SOME NONLINEAR SECOND ORDER EQUATION MODELLING
ROCKET MOTION

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Abstract. In this paper, we consider a nonlinear second order equation modelling
rocket motion in the gravitational field obstructed by the drag force. The proofs of the
main results are based on topological fixed point approach.

1. Introduction

In this paper we consider the following second order differential equation
\begin{equation}
\ddot{z}(t) = \alpha(t)(\dot{z}(t) + a)^2 \exp\left(-\frac{z(t)}{H}\right) + \beta(t)
\end{equation}
with suitably chosen functions \( \alpha, \beta \) to be defined below by (1.9) and some constants \( a, H \)
to be determined later and suitable initial or boundary conditions derived from some
model describing the motion of the rocket, cf. [4]. Specifically, it can be derived from the
one-dimensional first order differential equation
\begin{equation}
m(t) \dot{v}(t) = (w(t) - v(t)) \dot{m}(t) + f^{ext}(t).
\end{equation}
This equation can be used to model a rocket motion with decreasing mass \( m = m(t) \) and
under external force \( f^{ext}(t) \). The rocket should be equipped with the engine that emits
mass \( m_1 = m_1(t) \) with velocities \( w = w(t), t \in [t_0, t_1] \). Therefore, the difference between
the mass of the rocket \( m(t) \) and the mass of the gas emitted at time \( t \) is constant and
equal \( m_0 \) which is some positive value, i.e. \( m(t) = m_1(t) + m_0 \). In \( m_0 \) we include all
components of the rocket infrastructure like the engine, empty tank, guidance equipment,
payload and the like. The function \( f^{ext} = f^{ext}(t) \) denotes all external forces that stimulate

1991 Mathematics Subject Classification. 34B10, 34B15, 35A15, 35J25, 93C15.

Key words and phrases. Dirichlet problem, Tsiolkovskii equation, Meščerskii equation, rocket motion.

This work has been partially supported by the Polish Ministry of Science project N N201 418839.
the rocket motion for example the gravity force or the drag force and in fact may depend also on the position and the velocity of the rocket, cf. (1).

The expression $m\dot{v}(t)$ is the force acting on the rocket body which is given by the amount of mass expelled and the velocity of the mass relative to the rocket body, i.e. $w(t) - v(t)\dot{m}_1(t)$. The rocket moves forward since the rocket loses mass as it accelerates, so $\dot{m}_1(t)$ must be negative for any rocket. The term $w - v\frac{dm_1}{dt}$ is called the thrust of the rocket and can be interpreted as an additional force on the rocket due to the gas expulsion. The detailed derivation of the rocket equation one can find, among others, in [3], [4] and [5].

The equation (1.2) is so called the Meščerskii rocket equation, for details see [4], [3], where also the case of more that just one emitted mass was considered. If we assume, the external force to be zero and a constant relative velocity of the emitted mass, i.e. $f_{ext} = 0$ and $w(t) - v(t) =: c < 0$ for $t \in [t_1, t_2]$, then the equation (1.2) has the form of the well-known Tsiolkovskii rocket equation.

The corresponding solution to the Tsiolkovskii equation with the initial conditions $m(t_1) = \tilde{m}$ and $v(t_1) = \tilde{v}$ can be easily found after integration of (1.2) and reads as follows

$$v(t) = \tilde{v} + c \ln \frac{m(t)}{\tilde{m}}.$$  

The Tsiolkovskii rocket equation in the gravitational field, i.e. $f_{ext} = -mg$ with the initial conditions $m(t_1) = \tilde{m}$ and $v(t_1) = \tilde{v}$ has a solution of the form

$$v(t) = \tilde{v} - g(t - t_1) + c \ln \frac{m(t)}{\tilde{m}}.$$  

Assume that $f_{ext} = -mg - D$ with $D = \frac{1}{2}\rho v^2 AC_D$ where $C_D$ is a drag coefficient, $A$ is the cross sectional area of the rocket, $\rho$ is the air density that changes with altitude $x$, and can be approximated by $\rho = \rho_0 \exp \left(-\frac{x}{H}\right)$ where $H \approx 8000m$ is the so called "scale height" of the atmosphere, and finally $\rho_0$ is the air density at sea level. In that case it is impossible to solve the equation explicitly. However, it is interesting to note that the effect of drag loss is usually quite small and is often reasonable to ignore it in the first approach. In practice, the drag force is approximately only about 2% of the gravity force.
Except for the rocket equation, the Tsiolkovskii equation can also be applied to the one photon decay of the excited nucleus. Moreover, it is possible to identify the Meščerskii equation with the bremsstrahlung equation where the electron is in the permanent decay. Many other decays can be investigated from the point of view of the Meščerskii equation, as one can see in [5].

In the paper we will consider the equation (1.2), with a given relative velocity of emitted masses $c_i(t) = v_i(t) - v(t)$ for any $i = 1, ..., N$ and $t \in [t_1, t_2]$. Chemical rockets produce exhaust jets at velocities $|c_i|$ from 2500 m/s to 4500 m/s.

If we assume that the relative velocity is a given function $c(t) = w(t) - v(t)$, we obtain the following equation

$$m(t) \ddot{v}(t) = c(t) \dot{m}(t) + f^{\text{ext}}(t).$$

Let us put $v(t) = \dot{x}(t)$ where $x(t)$ denotes rocket position at time $t$. Furthermore, we assume that the external force depends not only on time $t$ but also on the rocket position $x(t)$ and its velocity $v(t)$ according to gravitational and drag forces, i.e.

$$f^{\text{ext}}(t) = f(t, x(t), v(t)) = -m(t) g - \frac{AC_D \rho_0}{2} v^2(t) \exp\left(-\frac{x(t)}{H}\right)$$

such that $f : [t_0, t_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Therefore, we get the equation

$$(1.3) \quad m(t) \ddot{x}(t) = c(t) \dot{m}(t) - m(t) g - \frac{AC_D \rho_0}{2} \dot{x}^2(t) \exp\left(-\frac{x(t)}{H}\right)$$

for any $t \in [t_0, t_1]$.

We provide our equation (1.3) with the boundary conditions

$$(1.4) \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

which guarantee that the rocket attains the given point in the required period of time.

After a shift $z(t) = x(t) - y(t)$ by a linear function

$$(1.5) \quad y(t) = a(t - t_0) + x_0.$$

where $a = \frac{x_1 - x_0}{t_1 - t_0}$ and equation (1.3) reads

$$(1.6) \quad \ddot{z}(t) = c(t) \frac{\dot{m}(t)}{m(t)} - g - \frac{AC_D \rho_0}{2} \frac{(\dot{z}(t) + a)^2}{m(t)} \exp\left(-\frac{z(t) + y(t)}{H}\right)$$
for any $t \in [t_0, t_1]$ and $z$ satisfies the homogenous Dirichlet boundary conditions

$$z(t_0) = z(t_1) = 0.$$  

Consequently, one can reduce considered equation to a aforementioned form

$$\ddot{z}(t) = \alpha(t) (\dot{z}(t) + a)^2 \exp \left(-\frac{z(t)}{H}\right) + \beta(t)$$

where

$$\alpha(t) = -\frac{ACD\rho_0}{2m(t)} \exp \left(-\frac{y(t)}{H}\right),$$
$$\beta(t) = c(t) \frac{m(t)}{m(t)} - g.$$  

2. Integral formulation

It should be noted that due to the presence of the derivative $\dot{z}$ the equation (1.6) is not in a variational form. Therefore our approach differs from the one employed in [1] (for extension to elliptic problems see [2]) and makes use of integral formulation approach. Specifically the boundary value problem (1.8) with (1.7) can be reformulated in the following way

$$z(t) = \frac{t_1}{t_0} \int_{t_0}^{t_1} G(t, s) F(s, z(s), \dot{z}(s)) \, ds + b(t)$$

where the nonlinear term $F$ is defined as

$$F(s, z, \dot{z}) = \alpha(s) (\dot{z} + a)^2 \exp \left(-\frac{z}{H}\right)$$

with the abuse of notation, namely $\dot{z}$ denotes in the above formula the third independent variable and the function $b$ is defined by

$$b(t) = \frac{t_1}{t_0} \int_{t_0}^{t_1} G(t, s) \beta(s) \, ds.$$  

Recall that $\alpha(t)$ and $\beta(t)$ are defined by formulas (1.9). The function $G$ is the Green function, i.e. the continuous symmetric function satisfying the following conditions.
a. the homogeneous equation, i.e. for any \( t \neq s \)
\[
\frac{\partial^2 G}{\partial t^2}(t, s) = 0,
\]
b. the boundary conditions, i.e. for any \( s \in [t_0, t_1] \)
\[
G(t_0, s) = G(t_1, s) = 0,
\]
c. the jump condition, i.e. for any \( t \in (t_0, t_1) \)
\[
\lim_{s \to t^+} \frac{\partial G}{\partial t}(t, s) - \lim_{s \to t^-} \frac{\partial G}{\partial t}(t, s) = 1.
\]

This function can be obtained from (1.8) by double integration and use of the the boundary conditions (1.7) and it reads
\[
G(t, s) = \begin{cases} 
\frac{1}{t_1-t_0} (t-t_0) (t_1-s), & t < s, \\
\frac{1}{t_1-t_0} (s-t_0) (t_1-t), & t > s.
\end{cases}
\]

with the derivative
\[
\frac{\partial G}{\partial t}(t, s) = \begin{cases} 
\frac{1}{t_1-t_0} (t_1-s), & t < s, \\
\frac{1}{t_1-t_0} (t_0-s), & t > s.
\end{cases}
\]

3. Main results

We shall work in the space \( C^1 = C^1([t_0, t_1], \mathbb{R}) \) of continuously differentiable functions equipped with the natural norm
\[
\|z\| = \max \{ |z|, |\dot{z}| \}
\]
where \( |z| = \sup_{t \in [t_0, t_1]} |z(t)| \) and \( |\dot{z}| = \sup_{t \in [t_0, t_1]} |\dot{z}(t)| \) for any \( z \in C^1 \). Next we can denote the right hand side of (2.1) as the value of the integral operator \( S \dot{z} \), i.e.
\[
S \dot{z}(t) = \int_{t_0}^{t_1} G(t, s) F(s, z(s), \dot{z}(s)) \, ds + b(t).
\]

Thus the existence of solution to (2.1) is reduced to finding a fixed point of the operator \( S \), i.e. such a \( z \in C^1 \) that
\[
S z = z.
\]
First of all, note that the operator $S$ is well-defined, continuous and compact. Indeed, for any $z \in C^1$ the function $Sz$ is continuous since the functions appearing under the integral sign \((3.1)\) are uniformly continuous and $(Sz)'$ is also continuous by the integrability of $\frac{\partial G}{\partial t}$, cf. \((2.3)\). The continuity of the operator $S$ follows from the smoothness of $F$ with respect both to $z$ and $\dot{z}$ and boundness of the functions $G$ and $\frac{\partial G}{\partial t}$. The compactness of the operator $S$ requires the straightforward application of the Ascoli-Arzela theorem.

### 3.1. Sign insensitive case.

In this subsection we provide crude but immediate estimates yielding by the Schauder fixed point theorem the existence of a fixed point for the operator $S$. To this end we start with simple estimates of the following sup norms $| \cdot |$

\begin{equation}
|Sz| \leq G_0 |\alpha| \left( |\dot{z}|^2 + 2a |\dot{z}| + a \right) + |b|
\end{equation}

\begin{equation}
|(Sz)'| \leq G_1 |\alpha| \left( |\dot{z}|^2 + 2a |\dot{z}| + a \right) + |b|
\end{equation}

where

\[
G_0 = \sup_{t \in [t_0, t_1]} \int_{t_0}^{t_1} G(t, s) \, ds,
\]

\[
G_1 = \sup_{t \in [t_0, t_1]} \int_{t_0}^{t_1} \left| \frac{\partial G}{\partial t}(t, s) \right| \, ds.
\]

Using formula \((2.2)\) one can easily derive the value of

\[
G_0 = \frac{3}{8} (t_1 - t_0)^2
\]

and

\[
G_1 = (t_1 - t_0)^2.
\]

**Theorem 3.1.** The boundary value problem \((1.7)-(1.8)\) admits at least one solution provided \((3.4)\) and \((3.3)\) are satisfied.

**Proof.** As announced we will make use of the Schauder fixed point theorem for the operator $S$, defined by \((3.1)\). We should prove that it maps some ball $B(0, R)$ in $C^1$ into itself. Using the preliminary estimates \((3.2)\) and defining

\[
G_2 = |\alpha| \max \{G_0, G_1\} = |\alpha| (t_1 - t_0)^2
\]
the invariance of the ball $B(0, R)$ under the action of $S$ can be reduced to the following condition

$$G_2 \left( R^2 + 2aR + a \right) + |b| \leq R$$

which can be satisfied by some $R > 0$ if

$$\Delta = (2aG_2 - 1)^2 - 4G_2 (G_2a + |b|) > 0,$$

i.e.

(3.3) $$|b| < \frac{4a(a - 1)G_2^2 - 4aG_2 + 1}{4G_2^2}$$

and the value in the nominator is positive which is guaranteed provided

(3.4) $$G_2 \in \left( 0, \frac{a - \sqrt{a}}{2a(a - 1)} \right) \cup \left( \frac{a + \sqrt{a}}{2a(a - 1)}, \infty \right), \ a > 1.$$  

\[\square\]

**Corollary 3.2.** The conditions (3.4) and (3.3) are satisfied if $\beta$ is sufficiently small.

**Example 3.1.** If $A = 0$, i.e. we neglect the drag force, accounting for negligible, in real world model, part, our assumptions are naturally satisfied. In such a context it is natural that one can attain the destination point in the desired period of time. In fact since the equation is linear one can derive the explicit formula for the solution.

3.2. **Sign sensitive case.** In this section we provide more subtle a priori estimates taking into regard the negative sign of some terms appearing in (1.8) and instead of using Schauder Theorem, we apply more subtle Schafer Theorem. First, we neglect negative term in (1.8) and integrate inequality

$$\ddot{z}(t) \leq \beta(t)$$

using boundary conditions. Thus we get by (1) the bound for $v$ and hence for $z$. Next we multiply by $\dot{z}$ both sides of (1.8) and integrate after dividing by $(\dot{z} + a)^2$... to get better a priori bounds.
REFERENCES

[1] Bors D., Walczak S., Multidimensional second order systems with controls, Asian Journal of Control 12 (2010), pp. 159–167

[2] Bors D., Superlinear elliptic systems with distributed and boundary controls, Control and Cybernetics 34 (2005), pp. 987–1004

[3] Kosmodemianskii A.A. (1966), The course of theoretical mechanics II, Moscow (in Russian)

[4] Meščerskii I.V. (1962) Works on mechanics of the variable mass, Moscow (in Russian)

[5] Pardy M., The rocket equation for decays of elementary particles, Preprint arXiv:hep-ph/0608161v1 (2008)

[6] Peraire J., Variable mass systems: the rocket equation, MIT OpenSourceWare, Massachusetts Institute of Technology, Available online (2004)

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