Research article

A novel formulation of the fuzzy hybrid transform for dealing nonlinear partial differential equations via fuzzy fractional derivative involving general order

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Abstract: The main objective of the investigation is to broaden the description of Caputo fractional derivatives (in short, CFDs) (of order 0 < α < r) considering all relevant permutations of entities involving t₁ equal to 1 and t₂ (the others) equal to 2 via fuzzifications. Under gH-differentiability, we also construct fuzzy Elzaki transforms for CFDs for the generic fractional order α ∈ (r − 1, r). Furthermore, a novel decomposition method for obtaining the solutions to nonlinear fuzzy fractional partial differential equations (PDEs) via the fuzzy Elzaki transform is constructed. The aforesaid scheme is a novel correlation of the fuzzy Elzaki transform and the Adomian decomposition method. In terms of CFD, several new results for the general fractional order are obtained via gH-differentiability. By considering the triangular fuzzy numbers of a nonlinear fuzzy fractional PDE, the correctness and capabilities of the proposed algorithm are demonstrated. In the domain of fractional sense, the schematic representation and tabulated outcomes indicate that the algorithm technique is precise and straightforward. Subsequently, future directions and concluding remarks are acted upon with the most focused use of references.

Keywords: fuzzy set theory; Elzaki transform; Adomian decomposition method; nonlinear partial differential equation; Caputo fractional derivative

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1. Introduction

The idea of differential and integral calculus is essential for stronger and more comprehensive descriptions of natural reality. It aids in the modelling of the early evolution and forecasting the future of the respective manifestations. Furthermore, thanks to its capability to express more fascinating ramifications of heat flux [1–3], neural network [4], hydrodynamics [5], circuit theory [6], aquifers [7], chemical kinetics [8], epidemics [9–11], simulations [12], inequality theory [13–15] and henceforth. Numerous researchers have subsequently been drawn to the investigation of fractional calculus [16–22].

Fractional calculus is particularly effective at modelling processes or systems relying on hereditary patterns and legacy conceptions, and traditional calculus is a restricted component of fractional calculus. This approach seems to be as ancient as a classical notion, but it has just subsequently been applied to the detection of convoluted frameworks by numerous investigators, and it has been demonstrated by various researchers [23–25]. Fractional calculus has been advocated by a number of innovators [26–29]. Li et al. [30] contemplated a novel numerical approach to time-fractional parabolic equations with nonsmooth solutions. She et al. [31] developed a transformed method for solving the multi-term time-fractional diffusion problem. Qin et al. [32] presented a novel scheme to capture the initial dramatic evolutions of nonlinear sub-diffusion equations. Many scholars analyze simulations depicting viruses, bifurcation, chaos, control theory, image processing, quantum fluid flow, and several other related disciplines using the underlying concepts and properties of operators shown within the framework of fractional calculus [33–40].

Fuzzy set theory (FST) is a valuable tool for modelling unpredictable phenomena. As a result, fuzzy conceptions are often leveraged to describe a variety of natural phenomena. Fuzzy PDEs are an excellent means of modelling vagueness and misinterpretation in certain quantities for specified real-life scenarios, see [41–43]. In recent years, FPDEs have been exploited in a variety of disciplines, notably in control systems, knowledge-based systems, image processing, power engineering, industrial automation, robotics, consumer electronics, artificial intelligence/expert systems, management, and operations research.

Because of its relevance in a wide range of scientific disciplines, FST has a profound correlation with fractional calculus [44]. Kandel and Byatt [45] proposed fuzzy DEs in 1978, while Agarwal et al. [46] were the first to investigate fuzziness and the Riemann-Liouville (RL) differentiability concept via the Hukuhara-differentiability (HD) concept. FST and FC both use a variety of computational methodologies to gain a better understanding of dynamic structures while reducing the unpredictability of their computation. Identifying precise analytical solutions in the case of FPDEs is a complicated process.

Due to the model’s intricacy, determining an analytical solution to PDEs is generally problematic. As a result, there is a developing trend of implementing mathematical approaches to get an exact solution. The Adomian decomposition method (ADM) is a prominent numerical approach that is widely used. Several researchers have employed different terminologies to address FPDEs. Nemati and Matinfar [47] constructed an implicit finite difference approach for resolving complex fuzzy PDEs. Also, to demonstrate the competence and acceptability of the synthesized trajectory, experimental investigations incorporating parabolic PDEs were provided. According to Allahviranloo and Kermani [48], an explicit numerical solution to the fuzzy hyperbolic and parabolic equations is
provided. The validity and resilience of the proposed system were investigated in order to demonstrate that it is inherently robust. Arqub et al. [49] expounded the fuzzy FDE via the non-singular kernel considering the differential formulation of the Atangana-Baleanu operator. Authors [50] contemplated the numerical findings of fuzzy fractional initial value problems utilizing the non-singular kernel derivative operator.

Integral transforms are preferred by investigators when it pertains to identifying results for crucial difficulties. The Elzaki transformation [51], proposed by T. Elzaki in 2011, was used on the biological population model, the Fornberg–Whitham Model, and Fisher’s models in [52–54].

The purpose of this study is to furnish a relatively effective Adomian decomposition approach [55] that can address complex partial fuzzy DEs by leveraging the fuzzy Elzaki transform. It can address the dynamics of partial fuzzy differential equations by utilizing the fuzzy Elzaki transform. A revolutionary computational concept is characterized by generating the result of a nonlinear fuzzy fractional PDE. To solve the nonlinear elements of the equation, the Adomian polynomial [56] methodology is implemented. The innovative decomposition approach is known as the “fuzzy Elzaki technique”.

In this research, CFDs of order \( \alpha \in (0, r) \) for a fuzzy-valued mapping by employing all conceivable configurations of objects with \( t_1 \) equal to 1 and \( t_2 \) (the others) equal to 2 are presented. Also, a new result in connection between Caputo’s fractional derivative and the Elzaki transform via fuzzification is also presented. Taking into consideration \( g^H \)-differentiability for a new algorithm, the fuzzy Elzaki decomposition process, which is intended to generate the parameterized representation of fuzzy functions, is regarded as a promising technique for addressing fuzzy fractional nonlinear PDEs involving fuzzy initial requirements. The Elzaki transform, implemented here, is generally a modification of the Laplace and Sumudu transforms. The varying fractional order and uncertainty parameter, \( \varphi \in [0, 1] \), are utilized to reveal a demonstration case for the suggested approach. Both 2D and 3D models illustrate the test’s superiority to previous approaches. As a result, the new revelation provides a couple of responses that are very identical to the earlier ones. We do, however, have the option of selecting the most suitable one. Ultimately, as part of our attempt to close remarks, we highlighted the information gathered during our investigation.

The following is a synopsis of the persisting sections with regard to introduction and implementation: Section 2 represents the fundamentals and essential details of fractional calculus and fuzzy set theory. Problem formulation, implementation, and execution were all used in Section 3. Section 4 utilized the Caputo fractional derivative formulation via fuzzification in generic order and some further results. Section 5 utilized numerical algorithms and mathematical debates with some tabulation and graphical results. Ultimately, Section 6 utilized concluding and future highlights.

2. Preliminaries

This section consists of some significant concepts and results from fractional calculus and FST. For more details, see [28, 57].

Here, there be a space of all continuous fuzzy-valued mappings \( \mathbb{C}^f[\tilde{a}, \tilde{b}] \) on \([\tilde{a}, \tilde{b}]\). Moreover, the space of all Lebesgue integrable fuzzy-valued mappings on the bounded interval \([\tilde{a}, \tilde{b}] \subset \mathbb{R}\) are represented by \( \mathbb{L}^f[\tilde{a}, \tilde{b}] \).

**Definition 2.1.** ([58]) A fuzzy number (FN) is a mapping \( f : \mathbb{R} \mapsto [0, 1] \), that fulfills the subsequent
also any ∇(A) FN

Definition 2.5. The distance between two FNs is presented as follows:

Definition 2.2. (\[59\]) The R

Consider the two FNs

\[f(x) = \begin{cases} \sqrt{x}, & x \in [\bar{a}, \bar{b}], \\ 0, & x \notin [\bar{a}, \bar{b}] \end{cases}\]

The set of all FNs is denoted by the letter \(E^1\). If \(\bar{a} \in \mathbb{R}\), it can be regarded as a FN; \(\bar{a} = \chi_{[\bar{a}]}\) is the characteristic function, and therefore \(\mathbb{R} \subset E^1\).

Definition 2.2. (\[59\]) The \(\varphi\)-level set of \(f\) is the crisp set \([f]^\varphi\), if \(\varphi \in [0, 1]\) and \(f \in E^1\), then

\[\{x \in \mathbb{R} : f(x) \geq \varphi\}\]

Also, any \(\varphi\)-level set is closed and bounded, signifies by \([f(\varphi), \bar{f}(\varphi)]\), \(\forall \varphi \in [0, 1]\), where \(f, \bar{f} : [0, 1] \mapsto \mathbb{R}\) are the lower and upper bounds of \([f]^\varphi\), respectively.

Definition 2.3. (\[59\]) For each \(\varphi \in [0, 1]\), a parameterize formulation of FN \(f\) is an ordered pair \(f = [f(\varphi), \bar{f}(\varphi)]\) of mappings \(f(\varphi)\) and \(\bar{f}(\varphi)\) that addresses the basic conditions:

(i) There be a bounded left continuous monotonic increasing mapping \(\bar{f}(\varphi)\) on \([0, 1]\);

(ii) There be a bounded left continuous monotonic decreasing \(f(\varphi)\) on \([0, 1]\);

(iii) \(f(\varphi) \leq \bar{f}(\varphi)\).

Furthermore, the addition and scalar multiplication of FNs \(f_1 = [f_1(\varphi), \bar{f}_1(\varphi)]\) and \(f_2 = [f_2(\varphi), \bar{f}_2(\varphi)]\) are presented as follows:

\[\left[f_1 \oplus f_2\right]^\varphi = [f_1]^\varphi + [f_2]^\varphi = [f_1(\varphi) + f_2(\varphi), \bar{f}_1(\varphi) + \bar{f}_2(\varphi)]\]

\[k \circ [f_1]^\varphi = \begin{cases} [kf(\varphi), k\bar{f}(\varphi)], & k > 0, \\ [k\bar{f}(\varphi), kf(\varphi)], & k < 0. \end{cases}\]  (2.2)

As a distance between FNs, we employ the Hausdorff metric.

Definition 2.4. (\[58\]) Consider the two FNs \(f_1 = [f_1(\varphi), \bar{f}_1(\varphi)]\) and \(f_2 = [f_2(\varphi), \bar{f}_2(\varphi)]\) defined on \(E^1\). Then the distance between two FNs is presented as follows:

\[d(f_1, f_2) = \sup_{\varphi \in [0, 1]} \max \{|f_1(\varphi) - f_2(\varphi)|, |\bar{f}_1(\varphi) - \bar{f}_2(\varphi)|\}.\]  (2.3)

Definition 2.5. (\[60\]) A FN \(f\) has the following forms:

(i) If \(f(1) \geq 0\), then \(f\) is positive;

(ii) If \(\bar{f}(1) > 0\), then \(f\) is strictly positive;

(iii) If \(\bar{f}(1) \leq 0\), then \(f\) is negative;

(iv) If \(\bar{f}(1) < 0\), then \(f\) is strictly negative.

The set of non-negative and non-positive FNs are indicated by \(E^+\) and \(E^-\), respectively.

Consider \(D\) be the set represents the domain of fuzzy-valued mappings \(f\). Define the mappings \(f(., ; \varphi), \bar{f}(., ; \varphi) : D \mapsto \mathbb{R}, \forall \varphi \in [0, 1]\). These mappings are known to be the left and right \(\varphi\)-level mappings of the map \(f\).
Definition 2.6. ([61]) A fuzzy valued mapping \( f : D \mapsto E^1 \) is known to be continuous at \((s_0, \xi_0) \in D\) if for every \( \epsilon > 0 \) exists \( \delta > 0 \) such that \( d(f(s, \xi), f(s_0, \xi_0)) < \epsilon \) whenever \(|s - s_0| + |\xi - \xi_0| < \delta\). If \( f \) is continuous for each \((s_1, \xi_1) \in D\), then \( f \) is said to be continuous on \( D \).

Definition 2.7. ([62]) Suppose \( x_1, x_2 \in E^1 \) and \( y \in E^1 \) such that the subsequent satisfies:

(i) \( x_1 = x_2 \oplus y \)

or

(ii) \( y = x_1 \oplus (-1) \odot x_2 \).

Then, \( y \) is known to be the generalized Hukuhara difference (\( g_{H} \)-difference) of FNs \( x_1 \) and \( x_2 \) and is denoted by \( x_1 \oplus g_{H} x_2 \).

Again, suppose \( x_1, x_2 \in E^1 \), then \( x_1 \oplus g_{H} x_2 = y \) \iff

(i) \( y = (x_1(\phi) - x_2(\phi), x_1(\phi) - x_2(\phi)) \)

or

(ii) \( y = (x_1(\phi) - x_2(\phi), x_1(\phi) - x_2(\phi)) \).

The association regarding the \( g_{H} \)-difference and the Hausdorff distance is demonstrated by the following lemma.

Lemma 2.8. ([62]) For all \( f_1, f_2 \in E^1 \), then

\[
\lim_{\epsilon \to 0} d(f_1, f_2) = \sup_{\phi \in [0, 1]} \| [f_1]^{\phi} \odot g_{H} [f_2]^{\phi} \|, \tag{2.4}
\]

where, for an interval \([\bar{a}, \bar{b}]\), the norm is \( \| [\bar{a}, \bar{b}] \| = \max |\bar{a}, |\bar{b}|\).

Definition 2.9. ([63]) Let \( f : D \mapsto E^1 \) and \((x_0, \xi) \in D\). A mapping \( f \) is known as the strongly strongly generalized Hukuhara differentiable on \((x_0, \xi)\) (\( g_{H} \)-differentiable for short) if there exists an element \( \frac{\partial f(x_0, \xi)}{\partial \xi} \in E^1 \), then the subsequent holds:

(i) The following \( g_{H} \)-differences exist, if \( \forall \ \epsilon > 0 \) sufficiently small, then

\[
f(x_0 + \epsilon, \xi) \ominus g_{H} f(x_0, \xi), \quad f(x_0, \xi) \ominus g_{H} f(x_0 + \epsilon, \xi),
\]

the following limits hold as:

\[
\lim_{\epsilon \to 0} \frac{f(x_0 + \epsilon, \xi) \ominus g_{H} f(x_0, \xi)}{\epsilon} = \frac{\partial f(x_0, \xi)}{\partial \xi}. \tag{2.5}
\]

(ii) The following \( g_{H} \)-differences exist, if \( \forall \ \epsilon > 0 \) reasonably small, then

\[
f(x_0, \xi) \ominus g_{H} f(x_0 + \epsilon, \xi), \quad f(x_0 - \epsilon, \xi) \ominus g_{H} f(x_0, \xi),
\]

the following limits hold as:

\[
\lim_{\epsilon \to 0} \frac{f(x_0, \xi) \ominus g_{H} f(x_0 + \epsilon, \xi)}{-\epsilon} = \frac{\partial f(x_0, \xi)}{\partial \xi}. \tag{2.6}
\]

Lemma 2.10. ([64]) Suppose a continuous fuzzy-valued mapping \( f : D \mapsto E^1 \) and \( f(x, \xi) = [f(x, \xi; \phi), \bar{f}(x, \xi; \phi)], \forall \phi \in [0, 1] \). Then
(i) If $f(x, \xi)$ is (i)-differentiable for $x$ under Definition 2.9(i), then we have the following:

$$\frac{\partial f(x_0, \xi)}{\partial x} = \left( \frac{\partial f(x_0, \xi)}{\partial \xi}, \frac{\partial \bar{f}(x_0, \xi)}{\partial x} \right).$$

(2.7)

(ii) If $f(x, \xi)$ is (ii)-differentiable for $x$ under Definition 2.9(ii), then we have the following:

$$\frac{\partial f(x_0, \xi)}{\partial x} = \left( \frac{\partial \bar{f}(x_0, \xi)}{\partial \xi}, \frac{\partial f(x_0, \xi)}{\partial x} \right).$$

(2.8)

Theorem 2.11. ([65]) Suppose $f : \mathbb{R}^+ \rightarrow \mathbb{E}$ and $\forall \varphi \in [0, 1]$.

(i) There be Riemann-integrable mappings $f(x; \xi; \varphi)$ and $\bar{f}(x; \xi; \varphi)$ on $[0, \bar{b}]$ for each $\bar{b} \geq 0$.

(ii) $\underline{M}(\varphi) > 0$ and $\bar{M}(\varphi) > 0$ are the constants, then

$$\int_{0}^{\bar{b}} |f(x; \xi; \varphi)| dx \leq \underline{M}(\varphi), \quad \int_{0}^{\bar{b}} |\bar{f}(x; \xi; \varphi)| dx \leq \bar{M}(\varphi), \quad \forall \bar{b} \geq 0.$$

Then, the mapping $f$ is improper fuzzy Riemann-integrable on $[0, \infty)$ and the subsequent satisfies:

$$\mathcal{F}\mathcal{R} \int_{0}^{\infty} f(x) dx = \left( \int_{0}^{\infty} f(x; \varphi) dx, \int_{0}^{\infty} \bar{f}(x; \varphi) dx \right).$$

(2.9)

Theorem 2.12. ([16]) Suppose there be a positive integer $r$ and a continuous mapping $D^{-1}f$ defined on $J = [0, \infty)$ and a collection of piece wise continuous mappings $C$ defined on $J' = (0, \infty)$ is integrable on finite sub-interval of $J = [0, \infty)$ and assume that $\nu > 0$. Then

(i) If $D^r f$ is in $C$, then

$$D^{-\nu} f(x) = D^{-\nu-r} [D^r f(x)] + X_r(x, \nu).$$

(ii) If there be a continuous mapping $D^r f$ on $J$, then for $x > 0$

$$D^r [D^{-\nu} f(x)] = D^{-\nu} [D^r f(x)] + X_r(x, \nu - r),$$

where

$$X_r(x, \nu) = \sum_{k=0}^{r-1} \frac{x^{\nu+k}}{\Gamma(\nu+k+1)} D^k f(0).$$

3. Fuzzy Elzaki transform

Definition 3.1. ([51]) Suppose a continuous fuzzy-valued mapping $f : \mathbb{R}^+ \rightarrow E$ and for $\omega > 0$, there be an improper fuzzy Riemann-integrable mapping $f(\xi) \ominus \exp(-\xi/\omega)$ defined on $[0, \infty)$. Then we have

$$\mathcal{F}\mathcal{R} \int_{0}^{\infty} \omega f(\xi) \exp(-\xi/\omega) d\xi, \quad \omega \in (p_1, p_2),$$

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which is known as the Fuzzy Elzaki transform and represented as

\[ W(\omega) = \mathbb{E}[f(\xi)] = \mathcal{F}\mathcal{R} \int_0^\infty \omega f(\xi) \exp(-\xi/\omega) d\xi. \]

The parameterized version of fuzzy Elzaki transform:

\[ \mathbb{E}[f(\xi)] = \left[ \mathcal{E}[f(\xi; \varphi)], \mathcal{E}[f(\xi; \psi)] \right], \]

where

\[ \mathcal{E}[f(\xi; \varphi)] = \int_0^\infty \omega f(\xi; \varphi) \exp(-\xi/\omega) d\xi, \]

\[ \mathcal{E}[f(\xi; \psi)] = \int_0^\infty \omega f(\xi; \psi) \exp(-\xi/\omega) d\xi. \]

4. Fuzzy Elzaki transform of the fuzzy CFDs of orders \( r - 1 < \alpha < r \)

This section consists of CFDs of the general fractional order \( 0 < \alpha < r \). Also, we obtain fuzzy Elzaki transform for CFD of the generic order \( r - 1 < \alpha < r \) for fuzzy valued mapping \( f \) under \( g\mathcal{H} \)-differentiability.

For the sake of simplicity, for \( 0 < \alpha < r \) and \( f(\mathbf{x}) \in \mathbb{C}^{C}[0, \hat{b}] \cap \mathbb{L}^{C}[0, \hat{b}] \), denoting

\[ \mathcal{G}(\mathbf{x}) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_0^\infty \frac{f(\xi)d\xi}{(\mathbf{x} - \xi)^{1-[\alpha]+\alpha}} \Theta \sum_{\kappa=0}^{[\alpha]-\alpha} \frac{\Gamma([\alpha] - \alpha + \kappa)}{\Gamma(1 + [\alpha] - \alpha + \kappa)} D^\alpha f(0) \mathbb{X}^{[\alpha]-\alpha+\kappa}, \]  

(4.1)

**Definition 4.1.** Suppose \( f(\mathbf{x}) \in \mathbb{C}^{C}[0, \hat{b}] \cap \mathbb{L}^{C}[0, \hat{b}] \) and \([\alpha]\) and \([\alpha]\) indicates \( \alpha \) values that have been rounded forward and descends to the closest integer value, respectively. It is clear that \( \mathcal{G}(\mathbf{x}) \) and the mappings \( \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{1}(\mathbf{x}_0) \) and \( \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{2}(\mathbf{x}_0) \) are stated as

\[ \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{1}(\mathbf{x}_0) = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{j_1,j_2,...,j_{\ell}}^{1}(\mathbf{x}_0 + \epsilon) \Theta \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{1}(\mathbf{x}_0)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{j_1,j_2,...,j_{\ell}}^{1}(\mathbf{x}_0 + \epsilon) \Theta \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{1}(\mathbf{x}_0 - \epsilon)}{-\epsilon}, \]  

(4.2)

\[ \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{2}(\mathbf{x}_0) = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{j_1,j_2,...,j_{\ell}}^{2}(\mathbf{x}_0) \Theta \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{2}(\mathbf{x}_0 + \epsilon)}{-\epsilon} = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{j_1,j_2,...,j_{\ell}}^{2}(\mathbf{x}_0 - \epsilon) \Theta \mathcal{G}_{j_1,j_2,...,j_{\ell}}^{2}(\mathbf{x}_0)}{\epsilon}, \]  

(4.3)

for \( \ell = 0, 1, 2, ..., r - 2 \) such that \( j_1, j_2, ..., j_{\ell} \) are all possible arrangements of \( \ell \) objects that represents the numbers in the following principal:

\[ \ell P_{t_1,t_2} = \frac{\ell!}{t_1!t_2!}, \]  

where \( t_1 \) of them equal 1 (means CD in the first version) and \( t_2 \) of them equal 2 (means CD in the second version). Also, \( j_1, j_2, ..., j_0 \).
Now, \( f(x) \) is the Caputo fractional type fuzzy differentiable mapping of order \( 0 < \alpha < r, \alpha \neq 1, 2, \ldots, r - 1 \) at \( x_0 \in (0, \tilde{b}) \) if there exists an \( \epsilon > 0 \) such that \( \forall \psi \in [0, 1] \) and \( \epsilon > 0 \) close to zero. Then

(i) If \( J_{(\alpha)} = 1 \), then

\[
(\mathcal{D}^{\psi}f)(x_0) = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0 + \epsilon) \Theta \mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0)}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0) \Theta \mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0 - \epsilon)}{\epsilon}. \tag{4.4}
\]

(ii) If \( J_{(\alpha)} = 2 \), then

\[
(\mathcal{D}^{\psi}f)(x_0) = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0) \Theta \mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0 + \epsilon)}{-\epsilon} = \lim_{\epsilon \to 0^+} \frac{\mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0 - \epsilon) \Theta \mathcal{G}_{J_1, J_2, \ldots, J_{(\alpha)}}(x_0)}{-\epsilon}, \tag{4.5}
\]

for \( \alpha \in (\kappa - 1, \kappa), \kappa = 1, 2, \ldots, r \) such that \( J_1, J_2, \ldots, J_{(\alpha)} \) are all the suitable arrangements of \( [\alpha] \) objects that have the following representation:

\[ [\alpha] = \frac{[\alpha]!}{t_1!t_2^!}, \ t_1 + t_2 = [\alpha]. \]

**Theorem 4.2.** Suppose \( f(x) \in \mathbb{C}^\ell[0, \tilde{b}] \cap \mathbb{L}\mathcal{F}[0, \tilde{b}] \) be a fuzzy-valued mapping such that \( f(x) = [f(x; \psi), \tilde{f}(x; \psi)] \) for \( \psi \in [0, 1] \), \( x_0 \in (0, \tilde{b}) \) and \( \mathcal{G}(x) \) is stated in (4.1).

Assume that \( 0 < \alpha < r \) and \( \ell \) is the number of repetitions of 2 among \( J_1, J_2, \ldots, J_{(\alpha)} \) for \( \alpha \in (\kappa - 1, \kappa), \kappa = 1, 2, \ldots, r \), say, \( J_{k_1}, J_{k_2}, \ldots, J_{k_\ell} \) such that \( k_1 < k_2 < \ldots < k_\ell \), i.e., \( J_{k_1} = J_{k_2} = \ldots = J_{k_\ell} = 2 \) and \( 0 \leq \ell \leq [\alpha] \). Then we have the following:

If \( \ell \) is even number, then

\[
(\mathcal{D}^{\alpha}f)(x_0) = \left[(\mathcal{D}^{\alpha}f)(x_0; \psi), (\mathcal{D}^{\alpha}\tilde{f})(x_0; \psi)\right]. \tag{4.6}
\]

If \( \ell \) is odd number, then

\[
(\mathcal{D}^{\alpha}f)(x_0) = \left[(\mathcal{D}^{\alpha}\tilde{f})(x_0; \psi), (\mathcal{D}^{\alpha}f)(x_0; \psi)\right], \tag{4.7}
\]

where

\[
(\mathcal{D}^{\alpha}\tilde{f})(x_0; \psi) = \left[\frac{1}{\Gamma([\alpha] - \alpha)} \int_0^x \mathcal{D}^{[\alpha]}\tilde{f}(\xi; \psi)(x - \xi)^{-[\alpha]+\alpha} d\xi\right]_{x=x_0}, \]

\[
(\mathcal{D}^{\alpha}f)(x_0; \psi) = \left[\frac{1}{\Gamma([\alpha] - \alpha)} \int_0^x \mathcal{D}^{[\alpha]}f(\xi; \psi)(x - \xi)^{-[\alpha]+\alpha} d\xi\right]_{x=x_0}, \tag{4.8}
\]

\[
\mathcal{D}^{\alpha}f(\xi) = \frac{d^\alpha f(\xi)}{d\xi^\alpha}.
\]

**Proof.** Let \( \ell \) is an even number and then \( \ell = 2s_1, s_1 \in \mathbb{N}. \) Here, we have two assumptions as follows:

The first assumption is \( (\mathcal{D}^{\alpha}f)(x_0) \) is the Caputo type fuzzy fractional differentiable mapping in the first form \( (J_{(\alpha)} = 1) \) and in view of (4.4) from Definition 4.1, we have

\[
\mathcal{G}_{J_1, \ldots, J_{(\alpha)}}(x_0 + \epsilon) \Theta \mathcal{G}_{J_1, \ldots, J_{(\alpha)}}(x_0) = \left[\mathcal{G}_{J_1, \ldots, J_{(\alpha)}}(x_0 + \epsilon; \psi) - \mathcal{G}_{J_1, \ldots, J_{(\alpha)}}(x_0; \psi), \tilde{\mathcal{G}}_{J_1, \ldots, J_{(\alpha)}}(x_0 + \epsilon; \psi) - \tilde{\mathcal{G}}_{J_1, \ldots, J_{(\alpha)}}(x_0; \psi)\right].
\]

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Conducting product on both sides by $1/\epsilon$, $\epsilon > 0$, and then applying $\epsilon \mapsto +$, yields
\[
\left( \overset{\mathcal{R}^\alpha}{d} \Phi \right) (x_0) = \left[ \frac{d}{dx} \mathcal{G}_{j_1, \ldots, j_{n-1}} (x_0; \varphi), \frac{d}{dx} \mathcal{G}_{j_1, \ldots, j_{n-1}} (x_0; \varphi) \right].
\]

Thus, $\mathcal{G}_{j_1, \ldots, j_{n-1}}$ is identical to the specified restrictions mentioned in (4.2) of Definition 4.1, then by employing (4.2) for $(k_1 - 1)$ times, we have that
\[
\mathcal{G}_{j_1, \ldots, j_{n-1}} (x_0) = \left[ \mathcal{G}^{(k_1-1)} (x_0; \varphi), \mathcal{G}^{(k_1-1)} (x_0; \varphi) \right].
\]

Since $\mathcal{G}_{j_1, \ldots, j_{n-1}} (x_0)$ is identical to the specified restrictions stated in (4.3) of Definition 4.1 then by employing (4.3), we have
\[
\mathcal{G}_{j_1, \ldots, j_{n-1}} (x_0) = \left[ \mathcal{G}^{(k_2-1)} (x_0; \varphi), \mathcal{G}^{(k_2-1)} (x_0; \varphi) \right].
\]

Since $\mathcal{G}_{j_1, \ldots, j_{n-1}} (x_0)$ is identical to the specified restrictions stated in (4.3) of Definition 4.1 then by employing (4.3), we have
\[
\mathcal{G}_{j_1, \ldots, j_{n-2}} (x_0) = \left[ \mathcal{G}^{(k_3-1)} (x_0; \varphi), \mathcal{G}^{(k_3-1)} (x_0; \varphi) \right].
\]

On the other hand, from (4.14) we notice that we will have a similar equation, following the application of (4.2) and (4.3) for any even number of $J_2, J_3, \ldots, J_{2m}$ of (4.14). Thus, for $\mathcal{G}_{j_1, \ldots, j_{2n}} (x_0)$, we have
\[
\mathcal{G}_{j_1, \ldots, j_{2n}} (x_0) = \left[ \mathcal{G}^{(1)} (x_0; \varphi), \mathcal{G}^{(1)} (x_0; \varphi) \right],
\]
where $2s_1$ is even number.

Consequently, since $\mathcal{G}_{j_1, \ldots, j_{2n}} (x_0)$ is identical to the specified restrictions stated in (4.2) of Definition 4.1 then by employing (4.2) for $(\alpha - k_{2s_1})$, we have
\[
\mathcal{G}_{j_1, \ldots, j_{2n}} (x_0) = \left[ \mathcal{G}^{(\alpha-1)} (x_0; \varphi), \mathcal{G}^{(\alpha-1)} (x_0; \varphi) \right],
\]
then, we have
\[
\mathcal{G}_{j_1, \ldots, j_{2n}} (x_0; \varphi) = \mathcal{G}^{(\alpha-1)} (x_0; \varphi), \mathcal{G}^{(\alpha-1)} (x_0; \varphi)\).
\]

Plugging (4.17) and (4.10) gives the subsequent
\[
(\overset{\mathcal{R}^\alpha}{D} \Phi) (x_0) = \left[ D^{(\alpha)} \mathcal{G} (x_0; \varphi), D^{(\alpha)} \mathcal{G} (x_0; \varphi) \right], \quad D = d/dx.
\]
Theorem 4.3. Assume that there be a fuzzy-valued mapping \( f \) virtue of Theorem 2.12, Thus, 

\[
D^\beta \left( \frac{1}{\Gamma([\alpha] - \alpha)} \int_0^x f(\xi; \varphi) d\xi \right) - \sum_{k=0}^{[\alpha]-\alpha} \frac{D^\beta f(0) x^{[\alpha]-\alpha+k}}{\Gamma(1+ [\alpha] - \alpha + k)} \bigg|_{x=x_0},
\]

where \( (D^{-(\beta)\alpha}) f)(x_0; \varphi) \) and \( (D^{-\beta\alpha}) \tilde{f})(x_0; \varphi) \) are the \( \mathcal{RL} \) fractional integrals of the mappings \( f(x_0; \varphi) \) and \( \tilde{f}(x_0; \varphi) \) at \( x = x_0 \), respectively. By the use of continuity of \( D^\beta \Gamma \) having \( r = [\alpha] \), \( \nu = [\alpha] - \alpha \) and by the virtue of Theorem 2.12, \( D^\beta x^\ell = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-r)} x^{\ell-r} \), we have 

\[
D^{\beta} f(x_0) = \left[ D^{-(\beta)\alpha} (D^{\alpha} \tilde{f})(x_0; \varphi)) + \frac{Q(x_0, -\alpha)}{D^{\alpha} \tilde{f}(0; \varphi) x^{\alpha-\alpha}} \right] x_0 = x_0,
\]

Thus, 

\[
(\cdot D^\alpha f)(x_0) = \left[ (D^{-\beta\alpha} (D^{\alpha} \tilde{f})(x_0; \varphi)) + \frac{Q(x_0, -\alpha)}{D^{\alpha} \tilde{f}(0; \varphi) x^{\alpha-\alpha}} \right],
\]

(4.24)

If \( \ell \) is an odd, solution is similar as we did before.

\( \square \)

**Theorem 4.3.** Assume that there be a fuzzy-valued mapping \( f(x) \in \mathbb{C}^\ell[0, \infty) \cap \mathbb{L}^\ell[0, \infty) \) such that \( f(x) = [f(x; \varphi), \tilde{f}(x; \varphi)] \) for \( \varphi \in [0, 1] \). Also, let \( r - 1 < \alpha < r \) and \( \ell \) is the quantity replicated of two amongst \( J_1, J_2, J_3, ..., J_r \) say \( J_1, J_2, J_3, ..., J_r \) such that \( k_1 < k_2 < ... < k_r \); i.e., \( J_1, J_2, J_3, ..., J_r \) and \( 0 \leq \ell \leq r \). Then, 

(1) If \( \ell \) is even number, then

\[
\mathbb{E}\left[ (\cdot D^\alpha f)(x_0) \right] = \omega^{-\alpha} \mathbb{E}[f(x)] \otimes \omega^{2-\alpha} f(0) \otimes \sum_{k=1}^{r-1} \omega^{2-\alpha+k} f^{(k)}(0),
\]

(4.23)

then

\[
\otimes = \begin{cases} \Theta, & \text{if such quantity is replication of two amongst } J_1, J_2, ..., J_{r-\ell+1} \text{ is an even number}, \\ \ominus, & \text{if such quantity is replication of two amongst } J_1, J_2, ..., J_{r-\ell+1} \text{ is an odd number}. \end{cases}
\]

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(2) If \( \ell \) is odd number, we have

\[
\mathbb{E}
\left[
\int \left( \mathcal{D}_f^{\alpha r} \right) \mathbb{D} J_1 J_2 \ldots J_r (x)
\right]
= -\omega^{-\alpha} f(0) \otimes (-\omega^{-\alpha}) \mathbb{E}[f(x)] \otimes \sum_{k=0}^{r-1} \omega^{2-\alpha+k} f(0),
\]

\( \otimes = \begin{cases} 
\otimes, & \text{if such quantity is replication of two amongst } J_1, J_2, \ldots J_r -(k+1) \text{ is an odd number,} \\
\ominus, & \text{if such quantity is replication of two amongst } J_1, J_2, \ldots J_r -(k+1) \text{ is an even number.}
\end{cases} \) \hspace{1cm} (4.26)

**Proof.** Considering \( \int \left( \mathcal{D}_f^{\alpha r} \right) \mathbb{D} J_1 J_2 \ldots J_r (x) \), that can be expressed as \( \int \left( \mathcal{D}_f^{\alpha r} \right) \mathbb{D} J_1 J_2 \ldots J_r (x) \). Also, assume that \( \ell \) is an odd number, then by means of Theorem 4.2 and \( r - 1 < \alpha < r \), we have

\[
(\mathcal{D}_f^{\alpha r}) (x; \varphi) = \left[ (\mathcal{D}_f^{\alpha r}) (x; \varphi), (\mathcal{D}_f^{\alpha r}) (x; \varphi) \right].
\]

Thus, we have

\[
(\mathcal{D}_f^{\alpha r}) (x; \varphi) = (\mathcal{D}_f^{\alpha r}) (x; \varphi),
\]

\[
(\mathcal{D}_f^{\alpha r}) (x; \varphi) = (\mathcal{D}_f^{\alpha r}) (x; \varphi).
\]

Using the fact of (4.28), we have

\[
\mathbb{E}
\left[
\int \left( \mathcal{D}_f^{\alpha r} \right) \mathbb{D} J_1 J_2 \ldots J_r (x)
\right]
= \mathbb{E}
\left[

\mathcal{E}[(\mathcal{D}_f^{\alpha r}) (x; \varphi)], \mathcal{E}[(\mathcal{D}_f^{\alpha r}) (x; \varphi)]
\right].
\]

In view of Elzaki transform of the Caputo fractional derivative of order \( \alpha \) ( [66]), we have

\[
\mathcal{E}[(\mathcal{D}_f^{\alpha r}) (x; \varphi)] = \omega^{-\alpha} \mathcal{E}[f(x; \varphi)] - \sum_{k=0}^{r-1} \omega^{2-\alpha+k} f^{(k)} (0; \varphi)
\]

\[
= \omega^{-\alpha} \mathcal{E}[f(x; \varphi)] - \omega^{2-\alpha} f(0; \varphi) - \sum_{k=1}^{r-1} \omega^{2-\alpha+k} f^{(k)} (0; \varphi).
\]

The aforementioned expression can be represented as

\[
\mathcal{E}[(\mathcal{D}_f^{\alpha r}) (x; \varphi)] = \omega^{-\alpha} \mathcal{E}[f(x; \varphi)] - \omega^{2-\alpha} f(0; \varphi)
\]

\[
- \sum_{k=1}^{k_1-1} \omega^{2-\alpha+k} f^{(k)} (0; \varphi) - \sum_{k=k_1}^{k_2-1} \omega^{2-\alpha+k} f^{(k)} (0; \varphi) - \ldots
\]

\[
- \sum_{k=k_{r-1}}^{k_{r-1}-1} \omega^{2-\alpha+k} f^{(k)} (0; \varphi) - \sum_{k=k_r}^{r-1} \omega^{2-\alpha+k} f(0; \varphi).
\]

Repeating the same process, we can write
\[ E[\langle \mathcal{D}^{\alpha} f \rangle(x; \varphi)] = \omega^{-\alpha} E[\mathcal{F}(x; \varphi)] - \omega^{-\alpha} \mathcal{F}(0; \varphi) \]
\[ - \sum_{k=1}^{\kappa_1-1} \omega^{2^{-\alpha+k}} \mathcal{F}(0; \varphi) - \sum_{k=\kappa_1}^{\kappa_2-1} \omega^{2^{-\alpha+k}} \mathcal{F}(0; \varphi) - \ldots \]
\[ d - \sum_{k=\kappa_{\ell-1}}^{\kappa_{\ell}-1} \omega^{2^{-\alpha+k}} \mathcal{F}(x; \varphi) - \sum_{k=\kappa_{\ell}}^{r-1} \omega^{2^{-\alpha+k}} \mathcal{F}(x; \varphi). \] (4.32)

Even though \( j_{\kappa_1} = j_{\kappa_2} = \ldots = j_{\kappa_r} = 2 \) and \( \ell \) is an odd number, then we have the subsequent forms
\[ f^{(k)}(0; \varphi) = f^{(k)}(0; \varphi), \forall k \in [1, \kappa_1 - 1], \]
\[ \mathcal{F}^{(k)}(0; \varphi) = \mathcal{F}^{(k)}(0; \varphi) \]
\[ f^{(k)}(0; \varphi) = f^{(k)}(0; \varphi), \forall k \in [\kappa_1, \kappa_2 - 1], \]
\[ \vdots \]
\[ f^{(k)}(0; \varphi) = f^{(k)}(0; \varphi), \forall k \in [\kappa_{\ell-1}, \kappa_{\ell} - 1], \]
\[ \mathcal{F}^{(k)}(0; \varphi) = \mathcal{F}^{(k)}(0; \varphi), \forall k \in [\kappa_{\ell}, r - 1]. \] (4.33)

When \( \ell \) is odd number and utilizing Theorem 4.2, we get the aforementioned equations.

In view of (4.32), (4.31) and (4.29) reduce to
\[ \mathbb{E}\left[ \langle \mathcal{D}^{\alpha}_{j_1, j_2, \ldots, j_r} f \rangle(x) \right] = -\omega^{2^{-\alpha}} f(0) \ominus (-\omega^{-\alpha}) \mathbb{E}[f(x)] \oplus \sum_{k=1}^{r-1} \omega^{2^{-\alpha+k}} f^{(k)}(0; \varphi). \] (4.34)

where \( \oplus \) defined in (4.26).

Adopting the same way, we can prove \( \ell \) to be even number on parallel lines. \( \square \)

**Corollary 1.** Assume that \( f(x) \in C^{\ell}[0, \infty) \cap L^{\infty}[0, \infty). \) Also, let \( \alpha \in (2, 3) \). Then we obtain the following:

If \( \langle \mathcal{D}^{\alpha}_{1,1} f \rangle(x) \) is \( \langle (i) - \alpha \rangle \)-differentiable fuzzy-valued mapping, then
\[ \mathbb{E}\left[ \langle \mathcal{D}^{\alpha}_{1,1,1} f \rangle(x) \right] = \omega^{-\alpha} \mathbb{E}[f(x)] \ominus \omega^{-\alpha+2} f(0) \ominus \omega^{-\alpha+3} f'(0) \ominus \omega^{-\alpha+4} f''(0). \]

If \( \langle \mathcal{D}^{\alpha}_{1,1} f \rangle(x) \) is \( \langle (ii) - \alpha \rangle \)-differentiable fuzzy-valued mapping, then
\[ \mathbb{E}\left[ \langle \mathcal{D}^{\alpha}_{1,1,1} f \rangle(x) \right] = -\omega^{\alpha+2} f(0) \ominus (-\omega^{-\alpha}) \mathbb{E}[f(x)] - \omega^{-\alpha+3} f'(0) - \omega^{-\alpha+4} f''(0). \]

If \( \langle \mathcal{D}^{\alpha}_{1,2} f \rangle(x) \) is \( \langle (i) - \alpha \rangle \)-differentiable fuzzy-valued mapping, then
\[ \mathbb{E}\left[ \langle \mathcal{D}^{\alpha}_{1,2,1} f \rangle(x) \right] = -\omega^{-\alpha+2} f(0) \ominus (-\omega^{-\alpha}) \mathbb{E}[f(x)] - \omega^{-\alpha+3} f'(0) \ominus \omega^{-\alpha+4} f''(0). \]
If \( (\mathbf{D}_{i}^{\alpha} \mathbf{f})(x) \) is \('[(ii) - \alpha']\)-differentiable fuzzy-valued mapping, then

\[
\mathbb{E}\left[ (\mathbf{D}_{i,2}^{\alpha} \mathbf{f})(x) \right] = \omega^{-\alpha} \mathbb{E}[\mathbf{f}(x)] \odot \omega^{-\alpha^{2} \mathbf{f}(0)} \odot \omega^{-\alpha^{3} \mathbf{f}'(0)} - \omega^{-\alpha^{4} \mathbf{f}''(0)}.
\]

If \( (\mathbf{D}_{i}^{\alpha} \mathbf{f})(x) \) is \('[(i) - \alpha']\)-differentiable fuzzy-valued mapping, then

\[
\mathbb{E}\left[ (\mathbf{D}_{i,2}^{\alpha} \mathbf{f})(x) \right] = -\omega^{-\alpha^{2} \mathbf{f}(0)} \odot (-\omega)^{-\alpha} \mathbb{E}[\mathbf{f}(x)] \odot \omega^{-\alpha^{3} \mathbf{f}'(0)} \odot \omega^{-\alpha^{4} \mathbf{f}''(0)}.
\]

If \( (\mathbf{D}_{i}^{\alpha} \mathbf{f})(x) \) is \('[(ii) - \alpha']\)-differentiable fuzzy-valued mapping, then

\[
\mathbb{E}\left[ (\mathbf{D}_{i,2}^{\alpha} \mathbf{f})(x) \right] = \omega^{-\alpha} \mathbb{E}[\mathbf{f}(x)] \odot \omega^{-\alpha^{2} \mathbf{f}(0)} - \omega^{-\alpha^{3} \mathbf{f}'(0)} \odot \omega^{-\alpha^{4} \mathbf{f}''(0)}.
\]

If \( (\mathbf{D}_{i}^{\alpha} \mathbf{f})(x) \) is \('[(i) - \alpha']\)-differentiable fuzzy-valued mapping, then

\[
\mathbb{E}\left[ (\mathbf{D}_{i,2}^{\alpha} \mathbf{f})(x) \right] = \omega^{-\alpha^{2} \mathbf{f}(0)} \odot (-\omega)^{-\alpha} \mathbb{E}[\mathbf{f}(x)] \odot \omega^{-\alpha^{3} \mathbf{f}'(0)} \odot \omega^{-\alpha^{4} \mathbf{f}''(0)}.
\]

If \( (\mathbf{D}_{i}^{\alpha} \mathbf{f})(x) \) is \('[(ii) - \alpha']\)-differentiable fuzzy-valued mapping, then

\[
\mathbb{E}\left[ (\mathbf{D}_{i,2}^{\alpha} \mathbf{f})(x) \right] = -\omega^{-\alpha^{2} \mathbf{f}(0)} \odot (-\omega)^{-\alpha} \mathbb{E}[\mathbf{f}(x)] \odot \omega^{-\alpha^{3} \mathbf{f}'(0)} - \omega^{-\alpha^{4} \mathbf{f}''(0)}.
\]

5. Fuzzy Elzaki decomposition method for finding solution of nonlinear fuzzy partial differential equation

In this note, we coupled the fuzzy Elzaki transform and the ADM for obtaining the solution of NFPDE. The generic form of NFPDE is presented as follows:

\[
\sum_{i=0}^{p} c_i \odot \mathbf{D}_{i}^{\alpha} \mathbf{f}(x, \xi) + \sum_{j=1}^{q} c_j \odot \frac{\partial^{j} \mathbf{f}(x, \xi)}{\partial \mathbf{x}^{j}} + \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta, \sigma} \odot \frac{\partial^{\eta} \mathbf{f}(x, \xi)}{\partial \mathbf{x}^{\eta}} \odot \frac{\partial^{\sigma} \mathbf{f}(x, \xi)}{\partial \mathbf{x}^{\sigma}} = \mathbf{g}(x, \xi),
\]

subject to initial conditions

\[
\frac{\partial \mathbf{f}(x, 0)}{\partial \xi^{\eta}} = \psi_{t}(x), \ t = 0, 1, ..., p - 1,
\]

where \( \mathbf{f}, \mathbf{g} : [0, \hat{b}] \times [0, \hat{d}] \mapsto E^{1}, \psi_{t} : [0, \hat{b}] \mapsto E^{1} \) are continuous fuzzy mappings and \( c_i, \ t = 1, 2, ..., p, \ c_j, j = 1, 2, ..., q, \ c_{\eta, \sigma}, \eta = 0, 1, 2, \sigma = 0, 1, 2, \) are non-negative constants.

Implementing the fuzzy Elzaki transform on (5.1), yields

\[
\sum_{i=0}^{p} c_i \odot \mathbb{E}[\mathbf{D}_{i}^{\alpha} \mathbf{f}(x, \xi)] + \sum_{j=1}^{q} c_j \odot \mathbb{E}\left[ \frac{\partial^{j} \mathbf{f}(x, \xi)}{\partial \mathbf{x}^{j}} \right] + \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta, \sigma} \odot \mathbb{E}\left[ \frac{\partial^{\eta} \mathbf{f}(x, \xi)}{\partial \mathbf{x}^{\eta}} \right] \odot \mathbb{E}\left[ \frac{\partial^{\sigma} \mathbf{f}(x, \xi)}{\partial \mathbf{x}^{\sigma}} \right] = \mathbb{E}[\mathbf{g}(x, \xi)].
\]

Consider \( \frac{\partial \mathbf{f}(x, \xi)}{\partial \xi^{\eta}}, \ \eta = 0, 1, 2 \) be a positive fuzzy-valued mappings.
In view of (5.4), then from (4.31) and (4.32) and IC, we have

\[ \sum_{i=0}^{p} c_i \mathcal{E}[D^p_x \mathbf{f}(x, \xi; \varphi)] + \sum_{j=1}^{q} c_j \mathcal{E} \left[ \frac{\partial^j \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^j} \right] + \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E} \left[ \frac{\partial^\eta \mathbf{f}(x, \xi; \varphi) \partial^\sigma \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^\eta \partial \mathbf{x}^\sigma} \right] = \mathcal{E}[\mathbf{g}(x, \xi; \varphi)], \]

and

\[ \sum_{i=0}^{p} c_i \mathcal{E}[D^p_x \mathbf{f}(x, \xi; \varphi)] + \sum_{j=1}^{q} c_j \mathcal{E} \left[ \frac{\partial^j \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^j} \right] + \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E} \left[ \frac{\partial^\eta \mathbf{f}(x, \xi; \varphi) \partial^\sigma \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^\eta \partial \mathbf{x}^\sigma} \right] = \mathcal{E}[\mathbf{g}(x, \xi; \varphi)]. \]

**Case I.** Consider the mapping \( \mathbf{f}(x, \xi; \varphi) \) be \([i - \alpha]\)-differentiable of the \( q \)-th order with respect to \( x \).

In view of (5.4), then from (4.31) and (4.32) and IC, we have

\[
\frac{1}{\omega^\rho} \sum_{i=0}^{p} c_i \mathcal{E}[\mathbf{f}(x, \xi; \varphi)] = \mathcal{E}[\mathbf{g}(x, \xi; \varphi)] + \sum_{i=1}^{p} \omega^2 \psi_i (x, \varphi) - \sum_{j=1}^{q} c_j \mathcal{E} \left[ \frac{\partial^j \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^j} \right]
- \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E} \left[ \frac{\partial^\eta \mathbf{f}(x, \xi; \varphi) \partial^\sigma \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^\eta \partial \mathbf{x}^\sigma} \right].
\]

It follows that

\[
\mathcal{E}[\mathbf{f}(x, \xi; \varphi)] = \left( \sum_{i=0}^{p} \frac{c_i}{\omega^\rho} \right)^{-1} \left[ \mathcal{E}[\mathbf{g}(x, \xi; \varphi)] + \sum_{i=1}^{p} \omega^2 \psi_i (x, \varphi) - \sum_{j=1}^{q} c_j \mathcal{E} \left[ \frac{\partial^j \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^j} \right]
- \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E} \left[ \frac{\partial^\eta \mathbf{f}(x, \xi; \varphi) \partial^\sigma \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^\eta \partial \mathbf{x}^\sigma} \right] \right].
\]

Now, employing the inverse Elzaki fuzzy transform to the aforementioned formulation, gives

\[
\mathbf{f}(x, \xi; \varphi) = \mathcal{E}^{-1} \left[ \left( \sum_{i=0}^{p} \frac{c_i}{\omega^\rho} \right)^{-1} \left( \mathcal{E}[\mathbf{g}(x, \xi; \varphi)] + \sum_{i=1}^{p} \omega^2 \psi_i (x, \varphi) \right)
- \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E} \left[ \frac{\partial^\eta \mathbf{f}(x, \xi; \varphi) \partial^\sigma \mathbf{f}(x, \xi; \varphi)}{\partial \mathbf{x}^\eta \partial \mathbf{x}^\sigma} \right] \right].
\]

In view of the Adomian decomposition technique, this approach has infinite series solution for the subsequent unknown mappings:

\[
\mathbf{f}(x, \xi; \varphi) = \sum_{r=0}^{\infty} \mathbf{f}_r (x, \xi; \varphi).
\]
The non-linearity is dealt by an infinite series of the Adomian polynomials $\Delta_p^{\eta \sigma}$, $\eta = 0, 1, 2$, $\sigma = 0, 1, 2$ has the subsequent representation:

$$
\frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\eta} \frac{\partial^\sigma f(x, \xi; \varphi)}{\partial x^\sigma} = \sum_{r=0}^{\infty} \mathcal{R}_r^{\eta \sigma},
$$

(5.8)

where

$$
\mathcal{R}_r^{\eta \sigma} = \begin{cases} 
\frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\eta}, & r = 0, \\
\frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\eta} + \frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\sigma}, & r = 1, \\
\frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\eta} + \frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\sigma} + \frac{\partial^\eta f(x, \xi; \varphi)}{\partial x^\sigma}, & r = 2, \\
\vdots 
\end{cases}
$$

(5.9)

Inserting (5.8) and (5.9) in (5.7) refers to the following equation:

$$
\sum_{r=0}^{\infty} f_r(x, \xi; \varphi) = \mathcal{E}^{-1} \left[ \left( \sum_{i=0}^{p} \frac{c_i}{\omega^i} \right)^{-1} \left( \mathcal{E}[g(x, \xi; \varphi)] + \sum_{i=1}^{p} \omega^2 \psi_i(x; \varphi) \right) \right] 
$$

$$
- \mathcal{E}^{-1} \left[ \left( \sum_{i=0}^{p} \frac{c_i}{\omega^i} \right)^{-1} \left( \sum_{j=1}^{q} c_j \mathcal{E} \left[ \sum_{r=0}^{\infty} \frac{\partial^r f(x, \xi; \varphi)}{\partial x^r} \right] \right) + \sum_{\sigma=0}^{2} c_{\sigma \rho} \mathcal{E} \left[ \sum_{r=0}^{\infty} \mathcal{R}_r^{\eta \sigma} \right] \right].
$$

(5.10)

The recursive terms of Elzaki decomposition method can be computed for $r \geq 0$ as follows:

$$
f_0(x, \xi; \varphi) = \mathcal{E}^{-1} \left[ \left( \sum_{i=0}^{p} \frac{c_i}{\omega^i} \right)^{-1} \left( \mathcal{E}[g(x, \xi; \varphi)] + \sum_{i=1}^{p} \omega^2 \psi_i(x; \varphi) \right) \right],
$$

$$
f_{r+1}(x, \xi; \varphi) = - \mathcal{E}^{-1} \left[ \left( \sum_{i=0}^{p} \frac{c_i}{\omega^i} \right)^{-1} \left( \sum_{j=1}^{q} c_j \mathcal{E} \left[ \sum_{r=0}^{\infty} \frac{\partial^r f(x, \xi; \varphi)}{\partial x^r} \right] \right) + \sum_{\sigma=0}^{2} c_{\sigma \rho} \mathcal{E} \left[ \sum_{r=0}^{\infty} \mathcal{R}_r^{\eta \sigma} \right] \right].
$$

(5.11)

**Case II.** Suppose the mapping $f(x, \xi; \varphi)$ is $[(i - \alpha)]$-differentiable of the $q$th order in regard to $x$ and $[(ii) - \alpha]$ differentiable of the 2$p$th order in regard to $\xi$. Then, the parametric version of (5.3) has the following representation:

$$
\sum_{i=0}^{p} c_{2i} \mathcal{E}[D^{2i}_\xi f(x, \xi; \varphi)] + \sum_{i=1}^{p} c_{2i-1} \mathcal{E}[D^{i}_\xi f(x, \xi; \varphi)] 
$$

$$
+ \sum_{j=1}^{q} c_j \mathcal{E} \left[ \frac{\partial^j f(x, \xi; \varphi)}{\partial x^j} \right] + \sum_{\sigma=0}^{2} \sum_{\rho=0}^{2} c_{\sigma \rho} \mathcal{E} \left[ \frac{\partial^\sigma f(x, \xi; \varphi)}{\partial x^\sigma} \frac{\partial^\rho f(x, \xi; \varphi)}{\partial x^\rho} \right] = \mathcal{E}[g(x, \xi; \varphi)],
$$

and

$$
\sum_{i=0}^{p} c_{2i} \mathcal{E}[D^{2i}_\xi f(x, \xi; \varphi)] + \sum_{i=1}^{p} c_{2i-1} \mathcal{E}[D^{i}_\xi f(x, \xi; \varphi)] 
$$

$$
+ \sum_{j=1}^{q} c_j \mathcal{E} \left[ \frac{\partial^j f(x, \xi; \varphi)}{\partial x^j} \right] + \sum_{\sigma=0}^{2} \sum_{\rho=0}^{2} c_{\sigma \rho} \mathcal{E} \left[ \frac{\partial^\sigma f(x, \xi; \varphi)}{\partial x^\sigma} \frac{\partial^\rho f(x, \xi; \varphi)}{\partial x^\rho} \right] = \mathcal{E}[g(x, \xi; \varphi)].
$$
Utilizing the fact of Theorem 4.3 and ICs, we have

\[ B \mathcal{E}[\tilde{f}(x, \xi; \varphi)] + C \mathcal{E}[D^\alpha_{\xi} \tilde{f}(x, \xi; \varphi)] = \mathcal{E}[\tilde{g}(x, \xi; \varphi) + F_1(x; \varphi)] - \sum_{j=1}^{q} c_j \mathcal{E}\left[ \frac{\partial \tilde{f}(x, \xi; \varphi)}{\partial x^j} \right] - 2 \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E}\left[ \frac{\partial^\eta \tilde{f}(x, \xi; \varphi)}{\partial x^\eta} \frac{\partial^\sigma \tilde{f}(x, \xi; \varphi)}{\partial x^\sigma} \right], \]  

(5.12)

and

\[ B \mathcal{E}[\tilde{f}(x, \xi; \varphi)] + C \mathcal{E}[\tilde{f}(x, \xi; \varphi)] = \mathcal{E}[\tilde{g}(x, \xi; \varphi) + F_2(x; \varphi)] - \sum_{j=1}^{q} c_j \mathcal{E}\left[ \frac{\partial \tilde{f}(x, \xi; \varphi)}{\partial x^j} \right] - 2 \sum_{\eta=0}^{2} \sum_{\sigma=\eta}^{2} c_{\eta \sigma} \mathcal{E}\left[ \frac{\partial^\eta \tilde{f}(x, \xi; \varphi)}{\partial x^\eta} \frac{\partial^\sigma \tilde{f}(x, \xi; \varphi)}{\partial x^\sigma} \right], \]  

(5.13)

where \( B = \sum_{i=0}^{p} c_{i} \omega^i \), \( C = \sum_{i=1}^{p} c_{2i-1} \omega^{2-i} \),

\[ F_1(x; \varphi) = \sum_{i=0}^{p} c_{2i} \left( \omega^{2-2i} \psi_0(x; \varphi) + \omega^{3-2i} \psi_1(x; \varphi) \right) + \sum_{i=0}^{p} c_{2i-1} \left( \omega^{3-2i} \psi_0(x; \varphi) + \omega^{2-2i} \psi_0(x; \varphi) \right), \]

and

\[ F_2(x; \varphi) = \sum_{i=0}^{p} c_{2i} \left( \omega^{2-2i} \psi_0(x; \varphi) + \omega^{3-2i} \psi_1(x; \varphi) \right) + \sum_{i=0}^{p} c_{2i-1} \left( \omega^{3-2i} \psi_0(x; \varphi) + \omega^{2-2i} \psi_0(x; \varphi) \right). \]

For the aforementioned Eqs (5.12) and (5.13), we obtain \( \mathcal{E}[f(x, \xi; \varphi)] \) and \( \mathcal{E}[\tilde{f}(x, \xi; \varphi)] \) similar to Case I, we find the the general solution \( f(x; \varphi) = [f(x, \xi; \varphi), \tilde{f}(x, \xi; \varphi)] \).

**Example 5.1.** Consider the fuzzy fractional partial differential equation as follows:

\[ D^\alpha_{\xi} f(x, \xi) \oplus \frac{\partial f(x, \xi)}{\partial x} \odot \frac{\partial f^2(x, \xi)}{\partial x^2} = g_3(x, \xi), \ x \geq 0, \ \xi > 0, \]  

(5.14)

subject to ICs

\[ f(x, 0) = \left( \frac{x^2}{2}, \frac{x^2}{2} (2 - \varphi) \right), \ f'(x, 0) = (0, 0), \ x > 0, \]  

(5.15)

and \( g_3(x, \xi) = (\varphi + x \varphi^2, 2 - \varphi + x (2 - \varphi)^2) \).

In order to find a solution of (5.14), we have the following three cases.

**Case I.** If \( f(x, \xi) \) is \([(i) - \alpha]\)-differentiable.

Employing the Elzaki transform on (5.14), then we have

\[ \frac{1}{\omega^{2i}} \mathcal{E}[f(x, \xi; \varphi)] = \omega^{2-2i} f(x, 0; \varphi) = \mathcal{E}\left[ g_3(x, \xi) - \frac{\partial f(x, \xi) \partial f^2(x, \xi)}{\partial x \partial x^2} \right]. \]
or equivalently, we have
\[
\mathcal{E}[f(x, \xi; \varphi)] - \omega^2 f(x, 0; \varphi) = \omega^{2\alpha} \mathcal{E}\left[g_3(x, \xi) - \frac{\partial f(x, \xi)}{\partial x} \frac{\partial^2 f(x, \xi)}{\partial x^2}\right].
\]

Further, implementing the inverse fuzzy Elzaki transform, we have
\[
f(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^2 f(x, 0; \varphi) + \omega^{2\alpha} \mathcal{E}\left[g_3(x, \xi) - \frac{\partial f(x, \xi)}{\partial x} \frac{\partial^2 f(x, \xi)}{\partial x^2}\right]\right].
\]

Also, applying the scheme described in Section 4, we have
\[
\sum_{r=0}^{\infty} f_r(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^2 f(x, 0; \varphi) + \omega^{2\alpha} \mathcal{E}\left[g_3(x, \xi) - \omega^{2\alpha} \mathcal{E}\left[\sum_{r=0}^{\infty} A_r\right]\right]\right].
\]

Utilizing the iterative procedure defined in (5.11), we have
\[
f_0(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^2 f(x, 0; \varphi) + \omega^{2\alpha} \mathcal{E}\left[g_3(x, \xi)\right]\right] = \varphi \frac{x^2}{2} + (\varphi + x\varphi^2) \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)},
\]

Also,
\[
f_{r+1}(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^{2\alpha} \mathcal{E}\left[\sum_{r=0}^{\infty} A_r\right]\right].
\]

Utilizing the first few Adomian polynomials mentioned in (5.9), we have
\[
f(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^{2\alpha}[A_0]\right] = -\varphi^2 x \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)} - \varphi^3 \frac{\xi^{4\alpha}}{\Gamma(4\alpha + 1)},
\]
\[
f_2(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^{2\alpha}[A_1]\right] = -\varphi^3 \frac{\xi^{4\alpha}}{\Gamma(4\alpha + 1)},
\]
\[
f_3(x, \xi; \varphi) = 0,
\]
\[\vdots.
\]

In a similar way we obtained the upper solutions as follows:
\[
\bar{f}_0(x, \xi; \varphi) = \frac{x^2}{2} (2 - \varphi) + (2 - \varphi + x(2 - \varphi)^2) \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)},
\]
\[
\bar{f}_1(x, \xi; \varphi) = -x(2 - \varphi)^2 \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)} - (2 - \varphi)^3 \frac{\xi^{4\alpha}}{\Gamma(4\alpha + 1)},
\]
\[
\bar{f}_2(x, \xi; \varphi) = -(2 - \varphi)^3 \frac{\xi^{4\alpha}}{\Gamma(4\alpha + 1)},
\]
\[
\bar{f}_3(x, \xi; \varphi) = 0,
\]
\[\vdots.
\]
The series form solution of Example 5.1 is presented as follows:

$$f(x, \xi) = \left(\left(\frac{x^2}{2} + \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)}\right)^{1/3}\varphi, \left(\frac{x^2}{2} + \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)}\right)(2 - \varphi)\right). \quad (5.21)$$

The numerical solution to the fuzzy fractional nonlinear PDE is presented in this section. Incorporating all of the data in regard to the numerous parameters involved in the related equation is a monumental task. Uncertain responses subject to Caputo fractional order derivatives have been considered, as previously said.

- Table 1 represents the obtained findings with $x = 0.4$ and $\xi = 0.7$. Table 1 also comprises the outcomes of a Georgieva and Pavlova [67]. As a consequence, the findings acquired by fuzzy Elzaki decomposition method are the same if $\alpha = 1$, as those reported by a Georgieva and Pavlova [67].

- Figure 1 demonstrates the three-dimensional illustration of the lower and upper estimates for different uncertainties $\varphi \in [0, 1]$.

- Figure 2 shows the fuzzy responses for different fractional orders.

- Figure 3 illustrates the fuzzy responses for different uncertainty parameters.

- The aforementioned representations illustrate that all graphs are substantially identical in their perspectives and have good agreement with one another, especially when integer-order derivatives are taken into account.

Finally, this generic approach for dealing with nonlinear PDEs is more accurate and powerful than the method applied by [67]. Our findings for the fuzzy Elzaki decomposition method, helpful for fuzzy initial value problems, demonstrate the consistency and strength of the offered solutions.

**Table 1.** Lower and upper solutions of Case I of Example 5.1 for various fractional orders in comparison with the solution derived by [67].

| $\varphi$ | $f(\alpha = 0.7)$ | $\tilde{f}(\alpha = 0.7)$ | $f(\alpha = 1)$ | $\tilde{f}(\alpha = 1)$ | $f$ [67] | $\tilde{f}$ [67] |
|-----------|------------------|--------------------------|-----------------|-------------------------|-----------|-------------|
| 0.1       | $1.9420 \times 10^{-2}$ | $3.6899 \times 10^{-1}$ | $9.0000 \times 10^{-3}$ | $1.7100 \times 10^{-1}$ | $9.0000 \times 10^{-3}$ | $1.7100 \times 10^{-1}$ |
| 0.2       | $3.8841 \times 10^{-2}$ | $3.4957 \times 10^{-1}$ | $1.8000 \times 10^{-2}$ | $1.6200 \times 10^{-1}$ | $1.8000 \times 10^{-2}$ | $1.6200 \times 10^{-1}$ |
| 0.3       | $5.8262 \times 10^{-2}$ | $3.30152 \times 10^{-1}$ | $2.7000 \times 10^{-2}$ | $1.5300 \times 10^{-1}$ | $2.7000 \times 10^{-2}$ | $1.5300 \times 10^{-1}$ |
| 0.4       | $7.7682 \times 10^{-2}$ | $3.1073 \times 10^{-1}$ | $3.6000 \times 10^{-2}$ | $1.4400 \times 10^{-1}$ | $3.6000 \times 10^{-2}$ | $1.4400 \times 10^{-1}$ |
| 0.5       | $9.7103 \times 10^{-2}$ | $2.9131 \times 10^{-1}$ | $4.5000 \times 10^{-2}$ | $1.3500 \times 10^{-1}$ | $4.5000 \times 10^{-2}$ | $1.3500 \times 10^{-1}$ |
| 0.6       | $1.1652 \times 10^{-2}$ | $2.71890 \times 10^{-1}$ | $5.4000 \times 10^{-2}$ | $1.2600 \times 10^{-1}$ | $5.4000 \times 10^{-2}$ | $1.2600 \times 10^{-1}$ |
| 0.7       | $1.3594 \times 10^{-2}$ | $2.5246 \times 10^{-1}$ | $6.3000 \times 10^{-2}$ | $1.1700 \times 10^{-1}$ | $6.3000 \times 10^{-2}$ | $1.1700 \times 10^{-1}$ |
| 0.8       | $1.5536 \times 10^{-2}$ | $2.3304 \times 10^{-1}$ | $7.2000 \times 10^{-2}$ | $1.0800 \times 10^{-1}$ | $7.2000 \times 10^{-2}$ | $1.0800 \times 10^{-1}$ |
| 0.9       | $1.7478 \times 10^{-2}$ | $2.1362 \times 10^{-1}$ | $8.1000 \times 10^{-2}$ | $9.9000 \times 10^{-2}$ | $8.1000 \times 10^{-2}$ | $9.9000 \times 10^{-2}$ |
| 1.0       | $1.9420 \times 10^{-1}$ | $1.9420 \times 10^{-1}$ | $9.0000 \times 10^{-2}$ | $9.0000 \times 10^{-2}$ | $9.0000 \times 10^{-2}$ | $9.0000 \times 10^{-2}$ |
Figure 1. Three-dimensional fuzzy responses of Example 5.1 for Case I at (a) $\varphi = 0.7$, (b) $\varphi = 0.9$ with fractional order $\alpha = 1$.

Figure 2. Two-dimensional fuzzy responses of Example 5.1 for Case I at (a) $\varphi = 0.7$ and $\xi = 0.7$, (b) $\varphi = 0.4$ and $\xi = 0.1$ with varying fractional orders.

Figure 3. Two-dimensional fuzzy responses of Example 5.1 for Case I at (a) $\alpha = 0.7$ and $\xi = 0.7$, (b) $\alpha = 0.4$ and $\xi = 0.1$ with varying uncertainty parameters $\varphi \in [0, 1]$. 

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Case II. If \( f(x, \xi) \) is \([(ii) - \alpha]\)-differentiable, taking into account (5.12) and (5.13), we find

\[
\frac{1}{\omega^{2\alpha}} \mathcal{E}[f(x, \xi; \varphi)] = \omega^{2-2\alpha} f(0, \varphi) + \mathcal{E}[g_3(x, \xi)] - \mathcal{E}\left[ \frac{\partial f(0, \xi)}{\partial x} \frac{\partial f^2(x, \xi)}{\partial x^2} \right],
\]

\[
\frac{1}{\omega^{2\alpha}} \mathcal{E}[f(x, \xi; \varphi)] = \omega^{2-2\alpha} f(0, \varphi) + \mathcal{E}[g_3(x, \xi)] - \mathcal{E}\left[ \frac{\partial f(0, \xi)}{\partial x} \frac{\partial f^2(x, \xi)}{\partial x^2} \right].
\]

(5.22)

Employing the inverse fuzzy Elzaki transform to the aforementioned equations and incorporation of Elzaki decomposition technique, we find the solution on same lines as we did in Case I.

Case III. If \( f(x, \xi) \) is \([(i) - \alpha]\)-differentiable and \( f'(x, \xi) \) is \([(ii) - \alpha]\)-differentiable, then

\[
\mathbb{E}(f'(x, \xi)) = [\mathcal{E}(f'(x, \xi; \varphi)), \mathcal{E}(\tilde{f}'(x, \xi; \varphi))]
\]

and

\[
\mathbb{E}(f''(x, \xi)) = [\mathcal{E}(\tilde{f}''(x, \xi; \varphi)), \mathcal{E}(f''(x, \xi; \varphi))].
\]

(5.23) 

(5.24)

In view of (5.11) and Theorem 4.3 with IC, we follow the iterative process:

Employing the Elzaki transform on (5.14), then we have

\[
\frac{1}{\omega^{2\alpha}} \mathcal{E}[f(x, \xi; \varphi)] - \omega^{2-2\alpha} f(x, 0; \varphi) = \mathcal{E}\left[ \tilde{g}_3(x, \xi) - \frac{\partial \tilde{f}(x, \xi)}{\partial x} \frac{\partial \tilde{f}^2(x, \xi)}{\partial x^2} \right],
\]

or equivalently, we have

\[
\mathcal{E}[f(x, \xi; \varphi)] - \omega^2 f(x, 0; \varphi) = \omega^{2\alpha} \mathcal{E}\left[ \tilde{g}_3(x, \xi) - \frac{\partial \tilde{f}(x, \xi)}{\partial x} \frac{\partial \tilde{f}^2(x, \xi)}{\partial x^2} \right].
\]

Further, implementing the inverse fuzzy Elzaki transform, we have

\[
\tilde{f}(x, \xi; \varphi) = \mathcal{E}^{-1}\left[ \omega^2 f(x, 0; \varphi) + \omega^{2\alpha} \mathcal{E}[\tilde{g}_3(x, \xi)] - \omega^{2\alpha} \mathcal{E}\left[ \sum_{r=0}^{\infty} \tilde{A}_r \right] \right].
\]

Also, applying the scheme described in Section 4, we have

\[
\sum_{r=0}^{\infty} \tilde{f}_r(x, \xi; \varphi) = \mathcal{E}^{-1}\left[ \omega^2 f(x, 0; \varphi) + \omega^{2\alpha} \mathcal{E}[\tilde{g}_3(x, \xi)] - \omega^{2\alpha} \mathcal{E}\left[ \sum_{r=0}^{\infty} \tilde{A}_r \right] \right].
\]

Utilizing the iterative procedure defined in (5.11), we have

\[
\tilde{f}_0(x, \xi; \varphi) = \mathcal{E}^{-1}\left[ \omega^2 f(x, 0; \varphi) + \omega^{2\alpha} \mathcal{E}[\tilde{g}_3(x, \xi)] \right] = \varphi \frac{x^2}{2} + [(2 - \varphi) + x(2 - \varphi)^2] \frac{\xi^{2\alpha}}{\Gamma(2\alpha + 1)}.
\]

Also,

\[
\tilde{f}_{r+1}(x, \xi; \varphi) = \mathcal{E}^{-1}\left[ \omega^{2\alpha} \mathcal{E}\left[ \sum_{r=0}^{\infty} \tilde{A}_r \right] \right].
\]
Utilizing the first few Adomian polynomials as follows:

\[
\mathcal{A}_r = \begin{cases} 
\frac{\partial f_0}{\partial x} \frac{\partial \xi}{\partial x}, & r = 0, \\
\frac{\partial f_1}{\partial x} \frac{\partial \xi}{\partial x} + \frac{\partial f_0}{\partial x} \frac{\partial \xi}{\partial x}, & r = 1, \\
\frac{\partial f_2}{\partial x} \frac{\partial \xi}{\partial x} + \frac{\partial f_1}{\partial x} \frac{\partial \xi}{\partial x} + \frac{\partial f_0}{\partial x} \frac{\partial \xi}{\partial x}, & r = 2, \\
\vdots & 
\end{cases}
\]

\[
f(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^{2\alpha}[A_0]\right] \\
= -\varphi^2(2 - \varphi) \frac{\xi^{4\alpha}}{(4\alpha + 1)} - x(2 - \varphi)^2 \frac{\xi^{2\alpha}}{(2\alpha + 1)} \\
f_2(x, \xi; \varphi) = \mathcal{E}^{-1}\left[\omega^{2\alpha}[A_1]\right] = \varphi^2(2 - \varphi) \frac{\xi^{4\alpha}}{(4\alpha + 1)} \\
f_3(x, \xi; \varphi) = 0, \\
\vdots
\]

In a similar way we obtained the upper solutions as follows:

\[
\tilde{f}_0(x, \xi; \varphi) = \frac{x^2}{2}(2 - \varphi) + (2 - \varphi + x(2 - \varphi)^2) \frac{\xi^{2\alpha}}{(2\alpha + 1)}, \\
\tilde{f}(x, \xi; \varphi) = -x(2 - \varphi)^2 \frac{\xi^{2\alpha}}{(2\alpha + 1)} - (2 - \varphi)^3 \frac{\xi^{4\alpha}}{(4\alpha + 1)}, \\
\tilde{f}_2(x, \xi; \varphi) = -(2 - \varphi)^3 \frac{\xi^{4\alpha}}{(4\alpha + 1)}, \\
\tilde{f}_3(x, \xi; \varphi) = 0, \\
\vdots
\]

The series form solution of Example 5.1 is presented as follows:

\[
f(x, \xi) = \left(\frac{x^2}{2} \varphi + (2 - \varphi) \frac{\xi^{2\alpha}}{(2\alpha + 1)}\right) \left(\frac{x^2}{2} (2 - \varphi) + \varphi \frac{\xi^{2\alpha}}{(2\alpha + 1)}\right)
\]

The results show that perfect fractional order precision and uncertainty for fuzzy numerical solutions of the function \(f(x, \xi)\) are highly correlated to stuffing time and the fractional order used, whereas additional precision solutions can be obtained by using more redundancy and iterative development.

- Table 2 represents the obtained findings with \(x = 0.4\) and \(\xi = 0.7\). Table 2 also comprises the outcomes of a Georgieva and Pavlova [67]. As a consequence, the findings acquired by fuzzy Elzaki decomposition method are the same if \(\alpha = 1\), as those reported by a Georgieva and Pavlova [67].
- Figure 4 demonstrates the three-dimensional illustration of the lower and upper estimates for different uncertainties \(\varphi \in [0, 1]\).
- Figure 5 shows the fuzzy responses for different fractional orders. Figure 6 illustrates the fuzzy responses for different uncertainty parameters.
The aforementioned representations illustrate that all graphs are substantially identical in their perspectives and have good agreement with one another, especially when integer-order derivatives are taken into account.

Finally, this generic approach for dealing with nonlinear PDEs is more accurate and powerful than the method applied by [67]. Our findings for the fuzzy Elzaki decomposition method, helpful for fuzzy initial value problems, demonstrate the consistency and strength of the offered solutions.

| \( \phi \) | \( \bar{f}(\alpha = 0.7) \) | \( \bar{f}(\alpha = 0.7) \) | \( \bar{f}(\alpha = 1) \) | \( \bar{f}(\alpha = 1) \) | \( \bar{f}[67] \) | \( \bar{f}[67] \) |
|---|---|---|---|---|---|---|
| 0.1 | 2.8799 \times 10^{-1} | 1.0042 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.2 | 2.7757 \times 10^{-1} | 1.1084 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.3 | 2.6715 \times 10^{-1} | 1.2126 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.4 | 2.5673 \times 10^{-1} | 1.3168 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.5 | 2.4631 \times 10^{-1} | 1.4210 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.6 | 2.3589 \times 10^{-1} | 1.5252 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.7 | 2.2546 \times 10^{-1} | 1.6294 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.8 | 2.1504 \times 10^{-1} | 1.7336 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 0.9 | 2.0462 \times 10^{-1} | 1.8378 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |
| 1.0 | 1.9420 \times 10^{-1} | 1.9420 \times 10^{-1} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} | 9.0000 \times 10^{-2} |

**Figure 4.** Three-dimensional fuzzy responses of Example 5.1 for **Case II** at (a) \( \phi = 0.7 \), (b) \( \phi = 0.9 \) with fractional order \( \alpha = 1 \).
Figure 5. Two-dimensional fuzzy responses of Example 5.1 for Case II at (a) $\varphi = 0.7$ and $\xi = 0.7$, (b) $\varphi = 0.4$ and $\xi = 0.1$ with varying fractional orders.

Figure 6. Two-dimensional fuzzy responses of Example 5.1 for Case II at (a) $\alpha = 0.7$ and $\xi = 0.7$, (b) $\alpha = 0.4$ and $\xi = 0.1$ with varying uncertainty parameters $\varphi \in [0, 1]$.

6. Conclusions

In this investigation, the fuzzy Caputo fractional problem formalism, homogenized fuzzy initial condition, partial differential equation, exemplification of fuzzy Caputo fractional derivative and numerical solutions under $g\mathcal{H}$ are the main significations of the following subordinate part.

- The generic formulation of fuzzy CFDs pertaining to the generic order of $0 < \alpha < r$ is derived by combining all conceivable groupings of items such that $t_1$ equals 1 and $t_2$ (the others) equals 2 and utilized for the first time.
- The generic formulas for CFDs regarding the order $\alpha \in (r - 1, r)$ are generated under the $g\mathcal{H}$-
difference.

- Under $H$-differentiability, a semi-analytical approach for finding the solution of nonlinear fuzzy fractional PDE has been applied. Besides that, this methodology offers a series of solutions as an analytical expression is its significant aspect.

- A test problem is solved to demonstrate the proposed approach. The simulation results can solve nonlinear partial fuzzy differential equations in a flexible and efficient manner, whilst, frame of numerical programming is natural and the computations are very swift in terms of fractional orders and uncertainty parameters $\varphi \in [0, 1]$.

- The results of the projected methodology are more general and fractional in nature than the results provided by [67].

- For futuristic research, a similar method can be applied to Fitzhugh-Nagumo-Huxley by formulating the Henstock integrals (fuzzy integrals in the Lebesgue notion) at infinite intervals [68, 69]. Furthermore, one can explore the implementation of this strategy for relatively intricate challenges, such as the spectral problem [70] and maximum likelihood estimation [71].

**Conflict of interest**

The authors declare that they have no competing interests.

**References**

1. M. Nazeer, F. Hussain, M. I. Khan, A. ur Rehman, E. R. El-Zahar, Y. M. Chu, et al., Theoretical study of MHD electro-osmotically flow of third-grade fluid in micro channel, *Appl. Math. Comput.*, **420** (2022), 126868. https://doi.org/10.1016/j.amc.2021.126868

2. Y. M. Chu, B. M. Shankaralingappa, B. J. Gireesha, F. Alzahrani, M. Ijaz Khan, S. U. Khan, Combined impact of Cattaneo-Christov double diffusion and radiative heat flux on bio-convective flow of Maxwell liquid configured by a stretched nano-material surface, *Appl. Math. Comput.*, **419** (2022), 126883. https://doi.org/10.1016/j.amc.2021.126883

3. Y. M. Chu, U. Nazir, M. Sohail, M. M. Selim, J. R. Lee, Enhancement in thermal energy and solute particles using hybrid nanoparticles by engaging activation energy and chemical reaction over a parabolic surface via finite element approach, *Fractal Fract.*, **5** (2021), 119. https://doi.org/10.3390/fractalfract5030119

4. T. H. Zhao, M. I. Khan, Y. M. Chu, Artificial neural networking (ANN) analysis for heat and entropy generation in flow of non-Newtonian fluid between two rotating disks, *Math. Methods Appl. Sci.*, 2021. https://doi.org/10.1002/mma.7310

5. Y. M. Chu, S. Bashir, M. Ramzan, M. Y. Malik, Model-based comparative study of magnetohydrodynamics unsteady hybrid nanofluid flow between two infinite parallel plates with particle shape effects, *Math. Methods Appl. Sci.*, 2022. https://doi.org/10.1002/mma.8234

6. S. A. Iqbal, M. G. Hafez, Y. M. Chu, C. Park, Dynamical analysis of nonautonomous RLC circuit with the absence and presence of Atangana-Baleanu fractional derivative, *J. Appl. Anal. Comput.*, **12** (2022), 770–789. https://doi.org/10.11948/20210324
7. F. Jin, Z. S. Qian, Y. M. Chu, M. ur Rahman, On nonlinear evolution model for drinking behavior under Caputo-Fabrizio derivative, *J. Appl. Anal. Comput.*, 12 (2022), 790–806. https://doi.org/10.11948/20210357

8. F. Z. Wang, M. N. Khan, I. Ahmad, H. Ahmad, H. Abu-Zinadah, Y. M. Chu, Numerical solution of traveling waves in chemical kinetics: Time-fractional fishers equations, *Fractals*, 30 (2022), 2240051. https://doi.org/10.1142/S0218348X22400515

9. S. Rashid, A. Khalid, S. Sultana, F. Jarad, K. M. Abualnaja, Y. S. Hamed, Novel numerical investigation of the fractional oncolytic effectiveness model with M1 virus via generalized fractional derivative with optimal criterion, *Results Phys.*, 37 (2022), 105553. https://doi.org/10.1016/j.rinp.2022.105553

10. T. H. Zhao, O. Castillo, H. Jahanshahi, A. Yusuf, M. O. Alassafi, F. E. Alsaadi, et al., A fuzzy-based strategy to suppress the novel coronavirus (2019-NCOV) massive outbreak, *Appl. Comput. Math.*, 20 (2021), 160–176.

11. S. Rashid, F. Jarad, A. G. Ahmad, K. M. Abualnaja, New numerical dynamics of the heroin epidemic model using a fractional derivative with Mittag-Leffler kernel and consequences for control mechanisms, *Results Phys.*, 35 (2022), 105304. https://doi.org/10.1016/j.rinp.2022.105304

12. S. N. Hajiseyedazizi, M. E. Samei, J. Alzabut, Y. M. Chu, On multi-step methods for singular fractional q-integro-differential equations, *Open Math.*, 19 (2021), 1378–1405. https://doi.org/10.1515/math-2021-0093

13. T. H. Zhao, M. K. Wang, Y. M. Chu, On the bounds of the perimeter of an ellipse, *Acta Math. Sci.*, 42 (2022), 491–501. https://doi.org/10.1007/s10473-022-0204-y

14. T. H. Zhao, M. K. Wang, G. J. Hai, Y. M. Chu, Landen inequalities for Gaussian hypergeometric function, *RACSAM*, 116 (2022), 53. https://doi.org/10.1007/s13398-021-01197-y

15. H. H. Chu, T. H. Zhao, Y. M. Chu, Sharp bounds for the Toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means, *Math. Slovaca*, 70 (2020), 1097–1112. https://doi.org/10.1515/ms-2017-0417

16. K. S. Miller, B. Ross, *Introduction to the fractional calculus and fractional differential equations*, New York: John Wiley & Sons, 1993.

17. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1998.

18. S. Rashid, E. I. Abouelmagd, S. Sultana, Y. M. Chu, New developments in weighted n-fold type inequalities via discrete generalized h-proportional fractional operators, *Fractals*, 30 (2022), 2240056. https://doi.org/10.1142/S0218348X22400564

19. S. Rashid, S. Sultana, Y. Karaca, A. Khalid, Y. M. Chu, Some further extensions considering discrete proportional fractional operators, *Fractals*, 30 (2022), 2240026. https://doi.org/10.1142/S0218348X22400266

20. S. Rashid, E. I. Abouelmagd, A. Khalid, F. B. Farooq, Y. M. Chu, Some recent developments on dynamical h-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels, *Fractals*, 30 (2022), 2240110. https://doi.org/10.1142/S0218348X22401107
21. H. M. Srivastava, A. K. N. Alomari, K. M. Saad, W. M. Hamanah, Some dynamical models involving fractional-order derivatives with the Mittag-Leffler type kernels and their applications based upon the Legendre spectral collocation method, *Fractal Fract.*, **5** (2021), 131. https://doi.org/10.3390/fractalfract5030131

22. H. M. Srivastava, K. M. Saad, Numerical Simulation of the fractal-fractional Ebola virus, *Fractal Fract.*, **4** (2020), 49. https://doi.org/10.3390/fractalfract4040049

23. S. Rashid, S. Sultana, N. Idrees, E. Bonyah, On analytical treatment for the fractional-order coupled partial differential equations via fixed point formulation and generalized fractional derivative operators, *J. Funct. Spaces*, **2022** (2022), 3764703. https://doi.org/10.1155/2022/3764703

24. M. Al Qurashi, S. Rashid, S. Sultana, F. Jarad, A. M. Alsharif, Fractional-order partial differential equations describing propagation of shallow water waves depending on power and Mittag-Leffler memory, *AIMS Math.*, **7** (2022), 12587–12619. https://doi.org/10.3934/math.2022697

25. M. Sharifi, B. Raesi, Vortex theory for two dimensional Boussinesq equations, *Appl. Math. Nonliner Sci.*, **5** (2020), 67–84. https://doi.org/10.2478/amns.2020.2.00014

26. T. A. Sulaiman, H. Bulut, H. M. Baskonus, On the exact solutions to some system of complex nonlinear models, *Appl. Math. Nonliner Sci.*, **6** (2020), 29–42. https://doi.org/10.2478/amns.2020.2.00007

27. S. Rashid, Y. G. Sánchez, J. Singh, K. M. Abualnaja, Novel analysis of nonlinear dynamics of a fractional model for tuberculosis disease via the generalized Caputo fractional derivative operator (case study of Nigeria), *AIMS Math.*, **7** (2022), 10096–10121. https://doi.org/10.3934/math.2022562

28. M. Caputo, *Elasticita e dissipazione*, Zanichelli, Bologna, 1969.

29. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769.

30. D. Li, W. Sun, C. Wu, A novel numerical approach to time-fractional parabolic equations with nonsmooth solutions, *Numer. Math. Theor. Methods Appl.*, **14** (2021), 355–376. https://doi.org/10.4208/nmtma.OA-2020-0129

31. M. She, D. Li, H. W. Sun, A transformed \(L1\) method for solving the multi-term time-fractional diffusion problem, *Math. Comput. Simulat.*, **193** (2022), 584–606. https://doi.org/10.1016/j.matcom.2021.11.005

32. H. Qin, D. Li, Z. Zhang, A novel scheme to capture the initial dramatic evolutions of nonlinear sub-diffusion equations, *J. Sci. Comput.*, **89** (2021), 65. https://doi.org/10.1007/s10915-021-01672-z

33. M. El-Borhamy, N. Mosalam, On the existence of periodic solution and the transition to chaos of Rayleigh-Duffing equation with application of gyro dynamic, *Appl. Math. Nonlinear Sci.*, **5** (2020), 93–108. https://doi.org/10.2478/amns.2020.1.00010

34. R. A. de Assis, R. Pazim, M. C. Malavazi, P. P. da C. Petry, L. M. E. da Assis, E. Venturino, A mathematical model to describe the herd behaviour considering group defense, *Appl. Math. Nonlinear Sci.*, **5** (2020), 11–24. https://doi.org/10.2478/amns.2020.1.00002

*AIMS Mathematics* Volume 7, Issue 8, 14946–14974.
35. J. Singh, Analysis of fractional blood alcohol model with composite fractional derivative, *Chaos Soliton. Fract.*, 140 (2020), 110127. https://doi.org/10.1016/j.chaos.2020.110127

36. P. A. Naik, Z. Jain, K. M. Owolabi, Global dynamics of a fractional order model for the transmission of HIV epidemic with optimal control, *Chaos Soliton. Fract.*, 138 (2020), 109826. https://doi.org/10.1016/j.chaos.2020.109826

37. A. Atangana, E. Alabaraoye, Solving a system of fractional partial differential equations arising in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller-Segel equations, *Adv. Differ. Equ.*, 2013 (2013), 94. https://doi.org/10.1186/1687-1847-2013-94

38. H. Günerhan, E. Çelik, Analytical and approximate solutions of fractional partial differential-algebraic equations, *Appl. Math. Nonlinear Sci.*, 5 (2020), 109–120. https://doi.org/10.2478/amns.2020.1.00011

39. F. Evirgen, S. Uçar, N. Özdemir, System analysis of HIV infection model with CD4+T under non-singular kernel derivative, *Appl. Math. Nonlinear Sci.*, 5 (2020), 139–146. https://doi.org/10.2478/amns.2020.1.00013

40. M. R. R. Kanna, R. P. Kumar, S. Nandappa, I. N. Cangul, On solutions of fractional order telegraph partial differential equation by Crank-Nicholson finite difference method, *Appl. Math. Nonliner Sci.*, 5 (2020), 85–98. https://doi.org/10.2478/amns.2020.2.00017

41. M. A. Alqudah, R. Ashraf, S. Rashid, J. Singh, Z. Hammouch, T. Abdeljawad, Novel numerical investigations of fuzzy Cauchy reaction-diffusion models via generalized fuzzy fractional derivative operators, *Fractal Fract.*, 5 (2021), 151. https://doi.org/10.3390/fractalfract5040151

42. S. Rashid, M. K. A. Kaabar, A. Althobaiti, M. S. Alqurashi, Constructing analytical estimates of the fuzzy fractional-order Boussinesq model and their application in oceanography, *J. Ocean Eng. Sci.*, 2022. https://doi.org/10.1016/j.joes.2022.01.003

43. S. Rashid, R. Ashraf, Z. Hammouch, New generalized fuzzy transform computations for solving fractional partial differential equations arising in oceanography, *J. Ocean Eng. Sci.*, 2021. https://doi.org/10.1016/j.joes.2021.11.004

44. Z. Li, C. Wang, R. P. Agarwal, R. Sakthivel, Hyers-Ulam-Rassias stability of quaternion multidimensional fuzzy nonlinear difference equations with impulses, *Iran. J. Fuzzy Syst.*, 18 (2021), 143–160.

45. A. Kandel, W. J. Byatt, Fuzzy differential equations, *Proceedings of the International Conference Cybernetics and Society*, Tokyo, 1978.

46. R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.: Theory Methods Appl.*, 72 (2010), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

47. K. Nemati, M. Matinfar, An implicit method for fuzzy parabolic partial differential equations, *J. Nonlinear Sci. Appl.*, 1 (2008), 61–71.

48. T. Allahviranloo, M. Afshar Kermani, Numerical methods for fuzzy partial differential equations under new definition for derivative, *Iran. J. Fuzzy Syst.*, 7 (2010), 33–50.
49. O. A. Arqub, M. Al-Smadi, S. Momani, T. Hayat, Application of reproducing kernel algorithm for solving second-order, two-point fuzzy boundary value problems, *Soft Comput.*, 21 (2017), 7191–7206. https://doi.org/10.1007/s00500-016-2262-3

50. O. A. Arqub, Adaptation of reproducing kernel algorithm for solving fuzzy Fredholm-Volterra integrodifferential equations, *Neural Comput. Applic.*, 28 (2017), 1591–1610. https://doi.org/10.1007/s00521-015-2110-x

51. T. M. Elzaki, S. M. Ezaki, Application of new transform “Elzaki transform” to partial differential equations, *Global J. Pure Appl. Math.*, 7 (2011), 65–70.

52. S. Rashid, K. T. Kubra, S. U. Lehre, Fractional spatial diffusion of a biological population model via a new integral transform in the settings of power and Mittag-Leffler nonsingular kernel, *Phys. Scr.*, 96 (2021), 114003.

53. S. Rashid, Z. Hammouch, H. Aydi, A. G. Ahmad, A. M. Alsharif, Novel computations of the time-fractional Fisher’s model via generalized fractional integral operators by means of the Elzaki transform, *Fractal Fract.*, 5 (2021), 94. https://doi.org/10.3390/fractalfract5030094

54. S. Rashid, R. Ashraf, A. O. Akdemir, M. A. Alqudah, T. Abdeljawad, S. M. Mohamed, Analytic fuzzy formulation of a time-fractional Fornberg-Whitham model with power and Mittag-Leffler kernels, *Fractal Fract.*, 5 (2021), 113. https://doi.org/10.3390/fractalfract5030113

55. G. Adomian, A review of the decomposition method in applied mathematics, *J. Math. Anal. Appl.*, 135 (1988), 501–544. https://doi.org/10.1016/0022-247X(88)90170-9

56. G. Adomian, R. Rach, On composite nonlinearities and the decomposition method, *J. Math. Anal. Appl.*, 113 (1986), 504–509. https://doi.org/10.1016/0022-247X(86)90321-5

57. S. S. L. Chang, L. Zadeh, On fuzzy mapping and control, *IEEE Trans. Syst. Man Cybern.*, 2 (1972), 30–34. https://doi.org/10.1109/TSMC.1972.5408553

58. R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus, *Fuzzy. Sets. Syst.*, 18 (1986), 31–43. https://doi.org/10.1016/0165-0114(86)90026-6

59. A. Kaufmann, M. M. Gupta, *Introduction to fuzzy arithmetic*, New York: Van Nostrand Reinhold Company, USA, 1991.

60. B. Bede, J. Fodor, Product type operations between fuzzy numbers and their applications in geology, *Acta Polytech. Hung.*, 3 (2006), 123–139.

61. A. Georgieva, Double fuzzy Sumudu transform to solve partial Volterra fuzzy integro-differential equations, *Mathematics*, 8 (2020), 692. https://doi.org/10.3390/math8050692

62. B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets Syst.*, 230 (2013), 119–141. https://doi.org/10.1016/j.fss.2012.10.003

63. B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets Syst.*, 151 (2005), 581–599. https://doi.org/10.1016/j.fss.2004.08.001

64. Y. Chalco-Cano, H. Román-Flores, On new solutions of fuzzy differential equations, *Chaos Solition. Fract.*, 38 (2008), 112–119. https://doi.org/10.1016/j.chaos.2006.10.043

65. H. C. Wu, The improper fuzzy Riemann integral and its numerical integration, *Inf. Sci.*, 111 (1998), 109–137. https://doi.org/10.1016/S0020-0255(98)00016-4
66. A. H. Sedeeg, A coupling Elzaki transform and homotopy perturbation method for solving nonlinear fractional heat-like equations, *Am. J. Math. Comput. Model.*, 1 (2016), 15–20
67. A. Georgieva, A. Pavlova, Fuzzy Sawi decomposition method for solving nonlinear partial fuzzy differential equations, *Symmetry*, 13 (2021), 1580. https://doi.org/10.3390/sym13091580
68. R. Henstock, *Theory of integration*, Butterworth, London, 1963.
69. Z. T. Gong, L. L. Wang, The Henstock-Stieltjes integral for fuzzy-number-valued functions, *Inf. Sci.*, 188 (2012), 276–297. https://doi.org/10.1016/j.ins.2011.11.024
70. L. Jäntsch, The Eigenproblem translated for alignment of molecules, *Symmetry*, 11 (2019), 1027. https://doi.org/10.3390/sym11081027
71. L. Jäntsch, D. Bálint, S. D. Bolboacă, Multiple linear regressions by maximizing the likelihood under assumption of generalized Gauss-Laplace distribution of the error, *Comput. Math. Methods Med.*, 2016 (2016), 8578156. https://doi.org/10.1155/2016/8578156

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