Generalized Hasse-Herbrand functions in positive characteristic

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Abstract

Let \(L/K\) be an extension of complete discrete valuation fields of positive characteristic, and assume that the residue field of \(K\) is perfect. The residue field of \(L\) is not assumed to be perfect.

In this paper, we show that the generalized Hasse-Herbrand function \(\psi_{L/K}^{ab}\) has properties similar to those of its classical counterpart. In particular, we prove that \(\psi_{L/K}^{ab}\) is continuous, piecewise linear, increasing, convex, and satisfies certain integrality properties.

1 Introduction

From classical ramification theory, if \(L/K\) is a finite Galois extension of local fields and \(G = G(L/K)\) is the Galois group of this extension, we know that there are lower and upper ramification subgroups \(G_t\) and \(G^t\) of \(G\), where \(t \in [0, \infty)\), related to each other by the classical Hasse-Herbrand \(\psi\)-function:

\[G_t = G_{\psi_{L/K}^t}.
\]

In [5], we considered an extension \(L/K\) of complete discrete valuation fields, where the residue field of \(K\) is perfect and of positive characteristic, but the residue field of \(L\) is possibly imperfect. We defined, for such an extension, a generalized Hasse-Herbrand \(\psi\)-function \(\psi_{L/K}^{ab}\), and showed that it coincides with the classical \(\psi_{L/K}\) when \(L/K\) is a finite Galois extension of local fields.

In this paper, we study these generalized Hasse-Herbrand \(\psi\)-functions for fields of positive characteristic, and show that they have properties similar to those of the classical \(\psi\)-function. In particular, we prove that \(\psi_{L/K}^{ab}\) is continuous, piecewise linear, increasing, convex, and satisfies certain integrality properties.

More precisely, we prove the following theorems:

**Theorem 1.1** (Theorem 3.9 and Theorem 3.11). Let \(L/K\) be an extension of complete discrete valuation fields of positive characteristic. Assume that the....
residue field of \( K \) is perfect. Then \( \psi_{L/K}^{ab} : [0, \infty) \to [0, \infty) \) is continuous, piecewise linear, increasing, and convex.

**Theorem 1.2 (Theorem 3.12).** Let \( L/K \) be an extension of complete discrete valuation fields of positive characteristic. Assume that the residue field of \( K \) is perfect. Then:

(i) For \( t \in \mathbb{Z}_{\geq 0} \), we have \( \psi_{L/K}^{ab}(t) \in \mathbb{Z}_{\geq 0} \).

(ii) For \( t \in \mathbb{Q}_{\geq 0} \), we have \( \psi_{L/K}^{ab}(t) \in \mathbb{Q}_{\geq 0} \).

(iii) The right and left derivatives of \( \psi_{L/K}^{ab} \) are integer-valued.

Furthermore, we compute \( \psi_{L/K}^{ab} \) for certain extensions \( L/K \) of complete discrete valuation fields of characteristic \( p > 0 \). Assume that the residue field of \( K \) is perfect. When \( e(L/K) \) is prime to \( p \), we show that

\[
\psi_{L/K}^{ab}(t) = e(L/K)t,
\]

where \( e(L/K) \) denotes the ramification index of \( L/K \).

More interestingly, we consider the case where there exists a field \( L_0 \) such that \( L/L_0 \) is a separable totally ramified cyclic extension of degree \( p \), and \( e(L_0/K) = 1 \). In Theorem 4.4 we prove that, in this case,

\[
\psi_{L/K}^{ab}(t) = \begin{cases} 
  t, & t \leq \delta_{\text{tor}}(L/K) \\
  pt - \delta_{\text{tor}}(L/K), & t > \frac{p - 1}{\delta_{\text{tor}}(L/K)} 
\end{cases}
\]

where \( \delta_{\text{tor}}(L/K) \) is the length of the torsion part of the completed \( \mathcal{O}_L \)-module of relative differential forms with log poles \( \hat{\Omega}_1^{\mathcal{O}_L/\mathcal{O}_K}(\log) \). This formula is a generalization of the classical formula obtained in [8, Chapter V, §3].

**Notation.** Through this paper, for a complete discrete valuation field \( K \), \( \mathcal{O}_K \) denotes its ring of integers, \( m_K \) the maximal ideal, \( \pi_K \) a prime element, and \( G_K \) the absolute Galois group. Lowercase \( k \) denotes the residue field of \( K \), and \( v_K \) the discrete valuation. When we say that \( K \) is a local field, we mean that \( K \) is a complete discrete valuation field with perfect (not necessarily finite) residue field.

We write

\[
\hat{\Omega}_1^{\mathcal{O}_K}(\log) = \lim_{m} \Omega_1^{\mathcal{O}_K}(\log)/m_K^{m}\Omega_1^{\mathcal{O}_K}(\log),
\]

where

\[
\Omega_1^{\mathcal{O}_K}(\log) = (\Omega_1^{\mathcal{O}_K} \oplus (\mathcal{O}_K \otimes_k K^\times))/(da - a \otimes a, a \in \mathcal{O}_K, a \neq 0).
\]

By completed free \( \mathcal{O}_L \)-module with basis \( \{e_\lambda\}_{\lambda \in \Lambda} \), we mean \( \lim_{m} M/m_M^{m}M \), where \( M \) is the free \( \mathcal{O}_L \)-module with basis \( \{e_\lambda\}_{\lambda \in \Lambda} \).
We shall denote by $P_{tor}$ the torsion part of an abelian group $P$. Let $L/K$ an extension of complete discrete valuation fields of positive characteristic $p > 0$. Throughout this paper, $e(L/K)$ shall denote the ramification index of $L/K$ and, when $K$ is of characteristic zero, $e_K$ shall denote the absolute ramification index of $K$. When $k$ is perfect and $L/K$ is separable, $\delta_{tor}(L/K)$ shall denote the length of $\left( \frac{\hat{\Omega}^1_{O_L}(\log)}{O_L \otimes_{O_K} \hat{\Omega}^1_{O_K}(\log)} \right)_{tor}$.

Following the notation in [4], we write, for $A$ a ring over $\mathbb{Q}$ or a smooth ring over a field of characteristic $p > 0$, and $n \neq 0$ possibly divisible by $p$,

$$H^q_n(A) = H^q((\text{Spec } A)_{et}, \mathbb{Z}/n\mathbb{Z}(q - 1))$$

and

$$H^q(A) = \lim_{\rightarrow} H^q_n(A).$$

2 Definition and previous results

Through this section, let $L/K$ be an extension of complete discrete valuation fields such that the residue field of $K$ is perfect and of characteristic $p > 0$. In [5], we defined generalizations $\psi_{AS}^{L/K}$ and $\psi_{ab}^{L/K}$ of the classical $\psi$-function for this case, in the sense that they both coincide with the classical $\psi_{L/K}$ when $L/K$ is a finite Galois extension of local fields ([5, Theorem 5.5]).

In this section, we review the definition of $\psi_{ab}^{L/K}$. For a detailed discussion, we refer to [5]. Assume first that the residue field $k$ of $K$ is algebraically closed. For $t \in \mathbb{Z}_{(p)}$, $t \geq 0$, define $\psi_{ab}^{L/K}(t) \in \mathbb{R}_{\geq 0}$ as

$$\psi_{ab}^{L/K}(t) = \inf \left\{ s \in \mathbb{Z}_{(p)} \mid \text{Im}(F_{e(K'/K)t}H^1(K') \to H^1(L')) \subset F_{e(L'/L)s}H^1(L') \right\},$$

for all finite, tame extensions $K'/K$ of complete discrete valuation fields such that $e(L'/L)s, e(K'/K)t \in \mathbb{Z}$.

where $L' = LK'$, and $F_nH^1(K)$ denotes Kato’s ramification filtration defined in [4]. Extend $\psi_{ab}^{L/K}$ to $\mathbb{R}_{\geq 0}$ by putting

$$\psi_{ab}^{L/K}(t) = \sup\{ \psi_{ab}^{L/K}(s) : s \leq t, s \in \mathbb{Z}_{(p)} \}.$$

When $k$ is not necessarily algebraically closed, define $\psi_{ab}^{L/K} = \psi_{ab}^{LK_{ur}/K_{ur}}$.

In a similar way, we can define $\phi_{ab}^{L/K}$, which is shown to be the inverse of $\psi_{ab}^{L/K}$ when the latter is bijective ([5, Proposition 5.1]). Further, we have established a formula for $\psi_{ab}^{L/K}(t)$ for sufficiently large $t \in \mathbb{R}_{\geq 0}$, which is given by the following theorem:
Theorem 2.1 ([5, Theorem 5.4]). Let $L/K$ be a separable extension of complete discrete valuation fields. Assume that $K$ has perfect residue field of characteristic $p > 0$. Let $t \in \mathbb{R}_{\geq 0}$ be such that

$$t \geq \frac{2e_K}{p-1} + \frac{1}{e(L/K)} + \left\lceil \frac{\delta_{\text{tor}}(L/K)}{e(L/K)} \right\rceil$$

if $K$ is of characteristic 0,

$$t > \frac{p}{p-1} \frac{\delta_{\text{tor}}(L/K)}{e(L/K)}$$

if $K$ is of characteristic $p$.

Then

$$\psi_{ab}^{L/K}(t) = e(L/K)t - \delta_{\text{tor}}(L/K).$$

We shall review some concepts that were necessary for obtaining Theorem 2.1 and will be used through the rest of this paper. For more detailed background on Kato’s Swan conductor, we recommend [4]. For an overview of classical ramification theory and modern advances, we suggest [9].

Let $L$ be a complete discrete valuation field of characteristic $p > 0$. Let $l$ be the residue field of $L$ and write $L = l((\pi_L))$ for some prime $\pi_L \in L$. Let $\{b_\lambda\}_{\lambda \in \Lambda}$ be a lift of a $p$-basis of $l$ to $\mathcal{O}_L$. Then $\hat{\Omega}_1^{\mathcal{O}_L}(\log)$ is the completed free $\mathcal{O}_L$-module with basis $\{db_\lambda, d\log \pi_L : \lambda \in \Lambda\}$. Write $\hat{\Omega}_1^L = L \otimes_{\mathcal{O}_L} \hat{\Omega}_1^{\mathcal{O}_L}(\log)$.

Denote by $W_s(L)$ the Witt vectors of length $s$. There is a homomorphism $d : W_s(L) \to \hat{\Omega}_1^L$ given by

$$a = (a_{s-1}, \ldots, a_0) \mapsto \sum_i a_i^{p^i-1} da_i.$$

Remark 2.2. In the literature, the operator $d : W_s(L) \to \hat{\Omega}_1^L$ is often denoted by $F^{s-1}d$.

Define, for $\omega \in \hat{\Omega}_1^L$ and $a \in W_s(L)$,

$$v^\log_L(\omega) = \sup \left\{ n : \omega \in \pi_L^n \otimes_{\mathcal{O}_L} \hat{\Omega}_1^{\mathcal{O}_L}(\log) \right\},$$

and

$$v_L(a) = -\max_i \{-p^i v_L(a_i)\} = \min_i \{p^i v_L(a_i)\}.$$
where $F$ is the endomorphism of Frobenius. Kato defined in [4] the filtration $F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z})$ as the image of $F_nW_s(L)$ under this map. We recall that, for $\chi \in H^1(L, \mathbb{Z}/p^s\mathbb{Z})$, the Swan conductor $Sw \chi$ is the smallest $n$ such that $\chi \in F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z})$.

**Definition 2.3.** Let $a \in W_s(L)$, and $n$ be the smallest non-negative integer such that $a \in F_nW_s(L)$. We say that $a$ is best if there is no $a' \in W_s(L)$ mapping to the same element as $a$ in $H^1(L, \mathbb{Z}/p^s\mathbb{Z})$ such that $a' \in F_{n'}W_s(L)$ for some non-negative integer $n' < n$.

When $v_L(a) \geq 0$, $a$ is clearly best. When $v_L(a) < 0$, $a$ is best if and only if there are no $a', b \in W_s(L)$ satisfying

$$a = a' + (F-1)b$$

and $v_L(a) < v_L(a')$.

Observe that $a \in F_nW_s(L) \setminus F_{n-1}W_s(L)$ is best if and only if $n = Sw \chi$, where $\chi$ is the image of $a$ under $F_nW_s(L) \to H^1(L, \mathbb{Z}/p^s\mathbb{Z})$. We remark that “best $a'$” is not unique.

We review the refined Swan conductor defined in [5], which shall be necessary for our proofs:

**Proposition 2.4 ([5 Proposition 2.8]).** Let $L$ be a complete discrete valuation field of characteristic $p > 0$.

(i) There is a unique homomorphism

$$rsw : F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \to F_n\hat{\Omega}_L^1/F_{\lceil n/p \rceil}\hat{\Omega}_L^1,$$

called refined Swan conductor, such that the composition

$$F_nW_s(L) \to F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \to F_n\hat{\Omega}_L^1/F_{\lceil n/p \rceil}\hat{\Omega}_L^1$$

coinsides with

$$d : F_nW_s(L) \to F_n\hat{\Omega}_L^1/F_{\lceil n/p \rceil}\hat{\Omega}_L^1.$$

(ii) For $\lfloor n/p \rfloor \leq m \leq n$, the induced map

$$rsw : F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z})/F_mH^1(L, \mathbb{Z}/p^s\mathbb{Z}) \to F_n\hat{\Omega}_L^1/F_m\hat{\Omega}_L^1$$

is injective.

In particular, for $\chi \in F_nH^1(L, \mathbb{Z}/p^s\mathbb{Z})$, if $rsw \chi \in F_n\hat{\Omega}_L^1/F_m\hat{\Omega}_L^1$ is non-trivial and the class of $\omega \in F_n\hat{\Omega}_L^1$, then $Sw \chi = -v_L^{log}(\omega)$.

**Remark 2.5.** Our refined Swan conductor $rsw$ is a refinement of the refined Swan conductor defined by Kato in [4, §5]. Related results were obtained by Yatagawa in [10], where the author compares the non-logarithmic filtrations of Matsuda ([6]) and Abbes-Saito ([1]) in positive characteristic.
3 Fundamental properties

In this section, we prove new properties of $\psi_{L/K}$ in the positive characteristic case. The central idea is to reduce the understanding of $\psi_{L/K}$ to the case of complete discrete valuation fields with perfect residue fields. This is achieved by studying the behavior of the Swan conductor after taking certain extensions of $L$ with perfect residue field.

Remark 3.1. The author would like to thank Kazuya Kato for pointing out that ramification and reduction to the perfect residue field case were also studied by Borger in [2].

We start with the following lemmas:

**Lemma 3.2.** Let $L'/L$ be an extension of complete discrete valuation fields of characteristic $p > 0$, and write $e = e(L'/L)$. We have the following commutative diagram:

$$
\begin{array}{c}
F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z})/F_{[\frac{p}{n}]} H^1(L, \mathbb{Z}/p^s \mathbb{Z}) \\
\Downarrow \\
F_{en} H^1(L', \mathbb{Z}/p^s \mathbb{Z})/F_{[\frac{p}{en}]} H^1(L', \mathbb{Z}/p^s \mathbb{Z})
\end{array}
\Rightarrow
\begin{array}{c}
F_n \hat{\Omega}_L^1/F_{[\frac{p}{n}]} \hat{\Omega}_L^1 \\
\Downarrow \\
F_{en} \hat{\Omega}_{L'}^1/F_{[\frac{p}{en}]} \hat{\Omega}_{L'}^1.
\end{array}
$$

**Proof.** The central point is to observe that the vertical arrows are well-defined, which follows from $e [\frac{p}{n}] \leq [\frac{p}{en}]$.

**Lemma 3.3.** Let $L$ be a complete discrete valuation field of characteristic $p > 0$, $\{T_\lambda : \lambda \in \Lambda\}$ be a lift of a $p$-basis of $l$ to $O_L$, and

$$L_\pi = \bigcup_{\lambda \in \Lambda} \bigcup_{j \in \mathbb{N}} L\left(T_\lambda^{1/p^j}\right).$$

Let $\chi \in F_n H^1(L, \mathbb{Z}/p^s \mathbb{Z}) \setminus F_{n-1} H^1(L, \mathbb{Z}/p^s \mathbb{Z})$, and write

$$rsw \chi = ad \log \pi_L + \sum_\lambda b_\lambda dT_\lambda.$$

Assume that $-v_L(a) = n > 0$. Then

$$Sw_{\chi_{\hat{\Lambda}_L}} = Sw \chi,$$

where $\chi_{\hat{\Lambda}_L}$ is the image of $\chi$ in $H^1(L_\pi, \mathbb{Z}/p^s \mathbb{Z})$.

**Proof.** Taking $L' = \widehat{L_\pi}$, we have a commutative diagram given by Lemma 3.2. Further, by the assumption, the image of the refined Swan conductor $rsw \chi$ in $F_n \hat{\Omega}_{L_\pi}^1/F_{[\frac{p}{n}]} \hat{\Omega}_{L_\pi}^1$ is non-trivial. More precisely, the image is

$$rsw \chi_{\hat{\Lambda}_L} = ad \log \pi_L = ad \log \pi_{\hat{\Lambda}_L}.$$

Thus the result follows from Proposition 2.4 (ii).
Lemma 3.4. Let $L$ be a complete discrete valuation field of characteristic $p > 0$, $m_i = p^i$ for $i \in \mathbb{Z}_{\geq 2}$, and $\{ T_\lambda : \lambda \in \Lambda \}$ be a lift of a $p$-basis of $l$ to $\mathcal{O}_L$. For $\lambda \in \Lambda$, write

$$L_{\lambda,i} = \left( \bigcup_{\gamma \in \Lambda \setminus \{ \lambda \}} \bigcup_{j \in \mathbb{N}} L \left( T_\gamma^{1/p^j} \right) \right) \cup \left( \bigcup_{j \in \mathbb{N}} L \left( \pi_L^{1/m_i} \right) \left( T_\lambda - \pi_L^{1/m_i} \right)^{1/p^j} \right).$$

Let $\chi \in \hat{F}_n \mathcal{H}^1(L, \mathbb{Z}/p^s \mathbb{Z}) \setminus \hat{F}_{n-1} \mathcal{H}^1(L, \mathbb{Z}/p^s \mathbb{Z})$, and write

$$\text{rsw } \chi = \text{ad log } \pi_L + \sum_{\gamma} b_\chi dT_\gamma.$$

Assume that $-v_L(b_\lambda) = n > 0$. Then

$$\text{Sw } \chi_{L_{\lambda,i}} = m_i \text{ Sw } \chi_L - 1,$$

where $\chi_{L_{\lambda,i}}$ is the image of $\chi$ in $\mathcal{H}^1(\hat{L}_{\lambda,i}, \mathbb{Z}/p^s \mathbb{Z})$.

Proof. The central idea is similar to that of Lemma 3.3. Taking $L' = \hat{L}_{\lambda,i}$, we have a commutative diagram given by Lemma 3.2 and, by the assumption, the image of the refined Swan conductor $\text{rsw } \chi$ in $\hat{F}_{n, n} \hat{\Omega}_{L_{\lambda,i}}^1 / \hat{F}_{m, n} \hat{\Omega}_{L_{\lambda,i}}^1$, is non-trivial. To see that, let $\omega \in \hat{F}_{n, n} \hat{\Omega}_{L_{\lambda,i}}^1$ be given by

$$\omega = \text{ad log } \pi_L + \sum_{\gamma} b_\lambda dT_\gamma.$$

Then the image $\omega'$ of $\omega$ in $\hat{\Omega}_{L_{\lambda,i}}^1$, is $\text{ad log } \pi_L + b_\lambda dT_\lambda$. Further, in $\hat{\Omega}_{L_{\lambda,i}}^1$,

$$dT_\lambda = d\pi_L^{1/m_i},$$

so we can write

$$\omega' = d \left( \frac{\left( \pi_L^{1/m_i} \right)^{m_i}}{\pi_L^{1/m_i} \pi_L^{1/m_i}} \right) + b_\lambda \pi_L^{1/m_i} d\pi_L^{1/m_i} = b_\lambda \pi_L^{1/m_i} d \log(\pi_L^{1/m_i}).$$

Since $n = -v_L(b_\lambda) \geq -v_L(b_\gamma)$ for $\gamma \in \Lambda$ and $n \geq -v_L(a)$, we have

$$-v_{L_{\lambda,i}}(\omega') = -v_{L_{\lambda,i}}(b_\lambda \pi_L^{1/m_i}) = -m_i v_L(b_\lambda) - 1 = m_i n - 1.$$

Observe that

$$m_i n - 1 - \left\lfloor \frac{m_i n}{p} \right\rfloor = p^i n - 1 - p^{i-1} n = p^{i-1} n (p - 1) - 1 > 0,$$

so the image of $\text{rsw } \chi$ in $\hat{F}_{m, n} \hat{\Omega}_{L_{\lambda,i}}^1 / \hat{F}_{m, n} \hat{\Omega}_{L_{\lambda,i}}^1$ is non-trivial. More precisely, we have

$$\text{rsw } \chi_{L_{\lambda,i}} = b_\lambda \pi_L^{1/m_i} d \log(\pi_L^{1/m_i})$$

and, from Proposition 2.4 (ii),

$$\text{Sw } \chi_{L_{\lambda,i}} = m_i \text{ Sw } \chi_L - 1.$$


**Proposition 3.5.** Let $L/K$ be an extension of complete discrete valuation fields of characteristic $p > 0$. Assume that $k$ is perfect. Then

$$\psi_{L/K}^{ab} = \sup \left\{ \frac{\psi_{M/K}^{ab}}{e(M/L)} : M \in S \right\},$$

where $S$ is the set of all extensions of complete discrete valuation fields $M/L/K$ such that the residue field of $M$ is perfect.

**Proof.** Let $\{T_\lambda : \lambda \in \Lambda\}$ be a lift of a $p$-basis of $l$ to $O_L$ and $L_\pi$ and $L_{\lambda,i}$ be as in Lemma 3.3 and Lemma 3.4, respectively. Put

$$S_L = \left\{ \hat{L}_\pi \right\} \cup \left\{ \hat{L}_{\lambda,i} : \lambda \in \Lambda, i \in \mathbb{N} \right\},$$

and let $\chi \in H^1(K, \mathbb{Z}/p^s\mathbb{Z})$, $rsw_{\chi_L} = ad \log \pi_L + \sum_\lambda b_\lambda dT_\lambda \in F_n\hat{\Omega}_L^1/F_1[p]\hat{\Omega}_L^1$, be the refined Swan conductor of $\chi_L$, and $n = Sw_{\chi_L} > 0$. We have that $n = \max \{\{-v_L(a)\} \cup \{-v_L(b_\lambda) : \lambda \in \Lambda\}\}$. From Lemma 3.3, we have

$$rsw_{\chi_{L_{\pi}}} = ad \log \pi_L$$

and $Sw_{\chi_L} \geq Sw_{\chi_{L_{\pi}}}$. From Lemma 3.4, we have, when $-v_L(a) = n$,

$$Sw_{\chi_L} = Sw_{\chi_{L_{\pi}}}.$$  

On the other hand, when $n = -v_L(b_\lambda)$, by Lemma 3.4 for $i \geq 2$,

$$Sw_{\chi_{L_{\lambda,i}}^\alpha} = m_i Sw_{\chi_L} - 1,$$

so

$$Sw_{\chi_L} = \frac{Sw_{\chi_{L_{\lambda,i}}^\alpha}}{m_i} + \frac{1}{m_i}.$$  

Therefore, in this case, we have

$$Sw_{\chi_L} = \lim_{i \to \infty} \frac{Sw_{\chi_{L_{\lambda,i}}^\alpha}}{m_i} = sup_i \left\{ \frac{Sw_{\chi_{L_{\lambda,i}}^\alpha}}{m_i} \right\}.$$  

From this it follows that, if $t = Sw_{\chi}$,

$$Sw_{\chi} = \sup\limits_{M \in S_L} \left\{ \frac{\psi_{M/K}^{ab}(t)}{e(M/L)} \right\} = sup\limits_{M \in S} \left\{ \frac{\psi_{M/K}^{ab}(t)}{e(M/L)} \right\}.$$  

Let $K'/K$ be a finite tame extension, and $L' = LK'$. Let $\chi \in H^1(K', \mathbb{Z}/p^s\mathbb{Z})$ be such that $Sw_{\chi} = e(K'/K)t$, where $t \in \mathbb{Z}_{(p)}$. From the same argument as...
before, we have that
\[
\frac{Sw_{\chi_{L'}}}{e(L'/L)} = \sup_{M \in S_{L'}} \left\{ \frac{\psi_{M/K}(e(K'/K)t)}{e(M/L)} \right\}
= \sup_{M \in S'} \left\{ \frac{\psi_{M/K}(t)}{e(M/L)} \right\} \leq \sup_{M \in S} \left\{ \frac{\psi_{M/K}(t)}{e(M/L)} \right\},
\]
where \(S'\) consists of all extensions of complete discrete valuation fields \(M/L'/K'\) such that the residue field of \(M\) is perfect. Since \(\psi_{L/K}^{ab} \geq \psi_{M/K}^{ab}\) for every \(M \in S\), we conclude that
\[
\psi_{L/K}^{ab} = \sup \left\{ \frac{\psi_{M/K}^{ab}}{e(M/L)} : M \in S \right\}.
\]

Assume that the residue field \(k\) of \(K\) is perfect, and \(L/K\) is a (possibly transcendental) extension of complete discrete valuation fields of positive characteristic. In what follows, we will reduce the study of the properties of \(\psi_{L/K}^{ab}\) to the case where \(L/K\) is a finite extension of local fields. In this case, \(\psi_{L/K}^{ab}\) is almost fully understood by the classical theory ([7]). However, the classical \(\psi\)-function is only defined for separable extensions, while we want to allow the presence of inseparability. For that reason, we shall first we investigate \(\psi_{L/K}^{ab}\) for purely inseparable finite extensions of local fields, and show that in this case \(\psi_{L/K}^{ab}\) is the identity map. From now on, we will write \(\psi_{L/K}^{ab}\) in the case of a finite extension of local fields, and call this function “classical” (even when the extension \(L/K\) has inseparability degree greater than 1).

**Lemma 3.6.** Let \(L/K\) be a finite extension of local fields of positive characteristic. Assume that \(L/K\) is purely inseparable. Then
\[
\psi_{L/K}(t) = t.
\]

**Proof.** Since \(L/K\) is purely inseparable, we can write \(L = K^{1/p^r}\) for some \(r \geq 0\). The map \(N_{L/K} : L \to K\) is given by \(x \mapsto x^{p^r}\) and is an isomorphism of complete discrete valuation fields. It induces an isomorphism \(G_K \simeq G_L\) which respects upper ramification subgroups, so it follows that \(Sw_{\chi_L} = Sw_\chi\) for any \(\chi \in H^1(K)\). The same argument holds for the extension \(LK'/K'\), where \(K'/K\) is a finite tamely ramified extension, so we have that
\[
\psi_{L/K}(t) = t. \quad \square
\]

We now have all the necessary tools to start the reduction to the classical case. We start by proving the following transitivity formula:

**Lemma 3.7.** Let \(L/K\) be an extension of complete discrete valuation fields of characteristic \(p > 0\), and assume that the residue fields of \(K\) and \(L\) are perfect. Then, if \(M/L\) is a finite extension of complete discrete valuation fields, we have
\[
\psi_{M/K}^{ab} = \psi_{M/L}^{ab} \circ \psi_{L/K}^{ab} = \psi_{M/L}^{ab} \circ \psi_{L/K}^{ab}.
\]

9
Then we have, from the definitions, for some finite, tame extension $\psi$ such that $e = \psi_{M/K}(t)$ and $\psi_{M/L} \circ \psi_{L/K}(t)$.

Let $s \in \mathbb{Z}(p)$ such that $s > \psi_{M/L}(\psi_{L/K}(t))$. Recall that $\psi_{M/L}$ coincides with the classical $\psi$-function ([12]). Therefore we can write $s = \psi_{M/L}(\psi_{L/K}(t) + \epsilon)$ for some $\epsilon > 0$. Then, for every $s' \in \mathbb{Z}(p)$ such that $0 \leq s' \leq \psi_{L/K}(t) + \epsilon$, we have

$$\text{Im}(F_{e(L'/L)} \circ H^1(L') \to H^1(ML')) \subset F_{e(ML'/M)} \circ H^1(L')$$

for all finite, tame extensions $L'/L$ of complete discrete valuation fields such that $e(ML'/M)s, e(L'/L)s^t \in \mathbb{Z}$. On the other hand, given $r \in \mathbb{Z}(p)$ such that $r \geq \psi_{L/K}(t)$,

$$\text{Im}(F_{e(K'/K)} \circ H^1(K') \to H^1(LK')) \subset F_{e(LK'/L)} \circ H^1(LK')$$

for all finite, tame extensions $K'/K$ of complete discrete valuation fields such that $e(LK'/L)r, e(K'/K)t \in \mathbb{Z}$. Taking $r \in \mathbb{Z}(p)$ such that $\psi_{L/K}(t) + \epsilon \geq r \geq \psi_{L/K}(t)$, we see that

$$\text{Im}(F_{e(K'/K)} \circ H^1(K') \to H^1(LK')) \subset F_{e(MK'/M)} \circ H^1(LK')$$

for all finite, tame extensions $K'/K$ of complete discrete valuation fields such that $e(ML'/M)s, e(K'/K)t \in \mathbb{Z}$. Hence $\psi_{M/K}(t) \leq \psi_{M/L} \circ \psi_{L/K}(t)$.

Now let $s \in \mathbb{Z}(p)$ such that $0 \leq s < \psi_{M/L}(\psi_{L/K}(t))$. By the properties of the classical $\psi$, we can write $s = \psi_{M/L}(\psi_{L/K}(t) - \epsilon) + \epsilon'$ for some $\epsilon, \epsilon' > 0$. We can assume that $\psi_{L/K}(t) - \epsilon \in \mathbb{Z}(p)$ by making an appropriate choice of $\epsilon, \epsilon'$. Then we have, from the definitions,

$$\text{Im}(F_{e(K'/K)} \circ H^1(K') \to H^1(LK')) \subset F_{e(MK'/M)} \circ H^1(LK')$$

and

$$\text{Im}(F_{e(K'/K)} \circ H^1(K') \to H^1(LK')) \subset F_{e(MK'/M)} \circ H^1(LK')$$

for some finite, tame extension $K'/K$ of complete discrete valuation fields such that $e(MK'/M)s, e(LK'/L)(\psi_{L/K}(t) - \epsilon), e(K'/K)t \in \mathbb{Z}$. It follows that $\psi_{M/K}(t) > s$. Thus $\psi_{M/K}(t) \geq \psi_{M/L} \circ \psi_{L/K}(t)$.

\[ \square \]

**Lemma 3.8.** Let $L/K$ be an extension of complete discrete valuation fields of characteristic $p > 0$. Assume that $k$ and $l$ are perfect. Let $L_0 = K \otimes_{W(k)} W(l)$. Then

$$\psi_{L/K} = \psi_{L/L_0} = \psi_{L/L_0}.$$

**Proof.** $L_0$ has perfect residue field $l$ and $e(L_0/K) = 1$. We have that $\psi_{L_0/K}$ is the identity map, and $L/L_0$ is a finite extension of local fields. Then the result follows from Lemma 3.7. \[ \square \]
Theorem 3.9. Let $L/K$ be an extension of complete discrete valuation fields of characteristic $p > 0$. Assume that $k$ is perfect. Then $\psi_{ab}^{L/K} : [0, \infty) \to [0, \infty)$ is continuous, increasing, and convex. In particular, it is bijective.

Proof. Let $S$ be as in Proposition 3.5, and

$$S = \left\{ \frac{\psi_{ab}^{M/K}}{e(M/L)} : M \in S \right\}.$$

By [8] Chapter IV, §3, Proposition 13, the classical Hasse-Herbrand $\psi$-functions are convex. Then, by Lemma 3.8, $S$ is an equicontinuous set of convex functions. It follows from Proposition 3.5 that $\psi_{ab}^{L/K}$ is continuous and convex.

To prove that $\psi_{ab}^{L/K}$ is increasing, observe that, since $\psi_{ab}^{M/K}(0) = 0$ for every $M \in S$, we have that $\psi_{ab}^{L/K}(0) = 0$. For $t \in (0, 1)$ and $y \in (0, \infty)$ we have, by convexity of $\psi_{ab}^{L/K}$,

$$t \psi_{ab}^{L/K}(y) = \psi_{ab}^{L/K}(0)(1 - t) + t \psi_{ab}^{L/K}(y) \geq \psi_{ab}^{L/K}(ty).$$

Further, for $y > 0$, clearly $\psi_{ab}^{L/K}(y) > 0$. Then, putting $t = x/y$ for $x, y \in (0, \infty)$ with $x < y$, we get

$$\psi_{ab}^{L/K}(y) > t \psi_{ab}^{L/K}(y) \geq \psi_{ab}^{L/K}(x),$$

so $\psi_{ab}^{L/K}$ is increasing.

Lemma 3.10. Let $L/K$ be an extension of complete discrete valuation fields of characteristic $p > 0$. Assume that $k$ is perfect. Then, for $t \in \mathbb{Z}_{\geq 0}$, we have $\psi_{ab}^{L/K}(t) \in \mathbb{Z}_{\geq 0}$.

Proof. Observe that

$$\inf\{s \in \mathbb{Z}_{\geq 0} : \text{Im}(F_{i}H^{1}(K) \to H^{1}(L)) \subset F_{i}H^{1}(L)\} \in \mathbb{Z}_{\geq 0}.$$

Let $m = \inf\{s \in \mathbb{Z}_{\geq 0} : \text{Im}(F_{i}H^{1}(K) \to H^{1}(L)) \subset F_{i}H^{1}(L)\}$. Then $m \leq \psi_{ab}^{L/K}(t)$. It is enough to show that, for a finite tame extension $K'/K$,

$$\inf\{s \in \mathbb{Z}_{\geq 0} : \text{Im}(F_{e(K'/K)}H^{1}(K') \to H^{1}(L')) \subset F_{e(L'/L)}H^{1}(L')\},$$

where $L' = LK'$.

From Proposition 3.5 we have, for $\chi \in F_{i}H^{1}(K, \mathbb{Z}/p^{s}\mathbb{Z})$,

$$\text{Sw}_{L \chi L} = \sup \left\{ \frac{\text{Sw}_{\chi M}}{e(M/L)} : M \in S \right\} = m,$$
where $S$ is the set of all extensions of complete discrete valuation fields $M/L/K$ such that the residue field of $M$ is perfect. For $\chi \in F_{e(K'/K)} H^1(K', \mathbb{Z}/p^s \mathbb{Z})$, 

$$\frac{Sw_{\chi L'}}{e(L'/L)} = \sup_{M \in S'} \left\{ \frac{\psi_{ab}^{M/K} (e(K'/K)t)}{e(M/L)} \right\}$$

$$= \sup_{M \in S'} \left\{ \frac{\psi_{ab}^{M/K} (t)}{e(M/L)} \right\}$$

$$\leq \sup_{M \in S} \left\{ \frac{\psi_{ab}^{M/K} (t)}{e(M/L)} \right\} = m,$$

where $S'$ consists of all extensions of complete discrete valuation fields $M/L'/K'$ such that the residue field of $M$ is perfect. Thus $\psi_{ab}^{L/K} (t) \in \mathbb{Z}$ and, furthermore, 

$$\psi_{ab}^{L/K} (t) = \inf \{ s \in \mathbb{Z} \geq 0 : \text{Im}(F_t H^1(K) \to H^1(L)) \subset F_s H^1(L) \}. \quad \square$$

**Theorem 3.11.** Let $L/K$ be an extension of complete discrete valuation fields of characteristic $p > 0$. Assume that $k$ is perfect. Then $\psi_{ab}^{L/K} : [0, \infty) \to [0, \infty)$ is piecewise linear.

**Proof.** First observe that, from Theorem 2.1, there is $\tilde{t} \in \mathbb{R}_{\geq 0}$ sufficiently large such that $\psi_{ab}^{L/K}$ is linear for $t > \tilde{t}$, so it is enough to prove that $\psi_{ab}^{L/K}$ is piecewise linear on the closed interval $[0, \tilde{t}]$.

Let $T_\lambda$, $L_\pi$, and $L_{\lambda, i}$ be as in Lemma 3.3 and Lemma 3.4. Denote $\psi_{ab}^\pi = \psi_{ab}^{L_{\pi}/K}$ and $\psi_{ab}^{\lambda, i} = \psi_{ab}^{L_{\lambda, i}/K}/e(\tilde{L}_{\lambda, i}/L)$. Let 

$$f_i = \sup \left\{ \{ \psi_{ab}^{\lambda, i} : \lambda \in \Lambda \} \cup \{ \psi_{ab}^\pi \} \right\}.$$ 

Here we go over an outline of the proof. The key idea is to show that, for sufficiently large $i$, there is a cover of $[0, \tilde{t}]$ by open intervals such that, on each open interval, $f_i$ is piecewise linear, which, by compactness, implies that $f_i$ is piecewise linear. Further, we shall show that, on the subintervals where $f_i$ is linear, $f_{i+1}$ is also linear. From $f_i \leq f_{i+1}$, we shall deduce that $\psi_{ab}^{L/K} = \sup_i \{ f_i \}$ is piecewise linear.

We start with a refinement of the argument used in the proof of Proposition 3.5. Let $\chi \in H^1(K', \mathbb{Z}/p^s \mathbb{Z})$ where $K'/K$ is a finite tame extension. For a field $M \supseteq K$, we shall denote $M' = MK'$. Write 

$$\text{rsw} \chi_{L'} = ad \log \pi_{L'} + \sum_\lambda b_\lambda dT_\lambda \in F_n \hat{\Omega}^1_L / F_n \hat{\Omega}^1_{L'},$$

where $n = Sw\chi_{L'}$. Using the same arguments from Lemma 3.3, Lemma 3.4, and Proposition 3.5, we have that, if $Sw\chi_{L'} = -v_{L'}(a)$,

$$\text{rsw} \chi_{L'/r} = ad \log \pi_{L'},$$
and, if \( Sw \chi_{L'} = -v_{L'}(b_\lambda) \) and \( i \) is sufficiently large,

\[
\text{rsw} \chi_{L',i}^{-1} = b_\lambda \pi_{L'}^{e(L'/L)/m_i} d \log(\pi_{L'}^{1/m_i}).
\]

From this we get:

(i) If \( Sw \chi_{L'} = -v_{L'}(a) \), then

\[
Sw \chi_{L'} = Sw \hat{\chi}_{L',i}.
\]

(ii) If \( Sw \chi_{L'} = -v_{L'}(b_\lambda) \), and \( i \) is sufficiently large, then

\[
Sw \hat{\chi}_{L',i} e(\hat{L}_{\lambda,i}/L') = Sw \chi_{L'} - e(L'/L) - e(L'/L') - e(L'/L) - e(K'/K) e(\hat{L}_{\lambda,i}/L)
\]

so

\[
\frac{1}{e(\hat{L}_{\lambda,i}/L)} Sw \chi_{L',i} = \frac{1}{e(L'/L)} Sw \chi_{L'} - \frac{1}{e(L'/L)}.\]

Therefore, for each \( t_0 \in \mathbb{R}_{\geq 0} \), either

\[
\psi_{ab L'/K}(t_0) = \psi_{ab L/K}(t_0) - \frac{1}{e(L'/L)}
\]

for some \( \lambda \in \Lambda \) and all sufficiently large \( i \) or

\[
\psi_{ab L'/K}(t_0) = \psi_{ab L/K}(t_0).
\]

Observe that these equalities can hold for only finitely many \( \lambda \in \Lambda \). Further, for \( t_0 \in \mathbb{Z}(p) \), \( 0 < t_0 < \tilde{t} \), let \( K'/K \) be a finite tame extension such that \( e(K'/K)t_0 \in \mathbb{Z} \). If

\[
\psi^{ab}_{L'_{\lambda',i}/K}(t_0) < \psi^{ab}_{L'/K}(t_0) - \frac{1}{e(L'/L)},
\]

then

\[
\psi^{ab}_{L'_{\lambda',i}/K}(e(K'/K)t_0) < e(\hat{L}_{\lambda,i}/L)\psi^{ab}_{L'/K}(e(K'/K)t_0) - e(L'/L).
\]

From Lemma 3.10

\[
\psi^{ab}_{L'_{\lambda',i}/K}(e(K'/K)t_0) - e(\hat{L}_{\lambda,i}/L)\psi^{ab}_{L'/K}(e(K'/K)t_0) - e(L'/L) - 1,
\]

so it follows that

\[
\psi^{ab}_{L'_{\lambda,i}/K}(t_0) \leq \psi^{ab}_{L'/K}(t_0) - \frac{1}{e(L'/L)} - \frac{1}{e(L'/L)e(\hat{L}_{\lambda,i}/L)} - \frac{1}{e(K'/K)e(L_{\lambda,i}/L)}.
\]
Since the slopes of $\psi_{L/K}^{ab}$ are bounded, it follows that there exists an open interval containing $t_0$ such that, for sufficiently large $i$,

$$f_i = \max \{ \{ \psi_{\lambda,i}^{ab} : \lambda \in \Lambda' \} \cup \{ \psi_\pi \} \}$$

on this interval, where $\Lambda' \subset \Lambda$ is finite. Therefore $f_i$ is piecewise linear on this interval. By compactness, for sufficiently large $i$, $f_i$ is piecewise linear on $[0, \tilde{\delta}]$.

To conclude the argument observe that, on an interval where

$$f_i(t) = \frac{\psi_{L_{\lambda,i}/K}^{ab}(t)}{e(\lambda, i/L)} = \psi_{L/K}^{ab}(t) - \frac{1}{e(\lambda, i/L)},$$

(respectively, $f_i(t) = \frac{\psi_{L_{\lambda,i+1}/K}^{ab}(t)}{e(\lambda, i+1/L)} = \psi_{L/K}^{ab}(t) - \frac{1}{e(\lambda, i+1/L)}$),

we also have

$$f_i(t) = \frac{\psi_{L_{\gamma, i}/K}^{ab}(t)}{e(\gamma, i/L)} = \psi_{L/K}^{ab}(t) - \frac{1}{e(\gamma, i/L)},$$

(respectively, $f_{i+1}(t) = \psi_{L_{\gamma, i+1}/K}^{ab}(t) = \psi_{L/K}^{ab}(t)$) for every $\gamma \in \Lambda$. On intervals where $f_i(t) = \psi_{L/K}^{ab}(t) - 1/e(\lambda, i/L)$, we have that $\psi_{\lambda,i}$ and $\psi_{\lambda,i+1}$ are linear on the same subintervals, so $f_i$ and $f_{i+1}$ are linear on the same subintervals. From $f_i \leq f_{i+1}$, we get the result. \qed

**Theorem 3.12.** Let $L/K$ be an extension of complete discrete valuation fields of positive characteristic. Assume that the residue field of $K$ is perfect. Then:

(i) For $t \in \mathbb{Z}_{\geq 0}$, we have $\psi_{L/K}^{ab}(t) \in \mathbb{Z}_{\geq 0}$.

(ii) For $t \in \mathbb{Q}_{\geq 0}$, we have $\psi_{L/K}^{ab}(t) \in \mathbb{Q}_{\geq 0}$.

(iii) The right and left derivatives of $\psi_{L/K}^{ab}$ are integer-valued.

**Proof.** (i) is given by [Lemma 3.10]. To prove (iii), it is sufficient to show that the slopes of $\psi_{L/K}^{ab}$ on intervals where $\psi_{L/K}^{ab}$ is linear are integers. Let $I$ be an open interval where $\psi_{L/K}^{ab}$ is linear and $\psi_{L/K}^{ab}(t) = \psi_{\lambda,i}^{ab}(t) + 1/p^i$ for $i$ sufficiently large or $\psi_{L/K}^{ab}(t) = \psi_\pi(t)$, in the notation of [Theorem 3.11]. If $\psi_{L/K}^{ab}(t) = \psi_\pi(t)$, then the slope of $\psi_{L/K}^{ab}$ is an integer on this interval, so there is nothing to prove. On the other hand, if

$$\psi_{L/K}^{ab}(t) = \psi_{\lambda,i}^{ab}(t) + 1/p^i = \frac{\psi_{L_{\lambda,i}/K}^{ab}(t)}{p^i} + \frac{1}{p^i},$$

then it is sufficient to prove that the slope of $\psi_{L_{\lambda,i}/K}^{ab}(t)$ on this interval is an integer divisible by $p^i$.

For $e \in \mathbb{Z}$ sufficiently large and prime to $p$, we can find $t_1, t_2 \in I$ such that $t_1 = \frac{t_1}{e}$, where $m_1 \in \mathbb{Z}$, and $t_2 = \frac{t_2}{e}$, where $m_2 \in \mathbb{Z}$ and $(m_2, p) = 1$. Let
$K'/K$ be a finite, totally tamely ramified extension with ramification index $e$, and write $L' = LK'$, $L_{\lambda,i}' = L_{\lambda,i}K'$. Then, for $t \in I$, we have

$$\frac{\psi_{ab}^{L'/K'}(et)}{e(L'/L)} = \frac{\psi_{ab}^{L_{\lambda,i}'/K'}(et)}{e(L'/L)p^i} + \frac{1}{p^i},$$

so

$$\psi_{ab}^{L'/K'}(et) = \frac{1}{p^i} \left( \psi_{ab}^{L_{\lambda,i}'/K'}(et) + e(L'/L) \right).$$

For $t \in I$,

$$\psi_{ab}^{L_{\lambda,i}'/K'}(et) = aet + b$$

for some $a \in \mathbb{Z}, b \in \mathbb{Q}$. We will show that $p^i | a$. From (i),

$$\mathbb{Z} \ni \psi_{ab}^{L'/K'}(et_1) = \frac{1}{p^i} \left( aet_1 + b + e(L'/L) \right) = am_1 + \frac{b + e(L'/L)}{p^i},$$

so $\frac{b + e(L'/L)}{p^i} \in \mathbb{Z}$. In the other hand,

$$\mathbb{Z} \ni \psi_{ab}^{L'/K'}(et_2) = \frac{1}{p^i} \left( am_2 + b + e(L'/L) \right) = \frac{am_2}{p^i} + \frac{b + e(L'/L)}{p^i},$$

so $\frac{am_2}{p^i} \in \mathbb{Z}$. Since $(m_2, p) = 1$, we have $p^i | a$. Further, for $t \in I$,

$$\psi_{ab}^{L/K}(t) = \frac{\psi_{ab}^{L'/K'}(et)}{e(L'/L)} = \frac{aet + b + e(L'/L)}{p^i e(L'/L)}.$$

Since $p^i | a$, $e(L'/L)$ is prime to $p$, and

$$\psi_{ab}^{L/K}(t) = \frac{\psi_{ab}^{L_{\lambda,i}'/K'}(t)}{p^i} + \frac{1}{p^i},$$

we have that $\frac{\psi_{ab}^{L_{\lambda,i}'/K'}(t)}{p^i} \in \mathbb{Z}$. Thus the slope of $\psi_{ab}^{L/K}$ on this interval is an integer. Finally, to prove (ii), consider again an interval $I$ where $\psi_{ab}^{L/K}$ is linear and $\psi_{ab}^{L/K}(t) = \psi_{ab}^{L_{\lambda,i}'/K'}(t) + 1/p^i$ or $\psi_{ab}^{L/K}(t) = \psi_{ab}^{\pi}(t)$. From the fact that $\psi_{ab}^{L_{\lambda,i}'/K'}(t)$ and $\psi_{ab}^{\pi}(t)$ have rational constant terms, the result follows.

### 4 Computation of $\psi_{ab}^{L/K}$ for basic cases

Through this section, $K$ shall denote a complete discrete valuation field of characteristic $p > 0$ with perfect residue field, and $L_0/K$ an extension of complete discrete valuation fields such that $e(L_0/K) = 1$. The residue field of $L_0$ is not assumed to be perfect.
In this section we will compute $\psi_{ab}^{L/K}$ in some cases. For a separable extension of complete discrete valuation fields $L/K$, and $a \in K$, define

$$\delta_{L/K}(a) := - v_K^\log(da) + \frac{v_L^\log(da)}{e(L/K)}.$$ 

Intuitively, we can see $\delta_{L/K}(a)$ as $\delta_{L/K}(a) = \text{pole}_K(da) - \text{pole}_L(da)$. Whenever there is no ambiguity, we write simply $\delta(a)$.

### 4.1 Tamely ramified extensions

In this subsection, we consider the simplest case, that of an extension of complete discrete valuation fields $L/K$ such that $e(L/K)$ is prime to $p$ (we say that such extension is tamely ramified). More generally, we consider extensions satisfying $\delta_{\text{tor}}(L/K) = 0$. In this case, from the formula obtained in Theorem 2.1 we have

$$\psi_{ab}^{L/K}(t) = e(L/K)t.$$ 

The formula for $\psi_{ab}^{L/K}(t)$ for a tamely ramified extension of complete discrete valuation fields $L/K$ is an immediate consequence of the fact that $\delta_{\text{tor}}(L/K) = 0$ in this case. To show that $\delta_{\text{tor}}(L/K) = 0$, denote $e(L/K) = e$. We have $\pi_K = u\pi_L^e$ for some $u \in U_L$, and

$$\delta_{\text{tor}}(L/K) = v_L^\log \left( \frac{d\pi_K}{\pi_K} \right) = v_L^\log \left( \frac{du}{u} + e \frac{d\pi_L}{\pi_L} \right) = 0.$$ 

Therefore, $\delta_{\text{tor}}(L/K) = 0$, so it immediately follows that

$$\psi_{ab}^{L/K}(t) = e(L/K)t.$$ 

### 4.2 Totally ramified cyclic extension $L/L_0$ of degree $p$

In this subsection, $L/L_0$ denotes a separable, totally ramified cyclic extension of degree $p$. Recall that, when $K_1/K$ is a separable, totally ramified cyclic extension of degree $p$, the classical Hasse-Herbrand $\psi$-function takes the following form ([8, Chapter V, §3]):

$$\psi_{K_1/K}(t) = \begin{cases} t, & t \leq \frac{\delta_{\text{tor}}(K_1/K)}{p-1} \\ pt - \delta_{\text{tor}}(K_1/K), & t > \frac{\delta_{\text{tor}}(K_1/K)}{p-1} \end{cases}$$ 

We will show that $\psi_{ab}^{L/K}$ is similar to $\psi_{K_1/K}$. 

16
Lemma 4.1. Let $\chi \in H^1(K, \mathbb{Z}/p\mathbb{Z})$, and $a \in K$ correspond to $\chi$ under Artin-Schreier-Witt theory. Assume that $a$ is best in $K$. Then

$$Sw\chi_L = \begin{cases} Sw\chi, & -v_L^{\log}(da) < p\frac{\delta(a)}{p-1} \\ p(Sw\chi - \delta(a)), & -v_L^{\log}(da) > p\frac{\delta(a)}{p-1} \end{cases}$$

Proof. If $v_K(a) \geq 0$ the result is clear, so assume $v_K(a) < 0$. Since $a \in K$ is best, we have that $Sw\chi = -v_K(a)$. Further, $p \nmid v_K(a)$, and $a$ is also best in $L_0$, so

$$Sw\chi_{L_0} = -v_{L_0}(a) = -v_{L_0}^{\log}(da).$$

If $a$ is best in $L$, then $\delta(a) = 0$ and $Sw\chi_L = pSw\chi$, so the formula holds. Assume that $a$ is not best in $L$. Write

$$a = b_0^p - b_0 + a_0,$$

where $a_0$ is best in $L$ (recall that we have $v_L(a) < v_L(a_0)$), and put $c_0 = -b_0 + a_0$. We have

$$a = b_0^p + c_0.$$

If $c_0$ is best in $L$, we put $b = b_0$ and $c = c_0$ to write $a = b^p + c$ with $c$ best. If not, observe that $v_L(a) < \min\{v_L(b_0), v_L(a_0)\} \leq v_L(c_0)$. Repeating the argument, write $c_0 = b_1^p + c_1$ with $v_L(c_0) < v_L(c_1)$, so that $a = (b_0 + b_1)^p + c_1$. By induction on $v_L(c_i)$, this process eventually stops, so we can write

$$a = b^p + c$$

for some $b, c \in L$ with $c$ best.

Observe that $v_L(b) = p^{-1}v_L(a) = v_K(a)$, and, since $da = dc$ and $c$ is best in $L$, we have $v_L^{\log}(da) = v_L^{\log}(dc) = v_L(c)$. Consider the following two cases:

(i) Assume $v_L(b) < v_L(c)$. This condition can be rewritten as $v_L(a) < v_L^{\log}(da)$, or

$$-p^{-1}v_L^{\log}(da) < \delta(a).$$

Write $a = b^p - b + (b + c)$. We claim that $b$ is best. Since $a \in K \subset L_0$ is best in $K$, $p \nmid v_L(a) = v_{L_0}(a)$. Since the extension is of degree $p$ and totally ramified, $v_L(a) = pv_K(a)$, so $p \mid v_L(a)$ but $p^2 \nmid v_L(a)$. Recall that $v_L(a) = pv_L(b)$. If $b$ were not best, then we would have $p \mid v_L(b)$. But then we would have $p^2 \mid v_L(a)$, a contradiction.

Then $b + c$ is best, and $Sw\chi_L = -v_L(b + c) = -v_L(b) = -v_K(a)$, so

$$Sw\chi_L = Sw\chi.$$
(ii) Assume \( v_L(b) > v_L(c) \). This condition can be rewritten as
\[
-\frac{p-1}{p} v_L^\log(da) > \delta(a).
\]

In this case, we have \( a = b^p - b + (b + c) \), with \( b + c \) best and
\[
v_L(b + c) = v_L(c) = v_L^\log(da) = p(\delta(a) + v_L(a)),
\]
so
\[
Sw \chi_L = p Sw \chi - p\delta(a). \quad \Box
\]

**Remark 4.2.** Observe that, for \( a \) as in [Lemma 4.1] if \( -v_K(a) < \frac{\delta(a)}{p-1} \), then we also have \( -v_L^\log(da) < p\frac{\delta(a)}{p-1} \).

**Proposition 4.3.** Assume that \( \delta(L/K) \neq 0 \). Then there exists a finite, tame extension \( K' \) of \( K \) and \( \chi_{K'} \in H^1(K', \mathbb{Z}/p\mathbb{Z}) \) such that
\[
Sw \chi_{L'} = Sw \chi_{K'} > 0,
\]
where \( L' = LK' \). Further, \( \chi_{K'} \) can be taken to satisfy \( 0 < Sw \chi_{K'} < \frac{\delta_{L/K}(a)}{p-1} \), where \( \hat{a} \in K' \) is best and corresponds to \( \chi_{K'} \).

**Proof.** Take \( a \in K \) such that \( a \) is best, \( v_K(a) < 0 \), and \( \delta_{L/K}(a) \neq 0 \). Such \( a \) exists since \( \delta(L/K) \neq 0 \). If \( -v_K(a) < \frac{\delta_{L/K}(a)}{p-1} \), then, by [Lemma 4.1] it correspond to \( \chi \in H^1(K, \mathbb{Z}/p\mathbb{Z}) \) such that \( Sw \chi_L = Sw \chi \). If not, let \( e, r \in \mathbb{Z}_{>0} \) be such that \( (e, p) = 1 \) and
\[
0 < -e \left( v_K(a) + \frac{\delta_{L/K}(a)}{p-1} \right) < p^r < -ev_K(a).
\]
This guarantees that \(-ev_K(a) - p^r > 0 \) and \(-ev_K(a) - p^r < e\frac{\delta_{L/K}(a)}{p-1} \). Let
\[
K' = K(\sqrt[p]{\pi_K}). \quad \text{It is a totally ramified tame extension with ramification index} \ e.
\]
Putting \( L_0' = L_0K' \) and \( L = LK' \), we have that \( e(L_0'/K') = 1 \) and \( L'/L_0' \) is a totally ramified cyclic extension of degree \( p \). We have \(-v_{K'}(a) - p^r > 0 \) and \(-v_{K'}(a) - p^r < e\frac{\delta_{L'/K'}(a)}{p-1} \).

Let \( m = p^r \) and \( \hat{a} = \pi_K^m a \). Observe that
\[
\delta_{L'/K'}(\hat{a}) = -v_{K'}(\pi_K^m da) + \frac{v_{L'}^*(\pi_K^m da)}{e(L'/K')}
\]
\[
= -v_{K'}(da) - m + \frac{v_{L'}^*(da)}{e(L'/K')} + m
\]
\[
= \delta_{L'/K'}(a).
\]
On the other hand, since $p \nmid v_K(a)$, we also have $p \nmid v_{K'}(\tilde{a})$. Then $\tilde{a}$ is best in $K$, and $v_{K'}(\tilde{a}) = m + v_{K'}(a)$. Thus $v_{K'}(\tilde{a}) < 0$, and

$$0 < -v_{K'}(\tilde{a}) < \frac{\delta_{L'/K'}(\tilde{a})}{p-1}.$$

Let $\chi_{K'} \in H^1(K', \mathbb{Z}/p\mathbb{Z})$ be the character corresponding to $\tilde{a}$ under Artin-Schreier-Witt theory. Then, by Lemma 4.1,

$$Sw_{\chi_{L'}} = Sw_{\chi_{K'}}.$$

**Theorem 4.4.** Assume that $L/L_0$ is a separable, totally ramified cyclic extension of degree $p$. Then

$$\psi^{ab}_{L/K}(t) = \begin{cases} t, & t \leq \frac{\delta_{\text{tor}}(L/K)}{p-1} \\ pt - \frac{\delta_{\text{tor}}(L/K)}{p-1}, & t > \frac{\delta_{\text{tor}}(L/K)}{p-1} \end{cases}$$

Proof. From Theorem 2.1 and the continuity of $\psi^{ab}_{L/K}$ (Theorem 3.9), we have that $\psi^{ab}_{L/K}(t) = pt - \delta_{\text{tor}}(L/K)$ for $t \geq \frac{\delta_{\text{tor}}(L/K)}{p-1}$. Observe also that $\psi^{ab}_{L/K}(0) = 0$, and

$$\psi^{ab}_{L/K}\left(\frac{\delta_{\text{tor}}(L/K)}{p-1}\right) = p\frac{\delta_{\text{tor}}(L/K)}{p-1} - \delta_{\text{tor}}(L/K) = \frac{\delta_{\text{tor}}(L/K)}{p-1}.$$

Since $\psi^{ab}_{L/K}$ is convex (Theorem 3.9), it follows that $0 \leq \psi^{ab}_{L/K}(t) \leq pt$ for $0 \leq t \leq \frac{\delta_{\text{tor}}(L/K)}{p-1}$. It suffices to show that $\psi^{ab}_{L/K}(t_0) \geq t_0$ for some $0 < t_0 < \frac{\delta_{\text{tor}}(L/K)}{p-1}$. Indeed, if $\psi^{ab}_{L/K}(t_0) \geq t_0$, then $\psi^{ab}_{L/K}(t_0) = t_0$. If $0 < \tilde{t} < t_0$ is such that $\psi^{ab}_{L/K}(\tilde{t}) < \tilde{t}$, then, by convexity, $\psi^{ab}_{L/K}(t) < t$ for $\tilde{t} < t < \frac{\delta_{\text{tor}}(L/K)}{p-1}$. This contradicts $\psi^{ab}_{L/K}(t_0) \geq t_0$. A similar argument can be made to show that $\psi^{ab}_{L/K}(\tilde{t}) \geq \tilde{t}$ for $0 < \tilde{t} < \frac{\delta_{\text{tor}}(L/K)}{p-1}$.

This shows that it is only necessary to prove that $\psi^{ab}_{L/K}(t_0) \geq t_0$ for some $0 < t_0 < \frac{\delta_{\text{tor}}(L/K)}{p-1}$, which follows from Proposition 4.3 and the definition of $\psi^{ab}_{L/K}$.

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