Dynamics of topological magnetic solitons*

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Abstract

A direct link between the topological complexity of magnetic media and their dynamics is established through the construction of unambiguous conservation laws for the linear and angular momenta as moments of a topological vorticity. As a consequence, the dynamics of topological magnetic solitons is shown to exhibit the characteristic features of the Hall effect of electrodynamics or the Magnus effect of fluid dynamics. The main points of this program are reviewed here for both ferromagnets and antiferromagnets, while a straightforward extension to the study of superfluids is also discussed briefly.

1. Introduction

Ferromagnetic (FM) bubbles are the best known examples of magnetic solitons and exhibit some distinct dynamical features due to their nontrivial topological structure. The inherent link between topology and dynamics was already apparent in the early work of Thiele [1] as well as in many investigations that followed [2]. The essence of the early work is best summarized by the experimentally observed skew deflection of FM bubbles under the influence of an applied magnetic-field gradient. The so-called golden rule of bubble dynamics relates the deflection angle $\delta$ to the winding number $Q$ by

$$\frac{g r^2}{2V} \sin \delta = Q,$$

(1)

where $g$ is the strength of the applied field gradient, $r$ is the bubble radius, and $V$ its speed. This relation is remarkable in two respects. First, it suggests that only topologically trivial ($Q = 0$) bubbles move in the direction of the gradient ($\delta = 0$), even though such a behavior would naively be expected for all FM bubbles; in fact, bubbles with a nonvanishing winding number ($Q = \pm 1, \pm 2, \ldots$) tend to be deflected in a direction nearly perpendicular ($\delta \sim \pm 90^\circ$) to the applied gradient, exactly so in

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the limit of vanishing dissipation. Second, eq. (1) implies some sort of a topological quantization in that it relates the integer-valued winding number to experimentally measured quantities that can, in principle, assume any values. A fresh look at this problem was initiated in ref. [3] where the link between topology and dynamics was made explicit by the construction of unambiguous conservation laws as moments of a suitable topological vorticity. The important qualitative features of bubble dynamics became then apparent. Thus, in the absence of external magnetic-field gradients or other perturbations, a bubble with a nonvanishing winding number cannot move freely but is always spontaneously pinned. On the other hand, in the absence of dissipation, a bubble with $Q \neq 0$ would be deflected at a right angle ($\delta = \pm 90^\circ$) with respect to an applied gradient, while its drift velocity can be calculated analytically in some important special cases and is generally consistent with eq. (1). The emerging picture is completely analogous to the Hall motion of an electron as well as to the Magnus effect of fluid dynamics. These analogies further suggest that the deflection angle should deviate from $90^\circ$ in the presence of dissipation. However an exact calculation of the deflection angle, i.e., a rigorous derivation of the golden rule, is no longer possible on the basis of conservation laws alone. Nonetheless the theoretical picture can be completed with more or less straightforward numerical simulations.

In this short review the emphasis is placed on some general dynamical features that enable one to detect in a systematic manner the existence (absence) of Hall or Magnus behavior in any field theory that bears topological solitons. The general framework is then briefly illustrated for ferromagnets (FM), antiferromagnets (AFM), and superfluids. In the process, we provide a complete list of references to the recent work where the issues involved are discussed in greater detail.

2. Vorticity and conservation laws

In this section we study a general field theory governed by the Hamilton equations

$$\dot{\Psi}_a = \frac{\delta W}{\delta \Pi_a}, \quad \dot{\Pi}_a = -\frac{\delta W}{\delta \Psi_a},$$

(2)

where $W$ is the Hamiltonian and $(\Psi_a, \Pi_a)$, with $a = 1, 2, \ldots, N$, is a set of $N$ canonically conjugate pairs of fields satisfying the standard Poisson bracket relations. We consider first a strictly two-dimensional (2D) theory defined in a plane with coordinates $\mathbf{x} = (x_1, x_2)$. One may then construct the scalar density

$$\gamma = \varepsilon_{\mu\nu} \partial_\mu \Pi_a \partial_\nu \Psi_a,$$

(3)

where each of the Greek indices $\mu$ and $\nu$ is summed over two distinct values corresponding to the two spatial coordinates $x_1$ and $x_2$, $\varepsilon_{\mu\nu}$ is the 2D antisymmetric
tensor, and the index $a$ is summed over all $N$ values counting the number of canonical pairs. The density $\gamma$ will be referred to as vorticity because it shares several formal properties with ordinary vorticity in fluid dynamics. The time derivative of the vorticity is then calculated from the Hamilton equations (2) to yield

$$\dot{\gamma} = -\varepsilon_{\mu\nu} \partial_\mu \tau_\nu,$$

(4)

where the vector density

$$\tau_\nu = \frac{\delta W}{\delta \Psi_a} \partial_\nu \Psi_a + \frac{\delta W}{\delta \Pi_a} \partial_\nu \Pi_a,$$

(5)

is formally analogous to the “force density” employed by Thiele [1] in the problem of FM bubbles. We proceed one step farther noting that $\tau_\nu$ may be written as a total divergence,

$$\tau_\nu = \partial_\lambda \sigma_{\nu\lambda},$$

(6)

where $\sigma_{\nu\lambda}$ is the stress tensor

$$\sigma_{\nu\lambda} = w \delta_{\nu\lambda} - \frac{\partial w}{\partial (\partial_\lambda \Psi_a)} \partial_\nu \Psi_a - \frac{\partial w}{\partial (\partial_\lambda \Pi_a)} \partial_\nu \Pi_a$$

(7)

defined in terms of the energy density $w$ identified from

$$W = \int \frac{1}{2} dx_1 dx_2.$$

Equation (4) then becomes

$$\dot{\gamma} = -\varepsilon_{\mu\nu} \partial_\mu \partial_\lambda \sigma_{\nu\lambda}$$

(8)

and proves to be fundamental for our purposes [3].

It should be noted that the preceding discussion makes no distinction between ordinary field theories and those endowed with nontrivial topological structure or related properties. However a clear distinction emerges when we consider the total vorticity

$$\Gamma = \int \gamma dx_1 dx_2 = \varepsilon_{\mu\nu} \int \partial_\mu \Pi_a \partial_\nu \Psi_a dx_1 dx_2,$$

(9)

which is conserved by virtue of eq. (8) for any field configuration with reasonable behavior at infinity. One may also write

$$\Gamma = \varepsilon_{\mu\nu} \int [\partial_\mu (\Pi_a \partial_\nu \Psi_a) - \Pi_a \partial_\mu \partial_\nu \Psi_a] dx_1 dx_2$$

(10)

to indicate that a vanishing value of the total vorticity is the rule rather than the exception. Indeed, under normal circumstances, the first term in eq. (10) is shown to vanish by transforming it into a surface integral at infinity and the second term also vanishes because $\varepsilon_{\mu\nu} \partial_\mu \partial_\nu \Psi_a = 0$ for any differentiable function $\Psi_a$. Yet the above conditions may not be met in a field theory with nontrivial topology, thus leading to
ambiguities in the canonical definitions of linear momentum $p = (p_1, p_2)$ and angular momentum $\ell$ given by

$$p_\mu = -\int \Pi_a \partial_\mu \Psi_a dx_1 dx_2, \quad \ell = -\int \Pi_a \epsilon_{\mu\nu} x_\mu \partial_\nu \Psi_a dx_1 dx_2. \quad (11)$$

In general, the above canonical conservation laws are rendered ambiguous when the total vorticity $\Gamma$ is different from zero.

Nevertheless unambiguous conservation laws can be constructed by returning to the fundamental relation (8) where the appearance of a double spatial derivative in the right-hand side implies that some of the low moments of the local vorticity $\gamma$ are conserved. Indeed the linear momentum is given by

$$p_\mu = -\epsilon_{\mu\nu} I_\nu, \quad I_\nu = \int x_\nu \gamma dx_1 dx_2, \quad (12)$$

and the angular momentum by

$$\ell = \frac{1}{2} \int \rho^2 \gamma dx_1 dx_2, \quad (13)$$

where $\rho^2 = x_1^2 + x_2^2$. The preceding identifications are made plausible by inserting the general expression for the vorticity of eq. (3) in eqs. (12) and (13) and by freely performing partial integrations to recover the canonical forms of linear and angular momenta quoted in eq. (11). However partial integrations should be performed with great care and are often unjustified when $\Gamma \neq 0$.

The effect of a nonvanishing total vorticity becomes apparent by considering the transformation of the moments $I_\nu$ of eq. (12) under a translation of coordinates $x \to x + c$ where $c = (c_1, c_2)$ is a constant vector:

$$I_\nu \to I_\nu + \Gamma c_\nu, \quad (14)$$

which implies a nontrivial transformation of the linear momentum (12) when $\Gamma \neq 0$. This is surely an unusual property and indicates that the moments $I_\nu$ provide a measure of position rather than momentum. Such a fact is made explicit by considering the guiding-center vector $\mathbf{R} = (R_1, R_2)$ with coordinates

$$R_\nu = \frac{I_\nu}{\Gamma} = \frac{1}{\Gamma} \int x_\nu \gamma dx_1 dx_2, \quad (15)$$

which transforms as $\mathbf{R} \to \mathbf{R} + c$ under a constant translation and is thus a measure of position of a field configuration with $\Gamma \neq 0$. Nevertheless, the vector $\mathbf{R}$ is conserved.

A related fact is that the familiar Poisson bracket algebra is significantly affected when $\Gamma \neq 0$. Using the canonical Poisson brackets

$$\{\Pi_a(x), \Psi_b(x')\} = \delta_{ab} \delta(x - x') \quad (16)$$

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and the general expression of the local vorticity (3) in the definition of the linear momentum (12), it is not difficult to establish the relations

$$\{p_1, p_2\} = \Gamma, \quad \{R_1, R_2\} = 1/\Gamma,$$

which are strongly reminiscent of the situation in the case of electron motion in a uniform magnetic field, the role of the latter being played here by the total vorticity $\Gamma$.

Similarly, the angular momentum (13) actually provides a measure of the soliton size, a fact made explicit by considering the mean squared radius defined from

$$r^2 = \frac{1}{\Gamma} \int (x - R)^2 \gamma dx_1 dx_2 = \frac{2\ell}{\Gamma} - R^2,$$

which is also conserved. Needless to say, the conservation laws (12) and (13) resume their ordinary physical significance at vanishing total vorticity ($\Gamma = 0$).

The observed transmutation in the physical significance of the conservation laws of linear and angular momenta implies a radical change in the dynamical behavior. For instance, a single soliton with $\Gamma \neq 0$ cannot be found in free translational motion ($\dot{R} = 0$). It is always spontaneously pinned or frozen within the medium, whereas translation invariance is preserved by the fact that spontaneous pinning can occur anywhere in the $(x_1, x_2)$ plane. Soliton motion is possible in the presence of external field gradients or other solitons, but the dynamical pattern is also expected to be unusual in that motion tends to take place in a direction perpendicular to the applied force. In other words, solitons with a nonvanishing total vorticity are expected to behave as electric charges in a uniform magnetic field or as ordinary vortices in a fluid. But one should keep in mind that a topological soliton does not necessarily carry a nonvanishing total vorticity. This and related issues can be settled only within a definite dynamical model, as discussed further in subsequent sections.

This section is completed with a brief discussion of a 3D generalization. Thus eq. (8) becomes

$$\dot{\gamma}_i = -\varepsilon_{ijk} \partial_j \partial_k \sigma_{kl},$$

where Latin indices assume three distinct values and $\varepsilon_{ijk}$ is the 3D antisymmetric tensor. The stress tensor $\sigma_{kl}$ is obtained by an obvious 3D extension of eq. (7) and the vorticity is now a vector density $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ given by

$$\gamma_i = \varepsilon_{ijk} \partial_j \Pi_a \partial_k \Psi_a.$$

Accordingly the conserved linear and angular momenta read

$$p = -\frac{1}{2} \int (r \times \gamma) dV, \quad \ell = -\frac{1}{3} \int |r \times (r \times \gamma)| dV,$$
where \( r = (x_1, x_2, x_3) \) and \( dV = dx_1 dx_2 dx_3 \). Again, if partial integrations are freely performed, eqs. (21) reduce to the standard (canonical) conservation laws at \( D = 3 \). However such integrations may not be justified for 3D field configurations with a nontrivial topology; e.g., configurations with a nonvanishing Hopf index [3, 4].

Finally we mention that the conservation laws (12) and (21) are formally identical to those derived in fluid dynamics, at least for incompressible fluids; see sect. 7 of ref. [5].

3. Ferromagnets

A ferromagnetic medium is described in terms of the density of magnetic moment or magnetization \( \mathbf{m} = (m_1, m_2, m_3) \) which is generally some function of position and time but has nearly constant magnitude for temperatures sufficiently below the Curie point. The dynamics is governed by the Landau-Lifshitz equation

\[
\dot{\mathbf{m}} = \mathbf{m} \times \mathbf{f}, \quad \mathbf{f} = -\frac{\delta W}{\delta \mathbf{m}}, \quad m^2 = 1,
\]

(22)

where \( W = W(\mathbf{m}) \) is a suitable energy functional and the constant magnitude of the magnetization is normalized to unity.

We first discuss some general features of the Landau-Lifshitz equation that do not depend on the details of the energy functional. For example, eq. (22) may be brought to the standard Hamiltonian form of eq. (2) by resolving the constraint \( m^2 = 1 \) through, say, the spherical parametrization \( m_1 = \sin \Theta \cos \Phi, \ m_2 = \sin \Theta \sin \Phi, \ m_3 = \cos \Theta \). It is then not difficult to show that eq. (22) may be written in the form (2) with a single pair of canonical variables \( \Psi = \Phi \) and \( \Pi = \cos \Theta \). For a strictly 2D theory the vorticity (3) reads

\[
\gamma = \varepsilon_{\mu \nu} \partial_\mu \Pi \partial_\nu \Psi = -\varepsilon_{\mu \nu} \sin \Theta \partial_\mu \Theta \partial_\nu \Phi = -\frac{1}{2} \varepsilon_{\mu \nu} (\partial_\mu \mathbf{m} \times \partial_\nu \mathbf{m}) \cdot \mathbf{m},
\]

(23)

where we recognize, in the third step, the familiar Pontryagin density. We further restrict our attention to field configurations \( \mathbf{m} = \mathbf{m}(x, t) \) that approach the ferromagnetic ground state \((0, 0, 1)\) at spatial infinity. Under such conditions the total vorticity of eq. (9) yields

\[
\Gamma = 4\pi Q,
\]

(24)

where \( Q = 0, \pm 1, \pm 2, \ldots \) is the integer-valued Pontryagin index or winding number. The FM bubbles alluded to in the Introduction are static solutions of the Landau-Lifshitz equation with a definite winding number. An abundance of such bubbles have been observed in practice [2] with values of the winding number ranging up to \(|Q| \sim 100\).
Therefore the current theory is a good example of application of the general theory developed in the preceding section. An immediate conclusion is that FM bubbles with \( Q \neq 0 \) cannot move freely but are always frozen within the ferromagnetic medium, in analogy with the 2D cyclotron motion of electrons in a uniform magnetic field. However bubble motion can occur either in the presence of external field gradients, which break translation invariance and hence the conservation of the guiding center \( \mathbf{R} \), or in the presence of other bubbles. We consider the two possibilities in turn.

Let us first assume that a static bubble with winding number \( Q \) is subjected to an external field \( \mathbf{h} = (0, 0, h) \) which is turned on at \( t = 0 \). The relevant dynamical question is then to predict the response of the bubble and could, in principle, be settled by solving the initial-value problem for eq. (22) extended according to \( \mathbf{f} \to \mathbf{f} + \mathbf{h} \) to include the applied field. However much can be said without actually solving the complete initial-value problem, taking advantage of the special nature of the conservation laws discussed in the preceding section. Hence, if the applied field were spatially uniform, translation invariance would be preserved and the guiding center of the bubble would remain fixed, even though finer details may acquire a nontrivial time dependence. A more interesting situation arises when the applied field is not uniform but is some prescribed function of position and time; i.e., \( \mathbf{h} = h(\mathbf{x}, t) \). The linear momentum (12) is no longer conserved but satisfies the implicit evolution equation [3]

\[
\dot{p}_\mu = \int (\partial_\mu h) (m_3 - 1) dx_1 dx_2.
\]  

The essential point is made apparent in the case of a highly idealized field \( h = gx_1 \), where \( g \) is a spatially uniform gradient that may still depend on time. Then eq. (25) applied for \( \mu = 1 \) and 2 yields

\[
\dot{p}_1 = mg, \quad \dot{p}_2 = 0; \quad m \equiv \int (m_3 - 1) dx_1 dx_2,
\]

where \( m \) is the total magnetization in the third direction, after subtracting off its trivial ground-state value. Equations (26) may be thought of as Newton’s law and would suggest that the bubble moves in the direction opposite to the gradient, also taking into account that \( m < 0 \). However this apparently straightforward conclusion is in sharp disagreement with the experimental fact that a bubble with \( Q \neq 0 \) actually moves in a direction perpendicular to the gradient.

On the other hand, the analogy with the Hall effect implied by the structure of the conservation laws suggests that a proper interpretation of eq. (25), when \( Q \neq 0 \), should proceed through the guiding-center coordinates of eq. (15) which are related to the linear momentum (12) simply by \( R_\mu = \varepsilon_{\mu
u} p_\nu / 4\pi Q \). Equation (15) is
then rewritten in terms of the drift velocity $V = \dot{R}$ to yield

$$V_\mu = \dot{R}_\mu = \frac{1}{4\pi Q} \int (\varepsilon_{\mu\nu} \partial_\nu h)(m_3 - 1) dx_1 dx_2,$$

(27)

or, in the case of a uniform gradient,

$$V_1 = 0, \quad V_2 = -\frac{mg}{4\pi Q}.$$  

(28)

The net conclusion is that the guiding center is indeed deflected at a right angle to the gradient irrespective of the time evolution of the finer details of the bubble. Furthermore the expression for the drift velocity (28) is fairly explicit and depends on the structural details of the bubble only through its total magnetization $m$.

In order to obtain a completely explicit expression for the drift velocity we must consider a specific form for the energy functional. For purposes of illustration we consider the 2D isotropic Heisenberg model defined by the Hamiltonian.

$$W = \int w dx_1 dx_2, \quad w = \frac{1}{2} (\partial_\mu m \cdot \partial_\mu m),$$

(29)

which leads to an effective field $f = \Delta m$ in eq. (22), where $\Delta$ is the 2D Laplacian. For completeness, we also quote an explicit expression for the stress tensor (7), namely

$$\sigma_{\nu\lambda} = w \delta_{\nu\lambda} - (\partial_\nu m \cdot \partial_\lambda m).$$

(30)

Static solutions of this model coincide with the Belavin-Polyakov (BP) instantons of the 2D Euclidean nonlinear $\sigma$ model [6]. Here we consider the special class of BP instantons given by

$$\Omega \equiv \frac{m_1 + im_2}{1 + m_3} = \left( \frac{a}{z} \right)^n \quad \text{or} \quad \left( \frac{a}{z} \right)^n,$$

(31)

where $z = x_1 + ix_2$, $a$ is an arbitrary complex constant, and $n$ is a positive integer. The special solutions (31) closely resemble realistic FM bubbles with winding numbers $Q = n$ or $-n$. We then calculate the radius $r$ from eq. (18) and the total moment $m$ from eq. (26) to obtain

$$r^2 = \left[ (\pi/Q) \csc(\pi/Q) \right] |a|^2, \quad m = -2\pi r^2.$$  

(32)

The first relation indicates that the current definition of the bubble radius differs significantly from the naive definition $r = |a|$, except in the limit $|Q| \to \infty$ which is sometimes referred to as the adiabatic limit. The second relation in (32) possesses a simple geometrical significance, for it could also be obtained by considering a crude model of a bubble in which the magnetization points toward the north pole, $m = (0, 0, 1)$, for $\rho > r$ and toward the south pole, $m = (0, 0, -1)$, for $\rho < r$. 

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Now the total moment $m$ is independently conserved in the isotropic Heisenberg model. We may then replace $m$ in eq. (28) by its initial value (32) to obtain a completely explicit, as well as exact, result for the drift velocity:

$$V_1 = 0, \quad V_2 = \frac{g r^2}{2Q}, \quad (33)$$

which is consistent with the golden rule (1) applied for $V = |V_2|$ and $\delta = \pm 90^\circ$, as is appropriate in the absence of dissipation. Note, however, that eq. (33) contains the sophisticated bubble radius of eq. (32) instead of the naive radius $r = |a|$.

The above results were confirmed in detail by a direct numerical simulation [7]. The effect of dissipation was also studied in models of increasing complexity [7,8]. The general conclusion was that golden rule (1) is correct in its gross features but not in its finer details.

The dynamics of interacting FM bubbles was also studied in refs. [8]. When two bubbles are brought to a relative distance $d$, one would naively expect that they would either converge toward each other or move off to infinity depending on whether their interaction potential is attractive or repulsive. In fact, two FM bubbles with like winding numbers orbit around each other, in analogy with the 2D motion of two interacting electrons in a uniform magnetic field or two like vortices in a fluid. Furthermore, two FM bubbles with opposite winding numbers undergo motion along roughly parallel trajectories, also in analogy with the 2D motion of an electron-positron pair in a uniform magnetic field or the familiar Kelvin motion of a vortex-antivortex pair in a fluid. Hence the analogy with the Hall effect of electrodynamics or the Magnus effect of fluid dynamics is essentially complete.

This brief discussion of ferromagnetic solitons is completed with two remarks. First, in practice, FM bubbles are not strictly 2D solitons but occur in thin ferromagnetic films [1,2]. However the current theoretical framework has been generalized to account for the quasi-2D nature of a film, with due caution on boundary effects including the effect of the long-range magnetostatic field [9]. Second, FM solitons of a different nature may occur in a strictly 3D ferromagnet and are characterized by the Hopf index. The potential implications of the present framework for Hopf solitons were discussed in ref. [4].

4. Antiferromagnets

Although direct experimental evidence for the existence of topological solitons in antiferromagnets is limited at this point, theoretical arguments suggest that static AFM solitons should exist for essentially the same reason as in ferromagnets. However their dynamics is significantly different and is now governed by suitable
extensions of the relativistic nonlinear $\sigma$ model instead of the Landau-Lifshitz equation. The relevance of the $\sigma$ model became apparent through standard hydrodynamic approaches [10-12] but detailed applications to soliton dynamics were carried out mostly in the Soviet literature reviewed in part in ref. [13]. We were thus sufficiently motivated to extend the preceding analysis to the case of layered or 2D antiferromagnets whose significance has increased in recent years in connection with high-$T_c$ superconductivity. Specifically, we elaborate on some work of Ivanov and Sheka [14] concerning the dynamics of AFM vortices in a uniform magnetic field [15,16].

The continuum dynamics of an antiferromagnet is described in terms of an order parameter $n$ of unit length ($n^2 = 1$) which satisfies the differential equation

$$n \times f = 0; \quad f = \ddot{n} - \Delta n + 2(h \times \dot{n}) + (h^2 + \alpha^2)n_3$$

Here the constant $\alpha^2$ is the strength of a crystal easy-plane anisotropy and $h = (0,0,h) = he$ is a uniform magnetic field applied along the symmetry axis. In the absence of an applied field ($h = 0$) eq. (34) reduces to the relativistic nonlinear $\sigma$ model extended to include anisotropy. The effect of the external field is twofold; it breaks Lorentz invariance, through the term $2(h \times \dot{n})$ in eq. (34), and also induces an effective easy-plane anisotropy of strength $h^2$. Finally it is useful to derive eq. (34) from an action principle, i.e.,

$$f = -\frac{\delta A}{\delta n}, \quad A = \int Ldx_1dx_2dt$$

where $A$ is the action and $L$ is the Lagrangian density

$$L = \frac{1}{2}[\dot{n}^2 - (\partial_{\mu}n \cdot \partial_{\mu}n)] + h \cdot (n \times \dot{n}) - \frac{1}{2}(h^2 + \alpha^2)n_3^2.$$ 

Hence we are armed with all the necessary information to carry out the general program of sect. 2.

Lagrangian (36) may be parametrized in terms of spherical variables ($n_1 = \sin \Theta \cos \Phi$, $n_2 = \sin \Theta \sin \Phi$, $n_3 = \cos \Theta$) to yield two pairs of canonically conjugate fields:

$$\Psi_1 = \Theta, \quad \Pi_1 = \dot{\Theta},$$

$$\Psi_2 = \Phi, \quad \Pi_2 = (h + \dot{\Phi}) \sin^2 \Theta.$$ 

We may then insert these fields in the general expression for vorticity given by eq. (3) to obtain

$$\gamma = \epsilon_{\mu\nu} \partial_\mu (n \cdot \partial_\nu n) + h\omega, \quad \omega = \frac{1}{2} \epsilon_{\mu\nu} \sin(2\Theta) \partial_\mu (2\Theta) \partial_\nu \Phi.$$
The first term is an uncomplicated total divergence and yields a vanishing contribution to the total vorticity of eq. (9) which is then written as \( \Gamma = h \int \omega dx_1 dx_2 \). However the last integral may be different from zero because \( \omega \) resembles the Pontryagin density (23) except for an overall factor \(-\frac{1}{2}\) and the replacement \( \Theta \to 2\Theta \). The latter suggests considering the three-component vector

\[
N_1 = 2n_3n_1 = \sin(2\Theta) \cos \Phi, \quad N_2 = 2n_3n_2 = \sin(2\Theta) \sin \Phi, \quad N_3 = 2n_3^2 - 1 = \cos(2\Theta),
\]

which is also a unit vector field \((N^2 = 1)\). The density \( \omega \) may be written as

\[
\omega = \frac{1}{4} \varepsilon_{\mu\nu}(\partial_\mu N \times \partial_\nu N) \cdot N
\]

and resembles the standard Pontryagin density given in the third step of eq. (23). Furthermore the field \( N \) satisfies the simple boundary condition \( N \to (0,0,-1) \) at spatial infinity, thanks to the condition \( n_3 \to 0 \) satisfied by all relevant configurations in the presence of an easy-plane anisotropy. The net conclusion is that \( \omega \) is actually the Pontryagin density for the field \( N \) and thus leads to an integer-valued total vorticity

\[
\Gamma = h \int \omega dx_1 dx_2 = 2\pi \kappa h; \quad \kappa = 0, \pm 1, \pm 2, \ldots,
\]

which is nonvanishing when both the applied field \( h \) and the vortex number \( \kappa \) are different from zero.

In the presence of either a crystal easy-plane anisotropy or an applied field, or both, the relevant topological solitons are AFM vortices that satisfy the boundary condition \( n_3 \to 0 \) at spatial infinity. If one insists on classifying these vortices by the standard winding number of sect. 3, one would obtain \( Q = -\frac{1}{2} \kappa \nu \) where \( \kappa = \pm 1 \) is the vortex number and \( \nu = \pm 1 \) the polarity. Hence \( Q \) is half integer for AFM vortices which may thus be called merons [17]. However the topological charge that is relevant for dynamics is not \( Q \) but the total vorticity \( \Gamma = 2\pi \kappa h \) which depends on both the vortex number and the applied field but not on the polarity. Therefore the general discussion of sect. 2 applied to the current example suggests that AFM vortices should exhibit Hall or Magnus behavior only when an external field is present, and that the general dynamical picture should be insensitive to the polarity.

The preceding general statement was thoroughly confirmed through detailed numerical simulations [15,16]. For \( h = 0 \), two vortices with the same vortex number \((\kappa_1 = \pm 1 = \kappa_2)\), initially at rest at a relative distance \( d \), move off to infinity, just as two ordinary particles would do when their interaction potential is repulsive. However, when a nonvanishing uniform field \( h \) is present, the two vortices actually orbit around other, again in complete analogy with two interacting electrons in a uniform magnetic field or two ordinary vortices in a fluid. Similarly, when a vortex
(κ₁ = 1) and an antivortex (κ₂ = −1) are initially at rest at a distance d, in the absence of an external field, they converge toward each other and eventually annihilate. However, when a field is present, the vortex-antivortex pair undergoes Kelvin motion along two roughly parallel lines that are perpendicular to the line connecting the vortex and the antivortex. One should further note that above dynamical picture was verified for either choice of relative polarity (ν₁ = 1 = ν₂ or ν₁ = 1 = −ν₂), also in agreement with the fact that the total vorticity (41) is independent of polarity.

Therefore, to the extent that vortices are relevant for the physics of a 2D antiferromagnet, the dynamical picture is changed significantly even by a very weak bias field. Perhaps the clearest manifestation of the effect of an applied field will emerge in the thermodynamics of an antiferromagnet. It is clear that much remains to be done in connection with the anticipated Berezinskii-Kosterlitz-Thouless (BKT) phase transition that relies on the dynamics of a gas of vortices and antivortices. Suffice it to say that the dynamics of vortex-antivortex pairs is radically affected by the applied field. Hence the BKT theory may have to be reformulated in a way that clearly reflects the fundamental change of behavior when a field is turned on. Finally we mention that we have thus far confined our attention to the classical approximation. However it is unlikely that the Hall behavior of classical AFM vortices described here will be averted by quantum effects, especially because the overall picture can be surmised directly from the conservation laws rather than a detailed solution of the equations of motion. On the contrary, one should expect that a full quantum treatment will lead to a richer picture, in analogy with the “classical” and “quantum” Hall effects of electrodynamics.

5. Superfluids

We finally discuss briefly a class of problems pertaining to superfluids. The simplest possibility is to consider the Hamiltonian dynamics defined from

\[ i \dot{\psi} = \frac{\delta W}{\delta \psi^*}, \quad i \dot{\psi}^* = -\frac{\delta W}{\delta \psi}, \quad (42) \]

where \( \psi = \psi(x,t) \) is an order parameter, \( \psi^* \) is its complex conjugate, and \( W = W(\psi, \psi^*) \) is some energy functional. A typical choice for \( W \) is the one that leads to the Gross-Pitaevski model, often used as a simple model for superfluid helium II [18].

A straightforward adaptation of the definition of vorticity given in eq. (20) yields

\[ \gamma = \frac{1}{i}(\nabla \psi^* \times \nabla \psi), \quad (43) \]

and the conserved linear and angular momenta are then obtained from the general relations (21). Again, if partial integrations are performed freely, these relations
reduce to the canonical conservation laws. However the latter are plagued by various ambiguities, as discussed by Jones and Roberts [19] in their study of the dynamics of vortex rings as well as of vortex-antivortex pairs. It is not difficult to see that the analysis of the above reference can be repeated using the current definition of conservation laws without encountering any ambiguities.

Actually the study of vortices also requires a 2D restriction of eq. (43) given by
\[ \gamma = \frac{1}{i} \epsilon_{\mu\nu} \partial_{\mu} \psi^* \partial_{\nu} \psi, \] (44)
which is the direct analog of eq. (3), while the corresponding conservation laws are again given by eqs. (12) and (13). The associated total vorticity reads
\[ \Gamma = \int \gamma \, dx_1 \, dx_2 = 2\pi \kappa, \] (45)
where the vortex number \( \kappa \) is an integer that can be extracted also from the asymptotic behavior of the order parameter at spatial infinity; \( \psi \sim e^{i\kappa \phi} \). Therefore the general discussion of sect. 2 applies here without modification and, not surprisingly, implies that the main dynamical features of superfluid vortices are similar to those of ordinary vortices. This fact is confirmed by the asymptotic analysis of Neu [20] which addresses the limit of widely separated vortices. Furthermore the so-called Hall-Vinen drag induced by the mutual friction between the superfluid and normal components can be studied within an extended model introduced by Carlson [21], pretty much along the lines of our discussion of skew deflection of ferromagnetic bubbles in sect. 3.

Since the study of vortex dynamics in an ordinary fluid is usually carried out in the idealized limit of an incompressible fluid [5], it is of interest to examine more closely our definition of vorticity for a superfluid which is inherently compressible. In particular, the superfluid density vanishes at the location of a vortex and thus eliminates coordinate singularities that would occur under the somewhat artificial assumption of incompressibility. Now, if we write \( \psi = \sqrt{\rho_s} e^{i\chi} \), where \( \rho_s = \psi^* \psi \) is the superfluid density, and further identify the velocity field from \( \mathbf{u} = \nabla \chi \), the vorticity (43) may be written as \( \gamma = \nabla \times (\rho_s \mathbf{u}) \). This expression differs from the standard definition of vorticity \( \mathbf{\omega} = \nabla \times \mathbf{u} \), except in the limit of an incompressible fluid where \( \rho_s = \text{const} \) and the two definitions differ only by an overall constant, namely, \( \gamma = \rho_s \mathbf{\omega} \). When the latter expression is inserted in the linear and angular momenta of eq. (21), one recovers precisely the conservation laws for an ordinary incompressible fluid given in eqs. (7.2.5) and (7.2.6) of ref. [5].

A related problem is that of the dynamics of Abrikosov vortices in a superconductor. Unfortunately this subject is controversial in that no general agreement exists on a suitable phenomenological model for the description of
the dynamics of the order parameter. Yet one should expect that some general features of vortex dynamics do not depend on the details of the model. In this spirit, a charged fluid was studied in refs. [22,23] that is described by a straightforward extension of the Gross-Pitaevski model to include electromagnetism. The corresponding Abrikosov vortices were then shown to exhibit dynamical properties analogous to those encountered in all models mentioned in this review. In particular, unambiguous expressions for the linear and angular momenta were obtained that are locally gauge invariant when expressed as moments of a suitable topological vorticity. One can further show that the same canonical structure persists in alternative models for a superconductor, such as a Chern-Simmons theory employed recently by Manton [24] to study the asymptotic dynamics of interacting Abrikosov vortices.

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