Critical regularity criteria for Navier–Stokes equations in terms of one directional derivative of the velocity

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In this paper, we consider the 3D Navier–Stokes equations in the whole space. We investigate some new inequalities and a priori estimates to provide the critical regularity criteria in terms of one directional derivative of the velocity field, namely, \( \partial_t u \in L^p((0, T); L^q(\mathbb{R}^3)) \), \( \frac{2}{p} + \frac{3}{q} = 2 \), \( 3 \leq q \leq 6 \). Moreover, we extend the range of \( q \) while the solution is axisymmetric, that is, the axisymmetric solution \( u \) is regular in \((0, T]\), if \( \partial_3 u_3 \in L^p((0, T); L^q(\mathbb{R}^3)) \), \( \frac{2}{p} + \frac{3}{q} = 2 \), \( \frac{3}{2} < q < \infty \).

KEYWORDS
Navier–Stokes equations, one directional derivative, Prodi–Serrin conditions, regularity, the decomposition of velocity

MSC CLASSIFICATION
35Q35; 35Q30; 76D03

1 | INTRODUCTION

In this paper, we consider the Cauchy problem of the 3D Navier–Stokes equations:

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \Pi &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \\
\nabla \cdot u &= 0, \\
u_{\mid t=0} &= u_0.
\end{aligned}
\] (1.1)

The solution \( u(t, x) = (u_1, u_2, u_3) \), \( \Pi(t, x) \), and \( u_0 \) denote the fluid velocity field, pressure, and the given initial data, respectively. These equations describe the flow of incompressible viscous fluid.

For given \( u_0 \in L^2(\mathbb{R}^3) \) with \( \text{div} u_0 = 0 \) in the sense of distribution, a global weak solution \( u \) to the Navier–Stokes equations was constructed by Leray\(^1\) and Hopf\(^2\) which is called Leray–Hopf weak solution. The regularity of such Leray–Hopf weak solution in three dimensions plays an important role in the mathematical fluid mechanics. One essential work is usually referred as Prodi–Serrin (P–S) conditions\(^3\)–\(^6\) that is, if the weak solution \( u \) satisfies

\[
u \in L^p((0, T); L^q(\mathbb{R}^3)),
\] (1.2)

with \( \frac{2}{p} + \frac{3}{q} \leq 1 \), \( 3 \leq q \leq \infty \), then the weak solution is regular in \((0, T]\).

An analogical result occurs for \( \nabla u \): the Leray–Hopf weak solution \( u \) is regular, if

\[
\nabla u \in L^p((0, T); L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} \leq q \leq +\infty.
\] (1.3)
Since then, many significant regularity criteria\textsuperscript{7–12} were established in terms of only partial components of the velocity field, or partial components of the gradient of the velocity field of the 3D Navier–Stokes equations. For instance, Chemin and Zhang\textsuperscript{8,9} and Han et al.\textsuperscript{11} proved the regularity of $u$ in $(0, T]$, if
\[
\int_0^T \|u_3\|_{\frac{p}{2} + \frac{3}{q}}^p dt < \infty, \quad p \geq 2.
\] (1.4)

In this paper, we focus on the regularity criteria in terms of one directional derivative of the whole velocity field. Kukavica and Zaine\textsuperscript{13} investigated the regularity criteria for the term $\partial_3 u$, which is scaling invariant,
\[
\partial_3 u \in L^p((0, T); L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{q}{2} < q \leq 3.
\] (1.5)

Later on, Cao,\textsuperscript{14} Namlyeyeva and Skalak,\textsuperscript{15} and Zhang\textsuperscript{16} extended the range of $q$ to $q \in [1.5620, 3]$.

In this paper, we develop some a priori estimates and extend the range of $q$ to $q \in (1.5, 6]$, which is optimal on the left side. Now, we state our main theorem.

**Theorem 1.** Let $u \in L^\infty((0, +\infty); L^2(\mathbb{R}^3)) \cap L^2((0, \infty); H^1(\mathbb{R}^3))$ be the Leray–Hopf weak solution of the Navier–Stokes equations (1.1) with the initial data $u_0 \in L^2(\mathbb{R}^3)$ and div $u_0 = 0$. The solution $u$ is regular in $(0, T]$, provided that
\[
\partial_3 u \in L^p((0, T); L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq 6.
\] (1.6)

If the solution $u$ is axisymmetric, then the solution $u$ is regular in $(0, T]$, provided that
\[
\partial_3 u_3 \in L^p((0, T); L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q < \infty.
\] (1.7)

**Remark 1.** Analogously, we can also prove that the solution $u$ is regular in $(0, T]$, if $\|\partial_3 u\|_{L^\infty((0, T); L^2(\mathbb{R}^3))} \ll 1$ is sufficiently small.

In recent years, it has been realized that the regularity problem for the axisymmetric Navier–Stokes equations is essentially a critical one under the standard scaling. Further investigations are well motivated.\textsuperscript{17–20}

For the regularity criteria (1.6), we give a brief overview of the proof and explain some main steps.

**Step 1.** Anisotropic decomposition of the velocity.

Inspired by Chemin et al.,\textsuperscript{8,9,11} we adopt a different type of the decomposition of the velocity field in (2.1).
\[
u_h = \Delta^{-1} \left( -\nabla_h \partial_3 u_3 + \partial_3^2 u_h + \nabla_h^3 \omega_3 \right).
\] (1.8)

The two-dimensional vorticity $\omega_3$ and $\partial_3 u$ are regarded as governing unknowns.

**Step 2.** Estimates of $\omega_3$ and $\partial_3 u$.

If we compute the time derivative of $\|\omega_3\|_2^2 + \|\partial_3 u\|_2^2$, we need to treat the nonlinear terms (see 3.2 and 3.13) such as
\[
\int -\partial_3 u_2 \partial_1 u_3 \omega_3 \, dx + \ldots = \int \partial_3 u_2 \partial_1 u_3 \omega_3 \, dx + \ldots,
\]

since $\nabla_h u_3$ is bad term in our analysis. Therefore, we have to deal with some a priori estimates involving the term $u_3$.

**Step 3.** Estimates of $u_3$.

This is the main part of our proof. We work with the norm $\|(u_3)^2\|_2$ when $\frac{3}{2} < q < 2$, which has the same scaling as $\|\omega_3\|_2^2$ and $\|\partial_3 u\|_2^2$, respectively. To get a glimpse into this, we assume $q = \frac{3}{2}$ in (1.6), and $\|\partial_3 u\|_{L^\infty((0, T); L^2(\mathbb{R}^3))} \ll 1$ is sufficiently
small. When computing the time evolution of \( \| (u_t)^2 \|_2^2 \), we need to estimate the term (see Section 3.2 for more details)

\[
J_2 = 2 \sum_{i=1,2,3, h=1,2} \int \Delta^{-1} \partial_t (\partial_t u_t u_h) (u_t)^3 \, dx.
\]

We adopt a new inequality (2.4) to control the term \( u_h \) by \( \| \Delta u_h \|_2 \). Therefore, it is quite natural to bound the above as

\[
J_2 \leq C \| \partial_t u \|_\infty \| u_h \|_3 \| u_3 \|_1 \frac{3}{7}
\]

\[
\leq C \| \partial_t u \|_{\frac{8}{3}n} \| \partial_t u_2 \|_{\frac{4}{3}} \| \Delta u_2 \|_3 \frac{3}{2} \| \nabla_h (u_2) \|_3 \frac{3}{2}
\]

\[
\leq C \| \partial_t u \|_{\frac{2}{3}} \left( \| \nabla \omega_3 \|_2^n + \| \nabla (u_3) \|_2^n + \| \nabla \partial_t u \|_2^n \right).
\]

Such estimates turn out to be crucial, and we obtain the uniform control of \( E_i(t) = \| \omega_3 \|_2^2 + \| (u_3)^2 \|_2^2 + \| \partial_t u \|_2^2 \) on the time interval \((0, T)\).

The rest of this paper is organized as follows. In Section 2, we set up some notations and collect a few useful lemmas. In Section 3, we obtain some a priori estimates of \( \omega_3, u_3 \), and \( \partial_t u \) when \( \frac{1}{3} < q < 2 \). In Section 4, we obtain some a priori estimates of \( \omega_3, u_3 \), and \( \partial_t u \) when \( 2 \leq q \leq 6 \). The final section is devoted to the proof of the main theorem.

## 2 | NOTATIONS AND PRELIMINARY

Given two comparable quantities, the inequality \( X \lesssim Y \) stands for \( X \leq CY \) for some positive constant \( C \). The dependence of the constant \( C \) on other parameters or constants are usually clear from the context, and we will often suppress this dependence. Moreover, we denote \( L^p_T = L^p([T_1, T_2]; L^q(\mathbb{R}^3)) \) and \( \| \cdot \|_r = \| \cdot \|_{L^r(\mathbb{R}^3)} \), for the sake of simplicity.

We shall adopt the following convention for the Fourier transform:

\[
\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} dx,
\]

\[
f(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{f}(\xi) e^{ix \cdot \xi} d\xi.
\]

For \( s \in \mathbb{R} \), the fractional Laplacian \( \Lambda^s \) then corresponds to the Fourier multiplier \( |\xi|^s \) defined as

\[
\Lambda^s f(\xi) = |\xi|^s \hat{f}(\xi),
\]

whenever it is well defined. Analogously, we also denote anisotropic fractional Laplacian \( \Lambda^s_h, \Lambda^s_3 \) as

\[
\Lambda^s_h f(\xi) = |\xi_h|^s \hat{f}(\xi),
\]

\[
\Lambda^s_3 f(\xi) = |\xi_3|^s \hat{f}(\xi),
\]

where \( \xi_h = (\xi_1, \xi_2) \) and \( \xi_3 \) are referred to the horizontal and vertical variables, respectively.

A remarkable idea introduced in Chemin and Zhang\(^8,^9\) and Han et al.\(^11\) is to use the decomposition of the velocity field along with horizontal and vertical directions, use the two-dimensional vorticity \( \omega_3 \) and \( u_3 \) as governing unknowns, where \( \omega_3 = \partial_1 u_2 - \partial_2 u_1 \).

We denote \( x_0 = (x_1, x_2, \nabla_h = (\partial_1, \partial_2), \nabla_3^h = (-\partial_2, \partial_1), \) and \( u_h = (u_1, u_2). \) To best illuminate our proof, we introduce a slightly different decomposition of the velocity field. Notice that

\[
\nabla \times \nabla \times \mathbf{v} = \nabla \text{div} \mathbf{v} - \Delta \mathbf{v}, \quad \mathbf{v} = (u_h, 0).
\]
Then, by using the Biot–Savart law, we get

$$u_h = \Delta^{-1} \left( -\nabla_h \partial_3 u_3 + \partial_3^2 u_h + \nabla_h^4 \omega_3 \right).$$  \hfill (2.1)

**Lemma 1.** For any $1 < r < \infty$, we have

$$\|\nabla u_h\|_r \lesssim \|\partial_3 u\|_r + \|\omega_3\|_r,$$  \hfill (2.2)

$$\|\nabla^2 u_h\|_r \lesssim \|\nabla \partial_3 u\|_r + \|\nabla \omega_3\|_r.$$  \hfill (2.3)

A key ingredient is introduced below.

**Lemma 2.** For any $1 < q < 2$, there exists a constant $C_q$ such that for any $f \in H^2(\mathbb{R}^3)$ and $\partial_3 f \in L^q(\mathbb{R}^3)$, we have $f \in L^b(\mathbb{R}^3)$ and

$$\|f\|_b \leq C_q \|\partial_3 f\|_q^s \|\Delta f\|_2^{1-s},$$  \hfill (2.4)

where $b = \frac{10q+12}{6-3q}$ and $s = \frac{4q}{5q+6}$.

**Proof.** Without loss of generality, we assume $f \in C^\infty_0(\mathbb{R}^3)$. By Sobolev embedding and Hölder inequality, we obtain

$$\|f\|_b \lesssim \left\|\Lambda_h^{\frac{m}{2}} f\right\|_2$$

$$\lesssim \left\|\xi_3^m |\xi|^{\frac{2m}{q}} f\right\|_2$$

$$\lesssim \left\|\xi_3^{\frac{2m}{q}} |\xi_3| f\right\|_2$$

$$\lesssim \left\|\xi_3^{-\frac{2m}{q}} \cdot |\xi| f\right\|_2 \left\|\xi| f\right\|_2^{1-s}$$

$$\lesssim \|\partial_3 f\|_q^s \|\Delta f\|_2^{1-s}.$$

\[\Box\]

We recall the following three-dimensional Sobolev–Ladyzhenskaya inequalities.\(^{14,16}\)

**Lemma 3.** For any $1 \leq q < \infty$, there exists a constant $C_q$ such that for any $f \in C^\infty_0(\mathbb{R}^3)$,

$$\|f\|_{3q} \leq C_q \|\partial_3 f\|_q \|\nabla h f\|_2^\frac{2}{q},$$  \hfill (2.5)

$$\|f\|_{2q} \leq C_q \|\partial_3 f\|_q \left\|\nabla h (|f|^2)\right\|_2^\frac{2}{q}.\hfill (2.6)$$

### 3 | A PRIORI ESTIMATES 1

In this section, we assume that

$$\partial_3 u \in L^p((0, T); L^q(\mathbb{R}^3)), \frac{2}{p} + \frac{3}{q} = 2, \frac{3}{2} < q < 2.$$

#### 3.1 | Estimate of $\omega_3$

Recall that $\omega_3$ satisfies the equation

$$\partial_t \omega_3 + (u \cdot \nabla) \omega_3 - \Delta \omega_3 = -\partial_3 u_2 \partial_1 u_3 + \partial_3 u_1 \partial_2 u_3 + \omega_3 \partial_3 u_3.$$  \hfill (3.1)

Taking $L^2$ inner product of Equation (3.1) with $\omega_3$, one has

$$\frac{1}{2} \frac{d}{dt} \|\omega_3\|_2^2 + \|\nabla \omega_3\|_2^2 = \int \partial_3 u_2 \partial_1 u_3 \omega_3 + \partial_3 u_1 \partial_2 u_3 \omega_3 + \omega_3 \partial_3 u_3 \omega_3 \, dx$$

$$= \int \partial_3 u_2 \partial_1 u_3 \partial_3 \omega_3 - \partial_3 u_1 \partial_3 u_3 \omega_3 + \frac{1}{2} \partial_3 u_3 (\omega_3)^2 \, dx$$

$$:= I_1 + I_2 + I_3.$$  \hfill (3.2)
According to Hölder, interpolation, and Cauchy–Schwarz inequalities, we have

\[ I_1 + I_2 \leq 2 \| \partial_3 u_h \|_{q[a]} \| u_3 \|_{q[a]} \| \nabla \omega_3 \|_2 \]

\[ \leq C \| \partial_3 u_h \|_q \| \nabla \partial_3 u_h \|_{q[a]} \| (u_3)^2 \|_2^{1-\frac{1}{r}} \| \nabla (u_3)^2 \|_2^{\frac{1}{r} - \frac{1}{2}} \| \nabla \omega_3 \|_2 \]

\[ \leq C \| \partial_3 u \|_p \| (u_3)^2 \|_2^2 + \frac{1}{16} \left( \| \nabla \omega_3 \|_2^2 + \| \nabla (u_3)^2 \|_2^2 + \| \nabla \partial_3 u \|_2^2 \right), \tag{3.3} \]

and

\[ I_3 \leq \frac{1}{2} \| \partial_3 u_3 \|_{q[a]} \| \omega_3 \|_{q[a]}^2 \]

\[ \leq C \| \partial_3 u_3 \|_q \| \omega_3 \|_{q[a]}^{2-\frac{1}{r}} \| \nabla \omega_3 \|_2^{\frac{1}{r}} \]

\[ \leq C \| \partial_3 u \|_p \| \omega_3 \|_2^2 + \frac{1}{16} \| \nabla \omega_3 \|_2^2 . \tag{3.4} \]

Summing up (3.2), (3.3), and (3.4), we get

\[ \frac{1}{2} \frac{d}{dt} \| \omega_3 \|_2^2 + \| \nabla \omega_3 \|_2^2 \leq C \| \partial_3 u \|_p \left( \| \omega_3 \|_2^2 + \| (u_3)^2 \|_2^2 \right) + \frac{1}{8} \left( \| \nabla \omega_3 \|_2^2 + \| \nabla (u_3)^2 \|_2^2 + \| \nabla \partial_3 u \|_2^2 \right). \tag{3.5} \]

### 3.2 Estimate of \( u_3 \)

The equation of \( u_3 \) is

\[ \partial_t u_3 + (a \cdot \nabla) u_3 - \Delta u_3 + \partial_3 \Pi = 0. \tag{3.6} \]

Taking inner product of the Equation (3.6) with \( (u_3)^3 \), we obtain

\[ \frac{1}{4} \frac{d}{dt} \| (u_3)^3 \|_2^2 + \frac{3}{4} \| \nabla (u_3)^2 \|_2^2 = - \int \partial_3 \Pi \cdot (u_3)^3 \, dx \]

\[ = 2 \sum_{i,j=1,2,3} \int \Delta^{-1} \partial_i \partial_j (\partial_3 u_i u_j) (u_3)^3 \, dx \]

\[ = 2 \sum_{i=1,2,3} \int \Delta^{-1} \partial_3 (\partial_3 u_i u_3) (u_3)^3 \, dx + 2 \sum_{i=1,2,3} \int \Delta^{-1} \partial_i (\partial_3 u_i u_j) (u_3)^3 \, dx \]

\[ := J_1 + J_2. \tag{3.7} \]

Applying Hölder, interpolation, and Cauchy–Schwarz inequalities, we have

\[ J_1 \leq C \| \partial_3 u \cdot u_3 \|_{q[a]} \| u_3 \|_{q[a]} \| (u_3)^3 \|_{q[a]} \]

\[ \leq C \| \partial_3 u \|_q \| (u_3)^2 \|_2^{\frac{1}{r}} \| u_3 \|_{q[a]} \]

\[ \leq C \| \partial_3 u \|_p \| (u_3)^2 \|_2^2 + \frac{1}{16} \| \nabla (u_3)^2 \|_2^2 , \tag{3.8} \]

which is similar to (3.4).

For \( \frac{3}{2} < q < 2 \), we pick \( s = \frac{4q}{5q+6} \), \( b = \frac{10q+12}{6-3q} \), and \( \kappa = \frac{5(3-q)}{7q-3}, \frac{1}{a} = \frac{20+15\kappa-5}{12\kappa+20}, \theta = \frac{3x+15-10s}{6x+10} \). By Lemmas 1, 2, and 3, we get

\[ J_2 \leq C \| \partial_3 u \|_a \| u_3 \|_b \| (u_3)^3 \|_2^{\frac{1-\kappa}{2}} \| u_3 \|_{5[a]}^{\frac{\kappa}{2}} \]

\[ \leq C \| \partial_3 u \|_a \| \partial_3 u u_3 \|_a \| \Delta u_3 \|_{2-\frac{1}{r}} \| (u_3)^2 \|_2^{\frac{1-\kappa}{2}} \| \partial_3 u \|_{a}^{\frac{\kappa}{2}} \| \nabla (u_3)^2 \|_2^{\frac{6\kappa}{2}} \]

\[ \leq C \| \partial_3 u \|_p \| \nabla \partial_3 u \|_2^{1+\frac{3q}{5q+6}-(1-\theta)} \| (u_3)^2 \|_2 + \| \nabla \partial_3 u \|_2^{1-\theta} \| (u_3)^2 \|_2^{\frac{3-\kappa}{2}} \| \nabla (u_3)^2 \|_2^{\frac{6\kappa}{2}} \]

\[ \leq C \| \partial_3 u \|_p \| (u_3)^2 \|_2^2 + \frac{1}{16} \left( \| \nabla \omega_3 \|_2^2 + \| \nabla (u_3)^2 \|_2^2 + \| \nabla \partial_3 u \|_2^2 \right) . \tag{3.9} \]
Summing up (3.7), (3.8), and (3.9), we obtain
\begin{equation}
\frac{1}{4} \frac{d}{dt} \left\| (u_3)^2 \right\|^2_2 + \frac{3}{4} \left\| \nabla (u_3)^2 \right\|^2_2 \leq C \| \partial_3 u \|_q^p \left( \| \omega_3 \|_2^2 + \left\| (u_3)^2 \right\|^2_2 + \| \partial_3 u \|_2 \right) + \frac{1}{8} \left( \| \nabla \omega_3 \|_2^2 + \left\| \nabla (u_3)^2 \right\|^2_2 + \| \nabla \partial_3 u \|_2^2 \right). \tag{3.10}
\end{equation}

### 3.3 Estimate of $\partial_3 u$

Taking inner product of Equation (1.1) with $-\partial_3^2 u$, we obtain
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \partial_3 u \|^2_2 + \| \nabla \partial_3 u \|^2_2 &= - \int \partial_3 u \cdot \nabla u \cdot \partial_3 u \, dx \\
&= - \sum_{j=1,2} \int \partial_3 u \cdot \nabla u_j \cdot \partial_3 u_j \, dx - \int \partial_3 u \cdot \nabla u_3 \cdot \partial_3 u_3 \, dx \\
&:= K_1 + K_2. \tag{3.11}
\end{align*}

By Hölder inequality and Lemma 1, we have
\begin{align*}
K_1 &\leq C \| \partial_3 u \|_q \| \nabla u_h \|_{\frac{2q}{q-1}} \| \partial_3 u \|_{\frac{2q}{q-1}} \\
&\leq C \| \partial_3 u \|_q \left( \| \omega_3 \|_{\frac{2q}{q-1}} + \| \partial_3 u \|_{\frac{2q}{q-1}} \right) \| \partial_3 u \|_{\frac{2q}{q-1}} \\
&\leq C \| \partial_3 u \|_q^p \left( \| \omega_3 \|_2^2 + \| \partial_3 u \|_2^2 \right) + \frac{1}{16} \left( \| \nabla \omega_3 \|_2^2 + \| \nabla \partial_3 u \|_2^2 \right), \tag{3.12}
\end{align*}

which is similar to (3.4). And by Hölder, interpolation, and Cauchy–Schwarz inequalities, we obtain
\begin{align*}
K_2 &= \int \partial_3 u \cdot u_3 \cdot \nabla \partial_3 u_3 \, dx \\
&\leq \| \partial_3 u \|_{\frac{2q}{q-2}} \| u_3 \|_{\frac{2q}{q-2}} \| \nabla \partial_3 u_3 \|_2 \\
&\leq C \| \partial_3 u \|_q^p \left( \| u_3 \|_2^2 + \left\| (u_3)^2 \right\|^2_2 + \| \partial_3 u \|_2 \right) + \frac{1}{16} \left( \| \nabla (u_3)^2 \|^2_2 + \| \nabla \partial_3 u \|_2^2 \right), \tag{3.13}
\end{align*}

which is similar to (3.3). Summing up (3.11), (3.12), and (3.13), we get
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \partial_3 u \|^2_2 + \| \nabla \partial_3 u \|^2_2 \leq C \| \partial_3 u \|_q^p \left( \| \omega_3 \|_2^2 + \left\| (u_3)^2 \right\|^2_2 + \| \partial_3 u \|_2 \right) + \frac{1}{8} \left( \| \nabla \omega_3 \|_2^2 + \left\| \nabla (u_3)^2 \right\|^2_2 + \| \nabla \partial_3 u \|_2^2 \right). \tag{3.14}
\end{equation}

### 4 A PRIORI ESTIMATES 2

In this section, we assume that
\[ \partial_3 u \in L^p((0, T); L^q(\mathbb{R}^3)), \frac{2}{p} + \frac{3}{q} = 2, 2 \leq q \leq 6. \]

#### 4.1 Estimate of $\omega_3$ and $\partial_3 u$

Recall that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \| \omega_3 \|^2_2 + \| \partial_3 u \|_2^2 \right) + \| \nabla \omega_3 \|_2^2 + \| \nabla \partial_3 u \|_2^2 = : I_1 + I_2 + I_3 + K_1 + K_2, \tag{4.1}
\end{equation}
in (3.2) and (3.11).

- For $2 \leq q < 3$.

\begin{equation}
I_1 + I_2 + K_2 \leq 2 \| \partial_3 u \|_q^2 \cdot \| \partial_3 u \|_2 \left\| (u_3)^2 \right\|^\frac{1}{2} \left( \| \nabla \omega_3 \|_2 + \| \nabla \partial_3 u \|_2 \right) \leq C \| \partial_3 u \|_q^2 \| \nabla \partial_3 u \|_2 \left\| (u_3)^2 \right\|_q^\frac{1}{2} \left( \| \nabla \omega_3 \|_2 + \| \nabla \partial_3 u \|_2 \right). \tag{4.2}
\end{equation}
Summing up (3.4), (3.12), (4.1), and (4.2), we obtain that for $0 < T_1 < T$,

\[
\begin{align*}
\|\omega_3\|_{L_{2,T_1,T}^2}^2 + \|\partial_3 u\|_{L_{2,T_1,T}^2}^2 + 2 \|\nabla \omega_3\|_{L_{2,T_1,T}^2}^2 + 2 \|\nabla \partial_3 u\|_{L_{2,T_1,T}^2}^2 &
\leq 2 \|\omega_3(T_1)\|_{L_{2,T_1,T}^2}^2 + 2 \|\partial_3 u(T_1)\|_{L_{2,T_1,T}^2}^2 + C \|\partial_3 u\|_{L_{2,T_1,T}^2}^2 \|\nabla \partial_3 u\|_{L_{2,T_1,T}^2}^\frac{3}{2} \|u_3\|_{L_{2,T_1,T}^2}^\frac{3}{2} + \frac{\lambda_3}{K_3} \left(\|\nabla \omega_3\|_{L_{2,T_1,T}^2}^2 + \|\nabla \partial_3 u\|_{L_{2,T_1,T}^2}^2\right) \\
&+ C \|\partial_3 u\|_{L_{2,T_1,T}^2}^2 \left(\|\omega_3\|_{L_{2,T_1,T}^2}^2 + \|\partial_3 u\|_{L_{2,T_1,T}^2}^2\right) + \frac{1}{2} \left(\|\nabla \omega_3\|_{L_{2,T_1,T}^2}^2 + \|\nabla \partial_3 u\|_{L_{2,T_1,T}^2}^2\right)
\leq 2 \|\omega_3(T_1)\|_{L_{2,T_1,T}^2}^2 + 2 \|\partial_3 u(T_1)\|_{L_{2,T_1,T}^2}^2 + C \left(\|\partial_3 u\|_{L_{2,T_1,T}^2}^2 + \|\partial_3 u\|_{L_{2,T_1,T}^2}^2\right) \left(\|\omega_3\|_{L_{2,T_1,T}^2}^2 + \|\partial_3 u\|_{L_{2,T_1,T}^2}^2\right) + \frac{1}{2} \left(\|\nabla \omega_3\|_{L_{2,T_1,T}^2}^2 + \|\nabla \partial_3 u\|_{L_{2,T_1,T}^2}^2\right).
\end{align*}
\]

(4.3)

- For $3 \leq q \leq 6$.

According to Hölder, interpolation, and Cauchy–Schwarz inequalities, we get

\[
\begin{align*}
I_1 + I_2 + K_2 &\leq 2 \|\partial_3 u\|_{L_q^{2-q}}^2 \|\partial_3 u\|_{L_q^{2+2}}^2 \left(\|\nabla \omega_3\|_{L_q^2}^2 + \|\nabla \partial_3 u\|_{L_q^2}^2\right) \\
&\leq C \|\partial_3 u\|_{L_q^{2-q}}^2 \|\partial_3 u\|_{L_q^{2+2}}^2 \left(\|\nabla \omega_3\|_{L_q^2}^2 + \|\nabla \partial_3 u\|_{L_q^2}^2\right) + \frac{1}{16} \left(\|\nabla \omega_3\|_{L_q^2}^2 + \|\nabla \partial_3 u\|_{L_q^2}^2\right).
\end{align*}
\]

(4.4)

Summing up (3.4), (3.12), (4.1), and (4.2), we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left(\|\omega_3\|_{L_q^2}^2 + \|\partial_3 u\|_{L_q^2}^2\right) + \|\nabla \omega_3\|_{L_q^2}^2 + \|\nabla \partial_3 u\|_{L_q^2}^2 &
\leq C \|\partial_3 u\|_{L_q^2}^2 \left(\|\omega_3\|_{L_q^2}^2 + \|\partial_3 u\|_{L_q^2}^2\right) + \frac{1}{16} \left(\|\nabla \omega_3\|_{L_q^2}^2 + \|\nabla \partial_3 u\|_{L_q^2}^2\right).
\end{align*}
\]

(4.5)

### 4.2 Estimate of $u_3$

Taking inner product of Equation (3.6) with $|u_3|^2 u_3$, we obtain

\[
\begin{align*}
\frac{2}{9} \frac{d}{dt} \left\|u_3\right\|_{L_q^2}^2 + \frac{56}{81} \left\|\nabla |u_3|^2 u_3\right\|_{L_q^2}^2 &= - \int \partial_3 \Pi \cdot \left|u_3\right|^2 u_3 \ dx \\
&= 2 \sum_{i,j=1,2,3} \int \Delta^{-1} \partial_i \partial_j \left(\partial_3 u_i u_j\right) \left|u_3\right|^2 u_3 \ dx \\
&= 2 \sum_{i=1,2,3} \int \Delta^{-1} \partial_i \partial_3 \left(\partial_3 u_i u_3\right) \left|u_3\right|^2 u_3 \ dx + 2 \sum_{j=1,2,3} \int \Delta^{-1} \partial_j \partial_3 \left(\partial_3 u_j u_3\right) \left|u_3\right|^2 u_3 \ dx \\
&:= J_1 + J_2.
\end{align*}
\]

Applying Hölder, interpolation, and Cauchy–Schwarz inequalities, we have

\[
\begin{align*}
J_1 &\leq C \|\partial_3 u \cdot u_3\|_{L_q^2} \left\|\left|u_3\right|^2 u_3\right\|_{L_q^2} \left\|\left|u_3\right|^2 u_3\right\|_{L_q^2} \\
&\leq C \|\partial_3 u\|_{L_q^2} \left\|u_3\right\|_{L_q^2}^2 + \frac{1}{16} \left\|\nabla |u_3|^2\right\|_{L_q^2}^2.
\end{align*}
\]

(4.7)
By Lemma 3, we obtain
\[ J_2 \leq C \| \partial_3 \mathbf{u} \|_q \| u_6 \|_{3q} \left\| u_3 \right\|_{3}^{14} \]
\[ \leq C \| \partial_3 \mathbf{u} \|_q \| \nabla u_6 \|_2 \left\| u_3 \right\|_{3}^{14} + \frac{\| \nabla |u_3|^{\frac{1}{q-1}} \|_2^{\frac{q}{q-1}}}{q(q-1)} \]
\[ \leq C \| \partial_3 \mathbf{u} \|_q \| \nabla u_6 \|_2 \left\| u_3 \right\|_{3}^{14} + \frac{1}{16} \| \nabla |u_3|^{\frac{1}{q-1}} \|_2^{2}. \]

(4.8)

Summing up (4.6), (4.7), and (4.8), we get
\[ \frac{d}{dt} \left\| u_3 \right\|_{3}^{2} + \left\| \nabla |u_3|^{\frac{1}{q-1}} \right\|_2^{2} \leq C \| \partial_3 \mathbf{u} \|_q \left( \left\| \nabla u_6 \right\|_2 \left\| u_3 \right\|_{3}^{14} + \left\| u_3 \right\|_{3}^{2} \right). \]

(4.9)

For \( 2 \leq q < 3 \).

By Lemma 1, we deduce that
\[ \left\| u_3 \right\|_{3}^{2} + \left\| u_3 \right\|_{\infty, \infty}^{2} \leq 2 \left\| u_3(T_1) \right\|_{3}^{2} + C \| \partial_3 \mathbf{u} \|_q \left( \left\| \nabla u_6 \right\|_2 \left\| u_3 \right\|_{3}^{14} + \left\| u_3 \right\|_{3}^{2} \right) \]
\[ \leq 2 \left\| u_3(T_1) \right\|_{3}^{2} + C \| \partial_3 \mathbf{u} \|_q \left( \left\| \nabla u_6 \right\|_2 \left\| u_3 \right\|_{3}^{14} + \left\| u_3 \right\|_{3}^{2} \right). \]

(4.10)

For \( 3 \leq q \leq 6 \).

By Lemma 1, we deduce that
\[ \frac{d}{dt} \left\| u_3 \right\|_{3}^{2} \leq C \| \partial_3 \mathbf{u} \|_q \left( \left\| \nabla u_6 \right\|_2 \left\| u_3 \right\|_{3}^{14} + \left\| u_3 \right\|_{3}^{2} \right) \]
\[ \leq C \| \partial_3 \mathbf{u} \|_q \left( \left\| \nabla u_6 \right\|_2 + \left\| \omega_3 \right\|_{2}^{2} \right) \leq C \| \partial_3 \mathbf{u} \|_q \left( \left\| \omega_3 \right\|_{2}^{2} + \left\| u_3 \right\|_{3}^{2} \right). \]

(4.11)

5 | PROOF OF THEOREM 1

For all \( s \in (0, T) \), there exists \( t_0 \in (0, s) \) such that \( \mathbf{u}(t_0, \cdot) \in H^1 \). Let \( \mathbf{u} \in C([t_0, T^*); H^1(\mathbb{R}^3)) \cap L^2_{\text{loc}}([t_0, T^*); H^2(\mathbb{R}^3)) \) be the unique solution of the Navier–Stokes equations, while \( T^* \) is the maximal existence time.

We are going to show \( T^* > T \) by contradiction method. Therefore, by (5.6), we obtain that \( \mathbf{u} \) is regular in \( (t_0, T] \). Due to the arbitrary of \( s \), the weak solution is regular in \( (0, T] \).

Denote
\[ E(t) = \left\| \omega_3 \right\|_{2}^{2} + \left\| \partial_3 \mathbf{u} \right\|_{2}^{2}, \]
\[ E_1(t) = E(t) + \left\| u_3 \right\|_{2}^{2}, \]
\[ E_2(t) = E(t) + \left\| u_3 \right\|_{3}^{2}. \]

We begin our proof of the regularity criteria (1.6).

Proof. Assume \( T^* \leq T \). From (1.6), there exists \( t_0 < T_1 < T^* \), such that \( \| \partial_3 \mathbf{u} \|_{3, 1} \leq \epsilon < 1 \), while the constant \( \epsilon \) is sufficiently small given in (5.3).
1. $\partial_3 u$ satisfies (1.6) with $\frac{3}{2} < q < 2$.

Summing up (3.5), (3.10), and (3.14) and then using Gronwall inequality, we have

$$\sup_{t_0 \leq t < T} E(t) \leq \sup_{t_0 \leq t < T} E_1(t) \exp C \int_{t_0}^{t} \|\partial_{i} u\|_q^q \, dr < +\infty. \quad (5.1)$$

2. $\partial_3 u$ satisfies (1.6) with $2 \leq q < 3$.

For the convenience of the readers, we give another proof for $2 \leq q < 3$, which is similar to Kukavica and Ziane.\textsuperscript{13}

Summing up (4.3) and (4.10), we obtain

$$\sup_{T_1 \leq t < T^*} E_2(t) \leq 2 E_2(T_1) + C_1 \epsilon \cdot \sup_{T_1 \leq t < T^*} E_2(t). \quad (5.2)$$

Picking $\epsilon$ sufficiently small such that

$$C_1 \epsilon < \frac{1}{2}, \quad (5.3)$$

then we have

$$\sup_{T_1 \leq t < T^*} E(t) \leq \sup_{T_1 \leq t < T^*} E_2(t) \leq 4 E_2(T_1) < +\infty. \quad (5.4)$$

3. $\partial_3 u$ satisfies (1.6) with $3 \leq q \leq 6$.

Summing up (4.5) and (4.11) and then using Gronwall inequality, we get

$$\sup_{t_0 \leq t < T^*} E(t) \leq \sup_{t_0 \leq t < T^*} E_2(t) \leq E_2(t_0) \exp C \int_{t_0}^{t} \|\partial_{i} u\|_q^q \, dr < +\infty. \quad (5.5)$$

Therefore, for $\frac{3}{2} < q \leq 6$, we obtain that $\sup_{T_1 \leq t < T^*} E(t) < \infty$.

Now we show the control of the terms $\|\nabla u\|_2$. Applying the spatial derivative $\nabla$ to the Navier–Stokes equations (1.1) and then taking $L^2$ inner product of the resulting equations with $\nabla u$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \|\nabla^2 u\|_2^2 = -\sum_{i,j,k=1,2,3} \int \partial_3 u_j \partial_{j} u_k \partial_{i} u_k \, dx$$

$$-\sum_{i,k=1,2,3} \int \partial_3 u_k \partial_{i} u_k \partial_{3} u_k \, dx - \sum_{i,k=1,2,3} \int \partial_3 u_k \partial_{i} u_k \partial_{3} u_k \, dx$$

$$\leq C \|\partial_3 u\|_2 \|\nabla u\|_3 \|\nabla u\|_6 + C \|\nabla u\|_2 \|\nabla u\|_3 \|\nabla u\|_6$$

$$\leq C (\|\partial_3 u\|_2 + \|\omega_3\|_2) \|\nabla u\|_3^\frac{1}{2} \|\nabla^2 u\|_2^\frac{1}{2}$$

$$\leq C (\|\partial_3 u\|_2 + \|\omega_3\|_2)^4 \|\nabla u\|_2^2 + \frac{1}{2} \|\nabla^2 u\|_2^2.$$

Applying Gronwall inequality, we obtain that

$$\sup_{T_1 \leq t < T^*} \|\nabla u(t)\|_2^2 \leq \|\nabla u(T_1)\|_2^2 \exp C \int_{T_1}^{t} \|\nabla u(r)\|_2^2 \, dr < +\infty. \quad (5.6)$$

We obtain that $u$ can be continued beyond $T^*$, which contradicts with the definition of $T^*$. Thus, $T^* > T$, which yields the regularity criterion (1.6).

Now, we prove the regularity criteria (1.7).

**Proof.** We assume the solution $u$ is axisymmetric and $\partial_3 u$, satisfies (1.7). By 1D Hardy inequality,\textsuperscript{17} we deduce that

$$\int_0^\infty \left( \frac{u_r}{r} \right)^q r \, dr < C(q)^q \int_0^\infty \left( \frac{\partial_r u_r}{r} \right)^q r \, dr,$$
\[ \| u_r \|_{L^q(\mathbb{R}^3)} < C(q) \| \partial_3 u_3 \|_{L^q(\mathbb{R}^3)}, \]
since \( \partial_3 u_3 = -\frac{\partial (ru_r)}{r} \). Then \( u \) is regular, by Theorem 1.1 in Kubica et al. \(^{19}\)

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\section*{CONFLICT OF INTEREST}

This work does not have any conflicts of interest.

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