The q-Binomial Coefficient for Negative Arguments 
and Some q-Binomial Summation Identities

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Abstract

Using a property of the q-shifted factorial, an identity for q-binomial coefficients is proved, which is used to derive the formulas for the q-binomial coefficient for negative arguments. The result is in agreement with an earlier paper about the normal binomial coefficient for negative arguments. Some new q-binomial summation identities are derived, and the formulas for negative arguments transform some of these summation identities into each other. One q-binomial summation identity is transformed into a new q-binomial summation identity.

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1 Definitions and Basic Identities

Let the following definition of the q-binomial coefficient, also called the Gaussian polynomial, be given.

\[
\binom{m + p}{m}_q = \prod_{j=1}^{m} \frac{1 - q^{p+j}}{1 - q^j}
\]  

(1.1)

Let the q-shifted factorial, also called the q-Pochhammer symbol, be given by [7]:

\[
(a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j)
\]  

(1.2)

Then the q-binomial coefficient for integer \( k \geq 0 \) is:

\[
\binom{n}{k}_q = \prod_{j=1}^{k-1} \frac{1 - q^{n-k+j}}{1 - q^j} = \frac{(q^{n-k+1}; q)_k}{(q; q)_k}
\]  

(1.3)

Let \( \Gamma_q(x) \) be the q-gamma function [2].

For complex \( x, y \):

\[
\binom{x}{y}_q = \frac{\Gamma_q(x+1)}{\Gamma_q(y+1)\Gamma_q(x-y+1)}
\]  

(1.4)
From this follows the symmetry identity:
For complex $x, y$:
\[
\binom{x}{y}_q = \binom{x}{x-y}_q \tag{1.5}
\]

The functional equation of the $\Gamma_q(x)$ function is \cite{2}:
\[
\Gamma_q(x + 1) = \frac{1 - q^x}{1 - q} \Gamma_q(x) \tag{1.6}
\]

Combination of (1.4) and (1.6) gives the absorption identity:
For complex $x, y$:
\[
\binom{x}{y}_q = 1 - q^x \binom{x-1}{y-1}_q \tag{1.7}
\]

From definition (1.3) follows:
For integer $n \geq 0$ and integer $k$:
\[
\binom{n}{k}_q = 0 \text{ if } k > n \tag{1.8}
\]

From this and (1.5) follows:
For integer $n \geq 0$ and integer $k$:
\[
\binom{n}{k}_q = 0 \text{ if } k < 0 \tag{1.9}
\]

2 The q-Binomial Coefficient for Negative Arguments

For deriving the q-binomial coefficient for negative arguments, the following theorem \cite{4, 7} is needed.

**Theorem 2.1.** For integer $k \geq 0$:
\[
(a; q)_k = (-a)^k q^{k(k-1)/2} \binom{q^{1-k}}{a; q}_k \tag{2.1}
\]

**Proof.** From:
\[
k(k-1)/2 = \sum_{j=0}^{k-1} j = \sum_{j=0}^{k-1} (k - j - 1) \tag{2.2}
\]

follows:
\[
q^{k(k-1)/2} = \prod_{j=0}^{k-1} q^j = \prod_{j=0}^{k-1} q^{k-j-1} \tag{2.3}
\]

This is used in:
\[
(-a)^k q^{k(k-1)/2} \prod_{j=0}^{k-1} \left(1 - \frac{q^{1-k}}{a} q^j\right) = q^{k(k-1)/2} \prod_{j=0}^{k-1} (q^{1-k+j} - a) = \prod_{j=0}^{k-1} (1 - a q^{k-j-1}) = \prod_{j=0}^{k-1} (1 - a q^j) \tag{2.4}
\]
Theorem 2.2. For integer \( k \geq 0 \):

\[
{n \choose k}_q = (-1)^k q^{nk-k(k-1)/2} \binom{-n+k-1}{k}_q
\]  

(2.5)

Proof. Using (1.3) and the previous theorem and \(-n = (-n + k - 1) - k + 1\):

\[
{n \choose k}_q = \frac{(q^{n-k+1}; q)_k}{(q; q)_k} = (-1)^k q^{k(n-k+1)+k(k-1)/2} \frac{(q^{-n}; q)_k}{(q; q)_k}
\]

\[
= (-1)^k q^{nk-k(k-1)/2} \binom{-n + k - 1}{k}_q
\]  

(2.6)

\[
\square
\]

Theorem 2.3. For integer \( k \leq n \):

\[
{n \choose k}_q = (-1)^{n-k} q^{(n-k)(n+k+1)/2} \binom{-k-1}{n-k}_q
\]  

(2.7)

Proof. In the previous theorem replacing \( k \) with \( n - k \), which makes it valid when \( n - k \geq 0 \) which is when \( k \leq n \), and then applying (1.5) to the left side:

\[
{n \choose n-k}_q = {n \choose k}_q
\]  

(2.8)

and using \( n(n-k) - (n-k)(n-k-1)/2 = (n-k)(n+k+1)/2 \) gives this theorem.  

\[
\square
\]

These two transformations can be used to transform one q-binomial summation identity into another, and can be used to express the q-binomial coefficient for negative integer \( n \) and integer \( k \) into the q-binomial coefficient for nonnegative integer \( n \) and \( k \).

Theorem 2.4. For negative integer \( n \) and integer \( k \):

\[
{n \choose k}_q = \begin{cases} 
(-1)^k q^{nk-k(k-1)/2} \binom{-n+k-1}{k}_q & \text{if } k \geq 0 \\
(-1)^{n-k} q^{(n-k)(n+k+1)/2} \binom{-k-1}{n-k}_q & \text{if } k \leq n \\
0 & \text{otherwise}
\end{cases}
\]  

(2.9)

Proof. The first two cases are identical to theorems 2.2 and 2.3.

For the third case, from (1.7) follows for integer \( n, k \):

\[
{n-1 \choose k-1}_q = \frac{1 - q^k}{1 - q^n} {n \choose k}_q
\]  

(2.10)

When \( k = 0 \) the right side is zero, so when \( n < 0 \) and \( k = 0 \) this identity produces zeros for \( n < k < 0 \), which is the third case. This identity does not produce zeros for all \( k < 0 \) because when \( n > 0 \) a point will be reached where \( n = 0 \) and this expression becomes \( (1 - q^k)0/0 \) which is undefined.  

\[
\square
\]
The normal binomial coefficients are the q-binomial coefficients with \( q = 1 \), in which case this theorem reduces to theorem 2.1 in \[5\].

For \( n = -1 \) this theorem results in:

\[
\binom{-1}{k}_q = \begin{cases} 
(-1)^{k}q^{-k(k+1)/2} & \text{if } k \geq 0 \\
(-1)^{k+1}q^{-k(k+1)/2} & \text{if } k \leq -1 
\end{cases} \tag{2.11}
\]

which is in agreement with example 1.4 in \[3\].

As an example of the second case of this theorem:

\[
\binom{-3}{-5}_q = q^{-7} \binom{4}{2} = q^{-7}(1 + q^2)(1 + q + q^2) \tag{2.12}
\]

which is in agreement with example 1.2 in \[3\].

The q-binomial coefficient polynomial is palindromic, which means that

\[
a_j = a_{k(n-k)-j}, \text{ from which follows that it is self-reciprocal, where } j \text{ is replaced by } k(n-k) - j:
\]

\[
\binom{n}{k}_q = \sum_{j=0}^{k(n-k)} a_j q^j = \sum_{j=0}^{k(n-k)} a_k(n(n-k)-j)q^j = \sum_{j=0}^{k(n-k)} a_j q^{k(n-k)-j}
\]

\[
= q^{k(n-k)} \sum_{j=0}^{k(n-k)} a_j q^{-j} = q^{k(n-k)} \binom{n}{k} q^{-1} \tag{2.13}
\]

The theorems above leave all q-binomial coefficients self-reciprocal.

For theorem 2.2

\[
q^{k(n-k)} \binom{n}{k} q^{-1} = (-1)^k q^{nk-k(k-1)/2} q^{-k(n+1)} \binom{-n+k-1}{k} q^{-1} \tag{2.14}
\]

and because \(-k(n-k) + nk - k(k-1)/2 - k(n+1) = -(nk - k(k-1)/2)\):

\[
\binom{n}{k} q^{-1} = (-1)^k q^{-(nk-k(k-1)/2)} \binom{-n+k-1}{k} q^{-1} \tag{2.15}
\]

For theorem 2.3

\[
q^{k(n-k)} \binom{n}{k} q^{-1} = (-1)^{n-k} q^{(n-k)(n+k+1)/2} q^{-(n-k)(n+1)} \binom{-k-1}{n-k} q^{-1} \tag{2.16}
\]

and because \(-k(n-k) + (n-k)(n+k+1)/2 - (n-k)(n+1) = -(n-k)(n+k+1)/2\):

\[
\binom{n}{k} q^{-1} = (-1)^{n-k} q^{-(n-k)(n+k+1)/2} \binom{-k-1}{n-k} q^{-1} \tag{2.17}
\]
3 Some q-Binomial Summation Identities

Some q-binomial summation identities are derived and it is shown how q-binomial coefficients with negative arguments transform one summation identity into another. The following identities are the q-binomial theorem and the q-binomial theorem for negative powers [10]:

\[
\prod_{k=0}^{n-1} (1 + xq^k) = \sum_{k=0}^{n} q^k(n)_k \frac{x^k}{q} \tag{3.1}
\]

\[
\frac{1}{\prod_{k=0}^{n-1} (1 - xq^k)} = \sum_{k=0}^{\infty} (n + k - 1) \frac{x^k}{n - 1} \tag{3.2}
\]

The following is an obvious product rule:

\[
\prod_{k=0}^{a-1} (1 + xq^k) \prod_{k=0}^{b-1} (1 + xq^{a+k}) = \prod_{k=0}^{a+b-1} (1 + xq^k) \tag{3.3}
\]

With these three identities some q-binomial summation identities are derived, using that the coefficients of a product of two polynomials are the convolutions of the coefficients of the two polynomials.

**Theorem 3.1.** The q-analog of the Chu-Vandermonde identity [7]:

\[
\sum_{k=0}^{n} q^{(a-k)(n-k)} \left( \begin{array}{c}
a \\
k
\end{array} \right)_q \left( \begin{array}{c}
b \\
k
\end{array} \right)_q = \left( \begin{array}{c}
a + b \\
n
\end{array} \right)_q \tag{3.4}
\]

**Proof.** Using (3.1) with (3.3):

\[
\left( \sum_{k=0}^{a} q^k \left( \begin{array}{c}
a \\
k
\end{array} \right)_q x^k \right) \left( \sum_{k=0}^{b} q^k \left( \begin{array}{c}
b \\
k
\end{array} \right)_q q^{ak} x^k \right) = \sum_{k=0}^{a+b-1} q^k \left( \begin{array}{c}
a + b \\
k
\end{array} \right)_q x^k \tag{3.5}
\]

The coefficients of both sides must be equal:

\[
\sum_{k=0}^{n} q^k \left( \begin{array}{c}
a \\
k
\end{array} \right)_q q \left( \begin{array}{c}
(n-k) \\
k
\end{array} \right)_q \left( \begin{array}{c}
b \\
k
\end{array} \right)_q q^{ak} = q^k \left( \begin{array}{c}
a + b \\
n
\end{array} \right)_q \tag{3.6}
\]

Because:

\[
\left( \begin{array}{c}
k \\
2
\end{array} \right) + \left( \begin{array}{c}
(n-k) \\
2
\end{array} \right) + a(n-k) - \left( \begin{array}{c}
n \\
2
\end{array} \right) = (a-k)(n-k) \tag{3.7}
\]

the theorem is proved. By replacing \( k \) with \( n - k \) and interchanging \( a \) and \( b \), the power of \( q \) in the summand can be replaced by \( q^{(b-n+k)k} \).

**Theorem 3.2.**

\[
\sum_{k=0}^{n} q^{(b+1)k} \left( \begin{array}{c}
a + k \\
aka
\end{array} \right)_q \left( \begin{array}{c}
b + n - k \\
bb
\end{array} \right)_q = \left( \begin{array}{c}
n + a + b + 1 \\
n
\end{array} \right)_q \tag{3.8}
\]
Proof. Using the reciprocal of (3.3) with $-x$:

$$\frac{1}{\prod_{k=0}^{a-1}(1-xq^k)} \cdot \frac{1}{\prod_{k=0}^{b-1}(1-xq^{a+k})} = \frac{1}{\prod_{k=0}^{a+b-1}(1-xq^k)} \tag{3.9}$$

which with (3.2) becomes:

$$\left(\sum_{k=0}^{\infty} \binom{a+k-1}{a-1} x^k\right) \cdot \left(\sum_{k=0}^{\infty} \binom{b+k-1}{b-1} q^{ak} x^k\right) = \sum_{k=0}^{\infty} \binom{a+b+k-1}{a+b-1} q^k x^k \tag{3.10}$$

The coefficients of both sides must be equal:

$$\sum_{k=0}^{n} \binom{a+k-1}{a-1} q^k \binom{b+n-k-1}{b-1} q^{a(n-k)} = \binom{a+b+n-1}{a+b-1} q^n = \binom{n+a+b-1}{n} q \tag{3.11}$$

Replacing $a$ by $a+1$ and $b$ by $b+1$, and then replacing $k$ by $n-k$ and interchanging $a$ and $b$ gives the theorem. By replacing $k$ with $n-k$ and interchanging $a$ and $b$, the power of $q$ in the summand can be replaced by $q^{(a+1)(n-k)}$. \qed

Theorem 3.3.

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{a}{k} \binom{b+n-k}{b} q^a = \begin{cases} q^{an}(n-a+b) & \text{if } a \leq b \\ (-1)^n q^{bn+n(n+1)/2}(a-b-1) & \text{if } a > b \end{cases} \tag{3.12}$$

Proof. From (3.3) replacing $a$ with $b$ and $b$ with $a-b$ gives:

$$\frac{1}{\prod_{k=0}^{b-1}(1-xq^k)} = \frac{a-b-1}{a-1} \prod_{k=0}^{a+b-1}(1 + xq^{b+k})$$

Therefore:

$$\left(\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{a}{k} \binom{b+n-k}{b} q^b \right) \cdot \sum_{k=0}^{\infty} \binom{b+k-1}{b-1} (-1)^k x^k = \sum_{k=0}^{a-b} q^{\binom{k}{2}} \binom{a-b}{k} q^b x^k \tag{3.14}$$

The coefficients of both sides must be equal:

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \binom{a}{k} \binom{b+n-k-1}{b-1} q^{bn+(n-k)} = q^{bn+n(n+1)/2} \binom{a-b}{n} \tag{3.15}$$

Replacing $b$ by $b+1$ gives the second case of the theorem. When $a \leq b$ application of theorem 2.2 to the right side of the second case gives the first case of the theorem. \qed

The special cases $b = a - 1$ and $a = n$, $b = 0$ of this theorem appear in exercise 3.9 in [7].
4 Transforming q-Binomial Summation Identities

From theorem 3.2 and theorem 2.3 the following theorem follows.

**Theorem 4.1.**

\[
\sum_{k=0}^{n} (-1)^k q^{(a-b)(n-k)+k(k+1)/2} \binom{a}{k} q^{b+n-k} = \begin{cases} 
q^{(a-b)n} \binom{n-a+b}{n} q^{n} & \text{if } a \leq b \\
(-1)^n q^{n(n+1)/2} \binom{a-b-1}{n} q & \text{if } a > b 
\end{cases}
\]  

(4.1)

**Proof.** In theorem 3.2 replacing \( a \) by \( -a \) and using theorem 2.3:

\[
\binom{-a+k}{-a} q = (-1)^k q^{ak+k(k+1)/2} \binom{a-1}{k} q
\]

(4.2)

and replacing \( a \) by \( a+1 \) gives the first case of this theorem. For the second case of the theorem, using theorem 2.2 when \( a > b \):

\[
\binom{n-a+b}{n} q = (-1)^n q^{n-a+b-n(n-1)/2} \binom{a-b-1}{n} q
\]

(4.3)

gives the second case of the theorem. This theorem is identical to theorem 3.3 except for the power of \( q \) in the summand.

Some summation identities from the previous section can be transformed into each other. Theorem 3.3 can be transformed into theorem 3.2 using theorem 2.2 by replacing \( b \) with \( -b \):

\[
\binom{-b+n-k}{-b} q = (-1)^{n-k} q^{(n-k)(-2b+n-k+1)/2} \binom{b-1}{n-k} q
\]

(4.6)

which with the second case of theorem 3.3 by replacing \( b \) by \( b+1 \) and using \( k(k-1)/2 + (n-k)(-2(b+1)+n-k+1)/2 + bn - n(n-1)/2 = k(b-n+k) \) gives:

\[
\sum_{k=0}^{n} q^{k(b-n+k)} \binom{a}{k} q^{b+n-k} = \binom{a+b}{n} q
\]

(4.7)

Replacing \( k \) by \( n-k \) and interchanging \( a \) and \( b \) gives theorem 3.1.
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