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Some remarks on Dedekind lattices

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Abstract In this paper, we prove that a principally generated C-lattice L is a Dedekind lattice if and only if L is a WI-lattice in which every invertible element is a finite meet of powers of prime elements.

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1 Introduction

By a C-lattice L we mean a not necessarily modular complete multiplicative lattice \((a(\vee x_i) = \vee ax_i)\) generated under joins by a multiplicatively closed subset \(C\) of compact elements, with least element 0 and compact greatest element 1, operating as the multiplicative identity. In any C-lattice multiplication defines a quotient operation by \(a: b = \vee \{x \in L \mid xb \leq a\}\). Obviously C-lattices arise as abstractions of ideal systems, in particular when considering rings with identity. There the principal ideals form a generating set of compact “elements” whereas the finitely generated ideals form the set of all compact elements.

The theory of C-lattices was initiated by Dilworth in his fundamental and ground breaking paper [6] based on the notion of a principal element \(e\). Recall that an element \(e \in L\) is said to be principal if it satisfies:

\[
(MP) \quad a \land be = ((a : e) \land b)e \\
(JP) \quad (ae \lor b) : e = (b : e) \lor a
\]

In case that \((MP)\) is satisfied, \(e\) is called “meet principal”; in case that \((JP)\) is satisfied, \(e\) is called “join principal”. If \(e\) satisfies \((MP)\) only for \(b = 1\), that is \(a \land e = (a : e)e\) for all \(a \in L\), then \(e\) is called “weak meet principal”. Finite products of meet (join) principal elements are again meet (join) principal [6, Lemmas 3.3 and 3.4]. Moreover in [2, Theorem 1.3], it is shown that principal elements are always compact. For more information on principal elements, the reader is referred to [5].

Throughout this paper \(L\) denotes a principally generated C-lattice. For the definitions of prime element, maximal element, minimal prime element, and primary element, the reader is referred to [1, 7]. An element \(a \in L\) is called a nonzero divisor if \((0 : a) = 0\) and \(a\) is called invertible if \(a\) is a principal nonzero divisor. An element \(a \in L\) is called regular if it contains an invertible element and \(a\) is called nilpotent if \(a^n = 0\) for

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some positive integer $n$. If 0 is the only nilpotent element, then $L$ is called reduced. For any $a, b \in L$, we say $a$ and $b$ are comaximal, if $a \lor b = 1$.

$L$-lattices can be localized. For any prime element $p$ of $L$, $L_p$ denotes the localization of $L$ at $F = \{x \in C \mid x \not\in p\}$. For details on $L$-lattices and their localization theory, the reader is referred to [7, 12].

$L$ is called a Prüfer lattice, if every compact element is principal. $L$ is called a $WI$-lattice if and only if $L$ is a reduced lattice, then $L$ is a $WI$-lattice, if every invertible element is a finite meet of primes $\forall a \in L$ is principal and $(0 : (0 : a)) \lor (0 : a) = 1$. Note that by definition, $(0 : (0 : a)) = (0 : a) = 0$.

Prüfer lattices have been studied in [2, 10]. A reduced lattice $L$ is called quasi-regular, if for any compact element $x$, there is a compact element $y$ such that $(0 : (0 : x)) = (0 : y)$. Quasi-regular lattices have been studied in [3]. Note that by [8, Theorem 4], $L$ is a $WI$-lattice if and only if $L$ is a quasi-regular lattice whose compact elements are principal. A reduced lattice $L$ is called a Dedekind lattice if every element not contained in any minimal prime is "weak meet principal". For various characterizations of $WI$-lattices and Dedekind lattices, the reader is referred to [8, 9, 11].

It is well known that $L$ is a Dedekind lattice if and only if $L$ is a $WI$-lattice in which every invertible element is a finite product of prime elements [11, Theorems 2.6 and 3.12]. In this paper we prove that $L$ is a Dedekind lattice if and only if $L$ is a $WI$-lattice in which every invertible element is a finite meet of powers of prime elements. For general background and terminology, the reader may consult [1, 2].

2 Nonminimal prime elements in $WI$-lattices

In this section we study nonminimal prime elements in $WI$-lattices in which every invertible element is a finite meet of powers of prime elements. Using these results, we establish that $L$ is a Dedekind lattice if and only if $L$ is a $WI$-lattice in which every invertible element is a finite meet of powers of prime elements.

We now prove some useful lemmas. It is well known that if $L$ is a reduced lattice, then $L$ is a Dedekind lattice if and only if $L$ is a $WI$-lattice in which every invertible element is a finite meet of powers of prime elements. For various characterizations of $WI$-lattices and their localization theory, the reader is referred to [7, 12].

**Lemma 2.1** Let $L$ be a $WI$-lattice. Then every nonminimal prime element of $L$ is the join of invertible elements.

**Proof** Let $p$ be a nonminimal prime element of $L$. As $L$ is quasi-regular, by [3, Theorem 2], there exists a compact element $x \leq p$ such that $(0 : x) = 0$. As $L$ is a $WI$-lattice, $x$ is principal, so $x$ is invertible, and hence $p$ is regular. Let $p_r = \lor\{y \in L \mid y \leq p \text{ and } y \text{ is invertible}\}$. Clearly, $p_r \leq p$. Suppose $p_r < p$. Choose any principal element $a \leq p$ such that $a \not\in p_r$. As $L$ is a $WI$-lattice, it follows that $x \lor a$ is invertible, so $x \lor a \leq p_r$, a contradiction. Therefore $p = p_r$ and hence every nonminimal prime element of $L$ is the join of invertible elements. This completes the proof of the lemma.

**Lemma 2.2** Let $L$ be a $WI$-lattice in which every invertible element is a finite meet of powers of prime elements. Let $m$ be a nonidempotent, nonminimal prime element of $L$. Then

(i) $m$ is minimal over an invertible element of $L$.

(ii) $m_m$ is invertible in $L_m$.

**Proof** (i) Since $m \neq m^2$, by Lemma 2.1, there exists an invertible element $a \leq m$ such that $a \not\leq m^2$. Choose any principal element $y \leq m$. As $L$ is a $WI$-lattice, $a \lor y^2$ is invertible, so by hypothesis, $a \lor y^2 = \land_{i=1}^{\alpha_i} p_i^{\alpha_i}$, where $p_i$’s are prime elements of $L$. As $a \not\leq m^2$, it follows that $\alpha_i = 1$ for all $p_i \leq m$. Again $(a \lor y^2)m = \land\{(p_i)m \mid p_i \leq m\} = (a \lor y)_m$, so by Nakayama’s lemma (see [1, Theorem 1.1] or [2, Theorem 1.4]), $y_m \leq a_m$ and hence $m_m = a_m$. Therefore $m$ is minimal over an invertible element $a$ of $L$.

(ii) Again since $m_m = a_m$ and $(0 : a) = 0$, it follows that $(0_m : m_m) = (0_m : a_m) = (0 : a)_m = 0_m$, so $m_m$ is invertible in $L_m$.

**Lemma 2.3** Let $L$ be a $WI$-lattice in which every invertible element is a finite meet of powers of prime elements. Let $p$ be a nonminimal prime which is minimal over an invertible element $y \in L$. Then $p^n$ is $p$-primary for all positive integers $n$.

**Proof** Let $n$ be a positive integer and let $r, s \in L$ be principal elements such that $rs \leq p^n$ and $s \not\leq p$. Since $y^n$ is invertible, by hypothesis, $r \lor y^n = \land_{i=1}^{\alpha_i} p_i^{\alpha_i}$, where $p_i$’s are prime elements of $L$. Since $p$ is minimal over $r \lor y^n$, it follows that $(r \lor y^n)_p = (rs \lor y^n)_p = (p_j^{\alpha_j})_p$ where $p = p_j$ for some $j \in \{1, 2, \ldots, m\}$. But $(p_j^{\alpha_j})_p \leq (p^n)_p$ since $rs \lor y^n \leq p^n$, so $\alpha_j \geq n$, therefore $p^{\alpha_j} \leq p^n$ and hence $r \leq p^n$. This shows that $p^n$ is $p$-primary for all positive integers $n$. This completes the proof of the lemma.
Lemma 2.4 Let $L$ be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Let $p$ be a nonidempotent, nonminimal prime element of $L$. Then

(i) $\{p^n\}_{n=1}^{\infty}$ is the set of all $p$-primary elements of $L$.
(ii) $p^\omega = \bigwedge_{n=1}^{\infty} p^n$ is a prime element of $L$.
(iii) If $q < p$ is a prime element of $L$, then $q \leq p^\omega$.

Proof (i) Note that by Lemmas 2.2 and 2.3, $p^n \neq p^{n+1}$ for all positive integers $n$ and $p^n$ is $p$-primary for all positive integers $n$. Suppose $q$ is $p$-primary. Then by [4, Lemma 3.2 (d)], $q = (p^n)p = p^\omega$, so (i) holds.

(ii) Since $p_p$ is invertible in $L_p$, by [4, Lemma 3.2 (c)], $p^{(\omega)} = \bigwedge_{n=1}^{\infty} (p^n)_p$ ($\omega$ is the meet in $L_p$) is a prime element of $L_p$. It can be easily verified that $p^\omega$ is a prime element of $L$.

(iii) Follows from [4, Lemma 3.2 (c)].

$\square$

Lemma 2.5 Let $L$ be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Then every invertible element is a finite meet of primary elements.

Proof The proof of the lemma follows from Lemma 2.4 and [3, Lemma 8].

Definition 2.6 A regular prime element $p$ of $L$ is said to be a minimal regular prime if for any prime $q < p$, $q$ is a nonregular prime element of $L$.

Lemma 2.7 Let $L$ be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. If $p$ is a nonidempotent, nonminimal prime element of $L$, then $p$ is a minimal regular prime element of $L$.

Proof Let $p$ be a nonidempotent, nonminimal prime element of $L$ and let $q < p$ be a prime element of $L$. Assume that $q$ is a regular prime element of $L$. Suppose $b \leq q$ and $(0 : b) = 0$ for some principal element $b \in L$. Choose an invertible element $a \leq p$ such that $p$ is minimal over $a$. Since $ab$ is invertible, by Lemma 2.5, $ab$ is a finite meet of primary elements of $L$. Let $ab = \bigwedge_{i=1}^{n} q_i$ be a normal primary decomposition of $L$. Let $q_i \leq p$ for $i = 1, 2, \ldots, k$ and $q_j \not\leq p$ for $j = k + 1, \ldots, n$. Then $(ab)_p = \bigwedge_{i=1}^{n} (q_i)_p$. By Lemma 2.4, we can assume that $\sqrt{q_i} \leq p^\omega$ for $i = 1, 2, \ldots, k$. Then $a \not\leq \sqrt{q_i}$ for $i = 1, 2, \ldots, k$, so $b \not\leq \bigwedge_{i=1}^{n} q_i$ and hence $a_pb_p = b_p$. Therefore, by Nakayama’s lemma, $b_p = 0_p$, a contradiction since $(0 : b) = 0$. This shows that $p$ is a minimal regular prime element of $L$. This completes the proof of the lemma.

$\square$

Lemma 2.8 Let $L$ be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Suppose $p$ is a prime minimal over an invertible element $y$ of $L$. Then $p \not= p^2$.

Proof If $p = p^2$, then by hypothesis, $p_p = y_p$, so by Nakayama’s lemma $p_p = 0_p$, hence $y_p = 0_p$, a contradiction, since $(0 : y) = 0$. This shows that $p \not= p^2$.

$\square$

Lemma 2.9 Let $L$ be a WI-lattice in which every invertible element is a finite meet of powers of prime elements. If $p$ is an idempotent prime, then $p$ is a minimal prime element of $L$.

Proof Suppose $p$ is an idempotent prime element of $L$. Assume that $p$ is nonminimal. Then there exists an invertible element $x \leq p$. By hypothesis, $x$ has only finitely many minimal primes, say $p_1, p_2, \ldots, p_n$. By Lemma 2.8, $p \not\leq p_i$ for all $i$. As $L$ is a WI-lattice, there exists a principal element $y \leq p$ such that $y \not\leq p_i$ for all $i$. Let $q \leq p$ be a prime minimal over $x \vee y$. If $q = q^2$, then by hypothesis, $(x \vee y)_q = q_q$, so by Nakayama’s lemma $q_q = 0_q$ and therefore $q$ is minimal, so by [3, Lemma 8], $x \not\leq q$, a contradiction. Therefore $q \neq q^2$ and nonminimal. By hypothesis and Lemma 2.7, $q$ is a minimal regular prime. Again since $x \leq q$, it follows that $p_i < q$ for some $i$. This contradicts the fact that $q$ is a minimal regular prime. Therefore $p$ is a minimal prime element of $L$.

$\square$

Theorem 2.10 $L$ is a Dedekind lattice if and only if $L$ is a WI-lattice in which every invertible element is a finite meet of powers of prime elements.

Proof If $L$ is a Dedekind lattice, then by [11, Theorem 2.6 (viii) and Theorem 3.12], $L$ is a WI-lattice in which every invertible element is a finite meet of powers of prime elements. Conversely, assume that $L$ is a WI-lattice in which every invertible element is a finite meet of powers of prime elements. We claim that every invertible element is a finite product of maximal prime elements. Let $a \in L$ be an invertible element and let $a = \bigwedge_{i=1}^{n} p_i^{m_i}$, where $p_i$’s are distinct prime elements of $L$. Note that by [3, Theorem 2], $p_i$’s are nonminimal
prime elements of $L$. Again by Lemmas 2.7 and 2.9, each $p_i$ is maximal and so they are pairwise comaximal. Consequently, $a$ is a finite product of maximal prime elements. Now the result follows from [11, Theorems 2.6 (viii) and 3.12]. This completes the proof of the theorem. □

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