An Algebraic Analysis of Conchoids to
Algebraic Curves. *

J. Rafael Sendra
Dep. de Matemáticas
Universidad de Alcalá
Alcalá de Henares, Madrid, Spain
rafael.sendra@uah.es

Juana Sendra
Dep. de Matemáticas
E.U.I.T. Telecomunicación
Univ. Politécnica de Madrid, Spain
jsendra@euitt.upm.es

Abstract

We study conchoids to algebraic curve from the perspective of
algebraic geometry, analyzing their main algebraic properties. We
introduce the formal definition of conchoid of an algebraic curve by
means of incidence diagrams. We prove that, with the exception of a
circle centered at the focus and taking $d$ as its radius, the conchoid
is an algebraic curve having at most two irreducible components. In
addition, we introduce the notions of special and simple components
of a conchoid. Moreover we state that, with the exception of lines
passing through the focus, the conchoid always has at least one simple
component and that, for almost every distance, all the components of
the conchoid are simple. We state that, in the reducible case, simple
conchoid components are birationally equivalent to the initial curve,
and we show how special components can be used to decide whether
a given algebraic curve is the conchoid of another curve.

*Both authors supported by the Spanish “Ministerio de Educación y Ciencia” under
the Project MTM2005-08690-C02-01
1 Introduction.

A conchoid is a curve derived from a fixed point, another curve, and a length in the following way. Let \( C \) be a plane curve (the base curve), \( A \) a fixed point in the plane (the focus), and \( d \) a non-zero fixed field element (the distance). Then, we consider the set of all points in the plane for which there exists a point \( P \in C \) such that the distance between \( P \) and \( Q \) is \( d \), being \( A, P, Q \) collinear (see Fig. 1). Such a geometric locus will be called the conchoid of \( C \) from the focus \( A \) at distance \( d \).

![Figure 1: Left: Conchoid Geometric Construction. Right: Conchoid of a circle with focus on it.](image)

The two classical and most famous conchoids are the Conchoid of Nicomedes (see Example 2.4 and Fig. 3) and the Limaçon of Pascal (see Example 3.6, Fig. 4) that appear when the base curve \( C \) is a line and a circle, respectively. Conchoid of Nicomedes was introduced by Nicomedes, around 200 B.C., to solve the problems of doubling the cube and of trisecting an angle.

Conchoids play an important role in many applications as construction of buildings (one can already find specific methods for producing Limaçons in Albert Dürer’s Underweisung der Messung), astronomy (see (6)), electro-
Although conchoids have been extensively used and applied in different areas, a deep theoretical analysis of the concept and its main properties is missing; at least from our point of view. In this paper, we consider conchoids from the perspective of algebraic geometry, and we study their main algebraic properties, with the aim of building a solid bridge from theory to practice that can be used for further theoretical and applied developments. More precisely, we introduce the formal definition of conchoid of an algebraic curve, over an algebraically closed field of characteristic zero, by means of incidence diagrams. We also introduce the notion of generic conchoid, and we show how elimination theory techniques, as Gröbner bases, can be applied to compute conchoids. We prove that, with the exception of a circle centered at the focus and taking $d$ as its radius, the conchoid is an algebraic curve having at most two irreducible components. Note that for the particular circle, mentioned above, the conchoid consists in two components: a circle of radius $2d$ and the zero-dimensional set formed by the focus. In addition, we introduce the notions of special and simple components of a conchoid. Essentially, a component of a conchoid is special if its points are generated for more than one point of the original curve. This phenomenon appears when one computes conchoids of conchoids. Moreover we state that, with the exception of lines passing through the focus, the conchoid always has at least one simple component. Furthermore we prove that, for almost every distance and with the exception of lines passing through the focus, all the components of the conchoid are simple. Simple and special components play an important role in the study of conchoids. On one hand, simple components are related to the birationality of the maps in the incidence diagram (for instance, if a conchoid has two components, its simple components are birationally equivalent to the initial curve) and, on the other hand, special components can be used to decide whether a given algebraic curve is the conchoid of another curve. A similar behavior of simple components holds for offsets (see (11)), and have allowed us to provide formulas for the genus (see (2)). We plan to investigate this, for the case of conchoids, in our future research.

The paper is structured as follows. In Section 2 we formally introduce the notion of conchoid of an algebraic curve. In Section 3 we state the basic algebraic properties of conchoids. Section 4 we introduce the notion
of simple and special component and we state its main properties. Section 5 studies how simple components are related to the birationality of the maps in the incidence diagram, and Section 6 shows how special components can be applied to detect whether a curve is the conchoid of another curve.

2 Definition of Conchoid.

Let \( K \) be an algebraically closed field of characteristic zero. We consider \( K^2 \) as the metric affine space induced by the inner product \( B((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 y_2 \). In this context, the circle of center \( (a_1, a_2) \in K^2 \) and radius \( d \in K^* \) is the plane curve defined by \( (x_1 - a_1)^2 + (x_2 - a_2)^2 = d^2 \). We will say that the distance between the points \( \bar{x}, \bar{y} \in K^2 \) is \( d \in K^* \) if \( \bar{y} \) is on the circle of center \( \bar{x} \) and radius \( d \) (notice that the distance is hence defined up to the sign). On the other hand, if \( \bar{x} \in K^2 \) is not isotropic we denote by \( \| \bar{x} \| \) any of the numbers such that \( \| \bar{x} \|^2 = B(\bar{x}, \bar{x}) \), and if \( \bar{x} \in K^2 \) is isotropic, then \( \| \bar{x} \| = 0 \). In this paper we usually work with both solutions of \( \| \bar{x} \|^2 = B(\bar{x}, \bar{x}) \). For this reason we use the notation \( \pm\| \bar{x} \| \).

In this situation, let \( C \) be the affine irreducible plane curve defined by the irreducible polynomial \( f(y_1, y_2) \in K[y_1, y_2] \), let \( d \in K^* \) be a non-zero field element, and let \( A = (a, b) \in K^2 \). In order to get a formal definition of the conchoid, one introduces the following incidence diagram:

\[
\mathcal{B}(C, A, d) \subset K^2 \times K^2 \times K \\
\pi_1 \quad \pi_2 \\
\pi_1(\mathcal{B}(C, A, d)) \subset K^2 \\
C \subset K^2
\]

where \( \mathcal{B}(C, A, d) \) is the algebraic set of \( K^2 \times K^2 \times K \) defined as

\[
\begin{cases}
(x, y, w) \in K^2 \times K^2 \times K \\
f(y_1, y_2) = 0 \\
(x_1 - y_1)^2 + (x_2 - y_2)^2 = d^2 \\
(y_2 - y_1) \cdot (x_1 - y_1) - (y_1 - a) \cdot (x_2 - y_2) = 0 \\
w \cdot ((y_1 - a)^2 + (y_2 - b)^2) = 1
\end{cases}
\]
and where $\pi_1, \pi_2$ are the natural projections

$$
\pi_1 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2, \quad \pi_2 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2
$$

($$\bar{x}, \bar{y}, w$$) $\longrightarrow \bar{x}$ \hspace{1cm} ($$\bar{x}, \bar{y}, w$$) $\longrightarrow \bar{y}$.

Observe that the first equation in $\mathcal{B}(C, A, d)$ corresponds to $C$, the second and third guarantee that the distance between $\bar{x}$ and $\bar{y}$ is $d$ and that $\bar{x}, \bar{y}, A$ are collinear. The last equation excludes isotropic points on $C$; i.e. point $\bar{y} \in C$ such that $(y_1 - a)^2 + (y_2 - b)^2 = 0$. Note that this phenomenon occurs, for instance, taking $f(y_1, y_2) = (y_1 - a) + \sqrt{-1}(y_2 - b)$. This type of situation will be analyzed in the next section. Then, we introduce the conchoid as follows.

**Definition 2.1** Let $C$ be an affine irreducible plane curve, $d \in \mathbb{K}^*$, and $A \in \mathbb{K}^2$. We define the conchoid of the base curve $C$ from the focus $A$ and distance $d$ as the algebraic Zariski closure in $\mathbb{K}^2$ of $\pi_1(\mathcal{B}(C, A, d))$, and we denote it by $C(C, A, d)$. That is,

$$
C(C, A, d) = \overline{\pi_1(\mathcal{B}(C, A, d))}.
$$

**Remark 2.2** Observe that:

(1) [Extension of the definition]. In Definition 2.1 we have considered irreducible curves. The same reasoning can be done for reducible curves, introducing the conchoid as the union of the conchoids of the irreducible components.

(2) [False components]. Let $C$ be a line (different to $y_1 \pm \sqrt{-1}y_2 = 0$), and $A \in C$ (similarly for any other irreducible curve). Then, following the “intuitive” geometric description of the conchoid, one might claim that $C(C, A, d)$ consists in $C$ and a circle centered at $A$ and radius $d$. However, attending to Definition 2.1, the circle (let us call it $D$) is not a component of the conchoid. Note that, although for every $\bar{x} \in D$ the pair $(\bar{x}, A)$ satisfies the three equations of $\mathcal{B}(C, A, d)$ it does not satisfies the last one.

(3) [Computation of the Conchoid]. Let $I$ be ideal in $\mathbb{K}[\bar{x}, \bar{y}, w]$ generated by the polynomials defining $\mathcal{B}(C, A, d)$. Then, by the Closure Theorem (see p. 122), one has that $C(C, A, d) = V(I \cap \mathbb{K}[\bar{x}])$. Hence elimination theory techniques, as Gröbner bases, provide the conchoid.
(4) [Generic Conchoid]. Reasoning as in Section 2 in \(\text{(14)}\), one may introduce the notion of generic conchoid. Let us consider \(d\) as a new variable. Now, \(\mathcal{B}(C, A, d)\) is seen as an algebraic set in \(K^2 \times K^2 \times K \times K\); we denote it by \(\mathcal{B}(C, A, d)_G\). Then, the generic conchoid is defined as
\[
\mathcal{C}(C, A, d)_G = \pi_1(\mathcal{B}(C, A, d)_G).
\]

Now, if \(I_G\) is the ideal in \(K[\bar{x}, \bar{y}, d, W]\) generated by the polynomials defining \(\mathcal{B}(C, A, d)_G\), by the Closure Theorem (see \(\text{(4)}\) p. 122), one has that \(\mathcal{C}(C, A, d)_G = V(I_G \cap K[\bar{x}, d])\). Moreover, reasoning as in Theorem 6 in \(\text{(14)}\) which is a direct consequence of Exercise 7, p. 283 in \(\text{(4)}\), one gets that for almost all values of \(d \in K^*\) the generic conchoid specializes properly (see Example 2.3). An example where the specialization improperly behaving is taking \(C\) as a circle centered at \(A\) and \(d\) its radius (compare to Theorem 3.4). A similar reasoning might be done with a generic focus, nevertheless we do not consider this situation here. \(\blacksquare\)

Let us illustrate the definition by two examples.

**Example 2.3** Let \(C\) be the parabola over \(\mathbb{C}\) defined by \(f(y_1, y_2) = y_2 - y_1^2\), let \(A = (0, -1)\) and \(d = 1/2\). Then \(\mathcal{B}(C, A, d)\) is defined by polynomials
\[
f(\bar{y}), \quad C(\bar{x}, \bar{y}) := (x_1 - y_1)^2 + (x_2 - y_2)^2 - \frac{1}{4},
\]
\[
L(\bar{x}, \bar{y}) := (y_1 + 1)(x_1 - y_1) - y_1(x_2 - y_2), \quad T(\bar{y}, W) := W(y_1^2 + (y_2 + 1)^2) - 1.
\]
Now, considering \(W > y_1 > y_2 > x_1 > x_2\), and computing a Gröbner basis w.r.t. the lex order, one gets that \(\mathcal{C}(C, A, d)\) is defined by the polynomial (see Fig. 2): 
\[
g(x_1, x_2) = 16x_1^8 + 32x_1^2x_2^6 + 16x_1^4x_2^4 + 32x_2x_1^6 - 32x_1^2x_2^5 + 24x_1^6 - 24x_1^4x_2^2 - 96x_1^2x_2^3 + 16x_2^6 - 8x_2x_1^4 - 120x_1^2x_2^2 + 64x_2^5 + 25x_1^4 - 68x_1^2x_2^2 + 92x_1^2 + 48x_3^2 + 12x_1^2 - 8x_2^2 - 16x_2 - 4,
\]
which is an irreducible curve over \(\mathbb{C}\). Similarly, one gets that \(\mathcal{C}(C, A, d)_G\) is given by
\[
g_G(x_1, x_2, d) = -d^2 + x_2^2 + 4x_2^3 - 2x_2x_1^2 - 4x_2d^2 - 6x_2x_1^2 - 6x_2^2d^2 + 6x_2^4 + 8x_1^2x_2d^2 - 8x_1^2x_2^3 + 3x_1^2d^2 + x_1^4 + 4x_2^3 - 2x_2x_1^2 - 6x_2^2x_1^2 + 7x_2^2x_2d^2 - 2x_1^4x_2d^2 + 2x_1^2 + 2x_1^2x_2 + 2x_1^2d^2 - 2x_1^2x_2^3 + 2x_1^2x_2^2 + x_1x_2^2 - 2x_2^2d^2x_1^2 + 2x_2^2d^2x_2^2 + x_2^6 - x_2^4d^2 - 2x_1^2d^2 + x_1^2 + x_1^2d^2 - 4x_3^2d^2.
\]
Note that \(g_G(\bar{x}, 1/2) = g(\bar{x})\). \(\blacksquare\)
Figure 2: $y_2 = y_1^2$ (in dots) and $C(y_2 - y_1^2, (0, -1), 1/2)$ (in continuous traced).

In the following example we consider that $C$ is a line which does not pass through the focus, appearing the well known Conchoid of Nicomedes.

Example 2.4 (Conchoid of Nicomedes) Let $C$ be the lined defined by $f(y_1, y_2) = y_2$ and let $A = (0, 1)$. Then, $C(C, A, 2)$ is defined by (see Fig. 3):

\[ g(x_1, x_2) = x_2^2 - x_1^2 + x_2^4 - 2x_2^3 - 3x_2^2 + 8x_2 - 4. \]

We finish this section studying the connection of conchoids to offsets (see [1] for further details on offsets). We represent the offset to $C$ at distance $d$ as $O_d(C)$.

Lemma 2.5 Let $C$ be a real irreducible curve, and let $A \in \mathbb{R}^2, d \in \mathbb{R}^*$. Then, $C(C, A, d) = O_d(C)$ if and only if $C$ is a circle centered at $A$.

Proof. If $C$ is a circle centered at $A$ the result is obvious. Let $C(C, A, d) = O_d(C)$, the result follows from Lemma 3 in [3].
3 Basic Properties of Conchoids.

In this section we state the first basic properties on conchoids. For this purpose, we assume that \( \mathcal{C} \) is irreducible and given by \( f(y_1, y_2) \), and that \( A = (a, b) \) and \( d \in \mathbb{K}^* \) are fixed. In addition, we consider the following Zariski open subset of \( \mathcal{C} \)

\[
\mathcal{C}_0 = \{(p_1, p_2) \in \mathcal{C} \mid (p_1 - a)^2 + (p_2 - b)^2 \neq 0\};
\]

i.e. \( \mathcal{C}_0 \) consists in those points \( P \in \mathcal{C} \) such that \( A - P \) is non-isotropic. \( \mathcal{C}_0 \) plays an important role in the conchoid construction. Note that if \( \mathcal{C}_0 = \emptyset \) then the last equation of \( \mathfrak{B}(\mathcal{C}, A, d) \) does not hold, and hence \( \mathfrak{E}(\mathcal{C}, A, d) \) is empty. This is the case of the lines \( (y_1 - a) \pm \sqrt{-1}(y_2 - b) = 0 \). The next proposition relates \( \mathcal{C}_0 \) with these lines, that we denote in the sequel as \( \mathcal{L}^+ \) and \( \mathcal{L}^- \).

**Proposition 3.1** It holds that:

1. \( \mathcal{C}_0 = \emptyset \) if and only if \( \mathcal{C} \) is either \( \mathcal{L}^+ \) or \( \mathcal{L}^- \).
2. \( \mathcal{C} \setminus \mathcal{C}_0 = \mathcal{C} \cap (\mathcal{L}^+ \cup \mathcal{L}^-) \). Moreover, \( \text{Card}(\mathcal{C} \setminus \mathcal{C}_0) \leq 2 \deg(\mathcal{C}) \)
3. \( A \notin \mathcal{C}_0 \).
(4) If $C$ is real then $C_0 \neq \emptyset$.

**Proof.** (1) The right-left implication is trivial. Conversely, let $C_0 = \emptyset$. Let $g(y_1, y_2) = (y_1 - a)^2 + (y_2 - b)^2$. Then, for every $P \in C$, $g(P) = 0$. That is, $g \in \mathcal{I}(C)$. Now, the proof ends taking into account that $f$ is irreducible. (2), (3) and (4) follows from (1). \qed

In the following, we analyze the maps appearing in the incidence diagram (see Section 2).

**Lemma 3.2** Let $\pi_1$ and $\pi_2$ be the projections in the incidence diagram associated with $C$.

(1) If $C$ is not the circle centered at $A$ and radius $d$ then it holds that

1.1) $\pi_1$ is, at most $(2 : 1)$, over all points in $\pi_1(\mathcal{B}(C, A, d)) \setminus \{A\}$.

1.2) If $A \in \pi_1(\mathcal{B}(C, A, d))$ then $\text{Card}(\pi_1^{-1}(A)) \leq 2 \deg(C)$.

(2) $\pi_2$ is $(2 : 1)$ over all points in $C_0$.

**Proof.** We consider the polynomials

\[
C(\bar{x}, \bar{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2, \\
L(\bar{x}, \bar{y}) = (y_2 - b)(x_1 - y_1) - (y_1 - a)(x_2 - y_2) \\
= (b - x_2)y_1 + (x_1 - a)y_2 - bx_1 + ax_2.
\]

Let us prove (1). For $\bar{x}^0 \in \pi_1(\mathcal{B}(C, A, d))$, let $D_{\bar{x}^0}$, $L_{\bar{x}^0}$ be the algebraic set in $\mathbb{K}^2$ defined by $C(\bar{x}^0, \bar{y})$, $L(\bar{x}^0, \bar{y})$, respectively. Note that $L(\bar{x}^0, \bar{y})$ is identically zero if and only if $\bar{x}^0 = A$. Now, if $\bar{x}^0 \neq A$ then $\pi_1^{-1}(\bar{x}^0)$ consists in those $(\bar{x}^0, \bar{y}, w^0)$ where $\bar{y} \in C \cap D_{\bar{x}^0} \cap L_{\bar{x}^0}$ and $w^0 = 1/((y_1 - a)^2 + (y_2 - b)^2)$. Thus, since $\deg(D_{\bar{x}^0}) = 2$ and $\deg(L_{\bar{x}^0}) = 1$, then $\text{Card}(\pi_1^{-1}(\bar{x}^0)) \leq 2$. If $\bar{x}^0 = A$ then $\pi_1^{-1}(\bar{x}^0)$ consists in those $(\bar{x}^0, \bar{y}, w^0)$ where $\bar{y} \in C \cap D_{\bar{x}^0}$ and $w^0 = 1/((y_1 - a)^2 + (y_2 - b)^2)$. Thus, since by hypothesis $C$ and $D_{\bar{x}^0}$ do not have common components, then $\text{Card}(\pi_1^{-1}(\bar{x}^0)) \leq 2 \deg(C)$.

Now we prove (2). Let $\bar{y}^0 \in C_0$, and let $D_{\bar{y}^0}$ and $L_{\bar{y}^0}$ be the algebraic sets defined by $C(\bar{x}, \bar{y}^0)$, and $L(\bar{x}, \bar{y}^0)$ respectively. $D_{\bar{y}^0}$ is the circle centered in $\bar{y}^0$ and radius $d$, and $L_{\bar{y}^0}$ is the line passing through $\bar{y}^0$ and $A$; note that by Prop. 3.1 $A \notin C_0$, and hence $\bar{y}^0 \neq A$. So, $L_{\bar{y}^0}$ is not tangent to $D_{\bar{y}^0}$. Moreover, $\pi_2^{-1}(\bar{y}^0)$ consists in those $(\bar{x}, \bar{y}^0, w^0)$ such that $\bar{x} \in D_{\bar{y}^0} \cap L_{\bar{y}^0}$ and $w^0 = 1/((y_1^0 - a)^2 + (y_2^0 - b)^2)$, with $\bar{y}^0 = (y_1^0, y_2^0)$. Therefore, $\text{Card}(\pi_2^{-1}(A)) = 2$. \qed
Remark 3.3  Because of the last equation of $\mathfrak{B}(\mathcal{C}, A, d)$, $\pi_2(\mathfrak{B}(\mathcal{C}, A, d)) \subset \mathcal{C}_0$, and by Lemma 3.2 (2), $\pi_2(\mathfrak{B}(\mathcal{C}, A, d)) = \mathcal{C}_0$. □

The following theorem essentially states that the conchoid is a curve with at most two irreducible components.

Theorem 3.4  Let $\mathcal{C}_0 \neq \emptyset$, then it follows that:

(1) All the components of $\mathfrak{B}(\mathcal{C}, A, d)$ have dimension 1.

(2) If $\mathcal{C}$ is not a circle centered at $A$ and radius $d$, all the components of $\mathfrak{C}(\mathcal{C}, A, d)$ have dimension 1.

(3) If $\mathcal{C}$ is a circle centered at $A$ and radius $d$, $\mathfrak{C}(\mathcal{C}, A, d)$ decomposes as the union of $\{A\}$ and the circle centered at $A$ and radius $2d$.

(4) $\mathfrak{C}(\mathcal{C}, A, d)$ has at most two components.

Proof.  Let us prove (1). By Remark 3.3, since $\mathcal{C}_0 \neq \emptyset$, one gets that $\mathfrak{B}(\mathcal{C}, A, d) \neq \emptyset$. Let $\Gamma$ be an irreducible component of $\mathfrak{B}(\mathcal{C}, A, d)$. Let $M \in \Gamma$ and $\bar{\pi} = \pi_2(M) \in \mathcal{C}_0$. Let $\mathcal{P}(t) = (P_1(t), P_2(t))$ be a place of $\mathcal{C}$ centered at $P$. We consider

$$Q^\pm(t) = \left( \mathcal{P}(t) \pm \frac{d}{\sqrt{\Delta(t)}}, (\mathcal{P}(t) - A), \mathcal{P}(t), \frac{1}{\Delta(t)} \right),$$

where $\Delta(t) = (P_1(t) - a)^2 + (P_2(t) - b)^2$. Note that $\Delta(t) = (p_1 - a)^2 + (p_2 - b)^2 + \cdots$, where $P = (p_1, p_2)$. Thus, since $P \in \mathcal{C}_0$, the above power series is a unit. Therefore, each component of $Q^\pm(t)$ can be written as a power series. Thus, $Q^\pm(t)$ parametrize locally, respectively, two curves contained in $\mathfrak{B}(\mathcal{C}, A, d)$ and passing through each of the two points in $\pi_2^{-1}(P)$. Let $Q^+(t)$ be the one centered at $M \in \pi_2^{-1}(P)$. Thus, $\dim \Gamma \geq 1$. Now, let us assume that $\dim(\Gamma) > 1$, and let $\tilde{\mathcal{C}}$ be an irreducible component of $\pi_2(\Gamma)$. By Theorem 7 pp. 76 in [12], there exists an open set $U \subset \tilde{\mathcal{C}}$ such that for every $\bar{y} \in U$ it holds that $\dim(\pi_2^{-1}(\bar{y})) \geq 1$. Then, taking $\bar{y} \in U \subset \mathcal{C}_0$ one gets a contradiction, since by Lemma 3.2 (2), $\dim(\pi_2^{-1}(\bar{y})) = 0$.

(2) follows as (1), using that $\pi_1$ is always finite (see Lemma 3.2 (1)).

(3) It is trivial.

For (4), the reasoning is analogous to Theorem 1 in [11], using statements (1), (2) and (3) in this theorem and Lemma 3.2. □

Next lemma follows from Lemma 3.2, Theorem 3.4 and the theorem on the dimension of fibres in [12] (see Theorem 7, pp.76).
Lemma 3.5  Let $C_0 \neq \emptyset$ and $\pi_1$ and $\pi_2$ the projections in the incidence diagram associated with $C$.

(1) If $\Omega$ is a non-empty open subset of $C$, then $\pi_1(\pi_2^{-1}(\Omega))$ is a non-empty Zariski dense subset of $\mathcal{C}(C, A, d)$.

(2) If $C$ is not the circle centered at $A$ and radius $d$, and $\Omega$ is a non-empty open subset of an irreducible component of $\mathcal{C}(C, A, d)$, then $\pi_2(\pi_1^{-1}(\Omega))$ is a non-empty Zariski dense subset of $C$.

Proof. (1) follows using that $\pi_2^{-1}(C_0) = \mathfrak{B}(C, A, d)$ (see Remark 3.3), and that $\pi_1(\mathfrak{B}(C, A, d))$ is constructible in $\mathcal{C}(C, A, d)$ (see Theorem 3.16. in [5]). In order to prove (2), let $M$ be an irreducible component of $\mathcal{C}(C, A, d)$, let $\emptyset \neq \Omega \subset M$ be open in $M$, and let $\Omega' = \Omega \cap \pi_1(\mathfrak{B}(C, A, d)))$. Since $\dim(M) = 1$ (see Theorem 3.4(2)) and $\pi_1$ is finite over $\Omega'$ (see Lemma 3.2), at least one component of $\pi_1^{-1}(\Omega')$ has dimension 1. Let $\Gamma \subset \pi_1^{-1}(\Omega')$ be irreducible of dimension 1. Then, since $\pi_2$ is finite over $C_0$ (see Lemma 3.2), by Theorem 7 (ii), pp. 76, in [12], one has that $\dim(\pi_2(\Gamma)) = 1$. Therefore, the result follows taking into account that $\pi_2(\Gamma) \subset \pi_2(\pi_1^{-1}(\Omega')) \subset \pi_2(\pi_1^{-1}(\Omega)) \subset C$ and that $C$ is irreducible. \hfill \Box

Example 3.6 (Limaçons of Pascal) Let $C$ be the circle centered at $(0,0)$ and radius $r = 2$. Then, the conchoid of $C$ with $A = (-2,0) \in C$ and $d = 1$ (Limaçon of Pascal at distance 1, see Fig. 4), is defined by the polynomial:

$$g(x_1, x_2) = x_1^4 + 2x_2^2x_1^2 - 9x_1^2 - 4x_1 - 9x_2^2 + 12 + x_2^4.$$  

On the other hand, if we move the focus to $A = (0,0) \notin C$, one observes that the conchoid at distance $d = 1$ has two irreducible components (two circles centered at $A$ and radius 1 and 3, respectively) defined by the irreducible factors $(x_1^2 - 9 + x_2^2) \cdot (x_1^2 - 1 + x_2^2)$. Note that, in this case, $\mathcal{C}(C, A, d)$ is the offset of $C$ at distance $d = 1$ (see Lemma 2.3). \hfill \Box

4  Simple and Special Components.

In Theorem 3.4, we have seen that if $C$ is not a circle centered at the focus, all components of the conchoid are curves. In the sequel, we assume that $C$ is not such a circle centered at the focus and that $C_0 \neq \emptyset$. Now, we introduce and analyze the notion of simple and special components of a conchoid.
Figure 4: Circle centered at $(0, 0)$ and radius $r = 2$ (in dots), and its conchoid from the focus $(-2, 0)$ and radius $d = 1$ (continuous traced).

Special and simple components provide information on the birationality of the projections in the incidence diagram (see Section 5), and they can be used to decide whether a curve is a conchoid (see Section 6). Essentially, one component of the conchoid is special if its points are generated for more than one point of the original curve. This phenomenon appears when one computes conchoids of conchoids (see Theorem 4.5). In addition, Theorem 4.12 states that, for almost every distance and with the exception of lines passing through the focus, all the components of the conchoid are simple.

Definition 4.1 An irreducible component $\mathcal{M}$ of $\mathcal{E}(\mathcal{C}, A, d)$ is called simple if there exists a non-empty Zariski dense subset $\Omega \subset \mathcal{M}$ such that if $Q \in \Omega$ then $\text{Card}(\pi_2(\pi_1^{-1}(Q))) = 1$. Otherwise $\mathcal{M}$ is called special. \hfill $\blacksquare$

Remark 4.2 An irreducible component $\mathcal{M}$ is special iff there exists a Zariski dense $\emptyset \neq \Omega \subset \mathcal{M}$ such that for $Q \in \Omega$, $\text{Card}(\pi_2(\pi_1^{-1}(Q))) > 1$. \hfill $\blacksquare$

Proposition 4.3 Let $\mathcal{M}$ be an irreducible component of $\mathcal{E}(\mathcal{C}, A, d)$. Then, it holds that:

1. The open subset $\mathcal{M}_0 \subset \mathcal{M}$ is not empty. (See Section 3 for the definition of $\mathcal{M}_0$.)
(2) $\mathcal{C}$ is a component of $\mathcal{E}(\mathcal{M}, A, d)$.

(3) Let $d' \in \mathbb{K}^*$ such that $d^2 \neq d'^2$. If $\mathcal{M}'$ is an irreducible component of $\mathcal{E}(\mathcal{M}, A, d')$, then $\mathcal{M}'$ is a component of $\mathcal{E}(\mathcal{C}, A, d + d')$ or of $\mathcal{E}(\mathcal{C}, A, d - d')$.

**Proof.** In order to prove (1), we suppose that $\mathcal{M}_0 = \emptyset$. By Proposition 3.1 (1), $\mathcal{M}$ is either $\mathcal{L}^+$ or $\mathcal{L}^-$. Say $\mathcal{M} = \mathcal{L}^+$, similarly if $\mathcal{M} = \mathcal{L}^-$. Then, $\Omega := \mathcal{L}^+ \cap \pi_1(\mathcal{B}(\mathcal{C}, A, d)) \neq \emptyset$ and dense in $\mathcal{L}^+$. So, there exists $P_0 \in \Omega \setminus \{A\}$.

Let $(P_0, Q_0, w_0) \in \pi_1^{-1}(P_0)$; note that obviously $\pi_1^{-1}(P_0) \neq \emptyset$. Now, observe that $A, P_0$ are different points on the line $\mathcal{L}^+$ and, because of the third equation of $\mathcal{B}(\mathcal{C}, A, d)$, $P_0, Q_0, A$ are collinear. Thus, $Q_0 \in \mathcal{L}^+$ which impossible because of the last equation of $\mathcal{B}(\mathcal{C}, A, d)$.

To prove (2), let $\pi_i$ and $\pi_i^*$ be the projections in the incidence diagram associated with $\mathcal{E}(\mathcal{C}, A, d)$ and $\mathcal{E}(\mathcal{M}, A, d)$, respectively. Let $\mathcal{C}_1 := \pi_2(\pi_1^{-1}(\mathcal{M}_0))$.

By the statement (1), $\mathcal{M}_0$ is a non-empty open of the irreducible component $\mathcal{M}$, and by Lemma 3.5 (2), $\mathcal{C}_1$ is a non-empty Zariski dense in $\mathcal{C}$; observe that we have assumed that $\mathcal{C}$ is not the circle centered at $A$ and radius $d$. Now, we see that for every $P \in \mathcal{C}_1$ there exists $Q_0 \in \mathcal{M}_0$ and $w_0 \in \mathbb{K}$ such that $(P, Q_0, w_0) \in \mathcal{B}(\mathcal{M}, A, d)$. Indeed, since $P \in \mathcal{C}_1$ there exists $Q_0 \in \mathcal{M}_0$ and $w_0 \in \mathbb{K}$ such that $(Q_0, P, w_0) \in \mathcal{B}(\mathcal{C}, A, d)$, and since $Q_0 \in \mathcal{M}_0$ there exists $w_1 \in \mathbb{K}$ such that $(P, Q_0, w_1) \in \mathcal{B}(\mathcal{M}, A, d)$. Therefore, $\mathcal{C}_1 \subset \pi_1(\mathcal{B}(\mathcal{M}, A, d))$, and taking closures $\overline{\mathcal{C}} \subset \mathcal{E}(\mathcal{M}, A, d)$.

Finally, we prove (3). Let $\pi_i, \pi_i^*$ and $\pi_i^\pm$ be the projections in the incidence diagram associated with $\mathcal{E}(\mathcal{C}, A, d)$, $\mathcal{E}(\mathcal{M}, A, d')$, and $\mathcal{E}(\mathcal{C}, A, d \pm d')$, respectively; note that $d^2 \neq d'$ and hence $\mathcal{E}(\mathcal{C}, A, d \pm d')$ is well defined.

We consider $\mathcal{M}'_1 = \pi_1^*(\pi_2^{-1}(\pi_1(\pi_2^{-1}(\mathcal{C}_0))) \cap \mathcal{M}_0) \cap \mathcal{M}_0$. Taking into account that $\mathcal{C}_0 \neq \emptyset, \mathcal{M}_0 \neq \emptyset, \mathcal{M}'_0 \neq \emptyset$, by Lemma 3.5 one gets that $\mathcal{M}'_1$ is a non-empty Zariski dense of $\mathcal{M}'$. Now, let $P \in \mathcal{M}'_1$. Then, there exists $Q_0 \in \pi_1(\pi_2^{-1}(\mathcal{C}_0)) \cap \mathcal{M}_0$ and $w_0 \in \mathbb{K}$ such that $(P, Q_0, w_0) \in \mathcal{B}(\mathcal{M}, A, d')$. Moreover, since $Q_0 \in \pi_1(\pi_2^{-1}(\mathcal{C}_0)) \cap \mathcal{M}_0$, there exists $Q_1 \in \mathcal{C}_0$ and $w_1 \in \mathbb{K}$ such that $(Q_0, Q_1, w_1) \in \mathcal{B}(\mathcal{C}, A, d)$. Let us see that either $(P, Q_1, w_1) \in \mathcal{B}(\mathcal{C}, A, d + d')$ or $(P, Q_1, w_1) \in \mathcal{B}(\mathcal{C}, A, d - d')$. If we would prove this then it would hold that $\mathcal{M}'_1 \subset \pi_1^*(\mathcal{B}(\mathcal{C}, A, d + d')) \cup \pi_1^*(\mathcal{B}(\mathcal{C}, A, d - d'))$. Hence, taking closures, $\mathcal{M}' \subset \mathcal{E}(\mathcal{C}, A, d + d') \cup \mathcal{E}(\mathcal{C}, A, d - d')$. Therefore, let us prove the claim; i.e. (i) $Q_1 \in \mathcal{C}_0$ (this is equivalent to the first and last equation of $\mathcal{B}(\mathcal{C}, A, d \pm d')$), (ii) $P$ is on the circle centered at $Q_1$ and radius either $d + d'$ or $d - d'$ (second equation of $\mathcal{B}(\mathcal{C}, A, d \pm d')$), (iii) $P, Q_1, A$ are collinear.
(third equation of $\mathfrak{B}(C, A, d \pm d')$). Indeed, we already now that $Q_1 \in C_0$. Moreover, since $(P, Q_0, w_0) \in \mathfrak{B}(\mathcal{M}, A, d')$ and $(Q_0, Q_1, w_1) \in \mathfrak{B}(C, A, d)$, then $P, A, Q_0$ are collinear and $Q_0, Q_1, A$ are also collinear. Therefore, since $A \not\in \{P, Q_0, Q_1\}$ (this follows from Prop. 3.1), because $P \in \mathcal{M}_0$, $Q_1 \in C_0$, and $Q_0 \in \mathcal{M}_0$, then $P, Q_0, Q_1, A$ are collinear; in particular (iii) holds. Also, since $(P, Q_0, w_0) \in \mathfrak{B}(\mathcal{M}, A, d')$ and $(Q_0, Q_1, w_1) \in \mathfrak{B}(C, A, d)$, then $P$ is on the circle centered at $Q_0$ radius $d'$ and $Q_1$ is on the circle centered at $Q_0$ and radius $d$. Thus, since $P, Q_0, Q_1$ are collinear (see above), then (ii) holds.

Remark 4.4 Note that, by Proposition 4.3 (2) and Lemma 2.5, if $C$ is not a circle centered at the focus, then for every $d \in K^*$ none component of $C(C, A, d)$ is a circle centered at $A$.

Next theorem shows that, similarly as in the offsetting construction (see (11)), special components appear only when computing conchoids.

Theorem 4.5 An irreducible component $\mathcal{M}$ of $C(C, A, d)$ is special if and only if $C(M, A, d) = C$.

Proof. Let $\mathcal{M}$ be special. We assume w.l.o.g. that $\mathcal{M}_0$ is the Zariski dense where the cardinality of the fiber is bigger than 1. By Prop. 4.3 (2), $C \subset C(M, A, d)$. In order to see that $C(M, A, d) \subset C$, let $\pi_i$ and $\pi_i^*$ the projections in the incidence diagram associated with $C$ and $\mathcal{M}$, respectively. Let $\Omega^* = \pi_i^*(\pi_i^{-1}(\mathcal{M}_0))$. By Lemma 3.4, $\Omega^*$ is dense in $C(M, A, d)$. Now, let $P \in \Omega^*$. Then, there exists $Q \in \mathcal{M}_0$ and $w_0 \in K$ such that $(P, Q, w_0) \in \mathfrak{B}(M, A, d)$. Moreover, using that $Q \in \mathcal{M}_0$, and that $\mathcal{M}$ is special, one gets that $\text{Card}(\pi_2(\pi_1^{-1}(Q))) > 1$. Let $P_1, P_2 \in \pi_2(\pi_1^{-1}(Q)) \subset C_0$. Then, there exists $w_1, w_2 \in K$ such that $(Q, P_1, w_1), (Q, P_2, w_2) \in \mathfrak{B}(C, A, d)$. Therefore, $Q, P_1, A$ are collinear, $Q, P_2, A$ are collinear, and $P_1, P_2$ are on the circle $\mathcal{D}$ centered at $Q$ and radius $d$. Furthermore, from $(P, Q, w_0) \in \mathfrak{B}(M, A, d)$, one gets that $P, Q, A$ are collinear and $P \in \mathcal{D}$. Since $Q \neq A$, because $Q \in \mathcal{M}_0$ (see Prop. 3.1), $P_1, P_2, P, Q, A$ are on the same line $\mathcal{L}$, and $\{P_1, P_2, P\} \subset L \cap \mathcal{D}$. Since the center of $\mathcal{D}$ is on $\mathcal{L}$, then $P = P_1$ or $P = P_2$. So, $P \in C$. Finally, since $\Omega^*$ is dense in $C(M, A, d)$, taking closures one gets $C(M, A, d) \subset C$.

Conversely, let $C(M, A, d) = C$. Let $\pi_i$ and $\pi_i^*$ as above. Let $\mathcal{M}_1 = \pi_2(\pi_1^{-1}(\mathcal{C}_0)) \cap \mathcal{M}_0$. Since $C(M, A, d) = C$, by Lemma 3.4, $\mathcal{M}_1$ is a non-empty Zariski dense in $\mathcal{M}$. We prove that for every $Q \in \mathcal{M}_1$, $\text{Card}(\pi_2(\pi_1^{-1}(Q))) > 1$.
and hence that $M$ is special. By Lemma 3.2(2), $\text{Card}(\pi_1(\pi_2^{-1}(Q))) = 2$. Let $P_1, P_2 \in \pi_1(\pi_2^{-1}(Q)) \subset C_0$, $P_1 \neq P_2$. Then, there exists $w_0 \in \mathbb{K}$ such that $(P_1, Q, w_0), (P_2, Q, w_0) \in \mathcal{B}(M, A, d)$. So, $P_1, P_2, Q$ satisfy the second and third equation of $\mathcal{B}(C, A, d)$. Moreover, since $P_i \in C_0$, there exists $w_i \in \mathbb{K}$ such that $(Q, P_i, w_i) \in \mathcal{B}(C, A, d)$, and therefore $P_1, P_2 \in \pi_2(\pi_1^{-1}(Q))$.

We illustrate the previous results by an example.

**Example 4.6** Let $C$ be the circle centered at $(0, 0)$ and radius $r = 1$ defined by the polynomial $f(y_1, y_2) = y_1^2 + y_2^2 - 1$. Let $A = (−1, 0) \in C$. First, we compute $\mathcal{C}(C, (−1, 0), 2)$, obtaining a Limaçon of Pascal, defined by the polynomial:

$$g(x_1, x_2) = x_1^4 + 2x_2^2x_1^2 - 8x_1 - 6x_2 - 3 + x_2^4.$$

Note that we get a cardioid. Let us denote it as $C'$, i.e. $C' = \mathcal{C}(C, A, 2)$ (see Fig. 5 left).

![Figure 5: Left: Circle centered at (0, 0) and radius r = 1 (in dots), and its Conchoid from the focus (-1, 0) and radius d = 2 (cardioid). Center: Cardioid (in dots), and its Conchoid from the focus (-1, 0) and radius d = 2 (continuous traced). Right: Cardioid (in dots), and its Conchoid from the focus (-1, 0) and radius d = 1 (continuous traced).](image-url)

Now, we compute the conchoid of the cardioid $C'$ from the same focus $A = (−1, 0)$ and the same distance $d = 2$. In this case, one gets a reducible...
curve, say \( C'' := \mathfrak{C}(C', A, 2) \), with two irreducible components defined by the irreducible factors of:

\[
(x_1^2 + x_2^2 - 1) \cdot (x_1^4 + 2 x_2^2 x_1^2 - 18 x_1^2 - 32 x_1 - 18 x_2^2 - 15 + x_2^4).
\]

Note that, one component of \( C'' \) is the initial circle \( C \) (and therefore it is an special component of \( C'' \); see Theorem 4.5), and the other component is a Limaçon of Pascal of \( C \) from the focus \( A \) and distance \( d = 4 \). See Fig. 5 center.

On the other hand, the conchoid of \( C' \), from the focus \( A \) but now taking distance \( d = 1 \), decomposes as the union of two irreducible components defined by the irreducible factors of the equation:

\[
(x_1^4 + 2 x_2^2 x_1^2 - 11 x_1^2 - 18 x_1 - 11 x_2^2 - 8 + x_2^4) \cdot (x_1^4 + 2 x_2^2 x_1^2 - 3 x_1^2 - 2 x_1 - 3 x_2^2 + x_2^4),
\]

that correspond to two Limaçons of Pascal of \( C \) from the focus \( A \) at distance \( d = 3 \) and \( d = 1 \) respectively, as indicates Proposition 4.3 (3); see Fig. 5 right.

Next theorem states the main property of the components of a conchoid.

**Theorem 4.7** Let \( C \) be different to a line passing through the focus, then \( \mathfrak{C}(C, A, d) \) has at least one simple component.

**Proof.** Recall that we have assumed that \( C \) is not a circle centered at \( A \). By Theorem 3.1 (4), \( \mathfrak{C}(C, A, d) \) has at most two irreducible components. So, we distinguish two cases: (i) \( \mathfrak{C}(C, A, d) = \mathcal{M} \) and (ii) \( \mathfrak{C}(C, A, d) = \mathcal{M}' \cup \mathcal{M}'' \); where all components are taken irreducible. Let us assume that all components of \( \mathfrak{C}(C, A, d) \) are special. By Theorem 4.5 one has that (i) \( \mathfrak{C}(\mathcal{M}, A, d) = C \) or (ii) \( \mathfrak{C}(\mathcal{M}', A, d) = \mathfrak{C}(\mathcal{M}'', A, d) = C \). Let \( r = \deg(C) \). We take \( P \in \mathcal{C}_0 \) such that \( \Delta := +\|P - A\| \not\in \{-n d \mid n \in \mathbb{N} \text{ and } 1 \leq n \leq 2r + 1\} \). Note that this is always possible because \( \mathcal{C}_0 \neq \emptyset \) and \( C \) is not a circle centered at \( A \). Also, let \( \pi_i \) be the projections in the incidence diagram of \( C \), and let \( \pi_i^* \) be the projections in the incidence diagram of \( \mathfrak{C}(\mathcal{M}, A, d) \) (if (i) happens) and \( \pi_i', \pi_i'' \) be the projections in the incidence diagrams of \( \mathfrak{C}(\mathcal{M}', A, d) \) and \( \mathfrak{C}(\mathcal{M}'', A, d) \), respectively (if (ii) happens). We consider the families of points

\[
\{P_n := P + \frac{2n d}{\Delta} (P - A)\}_{n \in \mathbb{N}, 0 \leq n \leq r}, \quad \{Q_n := P + \frac{(2n + 1) d}{\Delta} (P - A)\}_{n \in \mathbb{N}, 0 \leq n \leq r}.
\]
Finally, thus, \( Q \parallel \) In addition, reasoning as above \( A \neq Q \).
Furthermore, since 
Let Remark 4.9 holds that:
Furthermore, by construction \( P_0 \in C_0 \). Consequently, \( P_0 = P \in C_0 \).
Therefore, \((Q_0, P_0, 1/\Delta) \in \mathcal{B}(C, A, d)\), and hence \( Q_0 = \pi_1(Q_0, P_0, 1/\Delta) \in \mathcal{C}(C, A, d)\). Moreover, \( \|Q_0 - A\|^2 = (\Delta + d)^2 \), that is not zero because of the selection of \( P \) (see above), and hence \( Q_0 \in \mathcal{C}(C, A, d) \). Now, we prove it for \( P_n \) and \( Q_n \). Since \( P_n = Q_{n-1} + d/\Delta (P - A) \), then \( P_n, Q_{n-1}, A \) are collinear, and \( \|P_n - Q_{n-1}\| = d \). Moreover, by induction hypothesis, \( Q_{n-1} \in \mathcal{C}(C, A, d) \).

Now, if (i) happens, then \( Q_{n-1} \in M_0 \). Thus, there exists \( w \in K \) such that 
\((P_n, Q_{n-1}, w) \in \mathcal{B}(M, A, d)\). So, \( P_n = \pi_1(P_n, Q_{n-1}, w) \in \mathcal{C}(M, A, d) = C \).
In addition, \( \|P - A\|^2 = (\Delta + 2nd)^2 \neq 0 \), because of the selection of \( P \). Hence \( P_n \in C_0 \). On the other hand, if (ii) happens, then \( Q_{n-1} \in M'_0 \cup M''_0 \). Say \( Q_{n-1} \in M'_0 \), similarly if \( Q_{n-1} \in M''_0 \). Thus, there exists \( w \in K \) such that 
\((P_n, Q_{n-1}, w) \in \mathcal{B}(M', A, d)\). So, \( P_n = \pi_1'(P_n, Q_{n-1}, w) \in \mathcal{C}(M', A, d) = C \).
In addition, reasoning as above \( \|P - A\|^2 = (\Delta + 2nd)^2 \neq 0 \). Hence \( P_n \in C_0 \).
Furthermore, since \( Q_n = P_n + d/\Delta (P - A) \), then \( P_n, Q_n, A \) are collinear and \( \|Q_n - P_n\| = d \). Since, we already know that \( P_n \in C_0 \), there exists \( w \in K \) such that 
\((Q_n, P_n, w) \in \mathcal{B}(C, A, d)\). So, \( Q_n = \pi_1(Q_n, P_n, w) \in \mathcal{C}(C, A, d) \).
Finally, \( \|Q_n - A\|^2 = (\Delta + (2n + 1)d)^2 \neq 0 \), because of the selection of \( P \).
Thus, \( Q_n \in \mathcal{C}(C, A, d) \).

The next corollary follows from the previous theorem and Theorem 3.4.

**Corollary 4.8** Let \( C \) be different to a line passing through the focus, then it holds that:

1. If \( \mathcal{C}(C, A, d) \) is irreducible then is simple.

2. \( \mathcal{C}(C, A, d) \) has at least one simple component.

**Remark 4.9** Let \( C \) be a line different to \( L^+ \) and \( L^- \) (i.e. \( C_0 \neq \emptyset \)), and let \( A \in C \). Then \( \mathcal{C}(C, A, d) \) is irreducible (in fact \( \mathcal{C}(C, A, d) = C \)) and special for every \( d \neq 0 \). Indeed, let \( C \) be the line of equation 
\[ f = \lambda(y_1 - a) + \mu(y_2 - b). \]

Observe that, by Prop. 3.7, \( C_0 = C \setminus \{A\} \). Then, if \( \lambda = 0 \), one gets that 
\[ \pi_1(\mathcal{B}(C, A, d)) = \{(y_1 \pm d, b) | (y_1, y_2) \in C \setminus \{A\}\} \subseteq C, \]
17
and hence $\mathcal{E}(C, A, d) = C$. Moreover, if $Q = (x_1, x_2) \in \pi_1(\mathcal{B}(C, A, d)) \setminus \{(a \pm d, b)\}$ then $\pi_2(\pi_1^{-1}(Q)) = \{(x_1 - d, b), (x_1 + d, b)\}$. Thus $\mathcal{E}(C, A, d)$ is special. On the other hand, if $\lambda \neq 0$, one gets that

$$\pi_1(\mathcal{B}(C, A, d)) = \left\{ \left( y_1 \mp \frac{d \mu}{\sqrt{\lambda^2 + \mu^2}}, y_2 \pm \frac{d \lambda}{\sqrt{\lambda^2 + \mu^2}} \right) \mid (y_1, y_2) \in C_0 \right\} \subset C.$$ 

Note that $\lambda^2 + \mu^2 \neq 0$ because $C \neq \mathcal{L}^+$ and $C \neq \mathcal{L}^-$. Therefore, $\mathcal{E}(C, A, d) = C$. Moreover, if

$$Q = (x_1, x_2) \in \pi_1(\mathcal{B}(C, A, d)) \setminus \left\{ \left( a \mp \frac{d \mu}{\sqrt{\lambda^2 + \mu^2}}, b \mp \frac{d \lambda}{\sqrt{\lambda^2 + \mu^2}} \right) \right\}$$

then

$$\pi_2(\pi_1^{-1}(Q)) = \left\{ \left( x_1 \mp \frac{d \mu}{\sqrt{\lambda^2 + \mu^2}}, x_2 \mp \frac{d \lambda}{\sqrt{\lambda^2 + \mu^2}} \right) \right\}$$

and hence $\mathcal{E}(C, A, d)$ is special.

\[\square\]

**Corollary 4.10** The only curves for which the conchoid is irreducible and special are the lines passing through the focus.

**Proof.** It follows from Theorem 4.7 and the above Remark.

\[\square\]

**Lemma 4.11** Let $C$ be different to a line passing through the focus, then there exist, at most, a finite number of distances for which all the conchoids $\mathcal{E}(C, A, d)$ have one common component.

**Proof.** Let $D \subset \mathbb{K}^*$ be an infinite set and $\mathcal{M}$ an irreducible algebraic curve such that $\mathcal{M} \subset \bigcap_{d \in D} \mathcal{E}(C, A, d)$. Let $r = \deg(C)$, and $d_1, \ldots, d_{r+1} \in D$ such that $d_i^2 \neq d_j^2$, $\forall i \neq j$. Let $\Omega = \bigcap_{i=1}^{r+1} \pi_{1,i}(\pi_{2,i}^{-1}(C_0)) \cap \mathcal{M}_0$, where $\pi_{1,i}$ and $\pi_{2,i}$ are the projections of the incidence diagram associated to $\mathcal{E}(C, A, d_i)$. Since $\mathcal{M}$ is irreducible, since is $\mathcal{M}_0$ non-empty an open in $\mathcal{M}$, by Lemma 4.10 one has that $\Omega$ is a non-empty Zariski subset of $\mathcal{M}$. Now, let $Q \in \Omega$. Note that since $Q \in \mathcal{M}_0$ then $Q \neq A$. Then, for all $i \in \{1, \ldots, r+1\}$ there exist $P_i \in C_0$ and $w_i \in \mathbb{K}^*$ such that $(Q, P_i, w_i) \in \mathcal{B}(C, A, d_i)$. So, $Q, P_i, A$ are collinear and $\|Q - P_i\|^2 = d_i^2$. Furthermore, $P_i \neq P_j$ for all $i \neq j$ and $i, j \in \{1, \ldots, r+1\}$, since $\|Q - P_i\|^2 = d_i^2 \neq d_j^2 = \|Q - P_j\|^2$. Now, let $\mathcal{L}$ denote the line passing through $Q$ and $A$; note that $\mathcal{L}$ is well-defined since $Q \neq A$. Then, one has that $\{P_1, \ldots, P_{r+1}\} \subset \mathcal{L} \cap \mathcal{C}$. Thus, $\text{Card}(\mathcal{L} \cap \mathcal{C}) > \deg(C)$ and hence $C = \mathcal{L}$, in contradiction with the hypothesis.

\[\square\]
Theorem 4.12 Let $C$ be different to a line passing through the focus. Then for almost every distance $d \in \mathbb{K}^*$ all the components of the conchoid $\mathcal{C}(C, A, d)$ are simple.

**Proof.** Let us assume that there exists an infinite set $D \subset \mathbb{K}^*$ such that for all $d \in \mathbb{K}^*$, the conchoid $\mathcal{C}(C, A, d)$ has a special component. Let $r = \deg(C)$, $\delta_r := 1 + \left(\frac{1}{r}\right)$, and $d_1, \ldots, d_{\delta_r} \in D$ such that: $d_i^2 \neq d_j^2$, $C$ is not a component of $\mathcal{C}(C, A, d_i)$, and $\mathcal{M}_i \neq \mathcal{M}_j$ being $\mathcal{M}_i$ the special component of $\mathcal{C}(C, A, d_i)$. Note that this is always possible because $d_i^2 \neq d_j^2$ and lemma 4.11. Let $\mathcal{M}_{i,0} = \{P \in \mathcal{M}_i | \|P - A\|^2 \neq 0\}$, and let

$$\Delta_i = (\mathcal{M}_i \setminus \mathcal{M}_{i,0}) \cup (\mathcal{M}_i \cap C) \cup (\mathcal{M}_j \cap \mathcal{M}_j).$$

Since $\mathcal{M}_i \neq \mathcal{M}_j$, $\mathcal{M}_i \neq \mathcal{C}$, using that $\mathcal{M}_{i,0}$ is open and non-empty in $\mathcal{M}_i$ (see Proposition 4.3), and that $C_0$ is open and non-empty in $\mathcal{C}$, one has that

$$\Delta := \bigcup_{i=1}^{\delta_r} \Delta_i \cup (\mathcal{C} \setminus C_0)$$

is a finite set. In this situation, we take a line $\mathcal{L}$ passing through $A$ and such that $\mathcal{L} \cap \Delta \subset \{A\}$. Let $Q_i \in \mathcal{L} \cap \mathcal{M}_{i,0}$ for $i \in \{1, \ldots, \delta_r\}$. This, in particular, implies that $Q_i \neq A$. By construction Card($\{Q_1, \ldots, Q_{\delta_r}\}$) = $\delta_r$. Since $\mathcal{M}_i$ is special, by Theorem 4.3 one gets that $\mathcal{C}(\mathcal{M}_i, A, d_i) = \mathcal{C}$. Furthermore, by construction $Q_i \in \mathcal{M}_{i,0}$. Therefore, there exist $P_i', P_i \in \mathcal{C}$, $P_i' \neq P_i$, and $w_i$ such that $(P_i, Q_i, w_i), (P_i', Q_i, w_i) \in \mathcal{B}(\mathcal{M}_i, A, d_i)$. Thus, $\|Q_i - P_i\|^2 = \|Q_i - P_i'\|^2 = d_i^2$ and $Q_i, P_i, P_i' \in \mathcal{L}$. Let $C_i$ be the circle centered at $Q_i$ and radius $d_i$. Since $Q_i \neq Q_j$ we have $\delta_r$ different circles with all the centers at $\mathcal{L}$. This implies that Card($\mathcal{L} \cap \bigcup_{i=1}^{\delta_r} C_i$) $\geq r + 1$. On the other hand $\mathcal{L} \cap C_i = \{P_i, P_i'\} \subset \mathcal{C}$. Hence there exist, at least, $r + 1$ different points in $\mathcal{L} \cap \mathcal{C}$ and therefore $\mathcal{C} = \mathcal{L}$, in contradiction with the hypothesis. \[\square\]

5 The Role of Simple Components

Simple components (see previous section) play an important role, from the theoretical point of view, in the study of conchoids. Essentially, they provide information on the birationality of the maps in the incidence diagram, and hence they open the door for studying, in further research, algebraic and geometric properties of the conchoids.

**Lemma 5.1** Let $\pi_1$ and $\pi_2$ be the projections in the incidence diagram associated with $\mathcal{C}$. 

19
(1) Let $C$ be different to a circle centered at $A$ and radius $d$. If $\mathcal{C}(C, A, d)$ is reducible and $M$ is an irreducible component of $\mathcal{C}(C, A, d)$, then the restricted map

$$\tilde{\pi}_2 = \pi_2|_{\pi_1^{-1}(M)} : \pi_1^{-1}(M) \longrightarrow C$$

is birational.

(2) If $M$ is an irreducible component of $\mathcal{C}(C, A, d)$, then the restricted map

$$\tilde{\pi}_1 = \pi_1|_{\pi_1^{-1}(M)} : \pi_1^{-1}(M) \longrightarrow M$$

is birational iff $M$ is simple.

**Proof.** (1) Let $M'$ be the other component of $\mathcal{C}(C, A, d)$ (see Theorem 3.4). Let $\Sigma = M \cap M'$. By Lemma 3.5, $C_1 := \pi_2(\pi_1^{-1}(M \setminus \Sigma)) \cap \pi_2(\pi_1^{-1}(M' \setminus \Sigma)) \cap C_0$ is a non-empty Zariski dense of $C$. We prove that for $P \in C_1$ the cardinality of $\tilde{\pi}_2^{-1}(P)$ is 1. Since $P \in C_0$, by Lemma 3.2 $\text{Card}(\pi_2^{-1}(P)) = 2$. Moreover, since $P \in \pi_2(\pi_1^{-1}(M \setminus \Sigma)) \cap \pi_2(\pi_1^{-1}(M' \setminus \Sigma))$, there exist $H \in \pi_1^{-1}(M \setminus \Sigma)$ and $H' \in \pi_1^{-1}(M' \setminus \Sigma)$ such that $\pi_2(H) = P$ and $\pi_2(H') = P$; i.e. $\tilde{\pi}_2^{-1}(P) = \{H, H'\}$. Thus, there exist $Q \in M \setminus \Sigma, Q' \in M' \setminus \Sigma$ (note that this implies that $Q \in M$ but $Q' \notin M$) such that $H = (Q, P, w)$ and $H' = (Q', P, w)$. So, $H \in \pi_1^{-1}(M)$ and $H' \notin \pi_1^{-1}(M)$. Therefore $\tilde{\pi}_2^{-1}(P) = \{H\}$.

(2) follows from the notions of simple component and birational map. \qed

From this lemma, one directly deduces the following corollary.

**Corollary 5.2** Let $C$ be such that $\mathcal{C}(C, A, d)$ is reducible. Then, the simple components of $\mathcal{C}(C, A, d)$ are birationally equivalent to $C$.

In the following example we illustrate these results.

**Example 5.3** Let $C$ be the plane curve defined by

$$f(y_1, y_2) = -3 + 9 y_1^2 + 9 y_2^2 + 2 y_2 - 4 y_2^4 - 4 y_1^4 - 8 y_1^2 y_2^2.$$  

Let $A = (0, -1)$ and distance $d = 1/2$ then $\mathcal{C}(C, A, d) = M \cup N$ where $N$ is defined by $N(\bar{x}) := x_1^2 + x_2^2 - 1$ and $M$ by $M(\bar{x}) := x_1^4 + 2 x_2^2 x_1^2 - 3 x_1^2 - 3 x_2^2 - 2 x_2 + x_2^4$. Note that $N$ is special (i.e. $\mathcal{C}(N, A, d) = C$) and $M$ is simple. $\mathcal{B}(C, A, d)$ decomposes as $\mathcal{B}(C, A, d) = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 := \pi_1^{-1}(N)$
and $\Gamma_2 := \pi_1^{-1}(M)$. By Lemma 5.1 (2), $\pi_1|_{\Gamma_2} : \Gamma_2 \to \mathcal{M}$ is birational and $\pi_1|_{\Gamma_2} : \Gamma_1 \to N$ is not. Indeed

$$(\pi_1|_{\Gamma_2})^{-1} : \mathcal{M} \to \Gamma_2$$

$$(x_1, x_2) \mapsto \left( x_1, x_2, -\frac{x_1(x_1^2 - 6x_2^2 - 4x_2)}{4(x_2 + 1)}, -\frac{x_2^2 - 4x_2 + x_1^2 - 2}{4}, \frac{1}{8x_2 + 9} \right).$$

Thus, by Lemma 5.1 (1),

$$\varphi : \mathcal{M} \to \mathcal{C}$$

$$(x_1, x_2) \mapsto \left( -\frac{x_1(x_1^2 - 6 + x_2^2 - 4x_2)}{4(x_2 + 1)}, -\frac{x_2^2 - 4x_2 + x_1^2 - 2}{4} \right)$$

is birational (i.e. $\varphi = \pi_2|_{\Gamma_2} \circ (\pi_1|_{\Gamma_2})^{-1}$). In fact,

$$\varphi^{-1} : \mathcal{C} \to \mathcal{M}$$

$$(y_1, y_2) \mapsto \left( \frac{y_1(3 + 8y_2 + 4y_1^2 + 4y_2^2)}{8(y_2 + 1)}, \frac{y_2 - 5}{8} + \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 \right).$$

\[\square\]

6 Detecting Conchoids

In this section, we show how special components can be used to decide whether a given irreducible plane curve $D$ (being different to $L^+$ and $L^-$) is the conchoid of another curve. First observe that one can always find a curve $C$, and $A \in \mathbb{K}^2$, $d \in \mathbb{K}^*$ such that $D$ is a component of $\mathcal{C}(C, A, d)$. For this purpose, one simply has to take $C$ as an irreducible component of $\mathcal{C}(D, A, d)$. Then, by Proposition 4.3 (2), $D \subset \mathcal{C}(C, A, d)$. So, we are now interested in deciding whether there exist $A \in \mathbb{K}^2$, $d \in \mathbb{K}^*$ and $C$ such that $D = \mathcal{C}(C, A, d)$. By Theorem 4.5, this is equivalent to decide whether there exist $A \in \mathbb{K}^2$, $d \in \mathbb{K}^*$ such that $\mathcal{C}(D, A, d)$ has an special component; if so, $C$ is the special component. We proceed as follows

Finding the focus. Let $g(x)$ be the defining polynomial of $D$, and $A = (a, b)$ where $a$, $b$ are unknowns. We consider a line $L$ passing through $A$ and a generic point $Q = (z_1, z_2)$, expressed parametrically as $L(t) = A + t(Q - A)$. Now, we take two different points $P_1 := L(t_1), P_2 := L(t_2)$ on $L$, and we
consider the algebraic set

\[
S = \left\{ (a, b, z_1, z_2, t_1, t_2, \omega) \in \mathbb{K}^7 \mid \begin{array}{l}
\text{Eq}_1 := g(L(t_1)) = 0 \\
\text{Eq}_2 := g(L(t_2)) = 0 \\
\text{Eq}_3 := \|L(t_1) - Q\|^2 = \|L(t_2) - Q\|^2 \\
\text{Eq}_4 := \omega \cdot (t_1 - t_2) = 1.
\end{array} \right\}
\]

\text{Eq}_1 \text{ and } \text{Eq}_2 \text{ ensure that } P_1, P_2 \in \mathcal{D}, \text{ Eq}_3 \text{ requires that } P_1, P_2 \text{ are on the same circle centered at } Q, \text{ and } \text{Eq}_4 \text{ guarantees that } P_1 \neq P_2. \text{ Therefore, if } \pi : \mathbb{K}^7 \to \mathbb{K}^2 \text{ is the projection } \pi(a, b, z_1, z_2, t_1, t_2, \omega) = (a, b), \text{ then the possible focuses } A, \text{ such that there exists } d \text{ for which } \mathcal{C}(\mathcal{D}, A, d) \text{ has an special component, belong to } \pi(S); \text{ i.e. belong to } V(I \cap \mathbb{K}[a, b]), \text{ where } I \text{ is the ideal generated by } \{\text{Eq}_1, \ldots, \text{Eq}_4\} \text{ (see Closure Theorem in [4] p. 122).}

\textbf{Example 6.1} \text{ Let } \mathcal{D} \text{ be the line defined by } g(x_1, x_2) = \lambda x_1 + \mu x_2 + \rho. \text{ Then,}

\[
\begin{align*}
\text{Eq}_1 &= \lambda (a + t_1 (z_1 - a)) + \mu (b + t_1 (z_2 - b)) + \rho \\
\text{Eq}_2 &= \lambda (a + t_2 (z_1 - a)) + \mu (b + t_2 (z_2 - b)) + \rho \\
\text{Eq}_3 &= (a + t_1 (z_1 - a) - z_1)^2 + (b + t_1 (z_2 - b) - z_2)^2 - \\
&\quad - (a + t_2 (z_1 - a) - z_1)^2 + (b + t_2 (z_2 - b) - z_2)^2 \\
\text{Eq}_4 &= \omega (t_1 - t_2) - 1.
\end{align*}
\]

Moreover, 

\[
\{\mu b + \lambda a + \rho, \lambda z_1 + \mu z_2 + \rho, (-z_2 + b)^2 (t_1 + t_2 - 2), (-z_2 + b)^2 (-2 \omega + 1 + 2 \omega t_2) , \omega t_1 - \omega t_2 - 1\}
\]

is a Gröbner basis of \{\text{Eq}_1, \ldots, \text{Eq}_4\} w.r.t. lexorder, with \(\omega > t_1 > t_2 > z_1 > z_2 > a > b\). Hence the possible focuses \((a, b)\) satisfy \(\mu b + \lambda a + \rho = 0\); i.e. they are on the line \(\mathcal{D}\). Indeed all of them are valid (see Remark [4.9]). □

\textbf{Detecting the Conchoid.} \text{ Let } \mathcal{D} \text{ and } g(x) \text{ be as above, and let us assume that we are given know a focus } A = (a, b) \text{ and we want to decide whether } \mathcal{D} \text{ is a conchoid from } A. \text{ So, we need to decide whether there exists } d \in \mathbb{K}^* \text{ such that } \mathcal{C}(\mathcal{D}, A, d) \text{ has an special component. For this purpose, let } L^*(t), \text{ Eq}_1^*, \ldots, \text{Eq}_4^* \text{ be the polynomials } L, \text{ Eq} \text{ specialized at } A. \text{ The new algebraic set, defined by } \{\text{Eq}_1^*, \ldots, \text{Eq}_4^*\}, \text{ say } S^*, \text{ is in } \mathbb{K}^5. \text{ In this situation, if } \pi^* := \mathbb{K}^5 \to \mathbb{K}^2 \text{ where } \pi^*(z_1, z_2, t_1, t_2, \omega) = (z_1, z_2) \text{ and } I^* \text{ the ideal generated by } \{\text{Eq}_1^*, \ldots, \text{Eq}_4^*\}, \text{ reasoning similarly as above, the algebraic Zariski closure } \mathcal{H} := \pi^*(S^*) = V(I^* \cap \mathbb{K}[z_1, z_2]) \text{ contains the special components of } \mathcal{C}(\mathcal{D}, A, d). \text{ Thus, we may factor } \mathcal{H}, \text{ and for each irreducible component we compute its generic conchoid (see Remark [2.2]) to afterwards checking whether for some } d \in \mathbb{K}^*
we get $\mathcal{D}$. Note that in $\mathcal{H}$ we also may have the curve defining the geometric locus of those points $Q$ such that when intersecting $\mathcal{D}$ with the line passing through $A, Q$ we get at two points $P_1, P_2$ satisfying that $\|P_1 - Q\| = \|P_2 - Q\|$, being this quantity non-constant. On the other hand, we only know that for all, but finitely many exceptions, specializations of the generic conchoid we get the conchoid. These exceptions can be determined (see Example 2.3) but the computation may be too heavy.

**Example 6.2** Let $\mathcal{D}$ be defined by $g(\bar{x}) = x_1^4 + 2x_2^2x_1^2 - 9x_1^2 - 4x_1 - 9x_2^2 + 12 + x_2^4$ and $A = (-2, 0)$. Computing a Grobner basis of the ideal $I^*$ generated by $\{E_{1}^{*}, \ldots, E_{4}^{*}\}$ w.r.t. lexorder, with $\omega > t_1 > t_2 > z_1 > z_2$, we get that $\mathcal{H} := V(I^* \cap K[z_1, z_2])$ decomposes as the union of the circle $\mathcal{H}_1$ defined by $-4 + z_1^2 + z_2^2$ and the quartic $\mathcal{H}_2$ defined by $-4 - 4z_1 + 15z_1^2 + 16z_1^3 + 4z_1^4 - z_2^2 + 16z_2^2z_1 + 8z_2^2z_1^2 + 4z_2^4$. The generic conchoid $\mathfrak{C}(\mathcal{H}_1, A, d)$ is given by

$$G(\bar{x}, d) = -8x_1^2 - 8x_2^2 - 4d^2 + 16 - 4x_1d^2 + x_1^4 + 2x_2^2x_1^2 - x_1^2d^2 - x_2^2d^2 + x_2^4.$$  

Solving the algebraic system is $d$ provided by $g(\bar{x}) = G(\bar{x}, d)$ one gets that $d = \pm 1$. Indeed $\mathcal{D} = \mathfrak{C}(\mathcal{H}_1, A, 1)$. Performing the same computations with $\mathcal{H}_2$ one gets that $\mathcal{D}$ is not a conchoid of $\mathcal{H}_2$.

**References**

[1] Arrondo E., Sendra J., Sendra J. R. (1997). *Parametric Generalized Offsets to Hypersurfaces*. Journal of Symbolic Computation vol. 23, pp. 267–285.

[2] Arrondo E., Sendra J., Sendra J. R. (1999). *Genus Formula for Generalized Offset Curves*, Journal of Pure and Applied Algebra vol. 136, no. 3, pp. 199–209.

[3] Azzam R.M.A. (1992). *Limacon of Pascal locus of the complex refractive indices of interfaces with maximally flat reflectance-versus-angle curves for incident unpolarized light*. Journal of the Optical Society of America A: Optics, Image Science, and Vision, Vol 9. pp. 957–963.

[4] Cox D., Little J. and O'Shea D. (1997). *Ideals, Varieties, and Algorithms*. Springer-Verlag, New York.
[5] Harris J. (1992). *Algebraic Geometry, a First Course*. Springer-Verlag.

[6] Kerrick A.H. (1959). *The limacon of Pascal as a basis for computed and graphic methods of determining astronomic positions*. Journal of the Institute of Navigation. vol.6, No.5.

[7] Menschik F. (1997). *The hip joint as a conchoid shape*. Journal of Biomechanics. 1997 Sep; 30(9):971-3 9302622.

[8] MyungJin Kang (2004). *Hip joint center location by fitting conchoid shape to the acetabular rim images*. Engineering in Medicine and Biology Society, 2004. IEMBS'04. 26th Annual International Conference. Vol 2. pp. 4477-4480 vol 6. ISBN:0-7803-8439-3.

[9] San Segundo F., Sendra J. R. (2005). *Degree Formulae for Offset Curves*. Journal of Pure and Applied Algebra vol. 195, pp. 301–335.

[10] San Segundo F., Sendra J. R. (2006). *Partial Degree Formulae for Plane Offset Curves*. arXiv:math/0609137v1 [math.AG]

[11] Sendra J., Sendra J. R. (2000). *Algebraic Analysis of Offsets to Hypersurfaces*. Mathematische Zeitschrift vol. 234, pp. 697–719.

[12] Shafarevich R. I., *Basic Algebraic Geometry*. Springer, 1977, 2nd edition, 1994.

[13] Sultan A. (2005). *The Limaçon of Pascal: Mechanical Generating Fluid Processing*. Journal of Mechanical Engineering Science. vol. 219, Number 8/ 2005. pp. 813–822. ISSN.0954-4062

[14] Szmulowicz F. (1996). *Conchoid of Nicomedes from reflections and refractions in a cone*. American Journal of Physics, vol. 64, pp. 467–471.

[15] Weigan L., Yuang E., Luk K.M. (2001). *Conchoid of Nicomedes and Limaçon of Pascal as Electrode of Static Field and a Waveguide of High Frequency Wave*. Progress In Electromagnetics Research Symposium, PIER 30, 273–284.