Type I parametric down conversion of highly focused Gaussian beams in finite length crystals

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Abstract
This paper presents a study of the correlations in wave vector space of photon pairs generated by type I spontaneous parametric down conversion using a Gaussian pump beam. The analysis covers both moderate focused and highly focused regimes, paying special attention to the angular spectrum and the conditional angular spectrum. Simple analytic expressions are derived that allow a detailed study of the dependence of these spectra on the waist of the source and the length of the nonlinear crystal. These expressions are in good agreement with numerical expectations and reported experimental results. They are used to make a systematic search of optimization parameters that improve the feasibility of using highly focused Gaussian beams to generate idler and signal photons with predetermined mean values and spread of their transverse wave vectors.

Keywords: quantum optics, parametric down conversion, gaussian beam
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1. Introduction
The process of spontaneous parametric down conversion (SPDC) [1]- when a pump beam interacts in a non-linear crystal and individual photons of the pump decay in a pair of photons- is one of the most reliable sources of photons with predetermined properties [2]. In particular, the two-photon state may exhibit spatial entanglement [3–5] that can be manipulated via the modification of the spatial structure of the pump beam and/or the crystal length. Varying them, it is even possible to optimize the coupling efficiency to other optical elements like optical fibers [6–12].

Expressions for the main properties of the emission cone and their dependence on the crystal length have been the subject of intense research because of their potential use in the quantum engineering of SPDC [3, 13–19]. In this work, we study the spatial properties of the emitted photons via the angular spectrum (AS) and the conditional angular spectrum (CAS). Using the exact expressions for the birefringent dispersion relations, we numerically evaluate these spectra in both the moderate focused and the highly focused regimes of a Gaussian pump beam. When reliable approximations for the phase matching function are used, we also obtain simple analytical expressions that reproduce accurately the numerical expectations and reported experimental results; they allow a better understanding of the effects of the pump beam waist and crystal length in both the AS and CAS functions. We also study the use of these expressions for optimization of the SPDC process in the generation of idler and signal photons localized in predetermined regions of the wave vector space.

2. Theory
To first order in perturbation theory, the SPDC state is given by [2]:

$$|\Psi\rangle = |\text{vac}\rangle + \int d\omega_1 d\omega_2 d^2k_u d^2k_s F_{u,s} |\alpha; 1; 1\rangle$$

(1)
where $F_i = F(k_{i,p}, \omega_i, k_{i,s}, \omega_i)$ represents the joint amplitude expressed as a function of the frequency $\omega_i$ and the transversal wave vectors $k_{i,p}$ for the signal and idler modes $(b = s, i)$. It has the following structure:

$$F(k_{i,p}, \omega_i; k_{i,s}, \omega_i) = g \mathcal{E}(k_{i,p}, \omega_i)f(k_{i,s}, \omega_i; k_{i,s}, \omega_i)$$  \hspace{1cm} (2)

with $g$ the product of the effective second order susceptibility and the normalization factors of the quantized pump, signal, and idler photon states. Here, we take $\mathcal{E}$ as the amplitude function of a Gaussian coherent pump beam,

$$\mathcal{E}(k_{i,p}, \omega_i) \equiv \alpha(\omega_i) \cdot e^{-(i, x^2/2c^2)} e^{-(i, y^2/2c^2)} e^{-(i, z^2/2c^2)}$$  \hspace{1cm} (3)

with $\alpha(\omega_i)$ its spectral amplitude; $W_x$ and $W_y$ the values of waist pump along $x$-axis and $y$-axis respectively; $z_0$ determines the focal plane; and $z_0$ are the Rayleigh lengths along $x$-axis and $y$-axis. Energy conservation implies $\omega_p = \omega_s + \omega_i$. For a wide crystal, the conservation of transversal momenta yields the condition $k_{i,p} = k_{i,s} + k_{i,i}$. In equation (2), $f$ corresponds to the phase matching function:

$$f(k_{i,s}, \omega_i; k_{i,i}, \omega_i) = L \text{sinc}(L \Delta k_i/2) e^{ik_i^2/2}$$  \hspace{1cm} (4)

where $L$ is the crystal length and the mismatch term is $\Delta k_i = k_{i,p} - k_{i,s} - k_{i,i}$.

In this paper, we concentrate on type-I SPDC and degenerated emission, i.e., $\omega_s = \omega = 2\omega_i = 2\omega_p$. Thus, the emitted photons have ordinary polarization and their dispersion relation is

$$k_{i,s}^2 = \sqrt{c_0^2 \alpha_i}^2 4c^2 - k_{i,i}^2 \quad b = i, s$$  \hspace{1cm} (5)

with $c$ the velocity of light in vacuum; $c_0$ is the permeability function transversal to the ordinary plane defined by the optical axis $a = (0, a_x, a_y)$; the ordinary refraction index is $n_a = \sqrt{c_0}$. The permeability coefficient parallel to the optical axis $c_0$ yields the dispersion relation for the pump wave that evolves in the extraordinary plane:

$$k_{i,s}^2(k_{i,i}, \omega) = -\beta a \cdot k_{i,i} + \frac{\omega}{c} n_{i,s} \sqrt{1 - \frac{k_{i,i}^2}{\alpha_i^2} \eta}$$  \hspace{1cm} (6)

$$n_{i,s} = \sqrt{\frac{\epsilon_0 \epsilon_s}{\epsilon_0 + \epsilon_s + \Delta\epsilon_s}}$$  \hspace{1cm} (7)

$$\eta = \frac{1}{\epsilon_s + \Delta\epsilon_s}$$  \hspace{1cm} (8)

$\Delta\epsilon = \epsilon_0 - \epsilon_s$. The $\beta$ term is responsible for the deviation of the Poynting vector with respect to the pump wavefront inside the crystal, the so-called walk-off effect [22–25]. The $\eta$ term gives rise to astigmatic effects [24]. In the limit of normal incidence, this equation reduces to the expression of the effective refractive index experienced by a paraxial pump extraordinary wave, $k_{i,s}^2(k_{i} \sim 0) = n_{i,s} \omega/c$. The permeability coefficients $\epsilon_0$ and $\epsilon_s$ depend on the light frequency and some experimental set-up is implemented assuming $n_{\text{eff}}(\lambda) = n_{\text{eff}}(2\lambda)$. We shall consider the general case where this equation is not taken as guaranteed.

The distribution of the signal/idler photons in the wave vector domain is the AS:

$$R_s(k_{i,s}, \omega_s; k_{i,i}, \omega_i) = \int d^3k_{i,i} |F(k_{i,s}, \omega_s; k_{i,i}, \omega_i)|^2$$  \hspace{1cm} (9)

The CAS, which is a function of $k_{i,s}$ and $k_{i,i}$, is defined as:

$$R_c(k_{i,s}, \omega_i; k_{i,i}, \omega_i) = \int d^3k_{i,i} |F(k_{i,s}, \omega_s; k_{i,i}, \omega_i)|^2$$  \hspace{1cm} (10)

This function represents the probability to detect a signal photon with wave vector $k_{i,s}$ in coincidence with an idler photon with wave vector $k_{i,i}$. In the paraxial regime ($W_{k_{i,p}} \gg 1$) the pump beam can be approximated by a plane wave, and, if we consider a long crystal, the SPDC process results in a strict momentum conservation that correlates the observation of an individual signal photon with $k$-vector to a single idler photon with a well defined $k$-vector. In experimental situations involving the small but finite transverse dimensions of the pump beam and a standard wide but less long crystal, there is a set of relevant pump wave vectors $\{k_{i,p}\}$ that come close to satisfying the phase-matching condition $\Delta k_i = 0$ for given idler and signal wave vectors $k_{i,i}$. Notice that $R_s$ results are independent of the phase factor that contains the information about the Rayleigh lengths; that is, $R_s$ depends on just the amplitude of the Fourier content of the pump beam.

In order to obtain approximate expressions for the AS and CAS functions, we make a first-order Taylor description of the phase mismatch,

$$\Delta k_i \sim \kappa - d \cdot (k_{i,s} + k_{i,i})$$  \hspace{1cm} (11)

$$\kappa = (\omega/c)(n_{\text{eff}} - n_s) + (2c/n_s)k_{i,s}^2$$  \hspace{1cm} (12)

$$d = \beta a + (2c/n_s)k_{i,i}$$  \hspace{1cm} (13)

Note that in the absence of walk-off effects $d$ would point in the $k_{i,i}$ direction. Approximating the function $\text{sinc}(x)$ by a Gaussian function $\exp[-(\gamma x)^2]$, $\gamma = 0.4393$, the expression for the CAS becomes:

$$R_c(k_{i,s}, \omega_s; k_{i,i}, \omega_i) = \left|g L \alpha (\omega_s + \omega_i)\right|^2 \times e^{-\sigma^2/2} e^{-\sigma^2/2} e^{-\gamma^2/2} e^{-\gamma^2/2} \times e^{-\gamma^2/2} \times e^{-\sigma^2/2}$$

$$\alpha^2 = (W_s^2 + \gamma^2 L^2 d_i^2)/2$$  \hspace{1cm} (14)

The effective width of the Gaussian factor that modulates the CAS function has two contributions, one due solely to the pump geometry and another highly dependent on the crystal optics and geometry. The latter includes a term $\sim L (\Delta k_i a / (\epsilon_s + \Delta\epsilon_s^2))^2$. In the paraxial regime, the CAS
function will be determined by the geometry of the Gaussian pump beam whenever \( W_p > L \left( \Delta \epsilon \cdot a_o / \left( \epsilon_0 + \Delta \epsilon / a_o \right) \right) \). In fact, for long crystals, second-order terms proportional to \( k_j^2 \), in the extraordinary dispersion relation, equation (6), could become relevant. In such a case, a third term contributes to the effective width of the Gaussian pump beam:

\[
W_b^2 = W_p^2 + \frac{\gamma \mathcal{L} n_s^2}{\epsilon_0 + \Delta \epsilon / a_o} \left( \frac{n_{gr} - n_{gr}}{n_{gr}} \right), \quad b = x, y
\]

replaces \( W_p^2 \) in equation (14). In general, for a symmetric pump beam \( W_p = W_s = W \), the CAS function may exhibit an asymmetric profile due to the orientation of the birefringent axis as encoded in \( \mathbf{d} \).

We shall now illustrate the CAS function by: (i) choosing the value of the transversal wave vector of the idler photon \( k_{id} \) that maximizes the counts; and (ii) reporting the corresponding distribution of the signal transverse wave vector, \( k_{s},. \) Our parameters come from particular reported experimental situations [19]. This allows us to confirm the reliability of our numerical simulations. Thus we consider a 1 mm length BBO crystal, cut for type-I phase matching for degenerated emission with angle \( \theta_d = 29.3^\circ \) (the optical axis is defined by \( \mathbf{a} = (0, \sin \theta_d, \cos \theta_d) \)); the quasi monochromatic pump beam has a wavelength centered at 406.99 nm. Calculations are performed both by using the exact expressions for the birefringent dispersion relations for the ordinary wave, equation (5), and extraordinary wave, equation (6), as well as by using the approximate analytical equation (14).

Figure 1(a)(A) shows the numerical CAS function for a symmetric pump waist \( W_p = W_s = W = 185 \mu m \). As expected from equation (14), for a pump waist that validates the paraxial regime and \( W \gg \mathcal{L} |\mathbf{d}| \), the CAS describes highly localized momentum correlations determined mainly by the pump waist \( W \). Figure 1(a)(B) illustrates the CAS function calculated from the approximate equation (14). In order to make a clearer comparison between numeric and analytic calculations, figures 1(a)(C) and (D) show the marginal distributions \( M(k_{s,j}) = \int_{\mathbb{R}^2} R(k_{s,j}, \mathbf{a}_j; k_{id}^j, \mathbf{a}_j^0) \) for \( b, j = x, y \). The resulting agreement between the numerical and analytic calculations is also very good.

In figure 1(b)(A), the CAS is illustrated for a symmetric focused pump beam \( W_p = 35 \mu m \). Notice that the spread of the marginal distributions is, in this case, about twice that observed for a pump beam with \( W = 185 \mu m \). This is a direct consequence of the fact that higher focusing necessarily involves the incorporation of a wider distribution of pump wave vectors.

For a highly focused beam, as that shown in figure 1(c) where \( W = 5 \mu m \), the CAS function profile spreads even more and the conditional angular spectra becomes more sensitive to the crystal length and optical parameters as described by the analytical expression, equation (14). Notice that, although this approximate distribution does not reproduce in all details its numeric analog, it has the correct central point and width. The discrepancy is mainly due to the use of the approximate dispersion relation as described above. Another effect of the crystal is present in the small oscillations in the marginal distribution \( M(k_{s,j}) \) in figure 1(c)(D), which were lost in the replacement of the phase matching sinc by a Gauss function. Moreover, for this highly focused pump beam there is an observable difference between the absolute mean values of the y-component of the wave vector of the idler photon \( k_{id} = -0.51 \mu m^{-1} \) and those of the signal photon \( \langle k_{s,j} \rangle \approx 0.57 \mu m^{-1} \). The CAS for \( W = 185 \mu m \) and \( W = 35 \mu m \) can be successfully compared with figure 2 of [19].

In order to illustrate other interesting CAS features, we now consider the particular case in which the condition \( k_{s,j} = k_{id} = 0 \) is imposed in the idler and signal photons, and compare it to the case \( k_{s,j} = k_{id} = 0 \). This results, in a conditional distribution for \( k_{s,j} \) and \( k_{id} \) and a conditional distribution for \( k_{id} \). Each CAS is illustrated in figure 2, taking similar parameters as those used in previous examples. For \( W = 185 \mu m \), the signal and idler transverse wave vectors are restricted to almost opposite directions, so that, for \( k_{s,j} = k_{id} = 0 \), \( k_{s,j} \approx -k_{id} \) with \( \langle k_{id} \rangle \approx 0.5 \mu m^{-1} \). As the pump beam waist decreases, the anisotropic role of higher values of \( k_{id} \) in the phase-matching condition is more relevant. This has two consequences. One is the increasing width on the CAS distributions; the other that such distributions become clearly different for the condition \( k_{s,j} = k_{id} = 0 \) and for the condition \( k_{s,j} = k_{id} = 0 \).

An approximate closed expression for AS distribution can also be obtained. For simplicity, we consider a symmetric pump profile \( W_p = W_s = W \). Using the conservation of transverse momentum, writing the pump integration variable \( k_{id} \) in polar coordinates, and performing a rotation of the integration variable by an angle \( \theta_j = \arccos \left( d_j / |\mathbf{d}| \right) \), we obtain:

\[
R_j(k) = |gL| e^{-|k|^2 / (2 |\mathbf{d}|^2)} \int _0^{2\pi} d\theta \int _0^{\pi} dk |k| e^{-u}
\]

where

\[
\begin{align*}
\left( \begin{array}{c}
\sigma_{AS}^2 \\tilde{k}_s \\
\tilde{k}_i \\
\end{array} \right) &= \left( \begin{array}{c}
\frac{\omega_0^2 (c \epsilon_0^2)}{c^2} \left( 1 - \frac{n_{gr}}{n_{gr}} \right), \quad \sigma_{AS}^2 = \frac{2 |gL| n_{gr}}{1 + (gL |\mathbf{d}| / W)^2} \\
\frac{1}{c \sqrt{\epsilon_0}} \left[ c \sqrt{\epsilon_0} + \Delta \epsilon a_o / \cos \theta \right] \left( 1 + (gL |\mathbf{d}| / W)^2 \right) \\
\end{array} \right) \\
u &= \left( W^2 + (gL |\mathbf{d}|)^2 \right) (k \cos \theta - \zeta_\perp^2 + W k_\parallel^2 \sin^2 \theta) / 2.
\end{align*}
\]
Figure 1. Conditional angular spectra (CAS) for a 1 mm BBO crystal using a symmetric Gaussian pump with waist (a) \( W = 185 \mu m \); (b) \( W = 35 \mu m \); (c) \( W = 5 \mu m \). The transverse wave vectors of the corresponding idler photon are (a) \( k_x = 0.033 \mu m^{-1}, k_y = -0.485 \mu m^{-1} \); (b) \( k_x = 0.033 \mu m^{-1}, k_y = -0.485 \mu m^{-1} \); and (c) \( k_x = 0.000 \mu m^{-1}, k_y = -0.506 \mu m^{-1} \). These wave vectors were chosen so that the counts are maximized. In (A), CAS is calculated numerically, and in (B), analytically. Figures (C) and (D) are the marginal distributions calculated numerically (solid line) and analytically (dashed line).

Figure 2. In the first row the conditional spectrum that arise from demanding \( k_x = k_y = 0 \) is illustrated for pump waists (a) \( W = 185 \mu m \); (b) \( W = 35 \mu m \); (c) \( W = 5 \mu m \). In the second row, the conditional spectrum that arises from demanding \( k_x = k_y = 0 \) is illustrated for the same pump waists. The general parameters are the same as those reported in figure 1.
\[ R_s(k_{\text{w}}) = \left| g L \alpha (\omega_p) \right|^2 \frac{e^{-\sigma_2^2 \left( \frac{\theta_2^2}{\theta_p^2} \right)}}{2 \pi} \left[ \int_0^{2\pi} d\theta \frac{1 - e^{-u}}{\text{den} (\theta)} \right] + W_{\text{L}} \int_0^{2\pi} d\theta \cos \theta \frac{\cos \theta + W^2 \sin^2 \theta}{\text{den} (\theta)} \int_0^{\frac{\theta_2}{\theta_p}} dk e^{-\sigma_2^2 \left( \frac{\theta_2^2}{\theta_p^2} \right)} \right], \tag{17} \]

\text{den} (\theta) = \left( \frac{W + (\gamma L \alpha \theta_p)}{2} \right)^2 \cos^2 \theta + e^{-u} \theta_{\text{est}} \,
\text{den} (\theta) = \frac{1}{1 + (W L W)^2} \right竞争性。}

From the approximate analytical expression for the angular spectrum, we can study the marginal distributions on the \( k_{\text{w}} \) and \( k_{\text{w}} \) variables. Since the conditional angular spectrum shows that the idler and signal photons will be emitted nearby a cone with transversal radius \( r_{\text{AS}} \) defined in equation (16), it becomes interesting to analyze the relations between the angles of emission \( \theta_1 \) and \( \theta_2 \) of the idler and signal photons in that cone. We find it easier to make such an study in terms of the mean angle \( \theta_1 = (\theta_1 + \theta_2)/2 \) and the difference angle \( \theta_1 - \theta_2 \). The distribution
\[ \Re_{\text{AS}}(\theta_1, \theta_2) = \left| F \left( k_{\text{w}}, \omega_i; k_{\text{w}}, \omega_s \right) \right|^2 / \left| g L \alpha (\omega_p) \right|^2, \]
\[ k_{\text{w}} = r_{\text{AS}} \left( \cos \left( \theta_1 + \theta_2 / 2 \right), \sin \left( \theta_1 + \theta_2 / 2 \right) \right), \]
\[ k_{\text{w}} = r_{\text{AS}} \left( \cos \left( \theta_1 - \theta_2 / 2 \right), \sin \left( \theta_1 - \theta_2 / 2 \right) \right), \tag{19} \]

is illustrated for the pump beam waists \( W = 185 \mu m, W = 35 \mu m, \) and \( W = 5 \mu m \) in figure 4, taking the same general parameters used in the simulations reported before; in particular the crystal cut angle is \( \theta_c = 29.3^\circ \) yielding \( r_{\text{AS}} = 0.492 \mu m^{-1} \). The exact dispersion relations for the pump, idler, and signal photons were used. The calculation results coincide with those obtained using the approximate dispersion relations for \( W > 25 \mu m \) within the first three significant figures. Notice that the idler and signal photons are emitted mainly in opposite transverse directions since the maximum value of \( \Re_{\text{AS}}(\theta_1, \theta_2) \sim 1 \) is always taken for \( \theta_1 = 180^\circ \). However, because for \( W < 25 \mu m \) the width of the marginal distribution of \( \theta_1 \) increases as the waist decreases, there is a high possibility that the signal and idler photons are emitted with \( \theta_1 \in \{175^\circ, 185^\circ\} \). It can also be observed that the emission cone will have an almost isotropic distribution of photons for \( W > 100 \mu m \), since for those values of the pump waist, \( \Re_{\text{AS}} \) shows only a slight dependence on \( \theta_1 \). For lower values of \( W \), the largest values of the marginal distribution of \( \theta_2 \) are achieved for \( \theta_1 \in \{0^\circ, \pm \pi/2\} \). If \( \theta_1 \sim \pi \), this means that \( \theta_1 \sim \pm \pi/2 \), which is consistent with all the results for the CAS shown in last section.
The approximate analytical expression for the CAS distribution, equation (18), can be used to obtain approximate expressions for the mean aperture angle of the emission cone

\[ \Theta_{\omega} = -\sin \left( r_{\text{AS}} \right). \]

The spread in the distribution of this angle is determined by the spread in the distribution of the cone radius \( \sigma_{r_{\text{AS}}}^{2} \), given in equation (16), and it depends on the direction of the wave vectors \( k_{x} \). Its maximum (minimum) value is achieved for signal photons emitted at angles that maximize (minimize) the magnitude of the \( d \) vector. For a signal photon with \( |k| \sim r_{\text{AS}} \), these extreme values are given by

**Figure 3.** Angular spectrum (AS) for a Gaussian pump with waist (a) \( W = 185 \mu \text{m} \); (b) \( W = 35 \mu \text{m} \); (c) \( W = 5 \mu \text{m} \). In (A), the AS function is calculated numerically and, in (B), using the approximate analytical expression. (C) and (D) are the marginal distributions obtained numerically (solid line) and analytically (dashed line).

**Figure 4.** Angular correlation function \( R_{\theta_{\omega}}(\theta_{+}, \theta_{-}) \) for a Gaussian pump with waist (A) \( W = 185 \mu \text{m} \); (B) \( W = 35 \mu \text{m} \); (C) \( W = 5 \mu \text{m} \). (D) and (E) are the marginal distributions normalized to a maximum value 1 for the same waists of the pump beam. These distributions were obtained using the exact expressions of the dispersion relations and taking the same general parameters as in previous figures.
\[|d_{\text{eff}}| \sim \beta \cos \alpha \mp (2c/n_\alpha) r_{\text{AS}}\]
The corresponding extreme values of the aperture angle are
\[\Theta_{\text{AS}}^{\text{eff}} = \left( \Theta_{\text{AS}}^\mu + \Delta \Theta_{\text{AS}}^{\text{eff}} \right)\]
\[\Delta \Theta_{\text{AS}}^{\text{eff}} \sim \frac{1 + (\gamma L |d_{\text{eff}}| W)}{2^{5/4} r_{\text{AS}} \left( \sin \alpha / 2c \right)}.\] (20)

For the general parameters used up to now, \(\Theta_{\text{AS}}^\mu = 0.038\) rad and for \(W = 185\) μm the spread \(\Delta \Theta_{\text{AS}}^{\text{max}} = 0.006\) rad \(\sim \Delta \Theta_{\text{AS}}^{\text{min}}\), for \(W = 35\) μm the maximum spread increases to \(\Theta_{\text{AS}}^{\text{eff}} = 0.009\) rad and the minimum \(\Theta_{\text{AS}}^{\text{min}} = 0.007\) rad, and for the extreme condition \(W = 5\) μm, \(\Delta \Theta_{\text{AS}}^{\text{max}} = 0.022\) rad and \(\Delta \Theta_{\text{AS}}^{\text{min}} = 0.017\) rad.

### 4. SPDC optimization

It has been shown that SPDC with extremely focused Gaussian beams generally gives rise to an anisotropic distribution of photon pairs in an emission cone that has a mean radius independent of the pump waist but a spread that may increase as the pump waist diminishes. This may make the separation of the idler and signal effective modes difficult. The SPDC simulations reported in previous sections were worked out with parameters chosen to optimize a type I SPDC process for an ideal incoming plane wave. In this section, we show that parameters such as the cut angle of the nonlinear crystal or the wavelength of the pump beam can be easily optimized to increase the emission cone radius, preserve approximately the spread of the cone radius, and simultaneously preserve the anisotropic localization of the idler and signal wave vectors without a significant loss in the probability of coincidence detection of the emitted photons.

In standard experimental set-ups, the crystal cut angle is chosen so that an ideal pump beam with close to a plane wave front yields the possibility of creating almost collinear photon pairs with maximum probability. By increasing the cut angle, the effective extraordinary refraction index \(n_{\text{eff}}\) decreases and the mean radius of the emission cone \(r_{\text{AS}}\) increases. Thus, it is reasonable to study the behavior of SPDC for higher values of \(\theta_{\alpha}\) to decrease the overlap between the idler and signal effective modes. Notice however that \(d_{\text{eff}}\) also depends on \(\theta_{\alpha}\). We are interested in finding optimal values of \(\theta_{\alpha}\) that do not increase the width of the emission cone beyond the adequate values for the apertures of usual detecting elements. In most cases, the latter include optical fibers to which the signal and idler photons are required to couple.

Using the formalism presented in the previous section, the SPDC processes were roughly simulated and optimal angles were easily found. In figure (5), we report the behavior of the \(\Pi_{\text{AS}}(\theta_{\alpha}, \theta_{\beta})\) and its marginals for a pump beam with \(W = 5\) μm for \(\theta_{\alpha} = 29.3^\circ, 31.0^\circ,\) and \(33.0^\circ\). The mean emission cone radii are \(r_{\text{AS}}^{29.3^\circ} = 0.955\) μm\(^{-1}\) and \(r_{\text{AS}}^{33.0^\circ} = 1.313\) μm\(^{-1}\), so that \(\Theta_{\text{AS}}^{33.0^\circ} = 0.075\) rad and \(\Theta_{\text{AS}}^{31.0^\circ} = 0.103\) rad. The angular maximum and minimum widths are \(\Delta \Theta_{\text{AS}}^{\text{max},31.0^\circ} = 0.024\) rad and \(\Delta \Theta_{\text{AS}}^{\text{min},31.0^\circ} = 0.013\) rad, and \(\Delta \Theta_{\text{AS}}^{33.0^\circ} = 0.026\) rad and \(\Delta \Theta_{\text{AS}}^{\text{min},33.0^\circ} = 0.009\) rad. Hence, increasing the cut angle \(\Theta_{\alpha}\) slightly can take two or three times its original value while the maximum value of \(\Delta \Theta_{\text{AS}}\) is modified by less than 15%. Notice that the change in the maximum value of the marginal distributions evaluated at \(r_{\text{AS}}\) change by less than 15%.

Taking into account those results, we have performed a complete simulation of the SPDC process for the pump beam with \(W = 5\) μm and a cut angle \(\theta_{\alpha} = 33^\circ\). The corresponding AS and CAS are shown in figure (6). As predicted from the simulations described in the last paragraph, the spatial resolution of the idler and signal photons has greatly increased, whereas the location of the emitted photons is still concentrated around \(\theta_{\beta} = \pm \pi/2\). The numerical evaluation of the brightness calculated from the CAS function shows that it increases by 10% for the cut angle \(\theta_{\alpha} = 33^\circ\), compared to \(\theta_{\alpha} = 29.3^\circ\). Moreover, the rough estimates for the cone angle and its width given in the previous paragraph agree approximately with the exact numerical calculation.
We have also studied the possibility of maintaining the original cut angle $\theta_c = 29.3^\circ$ and optimizing the pump wavelength for highly focused Gaussian beams. The first calculation using the expression for $r_{AS}$ and $d$ shows that changing the pump wavelength from $\mu_4 = 0.406.99 \, \mu m$ to $\mu_4 = 0.436 \, \mu m$ (483 $\mu m$) yields the same results as changing to $\theta_c = 31^\circ$ ($\theta_c = 33^\circ$).

5. Conclusion

We have presented a study of the dependence of the AS and the CAS on the waist pump and the length of a nonlinear crystal used to achieve type I SPDC. Simple analytic expressions were derived for these spectra, equations (14–16). They make evident the relevance of the factor $\gamma L/W$, which includes information about the walk-off, to understand the general features of these spectra. Notice that under the conditions studied the CAS and the AS are independent of the position of the focus plane of the pump beam. However, it could lead to observable effects in other properties of the down-converted photons, such as the purity of state. The analytical expressions reproduce, with high accuracy, the predictions of numerical methods that do not make use of the approximations either of the dispersions relations or those of the phase matching function. Even more important, the results presented here are in excellent agreement with experimental measurements reported in the literature.

Having an analytical expression for the CAS allows the identification of explicit expressions for the mean traverse radii and the anisotropic width of the emission cone in the wave vector space. The first depends just on the linear dielectric response and the cut angle of the birefringent crystal. The anisotropy depends on the relation between the crystal length and beam waist through the factor $\gamma L/W$. Given a crystal length, a highly anisotropic SPDC process for extreme focused Gaussian pump beams is especially relevant since it allows the prediction of the direction at which idler and signal photons will be generated. This anisotropy has been already reported by Fedorov et al in [20, 21], where special care is taken of the theoretical and experimental analysis of the CAS and the AS along two lines; one of which is contained in a plane parallel to the optical axis; the other in a plane perpendicular to that axis. In [19], the anisotropy is also reported in terms of the AS and the CAS, so that the comparison between their experimental and our theoretical results could be directly performed.

We have also used our expressions to make estimates of the general properties of the distribution of idler and signal photons in wave vector space. They can be used to measure entanglement [20, 21] and for optimization of the SPDC process. That is, by simple calculations, which nevertheless include the exact expressions for $\Delta k = k_{id} - k_{ss} - k_{ii}$ evaluated on $r_{AS}$, the angular distribution of the emitted photons can be studied and conditions for a given angular localization can be found; using the expression of $\sigma_{SS}$ the maximum and minimum spread the idler and signal photons may be studied looking for values that could, e.g., optimize the coupling to optical fibers without a significant loss in the probability of coincidence detection of the emitted photons. Our analysis complements the studies reported in [6–12], providing a simple way to preselect the general experimental parameters in an optimization procedure. These parameters can then be used to make a detailed simulation of the process, previous to the experimental implementation.

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