The cohomology rings of homogeneous spaces

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Abstract

Let $G$ be a compact connected Lie group and $K$ a closed connected subgroup. Assume that the order of any torsion element in the integral cohomology of $G$ and $K$ is invertible in a given principal ideal domain $\mathbb{k}$. It is known that in this case the cohomology of the homogeneous space $G/K$ with coefficients in $\mathbb{k}$ and the torsion product of $H^*(BK)$ and $\mathbb{k}$ over $H^*(BG)$ are isomorphic as $\mathbb{k}$-modules. We show that this isomorphism is multiplicative and natural in the pair $(G, K)$ provided that 2 is invertible in $\mathbb{k}$. The proof uses homotopy Gerstenhaber algebras in an essential way. In particular, we show that the normalized singular cochains on the classifying space of a torus are formal as a homotopy Gerstenhaber algebra.

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1. Introduction

In 1950, Cartan gave the first uniform description of the cohomology of homogeneous spaces of Lie groups. Using a differential-geometric approach, he established the following result for a compact connected Lie group $G$ and a closed connected subgroup $K \subset G$ [7, Theorem 5].

**Theorem 1.1 (Cartan).** There is an isomorphism of graded algebras

$$H^*(G/K; \mathbb{R}) \cong \text{Tor}_{H^*(BG; \mathbb{R})}^*(\mathbb{R}, H^*(BK; \mathbb{R})).$$

A topological way to look at this formula is the following: One has a fibre bundle

$$G/K \hookrightarrow EG/K = BK \to BG,$$

(1.1)
and there is an associated Eilenberg–Moore spectral sequence

\[ E_2 = \text{Tor}^*_{H^*_{(BG; \mathbb{R})}}(\mathbb{R}, H^*(BK; \mathbb{R})) \Rightarrow H^*(G/K). \] (1.2)

In this language, Cartan’s result says that the spectral sequence collapses at the second page and that the product on that page agrees with the one on \( H^*(G/K) \).

The real cohomology of the classifying space of a connected Lie group is a polynomial algebra on even-degree generators. An obvious question is whether a result analogous to Cartan’s holds for other principal ideal domains \( k \) for which \( H^*(BG) \) and \( H^*(BK) \) have this property. An equivalent condition is that the orders of the torsion subgroups of \( H^*(G; \mathbb{Z}) \) and \( H^*(K; \mathbb{Z}) \) are invertible in \( k \), and we assume this throughout. It holds in many cases, for example, for \( U(n), SU(n) \) and \( Sp(n) \) over any \( k \), and for \( SO(n) \) and \( Spin(n) \) if \( 2 \) is invertible in \( k \).

In his 1952 thesis, Borel studied the case where \( G \) and \( K \) have the same rank and established a multiplicative isomorphism \[ H^*(G/K) \cong H^*(BK) / H^>0(BG) \cdot H^*(BK). \] (1.3)

The Leray–Hirsch theorem then implies that \( H^*(BK) \) is free over \( H^*(BG) \), so that Borel’s formula can be written as

\[ H^*(G/K) \cong \text{Tor}^*_H(BG)(k, H^*(BK)). \] (1.4)

Another step forward was achieved in 1968 by Baum, who proved that for field coefficients, the Eilenberg–Moore spectral sequence collapses at the second page for any \( G \) and \( K \) satisfying a certain ‘deficiency condition’ \[ 3, \text{Theorem 7.4}. \] This yields an additive isomorphism of the form (1.4). Shortly afterwards, May \[ 22, \text{p. 335} \] announced that the Eilenberg–Moore spectral sequence collapses for any \( k \), independently of the deficiency condition. Details appeared in Gugenheim–May \[ 15, \text{Theorem A} \], where additionally the extension problem was solved. This gives the following result.

**Theorem 1.2.** If \( H^*(BG) \) and \( H^*(BK) \) are polynomial algebras on even-degree generators, then there is an isomorphism of graded \( k \)-modules

\[ H^*(G/K) \cong \text{Tor}^*_H(BG)(k, H^*(BK)). \]

Munkholm \[ 26, \text{Theorem} \] provided a different proof of the isomorphism, and Husemoller–Moore–Stasheff \[ 18, \text{Theorem IV.8.2} \] a further one for the collapse of the Eilenberg–Moore spectral sequence. For field coefficients, yet another proof was published by Wolf \[ 35, \text{Theorem} \] extended Theorem 1.2 to generalized homogeneous spaces (see also Remark 12.10), and Barthel–May–Riehl \[ 1 \] put Gugenheim–May’s approach into a model-theoretic framework.

Apart from one special case \[ 3, \text{Corollary 7.5} \], the product structure is not addressed in any of the works after Borel. In their introduction \[ 15, \text{p. viii} \], Gugenheim and May remark:

*Multiplicatively, however, we are left with an extension problem; our results will compute the associated graded algebras of \( H^*(G/K) \) and [...] with respect to suitable filtrations. Refinements of our algebraic theory could conceivably yield precise procedures for the computation of these cohomology algebras. When \( k = \mathbb{Z}_2 \), there are examples where the extensions are non-trivial. There are no such examples known when \( k \) is a field of characteristic \( \neq 2 \).*

The examples alluded to are the projective unitary groups \( PU(n) = U(n)/U(1) \) for \( n \equiv 2 \pmod{4} \), see Remark 12.9. To the author’s knowledge, no progress on the multiplicative

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\[ ^1 \text{We have aligned the original notation with ours.} \]
structure has been made since these words were written. In the present paper, we prove the following:

**Theorem 1.3.** Assume that $2$ is invertible in $k$. If $H^*(BG)$ and $H^*(BK)$ are polynomial algebras, then there is an isomorphism of graded $k$-algebras

$$H^*(G/K) \cong \text{Tor}_{H^*(BG)}^*(k, H^*(BK)),$$

natural with respect to maps of pairs $(G, K) \to (G', K')$.

The central difficulty one faces when proving an isomorphism of the form (1.4) is the lack of commutativity of the singular cochain algebra. At some point, one has to pass from cochains to cohomology, and unlike in the case of differential forms, the assignment of representatives $a_i \in C^*(BG)$ to generators $x_i \in H^*(BG)$ does not extend to a morphism of differential graded algebras (dgas). To address this, all approaches after Baum resorted to some ‘up to homotopy’ structure, as suggested by Stasheff–Halperin [32, p. 575].

Munkholm, for example, further develops the idea of strongly homotopy commutative (shc) algebras introduced by Stasheff–Halperin. The only additional ingredient he then needs is that both $BG$ and $BK$ have polynomial cohomology, and his result holds more generally for the fibre of bundles where both the total space and the base have this property.

In contrast to this, Husemoller–Moore–Stasheff, Gugenheim–May and Wolf rely on the existence of a maximal torus $T \subset K$ to reduce the problem to that of a homogeneous space $G/T$. This was already done by Baum [3], who observed that $H^*(G/K)$ injects into $H^*(G/T)$, compare Lemma 12.4(ii). A crucial result in this direction, also used by Wolf, is the following [15, Theorem 4.1].

**Theorem 1.4** (Gugenheim–May). There is a quasi-isomorphism of dgas $C^*(BT) \to H^*(BT)$ annihilating all $\cup_1$-products.

We are going to extend Theorem 1.4 to homotopy Gerstenhaber algebras (hgas), which were introduced by Voronov–Gerstenhaber [34]. An hga structure on a dga $A$ is essentially a family of operations $E_k: A^\otimes (k+1) \to A$ that allow to define a product on the bar construction $BA$ compatible with the coalgebra structure. Based on a result of Baues [2], the former authors also noted that singular cochain algebras are endowed with this structure [11]. In this case, the first hga operation $E_1$ is the usual $\cup_1$-product, up to sign. We strengthen the Gugenheim–May result as follows.

**Theorem 1.5.** There is a quasi-isomorphism of dgas $C^*(BT) \to H^*(BT)$ annihilating all hga operations. In particular, $C^*(BT)$ is formal as an hga.

See Theorem 9.6. This seems to be the first time that the hga formality of a non-trivial space is established. The quasi-isomorphism from Theorem 1.5 actually annihilates even more operations, including the ones identified by Kadeishvili [19] to construct a $\cup_1$-product on $BC^*(BT)$. The only exception is the $\cup_2$-product on $C^*(BT)$, but we can show that also $\cup_2$-products of cocycles are in the kernel of the formality map provided that $2$ is invertible in $k$ (Proposition 9.7). We call an hga having a $\cup_2$-product as well as the other additional operations ‘extended’.

The following result from the companion paper [10] allows us to combine Theorem 1.5 with Munkholm’s techniques, see Theorem 6.3.

**Theorem 1.6.** Any extended hga is naturally an shc algebra in the sense of Munkholm.
In a nutshell, our strategy to prove Theorem 1.3 is the following: By the Eilenberg–Moore theorem, $H^*(G/K)$ is naturally isomorphic to the differential torsion product

$$\text{Tor}_{C^*(BG)}(k, C^*(BK)).$$

(1.5)

Kadeishvili–Saneblidze [20] observed that the hga structure on cochains permits to define a product on the one-sided bar construction underlying (1.5); the Eilenberg–Moore isomorphism then becomes multiplicative. Imitating mostly Munkholm, we first construct a $k$-module isomorphism

$$H^*(\Theta): \text{Tor}_{H^*(BG)}(k, H^*(BK)) \to \text{Tor}_{C^*(BG)}(k, C^*(BK)), \tag{1.6}$$

where we use the shc algebra structure given by Theorem 1.6. In order to show that our map is multiplicative and natural, we look at the composition

$$\text{Tor}_{C^*(HG)}(k, H^*(BK)) \xrightarrow{H^*(\Theta)} \text{Tor}_{C^*(BG)}(k, C^*(BK)) \xrightarrow{\sim} \text{Tor}_{C^*(BG)}(k, H^*(BT)). \tag{1.7}$$

The last map involves the quasi-isomorphism from Theorem 1.5 in the same way as Wolf applied the formality map constructed by Gugenheim–May. This leads to a dramatic simplification of the formulas and allows us to complete the proof of Theorem 1.3, see Section 12.

Along the way we exhibit an explicit homotopy between the two possible definitions of a tensor product of two $A_\infty$-maps (Proposition 4.1).

2. Preliminaries

2.1. Differential algebra

We work over a fixed commutative ring $k$ with unit, which will be assumed to be a principal ideal domain in Sections 8.5, 12 and 13. Since we will mostly deal with cohomological complexes, we assume a cohomological grading throughout this review section. The identity map on a complex $M$ is denoted $1_M$. The suspension map on a complex is denoted by $s$ and the desuspension by $s^{-1}$. All tensor products are over $k$ unless otherwise indicated.

Given two $\mathbb{Z}$-graded complexes $A$ and $B$, the complex $\text{Hom}(A, B)$ consists in degree $n \in \mathbb{Z}$ of all linear maps $f: A \to B$ raising degrees by $n$. The differential of such a map is

$$d(f) = df - (-1)^n f d.$$  

(2.1)

We write

$$T = T_{A: B}: A \otimes B \to B \otimes A, \quad a \otimes b \mapsto (-1)^{|a||b|} b \otimes a \tag{2.2}$$

for the transposition of factors in a tensor product. This illustrates the Koszul sign rule, according to which swapping two objects of degrees $m$ and $n$ incurs the sign $(-1)^{mn}$. Another incarnation of it is the definition

$$f \otimes g: A \otimes B \to C \otimes D, \quad a \otimes b \mapsto (-1)^{|a||b|} f(a) \otimes g(b) \tag{2.3}$$

of the tensor product of two maps $f: A \to C$ and $g: B \to D$. This implies that for maps $f_i: A_i \to A_{i+1}$ and $g_i: B_i \to B_{i+1}$, $i = 1, 2$, we have

$$(f_1 \otimes g_1)(f_2 \otimes g_2) = (-1)^{|f_2||g_1|} f_1 f_2 \otimes g_1 g_2. \tag{2.4}$$

We refer to [26, §§1.1, 1.2, 1.11] for the definitions of differential graded algebras (dgas) and dga maps as well as for differential graded coalgebras (dgc's), dgc maps and coalgebra homotopies. By an "ideal" $a$ of a dga $A$, we mean a two-sided differential ideal $a \triangleleft A$. We write
augmentations as $\varepsilon$ and coaugmentations as $\eta$. The augmentation ideal of a dga $A$ is denoted by $\bar{A}$; for any $a \in A$, we define $\bar{a} = a - \varepsilon(a) \in \bar{A}$. A dga $A$ is connected if it is $\mathbb{N}$-graded and $\eta_A : k \to A^0$ is an isomorphism; it is simply connected if additionally $A^1 = 0$. A connected or simply connected dgc $C$ is defined similarly.

For $n \geq 0$, we write

$$\mu^{[n]}_A : A^\otimes n \to A$$

(2.5)

for the iterated multiplication of a dga $A$, so that $\mu^{[0]}_A = \eta_A$, $\mu^{[1]}_A = 1_A$ and $\mu^{[2]}_A = \mu_A$. The iterations $\Delta^{[n]}$ are defined analogously. A dgc $C$ is cocomplete if for any $c \in C$ there is an $n \geq 0$ such that $(1_C - \varepsilon_C)^\otimes_n \Delta^{[n]}(c) = 0$. Any connected dgc is cocomplete.

Given two ideals $a \triangleleft A$ and $b \triangleleft B$ where $A$ and $B$ are dgas, we define the ideal

$$a \boxtimes b = a \otimes B + A \otimes b \triangleleft A \otimes B$$

(2.6)

as well as

$$a^{\boxtimes 0} = 0 \triangleleft A^{\boxtimes 0} = k, \quad a^{\boxtimes 1} = a, \quad a^{\boxtimes (n+1)} = a^{\boxtimes n} \boxtimes a \triangleleft A^{\boxtimes (n+1)}$$

(2.7)

inductively for $n \geq 1$.

We will make heavy use of the (reduced) bar construction

$$BA = \bigoplus_{k \geq 0} B_k A, \quad B_k A = (s^{-1}A)^{\otimes k}$$

(2.8)

of an augmented dga $A$, which is a cocomplete coaugmented dgc, connected if $A$ is simply connected, see [18, Section II.3] or [26, §1.6]. We write $1_{BA} = 1 \in k = B_0 A$ for the counit of $BA$. The canonical map

$$t_A : BA \to B_1 A = s^{-1}A \xrightarrow{s} A \hookrightarrow A$$

(2.9)

is a twisting cochain in the sense of the following definition.

For an augmented dga $A$ and a coaugmented dgc $C$, the complex $\text{Hom}(C, A)$ is an augmented dga with cup product

$$f \cup g = \mu_A (f \otimes g) \Delta_C,$$

(2.10)

unit element $\eta_A \varepsilon_C$ and augmentation $\varepsilon(f) = (\varepsilon_A f \eta_C)(1)$. Note that for $f$, $g$ as before and any dgc map $k : B \to C$, we have

$$(f \cup g) \circ k = (f \circ k) \cup (g \circ k).$$

(2.11)

A twisting cochain is an element $t \in \text{Hom}^1(C, A)$ such that

$$d(t) = t \cup t,$$

(2.12)

$$\varepsilon_A t = 0 \quad \text{and} \quad t \eta_C = 0.$$

(2.13)

If $C$ is cocomplete, then the assignment $f \mapsto t_A f$ sets up a bijection between the dgc maps $C \to BA$ and the twisting cochains $C \to A$, compare [26, Proposition 1.9].

Example 2.1. Let $A$ and $B$ be augmented dgas. The shuffle map

$$\nabla = \nabla_{A,B} : BA \otimes BB \to B(A \otimes B)$$

(2.14)

is the dgc map with associated twisting cochain $t_A \otimes \eta_B \varepsilon_B + \eta_A \varepsilon_B A \otimes t_B$,

$$[a_1|\ldots|a_k] \otimes [b_1|\ldots|b_l] \mapsto \begin{cases} a_1 \otimes 1 & \text{if } k = 1 \text{ and } l = 0, \\ 1 \otimes b_1 & \text{if } k = 0 \text{ and } l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2.15)
The shuffle map is associative and also commutative in the sense that the diagram
\[
\begin{array}{ccc}
BA \otimes BB & \xrightarrow{\nabla_{A,B}} & B(A \otimes B) \\
T_{B,A,BB} & \downarrow & \downarrow BT_{A,B} \\
BB \otimes BA & \xrightarrow{\nabla_{B,A}} & B(B \otimes A)
\end{array}
\] (2.16)

commutes.

If \( A \) is commutative, then the composition
\[
\mu_{BA} = B\mu_A \nabla_{A,A} : BA \otimes BA \to BA
\] (2.17)
turns \( BA \) into a \( dg \) bialgebra, that is, into a coaugmented \( dgc \) with an associative product that is a morphism of \( dgc \).

An element \( h \in \text{Hom}_0(C, A) \) is a twisting cochain homotopy from the twisting cochain \( t: C \to A \) to the twisting cochain \( u: C \to A \), in symbols \( h: t \simeq u \), if
\[
d(h) = t \cup h - h \cup u,
\] (2.18)
\[
\varepsilon_A h = \varepsilon_C \quad \text{and} \quad h\eta_C = \eta_A.
\] (2.19)

Assume again that \( C \) is cocomplete, and let \( f, g: C \to BA \) be two \( dgc \) maps. The assignment \( h \mapsto 1 + t_A h \) then is a bijection between the coalgebra homotopies from \( t_A f \) to \( g \) and the twisting cochain homotopies from \( t_A f \) to \( t A g \), see [26, §1.11].

Let \( h: C \to A \) be a twisting cochain homotopy, and let \( a \triangleleft A \) be an ideal. If \( h \) is congruent to \( 1 = \eta_A \varepsilon_C \) modulo \( a \triangleleft A \), we say that \( h \) as well as the associated coalgebra homotopy \( C \to BA \) is \( a\)-trivial.\(^\dagger\) By the first normalization condition (2.19) any twisting cochain homotopy \( h: C \to A \) is \( \bar{A} \)-trivial.

**Lemma 2.2.** Let \( a \triangleleft A \) be an ideal, and let \( C \) be a cocomplete \( dgc \). Being related by an \( a\)-trivial homotopy is an equivalence relation among twisting cochains \( C \to A \). More precisely:

(i) let \( h: t \simeq u \) and \( k: u \simeq v \) be \( a\)-trivial twisting cochain homotopies. Then \( h \cup k \) is an \( a\)-trivial homotopy from \( t \) to \( v \);

(ii) let \( h: t \simeq u \) be an \( a\)-trivial twisting cochain homotopy Then \( h \) is invertible in \( \text{Hom}_0(C, A) \), and its inverse
\[
h^{-1} = \sum_{n=0}^{\infty} (1 - h)^{\cup n} : C \to A
\]
is an \( a\)-trivial homotopy from \( u \) to \( t \).

In particular, we may unambiguously speak of an ‘\( a\)-trivial homotopy between twisting cochains \( t \) and \( u \)’ without specifying the direction of the homotopy.

**Proof.** The first part follows immediately from the definition of the cup product. Apart from the obvious \( a\)-triviality, the second claim is [26, §1.12]. \( \square \)

\(^\dagger\)The importance of this notion as well as that of \( b\)-strict shm maps defined in (3.12) below will only become evident in Section 12. Readers may wish to ignore them on a first reading.
2.2. Notation

The Koszul signs \((2.2)\) and \((2.3)\) quickly tend to clutter more complex formulas, as do the arguments of multilinear maps. For instance, in Section 6.1 we will encounter the formula

\[
E_k(a_1a_2; b_1, \ldots, b_k) = \sum_{l+m=k} (-1)^\varepsilon E_l(a_1; b_1, \ldots, b_l) E_m(a_2; b_{l+1}, \ldots, b_k)
\]  

\((2.20)\)

for certain multilinear operations \(E_k: A^{\otimes(k+1)} \rightarrow A\) of degree \(-k\) on a dga \(A\) and elements \(a_1, a_2, b_1, \ldots, b_k \in A\). Here the sign exponent is

\[
\varepsilon = |a_2| (|b_1| + \cdots + |b_l|) - m \left( |a_1| + |b_1| + \cdots + |b_l| \right),
\]

\((2.21)\)

and it is completely determined by the Koszul sign rule. Alternatively, one could dispense with the arguments and write \((2.20)\) more concisely as the identity of functions

\[
E_k (\mu_A \otimes 1^{\otimes k}) = \sum_{l+m=k} \mu_A (E_l \otimes E_m) \pi_l,
\]

\((2.22)\)

where \(\mu_A\) is the multiplication in \(A\) and \(\pi_l: A^{\otimes(k+2)} \rightarrow A^{\otimes(k+2)}\) the permutation of factors corresponding to the cycle \((l + 2, l + 1, \ldots, 2) \in S_{k+2}\). The advantage of such a notation is that it is easy to compute the differential of a map because compositions as well as tensor products of maps obey the graded Leibniz rule. For example, the differential of a term \(\mu_A (E_l \otimes E_m) \pi_l\) is

\[
d(\mu_A (E_l \otimes E_m) \pi_l) = \mu_A \left( d(E_l) \otimes E_m \right) \pi_l + (-1)^l \mu_A \left( E_l \otimes d(E_m) \right) \pi_l.
\]

\((2.23)\)

However, we feel that it is very hard to grasp the meaning of formulas of the form \((2.22)\) and \((2.22)\). We write maps in the form \((2.20)\), but without all Koszul signs involving the degrees of variables. To indicate that these signs need to be added, we write \(\overset{\varepsilon}{=}\) instead of an equality sign. For example, the identity \((2.20)\) is written as

\[
E_k (a_1a_2; b_1, \ldots, b_k) \overset{\varepsilon}{=} \sum_{l+m=k} E_l(a_1; b_1, \ldots, b_l) E_m(a_2; b_{l+1}, \ldots, b_k).
\]

\((2.24)\)

In this case, the sign to be added is exactly \((2.21)\). Stated differently, we really describe formulas of the form \((2.22)\), but use variables to specify the permutations of arguments like \(\pi_l\) that are to be applied before the maps that are spelt out.

To make the notation even more compact, we abbreviate sequences of variables with a bullet, as in

\[
E_k (a_1a_2; b_{\bullet}) \overset{\varepsilon}{=} \sum_{l+m=k} E_l(a_1; b_{\bullet}) E_m(a_2; b_{\bullet}).
\]

\((2.25)\)

The number of elements in each sequence is to be inferred from the maps and may be zero. Since \(E_k\) takes \(k\) arguments in addition to the leading \(a_1a_2\), the first occurrence of \(b_{\bullet}\) above stands for \(k\) arguments \(b_1, \ldots, b_k\). Throughout a product the order of ‘bullet variables’ is always maintained. Thus, the first \(b_{\bullet}\) on the right-hand side of \((2.25)\) stands for \(b_1, \ldots, b_l\) (as \(E_l\) takes \(l\) arguments in addition to \(a_1\)) and the last \(b_{\bullet}\) for the \(m\) arguments \(b_{l+1}, \ldots, b_k\). A tensor product like \(a_{\bullet} \otimes b_{\bullet}\) indicates a sequence of tensors \(a_1 \otimes b_1, a_2 \otimes b_2, \ldots\).

Composition of maps is distributed over tensor products, so that the linear order in which the maps appear in a formula is maintained when translating between our and the corresponding function notation. For example, the formula

\[
F(a, b) \overset{f_1(f_2(a)) \otimes g_1(g_2(b))}{\overset{\varepsilon}{=}} (2.26)
\]
stands for the identity of functions \( F = f_1 f_2 \otimes g_1 g_2 \). Note that our \( \sim \) notation does not incorporate the Koszul sign (2.4) involving only maps and no variables.

3. Strongly homotopy multiplicative maps

Our discussion is based on the treatment in \([26, \S 3.1]\) and \([35, \text{Section 1 (c)}]\).

Let \( A \) and \( B \) be augmented dgas. By definition, an \( A_\infty \)-map or strongly homotopy multiplicative (shm) map \( f: A \Rightarrow B \) is a twisting cochain \( f: BA \to B \). We write the corresponding dgc map as \( Bf: BA \to BB \). It is given by

\[
Bf([a_1|\ldots|a_n]) = \sum_{k \geq 0} \sum_{i_1 + \cdots + i_k = n} [f[a_1] \cdots [a_{i_1}] \cdots f[a_{n-k+1}] \cdots [a_n]],
\]

(3.1)

where the second sum is over all decompositions of \( n \) into \( k \) positive integers.

Following Munkholm \([26, \text{Appendix}]\), we define for \( n \geq 0 \) the map \( f_n: \bar{A} \otimes^n (s-1) \otimes^n B_n A \rightarrow B \)

(3.2)

of degree \( 1 - n \) and extend it to \( A \otimes^n \) by setting

\[
f(1) = 1,
\]

(3.3)

\[
f_n(a_1 \otimes \cdots \otimes a_n) = 0 \text{ if } n \geq 2 \text{ and } a_k = 1 \text{ for some } k.
\]

(3.4)

The twisting cochain conditions (2.12) and (2.13) for \( f \) translate into

\[
f(0) = \epsilon_B f_1 = 0,
\]

(3.5)

\[
d(f_n)(a_\bullet) \equiv \sum_{k=1}^{n-1} (-1)^k \left( f_k(a_\bullet) f_{n-k}(a_\bullet) - f_{n-1}(a_\bullet, a_k a_{k+1}, a_\bullet) \right)
\]

(3.6)

for all \( n \geq 1 \). In (3.6) we have used the symbol \( \sim \) to indicate the Koszul sign and also the notation ‘\( a_\bullet \)’ to denote a sequence of \( a \)-variables, ordered by their indices.

We call a family of multilinear functions

\[
f_n: A \otimes^n \rightarrow B
\]

(3.7)

of degree \( 1 - n \) satisfying (3.3)–(3.6) a twisting family. Twisting families correspond bijectively to shm maps \( A \Rightarrow B \) and therefore to dgc maps \( BA \rightarrow BB \). The dgc map determined by the twisting family \( f_n \) can be read off from (3.1), using the identity

\[
f([a_1|\ldots|a_n]) = (-1)\varepsilon f_n(a_1 \otimes \cdots \otimes a_n)
\]

(3.8)

for \( n \geq 0 \), where

\[
\varepsilon = \sum_{k=1}^{n} (n-k)(|a_k|-1).
\]

†We prefer the term ‘shm map’ used by Munkholm over the nowadays more popular terminology ‘\( A_\infty \)-map’ because it pairs better with the ‘shc algebras’ to be introduced in Section 5.

‡This definition leads to a sign convention different from Wolf’s [35, p. 319].
It follows from (3.6) that the component \( f_1 : A \to B \) is a chain map which is multiplicative up to homotopy since
\[
d(f_2) = f_1(\mu_A - \mu_B (f_1 \otimes f_1)).
\]
The map
\[
H^*(f) := H^*(f_1) : H^*(A) \to H^*(B)
\]
therefore is a morphism of graded algebras.

Any dga morphism \( f : A \to B \) induces an shm map \( \tilde{f} : A \to B \) with \( \tilde{f}_1 = f \) and \( \tilde{f}_n = 0 \) for \( n \geq 2 \). We call such an shm map \emph{strict}. Note that \( H^*(\tilde{f}) = H^*(f) \) in this case. We will not distinguish between a dga map and its induced strict shm map.

More generally, we say that an shm map \( f : A \to B \) is \emph{\( b \)-strict} for some \( b < B \) if
\[
f(n) \equiv 0 \pmod{b} \quad \text{for all } n \geq 2.
\]
Then \( f \) is 0-strict if and only if it is strict, and every \( f : A \to B \) is \( B \)-strict. Any \( b \)-strict shm map \( f : A \to B \) induces a strict map \( A \to B/b \).

A twisting cochain homotopy \( h : f \simeq g \) from an shm map \( f : A \to B \) to another shm map \( g : A \to B \) is called an \emph{shm homotopy}. Based on \( h \) we define the maps
\[
h(n) = h(s^{-1})^n : \tilde{A}^n \to B
\]
of degree \( -n \) for \( n \geq 0 \) and extend them to \( A^\otimes n \) by
\[
h(n)(a_1 \otimes \cdots \otimes a_n) = 0 \quad \text{if } a_k = 1 \text{ for some } k.
\]
The normalization conditions (2.19) mean
\[
h(0) = \eta_B \quad \text{and} \quad \varepsilon_B h(n) = 0 \quad \text{for } n \geq 1,
\]
and condition (2.18) is equivalent to
\[
d(h(n))(a_\bullet) \equiv \sum_{k=1}^{n-1} (-1)^k h(n-1)(a_\bullet, a_k a_{k+1}, a_\bullet)
\]
\[
\quad + \sum_{k=0}^{n} \left( f(k)(a_\bullet) h(n-k)(a_\bullet) - (-1)^k h(k)(a_\bullet) g(n-k)(a_\bullet) \right)
\]
for all \( n \geq 0 \). In particular, \( h(1) : g_1 \simeq f_1 \), so that \( H^*(f) = H^*(g) \).

We call a family of multilinear functions
\[
h(n) : A^\otimes n \to B
\]
of degree \( -n \) satisfying (3.14)–(3.16) a \emph{twisting homotopy family} from the twisting family \( f_\bullet \) to \( g_\bullet \). Twisting homotopy families correspond bijectively to homotopies between twisting cochains. We also write \( Bh : BA \to BB \) for the coalgebra homotopy induced by the twisting cochain homotopy \( h : BA \to B \).

A twisting homotopy family \( h_\bullet \) as above is called \emph{\( b \)-trivial} for some \( b \subset B \) if the twisting cochain homotopy \( BA \to B \) is so. Equivalently,
\[
h(n) \equiv 0 \pmod{b} \quad \text{for all } n \geq 1.
\]
Let \( f : A \Rightarrow B \) and \( g : B \Rightarrow C \) be shm maps. We define the composition
\[
g \circ f : A \Rightarrow C
\]
to be the twisting cochain \( g \mathbf{Bf} \) associated to the dgc map \( \mathbf{BgBf} : \mathbf{B} A \to \mathbf{B} C \). Using (3.1) and (3.8), one sees that the corresponding twisting family is given by

\[
(g \circ f)_{(n)}(a_\bullet) = \sum_{k \geq 1} \sum_{1 + \cdots + k = n} (-1)^k g(f(k)(a_\bullet), \ldots, f(i_k)(a_\bullet))
\]

for \( n \geq 0 \), where the second sum is over all decompositions of \( n \) into \( k \) positive integers and

\[
\varepsilon = \sum_{s=1}^{k} (k - s)(i_s - 1).
\]

Note that the composition of an shm map with the canonical twisting cochain (2.9) of a bar construction, considered as another shm map, is tautological in the sense that \( t_B \circ f = f \) and \( g \circ t_B = g \).

The composition of an shm map and an shm homotopy is similarly defined as the shm homotopy associated to the composition of the corresponding maps between bar constructions.

**Lemma 3.1.**

(i) Let \( f : A \Rightarrow B \) be a \( \mathfrak{b} \)-strict shm map, and let \( g : B \Rightarrow C \) a \( \mathfrak{c} \)-strict shm map. If \( g(1)(b) \subset c \), then \( g \circ f \) is \( \mathfrak{c} \)-strict.

(ii) Let \( h : C \to A \) be an \( \mathfrak{a} \)-trivial twisting cochain homotopy, and let \( f : A \to B \) be a \( \mathfrak{b} \)-strict shm map. If \( f(1)(a) \subset \mathfrak{b} \), then \( f \circ h \) is \( \mathfrak{b} \)-trivial.

(iii) Let \( h : C \to A \) be an \( \mathfrak{a} \)-trivial twisting cochain homotopy, and let \( g : D \to C \) be a map of coaugmented dgc. Then \( h \circ g \) is \( \mathfrak{a} \)-trivial.

**Proof.** The first two claims are readily verified, and the last one is trivial. \( \square \)

### 4. Tensor products of shm maps

In this section, \( A, B, A' \) and \( B' \) denote augmented dgas. We write \( a_\bullet \otimes b_\bullet \) for a sequence \( a_1 \otimes b_1, a_2 \otimes b_2, \ldots \) in \( A \otimes B \) whose length is given by the context.

Let \( f : A \Rightarrow A' \) be an shm map, and let \( g : B \Rightarrow B' \) be a dga map. Then

\[
(f \otimes g)_{(n)}(a_\bullet \otimes b_\bullet) \cong f_{(n)}(a_\bullet) \otimes g_\mu^{[n]}(b_\bullet)
\]

is a twisting family, hence defines an shm map

\[
f \otimes g : A \otimes B \Rightarrow A' \otimes B'.
\]

If \( h \) is an \( \mathfrak{a} \)-trivial homotopy from \( f \) to another shm map \( \tilde{f} \), then

\[
(h \otimes g)_{(n)}(a_\bullet \otimes b_\bullet) \cong h_{(n)}(a_\bullet) \otimes g_\mu^{[n]}(b_\bullet)
\]

defines an \( \mathfrak{a} \otimes B \)-trivial shm homotopy \( h \otimes g \) from \( f \otimes g \) to \( \tilde{f} \otimes g \).

Similarly, if \( f : A \to A' \) is a dga map and \( g : B \Rightarrow B' \) an shm map, then

\[
(f \otimes g)_{(n)}(a_\bullet \otimes b_\bullet) \cong f_\mu^{[n]}(a_\bullet) \otimes g_{(n)}(b_\bullet)
\]

defines an shm map

\[
f \otimes g : A \otimes B \Rightarrow A' \otimes B'.
\]

If \( h \) is a \( \mathfrak{b} \)-trivial homotopy from \( g \) to another shm map \( \tilde{g} \), then

\[
(f \otimes h)_{(n)}(a_\bullet \otimes b_\bullet) \cong f_\mu^{[n]}(a_\bullet) \otimes h_{(n)}(b_\bullet)
\]

defines an \( A \otimes \mathfrak{b} \)-trivial shm homotopy \( f \otimes h \) from \( f \otimes g \) to \( f \otimes \tilde{g} \).

Now let both \( f : A \Rightarrow A' \) and \( g : B \Rightarrow B' \) be shm maps. Then the two shm maps

\[
(f \otimes 1_{B'}) \circ (1_A \otimes g) \quad \text{and} \quad (1_{A'} \otimes g) \circ (f \otimes 1_B)
\]
are not equal in general. In fact, for any $n \geq 0$, one has
\[
((f \otimes 1) \circ (1 \otimes g))_{(n)}(a_\ast \otimes b_\ast) = \sum_{l \geq 1} \sum_{j_1 + \cdots + j_l = n} (-1)^\varepsilon F \otimes G
\]
where the sum is over all decompositions of $n$ into $l$ positive integers and
\[
F = f(l) \left( \mu^{[j_1]}(a_\ast), \ldots, \mu^{[j_1]}(a_\ast) \right),
\]
\[
G = \mu[g](b_\ast), \ldots, g(j_l)(b_\ast)),
\]
\[
\varepsilon = \sum_{l=1}^{l} (l - t)(j_t - 1),
\]
compare (3.20) and (3.21), while
\[
((1 \otimes g) \circ (f \otimes 1))_{(n)}(a_\ast \otimes b_\ast) = \sum_{k \geq 1} \sum_{i_1 + \cdots + i_k = n} (-1)^\varepsilon F \otimes G,
\]
where the sum is analogously over all decompositions of $n$ into $k$ positive integers and
\[
F = \mu^{[k]}(f(i_1)(a_\ast), \ldots, f(i_k)(a_\ast)),
\]
\[
G = \mu^g(\mu^{[i_1]}(b_\ast), \ldots, \mu^{[i_k]}(b_\ast)),
\]
\[
\varepsilon = \sum_{s=1}^{k} (s - 1)(i_s - 1).
\]
Note that if $f$ or $g$ is strict, then (4.8) and (4.12) coincide and agree with the formulas given previously. Following Munkholm [26, Proposition 3.3], we define
\[
f \otimes g = (f \otimes 1) \circ (1 \otimes g)
\]
in the general case and compare it to the other composition.

**Proposition 4.1.** Assume that $f$ is $a$-strict for some $a \lhd A'$ and that $g$ is $b$-strict for some $b \lhd B'$. Then the two shm maps
\[
f \otimes g = (f \otimes 1) \circ (1 \otimes g) \quad \text{and} \quad (1 \otimes g) \circ (f \otimes 1)
\]
are homotopic via an $a \otimes b$-trivial homotopy. In particular, if $f$ or $g$ is strict, then the two compositions agree.

**Proof.** Instead of using Munkholm’s theory of trivialized extensions [26, Section 2], we exhibit an explicit homotopy from $(1 \otimes g) \circ (f \otimes 1)$ to $(f \otimes 1) \circ (1 \otimes g)$. It is given by $h_{(0)} = \eta_{A'} \otimes \eta_{B'}$ and
\[
h_{(n)}(a_\ast \otimes b_\ast) = \sum_{k,l \geq 1} \sum_{i_1 + \cdots + i_k + j_1 + \cdots + j_l = n} (-1)^\varepsilon F \otimes G,
\]
for $n \geq 1$, where the second sum is over all decompositions of $n$ into $k + l$ positive integers,
\[
F = \mu^{[k]}(f(i_1)(a_\ast), \ldots, f(i_{k-1})(a_\ast), f(i_k + 1)(a_\ast), \mu^{[j_1]}(a_\ast), \ldots, \mu^{[j_l]}(a_\ast)),
\]
\[
G = \mu^{[l]}(g(k+j_1)(b_\ast), \ldots, g(j_{k-1})(b_\ast), g(j_k)(b_\ast), \ldots, g(j_l)(b_\ast)),
\]
\[
\varepsilon = \sum_{s=1}^{k} s(i_s - 1) + \sum_{t=1}^{l} (l - t)(j_t - 1) + k(l - 1) + 1.
\]
Verifying that $h$ is a homotopy as claimed is lengthy, but elementary, see Section A. That the homotopy is $a \otimes b$-trivial follows from the assumptions on $f$ and $g$ and the inequalities $i_k + l \geq 2$ and $k + j_1 \geq 2$. In particular, $h$ takes values in $A' \otimes B' \supset A' \otimes B'$ since $f$ and $g$ are $A$-strict and $B$-strict, respectively. This proves the second part of the normalization condition (3.15).

Let us verify the condition (3.14): Assume that $a_i = b_i = 1$ for some $i$ and consider a term $F \otimes G$ of the sum (4.17). Let $m$ be the index such that $a_i$ appears is the $m$th $f$-term of $F$. If $i_m > 1$ or $s = k$, this term vanishes by (3.6). Otherwise, the product inside $g(k+j_1)$ containing $b_m$ is $b_m$ itself, so that this term vanishes again by (3.6). In any case, we have $F \otimes G = 0$.

The last part of the statement is the special case $a = 0$ or $b = 0$ and has already been observed above.

Omitting the arguments $a_* \otimes b_*$, formula (4.17) looks as follows in small degrees.

$$h_{(1)} = 0,$$
$$h_{(2)} \equiv -f_{(2)}(a_1, a_2) \otimes g_{(2)}(b_1, b_2),$$
$$h_{(3)} \equiv -f_{(1)}(a_1) f_{(2)}(a_2, a_3) \otimes g_{(3)}(b_1, b_2, b_3) + f_{(3)}(a_1, a_2, a_3) \otimes g_{(2)}(b_1, b_2) g_{(1)}(b_3)$$
$$+ f_{(3)}(a_1, a_2, a_3) \otimes g_{(2)}(b_1 b_2, b_3) + f_{(2)}(a_1, a_2 a_3) \otimes g_{(3)}(b_1, b_2, b_3).$$

Remark 4.2. The summands appearing in (4.8) and (4.12) are in bijection with the vertices of an $(n-1)$-dimensional cube. For example, the vertex of $[0,1]^{n-1}$ corresponding to the decomposition $i_1 + \cdots + i_k = n$ is given by

$$\left(0, \ldots, 0, 1, \ldots, 0, 1, 0, \ldots, 0\right).$$

Similarly, the summands appearing in (4.17) are in bijection with the edges of an $(n-1)$-dimensional cube. Here the summand corresponding to the decomposition $i_1 + \cdots + i_k + j_1 + \cdots + j_l = n$ is identified with the edge

$$\left(0, \ldots, 0, 1, \ldots, 0, 1, 0, \ldots, 0, * , 0, \ldots, 0, 1, 0, \ldots, 0\right),$$

where * denotes the free parameter.

Corollary 4.3. Let $f_1 \colon A_0 \Rightarrow A_1$, $f_2 \colon A_1 \Rightarrow A_2$, $g_1 \colon B_0 \Rightarrow B_1$ and $g_2 \colon B_1 \Rightarrow B_2$ be shm maps. Assume that $f_1$ is $a_1$-trivial, $f_2$ $a_2$-trivial and $g_2$ $b_2$-trivial and that $(f_2)_{(1)}(a_1) \subset a_2$ for ideals $a_1 \subset A_1$, $a_2 \subset A_2$ and $b_2 \subset B_2$. Then the two shm maps

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) \quad \text{and} \quad (f_2 \circ f_1) \otimes (g_2 \circ g_1)$$

are homotopic via an $a_2 \otimes b_2$-trivial homotopy. If $f_1$ or $g_2$ are strict, then the two maps agree.

Proof. This follows by writing the maps as

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \otimes 1) \circ (1 \otimes g_2) \circ (f_1 \otimes 1) \circ (1 \otimes g_1)$$

and applying Proposition 4.1 and Lemma 3.1. The second identity above is a consequence of the formulas (4.1), (4.4) and (3.20).
**Lemma 4.4.** The shuffle map is natural with respect to shm maps. In other words, the diagram

\[
\begin{array}{c}
BA \otimes BB \xrightarrow{\nabla} B(A \otimes B) \\
\downarrow_{Bf \otimes Bg} \\
BA' \otimes BB' \xrightarrow{\nabla} B(A' \otimes B')
\end{array}
\]

commutes for all shm maps \( f : A \Rightarrow A' \) and \( g : B \Rightarrow B' \).

**Proof.** Since all morphisms involved are dgc maps and the bar construction cocomplete, it suffices to compare the associated twisting cochains. Let \( a \otimes b = [a_1|\ldots|a_k] \otimes [b_1|\ldots|b_l] \in B_kA \otimes B_lB \).

Assume \( g = 1_B \). Then both twisting cochains vanish on \( a \otimes b \) if \( k \geq 1 \) and \( l \geq 1 \). For \( l = 0 \), both twisting cochains yield \( f(a) \otimes 1 \), and for \( k = 0 \) they give \( 1 \otimes b_1 \) if \( l = 1 \) and 0 otherwise, compare Example 2.1.

The case \( f = 1_A \) is analogous, and the general case follows by combining the two and using the definition (4.16). \( \square \)

Now let \( f_i : A_i \Rightarrow B_i \) be a family of shm maps, \( 1 \leq i \leq m \). Generalizing (4.16), we define the shm map

\[
f_1 \otimes \cdots \otimes f_m = (f_1 \otimes 1 \otimes \cdots \otimes 1) \circ (1 \otimes f_2 \otimes 1 \otimes \cdots \otimes 1) \circ \cdots \circ (1 \otimes \cdots \otimes 1 \otimes f_m). \tag{4.28}
\]

If one of the maps is instead an shm homotopy \( f_i = h \), we use the same definition. The resulting map is an shm homotopy in this case. We observe that this convention is compatible with the definitions (4.3) and (4.6).

**Lemma 4.5.** Let \( h : A \Rightarrow B \) be an shm homotopy.

(i) For any dga map \( f : A' \Rightarrow B' \), we have

\[
h \otimes f = (1_B \otimes f) \circ (h \otimes 1_{A'}) \quad \text{and} \quad f \otimes h = (1_{B'} \otimes h) \circ (f \otimes 1_A).
\]

(ii) For any shm map \( g : C \Rightarrow A \) and any dga \( D \), we have

\[
(h \otimes 1_D) \circ (g \otimes 1_D) = (h \circ g) \otimes 1_D.
\]

**Proof.** The first part follows from inspection of the formulas (4.3) and (4.6). The second claim additionally uses that formula (3.20) remains valid for the shm homotopy \( h \) instead of the shm map \( f \). \( \square \)

5. **Strongly homotopy commutative algebras**

Let \( A \) be an augmented dga. According to Stasheff–Halperin [32, Definition 8], \( A \) is a strongly homotopy commutative (shc) algebra if:

(i) the multiplication map \( \mu_A : A \otimes A \Rightarrow A \) extends to an shm morphism

\[
\Phi : A \otimes A \Rightarrow A,
\]

where ‘extending’ means that \( \Phi(1) = \mu_A \).
Munkholm [26, Definition 4.1] additionally requires the following:

(i) The map $\eta_A$ is a unit for $\Phi$, that is,
\[
\Phi \circ (1_A \otimes \eta_A) = \Phi \circ (\eta_A \otimes 1_A) = 1_A : A \Rightarrow A.
\]

(ii) The map $\eta_A$ is a unit for $\Phi$, that is,
\[
\Phi \circ (1_A \otimes \eta_A) = \Phi \circ (\eta_A \otimes 1_A) = 1_A.
\]

(iii) The map $\Phi$ is homotopy associative, that is,
\[
\Phi \circ (\Phi \otimes 1) \simeq \Phi \circ (1 \otimes \Phi) : A \otimes A \otimes A \Rightarrow A.
\]

(iv) The map $\Phi$ is homotopy commutative, that is,
\[
\Phi \circ T_{A,A} \simeq \Phi : A \otimes A \otimes A \Rightarrow A.
\]

Whenever we speak of an shc algebra, we mean one satisfying all four properties unless otherwise indicated. Any commutative dga is canonically an shc algebra.

Let $A$ and $B$ be shc algebras, and let $b \triangleleft B$. A morphism of shc algebras is an shm map $f : A \Rightarrow B$ such that the diagram

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
\Phi_A & & \Phi_B \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes up to homotopy.† It is called $b$-strict if it is so as an shm map, and $b$-natural if there is a $b$-trivial homotopy making (5.1) commute.

Recall from [26, Proposition 4.2] that the tensor product of two shc algebras $A$ and $B$ is again an shc algebra with structure map

\[
\Phi_{A \otimes B} : A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes 1 \otimes \Phi_{A \otimes B}} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes 1 \otimes \Phi_{B \otimes A}} A \otimes B.
\]

The following is a variant of the result just cited.

**Lemma 5.1.** Let $f_i : A_i \rightarrow B_i$ be strict $b_i$-natural shc maps for $i = 1, 2$. Their tensor product $f_1 \otimes f_2 : A_1 \otimes A_2 \rightarrow B_1 \otimes B_2$ is a strict $b_1 \otimes b_2$-natural shc map.

**Proof.** We have to show that there is a $b_1 \otimes b_2$-trivial homotopy for the diagram (5.1), which in the present setting reads

\[
\begin{array}{ccc}
A_1 \otimes A_2 \otimes A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_2 \otimes f_1 \otimes f_2} & B_1 \otimes B_2 \otimes B_1 \otimes B_2 \\
\downarrow 1 \otimes 1 \otimes \Phi & & \downarrow 1 \otimes 1 \otimes \Phi \\
A_1 \otimes A_1 \otimes A_2 \otimes A_2 & \xrightarrow{f_1 \otimes f_1 \otimes f_2 \otimes f_2} & B_1 \otimes B_1 \otimes B_2 \otimes B_2 \\
\downarrow 1 \otimes 1 \otimes \Phi & & \downarrow 1 \otimes 1 \otimes \Phi \\
A_1 \otimes A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_1 \otimes f_2} & B_1 \otimes B_1 \otimes B_2 \\
\downarrow \Phi \otimes 1 & & \downarrow \Phi \otimes 1 \\
A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_2} & B_1 \otimes B_2.
\end{array}
\]

Since $f_1$ and $f_2$ are strict, the top square commutes. If $h_i$ denotes a $b_i$-trivial naturality homotopy for $f_i$, then $f_1 \otimes f_1 \otimes h_2$ is a $B_1 \otimes B_1 \otimes b_2$-natural homotopy making the middle...

†Munkholm also requires the identity $f \circ \eta_A = \eta_B$. Given the normalization condition (2.13), this holds automatically as both maps necessarily represent 0 as twisting cochains $k = Bk \rightarrow B$. 

diagram commute, and \( h_1 \otimes f_2 \) is a \( b_1 \otimes B_2 \)-natural one for the bottom square. Hence the cup product of

\[
(\Phi \otimes 1) \circ (f_1 \otimes f_1 \otimes h_2) \circ (1 \otimes T \otimes 1) \tag{5.4}
\]

and

\[
(h_1 \otimes f_2) \circ (1 \otimes 1 \otimes \Phi) \circ (1 \otimes T \otimes 1) \tag{5.5}
\]
yields the required homotopy by Lemmas 3.1 and 2.2(i).

Let \( A \) be an shc algebra with structure map \( \Phi : A \otimes A \Rightarrow A \). Following [26, p. 30], we define the shm map

\[
\Phi^{[n]} : A^{\otimes n} \Rightarrow A \tag{5.6}
\]

for \( n \geq 0 \) by

\[
\Phi^{[0]} = \eta_A, \quad \Phi^{[1]} = 1_A, \quad \Phi^{[2]} = \Phi, \quad \Phi^{[n+1]} = \Phi \circ (\Phi^{[n]} \otimes 1_A) \tag{5.7}
\]

for \( n \geq 2 \). Note that

\[
(\Phi^{[n]})^{(1)} = \mu^{[n]}_A. \tag{5.8}
\]

If \( \Phi \) is \( a \)-strict for some ideal \( a \triangleleft A \), then so is \( \Phi^{[n]} \) for any \( n \geq 0 \) by Lemma 3.1(i).

For the next result, compare [26, Proposition 4.6].

**Lemma 5.2.** Let \( A \) and \( B \) be shc algebras with ideals \( a \triangleleft A \) and \( b \triangleleft B \). Assume that \( \Phi_A \) is \( a \)-strict and \( \Phi_B \) \( b \)-strict. Let \( f : A \Rightarrow B \) be a \( b \)-strict and \( b \)-natural map of shc algebras such that \( f^{(1)}(a) \subset b \). Then the diagram

\[
\begin{array}{ccc}
A^{\otimes n} & \xrightarrow{f^{\otimes n}} & B^{\otimes n} \\
\downarrow \Phi^{[n]} & & \downarrow \Phi^{[n]} \\
A & \xrightarrow{f} & B
\end{array}
\]

commutes up to a \( b \)-trivial homotopy for any \( n \geq 0 \).

**Proof.** The claim is trivial for \( n \leq 2 \). Assume it proven for \( n \) and consider the diagram

\[
\begin{array}{ccc}
A^{\otimes n} \otimes A & \xrightarrow{1^{\otimes n} \otimes f} & A^{\otimes n} \otimes B & \xrightarrow{f^{\otimes n} \otimes 1} & B^{\otimes n} \otimes B \\
\downarrow \Phi^{[n]}_A \otimes 1 & & \downarrow \Phi^{[n]}_A \otimes 1 & & \downarrow \Phi^{[n]}_B \otimes 1 \\
A \otimes A & \xrightarrow{1 \otimes f} & A \otimes B & \xrightarrow{f \otimes 1} & B \otimes B \\
\downarrow \Phi_A & & \downarrow \Phi_B & & \downarrow \Phi_B \\
A & \xrightarrow{f} & B
\end{array} \tag{5.9}
\]

Since \( \Phi^{[n]}_A \) is \( a \)-strict, the top left square commutes up to an \( a \otimes B \)-trivial homotopy by Proposition 4.1. The composition of this homotopy with \( \Phi_B \circ (f \otimes 1) \) is \( b \)-trivial by Lemma 3.1 because \( \Phi_B \) and \( f \) are \( b \)-strict and \( f^{(1)}(a) \subset b \). By induction, the top right square commutes up to a \( b \otimes B \)-trivial homotopy, whose composition with \( \Phi_B \) is \( b \)-trivial. The bottom rectangle finally commutes up to a \( b \)-trivial homotopy since \( f \) is \( b \)-natural. The claim follows. \( \square \)
6. Homotopy Gerstenhaber algebras

6.1. Definition of an hga

Let $A$ be an augmented dga. We say that $A$ is a homotopy Gerstenhaber algebra (homotopy G-algebra, hga) if it is equipped with certain operations

$$E_k: A \otimes A^\otimes k \to A, \quad a \otimes b_1 \otimes \cdots \otimes b_k \mapsto E_k(a; b_1, \ldots, b_k)$$

(6.1)
of degree $|E_k| = -k$ for $k \geq 1$. To state the properties they satisfy, it is convenient to use the additional operation $E_0 = 1_A$. All $E_k$ with $k \geq 1$ take values in the augmentation ideal $\bar{A}$ and vanish if any argument is equal to 1. For $k \geq 1$ and all $a, b_1, \ldots, b_k \in A$, one has

$$d(E_k)(a; b_*) \equiv b_1 E_{k-1}(a; b_*) + \sum_{m=1}^{k-1} (-1)^m E_{k-1}(a; b_* b_m b_{m+1}, b_*) + (-1)^k E_{k-1}(a; b_*) b_k.$$  

(6.2)

For $k \geq 0$ and all $a_1, a_2, b_1, \ldots, b_k \in A$, one has

$$E_k(a_1 a_2; b_*) \equiv \sum_{k_1 + k_2 = k} E_{k_1}(a_1; b_*) E_{k_2}(a_2; b_*),$$

(6.3)

where the sum is over all decompositions of $k$ into two non-negative integers. Finally, for $k, l \geq 0$ and all $a, b_1, \ldots, b_k, c_1, \ldots, c_l \in A$, one has

$$E_l(E_k(a; b_*); c_*) \equiv \sum_{i_1 + \cdots + i_k + j_0 + \cdots + j_k = l} (-1)^\varepsilon E_n(a; c_{i_0}, E_{i_1}(b_1; c_*), c_{i_1}, \ldots, c_{i_{k-1}}, E_{i_k}(b_l; c_*), c_{j_k}),$$

(6.4)

where the sum is over all decompositions of $l$ into $2k + 1$ non-negative integers,

$$n = k + \sum_{t=0}^k \varepsilon = \sum_{i=1}^k i s + \sum_{l=0}^k j_l + \sum_{t=0}^k \varepsilon t j_l.$$

(6.5)

A morphism of hgas is a morphism $f: A \to B$ of augmented dgas that is compatible with the hga operations in the obvious way.

Given an hga $A$, we can define

$$E_{kl}: B_k A \otimes B_l A = (s^{-1} A)^{\otimes (k+l)} \to A$$

(6.6)

for $k, l \geq 0$ by

$$E_{kl}(s^{-1})^{\otimes (k+l)} = \begin{cases} 1_A & \text{if } k = 0 \text{ and } l = 1, \\ E_l & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(6.7)

The functions $E_{kl}$ assemble to a map

$$E: B A \otimes B A \to A,$$

(6.8)

which is a twisting cochain by (6.2) and (6.3) together with the normalization conditions. Moreover, the identity (6.4) implies that the induced dgc map

$$\mu_{BA}: B A \otimes B A \to B A$$

(6.9)

is associative and therefore turns $BA$ into a dg bialgebra. Conversely, a dg bialgebra structure on $BA$ whose associated twisting cochain $E$ is of the form (6.7) defines an hga structure on $A$ with operations $E_k$. 

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Remark 6.1. Our hga operations are related to the braces originally defined by Voronov and Gerstenhaber [34, §8], [11, Section 1.2], [33, Section 3.2] by the identity
\[ a\{b_1, \ldots, b_k\} = E_{k1} ([a] \otimes [b_1] \cdots [b_k]) = (-1)^{\varepsilon} E_k (a; b_1, \ldots, b_k) \] (6.10) for \( k \geq 1 \) where
\[ \varepsilon = k |a| + \sum_{m=1}^{k} (k - m) |b_m|. \] (6.11)
(Compare formula (3.9).) Our grading agrees with [33]; in [34] and [11], the degrees of the desuspended arguments are used.\(^*\)

We observe that the \( \cup \)-product
\[ a \cup_1 b = -E_1 (a; b) \] (6.12)
is a homotopy from the product with commuted factors to the standard one,
\[ d(a \cup_1 b) + da \cup_1 b + (-1)^{|a|} a \cup_1 db = ab - (-1)^{|a||b|} ba \] (6.13)
satisfying the Hirsch formula
\[ ab \cup_1 c = (-1)^{|a|} a(b \cup_1 c) + (-1)^{|b||c|} (a \cup_1 c) b \] (6.14)
for \( a, b, c \in A \). As a consequence, the cohomology \( H^*(A) \) is (graded) commutative and in fact a Gerstenhaber algebra with bracket
\[ \{ [a], [b] \} = [E_1 (a; b) - (-1)^{|a|-1} |b|-1] E_1 (b; a)] \] (6.15)
\[ = (-1)^{|a|-1} [ a \cup_1 b + (-1)^{|a||b|} b \cup_1 a \] for \( a, b \in A \), see [34, §10].

The main examples of hgas are the cochains on a simplicial set, see Section 8.2, and the Hochschild cochains of an algebra, see the references given above. Any commutative dga is canonically an hga by setting \( E_k = 0 \) for all \( k \geq 1 \). The induced multiplication on \( BA \) then is the shuffle product discussed in Example 2.1.

We say that an hga \( A \) is formal if it is quasi-isomorphic to its cohomology \( H^*(A) \), considered as an hga.

6.2. Extended hgas
In his study of \( \cup_i \)-products on \( BA \) for \( i \geq 1 \), Kadeishvili introduced operations \( E^i_{kl} \) for an hga \( A \) defined over \( k = Z_2 \) [19]. He called an hga equipped with these operations an ‘extended hga’. We will only need the family \( F_{kl} = E^1_{kl} \), but for coefficients in any \( k \). We therefore say that an hga is extended if it has a family of operations
\[ F_{kl} : A^{|k|} \otimes A^{|l|} \to A \] (6.16)
of degree \( |F_{kl}| = -(k + l) \) for \( k, l \geq 1 \), satisfying the following conditions. All operations \( F_{kl} \) take values in the augmentation ideal \( A \) and vanish if any argument equals \( 1 \in A \). Their differential is given by
\[ d(F_{kl}) (a_\star; b_\star) = A_{kl} + (-1)^k B_{kl} \] (6.17)
for all \( a_1, \ldots, a_k, b_1, \ldots, b_l \in A \), where
\[ A_{kl} = E_l (a_1; b_\star), \] (6.18)
\[ B_{kl} = E_k (a_\star; b_1). \]
Let \( A \) be an extended hga, and let \( \mathfrak{a} \) be the ideal generated by the values of all operations \( E_k \) with \( k \geq 1 \) as well as those of all operations \( F_{kl} \) with \( (k, l) \neq (1, 1) \).

(i) The extended hga \( A \) is canonically an shc algebra. The structure maps \( \Phi \), \( h^a \) and \( h^c \) commute with morphisms of extended hgas.

(ii) The shm map \( \Phi \) is \( a \)-strict. More generally, all iterations \( \Phi^{[n]} \) with \( n \geq 0 \) as well as the composition \( \Phi \circ (1 \otimes \Phi) \) are \( a \)-strict.

(iii) The homotopy \( h^a \) is \( a \)-trivial.

(iv) Modulo \( \mathfrak{a} \), we have for any \( n \geq 0 \) and any \( a_\bullet, b_\bullet \in A \) the congruence

\[
h^c_{(n)}(a_\bullet \otimes b_\bullet) \equiv \begin{cases} 1 & \text{if } n = 0, \\ \pm b_1 (a_1 \cup_2 b_2) a_2 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}
\]
Proof. The scf structure is constructed explicitly in the companion paper [10]. Inspection of the definition of \( \Phi \) there shows that it is a-strict. The case \( n = 0 \) of the iteration is void, and for \( n \geq 2 \) it is a consequence of Lemma 3.1(i) (observed already in Section 5), as is the case of the other composition. The statements about \( h^a \) and \( h^c \) follow again by looking at their definitions in [10]. □

7. Twisted tensor products

Let \( A \) be an augmented dga and \( C \) a coaugmented dgc. For any \( f \in \text{Hom}(C, A) \), we set

\[
\delta_f = (1_C \otimes \mu_A)(1_C \otimes f \otimes 1_A)(\Delta_C \otimes 1_A) : C \otimes A \to C \otimes A. \tag{7.1}
\]

The assignment

\[
\text{Hom}(C, A) \to \text{End}(C \otimes A), \quad f \mapsto \delta_f \tag{7.2}
\]

is a morphism of dgas. As a consequence, if \( t \in \text{Hom}(C, A) \) is a twisting cochain, then

\[
d_{\otimes} - \delta_t = (d_C \otimes 1_A + 1_C \otimes d_A) - \delta_t \tag{7.3}
\]

is a differential on \( C \otimes A \). The resulting complex is called a twisted tensor product and denoted by \( C \otimes \tilde{t} A \), compare [18, Definition II.1.4] or [17, Section 1.3].

**Lemma 7.1.** Let \( a \triangleleft A \), and let \( h : C \to A \) be an \( a \)-trivial homotopy from the twisting cochain \( t : C \to A \) to \( \tilde{t} : C \to A \). If \( \tilde{C} \) is cocomplete, then the map

\[
\delta_h : C \otimes {\tilde{t}} A \to C \otimes t A
\]

is an isomorphism of complexes, congruent to the identity map modulo \( C \otimes a \).

**Proof.** The inverse of \( \delta_h \) is given by \( \delta_{h^{-1}} \), see [17, Corollary 1.4.2]. The congruence to the identity map follows directly from the \( a \)-triviality. □

**Lemma 7.2.** Let \( t : C \to A \) be a twisting cochain, and let \( g : C' \to C \) be a map of coaugmented dgcs. Then \( t \circ g : C' \to A \) is a twisting cochain and

\[
g \otimes 1_A : C' \otimes {\text{log}} A \to C \otimes t A
\]

is a chain map.

**Proof.** This follows directly from the definitions. □

**Lemma 7.3.** Let \( t : C \to A \) and \( t' : C' \to A' \) be twisting cochains.

(i) Let \( f : A' \to A \) be a map of augmented dgas and \( g : C' \to C \) a map of coaugmented dgcs. If \( t g = f t' \), then

\[
g \otimes f : C' \otimes {t'} A' \to C \otimes t A
\]

is a chain map.

(ii) Let \( h : C' \to C \) be a coalgebra homotopy from \( g \) to another map \( \tilde{g} : C' \to C \) of coaugmented dgcs satisfying \( t \tilde{g} = f t' \). If \( th = 0 \), then \( h \otimes f \) is a homotopy from \( g \otimes f \) to \( \tilde{g} \otimes f \).

**Proof.** The first claim is again a direct consequence of the definitions. The second one follows from the identity \( \delta_t (h \otimes f) = -(h \otimes f) \delta_{t'} \), which uses the assumption \( th = 0 \). □
Let $f: A \to B$ be a map of augmented dgas. Then $f \circ t_A: BA \to B$ is a twisting cochain. The associated twisted tensor product
\[ \mathcal{B}(k, A, B) = BA \otimes_{f_{otA}} B \] (7.4)
is the one-sided bar construction. Usually, the map $f$ will be understood from the context and not indicated. We write the cohomology of the one-sided bar construction as the differential torsion product
\[ \text{Tor}_A(k, B) = H^*(\mathcal{B}(k, A, B)). \] (7.5)
Note that this is just a notation; we are not concerned with whether the bar construction leads to a proper projective resolution in case $k$ is not a field. However, if $A$ is free over $k$ and both $A$ and $B$ have zero differentials, then (7.5) is the usual torsion product.

Given an shm map $g: B \Rightarrow B'$, we define
\[ \Gamma_g: \mathcal{B}(k, A, B) = BA \otimes_{t_A} B \to BA \otimes_{g_{otA}} B', \] (7.6)
where for any $k \geq 0$ the map $g$ of degree 0 is defined as the composition
\[ g: B_k B \otimes B \xrightarrow{1 \otimes k \otimes s^{-1}} B_{k+1} B \xrightarrow{\eta} B'. \] (7.8)
The following is essentially taken from [35, Theorem 7], where also a version of Lemma 7.2 for two-sided bar constructions is given.

**Lemma 7.4.** Assume that $g: B \Rightarrow B'$ is $b$-strict for some $b \triangleleft B'$. Then $\Gamma_g$ as defined above is a chain map, congruent to $1_{BA} \otimes g(1)$ modulo $BA \otimes b$.

**Proof.** This is a direct computation. \(\square\)

**Remark 7.5.** Assume that all complexes involved are torsion-free over the principal ideal domain $k$ and (including the bar constructions) bounded below. If the map $g$ is a quasi-isomorphism, then the resulting maps in Lemmas 7.2 and 7.4 are quasi-isomorphisms. This follows from the Künneth theorem and a standard spectral sequence argument, compare the proof of Proposition 12.2(i).

Assume now that $A \to A'$ is a morphism of hgas. It is convenient to introduce the map
\[ \mathcal{E}: A' \otimes BA' \to A', \] (7.9)
of degree 0. Following [20], we can then define the map
\[ \circ: \mathcal{B}(k, A', B) \otimes \mathcal{B}(k, A, A') \to \mathcal{B}(k, A, A'), \] (7.10)
where $a = [a_1] \ldots [a_k]$, $b = [b_1] \ldots [b_l] \in BA$ and $a, b \in A'$. Observe that the summand for $m = l$ is the componentwise product
\[ (-1)^{|a||b|} a \circ b \otimes a b. \] (7.11)
Proposition 7.6 (Kadeishvili–Saneblidze). Assume the notation introduced above. Then $B(k, A, A')$ is naturally an augmented dga with unit $1_{B_A} \otimes 1_{A'}$, augmentation $\varepsilon_{B_A} \otimes \varepsilon_{A'}$ and product (7.10).

Proof. In [20, Corollaries 6.2 and 7.2], this is only stated for simply connected hgas. It is, however, a formal consequence of the defining properties of any hga. □

8. Simplicial sets

Our basic reference for this material is [23]. We write $[n] = \{0, 1, \ldots, n\}$.

8.1. Preliminaries

Let $X$ be a simplicial set. We call $X$ reduced if $X_0$ is a singleton and 1-reduced if $X_1$ is a singleton. We abbreviate repeated face and degeneracy operators as

$$\partial_i^j = \partial_i \circ \cdots \circ \partial_j, \quad \partial_i^{j-1} = \text{id}, \quad s_I = s_{i_m} \circ \cdots \circ s_{i_1}, \quad s_{\emptyset} = \text{id} \quad (8.1)$$

for $i \leq j$ and $I = \{i_1 < \cdots < i_m\}$.

We write $C(X)$ and $C^*(X)$ for the normalized chain and cochain complex of $X$ with coefficients in $k$, cf. [21, Section VIII.6]. Then $C(X)$ is a dgc with the Alexander–Whitney map as diagonal and augmentation induced by the unique map $X \to \ast$, and $C^*(X)$ is a dga with product $\Delta^*_C(X)$.

We say $X$ has polynomial cohomology (with respect to the chosen coefficient ring $k$) if $H^*(X)$ is a polynomial algebra on finitely many generators of positive even degrees. Note that $X$ is of finite type over $k$ in this case.

For $0 \leq k \leq n$, we define the ‘partial diagonal’

$$P^n_k : C_n(X) \to C_k(X) \otimes C_{n-k}(X), \quad (8.2)$$

$$\sigma \mapsto \partial^{n}_{k+1} \sigma \otimes \partial^{k-1}_{0} \sigma = \sigma(0, \ldots, k) \otimes \sigma(k, \ldots, n)$$

so that

$$\Delta c = \sum_{k=0}^{n} P^n_k(c) \quad (8.3)$$

for any $c \in C_n(X)$. Note that each $P^n_k$ is well defined on normalized chains.

For simplicial sets $X$ and $Y$, the shuffle map

$$\nabla = \nabla_{X,Y} : C(X) \otimes C(Y) \to C(X \times Y) \quad (8.4)$$

is a map of dgcs and also associative and commutative, cf. [9, Section 3.2]. Commutativity, for instance, means that the diagram

$$\begin{array}{ccc}
C(X) \otimes C(Y) & \xrightarrow{\nabla_{X,Y}} & C(X \times Y) \\
\tau_{C(X), C(Y)} \downarrow & & \downarrow \tau_{X,Y} \ast \\
C(Y) \otimes C(X) & \xrightarrow{\nabla_{Y,X}} & C(Y \times X)
\end{array} \quad (8.5)$$

commutes, where $\tau_{X,Y} : X \times Y \to Y \times X$ swaps the factors.

†Note that the definition of an hga in [20, Definition 7.1] uses Baues’ convention (see Footnote 8) and differs from ours (as does the definition of the differential on the bar construction [20, p. 208]). This results in a product on the one-sided bar construction $B(A', A, k)$.
8.2. The extended hga structure on cochains

Gerstenhaber and Voronov [11, Section 2.3] have constructed an hga structure on the non-normalized cochain complex of a simplicial set \( X \), which descends to the normalized cochain complex \( C^*(X) \). There it can be given in terms of the interval cut operations

\[
E_k = AW^*_e
\]

corresponding to the surjections

\[
e_k = (1, 2, 1, 3, 1, \ldots, 1, k + 1, 1),
\]

cf. [4, §1.6.6, Section 2]. Writing out the sign implicit in the transpose (8.6), we have

\[
E_k(a; b_1, \ldots, b_k)(c) = (-1)^k([a]+|b_1| + \cdots + |b_k|) (a \otimes b_1 \otimes \cdots \otimes b_k) AW^*_e(c).
\]

for \( a, b \in C^*(X) \) and \( c \in C(X) \).

The operations \( E_k \) vanish for \( k \geq 1 \) if any argument is of degree 0 and never return a non-zero cochain of degree 0. This implies that the normalization condition (2.13) is satisfied independently of the chosen augmentation \( C^*(X) \to \mathbb{k} \). This hga structure generalizes the multiplication on \( BC^*(X) \) previously defined by Baues [2, §IV.2] for 1-reduced \( X \).

Kadeishvili [19] has observed that \( C^*(X) \) is an extended hga with operations \( F_{kl} \) corresponding to the surjections

\[
f_{kl} = (k + 1, 1, k + 1, 2, k + 1, \ldots, k + 1, k, k + 1, k + 2, k, \ldots, k, k + l, k)
\]

for \( k, l \geq 1 \). The associated \( \cup_2 \)-product is \( \cup_2 = -AW^*_e(2, 1, 2, 1) \).

8.3. Simplicial groups

Let \( G \) be a simplicial group (for example, the singular simplices in a topological group) with multiplication \( \mu \). We write \( 1_p \in G \) for the identity element of the group of \( p \)-simplices. A loop in \( G \) is a 1-simplex \( g \in G \) such that \( \partial_0 g = \partial_1 g = 1_0 \).

The dgc \( C(G) \) is a dg bialgebra with unit given by the identity element of \( G \) and multiplication

\[
C(G) \otimes C(G) \xrightarrow{\nabla} C(G \times G) \xrightarrow{\mu} C(G).
\]

If \( G \) is commutative, then so is \( C(G) \).

Similarly, if \( G \) acts on the simplicial set \( X \), then \( C(G) \) acts on \( C(X) \). We write this action as \( a \ast c \) for \( a \in C(G) \) and \( c \in C(X) \). If the \( G \)-action is trivial, then the \( C(G) \)-action factors through the augmentation \( \varepsilon : C(G) \to \mathbb{k} \). (Remember that we use normalized chains.)

For any loop \( g \in G \) and any \( 0 \leq m \leq n \), we define the map

\[
A^g_m : C_n(X) \to C_{n+1}(X), \quad \sigma \mapsto (s_{[n] \setminus m} g) \cdot s_m \sigma
\]

(which is again well defined on normalized chains). By the definition of the shuffle map, we can write the action of the loop \( g \in C(G) \) on \( \sigma \in C(X) \) as

\[
g \ast \sigma = \sum_{m=0}^n (-1)^m A^g_m(\sigma).
\]

More precisely, Baues’ multiplication is obtained by transposing the factors of the product, so that \( E_{kl} \) vanishes for \( l \neq 1 \), except for \( E_{10} \). This also affects the components of the homotopy \( F \) from [10, Corollary 6.2].

Kadeishvili’s choice for \( F_{kl} \) [19, pp. 116, 123] does not lead to the formula (6.17) (or [19, Definition 2]) for \( d(F_{kl}) \), but to the one with \( a \)-variables and \( b \)-variables interchanged.
The diagonal of $C(X)$ is known to be $C(G)$-equivariant, cf. [9, Proposition 3.5]. For loops, a more refined statement is the following.

**Lemma 8.1.** Assume that $g \in G$ is a loop, and let $\sigma \in X_n$. Then

$$P_k^{n+1}(A^g_m(\sigma)) = \begin{cases} (-1)^k (1 \otimes A^g_{m-k}) P_k^n(\sigma) & \text{if } k \leq m, \\ (A^g_m \otimes 1) P_{k-1}^n(\sigma) & \text{if } k > m, \end{cases}$$

for any $0 \leq m \leq n$ and $0 \leq k \leq n + 1$.

**Proof.** We have

$$P_k^{n+1}(A^g_m(\sigma)) = \partial_{k+1}^{n+1} A^g_m(\sigma) \otimes \partial_0^{k+1} A^g_m(\sigma)$$

(8.13)

hence

$$P_k^{n+1}(A^g_m(\sigma)) = \partial_{k+1}^n s_m \sigma \otimes (\partial_0^{k} s_{\lceil n \rceil - k} g) \cdot (\partial_0^{k} s_m \sigma)$$

(8.15)

In the case $k > m$, we similarly find

$$\partial_0^{k} s_{\lceil n \rceil - k} g = 1 \in G_{n-k}$$

(8.16)

and

$$P_k^{n+1}(A^g_m(\sigma)) = (\partial_{k+1}^n s_m \sigma) \otimes (\partial_0^{k} s_{\lceil n \rceil - k} g) \cdot (\partial_0^{k} s_m \sigma)$$

(8.17)

as claimed. $\square$

8.4. Universal bundles

The standard reference for this material is [23, §21], where the notation $BG = \overline{W}(G)$ and $EG = W(G)$ is used.

Let $G$ be a simplicial group. Its classifying space is the simplicial set $BG$ whose $p$-simplices are elements of the Cartesian product

$$[g_{p-1}, \ldots, g_0] \in G_{p-1} \times \cdots \times G_0 = BG_p.$$ 

(8.18)

It is always reduced (with unique vertex $b_0 := [\cdot] \in BG_0$) and 1-reduced in case $G$ is reduced. The simplices in the total space of the universal $G$-bundle $\pi : EG \to BG$ are similarly given by

$$e = (g_p, [g_{p-1}, \ldots, g_0]) \in G_p \times BG_p = EG_p;$$  

(8.19)
the map \( \pi \) is the obvious projection. We write \( e_0 = (1_0, b_0) \in EG_0 \) for the canonical basepoint, which projects onto \( b_0 \). Our conventions for face and degeneracy maps can be obtained from [23, pp. 71, 87] by substituting the opposite group \( G^{op} \) for \( G \). More precisely, for \( EG \) they are given by

\[
\partial_k (g_p, [g_{p-1}, \ldots, g_0]) = (\partial_k g_p, [\partial_{k-1} g_{p-1}, \ldots, \partial_1 g_{p-k+1}, (\partial_0 g_{p-k}) g_{p-k-1}, g_{p-k-2}, \ldots, g_0]), \tag{8.20}
\]

for \( 0 \leq k \leq p \): for \( BG \), one drops the first component. Note that for \( k = 0 \) the right-hand side of formula (8.20) is interpreted as \(( (\partial_0 g_p) g_{p-1}, [g_{p-2}, \ldots, g_0])\) and for \( k = p \) as \((\partial_p g_p, [\partial_{p-1} g_{p-1}, \ldots, \partial_1 g_1])\). We consider \( EG \) as a left \( G \)-space via

\[
h \cdot (g_p, [g_{p-1}, \ldots, g_0]) = (hg_p, [g_{p-1}, \ldots, g_0]) \tag{8.22}
\]

for \( h \in G_p \).

There is a canonical map \( S : EG \to EG \) of degree 1 given by

\[
S(g_p, [g_{p-1}, \ldots, g_0]) = (1_{p+1}, [g_p, g_{p-1}, \ldots, g_0]) \tag{8.23}
\]

cf. [23, p. 88]. For all \( e \in EG_p \), one has

\[
\partial_0 Sc = e, \tag{8.24}
\]

\[
\partial_1 Sc = e_0 \quad \text{if } p = 0, \tag{8.25}
\]

\[
\partial_k Sc = S \partial_{k-1} e \quad \text{if } p > 0 \text{ and } k > 0. \tag{8.26}
\]

This implies that \( S \) induces a chain homotopy on \( C(EG) \), again called \( S \), from the projection to \( e_0 \) to the identity on \( EG \),

\[
(dS + Sd)(e) = \begin{cases} 
  e - e_0 & \text{if } p = 0, \\
  e & \text{if } p > 0, 
\end{cases} \tag{8.27}
\]

for any simplex \( e \in EG \), and that it additionally satisfies

\[
SS = 0 \quad \text{and} \quad Sc_0 = 0, \tag{8.28}
\]

compare [8, Proposition 2.7.1] or [9, Section 3.7].

**Lemma 8.2.** Let \( c \in C(EG) \) be of degree \( n \).

1. For any \( 0 \leq k \leq n + 1 \), one has

\[
P_{k+1}^n(Sc) = \begin{cases} 
  e_0 \otimes Sc & \text{if } k = 0, \\
  (S \otimes 1) P_{k-1}^n(c) & \text{if } k > 0.
\end{cases}
\]

2. One has

\[
\Delta Sc = (S \otimes 1) \Delta c + e_0 \otimes Sc.
\]

**Proof.** The first statement is immediate if \( c = e_0 \) or \( k = 0 \). For \( n > 0 \) and \( k > 0 \), it follows from the identities (8.24)–(8.26). Combining it with (8.3) gives the second claim, cf. [8, Proposition 2.7.1] or [9, Proposition 3.8]. \( \square \)
8.5. An Eilenberg–Moore theorem

In this section, we assume that \( k \) is a principal ideal domain.

The following result is suggested by work of Kadeishvili–Saneblidze [20, Corollary 6.2].

**Proposition 8.3.** Let \( F \xrightarrow{i} E \to B \) be a simplicial fibre bundle. If \( B \) is 1-reduced and of finite type over \( k \), then the map

\[
\mathcal{B}(k, C^*(B), C^*(E)) \to C^*(F), \quad [\gamma_1 \ldots \gamma_k] \otimes \gamma \mapsto \begin{cases} t^*(\gamma) & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}
\]

is a quasi-isomorphism of dgas. In particular, there is an isomorphism of graded algebras

\[
H^*(F) \cong \text{Tor}_{C^*(B)}(k, C^*(E)).
\]

**Proof.** By the usual Eilenberg–Moore theorem, the map is a quasi-isomorphism of complexes. For field coefficients, we can refer to [31, Theorem 3.2]. For general \( k \), it follows by dualizing the homological quasi-isomorphism [13, Section 6]

\[
C(F) \to \Omega(k, C(B), C(E)),
\]

where the target is the one-sided cobar construction.

Let us recall the argument: If we write \( G \) for the structure group of the bundle \( E \to B \), then \( C(F) \) is a left \( C(G) \)-module. By the twisted Eilenberg–Zilber theorem [13, Section 4], there is a twisting cochain \( t : C(B) \to C(G) \) and a homotopy equivalence

\[
C(E) \simeq C(B) \otimes_t C(F)
\]

of left \( C(B) \)-comodules. Under this isomorphism, the map \( t_* : C(F) \to C(E) \) corresponds to the canonical inclusion of \( C(F) \) into the twisted tensor product with the unique base point of \( B \) as first factor.

We therefore get a homotopy equivalence of complexes

\[
\Omega(k, C(B), C(E)) \simeq \Omega(k, C(B), C(B) \otimes_t C(F))
\]

\[
= \Omega(k, C(B), C(B)) \otimes_t C(F),
\]

between the one-sided cobar constructions, where we consider \( \Omega(k, C(B), C(B)) \) as a right \( C(B) \)-comodule, cf. [18, Definition II.5.1].

The canonical inclusion \( k \hookrightarrow \Omega(k, C(B), C(B)) \) is a homotopy equivalence [18, Proposition II.5.2], and \( \delta_t \) vanishes on its image. Because \( B \) is 1-reduced, a spectral sequence argument shows that the map \( C(F) \to \Omega(k, C(B), C(B)) \otimes_t C(F) \) is a quasi-isomorphism of complexes, hence so is the natural map

\[
C(F) \to \Omega(k, C(B), C(E)), \quad c \mapsto 1 \otimes 1_{\Omega C(B) \otimes \tau_t(c)}.
\]

Everything we have said so far is valid for any coefficient ring. Since \( k \) is a principal ideal domain and \( B \) of finite type, the canonical map

\[
(s^{-1}C^*(B))^\otimes k \otimes C^*(E) \to (k \otimes (s^{-1}C(B))^\otimes k \otimes C(E))^*
\]

is a quasi-isomorphisms for any \( k \geq 0 \) by the universal coefficient theorem, hence so is the composition

\[
\mathcal{B}(k, C^*(B), C^*(E)) \to \Omega(k, C(B), C(E))^* \to C^*(F).
\]

A look at Proposition 7.6 finally shows that the quasi-isomorphism is multiplicative because any cochain on \( B \) of positive degree restricts to 0 on \( F \). (Recall that we are working with normalized cochains.) \( \square \)
If we define an increasing filtration on $\mathcal{B}(k, C^*(B), C^*(E))$ by the length of elements, then we get an (Eilenberg–Moore) spectral sequence of algebras converging to $H^*(F)$ because the deformation terms in the product formula given in Proposition 7.6 lower the filtration degree. By the Künneth theorem, the second page of this spectral sequence is of the form

$$E_2 = \text{Tor}_{H^*(B)}(k, H^*(E))$$

(8.36)

with the usual product on Tor, provided that $H^*(B)$ is free over $k$.

**Remark 8.4.** Assume that the base $B$ has polynomial cohomology, say $H^*(B) = \mathbb{K}[y_1, \ldots, y_n]$. Let $b_1, \ldots, b_n \in C^*(B)$ be representatives of the generators, and let

$$\bigwedge (x_1, \ldots, x_n)$$

(8.37)

be the exterior algebra on generators $x_i$ of degrees $|x_i| = |y_i| - 1$. Since $\mathcal{B} C^*(Y)$ is a dg bialgebra and the elements $[b_i] \in \mathcal{B} C^*(Y)$ primitive, the assignment

$$\bigwedge (x_1, \ldots, x_n) \to \mathcal{B} C^*(Y), \quad x_{i_1} \wedge \cdots \wedge x_{i_k} \mapsto [b_{i_1}] \circ \cdots \circ [b_{i_k}]$$

(8.38)

is a dgc map (but not multiplicative in general) and in fact a quasi-isomorphism. Evaluating the product from the left to the right shows that the associated twisting cochain $t_{GM}$ is of the form $t_{GM}(x_i) = b_i$ and

$$t_{GM}(x_{i_1} \wedge \cdots \wedge x_{i_k}) = E_1(\cdots E_1(E_1(b_{i_1}; b_{i_2}); b_{i_3}); \cdots ; b_{i_k})$$

$$= (-1)^{k-1} \left( \left( (b_{i_1} \cup b_{i_2} \cup b_{i_3}) \cup_1 \cdots \cup_k \right) \cup b_{i_k} \right)$$

(8.39)

for $k \geq 2$ and $i_1 < \cdots < i_k$. A standard spectral sequence argument then implies that the twisted tensor product

$$\bigwedge (x_1, \ldots, x_n) \otimes t_{GM} C^*(E)$$

(8.40)

is quasi-isomorphic to $\mathcal{B}(k, C^*(B), C^*(E))$ as a complex, hence computes $H^*(F)$ as a graded $k$-module by the Eilenberg–Moore theorem. We thus recover the model constructed by Gugenheim–May [15, Example 2.2 & Theorem 3.3].

**Lemma 8.5.** Let $G$ be a connected simplicial group and $K \subset G$ a connected subgroup. Write $\tilde{G} \subset G$ for the reduced subgroup of simplices lying over $1 \in G_0$, and define $\tilde{K} \subset K$ analogously. Then the inclusion $\tilde{G}/\tilde{K} \hookrightarrow G/K$ is a homotopy equivalence, natural in the pair $(G, K)$.

**Proof.** The inclusions $\tilde{G} \hookrightarrow G$ and $\tilde{K} \hookrightarrow K$ are homotopy equivalences, compare [23, Theorem 12.5]. The long exact sequence of homotopy groups implies that the map $\tilde{G}/\tilde{K} \hookrightarrow G/K$ is also a homotopy equivalence. Injectivity follows from the identity $\tilde{K} = K \cap \tilde{G}$, and naturality is clear.

**Proposition 8.6.** Let $G$ be a reduced simplicial group and $K$ a reduced subgroup. There is an isomorphism of graded algebras

$$H^*(G/K) \cong \text{Tor}_{C^*(BG)}(k, C^*(BK)),$$

natural with respect to maps of pairs $(G, K)$.

**Proof.** The map $\pi: EG/K \to BG$ is a fibre bundle with fibre $G/K$. By Proposition 8.3, the dgas $C^*(G/K)$ and $\mathcal{B}(k, C^*(BG), C^*(EG/K))$ are naturally quasi-isomorphic. The homotopy
equivalence $BK = EK/K \to EG/K$ is a map over $BG$ and induces a quasi-isomorphism

$$B(k, C^*(BG), C^*(EG/K)) \to B(k, C^*(BG), C^*(BK)), \quad (8.41)$$

which is multiplicative by the naturality of the hga structure on cochains.

**Remark 8.7.** Let $G$ be a Lie group and $K \subset G$ a closed subgroup. Then the projection $G \to G/K$ is a principal $K$-bundle. Writing $S(X)$ for the simplicial set of singular simplices in a topological space $X$, we therefore have $S(G/K) = S(G)/S(K)$. The same holds if $G$ is only a topological group, but the closed subgroup $K$ has the structure of a Lie group, cf. [27, Section 4.1].

9. **Homotopy Gerstenhaber formality of $BT$**

9.1. **Dga formality**

Let $T$ be a simplicial torus of rank $n$. By this we mean a commutative simplicial group $T$ such that $H^*(T)$ is an exterior algebra on generators $x_1, \ldots, x_n$ of degree 1. For example, $T$ can be the compact torus $(S^1)^n$, the algebraic torus $(\mathbb{C}^\times)^n$ or the simplicial group $BZ^n$.

As mentioned in the introduction, Gugenheim–May [15, Theorem 4.1] have constructed a quasi-isomorphism of dgas

$$C^*(BT) \to H^*(BT) \quad (9.1)$$

annihilating all $\cup_1$-products. An alternative approach was given by the author in his doctoral dissertation [8, Proposition 2.2], see also [9, Proposition 5.3]. The goal of this section is to promote the latter construction to a quasi-isomorphism of hgas, that is, one that annihilates all operations $E_k$ with $k \geq 1$. We will see that also all operations $F_{k1}$ with the exception of the $\cup_2$-product are sent to 0.

We write $A = H(T)$ and $S = H(BT)$. The latter is the cocommutative coalgebra on cogenerators $y_i \in S_2$ that correspond to the generators $x_i$ under transgression. The cogenerators $y_i$ define a $k$-basis $y_{\alpha}$ of $S$ indexed by multi-indices $\alpha \in \mathbb{N}^n$. We also write $y_0 = 1$.

Let $t: S \to A$ be the (homological) twisting cochain that sends each $y_i$ to $x_i$ and vanishes in other degrees. The twisted tensor product

$$K = A \otimes_t S \quad (9.2)$$

is the Koszul complex. It is a dgc with $A$-equivariant diagonal given by the tensor product of the componentwise diagonals. For $a \in A$ and $c \in S$, we write $a \cdot c \in K$ instead of $a \otimes c$, reflecting the $A$-action. The differential on $K$ is given by

$$d(a \cdot y_\alpha) = (-1)^{|\alpha|} \sum_i a \cdot x_i \cdot y_{\alpha|i} = \sum_i x_i \wedge a \cdot y_{\alpha|i} \quad (9.3)$$

where the sum runs over all $i$ such that $\alpha_i > 0$, and ‘$\alpha|i$’ means that the $i$th component of $\alpha$ is decreased by 1.

Let $c_1, \ldots, c_n \in C_1(T)$ be linear combinations of loops in $G$ representing the generators $x_i$. They define a quasi-isomorphism of dg bialgebras

$$\varphi: A \to C(T), \quad x_{i_1} \wedge \cdots \wedge x_{i_k} \mapsto c_{i_1} \ast \cdots \ast c_{i_k} \quad (9.4)$$

for $i_1 < \cdots < i_k$. Moreover, let $\pi: ET \to BT$ be the universal $T$-bundle. Note that $C(ET)$ is a $A$-module via $\varphi$.

Our map (9.1) will be the transpose of a quasi-isomorphism $f: S \to C(BT)$. The construction of the latter is based on a map

$$F: K \to C(ET) \quad (9.5)$$
For the previous one if each chain map, we proceed by induction on the degree of the Alexander–Whitney map and induction, we get a case recursively defined by

\[ F(1) = e_0, \]

\[ F(a \cdot c) = a \ast F(c) \quad \text{if } |a| > 0, \]

\[ F(c) = SF(dc) \quad \text{if } |c| > 0 \]

for \( c \in S \) and \( a \in A \), where \( S \) is the homotopy defined in (8.23).

**Proposition 9.1.** The map \( F \) is a \( \Lambda \)-equivariant quasi-isomorphism of dgc s.

For the convenience of the reader, we adapt the proof given in [9, Proposition 4.3] to our slightly more general setting.

**Proof.** It is clear from the definition that \( F \) commutes with the \( \Lambda \)-action. To show that it is a chain map, we proceed by induction on the degree of \( a \cdot y \in K \). For \( a \cdot y = 1 \), this is obvious. For \( |a| > 0 \), we have by equivariance and induction

\[ dF(a \cdot y) = d(\varphi(a) \ast F(y)) = \varphi(da) \ast F(y) + (-1)^{|a|} \varphi(a) \ast dF(y) \]

\[ = F(da \cdot y + (-1)^{|a|} a \cdot dy) = F(a \cdot y). \]

For \( |y| > 0 \), we have by (8.27) and induction

\[ dF(y) = dSF(dy) = F(dy) - S dF(dy) = F(dy). \]

To show that \( f \) is a map of coalgebras, we proceed once more by induction on \( |a \cdot y| \), the case \( a \cdot y = 1 \) being trivial. If \( |a| > 0 \), then again by equivariance and induction we have

\[ \Delta F(a \cdot y) = \Delta(\varphi(a) \ast F(y)) = \Delta \varphi(a) \ast \Delta F(y) = \Delta \varphi(a) \ast (F \otimes F)\Delta y \]

\[ = (F \otimes F)(\Delta a \cdot \Delta y) = (F \otimes F)\Delta(a \cdot y). \]

For \( \alpha \neq 0 \), we therefore have by Lemma 8.2(ii) that

\[ \Delta F(y_\alpha) = \Delta SF(dy_\alpha) = (S \otimes 1) \Delta F(dy_\alpha) + e_0 \otimes SF(dy_\alpha) \]

\[ = \sum_i (S \otimes 1) \Delta F(x_i \cdot y_{\alpha|i}) + F(1) \otimes F(y_\alpha), \]

where the sum runs over the indices \( i \) such that \( \alpha_i \neq 0 \). Using again the equivariance of the Alexander–Whitney map and induction, we get

\[ \Delta F(y_\alpha) = \sum_i (S \otimes 1) \Delta c_i \ast \Delta F(y_{\alpha|i}) + F(1) \otimes F(y_\alpha) \]

\[ = \sum_i \sum_{\beta + \gamma = \alpha|i} (S \otimes 1) \Delta c_i \ast (F(y_\beta) \otimes F(y_\gamma)) + F(1) \otimes F(y_\alpha). \]

Now \( \Delta c_i = c_i \otimes 1 + 1 \otimes c_i \), and \( SF(y_\gamma) = 0 \) by (8.28), hence

\[ \Delta F(y_\alpha) = \sum_i \sum_{\beta + \gamma = \alpha|i} S(c_i \ast F(y_\beta)) \otimes F(y_\gamma) + F(1) \otimes F(y_\alpha). \]

\( \dagger \)Using [8, equation (2.12c)] or [9, equation (3.29a)], one can see that our new construction coincides with the previous one if each \( c_i \) lies entirely in the \( i \)th factor of a circle decomposition of \( T \).
We reorder the summands. For each $\gamma \neq \alpha$ whose components are all less than or equal to those of $\alpha$, we have one term of the form $c_i \cdot F(y_{\beta_i})$ for each $\beta = \alpha - \gamma$ and each $i$ such that $\beta_i \neq 0$. This gives
\[
\Delta F(y_\alpha) = \sum_{\beta + \gamma = \alpha, \gamma \neq \alpha} \sum_i S(c_i \cdot F(y_{\beta_i})) \otimes F(y_\gamma) + F(1) \otimes F(y_\alpha)
\]
\[
= \sum_{\beta + \gamma = \alpha, \gamma \neq \alpha} F(y_\beta) \otimes F(y_\gamma) + F(1) \otimes F(y_\alpha)
\]
\[
= \sum_{\beta + \gamma = \alpha} F(y_\beta) \otimes F(y_\gamma),
\]
as was to be shown.
That $F$ induces an isomorphism in homology is trivial. \hfill \Box

Since $C(G)$ acts trivially on $BT$, the composition $\pi_* F : K \to C(BT)$ descends to a map of dgcns
\[
f : S = \kappa \otimes_K K \to C(BT).
\]

**Proposition 9.2.** The transpose $f^* : C^*(BT) \to S^*$ is a morphism of dgcns that induces the identity in cohomology.

**Proof.** Being the transpose of a dgc map, $f^*$ clearly is a morphism of dgcns.
Let $1 \leq i \leq n$. By construction, $f(y_i)$ corresponds to $\varphi(x_i)$ under transgression: Let $t : T \to ET$ be the inclusion of the fibre (over $b_0$). Then $t_*(\varphi(x_i)) = dF(y_i)$ and $f(y_i) = \pi_* F(y_i)$. This means that $H(f)$ is the identity map on cogenerators, hence in general. By the universal coefficient theorem (or spectral sequence), the same conclusion holds for $H^*(f^*)$. \hfill \Box

### 9.2. Haga formality

We say that a (non-degenerate) simplex $\sigma \in ET$ appears in an element of $C(ET)$ if its coefficient in this chain is non-zero; an analogous definition applies to tensor products of chain complexes.

**Lemma 9.3.** Let $0 \leq k \leq n + 1$, $a \in A_1$ and $c \in S_n$. For any simplex $\sigma \in (ET)_{n+1}$ appearing in $F(a \cdot c)$, we have
\[
(S \otimes S) P_k^{n+1}(\sigma) = 0.
\]

**Proof.** By construction and formula (8.12), the simplex $\sigma$ is of the form $A^g_m(\tau)$ for some loop $g \in T_1$, some $0 \leq m \leq n$ and some $n$-simplex $\tau$ appearing in $F(c)$.
If $n = 0$, then $\tau = e_0$. Hence
\[
P^1_0(\sigma) = e_0 \otimes g * e_0 \quad \text{and} \quad P^1_1(\sigma) = g * e_0 \otimes e_0
\]
by Lemma 8.1, and our claim follows from the second identity in (8.28).
Now consider the case $n > 0$. The definition of the map $F$ implies that $\tau$ is of the form $S\rho$ where $\rho$ is a simplex appearing in $F(\tilde{a} \cdot \tilde{c})$ with $\tilde{a} \in A_1$ and $\tilde{c} \in S_{n-2}$.
Assume $k \leq m$. Then
\[
P^{n+1}_k(\sigma) = P^{n+1}_k(A^\varphi_m(S\rho)) = (-1)^k(1 \otimes A^\varphi_{m-k}) P^k_k(S\rho),
\]
where $\varphi$ is the dgc map that descends to $C(BT)$.
where we have once again used Lemma 8.1. In the case \( k = 0 \), we obtain

\[
(S \otimes S) P_0^{n+1}(\sigma) = S e_0 \otimes S A_m^g(S \rho) = 0
\]

by Lemma 8.2(i) and (8.28). If \( k > 0 \), then

\[
(S \otimes S) P_k^{n+1}(\sigma) = (-1)^k (S S \otimes S A_{m-k}^g) P_{k-1}^{n-1}(\rho) = 0
\]

again by Lemma 8.2(i) and the first identity in (8.28).

In the case \( k = m \), we have

\[
P_k^{n+1}(\sigma) = (A_m^g \otimes 1) P_{k-1}^{n}(S \rho).
\]

For \( k = 1 \), this gives

\[
(S \otimes S) P_1^{n+1}(\sigma) = -S A_m^g(e_0) \otimes S S \rho = 0.
\]

If \( k > 1 \), we finally get

\[
(S \otimes S) P_{k}^{n+1}(\sigma) = (S A_m^g S \otimes S) P_{k-2}^{n-1}(\rho)
\]

\[
= (S A_m^g \otimes 1) (S \otimes S) P_{k-2}^{n-1}(\rho) = 0
\]

by induction.

Proof. We proceed by induction on \( n \), the case \( n = 0 \) being void. For the induction step from \( n \) for \( n+1 \), we start by considering the case \(|a| = 0\), which entails \( n \geq 1 \). The definition of \( F \) then implies that \( \sigma \) is of the form \( \sigma = S \tau \) for some \( n \)-simplex \( \tau \in ET \) that appears in \( F(\tilde{a} \cdot \tilde{c}) \) for some \( \tilde{a} \in \Lambda_1 \) and some \( \tilde{c} \in S_{n-1} \).

If \( 1 \leq k < l \leq n+1 \), we get

\[
Q_{k,l}^n(\sigma) = \partial_{k+1}^{l-1} S \tau \otimes \pi_s \partial_{l+1}^{n+1} S \tau = S \partial_k^{l-2} \tau \otimes \pi_s \partial_{l+1}^{k-1} S \partial_l^n \tau
\]

\[
= S \partial_k^{l-2} \tau \otimes \pi_s \partial_{l+1}^{k-2} \partial_l^n \tau = (S \otimes 1) Q_{k-1,l-1}(\tau) = 0
\]

by induction.

For \( 0 < l \leq n+1 \), we have

\[
Q_{0,l}^n(\sigma) = \partial_{l+1}^{n+1} S \tau = S \partial_{l+1}^{-2} \tau \otimes \pi_s \partial_{l+1}^{n+1} S \tau
\]

\[
= S \partial_0^{l-1} S \tau \otimes \pi_s \partial_{l+1}^{n+1} S \tau = \pm (1 \otimes \pi_s) T (1 \otimes S) P_{l-1}^{n+1}(S \tau),
\]
where $T$ denotes the transposition of factors,
\[
\mp (1 \otimes \pi_s) T (S \otimes S) P_{l-1}^{n} (\tau) = 0
\]
by Lemmas 8.2(i) and 9.3.

Now we turn to the case $|a| > 0$. Then a simplex appearing in $F(a \cdot c)$ is of the form $\sigma = A_{2m}^{\gamma}(\tau)$ for some loop $g \in T_1$, some $n$-simplex $\tau \in ET$ appearing in $F(c)$ and some $0 \less m \less n$. We have
\[
Q_{k,l}^{n+1}(\sigma) = Q_{k,l}^{n+1}(A_{m}^{\gamma}(\tau)) = Q_{k,l}^{n+1}(s_{[n]} \cdot m g \cdot s_{m} \cdot m) \\
= \partial_{k+1}^{l} (s_{[n]} \cdot m g \cdot s_{m} \cdot m) \otimes \pi_s \partial_{0}^{k-1} \partial_{l+1}^{n+1} \cdot s_{m} \cdot m.
\]

Assume $l > m$. Then
\[
\partial_{0}^{k-1} \partial_{l+1}^{n+1} \cdot s_{m} \cdot m = \partial_{0}^{k-1} \cdot \partial_{l}^{n+1} \cdot s_{m} \cdot m = \begin{cases}
\partial_{0}^{k-1} \partial_{l}^{n+1} \cdot s_{m} \cdot m & \text{if } k \leq m, \\
\partial_{0}^{k-1} \partial_{l}^{n+1} \cdot s_{m} \cdot m & \text{if } k > m.
\end{cases}
\]
In the first case, we obtain a degenerate simplex, so that (9.28) vanishes. In the second case, we have
\[
Q_{k,l}^{n+1}(\sigma) = (s_{[n-l+k+1]} \cdot m g) \cdot (s_{m} \partial_{k}^{l-2} \cdot s_{m} \cdot m) \otimes \pi_s \partial_{0}^{k-2} \partial_{l}^{n+1} \cdot s_{m} \cdot m
\]
\[
= (A_{m}^{\gamma} \otimes 1) (\partial_{k}^{l-2} \otimes \pi_s \partial_{0}^{k-2} \partial_{l}^{n+1} \cdot s_{m} \cdot m)
\]
\[
= (A_{m}^{\gamma} \otimes 1) Q_{m-l-1}^{n+1}(\tau) = 0
\]
by induction.

Finally, consider $l \leq m$. Then
\[
Q_{k,l}^{n+1}(\sigma) = (\partial_{k+1}^{l-1} s_{[n]} \cdot m g) \cdot (\partial_{k+1}^{l-1} s_{m} \cdot m) \otimes \pi_s \partial_{0}^{k-1} \partial_{l+1}^{n+1} \cdot s_{m} \cdot m
\]
\[
= (s_{[n-l+k+1]} \cdot m-l+k+1 g) \cdot (s_{m+k-l+1} \partial_{k+1}^{l-1} s_{m+k-l+1}) \otimes \pi_s \partial_{0}^{k-1} \partial_{l+1}^{n+1} s_{m+k-l+1}
\]
\[
= A_{m-l+k+1}^{\gamma} (\partial_{k+1}^{l-1} s_{m+k-l+1}) \otimes \pi_s \partial_{0}^{k-1} \partial_{l+1}^{n+1} s_{m+k-l+1}
\]
\[
= (A_{m-l+k+1}^{\gamma} \otimes 1) Q_{m-l+1}^{n+1}(\tau) = 0
\]
by induction. This completes the induction step and the proof. \hfill \Box

We write $n = \{1, \ldots, n\}$. We say that a surjection $u : k + l \rightarrow l$ has an enclave between positions $i$ and $i' \geq i + 2$ if $u(i') = u(i)$ and if the values $u(j)$ for $i < j < i'$ do not also appear at positions $\leq i$ or $\geq i'$. For example, the surjection $(1, 2, 3, 2, 1, 4)$ has exactly two enclaves, namely between positions 1 and 5 and between 2 and 4.

**Proposition 9.5.** If the surjection $u : k + l \rightarrow l$ has an enclave, then $A W_u f = 0$.

**Proof.** We start with a general observation. Let $\sigma$ be a simplex in some simplicial set. If $u$ has an enclave, then it follows from the definition of interval cut operations [4, Section 2.2] that any tensor product of simplices appearing in $A W_u(\sigma)$ can be obtained from a term $\tau \otimes \rho$ appearing in $A W_{(1,2,1)}(\sigma)$ by applying an interval cut to $\tau$ (at one choice of positions, not at all positions as in [4, §2.2.6]), another one to $\rho$ and permuting the factors of the result.

Now let $c \in s_n$ for some $n \geq 0$. By definition and naturality, we have
\[
A W_u f(c) = A W_u \pi_s F(c) = (\pi_s \otimes \pi_s) A W_u F(c).
\]

(9.32)
Our previous remarks together with (9.25) show that it suffices to prove that $Q_{k,l}^c(\sigma)$ vanishes for any $\sigma \in ET$ appearing in $F(c)$ and any $0 \leq k < l \leq n$. But this has been done in Lemma 9.4.

**Theorem 9.6.** The map $f^*: C^*(BT) \to H^*(BT)$ is a quasi-isomorphism of hgas that additionally annihilates all extended hga operations $F_{kl}$ with $(k,l) \neq (1,1)$. In particular, $C^*(BT)$ is formal as an hga.

**Proof.** We know from Proposition 9.2 that $f^*$ is a quasi-isomorphism of dgas. The hga operations $E_k$ with $k \geq 1$ as well as the operations $F_{kl}$ with $(k,l) \neq (1,1)$ are defined by surjections having enclaves, see (8.7) and (8.9). Hence the claim follows by dualizing Proposition 9.5. □

**9.3. The case where 2 is invertible**

It would greatly simplify the discussion of the next sections if the formality map $f^*$ also annihilated the operation $F_{11} = -\cup_2$. However, this is impossible to achieve for the transpose of a quasi-isomorphism $f: H^*(BT) \to C^*(BT)$, independently of the coefficient ring $k$. This can be seen as follows.

Take a non-zero $y \in H_2(BT)$ and set $w = f(y) \in C^2(BT)$. Choose a cochain $a \in C^2(BT)$ such that $a(w) \neq 0$. Let $\sigma$ be a 2-simplex appearing in $w$ with coefficient $w_\sigma \neq 0$ and such that $a(\sigma) \neq 0$. Define $b \in C^2(BT)$ by $b(\sigma) = 1$ and $b(\tau) = 0$ for $\tau \neq \sigma$. Then

$$(a \cup_2 b)(w) = \sum_\tau w_\tau a(\tau) b(\tau) = w_\sigma a(\sigma) \neq 0,$$  

(9.33)

where we have used the identity $(a \cup_2 b)(\sigma) = a(\sigma) b(\sigma)$, cf. [4, §2.2.8]. Hence $f^*(a \cup_2 b) \neq 0$, and analogously $f^*(b \cup_2 a) \neq 0$. Note that $a$ may be a cocycle, but $b$ is not. (If $\sigma = [g\mid 1_0]$ for a loop $1_1 \neq g \in T_1$, then $b(d[sog^{-1}\mid g\mid 1_0]) \neq 0$.)

In general, one cannot even expect $f^*$ to annihilate all $\cup_2$-products of cocycles as they are related to Steenrod squares. For $k = \mathbb{Z}_2$ and any non-zero $[a] \in H^2(BT)$, one has

$$[a] = Sq^0[a] = [a \cup_2 a] \neq 0.$$  

(9.34)

The situation changes if we can invert 2.

**Proposition 9.7.** Assume that 2 is invertible in $k$. Then one can choose representatives $(c_i)$ such that $f^*$ additionally annihilates all $\cup_2$-products of cocycles.

**Proof.** Let $\iota: T \to T$ be the group inversion. Being a morphism of groups, it induces involutions of $ET$ and $BT$, which we denote by the same letter. Recall that $\iota_*$ changes the sign of all generators $x_i \in H_1(T)$ and all cogenerators $y_i \in H_2(BT)$. Starting from any set of representatives $(c_i)$, we set

$$\tilde{c}_i = \frac{1}{2} c_i - \frac{1}{2} \iota_* c_i,$$  

(9.35)

so that $\iota_* \tilde{c}_i = -\tilde{c}_i$. We construct $F$ and $f$ based on these representatives. The equivariance of $F$ with respect to the involutions follows inductively from the recursive definition, and it entails that of $f$.

Now let $a$ and $b$ be cocycles. By Lemma 6.2, the value $f^*(a \cup_2 b)$ only depends on the cohomology classes of $a$ and $b$. In particular, we may assume that $a$ is of even degree $2k$ and $b$ of degree $2l$. Then $a \cup_2 b$ is of degree $2(k + l - 1)$, whence

$$\iota^* f^*(a \cup_2 b) = (-1)^{k+l} f^*(a \cup_2 b).$$  

(9.36)
On the other hand, we have
\[ \iota^*(a \cup_2 b) = \iota^*(a) \cup_2 \iota^*(b) \] (9.37)
by naturality. Now \( \iota^*(a) \) is cohomologous to \((-1)^k a\) and \( \iota^*(b) \) cohomologous to \((-1)^l a\), which implies
\[ f^*(\iota^*(a \cup_2 b)) = (-1)^{k+l} f^*(a \cup_2 b). \] (9.38)
Since \( 2 \) is invertible in \( k \), this can only happen if the \( \cup_2 \)-product vanishes. \[ \square \]

10. The kernel of the formality map

The kernel of the formality map \( f^*: C^*(BT) \to H^*(BT) \) constructed in the previous section depends on the choice of representatives. It will be convenient to consider instead an ideal \( kX \triangleleft C^*(X) \) for any simplicial set \( X \) such that \( f_{BT} \subset \ker f^* \) independently of any choices and also \( \kappa^*(kY) \subset kX \) for any map \( \kappa: X \to Y \).

Let \( X \) be a simplicial set. Given elements \( b_0, \ldots, b_k \in C^*(X) \), we write the repeated \( \cup_1 \)-product as
\[ U_0(b_0) = b_0 \quad \text{and} \quad U_k(b_0, \ldots, b_k) = -U_{k-1}(b_0, \ldots, b_{k-1}) \cup_1 b_k \] (10.1)
for \( k \geq 1 \), compare the Gugenheim–May twisting cochain (8.39).

We define
\[ \mathfrak{t} = \mathfrak{t}_X \triangleleft C^*(X) \] (10.2)
to be the ideal generated by the following elements where \( a, b, a_*, b_* \in C^*(X) \):

(1) all elements of odd degree;
(2) all coboundaries;
(3) all elements \( E_k(a; b_1, \ldots, b_k) \) with \( k \geq 1 \);
(4) all elements \( F_{kl}(a_1, \ldots, a_k; b_1, \ldots, b_l) \) with \( (k, l) \neq (1, 1) \);
(5) all elements \( a \cup_2 E_k(b_1, c_1, \ldots, c_k) \) with \( k \geq 2 \);
(6) all elements \( a \cup_2 U_k(b_0, \ldots, b_k) \) with \( k \geq 0 \) where \( a \) and \( b_* \) are cocycles.

**Lemma 10.1.** Let \( \kappa: X \to Y \) be a map of simplicial sets. Then \( \kappa^*(kY) \subset kX \).

**Proof.** This follows directly from the naturality of the extended hga operations. \[ \square \]

Let us write
\[ [a, b] = ab - (-1)^{|a||b|} ba \] (10.3)
for the commutator of \( a, b \in C^*(X) \).

**Lemma 10.2.** For all \( a, b \in C^*(X) \),
\[ [a, b] \equiv 0 \pmod{\mathfrak{t}}. \]

**Proof.** This results from the definition of \( \mathfrak{t} \) and the identity
\[ d(\cup_1)(a; b) = [a, b]. \] (10.4) \[ \square \]

**Lemma 10.3.** The \( \cup_1 \)-product is a right derivation of the commutator. That is,
\[ [a, b] \cup_1 c = (-1)^{|a|}[a, b \cup_1 c] + (-1)^{|b||c|}[a \cup_1 c, b] \]
for all \(a, b, c \in C^*(X)\).

**Proof.** This is a consequence of the Hirsch formula (6.14). \[\square\]

**Lemma 10.4.** Let \(a, b, c \in C^*(X)\).

(i) Modulo \(\xi\), the \(\cup_2\)-product is both a left and a right derivation of the commutator. In other words,

\[
a \cup_2 (bc) \equiv (a \cup_2 b)(c + (-1)^{|a||b|} b(a \cup_2 c)) \pmod{\xi},
\]

\[
(a \cup_2 b) c \equiv (-1)^{|b||c|} (a \cup_2 c) b + a(b \cup_2 c) \pmod{\xi}.
\]

(ii) One has

\[
a \cup_2 [b,c] \equiv [a,b] \cup_2 c \equiv 0 \pmod{\xi}.
\]

**Proof.** The first part follows from the identities

\[
d(F_{12})(a; b, c) \equiv E_2(a; b, c) - F_{11}(a; b) c + F_{11}(a, b; c) - F_{11}(a; c), \tag{10.5}
\]

\[
d(F_{21})(a; b, c) \equiv a F_{11}(b; c) - F_{11}(a; b) c + F_{11}(a, c; b) - E_2(c; a, b), \tag{10.6}
\]

see (6.17). It implies the formulas

\[
a \cup_2 [b,c] \equiv [a \cup_2 b, c] + (-1)^{|a||b|}[b, a \cup_2 c] \pmod{\xi}, \tag{10.7}
\]

\[
[a, b] \cup_2 c \equiv (-1)^{|b||c|} [a \cup_2 c, b] + [a, b \cup_2 c] \pmod{\xi}, \tag{10.8}
\]

which together with Lemma 10.2 entail the second claim. \[\square\]

**Remark 10.5.** So far we have only used parts (1)–(4) of the definition of \(\xi\). The elements listed there are also contained in ker \(f^*\) by Theorem 9.6 and because \(H^*(BT)\) is concentrated in even degrees. Lemmas 10.2–10.4 therefore hold as well for \(X = BT\) and ker \(f^*\) instead of \(\xi\), for any choice of representatives \(c_i \in C_1(T)\).

**Lemma 10.6.** Let \(a, b, c_1, \ldots, c_k \in C^*(BT)\) with \(k \geq 2\). Then

\[
a \cup_2 E_k(b; c_1, \ldots, c_k) \equiv E_k(b; c_1, \ldots, c_k) \cup_2 a \equiv 0 \pmod{\text{ker } f^*}.
\]

**Proof.** When the surjection \(e_k\) with \(k \geq 2\) is split into two, then at least one part will have an enclave. By the composition rule for the surjection operad, this implies that each surjection appearing in \(f_{11} \circ_2 e_k\) or \(f_{11} \circ_1 e_k\) again has an enclave. This gives the desired identities by Proposition 9.5. \[\square\]

**Lemma 10.7.** Let \(a, b, c \in C^*(BT)\). If \(a\) is cocycle of degree \(|a| \leq 2\), then

\[
a \cup_2 (b \cup_1 c) \equiv (b \cup_1 c) \cup_2 a \equiv 0 \pmod{\text{ker } f^*}.
\]

**Proof.** We consider the element \(g_{12} = (2, 3, 1, 3, 1, 2, 1)\) in the surjection operad, following Kadeishvili [19, Remark 2]. \[\dagger\] It satisfies

\[
d g_{12} = e_1 \circ_1 f_{11} + (12) \cdot (e_1 \circ_2 f_{11}) - f_{11} \circ_2 e_1 - f_{12} + (23) \cdot f_{12} \tag{10.9}
\]

\[\dagger\]Kadeishvili takes \(g_{12} = (1, 2, 3, 1, 3, 2)\) and \(g_{21} = (1, 2, 3, 2, 3, 1, 3)\) instead. Assuming his definition of \(E^1_{pq}\) (see Footnote 9), this gives the formula for \(d(G_{12})\) stated in [19, Remark 2] with the term \((a \cup_2 c) \cup_1 b\) replaced by \((a \cup_2 b) \cup_1 c\). In the formula for \(d(G_{21})\), the double \(\cup_2\)-product should read \((a \cup_2 c) \cup_1 b\).
and the corresponding interval cut operation therefore

\[ d(G_{12})(a, b, c) \equiv \mp a \cup (b \cup c) \pmod{\ker f^*}. \quad (10.10) \]

Because there are three 1’s appearing in \( g_{12} \), the corresponding interval cut operation on a simplex \( \sigma \) has the property that the first simplex \( \sigma^{(1)} \) in the resulting tensor product involves three intervals, each of which contributes at least one vertex. Now the first occurrence of 1 in \( g_{12} \) is surrounded by two occurrences of 3. Hence the associated interval for this 1 must involve at least two vertices for otherwise the simplex \( \sigma^{(3)} \) made up of the 3-intervals would contain twice the same vertex and therefore be degenerate. Hence \( \sigma^{(1)} \) is of dimension at least 3.

Dually, \( G_{12}(a, b, c) \) vanishes for \( |a| \leq 2 \), which implies that the left-hand side of (10.10) is a coboundary if \( a \) is additionally a cocycle. This proves that the first term in the statement is congruent to 0.

The second part follows analogously by looking at \( g_{21} = (3, 1, 3, 2, 3, 2, 1) \), which satisfies

\[ d g_{21} = (23) \cdot (e_1 \circ_1 f_{11}) + e_1 \circ_2 f_{11} - f_{11} \circ_1 e_1 + f_{21} - (12) \cdot f_{21}. \quad (10.11) \]

\[ \square \]

**Proposition 10.8.** For all cocycles \( a, b_0, \ldots, b_k \in C^*(BT) \), \( k \geq 1 \), we have

\[ a \cup_2 U_k(b_0, \ldots, b_k) \equiv U_k(b_0, \ldots, b_k) \cup_2 a \equiv 0 \pmod{\ker f^*}. \]

**Proof.** We show that the first term in the statement lies in \( \mathfrak{k} \); the proof for the second is analogous.

Write \( b = U_k(b_0, \ldots, b_k) \) and assume first that \( a = dc \) is a coboundary. Then

\[ d(c \cup_2 b) = dc \cup_2 b \pm c \cup_2 db \pm c \cup_1 b \pm b \cup_1 c, \quad (10.12) \]

hence

\[ a \cup_2 b \equiv \mp c \cup_2 db \pmod{\ker f^*}. \quad (10.13) \]

Since \( b_1, \ldots, b_k \) are cocycles, we have

\[ db = \sum_{i=1}^k \pm U_{k-i}([U_{i-1}(b_0, \ldots, b_{i-1}), b_i], b_{i+1}, \ldots, b_k). \quad (10.14) \]

By a repeated application of Lemma 10.3, we see that each term on the right-hand side of (10.14) is a sum of commutators, so that the right-hand side of (10.13) vanishes by the analogue of Lemma 10.4(ii). This proves the claim for \( a = dc \).

As a consequence, we may replace \( a \) by any cocycle cohomologous to it. Because \( H^*(BT) \) is generated in degree 2, we may in particular assume that \( a \) is the product of cocycles of degree 2. By the analogue of Lemma 10.4(i), it is enough to consider the case where \( a \) is a single degree-2 cocycle, where Lemma 10.7 applies. \( \square \)

**Corollary 10.9.** Assume that 2 is invertible in \( k \), and let \( f^*: C^*(BT) \to H^*(BT) \) be a formality map as in Proposition 9.7. Then \( \mathfrak{k}_{BT} \subset \ker f^* \).

**Proof.** We have already observed in Remark 10.5 that \( \ker f^* \) contains the elements mentioned in parts (1)–(4) of the definition of \( \mathfrak{k}_{BT} \). Part (5) is covered by Lemma 10.6, and part (6) by Proposition 10.8 for \( k \geq 1 \) and by Proposition 9.7 for \( k = 0 \). \( \square \)

**Remark 10.10.** Corollary 10.9 remains valid for any \( k \) and any formality map constructed in Section 9 if part (6) in the definition of \( \mathfrak{k} \) is restricted to \( k \geq 1 \), that is, if \( \mathfrak{k} \) is only required to contain
(6') all elements $a \cup_k U_k(b_0, \ldots, b_k)$ with $k \geq 1$ where $a$ and $b_\bullet$ are cocycles.

11. Spaces and shc maps

Let $X$ be a simplicial set, and let $\mathfrak{t} = \mathfrak{t}_X \circ C^*(X)$ be the ideal defined in the preceding section. We want to relate $\mathfrak{t}$ to the canonical shc structure on $C^*(X)$. From Theorem 6.3, we conclude that the structure map $\Phi : C^*(X) \otimes C^*(X) \rightarrow C^*(X)$ is $\mathfrak{t}$-strict. Moreover, the associativity homotopy $h^a : \Phi \circ (\Phi \otimes 1) \simeq \Phi \circ (1 \otimes \Phi)$ is $\mathfrak{t}$-trivial, but the commutativity homotopy $h^c : \Phi \circ T\simeq \Phi$ is not in general. This failure requires extra attention.

We introduce the following terminology: Let $A$ be a simplicial set, and let $X = \mathfrak{t}$ be shm homotopies, $\mathfrak{b}$-trivial on cocycles. We need to extend this observation.

**Lemma 11.1.** Let $h, k : A^\otimes m \rightarrow B$ be shm homotopies, $\mathfrak{b}$-trivial on cocycles.

(i) The shm homotopies $h \cup k$ and $h^{-1}$ are again $\mathfrak{b}$-trivial on cocycles.

(ii) If $f : B \rightarrow C$ is a $\mathfrak{c}$-strict shm map such that $f(\mathfrak{t})(\mathfrak{b}) \subset \mathfrak{c}$, then $f \circ h$ is $\mathfrak{c}$-trivial on cocycles.

(iii) If $T : A^\otimes m \rightarrow A^\otimes m$ is some permutation of the factors, then $h \circ T$ is $\mathfrak{b}$-trivial on cocycles.

**Proof.** The first claim follows from the definition of the cup product and the formula for the inverse given in Lemma 2.2(ii). The second part is analogous to Lemma 3.1(ii), and the last claim is trivial. 

Because $\mathfrak{t}$ contains all $\cup_2$-products of cocycles, both $h^c$ and the homotopy $k^c = h^c \circ T$ in the other direction are $\mathfrak{t}$-trivial on cocycles. We need to extend this observation.

**Lemma 11.2.** For any $n \geq 0$, the shm homotopy

$h^c \circ (1 \otimes \Phi^{[n]}_l) : C^*(X) \otimes C^*(X)^\otimes n \rightarrow C^*(X)$

is $\mathfrak{t}$-trivial on cocycles, and the same holds with $k^c$ instead of $h^c$.

**Proof.** By naturality, we may assume that $\kappa$ is the identity map of $X = BT$.

Let $l \geq 0$, and let $b_1, \ldots, b_l$ be cocycles of $C^*(BT)^\otimes n$ with $b_i = b_{i,1} \otimes \cdots \otimes b_{i,n}$ where all $b_{i,j}$ are cocycles. We claim that $\Phi^{[n]}_l(b_1, \ldots, b_l)$ is a linear combination of products of terms of the following two kinds: Repeated $\cup_1$-products $U_k(c_0, \ldots, c_k)$ of cocycles with $k \geq 0$, or $E_k$-terms with $k \geq 2$ (and not necessarily cocycles as arguments).

This follows by induction: $\Phi^{[n]}_0(b_1, \ldots, b_l) = 0$, and $\Phi^{[n]}_1(b_1) = b_{1,1} \cdots b_{1,n}$ is a product of cocycles $b_{1,j} = U_0(b_{1,j})$. For the induction step, we observe from the formula for $\Phi$ and the composition formula (3.20) for shm maps that we get products of terms $E_m(\cdots)$ with some value $\Phi^{[n]}_l(b_1, \ldots, b_l)$ plugged into the first argument and cocycles into the remaining arguments.

We consider each factor $E_m$ separately. For $m = 0$, the induction hypothesis applies and for $m \geq 2$ there is nothing to show. So assume $m = 1$. By induction and the Hirsch formula (6.14), we may assume that the first argument is a repeated $\cup_1$-product of cocycles or a term $E_k$ with $k \geq 2$. In the former case, we get another repeated $\cup_1$-product of cocycles. In the
latter case, the identity (6.4) shows that we end up with a sum of terms $E_{k'}$ with $k' \geq k \geq 2$. This completes the proof of the claim.

Now consider $h_{(m)}^c$ for $m \geq 1$, or rather its description modulo $\mathfrak{k}$ given in Theorem 6.3(iv). We have to plug cocycles into the first arguments and values $\Phi_{(i)}^{[n]}(b_1, \ldots, b_l)$ as above into the second arguments. Because the $\cup_2$-product is a derivation modulo $\mathfrak{k}$ by Lemma 10.4(i), we only have to consider terms of the following two kinds in light of our previous discussion: firstly, terms $a \cup_2 b$ where $a$ is a cocycle and $b$ a repeated $\cup_1$-product of cocycles, and secondly, $\cup_2$-products where the second argument is an $E_k$-term with $k \geq 2$. Both kinds of terms are contained in $\mathfrak{k}$ by definition.

The proof for $k^c$ is analogous. \qed

**Proposition 11.3.** For any $n \geq 0$, the iteration $\Phi^{[n]}: C^*(X)^{\otimes n} \Rightarrow C^*(X)$ is an shc map that is $\mathfrak{k}$-natural on cocycles.

**Proof.** Munkholm [26, Proposition 4.5] has shown that $\Phi^{[n]}$ is an shc map. The non-trivial part of the proof is to construct a homotopy

$$h^{[n]}: \Phi \circ (\Phi^{[n]} \otimes \Phi^{[n]}) \simeq \Phi^{[n]} \circ \Phi \otimes \circ T_n,$$  

(11.2)

where $T_n: A^{\otimes n} \otimes A^{\otimes n} \rightarrow (A \otimes A)^{\otimes n}$ is the reordering of the $2n$ factors $A = C^*(X)$ corresponding to the permutation

$$
\begin{pmatrix}
1 & 2 & \ldots & n & n+1 & n+2 & \ldots & 2n
\end{pmatrix}.
$$

(11.3)

We follow Munkholm’s arguments and verify that in our setting they lead to a homotopy that is $\mathfrak{k}$-trivial on cocycles. There is nothing to show for $n \leq 1$.

We start with the case $n = 2$, see [26, p. 31]. The homotopies labelled $h_1$, $h_2$, $h_4$ and $h_5$ by Munkholm are $\mathfrak{k}$-trivial because $\Phi \circ (1 \otimes \Phi)$ is $\mathfrak{k}$-strict and the homotopy $h^a$ is $\mathfrak{k}$-trivial, see Lemmas 6.3, 11.1(ii) and 3.1. So consider the homotopy

$$h_3 = \Phi \circ (1_A \otimes \Phi) \circ (1_A \otimes k^c \otimes 1_A).$$

(11.4)

We have shown above that $k^c$ is $\mathfrak{k}$-trivial on cocycles. Together with the $\mathfrak{k}$-strictness of $\Phi \circ (1 \otimes \Phi)$, this implies by Lemma 11.1 that $h_3$ is $\mathfrak{k}$-trivial on cocycles, too, and therefore also the sought-after homotopy

$$h^{[2]} = h_1 \cup h_2 \cup h_3 \cup h_4 \cup h_5.$$  

(11.5)

For the induction step, we have another set of homotopies $h_1$ to $h_5$, see [26, p. 32]. The homotopies $h_1$ and $h_4$ are $\mathfrak{k}$-trivial by Corollary 4.3 and Lemma 3.1 because $\Phi$ is $\mathfrak{k}$-strict. The homotopy $h_3$ is actually not needed. In fact, the identity

$$(\Phi^{[n]} \otimes 1)_A \circ T_{A,A^{\otimes n}} = T_{A,A} \circ (1 \otimes \Phi^{[n]})$$

(11.6)

(see [26, §3.6 (iii)] and [26, Proposition 3.3 (ii)] (or Corollary 4.3) implies that

$$
(1_A \otimes T_{A,A} \otimes 1_A) \circ (\Phi^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes 1_A)
$$

$$
= (\Phi^{[n]} \otimes \Phi^{[n]} \otimes 1_A \otimes 1_A) \circ (1_A \otimes T_{A,A^{\otimes n}} \otimes 1_A),
$$

(11.7)

which means that the homotopy relation labelled $\Phi^{[n]}$ in [26, p. 32] is an equality. That the homotopy $h_5$ is $\mathfrak{k}$-trivial on cocycles uses that so is $h^{[n]}$ by induction, that $\Phi$ is $\mathfrak{k}$-strict and also Lemma 11.1.

To show that

$$h_2 = h^{[2]} \circ (\Phi^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes 1_A)$$

(11.8)
is $\mathfrak{f}$-trivial on cocycles, we may by (2.11) and Lemma 11.1(i) consider the composition with each factor in (11.5) separately. (Recall that $h \circ f = h \mathcal{B} f$ for an shm homotopy $h$ and an shm map $f$.) The homotopies $h_1, h_2, h_4$ and $h_5$ for the case $n = 2$ are $\mathfrak{f}$-trivial on all arguments and therefore pose no problem.

It remains to look at the homotopy

$$k_1 = h_3 \circ (\Phi^n \otimes 1_A \otimes \Phi^n \otimes 1_A)$$

(11.9)

$$= \Phi \circ (1_A \otimes \Phi) \circ (1_A \otimes k^c \otimes 1_A) \circ (\Phi^n \otimes 1_A \otimes \Phi^n \otimes 1_A).$$

We want to compare it to the homotopy

$$k_2 = \Phi \circ (1_A \otimes \Phi) \circ \left(\Phi^n \otimes (k^c \circ (1_A \otimes \Phi^n)) \otimes 1_A\right).$$

(11.10)

Denoting reduction modulo $\mathfrak{f}$ by a bar above a map, we have

$$\bar{k}_1 = \mu^{[3]}_{A/t} \circ (1_{A/t} \otimes \bar{k}^c \otimes 1_{A/t}) \circ \left(\Phi^n \otimes 1_A \otimes \Phi^n \otimes \bar{1}_A\right),$$

(11.11)

$$\bar{k}_2 = \mu^{[3]}_{A/t} \circ \left(\Phi^n \otimes (\bar{k}^c \circ (1_A \otimes \Phi^n)) \otimes \bar{1}_A\right).$$

(11.12)

Because $\bar{\Phi}^{[n]}$ is a dga map, $\bar{k}_1$ and $\bar{k}_2$ agree, see Lemma 4.5. Moreover, the homotopy $\bar{k}^c \circ (1 \otimes \Phi^{[n]})$ is 0-trivial (that is, trivial) on cocycles by Lemma 11.2, which together with Lemma 11.1(ii) implies that $k_2$ has the same property. Putting these facts together, we obtain that $k_1$ is $\mathfrak{f}$-trivial on cocycles. This completes the proof. \hfill $\Box$

Let $n \geq 0$ and choose cocycles $a_1, \ldots, a_n \in C^*(X)$ of even positive degrees. We write $a = (a_1, \ldots, a_n)$ and consider the shm map

$$\Lambda_a : k[x] := k[x_1] \otimes \cdots \otimes k[x_n] \xrightarrow{\lambda_a} C^*(X)^\otimes_n \xrightarrow{\Phi^{[n]}} C^*(X),$$

(11.13)

where $x_1, \ldots, x_n$ are indeterminates of degrees $|x_i| = |a_i|$ and $\lambda_a$ is the tensor product of the dga maps sending each $x_i$ to $a_i$.

Remark 11.4. The map $\Lambda_a$ can be expressed in terms of the hga operations on $C^*(X)$. It is not the same as Wolf’s explicit shm map [35, Section 3], which only uses $\cup_1$-products.

Proposition 11.5. If 2 is invertible in $k$, then the map $\Lambda_a : k[x] \to C^*(X)$ is a $\mathfrak{f}$-strict and $\mathfrak{f}$-natural shm map.

Proof. Since $\Phi^{[n]}$ is $\mathfrak{f}$-strict, so is $\Lambda_a$ by Lemma 3.1(i). It remains to consider the diagram

$$\begin{array}{ccc}
k[x] \otimes k[x] & \xrightarrow{\mu^{[n]} \otimes T_n} & k[x] \\
\downarrow{\lambda_a \otimes \lambda_a} & & \downarrow{\lambda_a} \\
C^*(X)^\otimes_n \otimes C^*(X)^\otimes_n & \xrightarrow{\Phi^{[n]} \otimes T_n} & C^*(X)^\otimes_n \\
\downarrow{\Phi^{[n]} \otimes \Phi^{[n]}} & & \downarrow{\Phi^{[n]}} \\
C^*(X) \otimes C^*(X) & \xrightarrow{\Phi} & C^*(X).
\end{array}$$

(11.14)

Each dga map $k[x_i] \to C^*(X)$, $x_i \mapsto a_i$ is in fact a $\mathfrak{f}$-natural shm map because we can choose $b = -\frac{1}{2} a_i \cup_2 a_i \in \mathfrak{f}$ in the statement of [10, Proposition 7.2]. Then the shm map $\lambda_a$ is $\Phi^{[n]}$-natural by Lemma 5.1 and induction. Because $\Phi^{[n]}$ is $\mathfrak{f}$-strict, its composition $h_1$ with the homotopy making the top diagram commute is $\mathfrak{f}$-trivial by Lemma 3.1(ii).
Composed with the top left arrow, the homotopy making the bottom square commute is $\mathfrak{t}$-trivial by Proposition 11.3. The cup product of this composed homotopy $h_2$ with $h_1$ then is $\mathfrak{t}$-trivial as well by Lemma 2.2(i). This proves the claim since

$$
(\lambda_\alpha \otimes \lambda_\alpha) \circ (\Phi^{[n]} \otimes \Phi^{[n]}) = (\lambda_\alpha \circ \Phi^{[n]} \otimes (\lambda_\alpha \circ \Phi^{[n]}) = \Lambda_\alpha \otimes \Lambda_\alpha
$$

by Corollary 4.3. \[\square\]

If $H^\ast(X) \cong k[x]$ is polynomial and each $a_i$ represents $x_i$ under this isomorphism, then $\Lambda_\alpha$ is a quasi-isomorphism. Note that it depends both on the choice of the generators $x_i$ and of their representatives $a_i$.

**Theorem 11.6.** Assume that 2 is invertible in $k$, and let $\varphi: Y \to X$ be a map of simplicial sets with polynomial cohomology. Let $\alpha$ and $\beta$ be representatives of some generators of $H^\ast(X)$ and $H^\ast(Y)$, respectively. Then the diagram

$$
\begin{array}{ccc}
H^\ast(X) & \xrightarrow{\delta^\ast} & H^\ast(Y) \\
\Lambda_\alpha \downarrow & & \downarrow \Lambda_\beta \\
C^\ast(X) & \xrightarrow{\delta^\ast} & C^\ast(Y)
\end{array}
$$

commutes up to a $\mathfrak{t}_Y$-trivial shm homotopy.

The corresponding result in [26, Section 7] is the heart of Munkholm’s paper, and for our proof of Theorem 1.3 in the next section Theorem 11.6 will also be crucial.

**Proof.** We write $f = \varphi^\ast$, $\alpha = (a_1, \ldots, a_n)$ and $\mathfrak{t} = \mathfrak{t}_Y$. By assumption, we have $H^\ast(X) = k[x_1, \ldots, x_n]$. We define $\tilde{\alpha} = (H^\ast(f)(a_1), \ldots, H^\ast(f)(a_n))$ and consider the diagram

$$
\begin{array}{ccc}
H^\ast(X) & \xrightarrow{\delta^\ast} & H^\ast(Y) \\
\Lambda_\alpha \downarrow & & \downarrow \Lambda_\beta \\
C^\ast(X)^{\otimes n} & \xrightarrow{f^{\otimes n}} & C^\ast(Y)^{\otimes n} & \xleftarrow{\Lambda_\alpha^{\otimes n}} & H^\ast(Y)^{\otimes n} \\
\Phi_\alpha^{[n]} \downarrow & & \Phi_\beta^{[n]} \downarrow & & \mu^{[n]} \\
C^\ast(X) & \xrightarrow{f} & C^\ast(Y) & \xleftarrow{\Lambda_\beta} & H^\ast(Y).
\end{array}
$$

The composition from $H^\ast(X)$ to $C^\ast(X)$ equals $\Lambda_\alpha$, and the one from $H^\ast(X)$ to $H^\ast(Y)$ is $H^\ast(f)$. The left square commutes strictly by the naturality of the hga structure. Lemma 5.2 implies that the right square commutes up to a $\mathfrak{t}$-trivial homotopy since $\Lambda_\beta$ is $\mathfrak{t}$-strict and $\mathfrak{t}$-natural by Proposition 11.5.

The composition

$$
k[x_i] \longrightarrow H^\ast(Y) \xrightarrow{\Lambda_\beta} C^\ast(Y)
$$

is a $\mathfrak{t}$-strict shm map, and the composition

$$
k[x_i] \longrightarrow C^\ast(X) \xrightarrow{f} C^\ast(Y)
$$

is the dga map sending $x_i$ to $f(a_i)$. Since both $(\Lambda_\beta)_{(1)}(\tilde{\alpha})$ and $f(\alpha_i)$ represent the even-degree element $\tilde{a}_i \in H^\ast(B)$ and $\mathfrak{t}$ contains all elements of odd degree, these two maps are homotopic via a $\mathfrak{t}$-trivial shm homotopy by [10, Proposition 7.1].

The two ways to go from $H^\ast(X)$ to $C^\ast(Y)^{\otimes n}$ in the diagram represent the tensor products of the shm maps just discussed. This implies by induction and Lemma 3.1 that also the triangle
commutes up to a $t^{2n}$-trivial homotopy. Its composition with $\Phi_n$ is $t$-trivial by Lemma 3.1 as $\Phi_n$ is $t$-strict. Lemma 2.2 concludes the proof.

\[\square\]

12. Homogeneous spaces

We are now ready to prove Theorem 1.3. We assume in this and the next section that $k$ is a principal ideal domain in which 2 is invertible.

Let $G$ be a connected Lie group and $\iota: K \hookrightarrow G$ a closed connected subgroup such that the order of the torsion subgroup of $H^*(G; \mathbb{Z})$ is invertible in $k$ and analogously for $K$. This implies that $BG$ and $BK$ have polynomial cohomology over $k$ (and in fact is equivalent to it), see [18, Remark IV.8.1]. While we will make use of a maximal torus $T \subset K$ in our proof, $G$ could more generally be any topological group such that $BG$ has polynomial cohomology in the sense of Section 8.1. By Lemma 8.5, we may assume both $BG$ and $BK$ to be 1-reduced. For simplicity, we denote the induced maps $C^*(BG) \to C^*(BK)$ and $H^*(BG) \to H^*(BK)$ both by $\iota^*$.

Our goal is to construct an isomorphism of graded algebras

\[H^*(G/K) \cong \text{Tor}_{H^*(BG)}(k, H^*(BK)),\]  

(12.1)

natural in the pair $(G, K)$. Recall from Proposition 8.6 that there is a natural isomorphism of graded algebras

\[H^*(G/K) \cong \text{Tor}_{C^*(BG)}(k, C^*(BK)),\]  

(12.2)

It suffices therefore to connect the two bar constructions underlying the torsion products in (12.1) and (12.2). We start by establishing an isomorphism of graded $k$-modules, proceeding in a way similar to Munkholm [26, §7.4]. Remember that we have defined one-sided bar constructions as twisted tensor products in (7.4).

Let $a$ and $b$ be representatives of generators of $H^*(BG)$ and $H^*(BK)$, respectively. We write the induced shm quasi-isomorphism $\Lambda_G: H^*(BG) \Rightarrow C^*(BG)$ introduced in (11.13) as $\Lambda_G$ and analogously $\Lambda^K_G = \Lambda_K: H^*(BK) \Rightarrow C^*(BK)$.

We define the map

\[\Theta_{G,K} : \text{B}(k, H^*(BG), H^*(BK)) \to \text{B}(k, C^*(BG), C^*(BK))\]  

(12.3)

as the composition of the chain maps

\[
\begin{align*}
\text{BH}^*(BG) \otimes_{\iota^* \circ \mu_{H^*(BG)}} H^*(BK) & \xrightarrow{\delta_h} \\
\text{BH}^*(BG) \otimes_{\Lambda^K_G \circ \mu_{H^*(BG)}} C^*(BK) & \xrightarrow{\delta_h} \\
\text{BH}^*(BG) \otimes_{\iota^* \circ \mu_{C^*(BG)}} C^*(BK) & \xrightarrow{\delta_h} \\
\text{BC}^*(BG) \otimes_{\iota^* \circ \mu_{C^*(BG)}} C^*(BK),
\end{align*}
\]

(12.4)

given by Lemmas 7.4, 7.1 and 7.2, respectively, where the $t_{BK}$-trivial twisting cochain homotopy $h$ in the second step comes from Theorem 11.6. Note that $\Theta_{G,K}$ depends on the chosen representative cocycles $a$ and $b$.

**Lemma 12.1.** Modulo $\text{BC}^*(BG) \otimes t_{BK}$, we have

\[\Theta_{G,K} \equiv \text{BA}_G \otimes \Lambda^K_{(1)}.\]
Recall that $\Lambda^K_{(1)}$ is the quasi-isomorphism of complexes

$$H^*(BK) \cong k[y_1, \ldots, y_n] \to C^*(BK), \quad y_1^{k_1} \cdots y_n^{k_n} \mapsto b_1^{k_1} \cdots b_n^{k_n}. \quad (12.5)$$

Proof. The congruence follows from Lemmas 7.1, 7.2 and 7.4, given that $\Lambda^K$ is a $\mathfrak{t}_{BK}$-strict shm map and $a$ a $\mathfrak{t}_{BK}$-trivial homotopy. $\square$

**Proposition 12.2.**

(i) The map

$$H^*(\Theta_{G,K}) : \text{Tor}_{H^*(BG)}(k, H^*(BK)) \to \text{Tor}_{C^*(BG)}(k, C^*(BK))$$

is an isomorphism of graded $k$-modules.

(ii) The Eilenberg–Moore spectral sequence for the fibration $G/K \hookrightarrow BK \to BG$ collapses at the second page.

Proof. Both $\Lambda^G$ and $\Lambda^K$ are quasi-isomorphisms, and so is $B\Lambda^G$. It follows from Lemma 12.1 as in Remark 7.5 that $\Theta_{G,K}$ induces an isomorphism between the second pages of these spectral sequences and therefore between the torsion products.

Because the spectral sequence for $B(k, H^*(BG), C^*(BK))$ collapses at this stage, so does the one for $B(k, C^*(BG), C^*(BK))$, which is the Eilenberg–Moore spectral sequence of the fibration. $\square$

**Remark 12.3.** We assume that 2 is a unit in $k$ to ensure that the shc maps $\Lambda^G$ and $\Lambda^K$ are natural with respect to $k_{BG}$ and $k_{BK}$, respectively, see the proof of Proposition 11.5. If this were not the case, then the congruence in Lemma 12.1 would still hold modulo $B_{BC^*(BG) \otimes C^>(BK)}$, and this is enough to prove Proposition 12.2. We thus recover Munkholm’s collapse theorem for spaces with polynomial cohomology [26, Theorem].

We now turn to the multiplicativity and naturality of $H^*(\Theta_{G,K})$. Here our approach is inspired by Wolf [35, p. 331]. Let $\kappa : T \to K$ be a morphism of simplicial groups where $T$ is some torus. We also choose a formality map $f^* : C^*(BT) \to H^*(BT)$ as in Proposition 9.7, keeping in mind that $f^*$ annihilates $\mathfrak{t}_{BT}$ by Corollary 10.9. Based on $\kappa$ and $f^*$ we define the map

$$\Psi_{\kappa} : B(k, C^*(BG), C^*(BK)) \to B(k, C^*(BG), H^*(BT)) \quad (12.6)$$

as the composition

$$\begin{array}{rcl}
BC^*(BG) \otimes_{\kappa^* \ast C^*(BG)} C^*(BK) & \xrightarrow{1 \otimes \kappa^*} & BC^*(BG) \otimes_{\kappa^* \ast C^*(BG)} C^*(BT) \\
& \downarrow & \downarrow \\
BC^*(BG) \otimes_{f^* \kappa^* \ast C^*(BG)} H^*(BT) & \xrightarrow{1 \otimes f^*} & BC^*(BG) \otimes_{f^* \kappa^* \ast C^*(BG)} H^*(BT). \quad (12.7)
\end{array}$$

**Lemma 12.4.**

(i) $\Psi_{\kappa}$ is a morphism of dgas.

(ii) If $\kappa$ is the inclusion of a maximal torus into $K$, then $H^*(\Psi_{\kappa})$ is injective, hence so is the map $H^*(G/K) \to H^*(G/T)$. 

The composition $\Psi_{K, G, K}$ is the map
\[
\begin{align*}
B\Lambda^G \otimes \kappa^* & \quad \longrightarrow \quad B\Lambda^G \otimes \kappa^* \\
\frac{B\Lambda^G \otimes \kappa^*}{\mathfrak{h}^*} & \quad \longrightarrow \quad \frac{B\Lambda^G \otimes \kappa^*}{\mathfrak{h}^*}.
\end{align*}
\]

The idea of reducing to a maximal torus goes back to Baum [3, Lemma 7.2]. Note that Lemma 7.3(i) confirms that $B\Lambda^G \otimes \kappa^*$ is a chain map because $B\Lambda^G$ is $kBG$-trivial, so that
\[
f^* \kappa^* \iota^* t_{H^* (BG)} \colon B\Lambda^G \otimes \kappa^* \longrightarrow B\Lambda^G \otimes \kappa^*.
\]

**Proof.** The first map in (12.7) is a dga map by naturality and the second one by inspection of the product formula (7.10). This proves the first claim.

If $T \subset K$ is a maximal torus, then $H^* (K / T)$ is concentrated in even degrees, as is $H^* (BK)$ by assumption. Hence the Serre spectral sequence for the fibration $K / T \to BT \to BK$ degenerates at the second page. By the Leray–Hirsch theorem, this implies that $H^* (BT)$ is a free module over $H^*(BK)$ with $\kappa^* (H^*(BK))$ being a direct summand.

As a consequence, the induced map
\[
\text{Tor}_{H^* (BG) \otimes k} \longrightarrow \text{Tor}_{H^* (BG) \otimes k}
\]
is injective. This is the map between the second pages of the Eilenberg–Moore spectral sequences for $G/K$ and $G/T$, respectively. Because these spectral sequences degenerate at this level by Proposition 12.2(ii), this implies that the map $1 \otimes \kappa^*$ in (12.7) is injective in cohomology.

Another standard spectral sequence argument shows that the map $1 \otimes f^*$ in (12.7) is a quasi-isomorphism since $f^*$ is so. Together with the naturality of the isomorphism (12.2), this shows the second claim.

The last part is a consequence of Lemma 12.1. □

**Theorem 12.5.** The isomorphism $H^* (\Theta_{G, K})$ is multiplicative.

**Proof.** Let $\kappa : T \to K$ be the inclusion of a maximal torus. By Lemma 12.4, it suffices to prove that the composition $\Psi_{K, G, K} = B\Lambda^G \otimes \kappa^*$ is multiplicative up to homotopy. Clearly, $\kappa^*$ is multiplicative.

We claim that $B\Lambda^G$ is multiplicative up to a $kBG$-trivial coalgebra homotopy
\[
h : B\Lambda^G \otimes B\Lambda^G \to B\Lambda^G.
\]

To see this, we consider the diagram
\[
\begin{align*}
B\Lambda^G \otimes B\Lambda^G & \quad \longrightarrow \quad B\Lambda^G \\
\frac{B\Lambda^G \otimes B\Lambda^G}{\mathfrak{h}^*} & \quad \longrightarrow \quad \frac{B\Lambda^G \otimes B\Lambda^G}{\mathfrak{h}^*} \\
\Lambda^G \otimes B\Lambda^G & \quad \longrightarrow \quad \Lambda^G \otimes B\Lambda^G
\end{align*}
\]

The composition along the top row is the multiplication in $B\Lambda^G$, and by [10, Prop. 4.3] the one along the bottom row equals the product in $B\Lambda^G$. The left square commutes by naturality of the shuffle map (Lemma 4.4). The right square commutes up to a $\xi_{BG}$-trivial coalgebra homotopy because $\Lambda^G$ is a $\mathfrak{h}_{BG}$-natural shc map by Proposition 11.5. The claim follows.

By transposing factors, we can pass from the tensor product
\[
(B\Lambda^G \otimes B\Lambda^G) \otimes (H^* (BK) \otimes H^* (BK))
\]

We then use naturality to show that the resulting map is multiplicative up to homotopy. This completes the proof of Theorem 12.5.
to the single twisted tensor product
\[ (BH^*(BG) \otimes BH^*(BK)) \otimes_{\nu'} (BH^*(BK) \otimes BH^*(BK)) \]
whose twisting cochain \( t' \) vanishes except for
\[ t'(x \otimes 1) = \iota^*(x) \otimes 1, \quad t'(1 \otimes x) = 1 \otimes \iota^*(x) \]
with \( x \in H^*(BG) \). We want to show that the two chain maps
\[ (B\Lambda G \otimes \kappa^*) (\mu_{BH^*(BG)} \otimes \mu_{H^*(BK)}) = B\Lambda G \mu_{BH^*(BG)} \otimes \kappa^* \mu_{H^*(BK)} \] 
and
\[ (\mu_{BC^*(BG)} \otimes \mu_{H^*(BK)}) (B\Lambda G \otimes B\Lambda G \otimes \kappa^* \otimes \kappa^*) = \mu_{BC^*(BG)} (B\Lambda G \otimes B\Lambda G) \otimes \kappa^* \mu_{H^*(BK)} \]
are homotopic. Given that the coalgebra homotopy \( h \) is \( \xi_{BG} \)-trivial, we can appeal to Lemma 7.3(ii). □

**Theorem 12.6.** Let \( \varphi: (G,K) \to (G',K') \) be a map of pairs, both satisfying our assumptions, and choose representatives \( a' \) and \( b' \) for generators of \( H^*(BG') \) and \( H^*(BK') \), respectively. Then the following diagram commutes.

\[
\begin{array}{ccc}
\text{Tor}_{H^*(BG')} \left( \mathbb{k}, H^*(BK') \right) & \xrightarrow{\text{Tor}_{\varphi^*}(1,\varphi^*)} & \text{Tor}_{H^*(BG)} \left( \mathbb{k}, H^*(BK) \right) \\
\downarrow & & \downarrow \\
H^*(G'/K') & \xrightarrow{1 \otimes \iota^*} & H^*(G/K)
\end{array}
\]

**Proof.** Let \( T \subset K \) again be a maximal torus. We consider the diagram

\[
\begin{array}{ccc}
B(\mathbb{k}, H^*(BG'), H^*(BK')) & \xrightarrow{B(1,\varphi^*,\varphi^*)} & B(\mathbb{k}, H^*(BG), H^*(BK)) \\
\downarrow{\Theta_{g',\kappa'}} & & \downarrow{\Theta_{g,\kappa}} \\
B(\mathbb{k}, C^*(BG'), C^*(BK')) & \xrightarrow{B(1,\varphi^*,\varphi^*)} & B(\mathbb{k}, C^*(BG), C^*(BK)) \\
\downarrow{1 \otimes \iota^* \varphi^*} & & \downarrow{1 \otimes \iota^*} \\
B(\mathbb{k}, C^*(BG'), H^*(BT)) & \xrightarrow{B(1,\varphi^*,1)} & B(\mathbb{k}, C^*(BG), H^*(BT)) \\
\downarrow{1 \otimes f^*} & & \downarrow{1 \otimes f^*} \\
B(\mathbb{k}, C^*(BG'), H^*(BT)) & \xrightarrow{B(1,\varphi^*,1)} & B(\mathbb{k}, C^*(BG), H^*(BT)).
\end{array}
\]

We have to show that the top square in the diagram commutes in cohomology. By Lemma 12.4(ii), it suffices to consider the prolongations of the maps in question to \( B(\mathbb{k}, C^*(BG), H^*(BT)) \).

The composition along the path via \( B(\mathbb{k}, H^*(BG), H^*(BK)) \) gives the map
\[ B\Lambda G \otimes B\varphi^* \otimes \kappa^* \varphi^* \]
by Lemma 12.4(iii). Since the middle square in (12.16) commutes by naturality and the bottom square by construction, the same result shows that the path via \( B(\mathbb{k}, C^*(BG'), C^*(BK')) \) gives
\[ B\varphi^* \otimes B\Lambda G \otimes \kappa^* \varphi^*. \]

By Theorem 11.6, there is a \( \xi_{BG} \)-trivial homotopy \( h \) between the shm maps \( \varphi^* \circ \Lambda G' \) and \( \Lambda G \circ \varphi^* \). In other words, \( Bh \) is a \( \xi_{BG} \)-trivial coalgebra homotopy between \( B\Lambda G \otimes B\varphi^* \)
and \( B \varphi^* B A^G \). This implies by Lemma 7.3(ii) that \( B h \otimes \kappa^* \varphi^* \) is a homotopy between the maps (12.17) and (12.18) and completes the proof. \( \Box \)

**Corollary 12.7.** The isomorphism (12.1) does not depend on the chosen representatives \( a \) and \( b \).

**Proof.** Take \( \varphi: (G, K) \to (G, K) \) to be the identity map in Theorem 12.6. \( \Box \)

**Remark 12.8.** Theorems 12.5 and 12.6 actually hold not just for principal ideal domains, but for all coefficient rings \( k \) in which 2 is invertible. (We remark that already Gugenheim–May [15, §4] allow Noetherian rings and Husemoller–Moore–Stasheff [18] arbitrary coefficients.) The only change required is to replace cochain complexes with chain complexes, which are now homotopy Gerstenhaber coalgebras. Imitating our arguments, one obtains quasi-isomorphisms of dgcs

\[
C(G/K) \to \Omega(k, C(BG), C(EG/K)) \leftarrow \Omega(k, C(BG), C(BK)) \to \Omega(k, H(BG), H(BK)). \tag{12.19}
\]

The cobar constructions are dgcs by the homological analogue of Proposition 7.6. The dual maps to (12.19) induce the isomorphism (12.1) by the universal coefficient spectral sequence since \( H(BG) \) and \( H(BK) \) are free of finite type over \( k \). We have chosen the cohomological setting in this paper because we expect it to be more accessible.

**Remark 12.9.** Baum [3, Example 4] has observed that there is no multiplicative isomorphism of the form (12.1) for the projective unitary group \( PU(n) = U(n)/U(1) \) with \( n \equiv 2 \) (mod 4) and \( k = \mathbb{Z}_2 \). This is readily verified for \( PU(2) \): Recall that the torsion product of graded commutative algebras is bigraded with the Tor-degree being non-positive. In the case at hand, one obtains

\[
\begin{array}{c|ccc}
 & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 \\
\mathbb{Z}_2 & 4 & 2 & 2 \\
-2 & -1 & 0 & .
\end{array}
\tag{12.20}
\]

Because the product respects bidegrees, the non-zero element in bidegree \((-1, 2)\) squares to 0. This does of course not happen for the generator \( x \in H^1(PU(2)) \) as \( PU(2) \approx SO(3) \approx \mathbb{RP}^3 \).

The same counterexample shows that one cannot expect the isomorphism (12.1) to be natural if 2 is not invertible in \( k \). Consider the diagonal map

\[
PU(2) = U(2)/U(1) \to (U(2) \times U(2))/U(1) \times U(1)) = PU(2) \times PU(2), \tag{12.21}
\]

which induces the cup product in cohomology. Naturality of the isomorphism (12.1) would predict that the image of \( x \otimes x \) in \( H^2(PU(2)) \) vanishes, which again is not the case.

**Remark 12.10.** May–Neumann [24] have observed that Theorem 1.2 extends to generalized homogeneous spaces, that is, to homotopy fibres of maps \( \varphi: Y \to X \) where \( X \) and \( Y \) take the roles of \( BG \) and \( BK \), respectively. The same is true for Theorem 12.5: Assume that \( X \) and \( Y \) have polynomial cohomology and that there is a map \( \kappa: BT \to Y \) where \( BT \) is the classifying space of some torus such that \( H^*(BT) \) is a free \( H^*(Y) \)-module. If \( X \) is, for example, 1-reduced, then the dga \( B(k, C^*(X), C^*(Y)) \) computes the cohomology of the homotopy fibre \( F \) of \( \varphi \), and
the same argument as before shows that in this case there is an isomorphism of graded algebras

\[ H^*(F) = \text{Tor}_{H^*(X)}(k, H^*(Y)) \]  

(12.22)

under our assumption that 2 is invertible in \( k \).

13. Examples

Like Cartan’s Theorem 1.1, our main result (Theorem 1.3) reduces the task of computing the cohomology ring of a homogeneous space \( G/K \) to a purely algebraic problem, provided that one understands the map \( H^*(BG) \to H^*(BK) \). We illustrate this with two examples that recover and (in the case \( n > n_1 + \cdots + n_k \) with \( k > 1 \)) generalize computations that can be found in [6], [12, Section XI.4], [16, Theorem 3.10] and [25, Chapter 3]. We continue to assume that 2 is invertible in the given principal ideal domain \( k \).

13.1. Unitary groups

We consider the homogeneous space

\[ U(n) / U(n_1) \times \cdots \times U(n_k), \]  

(13.1)

where \( k, n, n_1, \ldots, n_k \) are positive integers such that \( n \geq n_1 + \cdots + n_k \). For \( k = 1 \), this is a complex Stiefel manifold. For \( n = n_1 + \cdots + n_k \), we get a (complete or partial) complex flag variety, in particular a complex Grassmannian for \( k = 2 \).

Recall that

\[ H^*(BU(n)) = k[c_1, \ldots, c_n] \]  

(13.2)

is a polynomial ring in the Chern classes \( c_j \) of degree 2\( j \). The total Chern class \( c = 1 + c_1 + \cdots + c_n \) restricts to the product

\[ c^{(1)} \cdots c^{(k)} \in H^*(U(n_1) \times \cdots \times U(n_k)) \]  

(13.3)

of the total Chern classes of the factors. Hence

\[ \iota^*(c_j) = \sum_{j_1 + \cdots + j_k = j} c_{j_1}^{(1)} \cdots c_{j_k}^{(k)} \]  

(13.4)

for \( 1 \leq j \leq n \), cf. [6, Theorem 3.1] or [25, Theorem 3.5.8 (3)]. Here we allow \( j_i = 0 \) by setting \( c_{0_i} = 1 \).

Let \( r = n_1 + \cdots + n_k \) be the rank of \( K = U(n_1) \times \cdots \times U(n_k) \). If this equals the rank \( n \) of \( G = U(n) \), then \( H^*(G/K) \) is concentrated in even degrees [25, Theorem 7.3.21 (1)]. As in the proof of Lemma 12.4(ii), this implies by the Leray–Hirsch theorem that \( H^*(BK) \) is free over \( H^*(BG) \), so that

\[ H^*(G/K) = \text{Tor}_{H^*(BG)}(k, H^*(BK)) = k \otimes_{H^*(BG)} H^*(BK) \]  

(13.5)

\[ = k[c_{1}^{(1)}, \ldots, c_{n_1}^{(1)}, \ldots, c_{1}^{(k)}, \ldots, c_{n_k}^{(k)}] / \langle \iota^*(c_1), \ldots, \iota^*(c_r) \rangle \]

as a graded \( k \)-algebra.

If \( r < n \), then the only difference to the previous case is that we have \( \iota^*(c_j) = 0 \) for \( j > r \). Hence

\[ H^*(G/K) = \text{Tor}_k[c_1, \ldots, c_r](k, H^*(BK)) \]  

(13.6)

\[ = \text{Tor}_k[c_1, \ldots, c_r](k, H^*(BK)) \otimes \text{Tor}_k[c_{r+1}, \ldots, c_n](k, k) \]

\[ = R \otimes \bigwedge (x_{2r+1}, \ldots, x_{2n-1}), \]
where $R$ is the algebra from the last line of (13.5) and each exterior generator $x_i$ is of degree $i$. Note that $R = k$ if $k = 1$, that is, in case of a Stiefel manifold.

There is a canonical inclusion $\varphi: U(n) \hookrightarrow U(n')$ for $n \leq n'$. The case $k = 1$ of (13.4) means that each $c_j \in H^*(BU(n'))$ maps to $c_j \in H^*(BU(n))$ if $j \leq n$ and to 0 otherwise. Let $G'/K' = U(n')/U(n_1') \times \cdots \times U(n_k')$ be a second quotient with $n_i \leq n_i'$ for all $i$. If $\varphi$ restricts to the canonical inclusion $U(n_i) \hookrightarrow U(n_i')$ for each $1 \leq i \leq k$, then the naturality part of Theorem 1.3 implies that the induced map $H^*(G'/K') \to H^*(G/K)$ is given as follows: Each generator $c_j^{(i)} \in H^*(G'/K')$ is sent to its counterpart in $H^*(G/K)$ if $j \leq n_i$ and to 0 otherwise. Each exterior generator $x_{2j-1}$ is similarly sent to ‘itself’ if $j \leq r$ and to 0 otherwise.

### 13.2. Special orthogonal groups

We now turn to the homogeneous space

$$SO(n) / SO(n_1) \times \cdots \times SO(n_k),$$

(13.7)

where $k, n, n_1, \ldots, n_k$ are again positive integers such that $n \geq n_1 + \cdots + n_k$. As in the unitary case, we obtain real Stiefel manifolds, Grassmannians and other flag varieties as special cases. We set $G = SO(n)$ and $K = SO(n_1) \times \cdots \times SO(n_k)$. Depending on whether $n$ is even or odd, we write $n = 2m$ or $n = 2m + 1$, and similarly for the $n_i$. We assume that $n_1, \ldots, n_t$ are even and $n_{t+1}, \ldots, n_k$ odd. We finally abbreviate the rank $m_1 + \cdots + m_k$ of $K$ to $r$.

Since 2 is assumed to be a unit in $k$, we have

$$H^*(BSO(n)) = \begin{cases} k[p_1, \ldots, p_{m-1}, e] & \text{if } n \text{ is even,} \\ k[p_1, \ldots, p_m] & \text{if } n \text{ is odd} \end{cases}$$

(13.8)

where $p_j$ is the $j$th Pontryagin class of degree $4j$, and for even $n$ the Euler class $e$ of degree $n = 2m$ squares to $p_m$. The Künneth theorem gives $H^*(BK)$.

Analogously to the total Chern class, the total Pontryagin class $1 + p_1 + \cdots + p_m$ restricts to the product $p^{(1)} \cdots p^{(k)}$ of the total Pontryagin classes of the factors. In other words,

$$\iota^*(p_{j}) = \sum_{j_1 + \cdots + j_k = j} p^{(i)}_{j_1} \cdots p^{(k)}_{j_k}$$

(13.9)

for $1 \leq j \leq n$, where again we set $p^{(i)}_0 = 0$. For even $n$, the Euler class $e$ restricts to the product $e^{(1)} \cdots e^{(k)}$ of the Euler classes of the factors if $r = m$ and otherwise to 0 (since so does $p_m$). Compare [6, Corollary 7.3 (iii)].

We start with the case where $G$ and $K$ have the same rank $r = m$. This happens if and only if all $n_i$ are even and add up to $n$. As before, this implies that $H^*(BK)$ is free over $H^*(BG)$, so that we have

$$H^*(G/K) = H^*(BK) / \langle \iota^*(p_1), \ldots, \iota^*(p_{m-1}), \iota^*(e) \rangle.$$  

(13.10)

Now assume $r < m$. Then $\iota^*(p_{j}) = 0$ for $j > r$, and also $\iota^*(e) = 0$ if $n$ is even. Let $S \subset H^*(BK)$ be the subalgebra generated by all Pontryagin classes $p^{(i)}_j$ including $p^{(i)}_{m_i}$ for each even $n_i$. From (13.8), we see that $H^*(BK)$ is a free $S$-module with basis

$$e^I = \prod_{i \in I} e^{(i)},$$

(13.11)

where $I$ runs through the subsets of $\{1, \ldots, l\}$.

The polynomial algebra $k[p_1, \ldots, p_r] \subset H^*(BG)$ acts on $S$ in the same way as $H^*(BG)$ acted on $H^*(BK)$ in the unitary example from Section 13.1, except that all degrees are now doubled. This implies that $S$ is a free module over $k[p_1, \ldots, p_r]$, hence the same holds for $H^*(BK)$. For
odd \( n \), we therefore get
\[
H^*(G/K) = \text{Tor}_k[p_1, \ldots, p_r](k, H^*(BK)) \otimes \text{Tor}_k[p_{r+1}, \ldots, p_m](k, k) \quad (13.12)
\]
\[
= R \otimes \bigwedge(x_{4r+3}, \ldots, x_{4m-1}),
\]
where
\[
R = H^*(BK) / \langle \iota^*(p_1), \ldots, \iota^*(p_r) \rangle.
\]
As before, the subscripts of the exterior generators in (13.12) indicate degrees. For even \( n \), we similarly get
\[
H^*(G/K) = R \otimes \bigwedge(x_{4r+3}, \ldots, x_{4m-5}, y_{2m-1}), \quad (13.14)
\]
where the additional exterior generator is induced from the Euler class \( e \).

The behaviour of these isomorphisms under maps are analogous to the unitary case. We omit the details.

**Appendix A. Completing the proof of Proposition 4.1**

In this Appendix, we complete the proof of Proposition 4.1. Given two shm maps \( f : A \Rightarrow A' \) and \( g : B \Rightarrow B' \), we justify that the maps \( h_{(n)} \) introduced in that proof satisfy the defining identity (3.16) for a twisting homotopy family corresponding to an shm homotopy from \((1 \otimes g) \circ (f \otimes 1) \to (f \otimes 1) \circ (1 \otimes g)\). Explicitly, we have to show
\[
d(h_{(n)})(a_\bullet \otimes b_\bullet) \equiv \sum_{k=1}^{n-1} (-1)^k h_{(n-1)}(a_\bullet \otimes b_\bullet, a_k a_{k+1} \otimes b_k b_{k+1}, a_\bullet \otimes b_\bullet)
\]
\[
+ \sum_{k=0}^{n} ((1 \otimes g) \circ (f \otimes 1))_{(k)}(a_\bullet \otimes b_\bullet) h_{(n-k)}(a_\bullet \otimes b_\bullet)
\]
\[
- \sum_{k=0}^{n-1} (-1)^k h_{(k)}(a_\bullet \otimes b_\bullet) \left((f \otimes 1) \circ (1 \otimes g)\right)_{(n-k)}(a_\bullet \otimes b_\bullet)
\]
for \( n \geq 0 \) and \( a_\bullet \otimes b_\bullet \in A \otimes B \). Recall that \( h_{(0)} = \eta_{A'} \otimes \eta_{B'} \) and
\[
h_{(n)}(a_\bullet \otimes b_\bullet) \equiv \sum_{k,l \geq 0} \sum_{i_1 + \cdots + i_k + j_1 + \cdots + j_l = n} (-1)^\varepsilon F \otimes G
\]
for \( n \geq 1 \), where the second sum is over all decompositions of \( n \) into \( k+l \) positive integers,
\[
F = \mu^{[k]}(f_{(i_1)}(a_\bullet), \ldots, f_{(i_{k-1})}(a_\bullet), f_{(i_k+l)}(a_\bullet, \mu^{[j_1]}(a_\bullet), \ldots, \mu^{[j_l]}(a_\bullet))), \quad (A.3)
\]
\[
G = \mu^{[l]}(g_{(k+j_1)}(b_\bullet, \ldots, \mu^{[i_k]}(b_\bullet), b_\bullet), g_{(j_2)}(b_\bullet), \ldots, g_{(j_l)}(b_\bullet)), \quad (A.4)
\]
\[
\varepsilon = \sum_{s=1}^{k} s(i_s - 1) + \sum_{t=1}^{l} (l-t)(j_t - 1) + k(l-1) + 1. \quad (A.5)
\]
The twisting families corresponding to the compositions of shm maps \((1 \otimes g) \circ (f \otimes 1) \) and \((f \otimes 1) \circ (1 \otimes g)\) are given by (4.12) and (4.8), respectively. We only present a sketch of the proof. The computation is elementary, but lengthy because of the many cases to consider.

The case \( d(h_{(0)}) = 0 \) being clear, we have to compute \( d(h_{(n)})(a_\bullet \otimes b_\bullet) \) for \( n \geq 1 \). Recall from the discussion before (2.23) that both the composition and the tensor product of maps obey the graded Leibniz rule. The only maps in (4.17) that are not chain maps are the components \( f_{(n)} \)
and \(g_{(n)}\) of \(f\) and \(g\), respectively. From the formula (3.6), we see that two kinds of terms appear when applying the differential to \(f_{(n)}\): those obtained by splitting the arguments and those obtained by multiplying two of them. More precisely, we say that a term \(f_{(n)}(a_\bullet)\) is split at position \(m\) if it is split between the \(m\)th and the \((m+1)\)st argument, that is, if we consider the term \(f_{(m)}(a_\bullet)f_{(n-m)}(a_\bullet)\) of \(d(f_{(n)})(a_\bullet)\). We similarly say that two arguments are multiplied at position \(m\) if we consider the term \(f_{(n-1)}(a_\bullet,a_\bullet a_{m+1};a_\bullet)\) where the arguments at positions \(m\) and \(m+1\) are multiplied. The same applies to the components of \(g\). We claim that when computing \(d(h_{(n)})\), all terms on the right-hand side of (A.1) are indeed produced and all other terms that come up cancel in pairs.

Below is a list all terms that appear in the computation. In each case, we indicate whether the corresponding terms cancel against other terms or pair with terms on the other side of the equation. The notation ‘\(X \to Y\)’ means that the terms \(X\) cancel or pair with the terms \(Y\).

**Terms produced by \(d(h_{(n)})\)**

1. Splitting of a term \(f_{(i_s)}\)
   1.1. Term \(f_{(i_s)}\), \(1 \leq s < k\), at any position (if \(k \geq 2\)) \(\to 4.1.\)
   1.2. Term \(f_{(i_k+l)}\) at position \(1 \leq m < i_k\) (if \(i_k \geq 2\)) \(\to 4.2.\)
   1.3. Term \(f_{(i_k+l)}\) at position \(m = i_k\)
      1.3.1. \(j_1 = 1\) and \(l = 1 \to 10.1.\)
      1.3.2. \(j_1 = 1\) and \(l > 1 \to 2.2.2.\)
      1.3.3. \(j_1 > 1 \to 3.3.2.\)
   1.4. Term \(f_{(i_k+l)}\) at position \(i_k + 1 \leq m < i_k + l\) (if \(l \geq 2\)) \(\to 12.\)

2. Splitting of a term \(g_{(j_1)}\)
   2.1. Term \(g_{(k+j_1)}\) at position \(1 \leq m < k\) (if \(k \geq 2\)) \(\to 9.\)
   2.2. Term \(g_{(k+j_1)}\) at position \(m = k\)
      2.2.1. \(i_k = 1\) and \(k = 1 \to 11.1.\)
      2.2.2. \(i_k = 1\) and \(k > 1 \to 1.3.2.\)
      2.2.3. \(i_k > 1 \to 4.3.2.\)
   2.3. Term \(g_{(k+j_1)}\) at position \(k + 1 \leq m < k + j_1\) (if \(j_1 \geq 2\)) \(\to 3.4.\)
   2.4. Term \(g_{(j_1)}\), \(1 < t \leq l\), at any position (if \(l \geq 2\)) \(\to 3.5.\)

3. Multiplication of two arguments of a term \(f_{(i_s)}\)
   3.1. Term \(f_{(i_s)}\), \(1 \leq s < k\), at any position \(\to 5.\)
   3.2. Term \(f_{(i_k+l)}\) at position \(1 \leq m < i_k\) \(\to 6.\)
   3.3. Term \(f_{(i_k+l)}\) at position \(m = i_k\)
      3.3.1. \(i_k = 1\) and \(k = 1 \to 11.2.\)
      3.3.2. \(i_k = 1\) and \(k > 1 \to 1.3.3.\)
      3.3.3. \(i_k > 1 \to 4.3.3.\)
   3.4. Term \(f_{(i_k+l)}\) at position \(m = i_k + 1\) (if \(l \geq 2\)) \(\to 2.3.\)
   3.5. Term \(f_{(i_k+l)}\) at position \(i_k + 1 \leq m < i_k + l\) (if \(l \geq 3\)) \(\to 2.4.\)

4. Multiplication of two arguments of a term \(g_{(j_1)}\)
   4.1. Term \(g_{(k+j_1)}\) at position \(1 \leq m < k - 1\) (if \(k \geq 3\)) \(\to 1.1.\)
   4.2. Term \(g_{(k+j_1)}\) at position \(m = k + 1\) (if \(k \geq 2\)) \(\to 1.2.\)
   4.3. Term \(g_{(k+j_1)}\) at position \(m = k\)
      4.3.1. \(j_1 = 1\) and \(l = 1 \to 10.2.\)
      4.3.2. \(j_1 = 1\) and \(l > 1 \to 2.2.3.\)
      4.3.3. \(j_1 > 1 \to 3.3.3.\)
   4.4. Term \(g_{(k+j_1)}\) at position \(k + 1 \leq m < k + j_1\) \(\to 7.\)
   4.5. Term \(g_{(j_1)}\), \(1 < t \leq l\), at any position \(\to 8.\)
Terms appearing in $h_{(n-1)}(\ldots, a_m a_{m+1} \otimes b_m b_{m+1}, \ldots)$

5. $m \leq i_1 + \cdots + i_{k-1}$ (if $k \geq 2$) $\rightarrow$ 3.1.
6. $i_1 + \cdots + i_{k-1} < m \leq i_1 + \cdots + i_k$ $\rightarrow$ 3.2.
7. $i_1 + \cdots + i_k < m \leq i_1 + \cdots + i_k + j_1$ $\rightarrow$ 4.4.
8. $i_1 + \cdots + i_k + j_1 < m$ (if $l \geq 2$) $\rightarrow$ 4.5.

Terms appearing in $( (1 \otimes g) \circ (f \otimes 1) )_{(k)} \cdot h_{(n-k)}$

9. $k < n \rightarrow$ 2.1.
10. $k = n$
   10.1. $i_n = 1 \rightarrow$ 1.3.1.
   10.2. $i_n > 1 \rightarrow$ 4.3.1.

Terms appearing in $h_{(k)} \cdot ( (f \otimes 1) \circ (1 \otimes g) )_{(n-k)}$

11. $k = 0$
   11.1. $j_1 = 1 \rightarrow$ 2.2.1.
   11.2. $j_1 > 1 \rightarrow$ 3.3.1.
12. $k > 0 \rightarrow$ 1.4.

To illustrate how the proof proceeds, let us discuss two cases in detail. We write $i = (i_1, \ldots, i_k)$, $j = (j_1, \ldots, j_l)$ and $\varepsilon(i,j)$ for the sign exponent $\varepsilon$ from (4.20). We compute sign exponents modulo 2, indicated by $\equiv$.

Pair 1.1. $\leftrightarrow$ 4.1. The case 1.1. with a splitting of $f_{(i_s)}(a_*)$ at position $1 \leq p < i_s$ produces the term $(-1)^{\varepsilon'} F'' \otimes G'$ with

\[
F'' \equiv \mu^{|k+1|} \left( f_{(i_1)}(a_*), \ldots, f_{(p)}(a_*), f_{(q)}(a_*), \ldots, f_{(i_{k-1})}(a_*), f_{(i_k+l)}(a_*), \mu^{[i_1]}(a_*), \ldots, \mu^{[i_l]}(a_*) \right),
\]

(A.6)

where $p + q = i_s$.

\[
G' = G \equiv \mu^{|l|} \left( g_{(k+j_1)}(a_*), \ldots, g_{(j_2)}(a_*), \ldots, g_{(j_l)}(a_*) \right)
\]

(A.7)

and

\[
\varepsilon' \equiv \varepsilon(i,j) + (1 - i_1) + \cdots + (1 - i_{s-1}) + p
\]

\[
\equiv \varepsilon(i,j) + i_1 + \cdots + i_{s-1} + s + p + 1.
\]

(A.8)

On the other hand, the case 4.1. gives the term $(-1)^{\varepsilon''} F''' \otimes G''$ with

\[
F''' = F \equiv \mu^{|k|} \left( f_{(i_1)}(a_*), \ldots, f_{(i_{k-1})}(a_*), f_{(i_k+l)}(a_*), \mu^{[i_1]}(a_*), \ldots, \mu^{[i_l]}(a_*) \right)
\]

(A.9)

and

\[
G'' \equiv \mu^{|l|} \left( g_{(k-1+j_1)}(a_*), \ldots, g_{(j_2)}(a_*), \ldots, g_{(j_l)}(a_*) \right),
\]

(A.10)

\[
\varepsilon'' \equiv \varepsilon(i,j) + (1 - i_1) + \cdots + (1 - i_{k-1}) + (1 - i_k - l) + m + 1
\]

\[
\equiv \varepsilon(i,j) + i_1 + \cdots + i_k + k + l + m + 1.
\]
We rewrite this second case in terms of \( i'' = (i_1, \ldots, i_{s-1}, p, q, i_{s+1}, \ldots, i_k) \) (of length \( k'' = k + 1 \)) and \( s = m \). Then \( F'' = F', G'' = G' \) and
\[
\varepsilon(i'', j) \equiv \varepsilon(i, j) - s(p + q - 1) + s(p - 1) + (s + 1)(q - 1) \quad (A.12)
\]
\[
+ (i_{s+1} - 1) + \cdots + (i_k - 1) + (l - 1)
\]
\[
\equiv \varepsilon(i, j) + i_{s+1} + \cdots + i_k + p + k + l.
\]

Hence
\[
\varepsilon'' \equiv \varepsilon(i'', j) + i_1 + \cdots + p + q + \cdots + i_k + k'' + l + s + 1 \quad (A.13)
\]
\[
\equiv \varepsilon(i, j) + p + s + 1 \equiv \varepsilon' + 1,
\]
which means that the terms produced by these two cases have opposite signs and therefore cancel out.

Pair 2.1. \( \leftrightarrow \) 9. The case 2.1. produces the term \((-1)^s F' \otimes G'\) with
\[
F' = F
\]
\[
\equiv \mu^{[k]} \left( f_{(i_1)}(a_\bullet), \ldots, f_{(i_{k-1})}(a_\bullet), f_{(i_k+l)}(a_\bullet, \mu^{[j]}(a_\bullet), \ldots, \mu^{[j]}(a_\bullet)) \right),
\]
\[
G' \equiv \mu^{[l+1]} \left( g_{(m)}(\mu^{[i]}(b_\bullet), \ldots, \mu^{[i]}(b_\bullet), g_{(j_1)}(b_\bullet), \ldots, g_{(j)}(b_\bullet)) \right),
\]
\[
\varepsilon' \equiv \varepsilon(i, j) + (1 - i_1) + \cdots + (1 - i_{k-1}) + (1 - i_k - l) + m \quad (A.16)
\]
\[
\equiv \varepsilon(i, j) + i_1 + \cdots + i_k + k + l + m.
\]
We now consider the case 9, with \( k = m \). Taking (4.12) into account, we see that it produces terms of the form \((-1)^s F' \otimes G'\) with
\[
F'' \equiv \mu^{[m]} \left( f_{(i_1)}(a_\bullet), \ldots, f_{(i_m)}(a_\bullet) \right) \cdot \mu^{[k-m]} \left( f_{(i_{m+1})}(a_\bullet), \ldots, f_{(i_{k-1})}(a_\bullet, \mu^{[j]}(a_\bullet), \ldots, \mu^{[j]}(a_\bullet)) \right),
\]
\[
G'' \equiv g_{(m)}(\mu^{[i]}(b_\bullet), \ldots, \mu^{[i]}(b_\bullet), \mu^{[l]}(g_{(k-m+j_1)}(\mu^{[i]}(b_\bullet), \ldots, \mu^{[i]}(b_\bullet), g_{(j_2)}(b_\bullet), \ldots, g_{(j)}(b_\bullet))\right),
\]
\[
\varepsilon'' \equiv \sum_{s=1}^{m} (s - 1)(i_s - 1) + \varepsilon(i'', j) + (m - 1) \left( \sum_{s=m+1}^{k} (i_s - 1) + l \right) \quad (A.19)
\]
where \( i'' = (i_{m+1}, \ldots, i_k) \). The first summand of \( \varepsilon'' \) is (4.15), and last one arises because we have moved the second factor of \( F'' \) (which comes from \( h \)) past the first factor of \( G'' \) (which comes from \((1 \otimes g) \circ (f \otimes 1)\)). We have \( F'' = F', G'' = G' \) and
\[
\varepsilon'' \equiv \sum_{s=1}^{k} s(i_s - 1) + \sum_{s=1}^{m} (i_s - 1) + \sum_{s=m+1}^{k} m(i_s - 1) \quad (A.20)
\]
\[
+ \sum_{t=1}^{l} (l - t)(j_t - 1) + k(l - 1) - m(l - 1) + 1
\]
\[ + \sum_{s=m+1}^{k} (m-1)(i_s - 1) + l(m-1) \]
\[ \equiv \varepsilon(i,j) + \sum_{s=1}^{k} (i_s - 1) - m(l-1) + l(m-1) \]
\[ \equiv \varepsilon(i,j) + i_1 + \cdots + i_k + k + l + m. \]

Hence the terms produced by these two cases agree, including the signs.

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