Symmetric identities of higher-order degenerate Euler polynomials

Dae San Kim · Taekyun Kim

Received: date / Accepted: date

Abstract The purpose of this paper is to give some symmetric identities of higher-order degenerate Euler polynomials derived from the symmetric properties of the multivariate $p$-adic fermionic integrals on $\mathbb{Z}_p$.

Keywords Symmetry · Higher-order degenerate Euler polynomial · Identity

1 Introduction

Let $p$ be a fixed prime such that $p \equiv 1 \pmod{2}$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then $p$-adic fermionic integral on $\mathbb{Z}_p$ is defined by Kim as

$$\int_{\mathbb{Z}_p} f(x) \, d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p)$$

$$= \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x, \quad \text{(see [8])}.$$
From (1), we note that
\[
\int_{\mathbb{Z}} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}} f(x) d\mu_{-1}(x) = 2f(0),
\]
and
\[
\int_{\mathbb{Z}} f(x+n) d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}} f(x) d\mu_{-1}(x) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{(see [3]).}
\]

As is well known, the Euler polynomials are defined by the generating function
\[
\frac{2e^t}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [4];8).}
\]

When \(x = 0\), \(E_n = E_n(0)\) are called the Euler numbers. For \(r \in \mathbb{N}\), the higher-order Euler polynomials are given by
\[
\left( \frac{2}{e^t+1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [1,2,3,5,6,7,9,10])} \tag{5}
\]

In particular, \(x = 0\), \(E_n^{(r)}(0) = E_n^{(r)}(0)\) are called the Euler numbers of order \(r\).

From (2), we can easily derive the following equation:
\[
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} e^{(x_1+\cdots+x_r+x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left( \frac{2}{e^t+1} \right)^r e^{xt}
\]
\[
= \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.
\]

Thus, by (6), we get
\[
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} (x_1+\cdots+x_r+x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = E_n^{(r)}(x), \quad (n \geq 0).
\]

Carlitz introduced the degenerate Euler polynomials given by the generating function
\[
\left( \frac{2}{(1+\lambda t)^\xi+1} \right) (1+\lambda t)^\xi = \sum_{n=0}^{\infty} e_n(x | \lambda) \frac{t^n}{n!}, \quad \text{(see [5]).}
\]

When \(x = 0\), \(e_n(0 | \lambda) = E_n(\lambda)\) are the degenerate Euler numbers. Note that \(\lim_{\lambda \to 0} e_n(\lambda | \lambda) = E_n(x)\).
For \( r \in \mathbb{N} \), the higher-order degenerate Euler polynomials are also given by the generating function

\[
\left( \frac{2}{(1 + \lambda t) \frac{d}{dt} + 1} \right)^r \frac{1}{(1 + \lambda t)^{\frac{d}{dt}}} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)} (x | \lambda) \frac{t^n}{n!}.
\]

(9)

When \( x = 0 \), \( \mathcal{E}_n^{(r)} (0 | \lambda) = \mathcal{E}_n^{(r)} (\lambda) \) are called the higher-order degenerate Euler numbers (see [2]).

In [8], Kim and Kim showed that the degenerate Euler polynomials can be represented by a \( p \)-adic integral on \( \mathbb{Z}_p \). Recently, several researchers have studied the symmetric identities of higher-order Euler polynomials derived from the symmetric properties of \( p \)-adic integrals on \( \mathbb{Z}_p \) (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]).

In this paper, we investigate some properties of symmetry for the multivariate \( p \)-adic fermionic integrals on \( \mathbb{Z}_p \). From our investigation, we derive some identities of symmetry for the higher-order degenerate Euler polynomials.

### 2 Identities of symmetry for the higher-order degenerate Euler polynomials

In this section, we assume that \( \lambda, t \in \mathbb{C}_p \) with \( |\lambda t|_p < p^{-\frac{1}{r+1}} \). From [2], we can derive the following equation:

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{r+1} + \frac{r+1}{r} + \cdots + \frac{r}{1}} d\mu_{-1} (x_1) \cdots d\mu_{-1} (x_r)
\]

(10)

\[
= \left( \frac{2}{(1 + \lambda t)^{\frac{d}{dt}} + 1} \right)^r (1 + \lambda t)^{\frac{d}{dt}}
\]

\[
= \sum_{n=0}^{\infty} \mathcal{E}_n^{(r)} (x | \lambda) \frac{t^n}{n!}.
\]

Thus, by (11), we get, for \( n \geq 0 \),

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x | \lambda)_n d\mu_{-1} (x_1) \cdots d\mu_{-1} (x_r) = \mathcal{E}_n^{(r)} (x | \lambda),
\]

(11)

where

\[
(x | \lambda)_n = x (x - \lambda) \cdots (x - \lambda (n - 1)), \quad (n \geq 0).
\]

(12)

From [2], we note that

\[
(x | \lambda)_n = \sum_{l=0}^{n} S_1 (n, l) \lambda^{n-l} x^l, \quad (n \geq 0),
\]

(13)

where \( S_1 (n, l) \) is the Stirling number of the first kind.
From (3), we have, for $m \geq 0$,
\[
2 \sum_{l=0}^{n-1} (-1)^{n-1-l} (l \mid \lambda)_m
\]
\[
= \int_{Z_p} (x+n \mid \lambda)_m \; d\mu_{-1}(x) + (-1)^{n-1} \int_{Z_p} (x \mid \lambda)_m \; d\mu_{-1}(x).
\]

For $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$, we get
\[
2 \sum_{l=0}^{n-1} (-1)^l (l \mid \lambda)_m = E_m(n \mid \lambda) + E_m(\lambda), \quad (m \geq 0).
\]

Let us define $\tilde{S}_k(n \mid \lambda)$ as follows:
\[
\tilde{S}_k(n \mid \lambda) = \sum_{l=0}^{n} (-1)^l (l \mid \lambda)_k, \quad (k, n \geq 0).
\]
Then, we note that
\[
\lim_{\lambda \to 0} \tilde{S}_k(n \mid \lambda) = \sum_{l=0}^{n} (-1)^l l^k = \tilde{S}_k(n).
\]

Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{2}$. Then we get
\[
2 \int_{Z_p} (1 + \lambda t)^{\frac{n}{2}} \; d\mu_{-1}(x)
\]
\[
= \int_{Z_p} (1 + \lambda t)^{\frac{n-1}{2}} \; d\mu_{-1}(x) + \int_{Z_p} (1 + \lambda t)^{\frac{n}{2}} \; d\mu_{-1}(x)
\]
\[
= 2 \sum_{l=0}^{n-1} (-1)^l (1 + \lambda t)^{l} + \sum_{k=0}^{\infty} \tilde{S}_k(n-1 \mid \lambda) \frac{l^k}{k!}.
\]

Let $w_1, w_2 \in \mathbb{N}$, with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$. For $m \in \mathbb{N}$, we define
\[
K^{(m)}(w_1, w_2 \mid \lambda)
\]
\[
= \left(\frac{2}{1 + \lambda t^{\frac{n}{2}}} + 1\right)^m (1 + \lambda t^{\frac{n+1}{2}} (1 + \lambda t^{\frac{n}{2}} + 1)
\]
\[
\times \left(\frac{2}{1 + \lambda t^{\frac{n}{2}}} + 1\right)^m \frac{1}{2} (1 + \lambda t^{\frac{n+2}{2}} y).
\]
From (3), we note that
\[
K^{(m)}(w_1, w_2 | \lambda) = \frac{\int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1} (x_1 + \cdots + x_m + w_2 x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1 - 1} d\mu_{-1}(x)}
\]
(20)
\[
\times \int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1} (x_1 + \cdots + x_m + w_1 y) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m),
\]
where
\[
\int_{\mathbb{Z}_p} f(x_1, \ldots, x_m) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} f(x_1, \ldots, x_m) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m).
\]
(21)
It is easy to see that \(K^{(m)}(w_1, w_2 | \lambda)\) is symmetric in \(w_1\) and \(w_2\). Now, we observe that
\[
K^{(m)}(w_1, w_2 | \lambda) = (1 + \lambda)^{\sum w_1 - 1} \int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1} (x_1 + \cdots + x_m) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m)
\]
(22)
\[
\times \frac{\int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1} d\mu_{-1}(x_m)}{\int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1} d\mu_{-1}(x)}
\]
\[
\times (1 + \lambda t)^{\sum w_1} \int_{\mathbb{Z}_p^{m-1}} (1 + \lambda t)^{\sum w_1} (x_1 + \cdots + x_{m-1}) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_{m-1}).
\]
(23)
By (10), we get
\[
(1 + \lambda t)^{\sum w_2 x} \int_{\mathbb{Z}_p} (1 + \lambda t)^{\sum w_1} (x_1 + \cdots + x_m) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_m)
\]
(24)
\[
= \left( \frac{2}{(1 + \lambda t)^{\sum w_1 - 1} + 1} \right)^m (1 + \lambda t)^{\sum w_2 x}
\]
\[
= \sum_{n=0}^{\infty} \epsilon_n^{(m)} \left( \frac{w_2 x}{w_1} \right) \frac{\lambda}{w_1} \frac{w_1}{n!}.
\]
From (18), (22) and (23), we have
\[
K^{(m)}_t(w_1, w_2 | \lambda)
\]
(25)
\[
= \sum_{l=0}^{\infty} \epsilon_l^{(m)} \left( \frac{w_2 x}{w_1} \right) \frac{\lambda}{w_1} \frac{t^l}{l!} \sum_{k=0}^{\infty} \tilde{S}_k \left( \frac{w_1 - 1}{w_2} \right) \frac{w_2^2}{k!} t^k
\]
\[
\times \sum_{i=0}^{\infty} \epsilon_i^{(m-1)} \left( \frac{w_1 y}{w_2} \right) \frac{w_2^2}{l!} t^i
\]
In particular, if we take \( m \) and \( n \) with \( n \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( n \geq 0 \) and \( m \in \mathbb{N} \), we have

\[
\sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} \mathcal{E}_{n-j} \left( w_2 y \mid \frac{\lambda}{w_2} \right) S_k \left( w_1 - 1 \mid \frac{\lambda}{w_1} \right) E_{j-k}^{(m-1)} \left( w_1 y \mid \frac{\lambda}{w_1} \right) \frac{t^n}{n!}.
\]

On the other hand,

\[
K^{(m)} (w_1, w_2 \mid \lambda) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} w_1^j w_2^{n-j} \mathcal{E}_{n-j}^{(m)} \left( w_1 x \mid \frac{\lambda}{w_1} \right) \sum_{k=0}^{j} \mathcal{S}_k \left( w_2 - 1 \mid \frac{\lambda}{w_1} \right) E_{j-k}^{(m-1)} \left( w_2 y \mid \frac{\lambda}{w_1} \right) \frac{t^n}{n!}.
\]

Therefore, by (24) and (25), we obtain the following theorem.

**Theorem 1** For \( w_1, w_2 \in \mathbb{N} \), with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \), \( n \geq 0 \) and \( m \in \mathbb{N} \), we have

\[
\sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} \mathcal{E}_{n-j}^{(m)} \left( w_2 x \mid \frac{\lambda}{w_1} \right) \sum_{k=0}^{j} \mathcal{S}_k \left( w_1 - 1 \mid \frac{\lambda}{w_1} \right) E_{j-k}^{(m-1)} \left( w_1 y \mid \frac{\lambda}{w_1} \right) \frac{t^n}{n!}.
\]

Let \( y = 0 \) and \( m = 1 \) in Theorem 1. Then we have the following theorem.

**Theorem 2** For \( n \geq 0 \), \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 1 \), \( w_2 \equiv 1 \pmod{2} \), we have

\[
\sum_{j=0}^{n} \binom{n}{j} w_2^j w_1^{n-j} \mathcal{E}_{n-j} \left( w_2 x \mid \frac{\lambda}{w_1} \right) S_j \left( w_1 - 1 \mid \frac{\lambda}{w_2} \right)
\]

In particular, if we take \( w_2 = 1 \) in Theorem 2, then we obtain the following corollary.

**Corollary 3** For \( w_1 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( n \geq 0 \), we have

\[
\mathcal{E}_n \left( w_1 x \mid \lambda \right) = \sum_{j=0}^{n} \binom{n}{j} w_1^{n-j} \mathcal{E}_{n-j} \left( x \mid \frac{\lambda}{w_1} \right) S_j \left( w_1 - 1 \mid \lambda \right).
\]
we obtain the following theorem.

\[ K^{(m)}(w_1, w_2 | \lambda) \]

\[ = \left( \frac{2}{1 + \lambda t^{n+1}} \right)^m (1 + \lambda)^{\frac{w_1}{w_2} t^m + 1} \]

\[ \times \left( \frac{2}{1 + \lambda} \right)^{\frac{w_1}{w_2} t^{m+1} + 1} \left( \frac{2}{1 + \lambda} \right)^{m-1} (1 + \lambda)^{\frac{w_1}{w_2} t^{m-1} + 1} \]

\[ = \left( \frac{2}{1 + \lambda} \right)^{\frac{w_1}{w_2} t^{m+1} + 1} \sum_{i=0}^{w_1-1} (-1)^i (1 + \lambda)^{\frac{w_1}{w_2} t^m + \frac{w_1}{w_2} t^{m-i}} \]

\[ \times \sum_{i=0}^{\infty} E_i^{(m-1)} \left( w_1 y \left| \frac{\lambda}{w_1} \right. \right) w_2^{t^i} t^i \frac{n}{n!} \]

On the other hand, by the symmetric properties of \( K^{(m)}(w_1, w_2 | \lambda) \) in \( w_1 \) and \( w_2 \), we get

\[ K^{(m)}(w_1, w_2 | \lambda) \]

\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} w_2 w_1^{n-k} E_{n-k}^{(m-1)} \left( w_2 y \left| \frac{\lambda}{w_1} \right. \right) \]

\[ \times \sum_{i=0}^{w_2-1} (-1)^i E_k^{(m)} \left( w_1 y + w_1 y \left| \frac{\lambda}{w_2} \right. \right) \frac{n}{n!} \]

Therefore, by comparing the coefficients on the both sides of (26) and (27), we obtain the following theorem.

**Theorem 4** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 1, w_2 \equiv 1 \) (mod 2) and \( n \geq 0 \) and \( m \geq 1 \), we have

\[ \sum_{k=0}^{n} \binom{n}{k} w_1^k w_2^{n-k} E_{n-k}^{(m-1)} \left( w_1 y \left| \frac{\lambda}{w_2} \right. \right) \sum_{i=0}^{w_1-1} (-1)^i E_k^{(m)} \left( w_2 y + w_2 y \left| \frac{\lambda}{w_1} \right. \right) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} w_2^k w_1^{n-k} E_{n-k}^{(m-1)} \left( w_2 y \left| \frac{\lambda}{w_1} \right. \right) \sum_{i=0}^{w_2-1} (-1)^i E_k^{(m)} \left( w_1 y + w_1 y \left| \frac{\lambda}{w_2} \right. \right). \]
Let \( y = 0 \) and \( m = 1 \) in Theorem 4. Then we have the following corollary.

**Corollary 5** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 1, w_2 \equiv 1 \mod{2}, n \geq 0 \), we have

\[
\begin{align*}
\sum_{i=0}^{w_1-1} (-1)^i E_n \left( w_2 x + \frac{w_2}{w_1} \frac{\lambda}{w_1} \right) &= w_2^n \sum_{i=0}^{w_2-1} (-1)^i E_n \left( w_1 x + \frac{w_1}{w_2} \frac{\lambda}{w_2} \right).
\end{align*}
\]

Let us take \( w_2 = 1 \) in Corollary 5. Then we have

\[
\begin{align*}
\sum_{i=0}^{w_1-1} (-1)^i E_n \left( x + \frac{1}{w_1} \frac{\lambda}{w_1} \right) &= E_n (w_1 x | \lambda).
\end{align*}
\]

**References**

1. Araci, S., Acikgoz, M.: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. (Kyungshang) 22(3), 399–406 (2012)
2. Araci, S., Bagdasaryan, A., Özel, C., Srivastava, H.M.: New symmetric identities involving \( q \)-zeta type functions. Appl. Math. Inf. Sci. 8(6), 2803–2808 (2014). DOI 10.12785/amis/080616. URL http://dx.doi.org/10.12785/amis/080616
3. Bayad, A., Chikhi, J.: Apostol-Euler polynomials and asymptotics for negative binomial reciprocals. Adv. Stud. Contemp. Math. (Kyungshang) 24(1), 33–37 (2014)
4. Bayad, A., Kim, T.: Identities involving values of Bernstein, \( q \)-Bernoulli, and \( q \)-Euler polynomials. Russ. J. Math. Phys. 18(2), 131–143 (2011). DOI 10.1134/S1061920811020014. URL http://dx.doi.org/10.1134/S1061920811020014
5. Carlitz, L.: Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Math. 15, 51–88 (1979)
6. Duran, U., Acikgoz, M., Araci, S.: Symmetric identities involving weighed \( q \)-Genocchi polynomials under \( s_4 \). Proc. Jangjeon Math. 18(4), (in press) (2015)
7. He, Y.: Symmetric identities for Carlitz’s \( q \)-Bernoulli numbers and polynomials. Adv. Difference Equ. pp. 2013:246, 10 (2013). DOI 10.1186/1687-1847-2013-246. URL http://dx.doi.org/10.1186/1687-1847-2013-246
8. Kim, D.S., Kim, T.: Some identities of degenerate Euler polynomials arising from \( p \)-adic fermionic integrals on \( \mathbb{Z}_p \). Integral Transforms Spec. Funct. 26(4), 293–302 (2015). DOI 10.1080/10652469.2014.1002497. URL http://dx.doi.org/10.1080/10652469.2014.1002497
9. Kim, D.S., Lee, N., Na, J., Park, K.H.: Identities of symmetry for higher-order Euler polynomials in three variables (I). Adv. Stud. Contemp. Math. (Kyungshang) 22(1), 51–74 (2012)
10. Kim, D.S., Lee, N., Na, J., Park, K.H.: Abundant symmetry for higher-order Bernoulli polynomials (I). Adv. Stud. Contemp. Math. (Kyungshang) 23(3), 461–482 (2013)
11. Kim, T.: Symmetry of power sum polynomials and multivariate fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \). Russ. J. Math. Phys. 16(1), 93–96 (2009). DOI 10.1134/S1061920809010063. URL http://dx.doi.org/10.1134/S1061920809010063
12. Kim, T.: Symmetry properties of the generalized higher-order Euler polynomials. Proc. Jangjeon Math. Soc. 13(1), 13–16 (2010)
13. Kim, T., Dolgy, D.V., Jang, Y.S., Seo, J.J.: A note on symmetric identities for the generalized \( q \)-Euler polynomials of the second kind. Proc. Jangjeon Math. Soc. 17(3), 375–381 (2014)
14. Kim, T., Kim, D.S., Dolgy, D.V.: Degenerate $q$-Euler polynomials. Adv. Difference Equ. p. 2015:246 (2015). DOI 10.1186/s13662-015-0563-y. URL http://dx.doi.org/10.1186/s13662-015-0563-y

15. Kim, Y.H., Hwang, K.W.: Symmetry of power sum and twisted Bernoulli polynomials. Adv. Stud. Contemp. Math. (Kyungshang) 18(2), 127–133 (2009)

16. Moon, E.J., Rim, S.H., Jin, J.H., Lee, S.J.: On the symmetric properties of higher-order twisted $q$-Euler numbers and polynomials. Adv. Difference Equ. pp. Art. ID 765,259, 8 (2010)

17. Rim, S.H., Jeong, J.H., Lee, S.J., Moon, E.J., Jin, J.H.: On the symmetric properties for the generalized twisted Genocchi polynomials. Ars Combin. 105, 267–272 (2012)

18. Ryoo, C.S.: A note on the weighted $q$-Euler numbers and polynomials. Adv. Stud. Contemp. Math. (Kyungshang) 21(1), 47–54 (2011)

19. Şen, E.: Theorems on Apostol-Euler polynomials of higher order arising from Euler basis. Adv. Stud. Contemp. Math. (Kyungshang) 23(2), 337–345 (2013)