P-V criticality of first-order entropy corrected AdS black holes in massive gravity

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Abstract: We consider a massive black hole in four dimensional AdS space and study the effect of thermal fluctuations on the thermodynamics of the black hole. We consider thermal fluctuations as logarithmic correction terms in the entropy. We analyse the effect of logarithmic correction on thermodynamics potentials like Helmholtz and Gibbs which are found decreasing functions. We study critical points and stability and find that presence of logarithmic correction is necessary to have stable phase and critical point.

Keywords: Thermodynamics, Critical behavior, Massive black hole.
1 Introduction

In order to prevent the violation of the second law of thermodynamics one can associate a maximum entropy with black holes [1–5]. Otherwise, when an object with a finite entropy crossed the horizon, the entropy of the universe would spontaneously reduce and, therefore, violets the second law of thermodynamics. The scaling of the mentioned maximum entropy with the black hole horizon area led to the holographic principle [6, 7], which equates the degrees of freedom in any region of space to the degrees of freedom of the boundary. The holographic principle will be corrected near the Planck scale, as well as quantum gravity corrections modify the topology of space-time at this scale [8, 9]. As we know, the holographic principle is inspired by the entropy-area relation, hence the quantum gravity corrections will modify the entropy-area relation. In that case, the original black hole entropy is given by $S_0 = A/4$, where $A$ is the black hole event horizon area. Then, the corrected entropy-area relation of a black hole may be written as $S = S_0 + \alpha \log A + \gamma_1 A^{-1} + \gamma_2 A^{-2} \cdots$, where $\alpha, \gamma_1, \gamma_2 \cdots$, are the coefficients which depend on the black hole parameters. Also, the area dependence has been obtained by the specific models of the quantum gravity. In that case, the logarithmic correction of the form $\alpha \log A$ has already been used to study the corrected thermodynamics of some kinds of black holes such as Gödel like black hole [10]. We should note that the thermodynamics corrections of black holes can be studied by using the non-perturbative quantum general relativity [11], where the conformal blocks of the conformal field theory are used to study the behavior of the density states. The effects of quantum corrections to the black hole thermodynamics have already been studied with help of the Cardy formula [12]. The corrected thermodynamics of a black hole are also studied under the effect of matter fields around a black hole [13–15]. The thermodynamics corrections produced by the string theory have also been studied which are in agreement with the other approaches to quantum gravity [16–19]. The corrections to the thermodynamics of a dilatonic black hole have also been discussed and observed to
have the same universal manner [20]. The partition function of a black hole is very useful to study the corrected thermodynamics of a black hole [21]. It is also possible to use the generalized uncertainty principle to produce thermodynamics corrections, which yields to the logarithmic correction [22, 23], in agreement with the other approaches to quantum gravity. It should be noted that the Einstein equations in the Jacobson formalism are thermodynamics identities [24, 25]. Therefore, a quantum correction to the space-time topology would produce thermal fluctuations in the black holes thermodynamics, and it has the same universal form as expected from the quantum gravitational effects [26–28].

In fact, these corrections have already been considered to study several black geometries. For example, an AdS charged black hole has been studied under the effect of logarithmic correction of the entropy and has been found that the thermodynamics of the AdS black hole is modified due to the thermal fluctuations [29]. The effect of thermal fluctuations on the thermodynamics a black Saturn have also been studied in the Ref. [30]. It has been found that the thermal fluctuations do not have any major effect on the stability of the black Saturn. The thermal fluctuations for a modified Hayward black hole have been studied, where it has been found that thermal fluctuations reduce the pressure and internal energy of the Hayward black hole [31]. The effect of thermal fluctuations on the thermodynamics of a charged dilatonic black saturn has also been studied [32]. It was stated that the thermal fluctuations can be studied either using a conformal field theory or using the fluctuations in the energy of this system. However, it has been found that the fluctuations in the energy and the conformal field theory produce the same results for a charged dilatonic black saturn. This result may differ for the other black objects. Thermodynamics of a small singly spinning Kerr-AdS black hole under the effects of thermal fluctuations has been studied recently [33] with the conclusion that the logarithmic correction becomes important when the size of the black hole is sufficient small, which enable us to test the effects of quantum fluctuations on the black holes by analyzing the effects of thermal fluctuations for example on dumb holes (the analogous for black holes) to obtain the correct coefficient for the correction terms [34]. Such corrections may affect the critical behaviors of black object, for example a dyonic charged anti-de Sitter black hole, which is holographic dual of a Van der Waals fluid [35], is considered in the Ref. [36] where logarithm-corrected thermodynamics is investigated with the result that holographic picture is still valid. However, the van der Waals phase transitions of charged black holes in massive gravity without any order correction is discussed in Ref. [37]. An important application of such logarithmic correction can be found as the study of quark-gluon plasma properties by using AdS/CFT correspondence [38–41]. It may, for example, affect the shear viscosity to entropy ratio [42].

Massive gravity, overcoming its traditional problems, has found a resurgence of interest due to recent progress [43], yielding an avenue for addressing the cosmological constant naturalness problem. The possibility of a massive graviton has been studied first by Fierz and Pauli [44, 45]. Further, van Dam and Veltman [46] and Zakharov [47] had found the linear theory coupled to a source which discuss the curious fact that the theory makes predictions different from those of linear gravity theory even in the limit as the graviton mass goes to zero. Later, some specific nonlinear massive gravity theories have been studied.
which possess a ghostlike instability, known as the Boulware-Deser ghost. The significant progress has been made in construction of the massive gravity theories without such instability [50, 51]. The most straightforward way to construct the massive gravity theories is to simply add a mass term to the GR action, giving the graviton a mass in such a way that GR is recovered as mass vanishes. Recently, a charged BTZ black holes in the context of massive gravity’s rainbow has been studied [52]. The massive BTZ black holes in the presence of Maxwell and Born-Infeld electrodynamics in asymptotically (A)dS spacetimes is studied [53]. More recently, the higher order correction of the entropy and the thermodynamical properties of Schwarzschild-Beltrami-de Sitter black hole are studied [54]. In fact, the $P-V$ criticality of charged black holes in Gauss-Bonnet-massive gravity is also presented [55]. The van der Waals like phase transition [56] and $P-V$ criticality of AdS black holes in a general framework [57] are recently discussed.

Now, we would like to obtain the effect of the first-order (logarithmic) corrected entropy on the thermodynamics and $P-V$ criticality of black holes in AdS space-time of massive gravity. For this purpose, we consider the 4-dimensional charged black hole in massive gravity with a negative cosmological constant and discuss the effect of first-order correction on various thermodynamics quantities. For example, we derive the entropy, Hawking temperature, Helmholtz function, internal energy, pressure, enthalpy and Gibbs free energy. We analyse the Helmholtz free energy with respect to correction coefficient $\alpha$, which confirms that the effect of the logarithmic correction is important at small $r_+$ (or high temperature) and there exists a critical radius for which Helmholtz free energy vanishes. Also, we show that the logarithmic correction has no important effect on the pressure of the black hole with large event horizon radius. In fact, the internal energy, enthalpy and Gibbs free energy are found a decreasing function of correction parameter. We further discuss the holographic duality of logarithmic corrected AdS black hole in massive gravity with Van der Waals fluid for the large black hole and find that the thermal fluctuations have no important effect. In order to study the effect of thermal fluctuations on the critical points, we analyse $P-V$ behavior of the black hole. We discuss the effect of thermal fluctuations in view of critical point and stability of the model. From the plot, we find that the logarithmic correction will be helpful to remove instability of the black hole. For the stability of the model, we obtain a necessary requirement that the trace of Hessian matrix of the Helmholtz free energy must be non-negative.

This paper is organized as follow. In the next section, we recall the black holes of AdS space-time in massive gravity. In section 3, we introduce logarithmic corrected entropy as leading order of thermal fluctuations. In section 4, we discuss about holographic dual picture of the black hole. In section 5, we study critical point and stability of the black hole. Finally, in section 6, we discuss conclusion and summarize the results.
2 AdS black holes in massive gravity

Let us consider the following action for (3+1)-dimensional massive gravity with a Maxwell field
\[ S = \frac{1}{k^2} \int d^4x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F^2 + m^2 \sum_i c_i U_i \right], \quad (2.1) \]
where \( \Lambda = -3/l^2 \) is the cosmological constant and \( k = 1, 0, \) or \( 1, \) correspond to a sphere, Ricci flat, or hyperbolic horizon for the black hole, respectively. Here \( F_{\mu \nu} \) is the Maxwell field-strength tensor, \( c_i \) are constants, and \( U_i \) are symmetric polynomials of the eigenvalues of the matrix \( \sqrt{g^{\alpha \beta}} f_{\alpha \nu} \), where \( f_{\mu \nu} \) is a fixed symmetric tensor.

The action admits a static black hole solution with the space-time metric and reference metric as
\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 h_{ij} dx^i dx^j, \quad i, j = 1, 2. \quad (2.2) \]
Here, \( h_{ij} dx^i dx^j \) is the line element for an Einstein space with constant curvature. The metric function \( f(r) \) in terms of electric charge \( q \) is written by
\[ f(r) = k + \frac{r^2}{l^2} - \frac{m_0}{r} + \frac{q^2}{4r^2} + \frac{m^2 c_1 r^2}{2} + m^2 c_2. \quad (2.3) \]
The black hole horizon can be determined by setting \( f(r)|_{r=r_+} = 0 \), hence the mass parameter \( m_0 \) which is related to the total mass of the black hole is given by
\[ m_0 = (k + m^2 c_2) r_+ + \frac{r_+^3}{l^2} + \frac{q^2}{4r_+} + \frac{m^2 c_1 r_+^2}{2}. \quad (2.4) \]
where outer horizon \( r_+ \) is largest real root of the equation \( f(r) = 0 \), ie,
\[ k + \frac{r^2}{l^2} - \frac{m_0}{r} + \frac{q^2}{4r^2} + \frac{m^2 c_1 r^2}{2} + m^2 c_2 = 0. \quad (2.5) \]
For example by choosing the parameter as \( m_0 = 2, c_1 = 1, c_2 = 1, k = 1, l = 1, m = 0.2, \) and \( q = 1 \) one can obtain \( r_- = 0.1346110283 \) while \( r_+ = 0.9126757206 \) together two complex roots. There is also a chemical potential corresponding to the electrical charge \( q \) given by,
\[ \mu = \frac{q}{r_+}. \quad (2.6) \]

3 First-order corrected thermodynamics

The first order corrected entropy is given by [26],
\[ S = S_0 - \frac{\alpha}{2} \log(S_0 T_H^2), \quad (3.1) \]
where \( \alpha \) is a constant having dimension of length and the zeroth order entropy \( S_0 \) is given by [59],
\[ S_0 = \pi r_+^2. \quad (3.2) \]
Using the definition of Hawking temperature with relation to the surface gravity on the outer horizon \( r_+ \) \cite{58},

\[
T_H = \frac{1}{4\pi} \left[ \frac{k}{r_+} + \frac{3r_+}{l^2} - \frac{1}{4} \frac{q^2}{r_+^3} + m^2 c_1 + \frac{m^2 c_2}{r_+} \right].
\] (3.3)

Exploiting relations (3.1) and (3.3), the corrected entropy is given by,

\[
S = \pi r_+^2 - \alpha \log \left[ \frac{1}{4\sqrt{\pi}} \left( k + 3 \frac{r_+^2}{l^2} - \frac{q^2}{4r_+^3} + m^2 c_1 r_+ + m^2 c_2 \right) \right],
\] (3.4)

Using the entropy and temperature, we can find the Helmholtz function,

\[
F = -\int SdT_H,
\] (3.5)

as follow

\[
F = k \frac{r_+^2}{4} - \frac{r_+^3}{4l^2} + \frac{3q^2}{16r_+} + \frac{1}{4} m^2 c_2 r_+ + F_\alpha,
\] (3.6)

where

\[
F_\alpha = \frac{\alpha}{24\pi l^2} \left( \frac{l^2 q^2}{r_+^3} - 36r_+ \right) + \frac{\alpha}{16\pi l^2} \left( 12r_+ - \frac{q^2 l^2}{r_+^3} + \frac{4k l^2}{r_+} + \frac{4c_2 l^2 m^2}{r_+} \right) \times \log \left[ \frac{1}{4\sqrt{\pi}} \left( k + c_2 m^2 - \frac{q^2}{4r_+^3} + m^2 c_1 r_+ + \frac{3r_+^2}{l^2} \right) \right] + \frac{\alpha c_1 m^2}{4\pi} \log \left[ \frac{12r_+}{l_p} - \frac{q^2 l^2}{r_+^3 l_p} + \frac{4 l^2 k}{r_+ l_p} + \frac{4 l^2 m^2 (c_2 + c_1 r_+)}{r_+ l_p} \right],
\] (3.7)

and \( l_p \) is some constant of integration and has dimension as length. In the Fig. 1 we draw Helmholtz free energy in terms of horizon radius with variation of correction coefficient \( \alpha \).

As expected, \( F_\alpha \to 0 \) as \( r_+ \gg 1 \), which means that the effect of the logarithmic correction is important at small \( r_+ \). In the right plot of the Fig. 1, we can see the uncharged \((q = 0)\) case, for which the effect of logarithmic correction becomes significant at high temperature (infinitesimal \( r_+ \)). It should be noted that the cases of \( k = 0 \) and \( k = \pm 1 \) lead to the similar result. Also, the negative values of \( c_1 \) and \( c_2 \) have no important effect. Left plot of the Fig. 1 shows that there is a critical radius \( r_c \) where \( F_\alpha = F_{-\alpha} = 0 \) and we have \( r_- \leq r_c \leq r_+ \), where equality holds for the extremal black hole \((m_0 \approx q \text{ with } r_+ \approx 0.4)\). It should be noted that the value of the event horizon depends on the value of \( m_0 \).

Also, the second term of the rhs of Eq. (3.6) corresponds to an ordinary AdS black hole with thermodynamics pressure related to the cosmological constant,

\[
P_\Lambda(q = m = \alpha = 0) = -\frac{\Lambda}{16\pi} = \frac{3}{16\pi} \frac{1}{l^2},
\] (3.8)
Figure 1. Helmholtz free energy in terms of $r_+$ for $m = 0.2$ and we set unit values for all other parameters. $\alpha = 0, 1, -1$ are denoted by solid red, dash green, and dotted blue, respectively. Left plot with $q = 1$ and right plot with $q = 0$.

and thermodynamic volume,

$$V = \frac{4}{3} \pi r_+^3.$$  \hfill (3.9)

In order to calculate the internal energy, we use the well-known thermodynamics relation $E = F + TS$, and obtain

$$E = k \frac{r_+}{2} + \frac{r_+^3}{2l^2} + \frac{q^2}{8r_+} + \frac{1}{2} m^2 c_2 r_+ + \frac{1}{4} m^2 c_1 r_+^2 + E_\alpha$$  \hfill (3.10)

where

$$E_\alpha = \frac{\alpha c_1 m^2}{4\pi} \log \left[ \frac{16 \sqrt{\pi} l^2}{l_p} \left( \frac{12r_+^4 - q^2 l^2 + 4kl^2 r_+^2 + 4m^2 l^2 r_+^2 (c_2 + c_1 r_+)}{4kl^2 r_+^3 + 4c_2 m^2 l^2 r_+^3 - q^2 l^2 r_+ + 4m^2 c_1 l^2 r_+^3 + 12r_+^2} \right) \right] + \frac{\alpha}{24 \pi l^2} \left( \frac{q^2}{r_+^3} - 36r_+ \right).$$  \hfill (3.11)

It is clear that the correction parameter $\alpha$ decreases the value of the internal energy for all cases of $k = 0, \pm 1$, i.e., the charged, uncharged and extremal black hole, respectively.

As we know, modified pressure due to thermal fluctuation can be obtained using the derivative of the Helmholtz function with respect to the volume,

$$P = - \left( \frac{\partial F}{\partial V} \right)_T$$  \hfill (3.12)

which gives

$$P = - \frac{k}{16 \pi r_+^2} + \frac{3}{16 \pi l^2} + \frac{3q^2}{64 \pi r_+^4} - \frac{m^2 c_2}{16 \pi r_+^4} + P_\alpha,$$  \hfill (3.13)
where

\[ P_\alpha = \frac{3\alpha}{8\pi^2 r_+^4 l^2} + \frac{\alpha q^2}{32\pi^2 r_+^6 l^2} - \frac{\alpha}{64\pi^2 r_+^2 l^2} \left( \frac{12}{r_+^2} \frac{q^2 l^2}{r_+^2} - \frac{4k l^2}{r_+^2} - \frac{4c_2 l^2 m^2}{r_+^2} \right) \times \log \left[ \frac{1}{4\sqrt{\pi}} \left( k + c_2 m^2 - \frac{q^2}{4r_+^2} + m^2 c_1 r_+ + \frac{3r_+^2}{l^2} \right) \right] \]

As expected, pressure is decreasing function of \( r_+ \) while it is increasing function of \( \alpha \) for small radius. We find that logarithmic correction has no important effect on the pressure of the black hole with large event horizon radius.

Also, we can obtain enthalpy as,

\[ H = E + PV = \frac{3r_+^3}{4 l^2} + \frac{5}{12} kr_+ + \frac{3}{16} r_+^2 + \frac{5}{12} c_2 r_+ + \frac{1}{4} m^2 c_1 r_+ + H_\alpha, \quad (3.15) \]

where

\[ H_\alpha = \frac{\alpha q^2}{12\pi r_+^2} - \frac{\alpha r_+}{\pi l^2} - \frac{\alpha c_1 m^2 r_+}{12\pi} \left( \frac{12r_+^4 + l^2(3q^2 - 4kr_+^2 - 4c_2 m^2 r_+^2)}{12r_+^4 - q^2 l^2 r_+ + 4l^2 r_+^4 (k + m^2 c_2 + m^2 c_1 r_+)} \right) \]

\[ + \frac{\alpha c_1 m^2}{4\pi} \log \left[ \frac{16\sqrt{\pi} l^2}{l_p} \left( \frac{12r_+^4 - q^2 l^2 r_+ + 4l^2 r_+^4 (k + m^2 c_2 + m^2 c_1 r_+)}{4kl^2 r_+^3 + 4c_2 m^2 l^2 r_+^3 - q^2 l^2 r_+ + 4m^2 c_1 l^2 r_+^3 + 12r_+^5} \right) \right] \]

\[ - \frac{\alpha r_+}{24\pi l^2} \left( \frac{12r_+^4 - q^2 l^2 r_+ + 4l^2 r_+^4 (k + m^2 c_2 + m^2 c_1 r_+)}{12r_+^4 - q^2 l^2 r_+ + 4l^2 r_+^4 (k + m^2 c_2 + m^2 c_1 r_+)} \right) \]

\[ - \frac{\alpha c_1 m^2}{48\pi l^2} \left( 12 + \frac{3q^2 l^2}{r_+^2} - \frac{4c_2 m^2}{r_+^2} \right) \log \left[ \frac{1}{4\sqrt{\pi}} \left( k + c_2 m^2 - \frac{q^2}{4r_+^2} + m^2 c_1 r_+ + \frac{3r_+^2}{l^2} \right) \right] \]

\[ + \frac{m^2 c_1 r_+}{4\pi} \frac{3r_+^2}{l^2} \right) \right]. \quad (3.16) \]

We find that enthalpy in decreasing function of \( \alpha \) as well.

Gibbs free energy using the relation \( G = H - TS = F + PV \) is obtained as,

\[ G = \frac{1}{6} kr_+ + \frac{q^2}{4r_+} + \frac{1}{6} m^2 c_2 r_+ + G_\alpha, \quad (3.17) \]

where

\[ G_\alpha = \frac{\alpha q^2}{12\pi r_+^2} - \frac{\alpha r_+}{\pi l^2} - \frac{\alpha c_1 m^2 r_+}{12\pi} \left( \frac{12r_+^4 + l^2(3q^2 - 4kr_+^2 - 4c_2 m^2 r_+^2)}{12r_+^4 - q^2 l^2 r_+ + 4l^2 r_+^4 (k + m^2 c_2 + m^2 c_1 r_+)} \right) \]

\[ + \frac{\alpha c_1 m^2}{4\pi} \log \left[ \frac{12r_+^4 - q^2 l^2 r_+ + 4l^2 r_+^4 (k + m^2 c_2 + m^2 c_1 r_+)}{4kl^2 r_+^3 + 4c_2 m^2 l^2 r_+^3 - q^2 l^2 r_+ + 4m^2 c_1 l^2 r_+^3 + 12r_+^5} \right] \]

\[ - \frac{3q^2 l^2}{r_+^2} - \frac{8kl^2}{r_+^2} + \frac{8c_2 m^2}{r_+^2} \log \left[ \frac{1}{4\sqrt{\pi}} \left( k + c_2 m^2 - \frac{q^2}{4r_+^2} + m^2 c_1 r_+ + \frac{3r_+^2}{l^2} \right) \right] \]

\[ - \frac{\alpha c_1 m^2}{24\pi l^2} \left( \frac{l^2 q^2 - 4r_+^2 (k + c_2 m^2 + 3r_+^2)}{8r_+^2 (4r_+^2 - 4m^2 c_1 r_+^2)} \right) \]

\[ + \frac{m^2 c_1 r_+}{4\pi} \frac{3r_+^2}{l^2} \right) \right]. \quad (3.18) \]

which is decreasing function of \( \alpha \) like other thermodynamics potentials.
4 Holographic duality

It will be interesting if AdS black hole in massive gravity has a holographic dual of the form of Van der Waals fluid with the following equation of state

\[(P_W + \frac{a}{V^2})(V - b) = T,\]  

where we assumed $K_B = 1$ (unit of Boltzmann constant). Also, $a$ and $b$ are some positive constants in which the constant $a$ parameterizes the strength of the intermolecular interactions, while the constant $b$ accounts for the volume excluded owing to the finite size of molecules in fluid. If $a$ and $b$ are both set to zero, the equation of state for an ideal gas can be recovered. It means that

\[P_W = \frac{T}{V - b} - \frac{a}{V^2}.\]  

Now, AdS black hole in massive gravity with logarithmic correction is holographic dual of Van der Waals fluid if $P = P_W$, where $P$ given by the equation (3.13). By using numerical analysis we find that the mentioned duality holds for the large black hole ($V \gg 1$) and therefore thermal fluctuations have no important effect in this case. In the Fig. 2, we draw $\Delta P = P - P_W$ and find some regions where $\Delta P = 0$ corresponding to the large $V$. In this limit, the value of $\alpha$ is not important and thermal fluctuations have no key role to violate holographic dual picture. There exists a divergency also at critical volume. Hence, it is possible to have dual Van der Waals fluid in presence of logarithmic correction. In order to find the effect of thermal fluctuations on the critical points, we should analyze $P - V$ behavior of the black hole.

\[\Delta P = P - P_W\]  

in terms of $V$ for $m = 0.2, a = b = 4$ and $l = 2$, and we set unit values for all other parameters. Blue dashed line corresponding to $q = 1$ while solid red line corresponding to $q = 0$.

Therefore, we can study $P - V$ criticality via the following relations:

\[\left(\frac{\partial P}{\partial V}\right)_{T=T_c} = 0,\]

\[\left(\frac{\partial^2 P}{\partial V^2}\right)_{T=T_c} = 0.\]  

(4.3)
By using the relations (3.9) and (3.13), we can draw $P$ in terms of $V$ as illustrated by the Fig. 3. It is clear that critical point exists also in presence of logarithmic correction. Typical behavior of $P$ for the selected values of parameter shows critical point at $V \approx 3.5$ which means $T_c \approx 0.3$. At the $\alpha = 0$ limit, one can obtain the following condition to have the first condition of (4.3):

$$V_c = \frac{\pi q^3 \sqrt{6}}{(c_2 m^2 + k)^{\frac{3}{2}}}.$$  \hspace{1cm} (4.4)

while the second condition gives,

$$V_c = \frac{7\sqrt{35}}{25} \frac{\pi q^3 \sqrt{6}}{(c_2 m^2 + k)^{\frac{3}{2}}}.$$  \hspace{1cm} (4.5)

It is clear that both equations (4.4) and (4.5) never satisfy simultaneously. It means that without thermal fluctuations there is no critical point.

Figure 3. Pressure in terms of $V$ for $m = 0.2$, $q = 0.5$, and we set unit values for all other parameters. Blue dashed line corresponding to $\alpha = 1$, green dotted line corresponding to $\alpha = 0.6$, orange dash dotted line corresponding to $\alpha = 0.2$, solid red line corresponding to $\alpha = 0$.

5 Critical points and stability

As illustrated in the previous section, there is no critical point for AdS black hole of massive gravity in absence of logarithmic correction. Hence, we should consider the effect of thermal fluctuations to obtain the critical point and study the stability of the model. The first step is to study the specific heat which is given by,

$$C = T \left( \frac{dS}{dT} \right),$$  \hspace{1cm} (5.1)

which further yields to,

$$C = 2 \pi r_+^2 \frac{-12r_+^4 - 4c_1 l^2 m^2 r_+^4 - 4 (c_2 m^2 + k) l^2 r_+^2 + l^2 q^2}{-12r_+^4 + 4 l^2 (c_2 m^2 + k) r_+^2 - 3 q^2 l^2} + C_\alpha.$$  \hspace{1cm} (5.2)
where
\[
C_\alpha = \frac{(4c_1l^2m^2r_+^3 + 2q^2 l^2 + 24r_+^4)\alpha}{-12r_+^2 + 4l^2(c_2m^2 + k)r_+^3 - 3q^2 l^2}.
\] (5.3)

In the plots of the Fig. 4, we can see the typical behavior of the specific heat at any space curvatures \(k = 0, \pm 1\). We can see that some negative specific heat with \(\alpha = 0\) and \(\alpha > 0\), but specific heat is completely positive with negative \(\alpha\). It means that the logarithmic correction can remove instability of the black hole. For the positive correction coefficient \((\alpha = 1)\), the black hole has positive specific heat for \(r_+ > r_0\), where \(r_0\) gives the zero of the specific heat.

**Figure 4.** Specific heat in terms of \(r_+\) for \(m = 0.2\) and all possible values of \(k\). We set unit values for all other parameters. Green dashed line corresponding to \(\alpha = 1\), blue dotted line corresponding to \(\alpha = -1\), and solid red line corresponding to \(\alpha = 0\).

The important point is that for the charged black holes with chemical potential, the sign of specific heat is not enough to conclude stability of the model and more important test is required using of Hessian matrix of the Helmholtz free energy which we denoted by \(\mathcal{H}\), and given by the following matrix,

\[
\left(\begin{array}{cc}
\frac{\partial^2 F}{\partial T^2} & \frac{\partial^2 F}{\partial T\mu} \\
\frac{\partial^2 F}{\partial \mu T} & \frac{\partial^2 F}{\partial \mu^2}
\end{array}\right) = \mathcal{H}.
\] (5.4)

By using the relations (3.3), (3.7) and (3.9), one can find that the determinant of the matrix \(\mathcal{H}\) vanishes,

\[
\left(\frac{\partial^2 F}{\partial T^2}\right)\left(\frac{\partial^2 F}{\partial \mu^2}\right) = \left(\frac{\partial^2 F}{\partial T\mu}\right)\left(\frac{\partial^2 F}{\partial \mu T}\right).
\] (5.5)

It means that one of the eigenvalues is zero and we should consider the other one which is the trace of the matrix (5.4) given by,

\[
\tau \equiv Tr(\mathcal{H}) = \left(\frac{\partial^2 F}{\partial T^2}\right) + \left(\frac{\partial^2 F}{\partial \mu^2}\right).
\] (5.6)

Now, crucial condition to have stability is \(\tau \geq 0\). In the Fig. 5, we can see the behavior of \(\tau\) with \(r_+\). It is clear that positive region exists only for the case of positive \(\alpha\). Hence, we find that the presence of the logarithmic correction of the form (3.1) with positive \(\alpha\) is
essential to have critical point and stability at least for small values of the $r_+$. For example, we examine the special case with our selected values of parameters. In the case of $c_1 = 1$, $c_2 = 1$, $k = 1$, $l = 1$, $m = 0.2$, $q = 1$ we can see that stability exists approximately for $r_+ \leq 0.625$. These values of horizon radius obtained for $m_0 \leq 1.3$. Hence, we find suitable condition on the black hole mass where it is in the stable phase.

Figure 5. Trace of Hessian in terms of $r_+$ for $m = 0.2$ and we set unit values for all other parameters. Green dashed line corresponding to $\alpha = 1$, blue dotted line corresponding to $\alpha = -1$, and solid red line corresponding to $\alpha = 0$.

6 Conclusions

In this paper, we have considered a charged black hole solutions in 4-dimensional massive gravity with a negative cosmological constant and studied the first order corrected thermodynamics and phase structure of the black hole solutions. In particular, we have computed the first-order corrected entropy, Hawking temperature, Helmholtz function, internal energy, pressure, enthalpy and Gibbs free energy. We have plotted the Helmholtz free energy in terms of horizon radius with variation of correction coefficient $\alpha$, which confirms that the effect of the logarithmic correction is important at small $r_+$ (or high temperature) and, also, there exists a critical radius for which Helmholtz free energy vanishes. We found that the logarithmic correction has no important effect on the pressure of the black hole with large event horizon radius. However, the internal energy, enthalpy and Gibbs free energy are the decreasing function of correction parameter. Furthermore, we show that AdS black hole in massive gravity with logarithmic correction is holographic dual of Van der Waals fluid for the large black hole and, consequently, found that thermal fluctuations have no important effect. In order to find effect of thermal fluctuations on the critical points, we have analyzed $P - V$ behavior of the black hole. Furthermore, we have studied the effect of thermal fluctuations in order to obtain critical point and stability of the model. From the graphical analysis, we have found that the logarithmic correction can be used to remove instability of the black hole. However, for the stability of the model, we found remarkably that trace of Hessian matrix of the Helmholtz free energy must be non-negative.
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