ON THE THEORY OF SURFACES IN THE FOUR-DIMENSIONAL
EUCLIDEAN SPACE

GEORGI GANCHEV AND VELICHKA MILOUSHEVA

Abstract

For a two-dimensional surface $M^2$ in the four-dimensional Euclidean space $E^4$ we introduce an invariant linear map of Weingarten type in the tangent space of the surface, which generates two invariants $k$ and $\kappa$.

The condition $k = \kappa = 0$ characterizes the surfaces consisting of flat points. The minimal surfaces are characterized by the equality $\kappa^2 - k = 0$. The class of the surfaces with flat normal connection is characterized by the condition $\kappa = 0$. For the surfaces of general type we obtain a geometrically determined orthonormal frame field at each point and derive Frenet-type derivative formulas.

We apply our theory to the class of the rotational surfaces in $E^4$, which prove to be surfaces with flat normal connection, and describe the rotational surfaces with constant invariants.

1. Introduction

In [4] T. Ōtsuki introduced curvatures $\lambda_1, \lambda_2, \ldots, \lambda_n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) for a surface $M^2$ in a $(2 + n)$-dimensional Euclidean space $E^{2+n}$, defining a quadratic form in the normal space of the surface. In a suitable local frame of the normal space this quadratic form can be written in a diagonal form and the functions $\lambda_x$, $x = 1, \ldots, n$ are the coefficients in the diagonalized form ($\lambda_x$ is called the $x$-th curvature of $M^2$). These curvatures are closely related to the Gauss curvature $K$ of $M^2$:

$$K = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$  

The local cross-section, which diagonalizes the quadratic form is called a Frenet cross-section (Frenet-frame) of the surface.

For a surface $M^2$ in the four-dimensional Euclidean space $E^4$ the curvatures $\lambda_1$ and $\lambda_2$ are the maximum and minimum, respectively of the Lipschitz-Killing curvature of the surface [5]. The function $\lambda_1$ is called the principal curvature and the function $\lambda_2$—the secondary curvature of $M^2$ in $E^4$.

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Using the idea of the Frenet-frames, Shiohama [6] proved that a complete connected orientable surface $M^2$ in $E^4$ with curvatures $\lambda_1 = \lambda_2 = 0$ is a cylinder. The same result is proved in [7] for a surface in a higher dimensional space $E^{2+n}$.

Our aim is to find invariants of a surface $M^2$ in $E^4$, considering a geometrically determined linear map (of Weingarten type) in the tangent space of the surface, as well as to obtain a geometric Frenet-type frame field of $M^2$.

In Section 2 we define a geometrical linear map in the tangent space of a surface $M^2$ in $E^4$ and determine a second fundamental form $II$ of the surface. We find invariants $k$ and $\kappa$ of $M^2$ (which are analogous to the Gauss curvature and the mean curvature of a surface in $E^3$). These invariants divide the points of $M^2$ into four types: flat, elliptic, parabolic and hyperbolic.

In Section 3 we give a local geometric description of the surfaces consisting of flat points, proving that they are either planar surfaces (Proposition 3.1) or developable ruled surfaces (Proposition 3.2).

In Section 4 we characterize the minimal surfaces in $E^4$ in terms of the invariants $k$ and $\kappa$ (Proposition 4.1).

For the surfaces of general type (which are not minimal and which have no flat points) in Section 5 we obtain a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point of the surface and derive Frenet-type derivative formulas. The tangent frame field $\{x, y\}$ is determined by the defined second fundamental form $II$, while the normal frame field $\{b, l\}$ is determined by the mean curvature vector field of the surface.

We also characterize the surfaces with flat normal connection in terms of the invariant $\kappa$ (Theorem 5.1).

In the last section we apply our theory to the class of the rotational surfaces in $E^4$, which prove to be surfaces with flat normal connection, and describe the rotational surfaces with $k = \text{const}$.

2. The Weingarten map

We denote by $g$ the standard metric in the four-dimensional Euclidean space $E^4$ and by $\nabla'$ its flat Levi-Civita connection. All considerations in the present paper are local and all functions, curves, surfaces, tensor fields etc. are assumed to be of the class $C^\infty$.

Let $M^2 : z = z(u, v), (u, v) \in \mathcal{D} (\mathcal{D} \subset \mathbb{R}^2)$ be a 2-dimensional surface in $E^4$. The tangent space to $M^2$ at an arbitrary point $p = z(u, v)$ of $M^2$ is $\text{span}\{z_u, z_v\}$.

For an arbitrary orthonormal normal frame field $\{e_1, e_2\}$ of $M^2$ we have the standard derivative formulas:

\[
\begin{align*}
\nabla'_{z_u} z_u &= z_{uu} = \Gamma^1_{11} z_u + \Gamma^2_{11} z_v + c^1_{11} e_1 + c^2_{11} e_2; \\
\nabla'_{z_v} z_v &= z_{vv} = \Gamma^1_{12} z_u + \Gamma^2_{12} z_v + c^1_{12} e_1 + c^2_{12} e_2; \\
\nabla'_{z_v} z_u &= z_{uw} = \Gamma^1_{22} z_u + \Gamma^2_{22} z_v + c^1_{22} e_1 + c^2_{22} e_2,
\end{align*}
\]

where $\Gamma^k_{ij}$ are the Christoffel's symbols and $c^k_{ij}, i, j, k = 1, 2$ are functions on $M^2$. 
We use the standard denotations $E(u, v) = g(z_u, z_u)$, $F(u, v) = g(z_u, z_v)$, $G(u, v) = g(z_v, z_v)$ for the coefficients of the first fundamental form and set $W = \sqrt{EG - F^2}$. If $\sigma$ denotes the second fundamental tensor of $M^2$, then we have

$$
\sigma(z_u, z_u) = c_{11}^1 e_1 + c_{12}^2 e_2,
$$

$$
\sigma(z_u, z_v) = c_{12}^1 e_1 + c_{12}^2 e_2,
$$

$$
\sigma(z_v, z_v) = c_{22}^1 e_1 + c_{22}^2 e_2.
$$

We introduce the following functions:

$$
\Delta_1 = \begin{bmatrix} c_{11}^1 & c_{11}^2 \\
  c_{12}^1 & c_{12}^2 \end{bmatrix}; 
\Delta_2 = \begin{bmatrix} c_{11}^1 & c_{12}^2 \\
  c_{12}^1 & c_{22}^2 \end{bmatrix}; 
\Delta_3 = \begin{bmatrix} c_{12}^1 & c_{22}^2 \\
  c_{12}^1 & c_{22}^2 \end{bmatrix};
$$

$$
L(u, v) = \frac{2\Delta_1}{W}, 
M(u, v) = \frac{\Delta_2}{W}, 
N(u, v) = \frac{2\Delta_3}{W}.
$$

If

$$
u = u(\bar{u}, \bar{v}), 
v = v(\bar{u}, \bar{v}), 
(\bar{u}, \bar{v}) \in \mathcal{D}, \mathcal{D} \subset \mathbb{R}^2
$$

is a smooth change of the parameters $\{u, v\}$ on $M^2$ with $J = u_\bar{u}v_\bar{v} - u_\bar{v}v_\bar{u} \neq 0$, then

$$
z_\bar{u} = z_u u_\bar{u} + z_v v_\bar{u},
$$

$$
z_\bar{v} = z_u u_\bar{v} + z_v v_\bar{v}.
$$

Let

$$
\sigma(z_\bar{u}, z_\bar{u}) = \bar{c}_{11}^1 e_1 + \bar{c}_{11}^2 e_2,
$$

$$
\sigma(z_\bar{u}, z_\bar{v}) = \bar{c}_{12}^1 e_1 + \bar{c}_{12}^2 e_2,
$$

$$
\sigma(z_\bar{v}, z_\bar{v}) = \bar{c}_{22}^1 e_1 + \bar{c}_{22}^2 e_2.
$$

Differentiating (2.2) and taking into account (2.1) we find

$$
\bar{c}_{11}^k = u_\bar{u}^2 c_{11}^k + 2u_\bar{u} v_\bar{v} c_{12}^k + v_\bar{v}^2 c_{22}^k,
$$

$$
\bar{c}_{12}^k = u_\bar{u} u_\bar{v} c_{11}^k + (u_\bar{u} v_\bar{v} + u_\bar{v} v_\bar{u}) c_{12}^k + v_\bar{v}^2 c_{22}^k, 
(k = 1, 2)
$$

$$
\bar{c}_{22}^k = u_\bar{v}^2 c_{11}^k + 2u_\bar{v} v_\bar{v} c_{12}^k + v_\bar{v}^2 c_{22}^k.
$$

Using (2.3), we obtain

$$
\bar{\Delta}_1 = J(u_\bar{u}^2 \Delta_1 + u_\bar{u} v_\bar{v} \Delta_2 + v_\bar{v}^2 \Delta_3);
$$

$$
\bar{\Delta}_2 = J(2u_\bar{u} u_\bar{v} \Delta_1 + (u_\bar{u} v_\bar{v} + u_\bar{v} v_\bar{u}) \Delta_2 + 2v_\bar{v}^2 v_\bar{v} \Delta_3);
$$

$$
\bar{\Delta}_3 = J(u_\bar{v}^2 \Delta_1 + u_\bar{v} v_\bar{v} \Delta_2 + v_\bar{v}^2 \Delta_3).
$$
If $E = g(z_u, z_\theta)$, $F = g(z_\theta, z_\tau)$ and $G = g(z_\tau, z_\varsigma)$, then we have

$$E = u_\theta^2 E + 2u_\theta v_\theta F + v_\theta^2 G,$$

(2.5)

$$F = u_\theta u_\tau E + (u_\theta v_\tau + v_\theta u_\tau)F + v_\theta v_\tau G,$$

$$G = u_\tau^2 E + 2u_\tau v_\tau F + v_\tau^2 G$$

and

$$EG - F^2 = J^2(EG - F^2)$$

or

(2.6)

$$W = \varepsilon JW, \quad \varepsilon = \text{sign } J.$$

Taking into account (2.4) and (2.6), we find

$$L = \varepsilon(u_\theta^2 L + 2u_\theta v_\theta M + v_\theta^2 N),$$

(2.7)

$$M = \varepsilon(u_\theta u_\tau L + (u_\theta v_\tau + v_\theta u_\tau)M + v_\theta v_\tau N),$$

$$N = \varepsilon(u_\tau^2 L + 2u_\tau v_\tau M + v_\tau^2 N).$$

Further we denote

$$\gamma_1^1 = \frac{FM - GL}{EG - F^2}, \quad \gamma_2^1 = \frac{FL - EM}{EG - F^2},$$

(2.8)

$$\gamma_1^2 = \frac{FM - GM}{EG - F^2}, \quad \gamma_2^2 = \frac{FM - EN}{EG - F^2}$$

and consider the linear map

$$\gamma : T_pM^2 \rightarrow T_pM^2$$

determined by the conditions

$$\gamma(z_u) = \gamma_1^1 z_u + \gamma_1^2 z_\theta, \quad \gamma(z_\theta) = \gamma_1^1 z_u + \gamma_2^2 z_\theta,$$

(2.9)

$$\gamma(z_\tau) = \gamma_1^1 z_\theta + \gamma_2^2 z_\tau, \quad \gamma(z_\varsigma) = \gamma_1^1 z_\theta + \gamma_2^1 z_\tau.$$

Then a tangent vector $X = \lambda z_u + \mu z_\theta$ is transformed into the vector $X' = \gamma(X) = \lambda' z_u + \mu' z_\theta$ so that

$$\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \gamma'( \begin{pmatrix} \lambda \\ \mu \end{pmatrix} ).$$

We have

Lemma 2.1. The linear map $\gamma$ given by (2.9) is geometrically determined.

Proof. Let the change of the parameters be given by (2.2). Then we have

$$\begin{pmatrix} z_u \\ z_\theta \\ z_\tau \\ z_\varsigma \end{pmatrix} = T \begin{pmatrix} z_u \\ z_\theta \\ z_\tau \\ z_\varsigma \end{pmatrix}, \quad T = \begin{pmatrix} u_\theta & v_\theta \\ u_\tau & v_\tau \end{pmatrix}. $$
If we denote
\[ g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad h = \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \]
then the defining conditions (2.8) imply \( \gamma = -hg^{-1} \).

With respect to the new coordinates \((\bar{u}, \bar{v})\) the linear map \( \bar{g} \) is determined by the equality \( \bar{g} = -h \bar{g}^{-1} \).

On the other hand, the equalities (2.5) and (2.7) express that
\[ g = T g T^t, \quad h = e T h T^t. \]
Thus we obtain \( \bar{g} = -h \bar{g}^{-1} = e T \gamma T^{-1} \), which implies that \( \bar{g} = e \gamma \).

Further, let \( \{\bar{e}_1, \bar{e}_2\} \) be another orthonormal normal frame field of \( M^2 \). Then
\[ e_1 = \cos \theta \bar{e}_1 + \varepsilon' \sin \theta \bar{e}_2; \quad e_2 = -\sin \theta \bar{e}_1 + \varepsilon' \cos \theta \bar{e}_2; \quad \theta = \angle(\bar{e}_1, e_1), \]
and \( \varepsilon' = 1 \) (\( \varepsilon' = -1 \)) if the normal frame fields \( \{e_1, e_2\} \) and \( \{\bar{e}_1, \bar{e}_2\} \) have the same (opposite) orientation. The relation between the corresponding functions \( e^k_i \) and \( \bar{e}^k_i \), \( i, j, k = 1, 2 \) is given by the equalities
\[ \bar{e}^1_i = \cos \theta e^1_i - \sin \theta e^2_i; \quad \bar{e}^2_i = \varepsilon'(\sin \theta e^1_i + \cos \theta e^2_i); \]
and
\[ \Delta_i = \varepsilon' \Delta_i, \quad i = 1, 2, 3, \quad \text{and} \quad \bar{L} = \varepsilon' L, \quad \bar{M} = \varepsilon' M, \quad \bar{N} = \varepsilon' N, \]
which imply that \( \bar{g} = \varepsilon' g \).

The linear map \( g : T_p M^2 \to T_p M^2 \) is said to be the Weingarten map at the point \( p \in M^2 \). The following statement follows immediately from Lemma 2.1.

**Lemma 2.2.** The functions
\[
k := \det \gamma = \frac{LN - M^2}{EG - F^2}, \quad \kappa := -\frac{1}{2} \text{tr} \gamma = \frac{EN + GL - 2FM}{2(EG - F^2)}
\]
are invariants of the surface \( M^2 \).

It is clear that the sign of \( \kappa \) depends on the orientations of the tangent plane and the normal space of \( M^2 \), while \( k \) is an absolute invariant.

The characteristic equation of the Weingarten map \( \gamma \) in view of Lemma 2.2 is
\[
v^2 + 2\kappa v + k = 0.
\]
If \( X_1 \) and \( X_2 \) are two tangent vectors at a point \( p \in M^2 \), then \( g(\gamma(X_1), X_2) = g(\gamma(X_2), X_1) \), i.e. \( \gamma \) is a symmetric linear operator and hence
\[
\kappa^2 - k \geq 0.
\]
Using the defining equalities (2.10), it follows that
This equality implies that the condition \( \kappa^2 - k = 0 \) is equivalent to the equalities
\[
\gamma_1^2 = \gamma_2^2, \quad \gamma_1^2 = 0,
\]
i.e. to the conditions
\[
L = \rho E, \quad M = \rho F, \quad N = \rho G, \quad \rho \in \mathbb{R}.
\]
Thus we get the following equivalence at a point \( p \in M^2 \):
\[
L = M = N = 0 \iff k = \kappa = 0.
\]
As in the classical case (for a surface \( M^2 \) in \( \mathbb{E}^3 \)), the invariants \( k \) and \( \kappa \) divide the points of \( M^2 \) into four types. A point \( p \in M^2 \) is said to be:
- flat, if \( k = \kappa = 0 \);
- elliptic, if \( k > 0 \);
- parabolic, if \( k = 0, \ \kappa \neq 0 \);
- hyperbolic, if \( k < 0 \).

Let \( X = \lambda z_u + \mu z_v, \ (\lambda, \mu) \neq (0, 0) \) be a tangent vector at a point \( p \in M^2 \). The Weingarten map \( \gamma \) determines a second fundamental form of the surface \( M^2 \) at \( p \in M^2 \) as follows:
\[
II(\lambda, \mu) = -g(\gamma(X), X) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \quad \lambda, \mu \in \mathbb{R}.
\]
First we study the class of surfaces whose points are flat.

3. Surfaces consisting of flat points

In this section we consider surfaces \( M^2 : z = z(u, v), \ (u, v) \in \mathcal{D} \) consisting of flat points, i.e. surfaces satisfying the conditions
\[
k(u, v) = 0, \quad \kappa(u, v) = 0, \quad (u, v) \in \mathcal{D}.
\]
We give a local geometric description of these surfaces.

For the sake of simplicity, we shall assume that the parametrization of \( M^2 \) is orthogonal, i.e. \( F = 0 \). Denote the unit vector fields \( x = \frac{z_u}{\sqrt{E}}, \ y = \frac{z_v}{\sqrt{G}} \). Then we write (2.1) in the form
\[
\begin{align*}
\nabla'_x x &= \gamma_1 y + \frac{c_{11}}{E} e_1 + \frac{c_{21}}{E} e_2, \\
\nabla'_x y &= -\gamma_1 x + \frac{c_{12}}{\sqrt{EG}} e_1 + \frac{c_{12}}{\sqrt{EG}} e_2, \\
\nabla'_y x &= -\gamma_2 y + \frac{c_{12}}{\sqrt{EG}} e_1 + \frac{c_{22}}{\sqrt{EG}} e_2, \\
\nabla'_y y &= \gamma_2 x + \frac{c_{22}}{G} e_1 + \frac{c_{22}}{G} e_2
\end{align*}
\]
Obviously, the surface $M^2$ lies in a 2-plane if and only if $M^2$ is totally geodesic, i.e. $c^k_{ij} = 0$, $i, j, k = 1, 2$.

Now, let at least one of the coefficients $c^k_{ij}$ not be zero. Then

$$\text{rank} \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{22} \\ c^2_{11} & c^2_{12} & c^2_{22} \end{pmatrix} = 1$$

and the vectors $\sigma(x, x)$, $\sigma(x, y)$, $\sigma(y, y)$ are collinear. Let $\{b, l\}$ be a normal frame field of $M^2$, consisting of orthonormal vector fields, such that $b$ is collinear with $\sigma(x, x)$, $\sigma(x, y)$, and $\sigma(y, y)$. It is clear that the normal frame field $\{b, l\}$ is invariant. Then the derivative formulas of $M^2$ can be written as follows:

$$\begin{align*}
\nabla'_x x &= \gamma_1 y + v_1 b, & \nabla'_y b &= -v_1 x - \lambda y + \beta_1 l, \\
\nabla'_y y &= -\gamma_1 x + \lambda b, & \nabla'_y b &= -\lambda x - v_2 y + \beta_2 l, \\
\nabla'_y x &= -\gamma_2 y + \lambda b, & \nabla'_y l &= -\beta_1 b, \\
\nabla'_y y &= \gamma_2 x + v_2 b, & \nabla'_y l &= -\beta_2 b,
\end{align*}$$

(3.3)

for some functions $v_1$, $v_2$, $\lambda$, $\beta_1$, $\beta_2$, $\gamma_1$, $\gamma_2$ on $M^2$.

The Gauss curvature $K$ of $M^2$ is expressed by

$$K = v_1 v_2 - \lambda^2.$$  (3.4)

Further we denote $\beta = \beta_1^2 + \beta_2^2$. It follows immediately that $\beta$ does not depend on the change (2.2) of the parameters.

Since the curvature tensor $R'$ of the connection $V'$ is zero, then the equalities $R'(x, y, b) = 0$ and $R'(x, y, l) = 0$ together with (3.3) imply that either $K = 0$ or $\beta = 0$.

A surface $M^2$ is said to be planar if there exists a hyperplane $E^3 \subset E^4$ containing $M^2$. First we shall characterize the planar surfaces.

**Proposition 3.1.** A surface $M^2$ is planar if and only if

$$k = 0, \quad \kappa = 0, \quad \beta = 0.$$ 

**Proof.** I. Let $M^2 \subset E^3$ and $b$ be the usual normal to $M^2$ in $E^3$. Choosing $l$ to be the normal to the hyperplane $E^3$, from (3.3) we get $L = M = N = 0$ and $\beta = 0$.

II. Under the conditions $k = \kappa = \beta = 0$, from (3.3) it follows that $l = \text{const}$ and $M^2$ lies in a hyperplane $E^3$ orthogonal to $l$. $\square$

A ruled surface $M^2$ is a one-parameter system $\{g(v)\}$, $v \in J$ of straight lines $g(v)$, defined in an interval $J \subset R^1$. The straight lines $g(v)$ are called generators of $M^2$. A ruled surface $M^2 = \{g(v)\}$, $v \in J$ is said to be developable, if the tangent space $T_p M^2$ at all regular points $p$ of an arbitrary fixed generator $g(v)$ is one and the same.

Each ruled surface $M^2$ can be parameterized as follows:
(3.5) \[ z(u, v) = x(v) + ue(v), \quad u \in \mathbb{R}, \; v \in J, \]

where \( x(v) \) and \( e(v) \) are vector-valued functions, defined in \( J \), such that the vectors \( e(v) \) and \( x'(v) + ue'(v) \) are linearly independent for all \( v \in J \). The tangent space of \( M^2 \) is spanned by the vectors

\[
zu = e(v); \\
zv = x'(v) + ue'(v).
\]

The ruled surface \( M^2 \) determined by (3.5) is developable if and only if the vectors \( e(v), e'(v) \) and \( x'(v) \) are linearly dependent.

We shall characterize the developable ruled surfaces in terms of the invariants \( k, \kappa \) and the Gauss curvature \( K \).

**Proposition 3.2.** A surface \( M^2 \) is locally a developable ruled surface if and only if

\[
k = 0, \quad \kappa = 0, \quad K = 0.
\]

**Proof.** I. Let \( M^2 \) be a developable ruled surface, defined by the equality (3.5), where \( e(v), e'(v) \) and \( x'(v) \) are linearly dependent. Without loss of generality we assume that \( \|e(v)\| = 1 \). Then, the vector fields \( e(v) \) and \( e'(v) \) are orthogonal and the tangent space of \( M^2 \) is \( \text{span}\{e(v), e'(v)\} \). Since \( x'(v) \in \text{span}\{e(v), e'(v)\} \), then \( x'(v) \) is decomposed in the form \( x'(v) = p(v)e(v) + q(v)e'(v) \) for some functions \( p(v) \) and \( q(v) \). Hence, the tangent space of \( M^2 \) is spanned by

\[
zu = e; \\
zv = pe + (u + q)e'.
\]

Considering only the regular points of \( M^2 \) (where \( u \neq -q \)), we choose an orthonormal tangent frame field \( \{x, y\} \) of \( M^2 \) in the following way:

\[
x = e = zu; \\
y = \frac{e'}{\|e'\|} = -\frac{p}{(u + q)\|e'\|} zu + \frac{1}{(u + q)\|e'\|} zv.
\]

Since the tangent space of \( M^2 \) does not depend on the parameter \( u \), then the normal space of \( M^2 \) is spanned by vector fields \( b_1(v), b_2(v) \). With respect to the basis \( \{e(v), e'(v), b_1(v), b_2(v)\} \) the derivatives of \( b_1(v) \) and \( b_2(v) \) are decomposed in the form

\[
b'_1 = -c_1 e' + c_0 b_2, \\
b'_2 = -c_2 e' - c_0 b_1,
\]

where \( c_0, c_1, c_2 \) are functions of \( v \).

Then the equalities (3.6) and (3.7) imply
\[ \nabla_x' b_1 = 0, \]
\[ \nabla_y' b_1 = - \frac{c_1}{u + q} y + \frac{c_0}{(u + q)\|e'\|} b_2, \]
\[ \nabla_x' b_2 = 0, \]
\[ \nabla_y' b_2 = - \frac{c_2}{u + q} y - \frac{c_0}{(u + q)\|e'\|} b_1. \]

Consequently, \( L = M = N = 0 \) and \( K = 0 \).

II. Let \( M^2 \) be a surface for which \( L = M = N = 0 \) and \( K = 0 \). We consider an orthonormal frame field \( \{x, y, b, l\} \) of \( M^2 \), satisfying the equalities (3.3). Since \( K = 0 \), then \( v_1 v_2 - \lambda^2 = 0 \). If \( v_1 = v_2 = 0 \), then \( M^2 \) lies in a plane \( E^2 \). So we assume that there exists a neighborhood \( \mathcal{D} \subset \mathcal{D} \) such that \( v_2|_\mathcal{D} \neq 0 \) and we consider the surface \( M^2 = M^2_{\mathcal{D}} \).

Let \( \{\bar{x}, \bar{y}\} \) be the orthonormal tangent frame field of \( M^2 \), defined by
\[ \bar{x} = \cos \varphi x + \sin \varphi y; \]
\[ \bar{y} = -\sin \varphi x + \cos \varphi y, \]
where \( \tan \varphi = -\frac{\lambda}{v_2} \). Then \( \sigma(\bar{x}, \bar{x}) = 0, \sigma(\bar{x}, \bar{y}) = 0 \). So the formulas (3.3) take the form
\[ \nabla_{\bar{x}}' \bar{x} = \bar{\gamma}_1 \bar{y}, \quad \nabla_{\bar{x}}' b = -\bar{\beta}_1 l, \]
\[ \nabla_{\bar{x}}' \bar{y} = -\bar{\gamma}_1 \bar{x}, \quad \nabla_{\bar{x}}' b = -\bar{\gamma}_2 \bar{y} + \bar{\beta}_2 l, \]
\[ \nabla_{\bar{y}}' \bar{x} = -\bar{\gamma}_2 \bar{y}, \quad \nabla_{\bar{y}}' l = -\bar{\beta}_1 b, \]
\[ \nabla_{\bar{y}}' \bar{y} = \bar{\gamma}_2 \bar{x} + \bar{\gamma}_2 b, \quad \nabla_{\bar{y}}' l = -\bar{\beta}_2 b, \]
where \( \bar{v}_2 \neq 0 \).

Since the curvature tensor \( R' \) is zero, then the equalities \( R'(\bar{x}, \bar{y}, b) = 0 \) and \( R'(\bar{x}, \bar{y}, l) = 0 \) imply that
\[ \bar{\gamma}_1 = 0, \quad \bar{\beta}_1 = 0. \]
Hence,
\[ \nabla_{\bar{x}}' \bar{x} = 0, \quad \nabla_{\bar{x}}' b = 0, \]
\[ \nabla_{\bar{y}}' \bar{x} = 0, \quad \nabla_{\bar{y}}' l = 0. \]

Let \( p = z(\bar{a}_0, \bar{v}_0), (\bar{a}_0, \bar{v}_0) \in \mathcal{D} \) be an arbitrary point of \( \tilde{M}^2 \) and \( c_1 : z(\bar{a}) = z(\bar{a}, \bar{v}_0) \) be the integral curve of the vector field \( \bar{x} \), passing through \( p \). It follows from \( \nabla_{\bar{x}}' \bar{x} = 0 \) that \( c_1 \) is contained in a straight line. Hence, \( M^2 \) lies on a ruled surface. Moreover, since \( \nabla_{\bar{x}}' b = 0 \) and \( \nabla_{\bar{y}}' l = 0 \) then the normal space \( \text{span}\{b, l\} \) of \( M^2 \) is constant at the points of \( c_1 \) and hence, the tangent space \( \text{span}\{\bar{x}, \bar{y}\} \) of \( M^2 \) at the points of \( c_1 \) is one and the same. Consequently, \( M^2 \) is part of a developable surface.

From now on we exclude the flat points from our considerations.
4. Minimal surfaces

We recall that a surface \( M^2 \) is said to be minimal if the mean curvature vector \( H = 0 \). In this section we characterize the minimal surfaces in terms of the invariants \( k \) and \( \varkappa \).

**Proposition 4.1.** Let \( M^2 \) be a surface in \( E^4 \) without flat points. Then \( M^2 \) is minimal if and only if

\[
\varkappa^2 - k = 0.
\]

**Proof.** Without loss of generality we assume that \( F = 0 \) and denote the unit vector fields \( x = \frac{z_u}{\sqrt{E}} \), \( y = \frac{z_v}{\sqrt{G}} \). Then we have

\[
\begin{align*}
\nabla_x'y &= \gamma_1 y + \frac{c_{11}}{E} e_1 + \frac{c_{21}}{E} e_2, \\
\nabla_x'y &= -\gamma_2 x + \frac{c_{12}}{\sqrt{EG}} e_1 + \frac{c_{22}}{\sqrt{EG}} e_2, \\
\nabla_y'x &= -\gamma_2 y + \frac{c_{12}}{\sqrt{EG}} e_1 + \frac{c_{22}}{\sqrt{EG}} e_2, \\
\nabla_y'y &= \gamma_2 x + \frac{c_{22}}{G} e_1 + \frac{c_{22}}{G} e_2.
\end{align*}
\]

I. Let \( H = \frac{1}{2} (\sigma(x, x) + \sigma(y, y)) = 0 \). Then

\[
\Lambda_2 = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0, \quad \frac{\Lambda_2}{G} = \frac{\Lambda_1}{E}.
\]
Therefore

\[
L = \rho E, \quad M = \rho F, \quad N = \rho G,
\]
where \( \rho \) is a function on \( M^2 \). Hence \( \varkappa^2 - k = 0 \).

II. Let \( \varkappa^2 - k = 0 \). Then

\[
L = \rho E, \quad M = \rho F, \quad N = \rho G; \quad \rho \neq 0.
\]

The condition \( F = 0 \) implies that \( M = 0 \). Then \( \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0 \) and \( c_{22} = \tilde{\rho} c_{11}, \)

\( c_{22} = \tilde{\rho} c_{11} \). Further, the equality \( \frac{L}{E} = \frac{N}{G} \) implies that \( \tilde{\rho} = -\frac{G}{E} \). Hence \( \text{tr} \sigma = 0 \), i.e. \( H = 0 \).

5. Surfaces of general type

From now on we consider surfaces, satisfying the condition
and call them surfaces of general type.

As in the classical differential geometry of surfaces in \( \mathbb{E}^3 \) the second fundamental form determines conjugate tangents at a point \( p \in M^2 \). A tangent \( g : X = (\lambda z_u + \mu z_v) \) is said to be principal if it is perpendicular to its conjugate. The equation for the principal tangents at a point \( p \) is

\[
\begin{vmatrix}
E & F \\
L & M
\end{vmatrix} \lambda^2 + 
\begin{vmatrix}
E & G \\
L & N
\end{vmatrix} \lambda \mu + 
\begin{vmatrix}
F & G \\
M & N
\end{vmatrix} \mu^2 = 0.
\]

A line \( c : u = u(q), v = v(q); q \in \mathcal{J} \) on \( M^2 \) is said to be a principal curve if its tangent at any point is principal.

The surface \( M^2 \) is parameterized with respect to the principal lines if and only if

\[
F = 0, \quad M = 0.
\]

Let \( M^2 \) be parameterized with respect to the principal lines and denote the unit vector fields \( x = \frac{z_u}{\sqrt{E}}, \quad y = \frac{z_v}{\sqrt{G}} \).

Since the mean curvature vector field \( H \neq 0 \), we determine the unit normal vector field \( b \) by the equality \( \quad b = \frac{H}{\|H\|} \). Further we denote by \( l \) the unit normal vector field such that \( \{x, y, b, l\} \) is a positive oriented orthonormal frame field of \( M^2 \). Thus we obtain a geometrically determined orthonormal frame field \( \{x, y, b, l\} \) at each point \( p \in M^2 \). With respect to the frame field \( \{x, y, b, l\} \) we have the following Frenet-type derivative formulas:

\[
\begin{align*}
\nabla'_x x &= \gamma_1 y + v_1 b; & \nabla'_x b &= -v_1 x - \lambda y + \beta_1 l; \\
\nabla'_y y &= -\gamma_1 x + \lambda b + \mu l; & \nabla'_y b &= -\lambda x - v_2 y + \beta_2 l; \\
\nabla'_x y &= -\gamma_2 y + \lambda b + \mu l; & \nabla'_x l &= -\mu y - \beta_1 b; \\
\nabla'_y y &= \gamma_2 y + v_2 b; & \nabla'_y l &= -\mu x + \beta_2 b,
\end{align*}
\]

where \( \gamma_1 = -y(\ln \sqrt{E}), \quad \gamma_2 = -x(\ln \sqrt{G}) \) and \( \mu \neq 0 \).

Hence we have

\[
k = -4v_1 v_2 \mu^2, \quad \varepsilon = (v_1 - v_2) \mu, \quad K = v_1 v_2 - (\lambda^2 + \mu^2).
\]

Remark 1. We note that we determine the tangent frame field \( \{x, y\} \) by the Weingarten map (the second fundamental form \( II \)) and the normal frame field \( \{b, l\} \)—by the mean curvature vector field, while the Frenet-cross section in the sense of Otsuki diagonalizes a quadratic form in the normal space. In general the geometric frame field \( \{b, l\} \) is not a Frenet-cross section. Finding the relation between \( \{b, l\} \) and the Frenet-cross section of Otsuki we derive the following relation between the invariant \( k \) and the curvatures \( \lambda, \mu \) of Otsuki:
The same formula is valid in the cases of minimal surfaces and surfaces consisting of flat points.

Using (5.1) we find the length \( \|H\| \) of the mean curvature vector field and taking into account (5.2) we obtain the formula

\[
\|H\| = \frac{\sqrt{\kappa^2 - \mu^2}}{2|\mu|},
\]

which shows that \( |\mu| \) is expressed by the invariants \( k, \kappa \) and the mean curvature function.

Let \( z = g(z, x)x + g(z, y)y \) be an arbitrary tangent vector field of \( M^2 \). We define the one-form \( \theta \) by the equality

\[
\theta(z) = g(\nabla_z b, l).
\]

Then the formulas (5.1) imply that \( \theta(z) = g(\beta_1 x + \beta_2 y, z) \), which shows that the one-form \( \theta \) corresponds to the tangent vector field \( \beta_1 x + \beta_2 y \) and

\[
\|\theta\| = \sqrt{\beta_1^2 + \beta_2^2}.
\]

Using that \( R'(x, y, x) = 0, R'(x, y, y) = 0, R'(x, y, b) = 0 \) and \( R'(x, y, l) = 0 \), we get the following integrability conditions:

\[
\begin{align*}
v_1 v_2 - (\lambda^2 + \mu^2) &= x(\gamma_2) + y(\gamma_1) - ((\gamma_1)^2 + (\gamma_2)^2); \\
2\mu v_2 + v_1 \beta_2 - \lambda \beta_1 &= x(\mu); \\
\lambda \gamma_2 + \mu \beta_1 - (v_1 - v_2) \gamma_1 &= x(\lambda) - y(v_1); \\
2\lambda \gamma_2 + \mu \beta_1 + (v_1 - v_2) \gamma_2 &= -x(v_2) + y(\lambda); \\
\gamma_1 \beta_1 - \gamma_2 \beta_2 + (v_1 - v_2) \mu &= -x(\beta_2) + y(\beta_1).
\end{align*}
\]

At the end of this section we shall characterize the surfaces with flat normal connection in terms of the invariant \( \kappa \).

A surface \( M^2 \) is said to be of flat normal connection [3] if the normal curvature \( R^\perp \) of \( M^2 \) is zero. The equalities (5.1) imply that the normal curvature \( R^\perp \) of \( M^2 \) is expressed as follows:

\[
\begin{align*}
R^\perp_n(x, y) &= D_x D_y b - D_y D_x b - D_{[x, y]} b = (x(\beta_2) - y(\beta_1) + \gamma_1 \beta_1 - \gamma_2 \beta_2) l, \\
R^\perp_l(x, y) &= D_x D_y l - D_y D_x l - D_{[x, y]} l = -(x(\beta_2) - y(\beta_1) + \gamma_1 \beta_1 - \gamma_2 \beta_2) b.
\end{align*}
\]

Taking in mind (5.4) and the last equality of (5.3) we get:
\[ R_b^l(x, y) = -\kappa l, \quad R_l^l(x, y) = \kappa b, \]
i.e.
\[ \kappa = g(R_l^l(x, y), b) = g(R_l^l(x, y)l, b). \]

The function \( g(R_l^l(x, y)l, b) \) is the curvature of the normal connection \( D \) of \( M^2 \). Hence, the invariant \( \kappa \) is the curvature of the normal connection.

Thus the surfaces with flat normal connection are characterized by the following

**Proposition 5.1.** A surface \( M^2 \) in \( E^4 \) is of flat normal connection if and only if
\[ \kappa = 0. \]

Obviously, \( M^2 \) is a surface with flat normal connection if and only if \( v_1 = v_2 = v \). So, the Frenet-type formulas (5.1) of a surface \( M^2 \) with flat normal connection take the form:

\[
\begin{align*}
\nabla'_x x &= \gamma_1 y + vb; & \nabla'_l b &= -vx - \lambda y + \beta_1 l; \\
\nabla'_x y &= -\gamma_1 x + \lambda b + \mu l; & \nabla'_l h &= -\lambda x - vy + \beta_2 l; \\
\nabla'_y x &= -\gamma_2 y + \lambda b + \mu l; & \nabla'_l l &= -\mu y - \beta_1 b; \\
\nabla'_y y &= \gamma_2 x + vb; & \nabla'_l l &= -\mu x - \beta_2 b.
\end{align*}
\]

Hence the invariants \( k \) and \( K \) are expressed by
\[ k = -4v^2\mu^2, \quad K = v^2 - (\lambda^2 + \mu^2). \]

**Remark 2.** The curvature of the normal connection of a surface \( M^2 \) in \( E^4 \) is the Gauss torsion \( \kappa_G \) of \( M^2 \) [1]. The notion of the Gauss torsion is introduced by Ŋ. Cartan [2] for a \( p \)-dimensional submanifold of an \( n \)-dimensional Riemannian manifold and is given by the Euler curvatures. In case of a 2-dimensional surface \( M^2 \) in \( E^4 \) the Gauss torsion can be expressed in terms of the ellipse of normal curvature at a point \( p \in M^2 \).

According to the theorem of Rodrigues, a curve \( c \) on a surface \( M^2 \) in \( E^3 \) is a line of curvature if and only if the tangential component of the derivative of the normal vector field to \( M^2 \) along \( c \) is collinear with the tangent of \( c \). Using this geometric characterization of the lines of curvature for surfaces in \( E^3 \), Ŋ. Cartan generalized in [2] the notion of lines of curvature for a surface \( M^2 \) in \( E^4 \). However, the lines of curvature in the sense of Cartan exist only in the class of the surfaces with zero Gauss torsion \( (\kappa_G = 0) \), i.e. in the class of the surfaces with flat normal connection.
6. Rotational surfaces

Now we shall apply our theory to the class of the rotational surfaces in \( E^4 \).

We denote by \( Oe_1 e_2 e_3 \) a fixed orthonormal base of \( E^3 \). Let \( c: \tilde{z} = \tilde{z}(u) \), \( u \in J \) be a smooth curve in \( E^3 \), parameterized by

\[
\tilde{z}(u) = (x_1(u), x_2(u), r(u)); \quad u \in J.
\]

We denote by \( e_1 \) the projection of \( c \) on the 2-dimensional plane \( Oe_1 e_2 \).

Without loss of generality we can assume that \( c \) is parameterized with respect to the arc-length, i.e. \((x'_1)^2 + (x'_2)^2 + (r')^2 = 1\). We assume also that \( r(u) > 0 \), \( u \in J \). Let us consider the rotational surface \( M^2 \) in \( E^4 \) given by

\[
z(u, v) = (x_1(u), x_2(u), r(u) \cos v, r(u) \sin v); \quad u \in J, \ v \in [0; 2\pi).
\]

The tangent space of \( M^2 \) is spanned by the vector fields

\[
z_u = (x'_1, x'_2, r' \cos v, r' \sin v);
z_v = (0, 0, -r \sin v, r \cos v).
\]

Hence,

\[
E = 1; \quad F = 0; \quad G = r^2(u); \quad W = r(u).
\]

We consider the following orthonormal tangent vector fields

\[
\tilde{x} = (x'_1, x'_2, r' \cos v, r' \sin v);
\]

\[
\tilde{y} = (0, 0, -\sin v, \cos v),
\]

i.e. \( z_u = \tilde{x} \); \( z_v = r\tilde{y} \). The second partial derivatives of \( z(u, v) \) are expressed as follows

\[
z_{uu} = (x''_1, x''_2, r'' \cos v, r'' \sin v);
z_{uv} = (0, 0, -r' \sin v, r' \cos v);
z_{vv} = (0, 0, -r \cos v, -r \sin v).
\]

Let \( \kappa \) and \( \tau \) be the curvature and the torsion of the curve \( c \) (considered as a curve in \( E^3 \)). We consider the normal vector fields \( e_1 \) and \( e_2 \), defined by

\[
e_1 = \frac{1}{\kappa} (x''_1 x'_2 - x'_1 x''_2, r'' \cos v, r'' \sin v);
\]

\[
e_2 = \frac{1}{\kappa} (x'_2 r'' - x''_2 r', x''_1 r' - x'_1 r'', (x''_1 x'_2 - x'_1 x''_2) \cos v, (x'_1 x''_2 - x''_1 x'_2) \sin v).
\]

Now it is easy to calculate that

\[
L = 0; \quad M = -(x'_1 x''_2 - x''_1 x'_2); \quad N = 0.
\]

Hence,

\[
k = - \frac{(x'_1 x''_2 - x''_1 x'_2)^2}{r^2}; \quad \tau = 0.
\]

Applying Proposition 5.1 we get
Corollary 6.1. Any rotational surface $M^2$ in $E^4$, defined by (6.1), is a surface with flat normal connection.

Let us denote the curvature of the plane curve $c_1$ by $\kappa_1 = x_1'x_2'' - x_1''x_2'$. Then with respect to the frame field $\{x, y, e_1, e_2\}$ the derivative formulas of $M^2$ look like:

$$\nabla'_x x = \kappa e_1; \quad \nabla'_x e_1 = -\kappa x + \tau e_2;$$
$$\nabla'_x y = 0; \quad \nabla'_x e_1 = \frac{r''}{kr} y;$$
$$\nabla'_y x = \frac{r'}{r} y; \quad \nabla'_y e_2 = -\tau e_1;$$
$$\nabla'_y y = -\frac{r'}{r} x - \frac{r''}{kr} e_1 - \frac{\kappa_1}{kr} e_2; \quad \nabla'_y e_2 = \frac{\kappa_1}{kr} y.$$

So, the Gauss curvature of $M^2$ is:

$$K = -\frac{r''}{r}.$$

Obviously $M^2$ is not parameterized with respect to the principal lines. The principal tangents of $M^2$ are:

$$x = \frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y;$$
$$y = \frac{\sqrt{2}}{2} x - \frac{\sqrt{2}}{2} y.$$

With respect to the geometric frame field $\{x, y, b, l\}$ the Frenet-type formulas (5.5) hold good, where

$$\gamma_1 = \gamma_2 = -\frac{\sqrt{2}}{2} \frac{r'}{r}; \quad \nu = \frac{\sqrt{(k^2r - r'')^2 + (\kappa_1)^2}}{2kr};$$
$$\lambda = \frac{k^2r^2 - (r'')^2 - (\kappa_1)^2}{2kr} \sqrt{(k^2r - r'')^2 + (\kappa_1)^2}; \quad \mu = \frac{\kappa \kappa_1}{\sqrt{(k^2r - r'')^2 + (\kappa_1)^2}}.$$

Consequently, the invariants $k$, $\kappa$ and $K$ of the rotational surface $M^2$ are:

$$k = -\frac{(\kappa_1)^2}{r^2}; \quad \kappa = 0; \quad K = -\frac{r''}{r}.$$

At the end of the section we shall describe all rotational surfaces, for which the invariant $k$ is constant.

1. The invariant $k = 0$ if and only if $\kappa_1 = 0$, which means that the projection of the curve $c$ on the plane $Oe_1e_2$ lies on a straight line. There are two subcases:
1.1. If $K = 0$, i.e. $r'' = 0$, then $M^2$ is a developable ruled surface.

1.2. If $K \neq 0$, i.e. $r'' \neq 0$, then $M^2$ is a planar surface.

2. The invariant $k = \text{const}$ ($k \neq 0$) if and only if $r(u) = a(x_1 u' x_2 u' - x_1 u' x_2')$, $a = \text{const}$. Moreover, if $r(u)$ satisfies $r''(u) = cr(u)$, then the Gauss curvature $K$ is also a constant.

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Georgi Ganchev
BULGARIAN ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS AND INFORMATICS
ACAD. G. BONCHEV STR. BL. 8
1113 SOFIA
BULGARIA
E-mail: ganchev@math.bas.bg

Velichka Milousheva
BULGARIAN ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS AND INFORMATICS
ACAD. G. BONCHEV STR. BL. 8
1113, SOFIA
BULGARIA
E-mail: vmil@math.bas.bg