Abelian varieties as automorphism groups of smooth projective varieties

Davide Lombardo and Andrea Maffei

Abstract
We determine which complex abelian varieties can be realized as the automorphism group of a smooth projective variety.

1 Introduction
In this note we determine which complex abelian varieties \( A \) can be realized as the automorphism group of a complex smooth projective variety. Given an abelian variety \( A \), we denote by \( \text{Aut}_0(A) \) (respectively \( \text{Aut}(A) \)) the automorphism group of \( A \) as an algebraic group (respectively as a projective variety). We prove that if \( \text{Aut}_0(A) \) is infinite then \( A \) can never be realized as the automorphism group of a smooth projective variety (Theorem 2.1), while if \( \text{Aut}_0(A) \) is finite there exists a smooth projective variety \( Y \) of dimension \( 2 + \dim A \) such that \( \text{Aut}(Y) = A \) (Theorem 3.9).

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2 Abelian varieties with infinite automorphism group
In this section we show that no abelian variety with infinite \( \text{Aut}_0(A) \) can be realized as the automorphism group of a smooth projective variety:

Theorem 2.1. Let \( A \) be an abelian variety such that \( \text{Aut}_0(A) \), the automorphism group of \( A \) as an algebraic group, is infinite. Let \( X \) be a smooth projective variety on which \( A \) acts faithfully: then the automorphism group of \( X \) is strictly larger than \( A \).

The proof relies on the following result, due to Brion [Bri10, page 2]:

Theorem 2.2. Let \( X \) be a smooth projective variety on which an abelian variety \( A \) acts faithfully. There is a positive integer \( n \) and a \( A[n] \)-invariant closed subscheme \( Y \) of \( X \) such that there is an \( A \)-equivariant isomorphism

\[ X \cong Y \times A[n] \quad A. \]

Proof. (of Theorem 2.1) Let \( \iota : A \hookrightarrow \text{Aut}(X) \) be the given action of \( A \) on \( X \) and write \( X \cong Y \times A[n] \) as in Theorem 2.2. We can represent \( X \cong Y \times A[n] \) more explicitly as the quotient

\[ X \cong \frac{Y \times A}{A[n]}, \]

where \( t \in A[n] \) acts on \((y,a)\) as \( t \cdot (y,a) = (\iota(t)(y),a - t) \). This quotient is well-behaved, because \( A[n] \) is a finite group acting on \( Y \times A \) with no fixed points. In particular, in order to give an (invertible) map \( X \to X \) it is enough to give an (invertible) map \( Y \times A \to Y \times A \) that is compatible with the action of \( A[n] \). Notice that, since \( A[n] \) is finite and stable under the action of \( \text{Aut}_0(A) \),
there exists a nontrivial automorphism \( \varphi \in \text{Aut}_0(A) \) that acts trivially on \( A[n] \). We claim that the automorphism \( \psi \) of \( Y \times A \) given by \((y,a) \mapsto (y,\varphi(a))\) descends to an automorphism \( \overline{\psi} \) of \( X \); since \( \overline{\psi} \) is not in the image of \( \iota \), this proves that \( \text{Aut}(X) \) is strictly larger than \( \iota(A) \). To see that \( \overline{\psi} \) descends to \( X \), it suffices to check that for every \( t \in A[n] \) we have \( \psi(t \cdot (y,a)) = t \cdot \psi((y,a)) \), that is,
\[
\psi((\iota(t)y, a - t)) = t \cdot (y, \varphi(a)) \iff (\iota(t)y, \varphi(a) - t) = (\iota(t)y, \varphi(a) - t);
\]
this last equality holds since \( \varphi \) is a group homomorphism and \( t \) is in \( A[n] \), which \( \varphi \) fixes pointwise. \( \square \)

3 Abelian varieties with finite automorphism group

We will now prove that any abelian variety such that \( \text{Aut}_0(A) \) is finite can be realized as the automorphism group of a smooth projective variety \( Y \). We first make some remarks on the structure of abelian varieties with finite automorphism group.

**Lemma 3.1.** Let \( A \) and \( B \) two isogenous abelian varieties then \( \text{Aut}_0(A) \) is finite if and only if \( \text{Aut}_0(B) \) is finite.

*Proof.* Since being isogenous is a symmetric relation, it suffices to prove that if \( A \to B \) is an isogeny and \( \text{Aut}_0(A) \) is infinite, then so is \( \text{Aut}_0(B) \). Write \( B \cong A/H \), where \( H \) is a finite subgroup of \( A \), and assume that \( \text{Aut}_0(A) \) is infinite. Notice that every automorphism \( \varphi \) of \( A \) leaves \( H \) stable induces an automorphism \( \overline{\varphi} \) of \( B \), and that \( \overline{\varphi} \) is trivial if and only if \( \varphi \) is trivial. Let \( n \) be the order of \( H \); in particular, we have \( H \subset A[n] \). Any automorphism \( \varphi \) of \( A \) leaves \( A[n] \) stable, so, since \( \text{Aut}_0(A) \) is infinite and \( A[n] \) is finite, the subgroup of automorphisms \( \varphi \) which fix \( A[n] \) pointwise is infinite. Every such automorphism leaves \( H \) stable, hence it descends to an automorphism of \( B \), and since the map \( \varphi \mapsto \overline{\varphi} \) is injective we deduce that \( \text{Aut}_0(B) \) is infinite. \( \square \)

**Lemma 3.2.** Let \( A \) be an abelian variety such that \( \text{Aut}_0(A) \) is finite. Then any two simple abelian subvarieties \( A_1, A_2 \) of \( A \) are isogenous if and only if they coincide. Moreover, if \( A_1 \) is a simple abelian subvariety of \( A \), then \( \text{Aut}_0(A_1) \) is finite.

*Proof.* Suppose by contradiction that we can find two distinct but isogenous simple abelian subvarieties \( A_1, A_2 \) of \( A \). By Poincaré’s reducibility theorem, there is an abelian subvariety \( C \) of \( A \) such that the multiplication map \( A_1 \times A_2 \times C \to A \) is an isogeny. Let \( B \) be an abelian variety such that there exists isogenies \( \varphi_i : B \to A_i \) and define the isogeny
\[
\varphi : B^2 \times C \to A \quad \text{by} \quad \varphi(b_1, b_2, c) = \varphi_1(b_1) + \varphi_2(b_2) + c.
\]
Now notice that \( \psi(b_1, b_2, c) = (b_1, b_1 + b_2, c) \) defines an automorphism of \( B^2 \times C \) of infinite order, and by the previous Lemma we conclude that \( \text{Aut}_0(A) \) is also infinite, contradiction. The proof that for any simple abelian subvariety \( A_1 \) of \( A \) the group \( \text{Aut}_0(A_1) \) is finite is completely analogous. \( \square \)

From now on we fix an abelian variety \( A \) with finite automorphism group \( \text{Aut}_0(A) \). By the previous Lemma and Poincaré reducibility Theorem we know that there exist uniquely determined simple abelian subvarieties \( A_1, \ldots, A_h \) of \( A \) such that the sum
\[
\sigma : A_1 \times \cdots \times A_h \to A \quad \sigma(a_1, \ldots, a_h) = a_1 + \cdots + a_h
\]
is an isogeny. We denote by \( \Sigma \) the finite kernel of this map and denote by \( N \) its order. By Lemma 3.2, \( A_i \) and \( A_j \) are not isogenous if \( i \neq j \) and \( \text{Aut}_0(A_i) \) is finite for all \( i \). Finally, notice that any abelian variety constructed in this way has finite automorphism group.
3.1 Construction of the example

Let $A$ be as above and choose a prime number $p \geq 7$ such that

(*) for $i = 1, \ldots, h$, for any subgroup $H$ of $A_i$ contained in $A[N]$, and for any nontrivial $\varphi \in \text{Aut}_0(A_i/H)$, $p$ is larger than the order of $\langle x \in A_i/H : \varphi(x) = x \rangle$.

Notice that if $\varphi$ is a nontrivial automorphism of a simple abelian variety then $\varphi$ has only finitely many fixed points, so a prime number $p$ with this property exists.

Let $S/\mathbb{C}$ be a smooth hypersurface of degree $p$ in $\mathbb{P}^3$ with $\text{Aut}(S) \cong \mathbb{Z}/p\mathbb{Z}$ and such that every automorphism of $S$ acts on it without any fixed points; an explicit example of such a hypersurface is given in Theorem 3.12. Let $G = \text{Aut}(S) \cong \mathbb{Z}/p\mathbb{Z}$ and set $X := S/G$. We now proceed to describe some basic properties of $X$ ([3.11]), construct a certain smooth projective variety $Y$ of dimension $2 + \dim A$ ([3.12]), and prove that $Y$ has automorphism group isomorphic to $A$ (Theorem 3.9 in [4.2]).

3.1.1 Properties of $X$

**Lemma 3.3.** $X$ is a smooth projective variety.

**Proof.** $X$ is smooth since $G$ acts on $S$ without fixed points, and is projective since any quotient of a projective variety by a finite group of automorphisms is projective (see [Ser58, Remarque on page 51]).

**Lemma 3.4.** $X$ does not admit any nontrivial automorphisms.

**Proof.** Let $\varphi : X \to X$ be an automorphism. Composing with the natural projection $\pi : S \to X$, we obtain a map $\varphi \circ \pi : S \to X$ which, since $S$ is simply connected, lifts to a map $\tilde{\varphi} : S \to S$. Clearly $\tilde{\varphi}$ is algebraic, and it is easily seen to be a covering map, so it is an isomorphism since $S$ is connected and simply connected. It follows that $\tilde{\varphi} : S \to S$ is in $G$, hence (by passing to the quotient) it induces the identity on $X$. Since on the other hand $\tilde{\varphi}$ induces $\varphi$ on $X$, we get $\varphi = \text{id}_X$ as claimed.

**Lemma 3.5.** $X$ has Kodaira dimension 2.

**Proof.** Kodaira dimension is invariant under finite étale covers, so $\kappa(X) = \kappa(S)$. By adjunction, $K_S = O_S(p - 3 - 1)|_S$ is ample, so $\kappa(S) = \dim(S) = 2$.

**Lemma 3.6.** The Albanese variety of $X$ is trivial, therefore there are no non-constant maps from $X$ to any abelian variety.

**Proof.** Clearly $S$ is the universal cover of $X$, so $\pi_1(X)$ is isomorphic to $\text{Aut}(S \to X) \cong \mathbb{Z}/p\mathbb{Z}$ and in particular is finite. Since the Albanese variety of $X$ is dual to its Picard variety, one has $\dim \text{Alb}(X) = \dim H^1(X, O_X) = h^{1,0}(X)$; on the other hand, the fact that $\pi_1(X)$ is finite implies that $H_1(X, \mathbb{Q})$ is trivial, so $h^{1,0}(X) \leq h^1(X) = \dim H^1(X, \mathbb{C}) = 0$, hence $\text{Alb}(X)$ is trivial as claimed.

3.1.2 A nontrivial $A$-torsor $Y \to X$

**Definition 3.7.** Fix an isomorphism $\chi : G \to \mathbb{Z}/p\mathbb{Z}$ and a point $P$ such that

(**) $P$ is a $p$-torsion point of $A$ which is not contained in any proper abelian subvariety of $A$.

The abelian subvarieties of $A$ are all of the form $A_{i_1} + \cdots + A_{i_k}$, so a point with this property exists. We let $\mathbb{Z}/p\mathbb{Z}$ act on the group generated by $P$ in the obvious way (that is, for $n \in \mathbb{Z}$ the class of $n$ in $\mathbb{Z}/p\mathbb{Z}$ sends $P$ to $nP$). We set $Y = (S \times A)/G$, where the action of $G$ on the product $S \times A$ is given by

$$g \cdot (s, a) = (g \cdot s, a + \chi(g)P).$$
As in the proof of Lemma 3.4 it is easy to see that $Y$ is a smooth projective variety; moreover, $Y$ has a natural structure of principal space under $A$. Indeed for each $b \in A$, the translation map

\[ S \times A \to S \times A \]

\[(s, a) \mapsto (s, a + b)\]

commutes with the action of $G$, so it descends to an automorphism of $Y = (S \times A)/G$ that we denote by $y \mapsto b + y$ or by $\tau_b$. This defines an action of $A$ on $Y$ which is free and transitive along the fibers of the map $Y \to X$. Moreover, $Y \to X$ is an $A$-torsor in the analytic (and in fact even étale) topology: indeed, $S$ is an étale covering of $X$, and the pullback of $Y$ to $S$ is trivial.

**Lemma 3.8.** The map $Y \to X$ does not admit a section (in the analytic topology).

**Proof.** Notice that $Y \to X$ admits a section if and only if it is trivial as a torsor. Indeed if $Y \to X$ has a section $s$ then the map $A \times X \to Y$ given by $(a, x) \mapsto a + s(x)$ is an isomorphism of torsors. Let $\mathcal{A}$ be the sheaf of holomorphic functions on $X$ with values in $A$; $A$-torsors on $X$ are classified by $H^1(X, \mathcal{A})$, where the cohomology is taken in the analytic category. For any fixed $n > 0$, consider the exact sequence of sheaves on $X$

\[ 0 \to \mathcal{A}[n] \to \mathcal{A} \xrightarrow{[n]} \mathcal{A} \to 0 \]

and take cohomology to obtain the long exact sequence

\[ 0 \to H^0(X, \mathcal{A}[n]) \to H^0(X, \mathcal{A}) \xrightarrow{[n]} H^0(X, \mathcal{A}) \to H^1(X, \mathcal{A}[n]) \to H^1(X, \mathcal{A}). \]

By Serre’s GAGA principle, all maps from $X$ to $A$ are algebraic, so by Lemma 3.6 we have $H^0(X, \mathcal{A}) = A$, and $H^0(X, \mathcal{A}) \xrightarrow{[n]} H^0(X, \mathcal{A})$ is just $A \xrightarrow{[n]} A$, which is surjective. It follows in particular that the natural arrow

\[ H^1(X, \mathcal{A}[n]) \to H^1(X, \mathcal{A}) \]

is injective. Consider $Z := (S \times \langle P \rangle)/G \hookrightarrow Y$, where $\langle P \rangle$ denotes the order $p$ subgroup of $A(\mathbb{C})$ generated by $P$. By the injectivity of (1) (with $n = p$), proving that $Z$ is a nontrivial covering space of $X$ suffices to show that $Y \to X$ is a nontrivial torsor. But this is clear, because the natural map $S \to S \times A \to (S \times \langle P \rangle)/G$ is injective and surjective, hence (since $S$ is compact) a homeomorphism. It follows that $Z \cong S$ is a nontrivial cover of $X$ as desired. \hfill \Box

### 3.2 Determination of $\text{Aut}(Y)$

In this section we show:

**Theorem 3.9.** The automorphism group of $Y$ is isomorphic to $A$.

#### 3.2.1 Preliminaries on simple abelian varieties

We shall need the following basic fact about simple abelian varieties.

**Lemma 3.10.** Let $T$ be a projective complex torus. Let $A$ be the abelian variety obtained from $T$ by fixing an arbitrary origin; notice that $T$ is naturally a torsor under $A$. Finally let $\alpha$ be an automorphism of $T$ (as a projective variety) and assume that $A$ is simple. Then:

1. if $\alpha$ is translation by a point of $A$, then the determinant of $(1 - \alpha)_* : H_1(T, \mathbb{Q}) \to H_1(T, \mathbb{Q})$ is 0;

2. if $\alpha$ is not translation by a point of $A$, then $\alpha$ has at least one fixed point and the determinant of $(1 - \alpha)_* : H_1(T, \mathbb{Q}) \to H_1(T, \mathbb{Q})$ is the number of fixed points of $\alpha$. 

Proof. The statement of (1) is obvious, because translations induce the identity on $H_1(T, \mathbb{Q})$. Assume now that $\alpha$ is not a translation and identify $T$ with $A$ by choosing a point $t_0 \in T$ as the origin. We prove first that $\alpha$ has at least one fixed point. Letting $a = \alpha(t_0) - t_0$ we have $\alpha(t) = \varphi(t) + a$, where $\varphi \in \text{Aut}_0(A)$ is different from the identity. Let $\psi = \varphi - \text{id}_A : A \to A$; it is an endomorphism of $A$, and since $A$ is simple and $\varphi$ is nontrivial the image of $\psi$ is $A$ itself. One checks that $b \in T$ is a fixed point of $\alpha$ if and only if $\psi(b) = -a$. As $\psi$ is surjective, such $b$ exist, and there are only finitely many of them because the set $\{b : \psi(b) = -a\}$ is naturally a torsor under the finite group $\ker \psi$. We can then choose the origin $t_0$ to be a fixed point of $\alpha$, in which case $\alpha$ belongs to $\text{Aut}_0(A)$ and we have $\psi(t) = \alpha(t) - t$, so that $A^\varphi$ is equal to the kernel of $\psi$ and its order is the degree of $\psi$. The lemma follows from the fact that for a complex torus $H_n(\psi, \mathbb{Q}) = \det(\psi_* : H_1(T, \mathbb{Q}) \to H_1(T, \mathbb{Q}))$. \qed

3.2.2 Preliminaries on surfaces of Kodaira dimension 2

We shall need the following consequence of [DHP08].

Lemma 3.11. Let $S$ be a surface of Kodaira dimension 2 and $A$ be an abelian variety. The image of any morphism $f : A \to S$ is either a point or a (possibly singular) irreducible curve of geometric genus at most one.

Proof. Suppose by contradiction that $f$ is surjective. Then by [DHP08 Theorem 1.1] the surface $S$ admits a finite étale cover which is a product of projective spaces and an abelian variety; since Kodaira dimension is invariant under finite étale covers, this contradicts the fact that the Kodaira dimension of $S$ is 2, because any such product has non-positive Kodaira dimension. So the image of $f$ can only be a single point or a curve, which is then automatically irreducible since $A$ is.

Suppose that the image of $f$, call it $Z$, is a curve, and let $\tilde{Z}$ be its normalization. By the universal property of normalization, $f$ induces a (dominant, hence surjective) map from $A$ to $\tilde{Z}$; applying [DHP08 Theorem 1.1] again we obtain that $\tilde{Z}$ is covered by either $\mathbb{P}^1$ or an elliptic curve, which proves the statement about the genus. \qed

3.2.3 Proof of Theorem 3.9

We already noticed that $A$ injects into $\text{Aut}(Y)$. For the other inclusion let $\varphi$ be an automorphism of $Y$. We prove first that $\varphi$ preserve the fibers of the map $\pi : Y \to X$. For each $x \in X$, let $Y_x$ be the fiber of $\pi$ over $x$ and let

$$\varphi_x : Y_x \to Y \xrightarrow{\pi} Y \xrightarrow{\pi} X.$$ 

Suppose that for general $x$ the image of $Y_x$ is not reduced to a single point: then Lemma 3.11 implies that generically the image of $\varphi_x$ is a (possibly singular) curve of genus at most 1. By [BHPVdV04 Proposition VII.2.1], a surface of Kodaira dimension 2 admits no algebraic system (of positive dimension) of effective divisors whose general member is a (possibly singular) rational or elliptic curve. By Lemma 3.5 we know that $X$ is a surface of Kodaira dimension 2, so it follows that $\varphi_x$ is constant for all $x \in X$. In particular,

$$Y \xrightarrow{\varphi} Y \to A\setminus Y = X$$

descends to a map $\varphi_X : X \to X$, which is easily seen to be birational (its inverse being $(\varphi^{-1})_X$), and hence an automorphism. It follows from Lemma 3.4 that $\varphi_X$ is the identity, which implies that the equality $\varphi(Y_x) = Y_x$ holds for all $x \in X$. Thus we see that for every $x \in X$ the automorphism $\varphi$ of $Y$ induces an automorphism $\varphi|_{Y_x}$ of $Y_x$.

Thus, locally in the analytic topology, the automomorphism $\varphi$ can be described as follows. For each $x \in X$ we can choose an open connected neighborhood $U \subset X$ of $x$ such that $V = \pi^{-1}(U)$ can be identified with $U \times A$ (as an $A$-torsor) and $\varphi(u, a) = (u, \phi(u, a))$. Let $r : U \to A$ be defined by $r(u) = \phi(u, 0) - 0$. Then $a \mapsto \phi(u, a) - r(u)$ is an automorphism of $A$ as an algebraic group, and since $\text{Aut}_0(A)$ is finite it must be equal to an automorphism $\phi$ independent of $u$. Hence $\varphi(u, a) = (u, \phi(a) + r(u))$ and $\psi = \phi - \text{id}_A$ is an endomorphism of $A$ as an algebraic group.
Furthermore, since any two identifications of a fiber of $Y \to X$ with the trivial $A$-torsor differ only by a translation, we see that the endomorphism $\psi$ thus obtained is independent of our choice of $U$ and of the local trivialization $\pi^{-1}(U) \cong U \times A$.

We now prove the theorem by induction on $h$, the number of simple factors of $A$. Assume first that $h = 1$, so that $A$ is simple. For $x \in X$ we define

$$n(x) = \det \left((1 - \varphi|_{Y_x})_* \mid H_1(Y_x, \mathbb{Q})\right);$$

it is a continuous function on $X$. Since $X$ is connected and $\mathbb{Z}$ is discrete, it follows that $n(x)$ is actually constant: let $n$ be the common value of the various $n(x)$. We show that $n = 0$. Suppose by contradiction that $n > 0$. Let $\tilde{X} = Y^n$ and let $\tilde{\pi}$ be the restriction of $\pi$ to $\tilde{X}$. We prove that $\tilde{\pi}$ is an $n$-to-1 covering of $X$. The fact that it is $n$-to-1 follows from Lemma 3.6. The claim that it is a covering can be checked locally using the analytic topology: using the local description above we obtain $V^n = \{(u, a) : \psi(a) = r(u)\}$, which is a covering of $U$.

If at least one of the connected components of $\tilde{X}$ is the trivial cover of $X$, then this gives a section of the projection map $\pi : \tilde{Y} \to X$, contradicting Lemma 3.8. Otherwise, take a connected subcover of $\tilde{X}$: this is a connected $m$-to-1 cover of $X$ for some $m \leq n$ which is smaller than $b$ by our assumption (*). This contradicts the fact that $\#\pi_1(X) = p$.

It follows that $n(x) = n = 0$ for all $x$, hence by Lemma 3.10 $\varphi|_{Y_x}$ is translation by a point $a(x) \in A$ (recall that $Y_x$ is naturally a torsor under $A$, so it makes sense to identify translations of $Y_x$ with elements of $A$). Now $x \mapsto a(x)$ gives a map $X \to A$, which is necessarily constant by Lemma 3.6 hence $\varphi$ is globally a translation by a point of $A$.

We now prove the inductive step. Let $h > 1$. Since $\varphi$ preserves the fibers $Y_x$, composing with a translation by an element of $A$ we can assume that there exists $y_0 \in Y$ such that $\varphi(y_0) = y_0$. We want to prove that in this case $\varphi$ is the identity. Let $\pi : A \to A' := A/A_1$ be the natural projection and set $A'_i := \pi(A_i)$ for $i = 2, \ldots, h$. We let $P' = \pi(P)$ and write $\tilde{\pi} : A_1 \times \cdots \times A_h \to A'_2 \times \cdots \times A'_h$ for the homomorphism

$$\tilde{\pi}(a_1, \ldots, a_h) = (\pi(a_2), \ldots, \pi(a_h));$$

finally, we set $\Sigma' := \tilde{\pi}(\Sigma)$. One then checks that the sum $\sigma' : A'_2 \times \cdots \times A'_h \to A'$ is an isogeny with kernel $\Sigma'$.

Let $K = \ker(\Sigma \to \Sigma')$. For every $i = 2, \ldots, h$, the intersection $A_1 \cap A_i$ embeds naturally into $K$, so $N' : \#(A_1 \cap A_i) \mid N' \mid \#K = N$. It follows that every quotient of $A_1 = A_1/(A_1 \cap A_i)$ by a subgroup of $A'_i[N']$ is a quotient of $A_1$ by a subgroup of $A_1[N]$, so the analogue of condition (*) is satisfied by $A'$ and the prime $p$. It is immediate to check that (***) also holds for $A'$, $p$, and the point $P'$. In particular, by induction, the automorphism group of $Y' = S \times A'$ is equal to $A'$.

The projection map $S \times A \to S \times A' = G$-equivariant, so it induces a map $q : Y \to Y'$ which we prove to be a categorical quotient by the action of $A_1$. Indeed let $f : Y \to Z$ be an $A_1$-invariant map. It induces a $G \times A_1$-invariant map $f_1 : S \times A \to Z$ and therefore a map $f_2 : S \times (A_1 \times \cdots \times A_h) \to Z$ which is invariant by the action of both $A_1$ and $\Sigma$ on the second factor. Since the quotient of $A_1 \times \cdots \times A_h$ by the subgroup generated by $A_1$ and $\Sigma$ is $A'$, the map $f_2$ induces a regular map $g_2 : S \times A' \to Z$ such that $f_2 = g_2 \circ (id_S \times \pi')$, where $\pi' := \pi \circ \sigma$ is the natural map $A_1 \times \cdots \times A_h \to A'$. Since furthermore $f_2$ is $G$-invariant, $g_2$ is also $G$-invariant, hence it induces a map $g : Y' \to Z$ such that $f = g \circ q$. Moreover, as $q$ is surjective, the map $g$ is unique.

We can now prove that $\varphi$ is the identity. For $a \in A$ denote by $\tau_a$ the translation by $a$ in $Y$. Notice that for each $a$ and for each $x \in X$ there exists $\phi_x \in \text{Aut}_0(A)$ such that

$$\varphi \circ \tau_a \circ \varphi^{-1} = \tau_{\phi_x(a)} : Y_x \to Y_x.$$ 

In particular, if $a \in A_1$, then $\phi_x(a) \in A_1$. Being $Y'$ a categorical quotient of $Y$ by the action of $A_1$, we have that $\varphi$ induces a map $\varphi' : Y' \to Y'$, which is an automorphism since $(\varphi^{-1})'$ is its inverse. Moreover, the image of $y_0$ in $Y'$ is fixed by $\varphi'$, so $\varphi'$ is equal to the identity.

Hence $\varphi(y) - y \in A_1$ for all $y \in Y$. Arguing in the same way, but using $A_2$ instead of $A_1$, we obtain $\varphi(y) - y \in A_2$ for all $y$. So $\varphi(y) - y \in A_1 \cap A_2$ for all $y \in Y$, and since $A_1 \cap A_2$ is finite and $\varphi(y_0) = y_0$ we obtain $\varphi(y) = y$ for all $y$. \qed
3.3 A hypersurface in $\mathbb{P}^3$ with automorphism group $\mathbb{Z}/p\mathbb{Z}$

In this section we explicitly construct, for every prime $p \geq 7$, an algebraic surface in $\mathbb{P}^3$ of degree $p$ whose automorphism group is cyclic of order $p$:

**Theorem 3.12.** Let $p \geq 7$ be a prime number, and for $\lambda \in \mathbb{C}$ let $S_\lambda$ be the algebraic surface over $\mathbb{C}$ given by the zero locus in $\mathbb{P}^3$ of the homogeneous polynomial

$$f_\lambda(x_1, x_2, x_3, x_4) := x_1^p + x_2^p + x_3^p + x_4^p + \lambda(x_1^2 x_2^{p-4} x_3^2 + x_1^4 x_2^2 x_4^2).$$

The surface $S_\lambda$ is smooth for all but finitely many $\lambda \in \mathbb{C}$; if $\lambda \neq 0$, the automorphism group of $S_\lambda$ is cyclic of order $p$, generated by $[x_1 : x_2 : x_3 : x_4] \mapsto [x_1 : \zeta_p x_2 : \zeta_p^2 x_3 : \zeta_p^3 x_4]$, where $\zeta_p$ is a primitive $p$-th root of unity. Moreover, each nontrivial element of $\text{Aut}(S_\lambda)$ acts on $S_\lambda$ without any fixed points.

We start by noticing that for $\lambda = 0$ the surface $S_0$ is smooth. Since being smooth is a Zariski-open condition in the defining polynomial, this shows that $S_\lambda$ is smooth away from a proper Zariski-closed subset of $\mathbb{C}$, that is, $S_\lambda$ is smooth for all but finitely many values of $\lambda$. From now on fix a nonzero value of $\lambda$ such that $S_\lambda$ is smooth, and to simplify the notation write $S$ for $S_\lambda$ and $f(x_1, x_2, x_3, x_4)$ for $f_\lambda(x_1, x_2, x_3, x_4)$.

By [14, Theorem 2] we know that all the automorphisms of $S$ are induced by (linear) automorphisms of $\mathbb{P}^3$, so we only need to consider these. Let $L : \mathbb{P}^3 \to \mathbb{P}^3$ be a linear transformation that satisfies $L(S) = S$. We identify $L$ to the class $[M] \in \text{PGL}_4(\mathbb{C})$ of a matrix $M = (M_{ij}) \in \text{GL}_4(\mathbb{C})$. Furthermore, we let $e_1, \ldots, e_4$ be the canonical basis of $\mathbb{C}^4$ and denote by $\langle e_i \rangle$ the 1-dimensional $\mathbb{C}$-vector subspace of $\mathbb{C}^4$ generated by $e_i$. We shall show Theorem 3.12 in three steps: first we shall prove that $M$ either fixes or permutes the lines generated by $e_3$ and $e_4$; then we shall show that the same statement holds for the lines generated by $e_1$ and $e_2$; finally, we shall deduce from this that $M$ needs to be a diagonal matrix, at which point a direct computation concludes the proof. This approach is inspired by [14].

3.3.1 Step 1: $M$ permutes $\langle e_3 \rangle$ and $\langle e_4 \rangle$

The condition that $L(S) = S$ translates into the polynomial equality

$$f \circ M(x_1, \ldots, x_4) = \alpha f(x_1, \ldots, x_4)$$

for some $\alpha \in \mathbb{C}^\times$. Applying $\frac{\partial^2}{\partial x_i \partial x_j}$ to the two members of this equation and setting

$$H_{ij}(x_1, \ldots, x_4) := \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_4)$$

we find

$$\sum_k \sum_m M_{kj} M_{mi} H_{mk}(M(x_1, \ldots, x_4)) = \alpha H_{ij}(x_1, \ldots, x_4).$$

Let $u, v$ be two vectors in $\mathbb{C}^4$. Multiplying the previous identity by $u_i v_j$ and summing over $i$ and $j$ we get

$$\sum_{k,m} (Mu)_k (Mv)_m H_{mk}(M(x_1, \ldots, x_4)) = \alpha \sum_{i,j} H_{ij}(x_1, \ldots, x_4) u_i v_j. \quad (3)$$

We now define a bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}^4, \quad (u, v) \mapsto \sum_{i,j} H_{ij}(x_1, \ldots, x_4) u_i v_j,$$

so that Equation 3 reads

$$\langle Mu, Mv \rangle (M(x_1, \ldots, x_4)) = \alpha \langle u, v \rangle.$$

In particular, since $M$ is invertible we obtain:
Proposition 3.13. Let \( u, v \) be vectors in \( \mathbb{C}^4 \). The equalities \( \langle u, v \rangle = 0 \) and \( \langle Mu, Mv \rangle = 0 \) are equivalent.

Lemma 3.14. Let \( a, b \in \mathbb{C}^4 \) be two nonzero vectors such that \( \langle a, b \rangle = 0 \). Then there exist \( \lambda, \mu \in \mathbb{C}^x \) such that either \( a = \lambda e_3, b = \mu e_4 \), or \( a = \lambda e_4, b = \mu e_3 \) hold.

Proof. Write \( a = (a_1, a_2, a_3, a_4) \) and \( b = (b_1, b_2, b_3, b_4) \). By direct inspection, one checks that, for \( i = 1, 2, 3, 4 \), the only second derivative of \( f \) involving the monomial \( x_i^{p-2} \) is \( H_i \). This immediately implies that \( a_i b_i = 0 \) for \( i = 1, \ldots, 4 \), and by symmetry we can assume \( a_1 = 0 \). The coefficients of the monomials \( x_1 x_2^{p-5} x_3^1, x_1 x_2^{p-4} x_3^1 \) and \( x_1 x_2^{p-6} x_4^1 \) in \( \langle a, b \rangle \) are given by \( 2\lambda(p-4)(a_2 b_1 + a_1 b_2), 4\lambda(a_3 b_1 + a_1 b_3) \) and \( 8\lambda(a_4 b_1 + a_1 b_4) \) respectively, so under our assumptions \( \langle a, b \rangle = 0, \lambda \neq 0 \) and \( a_1 = 0 \) we obtain \( b_1 a_2 = b_1 a_3 = b_1 a_4 = 0 \). If we had \( b_1 \neq 0 \), this would imply \( a = (0, 0, 0, 0) \), contradicting our assumptions, so we must have \( b_1 = 0 \) as well. The situation is now again symmetric in \( a, b \), so we might assume \( a_2 = 0 \). Arguing as before (but looking at the monomials \( x_1 x_2^{p-5} x_3^1 \) and \( x_1 x_2^{p-7} x_4^1 \) one finds \( a_3 b_2 = a_4 b_2 = 0 \), so that \( b_2 = 0 \) as well. The conclusion now follows easily from the equalities \( a_3 b_3 = a_4 b_4 = 0 \).

Corollary 3.15. One of the following holds:

- \( M \langle e_3 \rangle = \langle e_3 \rangle \) and \( M \langle e_4 \rangle = \langle e_4 \rangle \);
- \( M \langle e_3 \rangle = \langle e_4 \rangle \) and \( M \langle e_4 \rangle = \langle e_3 \rangle \).

Proof. Apply Proposition 3.13 to \( u = e_3 \) and \( v = e_4 \); since \( \langle e_3, e_4 \rangle = H_{34} = 0 \) we obtain \( \langle Me_3, Me_4 \rangle = 0 \). The claim then follows from the previous lemma.

3.3.2 Step 2: \( M \) permutes \( \langle e_1 \rangle \) and \( \langle e_2 \rangle \)

Arguing as in the previous section, it is easily seen that if we let \( A : (\mathbb{C}^4)^p \to \mathbb{C} \) denote the multilinear form

\[
A : (u_1, \ldots, u_p) \mapsto \sum_{i_1, \ldots, i_p} \frac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_p}} (u_1)_{i_1} \cdots (u_p)_{i_p},
\]

where \( (u_i)_j \) is the \( j \)-th coordinate of \( u_i \), we have \( A(M u_1, \ldots, M u_p) = \beta A(u_1, \ldots, u_p) \) for some \( \beta \in \mathbb{C}^x \); notice that here we do not need to compose with \( M \) on the left hand side, because \( p \)-th derivatives of \( f \) are just scalars. Suppose that \( M \langle e_3 \rangle = \langle e_3 \rangle \) and \( M \langle e_4 \rangle = \langle e_4 \rangle \); the case \( M \langle e_3 \rangle = \langle e_4 \rangle \) and \( M \langle e_4 \rangle = \langle e_3 \rangle \) is completely analogous. Rescaling \( M \) if necessary (which we can do, since we are only interested in its projective class) we can assume \( Me_3 = e_3 \). Choosing \( u_1 = \cdots = u_{p-1} = e_3 \) and \( u_p = e_1 \) we have

\[
\beta A(\ldots, e_3, e_1) = \beta \frac{\partial^p f}{\partial x_3^{p-1} \partial x_1} = 0,
\]

from which we deduce

\[
0 = A(M e_3, \ldots, Me_3, Me_1) = A(e_3, \ldots, e_3, Me_1) = \sum_{i_p} \frac{\partial^p f}{\partial x_3^{p-1} \partial x_{i_p}} (Me_1)_{i_p};
\]

since the only nonvanishing partial derivative of the form \( \frac{\partial^p f}{\partial x_3^{p-1} \partial x_{i_p}} \) is \( \frac{\partial^p f}{\partial x_3^{p-1} \partial x_1} \), this implies \( M_{31} = 0 \). Similarly, the choice \( (e_4, \ldots, e_4, e_1) \) shows \( M_{41} = 0 \), while the choices \( (e_3, \ldots, e_3, e_2) \) and \( (e_4, \ldots, e_4, e_2) \) give \( M_{32} = M_{42} = 0 \). It follows that \( M \) sends the 2-plane \( \{ x_3 = x_4 = 0 \} \) to itself; in particular, \( M \) induces an automorphism of the finite set of points in \( \mathbb{P}^3 \) defined by the equations

\[
f(x_1, x_2, x_3, x_4) = 0, \quad x_3 = x_4 = 0 \iff x_3 = x_4 = 0, \quad x_1^3 + x_2^3 = 0.
\]

From this it is immediate to deduce:

Corollary 3.16. One of the following holds:

- \( M \langle e_1 \rangle = \langle e_1 \rangle \) and \( M \langle e_2 \rangle = \langle e_2 \rangle \);
- \( M \langle e_1 \rangle = \langle e_2 \rangle \) and \( M \langle e_2 \rangle = \langle e_1 \rangle \).
3.3.3 Step 3: determination of $\text{Aut}(S)$

Corollaries 3.15 and 3.16 tell us that $M$ either fixes or permutes the lines $\langle e_1 \rangle$, $\langle e_2 \rangle$, and that the same holds for the lines $\langle e_3 \rangle$, $\langle e_4 \rangle$. One checks easily that if $M$ exchanges $\langle e_1 \rangle$ with $\langle e_2 \rangle$, and/or it exchanges $\langle e_3 \rangle$ with $\langle e_4 \rangle$, then $f \circ M$ is not a scalar multiple of $f$, so that $M$ needs to be a diagonal matrix. Normalize $M$ so that $M_{11} = 1$ and write $M = \text{diag}(1, \mu_2, \mu_3, \mu_4)$: replacing in Equation (2) and comparing the coefficients of $x^p_1$ on the two sides we find $\alpha = 1$. Comparing the coefficients of $x^p_i$ for $i = 2, 3, 4$ we then obtain $\mu^p_i = 1$ for $i = 2, 3, 4$, so that $\mu_2, \mu_3, \mu_4$ are $p$-th roots of unity. It is now immediate to check that the only automorphisms of $S$ are represented by the powers of the (order $p$) matrix

\[
\begin{pmatrix}
1 & \zeta_p & \zeta_p^2 & \zeta_p^3 \\
\zeta_p & 1 & \zeta_p & \zeta_p^2 \\
\zeta_p^2 & \zeta_p & 1 & \zeta_p \\
\zeta_p^3 & \zeta_p^2 & \zeta_p & 1
\end{pmatrix}
\]

where $\zeta_p$ is a primitive $p$-th root of unity.

The fixed points (in $\mathbb{P}^3$) for the action of this matrix (or any of its powers, with the exception of the identity) are $[1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1]$, none of which lies on the hypersurface $f(x_1, x_2, x_3, x_4) = 0$. This concludes the proof of Theorem 3.12.

References

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. Compact complex surfaces, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 2004.

[Bri10] Michel Brion. Some basic results on actions of nonaffine algebraic groups. In Symmetry and spaces, volume 278 of Progr. Math., pages 1–20. Birkhäuser Boston, Inc., Boston, MA, 2010.

[DHP08] Jean-Pierre Demailly, Jun-Muk Hwang, and Thomas Peternell. Compact manifolds covered by a torus. J. Geom. Anal., 18(2):324–340, 2008.

[MM64] Hideyuki Matsumura and Paul Monsky. On the automorphisms of hypersurfaces. J. Math. Kyoto Univ., 3:347–361, 1963/1964.

[Poo05] Bjorn Poonen. Varieties without extra automorphisms. III. Hypersurfaces. Finite Fields Appl., 11(2):230–268, 2005.

[Ser58] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique $p$. In Symposium internacional de topología algebraica International symposium on algebraic topology, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.