Diagonal Stability for a Class of Interconnected Passive Systems

Murat Arcak
Department of Electrical, Computer, and Systems Engineering
Rensselaer Polytechnic Institute
Troy, NY 12180
Email: arcakm@rpi.edu

January 1, 2022

Abstract

We consider a class of matrices with a specific structure that arises, among other examples, in dynamic models for biological regulation of enzyme synthesis [6]. We first show that a stability condition given in [6] is in fact a necessary and sufficient condition for diagonal stability of this class of matrices. We then revisit a recent generalization of [6] to nonlinear systems given in [5], and recover the same stability condition using our diagonal stability result. Unlike the input-output based arguments employed in [5], our proof gives a procedure to construct a Lyapunov function. Finally we study static nonlinearities that appear in the feedback path, and give a stability condition that mimics the Popov criterion.

Main Result

The results of this note were triggered by the recent paper [5] and by several discussions with its author. We give our main diagonal stability result in Theorem 1 below, and present its implications for stability of a class of interconnected systems in the form of corollaries to this theorem.
Theorem 1 A matrix of the form

\[
A = \begin{bmatrix}
-1 & 0 & \cdots & 0 & -\gamma_1 \\
\gamma_2 & -1 & \cdots & 0 \\
0 & \gamma_3 & -1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \gamma_n & -1
\end{bmatrix}
\]

\[\gamma_i > 0, \ i = 1, \cdots, n, \quad (1)\]

is diagonally stable; that is, it satisfies

\[DA + A^T D < 0\]

for some diagonal matrix \(D > 0\), if and only if

\[\gamma_1 \cdots \gamma_n < \sec(\pi/n)^n. \quad (2)\]

The proof is given in an appendix. The results surveyed in [3, 2] for diagonal stability of various classes of matrices do not encompass the specific structure exhibited by (1). In particular, the sign reversal for \(\gamma_1\) in (1) rules out the “M-matrix” condition, which is applicable when all off-diagonal terms are nonnegative.

We now apply Theorem 1 to characterize the stability of the feedback interconnection in Figure 1. When each block \(H_i\) is a first-order linear system with transfer function \(H_i(s) = \gamma_i/(\tau_i s + 1), \ \gamma_i > 0, \ \tau_i > 0\), then a state-space representation of the interconnection would be obtained from the \(A\)-matrix in (1) by multiplying its \(i\)th row by \(1/\tau_i\) for \(i = 1, \cdots, n\). Since row multiplications by positive constants do not change diagonal stability, Theorem 1 recovers the result in [6] which states that if (2) holds then the feedback interconnection in Figure 1 is asymptotically stable. Theorem 1 further shows that stability can be proven with a Lyapunov function \(V = x^T P x\) in which \(P\) is diagonal.

The linear result in [6] has been extended in [3] to the situation where \(H_i\)’s are not restricted to be linear, and instead, characterized by the output feedback passivity (OFP) [4] (a.k.a. output strict passivity [7]) property:

\[-\beta \leq -\|y_i\|^2 + \gamma_i < u_i, y_i >\]

(3)
where $\|\cdot\|$ and $<\cdot,\cdot>$ denote, respectively, the norm and inner product in the extended $L_2$ space, and $\beta \geq 0$ represents the bias due to initial conditions. Using this property, [5] proves that the secant condition (2) insures stability of the feedback interconnection in Figure 1.

Unlike the input-output proof given in [5], we now assume that a storage function $V_i$ is available for each block in Figure 1 and show that a weighted sum of these $V_i$’s,

$$V = \sum_{i=1}^{n} d_i V_i,$$

(4)

where $d_i > 0$ are chosen following the procedure below, is a Lyapunov function for the closed-loop system. Indeed, a storage function verifying the OFP property (3) satisfies

$$\dot{V}_i \leq -y_i^2 + \gamma_i u_i y_i$$

(5)

which, when substituted in (4) along with the interconnection conditions

$$u_1 = -y_n, \quad u_i = y_{i-1}, \quad i = 2, \ldots, n,$$

results in

$$\dot{V} \leq -y^T D A y$$

(6)

where $A$ is as in (1), and $D$ is a diagonal matrix comprising of the coefficients $d_i$ in (4). It then follows from Theorem 1 that positive $d_i$’s that render the right-hand side of (6) negative definite indeed exist if (2) holds:

**Corollary 1** Consider the feedback interconnection in Figure 1 and let $u_i$, $x_i$ and $y_i$ denote the input, state vector, and output of each block $H_i$. Suppose, further, there exist $C^1$ storage functions $V_i(x_i)$, satisfying (4) with $\gamma_i > 0$ along the state trajectories of each block. Under these conditions, if (2) holds then there exist $d_i > 0$, $i = 1, \ldots, n$, such that the Lyapunov function (4) satisfies

$$\dot{V} = \sum_{i=1}^{n} d_i \dot{V}_i \leq -\epsilon |y|^2$$

for some $\epsilon > 0$.

Corollary 1 still holds when some of the blocks are static nonlinearities satisfying the sector condition

$$0 \leq -y_i^2 + \gamma_i u_i y_i, \quad \gamma_i > 0,$$

(7)
rather than the dynamic property (5). To see this we let $\mathcal{I}$ denote the subset of indices $i$ which correspond to dynamic blocks $H_i$ satisfying (5), and employ the Lyapunov function

$$V = \sum_{i \in \mathcal{I}} d_i V_i.$$  

(8)

For the static blocks, that is $H_i, i \notin \mathcal{I}$, we note from (7) that the sum

$$\sum_{i = 1}^{n} \sum_{i \notin \mathcal{I}} d_i (-y_i^2 + \gamma_i u_i y_i) d_i > 0$$

(9)

is nonnegative and, hence,

$$\dot{V} \leq \sum_{i \in \mathcal{I}} d_i \dot{V}_i + \sum_{i = 1}^{n} d_i (-y_i^2 + \gamma_i u_i y_i) \leq \sum_{i = 1}^{n} d_i (-y_i^2 + \gamma_i u_i y_i) = -y^T D A y$$

(10)

as in (6). Then, as in Corollary 1 condition (2) insures existence of a $D > 0$ such that $\dot{V} \leq -\epsilon |y|^2$ for some $\epsilon > 0$.

**A Popov Criterion**

A special case of interest is the feedback interconnection in Figure 2, where $H_i, i = 1, \ldots, n$, are dynamic blocks as in (5), and the feedback nonlinearity $\psi(t, \cdot)$ satisfies the sector property:

$$0 \leq y_n \psi(t, y_n) \leq \kappa y_n^2,$$

(11)
rewritten here as
\begin{equation}
0 \leq -\psi(t, y_n)^2 + \kappa \psi(t, y_n)y_n.
\end{equation}
If we treat the feedback nonlinearity as a new block \( y_{n+1} = \psi(t, y_n) \), and note from (12) that it satisfies (7) with \( \gamma_{n+1} = \kappa \), we obtain from Corollary 1 and the ensuing discussion the stability condition:
\begin{equation}
\kappa \gamma_1 \cdots \gamma_n < \sec(\pi/(n+1))^{(n+1)}.
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{The feedback interconnection for Corollary 2.}
\end{figure}

This condition, however, may be conservative because it does not exploit the static nature of the feedback nonlinearity. Indeed, using the Popov Criterion, the authors of [6] obtained a relaxed condition in which \( n + 1 \) in the right-hand side of (13) is reduced to \( n \) when \( H_i \)'s are first-order linear blocks of the form \( H_i(s) = \gamma_i/(\gamma_i s + 1) \) and the feedback nonlinearity is time-invariant.

To extend this result to the case where \( H_i \)'s are OFP as in [3], and not necessarily linear, we recall that the main premise of the Popov Criterion is that a time-invariant sector nonlinearity, when cascaded with a first-order, stable, linear block preserves its passivity properties. This means that, by only restricting \( H_n \) to be linear, and combining it with the feedback nonlinearity as in Figure 3, the relaxed sector condition of [6] holds even if \( H_1, \cdots, H_{n-1} \) are nonlinear:

**Corollary 2** Consider the feedback interconnection in Figure 2 where \( H_i, i = 1, \cdots, n - 1, \) satisfy [4] with \( C^1 \) storage functions \( V_i \) and \( \gamma_i > 0 \), \( H_n \) is a
linear block with transfer function

\[ H_n(s) = \frac{\gamma_n}{\tau_n s + 1}, \quad \tau_n > 0, \gamma_n > 0, \]  

(14)

the feedback nonlinearity \( \psi(\cdot) \) is time-invariant and satisfies the sector property (11). Under these assumptions, if

\[ \kappa\gamma_1 \cdots \gamma_n < \sec(\pi/n)^n, \]  

(15)

then there exists a Lyapunov function of the form

\[ V_n = \sum_{i=1}^{n-1} d_i V_i + d_n \int_0^{y_n} \psi(\sigma) d\sigma, \quad d_i > 0, \ i = 1, \cdots, n, \]  

(16)

satisfying

\[ \dot{V} \leq -\epsilon |(y_1, \cdots, y_{n-1}, \psi(y_n))|^2 \]

for some \( \epsilon > 0 \).

\[ \square \]

**Proof:** Rather than treat \( H_n \) and \( \psi(\cdot) \) as separate blocks, we combine them as in Figure 3:

\[ \tilde{H}_n : \begin{cases} \tau_n \tilde{y}_n &= -y_n + \gamma_n y_{n-1} \\ \tilde{y}_n &= \psi(y_n), \end{cases} \]  

(17)

and define

\[ V_n = \kappa \tau_n \int_0^{y_n} \psi(\sigma) d\sigma \]  

(18)

which, from (17), satisfies

\[ \dot{V}_n = -\kappa y_n \psi(y_n) + \kappa \gamma_n \psi(y_n) y_{n-1}. \]  

(19)

Because \( -\kappa y_n \psi(y_n) \leq -\psi(y_n)^2 \) from (12), we conclude

\[ \dot{V}_n \leq -\psi(y_n)^2 + \kappa \gamma_n \psi(y_n) y_{n-1} = -\tilde{y}_n^2 + \kappa \gamma_n \tilde{y}_n y_{n-1}, \]  

(20)

which shows that \( \tilde{H}_n \) is OFP as in [3], with \( \tilde{\gamma}_n = \gamma_n \kappa \). The result then follows from Corollary [1].

(21)

Corollary 2 can be further generalized to the situation where other nonlinearities exist in between the blocks \( H_i, \ i = 1, \cdots, n, \) in Figure 2. If such a nonlinearity is in the sector \([0, \kappa_{i+1}]\), and is preceded by a linear block \( H_i(s) = \frac{\gamma_i}{\tau_i s + 1} \), then the two can be treated as a single block with \( \tilde{\gamma}_i = \kappa_{i+1} \gamma_i \), thus reducing \( n \) in the right-hand side of (2).
The Shortage of Passivity in a Cascade of OFP Systems

When the blocks $H_1, \ldots, H_n$ each satisfy the OFP property (5), their cascade interconnection in Figure 4 inherits the sum of their phases and loses passivity. The following corollary to Theorem 1 quantifies the “shortage” of passivity in such a cascade:

**Corollary 3** Consider the cascade interconnection in Figure 4. If each block $H_i$ satisfies (5) with a storage function $V_i$ and $\gamma_i > 0$, then for any

$$\delta > \gamma_1 \cdots \gamma_n \cos\left(\frac{\pi}{(n+1)}\right)^{(n+1)},$$
(21)

the cascade admits a storage function of the form (4) satisfying

$$\dot{V} \leq -\epsilon |y|^2 + \delta u^2 + uy_n,$$
(22)

for some $\epsilon > 0.$

Inequality (22) is an input feedforward passivity (IFP) property where the number $\delta$ represents the gain with which a feedforward path, if added from $u$ to $y_n$ in Figure 4, would achieve passivity. Corollary 3 thus shows that the cascade of OFP systems (5) in which $\gamma_i > 0$ represents an “excess”
of passivity, satisfies the IFP property (22) with a “shortage” characterized by (21).

**Proof of Corollary 3.** Using (4), (5), and substituting \( u_i = y_{i-1}, i = 2, \ldots, n \), we rewrite (22) as

\[
d_1(-y_1^2 + \gamma_1 y_1 u) + \sum_{i=2}^{n} d_i(-y_i^2 + \gamma_i y_i y_{i-1}) + \delta(-u^2 - \frac{1}{\delta}u y_n) \leq -\epsilon|y|^2. \tag{23}
\]

To show that \( d_i > 0, i = 1, \ldots, n \), satisfying (23) indeed exist, we define

\[
\bar{A} = \begin{bmatrix}
-1 & 0 & \cdots & 0 & -\frac{1}{\delta} \\
\gamma_1 & -1 & \ddots & 0 \\
0 & \gamma_2 & -1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \gamma_n & -1
\end{bmatrix}
\]

and note that the left-hand side of (23) is

\[
[u \ y^T] \tilde{D} \bar{A} \begin{bmatrix} u \\ y \end{bmatrix}
\]

where \( \tilde{D} := \text{diag} \{ \delta, d_1, \ldots, d_n \} \). Because \( \bar{A} \) is of the form (11) with dimension \((n + 1)\), an application of Theorem 1 shows that a diagonal \( \tilde{D} \) rendering (25) negative definite exists if and only if \((\gamma_1 \cdots \gamma_{n/2}) < \sec(\pi/(n + 1))^{(n+1)}\). Because this condition is satisfied when \( \delta \) is as in (21), we conclude that such a \( \tilde{D} > 0 \) exists and, thus, (22) holds. \( \square \)
APPENDIX: Proof of Theorem \[1\]

Necessity follows because, as shown in \[6\], if \( (2) \) fails then \( A \) is not Hurwitz. To prove that \( (2) \) is sufficient for diagonal stability, we define

\[
  r := (\gamma_1 \cdots \gamma_n)^{1/n} > 0
\]

and note that

\[
  \Delta := \text{diag} \left\{ 1, \frac{-\gamma_2}{r}, \frac{-\gamma_3}{r^2}, \cdots, \frac{(-1)^{i+1}\gamma_i}{r^{i-1}}, \cdots, \frac{(1)^{n+1}\gamma_2 \cdots \gamma_n}{r^{n-1}} \right\}
\]

and note that

\[
  - \Delta^{-1} A \Delta = \begin{bmatrix}
  1 & 0 & \cdots & 0 & (-1)^{n+1} r \\
  r & 1 & \ddots & 0 \\
  0 & r & 1 & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & r & 1 
\end{bmatrix}
\]

Thus, with the choice

\[
  D = \Delta^{-2}
\]

we get

\[
  DA + A^T D = \Delta^{-1} (\Delta^{-1} A \Delta + \Delta A^T \Delta^{-1}) \Delta^{-1}
\]

which means that \( DA + A^T D < 0 \) holds if the symmetric part of \( (27) \), given by

\[
  \frac{1}{2} ( - \Delta^{-1} A \Delta - \Delta A^T \Delta^{-1} )
\]

is positive definite. To show that this is indeed the case, we note that \( (27) \) exhibits a \textit{circulant} structure \([1]\) when \( n \) is odd, and a \textit{skew-circulant} structure when \( n \) is even. In particular, it admits the eigenvalue-eigenvector pairs

\[
  \lambda_k = 1 + re^{i \frac{2\pi}{n} k} \quad v_k = \frac{1}{n} \left[ 1, e^{-i \frac{2\pi}{n} k}, e^{-2i \frac{2\pi}{n} k}, \ldots, e^{-i(n-1) \frac{2\pi}{n} k} \right]^T \quad k = 1, \ldots, n
\]

when \( n \) is odd; and

\[
  \lambda_k = 1 + re^{i \left( \frac{\pi}{n} + \frac{2\pi}{n} k \right)} \quad v_k = \frac{1}{n} \left[ 1, e^{-i \left( \frac{\pi}{n} + \frac{2\pi}{n} k \right)}, e^{-2i \left( \frac{\pi}{n} + \frac{2\pi}{n} k \right)}, \ldots, e^{-i(n-1) \left( \frac{\pi}{n} + \frac{2\pi}{n} k \right)} \right]^T
\]

when \( n \) is even. Since, in either case, \( (27) \) is diagonalizable with the unitary matrix \( V = [v_1 \cdots v_n] \), the eigenvalues of the symmetric part \( (30) \) coincide with the real parts of \( \lambda_k \)'s above. Finally, because

\[
  \min_{k=1,\ldots,n} \text{Re} \left\{ 1 + re^{i \frac{2\pi}{n} k} \right\} = \min_{k=1,\ldots,n} \text{Re} \left\{ 1 + re^{i \left( \frac{\pi}{n} + \frac{2\pi}{n} k \right)} \right\} = 1 - r \cos \left( \frac{\pi}{n} \right),
\]
we conclude that if (2) holds, that is $r < \sec(\pi/n)$, then all eigenvalues of (30) are positive and, hence, (30) is positive definite and (29) is negative definite. □

References

[1] P.J. Davis. *Circulant Matrices*. John Wiley & Sons, 1979.

[2] E. Kaszkurewicz and A. Bhaya. *Matrix Diagonal Stability in Systems and Computation*. Birkhäuser, Boston, 2000.

[3] R. Redheffer. Volterra multipliers - Parts I and II. *SIAM Journal on Algebraic and Discrete Methods*, 6(4):592–623, 1985.

[4] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonlinear Control*. Springer-Verlag, New York, 1997.

[5] E.D. Sontag. A generalization of the secant condition to passive systems. Submitted to the IEEE Conference on Decision and Control, 2005.

[6] J.J. Tyson and H.G. Othmer. The dynamics of feedback control circuits in biochemical pathways. In R. Rosen and F.M. Snell, editors, *Progress in Theoretical Biology*, volume 5, pages 1–62. Academic Press, New York, 1978.

[7] A. J. van der Schaft. *L2-gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, New York and Berlin, second edition, 2000.