\section*{Abstract}

The proof of a recent result by Guido and Longo establishing the equivalence of the KMS-condition with complete $\beta$-boundedness \cite{2} is shortcut and generalized in such a way that a covariant version of the theorem is obtained.

Recently it was proved by Guido and Longo in \cite{2} that the KMS-condition at a finite nonnegative temperature is equivalent to a condition called complete $\beta$-boundedness, which imposes a bound on the number of degrees of freedom in certain phase space regions and is a weak form of the Buchholz-Wichmann nuclearity condition \cite{1}, very similar to the (weaker) Haag-Swieca compactness criterion \cite{3}. On the other hand, it was shown in \cite{4} that the KMS-condition at nonnegative temperature in some (a priori unknown) inertial frame is equivalent to a condition called complete semipassivity.

Both proofs are variations of the classical result by Pusz and Woronowicz \cite{5}, who proved that the KMS-condition at nonnegative temperature is equivalent to complete passivity.

It is of interest to investigate bounds on the efficiency of thermodynamic cycles in less generic settings than that of a stationary and homogeneous state, as the extent to which the passivity condition is violated can be considered as a kind of a nonequilibrium state’s distance from thermodynamic equilibrium.
As a first step on this path, the result found by Guido and Longo is
generalized below in such a way that it characterizes semipassive states as
well. As a spinoff of this result, a shortcut of the Guido-Longo argument is
given first.

In what follows, the algebra of observables of the system under consid-
eration is a von Neumann algebra \( \mathcal{M} \) on a Hilbert space \( \mathcal{H} \), and the state \( \omega \) of \( \mathcal{M} \) under consideration is induced by a cyclic unit vector \( \Omega \). The time
evolution is generated by a selfadjoint operator \( H \) with the property that
\[
e^{itH} \mathcal{M} e^{-itH} = \mathcal{M} \quad \text{and} \quad H\Omega = 0.
\]
We fix a parameter \( \beta \geq 0 \) that will, even-
tually, estimate the inverse temperature of \( \omega \).

**Definition 1** The state \( \omega \) is called \( \beta \)-bounded with bound 1 if the linear
space \( \mathcal{M}\Omega \) is a subspace of the domain of \( e^{-\beta H} \) and if the set \( e^{-\beta H} \mathcal{M}_1 \Omega \)
consists of vectors with lengths \( \leq 1 \) where \( \mathcal{M}_1 := \{ A \in \mathcal{M} : \| A \| \leq 1 \} \).

\( \omega \) is called **completely \( \beta \)-bounded** if for each \( n \in \mathbb{N} \), the state \( \omega^\otimes n \) on
the algebra \( \mathcal{M} \otimes \cdots \otimes \mathcal{M} \) is \( \beta \)-bounded with bound 1.

**Theorem 2 (Guido, Longo)** \( \omega \) is completely \( \beta \)-bounded if and only if it
is a ground state or a KMS-state at an inverse temperature \( \geq 2\beta \).

**Proof.** The condition is sufficient by Lemma 1.2 in [2]. It remains to prove
that it is necessary.

If \( E \) denotes the orthogonal projection onto the closure of the subspace
\( \mathbf{h} := \mathcal{M}^\prime \Omega \) and if \( \Delta \) denotes the modular operator of \( \Omega \) with respect to the
von Neumann algebra \( \mathcal{N} \) obtained by restricting the elements of \( \mathcal{M} \) to \( \mathbf{h} \)
(note that \( \Omega \) is not only cyclic, but also separating with respect to this von
Neumann algebra on \( \mathbf{h} \)), then \( \beta \)-boundedness of \( \omega \) implies, by Cor. 1.8 in
[2], that

\[
e^{-2\beta H} \leq 1 + \Delta E.
\]

If \( \omega \) is faithful, then this implies that each point \( (\eta, \kappa) \) in the joint spec-
trum \( \sigma_{H,K} \) of the operators \( H \) and \( K := \log(\Delta) \) satisfies

\[
e^{-2\beta\eta} \leq 1 + e^\kappa,
\]

and as \( \omega \) is completely \( \beta \)-bounded, it follows by the arguments used in
[3] that this inequality must hold for the elements of the additive group
generated by the elements of \( \sigma_{H,K} \) as well.

Ineq. (1) can be rewritten as

\[
\eta \geq -\frac{1}{2\beta} \log(1 + e^\kappa) =: f(\kappa),
\]
and this estimate separates $\sigma_{H,K}$ from a region containing an open cone. Each subgroup in the admitted region must, therefore, be a subgroup of a one-dimensional subspace, and in particular, $\sigma_{H,K}$ and the additive subgroup it generates must be subsets of such a one-dimensional space $X$. As $f(\kappa)$ is defined everywhere, this subspace cannot be the $\eta$-axis, so $H = -\alpha K$ for some $\alpha \in \mathbb{R}$.

Since $f(\kappa)$ tends to zero as $\kappa \to -\infty$, $\alpha$ must be nonnegative, and since $f(\kappa)$ tends to $-\frac{\alpha}{2\beta}$ as $\kappa \to +\infty$, it follows that $\alpha \leq 1/(2\beta)$, so $\omega$ is a KMS-state at an inverse temperature $\geq 2\beta$ if $\alpha \neq 0$, and if $\alpha = 0$, then $\omega$ trivially is a ground state of $H = 0$.

It remains to consider the case that $\omega$ is not faithful. Then the elements of the space $h^\perp = (1 - E)H$ are eigenvectors of $\Delta E$ with the eigenvalue zero. As $\mathcal{M}'$ is invariant under the adjoint action of $e^{itH}$, the space $h$ is invariant under $e^{itH}$, so $H|_{h^\perp}$ is a self-adjoint operator in $h^\perp$, whose spectral projections are restrictions of the corresponding spectral projections of $H$, respectively. But this implies that $\sigma_{H,E}$ contains some point of the form $(\eta, 0)$.

Now let $(\eta', \delta')$ be an arbitrary point in $\sigma_{H,\Delta E}$. As $\omega$ satisfies complete $\beta$-boundedness, each point of the form $(n\eta' + \eta, n\delta' \cdot 0)$, $n \in \mathbb{N}$, must satisfy Eq. (1) as well (cf. [5, 4]), so $2\beta(n\eta' + \eta) \geq 0$ for all $n \in \mathbb{N}$, which can be only if $\eta' \geq 0$.

So we have proved that $\eta' \geq 0$ for all $(\eta', \delta') \in \sigma_{H,E}$, which implies $H \geq 0$ (cf. Lemma B.2 in [4]).

The above proof does not only provide a more direct argument than the original proof, it also admits a generalization of the theorem in the spirit of Thm. 3.3 in [4].

To this end, assume that $H$ generates, together with $s$ self-adjoint operators $P_1, \ldots, P_s$ collected in a vector operator $\mathbf{P}$, a strongly continuous representation of the $1 + s$-dimensional spacetime translation group that leaves the vector $\Omega$ invariant. $\omega$ is stationary in all inertial frames, whereas in the presence of matter, it can be a thermodynamic equilibrium state in that matter’s rest frame only. The reason is that the condition of passivity, which is a consequence of the Second Law, is violated in the other frames.

The appropriate weakening of the passivity condition is semipassivity. In the following definition, $U_t(\mathcal{M})$ denotes the group of all unitary operators in $\mathcal{M}$ that can be connected to the unit operator by a norm-continuous path of unitary operators in $\mathcal{M}$.

**Definition 3** $\omega$ is called semipassive if there exists a constant $E \geq 0$ such
that

\[-\langle W\Omega, HW\Omega \rangle \leq \mathcal{E}\langle W\Omega, |P|W\Omega \rangle\]

for each \(W \in \mathcal{U}_1(\mathcal{M})\) with \([H, W] \in \mathcal{M}\) and \([P_1, W], \ldots, [P_s, W] \in \mathcal{M}\).

Each constant \(\mathcal{E}\) satisfying this condition is called an efficiency bound of \(\omega\). \(\omega\) is called completely semipassive if all its tensorial powers are semipassive with respect to the same efficiency bound.

We recall Thm. 3.3 from [4]:

**Theorem 4** \(\omega\) is completely semipassive with efficiency bound \(\mathcal{E} \geq 0\) if and only if there exists a \(u \in \mathbb{R}^s\) with \(|u| \leq \mathcal{E}\) such that \(\omega\) is a ground state or a KMS-state (at finite \(\beta \geq 0\)) with respect to \(H + uP\).

This result describes a most generic example of a nonequilibrium state, and the question is whether bounds on the power of a cyclic process could be of interest in less generic situations. As far as such investigations are concerned, it is an obstacle of the above definition of semipassivity that the invariance of \(\omega\) is part of the definition and its motivation. While the problem addressed in Thm. 3.3 is nontrivial only if \(\omega\) is invariant under all spacetime translations, it would be of interest whether the semipassivity condition can be subdivided into this invariance property plus some additional condition that may be meaningful in other situations as well. Such a condition is semi-\(\beta\)-boundedness.

**Definition 5** The state \(\omega\) is called semi-\(\beta\)-bounded if there exists a damping factor \(\mathcal{E} \geq 0\) such that the linear space \(\mathcal{M}\omega\) is a subspace of the domain of \(e^{-\beta(H+\mathcal{E}|P|)}\) and the set \(e^{-\beta(H+\mathcal{E}|P|)}\mathcal{M}_1\omega\) consists of vectors with length \(\leq 1\).

It is called completely semi-\(\beta\)-bounded if for each \(n \in \mathbb{N}\), the state \(\omega^\otimes n\) on the algebra \(\mathcal{M} \otimes \cdots \otimes \mathcal{M}\) is semi-\(\beta\)-bounded with respect to one fixed damping factor \(\mathcal{E} \geq 0\).

As \(|P|\) is a positive operator, the operator \(e^{-\beta\mathcal{E}|P|}\) is bounded and provides an additional damping term, so \(\beta\)-boundedness implies semi-\(\beta\)-boundedness for all \(\mathcal{E} \geq 0\), i.e., semi-\(\beta\)-boundedness is the weaker assumption.

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1 As a quadratic form, the commutator of \(H\) and \(W\) is defined on the domain of \(H\). The condition \([H, W] \in \mathcal{M}\) means that this quadratic form is bounded and that its associated bounded operator is an element of \(\mathcal{M}\). \([P_1, W], \ldots, [P_s, W] \in \mathcal{M}\) is to be read accordingly.
Theorem 6 A stationary and homogeneous state $\omega$ is completely semi-$\beta$-bounded if and only if there exists a $u \in \mathbb{R}^s$ with $|u| \leq \mathcal{E}$ such that $\omega$ is a ground state or a KMS-state at an inverse temperature $\geq 2\beta$ with respect to $H + uP$.

Proof. As in the proof of Thm. 2, the only nontrivial part is the proof that the condition is necessary.

One checks that the proof of Cor. 1.8 in [2] still works if one replaces $H$ by $H + \mathcal{E}|P|$, and one obtains

\[ e^{-2\beta(H+\mathcal{E}|P|)} \leq 1 + \Delta E, \tag{2} \]

where $\Delta$ and $E$ are defined as in the proof of Thm. 4. Again, we distinguish the cases that $\omega$ is faithful and not faithful:

Lemma 7 If $\omega$ is faithful, then there exists a $u \in \mathbb{R}^s$ such that either

(i) $H + uP = 0$, or

(ii) $\omega$ is a KMS-state at an inverse temperature $\geq 2\beta$ with respect to $H + uP$.

Proof. Ineq. (2) implies that for each $(\eta, k, \kappa) \in \sigma_{H,P,K}$, one has

\[ \eta + \mathcal{E}|k| \geq \frac{1}{2\beta} \ln(1 + e^\kappa), \tag{3} \]

which expells the joint spectrum of $H$, $P$, and $K$ from a region containing an open cone. As in [3] it follows that $\sigma_{H,P,K}$ is a subset of a subspace $X$ of $\mathbb{R}^{s+2}$ with codimension $\geq 1$ whose elements satisfy Ineq. (3).

If $H$ is a linear function of $P$, then there exists a $u \in \mathbb{R}^s$ such that $H + uP = 0$. Inserting this into Ineq. (3), one finds that

\[ -uk + \mathcal{E}|k| \geq \frac{1}{2\beta} \ln(1 + e^\kappa) \]

whenever the pair $(k, \kappa)$ is in the joint spectrum of $P$ and $K$, and by complete semiboundedness, one also obtains

\[ -nuk + n\mathcal{E}|k| \geq \frac{1}{2\beta} \ln(1 + e^{n\kappa}) \tag{4} \]

for all $n \in \mathbb{N}$. If $J$ denotes the modular conjugation associated with $\mathcal{M}$ and $\Omega$, then $JHJ = -H$, $JPJ = -P$, and $JKJ = -K$ by elementary Tomita-Takesaki theory, so Ineq. (3) holds for all $n \in \mathbb{Z}$. But this immediately entails $|u| \leq \mathcal{E}$, proving Alternative (i) of the statement.
There remains the case that $H$ is not a linear function of $P$.

As $X$ cannot contain the $\kappa$-axis by Ineq. (3), $K$ is a linear function of $H$ and $P$, so there exists an $\alpha \in \mathbb{R}$ and a $v \in \mathbb{R}^s$ such that

$$K = -\alpha H + vP.$$  \hfill (5)

The vector $v$ is unique up to a component that is perpendicular to the smallest linear subspace $Y$ of $\mathbb{R}^s$ containing the joint spectrum of the components of $P$, so $v$ can be chosen in $Y$.

If $vP = 0$, then $K = -\alpha H$, and Ineq. (3) implies that $\alpha \leq 2\beta$, so $H$ generates a KMS-dynamics at an inverse temperature $\geq 2\beta$.

In the remaining case that $vP \neq 0$, the unit vector $e_v := |v|^{-1}v$ is in $Y$. If $\alpha \neq 0$, then Eq. (5) and the assumption that $H$ is not a function of $P$ entail $K \neq 0$, so for each $\kappa > 0$ and each $\lambda > 0$, one has $(\eta(\lambda, \kappa), \lambda e_v, \kappa) \in X$, where

$$\eta(\lambda, \kappa) := -\frac{1}{\alpha}(\kappa - \lambda e_v) = -\frac{1}{\alpha}(\kappa - \lambda |v|).$$

Ineq. (3) yields

$$-\frac{\kappa}{\alpha} - \lambda \left(\frac{|v|}{\alpha} - E\right) \geq -\frac{1}{2\beta} \ln(1 + e^\kappa)$$

for all $\kappa, \lambda > 0$, so $\alpha \leq 2\beta$ and $|\frac{\kappa}{\alpha}| \leq E$, and defining $u := \frac{v}{\alpha}$, one finds that $H + uP$ generates a KMS-dynamics at an inverse temperature $\geq 2\beta$, and one arrives at Alternative (ii) of the statement.

It remains to consider the case that $\alpha = 0$, i.e., that $K = vP$. As $H$ is not a linear function of $P$, while $K = vP$, $H$ cannot be a linear function of $K$ and $P$, so $X$ must contain the $\eta$-axis. But if $(\eta, k, \kappa) \in \sigma_{H,P,K}$, then so is $(\eta', k, \kappa)$ for all $\eta' \in \mathbb{R}$, and Ineq. (3) implies

$$\eta' + E|k| \geq -\frac{1}{2\beta} \ln(1 + e^{\epsilon k})$$

for all $\eta' \in \mathbb{R}$, which cannot be.

\[\square\]

**Lemma 8** If $\omega$ is not faithful, then there exists a $u \in \mathbb{R}^s$ with $|u| \leq E$ such that $\omega$ is a ground state with respect to $H + uP$.

**Proof.** The elements of the space $h^\perp = (1 - E)\mathcal{H}$ are eigenvectors of $\Delta E$ with the eigenvalue zero. As $\mathcal{M}'$ is invariant under the adjoint action of $e^{itH}$, the space $h$ is invariant under $e^{itP_1}$, $e^{itP_2}$, $\ldots$, $e^{itP_s}$, so $H|_{h^\perp}$ and
$P_1|_{\mathfrak{h}^\perp}, \ldots, P_s|_{\mathfrak{h}^\perp}$ are self-adjoint operators in $\mathfrak{h}^\perp$, whose spectral projections are restrictions of the corresponding spectral projections of $H$ and $P$, respectively. But this implies that $\sigma_{H,P,\Delta E}$ contains some point of the form $(\eta,k,0)$.

Now let $(\eta',k',\delta')$ be an arbitrary point in $\sigma_{H,P,\Delta E}$. As $\omega$ satisfies complete semi-$\beta$-boundedness, each point of the form $(\eta+n\eta',k+nk',0\cdot n\delta')$, $n \in \mathbb{N}$, must satisfy Ineq. (3) as well, so $\eta + n\eta' + \mathcal{E}|k + nk'| \geq 0$ for all $n \in \mathbb{N}$, which can be only if $\eta' \geq \mathcal{E}|k'|$.

So we have proved that
\begin{align}
\eta' \geq \mathcal{E}|k'|
\end{align}
for all $(\eta',k',\delta') \in \sigma_{H,P,\Delta E}$ and, hence, for all $(\eta',k') \in \sigma_{H,P}$ (use, e.g., Lemma B.2 in [4]). By complete semi-$\beta$-boundedness, the corresponding estimate should hold for all tensorial powers of $\omega$ as well, which implies that the joint spectrum of $H$ and $P$ is a subset of a sub-semigroup of $\mathbb{R}^{1+s}$ all of whose elements satisfy Ineq. (6). Such a semigroup must be a subset of a half space all of whose elements satisfy Ineq. (6). This implies that there exists a $u \in \mathbb{R}^s$ with $|u| \leq \mathcal{E}$ such that $H + uP$ is a positive operator (cf. Lemma B.2 in [4]).

With these two lemmas, the proof of Thm. 6 is complete as well.

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References

[1] Buchholz, D., Wichmann, E. H.: Causal Independence and the Energy-Level Density of States in Local Quantum Field Theory, *Commun. Math. Phys.* **106**, 321-344 (1986)

[2] Guido, D., Longo, R.: Natural Energy Bounds in Quantum Thermodynamics, *Commun. Math. Phys.* **218**, 513-536 (2001)

[3] Haag, R. Swieca, J. A.: When Does a Quantum Field Theory Describe Particles, *Commun. Math. Phys.* **1**, 308-320 (1965)
[4] Kuckert, B.: Covariant Thermodynamics of Quantum Systems, preprint hep-th/0107236, to appear in Ann. Phys. (N. Y.)

[5] Pusz, W., Woronowicz, S. L.: Passive States and KMS States for General Quantum Systems, Commun. Math. Phys. 58, 273-290 (1978)