Research Article

Sandwich Theorem of Cover Times

Yilun Shang

SUTD-MIT International Design Center, Singapore University of Technology and Design, Singapore 138682

Correspondence should be addressed to Yilun Shang; shylmath@hotmail.com

Received 26 June 2013; Accepted 22 July 2013

Academic Editors: I. Beg, P. E. Jorgensen, V. Makis, and O. Pons

Copyright © 2013 Yilun Shang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Based on a connection between cover times of graphs and Talagrand’s theory of majorizing measures, we establish sandwich theorems for cover times as well as blanket times.

1. Introduction

Let $G = (V, E)$ be a connected graph with $n$ vertices and $m$ edges. Consider a simple random walk on $G$: we start at a vertex $v \in V$; if at step $t$ we are at vertex $u$, then we move from $u$ to a neighbor at step $t + 1$ with probability $d_u^{-1}$, where $d_u$ is the degree of $u$. Let $E_v$ be the expectation operator governing the random walk started at vertex $v \in V$. The cover time of $G$ is defined as

$$ c(G) = \max_{v \in V} E_v \mu(G), $$

where $\mu(G)$ is the first time $t \geq 1$ that all vertices of $G$ have been traversed [1].

Another relevant quantity is the strong $\delta$-blanket time [2]. For $v \in V$, let $\pi_v = d_v/2m$ be the stationary measure of the random walk, and let $r_{uv}$ be the number of visits to $v$ up to time $t$. For $\delta \in (0, 1)$, the strong $\delta$-blanket time is defined as

$$ b(G, \delta) = \max_{v \in V} r_{\delta \pi_v} \mu(G), $$

where $\tau(\delta)$ is the first time $t \geq 1$ such that

$$ \frac{r_{uv}/\pi_u}{r_{uv}/\pi_v} \geq \delta, $$

holds for any two vertices $u$ and $v$. Clearly, we have $b(G, \delta) \geq c(G)$ for any $\delta \in (0, 1)$. We refer the reader to [1, 3] for more background information on random walks.

Recently, Ding et al. [4] established an important connection between cover times (blanket times) and the $\gamma_2$ functional from Talagrand’s theory of majorizing measures [5, 6]; see Theorem 1. We first review the $\gamma_2$ functional in brief. Consider a compact metric space $(X, d)$, and let $M_0 = 1, M_k = 2^k$ for $k \geq 1$. For a partition $\mathcal{P}$ of $X$ and an element $x \in X$, denote by $\mathcal{P}(x)$ the unique set $S \in \mathcal{P}$ containing $x$. An admissible sequence $\{\mathcal{A}_k\}_{k \geq 0}$ of partitions of $X$ is that $\mathcal{A}_k$ is a refinement of $\mathcal{A}_{k+1}$, and $|\mathcal{A}_k| \leq M_k$ for $k \geq 0$. The $\gamma_2$ functional is defined as

$$ \gamma_2(X, d) = \inf \sup_{\{\mathcal{A}_k\}_{k \geq 0}} \sum_{k=0}^{\infty} 2^{k/2} \text{diam}(\mathcal{A}_k(x)), $$

where $\text{diam}(S)$ represents the diameter of $S$, that is, $\text{diam}(S) = \sup_{x, y \in S} d(x, y)$. Throughout the paper, we view graph $G$ as a metric space with distance induced by the commute time $\kappa_G(u, v)$ between two vertices $u, v \in V$. Hence, $(V, \kappa_G)$ is a compact metric space.

**Theorem 1** (see [4]). For any graph $G = (V, E)$ and $\delta \in (0, 1)$,

$$ c(G) \approx (\gamma_2(V, \sqrt{\kappa_G}))^2 \kappa_G(G, \delta), $$

where $A \approx B$ means $c_1 B \leq A \leq c_2 B$ for some constants $C_1, C_2$, and furthermore, $A \approx b$ emphasizes that the constants may depend on $\delta$. Here, $\sqrt{\kappa_G}(u, v) = \sqrt{\kappa_G(u, v)}$.

A comparison theorem for cover times is also presented.

**Theorem 2** (see [4]). Suppose that graphs $G$ and $G'$ are on the same vertex set $V$, and $\kappa_G$ and $\kappa_G'$ are the distances induced by respective commute times. If there exists a number $L \geq 1$ such that $\kappa_G(u, v) \leq L \cdot \kappa_G'(u, v)$ for all $u, v \in V$, then

$$ c(G) \leq O(L) \cdot c(G'). $$
In this paper, we extend this nice comparison theorem and provide several applications.

2. Results

We have the following results.

**Theorem 3.** Suppose that three graphs $G, G_1$, and $G_2$ are on the same vertex set $V$ and that $\kappa_G, \kappa_{G_1}$, and $\kappa_{G_2}$ are the distances induced by respective commute times. If there exist $\alpha \in (0, 1)$, $L_1 > 0$, and $L_2 > 0$ such that $L_1 \cdot \kappa_{G_1}(u, v)^\alpha \leq \kappa_G(u, v) \leq L_2 \cdot \kappa_{G_2}(u, v)$ for all $u, v \in V$, then

$$O(L_1) \cdot c(G_1)^\alpha \leq c(G) \leq O(L_2) \cdot c(G_2). \quad (7)$$

**Proof.** From the assumption, we have

$$\sqrt{L_1} \cdot \kappa_{G_1}(u, v) \leq \sqrt{\kappa_G(u, v)} \leq \sqrt{L_2} \cdot \kappa_{G_2}(u, v), \quad (8)$$

for all vertices $u$ and $v$. Capitalizing on the $C_r$-inequality, which says that

$$\left(\sum_{i=1}^n x_i^{r}\right)^{\frac{1}{r}} \leq C_r \left(\sum_{i=1}^n x_i^{\frac{r}{r-1}}\right)^{\frac{1}{r-1}}, \quad (9)$$

where $C_r = 1$ if $r \in (0, 1]$ and $C_r = n^{r-1}$ if $r \in (1, \infty)$, we obtain

$$\sqrt{L_1} \cdot \gamma_2(V, \sqrt{\kappa_G}) \leq \gamma_2(V, \sqrt{\kappa_{G_1}}) \leq \sqrt{L_2} \cdot \gamma_2(V, \sqrt{\kappa_{G_2}}) \quad (10)$$

by using definition (4).

It follows from Theorem 1 that

$$O(L_1) \cdot c(G_1)^\alpha \leq c(G) \leq O(L_2) \cdot c(G_2) \quad (11)$$

as desired.

Generally, the conditions imposed on commute times in Theorem 3 are thorny, if possible, to test, especially for complex and large-scale graphs. However, commute times for recursive graphs are likely to be estimated (see, e.g., [7]).

Based on this comparison characterization, we have the following bounds regarding the ratio of cover times.

**Corollary 4.** Maintaining the notations of Theorem 3, if $G_1 = G_2$, one has

$$O(L_1) \cdot c(G_1)^\alpha \leq c(G) \leq O(L_2) \cdot c(G_1) \quad (12)$$

An upper bound of cover time [8] yields

$$\frac{O(L_1)}{(\max_{u,v \in V} H(u,v))^{1-\alpha} (1 + \ln n)^{\frac{1}{\alpha}}} \leq \frac{c(G)}{c(G_1)} \leq O(L_2), \quad (14)$$

where $n = |V|$ as mentioned before.

The following result is a "sandwich theorem" for monotonic graph sequences.

**Corollary 5.** Maintaining the notations of Theorem 3, if $G_1$ is a graph obtained by deleting an edge from $G$ and $G_2$ is obtained by adding an edge to $G$, one has

$$O\left(\frac{m}{m+1}\right) \cdot c(G_2) \leq c(G) \leq O\left(\frac{m}{m-1}\right) \cdot c(G_1), \quad (15)$$

where $m = |E|$ is the number of edges in $G$.

**Proof.** Since $G_1$ and $G_2$ have $m - 1$ and $m + 1$ edges, respectively, we obtain by [3, Theorem 2.9] that

$$\frac{m}{m+1} \kappa_{G_2}(u, v) \leq \kappa_G(u, v) \leq \frac{m}{m-1} \kappa_{G_1}(u, v), \quad (16)$$

for any $u, v \in V$. Thus, the result follows directly from Theorem 3 by taking $\alpha = 1$.

We mention that it is recently shown in [9] that $c(G_2)/4 \leq c(G)$. We conclude the paper with a result on $\delta$-strong blanket times analogous to our main theorem.

**Corollary 6.** Maintaining the notations of Theorem 3, for any $\delta \in (0, 1)$, one has

$$O_\delta(L_1) \cdot b(G_1, \delta)^\alpha \leq b(G, \delta) \leq O_\delta(L_2) \cdot b(G_2, \delta). \quad (17)$$

**Proof.** The same proof in Theorem 3 applies by using Theorem 1.

**Acknowledgment**

The author is thankful to the learned referees for the valuable suggestions.

**References**

[1] D. J. Aldous and J. Fill, *Reversible Markov Chains and Random Walks on Graphs*, book in preparation.

[2] J. Kahn, J. H. Kim, L. Lovász, and V. H. Vu, “The cover time, the blanket time, and the Matthews bound,” in *Proceedings of the 41st Annual Symposium on Foundations of Computer Science*, pp. 467–475, Redondo Beach, Calif, USA, 2000.

[3] L. Lovász, “Random walks on graphs: a survey,” in *Combinatorics, Paul Erdős is Eighty*, vol. 2, pp. 353–397, 1996.

[4] J. Ding, J. R. Lee, and Y. Peres, “Cover times, blanket times, and majorizing measures,” *Annals of Mathematics*, vol. 175, no. 3, pp. 1409–1471, 2012.
[5] M. Talagrand, ”Regularity of Gaussian processes,” *Acta Mathematica*, vol. 159, no. 1-2, pp. 99–149, 1987.

[6] M. Talagrand, ”Majorizing measures: the generic chaining,” *The Annals of Probability*, vol. 24, no. 3, pp. 1049–1103, 1996.

[7] Y. Shang, ”Mean commute time for random walks on hierarchical scale-free networks,” *Internet Mathematics*, vol. 8, no. 4, pp. 321–337, 2012.

[8] P. Matthews, ”Covering problems for Markov chains,” *The Annals of Probability*, vol. 16, no. 3, pp. 1215–1228, 1988.

[9] M. T. Barlow, J. Ding, A. Nachmias, and Y. Peres, ”The evolution of the cover time,” *Combinatorics, Probability and Computing*, vol. 20, no. 3, pp. 331–345, 2011.
Submit your manuscripts at http://www.hindawi.com