Dynamical Spin Response of Doped Two-Leg Hubbard-like Ladders

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We study the dynamical spin response of doped two-leg Hubbard-like ladders in the framework of a low-energy effective field theory description given by the SO(6) Gross-Neveu model. Using the integrability of the SO(6) Gross-Neveu model, we derive the low energy dynamical magnetic susceptibility. The susceptibility is characterized by an incommensurate coherent mode near $(\pi, \pi)$ and by broad two excitation scattering continua at other $k$-points. In our computation we are able to estimate the relative weights of these contributions.

All calculations are performed using form-factor expansions which yield exact low energy results in the context of the SO(6) Gross-Neveu model. To employ this expansion, a number of hitherto undetermined form factors were computed. To do so, we developed a general approach for the computation of matrix elements of semi-local SO(6) Gross-Neveu operators. While our computation of these form factors was performed in the context of the SO(6) Gross-Neveu model, we also consider the effects of perturbations away from an SO(6) symmetric model, showing that small perturbations at best quantitatively change the physics.

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I. INTRODUCTION

Ladder materials bridge the great divide between the well understood one dimensional (doped) Mott insulators and their perplexing two-dimensional analogs. The desire to understand their fascinating physical properties as well as the expectation that an understanding of ladder materials would culminate in the theoretical understanding of doped, two dimensional Mott insulators has led to intense, sustained theoretical interest in the problem. Analytical techniques have been used to deduce the zero temperature phase diagrams of ladder models. The dynamical properties are less well characterized. In fact, in contrast to the half-filled case, the accurate determination of dynamical response functions for two-leg ladder lattice models at finite doping remains a challenging open problem. One approach to calculating dynamical correlations at finite doping is to utilize numerical methods such as quantum Monte-Carlo techniques or exact diagonalization. The single-particle spectral function and the dynamical spin and charge susceptibilities for both undoped and doped Hubbard ladders were computed by quantum Monte-Carlo methods in Ref. 26 and compared to analytical random-phase approximation calculations for a two-leg Hubbard ladder. In Ref. 26 the dynamical structure factor was computed by exact diagonalization of a $2 \times 16$ t-J ladder model.

A different approach is to utilize field theory methods to describe the low-energy degrees of freedom. In Ref. 13 the structure of various dynamical response functions for the half-filled case was determined in the framework of a low-energy description in terms of the SO(8) Gross-Neveu model. Detailed calculations utilizing the integrability of the SO(8) Gross-Neveu model were carried out in Ref. 13. In the doped case it has been argued in Refs. 4 and 13 that two-leg ladders can be described by the SO(6) Gross-Neveu model $\otimes$ U(1) Luttinger liquid at low energies. The Luttinger liquid describes the gapless charge degrees of freedom on the ladder while the SO(6) Gross-Neveu model governs the behavior of the gapped degrees of freedom in the spin and orbital sectors. A prominent feature in the magnetic response due to bound state formation was analyzed in Ref. 25. It is important to go beyond the existing calculations in order to address questions raised by recent inelastic neutron scattering experiments on doped cuprate two-leg ladders. Furthermore, the dynamical susceptibilities of doped ladders can be used to develop a theory for systems of weakly coupled ladders, which in turn can be used as toy models for the striped phase in the cuprates.

In the present work we determine the dynamical structure factor of doped Hubbard-like two-leg ladders at low energies in the framework of the SO(6) Gross-Neveu description. We go beyond Ref. 25 in that (1) we consider all momentum space regions with non-vanishing low-energy response; and (2) we consider multiparticle contributions to the dynamical structure factor. This allows us to evaluate the contribution to the structure factor of the bound state identified in Ref. 25 relative to other contributions.

Given that we employ the SO(6) Gross-Neveu model as a low energy description of the doped ladders it is worthwhile dwelling briefly on the nature of this description. The SO(6) Gross-Neveu model represents the attractive fixed point of an RG flow of a doped ladder with weak generic interactions. The flow has been established to two-loop order. We point out that the flow to the SO(6)
Gross-Neveu does not assume that the bonding and anti-bonding bands of the Hubbard ladder have identical velocities. In the work of Ref. [4], it is shown that differing Fermi velocities will renormalize towards one another (an effect present only at two loops).

As the RG is perturbative in nature, the bare couplings must not be particularly strong. How strong they may be before the RG becomes unreliable is indicated in Ref. [3]. In this work, the related problem of half-filled ladders was studied where the RG flow takes the ladders to a sector where their low energy behavior is described by the SO(8) Gross-Neveu model. There it was found that $U/t$ ($U$ being representative of the strength of some generic short ranged interaction) may be as large as 6 before the RG flow is a poor indicator of the low energy physics. At larger values of $U$, it was found that the charge and spin/orbital sectors decoupled. This decoupling is already present in the U(1) Luttinger liquid model and so this scenario of false RG-induced symmetry breaking does not appear here.

Beyond a question of whether the bare couplings are sufficiently weak, it may be asked whether the RG flow accurately represents the physics in a wider sense. Where things may go wrong is illustrated by an appeal to the U(1) Thirring model. However the RG equations are misleading in the presence of an infinite series of conservation laws. This simplified scattering yields that generic eigenfunctions of the SO(6) Gross-Neveu model can always be reduced to a series of two body scattering amplitudes. This reduction occurs because of the presence of an infinite series of conservation laws. This non-integrable model would not be possible because of particle production and decay.

The particle content of SO(6) Gross-Neveu is relatively simple. The elementary excitations are found in two four dimensional representations, the kink (4) and the anti-kink (4). These excitations represent the elementary fermionic excitations of the ladder (as discussed in greater detail in Section II). An interesting feature of SO(6) Gross-Neveu is that two kinks or two anti-kinks can form bound states. These bound states transform as some distinct collection of the elementary excitations. In an non-integrable model this would not be possible because of particle production and decay.

With the spectrum of SO(6) Gross-Neveu in hand, we can readily write down the form factor expansion of any two-point correlation function (imaginary time ordered at $T = 0$) in the model:

$$G_T^O(x, \tau) = \langle 0|T(\mathcal{O}(x, \tau)\mathcal{O}^\dagger(0,0))|0\rangle = \sum_{n=0}^{\infty} \sum_{s_n} \langle n; s_n|\mathcal{O}(0,0)|0\rangle e^{-\tau E_{s_n}} x^{n}.$$
ing transition amplitudes between the ground state and multi-particle states.

The integrability of the $SO(6)$ Gross-Neveu model allows the computation of these matrix elements in principle. Each matrix element satisfies a number of algebraic constraints which can be solved. However as the particle number in the eigenstate $|n:s_n\rangle$ increases, solving these constraints becomes increasingly cumbersome. Fortunately, if one is only interested in the behavior of the low energy spectral function (i.e. the imaginary portion of the corresponding retarded correlator), it is only necessary to compute matrix elements involving a small number of particles. The spectral function is given by

$$-\frac{1}{\pi} \text{Im} G_T^\text{C}(x,-i\omega + \delta) = \sum_{n=0}^{\infty} \sum_{s_n} \left\{ \langle 0| \mathcal{O}(x,0)|n;s_n\rangle \langle n;s_n|\mathcal{O}^\dagger(0,0)|0\rangle \delta(\omega - E_{s_n}) - \epsilon \langle 0| \mathcal{O}^\dagger(0,0)|n;s_n\rangle \langle n;s_n|\mathcal{O}(x,0)|0\rangle \delta(\omega + E_{s_n}) \right\}, \quad (1.4)$$

where $\epsilon = \pm$ for fields, $\mathcal{O}$, that are bosonic/fermionic.

The delta functions present in the above equation guarantee that only eigenstates with energy, $\omega$, contribute to the spectral function. As each particle has a fixed mass, higher particle states will not contribute if the sum of their masses is greater than the energy, $\omega$, of interest.

In this article, we compute only the one and lowest two particle form-factors of appropriate fields. This allows us to compute the spin response exactly to an energy of $2\sqrt{2}$ times the spin gap in the region of $k$-space near $(\pi, \pi)$. In other regions of the Brillouin zone we are able to obtain exact results to energies twice the spin gap. However in these latter regions, the spectral intensity is considerably smaller than that found near $(\pi, \pi)$.

While our results are exact up to certain energies, it is likely that in certain cases (in particular, our computation of the spin response in the region of $(\pi, \pi)$) they remain extremely accurate beyond these thresholds. As a rule, form-factor expansions have been found to be extremely convergent with only the first terms yielding a significant contribution. As one example, in Ref. 47, the spectral function for the spin-spin correlation function in the Ising model was computed. The two-particle eigenstates produced a contribution 1000 times greater than the four-particle eigenstates and a contribution $10^8$ times greater than the six particle contribution. For practical purposes then the two particle form-factor in this case gave “exact” results for energies far in excess of the two particle threshold.

The outline of this paper is as follows. In Section II, we review the connection between doped Hubbard-like ladders and the $SO(6)$ Gross-Neveu model. In particular we outline the renormalization flow that takes generically interacting doped ladders to the $SO(6)$ Gross-Neveu model. We also delineate the connections between the excitations and operators in $SO(6)$ Gross-Neveu and their corresponding ladder lattice counterparts.

In Section III, we present our results for the dynamical magnetic susceptibility. We consider this quantity for a variety of dopings ranging from $1\%$ to $25\%$. We also compare the results derived from our field theoretic treatment to an RPA analysis as well as to non-interacting ladders. We find there are significant differences. In particular the RPA cannot reproduce accurately the most prominent feature in the spin response, the coherent mode near $(\pi, \pi)$.

In Section IV, we present our results on form-factors in the $SO(6)$ Gross-Neveu. The form-factors that are needed for the computation were not all known prior to this work. In part this came about because the spin response involve operators in $SO(6)$ Gross-Neveu which are not standard objects of field theoretic interest. These operators are unusual in that they are semi-local while not being $SO(6)$ currents. To derive the matrix of elements of these operators it was first necessary to establish the algebraic constraints satisfied by these form-factors. While it was not the primary intent of this work, we developed a general scheme by which the constraints for all semi-local operators in $SO(6)$ Gross-Neveu (and by straightforward extension, $SO(2N)$ Gross-Neveu with $N$ odd) may be written down. Portions of this developments are technical. The basics of the derivation of all the form factors may be found in Section IV with certain details related to the semi-local operators relegated to Appendix B. For those interested in only the end product, a summary of all the relevant form-factors may be found in Section IV H.

II. FROM DOPED LADDERS TO THE $SO(6)$ GROSS-NEVEU MODEL

Here we will briefly review the connection between doped Hubbard ladders and the $SO(6)$ Gross-Neveu model. We discuss the reduction of doped ladders to their field theoretic equivalent together with how to understand the excitation spectrum and fields of the $SO(6)$ Gross-Neveu model in terms of the original ladder model.
A. Relation of doped D-Mott Hubbard Ladders to the \( SO(6) \) Gross-Neveu model

We consider a weakly interacting ladder with generic repulsive interactions. It is well established that the system exhibits algebraically decaying d-wave pairing. We begin with non-interacting electrons hopping on a ladder:

\[
H_0 = - \sum_{x,\alpha} \left( ta^\dagger_{1\alpha}(x+1)a_{1\alpha}(x) + ta^\dagger_{2\alpha}(x+1)a_{2\alpha}(x) + t_L a^\dagger_{1\alpha}(x)a_{2\alpha}(x) + \text{h.c.} \right),
\]

(2.1)

Here the \( a_i^\dagger/a_i \) are the electron annihilation/creation operators for the electrons on rung \( i \) of the ladder, \( x \) is a discrete coordinate along the ladder, and \( \alpha = \uparrow, \downarrow \) describes electron spin. \( t \) and \( t_L \) describe respectively hopping between and along the ladder’s rung.

The first step in the map is to reexpress the \( a \)'s of \( H_0 \) in terms of bonding/anti-bonding variables:

\[
c_{j\alpha} = \frac{1}{\sqrt{2}}(a_{1\alpha} + (-1)^j a_{2\alpha}),
\]

(2.2)

(noting \( j = 1 = ab \) and \( j = 2 = b \)). With this transformation, the Hamiltonian can be diagonalized in momentum space in terms of two bands. As we are interested in the low energy behavior of the theory, the \( c_{j\alpha} \)'s are linearized about the Fermi surface, \( k_F \):

\[
c_{j\alpha} \sim c_{R\alpha} e^{ik_F x} + c_{L\alpha} e^{-ik_F x},
\]

(2.3)

where \( L, R \) corresponding to the right and left moving modes about the Fermi surface. With this \( H_0 \) becomes,

\[
H_0 = \int dx \sum_{j\alpha} v_{Fj} \left[ c^\dagger_{Lj\alpha} i \partial_x c_{Rj\alpha} - c^\dagger_{Rj\alpha} i \partial_x c_{Lj\alpha} \right].
\]

(2.4)

Away from half-filling, the \( v_{Fj} \) will be generically different. However for small to moderate dopings, the difference of the two Fermi velocities will be small. We thus set \( v_{Fj} = v_{Fb} \) in order to simplify matters.

The next step is to consider adding interactions to the system. We consider adding all possible allowed interactions to this Hamiltonian. To organize this addition, we introduce various \( SU(2) \) scalar and vector currents

\[
I^P_{ij} = c^\dagger_{p\alpha} c_{\sigma\alpha^\prime} c_{p\sigma^\prime}; \quad I^I_{ij} = \frac{1}{2} c^\dagger_{p\sigma^\prime} (\sigma^\prime \sigma)_{\tau\tau'} c_{p\tau'} c_{p\tau'},
\]

\[
J^P_{ij} = c^\dagger_{p\alpha} c_{\sigma\alpha^\prime}; \quad J^I_{ij} = \frac{1}{2} c^\dagger_{p\sigma^\prime} \sigma_{\tau\tau'} c_{p\tau'} c_{p\tau'}.
\]

(2.5)

Here \( P = R, L, \sigma \) are the Pauli matrices and \( \epsilon_{\sigma\sigma^\prime} \) is the \( \epsilon \)-tensor. The crystal-momentum conserving interactions divide themselves into forward and backward scattering terms:

\[
H_B = \sum_{i,j=1,2} b^2 c^\dagger_{ij} J^R_{ij} J^L_{ij}; \quad H_F = \sum_{i\neq j=1,2} f^2 c^\dagger_{ij} J^R_{ij} J^L_{ij} c^\dagger_{ij} J^L_{ij} J^R_{ij}.
\]

(2.6)

Here \( f \) and \( b \) are the forward and backward scattering amplitudes. From hermiticity and parity, we have \( b_{ij} = b_{ji} \) and \( f_{ij} = f_{ji} \). Furthermore, in the same spirit of taking the Fermi velocities of the two bands to be the same, \( b_{11} = b_{22} \). Thus we obtain six independent amplitudes. As we are working away from half-filling in the chains, we do not consider Umklapp interactions.

To facilitate the analysis of these interactions, we invoke a change to variables. We begin by bosonizing the \( c \)'s:

\[
c_{Pj\alpha} = \kappa_{j\alpha} e^{i\phi_{Pj\alpha}}, \quad P = +, - = R, L.
\]

(2.7)

Here \( \kappa_{j\alpha} \) are Klein factors satisfying

\[
\{\kappa_{j\alpha}, \kappa_{i\beta}\} = 2\delta_{ij}\delta_{\alpha\beta}.
\]

(2.8)

In terms of these four Bose fields, four new Bose fields are defined (effectively separating charge and spin):

\[
\phi_{P1} = \frac{1}{2}(\phi_{P1\uparrow} + \phi_{P1\downarrow} + \phi_{P2\uparrow} + \phi_{P2\downarrow}); \quad \phi_{P2} = \frac{1}{2}(\phi_{P1\uparrow} - \phi_{P1\downarrow} + \phi_{P2\uparrow} - \phi_{P2\downarrow});
\]

\[
\phi_{P3} = \frac{1}{2}(\phi_{P1\downarrow} - \phi_{P1\uparrow} - \phi_{P2\uparrow} + \phi_{P2\downarrow}); \quad \phi_{P4} = \frac{1}{2}(\phi_{P1\uparrow} + \phi_{P1\downarrow} - \phi_{P2\uparrow} - \phi_{P2\downarrow}).
\]

(2.9)

Note that \( \phi_{P4} \) has a relative sign between the right and left movers. This sign effectively masks the underlying \( SO(6) \) symmetry of the original Hamiltonian. The first and second bosons describe charge and spin fluctuations respectively. The latter two bosons, \( \phi_3 \) and \( \phi_4 \), are associated with less transparent quantum numbers (as we discuss shortly).

From these chiral bosons, one can define pairs of conjugate bosons in the standard fashion

\[
\varphi_i = \phi_{Ri} + \phi_{Li}; \quad \theta_i = \phi_{Ri} - \phi_{Li},
\]

(2.10)

which obey the commutation relations,

\[
[\varphi(x), \theta(x')] = -i4\pi\delta(x' - x).
\]

(2.11)

where \( \Theta(x - x') \) is the Heaviside step function.

In terms of these variables the free part of the Hamiltonian can be written

\[
H_0 = \frac{v_F}{8\pi} \sum_a \left\{ (\partial_x \varphi_a)^2 + (\partial_x \varphi_a)^2 \right\}.
\]

(2.12)

The momentum conserving interactions can be written as

\[
H_B + H_F = \frac{1}{16\pi^2} \sum_{a=1}^4 A_a \left\{ (\partial_x \varphi_a)^2 - (\partial_x \varphi_a)^2 \right\}
\]

\[
-2b_{12}^2 \cos(\theta_1) \cos(\theta_2) - (b_{12}^2 - 4b_{12}^2) \cos(\varphi_3) \cos(\theta_1)
\]

\[
+ \cos(\theta_2)(2b_{12}^2 \cos(\theta_3) + 2f_{12}^2 \cos(\varphi_3))
\]
where the coefficients $A_n$ are equal to $A_{1/4} = \pm 2(b_{11}^\dag + f_{12}^\dag)$ and $A_{2/3} = -(b_{11}^\dag \pm f_{12}^\dag)/2$.

For generically repulsive scattering amplitudes, the interaction Hamiltonian does not involve the boson governing the Neveu model. On the other hand we see that the inter-type excitations for the vector representation by $A_\alpha^\dagger$ are equal to $\kappa_1 = \kappa_2^\dagger$, $\kappa_2 = \kappa_1^\dagger$, $\kappa_3 = \kappa_1$, $\kappa_4 = \kappa_2^\dagger$.

We then find the free Hamiltonian can be written as

$$H_0 = \int dx \sum_a \left[ \Psi_{\alpha a}^\dagger \partial_x \Psi_{\alpha a} - \Psi_{La}^\dagger \partial_x \Psi_{La} \right],$$

where the Fermi velocity, $v_F$, has been set to 1, while the interaction Hamiltonian can be written as

$$H_{int} = -2g \left[ \sum_{a=2}^4 \left( i \Psi_{La}^\dagger \Psi_{Ra} - i \Psi_{Ra}^\dagger \Psi_{La} \right) \right]^2.$$  

This is precisely the interaction piece of the SO(6) Gross-Neveu model. It will sometimes prove convenient to recast the theory in terms of Majorana fermions, $\eta_{Pa}$. In terms of the Dirac fermions, $\Psi_{Pa}$, they are given by

$$\Psi_{Pa} = \frac{1}{\sqrt{2}} (\eta_{2a-2,Pa} + i\eta_{2a-3,Pa}), \quad (a = 2, 3, 4).$$

In this basis, $H_{int}$ can be recast as

$$H_{int} = 2g G_{R}^{ab} G_{L}^{ab}, \quad (a \neq b = 1, \ldots, 6),$$

where $G_{P}^{ab} = i\eta_{Pa}^\dagger \eta_{Pa}$ is one of the 15 SO(6) Gross–Neveu currents.

### B. Excitation spectrum of SO(6) Gross-Neveu model

The SO(6) Gross Neveu has a total of 14 different types of excitations. These excitations are organized into two four dimensional spinor representations and one six dimensional vector representation. The latter corresponds to the Majorana fermions introduced in the previous section. We denote the fermionic creation operators for the vector representation by $A_\alpha^\dagger$, $a = 1, \ldots, 6$. The excitations corresponding to the spinor representations are kinks and represent interpolations of the fields $\theta_i$, $b = 1, 2, 3$ between minima of the cosine potentials in Eqn. (2.15). There are eight different types of kinks, which we will denote by $A_\alpha^\dagger$. Here $\alpha$ is a “multi-index” of the form $\alpha = (\pm 1/2, \pm 1/2, \pm 1/2)$. The eight kinks divide themselves into two different four dimensional irreducible representations, something discussed in greater detail in Section 14.

SO(6) is a rank 3 algebra and the eigenvalues $N_i$, $i = 1, \ldots, 3$ of the three Cartan generators can be used as quantum numbers for the excitations. The combination of Majorana fermions

$$\eta_{2a-1} \pm i\eta_{2a},$$

generates excitations with quantum number $N_a = \pm 1$, $N_b = 0, b \neq a$. The quantum numbers carried by the kinks $A_\alpha^\dagger$ are directly encoded in $\alpha$. If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i = \pm 1/2$, then $A_\alpha^\dagger$ creates a kink with quantum numbers, $N_i = \alpha_i$.

The quantum numbers, $N_i$, are related to the physical quantum numbers of the system. In particular these
are the z-component of spin, $S_z$, the difference in z-component of spin of the two bands, $S_{12}$, and the “relative band chirality”, $P$, defined as $P = N_{R1} - N_{L1} - N_{R2} + N_{L2}$, where $N_{Rj}$ is the number of electrons in band $j$ with chirality $P$. In Ref. 12, it was found:

$$(N_1 = 1, 0, 0) \leftrightarrow (S_z = 1, S_{12} = 0, P_{12} = 0);$$

$$(0, N_2 = 1, 0) \leftrightarrow (S_z = 0, S_{12} = 1, P_{12} = 0);$$

$$(0, 0, N_3 = 1) \leftrightarrow (S_z = 0, S_{12} = 0, P_{12} = 2).$$

(2.23)

We note that these three quantum numbers are independent of the total charge, $Q$. Charge excitations all occur in the gapless sector governed by the boson $\theta_1$.

We can see that the vector representation of excitations corresponds to states of two electrons (particle-hole pairs) in the original formulation. For example, the fermion $\eta_1 \pm i \eta_2$ carries spin, $S_z = \pm 1$ and no charge and so is a magnon excitation. The spinor representations (the kinks) in turn are closely related to single particle excitations as their quantum numbers are combinations of $N_i/2$ (in fact they are single particle excitations modulo their charge component).

C. Identification of Physical Operators in terms of SO(6) Gross-Neveu Fields

In this section we provide a dictionary between the SO(6) Gross-Neveu model and the original fermions of the Hubbard ladders. As discussed previously, the kinks are closely related to single particle excitations and so the original electron operators. There are 16 kinks in total (counting both left and right movers) and sixteen electron operators, the $c$’s and $\bar{c}$’s (four for each of the four Fermi points). In terms of the four bosons $\phi_{R/L1} = \phi_{R/L\rho}$ and $\phi_{R/Li}$, $i = 2, 3, 4$, the electron operators are given by $c$’s as follows:

$$c_{Ri} = \epsilon^{\phi_{Ri}} e^{\frac{i}{2} (\phi_{R2} - \phi_{R3} - \phi_{R4})};$$

$$c_{R\bar{i}} = \epsilon^{\phi_{R\bar{i}}} e^{\frac{i}{2} (-\phi_{R2} + \phi_{R3} + \phi_{R4})};$$

$$c_{Rab} = \epsilon^{\phi_{Rab}} e^{\frac{i}{2} (\phi_{R2} + \phi_{R3} + \phi_{R4})};$$

$$c_{R\bar{a}b} = \epsilon^{\phi_{R\bar{a}b}} e^{\frac{i}{2} (-\phi_{R2} - \phi_{R3} + \phi_{R4})};$$

(even chirality)

$$c_{L\bar{i}i} = \epsilon^{\phi_{L\bar{i}i}} e^{\frac{i}{2} (-\phi_{L2} + \phi_{L3} + \phi_{L4})};$$

$$c_{L\bar{i}} = \epsilon^{\phi_{L\bar{i}}} e^{\frac{i}{2} (\phi_{L2} - \phi_{L3} + \phi_{L4})};$$

$$c_{L\bar{a}b} = \epsilon^{\phi_{L\bar{a}b}} e^{\frac{i}{2} (-\phi_{L2} - \phi_{L3} - \phi_{L4})};$$

$$c_{L\bar{a}b} = \epsilon^{\phi_{L\bar{a}b}} e^{\frac{i}{2} (\phi_{L2} + \phi_{L3} - \phi_{L4})};$$

(odd chirality).

(2.24)

The lattice fermions are all given by a charge piece, $e^{i\phi_{R1}}$, multiplied by a field creating an SO(6) kink, $c_{R\bar{i}i} = \phi_{R\bar{i}i}(\pm \phi_{R2} \pm \phi_{R3} \pm \phi_{R4})$. The first four fermions involve even chirality kink fields while the latter involve odd chirality kinks. We denote the group representation of the even chirality kinks by 4. Electrons transforming as even chirality kinks are those above in addition to $\{c_{L\bar{i}i}, c_{L\bar{i}i}^\dagger, c_{L\bar{a}b}^\dagger, c_{L\bar{a}b}\}$, that is the Hermitian conjugate of the above odd kinks. The odd chirality kinks transform under the 4 representation. Again, electrons transforming as such are the above as well as the Hermitian conjugate of the above even kinks, $\{c_{R\bar{i}i}^\dagger, c_{R\bar{i}i}, c_{R\bar{a}b}, c_{R\bar{a}b}^\dagger\}$.

The lack of symmetry between left and right movers reflects the parity sign in the definition of $\phi_{P4}$ in Eqn. (2.22).

To construct the spin-spin correlation function we need to consider the description of fermion bilinears in the language of the SO(6) Gross-Neveu model. Fermion bilinears can be of two types: one involving fermions associated with kinks of different chiralities and one involving fermions associated with kinks of the same chiralities. In the first case we want to consider fermion bilinears transforming under $4 \otimes 4 = 4 + 15$. We will want to focus on the fifteen dimensional representation, 15, which is no more than the set of SO(6) currents. In the second case we want to consider bilinears behaving as $4 \otimes 4 = 6 + 10$ (and similarly for $4 \otimes 4$). The six dimensional representation, 6, is associated with particle-hole excitations carrying unit quantum numbers such as the magnon. The particle-hole excitations corresponding to the 6 are formed by taking symmetric combinations excluding diagonal terms of the even electrons in Eqn. (2.24):

$$c_{R\bar{i}i} c_{L\bar{i}i}^\dagger + c_{R\bar{a}b} c_{L\bar{a}b}^\dagger.$$  

(2.25)

Such symmetric fermion combinations behave antisymmetrically under SO(6). As will be discussed, this correspondence between the 6 and symmetric combinations of fermion bilinears is most easily seen upon bosonization of the fermionic expressions. We will denote the fields creating these two particle excitations as $\psi^a$, $a = 1, \ldots, 6$. On the other hand, the ten dimensional representation, 10, arises from antisymmetric combinations of particle-hole excitations,

$$c_{R\bar{i}i} c_{L\bar{i}i}^\dagger - c_{R\bar{a}b} c_{L\bar{a}b}^\dagger;$$

together with diagonal combinations of fermions

$$c_{R\bar{i}i} c_{L\bar{i}i}.$$  

(2.26)

We will denote the fields creating these excitations by $\psi^{[abc]}_4$, where $a, b, c = 1, \ldots, 6$ are anti-symmetrized. The odd representation for these fields will facilitate the computation of their associated matrix elements.
III. SPIN-SPIN CORRELATION FUNCTIONS OF DOPED LADDERS

In this section we utilize the form factors we calculate in Section IV to determine the low-energy behavior of the dynamical structure factor of the doped two-leg ladder. For readers uninterested in the details of the form factor computations, we have summarized the results in Section IV H.

A. Preliminaries

In order to compute the spin response, we need to relate the lattice spin-spin correlation function to correlation functions in the SO(6) Gross-Neveu model. The dynamical magnetic susceptibilities of the doped ladder, \(\chi(\omega, q_x, q_y)\), is given in terms of two different spin correlators: one along a leg of the ladder

\[
\chi_{11}(x, t) \equiv \langle S_1(x, t) \cdot S_1(0, 0) \rangle_{\text{ret}}
\]

and one involving cross leg correlations

\[
\chi_{12}(x, t) \equiv \langle S_1(x, t) \cdot S_2(0, 0) \rangle_{\text{ret}} \equiv \chi_{21}(x, t).
\]

The correlators we need can then be written (using SU(2) invariance) as

\[
\chi_{ab}(\omega, k) = \int_{-\infty}^{\infty} d\tau dx e^{i(\omega + \epsilon)\tau - ikx} \langle T S_a(x, \tau) \cdot S_b(0) \rangle.
\]

The dynamical spin-spin correlation functions are expressed as

\[
\begin{align*}
I_1(\omega, k) &= \int dx \int dy \int dz \langle c_{a\dagger}^b c_{b\dagger}^d c_{a\dagger}^b c_{b\dagger}^d \rangle \\
I_2(\omega, k) &= \int dx \int dy \int dz \langle c_{ab\dagger}^b c_{bd\dagger}^a c_{ab\dagger}^b c_{bd\dagger}^a \rangle \\
I_3(\omega, k) &= \int dx \int dy \int dz \langle c_{ab\dagger}^b c_{bd\dagger}^a c_{ab\dagger}^b c_{bd\dagger}^a \rangle.
\end{align*}
\]

In the next step we take the low-energy limit, in which the fermion operators are expanded according to \(2\). This gives, for example

\[
\langle c_{a\dagger}^b c_{b\dagger}^d c_{a\dagger}^b c_{b\dagger}^d \rangle \sim \langle c_{Rb\dagger}^b c_{Rb\dagger}^d c_{Rb\dagger}^b c_{Rb\dagger}^d \rangle + \langle c_{Lb\dagger}^b c_{Lb\dagger}^d c_{Lb\dagger}^b c_{Lb\dagger}^d \rangle + ... + \langle e^{-2i K_F x} c_{Rb\dagger}^b c_{Rb\dagger}^d c_{Lb\dagger}^d c_{Lb\dagger}^b \rangle + \langle e^{2i K_F x} c_{Lb\dagger}^d c_{Lb\dagger}^b c_{Rb\dagger}^b c_{Rb\dagger}^d \rangle.
\]

Substituting \(3\) and its analogs into the expressions for \(I_1(\omega, k)\) we obtain sums of terms, each of which can be calculated using a form factor expansion as sketched in the introduction. For example in the imaginary piece of \(I_1(\omega, k)\) we have a contribution of the form

\[
\begin{align*}
\sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^{n-1} n!} &\langle c_{Rb\dagger}^b(0) c_{Rb\dagger}^d(0) \theta_1 \ldots \theta_n \rangle_{a_1 \ldots a_n}^2 \delta \left( \omega - \sum_{i=1}^{n} m_{a_i} \cosh \theta_i \right) \delta \left( k - \sum_{i=1}^{n} \frac{m_{a_i}}{v_F} \sinh \theta_i \right) \\
&\sim \frac{1}{(2\pi)^{n-1} n!} \langle c_{Rb\dagger}^b(0) c_{Rb\dagger}^d(0) \theta_1 \ldots \theta_n \rangle_{a_1 \ldots a_n}^2 \delta \left( \omega - \sum_{i=1}^{n} m_{a_i} \cosh \theta_i \right) \delta \left( k - \sum_{i=1}^{n} \frac{m_{a_i}}{v_F} \sinh \theta_i \right).
\end{align*}
\]
where the first sum $\sum_n$ runs over the number of particles in a particular term of the form factor expansion and the second sum $\sum_\alpha$ runs over the different possible particle types. The complete set of intermediate states used in the spectral representation is defined by

$$|\theta_1 \ldots \theta_n\rangle_{A_{\alpha}} = A_{\alpha} (\theta_n) \cdots A_{\alpha_1} (\theta_1) |0\rangle.$$ (3.10)

Carrying out the expansion for all terms in $\sum_\alpha$, we obtain

$$I_1(\omega, k) = \begin{cases} A_{11} J_1(\omega, k) & \text{if } k \approx 0 \\ A_{12} J_2(\omega, k \mp 2K_{Fb}) & \text{if } k \approx \pm 2K_{Fb}, \end{cases}$$

$$I_2(\omega, k) = \begin{cases} A_{21} J_1(\omega, k) & \text{if } k \approx 0 \\ A_{22} J_2(\omega, k \pm 2K_{Fab}) & \text{if } k \approx \pm 2K_{Fab}, \end{cases}$$

$$I_3(\omega, k) = \begin{cases} A_{31} J_1(\omega, k \mp K_{F-}) & \text{if } k \approx \pm K_{F-} \\ A_{32} J_3(\omega, k \mp K_{F+}) & \text{if } k \approx \pm K_{F+}. \end{cases}$$ (3.11-3.13)

Here we have defined

$$K_F^\pm = K_{Fab} \pm K_{Fb}.$$ (3.14)

The $J_i$’s correspond to the lowest (one and two) particle (bound state and kink) contributions to the spin response. Their explicit forms are derived in Appendix A.

At low energies, as explained in the introduction, these terms provide the sole contributions. The various $A$’s in the above are normalization constants. To fix them, we compare the spectral weight in the spin response obtained using our low energy field theoretic reduction to that obtained using a random phase approximation (RPA). While the RPA is a crude approximation, we believe that it will provide a rough estimate for the integrated spin response characterized by

$$\int d\omega \frac{\text{Im} \chi(\omega, k)}{w}.$$ (3.15)

Thus by equating the weight given by the two approaches, we are able to fix the values of the $A$’s. The details may be found in Appendix A.

### B. Spin Response at $U = t$

In this section we discuss the spin response at the relatively moderate value of $U = t$. At the top of Figure 1, we present an intensity plot from the field theoretic computation for the spin response function at 1% doping, both at $k_y = 0$ (the left panel) and $k_y = \pi$ (the right panel). The low energy spectral weight is found near values of $k_x$ related to $K_{Fab}$ and $K_{Fb}$, the bonding and anti-bonding wavevectors. As a function of doping, $\mu$, away from half-filling, the values of $K_{Fab}$ and $K_{Fb}$ are

$$K_{Fb} = \cos^{-1} \left[ \frac{-t_\perp + t_{2\perp}}{2t} + \frac{1}{2} \left( \frac{t_{2\perp}}{t^2} - 2 \frac{t_{2\perp}}{t^2} \sin^2(\pi \mu) \right)^{1/2} \right]$$

$$K_{Fab} = \pi(1 - \mu) - K_{Fb}. \quad (3.16)$$

At $k_y = \pi$, the low energy weight is present near $k_x = K_{Fab} + K_{Fb}, K_{Fb} - K_{Fab}, 2\pi - K_{Fab} - K_{Fb},$ and $2\pi - K_{Fab} + K_{Fb}$. We see that the greatest intensity is found at $k_x = K_{Fab} + K_{Fb}$ and $k_x = 2\pi - K_{Fb} - K_{Fab}$. Exactly at half-filling, these wave-vectors equal $\pi$, i.e. $K_{Fab} + K_{Fb} = \pi$. Thus as the system is doped the $(\pi, \pi)$ peaks splits into two incommensurate peaks. However at 1% doping, the splitting is barely resolved in Figure 1. For $k_y = \pi$, the spectral weight at $k_x = K_{Fab} - K_{Fb}$ is considerably smaller.

The splitting is barely resolved in Figure 1. For $k_y = 0$ at $k_x = 2K_{Fab}$, $2\pi - 2K_{Fab}, 2K_{Fb}, 2\pi - 2K_{Fb}$, and finally at $k_x = 0, 2\pi$. As we can see from the intensity scales, it is considerably less than that at $k_y = \pi$. We also see that the energy at which weight is first found is higher. This reflects the fact that the first contribution to the spectral weight is a two-particle contribution and so is at an energy a $\sqrt{2}$ times higher than that at $k_y = \pi$ where the initial contribution is made by a single coherent bound state.

In the lower and middle panels of Figure 1, we compare our field theoretic results for the spin response function both to the results for non-interacting ladders (see Appendix C) and to interacting ladders treated using an RPA approximation (see Appendix A 4). The RPA approximation sees a more diffuse distribution of spectral weight than that of the field theoretic treatment. In particular the RPA approximation does a poor job at deducing the existence of the contribution of a single bound state to the spin response near $(\pi, \pi)$. But both the RPA and field theoretic treatment predict a much larger low-energy response than that of non-interacting ladders. In this sense the ladders mimic the behavior of individual chains where the presence of interactions shifts spectral weight from high to low energies. The presence of low energy spectral weight in a single chain is captured by the Müller ansatz. It would be interesting if a similar ansatz might be developed for the spin response of doped ladders using our field theoretic treatment as a starting point.

To obtain a more quantitative understanding of the spin response, we have plotted constant wavevector cuts of the spectral function in Figure 2. Here we have done so at 10% doping so that the low energy spectral weight present near any of the values of $k_x$ related to $K_{Fab}$ or $K_{Fb}$ does not overlap with weight arising from a different such value of $k_x$. We first consider the cut at $K_{Fab} + K_{Fb}$. We see explicitly the onset of weight at the spin gap, $\Delta_s = \sqrt{2}t_n$, the excitation energy of the bound state. We note that contribution of the bound
FIG. 1: Plots of the spin response function computed in three different fashions: using SO(6) Gross-Neveu as a low-energy effective theory; RPA; and taking the ladders as non-interacting. The left-hand panels give the response at $k_y = 0$ while the right-hand panels correspond to $k_y = \pi$. In making these plots, we have assumed 1% doping and $U = t$. We have plotted $k_x$ in rationalized units of $2\pi$. 
FIG. 2: A number of constant wavevector cuts of the spin response function for 10% doping. We compare the computations done in the SO(6) Gross-Neveu framework with both non-interacting ladders and an RPA treatment.
state does not lead to a (theoretically) infinitely narrow feature in the spin response. While this would be so at half-filling, the gapless nature of the charge mode at finite doping smooths the feature out giving it a finite spread. Nonetheless there is a divergence at \((\omega^2-k^2/v_F^2) \to 0\) going as \((\omega^2-k^2/v_F^2)^{K/2-1}\). In the energy interval \((0,10n)\) we can also estimate the amount of spectral weight due to the single bound state. We find that it accounts for 37\% of the total spectral weight (with a Luttinger parameter, \(K = 0.93\)). Of this 37\%, 42\% of it occurs in the interval \((0, 2m)\), i.e. before the incoherent two kink contribution to the spectral function begins to be felt.

C. Spin Response at Larger U

As \(U\) becomes larger we expect that the Fermi velocity will renormalize significantly but that our effective low energy description will remain valid. This view is supported by the two loop RG computations of Fabrizio. There it was shown that the two-loop RG preserved the flow to SO(6) but that unlike one-loop RG equations, the Fermi velocity was renormalized downward.

To study the spin response at large \(U\) we thus suppose that the ratio \(v_F/m\) is much smaller than at weak coupling. The RPA breaks down at \(U\) in excess of \(t\) and so we cannot directly estimate the corresponding normalization constants, \(A\)'s. Instead we simply continue to use the values calculated for \(U = t\). We point out that while at best any comparison of low energy spectral weight at different wavevectors will then be qualitative, the spectral weight at any one value of \(k\) will be unaffected by this procedure.

In Figure 3 we plot the spin response for a value of \(U\) corresponding to \(v_F/m = 4\) and for three different values of doping, 5\%, 10\%, and 25\%. In comparison with the spin response in Figure 1 where \(v_F/m \gg 1\), the response at a given energy occurs over a much wider band of \(k_x\) values.

From Figure 3, we can observe how the spin response evolves as a function of doping. Two primary effects are observed. The first is increasingly significant incommensuration effects as doping is increased. Incommensuration effects are most noticeable near \((\pi, \pi)\), the region of greatest spin response. At small doping the sum of bonding and anti-bonding wavevectors remain close to \(\pi\) (see Eqn. (3.16)). Consequently at small dopings the spin response near \((\pi, \pi)\) appears akin to that at half-filling with only a single peak of intensity apparent. As doping is increased, however, two regions of intensity, one at \(K_{Fab}+K_{Fb}\) and one at \(2\pi-K_{Fab}-K_{Fb}\), become resolved. At 5\% doping, the two regions are becoming individually visible. As we increase doping further, the regions separate and become distinct over a wider and wider range of energy. In Figure 4, we plot the spin response in a narrow region about \((\pi, \pi)\) for the three different values of dopings. In this figure, the splitting of the \((\pi, \pi)\) peak as function of doping is more readily apparent.

An obvious signature of the incommensuration in a neutron scattering experiment would be found in a series of constant \(k\)-scans as a function of energy. In comparing such scans at \((\pi, \pi)\) and \((K_{Fab}+K_{Fb}, \pi)\), a greater intensity at the latter value of \(k_x\) would mark the presence of incommensuration.

Incommensuration effects will be less obvious at \(k_y = 0\) simply because the intensity present at this value of \(k_y = 0\) is markedly less than at \(k_y = \pi\). But the \(k_x\) points at which low energy intensity is found, i.e. \(k_x = 2K_{Fab}, 2K_{Fb}\), and 0 will shift relative to one another. This effect is convoluted with changing relative intensity at \(k_x = 2K_{Fab}\) and \(k_x = 2K_{Fb}\). For smaller dopings (i.e. 5\% and 10\%), the intensities found near these two points is roughly equal. For larger dopings, however, the intensity at \(2K_{Fab}\) becomes notably larger. In the Figure 3 panel for \(k_y = 0\) at 25\% doping, the contribution at \(k_x = 2K_{Fab}\) and \(k_x = 0\) overlap, while the intensity at \(k_x = 2K_{Fb}\) is not observable in the plot at hand.

The second notable effect of doping is that with increasing doping one sees a decrease in maximum intensity at \(k_y = \pi\) relative to that at \(k_y = 0\). At 5\% doping, the relative intensity at \(k_y = \pi\) is twelve times that at \(k_y = 0\). At 25\% doping, this ratio has been reduced from twelve to two.

D. Effects of Integrability Breaking Perturbations

In this section we consider the effects that integrability breaking perturbations will have upon our results. As we discussed in the introduction, this is important to consider because the renormalization group flow leaves a generic system more symmetric but does not necessarily render the low energy sector exactly equivalent to the SO(6) Gross-Neveu model. As was done for the SO(8) Gross-Neveu description of half-filled ladders, we consider this question in broad terms so as to demonstrate that small perturbations away from the SO(6) Gross-Neveu model do not drastically change the results. (For a more detailed analysis for general ladder models see Ref. 14.)

With this in mind we examine general possible perturbations of a U(1) Luttinger liquid \(\otimes\) SO(6) Gross-Neveu model. Possible perturbations can be read off from Eqn. (3.13), the Hamiltonian of the doped ladders with generic (bare) interactions. These interactions all take the form of

\[
H_{\text{pert}} = \lambda \cos(\theta_i) \cos(\theta_j), \ i \neq j
\]  

or

\[
H_{\text{pert}} = \lambda \cos(\varphi_3) \cos(\theta_j). \tag{3.18}
\]

In general these interaction preserve the charge, spin, and relative band chirality but not the difference in the z-component of spin between bands (owing to the presence of \(\cos(\varphi_3)\)).
FIG. 3: Plots of the spin response function for three different dopings, 5%, 10%, and 25%. The left-hand panels give the response at $k_y = 0$ while the right-hand panels correspond to $k_y = \pi$. These plots are made assuming a strongly renormalized Fermi velocity, $v_F/m = 4$. 
To determine the manner in which the above type of perturbations will affect the spectrum of the model is straightforward. We can immediately note that the various multiplets of SO(6) Gross Neveu will be split by the perturbation, but the single particle multiplets themselves will not be mixed. We expect also that the notion of single particle excitations will remain robust – no perturbation in Eqn. (3.19) will lead to kink confinement. Moreover no perturbation will gap out the U(1) Luttinger liquid.

To determine the splitting of any given multiplet of particles $\{A_i(\theta)\}$, we employ stationary state perturbation theory. In this sense we operate in the same spirit as Ref. (31) in treating the off-critical Ising model in a perturbation theory. In this sense we operate in the same spirit as Ref. (31) in treating the off-critical Ising model in a perturbation theory. Not only then does this correction leave unaffected the behavior of all relevant matrix elements at threshold appearing in the spin perturbation, but any matrix element involving $|O_0\rangle$ vanishes, for example

$$\langle j|H_{\text{pert}}(x,t)|i\rangle = -\langle j|H_{\text{pert}}(x,t)|i\rangle,$$

(3.21)

where $\sum_j$ represents schematically a sum over all states appearing in a resolution of the identity. We argue below that this correction will uniformly vanish for matrix elements, $|i\rangle$, involving two kinks or anti-kinks with minimal energy, 2$m$, and for which the matrix element, $\langle 0|O(0,0)|i\rangle$, itself vanishes. This implies that the leading correction leaves unaffected the behavior of all relevant matrix elements at threshold appearing in the spin response.

In the relevant cases where the matrix element itself vanishes, the vanishing is a consequence of the structure of the low energy S-matrix. The S-matrix at vanishing energies of two kinks is $-P$, where $P$ is the permutation matrix. Not only then does $\langle 0|O(0,0)|i\rangle$ vanish, but any matrix element involving $|i\rangle$ vanishes, for example $\langle j|H_{\text{pert}}(x,t)|i\rangle$ and $\langle 0|H_{\text{pert}}(x,t)|i\rangle$. The vanishing comes about through an application of the scattering axiom (see Section IV B). This axiom implies the following relation,

$$\langle j|H_{\text{pert}}(x,t)|i\rangle = -\langle j|H_{\text{pert}}(x,t)|i\rangle,$$

(3.22)

This then necessarily implies the matrix element vanishes regardless of the nature of $|j\rangle$ or $H_{\text{pert}}$. The computation of these matrix elements is straightforward in the context of integrability. Each matrix element is no more than a form factor which can easily be computed in the same fashion as done in Section IV. Moreover because the theory is massive, perturbation theory is controlled and so small perturbations should only introduce small quantitative changes to our results.

Deviations in the mass spectrum will have a two fold effect. The bound state producing the peak in the spin response at $\omega = \sqrt{2}m$ and $k = K_{F_{ab}} + K_{F_{b}}$ will, under the perturbation, be mixed into other members of the vector multiplet, leading to a number of particles that will couple to the spin operator. Consequently the peak will split. This splitting however will be obscured by the gapless charge excitations with their tendency to smear any sharp spectral features. This smearing will be enhanced by finite temperature. Unless the splitting is then large, it is unlikely the effects of the perturbation will be detectable experimentally.

Like the splitting of the multiplet of bound states, the two particle threshold will similarly split. Rather than occur at $\omega = 2m$, a number of thresholds will appear about this energy. This effect will be muted at points in k-space where the threshold opens smoothly from zero rather than appearing as a singularity. But the effect will be more pronounced at $k_x = 0, k_y = 0$ and $k_x = K_{F_{b}} - K_{F_{ab}}, k_y = \pi$, where the two particle threshold is marked by a van-Hove like square root singularity. However the relative weight of spectral intensity is small at these points (in comparison to $k_x = K_{F_{ab}} + K_{F_{b}}, k_y = \pi$) and so even here any splitting will presumably be difficult to detect.

We can also consider the effects of integrable breaking perturbations upon matrix elements. The behavior of matrix elements near threshold determine whether a threshold opens up smoothly (for example the two particle matrix element vanishing at threshold, i.e. $\langle 0\mid H^2/v_{\text{pert}}^2 \rangle \rightarrow 0$, extinguishing a van-Hove singularity) or whether it is marked by a discontinuity (the matrix element is finite at threshold). Small perturbations may then lead a vanishing matrix element to be finite forcing a qualitative change in the physics. However, we now argue that this does not happen at leading order in the perturbation.

To compute the leading order correction of a matrix element of some operator, $O(x = 0, t = 0)$, we employ the relation (arrived at from Keldysh perturbation theory)

$$\delta\langle 0|O(0,0)|i\rangle = -i \int_{-\infty}^{0} dt \int_{-\infty}^{0} dx \langle 0|O(0)\{H_{\text{pert}}(x,t)|i\rangle. \quad (3.20)$$

Here $|0\rangle$ is the vacuum state and $|i\rangle$ is some other state with some arbitrary number of particles. If we employ a form-factor expansion to evaluate the expectation values involving operator bilinears in the above expression we obtain

$$\delta\langle 0|O(0,0)|i\rangle = -i \int_{-\infty}^{0} dt \int_{-\infty}^{0} dx \sum_{j} \left( \langle 0|O(0)|j\rangle \langle j|H_{\text{pert}}(x,t)|i\rangle - \langle 0|H_{\text{pert}}(x,t)|j\rangle \langle j|O(0)|i\rangle \right), \quad (3.21)$$

(3.22)
IV. FORM FACTORS FOR THE SO(6) GROSS-NEVEU MODEL

A. S-matrices

In this section we lay out the S-matrices for scattering between the kinks. From these, we will be able to develop formulæ for the various form-factors needed to compute the spin-spin ladder correlation functions.

1. SO(6) Spinor Representations

The spinor representations are expressed in terms of SO(6) γ-matrices. The γ-matrices in turn are built out of the two-dimensional Pauli matrices, σi’s. As we are interested in SO(6), we consider three copies of the σi’s, σ1, σ2, σ3, each acting on a different two-dimensional space, i.e.:

\[ \sigma_2 |\alpha_1, \alpha_2, \alpha_3 \rangle = |\alpha_1, \alpha_2, \sigma_2 \alpha_3 \rangle. \]  

(4.2)

A particular basis vector, |α⟩, in the corresponding 8-dimensional vector space can be labeled by a series of three numbers \( \pm \frac{1}{2} \), i.e.:

\[ |\alpha⟩ \equiv |\alpha_1, \alpha_2, \alpha_3⟩ = |± \frac{1}{2}, ± \frac{1}{2}, ± \frac{1}{2}⟩. \]  

(4.3)

Physically, the kink associated with |α⟩ then carries quantum numbers \( N_i = ± \frac{1}{2} \) for the three U(1) symmetries corresponding to the Cartan elements of SO(6), discussed in the previous section.

In terms of the \( \sigma^i \)'s, the γ-matrices are defined by (following the conventions of Ref. 33),

\[ \gamma_{2n-1} = \sigma_n^1 \otimes \prod_{k=1}^{n-1} \sigma_3^k. \]  

(4.1)

for \( 1 \leq n \leq 3 \). These matrices satisfy the necessary Clifford algebra:

\[ \{ \gamma_a, \gamma_b \} = 2 \delta_{ab}. \]  

(4.5)

In this representation the Clifford-algebra generators, \( \gamma_n \), are imaginary and antisymmetric for \( n \) even, while for \( n \) odd, they are real and symmetric. The SO(6)-generators are represented by

\[ \sigma_{ab} = \frac{1}{2} (\gamma_a \gamma_b - \gamma_b \gamma_a), \]  

(4.6)

in analogy with the more familiar SO(4) case.

The 8-dimensional space of the γ’s decomposes into two 4-dimensional spaces, each of which forms one of the two irreducible SO(6) spinor representations. The decomposition is achieved explicitly via the hermitian chirality operator, Γ,

\[ \Gamma = (-i)^3 \prod_{a=1}^{6} \gamma_a = \sigma_1^1 \otimes \sigma_2^3 \otimes \sigma_3^3. \]  

(4.7)

Γ is such that it commutes with all SO(6) generators and is diagonal with eigenvalues ±1. If |α⟩ is a state with an even (odd) number of negative components (i.e. states with positive or negative isotopic chirality), then

\[ \Gamma |\alpha ±⟩ = ± |\alpha ±⟩. \]  

(4.8)

Thus the operators \( (1 ± \Gamma)/2 \) project onto the two irreducible subspaces. When it is necessary to make a distinction, we will denote kinks with an even number of negative components by \( \alpha \) and those with an odd number by \( \bar{\alpha} \).

The last item to be presented in this section is the charge conjugation matrix, \( C \). In terms of the γ’s, \( C \) is given by

\[ C = \gamma_1 \gamma_3 \gamma_5. \]  

(4.9)
C is completely off-diagonal (as expected). Moreover we note that C is skew symmetric \((C^T = -C, \; C^2 = -1)\). This fact will have important consequences in the determination of the form factors.

If \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\) is an even kink its corresponding anti-particle is denoted by \(\bar{\alpha} = (-\alpha_1, -\alpha_2, -\alpha_3)\). By the skew-symmetry of \(C\), we thus have

\[
|\bar{\alpha}\rangle = (-1)^{\alpha_2-1/2}C_{\alpha\beta}|\beta\rangle. \tag{4.10}
\]

2. S-Matrices for the Kinks

In order to describe factorizable scattering, we introduce Faddeev-Zamolodchikov (FZ) operators, \(A_n^\dagger(\theta)\), that create the elementary kinks. \(\theta\) is the rapidity which parameterizes a particle’s energy and momentum:

\[
P(\theta) = m \sinh(\theta) \quad \text{and} \quad E(\theta) = m \cosh(\theta). \tag{4.11}
\]

Because the \(SO(6)\) Gross-Neveu model is integrable, scattering is completely encoded in the two-body S-matrix. This S-matrix, in turn, is parameterized by the commutation relations of the Faddeev-Zamolodchikov operators:

\[
A_1^\dagger(\theta_1)A_2^\dagger(\theta_2) = S_{12}^{34}(\theta_{12})A_3^\dagger(\theta_2)A_4^\dagger(\theta_1). \tag{4.12}
\]

\(S_{12}^{34}(\theta_{12})\) is the amplitude of a process by which particles 1 and 2 scatter into 3 and 4. It is a function of \(\theta_{12} \equiv \theta_1 - \theta_2\) by reason of Lorentz invariance.

Kink-kink scattering takes the form

\[
S_{\alpha\beta}^{\gamma\delta}(\theta) = \frac{1}{8} \sum_{r=0}^{6} u_r(\theta) \sigma_{\alpha\beta}^{(r)} \sigma_{\gamma\delta}^{(r)}, \tag{4.13}
\]

where \(\sigma^{(r)}\) is a rank-\(r\) antisymmetric tensor:

\[
\sigma_{\alpha\beta}^{(r)} \equiv \sigma_{\alpha_1 \cdots \alpha_r} = (\gamma^{a_1} \cdots \gamma^{a_r})_A. \tag{4.14}
\]

Here A represents a complete anti-symmetrization of the gamma matrices. By \(\sigma_{\alpha\beta}^{(r)} \sigma_{\gamma\delta}^{(r)}\), we mean a trace over all possible rank-\(r\) antisymmetric tensors:

\[
\sigma_{\alpha\beta}^{(r)} \sigma_{\gamma\delta}^{(r)} = \sum_{a_{1 \cdots a_r}} \sigma_{\alpha\beta} \sigma_{\gamma\delta} \sigma_{\alpha_1 \cdots \alpha_r}. \tag{4.15}
\]

The generic form of \(S_{\alpha\beta}^{\gamma\delta}\) was determined in Ref. 33, while the specific forms of the \(u_s\’s\) were given in Ref. 33. There it was found

\[
u_{3+r}(\theta) = (-1)^r u_{3-r}(\theta);
\]

\[
u_{r+2}(\theta) = u_r(\theta) \left( \frac{1 - r^2}{2} - \left(1 + i\theta/\pi\right) \frac{1}{1 - r^2} + \left(1 + i\theta/\pi\right) \right);
\]

\[
u_2(\theta) = \frac{\Gamma(1 - \theta/2i\pi)\Gamma(\theta/2i\pi - 1/4)}{\Gamma(3/4 - \theta/2i\pi)\Gamma(\theta/2i\pi)};
\]

\[
u_1(\theta) = \frac{3/2 - \theta/\pi \Gamma(1/2 + \theta/2i\pi)}{1 - \theta/\pi \Gamma(1/2 - \theta/2i\pi)}
\]

We note that there is a sign difference in the definition of \(u_1\) between Ref. 33 and Eqn. (4.16). This sign difference arises as we give here the S-matrices of the physical particles (those with fractional statistics). \(S_{\alpha\beta}^{\gamma\delta}(\theta)\) satisfies a Yang-Baxter equation,

\[
S_{\alpha\beta}^{\gamma\delta}(\theta_{12})S_{\gamma\delta}^{\alpha\beta}(\theta_{13})S_{\gamma\delta}^{\alpha\beta}(\theta_{23}) = S_{\alpha\beta}^{\gamma\delta}(\theta_{23})S_{\gamma\delta}^{\alpha\beta}(\theta_{13})S_{\gamma\delta}^{\alpha\beta}(\theta_{12}). \tag{4.17}
\]

Physically, the Yang-Baxter equation encodes the equivalence of different ways of representing three-body interactions in terms of two-body amplitudes. Formally, it expresses the associativity of the Faddeev-Zamolodchikov algebra. The S-matrix, \(S_{\alpha\beta}^{\gamma\delta}(\theta)\), also satisfies both a crossing,

\[
S_{\alpha\beta}^{\gamma\delta}(i\pi - \theta) = S_{\alpha\beta}^{\gamma\delta}(\theta)C_{\gamma\delta}C_{\alpha\beta}. \tag{4.18}
\]

and unitarity relation,

\[
S_{\alpha\beta}^{\gamma\delta}(\theta)S_{\gamma\delta}^{\alpha\beta}(-\theta) = \delta_{\alpha\alpha} \delta_{\beta\beta}. \tag{4.19}
\]

We note that for the above kink-kink S-matrices, crossing is satisfied, as expected, with \(C = \gamma_5 \gamma_3\).

These constraints determine \(S_{\alpha\beta}^{\gamma\delta}\) up to a ‘CDD’-factor. Such factors allow additional poles to be added, if needed, in the physical strip, \(\text{Re}(\theta) = 0, \; 0 < \text{Im}(\theta) < 2\pi\), to the scattering matrix and so are indicative of bound states. Here we expect to find a pole in \(u_n\), \(n\) even, at \(\theta = i\pi/2\) reflecting the fact that two (same chirality) kinks can form a bound state (the fundamental fermions of the model) of mass \(\sqrt{2m}\). The overall sign of the S-matrix is determined by examining the residue of the pole in \(S_{\alpha\beta}^{\gamma\delta}\) at \(\theta = i\pi/2\). This pole is indicative of the formation of a mass \(\sqrt{2m}\) bound state in the s-channel and so should have positive imaginary residue.

The \(SO(6)\) Gross-Neveu model has an isotopic chirality conservation law. Thus opposite chirality kink scattering is determined solely by \(u_n\), \(n\) odd, while same chirality kink scattering is determined solely by \(u_n\), \(n\) even. For example, it can be shown that for even chirality kinks, the S-matrix reduces to

\[
S_{\alpha\beta}^{\gamma\delta}(\theta) = -u_2(\theta)\delta_{\gamma\delta} \delta_{\alpha\beta} + \frac{1}{4}(u_0(\theta) + u_2(\theta))\delta_{\gamma\delta}\delta_{\alpha\beta}. \tag{4.20}
\]

B. Basic Properties: Two Particle Form Factors

The two particle form factors of a field \(\psi(x)\) are defined as the matrix elements,

\[
f_{12}^\psi(\theta_1, \theta_2) = \langle 0 | \psi(0) A_2^\dagger(\theta_2) A_1^\dagger(\theta_1) | 0 \rangle. \tag{4.21}
\]
The form of $f^\psi_{12}(\theta_1, \theta_2)$ is constrained by integrability, braiding relations, Lorentz invariance, and hermiticity.

The constraint coming from integrability arises from the scattering of Faddeev-Zamolodchikov operators. For Eq. (4.22) to be consistent with Eq. (4.23), we must have

$$f^\psi_{12}(\theta_2, \theta_1) = S^\psi_{12}(\theta_{12}) f^\psi_{21}(\theta_1, \theta_2).$$  (4.22)

The second constraint can be thought of as a periodicity axiom. It reads

$$f^\psi_{12}(\theta_2, \theta_1) = Q_{12} R_{\psi_1} f^\psi_{21}(\theta_1 - 2\pi i, \theta_2).$$  (4.23)

Both $Q_{12}$ and $R_{\psi_1}$ are phases. $Q_{12}$ arises from the charge conjugation matrix in SO(6) Gross-Neveu being skew-symmetric. Its determination is discussed in some detail in Appendix B. The phase factor $R_{\psi_1}$, on the other hand, arises from the “semi-locality” of the fields in the SO(6) Gross-Neveu model. Working in Euclidean space, fields can be said to be semi-local if their product sees a phase change when the fields are taken around one another in the plane via the analytic continuation $z \rightarrow ze^{i2\pi}, \bar{z} \rightarrow ze^{-i2\pi}$. More specifically we have

$$\psi_1(z e^{i2\pi}, \bar{z} e^{-i2\pi}) \psi(0) = R_{\psi_1} \psi_1(z, \bar{z}) \psi(0).$$  (4.24)

Here $\psi_1$ is the field that is associated with the particle $A^i_1(\theta_1)$. The phase $R_{\psi_1}$ can be calculated from the operator product expansions of the operator $\psi$ with bosonic vertex operator representations of the kink fields. If $A_1$ is a kink, $|\alpha\rangle = (\pm 1/2, \pm 1/2, \pm 1/2)$, the corresponding representation of the field is

$$\psi_1 = e^{\pm \phi_1 \pm \phi_2 + \phi_3}$$  (4.25)

In contrast to the representation of the left and right components of the lattice fermions (Eqs. (4.22)-(4.24)), these fields are non-chiral. The form factor, $f^\psi_{12}$, must also satisfy Lorentz covariance. If $\psi$ has Lorentz spin, $s$, $f^\psi_{12}$ will take the form (at least in the cases at hand),

$$f^\psi_{12}(\theta_1, \theta_2) = e^{s(\theta_1 + \theta_2)/2} f(\theta_{12}),$$  (4.26)

where by virtue of $\theta_{12} \equiv \theta_1 - \theta_2$, $f$ is a Lorentz scalar.

The constraints Eqs. (4.22), Eq. (4.23), and Eq. (4.24), do not uniquely specify $f^\psi_{12}$. It is easily seen that if $f^\psi_{12}$ satisfies these axioms then so does

$$f^\psi_{12} g(\cosh(\theta_{12})), \text{ where } g(x) \text{ is some rational function.}$$

The strategy is then as follows. One first determines the minimal solution to the constraints, minimal in the sense that is has the minimum number of zeroes and poles in the physical strip, $\text{Re}(\theta) = 0, 0 < \text{Im}(\theta) < 2\pi$. Then one adds poles according to the theory’s bound state structure. If the S-matrix element describing scattering of particles 1,2 has a pole at $\theta = iu$, then $f^\psi_{12}(\theta_1, \theta_2)$ has a pole at

$$\theta_2 = \theta_1 + iu.$$  (4.27)

Insisting that $f^\psi_{12}$ has such poles and only such poles fixes $g(x)$ up to a constant.

The phase of this constant can be readily determined. Appealing to hermiticity gives us

$$\langle 0|\psi(0)A^1_1(\theta_2) A^1_1(\theta_1)|0\rangle^* = \langle 0|A_1(\theta_1) A_2(\theta_2) \psi(0)|0\rangle$$

$$= C_{11 \bar{C}}_{22} \langle 0|\psi(0)A^1_1(\theta_1 - i\pi) A^1_2(\theta_2 - i\pi)|0\rangle,$$  (4.28)

where the last line follows by crossing and so

$$f^\psi_{12}(\theta_1, \theta_2)^* = C_{11} \bar{C}_{22} f^\psi_{21}(\theta_2 - i\pi, \theta_1 - i\pi).$$  (4.29)

For the kinks the matrix $C$ equals the standard charge conjugation matrix, $C = \gamma_1 \gamma_3 \gamma_5$, while for the Majorana fermions $\bar{C} = 1$. (We note however that $C$ for the Majorana fermions is not identically 1. This difference between $C$ and $\bar{C}$ for the Majorana fermions arises because under Hermitean conjugation these fermions are invariant while under charge conjugation they undergo sign changes.) For the form factors we will examine, hermiticity will be enough to fix their overall phase.

As a final consideration we examine the large rapidity ($\theta \rightarrow \infty$) asymptotics of a form factor. These asymptotics are governed by the (chiral) conformal dimension, $\Delta$, of the operator appearing in the form factor. In particular, if we write

$$\lim_{|\theta| \rightarrow \infty} f^\psi(\theta_1, \theta_2) \sim e^{y^\psi|\theta|^\Delta},$$  (4.30)

the quantity, $y^\psi$, is bounded from above by $\Delta$. This bound will prove useful in checking the end result.

C. Basic Properties: One Particle Form Factors

One particle form factors are in a sense trivial; Lorentz covariance completely determines their form. If $\psi(x)$ has Lorentz spin, $s$, then

$$f^\psi_1(\theta) = \langle 0|\psi(0) A^i_1(\theta)|0\rangle = ce^{s\theta},$$  (4.31)

where $c$ is some constant. To determine $c$ we use the theory’s bound state structure.

If a particle $A_3$ is a bound state of two particles $A_1$ and $A_2$ we can represent it formally as product of two Faddeev-Zamolodchikov operators:

$$i g^3_{12} A^3_1(\theta) = \text{res}_{\delta=0} A^1_1(\theta + \delta + iv_{13}^2) A^1_2(\theta - iv_{23}^1).$$  (4.32)

Here $\text{res}_{\delta=0}$ denotes the residue at $\delta = 0$ and $\bar{I}$ is the charge conjugate particle of 1. Implicit to this relation is the particle normalization $\langle \theta|\theta'\rangle = 2\pi \delta(\theta - \theta')$. The $\bar{u}$’s are given by $\bar{u}_{13}^2 = \pi - u_{13}^2$, where $u_{13}^2$ is the location of the bound state pole of the S-matrix for particles 1,2 indicative of particle 3. $g^3_{12}$ is the amplitude to create particle 1 from particles 2 and 3. $g^3_{12}$ is defined by

$$S^3_{12}(\theta) \sim i g^3_{12} g^3_{12}^\psi \frac{\gamma_3}{\theta - i\gamma_3 v_{12}}.$$  (4.33)
Then we have
\[ i g^3 \partial_x \partial_y (\theta) = \text{res}_{\theta=0} f_{12}^\psi (\theta - i \bar{u}_2 \gamma_1, \theta + \delta + i \bar{u}_1 \gamma_1). \] (4.34)

Thus knowledge of the two particle form factors completely determines their one-particle counterparts.

D. Two particle form factors for the SO(6) currents

There are two possible two particle form factors for the SO(6) currents, one involving two fermions,
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = (C_{\alpha}^{ab} (0) A_\beta^a (\theta_2) A_\beta^b (\theta_1)), \] (4.35)
and one involving two kinks,
\[ \mu f_{\alpha \beta}^{\text{ka}} (\theta_1, \theta_2) = (C_{\alpha}^{ab} (0) A_\beta^a (\theta_2) A_\beta^b (\theta_1)), \] (4.36)
where the two kinks have opposite chirality. The contribution of the former to the spectral function is only felt at energies \( \omega > 2^{1/2} \Delta \) while the contribution of the second is seen at \( \omega > 2 \Delta \). We thus focus on the latter.

We begin by identifying the group theoretical structure of \( \mu f_{\alpha \beta}^{ab} \):
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = (C_{\sigma}^{ab} (0) f_{\mu} (\theta_1, \theta_2)), \] (4.37)
where \( C \) is the charge conjugation matrix introduced previously. \( (C_{\sigma}^{ab})_{\alpha \beta} \) is antisymmetric in \( a, b \). Now \( (C_{\sigma}^{ab})_{\alpha \beta} \) is not the only obvious choice of tensor. One also has \( (\sigma^{ab} C)_{\alpha \beta} \), antisymmetric combinations built up out of \( \gamma^a C \gamma^b \) or \( C \gamma^a \gamma^b C \), or some combination of all three (but not \( \sigma^{ab} \) as this choice violates obvious U(1) conservation). Perhaps the most natural choice is to make \( \mu f_{\alpha \beta}^{ab} \) explicitly symmetric in \( C \):
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = ((C_{\sigma}^{ab})_{\alpha \beta} + (\sigma^{ab} C)_{\alpha \beta}) f_{\mu} (\theta_1, \theta_2). \] (4.38)
However this forces \( \mu f_{\alpha \beta}^{(2a-1)(2b)} \) to zero, something we do not necessarily expect. We instead arrive at the choice as given above by explicitly checking the other possibilities for invariance under SO(6). Of the choices only \( (C_{\sigma}^{ab})_{\alpha \beta} \) is consistent as such.

Having specified the coupling of a kink+anti-kink to \( G_{\mu}^{ab} \), we use C-symmetry to specify the corresponding anti-kink+kink form-factor \( \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) \). Under the action of the charge conjugation matrix, \( C \), the currents \( G_{\mu}^{ab} \) transform as
\[ CG_{\mu}^{ab} C^\dagger = (-1)^{a+b} G_{\mu}^{ab}. \] (4.39)
We thus expect
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = -(-1)^{a+b} (-1)^{\bar{a}+\bar{b}} \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2). \] (4.40)
An ansatz for \( \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) \) satisfying this condition is
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = (C_{\sigma}^{ab})_{\alpha \beta} f_{\mu} (\theta_1, \theta_2). \] (4.41)
In particular we note the absence of any phase in the expression for \( \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) \) relative to that of \( \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) \).

As before Lorentz invariance demands
\[ f_{\mu} (\theta_1, \theta_2) = (e^{(\theta_1+\theta_2)/2} - (-1)^{\mu} e^{-(\theta_1+\theta_2)/2}) f_{\theta_12}. \] (4.42)
Then by the scattering axiom, \( f_{\theta_12} \) must satisfy
\[ f(-\theta) = u_1 (\theta) \frac{1 - \theta/i\pi}{3/2 - \theta/i\pi} f(\theta), \]
\[ = \exp \left[ \int_0^\infty \frac{dx}{x} G_c (x) \sinh \left( \frac{\theta x}{i\pi} \right) \right] f(\theta); \]
\[ G_c (x) = e^{\gamma/2} - 1 \sinh (x). \] (4.43)
The periodicity axiom takes the form
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = -\mu f_{\alpha \beta}^{ab} (\theta_1 - 2\pi i, \theta_2). \] (4.44)
The minus sign arises from \( Q_{\alpha \beta} R_{\alpha \beta} G_{ab} = -1 \). As shown in Appendix B, \( Q_{\alpha \beta} = -1 \) and \( R_{\alpha \beta} G_{ab} = 1 \) for neutral currents and vice-versa for currents carrying charge.

In terms of the scalarized form factor, \( f \), the periodicity axiom then reads
\[ f(-\theta) = f(\theta - 2\pi i). \] (4.45)
Two kinks of opposite chirality do not form a bound state. Thus the form factor should have no poles in the physical strip. Thus with Eqs. (4.42), (4.43), and (4.44), the form factor is
\[ \mu f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = \]
\[ = A_c ((C_{\sigma}^{ab})_{\alpha \beta} (e^{(\theta_1+\theta_2)/2} - (-1)^{\mu} e^{-(\theta_1+\theta_2)/2}) \]
\[ \times \exp \left[ \int_0^\infty \frac{dx}{x} G_c (x) \sin^2 \left( \frac{x}{2\pi} (i\pi + \theta_{12}) \right) \right]. \] (4.46)
The hermiticity condition for this current form-factor reads
\[ f_{\alpha \beta}^{ab} (\theta_1, \theta_2) = f_{\alpha \beta}^{ab} (\theta_2 - i\pi, \theta_1 - i\pi) G_{\alpha \beta} G_{\alpha \beta} \] (4.47)
Applying this constraint shows that \( A_c \) is some purely imaginary constant.

E. Two particle form factors for SO(6) symmetric tensor fields

Here we will compute the two particle form factor coupling to an operator that transforms as a scalar under Lorentz transformations and as the 10 dimensional symmetric representation under SO(6) (we indicate this operator as \( \psi^{(abc)} \) where \( a, b, c \) are anti-symmetrized). We also assume that the operator’s conformal dimension is \( \Delta = 3/8 \). Such an operator arises in the computation of the spin response for the ladders when considering
fermion bilinears of the form $c^\dagger_{R_{ab}} c_{L_{bc}}$ or antisymmetric combinations $c^\dagger_{R_{ab}} c_{L_{bc}} - c^\dagger_{R_{ab}} c_{L_{bc}}$. Each of the fermions transforms under the $4$ of $SO(6)$ ($SU(4)$). These bilinears are then in the $10$ appearing in $4_{R} \otimes 4_{L}$. Here the elements of $4_{R}$ are $(c_{R_{ab}}, c_{R_{bc}}, c_{R_{ba}})$ while the elements of $4_{L}$ are $(c^\dagger_{L_{bc}}, c^\dagger_{L_{bc}}, c^\dagger_{L_{ab}}, c^\dagger_{L_{bc}})$. We can see, for example, that $c^\dagger_{R_{ab}} c_{L_{bc}}$ lies in the $10$ as in its bosonized form,

$$c^\dagger_{L_{bc}} c_{R_{ab}} \sim e^{i(\theta_{a} - \theta_{b} - \theta_{c} + \phi_{+} + \phi_{-})} = e^{i(\theta_{1} - \phi_{2} + \phi_{3} - \phi_{4})},$$

it carries all three of the quantum numbers in $SO(6)$. In considering $\psi^{[abc]A}$, we rewrite the $SO(6)$ fields appearing in the bosonization as

$$e^{i\pm \phi/2} \sim \chi_{2} \pm i\chi_{1},$$

and so work in a Majorana like basis. In this basis we need to know how to construct the $10$ out of the $6$. From Ref. [46] we know the antisymmetric product of three $6$'s yields the $10$. We thus denote this field as $\psi^{[abc]A}$ where the $A$ indicates anti-symmetrization.

At the two particle level, two kinks of the same chirality will couple to such an operator. We will consider for the moment $\psi^{[abc]A}$ coupling to two even chirality kinks. This then leads to form factors of the type

$$f^{[abc]A}(\theta_{1}, \theta_{2}) = \langle \psi^{[abc]A}A_{\alpha}(\theta_{2})A_{\beta}(\theta_{1}) \rangle. \quad (4.48)$$

$SO(6)$ covariance suggests that $f^{[abc]A}(\theta_{1}, \theta_{2})$ take the form

$$f^{[abc]A}(\theta_{1}, \theta_{2}) = \left( c_{1}(C_{\alpha}^{[a}[\gamma^{\alpha}c^{\beta}c^{\gamma}]_{\alpha})_{\alpha\beta} + c_{2}(\gamma^{[a}C^{\beta}\gamma^{\gamma]c^{c}]_{\alpha\beta} + c_{3}(\gamma^{[a}c^{\beta}C^{\gamma]c^{c}]_{\alpha\beta} + c_{4}(\gamma^{[a}c^{[b]}c^{\gamma]c^{c}]_{\alpha\beta} f(\theta_{12}). \quad (4.49)$$

Here the $f^{[abc]A}$ is a function of $\theta_{1} - \theta_{2}$ because $\psi^{[abc]A}$ is a Lorentz scalar.

We point out that the $SO(6)$ tensors in the above expression are symmetric in $\alpha$ and $\beta$. To fix the values of the various coefficients we have two arguments available to us. By checking explicitly for $SO(6)$ covariance we see that all of the coefficients but $c_{1}$ are zero. We can also fix the values of $c_{1}$ relying on U(1) charge conservation alone. Suppose $|\alpha \rangle = (1/2, \pm 1/2, \pm 1/2)$ and $|\beta \rangle = (1/2, \pm 1/2, \pm 1/2)$. Then by rewriting the Majorana fields as Dirac fields, one readily finds that

$$f^{[abc]A}_{\alpha\beta} = -i f^{[abc]A}_{\beta\alpha}, \quad (4.50)$$

provided $b, c = 3, 4, 5, 6$. As can be checked, this yields $c_{2} = c_{3} = c_{4} = 0$. We thus obtain in the end

$$f^{[abc]A}(\theta_{1}, \theta_{2}) = (C_{\alpha}^{[a}[\gamma^{\alpha}c^{\beta}c^{\gamma}]_{\alpha})_{\alpha\beta} f(\theta_{1} - \theta_{2}). \quad (4.51)$$

where we have absorbed a constant into $f(\theta)$.

To determine $f(\theta)$ we apply the scattering and periodicity axioms. The scattering axiom constrains $f(\theta)$ as follows:

$$f(-\theta) = \left( \frac{1}{4} u_{0}(\theta) - \frac{3}{4} u_{2}(\theta) \right) f(\theta)$$

$$= -\frac{\Gamma(3/4 + \theta/2i\pi)}{\Gamma(3/4 - \theta/2i\pi)} \frac{\Gamma(1 - \theta/2i\pi)}{\Gamma(1 + \theta/2i\pi)} f(\theta)$$

$$= -\exp \left[ \int_{0}^{\infty} \frac{dx}{x} G_{s}(x) \sinh(x\theta/i\pi) \right] f(\theta);$$

$$G_{s}(x) = -2e^{-3x/4}\sinh(x/4) \sin(x). \quad (4.52)$$

The periodicity axioms reads

$$f(-\theta) = Q_{\alpha\beta} R_{\alpha\beta} f(\theta - 2\pi i). \quad (4.53)$$

Both phases, $Q_{\alpha\beta}$ and $R_{\alpha\beta}$, are equal to $1$ (Appendix B). With this we arrive at the following expression for $f(\theta)$

$$f(\theta) = \sinh(\theta/2) \exp \left( \int_{0}^{\infty} \frac{dx}{x} G_{s}(x) \sin^{2}(x/2i\pi) \right). \quad (4.54)$$

Although two kinks can form a bound state, they do so in the antisymmetric (vector) channel and so do not couple to $\psi^{[abc]A}$. Thus there is no need to add additional poles to the form factor. We then can put everything together leading to

$$f^{[abc]A}_{\alpha\beta}(\theta_{1}, \theta_{2}) = A_{\alpha}(\gamma^{[a}c^{\beta}\gamma^{c}]C)_{\alpha\beta} \sinh(\theta_{12}/2)$$

$$\times \exp \left[ \int_{0}^{\infty} \frac{dx}{x} G_{s}(x) \sin^{2}(x/2i\pi) \right], \quad (4.55)$$

where $A_{\alpha}$ is a real constant (as determined by hermiticity).

The UV asymptotics of this form factor are easily found to be

$$\lim_{\theta_{1} \rightarrow \infty} f^{[abc]A}_{\alpha\beta}(\theta_{1}, \theta_{2}) \sim e^{2\pi \theta_{1}}. \quad (4.56)$$

Given that the conformal dimension of $f^{[abc]A}$ is $\Delta = 3/8$, we see that the UV behavior of this form factor saturates its upper bound.

We now return to the situation of two odd kinks coupling to $f^{[abc]A}$. We thus want to consider the form factors

$$f^{[abc]A}_{\alpha\beta}(\theta_{1}, \theta_{2}) = \langle \psi^{[abc]A}A_{\beta}(\theta_{2})A_{\alpha}(\theta_{1}) \rangle. \quad (4.57)$$

Under charge conjugation, the fields, $f^{[abc]A}$, transform as

$$f^{[abc]A}(\theta_{1}, \theta_{2}) = -(-1)^{a+b+c} f^{[abc]A}(\theta_{1}, \theta_{2}) \quad (4.58)$$

We thus expect

$$f^{[abc]A}_{\alpha\beta}(\theta_{1}, \theta_{2}) = (-1)^{a+b+c}(-1)^{a_{2}+b_{2}} f^{[abc]A}_{\alpha\beta}(\theta_{1}, \theta_{2}) \quad (4.59)$$
We may satisfy this constraint by choosing \( f^{a[abc]}_{\alpha \beta}(\theta_1, \theta_2) \) to have the same form as \( f^{a[abc]}_{\alpha \beta}(\theta_1, \theta_2) \), i.e.

\[
f^{a[abc]}_{\alpha \beta}(\theta_1, \theta_2) = (C_{\gamma}^{[\alpha} b_{\gamma}^{\beta c]}\theta_{a})_{\alpha \beta} f(\theta_{12}).
\]  

(4.60)

Here \( f(\theta_{12}) \) is the same as before as all the other axioms governing the form factor remain unchanged.

**F. Two particle form factors for \( SO(6) \) vector fields**

In this section we are interested in computing the two particle form factors for an operator, \( \psi^{a} \), transforming as both a Lorentz scalar and a vector under \( SO(6) \). For specificity we consider such an operator arising from the following symmetric combinations of fermion bilinears \( c_{\beta} b_{\alpha} + c_{\alpha} b_{\beta} + c_{\beta} b_{\alpha} \). This operator has conformal dimension, \( \Delta = 3/8 \). At the two particle level, two kinks of the same chirality will again couple to such an operator. We first consider the case of two kinks of even chirality coupling to this operator.

We thus consider form factors, \( f^{a}_{\alpha\beta}(\theta_{1}, \theta_{2}) \), defined by

\[
f^{a}_{\alpha\beta}(\theta_{1}, \theta_{2}) = \langle \psi^{a}(0) A_{\beta}(\theta_{2}) A_{\alpha}(\theta_{1}) \rangle.
\]  

(4.61)

Covariance under \( SO(8) \) suggests that \( f^{a}_{\alpha\beta} \) takes the form

\[
f^{a}_{\alpha\beta}(\theta_{1}, \theta_{2}) = [c_{1}(\gamma \gamma C)_{\alpha\beta} + c_{2}(\gamma^{\gamma}_{\alpha}\gamma_{\beta})] f_{\epsilon}(\theta_{1}, \theta_{2}),
\]  

(4.62)

where \( c_{1} \) and \( c_{2} \) are constants. (That \( c_{1} \) and \( c_{2} \) are not more generally independent functions of \( \theta_{1} \) and \( \theta_{2} \) is easily seen; the constraints Eqn. (4.22) and Eqn. (4.23) do not allow it.) \( c_{1} \) and \( c_{2} \) can be fixed easily. Suppose \( |\alpha| = (1/2, 1/2, 1/2) \) and \( |\beta| = (1/2, -1/2, -1/2) \). Then by rewriting the Majorana fields as Dirac fields, one readily finds that

\[
f^{a}_{\alpha\beta}(\theta_{1}, \theta_{2}) = -i f^{a}_{\beta\alpha}(\theta_{1}, \theta_{2}).
\]  

(4.63)

This in turn forces \( c_{1} = 0 \). We then set \( c_{2} = 1 \) as we are uninterested at this point in an overall normalization.

As the operator is a Lorentz scalar, we must have

\[
f_{\epsilon}(\theta_{1}, \theta_{2}) = f_{\epsilon}(\theta_{12}).
\]  

(4.64)

\( f_{\epsilon}(\theta) \) is then constrained by the scattering axiom. Using the form of the kink-kink S-matrix for kinks of the same chirality together with the anti-symmetry of \( C_{\gamma}^{\alpha} \) we obtain

\[
f_{\epsilon}(\theta) = \frac{G_v(x)}{1 - e^{-2x}} (e^{-2\pi(1 - e^{\pi/2})} - e^{-5\pi/2(1 - e^{\pi/2})}).
\]  

(4.65)

We now apply the periodicity axiom. It takes the form

\[
f^{a}_{\epsilon\alpha\beta}(\theta) = Q_{\alpha\beta} R_{\epsilon\psi} f^{a}_{\alpha\beta}(\theta - 2\pi i).
\]  

(4.66)

As discussed in Appendix B, we find that for even kinks the product of phases \( Q_{\alpha\beta} R_{\psi} \) is equal to 1. With this the periodicity axiom reduces to

\[
f_{\epsilon}(\theta) = -f_{\epsilon}(\theta - 2\pi i).
\]  

(4.67)

The presence of a minus sign (compare Eqn. (4.58)) results from the antisymmetry of \( C_{\gamma}^{\alpha} \). The periodicity and scattering axioms then imply that \( f_{\epsilon}(\theta) \) must have the minimal form

\[
f_{\epsilon}(\theta) = \sinh(\theta) \exp \left[ \int_{0}^{\infty} \frac{dx}{x} \frac{G_v(x)}{\sinh(x)} \sin^{2} \left( \frac{x}{2\pi} (i\pi + \theta) \right) \right].
\]  

(4.68)

As two kinks of the same chirality form a fermionic bound state, \( f^{a}_{\alpha\beta} \) should have a pole at \( \theta_{2} = \theta_{1} + i\alpha_{\beta} = \theta_{1} + i\pi/2 \). Thus \( f^{a}_{\epsilon\alpha\beta} \) becomes

\[
f^{a}_{\epsilon\alpha\beta}(\theta_{1}, \theta_{2}) = A_{\epsilon}(C_{\gamma}^{\alpha})_{\alpha\beta} \frac{\sinh(\theta_{12})}{\cosh(\theta_{12})} \times \exp \left[ \int_{0}^{\infty} \frac{dx}{x} \frac{G_v(x)}{\sinh(x)} \sin^{2} \left( \frac{x}{2\pi} (i\pi + \theta_{12}) \right) \right].
\]  

(4.69)

Here \( A_{\epsilon} \) is the same normalization with mass dimension \( [m]^{3/8} \). We now consider the situation of two odd kinks coupling to \( \psi^{a} \). We thus want to consider form factors of the type

\[
f^{a}_{\epsilon\alpha\beta}(\theta_{1}, \theta_{2}) = \langle \psi^{a} A_{\beta}(\theta_{2}) A_{\alpha}(\theta_{1}) \rangle.
\]  

(4.70)

All but the periodicity axiom is of same form as for the two even kinks. For the two odd kinks, the periodicity axiom changes to

\[
f_{\epsilon}(\theta) = f_{\epsilon}(\theta - 2\pi i).
\]  

(4.71)

The difference with Eqn. (4.67) arises because the OPE of the odd kinks with the operator produces an additional sign. This sign changes the form factor involving two odd kinks to be

\[
f^{a}_{\epsilon\alpha\beta}(\theta_{1}, \theta_{2}) = \frac{A_{\epsilon}}{\sin(\pi/4)} (C_{\gamma}^{\alpha})_{\alpha\beta} \frac{\sinh(\theta_{12}/2)}{\cosh(\theta_{12})} \times \exp \left[ \int_{0}^{\infty} \frac{dx}{x} \frac{G_v(x)}{\sinh(x)} \sin^{2} \left( \frac{x}{2\pi} (i\pi + \theta_{12}) \right) \right].
\]  

(4.72)

Here \( A_{\epsilon} \) is the same normalization as for the even kinks. We can fix the normalization of \( f^{a}_{\epsilon\alpha\beta}(\theta_{1}, \theta_{2}) \) relative to
functions, the form factors involving both even and odd vector, we would arrive at the same results with the positive caveat that the form factors of the even and odd vector field will couple to a single Gross- Neveu fermion, a. So we consider the form factor,

$$f_0^a(\theta) = \langle \psi^a(0) A_0^a(\theta) \rangle, \quad (4.74)$$

which must take the form

$$f_0^a(\theta) = c \delta_0^a. \quad (4.75)$$

To determine the constant, we use the two particle form factor $f_{\alpha \beta}^a$. Let $|\alpha\rangle = (1/2, 1/2, 1/2)$ and $|\beta\rangle = (1/2, -1/2, -1/2)$. $A_{\alpha}$, $\alpha = 1$, can be written in terms of these two even chirality kinks:

$$i g_{\alpha \beta} A_{\alpha}(\theta) = \text{res}_{\delta} = A_{\alpha}(\theta + \delta + i \bar{u}_{\beta \alpha}) A_{\alpha}(\theta - i \bar{u}_{\alpha \alpha}). \quad (4.76)$$

Again the u’s mark out poles in the S-matrix indicative of bound states. Here they are given by

$$\bar{u}_{\beta \alpha} = \bar{u}_{\alpha \alpha} = \pi - u_{\alpha \alpha} = \pi/4. \quad (4.77)$$

$g_{\alpha \beta}^a$ can be determined up to a phase from the kink-kink S-matrix as before to be

$$g_{\alpha \beta}^a = \left( \frac{2}{3} \sqrt{\frac{\pi}{10}} \right)^{1/2}. \quad (4.78)$$

Given that,

$$i g_{\alpha \beta} f_{\alpha}^a(\theta) = \text{res}_{\delta} = \pm f_{\alpha \beta}^a(\theta - i \pi / 4, \theta + \delta + i \pi / 4), \quad (4.79)$$

together with the hermiticity constraint,

$$(f_0^a)^\ast(\theta) = f_0^a(\theta), \quad (4.80)$$

we find (up to a sign) the constant in Eqn. (4.75) to be

$$c = \frac{A_{\text{even}}}{g_{\alpha \beta}^a} \exp \left[ - \int_0^{\infty} \frac{dx}{x} \sin(x) \sin^2 \left( \frac{x}{2} \right) \right]. \quad (4.81)$$

If we repeat the above procedure using the two particle form-factor involving two odd chirality kinks, $f_{\bar{\alpha} \bar{\beta}}^a$, we obtain the same result.

### H. Summary of form factor results

In this section we summarize the results of this section for the various form factors.

#### Two particle form factors:

The form factors of the $SO(6)$ currents, $G_{\mu}^{ab}$, are:

$$\mu f_{\alpha \beta}^{\alpha \beta}(\theta_1, \theta_2) = \sum_{\gamma} \langle C_{\gamma}^{\alpha \beta} e^{i(\theta_1 + \theta_2)/2} - (-1)^{\mu} e^{i(\theta_1 - \theta_2)/2} \rangle \times \exp \left[ \int_0^{\infty} \frac{dx}{x} \sin(x) \sin^2 \left( \frac{x}{2\pi} (i\pi + \theta_2) \right) \right]. \quad (4.82)$$

The form factors of $SO(6)$ symmetric tensor fields, $\psi^{[abc]}_{\alpha}$, are:

$$\langle \psi^{[abc]}_{\alpha}(\theta_1, \theta_2) \rangle = \sum_{\gamma} \langle C_{\gamma}^{\alpha \beta} A_{\alpha}(\gamma^{[a} \gamma^{b} \gamma^{c]} C) \rangle \times \exp \left[ \int_0^{\infty} \frac{dx}{x} \sin^2 \left( \frac{x}{2\pi} (i\pi + \theta_2) \right) \right] \quad (4.83)$$

The form factors of $SO(6)$ vector fields, $\psi^{a}$, involving two even kinks are

$$f_{\alpha \beta}^a(\theta_1, \theta_2) = A_{\text{even}} (C_{\gamma}^{\alpha \beta} \frac{\sinh(\theta_1 2)}{\cosh(\theta_1 2)}) \times \exp \left[ \int_0^{\infty} \frac{dx}{x} \sin(x) \sin^2 \left( \frac{x}{2\pi} (i\pi + \theta_2) \right) \right]. \quad (4.84)$$

and for two odd kinks change to

$$f_{\alpha \beta}^a(\theta_1, \theta_2) = \frac{A_{\text{even}}}{\sin(\pi/4)} (C_{\gamma}^{\alpha \beta} \frac{\sinh(\theta_1 2)}{\cosh(\theta_1 2)}) \times \exp \left[ \int_0^{\infty} \frac{dx}{x} \sin(x) \sin^2 \left( \frac{x}{2\pi} (i\pi + \theta_2) \right) \right]. \quad (4.85)$$
We note that depending on the particular vector field considered the form factors of even and odd kinks may be swapped.

**One particle form factors:**

The one particle form factor for SO(6) vector fields is

\[ f^g_0(\theta) = c \delta^g_\theta, \quad (4.86) \]

where \( c \) is given by

\[ c = \frac{A_v}{g^{e\beta}_a} \exp \left[ - \int_0^\infty \frac{dx}{x} G_x(x) \sinh^2\left( \frac{x}{4} \right) \right]. \quad (4.87) \]

**V. SUMMARY AND CONCLUSIONS**

In this work we have calculated the dynamical magnetic susceptibilities of doped Hubbard-like ladders in the framework of a low-energy description by means of the SO(6) Gross-Neveu model. The most prominent feature of the low-energy spin response is a narrow peak at wave-vectors (along the leg direction) \( K_{F_a} + K_{F_b} \) and \( -K_{F_a} - K_{F_b} \). This peak corresponds to a fermionic bound state of two SO(6) kinks, which is broadened by the gapless bonding charge mode. This narrow peak sits on top of an incoherent scattering continuum of two SO(6) kinks. The low-energy magnetic response in the vicinity of the wave numbers \( \pm 2K_{F_b} \) and \( \pm 2K_{F_a} \) is rather featureless. The response at low momenta and in the vicinity of \( \pm (K_{F_a} - K_{F_b}) \) exhibits incoherent scattering continua with a threshold singularity.

Other dynamical correlation functions such as the single-particle Green’s function, density-density or superconducting correlators can be determined along the same lines.

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**APPENDIX A: CALCULATING THE DYNAMICAL SPIN-SPIN CORRELATORS**

In this appendix we present the detailed derivations of the functions \( J_{1,2,3}(\omega, k) \) in Eqs. (3.10)-(3.12) that determine the dynamical susceptibility. In subsection A 4 we estimate the amplitudes \( A_{ij} \) that determine the relative spectral weights of the various contributions to the structure factor.

1. **Calculation of \( J_1(\omega, k) \)**

The contribution \( J_1(\omega, k) \) involves fields arising from right/left symmetric combinations of fermion bilinears of the form \( c_{R\beta}^\dagger c_{R\beta} + c_{L\beta}^\dagger c_{L\beta} \). These fields in turn are expressible purely in terms of zeroth Lorentz component of the the SO(6) Gross-Neveu currents. For example

\[ c_{R\beta}^\dagger c_{R\beta} + c_{L\beta}^\dagger c_{L\beta} = \psi_{2R}^\dagger \psi_{3R} + \psi_{2L}^\dagger \psi_{3L} = \frac{1}{2} (-ig_{46}^0 + g_{45}^0 - g_{36}^0 - ig_{35}^0). \quad (A1) \]

We can then write down \( J_1 \) as

\[ J_1(\omega, k) = \int d\tau dx e^{i(-\omega + \tau) - ikx} (TG_{ed}(x, \tau)G_{cd}^0(0)). \quad (A2) \]

Here \( cd \) gives one SO(6) current. Given that the different currents that arise in expanding \( c_{R\beta}^\dagger c_{R\beta} + c_{L\beta}^\dagger c_{L\beta} \) give identical contributions (up to a constant) we simply write \( J_1 \) in terms of a single current, understanding that any degeneracy is fixed through absorption into the associated pre-factor, \( A_{j1}, j = 1, 2, 3 \). Under a form factor expansion, the lowest energy excitations contributing to \( J_1 \) are two kinks of opposite chirality. We first focus on the imaginary piece of \( J_1 \). For the real piece it is easiest to obtain via a Kramers-Kronig transformation.

The imaginary piece takes the form,

\[ \text{Im} \ J_1(\omega, k) = \pi^2 \sum \frac{d\theta_1 d\theta_2}{(2\pi)^2} |f^{ed}_{\alpha\beta}(\theta_1, \theta_2)|^2 \delta(\omega - E(\theta_1, \theta_2)) \delta(k - P(\theta_1, \theta_2)) , \quad (A3) \]

where \( R/L f^{ed}_{\alpha\beta}(\theta_1, \theta_2) \) is the kink-kink form factor for the current operator computed in Section 4 and

\[ E(\theta_1, \theta_2) = m \cosh \theta_1 + \cosh \theta_2 , \]

\[ P(\theta_1, \theta_2) = \frac{m}{v_F} \sinh \theta_1 + \sinh \theta_2 . \quad (A4) \]

We mean the indices, $\bar{\alpha}$ and $\beta$, of the sum, $\sum_{\bar{\alpha}\beta}$, to run over both kinks and anti-kinks. Performing the sum, we obtain

$$\text{Im} J_1(\omega, k) = 8\pi^2 \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} |f^0(\theta_1, \theta_2)|^2 \times \delta(\omega - E(\theta_1, \theta_2)) \delta(k - P(\theta_1, \theta_2)), \quad (A5)$$

where (up to a constant which will be absorbed into the appropriate $A_{i1}$),

$$|f^0(\theta_1, \theta_2)|^2 = \sinh^2\left(\frac{\theta_1 + \theta_2}{2}\right) \times \exp \left[ \int_0^\infty \frac{dx}{s(x)} \frac{G_c(x)}{x} \right] \right], \quad (A6)$$

and $G_c(x)$ is given in Eqn. $(3.4)$. We then perform the integrations over $\theta_1$ and $\theta_2$ obtaining

$$\text{Im} J_1(\omega, k) = \frac{8\pi^2}{2\pi} \frac{a_k^2}{k^2} \frac{\theta(\omega - \sqrt{k^2 + 4m^2})}{(\omega^2 - k^2 + 4m^2)^{3/2}} \times \exp \left[ \int_0^\infty \frac{dx}{s(x)} \frac{G_c(x)}{x} \frac{1 - c(x)}{s(x)} \cos\left(\frac{\theta_{12}x}{\pi}\right) \right], \quad (A7)$$

where $\theta_{12}$ is given by

$$\theta_{12} = \cosh^{-1}\left(\frac{\omega^2 - k^2 - 4m^2}{2m^2}\right), \quad (A8)$$

and $k = v_F k$. We plot $\text{Im} J_1(\omega, m/v_F)$ in Fig. 5.

2. Calculation of $J_2(\omega, k)$

We now consider the computation of $J_2$. This contribution arises from considering fermion bilinears of the form $\bar{c}_{R\bar{\alpha}L\bar{\beta}}^c c_{L\alpha}$ These bilinears are expressible as products of the gapless charge degrees of freedom with fields transforming under a 10 dimensional symmetric representation of $SO(6)$. For example

$$c_{L\bar{\alpha}}^a c_{R\bar{\beta}} c_{R\bar{\beta}}^b c_{L\bar{\alpha}} \sim e^{i\frac{2}{3} \theta_+} e^{-i\frac{2}{3} \theta_+ - \phi_+ - \phi_- - \theta_+}$$

$$\sim e^{i\frac{2}{3} \theta_+} \sum_{\alpha\beta} c_{\alpha\beta\gamma} \psi_{\bar{\alpha\beta\gamma}} \quad (A9)$$

Here the fields $\psi_{\bar{\alpha\beta\gamma}}$ are defined and discussed both in Sections 2 and 4. The coefficients $c_{\alpha\beta\gamma}$ do not need to be determined exactly as the contribution of each $\psi_{\bar{\alpha\beta\gamma}}$ to $J_2(\omega, k)$ will be identical up to some constant. Any undetermined constant will be fixed in the end by appealing to a sum rule.

$J_2(x, \tau)$ is then the product of two correlation functions, one coming from the gapless charge correlations and one coming from the gapped $SO(6)$ sector. We write

$$J_2(x, \tau) = J_{2g}(x, \tau) J_c(x, \tau), \quad (A10)$$

where

$$J_c(x, \tau) = \langle T e^{i\frac{2}{3} \theta_+ (x, \tau)} e^{i\frac{2}{3} \theta_+ (0, 0)} \rangle$$

$$= A_c(x^2 + \tau^2)^{K/4},$$

$$J_{2g}(x, \tau) = \langle T \psi_{\bar{\alpha\beta\gamma}}(x, \tau) \psi_{\bar{\alpha\beta\gamma}}(0) \rangle. \quad (A11)$$

Here $K$ is the Luttinger parameter of the gapless charge excitations and $A_c$ is some constant which we will absorb where appropriate into the $A$'s of Eqns. $(3.10-3.12)$. We evaluate $J_{2g}$ in the same fashion as $J_1$, namely we use employ a factor expansion that is truncated at the two kink level. We thus obtain

$$J_{2g}(x, \tau) = \frac{1}{2} \sum_{\alpha\beta} \int d\theta_1 d\theta_2 |f^{\alpha\beta\gamma}(\theta_1, \theta_2)|^2 \times e^{i\tau(-i\omega + \delta)} E(\theta_1, \theta_2) e^{ix(k - \delta^2)} \times K_0(2m \cos(\delta)) \frac{x}{v_F^2} + \tau^2)^{1/2}. \quad (A12)$$

Here $\alpha$ and $\beta$ are kinks of the same chirality. In the second line of the equation we have made a change of variables to $\theta_{\pm} = (\theta_1 \pm \theta_2)/2$ and performed the $\theta_+$ integral. Using the form factors of Section 4.4, we can write $|f^{\alpha\beta\gamma}(\theta_1, \theta_2)|^2$ as

$$|f^{\alpha\beta\gamma}(\theta_1, \theta_2)|^2 = \sinh^2\left(\frac{\theta_{12}}{2}\right) \times \exp \left[ \int_0^\infty \frac{dx}{s(x)} \frac{G_c(x)}{x} \frac{1 - c(x)}{s(x)} \cos\left(\frac{\theta_{12}x}{\pi}\right) \right]. \quad (A13)$$

Substituting Eqn. $(A13)$ and Eqn. $(A12)$ into Eqn. $(A11)$, we can carry out the $x - \tau$ integrals following Refs. 52 and 51. Up to some constant (which we absorb into $A_{12}/A_{22}$), we obtain
we have comes about from considering the correlations of fermion
special case \( K \).

We plot the imaginary part of \( J_2(\omega, 0) \) in Fig. 6 for the special case \( K = 1 \).

3. Calculation of \( J_3(\omega, k) \)

Finally we consider the calculation of \( J_3(\omega, k) \). \( J_3 \) comes about from considering the correlations of fermion bilinears of the form \( c_{Rab}^\dagger c_{Lb^\dagger} + c_{Rb^\dagger}^\dagger c_{Lab} \). In particular we have

\[
J_3(\omega, k) = \int d\tau \exp(-i\omega \tau - ikx) \langle \mathcal{O}_3(x, \tau) \mathcal{O}_3^\dagger(0, 0) \rangle ,
\]

(A15)

The contribution of the one-particle form-factor marks the most significant difference between \( J_3 \) and \( J_2 \). It indicates a propagating coherent mode contributes to the spectral function. We do note that this mode broadens out because of the gapless charge excitations. The contribution of this mode has been previously studied in Ref.
We now drop the sum, \( \sum_\omega |c_\omega|^2 \), as each term gives an identical contribution up to some constant. Using the form factors of Section 4.5 we can then write

\[
J_{3g,1}(x, \tau) = A_3^2 \frac{d}{\pi} K_0(\sqrt{2m} (\tau^2 + (x/v_F)^2))
\]

\[
d = \frac{1}{g^2} \exp[-2 \int_0^\infty \frac{dx}{x} \frac{G_v(x)}{s(x)} s^2(x/4)];
\]

\[
J_{3g,2}(x, \tau) = A_3^2 \frac{4}{\pi^2} \int d\theta_- \left( \frac{\tanh(2\theta_-)^2}{2} + \frac{\sinh(\theta_-)^2}{\cosh(2\theta_-)^2} \right)
\]

\[
\times K_0(2m \cosh(\theta_-)((x/v_F)^2 + \tau^2)^{1/2})
\]

We now turn to estimating the various normalization constants, i.e. the \( A \)'s appearing in Eqns 3.10-3.12. In the present context, perhaps the best way of determining them would be to compare the field theory results to numerical computations following the lines of B24. However, such computations are presently not available. A crude way of determining the order of magnitude of the \( A \)'s is to compare our results to the susceptibility calculated in a Random Phase Approximation. The latter is given by

\[
\chi^{RPA}(\omega, q) = \chi_{11}(\omega, q_z) + \chi_{22}(\omega, q_z) + 2 \cos(q_y) \chi_{12}(\omega, q_z)
\]

\[
\chi_{ab}(\omega, q_z) = \frac{\chi^{(0)}(\omega, q_z)}{1 + 2U \chi^{(0)}(\omega, q_z)} \bigg|_{ab}.
\]

Here \( \chi^{(0)}(\omega, q_z) \) is the susceptibility matrix of non-interacting electrons \((U = 0)\) on a two-leg ladder. In order to fix the constants \( A_{ij} \), we now require

\[
\int_0^\Lambda \frac{d\omega}{\omega} \text{Im} \chi^{RPA}(\omega, q) = \int_0^\Lambda \frac{d\omega}{\omega} \text{Im} \chi(\omega, q).
\]
However, we expect RPA to give a reasonable account of the static susceptibility (which corresponds to taking $\Lambda \to \infty$) and hence also of the integrated spectral weight at low frequencies. Field theory does not give a good account of the static susceptibility, which is reflected in the fact that one cannot take $\Lambda \to \infty$ in the field theory calculation as the resulting integral would diverge. We therefore arbitrarily set $\Lambda = 10m$. A useful consistency check is that the results for $A_{ij}$ are essentially unaffected by small changes in $\Lambda$. We choose to consider Eqn. (A23) rather than the integrated spectral weights in order to suppress as much as possible the neglected contributions that occur at higher energies in the field theory calculation. On the field theory side we have

\[
\frac{1}{\pi} \text{Im}\chi(\omega, q, \frac{\pi}{2}) \approx \frac{3(A_{11} + A_{21})}{4\pi} J_1(\omega, q),
\]

\[
\frac{1}{\pi} \text{Im}\chi(\omega, 2K_{Fb}, \frac{\pi}{2}) \approx \frac{3A_{12}}{4\pi} J_2(\omega, 0),
\]

\[
\frac{1}{\pi} \text{Im}\chi(\omega, 2K_{Fab}, \frac{\pi}{2}) \approx \frac{3A_{22}}{4\pi} J_2(\omega, 0),
\]

\[
\frac{1}{\pi} \text{Im}\chi(\omega, K_{Fb} - K_{Fab} + q, \frac{\pi}{2}) \approx \frac{3A_{31}}{4\pi} J_1(\omega, q),
\]

\[
\frac{1}{\pi} \text{Im}\chi(\omega, K_{Fb} + K_{Fab}, \frac{\pi}{2}) \approx \frac{3A_{32}}{4\pi} J_3(\omega, 0).
\]  

(A24)

The relevant frequency integrals are

\[
\int_0^{10m} \frac{d\omega}{\omega} J_1(\omega, \frac{5m}{v_F}) \approx 1.04 \frac{v_F}{m^2},
\]

\[
\int_0^{10m} \frac{d\omega}{\omega} J_2(\omega, \frac{5m}{v_F}) \approx 1.79 \frac{v_F}{m^2},
\]

\[
\int_0^{10m} \frac{d\omega}{\omega} J_3(\omega, \frac{5m}{v_F}) \approx 0.23 \frac{v_F}{m^2}.
\]  

(A25)

In order to compare to compare to the RPA calculation we need to estimate the gap $m$ as a function of $t$, $t_\perp$ and $U$. We do this in two steps. Firstly, the spin gap at half-filling $\Delta_s(1)$ has been determined numerically for small values of $U/t$ in Refs. [21,24]. For $t_\perp = 1.5t$ and $U = t$ it was found that

\[
\Delta_s(1) \approx 0.05t.
\]  

(A26)

In order to determine the kink gap $m$ we need to know how $\Delta_s$ evolves under doping. As numerical results are not available in the literature for small $U$, we instead use the results of Ref. [13], where the doping dependence of the spin gap was analyzed within the framework of a low-energy description of the undoped ladder in terms of the SO(8) Gross-Neveu model. At half-filling we have $m_{SO(8)} = \Delta_s(1)$. Assuming that the Fermi velocity is well approximated by the non-interacting value $v_F \approx 1.32t$ (for $t_\perp = 1.5t$), we find that the gap $m$ for SO(6) kinks at four different doping levels is roughly equal to

\[
\begin{array}{cccc}
\text{doping} & m_{SO(6)} & \text{(m1/2filling)} & K \\
1\% & .47 & .94 & \\
5\% & .29 & .915 & \\
10\% & .21 & .93 & \\
25\% & .13 & .945 & \\
\end{array}
\]

In this table we have additionally listed the corresponding Luttinger parameter for the different dopings, also available from Ref. [13]. Having fixed $v_F$ and $m$ in the low-energy effective SO(6) theory in terms of $t$ and $U$ we now can use the comparison between the field theory and RPA calculations of the integrated susceptibilities to estimate the amplitudes $A_{ij}$. We find

\[
\begin{array}{cccccccc}
\text{doping} & A_{11} & A_{21} & A_{12} & A_{22} & A_{31} & A_{32} \\
1\% & 0.24 & 0.13 & 0.10 & 0.14 & 6.61 & \\
5\% & 0.25 & 0.045 & 0.070 & 0.14 & 4.70 & \\
10\% & 0.26 & 0.12 & 0.091 & 0.22 & 3.24 & \\
25\% & 0.26 & 0.091 & 0.091 & 0.24 & 4.94 & \\
\end{array}
\]

A table of the field theoretic amplitudes, $A_{ij}$, computed for $U = t$ and $t_\perp = 1.5t$.

All values for the $A_i$'s are in units of $\frac{m^2}{v_F}$.

We thus have all the ingredients at hand to calculate the spin-spin correlation functions of the doped ladders.

APPENDIX B: DERIVATION OF FORM FACTOR AXIOMS FOR GENERAL OPERATORS IN SO(6) GROSS-NEVEU

In this appendix we derive the form-factor axioms for SO(6) Gross-Neveu for general semi-local operators. We take the following approach. We first consider the general form-factor axioms as written by Smirnov in Ref. [40] for the current and stress-energy operators. These are written for a particle basis which is not C-symmetric. In this basis the form-factor axioms are not easily generalized. To surmount this difficulty we transform to a particle basis which is C-symmetric. Here the form-factor axioms can be readily written down for all semi-local operators. In particular the semi-locality phases appearing in the axioms can be computed using operator product expansions. We then recover the form-factor axioms for general semi-local operators in the original basis by inverting the transformation. As an important consistency check of both the transformation and the operator product expansions, we recover the axioms for the currents and stress-energy tensor as originally stated.

1. The N-Particle Form Factor Axioms for the Currents and Stress-energy Tensor in SO(6) Gross-Neveu

The N-particle form factor axioms for the currents and the stress-energy tensor in SO(6) Gross-Neveu are given
by Smirnov as follows:

\[ f^O(\ldots, \theta_{j+1}, \theta_j, \ldots) = S^O_{\epsilon_{j+1}}(\theta_j) f^O(\ldots, \theta_j, \theta_{j+1}, \ldots) \]
\[ f^O(\theta_1, \ldots, \theta_n + 2\pi i) = (-1)^{\sum_{j=1}^n l[\epsilon_j]} f^O(\theta_1, \ldots, \theta_{n-1}) C_{\epsilon_n} \]

The first equation is the generalization of Eqn. 22 of scattering axiom to matrix elements with N-particles. The second equation is the N-particle generalization of Eqn. 24, the periodicity axiom. The phase that appears on the r.h.s. of the periodicity axiom is defined through

\[ l[\epsilon_j] = \begin{cases} 1 & \text{for even parity kinks} \\ 2 & \text{for fermions} \\ 3 & \text{for odd parity kinks} \end{cases} \]  

(B2)

where the \( l[\epsilon_j] \) are the rank of the representation of the particle, \( \epsilon_j \), falls in. For the operators to which these axioms apply, the phase is always either \( \pm 1 \) as \( \sum l[\epsilon_j] \) is always a multiple of 4. While the above axioms are completely general, we will concentrate solely on form factors involving particles which are either kinks or anti-kinks. The form factors have simple poles at the points \( \theta_n = \theta_j + i\pi \). The residues of these poles are the subject of the final axiom. Here \( C \) is the charge conjugation matrix and by necessity \( l[\epsilon_n] + l[\epsilon_{n-1}] = 4 \). The annihilation pole condition relates form factors with different numbers of particles. In the annihilation pole axiom a phase similar to that of the periodicity axiom appears.

From one viewpoint, the phases in the latter two axioms can be justified by insisting the currents and the stress-energy tensor satisfy certain properties under charge conjugation and appropriate commutation relations. From another, the phases can be seen as a result of the fact that the \( C \) is not symmetric, i.e. \( C^2 = -1 \), and the non-trivial semi-locality between the non-Cartan currents and the fundamental kink fields. Understanding the phases in this fashion provide the most ready path to generalize the axioms to other fields. But because the phases have two sources, this path needs to be indirect. The first step is to transform the particle basis to a basis where the corresponding charge conjugation matrix is symmetric. In this basis the form-factor axioms can be written down readily – the only additional phases that appear are semi-locality factors and these are well understood. To obtain the form-factor axioms in the original basis we then invert the transform.

2. Transformation to a C-symmetric basis

The path we then intend to follow is the one Ref. 40 uses for SU(2). There the transformation to a C-symmetric basis is straightforward because of the simplified particle content. Here, in contrast, the transformation is more involved. We define the transformation to a C-symmetric basis as follows. A n-particle state behaves under the transformation, \( T \), via

\[ \tilde{A}_{\epsilon_1} \cdots \tilde{A}_{\epsilon_n} |0\rangle = T A_{\epsilon_1} \cdots A_{\epsilon_n} |0\rangle = \left( \prod_{\epsilon_i < \epsilon_j} P_{\epsilon_i, \epsilon_j} \right) A_{\epsilon_1} \cdots A_{\epsilon_n} |0\rangle, \]

(B3)

where the \( P_{\epsilon_i, \epsilon_j} \) are given by

\[ P_{\alpha\beta} = P_{\beta\alpha} = P_{\bar{\alpha}\bar{\beta}}, \quad \alpha \neq \beta; \]
\[ P_{\beta\beta} = -P_{\beta\beta} = -1; \]
\[ P_{\alpha\alpha} = -P_{\bar{\alpha}\bar{\alpha}} = 1; \]
\[ P_{\alpha\bar{\beta}} = -P_{\bar{\beta}\alpha}; \]

(B4)

and

\[ P_{\alpha\bar{\beta}} P_{\beta\alpha} = -1 \quad \text{if} \ \alpha \neq \beta; \]
\[ P_{\alpha\bar{\beta}} P_{\beta\bar{\alpha}} = -1 \quad \text{if} \ \bar{\alpha} \neq \bar{\beta}. \]

To meet this last set of conditions, we arbitrarily set

\[ P_{+++,---} = P_{+++,-++} = P_{+-+,+++} = P_{-++,+--} = P_{-+-,--+} = -1. \]

Here \( \alpha, \beta \) correspond to even chirality kinks while \( \bar{\alpha}, \bar{\beta} \) correspond to odd chirality kinks. This choice of the transformation is not unique but is equivalent in effect to other possible choices.

In moving to this transformed basis, the S-matrix also undergoes transformation. The transformed S must satisfy

\[ \tilde{A}_{\epsilon_1} \tilde{A}_{\epsilon_2} = S_{\epsilon_1 \epsilon_2} \tilde{A}_{\epsilon_1} \tilde{A}_{\epsilon_2} \]

(B5)
This implies the relation
\[ \tilde{S}^{e^\prime}_{e_1 e_2} = \tilde{S}^{e^\prime}_{e_1 e_2} P_{e_1 e_2} P_{e^\prime} \]  \hspace{1cm} (B6)

By construction \( \tilde{S} \) is C-symmetric. In particular it satisfies
\[ \tilde{S}^{\alpha \beta} (i \pi - \theta) = \tilde{S}^{\alpha \beta} (\theta) \tilde{C}_{\gamma \delta} \tilde{C}_{\alpha \alpha \gamma} \]  \hspace{1cm} (B7)

where \( \tilde{C} = -i \gamma_4 \gamma_6 \), or equivalently, \( \tilde{C}_{\alpha \alpha \gamma} = C_{\alpha \alpha} P_{\alpha \alpha} \), is now a symmetric matrix.

This transformation, as constructed, satisfies an important consistency condition. Considering the relation involving three particle states,
\[ \tilde{A}_{e_1} \tilde{A}_{e_2} \tilde{A}_{e_3} = \tilde{S}^{e^\prime}_{e_1 e_2} \tilde{A}_{e^\prime_1} \tilde{A}_{e^\prime_2} \tilde{A}_{e^\prime_3} \]  \hspace{1cm} (B8)

we see that to transform this relationship covariantly to the original basis, i.e.
\[ A_{e_1} A_{e_2} A_{e_3} = \tilde{S}^{e^\prime}_{e_1 e_2} A_{e^\prime_1} A_{e^\prime_2} A_{e^\prime_3} \]  \hspace{1cm} (B9)

This transformation, as given above does indeed satisfy this condition. This condition guarantees both that \( \tilde{S} \) satisfies the Yang-Baxter equation and as we will see, that we can recover the axioms given in the original basis from the axioms stated in the C-symmetric basis.

3. Form Factor Axioms in C-Symmetric Basis

In the C-symmetric basis the axioms for arbitrary semi-local operators have the following form:

\[ \tilde{f}^O (\ldots, \theta_{j+1}, \theta_j, \ldots) = \tilde{S}^{e_j e_{j+1}} (\theta_j - \theta_{j+1}) \tilde{f}^O (\ldots, \theta_{j+1}, \ldots, \theta_j, \ldots) \]  \hspace{1cm} (B10)

holds for any \( e_1, e_2, e^\prime_1, e^\prime_2 \) such that \( \tilde{S}^{e_1 e_2} \) is non-zero. The transformation as above can be checked directly. With the phase in this form, the kink fields as described above uniformly leads to \( \tilde{S}^{e^\prime}_{e_1 e_2} \) that satisfies the Yang-Baxter equation.

\[ \tilde{f}^O (\theta_1, \ldots, \theta_n + 2 \pi i \epsilon_1, \ldots, \epsilon_n) = e^{2 \pi i \omega (O, \epsilon_n)} \tilde{f}^O (\theta_n, \theta_1, \ldots, \theta_{n-1}) \]  \hspace{1cm} (B11)

The phase factor \( e^{2 \pi i \omega (O, \epsilon)} \) arises from the semi-locality of the fields in the \( SO(6) \) Gross-Neveu model. Working in Euclidean space, fields can be said to be semi-local if their product sees a phase change when the fields are taken around one another in the plane via the analytic continuation \( z \to ze^{i 2 \pi}, \bar{z} \to \bar{z} e^{-i 2 \pi} \) (in conventions where right-moving fields depend only on \( \bar{z} \)). More specifically we have

\[ \psi (ze^{i 2 \pi}, \bar{z} e^{-i 2 \pi}) (O) = e^{2 \pi i \omega (O, \epsilon)} \psi (z, \bar{z}) (O) \]  \hspace{1cm} (B12)

Here \( \psi \) is the field that is associated with the particle \( A^i (\theta) \). \( \psi \) is chosen in such a fashion that it is both semi-local and bosonic. For the case at hand we represent the kinks through the vertex operators

\[ A_{\sigma_1, \sigma_2, \sigma_3} \to e^{\frac{1}{2} \epsilon (\sigma_1 \phi_2 + \sigma_2 \phi_3 + \sigma_3 \phi_1)} \]  \hspace{1cm} (B13)

where here \( \sigma = \pm 1 \) and \( \phi_i, i = 2, 3, 4 \), are the bosons as discussed in Section 2. The choice of field, \( \psi_\epsilon \), influences the phase, \( e^{2 \pi i \omega (O, \epsilon)} \), as different choices will have different operator product expansions (OPEs) with \( O \). Our choice passes an important check: the phase, \( e^{2 \pi i \omega (O, \epsilon)} \), so obtained from this assignment allows one to recover the form-factors axioms as stated in the original basis.

We now write down the semi-locality phases for the currents and the stress energy tensor. The stress energy tensor, \( T^{\mu \nu} \), is both chargeless and local. Its OPE with the kink fields as described above uniformly leads to \( e^{2 \pi i \omega (T^{\mu \nu}, \epsilon)} = 1 \). The phases for the currents can be written in a compact form. Any current, \( J^{\gamma \delta} \), can (group theoretically) be thought of as a product of an even and odd kink, \( \alpha \) and \( \tilde{\beta} \). If the \( \epsilon \) corresponds to a kink \( \gamma \) (of arbitrary chirality) the OPE then leads to the phase being given by

\[ e^{2 \pi i \omega (J^{\gamma \delta}, \epsilon)} = -P_{\gamma \alpha} P_{\delta \beta} P_{\gamma \beta} \]  \hspace{1cm} (B14)

This can be checked directly. With the phase in this form, we recover the form factors in their original form more readily.
4. Recovery of Form-Factor Axioms in Non-C-Symmetric Basis

We now show that we can recover the form-factor axioms for the currents and stress-energy tensor as given in the original basis (i.e. Eqn. (B11)) by transforming the form-factor axioms in the C-symmetric basis (i.e. Eqn. (B11)). This shows both that the transformation and our choice of fundamental fields to represent the kinks in the C-symmetric basis are consistent.

The demonstration that we recover the scattering axiom, the first equation in Eqn. (refeBi), is trivial. It follows immediately from the complementary fashion by which the S-matrix transforms relative to the basis (i.e. Eqn. (B10)) and the fact that the condition in Eqn. (B10) holds.

The periodicity axiom provides the first non-trivial check of the transformation. We will first consider the semi-locality phase is 1. If we apply the transformation to the second line of Eqn. (B11), we see that to recover the periodicity axiom as stated in its original form, we require that

\[ \prod_{i=1}^{n-1} P_{\epsilon_n \epsilon_i} P_{\epsilon_i \epsilon_n} = (-1)^{\sum_j f(\epsilon_j)}. \] (B15)

To establish this we first note the any set of particles coupling to a topologically neutral operator can be decomposed (not uniquely) into the following three subsets:

1. \( n_e \) complete sets of the four even kinks, \( \{(+ + +), (+ - -), (- + +), (- - +)\} \);
2. \( n_o \) complete sets of the four odd kinks, \( \{(- - -), (- + +), (+ - +), (+ + -)\} \);
3. \( n_p \) pairs of particle-antiparticles \( \{\alpha, \bar{\alpha}\} \).

We thus decompose the n-states as follows, \( n = 4n_e + 4n_o + 2n_p \). It is straightforward to show that the \( P'\)'s of the transformation satisfy the following conditions

\[ P_{\alpha \gamma} P_{\gamma \alpha} = -1; \]
\[ P_{\alpha \gamma} P_{\gamma \bar{\alpha}} = -1; \]
\[ P_{\alpha \gamma} P_{\gamma \alpha} P_{\bar{\alpha} \gamma} P_{\gamma \bar{\alpha}} = -1, \] (B17)

where \( \gamma \) is a kink of arbitrary chirality. The two sides of Eqn. (B16) can then be rewritten as

\[ \prod_{i=1}^{n-1} P_{\epsilon_n \epsilon_i} P_{\epsilon_i \epsilon_n} = (-1)^{n_e + n_o + n_p}; \]
\[ (-1)^{\sum_j f(\epsilon_j)} = (-1)^{n_e + 3n_o + n_p}. \] (B18)

These two are equal and we thus see that we recover the periodicity axiom for topologically neutral operators.

The case of the non-topologically neutral operators, specifically the non-Cartan currents, is dealt with similarly. Denote by \( J_{\alpha \beta} \) the particular current of concern. Any \( n \)-particle state coupling to it will consist of some topologically neutral combination of \( n_e \) complete sets of even kinks, \( n_o \) complete sets of odd kinks, and \( n_p \) particle-antiparticle pairs plus one of the following:

1. the kink pair \( \alpha \) and \( \beta \);
2. four even chirality kinks of the form \( \{\beta, \bar{\beta} \text{ and two even kinks } \gamma \neq \gamma', \gamma, \gamma' \neq \alpha, \beta\} \);
3. four odd chirality kinks of the form \( \{\alpha, \bar{\alpha} \text{ and two odd kinks, } \gamma \neq \gamma', \gamma, \gamma' \neq \alpha, \beta\} \).

(B19)

Each of the three combinations listed above carry the quantum numbers of \( J_{\alpha \beta} \). In case 1, \( n = 2 + 4n_e + 4n_o + 2n_p \). To recover the periodicity axiom in its initial form we must satisfy

\[ e^{i \pi \omega (J_{\alpha \beta}, \epsilon_n) \sum_{i=1}^{n-1}} P_{\epsilon_n \epsilon_i} P_{\epsilon_i \epsilon_n} = (-1)^{\sum_j f(\epsilon_j)}. \] (B20)

Using the expression for the semi-locality phase in Eqn. (B16) we see that

\[ e^{i \pi \omega (J_{\alpha \beta}, \epsilon_n) \sum_{i=1}^{n-1}} P_{\epsilon_n \epsilon_i} P_{\epsilon_i \epsilon_n} = (-1)^{n_e + n_o + n_p + 2} \] (B21)

while

\[ (-1)^{\sum_j f(\epsilon_j)} = (-1)^{n_e + 3n_o + n_p + 1}. \] (B22)

We thus see that in this case we recover Smirnov’s axiom for the non-Cartan currents. The other two cases in (B16) proceed identically.

The final axiom to consider is the annihilation pole axiom. If we transform the final equation of (B11) we obtain

\[ i \text{Res}_{\theta_n = \theta_{n-1} + 1 \pi} \int_0^\infty \frac{d\phi}{\phi} \int_0^{2\pi} d\theta_1 \cdots d\theta_n f^\infty (\theta_1, \ldots, \theta_n) \epsilon_1 \cdots \epsilon_n = f^\infty (\theta_1, \ldots, \theta_{n-2}) \epsilon_1' \cdots \epsilon_{n-2}' C_{\epsilon_n \epsilon_{n-1}} \prod_{i=1}^{n-2} P_{\epsilon_n \epsilon_i} P_{\epsilon_i \epsilon_{n-1}}. \]
We first demonstrate that
\[ \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} = 1 \] (B24)

Given that \( \epsilon_n \) and \( \epsilon_{n-1} \) are particle-antiparticle, the \( n-2 \) states remaining must be of the type given in Eqn. (B16) if the operator carries zero topological charge or of the form given in Eqn. (B14) if the operator is a non-Cartan current. In either case this product over \( \epsilon_i \) can be broken down into pairs of either two even chirality kinks, \( \{ \beta, \gamma \} \), two odd chirality kinks, \( \{ \beta, \gamma \} \), or one even kink and one odd kink, \( \{ \beta, \gamma \} \) (where \( \beta \) is not necessarily the anti-particle of \( \bar{\gamma} \)). That the above product is 1 then reduces to the following equalities which can easily be checked:
\[
\begin{align*}
P_{\alpha \beta} P_{\bar{\alpha} \beta} P_{\alpha \gamma} P_{\bar{\alpha} \gamma} & = 1; \\
P_{\alpha \beta} P_{\bar{\alpha} \beta} P_{\alpha \gamma} P_{\bar{\alpha} \gamma} & = 1; \\
P_{\alpha \beta} P_{\bar{\alpha} \beta} P_{\alpha \gamma} P_{\bar{\alpha} \gamma} & = 1;
\end{align*}
\] (B25)

where \( \beta \) and \( \gamma \) are arbitrary even kinks. Noting that \( \epsilon'_{n-1} = \epsilon_{n-1} \), we now demonstrate
\[
\prod_{i=1}^{n-2} P_{\epsilon_i \epsilon_{i+1}} = \prod_{i=1}^{n-2} P_{\epsilon_i \epsilon_{i+1}} \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} = \prod_{i=1}^{n-2} P_{\epsilon_i \epsilon_{i+1}} = 1.
\] (B26)

If we can do so, the axiom reverts to the form in Eqn. (B28) if we consider only the case \( \epsilon_{n-1} = \epsilon_{n-1} \)
\[
\frac{\delta_{\epsilon_{n-1}}}{\partial \epsilon_{n-1}} - e^{2\pi i \omega (O, \epsilon_{n-1})} \left( \prod_{i>j} P_{\epsilon_i \epsilon_j} P_{\epsilon'_i \epsilon'_j} \right) \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} \prod_{i=1}^{n-3} P_{\epsilon_{i+1} \epsilon_i} \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} (\theta_{n-1} - \theta_1) S_{\gamma_1 \epsilon_1}(\theta_{n-1} - \theta_2) \cdots \] (B23)

\[
\times S_{\gamma_1 \epsilon_1}(\theta_{n-1} - \theta_2) A_{\epsilon_{n-2}} \cdots A_{\epsilon_1} \cdots A_{\epsilon_1} (\theta_{n-1} - \theta_2).
\] (B28)

Applying the transformation, we know this must become
\[
A_{\epsilon_{n-1}} \cdots A_{\epsilon_1} = e^{2\pi i \omega (O, \epsilon_{n-1})} \left( \prod_{i>j} P_{\epsilon_i \epsilon_j} P_{\epsilon'_i \epsilon'_j} \right) \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} \prod_{i=1}^{n-3} P_{\epsilon_{i+1} \epsilon_i} \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} (\theta_{n-1} - \theta_1) S_{\gamma_1 \epsilon_1}(\theta_{n-1} - \theta_2) \cdots \] (B29)

For this to happen the following consistency condition must hold
\[
1 = \left( \prod_{i>j} P_{\epsilon_i \epsilon_j} P_{\epsilon'_i \epsilon'_j} \right) \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} \prod_{i=1}^{n-3} P_{\epsilon_{i+1} \epsilon_i} \prod_{i=1}^{n-2} P_{\epsilon_{i+1} \epsilon_i} (\theta_{n-1} - \theta_1) S_{\gamma_1 \epsilon_1}(\theta_{n-1} - \theta_2) \cdots \] (B30)

This consistency condition is equivalent to what we need to establish in Eqn. (B20) provided
\[
\prod_{i=1}^{n-2} P_{\epsilon_i \epsilon_{n-1}} = \prod_{i=1}^{n-2} P_{\epsilon'_i \epsilon_{n-1}} \] (B31)

If the \( \{ \epsilon_i \}_{i=1}^{n-2} \) are all kinks or anti-kinks then \( \{ \epsilon'_i \}_{i=1}^{n-2} \) is just a permutation and the above equality certainly holds. If there are kinks and anti-kinks, then a kink-anti-kink pair \( \alpha - \bar{\alpha} \) in \( \{ \epsilon_i \}_{i=1}^{n-2} \) may become \( \beta - \bar{\beta} \) in \( \{ \epsilon'_i \}_{i=1}^{n-2} \) but because
\[
P_{\alpha \epsilon_{n-1}} P_{\bar{\alpha} \epsilon_{n-1}} = P_{\beta \epsilon_{n-1}} P_{\bar{\beta} \epsilon_{n-1}},
\] (B32)
the above equality is unaffected.

5. Periodicity Axiom for Semi-local Operators of Section IV F

We are now able to write down the correct form for the periodicity axiom for the two-particle form-factors of the Sections III B 4. and 5. The periodicity axiom for the form factor of the scalar operator transforming as the 10 (Section III B 4.) in the C-symmetric basis is
\[
f_{[abc]}(\theta - 2\pi i) = f_{[abc]}(-\theta)
\] (B33)

The semi-locality phase in this case is 1. As this operator couples to two identical kinks, transforming to the non-C symmetric basis leaves the equality unchanged:
\[
f_{[abc]}(\theta - 2\pi i) = f_{[abc]}(-\theta)
\] (B34)
We now consider the periodicity axiom for the form factor of the scalar operator transforming as the $6$. For specificity we consider the operator

$$\psi_1 = e^{\frac{i}{2}(-\phi_2 + \theta_3 + \theta_4)}.$$  \hspace{1cm} (B35)

This operator couples to two same chirality kinks. In the $C$-symmetric basis that periodicity axiom reads

$$\tilde{f}_{\alpha\beta}(\theta - 2\pi i) = \begin{cases} f_{\alpha\beta}(\theta) & \text{for } \alpha, \beta \text{ even} \\ -f_{\alpha\beta}(\theta) & \text{for } \alpha, \beta \text{ odd} \end{cases}$$ \hspace{1cm} (B36)

We see that for this particular operator the even kinks coupling to the field have a non-trivial OPE whereas the odd kinks coupling do not. Transforming to the non-$C$ symmetric basis introduces a sign changes as $P_{\alpha\beta}P_{\beta\alpha} = -1$:

$$f_{\alpha\beta}(\theta - 2\pi i) = \begin{cases} f_{\alpha\beta}(\theta) & \text{for } \alpha, \beta \text{ even} \\ -f_{\alpha\beta}(\theta) & \text{for } \alpha, \beta \text{ odd} \end{cases}$$ \hspace{1cm} (B37)

If we had instead considered an operator of the form, $\psi_1 = e^{\frac{i}{2}(-\phi_2 - \theta_3 + \theta_4)}$ the role of even and odd kinks would be swapped. But regardless of which particular scalar operator transforming as the $6$ we consider, one type of kink will carry an additional sign in the periodicity axiom relative to the other.

### APPENDIX C: DERIVATION OF THE SUSCEPTIBILITY OF NON-INTERACTING LADDERS

The non-interacting susceptibility is given by

$$\chi_{11}^0(q_x, \omega) = \chi_{22}^0(q_x, \omega)$$

$$= -\frac{3}{4}(L_1(q_x) + L_2(q_x) + L_3(q_x) + L_4(q_x));$$

$$\chi_{12}^0(q_x, \omega) = \chi_{21}^0(q_x, \omega)$$

$$= -\frac{3}{4}(L_1(q_x) + L_2(q_x) - L_3(q_x) - L_4(q_x));$$

where the $L_i$’s are defined as follows:

\[
L_1(q_x, \omega) = \int d\tau dx e^{i\omega \tau - iq_x x} T \left( \left( \sum_{\sigma} \sigma c^\dagger_{\sigma \alpha}(x, \tau) c_{\sigma \alpha}(x, \tau) \right) \left( \sum_{\sigma} \sigma c^\dagger_{\sigma \beta}(0, 0) c_{\sigma \beta}(0, 0) \right) \right) \bigg|_{\omega \to -i\omega + \epsilon}
\]

\[
= -\frac{1}{2\pi} \left( \frac{1}{\Gamma(\omega, q_x)} \left( \tanh^{-1} \frac{4t \sin(q_x/2) \cos(K_{F\beta} + q_x/2)}{\Gamma(\omega, q_x)} \right) - \tanh^{-1} \frac{4t \sin(q_x/2) \cos(K_{F\beta} - q_x/2)}{\Gamma(\omega, q_x)} \right);
\]

\[
L_2(q_x, \omega) = \int d\tau dx e^{i\omega \tau - iq_x x} T \left( \left( \sum_{\sigma} \sigma c^\dagger_{\sigma \alpha}(x, \tau) c_{\sigma \alpha}(x, \tau) \right) \left( \sum_{\sigma} \sigma c^\dagger_{\sigma \beta}(0, 0) c_{\sigma \beta}(0, 0) \right) \right) \bigg|_{\omega \to -i\omega + \epsilon}
\]

\[
= -\frac{1}{2\pi} \left( \frac{1}{\Gamma(\omega, q_x)} \left( \tanh^{-1} \frac{4t \sin(q_x/2) \cos(K_{F\beta} + q_x/2)}{\Gamma(\omega, q_x)} \right) - \tanh^{-1} \frac{4t \sin(q_x/2) \cos(K_{F\beta} - q_x/2)}{\Gamma(\omega, q_x)} \right);
\]

\[
L_3(q_x, \omega) = \int d\tau dx e^{i\omega \tau - iq_x x} T \left( \left( \sum_{\sigma} \sigma c^\dagger_{\sigma \alpha}(x, \tau) c_{\sigma \alpha}(x, \tau) \right) \left( \sum_{\sigma} \sigma c^\dagger_{\sigma \beta}(0, 0) c_{\sigma \beta}(0, 0) \right) \right) \bigg|_{\omega \to -i\omega + \epsilon}
\]

\[
= L_{31}(\omega, q_x) + L_{32}(\omega, q_x) + L_{33}(\omega, q_x) + L_{34}(\omega, q_x);
\]

\[
L_{31}(\omega, q_x) = \frac{i}{4\pi} \int dq_1 \theta(2t \cos(q_1) + t_\perp) \frac{-i2t \cos(q_1) + i2t \cos(q_1 - q)}{(-i\omega + \epsilon - i2t_\perp)^2 + 4t^2(\cos(q_1) - \cos(q_1 + q))^2} = \frac{1}{2} L_1(\omega + 2t_\perp, q_x);
\]

\[
L_{32}(\omega, q_x) = \frac{i}{4\pi} \int dq_1 \theta(2t \cos(q_1) + t_\perp) \frac{-i2t \cos(q_1) + i2t \cos(q_1 - q)}{(-i\omega + \epsilon - i2t_\perp)^2 + 4t^2(\cos(q_1) - \cos(q_1 + q))^2} = \frac{1}{2} L_2(\omega + 2t_\perp, q_x);
\]

\[
L_{33}(\omega, q_x) = \frac{i}{4\pi} \int dq_1 \theta(2t \cos(q_1) + t_\perp) \frac{i\omega + i2t_\perp}{(-i\omega + \epsilon - i2t_\perp)^2 + 4t^2(\cos(q_1) - \cos(q_1 + q))^2} \left( \pi n_{K_{F\beta} + |q_x|/2} + \tan^{-1} \frac{\Gamma(\omega + 2t_\perp, q_x) \tan(K_{F\beta} + |q_x|/2)}{(-i\omega - i2t_\perp + \epsilon)} \right);
\]
\[ L_{34}(\omega, q_x) = \frac{i}{4\pi} \int d\Omega \theta(2t \cos(q_x) - t_\perp) \left( -iw + \epsilon - i2t_\perp \right)^2 + 4t^2 \cos(q_1, q_x)^2 \]
\[ = \frac{i}{4\pi} \Gamma(\omega + 2t_\perp, q_x) \left( \Gamma(\omega + 2t_\perp, q_x) \tan(\Gamma_{Fb} - |q_x|/2) \right) \left( -iw - i2t_\perp + \epsilon \right) \]
\[ + \text{sgn}(\Gamma_{Fb} - |q_x|/2) \left( \pi n \left| \Gamma_{Fb} - |q_x|/2 \right| + \tan^{-1} \left( \frac{\Gamma(\omega + 2t_\perp, q_x) \tan(\Gamma_{Fb} - |q_x|/2)}{(-i\omega - i2t_\perp + \epsilon)} \right) \right) \]
\[ L_4(\omega, q_x) = \int d\tau dx e^{i\omega t - iq_x x} \left( \sum_\sigma \sigma^a_{\alpha\beta}(x, \tau) c_{\alpha\beta}(x, \tau) \right) \left( \sum_\sigma \sigma^c_{\beta\sigma}(0, 0) c_{\sigma}(0, 0) \right) \big|_{\omega - i\epsilon} \]
\[ = L_3(\omega - 4t_\perp, q_x, K_{Fb} \leftrightarrow K_{Fab}), \quad (C2) \]

where \( \Gamma(\omega, q_x) = -(\omega + i\epsilon)^2 + 16t^2 \sin^2(q_x/2)^1/2 \) and \( n_q \) is an integer minimizing \( |n_q \pi - q| \).
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