Bubbles created from vacuum fluctuation*

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Abstract

We show that the bubbles $S^2 \times S^2$ can be created from vacuum fluctuation in certain De Sitter universe, so the space-time foam-like structure might really be constructed from bubbles of $S^2 \times S^2$ in the very early inflating phase of our universe. But whether such foam-like structure persisted during the later evolution of the universe is a problem unsolved now.

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J.A.Wheeler was the first one who pointed out that the space-time might have a foam-like structure around the Plank scale [1], though it is simply connected and smooth at large scale. But what is the constituents of the space-time foam is a problem long unsolved. Hawking himself oscillated between the wormhole pictured and bubble pictured foam-like structure [2]. However according to our point of view, the essential point is whether the bubble or wormhole could be really solutions of the semi-classical Einstein gravitational equation

\[ G_{\mu}^{\nu} = -\frac{8\pi G}{c^4} \langle 0 \mid T_{\mu}^{\nu} \mid 0 \rangle_{\text{ren}}, \]  

(1)

where \( G_{\mu}^{\nu} = R_{\mu}^{\nu} - \frac{1}{2} R \delta_{\mu}^{\nu} \) is the Einstein tensor of the bubbles or wormhole, \( \langle 0 \mid T_{\mu}^{\nu} \mid 0 \rangle_{\text{ren}} \) is however the renormalized matter stress-energy tensor of vacuum fluctuation in certain given classical background space-time, \( G \) is gravitational constant, \( c \) is velocity of light. The signature in our paper is -2.

Early in 1993, one of the authors (L.Liu) had found a transient Lorentzian wormhole solution [3]

\[ ds^2 = d(ct)^2 - l_p^2 \cosh^2(\frac{ct}{l_p})d\Omega_3^2, \]  

(2)

where \( l_p \) is the Planck length. From (1), this Lorentzian mini-wormhole creates at certain early time and annihilates later under vacuum fluctuation in a closed inflating de Sitter background space-time of metric

\[ ds^2 = d(ct)^2 - \alpha^2 e^{ct/\alpha} d\Omega_3^2, \]  

(3)

where \( \alpha = \sqrt{3/\Lambda} \), \( \Lambda \) is the cosmological constant of the background de Sitter universe. Now the problem is whether the Euclidean bubble \( S^2 \times S^2 \) can also be a solution of the semi-classical Euclidean Einstein field equation (1)? As is known, the Euclidean metric of \( S^2 \times S^2 \) is the Nariai instanton of metric [4]

\[ ds^2 = -\lambda^{-1}(\sin^2 \chi d \tilde{\psi}^2 + d\chi^2 + d\Omega_2^2), \]  

(4)

where \( \tilde{\psi} \) is the imaginary time and \( \sqrt{1/\lambda} \) is the radius of the 2-sphere \( S^2 \). \( \lambda \) in our paper is nothing to do with the so-called cosmological constant, though it is so in the degenerated De Sitter-Schwarzschild spacetime historically. The Lorentzian Nariai metric is [4]
\[ ds^2 = \lambda^{-1} \sin^2 \chi d\psi^2 - \lambda^{-1} d\chi^2 - \lambda^{-1} d\theta^2 - \lambda^{-1} \sin^2 \theta d\varphi^2, \tag{5} \]

where \( \psi = -i \bar{\psi} \) is a real time variable.

From the Lorentzian Nariai metric (5) and (1), the renormalized vacuum matter stress-energy tensor can be obtained straightforward as (Appendix)

\[ \langle 0 \mid T^\nu_\mu \mid 0 \rangle_{\text{ren}} = -\frac{c^4}{8\pi G} G^\nu_\mu = \frac{c^4}{8\pi G} \lambda \delta^\nu_\mu. \tag{6} \]

However expression (6) is just the well-known vacuum matter stress-energy tensor in one-loop approximation for scalar field

\[ \langle 0 \mid T^\nu_\mu \mid 0 \rangle_{\text{ren}} = \frac{\hbar c}{960\pi^2 \alpha^4} \delta^\nu_\mu = \frac{c^4 \Lambda^2 l_p^2}{8640\pi^2 G} \delta^\nu_\mu, \quad (\alpha \equiv \sqrt{\frac{3}{\Lambda}}, \Lambda \text{ is the cosmological constant}). \tag{7} \]

of the inflating flat de Sitter universe

\[ ds^2 = d(ct)^2 - e^{2\alpha} dx^i dx_i \tag{8} \]

or the inflating closed de Sitter universe

\[ ds^2 = d(ct)^2 - \alpha^2 \cosh^2 (ct/\alpha) d\Omega_3^2 \tag{9} \]

If we put

\[ \lambda = \frac{l_p^2 \Lambda^2}{1080\pi}, \tag{10} \]

if the bubbles \( S^2 \times S^2 \) are the result of the vacuum fluctuation of certain background de Sitter space-time, then a reasonable demand of (10) is \( \sqrt{\lambda^{-1}} \ll \sqrt{3\Lambda^{-1}}, \text{ or } \lambda \gg \Lambda/3 \), that is \( \Lambda \gg \frac{1080\pi l_p^{-2}}{3} \sim 10^{69} \), here we would like to point out again though \( \Lambda \) is the cosmological constant of the background De Sitter spacetime, \( \lambda \) is not, \( \lambda \) relates only with the radius of the two sphere \( S^2 \).

In concluding, we show unambiguously that the bubbles of topology \( S^2 \times S^2 \) can be really created from vacuum fluctuation in the background de Sitter universe of \( k = 0, 1 \).
However important comments should be given now, i.e., First, in our understanding, it seems the instanton $S^2 \times S^2$ is just certain kind of compact topological object with $\chi = 4$ and index $\tau = 0$, which have no connection with the creation of black hole pairs. In fact, we agree with Bousso and Hawking [4]: “Strictly speaking, it does not even contain a black hole, but rather two acceleration horizons.” Second, It seems either the wormhole pictured or the bubble pictured space-time foam-like structure can only be created from the vacuum fluctuation in the inflationary era of our universe. This remark had already been pointed out by Bousso and Hawking [4] and Hawking [2], but whether such spacetime foam-like structure stably exist during the later evolution of our universe is an interesting problem unsolved. So it seems that the stability of the above mentioned foam-like structure should be studied in order to confirm that Wheeler–Hawking’s conjecture is true or not.

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APPENDIX A: APPENDIX

The Lorentzian Nariai metric of $S^2 \times S^2$ is

$$ds^2 = \lambda^{-1} \sin^2 \chi d\psi^2 - \lambda^{-1} d\chi^2 - \lambda^{-1} d\theta^2 - \lambda^{-1} \sin^2 \theta d\varphi^2$$

namely $g_{00} = \lambda^{-1} \sin^2 \chi$, $g_{11} = -\lambda^{-1}$, $g_{22} = -\lambda^{-1}$, $g_{33} = -\lambda^{-1} \sin^2 \theta$, $g_{\alpha\beta} = 0$ ($\alpha, \beta = 0, 1, 2, 3; \alpha \neq \beta$). The determinant of the metric is $g = -\lambda^{-4} \sin^2 \chi \sin^2 \theta$.

From $g^{\mu\lambda} = \Delta^{\mu\lambda}/g$, we can compute contravariant components of the metric as follows:

$g^{00} = \lambda \sin^{-2} \chi$, $g^{11} = -\lambda$, $g^{22} = -\lambda$, $g^{33} = -\lambda \sin^{-2} \theta$, $g^{\alpha\beta} = 0$ ($\alpha, \beta = 0, 1, 2, 3; \alpha \neq \beta$).

From $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} (g_{\mu\lambda, \nu} + g_{\nu\lambda, \mu} - g_{\mu\nu, \lambda})$, we get the non-zero components of Christoffel which are $\Gamma^0_{01} = \Gamma^0_{10} = ct g \chi$, $\Gamma^1_{00} = \sin \chi \cos \chi$, $\Gamma^2_{33} = -\sin \theta \cos \theta$, $\Gamma^3_{23} = \Gamma^3_{32} = ct g \theta$. 

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Put the values of the above Christoffel into the following formula of Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \Gamma^\lambda_{\mu\lambda,\nu} - \Gamma^\lambda_{\mu\nu,\lambda} + \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\mu\nu}$, we can get the non-zero components of Ricci tensor as follows $R_{00} = \sin^2 \chi$, $R_{11} = -1$, $R_{22} = -1$, $R_{33} = -\sin^2 \theta$. We can also compute Ricci scalar as follows $R = g^{\mu\nu}R_{\mu\nu} = 4\lambda$. Through the formulas $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, we get the non-zero components of Einstein tensor which are $G_{00} = -\sin^2 \chi$, $G_{11} = 1$, $G_{22} = 1$, $G_{33} = \sin^2 \theta$. From the formulas $G'_{\mu} = g^{\nu\rho}G_{\rho\mu}$, we get $G'_{\mu} = -\lambda\delta^\nu_{\mu}$. 
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