Multiple solutions for a NLS equation with critical growth and magnetic field.

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Abstract

In this paper, we are concerned with the multiplicity of nontrivial solutions for the following class of complex problems

\[ (-i \nabla - A(\mu x))^2 u = \mu |u|^{q-2} u + |u|^{2^* - 2} u \text{ in } \Omega, \quad u \in H_0^1(\Omega, \mathbb{C}), \]

where \( \Omega \subset \mathbb{R}^N (N \geq 4) \) is a bounded domain with smooth boundary. Using the Lusternik-Schnirelman theory, we relate the number of solutions with the topology of \( \Omega \).

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1 Introduction

In this paper, we are concerned with the multiplicity of nontrivial solutions for the following class of complex problems

\[
\begin{cases}
(-i\nabla - A(\mu x))^2 u = \mu |u|^{q-2}u + |u|^{2^*-2}u \text{ in } \Omega \\
u \in H^1_0(\Omega, \mathbb{C}),
\end{cases}
\]

where \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N, N \geq 4 \), \( \mu \) is a positive parameter, \( 2 \leq q < 2^* = \frac{2N}{N-2} \) and \( A : \mathbb{R}^N \to \mathbb{R}^N \) is a magnetic field belonging to \( \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^N) \).

This class of problem is related with the existence of solitary waves, namely solutions of the form \( \psi(x, t) := e^{-iEht}u(x) \), with \( E \in \mathbb{R} \), for the nonlinear Schrödinger equation

\[
ih \frac{\partial \psi}{\partial t} = \left( \frac{h}{i} \nabla - A(z) \right)^2 \psi + U(z)\psi - f(|\psi|^2)\psi, \quad z \in \Omega,
\]

where \( t > 0, N \geq 2, h \) is the Planck constant and \( A \) is a magnetic potential associated to a given magnetic \( B \), \( U(x) \) is a real electric potential and the nonlinear term \( f \) is a superlinear function. A direct computation shows that \( \psi \) is a solitary wave for \( (NLS) \) if, and only if, \( u \) is a solution of the following problem

\[
\left( \frac{h}{i} \nabla - A(z) \right)^2 u + V(z)u = f(|u|^2)u, \quad \text{in } \Omega, \quad (1.1)
\]

where \( V(z) = U(z) - E \). It is important to investigate the existence and the shape of such solutions in the semiclassical limit, namely, as \( h \to 0^+ \). The importance of this study relies on the fact that the transition from Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck constant to zero.

There is a vast literature concerning the existence and multiplicity of bound state solutions for \( (1.1) \) with no magnetic field, namely \( A \equiv 0 \) and \( h = 1 \), which becomes an elliptic equation like

\[
\begin{cases}
-\Delta u = \mu |u|^{q-2}u + |u|^{2^*-2}u \text{ in } \Omega \\
u = 0 \quad \text{on } \partial\Omega.
\end{cases}
\]
Problem $(P)$ has received considerable attention in last years, after the seminal paper due to Brezis and Nirenberg [12], who investigated $(P)$ in the case $q = 2$. Motivated by that article, many authors have also considered a lot of problems involving critical growth in bounded and unbounded domains, see, for example, Struwe [26], Garcia Azorero and Peral Alonso [5, 6], Bahri and Coron [7], Rey [25], Benci and Cerami [8, 9, 10, 11], Coron [19], Alves and Ding [2], Alves [1] and references therein. This class of problem aroused the interest of all due to the lack of compactness in the inclusion of

$$H^1_0(Ω) \hookrightarrow L^2^*(Ω),$$

hence, the associated energy functionals do not satisfy in general the Palais-Smale condition.

Multiplicity of solutions to $(P)$ involving the geometry of $Ω$, precisely, the Lusternik-Schnirelman category $cat_Ω(Ω)$, was proved in [25] for $N \geq 5$ and in [21] for $N = 4$, cf. [28]. Other results of multiplicity involving subcritical growth and category $cat_Ω(Ω)$ can be found in [8, 9, 13]. Here, $cat_X(Y)$ denotes the Lusternik-Schnirelman category of $Y$ in $X$, namely the least number of closed and contractible sets in the topological space $X$ which cover the closed set $Y ⊂ X$.

If we now consider the magnetic case $A ≠ 0$, the first result was obtained by Esteban and Lions [20]. They have used the concentration-compactness principle and minimization arguments to obtain solution for $h > 0$ fixed and dimensions $N = 2$ or $N = 3$. More recently, Kurata [23] proved that the problem has a least energy solution for any $h > 0$ when a technical condition relating $V$ and $A$ is assumed. Under this technical condition, he proved that the associated functional satisfies the Palais-Smale compactness condition at any level. We also would like to cite the papers [17, 18, 14, 27, 15, 3, 4] for other results related to the problem (1.1) in the presence of magnetic field.

In view of the results of Rey [25] and Lazzo [21], it is natural to ask if the same kind of result holds for the problem with magnetic field. The main goal of this paper is to present a positive answer to this question. So, we relate the number of solution for $(P_μ)$ with topology of the set $Ω$ when the parameter $μ$ is small. We prove that, for small values of $μ$, the magnetic field does not play any role on the numbers of solutions of the equation $(P_μ)$ and therefore a result in the same spirit of [25] and [21] holds.

Our main result is:
Theorem 1.1 Let $2 \leq q < 2^*$. Then, there exists $\mu^* > 0$ such that, for each $\mu \in (0, \mu^*)$, problem $(P_\mu)$ has at least $\text{cat}_\Omega(\Omega)$ nontrivial solutions.

In the proof of Theorem 1.1, we apply variational methods and Ljusternik-Schnirelmann theory. We follow some arguments developed in [25], [21] and [2], where the non-magnetic case is handled. It is worthwhile to mention that, since we deal with different problems, where the function are complex, it is necessary to make a careful analysis in some estimates used in that papers.

The paper is organized as follows. In the next section we present the variational setting of the problem. In Section 3 we prove some preliminary results, and in the Section 4, we prove our main theorem.

2 Variational framework and notations

We shall denote by $H^1_0(\Omega, \mathbb{C})$ the Hilbert space obtained by the closure of $C_0^\infty(\Omega, \mathbb{C})$ under the scalar product

$$\langle u, v \rangle_{A_\mu} := \text{Re} \left( \int_\Omega \nabla_{A_\mu} u \overline{\nabla_{A_\mu} v} \, dx \right),$$

where $A_\mu(x) = A(\mu x) = (A_1(\mu x), A_2(\mu x), ..., A_N(\mu x))$, $\text{Re}(w)$ denotes the real part of $w \in \mathbb{C}$, $\overline{w}$ is its complex conjugated, $\nabla_{A_\mu} u := (D_1 u, D_2 u, ..., D_N u)$ and $D_j := -i \partial_j - A_j(\mu x)$, for $j = 1, \ldots, N$. The norm induced by this inner product is given by

$$\|u\|_{A_\mu} := \left( \int_\Omega |\nabla_{A_\mu} u|^2 \, dx \right)^{1/2}.$$

As proved by Esteban and Lions in [20], for any $u \in H^1_0(\Omega, \mathbb{C})$, there holds the diamagnetic inequality, namely

$$|\nabla |u||^2 = \left| \text{Re} \left( \nabla u \overline{u} \right) \right| = \left| \text{Re} \left( (\nabla u - iA_\mu u) \overline{u} \right) \right| \leq |\nabla_{A_\mu} u(x)|. \quad (2.1)$$

Thus, if $u \in H^1_0(\Omega, \mathbb{C})$, we have that $|u|$ belongs to the usual Sobolev space $H^1_0(\Omega, \mathbb{R})$. Moreover, the embedding $H^1_0(\Omega, \mathbb{C}) \hookrightarrow L^q(\Omega, \mathbb{C})$ is continuous for each $1 \leq q \leq 2^*$ and it is compact for $1 \leq q < 2^*$. 

From now on, we say that a function \( u \in H^1_0(\Omega, \mathbb{C}) \) is a weak solution of \((P_\mu)\) if
\[
\text{Re} \left( \int_\Omega \nabla A_\mu u \overline{\nabla A_\mu v} \, dx - \mu \int_\Omega |u|^{q-2} u \overline{v} \, dx - \int_\Omega |u|^{2^*-2} u \overline{v} \, dx \right) = 0,
\]
for all \( v \in H^1_0(\Omega, \mathbb{C}) \).

In this paper, the main tool used to prove Theorem 1.1 is the variational method, where the solutions to \((P_\mu)\) are obtained by looking for critical points of the functional
\[
I_\mu(u) = \frac{1}{2} \int_\Omega |\nabla A_\mu u|^2 \, dx - \frac{\mu}{q} \int_\Omega |u|^q \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx.
\]

A direct computation shows that \( I_\mu \in C^1(\dot{H}^1_0(\Omega, \mathbb{C})) \) with
\[
I'_\mu(u) v = \text{Re} \left( \int_\Omega \nabla A_\mu u \overline{\nabla A_\mu v} \, dx - \int_\Omega \mu |u|^{q-2} u \overline{v} \, dx - \int_\Omega |u|^{2^*-2} u \overline{v} \, dx \right).
\]
Thus the weak solutions of \((P_\mu)\) are precisely the critical points of \( I_\mu \).

Hereafter, we denote by \( \lambda_1 > 0 \) the best constant of the compact embedding
\[
H^1_0(\Omega, \mathbb{C}) \hookrightarrow L^2(\Omega, \mathbb{C})
\]
which is given by
\[
\lambda_1 = \inf_{u \in H^1_0(\Omega, \mathbb{C}) \setminus \{0\}} \frac{\int_\Omega |\nabla A_\mu u|^2 \, dx}{(\int_\Omega |u|^2 \, dx)^{\frac{1}{2}}}.
\]

Moreover, we denote by \( S \) the best Sobolev constant of the embedding
\[
H^1_0(\Omega, \mathbb{R}) \hookrightarrow L^{2^*}(\Omega, \mathbb{R})
\]
which is given by
\[
S = \inf_{u \in H^1_0(\Omega, \mathbb{R}) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |u|^{2^*} \, dx)^{2/2^*}}.
\]
It is well known that \( S \) is independent of \( \Omega \) and it is never achieved, except when \( \Omega = \mathbb{R}^N \). Moreover,
\[
S := \frac{\int_{\mathbb{R}^N} |\nabla U|^2 \, dx}{\left( \int_{\mathbb{R}^N} |U|^{2^*} \, dx \right)^{2/2^*}}.
\]
where \( U(x) = \frac{C_N}{(|x|^2 + 1)^{(N-2)/2}} \) and \( C_N \) is a constant such that
\[
-\Delta U = U^{2^* - 1} \text{ in } \mathbb{R}^N.
\]
A direct computation implies that for all \( \epsilon > 0 \) and \( y \in \mathbb{R}^N \) the function
\[
U_{\epsilon, y}(x) = \epsilon^{2-N} U\left(\frac{x - y}{\epsilon}\right)
\]
verifies the equality below
\[
\int_{\mathbb{R}^N} |\nabla U_{\epsilon, y}|^2 \, dx = \int_{\mathbb{R}^N} |U_{\epsilon, y}|^{2^*} \, dx = S^{N/2}.
\]

**Lemma 2.1** If
\[
S_{A_{\mu}} = \inf_{u \in H_0^1(\Omega, \mathbb{C})} \frac{\int_{\Omega} |\nabla A_{\mu} u|^2 \, dx}{\left( \int_{\Omega} |u|^{2^*} \, dx \right)^{2/2^*}},
\]
we have that \( S = S_{A_{\mu}} \).

**Proof.** First of all, we observe that by diamagnetic inequality,
\[
S \leq S_{A_{\mu}}.
\]
Now, we will prove that \( S \geq S_{A_{\mu}} \). To this end, we fix \( x_0 \in \Omega \). Thus, there exists \( r > 0 \) such that \( B_r(x_0) \subset \Omega \). Let \( \phi \) be a nonnegative smooth cutoff function, such that
\[
\phi(x) = 1 \quad \text{if } |x| < r, \phi(x) = 0 \quad \text{if } |x| > 2r,
\]
\[
u_{\epsilon} = \frac{u_{\epsilon}}{|u_{\epsilon}|^{2^*}}.
\]
From [12],
\[
\|v_{\epsilon}\|^2 = S + O(\epsilon^{\frac{N-2}{2}})
\]
and
\[ |v_\epsilon|^q \to 0 \text{ as } \epsilon \to 0 \quad \forall q \in [2, 2^*), \]
from where it follows that
\[ |v_\epsilon|^2 \to 0 \text{ as } \epsilon \to 0. \]
From definition of \( S_{A\mu} \), we derive that
\[ S_{A\mu} \leq \frac{\int_\Omega |\nabla A_\mu(e^{i\tau_{x_0}(x)}v)|^2dx}{\left(\int_\Omega |e^{i\tau_{x_0}(x)}v|^2dx\right)^{2^*}} \]
where \( \tau_{x_0}(x) := \sum_{j=1}^N A_j(\mu x_0)x^j \). This way,
\[ S_{A\mu} \leq \frac{\int_\Omega |\nabla v|^2dx + \int_\Omega (A(\mu x_0) - A(\mu x))|v|^2dx}{\left(\int_\Omega |v|^2dx\right)^{2^*}}, \]
or equivalently,
\[ S_{A\mu} \leq \|v\|^2 + \int_\Omega (A(\mu x_0) - A(\mu x))|v|^2dx. \]
Letting \( \epsilon \to 0 \) and using the fact that \( A \in L^\infty(\mathbb{R}^N) \), the above limits leads to
\[ S_{A\mu} \leq S; \]
finishing the proof.

3 Preliminary results

Next, we will show some lemmas related to the functional \( I_\mu \). Our first lemma is related to the fact that \( I_\mu \) verifies the mountain pass geometry. However, we omit its proof because it follows by using well known arguments.

Lemma 3.1 The functional \( I_\mu \) satisfies the following conditions:

(i) There exist \( \alpha, \rho > 0 \) such that:
\[ I_\mu(u) \geq \alpha \text{ with } \|u\|_{A\mu} = \rho, \]
for all \( \mu > 0 \) if \( 2 < q < 2^* \) and for all \( \mu \in (0, \lambda_1) \) if \( q = 2 \).

(ii) There exists \( e \in B^c_\rho(0) \subset H^1_0(\Omega, \mathbb{C}) \) such that \( I_\mu(e) < 0 \).
Applying the Mountain Pass Theorem without \((PS)\) condition (see Willem [28]), there exists a \((PS)\) sequence \((u_n) \subset H^1_0(\Omega, \mathbb{C})\), that is, a sequence satisfying

\[
I_\mu(u_n) \to b_\mu \quad \text{and} \quad I'_\mu(u_n) \to 0,
\]

where

\[
b_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t))
\]

and

\[
\Gamma = \{ \gamma \in C([0,1], H^1_0(\Omega, \mathbb{C})) : \gamma(0) = 0 \quad \text{and} \quad I_\mu(\gamma(1)) < 0 \}.
\]

By standard arguments, \((u_n)\) is bounded, and so, there exist a subsequence of \((u_n)\), still denoted by \((u_n)\), and \(u \in H^1_0(\Omega, \mathbb{C})\) such that

\[
u_n \rightharpoonup u \quad \text{in} \quad H^1_0(\Omega, \mathbb{C}) \quad \text{and} \quad u_n(x) \to u(x) \quad \text{a.e in} \quad \Omega.
\]

As in [24, Proposition 3.11], it is possible to prove that \(b_\mu\) verifies the following equalities

\[
b_\mu = \tilde{b}_\mu = \hat{b}_\mu,
\]

with

\[
\tilde{b}_\mu = \inf \left\{ \max_{t \geq 0} I_\mu(tu) : u \in H^1_0(\Omega, \mathbb{C}) \setminus \{0\} \right\}
\]

and

\[
\hat{b}_\mu = \inf \left\{ I_\mu(u) : u \in \mathcal{N}_\mu \right\}
\]

where \(\mathcal{N}_\mu\) denotes the Nehari manifold associated with \(I_\mu\) given by

\[
\mathcal{N}_\mu = \left\{ u \in H^1_0(\Omega, \mathbb{C}) \setminus \{0\} : I'_\mu(u)u = 0 \right\}.
\]

Next, we will prove that \(I_\mu\) satisfies the local Palais Smale condition.

**Lemma 3.2** Let \((u_n) \subset H^1_0(\Omega, \mathbb{C})\) be a sequence that \(I_\mu(u_n) \to c < \frac{1}{N} S^{N/2}\) and \(\|I'_\mu(u_n)\| = o_n(1)\). Then \(I_\mu\) satisfies the \((PS)_c\) condition for all \(\mu > 0\) if \(q > 2\) and for all \(\mu \in (0, \lambda_1)\) if \(q = 2\).
Proof. Let \((u_n) \subset H^1_0(\Omega, \mathbb{C})\) be a sequence satisfying

\[ I_{\mu}(u_n) \to c \quad \text{and} \quad I'_{\mu}(u_n) \to 0. \]

From a direct calculus, we have that \((u_n)\) is bounded in \(H^1_0(\Omega, \mathbb{C})\). Hence, by diamagnetic inequality, \(|u_n|\) is bounded in \(H^1_0(\Omega, \mathbb{R})\). Then, for some subsequence, there is \(u \in H^1_0(\Omega, \mathbb{C})\) such that \(u_n \rightharpoonup u\) in \(H^1_0(\Omega, \mathbb{C})\). We claim that

\[ \int_{\Omega} |u_n|^2^* \, dx \to \int_{\Omega} |u|^{2^*} \, dx. \tag{3.1} \]

In order to prove this claim, we suppose that

\[ |\nabla|u_n|^2| \to |\nabla|u|^2|+\sigma \quad \text{and} \quad |u_n|^{2^*} \to |u|^{2^*}+\nu \quad (\text{weak}^*\text{-sense of measures}). \]

Using the concentration compactness-principle due to Lions (cf. [22, Lemma 1.2]), we obtain a countable index set \(\Lambda\), sequences \((x_i) \subset \Omega\), \((\sigma_i)\), \((\nu_i) \subset (0, \infty)\), such that

\[ \nu = \sum_{i \in \Lambda} \nu_i \delta_{x_i}, \quad \sigma \geq \sum_{i \in \Lambda} \sigma_i \delta_{x_i} \quad \text{and} \quad S_{\nu_i}^{2/2^*} \leq \sigma_i, \tag{3.2} \]

for all \(i \in \Lambda\), where \(\delta_{x_i}\) is the Dirac mass at \(x_i \in \Omega\).

Now, for every \(\varrho > 0\), we set \(\psi_\varrho(x) := \psi((x-x_i)/\varrho)\) where \(\psi \in C^\infty_0(\mathbb{R}^N, [0, 1])\) is such that \(\psi \equiv 1\) on \(B_1(0)\), \(\psi \equiv 0\) on \(\mathbb{R}^N \setminus B_2(0)\) and \(|\nabla\psi|_\infty \leq 2\). Since \((\psi_\varrho u_n)\) is bounded in \(H^1_0(\Omega, \mathbb{C})\) and \(\psi_\varrho\) takes values in \(\mathbb{R}\), a direct calculation shows that

\[ I'_{\mu}(u_n)(\psi_\varrho u_n) \to 0 \]

and

\[ \nabla A_{\mu}(u_n \psi_\varrho) = iu_n \nabla \psi_\varrho + \psi_\varrho \nabla A_{\mu}u_n. \]

Therefore,

\[ \int_{\Omega} \psi_\varrho |\nabla A_{\mu}u_n|^2 \, dx + \operatorname{Re} \left( \int_{\Omega} iu_n \nabla A_{\mu}u_n \psi_\varrho \right) = \mu \int_{\Omega} |u_n|^q \psi_\varrho \, dx + \int_{\Omega} \psi_\varrho |u_n|^{2^*} \, dx + o_n(1). \]

It is not difficult to prove that

\[ \lim_{\varrho \to 0} \lim_{n \to \infty} \operatorname{Re} \left( \int_{\Omega} iu_n \nabla A_{\mu}u_n \psi_\varrho \, dx \right) = 0. \]
This way, by diamagnetic inequality
\[ \int_{\Omega} \psi_\theta |\nabla |u_n|^2 \, dx \leq \mu \int_{\Omega} |u_n|^q\psi_\theta \, dx + \int_{\Omega} \psi_\theta |u_n|^{2^*} \, dx + o_n(1). \]

Consequently, using the fact that \( u_n \to u \) in \( L^m(\Omega, \mathbb{R}) \) for all \( 1 \leq m < 2^* \) and \( \psi_\theta \) has compact support, we can let \( n \to \infty \) in the last inequality to obtain
\[ \int_{\Omega} \psi_\theta d\sigma \leq \int_{\Omega} \psi_\theta d\nu. \]

Letting \( \theta \to 0 \), it follows that \( \nu_i \geq \sigma_i \). Then, from (3.2)
\[ \nu_i \geq \frac{1}{N} S^{N/2}. \]

Next, we will prove that the inequality found in (3.3) cannot occur, and therefore the set \( \Lambda \) is empty. Indeed, arguing by contradiction, let us suppose that \( \nu_i \geq \frac{1}{N} S^{N/2} \) for some \( i \in \Lambda \). Once that
\[ c = I_\mu(u_n) - \frac{1}{2} I'_\mu(u_n)u_n + o_n(1), \]
it follows that
\[ c \geq \frac{1}{N} \int_{\Omega} |u_n|^{2^*} \, dx + o_n(1) \geq \frac{1}{N} \int_{B_\epsilon(x_i)} \psi_\theta |u_n|^{2^*} \, dx + o_n(1). \]

Letting \( n \to \infty \),
\[ c \geq \frac{1}{N} \sum_{i \in \Lambda} \psi_\theta(x_i)\nu_i = \frac{1}{N} \sum_{i \in \Lambda} \nu_i \geq \frac{1}{N} S^{N/2}, \]
which does not make sense. Hence, \( \Lambda \) is empty and the limit below holds
\[ \int_{\Omega} |u_n|^{2^*} \, dx \to \int_{\Omega} |u|^{2^*} \, dx. \]
The last limit implies that
\[ 0 \leq \|u_n - u\|^2_{A_\mu} = I'_\mu(u_n)u_n - I'_\mu(u_n)u + o_n(1) = o_n(1), \]
showing that \( u_n \to u \) in \( H^1_0(\Omega, \mathbb{C}) \). \[ \square \]

The next lemma is a key point in our arguments.
Lemma 3.3  The level $b_{\mu}$ verifies the inequality

$$0 < b_{\mu} < \frac{1}{N}S^{N/2},$$

for all $\mu > 0$ if $q > 2$ and for all $\mu \in (0, \lambda_1)$ if $q = 2$.

Proof. In the sequel, we fix $x_0 \in \Omega$ and $w_\epsilon(x) = \tau_{x_0}(x)v_\epsilon(x)$ for all $x \in \Omega$, where $\tau_{x_0}$ and $v_\epsilon$ were given in the proof of Lemma 2.1. Setting $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = I_\mu(w_\epsilon),$$

we have that

$$g(t) = \frac{t^2}{2} \int_\Omega |\nabla v_\epsilon|^2 \, dx + \frac{t^2}{2} \int_\Omega (A(\mu x_0) - A(\mu x))v_\epsilon^2 \, dx - \frac{\mu t^q}{q} \int_\Omega |v_\epsilon|^q \, dx - \frac{t^{2^*}}{2^{*}}.$$

Thus, there is $t_\epsilon > 0$ such that

$$g(t_\epsilon) = \max_{t \geq 0} g(t).$$

A direct computation shows that $(t_\epsilon)$ is bounded for $\epsilon$ small enough. Fixing

$$h(t) = \frac{t^2}{2} \int_\Omega |\nabla v_\epsilon|^2 \, dx - \frac{\mu t^q}{q} \int_\Omega |v_\epsilon|^q \, dx - \frac{t^{2^*}}{2^{*}},$$

and repeating the same arguments explored in [12], we obtain

$$\max_{t \geq 0} h(t) < \frac{1}{N}S^{N/2} \quad \text{for} \quad \epsilon \approx 0. \quad \text{(3.4)}$$

On the other hand, once that $A$ is a continuous function, $(t_\epsilon)$ is bounded, and $v_\epsilon \to 0$ in $L^2(\Omega)$, we have that

$$\frac{t^2}{2} \int_\Omega (A(\mu x_0) - A(\mu x))|v_\epsilon|^2 \, dx \to 0 \quad \text{as} \quad \epsilon \to 0. \quad \text{(3.5)}$$

Combining (3.4) and (3.5),

$$g(t_\epsilon) = \max_{t \geq 0} g(t) < \frac{1}{N}S^{N/2}$$
for \( \epsilon \) small enough. Now, from the definition of \( b_\mu \),
\[
b_\mu \leq g(t_\epsilon) \quad \forall \epsilon > 0,
\]
from where it follows that
\[
0 < b_\mu < \frac{1}{N} S^{N/2}.
\]

An immediate consequence of Lemmas 3.2 and 3.3 is the following result.

**Theorem 3.1**  On the hypotheses of Lemma 3.3, the mountain pass level \( b_\mu \) is a critical value of \( I_\mu \), that is, there is \( u_\mu \in H^1_0(\Omega, \mathbb{C}) \) such that
\[
I_\mu(u_\mu) = b_\mu \quad \text{and} \quad I'(u_\mu) = 0.
\]

From now on, we denote by \( c_0, c_\mu \) and \( M_0, M_\mu \) the mountain pass levels and the Nehari manifolds associated with the functionals
\[
J_0(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx \quad \forall u \in H^1_0(\Omega, \mathbb{R})
\]
and
\[
J_\mu(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\mu}{q} \int_\Omega |u|^q \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx \quad \forall u \in H^1_0(\Omega, \mathbb{R}),
\]
respectively.

**Lemma 3.4**  The minimax level \( c_0 \) is equal to \( \frac{1}{N} S^{N/2} \).

**Proof.** See proof in [2].

**Lemma 3.5**  If \( \lim_{n \to +\infty} \mu_n = 0 \), then \( \lim_{n \to +\infty} b_{\mu_n} = c_0 = \frac{1}{N} S^{N/2} \).

**Proof.** By Theorem 3.1 there is \( (u_n) \subset H^1_0(\Omega, \mathbb{C}) \) such that
\[
I_{\mu_n}(u_n) = b_{\mu_n} \quad \text{and} \quad I'_{\mu_n}(u_n) = 0.
\]
Choosing \( t_n > 0 \) such that \( t_n|u_n| \in M_0 \), we derive from diamagnetic inequality that
\[
c_0 \leq J_0(t_n|u_n|) \leq I_{\mu_n}(t_n u_n) + \frac{\mu_n t_n^q}{q} |u_n|^q,
\]
and so,
\[ c_0 \leq b_{\mu_n} + \frac{\mu_n t_n}{q} |u_n|^q. \]  
(3.6)

From Lemma 3.3, we have that \( b_{\mu_n} \leq \frac{1}{N} S^{N/2} \) for \( n \) large enough. Consequently, a direct computation implies that \((u_n)\) and \((t_n)\) are bounded sequences. This way, (3.6) leads to
\[ c_0 \leq \liminf_{n \to \infty} b_{\mu_n}. \]  
(3.7)

Now, from Lemmas 3.3 and 3.4, for \( n \) sufficiently large
\[ b_{\mu_n} < \frac{1}{N} S^{N/2} = c_0, \]  
(3.8)
leading to
\[ \limsup_{n \to \infty} b_{\mu_n} \leq c_0. \]  
(3.9)

From this, the lemma follows combining (3.7) with (3.9).

4 Technical lemmas

In this section, we recall some lemmas which are crucial in the proof of the main theorem. The next two lemmas are due to Lions [22] and can be found in Willem [28, Lemma 1.40].

**Lemma 4.1** Let \((u_n) \subset D^{1,2}(\mathbb{R}^N)\) a sequence such that
\[ u_n \rightharpoonup v \text{ in } D^{1,2}(\mathbb{R}^N), \]
\[ |\nabla(u_n - u)|^2 \rightharpoonup \sigma \text{ in } \mathcal{M}(\mathbb{R}^N), \]  
(4.1)
and
\[ |u_n - u|^{2^*} \rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N), \]  
(4.2)

and
\[ u_n \rightharpoonup u \text{ in } \mathbb{R}^N. \]

Then,
\[ \|\nu\|_{\frac{2}{2^*}} \leq S^{-1}\|\sigma\|, \]  
(4.3)
\[ \limsup_{n \to +\infty} |\nabla u_n|^2 = |\nabla u|^2 + \|\sigma\| + \sigma_{\infty}, \]  
(4.4)
\[ \limsup_{n \to +\infty} |u_n|_{2^*} = |u_n|_{2^*} + \|\nu\| + \nu_\infty, \quad (4.5) \]
and
\[ \nu_{2^*} \leq S^{-1}\sigma_\infty, \quad (4.6) \]
with
\[ \sigma_\infty = \lim_{R \to \infty} \limsup_{n \to +\infty} \int_{|x| \geq R} |\nabla u_n|^2 \, dx \]
and
\[ \nu_\infty = \lim_{R \to \infty} \limsup_{n \to +\infty} \int_{|x| \geq R} |u_n|^{2^*} \, dx. \]

Moreover, if \( u = 0 \) and \( \|\nu\|_{2^*} = S^{-1}\|\sigma\| \), the measures \( \nu \) and \( \sigma \) are concentrated at a single point.

**Lemma 4.2** Let \((u_n) \subset H^1_0(\Omega, \mathbb{R})\) be a sequence with \( |u_n|_{2^*} = 1 \) and \( \|u_n\|^2 = S + o_n(1) \). Then there exists a sequence \((y_n, \lambda_n) \subset \mathbb{R}^N \times \mathbb{R}\) such that
\[ v_n(x) := \lambda_n^{\frac{N-2}{2}} u_n(\lambda_n x + y_n) \]
contains a convergent subsequence, still denoted by itself, such that \( v_n \to v \in D^{1,2}(\mathbb{R}^N) \), \( \lambda_n \to 0 \) and \( y_n \to y \in \overline{\Omega} \).

An immediate consequence of the last lemma is the following corollary

**Corollary 4.1** Let \((u_n) \subset H^1_0(\Omega, \mathbb{R})\) a sequence with
\[ u_n \in \mathcal{M}_0 \quad \text{and} \quad J_0(u_n) \to c_0. \]

Then there exists a sequence \((y_n, \lambda_n) \subset \mathbb{R}^N \times \mathbb{R}\) such that \( v_n(x) := \lambda_n^{\frac{N-2}{2}} u_n(\lambda_n x + y_n) \) contains a convergent subsequence, still denoted by itself, such that \( v_n \to v \in D^{1,2}(\mathbb{R}^N) \), \( \lambda_n \to 0 \) and \( y_n \to y \in \overline{\Omega} \).

Since \( \Omega \) is a smooth bounded domain, we choose \( r > 0 \) small enough so that
\[ \Omega^+_r = \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) < r \} \]
and
\[ \Omega^-_r = \{ x \in \mathbb{R}^N : \text{dist}(x, \partial \Omega) > r \} \]
are homotopically equivalent to \( \Omega \).
From now on, we consider the functional $J_{\mu,B_r} : H^1_{\text{rad}}(B_r(0),\mathbb{R}) \to \mathbb{R}$ given by

$$J_{\mu,B_r}(u) = \frac{1}{2} \int_{B_r(0)} |\nabla u|^2 \, dx - \frac{\mu}{q} \int_{B_r(0)} |u|^q \, dx - \frac{1}{2} \int_{B_r(0)} |u|^{2^*} \, dx$$

where $H^1_{\text{rad}}(B_r(0),\mathbb{R}) = \{ u \in H^1_0(B_r(0),\mathbb{R}) : u \text{ is radial} \}$.

Moreover, we denote by $m(\mu)$ the mountain pass level associated with $J_{\mu,B_r}$, which can be characterized by

$$m(\mu) := \inf \{ J_{\mu,B_r}(u) : u \in \mathcal{M}_{\mu,B_r} \}$$

where

$$\mathcal{M}_{\mu,B_r} = \left\{ u \in H^1_{\text{rad}}(B_r(0),\mathbb{R}) \setminus \{ 0 \} : J'_{\mu,B_r}(u)u = 0 \right\}.$$

It is not difficult to check that Lemmas 3.2 and 3.3 also hold for $J_{\mu,B_r}$. This way, Theorem 3.1 is also true for $J_{\mu,B_r}$, from where it follows that there is a radial function $v_\mu \in \mathcal{M}_{\mu,B_r}$ satisfying

$$J_{\mu,B_r}(v_\mu) = m(\mu) \quad \text{and} \quad J'_{\mu,B_r}(v_\mu) = 0.$$

**Lemma 4.3** The level $m(\mu)$ converges to $c_0 = \frac{1}{N}S^{N/2}$ as $\mu \to 0$, that is,

$$\lim_{\mu \to 0} m(\mu) = c_0 = \frac{1}{N}S^{N/2}.$$

**Proof.** See proof in [2].

In what follows, we fix the map $\Psi : \Omega_r^- \to \mathcal{N}_\mu$ given by

$$\Psi_\mu(y)(x) = \begin{cases} t_{\mu,y}e^{i\tau_y(x)}v_\mu(|x-y|) & \text{if } x \in B_r(0) \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau_y(x) := \sum_{j=1}^N A_j(\mu y)x^j$ and $t_{\mu,y} \in (0, +\infty)$ is such that

$$t_{\mu,y}e^{i\tau_y(\cdot-y)}v_\mu(|\cdot-y|) \in \mathcal{N}_\mu.$$

Moreover, following the notation used in [2], we denote by $\beta : \mathcal{N}_\mu \to \mathbb{R}^N$ the barycenter function given by

$$\beta(u) := \frac{1}{\int_{\Omega} |u|^{2^*} \, dx} \int_{\Omega} x|u|^{2^*} \, dx.$$
Since $v_\mu$ is radial, for each $y \in \Omega^-$,
\[
(\beta \circ \Psi)(y) = \frac{1}{\int_\Omega v_\mu(|x-y|)^{2^*}} \int_\Omega x v_\mu(|x-y|)^{2^*} \, dx = y.
\]

**Lemma 4.1** Uniformly for $y \in \Omega^-$, there holds
\[
\lim_{\mu \to 0} I_\mu(\Psi_\mu(y)) = c_0.
\]

**Proof.** Given two sequences $\mu_n \to 0$ and $(y_n) \subset \Omega^-$, we shall prove that
\[
I_\mu(\Psi_\mu(y_n)) \to c_0 \quad \text{as} \quad n \to +\infty.
\]

Let $t_n := t_{\lambda_n,y_n}$ and $v_n = v_{\mu_n}$ be as in the definition of $\Psi_\mu$. Using the diamagnetic inequality, we have
\[
m(\mu_n) \leq I_\mu(\Psi_\mu(y_n))
\]
on the other hand,
\[
I_\mu(\Psi_\mu(y_n)) \leq m(\mu_n) + \frac{t_n^2}{2} \int_{B_r(y)} |A(\mu_n y_n) - A(\mu_n x)|^2 |v_n|^2 \, dx
\]
from where it follows that
\[
I_\mu(\Psi_\mu(y_n)) \leq m(\mu_n) + C_1 \left( \int_{\mathbb{R}^N} |A(\mu_n y_n) - A(\mu_n x)|^{\frac{N}{N-2}} |v_n|^{2^*} \, dx \right)^{\frac{N-2}{N}}.
\]
A direct computation implies that $(t_n)$ is bounded, hence
\[
I_\mu(\Psi_\mu(y_n)) \leq m(\mu_n) + C_1 \left( \int_{\mathbb{R}^N} |A(\mu_n y_n) - A(\mu_n x)|^{\frac{N}{N-2}} |v_n|^{2^*} \, dx \right)^{\frac{N-2}{N}}.
\]

From Corollary 4.1 there exist $(\lambda_n) \subset \mathbb{R}$ and $(z_n) \subset \mathbb{R}^N$ with $\lambda_n \to 0$ and $z_n \to z \in \overline{\Omega}$, such that the sequence $h_n(x) := \frac{\lambda_n}{\lambda_n^{N-2}} v_n(\lambda_n x + z_n)$ contains a convergent subsequence, still denoted by itself, that is, $h_n \to h$ in $D^{1,2}(\mathbb{R}^N, \mathbb{R})$. 


for some $h \in D^{1,2}(\mathbb{R}^N, \mathbb{R})$. Using the above notations,

$$I_{\mu_n}(\Psi_{\mu_n}(y_n)) \leq m(\mu_n) + C_1 \left( \int_{\mathbb{R}^N} |A(\mu_n y_n) - A(\mu_n \lambda_n x + \mu_n z_n)|^{\frac{N}{N-2}} |h_n|^{2^*} dx \right)^{\frac{N-2}{N}}$$

Once that $A$ is continuous and belongs to $L^\infty(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} |A(\mu_n y_n) - A(\mu_n \lambda_n x + \mu_n z_n)|^{\frac{N}{N-2}} |h_n|^{2^*} dx \to 0. \quad (4.9)$$

Combining (4.7), (4.8) and (4.9) with the limit $m(\mu_n) \to c_0$, we derive that

$$I_{\mu_n}(\Psi_{\mu_n}(y_n)) \to c_0,$$

finishing the proof.

Given $y \in \Omega_\lambda^-$, we have that $\Psi_\mu(y) \in \mathcal{M}_\mu$. Moreover, setting

$$g(\mu) := |I_\mu(\Psi_\mu(y)) - c_0|,$$

we have that $g(\mu) \to 0$ as $\mu \to 0$ and $I_\mu(\Psi_\mu(y)) - c_0 \leq g(\mu)$. Hence, the set

$$\mathcal{O}_\mu := \{u \in \mathcal{M}_\mu : I_\mu(u) \leq c_0 + g(\mu)\}$$

contains the function $\Psi_\mu(y)$, showing that $\mathcal{O}_\mu \neq \emptyset$.

**Lemma 4.4** There exists $\mu^* > 0$ such that, if $\mu \in (0, \mu^*)$ and $u \in \mathcal{O}_\mu$, then $\beta(u) \in \Omega^+_r$.

**Proof.** Suppose by contradiction that there exist $\mu_n \to 0$, $u_n \in \mathcal{N}_{\mu_n}$ and $I_{\mu_n}(u_n) \leq c_0 + g(\mu_n)$ such that $\beta(u_n)$ does not belong to $\Omega^+_r$.

From diamagnetic inequality, there is $t_n \in [0, 1]$ such that $v_n := t_n |u_n| \in \mathcal{M}_0$. Hence,

$$c_0 \leq J_0(t_n |u_n|) \leq I_{\mu_n}(t_n u_n) + \frac{\mu_n t_n}{q} \int_{\Omega} |u_n|^q dx \leq I_{\mu_n}(u_n) + o_n(1) \leq c_0 + o_n(1),$$

and so,

$$v_n \in \mathcal{M}_0, \quad \beta(v_n) = \beta(u_n) \notin \Omega^+_r \quad \text{and} \quad J_0(v_n) \to c_0.$$
Using Corollary 4.1, there exist \((\lambda_n) \subset \mathbb{R}\) and \((y_n) \subset \mathbb{R}^N\) with \(\lambda_n \to 0\) and \(y_n \to y \in \overline{\Omega}\), such that the sequence \(h_n(x) := \lambda_n^{-2} v_n(\lambda_n x + y_n)\) contains a convergent subsequence, still denoted by itself, that is,

\[
h_n \to h \quad \text{in} \quad D^{1,2}(\mathbb{R}^N, \mathbb{R})
\]

for some \(h \in D^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}\). Fixing \(\phi \in C^\infty_0(\mathbb{R}^N, \mathbb{R})\) with \(\phi(x) = x\) for all \(x \in \overline{\Omega}\), a simple computation gives

\[
\beta(v_n) = \frac{\int_{\mathbb{R}^N} \phi(x) |v_n(x)|^{2^*} \, dx}{\int_{\mathbb{R}^N} |v_n(x)|^{2^*} \, dx},
\]

or equivalently

\[
\beta(v_n) = \frac{\int_{\mathbb{R}^N} \phi(\lambda_n x + y_n) |h_n(x)|^{2^*} \, dx}{\int_{\mathbb{R}^N} |h_n(x)|^{2^*} \, dx}. \tag{4.11}
\]

Letting \(n \to +\infty\), we get

\[
\beta(v_n) \to y \in \overline{\Omega},
\]

which is a contradiction.

\[\Box\]

5 Proof of Theorem 1.1

By a direct computation, there exists \(C > 0\) such that

\[
\|u\|_{A_\mu} \geq C \quad \forall u \in \mathcal{N}_\mu. \tag{5.1}
\]

Since we are intending to consider the functional \(I_\mu\) constrained to \(\mathcal{N}_\mu\), we will need of the following result.

Lemma 5.1 The functional \(I_\mu\) constrained to \(\mathcal{N}_\mu\) satisfies the \((PS)_c\) condition with \(c < \frac{1}{N} S^{N/2}\) for \(\mu > 0\) if \(q > 2\) and \(\mu \in (0, \mu^*)\) for \(q = 2\).

Proof. Let \((u_n)\) be a \((PS)\)-sequence for \(I_\mu\) constrained to \(\mathcal{N}_\mu\). Then \(I_\mu(u_n) \to c\) and

\[
I'_\mu(u_n) = \theta_n G'_\mu(u_n) + o_n(1), \tag{5.2}
\]
for some $\theta_n \subset \mathbb{R}$, where $G_\mu : H^1_0(\Omega, \mathbb{C}) \to \mathbb{R}$ is given by

$$G_\mu(v) := \int_\Omega |\nabla A_\mu v|^2 \, dx - \mu \int_\Omega |v|^q \, dx - \int_\Omega |v|^2^* \, dx.$$ 

We recall that $G'_\mu(u_n)u_n \leq 0$. Moreover, standard arguments show that $(u_n)$ is bounded. Thus, up to a subsequence, $G'_\mu(u_n)u_n \to l \leq 0$. If $l \neq 0$, we infer from (5.2) that $\theta_n = o_n(1)$. In this case, we can use (5.2) again to conclude that $(u_n)$ is a (PS)$_c$ sequence for $I_\mu$ in $H^1_0(\Omega, \mathbb{C})$ and therefore $(u_n)$ has a strongly convergent subsequence. If $l = 0$, it follows that

$$\int_\Omega |u_n|^2^* \, dx \to 0.$$ 

Consequently, $\|u_n\|_{A_\mu} \to 0$, obtaining this way a contradiction with (5.1), finishing the proof of the lemma.

As a consequence of the above arguments, we obtain the following result.

**Corollary 5.1** If $u$ is a critical point of $I_\mu$ constrained to $N_\mu$, then $u$ is a nontrivial critical point of $I_\mu$ on $H^1_0(\Omega, \mathbb{C})$.

**Lemma 5.2** If $\mu^*$ is given by Lemma 4.4 then for each $\mu \in (0, \mu^*)$, there holds

$$\text{cat}_{\mathcal{O}_\mu}(\mathcal{O}_\mu) \geq \text{cat}_\Omega(\Omega).$$

**Proof.** Suppose that

$$\mathcal{O}_\mu = \Upsilon_1 \cup \ldots \cup \Upsilon_n,$$

where $\Upsilon_j$, $j = 1, \ldots, n$, is closed and contractible in $\mathcal{O}_\mu$. This means that there exists $h_j \in C([0,1] \times \Upsilon_j, \mathcal{O}_\mu)$ such that

$$h_j(0, u) = u, \quad h_j(1, u) = u_j, \quad \text{for each } u \in \Upsilon_j,$$

and some $u_j \in \Upsilon_j$ fixed. Consider the sets $B_j := \gamma^{-1}(\Upsilon_j)$, $j = 1, \ldots, n$, which are closed in $\Omega_r^-$ and satisfy

$$\Omega_r^- = B_1 \cup \cdots \cup B_n.$$

We define the deformation $g_j : [0,1] \times B_j \to \Omega_r^+$ given by

$$g_j(t, y) = \beta(h_j(t, \gamma(y))).$$
are well defined. A standard calculation show that these maps are contractions of the sets $B_j$ in $\Omega^+$. Hence that
\[ \text{cat}_\Omega(\Omega) = \text{cat}_{\Omega^+}(\Omega^-) \leq n, \]
and the proposition is proved.

We are now ready to prove our main result.

**Proof of Theorem 1.1** Arguing as in the proof of Proposition 5.1, we can check that $I_\mu$ satisfies the $(PS)_c$ condition on $\mathcal{N}_\mu$ for $c \in (0, \frac{1}{N}S^{N/2})$. Thus, we can apply standard Lusternik-Schnirelman theory and Lemma 5.2 to obtain $\text{cat}_{\mathcal{O}_\mu}(\mathcal{O}_\mu) \geq \text{cat}_\Omega(\Omega)$ critical points of $I_\mu$ restricted to $\mathcal{N}_\mu$. As in Corollary 5.1, each one of these critical points is a critical point of the unconstrained functional $I_\mu$, and therefore, a nonzero weak solution of the problem $(P_\mu)$.

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