Abstract

We propose a new interpretable measure of unfairness, that allows providing a quantitative analysis of classifier fairness, beyond a dichotomous fair/unfair distinction. We show how this measure can be calculated when the classifier’s conditional confusion matrices are known. We further propose methods for auditing classifiers for their fairness when the confusion matrices cannot be obtained or even estimated. Our approach lower-bounds the unfairness of a classifier based only on aggregate statistics, which may be provided by the owner of the classifier or collected from freely available data. We use the equalized odds criterion, which we generalize to the multiclass case. We report experiments on data sets representing diverse applications, which demonstrate the effectiveness and the wide range of possible uses of the proposed methodology. An implementation of the procedures proposed in this paper and as the code for running the experiments are provided in https://github.com/sivansabato/unfairness.

Keywords: Equalized odds, group fairness, unfairness measures, classifier statistics, multiclass classifiers

1. Introduction

As machine learning models increasingly affect people’s lives, it has become clear that it is not always sufficient to have accurate classifiers; in many cases, they should also be required to apply fair treatment to different sub-populations. The importance of auditing classifiers for fairness has been noted in many works (Goyal et al., 2022; Roth et al., 2022; Black et al., 2022, e.g.). However, in practice, in many cases it is difficult to audit classifiers for these properties, for instance due to lack of high-quality individual-level validation data, or because the classifier is considered proprietary by the developer in industry or government, and so direct access and use of the classifier is precluded.

In this work, we propose a new interpretable measure of unfairness, that allows providing a quantitative analysis of classifier fairness, beyond a dichotomous fair/unfair distinction. We first show how this measure can be calculated when the classifier’s conditional confusion matrices are known. We then turn to the more challenging task of auditing classifiers for
their fairness when the confusion matrices cannot be obtained or even estimated. We propose methods to lower-bound the unfairness of a classifier based only on aggregate statistics, which may be provided by the owner of the classifier or collected from freely available data. We focus on the fairness criterion of *equalized odds* (Hardt et al., 2016). This criterion was originally defined for binary classifiers. We generalize the criterion to the multiclass setting.

As a motivating example for bounding the unfairness of classifiers using aggregate statistics, suppose that a US-based health insurance company uses some unpublished method to decide whether a person should be classified as “at risk” for diabetes, for instance so as to offer diabetes screening. Two desirable properties of such a classifier are accuracy and fairness. For this example, consider fairness with respect to the state of residence. The only details that are publicly available from the insurance company are aggregate statistics on the number of people identified as “at risk” in each state of residence. In addition, true proportions of diabetes in each state are provided by government agencies. Can this information be used to gain quantifiable insights on the quality of this proprietary classifier, in terms of its fairness and accuracy? Moreover, suppose that the company publishes some additional information about the accuracy of the classifier, or about its fairness with respect to the state of residence. Can this information be used to draw any conclusions about the other property?

A similar challenge arises when a classifier is designed or learned with little or no labeled training data, or when the available training data is not representative of the target distribution. When labeled data is insufficient, it is still possible to construct a classifier using various methodologies, such as unsupervised learning, transfer learning, and hand-crafting rules based on domain expertise. The challenge is to then estimate the quality of the classifier without a validation set. Here too, we are interested in both the accuracy of the classifier and its fairness with respect to some attribute of interest. For instance, suppose that the goal is to predict how individuals will vote in the upcoming elections based on their votes in the previous elections. A straightforward approach, when detailed individual training data is not available, is to have the classifier predict that all individuals will vote for the same party as they did in the previous elections. Can we draw any conclusions on the quality, that is, the accuracy and fairness, of this classifier in individual predictions using only publicly available voting statistics? We show in this work that in many cases such conclusions can be derived.

To derive such conclusions, we exploit unavoidable trade-offs between accuracy and fairness. It is well known that fairness and accuracy in classification do not always co-exist: in some cases, a more accurate classifier must be less fair, and vice versa (Pleiss et al., 2017; Kleinberg et al., 2017; Menon and Williamson, 2018; Goel et al., 2018). We show that this property can be used to draw conclusions on classifier accuracy and fairness, based on partial aggregate information about the classifier.

We note that Menon and Williamson (2018) provide a method for calculating the degradation in accuracy, as measured by the cost-sensitive risk, that results from requiring classifier fairness for a given population distribution. This degradation is defined as the difference in risk between the Bayes-optimal classifier and the Bayes-optimal fair classifier. In contrast, we quantify the fairness and accuracy of a given classifier, which is not necessarily optimal, based on limited aggregate information.
We consider fairness in the sense of equalized odds (Hardt et al., 2016) (also termed disparate mistreatment; see, e.g., Zafar et al., 2017a). This notion, originally defined for binary classification, defines a classifier as fair if its false positive rate and its false negative rate, conditioned on the value of the fairness attribute, are the same for all attribute values. Here, we study both binary and multiclass classification. We generalize the notion of equalized odds to multiclass classification by requiring that a fair classifier have the same confusion matrix conditioned on each of the attribute values. Our main interest is in cases where the protected attribute for which fairness is desired is also multi-valued, as in attributes such as race, level of education, or state of residence.

Clearly, if the confusion matrix conditioned on each of the attribute values is known, then it is easy to check whether a classifier is fair and also to calculate its error. However, as in the scenarios described above, the confusion matrix may not be known. We study the conclusions that can be drawn based on label proportions: this a setting in which the only available information about the classifier is the proportions of predictions of each label in each sub-population. In addition, we assume access to the true label proportions in each sub-population and to the proportion out of the population of each sub-population. For instance, in the voting example, we have access to the fraction of votes for each party in the previous elections and in the predicted elections in each state, and the number of voters in each state.

We first discuss the case where the classifier is known to be fair. We show that in this case, a simple linear program can be applied to the known label proportions to obtain a lower bound on the error of the classifier. We then proceed to consider classifiers that are not necessarily completely fair. While complete fairness of a classifier is a desired property, it cannot always be guaranteed. It is thus of interest to be able to measure how unfair is a classifier: the least unfair, the better. Quantifying the unfairness of a classifier is an important and under-studied problem; Previous works that employed some type of fairness quantification (e.g., Donini et al., 2018; Calmon et al., 2017a; Alghamdi et al., 2020) used it in the context of regularization in a learning algorithm, so as to bias the algorithm towards fairer classifiers. However, these proposals were not justified by an underlying model and the resulting quantification had significant disadvantages. In this work, we propose a principled approach for quantifying unfairness. We model the amount of unfairness as the fraction of the population treated differently from an underlying baseline treatment. We use this definition to derive a well-defined unfairness measure with a clear quantifiable meaning. We show that this measure satisfies properties that are natural to expect from an unfairness measure and are not always satisfied by previous approaches.

First, we show that given the confusion matrices of the classifier in each sub-population, the unfairness measure that we define can be formulated as the solution to a minimization problem over the unobserved baseline confusion matrix. We show that this minimization problem can be solved exactly for binary classification. For multiclass classification, we propose practical algorithms that calculate lower and upper bounds on this measure.

Next, we study drawing conclusions on the unfairness and the error of a classifier given label proportion information, without access to the confusion matrices of the classifier in each sub-population. We define a measure called discrepancy, which balances unfairness and error. The minimal possible discrepancy for given label proportions captures the inherent accuracy-fairness trade off that can be inferred from these proportions. In the case of binary
classification, we provide an algorithm that calculates the exact minimal discrepancy value. For multiclass classification, we provide a local minimization algorithm that upper bounds this quantity. Combined with a lower bound based on the binary classification approach, this allows inferring a value range for the minimal discrepancy.

We report experiments that demonstrate their success in various scenarios. Our experiments further demonstrate diverse applications of these tools for real-world scenarios, in applications such as election prediction, disease diagnosis, and analysis of access to health care. An implementation of the procedures proposed in this paper and the code for running the experiments are provided in https://github.com/sivansabato/unfairness.

**Paper structure.** We discuss related work in Section 2. The setting and notation are defined in Section 3. We extend the equalized odds criterion to multiclass classifiers in Section 4. In Section 5, we discuss lower-bounding the error using label proportions when the classifier is known to be fair. In Section 6, we discuss possible ways of quantifying unfairness and derive our new unfairness measure. In Section 7, we show how to calculate the unfairness of a classifier when the confusion matrices of each sub-population are known. In Section 8, we provide algorithms for lower bounding the classifier discrepancy based on label proportions. Experiments are reported in Section 9. We conclude with a discussion in Section 10.

This work is an extended version of Sabato and Yom-Tov (2020).

2. Related work

Fairness in classification has been a highly studied topic of research in recent years, due to its importance in legal, financial, and medical decisions (Barocas et al., 2017). This importance has grown in parallel with the wide application of automated (and frequently, opaque) models in multiple areas affecting people. Various notions of fairness for binary classification have been proposed (see, e.g., Dwork et al., 2012; Grgic-Hlaca et al., 2016; Kusner et al., 2017; Berk et al., 2018; Verma and Rubin, 2018). In this work, we focus on the notion of equalized odds (Hardt et al., 2016), which requires the false positive rates to be the same in each sub-population, and the same for false negative rates. We are not aware of any previous works that generalize the notion of equalized odds to multiclass classification. A recent work of Denis et al. (2021) studies the generalization of another fairness notion, demographic parity, to multiclass classification.

Methods for learning fair binary classifiers under the equalized-odds definition have been proposed in many works (see, e.g., Feldman et al., 2015; Hardt et al., 2016; Goh et al., 2016; Zafar et al., 2017b,a; Woodworth et al., 2017; Wu et al., 2019). Learning methods that guarantee or approximate other definitions of fairness, such as equal opportunity and demographic fairness, have also been widely studied in recent years (e.g., Dwork et al., 2012; Zemel et al., 2013; Calmon et al., 2017b; Donini et al., 2018; Goel et al., 2018; Johndrow and Lum, 2019).

Estimating the fairness of the classifier is a crucial task in the pipeline of learning fair classifiers (Bellamy et al., 2019). Given access to the classifier and its individual predictions, Black et al. (2019) propose a method for fine-grained scrutiny of a classifier beyond group-fairness notions. McDuff et al. (2019) propose a simulation-based approach for interrogating the classifier. Kusner et al. (2017) define the property of “counterfactual fairness”, which can
be tested given access to the classifier or to individual classified examples. Fairness auditing has been studied in various contexts, including, for instance, assessment of algorithmic fairness of tax auditing (Black et al., 2022), testing for discrimination in candidate rankings (Roth et al., 2022), and identifying bias in visual systems (Goyal et al., 2022).

Learning classifiers based on both individual covariates and population statistics has been studied under the name “ecological inference”. Jackson et al. (2006, 2008) propose methods for regression based on both individual-level and aggregate-level statistics. Sun et al. (2017) infer voting patterns using aggregate statistics. We are not aware of any works that attempt to estimate properties of existing classifiers from aggregate statistics alone. In particular, we are not aware of such works in the context of fairness.

Approaches for quantifying unfairness for individual-fairness notions have been suggested by Speicher et al. (2018) and by Heidari et al. (2018). Speicher et al. (2018) propose a measure of unfairness based on a set of axioms that they suggest the measure should satisfy. Their approach is based on assigning a benefit to each individual as a function of its true label and its predicted label. Heidari et al. (2018) provide a family of measures for individual fairness that satisfy certain moral axioms and can be efficiently constrained. For group fairness, previous works (Donini et al., 2018; Calmon et al., 2017a; Alghamdi et al., 2020) suggest using constraints as regularizers for obtaining a classifier that is fair or almost fair. These constraints can be used to define a measure of (un)fairness. We discuss the resulting possible measures in Section 6.

Lamy et al. (2019) study fairness for binary fairness attributes under a noise model of Scott et al. (2013). In this model, it is assumed that the fairness attribute is noisy independently of the classifier’s prediction, leading to a model in which the confusion matrix conditioned on the attribute value is a mixture of the two true confusion matrices conditioned on the two attribute values. They then show how this model can be combined with previously proposed fairness constraints to yield an adapted constrained optimization formulation for this restricted model.

An earlier short version of this work appeared as Sabato and Yom-Tov (2020). That version studied the case of binary classification, but did not study the case of multiclass classification. As we show below, multiclass classification with more than two labels requires a different approach. The current work also provides a new fast minimization algorithm for the binary case, additional experiments, and significantly expanded and in-depth discussions.

3. Preliminaries

We consider a multiclass classification problem with \( k \) possible labels, in which each individual in the population has a true label in the label set \( \mathcal{Y} \equiv \{0, 1, \ldots, k-1\} \). We assume an attribute of interest (e.g., race, state of residence, age) that assigns a value for each individual. We denote the (finite) set of possible values of this attribute by \( \mathcal{G} \). We do not impose a limit the number of possible attribute values. For concreteness, we henceforth call the possible values of the attribute regions, alluding to location-based attributes such as state of residence or country of origin, although our work is not limited to this type of attributes. A sub-population is a subset of the population that includes all the individuals in the same region.
The object of our study is an existing classifier, which maps each individual from the population to a predicted label. The predicted label may or may not be equal to the true label of that individual. No assumptions are placed on the way this classifier was generated or on the classification mechanism. Denote by $\mathcal{D}$ uniform distribution over the population of the triplets of true label, predicted label, and region. A random triplet drawn according to $\mathcal{D}$ is denoted by $(Y, \hat{Y}, G)$, where $Y \in \mathcal{Y}$ is the true label, $\hat{Y} \in \mathcal{Y}$ is the predicted label, and $G \in \mathcal{G}$ is the region of the individual. Denote the probability of an event $E$ according to $\mathcal{D}$ by $\mathbb{P}[E]$, and the probability of an event $E$ conditioned on the individual being from region $g \in \mathcal{G}$ by $\mathbb{P}_g[E] := \mathbb{P}[E \mid G = g]$. Throughout this work, the value of a conditional-probability expression in which the probability of the condition is zero is treated as zero.

We define the following notation for properties of $\mathcal{D}$:

- The weight (prevalence) of each sub-population in the distribution is $w_g := \mathbb{P}[G = g]$; The vector of weights is $\mathbf{w} = (w_g)_{g \in \mathcal{G}}$.
- The proportion of true label $y \in \mathcal{Y}$ in sub-population $g \in \mathcal{G}$ is $\pi_g^y := \mathbb{P}_g[Y = y]$; The vector of these proportions is $\mathbf{\pi}_g := (\pi_g^y)_{y \in \mathcal{Y}}$.
- The proportion of predicted label $y \in \mathcal{Y}$ in sub-population $g \in \mathcal{G}$ is $\hat{p}_g^y := \mathbb{P}_g[\hat{Y} = y]$; The vector of these proportions is $\hat{\mathbf{p}}_g := (\hat{p}_g^y)_{y \in \mathcal{Y}}$.

| Notation | Meaning |
|----------|---------|
| $\mathbb{P}_g[E]$ | $\mathbb{P}[E \mid G = g]$ |
| $w_g$ | $\mathbb{P}[G = g]$ |
| $\mathbf{w}$ | $(w_g)_{g \in \mathcal{G}}$ |
| $\pi_g^y$ | $\mathbb{P}_g[Y = y]$ |
| $\mathbf{\pi}_g$ | $(\pi_g^y)_{y \in \mathcal{Y}}$ |
| $\hat{p}_g^y$ | $\mathbb{P}_g[\hat{Y} = y]$ |
| $\hat{\mathbf{p}}_g$ | $(\hat{p}_g^y)_{y \in \mathcal{Y}}$ |
| $\alpha_g^y$ | $\mathbb{P}_g[\hat{Y} = z \mid Y = y]$ |
| $\mathbf{\alpha}_g$ | $(\alpha_g^y)_{z \in \mathcal{Y}}$ |
| $A_g$ | $(\alpha_g^y)_{y, z \in \mathcal{Y}}$ |
| $\Delta$ | $\{\mathbf{v} \in \mathbb{R}^k_+ \mid \|\mathbf{v}\|_1 = 1\}$ |
| $\mathcal{M}_\Delta$ | $\{\mathbf{X} := (x_{yz})_{y, z \in \mathcal{Y}} \mid \forall y \in \mathcal{Y}, (x_{yz})_{z \in \mathcal{Y}} \in \Delta\}$ |
| $\mathbf{1}_d$ | all-one vector of dimension $d$ |
| $\mathbf{1}_{d_1 \times d_2}$ | all-one matrix of dimensions $d_1 \times d_2$ |
| $\mathbf{I}_d$ | identity matrix of dimensions $d \times d$ |

Table 1: Notation summary
The entries in the confusion matrix of the classifier on a sub-population \( g \in \mathcal{G} \) are denoted by
\[
\forall y, z \in \mathcal{Y}, \quad \alpha^{yz}_{g} := P_{g}[\hat{Y} = z | Y = y].
\]
Denote the indexed set of all such confusion matrices by \( \mathbf{A}_g := \{ \alpha^{yz}_{g} \}_{y,z \in \mathcal{Y}} \). Denote row \( y \) in \( \mathbf{A}_g \) by \( \alpha^y_g := \{ \alpha^{yz}_{g} \}_{z \in \mathcal{Y}} \).

Denote the simplex over \( \mathcal{Y} \) by \( \Delta := \{ \mathbf{v} \in \mathbb{R}^k_+ \mid \| \mathbf{v} \|_1 = 1 \} \). Denote the set of matrices whose rows are in the simplex over \( \mathcal{Y} \) by \( \mathcal{M}_\Delta := \{ \mathbf{X} := (x^{yz})_{y,z \in \mathcal{Y}} \mid \forall y \in \mathcal{Y}, (x^{yz})_{z \in \mathcal{Y}} \in \Delta \} \).

Denote the all-one vector of dimension \( d \) by \( \mathbf{1}_d \), the all-one matrix of dimensions \( d_1 \times d_2 \) by \( \mathbf{1}_{d_1 \times d_2} \), and the identity matrix of dimensions \( d \times d \) by \( \mathbf{I}_d \). The notation provided above is summarized in Table 1.

The population error of the classifier is given by:
\[
\text{error}(\mathbf{A}_g) := P[\hat{Y} \neq Y] = 1 - \sum_{g \in \mathcal{G}} w_g \pi_g^T \text{diag}(\mathbf{A}_g).
\]

In the label-proportions setting, we assume that the confusion matrices of the classifier are unknown, and the only available information on the labels is the proportions of the true labels and the predicted labels in each region. Denote the available information in the label-proportions setting by
\[
\text{Inputs} := (w, \{ (\pi_g, \hat{p}_g) \}_{g \in \mathcal{G}}).
\]

In the next section, we discuss the equalized odds fairness criterion and generalize it to the multiclass setting.

4. Equalized odds for multiclass classification

The fairness notion of equalized odds (Hardt et al., 2016), originally defined for binary classification, requires that the false positive rates of the classifier in each sub-population are the same, and similarly for the false negative rates. Formally, it was defined for the case \( \mathcal{Y} = \{0, 1\} \) and \( \mathcal{G} = \{0, 1\} \), as follows:
\[
\forall y \in \mathcal{Y}, \quad P[\hat{Y} = 1 \mid Y = y, G = 0] = P[\hat{Y} = 1 \mid Y = y, G = 1].
\]

Note that since there are only two labels in this case, we have
\[
\forall y \in \mathcal{Y}, g \in \mathcal{G}, \quad P[\hat{Y} = 1 \mid Y = y, G = g] = 1 - P[\hat{Y} = 0 \mid Y = y, G = g].
\]

Therefore, under these \( \mathcal{Y} \) and \( \mathcal{G} \), Eq. (3) is equivalent to requiring that for all \( y, z \in \mathcal{Y} \), the value of \( P[\hat{Y} = z \mid Y = y, G = g] \) is the same for all \( g \in \mathcal{G} \). We generalize the equalized odds criterion to multiclass classification, by applying the same requirement to multiclass problems, allowing both \( \mathcal{Y} \) and \( \mathcal{G} \) to include more than two values. Put differently, the multiclass fairness requirement is that that the confusion matrices are the same in each sub-population. Formally, we define the following multiclass equalized odds criterion.
### Definition 1 (Multiclass equalized odds)

We say that a classifier satisfies **multiclass equalized odds** if for each \( y, z \in \mathcal{Y} \), the value of \( \Pr[\hat{Y} = z \mid Y = y, G = g] \) is the same for all \( g \in \mathcal{G} \).

Under our notation (see Table 1), this is equivalent to requiring that for all \( y, z \in \mathcal{Y} \), the value of \( \alpha_{yz}^g \) be the same for all \( g \in \mathcal{G} \).

We note that weaker generalizations of the binary equalized odds could also be considered. The following are two options of interest:

1. One could generalize Eq. (3) by requiring that \( \forall y \in \mathcal{Y}, \Pr[\hat{Y} = 1 \mid Y = y, G = g] \) is the same for all \( g \). However, this would designate the label 1 with a special status, in which it is seen as more important or desirable than other labels. Unlike the case of the equal opportunity criterion, also defined in (Hardt et al., 2016), the equalized odds criterion was not suggested under the assumption that one label is preferred to others. Indeed, the use of \( Y = 1 \) and not \( Y = 0 \) in Eq. (3) can be seen as arbitrary, due to their symmetry in the binary case.

2. The binary definition in Eq. (3) is also consistent with requiring that the prediction accuracy conditioned on each \( y \) is the same across all sub-populations. Formally, the requirement would be that \( \text{diag}(\mathbf{A}_g) \) is the same for all \( g \in \mathcal{G} \). However, this would imply that all wrong predictions are considered to have the same effect on individuals. For instance, if a patient is provided with the wrong medical diagnosis, one could argue that it does matter which wrong diagnosis that patient received, and that a fair classifier should not make different types of mistakes for different sub-populations.

Our requirement in the multiclass equalized odds criterion is thus stronger than both of these requirements, encapsulating the strongest possible notion of group fairness that one

| fairness criterion          | \( g = 0 \)          | \( g = 1 \)          |
|-----------------------------|----------------------|----------------------|
| (binary) equalized odds     | \[
\begin{pmatrix}
0.2 & 0.8 \\
0.7 & 0.3
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.2 & 0.8 \\
0.7 & 0.3
\end{pmatrix}
\] |
| multiclass equalized odds   | \[
\begin{pmatrix}
0.1 & 0.6 & 0.3 \\
0.4 & 0.2 & 0.4 \\
0.05 & 0.25 & 0.7
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.1 & 0.6 & 0.3 \\
0.4 & 0.2 & 0.4 \\
0.05 & 0.25 & 0.7
\end{pmatrix}
\] |
| multiclass alternative 1    | \[
\begin{pmatrix}
0.1 & 0.6 & 0.3 \\
0.4 & 0.2 & 0.4 \\
0.05 & 0.25 & 0.7
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.1 & 0.4 & 0.5 \\
0.4 & 0.5 & 0.1 \\
0.05 & 0.3 & 0.65
\end{pmatrix}
\] |
| multiclass alternative 2    | \[
\begin{pmatrix}
0.1 & 0.6 & 0.3 \\
0.4 & 0.2 & 0.4 \\
0.05 & 0.25 & 0.7
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.1 & 0.1 & 0.9 \\
0.6 & 0.2 & 0.2 \\
0.3 & 0 & 0.7
\end{pmatrix}
\] |

Figure 1: Illustrating the multiclass equalized odds criterion, in comparison with the binary equalized odds criterion and the two alternative multiclass criteria discussed in Section 4. Bold numbers indicate the entries in the matrix that are required to be the same across both sub-population in a fair classifier under the listed criterion.
could consider for multiclass classification, while adhering to the paradigm that fairness is fully determined by the confusion matrices of each sub-population.

A demonstration of the difference between the possible definitions is provided in Figure 1. The figure illustrates a case with two sub-populations, $\mathcal{G} = \{0, 1\}$. In each line, a different criterion is listed. An example of confusion matrices for the two sub-population is provided, which is considered fair according to the listed criterion. The top row lists a binary classifier under the binary equalized odds criterion. Under this criterion, both confusion matrices must be identical for the classifier to be fair. The other rows illustrate the three multiclass criteria discussed above, all of which generalize the binary equalized odds criterion.

5. Lower-bounding the error of a fair classifier using label proportions

We first study the case where the studied classifier is known to be fair with respect to the attribute $G$, in the sense of multiclass equalized odds defined above. This could be the case, for instance, if it is known that a fairness-preserving procedure (e.g., Hardt et al., 2016; Woodworth et al., 2017) was used to generate the classifier. More generally, the fact that the classifier is fair may be disclosed to the public even if the classifier itself or its predictions are not accessible. In the label-proportions setting, the confusion matrices of the classifier are not available. In this case, the provided label proportions information can be used to draw conclusions on the error of the classifier.

As a warm-up, we describe a simple example, which shows how the information that the classifier is fair along with label-proportions information can be used to draw conclusions on the error of the classifier. Consider a simple binary classification problem with two regions (see Figure 2). In this problem, $\mathcal{Y} = \{0, 1\}$ and $\mathcal{G} = \{L, R\}$ (for “Left” and “Right”). The weights of both regions is the same: $w_L = w_R = 50\%$. In the left region, 30\% of the labels are positive ($\pi^1_L = 0.3$) and the classifier predicts a positive label for half of the individuals ($\hat{p}_L^1 = 0.5$). In the right region, 70\% of the labels are positive ($\pi^1_R = 0.7$), and this is also the fraction of labels predicted as positive by the classifier ($\hat{p}_R^1 = 0.7$).

If no additional information is provided, and we assume that the classifier has the minimal possible error given these label proportions, then this is consistent with a classification error of $(50\% - 30\%)/2 = 10\%$ (see case (I) in Figure 2). However, this can only be the case if the classifier is unfair. In particular, this error is obtained only if $\frac{2}{7} = \alpha^0_L \neq \alpha^0_R = 0$. In contrast, if it is known that the classifier is fair, then $\alpha^y_z^L = \alpha^y_z^R$ for all $y, z \in \{0, 1\}$. Under this constraint, only one configuration of the confusion matrices in each region is possible (see case (II) in Figure 2), with $\alpha^0_L = \alpha^0_R = 0.35$ and $\alpha^1_L = \alpha^1_R = 0.15$, giving a classification error of $\text{error} = 25\%$. Thus, in this example, by using the information that the classifier is fair, we are able to infer its error exactly.

More generally, the information that the classifier is fair can be used to infer a lower bound on the error of the classifier, since fairness imposes constraints on the confusion matrices. Combined with constraints derived from the given label proportions, a linear optimization problem can be derived.

First, given Inputs, the classifier must satisfy the following constraints on $A_g$:

$$\forall g \in \mathcal{G}, \ A_g \in \mathcal{M}_\Delta, \ \text{and} \ A_g^T \pi_g = \hat{p}_g.$$

(4)
Given label proportions:

|       | weight | true positive ($\pi^1_g$) | predicted positive ($\hat{p}^1_g$) |
|-------|--------|--------------------------|-----------------------------------|
| $g = \text{L(eft)}$ | 50%    | 30%                      | 50%                               |
| $g = \text{R(ight)}$ | 50%    | 70%                      | 70%                               |

(I) Assume that the classifier has the minimal possible error. Unique solution:

$$A_L = \begin{pmatrix} \frac{5}{7} & \frac{2}{7} \\ 0 & 1 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This solution is consistent with a classifier that has error $= 10\%$, but is unfair.

(II) Assume that the classifier is fair. Unique solution:

$$A_L = A_R = \begin{pmatrix} 0.65 & 0.35 \\ 0.15 & 0.85 \end{pmatrix}$$

This solution is consistent with a classifier that is fair, but has error $= 25\%$.

Figure 2: Two possible confusion matrices for a simple 2-region problem, with accompanying illustrations. Grid lines are displayed for scale. Given the label proportions provided in the table above, each of the two displayed assumptions leads to a different solution. The first assumes that the error is minimal, while the other assumes that the fairness is maximal (that is, that the classifier is completely fair). The details of the derivations are provided in Section 5.

In addition, if the classifier is known to be fair in the sense of multiclass equalized odds, then all matrices in $A_G$ are the same. The error of the classifier is determined by the diagonal entries of $(A_g)_{g \in G}$ (see Eq. (1)). Thus, a tight lower bound on the error of the classifier can be obtained by solving the following optimization problem, over a matrix $A$.
that represents the confusion matrix of each of the sub-populations.

\[
\text{Minimize}_A \quad \text{error}(A) \equiv 1 - \sum_{g \in G} w_g \pi^T g \text{diag}(A)
\]

\[\text{s.t.} \quad A \in \mathcal{M}_\Delta, \forall g \in G, y \in \mathcal{Y}, A^T \pi g = \hat{p}_g.\]

If this problem is infeasible, then the classifier cannot be fair.

We note that in practice, the values of the problem parameters are usually estimated from data, and are only accurate up to some margin of error. These margins of error can be incorporated into the optimization problem, for instance by converting the last constraint above into a set of two-sided inequalities.

In the next section, we propose a new measure of unfairness for unfair classifiers.

6. Quantifying unfairness

In the previous section, we considered a fair classifier, which means that the classifier satisfies the equalized odds requirements exactly. However, in many cases, exact fairness may not be guaranteed. For instance, the classifier may have been constructed using heuristic methods that attempt to obtain fairness (see, e.g., Goh et al., 2016; Zafar et al., 2017b). Nonetheless, being close to fairness as much as possible is a desired property. To draw conclusions about unfair classifiers, a meaningful measure of the unfairness of a classifier is required. In this section, we propose such an unfairness measure with respect to the multiclass equalized-odds fairness criterion. The guiding principle behind the derivation of this measure is that it is derived based on its intended meaning, which is rooted in a probabilistic interpretation. We provide this derivation in Section 6.1 below. In Section 6.2, we compare the proposed unfairness measure to possible measures that naturally arise from previous work, highlighting the advantages of our principled approach.

6.1 A model-based unfairness measure

We propose a measure of unfairness which is based on a clear probabilistic meaning, by defining the amount of unfairness as the fraction of the population on which the classifier behaves differently from its baseline behavior. This is analogous to the definition of error, which indicates the fraction of the population on which the classifier makes a mistake.

Formally, let \( y \in \mathcal{Y} \), and recall that \((Y, \hat{Y}, G) \sim D\). Denote by \( D^y_g \) the conditional distribution of \( \hat{Y} \mid Y = y, G = g \). For \( g \in G \), we model \( D^y_g \) as a mixture of a baseline distribution \( D^y_b \) and local “nuisance” distributions \( N^y_g \). Denote by \( \eta^y_g \) the probability, conditioned on \( Y = y, G = g \), that \( \hat{Y} \) is drawn from \( N^y_g \). Then we model \( D^y_g \) as the following mixture:

\[
D^y_g = (1 - \eta^y_g)D^y_b + \eta^y_g N^y_g. \tag{5}
\]

It is easy to see that if the classifier is fair, then \( D^y_g = D^y_b \) for all \( g \in G, y \in \mathcal{Y} \), hence in this case Eq. (5) holds for for all \( g \in G, y \in \mathcal{Y} \) for \( \eta^y_g = 0 \). However, if the classifier is not completely fair, any choice of \( D^y_b \) requires \( \eta^y_g > 0 \) for at least one \( g \in G \). Thus, an \( \eta^y_g \)-fraction of the individuals with \( Y = y, G = g \) suffer a different prediction behavior than the baseline prediction behavior defined by \( D^y_b \).
We define unfairness as the fraction of the population for which prediction behaves differently from the baseline prediction behavior. Since the decomposition to \( D_y \) and \( N_y \) is unobserved, we define this fraction with respect to the best possible decomposition. Formally, let \( \eta_g := (\eta_g^y)_{y \in Y} \). We define the unfairness measure for a classifier described by \( D \) as follows:

\[
\text{unfairness}(D) := \min_{\{\eta_g\}} \sum_{g \in G} w_g \pi_g^y \eta_g,
\]

where the minimum is over \( \{\eta_g\} \) such that Eq. (5) holds for some \( \{D_y^b, N_y^g\}_{g \in G} \) \( y \in Y \).

This definition of unfairness has several advantages: First and foremost, it is fully interpretable, since it has a specific and well defined meaning. The value of unfairness is always in \([0, 1]\), where unfairness \( (D) = 0 \) if and only if the classifier is completely fair. The value of unfairness is meaningful even if the label distribution is extremely unbalanced: Suppose that the target label concerns the diagnosis of a rare disease. This label is highly unlikely, yet if the disease is twice as likely to be wrongly diagnosed in one sub-population than in another, this would manifest as a high unfairness value. Moreover, since unfairness and error are both defined as fractions of the population, they can be compared and combined meaningfully, without the need for rescaling or other manipulations.

The above unfairness measure can be used to measure unfairness also with respect to laxer fairness definitions that do not require the entire confusion matrix to be identical across regions, as follows. Suppose that we consider the classifier to be fair if the rate of classification mistakes conditioned on the label is the same across all regions, regardless of the nature of those mistakes. In other words, all classifier mistakes are considered equivalent with respect to fairness. Alternatively, consider a more refined scenario, in which some types of mistakes are considered equivalent. For instance, in the application of vote prediction, one could consider predicting one of two similar parties as equivalent in terms of prediction fairness. This can be formulated within the proposed framework by defining an indexed set of label mappings conditioned on the true label \( y \), \( f := \{f_y\}_{y \in Y} \) where \( f_y : Y \rightarrow Y \), which map some predicted labels to other predicted labels. For instance, to model a case where all classifier mistakes are considered equivalent, one can set \( f_y(\hat{y}) := I[\hat{y} \neq y] \). For a distribution \( D \) with \( (Y, \hat{Y}, G) \sim D \), let \( D[f] \) be the distribution of the triplet \((Y, f_Y(\hat{Y}), G)\). Then, unfairness \( (D[f]) \) provides a measure of unfairness that does not penalize for making different mistakes in different regions.

This gives rise to a useful monotonicity property of unfairness: One would expect that defining some labels as equivalent for the sake of fairness evaluation would never increase the value of the unfairness measure. This is indeed the case for unfairness: For any \( f \) and any \( D \), unfairness \( (D[f]) \leq \text{unfairness}(D) \). This can be observed by noting that if Eq. (5) holds for \( D_y^b \) with \( \eta_y^b, D_y^b \) and \( N_y^g \), then Eq. (5) holds also for \( D_y^b[f] \) with \( \eta_y^b, D_y^b[f] \) and \( N_y^g[f] \). Thus, minimizing over \( \eta_y^b \) for \( D[f] \) can never result in a solution of a higher value. This natural monotonicity property further exemplifies the advantages of our approach to defining unfairness. In the next section, we discuss other possible unfairness measures and compare them to our approach.
6.2 Possible unfairness measures based on previous work

In this section, we compare the proposed unfairness measure to possible measures based on previous works. We are not aware of works that explicitly suggest a measure of unfairness for existing classifiers. However, several works propose ways to obtain a near-fair (binary) classifier based on constraints that relax the strict fairness requirements of equalized odds. One might suggest that these could be used to derive an unfairness measure. Below, we discuss these possibilities, and explain why they are disadvantageous compared to the proposed unfairness measure.

As discussed above, the proposed unfairness measure has several desired properties. These properties can all be attributed to the fact that unfairness is directly interpretable as a fraction of the population. This includes the fact that unfairness is always in $[0, 1]$, that it is useful for all distributions, including both balanced and unbalanced distributions, and that its value is directly comparable to the value of error, as they both use the same scale. We show below that possible measures based on previous works do not share these properties.

Relaxed fairness constraints typically require the classifier to have the same conditional distribution for different sub-populations, up to some small $\epsilon > 0$. In the context of this discussion, the most important difference between the different previous approaches is the measure of dissimilarity between conditional distributions. Some previous approaches constrain the absolute differences between conditional probabilities, while others constrain the ratios between conditional probabilities.

An absolute-difference approach was used in Donini et al. (2018), where they studied learning classifiers under fairness constraints for binary-valued fairness attributes. The proposed constraints, when adapted to equalized odds, would require that the absolute difference between the false positive rates in the two sub-populations is bounded by some small $\epsilon > 0$, and similarly for the false negative rates.

A ratio approach was used in Calmon et al. (2017a). In this work, constraints were proposed that require the conditional distributions of the labels given each of the attribute values to be similar, either by requiring the similarity to hold between each pair of attribute values, or by requiring the distribution for each attribute value to be similar to the same target distribution; the target distribution is to be selected based on societal considerations, or is otherwise set to the existing unconditional probability distribution of the labels predicted by the classifier. To measure the similarity between two conditional distributions, a ratio test was proposed, so that the distributions are constrained to a ratio close to 1 between the conditional probabilities. Ratio tests were considered also in Alghamdi et al. (2020).

Both of these approaches could be adapted to suggest a measure of unfairness, by replacing the hard constraints on the absolute differences or on the ratios with a sum of the values of each of the constrained quantities, and using this sum of as a measure of unfairness. For non-binary fairness attributes and possibly multiclass labels, this may require summing over all pairwise constraints, or comparing to an unconditional distribution, as proposed in Calmon et al. (2017a). However, applying this adaptation to either type of constraint would lead to undesirable results. If absolute-difference constraints were used, then the value of the unfairness measure in unbalanced cases, in which some labels are very rare,
would be meaningless, as it would be close to zero for all reasonable classifiers. As we show in our experiments, rare labels can be encountered, for instance, when dealing with a classifier for rare diseases. If ratio constraints were used, then the unfairness measure would become unbounded, leading to meaninglessly high values when there are significant differences between some of the conditional distributions. This issue has also been noted by Calmon et al. (2017a), who warn against using the ratio test when one of the rates is very small. As mentioned above, these issues can be attributed to the fact that neither of these approaches assigns a clear model-based meaning to the resulting measure. Our unfairness measure overcomes these issues because it is based on assigning a clear meaning to the measure.

7. Calculating unfairness when conditional confusion matrices are known

We first consider a scenario in which we have access to the confusion matrices of the classifier in each sub-population. For instance, this could be the case if we have a validation set, or if the confusion matrices are reported by the owner of the classifier.

We derive a useful formula for unfairness for a known distribution \( D \). Note that \( D \) is fully determined by \( \{w_g\}_{g \in G} \), the region weights, and \( A_G \), the confusion matrices of each region. Below, we assume some fixed \( w \) and denote unfairness \((A_G) := \text{unfairness}(D)\).

Consider \( \{\eta_g\} \) values that minimize the RHS in the definition of unfairness in Eq. (6). Suppose first that the distributions \( \{D_y^g\} \) that obtain these values are known, and consider nuisance distributions \( \{N_y^g\} \) which are consistent with this choice. Let the random variable \( N \) be equal to 1 if the triplet \((\hat{Y}, Y = y, G = g)\) was drawn from \( N_y^g \) and equal to 0 if it was drawn from \( D_y^g \), so that \( \eta_y^g = \mathbb{P}_g[N = 1 \mid Y = y] \). Denote the confusion matrix of \( \{D_y^g\} \) by \( A^y_b := (\alpha_{yg}^{yz})_{y,z \in Y} \), where

\[
\forall y \in \mathcal{Y}, \quad \alpha_{yg}^{yz} := \mathbb{P}_g[\hat{Y} = z \mid Y = y],
\]

Denote \( \alpha_y^y = (\alpha_{yg}^{yz})_{z \in \mathcal{Y}} \in \Delta \), where \( \Delta \) is the simplex over \( \mathcal{Y} \) (see Section 3). Denote \( \beta_y^y = \mathbb{P}_g[\hat{Y} = z \mid Y = y, N = 1] \) and note that the matrix \( B_g := (\beta_{yg}^{yz}) \) is in \( \mathcal{M}_\Delta \), where \( \mathcal{M}_\Delta \) is defined as in Section 3. The following holds for all \( g \in G \) and \( y, z \in \mathcal{Y} \):

\[
\alpha_{yg}^{yz} = \mathbb{P}_g[\hat{Y} = z \mid Y = y] = \mathbb{P}_g[\hat{Y} = z \mid Y = y, N = 0] \cdot \mathbb{P}_g[N = 0 \mid Y = y] + \mathbb{P}_g[\hat{Y} = z \mid Y = y, N = 1] \cdot \mathbb{P}_g[N = 1 \mid Y = y]
\]

\[
= \alpha_{yg}^{yz}(1 - \eta_y^g) + \beta_{yg}^{yz} \eta_y^g.
\]

We get \( \eta_y^y = \alpha_{yg}^{yz}(1 - \eta_y^g) + \beta_{yg}^{yz} \eta_y^g \). Thus,

\[
\forall y, z \in \mathcal{Y}, \quad \eta_{yz}^y = \frac{\alpha_{yg}^{yz} - \alpha_{yg}^{yz}}{\beta_{yg}^{yz} - \alpha_{yg}^{yz}} \text{ if } \beta_{yg}^{yz} \neq \alpha_{yg}^{yz} \text{, and } \eta_y^y = 0 \text{ otherwise.} \quad (7)
\]

It follows that given \( A_G \), we have

\[
\text{unfairness}(A_G) = \min_{\{\eta_g\}} \sum_{g \in G} w_g \pi_g^T \eta_g, \quad (8)
\]
where the minimization is over \( \eta_g \in [0,1]|\mathcal{Y}| \) such that Eq. (7) holds for some \({\mathbf{B}_g \in \mathcal{M}_\Delta}\). To calculate \( \text{unfairness}(\mathbf{A}_g) \), we first consider the optimal solution to the minimization in Eq. (8) given \( \mathbf{A}_b \). For a given \( y \in \mathcal{Y} \), unless \( \alpha_{yz}^y = \alpha_b^y \) for all \( z \in \mathcal{Y} \), the minimizer of \( \eta_g \) is obtained with an assignment of \( \beta_{gz}^y = 1 \) for some \( g,z,y \) such that \( \alpha_{g}^y > \alpha_{b}^y \), or \( \beta_{gz}^y = 0 \) for some \( g,z,y \) such that \( \alpha_{g}^y < \alpha_{b}^y \). It follows that there is some \( z \in \mathcal{Y} \) such that

\[
\alpha_{g}^y > \alpha_{b}^y \quad \text{and} \quad \eta_g = \frac{\alpha_g^y - \alpha_b^y}{1 - \alpha_b^y} = 1 - \frac{1 - \alpha_g^y}{1 - \alpha_b^y}.
\]

or

\[
\alpha_{g}^y < \alpha_{b}^y \quad \text{and} \quad \eta_g = \frac{\alpha_g^y - \alpha_b^y}{-\alpha_b^y} = 1 - \frac{\alpha_g^y}{\alpha_b^y}.
\]

Moreover, if \( \alpha_{g}^y > \alpha_{b}^y \) then \( \eta_g \geq 1 - \frac{1 - \alpha_{g}^y}{1 - \alpha_b^y} \) and if \( \alpha_{g}^y < \alpha_{b}^y \) then \( \eta_g \geq 1 - \frac{\alpha_{g}^y}{\alpha_b^y} \). It follows that \( \eta_y = \max_{z \in \mathcal{Y}} \max(1 - \frac{1 - \alpha_{g}^y}{1 - \alpha_b^y}, 1 - \frac{\alpha_{g}^y}{\alpha_b^y}) \). Define the function \( \eta : [0,1]^2 \to [0,1] \) as follows:

\[
\eta(a,b) = \begin{cases} 
1 - b/a & b < a \\
1 - (1 - b)/(1 - a) & b > a \\
0 & b = a.
\end{cases} \tag{9}
\]

The function \( \eta(a,b) \) for fixed \( a \) and for fixed \( b \) is illustrated in Figure 3. We get

\[
\text{unfairness}(\mathbf{A}_g) = \sum_{y \in \mathcal{Y}} \min_{\mathbf{b}_g \in \Delta} \sum_{g \in \mathcal{G}} w_g \pi_g^y \max_{z \in \mathcal{Y}} \eta(\alpha_b^y, \alpha_g^y). \tag{10}
\]

For \( y \in \mathcal{Y} \), define:

\[
\text{unfairness}_y(\alpha_g^y) := \min_{\alpha^y_g \in \Delta} \sum_{g \in \mathcal{G}} w_g \pi_g^y \max_{z \in \mathcal{Y}} \eta(\alpha_b^y, \alpha_g^y), \tag{11}
\]

where \( \alpha^y_g := \{ \alpha_g \}_{g \in \mathcal{G}} \). Then \( \text{unfairness}(\mathbf{A}_g) = \sum_{y \in \mathcal{Y}} \text{unfairness}_y(\alpha_g^y) \). Calculating this value requires solving the minimization problems in Eq. (11). In Section 7.1 we show that in the case of binary classification, this problem can be solved exactly. In Section 7.2 we consider the multiclass case, and derive procedures for obtaining upper and lower bounds on the \text{unfairness} value.
7.1 Binary classification

In the case of binary classification, where $Y = \{0, 1\}$, we have $\alpha_g^{y(1-y)} = 1 - \alpha_g^{yy}$ and the same for $\alpha_b$. Therefore,

$$\eta(\alpha_b^{y(1-y)}, \alpha_g^{y(1-y)}) = \eta(1 - \alpha_b^{yy}, 1 - \alpha_g^{yy}) = \eta(\alpha_b^{yy}, \alpha_g^{yy}).$$

We thus have

$$\text{unfairness}_y(\alpha_g^{y}) = \min_{\alpha_g^{yy} \in [0, 1]} \sum_{g \in G} w_g \pi_g^{y} \eta(\alpha_b^{yy}, \alpha_g^{yy}).$$

Define the function $\mu_{yz}(x) := \sum_{g \in G} w_g \pi_g^{y} \eta(x, \alpha_g^{yz})$. Then

$$\text{unfairness}_y(\alpha_g^{y}) = \min_{x \in [0, 1]} \mu^{yy}(x).$$

We now show that to minimize $\mu^{yy}$, it suffices to check the value of $\mu^{yy}(x)$ over a finite number of values to exactly minimize it.

First, suppose that for all $g \in G$, we have $\alpha_g^{yy} \in (0, 1)$. From the definition of $\eta$, it is easy to see that for any fixed $b \in (0, 1)$, $a \mapsto \eta(a, b)$ is concave on the interval $[0, b]$ and on the interval $[b, 1]$ (see illustration in Figure 3, Right). It follows that $\mu^{yy}(x)$ is concave on any closed interval $I \subseteq [0, 1]$ which is a subset of $[0, \alpha_g^{yy}]$ or of $[\alpha_g^{yy}, 1]$ for each $g \in G$. In each such interval, the minimizer of $\mu^{yy}(x)$ is one of the end points of the interval. Therefore, to minimize $\mu^{yy}(x)$ it suffices to consider the end points of maximal intervals which satisfy this property. In other words, $\min_{x \in [0, 1]} \mu^{yy}(x) = \min_{x \in \{0, 1\} \cup \{\alpha_g^{yy} \} \cup G} \mu^{yy}(x)$. Thus, only a small finite number of values for $a$ need to be checked in order to calculate this value exactly.

Now, suppose that for some $g \in G$, we have $\alpha_g^{yy} \in \{0, 1\}$. For a fixed $b \in \{0, 1\}$, we have $\eta(a, b) = 1_{a \neq b}$, thus it is concave in $a$ (in fact, constant) on $[0, 1] \setminus \{b\}$. Moreover, at the point of discontinuity $a = b$, $\eta(b, b) = 0 = \min_{a \in [0, 1]} \eta(a, b)$. Therefore, in this case as well, for any interval as defined above, even when one of its end points is the point of discontinuity $b$, $\mu^{yy}(x)$ is minimized on one of its end points.

Thus, we have the following equality, which shows that in the case of binary classification, unfairness can be calculated exactly, in time linear in $|G|$.

$$\text{unfairness}_y(\alpha_g^{y}) = \min_{x \in \{0, 1\} \cup \{\alpha_g^{yy} \} \cup G} \mu^{yy}(x).$$

(12)

7.2 Multiclass classification

In the general multiclass case, we are unaware of an efficient procedure for calculating the exact value of the minimization in Eq. (10). We propose instead methods for calculating a lower bound and an upper bound on the unfairness values of a given classifier.
A lower bound on $\text{unfairness}_y$ can be derived as follows:

$$\text{unfairness}_y(\alpha^y_b) = \min_{\alpha^y_b \in \Delta} \sum_{g \in G} w_g \pi^y_g \max_{z \in \mathcal{Y}} \eta(\alpha^y_b, \alpha^y_g)$$

$$\geq \min_{\alpha^y_b \in [0,1]^k} \sum_{g \in G} w_g \pi^y_g \max_{z \in \mathcal{Y}} \eta(\alpha^y_b, \alpha^y_g)$$

$$\geq \min_{\alpha^y_b \in [0,1]^k} \max_{z \in \mathcal{Y}} \sum_{g \in G} w_g \pi^y_g \eta(\alpha^y_b, \alpha^y_g)$$

$$= \max_{z \in \mathcal{Y}} \min_{x \in [0,1]} \sum_{g \in G} w_g \pi^y_g \eta(x, \alpha^y_g)$$

$$= \max_{z \in \mathcal{Y}} \min_{x \in [0,1]} \mu^y(z)$$

$$= \min_{x \in \{0,1\} \cup (\alpha^y_x)_{x \in \mathcal{Y}}} \mu^y(x). \quad (14)$$

The last equality follows from the analogous analysis to the binary case given in Section 7.1.

To derive an upper bound on $\text{unfairness}_y$, we propose below a local-minimization procedure for the minimization problem defined in Eq. (11). In Section 7.2.1, we provide a procedure for finding a favorable assignment for $\alpha^y_b$ using a greedy procedure. In Section 7.2.2, we provide a local optimization procedure for $\text{unfairness}_y$. This procedure can be initialized using the greedy procedure to obtain an even smaller upper bound.

### 7.2.1 A Greedy Initialization Procedure

An naive guess for an assignment for the baseline confusion matrix could be the weighted average of the confusion matrices of each region. However, this assignment does not necessarily even locally minimize the objective function. We provide a different approach for calculating an initial assignment; Our experiments in Section 9 demonstrate that it indeed obtains a significantly smaller objective value than the naive weighting approach.

The procedure is based on a greedy approach, whereby the entries in the baseline confusion matrix are optimized label-by-label, using the fact that the binary problem is easy to minimize. In the first iteration, all labels except for one are treated as the same label, and optimal confusion matrix values for the first labels are calculated. Thereafter, in each iteration one additional label is separated, and the assignment for this label is optimized given the assignments of the previous labels.

Formally, for $i \in [k - 1]$, let $f_i := (f_{i,y})_{y \in \mathcal{Y}}$ be an indexed set of label mappings, $f_{i,y} : \mathcal{Y} \rightarrow \mathcal{Y}$, defined as follows. Let $y_i$ be the $i$’th label in $\mathcal{Y}$ that is different from $y$. Denote $\mathcal{Y}_i = \{y, y_1, \ldots, y_i\}$. Note that $\mathcal{Y}_{k-1} = \mathcal{Y}$. For $i \in [k - 2]$, define

$$f_{i,y}(j) = \begin{cases} j & j \in \mathcal{Y}_i \\ y_i & \text{otherwise.} \end{cases}$$

Note that $f_{i,y}$ can be calculated from the image of $f_{i-1,y}$. Hence, $\mathcal{D}[f_i]$ is a refinement of $\mathcal{D}[f_{i-1}]$. In addition, $f_{k-1,y}$ is the identity. Thus, as observed in Section 6.1, the following monotonicity property holds:

$$\text{unfairness}(\mathcal{D}[f_1]) \leq \text{unfairness}(\mathcal{D}[f_2]) \leq \ldots \leq \text{unfairness}(\mathcal{D}[f_{k-1}]) = \text{unfairness}(\mathcal{D}).$$
Moreover, unfairness($f_1$) can be calculated exactly as in case of binary classification, since the range of $f_1$ includes only $y$ and $y_1$. Based on these observations, we define a greedy procedure for calculating an assignment to $\alpha^y_b$ for each $y \in \mathcal{Y}$.

Let $\alpha^y_G[i]$ be row $y$ of the confusion matrices of $\mathcal{D}[f_i]$. Then coordinates $j \in \mathcal{Y}_{i-1}$ of $\alpha^y_G[i]$ are the same as those of $\alpha^y_y$, and coordinate $y_i$ has the value $\alpha^y_{y_i} := \sum_{j=1}^{k-1} \alpha^y_{y_j}$. We have

$$\text{unfairness}_y(\alpha^y_G[1]) \leq \text{unfairness}_y(\alpha^y_G[2]) \leq \ldots \leq \text{unfairness}_y(\alpha^y_G[k-1])$$

$$= \text{unfairness}_y(\alpha^y_G).$$

The greedy procedure first calculates an assignment for $\alpha^y_G[1]$ that obtains the value of unfairness$_y(\alpha^y_G[1])$. This is a binary problem which can be solved exactly using Eq. (12). Then, at each iteration $i+1$ for $i \in [k-2]$, a local minimum $\alpha^y_G[i+1]$ for unfairness$_y(\alpha^y_G[i+1])$ is calculated by constraining $\alpha^y_G[i+1]$ to have the same coordinates as $\alpha^y_G[i]$ on $\mathcal{Y}_i$, and minimizing over $\alpha_{y_{i+1}}^y$, $\alpha_{y_{i+1}}$ such that their sum is equal to coordinate $y_i$ in $\alpha^y_G[i]$. This minimization can be solved accurately, as follows.

Denote the value of coordinate $y_i$ in $\alpha^y_G[i]$ by $\gamma = 1 - \sum_{j \in \mathcal{Y}_{i-1}} \alpha^y_{y_j}$. Minimizing the objective of unfairness$_y(\alpha^y_G[i+1])$ subject to the constraints resulting from $\alpha^y_G[i]$ is equivalent solving the following optimization problem:

Minimize $\alpha^y_{y_{i+1}}, \alpha^y_{y_{i+1}}$

$$\sum_{g \in \mathcal{G}} w_g \pi^y_g \max \{\max_{z \in \mathcal{Y}_i} \eta(\alpha^y_b, \alpha^y_g, \alpha^y_{z_{i+1}}, \alpha^y_{\gamma_{i+1}})\}

\text{s.t.} \quad \alpha^y_{y_{i+1}} \geq 0 \text{ and } \alpha^y_{y_{i+1}} \alpha^y_{\gamma_{i+1}} = \gamma.$$

Letting $v_g := \max_{z \in \mathcal{Y}_{i-1}} \eta(\alpha^y_b, \alpha^y_g, \alpha^y_z)$, this is equivalent to

Minimize $\sum_{g \in \mathcal{G}} w_g \pi^y_g \max \{v_g, \eta(\alpha^y_b, \alpha^y_g, \alpha^y_{y_{i+1}})\}$.

Similarly to the case of binary classification, this objective is one-dimensional, and concave in each of the intervals defined by the inflection points of the $\eta$ instances and the values for which any two of the expressions in the maximum are equal. Denote this set of points by $M^y$. Then the objective above is minimized by one of the values in the following set: $M^y \cup \{\alpha^y_{y_{i+1}}\}_{g \in \mathcal{G}} \cup \{\alpha^y_{\gamma_{i+1}}\}_{g \in \mathcal{G}} \cup \{0, \gamma\}$. Repeating this procedure until iteration $i = k - 1$, we obtain an assignment for $\alpha^y_G$ which can be used to calculate an upper bound for unfairness$_y(\alpha^y_G)$.

Since the ordering of the labels in the greedy procedure is arbitrary, it is possible to attempt several different orderings and select the one that obtains the smallest unfairness value. In our experiments, we tried 10 random orderings in each upper bound calculation.

Next, we propose a local optimization procedure which can be used to perform local minimization, starting from this greedy solution as an initial point.

7.2.2 The local optimization procedure

The upper bound that is derived from the assignment calculated above can be further improved by performing a local optimization on the objective function in Eq. (11). Given $y \in \mathcal{Y}$, we define the following notation:
The matrix \( H \in \mathbb{R}^{|G| \times k} \) includes the entries \( h_{gz} := \alpha_{yz}^g \). Note that the rows of \( H \) are in the simplex \( \Delta \).

The vector \( \vec{w} \in [0, 1]^{|G|} \) includes the entries \( w_g := w_g \pi_y \).

The optimization variables are denoted by the vector \( \alpha := \alpha_b \in \mathbb{R}^k \).

Our objective Eq. (11), including all its constraints, is thus given by:

\[
\begin{align*}
\text{Minimize} & \quad \alpha \in \mathbb{R}^k \quad \sum_g \vec{w}_g \max_z \{ \eta(\alpha_z, h_{gz}) \} \\
\text{s.t.} & \quad 0 \leq \alpha \leq 1, \\
& \quad \langle \alpha, 1_k \rangle = 1.
\end{align*}
\]

To solve the problem, we observe that its structure resembles a linear program (LP). That is, the objective is given by an inner product, and the constraints are linear. The only difference between an LP and Eq. (15) is the maximum term over the \( \eta \) values inside the inner product. Hence, to solve Eq. (15), we apply the sequential linear programming approach (Nocedal and Wright, 2006), which iteratively approximates Eq. (15) as a linear program and solves it.

First, we replace the maximum terms in the objective with another variable vector \( c \) and additional constraints, following Charalambous and Conn (1978):

\[
\begin{align*}
\text{Minimize} & \quad \alpha \in \mathbb{R}^k, c \in \mathbb{R}^{|G|} \quad \langle \vec{w}, c \rangle \\
\text{s.t.} & \quad 0 \leq \alpha, c \leq 1, \\
& \quad \langle \alpha, 1_k \rangle = 1, \\
& \quad \eta(\alpha_z, h_{gz}) \leq c_g, \forall z \in [k].
\end{align*}
\]

This problem is equivalent to Eq. (15), but has no maximum operation. However, now we have non-linear constraints, thus violating the definition of an LP. To correct this, we use a local approximation of \( \eta \) (around an iterate \( \alpha^{(t)} \)) using the Taylor series. Note that the function \( a \rightarrow \eta(a, b) \) is piecewise concave, non-negative, and smooth at \((0, 1)\) except at \( b \), where it is minimized and equals zero. Therefore, we do not expect the maximum over \( z \) in the objective Eq. (15) to fall on the discontinuity of \( \eta \) at \( b \), except for extreme cases. Hence, using a linear approximation we expect to get a good local approximation of \( \eta() \).

Overall, the sequential linear programming approach calculates the following iterates:

\[
(\alpha^{(t+1)}, c^{(t+1)}) = \text{LP solution of an approximation of Eq. (16) around } (\alpha^{(t)}, c^{(t)}).
\]

Given an iterate \((\alpha^{(t)}, c^{(t)})\) we first approximate \( \eta \) by a linear Taylor series in its first argument:

\[
\eta(\alpha + \epsilon_a, b) \approx \eta(\alpha, b) + \frac{\partial \eta(\alpha, b)}{\partial \alpha} \epsilon_a = \eta(\alpha, b) + \begin{cases} 
\frac{b}{\alpha} \epsilon_a & \alpha > b, \\
-\frac{1-b}{(1-\alpha)^2} \epsilon_a & \alpha < b, \\
0 & \alpha = b.
\end{cases}
\]

(17)
Algorithm 1 Local minimization of unfairness via sequential linear programming

Input: $\alpha^{(0)}, H, \bar{w}, \text{maxIter}, \varepsilon$
Output: The local minimizer $\alpha^*$

1: for $t = 0$ to $\text{maxIter}$ do
2: Define the LP approximation of Eq. (16) by computing $M_1, M_2,$ and $v$.
3: Define $\hat{\alpha}$ as the minimizer of the LP in Eq. (19).
4: Compute $e = \hat{\alpha} - \alpha^{(t)}$, and define $\alpha^{(t+1)} = \alpha^{(t)} + \mu e$,
5: where $\mu$ is obtained using backtracking line search to ensure a reduction in Eq. (15).
6: If $\|e\| < \varepsilon$, stop.
7: end for
8: return the last iterate $\alpha^{(t)}$.

We now locally approximate Eq. (16) around an iterate $\alpha^{(t)}$ via the following LP problem:

\[
\begin{aligned}
\text{Minimize}_{\alpha \in \mathbb{R}^k, c \in \mathbb{R}^{|G|}} & \quad \langle \bar{w}, c \rangle \\
\text{subject to} & \quad 0 \leq \alpha, c \leq 1, \\
& \quad \langle \alpha, 1_k \rangle = 1, \\
& \quad \forall g \in G, v_g + J_g(\alpha - \alpha^{(t)}) \leq 1_k c_g.
\end{aligned}
\]  

where $v_g \in \mathbb{R}^k$ is a vector with the coordinates $(v_g)_z = \eta(a_z^{(t)}, h_{gz})$ for all $z \in [k]$, and $J_g \in \mathbb{R}^{k \times k}$ is a diagonal Jacobian matrix s.t $(J_g)_{zz} = \frac{\partial \eta(a_z^{(t)}, h_{gz})}{\partial a_z}$. Bringing this into a more canonical form with a variable $\alpha = \alpha^{(t)} + e$ yields

\[
\begin{aligned}
\text{Minimize}_{\alpha \in \mathbb{R}^k, c \in \mathbb{R}^{|G|}} & \quad \langle \bar{w}, c \rangle \\
\text{subject to} & \quad 0 \leq \alpha, c \leq 1, \\
& \quad \langle \alpha, 1_k \rangle = 1, \\
& \quad M_1 c + M_2 \alpha - M_2 \alpha^{(t)} + v \leq 0,
\end{aligned}
\]

where $M_1 := -I_{|G|} \otimes 1_k$, $M_2 := [J_1; J_2; \ldots; J_{|G|}]$ is the stacking of all diagonal Jacobian matrices on top of each other, and $v := [v_1; \ldots; v_{|G|}]$ is the stacking of vectors. Eq. (19) is an LP in a canonical form, which is solved iteratively. Note that $M_1$ and $M_2$ are highly sparse, which can be exploited to easily solve large instances of the problem. The resulting local optimization algorithm is given in Alg. 1. The initial assignment can be set using the assignment found by the greedy procedure in Section 7.2.1.

A note on usage. By definition, the function $\eta$ in Eq. (9) is continuous in the open section $(0, 1)^2$, but at the boundaries, it is easy to see that $\eta(0, 0) = 0$, while $\lim_{\delta \rightarrow 0} \eta(\delta, 0) = 1$. Furthermore, the derivatives of $\eta$ when its first argument is close to 0 or 1 can be very large, causing numerical difficulties in the LP solvers. Thus, to avoid numerical instability, we make sure that all entries of $H$ are in $[\varepsilon, 1 - \varepsilon]$ (where we used $\varepsilon = 10^{-5}$) by replacing any value outside these boundaries by the relevant boundary value and renormalizing. Note
that since the input values in $H$ are calculated in practice based on finite data sets, values in $H$ are already noisy representations of the true population values, and so slightly changing them does not negatively affect the correctness of the method.

8. Lower-bounding classifier discrepancy from label proportions

In the previous section, we discussed the calculation of unfairness based on the confusion matrices of the classifier in each sub-population. However, as discussed in the Introduction, these confusion matrices might be unavailable, especially if we do not have access to the classifier or to individual-level validation data. Nonetheless, we show in this section that it is possible to provide meaningful conclusions on the properties of the classifier from label proportions only. Formally, we assume the label-proportions setting in which the only available information is the aggregate statistics in Inputs, defined in Eq. (2).

We define a combined measure for the accuracy and fairness of the classifier, which we term the discrepancy, denoted $\text{disc}_\beta$, where $\beta \in [0, 1]$ is a trade-off parameter:

$$\text{disc}_\beta(A_G) := \beta \cdot \text{unfairness}(A_G) + (1 - \beta) \cdot \text{error}(A_G).$$

Bounding the value of $\text{disc}_\beta$ with an appropriate value of $\beta$ can provide answers to various questions about the classifier of interest, such as:

- Is it possible that this classifier is fair? If not, how unfair must it be based on the provided label proportions?
- Suppose that the classifier is known to be nearly fair. How accurate can it be?
- Suppose that the classifier is known to be quite accurate. How fair can it be?
- Suppose that there is a penalty for each person who gets a wrong prediction, and for each person who is treated unfairly; what is the smallest possible overall cost of this classifier?

Each of the questions above is equivalent to asking for a lower bound on $\text{disc}_\beta$, for some $\beta \in [0, 1]$, or for a $\beta$ that optimizes a constraint. For instance, if it is known that $\text{unfairness} \leq U$ for some known $U$, then finding the best possible accuracy of the classifier is equivalent to minimizing $\text{disc}_\beta$ for the smallest $\beta$ such that the minimizing solution satisfies $\text{unfairness} \leq U$. Given $\beta$, we are therefore interested in the smallest possible $\text{disc}_\beta$ value under the constraints imposed by Inputs. Define:

$$\text{mindisc}_\beta(\text{Inputs}) := \min \{ \text{disc}_\beta(A_G) \mid \forall g \in G, A_g \in \mathcal{M}_\Delta, A_g^T \pi_g = \hat{p}_g \}. \quad (21)$$

Our goal is thus to calculate or bound $\text{mindisc}_\beta(\text{Inputs})$. Note that in the case of $\beta = 0$, $\text{disc}_0 = \text{error}$. Thus, it is easy to see that $\text{mindisc}_0(\text{Inputs}) = \sum_g w_g | \pi_g - \hat{p}_g |$. We henceforth consider $\beta \in (0, 1]$.

We show in Section 8.1 that in the case of binary classification, $\text{mindisc}_\beta(\text{Inputs})$ can be calculated exactly using an efficient procedure. In Section 8.2, we propose procedures for lower-bounding and upper-bounding $\text{mindisc}_\beta(\text{Inputs})$. We discuss the overall process for drawing conclusions using our proposed algorithms in Section 8.3. Note that a lower bound
on \( \text{mindisc}_\beta \) implies a lower bound on \( \text{disc}_\beta \), while an upper bound on \( \text{mindisc}_\beta \) implies a limitation on the conclusions that can be drawn on \( \text{disc}_\beta \) using the available information.

To derive the procedures for binary classification and for multiclass classification, we first define a useful decomposition of \( \text{disc}_\beta \). From Eq. (10) and Eq. (1), we have

\[
\text{disc}_\beta(A_G) = \beta \cdot \sum_{y \in \mathcal{Y}} \min_{a_k \in \Delta} \sum_{g \in \mathcal{G}} w_g \pi^y_g \max_{z \in \mathcal{Y}} \eta(\alpha^z_b, \alpha^z_g) + (1 - \beta) \cdot \sum_{g \in \mathcal{G}} w_g \sum_{y \in \mathcal{Y}} \pi^y_g (1 - \alpha^y_g).
\]  

(22)

Denote \( \tau_\beta(a, b) := \beta \eta(a, b) + (1 - \beta)(1 - b) \), and define

\[
D_{\beta,g}(A_b, A_g) := \sum_{y \in \mathcal{Y}} \pi^y_g \tau_\beta(\alpha^y_b, \alpha^y_g) + \beta \sum_{y \in \mathcal{Y}} \pi^y_g \max_{z \in \mathcal{Y} \setminus \{y\}} [\eta(\alpha^z_b, \alpha^z_g) - \eta(\alpha^y_b, \alpha^y_g)],
\]

where we used the notation \([x]_+ := \max(0, x)\). Then

\[
\text{disc}_\beta(A_G) = \min_{A_b \in \mathcal{M}_\Delta} \sum_{g \in \mathcal{G}} w_g \cdot D_{\beta,g}(A_b, A_g).
\]  

(23)

We now turn to the calculation of \( \text{mindisc}_\beta \) in the two classification settings.

### 8.1 Binary Classification

In this section, we show how to calculate \( \text{mindisc}_\beta \) for the case of binary classification. We show that in this case, the set of possible solutions for \( A_G \) can be sufficiently restricted, so that it is computationally tractable to consider them all, and to globally minimize \( A_b \) with respect to each of these possible solutions. Thus, the value of \( \text{mindisc}_\beta \) can be calculated accurately.

In the case of binary classification, the second term in the definition of \( D_{\beta,g} \) is always zero, since there is only one \( z \in \mathcal{Y} \setminus \{y\} \) and we have

\[
\eta(\alpha^y_b, \alpha^y_g) = \eta(1 - \alpha^y_b, 1 - \alpha^y_g) = \eta(\alpha^y_b, \alpha^y_g).
\]

Since there are only two labels, a confusion matrix is fully determined by the false positive rate and the false negative rate, which we denote by \( \alpha^0_g, \alpha^1_g \), respectively, where \( \alpha^1_g := \alpha^y_g(1 - y) \). The constraint \( A_g^T \pi_g = \tilde{\pi}_g \) implies \( \tilde{\pi}^1_g = \pi^0_g \alpha^0_g + \pi^1_g (1 - \alpha^1_g) \). Define the set \( \mathcal{G}^1 = \{g \in \mathcal{G} \mid \pi^1_g > 0\} \). For \( g \in \mathcal{G}^1 \), \( \alpha^1_g = 1 - \tilde{\pi}^1_g/\pi^1_g + (\pi^0_g/\pi^1_g) \alpha^0_g \). We denote \( r_g := 1 - \tilde{\pi}^1_g/\pi^1_g \) and \( q_g := \pi^0_g/\pi^1_g \), so that

\[
\forall g \in \mathcal{G}^1, \quad \alpha^1_g = r_g + q_g \alpha^0_g.
\]  

(24)

Note that for \( g \notin \mathcal{G}^1 \), \( \pi^0_g = 1 \) and so \( \alpha^0_g = \tilde{\pi}^1_g \) and \( \alpha^1_g = 0 \). Denote \( \alpha^y_g := \alpha^y_g(1 - y) \) and \( \alpha_b := (\alpha^0_b, \alpha^1_b) \). We thus have that in the binary case,

\[
\forall g \in \mathcal{G}, \quad \min_{A_b \in \mathcal{M}_\Delta} D_{\beta,g}(A_b, A_g) \equiv D_{\beta,g}(\alpha_b, \alpha^0_g) = \pi^0_g \tau_\beta(\alpha^0_b, \alpha^0_g) + \pi^1_g \tau_\beta(\alpha^1_b, r_g + q_g \alpha^0_g),
\]

where we define the second term to be zero if \( \pi^1_g = 0 \). Since \( \alpha^y_g \in [0, 1] \), it follows from Eq. (24) that the feasible domain of \( \alpha^0_g \) for \( g \in \mathcal{G}^1 \) is

\[
\text{dom}_g := [\max\{-r_g/q_g, 0\}, \min\{(1 - r_g)/q_g, 1\}].
\]
If $g \notin \mathcal{G}^1$, define $\text{dom}_g := \{ \hat{p}^0_g \}$. From Eq. (21) and Eq. (23), we thus have

$$\text{mindisc}_\beta(\text{Inputs}) = \min_{\alpha_b \in [0,1]^2} \left( \sum_{g \in \mathcal{G}} w_g \cdot \min_{\alpha_0^g \in \text{dom}_g} D_{\beta,g}(\alpha_b, \alpha_0^g) \right). \quad (25)$$

Below, we show that this minimization problem can be reduced to a small set of one-dimensional minimization problems. We further provide a procedure for minimizing these one-dimensional problems. Combining the two, this gives a procedure for calculating $\text{mindisc}_\beta$ accurately in the case of binary classification. For the sake of presentation, we first provide the procedure and a sketch of its derivation in Section 8.1.1, and then provide the full derivation, by first showing the reduction to few one-dimensional problems in Section 8.1.2, and then providing the procedure for minimizing these in Section 8.1.3.

8.1.1 Calculating $\text{mindisc}_\beta$

The algorithm for calculating $\text{mindisc}_\beta$ up to a given tolerance is listed as Alg. 2. Alg. 2 gets as input the label proportions $\text{Inputs}$, the value of $\beta$ that specifies the required minimization instance of Eq. (25), and a tolerance parameter $\gamma > 0$. It outputs a discrepancy $V \in \mathbb{R}$ which is the value of $\text{mindisc}_\beta(\text{Inputs})$ up to a tolerance of the input parameter $\gamma$. It further provides the values of unfairness and error that obtain the discrepancy $V$. We give here an overview of Alg. 2. For simplicity of presentation, we defer some of the definitions used by the algorithm to Section 8.1.2.

Denote $\mathcal{G}^+ := \{ g \in \mathcal{G} | \forall y \in \mathcal{Y}, \pi_y^g \neq 1 \}$. Alg. 2 first runs $\text{ArgMin1D}$, a one-dimensional minimization procedure, for each $g \in \mathcal{G}^+$. $\text{ArgMin1D}$ is provided in Section 8.1.3 below. Running $\text{ArgMin1D}$ gives a set of values $\{ v_g \}_{g \in \mathcal{G}^+}$, which are then used along with two other sets $V_0$ and $V_1$ to construct a set $P$ of possible assignments to $\alpha_b$. We show below that $P$ necessarily includes an assignment which minimizes $\text{disc}_\beta$, subject to the provided label proportions in $\text{Inputs}$, up to a tolerance of $\gamma$. Alg. 2 then finds a minimizing assignment in $P$ using a proxy function $F$, which is defined in Section 8.1.2 below. This function is easy to calculate and we show below that minimizing it is equivalent to minimizing the objective. The minimizing assignment is then used to calculate the value of the objective, as well as the value of the unfairness and error that obtain this objective. The next two sections provide the full derivation of Alg. 2.

8.1.2 A reduction to one-dimensional minimization problems

We first show that given a fixed assignment for $\alpha_b \in [0,1]^2$, $\arg \min_{\alpha_0^g \in \text{dom}_g} D_{\beta}(\alpha_b, \alpha_0^g)$ belongs to a small set of possible values.

**Lemma 2** Fix some set of parameters $\text{Inputs}$. For $g \in \mathcal{G}, \alpha_b \in [0,1]^2$, define

$$S_g(\alpha_b) := \{ s^i_g(\alpha_b) \}_{i \in [4]} \cap \text{dom}_g,$$

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Algorithm 2 Calculating $\text{mindisc}_\beta$ for binary classification

Input: Inputs, $\beta \in [0, 1]$, tolerance $\gamma > 0$

Output: $V \in [\text{mindisc}_\beta(\text{Inputs})]$, $\text{mindisc}_\beta(\text{Inputs}) + \gamma]$; $(u, \epsilon)$, the values of unfairness and error that obtain the discrepancy $V$.

1: For each $g \in G^+$, $v_g \leftarrow \text{ArgMin1D}(g, \gamma, \beta, \text{Inputs})$. #Get minimizer of $H^g$
2: $V_0 \leftarrow \{0, 1\} \cup \{-r_g/q_g, (1-r_g)/q_g\}_g \in G^+$ \cup $\{\hat{p}_g\}_g \in G$.
3: $V_1 \leftarrow \{0, 1\} \cup \{r_g, r_g + q_g\}_g \in G^1$.
4: $P \leftarrow (V_0 \times V_1) \cup \{(v_g, r_g + q_g v_g) \mid g \in G^+\}$.
5: $\alpha_b \leftarrow \text{argmin}_{\alpha_b \in P} F(\alpha_b)$
6: $V \leftarrow F(\alpha_b)$
7: $\forall g \in G$, $\hat{\alpha}_g^0 \leftarrow \text{argmin}_{\alpha_g \in S_g(\alpha_b)} D_{\beta,g}(\alpha_b, \alpha_g^0)$
8: $\forall g \in G^1$, $\hat{\alpha}_g^1 \leftarrow r_g + q_g \hat{\alpha}_g^0$
9: $u \leftarrow \sum_{g \in G} w_g \sum_g \pi_g \eta(\hat{\alpha}_b^0, \hat{\alpha}_g^y)$
10: $\epsilon \leftarrow \sum_{g \in G} w_g \sum_g \pi_g \hat{\alpha}_g^y$
11: Return $V$, $(u, \epsilon)$

where for $g \in G^1$,

$$s_1^1(\alpha_b) \equiv s_1^g := \max\{0, -r_g/q_g\},$$

$$s_2^1(\alpha_b) \equiv s_2^g := \min\{1, (1-r_g)/q_g\},$$

$$s_3^1(\alpha_b) \equiv s_3^g(\alpha_g^0) := \alpha_g^0,$$

$$s_4^1(\alpha_b) \equiv s_4^g(\alpha_g^1) := (\alpha_g^1 - r_g)/q_g.$$

For $g \in G \setminus G^1$, define $s_i^g(\alpha_b) = \{\hat{p}_g^i\}$ for all $i \in [4]$. Then

$$\min_{\alpha_g \in \text{dom}_g} D(\alpha_b, \alpha_g^0) = \min_{\alpha_g \in S_g(\alpha_b)} D(\alpha_b, \alpha_g^0).$$

**Proof** Assume that $\pi_g^0 \in (0, 1)$, since the other case is trivial. Consider the function $b \mapsto \tau_\beta(a, b)$ for a fixed $a$. It is a convex combination of $\eta(a, b)$ and of $b$. Now, for a fixed $a \in [0, 1]$, $b \mapsto \eta(a, b)$ is a convex piecewise-linear function: it is linear and decreasing over $[0, a]$ and linear and increasing over $[a, 1]$, as can be verified from the definition of $\eta$ (see Figure 3 (left) for illustration). Since by Eq. (24), $\alpha_g^1$ is linear in $\alpha_g^0$, it follows that $\rho(\alpha_g^0)$ is convex and piecewise-linear, with inflection points at $\alpha_g^0 = \alpha_g^0 =: p_1$, and at $\alpha_g^1 = \alpha_g^1$, that is at $\alpha_g^0 = (\alpha_g^1 - r_g)/q_g =: p_2$ (see Figure 4 for illustration). It follows that the minimizer of $\tau_\beta(\alpha_g^0)$ must be one of $p_1, p_2$ or the end points of dom$_g$. These are exactly the values in $S_g(\alpha_b)$ defined above. $\blacksquare$

Define

$$F(\alpha_b) := \sum_{g \in G} w_g \min_{\alpha_g \in S_g(\alpha_b)} D_{\beta,g}(\alpha_b, \alpha_g^0).$$
Figure 4: $\rho(\alpha_g^0)$ and the inflection points $p_1, p_2$ for $\alpha_b^0 = 0.2, \alpha_b^1 = 0.6, \pi_g^1 = 0.1, \hat{p}_g^1 = 0.1$.

Note that $F(\alpha_b)$ is easy to calculate, since the minimizations are over $S_g(\alpha_b)$, which is a small finite set of values. From Lemma 2 and Eq. (25), it follows that

$$\text{mindisc}_\beta = \min_{\alpha_b \in [0, 1]^2} F(\alpha_b).$$  \hspace{1cm} (26)

While this two-dimensional minimization could be approached using grid-search, the grid resolution might be prohibitively small. This is especially evident when data is extremely unbalanced, as in the case of cancer occurrence statistics, on which we report experiments in Section 9. In such unbalanced cases, the necessary grid size can be prohibitive. For instance, cancer occurrence statistics can be of the order $10^{-6}$. A two-dimensional grid search of this resolution would be of order $10^{12}$, which could be impractical to perform. Thus, a further reduction of the search size is necessary. We now show that this two-dimensional problem can be reduced to a small set of one-dimensional minimization problems.

**Theorem 3** Let $V_0$ and $V_1$ be as defined in Alg. 2. Define the following set of vectors in $\mathbb{R}^2$:

$$\text{Pairs} := (V_0 \times V_1) \cup \{(v, r_g + q_g v) \mid v \in \text{dom}_g, g \in G^+\}.$$  

Then

$$\text{mindisc}_\beta = \min_{\alpha_b \in \text{Pairs}} F(\alpha_b).$$

The proof of Theorem 3 is given in Appendix A. In the proof, we show that given $\alpha_b^0$, it is easy to solve $\min_{\alpha_b^1 \in [0, 1]} F((\alpha_b^0, \alpha_b^1))$, since there is a small finite number of possible minimizers. This is because the domain $[0, 1]$ can be partitioned into intervals that depend on $\alpha_b^0$, such that $\alpha_b^1 \to F(\alpha_b)$ is concave in each of these intervals.

By Theorem 3, finding $\text{mindisc}_\beta$ in the case of binary classification can be done by minimizing $F(\alpha_b)$ over the set $\text{Pairs}$, which includes $O(|G|^2)$ discrete solutions $V_0 \times V_1$, and $|G^+|$ one-dimensional solution sets of the form $(v, r_g + q_g v)$ for some $g \in G^+$. Define, for $g \in G^+$,

$$H^g(v) := F(v, r_g + q_g v), \text{ and } V_g := \min_{v \in \text{dom}_g} H^g(v).$$

Then, we have

$$\text{mindisc}_\beta = \min \{ \min_{\alpha_b \in V_0 \times V_1} F(\alpha_b), \min_{g \in G^+} V_g \}.$$
Finding $V_g$ is a one-dimensional minimization problem. Due to the irregularity of $H^g$ and its possibly large absolute derivatives, especially in unbalanced cases, obtaining a reasonable solution accuracy using a naive grid search could be prohibitively expensive. In the next section we provide a procedure, termed ArgMin1D, that searches over a smaller number of values when minimizing $H^g$, making the procedure practical even in extremely unbalanced cases. ArgMin1D gets as input the region identifier $g$, the solution value tolerance $\gamma$, the discrepancy coefficient $\beta$, and the set of inputs Inputs. It returns the minimizer of $H^g$.

8.1.3 Fast minimization of $H^g$

Fix $g \in \mathcal{G}^+$. We give the procedure ArgMin1D that calculating $\arg\min_{v \in \text{dom}_g} H^g(v)$ up to a given tolerance. Let $\text{dom}_g^0$ be the open interval which is the interior of the closed interval $\text{dom}_g$. Note that $H^g$ is continuous on $\text{dom}_g^0$. To minimize $H^g$ on $\text{dom}_g^0$, it suffices to find a constant $L > 0$ such that $H^g$ is $L$-Lipschitz on $\text{dom}_g^0$, that is,

$$\forall a, b \in \text{dom}_g^0, |H^g(a) - H^g(b)| \leq L|a - b|.$$ 

Then, to find an optimal solution up to a tolerance of $\gamma$, one can perform a grid search with a resolution of $L/\gamma$. However, $H$ can have very large derivatives in small intervals, leading to a prohibitively large value of $L$. We therefore partition $\text{dom}_g^0$ into separate intervals, and derive a separate Lipschitz upper bound for each interval. We show that the size of intervals with a large Lipschitz constant is small; Thus, the resulting search size is significantly smaller than $L/\gamma$.

$H^g$ is a linear combination of minimizations over applications of $\tau_\beta$. For $x \in [0, 1], g \in \mathcal{G}$, denote $[x]_g = (x, r_g + q_g x)$. Where $H^g$ is differentiable, its derivative can be upper bounded by

$$\frac{dH^g(v)}{dv} \leq \sum_{z \in \mathcal{G}} w_z \max_{x \in S_z([v]_g)} \sum_{y \in \{0, 1\}} \pi^y_z \left| \frac{\partial \tau_\beta([v]_g(y), [x]_z(y))}{\partial v} \right|. \quad (27)$$

$\tau_\beta(a, b)$ is differentiable except at $a \in \{0, 1\}, b \in \{0, 1\}$ and $a = b$. Based on this observation, we partition the search interval $[0, 1]$ into sub-intervals such that all the $\tau_\beta$ terms in the sum are differentiable in the interior of each of the sub-intervals.

Then the Lipschitz constant $L_I$ in sub-interval $I$ is upper bounded by

$$L_I \leq \sum_{z \in \mathcal{G}} w_z \sum_{y \in \{0, 1\}} \pi^y_z \max_{x \in S_z([v]_g)} \max_{v \in I} \left| \frac{\partial \tau_\beta([v]_g(y), [x]_z(y))}{\partial v} \right|. $$

Note that for $x \in S_z([v]_g)$, $x$ is a linear transformation of $v$. Therefore, all the maximizations above are of the form

$$\max_{v \in I} \left| \frac{\partial \tau_\beta(a + bv, c + dv)}{\partial v} \right|,$$

for some known real values $a, b, c, d$. The following lemma provides an upper bound for this expression, depending on the values of $a, b, c, d$. 

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Lemma 4 Let \( b > 0, c \leq 1, a \leq 1, d \geq 0 \). Define \( \psi(x) := \tau^\beta(a + bx, c + dx) \), defined over \( x \) such that \( a + bx, c + dx \in [0, 1] \). Then \( \psi \) is differentiable in the interior of its domain, except perhaps for \( x \) such that \( a + bx = c + dx \). Define

\[
D(a, b, c, d) := \begin{cases}
    b/c & (d = 0) \land (c > 0) \\
    d^2/(bc - ad) & (d > 0) \land (a/b > c/d) \\
    |bc - ad|/(a + b(c - a)/(b - d))^2 & (d > 0) \land (a/b < c/d) \land (b > d) \\
    0 & \text{otherwise}.
\end{cases}
\]

Then for any \( x \) in the interior of the domain of \( \psi \) such that \( a + bx \neq c + dx \),

\[
|\frac{d\psi(x)}{dx}| \leq \begin{cases}
    D(a, b, c, d) & (d > b \land x < \frac{c - a}{b - d}) \lor (d < b \land x > \frac{a - c}{b - d}) \\
    D(1 - a - b, b, 1 - c - d, d) & (d > b \land x > \frac{c - a}{b - d}) \lor (d < b \land x < \frac{a - c}{b - d}) \\
    \frac{a - c}{|a - c|} & d = b, a \neq c \\
    0 & \text{otherwise}.
\end{cases}
\]

The proof is provided in Appendix B. The procedure \texttt{ArgMin1D} calculates an upper bound on \( L_I \) for each sub-interval \( I \) using the upper bound provided in Lemma 4, and then searches with a grid resolution of \( L_I/\gamma \) in the sub-interval. In addition, it searches each of the end points of all sub-intervals. This provides a minimizer up to a tolerance of \( \gamma \) using a limited number of search points compared to a naive fixed-resolution search.

### 8.2 Multiclass Classification

In the multiclass case, we are not aware of an efficient algorithm that can calculate the value \( \text{mindisc}^\beta_c \) in Eq. (21) exactly. A lower bound can be obtained by calculating the term in \( D_{\beta,g} \) that is shared with the binary case using the procedure in Section 8.1. This lower bound in effect calculates the minimal discrepancy under a less stringent multiclass fairness requirement that only the total mistake proportions to be the same in all sub-populations, regardless of the type of mistake (see the discussion in Section 6.1).

To obtain an upper bound, we use a sequential linear programming approach similarly to our solution to Eq. (15) in Section 7.2.2. First, we extend the notation above Eq. (15) to explicitly denote the value of \( y \in \mathcal{Y} \): The matrix \( \mathbf{H}^y \in \mathbb{R}^{[\mathcal{G}] \times k} \) is defined such that \( h_{yz}^y = \alpha^y_{yz} \); The vector \( \mathbf{w}^y \in [0, 1]^{[\mathcal{G}]} \) includes the entries \( w_{yz}^y := w_y \pi_i^y \); The optimization variables are denoted by the vectors \( \mathbf{\alpha}^y := \mathbf{\alpha}_y^y \in \mathbb{R}^k \). We replace the maximum terms in the problem in Eq. (22) with the vectors \( \mathbf{c}^y \in \mathbb{R}^k \), similarly to the transformation from Eq. (15) to Eq. (16). Define the matrix \( \mathbf{F}^y \in \mathbb{R}^{[\mathcal{G}] \times k} \), in which the \( y \)-th column is all 1’s and the rest of the entries are zeros. In the objective below, the notation \( \langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{A}^\top \mathbf{B}) \) corresponds to the standard matrix inner product for any two matrices \( \mathbf{A}, \mathbf{B} \). The variables for the optimization problem are the collection of the vectors and matrices

\[
\mathbf{x} = \{ \mathbf{\alpha}^1, \mathbf{H}^1, \mathbf{c}^1, \ldots, \mathbf{\alpha}^k, \mathbf{H}^k, \mathbf{c}^k \},
\]

which amount to \( k^2 + k^2 \cdot [\mathcal{G}] + k \cdot [\mathcal{G}] \) scalar unknowns in total. We get the following new problem, which is equivalent to the problem defined in Eq. (20)-(22) including its constraints:
Minimize \( \{\alpha^y\}, \{H^y\}, \{c^y\} \)  
\[
\sum_y \beta \langle \mathbf{w}^y, \mathbf{c}^y \rangle + (1 - \beta) \langle \mathbf{F}^y, \mathbf{1}_{k \times |G|} - \mathbf{H}^y \rangle 
\]  
\[
\text{s.t.} \\
0 \leq \alpha^y \leq 1 \quad \forall y \in [k], \\
0 \leq H^y \leq 1 \quad \forall y \in [k], \\
0 \leq c^y \leq 1 \quad \forall y \in [k], \\
\langle \alpha^y, 1 \rangle = 1 \quad \forall y \in [k], \\
H^y \mathbf{1}_k = \mathbf{1}_{|G|} \quad \forall y \in [k], \\
\eta(\alpha^y, h^y_{g,z}) \leq c^y_g \quad \forall y, z \in [k], \ g \in [|G|], \\
\sum_{y=1}^k \pi^y_g h^y_{g,z} = \hat{p}^z_g \quad \forall z \in [k], \ g \in [|G|]. 
\]  

Note that the objective in Eq. (29) is linear, and its constraints are linear as well, except the ones involving \( \eta() \), similarly to Eq. (16). Also note that all of the linear constraints are separable in \( y \), except for the last one, which introduces a coupling between the \( H^y \) of different \( y \)'s.

To solve the problem we again use the sequential linear programming approach, as the structure of the problem again resembles a linear program. Given the \( t \)-th iterate \( \mathbf{x}^{(t)} \) for all the variables defined in Eq. (28), we will first approximate \( \eta \) by a linear Taylor series, this time in both of its arguments, since now our optimization variables appear in both. Note that the function \( \eta(\alpha, b) \) is linear in \( b \), hence the Taylor series is exact with respect to that parameter, which is appealing for our approach. However, decreasing the objective with respect to both the parameters \( h^y_{g,z} \) and \( \alpha^y_g \) leads to the minimum points of \( \eta \). These points are the singularity points (see Fig. 3), where \( \eta \) is not approximated well by a linear function. This results in a slow convergence of the sequential linear programming algorithm. To solve this, we split \( \eta \) and define it as the maximum between the two continuous functions:
\[
\eta(\alpha, b) = \max \{ \eta_1(\alpha, b), \eta_2(\alpha, b) \}, \tag{30}
\]

where
\[
\eta_1(\alpha, b) = 1 - \frac{b}{\alpha}, \quad \eta_2(\alpha, b) = 1 - \frac{1 - b}{1 - \alpha}.
\]

This split is an alternative formulation to the original definition of \( \eta \) in Eq. (9), only now it has no singularity point, and can be treated in the same way that we treat the maximization over all the terms involving \( \eta \). That is, the relevant constraint satisfies:
\[
\eta(\alpha, b, y, z, h) \leq c \iff \eta_1(\alpha, b, y, z, h) \leq c \quad \text{and} \quad \eta_2(\alpha, b, y, z, h) \leq c,
\]

and now the constraints does not include singularity points, and can be effectively locally approximated by two linear functions, one for \( \eta_1 \) and one for \( \eta_2 \) (see Figure 5). Explicitly, the Taylor approximation of \( \eta_i \) for \( i = 1, 2 \) are now given by:
\[
\eta_i(\alpha + \epsilon_a, b + \epsilon_b) \approx \eta_i(\alpha, b) + \frac{\partial \eta_i}{\partial \alpha}(\alpha - \tilde{\alpha}) + \frac{\partial \eta_i}{\partial b}(b - \tilde{b}), \tag{31}
\]

where:
\[
\frac{\partial \eta_1}{\partial \alpha} = \frac{b}{\alpha^2}, \quad \frac{\partial \eta_1}{\partial b} = -\frac{1}{\alpha}, \quad \frac{\partial \eta_2}{\partial \alpha} = -\frac{1 - b}{(1 - \alpha)^2}, \quad \frac{\partial \eta_2}{\partial b} = \frac{1}{1 - \alpha}.
\]

Given the first order Taylor approximations above, we form the LP approximation of Eq. (29) around an iterate \( x^{(t)} = \{\tilde{\alpha}^y, \tilde{H}^y, \tilde{c}^y\}_{y=1}^k \) as follows:
\[
\text{Minimize}_{\{\alpha^y\}, \{H^y\}, \{c^y\}} \sum_y \beta(\tilde{w}^y, c^y) + (1 - \beta)(F^y, 1_{k \times |G|} - H^y) \tag{32}
\]

s.t.
\[
\begin{align*}
0 \leq \alpha^y &\leq 1 & \forall y &\in [k], \\
0 \leq H^y &\leq 1 & \forall y &\in [k], \\
0 \leq c^y &\leq 1 & \forall y &\in [k], \\
\langle \alpha^y, 1_k \rangle &= 1 & \forall y &\in [k], \\
H^y 1_k &= 1_{|G|} & \forall y &\in [k], \\
\eta_i(\alpha^y, h^y) + \frac{\partial \eta_i}{\partial \alpha}(\alpha^y - \tilde{\alpha}^y) + \frac{\partial \eta_i}{\partial b}(h^y - \tilde{h}^y) &\leq c^y & \forall y, z &\in [k], g &\in [|G|], i = 1, 2, \\
\sum_{y=1}^k \pi^y h^y_{z,g} &= p^z_{g} & \forall z &\in [k], g &\in [|G|].
\end{align*}
\]

Eq. (32) is an LP problem which is solved at each iteration by an LP solver, for which we use the `scipy.optimize` library. In addition to the box constraints of \([0, 1]\) for all variables, it includes \( 2 \cdot k \cdot |G| + k \) equality constraints, and \( 2 \cdot k^2 \cdot |G| \) inequality constraints. It can be seen that the number of inequality constraints is rather high, and as a result, so is the computational complexity of the algorithm if run as is. However, many of these constraints are not active in the solution, and in any case we limit the step size of our algorithm. Hence,
we can ease the difficulty of the LP problem by both limiting the search space of the LP solver, and removing the inequality constraints that seem to be inactive in the solution. To this end, we remove the inequality constraints where \( \eta_i(\tilde{\alpha}^y, \tilde{h}^y_{\beta, z}) < -1 \), as we expect these not to be active at the solution. Furthermore, we use a maximum step size \( \tau \) so that

\[
\|\alpha^y - \tilde{\alpha}^y\|_{\infty} < \tau, \quad \|H^y - \tilde{H}^y\|_{\infty} < \tau,
\]

where \( \| \cdot \|_{\infty} \) is the maximum norm. This condition is trivially incorporated in the box constraints of Eq. (32). Finally, given the LP approximation detailed above, the algorithm for solving Eq. (29) follows the same lines as Alg. 1.

As in Section 7.2.2, here too numerical instabilities arise when \( \alpha^y_{\beta} \) is too close to 0 or 1. Thus, here as well we restrict these values to be in the segment \([\varepsilon, 1 - \varepsilon]\), this time as optimization variables through the box constraints. Also, we update the values \( p^z_{\beta} \) to be \( p^z_{\beta} = (1 - k\epsilon)p^z_{\beta} + \epsilon \) to guarantee a solution for Eq. (29) after updating the box constraints for \( \alpha^y_{\beta} \).

### 8.3 Drawing conclusions: the overall process

In Section 8.1 and Section 8.2, we provided algorithms to calculate \( \text{mindisc}_{\beta} \) (in the case of binary classification) or to estimate it (in the case of multiclass classification). These procedures can be used to draw conclusions by using the following process.

1. Obtain the necessary information about the classifier: The prevalence of each sub-population; The true frequency of each of the possible labels in each sub-population; The frequency of prediction of each of the possible labels in each sub-population. Set Inputs using this information.

2. Select a desired tolerance \( \gamma \), based on accuracy requirements and amount of computation resources.

3. For each relevant value of \( \beta \) (see below)
   - If there are only two possible labels, run Alg. 2 on Inputs with parameters \( \gamma \) and \( \beta \). Get the value of \( \text{mindisc}_{\beta} \) up to a tolerance of \( \gamma \).
   - Otherwise (more than two possible labels), calculate an upper bound on \( \text{mindisc}_{\beta} \) by running the procedure described in Section 8.2 for solving Eq. (32).

4. Draw a conclusion based on the calculated values and the value of \( \beta \) (see below).

The choice of the value of \( \beta \) and the use of the results to draw conclusions depend on the goal of the procedure. In some cases, a search over several values \( \beta \) is necessary for obtaining the goal.

- If the goal is to obtain information about \( \text{disc}_{\beta} \) for a given \( \beta \), then this value of \( \beta \) should be used in the procedure above. For instance, this would be the case if there is a cost associated with each percentage point of error and a (possibly different) cost associated with each percentage point of unfair treatment by the studied classifier. As an example, if the cost for error is twice the cost for unfairness, then one should set \( \beta = 1/3 \). The procedure above provides a lower bound on this cost for the studied classifier.
• If the goal is to draw conclusions about the amount of unfairness of the classifier, then one should set $\beta = 1$. In this case, $\text{disc}_\beta$ is simply the unfairness of the classifier, and procedure above provides the amount of unfairness that the classifier must have, based on the Inputs values.

• If there is additional information on the unfairness or on the error of the classifier, then this information can be used to obtain a lower bound on the other measure by searching over values of $\beta$. For instance, suppose that in addition to Inputs, it is known that this classifier has $\text{error} \leq \epsilon$ for some given $\epsilon \in [0,1)$. To find the tightest lower bound for unfairness based on this information, one should search (e.g., using binary search) for the largest value of $\beta$ such that the solution for $\text{mindisc}_\beta$ has $\text{error} \leq \epsilon$. The value of unfairness in this solution indicates the minimal possible unfairness under the constraint that $\text{error} \leq \epsilon$. A symmetric process should be used if an upper bound of unfairness is known and the goal is to find a lower bound on error.

• If the goal is to generate a Pareto-curve of all the possible combinations of unfairness and error, then one should run the procedure above for the full range of values of $\beta \in [0,1]$, setting the number of values based on the available computational resources. Each pair of unfairness and error that is the solution for one $\text{mindisc}_\beta$ provides one point in a lower-bound Pareto-curve for the possible combinations of unfairness and error.

In all of the cases above, we get an accurate result (up to tolerance $\gamma$) in the case of two labels. In the case of more than two labels, we get a heuristic result, since our minimization procedure is not guaranteed to converge to $\text{mindisc}_\beta$. In this case, if the resulting value is large, it can be used as a preliminary indication that this classifier may be unfair. In contrast, if the resulting value is small, then since this value is necessarily an upper bound on $\text{mindisc}_\beta$, it can be concluded with certainty that the information in Inputs is insufficient to prove that the classifier is unfair beyond that value.

9. Experiments

In this section, we report experiments demonstrating the various algorithms proposed in this work, as well as the diversity of possible applications of bounding fairness from aggregate data. In Section 9.1, we report experiments with binary classifiers. In Section 9.2, we report experiments with multiclass classifiers. The procedures proposed in this paper and the code for running the experiments are provided in https://github.com/sivansabato/unfairness.

9.1 Binary classification

We report three sets of experiments for binary classifiers. In the first set of experiments, reported in Section 9.1.1, we use data sets for which labeled data is available and can be used for validation. For various classifiers, we calculate the value of $\text{mindisc}_\beta$ using our proposed procedure, and compare it to the true discrepancy $\text{disc}_\beta$ of the classifier, which we calculate using the labeled data. This provides an evaluation of the tightness of $\text{mindisc}_\beta$. 
as a lower bound for $\text{disc}_\beta$. Since $\text{mindisc}_\beta$ is calculated using only the limited information of aggregate statistics, we do not expect this lower-bound to always be tight. However, our experiments show that in many cases, it is quite close to the actual discrepancy.

In the second set of experiments, reported in Section 9.1.2, we demonstrate possible uses and outcomes of the calculation of $\text{mindisc}_\beta$ in various applications. We study several classifiers for which we only have aggregate statistics, calculate lower-bound Pareto curves of the trade-off between unfairness and error for each classifier, and discuss how these curves can help in decision making. In all the experiments below, we considered classification of US-based individuals, and the fairness attribute was the state of residence (or work) of the individual.

The experiments in Section 9.1.1 and Section 9.1.2 focus on classification problems in which there are many sub-populations with non-negligible probability mass. In these cases, the derived bounds are usually relatively tight. In contrast, in Section 9.1.3, we discuss and demonstrate phenomena that can occur when minimizing the discrepancy for binary classification over distributions in which all or almost all of the probability mass is concentrated on only two sub-populations. While the bounds remain valid in this regime, we demonstrate that they can sometimes be very loose.

9.1.1 Comparing the minimal discrepancy to the actual discrepancy

For this comparison, we used the US Census (1990) data set (Dua and Graff, 2019) to generate over 800 classifiers, on which we tested the tightness of the lower bound $\text{mindisc}_\beta$ by comparing it to the true discrepancy $\text{disc}_\beta$.

The US Census data set includes approximately 2.5 million records, with 124 attributes for each record. We studied fairness with respect to the “state of work” attribute, which specifies the state in which the individual worked. We used for the experiment the approximately 1.1 million records in which this attribute was present. We split this data into two halves at random, using one half as a training set to generate classifiers, and the other half as a test set to calculate the aggregate statistics of the classifier, as well as its true unfairness and error. For each attribute in the data set except for the state of work, and for each value $v$ of the attribute, we generated a binary classification problem in which the examples are the records without this attribute and without the “state of work” attribute, and the label of a record was 1 if the attribute had value $v$. For attributes with more than 10 values, we binned the values of the attribute to 10 bins, and the label was 1 if the attribute had value at least $v$. If the resulting classification problem had more than 99% of the examples assigned to the same label, this classification problem was discarded. This process resulted in 410 classification problems. For each classification problem, we generated classifiers using two learning methods: a linear fit and a decision tree learning, using standard Matlab packages.

We ran Alg. 2 on Inputs, as calculated for each of these classifiers on the test set. The vector $w$ that holds the fraction of the population in each state was also calculated based on the test set. First, we used Alg. 2 to calculate $\text{mindisc}_1$, which is a lower bound on $\text{disc}_1 \equiv \text{unfairness}$. We then calculated the ratio between the true unfairness (as calculated on the same test set) and the calculated $\text{mindisc}_1$. Figure 6 shows the fraction of the classifiers that achieved a ratio above a range of thresholds. For the linear method,
Figure 6: US Census experiments. We report the fraction of classifiers for which unfairness/mindisc is larger than a given threshold. A ratio of 100\% indicates that the lower bound is equal to the true unfairness.

Figure 7: US Census experiments. For 10 classifiers in each method, the ratio between disc lower bound and the true disc, as a function of \( \beta \).
the median ratio was 48%; it was 23% for the tree method. 97% of all classifiers obtained a ratio of at least 10%. We conjecture that the difference between the two learning methods might be due to the local nature of tree methods, which may make them inherently less fair (see a related discussion in Valdivia et al., 2021).

Next, we used Alg. 2 with a range of values of $\beta$, for 10 randomly selected classifiers. Figure 7 plots the ratio between the true $\text{disc}_\beta$ and our obtained lower bound for $\beta \in [0,1]$ in each of the classifiers. Here too, it can be observed that in most cases, $\text{mindisc}_\beta$ was quite similar to $\text{disc}_\beta$.

9.1.2 Pareto curves for classifiers with unknown confusion matrices

In the next set of experiments for binary classifiers, we demonstrate possible uses and outcomes of the calculation of $\text{mindisc}_\beta$. We study several classifiers for which we only have aggregate statistics, and generate Pareto curves of the resulting lower bounds which provide the best possible trade-off between unfairness and error. We further demonstrate answering the types of questions in Section 8 using $\text{mindisc}_\beta$ and the resulting Pareto curves. The values in the weight vector $w$, which describe the fraction of the population in each state, were obtained from US Census Bureau, 2019.

In the first experiment, we consider a classification problem of identifying people diagnosed with a certain type of cancer from their queries in the Microsoft Bing search engine. The end goal was to identify possible participants for an anonymous patient study. We studied the 18 cancer types listed in CDC and NCI., 2019, which reports their true-positive rate (incidence rate) in each state, giving the values in $\{\pi_g\}_{g \in G}$. For each cancer type, we constructed a classifier that predicted which anonymous US-based Bing users were likely ill with a specific cancer. Note that we did not have individual validation data connecting users to a medically-validated diagnostic status. A user was predicted positive (having cancer) if they mentioned in their queries made between January 1st and June 30th, 2019 that they are suffering from a specific type of cancer. Following the methodology in Yom-Tov et al. (2024); Shaklai et al. (2021); Hochberg et al. (2019), a user was predicted positive if they made at least one query that included the terms “I have [condition]” or “diagnosed with [condition]”, where [condition] is one of the 18 specific cancer types. We excluded queries where the following phrases were included: “how do I know if I have [condition]”, “do I have [condition]”, “I think I have [condition]”, “did I have [condition]” or the words “sister”, “brother”, “wife”, “husband”, “father”, “mother”, “dog”, “cat”.

For each classifier, we wish to discover a best-case accuracy-fairness trade-off, so as to identify classifiers that might be useful for a future, more detailed study. We note that unfairness may ensue using this classification method due to differences in health literacy between states. Out of the 18 cancer types, classifiers for 8 types had an overall positive prediction rate of more than twice or less than half of the true positive rate. We removed these classifiers from the experiment, since they were clearly inadequate and further analysis was not needed. For the remaining 10 classifiers, we ran Alg. 2 for values of $\beta \in [0,1]$.

We started with 10 evenly-spaced values for $\beta$, and added more values where the changes with respect to $\beta$ were large. The pairs of (unfairness, error) that minimized $\text{disc}_\beta$ for each of the values of $\beta$ for each classifier (cancer type) define a Pareto curve, representing the best-possible combinations for these classifiers. Figure 8 plots these curves. To make the
Figure 8: Pareto curves, cancer by search queries.

Figure 9: Pareto curves, pre-election polls.

Figure 10: Pareto curves, cancer diagnosis and mortality.
values of different cancer types comparable in the plot, all the values for a given classifier are normalized in the plot by the overall occurrence rate of the cancer type it predicts. Thus, values close to 1 indicate a poor classifier. Recall that we do not have validation data these classifiers, thus we cannot compare to their true unfairness and error values. However, since \( \text{mindisc}_\beta \) is a lower bound on \( \text{disc}_\beta \), the true (unfairness, error) pair of each classifier is necessarily above its Pareto curve. We concluded from the results in Figure 8 that the classifier for lung cancer is the most promising for a future study, while many of the other classifiers are guaranteed to perform poorly on both measures.

The numerical values that generated the Pareto-curves in Figure 8 can further be used to answer many questions of the forms listed in Section 8. We demonstrate with a few examples below, based on the numerical results available at the linked github repository. All unfairness and error values below are normalized by the overall prevalence of true positives.

- “Is it possible that this classifier is fair? If not, how unfair must it be based on the provided label proportions?” This question can be answered by checking the leftmost point (minimal unfairness, \( \beta = 1 \)) in the Pareto-curve. This is the minimal possible value of unfairness for this classifier, regardless of its true accuracy. For instance, the Breast Cancer classifier has a minimal possible unfairness value of 12%, and the Stomach Cancer classifier has a minimal possible unfairness value of 35%.

- “Suppose that the classifier is known to be nearly fair. How accurate can it be?” If it is known that the classifier’s unfairness is smaller than some value, then the minimal possible error of the classifier is the smallest value of the error lower bound in the part of the Pareto curve that is to the left of this unfairness value. For instance, if the Thyroid Cancer classifier has unfairness \( \leq \%18 \), then its minimal possible error value is 35%, while for the Bladder Cancer classifier its minimal possible error value is 23%.

- “Suppose that the classifier is known to be quite accurate. How fair can it be?” If it is known that the classifier’s error is smaller than some value, then the minimal possible unfairness of the classifier is the smallest value of the unfairness lower bound in the part of the Pareto curve that is below this error value. For instance, if the Thyroid Cancer classifier has error \( \leq 35\% \), then its unfairness is at least 18%. For the Skin Cancer classifier, under the same upper bound on error, its unfairness is at least 28%.

For the last question, “Suppose that there is a penalty for each person who gets a wrong prediction, and for each person who is treated unfairly; what is the smallest possible overall cost of this classifier?”, under any given trade-off between the cost of unfairness and the cost of error, the lowest possible cost of the classifier is \( \text{mindisc}_\beta \), where \( \beta \) is set to the desired trade-off between the two quantities. For instance, setting \( \beta = 3/4 \) assigns unfairness to have triple the cost of error. Under this setting, the cost of the Breast Cancer classifier is \( 0.13 \cdot 0.75 + 0.53 \cdot 0.25 = 0.23 \) and the cost of the Stomach Cancer classifier is \( 0.35 \cdot 0.75 + 0.62 \cdot 0.25 = 0.42 \).

In the next experiment of this section, we study pre-election polls, and use them to provide the value of \( \text{mindisc}_\beta \) for the classifiers that they might represent. Statistics on
10 pre-election polls of the 2016 US Presidential elections were obtained from Five Thirty Eight, 2016. The label was set to 1 if the individual voted for the Democratic candidate and to 0 otherwise. Each poll predicted a voting rate for the candidate in each state. The true positive rate was obtained from the actual results of the presidential elections in each state (Federal Elections Commission, 2016). Treating each poll prediction as the aggregate statistics of an unknown classifier, we calculated the Pareto curves representing the best-possible combinations of unfairness and error for these classifiers, using Alg. 2 and values of $\beta \in [0.01, 0.99]$, as in the experiment described above. The resulting Pareto curves, shown in Figure 9 by date of poll publication, can be used to compare the classifiers that could be underlying the different polls. For instance, although some of the more accurate polls are also the latest ones, this is not always the case. Moreover, some pairs of polls are incomparable, since they perform better in different parts of the curve. For instance, the Nov-06(b) poll has a lower error bound than the Nov-06(a) poll if its unfairness is larger than 0.05, but it cannot obtain unfairness smaller than 0.05, while the other poll can, hinting perhaps at a more biased polling methodology. This information can be used to further analyze the polling and prediction strategies employed in the various polls.

In the last experiment, we use the proposed method to explore the variation in cancer mortality rates across US states. Rates of cancer diagnosis and mortality in each US state, for 10 cancer types, were taken from the data published in CDC and NCI, 2019. The true occurrence rates $\{\pi_g\}_{g \in G}$ for each cancer type were set to the fraction of people in each state who died from the given cancer. We generated aggregate statistics for a classifier that would predict mortality of a diagnosed individual in each state with a probability equal to the overall mortality rate, which is the ratio between the overall mortality and the overall diagnosis rate. Thus, this classifier is based on the premise that in each state, the mortality rate is the same. Any deviation from this premise would mean that the classifier cannot be fully accurate or fair. In this context, a high unfairness value may be interpreted as a large fraction of individuals who have a non-typical manifestation of the disease or are treated differently by the health care system, while a high error may indicate a large fraction of the population whose mortality differs from the expected mortality, perhaps due to a difference in access to health services such as screening and care. The Pareto curves that we calculated for these classifiers are shown in Figure 10; values for each cancer type are normalized by the occurrence rate of that cancer. It can be observed, for instance, that in several of the cancer types, allowing a small amount of unfairness, interpreted as modeling a fraction of the population as having non-typical variations of the disease, leads to a significantly smaller error, interpreted as a small difference in mortality rate between the states on the population with the typical variation of the disease. This use of our method sheds light on the diversity of possibilities of using such a tool in various ways for studying real-world phenomena.

We note that conclusions on unfairness can be combined with other available information to provide possible explanations for the unfairness. For instance, consider the value of the unfairness lower bound when minimizing the error. This is the rightmost point in the Pareto curve of the classifier. Assuming that the classifier is relatively accurate, this value can serve as a possible indication of its true unfairness. For the cancer diagnosis classifiers

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2. In cases where data from some states was missing, we removed these states from the list of regions and renormalized the weights of the other states.
that were based on search queries (Figure 8), we compared this unfairness value to the average cost of care for each of the cancer types, as taken from American Cancer Society (2019). As Figure 11 shows, there is a strong correlation \( R = 0.5 \) between the cost of care and the unfairness value of the classifier. This may indicate that there are large differences in awareness and treatment between states when the cost of care is higher. As another example, we compared the unfairness of the cancer mortality classifiers (Figure 10), with the 5-year relative survival rate for each type of cancer, as taken from American Cancer Society (2019). As Figure 12 shows, there is a very strong correlation \( R = 0.79 \) between the survival rate and the unfairness of the mortality predictor. We speculate that this is because in cancers with higher survival rates, the influence of quality treatment on survival is significant. These two examples demonstrate how unfairness analysis may be used to inform exploratory studies of various social phenomena.

![Figure 11: The correlation between the (normalized) unfairness of the search-query cancer classifier and the average cost of care for the cancer type.](image)

9.1.3 Minimizing the discrepancy when there are only two sub-populations

In this section, we discuss and demonstrate two phenomena that can arise when minimizing the discrepancy for binary classification over distributions in which all or almost all of the probability mass is concentrated on only two sub-populations. These phenomena can sometimes limit the usefulness of lower-bounding the discrepancy in these cases, since the resulting lower bound may be significantly lower than the true unfairness. This is a direct result of the small number of meaningful aggregate statistics that are available in this case. Nonetheless, any obtained lower bounds are still valid.

As discussed in Section 5, a fair classifier has a confusion matrix \( A \) which is the same for each of the sub-populations. A given set of Inputs statistics is consistent with a fair classifier if and only if there exists a matrix \( A \in M_\Delta \) such that \( A^T \pi_g = \hat{p}_g \) for each \( g \in G \). In the case of binary classification, this describes two linear constraints for each \( g \in G \). However, since \( A \in M_\Delta, \pi_g, \hat{p}_g \in \Delta \), it can be easily verified that these two constraints are equivalent. Thus, in the case of binary classification, each sub-population induces a single
Figure 12: The correlation between the (normalized) unfairness of the mortality prediction classifier and the survival rate of the cancer type.

linear constraint. $A$ has two degrees of freedoms. Therefore, if there are only two sub-populations, this problem is always feasible, unless the solution is outside of the legal range of $[0, 1]$. This means that in general, in problems in which almost all of the probability mass is concentrated on two sub-populations, it should be unsurprising if a classifier’s minimal unfairness based on Inputs is found to be zero or close to zero.

A second phenomenon that can sometimes be observed in data sets with only two sub-populations, is the possible irrelevance of the value of $\beta$ when minimizing $\text{mindisc}_\beta$. In cases where the same confusion matrix minimizes both the unfairness and the error, $\text{mindisc}_\beta$ would have the same solution regardless of $\beta$. This can be the case, for instance, if in all sub-populations, the frequency of positive predictions is larger than the frequency of true positives, or vice versa. Formally, consider a case of Inputs in which the following holds:

$$\forall g \in G, \hat{p}^1_g \geq \pi^1_g \quad \text{or} \quad \forall g \in G, \hat{p}^1_g \leq \pi^1_g.$$  \hspace{1cm} (33)

Note that Eq. (33) is much more likely when there is a small number of possible sub-populations. Consider a classifier that satisfies the former inequality of Eq. (33); the latter inequality is symmetric. In this case, the minimal error for the given Inputs is obtained by assuming that there are no false negatives in either of the sub-populations. More specifically, the error is minimized when assuming the following confusion matrix for each sub-population $g \in G$:

$$A_g = \begin{pmatrix} \frac{1-\pi^1_g}{\hat{p}^1_g} & \frac{\pi^1_g - \hat{p}^1_g}{\hat{p}^1_g} \\ \frac{\pi^0_g}{\hat{p}^1_g} & 1 \end{pmatrix}.$$  

Note that in this solution, the positive label does not induce any unfairness, since the second row is the same for all sub-populations. This may be the reason that when Eq. (33) holds, it is common to have no effect of $\beta$ on the minimizing solution for $\text{mindisc}_\beta$.

We demonstrate the two phenomena described above with two data sets that have been widely used in the fairness literature. The first is the UCI Adult data set (Becker and
Table 2: Values of Inputs parameters for the four tested classifiers. The table uses the following shorthand: AIE for Amer-Indian-Eskimo, API for Asian-Pac-Islander, Afr-Am for African-American, Cauc. for Caucasian, log-reg for logistic regression. The last column provides an experiment number, which is used in the tables below to refer to a specific experiment. It can be seen that in experiments I, II, III, V, $\forall g \in \mathcal{G}, \hat{p}_g^1 \leq \pi_g^1$. In contrast, this is not the case in experiments IV, VI. This distinction is relevant to the effect of $\beta$ on the solution for $\text{mindisc}_\beta$, as we discuss in Section 9.1.3.

| Data set | Classifier | Protected Attribute | Sub-population | $w_g$ | $\pi_g^1$ | $\hat{p}_g^1$ | #Exp |
|----------|------------|---------------------|----------------|-------|---------|---------|------|
| Adult    | vanilla $k$-NN | Sex                  | Female         | 33.30%| 10.88%  | 8.45%  | I    |
|          |            | Male                 | 66.70%         | 29.98%| 27.21%  |         |      |
|          |            | Race                 | AIE            | 0.98% | 11.95%  | 6.29%  |      |
|          |            |                      | API            | 2.95% | 27.71%  | 23.33% | II   |
|          |            |                      | Black          | 9.59% | 11.47%  | 9.48%  |      |
|          |            |                      | Other          | 0.83% | 18.52%  | 7.41%  |      |
|          |            |                      | White          | 85.66%| 25.03%  | 22.47% |      |
|          | weighted $k$-NN | Sex                 | Female         | 33.30%| 10.88%  | 11.20% | III  |
|          |            | Male                 | 66.70%         | 29.98%| 35.11%  |         |      |
|          |            | Race                 | AIE            | 0.98% | 11.95%  | 7.55%  |      |
|          |            |                      | API            | 2.95% | 27.71%  | 27.92% |      |
|          |            |                      | Black          | 9.59% | 11.47%  | 13.45% | IV   |
|          |            |                      | Other          | 0.83% | 18.52%  | 14.81% |      |
|          |            |                      | White          | 85.66%| 25.03%  | 29.00% |      |
| COMPAS   | vanilla log-reg | Race                | Afr-Am         | 59.01%| 54.57%  | 51.56% | V    |
|          |            | Cauc.                | 40.99%         | 37.96%| 28.15%  |         |      |
|          | weighted log-reg | Race                | Afr-Am         | 59.01%| 54.57%  | 57.30% | VI   |
|          |            | Cauc.                | 40.99%         | 37.96%| 32.91%  |         |      |

Kohavi, 1996). This data set is used in many algorithmic fairness works (see, e.g. Calders et al., 2009; Turgut, 2023; Kumar et al., 2023). It includes data on over 48,000 individuals. For each individual, 14 demographic features are provided. The goal is to predict whether an individual’s income is below or above $50,000. The protected attribute can be set to the sex or the race of the individual. The sex attribute has two possible values, matching the discussion on distributions with only two-sub-populations. The race attribute has 5 possible values. However, more than 96% of the examples have one of only two races, Black and White. Thus, this is also a relevant example for this discussion.

The second data set is the Propublica COMPAS Recidivism data set (ProPublica, 2016), also used in many fairness works (see, e.g., Zafar et al., 2017b; Rudin et al. 2020). This data set includes demographic and historical data on over 5,000 defendants from Broward County, Florida, from 2013 and 2014. The label indicates whether these individual have re-offended within the next two years. The protected attribute in this data set is race, and only two races are represented in the data: African-American and Caucasian.
Table 3: Results of unfairness and error minimization for the experiments in Table 2. The last column indicates whether the value of $\beta$ when calculating $\text{mindisc}_\beta$ has any effect on the resulting minimizing solution.

| #Exp | unfairness (mindisc$_1$) | Min. unfairness (mindisc$_0$) | error | Min. error | Eq. (33) holds? | $\beta$ affects solution? |
|------|--------------------------|-------------------------------|-------|------------|-----------------|--------------------------|
| I    | 5.10%                    | 0.52%                         | 16.18%| 2.66%      | Yes             | No                       |
| II   | 4.79%                    | 0.27%                         | 16.18%| 2.66%      | Yes             | No                       |
| III  | 7.49%                    | 2.07%                         | 17.33%| 3.53%      | Yes             | Yes                      |
| IV   | 6.00%                    | 0.52%                         | 17.33%| 3.67%      | No              | Yes                      |
| V    | 10.44%                   | 3.34%                         | 33.93%| 5.80%      | Yes             | No                       |
| VI   | 10.91%                   | 3.59%                         | 33.98%| 3.67%      | No              | Yes                      |

Table 4: For those experiments in Table 3 in which $\beta$ affects the solution, this table provides the values of the found solutions and the ranges of $\beta$ in which each value was obtained.

| #Exp | $\beta$ range | solution unfairness | solution error |
|------|---------------|---------------------|----------------|
| III  | 0 – 0.78      | 3.27%               | 3.53%          |
|      | 0.79 – 1.0    | 2.07%               | 7.66%          |
| IV   | 0.01 – 0.05   | 3.67%               | 3.67%          |
|      | 0.06 – 0.1    | 3.47%               | 3.68%          |
|      | 0.06 – 0.32   | 2.16%               | 3.83%          |
|      | 0.33 – 1.0    | 0.52%               | 4.57%          |
| VI   | 0 – 0.98      | 3.67%               | 3.67%          |
|      | 0.99 – 1.00   | 3.59%               | 6.72%          |

We generated two classifiers for each data set. For each data set, the first classifier was a simple classifier, which turned out to satisfy Eq. (33). To obtain also classifiers in which Eq. (33) does not hold, we generated a second classifier for each data set, in which the positive label was assigned a higher weight than its default weight based on the training data set. We report in Table 2 the values of Inputs for each valid combination of data set, classifier, and protected attribute.

The first classifier for the Adult data set was a vanilla $k$-Nearest-Neighbors classifier with 9 neighbors and a normalized Euclidean distance, as implemented by Matlab’s fitcknn procedure. The first classifier for the COMPAS data set was a vanilla logistic regression classifier, as implemented by python’s sklearn package. It can be seen in Table 2 that in both data sets, the vanilla classifier (#Exp I, II, V) satisfies Eq. (33). For the Adult data set, a weighted variant of the $k$-Nearest-Neighbor classifier was generated by initializing the classifier to use a different prior distribution of the labels, assigning 30% to positive labels and 70% to negative labels. For the COMPAS data set, a weighted logistic regression classifier was generated by setting the weight ratio of the positive labels to negative labels to 1.1. This has led to two classifiers that did not satisfy Eq. (33) (#Exp IV, VI) and one that did (#Exp III).
Table 3 reports the values of $\text{mindisc}_\beta$ that correspond to the minimal unfairness value and the minimal error value for each of the classifiers given Inputs, using Alg. 2. These values are compared to the true unfairness and error, reported in the same table. It can be seen that indeed, in some cases (#Exp I,II,IV) the minimal unfairness is close to zero although the true unfairness is not, demonstrating the first phenomenon discussed above. Nonetheless, note that in the other experiments, the ratio between the true unfairness and the lower bound is non-trivial, ranging between 3.1 and 3.6. Thus, in these cases minimizing the discrepancy is more informative.

The second phenomenon, in which the value of $\beta$ does not affect the solution to $\text{mindisc}_\beta$, can also be observed in the last two columns of Table 3. It can be seen that $\beta$ has no effect in 3 out of 4 experiments in which Eq. (33) holds, but does have an effect in all of the experiments in which Eq. (33) does not hold. For the experiments in which $\beta$ had an effect on the solution, Table 4 reports the ranges of $\beta$ that led to each of the identified solutions to $\text{mindisc}_\beta$. The small number of solutions across the range of possible $\beta$ is again related to the small number of sub-populations in these experiments, which results in a small number of meaningful constraints on the minimization problem.

We conclude that in general, while calculating $\text{mindisc}_\beta$ is always possible, its value may be less informative in some distributions that are concentrated on two sub-populations.

9.2 Multiclass classification

In this section we report experiments on multiclass classifiers. In Section 9.2.1, we report experiments in which we calculate a lower bound and an upper bound on the unfairness of a classifier when confusion matrices are available, using the algorithm proposed in Section 7.2. In Section 9.2.2, we report experiments in which we derive an upper bound on $\text{mindisc}$ using the local optimization algorithm provided in Section 8.2.

9.2.1 Bounding the unfairness for classifiers with known confusion matrices

In this section, we report the results of calculating an upper bound and a lower bound on the unfairness of a multiclass classifier when the confusion matrices in each sub-population are available; for instance, they can be calculated based on a validation set.

In Section 7.2, a simple approach for calculating a lower bound on unfairness was provided, as well as a two-stage approach for calculating an upper bound. In the experiments below, we compared our proposed approach to other approaches for finding an assignment for the upper bound. In total, we tested the following four approaches:

- Average: the weighted average of the confusion matrices of each region.
- Greedy: the greedy procedure proposed in Section 7.2.1.
- Average+LM: Initializing with the weighted average, then running the local optimization procedure of Section 7.2.2.
- Greedy+LM: Initializing with the greedy procedure of Section 7.2.1, then running the local optimization procedure of Section 7.2.2.
The last approach is the one that we propose to use; Our results below show that it indeed performs the best. We also report the ratio between the best upper bound and the lower bound. A ratio close to 1 indicates that in this case, the bounds are tight.

**US Census data set.** We used the US Census data set (Dua and Graff, 2019) to generate multiclass classifiers. We generated classifiers to predict each of the attributes that had between 3 and 10 values. We generated two types of classifiers for each target attribute: a decision tree classifier and a nearest neighbor classifier using standard Matlab libraries. The fairness attribute was the state of work, as in the binary classification experiments. Table 5 and Table 6 list the fairness lower bound calculated for each of these classifiers, as well as the upper bound obtained by each of the tested methods, and the ratio between the smallest upper bound and the lower bound. In some of the experiments the lower bound is equal to the upper bound, giving the exact value of the unfairness of this classifier. In all of the experiments, our proposed approach provides the tightest upper bound. The ratio between the upper bound and the lower bound ranges between 1.00 and 2.53, showing that for these classifiers the unfairness can be estimated quite well.

**Natality data set.** In this experiment we used data about births in the United States (CDC, 2017), which provides detailed information about each birth that occurred during 2017. The data set includes approximately 3.8 million data points and 50 attributes. Using only attributes that are known before the labor, we generated two classifiers that attempt to predict the type of labor out of the five possible options (e.g., spontaneous, Cesarean): a decision tree classifier and a $k$-Nearest Neighbor classifier. We note that the error of these classifiers was high; Nonetheless, our goal is to study their fairness. We studied the fairness of the classifier with respect to each of the following attributes: the type of labor attendant (e.g., midwife, doctor), the race of the father, the race of the mother, and the type of service payer (e.g., Medicare, private). Table 7 and Table 8 report the unfairness lower bound and upper bounds for each of these fairness attributes. Here too, our proposed approach provides the tightest upper bounds. The ratio between the upper bound and the lower bound ranges between 1.09 and 1.58, showing that for these classifiers as well, the unfairness can be estimated quite well.

9.2.2 Upper bounding the minimal discrepancy with unknown confusion matrices

In our last set of experiments, we calculate an upper bound on $\text{mindisc}_1$ (the minimal possible fairness) given only aggregate statistics on label proportions. If this upper bound is smaller than a threshold, then it can be inferred that the label proportions statistics of this classifier cannot prove that the classifier is more unfair than this threshold. Cases in which the upper bound is large may require problem-specific analysis.

In the first two sets of experiments, we calculated the upper bound on $\text{mindisc}_1$ for the same labeled data sets studied in Section 9.2.1, and compared it to the fairness range that was calculated for this classifier in Section 9.2.1 based on the labeled data. Table 9 provides results for the US Census data set, and Table 10 provides results for the Natality data set. Except for one case (6 labels in Table 9), in all other cases the $\text{mindisc}_1$ upper bound was lower than the true unfairness range of the classifier.
Table 5: Bounding the unfairness for multiclass classifiers with known confusion matrices; US Census data set with decision tree classifiers

| # Labels | Error Bound | Lower | Upper Bounds | Best Bound | Ratio |
|----------|-------------|-------|--------------|------------|-------|
|          |             | Bound | Average Greedy | Average+LM | Greedy+LM | Ratio |
| 3        | 12.15%      | 5.64% | 29.14% 9.26% | 15.13%     | 8.27%   | 1.47 |
| 3        | 5.39%       | 4.28% | 33.44% 7.35% | 17.47%     | 5.47%   | 1.28 |
| 3        | 4.03%       | 3.84% | 39.16% 5.92% | 20.43%     | 3.84%   | 1.00 |
| 3        | 4.85%       | 4.71% | 35.83% 6.44% | 12.85%     | 4.74%   | 1.01 |
| 3        | 3.32%       | 3.32% | 47.23% 5.25% | 23.51%     | 3.32%   | 1.00 |
| 3        | 1.63%       | 1.63% | 57.34% 6.95% | 13.45%     | 1.63%   | 1.00 |
| 3        | 1.70%       | 1.69% | 51.85% 6.75% | 17.53%     | 1.70%   | 1.00 |
| 3        | 1.94%       | 1.94% | 59.66% 4.52% | 6.61%      | 1.94%   | 1.00 |
| 3        | 14.11%      | 6.01% | 29.91% 10.00% | 12.02% | 8.09% | 1.35 |
| 4        | 3.81%       | 2.29% | 16.97% 10.34% | 4.04%      | 2.29%   | 1.00 |
| 5        | 5.78%       | 2.28% | 12.33% 42.96% | 5.32%      | 3.95%   | 1.73 |
| 5        | 11.51%      | 4.29% | 29.05% 32.91% | 22.88%     | 7.47%   | 1.74 |
| 6        | 0.77%       | 0.77% | 19.66% 28.34% | 14.15%     | 0.77%   | 1.00 |
| 8        | 22.36%      | 8.87% | 82.06% 40.07% | 29.24%     | 19.74%  | 2.22 |
| 9        | 22.07%      | 6.43% | 79.64% 15.13% | 27.63%     | 8.12%   | 1.26 |

Table 6: Bounding the unfairness for multiclass classifiers with known confusion matrices; US Census data set with nearest neighbor classifiers

| # Labels | Error Bound | Lower | Upper Bounds | Best Bound | Ratio |
|----------|-------------|-------|--------------|------------|-------|
|          |             | Bound | Average Greedy | Average+LM | Greedy+LM | Ratio |
| 3        | 27.61%      | 6.08% | 18.67% 12.27% | 11.28%     | 10.18%   | 1.67 |
| 3        | 28.66%      | 5.76% | 17.04% 10.57% | 10.44%     | 9.28%   | 1.61 |
| 3        | 27.15%      | 6.71% | 19.46% 11.49% | 12.04%     | 10.68%   | 1.59 |
| 3        | 24.80%      | 6.46% | 19.06% 10.71% | 10.38%     | 8.24%   | 1.28 |
| 3        | 24.81%      | 6.30% | 19.51% 11.28% | 12.38%     | 10.08%   | 1.60 |
| 3        | 23.93%      | 6.51% | 19.36% 10.56% | 11.49%     | 8.95%   | 1.38 |
| 3        | 23.75%      | 6.33% | 19.62% 10.40% | 12.01%     | 9.33%   | 1.47 |
| 3        | 23.84%      | 6.63% | 18.77% 10.58% | 11.35%     | 9.49%   | 1.43 |
| 3        | 19.51%      | 5.39% | 27.54% 10.11% | 11.86%     | 6.98%   | 1.29 |
| 4        | 27.33%      | 6.59% | 56.32% 10.87% | 30.02%     | 9.75%   | 1.48 |
| 5        | 14.09%      | 8.95% | 72.84% 13.10% | 33.99%     | 9.57%   | 1.07 |
| 5        | 18.66%      | 10.98% | 64.83% 14.09% | 30.61%     | 13.86%   | 1.26 |
| 5        | 49.69%      | 9.24% | 44.65% 20.90% | 28.76%     | 18.73%   | 2.03 |
| 5        | 53.58%      | 8.64% | 38.75% 24.09% | 27.54%     | 19.99%   | 2.31 |
| 6        | 51.59%      | 10.25% | 54.57% 24.56% | 34.50%     | 20.98%   | 2.05 |
| 8        | 42.84%      | 9.66% | 69.98% 39.24% | 46.36%     | 24.47%   | 2.53 |
| 9        | 39.41%      | 10.30% | 93.53% 22.68% | 34.42%     | 17.35%   | 1.69 |
Lastly, we provide results for two experiments where we do not have ground truth labels to compare to. These results demonstrate how this method can be used for studying various empirical questions.

In the first experiment, we studied the general elections in the UK from 1918 to 2019 (Watson et al., 2020). Here, we studied the changes in voting patterns between elections by studying the \( \text{mindisc}_1 \) value of a hypothetical classifier that would predict the vote of individuals in one general elections to be the same as their vote in the previous general elections (neglecting the change in population between elections). We studied fairness with respect to the geographic region of the voters (e.g., London, West Midlands, Scotland), as reported in the data set. A high unfairness value of such a classifier would indicate a possibly large difference between voting pattern changes across regions. We studied such hypothetical classifiers between each two consecutive elections in the data set. The results are reported in Table 11. Figure 13, which portrays the value of \( \text{mindisc}_1 \) by election year, shows clear trends in different periods of the last century.

In the second experiment, we studied a data set on US education (USDA Economic Research Service, 2021), which provides the percentage of various levels of education attainment (e.g., high school, college) in each US state in each decade. Here too, we calculated \( \text{mindisc}_1 \) for a hypothesized classifier that predicts the education level to be distributed the same in each state in each decade. Table 12 provides our results. Here, we found no significant differences in the fairness of change patterns in different decades.

10. Discussion

In this work, we showed that aggregate statistics on classifiers can be used to draw useful conclusions about the fairness and the accuracy of the classifiers. We defined the new unfairness measure, which facilitates a quantifiable study of classifiers that are not completely fair, and provided several procedures for bounding unfairness, error and the trade-offs between them in binary and multiclass classification. Our experiments demonstrate how such analyses can be useful in various areas of study. In addition, such analyses can help in classifier design, when individual labeled data is not available.

As classifiers become more abundant and influential in our daily lives, studying their properties from limited information becomes more and more important. This work shows that this is possible and can provide meaningful information. We expect that future work will expand on this vision and extend this approach to other performance metrics and other types of classification tasks.
Table 7: Bounding the unfairness for multiclass classifiers with known confusion matrices; Natality data set with a decision tree classifiers. Classifier error: 30.8%.

| Test      | Lower Bound | Upper Bounds | Best Ratio |
|-----------|-------------|--------------|------------|
|           | Average     | Greedy       | Average+LM | Greedy+LM   |
| Attendant | 1.91%       | 2.34%        | 2.10%      | 2.08%       | 1.09        |
| Father Race | 0.92%    | 1.50%        | 1.32%      | 1.28%       | 1.40        |
| Mother Race | 0.65%    | 1.31%        | 1.14%      | 1.12%       | 1.71        |
| Payer     | 1.74%       | 1.96%        | 1.97%      | 1.89%       | 1.09        |

Table 8: Bounding the unfairness for multiclass classifiers with known confusion matrices; Natality data set with a k-Nearest-Neighbor classifier. Classifier error: 24.26%.

| Test      | Lower Bound | Upper Bounds | Best Ratio |
|-----------|-------------|--------------|------------|
|           | Average     | Greedy       | Average+LM | Greedy+LM   |
| Attendant | 1.80%       | 28.53%       | 1.82%      | 1.82%       | 1.01        |
| Father Race | 1.17%    | 46.54%       | 1.18%      | 1.18%       | 1.01        |
| Mother Race | 0.61%    | 21.47%       | 0.62%      | 0.62%       | 1.02        |
| Payer     | 1.73%       | 54.61%       | 1.76%      | 1.75%       | 1.01        |

Table 9: Comparing mindisc1 upper bound (UB) to the unfairness range calculated for US Census classifiers. The ranges are derived from the results of Table 5 and Table 6.

| # Labels | Decision tree | Nearest Neighbor |
|----------|---------------|------------------|
|          | mindisc1 UB   | true unfairness  | mindisc1 UB | true unfairness  |
| 3        | 1.64%         | 5.64% – 8.27%    | 2.70%       | 6.08% – 10.18%   |
| 3        | 1.01%         | 4.28% – 5.47%    | 2.97%       | 5.76% – 9.28%    |
| 3        | 0.92%         | 3.84%            | 2.84%       | 6.71% – 10.68%   |
| 3        | 1.05%         | 4.71% – 4.74%    | 2.78%       | 6.46% – 8.24%    |
| 3        | 0.76%         | 3.32%            | 2.40%       | 6.30% – 10.08%   |
| 3        | 0.46%         | 1.63%            | 2.74%       | 6.51% – 8.95%    |
| 3        | 0.70%         | 1.69% – 1.70%    | 2.50%       | 6.33% – 9.33%    |
| 3        | 0.58%         | 1.94%            | 2.24%       | 6.63% – 9.49%    |
| 3        | 2.04%         | 6.01% – 8.09%    | 2.32%       | 5.39% – 6.98%    |
| 4        | 2.17%         | 2.29%            | 3.74%       | 3.60% – 9.75%    |
| 5        | 1.79%         | 2.28% – 3.95%    | 2.82%       | 8.95% – 9.57%    |
| 5        | 1.81%         | 4.29% – 7.47%    | 3.73%       | 10.98% – 13.86%  |
| 5        | –             | –                | 6.10%       | 9.24% – 18.73%   |
| 5        | –             | –                | 5.48%       | 8.64% – 19.99%   |
| 6        | 3.20%         | 0.77%            | 6.80%       | 10.25% – 20.98%  |
| 8        | 4.24%         | 8.4% – 19.74%    | 5.91%       | 9.66% – 24.47%   |
| 9        | 3.15%         | 6.43% – 8.12%    | 5.92%       | 10.30% – 17.35%  |
Table 10: Comparing $\text{mindisc}_1$ upper bound (UB) to the unfairness ranges calculated for Natality classifiers. The ranges are derived from the results of Table 7 and Table 8.

| Test          | Decision tree | $\text{mindisc}_1$ UB | true unfairness | $\text{mindisc}_1$ UB | true unfairness |
|---------------|---------------|------------------------|-----------------|------------------------|-----------------|
| Attendant     | 0.75%         | 1.91% - 2.08%          | 0.74%           | 1.80% - 1.82%         |
| Father Race   | 0.26%         | 0.92% - 1.28%          | 0.32%           | 1.17% - 1.18%         |
| Mother Race   | 0.12%         | 0.65% - 1.12%          | 0.14%           | 0.61% - 0.62%         |
| Payer         | 0.05%         | 1.74% - 1.89%          | 0.59%           | 1.73% - 1.75%         |

Table 11: Calculated $\text{mindisc}_1$ upper bounds for the UK elections data.

| Elections         |          | # regions | $\text{mindisc}_1$ upper bound |
|-------------------|----------|-----------|---------------------------------|
| 1918               | 1922     | 12        | 6.36%                           |
| 1922               | 1923     | 13        | 8.57%                           |
| 1923               | 1924     | 13        | 3.26%                           |
| 1924               | 1929     | 13        | 4.92%                           |
| 1929               | 1931     | 13        | 5.14%                           |
| 1931               | 1935     | 13        | 3.11%                           |
| 1935               | 1945     | 13        | 5.69%                           |
| 1945               | 1950     | 12        | 4.75%                           |
| 1950               | 1951     | 12        | 3.79%                           |
| 1951               | 1955     | 12        | 5.54%                           |
| 1955               | 1959     | 12        | 2.40%                           |
| 1959               | 1964     | 12        | 2.16%                           |
| 1964               | 1966     | 12        | 1.37%                           |
| 1966               | 1970     | 12        | 1.32%                           |
| 1970               | 1974 (Feb) | 12       | 2.69%                           |
| 1974 (Feb)         | 1974 (Oct) | 12      | 6.28%                           |
| 1974 (Oct)         | 1979     | 12        | 3.67%                           |
| 1979               | 1983     | 11        | 3.23%                           |
| 1983               | 1987     | 12        | 1.43%                           |
| 1987               | 1992     | 12        | 4.21%                           |
| 1992               | 1997     | 11        | 5.25%                           |
| 1997               | 2001     | 12        | 5.25%                           |
| 2001               | 2005     | 12        | 4.83%                           |
| 2005               | 2010     | 12        | 5.26%                           |
| 2010               | 2015     | 12        | 3.96%                           |
| 2015               | 2017     | 12        | 5.54%                           |
| 2017               | 2019     | 11        | 5.24%                           |
Figure 13: The value of $\text{mindisc}_1$ as a function of the year of the predicted elections.

Table 12: Calculated $\text{mindisc}_1$ upper bounds for the US education data set.

| Year       | Baseline | Predicted | $\text{mindisc}$ upper bound |
|------------|----------|-----------|-----------------------------|
| 1970       | 1980     |           | 2.38%                       |
| 1980       | 1990     |           | 2.94%                       |
| 1990       | 2000     |           | 2.22%                       |
| 2000       | 2015-2019|           | 2.32%                       |
References

W. Alghamdi, S. Asoodeh, H. Wang, F. P. Calmon, D. Wei, and K. N. Ramamurthy. Model projection: Theory and applications to fair machine learning. In 2020 IEEE International Symposium on Information Theory (ISIT), pages 2711–2716, 2020. doi: 10.1109/ISIT44484.2020.9173988.

American Cancer Society. Cancer facts and statistics, table 8, 2019. URL https://www.cancer.org/content/dam/cancer-org/research/cancer-facts-and-statistics/annual-cancer-facts-and-figures/2019/cancer-facts-and-figures-2019.pdf.

Solon Barocas, Moritz Hardt, and Arvind Narayanan. Fairness in machine learning. NIPS Tutorial, 2017.

Barry Becker and Ronny Kohavi. Adult. UCI Machine Learning Repository, 1996. DOI: https://doi.org/10.24432/C5XW20.

Rachel KE Bellamy, Kuntal Dey, Michael Hind, Samuel C Hoffman, Stephanie Houde, Kalapriya Kannan, Pranay Lohia, Jacquelyn Martino, Sameep Mehta, A Mojsilović, et al. Ai fairness 360: An extensible toolkit for detecting and mitigating algorithmic bias. IBM Journal of Research and Development, 63(4/5):4–1, 2019.

Richard Berk, Hoda Heidari, Shahin Jabbari, Michael Kearns, and Aaron Roth. Fairness in criminal justice risk assessments: The state of the art. Sociological Methods & Research, page 0049124118782533, 2018.

Emily Black, Samuel Yeom, and Matt Fredrikson. Fliptest: Fairness auditing via optimal transport. CoRR, abs/1906.09218, 2019. URL http://arxiv.org/abs/1906.09218.

Emily Black, Hadi Elzayn, Alexandra Chouldechova, Jacob Goldin, and Daniel Ho. Algorithmic fairness and vertical equity: Income fairness with irs tax audit models. In Proceedings of the 2022 ACM Conference on Fairness, Accountability, and Transparency, FAccT ’22, page 1479–1503, New York, NY, USA, 2022. Association for Computing Machinery. ISBN 9781450393522.

Toon Calders, Faisal Kamiran, and Mykola Pechenizkiy. Building classifiers with independence constraints. In 2009 IEEE international conference on data mining workshops, pages 13–18. IEEE, 2009.

Flavio Calmon, Dennis Wei, Bhanu Vinzamuri, Karthikeyan Natesan Ramamurthy, and Kush R Varshney. Optimized pre-processing for discrimination prevention. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems 30, pages 3992–4001. Curran Associates, Inc., 2017a. URL http://papers.nips.cc/paper/6988-optimized-pre-processing-for-discrimination-prevention.pdf.

Flavio Calmon, Dennis Wei, Bhanukiran Vinzamuri, Karthikeyan Natesan Ramamurthy, and Kush R Varshney. Optimized pre-processing for discrimination prevention. In
I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, *Advances in Neural Information Processing Systems 30*, pages 3992–4001. Curran Associates, Inc., 2017b.

CDC. United states natality public use file, 2017. URL https://ftp.cdc.gov/pub/Health_Statistics/NCHS/Datasets/DVS/natality/Nat2017us.zip

CDC and NCI. United states cancer statistics: Data visualizations, 2019. URL https://gis.cdc.gov/Cancer/USCS/DataViz.html.

Conn Charalambous and AR Conn. An efficient method to solve the minimax problem directly. *SIAM Journal on Numerical Analysis*, 15(1):162–187, 1978.

Christophe Denis, Romuald Elie, Mohamed Hebiri, and François Hu. Fairness guarantee in multi-class classification. *arxiv preprint arXiv:2109.13642*, 2021.

Michele Donini, Luca Oneto, Shai Ben-David, John S Shawe-Taylor, and Massimiliano Pontil. Empirical risk minimization under fairness constraints. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, *Advances in Neural Information Processing Systems 31*, pages 2791–2801. Curran Associates, Inc., 2018.

Dheeru Dua and Casey Graff. UCI machine learning repository, 2019. URL http://archive.ics.uci.edu/ml.

Cynthia Dwork, Moritz Hardt, Toniann Pitassi, Omer Reingold, and Richard Zemel. Fairness through awareness. In *Proceedings of the 3rd innovations in theoretical computer science conference*, pages 214–226. ACM, 2012.

Federal Elections Commission. Federal elections 2016, 2016. URL https://transition.fec.gov/pubrec/fe2016/federalelections2016.pdf.

Michael Feldman, Sorelle A Friedler, John Moeller, Carlos Scheidegger, and Suresh Venkatasubramanian. Certifying and removing disparate impact. In *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 259–268. ACM, 2015.

Five Thirty Eight. National presidential polls, november 8th, 2016, 2016. URL https://projects.fivethirtyeight.com/2016-election-forecast/national-polls/.

Naman Goel, Mohammad Yaghini, and Boi Faltings. Non-discriminatory machine learning through convex fairness criteria. In *Thirty-Second AAAI Conference on Artificial Intelligence*, 2018.

Gabriel Goh, Andrew Cotter, Maya Gupta, and Michael P Friedlander. Satisfying real-world goals with dataset constraints. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, *Advances in Neural Information Processing Systems 29*, pages 2415–2423. Curran Associates, Inc., 2016.
Fairness and Unfairness in Binary and Multiclass Classification

Priya Goyal, Adriana Romero Soriano, Caner Hazirbas, Levent Sagun, and Nicolas Usunier. Fairness indicators for systematic assessments of visual feature extractors. In Proceedings of the 2022 ACM Conference on Fairness, Accountability, and Transparency, FAccT ’22, page 70–88, New York, NY, USA, 2022. Association for Computing Machinery. ISBN 9781450393522.

Nina Grgic-Hlaca, Muhammad Bilal Zafar, Krishna P Gummadi, and Adrian Weller. The case for process fairness in learning: Feature selection for fair decision making. In NIPS Symposium on Machine Learning and the Law, volume 1, page 2, 2016.

Moritz Hardt, Eric Price, and Nati Srebro. Equality of opportunity in supervised learning. In Advances in neural information processing systems, pages 3315–3323, 2016.

Hoda Heidari, Claudio Ferrari, Krishna Gummadi, and Andreas Krause. Fairness behind a veil of ignorance: A welfare analysis for automated decision making. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, and R. Garnett, editors, Advances in Neural Information Processing Systems 31, pages 1265–1276. Curran Associates, Inc., 2018.

Irit Hochberg, Deeb Daoud, Naim Shehadeh, and Elad Yom-Tov. Can internet search engine queries be used to diagnose diabetes? analysis of archival search data. Acta Diabetologica, 56:1149–1154, 2019.

Christopher Jackson, Nicky Best, and Sylvia Richardson. Improving ecological inference using individual-level data. Statistics in medicine, 25(12):2136–2159, 2006.

Christopher Jackson, Nicky Best, and Sylvia Richardson. Hierarchical related regression for combining aggregate and individual data in studies of socio-economic disease risk factors. Journal of the Royal Statistical Society: Series A (Statistics in Society), 171(1):159–178, 2008.

James E Johndrow and Kristian Lum. An algorithm for removing sensitive information: application to race-independent recidivism prediction. The Annals of Applied Statistics, 13(1):189–220, 2019.

Jon Kleinberg, Sendhil Mullainathan, and Manish Raghavan. Inherent trade-offs in the fair determination of risk scores. In 8th Innovations in Theoretical Computer Science Conference (ITCS 2017). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

Deepak Kumar, Tessa Grosz, Navid Rekabsaz, Elisabeth Greif, and Markus Schedl. Fairness of recommender systems in the recruitment domain: an analysis from technical and legal perspectives. Frontiers in big Data, 6, 2023.

Matt J Kusner, Joshua Loftus, Chris Russell, and Ricardo Silva. Counterfactual fairness. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems 30, pages 4066–4076. Curran Associates, Inc., 2017.
Alex Lamy, Ziyuan Zhong, Aditya K Menon, and Nakul Verma. Noise-tolerant fair classification. In Advances in Neural Information Processing Systems 32, pages 294–306. Curran Associates, Inc., 2019. URL http://papers.nips.cc/paper/8322-noise-tolerant-fair-classification.pdf.

Daniel McDuff, Shuang Ma, Yale Song, and Ashish Kapoor. Characterizing bias in classifiers using generative models. In Advances in Neural Information Processing Systems 32, pages 5404–5415. Curran Associates, Inc., 2019.

Aditya Krishna Menon and Robert C Williamson. The cost of fairness in binary classification. In Conference on Fairness, Accountability and Transparency, pages 107–118, 2018.

Jorge Nocedal and Stephen Wright. Numerical optimization. Springer Science & Business Media, 2006.

Geoff Pleiss, Manish Raghavan, Felix Wu, Jon Kleinberg, and Kilian Q Weinberger. On fairness and calibration. In Advances in Neural Information Processing Systems, pages 5680–5689, 2017.

ProPublica. Compas analysis github repository. https://github.com/propublica/compas-analysis, 2016.

Jonathan Roth, Guillaume Saint-Jacques, and YinYin Yu. An outcome test of discrimination for ranked lists. In Proceedings of the 2022 ACM Conference on Fairness, Accountability, and Transparency, FAccT ’22, page 350–356, New York, NY, USA, 2022. Association for Computing Machinery. ISBN 9781450393522.

Cynthia Rudin, Caroline Wang, and Beau Coker. The age of secrecy and unfairness in recidivism prediction. Harvard Data Science Review, 2(1):1, 2020.

Sivan Sabato and Elad Yom-Tov. Bounding the fairness and accuracy of classifiers from population statistics. In Hal Daumé III and Aarti Singh, editors, Proceedings of the 37th International Conference on Machine Learning, volume 119 of Proceedings of Machine Learning Research, pages 8316–8325. PMLR, 13–18 Jul 2020.

Clayton Scott, Gilles Blanchard, and Gregory Handy. Classification with asymmetric label noise: Consistency and maximal denoising. In Shai Shalev-Shwartz and Ingo Steinwart, editors, Proceedings of the 26th Annual Conference on Learning Theory, volume 30 of Proceedings of Machine Learning Research, pages 489–511, Princeton, NJ, USA, 12–14 Jun 2013. PMLR.

Sigal Shaklai, Ran Gilad-Bachrach, Elad Yom-Tov, and Naftali Stern. Detecting impending stroke from cognitive traits evident in internet searches: analysis of archival data. Journal of Medical Internet Research, 23(5):e27084, 2021.

Till Speicher, Hoda Heidari, Nina Grgic-Hlaca, Krishna P Gummadi, Adish Singla, Adrian Weller, and Muhammad Bilal Zafar. A unified approach to quantifying algorithmic unfairness: Measuring individual &group unfairness via inequality indices. In Proceedings
of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, pages 2239–2248, 2018.

T. Sun, D. Sheldon, and B. O’Connor. A probabilistic approach for learning with label proportions applied to the us presidential election. In 2017 IEEE International Conference on Data Mining (ICDM), pages 445–454, Nov 2017. doi: 10.1109/ICDM.2017.54.

Ozgu Turgut. Fair classification with ensembles. In 2023 Innovations in Intelligent Systems and Applications Conference (ASYU), pages 1–4. IEEE, 2023.

US Census Bureau. Annual estimates of the resident population for the united states, regions, states, and puerto rico: April 1, 2010 to july 1, 2019, 2019. URL https://www2.census.gov/programs-surveys/popest/tables/2010-2019/state totals/nst-est2019-01.xlsx.

USDA Economic Research Service. Highest level of educational attainment, 2021. URL https://data.ers.usda.gov/reports.aspx?ID=17829.

Ana Valdivia, Javier Sánchez-Monedero, and Jorge Casillas. How fair can we go in machine learning? assessing the boundaries of accuracy and fairness. International Journal of Intelligent Systems, 36(4):1619–1643, 2021. doi: https://doi.org/10.1002/int.22354. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/int.22354.

Sahil Verma and Julia Rubin. Fairness definitions explained. In 2018 IEEE/ACM International Workshop on Software Fairness (FairWare), pages 1–7. IEEE, 2018.

Christopher Watson, Elise Uberoi, and Philip Loft. General election results from 1918 to 2019, 2020. URL https://commonslibrary.parliament.uk/research-briefings/cbp-8647/.

Blake Woodworth, Suriya Gunasekar, Mesrob I. Ohannessian, and Nathan Srebro. Learning non-discriminatory predictors. In Satyen Kale and Ohad Shamir, editors, Proceedings of the 2017 Conference on Learning Theory, volume 65 of Proceedings of Machine Learning Research, pages 1920–1953, Amsterdam, Netherlands, 07–10 Jul 2017. PMLR.

Yongkai Wu, Lu Zhang, and Xintao Wu. On convexity and bounds of fairness-aware classification. In The World Wide Web Conference, pages 3356–3362. ACM, 2019.

Elad Yom-Tov, Indu Navar, Ernest Fraenkel, and James D Berry. Identifying amyotrophic lateral sclerosis through interactions with an internet search engine. Muscle & Nerve, 69 (1):40–47, 2024.

Muhammad Bilal Zafar, Isabel Valera, Manuel Gomez Rodriguez, and Krishna P Gummad. Fairness beyond disparate treatment & disparate impact: Learning classification without disparate mistreatment. In Proceedings of the 26th International Conference on World Wide Web, pages 1171–1180. International World Wide Web Conferences Steering Committee, 2017a.
Appendix A. Proof of Theorem 3

In this section, we prove Theorem 3. We start with the observation that each of the functions $s^i_g(\alpha_b)$ defined in Lemma 2 is continuous in $\alpha_b$. Thus, $F$ can be redefined using continuous functions. Define

$$\forall g \in G^+, i \in [4], f^i_g(\alpha_b) := \begin{cases} \pi_0^g \tau_3(\alpha^0_b, s^i_g(\alpha_b)) + \pi_1^g \tau_\beta(\alpha^1_b, r_g + q_g s^i_g(\alpha_b)) & s^i_g(\alpha_b) \in \text{dom}_g, \\ \infty & \text{otherwise}. \end{cases}$$

In addition, for $g \in G \setminus G^+$, define

$$f^i_g(\alpha_b) := \begin{cases} \sum_{y: \pi^g_y = 1} \tau_\beta(\alpha^0_b, 1 - \hat{p}^i_g) & i = 1 \\ \infty & \text{otherwise}. \end{cases}$$

Then, it easily follows that $F(\alpha_b) = \sum_{g \in G} w_g \min_{i \in [4]} f^i_g(\alpha_b)$.

The following lemma shows that each of the functions $f^i_g$ is concave in $\alpha^1_b$ on an appropriate segmentation of its domain. A technical detail is that in some cases, $f^i_g(\alpha_b)$ might not be continuous in $\alpha^1_b$ at $\alpha^1_b = 0$ or at $\alpha^1_b = 1$. This is because the function $\eta$ is not continuous in its first argument in specific cases: For $b \in \{0, 1\}$, we have $1 = \lim_{a \to b} \eta(a, b) \neq \eta(b, b) = 0$. Thus, in some cases the concavity holds on a restricted domain, which does not include 0 or 1.

**Lemma 5** Fix $\alpha^0_b \in [0, 1]$. For $g \in G$, $i \in [4]$, consider the function $\alpha^1_b \mapsto f^i_g(\alpha^0_b, \alpha^1_b)$ and the domain $I^i_g$ on which it is finite. Define $\theta^i_g(\alpha^0_b)$ as follows:

$$\theta^i_g(\alpha^0_b) = \begin{cases} \max\{r_g, 0\} & i = 1, g \in G^+; \\ \min\{r_g + q_g, 1\} & i = 2, g \in G^+; \\ r_g + q_g \alpha^0_b & i \in \{3, 4\}, g \in G^+, \alpha^0_b \in \text{dom}_g; \\ 1 - \hat{p}^i_g & i = 1, g \in G, \pi^g_1 = 1; \\ 0 & \text{otherwise}. \end{cases}$$

Then $\alpha^1_b \mapsto f^i_g(\alpha^0_b, \alpha^1_b)$ is concave on each of the intervals $(0, \theta^i_g(\alpha^0_b)) \cap I^i_g$ and $[\theta^i_g(\alpha^0_b), 1) \cap I^i_g$. Moreover, if $\theta^i_g(\alpha^0_b) \in (0, 1)$, then this holds also for the closed intervals.

**Proof** For given $g \in G$, $i \in [4]$, define $v^0 := s^i_g(\alpha_b)$, and $v^1 := r_g + q_g v^0$. First, let $i \in [3]$, $g \in G^+$, and assume $s^i_g(\alpha^0_b) \in \text{dom}_g$. In this case, $f^i_g(\alpha_b) = \pi^0_g \tau_3(\alpha^0_b, v^0) + \pi^1_g \tau_\beta(\alpha^1_b, v^1)$. 


Observe that $s_i^1, s_i^2, s_i^3$ are constant in $\alpha_b^1$, thus $v^0$ does not depend on $\alpha_b^1$, and the same holds for $\tau_\beta(\alpha_b^0, v^0)$. Therefore, from the definition of $\tau_\beta$, the function $\alpha_b^1 \mapsto f_g^i(\alpha_b)$ is an affine function of $\eta(\alpha_b^1, v^1)$ (with non-negative coefficients). Note that $v^1$ does not depend on $\alpha_b^1$. It is easy to verify that for any fixed $v^1 \in (0, 1)$, $\alpha_b^1 \mapsto \eta(\alpha_b^1, v^1)$ is concave on the interval $[0, v^1]$ and on the interval $[v^1, 1]$ (see illustration in Figure 3 (right)). Therefore, $\alpha_b^1 \mapsto f_g^i(\alpha_b)$ is also concave on the same intervals. Substituting $v^1$ with its definition, we get $\theta_g^i(\alpha_b^0)$ as defined above. If $v^1 \in \{0, 1\}$ then the function is concave on $(0, v^1]$ and on $[v^1, 1]$.

Next, consider $i = 4$, $g \in G^+$. Note that we have $I_g^4 = [r_g, q_g + r_g] \cap [0, 1]$, since $\alpha_b^1 \in I_g^4$ iff $v^0 = s_g^4(\alpha_b^1) \in \text{dom}_g$. In this case, we have $v^1 = \alpha_b^1$, thus $\eta(\alpha_b^1, v^1) = 0$ and so $\tau_\beta(\alpha_b^1, v^1)$ is linear in $\alpha_b^1$. In addition, $\tau_\beta(\alpha_b^0, v^0)$ is an affine function of $\eta(\alpha_b^0, v^0) = \eta(\alpha_b^0, (\alpha_b^1 - r_g)/q_g)$. This function is linear in $\alpha_b^1$ on each of the intervals $[0, r_g + q_g \alpha_b^0] \cap I_g^4$ and $[r_g + q_g \alpha_b^0, 1] \cap I_g^4$. Thus, $\alpha_b^1 \mapsto f_g^i(\alpha_b) = \eta(\alpha_b^1, v^1)$ is linear on each of these intervals.

Lastly, consider $g \in G \setminus G^+$. In this case, only $i = 1$ is finite. If $\pi_g^1 = 1$, then $\alpha_b^1 \mapsto f_g^1(\alpha_b)$ is constant. If $\pi_g^1 = 0$, then $f_g^1$ is an affine function with positive coefficients of $\eta(\alpha_b^1, 1 - p_g^1)$. Similarly to the first case, this implies that $\alpha_b^1 \mapsto f_g^i(\alpha_b)$ is concave on $[0, 1 - p_g^1]$ and on $[1 - p_g^1, 1]$.

From the concavity property proved in Lemma 5, we can conclude that it is easy to minimize $F(\alpha_b)$ over $\alpha_b^1$ when $\alpha_b^0$ is fixed, since there is only a small finite number of solutions. This is proved in the following lemma.

**Lemma 6** Let $\alpha_b^0 \in [0, 1]$, and define

$$\Theta(\alpha_b^0) := \{\theta_g^i(\alpha_b^0)\}_{g \in G, i \in [4]} \cup \{0, 1\}.$$ 

Then,

$$\min_{\alpha_b^1 \in [0, 1]} F(\alpha_b) = \min_{\alpha_b^1 \in \Theta(\alpha_b^0)} F(\alpha_b).$$

**Proof** Fix $\alpha_b^0$. Ordering the points in $\Theta(\alpha_b^0)$ in an ascending order and naming them $0 = p_1 < p_2 < \ldots < p_N = 1$, where $N = |\Theta(\alpha_b^0)|$, define the set of intervals $\mathcal{I} = \{[p_i, p_{i+1}] \setminus \{0, 1\}\}_{i \in [N - 1]}$. Observe that the intervals in $\mathcal{I}$ partition $(0, 1)$ (with overlaps at end points). Moreover, for all $g \in G^+$, $i \in [4]$, each of the intervals $I_i \subseteq \{0, \theta_g^i(\alpha_b^0)\} \cap I_g^i$ or $I_i \subseteq [\theta_g^i(\alpha_b^0), 1) \cap I_g^i$. By Lemma 5, it follows that for all $g \in G^+, i \in [4]$, $\alpha_b^1 \mapsto f_g^i(\alpha_b)$ is concave on $I_i$.

$F$ is a conic combination of minima of concave functions on $I$, thus it is concave on $I$. Thus, on each $I \in \mathcal{I}$ which is a closed interval (that is, with endpoints other than 0 or 1), $F(\alpha_b)$ is minimized by one of the endpoints of $I$. Now, consider $I$ with an endpoint 0 or 1. For simplicity, assume $I = (0, p_2]$; the other case is symmetric. $F(\alpha_b)$ is concave on $I$. Thus, it is either minimized on $I$ at $p_1$ or satisfies $\lim_{\alpha_b^1 \to 0} F(\alpha_b) \leq \inf_{\alpha_b^1 \in I} F(\alpha_b)$. In the latter case, there are two options: if $\alpha_b^1 \mapsto F(\alpha_b)$ is continuous at 0, then $F(\alpha_b^0, 0) = \lim_{\alpha_b^1 \to 0} F(\alpha_b)$, implying that the function is minimized on $I$ by $\alpha_b^1 = 0$. If it is not continuous at 0, this can only happen if $r_g + q_g s_g^0(\alpha_b) = 0$, which leads to $\eta(\alpha_b^0, r_g + q_g s_g^0(\alpha_b)) = 1$. Thus, in this case, $F(\alpha_b^0, 0) \leq \lim_{\alpha_b^1 \to 0} F(\alpha_b)$.

It follows that in this case as well, $F$ is minimized on $I$ by 0.
It follows that the minimizer of $\alpha_b^1 \mapsto F(\alpha_b)$ must be one of the end points of $\Theta(\alpha_0^0)$, as claimed. \hfill $\blacksquare$

Now, due to the complete symmetry of the problem definition between the two labels, Lemma 6 implies also the symmetric property, that for a fixed $\alpha_b^1 \in [0,1]$, $\min_{\alpha_0^0\in[0,1]} F(\alpha_b) = \min_{\alpha_b^1\in\tilde{\Theta}(\alpha_b^1)} F(\alpha_b)$, where $\tilde{\Theta}$ is symmetric to $\Theta$ and is obtained by switching the roles of $y = 1$ and $y = 0$ in all definitions. By explicitly calculating the resulting values, we get that

$$\tilde{\Theta}(\alpha_b^1) = V_0 \cup \{(\alpha_b^1 - r_g)/q_g\}_{g \in G^+},$$
$$\Theta(\alpha_b^1) = V_1 \cup \{(r_g + q_g\alpha_0^0)\}_{g \in G^+},$$

Where $V_0, V_1$ are defined as in Theorem 3. Using the above results, we can now prove Theorem 3.

**Proof** [of Theorem 3] By Eq. (26), $\min\text{disc}_y = \min_{\alpha_b^0 \in [0,1]^2} F(\alpha_b)$. Thus, we have left to show that there exists a minimizer for $F(\alpha_b)$ in the set $\text{Pairs}$ defined in Theorem 3. From Lemma 6, it follows that there exists a minimizing solution $(a^0, a^1) \in \arg\min_{\alpha_b^0 \in [0,1]^2} F(\alpha_b)$ such that $a^1 \in \Theta(a^0)$. The symmetric counterpart of Lemma 6 implies that there exists some $b^0 \in \tilde{\Theta}(a^1)$ such that $(b^0, a^1) \in \arg\min_{\alpha_b^0 \in [0,1]^2} F(\alpha_b)$. Thus, at least one of the following options must hold:

- $(b^0, a^1) \in V_0 \times V_1$.
- $a^1 \notin V_1$; this implies that $a^1 = r_g + q_g a^0$ for some $g \in G^+$.
- $b^0 \notin V_0$; this implies that $b^0 = (a^1 - r_g)/q_g$ for some $g \in G^+$.

In the first case, there exists a minimizer in $V_0 \times V_1$. In the second and third case, there exists a minimizer in the set $\{(v, r_g + q_g v) \mid v \in [0,1], g \in G^+\}$. Thus, in all cases there is a minimizer in the set $\text{Pairs}$, as required. \hfill $\blacksquare$

**Appendix B. Proof of Lemma 4**

**Proof** [of Lemma 4] Let $\phi[a, b, c, d](x) := 1 - (c + dx)/(a + bx)$. Denote $a^\prime = 1 - a - b$, $c^\prime = 1 - c - d$, $x^\prime = 1 - x$. Then, it can be easily verified that

$$\eta(a + bx, c + dx) = \begin{cases} 
\phi[a, b, c, d](x) & a + bx > c + dx \\
\phi[a^\prime, b, c^\prime, d](x^\prime) & a^\prime + bx^\prime > c^\prime + dx^\prime.
\end{cases}$$

Note that $a + bx > c + dx \iff a^\prime + bx^\prime < c^\prime + dx^\prime$. Therefore, it suffices to find an upper bound on $|d\phi[a, b, c, d](x)/dx| = \frac{|bc - ad|}{(a + bx)^2}$, for $a, b, c, d$ such that $b > 0$ and $d \geq 0$, on $x \in [0,1]$ such that $a + bx > c + dx$, and $a + bx, c + dx \in [0,1]$.

We consider the possible cases.

- Suppose $d = 0$
  - If $c > 0$ then $a + bx > c$, thus $|d\phi[a, b, c, d](x)/dx| = \frac{|bc - ad|}{(a + bx)^2} \leq \frac{|bc|}{c^2} = b/c$. 

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– If \( c = 0 \), then \( bc - ad = 0 \), thus the derivative is zero.
– If \( c < 0 \), then \( c + dx < 0 \), thus the domain is empty.

• Otherwise, \( d > 0 \). Let \( x_1 \) such that \( a + bx_1 = 0 \) and \( x_2 \) such that \( c + dx_2 = 0 \). Then \( x_1 = -a/b \), \( x_2 = -c/d \). Note that we must have \( x \geq \max\{x_1, x_2\} \).

– If \( x_1 < x_2 \) \((-a/b < -c/d)\) then \( a + bx \geq a + bx_2 > a + bx_1 = 0 \). Since \( |a + bx_2| = |a - bc/d| = |bc - ad|/d \), we have \( |d\phi[a, b, c, d](x)|/dx = \frac{|bc-ad|}{(a+bx)^2} \leq \frac{d^2}{|bc-ad|} \).

– If \( x_1 > x_2 \) \((-a/b > -c/d)\), then \( c + dx_1 > c + dx_2 = 0 = a + bx_1 \). Thus, \( c-a+(d-b)x_1 > 0 \). Let \( x \) be any value in the feasible domain. Then \( a + bx \geq c + dx \), hence \( c - a + (d-b)x \leq 0 \).

  * If \( d-b = 0 \) then this contradicts the property of \( x_1 \), thus there is no such \( x \).
  * If \( d-b > 0 \), then \( x < x_1 \), thus \( a + bx < 0 \), again implying that \( x \) is not in the domain. Thus, in this case the domain is empty as well.
  * If \( d-b < 0 \), then \( x \geq (c-a)/(b-d) > x_1 \), thus \( a + bx \geq a + b(c-a)/(b-d) > a + bx_1 \geq 0 \). Therefore, \( |d\phi[a, b, c, d](x)|/dx = \frac{|bc-ad|}{(a+bx)^2} \leq \frac{|bc-ad|}{a+b(c-a)/(b-d)} \).

– If \( x_1 = x_2 \), Then \(-c/d = -a/b\), which means that \( ad = bc \), thus the derivative in this case is zero.

Combining the cases gives the upper bound for the derivative of \( \psi \) as given by \( D(a, b, c, d) \), for \( x \) such that \( a + bx > c + dx \). In the special case that \( b = d \), this holds for all \( x \) if \( a > c \) and for no \( x \) otherwise. In this special case, if \( a > c \), then \( D(a, b, c, d) = d^2/|bc-ad| = d/|a-c| \).

For the other case in the definition of \( \eta \), the upper bound is obtained by \( D(a', b, c', d) \) as described above, including the special case \( b = d \) and \( a' > c' \), that is \( a < c \).