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Abstract: This paper presents an analytical approach to investigate the nonlinear dynamic response and vibration of thick functionally graded material (FGM) plates using both of the first-order shear deformation plate theory and stress function with full motion equations (not using Volmir's assumptions). The FGM plate is assumed to rest on elastic foundation and subjected to mechanical, thermal, and damping loads. Numerical results for dynamic response of the FGM plate are obtained by Runge–Kutta method. The results show the material properties, the elastic foundations, mechanical and thermal loads on the nonlinear dynamic response of functionally graded plates.

Keywords: nonlinear vibration; nonlinear dynamic response; thick functionally graded plate; the first-order shear deformation plate theory; stress function

1. Introduction

Functionally graded materials (FGMs) are composite and microscopically in homogeneous with mechanical and thermal properties varying smoothly and continuously from one surface to the other. Typically, these materials are made from a mixture of metal and ceramic or a combination of

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PUBLIC INTEREST STATEMENT

In recent years, there has been significant interest in the development of functionally graded materials (FGMs) for engineering applications. FGM materials have been used in aerospace, nuclear, and microelectronics engineering applications, where the materials are required to work in extreme temperature environments. It is also important for these materials to maintain their structural integrity, with minimum failures due to material mismatch. The focus of this manuscript is on a theoretical analysis on the nonlinear dynamic analysis and vibration of thick FGM plates using both of the first order shear deformation theory and stress function. The FGM plate is assumed to rest on elastic foundations and subjected to mechanical, thermal, and damping loads. The influences of the elastic foundations, mechanical loads, and temperature on the nonlinear dynamic response and nonlinear vibration of thick FGM plates are examined in detail. The outcomes from this work are important to composite engineers and designers.
different metals by gradually varying the volume fraction of the constituent materials. The properties of FGM plates and shells are assumed to vary through the thickness of the structures. Due to the high heat resistance, FGMs have many practical applications, such as reactor vessels, aircrafts, space vehicles, defense industries, and other engineering structures. As a result, in recent years, many investigations have been carried out on the dynamic and vibration of FGM plates.

In 2004, Vel and Batra (2004) investigated the three-dimensional exact solution for the vibration of FGM rectangular plates. Ferreira, Batra, and Roque (2006) received natural frequencies of FGM plates by meshless method. Woo, Meguid, and Ong (2006) investigated the nonlinear free vibration behavior of functionally graded plates. Also in this year (2006), Wu, Shukla, and Huang (2006) published their results on nonlinear static and dynamic analysis of functionally graded plates. Natural frequencies and buckling stresses of FGM plates were analyzed by Matsunaga (2008) using 2-D higher order deformation theory. Allahverdizadeh, Naei, and Nikkhah Bahrami (2008) studied nonlinear free and forced vibration analysis of thin circular functionally graded plates. Shen (2009) published a valuable book Functionally Graded Materials, Nonlinear Analysis of Plates and Shells, in which the results about nonlinear vibration of shear deformable FGM plates are presented. Fakhari and Ohadi (2010) studied nonlinear vibration control of functionally graded plate with piezoelectric layers in thermal environment using finite element method. In their study, the material properties of FGM have been also assumed to be temperature-dependent and graded in the thickness direction according to a simple power law distribution in terms of the volume fractions of the constituents. Talha and Singh (2010) studied static response and free vibration of unsymmetrical FGM plates using first-order shear deformation plate theory with finite element method.

It can be seen that nonlinear dynamic analysis of FGM plates is currently taken much attention of many researchers. A number of recent publications focused on the dynamic of FGM plates using the stress function and Volmir’s hypothesis but they all used classical plate theory (Duc, 2013; Duc & Cong, 2013a) for thin structures. When the higher order shear deformation plate theory is applied for nonlinear dynamic analysis of thick plates, Volmir’s hypothesis is useless. Hence, to solve the dynamic problem for the plate when using the first-order shear deformation plate theory, the other authors often use finite element method with displacement functions (Hosseini-Hashemi, Rokni Damavandi Taher, Akhavan, & Omidi, 2010; Zhao, Lee, & Liew, 2009). There has not been any publication using analytical approach to study dynamic for thick FGM plates. This paper presents an analytical approach to investigate the nonlinear dynamic response and nonlinear vibration of thick FGM plates using both of the first-order shear deformation plate theory and stress function. Numerical results for dynamic response of the FGM plate are obtained by fourth-order Runge–Kutta method.

2. Governing equations
Consider a rectangular FGM plate on elastic foundations. The plate is referred to a Cartesian coordinate system $x, y, z$, where $xy$ is the midplane of the plate and $z$ is the thickness coordinator, $-h/2 \leq z \leq h/2$. The length, width, and total thickness of the plate are $a, b,$ and $h$, respectively (Figure 1).

The elastic modulus $E$, thermal expansion coefficient $\alpha$, and density $\rho$ of the FGM plate can be written as follows (Tung & Duc, 2010):

$$[E(z), \alpha(z), \rho(z)] = [E_m, \alpha_m, \rho_m] + [E_c, \alpha_c, \rho_c] \left(\frac{2z + h}{2h}\right)^N$$  \hspace{2cm} (1)

in which subscripts $m$ and $c$ stand for the metal and ceramic constituents, respectively, and

$$E_{cm} = E_c - E_m, \quad \alpha_{cm} = \alpha_c - \alpha_m, \quad \rho_{cm} = \rho_c - \rho_m$$  \hspace{2cm} (2)

the Poisson ratio $\nu(z)$ is assumed to be constant $\nu(z) = \nu = \text{const.}$
2.1. Nonlinear dynamic FGM plates

Suppose that the FGM plate is subjected to a transverse load and compressive axial loads. In the present study, the first-order shear deformation plate theory is used to obtain the motion, compatibility equations (Reddy, 2004). At the same time, the stress function is applied for determining the nonlinear dynamic response and vibration of the FGM plate.

The forces and moments of the plate can be written as follows (Duc & Tung, 2010):

\[
(N_x, M_x) = \frac{1}{1 - \nu^2} \left[ (E_1, E_2) \left( \epsilon_x^0 + \nu \epsilon_y^0 \right) + (E_2, E_3) \left( \chi_x + \nu \chi_y \right) - (1 + \nu) (\Phi_x, \Phi_y) \right]
\]

\[
(N_y, M_y) = \frac{1}{1 - \nu^2} \left[ (E_1, E_2) \left( \epsilon_y^0 + \nu \epsilon_x^0 \right) + (E_2, E_3) \left( \chi_y + \nu \chi_x \right) - (1 + \nu) (\Phi_x, \Phi_y) \right]
\]

\[
(N_{xy}, M_{xy}) = \frac{1}{2(1 + \nu)} \left[ (E_1, E_2) \chi_{xy} + (E_2, E_3) \chi_{xy} \right]
\]

\[
(Q_x, Q_y) = \frac{KE_1}{2(1 + \nu)} (\chi_{sx}, \chi_{sy})
\]

in which the explicit analytical expressions of \( E_i, i = 1, 2, 3 \) \( \Phi_x, \Phi_y \) are calculated and given in the reference Duc and Tung (2010).

According to first-order shear deformation plate theory, the equations of motion are (Reddy, 2004):

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2}, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2}
\]

(4a)

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - K_{1w} + K_{2w} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + q = I_0 \frac{\partial^2 w}{\partial t^2}
\]

\[+ 2\epsilon I_0 \frac{\partial \omega}{\partial t} \]

(4b)

\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2}
\]

(4c)

where

\[
I_i = \int_{-h/2}^{h/2} \rho(z)z'dz, \quad (i = 0, 1, 2)
\]

\[
I_0 = \rho_m h + \frac{\rho_{cm} h}{N + 1}; \quad I_1 = \rho_{cm} h^2 \left( \frac{1}{N + 2} - \frac{1}{2(N + 1)} \right)
\]
\[ I_2 = \frac{\rho_m h^3}{12} + \rho_{cm} h^3 \left[ \frac{1}{N+3} - \frac{1}{N+2} + \frac{1}{4(N+1)} \right] \]  

(5)

In which \( q \) is an external pressure uniformly distributed on the surface of the plate, \( \varepsilon \) is the damping coefficient.

The stress function \( f(x, y, t) \) is introduced as:

\[ N_x = \frac{\partial^2 f}{\partial y^2}, \quad N_y = \frac{\partial^2 f}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 f}{\partial x \partial y} \]  

(6)

Substituting Equation 6 into 4a and then into Equations 4b, 4c, Equations 4b, 4c can be rewritten as follows:

\[
\begin{align*}
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - K_1 w + K_2 \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + q \\
= I_0 \frac{\partial^2 w}{\partial t^2} + 2\varepsilon I_0 \frac{\partial w}{\partial t}
\end{align*}
\]

(7)

The compatibility equation of the plate can be written as follows (Duc & Tung, 2010):

\[
\frac{1}{E_1} \left( \frac{\partial^4 f}{\partial x^4} + 2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} \right) = \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}
\]

(8)

By substituting Equation 3 into Equation 7, the system of motion Equation 7 is rewritten as follows:

\[
\begin{align*}
L_{11}(w) + L_{12}(\phi_x) + L_{13}(\phi_y) + P(w, f) + q & = I_0 \frac{\partial^2 w}{\partial t^2} + 2\varepsilon I_0 \frac{\partial w}{\partial t} \\
L_{21}(w) + L_{22}(\phi_x) + L_{23}(\phi_y) & = \left( I_2 - \frac{I_1}{I_0} \right) \frac{\partial^2 \phi_x}{\partial t^2} \\
L_{31}(w) + L_{32}(\phi_x) + L_{33}(\phi_y) & = \left( I_3 - \frac{I_1}{I_0} \right) \frac{\partial^2 \phi_y}{\partial t^2}
\end{align*}
\]

(9)

Where the linear operators \( L_j (i = 1-3, j = 1-3) \) and the nonlinear operator \( P \) are defined below

\[
\begin{align*}
L_{11}(w) & = \frac{KE_1}{2(1+\nu)} \frac{\partial^2 w}{\partial x^2} + \frac{KE_1}{2(1+\nu)} \frac{\partial^2 w}{\partial y^2} - K_1 w + K_2 \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\
L_{12}(\phi_x) & = \frac{KE_1}{2(1+\nu)} \frac{\partial \phi_x}{\partial x} \\
L_{13}(\phi_y) & = \frac{KE_1}{2(1+\nu)} \frac{\partial \phi_y}{\partial y}
\end{align*}
\]
\[ P(w, f) = \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \]

\[ L_{21}(w) = -\frac{K_E}{2(1+\nu)} \frac{\partial w}{\partial x} \]

\[ L_{22}(\phi_x) = \frac{1}{1-\nu^2} \left( E_3 - \frac{E_2}{E_1} \right) \frac{\partial^2 \phi_x}{\partial x^2} + \frac{1}{2(1+\nu)} \left( E_3 - \frac{E_2}{E_1} \right) \frac{\partial^2 \phi_y}{\partial y^2} - \frac{K_E}{2(1+\nu)} \phi_x \]

\[ L_{23}(\phi_y) = \frac{1}{2(1-\nu)} \left( E_3 - \frac{E_2}{E_1} \right) \frac{\partial^2 \phi_y}{\partial x \partial y} \]

\[ L_{31}(w) = -\frac{K_E}{2(1+\nu)} \frac{\partial w}{\partial y} \]

\[ L_{32}(\phi_x) = \frac{1}{2(1-\nu)} \left( E_3 - \frac{E_2}{E_1} \right) \frac{\partial^2 \phi_x}{\partial x \partial y} \]

\[ L_{33}(\phi_y) = \frac{1}{2(1+\nu)} \left( E_3 - \frac{E_2}{E_1} \right) \frac{\partial^2 \phi_y}{\partial x^2} + \frac{1}{1-\nu^2} \left( E_3 - \frac{E_2}{E_1} \right) \frac{\partial^2 \phi_y}{\partial y^2} - \frac{K_E}{2(1+\nu)} \phi_y \quad (10) \]

The system of four Equations 8–9 combined with boundary conditions and initial conditions could be used for nonlinear dynamical analysis of thick FGM plates in the next section.

2.2. Nonlinear dynamical analysis

Depending on the in-plane behavior at the edges, there are two boundary conditions - unable to move and able to move labeled Case 1 and Case 2, respectively, will be considered (Duc & Tung, 2010).

The approximate solutions of the system of Equations 8 and 9 satisfying the boundary conditions can be written as:

\[ w(x, y, t) = W(t) \sin \alpha x \sin \beta y \]

\[ \phi_x(x, y, t) = \Phi_x(t) \cos \alpha x \sin \beta y \quad (11a) \]

\[ \phi_y(x, y, t) = \Phi_y(t) \sin \alpha x \cos \beta y \]

\[ f(x, y, t) = A_1(t) \cos 2\alpha x + A_2(t) \cos 2\beta y + \frac{1}{2} N_{x_1} y^2 + \frac{1}{2} N_{y_1} x^2 \quad (11b) \]

where \( \alpha = \frac{m\pi}{L_x}, \beta = \frac{n\pi}{L_y}, m, n = 1, 2, \ldots \) are the natural numbers of half waves in the corresponding direction \( x, y \), and \( W(t), \Phi_x, \Phi_y \)—the amplitudes which are functions dependent on time. The coefficients \( A_i (i = 1 - 2) \) are determined by the substitution of Equations 11a and 11b into Equation 8 as

\[ A_1 = \frac{E_1 \beta^2}{32\alpha^2} W^2, \quad A_2 = \frac{E_4 \alpha^2}{32\beta^2} W^2 \quad (11c) \]

Replacing Equation 11 into the equations of motion 9 and then applying Galerkin method we obtain

\[ \left( 1_{11} + n_2 N_{x_1} + n_4 N_{y_1} \right) W + l_{12} \Phi_x + l_{13} \Phi_y + n_1 W^3 + n_3 q = I_o \frac{\partial^2 W}{\partial t^2} + 2\epsilon I_o \frac{\partial W}{\partial t} \]
\begin{align*}
l_{21}W + l_{22}\Phi_x + l_{23}\Phi_y &= \rho_1 \frac{\partial^2 \Phi_x}{\partial t^2} \\
l_{31}W + l_{32}\Phi_x + l_{33}\Phi_y &= \rho_1 \frac{\partial^2 \Phi_y}{\partial t^2}
\end{align*}

in which
\begin{align*}
l_{11} &= -\left[ \frac{KE_1}{2(1 + \nu)} \left( \frac{m\pi}{a} \right)^2 + \frac{KE_1}{2(1 + \nu)} \left( \frac{n\pi}{b} \right)^2 + K_1 + K_2 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right) \right] \\
l_{12} &= -\frac{KE_1}{2(1 + \nu)} \frac{m\pi}{a} \quad ; \quad l_{13} = -\frac{KE_1}{2(1 + \nu)} \frac{n\pi}{b} \\
n_1 &= \frac{E_1}{16} \left( \left( \frac{m\pi}{a} \right)^4 + \left( \frac{n\pi}{b} \right)^4 \right) \quad ; \quad n_2 = -\left( \frac{m\pi}{a} \right)^2 \\
n_3 &= \frac{16}{mn\pi^2} \quad ; \quad n_4 = -\left( \frac{n\pi}{b} \right)^2 \\
l_{21} &= \frac{KE_1}{2(1 + \nu)} \frac{m\pi}{a} \\
l_{22} &= \frac{1}{1 - \nu^2} \left( E_3 - \frac{E_2^2}{E_1} \right) \left( \frac{m\pi}{a} \right)^2 + \frac{1}{2(1 + \nu)} \left( E_3 - \frac{E_2^2}{E_1} \right) \left( \frac{n\pi}{b} \right)^2 + \frac{KE_1}{2(1 + \nu)} \\
l_{23} &= \frac{1}{2(1 - \nu)} \left( E_3 - \frac{E_2^2}{E_1} \right) \frac{mn\pi^2}{ab} \\
\rho_1 &= \frac{I_1}{I_0} - I_2 \\
l_{31} &= \frac{KE_1}{2(1 + \nu)} \frac{n\pi}{b} \\
l_{32} &= \frac{1}{2(1 - \nu)} \left( E_3 - \frac{E_2^2}{E_1} \right) \frac{mn\pi^2}{ab} \\
l_{33} &= \frac{1}{2(1 + \nu)} \left( E_3 - \frac{E_2^2}{E_1} \right) \left( \frac{m\pi}{a} \right)^2 + \frac{1}{1 - \nu^2} \left( E_3 - \frac{E_2^2}{E_1} \right) \left( \frac{n\pi}{b} \right)^2 + \frac{KE_1}{2(1 + \nu)}
\end{align*}

### 2.3. Nonlinear dynamic analysis with effect of pre-loaded axial compression

Consider the FGM plate hinges on four edges which are simply supported and freely movable (corresponding to Case 1, all edges FM). Assume that the FGM plate is loaded under uniform compressive forces $P_x$ and $P_y$ (Pascal) on the edges $x = 0, a$, and $y = 0, b$, where

\begin{align*}
N_{x_0} &= -P_x h, & N_{y_0} &= -P_y h
\end{align*}
Substituting Equation 14 into Equation 12 leads to the system of differential equations for studying the nonlinear dynamic response of the plate

\[
\begin{bmatrix}
\frac{1}{11} - \left( n_2 P_x + n_4 P_y \right) h \\
\frac{1}{21} + \frac{1}{31} \\
\frac{1}{22} + \frac{1}{32} \\
\frac{1}{23} + \frac{1}{33} + \rho_1 \omega^2
\end{bmatrix}
\begin{bmatrix}
W + l_{12} \Phi_x + l_{13} \Phi_y + n_1 W^3 + n_3 q = I_0 \frac{\partial^2 W}{\partial t^2} + 2 \epsilon I_0 \frac{\partial W}{\partial t} \\
\frac{\partial^2 \Phi_x}{\partial t^2} \\
\frac{\partial^2 \Phi_x}{\partial t^2} + \rho_1 \frac{\partial^2 \Phi_y}{\partial t^2}
\end{bmatrix} = 0
\]

(15)

Taking linear parts of the set of Equation 15 and putting \( q = 0 \), the natural frequencies of the plate can be determined directly by solving determinant

\[
\begin{bmatrix}
\frac{1}{11} - \left( n_2 P_x + n_4 P_y \right) h + I_0 \omega^2 \\
\frac{1}{21} + \frac{1}{31} \\
\frac{1}{22} + \rho_1 \omega^2 \\
\frac{1}{23} + \frac{1}{33} + \rho_1 \omega^2
\end{bmatrix}
= 0
\]

(16)

Solving Equation 16 yields three angular frequencies, the smallest one is being considered.

### 2.4. Nonlinear dynamic analysis with the effect of temperature dependent

Consider the FGM plate with all edges which are simply supported and immovable (corresponding to Case 2, all edges IM) under thermal and mechanical loads. The condition expressing the immovability on the edges, \( u = 0 \) (on \( x = 0, a \)) and \( v = 0 \) (on \( y = 0, b \)), is satisfied in an average sense as (Duc & Cong, 2013b; Duc & Tung, 2010)

\[
\int_0^b \int_0^a \frac{\partial u}{\partial x} \, dx \, dy = 0, \quad \int_0^b \int_0^a \frac{\partial v}{\partial y} \, dx \, dy = 0
\]

(17)

Equation 17 and Equation 12 lead to the basic equations used to investigate the nonlinear dynamic response of the plates in the case all IM edges

\[
\begin{bmatrix}
\frac{1}{11} - \left( n_2 P_x + n_4 P_y \right) \frac{\Delta T}{1 - \nu} \\
\frac{1}{21} + \frac{1}{31} \\
\frac{1}{22} + \frac{1}{32} \\
\frac{1}{23} + \frac{1}{33} + \rho_1 \omega^2
\end{bmatrix}
\begin{bmatrix}
W + l_{12} \Phi_x + l_{13} \Phi_y + l_{14} \Phi_x W + l_{15} \Phi_y W + l_{16} W^3 + n_3 q \\
\frac{\partial^2 \Phi_x}{\partial t^2} \\
\frac{\partial^2 \Phi_x}{\partial t^2} + \frac{\partial^2 \Phi_y}{\partial t^2}
\end{bmatrix} = I_0 \frac{\partial^2 W}{\partial t^2} + 2 \epsilon I_0 \frac{\partial W}{\partial t}
\]

(18)

in which

\[
l_{14} = - (n_2 + v n_4) \frac{4E_2}{an \pi \left( 1 - \nu^2 \right)}; \quad l_{15} = - (n_2 \nu + n_4) \frac{4E_2}{bm \pi \left( 1 - \nu^2 \right)}
\]
\[ l_{16} = \left( \frac{na}{b} \right)^2 \frac{(\nu n_2 + n_4) E_1}{8 \left( 1 - \nu^2 \right)} + \left( \frac{m\pi}{a} \right)^2 \frac{(n_2 + \nu n_4) E_1}{8 \left( 1 - \nu^2 \right)} + n_1 \]

\[ p = h \left( E_m \alpha_m + \frac{E_m \alpha_{cm} + E_{cm} \alpha_m}{N + 1} + \frac{E_{cm} \alpha_{cm}}{2N + 1} \right) \]

Taking linear parts of Equation 18 and putting \( q(t) = 0 \), the natural frequencies of the plate under thermal loads can be determined directly by solving determinant

\[ \begin{vmatrix} l_{11} - (n_2 + n_4) \frac{\rho \Delta T}{1 - \nu} + I_0 \omega^2 & l_{12} & l_{13} \\ l_{21} & l_{22} + \rho_1 \omega^2 & l_{23} \\ l_{31} & l_{32} & l_{33} + \rho_1 \omega^2 \end{vmatrix} = 0 \]  \hspace{1cm} (19)

2.5. Nonlinear dynamic analysis using simplified assumption

For further research we consider the hypothetical case when rotations \( \Phi_x, \Phi_y \) exist, but the inertial forces caused by the rotation \( \rho_1 \frac{\partial^2 \Phi}{\partial t^2}, \rho_1 \frac{\partial^2 \Phi}{\partial t} \) are small and can be ignored.

Solving the second and third Equation 18 with respect to \( \Phi_x, \Phi_y \) then substituting the obtained results into first Equation 18, yields. The system of Equation 18 can be rewritten as:

\[ I_0 \frac{d^2 W}{dt^2} + 2\epsilon I_0 \frac{dW}{dt} - m_1 W - m_2 W^2 - l_{16} W^3 = \frac{16}{mn^2} q \]  \hspace{1cm} (20)

where

\[ m_1 = l_{11} - (n_2 + n_4) \frac{\rho \Delta T}{1 - \nu} + l_{12} \frac{l_{21} l_{33} + l_{23} l_{31} - l_{22} l_{33} - l_{32} l_{33}}{l_{32}} + l_{13} \frac{-l_{22} l_{31} + l_{32} l_{23}}{l_{32}} \]

\[ m_2 = l_{14} \frac{-l_{21} l_{33} + l_{23} l_{31} - l_{22} l_{33} - l_{32} l_{33}}{l_{32}} + l_{15} \frac{-l_{22} l_{31} + l_{32} l_{23}}{l_{32}} \]

Equation 20 is used to analyze the nonlinear dynamic response of the FGM plates under uniform temperature rise in the case all the edges of the plate cannot move freely (all edges IM).

In other hand, from Equation 20 the fundamental frequencies of the plate can be determined approximately by explicit expression

\[ \omega_{mn} = \sqrt{\frac{-m_1}{I_0}} \]  \hspace{1cm} (21)

Consider nonlinear vibration of a plate under an uniformly distributed transverse load \( q = Q \sin \Omega t \), Equation 20 has of the form

\[ \frac{\partial^2 W}{\partial t^2} + 2\epsilon \frac{\partial W}{\partial t} + \omega_{mn}^2 \left( W - MW^2 + NW^3 \right) - F \sin \Omega t = 0 \]  \hspace{1cm} (22)

where \( \omega_{mn}^2 = -m_1/I_0 \) is the fundamental frequency of linear vibration of the FGM plate and \( M = -\frac{m_2}{m_1} \), \( N = \frac{l_{16}}{m_1} F = \frac{16Q}{l_{1,mne}} \).
For seeking amplitude–frequency characteristics based on the method of harmonic balance of nonlinear vibration, we chose

\[ W(t) = A \sin \Omega t \]

and substitute it into Equation 22 to give

\[ X = -A\Omega^2 \sin \Omega t + 2\varepsilon A\Omega \cos \Omega t + \omega_{mm}^2 \left( A \sin \Omega t - M (A \sin \Omega t)^2 + N (A \sin \Omega t)^3 \right) - F \sin \Omega t = 0 \]

applying procedure like Gakerkin method \( \int_{-\pi/2}^{\pi/2} X \sin \Omega t \, dt = 0 \), the frequency–amplitude relation of nonlinear forced vibration is obtained

\[ \lambda^2 = \frac{4\varepsilon}{\pi \omega_{mn}^2} \lambda - \left( 1 - \frac{8}{3\pi} MA + \frac{3N}{4} A^2 \right) = -\frac{F}{A\omega_{mn}^2} \tag{23} \]

in which \( \lambda^2 = \Omega^2 / \omega_{mn}^2 \).

For the nonlinear vibration of the shell without damping, this relation leads to

\[ \Omega^2 - \omega_{mn}^2 \left( 1 - \frac{8}{3\pi} MA + \frac{3N}{4} A^2 \right) = -\frac{F}{A} \]

or

\[ \lambda^2 - \left( 1 - \frac{8}{3\pi} MA + \frac{3N}{4} A^2 \right) = -\frac{F}{A\omega_{mn}^2} \tag{24} \]

If \( F = 0 \), i.e. no external force acts on the shell, the frequency–amplitude relation of free nonlinear vibration is obtained

\[ \omega_{nn}^2 = \omega_{mn}^2 \left( 1 - \frac{8}{3\pi} MA + \frac{3N}{4} A^2 \right) \tag{25} \]

3. Numerical results and discussion

Consider a FGM plate acted on by an uniformly distributed transverse load \( q = Q \sin \Omega t \) (\( Q \) is the amplitude of uniformly excited load, \( \Omega \) is the frequency of the load). The fourth-order Runge–Kutta method is used to solve Equations 15 and 18.

In order to illustrate the present approach, we consider a ceramic–metal FGM plate that consists of aluminum (metal) and alumina (ceramic) with the following properties (Duc & Cong, 2013b)

\[
\begin{align*}
E_c &= 380 \times 10^9 \text{N/m}^2, \quad \rho_c = 3800 \text{kg/m}^3 \\
E_m &= 70 \times 10^9 \text{N/m}^2, \quad \rho_m = 2702 \text{kg/m}^3 \\
\alpha_m &= 23 \times 10^{-6} \text{C/}^\circ \text{C}, \quad \alpha_c = 7.4 \times 10^{-6} \text{C/}^\circ \text{C} \\
\nu &= 0.3
\end{align*}
\]

Table 1 presents a comparison of the fundamental frequency parameter established in this paper with the results of Hosseini-Hashemi et al. (2010) and Zhao et al. (2009). In these two papers, the authors applied the first order shear deformation theory, in addition Hosseini-Hashemi et al. (2010) used the displacement functions, Zhao et al. (2009) used the numerical solution element-free kp-Ritz method. From Table 1, as can be seen that the present values are not significantly different from the results of Hosseini-Hashemi et al. (2010) and by Zhao et al. (2009).
Effect of elastic foundations on the natural frequency of the FGM plate is shown in Table 2. The value of the natural frequency increases when the values $K_1$ and $K_2$ increase. Furthermore, the Pasternak elastic foundation influences on the natural frequency larger than the Winkler foundation. Table 2 also shows that the lowest natural frequency corresponds to mode $(m, n) = (1, 1)$.

Table 1. Comparison of fundamental frequency parameter $\gamma = \omega h \sqrt{\frac{E}{h}}$ for Al/Al$_2$O$_3$ square plates without elastic foundations and $a/b = 1$, $(m, n) = (1, 1), P_x = 0, P_y = 0, \Delta T = 0$

| $h/a$ | References | 0   | 0.5 | 1   | 4   | 10  |
|-------|------------|-----|-----|-----|-----|-----|
| 0.05  | Hosseini-Hashemi et al. (2010) | 0.0148 | 0.0128 | 0.0115 | 0.0101 | 0.0096 |
|       | Zhao et al. (2009) | 0.0146 | 0.0124 | 0.0112 | 0.0097 | 0.0093 |
|       | Present study | 0.0148 | 0.0125 | 0.0113 | 0.0101 | 0.0094 |
| 0.1   | Hosseini-Hashemi et al. (2010) | 0.0577 | 0.0492 | 0.0445 | 0.0383 | 0.0363 |
|       | Zhao et al. (2009) | 0.0567 | 0.0482 | 0.0435 | 0.0376 | 0.0359 |
|       | Present study | 0.0577 | 0.0490 | 0.0442 | 0.0383 | 0.0366 |
| 0.2   | Hosseini-Hashemi et al. (2010) | 0.2112 | 0.1806 | 0.1650 | 0.1371 | 0.1304 |
|       | Zhao et al. (2009) | 0.2055 | 0.1757 | 0.1587 | 0.1356 | 0.1284 |
|       | Present study | 0.2112 | 0.1806 | 0.1634 | 0.1403 | 0.1328 |

Table 2. Effect of elastic foundations on natural frequencies (s$^{-1}$) of FGM plates with $a/b = 1$, $a/h = 20$, $N = 1$, $P_x = 0, P_y = 0, \Delta T = 0$

| $(m, n)$ | $K_1$ (GPa/m), $K_2$ (GPa.m) | 0, 0 | 0.3, 0 | 0.3, 0.02 | 0, 0.04 |
|----------|-------------------------------|------|--------|-----------|---------|
| (1, 1)   | (2,261.80)*                   | 2,637.05 | 3,061.55 | 3,154.98 |
| (1, 3)   | 10,975.39                     | 11,057.66 | 11,584.60 | 12,013.00 |
| (1, 5)   | 27,048.98                     | 27,081.76 | 27,363.65 | 28,148.73 |
| (3, 5)   | 34,526.63                     | 34,552.10 | 35,117.12 | 35,648.51 |
| (5, 5)   | 48,575.06                     | 48,592.93 | 49,177.31 | 49,737.35 |

*The obtained results are the same with Zhao et al. (2009).*

Effect of elastic foundations on the natural frequency of the FGM plate is shown in Table 2. The value of the natural frequency increases when the values $K_1$ and $K_2$ increase. Furthermore, the Pasternak elastic foundation influences on the natural frequency larger than the Winkler foundation. Table 2 also shows that the lowest natural frequency corresponds to mode $(m, n) = (1, 1)$.

Figure 2 gives the effect of the power law index $N$ on the nonlinear dynamic response of FGM plates in the case of all FM edges with $a/b = 1$, $a/h = 20$, $P_x = 0, P_y = 0, \Delta T = 0$, $q$ is far away from the natural frequency of the FGM plate with $N = 0, 1, 3$. It can be seen that the amplitude of the nonlinear dynamic response of FGM plate increases when increasing the power law index $N$.

Figures 3 and 4 show the effect of elastic foundations on the nonlinear dynamic response of the FGM plate with $a/b = 1$, $a/h = 20$, $N = 1$, $P_x = 0, P_y = 0$. Figure 3 shows the effect of the Winkler foundation. It is clear that the amplitude of the plate decreases when the module $K_1$ of Winkler foundation increases. The Pasternak foundation with parameter $K_2$ also has a similar behavior. The graphs in Figures 3 and 4 show the beneficial effects of elastic foundations on the nonlinear dynamic response of FGM plates, namely the amplitude of the plate decreases when it is resting on elastic foundations, and the beneficial effect of the Pasternak foundation is better than the Winkler one.

Figure 5 indicates the effect of excited force amplitude on nonlinear dynamic response in the case of $Q = (1,000, 2,000, 3,000$ N/m$^2$) and $P_x = 0, P_y = 0$. The FGM plate's amplitude increases when the excited force amplitude increases. Figure 6 shows the effect of pre-loaded axial compression $P_x$ on the nonlinear dynamic response (with $P_y = 0$). This figure also indicates that the nonlinear dynamic response amplitude of the FGM plate increases when the value of the pre-loaded compressive force increases.
Figure 2. Effect of power law index $N$ on nonlinear dynamical response of the FGM plates (all FM edges).

Figure 3. Effect of the linear Winkler foundation on nonlinear dynamical response of the FGM plate (all FM edges).

Figure 4. Effect of the Pasternak foundation on nonlinear dynamical response of the FGM plate (all FM edges).
Figure 5. Nonlinear dynamic responses of the FGM plate with different loads (all FM edges).

![Nonlinear dynamic responses of the FGM plate with different loads](image)

Figure 6. Effect of pre-loaded axial $P_x$ compression on nonlinear response of the FGM plate (all FM edges).

![Effect of pre-loaded axial $P_x$ compression on nonlinear response of the FGM plate](image)

Figure 7. Effect of temperature on nonlinear response of the FGM plate (all IM edges).

![Effect of temperature on nonlinear response of the FGM plate](image)
$P_i$ increases. Figure 7 shows the effects of temperatures $\Delta T = (0, 50, 150^\circ C)$ on the nonlinear dynamic response of FGM plates. The dynamic response amplitude increases when the temperature $\Delta T$ increases.

The comparison in Table 3 shows the natural frequencies calculated by Equation 19 of full order motion by Equation 18 and by Equation 21 of simplified motion, Equation 20 quite close to each other.

Comparison of nonlinear dynamic responses of the plate calculated by Equation 18 and Equation 20 is shown in Figures 8 and 9. It can be seen that there is no much difference between the nonlinear dynamic response of two cases also.

From Table 3 and Figures 8 and 9, we can see that the assumption that the inertial forces caused by two rotations $\Phi_x, \Phi_y$ are too small, is reasonable. Therefore, the simplified assumption can be used for nonlinear dynamical analysis with an acceptable accuracy.

Figures 10 and 11 show the effects of uniform rising temperatures and elastic foundations on the frequency–amplitude relations of the nonlinear free vibration of FGM plate. Figure 10 shows that with the same amplitude, when the temperature increases, the frequency of vibration becomes larger. Figure 11 shows that with the same frequency, FGM plates on elastic foundations have smaller amplitude than the plates without elastic foundations, and the Pasternak foundation influences faster and more powerful than Winkler foundation on the frequency–amplitude relations of the plate. In Duc (2013), Duc and Cong (2013a) the similar conclusion also has drawn.

| $N$ | Natural frequencies obtained from Equation 19 – full order motion Equation 18 | Natural frequencies obtained from Equation 21 – motion Equation 20 |
|-----|--------------------------------------------------------------------------|----------------------------------------------------------------|
| 0   | 2,959.9                                                                 | 2,965.8                                                                 |
| 0.5 | 2,508.9                                                                 | 2,513.9                                                                 |
| 1   | 2,261.8                                                                 | 2,266.3                                                                 |
| 3   | 1,991.0                                                                 | 1,995.1                                                                 |
| 2   | 2,056.5                                                                 | 2,060.6                                                                 |
| 5   | 1,947.7                                                                 | 1,951.8                                                                 |
| $\infty$ | 1,506.6                                                                | 1,509.6                                                                |

**Figure 8.** Comparison of dynamic responses in the two cases between Equations 18 and 20 when the frequency of external load is away far from the natural frequencies of the plate.
Figure 9. Comparison of dynamic responses in the two cases between Equations 18 and 20 when the frequency of the excitation is near to the natural frequency of the plate.

Figure 10. Effect of the temperature field on frequency–amplitude curve of FGM plates in case of free vibration and no damping.

Figure 11. Effect of the elastic foundations on frequency–amplitude curve of FGM plates in case of free vibration and no damping.

Figure 12 presents the effects of the external forces on frequency–amplitude is the case of forced vibration. The dashed line is the case $F = 0$ (the free vibration), the solid line is the case when $F = 2,000$ and the dotted line is $F = 4,000$ case. It can be seen with the same amplitude of external force, the frequency of the plate will be increased with increasing of the coefficients $K_1, K_2$ of elastic foundations.
4. Conclusions

This paper presents an analytical approach to investigate the nonlinear dynamic response and vibration of thick FGM plates using both of the first-order shear deformation plate theory and stress function with full motion equations (not using Volmir’s assumptions). The FGM plate is assumed to rest on elastic foundation and subjected to mechanical, thermal, and damping loads. Numerical results for dynamic response of the FGM plate are obtained by Runge–Kutta method. The results show the influences of the material properties, the elastic foundations, pre-loaded axial compression, and thermal loads on the nonlinear dynamic behavior of functionally graded plates.

Especially, this paper tested the differences between nonlinear dynamic responses of the FGM plates when the inertial forces caused by rotations $\Phi_x$, $\Phi_y$ in motion equations are ignored with nonlinear dynamic responses of the FGM plates using full motion equations. The tested results show that the assumption that the inertial forces caused by two rotations $\Phi_x$, $\Phi_y$ are two small, is reasonable. Therefore, this assumption can be used for nonlinear dynamical analysis with an acceptable accuracy.

Nomenclature

- $\varepsilon_x^0$, $\varepsilon_y^0$: the normal strains
- $\gamma_{xy}^0$: the shear strain on the midplane of the plate
- $\gamma_{xz}$, $\gamma_{yz}$: the transverse shear strains
- $u$, $v$: the displacement components along the $x$, $y$ directions
- $w$: the deflection of the plate
- $\phi_x$, $\phi_y$: the rotation angles in the $xz$ and $yz$ planes
- $N$: the volume-fraction index
- $m$: number of half waves axial direction
- $n$: number of wave in circumferential direction
- $a$: length of FGM plate
- $b$: width of FGM plate
- $h$: thickness of FGM plate
- $K_1$: Winkler foundation modulus
- $K_2$: the layer foundation stiffness of Pasternak model
- $K$: the correction factors and are taken as $K = 5/6$
References

Allahverdizadeh, A., Naei, M. H., & Nikkhah Bahrami, M. (2008). Nonlinear free and forced vibration analysis of thin circular functionally graded plates. Journal of Sound and Vibration, 310, 966–984. http://dx.doi.org/10.1016/j.jsv.2007.08.011

Duc, N. D. (2013). Nonlinear dynamic response of imperfect eccentrically stiffened FGM double curved shallow shells on elastic foundation. Composite Structures, 102, 306–314. http://dx.doi.org/10.1016/j.compstruct.2013.03.009

Duc, N. D., & Cong, P. H. (2013a). Nonlinear dynamic response of imperfect symmetric thin S-FGM plate with metal–ceramic–metal layers on elastic foundation. Journal of Vibration and Control, 21, 637–646. doi:10.1177/1077546313489717

Duc, N. D., & Cong, P. H. (2013b). Nonlinear postbuckling of symmetric S-FGM plates resting on elastic foundations using higher order shear deformation plate theory in thermal environments. Composite Structures, 100, 566–574. http://dx.doi.org/10.1016/j.compstruct.2013.01.006

Duc, N. D., & Tung, H. V. (2010). Mechanical and thermal postbuckling of shear-deformable FGM plates with temperature-dependent properties. Mechanics of Composite Materials, 46, 461–476. http://dx.doi.org/10.1007/s11029-010-9163-9

Fakhari, V., & Ohadi, A. (2010). Nonlinear vibration control of functionally graded plate with piezoelectric layers in thermal environment. Journal of Vibration and Control, 17, 449–469. doi:10.1177/1077546309356970

Ferreira, A. J. M., Batro, R. C., & Roque, C. M. C. (2006). Natural frequencies of functionally graded plates by a meshless method. Composite Structures, 75, 593–600. http://dx.doi.org/10.1016/j.compstruct.2006.04.018

Hosseini-Hashemi, Sh., Rokni Damavandi Tehr, H., Ashkavan, H., & Omidi, M. (2010). Free vibration of functionally graded rectangular plates using first-order shear deformation plate theory. Applied Mathematical Modelling, 34, 1276–1291. http://dx.doi.org/10.1016/j.apm.2009.08.008

Matsumoto, H. (2008). Free vibration and stability of functionally graded plates according to a 2-D higher-order deformation theory. Composite Structures, 82, 499–512. http://dx.doi.org/10.1016/j.compstruct.2007.01.030

Reddy, J. N. (2006). Mechanics of laminated composite plates and shells: Theory and analysis. Boca Raton, FL: CRC Press.

Shen, H.-S. (2009). Functionally graded materials. London: CRC Press; New York, NY: Taylor & Francis Group. http://dx.doi.org/10.1201/9781420092578

Talha, M., & Singh, B. N. (2010). Static response and free vibration analysis of FGM plates using higher order shear deformation theory. Applied Mathematical Modelling, 34, 3991–4011. http://dx.doi.org/10.1016/j.apm.2010.03.034

Tung, H. V., & Duc, N. D. (2010). Nonlinear analysis of stability for functionally graded plates under mechanical and thermal loads. Composite Structures, 92, 1184–1191. http://dx.doi.org/10.1016/j.compstruct.2009.10.015

Vel, S. S., & Botra, R. C. (2004). Three-dimensional exact solution for the vibration of functionally graded rectangular plates. Journal of Sound and Vibration, 272, 703–730. http://dx.doi.org/10.1016/S0022-460X(03)00412-7

Woo, J., Meguid, S. A., & Ong, L. S. (2006). Nonlinear free vibration behavior of functionally graded plates. Journal of Sound and Vibration, 289, 595–611. http://dx.doi.org/10.1016/j.jsv.2005.02.031

Wu, T.-L., Shuik, K.-K., & Huang, J. H. (2008). Nonlinear static and dynamic analysis of functionally graded plates. International Journal of Applied Mechanics and Engineering, 11, 679–698.

Zhao, X., Lee, Y. Y., & Liew, K. M. (2009). Free vibration analysis of functionally graded plates using the element-free kp-Ritz method. Journal of Sound and Vibration, 319, 918–939. http://dx.doi.org/10.1016/j.jsv.2008.06.025