Taming the $b$ antighost with Ramond-Ramond flux

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In the pure spinor formalism for the superstring, the $b$ antighost is necessary for multiloop amplitude computations and is a composite operator constructed to satisfy $\{Q, b\} = T$ where $Q$ is the BRST operator and $T$ is the holomorphic stress-tensor. In superstring backgrounds with only NS-NS fields turned on, or in flat space, one needs to introduce “non-minimal” variables in order to construct the $b$ antighost. However, in Type II backgrounds where the Ramond-Ramond bispinor field-strength satisfies certain conditions, the $b$ antighost can be constructed without the non-minimal variables. Although the $b$ antighost in these backgrounds is not holomorphic, its antiholomorphic derivative is BRST-trivial. We discuss the properties of this operator both in the $AdS_5 \times S^5$ background and in a generic curved background.
1. Introduction

Over the last ten years, the pure spinor formalism for the superstring has been used successfully to compute superstring scattering amplitudes (see [1] for a recent review). A major advantage over computations using the Ramond-Neveu-Schwarz formalism is that spacetime supersymmetry is manifest in the pure spinor formalism so one does not need to sum over spin structures to see cancellations in the multiloop amplitudes.

Since $(b, c)$ and $(\hat{b}, \hat{c})$ reparameterization ghosts are not fundamental worldsheet variables in the pure spinor formalism, $g$–loop scattering amplitudes $\mathcal{A}_g$ are defined as in topological string theory where the left-moving $b$ antighost and right-moving $\hat{b}$ antighost are composite fields constructed to satisfy

$$\{Q, b\} = T, \quad \{\hat{Q}, \hat{b}\} = \hat{T},$$

where $Q$ and $\hat{Q}$ are the left and right-moving BRST operators and $T$ and $\hat{T}$ are the left and right-moving stress tensors. As in topological string theory, the integration measure is then defined by contracting $(3g-3)$ composite $b$ antighosts with the Beltrami differentials $\mu$ corresponding to the $(3g-3)$ Teichmüller moduli $\tau$ of the genus $g$ Riemann surface

$$\mathcal{A}_g = \int d^{3g-3}\tau \int d^{3g-3}\hat{\tau} ((\int \mu b)^{3g-3} (\int \hat{\mu} \hat{b})^{3g-3} \prod_{i=1}^{N} \int d^2 z_i V_i(z_i)) . \quad (1.1)$$

In a flat background, the construction of the $b$ antighost satisfying $\{Q, b\} = T$ is complicated and requires the introduction of non-minimal worldsheet variables. In the “minimal” pure spinor formalism, one has the usual $(x^m, \theta^\alpha, \hat{\theta}^{\hat{\alpha}})$ Type II superspace variables as well as the left and right-moving bosonic pure spinor ghosts $(\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}})$ satisfying

$$\lambda^\gamma m \lambda = \hat{\lambda}^{\gamma m} \hat{\lambda} = 0,$$

where $\alpha = 1$ to 16 and $\hat{\alpha} = 1$ to 16 are ten-dimensional spinor indices which have the same chirality for Type IIB and opposite chirality for Type IIA. To construct the $b$ antighost in a flat background, one then needs to include non-minimal variables consisting of a new set of left and right-moving bosonic pure spinors, $(\bar{\lambda}_\alpha, \hat{\bar{\lambda}}_{\hat{\alpha}})$, as well as a set of left and right-moving constrained fermions, $(r_\alpha, \hat{r}_{\hat{\alpha}})$. These non-minimal variables satisfy the constraints

$$\bar{\lambda}^\gamma m \bar{\lambda} = \hat{\bar{\lambda}}^{\gamma m} \hat{\bar{\lambda}} = 0$$

and

$$\bar{\lambda}^\gamma r = \hat{\bar{\lambda}}^{\gamma m} \hat{r} = 0,$$

so that there are an equal number of non-minimal bosonic and fermionic degrees of freedom. After modifying the BRST operator to include the standard non-minimal term, these new variables decouple from the cohomology and the physical spectrum.
In addition to allowing construction of a $b$ antighost satisfying $\{Q, b\} = T$, these non-minimal variables also allow functional integration over the pure spinor ghosts, where $\overline{\lambda}_\alpha$ is interpreted as the complex conjugate of $\lambda^\alpha$ and $\hat{\lambda}_{\hat{\alpha}}$ is interpreted as the complex conjugate of $\hat{\lambda}^{\hat{\alpha}}$. Although scattering amplitudes have been computed using this prescription only in a flat background, it is natural to ask if this construction of the $b$ antighost generalizes to curved supergravity backgrounds. One interesting background to consider is the $AdS_5 \times S^5$ background.

In a recent paper [2] by one of the authors, it was argued that unlike in a flat background, non-minimal variables are not needed to construct the $b$ antighost in an $AdS_5 \times S^5$ background. Instead of introducing new non-minimal variables ($\overline{\lambda}_\alpha$, $\hat{\lambda}_{\hat{\alpha}}$) to play the role of the complex conjugates of ($\lambda^\alpha$, $\hat{\lambda}^{\hat{\alpha}}$), one can simply define

$$
\overline{\lambda}_\alpha \equiv \gamma^{01234}_\alpha \lambda^{\hat{\alpha}}, \quad \hat{\lambda}_{\hat{\alpha}} \equiv \gamma^{01234}_{\alpha\hat{\alpha}} \lambda^\alpha, \quad (1.2)
$$

where $\gamma^{01234}_{\alpha\hat{\alpha}}$ is the five-form gamma matrix in the direction of the five-form Ramond-Ramond flux. So after multiplying by $\gamma^{01234}$, the original left and right-moving pure spinor ghosts can be interpreted as complex conjugates of each other. In a flat background, this interpretation is not possible since $\lambda^\alpha \overline{\lambda}_\alpha = \gamma^{01234}_\alpha \lambda^\alpha \hat{\lambda}^{\hat{\alpha}}$ is BRST-trivial, so it cannot be interpreted as a positive-definite quantity. But in an $AdS_5 \times S^5$ background, $\gamma^{01234}_\alpha \lambda^\alpha \hat{\lambda}^{\hat{\alpha}}$ is in the BRST cohomology: it is the vertex operator for the radius modulus. So it is consistent to interpret $\gamma^{01234}_\alpha \lambda^\alpha \hat{\lambda}^{\hat{\alpha}}$ as a positive-definite quantity since it cannot be gauged away. After interpreting the complex conjugate of the pure spinor variables as in (1.2), the construction of the $b$ antighost in an $AdS_5 \times S^5$ background is straightforward.

In the first part of this paper, this construction of the $b$ antighost in an $AdS_5 \times S^5$ background will be shown to satisfy the necessary properties for consistency of the amplitude prescription of (1.1). In addition to satisfying $\{Q, b\} = T$, it will be shown that the $b$ antighost also satisfies $\{\hat{Q}, b\} = 0$. However, unlike the left-moving $b$ antighost in a flat background, the $b$ antighost in an $AdS_5 \times S^5$ background is not holomorphic, i.e. it does not satisfy $\overline{\partial} b = 0$. Instead it satisfies

$$
\overline{\partial} b = [\hat{Q}, \mathcal{O}] \quad (1.3)
$$

where $\mathcal{O}$ is defined by taking the antiholomorphic contour integral of $\hat{b}$ around $b$. One similarly finds that the $\hat{b}$ antighost is not antiholomorphic and instead satisfies

$$
\partial \hat{b} = [Q, \hat{\mathcal{O}}] \quad (1.4)
$$
where \( \hat{O} \) is defined by taking the holomorphic contour integral of \( b \) around \( \hat{b} \).

To prove (1.3), one uses the properties

\[
\{Q, b\} = T, \quad \{\hat{Q}, b\} = 0, \quad \{\hat{Q}, \hat{b}\} = \hat{T}, \quad \{Q, \hat{b}\} = 0
\]

(1.5)
to show that

\[
\bar{\partial} b = [\hat{T}_{-1}, b] = [\{\hat{Q}, \hat{b}_{-1}\}, b] = [\hat{Q}, \hat{O}]
\]

(1.6)
where \([\hat{T}_{-1}, X] \) and \( \{\hat{b}_{-1}, X\} \) denote the antiholomorphic contour integral of \( \hat{T} \) and \( \hat{b} \) around \( X \), and \( \hat{O} \equiv \{\hat{b}_{-1}, \hat{b}\} \). One can similarly use (1.5) to prove (1.4) where \( \hat{O} \equiv \{b_{-1}, \hat{b}\} \) and \( \{b_{-1}, X\} \) denotes the holomorphic contour integral of \( b \) around \( X \).

Although this non-holomorphic structure of the \( b \) and \( \hat{b} \) antighosts is unusual, (1.3) and (1.4) should be enough for consistency of the amplitude prescription of (1.1). In order that \( \int \mu b \) in (1.1) is invariant under the shift \( \mu \rightarrow \mu + \bar{\partial} \nu \) for any \( \nu \), one usually requires that \( \bar{\partial} b = 0 \). However, if one can ignore surface terms coming from the boundary of Teichmuller moduli space, it is sufficient to require the milder condition

\[
\bar{\partial} b = [\hat{Q}, \hat{O}].
\]

(1.7)
This can be shown by pulling \( \hat{Q} \) off of \( \hat{O} \) and using \([\hat{Q}, V] = \{\hat{Q}, b\} = 0 \) and \( \{\hat{Q}, \hat{b}\} = \hat{T} \) to obtain terms which are total derivatives in the Teichmuller moduli. If one can ignore surface terms from the boundary of moduli space, these total derivatives do not contribute. For backgrounds such as \( AdS_5 \times S^5 \) which preserve spacetime supersymmetry, one does not expect the integrand of the scattering amplitude to diverge near the boundary of moduli space, so it should be OK to ignore these surface terms. However, the role of such terms in the \( AdS_5 \times S^5 \) Ramond-Ramond background deserves further investigation.

In the second part of the paper, we show that a similar construction of the \( b \) antighost is possible whenever the supergravity background includes a Ramond-Ramond field strength which, when expressed in bispinor notation as \( P^{\alpha \dot{\beta}} \), obeys certain conditions. In the type II superstring, the dependence of the superfield \( P \) on the Ramond-Ramond \( p \)-form field strengths \( F_p \) is

\[
\begin{align*}
\text{IIB:} & \quad \frac{1}{g_s} P = \gamma^{a_1} F_{a_1} + \frac{1}{3!} \gamma^{a_1 a_2 a_3} F_{a_1 a_2 a_3} + \frac{1}{2 \cdot 5!} \gamma^{a_1 \ldots a_5} F_{a_1 \ldots a_5}, \\
\text{IIA:} & \quad \frac{1}{g_s} P = F_0 + \frac{1}{2!} \gamma^{a_1 a_2} F_{a_1 a_2} + \frac{1}{4!} \gamma^{a_1 \ldots a_4} F_{a_1 \ldots a_4}.
\end{align*}
\]

(1.8)
For example, in the type IIB $AdS_5 \times S^5$ background, $P^{\alpha \dot{\alpha}} = \gamma^{01234}_{\alpha \dot{\alpha}}$ whose inverse is $(P^{-1})_{\alpha \dot{\alpha}} = \gamma_{01234}^{\alpha \dot{\alpha}}$. We will show that when the R-R superfield is covariantly constant and invertible, the state $(P^{-1})_{\alpha \dot{\alpha}} \lambda^\alpha \dot{\lambda}^\dot{\alpha}$ is in the BRST cohomology and one can redefine

$$
\lambda_\alpha \equiv (P^{-1})_{\alpha \dot{\alpha}} \dot{\lambda}^\dot{\alpha}, \quad \lambda_{\dot{\alpha}} \equiv (P^{-1})_{\alpha \dot{\alpha}} \lambda^\alpha,
$$

and construct a $b$ antighost such that $\{Q, b\} = T$. In backgrounds where the Ramond-Ramond field strength satisfies the additional requirement

$$
P_{\gamma}^k P = f^k_m \gamma^m,
$$

where $f^k_m$ is a superfield, it will be shown that (1.5) is still satisfied, which implies (1.3) and (1.4). We will solve the condition (1.10) explicitly in terms of the type II R-R fluxes. Surprisingly, the construction of the $b$ antighost in a curved NS-NS background is more complicated than in a R-R background since it requires non-minimal variables.

The fact that non-minimal variables in the pure spinor formalism are not necessary in backgrounds where the Ramond-Ramond superfield is covariantly constant and invertible should have consequences for scattering amplitudes in these backgrounds. Firstly, it would be interesting to know if there are any examples of such backgrounds besides $AdS_5 \times S^5$. Since non-minimal variables play an important role in the proof of non-renormalization theorems in a flat background, it would be interesting to study non-renormalization theorems in these Ramond-Ramond backgrounds.

Another interesting feature of this paper is the construction of $O$ in (1.3) in terms of the single pole between the $b$ and $\dot{b}$ antighost. We are not aware of any previous discussion of such a construction, and there should be a natural geometrical interpretation of $O$ in backgrounds where $b$ is not holomorphic but (1.5) is satisfied.

2. $AdS_5 \times S^5$ Background

Superstring propagation in the $AdS_5 \times S^5$ background is described by a non-linear sigma model defined on the supercoset $PSU(2, 2|4)/SO(1, 4) \times SO(5)$. To set the notation we briefly collect some facts about the pure spinor sigma model.

A coset representative $g(\sigma)$ transforms as $g'(\sigma) = g_0 g(\sigma) h(\sigma)$, where $g_0$ is an element of the global $PSU(2, 2|4)$ and $h(\sigma)$ is an element of the local $SO(1, 4) \times SO(5)$ Lorentz
group. The left-invariant currents \( J = g^{-1}dg \) can be decomposed according to the \( Z_4 \) automorphism of the super Lie algebra \( PSU(2, 2|4) \) as

\[
J_0 = (g^{-1} dg)^{[ab]} T_{[ab]}, \quad J_1 = (g^{-1} dg)^{a} T_{a}, \quad J_2 = (g^{-1} dg)^{a} T_{a}, \quad J_3 = (g^{-1} dg)^{\hat{a}} T_{\hat{a}},
\]

where \( T_{A} \) are the super Lie algebra generators. They satisfy the Maurer-Cartan equations

\[
\partial J + \bar{\partial} J + [J, J] = 0,
\]

which can be conveniently split according to the \( Z_4 \) grading. We will need the left and right-moving ghosts and their conjugate momenta \((\lambda^\alpha, w_\alpha)\) and \((\hat{\lambda}^\hat{\alpha}, \hat{w}_{\hat{\alpha}})\). As anticipated in (1.3), it will be convenient to redefine the hatted worldsheet quantities by introducing a factor of the constant Ramond-Ramond superfield \( P^{\alpha \hat{\alpha}} = (\gamma_{01234})^{\alpha \hat{\alpha}} \) and its inverse \( P_{\alpha \hat{\alpha}} = (\gamma_{01234})_{\alpha \hat{\alpha}} \)

\[
\hat{\lambda}_\alpha \equiv P_{\alpha \hat{\alpha}} \hat{\lambda}^{\hat{\alpha}}, \quad \hat{w}^\alpha \equiv P^{\alpha \hat{\alpha}} \hat{w}_{\hat{\alpha}}, \quad (J_3)_\alpha \equiv P_{\alpha \hat{\alpha}} J^{\hat{\alpha}}.
\]

The worldsheet action reads

\[
S = \frac{R^2}{2\pi} \int d^2z \left( \frac{1}{2} \eta_{ab} J^a J^b + 3 \eta_{ab} J^a J^b - \frac{3}{4} (J_3)_\alpha J_1^\alpha - \frac{1}{4} J_1^\alpha (J_3)_\alpha 
\right.
\]

\[
\left. \quad + w_\alpha (\nabla_\lambda^\alpha) + \hat{w}^\alpha (\nabla_\hat{\lambda}^\alpha) - \frac{1}{2} \eta_{(ab)} N^{(cd)} N_{(ab)} N_{(cd)} \right),
\]

where \( N^{ab} \) and \( \hat{N}^{ab} \) are the \( SO(1, 4) \times SO(5) \) Lorentz generators of the pure spinors and \( \eta_{[ab][cd]} = (\eta_{[\bar{a}\bar{b}]} \eta_{\bar{d}\bar{e}], -\delta_{[\bar{a}\bar{b}} \delta_{\bar{d}\bar{e}]}}, \) where \( \bar{a} = 0, \ldots, 4 \) and \( \bar{a} = 5, \ldots, 9 \) are the \( AdS_5 \) and \( S^5 \) directions respectively. We introduced the covariant derivatives

\[
(\nabla_\lambda^\alpha) = \partial \lambda^\alpha + \frac{1}{2} J^a_0 (\gamma_{ab} \lambda)^\alpha, \quad (\nabla_\hat{\lambda}^\alpha) = \bar{\partial} \hat{\lambda}^\alpha - \frac{1}{2} J_0^{\hat{b}} (\gamma_{ab} \hat{\lambda})^\alpha.
\]

The physical states are vertex operators in the cohomology of the nilpotent BRST charge \( Q + \hat{Q} \)

\[
Q = \oint d\sigma \lambda^\alpha (J_3)_\alpha, \quad \hat{Q} = \oint d\sigma \hat{\lambda}^\alpha \bar{J}^{\bar{\alpha}}.
\]

that generate the following BRST transformations [3]

\[
Q J_1^\alpha = (\nabla_\lambda^\alpha), \quad Q J_2^\alpha = (\lambda^\gamma_a J_1^a), \quad Q (J_3)_\alpha = - (\lambda^\gamma_a) (J_2^a),
\]

\[
\hat{Q} J_1^\alpha = - (\gamma_{a \hat{\lambda}}) J_2^a, \quad \hat{Q} J_2^\alpha = (\hat{\lambda}^\gamma_a J_3^a), \quad \hat{Q} (J_3)_\alpha = (\nabla_\hat{\lambda})^\alpha.
\]
\[ Q w_\alpha = (J_3)_\alpha , \quad \hat{Q} \hat{\hat{w}}^\alpha = - \hat{J}_1^\alpha , \quad \hat{Q} w_\alpha = Q \hat{\hat{w}}^\alpha = 0 , \]
\[ QN^{ab} = \frac{1}{2} (J_3 \gamma^{ab} \lambda) , \quad \hat{Q} \hat{N}^{ab} = \frac{1}{2} (\hat{\lambda} \gamma^{ab} \hat{J}_1) . \]

In terms of the \( PSU(2,2|4) \) super Lie algebra, the grading one and the grading three subspaces are related by hermitian conjugation which implies

\[ (\lambda^\alpha)^\dagger = \hat{\lambda}_\alpha . \]

The stress tensor of the worldsheet theory is

\[ T = - \frac{1}{2} J_2^a J_2^b \eta_{ab} + J_3^\alpha J_3^\alpha - w_\alpha \nabla \lambda^\alpha , \quad (2.7) \]

and it is easy to check that it satisfies \( \{ Q, T \} = \{ \hat{Q}, T \} = 0 \). The consistency of the theory at the quantum level has been checked in [4][5][6].

2.1. The antighost

Before we consider the \( b \) antighost, let us take a quick detour and introduce a useful projection operator. The conjugate momentum to the pure spinor variable, that we denoted \( w \), may only appear in expressions that are gauge invariant with respect to the local symmetry

\[ \delta_w w_\alpha = (\gamma^a \lambda)_\alpha \Lambda_a , \quad (2.8) \]

which is generated by the pure spinor constraint. As in flat space, the only gauge invariant combinations of \( w \) are the \( SO(1,9) \) Lorentz generators \( N^{ab} \) and the ghost number current \( J_{gh} \). However, instead of working with \( N^{ab} \) and \( J_{gh} \), it will be convenient to define a projection operator \( (1 - K)^\alpha_\beta \) which selects out the gauge-invariant components of \( w_\alpha \). In other words, \( (1 - K)^\alpha_\beta \delta_w w_\alpha = 0 \) under (2.8).

Consider the following projection operator,\(^1\) built out of the inverse power of \( (\lambda \hat{\lambda}) \equiv P_{\alpha \dot{\alpha}} \lambda^\alpha \hat{\lambda}^{\dot{\alpha}} \)

\[ K^\alpha_\beta = \frac{1}{2 (\lambda \hat{\lambda})} (\gamma^a \hat{\lambda})^\alpha (\lambda \gamma_a)_\beta = \frac{1}{2 (\lambda \hat{\lambda})} (\gamma^a \lambda)_\beta (\hat{\lambda} \gamma_a)^\alpha , \quad (2.9) \]

\(^1\) A brief historical comment. The projection operator \( K^\alpha_\beta \) was first introduced in [7][8] in a flat background, in the context of a semiclassical derivation of the pure spinor formalism from a Green-Schwarz type action. However, the \( \hat{\lambda} \) variable in [7][8] is a fixed spinor so the formalism in [7][8] is not manifestly Lorentz covariant, but is only valid in a patch of the pure spinor manifold. The Lorentz variation of the non-covariant \( b \) antighost constructed in [8] is BRST exact.
with the following properties

\[(1 - K)\gamma^a \lambda = 0, \quad K\gamma^a \gamma^b \lambda = 0, \quad K\nabla \lambda = 0,\]

\[(1 - K)\gamma^a \hat{\lambda} = 0, \quad K\gamma^a \gamma^b \hat{\lambda} = 0, \quad K\nabla \hat{\lambda} = 0,\]  

(2.10)

and its traces over the spinor indices are \(\text{Tr} K = 5\) and \(\text{Tr} (1 - K) = 11\). By means of the projector \(K^\beta_\alpha\) we can introduce the new quantity

\[w_\alpha (1 - K)^\alpha_\beta,\]  

(2.11)

which is invariant under (2.8).

In a flat background, one can construct a similar \(K^\beta_\alpha\) by replacing \(\hat{\lambda}_\alpha\) with the non-minimal variable \(\bar{\lambda}_\alpha\). If one interprets \(\bar{\lambda}_\alpha\) as the complex conjugate of \(\lambda^\alpha\), \((\lambda\bar{\lambda})^{-1}\) is formally well-defined after the point \(\lambda^\alpha = 0\) for all sixteen components is removed from the theory. However, as discussed in [9][10], there are problems if the negative powers of \((\lambda\bar{\lambda})\) accumulate beyond 11. In an \(AdS_5 \times S^5\) background, one expects similar problems if the negative powers of \((\lambda\hat{\lambda})\) accumulate beyond a certain amount. However, as in a flat background, we expect that our construction of the \(b\) antighost does not contain enough negative powers of \((\lambda\hat{\lambda})\) to cause problems.

After using the ten dimensional identity

\[(\gamma_{ab})_\alpha^\beta (\gamma_{ab})^\gamma_\delta = 4(\gamma_a)^\beta_\alpha (\gamma_a)^\gamma_\delta - 2(\delta_\alpha^\beta \delta_\gamma^\delta - 8(\delta_\delta^\alpha \delta_\delta^\gamma),\]  

(2.12)

the expression for the \(AdS_5 \times S^5\) antighost in [2] can be written in terms of the \((1 - K)^\alpha_\beta\) projector as

\[b = \frac{(\hat{\lambda}\gamma_a J_3)J_3^a}{2(\lambda\bar{\lambda})} - w_\alpha (1 - K)^\alpha_\beta J_1^\beta.\]  

(2.13)

Using the BRST transformations in (2.4), it was shown in [2] that \(\{Q, b\} = T\). Note that as in [2], we will be ignoring possible normal-ordering corrections to the \(b\) antighost throughout this paper and will only be considering the terms in \(b\) which are lowest order in \(\alpha'\).

The other crucial property of the \(b\) antighost is

\[\{\hat{Q}, b\} = 0.\]

Let us prove it. The variation of the first term in (2.13) is

\[\frac{1}{2(\lambda\bar{\lambda})} \left( (\hat{\lambda}\gamma_a \overrightarrow{\nabla} \hat{\lambda})J_2^a - (\hat{\lambda}\gamma_a J_3^a)(\hat{\lambda}\gamma^a J_3) \right),\]  

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which vanishes because of the pure spinor constraint and the properties of ten dimensional gamma matrices. The variation of the second term in (2.13) is

\[ w_\alpha (1 - K)_\beta^\alpha (\gamma_a \hat{\lambda})^\beta J_2^a , \]

which vanishes due to the properties of the projector (2.10).

An analogous construction carries over to the right-moving sector. The right-moving stress tensor and antighost are

\[
\hat{T} = -\frac{1}{2} \bar{J}_2^a \bar{J}_2^{b\eta_{ab}} + \bar{J}_1^a \bar{J}_3^\alpha - \hat{w} \hat{\nabla} \hat{\lambda} ,
\]

(2.14)

\[
\hat{b} = -\frac{1}{2\lambda \hat{\lambda}} (\lambda \gamma_a \bar{J}_1^a) \bar{J}_2^\alpha - \hat{w}^\alpha (1 - K)_\alpha^\beta (\bar{J}_3)^\beta .
\]

(2.15)

One can check that \( \{ \hat{Q}, \hat{b} \} = \hat{T} \) and \( \{ Q, \hat{b} \} = 0. \)

2.2. Conservation of the antighost

We can apply the argument given in the introduction to show that the \( b \) antighost is conserved up to BRST exact terms. Let us rewrite (2.13) in the convenient form

\[
b = \frac{\hat{\lambda}_\alpha}{(\lambda \hat{\lambda})} G^\alpha ,
\]

(2.16)

\[
G^\alpha = -\frac{1}{2} (\gamma_a J_3)^\alpha J_2^a - \lambda^\alpha (w J_1) + \frac{1}{2} (\gamma^a w)^\alpha (\lambda \gamma_a J_1) .
\]

Since the \( b \) antighost is a Lorentz scalar, we have that \( \bar{\partial} b = \hat{\nabla} b \) and

\[
\hat{\nabla} b = \hat{\nabla} \left( \frac{\hat{\lambda}_\alpha}{(\lambda \hat{\lambda})} \right) \left( -\frac{1}{2} (\gamma_a J_3)^\alpha J_2^a + \frac{1}{2} (\gamma^a w)^\alpha (\lambda \gamma_a J_1) \right) + \frac{\hat{\lambda}_\alpha}{(\lambda \hat{\lambda})} \hat{\nabla} G^\alpha .
\]

(2.17)

Let us look at the second term in (2.17). By using the equations of motion of the action (2.4) and the Maurer-Cartan equations (2.2) we find

\[
\frac{\hat{\lambda}_\alpha}{(\lambda \hat{\lambda})} \hat{\nabla} G^\alpha = b_0 + b_w + b_{ww} + b_{ww} ,
\]

\[
b_0 = \frac{1}{2(\lambda \hat{\lambda})} (\hat{\lambda} \gamma_a J_3)(J_3 \gamma^a \bar{J}_3) ,
\]

\[
b_w = \left[ w (1 - K) \gamma_a \right]^\alpha (J_3)^\alpha J_2^a - (\bar{J}_3)^a J_2^\alpha \\
+ \frac{1}{2(\lambda \hat{\lambda})} \left( \frac{1}{2} (\bar{J}_3 \gamma_{ab} \gamma_c \hat{\lambda}) N_{ab} J_2^c + 2(\hat{\lambda} \gamma_a J_3)(\bar{J}_2) b N_{ab} \right) ,
\]

(2.18)

\[
b_{ww} = -\frac{1}{2} \left[ w (1 - K) \gamma_{ab} \bar{J}_1 \right] N_{ab} ,
\]

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where the subscript indicates the number of $w$’s and $\hat{w}$’s present in each term. The term $b_{w\hat{w}}$ is proportional to $\eta_{[ab][cd]}\hat{N}^{ab}(\hat{\lambda}\gamma^{cd})_\alpha$ which vanishes on the pure spinor constraint.

Let us show that $\bar{\nabla}b$ is BRST exact. Consider the operator $\mathcal{O}$ of weight $(2,1)$, defined as the coefficient of the single pole in the OPE of the hatted and unhatted antighosts

$\hat{b}(z,\bar{z})b(0) = \ldots + \frac{\mathcal{O}_{zz}(0)}{\bar{z}} + \ldots$  \hspace{1cm} (2.19)

Since $\{\hat{Q},\hat{b}\} = \hat{T}$ and $\{\hat{Q},b\} = 0$, by applying $\hat{Q}$ to (2.19) we conclude that

$\{\hat{Q},\mathcal{O}\} = \bar{\nabla}b$  \hspace{1cm} (2.20)

Since the pure spinor superstring in $AdS_5 \times S^5$ is an interacting two–dimensional conformal field theory, the OPE (2.19) has to be computed in the worldsheet perturbation theory. In this paper, we are only interested in the leading order result that we obtain using the tree level algebra of OPE’s between the left invariant currents, which was derived in [11][12]. One finds

$\mathcal{O} = A_0 + A_w + A_{ww}$,  \hspace{1cm} (2.21)

where

$A_0 = \frac{1}{2\lambda\bar{\lambda}} \left( J_2^a(J_3\gamma_a KJ_3) - J_2^a(\bar{J}_3(1 - K)\gamma_a J_3) \right)$

$+ \frac{2}{(2\lambda\bar{\lambda})^2} \left( -\bar{J}_2^a(\lambda\bar{J}_3)(\hat{\lambda}\gamma_a J_3) + J_2^a(\lambda\bar{J}_3)(\hat{\lambda}\gamma_a J_3) \right)$,  \hspace{1cm} (2.22)

$A_w = \frac{1}{(2\lambda\bar{\lambda})^2} \left( \frac{1}{2}(\lambda\gamma_a \gamma_{ef}\gamma_b\lambda)\bar{J}_2^a J_2^b N_{ef} - 2(\lambda\gamma_a \bar{J}_1)(\hat{\lambda}\gamma_b J_3)N^{ab} \right)$

$+ \frac{1}{2\lambda\bar{\lambda}} \left( -w\gamma_a (1 - K)\bar{J}_3[\lambda\gamma_a J_1] - 2(\lambda\bar{J}_3)(wKJ_1) \right)$

$+ [w\gamma_a (1 - K)\bar{J}_3(\lambda\gamma_a J_1) - 2(\lambda\bar{J}_3)(wKJ_1)]$,  \hspace{1cm} (2.23)

$A_{ww} = \frac{1}{2} [w(1 - K)\gamma_{ef}(1 - K)\hat{w}]N^{ef}$.  \hspace{1cm} (2.24)

The proof that (2.21) satisfies (2.20) is postponed to the Appendix.

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2 The one-loop correction to the classical OPE’s have been computed in [13]. It would be interesting to use them to compute the normal ordering terms in the antighost (2.13) and the quantum corrections to the operator $\mathcal{O}$. 

9
3. Type II supergravity background

In this Section we will introduce the action for a generic type II pure spinor superstring sigma model and show that the $b$ antighost is conserved in the classical BRST cohomology, in a similar way to the $AdS_5 \times S^5$, whenever the background satisfies certain conditions. At the end we will comment on the case when the supergravity background does not satisfy such conditions.

The sigma model action for the type II pure spinor superstring in a generic supergravity background

\[ S = \frac{1}{2\pi \alpha'} \int d^2 z \left[ \frac{1}{2} \Pi^a \Pi^b \eta_{ab} + \frac{1}{2} \Pi^A \Pi^B B_{AB} + d_\alpha \Pi^\alpha + \hat{d}_\hat{\alpha} \hat{\Pi}^{\hat{\alpha}} \right] + w_\alpha \nabla \lambda^\alpha + \hat{w}_{\hat{\alpha}} \nabla \hat{\lambda}^{\hat{\alpha}} \]

\[ + d_\alpha \dot{d}_\dot{\alpha} P^{\alpha \dot{\alpha}} + \lambda^\alpha w_{\beta \dot{\gamma}} \tilde{C}_\alpha^{\beta \dot{\gamma}} + \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta} \hat{\gamma}} \tilde{C}_{\hat{\alpha}}^{\hat{\beta} \hat{\gamma}} + \lambda^\alpha w_{\beta \dot{\gamma}} \hat{\lambda}^{\hat{\alpha}} \hat{w}_{\hat{\beta}} \tilde{S}^{\alpha \beta \dot{\gamma}} + \alpha' R \Phi(Z) \]  

(3.1)

has been studied in [14], to which we refer the reader for the details. Here we will only describe some features relevant for the present discussion. The worldsheet matter fields are the pullback of the target space super-vielbein $\Pi^A = E_M^A dZ^M$, where $A = (a, \alpha, \dot{\alpha})$ is a tangent space superspace index and $M = (m, \mu, \dot{\mu})$ a curved superspace index. The ghost content is the same as in the previous case and the covariant derivative on $\lambda (\hat{\lambda})$ is defined using the pullback of the left-moving (right-moving) spin connection $\Omega^\beta_M = dZ^M \Omega_M^{\alpha \beta}$ ($\hat{\Omega}_{\dot{\alpha}}^{\dot{\beta}} = dZ^M \hat{\Omega}_{M \dot{\alpha}}^{\dot{\beta}}$) as

\[ (\nabla \lambda)^\alpha = \partial \lambda^\alpha + \Omega^\alpha_M \lambda^M, \quad (\nabla \hat{\lambda})^{\dot{\alpha}} = \partial \hat{\lambda}^{\dot{\alpha}} + \hat{\Omega}^{\dot{\alpha}}_\dot{\mu} \hat{\lambda}^{\dot{\mu}}. \]

The background superfield $B_{AB}$ appearing in (3.1) is the superspace two-form potential; the lowest components of $C_\alpha^{\beta \dot{\gamma}}$ and $\tilde{C}_{\dot{\alpha}}^{\beta \gamma}$ are related to the gravitini and dilatini; the lowest component of $P^{\alpha \dot{\alpha}}$ is the Ramond-Ramond bispinor field strength (1.8); $S^{\alpha \beta \dot{\gamma}}$ is related to the Riemann curvature. The left- and right- moving BRST charges are

\[ Q = \oint d\sigma \lambda^\alpha d_\alpha, \quad \hat{Q} = \oint d\sigma \hat{\lambda}^{\dot{\alpha}} \hat{d}_{\dot{\alpha}}. \]

(3.2)

where $d$ and $\dot{d}$ are the pullback of the spacetime supersymmetric derivatives. Conservation of $Q$ and $\hat{Q}$ and nilpotency of $Q + \hat{Q}$ imply a set of type IIA/B supergravity constraints, that put the background onshell [14]. It was shown in [15] that one-loop conformal invariance of the worldsheet action is implied by such constraints. In the following we will recall some of those constraints when needed.
The stress tensor for the pure spinor action in a generic type II supergravity background reads

\[ T = -\frac{1}{2} \Pi^a \Pi^b \eta_{ab} - d_\alpha \Pi^\alpha - w_\alpha (\nabla \lambda)^\alpha , \quad (3.3) \]

When the R-R superfield \( P^{\alpha \dot{\alpha}} \) is an invertible matrix, we can simplify the sigma model action. The variables \( d \) and \( \hat{d} \) couple to the R-R field strength through the term \( d_\alpha \hat{d}_{\alpha} P^{\alpha \dot{\alpha}} \rightarrow d_\alpha \hat{d}^\alpha \) in the action (3.1). If \( P \) is invertible we can integrate \( d \) and \( \hat{d} \) out upon their equations of motion

\[ \begin{align*}
  d_\alpha &= \hat{\Pi}_\alpha + \lambda^\rho w_\beta P_{\alpha \dot{\alpha}} C^{\sigma \dot{\alpha}} C^{\sigma \dot{\alpha}} , \\
  \hat{d}^\alpha &= - \Pi^\alpha - \hat{\lambda}_\rho \hat{w}^\sigma P_{\alpha \dot{\alpha}} P^{\rho \dot{\beta}} P_{\sigma \dot{\gamma}} C^{\dot{\beta} \dot{\gamma}} .
\end{align*} \quad (3.4) \]

Substituting (3.4) into the stress tensor (3.3) we find

\[ T = -\frac{1}{2} \Pi^a \Pi^b \eta_{ab} - (\hat{\Pi}_\gamma + \lambda^\rho w_\beta P_{\gamma \dot{\gamma}} C^{\sigma \dot{\alpha}} \Pi^\gamma - w_\alpha (\nabla \lambda)^\alpha . \quad (3.5) \]

The proof that the stress tensor (3.3) is separately invariant under the BRST transformations generated by the left and right-moving BRST charges

\[ \{ Q, T \} = \{ \hat{Q}, T \} = 0 , \quad (3.6) \]

involves the supergravity constraints of [14] and is postponed to the Appendix.

3.1. Some particular backgrounds

We would like to specialize to a background where the R-R superfield is covariantly constant, namely

\[ \nabla_\alpha P^{\beta \dot{\beta}} = \nabla_{\dot{\alpha}} P^{\beta \dot{\beta}} = 0 . \quad (3.7) \]

This first requirement on the background ensures that the \( b \) antighost we will momentarily introduce satisfies \( \{ Q, b \} = T \).

We will also have a second requirement on the background, namely that the R-R superfield \( P^{\alpha \dot{\alpha}} \) is such that

\[ P^{\alpha \dot{\alpha}} \tilde{\gamma}^k \gamma_{\alpha \beta} P^{\beta \dot{\beta}} = f^k_m (\gamma^m)_{\dot{\alpha} \dot{\beta}} \quad (3.8) \]

for some tensor superfield \( f^k_m \). By expanding the superfield \( P \) in the basis (1.8) we can find the general solution of the condition (3.8) in terms of the R-R \( p \)-form fluxes. We find
that it holds when we turn on a single species of $p$-form flux with all nonzero components sharing $p-1$ legs. For example, the $p$-form flux

$$F_{a_1...a_p} = \delta^1_{a_1} \delta^2_{a_2} ... \delta^{p-1}_{a_p} c_{a_p}$$

is a solution of (3.8) for any choice of $c_{a_p}$. In the next subsections, we will show that condition (3.8) ensures that $\{\hat{Q}, b\} = 0$ and the $b$ ghost is conserved in the BRST cohomology.

When the R-R superfield obeys (3.8), we have that $\hat{\lambda} P \gamma^m P \hat{\lambda} = 0$ and we can redefine the hatted torsion as $\hat{T}^{\alpha}{}_{\hat{\beta}} = P^{\alpha}{}^\hat{\alpha} \hat{T}^{\hat{\alpha}}{}_{\hat{\beta}}$. Since the R-R superfield $P^{\alpha}{}^\hat{\alpha}$ is invertible, by a combined local Lorentz and scale transformation we can reabsorb it into a redefinition of the hatted spinor indices, just as we did in the AdS case in (2.3), namely

$$\hat{\lambda}_{\alpha} \equiv P^{\alpha}{}^\hat{\alpha} \lambda^\hat{\alpha}, \quad \hat{w}^{\alpha} \equiv P^{\alpha}{}^\hat{\alpha} \hat{w}_{\hat{\alpha}}, \quad \hat{\Pi}_{\alpha} \equiv P^{\alpha}{}^\hat{\alpha} \hat{\Pi}^\hat{\alpha}, \quad \hat{d}^{\alpha} \equiv P^{\alpha}{}^\hat{\alpha} \hat{d}_{\hat{\alpha}}.$$

When acting on scalar operators such as the stress tensor and the antighost, the BRST transformations can be cast into the following convenient form

$$Q \Pi^{\alpha} = \lambda^{\alpha} \Pi, \quad \hat{Q} \Pi^{\alpha} = \hat{\lambda}^{\alpha} \hat{\Pi},$$
$$Q \Pi^{\alpha} = \nabla \lambda^{\alpha}, \quad \hat{Q} \Pi^{\alpha} = -\hat{\lambda}^{\alpha} \Pi^{\alpha} \gamma_{\alpha}^{\beta},$$
$$Q \hat{\Pi}_{\beta} = -\lambda^{\alpha} \Pi^{\alpha} \gamma_{\alpha}^{\beta}, \quad \hat{Q} \hat{\Pi}_{\alpha} = \nabla \hat{\lambda}_{\alpha},$$
$$Q \lambda^{\alpha} = Q \hat{\lambda}_{\alpha} = Q \hat{w}^{\alpha} = 0, \quad \hat{Q} \lambda^{\alpha} = \hat{Q} \hat{\lambda}_{\alpha} = \hat{Q} w_{\beta} = 0,$$
$$Q w_{\beta} = d_{\alpha}, \quad \hat{Q} \hat{w}^{\alpha} = \hat{d}^{\alpha},$$

where the background fields $R$ and $\hat{R}$ are the Riemann curvatures of the left and right-moving Lorentz connections respectively. The BRST transformation of a background tensor superfield is

$$\{Q, \Phi(Z)^A_B\} = \lambda^{\alpha} [\nabla_{\alpha} \Phi(Z)]^A_B = \lambda^{\alpha} (\partial_{\alpha} \Phi(Z)^A_B + \Omega_{\alpha C} A \Phi(Z)^A_B - \Omega_{\alpha B C} \Phi(Z)^A_C),$$
$$\{\hat{Q}, \Phi(Z)^A_B\} = \hat{\lambda}_{\alpha} [\nabla^{\alpha} \Phi(Z)]^A_B = \hat{\lambda}_{\alpha} (P^{\alpha \hat{\alpha}} \partial_{\alpha} \Phi(Z)^A_B + P^{\alpha \hat{\alpha}} \Omega_{\hat{\alpha} C} A \Phi(Z)^A_B - P^{\alpha \hat{\alpha}} \Omega_{\hat{\alpha} B C} \Phi(Z)^A_C)$$

where $\Omega_{\alpha B C}$ and $\Omega_{\hat{\alpha} B C}$ are hatted or unhatted spin connections depending if $(B, C)$ are hatted or unhatted spinor indices.

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3 In reducing the BRST transformations in the Appendix to the formulas below, one needs to verify that the contributions of the unhatted and hatted spin connections in the transformations of the Appendix cancel independently. This independent cancellation is easily verified for the stress tensor and antighost. A similar approach has been previously discussed in [16] [17].
3.2. Antighost

As anticipated in the introduction, if the R-R superfield is covariantly constant and invertible, we can follow the same steps as in $\text{AdS}_5 \times S^5$. The operator

$$ (\lambda \hat{\lambda}) \equiv \lambda^\alpha P_{\alpha \hat{\alpha}} \hat{\lambda}^{\hat{\alpha}}, $$

(3.12)
is in the BRST cohomology and we can use the inverse of this operator to construct the antighost

$$ b = \frac{1}{2(\lambda \hat{\lambda})} (\hat{\lambda} \gamma_a \hat{\Pi}) \Pi^a - w_\alpha (1 - K)^\beta_\alpha \Pi^\beta, $$

(3.13)
where we use the curved space projector $K$

$$ K^\alpha_\beta = \frac{1}{2(\lambda \hat{\lambda})} (\gamma^a \hat{\lambda}^\beta (\gamma_a \lambda)_\alpha), $$

(3.14)
which satisfies the same properties as in (2.10) and is annihilated by both BRST charges.

Let us show that the variation of (3.13) with respect to the holomorphic BRST charge $Q$ satisfies

$$ \{Q, b\} = T, $$

where $T$ is given in (3.5). The variation of the first term in (3.13) is

$$ -\frac{1}{2} \eta_{ab} \Pi^a \Pi^b - \frac{1}{2(\lambda \hat{\lambda})} (\hat{\lambda} \gamma_a \hat{\Pi}) (\lambda^a \Pi), $$

(3.15)
while the second term gives

$$ Q(-w(1 - K)\Pi) = -\hat{\Pi}^\beta_\alpha (1 - K)^\beta_\alpha \Pi^\alpha - w_\alpha (1 - K) \nabla \lambda^\alpha. $$

(3.16)
The BRST holomorphicity constraint

$$ \nabla_\alpha P^\beta_\gamma + C^\beta_\gamma = 0, $$

(3.17)

\footnote{We thank Sebastian Guttenberg for pointing out the necessity of including additional terms in the $b$ ghost when $P^{\alpha \hat{\alpha}}$ is not covariantly constant.}
together with the condition (3.7), implies that $C = 0$. And the term $-\hat{\Pi} K \Pi$ in (3.16) cancels with the second term in (3.15) so we proved that $\{Q, b\} = T$.

If the Ramond-Ramond superfield is invertible but is not covariantly constant, the $b$ ghost of (3.13) will instead satisfy $\{Q, b\} = T + f_{a\hat{\alpha}} (\lambda^\beta \nabla_\beta P^{a\hat{\alpha}})$ for some $f_{a\hat{\alpha}}$. It should be possible to modify $b \rightarrow b - b'$ such that $\{Q, b'\} = f_{a\hat{\alpha}} (\lambda^\beta \nabla_\beta P^{a\hat{\alpha}})$, but we will not attempt to construct $b'$ in this paper.
3.3. Conservation of the antighost

We need to show that

\[ \hat{Q}b = 0, \]  

so that the argument (1.6) for the conservation of the antighost in the BRST cohomology carries over. We have

\[ \hat{Q} \left( \frac{1}{2(\lambda\lambda)} (\hat{\lambda}_a \hat{\Pi})^{\alpha} \right) = \frac{1}{2(\lambda\lambda)} (\hat{\lambda}_a)^{\alpha} \left( \nabla \hat{\lambda}_a \Pi^a - \hat{\Pi}_a (\hat{\lambda}_a \hat{\Pi}) \right), \]  

(3.19)

\[ \hat{Q}(-w_{\beta}(1-K)^\beta_\alpha \Pi^\alpha) = w_\alpha(1-K)^\alpha_\beta \hat{\lambda}^{\hat{\alpha}} (\gamma_a)_{\hat{\alpha}\hat{\beta}} P_{\beta}^\beta \Pi^a, \]  

(3.20)

The right hand side of (3.19) vanishes on the pure spinor constraint, while the right hand side of (3.20) vanishes due to the properties (2.10) of the projector $K$, when the background satisfies (3.8). Hence, we proved (3.18) and the conservation of the antighost up to BRST exact terms.

3.4. Antighost in a generic type II background

In this subsection, we discuss the complications in constructing the antighost in a generic supergravity background. If we relax the assumption that the R-R superfield $P^{\alpha\hat{\alpha}}$ be invertible, we cannot integrate out $d_\alpha$ and $\hat{d}_{\hat{\alpha}}$ using their equations of motion (3.4). On top of this, we are forced to introduce the non-minimal variables as in a flat background. They consist of a new set of left and right-moving bosonic pure spinors $(\hat{\lambda}_\alpha, \hat{\hat{\lambda}}_{\hat{\alpha}})$ and their conjugate momenta $(\hat{w}_\alpha, \hat{\hat{w}}_{\hat{\alpha}})$, as well as a set of left and right-moving constrained fermions $(r_{\alpha}, \hat{r}_{\hat{\alpha}})$ and their conjugate momenta $(s_\alpha, \hat{s}_{\hat{\alpha}})$. These non-minimal variables satisfy the constraints $\hat{\lambda}_\gamma^m \hat{\lambda} = \hat{\hat{\lambda}}_\gamma^m \hat{\hat{\lambda}} = 0$ and $\hat{\lambda}_\gamma^m r = \hat{\hat{\lambda}}_\gamma^m \hat{\hat{r}} = 0$, so that there are an equal number of non-minimal bosonic and fermionic degrees of freedom. After modifying the BRST operator to include the standard non-minimal term, these new variables decouple from the cohomology and the physical spectrum.

The first step in constructing the $b$ antighost in a generic background would be to find an expression satisfying \( \{Q, b_0\} = T \) where

\[ Q = \int dz (\lambda^\alpha d_\alpha + \hat{w}^\alpha r_\alpha), \]  

(3.21)

\[ T = -\frac{1}{2} \Pi^a \Pi^b \eta_{ab} - d_\alpha \Pi^\alpha - w_\alpha (\nabla \lambda)^\alpha + s_\alpha (\nabla r)_\alpha - \hat{w}^\alpha (\nabla \hat{\lambda})_\alpha. \]
Note that in a curved background, one introduces couplings of the non-minimal variables to the spin connection in order to make the action invariant under local Lorentz transformations. This can be done in a BRST-invariant manner by adding the BRST-trivial term

$$\{Q, -s^\alpha(\nabla \bar{\lambda})_\alpha\} = s^\alpha(\nabla r)_\alpha - \bar{w}^\alpha(\nabla \bar{\lambda})_\alpha + \lambda^\alpha \Pi^A s^\beta \bar{\lambda}_\gamma R_{\alpha A \beta \gamma} \tag{3.22}$$

to the minimal action. Note that the non-minimal Lorentz current $\bar{N}_{ab} = \frac{1}{2}(-\bar{w}_a \bar{\lambda} + s_{\gamma ab} \bar{\lambda})$ is equal to $\{Q, -\frac{1}{2}(s_{\gamma ab} \bar{\lambda})\}$, so the non-minimal action includes the usual coupling of the spin connection to the Lorentz current.

The nilpotent BRST transformations on the non-minimal variables which follow from the curved action are

$$Q\bar{\lambda}_\alpha = \bar{\lambda}_\beta (\lambda^\gamma \Omega_{\gamma \alpha \beta}) + r_\alpha, \tag{3.23}$$

$$Q\bar{w}^\alpha = \bar{w}^\beta (-\lambda^\gamma \Omega_{\gamma \beta}^\alpha) + \lambda^\beta \lambda^\gamma s^\delta R_{\beta \gamma \delta}^\alpha,$$

$$Qr_\alpha = r_\beta (\lambda^\gamma \Omega_{\gamma \alpha}^\beta) + \lambda^\beta \lambda^\gamma \bar{\lambda}_\delta R_{\beta \gamma \delta}^\alpha,$$

$$Qs^\alpha = s^\beta (-\lambda^\gamma \Omega_{\gamma \beta}^\alpha) + \bar{w}_\alpha,$$

where the second term in $Q\bar{w}^\alpha$ and $Qr_\alpha$ comes from the last term in (3.22). When acting on scalars, the spin connection $\Omega_{\gamma \alpha \beta}$ can be dropped and the non-minimal BRST transformations simplify to

$$Q\bar{\lambda}_\alpha = r_\alpha, \quad Q\bar{w}^\alpha = \lambda^\beta \lambda^\gamma s^\delta R_{\beta \gamma \delta}^\alpha, \tag{3.24}$$

$$Qr_\alpha = \lambda^\beta \lambda^\gamma \bar{\lambda}_\delta R_{\beta \gamma \delta}^\alpha, \quad Qs^\alpha = \bar{w}_\alpha.$$

Using the above non-minimal BRST transformations together with the minimal BRST transformations of (3.10), one expects that $b_0$ satisfying $\{Q, b_0\} = T$ will be a generalization of the flat-space expression which is

$$b_0^{flat} = s^\alpha \nabla \bar{\lambda}_\alpha + \frac{1}{2\lambda \lambda} (\bar{\lambda}_\gamma a)^\alpha d_\alpha \Pi^a - w_\alpha (\delta_\beta^\alpha - \bar{K}_\beta^\alpha) \Pi^\beta \tag{3.25}$$

$$+ \frac{(\bar{\lambda}_\gamma a c)(d \gamma_{abc} d + 24N_{ab} \Pi_c)}{192(\lambda \lambda)^2} - \frac{(r_\gamma a c d)(\bar{\lambda}_\gamma a d) N^{bc}}{16(\lambda \lambda)^3} + \frac{(r_\gamma a c d)(\bar{\lambda}_\gamma c d e r) N^{ab} N_{de}}{128(\lambda \lambda)^4}.$$
depending on this curvature. Moreover, note that the non-minimal version of the projector \((2.3)\) used in \((3.25)\) is
\[
\tilde{K}^\beta_\alpha = \frac{1}{2(\lambda \bar{\lambda})} (\gamma_a \bar{\lambda})^\beta (\lambda \gamma_a)_\alpha .
\]
which has the important difference with the expression in \((3.13)\) that the hatted pure spinor has been replaced by the barred non-minimal pure spinor.

Because the hatted pure spinor has been replaced with the non-minimal pure spinor, the right-moving BRST operator \(\hat{Q}\) is no longer expected to anticommute with \(b_0\). Although it will not be proven here, we conjecture that \(\{\hat{Q}, b_0\} = -\{Q, b_1\}\) for some \(b_1\). In other words, we conjecture that its anticommutator with \(\hat{Q}\) is BRST-trivial with respect to \(Q\). Furthermore, we conjecture that \(\{\hat{Q}, b_1\} = -\{Q, b_2\}\) for some \(b_2\), etc. Note that \(b_n\) has left-moving ghost-number \((-1-n)\) and right-moving ghost-number \(n\).

If one assumes this conjecture and defines \(b = b_0 + b_1 + ...\), one finds that
\[
\{Q + \hat{Q}, b\} = T. \tag{3.26}
\]
Repeating these arguments, one can construct \(\hat{b} = b_0 + \hat{b}_1 + ...\) such that \(\{Q + \hat{Q}, \hat{b}\} = \hat{T}\).

Using the amplitude prescription of \((1.1)\), one can now insert these composite \(b\) and \(\hat{b}\) antighosts. Although the \(b\) and \(\hat{b}\) antighosts do not have fixed (left, right) ghost-numbers, the prescription is invariant (up to possible surface terms) under BRST transformations generated by \((Q + \hat{Q})\).

As in Ramond-Ramond curved backgrounds, the \(b\) antighost does not necessarily satisfy \(\bar{\partial}b = 0\). In a Ramond-Ramond background, \(\bar{Q}b = 0\) implied that \(\bar{\partial}b = \{\bar{Q}, \Omega\}\), which was sufficient for the consistency of \((1.1)\). However, in a generic curved background, one needs to use non-minimal variables and \(\{\bar{Q}, b\}\) may be non-zero. Nevertheless, since \(\{Q + \hat{Q}, b\} = T\) and \(\bar{\partial}T = 0\), it might be possible to show that \(\bar{\partial}b = \{(Q + \hat{Q}), \bar{\Omega}\}\) for some \(\bar{\Omega}\) (and similarly, \(\bar{\partial}\hat{b} = \{(Q + \hat{Q}), \hat{\Omega}\}\) for some \(\hat{\Omega}\)). If this can be shown, the amplitude prescription of \((1.1)\) would be consistent, not only for invertible Ramond-Ramond superfields, but for any curved background, since \(Q + \hat{Q}\) can be pulled off of \(\Omega\) and would only generate possible surface terms.

**Acknowledgements:** We would especially like to thank S. Guttenberg for pointing out an error in the first version of the paper that construction of the \(b\) ghost requires additional conditions on \(P^{\alpha \beta}\). We would also like to thank O. Chandia and N. Nekrasov for discussions. NB would like to thank the Simons Center for Geometry and Physics where part of this research was done and FAPESP grant 09/50639-2 and CNPq grant 300256/94-9 for partial financial support.
Appendix A. Some results in $\text{AdS}_5 \times S^5$

A.1. Proof of conservation of $b$ in AdS

Let us check that the BRST transformation of the operator $O$ in (2.20) is equal to $\nabla b$ in (2.17) and (2.18). The BRST transformations of the various terms are

\[ \{ \hat{Q}, A_0 \} = C_{333} + C_{23} + C_{23} + C_{23}, \]

\[ C_{333} = b_0, \]

\[ C_{23} = \frac{2}{2\lambda\lambda} (\lambda\gamma_3 J_3) J_2 \rightleftharpoons \frac{2}{(2\lambda\lambda)^2} (\lambda\gamma_3 \gamma_a \lambda) (\lambda\gamma_3 J_3) J_2 \rightleftharpoons N^{ef}, \]

(A.1)

\[ C_{23} = -\nabla \left( \frac{\lambda_\alpha}{2\lambda\lambda} \right) (\gamma_a J_3) J_2, \]

\[ C_{23} = \frac{1}{2\lambda\lambda} \left( (\lambda J_3 \gamma_3 \gamma_a \lambda) J_2 \rightleftharpoons \lambda J_2 N^{ef} - (\lambda J_3 K \gamma_3 \gamma_a \lambda) J_2 \rightleftharpoons N^{ef}, \]

\{ \hat{Q}, A_w \} = B_{23} + B_{23} + B_1 + B_1,

\[ B_{23} = \frac{1}{2\lambda\lambda} (\lambda J_3 K \gamma_3 \gamma_a \lambda) J_2 \rightleftharpoons N^{ef} - [w(1 - K) \gamma_a J_3] J_2 \rightleftharpoons, \]

\[ B_{23} = [w(1 - K) \gamma_b J_3] J_2 - \frac{2}{2\lambda\lambda} (\lambda\gamma_3 \gamma_a \lambda) (\lambda\gamma_3 J_3) J_2 \rightleftharpoons N^{ef}, \]

(A.2)

\[ B_1 = \nabla \left( \frac{\lambda_\alpha}{2\lambda\lambda} \right) (w\gamma^a) \rightleftharpoons (\lambda\gamma_a J_1), \]

\[ B_1 = -\frac{1}{2} (w\gamma_3 K J_1) N^{ef} + \frac{2}{2\lambda\lambda} (\lambda\gamma_3 J_1) N^{ef}, \]

\{ \hat{Q}, A_{ww} \} = b_{ww} + \frac{1}{2} [w\gamma_3 K J_1] N^{ef} - \frac{2}{2\lambda\lambda} (w\gamma_3 \lambda) (\lambda\gamma_3 J_1) N^{ef}. \]

(A.3)

Summing up we find

\[ \{ \hat{Q}, \Omega \} = b_0 + b_w + b_{ww} + \nabla \left( \frac{\lambda_\alpha}{2\lambda\lambda} \right) (-(\gamma_a J_3) \rightleftharpoons J_2 \rightleftharpoons + (w\gamma^a) \rightleftharpoons (\lambda\gamma_a J_1)), \]

(A.4)

\[ = \nabla b. \]

Appendix B. Some results in type IIA/B curved backgrounds

We consider the case where the R-R superfield is invertible and we have integrated out $d$ upon its equation of motion (3.4). The BRST transformations of the worldsheet fields
are generated by the BRST charge \( Q + \hat{Q} \) and we will consider the separate left and right-moving BRST transformations.\(^5\) We assume the background type II supergravity is onshell and we use the holomorphicity and nilpotency constraints of \([14]\) to simplify the transformations. We also use the gauge choice of \([14]\) where \( T_{\alpha \beta}^\gamma = T_{\dot{\alpha} \dot{\beta}}^{\dot{\gamma}} = 0 = T_{\alpha a}^\beta = T_{\dot{a} \alpha}^\dot{\beta} = 0 \) and where \( T_{\alpha a}^\beta = \hat{T}_{\alpha a}^\beta = 0 \). As explained in \([14]\), it is convenient to introduce both left and right-moving spin connections, \( \Omega_{A\beta}^\gamma \) and \( \hat{\Omega}_{A\dot{\beta}}^{\dot{\gamma}} \) which act respectively on unhatted and hatted spinor indices. On vector indices, one can use either of these connections and \( T_{\alpha a}^\beta \) is defined using \( \Omega_A \) whereas \( \hat{T}_{\alpha a}^\beta \) is defined using \( \hat{\Omega}_A \).

The nilpotent BRST transformations are given by:

\[
Q \Phi(Z)^{A_1 \cdots A_M}_{B_1 \cdots B_N} = \lambda^a \partial_a \Phi(Z)^{A_1 \cdots A_M}_{B_1 \cdots B_N}, \quad \hat{Q} \Phi(Z)^{A_1 \cdots A_M}_{B_1 \cdots B_N} = \hat{\lambda}^\dot{a} \partial_{\dot{a}} \Phi(Z)^{A_1 \cdots A_M}_{B_1 \cdots B_N},
\]

\[
Q Z^M = \lambda^a E^M_a, \quad \hat{Q} \hat{\Pi}^a = \Pi^\beta (\lambda^\gamma \hat{\Omega}^{\gamma \dot{\beta}}_\beta) - \lambda^a \Pi^a (\gamma_\alpha)_{\beta} P^{\alpha \dot{\beta}},
\]

\[
Q \Pi^a = \Pi^b (\lambda^\gamma \hat{\Omega}^{\gamma \gamma}_a) + \lambda^\gamma \Pi^a, \quad \hat{Q} \Pi^a = \Pi^\beta (\lambda^\gamma \Omega^{\gamma \beta}_a) + \lambda^\gamma \Pi^a + \lambda^\alpha \lambda^\beta w_\delta R_{\gamma \alpha \beta}^\delta,
\]

\[
Q d_\gamma = -d_\alpha (\lambda^\gamma \Omega^{\gamma \gamma}_a) - (\lambda_\gamma)_{\alpha} \Pi^a + \lambda^\alpha \lambda^\beta w_\delta R_{\gamma \alpha \beta}^\delta, \quad Q d_{\dot{\gamma}} = -d_\dot{\alpha} (\lambda^\gamma \hat{\Omega}^{\gamma \dot{\gamma}}_{\dot{\beta}}) + \lambda^\gamma \hat{\Omega}^{\gamma \dot{\gamma}}_{\dot{\beta}}, \quad Q w_\beta = -w_\alpha (\lambda^\gamma \Omega^{\gamma \beta}_a) + d_\alpha,
\]

\[
Q \hat{\lambda}^\dot{a} = \hat{\lambda}^\dot{b} (\lambda^\gamma \hat{\Omega}^{\gamma \dot{\beta}}_{\dot{\beta}}), \quad Q \hat{w}_\beta = -w_\dot{\alpha} (\lambda^\gamma \hat{\Omega}^{\gamma \dot{\beta}}_{\dot{\beta}}).
\]

The background fields \( R \) and \( \hat{R} \) are the Riemann curvatures of the left and right-moving Lorentz connections \( \Omega \) and \( \hat{\Omega} \) respectively. In writing the BRST transformation of \( \Pi^a \), one can either use the unhatted or hatted spin connection. Since \( \hat{T}_{\alpha a}^\beta = T_{\alpha a}^\beta = 0 \), it is convenient to use the hatted spin connection in the definition of \( Q \Pi^a \) and the unhatted spin connection in the definition of \( \hat{Q} \Pi^a \). Of course, one can also write \( Q \Pi^a \) in terms of the unhatted spin connection using the relation \( \hat{\Omega}_{\alpha a}^\beta = \Omega_{\alpha a}^\beta - T_{\alpha a}^\beta \).

\(^5\) The BRST transformations of the heterotic pure spinor superstring in a SYM and SUGRA background have been presented in \([18]\).
We can check nilpotency of these BRST transformations using the supergravity constraints \[18\]. For example, to check that \( Q^2 = 0 \),

\[
\epsilon_1 Q(\epsilon_2 Q Z^M) = \epsilon_1 \epsilon_2 \lambda^\beta \gamma \left( \partial_{(\beta} E^{M}_{\gamma)} + \Omega_{\beta \gamma}^\alpha E^{M}_{\alpha} \right) \]

\[
= \epsilon_1 \epsilon_2 \lambda^\beta \lambda^\gamma T^A_{\beta \gamma} E^M_A ,
\]

which vanishes because of the torsion constraint \( \lambda^\beta \lambda^\gamma T^A_{\beta \gamma} = 0 \). We also have

\[
\epsilon_1 Q(\epsilon_2 Q \lambda^\alpha) = \epsilon_1 \epsilon_2 \lambda^\beta \lambda^\gamma \lambda^\rho \left( \partial_{(\rho} \Omega^{\gamma \beta)_\alpha} - \Omega^{(\gamma \rho}_T \Omega_{\tau | \beta)\alpha} - \Omega^{(\gamma \rho}_T \Omega_{\beta)\tau} \right) \]

\[
= \epsilon_1 \epsilon_2 \lambda^\beta \lambda^\gamma \lambda^\rho R^\rho_{\gamma \beta} \alpha ,
\]

which vanishes due to the constraint \( \lambda^\beta \lambda^\gamma \lambda^\rho R^\rho_{\gamma \beta \alpha} = 0 \). We can similarly check that the supergravity constraints imply that \( \hat{Q}^2 = 0 \) and \( \{ Q, \hat{Q} \} = 0 \).

When acting on a target space scalar operator, we can rearrange the BRST transformations by removing the Lorentz spin connection from the transformation of the worldsheet fields (B.2) and (B.3) and covariantizing the BRST transformations of the background superfields (B.1) as

\[
\{ Q, \Phi(Z)^A \} = \lambda^\alpha [\nabla_\alpha \Phi(Z)^A]_B = \lambda^\alpha \left( \partial_\alpha \Phi(Z)^A + \Omega_{\alpha \gamma} \Phi(Z)^B - \Omega_{\alpha B} \Phi(Z)^C \right) ,
\]

\[
\{ \hat{Q}, \Phi(Z)^A \} = \hat{\lambda}^\alpha [\nabla_{\hat{\alpha}} \Phi(Z)^A]_B = \hat{\lambda}^\alpha \left( \partial_{\hat{\alpha}} \Phi(Z)^A + \Omega_{\hat{\alpha} \gamma} \Phi(Z)^B - \Omega_{\hat{\alpha} B} \Phi(Z)^C \right) .
\]

Using these redefined transformations, one can check that the stress tensor is BRST invariant.

When the R-R superfield satisfies (3.8), we can further redefine the worldsheet fields as in (3.9) and finally obtain (3.10) and (B.6). The only subtlety is that both the unhatted and hatted spin connections appear in (3.2) and (3.3), so one needs to verify that they cancel independently in the transformation of the scalar operator. Fortunately, this is easily verified for the stress tensor and antighost of (3.3) and (3.13). The unhatted and hatted spin connections appearing in the BRST transformations of unhatted and hatted spinor fields are easily shown to cancel. And the hatted spin connection appearing in the BRST transformation \( Q \Pi^a \) cancels since \( \Pi^a \) only appears in the combinations \( \eta_{ab} \Pi^a \Pi^b \) and \( \hat{\lambda}^\gamma_a \hat{\Pi}^b_{\gamma} \Pi^a \) in (3.3) and (3.13).
B.1. BRST invariance of the stress tensor

Let us check that the stress tensor (3.5) is BRST invariant. Since we are acting on a scalar operator, we can use the redefined BRST transformations (B.6). First consider the left-moving BRST variation \( \{Q, T\} \). We find

\[
Q \left( -\frac{1}{2} \Pi^a \Pi^b \eta_{ab} \right) = -\eta_{ab} \Pi^a (\lambda^b \Pi) ,
\]

(B.7)

\[
Q \left( -P_{\gamma \tilde{\gamma}} \hat{\Pi}^{\gamma} \Pi^\gamma \right) = -\lambda^\rho (\nabla_\rho P_{\gamma \tilde{\gamma}}) \hat{\Pi}^{\gamma} \Pi^\gamma + \lambda^\alpha \Pi^a (\gamma_a)^{\alpha \rho} \Pi^\rho + P_{\gamma \tilde{\gamma}} \hat{\Pi}^{\gamma} \nabla \lambda^\gamma ,
\]

(B.8)

\[
Q \left( -P_{\gamma \tilde{\gamma}} \lambda^\rho \omega_\beta C^\beta \gamma \Pi^\gamma \Pi^\gamma \right) = -\lambda^\rho (\nabla_\rho P_{\gamma \tilde{\gamma}}) \lambda^\alpha \omega_\beta C^\beta \gamma \Pi^\gamma - P_{\gamma \tilde{\gamma}} \lambda^\alpha \lambda^\rho \omega_\sigma C^\sigma \beta C^\beta \gamma \Pi^\gamma
- P_{\gamma \tilde{\gamma}} \lambda^\alpha \Pi^a (\gamma_a)^{\alpha \beta} \Pi^\beta + P_{\gamma \tilde{\gamma}} \lambda^\alpha \omega_\beta \lambda^\rho \nabla \rho C^\rho \gamma \Pi^\gamma \nabla \lambda^\gamma ,
\]

(B.9)

\[
Q (-w_\alpha \nabla \lambda^\alpha) = -P_{\alpha \tilde{\gamma}} \hat{\Pi}^{\gamma} + \lambda^\sigma w_\rho C^\rho \gamma \Pi^\gamma \nabla \lambda^\alpha + w_\beta \lambda^\alpha \Pi^a \lambda^\delta R_{\rho \delta \beta} \alpha ,
\]

(B.10)

where in the last equation we used the fact that \( \Omega_{\alpha \beta} = \Pi^A \Omega_{A \alpha \beta} \) and

\[
Q \Omega_{\alpha \beta} = \Pi^A \lambda^\gamma R_{A \gamma \alpha \beta} ,
\]

(B.11)

and \( \lambda^\gamma \lambda^\rho R_{\alpha \gamma \beta} \rho = \lambda^\gamma \lambda^\rho R_{\alpha \gamma \beta} \rho = 0 \) from the BRST nilpotency contraints. Let us simplify the previous expressions, noting that

\[
\nabla_\rho P_{\gamma \tilde{\gamma}} = -P_{\gamma \tilde{\gamma}} (\nabla_\rho P^{\alpha \tilde{\gamma}}) P_{\alpha \gamma} .
\]

Due to the holomorphicity constraint of (3.17) the first line in (B.9) vanishes, while the first term in (B.8) cancels against the second line in (B.9). The last term in (B.8) cancels against the last term in (B.9) plus the first term in (B.10). Finally, using the BRST holomorphicity constraint \( \lambda^\alpha \lambda^\beta (\nabla_\alpha C^\gamma \beta \gamma - P^\delta \gamma R_{\alpha \beta} \gamma) = 0 \), the third line in (B.9) cancels against the last term in (B.10). Hence the result \( \{Q, T\} = 0 \).

Let us check that the right-moving BRST variation vanishes as well. The various terms in (3.3) transform as

\[
\hat{Q} \left( -\frac{1}{2} \Pi^a \Pi^b \eta_{ab} \right) = -\eta_{ab} \Pi^a (\lambda^b \Pi) ,
\]

(B.13)

\[
\hat{Q} \left( -P_{\gamma \tilde{\gamma}} \Pi^\gamma \hat{\Pi}^\gamma \right) = -\hat{\lambda}^\rho \nabla_\rho P_{\gamma \tilde{\gamma}} \Pi^\gamma \hat{\Pi}^\gamma + P_{\gamma \tilde{\gamma}} \Pi^\gamma \nabla \hat{\lambda}^\gamma
\]

(B.14)
\[
\dot{Q} \left( - P_{\gamma \delta} \lambda^\rho w_\beta C^\beta_\gamma \Pi_\lambda \right) = -\hat{\lambda}^\delta \nabla_\delta P_{\gamma \gamma} \lambda^\alpha w_\beta C^\beta_\alpha \Pi_\gamma \\
- P_{\gamma \gamma} \lambda^\alpha w_\beta \hat{\lambda}^\delta \nabla_\delta C^\beta_\alpha \Pi_\gamma \\
- \lambda^\alpha w_\beta C^\beta_\gamma \lambda^\delta \Pi_\alpha \gamma \hat{\Pi}_\delta \beta ,
\]


\[
\dot{Q}(- w_\alpha \nabla \lambda^\alpha) = - w_\alpha \hat{\lambda}^\alpha \lambda^\beta \Pi^\alpha R_{\alpha \beta} \gamma - w_\alpha \hat{\lambda}^\alpha \lambda^\beta \Pi_\gamma R_{\gamma \alpha \beta} \gamma .
\]

The first term \(B.13\) cancels against the second line of \(B.14\). Using the BRST holomorphicity constraint

\[
\nabla_\alpha C^\gamma_\beta = P^\rho_\delta \hat{R}_{\rho \delta \alpha} \gamma - S_{\beta \alpha} \gamma = 0 ,
\]

we can recast the second line of \(B.13\) into

\[
- P_{\gamma \gamma} \lambda^\alpha w_\beta \hat{\lambda}^\delta \nabla_\delta C^\beta_\alpha \Pi_\gamma = \lambda^\alpha w_\beta \hat{\lambda}^\alpha \Pi_\gamma R_{\gamma \alpha \beta} \gamma + \lambda^\alpha w_\beta \hat{\lambda}^\alpha \Pi_\gamma P_{\gamma \gamma} S_{\alpha \beta} \gamma ,
\]

then we see that the first term in \(B.17\) cancels against the last term in \(B.16\) while using the holomorphicity constraint \(R_{\alpha \beta \gamma} = C^\delta_\beta \hat{T}_{\delta \alpha \alpha\gamma}\) we find that the first term in \(B.16\) cancels against the last line in \(B.15\). The remaining terms vanish due to the equations of motion for \(\hat{\lambda}\) in a curved background

\[
\nabla \hat{\lambda}^\alpha = - \hat{\lambda}^\beta P_{\gamma \gamma} (\hat{\Pi}^\gamma + \lambda^\rho w_\sigma C^\sigma_\gamma) \hat{C}^\gamma_\beta - \hat{\lambda}^\beta \lambda^\alpha w_\beta S_{\alpha \beta} \gamma ,
\]

by using the holomorphicity constraint \(\hat{C}^\gamma_\beta - \nabla_\alpha P^\alpha \gamma = 0\) in the gauge \(P^{\beta \gamma} T_{\beta \alpha} \gamma = 0\). Hence we proved that \(\{\dot{Q}, T\} = 0\).
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