DEFORMATIONS OF FUNCTIONS ON SURFACES
BY ISOTOPIC TO THE IDENTITY Diffeomorphisms

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Abstract. Let $M$ be a smooth compact connected surface, $P$ be either the real line $\mathbb{R}$ or the circle $S^1$, $f: M \to P$ be a smooth map, $\Gamma(f)$ be the Kronrod-Reeb graph of $f$, and $O(f)$ be the orbit of $f$ with respect to the right action of the group $D(M)$ of diffeomorphisms of $M$. Assume that at each of its critical point the map $f$ is equivalent to a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors. In a series of papers the author proved that $\pi_n O(f) = \pi_n M$ for $n \geq 3$, $\pi_2 O(f) = 0$, and that $\pi_1 O(f)$ contains a free abelian subgroup of finite index, however the information about $\pi_1 O(f)$ remains incomplete.

The present paper is devoted to study of the group $G(f)$ of automorphisms of $\Gamma(f)$ induced by the isotopic to the identity diffeomorphisms of $M$. It is shown that if $M$ is orientable and distinct from sphere and torus, then $G(f)$ can be obtained from some finite cyclic groups by operations of finite direct products and wreath products from the top. As an application we obtain that $\pi_1 O(f)$ is solvable.

1. Introduction

Study of groups of automorphisms of discrete structures has a long history. One of the first general results was obtained by A. Cayley (1854) and claims that every finite group $G$ of order $n$ is a subgroup of the permutation group of a set consisting of $n$ elements, see also E. Nummela [35] for extension this fact to topological groups and discussions. C. Jordan [14] (1869) described the structure of groups of automorphisms of finite trees and R. Frucht [11] (1939) shown that every finite group can also be realized as a group of symmetries of certain finite graph.

Given a closed compact surface $M$ endowed with a cellular decomposition $\Xi$, (e.g. with a triangulation) one can consider the group of “combinatorial” automorphisms of $M$. More precisely, say that a homeomorphism $h: M \to M$ is cellular or $\Xi$-homeomorphism, if it maps $i$-cells to $i$-cells, and $h$ is $\Xi$-trivial if it preserves every cell and its orientation. Then the group of combinatorial automorphisms of $\Xi$ is the group of $\Xi$-homeomorphisms modulo $\Xi$-trivial ones. Denote this group by $\text{Aut}(\Xi)$. It was proved by R. Cori and A. Machi [7] and J. Širáň and M. Škoviera [39] that every finite group is isomorphic with $\text{Aut}(\Xi)$ for some cellular decomposition of some surface which can be taken equally either orientable or non-orientable.

Notice that the 1-skeleton $M^{(1)}$ of $\Xi$ can be regarded as a graph. Suppose each vertex $M^{(1)}$ has even degree. Then in many cases (e.g. when $M$ is orientable) one can construct a smooth function $f: M \to \mathbb{R}$ such that $M^{(1)}$ is a critical level containing all saddles.
(i.e. critical points being not local extremes), and the group Aut(Ξ) can be regarded as the group of “combinatorial symmetries” of f.

Such a point of view was motivated by work of A. Fomenko on classification of Hamiltonian systems, see [9, 10]. The group Aut(Ξ) is called the group of symmetries of an “atom” of f. The group of symmetries of Morse functions f was studied by A. Fomenko and A. Bolsinov [8, A. Oshemkov and Yu. Brailov [4], Yu. Brailov and E. Kudryavtseva [5], A. A. Kadubovsky and A. V. Klimchuk [15], and A. Fomenko, E. Kudryavtseva and I. Nikonov [22].

In [26] the author gave sufficient conditions for a Ξ-homeomorphism to be Ξ-trivial, and in [29] estimated the number of invariant cells of a Ξ-homeomorphism, see Lemma 8.1.1.

It was proved by A. Fomenko and E. Kudryavtseva [20, 21] that every finite group is the group of combinatorial symmetries of some Morse function f having critical level containing all saddles.

In general, if f : M → R is an arbitrary smooth function having many critical values, then a certain part of its “combinatorial symmetries” is reflected by a so-called Kronrod-Reeb graph Γ(f), see e.g. [18, 2, 23, 19, 38].

The main result of the present paper describes the structure of the group of automorphisms of the Kronrod-Reeb graph Γ(f) induced by isotopic to the identity diffeomorphisms of M. It is based on the observation that the group of combinatorial automorphisms leaving invariant 2-cell is either cyclic or dihedral.

The paper continues the series of papers [26, 27, 28, 29, 31] by the author on computation of homotopy types of stabilizers, S(f), and orbits, O(f), of smooth functions on compact surfaces with respect to the right action of the group of diffeomorphisms of M. As an application we obtain that the fundamental group π1O(f) is solvable.

1.1. Preliminaries. Let M be a smooth compact connected surface and P be either the real line R or the circle S1. Consider a natural right action

\[ \gamma : C^\infty(M, P) \times D(M) \rightarrow C^\infty(M, P) \]

of the group of diffeomorphisms D(M) on M on the space C^\infty(M, P) defined by

\[ \gamma(f, h) = f \circ h, \]

for f ∈ C^\infty(M, P) and h ∈ D(M). Let also

\[ S(f) = \{ h \in D(M) \mid f \circ h = f \}, \quad O(f) = \{ f \circ h \mid h \in D(M) \} \]

be respectively the stabilizer and the orbit of f.

We will endow D(M) and C^\infty(M, P) with C^\infty Whitney topologies. These topologies induce certain topologies on the spaces S(f) and O(f). Denote by D_{id}(M) and S_{id}(f) the identity path components of D(M) and S(f) respectively.

Let f ∈ C^\infty(M, P), z ∈ M, and g : (R^2, 0) → (R, 0) be a germ of smooth function. Then f is said to be equivalent at z to g is there exist germs of diffeomorphisms h : (R^2, 0) → (M, z) and φ : (R, 0) → (P, f(z)) such that φ ∘ g = f ∘ h.

Since both R^1 and S^1 are orientable, one can say about local minimums and local maximums of f : M → P with respect to that orientation. Also for any a, b ∈ S^1 we will denote by [a, b] the arc in between a and b whose orientation from a to b agrees with orientation of S^1.
Denote by $C^\infty_\partial(M, P)$ the subset of $C^\infty(M, P)$ consisting of maps $f$ satisfying the following axiom:

**Axiom (B).** The map $f : M \to P$ takes a constant value at each connected component of $\partial M$, and the set $\Sigma_f$ of critical points of $f$ is contained in the interior $\text{Int} M$.

Let Morse$(M, P) \subset C^\infty_\partial(M, P)$ be a subset consisting of Morse maps, i.e. maps having only non-degenerate critical points. It is well known that Morse$(M, P)$ is open and everywhere dense in $C^\infty_\partial(M, P)$.

Let also $F(M, P)$ be the subset of $C^\infty_\partial(M, P)$ consisting of maps $f$ satisfying additional axiom:

**Axiom (L).** For every critical point $z$ of $f$ the germ of $f$ at $z$ is equivalent to some homogeneous polynomial $f_z : \mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

By Morse lemma a non-degenerate critical point of a function $f : M \to P$ is equivalent to a homogeneous polynomial $\pm x^2 \pm y^2$ which, evidently, has no multiple factors, and so satisfies (L). Hence

$$\text{Morse}(M, P) \subset F(M, P).$$

In [26, 27, 28, 29, 31] the author computed the homotopy types of $S(f)$ and $O(f)$ for all $f \in F(M, P)$.

It was proven that if $f \in F(M, P)$, then, except for few cases, $S_{id}(f)$ is contractible, $\pi_n O(f) = \pi_n M$ for $n \geq 3$ and $\pi_2 O(f) = 0$.

It was also established that the fundamental group $\pi_1 O(f)$ contains a normal subgroup isomorphic to $\pi_1 D_{id}(M) \oplus \mathbb{Z}^k$ for some $k \geq 0$, and that the factor group

$$G(f) = \pi_1 O(f) / (\pi_1 D_{id}(M) \oplus \mathbb{Z}^k)$$

is finite.

In a typical situation, when $f$ is generic, i.e. a Morse map taking distinct values at distinct critical points, the group $G(f)$ is trivial and $k$ can be easily expressed via the number of critical points of $f$, [26, Theorem 1.5]. In particular, we get an isomorphism $\pi_1 O(f) \cong \pi_1 D(M) \oplus \mathbb{Z}^k$.

However, if $f$ admits certain “non-trivial symmetries”, the structure of $\pi_1 O(f)$ is not well-understood, though some partial results were obtained.

In the present paper we characterize the class of all finite groups that arise as the group $G(f)$ in (1.2) for the case when $M$ is orientable and differs from sphere and torus, see Theorem 1.5.

For its proof we first show that a finite group $G(f)$ has the form (1.2) for some $f \in F(M, P)$ if and only if $G(f)$ admits a special action on a finite tree.

Further, in Theorem 6.6 we prove that groups having that actions are exactly the ones obtained from finite cyclic and dihedral groups by operations of direct product and wreath product from the top.

The latter statement can be regarded as an extension of an old result of C. Jordan mentioned above about characterization of groups of automorphisms of finite trees.

As a consequence we also obtain that $\pi_1 O(f)$ is always solvable, see Theorem 1.6.
1.2. Main results. In [2] we will recall from [30] the definitions of an enhanced Kronrod-Reeb graph \( \Gamma(f) \) of \( f \in \mathcal{F}(M, P) \) and how \( \mathcal{S}(f) \) acts on \( \Gamma(f) \). If \( f \) is Morse, then \( \Gamma(f) \) coincides with the usual Kronrod-Reeb graph of \( f \). The enhanced graph includes additional edges describing structure of degenerate local extremes of \( f \).

The action of \( \mathcal{S}(f) \) on \( \Gamma(f) \) gives a homomorphism

\[
\lambda : \mathcal{S}(f) \longrightarrow \text{Aut}(\Gamma(f)).
\]

Put

\[
G(f) := \lambda(\mathcal{S}(f) \cap D_{\text{id}}(M)).
\]

(1.3)

So \( G(f) \) is a group of all automorphisms of the enhanced KR-graph \( \Gamma(f) \) of \( f \) induced by isotopic to \( \text{id}_M \) diffeomorphisms belonging to \( \mathcal{S}(f) \). This group in fact coincides with \([1,2]\).

Our main result describes the structure of \( G(f) \).

**Definition 1.3** (Class \( \mathcal{R}(\mathcal{G}) \)). Let \( \mathcal{G} = \{ G_i \mid i \in \Lambda \} \) be a family of finite groups containing the unit group \{1\}. Define inductively the following class of groups \( \mathcal{R}(\mathcal{G}) \):

(i) if \( A, B \in \mathcal{R}(\mathcal{G}) \), then their direct product \( A \times B \) belongs to \( \mathcal{R}(\mathcal{G}) \);

(ii) if \( A \in \mathcal{R}(\mathcal{G}) \) and \( G \in \mathcal{G} \), then the wreath product \( A \wr G \) belong to \( \mathcal{R}(\mathcal{G}) \), see \([4]\).

Since \( 1 \wr G = G \), it follows from (ii) that \( \mathcal{G} \subset \mathcal{R}(\mathcal{G}) \). Thus \( \mathcal{R}(\mathcal{G}) \) is the minimal class of groups containing \( \mathcal{G} \) and closed with respect to finite direct products and wreath products “from the top” by groups belonging to \( \mathcal{G} \).

**Remark 1.4.** Emphasize, that if \( A, B \in \mathcal{R}(\mathcal{G}) \) and \( B \notin \mathcal{G} \), then \( A \wr B \) does not necessarily belong to \( \mathcal{R}(\mathcal{G}) \), see Example \([1,2]\) below.

**Theorem 1.5.** Let \( M \) be a connected compact orientable surface distinct from the 2-sphere \( S^2 \) and 2-torus \( T^2 \), and \( P \) be either \( \mathbb{R} \) or \( S^1 \). Then the following classes of groups coincide:

\[
\{ G(f) \mid f \in \text{Morse}(M, P) \} = \{ G(f) \mid f \in \mathcal{F}(M, P) \} = \mathcal{R}(\{ \mathbb{Z}_n \}_{n \geq 1}).
\]

In other words, if \( M \) is the same as in Theorem \([1,5]\) then for every \( f \in \mathcal{F}(M, P) \) the group of automorphisms of \( \Gamma(f) \) induced by isotopic to the identity diffeomorphisms \( h \in \mathcal{S}(f) \) is obtained from finite cyclic groups by the operations of direct product and wreath product from the top.

Conversely, every finite group having the above structure can be realized as the group \( G(f) \) for some \( f \in \mathcal{F}(M, P) \) or even \( f \in \text{Morse}(M, P) \).

As a consequence of this theorem we will obtain the following statement about the algebraic structure of \( \pi_1 \mathcal{O}(f) \).

Recall that a group \( G \) is solvable if there are subgroups \( \{1\} = G_0 \leq \cdots \leq G_k = G \) such that \( G_{i-1} \) is normal in \( G_i \), and \( G_i/G_{i-1} \) is an abelian group, for \( i = 1, 2, \ldots, k \). It is well known and is easy to see that

1. every abelian group is solvable;
2. if \( A, B \) are two solvable groups, then so is \( A \times B \);
3. if \( A \) is a finite solvable, and \( B \) is solvable, then \( A \wr B \) is also solvable;
4. if \( A \) is a normal subgroup of a group \( B \), and two of three groups \( A, B, A/B \) are solvable, then so is the third one.
In particular, it follows from (1)-(3) that each group $G \in \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$ is solvable.

**Theorem 1.6.** Suppose $M$ is a connected compact orientable surface distinct from $S^2$ and $T^2$. Then for every $f \in \mathcal{F}(M, P)$ the fundamental group $\pi_1\mathcal{O}(f)$ is solvable.

1.7. Structure of the paper. §2 contains the definition of the enhanced Kronrod-Reeb graph of $f \in \mathcal{F}(M, P)$ and some other groups related with a map $f \in \mathcal{F}(M, P)$. In §3 we will show how to deduce Theorem 1.6 from Theorem 1.5.

§4 recalls the notion of wreath products of groups. It is well known that if a group $G$ acts on a set $A$ and a group $H$ acts on a set $B$, then their wreath product $G \wr H$ acts in a special way on $A \times B$ and preserves partition of the form $\{A \times b\}_{b \in B}$. In §5 this observation is extended to group actions preserving arbitrary partitions, see Lemma 5.3.

On the other hand, it is also well known that wreath products arise as groups of automorphisms of certain trees. Let $\mathcal{G} = \{G_i \mid i \in \Lambda\}$ be a family of finite groups containing the unit group $\{1\}$. In §6 we characterize the class $\mathcal{R}(\mathcal{G})$ in terms of special actions on finite trees, see Theorem 6.6. In fact, the family $\mathcal{G}$ will be extended to a certain class $\mathcal{T}(\mathcal{G})$ of finite groups acting on finite trees (see Definition 6.5.1) and then it will be shown that classes $\mathcal{R}(\mathcal{G})$ and $\mathcal{T}(\mathcal{G})$ coincide.

In §7 we study the structure of level sets of $f$ and then in §8 prove Theorem 1.5 for the case when $M$ is either a 2-disk or a cylinder.

Further in §9 we deduce Theorem 1.5 for all other surfaces from the case when $M$ is either a 2-disk or a cylinder. The proof is based on the results of [29].

**2. Enhanced KR-graph**

Let $f \in \mathcal{F}(M, P)$. In this section we recall the notion of KR-graph of $f$, “enhance” it by adding additional edges, and define the action of $\mathcal{S}(f)$ on the obtained “enhanced” graph.

2.1. Notations. Let $h : W \to W$ be a homeomorphism of an orientable manifold $W$. Then we write $h(W) = +W$ if $h$ preserves orientation and $h(W) = -W$ otherwise.

Similarly, let $V$ be a graph, $\omega$ be a path in $V$ (in particular, a cycle or even an edge), and $h : V \to V$ be an automorphism. Then we write that $h(\omega) = \pm \omega$ if $h$ leaves $\omega$ invariant and preserves or, respectively, reverses its orientation.

2.2. Partition $\Theta_f$ and KR-graph. Let $\Theta_f$ the partition of $M$ whose elements are connected components of level-sets $f^{-1}(c)$ of $f$, where $c$ runs over all $P$. An element $\omega \in \Theta_f$ is called critical if it contains a critical point of $f$, otherwise, $\omega$ is regular, see Figure 2.1.

![Figure 2.1. Partition $\Theta_f$ and Kronrod-Reeb graph $M/\Theta_f$.](image-url)
Let $\Sigma(\Theta_f)$ be the union of all critical elements of $\Theta_f$ and all connected components of $\partial M$. Then the connected components of $M \setminus \Sigma(\Theta_f)$ will be called regular components of $\Theta_f$. Notice that every local extreme $z$ of $f$ is an element of $\Theta_f$.

It is well known that the factor space $M/\Theta_f$ has a natural structure of a finite graph, called Kronrod-Reeb graph or simply KR-graph of $f$: the vertices of $M/\Theta_f$ are critical elements of $\Theta_f$, and the open edges of $M/\Theta_f$ are regular components of $\Theta_f$.

This graph plays an important role in understanding the structure of degenerate local extremes of $f$: the vertices of $M/\Theta_f$ are critical elements of $\Theta_f$, and the open edges of $M/\Theta_f$ are regular components of $\Theta_f$.

For our purposes of study maps from class $\mathcal{F}(M, P)$ we will add to $M/\Theta_f$ some edges describing structure of degenerate local extremes of $f$. That new graph will be called the enhanced KR-graph of $f$.

Let us mention that $f$ induces a unique continuous map $f : M/\Theta_f \to P$ such that

$$f = f \circ p : M \to M/\Theta_f \overset{f}{\longrightarrow} P.$$ 

2.3. Homogeneous polynomials. It is well-known that each homogeneous polynomial $g : \mathbb{R}^2 \to \mathbb{R}$ splits into a product of linear and irreducible over $\mathbb{R}$ quadratic factors, so

$$g = L^m_1 \cdots L^m_{\alpha_m} \cdot Q_{1}^{n_1} \cdots Q_{2}^{n_2},$$  

where $L_i(x, y) = a_{i}x + b_{i}y$ is a linear function, $Q_j(x, y) = c_{j}x^2 + d_{j}xy + e_{j}y^2$ is an irreducible over $\mathbb{R}$ quadratic form, $L_i/L_{i'} \neq \text{const}$ for $i \neq i'$ and $Q_j/Q_{j'} \neq \text{const}$ for $j \neq j'$.

Let $S^+(g)$ denote the group of germs at $0$ of orientation preserving $C^\infty$ diffeomorphisms $h : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ such that $g \circ h = g$. Then $h(0) = 0$ for each $h \in S^+(g)$, and so we have a homomorphism $J : S^+(g) \to \text{GL}^+(2, \mathbb{R})$ associating to $h \in S^+(g)$ its Jacobi matrix $J(h)$ at $0$.

**Lemma 2.4.** Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial without multiple factors of degree $\deg(g) \geq 2$. Then the origin $0 \in \mathbb{R}^2$ is a unique critical point of $g$. This point is non-degenerate if and only if $\deg(g) = 2$.

1. Suppose the origin $0$ is a non-degenerate local extreme of $g$, so $g$ is an irreducible over $\mathbb{R}$ quadratic form. Then $J(S^+(g))$ is conjugated in $\text{GL}^+(2, \mathbb{R})$ to $\text{SO}(2)$.

2. In all other cases, i.e. when $0$ is either not an extreme or a degenerate extreme, $J(S^+(g))$ is conjugated to a subgroup of $\text{SO}(2)$ generated by the rotation by the angle $2\pi/n$, for some even number $n = 2k \geq 2$. In particular, $J(S^+(g)) \cong \mathbb{Z}_{2k}$. The number $n = 2k$ will be called the symmetry index of $0$. □

Let $f \in \mathcal{F}(M, P)$ and $z$ be a critical point of $f$, so due to Axiom (L) in some local coordinates at $z$ the map $f$ is equivalent to a homogeneous polynomial $f_z : \mathbb{R}^2 \to \mathbb{R}$ without multiple factors. Then by Lemma 2.4 $z$ is isolated.

Moreover, if $z \in \Sigma_f$ is not a non-degenerate local extreme of $f$, then by (2) of Lemma 2.4 we have a well-defined symmetry index of $z$ which will be denoted by $n_z$.

Let $\Sigma_f^*$ be the set of all degenerate local extremes of $f$. Evidently, $h(\Sigma_f^*) = \Sigma_f^*$ and $n_{h(z)} = n_z$ for all $h \in S(f)$ and $z \in \Sigma_f^*$. 


Definition 2.5. A **framing** for \( f \) is a family of tangent vectors \( F = \{ \xi_z^0, \ldots, \xi_z^{n_z-1} \in T_zM \mid z \in \Sigma_f^* \} \) satisfying the following conditions.

(a) For each \( z \in \Sigma_f^* \) the vectors \( \xi_z^0, \ldots, \xi_z^{n_z-1} \in T_zM \) are cyclically ordered, so that there are local coordinates at \( z \) in which \( \xi_z^k \) is obtained from \( \xi_z^0 \) by rotation by angle \( \frac{2\pi k}{n_z} \).

(b) \( F \) is \( S(f) \)-invariant, that is \( T_z h(\xi_z^k) \in F \) for each \( h \in S(f) \), \( z \in \Sigma_f^* \), and \( k = 0, \ldots, n_z - 1 \). In this case \( T_z h \) maps \( \{ \xi_z^0, \ldots, \xi_z^{n_z-1} \} \) onto \( \{ \xi_{h(z)}^0, \ldots, \xi_{h(z)}^{n_z-1} \} \) preserving or reversing the cyclic orders on these vectors.

The following lemma is not hard to prove, see [30].

**Lemma 2.6.** Every \( f \in F(M, P) \) admits a framing (not necessarily unique). \( \square \)

Let \( F = \{ \xi_z^0, \ldots, \xi_z^{n_z-1} \in T_zM \mid z \in \Sigma_f^* \} \) be a framing for \( f \). Notice that each \( z \in \Sigma_f^* \) is also a vertex of \( M/\Theta_f \). Regard tangent vectors \( \xi_z^0, \ldots, \xi_z^{n_z-1} \) as new \( n_z \) vertices of \( M/\Theta_f \) and connect them by edges with vertex \( z \). Thus for every \( z \in \Sigma_f^* \) we add to the KR-graph \( M/\Theta_f \) new \( n_z \) edges

\[
\xi_z^0, \xi_z^1, \ldots, \xi_z^{n_z-1}.
\]

The obtained graph will be called the **enhanced** KR-graph of \( f \) and denoted by \( \Gamma(f) \), see Figure [2.2]

\[\text{Figure 2.2. Enhanced KR-graph } \Gamma(f).\]

2.7. **Action of** \( S(f) \) **on** \( \Gamma(f) \). Let \( h \in S(f) \), so \( f \circ h = f \). Then \( h(f^{-1}(c)) = f^{-1}(c) \) for each \( c \in P \). Hence \( h \) permutes connected components of \( f^{-1}(c) \) being elements of \( \Theta_f \). This implies that \( h \) induces a well-defined homeomorphism \( \lambda(h) : M/\Theta_f \to M/\Theta_f \).

Moreover, \( \lambda(h) \) extends to a homeomorphism of the enhanced KR-graph \( \Gamma(f) \) as follows. Let \( h \in S(f) \), \( z \in \Sigma_f^* \), and \( \xi_z^i \in F \) regarded as a vertex of \( \Gamma(f) \), so \( z\xi_z^i \) is an edge of \( \Gamma(f) \). Then \( h(z) \in \Sigma_f^* \) and by definition of framing we also have that \( T_z h(\xi_z^i) \in F \). In particular, \( \Gamma(f) \) contains the edge with vertices \( h(z) \) and \( T_z h(\xi_z^i) \), and so we define

\[
\lambda(h)(\xi_z^i) = T_z h(\xi_z^i),
\]

where \( \xi_z^i \) and \( T_z h(\xi_z^i) \) are now regarded as vertices of \( \Gamma(f) \).
Remark. This action can be understood as follows. Let \( z \in \Sigma^*_f \). By assumption tangent vectors at \( z \) belonging to the framing \( F \) are cyclically ordered. Moreover, every \( h \in \mathcal{S}(f) \) interchanges collections of such cyclically ordered vectors also preserving or reversing their cyclic order. Constructing \( \Gamma(f) \) we just attached these cyclically ordered collections to the corresponding vertices of \( M/\Theta_f \), and the action of \( \mathcal{S}(f) \) on \( \Gamma(f) \) mimics the corresponding action on \( F \).

Thus \( \mathcal{S}(f) \) acts on \( \Gamma(f) \), and so the correspondence

\[
\lambda : \mathcal{S}(f) \longrightarrow \text{Aut}(\Gamma(f)), \quad h \mapsto \lambda(h),
\]

is a homomorphism into the group of automorphisms of \( \Gamma(f) \).

Extend \( f: M/\Theta_f \rightarrow P \) to the map \( f: \Gamma(f) \rightarrow P \) by setting \( f(z \xi^i) = f(z) \) for every edge \( z \xi^i \). Also notice that for each \( h \in \mathcal{S}(f) \) the following diagram is commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{p} & M/\Theta_f \\
\downarrow{h} & \nearrow{\lambda(h)} & \downarrow{\lambda(h)} \\
M & \xrightarrow{p} & M/\Theta_f \\
\end{array}
\quad \begin{array}{ccc}
\Gamma(f) & \xrightarrow{f} & P \\
\downarrow{h} & \nearrow{\lambda(h)} & \downarrow{\lambda(h)} \\
\Gamma(f) & \xrightarrow{f} & P \\
\end{array}
\quad \begin{array}{ccc}
P & \xrightarrow{f} & P \\
\end{array}
\] (2.3)

2.8. Partition \( \Delta_f \). Define another partition \( \Delta_f \) of \( M \) by the rule: a subset \( \omega \subset M \) is an element of \( \Delta_f \) if and only if \( \omega \) is either a critical point of \( f \) or a connected component of the set \( f^{-1}(c) \setminus \Sigma(f) \), for some \( c \in P \), see Figure 2.3.

In other words, \( \Delta_f \) is obtained from \( \Theta_f \) by dividing critical elements of \( \Theta_f \) with critical points of \( f \).

**Figure 2.3.** Difference between partitions \( \Theta_f \) and \( \Delta_f \)

2.9. Automorphisms of \( \Theta_f \) and \( \Delta_f \). Let \( \mathcal{D}(\Theta_f) \) be the group of all diffeomorphisms \( h \in \mathcal{D}(M) \) such that

(i) \( h(\omega) = \omega \) for every \( \omega \in \Theta_f \);
(ii) if \( \omega \) is a regular element of \( \Theta_f \), then \( h \) preserves orientation of \( \omega \);
(iii) for every degenerate local extreme \( z \in \Sigma^*_f \) of \( f \) the tangent map \( T_z h : T_z M \rightarrow T_z M \) is the identity.

Let also

\[
\mathcal{D}(\Delta_f) = \{ h \in \mathcal{D}(\Theta_f) \mid h(\omega) = \omega \text{ for each } \omega \in \Delta_f \}.
\]

Remark. In [30] \( \mathcal{D}(\Delta_f) \) denoted the group satisfying only (i) and (ii), while its subgroup satisfying (iii) was denoted by \( \mathcal{D}^N(\Delta_f) \). We will use here simplified notation \( \mathcal{D}(\Delta_f) \).
Evidently,
\[ D(\Delta f) \subset D(\Theta f) \subset \ker \lambda \subset \mathcal{S}(f). \]  
(2.4)

We will now describe the difference between these four groups. Let \( c \in P \), \( X_c = f^{-1}(c) \) be the corresponding level set of \( f \), and
\[ \rho_1, \ldots, \rho_a, \sigma_1, \ldots, \sigma_b, z_1, \ldots, z_r \]  
(2.5)
be all the connected components of \( X_c \) being elements of \( \Theta_f \) such that

* each \( \rho_i \) is a regular element (i.e. a simple closed curve);
* each \( \sigma_j \) is a critical element being not a degenerate local extreme of \( f \), so \( \sigma_j \) is a connected graph each of whose vertices has even degree (possibly zero in which case \( \sigma_j \) is a non-degenerate local extreme of \( f \));
* and each \( z_k \) is a degenerate local extreme.

Moreover, for every \( z_k \) we also have \( n_{z_k} \) tangent vectors \( \xi_{z_k}^0, \ldots, \xi_{z_k}^{n_{z_k}-1} \in T_{z_k}M \) belonging to the framing \( F \), see Figure 2.4. Then the following statements hold true.

- Every \( h \in \mathcal{S}(f) \) preserves \( X_c \), that is \( h(X_c) = X_c \), and so \( h \) can interchange elements of (2.5).
- Every \( h \in \ker \lambda \) preserves each element in (2.5) and the corresponding vectors in the framing, i.e. \( h(\rho_i) = \rho_i, h(\sigma_j) = \sigma_j, h(z_k) = z_k \), and \( T_{z_k} h(\xi_{z_k}^l) = \xi_{z_k}^l \).

  Recall that by Lemma 2.4 in some local coordinates at \( z_k \) the tangent map \( T_{z_k} h \) is a rotation. On the other hand \( T_{z_k} h \) has \( n_z \geq 2 \) fixed vectors, which implies that \( T_{z_k} h \) is the identity map of \( T_{z_k} M \).

  However, \( h \) can change orientations of \( \rho_i \) and also can induce non-trivial automorphisms of \( \sigma_j \), that is interchange its vertices and edges.

- Every \( h \in D(\Theta_f) \) additionally preserves orientation of circles \( \rho_i \).
- Every \( h \in D(\Delta f) \) additionally fixes every vertex of \( \sigma_j \) and leaves invariant every edge of \( \sigma_j \). Since each edge \( e \) of \( \sigma_j \) belongs to the closure of the union of regular elements of \( \Delta f \) and \( h \) preserves orientations of that elements, it follows \( h \) also preserves orientation of \( e \).
Let \( \mathcal{D}(M, \partial M) \) be the group of diffeomorphisms of \( \mathcal{D}(M) \) fixed on \( \partial M \). Let also \( \mathcal{D}_{id}(\Theta_f) \) (resp. \( \mathcal{D}_{id}(\Delta_f) \), \( (\ker \lambda)_{id} \), \( \mathcal{D}_{id}(M, \partial M) \)) be the identity path component of \( \mathcal{D}(\Theta_f) \) (resp. \( \mathcal{D}(\Delta_f) \), \( \ker \lambda \), \( \mathcal{D}(M, \partial M) \)) in the \( C^\infty \) Whitney topology. The following statement is proved in [30].

**Lemma 2.10.** [30]. Let \( f \in \mathcal{F}(M, P) \). Then
1. \( \mathcal{D}_{id}(\Delta_f) = \mathcal{D}_{id}(\Theta_f) = (\ker \lambda)_{id} = S_{id}(f) \) and this group is contractible.
2. \( \pi_0(\mathcal{D}(\Delta_f) \cap \mathcal{D}(M, \partial M)) \) are finitely generated free abelian groups.
3. If \( M \) is orientable, then

\[
\mathcal{D}(\Delta_f) \cap \mathcal{D}_{id}(M) = \mathcal{D}(\Theta_f) \cap \mathcal{D}_{id}(M) = \ker \lambda \cap \mathcal{D}_{id}(M).
\]

**Remark.** Statement similar to (2.6) holds for non-orientable surfaces, however one should modify enhanced KR-graph of \( f \) by adding more information and thus changing the definition of homomorphism \( \lambda \), see [30] for details. In that case we will even have that \( \mathcal{D}(\Theta_f) = \ker \lambda \). However, since our main result concerns only orientable surfaces, Lemma 2.10 is formulated just for them.

3. Proof of Theorem 1.6

Let \( M \) be a connected compact orientable surface distinct from \( S^2 \) and \( T^2 \) and \( f \in \mathcal{F}(M, P) \). Since \( M \) is connected, compact, and distinct from \( S^2 \) and \( T^2 \), the group \( \mathcal{D}_{id}(M, \partial M) \) contractible, see e.g. [12]. In this case by [31, Theorem 2.6] we have an isomorphism

\[
\partial : \pi_1 \mathcal{O}(f) \cong \pi_0 \mathcal{S}'(f, \partial M).
\]

Therefore we need to show solvability of \( \pi_0 \mathcal{S}'(f, \partial M) \). This will be deduced from Theorem 1.5. Denote

\[
\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{id}(M), \quad \mathcal{S}'(f, \partial M) = \mathcal{S}(f) \cap \mathcal{D}_{id}(M, \partial M).
\]

First we establish one lemma.

**Lemma 3.1.** \( G(f) \equiv \lambda(\mathcal{S}'(f)) = \lambda(\mathcal{S}'(f, \partial M)) \).

**Proof.** Since \( \mathcal{S}'(f, \partial M) \subset \mathcal{S}'(f) \), it suffices to prove that for each \( h \in \mathcal{S}'(f) \) there exists \( h_1 \in \mathcal{S}'(f, \partial M) \) such that \( \lambda(h) = \lambda(h_1) \).

Let \( C_1, \ldots, C_k \) be all the connected components of \( \partial M \). Since a regular neighborhood of \( \partial M_i \), \( i = 1, \ldots, k \), is diffeomorphic to \( S^1 \times I \) in such a way that the sets \( S^1 \times \{t\} \) correspond to level-sets of \( f \), there exists a Dehn twist \( \tau_i \in \mathcal{S}(f) \) fixed on \( \partial M \), see e.g. [25, §6.1] or [26, §6.1]. In particular, \( \lambda(\tau_i) = \text{id}_{r(f)} \) for each \( i = 1, \ldots, k \).

Now let \( h \in \mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{id}(M) \). Since \( h \) is isotopic to \( \text{id}_M \), it follows that \( h(C_i) = +C_i \) for all \( i \). Hence by [31, Corollary 6.4] \( h \) is isotopic in \( \mathcal{S}(f) \) to a diffeomorphism \( h_0 \) fixed on \( \partial M \). In particular, we have that \( \lambda(h_0) = \lambda(h) \).

However, in general, \( h_0 \) is not isotopic to \( \text{id}_M \) rel. \( \partial M \), that is \( h_0 \in \mathcal{S}(f) \cap \mathcal{D}(M, \partial M) \) but not necessarily to \( \mathcal{S}(f) \cap \mathcal{D}_{id}(M, \partial M) = \mathcal{S}'(f, \partial M) \). Nevertheless, it easily follows from [3, Lemma 6.1] that one can find integers \( m_1, \ldots, m_k \) such that

\[
h_1 = h_0 \circ \tau_1^{m_1} \circ \cdots \circ \tau_k^{m_k}
\]
is isotopic to \( \text{id}_M \) relatively \( \partial M \). In other words, \( h_1 \in S'(f, \partial M) \). It remains to note that

\[
\lambda(h_1) = \lambda(h_0) \circ \lambda(\tau_1^m) \circ \cdots \circ \lambda(\tau_k^m) = \lambda(h_0) = \lambda(h),
\]

whence \( \lambda(S'(f)) = \lambda(S'(f, \partial M)) \).

Consider the natural factor map \( q : S'(f, \partial M) \to \pi_0 S'(f, \partial M) \). Evidently, the homomorphism \( \lambda : S(f) \to \text{Aut}(\Gamma(f)) \) is continuous and its image is discrete (in fact finite). Hence there exists an epimorphism \( \hat{\lambda} : \pi_0 S'(f, \partial M) \to G(f) \) such that \( \lambda = \hat{\lambda} \circ q \), that is the following diagram is commutative:

\[
\begin{array}{ccc}
S'(f, \partial M) & \xrightarrow{\lambda} & \lambda(S'(f, \partial M)) = \lambda(S'(f)) \equiv G(f) \\
q \downarrow & & \hat{\lambda} \downarrow \\
\pi_0 S'(f, \partial M) & \xrightarrow{\hat{\lambda}} & \\
\end{array}
\]

By Theorem \([1.5]\) the group \( G(f) \in \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1}) \) is solvable. We claim that \( \ker(\hat{\lambda}) \cong \pi_0 (\mathcal{D}_{\text{id}}(M, \partial M) \cap \mathcal{D}(\Delta f)) \).

By (2) of Lemma \([2.10]\) the latter group is finitely generated free abelian. This will imply that \( \pi_0 S'(f, \partial M) \) and therefore \( \pi_1 \mathcal{O}(f) \) are solvable as well.

**Lemma 3.1.1.** There exists a natural isomorphism

\[
\hat{q} : \pi_0 (S'(f, \partial M) \cap \ker \lambda) \longrightarrow \ker(\hat{\lambda}), \quad \hat{q}([h]) := q(h),
\]

where \([h] \in \pi_0 (S'(f, \partial M) \cap \ker \lambda) \) is an isotopy class of some \( h \in S'(f, \partial M) \cap \ker \lambda \).

The proof is straightforward and we left it for the reader. Now notice that from \( \mathcal{D}_{\text{id}}(M, \partial M) \subset \mathcal{D}_{\text{id}}(M) \) and \( \mathcal{D}(\Delta f) \subset \ker \lambda \subset S(f) \) it follows that

\[
S'(f, \partial M) \cap \ker \lambda = S(f) \cap \mathcal{D}_{\text{id}}(M, \partial M) \cap \ker \lambda
= \mathcal{D}_{\text{id}}(M, \partial M) \cap \mathcal{D}_{\text{id}}(M) \cap \ker \lambda
\]

\[
\equiv \mathcal{D}_{\text{id}}(M, \partial M) \cap \mathcal{D}(\Delta f)
= \mathcal{D}_{\text{id}}(M, \partial M) \cap \mathcal{D}(\Delta f),
\]

whence

\[
\ker(\hat{\lambda}) \cong \pi_0 (S'(f, \partial M) \cap \ker \lambda) \equiv \pi_0 (\mathcal{D}_{\text{id}}(M, \partial M) \cap \mathcal{D}(\Delta f)).
\]

Theorem \([1.6]\) is completed.

### 4. Wreath Products

Let \( B \) be a set and \( H \) be a subgroup of the group of permutations \( \Sigma(B) \) on \( B \), and \( S \) be yet another group. Denote by \( \text{Map}(B, S) \) the group of all maps \( B \to S \) with respect to the point-wise multiplication. Then \( H \) acts on \( \text{Map}(B, S) \) by following rule:

\[
\alpha \cdot h = \alpha \circ h : B \xrightarrow{h} B \xrightarrow{\alpha} S,
\]
for $\alpha \in \text{Map}(B, S)$ and $h \in H$. Using this action one can define the semi-direct product $\text{Map}(B, S) \rtimes H$ which is denoted by $S \wr_B H$ and called the wreath product of $S$ and $H$ over $B$. Thus

$$S \wr_B H := \text{Map}(B, S) \rtimes H$$

is the Cartesian product $\text{Map}(B, S) \times H$ with the multiplication given by the formula:

$$(\alpha_1, h_1) \cdot (\alpha_2, h_2) = (\alpha_1 \circ h_2) \cdot \alpha_2, h_1h_2).$$ (4.1)

for $(\alpha_1, h_1), (\alpha_2, h_2) \in \text{Map}(B, S) \times H$.

The group $H$ is called the top group, and $\text{Map}(B, S)$ is the base of $S \wr_B H$.

Let $\varepsilon : B \to S$ be the constant map into the unit of $S$. Then the pair $(\varepsilon, \text{id}_B)$ is the unit of $S \wr_B H$. Moreover, let $(\alpha, h) \in S \wr_B H$ and $\bar{\alpha} \in \text{Map}(B, S)$ be the point-wise inverse from $\alpha$, i.e. $\bar{\alpha}(b) = (\alpha(b))^{-1} \in S$ for all $b \in B$. Then $(\bar{\alpha} \circ h^{-1}, h^{-1})$ is the inverse to $(\alpha, h)$ in $S \wr_B H$.

Further, we have the following exact sequence

$$1 \to \text{Map}(B, S) \xrightarrow{i} S \wr_B H \xrightarrow{\pi} H \to 1,$$ (4.2)

where $i(\alpha) = (\alpha, e)$ and $\pi(\alpha, h) = h$, in which the map $\pi$ admits a section:

$$s : H \to S \wr_B H, \quad s(h) = (\varepsilon, h).$$

Also notice that the group $H$ naturally acts on itself by right shifts. Hence we can also define the wreath product $S \wr_H H$, which in this particular case is denoted by $S \wr H$. So

$$S \wr H := S \wr_H H = \text{Map}(H, S) \rtimes H,$$

where $\text{Map}(H, S)$ is the set of all maps $H \to S$ (not necessarily homomorphisms).

It is known, that the operation of wreath product of groups is associative, that is for any groups $F, G, H$ the groups $(F \wr G) \wr H$ and $F \wr (G \wr H)$ are canonically isomorphic. Therefore we will denote each of them by $F \wr G \wr H$.

Wreath products play an outstanding role in group theory. They seem to be independently appeared in papers by C. Jordan [14], A. Loewy [24], and L. M. Kaluzhin [16], see also G. Pólya [37, §27] and B. H. Neumann [33] for discussions.

The following theorem describes two properties about the structure of wreath products. They can be used to check whether a certain group belongs to a class $\mathcal{R}(\mathcal{G})$.

**Theorem 4.1.** [34] Let $W = A \wr B$. Then the following statements hold true.

1. $W \cong P \times Q$ for some non-trivial groups $P$ and $Q$, if and only if $B$ is finite, of order $n$, say, and $A$ has a non-trivial abelian direct factor $C_1$ (which may coincide with $A$) having unique $n$-th roots for all of its elements, [34, Theorem 7.1].

2. $W \cong P \wr Q$ if and only if $P \cong A$ and $Q \cong B$, [34, Theorem 10.3].

**Example 4.2.** Let $\mathcal{G} = \{\mathbb{Z}_n\}_{n \geq 1}$ be the class of all finite cyclic groups. Then

$$W = \mathbb{Z}_q \wr (\mathbb{Z}_q \times \mathbb{Z}_{rq}) \not\in \mathcal{R}(\mathcal{G}), \quad q \geq 2, \ r \geq 1.$$  

**Proof.** Suppose $W \in \mathcal{R}(\mathcal{G})$. Then either $W = P \times Q$ for some non-trivial groups $P, Q \in \mathcal{R}(\mathcal{G})$ or $W = P \wr Q$ for some non-trivial groups $P \in \mathcal{R}(\mathcal{G})$ and $Q \in \mathcal{G}$.

Assume that $W = \mathbb{Z}_q \wr (\mathbb{Z}_q \times \mathbb{Z}_r) \cong P \times Q$. Then by (1) of the above Theorem 4.1 each element of $\mathbb{Z}_q$ must have a unique root of order $|\mathbb{Z}_q \times \mathbb{Z}_{rq}| = q^2r$. However, $q^2r \cdot x = 0$
for all $x \in \mathbb{Z}_q$, and so each non-zero element of $\mathbb{Z}_q$ has no roots of order $q^2r$. Hence $W$ is not decomposable.

On the other hand, if $W = \mathbb{Z}_q \wr (\mathbb{Z}_q \times \mathbb{Z}_{qr}) \cong P \wr Q$, then by (2) of Theorem 4.1, $P \cong \mathbb{Z}_q$ and $Q \cong \mathbb{Z}_q \times \mathbb{Z}_{qr}$. But we should also have that $Q \in \mathcal{G}$, i.e. $\mathbb{Z}_q \times \mathbb{Z}_{qr}$ must be a cyclic group which is not true since $q$ divides $qr$.

Thus both cases of $W$ are impossible, and so $W \notin \mathcal{R}(\mathcal{G})$. \hfill \Box

As noted in [37] Nr. 54] the following theorem is a consequence of an old result by C. Jordan [14] characterizing groups of automorphisms of finite trees, see also [1] Proposition 1.15] and [17] Lemma 5].

**Theorem 4.3.** (C. Jordan’ 1869). Let $\Sigma_n$, $n \geq 1$, be the permutation group of $n$ elements. Then a group $G$ is isomorphic to a group of automorphisms of a finite tree if and only if $G$ belongs to the class $\mathcal{R}(\{\Sigma_n\}_{n \geq 1})$.

Recall that our main result (Theorem 1.5) characterizes class $\mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$ generated by cyclic groups. In [10] this class will be described as a class of groups having special actions on finite trees. See also Remark 6.7.

### 5. Partition preserving group actions

Let $A$ be a set, $\tau = \{T_b\}_{b \in B}$ be a partition of $A$ and $q : A \to B$ be the natural projection, so $q(a) = b$ for some $a \in A$ and $b \in B$ if and only if $a \in T_b$.

Let $G$ be a group effectively acting on $A$ and preserving partition $\tau$, that is $g(T) \in \tau$ for all $g \in G$ and $T \in \tau$. Then this action reduces to the action of $G$ on $B$: we have a unique homomorphism $p : G \to \Sigma(B)$ into the group of permutation of $B$ such that

$$g(T_b) = T_{p(g)(b)}$$

for every $g \in G$ and $b \in B$. In particular, the following diagram is commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{g} & A \\
\downarrow q & & \downarrow q \\
B & \xrightarrow{p(g)} & B
\end{array}
$$

For each $b \in B$ define the following subgroup of $\Sigma(T_b)$:

$$S_b = \{ g|_{T_b} \mid g \in G, \ g(T_b) = T_b \},$$

so $S_b$ consists of all permutations of $T_b$ induced by elements of $G$ preserving $T_b$. Let also

$$H = p(G)$$

be the image of $G$ in $\Sigma(B)$ under $p$. Evidently, if $g(T_b) = T_{b'}$ for some $b, b' \in B$, i.e. $b$ and $b'$ belongs to the same $H$-orbit, then the groups $S_b$ and $S_{b'}$ are isomorphic.

Let $\{B_{\lambda}\}_{\lambda \in B/H}$ be the partition of $B$ by orbits of the action of $H$. For each $\lambda \in B/H$ fix some element $b_\lambda \in B_{\lambda}$ and consider the group $\text{Map}(B_{\lambda}, S_{b_\lambda})$ of all maps $B_{\lambda} \to S_{b_\lambda}$ with respect to a point-wise multiplication. Then we can define the standard right action of $H$ on the product $\times_{\lambda \in B/H} \text{Map}(B_{\lambda}, S_{b_\lambda})$ as follows. Let

$$\alpha = \{\alpha_\lambda\}_{\lambda \in B/H} \in \times_{\lambda \in B/H} \text{Map}(B_{\lambda}, S_{b_\lambda})$$
where each $\alpha_\lambda : B_\lambda \to S_{b_\lambda}$ is a map, and $h \in H$ is regarded as a bijection $h : B_\lambda \to B_\lambda$. Then the result of the action of $h$ on $\alpha$ is

$$\alpha \circ h := \{ \alpha_\lambda \circ h|_{B_\lambda} \}_{\lambda \in B/H},$$

where

$$\alpha_\lambda \circ h|_{B_\lambda} : B_\lambda \xrightarrow{h|_{B_\lambda}} B_\lambda \xrightarrow{\alpha} S_{b_\lambda}$$

is a usual composition map. Therefore we have a well-defined semi-direct product

$$C := \left( \times_{\lambda \in B/H} \text{Map}(B_\lambda, S_{b_\lambda}) \right) \rtimes H$$

with respect to this action, and so $C$ is a product of sets

$$\left( \times_{\lambda \in B/H} \text{Map}(B_\lambda, S_{b_\lambda}) \right) \times H$$

with the following operation: if $(\alpha, h_1), (\beta, h_2) \in C$, then

$$(\alpha, h_1) \cdot (\beta, h_2) = ((\alpha \circ h_2) \cdot \beta, h_1 h_2).$$

Also notice that $\times_{\lambda \in B/H} \text{Map}(B_\lambda, S_{b_\lambda})$ can be regarded as the set of maps

$$\alpha : B \longrightarrow \bigsqcup_{\lambda \in B/H} S_{b_\lambda}$$

into a disjoint union of $S_{b_\lambda}$, such that $\alpha(B_\lambda) \subseteq S_{b_\lambda}$ for all $\lambda \in B/H$, and the action of $H$ on this product is induced from the standard right action of $H$ on $\text{Map}(B, \bigsqcup_{\lambda \in B/H} S_{b_\lambda})$.

**Lemma 5.1.** 1) If $H$ acts transitively on $B$, and $b \in B$, then

$$C \cong S_b \wr B \, H.$$

2) Suppose the action of $H$ is free, then

$$C \cong \left( \times_{\lambda \in B/H} S_{b_\lambda} \right) \rtimes H. \tag{5.1}$$

3) Let $F$ be the set of all orbits in $B/H$ corresponding to fixed points, and $N$ be its complement in $B/H$. Then

$$C \cong \left( \times_{\lambda \in F} S_{b_\lambda} \right) \times \left( \times_{\lambda \in N} \text{Map}(B_\lambda, S_{b_\lambda}) \right) \rtimes H.$$

**Proof.** Statement 1) is trivial.

2) Suppose the action of $H$ is free. Then for each $\lambda \in B/H$ the correspondence $H \ni h \mapsto h(b_\lambda) \in B_\lambda$ is a bijection which leads to the following identifications:

$$\times_{\lambda \in B/H} \text{Map}(B_\lambda, S_{b_\lambda}) \cong \times_{\lambda \in B/H} \text{Map}(H, S_{b_\lambda}) \cong \text{Map}(H, \times_{\lambda \in B/H} S_{b_\lambda}).$$

Therefore $C$ is a certain semi-direct product

$$\text{Map}(H, \times_{\lambda \in B/H} S_{b_\lambda}) \rtimes H.$$

It is easy to check that the multiplication in $C$ coincides with the wreath product multiplication which gives an isomorphism \[5.1\].
3) By assumption every orbit \( \lambda \in F \) consists of a unique point. Therefore \( B_\lambda = \{b_\lambda\} \), \( \text{Map}(B_\lambda, S_{b_\lambda}) \cong S_{b_\lambda} \), and the right action of \( H \) on \( \text{Map}(B_\lambda, S_{b_\lambda}) \) is trivial. This easily implies that the following correspondence

\[
(\{\alpha_\lambda\}_{\lambda \in B/H}, h) \mapsto (\{\alpha_\lambda\}_{\lambda \in F}, (\{\alpha_\lambda\}_{\lambda \in N}, h))
\]

is an isomorphism of \( C \) onto \( \left( \times_{\lambda \in F} S_{b_\lambda} \right) \times \left( \left( \times_{\lambda \in N} \text{Map}(B_\lambda, S_{b_\lambda}) \right) \times H \right) \). We leave the details for the reader. \( \square \)

**Definition 5.2.** Let \( X \subset B \) be a subset, \( \tau = \{T_b\}_{b \in B} \) be a partition on \( A \). We will say that a \( \tau \)-preserving action of \( G \) on \( A \) is \( \tau \)-decomposable over \( X \) if for any family \( \{g_b\}_{b \in X} \) of (not necessarily mutually distinct) elements of \( G \) enumerated by points of \( X \) such that \( g_b(T_b) = T_b \) for all \( b \in X \) the following permutation \( g : A \to A \) of \( A \) defined by

\[
g(a) = \begin{cases} 
  g_b(a), & a \in T_b, b \in X, \\
  a, & a \in A \setminus \cup_{b \in X} T_b = A \setminus q^{-1}(X).
\end{cases}
\]

belongs to \( G \). An action which is \( \tau \)-decomposable over any subset \( X \subset B \) will be called \( \tau \)-decomposable.

The following lemma characterizes \( \tau \)-decomposable actions.

**Lemma 5.3.** (1) There exists a monomorphism \( \nu : G \to C \) making commutative the following diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\nu} & C = \left( \times_{\lambda \in B/H} \text{Map}(B_\lambda, S_{b_\lambda}) \right) \times H \\
\downarrow p & & \downarrow \pi \\
H & &
\end{array}
\]

where \( \pi \) is the natural projection.

(2) The following conditions are equivalent:

(a) \( \nu \) is an isomorphism;

(b) the action of \( G \) on \( A \) is \( \tau \)-decomposable.

(c) the subgroup \( \pi^{-1}(\text{id}_B) = \left( \times_{\lambda \in B/H} \text{Map}(B_\lambda, S_{b_\lambda}) \right) \times \text{id}_B \) is contained in \( \nu(G) \);

**Proof.** (1) Since \( H \) transitively acts on \( B_\lambda \), for each \( u \in B_\lambda \) we can fix some \( \phi_u \in G \) such that \( \phi_u(T_{b_\lambda}) = T_u \).

Let \( g \in G \). For each \( \lambda \in B/H \) define the map \( \alpha_\lambda : B_\lambda \to S_{b_\lambda} \) by the formula:

\[
\alpha_\lambda(u) = (\phi_{p(g)(u)}^{-1} \circ g \circ \phi_u)|_{T_{b_\lambda}}, \quad u \in B_\lambda.
\]

Notice that we have the following commutative diagram:

\[
\begin{array}{ccc}
T_u & \xrightarrow{g} & T_{p(g)(u)} \\
\phi_u \uparrow & & \phi_{p(g)(u)} \uparrow \\
T_{b_\lambda} & \xrightarrow{\alpha_\lambda(u)} & T_{b_\lambda}
\end{array}
\]
Now define \( \nu : G \to C \) by the formula:
\[
\nu(g) = (\{\alpha_\lambda\}_{\lambda \in B/H}, p(g)).
\]
First we prove that \( \nu \) is a homomorphism. Let \( g_1, g_2 \in G \), and
\[
\nu(g_1) = (\{\alpha_\lambda\}_{\lambda \in B/H}, p(g_1)), \quad \nu(g_2) = (\{\beta_\lambda\}_{\lambda \in B/H}, p(g_2)),
\]
\[
\nu(g_1 g_2) = (\{\gamma_\lambda\}_{\lambda \in B/H}, p(g_1 g_2)).
\]
Due to (4.1) we need to show that
\[
\gamma_\lambda(u) = \alpha_\lambda(h_2(u)) \cdot \beta_\lambda(u), \quad h = h_1 \circ h_2,
\]
for all \( u \in B_\lambda \). Since \( p \) is a homomorphism, we get
\[
h = p(g_1 g_2) = p(g_1) \circ p(g_2) = h_1 \circ h_2.
\]
Moreover,
\[
\alpha_\lambda(h_2(u)) = (\phi^{-1}_{h_1 \circ h_2(u)} g_1 \phi_{h_2(u)})|_{T_\lambda} = (\phi^{-1}_{h(u)} g_1 \phi_{h_2(u)})|_{T_\lambda},
\]
\[
\beta_\lambda(u) = (\phi_{h_2(u)} g_2 \phi_u)|_{T_\lambda},
\]
\[
\gamma_\lambda(u) = (\phi^{-1}_{h(u)} g_1 g_2 \phi_u)|_{T_\lambda}.
\]
Hence \( \gamma_\lambda(u) = \alpha_\lambda(h_2(u)) \cdot \beta_\lambda(u) \), and so \( \nu \) is a homomorphism.

Let us prove that the kernel of \( \nu \) is trivial. Let \( \varepsilon : B_\lambda \to S_{b_\lambda} \) be the constant map into the unit \( \varepsilon \) of \( S_{b_\lambda} \), \( \varepsilon = \{\varepsilon_\lambda\}_{\lambda \in B/H} \), and \( \text{id}_B : B \to B \) be the identity map, so the pair \( (\varepsilon, \text{id}_B) \) is the unit in \( C \).

Suppose \( \nu(g) = (\varepsilon, \text{id}_B) \). Then \( p(g) = \text{id}_B \), i.e. \( g(T_u) = T_u \) for all \( u \in B \). Moreover,
\[
\varepsilon_\lambda(u) = (\phi_{h(u)}^{-1} g \circ \phi_u)|_{T_\lambda} = \varepsilon_\lambda \in S_{b_\lambda}, \quad u \in B_\lambda,
\]
whence \( g|_{T_u} : T_u \to T_u \) is the identity map, and so \( g \) trivially acts on all of \( A \). Since \( G \) is effective, we see that \( g \) is a unit of \( G \). Thus \( \nu \) is a monomorphism.

(2) We should establish equivalence of (a)-(c).

(a)\(\Rightarrow\)(b) Suppose \( \nu \) is an isomorphism. Let \( X \subset B \) be any subset, \( \{g_b\}_{b \in X} \) be any family of elements of \( G \) such that \( g_b(T_b) = T_b \), and \( g : A \to A \) be the permutation defined by the rule: \( g|_{T_b} = g_b|_{T_b} \) for \( b \in X \), and \( g = \text{id} \) on \( A \setminus q^{-1}(X) \). We should prove that \( g \in G \). This will imply that the action of \( G \) on \( A \) is \( \tau \)-decomposable.

For each \( \lambda \in B/H \) define the map \( \alpha_\lambda : B_\lambda \to S_{b_\lambda} \) by the formula
\[
\alpha_\lambda(b) = \begin{cases} 
\phi_b^{-1} \circ g_b \circ \phi_b|_{T_\lambda}, & b \in B_\lambda \cap X, \\
\text{id}_{T_\lambda}, & b \in B_\lambda \setminus X,
\end{cases}
\]
and consider the following element
\[
\gamma = (\{\alpha_\lambda\}_{\lambda \in B/H}, \text{id}_B) \in C.
\]
Then it is easy to see that \( \nu^{-1}(\gamma) \) coincides with \( g \). Thus \( g \in G \), and so the action of \( G \) on \( A \) is \( \tau \)-decomposable.

(b)\(\Rightarrow\)(c). Suppose the action of \( G \) on \( A \) is \( \tau \)-decomposable. Let \( \gamma = (\{\alpha_\lambda\}_{\lambda \in B/H}, \text{id}_B) \) be any element of \( \pi^{-1}(\text{id}_B) \). We should prove that \( \gamma = \nu(g) \) for some \( g \in G \).
By definition \( S_b \) consists of restrictions of elements from \( G \) to \( T_b \), whence for each \( b \in B \) there exists \( g_b \in G \) such that \( g_b|T_b = \alpha_\lambda(b) \). In particular,

\[
\phi_b \circ g_b \circ \phi_b^{-1}(T_b) = T_b.
\]

(5.3)

Consider now the family of elements of \( G \):

\[
\{ \phi_b \circ g_b \circ \phi_b^{-1} \}_{b \in B}.
\]

As \( G \) is \( \tau \)-decomposable, it follows from (5.3) that there exists \( \hat{g} \in G \) such that \( \hat{g}|T_b = \phi_b \circ g_b \circ \phi_b^{-1}|T_b \).

Now we get from (5.2) that \( \nu(\hat{g}) = \nu(g) \).

(c) \( \Rightarrow \) (a). Let \( \gamma = \{ \{ \alpha_\lambda \}_{\lambda \in B/H}, h \} \) be any element of \( C \). Since \( p : G \rightarrow H \) is surjective, there exists \( g \) such that \( h = p(g) \). Then it follows from the multiplication rule in \( C \) that the element \( \delta = \nu(g) \cdot \gamma \in C \) has the following form:

\[
\delta = \{ \{ \alpha_\lambda \}_{\lambda \in B/H}, id_B \},
\]

i.e. belongs to \( \pi^{-1}(id_B) \). But by (c), \( \pi^{-1}(id_B) \subset \nu(G) \), so \( \delta = \nu(\hat{g}) \) for some \( \hat{g} \in G \), whence \( \gamma = \nu(g \circ \hat{g}) \). Thus \( \nu \) is surjective, and therefore it is an isomorphism.

□

6. Groups acting on finite trees

Let \( T \) be a finite graph and \( G \) be a subgroup of \( \text{Aut}(T) \). If \( g \in G \), then an edge \( e \) of \( T \) is fixed under \( g \) if \( g(e) = e \) and \( g \) preserves orientation of \( e \). We will denote by \( \text{Fix}(G) \) the subgraph consisting of common fixed vertices and edges \( G \).

For each vertex \( u \in T \) let

\[
G_u := \{ g \in G \mid g(u) = u \}
\]

be the stabilizer of \( u \) with respect to \( G \). Similarly, for an edge \( uv \) of \( T \) we define its stabilizer by

\[
G_{uv} := G_u \cap G_v = \{ g \in G \mid g(u) = u, g(v) = v \}.
\]

Regarding \( T \) as a CW-complex, let \( \text{Star}(u) \) be a small regular \( G_u \)-invariant neighborhood of \( u \) containing no other vertices of \( T \) except for \( u \), see Figure 6.1.

Then the group of restrictions of \( G_u \) to \( \text{Star}(u) \) will be called the local stabilizer of \( u \) with respect to \( G \) and denoted by \( G_u^{\text{loc}} \), that is

\[
G_u^{\text{loc}} := \{ g|_{\text{Star}(u)} \mid g \in G_u \} = \{ g|_{\text{Star}(u)} \mid g \in G, g(u) = u \}.
\]

Let also \( e \) be an edge of \( \text{Star}(u) \). Then the stabilizer of \( e \) with respect to \( G_u^{\text{loc}} \) will be denoted by \( G_{u,e}^{\text{loc}} \), so

\[
G_{u,e}^{\text{loc}} := \{ \gamma \in G_u^{\text{loc}} \mid \gamma(e) = e \} = \{ g|_{\text{Star}(u)} \mid g \in G, g(u) = u, g(e) = e \}.
\]
The following lemma is evident.

**Lemma 6.1.** Let \( X \subset T \) be a \( G \)-invariant subgraph, so \( G \) also acts on \( X \). Suppose \( w \in X \) is a vertex such that \( \text{Star}(w) \subset X \). Then the local stabilizers of \( w \) with respect to the action of \( G \) on \( T \) and on \( X \) coincide. \( \square \)

From now on we will assume that \( T \) is a finite tree.

**Definition 6.2.** For an edge \( uv \) of \( T \) let \( T_{uv} \) be the closure of the connected component of \( T \setminus \{u\} \) containing \( v \). We will call \( T_{uv} \) the subtree **growing out of the vertex** \( u \) and containing \( uv \), see Figure 6.2.

Let also \( \hat{T}_{uv} \) be the subtree tree obtained from \( T_{uv} \) by removing the vertex \( u \) and the open edge \( uv \).

![Figure 6.2. Subtrees \( T_{uv}, \hat{T}_{uv}, T_{vu}, \) and \( \hat{T}_{vu} \).](image)

### 6.3. \( t \)-decomposable actions.

Let \( uv \) be an edge of \( T \) and \( g \in G_{uv} \), so \( g(u) = u \) and \( g(v) = v \). Then \( g(T_{uv}) = T_{uv} \) and \( g(T_{vu}) = T_{vu} \). Therefore we can define the following automorphism \( r_{uv}(g) \) of \( T \) by the rule:

\[
 r_{uv}(g) = \begin{cases} 
 g & \text{on } T_{uv} \\
 \text{id} & \text{on } T_{vu}.
\end{cases} \tag{6.1}
\]

Notice that, in general, \( r_{uv}(g) \) does not belong to \( G \). It is also evident, that the correspondence \( g \mapsto r_{uv}(g) \) is a homomorphism:

\[
 r_{uv} : G_{uv} \rightarrow \text{Aut}(T). \tag{6.2}
\]

**Definition 6.3.1.** The action of \( G \) on \( T \) will be called **\( t \)-decomposable** if

\[
 r_{uv}(G_{uv}) \subset G
\]

for every edge \( uv \) of \( T \).

**Lemma 6.3.2.** Suppose \( r_{uv}(G_{uv}) \subset G \) for some edge \( uv \). Then \( r_{vu}(G_{vu}) \subset G \) as well.

**Proof.** It is evident that \( g = r_{uv}(g) \circ r_{vu}(g) \) for any \( g \in G_{vu} = G_{uv} = G_u \cap G_v \). By assumption \( g, r_{uv}(g) \in G \). Therefore \( r_{vu}(g) = g \circ r_{uv}(g)^{-1} \in G \) as well. \( \square \)

For each edge \( uv \) define the following subgroup of \( G \):

\[
 S_{uv} = \{ g \in G \mid \text{supp}(g) \subset \hat{T}_{uv} \},
\]

Evidently, \( S_{uv} \subset G_{uv} \).
Lemma 6.3.3. Suppose the action of $G$ on $T$ is $t$-decomposable. Let also $uv$ be an edge of $T$. Then

1. $r_{uv}(G_{uv}) = S_{uv}$, and the induced map $r_{uv} : G_{uv} \to S_{uv}$ is a retraction, that is $r_{uv} \circ r_{uv} = r_{uv}$ and the restriction of $r_{uv}$ to $S_{uv}$ is the identity.
2. Let $g \in G_{uv}$ and $A \subset T$ be a subset such that $g(A) = A$. Then $r_{uv}(g)(A) = A$. In particular, $r_{uv}(G_{uv} \cap G_{uv}) \subset G_{uv}$ for any vertex $w \in T$.
3. $r_{uv}$ does not increase supports, that is $\operatorname{supp}(r_{uv}(g)) \subset \operatorname{supp}(g)$ for each $g \in G_{uv}$.
4. The induced action of $S_{uv}$ on $T$ is $t$-decomposable.
5. If $G = S_{uv}$, so $T_{uv} \subset \operatorname{Fix}(G)$, then the induced action of $G$ on $\hat{T}_{uv}$ is also $t$-decomposable.

Proof. Statements (1)-(3) are easy and we left them for the reader.

Let $wz$ be an edge in $T$. We should prove that $r_{wz}((S_{uv})_{wz}) \subset S_{uv}$, that is for every $g \in G$ satisfying $\operatorname{supp}(g) \subset \hat{T}_{uv}$, $g(w) = w$, and $g(z) = z$ we have that $\operatorname{supp}(r_{wz}(g)) \subset \hat{T}_{uv}$ as well. But this follows from (3).

Let $wz$ be an edge of $\hat{T}_{uv}$, $g \in G_{wz}$, and $\hat{r}_{wz}(g)$ be an automorphism of $\hat{T}_{uv}$ defined similarly to (6.2), so

$$\hat{r}_{wz}(g) = \begin{cases} g & \text{on } T_{wz} \cap \hat{T}_{uv}, \\ \text{id} & \text{on } T_{zw} \cap \hat{T}_{uv}. \end{cases}$$

We have to show that $\hat{r}_{wz}(g) = h|_{\hat{T}_{uv}}$ for some $h \in G$. In fact, one can put $h = r_{wz}(g)$. \hfill \Box

6.4. Relation between $\tau$- and $t$-decomposability. Let $u$ be a vertex of $T$ and $y_1, \ldots, y_c$ be all the vertices adjacent to $u$. Then the set $T \setminus \{u\}$ admits a partition

$$\tau_u = \{T_u y_1 \setminus \{u\}\}_{i=1,\ldots,c}.$$

Notice that $G_u(T \setminus \{u\}) = T \setminus \{u\}$ and $G_u$ preserves partition $\tau_u$.

Since $T$ is a tree, there is a bijection between the edges in $\operatorname{Star}(u)$ and vertices adjacent to $u$. In particular, one can assume that $G_u^{\text{loc}}$ acts on vertices adjacent to $u$.

The following lemma is a direct consequence of definitions.

Lemma 6.4.1. The action of $G$ on $T$ is $t$-decomposable if and only if for every vertex $u \in T$ the action of the stabilizer $G_u$ on $T \setminus \{u\}$ is $\tau_u$-decomposable in the sense of Definition 5.2. \hfill \Box

Corollary 6.4.2. Let $u$ be a vertex of $T$, $v_1, \ldots, v_a, w_1, \ldots, w_b$ be the set of all vertices adjacent to $u$ such that $v_1, \ldots, v_a$ are all vertices fixed with respect to the local stabilizer $G_u^{\text{loc}}$. Assume that the restriction of $G_u^{\text{loc}}$ to the set $W = \{w_1, \ldots, w_b\}$ has exactly $p \leq b$ orbits $W_1, \ldots, W_p$ so that $w_j \in W_j$ for $j = 1, \ldots, p$. If the action of $G$ on $T$ is $t$-decomposable, then

$$G_u \cong \left( \bigtimes_{i=1}^a S_{uv_i} \right) \times \left( \left( \bigtimes_{j=1}^p \operatorname{Map}(W_j, S_{uw_j}) \right) \rtimes G_u^{\text{loc}} \right), \quad (6.3)$$

where the multiplication in the semi-direct product is induced by the right action of $G_u^{\text{loc}}$ on each set $W_j$. In particular, if the action of $G_u^{\text{loc}}$ on $W$ is free, then

$$G_u \cong \left( \bigtimes_{i=1}^a S_{uv_i} \right) \times \left( \left( \bigtimes_{j=1}^p S_{uw_j} \right) \rtimes G_u^{\text{loc}} \right). \quad (6.4)$$
Proof. By Lemma 6.4.1 the action of the stabilizer $G_u$ on $T \setminus \{u\}$ is $\tau_u$-decomposable in the sense of Definition 5.2. Therefore by Lemma 5.3 we have an isomorphism (6.3). Relation (6.4) follows from statements 2) and 3) of Lemma 5.1. □

6.5. Class $\mathcal{T}(\mathcal{G})$. Recall that an effective action of a group $G$ on a set $X$ is semi-free if the restriction of this action to the set of non-fixed points, $X \setminus \text{Fix}(G)$, is free. Equivalently, the action of $G$ is semi-free if $\text{Fix}(g) = \text{Fix}(h)$ for all $g, h \in G \setminus \{1\}$.

Evidently, that in this case for any subset $F \subset \text{Fix}(G)$, the set $X \setminus F$ is $G$-invariant, and the induced action of $G$ on $X \setminus F$ is semi-free as well.

Definition 6.5.1. Let $\mathcal{G} = \{G_i \mid i \in \Lambda\}$ be a family of finite groups containing the unit group $\{1\}$. We will say that a group $G$ belongs to the class $\mathcal{T}(\mathcal{G})$ if there exists an effective action of $G$ on a finite tree $T$ satisfying the following conditions:

(a) $\text{Fix}(G) \neq \emptyset$;
(b) the action of $G$ is $t$-decomposable;
(c) for every vertex $u$ of $T$ the local stabilizer $G_u^{\text{loc}}$ of $u$ belongs to $\mathcal{G}$ and its action on the set of edges of $\text{Star}(u)$ is semi-free.

Lemma 6.5.2. Suppose $G$ acts on $T$ so that the conditions (a)-(c) of Definition 6.5.1 hold, and $uv$ is an edge of $T$ such that $G_u \subset G_v$ for every vertex $w$ of $\hat{T}_{uv}$. Then

(i) the action of $S_{uv}$ on $T$ and
(ii) the induced action of $S_{uv}$ to $\hat{T}_{uv}$
also satisfy conditions (a)-(c) of Definition 6.5.1 for the same family of groups $\mathcal{G}$.

Proof. (a) Notice that $v \in \text{Fix}(S_{uv}) \cap \hat{T}_{uv} \neq \emptyset$. This proves that both actions of $S_{uv}$ have non-empty set of fixed points.

Condition (b) for the actions of $S_{uv}$ on $T$ and on $\hat{T}_{uv}$ follow from (4) and (5) of Lemma 6.3.3 respectively.

(c) First we check this condition for the action of $S_{uv}$ on $T$.

Let $w$ be any vertex of $T$. We should prove that the local stabilizer $(S_{uv})_{w}^{\text{loc}}$ of $w$ with respect to $S_{uv}$ belongs to $\mathcal{G}$ and its action on $\text{Star}(w)$ is semi-free.

If $w \in \hat{T}_{uv}$, then $\text{Star}(w) \subset T_{uv}$. Hence any $g \in S_{uv}$ is fixed on $\text{Star}(w)$. This means that $(S_{uv})_{w}^{\text{loc}}$ is a trivial group, and so it belongs to $\mathcal{G}$. In particular, the action of $(S_{uv})_{w}^{\text{loc}}$ on the set of edges in $\text{Star}(w)$ is free.

Suppose $w \in \hat{T}_{uv}$, so $\text{Star}(w) \subset T_{uv}$. It suffices to show that in this case $(S_{uv})_{w}^{\text{loc}} = G_{w}^{\text{loc}}$. Then condition (c) for $(S_{uv})_{w}^{\text{loc}}$ will follow from the condition (c) for $G_{w}^{\text{loc}}$. As $S_{uv} \subset G$, it suffices to show that $(S_{uv})_{w}^{\text{loc}} \supseteq G_{w}^{\text{loc}}$, that is for each $g \in G_{w}$ there exists $\hat{g} \in (S_{uv})_{w}$ such that $g|_{\text{Star}(w)} = \hat{g}|_{\text{Star}(w)}$.

Let $g \in G_{w}$. By assumption $G_{w} \subset G_{v}$, so $g(u) = u$.

We claim that $g(v) = v$ as well, i.e. $g \in G_{v}$. Indeed, let $u = z_0, z_1, \ldots, z_k = w$ be a unique simple path in $T_{uv}$ between $u$ and $z$. Since $uw$ is a unique edge in $T_{uv}$ adjacent to $u$, it follows that $z_1 = v$. Moreover, as $g$ fixes $u$ and $w$, it follows from uniqueness of this path that $g(z_i) = z_i$ for all $i$. In particular, $g(v) = v$.

Thus $g \in G_{u} \cap G_{v}$. Put $\hat{g} = r_{uv}(g)$. Then $\hat{g} \in S_{uv}$, and $\hat{g} = g$ on $T_{uv}$. In particular, $\hat{g} = g$ on $\text{Star}(w)$.
Now let us check condition (c) for the restriction of $\mathcal{S}_{u,v}$ to $\hat{T}_{u,v}$. Let $w \in \hat{T}_{u,v}$. Consider two cases.

1) Suppose $w \neq v$. Then $\text{Star}(w) \subset \hat{T}_{u,v}$, whence by Lemma 6.1 the local stabilizers of $w$ with respect to the actions of $\mathcal{S}_{u,v}$ on $T$ and $\hat{T}_{u,v}$ coincide.

2) Suppose $w = v$. Then $\text{Star}(w) \setminus uv$ is a regular neighborhood of $v$ in $\hat{T}_{u,v}$. As the edge $uv$ is fixed under $\mathcal{S}_{u,v}$, it follows that the local stabilizers of $v$ with respect to the actions of $\mathcal{S}_{u,v}$ on $T$ and $\hat{T}_{u,v}$ are isomorphic, and the action of the local stabilizer of $v$ with respect to the action of $\mathcal{S}_{u,v}$ on $\hat{T}_{u,v}$ is also semi-free. □

Our main result claims that classes $\mathcal{R}(\mathcal{G})$ and $\mathcal{T}(\mathcal{G})$ coincide, see Definition 1.3.

**Theorem 6.6.** $\mathcal{R}(\mathcal{G}) = \mathcal{T}(\mathcal{G})$ for any family $\mathcal{G} = \{G_i \mid i \in \Lambda\}$ of finite groups containing the unit group $\{1\}$.

**Proof.** First we show that $\mathcal{T}(\mathcal{G}) \subset \mathcal{R}(\mathcal{G})$. Let $G \in \mathcal{T}(\mathcal{G})$, so there is an effective action of $G$ on a finite tree $T$ satisfying conditions (a)-(c) of Definition 6.5.1. We should prove that $G \in \mathcal{R}(\mathcal{G})$.

Let us use the induction on the number of vertices in a tree $T$ on which $G$ can act satisfying that conditions (a)-(c).

If $T$ is a vertex, the group $G$ is trivial, and so it belongs to $\mathcal{R}(\mathcal{G})$.

In general case, due to (a) there exists a fixed vertex $u \in \text{Fix}(G) \neq \emptyset$, whence $G = G_u$. Then using notations of Corollary 6.4.2 we have an isomorphism (6.4):

$$G = G_u \cong \left( \times_{i=1}^{\ell} S_{u,v_i} \right) \times \left( \left( \times_{j=1}^{p} S_{u,w_j} \right) \wr G_u^{\text{loc}} \right),$$

where $G_u^{\text{loc}} \in \mathcal{G}$. We claim that $\mathcal{S}_{u,z} \in \mathcal{R}(\mathcal{G})$ for each $z \in \{v_1, \ldots, v_a, w_1, \ldots, w_p\}$. This will imply that $G \in \mathcal{R}(\mathcal{G})$ as well.

Notice that $G_x \subset G_u = G$ for every vertex $x$ of $T_{u,z}$. Therefore by Lemma 6.5.2 the action of $\mathcal{S}_{u,z}$ on $\hat{T}_{u,z}$ satisfies conditions (a)-(c) of Definition 6.5.1. But $u \notin \hat{T}_{u,z}$, so $\hat{T}_{u,z}$ contains less vertices than $T$, whence by inductive assumption $\mathcal{S}_{u,z} \in \mathcal{R}(\mathcal{G})$.

Now we will establish the inverse inclusion $\mathcal{R}(\mathcal{G}) \subset \mathcal{T}(\mathcal{G})$.

(i) Suppose $A_1, A_2 \in \mathcal{T}(\mathcal{G})$. Then $A_i$, $i = 1, 2$, acts on some tree $T_i$ so that the conditions (a)-(c) of Definition 6.5.1 hold true. We should prove that $A_1 \times A_2 \in \mathcal{T}(\mathcal{G})$.

For $i = 1, 2$ choose a fixed point $u_i \in \text{Fix}(A_i)$. Not losing generality, one can assume that $u_i$ has degree 1 in $A_i$. Let $T = T_1 \vee_{u_1 = u_2} T_2$ be the tree obtained from $T_1$ and $T_2$ by identifying $u_1$ with $u_2$. Then each $(g_1, g_2) \in A_1 \times A_2$ defines a unique automorphism $f_{g_1, g_2}$ of $T$ equal to $g_i$ on $A_i$, $i = 1, 2$.

It is easy to check that the obtained action of $A_1 \times A_2$ satisfies conditions (a)-(c) of Definition 6.5.1. We left the details for the reader.

(ii) Suppose $A \in \mathcal{T}(\mathcal{G})$ and $G \in \mathcal{G}$. We have to show that $A \wr G \in \mathcal{T}(\mathcal{G})$.

By assumption $A$ acts on some tree $T$ so that the conditions (a)-(c) of Definition 6.5.1 hold true. Let $u$ be any fixed point of this action. Again one can assume that $\deg u = 1$.

For each $g \in G$ take a copy $T_g$ of the tree $T$ and fix an isomorphism $\phi_g : T \to T_g$. Let also $u_g = \phi_g(u)$ be a vertex of $T_g$ corresponding to $u$. 

Let \( S = \bigvee_{\{u_g \mid g \in G\}} T_g \) be a tree obtained from disjoint union of all \( T_g \) by identifying all points \( u_g \). Now for every \((\alpha, g) \in A \wr G = \text{Map}(G, A) \rtimes G\) define an automorphism \( f_{\alpha, g} \) of \( S \) by the following rule:

\[
f_{\alpha, g}(T_h) = T_{gh}
\]

and

\[
f_{\alpha, g}|_{T_h} = \phi_{gh} \circ \alpha(h) \circ \phi_h^{-1}.
\]

Again it is not hard to verify that the latter formula gives an action of \( A \wr G \) on \( T \) satisfying conditions (a)-(c) of Definition 6.5.1, i.e. \( A \wr G \in \mathcal{T}(G) \).

Thus \( \mathcal{T}(G) = \mathcal{R}(G) \). Theorem 6.6 completed. \( \square \)

Remark 6.7. Let \( T \) be a finite tree and \( G = \text{Aut}(T) \) be the group of its automorphisms.

Then by Jordan’s Theorem 4.3 and by Theorem 6.6

\[
G \in \mathcal{R}(\{S_n\}_{n \geq 1}) = \mathcal{T}(\{S_n\}_{n \geq 1}).
\]

In particular, \( G \) admits an action on a finite tree \( T' \) satisfying conditions (a)-(c) of Definition 6.5.1. On the other hand we have a natural action of \( G \) on \( T \). It is easy to see that this action is \( t \)-decomposable, however other conditions (a) and (c) of Definition 6.5.1 can fail. Therefore, in general, \( T' \neq T \).

7. Critical level-sets

Let \( f \in \mathcal{F}(M, P) \). For every subset \( Y \subset M \) define the following groups

\[
S(f, Y) := \{ h \in S(f) \mid h(Y) = Y \}, \quad S'(f, Y) := S(f, Y) \cap \text{D}_\text{id}(M).
\]

If \( Y \) is an orientable submanifold of \( M \), then we also set

\[
S(f, +Y) := \{ h \in S(f) \mid h(Y) = +Y \}.
\]

Let \( U \) be a critical element of partition \( \Theta_f \) and \( c = f(U) \in P \) be the value of \( f \) on \( U \).

Fix \( \varepsilon > 0 \) such that the closed interval \([c - \varepsilon, c + \varepsilon] \subset P \) contains no critical values of \( f \) except for \( c \), and denote by \( N \) the connected component of \( f^{-1}[c - \varepsilon, c + \varepsilon] \) containing \( U \), see Figure 7.1.

If

\[
N \cap \partial M = \emptyset \quad (7.1)
\]

then we will call \( N \) a regular \( \varepsilon \)-neighborhood of \( U \). Such a neighbourhood is often called an atom of \( U \), see e.g. [9, 10, 2]. In fact, if \( \varepsilon \) is sufficiently small, then (7.1) always hold.

So assume that \( N \) is a regular \( \varepsilon \)-neighborhood of \( U \). Denote

\[
\partial_- N := N \cap f^{-1}(c - \varepsilon), \quad \partial_+ N := N \cap f^{-1}(c + \varepsilon).
\]

Then

\[
\partial N = \partial_- N \cup \partial_+ N.
\]

Lemma 7.1. Let \( B \) be a connected component of \( \partial N \). Then

\[
S(f, B) \subset S(f, U) = S(f, N).
\]
Proof. First we prove that
\[ S(f, B), S(f, U) \subset S(f, N). \]
Denote \( Q = f^{-1}[c - \varepsilon, c + \varepsilon] \). Then \( h(Q) = Q \) for every \( h \in S(f) \), as \( f \circ h = f \). Notice that \( N \) is a connected component of \( Q \) containing \( U \) and \( B \). Hence if \( h \in S(f) \) is such that either \( h(V) = V \) or \( h(B) = B \), then \( h(N) = N \). This completes (7.2).

It remains to show that \( S(f, N) \subset S(f, U) \). Let \( h \in S(f, N) \). From \( U = N \cap f^{-1}(c) \), it follows that
\[ h(U) = h(N \cap f^{-1}(c)) = h(N) \cap h(f^{-1}(c)) = N \cap f^{-1}(c) = U, \]
i.e. \( h \in S(f, U) \). Thus \( S(f, N) \subset S(f, U) \). \( \square \)

7.2. Action of \( S(f, N) \). Let \( \lambda : S(f) \to \text{Aut}(\Gamma(f)) \) be the action homomorphism of \( S(f) \) on the enhanced KR-graph \( \Gamma(f) \) of \( f \), and \( u = p(U) \) be the vertex of \( \Gamma(f) \) corresponding to \( U \). Denote
\[ G := \lambda(S(f)), \quad G' := \lambda(S')(f)). \]

Evidently, \( N \) is the union of some elements of the partition \( \Theta_f \), and the restriction \( f|_N \) of \( f \) to \( N \) belongs to \( \mathcal{F}(N, P) \). Therefore image of \( N \) under \( p \) coincides with the enhanced KR-graph of \( f|_N \), that is
\[ \Gamma(f|_N) = p(N). \]
It is easy to see that \( \Gamma(f|_N) \) is a closed neighborhood of \( u \) in \( \Gamma(f) \), and so it can be regarded as a regular neighborhood \( \text{Star}(u) \) of \( u \) in \( \Gamma(f) \), see Figure 7.1 where \( \Gamma(f|_N) \) is drawn with bold lines.

In particular, one can define the partition \( \Delta_{f|_N} \) which clearly consists of all elements of \( \Delta_f \) contained in \( N \). Let
\[ r_N : S(f, N) \to S(f|_N), \quad r_N(h) = h|_N, \]
be “the restriction to \( N \) map” and
\[ \lambda_N : S(f|_N) \to \text{Aut}(\Gamma(f|_N)) \equiv \text{Aut}(\text{Star}(u)) \]
be the corresponding action homomorphism.

Notice that \( p \) induces a bijection between the connected components of \( \partial N \) and edges of \( \Gamma(f|_N) \). Let \( B \) be a boundary component of \( \partial N \) and \( e \) be the edge of \( \Gamma(f|_N) \) corresponding to \( B \). The following lemma is easy and we left it for the reader:
Lemma 7.2.1. The following identities hold true:

\[ \lambda(S(f, U)) = G_u \equiv \{ g \in G \mid g(u) = u \}, \]
\[ \lambda_N \circ r_N(S(f, U)) = G^\text{loc}_{u} \equiv \{ g|_{r(f|_N)} \mid g \in G_u \}, \]
\[ \lambda_N \circ r_N(S(f, B)) = G^\text{loc}_{u,e} \equiv \{ \hat{g} \in G^\text{loc}_u \mid \hat{g}(e) = e \}, \]

and similarly for the group \( G \):

\[ \lambda(S'(f, U)) = G_u, \quad \lambda_N \circ r_N(S'(f, U)) = G^\text{loc}_{u}, \quad \lambda_N \circ r_N(S'(f, B)) = G^\text{loc}_{u,e}. \]

7.3. Cycle \( \omega(B) \). Let \( N_B \) be the connected component of \( N \setminus U \) containing \( B \), and let

\[ C(B) := U \cap \overline{N_B}, \]

see Figure 7.2.

![Figure 7.2. Cycle \( \omega(B) \).](image)

**Figure 7.2. Cycle \( \omega(B) \).**

Lemma 7.3.1. 1) \( C(B) \) is a connected subcomplex of \( U \) and for every open edge \( \delta \) of \( U \) there exists a unique connected component \( B_- \) of \( \partial_- N \) such that \( \delta \subset C(B_-) \) and a unique connected component \( B_+ \) of \( \partial_+ N \) such that \( \delta \subset C(B_+) \).

Suppose \( U \) contains more than one point, so it is not a local extreme. Then there exists an oriented Eulerian cycle \( \omega(B) \) in \( C(B) \) (i.e. a closed path visiting every edge exactly once), see Figure 7.2, such that the following statements hold.

2) Let \( h : N \to N \) be a homeomorphism such that \( h(U) = U \), \( B_0, B_1 \) be two oriented connected components of \( \partial N \), and \( \varepsilon = + \) or \(-\). Then \( h(B_0) = \varepsilon B_1 \) if and only if \( h(\omega(B_0)) = \varepsilon(\omega(B_1)) \).

3) Let \( h : N \to N \) be a homeomorphism such that \( h(U) = U \) and \( h(\partial_- N) = \partial_+ N \). Suppose there exists an open edge \( \delta \) of \( U \) such that \( h(\delta) = +\delta \). Then \( h \) is a trivial automorphism of \( U \), whence, by 2), \( h(B) = +B \) for every connected component of \( \partial N \). Moreover, if \( f \circ h = f \), then \( h \) preserves each element of \( \Delta_f \) with its orientation.

Proof. Statements 1) and 2) are well-known. They are widely used e.g. in [13, 6, 36, 19, 38, 15, 32] and others. Statement 3) is contained in [26, Claim 7.1.1].

7.4. Critical levels with more than one point. Assume that \( U \) contains more than one point.

Let \( \text{Aut}(\omega(B)) \) be the group of combinatorial automorphisms of the graph \( C(B) \) which preserve or reverse the cyclic order of edges in the cycle \( \omega(B) \), and \( \text{Aut}^+(\omega(B)) \) be the subgroups of \( \text{Aut}(\omega(B)) \) consisting of automorphism preserving orientation of \( \omega(B) \).

Then \( \text{Aut}(\omega(B)) \) is a subgroup of a dihedral group \( D_n \), where \( n \) is the number of edges in \( C(B) \). In particular, \( \text{Aut}(\omega(B)) \) is either a cyclic or a dihedral group, while \( \text{Aut}^+(\omega(B)) \) is a cyclic group, and the orders of both groups divide \( n \).
As noted above, \( f|_N \in \mathcal{F}(M, P) \), so we can define the partition \( \Delta_{f|_N} \). Clearly, \( \Delta_{f|_N} \) consists of all elements of \( \Delta_f \) contained in \( N \). Let \( \mathcal{D}(\Delta_{f|_N}) \) be the group of its automorphisms, see \( \text{[2.9]} \).

**Lemma 7.4.1.** 1) There exists a homomorphism \( \alpha : \mathcal{S}(f|_N, B) \longrightarrow \text{Aut}(\omega(B)) \) such that

(a) \( \ker(\alpha) = \mathcal{D}(\Delta_{f|_N}) \);

(b) \( \alpha(\mathcal{S}(f|_N, +B)) \subset \text{Aut}^+(\omega(B)) \).

Hence \( \alpha(\mathcal{S}(f|_N, B)) \) is either finite cyclic or dihedral, while \( \alpha(\mathcal{S}(f|_N, +B)) \) is cyclic.

2) Moreover, there exists an epimorphism \( \beta : \alpha(\mathcal{S}(f|_N, B)) \longrightarrow \lambda_N(\mathcal{S}(f|_N, B)) \) such that \( \lambda_N|_{\mathcal{S}(f|_N, B)} = \beta \circ \alpha \), so the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{S}(f, B) & \xrightarrow{\mathcal{r}_N} & \mathcal{S}(f|_N, B) \\
\alpha & & \beta \\
\lambda_N(\mathcal{S}(f|_N, B)) & & \alpha(\mathcal{S}(f|_N, B))
\end{array}
\]

Hence

(i) the group \( \lambda_N \circ \mathcal{r}_N(\mathcal{S}(f, B)) \) is either finite cyclic or dihedral, while

(ii) the group \( \lambda_N \circ \mathcal{r}_N(\mathcal{S}(f, +B)) \) is always cyclic.

**Proof.** 1) Let \( h \in \mathcal{S}(f|_N, B) \). Then due to 2) of Lemma \( \text{[7.3.1]} \), \( h(\omega(B)) = \omega(B) \), so the restriction \( h|_{\omega(B)} \) induces a combinatorial automorphism of \( \omega(B) \) which we will denote by \( \alpha(h) \). Evidently, the correspondence \( h \mapsto \alpha(h) \) is the desired homomorphism \( \alpha \).

(a) Since the vertices and edges of \( \omega(B) \) are elements of \( \Delta_f \), we see that \( \mathcal{D}(\Delta_{f|_N}) \subset \ker(\alpha) \).

Conversely, let \( h \in \ker(\alpha) \), so \( h \) preserves every vertex and every edge of \( \omega(B) \) with their orientation. Then, by 3) of Lemma \( \text{[7.3.1]} \), \( h \in \mathcal{D}(\Delta_{f|_N}) \).

(b) If \( h \in \mathcal{S}(f|_N, +B) \), so \( h \) preserves orientation of \( B \), then by 2) of Lemma \( \text{[7.3.1]} \), \( \alpha(h) \in \text{Aut}^+(\omega(B)) \).

2) Existence of \( \beta \) follows from the relation \( \ker(\alpha) \xrightarrow{(a)} \mathcal{D}(\Delta_{f|_N}) \xrightarrow{(b)} \ker(\lambda_N) \).

Statements (i) and (ii) follow from the description of \( \alpha(\mathcal{S}(f|_N, B)) \) and \( \alpha(\mathcal{S}(f|_N, +B)) \) given in 1).

**Corollary 7.4.2.** 1) The group \( G_{u,e}^{\text{loc}} = \lambda_N \circ \mathcal{r}_N(\mathcal{S}(f, B)) \) is either cyclic or dihedral.

2) If \( M \) is orientable, then the group \( G_{u,e}^{\text{loc}} = \lambda_N \circ \mathcal{r}_N(\mathcal{S}(f, B)) \) is cyclic.

**Proof.** Statement 1) coincides with 2(i) of Lemma \( \text{[7.4.1]} \).

2) Suppose \( M \) is orientable. We claim that then

\[
\mathcal{S}'(f, B) \subset \mathcal{S}(f, +B)
\]  \hspace{1cm} (7.3)

which will imply by 2(ii) of Lemma \( \text{[7.4.1]} \) that \( G_{u,e}^{\text{loc}} \) is a subgroup of the cyclic group \( \lambda_N \circ \mathcal{r}_N(\mathcal{S}(f, +B)) \).
Let $q = f(B)$, $W$ be a small cylindrical neighborhood of $B$, such that $W \setminus B$ consists of two connected components $W_1$ and $W_2$ with $f(W_1) \cap f(W_2) = \emptyset$. We can assume that $W$ is a connected component of a set $f^{-1}[q - \mu, q + \mu]$ for some small $\mu$.

Let $h \in S'(f, B) = S(f, B) \cap D_{id}(M)$. Since $f \circ h = f$ and $h(B) = B$, it follows that $h(W) = W$, and $h(W_i) = W_i$, $i = 1, 2$. Moreover, as $h$ is isotopic to $id_M$, it preserves orientation of $M$, and so we must have that $h(B) = +B$. This proves (7.3). □

7.5. **Local extremes.** Suppose $U = \{z\}$ is a local extreme of $f$. Then the vertex $u = p(z)$ has degree 1 in the KR-graph $M/\Theta_f$. Let also $v$ be a unique vertex of $M/\Theta_f$ adjacent to $u$.

**Lemma 7.6.**

1) If $z$ is a non-degenerate local extreme, then the local stabilizers $G_{u}^{loc}$ and $G_{u}$ are trivial.

2) Suppose $z$ is a degenerate local extreme having symmetry index $n$. Then $G_{u}$ is either a cyclic group $\mathbb{Z}_n$ or a dihedral group $\mathbb{D}_n$.

If $M$ is orientable, then $G_{u}^{loc} \cong \mathbb{Z}_n$ and its action on $Star(u)$ is semi-free.

**Proof.** 1) If $z$ is a non-degenerate local extreme, then $u$ has degree 1 in the enhanced KR-graph $\Gamma(f)$. Therefore its $Star(u)$ consists of a unique edge $uv$, and so both groups $G_{u}^{loc} \subseteq G_{u}$ are trivial.

2) Suppose $z$ is a degenerate local extreme having symmetry index $n$. Thus besides $v$ there are also $n$ vertices adjacent to $u$ in the enhanced KR-graph $\Gamma(f)$:

$$\xi_z^0, \ldots, \xi_z^{n-1}, \tag{7.4}$$

where each $\xi_z^i$ is regarded as a tangent vector in $T_z M$ belonging to the framing $F$.

By Definition 2.5 for each $h \in S(f, U)$ its tangent map $T_z h$ preserves or reverses cyclic order of vectors (7.4), whence $g = \lambda(h) \in G_u$ preserves cyclic order of vertices (7.4), and so $g(v) = v$. Moreover, by (2) of Lemma 2.4 for each $i = 1, \ldots, n - 1$ there exists $h \in S(f, U)$ such that $T_z h(\xi_z^0) = \xi_z^i$, so $G_u$ acts on vertices (7.4) transitively.

This implies that $G_{u}^{loc} \cong \mathbb{D}_n$ if there exists $h \in S(f, U)$ reversing cyclic order of (7.4) and $G_{u}^{loc} \cong \mathbb{Z}_n$ otherwise.

Suppose $M$ is orientable. Then each $h \in S'(f, U)$ being isotopic to $id_M$ preserves cyclic order of vectors (7.4). We will find $\hat{h} \in S'(f, U)$ such that $T_z \hat{h}(\xi_z^0) = \xi_z^1$. This will imply that $G_{u}^{loc} \cong \mathbb{Z}_n$.

Let $h \in S(f, U)$ be any diffeomorphism such that $T_z h$ preserves cyclic order of vectors (7.4), and $T_z h(\xi_z^i) = \xi_z^i$. Let $N$ be a regular $\varepsilon$-neighborhood of $z$. Then $N$ is a 2-disk, $h(\partial N) = \partial N$, and $h$ preserve orientation of $\partial N$. Therefore $h$ is isotopic in $S(f)$ to a diffeomorphism $\hat{h}$ fixed on some neighborhood of $\partial N$. Change $\hat{h}$ on $M \setminus N$ by the identity. Then $\hat{h} \in S'(f, U)$ and $\hat{h} = h$ on $N$.

Since each non-trivial element $g \in G_{u}^{loc}$ cyclically permutes vertices (7.4) and fixes $v$, we see that the action of $G_{u}^{loc}$ on $Star(u)$ is semi-free. □

8. **Proof of Theorem 1.5.** **Case** $M = D^2$ and $S^1 \times I$

Let $M$ be either a 2-disk $D^2$ or a cylinder $S^1 \times I$. We will prove that

$$\mathcal{R}([\mathbb{Z}_n]_{n \geq 1}) = \{ G(f) \mid f \in \mathcal{F}(M, P) \}.$$
The proof consists of two Lemmas 8.1 and 8.2 showing that each of the classes contains the other.

**Lemma 8.1.** Let $M$ be either a 2-disk $D^2$ or a cylinder $S^1 \times I$, and let $f \in \mathcal{F}(M, P)$. Then the action of the group $G(f)$ on $\Gamma(f)$ satisfies conditions (a)-(c) of Definition 6.5.1 for $\mathcal{G} = \{\mathbb{Z}_n\}_{n \geq 1}$ being the family of all finite cyclic groups. In particular, by Theorem 6.6

$$G(f) \in \mathcal{T}(\{\mathbb{Z}_n\}_{n \geq 1}) = \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1}),$$

and so $\{G(f) \mid f \in \mathcal{F}(M, P)\} \subset \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$.

**Proof.** (a) Since $M$ is either $D^2$ or $S^1 \times I$, the enhanced KR-graph $\Gamma(f)$ is a finite tree. Then only one of the following two possibilities hold, see e.g. [14] or [37, §46]:

(i) either $\text{Fix}(G(f)) \neq \emptyset$,

(ii) or there exists a common invariant edge $uv$ for all elements of $g \in G$, so $g(uv) = uv$ or $g(uv) = vu$ and both cases are realized.

Notice that the edges of $\Gamma(f)$ are oriented via orientation of $P$, and $G(f)$ preserves those orientations. This implies, we see that the case (ii) is impossible. Hence $\text{Fix}(G(f)) \neq \emptyset$.

(b) We should prove that the action of $G(f)$ on $\Gamma(f)$ is $t$-decomposable.

Let $uv$ be an edge of enhanced KR-graph $\Gamma(f)$, $T_{uv}$ be the tree growing out of $u$, $g \in G(f)$ be such that $g(u) = u$ and $g(v) = v$, $h \in S'(f)$ be such that $g = \lambda(h)$, and $\hat{g} = r_{uv}(g)$. We have to show that $\hat{g} \in G(f)$ as well, that is there exists $\hat{h} \in S'(f)$ such that $\hat{g} = \lambda(\hat{h})$.

Due to Lemma 6.3.2 we can exchange roles of vertices $u$ and $v$. Therefore we can assume that $u$ is a vertex of the (usual) KR-graph $M/\Theta_f$ of $f$, i.e. it corresponds to a critical element of $\Theta_f$ but not to a tangent vector belonging the framing $F$. Let also $N$ be a regular $\varepsilon$-neighborhood of $U$.

Consider two cases.

(b1) Suppose $v$ corresponds to a tangent vector of the framing $F$. In other words, $u$ is a vertex corresponding to a degenerate local extreme $z$ of $f$ and $v = \xi^i_z$ for some $i$. Hence $T_{uv}$ is the edge $uv$, and therefore $\hat{g} = r_{uv}(g) = \text{id}_{\Gamma(f)} \in G(f)$.

(b2) Suppose $v$ is a vertex of KR-graph $M/\Theta_f$, and $B$ be the boundary component of $N$ corresponding to the edge $uv$. Notice that $B$ separates $M$. Denote by $Q$ the closure of the connected component of $M \setminus B$ that does not contain $N$. Then $h(Q) = Q$. Not loosing generality one can assume that $h$ is fixed on some neighborhood of $B$. Define a diffeomorphism $\hat{h} : M \to M$ by the rule: $\hat{h} = h$ on $Q$ and $\hat{h} = \text{id}$ on $M \setminus Q$. Then $\hat{h} \in S(f)$, and it is evident that $\lambda(\hat{h}) = \hat{g}$.

Therefore it remains to show that $\hat{h}$ is isotopic to $\text{id}_M$, which will imply that $\hat{h} \in S'(f) = S(f) \cap \mathcal{D}_{\text{id}}(M)$.

Notice that $\hat{h}$ is fixed on the non-empty open set $M \setminus Q$, whence it preserves orientation of $M$. Moreover, $\hat{h}$ preserves connected components of $\partial M$. As $M$ is either $D^2$ or $S^1 \times I$, it follows that $\hat{h}$ is isotopic to $\text{id}_M$. 


(c) Let \( u \) be a vertex of the enhanced KR-graph \( \Gamma(f) \). We should prove that its local stabilizer \( G^{loc}_u \) is a cyclic group and its action on the set of adjacent edges to \( u \) is semi-free.

Consider three cases.

(c1) Suppose \( u = \xi^*_i z \) corresponds to a tangent vector at some degenerate local extreme \( z \) of \( f \). Then \( \text{Star}(u) \) consists of a unique edge \( uz \), whence \( G^{loc}_u \) is a unit group. In particular, this group is cyclic and its action on \( \text{Star}(u) \) is semi-free.

(c2) Suppose \( u \) corresponds to a degenerate local extreme \( z \) of \( f \). Then our statement is contained in 2) of Lemma 7.6.

(c3) Now assume that \( u \) is a vertex of \( \Gamma(f) \) corresponding to a critical element \( U \) of \( \Theta f \) being not a degenerate local extreme.

First we recall one result from [29]. Let \( \Xi \) be a cellular decomposition of a closed surface \( N \) and \( h : N \to N \) be a homeomorphism. If \( h(\alpha) = \alpha \) for some cell \( \alpha \) of \( \Xi \), then we will say that \( \alpha \) is \( h \)-invariant. Moreover, \( \alpha \) will be called \( h^+\)-invariant if either \( \dim \alpha = 0 \), or \( \dim \alpha \geq 1 \) and \( h \) preserves orientation of \( \alpha \). We will say that \( h \) is a \( \Xi \)-homeomorphism if for every cell \( \alpha \) of \( \Xi \) its image \( h(\alpha) \) is also a cell of \( \Xi \). We also say that a homeomorphism \( h \) is \( \Xi \)-trivial if every cell of \( \Xi \) is \( h^+\)-invariant.

Lemma 8.1.1. [29] Proposition 5.4.] Let \( N \) be a closed, orientable surface endowed with some cellular decomposition \( \Xi \) and \( h : N \to N \) a \( \Xi \)-homeomorphism preserving orientation of \( N \). Then either

1. \( h \) is \( \Xi \)-trivial, or
2. the number of \( h \)-invariant cells of \( \Xi \) is equal to the Lefschetz number \( L(h) \) of \( h \), so in this case \( L(h) \geq 0 \).

Let \( \hat{M} \) be a closed surface obtained by gluing each boundary component of \( M \) with a 2-disk, so \( \hat{M} \) is a 2-sphere.

Let also \( U \) be the critical element of \( \Theta f \) corresponding to \( u \) and \( N \) be a regular \( \varepsilon \)-neighborhood of \( U \), and \( \Xi \) be the CW-subdivision of \( \hat{M} \) by vertices and edges of \( U \) and connected components of \( \hat{M} \setminus U \). Evidently, there is a natural bijection between 2-cells of \( \Xi \) and edges of \( \text{Star}(u) \).

Let \( g \in G^{loc}_u = \lambda_N \circ r_N(S'(f,U)) \) be a non-unit element, and \( h \in S'(f,U) \) be such that

\[
g = \lambda_N \circ r_N(h).
\]

Since \( h \) is isotopic to \( \text{id}_M \), it leaves invariant every connected component of \( \partial M \) and preserves its orientation. Therefore we can extend \( h \) onto the union \( \hat{M} \setminus M \) of glued 2-disks to some homeomorphism \( \hat{h} \) of \( \hat{M} \). As \( h(U) = \hat{h}(U) = U \), \( \hat{h} \) induces a certain permutation of cells of \( \Xi \), whence \( \hat{h} \) is a \( \Xi \)-homeomorphism.

Evidently, fixed edges of \( g \) correspond to \( \hat{h} \)-invariant 2-cells of \( \Xi \) and vice versa. In particular, as \( g \) is a non-unit element, we see that \( \hat{h} \) is not \( \Xi \)-trivial.

Moreover, \( \hat{h} \) preserves orientation of \( \hat{M} = S^2 \), and therefore it is isotopic to \( \text{id}_{\hat{M}} \). Hence we get from (ii) of Lemma 8.1.1 that the number of \( \hat{h} \)-invariant cells is equal to

\[
L(\hat{h}) = L(\text{id}_{S^2}) = \chi(S^2) = 2.
\]
Let $V$ be a connected component of $\partial M$, $\alpha$ be a 2-cell of $\Xi$ containing $V$, and $e$ be the edge of $\text{Star}(u)$ corresponding to $\alpha$. Then $\hat{h}(\alpha) = \alpha$, and so $g(e) = e$. Since $\alpha$ does not depend on $g$, we see that $e$ is a fixed edge of $G^\text{loc}_u$, whence by Corollary 7.4.2 $G^\text{loc}_u = G^\text{loc}_{u,e} \cong \mathbb{Z}_n$ for some $n \geq 1$.

It remains to show that the action of $G^\text{loc}_u$ on the set of edges of $\text{Star}(u)$ is semi-free. For $g \in G^\text{loc}_u$ let $\text{Fix}(g)$ be the set of fixed edges of $g$ on $\text{Star}(u)$. Then it suffices to establish that $\text{Fix}(g) = \text{Fix}(g_1)$ for all $g, g_1 \in G^\text{loc}_u \setminus \{1\}$.

Using the notation above assume that $g$ is a generator of $G^\text{loc}_u \cong \mathbb{Z}_n$. Let $\alpha_1$ be another $\hat{h}$-invariant cell of $\Xi$. Then for $k = 1, \ldots, n - 1$ the homeomorphism $\hat{h}_k$ is not $\Xi$-trivial, and by the same arguments as above has exactly two invariant cells. Therefore these cells must be $\alpha$ and $\alpha_1$. Thus $\text{Fix}(g_k) = \{e, e_1\}$ if $\dim \alpha_1 = 2$, and $\text{Fix}(g_k) = \{e\}$ otherwise.

\[ \square \]

**Lemma 8.2.** Let $M = D^2$ or $S^1 \times I$. Denote $\mathcal{M} = \{ G(f) \mid f \in \text{Morse}(M, P) \}$. Then

1) $\mathbb{Z}_n \in \mathcal{M}$ for each $n \geq 1$;
2) if $G_0, G_1 \in \mathcal{M}$, then $G_0 \times G_1 \in \mathcal{M}$;
3) if $G \in \mathcal{M}$ and $n \geq 1$, then $G \wr \mathbb{Z}_n \in \mathcal{M}$ as well.

Hence $\mathcal{R}(\{ \mathbb{Z}_n \}_{n \geq 1}) \subset \mathcal{M} \subset \{ G(f) \mid f \in \mathcal{F}(M, P) \}$.

**Proof.**

1) Structure of critical level-sets of the function $f$ satisfying $G(f) \cong \mathbb{Z}_n$ is shown in Figure 8.1 (a) for $M = D^2$ and in (b) for $M = S^1 \times I$.

![Figure 8.1](image)

**Figure 8.1.** Maps $f : M \to P$ with $G(f) \cong \mathbb{Z}_n$, for $n = 3$.

2) Suppose $G_0, G_1 \in \mathcal{M}$. We have to construct a mapping $f \in \text{Morse}(M, P)$ such that $G(f) \cong G_0 \times G_1$. Such a map schematically shown in Figure 8.2. Let us describe its structure in details.

![Figure 8.2](image)

**Figure 8.2.** Maps $f : M \to P$ with $G(f) \cong G(f_1) \times G(f_2)$

Suppose $M = D^2$. Let $f \in \text{Morse}(M, P)$ be a Morse function reaching its minimum on $\partial M$, having 1 saddle and two local maximums $z_0$ and $z_1$ and such that $f(z_0) \neq f(z_1)$. Let also $Y_i, i = 0, 1$, be a small neighborhood of $z_1$ diffeomorphic to 2-disk and such that
\[ \partial Y_i \text{ is a connected component of some level-set of } f, \text{ and } Y_0 \cap Y_1 = \emptyset. \] Replace \( f \) on \( Y_i \) so that \( G(f|_{Y_i}) = G_i \), and \( f(Y_0) \cap f(Y_1) = \emptyset \), see Figure 8.2(a).

If \( M = S^1 \times I \), then let \( f \in \text{Morse}(M, P) \) without critical points, so we can assume that \( f \) takes its minimum of \( S^1 \times 0 \) and its maximum on \( S^1 \times 1 \). Let \( Y_i, i = 0, 1 \), be a regular neighborhood of \( S^1 \times i \) diffeomorphic to a cylinder and such that \( \partial Y_i \) consists of connected components of some level-set of \( f \), and \( Y_0 \cap Y_1 = \emptyset \). Replace \( f \) on \( Y_i \) so that \( G(f|_{Y_i}) = G_i \) and \( f(Y_0) \cap f(Y_1) = \emptyset \), see Figure 8.2(b).

We claim that in both cases \( G(f) \cong G(f_0) \times G(f_1) \). Indeed, first notice that we can regard \( \Gamma(f_i) \) as a subgraph of \( \Gamma(f) \). Let \( \lambda_i : S(f_i) \to \text{Aut}(\Gamma(f_i)) \) be the corresponding action homomorphism, and \( \lambda : S(f) \to \text{Aut}(\Gamma(f)) \).

Let \( h \in S(f) \) and \( g = \lambda(h) \). Since \( f(Y_0) \cap f(Y_1) = \emptyset \), it follows that \( g \) is supported in \( \Gamma(f_0) \cup \Gamma(f_1) \), and
\[
\lambda(g|_{\Gamma(f_0)}) = \lambda_i(h|_{Y_i}), \quad i = 0, 1. 
\]

Therefore we have a homomorphism
\[
\eta : G(f) \longrightarrow G(f_0) \times G(f_1), \quad \eta(g) = (g|_{\Gamma(f_0)}, g|_{\Gamma(f_1)}).
\]

Let us show that \( \eta \) is an isomorphism.

As \( g \) is supported in \( \Gamma(f_0) \cup \Gamma(f_1) \), we see that \( g \) is a monomorphism.

To show that \( \eta \) is surjective let \( g_i \in G(f_i) = \lambda_i(S'(f_i, \partial Y_i)) \), \( i = 0, 1 \). Thus \( g_i = \lambda_i(h_i) \), where \( h_i \in S'(f_i, \partial Y_i) \), so we can assume that \( h_i \) is fixed on some neighborhood of \( \partial Y_i \). Now define \( h \in S(f) \) by \( h = h_i \) on \( Y_i \), and \( h = \text{id} \) on \( M \setminus (Y_0 \cup Y_1) \). Then \( h \) preserves orientation of \( M \) and leaves invariant every connected component of \( \partial Y_i \). Therefore \( h \) is also isotopic to \( \text{id}_M \), that is \( h \in S(f) \cap D_{\text{id}}(M) = S'(f) \). Denote \( g = \lambda(h) \). Then it follows that \( \eta(g) = (g_0, g_1) \). Hence \( \eta \) is an isomorphism.

3) Let \( G \in \mathcal{M} \) and \( n \geq 1 \). We have find a mapping \( f \in \text{Morse}(M, P) \) such that \( G(f) \cong G \wr \mathbb{Z}_n \). Construction of such a map \( f \) is shown in Figure 8.3.

![Figure 8.3](image)

Let \( g \in \text{Morse}(M, P) \) be a map constructed in 1) such that \( G(g) \cong \mathbb{Z}_n \). This map has \( n \) local maximums \( z_1, \ldots, z_n \) such that \( z_i \) is contained in the domain denoted by \( i \), see Figure 8.4. One can also assume that there exists \( h \in S(f) \) such that \( h \) preserves orientation of \( M \), \( h^i(z_i) = z_i \), and \( h^n = \text{id}_M \). Now let \( Y_i \) be a small disk neighborhood of \( z_i \) such that \( \partial Y_i \) is a connected component of a level set of \( f \), and \( Y_i \) does not contain other critical values except for \( f(z_i) \). Then \( Y_i = h^i(Y_i) \) is a neighborhood of \( z_i \) with similar properties.

Define a new function \( f \in \text{Morse}(M, P) \) such that \( G(f|_{Y_i}) \cong G \), \( f|_{Y_i} = g \circ h^i|_{Y_i} \), \( i = 2, \ldots, n \), and \( f = g \) on \( M \setminus (\cup_{i=1}^n Y_i) \). Then it is easy to check that \( G(f|_{Y_i}) \cong G \) for all \( i \), and \( G(f) \cong G \wr \mathbb{Z}_n \). We leave the details for the reader. \( \Box \)
9. Proof of Theorem 1.5. All other surfaces

It remains to consider the case when $M$ is a connected compact orientable surface distinct from $S^2$, $T^2$, $D^2$ and $S^1 \times I$, and so $\chi(M) < 0$. We have to prove that

$$\mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1}) = \{G(f) \mid f \in \mathcal{F}(M, P)\}.$$  

The proof reduces the situation to the case $M = D^2$ and $S^1 \times I$ considered in §8.

**Lemma 9.1.** Let $f \in \mathcal{F}(M, P)$. Then there exist finitely many surfaces $Y_1, \ldots, Y_k$ each diffeomorphic either to a 2-disk $D^2$ or to a cylinder $S^1 \times I$, and maps $f_i \in \mathcal{F}(Y_i, P)$ such that

$$G(f) \cong G(f_1) \times \cdots \times G(f_k),$$

whence $G(f) \in \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$, and so $\{G(f) \mid f \in \mathcal{F}(M, P)\} \subset \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$.

**Proof.** Since $\chi(M) < 0$, it follows from [29, Theorem 1.7] that there exists finitely many disjoint subsurfaces $Y_1, \ldots, Y_k$ having the following properties. Denote $Y = \bigcup_{i=1}^k Y_i$. Then

(a) each $Y_i$ is diffeomorphic either to $D^2$ or to $S^1 \times I$;
(b) the restriction $f|_{Y_i}$ satisfies axioms (B) and (L) for each $i = 1, \ldots, k$;
(c) every $h \in S'(f)$ is isotopic in $S'(f)$ to a diffeomorphism $h_1$ is supported in $\text{Int} Y_i$ and, in particular, $\lambda(h) = \lambda(h_1)$;
(d) if $h \in S'(f)$ is supported in $\text{Int} Y_i$, then for each $i = 1, \ldots, k$ the restriction $h|_{Y_i}$ is isotopic in $D(Y_i)$ to $\text{id}_{Y_i}$ with respect to some neighborhood of $\partial Y_i$. In other words, $h|_{Y_i} \in S'(f|_{Y_i}, \partial Y_i)$.

Let $f_i = f|_{Y_i} : Y_i \to P$ be the restriction of $f$ to $Y_i$, $\Gamma_i = \Gamma(f_i)$ be the enhanced KR-graph of $f_i$,

$$\lambda_i : S(f_i) \to \text{Aut}(\Gamma_i)$$

be the corresponding action homomorphism, and

$$G_i := G(f_i) \cong \lambda_i(S'(f_i)) \xrightarrow{\text{Lemma 3.1}} \lambda_i(S'(f_i, \partial Y_i)).$$

It follows from (a) that $\Gamma_i$ is a subtree of $\Gamma(f)$ with respect to some CW-partition. To complete Lemma 9.1, it suffices to prove that there is an isomorphism

$$\eta : G(f) \cong G_1 \times \cdots \times G_k.$$  

Let $g \in G(f)$, so $g = \lambda(h)$ for some $h \in S'(f)$. Then it follows from (c) that $g$ is supported in $\bigcup_{i=1}^k \Gamma_i$, and by (d) the restriction of $g$ to $\Gamma_i$ belongs to $G_i$. Hence we have a well-defined homomorphism:

$$\eta : G(f) \to G_1 \times \cdots \times G_k, \quad \eta(g) = (g|_{\Gamma_1}, \ldots, g|_{\Gamma_k}).$$

We claim that $\eta$ is an isomorphism.

Since each $g \in G(f)$ is supported in $\bigcup_{i=1}^k \Gamma_i$, it follows that $\eta$ is a monomorphism.

Let $(g_1, \ldots, g_k) \in G_1 \times \cdots \times G_k$ and for each $i = 1, \ldots, k$ let $h_i \in S'(f_i, \partial Y_i)$ be such that $\lambda_i(h_i) = g_i$. Since $h_i$ is fixed near $\partial Y_i$, it extends by the identity on $M \setminus Y_i$ to a diffeomorphism supported in $Y_i$ and isotopic to $\text{id}_M$. 

Denote $h = h_1 \circ \cdots \circ h_k \in S'(f)$. Then
\[
\eta(\lambda(h)) = (\lambda_1(h_1), \ldots, \lambda_k(h_k)) = (g_1, \ldots, g_k),
\]
and so $\eta$ is surjective, and therefore it is an isomorphism.

Now by Lemma 8.1 each $G_i \in \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$, and so $G(f) \in \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$ as well. □

**Lemma 9.2.** Let $M$ be a connected compact orientable surface with $\chi(M) < 0$. Then
\[
\mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1}) \subset \{G(f) \mid f \in \text{Morse}(M, P)\}.
\]

**Proof.** Let $G \in \mathcal{R}(\{\mathbb{Z}_n\}_{n \geq 1})$. Take a generic map $f \in \text{Morse}(M, P)$ having a unique local maximum $z$ and such that the group $G(f)$ is trivial, see e.g. Figure 9.1.

![Figure 9.1](image)

**Figure 9.1.** Maps $f : M \to P$ with $G(f) \cong G(f|_Y) \cong G$.

Let $Y$ be a disk-neighborhood of $z$ such that $\partial Y$ is a connected component of some level set of $f$ and $Y$ contains no critical points except for $z$.

By Lemma 8.2 we can change $f$ on the disk $Y$ so that $G(f|_Y) \cong G$. We claim that then $G(f) \cong G$.

The proof is similar to 3) of Lemma 8.2. Let $h \in S'(f)$ and $g = \lambda(h)$. Since $h$ is isotopic to $\text{id}_M$ and the factor map $p : M \to \Gamma(f)$ induces a surjection on the first homology groups $p_* : H_1(M, \mathbb{Z}) \to H_1(\Gamma(f), \mathbb{Z})$, it follows that $g$ induces the identity automorphism of $H_1(\Gamma(f), \mathbb{Z})$. This implies that $g$ is supported in $\Gamma(f|_Y)$. Now let
\[
\eta : G(f) \to G(f|_Y) \cong G, \quad g \mapsto g|_{\Gamma(f|_Y)}
\]
be the restriction map. We claim that $\eta$ is an isomorphism.

Indeed, since $g$ is supported in $\Gamma(f|_Y)$, we obtain that $\eta$ is a monomorphism.

To prove that $\eta$ is surjective, let $\hat{\lambda} : S(f|_Y) \to \text{Aut}(\Gamma(f|_Y))$ be the corresponding action homomorphism. Let $\hat{g} \in G(f|_Y) = \hat{\lambda}(S'(f|_Y, \partial Y))$, so $\hat{g} = \hat{\lambda}(\hat{h})$ for some $\hat{h} \in S'(f|_Y, \partial Y)$. We can even assume that $\hat{h}$ is fixed near $\partial Y$. Then it extends by the identity on $M \setminus Y$ to a diffeomorphism $h \in S(f)$. Moreover, as $h$ is supported in a disk $Y$, it is isotopic to $\text{id}_M$, and so $h \in S'(f)$. It remains to note that $\eta(\lambda(h)) = \hat{g}$. Thus $\eta$ is an isomorphism. □

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