T_0$ be small. Denote by $\Delta(0,T)$ the intersection of the domain of influence of $\Gamma$ with $\partial \Omega_0 \times [0,T]$. We assume that the domain of influence of $\Gamma$ does not intersect $\overline{\Omega_p} \times [0,T]$.

**Lemma 2.1.** Suppose $\Lambda^{(1)} = \Lambda^{(2)}$ on $\Delta(0,T)$. There exist neighborhoods $U^{(p)} \subset \Omega^{(p)}, p = 1, 2, U^{(p)} \cap \partial \Omega_0 = \overline{\Gamma}$ and the diffeomorphism $\varphi : U^{(1)} \to U^{(2)}$ such that $\varphi |_\Gamma = I$ and $\|g_2^{jk}\| = \varphi \circ \|g_1^{jk}\|$. Moreover $A_j^{(1)}, 1 \leq j \leq n, V^{(1)}$ and $\varphi \circ A_j^{(2)}, 1 \leq j \leq n, \varphi \circ V^{(2)}$ are gauge equivalent in $U^{(1)}$, i.e. there exists $g(x) \in G_0(U^{(1)}), g(x) = I$ on $\Gamma$ such that (1.4) holds in $\overline{U^{(1)}}$.

The proof of Lemma 2.2 is the same as the proof of Lemma 2.1 in [E1]. One should replace only the inner products of the form $\int u(x,t) \overline{v(x,t)} \ dx \ dt$ by $\int Tr(uv^*) \ dx \ dt$ where $v^*$ is the adjoint matrix to $v(x,t)$. We do not assume
that matrices $A_j^{(p)}, V^{(p)}$ are self-adjoint. In the latter case Lemma 2.1 can be obtained by the BC-method (see [B], [KKL]). Extend $\varphi^{-1}$ from $U_2$ to $\tilde{\Omega}^{(2)}$ in such a way that $\varphi = I$ on $\partial\Omega_0$ and $\varphi$ is a diffeomorphism of $\tilde{\Omega}^{(2)}$ and $\tilde{\Omega}^{(2)} = \varphi^{-1}(\Omega^{(2)})$. Also extend $g(x)$ from $U_1$ to $\tilde{\Omega}^{(2)}$ so that $g(x) \in G_0(\tilde{\Omega}_2)$, $g = I$ on $\partial\Omega_0$. Then we get that $L^{(2)} = g \circ \varphi \circ L^{(2)} = L^{(1)}$ in $U^{(1)}$.

**Lemma 2.2.** Let $L^{(1)}$ and $L^{(2)}$ be the operators of the form (2.4) in $\Omega^{(p)} = \Omega_0 \setminus \overline{\Omega}_p$, $p = 1, 2$. Let $B \subset \Omega^{(1)} \cap \Omega^{(2)}$ be simply-connected, $\partial B \cap \partial\Omega_0 = \Gamma$ be open and connected, and $\Omega^{(p)} \setminus B$ be smooth. Suppose $L^{(2)} = L^{(1)}$ in $B$ and $\Lambda^{(1)} = \Lambda^{(2)}$ on $\partial\Omega_0 \times (0,T_0)$ where $\Lambda^{(p)}$ are the D-to-N operators corresponding to $L^{(p)}$, $p = 1, 2$. Then $\tilde{\Lambda}^{(1)} = \tilde{\Lambda}^{(2)}$ where $\tilde{\Lambda}^{(p)}$ are the D-to-N operators corresponding to $L^{(p)}$ in the domains $(\Omega^{(p)} \setminus B) \times (\delta,T_0 - \delta)$, $\delta = \max_{x \in \overline{B}} d(x,\partial\Omega_0)$, $\delta(T_0 - \delta)$. Combining Lemmas 2.1 and 2.2 we can prove that for any $x^{(0)} \in \Omega^{(1)}$ there exist a simply-connected domain $B_1 \subset \Omega^{(1)}$, $x^{(0)} \in B_1$, a diffeomorphism $\varphi$ of $\tilde{\Omega}^{(2)}$ onto $\Omega^{(2)}$, $\varphi = I$ on $\partial\Omega_0$, such that $g \in G_0(\tilde{\Omega}^{(2)})$ such that $\tilde{L}^{(2)} = g \circ \varphi \circ L^{(2)} = L^{(1)}$ in $B_1$. To prove the global gauge equivalence and global diffeomorphism in the case when $\Omega^{(1)}$ is not simply-connected we shall use some additional global quantities determined by the D-to-N operator (c.f. [E2]).

### 3 Global gauge equivalence.

In this section we shall prove Theorem 2.1. Fix arbitrary point $x^{(0)} \in \partial\Omega_0$. Let $\gamma$ be a path in $\Omega$ starting at $x^{(0)}$ and ending at $x^{(1)} \in \overline{\Omega}$, $\gamma(\tau) = x(\tau)$ is the parametric equation of $\gamma$, $0 \leq \tau \leq \tau_1$, $x^{(0)} = x(0)$, $x^{(1)} = x(\tau_1)$. Denote by $c^{(p)}(\tau,\gamma)$, $p = 1, 2$, the solution of the system of differential equations

\[
(3.1) \quad i \frac{\partial c^{(p)}(\tau,\gamma)}{\partial \tau} = \dot{\gamma}(\tau) \cdot A^{(p)}(x(\tau))c^{(p)}(\tau,\gamma),
\]

where

\[
(3.2) \quad c^{(p)}(0,\gamma) = I_m, \quad p = 1, 2, \quad 0 \leq \tau \leq \tau_1,
\]
(3.3) \quad \dot{\gamma}(\tau) = \frac{dx}{d\tau}.

Denote \( c^{(p)}(x^{(1)}, \gamma) = c^{(p)}(\tau_1, \gamma), \ p = 1, 2. \)

**Lemma 3.1.** Suppose \( A^{(1)} \) and \( A^{(2)} \) are locally gauge equivalent. Then the matrix \( c^{(2)}(x^{(1)}, \gamma)(c^{(1)}(x^{(1)}, \gamma))^{-1} \) depends only on the homotopy class of the path \( \gamma \) connecting \( x^{(0)} \) and \( x^{(1)}. \)

**Proof** Let \( \gamma_1 \) and \( \gamma_2 \) be two homotopic paths connecting \( x^{(0)} \) and \( x^{(1)}. \)

Consider the path \( \gamma_0 = \gamma_1 \gamma_2^{-1} \) that starts and ends at \( x^{(0)}. \) It follows from \( 3.1. \) that \( c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1} \) satisfies the following system of differential equations:

\[
\begin{align*}
    i \frac{\partial}{\partial \tau} (c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1}) &= \dot{\gamma}(\tau) \cdot A^{(2)}(x(\tau))(c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1}) \\
    -(c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1}) A^{(1)}(x(\tau)) \cdot \dot{\gamma}(\tau).
\end{align*}
\]

(3.4)

Let \( b(\tau, \gamma) = c^{(2)}(\tau, \gamma)(c^{(1)}(\tau, \gamma))^{-1}, \) \( b(x^{(1)}, \gamma_1) = b(\tau_1, \gamma_1), b(x^{(1)}, \gamma_2) = b(\tau_2, \gamma_2), \) where \( x = x^{(p)}(\tau) \) are parametric equations of \( \gamma^{(p)} \), \( 0 \leq \tau \leq \tau_p, \ p = 1, 2. \)

We have that \( b(x^{(1)}, \gamma_1) = b(x^{(1)}, \gamma_2) \) iff \( b(x^{(0)}, \gamma_0) = I_m, \) where \( x^{(0)} \) is the endpoint of path \( \gamma_0 = \gamma_1 \gamma_2^{-1} \) and \( b(x^{(0)}, \gamma_2) \) is the value at the endpoint of the solution of \( 3.4. \) along \( \gamma_0 \) with the initial value \( 3.2. \). If \( \gamma_0 \) can be contracted to a point in \( \overline{\Omega} \) there exists closed paths \( \sigma_1, ..., \sigma_N \) such that \( \gamma_0 = \sigma_1...\sigma_N \) and each \( \sigma_j \) is contained in a neighborhood \( U_j \subset \Omega \) where \( A^{(1)} \) and \( A^{(2)} \) a gauge equivalent (see Lemma 2.1). We shall show that \( b_j(\tau, \sigma_j) \) is continuous on \( \sigma_j \) where \( b_j(\tau, \sigma_j) \) is the solution of \( 3.4. \) with \( \gamma \) replaced by \( \sigma_j \) \( \sigma_j(\tau) = x^{(j)}(\tau), \ 0 \leq \tau \leq \tau_j, \) is the parametric equation of \( \sigma_j \) \( \sigma_j(0) = \sigma_j(\tau_j) \). The continuity on \( \sigma_j \) means that \( b(0, \sigma_j) = b(\tau_j, \sigma_j). \)

Since \( A^{(1)} \) and \( A^{(2)} \) are gauge equivalent in \( U_j \) there exists \( g_j(x) \in C^\infty(\overline{U_j}) \) such that \( 1.4. \) holds in \( U_j. \) It follows from \( 3.4. \) and \( 1.4. \) that

\[
(3.5) \quad i \frac{\partial}{\partial \tau} (b_j(\tau, \sigma_j)g_j^{-1}(x^{(j)}(\tau))) = 0 \quad \text{for} \quad 0 \leq \tau \leq \tau_j.
\]

Therefore \( b_j(\tau, \sigma_j)g_j^{-1}(x^{(j)}(\tau)) = C \) on \( \sigma_j \). We have \( b_j(0, \sigma_j) = b_j(\tau_j, \sigma_j) = C \ g_j(x^{(j)}(0)). \) Since \( b_j(\tau, \sigma_j) \) is continuous on each \( \sigma_j, \ 1 \leq j \leq N, \) we get that \( b(\tau, \gamma_0) \) is continuous on \( \gamma_0, \) in particular, \( b(x^{(0)}, 0, \gamma_0) = I_m. \) Therefore \( b(x^{(1)}, \gamma_1) = b(x^{(1)}, \gamma_2). \) \qed
Now we shall prove that \( b(x^{(1)}, \gamma_1) = b(x^{(1)}, \gamma_2) \) for any two paths connecting \( x^{(0)} \) and \( x^{(1)} \). As in the case of Lemma 3.1, it is enough to prove that \( b(\tau, \gamma) \) is continuous on \( \gamma_0 = \gamma_1 \gamma_2^{-1} \) where \( b(\tau, \gamma_0) \) is the solution of (3.4) for \( \gamma_0 \).

We say that \( \tilde{\gamma} = \tilde{\gamma}_1, \ldots, \tilde{\gamma}_N \) is a broken ray in \( \overline{\Omega} \times [0, T_0] \) with legs \( \tilde{\gamma}_j, 1 \leq j \leq N \), if it starts at some point \( (x^{(1)}, t^{(1)}) \in \partial \Omega_0 \times [0, T_0] \), \( x = x^{(1)} + \tau \omega \), \( t = t^{(1)} + \tau \) is the parametric equation of \( \tilde{\gamma}_1 \) for \( 0 \leq \tau < \tau_1 \). Then \( \tilde{\gamma} \) makes \( N - 1 \) nontangential reflections at \( \partial \overline{\Omega} \times [0, T_0] \), where \( \overline{\Omega} = \bigcup_{j=1}^N \Omega_j \) and ends at \( \partial \Omega_0 \times [0, T_0] \). Denote by \( \gamma = \gamma_1 \ldots \gamma_N \) the projection of \( \tilde{\gamma} \) onto the \( x \)-plane. Let \( c^{(p)}(\tau, \gamma) \) be the solution of the system

\[
(3.6) \quad i \frac{\partial}{\partial \tau} c^{(p)}(\tau, \gamma) = A^{(p)}(\gamma(\tau)) \cdot \dot{\gamma}(\tau) c^{(p)}(\tau, \gamma),
\]

\[
(3.7) \quad c^{(p)}(0, \gamma) = I_m, \quad p = 1, 2,
\]

where \( \gamma(\tau) \) is the parametric equation of broken ray, \( 0 \leq \tau \leq \tau_N \), \( \dot{\gamma} = \frac{dx}{d\tau} \) is the direction of the broken ray, \( c^{(p)}(\tau, \gamma) \) is continuous on \( \gamma(\tau) \). Note that \( \frac{dx}{d\tau} \) is constant on \( \gamma_j, \tau_{j-1} \leq \tau \leq \tau_j \).

The following lemma is the generalization of Theorem 2.1 in [E2]:

**Lemma 3.2.** Let \( (x^{(N)}, t^{(N)}) \) be the endpoint of the broken ray \( \tilde{\gamma} : \tilde{\gamma}(\tau_N) = (x^{(N)}, t^{(N)}) \). Denote \( c^{(p)}(x^{(N)}, \gamma) = c^{(p)}(\tau_N, \gamma), \quad p = 1, 2 \). Then \( c^{(1)}(x^{(N)}, \gamma) = c^{(1)}(x^{(N)}, \gamma) \) assuming that \( A^{(1)} = A^{(2)} \) on \( \partial \Omega_0 \times (0, T_0) \).

Assuming that Lemma 3.2 is proven we shall complete the proof of Theorem 2.1.

Let \( \gamma \) be a broken ray starting at \( x^{(1)} \) and ending at \( x^{(N)}, x^{(1)} \in \partial \Omega_0, x^{(N)} \in \partial \Omega_0 \).

Let \( c^{(p)}(\tau, \gamma) \) be the solution of (3.6), (3.7). Let \( \alpha_1 \) be a path on \( \partial \Omega_0 \) connecting \( x^{(0)} \) and \( x^{(1)} \) and let \( \alpha_2 \) be a path on \( \partial \Omega_0 \) connecting \( x^{(N)} \) and \( x^{(0)} \). Therefore \( \alpha = \alpha_1 \gamma \alpha_2 \) is a closed path starting and ending at \( x^{(0)} \). If \( U_j \cap \partial \Omega_0 \neq 0 \) then the gauge \( g_j = I_m \) on \( \partial \Omega_0 \). Therefore \( \frac{\partial}{\partial \tau} g_j \cdot \vec{l} = 0 \) on \( U_j \cap \partial \Omega_0 \) for any vector \( \vec{l} \in \mathbb{R}^n \) tangent to \( \partial \Omega_0 \). Then (1.4) implies that \( A^{(2)} \cdot \vec{l} = A^{(1)} \cdot \vec{l} \). It follows from (3.1), (3.2) that \( c^{(1)}(\tau, \alpha_1) = c^{(2)}(\tau, \alpha_1) \) and \( c^{(1)}(\tau, \alpha_2) = c^{(2)}(\tau, \alpha_2) \). Therefore \( b(\tau, \alpha) \) is continuous on \( \alpha = \alpha_1 \gamma \alpha_2 \). We shall call \( \alpha = \alpha_1 \gamma \alpha_2 \) an extended broken ray.

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We shall assume for simplicity that extended broken rays generate the homotopy group of \( \Omega \). Otherwise we can, as in the end of \( \S 2 \), construct a simply-connected domain \( \Omega(0) \) such that \( \overline{\Omega(0)} \subset \Omega, \partial \Omega(0) \supset \partial \Omega, \partial \Omega(0) \cap \overline{\Omega} = \emptyset \) and \( \Omega \setminus \Omega(0) \) is "thin", i.e. the volume of \( \Omega \setminus \Omega(0) \) is small. Since \( \Omega(0) \) is homotopic to \( \partial \Omega \) we get, using Lemmas 2.1 and 2.2 that potentials \( A^{(1)} \) and \( A^{(2)} \) are globally gauge equivalent in \( \Omega(0) \). Therefore the proof of global gauge equivalence in \( \overline{\Omega} \) can be reduced to the proof of the global gauge equivalence in \( \Omega \setminus \Omega(0) \). It is clear that the extended broken rays in \( \Omega \setminus \Omega(0) \) generate the fundamental group \( \pi_1(\overline{\Omega} \setminus \Omega(0)) \). Note that rays without reflections also generated the fundamental group \( \pi_1(\overline{\Omega} \setminus \Omega(0)) \). Then the closed path \( \gamma_0 \) is homotopic to \( \alpha^{(1)}\ldots \alpha^{(N_1)} \) where \( \alpha^{(j)} \) are extended broken rays. Since \( b(\tau, \alpha^{(j)}) \) is continuous on \( \alpha^{(j)}, j = 1, \ldots, N_1 \), we get that \( b(\tau, \gamma_0) \) is continuous on \( \gamma_0 \). It follows from Lemma 3.2 that \( b(\tau, \sigma_j) = I_m \) on \( \partial \Omega \) and hence \( b(\tau, \gamma_0) = I_m \) on \( \partial \Omega \). Therefore we proved that \( \hat{c}^{(2)}(x^{(1)}, \gamma)(c^{(1)}(x^{(1)}, \gamma))^{-1} \) does not depend on the path \( \gamma \) connecting \( x^{(0)} \) and \( x^{(1)} \). Denote \( g(x^{(1)}) = e^{(2)}(x^{(1)}, \gamma)(c^{(1)}(x^{(1)}, \gamma))^{-1} \). We have that \( g(x) \) is a single-valued matrix on \( \overline{\Omega}, g(x) = I_m \) on \( \partial \Omega \) and \( g(x) \) is nonsingular since \( c^{(p)}(x, \gamma) \) are nonsingular, \( p = 1, 2 \). We have for arbitrary \( x^{(1)} \):

\[
\tag{3.8}
A^{(2)}(x^{(1)}) = i \frac{\partial c^{(2)}(\tau, \gamma)}{\partial \tau} (c^{(2)}(\tau, \gamma))^{-1} - \frac{\partial c^{(1)}}{\partial \tau} (c^{(1)}(\tau, \gamma))^{-1} g^{-1}
\]

\[
= i \frac{\partial}{\partial x} g \cdot \gamma g^{-1} + g A^{(1)} \cdot \gamma g^{-1} = (i \frac{\partial}{\partial x} g g^{-1} + g A^{(1)} g^{-1}) \cdot \gamma.
\]

Since we can choose \( \gamma(\tau) \) such that \( \gamma(\tau_1) = x^{(1)} \) and \( \gamma(\tau) \) is arbitrary at \( \tau = \tau_1 \), we get that

\[
\tag{3.9}
A^{(2)}(x^{(1)}) = i(\frac{\partial}{\partial x} g) g^{-1} (x^{(1)}) + g(x^{(1)}) A^{(1)}(x^{(1)}) g^{-1} (x^{(1)}),
\]

i.e. \( A^{(2)} \) is gauge equivalent to \( A^{(1)} \) in \( \overline{\Omega} \). We can change \( A^{(1)} \) to \( A' = g A^{(1)} g^{-1} + i \frac{\partial}{\partial x} g g^{-1} \). Then we will have \( A^{(2)} = A^{(1)} \). Therefore applying Lemmas 2.1 and 2.2 we get that \( V^{(2)} = V' \) in \( \overline{\Omega} \). Therefore \( V^{(2)} = g V^{(1)} g^{-1} \) where \( g(x) \) is the same as in (3.9).

It remains to prove Lemma 3.2. In the case when the broken ray \( \gamma = \gamma_1 \ldots \gamma_M \) does not contain caustics points the proof of Lemma 3.2 is the same as the proof of Theorem 2.1 in [E2]. We shall consider the case when \( \gamma \) has
some caustics points and we shall simplify also the proof of Theorem 2.1 in [E2]. However in this paper we shall not use rays having caustics points.

Consider, for simplicity, the case \( n = 2 \) and \( x_* \in \gamma_M \) is the only caustics point on \( \gamma \). We also assume that the caustics point is generic (see [V]). Note that if \( x_* \) is not generic but the broken ray \( \gamma \) can be approximated by a sequence of broken rays having only generic caustics points, then Lemma 3.2 holds for such \( \gamma \) too. This fact suggests that Lemma 3.2 is likely true for any broken ray.

Let \( \chi_0(y') \in C_0^\infty(\mathbb{R}^2), \ y' = (y_1, y_2), \ \chi_0(y') \geq 0, \ \chi_0(y') = 0 \) for \(|y'| > 1\), \( \chi_0(y') = 1 \) for \(|y'| < \frac{1}{2}\), \( \int_{\mathbb{R}^2} \chi_0^2(y')dy' = 1 \). Denote

\[
\chi(y') = \frac{1}{\varepsilon} \chi_0 \left( \frac{y'}{\varepsilon} \right). \tag{3.10}
\]

We shall heavily use the notations of [E2, §2]. The difference with [E2] is that in this paper we consider the broken ray \( \tilde{\gamma} \) in \( \Omega \times [0, T_0) \) and its projection on \( \Omega \) will be the broken ray \( \gamma \) considered in [E2].

Let \( \Pi \) be a plane in \( \mathbb{R}^2 \times \mathbb{R}, \ (x, t) \in \Pi \) if \( x = x_0^{(0)} + y_1\omega_\perp, \ t = y_2 + t^{(0)} \), where \( \omega_\perp \cdot \omega = 0, \ x^{(0)} \notin \Omega \) and the plane \( \Pi \) does not intersect \( \overline{\Omega} \times \mathbb{R} \). We denote by \( \tilde{\gamma}(y') = \tilde{\gamma}_0(y')\tilde{\gamma}_1(y')...\tilde{\gamma}_M(y') \) the broken ray starting at \((x^{(0)} + y_1\omega_\perp, t^{(0)} + y_2)\) in the direction \((\omega, 1)\), \( y' = (y_1, y_2) \). Then the equation of \( \tilde{\gamma}_0(y') \) is \( x = x^{(0)} + y_1\omega_\perp + t_0\omega, \ t = t^{(0)} + y_2 + t_0, \ 0 \leq t_0 \leq t_0(y_1) \), where \( (x^{(0)} + y_1\omega_\perp + t_0(y_1)\omega, t^{(0)} + y_2 + t_0(y_1)) = \tilde{P}_1 \) is the point where \( \tilde{\gamma}_0 \) hits \( \partial \Omega' \times (0, T_0) \). As in [E2] we introduce “ray coordinates” \((s_p, t_p)\) in the neighborhood of \( \gamma_p, \ 0 \leq p \leq M \). Denote by \( D_j(x(s_j, t_j)) \) the Jacobian of the change of coordinates \( x = x^{(j)}(s_j, t_j) \). Let \( \tilde{P}_j \) be the points of reflections of \( \tilde{\gamma}(y') \) at \( \partial \Omega' \), \( 1 \leq j \leq M \). Denote by \( P_j \) the projection of \( \tilde{P}_j \) on the \( x \)-plane. Note that the time coordinate of \( \tilde{P}_j \) is \( t^{(j)} = t^{(0)} + y_2 + \sum_{r=0}^j t_r(y_1) \) where \( t_r(y_1) \) is the distance between \( P_r \) and \( P_{r-1} \). Note that \( t = t^{(j)} + t_j \) on \( \tilde{\gamma}_j, \ 0 \leq t_j \leq t_j(y_1) \).

Let \( L^{(p)} \) be the same as in \( \textbf{(2.4)}, \ p = 1, 2 \). We construct a solution of \( L^{(1)}u = 0 \) of the form (c.f. \( \textbf{(2.1)}, \textbf{(2.9)} \) in [E2], see also the earlier work [l]):

\[
u(x, t, \omega) = \sum_{j=0}^{M-1} u_j(x, t, \omega) + u_{M1} + u_{M2} + u_{M3} + u^{(1)}, \tag{3.11}\]

where the principal part of \( u_j \) has a form

\[
u_{j0} = a_{j0}(x, t, \omega)e^{ik(\psi_j(x, \omega) - t)}, \tag{3.12}\]

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ψ_j(x,ω) are the same as in (2.2), (2.3), (2.4) in [E2] and

\[(3.13) \quad a_{j0} = (a_{j-1,0}|D_j|^{\frac{1}{2}})|p_j| \frac{1}{|D_j|^{\frac{1}{2}}}c_j(x,\omega),\]

where \(c_j(x,\omega)\) is the solution of the system

\[(3.14) \quad i\theta_j \cdot \nabla c_j = (A^{(1)} \cdot \theta_j) c_j, \quad t_{j-1}(y_1) \leq t_j \leq t_j(y_1), \quad c_j|_{p_j} = I_m,\]

0 ≤ j ≤ M, \(\nabla = \frac{\partial}{\partial x}\), \(\theta_j\) is the direction of \(\gamma_j\), \(\theta_0 = \omega\).

We shall assume that

\[(3.15) \quad u_0(x,t,\omega) = \chi(y')\alpha_0\]
on the plane \(\Pi\), i.e. when \(t_0 = 0\) and \(x = x^{(0)} + y_1\omega_\perp\), \(t = t^{(0)} + y_2\). Here \(\alpha_0\) is an arbitrary constant matrix.

Let \((x_*,t_*) \in \tilde{\gamma}_M\) be such that \(x_*\) is the caustics point in the \(x\)-plane. Note that \(u_{M1}\) has the same form as \(u_{M-1}\) for \(t < t_* - C\varepsilon\), where \(\varepsilon\) is the same as in \((3.10)\), solution \(u_{M2}\) is defined in a \(C\varepsilon\)-neighborhood \(U_\varepsilon\) of \((x_*,t_*)\).

We will not write the explicit form of \(u_{M2}\) (see, for example, [V]) since we will only need an estimate

\[(3.16) \quad |u_{M2}| \leq \frac{Ck^{\frac{\beta}{2}}}{1 + k^{\frac{\beta}{2}}d^\beta(x)},\]

where \(d(x)\) is the distance from \(x \in U_{0,\varepsilon}\) to the caustics curve. Such estimate holds in the generic case (see [V]). Moreover,

\[|\nabla u_{M2}| \leq \frac{Ck^{\frac{\beta}{2}}}{1 + k^{\frac{\beta}{2}}d^\beta(x)}.\]

Finally, \(u_{M3}\) is defined for \(t > t_* + C\varepsilon\) and it has the same form as \(u_{M1}\).

The main difference is that the amplitude of \(u_{M3}\) has an extra factor \(e^{i\beta}\) where \(\beta\) is real. The construction and the estimate of \(u^{(1)}\) in \((3.11)\) is similar to [E2, Lemma 2.1] with the simplification that we consider the hyperbolic initial-boundary value problem with the zero initial conditions when \(t = 0\) and zero boundary conditions on \(\partial\Omega \times (0,T_0)\) instead of \((2.9)\) in [E2]. Since we assumed that \(T_0\) is large enough we get that the endpoint of \(\tilde{\gamma}_M\) belongs to \(\partial\Omega_0 \times (0,T_0)\).
We construct a solution \( v(x,t,\omega) \) of \( L^*_\omega v = 0 \) similar to (3.11) with the same initial data as (3.15) for \( v_0 \) with \( \alpha_0 \) replaced by \( \beta_0 \) where \( \beta_0 \) is an arbitrary constant matrix and with the same phase function \( \psi_j(x,\omega), \ 0 \leq j \leq M \), as in (3.12): We have

\[
(3.17) \quad v = \sum_{j=0}^{M-1} v_j(x,t,\omega) + v_{M1} + v_{M2} + v_{M3} + v^{(1)}(x,t,\omega),
\]

where the principal term of \( v_j \) has the following form:

\[
(3.18) \quad v_{j0} = b_{j0}(x,t,\omega)e^{ik(\psi_j(x,\omega)-t)},
\]

where \( b_{j0} \) are the same as \( a_{j0} \) with \( c_{j1}(x,\omega) \) replaced by \( c_{sj}(x,\omega) \) where \( c_{sj} \) is the solution of the system

\[
(3.19) \quad i\theta_j \cdot \nabla c_{sj} = (\theta_j \cdot (A^{(2)})^*)c_{sj}.
\]

Taking the adjoint of (3.19) we get

\[
(3.20) \quad -i\theta_j \cdot \nabla c_{sj}^* = c_{sj}^*(\theta_j \cdot A^{(2)}).
\]

Denote

\[
(3.21) \quad c_{j2} = (c_{sj}^*)^{-1}.
\]

Then (3.20) implies that

\[
(3.22) \quad i\theta_j \cdot \nabla c_{j2} = (\theta_j \cdot A^{(2)})c_{j2}.
\]

We assume that \( v^{(1)} \) satisfies zero initial conditions when \( t = T, \ x \in \Omega \), and zero boundary conditions on \( \partial \Omega \times (0,T_0) \). Substitute (3.11) instead of \( u^{(1)} \) and (3.17) instead of \( v^{(2)} \) in the Green’s formula. Dividing by \( 2k \) and passing to the limit when \( k \to \infty \) we obtain (c.f. [E2]):

\[
(3.23) \quad 0 = \sum_{j=0}^{M-1} \int_\Omega \int_0^T ((A^{(1)} - A^{(2)}) \cdot \nabla (\psi_j - t)a_{j0},b_{j0})dxdt + I_M,
\]

where \( I_M \) is the integral over a neighborhood of \( \gamma_M \). We make a series of changes of variables as in (2.43) in [E2].
Note that the Jacobian $D_M(x^{(M)}(s_M, t_M))$ vanishes on the caustics set and therefore $D_M^{-1}$ has a singularity there. However when we make changes of variables this singularity in $u_{M1}, v_{M1}$ and in $u_{M3}, v_{M3}$ cancels. Note also that the estimate (3.16) implies that the integral over the neighborhood $U_\varepsilon$ is $O(\sqrt{\varepsilon})$. Therefore taking into account that $\alpha_0$ and $\beta_0$ are arbitrary matrices we get

\begin{equation}
\sum_{j=0}^{M} \int_{\mathbb{R}^2} \int_{\tilde{\gamma}(y')} \chi^2(y') c_j^{-1} (A^{(1)} - A^{(2)}) \cdot \theta_j c_j dt_j dy' + O(\sqrt{\varepsilon}) = 0,
\end{equation}

where $\tilde{\gamma}(y')$ is the broken ray starting at $(x^{(0)} + y_1 \omega_1, t^{(0)} + y_2)$ and we use in (3.24) that $c_j^* = c_j^{-1}$ (see (3.21)). Note that

\begin{equation}
c_j^{-1} (A^{(1)} - A^{(2)}) \cdot \theta_j c_j = i \theta_j \cdot \nabla (c_j^{-1} c_j),
\end{equation}

since $-c_j^{-1} (A^{(2)} \cdot \theta_j) = i \theta_j \cdot \nabla c_j^{-1}$. After changes of variables $c_{j1}$ and $c_{j2}$ in (3.24) satisfy the differential equations (3.14), (3.22) but the initial conditions are different:

\begin{equation}
c_{j_i} \big|_{P_j} = c_{j-1,i} \big|_{P_j}, \quad 1 \leq j \leq M, \quad i = 1, 2.
\end{equation}

We kept the same notation for the simplicity. Taking the limit in (3.24) when $\varepsilon \to 0$ we get

$$
\sum_{j=0}^{M} \int_{\tilde{\gamma}} \theta_j \cdot \nabla (c_j^{-1} c_j) dt_j = \sum_{j=0}^{M} \left[ (c_j^{-1} c_j) \big|_{P_j} - (c_j^{-1} c_j) \big|_{P_{j-1}} \right] = 0.
$$

Since $c_{0i} \big|_{P_0} = I_m$ and (3.26) holds we get that $c_M^{-1} c_M \big|_{P_M} = I_m$, i.e. $c_M \big|_{P_M} = c_{M2} \big|_{P_M}$. Lemma 3.2 is proven.

4 Global diffeomorphism.

Let $L^{(1)} u^{(1)} = 0$ and $L^{(2)} u^{(2)} = 0$ be equations of the form (2.4) in domains $\Omega^{(p)} = \Omega_0 \setminus \Omega'_p$, where $\Omega'_p = \cup_{j=1}^{r_p} \Omega_{jp}$, $p = 1, 2$. We assume that the initial conditions (2.2) in $\Omega^{(p)}$, $p = 1, 2$ and the boundary conditions (2.3) with $\Omega_j$ replaced by $\Omega_{jp}$, $p = 1, 2$, are satisfied.
Theorem 4.1. Suppose $\Lambda^{(1)} = \Lambda^{(2)}$ on $\partial \Omega_0$, where $\Lambda^{(p)}$ are the D-to-N operators corresponding to $L^{(p)}$, $p = 1, 2$. Suppose

$$T_0 > 2 \min_p \max_{x \in \Omega(p)} d_p(x, \partial \Omega_0)$$

where $d_p$ is the distance with respect to the metric tensor $\|g_p^{jk}\|^{-1}$. Then there exists a diffeomorphism $\varphi$ of $\overline{\Omega^{(1)}}$ onto $\overline{\Omega^{(2)}}$ such that $\varphi = I$ on $\partial \Omega_0$ and $\|g_2^{jk}\| = \varphi \circ \|g_1^{jk}\|$.

We shall sketch the proof of Theorem 4.1 assuming for the simplicity that $m = 1$, $A_j^{(p)} \equiv 0$, $1 \leq j \leq r$, $p = 1, 2$, and $T_0 = \infty$. By using Lemmas 2.1 and 2.2 we can get a simply-connected domain $\Omega^{(0)} \subset \Omega^{(1)}$ such that $\Omega^{(1)} \setminus \Omega^{(0)}$ has a small volume. Moreover there exists a diffeomorphism $\tilde{\varphi}$ of $\Omega^{(2)}$ onto $\overline{\Omega^{(2)}} = \tilde{\varphi}^{-1}(\Omega^{(2)})$, $\tilde{\varphi} = I$ on $\partial \Omega_0$ such that $\tilde{L}^{(2)} \equiv \tilde{\varphi} \circ L_2$ is equal to $L^{(1)}$ in $\Omega^{(0)}$. Note that $\Omega^{(0)} \subset \Omega^{(1)} \cap \tilde{\Omega}^{(2)}$. We also get from Lemma 2.2 that $\Lambda^{(1)} = \tilde{\Lambda}^{(2)}$ on $\partial \Omega^{(0)} \setminus \partial \Omega_0$ where $\tilde{\Lambda}^{(2)}$ is the D-to-N operator corresponding to $\tilde{L}^{(2)}$. Since $\Omega^{(1)} \setminus \Omega^{(0)}$ is thin, there is an open subset $\Gamma_1$ of $\partial \Omega^{(0)}$ such that the endpoints of geodesics corresponding to $L^{(1)}$ in $\Omega^{(1)} \setminus \Omega^{(0)}$, orthogonal to $\Gamma_1$, form an open subset $\Gamma_2 \subset \partial \Omega^{(0)}$. Denote by $D_1 \subset \Omega^{(1)} \setminus \overline{\Omega^{(0)}}$ the union of these geodesics. It follows from the proof of Lemma 2.1 [see (E1)] that $\Lambda^{(1)}$ on $\Gamma_1$ uniquely determines the metric tensor $\|g_1^{jk}\|^{-1}$ in the semi-geodesic coordinates in $D_1$. Denote by $\psi_1$ the map of $D_1$ on $\tilde{D}_1 = \psi_1(D_1)$ such that $\psi_1(x)$ are the semi-geodesic coordinates in $\tilde{D}_1$. Analogously let $D_2$ be the union of all geodesics of $\tilde{L}^{(2)}$ orthogonal to $\Gamma_1$ and let $\Gamma_2 \subset \partial \Omega^{(0)}$ be the set of its endpoints. Denote by $\psi_2(x)$ the semi-geodesic coordinates for $\tilde{L}^{(2)}$ and let $\tilde{D}_2 = \psi_2(D_2)$. By Lemma 2.1 $\psi_1 \circ L^{(1)} = \psi_2 \circ \tilde{L}^{(2)}$ in $\tilde{D}_1 \cap \tilde{D}_2$. It follows from Lemma 2.1 that $\psi_j = I$ on $\Gamma_j$. Note that $\Omega^{(2)} \setminus \Omega^{(0)}$ coincide with $\Omega^{(1)} \setminus \Omega^{(0)}$ near $\Gamma_1$.

Lemma 4.1. The following equalities hold: $\tilde{D}_1 = \tilde{D}_2$ and $\psi = \psi_2^{-1}\psi_1 = I$ on $\Gamma_2$.

Proof: Since we assume that $T_0 = \infty$ we can switch to the inverse problem for the equations of the form (1.1). Choose parameter $k \in \mathbb{C}$ such that the boundary value problem of the form (1.1), (1.2), (1.3) has a unique solution $u_p$ for any $f \in H_2(\partial \Omega^{(0)} \setminus \partial \Omega_0)$ where $f$ is the same for $p = 1$ and $p = 2$. Choose $f$ nonsmooth. Denote $\tilde{\Gamma}_2 = \psi_1^{-1}(\Gamma_2)$, $\tilde{\Gamma}_2' = \psi_2^{-1}(\Gamma_2')$. It follows from the unique continuation theorem that $\psi_1 \circ u_1 = \psi_2 \circ u_2$ in $\tilde{D}_2 \cap \tilde{D}_1$ since
the Cauchy data of $u_1$ and $u_2$ coincide on $\Gamma$. Here $L^{(p)}u_p = 0$, $p = 1, 2$. If $\tilde{\Gamma}_2 \neq \tilde{\Gamma}_2'$ we get a contradiction since $\psi_2 \circ u_1$ is $C^\infty$ outside $\tilde{\Gamma}_2$ and $\psi_1 \circ u_2$ is $C^\infty$ outside $\tilde{\Gamma}_2'$. Therefore $\tilde{D}_1 = \tilde{D}_2$ and $\psi = \psi_2^{-1}\psi_1$ is a diffeomorphism of $D_1$ onto $D_2$. Since $\tilde{\Gamma}_2 = \tilde{\Gamma}_2'$ we have $\Gamma_2' = \psi(\Gamma_2)$. Since $\psi_1 \circ u_1 = \psi_2 \circ u_2$ in $\tilde{D}_1 = \tilde{D}_2$ and $u_1 = u_2 = f(x)$ on $\Gamma_2$ we get $f(x) = f(\psi(x))$ on $\Gamma_2$. Since $f$ is arbitrary this implies that $\psi = I$ on $\Gamma_2$.

Therefore $\psi = I$ on $\partial D_1 \cap \partial \Omega^{(0)}$. Define $\varphi^{(1)} = \tilde{\varphi}$ on $\Omega^{(0)}$, $\varphi^{(2)} = \psi \circ \tilde{\varphi}$ on $D_1$. We get that $\varphi^{(1)} \circ L^{(2)} = L^{(1)}$ in $\Omega^{(0)} \cup \overline{D_1}$.

Applying Lemma 2.2 to $\Omega^{(\mu)} \setminus (\Omega^{(0)} \cup \overline{D_1})$ and using again Lemmas 4.1 and 2.2 we prove Theorem 4.1.

Remark 4.1 (c.f. [E1]). We shall show now that the obstacles can be recovered up to the diffeomorphism. Let $\gamma_0$ be an open subset of $\partial \Omega^{(0)}$ close to the obstacle $\Omega_1'$. Denote by $\Delta_1$ the union of all geodesics in $\Omega^{(1)}$ orthogonal to $\gamma_0$ and ending on $\Omega_1'$. Denote by $\gamma_1$ the intersection of $\overline{\Delta_1}$ and $\overline{\Omega_1'}$. Introduce semi-geodesic coordinates for $L^{(1)}$ in $\Delta_1$. Let $\varphi_1$ be the change of variables to the semi-geodesic coordinates and let $\tilde{\Delta}_1 = \varphi_1(\Delta_1)$. Let $\varphi_2$ be the change of variables to the semi-geodesic coordinates for $L^{(2)}$ in $\Delta_2$ where $\Delta_2$ is the union of all geodesics of $L^{(2)}$ orthogonal to $\gamma_0$ and ending on $\Omega_2'$. Let $\gamma_2 = \overline{\Delta_2} \cap \partial \Omega_2'$, $\tilde{\gamma}_2 = \varphi_2(\gamma_2)$, $\tilde{\Delta}_2 = \varphi_2(\Delta_2)$. Let $L^{(1)}u_1 = 0$ be a geometric optics solution in $\Omega^{(1)} \setminus \Omega^{(0)}$ similar to constructed in §3 that starts on $\gamma_0$, reflects at $\partial \Omega_1'$ and leaves $\Omega^{(1)} \setminus \Omega^{(0)}$ again on $\gamma_0$. Let $u_2$ be the solution of $L^{(2)}u_2 = 0$ in $\Omega^{(2)} \setminus \Omega^{(0)}$ having the same boundary data as $u_1$. Since $\varphi_1 \circ L^{(1)} = \varphi_2 \circ L^{(2)}$ in $\tilde{\Delta}_1 \cap \tilde{\Delta}_2$ and since $\varphi_1 \circ u_1$ and $\varphi_2 \circ u_2$ have the same Cauchy data on $\gamma_0$ we get by the uniqueness continuation theorem that $\varphi_1 \circ u_1 = \varphi_2 \circ u_2$. If $\tilde{\gamma}_1 \neq \tilde{\gamma}_2$ then we can find $u_1$ such that $\varphi_1 \circ u_1$ and $\varphi_2 \circ u_2$ will have different point of reflection and this will contradict that $\varphi_1 \circ u_1 = \varphi_2 \circ u_2$. Since $\tilde{\gamma}_1 = \tilde{\gamma}_2$ we get that $\varphi(\gamma_1) = \gamma_2 \subset \partial \Omega_2'$ and $\varphi(\Delta_1) = \Delta_2$ where $\varphi = \varphi_2^{-1}\varphi_1$. □

Remark 4.2 Note that Lemma 2.1 allows to consider the inverse problems in multi-connected domains $\Omega$ with the D-to-N operator given on a not connected part $\Gamma_0$ of $\partial \Omega$.

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