Smooth torus quotients of Richardson varieties in the Grassmannian

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Abstract

Let \(k\) and \(n\) be positive coprime integers with \(k < n\). Let \(T\) denote the subgroup of diagonal matrices in \(SL(n, \mathbb{C})\). We study the GIT quotient of Richardson varieties \(X_{v}^{w}\) in the Grassmannian \(\text{Gr}_{k,n}\) by \(T\) with respect to a \(T\)-linearised line bundle \(L\) corresponding to the Plücker embedding. We give necessary and sufficient combinatorial conditions for the quotient variety \(T\backslash (X_{v}^{w})^{ss}_{T}(L)\) to be smooth.

1 Introduction

A Richardson variety is the intersection of a Schubert variety and an opposite Schubert variety in a generalised flag variety. They first appeared in the work of Richardson [16]. Since its introduction there have been many applications of Richardson varieties. For instance, they were considered by Lakshmibai–Brion [6] to obtain a geometric construction of a standard monomial theoretic basis of the global sections of line bundle on flag variety. They also appeared in the studies of \(K\)-theory for the flag varieties ([5, 14]). In this paper, we will study geometric invariant theoretic (GIT) quotients of Richardson varieties in the Grassmannian by a maximal torus for a polarised line bundle and obtain a combinatorial description of the smooth quotients.

Let \(\text{Gr}_{k,n}\) denote the Grassmannian variety of \(k\)-subspaces in complex \(n\)-space. Let \(G = SL(n, \mathbb{C})\). Let \(T\) be the subgroup of all diagonal matrices in \(G\), and \(B\) the subgroup of all upper triangular matrices in \(G\) and \(B^{-}\) the subgroup of all lower triangular matrices in \(G\). We know that \(G\) acts transitively on \(\text{Gr}_{k,n}\) by left multiplication. Let \(e_{1}, \ldots, e_{n}\) be the standard basis of \(\mathbb{C}^{n}\). Let \(P\) be the stabiliser \(\langle e_{1}, e_{2}, \ldots, e_{r} \rangle\) in \(G\). So the Grassmannian variety can also be realised as the homogeneous space \(G/P\). We note that \(P\) is a parabolic subgroup containing \(B\). Let \(W_{P}\) denote the Weyl group of \(P\). Let \(W_{P}^{P} = W/W_{P}\) denote the set of minimal length coset representative. The \(T\)-fixed points in \(G/P\) are \(e_{w} = wP/P\) with \(w \in W^{P}\). The \(B\)-orbit \(C_{w}\) of \(e_{w}\) is called a Schubert cell and it is an affine space of dimension \(l(w)\). The closure of \(C_{w}\) in \(G/P\) is the Schubert variety \(X(w)\). Dually, one defines the opposite Schubert cell \(C^{v}\) to be the \(B^{-}\)-orbit through \(e_{v}\) and the closure of the opposite Schubert cell is called the opposite Schubert variety. The Richardson variety \(X_{w}^{v}\) is the intersection of the Schubert variety \(X(w)\) with the opposite Schubert variety \(X^{v}\). It is shown to be non empty if and only if \(v \leq w\) in the Bruhat order. Let \(S_{d}\) denote the symmetric group in \(d\) letters. We observe that \(W_{P} = S_{k} \times S_{n-k}\), so the minimal length coset representatives of \(W_{P}\) can be identified with

\[
\{w \in W|w(1) < w(2) < \ldots < w(k), w(k + 1) < w(k + 2) < \ldots < w(n)\}.
\]

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Let
\[ I(k, n) = \{(i_1, i_2, \ldots, i_k) | 1 \leq i_1 < i_2 \cdots < i_k \leq n \}. \]

Then there is a natural identification of \( W^P \) with \( I(k, n) \) sending \( w \) to \((w(1), w(2), \ldots, w(k))\).

In [12], Lakshmibai–Kreiman gave a self-contained presentation of standard monomial theory for unions of Richardson varieties in the Grassmannian. In the same paper, they determine a basis for the tangent space and gave a criteria for smoothness for \( X_w^v \) at any \( T \)-fixed point \( e_T \). Billey–Coskun [3] introduces a generalisation of Richardson varieties called the intersection varieties. They characterize the smooth Richardson varieties in terms of vanishing conditions on certain products of cohomology classes for Schubert varieties. Richardson varieties in the Grassmannian are also studied in [18], where these varieties are called skew Schubert varieties.

Let \( p : Gr_{k,n} \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n) \) denote the Plücker embedding where a \( k \)-dimensional vector space is sent to its \( k \) wedges. Let \( \mathcal{M} \) denote the pullback \( p^*\mathcal{O}(1) \). We note that \( \mathcal{M} \) is a \( T \)-linearised line bundle on \( Gr_{k,n} \). In [13], Kumar showed that \( \mathcal{L} := \mathcal{M}^n \) descends to the GIT quotient of the \( Gr_{k,n} \) by the maximal torus \( T \). Let \((Gr_{k,n})^s_T(\mathcal{L})\) denote the set of all semistable (respectively, stable) points with respect to the \( T \)-linearized line bundle \( \mathcal{L} \). When \( k \) and \( n \) are coprime, Skorobogatov [17] and independently Kannan [7] have shown that all semistable points are stable. They further prove that the GIT quotient \( T\backslash(Gr_{k,n}^s)T(\mathcal{L}) \) is smooth.

In [10], Kannan–Sardar showed that there is a unique minimal Schubert Variety \( X(w_{k,n}) \) in \( Gr_{k,n} \) admitting semistable points with respect to the line bundle \( \mathcal{L} \) whenever \( k \) and \( n \) are coprime. However, there was a computational error in their final result which has been corrected in Bakshi–Kannan–Subrahmanyam [2] while obtaining a simpler proof. Let \( w_{k,n} = (a_1, a_2, \ldots, a_k) \). In [9], Kannan–Paramasamy–Pattanayak–Upadhyay gave a condition on \( v \) and \( w \) for which the semistable locus in the Richardson variety \( X_w^v \) in \( Gr_{k,n} \) is non empty. The GIT quotient \( T\backslash X(w_{k,n})^s_T(\mathcal{L}) \) is shown to be smooth in [2] by Bakshi–Kannan–Subrahmanyam. In the same paper, they showed that there exist a unique minimal Schubert variety \( X_{w_{k,n}}^{v_{k,n}} \) in \( Gr_{k,n} \) admitting semistable points with respect to the line bundle \( \mathcal{L} \). They show \( v_{k,n} = (1, a_1, \ldots, a_{k-1}) \). In [1], Bakshi–Kannan–Subrahmanyam gave a combinatorial criteria on \( w \) for which the GIT quotient \( T\backslash X(w)^s_T(\mathcal{L}) \) is smooth. More recently, a similar criteria is also obtained by Chary–Pattanayak [4].

In this paper, we study the GIT quotients of Richardson varieties \( X_w^v \) in the Grassmannian generalising the result of Bakshi–Kannan–Subrahmanyam in [1]. We obtain a condition on \( v \) and \( w \) for which \( T\backslash(X_w^v)^s_T(\mathcal{L}) \) is smooth. Our main result is the following:

**Theorem 1.1.** Let \( k \) and \( n \) be coprime. Let \( v = (c_1, c_2, \ldots, c_k) \) and \( w = (b_1, b_2, \ldots, b_k) \) be such that \( v \leq v_{k,n} \) and \( w \geq w_{k,n} \). Let \( X_{w_1}^{v_1}, \ldots, X_{w_r}^{v_r} \) be \( r \) components in the singular locus of \( X(w) \). Then the following are equivalent

1. \( T\backslash(X_w^v)^s_T(\mathcal{L}) \) is smooth.
2. We have \( w_i \not\in w_{k,n} \) and \( v_i \not\in v_{k,n} \) for all \( i \).
3. Whenever \( b_j \geq b_{j-1} + 2 \), we have \( a_j \geq b_{j-1} + 1 \), and whenever \( c_j \geq c_{j-1} + 2 \), we have \( a_{j-1} \leq c_j + 1 \).
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3 Singular locus of Richardson varieties

Let \( 1 \leq k < n \). Let \( \text{Gr}_{k,n} \) be the Grassmannian variety of \( k \) dimensional subspace in \( n \) dimensional complex vector space. We recall that the Plücker embedding

\[
p : \text{Gr}_{k,n} \longrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)
\]

takes a \( k \)-dimensional vector space \( V \) and maps it to \([\bigwedge^k(V)]\).

The map \( p \) gives a closed embedding of \( \text{Gr}_{k,n} \) inside \( \mathbb{P}(\bigwedge^k \mathbb{C}^n) \), hence giving a projective variety structure to the Grassmannian. Let

\[
I(k,n) = \{(i_1,i_2,\ldots,i_k)|1 \leq i_1 < i_2 \cdots < i_k \leq n\}.
\]

We recall that \( G = \text{SL}(n,\mathbb{C}) \) acts transitively on \( \text{Gr}_{k,n} \). Denoting \( e_1,\ldots,e_n \) as the standard basis of \( \mathbb{C}^n \) we observe that the stabiliser of \( \langle e_1,e_2,\ldots,e_k \rangle \) is given by the subgroup \( P = \left[ \begin{array}{cc} * & * \\ 0_{n-k,k} & * \end{array} \right] \). Thus \( \text{Gr}_{k,n} \) gets identified with \( G/P \). Let \( T \) be a maximal torus consisting of the diagonal matrices in \( G \). Let \( B \) denote the Borel subgroup consisting of all upper triangular matrices in \( G \) and \( B^- \) the opposite Borel subgroup consisting of all lower triangular matrices in \( G \). Let \( S_d \) denote the symmetric group in \( d \) letters. We know that the Weyl group of \( G \) with respect to \( T \) is \( S_n \). The Weyl group \( W_P \) of the parabolic subgroup \( P \) is given by \( S_k \times S_{n-k} \). The set of minimal length coset representatives \( W_P = W/W_P \) gets identified with the set \( I(k,n) \): \( w \mapsto (w(1),w(2),\ldots,w(r)) \). For \( w \in I(k,n) \), let \( C_w := BwP/P \) (respectively, \( C^w := B^-wP/P \)) denote the Schubert cell (respectively, opposite Schubert cell). The Schubert variety \( X(w) \) (respectively, opposite Schubert variety \( X^w \)) is \( \overline{C_w} \) (respectively, \( \overline{C^w} \)) inside \( \text{Gr}_{k,n} \). For \( v, w \in I(k,n) \), the Richardson variety \( X_w^v \) is defined as \( X(w) \cap X^v \). Note that this intersection is a scheme-theoretic intersection.

Let \( v, w \in I(k,n) \). Let \( w = (w_1,w_2,\ldots,w_k) \) and \( v = (v_1,v_2,\ldots,v_k) \). We define a partial order \( \leq \) on \( I(k,n) \) as \( v \leq w \) if and only if \( v_t \leq w_t \) for all \( 1 \leq t \leq k \). This order is called the Bruhat order on \( I(r,n) \). We recall from [12] that \( X^v_w \) is non empty if and only \( v \leq w \). One also notes that \( X^v_{w_1} \subset X^v_{w_2} \) if and only if \( w_1 \leq w_2 \) and \( v_1 \geq v_2 \) in the Bruhat order.

The singular loci of Schubert varieties in \( \text{Gr}_{k,n} \) were determined by Lakshmibai and Weyman [15]. We briefly recall their description of the singular locus. Let \( w = (w_1,w_2,\ldots,w_k) \in I(k,n) \). We associate to each \( w \) a weakly increasing sequence \( w = (w_1,w_2,\ldots,w_k) \) where \( w_i = w_i - i \), so \( 0 \leq w_1 \leq w_2 \leq \ldots \leq w_k \leq n - k \). Each such \( w \) can be represented by Young diagram \( Y(w) \), in a \( k \times (n-k) \) rectangle by putting \( w_i \) boxes in the \( i \)-th row from bottom. Note the filling of the boxes are from left to right.

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Recall the following theorem from [15].

**Theorem 3.1 (Theorem 5.3 [15]).** Let $X(w)$ be a Schubert variety in the Grassmannian. Let $w = (p_1^{q_1}, \ldots, p_r^{q_r}) = (p_1, \ldots, p_1, \ldots, p_r, \ldots, p_r)$ be the non-zero parts of the increasing sequence $w$ with $1 \leq p_1 < p_2 \cdots < p_r \leq n - k$. The singular locus $X(w)$ consists of $r - 1$ components. The components are given by the Schubert varieties corresponding to the Young diagrams $\mathcal{Y}(w_1), \mathcal{Y}(w_2), \ldots, \mathcal{Y}(w_{r-1})$ where the sequences $w_i$ are given by

$$w_i = (p_1^{q_1}, \ldots, p_{i-1}^{q_{i-1}}, (p_i - 1)^{q_i+1}, p_{i+1}^{q_{i+1} - 1}, p_{i+2}^{q_{i+2}}, \ldots, p_r^{q_r}),$$

for $1 \leq i \leq r - 1$ and $1 \leq p_i < p_{i+1}$.

A box in a Young diagram is said to be a **peak** if it does not have any box to its **south** or **east**. A box is said to be a **valley** if there are boxes in its south and east but there is no box to its **southeast** in the Young diagram. An easy way to remember description of the irreducible components of the singular locus of $X(w)$ is as follows: they are the Schubert varieties in correspondence with Young diagram $\mathcal{Y}(w_i)$ obtained from $\mathcal{Y}(w)$ by removing the hook which consists of two adjacent peaks and one valley.

**Example 3.2.** Let us consider the Schubert variety $X((3,5,7,9))$ in $\text{Gr}_{4,9}$. The Young diagram corresponding to this Schubert variety looks like

![Young diagram](image)

The singular locus obtained by removing the hooks consists of Schubert varieties $X((2,3,7,9)), X((3,4,5,7))$ and $X((3,5,6,7))$. The corresponding Young diagrams are given by

![Young diagrams](image)

We can similarly give a description of the singular locus of opposite Schubert varieties in the Grassmannian. Let $v = (v_1, v_2, \ldots, v_k) \in I(k, n)$. Let $v'_i = n + 1 - v_{k-i}$. Let $v' = (v'_1, v'_2, \ldots, v'_k)$. We first observe that $v' \in I(k, n)$. Let $w_0$ be the unique longest element of $W$. We recall from [12] that since $X^v$ is isomorphic to $w_0X(v)$, we have $X^v$ is isomorphic to the Schubert variety $X(v')$.

We associate an **opposite Young diagram** $\mathcal{Y}^v$ by removing the Young diagram $\mathcal{Y}(v)$ from the $k \times (n - k)$ rectangle. So the opposite Young diagram $\mathcal{Y}^v$ will have $n - k - v_i + i$ boxes in the $i$-th row from bottom. A box in an opposite Young diagram is said to be a **peak** if it does not have any box to its **north** or **west**. A box is said to be a **valley** if there are boxes in its north and west and if there is no box to its **northwest** in the opposite Young diagram. The irreducible components of the singular locus of the opposite Schubert variety $X^v$ are the opposite Schubert varieties $X^v$ such that the associated opposite Young diagram $\mathcal{Y}^v$ are obtained from $\mathcal{Y}^v$ by removing the hook which consists of two adjacent peaks and one valley.

\[^1\]Note that the notation we use is different from that used in [15], they work with non-increasing sequences.
Example 3.3. Let us consider the opposite Schubert variety $X^{(2,4,5,7)}$ in $Gr_{4,9}$. We first mark the Young diagram $Y(w)$ in red inside the $4 \times 5$ rectangle.

After removing the red rectangle, we obtain the following opposite Young diagram $Y^v$.

The irreducible components of the singular locus obtained by removing the hooks are the opposite Schubert varieties $X^{(4,5,6,7)}, X^{(2,4,7,8)}$ whose opposite Young diagrams are given by the following.

We now give a description for the singular locus of Richardson varieties in the Grassmannian in terms of diagrams. Using Kleiman’s transversality theorem [11], Billey–Coskun characterized the singular locus of Richardson varieties. We recall the following theorem from Billey–Coskun [3], which we are going to use in this paper.

Theorem 3.4. [3, Corollary 2.10] Let $X^v_w$ be a non-empty Richardson variety in $Gr_{k,n}$. Let $X(w)_{\text{sing}}$ and $X^v_{\text{sing}}$ denote the singular loci of the Schubert variety $X(w)$ and the opposite Schubert variety $X^v$ respectively. Then the singular locus of $X^v_w$ is the union of the Richardson varieties

$$(X^v_w)_{\text{sing}} = (X(w)_{\text{sing}} \cap X^v) \cup (X^v_{\text{sing}} \cap X(w)).$$

Let $w = (w_1, w_2, \ldots, w_k)$ and $v = (v_1, v_2, \ldots, v_k) \in I(k, n)$ be such that $v \leq w$. Let $X^v_w$ be a Richardson variety. We have associated a Young diagram $Y(w)$ to the Schubert variety $X(w)$ and an opposite Young diagram $Y^v$ to the opposite Schubert variety $X^v$. We now associate a skew Young diagram $Y^v_w$ to the Richardson variety $X^v_w$. It is obtained by removing $Y^v$ from $Y_w$. A box in a skew Young diagram is said to be a peak if either it does not have any box to its south or east or it does not have any box to its north or west. A box in a skew Young diagram is said to be a valley if either there are boxes in its south and east and it has no box in its southeast or if there are boxes in its north and west and it has no box in its northwest in the skew Young diagram. The irreducible components of the singular locus of the Richardson variety $X^v_w$ are the Richardson varieties $X^v_{w_i}$ such that the associated skew Young diagram $Y^v_{w_i}$ are obtained from $Y^v$ by removing the hook which consists of two adjacent peaks and one valley.
Example 3.5. Let $X(w), X^v$ and $X^w_v$ denote the Schubert variety, opposite Schubert variety and the Richardson variety in $\text{Gr}_{4,9}$ for $v = (2, 4, 5, 7)$ and $w = (3, 5, 7, 9)$. The Young diagram $\mathcal{Y}(w)$ corresponding to the Schubert variety $X(w)$ is the subdiagram of $4 \times 5$ rectangle consisting of the blue and red boxes. The opposite Young diagram $\mathcal{Y}^v$ corresponding to the opposite Schubert variety $X^v$ is the subdiagram consisting of the blue and green boxes. And the skew Young diagram $\mathcal{Y}_w^v$ corresponding to the Richardson variety $X^w_v$ corresponds to the region enclosed by the blue boxes.

We first note that there are three Schubert varieties which are in the singular locus of the Schubert variety $X(w)$ namely $X((2, 3, 7, 9)), X((3, 4, 5, 9))$ and $X((3, 5, 6, 7))$ and two opposite Schubert varieties which are in the singular locus of the opposite Schubert variety $X^{(4, 5, 6, 7)}$ and $X^{(2, 4, 7, 8)}$. So the singular locus of the Richardson variety is the union of the following Richardson varieties $X^{(2, 3, 7, 9)}, X^{(2, 4, 5, 7)}, X^{(2, 4, 5, 7)}, X^{(4, 5, 6, 7)}$ and $X^{(2, 4, 7, 8)}$. But we observe that $X^{(2, 4, 5, 7)}$ and $X^{(4, 5, 6, 7)}$ is empty since $(2, 4, 5, 7) \not\leq (2, 3, 7, 9)$ and $(4, 5, 6, 7) \not\leq (3, 5, 7, 9)$ in the Bruhat order. So the non empty Richardson varieties in the singular locus are $X^{(2, 4, 5, 7)}, X^{(2, 4, 5, 7)}, X^{(2, 4, 5, 7)}$ and $X^{(2, 4, 7, 8)}$. Indeed, we note that the skew Young diagram associated with $X^w_v$ has three hooks which consists of two adjacent peaks and one valley. The skew Young diagram corresponding to $X^{(2, 4, 5, 7)}, X^{(2, 4, 5, 7)}$ and $X^{(2, 4, 7, 8)}$ are marked by the blue boxes in the $4 \times 5$ rectangle.

4 Minimal dimensional Richardson varieties admitting semistable points

We recall from [2], that there exists a unique minimal dimensional Schubert variety $X(w_{k,n})$ in $\text{Gr}_{k,n}$ which admits semistable point for $k$ and $n$ coprime. The existence of the unique Schubert variety was first shown by Kannan–Sardar [10], however there was a computational error in their description of $w_{k,n}$. Later Bakshi–Kannan–Subrahmanyam [2] corrected the error and came up with a simpler proof for the same.

Theorem 4.1. [2, Proposition 2.2] Let $k$ and $n$ be coprime. Then $w_{k,n} = (a_1, a_2, \ldots, a_k)$ where $a_i$ is the smallest integer such that $a_i \cdot k \geq i \cdot n$.

In [9], Kannan–Paramasamy–Pattanayak–Upadhyay gave a condition on $v$ and $w$ for which the semistable locus in the Richardson variety $X^w_v$ in $\text{Gr}_{k,n}$ is non empty. Bakshi–Kannan–
Subrahmanyam [2], showed that there is a unique minimal dimensional Richardson variety $X_{w_{k,n}}^v$ in the $\text{Gr}_{k,n}$ which admits a semistable point using standard monomials. We recall their theorem.

**Theorem 4.2.** [2, Proposition 3.2] Let $k$ and $n$ be coprime. Let $v_{k,n}$ be such that $X_{w_{k,n}}^v$ is the smallest Richardson variety in $X(v_{k,n})$ admitting semistable points. Then $v_{k,n} = (1, a_1, \ldots, a_{k-1})$ with the $a_i$ defined as the smallest integer satisfying $a_i \cdot k \geq i \cdot n$.

**Example 4.3.** Let us consider our running example of $\text{Gr}_{4,9}$. Using the above criteria, one checks that $w_{4,9} = (3, 5, 7, 9)$ and $v_{4,9} = (1, 3, 5, 7)$. The skew Young diagram that one associates with the Richardson variety is the blue region in the rectangle below

![Diagram](image)

**Remark 4.4.** We note from 4.2 that a Richardson variety $X_w^v$ contains semistable points iff $X_{w_{k,n}}^v \subset X_w^v$ iff $v \leq v_{k,n}$ and $w \geq w_{k,n}$. For instance, the Richardson variety $X_{(1,2,4,7)}^{(1,2,4,7)}$ contains semistable points, however $X_{(3,6,7,9)}^{(1,2,6,7)}$ does not contain any semistable point.

## 5 Smooth quotients of Richardson varieties

Let $k$ and $n$ be coprime. The following lemma is going to be the key for our main theorem. The proof of this lemma follows along the lines described in [8, Example 3.3].

**Lemma 5.1.** Let $v, w \in I(k, n)$. Then $T \backslash (X_w^v)^{ss} (\mathcal{L})$ is smooth if and only if $(X_w^v)^{ss} \subseteq (X_w^v)_{sm}$.

**Proof.** We first observe that $(X_w^v)^{ss} = (X_w^v)^s$, since $\gcd(k, n) = 1$ (see [7, Theorem 3.3], [17, Corollary 2.5]). Let $T \backslash (X_w^v)^{ss} (\mathcal{L})$ be smooth. Let $x \in (X_w^v)^{ss}$. If $x$ is not a smooth point in $X_w^v$ then its image in the quotient will not be a smooth point as well, violating our assumption.

Conversely, let us assume $(X_w^v)^{ss} \subseteq (X_w^v)_{sm}$. Let $x \in (X_w^v)^{ss}$. Let $x \in B\tau P/P$ for $v \leq \tau \leq w$. Let $\beta_1, \ldots, \beta_r$ be a subset of positive roots such that $x = u_{\beta_1}(t_1) \ldots u_{\beta_r}(t_r)\tau P/P$ with $u_{\beta_j}(t_j)$ in the root subgroup $U_{\beta_j}$, $t_j \neq 0$ for $j = 1, \ldots, r$. The isotropy group $T_x$ is $\cap_{i=1}^r \ker(\beta_j)$. We note from [8, Example 3.3], $T_x$ is finite and $T_x = Z(G)$, the center of $G$. Working with the adjoint group we may assume that the stabler is trivial. So $T \backslash X(w)^{ss} (\mathcal{L}(\omega_r))$ is smooth.

We recall from §3 that $w_{k,n} = (a_1, a_2, \ldots, a_k)$ and $v_{k,n} = (1, a_1, \ldots, a_{k-1})$, where $a_i$ is the smallest integer satisfying $a_i \cdot k \geq i \cdot n$. We recall the main result of Bakshi–Kannan–Subrahmanyam [1].

**Theorem 5.2.** [1, Theorem 3.2] Let $w = (b_1, b_2, \ldots, b_k) \in I(k, n)$ with $b_i \geq a_i$, for all $1 \leq i \leq k$. Let $X(v_1), \ldots, X(v_r)$ be $r$ components in the singular locus of $X(w)$. Then the following are equivalent

1. $T \backslash X(w)^{ss} (\mathcal{L})$ is smooth.
(2) We have \( v_i \not\in w_{k,n} \) for all \( i \).

(3) Whenever \( b_j \geq b_{j-1} + 2 \) we have \( a_j \geq b_{j-1} + 1 \).

Our goal in this paper is to extend the above result for Richardson varieties. We now prove the main theorem of this paper. This builds on the proof of 5.2 from [1].

**Theorem 5.3.** Let \( k \) and \( n \) be coprime. Let \( v = (c_1, c_2, \ldots, c_k) \) and \( w = (b_1, b_2, \ldots, b_k) \) be such that \( v \leq v_{k,n} \) and \( w \geq w_{k,n} \). Let \( X_{w_1}^v, \ldots, X_{w_r}^v \) be \( r \) components in the singular locus of \( X(w) \). Then the following are equivalent

1. \( T \backslash (X_w^v)_{T}^{ss}(L) \) is smooth.
2. We have \( w_i \neq w_{k,n} \) and \( v_i \neq v_{k,n} \) for all \( i \).
3. Whenever \( b_j \geq b_{j-1} + 2 \), we have \( a_j \geq b_{j-1} + 1 \), and whenever \( c_j \geq c_{j-1} + 2 \), we have \( a_{j-1} \leq c_j + 1 \).

**Proof.** We first observe that \((X_w^v)_{T}^{ss}(L)\) is nonempty since \( v \leq v_{k,n} \) and \( w \geq w_{k,n} \). We know from 5.1 that \( T \backslash (X_w^v)_{T}^{ss}(L) \) is smooth if and only if \((X_w^v)^{ss} \subseteq (X_w^v)_{sm} \), where \((X_w^v)_{sm} \) denotes the smooth locus of \((X_w^v)\). So whenever \( w_i \neq w_{k,n} \) and \( v_i \neq v_{k,n} \), we observe that the singular locus of \((X_w^v)\) does not contain a semistable point. Hence, \( T \backslash (X_w^v)_{T}^{ss}(L) \) is smooth if and only if \((X_w^v)^{ss} \subseteq (X_w^v)_{sm} \), which further implies that the singular locus of \((X_w^v)\) cannot contain a semistable point. Hence, (1) and (2) are equivalent.

We note from 3.4 that the singular locus of the Richardson variety \( X_w^v \) is the union of Richardson varieties which are either of the form \( X_w^{v_j} \) or of the form \( X_w^{v'} \), where \( X(w') \) (respectively, \( X_w^{v'} \)) are the Schubert varieties (respectively, opposite Schubert varieties) in the irreducible component of the singular locus of the Schubert variety \( X(w) \) (respectively, \( X_w^v \)). Let \( X_w^{v_j} \) be a Richardson variety which is in the singular locus of \( X_w^v \), where \( X(w') \) is the Schubert variety in the singular locus of \( X(w) \) obtained by removing a hook at a valley which lies in \( j \)-th row from bottom of the Young diagram \( Y(w) \). Now from our hypothesis and 4.4, we note that \( X_w^{v_j} \) contains a semistable point if and only if \( X_{w_k,n}^{v_j} \subseteq X_{w}^{v_j} \) if and only if \( w_{k,n} \leq w' \). Let \( w' = (b'_1, b'_2, \ldots, b'_k) \). Recall \( w_{k,n} = (a_1, a_2, \ldots, a_k) \). By assumption, we have \( b_j \geq b_{j-1} + 2 \). Let \( P \) denote the set of all \( t \) such that \( b'_t \neq b_t \). Clearly, for \( t > j \) we have \( b'_t = b_t \). Let \( P = \{m, m+1, \ldots, j\} \), where \( 1 \leq m \leq j \). We have \( a_t \leq b'_t \) for all \( t \in \{1, 2, \ldots, k\} \). We also have \( a_t + 2 \leq a_{t+1} \) for all \( t \). Therefore, we have \( a_t \leq b'_t \) for all \( t \) if and only if \( a_j < b_{j-1} + 1 = b'_j \). Equivalently, \( w_{k,n} \leq w' \). So, we have proved that \( w_i \not\in w_{k,n} \) for all \( i \) if and only if \( b_j \geq b_{j-1} + 2 \), we have \( a_j \geq b_{j-1} + 1 \).

Let \( X_w^{v_j} \) be a Richardson variety which lies in the singular locus of \( X_w^v \), where \( X_w^{v_j} \) is the opposite Schubert variety in the singular locus of \( X_w^v \) which is obtained by removing a hook at a valley which lies in \( j \)-th row from bottom of the opposite Young diagram \( Y^{v_j} \). As in the previous paragraph, we note that \( X_w^{v_j} \) contains a semistable point if and only if \( X_{w_k,n}^{v_j} \subseteq X_{w}^{v_j} \) if and only if \( v_{k,n} \geq v' \). Let \( v' = (c'_1, c'_2, \ldots, c'_k) \). Recall \( v_{k,n} = (1, a_1, \ldots, a_k) \). Let \( a_0 = 1 \). By assumption, we have \( c_j \geq c_{j-1} + 2 \). In this case, we have \( c'_t = c_t \) whenever \( t < j \). Let \( Q \) denote the set of all \( t \) such that \( c'_t \neq c_t \). Let \( Q = \{j, j+1, \ldots, m\} \) where \( 1 \leq m \leq k \). We have \( a_t \geq b'_t \) for all \( t \in \{1, 2, \ldots, k\} \). So, we will have \( a_{t-1} \geq c'_t \) for all \( 1 \leq t \leq k-1 \) if and only if \( a_j \geq c_j + 1 = c'_j \). Equivalently,
we have proved that \( v_i \not\in v_{k,n} \) for all \( i \) iff whenever \( c_j \geq c_{j-1} + 2 \), we have \( a_{j-1} \leq c_j + 1 \). This completes the proof.

\[ \square \]

6 Examples and Non-examples

In this section we discuss some examples of smooth and singular quotients of Richardson varieties in \( \text{Gr}_{4,9} \).

Example 6.1. We know that \( X_{(3,5,7,9)}^{(1,3,5,7)} \) is the minimal dimensional Richardson variety admitting a semistable point. Hence, \( T\sslash(X_{(3,5,7,9)}^{(1,3,5,7)})^{ss}(L) \) is smooth.

Example 6.2. Let us consider the Richardson variety \( X_{(3,5,7,9)}^{(1,3,4,6)} \). The singular locus consists of the following Richardson varieties: \( X_{(3,5,7,9)}^{(1,3,7,9)} \), \( X_{(3,5,7,9)}^{(1,3,4,6)} \) \( X_{(3,4,5,6)}^{(1,3,4,6)} \), \( X_{(3,4,5,6)}^{(3,4,5,9)} \) \( X_{(3,5,7,9)}^{(3,4,5,9)} \), \( X_{(3,5,7,9)}^{(3,5,6,7)} \), and \( X_{(3,5,7,9)}^{(1,3,4,6)} \). Using 5.3 we observe that none of the Richardson varieties which lies in the singular locus contains semistable points. Hence, \( T\sslash(X_{(3,5,7,9)}^{(1,3,5,7)})^{ss}(L) \) is smooth.

Example 6.3. Consider the Richardson variety \( X_{(3,5,7,9)}^{(1,2,3,5)} \). The singular locus consists of the following Richardson varieties: \( X_{(3,5,7,9)}^{(1,2,5,6)} \), \( X_{(2,3,7,9)}^{(1,2,3,5)} \), \( X_{(3,4,5,9)}^{(1,2,3,5)} \), and \( X_{(3,5,6,7)}^{(1,2,3,5)} \). Note that the Richardson variety \( X_{(3,5,7,9)}^{(1,2,5,6)} \) admits semistable points. So \( T\sslash(X_{(3,5,7,9)}^{(1,2,4,6)})^{ss}(L) \) is not smooth.

Example 6.4. Consider the Richardson variety \( X_{(5,7,8,9)}^{(1,3,4,6)} \). We observe that the singular locus contains the Richardson variety \( X_{(4,5,8,9)}^{(1,3,4,6)} \) which contains semistable points. Hence \( T\sslash(X_{(5,7,8,9)}^{(1,3,4,6)})^{ss}(L) \) is not smooth either.

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