Cognitive map: strategies for tracking error patterns in mathematics proof

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Abstract. The difficulty of students in constructing mathematical proof is often seen in the form of solving errors they make. In this case, students' mistakes in mathematical proof are a reflection of their way of thinking. If these errors are not resolved, it will have an impact on students' thinking when working on further mathematical proofs. The patterns of student error in constructing mathematical evidence can be identified, namely: 1) proving statements by providing numbers or examples, 2) manipulating incorrect algebra, 3) verifying numerical proof after formal proof, and 4) inability to understand the definition of the statement. Of course, this error pattern needs to be traced more deeply using cognitive maps. Cognitive maps are techniques for representing how subjects think about a particular problem or situation so that researchers can act for the next step. This is, of course, a cognitive map is a person's perspective on the subject, which is described qualitatively by connecting concepts to be able to predict causal behaviour. Therefore, students' thought processes of constructing mathematical evidence can be traced using cognitive map.

1. Introduction

The basic course that can be used to construct proof in the mathematics education department is calculus. Almost all students in the mathematics education department learn calculus in the first semester. If students can learn to mathematics proof well, they can be construction proof on advanced mathematics. Otherwise, if students experience errors in construct proof, then will have an impact on the next proof material, for example, proof in real analysis and algebraic. Therefore, if the error is not be resolving, then will have an impact on the next mathematical proof. The learning proof is a simple concept that can lead students to develop next mathematics proof [1]. Some researchers have examined the difficulties of students and thinking processes in the construction of mathematical proofs, as the results of research [2], [3] showed the difficulties students when writing proof because of deficient of understanding of mathematical concepts. Further, the failure of the students construct mathematical proof because they could not use the knowledge they have to strategy in resolving [4].

Student difficulties in mathematical construction proof often seen in the form of his error [5], [6]. The student error in mathematical proof was a reflection of his thinking. If the student error not prevented, then it will have an impact on the thinking of students when working on advanced mathematical proof. Thinking done by students is often reflected in the results of completion. Therefore, the thinking done by students in solving math problems can be seen from their behavior in the form of task completion results [7], [8]. In addition, thinking can be traced through the results of student work in constructing and
solving mathematical problems \cite{9}. To be able to trace students’ thinking, knowledge is needed about how to capture student thinking. In the results of his research \cite{10} explained that to deeply study the uniqueness of the thought process, cognitive map / cognitive style can be used. This shows that using a cognitive map can be traced to how students think in solving math problems.

Some researchers traced students' thought processes using a cognitive map. From the results of his research, it was found that cognitive maps can describe the causal relationship of various phenomena and concepts and can be model \cite{11}; Cognitive maps can be used as a guide to the next step in order to obtain the next direction of thinking \cite{12}; cognitive style or thinking style is used as a mediator for student work in solving geometric problems with Van Hielle's theory \cite{13}; and cognitive maps can be used to track students' mistakes in constructing mathematical concepts \cite{9}.

Cognitive map or also called cognitive mapping is a technique to represent how the subject thinks about a particular problem or situation, so that researchers can act for the next step \cite{14}. Cognitive map is a person's perspective on the subject, which is described qualitatively by connected concepts to predict causal behavior. Therefore, students' thought processes in solving math problems can be traced by using cognitive maps. Cognitive map or Cognitive map can be described as follows.

![Figure 1. Cognitive Map](image1.png)

2. Research Methods
The approach used in this study is a mixed method, which combines quantitative and qualitative approaches. The quantitative approach is made to process data related to how much students make mistakes in proving mathematics, while a qualitative approach is carried out to trace how the students' errors occurred in constructing a mathematical proof.

Data collection in this research by giving the problem of proof to the student to be solved by think aloud \cite{15}. This research will be conducted in three stages, but in this study, the procedure for collecting data is in the first stage. In this step, the researcher records and records student behaviour when solving proof problems. At this stage, student construction of mathematics proof will obtain. If the subject has an error in proving, then the subject is grouped based on the error. In this case, mistakes made by students when working on the evidence will be group and identified as a pattern of errors.

The subject of this study takes from the second generation of mathematics education students at Wisnuwardhana University and Kanjuruhan University. The results of collecting data obtained are student work, interviews, and field notes then analyzed by processing data, reading the entire data, coding and interpreting data in making conclusions \cite{16}.

3. Result and Discussion
In proof of real numbers presented two constructions: the first construction, students are asked to prove rational numbers. While the other construction, proof of odd and even numbers. When students got questions about proof of rational numbers, most (66%) students experience errors. The question of proof given to students is as follows.

![Suppose a rational number \( \frac{a}{b} \), a and b are integers with \( b \neq 0 \). Prove that the sum of two rational numbers is rational](image2.png)

Figure 2. Research Instrument

In resolving such proof, many students responded that rational numbers by using the analogy of numbers. In this article, the author only presents the students interpretation rational number with
number \( \left( \frac{2}{6} \text{ or } \frac{1}{3} \right) \). When there were statements about the number of two rational numbers, students directly operate two rational numbers. In this case, students write \( \frac{5}{6} + \frac{5}{6} = \frac{10}{6} \). Likewise, with other students, assume that in proving the number of two rational numbers is rational can be proved by \( \frac{1}{3} + \frac{2}{5} = \frac{11}{15} \). The students' answers to the proof are as follows.

Translate
Rational number = \( \frac{a}{b} \), \( a \& b \) is integer, \( b \neq 0 \)
the sum of two numbers is rational
\( \frac{a}{b} + \frac{a}{b} = \frac{2a}{b} \)
Suppose, \( a = 5, b = 6 \), so that \( \frac{5}{6} + \frac{5}{6} = \frac{10}{6} \) is rational number.

**Figure 3.** Result of Student Work When Proving the Sum of Two Rational Numbers

Students begin proof by thinking about what is known, namely rational number \( \frac{2}{b} \), \( b \neq 0 \) with \( a \) and \( b \) are integers and then add them together. So that, they write \( \frac{a}{b} + \frac{a}{b} = \frac{2a}{b} \). In this step, students were unable to provide the right reason that \( \frac{2a}{b} \) is a rational number. Students have not provided answers that are considered sure. This case, indicated by the question of researchers towards student.

\[ P : \text{Were } \frac{2a}{b} \text{ a rational number?} \]

\[ S_1 : \text{No, because real number is } \frac{a}{b} \text{ while, } 2 \text{ is natural number. So that } \frac{2a}{b} \text{ not rational number.} \]

They assume that \( \frac{2a}{b} \) not a rational number. It is because in numbers \( \frac{2a}{b} \) consists of natural numbers (i.e., 2) and rational numbers \( \frac{a}{b} \). This course, the student experienced errors in thinking about rational numbers. They have not understood the definition of rational numbers \( \frac{a}{b} ; a \text{ and } b \) integers). If explored in debt that \( 2a \) is an integer because 2 and \( a \) are integers. It is, of course, that \( \frac{2a}{b} \) is a rational number. Another inability to understand the definition of rational numbers indicated by specifying \( a = 5 \) and \( b = 6 \).

In the next verification step, students thinking that \( \frac{a}{b} \) can be replaced with \( \frac{5}{6} \), "five-sixths that students were thinking are rational numbers defined as 5 and 6 integers. The next process, they continue by written \( \frac{5}{6} + \frac{5}{6} = \frac{10}{6} \). Because 10 and 6 are integers and 6 are not the same with zero, \( \frac{10}{6} \) included as a rational number. Thus, students conclude that the number of two rational numbers is a rational number. In this case, students prove the proof of rational numbers by presenting examples of a number, assuming that examples of numbers can resolve for proof. From the way of thinking, it can see that they have an inability to understand the definition of rational numbers. Students' errors in constructing mathematical proof also occur in the form of examples (i.e., \( \frac{5}{6} \)). Thus, the case is used by students in compiling mathematical proof. As a verification process, the verification is only valid for \( \frac{5}{6} \). They don't understand that this proof applies in general. Students' thinking mistakes when portrayed using a cognitive map as follows.
4. Conclusion
From the results of student work, it looks that students experience an inability to understand the definition of rational numbers. S1 errors in constructing mathematical proof occur in the form of an example (i.e., 5/6). So, the case is used by students in compiling mathematical proof. In understanding S2 on the definition of rational numbers, students are not able to interpret the form of a/b in numbers. They only understand that 1/2 or 5/10 are rational numbers, while 2 or 0 whose divisors are 1 are not rational numbers. Based on this cognitive map, thinking errors in mathematical proof can be traced. In addition, the cognitive map can also trace the location of thinking mistakes, failure to think and some math concepts developed by students.

References
[1] K. Weber and L. Alcock, “Semantic and Syntactic Proof Productions,” Educ. Stud. Math., vol. 56, no. 3, pp. 209–234, 2004.
[2] C. Batanero, J. D. Godino, and R. Roa, “Training Teachers to Teach Probability,” J. Stat. Educ., vol. 12, no. 1, Jan. 2004.
[3] A. Izsak, E. Tillema, and Z. Tunc-Pekkan, “Teaching and learning fraction addition on number lines,” J. Res. Math. Educ., vol. 39, no. 1, pp. 33–62, 2008.
[4] A. Samkof, Y. Lai, and K. Weber, “On The Different Ways That Mathematicians Use Diagrams In Proof Construction,” Res. Math. Educ., vol. 14, no. 1, pp. 49–67, 2012.
[5] A. Prayitno, A. Rossa, F. D. Widayanti, S. Rahayuningsih, A. Hamid, and M. Baidawi, “Characteristics of Students’ Proportional Reasoning in Solving Missing Value Problem,” in Journal of Physics: Conference Series, 2018, vol. 1114, no. 1.
[6] A. Prayitno et al., “Performance of Understanding Students’ Construction In The Naming Fraction of The Three Representation,” J. Phys. Conf. Ser., vol. 1114, no. 1, p. 012022, Nov. 2018.
[7] A. Prayitno, “Proses Berpikir refraktif Mahasiswa dalam Menyelesaikan Masalah Matematika,” Universitas Negeri Malang (Disertasi UM), 2015.
[8] A. Prayitno, Subanji, and M. Muksar, “Refractive Thinking with Dual Strategy in Solving Mathematics Problem,” IOSR J. Res. Method Educ. Ver. III, vol. 6, no. 3, pp. 49–56, 2016.
[9] Subanji and T. Nusantara, “Karakterisasi Kesalahan Berpikir Siswa Dalam Mengonstruksi

**Figure 4. Result of Student Work by cognitive map**
Konsep Matematika,” *J. Ilmu Pendidik.*, vol. 19, no. 2, pp. 208–217, 2013.

[10] A. Gutiérrez, A. Jaime, and J. M. Fortuny, “An An Alternative the Paradigm To Evaluate the Acquisition of the Van Hiele Levels,” *J. Res. Math. Educ.*, vol. 22, no. 3, pp. 237–251, 1991.

[11] A. Pena, H. Sossa, and A. Gutierrez, “Cognitive Maps: an Overview and their Application for Student Modeling,” *Comput. y Sist.*, vol. 10, no. 3, pp. 230–250, 2007.

[12] L. F. Jacobs and F. Schenk, “Unpacking the Cognitive Map: The Parallel Map Theory of Hippocampal Function,” *Psychol. Rev.*, vol. 110, no. 2, pp. 285–315, 2003.

[13] S. C. Perdikaris, “Using the Cognitive Styles to Explain an Anomaly in the Hierarchy of the van Hiele Levels,” *J. Math. Sci. Math. Educ.*, vol. 6, no. 2, pp. 35–43, 2011.

[14] F. Ackermann, C. Eden, S. Cropper, and S. Cropper, “Getting Started with Cognitive Mapping,” in *The 7th Young OR Conference*, 2004, pp. 1–14.

[15] A. Prayitno, “The Characteristics of Students’ Refractive Thinkingabout Data,” in *Proceeding of 3rd International Conference On Research, Implementation And Education Of Mathematics And Science (ICRIEMS)*, 2016, p. ME.29-ME.38.

[16] J. W. Creswell, *Research Design Qualitative, Quantitative, and Mixed Approaches*, 3rd ed. United Kingdom: SAGE, 2009.