A generalised ansatz for continuous Matrix Product States

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Recently it was shown that continuous Matrix Product States (cMPS) cannot express the continuum limit state of any Matrix Product State (MPS), according to a certain natural definition of the latter. The missing element is a projector in the transfer matrix of the MPS. Here we provide a generalised ansatz of cMPS that is capable of expressing the continuum limit of any MPS. It consists of a sum of cMPS with different boundary conditions, each attached to an ancilla state. This new ansatz can be interpreted as the concatenation of a state which is at the closure of the set of cMPS together with a standard cMPS. The former can be seen as a cMPS in the thermodynamic limit, or with matrices of unbounded norm. We provide several examples and discuss the result.

1. INTRODUCTION

Tensor networks are a powerful ansatz to describe and simulate quantum many-body systems in an efficient way [1, 2], which originated in the context of condensed matter physics, but has nowadays percolated to other areas—for a recent review see [2]. The class of Matrix Product States (MPS) has been extremely successful in capturing ground states of gapped Hamiltonians in one spatial dimension [3–6], and many generalisations to different settings have been considered: to higher dimensional lattices [7], to describe mixed states [8, 9], to describe fermionic systems [10], in the context of conformal field theories [11], or to describe ground states of critical systems [12], to cite a few. The generalisation that is relevant for this work is the one to continuous systems, that is, where the lattice is replaced by a segment of the reals \( R = [x, y] \), but the bond dimension remains finite. Specifically, continuous MPS (cMPS) were proposed in 2010 as a tensor network ansatz to describe quantum field states in one spatial dimension [13], and they were shown to be a natural continuous version of MPS. In this work, “tensor network ansatz” is synonymous with “matrix product ansatz”, which means that the coefficients of the state can be expressed as a product of matrices in some basis. Further studies about cMPS can be found in Refs. [14–18].

Recently this topic was studied from the opposite perspective [19]. Namely, one asked the question: given a translationally invariant MPS, does it have a continuum limit? And, if it does, what does the state at the continuum look like? These questions obviously depend on how one defines a continuum limit—a very subtle question without an unambiguous answer. One could argue, though, that the choice made in Ref. [19] is fairly natural: it was defined as the limit of repeatedly applying the inverse of the renormalisation operation considered in Ref. [20], together with a regularisation condition in the limit. (Henceforth, when we refer to continuum limit of an MPS we mean the one defined in [19].) Ref. [19] fully characterised the set of MPS with a continuum limit, and found that the state at the continuum can generally not be expressed as a cMPS. In other words, the class of cMPS was too narrow to express all continuum limits of MPS.

To be more explicit, Ref. [19] showed that a translationally invariant MPS has a continuum limit if and only if its transfer matrix \( E \) is an infinitely divisible quantum channel. The latter are channels of the form \( E = Pe^L \), where \( P^2 = P \) is a projector quantum channel, \( L \) is a Liouvillian of Lindblad form, and \( PL = PLP \). Since the transfer matrix of a (homogeneous) cMPS is given by \( e^L \), this class can only express the continuum limits of MPS with a trivial projector, i.e. \( P = I \), the identity matrix. The new element that needs to be represented, thus, is the projector \( P \). Note that a non-trivial \( P \) implies that the transfer matrix has eigenvalues 0 (in particular, the presence of \( P \) is unrelated to the degeneracy of the eigenvalue 1, which is related to the (lack of) injectivity of an MPS [6, 21]). A simple example of an MPS with a continuum limit is the superposition of ferromagnetic states \( |0 \ldots 0\rangle + |1 \ldots 1\rangle \); its transfer matrix is \( E = P = |00\rangle\langle 00| + |11\rangle\langle 11| \), and can thus not be expressed as a cMPS. This gives rise to the question: Is there a matrix product ansatz that can express the continuum limit of any MPS?

In this work, we present a generalised ansatz of cMPS that is capable of expressing the continuum limit of any MPS (Theorem 8). Our ansatz consists of a sum of cMPS, each with a different boundary operator, and attached to an ancilla state. The boundary operators are given by the Kraus operators of \( P \). We show that this new ansatz can be understood as the concatenation of a state at the closure of the set of cMPS (which will contribute \( P \) to the transfer matrix; Proposition 10), and a standard cMPS on a finite segment (which will contribute \( e^{PL} \) to the transfer matrix). The former can be thought of as a cMPS with matrices of unbounded norm, or a cMPS on a segment of unbounded length, that is, in the thermodynamic limit. Informally, this can be understood as the enforcement of some “superselection rules” which correspond to the zeros in the transfer matrix brought in by \( P \). Our ansatz thus provides a way to directly enforce them without the need to resort to cMPS with matrices of very large norm, or to very large system sizes.

This paper is organized as follows. In Section 2, Section 3 and Section 4 we present the preliminary material for this work: the preliminaries on quantum channels, on Matrix Product States, and on continuous Matrix Product States, respectively. The core of this work is presented in Section 5, where we present the ansatz of generalised cMPS, we discuss it, provide an interpretation and some examples. Finally, we conclude and present an outlook in Section 6. Appendix A contains the proof of Proposition 10, and Appendix B a characterisation of the condition \( PL = PLP \) for a special case.
2. PRELIMINARIES ON QUANTUM CHANNELS

In this section we present the background on quantum channels. We will first present the basic definitions (Section 2.1), then characterise Markovian channels (Section 2.2), projector quantum channels (Section 2.3), and finally infinitely divisible channels (Section 2.4).

2.1. Basic definitions

Throughout this paper we let \( M_D \) denote the set of \( D \times D \) complex matrices. \( A \geq 0 \) denotes that \( A \) is positive semidefinite, i.e., Hermitian and with non-negative eigenvalues, and \( A > 0 \) that it is positive definite, i.e., Hermitian with positive eigenvalues.

A quantum channel \( \mathcal{E} : M_D \to M_D \) is a linear, completely positive, trace-preserving map. It has a Kraus decomposition \( \mathcal{E}(X) = \sum_{i=1}^d A_i^X A_i^\dagger \), where \( ^\dagger \) denotes complex conjugate. The minimal number of Kraus operators \([A_i]\) needed in this decomposition is called the Kraus rank of \( \mathcal{E} \).

The superoperator \( \mathcal{E} \) can be represented as a matrix \( E \) by expressing \( M_D \) as \( \mathbb{C}^{D^2} \). Specifically, a matrix \( X = \sum_{i,j} X_{i,j} |i\rangle \langle j| \) is expressed as \( |X\rangle = \sum_{i,j} X_{i,j} |i\rangle \langle j| \) where \( ^\dagger \) denotes complex conjugate transpose. The minimal number of Kraus operators \([A_i]\) needed in this decomposition is called the Kraus rank of \( \mathcal{E} \).

Finally, we will denote the identity matrix of size \( n \times n \) by \( I_n \) (or simply \( I \), when clear from the context), and the Pauli matrices by \( \sigma_x = |0\rangle \langle 1| + |1\rangle \langle 0| \) and \( \sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1| \).

2.2. Markovian channels

A quantum channel \( \mathcal{E} \) is called Markovian (sometimes also called a quantum dynamical semigroup) if \( \mathcal{E} = e^{L t} \) where \( L \) is a Liouvillian of Lindblad form [22]. There are several equivalent representations of \( L \) [23], and here we present one of them.

**Definition 1 (Liouvillian)** A Liouvillian of Lindblad form \( L : M_D \to M_D \) is a superoperator of the form

\[
L[Q, \{ R_n \}] = Q\rho + \rho Q^\dagger + \sum_{\alpha=1}^q R_{\alpha} \rho R_{\alpha}^\dagger,
\]

where

\[
Q = -iH - \frac{1}{2} \sum_{\alpha=1}^q R_{\alpha}^\dagger R_{\alpha},
\]

and \( H \) is a Hermitian operator.

In this context, \( H \) is called the Hamiltonian, and \( R_n \) the jump operators of \( L \).

2.3. Projector quantum channels

We say that \( \mathcal{P} : M_D \to M_D \) is a projector quantum channel if it is a quantum channel and it fulfills \( \mathcal{P}(\rho) = \mathcal{P}(\rho) \) for all \( \rho \in M_D \). Examples are the identity channel, \( \mathcal{E}(\rho) = \rho \), the pinching map \( \mathcal{E}(\rho) = \sum_{i=1}^d |i\rangle \langle i| \rho |i\rangle \langle i| \), where \( |i\rangle \) is some orthonormal basis, and the completely depolarising map \( \mathcal{E}(\rho) = tr(\sigma_2) \rho \) where \( \sigma_2 > 0 \) and \( tr(\sigma_2) = 1 \). In fact, these three are the building blocks of any projector quantum channel, as we will see in Proposition 3. To this end, let us first review a characterisation of the fixed point set of a quantum channel.

**Theorem 2 (Theorem 6.14 of [23])** Let \( \mathcal{E} : M_D \to M_D \) be a completely positive, trace-preserving, positive, linear map. Then there is a unitary \( U \in M_D \) and a set of positive definite density matrices \( \sigma_k \in M_m \) such that the fixed point set of \( \mathcal{E} \), \( \mathcal{F}_E := \{ X \in M_D \mid \mathcal{E}(X) = X \} \), is given by

\[
\mathcal{F}_E = U(0 \oplus \bigoplus_{k=1}^n M_{D_k} \otimes \sigma_k) U^\dagger,
\]

for an appropriate basis of the Hilbert space \( \mathbb{C}_D = \mathbb{C}_{D_0} \oplus \bigoplus_{k=1}^n \mathbb{C}_{D_k} \otimes \mathbb{C}_{m_k} \).

Since the fixed point set of a projector quantum channel \( \mathcal{P} \) equals its image, we now use Theorem 2 to characterise the action of \( \mathcal{P} \) on a general element \( \rho \in M_D \). So, define orthogonal projectors \( \pi_k \) (of dimension \( M_{D_0} \) for \( k = 0 \), and \( M_{D_k} \otimes M_{m_k} \) for \( k > 0 \)) so that for any

\[
\rho = \sum_{k,m=0}^n \pi_k \rho \pi_m \in M_D
\]

we have that \( \mathcal{P}(\rho) = \sum_{k=1}^n \pi_k \rho \pi_k \). Denote \( \rho_k = \pi_k \rho \pi_k \), and consider its operator Schmidt decomposition (see e.g. [24]),

\[
\rho_k = \sum_{l=1}^{r_k} \rho_{k,l}^{(1)} \otimes \rho_{k,l}^{(2)},
\]

where \( \rho_{k,l}^{(1)} \in M_{D_k}, \rho_{k,l}^{(2)} \in M_{m_k}, \) and \( r_k \) is the operator Schmidt rank of \( \rho_k \) [30]. Then the action of \( \mathcal{P} \) on this \( \rho \) is given by

\[
\mathcal{P}(\rho) = \sum_{k,l=1}^n \pi_k \left[ \sum_{l=1}^{r_k} \pi_k \left[ \sum_{l=1}^{r_k} \right] \right] \pi_k
\]

In summary, we have shown the following.

**Proposition 3 (Projector quantum channel)** Let \( \mathcal{P} : M_D \to M_D \) be a projector quantum channel. Then there is a decomposition of \( M_D = M_{D_0} \oplus \bigoplus_{k=1}^n M_{D_k} \otimes M_{m_k} \) such that, for the \( \rho \in M_D \) defined in (3) and (4), we have that

\[
\mathcal{P}(\rho) = \sum_{k,l=1}^n \pi_k \left[ \sum_{l=1}^{r_k} \pi_k \left[ \sum_{l=1}^{r_k} \right] \right] \pi_k
\]

where \( \sigma_k > 0 \) and \( tr(\sigma_k) = 1 \).

Note that the identity map corresponds to \( n = 1, D_1 = D \) and \( m_1 = 1 \); the pinching map corresponds to \( n = D \) with \( D_k = m_k = 1 \) for all \( k \); and the completely depolarising map corresponds to \( n = 1, D_1 = 1 \) and \( m_1 = D \).
2.4. Infinitely divisible quantum channels

Having defined Markovian channels and projector quantum channels, we are now ready to define infinitely divisible channels. A quantum channel \( E \) is called infinitely divisible [25] (see also Refs. [26, 27]) if for every natural \( n \) there exists a quantum channel \( E_n \) such that \( E = (E_n)^n \), where the power \( (E_n)^n \) stands for \( n \)-fold composition, \( E_n \circ \cdots \circ E_n \) (\( n \) times). We have the following characterisation.

**Theorem 4 (Infinitely divisible channels [19, 26, 27])** Let \( E \) be a quantum channel. The following are equivalent:

1. \( E \) is infinitely divisible, i.e. \( E = (E_n)^n \) for all \( n \in \mathbb{N} \).
2. There is a \( p \geq 1 \) such that \( E = (E_p^p)^\ell \) for all \( \ell \in \mathbb{N} \), and for all \( (n_k/p^k) \to 0 \) it holds that \( (E_n)^{p^k} \to P \), where \( P \) is a projector quantum channel.
3. \( E = P e^L \) where \( P \) is a projector quantum channel and \( L \) is a Liouvillian of Lindblad form such that \( PL = P LP \).

Therefore, Markovian quantum channels are infinitely divisible channels with \( P = \text{id} \), the identity channel.

3. PRELIMINARIES ON MATRIX PRODUCT STATES

In this section we present the background about Matrix Product States (MPS) and their continuum limits. We will first present the basic definitions of MPS (Section 3.1) and then review the results on their continuum limits (Section 3.2).

3.1. Matrix Product States

In this work we consider exclusively translationally invariant MPS. Namely, we are given a tensor \( A = |\alpha_i\beta_j\rangle \), where \( A_{\alpha_i\beta_j} \) is a complex number. The index \( i = 1, \ldots, d \) (or \( i = 0, 1, \ldots, d-1 \), depending on the example) is called the physical index and \( d \) the physical dimension. The indices \( \alpha, \beta = 1, \ldots, D \) (or \( \alpha, \beta = 0, 1, \ldots, D-1 \), depending on the example) are called virtual indices and \( D \) the bond dimension. This tensor defines a translationally invariant MPS [31]

\[
|V_N(A)\rangle = \sum_{a_1, \ldots, a_N=1}^d \text{tr}(A_{i_1}^1 A_{i_2}^2 \cdots A_{i_N}^N) |i_1, i_2, \ldots, i_N\rangle,
\]

and a family of MPS

\[ V(A) = \{|V_n(A)\rangle\}_N. \]

It should be understood that \( |V_n(A)\rangle \) describes the state of, for example, a spin chain with \( N \) spins, each of which has dimension \( d \). Among the spins there is a fixed lattice spacing \( a \), which is arbitrary but fixed. That is, the length on which \( |V_n(A)\rangle \) is defined is given by \( l_N := Na \). Note that what is common in the family \( V(A) \) is the lattice spacing \( a \).

The transfer matrix associated to \( V(A) \) is given by \( E_a = \sum_{\alpha=1}^d A_{\alpha}^\dagger A_{\alpha} \). The subindex \( \alpha \) emphasises the length over which it is defined. \( E_a \) is the matrix representation of the completely positive map \( E_a(\rho) = \sum_{\alpha=1}^d \rho A_{\alpha}^\dagger A_{\alpha} \), and one can assume without loss of generality that this is trace preserving [19, 28]. Hence we will refer to \( E_a \) as a quantum channel.

3.2. The continuum limit of an MPS

We review here some definitions and results of Ref. [19]. First, a family of MPS \( V(A) \) can be \( p \)-refined if there a tensor \( B \) and an isometry \( W : \mathbb{C}^d \to (\mathbb{C}^d)^{\otimes p} \) such that

\[ W^{\otimes N}|V_N(A)\rangle = |V_{pN}(B)\rangle \quad \forall N. \]

Thus, one \( p \)-refinement step divides the lattice spacing by \( p \), \( a \to a/p \), and multiplies the number of particles by \( p \), namely \( N \to Np \), so that \( l_N = aN \) remains unchanged. This happens for all \( N \) in parallel.

As argued in Ref. [19], it is not satisfactory to define the continuum limit as the infinite iteration of the \( p \)-refining procedure, but one additionally needs to impose that the limit is stable under the blocking of a few spins.

**Definition 5 (Continuum limit of an MPS [19])** A family of MPS \( V(A) \) has a continuum limit if there is a \( p \geq 1 \) such that \( V(A) \) can be \( p \)-refined infinitely many times, and blocking \( n_k \) after \( k \) \( p \)-refining steps yields the same result, as long as \( (n_k/p^k) \to 0 \).

The set of MPS with a continuum limit is fully characterised as follows.

**Theorem 6 (Continuum limit of an MPS [19])** \( V(A) \) has a continuum limit if and only if its transfer matrix \( E_a \) is infinitely divisible.

Using the characterisation of Theorem 4, and choosing the appropriate normalisation of \( L \), we thus have that \( V(A) \) has a continuum limit if and only if \( E_a = P e^L \), where \( P^2 = P \), and \( PL = P LP \).

4. PRELIMINARIES ON CONTINUOUS MATRIX PRODUCT STATES

In this section we present the background on cMPS, mainly by following Ref. [14]. We will first present the mathematical setting (Section 4.1) and then the definition of continuous MPS (Section 4.2).

4.1. Mathematical setting

We first define a segment \( \mathcal{R} \) as an interval of the reals \( \mathcal{R} = [x, y] \), where \( x < y \) and \( x, y \in \mathbb{R} \). We will consider a quantum system defined on \( \mathcal{R} \), which accommodates \( q \) bosonic or fermionic particle species, which are labeled by the index \( \alpha = 1, \ldots, q \). The \( N \)-fold cartesian product of \( \mathcal{R} \) is denoted \( \mathcal{R}^N \). The symmetric (antisymmetric) subspace of \( \mathcal{R}^N \) is denoted \( \mathcal{R}^N_\text{s} \) (\( \mathcal{R}^N_\text{as} \)). If particle of type \( \alpha \) is bosonic (fermionic), then a state
of \( N_\alpha \) particles of type \( \alpha \) is described by a square integrable function on \( \mathcal{R}^{(N_\alpha)}_\eta \), where \( \eta_\alpha = +1(-1) \), denoted \( L^2(\mathcal{R}^{(N_\alpha)}_\eta) \). Thus a state with \( N_\alpha \) particles of type \( \alpha \), for \( \alpha = 1, \ldots, q \), is an element of

\[
\mathbb{H}_\mathcal{R}^{(N_1, \ldots, N_q)} = L^2(\prod_{\alpha=1}^{q} \mathcal{R}^{(N_\alpha)}_\eta).
\]

An arbitrary state of the system is an element of the Fock space

\[
\mathbb{H}_\mathcal{R} = \bigoplus_{N_1=0}^\infty \cdots \bigoplus_{N_q=0}^\infty \mathbb{H}_\mathcal{R}^{(N_1, \ldots, N_q)}.
\]

We refer to this space as the physical space. \( \{ \Omega_\mathcal{R} \} \) denotes the vacuum state, i.e. \( \{ \Omega_\mathcal{R} \} \in \mathbb{H}_\mathcal{R}^{(N_\alpha=0)} \).

Now, a particle of type \( \alpha \) is created or annihilated at position \( x \in \mathcal{R} \) with the operators \( \hat{\phi}_\alpha^c(x) \) and \( \hat{\phi}_\alpha^a(x) \), respectively. These satisfy the commutation or anticommutation relations

\[
\begin{align*}
\hat{\phi}_\alpha^c(x)\hat{\phi}_\beta^a(y) &- \eta_{\alpha,\beta}\hat{\phi}_\beta^a(y)\hat{\phi}_\alpha^c(x) = 0, \\
\hat{\phi}_\alpha^a(x)\hat{\phi}_\beta^c(y) &- \eta_{\alpha,\beta}\hat{\phi}_\beta^c(y)\hat{\phi}_\alpha^a(x) = \delta_{\alpha,\beta}\theta(x-y),
\end{align*}
\]

where \( \eta_{\alpha,\beta} = -1 \) if both \( \alpha \) and \( \beta \) represent fermionic particles, and \( \eta_{\alpha,\beta} = 1 \) when at least one of the two particles is bosonic. Clearly, \( \eta_{\alpha,\alpha} = \eta_\alpha \).

The auxiliary space is \( \mathbb{C}^D \), where \( D \) is the bond dimension. The variational parameters of the cMPS (Definition 7) will correspond to the functions \( \phi, R_a : \mathcal{R} \rightarrow \mathcal{B}(\mathcal{M}_D) \), that take value in \( \mathcal{B}(\mathcal{M}_D) \), the space of bounded linear operators acting on the auxiliary space. In addition, the boundary operator \( B \in \mathcal{B}(\mathcal{M}_D) \) will encode the boundary conditions.

### 4.2. Definition of continuous MPS

In this work we focus exclusively on homogeneous cMPS [14], and we will refer to them simply as cMPS. In this case, the matrices \( \phi \) and \( \{ R_a \} \) do not depend on the position \( x \).

**Definition 7 (cMPS)** A cMPS on a segment \( \mathcal{R} \) is defined as

\[
\begin{align*}
| \phi_\mathcal{R}[B, Q, \{ R_a \}] \rangle &= \text{tr}_{\text{aux}} \{ B \mathcal{T} \exp \left[ \int_{\mathcal{R}} dx \left( Q \otimes I + \sum_{\alpha=1}^{q} R_\alpha \otimes \hat{\psi}_\alpha^c(x) \right) \right] \}| \Omega_\mathcal{R} \rangle,
\end{align*}
\]

where \( B, Q, R_a \in \mathcal{B}(\mathcal{M}_D) \) and \( \mathcal{T} \exp \) denotes the path ordered exponential.

Note that \( | \phi_\mathcal{R}[B, Q, \{ R_a \}] \rangle \in \mathbb{H}_\mathcal{R} \), which is the physical space. We also define the corresponding operator which lives in the auxiliary and physical space \( \phi_\mathcal{R}[B, Q, \{ R_a \}] \in \mathcal{B}(\mathcal{M}_D) \otimes \mathbb{H}_\mathcal{R} \) as

\[
\phi_\mathcal{R}[B, Q, \{ R_a \}] = (B \otimes I) \mathcal{T} \exp \left[ \int_{\mathcal{R}} dx \left( Q \otimes I + \sum_{\alpha=1}^{q} R_\alpha \otimes \hat{\psi}_\alpha^c(x) \right) \right]| \Omega_\mathcal{R} \rangle,
\]

so that

\[
| \phi_\mathcal{R}[B, Q, \{ R_a \}] \rangle = \text{tr}_{\text{aux}} (\phi_\mathcal{R}[B, Q, \{ R_a \}]).
\]

We will say that \( \phi_\mathcal{R}[B, Q, \{ R_a \}] \) has “open auxiliary indices.” We also define an inner product for these objects

\[
(\cdot, \cdot) : (\mathcal{M}_D \otimes \mathbb{H}_\mathcal{R}, \mathcal{M}_D \otimes \mathbb{H}_\mathcal{R}) \rightarrow \mathcal{M}_D \otimes \mathcal{M}_D,
\]

as

\[
(\phi_\mathcal{R}[B', Q', \{ R'_a \}], \phi_\mathcal{R}[B, Q, \{ R_a \}]) =

(B' \otimes B) \exp \left[ \mathcal{R} \left( Q' \otimes I + I \otimes Q + \sum_{a=1}^{q} R'_a \otimes R_a \right) \right].
\]

With this, and by analogy with the discrete case, we can define the transfer matrix of the state \( | \phi_\mathcal{R} \rangle \) for a length \( | \mathcal{R} | = a \) as

\[
E_a = (\phi_\mathcal{R}[B, Q, \{ R_a \}], \phi_\mathcal{R}[B, Q, \{ R_a \}]) =

(B \otimes B) \exp [\mathcal{R} Q \{ R_a \}],
\]

where \( L \{ Q, \{ R_a \} \} \) is the matrix version of the Liouvillian of Lindblad form of Eq. (1). Note that the transfer matrix of a cMPS (with boundary conditions \( B = I \)) is a Markovian quantum channel, where the jump operators of the Liouvillian are precisely \( \{ R_a \} \) and the Hamiltonian is determined by \( Q \) via (2).

### 5. REPRESENTING THE CONTINUUM LIMIT OF AN MPS

In this section we present the core of this work, namely a generalised ansatz of cMPS. First we will state the problem precisely (Section 5.1), then we will present a generalised ansatz of cMPS addressing this problem (Section 5.2), we will discuss it (Section 5.3), provide an interpretation thereof (Section 5.4) and finally give some examples (Section 5.5).

#### 5.1. Statement of the problem

We saw in Theorem 6 that a family of MPS \( \mathcal{V}(A) \) has a continuum limit if and only if its transfer matrix is infinitely divisible, i.e. of the form \( E_a = P \exp PL \), where \( P^2 = P \) and \( PL = PLP \).

On the other hand, we saw in Eq. (8) that the transfer matrix of a cMPS is Markovian. Thus, cMPS can only represent the continuum limit of MPS whose transfer matrix is Markovian. An example of this lack of generality is the the equal superposition of two ferromagnetic states \( |0, 0, \ldots, 0 \rangle + |1, 1, \ldots, 1 \rangle \), whose transfer matrix is \( E_a = P = |0, 0\rangle (0, 0\rangle + |1, 1\rangle |1, 1\rangle \), and thus its continuum limit that cannot be expressed as a cMPS. A generalised cMPS (to be presented next) is a matrix product ansatz, defined directly in the continuum, and capable of expressing the continuum limit of any MPS.

#### 5.2. A generalised ansatz of cMPS

We now present the central result of this work.
Theorem 8 (Main result) Let $\mathcal{V}(A) = \{\mathcal{V}_N(A)\}_{N}$ be a family of MPS which has a continuum limit according to Definition 5, and let its transfer matrix be given by $E_a = P e^{\alpha L}[Q,\{R_a\}]$, where $P^2 = P$ is a projector quantum channel, $L[Q,\{R_a\}]$ a Liouvillian of Lindblad form, and $PL = PLP$. Consider a Kraus decomposition of $P$

$$P = \sum_{i=1}^{K} B_i \otimes B_i,$$

where $K$ is the Kraus rank of $P$. Then, for any $N$, the continuum limit state of $\{\mathcal{V}_N(A)\}$ can be represented by the generalised cMPS

$$|\Phi_{\mathcal{K}}(\{v_i\}, \{B_i\}, Q, \{R_a\})\rangle = \sum_{i=1}^{K} |v_i\rangle \otimes |\phi_{\mathcal{K}}(B_i, Q, \{R_a\})\rangle,$$

where $\{|v_i\rangle\}_{i=1}^{K}$ is an orthonormal basis of $\mathbb{C}^K$, and $|\phi_{\mathcal{K}}(B_i, Q, \{R_a\})\rangle \in \mathbb{H}_R$ is a cMPS (Definition 7).

Note that $|\Phi_{\mathcal{K}}(\{v_i\}, \{B_i\}, Q, \{R_a\})\rangle \in \mathbb{C}^K \otimes \mathbb{H}_R$. In words, the generalised cMPS is a sum of cMPS, all with the same $Q$ and $\{R_a\}$, but with boundary matrices $B_i$ given by the Kraus operators of $P$. In addition, each such cMPS is attached to a state $|v_i\rangle \in \mathbb{C}^\otimes$, the ancilla space. (Note that $\mathbb{C}^\otimes$ is called the auxiliary space, and this is already present in a cMPS, cf. Section 4.2). This ancilla space is the novelty in comparison with cMPS. Indeed, for $P = I$ (the identity channel), we have $K = 1$ and thus recover the case of a cMPS with periodic boundary conditions ($B = I$). Note also that $|\Phi_{\mathcal{K}}\rangle$ is a matrix product, as can be seen by expanding the time ordered exponential as a sum of states in $\mathbb{H}_R$, each of which has a finite number of excitations with respect to the vacuum.

To keep the notation short, sometimes we do not express the dependencies of the state and write $|\Phi_{\mathcal{K}}\rangle$ or $\Phi_{\mathcal{K}}$ (see following lines).

Proof: To compute the transfer matrix over a segment $\mathcal{R}$ of length $|\mathcal{R}| = a$, we define $\Phi_{\mathcal{K}}$ with open auxiliary indices as

$$\Phi_{\mathcal{K}} = \sum_{i=1}^{K} |v_i\rangle \otimes \phi_{\mathcal{K}}(B_i, Q, \{R_a\}),$$

where $\phi_{\mathcal{K}}(B_i, Q, \{R_a\}) \in \mathcal{B}(\mathcal{M}_D) \otimes \mathbb{H}_R$ is a cMPS with open auxiliary indices [Eq. (6)]. Note that $\Phi_{\mathcal{K}} \in \mathbb{C}^K \otimes \mathcal{M}_D \otimes \mathbb{H}_R$. Define the scalar product $\langle \cdot, \cdot \rangle$ as

$$(\mathbb{C}^K \otimes \mathcal{M}_D \otimes \mathbb{H}_R) \times (\mathbb{C}^K \otimes \mathcal{M}_D \otimes \mathbb{H}_R) \rightarrow \mathcal{M}_D \otimes \mathcal{M}_D$$

as the usual scalar product in quantum mechanics $\mathbb{C}^K \times \mathbb{C}^K \rightarrow \mathbb{C}$ times the scalar product $(\mathcal{M}_D \otimes \mathbb{H}_R) \times (\mathcal{M}_D \otimes \mathbb{H}_R) \rightarrow \mathcal{M}_D \otimes \mathcal{M}_D$ defined in (7), we obtain

$$E_a = (\Phi_{\mathcal{K}}, \Phi_{\mathcal{K}})$$

$$= \sum_{i,j=1}^{K} \langle v_i | v_j \rangle \langle B_j \otimes B_i | \phi_{\mathcal{K}}(B_j, Q, \{R_a\}), \phi_{\mathcal{K}}(B_i, Q, \{R_a\}) \rangle$$

$$= \left( \sum_{i=1}^{K} B_i \otimes B_i \right) e^{\alpha L}[Q,\{R_a\}],$$

where we have used that $|v_i\rangle$ is an orthonormal basis. Recalling (9) we obtain that the transfer matrix $E_a = P e^{\alpha L}[Q,\{R_a\}]$. $\square$

5.3. Discussion

We now discuss the ansatz of generalised cMPS presented above. One first central observation is that, in our setting, the transfer matrix is more important than the state at the continuum itself. For this reason, it is natural that given a family of MPS $\mathcal{V}(A)$ with transfer matrix $E_a = P e^{\alpha L}$, there are many states in the continuum $|\Phi_{\mathcal{K}}\rangle$ whose transfer matrix is $(\Phi_{\mathcal{K}}, \Phi_{\mathcal{K}}) = E_{|\mathcal{R}|}$. In the following we characterise this freedom.

First, concerning the choice of $P$ and $L$ we have that:

(a) Given $E_a$, the projector $P$ is fixed, because it determines the kernel of $E_a$, as $e^{\alpha L}$ has no kernel.

(b) If there is Liouvillian of Lindblad form $L$ such that $E_a = P e^{\alpha L}$ and $PL = PLP$, then any other Liouvillian $L'$ such that $PL = PL'$ will satisfy the same conditions.

Second, once $P$ and $L$ are fixed, there is the following freedom:

(a) Given $P$, there is the freedom of the choice of Kraus operators of $P$. Since we only admit decompositions with the minimal number of terms $K$, all such Kraus operators are related by a unitary in the physical index, i.e. $B_j = \sum_{i=1}^{K} U_{i,j} B_i$.

(b) Given $L$, there is the freedom of the choice of $Q$ and $\{R_a\}$, characterised in [23, Proposition 7.4.]. Here we only consider decompositions where the number of jump operators $q$ is minimal, as this corresponds to the number of particle species of the cMPS.

In addition, once these features of $P$ and $L$ are chosen, there is the following freedom in choosing the space where $|\Phi_{\mathcal{K}}\rangle$ lives:

(a) The ancilla space associated to $P$ is $\mathbb{C}^K$, which is uniquely defined. However, the basis choice $|v_i\rangle$, is arbitrary.

(b) The physical space associated to $e^{\alpha L}$ is the Fock space $\mathcal{H}_R$. This is a not uniquely defined—one can essentially choose any Fock space, i.e. one has to choose a vacuum and excitations on top. The number of different excitations $q$ is uniquely fixed, as it is the minimal number of jump operators of $L$.

In summary, given a family of MPS $\mathcal{V}(A)$ with transfer matrix $E_a = P e^{\alpha L}$, its continuum limit can be written as a generalised cMPS, for which we have to choose:

1. A Liouvillian of Lindblad form $L$ such that $PL = PLP$ and such that $E_a = P e^{\alpha L}$.
2. The Kraus operators $\{B_i\}_{i=1}^{K}$ of $P$.
3. $Q$ [satisfying (2)] and $\{R_a\}_{a=1}^{q}$ such that $L = Q \otimes I + I \otimes \hat{Q} + \sum_{a=1}^{q} R_a \otimes \hat{R}_a$.
4. An orthonormal basis $|v_i\rangle_{i=1}^{K}$ of $\mathbb{C}^K$.
5. A Fock space $\mathbb{H}_R$.

A second comment concerns Definition 5, i.e. the definition of continuum limit. As mentioned above, the equal superposition of ferromagnetic states $|0, 0, \ldots, 0\rangle + |1, 1, \ldots, 1\rangle$ has a continuum limit. However, this state is distinguishable from a probabilistic mixture of $|0, \ldots, 0\rangle |0, \ldots, 0\rangle$ and $|1, 1, \ldots, 1\rangle |1, 1, \ldots, 1\rangle$ only if one has access to completely non-local (i.e. global) observables, whereas in defining and analysing continuum limits, one usually assumes that only the algebra of quasi-local observables is accessible. In Definition 5, we ask that there be an infinite sequence of isometries $W_1, W_2, \ldots$ that p-refine $|V_N(A)\rangle$ (and the same sequence of isometries can be applied to all $N$ uniformly; and additionally there is the regularisation condition in the limit). This implies that the continuum limit state of $|V_N(A)\rangle$ has the same information as $|V_N(A)\rangle$ itself, as the refining process consists of a sequence of bases changes. For this reason the global information is not lost in the process. For further interpretations of our continuum limit, see Section 5.4.

A third comment is that the condition $PL = PLP$ is not incorporated into the ansatz of (10), since the $\{R_i\}$, $Q$ and $\{R_e\}$ need not obey any condition. For this reason, the class of generalised cMPS is broader than the set of continuum limit states of MPS. We leave as an open question in the Outlook whether there is a narrower class in which the condition $PL = PLP$ is built-in.

Example 9 (Superposition of ferromagnetic states)

Consider the superposition of two ferromagnetic states $|V_N(A)\rangle = |0, 0, \ldots, 0\rangle + |1, 1, \ldots, 1\rangle$. The transfer matrix is $E_a = P = |0, 0, 1, 1\rangle |1, 1, 1\rangle$ thus $Q = R_a = 0$. If we choose the Kraus operators of $P$ to be $B_i = |i\rangle\langle i|$ for $i = 0, 1$, and some basis of the ancilla space, $|v_0\rangle, |v_1\rangle$ then the continuum limit state can be written as a generalised cMPS

$$|\Phi_R\rangle = (|v_0\rangle + |v_1\rangle) \otimes |\Omega_R\rangle.$$ 

If we choose the Kraus operators of $P$ to be $B_0 = I$ and $B_1 = \sigma_z$, and some other basis of the ancilla space, $|w_0\rangle, |w_1\rangle$ then the generalised cMPS is given by

$$|\Phi_R\rangle = |w_0\rangle \otimes |\Omega_R\rangle.$$ 

We thus see that if $L = 0$ then not the entire space $\mathbb{C}^K$ may be occupied, but only part of it (i.e. here we effectively have $K = 1$). In Example 13 we will consider a channel $E_a = Pe^{IL}$ with the same $P$ as here but $L \neq 0$, in which the entire space will be occupied (Eq. (14)).

5.4. Interpretation of the ansatz

We now show that every projector quantum channel $P$ can be obtained as the projection onto the fixed point subspace of a Markovian channel $e^{IL}$. (We call this Liouvillian of Lindblad form $\tilde{L}$ to distinguish it from the $L$ of the infinitely divisible channel, $E_a = Pe^{IL}$.) We will prove this statement for the corresponding superoperator versions, i.e. for $P$ and $\tilde{L}$ (see Section 2.1). To this end, consider the projector quantum channel $\mathcal{P}$ given by (5). Now consider a Liouvillian of Lindblad form $\tilde{L}[Q, \{R_{k,1}, \ldots, R_{k,2^m-1}\}]$ [see Eq. (1)] given by $H = 0$,

$$R_{k,1} = |k\rangle \langle k| \otimes I_{D_k} \sum_{i=1}^{m-1} \theta_{k,i}|v_{k,i}\rangle |v_{k,i+1}\rangle,$$  

(11a)

$$R_{k,2} = |k\rangle \langle k| \otimes I_{D_k} \sum_{i=1}^{m-1} \theta_{k,i+1}|v_{k,i}\rangle |v_{k,i+1}\rangle,$$  

(11b)

where $|\langle k|\rangle$ is the computational basis, $\theta_{k,i} > 0$, and $\{|v_{k,i}\rangle\}$ is an orthonormal basis. Recall that $Q$ is determined by $H$ and $\{R_{k,1}, \ldots, R_{k,2^m-1}\}$ as specified in (2). See the text immediately after Proposition 3 to see how to recover familiar projector quantum channels by considering particular cases of $n, D_k$ and $m_k$.

Proposition 10 Let $\mathcal{P}$ be the projector quantum channel given by (5). Then the Liouvillian of Lindblad form $\tilde{L}$ given by (11) is such that

$$\mathcal{P} = \lim_{t \to \infty} e^{t\tilde{L}[Q, \{R_{k,1}, \ldots, R_{k,2^m-1}\}]},$$

The proof of Proposition 10 can be found in Appendix A. Note that, given $\mathcal{P}$, $\tilde{L}$ in Proposition 10 is highly not unique. Since $\mathcal{P}$ is a projection to the kernel of $\tilde{L}$, Ker($\tilde{L}$), any other $\tilde{L}$ with the same kernel will be equally valid.

Note also that if we have a channel $E_a = Pe^{IL}$, and $P = \lim_{t \to \infty} e^{t\tilde{L}}$, then $L$ and $\tilde{L}$ do not necessarily commute. This is because $PL = PLP$ implies that $L$ is lower block triangular in the basis given by Ker($\tilde{L}$), (Ker($\tilde{L}$))$^\perp$. In Appendix B we characterise $P$ and $L$ such that $PL = PLP$ for a special case.

Now, the limit $\lim_{t \to \infty} e^{t\tilde{L}}$ in Proposition 10 can be interpreted in two different ways. The first interpretation is that the limit of $\tilde{L}$ diverges, i.e. at least one of the matrices $Q$ or $\{R_{k,i}\}$ has an unbounded norm. The second interpretation is that $\tilde{L}$ is normalised, but the length of the segment where the corresponding cMPS is defined, i.e. $|R|$, diverges. In this case we would have a cMPS in the thermodynamic limit. In either case, the corresponding state is at the closure of the set of cMPS, not a cMPS itself.

In summary, Theorem 8 together with Proposition 10 show that the state $|\Phi_R\rangle$ can be seen as the concatenation of an element at the closure of the set of cMPS, and a cMPS. Taking the second interpretation of Proposition 10, one can imagine that the first state is a cMPS in the thermodynamic limit (which gives rise to $\mathcal{P}$), whereas the second cMPS is defined on a region of length $|R|$ (which gives rise to $e^{t\tilde{L}}$).

5.5. Examples

Let us now consider two examples of Proposition 10, and one example of both Proposition 10 and Theorem 8.

Example 11 (Superposition of ferromagnets) We revisit the equal superposition of ferromagnetic states (Example 9), this time to illustrate Proposition 10. We consider in this case the equal superposition of $K$ ferromagnetic states, $|V_N(A)\rangle = \sum_{m=1}^{2^K}(m)\otimes Y$. Its transfer matrix is given by

$$

|v_0\rangle = (1, 1, \ldots, 1) \otimes |\Omega_R\rangle, \\

|v_1\rangle = (1, 1, \ldots, 1) \otimes |\Omega_R\rangle.

$$
$E_a = P = \sum_{m=1}^{K} |m\rangle\langle m| \otimes |m\rangle\langle m|$. This state has a continuum limit, and according to Proposition 10, the state at the continuum can be obtained as $l \to \infty$ of a cMPS with $H = 0$ and jump operators $R_m = |m\rangle\langle m|$ for $m = 1, \ldots, K$. Explicitly, this gives rise to the following cMPS with open indices on a segment $R$ of length $|R| = l$,

$$\phi_R = e^{-|R|/2} \left[ I \otimes I + \sum_{m=1}^{K} R_m \otimes (T \exp \int_{R} dx \hat{\psi}_m(x) - I) \right] |\Omega_R\rangle,$$

which satisfies that

$$E_{|R|} = (\phi_R, \phi_R) = e^{-|R|} [I \otimes I + \sum_{m=1}^{K} R_m \otimes R_m (e^{-|R|} - 1)].$$

This verifies that $\lim_{|R| \to \infty} E_{|R|} = P$.

**Example 12 (Completely depolarising map)** Consider the MPS given by matrices

$$A^0 = \frac{1}{\sqrt{2}} |0\rangle\langle 0|, \quad A^1 = \frac{1}{\sqrt{2}} |0\rangle\langle 1|,$$

$$A^2 = \frac{1}{\sqrt{2}} |1\rangle\langle 0|, \quad A^3 = \frac{1}{\sqrt{2}} |1\rangle\langle 1|.$$

The corresponding transfer matrix is the completely depolarising map $P(\rho) = \text{tr}(\rho)(1/2)$, which is a projector quantum channel. This state has a continuum limit, and according to Proposition 10, the state at the continuum can be obtained as the $l \to \infty$ limit of a cMPS with $H = 0$ and jump operators $R_0 = (1/\sqrt{2}) |0\rangle\langle 1|$ and $R_1 = (1/\sqrt{2}) |1\rangle\langle 0|$. This gives rise to the following cMPS with open indices on a segment $R$ of length $|R| = l$,

$$\phi_R = e^{-|R|/2} [I \otimes I + \sum_{a=0}^{2} R_a \otimes R_a \times$$

$$\int_{0 \leq x_1 \leq \ldots \leq x_N} dx_1 \ldots dx_N \hat{\psi}_a^\dagger(x_1) \hat{\psi}_a(x_1) \hat{\psi}_a^\dagger(x_2) \ldots \hat{\psi}_a(x_N)$$

$$+ \sum_{\alpha = 0}^{2} R_{\alpha+1} \sum_{N \geq 2, \text{even}} \frac{1}{2(N-2)!} \times$$

$$\int_{0 \leq x_1 \leq \ldots \leq x_N} dx_1 \ldots dx_N \hat{\psi}_\alpha^\dagger(x_1) \hat{\psi}_{\alpha+1}(x_2) \hat{\psi}_{\alpha+1}(x_3) \cdots \hat{\psi}_{\alpha+1}(x_N) |\Omega_R\rangle,$$

where the creation operators are of type $a, a+1, a+1, \ldots$ ($N$ times), and the sum on $a$ is modulo 2. This gives rise to

$$E_{|R|} = (\phi_R, \phi_R) = e^{-|R|/2} [I \otimes I + 2 \sinh(l/2) \sum_{a=0}^{2} R_a \otimes R_a$$

$$+ 4(\cosh(l/2) - 1) \sum_{\alpha = 0}^{2} R_{\alpha+1} \otimes R_{\alpha+1}] |\Omega_R\rangle.$$

We again see that $\lim_{|R| \to \infty} E_{|R|} = P$.

**Example 13 (The bracket state)** Consider the transfer matrix $E_a = Pe^{\gamma L}$ where

$$P = I \otimes I + \sigma_z \otimes \sigma_z,$$

$$L(\rho) = \frac{\gamma}{\alpha} (\sigma_z \rho \sigma_z - \rho).$$

One can verify that $PL = PLP$. This results in the infinitely divisible channel

$$E_a = Pe^{\gamma L} = I \otimes I + e^{2\gamma} \sigma_z \otimes \sigma_z.$$

Note that $E_a$ has a non-degenerate eigenvalue 1, and two eigenvalues 0. This channel can be seen as a convex combination of the channel corresponding to the equal superposition of ferromagnetic states, $E_a^{(1)} = |0\rangle\langle 0| + |1\rangle\langle 1|$, and the one corresponding to the superposition of antiferromagnetic states, $E_a^{(ad)} = |0\rangle\langle 1| + |1\rangle\langle 0|$, namely

$$E_a = pE_a^{(f)} + qE_a^{(ad)},$$

where

$$p = \frac{1 + e^{-2\gamma}}{2}, \quad q = \frac{1 - e^{-2\gamma}}{2}.$$

One possible family of MPS $\mathcal{V}(A)$ whose transfer matrix is this channel is given by the matrices

$$A^0 = \sqrt{p} |0\rangle\langle 0|, \quad A^1 = \sqrt{q} |1\rangle\langle 1|,$$

$$A^2 = \sqrt{p} |1\rangle\langle 0|, \quad A^3 = \sqrt{q} |0\rangle\langle 1|.$$

The state is

$$|V_N(A)\rangle = \sum_{i_0, \ldots, i_N} p^{(n_1 + n_2)/2} q^{(n_2 + n_3)/2} |i_1, \ldots, i_N\rangle$$

where $\cdot$ indicates that the sum is over allowed states only. To understand graphically which are the allowed states, we represent 0 by a dot, 1 by a dash, -1 by an opening bracket, -3 by a closing bracket). Now, the only allowed states are those such that every bracket which is opened is closed before opening another bracket, and which contain zero or more dots outside the brackets, and zero or more dashes inside the brackets. Examples of allowed states are $\ldots(-)(-)\ldots(-)(-)\ldots$, and $\ldots-\ldots-\ldots$. Every bracket contributes with $\sqrt{q}$ (and contains the ‘antiferromagnetic character’ of the state) and every point or dash contributes with $\sqrt{p}$ (and contains the ‘ferromagnetic character’ of the state).

The continuum limit of $|V_N(A)\rangle$ can be represented by a generalised cMPS with $|v_1\rangle = |\psi\rangle$, $r_0 = |0\rangle\langle 0|$, $r_1 = |1\rangle\langle 1|$, $R = \sqrt{q} \sigma_z$, with $\gamma' := \gamma/\alpha$, and $|R| = Na$, namely

$$|\Phi_R\rangle = |0\rangle \otimes \text{tr}_{\text{max}} \left( |0\rangle\langle 0| T \exp \left[ \int_R -\gamma' I \otimes I \right.ight.$$

$$\left. + \sqrt{q} \sigma_z \otimes \hat{\psi}^\dagger(x) dx \right] |\Omega_R\rangle)$$

$$+ |1\rangle \otimes \text{tr}_{\text{max}} \left( |1\rangle\langle 1| T \exp \left[ \int_R -\gamma' I \otimes I \right.$$

$$\left. + \sqrt{q} \sigma_z \otimes \hat{\psi}^\dagger(x) dx \right] |\Omega_R\rangle \right) .$$

In addition, according to Proposition 10, $P$ can be obtained as $P = \lim_{r \to \infty} e^{\gamma L}$, where

$$\tilde{L}(\rho) = \gamma \rho \sigma_z - \rho.$$

Note that in this case $L$ and $Lz$ commute, $L^z \tilde{L} = L^z L$.
6. CONCLUSIONS AND OUTLOOK

In summary, we have proposed a generalised ansatz of continuous MPS which can express the continuum limit of any MPS (Theorem 8), according to the continuum limit given in Definition 5. The ansatz consists of a sum of cMPS, each with a different boundary operator, and attached to an ancilla state. The boundary operators are given by the Kraus operators of the projector quantum channel \( P \). We have shown that this ansatz can be interpreted as the concatenation of an element which is at the closure of the set of cMPS (which can be thought of as a cMPS in the thermodynamic limit, or a cMPS with matrices of unbounded strength), and a cMPS (Proposition 10).

As mentioned in Section 5.3, there is, on the one hand, quite a lot of freedom in the choice of the generalised cMPS, and, on the other hand, our ansatz does not impose explicitly the fact that \( PL = PLP \). Hence, there might exist another ansatz that incorporates this condition and reduces some of that freedom. A good place to seek inspiration may be the situation for \( G \)-injective MPS [21]. These have a parent hamiltonian whose degeneracy is given by the number of conjugacy classes of \( G \), and the ground state subspace of the parent hamiltonian is obtained precisely by considering superpositions of MPS with different boundary conditions, which is reminiscent of our ansatz. Whether the two ideas are in fact connected is a matter of future work.

Another important question is what is the physical nature, or how are we supposed to understand, the ancilla Hilbert space \( \mathbb{C}^K \) in the ansatz of generalised cMPS. It would be very interesting to understand what is the nature of the observables that give access to it.

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Appendix A: Proof of Proposition 10

In this Appendix we prove Proposition 10. We start by restating it.

Proposition (Proposition 10) Let \( P \) be the projector quantum channel given by (5). Then the Liouvillian of Lindblad form \( \tilde{L} \) given by (11) is such that

\[
P = \lim_{t\to\infty} e^{t\tilde{L}} |Q,R_{i_1},R_{i_2}|_{\sigma}.
\]

Proof

Throughout the proof we will write \( \tilde{L} \) instead of \( \tilde{L} \) in order to simplify the notation. We will prove the statement for the following cases:

(i) The completely depolarising map, which has \( n = 1 \), \( D_1 = 1 \) and \( m_1 = D \).

(ii) Case \( n = 1 \) (and \( D_1 \) and \( m_1 \) unfixed).

(iii) Case \( D_k = 1 \) for all \( k \) (and \( n \) and \( m_k \) unfixed).

Prove that \( \tilde{L} \) is linear and trace preserving, i.e.,

\[
\tilde{L} \rho = \text{tr}(\tilde{L} \rho)\rho, \quad \text{tr}(\rho) = 1, \quad \text{and} \quad |Q,R_{i_1},R_{i_2}|_{\sigma} = 0 \text{ tr}(\sigma) = 1.
\]

We claim that this can be written as \( \tilde{L} = \lim_{t\to\infty} e^{t\tilde{L}} |Q,R_{i_1},R_{i_2}|_{\sigma} \) with \( H = 0 \),

\[
R_1 = \sum_{i=1}^{D-1} \theta_i |v_i\rangle \langle v_{i+1}|, \quad R_2 = \sum_{i=1}^{D-1} \theta_{i+1} |v_{i+1}\rangle \langle v_i|.
\]
We will show that $\sigma$ is the only fixed point of $e^{L}$, since, by [23, Proposition 7.5], this implies that the map is primitive (i.e. $e^{L}$ does not have any other eigenvalue of modulus 1). First note that $L(\sigma) = 0$, so that we only have to see that $L$ has no other 0 eigenvalue. Observe that, for any $\rho$ which is orthogonal to $\sigma$, i.e., $\text{tr}(\sigma^\dagger \rho) = 0$, we have that $L(\rho) \neq 0$, as can be easily seen using the form of $L$. If $\rho$ has only off-diagonal terms, i.e., $\rho = \sum_{k \neq l}^{D} \alpha_{k,l} \langle v_{k} | v_{l} \rangle$ for $k \neq l$, then it is immediate to see that $L(\rho) \neq 0$. If $\rho$ is diagonal but different from $\sigma$, i.e., $\rho = \sum_{l=1}^{D} \alpha_{l} | v_{l} \rangle \langle v_{l} |$, then $L(\rho) \neq 0$. Finally, if $\rho$ is a combination of diagonal and off-diagonal terms, $\rho = \sum_{k,l=1}^{D} \alpha_{k,l} | v_{k} \rangle \langle v_{k} | + \beta_{l} | v_{l} \rangle \langle v_{l} |$, it can be easily verified that $L(\rho) \neq 0$ as well.

(ii) Now consider the case $n = 1$, so that $M_{D} = M_{D_{b}} \otimes M_{m_{1}}$, so that the projector quantum channel is $P(\rho^{(1)} \otimes \rho^{(2)}) = \text{id}(\rho^{(1)}) \otimes \text{tr}(\rho^{(2)}) \sigma_{1}$. It is straightforward to see that this can be written as $P = \lim_{n \rightarrow \infty} e^{L_{[0,1;\sigma_{1}]}}$ with $H = 0$

$$R_1 = I_{D_{b}} \otimes \sum_{i=1}^{m_{1}-1} \theta_{i} | v_{i} \rangle \langle v_{i+1} |,$$

$$R_2 = I_{D_{b}} \otimes \sum_{i=1}^{m_{1}-1} \theta_{i} \otimes | v_{i+1} \rangle \langle v_{i} |.$$

(iii) Now consider the $D_{k} = 1$ for all $k$, so that $P(\rho) = \bigoplus_{k=1}^{n} \text{tr}(\rho_{k}) \sigma_{k}$. It can be written as $P = \lim_{n \rightarrow \infty} e^{L_{[0,1;\sigma_{k}]}}$ with

$$R_{k,1} = | k \rangle \langle k | \otimes \sum_{i=1}^{m_{k}-1} \theta_{i,k} | v_{k,i} \rangle \langle v_{k,i+1} |,$$

$$R_{k,2} = | k \rangle \langle k | \otimes \sum_{i=1}^{m_{k}-1} \theta_{i+1,k} | v_{k,i+1} \rangle \langle v_{k,i} |.$$ 

where $k = 1, \ldots, n$.

Finally, putting these three building blocks together, it is immediate to see that the statement holds for a generic $P$ with unfixed $n, D_{b}$ and $m_{k}$.

**Appendix B: Characterisation of $PL = PLP$ for a special case**

In this Appendix we characterise the implications of the condition $PL = PLP$ for a special case of $P$ and $L$. Namely, we consider a special case of the projector quantum channel $P : M_{D} \rightarrow M_{D}$ given by (5) in which $D_{0} = 0, D_{k} = D_{1}$ and $m_{k} = m_{1}$ for all $k > 0$, so that $M_{D} = \bigoplus_{k=1}^{n} (M_{D_{1}} \otimes M_{m_{1}})$, which is equivalent to $I_{n} \otimes M_{D_{1}} \otimes M_{m_{1}}$.

In addition, we assume that $L$ has a single jump operator $R$, with the tensor product structure

$$R = S \otimes T \otimes V,$$

with $S \in M_{m}, T \in M_{D_{1}}$, and $V \in M_{m_{1}}$. Similarly, we assume that the Hamiltonian $H$ consists of a single term with the same structure,

$$H = A \otimes B \otimes C,$$

with $A \in M_{n}, B \in M_{D_{1}},$ and $C \in M_{m_{1}}$.

**Proposition 14** Consider a projector quantum channel $P : M_{D} \rightarrow M_{D}$ given by (5), and $L_{[0,1;\sigma]}$ given by (1) with $H = A \otimes B \otimes C$ and $R = S \otimes T \otimes V$. If $PL = PLP$, then

(i) $A$ is diagonal and either $B \propto I$ or $C \propto I$.

For $R$, either

(a) $S$ satisfies

$$S_{k,l} S_{k,m} = 0,$$

(B1a)

$$S_{k,l} S_{l,j} = S_{k,l} S_{l,j},$$

(B1b)

$$T \propto I \text{ and } V \propto U,$$

or

(b) $S$ has one non-zero element per row, $T \propto I$ and $V \propto U$, or

(c) $S$ is diagonal, $T \propto I$ and $V \propto U$.

where $U$ is a unitary.

Recall that $I$ denotes the identity matrix.

If $n, D_{1}$ or $m_{1}$ are 1, then the only cases that hold are the ones in which the corresponding condition is trivial. That is, if $n = 1$, then all three cases are possible. If $D_{1} = 1$, then case (a) and (c) are possible. If $D_{1} = 2$, then case (b) and (c) are possible. If $D_{1} = 1 = 1$ then case (a) is possible. If $n = D_{1} = 1$ then $V$ must be a unitary. If $n = m_{1} = 1$ then $T$ must be the identity.

Note that Eq. (B1a) implies that $S$ has at most one non-zero off-diagonal element in every row. Examples would be a diagonal $S$, or $S$ being a permutation matrix times a diagonal matrix, or a matrix which is zero everywhere except for a column, e.g. $S = \sum_{j=1}^{n} | j \rangle \langle 1 |$. In the latter two cases, $S$ can additionally have non-zero diagonal elements, as long as the symmetry conditions of (B1b) are fulfilled. For example, consider $\rho_{1}, \rho_{2} \in M_{2},$ and the projector

$$P(\rho_{1} \otimes \rho_{2}) = (| 0 \rangle \langle 0 | \rho_{1} | 0 \rangle \langle 0 | + | 1 \rangle \langle 1 | \rho_{1} | 1 \rangle \langle 1 |) \otimes \text{tr}(\rho_{2}) \frac{I}{2}. $$

Then we could have, for example, $R = \sigma_{x} \otimes \sigma_{x}$ corresponding to case (a), or $R = \sigma_{z} \otimes \sigma_{z}$ corresponding to case (c).

**Proof** Throughout the proof we will use that $\sigma_{k}$ is full rank and hence can be inverted, and that $\text{tr}(\sigma_{k}) = 1$. We also consider an input state of the form $\rho = \rho_{1} \otimes \rho_{2} \otimes \rho_{3}$.

For the imaginary part of $PL = PLP$, the component $(k,k) \ldots (l,k)$ (with $l \neq k$) of the first tensor product implies that

$$-i A_{k,l} (\rho_{1})_{k,l} B_{l} C \text{tr}(C \rho_{3}) = 0,$$

which implies that $A$ has to be diagonal. The rest of the components yield trivial identities, since $B$ and $C$ are Hermitian, thus proving (i).

For the real part of $PL = PLP$, the component $(k,k) \ldots (l,m)$ of the first tensor product, with $k \neq l, k \neq m$ and $m \neq l$ gives

$$S_{k,l} T_{k,l} \text{tr}(V \rho_{3} V^\dagger) = 0.$$
for all $\rho_3$. This implies (B1a). The components $\langle k, k \ldots | k, k \rangle$, $\langle k, k \ldots | l, k \rangle$, and $\langle k, k \ldots | l, l \rangle$ of the first tensor product give, respectively,

$\frac{1}{2} \left\{ \left| S_{k,k} \right|^2 (T \rho_2 T^\dagger) - \frac{1}{2} (S^\dagger S)_{k,k} (T^\dagger T \rho_2) + (\rho_2 T^\dagger T) \right\} \times \left[ \text{tr}(V \rho_3 V^\dagger) - \text{tr}(V \sigma_k V^\dagger) \text{tr}(\rho_3) \right] = 0, \quad (B2)$

$\left\{ S_{k,k} \tilde{S}_{k,k} (T \rho_2 T^\dagger) - \frac{1}{2} (S^\dagger S)_{k,k} (T^\dagger T \rho_2) \right\} \text{tr}(V \rho_3 V^\dagger) = 0, \quad (B3)$

$\left| S_{k,l} \right|^2 (T \rho_2 T^\dagger) \left[ \text{tr}(V \rho_3 V^\dagger) - \text{tr}(V \sigma_k V^\dagger) \text{tr}(\rho_3) \right] = 0, \quad (B4)$

for all $\rho_2$ and $\rho_3$. First note that if $T \not\propto I$ and $V \not\propto U$ then Eqs. (B2) and Eq. (B4) imply that $S = 0$, which is false by assumption. Hence there are three cases:

(a) If $T \propto I$ and $V \propto U$ then Eq. (B3) implies that $S_{k,k} \tilde{S}_{k,k} = \tilde{S}_{k,k} S_{k,k}$, which together with (B1b) means that $S$ satisfies the conditions of the pinching map, (B1).

(b) If $T \not\propto I$ and $V \propto U$ then Eq. (B3) implies that $S_{k,k} \tilde{S}_{k,k} = \tilde{S}_{k,k} S_{k,k} = 0$, i.e. $S$ only has one non-zero element per row.

(c) If $T \propto I$ and $V \not\propto U$, then Eq. (B4) implies that $S_{k,l} = 0$, i.e. $S$ is diagonal. \hfill \Box