BASE-POINT-FREE PENCILS ON TRIPLE COVERS OF SMOOTH CURVES

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Abstract. Let $X$ be a smooth algebraic curve. Suppose that there exists a triple covering $f : X \to Y$ where $Y$ is a smooth algebraic curve. In this paper, we investigate the existence of morphisms from $X$ to the projective line $P^1$ which do not factor through the covering $f$. For this purpose, we generalize the classical results of Maroni concerning base-point-free pencils on trigonal curves to the case of triple covers of arbitrary smooth irrational curves.

1. Introduction

Let $X$ be a smooth algebraic curve of genus $g_x$. Suppose that there exists a covering $f : X \to Y$ of degree $k$ where $Y$ is a smooth algebraic curve of genus $g_y$. What kind of morphisms from the cover $X$ to the projective line $P^1$ exist? Clearly, there are morphisms $X \to P^1$ induced from the base curve $Y$, that is, morphisms of the form $h' \circ f : X \to P^1$ for some $h' : Y \to P^1$. Furthermore, by the following application of the Castelnuovo-Severi inequality, every morphism $h : X \to P^1$ whose degree is small enough compared with the genus of $X$ is induced from the base curve $Y$ if $\deg f$ is a prime number.

Castelnuovo-Severi inequality (cf. [1, p. 366]). Suppose that $k$ is a prime number. Let $h : X \to P^1$ be a morphism of degree $d$. If
\[
d < \frac{g_x - k g_y + k - 1}{k - 1},
\]
then $h$ factors through the covering $f$.

In view of the Castelnuovo-Severi inequality, it is natural to raise the following question.

Definition. A morphism $h : X \to P^1$ is said to be nontrivial if there is no morphism $h' : Y \to P^1$ such that $h = h' \circ f$.

Question A. Suppose that $k$ is a prime number. For every integer $d$ with
\[
d \geq \frac{g_x - k g_y + k - 1}{k - 1},
\]
does there exist a nontrivial morphism $h : X \to P^1$ of degree $d$?

For double covering case ($k = 2$), Question A has been answered affirmatively in [8]. For trigonal curve case ($k = 3, g_y = 0$), Question A has been completely answered by the classical results of Maroni; cf. [10]. However, if $k \geq 3$ and $g_y > 0$,

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then one cannot expect an affirmative answer for all that $d$'s: In the case of an affirmative answer we at least must have
\[
\frac{g_x - kg_y + k - 1}{k - 1} > \text{gon} X - 1,
\] (1.1)
where gon $X$ is the gonality of $X$, and the bound (1.1) is a restriction on (the gonality of) $X$ unless $k = 2$ or $(k, g_y) = (3, 0)$.

On the other hand, if \( \frac{g_x - kg_y + k - 1}{k - 1} > k \cdot \frac{g_y + 3}{2} - 1 \), then the inequality (1.1) is no longer obstrucional since gon $X \leq k \text{gon} Y \leq k \cdot \frac{g_y + 3}{2}$. Thus it seems senseful to ask Question A with $g_x$ which is sufficiently large compared with $g_y$. For instance, it has been known that, if $X$ is a triple cover of an irrational curve $Y$ ($k = 3$, $g_y > 0$) and $g_x \geq 2[(3g_y + 1)/2] + 1) \left\lceil \left( (3g_y + 1)/2 \right) + 1 \right\rceil$, then there exist nontrivial morphisms of degree $d$ for all integers $d$ with $d \geq g_x - \left\lceil (3g_y + 1)/2 \right\rceil$; [7, Theorem A]. However the degree bound $d \geq g_x - \left\lceil (3g_y + 1)/2 \right\rceil$ is rather far from the plausible bound $d \geq (g_x - 3g_y + 2)/2$ given in Question A and it is assumed in [7, Theorem A] that the base curve $Y$ is general in the sense of Brill-Noether theorem.

In this paper, we investigate Question A for triple covers of arbitrary (not necessarily general) irrational smooth curves with a mild condition on the genera of the curves to be sure that the bound (1.1) is no longer obstrucional. The following Main Theorems A and B give a partial answer to Question A in case of triple covers, which may be considered as generalizations of the results of Maroni to the case of triple covers of irrational smooth curves; see Remark 2.13. Recall that a certain rank two vector bundle, so-called Tschirnhausen module, can be associated to every triple covering; cf. [11,1]

**Main Theorem A** (Theorem 2.1). Let $X$ be a smooth irreducible curve of genus $g_x$. Suppose that there exists a triple covering $f : X \to Y$ where $Y$ is a smooth irreducible curve of genus $g_y$ and suppose that $g_x \geq 9g_y + 4$. Let $E'$ be the Tschirnhausen module for the triple covering $f$ and $e$ the $e$-invariant of the ruled surface $P(E)$. For every integer $d$ with
\[
d \geq \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2} + 4g_y,
\]
there exists a nontrivial morphism $h : X \to P^1$ of degree $d$.

**Main Theorem B** (Theorem 2.12). Let $X$ be a smooth irreducible curve of genus $g_x$. Suppose that there exists a triple covering $f : X \to Y$ where $Y$ is a smooth irreducible curve of genus $g_y$. Let $E'$ be the Tschirnhausen module for the triple covering $f$ and $e$ the $e$-invariant of the ruled surface $P(E)$. If there exists a nontrivial morphism $h : X \to P^1$ of degree $d$, then
\[
d \geq \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2}.
\]

Note that Question A cannot be answered affirmatively for triple coverings with $e \neq 0$ by Main Theorem B; hence we raise a weaker question as follows.

**Question B.** Let $Y$ be an irreducible smooth curve of genus $g_y$. Let $k \geq 2$ be a prime number and $g_x$ an integer with $g_x - kg_y + k - 1 > 0$. For every integer $d$ with
\[
d \geq \frac{g_x - kg_y + k - 1}{k - 1},
\]
does there exist an irreducible smooth curve $X$ of genus $g_x$ together with a degree $k$ covering $f : X \to Y$ such that $X$ admits a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $d$?

It is clear that we must have further conditions on $g_x$ provided that Question B has an affirmative answer: Since $d \geq \text{gon}(X) \geq \text{gon}(Y)$, we have $\frac{g_x-kg_y+k-1}{k-1} > \text{gon}(Y)-1$, and since $\text{gon}(Y) \geq \frac{g_x+2}{k-2}$ is possible, one obtains, for an affirmative answer for any $Y$, a bound on $g_x$ which is worse than that given by the condition $g_x - kg_y + k - 1 > 0$.

For double covering case ($k = 2$), [2] gives an affirmative answer to Question [3] for all $g_x \geq 4g_y + 5$ and $d \geq g_x - 2g_y + 1$. For trigonal curve case $(k, g_y) = (3, 0)$, it has been known that there exist trigonal curves together with nontrivial morphisms to $\mathbb{P}^1$ of degree $d$ for all integer $d \geq (g_x + 2)/2$ if $g_x \geq 5$; cf. [10]. In case of $k = 3$ and $g_y > 0$, in [4, Example 3.3], the authors proved that, if $g_x \geq 7g_y - 4$, then, for every integer $d$ with

$$\frac{g_x - 3g_y + 2}{2} \leq d \leq (g_x - 3g_y + 2)$$

and $d \equiv 2g_x - 2 \mod 3$,

there exists an irreducible smooth curve $X$ of genus $g_x$ and a cyclic triple covering $f : X \to Y$ such that $X$ admits a nontrivial morphism $X \to \mathbb{P}^1$ of degree $d$; however, the range of the degrees $d$ is somewhat restricted.

The following theorem provides a partial answer to Question [3] for triple covers of irrational smooth curves, which may be considered as a generalization of trigonal curve case.

**Main Theorem C (Theorem 4.3).** Let $Y$ be an irreducible smooth curve of genus $g_y \geq 1$, and let $g_x$ be an integer with $g_x \geq 37g_y - 2$. For every integer $d$ with

$$d \geq \frac{g_x - 3g_y + 2}{2} + g_y,$$

there exists an irreducible smooth curve $X$ of genus $g_x$ together with a triple covering $f : X \to Y$ such that $X$ admits a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $d$. If $g_y \geq 5$, we get sharper lower bound for $d$:

$$d \geq \frac{g_x - 3g_y + 2}{2} + \frac{g_y + 3}{2}.$$

The paper is organized as follows. In Section 2, we will prove Main Theorem A and Main Theorem B. In Section 3, we will investigate what conditions may be imposed on the Tschirnhausen module for a triple cover $f : X \to Y$ if $X$ admits a nontrivial morphism $X \to \mathbb{P}^1$ of minimal possible degree given by Main Theorem B. If a triple cover $X$ admits a nontrivial morphism $X \to \mathbb{P}^1$ of minimal possible degree, then its Tschirnhausen module must be decomposable; Proposition 3.2. Finally, in Section 4, we will prove Main Theorem C.

1.0.1. **Ideas of proofs.** Let $X$ and $Y$ be irreducible smooth curves. Assume that there exists a triple covering $f : X \to Y$. Let $\mathcal{E}'$ be the Tschirnhausen module for the triple covering $f$. It has been known that there exists an embedding $i : X \hookrightarrow \mathbb{P}(\mathcal{E})$ of $X$ into the ruled surface $\mathbb{P}(\mathcal{E})$; [3, Theorem 2.1].

For proving Main Theorem A, we first choose a linear series $\mathcal{L}$ on $\mathbb{P}(\mathcal{E})$ so that $\mathcal{L}$ is very ample or separates points in the same fiber of the triple covering $f$. Then $\mathcal{L}$ cuts out a base-point-free linear series on $X$ that is not induced from the base curve $Y$. The idea is based on the proof of [10, Lemma 1].
The main tools for proving Main Theorem [3] are elementary transformations. It has been known that the ruled surface $\mathbf{P}(\mathcal{E})$ (in fact, every ruled surface) can be transformed to $Y \times \mathbf{P}^1$ by a finite sequence of elementary transformations; cf. [6, V, Ex. 5.5]. Let $\text{elm} : \mathbf{P}(\mathcal{E}) \to Y \times \mathbf{P}^1$ be a sequence of elementary transformations that transforms $\mathbf{P}(\mathcal{E})$ to $Y \times \mathbf{P}^1$. Then it is clear that the morphism

$$h = p_2 \circ \text{elm}|_X : X \to \mathbf{P}^1$$

is nontrivial, where $p_2 : Y \times \mathbf{P}^1 \to \mathbf{P}^1$ is the second projection. The main ingredient of the proof of Main Theorem [3] is Proposition [2,10]. Every nontrivial morphism $h : X \to \mathbf{P}^1$ is, in fact, of the form $p_2 \circ \text{elm}|_X$ for some sequence of elementary transformations $\text{elm} : \mathbf{P}(\mathcal{E}) \to Y \times \mathbf{P}^1$ that transforms $\mathbf{P}(\mathcal{E})$ to $Y \times \mathbf{P}^1$. We prove Main Theorem [3] by investigating behaviors of ruled surfaces and their $\text{e}$-invariants under elementary transformations.

We prove Main Theorem [3] as follows. It has been known that every triple covering $f : X \to Y$ of a given curve $Y$ is determined by a rank two vector bundle $\mathcal{E}$ on $Y$ and a smooth zero locus $X$ of a section in $H^0(\mathbf{P}(\mathcal{E}), \pi^* \det \mathcal{E}^{-1}(3))$, where $\pi : \mathbf{P}(\mathcal{E}) \to Y$ is the projection; [3, Theorem 3.4]. We choose a rank two vector bundle $\mathcal{E}$ on the base curve $Y$ which is decomposable and we take some points $P_1, \ldots, P_e \in \mathbf{P}(\mathcal{E})$ so that the elementary transformation $\text{elm}$ with $P_1, \ldots, P_e$ as centers transform $\mathbf{P}(\mathcal{E})$ into $Y \times \mathbf{P}^1$. We then prove that there exists an irreducible smooth zero locus $X$ of a section in $H^0(\mathbf{P}(\mathcal{E}), \pi^* \det \mathcal{E}^{-1}(3))$ passing through the given points $P_1, \ldots, P_e$ and we compute the degree of the morphism $h = p_2 \circ \text{elm}|_X : X \to \mathbf{P}^1$ by investigating behaviors of trisections of ruled surfaces under elementary transformations.

1.1. Preliminaries. We collect some results concerning triple coverings, especially results related to Tschirnhausen modules; cf. [3] and [12] for detail. We briefly review basics of sections, especially minimal degree sections, of ruled surfaces; cf. [11]. Finally, we recall the definition of elementary transformation; cf. [6, V,5.7.1] and [11].

1.1.1. Triple coverings. Let $X$ and $Y$ be irreducible smooth curves of genus $g_x$ and $g_y$. Assume that there exists a triple covering $f : X \to Y$. One can associate a split exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{f^*} f_* \mathcal{O}_X \to \mathcal{E}^\vee \to 0,$$

where $\mathcal{E}^\vee$ is a locally free $\mathcal{O}_Y$-sheaf of rank 2 called, according to R. Miranda [12], the Tschirnhausen module for the triple covering $f$; cf. [3], [12].

The branch locus of the triple covering $f$ is a divisor whose associated line bundle is $(\det \mathcal{E})^\otimes 2$; [12, Proposition 4.7]. Setting

$$\mathcal{O}_Y(B) = \det \mathcal{E},$$

we have

$$b := \deg B = \deg \mathcal{E} = g_x - 3g_y + 2$$

by the Riemann-Hurwitz formula.

There exists an embedding $\iota : X \hookrightarrow \mathbf{P}(\mathcal{E})$ of $X$ into the ruled surface $\mathbf{P}(\mathcal{E})$ such that $f = \pi \circ \iota$, where $\pi : \mathbf{P}(\mathcal{E}) \to Y$ is the projection; cf. [3]. Furthermore, such an embedding is unique.

Lemma 1.1 ([3, Theorem 2.1(i)]) If there is an embedding $j : X \hookrightarrow \mathbf{P}$ into another ruled surface $\pi' : \mathbf{P} \to Y$ such that $f = \pi' \circ j$, then $\mathbf{P} \cong \mathbf{P}(\mathcal{E})$. 

1.1.2. Sections of ruled surfaces. Let \( F \) be any rank two vector bundle on a curve \( Y \) and let \( \pi : P(F) \to Y \) be the projection. Suppose \( F_0 = F \otimes O_Y(-N) \) is normalized. A section \( S_0 \) of \( P(F) \) is called a minimal degree section of \( P(F) \) if \( S_0 \) is a section whose self intersection number is minimum among all sections on \( P(F) \). Let \( e \) be the \( e \)-invariant of \( P(E) \). Since \( \det F_0 = \det F \otimes O_Y(-2N) \), we have

\[
e = - \deg F_0 = 2 \deg N - \deg B = 2n - b = -S_0^2. \tag{1.4}
\]

Giving a section of \( P(F) \) is equivalent to giving a subbundle of \( F \); cf. [6, V, 2.8]. For a canonical divisor \( K \) where \( D \) is normalized; cf. [6, V, 2.8]. For a canonical divisor \( K \) associated to the section \( S_0 \) of \( E \), then we have an exact sequence

\[
0 \to L_S \to F \to L \to 0
\]

for some subbundle \( L \). The subbundle \( L_{S_0} \) corresponding to the minimal degree section \( S_0 \) is a maximal degree subbundle of \( F \), and vice versa; cf. [11] Theorem 1.16. Especially, we have the following lemma.

**Lemma ([11] Theorem 1.16]).** Let \( S \) be a section of \( P(F) \) and \( L_S \) the subbundle of \( E \) associated to the section \( S \). Then we have

\[
\pi_*(S \cdot S) = \text{Divisor of } (\det E \otimes L_S^{-2}), \tag{1.5}
\]

where \( (S \cdot S) \) denotes the intersection divisor on the smooth curve \( S \).

If \( F \) is decomposable, then there exist two disjoint sections; cf. [6] V, Ex. 2.2. We record this fact for the convenience of the reader.

**Lemma 1.2.** Let \( F = O_Y(B_1) \oplus O_Y(B_2) \) be a decomposable rank two vector bundle on a smooth curve \( Y \) with \( \deg B_1 \leq \deg B_2 \). Let \( S_0 \) be a minimal degree section of \( P(F) \) and \( \pi : P(F) \to Y \) the projection. Then there exists a section \( S \in \{ S_0 + \pi^*(B_2 - B_1) \} \) such that \( S_0 \cap S = \emptyset \). Conversely, if \( S \) is a section such that \( S \cap S_0 = \emptyset \), then \( S \in \{ S_0 + \pi^*(B_2 - B_1) \} \).

**Proof.** Let \( S \) be a section corresponding to the subbundle \( O_Y(B_1) \) of \( F \). Consider \( (S \cdot S_0) \) as a divisor on the section \( S_0 \). According to [11] Remark 1.19 and by (1.5), we have

\[
2\pi_*(S \cdot S_0) \sim \pi_*(S \cdot S) + \pi_*(S_0 \cdot S_0) \sim (B_2 - B_1) + (B_1 - B_2) \sim 0;
\]

hence \( S \cap S_0 = \emptyset \). Conversely, suppose that \( S \) is a section such that \( S_0 \cap S = \emptyset \). Then \( \pi_*(S \cdot S_0) \sim 0 \). By (1.5), we have

\[
\pi_*(S_0 \cdot S_0) = \text{Divisor of } (\det F \otimes L_S^{-2}) \sim B_1 - B_2.
\]

Set \( S \sim S_0 + \pi^*D \) for some divisor \( D \) on \( Y \). Then we have

\[
0 \sim \pi_*(S \cdot S_0) \sim \pi_*(S_0 + \pi^*D) \cdot S_0) \sim (B_1 - B_2) + D;
\]

therefore \( D \sim B_2 - B_1 \); hence \( S \in \{ S_0 + \pi^*(B_2 - B_1) \} \).

Return to the triple covering case. Suppose that

\[
\mathcal{E}_0 = \mathcal{E} \otimes O_Y(-N) \tag{1.6}
\]

is normalized; cf. [6] V, 2.8. For a canonical divisor \( K_{P(E)} \) of \( P(E) \) and for the triple cover \( X \) regarded as a divisor of \( P(E) \), we have

\[
K_{P(E)} \sim -2S_0 + \pi^*(B - 2N + K_Y) \text{ and } X \sim 3S_0 + \pi^*(3N - B), \tag{1.7}
\]

where \( K_Y \) is a canonical divisor of \( Y \); cf. [6] V, 2.10, [3] Theorem 2.1.
1.1.3. Elementary transformations. Let \( W \) be a ruled surface with the projection \( \pi : W \to Y \). Let \( P \in W \) and \( F = \pi^{-1}(\pi(P)) \). Let \( \varphi : \tilde{W} \to W \) be the blowing up of \( W \) at \( P \). The strict transform \( \tilde{F} \) of the fiber \( F \) by the blowing up \( \varphi \) on \( \tilde{W} \) is isomorphic to \( \mathbb{P}^1 \) and \( \tilde{F}^2 = -1 \). Therefore we can blow down \( \tilde{W} \) along \( \tilde{F} \); in other words, there is a morphism \( \psi : \tilde{W} \to W' \) to a surface \( W' \) which is the blowing up of \( W' \) contracting \( \tilde{F} \) to a point \( P' \). It is easy to prove that the surface \( W' \) is again a ruled surface over \( Y \). The new ruled surface \( W' \) is called the \textit{elementary transformation of} \( W \) \textit{with center} \( P \). We denote the birational map \( \psi \circ \varphi^{-1} \) by \( \elm_P \) and call it the \textit{elementary transform with center} \( P \). The surface \( W' \) is denoted by \( \elm_P(W) \) and, if no confusion is likely to occur, we denote the projection \( \elm_P(W) \to Y \) by \( \pi \) again. Note that \( \elm_P \) is the inverse of \( \elm_P \).

Let \( P_1, \ldots, P_n \) be distinct points in \( W \) such that \( \pi(P_i) \neq \pi(P_j) \) for \( i \neq j \). We inductively define \( \elm_{P_1, \ldots, P_n} \) by

\[
\elm_{P_1, \ldots, P_n} = \elm_{P_n} \circ \elm_{P_1, \ldots, P_{n-1}},
\]

where \( P_n \) is considered as the point \( \elm_{P_1, \ldots, P_{n-1}}(P_n) = \elm_{P_1, \ldots, P_{n-1}}(W) \). Note that \( \elm_{P_1, P_2} = \elm_{P_2, P_1} \) if \( \pi(P) \neq \pi(Q) \). Let \( S \) be a section of the ruled surface \( W \). Then the image \( \elm_{P_1, \ldots, P_n}(S) \) of \( S \) is a section of \( \elm_{P_1, \ldots, P_n}(W) \). Suppose \( P \in S \).

We define \( \elm_{m_P} \) for \( n \in \mathbb{N} \) by

\[
\elm_{m_P} = \elm_{P_1, P_2, \ldots, P_m},
\]

where \( P_1 = P \) and \( P_{m+1} (m = 1, \ldots, n-1) \) is the unique point in \( \elm_{m_P}(S) \cap \pi^{-1}(\pi(P)) \). It is clear how to define \( \elm_D \) for an arbitrary effective divisor \( D \) on a section \( S \) of \( W \).

1.1.4. Notations and conventions. We adopt and use almost all the notations and conventions in [11] and [6]. If no confusion is likely to occur, we denote \( H^i(C, \mathcal{O}_C(D)) \) by \( H^i(C, D) \) and \( \dim H^i(C, \mathcal{O}_C(D)) \) by \( h^i(C, D) \). For a linear series \( |D|, \text{Bp}(|D|) \) denotes the set of all base points of \( |D| \); possibly \( \text{Bp}(|D|) = \emptyset \).

If we denote \( \elm_P = \psi \circ \varphi^{-1} \) for \( \elm_P \), then the map \( \varphi : \tilde{W} \to W \) always denotes the blowing up of \( W \) at \( P \) and the map \( \psi : \tilde{W} \to W' \) always denotes the blowing down of \( \tilde{W} \) along the strict transform \( \tilde{F} \) under \( \varphi \) of the fiber \( F = \pi^{-1}(P) \). For a curve \( C \subset W \), the curve \( \tilde{C} \subset \tilde{W} \) denotes the \textit{strict transform} of \( C \) by the blowing up \( \varphi \).

The integer \( r_P \) is the \textit{multiplicity} of \( C \) at \( P \) and the integer \( \delta_P \) is the \textit{measure of a curve singularity}, that is, \( \delta_P = \text{length}(\mathcal{O}_{C,P}/\mathcal{O}_{C,P}) \), where \( \mathcal{O}_{C,P} \) is the integral closure of \( \mathcal{O}_{C,P} \).

2. Base-point-free pencils on triple covers

Let \( X \) and \( Y \) be irreducible smooth curves of genus \( g_x \) and \( g_y \) and assume that there exists a triple covering \( f : X \to Y \). Let \( \mathcal{E}' \) be the Tschirnhausen module for the triple covering \( f, e \) the \( e \)-invariant of \( \mathcal{P}(\mathcal{E}) \), and \( \pi : \mathcal{P}(\mathcal{E}) \to Y \) the projection. Finally, let \( p_1 : Y \times \mathbb{P}^1 \to Y \) and \( p_2 : Y \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the projections.

For convenience, we define the following invariant which will be used several times; cf. Remark 2.13.
**Definition 2.1.** The $M$-invariant $m$ of the triple covering $f$ is defined by the number

$$m = \frac{g_x - 3g_y + 2}{2} - \frac{|e|}{2} - 2.$$

**Remark 2.1.1.** The number $m$ is an integer since, by (1.3) and (1.4), we have

$$m = \deg N - \frac{e + |e|}{2} - 2. \quad (2.1)$$

2.1. **Existence of nontrivial morphisms.** We will prove the existence of nontrivial morphisms, which give a partial answer to Question [A] Theorem 2.6. We need the following lemma.

**Lemma 2.2.** Let $e$ be the $e$-invariant of $\mathbb{P}(E)$. Then

$$-g_y \leq e \leq \max \left\{ 2g_y - 2, \, \frac{g_x - 3g_y + 2}{3} \right\}. \quad (2.2)$$

**Proof.** Suppose that the Tschirnhausen module $E^\vee$ is indecomposable; then the $e$-invariant $e$ (in fact, any $e$-invariants of indecomposable ruled surfaces) satisfies the inequalities

$$-g_y \leq e \leq 2g_y - 2 \quad (2.3)$$

by Nagata’s Theorem; cf. [6] V, Ex.2.5. Suppose that the Tschirnhausen module $E^\vee$ is decomposable with $E = L \oplus M$ and $\deg L \leq \deg M$. Since $(\det E)^\otimes 2 = L^2 \otimes M^2$, it follows that

$$\deg L + \deg M = g_x - 3g_y + 2 \quad (2.4)$$

by (1.3). We also have $2\deg L \geq \deg M$ and $2\deg M \geq \deg L$ by [12] P.1145. Therefore we have

$$\frac{g_x - 3g_y + 2}{3} \leq \deg L \leq \frac{g_x - 3g_y + 2}{2} \leq \deg M \leq \frac{2(g_x - 3g_y + 2)}{3}. \quad (2.5)$$

Since $e = \deg M - \deg L$, it follows that

$$0 \leq e \leq \frac{g_x - 3g_y + 2}{3}. \quad (2.6)$$

Hence (2.2) follows from (2.3) and (2.5). □

To get base-point-free pencils on $X$, we first take certain linear series on $\mathbb{P}(E)$.

**Lemma 2.3.** Assume that $g_x \geq 9g_y + 4$. Let $A \in \text{Div}(Y)$ be an effective divisor with $\deg A \geq 2g_y - 1$ and let $m$ be the $M$-invariant of the triple covering $f$. If $\deg A \leq m$, then the linear series $|K_X - f^*A|$ on $X$ is cut out by the linear series $|K_{\mathbb{P}(E)} + X - \pi^*A|$ on $\mathbb{P}(E)$.

**Remark 2.3.1.** By (2.2), we have $e \leq (g_x - 3g_y + 2)/3$; hence it follows by the assumption $g_x \geq 9g_y + 4$ that $m \geq 2g_y$. Therefore there exists a divisor $A \in \text{Div}(Y)$ with $2g_y - 1 \leq \deg A \leq m$.

**Proof of Lemma 2.3.** Let $S_0$ be a minimal degree section of $\mathbb{P}(E)$ and set $a = \deg A$. To begin with, we count the dimension of $|K_{\mathbb{P}(E)} + X - \pi^*A|$. By (1.7), we have

$$K_{\mathbb{P}(E)} + X - \pi^*A \sim S_0 + \pi^*(N + K_Y - A). \quad (2.6)$$

By [6] V, 2.4 and [6] II, 7.11, we have

$$h^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(S_0)) = h^0(Y, \pi_*\mathcal{O}_{\mathbb{P}(E)}(S_0)) = h^0(Y, \mathcal{E}_0).$$
It follows by the projection formula that
\[
h^0(\mathcal{P}(E), K_{\mathcal{P}(E)} + X - \pi^*A) = h^0(Y, \pi_*(\mathcal{O}_{\mathcal{P}(E)}(S_0) \otimes \pi^*\mathcal{O}_{\mathcal{P}(E)}(N + KY - A)))
\]
\[= h^0(Y, E_0 \otimes \mathcal{O}_Y(N + KY - A)). \tag{2.7}
\]
Since \(\deg(E_0 \otimes \mathcal{O}_Y(N + KY - A)) = -e + 2(n + 2g_y - 2 - a)\), it follows by Riemann-Roch Theorem, \((1.3)\) and \((1.4)\), and Serre duality that
\[
h^0(\mathcal{P}(E), K_{\mathcal{P}(E)} + X - \pi^*A)
\]
\[= -e + 2(n + 2g_y - 2 - a) + 2(1 - g_y) + h^1(Y, E_0 \otimes \mathcal{O}_Y(N + KY - A))
\]
\[= g_x - g_y - 2a + h^0(Y, E_0^\vee \otimes \mathcal{O}_Y(-N + A)). \tag{2.9}
\]
To compute \(h^0(Y, E_0^\vee \otimes \mathcal{O}_Y(-N + A))\), we have
\[
E_0^\vee \cong E_0 \otimes \det E_0^{-1} \cong E_0 \otimes (\det E_0^{-1} \otimes \mathcal{O}_Y(2N)) = E_0 \otimes \mathcal{O}_Y(-B + 2N). \tag{2.8}
\]
Therefore it follows by \((2.7)\) and \((2.8)\) that
\[
h^0(\mathcal{P}(E), K_{\mathcal{P}(E)} + X - \pi^*A)
\]
\[= g_x - g_y - 2a + h^0(Y, E_0 \otimes \mathcal{O}_Y(-B + N + A)). \tag{2.9}
\]
By \((1.4)\) and \((2.1)\), we have
\[
\deg(-B + N + A) \leq -b + n + m \leq -2 - \frac{|e| - e}{2} < 0.
\]
Since \(E_0\) is normalized, we have
\[
h^0(Y, E_0 \otimes \mathcal{O}_Y(-B + N + A)) = 0. \tag{2.10}
\]
Therefore it follows by \((2.9)\) and \((2.10)\) that
\[
h^0(\mathcal{P}(E), K_{\mathcal{P}(E)} + X - \pi^*A) = g_x - g_y - 2a. \tag{2.11}
\]
We now prove that
\[
h^0(X, K_X - f^*A) = h^0(\mathcal{P}(E), K_{\mathcal{P}(E)} + X - \pi^*A) = g_x - g_y - 2a.
\]
Since \(f\) is finite, we have \(h^0(X, \mathcal{O}_X(f^*A)) = h^0(Y, f_*(\mathcal{O}_X(f^*A)))\) by \([6, III, Ex.4.1]\).
Hence, by the projection formula, we have
\[
h^0(X, \mathcal{O}_X(f^*A)) = h^0(Y, f_*(\mathcal{O}_X(f^*A) \otimes \mathcal{O}_X)) = h^0(Y, \mathcal{O}_Y(A) \otimes f_*\mathcal{O}_X)
\]
\[= h^0(Y, \mathcal{O}_Y(A) \otimes (\mathcal{O}_Y \oplus E^\vee)) = h^0(Y, \mathcal{O}_Y(A)) + h^0(Y, E^\vee \otimes \mathcal{O}_Y(A)) \tag{2.12}
\]
\[= a - g_y + 1 + h^0(Y, E^\vee \otimes \mathcal{O}_Y(A)),
\]
where \(h^0(Y, \mathcal{O}_Y(A)) = 0\) by the assumption \(\deg A \geq 2g_y - 1\). It remains to count \(h^0(Y, E^\vee \otimes \mathcal{O}_Y(A))\). We have \(E_0^\vee \cong E_0 \otimes \mathcal{O}_Y(-B + 2N)\) by \((2.8)\), but \(E_0 = E \otimes \mathcal{O}_Y(-N)\); hence \(E^\vee \cong E_0 \otimes \mathcal{O}_Y(-B + N)\). Therefore it follows that
\[
h^0(Y, E^\vee \otimes \mathcal{O}_Y(A)) = h^0(Y, E_0 \otimes \mathcal{O}_Y(-B + N + A)) = 0
\]
by \((2.10)\). Then we have
\[
h^0(X, \mathcal{O}_X(f^*A)) = a - g_y + 1
\]
by \((2.12)\); hence
\[
h^0(X, K_X - f^*A) = g_x - g_y - 2a. \tag{2.13}
\]
Thus it follows by (2.11) and (2.13) that
\[ h^0(\mathcal{P}(\mathcal{E}), K_{\mathcal{P}(\mathcal{E})} + X - \pi^* A) = h^0(X, K_X - f^* A) = g_x - g_y - 2a. \] (2.14)

Finally, we claim that the following restriction map \( \gamma \) is injective:
\[ \gamma : H^0(\mathcal{P}(\mathcal{E}), K_{\mathcal{P}(\mathcal{E})} + X - \pi^* A) \to H^0(X, K_X - f^* A). \]
Let \( C, C' \in |K_{\mathcal{P}(\mathcal{E})} + X - \pi^* A| \). By (2.11), \( m \leq n - e - 2 \); hence
\[ a \leq m \leq n - e - 2 \leq n - e + 2g_y - 2. \] (2.15)
We have \( S_0^2 = b - 2n \) by (1.3); it follows by (1.3) and (2.6) that
\[ C.X - C.C' = 2n - e + 2g_y - 2 - a. \]
By (1.3) and (1.4), we have \( e = 2n - (g_x - 3g_y + 2) \), and, by (2.2), we have \( e \geq -g_y \).
Therefore it follows that \( 2n - (g_x - 3g_y + 2) \geq -g_y \); hence it follows by the assumption \( g_x \geq 9g_y + 4 \) that \( n = \deg N > 0 \). Furthermore we have \( n - e + 2g_y - 2 - a \geq 0 \) by (2.15); hence it follows that
\[ C.X - C.C' = n + (n - e + 2g_y - 2 - a) > 0. \]
Therefore the restriction map \( \gamma \) is injective as asserted; hence the linear series \( |K_X - f^* A| \) is cut out by the linear series \( |K_{\mathcal{P}(\mathcal{E})} + X - \pi^* A| \).

We will get base-point-free linear series on \( X \) which are not induced from the base curve \( Y \).

**Lemma 2.4.** Assume that \( g_x \geq 9g_y + 4 \). Let \( A \in \text{Div}(Y) \) be an effective divisor with \( \deg A \geq 2g_y - 1 \) and let \( m \) be the \( M \)-invariant of the triple covering \( f \). If \( \deg A \leq m - 1 \), then the linear series \( |K_X - f^* A| \) on \( X \) is very ample. If \( \deg A = m \), then \( |K_X - f^* A| \) is base-point-free and separates points in the same fiber of the triple covering \( f \).

**Proof.** Set \( \mathcal{L} = |K_{\mathcal{P}(\mathcal{E})} + X - \pi^* A| \). By (1.7), we have \( \mathcal{L} = |S_0 + \pi^*(N + K_Y - A)| \).
We first prove that \( \mathcal{L} \) is very ample if \( \deg A \leq m - 1 \), or \( \mathcal{L} \) is base-point-free and separates points in the same fiber of the triple covering \( f \) if \( \deg A = m \).

Suppose that \( a = \deg A \leq m - 1 \). We have \( m = n - 2 - (e + |e|)/2 \) and \( e = 2n - b \) by (1.4) and (2.1); it follows that
\[ \deg(N + K_Y - A) \geq 2g_y + 1 + \frac{|e| + e}{2} \geq 2g_y + 1, \]
\[ \deg(N + K_Y - A + (B - 2N)) \geq 2g_y + 1 + \frac{|e| - e}{2} \geq 2g_y + 1. \]
Therefore \( |N + K_Y - A| \) and \( |N + K_Y - A + (B - 2N)| \) are very ample, and \( |N + K_Y - A - P| \) and \( |N + K_Y - A + (B - 2N) - P| \) are nonspecial for every \( P \in Y \). Thus \( \mathcal{L} \) is very ample by [6, V, Ex.2.11].

Suppose that \( a = \deg A = m \). Then we have
\[ \deg(N + K_Y - A) \geq 2g_y + \frac{|e| + e}{2} \geq 2g_y, \]
\[ \deg(N + K_Y - A + (B - 2N)) \geq 2g_y + \frac{|e| - e}{2} \geq 2g_y, \]
by (1.3), (1.4), and (2.1). Therefore \( |N + K_Y - A| \) and \( |N + K_Y - A + (B - 2N)| \) have no base points, and \( |N + K_Y - A| \) is nonspecial. Thus \( \mathcal{L} \) is base-point-free.
by \[6\] V, Ex.2.11. Furthermore \(L\) separates points in the same fiber of the triple covering \(f\) because \(|S_0 + \pi^*(N + K_Y - A)|\) contains a section by \[6\] V, Ex.2.11.

Since \(L\) is base-point-free for any effective divisor \(A \in \text{Div}(Y)\) with \(2g_y - 1 \leq \text{deg} A \leq m\), the linear series \(|K_X - f^*A|\) on \(X\) is also base-point-free because \(|K_X - f^*A|\) is cut out by \(L\) by Lemma 2.2.1. Set

\[ r = \dim |K_X - f^*A| = \dim |K_P(\mathcal{E}) + X - \pi^*A|; \]

cf. (2.14). Let \(\alpha : X \to \mathbf{P}^r\) and \(\beta : \mathbf{P}(\mathcal{E}) \to \mathbf{P}^r\) be the morphisms associated to the base-point-free linear series \(|K_X - f^*A|\) and \(L\), respectively. The morphism \(\beta\) is an embedding for \(a \leq m - 1\) or a morphism which separates points in the same fiber of the triple covering \(f\) for \(a = m\); but, we have \(\alpha = \beta \circ i\), where \(i : X \to \mathbf{P}(\mathcal{E})\) is the inclusion. Therefore the morphism \(\alpha\) is also an embedding for \(a \leq m - 1\) or a morphism which separates points in the same fiber of the triple covering \(f\) for \(a = m\). \(\square\)

We need the following lemma.

**Lemma 2.5** ([3] Lemma 2.2.1). Let \(f : X \to Y\) be a covering of degree \(k \geq 2\) of smooth curves and let \(g_\alpha^k(r \geq 1)\) be a base-point-free linear series on \(X\) which is not induced by \(Y\). Let \(P_1, \ldots, P_{r-1}\) be \(r - 1\) general points on \(X\). Then the base-point-free part of the pencil \(g_\alpha^k(-P_1 - \cdots - P_{r-1})\) is not induced by \(Y\).

We now prove the existence theorem.

**Theorem 2.6.** Let \(X\) be a smooth irreducible curve of genus \(g_x\). Suppose that there exists a triple covering \(f : X \to Y\) where \(Y\) is a smooth irreducible curve of genus \(g_y\) and suppose that \(g_x \geq 9g_y + 4\). Let \(\mathcal{E}^*\) be the Tschirnhausen module for the triple covering \(f\) and \(e\) the \(e\)-invariant of the ruled surface \(\mathbf{P}(\mathcal{E})\). For every integer \(d\) with

\[ d \geq \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2} + 4g_y, \]

there exists a nontrivial morphism \(h : X \to \mathbf{P}^{1}\) of degree \(d\).

**Proof.** It is obvious for \(d \geq g_x + 1\); just take a general \(|D| \in \text{Pic}(X)\) which is nonspecial. From now on, assume that \(d \leq g_x\). Let \(m\) be the \(M\)-invariant of the triple covering \(f\). Choose an effective divisor \(A \in \text{Div}(Y)\) of degree \(a\) with \(2g_y - 1 \leq a \leq m\). By Lemma 2.4, the linear series \(|K_X - f^*A|\) is not composed with the triple covering \(f\). By (2.13), we have

\[ \dim |K_X - f^*A| = g_x - g_y - 2a - 1. \]

Therefore, subtracting general \(g_x - g_y - 2a - 2\) points from \(|K_X - f^*A|\), we have a base-point-free pencil of degree \(g_x + g_y - a\), which is not composed with the triple covering \(f\) by Lemma 2.5. Since

\[ 2g_y - 1 \leq a \leq m = \frac{g_x - 3g_y + 2}{2} - \frac{|e|}{2} - 2, \]

we have

\[ \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2} + 4g_y \leq g_x + g_y - a \leq g_x - g_y + 1. \]

Therefore, for every integer \(d\) with

\[ \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2} + 4g_y \leq d \leq g_x - g_y + 1, \]
there exists a base-point-free pencil of degree $d$ which is not composed with the triple covering $f$.

If $g_y = 0$, the proof is done. Assume that $g_y \geq 1$. We need the following lemma; here we adopt the conventions and notation used in [1].

Lemma ([2] Lemma 3.2). Fix an integer $s \geq 1$. Let $C$ be a smooth curve of genus $g \geq 4s - 4$ defined over an algebraically closed field of characteristic zero. For an integer $d$, let $\Sigma^1_d$ be the union of components of $W^1_d(C)$ whose general element is base-point-free and complete. If $\Sigma^1_{g-s+1} \neq \emptyset$, then we have $\Sigma^1_{g-s+2} \neq \emptyset$.

Taking $C = X$ and $s = g_y$ in the above Lemma, it follows that $\Sigma^1_{g_y-g_y+1} \neq \emptyset$; hence $\Sigma^1_{g_y-g_y+2} \neq \emptyset$. By taking $s' = g_y - 1$, we again have $\Sigma_{g_x-s'+2} = \Sigma_{g_x-g_y+3} \neq \emptyset$; note that $g_x \geq 4g_y - 4 > 4s' - 4$ by the hypothesis $g_x \geq 9g_y + 4$. We may continue this process by taking smaller $s$’s until $s = 2$ and we stop.

Remark 2.6.1. Let $f : X \to Y$ be a covering of degree $k$. For prime $k$, assume that $g_x \geq k^2g_y + (k-1)^2$ which for $k = 3$ is the bound of the above theorem. Then it follows that

$$ \frac{g_x - g_y k - 1}{k} \geq k(g_y + 1) \geq k \text{gon}(Y) \geq \text{gon}(X). $$

If $\frac{g_x - g_y k - 1}{k} > \text{gon}(X)$, then any morphism of degree $\text{gon}(X)$ onto $\mathbb{P}^1$ factors through $f$; in particular, $\text{gon}(X) = k \cdot \text{gon}(Y)$. On the other hand, if $\frac{g_x - g_y k - 1}{k} = \text{gon}(X)$, then $\text{gon}(X) = k(g_y + 1)$; hence

$$ k \cdot \text{gon}(Y) = \text{gon}(X) = k(g_y + 1) $$

i.e., $\text{gon}(Y) = g_y + 1$ which is only possible for $g_y \leq 1$. But for $g_y \leq 1$ the above equality shows that $\text{gon}(X) = k \text{gon}(Y)$, again. Therefore, assuming $g_x \geq 9g_y + 4$ in Main Theorem [A] the gonality of $X$ is fixed by that of $Y$. Furthermore, the bound implies that $Y$ is the only curve of genus at most $g_y$ triply covered by $X$.

2.2. Castelnuovo-Severi inequality for triple coverings. We will improve the Castelnuovo-Severi inequality in case of triple coverings: Theorem 2.12. As a by-product, we may conclude that Question [A] does not have an affirmative answer for certain triple coverings.

First, we prove that every nontrivial morphism $X \to \mathbb{P}^1$ is determined by a certain finite sequence of elementary transformations; Proposition 2.10. Recall that trisections are irreducible curves on ruled surfaces whose intersection number with a fiber is 3.

Remark 2.7. Let $h : X \to \mathbb{P}^1$ be a nontrivial morphism and set $X' = (f \times h)(X) \subset Y \times \mathbb{P}^1$. Since $\text{deg} f = 3$ and $\text{deg} h = d$, we have $X' \equiv 3S + dF$, where $S = p^{-1}_2(s)$ for some $s \in \mathbb{P}^1$ and $F = p^{-1}_1(y)$ for some $y \in Y$. Therefore $X'$ is a trisection of $\mathbb{P}^1(\mathcal{E})$. Let $p_a(X')$ be the arithmetic genus of $X'$. By the adjunction formula, we have $p_a(X') = 2d + 3g_y - 2$. Since the morphism $h$ does not factor through the triple covering $f$, the image $X'$ is birational to $X$; hence $X$ is the normalization of $X'$. Therefore it follows that

$$ p_a(X') = 2d + 3g_y - 2 = g_x + \sum_{P \in X'} \delta_P. \quad (2.16) $$
Lemma 2.8. Let $C$ be a trisection on a ruled surface $W$ over $Y$ and let $\pi : W \to Y$ be the projection. For $P \in C \subset W$, set $elm_P = \psi \circ \varphi^{-1}$. Suppose that $P$ is a singular point of $C$ with multiplicity $r_P$ and suppose that there exists a infinitely near singular point $Q$ of $C$ lying over $P$. Then $Q$ is the unique singular point of $C$ among points lying over $P$, and $P' = \psi(Q)$ is the unique singular point of $C'$ among points on $\pi^{-1}(\pi(P')) \cap C'$. Furthermore, if $r_P = 2$ then $r_{P'} = 2$ and $\delta_{P'} = \delta_P - 1$, and if $r_P = 3$ then $\delta_{P'} = \delta_P - 3$.

Proof. Let $E$ be the exceptional divisor of the blowing up $\varphi$. Set $F = \pi^{-1}(\pi(P))$, $W' = elm_P(W)$, $C' = elm_P(C)$, and $E' = \psi(E)$. Suppose that $r_P = 2$, then $E.C = 2$; but $Q \in E \cap C$ and $r_Q \geq 2$, hence the point $Q$ is the unique singular point of $C$ lying over $P$ and $r_Q = 2$. Since $F.C = 1$, it follows that $Q \notin F$. Therefore $P' = \psi(Q)$ is a singular point of $C'$ with $r_{P'} = 2$ and $\delta_{P'} = \delta_Q = \delta_P - 1$.

Note that $E'.C' = 3$ and $r_{P'} = 2$. Therefore there is no singular points of $C'$ other than $P'$ on $\pi^{-1}(\pi(P)) \cap C'$.

Suppose that $r_P = 3$, then $E.C = 3$; but $Q \in E \cap C$ and $r_Q \geq 2$, hence the point $Q$ is the unique singular point of $C$ lying over $P$. Since $F.C = 0$, we have $Q \notin F$. Therefore $P'$ is a singular point of $C'$ and $\delta_{P'} = \delta_Q = \delta_P - 3$.

Furthermore, since $E'.C' = 3$ and $r_{P'} \geq 2$, it follows that there is no singular points of $C'$ other than $P'$ on $\pi^{-1}(\pi(P)) \cap C'$.

The following Lemma is not difficult; but we can’t find references.

Lemma 2.9. The singularities of a trisection $C$ can be resolved by a finite sequence of elementary transformations consisting of at most $\sum_{P \in C} \delta_P$-elementary transformations.

Proof. Let $C$ be a trisection on a ruled surface. Suppose that $P \in X$ is a singular point of $X$. By Lemma 2.8 we have $\delta_{P'} < \delta_P$. Therefore, after applying finitely many elementary transformations, we get $\delta_{P'} = 0$, which means $P'$ is a nonsingular point of $X'$. According to Lemma 2.8 no singular points other than infinitely near singular points arise during applying elementary transformations. Therefore, applying this process to all singular points of $X$, one can resolve the singularities of $C$.

Proposition 2.10. Suppose that there exists a nontrivial morphism $h : X \to \mathbb{P}^1$. Then

$$h = p_2 \circ elm|_X,$$

where $p_2 : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection and $elm$ is a finite sequence of elementary transformations which transforms $\mathbb{P}(\mathcal{E})$ to $Y \times \mathbb{P}^1$.

Proof. Set $X' = (f \times h)(X) \subset Y \times \mathbb{P}^1$. The image $X'$ is a trisection of $Y \times \mathbb{P}^1$. According to Lemma 2.9 the singularities of $X'$ can be resolved by a finite sequence of elementary transformations. Let $u$ denote a sequence of elementary transformations which resolves the singularities of $X'$. Set $W = u(Y \times \mathbb{P}^1)$ and let
\[ \pi : W \to Y \] be the projection. Note that \( \pi \circ u = p_1 \), where \( p_1 : Y \times P^1 \to Y \) be the first projection; hence it follows that
\[ \pi \circ u \circ (f \times h) = f. \]
Since the morphism \( h \) does not factor through the triple covering \( f \), the image \( X' \) is birational to \( X \). Therefore \( u(X') \) is isomorphic to \( X \). Therefore there is an embedding
\[ j = u \circ (f \times h) : X \hookrightarrow W \]
such that \( f = \pi \circ j \). By Lemma 1.1 such an embedding is unique. Therefore it follows that \( W \cong P(\mathcal{E}) \) and hence \( h = p_2 \circ u^{-1}|_X \). Note that \( u^{-1} \) is also a finite sequence of elementary transformations which transforms \( P(\mathcal{E}) \) to \( Y \times P^1 \). □

We need the following lemma several times.

**Lemma 2.11** ([13] Lemma 7). Let \( W \) be a ruled surface with \( e \)-invariant \( e \) and let \( P \in W \). Let \( e' \) be the \( e \)-invariant of \( W' = \text{elm}_P(W) \). If \( P \) is contained in a minimal degree section of \( W \) then \( e'=e+1 \), and, if \( P \) is not contained any minimal degree sections of \( W \), then \( e'=e-1 \).

We now improve Castelnuovo-Severi inequality for triple coverings.

**Theorem 2.12.** Let \( X \) be a smooth irreducible curve of genus \( g_x \). Suppose that there exists a triple covering \( f : X \to Y \) where \( Y \) is a smooth irreducible curve of genus \( g_y \). Let \( \mathcal{E} \) be the Tschirnhausen module for the triple covering \( f \) and \( e \) the \( e \)-invariant of the ruled surface \( P(\mathcal{E}) \). If there exists a nontrivial morphism \( h : X \to P^1 \) of degree \( d \), then
\[ d \geq \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2}. \]

*Proof.* Set \( X' = (f \times h)(X) \subset Y \times P^1 \). Let \( P \) be a singular point of \( X' \). According to Lemma 2.9 there exists a finite sequence of elementary transformations which resolves the singularities of \( X' \). Set
\[ \alpha = \min \{ l : \text{elm}_{P_1, \ldots, P_l} \text{ is a resolution of singularities of } X' \}. \]

Let \( \text{elm}_{P_1, \ldots, P_\alpha} \) be a minimal sequence of elementary transformations that is a resolution of singularities of \( X' \). By Lemma 2.9 we have
\[ \alpha \leq \sum_{P \in X'} \delta_P. \quad (2.17) \]

We now prove that \( |e| \leq \alpha \). Since \( X \) is a normalization of \( X' \), we have an isomorphism \( \phi : X \to \text{elm}_{P_1, \ldots, P_\alpha}(X') \). Therefore there exists an embedding
\[ j = j' \circ \phi : X \hookrightarrow \text{elm}_{P_1, \ldots, P_\alpha}(Y \times P^1) \]
such that \( f = \pi \circ j \), where \( j' : \text{elm}_{P_1, \ldots, P_\alpha}(X') \hookrightarrow \text{elm}_{P_1, \ldots, P_\alpha}(Y \times P^1) \) is the inclusion and \( \pi : \text{elm}_{P_1, \ldots, P_\alpha}(Y \times P^1) \to Y \) is the projection. However such an embedding \( j \) is unique by Lemma 1.1 hence it follows that \( \text{elm}_{P_1, \ldots, P_\alpha}(Y \times P^1) \cong P(\mathcal{E}) \). The \( e \)-invariant of \( P(\mathcal{E}) \) is equal to \( e \), but the \( e \)-invariant of \( Y \times P^1 \) is equal to zero. Therefore, by Lemma 2.11 it follows that
\[ |e| \leq \alpha. \quad (2.18) \]
By (2.17) and (2.18), we have $|e| \leq \sum_{P \in X'} \delta_P$. Hence it follows by (2.16) that

$$2d = g_x - 3g_y + 2 + \sum_{P \in X'} \delta_P \geq g_x - 3g_y + 2 + |e|. \quad \square$$

Remarks 2.13.

(a) Theorem 2.6 and 2.12 give a partial answer to Question A in case of triple covers. On the other hand, as we already remarked in the introduction, Question A has been completely answered in case of trigonal covers by the following results of Maroni.

Results of Maroni ([9], cf. [10] Proposition 1, Corollary 1). Let $X$ be a trigonal curve of genus $g_x > 4$ with the triple covering $f : X \to \mathbb{P}^1$ and let $m_0$ be the so-called Maroni invariant of $X$. For every integer $d$ with $d \geq g_x - m_0$, there exists a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $d$. Conversely, if there exists a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $d$, then $d \geq g_x - m_0$.

(b) If $X$ is a trigonal curve, then it is not difficult to prove that the $M$-invariant of $X$ is equal to the Maroni invariant of $X$. Therefore Theorem 2.6 and 2.12 may be regarded as a generalization of the results of Maroni to the case of triple covers of smooth irrational curves.

3. Nontrivial morphisms of minimal possible degree

We investigate what conditions may be imposed on $E^\vee$ if $X$ admits a nontrivial morphism $X \to \mathbb{P}^1$ of minimal possible degree $\frac{g_x - 3g_y + 2}{2} + |e|$. Proposition 3.2. As a corollary, we will show that certain triple covers do not admit nontrivial morphisms of minimal possible degree; Corollary 3.3. We need the following proposition.

Proposition 3.1 (cf. [8] or [5]). Let $F = \mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2)$ be a decomposable rank two vector bundle on a curve $Y$. Let $S_F$ be a minimal degree section of $\mathbb{P}(F)$ and let $S$ be a section of $\mathbb{P}(F)$ such that $S \cap S_F = \emptyset$. Let $P \in S \cup S_F$. Set $P(G) = \text{elm}_P(\mathbb{P}(F))$.

(a) If $P \in S$ and $\deg(-B_2 + B_1 + p) \leq 0$, then $\text{elm}_P(S_F)$ is a minimal degree section of $\mathbb{P}(G)$ and

$$\text{P}(G) \cong \text{P}(\mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2 - p)) \cong \text{P}(\mathcal{O}_Y(B_1 + p) \oplus \mathcal{O}_Y(B_2)).$$

(b) If $P \in S_F$, then $\text{elm}_P(S_F)$ is a minimal degree section of $\mathbb{P}(G)$ and

$$\text{P}(G) \cong \text{P}(\mathcal{O}_Y(B_1 - p) \oplus \mathcal{O}_Y(B_2)) \cong \text{P}(\mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2 + p)).$$

Proposition 3.2. Assume that $e \geq 0$. Suppose that there exists a nontrivial morphism $h : X \to \mathbb{P}^1$ of the minimal possible degree $\frac{g_x - 3g_y + 2}{2} + \frac{e}{2}$. Then the Tschirnhausen module $E^\vee$ is decomposable.

Proof. Set $X' = (f \times h)(X) \subset Y \times \mathbb{P}^1$ and set $d_0 = \frac{g_x - 3g_y + 2}{2} + \frac{e}{2}$. It follows by Remark 2.7 that

$$\sum_{P \in X} \delta_P = e. \quad (3.1)$$

Step 1. There exists a sequence $\text{elm}_{P_1, \ldots, P_e} : Y \times \mathbb{P}^1 \to \mathbb{P}(E)$ consisting of exactly $e$ elementary transformations that is a resolution of singularities of $X'$. 

Proof of Step 1. According to Lemma 2.9, the singularities of $X'$ can be resolved by a finite sequence of elementary transformations with the singular points as centers. Set

$$\alpha = \min\{l : elm_{P_1, \ldots, P_l} \text{ resolves the singularities of } X'\}. $$

We will prove that $\alpha = e$. By Lemma 2.9 we have

$$\alpha \leq \sum_{P \in X'} \delta_P = e. \quad (3.2)$$

According to [6, V, Ex.5.5], there is a finite sequence of elementary transformations which transform $Y \times P_1$ into $P(E)$; set

$$\beta = \min\{l : elm_{P_1, \ldots, P_l}(Y \times P_1) = P(E)\}. $$

Since the $e$-invariants of $Y \times P_1$ and $P(E)$ are 0 and $e$, respectively, it follows by Lemma 2.11 that

$$e \leq \beta. \quad (3.3)$$

Let $elm$ be a sequence of elementary transformations that resolves the singularities of the image $X'$. Then $elm(X') \cong X$; hence

$$elm(Y \times P_1) \cong P(E) \quad (3.4)$$

by Lemma 1.1. Therefore we have

$$\beta \leq \alpha. \quad (3.5)$$

By (3.2), (3.3), and (3.5), it follows that

$$\alpha = \beta = e. \quad (3.6)$$

Therefore there exists a sequence of elementary transformations consisting of exactly $e$ elementary transformations which resolves the singularities of $X'$. □

Step 2. For some $t \in P^1$, we have

$$P_i \in Y \times \{t\}, \quad P_{i+1} \in elm_{P_1, \ldots, P_i}(Y \times \{t\})$$

for all $i = 1, \ldots, e - 1$.

Proof of Step 2. Let $elm_{P_1, \ldots, P_i}$ be a minimal sequence of elementary transformations that resolves the singularities of $X'$, which consists of $e$-elementary transformations. Set

$$X_i = elm_{P_1, \ldots, P_i}(X'), \quad W_i = elm_{P_1, \ldots, P_i}(Y \times P_1), \quad X_0 = X', \quad W_0 = Y \times P_1. $$

Let $e_i$ be the $e$-invariant of $elm_{P_1, \ldots, P_i}(Y \times P_1)$. It follows by Lemma 2.11 that $e_i \leq i$, where the equality holds if and only if $P_j$ is contained in a minimal degree section of $W_{j-1}$ for all $j$ with $1 \leq j \leq i$. Note that $elm_{P_1, \ldots, P_i}$ consists of exactly $e$ elementary transformations and the $e$-invariant of $P(E)$ is equal to $e$; but we have

$$elm_{P_1, \ldots, P_i}(Y \times P_1) \cong P(E)$$

by (3.2). Therefore it follows that $e_i = i$ for all $i$; hence $P_{i+1}$ is contained in a minimal degree section of $W_i$ for all $i = 0, \ldots, e - 1$.

Suppose that $P_t \in Y \times \{t\} \subset Y \times P^1$ for some $t \in P^1$. Since $Y \times \{t\}$ is a minimal degree section of $Y \times P^1$, it follows that the section $elm_{P_t}(Y \times \{t\})$ is also a minimal degree section of $W_1$ by Proposition 3.1. However, since $P_t$ is contained in the minimal degree section $Y \times \{t\}$, the ruled surface $W_1$ is decomposable by Proposition 3.1. Therefore there exists a unique minimal degree section of $W_1$.
by Corollary 1.17, which is equal to $elm_{P_i}(Y \times \{t\})$; hence we have $P_2 \in elm_{P_i}(Y \times \{t\})$. Repeating this process, we have $P_{i+1} \in elm_{P_i,\ldots,P_i}(Y \times \{t\})$ for all $i = 1, \ldots, e - 1$.

Continue the proof of Proposition 3.2. Note that $elm_{P_i,\ldots,P_i}(Y \times P^1) \cong P(E)$ by (3.4). By Proposition 3.1, the sections $elm_{P_i,\ldots,P_i}(Y \times \{t\})$ of $elm_{P_i,\ldots,P_i}(Y \times P^1)$ are minimal degree sections for all $i$; hence the ruled surface $P(E)$ is decomposable by Proposition 3.1. Therefore $E$ is decomposable.

Corollary 3.3. Assume that $e \geq 0$ and $E$ is indecomposable. If there exists a nontrivial morphism $h : X \rightarrow P^1$ of degree $d$, then we have

$$d \geq \frac{g_x - 3g_y + 2}{2} + \frac{e}{2}.$$ 

We characterize nontrivial morphisms $X \rightarrow P^1$ of minimal possible degree.

Proposition 3.4. Assume that the Tschirnhausen module $E^\ell$ is decomposable; set $\mathcal{E} = \mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2)$ with $\deg B_1 \leq \deg B_2$. Suppose that there exists a nontrivial morphism $h : X \rightarrow P^1$ of degree $\frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2}$. Then there exist a section $S$ of $P(E)$ which is not a minimal degree section and $e$ points $Q_1, \ldots, Q_e \in X \cap S \subset P(E)$ with $\pi_* (\sum Q_i) \in |B_2 - B_1|$ as a divisor on $S$ such that

$$h = p_2 \circ elm_{Q_1,\ldots,Q_e}|X,$$

where $p_2 : Y \times P^1 \rightarrow P^1$ is the projection.

Proof. Set $d_0 = \frac{g_x - 3g_y + 2}{2} + \frac{|e|}{2}$. Let $X' = (f \times h)(X) \subset Y \times P^1$. According to Step 1 of the proof of Proposition 3.2 there exists a sequence of elementary transformations

$$elm_{P_i,\ldots,P_i} : Y \times P^1 \rightarrow P(E)$$

that is a resolution of singularities of $X'$. Set

$$X_i = elm_{P_i,\ldots,P_i}(X'), \quad W_i = elm_{P_i,\ldots,P_i}(Y \times P^1), \quad X_0 = X', \quad W_0 = Y \times P^1.$$ 

We may assume that $P_{i+1}$ is a singular point of $X_i$ for all $i = 0, \ldots, e - 1$. Let $r_{i+1}$ be the multiplicity of $X_i$ at $P_{i+1}$.

First, we will prove that $r_{i+1} = 2$ for all $i$. Suppose $r_{i+1} = 3$ but $r_j = 2$ for all $j \leq i$. If we apply $elm_{P_{i+1}}$ to $X_i$, then, by Lemma 2.8 we have

$$\sum_{P \in X_{i+1}} \delta_P = \sum_{P \in X_i} \delta_P - 3 = \sum_{P \in X_0} \delta_P - i - 3 = e - i - 3.$$ 

Therefore we need at most $(e - i - 3)$ elementary transformations to resolve the singularities of $X_{i+1}$ by Lemma 2.8. However it contradicts the assumption: $elm_{P_{i+1,\ldots,P_{i+1}}}$ is the minimal sequence. Therefore we have $r_{i+1} = 2$ for all $i$.

Set $S_0 = elm_{P_{i+1,\ldots,P_{i+1}}}(Y \times \{t\})$, which is the minimal degree section of $P(E)$, and set $Q_{i+1} = elm_{P_{i+1,\ldots,P_{i+1}}}(F_{i+1})$, where $F_{i+1} = p_i^{-1}(p_i(P_{i+1}))$ is the fiber of the first projection $p_1 : elm_{P_{i+1,\ldots,P_i}}(Y \times P^1) \rightarrow Y$ over $p_1(P_{i+1})$. It is clear that $elm_{Q_{i+1,\ldots,Q_{i+1}}}$ is the inverse of $elm_{P_{i+1,\ldots,P_i}}$; cf. [11]. Let $S = elm_{P_{i+1,\ldots,P_i}}(Y \times \{t'\})$ for some $t' \neq t$. Then $S$ is a section of $P(E)$ such that $S \cap S_0 = \emptyset$ by Proposition 3.1.

Let $elm_{P_{i+1}} = \psi_{i+1} \circ \varphi_{i+1}^{-1}$. Since $r_{i+1} = 2$ and $F_{i+1}X_i = 3$ for all $i$, we have $\varphi_{i+1} X_i \neq \emptyset$, where $F_{i+1}$ and $X_i$ are strict transforms under the blowing up $\varphi_{i+1}$. Therefore it follows that

$$Q_{i+1} \in X \cap elm_{P_{i+1,\ldots,P_{i+2}}}(Y \times \{t'\}).$$
Lemma 1.2, it follows by (1.5) that

$$elm_{Q_1,\ldots,Q_e}(\mathcal{P}(\mathcal{E})) \equiv \mathcal{P}(\mathcal{O}_Y(B_1 + q_1 + \cdots + q_e) \oplus \mathcal{O}_Y(B_2)),$$

where $q_i = \pi(Q_i)$. Since $elm_{Q_1,\ldots,Q_e}(\mathcal{P}(\mathcal{E})) \equiv Y \times \mathbb{P}^1$, it follows that

$$\mathcal{P}(\mathcal{O}_Y(B_1 + q_1 + \cdots + q_e) \oplus \mathcal{O}_Y(B_2)) \equiv \mathcal{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y);$$

hence $\sum q_i \sim B_2 - B_1$; that is, $\pi_*(\sum Q_i) \sim B_2 - B_1$. It is clear that $h = p_2 \circ elm_{Q_1,\ldots,Q_e}|_X$. 

Corollary 3.5. Let $Y$ be an irreducible smooth curve of genus $g_y > 1$. For every integer $g_x$ with $g_x \geq 13g_y$ and $e$ with $2g_y - 3 \geq e \geq 0$, there exist an irreducible smooth curve $X$ of genus $g_x$ and a triple covering $f : X \to Y$ such that $X$ does not admit any nontrivial morphisms $X \to \mathbb{P}^1$ of degree $2x - 3g_y + 2 + \frac{|e|}{2}$ and $e$ is the $e$-invariant of the ruled surface $\mathcal{P}(\mathcal{E})$ associated to the Tschirnhausen module $\mathcal{E}^e$ for the triple covering $f$.

Proof. Choose an effective divisor $D = Q_1 + \cdots + Q_e \in \text{Div}(Y)$ so that $Q_i \neq Q_j$ for $i \neq j$, $\deg D = e \leq 2g_y - 3$, and $H^0(Y, D) = 1$. Choose an effective divisor $B_1 \in \text{Div}(Y)$ such that

$$2 \deg B_1 + \deg D = g_x - 3g_y + 2.$$

Set $B_2 = B_1 + D$ and $\mathcal{E} = \mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2)$. Let $\pi : \mathcal{P}(\mathcal{E}) \to Y$ be the projection. We have

$$S^3\mathcal{E} \otimes \det^{-1} \mathcal{E}^e = \mathcal{O}_Y(2B_1 - B_2) \oplus \mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2) \oplus \mathcal{O}_Y(2B_2 - B_1).$$

By the assumptions $g_x \geq 13g_y$ and $\deg D = e \leq 2g_y - 3$, it follows that

$$\deg(2B_1 - B_2) \geq 2g_y.$$

Hence $S^3\mathcal{E} \otimes \det^{-1} \mathcal{E}^e$ is generated by global sections. Furthermore $H^0(Y, \mathcal{E}^e) = 0$. According to [3, Theorem 3.6], the zero locus of a general section contained in $H^0(\mathcal{P}(\mathcal{E}), \pi^*\det \mathcal{E}^{-1}(3))$ is an irreducible smooth triple cover of $Y$ with genus $g_x$.

Let $S_0$ be the unique minimal degree section of $\mathcal{P}(\mathcal{E})$ and $S$ a section of $\mathcal{P}(\mathcal{E})$ such that $S_0 \cap S = \emptyset$. We may choose a section $\delta \in H^0(\mathcal{P}(\mathcal{E}), \pi^*\det \mathcal{E}^{-1}(3))$ so that the zero locus $X$ of $\delta$ is an irreducible smooth triple cover of $Y$ and $X$ does not pass through the points in $S \cap \pi^{-1}([Q_1, \ldots, Q_e])$.

Let $T$ be a section of $\mathcal{P}(\mathcal{E})$ such that $T \cap S_0 = \emptyset$. Since $S, T \in |S_0 + \pi^* D|$ by Lemma [2, it follows by (1.5) that

$$\pi_*(S \cdot T) \in |\text{det} \mathcal{E} \otimes \mathcal{L}^{-1}_S \otimes \mathcal{L}^{-1}_T| = |D|.$$

Therefore $S \cap T = \{Q_1, \ldots, Q_e\}$ because $H^0(Y, D) = 1$.

Suppose that there exists a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $\frac{2z - 3g_y + 2 + |e|}{2}$. By Proposition [2], it follows that $Q_1, \ldots, Q_e \in X \cap S$, but which contradicts the choice of $X$. Therefore $X$ does not admit a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $\frac{2z - 3g_y + 2 + |e|}{2}$. 

\[\square\]
4. Existence of triple covers

We give a partial answer to Question [3] the existence of triple covers, which admit nontrivial morphisms of certain degrees, of a given base curve. We first investigate behaviors of trisections under elementary transformations. Let $C$ be a trisection on a ruled surface $\mathbb{P}(\mathcal{F})$. Assume that $C \sim 3S + \pi^*Z$ as a divisor in $\mathbb{P}(\mathcal{F})$, where $S$ is a (not necessarily minimal degree) section of $\mathbb{P}(\mathcal{F})$, $\pi : \mathbb{P}(\mathcal{F}) \to Y$ is the projection, and $Z \in \text{Div} Y$. Let $P \in \mathbb{P}(\mathcal{F})$. Set

$$p = \pi(P), \quad S' = \text{elm}_P(S), \quad C' = \text{elm}_P(C).$$

**Proposition 4.1.** Assume that $P \in S$. If $P \notin C$ then $C' \sim 3S' + \pi^*(Z + 3p)$, and if $P \in C$ then $C' \sim 3S' + \pi^*(Z + 2p)$. Assume that $P \notin S$. If $P \notin C$ then $C' \sim 3S' + \pi^*Z$, and if $P \in C$ then $C' \sim 3S' + \pi^*(Z - p)$.

**Proof.** We use similar techniques in [8] which deals the behavior of double covers under elementary transformations. Let admit nontrivial morphisms of certain degrees, of a given base curve. We first need the following Lemma. We give a partial answer to Question B; the existence of triple covers, which

The following theorem provides a partial answer to Question [3].

**Lemma 4.2.** Let $S$ and $T$ be sections of a ruled surface $\mathbb{P}(\mathcal{F})$ over $Y$ with $T \sim S + \pi^*Z$ for some $Z \in \text{Div}(Y)$, where $\pi : \mathbb{P}(\mathcal{F}) \to Y$ is the projection. Let $\text{elm}_P$ be the elementary transformation with center $P \in S$ and let $S', T'$ be the image $\text{elm}_P(S), \text{elm}_P(T)$, respectively. Then we have $T' \sim S' + \pi^*(Z + \pi(P))$.

**Proof.** Set $\text{elm}_P = \psi \circ \phi^{-1}$ and $P' = \psi(F)$, where $F = \pi^{-1}(\pi(P))$. Then we have

$$\psi^*T' = \widetilde{T} + \widetilde{F} = \phi^*T + \widetilde{F} \sim \phi^*S + \phi^*(\pi^*Z) + \widetilde{F} = \widetilde{S} + \phi^*(\pi^*Z) + \widetilde{F} = \psi^*S' + \psi^*(\pi^*(Z + p)) + \widetilde{F} = \psi^*(S' + \psi^*(Z + p)).$$

The following theorem provides a partial answer to Question [3].
Theorem 4.3. Let $Y$ be an irreducible smooth curve of genus $g_y \geq 1$, and let $g_x$ be an integer with $g_x \geq 37g_y - 2$. For every integer $d$ with
\[
d \geq \frac{g_x - 3g_y + 2}{2} + g_y,
\]
there exists an irreducible smooth curve $X$ of genus $g_x$ and a triple covering $f : X \to Y$ such that $X$ admits a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $d$. If $g_y \geq 5$, we get sharper lower bound for $d$:
\[
d \geq \frac{g_x - 3g_y + 2}{2} + \frac{g_y + 3}{2}.
\]

Lemma 4.4. Let $\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(D)$ be a rank $2$ vector bundle on an irreducible smooth curve $Y$ and let $E = y_1 + \cdots + y_n \in |D|$ be an effective divisor consisting of distinct points. Let $S$ be a section of $\mathcal{P}^{(\mathcal{E})}$ with $S \cap S_0 = \emptyset$, where $S_0$ is a minimal degree section. Set $P_i = S \cap \pi^{-1}(y_i)$. Then, for the pair $(S, \{P_1, \ldots, P_n\})$, there exists an one-dimensional family $V$ of sections of $\mathcal{P}^{(\mathcal{E})}$ such that, for all $T \in V$,
\[
S \in V, \quad P_i \in T \text{ for all } i, \quad T \in |S_0 + \pi^*D|.
\]

Proof. Let $\text{elm}_{P_1} = \psi_1 \circ \phi_1^{-1}$ be the elementary transformation of $\mathcal{P}^{(\mathcal{E})}$ with center $P_i$. Set $Z_i = \psi(\overline{F}_i)$ where $F_i = \pi^{-1}(P_i)$ and $\overline{F}_i$ is the strict transform of $F_i$ under the blowing-up $\varphi$. By Proposition 3.1, we have
\[
\text{elm}_{P_1, \ldots, P_n}(\mathcal{P}^{(\mathcal{E})}) = Y \times \mathbb{P}^1, \quad \text{elm}_{Z_1, \ldots, Z_n}(Y \times \mathbb{P}^1) = \mathcal{P}^{(\mathcal{E})}.
\]

Note that
\[
\text{elm}_{P_1, \ldots, P_n} S = Y \times \{a\}, \quad \text{elm}_{P_1, \ldots, P_n} S_0 = Y \times \{b\}
\]
for some $a, b \in \mathbb{P}^1$ because they are sections of $Y \times \mathbb{P}^1$. Set
\[
V = \{\text{elm}_{Z_1, \ldots, Z_n}(Y \times \{c\}) | c \in \mathbb{P}^1, c \neq b\}.
\]
It is clear that $S \in V$. Let $T = \text{elm}_{Z_1, \ldots, Z_n}(Y \times \{c\}) \in V$. Since $Z_i \notin Y \times \{c\}$ for all $i$, we have $P_i \in T$ for all $i$. By Lemma 1.2, we have $T \sim S_0 + \pi^*E$. Therefore $V$ is the desired family. \qed

Remark 4.4.1. For any $S_1, S_2 \in V$ we have $S_1 \cap S_2 = \{P_1, \ldots, P_n\}$ since $S_1 \cdot S_2 = n$.

Proof of Theorem 4.3. Set
\[
d_0 = \begin{cases} 
(g_x - 3g_y + 2)/2 & \text{if } g_x - 3g_y + 2 \equiv 0 \pmod{2}, \\
(g_x - 3g_y + 2)/2 + 1/2 & \text{if } g_x - 3g_y + 2 \equiv 1 \pmod{2}.
\end{cases}
\]
Suppose that $g_x - 3g_y + 2 \equiv 0 \pmod{2}$. Let $t$ be an integer with
\[
g_y \leq t \leq \frac{d_0 - 2g_y}{3}.
\]
Choose a divisor $D = y_1 + \cdots + y_2$, deg $D = 2t$ of $Y$ consisting of distinct points. Choose an effective divisor $D'$ of $Y$ with deg $D' = d_0 - 3t$. Set
\[
B_1 = D + D', \quad B_2 = 2D + D', \quad \mathcal{E} = \mathcal{O}_Y(B_1) \oplus \mathcal{O}_Y(B_2).
\]
Let $\pi : \mathcal{P}^{(\mathcal{E})} \to Y$ be the projection. Since $\mathcal{E}$ is decomposable, there exists a unique minimal degree section $S_0$ of $\mathcal{P}^{(\mathcal{E})}$ by [11] Corollary 1.17 and there exists a section $S \in |S_0 + \pi^*D|$ with $S \cap S_0 = \emptyset$ by Lemma 1.2. Set
\[
S \cap \pi^{-1}(y_i) = \{P_i\}.
\]
By Remark 4.4.1, we have

Let $V$ be the one dimensional family of sections given by Lemma 4.4 corresponding to $(E, \{P'_1, \ldots, P'_{2t}\})$. Let $S_1, S_2 \in V$ be sections with $S_1 \neq S_2$ and $S_i \neq S$ for $i = 1, 2$. Set

$$S_1 \cap \pi^{-1}(y_i) = \{Q_i\}, \quad S_2 \cap \pi^{-1}(y_i) = \{R_i\}.$$

By Remark 4.4.1 we have $Q_i \neq R_i$; cf. Figure 1.

Define a linear system $H$ on $\mathbb{P}(E)$ by

$$H = \{X \in |3S_0 + \pi^*(2B_2 - B_1)| : P_i, Q_i, R_i \in X \text{ for all } i\}.$$

**Step 1.** There exist irreducible smooth curves $S_Q, S_R$ such that $S_Q, S_R \in |S_0 + \pi^*B_1|$ and $Q_i \in S_Q$, $R_i \in S_R$ for all $i = 1, \ldots, 2t$.

**Proof of Step 1** Let $elm_{Q_i} = \psi_i \circ \varphi_i^{-1}$ be the elementary transformation with center $Q_i$. Set $Q'_i = \psi_i(F)$, where $F$ is the strict transform by $\varphi_i$ of the fiber $F = \pi^{-1}(Q_i)$. Set

$$T_0 = elm_{Q_1, \ldots, Q_{2t}}(S_0).$$

Since $S_1$ is a section of $\mathbb{P}(E)$ such that $S_1 \cap S_0 = \emptyset$, it follows by Proposition 4.4 that

$$elm_{Q_1, \ldots, Q_{2t}}(\mathbb{P}(E)) \cong Y \times \mathbb{P}^1;$$

hence we have $T_0$ is a minimal degree section of $Y \times \mathbb{P}^1$. Define a subset $H_Q$ of the linear series $|T_0 + \pi^*(D')|$ on $Y \times \mathbb{P}^1$ by

$$H_Q = \{S'_Q \in |T_0 + \pi^*(D')| : Q'_i \notin S'_Q \text{ for all } i\}.$$

Since $\deg D' \geq 2g_y + 1$, the linear series $|S_0 + \pi^*(D')|$ is very ample by [6, V. Ex. 2.11]. Therefore there exists an irreducible smooth curve $S'_Q \in H_Q$. Set

$$S_Q = elm_{Q_1, \ldots, Q_{2t}}(S'_Q)$$

By Lemma 4.2 we have $S_Q \in |S_0 + \pi^*B_1|$. Since $Q'_i \notin S'_Q$, it is clear that $Q_i \in S_Q$ for all $i = 1, \ldots, 2t$. Therefore we get the desired $S_Q$. Applying the same method to $R_i$, we get the desired $S_R$. \qed

**Step 2.** A general member $X$ of $H$ is an irreducible smooth triple cover of $Y$ of genus $g_x$. 

---

**Figure 1. Sections on $\mathbb{P}(E)$**

Since $\deg D \geq 2g_y$, there exists a divisor $E = y'_1 + \cdots + y'_{2t} \in |D|$ such that $y'_i \neq y'_j$ for $i \neq j$ and $\text{Supp } E \cap \text{Supp } D = \emptyset$. Set

$$S \cap \pi^{-1}(y'_i) = \{P'_i\}.$$
Proof of Step 2. Let $V_P$, $V_Q$, $V_R$ be the one dimensional families of sections corresponding to $(D, \{P_1, \ldots, P_{2t}\})$, $(D, \{Q_1, \ldots, Q_{2t}\})$, $(D, \{R_1, \ldots, R_{2t}\})$ given by Lemma 4.4 respectively. Then, for any $S \in V$, $S_1 \in V_Q$, $S_2 \in V_R$, we have

$$S' + S_1 + S_R \in H$$

and

$$S' + S_Q + S_2 \in H.$$  

By Remark 4.4.1, it follows that

$$\text{Bp}(V) = \{P_1, \ldots, P_{2t}\},$$

$$\text{Bp}(V_Q) = \{Q_1, \ldots, Q_{2t}\},$$

$$\text{Bp}(V_R) = \{R_1, \ldots, R_{2t}\}.$$ 

Therefore we have

$$\text{Bp}(H) = \{P_1, \ldots, P_{2t}, Q_1, \ldots, Q_{2t}, R_1, \ldots, R_{2t}\}.$$ 

By Bertini theorem of characteristic zero (cf. [6, III, 10.9]), general members of $H$ is smooth outside $\text{Bp}(H)$. Since $S' + S_1 + S_R$ and $S' + S_Q + S_2$ do not contain any fiber of $\pi$, they are singular triple coverings of $Y$. Note that

$$S^3\mathcal{E} \otimes \det \mathcal{E}^{-1} = O_Y(2B_2 - B_1) \oplus O_Y(B_2) \oplus O_Y(B_1) \oplus O_Y(2B_1 - B_2),$$

where

$$\deg(2B_2 - B_1), \deg(B_2), \deg(B_1), \deg(2B_1 - B_2) > 2g_y - 1$$

by the assumptions $g_x \geq 37g_y - 2$ and the choice of $d': d' \geq 2g_y$. Therefore $S^3\mathcal{E} \otimes \det \mathcal{E}^{-1}$ is globally generated; hence, by [3, Theorem 3.6], general members of $H$ are (possibly singular) triple coverings of $Y$. Since $h^0(Y, \mathcal{E}^*) = 0$, it follows by [3, Theorem 3.6] that every triple covers in $H$ are connected. Therefore we proved that general members of $H$ are connected triple covers of $Y$ which are smooth outside $\text{Bp}(H)$.

Let $X$ be a general member of $H$ which is smooth outside $\text{Bp}(H)$. Since

$$X \cap \pi^{-1}(y_i) = \{P_i, Q_i, R_i\},$$

it follows that $X$ cannot have a singular point on $P_i$, $Q_i$, and $R_i$; hence $X$ is smooth. Thus a general member of $H$ is an irreducible smooth triple cover of $Y$. Let $X \in H$ be an irreducible smooth triple covering of $Y$ with the triple covering map $f : X \to Y$. By [12, Proposition 8.1], the vector bundle $\mathcal{E}'$ is the Tschirnhausen module for $f : X \to Y$. Therefore it follows by [12, Proposition 4.7] and Riemann-Hurwitz formula that the genus of $X$ is equal to $g_x$. 

Step 3. Let $X \in H$ be a smooth triple cover of $Y$ with the triple covering $f : X \to Y$. For every integer $d$ with

$$d = d_0 + t, \quad d \geq d_0 + t + 4g_y,$$

there exists a nontrivial morphism $h : X \to \mathbb{P}^1$ of degree $d$.

Proof of Step 3 Set

$$T = P_1 + \cdots + P_{2t} \in \text{Div}(S).$$

Choose an effective divisor $T_1$ on $S$ satisfying the followings: $T_1 \leq S \cap X$, and if $aP \leq T_1$ and $bP \leq T$ for some $P \in S$ and $a, b \geq 0$, then $(a + b)P \leq S \cap \pi^*E$. Choose an effective divisor $T_2$ on $S$ satisfying

$$\text{Supp} T_2 \subset S - S \cap X$$

and

$$\deg T_1 + \deg T_2 \geq 2g_y.$$
Choose an effective divisor $T_3$ on $S_0$ satisfying
\[ \pi_* T_3 \sim \pi_*(T_1 + T_2), \quad \text{Supp} \ T_3 \cap \text{Supp}(T_1 + T_2) = \emptyset, \]
which is possible because $\deg T_1 + \deg T_2 \geq 2g_y$. Set $t_i = \deg T_i (i = 1, 2)$.

First of all, consider $elm_T$. By Proposition 3.1, $elm_T(\mathcal{P}(\mathcal{E})) \cong Y \times \mathbb{P}^1$. Set $X' = elm_T X$. By Proposition 4.1
\[ X' \sim 3Y_0 + \pi^*(2B_2 - B_1 - D) \equiv 3Y_0 + (d_0 + t)F. \]
Set $h = p_2 \circ elm_T | X$. Then $h$ is a nontrivial morphism of degree $d_0 + t$.

Consider now $elm_{T_3,T_1+T_2,T}$. By Proposition 5.1 it follows that
\[ elm_{T_3,T_1+T_2,T}(\mathcal{P}(\mathcal{E})) \cong Y \times \mathbb{P}^1. \]
Set $X' = elm_{T_3,T_1+T_2,T}(X)$. By Proposition 4.1
\[ X' \sim 3Y_0 + \pi^*(2B_2 - B_1 - T - T_1 + 3T_3) \equiv 3Y_0 + (d_0 + t + 2t_1 + 3t_2)F. \]
Set $h = p_2 \circ elm_T | X$. Then $h$ does not factor through the triple covering $f$ and we have
\[ \deg h = d_0 + t + 2t_1 + 3t_2. \]
Note that, by the assumption $g_x \geq 37g_y - 2$, we have
\[ 2g_y \leq t_1 \leq 2b_2 - b_1. \]
We may choose $T_2$ such that $\deg T_2$ is arbitrary large. Therefore any integer greater than or equal to $4g_y$ can be represented by $2t_1 + 3t_2$ for some $t_1$ and $t_2$.

Note that we have $\frac{d_0 - 2g_y}{3} \geq 5g_y$ by the assumption $g_x \geq 37g_y - 2$. Therefore it follows that
\[ \{d : d \geq d_0 + g_y \} = \left\{ d_0 + t : g_y \leq t \leq \frac{d_0 - 2g_y}{3} \right\} \cup \{d : d \geq d_0 + 5g_y \}. \]
Suppose that $g_x - 3g_y + 2 \equiv 1 \pmod{2}$. Let $t$ be an integer with $g_y + 1 \leq t \leq \frac{d_0 - 2g_y}{3}$ and fix a divisor $D = y_1 + \cdots + y_{2t-1}$ of $Y$ with $\deg D = 2t - 1$ which consists of distinct points and then repeat the above proof. \hfill $\square$

\textit{Continue the proof of Theorem 4.3} Hence far, we proved the existence of triple coverings and the existence of nontrivial morphisms for $d \geq \frac{g_x - 3g_y + 2}{2} + g_y$. It remains to prove the existence for $d \geq \frac{g_x - 3g_y^2 + 2}{2} + \frac{g_y + 3}{2}$ if $g_y \geq 5$. Suppose that $g_y \geq 2$. A classical theorem of Halphen says that a curve $Y$ of genus $g_y \geq 2$ has a nonspecial very ample divisor $D$ with $\deg D \geq g_y + 3$. Therefore, in the above proof, we can take $t$ with $t \geq \frac{g_y + 3}{2}$ and take this divisor $D$ instead of arbitrary divisor of degree $2t$. Then we get better lower bound $d \geq d_0 + \frac{g_y + 3}{2}$. \hfill $\square$

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References

[1] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, Geometry of algebraic curves. Vol. I, Springer-Verlag, 1985. MR 86h:14019
[2] E. Ballico, C. Keem, and S. Park, Double covering of curves, Proc. Amer. Math. Soc. 132 (2004), no. 11, 3153–3158 (electronic). MR 2005a:14037
[3] G. Casnati and T. Ekedahl, Covers of algebraic varieties. I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces, J. Algebraic Geom. 5 (1996), no. 3, 439–460. MR 97c:14014
[4] M. Coppens, C. Keem, and G. Martens, Primitive linear series on curves, Manuscripta Math. 77 (1992), no. 2-3, 237–264. MR 93j:14028
[5] L. Fuente and M. Pederia, The projective theory of ruled surfaces, Note Mat. 24 (2005), no. 1, 25–63. MR 2199622
[6] R. Hartshorne, Algebraic geometry, Springer-Verlag, 1977, Graduate Texts in Mathematics, No. 52. MR 57 #3116
[7] T. Kato, C. Keem, and A. Ohbuchi, On triple coverings of irrational curves, Tsukuba J. Math. 21 (1997), no. 2, 421–441. MR 98m:14031
[8] C. Keem and A. Ohbuchi, On the Castelnuovo-Severi inequality for a double covering, preprint, 2004.
[9] A. Maroni, Le serie lineari speciali sulle curve trigonali, Ann. Mat. Pura Appl. (4) 25 (1946), 343–354. MR 0024182 (9,463j)
[10] G. Martens and F.-O. Schreyer, Line bundles and syzygies of trigonal curves, Abh. Math. Sem. Univ. Hamburg 56 (1986), 169–189. MR 88d:14019
[11] M. Maruyama, On classification of ruled surfaces, Lectures in Mathematics, Department of Mathematics, Kyoto University, vol. 3, Kinokuniya Book-Store Co. Ltd., Tokyo, 1970. MR 43 #1990
[12] R. Miranda, Triple covers in algebraic geometry, Amer. J. Math. 107 (1985), no. 5, 1123–1158. MR 86k:14008
[13] W. Seiler, Deformations of ruled surfaces, J. Reine Angew. Math. 426 (1992), 203–219. MR 93b:14063

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