AS-REGULAR ALGEBRAS FROM ACYCLIC SPHERICAL HELICES

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Abstract. We discuss a construction of AS-regular algebras from acyclic spherical helices, generalizing works of Bondal and Polishchuk [BP93] and Van den Bergh [VdB11].

Let \( \mathcal{C} \) be an idempotent-complete pretriangulated dg category over a field \( k \). See e.g. [Kel06] and references therein for basic definitions on dg categories. For \( X, Y \in \mathfrak{D}b(\mathcal{C}) \), we write the dg vector space of morphisms in \( \mathcal{C} \) and its \( i \)-th cohomology as \( \text{hom}(X, Y) \) and \( \text{Hom}^i(X, Y) \) respectively.

A \( \mathcal{C} \)-module is a dg functor \( \mathcal{C}^{op} \to \text{Mod} k \) to the dg category of dg vector spaces. A dg functor \( F: \mathcal{C} \to \mathcal{C}' \) will be identified with its graph bimodule \( \Gamma_F \in \text{Mod}(\mathcal{C}^{op} \otimes \mathcal{C}') \).

Assume that \( \mathcal{C} \) is proper in the sense that \( \text{hom}(X, Y) \in \text{per} k \) for any \( X, Y \in \mathfrak{D}b(\mathcal{C}) \). Assume further that \( \mathcal{C} \) has a Serre functor \( S \) in the sense of [BKS9]. The graph of the Serre functor is given by

\[
\text{hom}(Y, S(X)) = \text{hom}(X, Y)^*,
\]

where \((-)^* := \text{hom}_k(-, k): (\text{Mod} k)^{op} \to \text{Mod} k\).

A prototypical example of an idempotent-complete proper pretriangulated dg category admitting a Serre functor is the category of perfect complexes on a proper Gorenstein scheme \( X \). The Serre functor is the tensor product with the canonical bundle followed by the shift by the dimension of \( X \); \( S(-) = (-) \otimes \omega_X[\text{dim} X] \).

Let \( d \) be a natural number. An object \( S \) of \( \mathcal{C} \) is said to be spherical of dimension \( d \) if \( S(S) \simeq S[d] \) and

\[
\text{Hom}^i(S, S) \cong \begin{cases} k & i = 0, d, \\ 0 & \text{otherwise.} \end{cases}
\]

The spherical twist along \( S \) is the functorial cone \( T_S \) over the evaluation morphism \( \text{ev}: \text{hom}(S, -) \otimes S \to \text{id}_\mathcal{C} \). It is an autoequivalence of the dg category per \( \mathcal{C} \) of perfect \( \mathcal{C} \)-modules, whose inverse is given by the dual twist \( T_S^\vee \) defined as the cone over the coevaluation morphism \( \text{coev}: \text{id}_\mathcal{C} \to \text{hom}(-, S)^\vee \otimes S \) shifted by \(-1 \) [ST01].

Given a sequence \( S = (S_i)_{i=1}^\ell \) of spherical objects, let \( \mathcal{B} = \mathcal{B}_S \) be the full subcategory of \( \mathcal{C} \) consisting of \( \{S_i\}_{i=1}^\ell \). We identify per \( \mathcal{B} \) with the full idempotent-complete pretriangulated subcategory of \( \mathcal{C} \) generated by \( \mathcal{B} \).

Definition 1. A sequence \( S = (S_i)_{i=1}^\ell \) of spherical objects is a spherical collection if the Serre functor of \( \mathcal{C} \) preserves per \( \mathcal{B} \) and acts as the shift functor there; \( S|_{\text{per} \mathcal{B}} \simeq [d] \).

The number \( \ell \) of objects in the collection is called the length of the collection.

The directed subcategory \( \mathcal{A} = \mathcal{A}_S \) of \( \mathcal{B} \) is the dg category whose set of objects is \( \{E_i\}_{i=1}^\ell \) and whose dg vector spaces of morphisms are given by

\[
\text{hom}_\mathcal{A}(E_i, E_j) := \begin{cases} \text{hom}_\mathcal{B}(S_i, S_j) & i < j, \\ k \cdot \text{id}_{E_i} & i = j, \\ 0 & i > j. \end{cases}
\]

The differentials and the compositions of morphisms in \( \mathcal{A} \) are inherited from that of \( \mathcal{B} \) in the obvious way.

Since the collection \( S \) is spherical, the \( \mathcal{A} \)-bimodule \( \mathcal{B}/\mathcal{A} \), defined as the cone of the natural morphism \( \mathcal{A} \to \mathcal{B} \) of \( \mathcal{A} \)-bimodules, is isomorphic to the shift \( \mathcal{A}^\ast[-d] \) of the graph \( \mathcal{A}^\ast := \text{hom}_k(\mathcal{A}, k) \) of the Serre functor. In other words, the category \( \mathcal{B} \) is a noncommutative anticanonical divisor of \( \mathcal{A} \) in the sense of [Sei17].
The sequence \((E_i)_{i=1}^\ell\) is a full exceptional collection of \(\mathsf{per}\ A\). The existence of a full exceptional collection implies that per \(A\) is proper and smooth (in the sense that the diagonal bimodule is perfect), so that per \(A\) has a Serre functor.

Let \(\iota_*: \mathsf{per} B \to \mathsf{per} A\) be the dg functor sending a \(B\)-module to the same module considered as an \(A\)-module by the embedding \(A \hookrightarrow B\). The graph of the left adjoint functor \(\iota^* : \mathsf{per} A \to \mathsf{per} B\) is \(B\) considered as an \(A^{op} \otimes B\)-module. The right adjoint functor is given by \(\iota^* = \mathcal{S}_B \circ \iota^* \circ \mathcal{S}^{-1}_A \simeq \iota^* \circ \mathcal{S}^{-1}_A [d]\).

The dual twist \(T_i^\vee\) along \(\iota_*\) is defined as the shift by \(-1\) of the cone over the unit \(\iota^* \circ \iota^*\) of the adjunction. The graph of the dual twist is the \((A\otimes B)^{op}\)-module. The right adjoint functor is given by \(\iota^* = \mathcal{S}_B \circ \iota^* \circ \mathcal{S}^{-1}_A [d]\).

The dual cotwist \(C_i^\vee\) along \(\iota_*\), defined as the cone over the counit \(\iota^* \circ \iota^* \to \text{id}_A\) of the adjunction, is right adjoint to \(T_i^\vee\) by [AL17] Proposition 5.3, and hence quasi-isomorphic to \(\mathcal{S}^{-1}[d+1]\).

The dual cotwist \(C_i^\vee\) along \(\iota_*\), defined as the cone over the counit \(\iota^* \circ \iota^* \to \text{id}_B\), is described explicitly as the (iterated) cone of
\[
\begin{align*}
\text{hom}(S_\ell, -) \otimes \text{hom}(S_{\ell-1}, S_\ell) \otimes \cdots \otimes \text{hom}(S_1, S_2) \otimes S_1 & \to \cdots \\
& \to \bigoplus_{1 \leq i_1 < i_2 \leq \ell} \text{hom}(S_{i_1}, S_{i_2}) \otimes S_{i_1} \to \bigoplus_{i=1}^\ell \text{hom}(S_i, -) \otimes S_i \to \text{id}_B,
\end{align*}
\]
which is quasi-isomorphic to the composition of spherical twists;
\[
C_i^\vee \simeq T_{S_1} \circ T_{S_2} \circ \cdots \circ T_{S_\ell}.
\]
Since both \(T_i\) and \(C_i^\vee\) are equivalences, the functor \(\iota_*\) is spherical in the sense of [AL17] Theorem 1.1. We expect that \(\iota_*\) has a relative Calabi–Yau structure in the sense of [BD19], which is a slight strengthening of an identification of the spherical twist with a shift of the inverse Serre functor.

For \(1 \leq i \leq \ell - 1\), the right mutation of a spherical collection \(S = (S_j)_{j=1}^\ell\) at position \(i\) is defined by
\[
R_i(S) := (S_1, \ldots, S_{i-1}, S_{i+1}, T_{S_{i+1}}^\vee S_i, S_{i+2}, \ldots, S_\ell).
\]
The inverse operation is the left mutation defined by
\[
L_i(S) := (S_1, \ldots, S_{i-1}, T_{S_{i}} S_{i+1}, S_{i+2}, \ldots, S_\ell).
\]
The left mutation \(L_i(A)\) of \(A\) at position \(i\) is the full subcategory of per \(A\) consisting of \(E_j\) for \(j \neq i + 1\) and
\[
L_E E_{i+1} := \text{Cone} \left( \text{hom}(E_i, E_{i+1}) \otimes E_i \xrightarrow{e_i} E_{i+1} \right).
\]
The passage to the directed subcategory commutes with mutations; \(A_{L_i(S)} \simeq L_i(A_S)\).

The helix generated by an exceptional collection \((E_i)_{i=1}^\ell\) is the sequence \((E_i)_{i\in \mathbb{Z}}\) of exceptional objects of per \(A\) satisfying
\[
E_{i-\ell} = L_{E_i \leftarrow E_{i+1}} \circ L_{E_{i+2} \leftarrow E_{i+1}} \circ \cdots \circ L_{E_{i-\ell-1} \leftarrow E_{i-\ell}}[d-1],
\]
for any \(i \in \mathbb{Z}\). The shift is chosen in such a way that \(E_{i-\ell} \simeq \mathcal{S}(E_i)[-d-1]\), so that if per \(A \simeq D^b \text{coh} X\) for a smooth projective variety \(X\) of dimension \(d + 1\), then one has \(E_{i-\ell} \simeq E_i \otimes \omega_X\). The length \(\ell\) of the exceptional collection generating the helix is called the period of the helix. For any \(i \in \mathbb{Z}\), the exceptional collection \((E_j)_{j=i}^{i+\ell-1}\) consisting of \(\ell\) consecutive objects in the helix is called a foundation of the helix. We say that a helix \((E_i)_{i\in \mathbb{Z}}\) is acyclic if \(\text{Hom}^\neq(E_i, E_j) = 0\) for any \(-\infty < i < j < \infty\).

Given an invertible \(A\)-bimodule \(\theta\), the tensor algebra over \(A\) is defined by
\[
T_A(\theta) := \bigoplus_{i=0}^\infty \theta^i \otimes A^i
\]
\[
= A \oplus \theta \oplus (\theta \otimes A \theta) \oplus \cdots
\]
\[
\simeq \bigoplus_{i=0}^\infty \bigoplus_{j,k=1}^{\ell} \text{hom}(E_j, \theta^i(E_k)).
\]
In addition to the cohomological grading coming from that of $\mathcal{A}$ and $\theta$, the tensor algebra $T_\mathcal{A}(\theta)$ has an additional grading such that $\theta \otimes A^i$ has degree $i$. The positively graded part $T_\mathcal{A}(\theta)^+ := \bigoplus_{i=1}^{\infty} \theta \otimes A^i \simeq \theta \otimes A T_\mathcal{A}(\theta)$ is the kernel (the cone shifted by $-1$) of the natural morphism $\sigma^i : T_\mathcal{A}(\theta) \to \mathcal{A}$.

The category $\text{per} T_\mathcal{A}(\theta)$ is a generalization of the derived category of coherent sheaves on the total space of a line bundle, and the pull-back $\sigma_* : \text{per} \mathcal{A} \to \text{per} T_\mathcal{A}(\theta)$ along $\sigma^i$ is a generalization of the push-forward along the zero section. Its left adjoint $\sigma^* : \text{per} T_\mathcal{A}(\theta) \to \text{per} \mathcal{A}$ is given by tensoring $\mathcal{A}$ over $T_\mathcal{A}(\theta)$. Since the graph of $\sigma^* \circ \sigma_*$ is

$$\text{Cone}(\theta \otimes A T_\mathcal{A}(\theta) \to T_\mathcal{A}(\theta)) \otimes T_\mathcal{A}(\theta) \mathcal{A} \simeq \text{Cone}(\theta \to \mathcal{A}),$$

the dual cotwist along $\sigma_*$ is the autoequivalence $\theta[1]$ of $\text{per} \mathcal{A}$. See also [Seg18] for a related construction.

The push-forward $\pi^* := (-) \otimes A T_\mathcal{A}(\theta) : \text{per} \mathcal{A} \to \text{per} T_\mathcal{A}(\theta)$ along the structure morphism $\pi^i : \mathcal{A} \to T_\mathcal{A}(\theta)$ of the tensor algebra is a generalization of the pull-back along the projection to the base space. For any pair $(E, F)$ of objects of $\text{per} \mathcal{A}$, one has

$$\text{hom}(\sigma_* E, \pi^* F) \simeq \text{hom}(\pi^* E \otimes T_\mathcal{A}(\theta) \text{Cone}(\theta \otimes A T_\mathcal{A}(\theta) \to T_\mathcal{A}(\theta)), \pi^* F))$$

$$\simeq \text{hom}(\pi^* E, \pi^* F \otimes T_\mathcal{A}(\theta) \text{Cone}(T_\mathcal{A}(\theta) \to \theta^{-1} \otimes A T_\mathcal{A}(\theta))[-1]))$$

$$\simeq \text{hom}(\pi^* E, \sigma_*(\theta^{-1}(F))[1])$$

$$\simeq \text{hom}(E, \theta^{-1}(F))[-1].$$

The dg category $\text{gr} T_\mathcal{A}(\theta)$ of graded perfect $T_\mathcal{A}(\theta)$-modules is a generalization of the derived category of equivariant coherent sheaves on the total space of a line bundle with respect to the $\mathbb{G}_m$-action by fiberwise dilation. The dg $\mathbb{Z}$-algebra $A = \bigoplus_{i,j=-\infty}^{\infty} A_{ij}$ defined by

$$A_{i+m\ell,j+n\ell} = \text{hom}(\theta^m(E_j), \theta^n(E_i))$$

for $1 \leq i, j \leq \ell$ and $-\infty < m \leq k < \infty$ is $\ell$-periodic, and the 1-periodic dg $\mathbb{Z}$-algebra $A' = \bigoplus_{i,j=-\infty}^{\infty} A'_{ij}$ defined by

$$A'_{km} = \bigoplus_{i=(k-1)\ell+1}^{k\ell-1} \bigoplus_{j=(m-1)\ell+1}^{m\ell} A_{ij}$$

is the dg $\mathbb{Z}$-algebra associated to the $\mathbb{Z}$-graded algebra $T_\mathcal{A}(\theta)$. One has quasi-equivalences

$$\text{gr} T_\mathcal{A}(\theta) \simeq \text{mod} A' \simeq \text{mod} A$$

of dg categories.

Let $(F_{ij})^1_{i=\ell}$ be the exceptional collection in $\text{per} \mathcal{A}$ right dual to $(E_i)^\ell_{i=1}$ so that $\text{hom}(E_i, F_j) \simeq k$ if $i = j$ and 0 otherwise [Bon89, Section 7]. Then $E_i$ (resp. $F_i$) represents the $i$-th projective (resp. simple) $\mathcal{A}$-module, and $\pi^i E_i$ (resp. $\sigma_i F_i$) represents the $i$-th projective (resp. simple) $T_\mathcal{A}(\theta)$-module.

The dg algebra $T_\mathcal{A}(\theta)$ in the case $\theta = S^{-1}[d + 1]$ is known as the derived $(d + 1)$-preprojective algebra, which is a model of the trivial Calabi–Yau completion [Kel11];

$$\Pi_{d+1}(\mathcal{A}) := T_\mathcal{A}(S^{-1}[d + 1]).$$

It is given explicitly as

$$\bigoplus_{i=0}^{\infty} \bigoplus_{j,k=1}^{\ell} \text{hom}(E_j, S^{-i}(E_k)[i(d + 1)]) \simeq \bigoplus_{j=1}^{\infty} \bigoplus_{k=1}^{\infty} \text{hom}(E_j, E_k),$$

and also known as the rolled-up helix dg algebra. The dg $\mathbb{Z}$-algebra $A$ in this case will be called the helix dg algebra.
For $1 \leq i, j \leq \ell$, one has

\begin{align}
(23) \quad & \hom (\sigma_+ E_i, \pi^* E_j) \simeq \hom (F_i, S(E_j)[-d - 1]) [-1] \\
(24) \quad & \simeq \hom (E_j, F_i)[-d - 2] \\
(25) \quad & \simeq \begin{cases} 
k[-d - 2] & i = j, \\
0 & \text{otherwise.}
\end{cases}
\end{align}

An object of $\text{gr } T_A(\theta)$ is said to be a torsion if the grading is bounded. Let $\text{tor } T_A(\theta)$ be the full subcategory of $\text{gr } T_A(\theta)$ consisting of torsion modules, and $\text{qgr } T_A(\theta) := \text{gr } T_A(\theta)/\text{tor } T_A(\theta)$ be the dg quotient.

**Proposition 2.** One has an equivalence $\text{qgr } T_A(\theta) \simeq \per A$.

**Proof.** One has a semiorthogonal decomposition

\begin{equation}
\text{tor } T_A(\theta) = \langle S_i \rangle_{i \in \mathbb{Z}},
\end{equation}

where $S_i \simeq \per A$ is the admissible subcategory consisting of graded modules concentrated in degree $i$. One has

\begin{equation}
\text{gr } T_A(\theta) = \langle \ldots, S_{-2}, S_{-1}, (\text{gr } T_A(\theta))_{\geq 0} \rangle
\end{equation}

just as in [Orl09, Lemma 14], where $(\text{gr } T_A(\theta))_{\geq 0}$ is the full subcategory of $\text{gr } T_A(\theta)$ consisting of non-negatively graded modules. It follows from the quasi-isomorphism

\begin{equation}
\mathcal{A} \simeq \text{Cone}(T_A(\theta)^+ \to T_A(\theta))
\end{equation}

of graded $T_A(\theta)$-modules that the free module $T_A(\theta)$ is right orthogonal to $\langle S_i \rangle_{i \geq 0}$, and that its shift $T_A(\theta)(i)$ for any $i < 0$ is in the full pretriangulated subcategory generated by $T_A(\theta)$ and $\langle S_i \rangle_{i \geq 0}$. Hence $T_A(\theta)$ generates the right orthogonal to $\langle S_i \rangle_{i \geq 0}$ in $\langle \text{gr } T_A(\theta) \rangle_{\geq 0}$, so that $\text{qgr } T_A(\theta)$ is equivalent to the full subcategory $\langle T_A(\theta) \rangle$ of $\text{gr } T_A(\theta)$, which in turn is equivalent to per $\mathcal{A}$. □

**Definition 3.** A spherical helix generated by a spherical collection $(S_i)_{i=1}^\ell$ is a sequence $(S_i)_{i \in \mathbb{Z}}$ of spherical objects of $\mathcal{C}$ satisfying

\begin{equation}
S_{i-\ell} = T_{S_{i-\ell+1} \circ T_{S_{i-\ell+2} \circ \cdots \circ T_{S_{i-1}}}}[-d - 1]
\end{equation}

for any $i \in \mathbb{Z}$.

The length $\ell$ of the spherical collection generating the spherical helix is called the period of the spherical helix. For any $i \in \mathbb{Z}$, the spherical collection $(S_j)_{j=1}^{i+\ell-1}$ consisting of $\ell$ consecutive objects in the spherical helix is called a foundation of the spherical helix $\mathcal{H}$.

**Definition 4.** A spherical helix $(S_i)_{i \in \mathbb{Z}}$ is acyclic if $\text{Hom}^{\neq 0}(S_i, S_j) = 0$ for any $-\infty < i < j < \infty$.

One has

\begin{align}
(30) \quad & \hom(S_i, S_j) \simeq \hom(t^* E_i, t^* E_j) \\
(31) \quad & \simeq \hom(E_i, t_* t^* E_j) \\
(32) \quad & \simeq \hom(E_i, \text{Cone}(E_{j-\ell} \to E_j)),
\end{align}

which immediately yields the following:

**Theorem 5.** If the spherical helix $(S_i)_{i \in \mathbb{Z}}$ is acyclic, then the helix $(E_i)_{i \in \mathbb{Z}}$ is acyclic.

Recall that a connected $\mathbb{Z}$-algebra $A = \bigoplus_{i,j=-\infty}^{\infty} A_{ij}$ over $k$ is said to be AS-Gorenstein if

\begin{equation}
\sum_{j,k=-\infty}^{\infty} \dim \text{Hom}^k(S_i A, P_{j, A}) = 1
\end{equation}

for any $i \in \mathbb{Z}$, where $S_{i, A}$ and $P_{j, A}$ are the $i$-th simple and projective $A$-modules respectively. It follows from (25) that the acyclicity of the helix $(E_i)_{i \in \mathbb{Z}}$ implies the AS-Gorenstein property of the helix $\mathbb{Z}$-algebra $A$. Since $\mathcal{A}$ is smooth and $\theta$ is perfect, the tensor algebra $T_A(\theta)$ is smooth, so
that $A$ has finite global dimension (see e.g. [Kel11], Theorem 4.8]). The Hilbert polynomial of $A$ is determined by the projective resolutions of the simples.

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