Violation of horizon by topological quantum excitations

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One of the fundamental principles of relativity is that a physical observable at any space-time point is determined only by events within its past light-cone. In nonequilibrium quantum field theory this is manifested in the way correlations spread through space-time: starting from an initially short-range correlated state, measurements of two observers at distant space-time points are expected to remain independent until their past light-cones overlap, which is usually called the “horizon effect”. Surprisingly, we find that in the presence of topological excitations correlations can develop outside of horizon - even between infinitely distant points. We demonstrate this effect in the sine-Gordon model, showing that it can be attributed to the non-local nature of its topological excitations and interpret it as dynamical emergence of entanglement between distant regions of space.

Horizon effect in quantum field theory. Consider the following thought experiment: A system described by a quantum field theory (QFT) is prepared in some initial state and let to evolve under unitary relativistic dynamics. This can be realised by a paradigmatic protocol known as a “quantum quench” [1], in which a closed quantum system initially prepared in an equilibrium state undergoes a sudden change of its Hamiltonian at time \( t = 0 \). Let us suppose that the initial state is characterised by short-range correlations of local quantum fields with a small correlation length \( \xi \); such states are rather common and include ground and thermal states of massive QFTs. As a result, measurements made by two observers separated by a distance \( r \) will be independent until time \( t \sim r/c \), where \( c \) is the finite maximum speed at which information propagates [1][2]. This is known as the “horizon effect” and can be justified by a semiclassical interpretation where the correlations propagate by pairs of entangled quasiparticles emitted from initially correlated nearby points (at a distance \( \lesssim \xi \)) that travel to opposite directions with velocities limited by \( c \). The horizon effect can be proved in the case of non-interacting relativistic dynamics provided the initial state satisfies the cluster decomposition property [3], and also holds in lattice systems with local interactions due to the finite maximum velocity of quasi-particles [4]. It has also recently been observed in cold atom experiments [5][6].

But what happens in the case of non-trivial interactions? It is known that interactions can have significant effects on the speed of propagation [8] or even fully suppress the spreading of correlations [9]. However, their interplay with the horizon effect remains incompletely understood. Here we address this question in the context of the sine-Gordon (SG) model described by the Hamiltonian (written in units with \( c = 1 \))

\[
H_{SG} = \int \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \Phi)^2 - \lambda \cos \beta \Phi \right) dx
\]

where \([\Phi(x), \Pi(x')] = i \delta(x - x')\). The SG model is a prototypical example of a 1+1 dimensional relativistic QFT with rich physics governed by topological excitations i.e. solitons (and antisolitons). They appear due to the periodic cosine potential compactifying the field to a circle topology \( \Phi \sim \Phi + 2\pi/\beta \), and can also form neutral bound states called breathers when the coupling parameter \( \beta \) is in the attractive regime \( \beta^2 < 4\pi \). The SG model has a non-trivial phase diagram with a transition of the Berezinskii-Kosterlitz-Thouless type at \( \beta^2 = 8\pi \). It is an integrable model [10][11] and describes the dynamics of numerous condensed matter systems [12]. The SG model has recently been realised in an ultracold atom experiment which enables the study of its correlation functions and non-equilibrium dynamics [13][14]. Its dynamics has attracted considerable recent interest [15][20].

We prepare the system to be in the ground state \( |\Omega\rangle \) of the massive Klein-Gordon (KG) Hamiltonian

\[
H_{KG} = \int \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \Phi)^2 + \frac{1}{2} m_0^2 \Phi^2 \right) dx
\]

for which the correlations of \( \Phi(x) \) have finite range \( \xi = 1/m_0 \). At time \( t = 0 \) we switch on the SG model interactions and study the dynamics of connected correlations

\[
C_O(x, y; t) := \langle \Omega | e^{+iH_{SG}t} O(x) O(y) e^{-iH_{SG}t} |\Omega\rangle_{\text{con}}
\]

of local observables \( O \). We focus on two physically relevant observables: the field \( \Phi(x) \) and \( \partial_x \Phi \) which corresponds to the soliton density. We stress that despite being an angular variable, correlations of the field \( \Phi(x) \) are experimentally observable [13].

A surprise: out-of-horizon spreading of correlations. In order to study the dynamics of correlations we employ a recent variant [21] of a numerical method known as “Truncated Conformal Space Approach” (TCSA) [22] which is based on Renormalisation Group and Conformal Field Theory. TCSA is especially suited for the study of interacting (1+1) dimensional QFTs and efficiently captures non-perturbative effects (cf. Methods).

Fig. [1] shows the spreading of correlations under SG dynamics for increasing values of the interaction \( \beta \), compared to non-interacting KG dynamics. In marked contrast to the free time evolution, SG dynamics leads to strong violations of horizon for both observables. While for small values of \( \beta \) the spreading of correlations is similar to the KG case, for
larger $\beta$ they quickly spread outside of the light-cone. The correlations also develop temporal oscillations at a frequency increasing with $\beta$.

The effect of horizon violation is most drastic for the $\partial_x \Phi$ field. In this case, the out-of-horizon correlations are spatially uniform and display no visible decay with distance. This excludes an explanation of the effect by the periodic ambiguity of the $\Phi$ field since $\partial_x \Phi$ is an unambiguously defined local observable.

**Explanation.** To verify the validity of the above unexpected numerical observation and shed light on the origins of the out-of-horizon effect, we perform an analytical calculation of the dynamics at a convenient value of the coupling. We exploit the powerful analytical tool of bosonisation [23, 24], which is a mapping between a fermionic and a bosonic QFT via an exact non-linear and non-local correspondence in 1+1 dimensions (cf. Methods).

Bosonisation relates the SG model to an interacting fermionic QFT: the massive Thirring model [25] (cf. Eq. (5) in Methods). In particular, for $\beta^2 = 4\pi$ which is deep in the strongly interacting regime of SG model, the fermion interaction of the Thirring model vanishes and the SG model becomes equivalent to the free massive Dirac theory

$$H_{MF} = \int \Psi \left(-i\gamma^1 \partial_x + M\right) \Psi \, dx,$$

which enables analytic calculation of the time evolution.

With the help of bosonisation we compute the dynamics of SG correlations by the following steps: (1) fermionising the theory - expressing the observables $O$ in terms of the fermion field $\Psi$, which is a mapping between a fermionic and a bosonic QFT via an exact non-linear and non-local correspondence in 1+1 dimensions (cf. Methods).

Bosonisation relates the SG model to an interacting fermionic QFT: the massive Thirring model [25], whereas the nonlocal soliton-antisoliton fields of SG are identified with the fermion fields [26] (cf. Eq. (5) in Methods). In particular, for $\beta^2 = 4\pi$ which is deep in the strongly interacting regime of SG model, the fermion interaction of the Thirring model vanishes and the SG model becomes equivalent to the free massive Dirac theory

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which enables analytic calculation of the time evolution.
fermion point obtained this way is shown in Fig. 2. The analytical results confirm the dynamical emergence of out-of-horizon connected correlations persisting for infinite separations. These long range correlations exhibit temporal oscillations decaying algebraically with time.

The calculation outlined above also provides a transparent explanation of the effect. Using bosonisation, the correlation function $C_{\phi \bar{\phi}}(x, y; t)$ can be expressed as a quadruple convolution of initial fermionic 4-point and 2-point functions $(C^F_{\sigma \rho \pi \sigma}, (x_1, x_2, y_1, y_2))$ and $C^F_{\rho \sigma}, (x_1, x_2)$ respectively) with the free fermion propagator $G_{\sigma \rho}(x-x', t)$ (cf. eq. (11) and (10) in Methods). The indices $\sigma, \rho, \pi$ correspond to the fermion field components labelled by $\pm$. The fermion propagators vanish identically outside of the light-cone i.e. for $|x - x'| > t$ and therefore cannot be responsible for the observed effect.

The initial correlations, however, display some unexpected behaviour. For any state that lives in the vacuum sector of the SG Hilbert space such as our initial state $|\Omega\rangle$, fermion fields $\Psi^+_i$ and antifermion fields $\Psi^-_i$ can only be present in equal numbers for each of the two components $\sigma = \pm$ separately. This imposes selection rules on the initial correlations: the only allowed 2-point functions are $C^F_{++}$ and $C^F_{--}$, while the non-zero 4-point functions are $C^F_{++-+}$, $C^F_{+-+-}$, $C^F_{++-+}$ and $C^F_{+-+-}$. The initial fermionic correlations $C^F_{++-+}$ and $C^F_{+-+-}$ are found to tend to a nonzero value at infinite distance, violating the cluster decomposition principle. Note that clustering requires that for large separation between the $x_1, x_2$ and $y_1, y_2$ (cf. Fig. 3), $C^F_{++-+}$ and $C^F_{+-+-}$ correlations should decay with separation since the product $C^F_{++-+} C^F_{+-+-}$ vanishes. Due to the non-locality of the fermionic fields such correlations are undetectable by local measurements in the initial state, which has short-range correlations for the observable bosonic local fields. However, the long-range correlations of the fermionic (i.e. solitonic) degrees of freedom are made manifest in the correlations of local fields for $t > 0$ due to the interacting dynamics of the SG model and so they dominate the asymptotic correlations of $\partial_t \Phi$ (and thus $\bar{\Phi}$) for $t > 0$.

Our derivation also identifies the oscillation frequency of long-range correlations as four times the fermion mass $M$. Moreover, the oscillations are found to decay with time as $1/t$, which is due to the continuous spectrum of the soliton-antisoliton pairs which leads to dephasing $[1]$. This picture is expected to stay valid beyond the free fermion point and into the repulsive regime $\beta^2 > 4\pi$. In the attractive regime $\beta^2 < 4\pi$ shown in Fig. 1 the soliton-antisoliton pairs are dominantly created as breather bound states, therefore it is expected that the oscillation is dominated by twice the frequency of the lowest-lying breather $B_1$, consistently with Fig. 1. The suppression of the dephasing effect in the attractive regime is consistent with the breather being an isolated bound state below the two-particle continuum.

Discussion. We stress that the effect is entirely compatible with all relevant physical principles: there is no contradiction with relativistic causality, while in the initial equilibrium state, the cluster decomposition property is only violated for a non-local (fermionic) field. For $t > 0$ it is also violated for the local field $\partial_x \Phi$; however, the system is no longer at equilibrium so clustering does not necessarily apply.

While the horizon effect is typically understood as a feature of the dynamics, in fact it is a consequence of two equally important factors: relativistically invariant dynamics and exponential clustering of initial correlations. For dynamics that is free in terms of the local fields, the above two conditions are sufficient to prove the presence of horizon $[3]$. However, due to their topological nature the quasi-particle excitations of the SG are non-local in terms of the bosonic fields, therefore short range bosonic correlations in the initial state do not preclude long-distance entanglement of the quasi-particle pairs created during the quantum quench.

It might appear counter-intuitive that two observers located far away from each other experience correlated measurements right after the initial time, i.e. all points in space become immediately correlated for $t > 0$. This can be understood as a manifestation of long-distance entanglement of the Einstein-Podolsky-Rosen type $[27]$, symbolised by the coloured lines connecting the correlated $\pm$ indices of the cluster violating initial correlations in Fig. 5.

The out-of-horizon spreading of correlations uncovered here is a concrete prediction for experiments realising the SG model dynamics such as in $[13]$, and is also expected to be a general feature of non-equilibrium dynamics in topological phases of quantum condensed matter systems and other QFTs with non-trivial target space topology.
Acknowledgments. This work was partially supported by the Advanced Grant of European Research Council (ERC) 694544 – OMNES, by the Slovenian Research Agency under grants N1-0055 (OTKA-ARRS joint grant) and P1-0402, as well as by the National Research Development and Innovation Office of Hungary within the Quantum Technology National Excellence Program (Project No. 2017-1.2.1-NKP-2017-00001) and under grants OTKA No. SNN118028 and K-16 No. 119204. S.S. acknowledges support by the Slovenian Research Agency under grant N1-0109 (QTE). The work of G.T. was also partially supported by the BME-Nanotechnology FIKP grant of EMMI (BME FIKP-NAT). I.K. is grateful to Martin Horvat for support in high-performance computing.

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**Methods.** The dynamics of correlations was computed using two complementary methods, the numerical “Truncated Conformal Space Approach” (TCSA) and the analytical approach of bosonisation.

*Truncated Conformal Space Approach.* The TCSA [22] is a numerical method for the non-perturbative study of (1+1)-dimensional QFT models (cf. [28] for a recent review) and recently extended to non-equilibrium time evolution [29]. It assumes that the Hamiltonian can be written as $H = H_0 + \lambda \Delta H$ where $H_0$ is exactly solvable, while $\Delta H$ is an operator with known matrix elements between eigenstates of $H_0$. If $\Delta H$ does not couple the low energy part of the spectrum strongly with high energy states, the space of states can be truncated at an energy cut-off. In finite volume $L$ the truncated Hilbert space is finite dimensional and so $H$ becomes a finite matrix, reducing computations to simple matrix manipulations.

For the SG model [1] the reference Hamiltonian $H_0$ is the massless free boson conformal field theory (CFT) $H_{CFT} = \int (\frac{1}{2} \Pi^2 + \frac{1}{2} (\partial \Phi)^2) dx$ and the perturbing operator is $\Delta H = \int \cos \beta \Phi \ dx$ [30]. For $\beta^2 < 8\pi$ the operator $\cos \beta \Phi$ is a relevant scaling operator and so the effective coupling flows to zero at high energy, therefore the numerical results are expected to converge with the cut-off, with a rate dependent on $\beta$; for sufficiently attractive couplings the method converges very fast. To avoid infrared singularities related to bosonic zero modes we used Dirichlet boundary conditions $\Phi(x = 0) = \Phi(x = L) = 0$, which allows for the KG Hamiltonian to be represented in the same CFT basis as the SG Hamiltonian, and also preserves the conservation of the topological charge. The evolution is followed for times up to $t = L/2$ to avoid boundary effects arising from the finite volume.

The numerical simulation consisted of the following steps:

1. Constructing truncated matrix representations of $H_{KG}$ and $H_{SG}$
2. Computing the KG ground state $|\Omega\rangle$ in the CFT basis, by computation of the lowest energy eigenvector of $H_{KG}$;
3. Computing the time evolution operator $e^{-iH_{SG}t}$ by matrix exponentiation;
4. Constructing $\Phi$ and $\partial_x \Phi$ at different positions in space using mode expansions in the CFT basis;
5. Evaluating the correlation functions as expectation values $\langle \Omega e^{+iH_{SG}t} O(x) O(y) e^{-iH_{SG}t} |\Omega\rangle$, and determining the connected correlations $C_O(x, y; t)$.

**Bosonisation/fermionisation.** As discovered in [25], the massive Thirring model with the Hamiltonian

$$H_{MT} = \int \left[ \bar{\Psi} \left( -i\gamma^j \partial_x + M \right) \Psi + \frac{1}{2} g \left( \bar{\Psi} \gamma^\mu \Psi \right) \left( \bar{\Psi} \gamma_\mu \Psi \right) \right] \ dx$$

is equivalent to the SG model [1] with

$$\beta^2 = \frac{1}{4\pi \left( 1 + g/\pi \right)} ,$$

where the fermion mass $M$ is identified with the SG soliton mass. The interacting fermion fields $\Psi_{\pm}$ create the soliton/anti-soliton excitations and can be written as a non-local expression of the SG fields $\Phi(x)$ and $\Pi(x)$ [26]:

$$\Psi_{\pm}(x) = N : e^{\pm \sqrt{\beta \pi} \Phi_{\pm}(x)} :$$

where $N$ is a normalisation constant, and the semicolon denotes normal ordering. For $\beta^2 = 4\pi$ the fermion interaction $g$ vanishes and [5] simplifies to

$$\psi_{\pm}(x) = N : e^{\pm \sqrt{\beta \pi} \Phi_{\pm}(x)} :$$

with the inverse relation given by

$$\partial_x \Phi_{\pm}(x) = \pm \frac{1}{\sqrt{4\pi}} : \psi^\dagger_{\pm}(x) \psi_{\pm}(x) :$$

Our analytical calculation consisted of

1. Rewriting the observables in terms of the fermionic fields. In particular, $\partial_x \Phi$ is obtained from [7] and [8].
2. Deriving the time evolved correlations $C_{\Phi}(x, y; t)$ from the initial fermionic correlations by:

(a) solving the free fermion time evolution

$\Psi_{\sigma} = \sigma (\partial_x \Psi_{\sigma} + M \Psi_{\sigma})$

in terms of initial conditions for the field as

$\Psi_{\sigma}(x, t) = \sum_{\sigma' = \pm} \int dx' G_{\sigma \sigma'}(x - x', t) \Psi_{\sigma'}(x', 0)$

where $G_{\sigma \sigma'}(x - x', t)$ is the retarded Green’s function;
(b) using (9) to propagate the initial fermionic correlations:

\[ C_{\phi\psi}(x, y; t) = \sum_{\sigma, \rho=\pm} \sigma_0 \rho_0 \int \! dx_1 dx_2 dy_1 dy_2 \]
\[ \times G_{\sigma_0 \sigma_1}^*(x - x_1, t) G_{\sigma_0 \sigma_2}(x - x_2, t) G_{\rho_0 \rho_1}^*(y - y_1, t) \]
\[ \times G_{\rho_0 \rho_2}(y - y_2, t) \left( \sum_{\sigma_1, \rho_1, \sigma_2, \rho_2} C_{\sigma_1 \sigma_2 \rho_1 \rho_2}^F (x_1, x_2; y_1, y_2) \right) \]
\[ - C_{\sigma_1 \sigma_2}^F (x_1, x_2) C_{\rho_1 \rho_2}^F (y_1, y_2) \right) \]

(10)

3. Deriving the initial fermionic correlations

in the KG ground state \(|\Omega\rangle\) as follows:

(a) identifying the index combinations \(\sigma, \sigma', \rho, \rho'\) allowed by fermionic superselection rules;

(b) expressing fermionic correlations \(C_{\sigma\sigma'\rho\rho'}^F (x, x', y, y')\) in terms of bosonic ones \(\langle \Omega | \Phi_\sigma^\dagger (x) \Phi_{\sigma'} (x') \Phi_\rho^\dagger (y) \Phi_{\rho'} (y') | \Omega \rangle \) using (6) and exploiting the Gaussianity of \(|\Omega\rangle\) in terms of the bosonic fields via Wick's theorem;

(c) obtaining correlators \(\langle \Omega | \Phi_\sigma^\dagger (x) \Phi_{\sigma'} (x') | \Omega \rangle \) using (7).

Finally, after putting all the building blocks together, we integrate numerically (10) to get \(C_{\phi\psi}(x, y; t)\) for finite \(r = |x - y|\) and compute analytically the asymptotics at large \(r\) for any time \(t\) to verify the horizon violation effect explicitly.

The interested reader can find further details in the SI.
SUPPLEMENTARY INFORMATION

Proof of horizon effect for free dynamics and local initial states

The horizon or lightcone effect after a quantum quench was pointed out in [1, 2, 31] and explained through a quasiparticle interpretation. In this description the quench initial state acts as a source of entangled pairs of quasiparticles that originate from points at a distance smaller or of the order of the initial correlation length and travel ballistically to opposite directions, spreading correlations through the system. Therefore correlations between two points start developing only when the fastest pair from points at a distance smaller or of the order of the initial correlation length and travel ballistically to opposite directions, interpretation. In this description the quench initial state acts as a source of entangled pairs of quasiparticles that originate with information propagation [4], and it has also been observed in experiments [5–7].

The horizon or lightcone effect after a quantum quench was pointed out in [1, 2, 31] and explained through a quasiparticle interpretation. In this description the quench initial state acts as a source of entangled pairs of quasiparticles that originate from points at a distance smaller or of the order of the initial correlation length and travel ballistically to opposite directions, spreading correlations through the system. Therefore correlations between two points start developing only when the fastest pair from points at a distance smaller or of the order of the initial correlation length and travel ballistically to opposite directions, interpretation. In this description the quench initial state acts as a source of entangled pairs of quasiparticles that originate with information propagation [4], and it has also been observed in experiments [5–7].

Here we show that the horizon effect is always present in the special case of free relativistic dynamics for any initial state $|\Omega\rangle$ that exhibits exponential clustering of correlations. More precisely the condition is that the dynamics is free in terms of some choice of local fields and the initial state satisfies exponential clustering in terms of the same fields. The horizon effect can be stated mathematically as follows

$$|C_{\phi}(x, y; t)| = |\langle \Omega | \mathcal{O}(x, t) \mathcal{O}(y, t) | \Omega \rangle - \langle \Omega | \mathcal{O}(x, t) | \Omega \rangle \langle \Omega | \mathcal{O}(y, t) | \Omega \rangle| < A e^{-(|x-y|-2t)/\xi_h}$$

(12)

where the length $\xi_h$ can be called “horizon thickness”, $A$ is independent of $x$ and $y$ and as usual we have set the speed of light equal to unit $c = 1$. To demonstrate this relation we focus on the example of Klein-Gordon dynamics, on translationally invariant initial states and choose as local observable the field $\Phi$ itself, even though the reasoning holds more generally. Because the Hamiltonian that describes the dynamics is free, the Heisenberg equations of motion are linear which means that they can be solved for general initial conditions using Green’s functions

$$\Phi(x, t) = \int dx' (\partial_t G(x - x', t) \Phi(x') + G(x - x', t) \Pi(x'))$$

$$\equiv \sum_{i=0,1} \int dx' G_i(x - x', t) \phi_i(x')$$

(13)

where we denote $\phi_0 = \Phi, \phi_1 = \Pi$, $G_0 = \partial_t G$, $G_1 = G$ and the Green’s function is

$$G(x - x', t) = \int \frac{dk}{2\pi} e^{ik(x-x')} \frac{\sin E_k t}{E_k}$$

(14)

with $E_k = \sqrt{k^2 + m^2}$. Therefore one obtains

$$C_{\phi}(x, y; t) = \sum_{i,j=0,1} \int \int dx'dy' G_i(x - x', t) G_j(y - y', t) C_{\phi_i,\phi_j}(x', y'; 0)$$

(15)

which can be depicted schematically as shown in Fig. 4.

Since the dynamics corresponds to a local relativistically invariant theory, the commutators $[\Phi(x, t), \phi_i(x', 0)]$ vanish outside of the past lightcone i.e. for $|x - x'| > t \geq 0$, which means equivalently that the retarded Green’s function $G(x - x', t)$ has support only in the interval $x' \in [x - t, x + t]$. This can be easily verified from (14) by application of Cauchy’s theorem and noticing that the integrand is an analytic function of $k$ in the complex $k$–plane and decays exponentially in the upper or lower half $k$–plane for $x - x' > t$ or $x - x' < -t$ respectively. Because the initial state satisfies exponential clustering for local field

![Figure 4. Schematic explanation of the horizon effect](image-url)
correlations we have
\[ |C_{\phi_i, \phi_j}(x', y'; 0)| = |\langle \Omega | \phi_i(x') \phi_j(y') | \Omega \rangle - \langle \Omega | \phi_i(x') | \Omega \rangle \langle \Omega | \phi_j(y') | \Omega \rangle| < c_{ij} e^{-|x'-y'|/\xi_0} \]
where \( c_{ij} \) is some constant and \( \xi_0 \) is the correlation length characterising the initial state. Substituting in (15) and taking into account the support of the functions \( G_i \) we deduce that
\[ |C_{\phi}(x, y; t)| < A e^{-|x-y|/2t}/\xi_0 \]
where \( A = \sum_{i,j=0,1} c_{ij} \int_0^L dx' |G_i(x', t)| \int_0^L dy' |G_j(y', t)| \) is independent of \( x \) and \( y \).

This proves (12) and shows that for the free case the horizon thickness \( \xi_h \) is equal to the initial correlation length \( \xi_0 \).

**Truncated Conformal Space Approach for the sine-Gordon correlation functions**

In this section we briefly outline our implementation of the Truncated Conformal Space Approach (TCSA) to compute the time dependence of Klein-Gordon correlation functions, and of sine-Gordon correlation functions for general values of coupling in the attractive regime, based on [21].

**Hilbert space**

The choice for the basis Hamiltonian is the massless free boson:
\[ H_{\text{CFT}} = \int_0^L dx \left[ 2\pi (\Pi(x))^2 + \frac{1}{8\pi} (\partial_x \Phi(x))^2 \right] \] (16)
with \([\Phi(x), \Pi(y)] = i \delta(x - y)\). For convenience here we use the CFT normalisation of the fields related to the one in the main text by a field redefinition \( \Phi \rightarrow \sqrt{4\pi} \Phi, \Pi \rightarrow \Pi/\sqrt{4\pi} \).

The TCSA requires a finite volume system, which we implement with Dirichlet boundary conditions:
\[ \Phi(0) = \Phi(L) = 0 . \]

Accordingly, the field can be expanded as
\[ \Phi(x) = 2 \sum_{k \neq 0} \frac{a_k}{\kappa} \sin \left( \frac{\kappa \pi}{L} x \right) \]
\[ \Pi(x) = -\frac{i}{2L} \sum_{k \neq 0} a_k \sin \left( \frac{\kappa \pi}{L} x \right) \]
(17)
with the modes \( a_k \) \((k \in \mathbb{Z})\) satisfying
\[ [a_k, a_l] = k \delta_{k+l} \] (18)

The vacuum state is defined by
\[ a_k |0\rangle = 0 \quad \forall k > 0 , \]
and the computational basis in the Hilbert space is given by the states
\[ |\vec{r}\rangle = |r_1, r_2, \ldots, r_k, \ldots\rangle := \frac{1}{N_{\vec{r}}} \prod_{k>0} a_{r_k} |0\rangle \] (19)
with normalization
\[ N_{\vec{r}}^2 = \prod_{k>0} r_k! r_k^{r_k} . \]
Operator matrix elements

Free boson Hamiltonian

The free massless boson Hamiltonian (16) is diagonal in the basis (19):

\[ \langle \vec{r}' | H_{\text{CFT}} | \vec{r} \rangle = \frac{\pi}{L} \left( \sum_{k=1}^{\infty} kr_k - \frac{1}{24} \right) \delta (\vec{r}', \vec{r}) , \]

where we used \( \zeta \)-function regularisation to replace \( \sum_{n=1}^{\infty} n \to -\frac{1}{12} \) and denoted

\[ \delta (\vec{r}', \vec{r}) = \prod_{k>0} \delta_{r_k', r_k} . \]

Klein-Gordon Hamiltonian

The KG Hamiltonian

\[ H_{\text{KG}} = H_{\text{CFT}} + \frac{1}{8\pi} m^2 \int_0^L dx x^2 \]

has the following matrix elements:

\[ \langle \vec{r}' | H_{\text{KG}} | \vec{r} \rangle = \langle \vec{r}' | H_{\text{CFT}} | \vec{r} \rangle + \frac{\pi}{L} \left( \frac{mL^2}{4\pi^2} \right)^2 \left( 2\delta (\vec{r}', \vec{r}) \sum_{k=1}^{\infty} \frac{r_k}{k} + \right.
\]

\[ \left. + \sum_{k=1}^{\infty} \left( \prod_{n=1}^{\infty} \delta_{r_k', r_n} \right) \frac{1}{k^2} \left( \sqrt{rk}\sqrt{(rk-1)} \delta_{r_k', 2r_k} + \sqrt{(rk+2)}\sqrt{(rk+1)} \delta_{r_k', 2r_k} \right) \right) . \]

Vertex operators

In the sine-Gordon Hamiltonian

\[ H_{SG} = \int \left( \frac{\partial^2}{\partial x^2} + \frac{1}{8\pi} (\partial_x \Phi)^2 - \lambda \cos \left( \frac{\beta}{\sqrt{4\pi}} \Phi \right) \right) dx \]

the interaction is composed of exponential fields

\[ e^{iq\Phi(x)} . \]

After quantisation these fields are divergent and require a suitable regularisation which is described in [36]. This is performed by first mapping the strip to the upper half plane using \( z = e^{\frac{\pi}{2}(r-i\pi)} \) where the operators are normal ordered, resulting in the so-called vertex operators

\[ V_q(x) = \left( \frac{2L}{\pi} \sin \left( \frac{\pi x}{L} \right) \right) ^{-q^2} e^{iq\Phi(z_0, \bar{z}_0)} ; \]

which are scaling fields with weight \( q^2 \), where \( z_0 \equiv e^{\frac{\pi}{2}x} \). The matrix elements of these operators are given by

\[ \langle \vec{r}' | V_q(x) | \vec{r} \rangle = N_F^{-1} N_{\bar{F}}^{-1} \left[ \frac{2L}{\pi} \sin \left( \frac{\pi x}{L} \right) \right]^{-q^2} \prod_{k=1}^{\infty} \langle 0 | a_k^r e^{-\frac{q}{2} (z_k^0 - z_k^0)} e^{\frac{q}{2} (z_k^0 - \bar{z}_k^0)} a_k^l | 0 \rangle , \]

where

\[ \langle 0 | a_k^r e^{-\frac{q}{2} (z_k^0 - z_k^0)} e^{\frac{q}{2} (z_k^0 - \bar{z}_k^0)} a_k^l | 0 \rangle = \sum_{j'=0}^{\infty} \sum_{j=0}^{\infty} \frac{q}{k} ^{j+j} \langle \bar{z}_k^0 - z_k^0 \rangle ^{j+j} \langle 0 | a_k^r a_k^l a_k^l a_k^r | 0 \rangle \]
To obtain the matrix element of the Hamiltonian it is necessary to integrate over $x$ which can be performed using the identity
\[
\int_0^\pi d\theta [2 \sin (\theta)]^{-q^2} e^{-ik\theta} = \frac{\pi e^{-i\frac{\pi}{4} k}}{(1-q^2)B \left( \frac{1}{2}(2-q^2-k), \frac{1}{2}(2-q^2+k) \right)},
\]
where $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler beta function.

**Sine-Gordon Hamiltonian**

Using the vertex operators the sine-Gordon Hamiltonian [24] can be rewritten as
\[
H_{\text{SG}} := H_{\text{CFT}} - \frac{\kappa(\Delta)}{2} M^{2-2\Delta} \int_0^L dx \left( V_{\frac{x}{\sqrt{\pi}}} (x) + V_{\frac{x}{\sqrt{\pi}}} (x) \right).
\]

Note that the redefinition of the exponential fields also renormalises the coupling which acquires an anomalous dimension $2\Delta$ where
\[
\Delta = \frac{\beta^2}{8\pi}.
\]

The renormalised coupling can be expressed in units of where $M^{2-2\Delta}$ is the soliton mass, with the exact coefficient given by [37]
\[
\kappa(\Delta) = \frac{2}{\pi} \frac{\Gamma(\Delta)}{\Gamma(1-\Delta)} \left[ \frac{\sqrt{\pi} \Gamma \left( \frac{1}{2-2\Delta} \right)}{2^\Delta \Gamma \left( \frac{\Delta}{2-2\Delta} \right)} \right]^{2-2\Delta}.
\]

For the numerical simulation all quantities are expressed in units of the appropriate power of $M$. In particular, the dimensionless SG Hamiltonian is
\[
H_{\text{SG}}/M = H_{\text{CFT}}/M - \frac{1}{2} \kappa(\Delta) \left( \frac{\pi}{ML} \right)^{2\Delta-1} \int_0^\pi d\theta [2 \sin (\theta)]^{-2\Delta} \left( \phi^{i\sqrt{2\Delta} \Phi(e^{i\phi}, e^{-i\phi})} + \phi^{-i\sqrt{2\Delta} \Phi(e^{i\phi}, e^{-i\phi})} \right). \tag{26}
\]

**The observables $\Phi(x)$ and $\partial_x \Phi(x)$**

We also need the matrix elements of our observables in the computational basis:
\[
\langle \vec{r}' | \Phi(x) | \vec{r} \rangle = 2 \sum_{k=1}^\infty \prod_{\substack{l,j=1 \atop j \neq k}}^\infty \delta_{r'_l - k r_j} \left( \sqrt{\frac{r_k}{k}} \delta_{r'_k+1, r_k} + \sqrt{\frac{r_k}{k}+1} \delta_{r'_k-1, r_k} \right) \sin \left( \frac{k \pi}{L} x \right) \tag{27}
\]
and
\[
\langle \vec{r}' | \partial_x \Phi(x) | \vec{r} \rangle = 2 \frac{\pi}{L} \sum_{k=1}^\infty \prod_{\substack{l,j=1 \atop j \neq k}}^\infty \delta_{r'_l - k r_j} \left( \sqrt{k r_k} \delta_{r'_k+1, r_k} + \sqrt{k (r_k+1)} \delta_{r'_k-1, r_k} \right) \cos \left( \frac{k \pi}{L} x \right). \tag{28}
\]

**TCSA simulations**

To obtain finite dimensional matrices from the matrix elements given above, we introduce a high-energy truncation by keeping only those states $|\vec{r}\rangle$ in the Hilbert space with energies below a fixed cut-off:
\[
\mathcal{H}_{\text{cut}} = \text{span} \left( \{|\vec{r}\rangle \}_{\vec{r}} \right) \quad \text{with} \quad \begin{cases} H_{\text{CFT}} |\vec{r}\rangle \leq E_{\text{cut}} \end{cases}
\]
which can be parametrised as

\[ E_{\text{cut}} = \frac{\pi}{L} \left( k_{\text{cut}} - \frac{1}{24} \right). \]

The dimension of such truncated Hilbert space is equal to

\[ \sum_{k=0}^{k_{\text{cut}}} P(k) \]

where

\[ P(k) \]

is the number of integer partitions of \( k \) and for the range of \( k_{\text{cut}} \) used in this work it is:

| \( k_{\text{cut}} \) | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|------------------|----|----|----|----|----|----|----|----|----|----|----|
| \( \dim(H_{\text{cut}}) \) | 2714 | 3506 | 4508 | 5763 | 7338 | 9296 | 11732 | 14742 | 18460 | 23025 | 28629 |

The initial state for all quenches we studied was the ground state \( |\Omega\rangle \) of the Klein-Gordon Hamiltonian (22) with a fixed mass \( m_0 \), computed by numerical diagonalisation. The time evolution with the Hamiltonian \( H \) of interest was performed in real time

\[ |\Omega(t)\rangle = e^{-iHt}|\Omega\rangle \]

by numerical exponentiation. Time dependent correlation functions were computed in the Schrödinger picture as

\[ C_\mathcal{O}(x, y; t) := \langle \Omega(t)|\mathcal{O}(x)\mathcal{O}(y)|\Omega(t)\rangle_c \]

where the connected part of the correlation function is defined as

\[ \langle AB\rangle_c = \langle AB \rangle - \langle A \rangle \langle B \rangle \]

The quenches can be parameterised as follows:

1. For the Klein-Gordon case it is enough to specify the ratio \( m/m_0 \) of the post-quench mass \( m \) with the pre-quench \( m_0 \);

2. For the sine-Gordon model it is necessary to specify both the ratio \( M/m_0 \) of the post-quench soliton mass \( M \) to the pre-quench Klein-Gordon mass \( m_0 \) and the sine-Gordon coupling \( \beta \) or equivalently the parameter \( \Delta = \beta^2 / 8\pi \).

Truncation introduces several artifacts, which must be controlled properly. The high energy cut-off leads to cut-off effects in the spectrum and correlation functions. Since the perturbing fields for both the Klein-Gordon and the sine-Gordon model are relevant, these truncation errors decrease when the cut-off \( k_{\text{cut}} \) is increased. The rate of convergence depends on the parameters of the quench: it is better for smaller quenches, and also improves if the post-quench interaction term is more relevant. The convergence properties can be studied using renormalisation group methods; in the present case the rate of convergence was fast enough so that we performed a simple numerical verification that the results have sufficiently converged with the cut-off (cf. also (21)).

Another artifact is the appearance of vertical stripes in the \( x \)-dependence of correlations due to the truncation of the Fourier series (27) and (28) (cf. Fig. 1 of the main text). This is only visible for correlations of \( \partial_x \Phi \) due to the derivative enhancing the contribution of short wavelength modes. To suppress this artifact we applied a filter by averaging over a running window with width equal to the wavelength corresponding to the cut-off \( k_{\text{cut}} \), as illustrated in Fig. 5. Outside the central region the procedure suppresses the vertical stripe artefacts, while keeping all the essential features intact. The filter affects more the central part, which is within the range of the initial correlation length, however this is not essential for the horizon effect. In TCSA plots of \( \Phi \) correlations, as well as in any of the figures showing the analytical results such filtering was not necessary and we present the original raw data.

![Figure 5. The effect of the averaging procedure used to eliminate cut-off artefacts in the \( x \) dependence. Left: raw TCSA data; center: filtered TCSA data; right: difference between raw and filtered data.](image_url)

The truncation errors in the spectrum also lead to the unitary time evolution getting out of phase with time since it is governed by frequencies determined by the energy level differences. However, in this work we are interested in short time scales limited by \( t \leq L/2 \), where this effect does not play an important role.
Analytical demonstration of the out-of-horizon effect

In this section we give more details about the analytical solution of the quench from the Klein-Gordon model to the sine-Gordon model that we presented in the main text, following the three conceptual steps outlined in the Methods section:

1. Fermionising the sine-Gordon model,
2. Computing the dynamics of the correlation functions in terms of initial fermionic correlations,
3. Deriving the initial fermionic correlations from the bosonic ones.

Finally we want to discuss how to extract exact asymptotic expressions for the connected correlations functions at large distances and clarify in this way the origin of the observed out-of-horizon effect.

Fermionising the sine-Gordon model

Due to strong coupling, the dynamics of the sine-Gordon theory cannot be accessed perturbatively. The model is integrable but the current state of the art methods of the theory of integrability do not allow for computation of dynamical multi-point correlation functions. Our solution therefore relies on a powerful analytical tool, the theory of bosonisation.

Bosonisation is one of the first examples of QFT dualities. Dualities were first proposed as a method to solve strongly interacting QFTs, like QCD, for which, unlike in QED, perturbation theory is not applicable. The idea is to introduce a nonlinear field transformation, such that the original strongly-interacting model is mapped into a weakly or non-interacting model in terms of the new fields. In modern theoretical physics, dualities play a central role in understanding the physics of quantum fields and unveiling the deeper symmetries of Nature. Bosonisation establishes a mapping between two different (1+1)-dimensional QFTs, one of which is bosonic and the other fermionic. This is achieved through an isomorphism between the Hilbert spaces and the operators of the theories. This isomorphism can be rigorously proven for a finite system of size $L$. After deriving all expressions, however, one can take the thermodynamic limit $L \to \infty$ and obtain exact finite expressions. In order to perform the calculation rigorously, this is the procedure that we shall adopt here.

The sine-Gordon model, which is a bosonic theory described by

$$H_{SG} = \int \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \Phi)^2 - \lambda \cos \beta \Phi \right) dx$$

(30)

can be mapped via bosonisation to a fermionic theory called the massive Thirring model:

$$H_{MT} = \int \left[ \Psi \left( -i \gamma^1 \partial_x + M \right) \Psi + \frac{1}{2} g \left( \overline{\Psi} \gamma^\mu \Psi \right) \left( \overline{\Psi} \gamma_\mu \Psi \right) \right] dx$$

(31)

The relation between the couplings is given by Coleman’s formula:

$$\beta^2 \frac{4 \pi}{g^2} = \frac{1}{1 + g^{4 \pi}} ,$$

At a particular point, the so-called "free fermion point" $\beta = \sqrt{4 \pi}$ the Thirring interaction vanishes and the dual theory of the SG model is thus the theory of a free (two component) fermion field governed by the Dirac Hamiltonian:

$$H_{MF} = \int \left( -i \gamma^1 \partial_x + M \right) \Psi \Psi dx ,$$

(32)

The Dirac dynamics can be calculated analytically, thus we want to use this correspondence in order to solve our quench problem.

First we want to discuss how bosonisation is performed and introduce the bosonisation dictionary.

Equivalence of Hilbert spaces

Let us begin by briefly discussing the equivalence between the bosonic and fermionic Hilbert spaces in 1+1 D which is the reason why such a mapping is possible. We assume that the fermionic theory is described by an unbounded set of momentum modes $c_k, k \in \mathbb{Z}, \sigma = \pm$, satisfying canonical anti-commutation relations. Further, we assume that the vacuum of the theory is given by the Fermi sea. This is the state where all momentum modes with $k \leq 0$ are occupied. Then it can be shown that any excitation on top of the Fermi sea can be decomposed into a part that purely changes the expectation value of the fermionic number operator and a part which is a function of only particle-hole excitations. The particle-hole excitation operators:

$$a_k^\dagger = \frac{i}{\sqrt{|n_k|}} \sum_{n_q = -\infty}^\infty c_{q+|k|, \sigma}^\dagger c_{q, \sigma} \quad \text{with} \quad \sigma = -\text{sign}(k)$$

(33)
have all the algebraic properties of bosonic excitation operators. The fermionic Hilbert space can thus be decomposed as:

$$\mathcal{H}_{\text{Fermi}} = \mathcal{H}_{N_-,N_+} \otimes \mathcal{H}_{\text{Bose}}$$ (34)

where the states in $\mathcal{H}_{N_-,N_+}$ correspond to sectors with different expectation values of the number operator $N_\sigma$ and $\mathcal{H}_{\text{Bose}}$ is spanned by all possible particle-hole, i.e. bosonic, excitations. The bosonic character of particle-hole excitations also enables us to construct operator identities between fermionic and bosonic fields.

**Bosonisation identities**

We use the following conventions:

For the boson field:

$$\Phi(x) = -\frac{1}{\sqrt{4\pi}} \sum_{n_k = -\infty}^{\infty} \frac{1}{\sqrt{|n_k|}} \left( a_k + a_k^\dagger \right) e^{ikx}$$ (35)

with $k = \frac{2\pi}{L} n_k$ and $[a_k, a_l^\dagger] = \delta_k,l$. It can be decomposed into two ($\sigma = \pm$) components:

$$\Phi_\sigma(x) = -\frac{1}{\sqrt{4\pi}} \sum_{n_k = 1}^{\infty} \frac{1}{\sqrt{|n_k|}} \left( a_{-\sigma k} e^{-\sigma ikx} + a_{-\sigma k}^\dagger e^{\sigma ikx} \right)$$ (36)

The $\sigma = \pm$ components can be easily identified as the left/right moving components of the $\Phi$ field when it is time evolved under the free massless boson Hamiltonian $H_{\text{CFT}}$: $e^{+iH_{\text{CFT}}t} \Phi_\sigma(x)e^{-iH_{\text{CFT}}t} = \Phi_\sigma(x + \sigma t)$.

We use a two component fermion field:

$$\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}$$

satisfying canonical anti-commutation relations and anti-periodic boundary conditions (Neveu-Schwarz sector). The mode expansions are:

$$\Psi_\sigma(x) = \sqrt{\frac{2\pi}{L}} e^{\sigma i \frac{2\pi}{L} x} \sum_{n_k = -\infty}^{\infty} c_{k,\sigma} e^{-\sigma ikx}$$ (37)

with $\sigma = \pm$, $k = \frac{2\pi}{L} n_k$ and $\{c_{k,\sigma}, c_{l,\rho}^\dagger\} = \delta_{\sigma,\rho}\delta_{k,l}$. If evolved with the massless Dirac Hamiltonian $H_{0F} = -i \int \overline{\Psi} \gamma^1 \partial_x \Psi \, dx$ the $\sigma = \pm$ modes are the left and right-moving components: $e^{+iH_{0F}t} \Psi_\sigma(x)e^{-iH_{0F}t} = \Psi_\sigma(x + \sigma t)$. Let us also define the fermionic number operator:

$$N_\sigma \equiv \sum_{n_k = -\infty}^{\infty} c_{k,\sigma}^\dagger c_{k,\sigma};$$

It acts on the states in $\mathcal{H}_{N_-,N_+}$ as $N_\sigma \ket{n_-,n_+} = n_\sigma \ket{n_-,n_+}$.

We now introduce the bosonisation identity, an exact operator identity between boson and fermion fields that follows from (34):

$$\Psi_\sigma(x) = \sqrt{\frac{2\pi}{L}} F_\sigma e^{-i\frac{2\pi}{L} \left(N_\sigma - \frac{1}{2}\right)x} e^{-\sigma i\sqrt{4\pi}\Phi_\sigma(x)}.$$ (38)

To ensure that the expression in the r.h.s. of the equation acts in the Hilbert space (34) identically as the fermion field (37), it is composed of a bosonic part ($e^{-\sigma i\sqrt{4\pi}\Phi_\sigma(x)}$) that acts upon $\mathcal{H}_{\text{Bose}}$ and a part ($F_\sigma e^{-i\frac{2\pi}{L} \left(N_\sigma - \frac{1}{2}\right)x}$) that acts upon $\mathcal{H}_{N_-,N_+}$. The exponentials of the bosonic fields are well known as vertex operators. The operators $F_\sigma$ are known as Klein factors and act as hopping operators between different $N_\sigma$ sectors. They are defined as:

$$F_\sigma^{\dagger} \ket{n_-,n_+} \otimes \ket{\psi}_{\text{Bose}} = (-1)^{n_-+\delta_+,n_+} \ket{n_-+\delta_-,n_++\delta_+,\sigma} \otimes \ket{\psi}_{\text{Bose}}$$
and have the following algebraic properties:

\[ [F_{\sigma}, a_k^\dagger] = [F_{\sigma}, a_k] = 0 \]

\[ \{F_{\sigma}^\dagger, F_{\rho}\} = 2\delta_{\sigma,\rho}, \quad \forall \sigma, \rho \quad (\text{with } F_{\rho}F_{\rho}^\dagger = F_{\rho}^\dagger F_{\rho} = 1) \]

\[ \{F_{\sigma}^\dagger, F_{\rho}^\dagger\} = \{F_{\sigma}, F_{\rho}\} = 0, \quad \sigma \neq \rho \]

\[ [N_{\sigma}, F_{\rho}^\dagger] = \delta_{\sigma,\rho} F_{\rho}^\dagger, \quad [N_{\sigma}, F_{\rho}] = -\delta_{\sigma,\rho} F_{\rho} \]

We have denoted the normal ordering with respect to the bosonic vacuum as \( \langle \! \langle \ldots \rangle \! \rangle \). Formally, the bosonisation identity is proven for systems of finite size \( L \). The fermionic canonical anti-commutation relations are provided by the Klein factors for \( \sigma \neq \rho \) and by an interplay between anti-commutation relations of vertex operators and those of Klein factors and number operators for \( \sigma = \rho \).\[ 23 \]

The inverse relation of (39) expressing the bosonic fields in terms of the fermionic ones is given by:

\[ \partial_x \Phi_\sigma(x) = \sigma \frac{1}{\sqrt{4\pi}} : \Psi_\sigma^\dagger(x) \Phi_\sigma(x) : - \sigma \frac{\sqrt{\pi}}{L N_D} \]

(40)

It is easy to see that the r.h.s. satisfies the bosonic canonical commutation relations.

**Sine-Gordon correlations at the free fermion point**

Using the above relations we can fermionise the sine-Gordon model (30) at \( \beta = \sqrt{4\pi} \) to the free massive Dirac Hamiltonian (32). The \( \partial_x \Phi \) connected correlation function is expressed using (40) as:

\[
\langle \Omega | \partial_x \Phi(x, t) \partial_y \Phi(y, t) | \Omega \rangle_c = \langle \Omega | \partial_x \Phi(x, t) \partial_y \Phi(y, t) | \Omega \rangle - \langle \Omega | \partial_x \Phi(x, t) | \Omega \rangle \langle \Omega | \partial_y \Phi(y, t) | \Omega \rangle
\]

\[
= \frac{1}{4\pi} \sum_{\sigma, \rho = \pm} \sigma \rho \left( \langle \Omega | : \Psi_\sigma^\dagger(x, t) : \Psi_\sigma(x, t) : : \Psi_\rho^\dagger(y, t) : \Psi_\rho(y, t) : | \Omega \rangle - \langle \Omega | : \Psi_\sigma^\dagger(x, t) : \Psi_\sigma(x, t) : | \Omega \rangle \langle \Omega | : \Psi_\rho^\dagger(y, t) : \Psi_\rho(y, t) : | \Omega \rangle \right),
\]

(41)

The \( N_D \) expectation values on the state \( | \Omega \rangle \) vanish as we shall see soon. The correlations of \( \Phi \) can be derived by integration of those of \( \partial_x \Phi \):

\[
\langle \Omega | \Phi(x, t) \Phi(y, t) | \Omega \rangle_c = \int_{-\infty}^{x} dx \int_{-\infty}^{y} dy \langle \Omega | \partial_x \Phi(x', t) \partial_y \Phi(y', t) | \Omega \rangle_c.
\]

We also use the identity:

\[
\langle \Omega | A : B | \Omega \rangle = \langle \Omega | A | \Omega \rangle \langle \Omega | B | \Omega \rangle = \langle \Omega | A B | \Omega \rangle - \langle \Omega | A | \Omega \rangle \langle \Omega | B | \Omega \rangle
\]

(42)

to drop the normal ordering of \( : \Psi_\sigma^\dagger : \) pairs in fermionised expressions for correlation functions. This equality can be easily seen using \( : A : = A - \langle 0 | A | 0 \rangle \) where \( | 0 \rangle \) is the vacuum state.

**Computing the dynamics of correlation functions**

**Free fermion dynamics**

The equations of motion of the free massive fermion, following from (32)

\[
\dot{\Psi}_\sigma = \sigma (\partial_x \Psi_\sigma + M \Psi_{-\sigma})
\]

can be solved exactly for arbitrary initial conditions. The solution in infinite volume can be expressed in the form where the initial fields are propagated with the retarded Green’s functions

\[ \Psi_\sigma(x, t) = \sum_{\sigma' = \pm} \int_{-\infty}^{\infty} dx' G_{\sigma\sigma'}(x - x', t) \Psi_{\sigma'}(x') \]

(43)

The retarded Green’s function can be shown to be

\[ G_{\sigma\sigma'}(x, t) = \Theta(t) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \left[ \delta_{\sigma,\sigma'} \cos (E_k t) + (\sigma k \delta_{\sigma,\sigma'} + \sigma M \delta_{\sigma,-\sigma'}) \sin (E_k t) \right] \]

(44)
where \( E_k = \sqrt{k^2 + M^2} \). We have taken the thermodynamic limit \( L \to \infty \) by substituting
\[
\frac{1}{L} \sum_{n_k=-\infty}^{\infty} \to \int_{-\infty}^{\infty} \frac{dk}{2\pi}
\]
Notice that the Green's function \( G(x-x',t) \) is a real function, it is Lorentz invariant, it vanishes out of the light-cone i.e. for \( |x-x'| > t \). Also notice that at \( t = 0 \) it is \( G(x-x',0) = \delta(x-x') \) as it should. For \( t > 0 \) the off-diagonal terms become nonzero which will uncover the hidden long-range correlations. The propagator can be evaluated for \( t > 0 \) into [38, 39]:
\[
G_{\sigma \sigma'}(x,t) = \Theta(t)\Theta(t^2-x^2) \frac{1}{2} \left[ \delta_{\sigma,\sigma'} (\partial_t - \partial_x \partial_y) + \sigma M \delta_{\sigma,\sigma'} \right] J_0 \left( M \sqrt{t^2-x^2} \right),
\]
where \( J_0 \) is the 0th order Bessel function.

**Dynamics of correlation functions**

Substituting (43) into (41) and using (42) we find the formula for the time evolution of correlation functions:
\[
\langle \Omega | \partial_x \Phi(x,t) \partial_y \Phi(y,t) | \Omega \rangle_e = \frac{1}{4\pi} \sum_{\sigma,\rho, \sigma', \rho' = \pm} \sigma \rho \int dx_1 dx_2 dy_1 dy_2 G_{\sigma \sigma_1}^*(x-x_1,t) G_{\sigma \sigma_2} (x-x_2,t) G_{\rho\rho_1}^*(y-y_1,t) G_{\rho \rho_2} (y-y_2,t) \times
\]
\[
\times \left( \langle \Omega | \Psi_{\sigma_1}^\dagger (x_1) \Psi_{\sigma_2} (x_2) \Psi_{\rho_1}^\dagger (y_1) \Psi_{\rho_2} (y_2) | \Omega \rangle - \langle \Omega | \Psi_{\sigma_1}^\dagger (x_1) \Psi_{\rho_1} (y_1) \Psi_{\sigma_2} (x_2) \Psi_{\rho_2} (y_2) | \Omega \rangle \right).
\]

Figure 6. Diagrammatic representation of the convolution formula (45) for the derivation of dynamics of correlations at the free fermion point. The dashed red lines denote the propagators, the blue and green lines denote the Klein factor contractions corresponding to the initial fermionic correlations.

**Initial fermionic correlation functions**

The last step is to calculate the initial fermionic correlation functions
\[
C_{\sigma_1,\rho_1,\rho_2}(x_1,x_2,y_1,y_2) = \langle \Omega | \Psi_{\sigma_1}^\dagger (x_1) \Psi_{\sigma_2} (x_2) \Psi_{\rho_1}^\dagger (y_1) \Psi_{\rho_2} (y_2) | \Omega \rangle
\]
\[
C_{\sigma_1,\sigma_2}(x_1,x_2) = \langle \Omega | \Psi_{\sigma_1}^\dagger (x_1) \Psi_{\sigma_2} (x_2) | \Omega \rangle.
\]

entering (45).

**Initial state**

The state |\( \Omega \rangle \) that we are quenching from is the ground state of the Klein-Gordon model
\[
H_{KG} = \int \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \Phi)^2 + \frac{1}{2} m_0^2 \Phi^2 \right) dx.
\]

Seen in the Hilbert space (34), |\( \Omega \rangle \) thus only has excitations of bosonic particle-hole type:
\[
|\Omega \rangle \in \mathcal{H}_{Fermi} = |0,0\rangle \otimes |\Omega \rangle.
\]

In (41) and all other fermionic expressions we drop the index \( \mathcal{H}_{Fermi} \) for simplicity of notation but always have |\( \Omega \rangle \in \mathcal{H}_{Fermi} \) in mind when we write |\( \Omega \rangle \).
Fermionic superselection rules

We can now use the bosonisation identity (38) to express the initial fermionic correlations (46) in terms of bosonic fields. We start by collecting together all Klein factors entering in $C_{\sigma_1 \sigma_2 \rho_1 \rho_2}(x_1, x_2, y_1, y_2)$ and $C_{\sigma_1 \sigma_2}(x_1, x_2)$ using the algebraic relations (39). Commuting the Klein factors past the $\Phi$ factors from (38) using the identity $e^{A B} = Be^{A + c}$ if $[A, B] = cB$ for $c \in \mathbb{C}$ gives the following formula for exponential operators:

\[ \Theta_{\sigma_1 \sigma_2 \rho_1 \rho_2}(x_1, x_2, y_1, y_2) = e^{i \frac{2}{N_+ \sqrt{N_-}} \int_{x_1}^{x_2} dx \phi(x) \phi(x)} \]

In case of $C_{\sigma_1 \sigma_2 \rho_1 \rho_2}$ and $C_{\sigma_1 \sigma_2}(x_1, x_2)$ the Klein factors become identical equal to 1 once we take the thermodynamic limit $L \to \infty$.

The Klein factors, however, impose the following superselection rules when contracted with the $H_F$ part of $|\Omega\rangle$:

\[ \langle 0, 0 | F^\dagger_{\sigma_1} F_{\sigma_2} | 0, 0 \rangle = \delta_{\sigma_1, \sigma_2} \]

That is, for any fermion $F_{\sigma}$ appearing in the expression, there has to be an antifermion $F^\dagger_{\sigma}$ of the same type $\sigma$ annihilating it. Put in other words, the string of operators have to preserve the total $N_+ = 0$ and $N_- = 0$. We can either contract the first pair, $\Psi_{\sigma_1}, \Psi_{\sigma_2}$, with each other and the second pair, $\Psi^\dagger_{\sigma_1}, \Psi^\dagger_{\sigma_2}$, with each other. Or we can contract the antifermion $\Psi^\dagger_{\sigma_1}$ from the first pair with the fermion $\Psi_{\sigma_2}$ from the second pair and the remaining two operators $\Psi^\dagger_{\sigma_1}, \Psi_{\sigma_2}$ between each other. Therefore the only non-vanishing combinations are $C_{++++}, C_{+++}, C_{++-}$ and those with the opposite signs. For the two-point functions we have:

\[ \langle 0, 0 | F^\dagger_{\sigma_1} F_{\sigma_2} | 0, 0 \rangle = \delta_{\sigma_1, \sigma_2} \]

Thus only combinations with matching types, $C_{++}$ and $C_{--}$, are allowed.

Contracting vertex operators

We therefore end up with the following expressions:

\[ C_{\sigma_1 \sigma_2 \rho_1 \rho_2}(x_1, x_2, y_1, y_2) = \left(\frac{2\pi}{L}\right)^2 \left\{ \delta_{\sigma_1, \sigma_2} \delta_{\rho_1, \rho_2} + \delta_{\sigma_1, \rho_2} \delta_{\sigma_2, \rho_1} (1 - \delta_{\sigma_1, \sigma_2}) \right\} \Theta_{\sigma_1 \sigma_2 \rho_1 \rho_2}(x_1, x_2, y_1, y_2) \times \]

\[ \times \left\langle \Omega \left| e^{+\sigma_1 \sqrt{4\pi i} \phi(x_1) : e^{-\sigma_2 \sqrt{4\pi i} \phi(x_2) : e^{+\rho_1 \sqrt{4\pi i} \phi(y_1) : e^{-\rho_2 \sqrt{4\pi i} \phi(y_2)}} \right| \right. \Omega \right\rangle \]

\[ C_{\sigma_1 \sigma_2}(x_1, x_2) = \frac{2\pi}{L} \delta_{\sigma_1, \sigma_2} \Theta_{\sigma_1 \sigma_2}(x_1, x_2) \left\langle \Omega \left| e^{+\sigma_1 \sqrt{4\pi i} \phi(x_1) : e^{-\sigma_2 \sqrt{4\pi i} \phi(x_2) : e^{+\rho_1 \sqrt{4\pi i} \phi(y_1) : e^{-\rho_2 \sqrt{4\pi i} \phi(y_2)}} \right| \right. \Omega \right\rangle \]

In order to evaluate these correlation functions of vertex operators - exponentials of $\Phi_\sigma(x)$, we exploit the fact that since the initial state $|\Omega\rangle$ is the ground state of Klein Gordon model it is Gaussian in terms of the bosonic field. For such Gaussian states Wick’s theorem gives the following formula for exponential operators:

\[ \left\langle \prod_i e^{ia_i \phi(x_i)} \right\rangle = e^{-\frac{1}{2} \sum_i a_i^2 \langle \phi^2(x_i) \rangle - \sum_{i < j} a_i a_j \langle \phi(x_i) \phi(x_j) \rangle} \]

(48)

Using the Baker-Campbell-Hausdorff formula and after some algebra we obtain an analogous formula for normal-ordered exponentials

\[ \left\langle \prod_i e^{ia_i \phi(x_i)} \right\rangle = e^{-\frac{1}{2} \sum_i a_i^2 \langle \phi^2(x_i) \rangle - \sum_{i < j} a_i a_j \langle \phi(x_i) \phi(x_j) \rangle} \]

(49)

Applying it to our problem we have

\[ \left\langle \Omega \left| e^{+\sigma_1 \sqrt{4\pi i} \phi(x_1) : e^{-\sigma_2 \sqrt{4\pi i} \phi(x_2) : e^{+\rho_1 \sqrt{4\pi i} \phi(y_1) : e^{-\rho_2 \sqrt{4\pi i} \phi(y_2)}} \right| \right. \Omega \right\rangle = \exp \left[ -2\pi \sum_{i=1}^4 \left( \langle \Phi^2_{\sigma_i} \rangle - 4\pi \left( -\sigma_1 \sigma_2 \langle \Phi_{\sigma_1}(x_1) \Phi_{\sigma_2}(x_2) \rangle - \sigma_1 \sigma_4 \langle \Phi_{\sigma_1}(x_1) \Phi_{\sigma_4}(x_4) \rangle - \sigma_2 \sigma_3 \langle \Phi_{\sigma_2}(x_2) \Phi_{\sigma_3}(x_3) \rangle - \sigma_3 \sigma_4 \langle \Phi_{\sigma_3}(x_3) \Phi_{\sigma_4}(x_4) \rangle \right) \right] \]

\[ \left. - \sigma_3 \sigma_4 \langle \Phi_{\sigma_3}(x_3) \Phi_{\sigma_4}(x_4) \rangle + \sigma_1 \sigma_3 \langle \Phi_{\sigma_1}(x_1) \Phi_{\sigma_3}(x_3) \rangle + \sigma_2 \sigma_4 \langle \Phi_{\sigma_2}(x_2) \Phi_{\sigma_4}(x_4) \rangle \right] \]
\[
\left< \Omega \left| e^{\pm \sigma_1 \sqrt{\pi} \Phi_{\sigma_1}(x_1)} e^{\pm \sigma_2 \sqrt{\pi} \Phi_{\sigma_2}(x_2)} \right| \Omega \right> = \exp \left[ -2\pi \sum_{i=1}^{2} \left< \Phi^2_{\sigma_i} \right> + 4\pi \sigma_1 \sigma_2 \left< \Phi_{\sigma_1}(x_1) \Phi_{\sigma_2}(x_2) \right> \right]
\]

where we have denoted \( \left< \Omega \right| \bullet \left| \Omega \right> \) by \( \left< \bullet \right> \) for brevity.

### Initial two-point functions of bosonic fields

The problem now reduces to calculating the initial two-point correlation functions \( \left< \Omega \left| \Phi_{\sigma_1}(x_1) \Phi_{\sigma_2}(x_2) \right| \Omega \right> \), which are correlations between the chiral components of the bosonic field evaluated in the Klein-Gordon ground state. They can be computed using a Bogoliubov transformation from massive to massless boson modes giving

\[
\left< \Omega \left| \Phi_{\sigma_1}(x_1) \Phi_{\sigma_2}(x_2) \right| \Omega \right> = \frac{1}{L} \sum_{n_k=1}^{\infty} \left\{ \delta_{\sigma_1,\sigma_2} \left[ \frac{1}{2k} e^{-\sigma_1 k(x_1-x_2)} + \frac{1}{4k} \left( \frac{E_{0k}}{k} + \frac{k}{E_{0k}} - 2 \right) \cos k(x_1-x_2) \right] 
- \delta_{\sigma_1,-\sigma_2} \frac{1}{4k} \left( \frac{E_{0k}}{k} - \frac{k}{E_{0k}} \right) \cos k(x_1-x_2) \right\}
\]

where \( E_{0k} = \sqrt{k^2 + m_0^2} \) and the sum runs over discrete momenta, \( k = 2\pi n_k/L \), for positive integers \( n_k \). It is easy to verify that for \( m_0 = 0 \) the only term that does not vanish is the first one: this equals \( -\frac{1}{4\pi} \log \left(1 - e^{-\sigma \pi x/L}\right) \) and is the one that results in the standard CFT ground state correlations.

### Putting the building blocks together

Substituting (51) in (50) and then back to (47) we finally find explicit formulae for the non vanishing initial four-point fermionic correlations

\[
C_{\sigma\sigma\sigma\sigma}(x_1, x_2, y_1, y_2) = \Theta_{\sigma\sigma\sigma\sigma}(x_1, x_2, y_1, y_2) \times \bigg( 4\pi \left[ I_1(x_1 - x_2) + I_1(y_1 - y_2) + I_1(x_1 - y_2) + I_1(x_2 - y_1) \right] \bigg)
\]

\[
C_{\sigma(-\sigma)(-\sigma)(\sigma)}(x_1, x_2, y_1, y_2) = \Theta_{\sigma(-\sigma)(-\sigma)(\sigma)}(x_1, x_2, y_1, y_2) \times \bigg( 4\pi \left[ I_2(x_1 - x_2) + I_2(x_1 - y_2) + I_2(y_1 - x_2) + I_2(y_1 - y_2) \right] \bigg)
\]

\[
C_{\sigma(-\sigma)(-\sigma)(\sigma)}(x_1, x_2, y_1, y_2) = \Theta_{\sigma(-\sigma)(-\sigma)(\sigma)}(x_1, x_2, y_1, y_2) \times \bigg( 4\pi \left[ I_2(y_1 - x_2) + I_2(y_1 - y_2) + I_2(x_1 - x_2) + I_2(x_1 - y_2) \right] \bigg)
\]

where \( C^0_{\sigma_1,\sigma_2,\rho_1,\rho_2}(x_1, x_2, y_1, y_2) \) denotes the corresponding CFT part of the correlations:

\[
C^0_{\sigma\sigma\sigma\sigma}(x_1, x_2, y_1, y_2) = \left( \frac{2\pi}{L} \right)^2 \exp \left[ 4\pi \left( I^0_0(x_1 - x_2) + I^0_0(y_1 - y_2) + I^0_0(x_1 - y_2) + I^0_0(x_2 - y_1) \right) \right]
\]

\[
C^0_{\sigma(-\sigma)(-\sigma)(\sigma)}(x_1, x_2, y_1, y_2) = \left( \frac{2\pi}{L} \right)^2 \exp \left[ 4\pi \left( I^0_0(x_1 - x_2) + I^0_0(y_1 - y_2) \right) \right]
\]

\[
C^0_{\sigma(-\sigma)(-\sigma)(\sigma)}(x_1, x_2, y_1, y_2) = \left( \frac{2\pi}{L} \right)^2 \exp \left[ 4\pi \left( I^0_0(x_1 - x_2) + I^0_0(x_1 - y_2) \right) \right]
\]

For the non vanishing two-point correlations we have:

\[
C_{\sigma\sigma}(x_1, x_2) = \Theta_{\sigma\sigma}(x_1, x_2) C^0_{\sigma\sigma}(x_1, x_2) \exp \left[ 4\pi I_1(x_1 - x_2) \right],
\]

with

\[
C^0_{\sigma\sigma}(x_1, x_2) = \left( \frac{2\pi}{L} \right) \exp \left[ 4\pi I^0_0(x_1 - x_2) \right].
\]
The functions $I_0^{-}, I_1$ and $I_2$ are given by

$$I_1(x) := \frac{1}{L} \sum_{n_k=1}^{\infty} \frac{1}{4k} \left( \frac{E_{0k}}{k} + \frac{k}{E_{0k}} - 2 \right) (\cos kx - 1)$$

$$I_2(x) := \frac{1}{L} \sum_{n_k=1}^{\infty} \frac{1}{4k} \left( \frac{E_{0k}}{k} - \frac{k}{E_{0k}} \right) (\cos kx - 1)$$

$$I_0^{-}(x) := \frac{1}{L} \sum_{n_k=1}^{\infty} \frac{1}{2k} e^{-\sigma kx}$$  \tag{55}

It can be verified that the above formulae reproduce the known fermionic correlations for the massless free fermion (CFT) ground state. In particular, the fermionic antisymmetry property is granted by the canonical anticommutation relations of the fermion field as defined by $\{b_i, b_j^\dagger \} = \delta_{ij}$. The $I_0^{-}(x)$, when evaluated, gives terms $\propto \log L$ that cancel the $\frac{1}{L}$ factors in front of the initial correlations.

We are now ready to take the thermodynamic limit $L \to \infty$ of the expressions and thus obtain exact formulae for the infinite size system. In this limit the sums in the functions (55) are replaced by integrals: $\sum_{n_k=1}^{\infty} \to \int_0^\infty dk$, unless there is an infrared singularity. The phases $\Theta_{\sigma_1 \sigma_2 \rho_1 \rho_2}$ and $\Theta_{\sigma_1 \sigma_2}$ become identically equal to one.

The integrals corresponding to $I_1(x)$ and $I_2(x)$ are infrared and ultraviolet convergent and can be evaluated easily numerically. The expression $I_0^{-}(x)$ has an infrared divergence and should be kept in the sum form until the end of the calculation. This part is the only one that does not vanish in the case $m_0 = m$ and it is responsible for the CFT ground state correlations. Specifically it can be shown that upon exponentiation

$$\lim_{L \to \infty} \frac{2\pi}{L} \exp \left( 4\pi I_0^{-}(x) \right) = -\sigma i / x,$$  \tag{56}

which gives the standard algebraically decaying CFT correlations

$$C_{\sigma \sigma}(x_1, x_2) = -\sigma i \frac{1}{x_1 - x_2}.$$  \tag{57}

It is worth noticing that, while for $|x| \to 0$ both $I_1(x)$ and $I_2(x)$ vanish and therefore do not alter the short distance behaviour of any correlator (which should indeed be controlled by the CFT scaling laws), at large distances instead their asymptotic behaviour is

$$I_1(x) \sim -\frac{m_0 |x|}{16} + \frac{1}{4\pi} \log |x| + c_1$$

$$I_2(x) \sim -\frac{m_0 |x|}{16} + c_2$$  \tag{58}

where $c_1, c_2$ are numerical constants. These asymptotics are precisely the right ones to cancel the algebraic decay of CFT correlations and switch it to exponential, as expected for a massive KG ground state $|\Omega\rangle$.

Combining (52), (54) and (55) we can compute the initial fermionic correlations and then substituting in (45) and using (44) we can finally compute the time evolution of the two-point connected correlation function $C_{\sigma \rho \Phi}(x, y; t)$. The result of numerical evaluation is shown in Fig. 2 of the main text. Note that while the numerical computation is efficient for $|x - y| > 2t$, this is not true for $|x - y| < 2t$ due to the presence of singularities at $x_1 = y_2$ and $x_2 = y_1$. In this region it is necessary to split the integrands into their singular and non-singular parts and evaluate them separately, noticing that the singular part can be expressed as a sum of products of double integrals (instead of quadruple), which speeds up its evaluation.

**Explanation of the out-of-horizon effect**

Using the same formulas (52), (54), (55), (45) and (44) we can derive analytically the asymptotics of connected correlations at any time $t$ and large distance $r = |x - y|$. From (45) and the fact that the fermionic propagators have support only inside the lightcones we easily conclude that the large distance asymptotics outside of the horizon lines $r = 2t$ are determined by the asymptotics of the initial four-point fermionic correlations as the distance between the pairs of coordinates $x_1, x_2$ and $y_1, y_2$ becomes large. Based on the asymptotic expansions (58) we now notice that, unlike the $C_{\sigma \sigma \sigma \sigma}$ and $C_{\sigma \sigma \rho \rho}$ correlations, which factorise at large distances to $C_{\sigma \rho} C_{\sigma \rho}$ and $C_{\sigma \rho} C_{\rho \rho}$ respectively, the third type of correlations, the cross-terms $C_{\rho \rho}$ do not tend to $C_{\sigma \rho} C_{\sigma \rho}$ (which vanishes due to the Klein factor superselection rules), but they have a nonzero
limit instead
\[
\lim_{r \to \pm \infty} C_{\sigma(-\sigma)(-\sigma)}(x_1, x_2, y_1 + r, y_2 + r) = A \exp \left[ 4\pi \left( I_2(x_1 - x_2) + I_2(y_1 - y_2) \right) \right] \neq C_{\sigma(-\sigma)}(x_1, x_2)C_{(-\sigma)\sigma}(y_1, y_2) = 0
\]

(59)

where \( A = \exp[8\pi(c_1 - c_2)] \) and \( c_1, c_2 \) are the constants entering in (58). This violation of clustering has as direct consequence the dynamical emergence of infinite range correlations presented in the main text. More specifically, from (45) and (59) the asymptotic value of the connected correlation function \( C_{\partial \Phi}(x, y; t) \) at large distance \( r = |x - y| \) is

\[
\lim_{r \to \infty} C_{\partial \Phi}(0, r; t) = A \left( \sum_{\sigma} \int dx_1 dx_2 G_{\sigma,+1}(x_1, t)G_{\sigma,-1}(x_2, t) e^{4\pi I_2(x_1 - x_2)} \right)^2
\]

(60)

At \( t = 0 \) the Green’s functions (given in (44)) corresponding to the time evolution of the cross-terms vanish, so these terms do not contribute to the initial state; they are hidden. For \( t > 0 \) they get uncovered by the off-diagonal part of the Green’s functions. As we can easily see, the above expression is oscillatory and decays with time as \( \sim \sin^2 (2Mt) / t \). We can also derive the leading correction to this asymptotic value at large distances which turns out to decay exponentially with the distance.