CNOT operator and its similar matrices in quantum computation

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Abstract

We present the theoretical result, which is based on the linear algebra theory (similar operators). The obtained theoretical results optimize the experimental technique to construct quantum computer e.g., reduces the number of steps to perform the logical CNOT (XOR) operation. The present theoretical technique can also be generalized to the other operators in quantum computation and information theory.

Keywords: quantum computing, similar operators, CNOT or XOR, nuclear magnetic resonance.

1 Introduction

The linear algebra, which is the nucleus of quantum theory, has many interesting properties in the theoretical predictions of physical processes. Tensor product \cite{3}, which is very widely, used in quantum theory plays an important role not only for obtaining the higher dimensional Hilbert space but also to optimize the experimental technique. Since, the tensor product is non-commutative in nature i.e., it can not be applied freely to any operator without any knowledge of the constituents of the physical system. One very important property of tensor product is similar matrices \cite{3}, which have two optimization properties: reduces the number of pulses to realize some operator in physical experiment, and provides different choices of pulse propagations along different directions of axes. These two properties will be discussed in section 2 with concrete physical realization of CNOT operator and how we can optimize its realization.

2 CNOT gates and CNOT’s similar matrices in quantum computation

CNOT gates or similar matrices play an important role in the construction of quantum computer and computation. All the complex quantum algorithms are based on the combination of NOT and CNOT logical gates or matrices in quantum computation. Here, we will not discuss the NOT logical gate or matrix, which is very simple negation operation. We will go in deep the mathematical
and physical nature of CNOT logical gates. The other complex gates can be constructed on the basis of CNOT logical gates. Moreover, the construction of CNOT matrices plays an important role in quantum computation. Here, we will show that the CNOT matrices, which were used by [1,2] can be optimized by using the similar matrices i.e., we can find different CNOT matrices with the same mathematical properties but some different experimental or physical realization. For better insight to understand the physical nature of similar matrices, we will discuss in the whole article the system of two spins \( \hat{\sigma}_1 = \frac{1}{2} \) and \( \hat{\sigma}_2 = \frac{1}{2} \) with slightly different resonance frequencies \( \omega_1 \) and \( \omega_2 \) and having scalar coupling \( \omega_{12} \). The Hamiltonian of the two spins aligned along the \( z \)-axis with the constant magnetic field

\[
\hat{H} = \hbar \omega_1 \hat{\sigma}_{1z} \otimes \hat{e}_2 + \hbar \omega_2 \hat{\sigma}_{2z} + \hbar \omega_{12} \hat{\sigma}_{1z} \otimes \hat{\sigma}_{2z}.
\]  

(1)

Where \( \hat{e}_{i(1,2)} \) - identity matrix with dimensions 2 \( \times \) 2 and \( \hbar \) - is Plank’s constant.

2.1 Physical differences between the CNOT matrices obtained by Gershenfeld and Cory

The CNOT matrix used by [2] contains additional spin \( \hat{\sigma}_1 \) rotation around the \( z \)-axis, which is unnecessary and which complicate the experiment realization of CNOT as compare to the CNOT matrix used by Cory [1]. In the mathematical point of view, CNOT matrices obtained by Gershenfeld and Cory have different mathematical properties i.e., CNOT matrices (2) and (4,5) are not similar matrices due to the additional rotation of spin \( \hat{\sigma}_{z1} \) around \( z \)-axis in (3) as compare to the (6,7).

Moreover, additional rotation takes more time to fulfill the CNOT operation, which slows down the computation of quantum algorithms. The CNOT matrix for Hamiltonain (1) obtained by Gershenfeld [2] is

\[
C_g = \sqrt{-i} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 
\end{pmatrix}.
\]

(2)

The \( C_g \) matrix was obtained by the following pulse sequences

\[
C_g = R_{y2}(\pi/4)R_{z1}(\pi/4)R_{z2}(\pi/4)R_{z12}(\pi/4)R_{y2}(\pi/4) \\
= e^{-i \hat{e}_1 \otimes \hat{\sigma}_{y2}} e^{-i \hat{\sigma}_{z1} \otimes \hat{e}_2} e^{-i \hat{e}_1 \otimes \hat{\sigma}_{z2}} e^{-i \hat{\sigma}_{z1} \otimes \hat{\sigma}_{z2}} e^{i \hat{e}_1 \otimes \hat{\sigma}_{z2}}.
\]

(3)

Where \( R \) - is rotation matrix around the different axes along with the different angles around different axes. The physical interpretation of matrix \( R \) is just the rotation of our physical reference system or quantum tomography [4].

The two CNOT matrices for Hamiltonain (1) obtained by Cory [1] are

\[
C_{c1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 
\end{pmatrix}.
\]

(4)
\[
C_{c2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\] (5)

The \(C_{c1}\) and \(C_{c2}\) matrices were obtained with the following pulse sequences

\[
C_{c1} = R_{x2}(-\pi/4)R_{z2}(-\pi/4)R_{z12}(\pi/4)R_{x2}(\pi/4)
= e^{-i\hat{\sigma}_x \otimes \hat{e}_1} e^{-i\hat{\sigma}_z \otimes \hat{e}_2} e^{i\hat{\sigma}_z \otimes \hat{e}_1} e^{i\hat{\sigma}_x \otimes \hat{e}_2}.
\] (6)

\[
C_{c2} = R_{x1}(-\pi/4)R_{z1}(-\pi/4)R_{z12}(\pi/4)R_{x1}(\pi/4)
= e^{-i\hat{\sigma}_x \otimes \hat{e}_2} e^{-i\hat{\sigma}_z \otimes \hat{e}_1} e^{i\hat{\sigma}_z \otimes \hat{e}_1} e^{i\hat{\sigma}_x \otimes \hat{e}_2}.
\] (7)

2.2 Mathematical differences between the CNOT matrices obtained by Gershenfeld and Cory

Since, the Hamiltonian of the CNOT matrices (2) and (4,5) are the same but they are not similar matrices. To check the mathematical nature of (2) and (4,5), first of all we will write six mathematical properties of similar square matrices \(A\) and \(B\) of dimensions \(4 \times 4\) on the Hilbert space \(F\).

- Determinant of \(A\) is equal to determinant of \(B\).
- Trace of \(A\) is equal to trace of \(B\).
- If \(A\) and \(B\) are nonsingular than \(A^{-1}\) and \(B^{-1}\) are also similar matrices.
- \(A\) and \(B\) are similar matrices, if there exist nonsingular matrix \(P\) such that \(B = P^{-1}AP\) or \(PBP^{-1} = A\).
- Matrices \(A\) and \(B\) have the same eigenvalues.
- \(PB_{evvec} = A_{evvec}\), where \(B_{evvec}\) and \(A_{evvec}\) are the eigenvectors of the matrices \(B\) and \(A\).

Here, we will not proof the properties of similar matrices. The proof of above six properties are very simple and can be found in the course of linear algebra [5].

By applying six properties of similar matrices to the CNOT matrices (2), (4,5), we will find that that the properties second, fourth, fifth and sixth are not satisfied between CNOT matrices (2) and (4,5). So, the matrices (2) and (4,5) are not similar.

2.3 Physical interpretation of similar matrices

The similar matrices are very important part of the linear algebra in quantum computation. For example, by finding all the similar matrices, we can get all the CNOT matrices or operators, which will reduce our mathematical and physical realization of quantum computation. If we apply all the six properties of similar
matrices to CNOT matrices (4) and (5), we will see that matrices (4) and (5) are similar matrices.

Now, if we apply operators $C_{c1}$ and $C_{c2}$ to the state $|\phi> = a|\uparrow,\uparrow> + b|\uparrow,\downarrow> + c|\downarrow,\uparrow> + d|\downarrow,\downarrow>$, of Hamiltonian (1). We will obtain

$$|\phi_1> = C_{c1}|\phi> = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a(\uparrow,\uparrow) \\ b(\uparrow,\downarrow) \\ c(\downarrow,\uparrow) \\ d(\downarrow,\downarrow) \end{pmatrix} = \begin{pmatrix} a(\uparrow,\uparrow) \\ b(\uparrow,\downarrow) \\ -c(\downarrow,\uparrow) \\ d(\downarrow,\downarrow) \end{pmatrix}. \quad (8)$$

$$|\phi_2> = C_{c2}|\phi> = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a(\uparrow,\uparrow) \\ b(\uparrow,\downarrow) \\ c(\downarrow,\uparrow) \\ d(\downarrow,\downarrow) \end{pmatrix} = \begin{pmatrix} a(\uparrow,\uparrow) \\ b(\uparrow,\downarrow) \\ c(\downarrow,\uparrow) \\ -b(\downarrow,\downarrow) \end{pmatrix}. \quad (9)$$

the state $|\phi_{1,2}>$, which is obtained by operating $C_{c1}$ and $C_{c2}$, which gives us simple picture of the state before and after the CNOT operation. In the experiment of nuclear magnetic resonance, the CNOT matrices (4,5) are called Pound-Overhauser operators [1] i.e., transformation of spin polarization from one spin to other spin.

How can we find all the CNOT matrices or similar matrices of (4,6)? Is there any method to get all the similar matrices? One of the main purpose of this paper is to find answer to these questions.

We are not interested in finding similar matrices of CNOT matrix (2) because it has an additional rotation, which complicates CNOT operation and which is not required for the CNOT operation. So, we are interested in finding the similar matrices of (4,5), which minimize CNOT operation in quantum computation.

### 2.4 How we can find CNOT similar matrices and how many are they?

As it is seen from the (8) and (9) that the state $|\uparrow,\uparrow>$ in $|\phi>$ is not used by the CNOT matrix. It means, we have still more options to use the possibility to get other CNOT matrices or similar matrices. This can be done by considering the six similar matrix properties.

$$C_{c11} = R_{y2}(-\pi/4) R_{z12}(\pi/4) R_{z2}(-\pi/4) R_{y2}(\pi/4) = e^{-i \hat{\sigma}_y \otimes \hat{\sigma}_y} e^{i \hat{\sigma}_z \otimes \hat{\sigma}_z} e^{-i \hat{\sigma}_z \otimes \hat{\sigma}_z} e^{i \hat{\sigma}_y \otimes \hat{\sigma}_y}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}. \quad (10)$$

$$C_{c22} = R_{y1}(-\pi/4) R_{z12}(\pi/4) R_{z1}(-\pi/4) R_{y1}(\pi/4) = e^{-i \hat{\sigma}_y \otimes \hat{\sigma}_y} e^{i \hat{\sigma}_z \otimes \hat{\sigma}_z} e^{-i \hat{\sigma}_z \otimes \hat{\sigma}_z} e^{i \hat{\sigma}_y \otimes \hat{\sigma}_y}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$
\[ C_{c31} = R_{y1}(-\pi/4)R_{z12}(-\pi/4)R_{z1}(-\pi/4)R_{y1}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & 1 & 0 \\
0 & -i & 0 & 0 \\
\end{pmatrix}.
\]  
(11)

\[ C_{c32} = R_{y2}(-\pi/4)R_{z12}(-\pi/4)R_{z2}(-\pi/4)R_{y2}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
0 & -i & 0 & 0 \\
-i & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]  
(12)

\[ C_{c41} = R_{z1}(-\pi/4)R_{z12}(-\pi/4)R_{z1}(-\pi/4)R_{z11}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]  
(13)

\[ C_{c42} = R_{x2}(-\pi/4)R_{z12}(-\pi/4)R_{z2}(-\pi/4)R_{x21}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]  
(14)

\[ C_{c51} = R_{y1}(-\pi/4)R_{z12}(-\pi/4)R_{z1}(-\pi/4)R_{y11}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 1 & 0 \\
0 & i & 0 & 0 \\
\end{pmatrix}.
\]  
(15)

\[ C_{c52} = R_{y2}(-\pi/4)R_{z12}(-\pi/4)R_{z2}(-\pi/4)R_{y21}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & i & 0 \\
\end{pmatrix}.
\]  
(16)

\[ C_{c61} = R_{z1}(-\pi/4)R_{z12}(-\pi/4)R_{z1}(-\pi/4)R_{z11}(\pi/4) = e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{-i\frac{\pi}{4} \sigma_z} e^{i\frac{\pi}{4} \sigma_z} \]
\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & i & 0 \\
\end{pmatrix}.
\]  
(17)
different axes according to our convenience. The CNOT operation on qubits with the different pulse rotations around the obtained by using the similar matrices properties. It means we can perform We have obtained 16 CNOT matrices or operators i.e., (4-5, 10-23), which are

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, 
\]

\(C_{c62} = R_{x2}(-\pi/4)R_{z12}(-\pi/4)R_{z2}(\pi/4)R_{x2}(\pi/4) = e^{-i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{-i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}. 
\]

\(C_{c71} = R_{x2}(-\pi/4)R_{z12}(\pi/4)R_{z2}(\pi/4)R_{x2}(\pi/4) = e^{-i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2}
\]

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. 
\]

\(C_{c72} = R_{x1}(-\pi/4)R_{z12}(\pi/4)R_{z1}(\pi/4)R_{x1}(\pi/4) = e^{-i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2}
\]

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. 
\]

\(C_{c81} = R_{y2}(-\pi/4)R_{z12}(\pi/4)R_{z2}(\pi/4)R_{y2}(\pi/4) = e^{-i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2}
\]

\[
\begin{pmatrix}
0 & i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. 
\]

\(C_{c82} = R_{y1}(-\pi/4)R_{z12}(\pi/4)R_{z1}(\pi/4)R_{y1}(\pi/4) = e^{-i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2} e^{i\frac{\pi}{4}\sigma_1 \otimes \sigma_2}
\]

\[
\begin{pmatrix}
0 & 0 & i & 0 \\
0 & 1 & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. 
\]

We have obtained 16 CNOT matrices or operators i.e., (4-5,10-23), which are obtained by using the similar matrices properties. It means we can perform the CNOT operation on qubits with the different pulse rotations around the different axes according to our convenience.
3 Conclusion

By using the properties of similar matrices we can reduce to some extent the physical realization of construction of quantum computer by having different number of pulse choices around different axes or different number of CNOT matrices. For example, the nuclear magnetic resonance (NMR) apparatus does not allow us to send pulse to some concrete direction, if we have one CNOT matrix. It means we have to modify our apparatus or by new one. But when we have 16 number of CNOT matrices, we can find compromise with our apparatus.

Moreover, the technique of similar matrices are very closely related to the tensor product, which was observed by us earlier [3]. The similar matrices method can be applied to the other branches of quantum theory.

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