Douglas–Rachford splitting for a Lipschitz continuous and a strongly monotone operator

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Abstract

The Douglas–Rachford method is a popular splitting technique for finding a zero of the sum of two subdifferential operators of proper closed convex functions; more generally two maximally monotone operators. Recent results concerned with linear rates of convergence of the method require additional properties of the underlying monotone operators, such as strong monotonicity and cocoercivity. In this paper, we study the case when one operator is Lipschitz continuous but not necessarily a subdifferential operator and the other operator is strongly monotone. This situation arises in optimization methods which involve primal-dual approaches. We provide new linear convergence results in this setting.

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1 Introduction

The Douglas–Rachford splitting algorithm, introduced by Lions and Mercier [22], is a fundamental algorithm for solving monotone inclusion problems that involves finding a zero of the sum of two maximally monotone operators $A$ and $B$. (See Section 2 for a review of these and other definitions used in the paper.) Monotone inclusions can be used to formulate primal, dual, and primal-dual optimality conditions of convex optimization problems, equilibrium conditions in convex-concave games, monotone variational inequalities, and monotone complementarity problems. The Douglas–Rachford algorithm is useful for all these applications, provided that the operator in the inclusion problem can be written as a sum of two operators, as in (29), with resolvents that are easily computed. This is often the case in large scale applications; see, e.g., [6, 8, 9, 11, 12, 30, 31, 29, 34, 35], and the references therein. The Douglas–Rachford method

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can also be used to derive other important splitting methods, such as the Alternating Direction Method of Multiplier or ADMM [18, 17], Spingarn’s method of partial inverses [17], the primal-dual hybrid gradient method [26], and linearized ADMM [26].

Under additional assumptions on the operators A and B, linear rates of convergence are possible. In their seminal work [22], Lions and Mercier proved linear convergence of the Douglas–Rachford iteration when one operator is strongly monotone and cocoercive. Recent works concerned with linear rates of convergence of Douglas–Rachford method include [3, 5, 27] for linear rates in convex feasibility settings, [19, 20, 22] for linear rates under strong convexity assumptions, [16] for linear rates in basis pursuit setting, and [1, 23, 24] for local linear rates in more general settings. In the recent work [19], Giselsson studied and proved tight linear rates of convergence of Douglas–Rachford in the following three cases: (a) A is strongly monotone and B is cocoercive (see [19, Theorem 5.6]), (b) A is strongly monotone and Lipschitz continuous (see [19, Theorem 6.5]), and (c) A is strongly monotone and cocoercive (see [19, Theorem 7.4]). Giselsson’s results are independent of the order of A and B, and therefore actually cover six cases.

The main contribution of this paper is to supplement Giselsson’s results with a linear convergence result for the case when A is Lipschitz continuous and B is strongly monotone. Unlike the results in [19], our linear convergence result is not symmetric in A and B, and does not apply to the case where A is strongly monotone and B is Lipschitz continuous, except in the important special case when B is a linear mapping. When A is the subdifferential of a convex function, Lipschitz continuity and cocoercivity are equivalent properties. However, for general monotone operators, Lipschitz continuity is a much weaker condition than cocoercivity, so the case studied in this paper is an important extension of [19, Theorem 5.6].

As an application, we discuss the Douglas–Rachford splitting method applied to the primal-dual optimality conditions of a convex problem, formulated as an inclusion problem (29) in which one of the operators is a skew-symmetric linear mapping and not a subdifferential, see, e.g., [7, 10, 14, 15, 25].

This paper is organized as follows. Section 2 presents a collection of useful properties of reflected resolvents of monotone operators under additional assumptions on the operator. Section 3 provides a high level overview of relevant linear convergence results. Our main results appear in Section 4 where we prove linear convergence of Douglas–Rachford iteration when applied to find a zero of the sum of maximally monotone operators A and B when A is Lipschitz continuous and B is strongly monotone. Finally, in Section 5 we present an application of our results to the primal dual Douglas–Rachford method.

2 Contraction properties of reflected resolvents

Throughout the paper, X is a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We use the notation $A : X \rightrightarrows X$ to indicate that A is a set-valued operator on X. The domain of A is $\text{dom}(A) = \{x \in X \mid Ax \neq \emptyset\}$ and the graph of A is $\text{gra}(A) = \{(x, u) \in X \times X \mid u \in Ax\}$. We use the notation $A : X \to X$ to indicate that A is a single-valued operator on X and $\text{dom} A = X$. The inverse of A, denoted by $A^{-1}$, is the operator with graph $\text{gra}(A^{-1}) = \{(u, x) \mid (x, u) \in \text{gra}(A)\}$. Let C be a convex closed nonempty subset of X. We use $N_C$ to denote the normal cone of C defined as $N_C(x) := \{u \in X \mid \langle u, y - x \rangle \leq 0 \text{ for all } y \in C\}$, if $x \in C$; and $N_C(x) := \emptyset$, otherwise; and $P_C$ to denote that orthogonal projection onto C (this is also known as the closest point mapping) defined at every $x \in X$ by $P_C(x) := \text{argmin}_{c \in C} \|x - c\|$. 


are related to the graph of $A$. They can be defined equivalently as properties of the resolvent, given by (9) with the matrix $M$ mapping $R$ nonexpansive $A$. 1-Lipschitz continuous operator is called monotonicity. shown in the table. Taking $\mu = 0$ in the first column also gives three equivalent definitions of monotonicity.

Table 1: Each of the four operator properties is defined as (5) for the matrix $L$ shown in the table. They can be defined equivalently as properties of the resolvent, given by (9) with the matrix $M$ shown in the table, and as properties of the reflected resolvent, given by (10) for the matrix $N$ shown in the table. Taking $\mu = 0$ in the first column also gives three equivalent definitions of monotonicity.

| $\mu$-strong monotonicity | $\beta$-Lipschitz continuity | $(1/\beta)$-cocoercivity | $\alpha$-averagedness |
|---------------------------|-----------------------------|---------------------------|--------------------------|
| $L$                       | $\begin{bmatrix} -2\mu & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} \beta^2 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & \beta \\ \beta & -2 \end{bmatrix}$ | $\begin{bmatrix} 2\alpha - 1 & 1 - \alpha \\ 1 - \alpha & -1 \end{bmatrix}$ |
| $M$                       | $\begin{bmatrix} 0 & 1 \\ 1 & -2\mu - 2 \end{bmatrix}$ | $\begin{bmatrix} -1 & 1 \\ 1 & \beta^2 - 1 \end{bmatrix}$ | $\begin{bmatrix} -2 & \beta + 2 \\ \beta + 2 & -2\beta - 2 \end{bmatrix}$ | $\begin{bmatrix} -1 & 2 - \alpha \\ 2 - \alpha & 4\alpha - 4 \end{bmatrix}$ |
| $N$                       | $\begin{bmatrix} 1 - \mu & -\mu \\ -\mu & -1 - \mu \end{bmatrix}$ | $\begin{bmatrix} \beta^2 - 1 & \beta^2 + 1 \\ \beta^2 + 1 & \beta^2 - 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & \beta - 1 \\ \beta - 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & \alpha \\ \alpha & -2(1 - \alpha) \end{bmatrix}$ |

An operator $A$ on $X$ is $\beta$-Lipschitz continuous if it is single-valued on $\text{dom}(A)$ and

$$\|Ax - Ay\| \leq \beta\|x - y\| \quad \forall x, y \in \text{dom}(A).$$  \hspace{1cm} (1)

A 1-Lipschitz continuous operator is called nonexpansive. An operator $A$ is $\alpha$-averaged, with $\alpha \in [0,1]$, if it can be expressed as $A = (1 - \alpha)\text{Id} + \alpha N$ where $\text{Id}$ is the identity operator, $N$ is nonexpansive. An operator $A$: $X \rightrightarrows X$ is monotone if

$$\langle x - y, u - v \rangle \geq 0 \quad \forall (x,u), (y,v) \in \text{gra}(A).$$  \hspace{1cm} (2)

A monotone operator $A$ is maximally monotone if its graph admits no proper extension that preserves the monotonicity of $A$. An operator $A$ is $\mu$-strongly monotone, with $\mu > 0$, if

$$\langle x - y, u - v \rangle \geq \mu\|x - y\|^2 \quad \forall (x,u), (y,v) \in \text{gra}(A).$$  \hspace{1cm} (3)

Equivalently, $A - \mu\text{Id}$ is monotone. The operator $A$ is $(1/\beta)$-cocoercive, with $\beta > 0$, if its inverse $A^{-1}$ is $(1/\beta)$-strongly monotone, i.e.,

$$\langle x - y, u - v \rangle \geq \frac{1}{\beta}\|u - v\|^2 \quad \forall (x,u), (y,v) \in \text{gra}(A).$$  \hspace{1cm} (4)

Note that this implies that $A$ is single-valued on its domain, and (by the Cauchy-Schwarz inequality) that $A$ is $\beta$-Lipschitz continuous. A 1-cocoercive operator is also called firmly nonexpansive. We note that all these properties are defined as quadratic inequalities on the graph of the operator. Table 1 summarizes the definitions. Each of the four properties in the table is defined as

$$L_{11}\|x - y\|^2 + 2L_{12}\langle x - y, u - v \rangle + L_{22}\|u - v\|^2 \geq 0 \quad \forall (x,u), (y,v) \in \text{gra}(A),$$  \hspace{1cm} (5)

where $L$ is the $2 \times 2$ matrix shown on row 2 of the table.

The resolvent of an operator $A$ is the mapping $J_A = (\text{Id} + A)^{-1}$. The reflected resolvent is the mapping $R_A = 2J_A - \text{Id}$. The graphs of the resolvent and reflected resolvent of an operator $A$ are related to the graph of $A$ by invertible linear transformations:

$$\text{gra} A = \{(u, x - u) \mid (x, u) \in \text{gra}(J_A)\}$$  \hspace{1cm} (6)
\[
\left\{ \frac{1}{2}(x + u, x - u) \mid (x, u) \in \text{gra}(R_A) \right\}.
\]

Hence, if we define
\[
M = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} L \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} L \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},
\]
then the property (5) is equivalent to
\[
M_{11} \|x - y\|^2 + 2M_{12}(x - y, u - v) + M_{22}\|u - v\|^2 \geq 0 \quad \forall (x, u), (y, v) \in \text{gra}(J_A),
\]
and also to
\[
N_{11} \|x - y\|^2 + 2N_{12}(x - y, u - v) + N_{22}\|u - v\|^2 \geq 0 \quad \forall (x, u), (y, v) \in \text{gra}(R_A).
\]

For each property in Table 1, we therefore have equivalent definitions of the form (9) and (10). The matrices \(M\) and \(N\) are shown in the third and fourth rows of the table, respectively.

In Proposition 2.1 below we collect some useful properties of the reflected resolvent \(R_A\). We point out that items (iv) & (v) below provide short proofs for Theorems 7.2 and 6.3 in [19].

**Proposition 2.1.** The following statements hold for any operator \(A\).

(i) Suppose \(\mu > 0\) and \(\beta > 0\). If \(A\) is \(\mu\)-strongly monotone and \(\beta\)-Lipschitz continuous, then \(A\) is \((\mu/\beta^2)\)-cocoercive.

(ii) \(A\) is monotone and nonexpansive if and only if \(J_A\) is \(\frac{1}{2}\)-strongly monotone, and if and only if \(R_A\) is monotone.

(iii) Suppose \(\mu > 0\). \(A\) is \(\mu\)-strongly monotone if and only if \(J_A\) is \((1 + \mu)\)-cocoercive, and if and only if \(-R_A\) is \((1 + \mu)^{-1}\)-averaged.

(iv) Suppose \(\beta \geq \mu > 0\). If \(A\) is \(\mu\)-strongly monotone and \((1/\beta)\)-cocoercive, then \(R_A\) is \(\kappa\)-Lipschitz continuous with
\[
\kappa = \left( \frac{1 - 2\mu + \mu\beta}{1 + 2\mu + \mu\beta} \right)^{1/2}.
\]

(v) Suppose \(\beta \geq \mu > 0\). If \(A\) is \(\mu\)-strongly monotone and \(\beta\)-Lipschitz continuous, then \(R_A\) is \(\kappa\)-Lipschitz continuous with
\[
\kappa = \left( \frac{1 - 2\mu + \beta^2}{1 + 2\mu + \beta^2} \right)^{1/2}.
\]

(vi) Suppose \(0 < \mu < 1\) and \(0 < \alpha < 1\). If \(A\) is \(\mu\)-strongly monotone and \(\alpha\)-averaged, then \(R_A\) is \(\kappa\)-Lipschitz continuous with
\[
\kappa = \left( \frac{\alpha(1 - \mu)}{\alpha(1 - \mu) + 2\mu} \right)^{1/2}.
\]

**Proof (i):** This follows from Table 1 and
\[
\begin{bmatrix} 0 & \beta^2/\mu \\ \beta^2/\mu & -2 \end{bmatrix} = \frac{\beta^2}{\mu} \begin{bmatrix} -2\mu & 1 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} \beta^2 & 0 \\ 0 & -1 \end{bmatrix}.
\]
(ii): Use Table 1 for the case when \( A \) is 1-Lipschitz continuous. (iii): This is clear from Table 1. (iv): This follows from Table 1 and
\[
\begin{bmatrix}
1 - 2\mu + \mu\beta & 0 \\
0 & -(1 + 2\mu + \mu\beta)
\end{bmatrix} = \begin{bmatrix}
1 - \mu & -\mu \\
-\mu & 1 - \mu
\end{bmatrix} + \mu \begin{bmatrix}
\beta - 1 & 1 \\
1 & -\beta - 1
\end{bmatrix}.
\] (15)

(v): This follows from Table 1 and
\[
\begin{bmatrix}
1 - 2\mu + \beta^2 & 0 \\
0 & -(1 + 2\mu + \beta^2)
\end{bmatrix} = (\beta^2 + 1) \begin{bmatrix}
1 - \mu & -\mu \\
-\mu & 1 - \mu
\end{bmatrix} + \mu \begin{bmatrix}
\beta^2 - 1 & \beta^2 + 1 \\
\beta^2 + 1 & \beta^2 - 1
\end{bmatrix}.
\] (16)

Alternatively, combine (i) and (iv) applied with \( \beta \) replaced by \( \beta^2 / \mu \) to learn that \( R_A \) is \( \kappa \)-Lipschitz continuous with
\[
\kappa = \left( \frac{1 - 2\mu + \mu(\beta^2 / \mu)}{1 + 2\mu + \mu(\beta^2 / \mu)} \right)^{1/2} = \left( \frac{1 - 2\mu + \beta^2}{1 + 2\mu + \beta^2} \right)^{1/2}.
\] (17)

(vi): This follows from Table 1 and the identity
\[
\begin{bmatrix}
\alpha(1 - \mu) & 0 \\
0 & -(\alpha(1 - \mu) + 2\mu)
\end{bmatrix} = \alpha \begin{bmatrix}
1 - \mu & -\mu \\
-\mu & 1 - \mu
\end{bmatrix} + \mu \begin{bmatrix}
0 & \alpha \\
\alpha & -2(1 - \alpha)
\end{bmatrix}.
\] (18)

**Remark 2.2.** The contraction factors of the reflected resolvents are important in the linear convergence proofs in [19]. Proposition 2.1(v) gives the contraction factor of the reflected resolvent of a strongly monotone and Lipschitz continuous operator. As indicated in the proof, this result can be derived in two ways. In the second approach, we use Proposition 2.1(i) to derive the contraction factor from the result for strongly monotone and cocoercive operators.

We conclude this section with the following lemma.

**Lemma 2.3.** Suppose that \( A : X \to X \) is monotone and \( \beta \)-Lipschitz continuous with \( \beta > 0 \). Let \( (x, y) \in X \times X \). Then the following hold:

(i) \( \|x - y\| \leq (1 + \beta)\|J_A x - J_A y\| \).

(ii) \( \text{Id} - J_A \) is a Banach contraction with constant \( \frac{\beta}{\sqrt{1 + \beta^2}} \).

(iii) \( J_A \) is \( \left( \frac{1}{2(1 + \beta^2)} + \frac{1}{2(1 + \beta^2)} \right) \)-strongly monotone.

(iv) \( \langle x - y, R_A x - R_A y \rangle \geq -\lambda \|x - y\|^2 \) where
\[
\lambda = \left( 1 - \frac{1}{(1 + \beta^2)} - \frac{1}{1 + \beta^2} \right) \in [-1, 1[.
\]

**Proof (i):** This follows from entry (2, 2) of Table 1 and
\[
\begin{bmatrix}
-1 & 0 \\
0 & (1 + \beta)^2
\end{bmatrix} \preceq \frac{\beta + 1}{\beta} \begin{bmatrix}
-1 & 1 \\
1 & \beta^2 - 1
\end{bmatrix}.
\] (19)

\[\text{This property also means that } R_A \text{ is hypomonotone, see [29, Example 12.28].}\]
(ii): Let \( P \) be a \( 2 \times 2 \) matrix satisfying
\[
P_{11}\|x - y\|^2 + 2P_{12}(x - y, u - v) + P_{22}\|u - v\|^2 \geq 0 \quad \forall (x, u), (y, v) \in \text{gra}(J_{A^{-1}}).
\] (20)

On the one hand, it follows from (5) that each of the four properties in the first row of Table 1 correspond to
\[
L_{22}\|x - y\|^2 + 2L_{12}(x - y, u - v) + L_{11}\|u - v\|^2 \geq 0 \quad \forall (x, u), (y, v) \in \text{gra}(A^{-1}).
\] (21)

Therefore, using the \((1, 2)\) entry of Table 1, the first equation in (8) (applied to \( A^{-1} \)) and (21) we learn that the \( \beta \)-Lipschitz continuity of \( A \) corresponds to the matrix
\[
P = \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & \beta^2
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix}
= \begin{bmatrix}
\beta^2 & -\beta^2 \\
-\beta^2 & \beta^2 - 1
\end{bmatrix}.
\] (22)

Similarly, we learn from the \((2, 1)\) entry of Table 1 (applied with \( \mu = 0 \)), that the monotonicity of \( A \) (equivalently, the monotonicity of \( A^{-1} \)) corresponds to the matrix
\[
P = \begin{bmatrix}
0 & 1 \\
1 & -2
\end{bmatrix}.
\] (23)

The conclusion then follows from (22) and (23) in view of (20) by noting that
\[
\begin{bmatrix}
\beta^2 & 0 \\
0 & -(1 + \beta^2)
\end{bmatrix}
= \begin{bmatrix}
\beta^2 & -\beta^2 \\
-\beta^2 & \beta^2 - 1
\end{bmatrix} + \beta^2 \begin{bmatrix}
0 & 1 \\
1 & -2
\end{bmatrix}.
\] (24)

(iii): This follows from the entries \((2, 1)\) (applied with \( \mu = 0 \)) and \((2, 2)\) in Table 1, (i) and
\[
\begin{bmatrix}
-1/(1 + \beta^2) - 1/(1 + \beta^2) & 1 \\
1 & 0
\end{bmatrix}
= \frac{1}{\beta^2 + 1} \begin{bmatrix}
-1 & 1 \\
1 & \beta^2 - 1
\end{bmatrix} + \frac{\beta^2}{1 + \beta^2} \begin{bmatrix}
0 & 1 \\
1 & -1
\end{bmatrix} + \frac{1}{(1 + \beta^2)} \begin{bmatrix}
-1 & 0 \\
0 & (1 + \beta^2)
\end{bmatrix},
\] (25)
in view of (20).

(iv): One can readily verify that
\[
\text{gra}(J_A) = \frac{1}{2} \begin{bmatrix}
2 & 0 \\
1 & 1
\end{bmatrix} \text{gra}(R_A).
\] (26)

Now the conclusion follows from (26) and (iii) where \(-1/(1 + \beta^2) - 1/(1 + \beta^2) = \lambda - 1 \) and
\[
\begin{bmatrix}
2 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\lambda - 1 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
2 & 0 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
4\lambda & 2 \\
2 & 0
\end{bmatrix} =: Q,
\] (27)

by noting that
\[
Q_{11}\|x - y\|^2 + 2Q_{12}(x - y, u - v) + Q_{22}\|u - v\|^2 \geq 0 \quad \forall (x, u), (y, v) \in \text{gra}(R_A).
\] (28)
3 Linear rates of convergence: three cases

The Douglas–Rachford splitting algorithm, introduced by Lions and Mercier [22], is a fundamental algorithm for solving monotone inclusion problems of the form

\[
\text{Find } x \in X \text{ such that } 0 \in Ax + Bx, \quad (29)
\]

where \(A: X \rightrightarrows X\) and \(B: X \rightrightarrows X\) are maximally monotone operators. The algorithm is based on the iteration

\[
u_{n+1} = T_{\text{DR}}u_n = \frac{1}{2} \left( \text{Id} + R_B R_A \right) u_n, \quad (30)
\]

starting at arbitrary \(u_0 \in X\), where \(R_A\) and \(R_B\) are the reflected resolvents of \(A\) and \(B\). If the inclusion problem (29) has a solution, then the iterates of \((u_k)_{k \in \mathbb{N}}\) can be shown to converge weakly to some point \(u \in X\), where \(u = T_{\text{DR}}u\) and \(x = J_Au\) solves (29), see, e.g., [6, 13, 32].

In this section we review the results from [19] on contraction properties of the Douglas–Rachford operator \(T_{\text{DR}} = (1/2)(\text{Id} + R_B R_A)\). These results are summarized in Corollary 3.2. The following lemma shows that the three cases in Corollary 3.2 all have in common that \(T_{\text{DR}}\) is the resolvent of a strongly monotone operator (hence, in view of Proposition 2.1(iii), a contraction). We will see that this is a key difference with the new result in Section 4.

Lemma 3.1. Let \(T_1: X \to X\) and \(T_2: X \to X\) be nonexpansive. Define \(T = \frac{1}{2}(\text{Id} + T_2 T_1)\) and \(C = T^{-1} - \text{Id}\). Let \(\alpha \in ]0, 1[.\) Consider the following statements.

(a) \(C\) is \(((1 - \alpha)/\alpha)\)-strongly monotone.

(b) \(-T_2 T_1\) is \(\alpha\)-averaged\(^2\).

(c) \(T\) is \((1/\alpha)\)-cocoercive.

(d) \(T\) is a Banach contraction with a constant \(\alpha\).

Then \((a) \iff (b) \iff (c) \implies (d)\).

Proof We first note that \(T = J_C\) and \(T_2 T_1 = R_C\), by definition of \(T\) and \(C\). Hence \(T\) is firmly nonexpansive by [21, Theorem 2.1] and \(C\) is maximally monotone by [17, Theorem 2].

\((a) \iff (b) \iff (c)\): This follows from Proposition 2.1(iii) applied with \(A\) replaced by \(C\) and \(\mu\) replaced by \((1 - \alpha)/\alpha\) and the fact that \(T_2 T_1 = R_C\) (see also [19, Proposition 5.4]). \((c) \implies (d)\): This follows from [6, Proposition 23.13] (see also the comment after (4)).

Reference [19, Sections 5, 6 & 7] contains a comprehensive analysis of the rates of linear convergence of the Douglas–Rachford method with optimal relaxation parameters and step lengths, for the three cases presented in the next corollary. The key idea is that in each case, the Douglas–Rachford operator is a contraction, as summarized below.

Corollary 3.2. Let \(\beta \geq \mu > 0\). Suppose that one of the following properties is satisfied.

(a) \(A\) is \((1/\beta)\)-cocoercive and \(B\) is \(\mu\)-strongly monotone.

(b) \(A\) is \((1/\beta)\)-cocoercive and \(\mu\)-strongly monotone.

(c) \(A\) is \(\beta\)-Lipschitz continuous and \(\mu\)-strongly monotone.

\(^2\)This is also known as negative averagedness of the operator \(T_2 T_1\) [19].
Then

(i) \(-R_BR_A\) is \(\alpha\)-averaged for some \(\alpha \in ]0, 1[\).

(ii) \(T_{DR} = (1/2)(\text{Id} + R_BR_A)\) is a Banach contraction with a contraction factor \(\kappa \in ]0, 1[\).

The expressions for \(\alpha\) and \(\kappa\) are as follows.

Case (a): \(\alpha = \kappa = \frac{1 + \mu \beta}{1 + \mu + \mu \beta}\)

Case (b): \(\alpha = \kappa = \frac{1}{2} + \frac{1}{2} \left( \frac{1 - 2\mu + \mu \beta}{1 + 2\mu + \mu \beta} \right)^{1/2}\)

Case (c): \(\alpha = \kappa = \frac{1}{2} + \frac{1}{2} \left( \frac{1 - 2\mu + \beta^2}{1 + 2\mu + \beta^2} \right)^{1/2}\).

Proof We first discuss (i). From [19, Proposition 5.5], Assumption (a) implies that \(-R_BR_A\) is \(\alpha\)-averaged. If Assumption (b) holds, then \(R_A\) is a Banach contraction with factor

\[\kappa_1 = \left( \frac{1 - 2\mu + \mu \beta}{1 + 2\mu + \mu \beta} \right)^{1/2}\]

(see [19, Theorem 6.3] or Proposition 2.1(iv)). If Assumption (c) holds, then \(R_A\) is a Banach contraction with factor

\[\kappa_2 = \left( \frac{1 - 2\mu + \beta^2}{1 + 2\mu + \beta^2} \right)^{1/2}\]

(see [19, Theorem 7.2] or Proposition 2.1(v)). In both cases ((b) and (c)), this implies that the compositions \(R_BR_A\) and \(-R_BR_A\) are Banach contractions with factors \(\kappa_1\) and \(\kappa_2\) respectively. Hence \(-R_BR_A\) is ((\(\kappa_1 + 1)/2\))-averaged (respectively ((\(\kappa_2 + 1)/2\))-averaged) by [6, Proposition 4.38].

The second part (ii) is proved by combining (i) and Lemma 3.1 applied with \(T_1 = R_A\) and \(T_2 = R_B\), and using the triangle inequality. \(\square\)

4 Main results

We now consider the Douglas–Rachford iteration under the following assumptions:

\[A: X \rightarrow X\) is \(\beta\)-Lipschitz continuous and monotone, and \(\beta > 0\)\] (32)

and that

\[B: X \rightrightarrows X\) is maximally monotone and \(\mu\)-strongly monotone, and \(\mu > 0\).\] (33)

This case is not covered by Corollary 3.2, and is significantly different in nature, because these two properties in (32) and (33) do not imply that \(-R_BR_A\) is averaged, as shown by the following example.

Example 4.1. Suppose that \(X = \mathbb{R}^2\) and define

\[A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = N_{\{0\}}.\] (34)

Then \(A\) is monotone and nonexpansive (hence 1-Lipschitz continuous), \(B\) is maximally monotone and \(\mu\)-strongly monotone for every \(\mu > 0\), \(R_A = -A\) and \(R_B = -\text{Id}\). Hence, \(-R_BR_A = -A\) which is not averaged.
The main results in this section are Theorem 4.3 and Theorem 4.4 below. We first prove a more general result on an averaged composition of a $\beta$-Lipschitz continuous operator and an averaged operator.

**Proposition 4.2.** Let $R: X \to X$ be such that $-R$ is $\alpha$-averaged, with $\alpha \in [0,1]$. Let $M: X \to X$ be nonexpansive such that $(\forall (x,y) \in X \times X)$

$$\langle x - y, Mx - My \rangle \geq -\lambda \| x - y \|^2, \text{ with } \lambda \in [-1,1[.$$  \hspace{1cm} (35)

Define

$$T = \frac{1}{2}(\text{Id} + RM), \quad \tilde{T} = \frac{1}{2}(\text{Id} + MR).$$  \hspace{1cm} (36)

Then the following hold:

(i) $\text{Id} + (\alpha - 1)M$ is Lipschitz continuous with constant $\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < 2 - \alpha < 2$.

(ii) $T$ is Lipschitz continuous with constant

$$\frac{1}{2}\left(\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha}\right) < 1.$$  \hspace{1cm} (37)

Hence, $T$ is a Banach contraction and $\text{Fix } T$ is a singleton.

(iii) If $M$ is linear, then $\tilde{T}$ is Lipschitz continuous with constant given in (37). Hence, $\tilde{T}$ is a Banach contraction and $\text{Fix } \tilde{T}$ is a singleton.

**Proof (i):** Set $S = \text{Id} + (\alpha - 1)M$ and let $(x,y) \in X \times X$. Then

$$\|Sx - Sy\|^2 = \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 - 2(1 - \alpha)\langle x - y, Mx - My \rangle$$  \hspace{1cm} (38a)

$$\leq \|x - y\|^2 + (1 - \alpha)^2\|Mx - My\|^2 + 2\lambda(1 - \alpha)\|x - y\|^2$$  \hspace{1cm} (38b)

$$\leq (1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha))\|x - y\|^2.$$  \hspace{1cm} (38c)

The first inequality follows from (35) and the second inequality follows from the nonexpansiveness of $M$. Finally note that, because $-1 \leq \lambda < 1$, we learn that $\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha)} < \sqrt{1 + (1 - \alpha)^2 + 2(1 - \alpha)} = \sqrt{1 + 2(1 - 2\alpha + \alpha^2)} = \sqrt{2 - \alpha} < 2$.

(ii): Since $-R$ is $\alpha$-averaged, we have $R = (\alpha - 1)\text{Id} + \alpha N$ for some nonexpansive $N: X \to X$. Substituting this in the definition of $T$, we get $T = \frac{1}{2}(\text{Id} + (\alpha - 1)M + \alpha NM)$. It follows from the triangle inequality, (i), and the nonexpansiveness of $M$ and $N$ that $T$ is Lipschitz continuous with a constant

$$\frac{1}{2}\left(\sqrt{1 + (1 - \alpha)^2 + 2\lambda(1 - \alpha) + \alpha}\right) < \frac{1}{2}(2 - \alpha + \alpha) = 1.$$  \hspace{1cm} (39)

(iii): As in (ii), we write $R$ as $R = (\alpha - 1)\text{Id} + \alpha N$ with $N$ nonexpansive. Then

$$\tilde{T} = \frac{1}{2}(\text{Id} + M((\alpha - 1)\text{Id} + \alpha N)) = \frac{1}{2}(\text{Id} + (\alpha - 1)M + \alpha MN).$$

The second identity follows from linearity of $M$. Now the proof of (iii) is similar to (ii). \qed

We are now ready for our main results.

**Theorem 4.3.** Suppose that $A: X \to X$ is monotone and $\beta$-Lipschitz continuous with $\beta > 0$, and that $B: X \Rightarrow X$ is maximally monotone and $\mu$-strongly monotone with $\mu > 0$. Let $x_0 \in X$, let

$$T = \frac{1}{2}\left(\text{Id} + R_B R_A\right).$$

Then the following hold:
(i) \((x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}\) converges strongly to some \(\bar{x} \in X\), with a linear rate \(r\), where
\[
 r = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1 + 2 \left( 1 - \frac{1}{1+\beta^2} - \frac{1}{1+\beta^2} \right)} \mu(1+\mu) + 1 \right) < 1. \tag{40}
\]

(ii) \((J_A x_n)_{n \in \mathbb{N}}\) converges strongly to \(J_A \bar{x}\) with a linear rate \(r\) given in (40).

Moreover, \(\text{Fix } R_B R_A = \text{Fix } T = \{\bar{x}\}\), and \(\text{zer}(A + B) = \{J_A \bar{x}\}\).

**Proof** Since \(A\) is monotone and \(\beta\)-Lipschitz continuous, we have \(R_A\) is nonexpansive and \((\forall (x, y) \in X \times X)\)
\[
\langle x - y, R_A x - R_A y \rangle \geq - \left( 1 - \frac{1}{1+\beta^2} - \frac{1}{1+\beta^2} \right) \|x - y\|^2, \tag{41}
\]
by Lemma 2.3(ii). Since \(B\) is \(\mu\)-strongly monotone, \(-R_B\) is \((1 + \mu)^{-1}\)-averaged (see [19, Proposition 5.4] or Proposition 2.1(iii)). (i): The claim of strong convergence follows from [6, Theorem 26.11(vi)(a)]. The rate \(r\) follows from Proposition 4.2(ii) applied with \(\alpha = (1 + \mu)^{-1}\), \(\lambda = (1 - (1/(1+\beta)^2) - (1/1+\beta^2))\), \(M = R_A\), and \(R = R_B\). (ii): This is a direct consequence of (i) and the fact that \(J_A\) is (firmly) nonexpansive. 

When \(A\) is linear, similar conclusion to that of Theorem 4.3 holds if we switch the order of the operators in the Douglas–Rachford iteration.

**Theorem 4.4.** Suppose that \(A : X \to X\) is monotone, \(\beta\)-Lipschitz continuous with \(\beta > 0\) and linear, and that \(B : X \rightrightarrows X\) is maximally monotone and \(\mu\)-strongly monotone with \(\mu > 0\). Let \(\bar{x}_0 \in X\), and let \(T = \frac{1}{2} \left( \text{Id} + R_A R_B \right)\). Then the following hold:

(i) \((\bar{x}_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}\) converges strongly to some \(\hat{x}\) with a linear rate \(r\) given in (40).

(ii) \((J_B \bar{x}_n)_{n \in \mathbb{N}}\) converges strongly to \(J_B \hat{x}\) with a linear rate \(r\) given in (40).

Moreover, \(\text{Fix } R_A R_B = \text{Fix } T = \{\hat{x}\}\) and \(\text{zer}(A + B) = \{J_B \hat{x}\}\).

**Proof** Proceed as in the proof of Theorem 4.3(i)–(ii), but use Proposition 4.2(iii) and the fact that \(R_A\) is linear. 

We conclude this section with the following remark.

**Remark 4.5.** It is not clear whether or not the conclusion of Proposition 4.2(iii) remains true if we drop the assumption of linearity. Any counterexample to show failure of the conclusion in the absence of linearity must feature nonexpansive operator that satisfies (35) which is neither linear nor averaged, because if \(M\) is averaged we have that \(T\) is a Banach contraction by [19, Proposition 3.9]. Note, however, that the result for linear \(M\) covers important applications, such as the the primal-dual Douglas–Rachford method discussed in Section 5.

## 5 The linear skew case and application to primal-dual Douglas–Rachford method

The main goal of this section is to prove linear convergence of the primal-dual Douglas–Rachford method discussed in [25, Sections 3.1&3.2] (see also [10] for a more general framework) when applied to solve the monotone inclusion (53) below, under additional assumptions on the underlying operators.
In the following we show that when $A$ is linear and skew then the rate in (40) is improved. We first start with the following lemma which shows that when $A$ is linear and skew, the bounds in Lemma 2.3 can be tightened.

**Lemma 5.1.** Suppose that $A : X \to X$ is linear, skew, i.e., $A = -A^*$ and $\beta$-Lipschitz continuous with $\beta > 0$. Let $x \in X$. Then the following hold:

(i) $R_A$ is an isometry, i.e., $\|R_A x\| = \|x\|$.  

(ii) $\|x\|^2 \leq (1 + \beta^2) \|JAx\|^2$.  

(iii) $J_A$ is $\frac{1}{\beta^2+1}$-strongly monotone.  

(iv) $\langle x, R_A x \rangle \geq (\frac{2}{1+\beta^2} - 1) \|x\|^2$.  

**Proof** Set $u = JAx$ and note that $Au = x - u$. Now, since $A$ is skew, in view of (6) we have  
\[
\langle u, x - u \rangle = \langle u, Au \rangle = 0. \tag{42}
\]

(i): Using (42) we have  
\[
\|R_A x\|^2 = \|JAx - J_A^{-1}x\|^2 = \|u - Au\|^2 \tag{43a}
\]
\[
= \|u\|^2 - 2 \langle u, Au \rangle + \|Au\|^2 \tag{43b}
\]
\[
= \|u\|^2 + 2 \langle u, Au \rangle + \|Au\|^2 \tag{43c}
\]
\[
= \|u + Au\|^2 = \|x\|^2. \tag{43d}
\]

(ii): Indeed, using (42) we have  
\[
\|x\|^2 = \|u\|^2 + 2 \langle u, Au \rangle + \|Au\|^2 = \|u\|^2 + \|Au\|^2 \tag{44a}
\]
\[
\leq \|u\|^2 + \beta^2 \|u\|^2 = (1 + \beta^2) \|u\|^2, \tag{44b}
\]

where the inequality follows from the $\beta$-Lipschitz continuity of $A$.

(iii): It follows from Lemma 2.3(iv) that $(1 + \beta^2) \|x - JAx\|^2 \leq \beta^2 \|x\|^2$. Expanding yields  
\[
(1 + \beta^2)(\|x\|^2 + \|JAx\|^2 - 2 \langle x, JAx \rangle) \leq \beta^2 \|x\|^2. \tag{45}
\]
Equivalently,  
\[
2(1 + \beta^2) \langle x, JAx \rangle \geq \|x\|^2 + (1 + \beta^2) \|JAx\|^2. \tag{46}
\]

(iv): We have  
\[
\langle x, R_A x \rangle = \langle x, 2JAx - x \rangle = 2 \langle x, JAx \rangle - \|x\|^2 \geq (\frac{2}{1+\beta^2} - 1) \|x\|^2,
\]
where the inequality follows from (iii). \qed

**Theorem 5.2.** Suppose that $A : X \to X$ is linear, skew (hence monotone) and $\beta$-Lipschitz continuous with $\beta > 0$, and that $B : X \rightrightarrows X$ is maximally monotone and $\mu$-strongly monotone with $\mu > 0$. Let $x_0 \in X$, let $T = \frac{1}{2}(\Id + R_B R_A)$ and let $\tilde{T} = \frac{1}{2}(\Id + R_A R_B)$. Then the following hold:

(i) $(x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}$ converges strongly to some $\bar{x} \in X$, with a linear rate $r$, where
\[
r(\beta, \mu) = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1} + 2 \left( 1 - \frac{2}{1+\beta^2} \right) \mu(1+\mu) + 1 \right) < 1. \tag{45}
\]

(ii) $(JAx_n)_{n \in \mathbb{N}}$ converges strongly to $J_A \bar{x}$ with a linear rate $r$ given in (45).
(iii) \((x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}\) converges strongly to some \(x\) with a linear rate \(r\) given in (45).

(iv) \((J_B x_n)_{n \in \mathbb{N}}\) converges strongly to \(J_B x\) with a linear rate \(r\) given in (45).

Moreover, Fix \(R_B R_A = \text{Fix} T = \{x\}\), Fix \(R_A R_B = \text{Fix} T = \{x\}\) and \(\text{zer}(A + B) = \{J_A x\} = \{J_B x\}\).

Proof Proceed as in the proof of Theorem 4.3 for (i)–(ii) (respectively Theorem 4.4 for (iii)–(iv)) in view of Lemma 5.1(iv).

The contraction factor in (45) is sharp as we illustrate in Example 5.3 below.

Example 5.3 (sharpness of the contraction factor). Let \(\beta > 0\) and let \(\mu > 0\). Suppose that \(X = \mathbb{R}^2\),

\[
A = \beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \mu \text{Id} + N_{\{0\} \times \mathbb{R}}.
\]

Then \(A\) is \(\beta\)-Lipschitz continuous and monotone, \(B\) is \(\mu\)-strongly monotone and

\[
R_A = \begin{bmatrix} \frac{2}{\beta^2+1} & -1 & -\frac{2\beta}{\beta^2+1} \\ \frac{2}{\beta^2+1} & 2 & -1 \end{bmatrix}, \quad R_B = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{1+\mu} \end{bmatrix}.
\]

Therefore,

\[
T = \frac{1}{2} (\text{Id} + R_B R_A) = \frac{1}{\beta^2 + 1} \begin{bmatrix} \beta^2 & \beta \\ \beta & \frac{1}{1+\mu} \end{bmatrix},
\]

\[
\|T\| = \frac{1}{2(1+\mu)} \left( \sqrt{2\mu^2 + 2\mu + 1} + 2 \left(1 - \frac{2}{1+\beta^2}\right) \mu(1+\mu) + 1 \right).
\]

Proof The claim about \(R_A\) is straightforward to verify. By [6, Example 23.4 and Corollary 3.24(iii)], we have

\[
J_B = ((1+\mu) \text{Id} + N_{\{0\} \times \mathbb{R}})^{-1} = ((1+\mu)(\text{Id} + N_{\{0\} \times \mathbb{R}}))^{-1} = J_{N_{\{0\} \times \mathbb{R}}} \circ \frac{1}{1+\mu} \text{Id} = \frac{1}{1+\mu} P_{\{0\} \times \mathbb{R}} = \frac{1}{1+\mu} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.
\]

The expression for \(R_B\) in (47) and the formula for \(T\) readily follows. A routine calculation yields that the eigenvalues of \(TT^T\) are

\[
2\beta^2 \mu^2 + 2\beta^2 \mu + \beta^2 + 1 \pm \sqrt{(2\beta^2 \mu^2 + 2\beta^2 \mu + \beta^2 + 1)^2 - 4(1+\mu)^2(1+\beta^2)\beta^4 \mu^2}.
\]

Hence,

\[
\|T\| = \|TT^T\|^{1/2} = \sqrt{\frac{2\beta^2 \mu^2 + 2\beta^2 \mu + \beta^2 + 1 + \sqrt{(2\beta^2 \mu^2 + 2\beta^2 \mu + \beta^2 + 1)^2 - 4(1+\mu)^2(1+\beta^2)\beta^4 \mu^2}}{2(1+\mu)^2(1+\beta^2)}}.
\]
Figure 1: A Mathematica [33] snapshot. Shown is the rate \( r = r(\beta, \mu) \) given in (45) for the case when \( A \) is \( \beta \)-Lipschitz continuous and monotone and \( \beta > 0 \) and \( B \) is \( \mu \)-strongly monotone.

\[
= \frac{1}{2(1 + \mu)} \left( \sqrt{2\mu^2 + 2\mu + 1 + 2 \left( 1 - \frac{2}{1 + \beta^2} \right) \mu (1 + \mu) + 1} \right). \quad (52c)
\]

In Figure 1 we provide a plot of the rate in (45) as a function of \( \beta \) and \( \mu \). Figure 2 provides plots of the rate in (45) as a function of \( \mu \) (respectively \( \beta \)) for some concrete values of \( \beta \) (respectively \( \mu \)).

Remark 5.4. Suppose that \( \gamma > 0 \). Then, in the setting of Theorem 5.2, the rate obtained when iterating \( T = (1/2)(\text{Id} + R_{\gamma B} R_{\gamma A}) \) or \( T = (1/2)(\text{Id} + R_{\gamma A} R_{\gamma B}) \) is \( r(\gamma \beta, \gamma \mu) \), where \( r \) is defined in (45). However, unlike the rates presented in [19], the rate given in (45) cannot be easily optimized as a function of the step-length \( \gamma \). Indeed, suppose that \( \beta = \mu = 1 \). Then \( r(\gamma \beta, \gamma \mu) = h(\gamma) = \frac{1}{2(1 + \gamma)} \left( \sqrt{2\gamma^2 + 2\gamma + 1 + 2 \left( 1 - \frac{2}{1 + \gamma^2} \right) \gamma (1 + \gamma) + 1} \right) \). One can show that \( h'(\gamma) = 0 \) reduces to solving the quintic \( 4\gamma^5 + 5\gamma^4 + 12\gamma^3 + 2\gamma^2 - 3 = 0 \). Of course we can solve numerically for the optimal value of \( \gamma \), which in this case yields \( \gamma \approx 0.4815 \).

Throughout the remainder of this section, we assume that \( Y \) is a real Hilbert space\(^3\), that \( L: X \to Y \) is nonzero and linear, that\(^4\) \( f: X \to [\infty, +\infty] \) is \( \sigma \)-strongly convex and closed, and that \( g: X \to \mathbb{R} \) is convex and \( \nabla g \) is \( \beta \)-Lipschitz continuous for some \( \beta > 0 \).

Consider the monotone inclusion:

\[
\text{Find } (x, y) \in X \times Y \text{ such that } 0 \in A(x, y) + B(x, y), \quad (53)
\]

where\(^5\)

\[
A: X \times Y \to X \times Y: (x, y) \mapsto (L^* y, -Lx), \quad B: X \times Y \rightrightarrows X \times Y: (x, y) \mapsto \partial f(x) \times \partial g^*(y). \quad (54)
\]

---

\(^3\)A finite-dimensional example is \((X, Y) = (\mathbb{R}^n, \mathbb{R}^m)\).

\(^4\)A closed function is also known as lower semicontinuous.

\(^5\)Here and elsewhere we use \( f^* \) to denote the convex conjugate (this is also known as the Fenchel or Legendre conjugate) of \( f \) defined at \( u \in X \) as \( f^*(u) = \sup_{x \in X} \{ \langle u, x \rangle - f(x) \} \).
Figure 2: Left: Shown are the optimal rates of convergence given in (45) as functions of $\mu$ for $\beta = 0.2$ (black loosely-dotted line), $\beta = 0.5$ (blue solid line), $\beta = 1$ (brown densely-dotted line), $\beta = 2$ (red dashed line) and $\beta = 5$ (green dash-dotted line). Right: Shown are the optimal rates of convergence given in (45) as functions of $\beta$ for $\mu = 0.2$ (black loosely-dotted line), $\mu = 0.5$ (blue solid line), $\mu = 1$ (brown densely-dotted line), $\mu = 2$ (red dashed line) and $\mu = 5$ (green dash-dotted line).

One can check that $\|A\| = \|L\| \neq 0$. Hence,

\[ A \text{ is Lipschitz continuous with the sharp constant } \|L\|. \quad (55) \]

Note that $\partial f$ is maximally monotone and $\sigma$-strongly monotone by e.g., [28, Theorem A], and [6, Example 22.4(iv)]. Moreover, we have $(\nabla g)^{-1} = \partial g^*$ by [28, Remark on page 216]. Therefore, in view of [2, Corollaire 10] (see also [4]) we learn that

\[ B \text{ is maximally monotone and } \mu \text{-strongly monotone and } \mu = \min \{\sigma, 1/\beta\}. \quad (56) \]

The inclusion in (53) arises in primal-dual optimality conditions of the primal problem (P) and its Fenchel–Rockafellar dual (D) given by:

\[
\min_{x \in X} f(x) + g(Lx), \quad (P)
\]

\[
\min_{y \in Y} f^*(-L^*y) + g^*(y), \quad (D)
\]

under appropriate assumptions on $f$, $g$ and $L$.

We are now ready for the main result in this section.

**Theorem 5.5 (application to primal-dual Douglas–Rachford method).** Let $\mu = \min \{\sigma, 1/\beta\}$. Suppose $A$ and $B$ are as defined in (54). Set

\[
T = \frac{1}{2} \left( \text{Id} + R_B R_A \right), \quad \tilde{T} = \frac{1}{2} \left( \text{Id} + R_A R_B \right). \quad (57)
\]
Let $x_0 \in X \times Y$, let $(x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}$, let $(y_n)_{n \in \mathbb{N}} = (J_A T^n x_0)_{n \in \mathbb{N}}$, let $(\tilde{x}_n)_{n \in \mathbb{N}} = (\tilde{T}^n x_0)_{n \in \mathbb{N}}$, and let $(\tilde{y}_n)_{n \in \mathbb{N}} = (J_B T^n x_0)_{n \in \mathbb{N}}$. Then there exists $\bar{x} \in X \times Y$, $\{\bar{x}\} = \text{Fix} T = \text{Fix} R_B R_A$, there exists $\hat{x} \in X \times Y$, $\{\hat{x}\} = \text{Fix} R_A R_B = \text{Fix} \tilde{T}$, such that $\text{zer}(A + B) = \{J_A x\} = \{J_B \hat{x}\}$. Moreover, the following hold:

(i) $(x_n)_{n \in \mathbb{N}}$ converges strongly to $\bar{x}$ with a linear rate $r$, where

$$r = \frac{1}{2(1 + \mu)} \left( \sqrt{2 \mu^2 + 2 \mu + 1 + 2(1 - 2(1 + \|L\|^2)^{-1}) \mu(1 + \mu) + 1} \right).$$

(ii) $(y_n)_{n \in \mathbb{N}}$ converges strongly to $J_A \bar{x}$ with a linear rate $r$ given in (58).

(iii) $(\tilde{x}_n)_{n \in \mathbb{N}}$ converges strongly to $\hat{x}$ with a linear rate $r$ given in (58).

(iv) $(\tilde{y}_n)_{n \in \mathbb{N}}$ converges strongly to $J_B \hat{x}$ with a linear rate $r$ given in (58).

Proof Note that $A$ is $\|L\|$-Lipschitz continuous and $B$ is $\mu$-strongly monotone by (55) and (56) respectively. The proof of (i)–(iv) follows from Theorem 5.2 applied with $X$ replaced by $X \times Y$, $\beta$ replaced by $\|L\|$, $A$ replaced by $A$ and $B$ replaced by $B$. \qed

6 Conclusion

In this paper we prove that the Douglas–Rachford method converges linearly with a sharp rate when applied to solve (29) in the case $A$ is Lipschitz continuous (but not necessarily a subdifferential operator) and $B$ is strongly monotone. We also discuss an important application of the results to primal-dual Douglas–Rachford method. In this case we get sharp rate. As a byproduct of our work, we obtain useful equivalences between the operator properties and the properties of the corresponding resolvent and reflected resolvent.

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