Squeezing of thermal noise in a parametrically-driven oscillator

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Abstract. In this article we report a theoretical model based on Green’s functions and averaging techniques that describes the dynamics of parametrically-driven mechanical resonators under the action of thermal noise. Quantitative estimates for quadrature thermal noise squeezing near the first parametric instability zone of the oscillator are given. Furthermore, the parameter space where these phenomena occur is presented. Very good agreement between analytical estimates and numerical results is achieved.

1. Introduction

The field of microelectromechanical devices (MEMs) has been developing rapidly since the advent of the atomic force microscope in mid 80’s [1]. MEMs have been used primarily for measuring small forces and as ultrasensitive mass detectors. Further detection techniques have been developed in the early 90’s that use mechanical parametric amplification (before transduction) to improve the sensitivity of measurements. This amplification method works by driving the parametrically-driven oscillator on the verge of parametric unstable zones. Rugar and Grüter [2] have shown ways, using this method, to obtain linear parametric gain. They were also looking for ways of reducing noise and increasing precision in a detector for gravitational waves, when they experimentally found classical thermomechanical squeezing. DiFilippo et al. [3, 4] have proposed explanations for the noise squeezing phenomenon, but their models do not treat noise directly in the equations of motion. They use a Gaussian (thermal) distribution of initial states and evolve it deterministically. In this evolution amplitude squeezing of the signal was observed, at the same time higher dephasing was obtained. In Ref. [4], the authors proposed a theoretical model for explaining the cause of thermomechanical squeezing in which the system parameters are chosen to be inside the first zone of parametric instability. Such approach is not correct, since, with this choice, the solutions diverge exponentially. One would have to use a nonlinear model, based on the Duffing oscillator [5, 6], instead of the linear model used to prevent the divergence. Furthermore, detuning, dissipation, and stochastic dynamics are all neglected. Basically, there is no treatment of the dynamics due to noise, apart from the probabilistic initial state distribution. In Ref. [7], a nonlinear noise squeezing procedure is experimentally observed in a driven Duffing oscillator, again the theoretical model used does not treat noise dynamically.

Here, we investigate the effect of adding thermal noise to the fundamental mode of a doubly-clamped beam resonator that is axially loaded (i.e., its first normal mode is equivalent to a parametrically-driven oscillator) with the objective of explaining thermomechanical noise squeezing. This model can also be applied to the linear response of non-linear driven oscillators, in which, due to the linearization, the response is given by parametrically-driven oscillations. With our model we obtain analytical quantitative estimates of the amount of quadrature thermal-noise squeezing. In the following section we use a Green’s
function approach to solve the Langevin equation and, aligned with averaging techniques, we obtain analytical estimates of the oscillator fluctuations showing how thermal-noise squeezing occurs.

2. Theoretical model
The equation for the parametrically-driven oscillator (in dimensionless format) is given by
\[ \ddot{x} + \omega_0^2 x = -\gamma \dot{x} + F_p \cos(2\omega t) x, \]
in which \( \gamma \) and \( F_p \sim O(\varepsilon) \), where \( \varepsilon \ll 1 \). Since we want to apply the averaging method (AM) [8, 9] to situations in which we have detuning, it is convenient to rewrite Eq. (1) in a more appropriate form with the notation \( \Omega = \omega_0^2 - \omega^2 \), where we also have \( \Omega \sim O(\varepsilon) \). With this substitution we obtain
\[ \ddot{x} + \omega^2 x = -\Omega x - \gamma \dot{x} + F_p \cos(2\omega t) x. \]
We then rewrite this equation in the form \( \dot{y} = -\omega^2 x + f(x, y, t) \), where \( f(x, y, t) = -\Omega x + F_p \cos(2\omega t) x - \gamma y \). We now set the above equation in slowly-varying form with the Van der Pol transformation
\[ (x \ y) = \begin{pmatrix} x \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \omega \sin \omega t & -\omega \cos \omega t \end{pmatrix} (U \ V). \]
and obtain
\[ \begin{pmatrix} \dot{U} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & -\omega \cos \omega t \end{pmatrix} \begin{pmatrix} 0 \\ f(x, y, t) \end{pmatrix} = -\frac{1}{\omega} \begin{pmatrix} \sin \omega t f(x, y, t) \\ \omega \cos \omega t f(x, y, t) \end{pmatrix}. \]

After an application of the averaging method (in which, in our case, basically, we filter out oscillating terms at and near 2\( \omega \)), we obtain
\[ \begin{align*}
\dot{u} &= -\frac{1}{2\omega} \left[ \gamma \omega u + \left( \Omega + \frac{F_p}{2} \right) v \right], \\
\dot{v} &= -\frac{1}{2\omega} \left[ \left( -\Omega + \frac{F_p}{2} \right) u + \gamma \omega v \right],
\end{align*} \]
where the functions \( U(t) \) and \( V(t) \) were replaced by their slowly-varying averages \( u(t) \) and \( v(t) \), respectively. It is known that, by integration of Eq. (2), one finds the first parametric resonance, i.e. the boundary between the stable and unstable regions. It is given by
\[ (\gamma \omega)^2 = (F_p/2)^2 - \Omega^2. \]

This result is valid for \( \omega \approx \omega_0 \) even in the presence of noise. In Fig. (1) we find very good agreement between results obtained from numerical integration of Eq. (1) and the boundary given by the averaging technique.

We now will investigate the effect of noise on the parametric amplification mechanism [2, 3, 4, 7]. We start by adding noise to Eq.(1) and obtain
\[ \ddot{x} = -\omega_0^2 x - \gamma \dot{x} + F_p \cos(2\omega t) x + R(t), \]
where \( R(t) \) is a random function that satisfies the statistical averages \( \langle R(t) \rangle = 0 \) and \( \langle R(t)R(t') \rangle = 2T \gamma \delta(t - t') \), according to the fluctuation-dissipation theorem [10]. \( T \) is the heat-bath temperature of the surrounding medium in which the oscillator is embedded. Once we integrate these equations of motion we can show how classical mechanical noise squeezing occurs.
Figure 1. (color online) Comparison between numerical and averaging method predictions for the boundary of the first instability zone of the damped parametrically-driven oscillator of Eq. (1). In the region above the green line lies the unstable zone obtained by numerical computation, while the region above the red line is the analytical prediction for the unstable zone. The numerical results are obtained by numerically calculating the corresponding Floquet multipliers, when at least one of them has modulus equal to 1. The averaging predictions are given by Eq. (3). The fixed parameters of the equations of motion are $\gamma = 0.1, \omega_0 = 1.0$. These parameters were also used to obtain the results portrayed in the following figures.

2.1. Green’s function method

Equation (4) can be solved exactly by using Floquet theory and Green’s functions methods [11]. The equation for the corresponding Green’s function is given by

$$\left[\frac{\partial^2}{\partial t^2} + \omega_0^2 + \frac{\gamma}{\partial t} - F_p \cos(2\omega t)\right] G(t,t') = \delta(t-t')$$  \hspace{1cm} (5)

Using the Green’s function we obtain the solution $x(t)$ of Eq. (4) in the presence of noise $R(t)$

$$x(t) = x_h(t) + \int_{-\infty}^{\infty} dt' \ G(t,t') R(t'),$$

where $x_h(t)$ is the homogeneous solution, which in the stable zone decays exponentially with time; since we assume the pump has been turned on for a long time, $x_h(t) = 0$. By statistically averaging the fluctuations as a function of time we obtain

$$\langle x^2(t) \rangle = \int_{-\infty}^{\infty} dt' \ dt'' G(t,t')G(t,t'') \langle R(t')R(t'') \rangle \hspace{1cm}$$

$$= 2T\gamma \int_{-\infty}^{t} dt' G(t,t')^2 = 2T\gamma \int_0^{\infty} d\tau \ G(t,t-\tau)^2, \hspace{1cm} (6)$$

where $\tau = t - t'$.

Instead of using Floquet theory, which, though exact, yields no simple analytical solutions, we obtain fairly simple analytical approximations to the Green’s functions using the averaging method, in which we solve Eq. (2), where the initial conditions at $t = t'$ are given by $u(t') = -\sin(\omega t')/\omega$ and
\[ v(t') = -\cos(\omega t')/\omega. \]

The fundamental matrix of this system is given by

\[
e^{A(t-t')} = e^{-\gamma(t-t')/2} \times \left( \begin{array}{cc}
\cosh(\kappa (t-t')) & \frac{\beta - \delta}{\kappa} \sinh(\kappa (t-t')) \\
\frac{\beta + \delta}{\kappa} \sin(\kappa (t-t')) & \cosh(\kappa (t-t'))
\end{array} \right),
\]

where \( \kappa = \sqrt{\beta^2 - \delta^2}, \beta = -F_p/4\omega \) and \( \delta = \Omega/2\omega \). The approximate Green’s function is given by

\[
G(t, t') \approx e^{-\gamma(t-t')/2} \left[ \cos(\omega t) \left( \cosh(\kappa (t-t'))u(t') + \frac{\beta - \delta}{\kappa} \sinh(\kappa (t-t'))v(t') \right) \\
- \sin(\omega t) \left( \frac{\beta + \delta}{\kappa} \sinh(\kappa (t-t'))u(t') + \cosh(\kappa (t-t'))v(t') \right) \right]
\] (7)

for \( t > t' \) and \( G(t, t') = 0 \) for \( t < t' \). In the stable zone of the parametrically-driven oscillator, when \( |\beta| < |\delta| \), we can rewrite the Green’s function, replacing explicitly the initial conditions and using simplifying trigonometrical identities. We obtain

\[
G(t, t') \approx -\frac{e^{-\gamma(t-t')/2}}{\omega} \left\{ \cos(|\kappa| \tau) \sin(\omega \tau) + \frac{\delta}{|\kappa|} \sin(|\kappa| \tau) \cos(\omega \tau) - \frac{\beta}{|\kappa|} \sin(|\kappa| \tau) \cos(2\omega \tau) + \sin(\omega \tau) \sin(2\omega \tau) \right\}.
\] (8)

The squared Green’s function is given by

\[
G(t, t - \tau)^2 \approx \frac{e^{-\gamma\tau/2}}{\omega^2} \left\{ \cos(|\kappa| \tau) \sin(\omega \tau) + \frac{\delta}{|\kappa|} \sin(|\kappa| \tau) \cos(\omega \tau) \right\}^2 + \frac{\beta^2}{2|\kappa|^2} \sin^2(|\kappa| \tau)
- \frac{2\beta \sin(|\kappa| \tau)}{|\kappa|} \left[ \cos(|\kappa| \tau) \sin(\omega \tau) + \frac{\delta}{|\kappa|} \sin(|\kappa| \tau) \cos(\omega \tau) \right] \cos(2\omega \tau) \cos(\omega \tau) + \sin(2\omega \tau) \sin(\omega \tau)
+ \frac{\beta}{2|\kappa|^2} \sin^2(|\kappa| \tau) \cos(2\omega \tau) \cos(4\omega \tau) + \sin(2\omega \tau) \sin(4\omega \tau) \right\}.
\]

To obtain the squared Green’s function when \( |\beta| > |\delta| \) one should replace \(|\kappa| \) by \( i\kappa \) in the above expression. When there is no external parametric driving, \( F_p = 0 \), and \( \Omega = 0 \), we obtain

\[
G(t, t - \tau) \approx \frac{e^{-\gamma\tau/2}}{\omega_0} \sin(\omega_0 \tau),
\]

which is within an error of order \( O(\epsilon^2) \) of the simple harmonic oscillator Green’s function. Also, the average fluctuations at \( F_p = \Omega = 0 \) become

\[
\langle x^2 \rangle = T/\omega_0^2 \left[ 1 - \frac{\gamma^2}{\gamma^2 + 4\omega_0^2} \right],
\]
reproducing the equipartition within an error of $O(\epsilon^2)$. As we are using an approximation of the Green’s function with an error of $O(\epsilon)$, as stated below Eq. (1), this result is within the expected error bounds.

An estimate of the time average of the thermal fluctuations, when $|\beta| < |\delta|$, is given by

$$\langle x^2(t) \rangle = \frac{2T\gamma}{\omega^2} \int_0^\infty e^{-\gamma \tau} \left\{ \cos(\kappa|\tau|) \sin(\omega \tau) + \frac{\delta}{\kappa} \sin(\kappa|\tau|) \cos(\omega \tau) \right\} d\tau$$

$$= \frac{2T\gamma}{\omega^2} \left\{ \frac{\beta^2}{\gamma} \right\} \left[ \frac{1}{\gamma^2 + 4(\kappa|\tau| - \omega)^2} - \frac{1}{\gamma^2 + 4(\kappa|\tau| + \omega)^2} \right]$$

$$+ \frac{1}{4} \left[ \frac{1}{\gamma^2 + 4\kappa^2} - \gamma^2 - 2\omega^2 - \frac{\gamma}{2} \left( \frac{1}{\gamma^2 + 4(\kappa|\tau| - \omega)^2} + \frac{1}{\gamma^2 + 4(\kappa|\tau| + \omega)^2} \right) \right]$$

$$\int_0^\infty e^{-\gamma \tau} \left\{ \cos(\kappa|\tau|) \sin(\omega \tau) + \frac{\delta}{\kappa} \sin(\kappa|\tau|) \cos(\omega \tau) \right\} d\tau$$

$$= \frac{2T\gamma}{\omega^2} \left[ I_1 + I_2 + \frac{\delta^2 I_3}{2} + \delta I_4 \right],$$

where the integrals are given by

$$I_1 = \frac{\beta^2}{2\kappa^2} \int_0^\infty e^{-\gamma \tau} \sinh^2(\kappa \tau) d\tau = \frac{\beta^2}{\gamma (\gamma^2 - 4\kappa^2)}$$

$$I_2 = \int_0^\infty e^{-\gamma \tau} \cosh^2(\kappa \tau) \sin(\omega \tau) d\tau$$

$$= \frac{1}{2} \left( \frac{1}{2\gamma} - \frac{\gamma}{2(\gamma^2 + 4\omega^2)} + \frac{\gamma}{2(\gamma^2 - 4\kappa^2)} - \frac{1}{4} \text{Re} \left[ \frac{1}{\gamma - 2\kappa - 2i\omega} + \frac{1}{\gamma + 2\kappa - 2i\omega} \right] \right)$$

$$I_3 = \frac{1}{\kappa^2} \int_0^\infty e^{-\gamma \tau} \sinh^2(\kappa \tau) \cos(\omega \tau) d\tau$$

$$= \frac{1}{\kappa^2} \left( \frac{\gamma}{\gamma^2 - 4\kappa^2} + \frac{1}{8} \text{Re} \left[ \frac{1}{\gamma - 2\kappa - 2i\omega} + \frac{1}{\gamma + 2\kappa - 2i\omega} - \frac{\gamma}{4(\gamma^2 + 4\omega^2)} \right] \right)$$

$$I_4 = \frac{1}{2\kappa^2} \int_0^\infty e^{-\gamma \tau} \sin(2\kappa \tau) \sin(2\omega \tau) d\tau = \frac{\gamma}{4i\kappa} \left[ \frac{1}{\gamma^2 - 4(\kappa^2 + i\omega^2)} + \frac{1}{\gamma^2 - 4(\kappa^2 - i\omega^2)} \right]$$

An estimate of the thermal fluctuations is given by

$$\langle x^2(t) \rangle \approx \langle x^2(t) \rangle + A_{2\omega} \cos(2\omega t) + B_{2\omega} \sin(2\omega t) + A_{4\omega} \cos(4\omega t) + B_{4\omega} \sin(4\omega t).$$

From Eqs. (6) and (7), we find that the coefficients necessary for the calculation of the fluctuations are given by

$$A_{2\omega} = -\frac{4\beta T\gamma}{\omega^2} \int_0^\infty d\tau e^{-\gamma \tau} \sinh(\kappa \tau) \left[ \cos(\kappa \tau) \sin(\omega \tau) + \frac{\delta}{\kappa} \sin(\kappa \tau) \cos(\omega \tau) \right] \cos(\omega \tau),$$

$$B_{2\omega} = -\frac{4\beta T\gamma}{\omega^2} \int_0^\infty d\tau e^{-\gamma \tau} \sinh(\kappa \tau) \left[ \cos(\kappa \tau) \sin(\omega \tau) + \frac{\delta}{\kappa} \sin(\kappa \tau) \cos(\omega \tau) \right] \sin(\omega \tau),$$

$$A_{4\omega} = \frac{\beta^2 T\gamma}{\omega^2 \kappa^2} \int_0^\infty d\tau e^{-\gamma \tau} \sinh^2(\kappa \tau) \cos(2\omega \tau),$$

$$B_{4\omega} = \frac{\beta^2 T\gamma}{\omega^2 \kappa^2} \int_0^\infty d\tau e^{-\gamma \tau} \sinh^2(\kappa \tau) \sin(2\omega \tau).$$
The following results are found when $|\beta| > |\delta|$.

$$A_{2\omega} = -\frac{\beta T \gamma}{2\omega^2} \left\{ \Im \left[ \frac{1}{\kappa} \left( \frac{1}{\gamma - 2\kappa} + 2i\omega - \frac{1}{\gamma + 2\kappa} + 2i\omega \right) \right] \right\} + \frac{\delta}{\kappa^2} \Re \left[ \frac{1}{(\gamma - 2\kappa) + 2i\omega} + \frac{1}{(\gamma + 2\kappa) + 2i\omega} + \frac{2\delta}{\kappa^2} \right] \left[ -\frac{1}{\gamma} + \frac{\gamma(\kappa^2 + \omega^2)}{(\gamma^2 + 4\omega^2)(\gamma^2 - 4\kappa^2)} \right],$$

$$B_{2\omega} = -4\frac{\beta T \gamma}{\omega^2} \left\{ \frac{1}{2(\gamma^2 - 4\kappa^2)} + \frac{1}{8} \Re \left[ \frac{1}{\kappa} \left( \frac{1}{\gamma + 2\kappa} + 2i\omega - \frac{1}{\gamma - 2\kappa} + 2i\omega \right) \right] \right\} + \frac{\delta}{8\kappa^2} \Im \left[ \frac{1}{(\gamma + 2\kappa) - 2i\omega} + \frac{1}{(\gamma - 2\kappa) - 2i\omega} + \frac{2}{\gamma + 2i\omega} \right],$$

$$A_{4\omega} = -\frac{\beta T \gamma}{4\omega^2} \Re \left[ \frac{1}{(\gamma + 2\kappa) - 2i\omega} + \frac{1}{(\gamma - 2\kappa) - 2i\omega} + \frac{2}{\gamma + 2i\omega} \right],$$

$$B_{4\omega} = -\frac{\beta T \gamma}{4\omega^2} \Im \left[ \frac{1}{(\gamma + 2\kappa) - 2i\omega} + \frac{1}{(\gamma - 2\kappa) - 2i\omega} + \frac{2}{\gamma + 2i\omega} \right].$$

When $|\beta| < |\delta|$, replace $\kappa$ by $i|\kappa|$ in the expressions above for coefficients of the thermal fluctuations.

3. Results and discussion

In Fig. (2) we plot the Green’s functions obtained directly from the numerical integration of Eq. (5) alongside analytical approximation results given by Eq. (7), if $|\beta| > |\delta|$, or by Eq. (9) if $|\beta| < |\delta|$. We obtain excellent agreement between the two methods, what implies that our analytical estimates of $\langle x^2(t) \rangle$ are accurate. The numerical integration was performed using a RK4 algorithm with a time step given by $\pi/(512\omega)$.

In Fig. (3) we show a time series of $\langle x^2(t) \rangle$ given by Eqs. (12). By looking at the time average values (dynamical temperatures) we see that cooling occurs at positive detuning (here at $\omega/\omega_0 = 1.1$), heating (at $\omega/\omega_0 = 0.9$ and 1.0), and thermal noise quadrature squeezing (at all parameters). It is noteworthy to mention that heating or cooling are related to the decay rate of the oscillations of the Green’s function, as can be seen in Eq. (6). Lower decay rates imply more heating, while higher decay rates imply less heating or even cooling.

In Fig. (4), a 2D color intensity plot displays the amplitude of the thermal noise squeezing as a function of half the pump frequency, namely $\omega_0$, and the pump amplitude. We notice that the closer one gets to the first instability zone the higher the amplitude of the squeezing, i.e. the amplitude of the oscillations of the thermal fluctuations $\langle x^2(t) \rangle$. The oscillations occur basically at $2\omega$, the superharmonics at $4\omega$ are too small and are thus neglected in this plot. For low dynamical temperatures, the squeezing amplitude has to be small too. For higher dynamical temperatures, the squeezing amplitude tends to increase, although there is also dependence on detuning. From this figure one sees that the larger the detuning, the smaller the squeezing effect, although the effect is not symmetric as one can clearly see. This is related to terms that depend linearly on detuning in the Green’s functions.

The results given by Eq. (12) imply that by varying the pump amplitude $F_p$ and the detuning $\Omega$, we can create a continuous family of classical thermo-mechanical squeezed states, generalizing the experimental results of Rugar and Grüter [2].
Figure 2. (color online) In the frames above we show several Green’s functions with equally-spaced in time initial conditions in one given period of the parametric driving. They are vertically spaced for clarity, since all asymptotes are zero. In each frame we have a comparison between numerical results given by the numerical integration of Eq. (5) and the analytical approximate results given by Eqs. (7) or (9). We have a) $\omega = 0.9 \omega_0$, b) $\omega = 1.0 \omega_0$, and c) $\omega = 1.1 \omega_0$. The initial values of the Green’s functions are $G(t, t') = 0$ and $\frac{\partial}{\partial t} G(t, t') = 1.0$ when $t = t' + 0^+$. The pump amplitude used was $F_p = 0.15$. 
Figure 3. (color online) Steady-state time evolution of the mean square displacement $\langle x^2(t) \rangle$ versus time as defined in Eq. (6), with the Green function given approximately by Eqs. (7) or (9). The same results are obtained by using Eqs. (12), indicative that the expressions for the analytical estimates of $\langle x^2(t) \rangle$ are correct. We obtain cooling when $\omega = \omega_0$ and heating at the other frequencies. The horizontal solid green line is obtained from the equipartition theorem, it indicates the expected equilibrium value of the fluctuations (for $F_p = 0$) when the heat-bath temperature is at $T = 1.0$, while the other straight lines represent the corresponding dynamical temperatures, i.e. time-averaged values $\langle x^2(t) \rangle$ as given by Eq. (10) or by Eq. (11) for $F_p = 0.15$.

Figure 4. (color online) Contour plot of the amplitude of squeezing at $2\omega$ of the mean square displacement $\langle x^2(t) \rangle$ as given by $\sqrt{|A_{2\omega}|^2 + |B_{2\omega}|^2}$, in which $A_{2\omega}$ and $B_{2\omega}$ are defined in Eqs. (12). The amplitude at $4\omega$ is very small for the pump amplitudes $F_p$ used here.
4. Conclusions
The ability to reduce thermal noise in MEMS devices, in addition to parametric amplification mechanisms, can greatly improve the accuracy and precision of measuring small masses, weak forces, and, furthermore, may contribute to the effort of measuring gravitational waves. Our main contribution resides in developing a relatively simple and general theoretical framework that provides us with quantitative estimates of cooling, heating, and thermal noise squeezing in mechanical resonators. We believe these estimates can well serve, with the appropriate modifications to adjust to particular systems, as a guide to the experimentalist in search for such phenomena. Further development to our model may be achieved by including more details about the noise model, such as memory effects [12, 13], by taking into account the coupling to a heat bath that could be made out of photons, as in radiation-pressure cooling, or via coupling to phonons. Finally, we note that here we provided a general framework for studying classical thermomechanical squeezing that could be applied to several different systems in the presence of noise, such as ac-driven doubly-clamped beams [14], ions in ion traps [15], opto-mechanical cavities [16], and to magnetic nanoparticles driven by ac magnetic fields.

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