The Z-eigenpairs of orthogonally diagonalizable symmetric tensors

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Abstract

In this paper, we focus on a special class of symmetric tensors, which can be orthogonally diagonalizable, and investigate their Z-eigenpairs problem. We show that the eigenpairs can be uniformly expressed using several basic eigenpairs, and the number of all the eigenpairs is uniquely determined by the order and rank of the symmetric tensor. In addition, we exploit the local optimality of each eigenpair by checking the second-order necessary condition.

Keywords: Z-eigenpairs, orthogonally diagonalizable, symmetric tensors, projected Hessian, local optimality

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1. Introduction

As the high-order generalization of matrix, tensor analysis and applications have been given more attentions in recent years \cite{1, 2}. Many concepts have been naturally extended from matrices to tensors, such as the inner product of the tensor\cite{1}, tensor norm\cite{2}, tensor rank\cite{3, 4, 5}, high-order Singular value decomposition (HOSVD)\cite{6}, etc. However, it has also been analyzed that most of tensor problems are NP hard\cite{7}, one of which is to obtain all the eigenpairs of symmetric tensors\cite{8, 9}. Different from the matrix case, there are several definitions for eigenpairs of symmetric tensors, such as D-eigenpairs\cite{10}, H-eigenpairs\cite{8}, Z-eigenpairs\cite{8}, etc. In this paper, we mainly focus on one of them — E-eigenpairs and when the corresponding eigenvector is real, it is also called Z-eigenpairs.

There are many algorithms and numerous applications that have been investigated concerning this subject. see for example \cite{11, 12, 13, 14, 15, 16, 17, 18, 19}. In this paper, we mainly investigate the Z-eigenpairs problem of a special class of symmetric tensors, which is orthogonally diagonalizable. We first prove that the eigenpairs of such a type of tensors can be enumerated in a linear-combination way using several basic eigenvectors, and the number of eigenpairs can

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be uniquely determined by the order and rank of the symmetric tensor. In addition, the local optimality of each eigenpair is also analyzed by checking the second-order necessary condition.

2. Preliminaries

We start by defining some notations. Let $\mathbb{C}$ and $\mathbb{R}$ be the complex and real field. High-order tensors are denoted boldface Euler script letters, e.g., $\mathcal{A}$. An $d$th-order tensor is denoted $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_d}$, where $d$ is the order (or the way, or the mode) of $\mathcal{A}$, and $I_j$ ($j \in \{1, 2, \ldots, d\}$) is the dimension of $j$th-mode. The element of $\mathcal{A}$, which is indexed by integer tuples $(i_1, i_2, \ldots, i_d)$, is denoted $(a_{i_1, i_2, \ldots, i_d})_{1 \leq i_1 \leq I_1, 1 \leq i_2 \leq I_2, \ldots, 1 \leq i_d \leq I_d}$. When $d = 2$, a tensor reduces to a matrix. When $d = 1$, it is a vector.

A tensor is called symmetric (or supersymmetric) if its elements remain invariant under any permutation of the indices $(i_1, i_2, \ldots, i_d)$\cite{1}. A symmetric tensor of order $m$ and dimension $n$ is denoted $\mathcal{S} \in \mathbb{R}^{n \times n \times \cdots \times n}$, whose element is

$$
(\mathcal{S} \times_j \mathcal{B})_{i_1 \ldots i_{n-1} j i_{n+1} \ldots i_d} = \sum_{i_j=1}^{I_j} a_{i_1, i_2, \ldots, i_d} b_{ji_n}.
$$

For different modes in series of multiplications, the order is commutative\cite{1}, i.e.,

$$
\mathcal{A} \times_j \mathcal{B} \times_k \mathcal{C} = \mathcal{A} \times_k \mathcal{C} \times_j \mathcal{B} \quad (j \neq k).
$$

If the modes are the same, it holds that

$$
\mathcal{A} \times_j \mathcal{B} \times_j \mathcal{C} = \mathcal{A} \times_j (\mathcal{C} \mathcal{B}).
$$

Given a $d$th-order tensor $\mathcal{A}$ and a series of matrices $\mathcal{B}^{(i)}$ ($i = 1, 2, \ldots, d$), it is simply denoted

$$
\mathcal{A} \times_1 \mathcal{B}^{(1)} \times_2 \mathcal{B}^{(2)} \times_3 \cdots \times_d \mathcal{B}^{(d)} = [\mathcal{A}; \mathcal{B}^{(1)}, \mathcal{B}^{(2)}, \ldots, \mathcal{B}^{(d)}],
$$

which is a notation introduced by Kolda in \cite{1}.

A tensor is called symmetric (or supersymmetric) if its elements remain invariant under any permutation of the indices $(i_1, i_2, \ldots, i_d)$\cite{1}. A symmetric tensor of order $m$ and dimension $n$ is denoted $\mathcal{S} \in \mathbb{R}^{n \times n \times \cdots \times n}$, whose element is

$$
\mathcal{S} = (s_{i_1, i_2, \ldots, i_m})_{1 \leq i_1 \leq n, 1 \leq i_2 \leq n, \ldots, 1 \leq i_m \leq n}, i_j \in \{1, 2, \ldots, n\}, j = 1, 2, \ldots, m.
$$

Let $T^m(\mathbb{R}^n)$ denote the space of all such real symmetric tensors. Given a vector $\mathbf{u} \in \mathbb{C}^n$ and a symmetric tensor $\mathcal{S} \in T^m(\mathbb{R}^n)$, a series of multiplication along different modes can be simply denoted as follows:

$$
\mathcal{S} \times_1 \mathbf{u}^T \times_2 \mathbf{u}^T \times_3 \cdots \times_m \mathbf{u}^T = \mathcal{S} \mathbf{u}^m = \sum_{i_1, i_2, \ldots, i_m=1}^{n} s_{i_1, i_2, \ldots, i_m} u_{i_1} \cdots u_{i_m}.
$$

And in a similar way, $\mathbf{u}^{m-1}$ denotes an $n$-dimensional column vector, whose $j$th element is

$$
(\mathbf{u}^{m-1})_j = \sum_{i_2, \ldots, i_m=1}^{n} s_{j, i_2, \ldots, i_m} u_{i_2} \cdots u_{i_m}.
$$
Furthermore, $S^m u^{-2}$ is an $n \times n$ matrix, whose $(i,j)$th element is

$$(S^m u^{-2})_{i,j} = \sum_{i_3, \ldots, i_m=1}^n s_{i_3,i_4,\ldots,i_m} u_{i_3} \cdots u_{i_m}. \quad (8)$$

Given a matrix $A \in \mathbb{R}^{n \times k}$ and assuming that $A$ is full column rank ($k \leq n$), the orthogonal complement projection matrix of $A$ is denoted $P_A = I_n - A(A^T A)^{-1} A^T$, where we use $I_n$ to denote an $n \times n$ identity matrix.

Given $d$ vectors $a^{(i)} \in \mathbb{R}^{l_i \times 1}$ ($i = 1, 2, \ldots, d$), their outer product $a^{(1)} \circ a^{(2)} \circ \cdots \circ a^{(d)}$ is a $d$th-order tensor, denoted $X \in \mathbb{R}^{l_1 \times l_2 \times \cdots \times l_d}$, whose element is the product of the corresponding vector element:

$$x_{i_1, i_2, \ldots, i_d} = a^{(1)}_{i_1} a^{(2)}_{i_2} \cdots a^{(d)}_{i_d}, \quad 1 \leq i_1 \leq I_1, 1 \leq i_2 \leq I_2, \ldots, 1 \leq i_d \leq I_d. \quad (9)$$

The tensor $X$ is called to be rank-one if it can rewritten as the outer product of $d$ vectors. When $a^{(1)} = a^{(2)} = \cdots = a^{(d)} = a$ ($I_1 = I_2 = \cdots = I_d$), we use the notation $X = a^{\otimes d}$ for simplicity, where $X$ is a symmetric tensor of order $d$ and dimension $I$. Let $A$ be the set of $k$ ($1 \leq k \leq r$) integers randomly selected from the set of $r$ integers $\{1, 2, \ldots, r\}$. $|A| = k$ denotes the number of the elements in $A$.

In this paper, the following optimization model is considered:

$$\begin{aligned}
\max_{u} & \quad S^m u^m \\
\text{s.t.} & \quad u^T u = 1.
\end{aligned} \quad (10)$$

The Lagrangian function of (10) is defined as:

$$L(u, \lambda) = \frac{1}{m} S^m u^m - \frac{\lambda}{2} (u^T u - 1). \quad (11)$$

When the gradient of $L(u, \lambda)$ to $u$ is 0, the eigenpair of a symmetric tensor can be deduced, which was independently defined by Lim and Qi in 2005:

**Definition 1.** Given a tensor $S \in T^n(\mathbb{R}^n)$, a pair $(\lambda, u)$ is an $S$-eigenpair of $S$ if

$$S^m u^{-1} = \lambda u, \quad (12)$$

where $\lambda \in \mathbb{C}$ is the eigenvalue and $u \in \mathbb{R}^{n \times 1}$ is the corresponding eigenvector satisfying $u^T u = 1$.

Assuming that $(\lambda, u)$ is an $S$-eigenpair of $S \in T^n(\mathbb{R}^n)$, and it is easily checked that so is $(\lambda', t u')$ for $t \in \mathbb{C}\setminus\{0\}$. This means that the solution of (12) consists of different equivalence classes. Such an equivalence class is denoted as follows:

**Definition 2.** Let $(\lambda, u)$ be an $S$-eigenpair of $S \in T^n(\mathbb{R}^n)$, the equivalence class of $(\lambda, u)$ is denoted

$$[(\lambda, u)] := \{(\lambda', t u') | \lambda' = t^{m-2} \lambda, u' = t u, t \in \mathbb{C}\setminus\{0\}\}. \quad (13)$$

Assume that (10) or (12) has finite solutions. Then, the following theorem provides a theoretical upper bound for the number of $S$-eigenpairs:
Theorem 1. [20] If a tensor $S \in T^{m}(\mathbb{R}^{n})$ has finitely many equivalence classes of $Z$-eigenpairs over $\mathbb{C}$, then their number, counted with multiplicity, is bounded by

$$M(m,n) = \frac{(m-1)^{n} - 1}{m-2}. \quad (14)$$

Cartwright and Sturmfels firstly proves the above theorem [20]. In the literature of [18], the authors also provided another version of the proof, which considers various types of tensor eigenvalues and provide an unified results.

The first-order gradient derivation (12) can be used to obtain all stationary points of (10). While the second-order derivation information plays an important role in identifying whether a stationary point is locally optimal given an optimization model. The second-order derivation of $L(u,\lambda)$ to $u$, i.e., the Hessian matrix of (11), is denoted

$$H(u) = (m-1)Su^{m-2} - \lambda I_{n}, \quad (15)$$

where $I_{n}$ is an $n \times n$ identity matrix. The following theorem is well established for the constrained optimization problem to identify the locally optimal solutions (Page 332 in [21]):

Theorem 2 (Second-order necessary condition). [21] Suppose that for any vector $w \in V$, if

$$w^{T}H(u)w \leq 0 \quad (16)$$

holds, then $u$ is a local maximum solution of (10). And for (11), the set $V$ is defined as

$$V = \{ w \in \mathbb{R}^{n}|(\nabla g)^{T}w = 0 \} = \text{Null}[(\nabla g)^{T}],$$

where $\text{Null}(A)$ denotes the null space of $A$ and $\nabla g = u$ denotes the gradient of the constraint: $g(u) = u^{T}u - 1 = 0$.

If a stronger condition, i.e., $w^{T}H(u)w < 0$, is satisfied, then, $u$ is a strict local maximum solution of (10).

In addition, instead of directly utilizing Theorem 2 it is more preferred to identify the local extremum by checking the positive or negative definiteness of the projected Hessian matrix (denoted $P \in \mathbb{R}^{(n-1) \times (n-1)}$). It is calculated by $P = Q_{2}^{T}H(u)Q_{2}$, where $Q_{2}$ is obtained by QR factorization of $\nabla g$:

$$\nabla g = Q \begin{bmatrix} R \\ 0 \end{bmatrix} = [Q_{1} \ Q_{2}] \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_{1}R, \quad (17)$$

where $Q$ is an $n \times n$ orthogonal matrix, and $R$ is a square upper triangular matrix. In this case, $R$ is reduced to a scalar since $\nabla g$ is a column vector and 0 is an $(n-1) \times 1$ vector with all elements equal to 0. $Q_{1}$ and $Q_{2}$ are $n \times 1$, $n \times (n-1)$ matrix, respectively. See more details for this part in Page 337 of [21].

The concept of the projected Hessian matrix has been widely researched for tensor eigenvalues problem. In [11], T.G.Kolda defines that an eigenvector $u$ was termed positive-stable if the corresponding projected Hessian matrix $P$ is positive-definite, and negative-stable if $P$ is negative-definite. The authors in [15] proposed an algorithm termed Orthogonal Newton correction method (ONCM), where the projected Hessian matrix is calculated in each iteration update step.

Using these preliminaries, we are interested in analyzing the $Z$-eigenpair problem of a special class of symmetric tensors, which can be orthogonally diagonalizable. And the definition is as follows:
Definition 3. [24] Given a symmetric tensor \( S \in T^m(\mathbb{R}^n) \), if there exists a matrix \( U = [u_1, u_2, \ldots, u_r] \in \mathbb{R}^{n \times r} \), composed of \( r \) (\( \leq n \)) orthonormal vectors \( u_i \) \((i = 1, 2, \ldots, r)\), and a diagonal tensor \( D \in T^m(\mathbb{R}^r) \) with \( D_{ii} = \lambda_i > 0 \), and it holds

\[
S = [D; U, U, \ldots, U] = \sum_{i=1}^{r} \lambda_i u_i^o \tag{18}
\]

then \( S \) is called orthogonally diagonalizable symmetric tensor, which is the summation of \( r \) rank-one tensors \( u_i^o \in \mathbb{R}^n \) \((i = 1, 2, \ldots, r)\).

Remark 1. The rank of a tensor \( X \) is defined as the smallest number of rank-one tensors that generate \( X \) as their sum. For the symmetric tensor \( S \) which can be orthogonally diagonalizable, it holds that \( \text{rank}(S) = \text{rank}(U) = r \).

Due to its special structural property, the following lemma holds:

Lemma 1. Let \( S \in T^m(\mathbb{R}^r) \), \( D \in \mathbb{R}^{n \times r} \) be a matrix composed of \( r \) orthonormal vectors \( u_i \) \((i = 1, 2, \ldots, r)\), and \( \sum_{i=1}^{r} \lambda_i u_i^o \in T^m(\mathbb{R}^n) \) holds. Then, \((\lambda_i, u_i)\) is an Z-eigenpair of the orthogonally diagonalizable symmetric tensor \( S \)

Proof. Since \( u_i^T u_j = 0 \) for \( \forall \ i \neq j \), thus

\[
S u_i^{m-1} = (\sum_{i=1}^{r} \lambda_i u_i^o) u_i^{m-1} = (\lambda_i u_i^o) u_i^{m-1} = \lambda_i (u_i^T u_i)^{m-1} u_i = \lambda_i u_i \tag{19}
\]

which implies that the conclusion holds.

3. Main results

Based on Lemma 1, we first show that the Z-eigenpair of \( S \) can be uniformly expressed as follows:

Lemma 2. Let \( S \in T^m(\mathbb{R}^r) \) with \( D_{ii} = \lambda_i > 0 \). Let \( U \in \mathbb{R}^{n \times r} \) be a matrix composed of \( r \) orthonormal vectors \( u_i \) \((i = 1, 2, \ldots, r)\), and \( \sum_{i=1}^{r} \lambda_i u_i^o \in T^m(\mathbb{R}^n) \) holds. For \( i \in \mathbb{A} \), we assume that \( \tilde{u} = \sum_{i \in \mathbb{A}} \tilde{c}_i u_i \) is a linear combination of \( u_i \), where the coefficients \( \tilde{c}_i = \frac{m-2}{\sqrt{\lambda_i}} \). Denote \( l = \| \tilde{u} \| = \sqrt{\sum_{i \in \mathbb{A}} \tilde{c}_i^2} \), then \((\lambda = \frac{1}{m-2}, u = \frac{\tilde{u}}{l})\) is an eigenpair of \( S \).

Proof. We start by computing

\[
S \tilde{u}^{m-1} = S(\sum_{i \in \mathbb{A}} \tilde{c}_i u_i)^{m-1} = \sum_{i \in \mathbb{A}} \tilde{c}_i^{m-1} S u_i^{m-1} = \sum_{i \in \mathbb{A}} \tilde{c}_i^{m-1} \lambda_i u_i \tag{20}
\]

When

\[
\tilde{c}_i^{m-1} \lambda_i = \tilde{c}_i \tag{21}
\]

is satisfied, i.e., \( \tilde{c}_i = \frac{m-2}{\sqrt{\lambda_i}} \), it holds that \( S \tilde{u}^{m-1} = \tilde{u} \). Normalize it into a unit length, it can be derived

\[
S \left( \frac{\tilde{u}}{l} \right)^{m-1} = \frac{1}{lm-2} \frac{\tilde{u}}{l} \tag{22}
\]

which implies that \((\lambda = \frac{1}{m-2}, u = \frac{\tilde{u}}{l})\) is an Z-eigenpair of \( S \).
An equivalent matrix expression is:

\[ \mathbf{u} = \mathbf{U}_c \]

where \( \mathbf{c} = \begin{bmatrix} \mathbf{c}_A \\ \mathbf{0}_{r-k} \end{bmatrix} \in \mathbb{R}^{r \times 1} \), \( \mathbf{c}_A \in \mathbb{R}^{k \times 1} \), \( \mathbf{0}_{r-k} \in \mathbb{R}^{(r-k) \times 1} \), and

\[ c_i = \frac{\tilde{c}_i}{\ell} = \frac{m-2}{\ell} \sqrt{\frac{1}{\lambda_i}}, \quad i \in \mathbb{A}. \]

Lemma 2 shows that the eigenpairs of \( S \) can be uniformly expressed using \( (\lambda_i, \mathbf{u}_i) (i = 1, 2, \ldots, r) \), which is termed the basic eigenpairs for the symmetric tensor \( S \). And all the eigenvectors is a linear combination of the basic eigenvectors where the \( k \) coefficients are given in (24) while the left \( r-k \) coefficients are equal to 0. Here, for convenience in the later analysis, we do not arrange the \( r \) eigenvectors from \( \mathbf{u}_1 \) to \( \mathbf{u}_r \), but always preferentially arrange the \( k \) participated eigenvector. It is easily checked that

\[ \lambda_i c_i^{m-2} = \lambda_i \left( \frac{m-2}{\ell} \right)^{m-2} = \lambda > 0, \quad i \in \mathbb{A}. \]

When \( |\mathbb{A}| = k = 1 \), it falls into \( (\lambda_i, \mathbf{u}_i) (i = 1, 2, \ldots, r) \). Naturally, the following question to be answered is how much is the number of all the eigenpairs, and we have the following lemma:

**Lemma 3.** Given a symmetric tensor \( S \in T^m(\mathbb{R}^n) \), if there exists a matrix \( \mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r] \in \mathbb{R}^{n \times r} \), composed of \( r \) (\( r \leq n \)) orthonormal vectors \( \mathbf{u}_i \) (\( i = 1, 2, \ldots, r \)), and a diagonal tensor \( \mathbf{D} \in T^m(\mathbb{R}^r) \) with \( \mathbf{D}_{i, i, \ldots, i} = \lambda_i > 0 \), and (18) holds, then, the number of all the eigenpairs is given by \( \frac{(m-1)^r - 1}{m-2} \).

**Proof.** It can be seen from (24) that \( c_i \) can be chosen to be one of the \( (m-2) \) roots of \( \frac{1}{\ell} \) (\( i = 1, 2, \ldots, r \)). Since \( k \) is randomly chosen form \( \{1, 2, \ldots, r\} \), the total number of the combinations is

\[ \sum_{k=1}^{r} \binom{r}{k} (m-2)^k = (m-2+1)^r - \binom{r}{0} = (m-1)^r - 1. \]

Based on Definition 13, there are \( m-2 \) distinct members of every equivalence class, and it is enough for each equivalence class to find one of them (Theorem 5 in [23]). So the number of all the eigenpairs is given by \( \frac{(m-1)^r - 1}{m-2} \), which is determined by the order and rank of the symmetric tensor.

**Remark 2.** When \( \text{rank}(S) = \text{rank}(\mathbf{U}) = r = n \), it is easily checked that the number of all eigenpairs is given by \( \frac{(m-1)^n - 1}{m-2} = M(m, n) \), which will reach at the theoretical upper bound for the number of eigenpairs of the symmetric tensor.

So far, we has answered the questions for an orthogonally diagonalizable symmetric tensor \( S \in T^m(\mathbb{R}^n) \):

- what is each eigenpair?
- how much is the number of all eigenpairs?
Next, we turn to analyzing Theorem 2, i.e., the local optimality of each eigenpair of the orthogonally diagonalizable symmetric tensor $S \in T^m(\mathbb{R}^n)$. The first difficulty that needs to be solved is how to obtain the matrix $Q_2$. Since it is derived via QR factorization, it can only be numerically computed but not suitable to be theoretically analyzed. To deal with this issue, the following lemma is presented:

**Lemma 4.** Let $u \in \mathbb{R}^{n \times 1}$ and $P = Q_2^T H(u) Q_2 \in \mathbb{R}^{(n-1) \times (n-1)}$ be the projected Hessian matrix at the point $u$, where $H(u)$ is defined in (15), and $Q_2$ is calculated by (17). Define a new matrix $M = P_u^\perp H(u) P_u^\perp$, where $P_u^\perp$ is the orthogonal complement projector of $u$. Then, it holds

$$P \preceq 0 \iff M \preceq 0. \tag{27}$$

**Proof.** First, denote the eigen-decomposition of $P$ as

$$P = V \Lambda V^T, \tag{28}$$

where

$$\Lambda = diag(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}) \tag{29}$$

is the eigenvalue matrix, and $V$ is an $(n-1) \times (n-1)$ orthogonal matrix. Furthermore, it can be easily checked that

$$P_u^\perp = I - uu^T = QQ^T - Q_1 R R^T Q_1^T$$

$$= [Q_1 \ Q_2] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} - Q_1 Q_1^T = Q_2 Q_2^T \tag{30}$$

then it holds

$$M = P_u^\perp H(u) P_u^\perp = Q_2 Q_2^T H(u) Q_2 Q_2^T$$

$$= Q_2 P Q_2^T = Q_2 V \Lambda (Q_2 V)^T = W \Lambda W^T \tag{31}$$

where $W = Q_2 V$ is an $n \times (n-1)$ orthogonal matrix. Since $W$ is singular, whose rank is $n-1$, we rewrite (31) as

$$M = W \Lambda W^T = \begin{bmatrix} W & v \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ W^T \\ v^T \end{bmatrix} \tag{32}$$

where $v \in \mathbb{R}^{n \times 1}$ is a unit vector, which lies in the null space of $W^T$, i.e., $W^T v = 0$, and $v^T v = 1$. Then it can proved that $V = \begin{bmatrix} W & v \end{bmatrix}$ is an orthogonal matrix, and thus, the eigenvalues matrix of $P_u^\perp H(u) P_u^\perp$ are

$$\tilde{\Lambda} = diag(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, 0). \tag{33}$$

In this way, based on (29) and (33), we can conclude that judging the positive or negative semidefiniteness of $P$ is equivalent to judging that of $M$, and vice verse. \hfill \Box

By using Lemma 4, we can avoid calculating the QR factorization of $u$, and turn to analyzing the positive or negative semidefiniteness of $M$, which can be explicitly expressed. Now, we are interested in identifying that which eigenpair corresponds to the local extremum of (10). In [24], the authors has considered the case of $m = 3$. 

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Theorem 3 (Theorem 4.2 in [24]). Let $\mathcal{T} \in T^3(\mathbb{R}^n)$ have an orthogonal decomposition as given in the form of

$$
\mathcal{T} = \sum_{i=1}^{r} \lambda_i \mathbf{u}_i^3,
$$

and consider the optimization problem

$$
\max_{\mathbf{u}} \quad \mathcal{T} \mathbf{u}^3 \quad \text{s.t.} \quad \mathbf{u}^T \mathbf{u} \leq 1.
$$

Then, 1): the stationary points are eigenvectors of $\mathcal{T}$, and 2): a stationary point $\mathbf{u}$ is an isolated local maximizer if and only if $\mathbf{u} = \mathbf{u}_i$ ($i = 1, 2, \ldots, r$).

The detailed proof can refer to [24]. However, the authors only consider the case of $m = 3$. Now, we are interested in what is the result for the cases of $m$ that which eigenvector is locally maximized. What is the local optimality of the other eigenpairs? Here, we provide a more generalized results for this issue. And the following theorem is presented:

Theorem 4. Let $\mathcal{S} \in T^m(\mathbb{R}^n)$ be an orthogonally diagonalizable symmetric tensor as given in the form of (37). Consider the optimization model in (10), for $|A| = k(1 \leq k \leq r)$, we have

case 1: when $k = 1$, $(\lambda_i, \mathbf{u}_i)$ ($i = 1, 2, \ldots, r$) is the isolated local maximum solution of (10).

case 2: when $1 < k \leq r < n$, a linear combination of $(\lambda_i, \mathbf{u}_i)$ is the saddle points of (10).

case 3: when $k = n$, a linear combination of $(\lambda_i, \mathbf{u}_i)$ ($i = 1, 2, \ldots, n$) is the only one isolated local minimum solution of (10).

Before proceeding our proof, we would like to re-express (23) as the following form:

$$
\mathbf{u} = \mathbf{U} \mathbf{c} = \begin{bmatrix} \mathbf{U} & \mathbf{U}_⊥ \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_{n-r} \end{bmatrix} = \hat{\mathbf{U}} \hat{\mathbf{c}},
$$

where $\mathbf{U}_⊥ = [\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \ldots, \mathbf{u}_n] \in \mathbb{R}^{n \times (n-r)}$ is a matrix that lies in the null space of $\mathbf{U}^T$, i.e., $\mathbf{U}^T \mathbf{u}_j = \mathbf{0}_r$, and the column vectors of $\mathbf{U}_⊥$ satisfy $\mathbf{u}_j^T \mathbf{u}_j = 1$, $\mathbf{u}_j^T \mathbf{u}_k = 0$ ($j \neq k$) for $j, k = r + 1, r + 2, \ldots, n$. $\mathbf{0}_{n-r} \in \mathbb{R}^{(n-r) \times 1}$ is a vector with all elements are equal to 0. $\hat{\mathbf{U}} = [\mathbf{U} \mathbf{U}_⊥] = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \ldots, \mathbf{u}_n] \in \mathbb{R}^{n \times n}$ is an orthogonal matrix that satisfies $\hat{\mathbf{U}}^T \hat{\mathbf{U}} = \hat{\mathbf{U}}^T \hat{\mathbf{U}} = \mathbf{I}_n$. $\hat{\mathbf{c}} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_{n-r} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_A \\ \mathbf{0}_{n-k} \end{bmatrix}$.

Proof. Based on the notation used in (23), and by utilizing properties (22) (35), we have

$$
\mathbf{S} \mathbf{u}^{m-2} = (\mathcal{D} \times_1 \mathbf{U} \times_2 \mathbf{U} \times_3 \cdots \times_m \mathbf{U}) \times_3 (\mathbf{U} \mathbf{c})^T \times_4 (\mathbf{U} \mathbf{c})^T \times_5 \cdots \times_m (\mathbf{U} \mathbf{c})^T
$$

$$
= \mathcal{D} \times_1 \mathbf{U} \times_2 \mathbf{U} \times_3 (\mathbf{c}^T \mathbf{U}^T \mathbf{U}) \times_4 (\mathbf{c}^T \mathbf{U}^T \mathbf{U}) \cdots \times_m (\mathbf{c}^T \mathbf{U}^T \mathbf{U})
$$

$$
= \mathcal{D} \times_1 \mathbf{U} \times_2 \mathbf{U} \times_3 (\mathbf{c}^T) \times_4 (\mathbf{c}^T) \cdots \times_m (\mathbf{c}^T)
$$

$$
= [\mathcal{D} \times_3 (\mathbf{c}^T) \times_4 (\mathbf{c}^T) \cdots \times_m (\mathbf{c}^T)] \times_1 \mathbf{U} \times_2 \mathbf{U}
$$

$$
= \mathbf{U} (\mathbf{D} \mathbf{c}^{m-2}) \mathbf{U}^T
$$

$$
= \mathbf{U} \Sigma \mathbf{U}^T.
$$

where $\Sigma$ is an $r \times r$ diagonal matrix, and based on (28), it holds

$$
\Sigma_{ii} = (\mathbf{D} \mathbf{c}^{m-2})_{ii} = \begin{cases} 
\mathcal{D}_{i,i} \mathbf{c}_i^{m-2} = \lambda_i \mathbf{c}_i^{m-2} = \lambda, & i \in \mathcal{A} \\
0, & i \notin \mathcal{A}
\end{cases}
$$

(37)
Considering (33), it can be further rewritten as

$$ Su^{m-2} = U \Sigma U^T = [U \ U_{\perp}] [\Sigma \ 0] [U \ U_{\perp}]^T = \hat{U} [\Sigma \ 0] \hat{U}^T. \quad (38) $$

Thus, we can derive

$$ (m-1)Su^{m-2} - \lambda I_n = (m-1)\hat{U} [\Sigma \ 0] \hat{U}^T - \lambda \hat{U} \hat{U}^T $$

$$ = \hat{U}[(m-1) [\Sigma \ 0] - \lambda I_n] \hat{U}^T = \hat{U} \Sigma \hat{U}, \quad (39) $$

where the elements of $\hat{\Sigma}$ satisfy

$$ \hat{\Sigma}_{ii} = [(m-1) [\Sigma \ 0] - \lambda I_n]_{ii} = \begin{cases} (m-2)\lambda, & i \in \hat{A} \\ -\lambda, & i \notin \hat{A}. \end{cases} \quad (40) $$

Therefore, it can be further divided into two blocks, which can be expressed as

$$ (m-1)Su^{m-2} - \lambda I_n = \hat{U} \hat{\Sigma} \hat{U} = \hat{U} \begin{bmatrix} \hat{\Sigma}_{\hat{A}} & 0 \\ 0 & \hat{\Sigma}_{\hat{B}} \end{bmatrix} \hat{U}^T, \quad (41) $$

where $\Sigma_{\hat{A}} = (m-2)\lambda I_k$, $\Sigma_{\hat{B}} = -\lambda I_{n-k}$, where $I_k$ and $I_{n-k}$ is an $k \times k$, $(n-k) \times (n-k)$ identity matrix.

Next, we consider $P_u^\perp$ in a similar way.

$$ P_u^\perp = I_n - uu^T = I_n - (\hat{U}c)(\hat{U}c)^T = \hat{U}\hat{U}^T - (\hat{U}c)(\hat{U}c)^T = \hat{U}(I_n - cc^T)\hat{U}^T $$

$$ = \hat{U}(I_n - \begin{bmatrix} c_{\hat{A}} \\ 0_{n-k} \end{bmatrix} \begin{bmatrix} c_{\hat{A}}^T \\ 0_{n-k} \end{bmatrix})\hat{U}^T = \hat{U} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{U}^T, \quad (42) $$

Then, it is derived

$$ M = P_u^\perp H(x)P_u^\perp $$

$$ = \hat{U} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{\Sigma} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{U}^T $$

$$ = \hat{U} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{\Sigma}_{\hat{A}} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{U}^T. \quad (43) $$

$$ = \hat{U} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{\Sigma}_{\hat{A}} \begin{bmatrix} I_k - c_{\hat{A}}c_{\hat{A}}^T & 0 \\ 0 & I_{n-k} \end{bmatrix} \hat{U}^T $$

Note that $I_k - c_{\hat{A}}c_{\hat{A}}^T$ is a projection matrix with rank $k - 1$, and it can be rewritten as the form of $Q_{\hat{A}}\Gamma Q_{\hat{A}}^T$, where $Q_{\hat{A}}$ is an $k \times k$ orthogonal matrix, $\Gamma = diag(1,1,\ldots,1,0)$ is an $k \times k$ diagonal matrix, where the number of the eigenvalue of 1 is $k - 1$. Then, (43) can be further denoted as

$$ M = \hat{U} \begin{bmatrix} Q_{\hat{A}} & 0 \\ 0 & Q_{\hat{B}} \end{bmatrix} \hat{\Sigma}_{\hat{A}} \Gamma \begin{bmatrix} Q_{\hat{A}}^T & 0 \\ 0 & Q_{\hat{B}}^T \end{bmatrix} \hat{U}^T, \quad (44) $$
where $Q_A$ is an $(n - k) \times (n - k)$ orthogonal matrix. Thus, (44) is the eigen-decomposition form of the matrix $M$, where $\hat{U} \begin{bmatrix} Q_A & 0 \\ 0 & Q_B \end{bmatrix}$ is the eigenvector matrix, and $\begin{bmatrix} \Sigma_A & 0 \\ 0 & \Sigma_B \end{bmatrix}$ is the eigenvalue matrix. Based on (40), the $n$ eigenvalues are 

$$-\lambda, -\lambda, \ldots, -\lambda, (m - 2)\lambda, (m - 2)\lambda, \ldots, (m - 2)\lambda, 0.$$ (45)

Then, we separately consider the sign of the eigenvalues in (45) according to the value of $k$:

**case 1:** when $k = 1$, the eigenpair falls into one of $(\lambda_i, u_i)(i = 1, 2, \ldots, r)$. Since $m \geq 3$, so $n$ eigenvalues are $-\lambda$ (the number is $n - 1$) and 0, which are all non-positive. This implies that $M \preceq 0$. Since $\lambda$ is positive, we conclude $P \succ 0$. Thus, $(\lambda_i, u_i)(i = 1, 2, \ldots, r)$ is the isolated local maximum solution of (10).

**case 2:** when $1 < k < r$, eigenvalues are either negative or positive, and both $M$ and $P$ are uncertain. So a linear combination of $(\lambda_i, u_i)$ is the saddle points of (10).

**case 3:** when $\text{rank}(U) = n$ and $k = n$, all eigenvalues are non-negative, indicating that $M \succeq 0$ and $P \succ 0$. a linear combination of $(\lambda_i, u_i)(i=1,2,\ldots,n)$ is the only one isolated local minimum solution of (10).

The proof is completed. \(\square\)

4. conclusion

In this paper, the Z-eigenpairs problem of orthogonally diagonalizable symmetric tensors is investigated. We first show that the eigenpairs can be expressed in an unified way and the number of all the eigepairs is determined by the order and rank of the symmetric tensor. Equipped with some theoretical analysis, we eventually provide an unified proof for the local optimality of the eigenpairs, which generalizes the result in [24].

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