Maximally entangled states are of utmost importance to quantum communication, dense coding, and quantum teleportation. With a trapped ion placed inside a high finesse optical cavity, interacting with field of an external laser and quantized cavity field, a scheme to generate a maximally entangled three qubit GHZ state, is proposed. The dynamics of tripartite entanglement is investigated, using negativity as an entanglement measure and linear entropy as a measure of mixedness of a state.

Consider a trapped two-level ion in an optical cavity interacting with an external laser and the quantized cavity field. The quantum state of tripartite system (ABC) contains information about the internal state of the ion (subsystem A), the vibrational state of ionic center of mass state, and cavity state is investigated using negativity as an entanglement measure and linear entropy as a measure of mixedness of a state.

I. TRIPARTITE SYSTEM AND INTERACTION HAMILTONIAN

Maximally entangled states are of utmost importance to quantum communication, dense coding, and quantum teleportation. It is now possible to prepare a cold trapped ion in a given initial state with the trap placed inside a high finesse optical cavity. We propose an experimental scheme, using ion trap in an optical cavity, to generate a maximally entangled three qubit GHZ state. For different sets of initial states of the system, entanglement dynamics of ionic internal state, ionic center of mass state, and cavity state is investigated using negativity as an entanglement measure and linear entropy as a measure of state purity.

Dynamics of tripartite entanglement

S. Shelly Sharma
Depto. de Física, Universidade Estadual de Londrina, Londrina 86051-990, PR Brazil

N. K. Sharma
Depto. de Matemática, Universidade Estadual de Londrina, Londrina 86051-990, PR Brazil

The vibrational (photonic) number states are denoted by \(|\nu\rangle\), whereas \(|\sigma\rangle\) represents ion in ground state \((i = g)\) and excited state \((i = e)\). The vibrational (photonic) number states are denoted by \(|m\rangle\), where \(|m\rangle, m(n) = 0, 1, 2, ... \infty\). The Hamiltonian due to interaction of trapped two-level ion of internal frequency \(\omega_0\), with resonant external laser field of frequency \(\omega_L = \omega_0\), and with the cavity field tuned to red sideband of ionic vibrational motion that is \(\omega_0 - \omega_c = \nu\), in interaction picture and rotating wave approximation is given by

\[
H_I = \hbar \Omega [\sigma_+ \hat{O}_b^L + \sigma_- \hat{O}_b^L] + \hbar g \left[ \eta c \sigma_+ \hat{b} \hat{a}^L + h.c. \right].
\]

Here \(\hat{a}^\dagger (\hat{a})\) and \(\hat{b}^\dagger (\hat{b})\) are creation(destruction) operators for vibrational phonon and cavity field photon, respectively, and \(\nu\) is trap frequency. The ion phonon and ion-cavity coupling constants are \(\Omega\) and \(g\), whereas \(\sigma_k (k = z, +, -)\) are the Pauli operators qualifying the internal state of the ion. The operator \(\hat{O}_k\) is defined as

\[
\hat{O}_k = \exp \left( \frac{\eta_k^2}{2} \right) \sum_{p=0}^{\infty} \frac{(i n)^2 p^p \hat{a}^p \hat{a}^\dagger p^p}{p! (p + k)!}.
\]

with Lamb-Dicke (LD) parameters relative to the laser field and the cavity field denoted by \(\eta = \eta_L\) and \(\eta = \eta_c\) respectively. In the limiting case \(\eta_L \ll 1\) and \(\eta_c \ll 1\), \(\hat{O}_{k=0,1} \rightarrow 1\), as such the relevant part of interaction picture Hamiltonian reduces to

\[
\hat{H}_I = \hbar \Omega [\sigma_+ + \sigma_-] + \hbar g \eta_c \left[ \sigma_+ \hat{b} \hat{a} + \sigma_- \hat{b}^\dagger \hat{a} \right].
\]
To obtain unitary time evolution of the system we work in the basis, $|g,m,n\rangle$, $|e,m,n\rangle$, $|g,m-1,n-1\rangle$, and $|e,m-1,n-1\rangle$. Eigen values and eigen vectors of $\hat{H}_I$ are obtained analytically. Starting from a given initial state of the system, written in terms of eigen functions of $\hat{H}_I$, the pure state $\Psi(t)$ for the composite system is obtained by solving the time dependent Schrödinger equation,

$$\hat{H}_I\Psi(t) = i\hbar \frac{d}{dt}\Psi(t).$$

(4)

The density operator for the composite system defined as $\hat{\rho}_{ABC}(t) = |\Psi(t)\rangle \langle \Psi(t)|$, can be used to obtain reduced density operators $\hat{\rho}_A(t)$, $\hat{\rho}_B(t)$, and $\hat{\rho}_C(t)$ for subsystems A, B, and C, respectively.

II. INITIAL STATES OF THE SYSTEM AND GHZ STATE GENERATION

Consider that an ion is prepared initially in state

$$\Psi(0) = (\cos(\theta)|g\rangle + \sin(\theta)|e\rangle)|m = 0, n = 0\rangle,$$

(5)

with the center of mass in lowest energy trap level ($m = 0$), cavity in it’s vacuum state ($n = 0$), and $0 \leq \theta \leq \pi$. Defining, $a = \frac{1}{2}g\eta_c$, and $\mu = \sqrt{a^2 + \Omega^2}$, we solve Eq.(4) and verify that for interaction time $t_p$ such that $\mu t_p = p\pi$, $p = 1, 2, ...$,

$$\Psi(t_p) = (-1)^p (\cos(\theta)\cos(at_p)|g,0,0\rangle + \sin(\theta)\cos(at_p)|e,0,0\rangle$$

$$-i\sin(\theta)\sin(at_p)|g,1,1\rangle - i\cos(\theta)\sin(at_p)|e,1,1\rangle).$$

(6)

For the choice $\mu = 4a$, $\theta = q\pi(q = 0, 1)$ at instant $t_1 = [\pi/(4a)]$ the system is found to be in state

$$\Psi(t_1) = \frac{(-1)^{1+q}}{\sqrt{2}}(|g,0,0\rangle - i|e,1,1\rangle),$$

(7)

which is a maximally entangled GHZ state of the trionpartite two mode system.

FIG. 1: Contour plots of negativity $N^B (= N^C)$ and linear entropy $S^B (= S^C)$ as a function of $\theta$ and scaled time $T(= at)$, for subsystems B and C. Initial states are of type (i) that is separable at $T = 0$. Variables $\theta$ and $T$ are in degrees.
To get further insight into tripartite entanglement, the time evolution of entanglement for the following sets of initial states is investigated:

(i) \(\Psi(0) = (\cos(\theta) |g\rangle + \sin(\theta) |e\rangle) |0,0\rangle, 0 \leq \theta \leq \pi\), a separable state.

(ii) \(\Psi(0) = (\cos(\theta) |g,1\rangle + \sin(\theta) |e,0\rangle) |0\rangle, 0 \leq \theta \leq \pi\), cavity state is separable at \(t = 0\).

(iii) \(\Psi(0) = (\cos(\theta) |g,\beta\rangle + \sin(\theta) |e,-\beta\rangle) |0\rangle, 0 \leq \theta \leq \pi\), cavity state is separable at \(t = 0\), with the ion prepared in a Schrödinger cat state. The coherent state \(|\beta\rangle\), is given by

\[
|\beta\rangle = \exp\left(-|\beta|^2/2\right) \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} |m\rangle,
\]

where \(|\beta|\) is the average number of vibrational quanta associated with center of mass motion.

### III. MEASURES OF ENTANGLEMENT AND PURITY OF A STATE

For a given density matrix operator \(\hat{\rho}^{ABC}(t)\) at instant \(t\), acting on composite Hilbert space associated with system \((ABC)\), reduced density operator for subsystem \(A\) is defined as, \(\hat{\rho}^A(t) = Tr_{BC}(\hat{\rho}^{ABC}(t))\). The partial transpose of density operator \(\hat{\rho}^A\) with respect to subsystem \(A\) reads as

\[
\hat{\rho}^{TA} = \sum_{i,j=1}^{d_A} \sum_{m,r=1}^{d_B} \sum_{n,s=1}^{d_C} \langle i, m, n | \hat{\rho}^A | j, r, s \rangle | j, m, n \rangle \langle i, r, s |.
\]

Negativity \([7]\) for subsystem \(A\) defined in terms of trace norm of partial transpose of density matrix as

\[
N^A = \frac{\|\rho^{TA}\| - 1}{2},
\]

is equal to the modulus of sum of negative eigenvalues of operator \(\rho^{TA}\). \(N^A\) is a measure of entanglement of subsystem \(A\) with its complement \((RC)\) in the composite quantum system \((ABC)\). Measures of entanglement, \(N^B\) and \(N^C\), for

![Graph](image-url)

**FIG. 2**: \(S_i\) versus scaled time \(T(= at)\), for initial states of the type (i) separable at \(T = 0\), (ii) cavity state separable at \(T = 0\), and (iii) cavity state separable and ion in a Schrödinger cat state at \(T = 0\), for the choice \(\theta = 90^\circ, \beta = 1\).
FIG. 3: Contour plots of linear entropy $S_A^l$, as a function of $\theta$ and $T(=at)$, for initial states of the types (i) separable at $T = 0$, (ii) cavity state separable at $T = 0$, and (iii) cavity state separable and ion in a Schrödinger cat state ($\beta = 1$) at $T = 0$.

Although matrix $\rho^{ABC}(t)$ represents a pure state, the reduced density matrices, $\rho^A(t), \rho^B(t)$ and $\rho^C(t)$ do not necessarily do so. Linear Entropy, $S_l$ defined as

\[ S_l = \frac{d}{d-1} \left( 1 - tr(\rho^2) \right), \tag{11} \]

is used as a measure of purity of a state. For a pure state $S_l = 0$ while for a maximally mixed state $S_l = 1$. Here $d$ is the number of modes available to a subsystem.

IV. RESULTS AND CONCLUSIONS

For the three sets of states described in section (2), we have obtained analytic expressions for $\Psi^{ABC}(t)$. These expressions have been used for numerical calculations of entanglement measures $N^A, N^B$ and $N^C$, for the system parameter choice of $\mu/a = 4$. We have also calculated the reduced density matrices, $\rho^A(t), \rho^B(t)$ and from these the linear entropies $S_A^l, S_B^l$ and $S_C^l$. Some of the results are displayed in Figs. (1) to (3). Fig. (1) shows contour plots of negativity $N^B (= N^C)$ and linear entropy $S_B^l (= S_C^l)$ as a function of $\theta(0 \leq \theta \leq \pi)$, and scaled time $T(=at)$. Initial states are separable at $T = 0$ (type (i)). A comparison of contour shapes denounces the fact that negativity and linear entropy do not produce the same ordering of states. However at zero and maximum entanglement, the states correspond to pure states and maximally mixed states, respectively. We find the subsystems $A$ and $B$ to be maximally entangled and maximally mixed at $T = 45^\circ$ and $135^\circ$ for all possible choices of $\theta$.

Fig (2) displays for the choice $\theta = 90^\circ$, linear entropy $S_l$ versus scaled time $T(=at)$ in degrees, for subsystems $A$, $B$ and $C$. Parts (i), (ii) and (iii) in the figure refer to initial states of the type (i), (ii), and (iii), as in section (2). For initial state (i), all three subsystems are found to be in maximally mixed state at $T = 45^\circ$ and $135^\circ$, hence are maximally entangled. For initial state (ii), although subsystems $A$ and $C$ become maximally entangled at $T = 45^\circ$ and $135^\circ$, system $B$ fails to reach maximum entanglement. For case (iii), when ion’s vibrational state is a coherent state ($\beta = 1$) at $t = 0$, entanglement is seen to increase with time for all the three subsystems. As expected, the initial state temperature has strong influence on generation of maximally entangled tripartite state.

Contour plots of linear entropy $S_A^l$, as a function of $\theta$ and $T(=at)$, for initial states of the types (i), (ii), and (iii) are displayed in Fig. (3). For initial states (i) subsystem $B$ is a qubit, for initial states (ii) subsystem $B$ is a qutrit, and for initial states (iii) a large number of modes are available to system $B$, while $A$ and $C$ remain as two mode
systems. A comparison of parts (i), (ii), and (iii) of Fig. (3) shows that for a given choice of $\theta$, the time evolution of purity and entanglement of subsystem $A$ depends strongly on the number of modes available to subsystem $B$. Time evolution of $S_l$ for system $A$ in Figure (2) is a particular case of Fig. (3). We conclude that (a) the number of modes available to a subsystem not only determines the maximum entanglement of that subsystem but also has a strong influence on entanglement of other subsystems, b) at entanglement maxima and minima, linear entropy and negativity uniquely determine the nature of state, but the two measures do not induce the same ordering of states, c) for a special choice of system parameters maximally entangled tripartite two mode GHZ state is generated. The scheme presented for GHZ state generation is a single step process and is reduction free.

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