A Note on Quantum Bell Nonlocality and Quantum Entanglement for High Dimensional Quantum Systems

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Abstract
We study the Bell nonlocality of high dimensional quantum systems based on quantum entanglement. A quantitative relationship between the maximal expectation value $B$ of Bell operators and the quantum entanglement concurrence $C$ is obtained for even dimension pure states, with the upper and lower bounds of $B$ governed by $C$.

Keywords Quantum nonlocality · Quantum entanglement · Concurrence

1 Introduction
Quantum nonlocality, such as that revealed by the violation of Bell inequalities by quantum entangled states [1], is one of the most startling predictions of quantum mechanics. Recently, as confirmed in loophole-free experiments [2–4], nonlocality has been proven to be useful in many quantum tasks such as device-independent cryptography [5] and randomness certification [6, 7].

According to Gisin theorem, all entangled pure states can display nonlocal correlations [8–11], but some mixed entangled states can provably satisfy local hidden variable models [12–15]. Namely, there exist entangled mixed states that never lead to nonlocality by any local POVM measurements [13]. Quantum nonlocality is usually associated with entangled states that violate at least one of the Bell inequalities. However, separable bipartite states can also show some nonlocal properties [16, 17]. Despite all these progresses, the precise relationship between entanglement and Bell violations has remained less known, particularly for high dimensional bipartite and multipartite cases.

Different ways to quantify the quantum nonlocality have been presented [18–26], for examples, the volume of the violation of Bell-type inequalities [21, 22]. By employing

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the probability of violation of local realism under random measurements [27], in [24]
the authors investigated the nonlocality of entangled qudits with dimensions ranging from
d = 2 to d = 10. In [25] the authors proposed a machine learning approach for detection
and quantification of nonlocality. Quantifying Bell nonlocality by the trace distance has
been studied in [23].

For two-qubit states, the well-known CHSH inequality [28] has been used to detect the
nonlocality. The corresponding operator is given by \( B = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2 \). The mean value \( B \equiv \langle B_\rho \rangle \). For separable bipartite pure states satisfies the CHSH
inequality, \( B \leq 2 \), where \( A_i = \vec{a}_i \cdot \vec{\sigma}, B_j = \vec{b}_j \cdot \vec{\sigma}, \vec{a}_i \) and \( \vec{b}_j \) are three-dimensional real unit
vectors, \( i, j = 1, 2, \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) with \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) the standard Pauli matrices.

In the following we use concurrence \( C \) [29, 30] as the measure of quantum entanglement.
Let \( H_i \) denote the Hilbert space associated with the \( i \)th subsystem. For a pure state \( |\psi\rangle \in
H_1 \otimes H_2 \), the concurrence is defined by [31–33], \( C(|\psi\rangle) = \sqrt{2(1 - Tr\rho_1^2)} \), where the reduced density matrix \( \rho_1 = Tr_2|\psi\rangle\langle\psi| \) is obtained by tracing over the second subsystem. The concurrence is then extended to mixed states \( \rho \) by convex roof,

\[
C(\rho) \equiv \min_{p_i, \psi_i} \sum_i p_i C(|\psi_i\rangle),
\]

where the minimization goes over all possible ensemble realizations \( \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \),
\( p_i \geq 0 \) and \( \sum_i p_i = 1 \). For two-qubit states \( C \) can be calculated directly [29]. For high
dimensional quantum states one has no general results [34, 35].

For two-qubit states, the relationship between the concurrence \( C \) and the mean value \( B \)
satisfy the relation: \( 2\sqrt{2}C \leq B \leq 2\sqrt{1 + C^2} \) [36]. More recently, [37] obtain the necessary
and sufficient condition that the upper bound can be reached. Therefore, the Bell inequality
is violated if \( C > 1/\sqrt{2} \). Such relations have been also investigated by using randomly
generated two-qubit states [38]. The Bell inequality for three-qubit states has been also studied [39]. The relationship between tripartite entanglement and genuine tripartite nonlocality
for three qubit Greenberger-Horne-Zeilinger class is also investigated [40]. The authors in
[41, 42] studied the relation between the upper bound of Bell violation and a generalized
concurrence for some \( n \)-qubit states. In [43] the nonlocality distributions among multiqubit
systems have been studied based on the maximal violations of the CHSH inequality of
reduced pairwise qubit systems. Furthermore, from the reduced three-qubit density matrices
of the four-qubit generalized Greenberger-Horne-Zeilinger (GHZ) states and W-states,
a trade-off relation among the mean values of the Svetlichny operators associated with these
reduced states has been presented [44].

For high dimensional quantum systems less is known about the relationship between concurrence and Bell violations. The main difficulty lies in finding the mean value of suitable
Bell operators. In this paper, we explore the quantitative relationship between concurrence
\( C \) and the Bell value \( B \) for high dimensional quantum systems.

## 2 Bell Nonlocality and Concurrence of Bipartite Quantum States

Based on Bell’s idea [1], for any given \( n \times n \) real matrix \( N \) with entries \( N_{ij} \), one can define a classical quantity,

\[
J(N) = \sup_{i,j=1}^n \left| \sum_{i,j=1}^n N_{ij} a_i b_j \right|,
\]
where the supremum is taken over all possible assignment $a_i, b_j \in \{-1, 1\}, 1 \leq i, j \leq n$. For any bipartite state $\rho$, the corresponding Bell operator is defined by

$$B(N) = \sum_{i,j=1}^{n} N_{ij} A_i \otimes B_j,$$

where $A_i$ and $B_j$ are arbitrary observables whose absolute values of all eigenvalues are less or equal to one.

A state $\rho$ is said to be nonlocal if it violates the following Bell inequality,

$$B(N) \leq J(N),$$

where $B(N) = tr(B(N)\rho)$ is the mean value of the Bell operator. If one takes $N = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, one gets the CHSH inequality with $J(N) = 2$.

A pure $m \otimes n (m \leq n)$ quantum state has the standard Schmidt form,

$$|\psi\rangle = \sum_{i=1}^{m} c_i |a_i b_i\rangle,$$

where $c_i (i = 1, \cdots, m)$ are the Schmidt coefficients, and they are in descending order, $|a_i\rangle$ and $|b_i\rangle$ are the orthonormal bases in $H_1$ and $H_2$, respectively. The concurrence of $|\psi\rangle$ is given by

$$C = 2 \sqrt{\sum_{i<j} c_i^2 c_j^2},$$

which varies from 0 for pure product states to $\sqrt{2(m-1)/m}$ for maximally entangled pure states [33].

By selecting the matrix $N = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ as that from the CHSH inequality, and Bell operators such as

$$A_0 = I_2 \otimes (\cos \theta \sigma_3 + \sin \theta \sigma_1),$$

$$A_1 = I_2 \otimes (\cos \theta \sigma_3 - \sin \theta \sigma_1),$$

$$B_0 = I_2 \otimes \sigma_3,$$

$$B_1 = I_2 \otimes \sigma_1,$$

for even $m, n$ or

$$A_0 = \begin{pmatrix} I_{[\frac{m}{2}]} \otimes (\cos \theta \sigma_3 + \sin \theta \sigma_1) & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} I_{[\frac{m}{2}]} \otimes (\cos \theta \sigma_3 - \sin \theta \sigma_1) & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} I_{[\frac{m}{2}]} \otimes \sigma_3 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} I_{[\frac{m}{2}]} \otimes \sigma_1 & 0 \\ 0 & 1 \end{pmatrix}.$$
for odd \( m, n \), then for any bipartite pure state \( |\psi\rangle \) as given by (1), it has been shown that [45],

\[
B = 2\sqrt{(1 - \gamma)^2 + K^2 + 2\gamma},
\]

where \( K = 2(c_1c_2 + c_3c_4 + \cdots) \), \( \gamma = c_m^2 \) for odd \( m \), and \( \gamma = 0 \) for even \( m \).

In the following we use (3) to obtain the following facts, which does not depend on the optimality of (3).

**Theorem 1** For any pure \( m \otimes n (m \leq n) \) quantum state \( |\psi\rangle \) as given by (1), we have

\[
B \leq 2\sqrt{1 + C^2}
\]

for even \( m \).

Proof: From (3) we have for even \( m \),

\[
B = 2\sqrt{1 + K^2}.
\]

To prove the theorem, we only need to prove that

\[
C^2 \geq K^2,
\]

namely

\[
\sum_{i<j} c_i^2 c_j^2 \geq (c_1c_2 + c_3c_4 + \cdots)^2.
\]

We prove (6) by induction. For \( m = 2 \), we have

\[
C^2 = 4c_1^2 c_2^2 \geq K^2 = 4c_1^2 c_2^2,
\]

and the inequality (6) holds in this case. Assume that the inequality (6) is true for \( m = 2(k - 1) \) for arbitrary positive integer \( k \). For \( m = 2k \) we have

\[
\sum_{i<j} c_i^2 c_j^2 = \sum_{i<j} c_i^2 c_j^2 \geq \sum_{i} c_i^2 c_{2k-1}^2 + \sum_{i} c_i^2 c_{2k}^2
\]

and

\[
(c_1c_2 + \cdots + c_{2(k-1)-1}c_{2(k-1)} + c_{2k-1}c_{2k})^2
\]

\[
= (c_1c_2 + \cdots + c_{2(k-1)-1}c_{2(k-1)})^2
\]

\[
+ 2(c_1c_2 + \cdots + c_{2(k-1)-1}c_{2(k-1)})c_{2k-1}c_{2k}
\]

\[+ c_{2k-1}^2 c_{2k}^2.
\]

According to the assumption for \( m = 2(k - 1) \), we have

\[
\sum_{i<j} c_i^2 c_j^2 \geq (c_1c_2 + \cdots + c_{2(k-1)-1}c_{2(k-1)})^2.
\]

Therefore, we have

\[
\sum_{i} c_i^2 c_{2k-1}^2 + \sum_{i} c_i^2 c_{2k}^2 \geq \sum_{j=1}^{k-1} c_{2j-1}^2 c_{2k-1}^2 + \sum_{j=1}^{k-1} c_{2j}^2 c_{2k}^2
\]

\[
\geq 2(c_1c_2 + \cdots + c_{2(k-1)-1}c_{2(k-1)})c_{2k-1}c_{2k}.
\]

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where the second inequality is due to the inequality $a^2 + b^2 \geq 2ab$ for $a \geq 0$ and $b \geq 0$. According to the inequalities (7), (8), (9) and (10), we obtain the inequality (6), which completes the proof of the inequality (4).

The upper bound of $B$ provided in inequality (4) consists with the result of two-qubit case [36]. Therefore, the inequality (4) can be regarded as the extension of the relationship between $B$ and $C$ from two-qubit to high dimensional quantum states. Besides, the inequality (4) is saturated when only two $c_i$ are not zero.

Nevertheless, the lower bound of $B$ is generally different from the one for two-qubit case. For high dimensional systems we have the following general result.

**Theorem 2** For any pure $m \otimes n (m \leq n)$ quantum state $|\psi\rangle$, with the standard Schmidt form (1), then for even $m$, we have

$$B \geq \sqrt{2[1 + C^2]}.$$  \hspace{1cm} (11)

**Proof** According to (2) and (3), the inequality (11) is equivalent to $1 + 2K^2 \geq C^2$, namely, we need to prove

$$1 + 2 \left[ 4(c_1c_2 + c_3c_4 + \cdots + c_{m-1}c_m) \right] \geq 4 \sum_{i<j} c_i^2 c_j^2.$$  

That is, for odd $k, l$, we have

$$1 + 2 \left[ 4 \sum_{k=1,l=1}^{m-1} (c_{k}c_{k+1}c_{l}c_{l+1}) \right] \geq 2 \sum_{i \neq j} c_i^2 c_j^2. \hspace{1cm} (12)$$  

Without loss of generality, we assume that the Schmidt coefficients in (1) satisfy $c_i \geq c_{i+1}, i = 1, 2, \cdots, m$. Then we have the following facts, $c_kc_{k+1}c_{l}c_{l+1} \geq c_k^2c_{l+1}^2$, $c_kc_{k+1}c_{l}c_{l+1} \geq c_{k+1}^2c_{l+2}^2$, $c_kc_{k+1}c_{l}c_{l+1} \geq c_{k+2}^2c_{l+1}^2$ and $c_kc_{k+1}c_{l}c_{l+1} \geq c_{k+2}^2c_{l+2}^2$. Hence, we have

$$4 \sum_{k=1,l=1,odd}^{m-1} (c_{k}c_{k+1}c_{l}c_{l+1}) \geq \sum_{k=1,l=1,odd}^{m-1} [c_k^2c_{l+1}^2 + c_{k+1}^2c_{l+2}^2] + \sum_{k=1,l=1,odd}^{m-1} (c_{k}^2c_{l+1}^2 + c_{k+2}^2c_{l+2}^2)$$  

and

$$1 = \left( c_1^2 + \sum_{i \neq 1}^{m} c_i^2 \right)^2 \geq 4c_1^2 \left( \sum_{i \neq 1}^{m} c_i^2 \right).$$

Combining above relations we obtain

$$1 + 2 \left[ 4 \sum_{k=1,l=1}^{m-1} (c_{k}c_{k+1}c_{l}c_{l+1}) \right] \geq 2 \sum_{k=1,l=1,odd}^{m-1} [c_k^2c_{l+1}^2 + c_{k+1}^2c_{l+2}^2] + \sum_{k=1,l=1,odd}^{m-1} (c_{k}^2c_{l+1}^2 + c_{k+2}^2c_{l+2}^2) + 4 \sum_{i \neq 1}^{m} c_i^2 c_j^2 \geq 2 \sum_{i \neq j} c_i^2 c_j^2.$$
which gives rise to the inequality (12), and proves the inequality (11).

From Theorem 2, it is obvious that if $C > 1$, the state $|\psi\rangle$ shows nonlocality.

3 Conclusions and Discussions

Quantum nonlocality is a fundamental feature in quantum mechanics. We have investigated the relation between the maximal expectation value of Bell operators $B$ and entanglement concurrence $C$. The upper and lower bounds of $B$ have been derived based on $C$. Such the relations between $C$ and $B$ play important roles in judging nonlocality from entanglement. Moreover, determining the non-locality of high dimensional quantum states has been a difficult problem in the theory of quantum information. Our results may highlight further researches on the quantum nonlocality and related the quantum correlations such as quantum steerability.

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Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations All partial information is available.

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