Research Article

New Zero-Free Regions for Hypergeometric Zeta and Fractional Hypergeometric Zeta Functions

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We describe and demonstrate zero-free regions for the families called “hypergeometric zeta functions” and “fractional hypergeometric zeta functions”. These zero-free regions are vertical strips in the right half of the complex plane. In the proofs, we utilize positivity results on the oscillatory integrals and monotonicity of real-valued functions to demonstrate these zero-free regions for both families.

1. Introduction

Many papers have been devoted to the study of the locations of zeros and zero-free regions of zeta function [1–3] and its generalizations such as the Hurwitz zeta function [4–6], multiple zeta functions [7], hypergeometric zeta functions [8], and fractional hypergeometric zeta functions [9, 10]. The families of hypergeometric zeta functions and fractional hypergeometric zeta functions are known only in their integral representations as generalizations of the classical Riemann zeta function via integral representation. A study of these families of zeta functions was initiated in [10, 11], respectively. In the same papers [10, 11], the authors discovered that both the hypergeometric zeta functions of order \( N \), \( \zeta_N(s) \) and the fractional hypergeometric zeta functions of order \( a \), \( \zeta_a(s) \) can be continued analytically to the whole complex plane, except for finite simple poles on the real axis at \( 1, 0, \ldots, 2 - N \) for the hypergeometric zeta functions and except for infinite simple poles on the real axis at \( 1, 0, -1, -2, \ldots \) for fractional hypergeometric zeta functions. It was also discovered in [10] that \( \zeta_a(s) \) has also simple zeros on the real axis at \( 1 - a, -a, -(1 + a), -(2 + a), \ldots \) where \( a \) is a fixed positive real number. These zeros are called the trivial zeros for fractional hypergeometric zeta function. Regarding these families, the following open problems were raised in [8–10] as follows:

(1) Do the families admit the Dirichlet series?
(2) Do the families admit functional equation?
(3) Do they admit product formula?
(4) Do they have nontrivial zeros?
(5) Do they have zero-free regions?

Concerning zero-free regions for hypergeometric zeta function of order 2, some results were known, and some problems were left open, see [8, 9].

In this paper, we investigate zero-free regions for the families of hypergeometric zeta functions of order \( N \) denoted by \( \zeta_N(s) \), where \( N \) is a positive integer, and the families of fractional hypergeometric zeta functions of order \( a \) denoted by \( \zeta_a(s) \), where \( a \) is a positive real number, which is defined by the integral representations, respectively, as follows:

\[
\zeta_N(s) = \frac{1}{s(s+N-1)} \int_0^\infty x^{s-N-2} e^x - T_{N-1}(x) \, dx. \tag{1}
\]

For \( \Re(s) > 1 \), where \( T_N(x) = \sum_{n=0}^N \frac{x^n}{n!} \) is the Taylor (Maclaurin) polynomial of the exponential function \( e^x \) and
are the same, and in particular, if \( \int_0^\infty x^{s-1} e^{-x} \, dx \) then, \( \int_0^\infty x^{s-1} e^{-x} \, dx \) is the Gamma function defined by

\[
\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx.
\]

For \( \Re(s) > 1 \) and for all positive real number \( a \), where \( \Gamma(s) \) is the Gamma function defined by

\[
\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx.
\]

We observe that if \( a \) is a natural number, both families are the same, and in particular, if \( N = a = 1 \), the families represent the classical Riemann zeta function.

Finding zeros and zero-free regions for these families becomes difficult as compared to the classical Riemann zeta functions which are known in different representations, such as Dirichlet series, Euler product formula, and functional equations. In the absence of such different representations, finding zeros and zero-free regions lie entirely on their integral representations. Therefore, in the present paper, we describe and demonstrate zero-free regions for \( \zeta_N(s) \) for fixed but arbitrary natural number \( N \) and \( \zeta_a(s) \) for fixed but arbitrary \( a \) such that \( 0 < a < 1 \) in the right half of the complex plane by utilizing positivity results on oscillatory integrals and monotonicity of real-valued functions.

Our main results are the following.

**Theorem 1.** For fixed natural number \( N \), let \( V_N = \{ s = \sigma + it \in \mathbb{C} : 1 \leq \sigma < 2 \} \). Then, \( \zeta_N(s) \neq 0 \) on \( V_N \).

**Theorem 2.** For fixed positive real number \( a \) with \( 0 < a < 1 \), let \( V_a = \{ s = \sigma + it \in \mathbb{C} : 1 \leq \sigma < 2 - a \} \). Then, \( \zeta_a(s) \neq 0 \) on \( V_a \).

The structure of the present work is as follows. In Section 2, we review some of the main results obtained so far regarding zeta function, hypergeometric zeta functions, fractional hypergeometric zeta functions, and positivity results on oscillatory integrals and monotonicity of real-valued functions. In Section 3, we reveal and prove our main results and demonstrate that both \( \zeta_N(s) \) and \( \zeta_a(s) \) are zero free on the right half-plane in vertical strips \( V_N \) and \( V_a \), respectively. In Section 4, we give some concluding remarks and discussion.

### 2. Preliminaries

As our work is a continuation of the hypergeometric zeta functions [11] and fractional hypergeometric zeta functions [10], first, we review some basic results concerning these families of zeta functions, and second, we review positivity results on oscillatory integrals and monotonicity of real-valued functions, in order to state and prove our results in perspective.

#### 2.1. An Overview of Some Families of Zeta Functions

**Definition 1.** The Riemann zeta function \( \zeta(s) \) is a function of a complex variable \( s = \sigma + it \) defined as an infinite series (Dirichlet series) given, convergent for all \( \sigma > 1 \),

\[
\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.
\]

For \( \sigma > 1 \), the classical zeta function \( \zeta(s) \) is also defined by the integral given

\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-x} \, dx,
\]

where \( \Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx \) is the Gamma function. For \( \sigma > 1 \), \( \zeta(s) \) is also defined by

\[
\zeta(s) = \prod_p (1 - p^{-s})^{-1},
\]

where the product runs over all prime numbers \( p \). This product formula is called Euler’s product formula for the Riemann’s zeta function. Since a convergent infinite product of nonzero factors is not zero, the zeta function does not vanish in the right half of the complex plane. Therefore, this product formula is one of the important tools to show that the Riemann zeta function is zero free in the right half of the complex plane (actually for \( \sigma > 1 \)).

The celebrated functional equation of the Riemann zeta function is also another important representation to locate the zeros of the zeta function on the left half of the complex plane. It is given by

\[
\zeta(s) = 2(2\pi)^{s-1} \sin(\pi s) \Gamma(1-s) \zeta(1-s) \Gamma\left(\frac{1-s}{2}\right).
\]

**Definition 2.** The points \( s = -2, -4, -6, \ldots \) are called the “trivial” zeros of the zeta function \( \zeta(s) \), and the strip \( \{ s \in \mathbb{C} : 0 \leq \sigma \leq 1 \} \) is called the critical strip.

Regarding the zeros inside the critical strip, it is conjectured that the nontrivial zeros all must be located on the “critical line” \( \Re(s) = 1/2 \). This conjecture is known as Riemann’s hypothesis. As a generalization of the Riemann zeta function \( \zeta(s) \) via integral representation, we have the following definition.

**Definition 3** (see [11], Definition 2.1). Let \( N \) be a natural number and \( T_N(x) = \sum_{n=0}^N x^n/n! \) be the Taylor (Maclaurin) polynomial of the exponential function \( e^x \). For \( \sigma > 1 \), the \( N^{th} \) order hypergeometric zeta functions \( \zeta_N(s) \) is defined as

\[
\zeta_N(s) = \frac{1}{\Gamma(s + N - 1)} \int_0^\infty x^{s+N-2} T_{N-1}(x) e^{-x} \, dx.
\]
Observe that when \( N = 1 \), we get the classical zeta function \( \zeta(s) \). It has been shown in [11] that the hypergeometric zeta function \( \zeta_N(s) \) can be extended analytically to the entire complex plane, except for \( N \) simple poles at \( s = 1, 0, -1, \ldots, 2 - N \). The hypergeometric zeta functions have zeros on the negative real axis, and these zeros are called the trivial zeros for the hypergeometric zeta functions.

Concerning zero-free regions for hypergeometric zeta function of order 2, the following results are known.

**Theorem 3** (see [8], Theorem 1.1). \( \zeta_2(s) \) has no zeros in the left-half complex plane \( \{ s = \sigma + it \in \mathbb{C}; \sigma < \sigma_2 \} \), except for infinitely many “trivial” zeros on the negative real axis, one in each of the intervals

\[
S_m = \left[ \sigma_{m+1}, \sigma_m \right],
\]

where \( m \geq 2 \) is a positive integer, \( z_k = r_k e^{i\theta_k} \) are the nonzero roots of \( e^x - T_N^{-1}(x) \) and \( \sigma_{m} = 1 - \left( \pi/\pi - \sigma_1 \right)m \).

**Theorem 4** (see [9], Theorem 3.4.2 and Corollary 3.4.3). Let \( s = \sigma + it \) with \( \sigma > 1 \) and \( C > 1 \). Then, \( \zeta_2(s) \neq 0 \) for \( |s| < Ca \).

**Definition 4** (see [10], Definition 2.1). The fractional hypergeometric zeta function \( \zeta_a(s) \) is defined for all positive real number \( a \) and \( \sigma > 1 \) as

\[
\zeta_a(s) = \frac{\Gamma(a + 1)}{\Gamma(s + a - 1)} \int_0^\infty \frac{x^{a-2} e^{-x}}{\alpha y(a, x)} dx,
\]

where \( \Gamma(s) \) is the Gamma function defined by

\[
\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx,
\]

and \( y(a, x) \) is the lower incomplete Gamma function of the form

\[
y(a, x) = \int_0^x e^{-t} t^{a-1} dt.
\]

Observe that when \( a = N \), natural number, we get the Hypergeometric zeta functions \( \zeta_N(s) \).

### 2.2. Positivity Properties of Integrals

**Proposition 1** (see [1], Proposition 3.1). Let the function \( h(r) \geq 0 \) on \( (0, \infty) \), \( h \in L_1^{1,0}(0, \infty) \), \( h \) decreasing and strictly decreasing on some open subinterval of the form \( [(kn/t), ((k + 1)n/t)) \) for some nonnegative integer \( k \), and \( t \) positive real number. Then,

\[
\int_0^\infty h(r) \sin (tr) dr > 0.
\]

Moreover,

\[
\int_0^T h(r) \sin (tr) dr > 0.
\]

For any \( T > 0 \) provided, \( h \) satisfies the above assumptions with \( (0, \infty) \) replaced by \( (0, T) \).

**Corollary 1** (see [1], Corollary 3.2). Let \( t \) be a positive real number and \( x_{t,k} \geq 1 \) be such that \( \theta t x_{t,k} = 2\pi k \), for some positive integer \( k \). Let \( h \geq 0 \) on \( [x_{t,k}, \infty) \), \( h \in L_1^{1,0}(0, \infty) \), \( h \) decreasing, strictly decreasing on some interval of the form \( ((jn/t), ((j + 1)n/t)) \) for some positive integer \( j \) such that \( j(n/t) \geq x_{t,k} \). Then,

\[
\int_0^\infty h(r) \sin (tr) dr > 0.
\]

Moreover,

\[
\int_0^T h(r) \sin (tr) dr > 0.
\]

For some \( T > x_{t,k} \) if \( T > (j + 1)(n/t) \geq x_{t,k} \).

**Remark 1.** Given \( t > 0 \), we can take a \( y = e^{2\pi n/t} \geq 1 \) such that

\[
\int_0^\infty h(r) \sin (tr) dr > 0,
\]

where \( h \geq 0 \) on \( [\ln y, \infty) \), \( h \in L_1^{1,0}(0, \infty) \), \( h \) decreasing, strictly decreasing on some interval of the form \( ((jn/t), ((j + 1)n/t)) \) for some \( j \in \mathbb{N} \) such that \( j(n/t) \geq x_{t,k} \).

**Corollary 2** (see [1], Corollary 3.3). Let \( t > 0 \) and \( x_{t,k} \geq 1 \) be such that \( \theta t x_{t,k} = 2\pi k \), for some positive integer \( k \) and let \( g \geq 0 \) on \( [x_{t,k}, \infty) \), \( tgn \in qL_1^{1,0}(x_{t,k}, \infty) \) such that \( x_{t,k} \rightarrow xg(x) \) is decreasing on \( [x_{t,k}, \infty) \), \( x_{t,k} \rightarrow xg(x) \) strictly decreasing on \( [(jn/t), ((j + 1)n/t)) \) for some positive integer \( j \). Then,

\[
\int_0^\infty g(x) \sin (\ln x) dx > 0.
\]

Moreover,

\[
\int_0^T g(x) \sin (\ln x) dx > 0.
\]

For any \( T > x_{t,k} \), whenever \( xg(x) \) is decreasing in \( [x_{t,k}, T] \), strictly decreasing on \( (\ln (jn/t), (j + 1)n/t)) \) for some positive integer \( j \) such that \( \ln (jn/t) \geq x_{t,k} \) and \( \ln ((j + 1)n/t) \leq T \).

**Lemma 1.** Let \( f(x) \) and \( g(x) \) be nonnegative differentiable real-valued functions on some domain \( D \) such that \( f(x) \) is decreasing on \( D \) and \( g(x) \) is increasing on \( D \). Then, the function

\[
h(x) = \frac{f(x)}{g(x)}
\]

is decreasing on \( D \).

**Proof.** Since \( f \) is decreasing and \( g \) is increasing, we have \( f(x) \leq 0 \) on \( D \) and \( g'(x) \geq 0 \) on \( D \), respectively. Thus,

\[
f'(x) = \left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \leq 0.
\]
On \( D \), because \( f'(x)g(x) \leq 0, f(x)g'(x) \geq 0 \) and their difference \( f'(x)g(x) - f(x)g'(x) \leq 0 \). Hence, \( h(x) \) is decreasing on the domain \( D \).

\[ \square \]

Remark 2. If \( f(x) \) and \( g(x) \) are strictly decreasing and strictly increasing, respectively, which are nonnegative on some domain \( D \), then \( h(x) = f(x)/g(x) \) is also strictly decreasing on the same domain \( D \).

3. Zero-Free Regions for \( \zeta_N(s) \) and \( \zeta_a(s) \)

In this section, we demonstrate zero-free regions for hypergeometric zeta function \( \zeta_N(s) \) and fractional hypergeometric zeta function \( \zeta_a(s) \).

**Theorem 5.** Let \( N \) be a fixed natural number. Then, \( \zeta_N(s) \neq 0 \) on the vertical strip \( V_N = \{ s = a + it \in \mathbb{C} : 1 \leq a < 2 \} \).

**Proof.** For \( s > 1 \),

\[ \zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^\infty \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} \, dx. \]  

Rearranging the above equation, we get

\[ \Gamma(s+N-1)\zeta_N(s) = \int_0^\infty \frac{x^{s-2}}{\left( (e^x - T_{N-1}(x))/x^N \right)^N} \, dx. \]  

To apply positivity properties of oscillatory integrals, put \( h(x) = f(x)/g(x) \), where

\[ f(x) = x^{s-2} \quad \text{and} \quad g(x) = \frac{e^x - T_{N-1}(x)}{x^N}, \]  

Since the function \( f(x) = x^{s-2} \) is decreasing on \((0, \infty)\) and the function \( g(x) = (e^x - T_{N-1}(x))/x^N \) is increasing on \((0, \infty)\) for \( s < 2 < 0 \), by Lemma 1, \( h(x) \) is decreasing on \((0, \infty)\). Moreover, the function \( h(x) \) is nonnegative, \( h(x) \in L^1_{\text{loc}}(0, \infty) \) and strictly decreasing on any subinterval of \((0, \infty)\) as it is actually strictly decreasing on \((0, \infty)\). Thus, for any positive real number \( t \),

\[ \int_0^\infty \frac{x^{s-2}}{\left( (e^x - T_{N-1}(x))/x^N \right)^N} \sin(t \ln x) \, dx > 0. \]  

This implies that

\[ \Im \left( \Gamma(s+N-1)\zeta_N(s) \right) > 0. \]  

Hence, \( \zeta_N(s) \neq 0 \). But, then \( \zeta_N(s) = \overline{\zeta_N(\overline{s})} \), so that it holds also for \( t < 0 \).

For \( t = 0 \),

\[ \Gamma(s+N-1)\zeta_N(s) = \int_0^\infty \frac{x^{s-2}}{\left( (e^x - T_{N-1}(x))/x^N \right)^N} \, dx. \]  

We know that

\[ x^{s-2} \frac{\Gamma(s+N-1)(x)}{(e^x - T_{N-1}(x))/x^N)} > 0. \]  

For all \( x \in (0, \infty) \) and continuous on \((0, \infty)\). Hence,

\[ \Gamma(s+N-1)\zeta_N(s) = \int_0^\infty \frac{x^{s-2}}{\left( (e^x - T_{N-1}(x))/x^N \right)^N} \, dx > 0. \]  

Therefore, \( \zeta_N(s) \neq 0 \) for any real number \( t \) where \( 1 < s < 2 \). Therefore, \( \zeta_N(s) \neq 0 \) on \( V_N = \{ s = a + it \in \mathbb{C} : 1 \leq a < 2 \} \). This vertical strip is shown roughly in Figure 1.

\[ \square \]

**Theorem 6.** Let \( 0 < a < 1 \) be fixed. Then, \( \zeta_a(s) \neq 0 \) in the vertical strip \( V_a = \{ s = a + it \in \mathbb{C} : 1 \leq a < 2 - a \} \).

**Proof.** For \( s > 1 \),

\[ \zeta_a(s) = \frac{\Gamma(s+1)}{\Gamma(s+a+1)} \int_0^\infty \frac{x^{a+2}}{e^x a(y(x,a))} \, dx, \]  

is defined and analytic to the right half of \( s = 1 \). Rearranging the above equation, we get

\[ \Gamma(s+a+1) \zeta_a(s) = \int_0^\infty \frac{x^{a+2}}{\Gamma(\gamma(x,a))} \, dx. \]  

To apply positivity properties of the oscillatory integral, for \( x \in (0, \infty) \), put

\[ h(x) = \frac{x^{a+2}}{e^x a(y(x,a))} > 0, \]  

and consider the case where \( 1 < s < 2 - a \).

In this case, the function \( f(x) = x^{a+2} \) is decreasing on \((0, \infty)\), and the function \( g(x) = e^x a(y(x,a)) \) is increasing on \((0, \infty)\). Hence, by Lemma 1, \( h(x) \) is decreasing on \((0, \infty)\). Moreover, the function \( h(x) \) is nonnegative, \( h(x) \in L^1_{\text{loc}}(0, \infty) \) and strictly decreasing on any subinterval of \((0, \infty)\). Thus, for any \( t > 0 \),

\[ \int_0^\infty \frac{x^{a+2}}{\Gamma(\gamma(x,a))} \sin(t \ln x) \, dx > 0. \]  

This implies that

\[ \Im \left( \Gamma(s+a+1) \zeta_a(s) \right) > 0, \]  

which implies that \( \zeta_a(s) \neq 0 \). But, then \( \zeta_a(s) = \overline{\zeta_a(\overline{s})} \), so that it holds also for \( t < 0 \).

For \( t = 0 \),

\[ \Gamma(s+a+1) \zeta_a(s) = \int_0^\infty \frac{x^{a+2}}{\Gamma(\gamma(x,a))} \, dx. \]  

We know that \( x^{a+2}/(\gamma(x,a)e^x) > 0 \) for all \( x \in (0, \infty) \) and continuous on \((0, \infty)\). Thus,
The zero-free regions are vertical strips. Figure 2: Zero-free region for fractional hypergeometric zeta function of order $a$.

\[
\int_0^\infty x^{\sigma+a-2} e^{-x} \frac{1}{y(a,x)} - dx > 0.
\]

Hence, $\zeta_a(s) \neq 0$, for any $t \in \mathbb{R}$ and $\{1 < \sigma < 2 - a\}$. Therefore, $\zeta_a(s) \neq 0$ on $V_a = \{s = \sigma + it \in \mathbb{C}: 1 \leq \sigma < 2\}$. This vertical strip is shown roughly in Figure 2.

Remark 3. The widths of the vertical strips $V_N$ and $V_a$ have the following properties.

1. The width of the vertical strip $V_N$ is independent of $N$ for $N > 1$. For $N = a = 1$, we have the classical Riemann zeta function, and in this case, actually, we have a zero-free region to the right of $\sigma = 1$.

2. The width of the vertical strip $V_a$ depends on $a$. Observe that as “$a$” increases from 0 to 1 excluding both zero and one, the vertical strip becomes narrower and narrower and eventually becomes a vertical line.

3. As $a$ decreases from 1 to 0 excluding both zero and one, the vertical strip $V_a$ becomes wider and wider, and eventually, the vertical strip $V_a$ becomes $\{s = \sigma + it \in \mathbb{C}: 1 \leq \sigma < 2\}$.

4. Conclusion

In this paper, zero-free regions were described and demonstrated for both families of hypergeometric zeta functions $\zeta_N(s)$ and fractional hypergeometric zeta functions $\zeta_a(s)$. The zero-free regions are vertical strips $V_N$ for $\zeta_N(s)$ and $V_a$ for $\zeta_a(s)$. The width of $V_N$ is independent of the parameter $N$, and the width of $V_a$ depends on the parameter $a$. We have some evidence but no proof at the present time that it is possible to extend the zero-free regions for both $\zeta_N(s)$ and $\zeta_a(s)$ to both the left half and right half of the complex plane. We do not know whether these families of functions have nontrivial zeros, but we have some evidence but no proof at present especially the fractional hypergeometric zeta functions may have nontrivial zeros which may coincide with the nontrivial zeros of the classical Riemann zeta functions.

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Data Availability

The data used to support the findings of this study are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Albeverio and C. Cebulla, “M’untz formula and zero free regions for the Riemann zeta function,” Bulletin des Sciences Mathematiques, vol. 131, no. 1, pp. 12–38, 2007.
[2] T. M. Apostol, Introduction to Analytic Number Theory, Springer-Venag, New York Heidelberg Berlin, 1976.
[3] L. Fekih-Ahmed, "On the zeros of the Riemann Zeta function," arXiv, 2011.
[4] R. Garunkstis and J. Steuding, “On the distribution of zeros of the Hurwitz zeta-function,” Mathematics of Computation, vol. 76, no. 257, pp. 323–338, 2007.
[5] H. D. Nguyen and A. Hassen, “Moments of hypergeometric hurwitz zeta functions,” Pre-Print, 2010.
[6] R. Spira, “Zeros of Hurwitz zeta functions,” Mathematics of Computation, vol. 30, no. 136, pp. 863–866, 1976.
[7] J. Zhao, “Analytic continuation of multiple zeta functions,” Proceedings of the American Mathematical Society, vol. 128, no. 5, pp. 1275–1283, 1999.
[8] A. Hassen and H. D. Nguyen, “Zero free regions for hypergeometric zeta functions,” International Journal of Number Theory, vol. 4, no. 4, pp. 1033–1043, 2011.
[9] H. L. Geleta, Fractional hypergeometric zeta function, http://etd.aau.edu.et/handle/123456789/8790 Ph.D. Thesis, Addis Ababa University, Addis Ababa, Ethiopia, 2014, http://etd.aau.edu.et/handle/123456789/8790 Ph.D. Thesis.
[10] H. L. Geleta and A. Hassen, “Fractional hypergeometric zeta functions,” The Ramanujan Journal, vol. 41, no. 1-3, pp. 421–436, 2016.
[11] A. Hassen and H. D. Nguyen, “Hypergeometric zeta functions,” International Journal of Number Theory, vol. 6, no. 1, pp. 99–126, 2010.