Solvability of Time-Varying Infinite-Dimensional Linear Port-Hamiltonian Systems

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Abstract—Thirty years after the introduction of port-Hamiltonian systems, interest in this system class still remains high among systems and control researchers. Very recently, Jacob and Laars obtained strong results on the solvability and well-posedness of time-varying linear port-Hamiltonian systems with boundary control and boundary observation. In this article, we complement their results by discussing the solvability of linear, infinite-dimensional time-varying port-Hamiltonian systems not necessarily of boundary control type. The theory is illustrated on a system with a delay component in the state dynamics.

Index Terms—Evolution family, hilbert space, linear systems, port-Hamiltonian systems, time-varying systems.

I. INTRODUCTION

Ever since they were introduced by Maschke and van der Schaft in [1], the study of port-Hamiltonian systems has inspired intensive research. For a nice overview of the fundamentals and early history of port-Hamiltonian systems, see the introduction in [2]. In his Ph.D. thesis [3], Golo started the study of infinite-dimensional Dirac structures, which describe the structure of a given port-Hamiltonian system. The methods of the semigroup theory were first put to use in the study of port-Hamiltonian systems by Gorrec et al. in [4]. This direction was further developed in [5] and [6], the latter being the first treatment of a port-Hamiltonian system with a spatial dimension larger than one. Skrepek [7] studied boundary controlled port-Hamiltonian systems on a high-dimensional spatial domain. Also very recently, the authors in [8] and [9] studied solvability, well-posedness, and exponential stability for time-varying boundary controlled port-Hamiltonian systems, hence starting the study of time-varying infinite-dimensional port-Hamiltonian systems. In the aforementioned references, one can find applications to the string and wave equation, beam and plate equations, and Maxwell’s equations.

In this short note, we will study the solvability of time-varying, linear but in general infinite-dimensional, port-Hamiltonian systems, which are not necessarily of boundary control type. To the author’s knowledge, this has not yet been researched.

In Section VI, we will illustrate the theory using a time-varying finite-dimensional system with a delay \( \tau > 0 \) in the state

\[
\begin{align*}
\dot{z}(t) &= (J - R)H(t)z(t) - A_I H(t - \tau)z(t - \tau) + Ec(t) \\
\dot{f}(t) &= E^* H(t)z(t), \quad t \geq 0, \quad (Hx)[t, t] = w_0 \quad \text{given} \quad (1)
\end{align*}
\]

where, for simplicity, \( J = -J^*, R \geq 0, A_I, E, H(t) > 0 \) are all bounded operators, with some additional properties, which will be described later; \( H > 0 \) means that \( H \) is self-adjoint and coercive: for some \( \gamma > 0 \), all \( z \in \text{dom}(H), \langle Hz, z \rangle \geq \gamma \|z\|^2 \).

In Section II, we introduce the Dirac node, an infinite-dimensional analogue of the connecting matrix \( [\begin{array}{cc} A & B \\ C & D \end{array}] \) of a linear time-invariant system and an operator version of the so-called Dirac structure [5] so that (1) can be written in a form resembling

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{f}(t)
\end{bmatrix} = \begin{bmatrix} A + G(t)P(t)^{-1}B & B \\
B^* & 0 \end{bmatrix} \begin{bmatrix} P(t)x(t) \\
e(t(t)
\end{bmatrix}, \quad t \geq 0. \quad (2)
\]

We also touch on the solvability properties of a Dirac node.

In Section III, we make a precise definition of a time-varying port-Hamiltonian system, and define its external scattering representation, giving some of its properties. In Section IV, we apply the theory of time-varying well-posed systems to the scattering representation, and Section V ends the theoretical part with discussion on the solvability of the impedance representation (2). Our main contributions are Sections II, V, and VI.

II. DIRAC NODES AND SYSTEM NODES

Let the space \( X \), the space of external efforts \( E \), and the space of external flows \( F \) all be (complex or real) Hilbert spaces. Assume that \( E \) and \( F \) are dual, and let \( \psi : F \rightarrow E \) be the associated unitary duality map, which satisfies

\[
\langle \psi f, e \rangle_E = \langle f, e \rangle_F, \quad f \in F, \quad e \in E.
\]

Definition II.1: By a Dirac node on \((E, X, F)\), we mean an unbounded linear operator

\[
[\begin{array}{cc}
A & B \\
C & D
\end{array}]: \begin{bmatrix} X \\
E
\end{bmatrix} \supset \text{dom} \begin{bmatrix} A & B \\
C & D
\end{bmatrix} \rightarrow \begin{bmatrix} X \\
F
\end{bmatrix}
\]

such that \( \begin{bmatrix} A & B \\
C & D
\end{bmatrix} \) is dissipative on \( [X, E] \) and

\[
\begin{bmatrix} I_x & 0 \\
0 & \beta I_E
\end{bmatrix} + \begin{bmatrix} 0 \\
\psi C & D
\end{bmatrix}
\]

(3)

is injective for some \( \beta \in \mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Re} z > 0\} \).

We next give two examples that show that most of the currently known linear port-Hamiltonian systems are governed by Dirac nodes. First consider the finite-dimensional system

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bf(t), \\
f(t) &= Cz(t) + Df(t).
\end{align*}
\]

Example II.2: Let \( X = \mathbb{K}^n \) and \( E = \mathbb{K} \), and \( F = \mathbb{K}^m \), where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \); then \( \psi : E \rightarrow E \) is the identity matrix. Moreover,\n
\[
\begin{bmatrix} A & B \\
-\psi C & D
\end{bmatrix}
\]

is maximal dissipative if and only if

\[
A + A^* \leq 0, \quad B = C^*, \quad D + D^* \geq 0.
\]
Then, \( I + D \) is invertible, and

\[
\begin{bmatrix}
I_x & 0 \\
0 & I_x
\end{bmatrix}
+ \begin{bmatrix}
0 & \psi C & D \end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I + D
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
= 0
\]

implies \( x = 0 \) so that, moreover, \( (I + D)e = 0 \Rightarrow e = 0 \).

The next example is an abstract class of boundary control sys-

be a boundary triplet

For more background on the example, see [5, Sec. 4] and [11, Sec. 5].

Example II.3: Let \( A_0 \) be a closed, densely defined and symmet-

Letting

\[
\mathfrak{A} := iA_0^*, \quad \mathfrak{B} := \Gamma_1, \quad \mathfrak{C} := -i\Gamma_2
\]

we get that the operator defined by

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
:= \begin{bmatrix}
\mathfrak{A} \\
\psi e
\end{bmatrix}
\begin{bmatrix}
I_x \\
\mathfrak{B}
\end{bmatrix}
\text{dom}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
:= \begin{bmatrix}
I_x \\
\text{dom}(\mathfrak{A})
\end{bmatrix}
\]

is a Dirac node:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1}
= \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1}
\]

and it is easy to see that (3) with \( \beta = 1 \) is injective.

A Dirac node describes the relationship between internal dynamics

and external efforts and flows via

\[
\begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix}
= \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix}, \quad t \geq 0.
\]

We will now see that it does in general not induce a “system” in a very

strong sense. In order for (4) to be solvable, the connecting operator

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

should have the structure of a system node [13, Lemma 4.7.7].

Definition II.4: By a system node on \((E, X, F)\), we mean a closed

linear operator

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
: \begin{bmatrix}
X \\
E
\end{bmatrix}
\supset \text{dom}\left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right)
\rightarrow \begin{bmatrix}
X \\
F
\end{bmatrix}
\]

with the following properties:

1) the operator \( A & B : \begin{bmatrix}
X \\
E
\end{bmatrix}
\supset \text{dom}(A & B) \rightarrow X \) is closed, where

\[
\text{dom}(A & B) = \text{dom}\left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right);
\]

2) the main operator \( A : X \supset \text{dom}(A) \rightarrow X \) defined by

\[
Ax := A & B \begin{bmatrix}
x \\
0
\end{bmatrix}, \quad \begin{bmatrix}
x \\
0
\end{bmatrix} \in \text{dom}\left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right)
\]

generates a \( C_0 \)-semigroup \( T \) on \( X \);

3) for every \( e \in E \), there exists an \( x \in X \), such that \( \begin{bmatrix}
x \\
e
\end{bmatrix} \in \text{dom}\left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right) \).

These three conditions essentially boil down to “A generating a

solution, while at the same time being the most unbounded part

of the node.” There is unfortunately no reason to believe that this should

be the case for a general Dirac node. We have the following solvability

result [13, Th. 4.6.11].

Theorem II.5: Let \( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\) be a system node, let \( e \in H^1_{\text{loc}}(\mathbb{R}_+; \mathfrak{C}) \)

and \( \begin{bmatrix}
x_0 \\
e(0)
\end{bmatrix} \in \text{dom}\left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right) \). Then, there exist \( x \in C^1(\mathbb{R}_+; X) \) and

\[
f \in H^1_{\text{loc}}(\mathbb{R}_+; \mathfrak{F}),
\]

such that (4) holds.

The triple \((e, x, f)\) in Theorem II.5 is called a classical solution of

(4). A system node is impedance passive if it is a Dirac node; see in [14, Th. 4.2 and Lemma 4.3]. It is well-posed if there are \( t > 0 \) and \( K_t \), such that all classical solutions satisfy

\[
\| x(t) \|^2 + \int_0^t \| f(s) \|^2 \, ds \leq K_t \left( \| x(0) \|^2 + \int_0^t \| e(s) \|^2 \, ds \right)
\]

if this inequality holds with \( K_t = 1 \), then the system node is said to be scattering passive [13].

III. TIME-VARYING PORT-HAMILTONIAN SYSTEMS

In order to define the time-varying port-Hamiltonian system

associated to a Dirac node, we introduce two functions \( P, G \), defined on \( \mathbb{R}_+ \), with values being bounded operators on the state space, which have the following properties. For all \( t \geq 0 \)

1) \( P(t) = P(t)^* \geq 0 \),

2) \( P(t) \) has an inverse in \( \mathcal{L}(X) \), and for all \( z \in X : \)

3) \( P(z)z \in C^2(\mathbb{R}_+; X) \) and

4) \( P(z)z \in C^1(\mathbb{R}_+; X) \).

(6)

Here, \( \mathcal{L}(X) \) is the space of bounded linear operators on \( X \).

Definition III.1: Let \( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\) be a Dirac node and let \( P(\cdot) \) and \( G(\cdot) \) be as mentioned.

By the associated time-varying \( P \)-\( G \)-system, we mean the system

\[
\begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix}
= \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
G(t)P(t)^{-1} \\
0
\end{bmatrix}
\begin{bmatrix}
P(t)x(t) \\
e(t)
\end{bmatrix}, \quad t \geq 0.
\]

(7)

By a classical solution of (7) on \( \mathbb{R}_+ \), we mean a triple \((e, x, f)\) in \( H^1_{\text{loc}}(\mathbb{R}_+; \mathfrak{C}) \times C^1(\mathbb{R}_+; X) \times H^1_{\text{loc}}(\mathbb{R}_+; \mathfrak{F}) \), such that \( \begin{bmatrix}
P(t)x(t) \\
e(t)
\end{bmatrix} \in \text{dom}\left( \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \right) \) and (7) holds for all \( t \geq 0 \).

The multiplicative perturbation \( P(t) \) before \( x(t) \) in (7) encodes time-

varying physical parameters of the system. The additive perturbation

\( G(t)P(t)^{-1} \) collects time derivatives of physical parameters (which do not appear in the time invariant case), and in the case of a wave equation, it can also encode a time-varying damper inside of the spatial domain. In Theorem V.2 later, \( G \) will be used in a perturbation argument, in order to prove an existence result for classical solutions of (7).

Proposition III.2: Assuming (6), every classical solution of (7) is

uniquely determined by the initial state \( x(0) \) and the effort signal \( e \). For such solutions, the power inequality

\[
\frac{d}{dt} H(x, t) \leq 2 \text{Re} \left( (f(t), e(t))_{\mathfrak{F}, \mathfrak{E}} + \left( \bar{P}(t)x(t), x(t) \right) \right) + 2 \text{Re} \left( P(t)x(t), G(t)x(t) \right), \quad t \geq 0
\]

(8)
holds, with equality if \[ \begin{bmatrix} A & B \\ -\psi C & D \end{bmatrix} \subset \begin{bmatrix} A & B \\ -\psi C & D \end{bmatrix}^\ast. \]

**Proof:** The power inequality will be motivated after Theorem IV.2. In order to obtain uniqueness, we first observe that the main operator \( A \) of a Dirac node is dissipative: for \( x \in \text{dom}(A) \):

\[ 2\text{Re} \langle Ax, x \rangle = 2\text{Re} \left( \begin{bmatrix} A & B \\ -\psi C & D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right) \leq 0. \]

From (6) and the uniform boundedness principle, it follows that \( P(\cdot), P(\cdot)^{-1}, P(\cdot)^{\ast} \) are all uniformly bounded on the compact subinterval \([0, T] \subseteq \mathbb{R}_+\), where \( T > 0 \) is fixed arbitrarily. Denote a common bound for these by \( U \).

Now let \((e, x, f)\) and \((e, z, g)\) be classical solutions of (7) with \( x(0) = z(0) \) and set \( w := x - z \). Then

\[ w(t) = (AP(t) + G(t))w(t), \quad t \geq 0, \quad w(0) = 0. \]

Next, we define the auxiliary function

\[ v(t) := e^{-\sigma t} H(w(t), t) \geq 0, \quad t \geq 0 \]

where \( \sigma := 2U^2 + U^2 \). Then, \( v(0) = 0 \) and we next prove that \( v(t) \leq 0 \) for all \( t \in [0, T] \); then, \( v(0) = 0 \), i.e., \( x = z \), and consequently, \( f = g \) by (7), on \( \mathbb{R}_+ \). Indeed, for \( t \in [0, T] \)

\[ (P(t)v(t), w(t)) \geq \|w(t)\|^2/\|P(t)^{-1}\|, \quad \text{and then} \]

\[ e^{\sigma t}v(t) = \left( 2\text{Re} \langle P(t)v(t), w(t) \rangle + \langle \dot{P}(t)v(t), w(t) \rangle \right) - \sigma \langle P(t)v(t), w(t) \rangle \]

\[ = (2\text{Re} \langle P(t)v(t), AP(t)v(t) \rangle - \sigma \langle P(t)v(t), w(t) \rangle) + 2\text{Re} \langle P(t)v(t), G(t)v(t) \rangle + \langle \dot{P}(t)v(t), w(t) \rangle \]

is nonpositive due to the dissipativity of \( A \).

Within the port-Hamiltonian framework, the external efforts and flows can be combined into inputs and outputs in a flexible way, and we will next consider the following *external scattering representation* of a port-Hamiltonian system, which has better solvability properties than the *impedance representation* (7). First fix some \( \beta \in \mathbb{C}_+ \), such that (3) is injective. Then, pick the input \( u \) to the system and the output \( y \) as

\[ \begin{bmatrix} u \\ y \end{bmatrix} := \frac{1}{\sqrt{2\text{Re} \beta}} \begin{bmatrix} \beta I & \psi \\ \bar{\beta} & -\bar{\psi} \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} \]

\[ \begin{bmatrix} e \\ f \end{bmatrix} := \frac{1}{\sqrt{2\text{Re} \beta}} \begin{bmatrix} I & I \\ \beta & -\bar{\beta} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}. \]

The reader may verify that the resulting transform of (7) is

\[ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \sqrt{2\text{Re} \beta} A & B \\ C & D \end{bmatrix} \begin{bmatrix} G(t)P(t)^{-1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P(t)x(t) \\ u(t) \end{bmatrix} \]

\[ t \geq 0, \quad \text{where} \]

\[ \begin{bmatrix} \sqrt{2\text{Re} \beta} A & B \\ C & D \end{bmatrix} \in \text{dom}(A) \]

is the usual *external Cayley transform* (or diagonal transform [15, (6.2)]) of the Dirac node \([A^{\ast}, B^{\ast}, C^{\ast}, D^{\ast}] \) of the port-Hamiltonian system do not interact with the external Cayley transform.

**Theorem III.3:** For \([A^{\ast}, B^{\ast}, C^{\ast}, D^{\ast}] \) a Dirac node, \([A^{\ast}, B^{\ast}, C^{\ast}, D^{\ast}]^{\ast} \) is a scattering passive system node. There exist a Hilbert space \( X_{1}^{\ast} \supset X \), and operators \( A_{1}^{\ast} \in \mathcal{L}(X, X_{1}^{\ast}), B^{\ast} \in \mathcal{L}(\mathcal{E}, X_{1}^{\ast}), C^{\ast} : X \supset \text{dom}(C^{\ast}) \to \mathcal{E}, \) and \( D^{\ast} \in \mathcal{L}(\mathcal{E}) \), such that

\[ \begin{bmatrix} A^{\ast} \\ C^{\ast} \\ D^{\ast} \\
B^{\ast} \end{bmatrix} \]

is dissipative in the sense of [16]; hence \([A^{\ast}, B^{\ast}, C^{\ast}, D^{\ast}]^{\ast} \) is a scattering-passive system node by [16, Th. 2.5].

By [13, Lemma 11.1.3], all scattering-passive systems induce well-posed systems, and since we work on Hilbert spaces, \([A^{\ast}, B^{\ast}, C^{\ast}, D^{\ast}]^{\ast} \) is then compatible by [13, Th. 5.1.12], and so, (11) holds.

We will next see how the system (10) can be associated to a *time-varying well-posed system* in the sense of [17].

**IV. Time-Varying Well-Posed Systems**

In the time-invariant case, one observes that (5) expresses that the final state and the output signal are both bounded by the initial state and the input signal, and one then, defines families of bounded operators \( T, \Phi, \Psi, \text{and} F, \) such that \( \begin{bmatrix} T \Phi \Psi \end{bmatrix} \) maps \( \begin{bmatrix} x(0) \\ e_{[0,t]} \end{bmatrix} \) into \( \begin{bmatrix} x(t) \\ f_{[0,t]} \end{bmatrix} \).

In the time-varying case, one gets the following.

**Definition IV.1:** A time-varying well-posed system on \( \mathbb{R}_+ \), with Hilbert input, state, and output spaces \((U, \mathcal{X}, \mathcal{Y})\), is a quadruple of linear operator families defined for \( (t, r) \in \Delta := \{(s, \sigma) \in \mathbb{R}_+^2 \mid s \geq \sigma \}, \)

\[ \begin{align*}
T(t, r) : & \mathcal{X} \to \mathcal{X}, \\
F(t, r) : & L^2(\mathbb{R}_+; U) \to L^2(\mathbb{R}_+; \mathcal{Y}) \\
\Phi(t, r) : & L^2(\mathbb{R}_+; U) \to \mathcal{X}, \\
\Psi(t, r) : & \mathcal{X} \to L^2(\mathbb{R}_+; \mathcal{Y})
\end{align*} \]

boundedly, which have the following additional properties:

1) \( \mathcal{X} \) is an evolution family on \( \mathcal{X} \) with time interval \( \mathbb{R}_+ \);
2) the other families are *causal* in the sense that
\[ \Phi(t, r) = \Phi(t, r) \mathcal{P}_{[t, r]} \]
\[ \Psi(t, r) = \mathcal{P}_{[t, r]} \Psi(t, r) \]
\[ \mathcal{F}(t, r) = \mathcal{P}_{[t, r]} \mathcal{F}(t, r) = \mathcal{F}(t, r) \mathcal{P}_{[0, t]} \]

for \( (t, r) \in \Delta \);
3) all four families are locally uniformly bounded;
they encode the linearity of the system so that for all \( t \geq s \geq r \geq 0 \)
\[
\Phi(t, r) = \Phi(t, s) + \Psi(t, s)\Phi(s, r)
\]
\[
\Psi(t, r) = \Psi(t, s)\Phi(s, r) + \Psi(s, r)
\]
\[
F(t, r) = F(t, s) + F(s, r) + \Psi(t, s)\Phi(s, r).
\]
(12)

By a mild solution of a well-posed system on \( \mathbb{R}_+ \) with initial state \( x_0 \in \mathcal{X} \) at time 0 and input \( u \in L^2_{s.c.}(\mathbb{R}_+, \mathcal{U}) \) we mean the triple \((u, x, y) \in L^2_{s.c.}(\mathbb{R}_+, \mathcal{X}) \times C(\mathbb{R}_+, \mathcal{X}) \times L^2_{s.c.}(\mathbb{R}_+, \mathcal{Y})\)
\[
x(t) = T(t, 0)x_0 + \Phi(t, 0)u
\]
and the definitions of \( P(t)z(t) \) and \( G(t)z(t) \). The following theorem shows that, under assumptions (6), there is a unique classical solution \( x \in C^1(\mathbb{R}_+, \mathcal{X}) \) for all \( z \in \mathcal{X} \).

Theorem II.5: Let \( \Sigma \) be a scattering passive system node and assume (6). Then, there is a time-varying well-posed system with time interval \( \mathbb{R}_+ \), such that the following.

1) For every \( \left[\begin{array}{c} x_0 \\ u \end{array}\right] \in V^u \), there is a unique classical solution \( (u, x, y, \hat{u}) \) of (10) on \( \mathbb{R}_+ \) with \( x(0) = x_0 \) and the given input signal \( u \). The corresponding output satisfies \( y \in H^1_{s.c.}(\mathbb{R}_+, \mathcal{Y}) \), and \((u, x, y, \hat{u}) \) is also the unique mild solution of \( \left[\begin{array}{c} T \\ \Phi \\ F \end{array}\right] \) with \( x(0) = x_0 \) and the given input signal \( u \).

2) Every classical solution of (10) on \( \mathbb{R}_+ \) satisfies the power inequality
\[
\frac{d}{dt} \langle P(t)x(t), x(t) \rangle_{\mathcal{X}} + \text{Re} \left\langle \dot{P}(t)x(t), G(t)x(t) \right\rangle_{\mathcal{Y}} + \|u(t)\|^2_G - \|y(t)\|^2_F, \quad t \geq 0.
\]
(15)

Equality holds in (15) if
\[
\begin{bmatrix} A & B \\ -\psi C & D \end{bmatrix} \in \mathbb{C}^{2 \times 2}
\]

Remark IV.3: Theorem IV.2 and Proposition III.2 remain true if (7) is replaced by a port-Hamiltonian system in “Berlin form,”
\[
\begin{bmatrix} A & B \\ -\psi C & D \end{bmatrix} \in \mathbb{C}^{2 \times 2}
\]
and the same modification is made for (10). Then, (6) can be relaxed to \( P(t)z(t) \in C^1(\mathbb{R}_+, \mathcal{X}) \); see [17, Th. VI.1].

We have the following explicit formulas for the operator families \( \Phi, F, I \), whose existence is given in Theorem IV.2.

Theorem IV.4: Under the assumptions of Theorem IV.2, for all \( x \in P(r)^{-1} \text{dom}(A^+) \)
\[
\Phi(t, r)x_r = s \in C^\infty P(s)\Phi(s, r)x_r, \quad r \leq s \leq t.
\]

If additionally \( B^*: \mathcal{E} \rightarrow \mathcal{X} \) is bounded and \( G(\cdot)^* \in C^1(\mathbb{R}_+, \mathcal{X}) \) for all \( z \in \mathcal{X} \), then
1) for all \( u \in L^2_{s.c.}(\mathbb{R}_+, \mathcal{U}) \)
\[
\Phi(t, r)u = \int_t^r T(t, s)B^*u(s)\, ds, \quad (t, r) \in \Delta
\]
2) for all \( u \in H^1_{s.c.}(\mathbb{R}_+, \mathcal{E}) \)
\[
(F(t, r)u)(s) + C^\infty P(s)\int_t^s T(s, \sigma)B^*u(\sigma)\, d\sigma + D^*u(s), \quad r \leq s \leq t.
\]

The representation formulas in items 1) and 2) of Theorem IV.4 look almost the same in the case where \( B^* \) is unbounded, but they require a discussion on the extrapolation space, and here, we would not benefit from getting into the details of that.

V. Existence of Classical Solutions for Impedance Representations

We can now examine the existence of classical solutions of a port-Hamiltonian system in impedance form (7), where the external effort \( e \) is considered the input and the external flow \( f \) is taken as the output. The question is for which combinations of initial state \( x_0 \) and effort signal \( e \) we can expect a classical solution of (7). The impedance representation is in general not well-posed even in the time-invariant setting [14, Theorem 5.1], and then, the concept “mild solution of (7)” is not defined either. The discussion here is mostly relevant for port-Hamiltonian systems, which are not of boundary control type, because Jacob and Laasri have already obtained better results for the boundary control case in [9].

First define (with the limit taken in \( \mathcal{X} \times L^2_{s.c.}(\mathbb{R}_+, \mathcal{E}) \))
\[
V := \lim_{t \to \sup_{\mathbb{R}_+}} \left[ \sqrt{2\text{Re} \cdot \Psi^T(t, 0) I + F^T(t, 0)P_{[0,t]} \Psi(t, 0) \cdot \Psi(t, 0)} \right]^u.
\]
(16)

Then, we have the following corollary of Theorem IV.2.

Corollary V.1: Under the assumptions of Theorem IV.2, for every \( x_0 \in V \), there is a unique classical solution \( (e, x, f) \) of (7) on \( \mathbb{R}_+ \) with \( x(0) = x_0 \) and the given effort signal \( e \). (The flow satisfies \( f \in H^1_{s.c.}(\mathbb{R}_+, \mathcal{E}) \)).

Proof: By (13) and the definitions of \( V^u \) and \( V \), the condition
\[
\left[\begin{array}{c} x_0 \\ e \end{array}\right] \in V \text{ means that } e \text{ is the effort associated to some smooth mild solution } (u, x, y) \text{ of } \left[\begin{array}{c} T \\ \Psi \\ F \end{array}\right] \text{ with } x(0) = x_0, \text{ via } e = (u + y)/\sqrt{2\text{Re} \cdot \Psi^T \cdot \Psi}.
\]
By Theorem IV.2, \((u, x, y)\) is also a classical solution of (10). Setting
\[
\begin{align*}
  f := \frac{\beta \psi u - \beta \psi y}{\sqrt{2 \text{Re} \beta}} &\in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E})
\end{align*}
\]
we then get that \((e, x, f)\) is a classical solution of (7). Uniqueness was established already in Proposition III.2.

The difficulty is of course to characterize \(\mathcal{V}\). Indeed, computing \(\mathcal{F}\) using Theorem IV.4 requires the evolution family \(\mathcal{T}\), and in general, one has no analytic formula for that.

We next characterize \(\mathcal{V}\) for the important special case of distributed control and observation, leaving the general case as an open problem.

**Theorem V.2:** Assume that \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) is a Dirac node with bounded control, observation, and feedthrough, i.e., \(\mathcal{F} = \mathcal{E}, \psi = I, \forall \in \text{dom}(A) \subseteq \mathcal{E}, \text{ and } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) where \(B \in \mathcal{L}(\mathcal{E}, X), C \in \mathcal{L}(X, \mathcal{F}), \) and \(D \in \mathcal{L}(\mathcal{E})\). Then \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) is a system.

If additionally (6) holds, then
\[
\mathcal{V} = \mathcal{V}^\infty = \left\{ P(0)^{-1}\text{dom}(A) H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E}) \right\}
\]
i.e., for all \(x_0 \in P(0)^{-1}\text{dom}(A)\) and \(\epsilon \in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E})\), there exist unique \(x\) and \(f\), such that \((e, x, f)\) is a classical solution of (7) with \(x(0) = x_0\).

**Proof:** The dissipative operator \(A\) inherits closedness and denseness of its domain from \(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \) which necessarily have these properties due to maximal dissipativity. Moreover, \(A^*\) inherits dissipativity from \(\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^* & -C^* \\ -B^* & D^* \end{bmatrix} \) so that \(A\) generates a contraction semigroup by the Lumer–Phillips theorem. The other system node properties are easily checked.

Due to the boundedness of \(D\), it is easy to find a \(\beta \in \mathbb{C}_+\) such that \((\beta I + D)^{-1} : \mathcal{F} \to \mathcal{E}\) is bounded. A calculation gives the external Cayley transform
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \xrightarrow{\text{Cayley transform}} \begin{bmatrix} A - B(\beta I + D)^{-1}C & \sqrt{2 \text{Re} \beta}B(\beta I + D)^{-1} \\ -\sqrt{2 \text{Re} \beta}(\beta I + D)^{-1}C & \beta I(D I)(\beta I + D)^{-1} \end{bmatrix}
\]
with the same domain as \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) so that
\[
A^* := A - B(\beta I + D)^{-1}C, \quad \text{dom}(A^*) = \text{dom}(A)
\]
is a bounded perturbation of \(A\). From \(\text{dom}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\) being bounded perturbation of \(A\), we have
\[
\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \text{dom}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \iff \begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in P(0)^{-1}\text{dom}(A^*)
\]
and this establishes that \(\mathcal{V}^\infty\) in (14) satisfies (17).

Next, we prove that \(\mathcal{V} = \mathcal{V}^\infty\). First pick \(\begin{bmatrix} x_0 \\ e \end{bmatrix} \in \mathcal{V}\) arbitrarily as in the proof of Corollary VI.1, there exists some classical solution \((u, x, y)\) of (10), with \(x(0) = x_0\) and \(u, y \in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E})\), and moreover, \(e = (u + y)/\sqrt{2 \text{Re} \beta} \in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E})\). Hence, \(\begin{bmatrix} x_0 \\ e \end{bmatrix} \in \mathcal{V}^\infty\), and so \(\mathcal{V} \subset \mathcal{V}^\infty\).

Now let \(\begin{bmatrix} x_0 \\ e \end{bmatrix} \in \mathcal{V}^\infty\) be arbitrary and define \(v := (\beta I + D)e/\sqrt{2 \text{Re} \beta}\) so that \(\begin{bmatrix} x_0 \\ v \end{bmatrix} \in \mathcal{V}^\infty\).

With \(\tilde{G}(t) := G(t) + B(\beta I + D)^{-1}CP(t), \ t \geq 0\)

we get \(\tilde{G}(t)z \in C^1(\mathbb{R}^+; \mathcal{E})\), and so
\[
\dot{x}(t) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tilde{G}(t)P(t)^{-1} \begin{bmatrix} 0 \\ v(t) \end{bmatrix}
\]
has a unique solution \((x, v, w)\) with \(x(0) = x_0\) and \(v, w \in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E})\). However, this \(x\) also solves
\[
\dot{x}(t) = (A^* P(t) + G(t) + B(\beta I + D)^{-1}CP(t))x(t) + B^* v(t)
\]
and we get a classical solution \((e, x, f)\) of (7), because
\[
f(\cdot) := CP(\cdot)x(t) + De(\cdot) \in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E}).
\]

Setting \(u := (\beta e + f)/\sqrt{2 \text{Re} \beta} \in H^1_{\text{loc}}(\mathbb{R}^+; \mathcal{E})\), we then get \(\begin{bmatrix} x_0 \\ u \end{bmatrix} \in \mathcal{V}^\infty\) so that \(\begin{bmatrix} x_0 \\ e \end{bmatrix} \in \mathcal{V}\). Hence, \(\mathcal{V}^\infty \subset \mathcal{V}\), and we also constructed a classical solution \((e, x, f)\) of (7).

If the Dirac node \(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\) is at the same time, a scattering passive system node. Then, we can apply Theorems IV.2 and IV.4 directly to the impedance representation (7) rather than to the scattering representation (10). We end this article with an example that shows that such Dirac nodes actually exist.

**VI. Example**

Denote the state space of (1) by \(\mathcal{Z}\) and assume that this is a Hilbert space. The delay in (1) can be implemented by adding a delay line to the system state, leading to the system
\[
\begin{align*}
  \dot{w}(t, \xi) &\equiv \frac{\partial}{\partial \xi} w(t, \xi), \ -\tau < \xi < 0, \ t \geq 0 \\
  w(t, 0) &\equiv H(t)(t) \\
  z(t) &\equiv (J - H)z(t) - A_1 w(t, -\tau) + E e(t) \\
  f(t) &\equiv E^* H(t)z(t)
\end{align*}
\]
observe that the delay line is filled with past \(H(t)z(\cdot)\), not with past \(z(\cdot)\). The state space of (18) is \(\mathcal{X} := L^2(-\tau, 0; \mathcal{Z})\) which we equip with the Hilbert-space norm given by
\[
\left\| \begin{bmatrix} x \\ z \end{bmatrix} \right\|_\mathcal{X}^2 := \int_{-\tau}^0 \langle H_0 w(\xi), w(\xi) \rangle_\mathcal{Z} d\xi + \| z \|^2_\mathcal{Z}
\]
for a bounded \(H_0 > 0\). The space \(\mathcal{F}\) equals \(\mathcal{E}, \psi = I_\mathcal{E}\). Set
\[
P(t) := \begin{bmatrix} I & 0 \\ 0 & H(t) \end{bmatrix}, \ t \geq 0
\]
to get the associated Dirac node candidate
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} A & 0 \\ 0 & E^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \end{bmatrix} \circ \text{dom}(A) \rightarrow \begin{bmatrix} \mathcal{E} \end{bmatrix}
\]
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} A & 0 \\ 0 & E^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \end{bmatrix} \circ \text{dom}(A) \rightarrow \begin{bmatrix} \mathcal{E} \end{bmatrix}
\]
where $A : \mathcal{X} \ni \text{dom}(A) \to \mathcal{X}$ is defined as

$$
A := \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
-A_1 \tau & J - R
\end{bmatrix}
$$

and

$$
\text{dom}(A) := \left\{ \begin{bmatrix} w \\ z \end{bmatrix} \in H^1(-\tau, 0; \mathbb{Z}) \mid \epsilon_0 w = z \right\}
$$

where $\epsilon_0$ denotes point evaluation at $\xi \in [-\tau, 0]$.

**Theorem VI.1:** The operator $[A; B]$ constructed previously is a Dirac node if and only if

$$
M := \begin{bmatrix} H_0 & A_1^\dagger \\ A_1 & 2R - H_0 \end{bmatrix} \geq 0.
$$

In this case, for all $\beta \in \mathbb{C}_+$, all statements in Theorem V.2 hold.

The operator $[A; B]$ is a scattering passive system node if and only if

$$
N := \begin{bmatrix} H_0 & A_1^\dagger \\ A_1 & 2R - H_0 - EE^* - E \end{bmatrix} \geq 0.
$$

Assume $H(\cdot) > 0$ and $H(\cdot), H(\cdot)^{-1} \in C^1(\mathbb{R}_+; \mathbb{Z})$ strongly. For all $w_0 \in H^1(-\tau, 0; \mathbb{Z})$, $e \in H^1(\mathbb{R}_+; \mathcal{E})$, there are unique $z$ and $f$, such that $(e, z, f)$ solves (1) classically.

We point out that $M \geq 0$ is possible only if $0 < H_0 \leq 2R$, i.e., if there is delay in the system, then we can have a Dirac node only if there is also some internal damping.

**Proof:** We first prove that if $M \geq 0$, then $[A; B]$ is a Dirac node. It is clear that (3) is injective for $\beta = 1$, so it only remains to verify that $[A; B]$ is maximal dissipative. For all $w \in \text{dom}(A)$ and $e \in \mathcal{E}$, we indeed have

$$
2\text{Re} \left\langle \begin{bmatrix} A & B \\ -C & D \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \begin{bmatrix} w \\ z \end{bmatrix} \right\rangle_{\mathcal{X}}
$$

$$
= 2\text{Re} \int_{-\tau}^0 \langle H_0 w(\xi), w'(\xi) \rangle d\xi - 2\text{Re} \langle E^* z, e \rangle
$$

$$
+ 2\text{Re} \left\langle (J - R)z - A_1 w(-\tau) + Ee, z \right\rangle
$$

$$
= -\left\langle M \begin{bmatrix} w(-\tau) \\ z \end{bmatrix}, \begin{bmatrix} w(-\tau) \\ z \end{bmatrix} \right\rangle
$$

which establishes dissipativity if $M \geq 0$.

Next, let $\begin{bmatrix} v \\ x \end{bmatrix} \in \mathcal{X}$ and $f \in \mathcal{E}$ be arbitrary. We need to find $\lambda \in \mathbb{C}_+$,

$$
\begin{bmatrix} w \\ z \end{bmatrix} \in \text{dom}(A) \text{ and } e \in \mathcal{E}, \text{ such that }
$$

$$
\begin{bmatrix} \lambda I - [A & B] \\ -C & D \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} v \\ x \end{bmatrix}. \tag{20}
$$

Straightforward calculations show that (20) is equivalent to

$$
w(\xi) = e^{i\xi} w(0) - \int_0^\xi e^{i(\xi - \eta)} v(\eta) d\eta \tag{21}
$$

$$
e = f/\lambda - E^* w(0)/\lambda \quad \text{and} \quad (A_1 e^{-\lambda \tau} + \lambda I - J + R + EE^* \lambda) w(0)
$$

$$
= x + Ef/\lambda - A_1 \int_{-\tau}^\xi e^{-i(\tau + \eta)} v(\eta) d\eta. \tag{23}
$$

Due to the boundedness of the operators, for $\lambda \in \mathbb{R}_+$ large enough, (23) can be solved for $w(0)$. Defining $w$ by (21) and setting $z := w(\tau)$, we then get $\begin{bmatrix} w \\ z \end{bmatrix} \in \text{dom}(A)$. Defining $e$ by (22), we have that $[A; B]$ is a Dirac node if $M \geq 0$.

Now we prove the converse, so let $\omega, z \in \mathcal{B}$ be such that

$$
\left\langle M \begin{bmatrix} \omega \\ \omega \end{bmatrix}, \begin{bmatrix} \omega \\ \omega \end{bmatrix} \right\rangle < 0.
$$

By the definition of $\text{dom}(A)$ and (19), in order to prove that $[A; B]$ is not dissipative, all we need to do is to find a $w \in H^1(-\tau, 0; \mathbb{Z})$, such that $w(\tau) = w$ and $w(0) = z$. Take for instance $w(\xi) := z - \xi/\tau \cdot (w(\tau), 0 \leq \xi \leq \tau$.

Adding $2\text{Re} \langle e, E^* z \rangle - \|e\|^2 + \|E^* z\|^2$ to (19), we get that $[A; B]$ is scattering dissipative if and only if $M \geq 0$ with an argument almost identical to the above.

Finally, the smoothness assumptions on $H(\cdot)$ imply that (6) holds with $G$ zero so that Theorem V.2 is applicable. Now define $z_0 := H(0)^{-1} w_0(0)$ to get $P(0) \begin{bmatrix} w_0 \\ z_0 \end{bmatrix} \in \text{dom}(A)$. By Theorem V.2, a classical solution $(e, z, f)$ with $x(0) = \left(\begin{bmatrix} H_2 \end{bmatrix} \begin{bmatrix} w_0 \\ z_0 \end{bmatrix} \right)$ exists, and it is unique by Theorem III.2.

Curiously, scattering passivity implies impedance passivity: $N \geq 0 \Rightarrow M \geq 0$. We get an example where $M, N \geq 0$ in Theorem VI.1 by choosing $H_0 := E := 1, A_1 := 10$, and $R := 100$. It seems like $N \geq 0$ only occurs when there is rather strong damping, i.e., when $R$ is large.

**VII. Conclusion**

This article exhibits how time-varying port-Hamiltonian systems, which are not necessarily of boundary-control type, can be solved using the theory of time-varying well-posed linear systems, which was presented in [17]. The external Cayley transform is the tool for passing between the two system classes, and the theory is illustrated with a few examples.

Currently, the theory of evolution families on Hilbert spaces is limited, and it would need some further development in order to get much farther in the direction of this article. Then, progress could be made on time-varying port-Hamiltonian systems, using techniques from [9] and [17].

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