Algebraic Characterization of $\mathbb{C}$-Regular Fractions Under Level Permutations

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Abstract
In this paper we study the behavior of the fractions of a factorial design under permutations of the factor levels. We code the $s$ levels of a factor with the $s$-th roots of the unity. We focus on the notion of regular fraction in the complex coding, called $\mathbb{C}$-regularity. We introduce methods to check whether a given symmetric orthogonal array can or cannot be transformed into a regular fraction by means of suitable permutations of the factor levels. The proposed techniques take advantage of the complex coding of the factor levels and of some tools from polynomial algebra. Several examples are described, mainly involving factors with five levels.

Keywords Algebraic statistics · Complex coding · Fractional factorial designs · Indicator function · Isomorphic fractions · Orthogonal arrays · Regular fractions

Mathematics Subject Classification 62K15 · 05B15 · 13P10

1 Introduction
In design of experiments the use of fractions of a full factorial design plays an important role when the observation of the response variable at each of the possible level combinations of the factors is impracticable. In general, the selected level combinations must satisfy optimality criteria to conveniently measure the impact of factors and their interactions on the mean and on the variability of the response variable. In this framework, orthogonal arrays and regular fractions are widely used and both are based on properties of orthogonality among factors. The notion
of orthogonality has two main interpretations that coincide in the two level cases: vector orthogonality and combinatorial orthogonality. Vector orthogonality allows one to construct linear models with uncorrelated factor effect estimators, but this concept is relevant only with quantitative factors. Combinatorial orthogonality easily applies both in qualitative and in quantitative cases. Here we say that two factors or interactions are combinatorially orthogonal if all the pairs of level combinations appear equally often in their interaction (i.e., their product).

In this paper we consider the complex coding of the factor levels, extensively studied in Pistone and Rogantin [19] for asymmetric and multilevel designs. With such a coding, a definition of regularity, called $\mathbb{C}$-regularity, is introduced, see Definition 3 in Sect. 2. Given a $\mathbb{C}$-regular fraction, there is a class of permutations of the factor levels for which combinatorial orthogonality always implies vector orthogonality, see Proposition 4. This equivalence between the two orthogonalities is always true for symmetric designs with up to three levels under the complex coding. The last property is no longer true for other codings. A simple counterexample on a $3^3$ design is described in Cheng and Ye [3] where the integer coding is used.

We study how the orthogonal properties of a fraction change in the presence of permutations of the factor levels, and in particular we give some methods to check whether a given fraction with qualitative factors is or not isomorphic (or equivalent) to a $\mathbb{C}$-regular fraction up to permutations of the factor levels. In a $\mathbb{C}$-regular fraction the factors and their interactions are either orthogonal or totally confounded, so that regular fractions are special orthogonal arrays.

Two fractional factorial designs are said to be isomorphic if one of them can be obtained from the other by reordering the runs, relabeling the factors and/or permuting the levels of one or more factors. If a factor is quantitative, only the reverse permutation is allowed. Usually, the isomorphism is referred as geometric when the factors are quantitative, and as combinatorial when the factors are qualitative. Mixed situations can occur in practice, see Katsaounis [13].

Different methods to check the isomorphism between fractions have been introduced in the literature. Clark and Dean [4, 14] and Katsaounis [13] give necessary and sufficient conditions for isomorphism, based on the Hamming distance between two fractions, while Katsaounis et al. [15] use the singular value decomposition of the design matrix for binary designs. Ma et al. [17] and Pang and Liu [18] consider an approach using the centered $L_2$ discrepancy for geometric isomorphism. In the latter paper, an algorithm with low complexity is presented. We emphasize that most of the algorithms in the literature are specific for binary factors, and even when defined in general for multilevel factors they are exemplified in the three levels case.

A necessary condition for isomorphism between two fractions is that they have the same generalized word length pattern (GWLP) and this condition does not depend on the level coding, see Wu and Hamada [23]. Fontana et al. [9] derive a formula, computationally easy and of clear interpretation, for computing the GWLP of a fraction, based on the mean aberration. In recent years, the research of coding-invariant quantities to analyze mixed factorial designs is increasing. For instance, Grömping [10] accounts for a new coding-invariant decomposition of the generalized word counts for screening combinatorial equivalence of designs with qualitative factors.
Also different definitions of regularity have been proposed in the literature. All of them are based on specific codings for the factor levels, integer coding, Galois field coding, complex coding, eigenvector coding. Some of them have limitation on the numbers of levels (prime or prime power numbers). In Pistone and Rogantin [20] the nonequivalence between Galois field regularity and \( \mathbb{C} \)-regularity is discussed. Grömping and Bailey [11] introduce different definitions of regularity: geometric regularity, \( R^2 \)-regularity and CC-regularity and discuss their mutual relationships. Tang and Xu [22] essentially deal with an inverse problem: to find factor level permutations of a regular fraction to reduce contamination of nonnegligible interactions on the estimation of linear effects without increasing the run size.

In this paper we use tools from algebraic statistics, and in particular the polynomial indicator function, to specify a fraction and its orthogonality and regularity properties, the polynomial representation of the permutations of the factor levels, and the representation through Latin squares of orthogonal arrays of strength two. In this framework, the coding of the \( s \) levels of a factor by the \( s \)-th roots of the unity is essential. The methodology introduced here to detect isomorphism between fractions applies to symmetric designs with \( s \) prime and the given examples consider \( s \geq 5 \).

The paper is organized as follows. In Sect. 2 some relevant results of the algebraic theory of fractional factorial designs are presented. In Sect. 3 the polynomial representation of factor level permutations is analyzed. Using polynomial conditions on the coefficients of such a representation, an algorithm to check whether two fractions are isomorphic is given and some examples, implemented in the symbolic software CoCoA-5, are shown. In particular fractions of a \( 5^3 \) design are checked to be isomorphic or not to a \( \mathbb{C} \)-regular one. In Sect. 4 the investigation if a fraction is isomorphic to a \( \mathbb{C} \)-regular one or not is approached using the Latin square representation of orthogonal arrays of strength 2. When the isomorphism exists, the relevant level permutations and the generating equations of the \( \mathbb{C} \)-regular fraction are recovered, by exploiting the properties of the complex coding of the levels. Such a check is based on the nullity of all the \( 2 \times 2 \) minors of the multilayer Latin squares in the numeric complex field. Several examples are discussed to show how to actually apply the proposed method. Finally, some further developments are illustrated in Sect. 5.

## 2 Algebraic Characterization of Fractional Designs

In this section we present some relevant results of the algebraic theory of fractional designs. The interested reader can find further information, including the proofs of the propositions, in Pistone and Rogantin [19] and in Fontana et al. [9].

Let us consider an experiment with \( m \) factors observed at \( s \) levels each, with \( s \) a prime number.

Let us code the \( s \) levels with the \( s \)-th roots of the unity \( \omega_k = \exp\left(\sqrt{-1} \frac{2\pi}{s} k\right) \), \( k = 0, \ldots, s - 1 \). We denote the level set by \( \Omega_s = \{\omega_0, \ldots, \omega_{s-1}\} \).
As $\alpha = \beta \mod s$ implies $\omega_k^\alpha = \omega_k^\beta$, it is useful to introduce the residue class ring $\mathbb{Z}_s$ and the notation $[k]$, for the residue of $k \mod s$. For integer $\alpha$, we obtain $(\omega_k^\alpha)^{\alpha} = \omega_{\alpha k}$. We also have $\omega_h \omega_k = \omega_{[h+k]}$. We drop the sub-$s$ notation when there is no ambiguity.

We denote by $D$ the full factorial design with complex coding: $D = \Omega_s^m$; the cardinality of the full factorial design is $#D = s^m$.

We denote by $L$ the exponent set of the complex coded design: $L = \mathbb{Z}_s^m$. Notice that $L$ is both the exponent set of the complex coded design and the integer coded design. The elements of $L$ are denoted in vector notation by $\alpha, \beta, \ldots$:

$$L = \{\alpha = (\alpha_1, \ldots, \alpha_m) : \alpha_j = 0, \ldots, s - 1, j = 1, \ldots, m\};$$

$[\alpha - \beta]$ is the $m$-tuple $([\alpha_1 - \beta_1], \ldots, [\alpha_m - \beta_m])$.

In order to use polynomials to represent all the functions defined over $D$, we define

- $X_j$, the $j$-th component function, which maps a point of $D$ to its $j$-th component, $X_j : D \ni (\xi_1, \ldots, \xi_m) \mapsto \xi_j$. The function $X_j$ is a simple term or, by abuse of terminology, a factor.
- $X^\alpha = X_{\alpha_1}^\alpha_1 \cdot \ldots \cdot X_{\alpha_m}^\alpha_m$, $\alpha \in L = \mathbb{Z}_s^m$, i.e., the monomial function $X^\alpha : D \ni (\xi_1, \ldots, \xi_m) \mapsto \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_m^{\alpha_m}$. The function $X^\alpha$ is an interaction term. As $s$ is a prime number, the interaction $X^\alpha$ takes values in $\Omega_s$.

The set of monomials $\{X^\alpha : \alpha \in L\}$ is an orthonormal basis of all the complex functions defined over $D$, i.e., the complex coding we use is a normalized orthogonal coding, as defined in Xu and Wu [24].

Since we will make use occasionally of factors with a nonprime number of levels, the behavior of the factors and of the interactions is summarized below.

- Let $X_i$ be a simple term with level set $\Omega_i$. Let us define $h = s / \text{gcd}(r, s)$, and let $\Omega_h$ be the set of the $h$-th roots of the unity. The term $X_i^h$ takes all the values of $\Omega_h$ equally often over $D$.
- Let $X^\alpha = X_{\alpha_{j_1}}^{\alpha_{j_1}} \cdot \ldots \cdot X_{\alpha_{j_k}}^{\alpha_{j_k}}$ be an interaction term of order $k$ where $X_{\alpha_{j_i}}^{\alpha_{j_i}}$ takes values in $\Omega_{h_i}$. Let us define $h = \text{lcm}\{h_{j_1}, \ldots, h_{j_k}\}$. The interaction $X^\alpha$ takes all the values of $\Omega_h$ equally often over $D$.

Let $F$ be a subset of the full factorial design $D$. We consider here only fractions without replications.

**Definition 1** The indicator function $F$ of a fraction $F$ is a complex polynomial defined over $D$ such that for each $\zeta \in D$, $F(\zeta)$ is equal to 1 if $\zeta \in F$ and equal to 0 if $\zeta \in D \setminus F$. We denote by $b_\alpha$ the coefficients of the representation of $F$ on $D$ using the monomial basis $\{X^\alpha, \alpha \in L\}$.
\[ F(\zeta) = \sum_{a \in L} b_a X^a(\zeta), \quad \zeta \in D, \quad b_a \in \mathbb{C}. \quad (2.1) \]

**Proposition 1** Let \( P \) be a fraction with indicator function \( F \).

1. The coefficients \( b_a \) of \( F \) are given by:

\[
b_a = \frac{1}{\#D} \sum_{\zeta \in P} X^{[a]}(\zeta) = \frac{1}{\#D} \sum_{h=0}^{s-1} n_{a,s-h} \omega_h
\]

where \( n_{a,h} \) is the number of the occurrences of \( \omega_h \) in \( \{X^a(\zeta) : \zeta \in P\} \).

In particular \( b_0 = \#P/\#D \).

2. The term \( X^a \) is centered on \( P \) if, and only if, \( b_a = b_{[-a]} = 0 \).
3. The terms \( X^\alpha \) and \( X^\beta \) are orthogonal on \( P \) if, and only if, \( b_{[\alpha - \beta]} = 0 \);
4. If \( X^a \) is centered then, for each \( \beta \) and \( \gamma \) such that \( \alpha = [\beta - \gamma] \) or \( \alpha = [\gamma - \beta] \), \( X^\beta \) is orthogonal to \( X^\gamma \).
5. Let \( s \) be prime. Then, the term \( X^a \) is centered on \( P \) if, and only if, its \( s \) levels appear equally often: \( n_{a,0} = \cdots = n_{a,s-1} \).

As stated in the proposition above, the coefficients \( b_a \) encode interesting properties of the fraction such as orthogonality among factors and interactions.

A common choice to select an experiment is to use orthogonal array. The assumption that interactions greater than a specified order are not present is translated into a combinatorial property on the frequency of the levels in the fraction. Let us denote with \( \text{OA}(n, s^m, t) \) a symmetric orthogonal array with \( n \) rows and \( m \) columns, in which each column has \( s \) symbols, and of strength \( t \), as defined, e.g., in Wu and Hamada [23]. Strength \( t \) means that for every selection of \( t \) columns of the matrix, all the elements of the relevant product space appear equally often in the \( t \) columns.

**Definition 2** Let \( I \) be a nonempty subset of \( \{1, \ldots, m\} \). A fraction \( P \) factorially projects on the \( I \)-factors if the projection of \( P \) on the \( I \)-factors is a full factorial design where each point appears \( k \) times. A fraction \( P \) is an orthogonal array of strength \( t \) if it factorially projects on any \( I \)-factors with \( \#I = t \).

The proposition below shows a connection between the combinatorial definition of orthogonal array introduced above and the coefficients of the indicator function in Eq. (2.1).

**Proposition 2** A fraction is an orthogonal array of strength \( t \) if, and only if, all the coefficients of the indicator function up to the order \( t \) are zero:

\[ b_a = 0 \quad \forall \ \alpha \text{ of order up to } t, \ \alpha \neq (0, \ldots, 0). \]
Regular fractions are a subset of the orthogonal arrays. In a regular fraction any two simple or interaction terms are either orthogonal or totally confounded.

**Definition 3** (C-regularity) A fraction $\mathcal{F}$ is C-regular if there exist a sup-group $\mathcal{L}$ of $L$, a group homomorphism $e$ from $\mathcal{L}$ to $\Omega_s$ such that the set of equations

$$\{X^\alpha = \omega_{e(\alpha)} : \alpha \in \mathcal{L}\}$$

(2.3)

defines the fraction $\mathcal{F}$. If $\mathcal{H}$ is a minimal generator of the group $\mathcal{L}$, the set of equations $\{X^\alpha = \omega_{e(\alpha)} : \alpha \in \mathcal{H}\}$ is called the set of defining equations of $\mathcal{F}$.

**Proposition 3** Let $\mathcal{F}$ be a fraction. The following statements are equivalent:

1. The fraction $\mathcal{F}$ is C-regular according to Definition 3.
2. The indicator function of the fraction has the form

$$F(\xi) = \frac{1}{\#\mathcal{L}} \sum_{\alpha \in \mathcal{L}} \omega_{e(\alpha)} X^\alpha(\xi) \quad \xi \in D.$$ 

where $\mathcal{L}$ is a given subset of $L$ and $e : \mathcal{L} \rightarrow \Omega_s$ is a given mapping.
3. For each $\alpha, \beta \in L$ the interactions represented on $\mathcal{F}$ by the terms $X^\alpha$ and $X^\beta$ are either orthogonal or totally aliased.

Finally, we recall two basic definitions of isomorphic fractions. For details see, e.g., Dean et al. [7].

**Definition 4** Two fractions are

- geometrically isomorphic if one design can be obtained from the other by reordering the runs, relabeling the factors and/or reversing the level order of one or more factors;
- combinatorially isomorphic if one can be obtained from the other by reordering the runs, relabeling the factors and/or switching the levels of one or more factors.

The geometric isomorphism pertains to quantitative factors, while the combinatorial isomorphism pertains essentially to qualitative factors. In this paper we focus on the notion of combinatorial isomorphism.

From the definition of the indicator function, it follows immediately that a reordering of the runs does not affect the polynomial representation of the indicator function. Moreover, a relabeling of the factors simply permutes the subscripts. From the expression of the indicator function, it is relatively simple to find the relevant relabeling. Therefore, the most interesting task in analyzing the isomorphism of fractions is to study the behavior of the fractions under permutations of the factor levels.
3 Polynomial Representation of the Factor Level Permutations

In this section, first we give an account of the polynomial representation of the level permutations for a single factor. Then, we extend this representation to several factors. As shown below, for factors with up to three levels, the \( \mathbb{C} \)-regularity is not affected by level permutations. As a consequence, when factors with \( s > 3 \) levels are involved, it is enough to check \((s - 2)!\) permutations for each factor, with a notable saving of computational time with respect to other choices of coding. Finally, we make explicit the polynomial representation of such permutations and we show that they have simple expressions as functions of an inverse Vandermonde matrix, easy to compute. This allows us to use symbolic software in order to check isomorphism and in particular isomorphism to a \( \mathbb{C} \)-regular fraction. The use of symbolic software in this kind of algorithms is not competitive, in terms of CPU time, with respect to other combinatorial algorithms for isomorphism check, and nevertheless a polynomial algorithm may have a noticeable mathematical interest and may represent the basis for further developments.

A level permutation is a function from \( \Omega_s \) to \( \Omega_s \), and it always admits a polynomial representation. In special cases, such polynomial reduces to a monomial, and therefore a permutation of the levels does not affect the \( \mathbb{C} \)-regularity of a fraction.

**Proposition 4** A \( \mathbb{C} \)-regular fraction is transformed into a \( \mathbb{C} \)-regular fraction by the group of transformations generated by the following level permutations on the factor \( X_j \):

1. Cyclical permutations \( X_j \rightarrow \omega_k X_j \) with \( k = 0, \ldots, s - 1 \).
2. Power permutations, for \( s \) prime, \( X_j \rightarrow X_j^h \) with \( h = 1, \ldots, s - 1 \).

Permutations of types 1 and 2, \( X_j \rightarrow \omega_k X_j^h \), produce \( s(s - 1) \) permutations on the factor \( X_j \), and produce the following transformed monomial on the term \( X^a \):  

\[
\prod_{j=1}^{m} \omega_{[a_i, k_j]} X_j^{[a_i, h_j]}.
\]  

(3.1)

The proof of Proposition 4 is in Pistone and Rogantin [19]. We observe only that under monomial permutations the absolute values of the coefficients of the indicator function do not change, so that a \( \mathbb{C} \)-regular fraction is transformed into a \( \mathbb{C} \)-regular fraction.

We highlight again that, for factors with two or three levels, all the level permutations have a monomial representation.

**Remark 1** From Proposition 4 the monomial representation of a geometric isomorphism follows. In fact, for quantitative factors, the two admissible level permutations
are both in monomial form: \( Y_j = X_j \) (the identity) and \( Y_j = \omega_{s-1} X_j^{s-1} \) (the reversing of the factor levels).

Now we characterize the polynomial representation of the permutations for general multilevel factors. Such a characterization leads to a criterion to actually check whether a given fraction may be or may be not transformed into a \( \mathbb{C} \)-regular fraction after permutation of the levels of one or more factors.

Let \( X_j \) be a factor with \( s \) levels, and let \( \pi \) be a permutation of the level set \( \Omega_s \). We denote by \( Y_j \) the transformed factor, \( Y_j = \pi(X_j) \). The polynomial representation of \( Y_j \) is

\[
Y_j = \sum_{h=0}^{s-1} u_h X_j^h \quad u_h \in \mathbb{C}
\]  

(3.2)

with \( \pi(\omega_i) = \sum_{h=0}^{s-1} u_h \omega_i^h \). The \( u_h \)'s coefficients are the solutions of the linear system

\[
\begin{pmatrix}
\omega_0^0 & \omega_0^1 & \cdots & \omega_0^{s-1} \\
\omega_1^0 & \omega_1^1 & \cdots & \omega_1^{s-1} \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{s-1}^0 & \omega_{s-1}^1 & \cdots & \omega_{s-1}^{s-1}
\end{pmatrix}
\begin{pmatrix}
u_0 \\ u_1 \\ \vdots \\ u_{s-1}
\end{pmatrix} =
\begin{pmatrix}
\pi(\omega_0) \\ \pi(\omega_1) \\ \vdots \\ \pi(\omega_{s-1})
\end{pmatrix}
\]

(3.3)

The matrix appearing in Eq. (3.3) is a Vandermonde matrix \( V \). If we denote by \( v_{h+1,k+1} \), with \( h, k = 0, \ldots, s - 1 \), the generic element of \( V \), it is known that the inverse \( V^{-1} \) of \( V \) has generic element \( v_{h+1,k+1}^{-1} = s \cdot \omega_{[s\cdot k]} \). In our case, from the results in Sect. 2:

\[
v_{h+1,k+1} = \omega_h^k = \omega_{[hk]} \quad \text{and} \quad v_{h+1,k+1}^{-1} = \frac{1}{s} \omega_{[s\cdot k]}
\]

and the resulting system is

\[
\begin{pmatrix}
u_0 \\ u_1 \\ \vdots \\ u_{s-1}
\end{pmatrix} = \frac{1}{s}
\begin{pmatrix}
\omega_0 & \omega_0 & \cdots & \omega_0 \\
\omega_1 & \omega_1 & \cdots & \omega_1 \\
\vdots & \vdots & \ddots & \vdots \\
\omega_{s-1} & \omega_{s-1} & \cdots & \omega_{s-1}
\end{pmatrix}
\begin{pmatrix}
\pi(\omega_0) \\ \pi(\omega_1) \\ \vdots \\ \pi(\omega_{s-1})
\end{pmatrix}
\]

(3.4)

Full details on the Vandermonde matrices for the roots of the unity, their properties and applications to complex interpolations can be found, e.g., in Corless and Fillion [5].

Constraints on the coefficients \( u_0, \ldots, u_{s-1} \) derive from the expression of \( V^{-1} \) and from the fact that \( \pi \) is a permutation, as in the proposition below.

**Proposition 5** The coefficients \( u_i \)'s must satisfy the following equations:
(i) \( u_0 = 0 \);
(ii) for all \( q = 2, \ldots, s - 1 \),
\[
\sum_{h_1, \ldots, h_{s-1}=0}^{s-1} u_{h_1} \cdots u_{h_{q-1}} u_{[-h_1, \ldots, -h_{q-1}]} = 0; \tag{3.5}
\]
(iii) given a permutation \( \pi \), we have \( \sum_{h=1}^{s-1} u_h = \pi(\omega_0) \),
(iv) \( (\sum_{h=1}^{s-1} u_h)^s - 1 = 0 \).

Proof Let \( \pi \) be a permutation of \( \Omega_s \). For item (i), observe that the first row in system (3.4) leads to
\[
u_0 = \frac{1}{s} \sum_{i=0}^{s-1} \pi(\omega_i) = \frac{1}{s} \sum_{i=0}^{s-1} \omega_i = 0.
\]
The constraints in item (ii) are derived in the same way, but using the powers \( Y^2, \ldots, Y^{s-1} \). For \( Y^2 \), the term of degree zero is \( \sum_{h=0}^{s-1} u_h u_{s-h} \) which is the left-hand side in Eq. (3.5) when \( q = 2 \). In the same way the corresponding degree zero terms for \( q = 3, \ldots, s - 1 \) can be derived. Now, if \( s \) is prime, all the powers \( Y^2, \ldots, Y^{s-1} \) are permutations of \( \Omega_s \) and the result follows from item (i). For general \( s \) (not prime necessarily), define \( m = \gcd(q, s) \) and note that \( Y^q \) contains \( m \) times all the elements of the set \( \Omega_{[s/m]} \), whose sum is again zero. For item (iii) it is enough to write
\[
\sum_{h=1}^{s-1} u_h = \frac{1}{s} \sum_{h=0}^{s-1} \frac{1}{s} \sum_{k=0}^{s-1} \omega_k^{-h} \pi(\omega_k) = \frac{1}{s} \sum_{k=0}^{s-1} \pi(\omega_k) \sum_{h=0}^{s-1} \omega_{[-kh]}.
\]
where the inner sum is always equal to zero except for \( k = 0 \), and thus
\[
\sum_{h=1}^{s-1} u_h = \frac{1}{s} s \pi(\omega_0) = \pi(\omega_0).
\]
Finally, item (iv) follows from (iii) by noting that the value of \( \pi(\omega_0) \) may take any values in \( \Omega_s \). \( \square \)

It is interesting to write explicitly the equations in items (i), (ii) and (iv) for small \( s \).

For \( s = 2 \), we have \( u_0 = 0 \) and \( u_2^2 - 1 = 0 \), and such two equations characterize the only two possible permutations of \( \Omega_2 \). The same holds for \( s = 3 \), where we obtain:
From the second equation, we conclude that one among $u_1$ and $u_2$ is zero, providing an alternative proof to the fact that all the level permutations have a monomial representation for factors with three levels.

We illustrate now an example with $s = 4$, i.e., a nonprime $s$. The conditions become:

$$
\begin{align*}
    u_0 &= 0; \\
    u_1 u_2 &= 0; \\
    (u_1 + u_2)^3 - 1 &= 0.
\end{align*}
$$

In this case, not all the monomial maps of the form $Y = c_0 h X^k$, $h = 0, \ldots, 3$, $k = 1, \ldots, 3$ are the polynomial representation of a permutation. Take for example the monomial map $Y = X^2$. This correspond to the transformation with coefficients $u_0 = u_1 = u_2 = 0$ and $u_3 = 1$. This is not a solution of the above equations, since the second equation is not satisfied.

When $s$ increases, the situation becomes computationally less simple, since from Proposition 5 there are $s$ polynomial equations with degrees $1, \ldots, s$. Therefore, the degree of the polynomial system is $s!$, which is exactly the number of the permutations of $\Omega_s$.

To check whether the system has a finite number of solutions one can apply a known result in polynomial algebra, namely the finiteness theorem, see, e.g., Cox et al. [6]. It is enough to compute a Gröbner basis of the ideal generated by the $s$ equations in Proposition 5 and check whether all the terms $u_0^{c_0}, \ldots, u_{s-1}^{c_{s-1}}$ are all leading terms of polynomials in the Gröbner basis for some exponents $c_0, \ldots, c_{s-1}$.

**Example 1** For $s = 5$ the system in Eq. (3.4) yields 5 equations, and the Gröbner basis of the corresponding polynomial ideal is formed by 28 polynomials. Among them, the five polynomials displayed below have leading term of the form $u[i]^c[i]$ for appropriate exponents $c[i]$ for all $i$, and therefore the finiteness theorem applies. Then the polynomial system has a finite number of solutions.

\[
\begin{align*}
    u[0], \\
    u[1]^5 + u[2]^5 + u[3]^5 + (-20)u[1]^3 u[3] u[4] + \\
    + (-20) u[1] u[2] u[4]^3 + u[4]^5 + (-1), \\
    u[2]^6 + (15) u[1]^4 u[4]^2 + (16) u[1]^2 u[3] u[4]^3 + \\
    + (15) u[1]^2 u[4]^4 + (20) u[2] u[4]^5 + (-1) u[2], \\
    u[3]^6 + (40) u[1] u[4]^2 + (26) u[2]^3 u[4]^3 + \\
    + (-81) u[1]^2 u[4]^4 + (27) u[3] u[4]^5 + (-1) u[3], \\
    u[4]^6 + (-12628/625) u[1]^3 u[3] u[4]^2 + (-77/625) u[1] u[3]^2 u[4]^3 + \\
    + (-12639/625) u[1] u[2] u[4]^4 + (-121/625) u[4]^6 + (-504/625) u[4].
\end{align*}
\]

Finally, an interesting property of the coefficients $u_i$’s concerns their expression in terms of the roots of the unity.
Proposition 6  Up to the constant \(1/s\), the coefficients \(u_i\) are integer nonnegative combinations of the \(s\)-th roots of the unity:

\[
u_i = \frac{1}{s} \sum_{r=0}^{s-1} v_r \omega_r, \quad v_r \in \mathbb{N}
\]

When the number of levels \(s\) is prime, for all permutations \(\pi\), such representation of the coefficients is unique up to an additive integer constant.

Proof The first part follows directly from the expression of the inverse of the Vandermonde matrix in Eq. (3.4). For the uniqueness, see, e.g., Pistone and Rogantin [19]. □

3.1 A Computer Algebra Algorithm to Check \(\mathbb{C}\)-Regularity

Now we show how to use the equations above in order to study the isomorphism between two fractions, by merging polynomial constraints on the support of a full factorial design and the polynomial constraints in Proposition 5. We compute several examples using CoCoA-5, see Abbott et al. [2], and we determine the coefficients of the polynomial transformations using the technique described in Abbott [1]. Also in these examples we make use of basic tools from Computational Commutative Algebra, such as polynomial ideal, Gröbner basis, normal form. For basic definitions and results see, e.g., Cox et al. [6].

We write the polynomial indicator function of the two fractions under investigation, and we do some algebraic manipulations in order to obtain the coefficients of the (possible) permutations needed to transform the first fraction into the second one. In particular, we check whether a fraction is isomorphic to a \(\mathbb{C}\)-regular one. Our examples are given in the \(S^3\) case, where there is only one \(\mathbb{C}\)-regular orthogonal array with strength 2 up to monomial transformations. There are several online databases of orthogonal arrays. The examples analyzed here are taken from Eendebak [8], generated with the algorithm introduced in Schoen et al. [21].

Let \(F_0\) and \(F_1\) be the two fractions to compare, with polynomial indicator functions \(F_0\) and \(F_1\) respectively. Consider a generic transformation \(\pi = (\pi_1, \pi_2, \pi_3)\) where \(\pi_j\) acts on the factor \(X_j\):

\[
\pi_j : X_j \rightarrow \sum_{k=0}^{4} u_{kj} X_j^k.
\]

In particular, if we want to check the \(\mathbb{C}\)-regularity of the fraction \(F_1\), then \(F_0\) is the indicator function of the \(\mathbb{C}\)-regular fraction with defining equation \(X_1X_2X_3 = \omega_0\), namely \(F_0 = F_0^{(r)} = \frac{1}{5} \sum_{k=0}^{4} \left( X_1 X_2 X_3 \right)^k \)

1. Consider the ring of the indeterminates
1. \( x[1], x[2], x[3] \), the factors;
2. \( u[0..4, 1..3] \), the 5 \( \times \) 3 transformation coefficients;
3. \( w \), the 5-th primitive root of the unity, satisfying the equation 
   \( 1 + w + w^2 + w^3 + w^4 = 0 \). The indeterminate \( w \) is considered here as a parameter.

2. Input \( F_0 \) and \( F_1 \), the indicator functions of the two fractions to be compared (minus one).

3. Consider \( I \), the ideal generated by the polynomials defining the full factorial design \((x^i - 1, i = 1, 2, 3)\) and the 3 \( \times \) 5 polynomials with the conditions for the transformation coefficients. The CoCoA code for the last polynomials is:

   ```cocoa
   L:=NewList(5);
   L[1]:=[u[0,j] | j in 1..3];
   L[2]:=[(Sum{|u[i,j]| i in 1..4})^5 -1 | j in 1..3];
   L[3]:=[Sum{|u[i,j]*u[Mod(-i,5),j]| i in 1..4} | j in 1..3];
   L[4]:=[Sum{|u[i,j]*u[h,j]*u[Mod(-i-h,5),j] | i in 1..4} | j in 1..4] +
   (h in 1..4) | j in 1..3];
   L[5]:=[Sum(Sum{|u[i,j]*u[h,j]*u[k,j]*u[Mod(-i-h-k,5),j]| i in 1..4} | h in 1..4} | k in 1..4) | j in 1..3];
   ```

4. Compute \( NF_P F_0 \), the normal form of the transformation of \( F_0 \) by \( \pi \) in the quotient space \( K/I \) and compute \( Coe_F0 \), the list of the coefficients of the terms in \( x[1], x[2], x[3] \) appearing in \( NF_P F_0 \).

5. Compute \( Coe_F1 \), the list of the coefficients of the terms in \( x[1], x[2], x[3] \) appearing in \( F_1 \).

6. Compute \( Coe_Diff \), the difference between the coefficients \( Coe_F1 \) and the coefficients \( Coe_F0 \), for all the terms in \( Coe_F0 \).

7. Compute the Gröbner basis of the ideal generated by the polynomials in \( Coe_Diff \) and the polynomials in \( L[0..4] \) with the conditions for the transformation coefficients.

If the Gröbner basis is empty, then the two fractions are not isomorphic; otherwise, the Gröbner basis contains equations on the transformation coefficients that allow us to find the permutations.

Notice that, if \( F_0^{(r)} \) is the indicator function with generating equation 
\( X_1X_2X_3 = 1 \), then the number of terms of \( NF_P F_0 \) is 6401, while, obviously, the length of \( Coe_F0 \) is 65, the number of the interactions of order 3 plus the constant term.

**Example 2** Let \( F_A, F_B \) and \( F_C \) be three fractions of a 5\(^3\) factorial design. The first two fractions are listed in Eendebak [8]. The fraction \( F_B \) is a \( \mathbb{C} \)-regular fraction.

Using the previous algorithm we checked whether they are isomorphic to the \( \mathbb{C} \)-regular fraction \( F_0^{(r)} \) above. In the first case, the Gröbner basis of step 7 has only element 1; then \( F_A \) is not isomorphic to any \( \mathbb{C} \)-regular fraction of a 5\(^3\) factorial design.
In the last case the Gröbner basis has been computed in 29 secs. of CPU time and contains 91 elements. A solution is:
that corresponds to no permutation on the first factor, a power permutation on the third factor and the switch between the levels $\omega_0$ and $\omega_1$ on the second factor.

**Remark 2** The CPU time in Example 2 is reasonable, and other problems with the same size have approximately the same execution time. However, it is well known that the computational complexity of the symbolic algorithms increases fast with the dimension of the problem, and therefore the computation becomes rapidly very hard, and even unfeasible, for large problems. This issue could be partially mitigated through some optimizations of the symbolic algorithms, but the curse of dimensionality is largely inherent in the symbolic approach. Nevertheless, the algorithm above has its own mathematical interest and may be the basis of future implementations.

### 4 $\mathbb{C}$-Regularity Check of Multilevel Orthogonal Arrays

In this section we approach the problem of checking the $\mathbb{C}$-regularity of a multilevel orthogonal array under a different perspective. The technique takes advantage of the connections between orthogonal arrays and Latin squares, see for instance Keedwell and Dénes [16] and Hedayat et al. [12]. This methodology is based on the generating equations of the $\mathbb{C}$-regular fraction rather than on the whole indicator function.

First, we focus on generating equations involving interactions of order three, and thus we limit our analysis to orthogonal arrays of strength two. Then, we extend the methodology to the general case.

#### 4.1 Generating Equations Involving Interactions of Order Three

Let $F$ be an $OA(n,s^m,2)$ with $s$ prime. A $\mathbb{C}$-regular fraction with strength 2 has at least one generating equation involving only 3 factors. Therefore we first look at generating equations involving three factors. Without loss of generality, let us consider the factors $X_1, X_2, X_3$. Let

$$X_1^{a_1}X_2^{a_2}X_3^{a_3} = \omega_k$$

be a generating equation of $F$, with $a_1, a_2, a_3 \in \{1, \ldots, s-1\}$ and $\omega_k \in \Omega_s$. For brevity, we say that $X_1, X_2, X_3$ form a generating equation of the orthogonal array $F$.

Let $X_3$ be a function of $X_1$ and $X_2$, so that we can consider the $s \times s$ table $C = X_3(X_1, X_2)$ containing the values of $X_3$ as a function of $X_1$ and $X_2$, i.e., $C_{i,j_2} = x_3$ given $x_1 = \omega_{j_1}$ and $x_2 = \omega_{j_2}$. Since the strength of the orthogonal array is 2, table $C$
may be regarded as a $s \times s$ Latin square with values in $\Omega_s$. The main result can be stated as follows.

**Theorem 1** Let $X_1, X_2, X_3$ be three factors of an OA($n, s^n, 2$), with $s$ prime. Assume that $X_3$ is a function of $X_1$ and $X_2$, and let $X_3(X_1, X_2)$ be the corresponding Latin square.

(a) If $X_1, X_2, X_3$ form a generating equation, then $X_3(X_1, X_2)$ has rank 1 in $\mathbb{C}$, i.e., all $2 \times 2$ minors of $X_3(X_1, X_2)$ vanish in $\mathbb{C}$;

(b) If there is a permutation $\pi_3$ of $\Omega_s$ such that $(\pi_3(X_3))(X_1, X_2)$ is a Latin square with rank 1 in $\mathbb{C}$, then there exist permutations $\pi_1$ and $\pi_2$ such that $\pi_1(X_1), \pi_2(X_2), \pi_3(X_3)$ form a generating equation.

**Proof** (a) By hypothesis there exist $\alpha_1, \alpha_2, \alpha_3 \in \{1, \ldots, s-1\}$ and $\omega_k \in \Omega_s$ such that $X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} = \omega_k$. Since $s$ is prime, we can assume $\alpha_3 = 1$. In fact, given a generating equation $X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} = \omega_k$, there exists $r$ such that $[ra_3] = 1$ and the equation

$$X_1^{[ra_1]} X_2^{[ra_2]} X_3 = \omega_{[rk]}$$

is also a generating equation of the fraction. Consider the Latin square $C = X_3(X_1, X_2)$. The entry $C_{j_1, j_2}$ of $C$ is $C_{j_1, j_2} = \omega_{[rk]} \omega_{[rj_1]}^{-[ra_1]} \omega_{[rj_2]}^{-[ra_2]}$ for $j_1, j_2 = 0, \ldots, s-1$, and the generic $2 \times 2$ minor of $C$ is

$$\omega_{[2rk]} (\omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]} - \omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]} - \omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]} - \omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]} - \omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]} - \omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]} - \omega_{[j_1_1]} \omega_{[j_2_1]} \omega_{[j_1_2]} \omega_{[j_2_2]})$$

that equals 0 for all pairs of distinct indices $j_1, j_2 \in \{0, \ldots, s-1\}$ for the rows and for all pairs of distinct indices $j_{21}, j_{22} \in \{0, 1, \ldots, s-1\}$ for the columns.

(b) Suppose that there is a permutation $\pi_3$ of $X_3$ such that the table $C = \pi_3(X_3)(X_1, X_2)$ is a Latin square with all $2 \times 2$ minors equal to 0. Then apply suitable permutations $\pi_1$ and $\pi_2$ to $X_1$ and $X_2$, respectively, in order to obtain a Latin square in reduced form, i.e., with the first row and column lexicographically ordered. Now it is immediate to check that $\pi_3(X_3) = \pi_1(X_1) \pi_2(X_2)$ and therefore a defining equation after the level permutations is $\pi_1(X_1) \pi_2(X_2) \pi_3(X_3)^{-[1]} = \omega_0$. \qed

Some remarks on part (b) of the theorem above are now in order. We can exploit the monomial representation of the permutations in Proposition 4 to reduce considerably the computational cost. First, observe that the permutations to be checked on $X_3$ are at most $(s-2)!$. In fact, we can exclude the powers (to each defining equation corresponds other $(s-2)$ equivalent ones) and the $s$ cyclic permutations (they only affect the constant term). For instance, if $s = 5$, there are 120 level permutations, but only six of them are to be checked. Secondly, the relevant permutations of $X_1$ and $X_2$ are the permutations needed to put the Latin square in reduced form. Additionally, if the permutations $\pi_1$ and $\pi_2$ can be
expressed in monomial form (powers and/or cyclic permutations), then the defining equation can be written without actual permutations on \(X_1\) and \(X_2\).

Finally, note that the permutations of the factors are not uniquely defined: For instance, a shift of the form \(\omega_{h}X_j\) and a shift of the form \(\omega_{h}X_i\), with \(j \neq i\), produce the same transformation in the generating equation.

Before analyzing the general case of symmetric multilevel designs, we present some applications of the theorem above in the simple case of orthogonal arrays with 3 factors and strength 2, so that there is only one defining equation.

**Example 3** In the framework of the \(5^3\) full factorial design, consider the three orthogonal arrays with strength 2 identified by the three Latin squares in Fig. 1. To ease the readability of the tables, we write \(k\) in place of \(\omega_k\). The first two designs are the fractions \(F_A\) and \(F_B\) in Example 2, while the last design is a new fraction \(F_D\) obtained from \(F_B\) with permutation the levels of the factors. In this case of fractions of the \(5^3\) design with 25 runs, it is known that there are only two classes of nonisomorphic orthogonal arrays.

---

**Fig. 1** The three orthogonal arrays of Example 3

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 0 | 3 | 4 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 4 | 1 | 2 |
| 4 | 4 | 2 | 0 | 1 |

(a)

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 3 | 4 | 0 |
| 3 | 3 | 4 | 0 | 1 |
| 4 | 4 | 0 | 1 | 2 |

(b)

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 4 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 4 | 1 | 2 |
| 4 | 4 | 2 | 0 | 1 |

(c)

---

**Fig. 2** The five Latin squares obtained from the orthogonal array (a) of Example 3 after the five nonidentical relevant permutations

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 0 | 1 | 4 | 2 |
| 1 | 1 | 4 | 2 | 3 |
| 2 | 2 | 3 | 1 | 4 |
| 3 | 3 | 2 | 1 | 4 |
| 4 | 4 | 0 | 1 | 2 |

---

---
(a) The Latin squares after the 5 relevant permutations of $X_3$ are shown in Fig. 2. The first minor in all tables is $\omega_0^2 - \omega_1^2 = 1 - \omega_2 \neq 0$, and this is enough to conclude that this fraction cannot be transformed into a $\mathbb{C}$-regular one.

(b) All the $2 \times 2$ minors vanish in $\mathbb{C}$. The defining equation of this fraction, without any permutations, is $X_1X_2 = X_3$ or equivalently $X_1X_2X_3^5 = \omega_0$.

(c) Remember that this fraction has been built from the previous one by applying permutations to the three factors.

Now we consider the factor $X_3$ and the five possible permutations (other than the identity) of its levels. We observe that the permutation $\pi_3(X_3) = (\omega_4, \omega_3, \omega_0, \omega_2, \omega_1)$ produces the Latin square in Fig. 3, where all the $2 \times 2$ minors vanish in $\mathbb{C}$.

Looking at the column and the row beginning with $\omega_0$, we can read easily the permutations of the levels of $X_1$ and $X_2$ needed to obtain a Latin square in reduced form:

$$
\pi_1(X_1) = (\omega_0, \omega_4, \omega_2, \omega_1, \omega_3) \quad \pi_2(X_2) = (\omega_0, \omega_1, \omega_4, \omega_3, \omega_2).
$$

With such permutations, we obtain the defining equation

$$
\pi_1(X_1)\pi_2(X_2)\pi_3(X_3)^4 = \omega_0.
$$

In this example, the relevant permutations $\pi_i$ cannot be expressed in monomial form, and thus the permutations of the factor levels are unavoidable.

Remark 3 In part (a) of the previous example, note that the $2 \times 2$ upper-left matrix has nonzero determinant for all permutations of the levels of $X_3$. In fact, if the permutation $\pi_3$ maps $\omega_0$ into $\omega_{j_0}$ and $\omega_1$ into $\omega_{j_1}$, with $j_0 \neq j_1$, one obtains the minor $\omega_{j_1}^2 - \omega_{j_0}^2 = \omega_{(2j_1-2j_0)} \neq 0$. This remark can also be used to build non-$\mathbb{C}$-regular
orthogonal arrays in case of a large number of levels, as illustrated in the example below.

**Example 4** The Latin square in Fig. 4 represents an orthogonal array of strength 2 of the $7^3$ design. It has been defined starting from the upper-left $2 \times 2$ sub-matrix and then completed in the remaining entries. By construction, it is a non-$\mathbb{C}$-regular design even under permutations of the factor levels.

### 4.2 The General Case

Theorem 1 can be applied recursively layer by layer for constructing an algorithm to check the $\mathbb{C}$-regularity of orthogonal arrays with an arbitrary number of factors. First, consider an orthogonal array with one defining equation. Then we show an example with two defining equations to point the way to a complete generalization.

We consider a design with four factors and an orthogonal array of strength three. The factors $X_1, X_2, X_3, X_4$ form a generating equation if and only if

$$X_1^{a_1}X_2^{a_2}X_3^{a_3}X_4 = \omega_k$$

for some $a_1, a_2, a_3 \in \{1, \ldots, s-1\}$ and $\omega_k \in \Omega_s$. Thus, for each $x_4 = \omega_{j_4}$, $j_4 = 0, \ldots, s-1$, the equation

$$X_1^{a_1}X_2^{a_2}X_3^{a_3} = \omega_{[k-j_4]}$$

is satisfied. Conversely, assume that Theorem 1 applies to each layer corresponding to $x_4 = \omega_{j_4}$, $j_4 = 0, \ldots, s-1$, with the same permutation, and assume that there exist permutations $\pi_1, \pi_2, \pi_3$ on $X_1, X_2, X_3$ respectively such that

$$\pi_1(X_1)^{a_1}\pi_2(X_2)^{a_2}\pi_3(X_3)^{a_3} = \omega_{k[j_4]}$$

with different $\omega_{k[j_4]}$, for $j_4 = 0, \ldots, s-1$. Then $k(j_4)$ defines a permutation $\pi_4$ for $X_4$ and

$$\pi_1(X_1)^{a_1}\pi_2(X_2)^{a_2}\pi_3(X_3)^{a_3}\pi_4(X_4)^{[s-1]} = \omega_0$$

is a defining equation for the orthogonal array.

For instance, for orthogonal arrays with strength 2 and $m$ factors, we check the possible defining equations in the following order.

1. First, check all the 3-tuples.
2. Then, if the defining equations with 3 factors are not sufficient to define the orthogonal array, check the defining equations with 4 or more factors.

Notice that the number of (independent) defining equations is $\#D/\#F$.

The regularity check is based on the following property of $\mathbb{C}$-regular fractions. The proof of this result is based on the properties of the elimination ideals, see, e.g., Cox et al. [6].
Proposition 7 Let $\mathcal{F}$ be a $\mathbb{C}$-regular fraction of a $s^m$ design, $s$ prime, and let $I \subset \{1, \ldots, m\}$. Denote with $\mathcal{F}_I$ the projection of $\mathcal{F}$ onto the $I$-factors. Apart from the multiplicity, $\mathcal{F}_I$ is either a full factorial design or a $\mathbb{C}$-regular fraction.

Proof Let $\bar{I} = \{1, \ldots, m\} \setminus I$. Let us define the ideal $I(\mathcal{F})$ as the ideal in $\mathbb{C} [x_1, \ldots, x_m]$ generated by the binomials $x_j^s - 1 = 0$, $j = 1, \ldots, m$ and by the generating equations of $\mathcal{F}$. The ideal $I(\mathcal{F})$ is a binomial ideal. In fact, two factors or interactions are either orthogonal or totally confounded and this yields only binomial equations. The ideal $I(\mathcal{F}_I)$ is the elimination ideal of $I(\mathcal{F})$ with respect to the variables $x_j$, $j \in \bar{I}$. From the results in Chapter 3 of Cox et al. [6], $I(\mathcal{F}_I)$ is also a binomial ideal and two cases may arise: (a) The binomials $x_j^s - 1 = 0$, $j \in I$ generate $I(\mathcal{F}_I)$, and this means that $\mathcal{F}_I$ is a full factorial design on the $I$-factors; (b) there are other generators. From the definition of ideal, such polynomials define $\mathcal{F}_I$ as a $\mathbb{C}$-regular fraction. □

Remark 4 To ease computations, remember that a defining equation with a given number of factors cannot include simultaneously all the factors of a defining equation with a smaller number of factors.

Example 5 Consider the $5^{5-2}$ $\mathbb{C}$-regular fraction defined by

$$X_1^2X_2X_3 = \omega_1 \quad \text{and} \quad X_1X_2X_4X_5 = \omega_1 \quad (4.2)$$

and a fractions obtained by permuting the levels of two factors. The indicator function of this permuted fraction $\mathcal{F}_E$ has 289 nonzero monomials. We apply our technique to the new fraction, and we show how to recover defining equation (4.2) and the correct permutations starting from the fraction points.

First we check whether there are defining equations with 3 factors. We obtain a valid Latin square only with $X_1, X_2, X_3$ as in Fig. 5.

Here the minors are all zero, and therefore no permutation on $X_3$ is needed. Now, from the column beginning with $\omega_0$ we read the permutation of $X_1$: $\pi_1(X_1) = (\omega_0, \omega_2, \omega_4, \omega_1)$. From the row beginning with $\omega_0$ we obtain the permutation $X_2^4$. Thus we have $X_3 = \omega_1 \pi_1(X_1) X_2^4$, or equivalently $\pi_1(X_1)^2 X_2 X_3 = \omega_1$. Notice that this constant term $\omega_1$ can be easily recovered from the Latin square above, since it is the symbol in the upper-left position, where $\pi_1(X_1) = X_2 = \omega_0$ and therefore $X_3$ is equal to the constant term of the defining equation.

Fig. 5 The Latin square $X_3(X_1, X_2)$ for Example 5

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 0 | 4 | 3 | 2 |
| 1 | 4 | 3 | 2 | 1 | 0 |
| 2 | 3 | 2 | 1 | 0 | 4 |
| 3 | 0 | 4 | 3 | 2 | 1 |
| 4 | 2 | 1 | 0 | 4 | 3 |
As no other defining equations with 3 factors can be obtained, we move to the interactions of order 4. There are few checks to do at this stage, because there are only 5 subsets with 4 factors and two of them, \{X_1, X_2, X_3, X_4\} and \{X_1, X_2, X_3, X_5\}, can be excluded as they contain \{X_1, X_2, X_3\}, see Remark 4. Consider the 4-tuple \{X_2, X_3, X_4, X_5\}. We look at the layers defined by \(X_5\), and we obtain the five Latin squares in Fig. 6.

First we observe that the five Latin squares are all equal up to permutations of rows and columns. In each of them, the 2 \(\times\) 2 minors are all zero, and thus no permutation is needed on \(X_4\). In the column and in the row beginning with 0, we read the permutation \((\omega_0, \omega_2, \omega_4, \omega_1, \omega_3)\) for \(X_2\), corresponding to \(X_2^5\), and the permutation \((\omega_0, \omega_3, \omega_1, \omega_4, \omega_2)\) for \(X_3\), corresponding to \(X_3^5\). Looking at the upper-left cell we have the constant term of the defining equations. From the first Latin square we find \(\omega_1X_4 = X_2^5X_3^5\) or \(X_2^5X_3^5\omega_4 = \omega_0\). Analogously, from the other Latin squares we find \(X_3^5X_2^5\omega_2 = \omega_0\), \(X_2^5X_3^5\omega_3 = \omega_0\), \(X_2^5X_3^5\omega_4 = \omega_0\), \(X_2^5X_3^5\omega_5 = \omega_0\), respectively. Then, the permutation on \(X_5\) is such that

\[
X_3^5X_2^5\pi_5(X_5) = \omega_0 \quad \text{so that} \quad \pi_5(X_5) = (\omega_4, \omega_2, \omega_3, \omega_0, \omega_1)
\]

Finally, we check that this defining equation corresponds to the second equation used in Eq. (4.2) to define the array. Indeed, from \(\pi_1(X_1)^2X_2X_3 = \omega_1\) we have \(X_3 = \omega_1\pi_1(X_1)^3X_2^4\), and replacing this expression of \(X_3\) into the previous equation one obtains immediately the defining equation

\[
\pi_1(X_1)X_2X_4\pi_5(X_5) = \omega_3.
\]

Notice that the constant term is different, but this can be incorporated in \(\pi(X_5)\) through a monomial transformation.

The example above shows that the proposed methodology is recursive and it can be applied to a larger number of factors and defining equations.

---

**Fig. 6** The layers \(X_4(X_2, X_3)\) given \(X_5\) for Example 5

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
0 & 1 & 4 & 2 & 0 & 3 \\
1 & 3 & 1 & 4 & 2 & 0 \\
2 & 0 & 3 & 1 & 4 & 2 \\
3 & 2 & 0 & 3 & 1 & 4 \\
4 & 4 & 2 & 0 & 3 & 1 \\
\end{array}
\]

\[
X_5 = \omega_0
\]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
0 & 1 & 2 & 3 & 4 & \\
1 & 2 & 0 & 3 & 1 & 4 \\
2 & 4 & 2 & 0 & 3 & 1 \\
3 & 1 & 4 & 2 & 0 & 3 \\
4 & 3 & 1 & 4 & 2 & 0 \\
\end{array}
\]

\[
X_5 = \omega_3
\]

---

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & \\
0 & 1 & 2 & 3 & 4 & \\
1 & 2 & 0 & 3 & 1 & 4 \\
2 & 4 & 2 & 0 & 3 & 1 \\
3 & 1 & 4 & 2 & 0 & 3 \\
4 & 3 & 1 & 4 & 2 & 0 \\
\end{array}
\]

\[
X_5 = \omega_4
\]
5 Final Remarks

In this paper we addressed the problem of level permutations for qualitative factors. In particular we presented two tools to check whether a fraction of a $s^n$ factorial design is isomorphic or not to a $\mathbb{C}$-regular fraction by permutations of factor levels. Such a problem is very important in the applications because the $\mathbb{C}$-regular fractions have the special property of not partial confounding. In this framework, the coding of levels by the $s$-th roots of the unity and some tools of algebraic statistics are essential.

Future works will concern the case of designs with nonprime number of levels and mixed designs, where several properties of the roots of the unity do not hold, and therefore for this class of designs a different approach must be implemented.

Moreover, we want to deepen and better define the concept of mean aberration, already introduced in Fontana et al. [9]. In particular, we want to limit the mean only to permutations compatible with the design matrix of the fraction under investigation. In fact, the aberrations are calculated through the level counts, and they are connected to each other by a convolution formula presented in the aforementioned article. This new definition could allow us to clarify which aberrations are compatible with those of permuted $\mathbb{C}$-regular fractions.

Finally, in order to generalize the algorithms presented in this paper for specific examples, we want to provide efficient packages, both in symbolic and in statistical software (i.e., CoCoA and R respectively), to make actual computations regarding $\mathbb{C}$-regularity and isomorphism checks under level permutations in a general setting.

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References

1. Abbott J (2002) Sparse squares of polynomials. Math Comput 71(237):407–413
2. Abbott J, Bigatti AM, Lagorio G (2015) CoCoA-5: a system for doing computations in commutative algebra. http://cocoa.dima.unige.it. Accessed 19 Oct 2018
3. Cheng S-W, Ye KQ (2004) Geometric isomorphism and minimum aberration for factorial designs with quantitative factors. Ann Stat 32(5):2168–2185
4. Clark JB, Dean AM (2001) Equivalence of fractional factorial designs. Stat Sin 11(2):537–547
5. Corless RM, Fillion N (2013) A graduate introduction to numerical methods. Springer, Berlin
6. Cox D, Little J, O’Shea D (2015) Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra. Springer, Berlin
7. Dean A, Morris M, Stufken J, Bingham D (2015) Handbook of design and analysis of experiments. CRC Press, Boca Raton
8. Eendebak P (2018) Complete series of non-isomorphic orthogonal arrays. http://pietereneebak.nl/oapage/. Accessed 19 July 2018
9. Fontana R, Rapallo F, Rogantin M-P (2016) Aberration in qualitative multilevel designs. J Stat Plan Inference 174:1–10
10. Grömping U (2018) Coding invariance in factorial linear models and a new tool for assessing combinatorial equivalence of factorial designs. J Stat Plan Inference 193(1):1–14
11. Grömping U, Bailey RA (2016) Regular fractions of factorial arrays. In Kunert J, Müller CH, Atkinson AC (eds) mODa 11—advances in model-oriented design and analysis: proceedings of the 11th international workshop in model-oriented design and analysis held in Hamminkeln, Germany, June 12–17, 2016, pp 143–151. Cham: Springer
12. Hedayat AS, Sloane NJA, Stufken J (1999) Orthogonal Arrays: Theory and Applications. Springer, New York
13. Katsaounis TI (2012) Equivalence of factorial designs with qualitative and quantitative factors. J Stat Plan Inference 142(1):79–85
14. Katsaounis TI, Dean AM (2008) A survey and evaluation of methods for determination of combinatorial equivalence of factorial designs. J Stat Plan Inference 138(1):245–258
15. Katsaounis TI, Dean AM, Jones B (2013) On equivalence of fractional factorial designs based on singular value decomposition. J Stat Plan Inference 143(11):1950–1953
16. Keedwell A D, Dénes J (2015) Latin Squares and their Applications, 2nd edn. Elsevier, North-Holland
17. Ma C-X, Fang K-T, Lin DKJ (2001) On the isomorphism of fractional factorial designs. J Complex 17(1):86–97
18. Pang F, Liu M-Q (2011) Geometric isomorphism check for symmetric factorial designs. J Complex 27(5):441–448
19. Pistone G, Rogantin M-P (2008) Indicator function and complex coding for mixed fractional factorial designs. J Stat Plan Inference 138(3):787–802
20. Pistone G, Rogantin M-P (2010) Regular fractions and indicator polynomials. In: Viana MAG, Wynn HP (eds) Algebraic methods in statistics and probability II: contemporary mathematics, vol 516. American Mathematical Society, Providence, pp 285–304
21. Schoen ED, Eendebak PT, Nguyen MVM (2010) Complete enumeration of pure-level and mixed-level orthogonal arrays. J Comb Des 18(2):123–140
22. Tang Y, Xu H (2014) Permuting regular fractional factorial designs for screening quantitative factors. Biometrika 101(2):333–350
23. Wu CFJ, Hamada M (2000) Experiments: Planning, Analysis, and Parameter Design Optimization. Wiley, New York
24. Xu H, Wu CFJ (2001) Generalized minimum aberration for asymmetrical fractional factorial designs. Ann Stat 29(4):1066–1077