Analytic approximation for the collapse of viscous tubes driven by surface tension and pressure difference

Received: 8 August 2021 / Accepted: 16 February 2022
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Abstract In this work, an approximate analytic formula is developed which accurately models the one-dimensional collapse kinetics of viscous glass tubes, driven by surface tension and low-to-moderate pressure differences. This is in contrast to existing analytic approaches from the literature where either surface tension is the only driving force, or extremely high pressure differences are assumed. Extensive model validation is provided against numerical computation of the exact one-dimensional and two-dimensional models for cross-sectional collapse, as well as with experimental data from the literature. Practical utility of this formula is demonstrated for effortlessly solving the inverse problem for determining the viscosity and surface tension of glass tubes.

1 Introduction

The prediction of collapse times of heat softened glass tubes is important during the manufacture of fibre optic preforms [1, 2]. Typically, fibre optic preforms are manufactured by collapsing a silica glass tube, with an inner deposit, in a glass lathe with an oxy-hydrogen gas flame [3]. The total collapse process for these large preform tubes can take as long as 24 hours [4]. Through a system which controls the gas flow to the flame and the internal pressure of the tube [5, 6], accurate tuning of the pressure differential ($\Delta p$) can be achieved, in order to reduce the total collapse duration [1], while not introducing distortions to the preform geometry by exceeding critical pressures [2]. For such processes, it is useful to have a simplified analytic relationship which accurately predicts as well as readily lends insight into the dependence of the collapse times with pressure differences, without immediately needing to resort to less intuitive numerical computations. A similar argument is also applicable during the manufacture of more complex fused optical fibre bundles [7, 8], where a glass tube holding many fibres usually requires a partial collapse process to enable a better fit with the internal fibres. The experimental collapse of glass tubes combined with an accurate model of this process can also provide a powerful method for determining the surface tension [5] and the viscosity [6, 9] of glasses at high operating temperatures.

In Lewis’s seminal publication [1], an accurate one-dimensional differential equation describing the collapse of axisymmetric glass tubes in the presence of both surface tension and pressure difference was derived. This model accounted generally for the collapse of composite tubes consisting of two different layers with piecewise constant viscosity [1]. In that work, approximate analytic solutions were found from the governing equation using an ad hoc asymptotic approach for situations where the pressure difference (external over-pressures) is extremely large in comparison with the surface tension stresses. Later, Kirchhof provided...
Table 1 Literature survey of the various types of models addressing the collapse of viscous glass tubes including where our analytic model fits in

|                      | One-dimensional                                                                 | Two-dimensional                                                                 |
|----------------------|----------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| **Numerical**        | Lewis [1]                                                                         | Non-axisymmetric tubes: Geyling et al. [2], Yarin et al. [10]                    |
|                      |                                                                                 | Cross-sectional & axial flow (drawing of tube): Fitt et al. [11], Chen et al. [12] |
| **Analytic**         | Pressure difference >> Surface tension: Lewis [1]                                | Purely surface tension driven (cross-sectional & axial flow): Klupsch et al.     |
|                      | Small radial changes (< 1 − 5%): Kirchhof [5]                                    | (finite collapse zone) [13], Stokes (drawing of tube) [14]                      |
|                      | Purely surface tension driven: Makovetskii et al. [9]                            |                                                                                 |
|                      | Moderate pressure differences: Sanders et al. (this work)                       |                                                                                 |
Analytic approximation for the collapse of viscous tubes

Fig. 1 Cross-section of a glass tube with inner radius $a$ and outer radius $b$ and constant viscosity $\mu$ in the glass region. The viscous glass tube is acted upon by external pressure outside of the tube, $p_{out}$, internal pressure, $p_{in}$, and surface tension stresses. The stress boundary conditions, $\tau \hat{n}(r = a)$ and $\tau \hat{n}(r = b)$, and the kinematic (velocity) conditions, $v(r = a)$ and $v(r = b)$, are indicated in the figure.

Experimental validation to this one-dimensional model for specifically homogenous tubes where small radial changes occurred [5]. An exact analytic solution to the one-dimensional model where collapse is driven only by surface tension was developed by Makovetskii et al [9]. They demonstrated the utility of this formula for estimating the viscosity of glass tubes [9]. As depicted in Table 1, significant strides have also been made to develop complex two-dimensional models for the collapse of viscous tubes, in order to account for deviations from tube axisymmetry [2,10], or to account for axial flows [11–14]. However, till date, there are no analytic formulations which account for collapse driven by both surface tension and the low pressure differences relevant when a gaseous flame impinges the surface of the tube [6].

In this work, an analytic approximation to the one-dimensional model is developed, for describing the collapse of homogenous tubes under the presence of both small pressure differentials and surface tension. This analytic formula is extensively validated against the numerical solution of the exact one-dimensional model, two-dimensional finite volume method (FVM) simulations using ANSYS Polyflow, and the limited experimental data available from the literature. The aim of developing the simplified analytic solution is to readily provide a practitioner with insight into the dependence of the collapse times with the parameter space, without the need to immediately resort to less intuitive numerical computations. In fact, the formula provided in this work can effortlessly provide predictions in regions of interest, requiring only a standard handheld scientific calculator. Through two case studies, it is demonstrated that this model provides utility for inferring the viscosity and surface tension of glass tubes at the typical processing temperatures used in various fibre optic fabrication processes.

2 Problem description

The physical system of interest in this work is depicted in Fig. 1. Collapse of the tube is driven by the action of surface tension stresses, $\gamma/a$ and $\gamma/b$, where $\gamma$ is the coefficient of surface tension, and a pressure difference, $\Delta p = p_{out} - p_{in}$, between the outside and inside of the tube. It is assumed that the tube is homogeneous (a single material) and is uniformly heated to achieve a constant temperature and is therefore at a constant viscosity, $\mu$.

In Lewis’s 1976 publication [1], an accurate one-dimensional representation of the collapse kinetics of glass tubes in the radial direction was developed. This model was derived from the Navier–Stokes momentum and continuity equations for incompressible Newtonian fluids, and using the stress and kinematic (velocity)
conditions depicted in Fig. 1. The model assumes no axial flow or out-of-plane dynamics in Fig. 1. This is a reasonable assumption in applications of interest [3,6,7], where the collapse region is much longer in comparison with the outer diameter of the tube [2]. The equations were further simplified and obtained by assuming the tube is axisymmetric and that the inertial terms in the momentum equation can be neglected, as they are extremely small in comparison with the viscous terms. Through initial dimensional analysis, it is observed that two characteristic timescales influence the collapse kinetics, one due to surface tension and the other due to pressure difference. Therefore, if surface tension is the dominant driving mechanism, the characteristic collapse velocity for the problem is defined by: \( v_c = a_0/t_{\gamma} \). For typical viscosity values of order \( 10^5 \text{ Pa.s} \) [15], surface tension of 0.3 Nm\(^{-1} \) [16], and density of 2.1 g cm\(^{-3} \) [17], for silica glass at processing temperatures of approximately 2000°C, extremely small Reynolds numbers of order \( Re = \rho v_c a_0/\mu \sim 10^{-11} \) and \( Re = \rho v_c a_0/\mu \sim 10^{-10} \) are determined, for millimetre and centimetre diameter tubes, respectively. Due to the extremely small Reynolds numbers, it is justifiable to neglect the inertial terms from the momentum equation. With these simplifications, the collapse kinetics is described by

\[
\gamma \left( \frac{1}{a} + \frac{1}{b_f^2 + a^2} \right) + 2 \frac{da}{dt} \mu \left( \frac{1}{a} - \frac{a}{b_f^2 + a^2} \right) = -\Delta p,
\]

where \( a \) is the inner radius which is a function of time, \( t \), and \( b_f \) is the final outer collapse radius of the tube (i.e. when \( a = 0 \), the inner radius vanishes). The above differential equation can be obtained without derivation from first principles by simply substituting \( \nu = 1 \) in Lewis’s differential form for composite tubes given by Equation (6) in [1]. Due to the requirement for conservation of mass in the plane, the final collapse radius is related to the outer and inner radius as

\[
b_f^2 = b(t)^2 - a(t)^2 = b_0^2 - a_0^2,
\]

where \( a_0 \) and \( b_0 \) are the initial values for the inner and outer radius, respectively.

### 3 Scaling and exact solutions to the 1D model

Typically in set-ups where a mixed oxy-hydrogen flame is used, the outside of the tube only experiences an excess pressure in the order of tens of Pascals on average [6]. At these scales, surface tension is the dominant driving mechanism for the collapse. A slight increase in the pressure difference will reduce the final collapse time, below that expected for a purely surface tension driven system. With this in mind, the time variable is scaled by the surface tension timescale as, \( t' = t/t_\gamma \). Furthermore, as it is the change in the inner radius with time we are primarily interested in during the collapsing process, the length variables are scaled by the initial inner radius. For instance, we have \( a' = a/a_0 \) and \( b' = b/a_0 \) for the scaled versions of the inner and outer radius, respectively. Using these length and time scales (1) is recast into the following dimensionless form

\[
\frac{1}{2} \left( \frac{1}{a'} + \frac{1}{(b_f^2 + a'^2)} \right) + \frac{da'}{dt'} \mu \left( \frac{1}{a'} - \frac{a'}{(b_f^2 + a'^2)} \right) = -q,
\]

The primed notation refers to the scaled (dimensionless) variables and is not to be confused with derivatives. In reaching (3), two dimensionless quantities represented by \( \Lambda \) and \( q \) are identified. Considering that

\[
\Lambda = b_f^2 = \frac{b_f^2}{a_0^2} = \frac{b_0^2}{a_0^2} - 1,
\]

and

\[
q = \frac{\Delta p a_0}{2\gamma} = \frac{t_\gamma}{t_{\Delta p}}
\]

\( \Lambda \) is the ratio of the total fluid area (grey region) in Fig. 1 to the area of the initial inner void, and \( q \) (the q-number) is the scaled pressure term or the ratio of the two time scales involved.
The solution to (3) can be expressed by the following integral

\[ t' = -2\Lambda \int_1^{a'} \frac{d\alpha}{f(\alpha)(f(\alpha) + \alpha)(1 + 2qg(\alpha))}, \]  

(6)

where

\[ f(a') = \sqrt{\Lambda + a'^2} = b', \]

(7)

and

\[ g(a') = a' \frac{f(a')}{f(a') + a'} = a' \frac{b'}{b' + a'}. \]

(8)

Equation (7) is essentially the scaled version of (2) for mass conservation. Isolating the function, \( g(\alpha) \), in the integral above is a key step in making later approximations to the integrand. In general, (6) needs to be numerically integrated as it cannot be algebraically evaluated, except in the limited case for zero pressure difference (\( q = 0 \)). The exact algebraic solution for purely surface tension driven collapse is then given by

\[ t' = 2\left( f(a') - f(1) + 1 - a' \right). \]

(9)

This limited case was used by Makovetskii et al. to form the basis of a new method to estimate glass viscosity [9]. We observe that while their final analysis was correct, typographical errors in the analytic solution they presented occurred in Equation (3) in that work [9]. The correct solution for the case where \( \Delta p = 0 \) (i.e. \( q = 0 \)) is given here by (9). We shall refer to (6) as well as the numerical evaluation of this integral depicted in later figures, as simply the exact one-dimensional model.

4 Simplified analytic approximation

We now proceed to make a number of approximations to the integral, (6), to render it analytically solvable for low-to-moderate pressure difference ranges, including the ranges relevant when a gaseous flame is used. The first approximation that is made to the integrand comes from observing that \( g(a') \) is linear in the following two asymptotic limits for the aspect ratio:

\[ \lim_{b_0/a_0 \to 1} g(a') = a' \]

(10)

for the ultrathin wall limit, and

\[ \lim_{b_0/a_0 \to \infty} g(a') = a' \]

(11)

for the extremely thick wall limit. Therefore, for any value of the aspect ratio, \( b_0/a_0 \), we make the following linear approximation

\[ g(a') \approx \beta a', \]

(12)

where

\[ \beta = \frac{b_0}{b_0 + a_0}. \]

(13)

As depicted in Fig. 2, this approximation ensures convergence to the correct linear forms at the two limits for the aspect ratio. The gradient of the approximation varies between 0.5 and 1. There are discrepancies in the approximation made for aspect ratios between these limits as shown in Fig. 2. However, as will be shown, these discrepancies have minor impact under the integration in (6), provided q-numbers are below 1 (i.e. pressure differences are low to moderate). As can be observed in (6), the q-number is a coefficient of \( g(\alpha) \). Therefore,
Fig. 2 Plots of $g(a')$ defined by (8), along with the linear approximation (dotted line) defined by (12), for different aspect ratios.

Fig. 3 a Colour plot of the absolute percentage error in computing the final collapse time between the analytic approximation, (15), and the exact one-dimensional model, (6), as a function of the aspect ratio and q-number. b Plot of the error along $q = 1$.

high pressure differences can amplify errors with approximations made to this function. Substitution of (12) into (6) leads to the simplification

$$t' \approx \frac{2\alpha}{f(\alpha) (1 + 2q\beta\alpha)} + \frac{1}{\beta q} \ln \left( \frac{2\beta q + 1}{2\beta q a' + 1} \right).$$

This approximation enables the extraction of the logarithmic term and a simpler integral term. It is observed from (7) that for extremely thick wall tubes where $b_0/a_0 \to \infty$ (also $\Lambda \to \infty$) that also $f(\alpha) \approx b_0/a_0 \to \infty$. Therefore, the integrand in the first term in (14) becomes negligibly small. Furthermore, by substituting $a' = 0$ for the final collapse time, (14) reduces to $t' \approx \frac{1}{q} \ln (2q + 1)$ in the limit $b_0/a_0 \to \infty$. This form matches an earlier approximation made by Lewis [1], for the final collapse time of extremely thick wall tubes. On the other hand, for extremely thin wall tubes where $b_0/a_0 \to 1$ (also $\Lambda \to 0$) we have $f(\alpha) \to \alpha$. In this thin wall limit, the first integral term in (14) evaluates analytically as the negative of the second term, giving the expected zero-valued solution for the collapse time (i.e. instantaneous collapse) where there is an absence of viscous material.

The first integral term in (14) cannot be analytically evaluated. However, it is solvable when we only replace the linear $\alpha$ term in the denominator of the integrand by a value invariant with the integration variable. A reasonable approximation to the integrand, as will be shown for regions of interest, is obtained by using the midpoint value across the integration bounds of $\bar{\alpha} = \frac{a' + 1}{2}$, in place of the linear $\alpha$ term in the denominator.
Analytic approximation for the collapse of viscous tubes

This approximation is reasonable provided pressure differences are not too large such that the pre-multiplying q-number for this term is below 1, and the tube walls are not too thin such that aspect ratio \( \frac{b_0}{a_0} \) is extremely close in value to 1. The tube walls cannot be too thin as this approximation will prevent the first term in (14) from exactly cancelling out the effect of the second term in the limit \( \frac{b_0}{a_0} \to 1 \), where the solution is expected to be zero-valued. With this approximation, the following analytic formula is obtained:

\[
t' \approx 2 \left[ \frac{f(a') - f(1)}{1 + \beta(qa' + 1)} + \frac{1}{2\beta q} \ln \left( \frac{2\beta q + 1}{2\beta qa' + 1} \right) \right].
\] (15)

It is observed from (15) that in the low pressure difference limit, \( q \to 0 \), (15) reduces to the exact expression, (9), where the sole driving force is surface tension. Furthermore, we still obtain the correct degenerate form for the final collapse time for extremely thick wall tubes, \( t' \sim \frac{1}{q} \ln (2q + 1) \). In fact, the effect of tube wall thickness on collapse time is intuitively observed from the formula by re-expressing (15) as

\[
t' \approx \frac{2\Delta b'}{1 + \beta(qa' + 1)} + \frac{1}{\beta q} \ln \left( \frac{2\beta q + 1}{2\beta qa' + 1} \right),
\] (16)

where \( \Delta b' = b' - b_0/a_0 \) is the total change in the scaled outer radius, which is negative in value during collapse. For extremely thick wall tubes, it is inferred from the expressions for mass conservation that \( \Delta b' \approx 0 \). Therefore, the time for collapse to reach any inner radius value for extremely thick wall tubes is determined by the second term in (16), with \( \beta = 1 \), giving \( t' = \frac{1}{q} \ln \left( \frac{2q + 1}{2qa' + 1} \right) \). For thinner wall tubes, this collapse time is reduced and is corrected for by the first term in (16), which subtracts from the effect of the dominant (larger) logarithmic term. Also, the \( \beta \) parameter varies from 1, for thick, down to \( \frac{1}{2} \), for thin wall tubes. However, as identified earlier, in the ultrathin wall limit \( (b_0/a_0 \to 1) \) the first term will not exactly cancel out the effect of the logarithmic term to give the expected zero-valued solution (i.e. instantaneous collapse), where there is an absence of viscous material. In order to illustrate the behaviour described, in Fig. 4, the final collapse time is plotted as a function of the aspect ratio, at selected q-numbers, for both the exact and analytic one-dimensional solution.

For most applications of interest such as for fabrication of optical fibre preforms [3,6], or for purposes of estimating glass viscosity through the collapse process [9], ultrathin wall glass tubes are generally not used nor desired. For many of these applications, tubes typically have an aspect ratio \( (b_0/a_0) \) of 1.2 or greater [5,7,9]. This is due to the fragility of working initially with ultrathin wall glass tubes, and the increased sensitivity of collapse times in this thin wall region. Therefore, the analytic formula developed in this work is still broadly applicable.
Fig. 5 Comparison between two-dimensional FVM simulations (points) and the analytic one-dimensional formula (solid line) for the change in inner radius with time. Results presented for $\Delta p = 0$ Pa $(q = 0)$ and $\Delta p = 600$ Pa $(q = 0.5)$ for tube dimensions $b_0/a_0 = 0.6 \, \text{mm}/0.5 \, \text{mm}$. Scaled (dimensionless) quantities are also shown on separate axes.

applicable. Figure 3a shows that the absolute percentage error between the approximate analytic solution and the exact one-dimensional model in computing the final collapse time is negligible in the region of interest. For typical tube aspect ratios ($b_0/a_0$) above 1.2 [3, 6, 7] and q-numbers below 1, the error is bounded below 3% as depicted in Fig. 3a-b. However, depending on the level of accuracy required the formula need not be limited to this region given all of the blue region in Fig. 3a indicates an error below 5% with the exact one-dimensional model. As shown in Fig. 3a, the error between (15) and (6) generally tends to decrease with the q-number and for increasing aspect ratios. This analytic formula therefore covers the low pressure differences relevant for instance when an oxy-hydrogen flame is used as a heat source [6], which previous literature-based analytic models have not addressed [1, 9, 13, 14].

5 Comparisons with numerical computations

Here the analytic formula, (15), is validated against comprehensive two-dimensional FVM simulations using the ANSYS Polyflow package, as depicted in Fig. 5. The FVM simulations were conducted by retaining the inertial terms. For the simulations and results presented here, dimensions are reintroduced by using the material parameters for silica glass at typical processing temperatures of approximately 2000° C. The material parameters used in the simulations are: $\mu = 10^5$ Pa.s, $\gamma = 0.3$ Nm$^{-1}$, and $\rho = 2.1$ gcm$^{-3}$. Furthermore, the simulation results are presented in Fig. 5 for a $b_0/a_0 = 0.6 \, \text{mm}/0.5 \, \text{mm}$ tube. These dimensions are similar to that used previously for fabricating fused optical fibre bundles for imaging [8], as well as for purposes of estimating the viscosity of silica at high temperatures through the collapse process [9]. For the general application of the results, separate axes depicting the corresponding dimensionless variables are shown.

In order to provide an initial demonstration that the analytic formula is applicable across a wide range of pressure differences, in Fig. 4 results are depicted for the situation where there is no pressure difference, and where there is a relatively high pressure difference of $\Delta p = 600$ Pa $(q = 0.5)$ in the system. The latter is considered relatively high in this context, since typically the impact pressures exerted by gaseous flames are only in the order of tens of Pascals [6]. However, as mentioned earlier, in practice the pressure difference in the system can be increased through controlling the flow of gases to the flame and the internal pressure in the tube for achieving the optimal controlled conditions for the application. The FVM simulations were set up using approximately 13,000 mesh elements for the fluid domain (grey region in Fig. 1), which took approximately 30 minutes of simulation time using a computer with an Intel (R) Core (TM) I9-7900X ten core processor. In contrast, the calculations using the analytic formula are rapid, and can essentially be performed using a standard handheld scientific calculator since no numerical integration is required.

The analytic model exhibits excellent agreement with the computationally demanding FVM simulations shown in Fig. 5. The root-mean-squared (RMS) value of the errors between the models, in terms of the collapse times to reach any inner radius, is only $\sim 0.064$ s or 0.04% of the final collapse time for the no pressure difference
Analytic approximation for the collapse of viscous tubes

Fig. 6 Comparison between the approximate analytic formula (solid), (15), the numerical integration of the exact 1D model (dashed), (6), and 2D FVM simulations (points) for two different tubes. For these simulations, the inner radius is fixed at $a_0 = 0.5$ mm. Scaled (dimensionless) quantities are also shown on separate axes.

case. Whereas, for the $\Delta p = 600$ Pa ($q = 0.5$) case the RMS error between models is only $\sim 0.77$ s or 0.61% of the final collapse time. This indicates that the simplifying assumptions in the one-dimensional model are well justified. In particular, it is not surprising that the inertial term can be neglected from the momentum equation, given the extremely small Reynolds number of order $10^{-11}$ for millimetre diameter tubes. Furthermore, this provides validation to the approximations made for obtaining the final analytic formula.

Figure 5 shows that an increase in pressure difference results in an increased collapse rate, and therefore reduced times to reach any inner radius during collapse. It is shown that the millimetre diameter tube completely collapses in $\sim 126$ s when a pressure difference of 600 Pa is imposed. This is 18% below the time for purely surface tension driven collapse. If these results were instead applied to a larger tube with 1 cm inner diameter, typically used for fabrication of fibre preforms, and if we retain the same aspect ratio of 1.2, a lower pressure difference of 60 Pa would give the same q-number of 0.5. This would also result in the same 18% difference in final collapse times with the purely surface tension driven case. However, the final collapse times for the large centimetre diameter tube would increase by a single order of magnitude in comparison with the small millimetre diameter tube. This exercise illustrates that even relatively low pressure differences experienced by large diameter tubes can have a non-negligible impact on the collapse times, which must be accounted for in the model.

To directly show the effect of pressure on collapse times, the change in the final collapse time, $t_f$, with increasing pressure difference, $\Delta p$, is depicted in Fig. 6. Here the analytic formula, (15), is validated against the numerical computation of the exact one-dimensional model, (6), as well as two-dimensional FVM computations for selected points. Figure 6 shows that the exact one-dimensional model is aligned with the two-dimensional FVM simulations. Furthermore, the analytic formula provides good agreement with the exact models for lower pressure difference ranges. As identified earlier, the absolute percentage error between the analytic approximation and the exact one-dimensional model in computing the final collapse time is bounded below 3% for aspect ratios above 1.2 and q-numbers below 1. In contrast, the final formula developed by Lewis [1] is only valid for extremely high pressure differences, as indicated in Fig. 7a. For Lewis’s final formula, q-numbers greater than 10 are required for similarly low percentage errors with the exact models, as shown in Fig. 7b. These high pressure difference ranges are not applicable where a gaseous flame is used [6]. Such extreme pressure conditions can also result in the rapid growth of any initial perturbation from tube axisymmetry [2].
6 Case studies from literature

6.1 Case study 1

In this section, validation of the analytic formula is provided with the limited experimental data available from the literature [4, 5]. The presented case studies also assist to reveal the utility of the analytic formula developed in this work for determining unknown material parameters. In the first case study, experimental data from [4], for the collapse evolution of a $b_0/2a_0 = 25$ mm/19 mm homogenous fused quartz tube, is fitted with the analytic formula developed in this work, as depicted in Fig. 8. It is immediately identified, that a typographical error occurred in the presentation of the experimental data in that work, by incorrectly specifying the range of measurements for the outer radius as $9.5$ mm down to $6.5$ mm for the specified tube. Therefore, the data points are corrected by shifting vertically by $3$ mm to ensure the correct initial outer radius of $12.5$ mm for the specified tube is depicted at $t = 0$ in Fig. 8. In that work, the pressure difference was maintained at $11$ Pa at a temperature close to $2000^\circ$ C, utilising a glass lathe with an oxy-hydrogen burner [4].

In Fig. 8, the viscosity is the fitting parameter, and a constant surface tension value of $0.3$ Nm$^{-1}$ assumed [16]. The analytic formula is fitted to the experimental data to obtain the least squared errors in the time variable using the Levenberg–Marquardt algorithm [18, 19]. When fitting all the data (red solid curve) a
Fig. 9 Experimental data from [5] with linear fit (dotted lines) and also our model results (solid curves) based on the analytic formula, (15). a Model validation within range of experimental data. b Depicting the extended pressure difference and radial change range for our model.

viscosity value of $5.7 \times 10^4$ Pa.s is estimated for the tube, with an RMS error with all the data points of 18 s (4% of duration of experiment), as indicated in Fig. 8. The relatively low RMS error provides experimental validation to the analytic formula. If the same data were instead fitted with the exact one-dimensional model, (6), a viscosity value which is only 0.35% larger than the previous estimation is obtained and the curves are nearly indistinguishable, providing further confidence in the predictions made by (15).

Figure 8 shows that the errors with the solid red curve is greatest in the time interval between approximately 100 to 250 s. Since tube collapse is extremely sensitive to variations in viscosity, and in turn the viscosity of silica glass is extremely sensitive to fluctuations in temperature [15], it is likely that a variation in the viscosity is the primary cause for the observed errors. According to the one-dimensional model an $x\%$ uncertainty in viscosity would yield the same percentage uncertainty in the collapse times. By fitting the data only for time $\leq 250$ s (blue dashed curve), an improved fit with the first few data points is obtained with a lower viscosity value of $5.1 \times 10^4$ Pa.s, corresponding to a slightly lower temperature. Both of the values estimated for the viscosity are in the correct order of magnitude, and similar to the value of $5 \times 10^4$ Pa.s originally estimated in [4]. The experimental data suggests there is at least a 17% uncertainty in the viscosity value which leads to a 17% error in the times for collapse (see error bars in Fig. 8), which at these high operating temperatures would correspond to approximately a $\pm 15^\circ$ C error in temperature, determined using the Arrhenius relationship in [15]. Maintaining a temperature of $\pm 15^\circ$ C in the vicinity of 2000$^\circ$ C utilising an oxy-hydrogen burner is extremely difficult to achieve in practice, and the true uncertainty is likely greater and can only only be discerned if the entire experiment is repeated a greater number of times. The analytic formula is demonstrated to be effective in accurately solving the inverse problem for unknown parameters, and for estimating the uncertainty for the extracted parameters.

It is also important to identify that for these extremely low pressure differences in the experiment, the q-number is also quite low at $q \approx 0.17$, which indicates collapse is primarily driven by surface tension. However, even for these low pressure differences, the effect of pressure should not be ignored and must be accounted for (see Fig. 8). As demonstrated in Fig. 8, the corresponding zero pressure difference model (thin dotted curve), evaluated at the viscosity value which fitted the entire data set, overestimates the predicted times for each radius in the collapse evolution, with a larger RMS error of around 10% over the duration.

6.2 Case study 2

In this case study experimental data from Kirchhof’s work [5] is presented in Fig. 9, and studied through application of the analytic formula developed in this work. Experimental data in Fig. 9 was collected for a
fused quartz tube with initial dimensions, \( b_0/a_0 = 8 \text{ mm}/6 \text{ mm} \). In Kirchhoff’s work, an effective method for extracting the surface tension parameter for fused quartz tubes was developed. The basis of this method was recognising that for small radial changes (1) can be linearised in terms of the axes variables depicted in Fig. 8. The linearised relationship is

\[
\Delta a \left(1 - \frac{a}{b}\right) \approx \frac{t}{2\mu} \left(\frac{\Delta p}{b^{1/a} + a^{1/b}} - \gamma\right).
\]

where \( \Delta a = a - a_0 \) is the change in the inner radius. In that work, measurements were taken of the change in inner radius with varying degree of internal pressurisation (\( \Delta p < 0 \)), while the duration and the temperature were held constant. From (17), it is evident that the x-intercept of the line of best fit to the data (see dotted lines in Fig. 9) must be the coefficient of surface tension for the tube. Furthermore, the gradient of the line is simply related to the fixed duration of the experiments and tube viscosity by: gradient \( = t / 2\mu \). Therefore, if the duration for each experiment is known, the average viscosity for the tube can also be determined from this single experiment.

To provide further validation to the analytic formula, the extracted values from the linear fit (i.e. the \( \gamma \) and \( t / 2\mu \)) are fed into our model, and the model results are depicted in Fig. 9 by the solid curves. Since, the analytic formula, (15), cannot be symbolically inverted to represent the inner radius as the independent variable, in order to determine the change in inner radius with pressure difference, while \( t \) is held constant, a transcendental equation needs to be solved numerically. With dimensions reintroduced into (15), the transcendental equation which is numerically solved for the inner radius is

\[
\frac{t}{2\mu} \approx 2a_0 \left[ \frac{f(a/a_0) - f(1)}{2\gamma + \beta \Delta p(a + a_0)} \right] + \frac{1}{\beta \Delta p} \ln \left( \frac{\beta \Delta pa_0 + \gamma}{\beta \Delta pa + \gamma} \right).
\]

The above equation is solved numerically using Newton’s method for the inner radius for different values of the pressure difference. Therefore, in this manner we can determine the total change in inner radius, \( \Delta a \), for different applied pressure differences, \( \Delta p \), and depict our model results (solid curves) in terms of the axes variables in Fig. 9. Our model results are depicted for both small degrees of internal pressurisation (\( \Delta p < 0 \)), within the range of the experimental data as shown in Fig. 9a, and for a wide range of external over-pressures (i.e. \( \Delta p > 0 \)), which are not within the range of the data as shown in Fig. 9b.

As depicted in Fig. 9a, our analytic formula agrees with the experimental data and the approximate linearity for small radial changes. Even though our analytic formula was not developed in general to account for the swelling (increase in radii) of tubes, caused by sufficiently large internal pressurisation (i.e. where it overcomes surface tension), the results indicate that the formula is applicable to the description of such behaviour provided that the increase in radius is within 5% of the initial values. In contrast to Kirchhof’s linearised model, our formula is also applicable over a relatively wide range of positive pressure differences, which favours the collapse as opposed to the swelling of the tube. Our analytic model is only plotted in Fig. 9b for \( \Delta p < 150 \text{ Pa} \) or equivalently where q-numbers are approximately below 1 for the specified tube, which is within the range of applicability of the formula. As depicted from our model results over the extended pressure difference range in Fig. 9b, there is a clear deviation from the linear behaviour where large reductions in the inner radius occur.

Theoretically, similar experimental data can be collected for moderate positive pressure differences where large reductions in the inner radius occur, and through a fitting process with our formula, the average values for both surface tension and viscosity estimated. The primary advantage of using this analytic formula is that experimental measurements need not be limited only to small radial changes, where expensive high precision and accurate instruments are required to make the measurements for these small changes. Furthermore, use of the analytic result from this work requires very little computational effort, with no numeric integration required.

### 7 Conclusion

In this work, a simple analytic solution to the one-dimensional model is obtained, for describing the collapse of viscous homogenous tubes due to surface tension and moderate pressure differentials. This formula is extensively validated against numerical solutions of the exact models, and experimental data from the literature. It is shown that this formula provides less than 3% error with the exact models for determining the final collapse time where tube aspect ratios are greater than 1.2 and the q-number is below 1. The analytic formula readily lends insight into the dependence of collapse time with the parameter space and accurate predictions can be made.
through hand calculations in regions of practical interest, without the need to resort to less intuitive numerical computations. In this work, we demonstrate the application of this analytic formula for effortlessly estimating the surface tension and viscosity parameters from experimental data. This work addresses an important gap between those analytic models where surface tension is the sole driving factor for collapse [9,13,14], and those approximations requiring the pressure difference to dominate the collapse kinetics over surface tension [1]. Therefore, this work provides an incremental contribution to the prior art.

Acknowledgements The authors gratefully acknowledge the early discussions with Professor Neville Fowkes on the mathematics and physics involved in this work which assisted with finding a novel solution to the problem.

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Funding Funding Open Access funding enabled and organized by CAUL and its Member Institutions.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Lewis, J.A.: The collapse of a viscous tube. J. Fluid. Mech. 81(1), 129–135 (1977). https://doi.org/10.1017/S0022112077001943
2. Geyling, F., Walker, K., Csencsits, R.: A two-dimensional asymptotic model for capillary collapse. J. Appl. Mech. 50, 303–310 (1983). https://doi.org/10.1115/1.3164078
3. Mazzarrese, D., Oulundsen III, G.E., Mcmahon II, T.F., Owsiiany, M.T.: Method of collapsing a tube for an optical fiber preform (2004). http://www.freepatentonline.com/6718800.html
4. Kjolbro, J., Hendricks, E., Holst, J.: Identification of a preform collapse process. Int. J. Modell. Simulat. 6(4), 141–145 (1986). https://doi.org/10.1080/02286203.1986.1175997
5. Kirchhof, J.: A hydrodynamic theory of the collapsing process for the preparation of optical waveguide preforms. Physica. Status Solid. (a) 60(2), K127–K131 (1980). https://doi.org/10.1002/pssa.2210600245
6. Kirchhof, J., Unger, S.: Viscous behavior of synthetic silica glass tubes during collapsing. Opt. Mater. Exp. 7(2), 386–400 (2017). https://doi.org/10.1080/02286596.2017.1100386
7. Bland-Hawthorn, J., Bryant, J., Robertson, G., Gillingham, P., O’Byrne, J., Cecil, G., Haynes, R., Croom, S., Ellis, S., Maack, M., Skovgaard, P., Nordegraaf, D.: Hexabundles: imaging fiber arrays for low-light astronomical applications. Opt. Exp. 19(3), 2649–2661 (2011). https://doi.org/10.1364/OE.19.002649
8. Sanders, T.M., Lamb, C., Putrino, G., Keating, A.J.: Method for increasing the core count and area of high density optical fiber bundles. IEEE J. Select. Topics Quant. Electr. 26(4), 1–8 (2020). https://doi.org/10.1109/JSTQE.2020.2984562
9. Makovetskii, A.A., Zamyatin, A.A., Ivanov, G.A.: Technique for estimating the viscosity of molten silica glass on the kinetics of the collapse of the glass capillary. Glass. Phys. Chem. 40(5), 526–530 (2014). https://doi.org/10.1134/S1087659614050083
10. Yarin, A.L., Bernat, V., Doupovec, J., Miklos, P.: The viscous collapse of radial nonsymmetric composite tubes. J. Lightw. Tech. 11(2), 198–204 (1993). https://doi.org/10.1109/50.212527
11. Fitt, A., Furusawa, K., Monro, T., Please, C., Richardson, D.: The mathematical modelling of capillary drawing for holey fibre manufacture. J. Eng. Math. 43(2), 201–227 (2002)
12. Chen, M., Stokes, Y., Buchak, P., Crowdy, D., Ebendorff-Heidepriem, H.: Microstructured optical fibre drawing with active channel pressurisation. J Fluid Mech 783, 137–165 (2015)
13. Klupsch, T., Pan, Z.: Collapsing of glass tubes: analytic approaches in a hydrodynamic problem with free boundaries. J. Eng. Math. 106, 143–168 (2017). https://doi.org/10.1007/s10665-017-9954-2
14. Stokes, Y.M.: A two-dimensional asymptotic model for capillary collapse. J. Fluid Mech. 909, A5 (2021). https://doi.org/10.1017/jfm.2020.954
15. Doremus, R.H.: Viscosity of silica. J. Appl. Phys. 92(12), 7619–7629 (2002). https://doi.org/10.1063/1.1515132
16. Boyd, K., Ebendorff-Heidepriem, H., Monro, T.M., Munch, J.: Surface tension and viscosity measurement of optical glasses using a scanning co2 laser. Opt. Mater. Exp. 2(8), 1101–1110 (2012). https://doi.org/10.1364/OME.2001010
17. Aksay, I.A., Fask, J.A., Davis, R.F.: Densities of $Si + O_2 - Al_2 + O_3$ melts. J. Am. Cerum. Soci. 62(7–8), 332–336 (1979). https://doi.org/10.1111/j.1151-2916.1979.tb19071.x
18. Levenberg, K.: A method for the solution of certain non-linear problems in least squares. Quart. Appl. Math. 2(2), 164–168 (1944). http://www.jstor.org/stable/43633451
19. Marquardt, D.W.: An algorithm for least-squares estimation of nonlinear parameters. J. Soci. Ind. Appl. Math. 11(2), 431–441 (1963). http://www.jstor.org/stable/2098941

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