From exact-WKB towards singular quantum perturbation theory

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Abstract

We use exact WKB analysis to derive some concrete formulae in singular quantum perturbation theory, for Schrödinger eigenvalue problems on the real line with polynomial potentials of the form $(q^M + gq^N)$, where $N > M > 0$ even, and $g > 0$. Mainly, we establish the $g \to 0$ limiting forms of global spectral functions such as the zeta-regularized determinants and some spectral zeta functions.

1 Introduction

The purpose of this work is to set up a path to obtain precise statements of a quantum perturbative nature with the help of exact WKB analysis. The RIMS has always played a major and pioneering role in the inception and growth of exact asymptotic analysis, and earlier, in the development of some of its fundamental tools (such as hyperfunctions and holomorphic microlocal analysis). This influence is testified by the Proceedings volume of a recent Kyoto conference, which offers a very complete view of the subject [5]. It is therefore a great honor and a proper tribute to RIMS to write about exact WKB analysis in this anniversary issue.

A prototype problem in quantum perturbation theory is the quartic anharmonic oscillator,

$$\left(-\frac{d^2}{dq^2} + q^2 + gq^4 - E\right)\Psi(q) = 0, \quad q \in \mathbb{R}, \ g \geq 0. \quad (1)$$

This problem has a purely discrete eigenvalue spectrum $\{E_k(g)\}$ for all $g \geq 0$. A typical task in (Rayleigh–Schrödinger) perturbation theory is to compute individual eigenvalues
\( E_k(g) \) (or their eigenfunctions) as formal power series of the coupling constant \( g \). This is practically important when the unperturbed \( (g = 0) \) problem is exactly solvable, here a harmonic oscillator; a major drawback is however that the coupling term has the higher degree, hence the formalism is singular. Thus, the perturbation series converges for no \( g \neq 0 \); it only gives an asymptotic expansion for \( g \to 0 \), moreover non-uniform in the quantum number \( k \).

As our theoretical discussion can readily include all binomials potentials, we will actually study the more general Schrödinger equation

\[
\left(-\frac{d^2}{dq^2} + U_g(q) - E\right) \Psi(q) = 0, \quad U_g(q) \overset{\text{def}}{=} q^M + gq^N, \quad q \in \mathbb{R}, \quad N > M \geq 2 \text{ even}, \quad g \geq 0;
\]

we keep \( E_k(g) \) as a generic notation for the eigenvalues of this problem ((\( N, M \))-dependences now being implied).

(Exact WKB formalisms accommodate non-even potentials as well \[2, 7\]; for instance, eq.(2) could be considered with odd \( N \) or \( M \) but on the half-line \([0, +\infty)\] \[7\]; however, this extension is not essential here while it does complicate the classification when \( M = 1 \), so we omit it in the present work.)

A very basic fact (Symanzik scaling property) is that a simple coordinate dilation, \( q \mapsto g^{-1/(N+2)}q \), establishes a unitary equivalence between the two Schrödinger operators

\[
\begin{align*}
    \hat{H}_v &\overset{\text{def}}{=} -\frac{d^2}{dq^2} + U_v(q), \\
    \text{where} \quad v &\equiv g^{-(M+2)/(N+2)} \\
    V_v(q) &\overset{\text{def}}{=} q^N + q^M.
\end{align*}
\]

Thus, eq.(2) is equivalent to

\[
\left(-\frac{d^2}{dq^2} + q^N + q^M + \lambda\right) \Psi(q) = 0, \quad v \equiv g^{-(M+2)/(N+2)}, \lambda \equiv -v^{2/(M+2)}E. \quad (5)
\]

In this transformed Schrödinger equation, the interaction term is now \( vq^M \) and has the lower degree, so that \( v \) can act as a regular deformation parameter; the former perturbative regime \( g \to 0 \) translates as the asymptotic \( v \to +\infty \) regime. However, at no finite \( v \) is the problem \( 5 \) solvable in any traditional sense, and this has severely limited practical uses of this reparametrization. On the other hand, this deformation can be fully studied by exact WKB analysis, which now handles general (1D) polynomial potentials. One earlier detailed study of this sort is based on resurgence theory \[2\]. Another such path from exact WKB to perturbation theory lies in proving the Zinn-Justin conjectures about multi-instantons \[3, 8\]. Here, continuing a different type of study initiated in \[7\] (Secs. 3–4) (within an exact WKB framework built upon Sibuya’s formalism \[6\]), we seek to specify how the spectral determinants themselves (and related spectral functions) asymptotically depend on the coupling parameter \( v \to +\infty \) (or \( g \to 0 \)). Spectral functions being symmetric functions of all eigenvalues \( E_k(g) \) together, the non-uniformity in \( k \) of perturbative approximations must show up somehow, and the \( g \to 0 \) behavior of such objects might not be obviously traceable to existing (fixed-\( k \)) perturbative results.

As a very concrete example, we may ask: how do the spectral zeta functions \( Z_g(s) = \sum_{k=0}^{\infty} E_k(g)^{-s} \) precisely behave for \( g \to 0 \)? Specially at \( s = 1 \) when \( M = 2 \): then, that
The series converges ∀g > 0 but term by term it becomes the divergent (odd) harmonic series \( \sum_k (2k+1)^{-1} \) at g = 0. The latter admits one fundamental regularization by means of a sharp summation cutoff \( K \):

\[
\sum_{k=0}^{K-1} \frac{1}{2k+1} \sim \frac{1}{2} (\log K + \gamma + 2 \log 2), \quad K \to +\infty,
\]

which is (in just a slight variant form) the basic definition of Euler’s constant \( \gamma \). Now, the eigenvalues themselves obey \( E_k(g) \sim 2k+1 \) for \( g \to 0 \) (\( k \) fixed) by perturbation theory, but \( E_k(g) \propto [g(2k+1)^N]^{2/(N+2)} \) for \( k \to +\infty \) (\( g \) fixed) by the asymptotic Bohr–Sommerfeld condition, the crossover zone being roughly located by \( 2K_g + 1 \propto g^{-2/(N-2)} \). Hence the series for \( Z_g(1) \) is another natural, “soft” regularization of the odd harmonic series, beginning to act around \( k \propto K_g \). Substituting \( K = K_g \) into eq.(6), we are led to expect \( Z_g(1) \sim -\frac{1}{N-2} \log g + C_N \). Such an intuitive approach may work for the logarithmic slope, but not for identifying the additive constant \( C_N \). By contrast, exact WKB analysis can yield a precise asymptotic prediction for this (and other) zeta-values: see our final formula.

The outline of the paper is as follows. Sec. 2 recalls essential prerequisites and definitions for the exact WKB approach to be used here. Sec. 3 presents the asymptotic problem and its conceptual resolution by exact WKB theory as eq.(34). Sec. 4 performs the key computational steps: a class of specific improper action integrals \( \int_0^{+\infty} \Pi(q) \, dq \) (where \( \Pi(q) \) are essentially classical momentum functions, and the integrals are primitively very divergent) are explicitly evaluated. Finally, Sec. 5 processes all intermediate calculations into concrete formulae, mainly eqs.(53),(58).

2 Some notions from exact WKB theory

We recall the essential facts and notations to be used later concerning the exact WKB treatment of Schrödinger operators on \( L^2(\mathbb{R}) \), of the form \( \hat{H} \equiv -\frac{d^2}{dq^2} + V(q) \), with a polynomial potential \( V(q) = +q^N + [\text{lower-degree terms}] \), here taken real and even. Details and justifications have to be omitted (cf. Sec. 1, and references therein). Such operators are self-adjoint, have compact resolvents and commute with the parity operator \( (q \mapsto -q) \). A frequently needed quantity (which we call the order of the problem) is

\[
\mu(N) \equiv \frac{1}{2} + \frac{1}{N}.
\]

As standard notations, we will also use \( \psi(z) \equiv \left[ \Gamma' / \Gamma \right](z) \), and \( \gamma = \text{Euler’s constant} \).

2.1 Improper action integral, and residue function

Important quantities enter at the classical dynamical level around the (complexified) momentum function,

\[
\Pi_\lambda(q) \equiv (V(q) + \lambda)^{1/2},
\]
where the constant \((-\lambda)\) stands for the classical energy and, say, \(\lambda > -\inf V\) (initially). Next, improper action integrals over semi-infinite paths prove very useful: \(\int_{q}^{+\infty} \Pi_{\lambda}(q') \, dq'\) (primatively divergent) is very naturally redefined as the analytical continuation to \(s = 0\), when this is finite, of

\[
I_{q}(s, \lambda) \overset{\text{def}}{=} \int_{q}^{+\infty} (V(q') + \lambda)^{-s+1/2} \, dq' \quad \text{(convergent for Re}\,(s) > \mu(N)). \tag{9}
\]

Now, the \(s\)-plane singularities of \(I_{q}(s, \lambda)\) entirely come from the large-\(q\) behavior of the integrand. Specifically, the \(q \to +\infty\) expansion (explicitly computable order by order)

\[
(V(q) + \lambda)^{-s+1/2} \sim \sum_{\rho} \beta_{\rho}(s) q^{\rho-Ns} \quad (\rho = N/2, \ N/2 - 1, \cdots) \tag{10}
\]

implies the singular decomposition

\[
I_{q}(s, \lambda) \sim -\sum_{\rho} \beta_{\rho}(s) q^{\rho+1-Ns}; \tag{11}
\]

hence at \(s = 0\), \(I_{q}(s, \lambda)\) has at most a simple pole, generated by the \(\rho = -1\) term (if any):

\[
\text{Res}_{s=0} I_{q}(s, \lambda) = \beta_{-1}(0)/N; \tag{12}
\]
a value actually independent of \(\lambda\) (save when \(N = 2\)) and of \(q\).

A central distinction sets in at this point: if the “residue function” \(\beta_{-1}(s) \equiv 0\), the Schrödinger problem \((\hat{H} + \lambda)\Psi = 0\) will behave more simply (“normal” type, \(N\)); otherwise, “anomaly” corrections will enter (type \(A\)). Wholly generic polynomials \((V(q) + \lambda)\) are of type \(A\); still, in a sense, “a majority” of them have \(\beta_{-1}(s) \equiv 0\): among the even ones, already all those having a degree \(N\) multiple of \(4\) (and, among the non-even ones, all those of odd degree).

Thus, for the \(N\) type, \(\int_{q}^{+\infty} \Pi_{\lambda}(q') \, dq'\) can be readily defined as the analytical continuation of \(I_{q}(s, \lambda)\) to the (regular) point \(s = 0\); whereas for the general (\(A\)) type, the best specification is not the bare finite part of \(I_{q}(s, \lambda)\) at the pole \(s = 0\) (denoted \(\text{FP}_{s=0} I_{q}(s, \lambda)\)), but rather \((\text{[7]}, \text{eq.}(32))\)

\[
\int_{q}^{+\infty} \Pi_{\lambda}(q') \, dq' \overset{\text{def}}{=} \text{FP}_{s=0} I_{q}(s, \lambda) + 2(1 - \log 2) \beta_{-1}(0)/N \tag{13}
\]
in order to preserve the basic identities \((\text{[2]}\text{]) below. Additivity is also maintained:

\[
\int_{q}^{+\infty} \Pi_{\lambda}(q') \, dq' = \int_{q}^{q''} \Pi_{\lambda}(q') \, dq' + \int_{q''}^{+\infty} \Pi_{\lambda}(q') \, dq' \quad \text{for all } q, q'' \text{ finite} \tag{14}
\]
(because a finite integral \(\int_{q''}^{q''} (V(q') + \lambda)^{-s+1/2} \, dq'\) is entire in \(s\)).

Remarks: (i) \(\int_{q}^{+\infty} \Pi_{\lambda}(q') \, dq'\) is an “Agmon distance from \(q\) to \(+\infty\), suitably renormalized; (ii) this procedure is a classical counterpart to zeta-regularization at the quantum level; (iii) just as the extra term in eq.\((\text{[13]}\), all anomaly terms here will simply be proportional to the value \(\beta_{-1}(0)\), but more general forms occur elsewhere \([\text{7}]\).
2.2 Spectral functions

An operator $\hat{H}$ as above has a purely discrete real spectrum \( \{\lambda_0 < \lambda_1 < \lambda_2 < \cdots\} \), \( (\lambda_k \uparrow +\infty) \), where even (resp. odd) \( k \) correspond to eigenfunctions of even (resp. odd) parity. Parity-split spectral zeta functions (à la Hurwitz) can be defined as

\[
Z^\pm(s, \lambda) \overset{\text{def}}{=} \sum_{k \text{ even/odd}} (\lambda_k + \lambda)^{-s} \quad \text{for } \Re s > \mu(N) \tag{15}
\]

and, say, \( \lambda > -\lambda_0 \); many results however take a sharper form upon a skew versus a full zeta function, respectively defined as

\[
Z^\pm \equiv Z^+ - Z^-, \quad Z \equiv Z^+ + Z^-.
\tag{16}
\]

Spectral determinants \( D^\pm(\lambda) \equiv \det^\pm(\hat{H} + \lambda) \) are defined by zeta-regularization, as

\[
D^\pm(\lambda) \overset{\text{def}}{=} \exp[-\partial_s Z^\pm(s, \lambda)]_{s=0} \quad \text{(and } D^P \equiv D^+/D^-, \ D \equiv D^+D^-), \tag{17}
\]

where \( s = 0 \) is reached by analytical continuation from \( \{\Re s > \mu(N)\} \). More constructive specifications are

- their Weierstrass infinite products (written for \( \mu(N) < 2 \), which is true here):

\[
D^\pm(\lambda) \equiv D^\pm(0) e^{\sum_{k \text{ even/odd}} Z^\pm(s,0) \lambda} \prod_k (1 + \lambda/\lambda_k) e^{-\lambda/\lambda_k}, \tag{18}
\]

\[
eq D^\pm(0) \prod_{k \text{ even/odd}} (1 + \lambda/\lambda_k) \quad \text{when } \mu(N) < 1, \ i.e., \ N > 2 \tag{19}
\]

and likewise for \( D, D^P \); this shows that the determinants continue to entire functions (of order \( \mu(N) \)) in the variable \( \lambda \) (except for \( D^P \), meromorphic);

- the basic identities of the exact-WKB method: let \( \Psi_\lambda(q) \) be the canonical recessive solution of the differential equation, specified through its \( q \rightarrow +\infty \) asymptotic form

\[
\Psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} e^{q \int_q^{+\infty} \Pi_\lambda(q')dq'}, \tag{20}
\]

where \( \int_q^{+\infty} \Pi_\lambda(q')dq' \) is fixed according to eq.(13); then, under that precise normalization,

\[
D^-(\lambda) \equiv \Psi_\lambda(0), \quad D^+(\lambda) \equiv -\Psi'_\lambda(0), \tag{21}
\]

(also valid for a rescaled potential, i.e., \( V(q) = uq^N + \cdots \), with \( u > 0 \)). Remark: the solutions obeying \( \Psi_\lambda \) are close to Sibuya’s subdominant solutions \( 6 \), but the two normalizations fully coincide only when the type is \( N \).

Finally, we will need the transformation rules for spectral functions under a global spectral dilation \( (\lambda_k \mapsto r\lambda_k, \ r = \text{cst.} > 0) \). Obviously,

\[
Z^\pm(s, \lambda) \mapsto r^{-s} Z^\pm(s, \lambda/r) \quad \text{for } \Re s > \mu(N) \tag{22}
\]

(and likewise for \( Z, Z^P \)); hence upon continuation to \( s = 0 \), and applying eq.(17),

\[
D(\lambda) \mapsto r^{Z(0,\lambda/r)} D(\lambda/r), \quad D^P(\lambda) \mapsto r^{Z^P(0,\lambda/r)} D^P(\lambda/r) \tag{23}
\]

where, moreover, \( 7 \), eqs.(27),(37))

\[
Z(0, \lambda) \equiv -2\beta_{-1}(0)/N, \quad Z^P(0, \lambda) \equiv 1/2. \tag{24}
\]
3 The asymptotic $v \to +\infty$ problem

We now return to the Schrödinger operator $\hat{H}(v) = -d^2/dq^2 + q^N + vq^M$, as in eq. (5) ($N > M \geq 2$ both even, $v > 0$). We will find the asymptotic behaviors of its spectral determinants in the regime of singular perturbation theory for eq. (1):

$$D^\pm(\lambda, v) \equiv \det^\pm(\hat{H}(v) + \lambda) \quad (\lambda > 0), \quad v \equiv g^{-(M+2)/(N+2)} \to +\infty. \quad (25)$$

To lowest-order in the $g \to 0$ perturbation theory, the individual eigenvalues $\lambda_k(v)$ of $\hat{H}(v)$ become asymptotic to those of $\hat{H}_0(v) = -d^2/dq^2 + vq^M$. We then expect $\det^\pm(\hat{H}(v) + \lambda)$ to become somehow asymptotically proportional to $\det^\pm(\hat{H}_0(v) + \lambda)$ as $v \to +\infty$, but the latter regime is singular and moreover non-uniform in $k$; hence the actual behavior of the determinants cannot be taken for granted. In [7] (Secs. 3–4), we tackled it for a few binomial potentials and exclusively at $\lambda = 0$; now we will do it in full generality.

3.1 Detailed anomaly types

As argued in Sec. 2.1, it is essential to distinguish between normal (zero-residue) and anomalous (non-zero residue) cases, but this now applies independently to the coupled ($= \hat{H}(v)$) and the uncoupled ($= \hat{H}_0(v)$) problems.

- the coupled problem $(\Pi_\lambda(q)^2 = q^N + vq^M + \lambda)$: the residue function is the coefficient of $q^{-1-Ns}$ in the generalized binomial expansion for $q^N(1/q^s)(1 + vq^{M-N} + \lambda q^{-N})^{1/2-s}$. When $N > 2$ as here, the residue function cannot involve $\lambda$; specifically,

$$\beta_{-1}(s) \equiv 0 \quad \text{unless} \quad \frac{N+2}{2(N-M)} = j \in \mathbb{N}^* \quad \text{("anomaly condition $A_j$ of level $j$")}; \quad (26)$$

thus, anomalies attach to specially correlated exponents $N$, $M$ only:

(level $j$ : ) $N = 2jm - 2$, $M = N - m$ for $m \in \mathbb{N}^*$ (with $m$ even for even potentials);

and then

$$\beta_{-1}(s) \equiv (-1)^j \frac{\Gamma(s+j-1/2)}{\Gamma(s-1/2)} j! v^j \left[ \beta_{-1}(0) = (-1)^{j-1} \frac{(2j-2)!}{2^{2j-1}(j-1)! j!} v^j \right]. \quad (28)$$

- the uncoupled problem $(\Pi_{0,\lambda}(q)^2 = vq^M + \lambda)$: the same calculation now simply yields

$$\beta_{-1}(s) \equiv v^{-1/2} \lambda (1/2 - s) \quad \text{if} \quad M = 2 \quad [A_1 \text{ for } \lambda \neq 0], \quad \text{otherwise} \quad \beta_{-1}(s) \equiv 0 \quad [N]. \quad (29)$$

The harmonic oscillator $(\Pi(q)^2 = vq^2 + \lambda)$ thus gives the prime example of anomaly, actually the unique case (among all potentials) where the residue depends on the spectral parameter; all other binomials $\{vq^M + \lambda\}$ ($M \neq 2$) are of type $N$.

The type can abruptly change either way in the $v \to +\infty$ limit, giving birth to four distinct variants (the “basic” example of eq. (1) is not the simplest!):

$\mathbf{N} \to \mathbf{N}$: e.g., $V(q) = q^8 + vq^4$;
$\mathbf{A}_j \to \mathbf{N}$: e.g., $V(q) = q^6 + vq^4$, of level $j = 2$;
$\mathbf{N} \to \mathbf{A}_1$: e.g., $V(q) = q^4 + vq^2$ (the “basic” example) when $\lambda \neq 0$;
$\mathbf{A}_j \to \mathbf{A}_1$: only one case, $V(q) = q^6 + vq^2$ when $\lambda \neq 0$, for which $j = 1$. 

3.2 The main estimate

We can relate the coupled and uncoupled spectral determinants very easily through a key result of exact WKB theory, the basic identities \cite{21}. These are to be written for both (coupled and uncoupled) problems independently:

\begin{equation}
\begin{aligned}
\det^-(\hat{H}(v) + \lambda) &\equiv \Psi_{\lambda}(0, v), &\det^+(\hat{H}(v) + \lambda) &\equiv -\Psi_{\lambda}'(0, v), \\
\det^-(\hat{H}_0(v) + \lambda) &\equiv \Psi_{0,\lambda}(0, v), &\det^+(\hat{H}_0(v) + \lambda) &\equiv -\Psi_{0,\lambda}'(0, v),
\end{aligned}
\end{equation}

where \(\Psi_{\lambda}(q, v)\), resp. \(\Psi_{0,\lambda}(q, v)\) are the canonical recessive solutions of \((\hat{H}(v) + \lambda)\Psi = 0\), resp. \((\hat{H}_0(v) + \lambda)\Psi_0 = 0\). So, the problem boils down to relating \(\Psi_{\lambda}(q, v)\) and \(\Psi_{0,\lambda}(q, v)\) near \(q = 0\) as \(v \to +\infty\).

Now, as soon as \(|q|^{N-M} \ll v\), the term \(q^N\) becomes a negligible perturbation of \(vq^M\) within the Schrödinger equation, hence the recessive solution \(\Psi_{\lambda}(q, v)\) has to become asymptotically proportional to \(\Psi_{0,\lambda}(q, v)\) in that regime (given that the WKB form \(20\) holds asymptotically for \(\Pi_{\lambda}(q) \to +\infty\) whichever way the limit takes place, including \(v \to +\infty\) at fixed \(q\)). The only problem is then to determine the asymptotic ratio \(\Psi_{\lambda}(q, v)/\Psi_{0,\lambda}(q, v) \sim C(\lambda, v)\) \((q\text{-independent})\) as \(v \to +\infty\) at fixed \(q\). By contrast, the alternative normalization of recessive solutions based at \(q = 0\),

\begin{equation}
\Psi_{\lambda}(q, v) \sim \Pi_{\lambda}(q, v)^{-1/2} e^{-\int_0^q \Pi_{\lambda}(q', v) \, dq'}, &\Psi_{0,\lambda}(q, v) \sim \Pi_{0,\lambda}(q, v)^{-1/2} e^{-\int_0^q \Pi_{0,\lambda}(q', v) \, dq'}
\end{equation}

(for \(q \to +\infty\)) immediately entails

\begin{equation}
\Psi_{\lambda}(q, v) \sim \Psi_{0,\lambda}(q, v) \quad \text{for } v \to +\infty, \quad |q|^{N-M} \ll v,
\end{equation}

because the asymptotic equivalence \(\Pi_{\lambda}(q', v) \sim \Pi_{0,\lambda}(q', v)\) can be used all over the bounded interval \([0, q]\).

The final issue is to relate the two normalizations, the canonical one of eq. \(20\) \("based at \(q = +\infty\)"") and the latter one based at \(q = 0\). Thanks to eq. \(14\), the answer is simply

\begin{equation}
\Psi_{\lambda}(q, v) \equiv e^{\int_0^{q} \Pi_{\lambda}(q', v) \, dq'} \Psi_{\lambda}(v, q) \quad \text{(and likewise for } \Psi_{0,\lambda} \text{ with } \Pi_{0,\lambda}).
\end{equation}

Finally, putting together eqs. \(30\) - \(33\), we end up with the comparison formula

\begin{equation}
\det^\pm(\hat{H}(v) + \lambda) \sim e^{S(\lambda, v)} e^{-S_0(\lambda, v)} \det^\pm(\hat{H}_0(v) + \lambda) \quad (v \to +\infty)
\end{equation}

(stated in most general terms), where

\begin{equation}
S(\lambda, v) = \int_0^{+\infty} \Pi_{\lambda}(q, v) \, dq, \quad \text{resp. } S_0(\lambda, v) = \int_0^{+\infty} \Pi_{0,\lambda}(q, v) \, dq,
\end{equation}

are coupled, resp. uncoupled, improper action integrals. Specifically here,

\begin{equation}
S(\lambda, v) = \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2} \, dq, \quad \text{resp. } S_0(\lambda, v) = \int_0^{+\infty} (vq^M + \lambda)^{1/2} \, dq.
\end{equation}

The problem has thus been decomposed and reduced to the asymptotic \((v \to +\infty)\) evaluation of the two action integrals of eq. \(36\) separately.
4 Explicit formulae for improper action integrals

This Section constitutes a kind of technical digression, but the effective computations of improper action integrals to be presented might also be of autonomous interest and applicability.

4.1 Binomial \( \Pi(q)^2 \) : exact evaluation

We compute the improper action integral \( \int_0^{+\infty} \Pi(q) \, dq \) exactly for a binomial \( \Pi(q)^2 = uq^N + vq^M \), in the rather general setting \( N > M \geq 0, u, v > 0 \), resulting in the multipurpose formulae (10) and (11) (here, \( N \) and \( M \) might even be non-integers).

At the core, \( \int_0^{+\infty} \Pi(q) \, dq = \lim_{s \to 0} I_0(s) \) where

\[
I_0(s) \overset{\text{def}}{=} \int_0^{+\infty} (uq^N + vq^M)^{1/2-s} \, dq \quad (\text{Re} \, s > \frac{1}{2} + \frac{1}{N}), \tag{37}
\]

as long as the limit (understood as the analytical continuation to \( s = 0 \)) is finite. Now the right-hand side reduces to a Eulerian integral, of the form

\[
\int_0^{+\infty} (ax + b)^{1/2-s} x^{\alpha-1} \, dx = a^{-\alpha} b^{1/2 + \alpha - s} \Gamma(\alpha) \Gamma(s - \alpha - 1/2) / \Gamma(s - 1/2) \tag{38}
\]

(under the change of variable \( q^{N-M} = u^{-1} v x \); here, \( \alpha = [M(1-2s)+2]/[2(N-M)] \)); more precisely,

\[
I_0(s) \equiv \frac{\Gamma\left(\frac{M(1-2s)+2}{2(N-M)}\right)}{(N-M) \Gamma(s-1/2)} u^{-\frac{M+2}{2(N-M)}} v^{\frac{N+2}{2(N-M)}}. \tag{39}
\]

Consequently, at \( s = 0 \),

\[
\int_0^{+\infty} (uq^N + vq^M)^{1/2} \, dq = \frac{\Gamma\left(\frac{M+2}{2(N-M)}\right) \Gamma\left(\frac{-N+2}{2(N-M)}\right)}{(N-M) \Gamma(-1/2)} u^{-\frac{M+2}{2(N-M)}} v^{\frac{N+2}{2(N-M)}} \tag{40}
\]

in the normal case, i.e., when the right-hand side is finite, meaning here \( \frac{N+2}{2(N-M)} \notin \mathbb{N} \).

As concrete examples of this \( \mathbb{N} \) type:

\[
\int_0^{+\infty} (q^4 + vq^2)^{1/2} \, dq = -v^{3/2} / 3 \tag{41}
\]

\[
\int_0^{+\infty} (uq^N + \lambda)^{1/2} \, dq = -(2\sqrt{\pi})^{-1} \Gamma\left(1 + \frac{1}{N}\right) \Gamma\left(-\frac{1}{2} - \frac{1}{N}\right) u^{-\frac{1}{N}} \lambda^{1/2 + \frac{1}{N}} \quad (N \neq 2). \tag{42}
\]

Now, the right-hand side of eq.(10) turns infinite whenever \( (2j-1)N = 2(jM+1) \) for some \( j \in \mathbb{N}^* \) (\( j = 0 \) cannot occur); this is precisely the anomaly condition \( \mathbf{A}_j \) of level \( j \). The binomials of any type \( \mathbf{A}_j \) can be readily (albeit tediously) handled by applying eq.(13) to \( I_0(s) \). (The cases with \( j = 1 \) as well as the action integral of eq.(11) were implicitly evaluated in [7], by a different route.) First, the residue is

\[
\beta_{-1}(0) = (-1)^{j-1} \frac{(2j-2)!}{2^{2j-1}(j-1)!j!} u^{1/2-j} v^j; \tag{43}
\]
then, the finite part at \( s = 0 \) of eq. (39) gets extracted as

\[
\beta_{-1}(0) \left( \frac{2j}{N+2} \left[ \psi(j+1) - \log v + \frac{M}{N} \left( -\psi(j-1/2) + \log u \right) \right] - \frac{1}{N} \psi(-1/2) \right);
\]

so that finally, when \( M = [(j-1/2)N - 1]/j \) (condition \( A_j \)), eq. (13) yields

\[
\int_{0}^{+\infty} (uv^{N} + uq^{M})^{1/2} dq = \frac{2j \beta_{-1}(0)}{N+2} \left[ -\log v + \frac{j}{m+1} + \frac{2M}{N} \left( \log 2 + \frac{1}{2} \log u - \frac{j-1}{2m-1} \right) \right].
\]

The cases with \( j = 1 \) are of special interest. Besides the harmonic oscillator, the general binomials of type \( A_1 \) are just the supersymmetric potentials (at zero energy):

\[
\Pi(q)^{2} = uq^{N} + vq^{M} \quad \text{with } N = 2M + 2 \quad (M > 0),
\]

and eq. (44) distinctly simplifies to

\[
\int_{0}^{+\infty} (uv^{N} + uq^{N/2-1})^{1/2} dq = \frac{u^{-1/2}v}{N+2} \left[ -\log v + 1 + \frac{N-2}{N} \left( \log 2 + \frac{1}{2} \log u \right) \right] \quad (j = 1).
\]

In particular, for the harmonic oscillator \((N = 2)\) at a general energy value \((-\lambda)\),

\[
\int_{0}^{+\infty} (vq^{2} + \lambda)^{1/2} dq = \frac{1}{4}v^{-1/2}\lambda(1 - \log \lambda).
\]

### 4.2 Trinomial \( \Pi(q)^{2} \): asymptotic \( v \to \infty \) evaluation

We now consider a trinomial \( \Pi(q)^{2} \) of the form \( q^{N} + vq^{M} + \lambda \), with even \( N > M > 0 \), and a systematically constant third term: the spectral parameter itself, \( \lambda \) \((>0)\) \((=\) minus the total energy\). One of the coefficients can always be scaled out to unity, and we have done this for the highest power initially.

In this case we can no longer compute the action integral \( \int_{0}^{+\infty} \Pi(q) dq \) exactly. In view of eq. (34), however, we mainly need its large-\( v \) behavior, specially for \( v \to +\infty \) in order to recover singular perturbation theory according to eq. (3) (but as in [7], we expect the results to remain valid over suitable sectors in the complex \( v \)-plane).

According to the zeta-regularization idea, we must start from the large-\( v \) behavior of \( I(s; \lambda, v) \) \( \overset{\text{def}}{=} \int_{0}^{+\infty} (q^{N} + vq^{M} + \lambda)^{1/2-s} dq \); this problem is rather delicate, so any brute-force expansion scheme is dubious. Instead, we apply the following general idea: if the function \( I(v) \) under study is an inverse Mellin transform,

\[
I(v) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \tilde{I}(\sigma)v^{\sigma} d\sigma,
\]

then the singularities of \( \tilde{I}(\sigma) \) in the half-plane \( \{ \text{Re} \ \sigma < c \} \) encode the large-\( v \) behavior of \( I(v) \). Thus (by the residue calculus) any polar part of the form \( A(\sigma - \sigma_{0})^{-2} + B(\sigma - \sigma_{0})^{-1} \) in \( \tilde{I}(\sigma) \) represents an asymptotic contribution \( v^{\sigma_{0}}(A \log v + B) \) to \( I(v) \). This is particularly valuable for \( I(s; \lambda, v) \), because its direct Mellin transform \( \tilde{I}(s; \lambda, \sigma) \) \( \overset{\text{def}}{=} \)
Figure 1: Schematic plot (using some non-integer $N, M$) of the poles $\sigma(s)$ of the Mellin transform $\tilde{I}(s; \lambda, \sigma)$ in eq. (49). The two main contributing poles (in the $v \to +\infty$ limit) are drawn with bold lines; non-contributing poles are drawn with dashed lines.

\[
\int_0^{+\infty} I(s; \lambda, v) v^{-\sigma-1} dv \text{ is exactly computable (using the same formula as for the exact action integral of a binomial but now twice in succession), and it is a meromorphic function of } \sigma: \text{ formally,}
\]

\[
\int_0^{+\infty} dv \, v^{-(\sigma-1)} (q^N + vq^M + \lambda)^{1/2-s} = \frac{\Gamma(-\sigma) \Gamma(s+\sigma-1/2)}{\Gamma(s-1/2)} q^{M\sigma}(q^N + \lambda)^{1/2-s} \sigma
\]

\[
\Rightarrow \quad \tilde{I}(s; \lambda, \sigma) = \frac{\Gamma(-\sigma) \Gamma(-\frac{M+1}{N}) \Gamma(s+\sigma-1/2)}{\Gamma(s-1/2)} \lambda^{-s-\frac{N-M}{M} \sigma + 1/2 \sigma + \frac{1}{2}}
\]

(using the change of variable $q^N = \lambda r$ for the $q$-integration). Now this Mellin transform also has to genuinely exist somewhere; here, all integrations converge in some strip $\sigma' < \text{Re } \sigma < 0$ provided Re $s > \mu(N)$, and the inverse transformation applies with $c = -1$. Consequently, the poles $\sigma(s)$ relevant to the current asymptotic problem are those which lie in $\{\text{Re } \sigma < 0\}$ when Re $s > \mu(N)$, and their contributions are then to be analytically continued to $s = 0$. Overall, the poles in eq. (49) form three arithmetic progressions, one for each Gamma factor in numerator; they are real for real $s$ (fig. 1). At $s = 0$, any pole $\sigma(s)$ will contribute an asymptotic term of degree $d_v = \sigma(0)$ in $v$ (on general grounds) and of degree $d_\lambda = \frac{1}{2} + \frac{1}{N} - \frac{N-M}{N} \sigma(0)$ in $\lambda$ (by examination of eq. (49)). At the end, we plan to keep the terms of degree $d_g \leq 0$ in the perturbative coupling constant $g$ (discarding $o(1)$ terms when $g \to 0$); now the Symanzik scaling (eq. (5) at fixed $E$) entails $d_g \equiv \frac{-M+2}{N+2} (d_v + \frac{2}{M+2} d_\lambda) = -(M\sigma(0)+1)/N$; altogether, $d_g \leq 0$ then amounts to keeping only the poles for which $\sigma(0) \geq -1/M$.

When $M \geq 2$ (as here), only two poles $\sigma(s)$ satisfy both criteria, (in real form) $\sigma(s) < 0$ for $s > \mu(N)$ and $\sigma(0) \geq -1/M$: they are, in decreasing order (at $s = 0$),

\[
\sigma_0(s) \equiv \frac{N}{N-M} \left( \frac{1}{2} + \frac{1}{N} - s \right) \quad \text{(leading)} \quad \text{and} \quad \sigma_1(s) \equiv -\frac{1}{M} \quad \text{(subleading). (50)}
\]

They are generically simple, with two exceptions at $s = 0$: $\sigma_0(0) = \frac{N+2}{2(N-M)}$ becomes confluent with the (fixed) pole $\sigma = +j$ when the coupled problem is of anomalous type.
A_j; and independently, \( \sigma_1 \) becomes confluent with the (next mobile) pole \( \frac{N}{N-M} \left( \frac{1}{2} + \frac{1}{N-M} - s - 1 \right) \) when the uncoupled problem is anomalous, i.e., \( M = 2 \). The latter will induce a usual double-pole contribution; the former confluence is worse, making the inverse-Mellin representation singular as the integration path gets pinched between the two confluent poles.

We now specifically evaluate the two dominant polar contributions, from \( \sigma_0 \) and \( \sigma_1 \).
- the leading pole \( \sigma_0(0) = \frac{\lambda}{2(N+1-2M)} \): if the coupled problem is of type \( N \) this pole remains simple, and its asymptotic contribution \( \text{Res}_{\sigma_0} I(s; \lambda, \sigma) \) turns out (by inspection) to be just \( \int_0^{+\infty}(q^N + vq^M)^{1/2} dq \) (as given by eq.(40) at \( u = 1 \)). Furthermore, \( \partial_\lambda I(s; \lambda, \sigma) \propto I(s+1; \lambda, \sigma) \), an operation which precisely annihilates this leading pole part in all cases, so the latter has to be a constant in \( \lambda \); then, its computation at \( \lambda = 0 \) precisely yields \( \int_0^{+\infty}(q^N + vq^M)^{1/2} dq \), now including the confluent cases (\( A_j \)).
- the subleading pole \( \sigma_1 = -1/M \): if \( M > 2 \) this pole remains simple, and its asymptotic contribution \( \text{Res}_{\sigma_1} I(s; \lambda, \sigma) \) coincides with \( \int_0^{+\infty}(vq^M + \lambda)^{1/2} dq \) as given by eq.(41). Under confluence \( (M = 2) \), the contribution becomes that of the double pole of eq.(41) at \( \sigma = -1/2 \): this is \( \int_0^{+\infty}(vq^2 + \lambda)^{1/2} dq + \frac{N}{N-2}A_1(\lambda, v) \) where the action integral is given by eq.(47), and

\[
A_1(\lambda, v) = \frac{1}{4} v^{-1/2} \lambda (\log v + 2 \log 2).
\]

(51)

All in all, the asymptotic \( v \to +\infty \) formula for the trinomial action integral is then

\[
\int_0^{+\infty}(q^N + vq^M + \lambda)^{1/2} dq \sim \int_0^{+\infty}(q^N + vq^M)^{1/2} dq + \int_0^{+\infty}(vq^M + \lambda)^{1/2} dq + \delta_{M,2} \frac{N}{N-2}A_1(\lambda, v),
\]

(52)

where \( \int_0^{+\infty}(q^N + vq^M)^{1/2} dq \) is specified through eq.(40) if the coupled problem is of type \( N \), or else eq.(41) if the coupled problem is of type \( A_j \) (i.e., if \( \frac{N+2}{2(N-1)} = j \in \mathbb{N}^* \)); whereas \( \int_0^{+\infty}(vq^M + \lambda)^{1/2} dq \) is given by eq.(42) if \( M > 2 \), or eq.(47) if \( M = 2 \) (and \( \delta_{M,2} \) is a Kronecker delta symbol).

5 Asymptotic behaviors of spectral functions

The theoretical results of Secs. 3–4 translate into concrete formulae for spectral functions in the \( v \to +\infty \) regime.

5.1 The spectral determinants

Upon substituting the explicit formulae of Sec. 4 into eq.(42), \( \int_0^{+\infty}(vq^M + \lambda)^{1/2} dq \) cancels out, and a slightly simpler \( v \to +\infty \) formula results:

\[
\text{det}^\pm(-d^2/dq^2 + q^N + vq^M + \lambda) \sim e^{\int_0^{+\infty}(q^N + vq^M)^{1/2} dq + \delta_{M,2} \frac{N}{N-2}A_1(\lambda, v)} \text{det}^\pm(-d^2/dq^2 + vq^M + \lambda),
\]

(53)

where \( \int_0^{+\infty}(q^N + vq^M)^{1/2} dq \) is given through eq.(40) if the coupled problem is of type \( N \), or eq.(41) if it is of type \( A_j \), and \( A_1(\lambda, v) \) by eq.(51).

Being homogeneous, the uncoupled potentials obey a simpler form of the scaling eq.(8): \( (-d^2/dq^2 + vq^M) \) is unitarily equivalent to \( v^{2/(M+2)}(-d^2/dq^2 + q^M) \); then the scaling laws
apply with \( r = v^{2/(M+2)} \); as these laws are more awkward for the \( \det^\pm \) than for the full and skew determinants \( \det \) and \( \det^P \), we now switch to the latter combinations and explicitly get

\[
\begin{align*}
\det(-d^2/dq^2 + vq^M + \lambda) & \equiv \det(-d^2/dq^2 + q^M + v^{-2/(M+2)}\lambda) \quad (M \neq 2) \quad (54) \\
\det(-d^2/dq^2 + vq^2 + \lambda) & \equiv v^{-1/2}\lambda^{4/3} \det(-d^2/dq^2 + q^2 + v^{-1/2}\lambda) \quad (55) \\
\det^P(-d^2/dq^2 + vq^M + \lambda) & \equiv \lambda^{1/(M+2)} \det^P(-d^2/dq^2 + q^M + v^{-2/(M+2)}\lambda). \quad (56)
\end{align*}
\]

Remark: the harmonic-oscillator determinants are actually known in closed form ([7], eqs.(155)); e.g., eq.(55) has the fully explicit form (to be used later)

\[
\det(-d^2/dq^2 + vq^2 + \lambda) \equiv v^{-\nu-1/2}\lambda^{4/3}2^{-\nu-1/2}\lambda^{2}\sqrt{2\pi}/\Gamma(\frac{1}{2}(1+v^{-1/2}\lambda)). \quad (57)
\]

Thus, eqs.(53) plus (54)–(56) supply the \( v \to +\infty \) behaviors at fixed \( \lambda \) of the coupled determinants in terms of the corresponding uncoupled determinants at \( \lambda = 0 \) (which are computable numbers, cf. [7], eq.(136)).

However, our main concern is rather the singular perturbation limit: \( v \equiv g^{-(M+2)/(N+2)} \to +\infty \) with \( v^{-2/(M+2)}\lambda \) \( \overset{\text{def}}{=} (E) \) fixed, according to eq.(5). The explicit final results, deduced from eqs.(53)–(56) after rescaling both sides, are then

\[
\begin{align*}
\det(-d^2/dq^2 + q^M + gq^N - E) / \det(-d^2/dq^2 + q^M - E) & \sim e^{2\int_0^{\infty} (q^N + vq^M)^{1/2}dq} e^{-\delta_{M,2} \frac{v}{2} \log v + N \log 2} E \quad \text{for type } N \quad (58) \\
& \sim v^{-1/2} e^{2\int_0^{\infty} (q^N + vq^M)^{1/2}dq} e^{-\delta_{M,2} \frac{v}{2} \log v + N \log 2} E \quad \text{for type } A_j \quad (59)
\end{align*}
\]

\((\beta_{-1}(0) \text{ (given by eq.(133))})\), and type, both refer to the coupled problem); whereas the skew determinants always behave straightforwardly:

\[
\det^P(-d^2/dq^2 + q^M + gq^N - E) \sim \det^P(-d^2/dq^2 + q^M - E). \quad (59)
\]

The main result here is the explicit non-trivial prefactor in eq.(58). Its essential singularity for \( g \to 0 \) should relate to the non-uniformity of this limit with respect to the quantum number \( k \). By contrast, the dependence of its logarithm upon \( E \) is elementary, consisting only of (a) constant term(s) (already determined in [7] for some cases), then a linear term, and nothing else. The basic example \( (\Omega) \), being of type \( N \), thus gives

\[
\begin{align*}
\det(-d^2/dq^2 + q^2 + gq^4 - E) & \sim e^{-2/3g} e^{(\log g/2 - 2\log 2)E} \det(-d^2/dq^2 + q^2 - E). \quad (60)
\end{align*}
\]

(Note the “instanton-like” structure of the first prefactor, computed by eq.(44).)

### 5.2 The spectral zeta functions

Over the spectrum \( \{E_k(g)\} \) of the rescaled operator \( (-d^2/dq^2 + q^M + gq^N) \), we can consider the full and skew spectral zeta functions

\[
Z_g(s; E) \overset{\text{def}}{=} \sum_{k=0}^{\infty} (E_k(g) - E)^{-s}, \quad Z_g^P(s; E) \overset{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^k (E_k(g) - E)^{-s} \quad (61)
\]
for, say, integer $s \in \mathbb{N}^*$, in which case they converge for $g > 0$ and relate to the spectral determinants in a simpler way than for general $s$,

$$Z_g(s; E) \equiv -\frac{1}{(s-1)!} \frac{\partial^s}{\partial E^s} \log \det(-d^2/dq^2 + q^M + gq^N - E), \quad (62)$$

(obtained from eq.(19) upon rescaling; and likewise for $(Z^P, \det^P)$).

Assuming all previous estimates are stable under $E$-differentiations (as is usually the case in WKB theory), the preceding formulae imply the regular behaviors (see fig. 2, left)

$$Z_g(s; E) \sim Z_0(s; E), \quad Z_g^P(s; E) \sim Z_0^P(s; E) \quad (g \to 0), \quad (63)$$

except for $Z_g(1; E)$ (the resolvent trace) when $M = 2$, which gives the singular case ($Z_0(1; E)$ infinite, while $Z_0^P(1; E)$ stays finite). Those patterns were conjectured in [7] (Sec. 3), but not the precise divergent behavior of $Z_g(1; E)$, which requires to know the $E$-linear term in the exponent of the determinant ratio (58). For $s = 1$, eq.(62) needs to be regularized at $g = 0$, as

$$- (d/dE) \log \det(-d^2/dq^2 + q^2 - E) \equiv -\frac{1}{2} [\psi(\frac{1}{2}(1 - E)) + \log 2] \quad (64)$$

(using the known closed form (57) of the harmonic-oscillator determinant). Then the logarithmic differentiation of eqs.(58) for $M = 2$ yields the $g \to 0$ behavior of $Z_g(1; E)$ for all potentials $q^2 + gq^N$ (irrespective of type), as the following singular expression:

$$Z_g(1; E) \sim \frac{1}{N-2} (-\log g + N \log 2) - \frac{1}{2} [\psi(\frac{1}{2}(1 - E)) + \log 2]. \quad (65)$$
For instance, at $E = 0$ this gives (see fig. 2, right)
\[
\sum_{k=0}^{\infty} E_k(g)^{-1} \sim -\frac{1}{N-2} \log g + \frac{1}{2} \left( \gamma + \frac{3N-2}{N-2} \log 2 \right) \quad (g \to 0)
\]
\[
\sim -\frac{1}{2} \log g + \frac{1}{2} (\gamma + 5 \log 2) \quad \text{for } N = 4,
\]
\[
-\frac{1}{4} \log g + \frac{1}{2} \gamma + 2 \log 2 \quad \text{for } N = 6, \ldots
\]
(to be compared with the sharp cutoff regularization of eq. (6)).

5.3 Concluding remarks

We have completed here one “exercise in exact quantization” begun in [7]: we gave the $g \to 0$ behavior of the spectral determinants $\det^\pm (-d^2/dq^2 + q^M + gq^N - E)$, now for general parameter values. While it may appear wasteful to use a wholly exact approach for perturbative calculations, exact WKB analysis actually proved quite efficient for the task; inversely, such problems help to strengthen the practical sides of that field, which still need further development.

We are also confident that the above approach can be extended further, both to complex parameter asymptotics and towards higher orders in powers of $g$.

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