ON DIRECTIONAL ENTROPY OF A $\mathbb{Z}^2$-ACTION

HASAN AKIN

Abstract. Consider the cellular automata (CA) of $\mathbb{Z}^2$-action $\Phi$ on the space of all doubly infinite sequences with values in a finite set $\mathbb{Z}_r, r \geq 2$ determined by cellular automata $T_{F[-k,k]}$ with an additive automaton rule

$$F(x_{n-k},...,x_{n+k}) = \sum_{i=-k}^{k} a_i x_{n+i} (mod r).$$

It is investigated the concept of the measure theoretic directional entropy per unit of length in the direction $\omega_0$. It is shown that $h_\mu(T_{F[-k,k]}) = u h_\mu(T_{F[-k,k]})$, $h_\mu(\Phi^u) = u h_\mu(\Phi)$ and $h_{\vec{v}}(\Phi^u) = u h_{\vec{v}}(\Phi)$ for $\vec{v} \in \mathbb{Z}^2$ where $h$ is the measure-theoretic entropy.

1. Introduction

In the present paper we study directional entropy of $\mathbb{Z}^2$-action generated by an additive cellular automata (CA). CA initialed by Ulam and von Neumann has been investigated by Hedlund [4]. He systematically studied purely mathematical point of view. In Hedlund’s work are given current problems of symbolic dynamics. In [7], Shereshevsky has investigated ergodic properties of CA, and also defined the n-th iteration of a permutative cellular automata.

The concept of the directional entropy of a $\mathbb{Z}^2$-action has first been introduced by Milnor [5]. Milnor defined the concept of the directional entropy function for $\mathbb{Z}^2$-action generated by a full shift and a block map. This concept was also studied in [2], [6] and [8].

In [2], Courbage and Kaminski have calculated the directional entropy for any cellular automata (CA) of $\mathbb{Z}^2$-action $\Phi$ on the space of all doubly infinite sequences with values in a finite set $A$, determined by an automaton rule $F[l, r], l, r \in \mathbb{Z}, l \leq r$, and any $\Phi$-invariant Borel probability measure. In [6], Park expressed the directional entropy in an integral form.

In [1], the author calculated the measure entropy of additive one-dimensional cellular automata with respect to uniform Bernoulli measure. In [3], Coven and Paul investigated some properties of the endomorphisms of irreducible subshifts of finite type and n-block maps.

The shift $\sigma$ and $T_{F[-k,k]}$ are commuted and if $T_{F[-k,k]}$ is non-invertible, they generate a $\mathbb{Z} \times \mathbb{N}$ action $\Phi^{(p,q)} = \sigma^p T_{F[-k,k]}^q$ which can be extended to $\mathbb{Z}^2$-action on $\Omega$. Notice that $\sigma^i T_{F[-k,k]} = T_{F[-k,k]} \sigma^i = T_{F[-k+i,k+i]}$ for all $i \in \mathbb{Z}$. We suppose that $\mu$ is a probability ergodic measure which is invariant under the action $\Phi$ of $\mathbb{Z} \times \mathbb{N}$. Let $\vec{v}$ be an arbitrary vector of $\mathbb{Z}^2$. Denote by $h_{\vec{v}}(\Phi)$ the directional entropy of $\Phi$ [2]. The measure-theoretic entropy of $\Phi^{(p,q)}$ with respect to $\mu$ is denoted by $h_{p,q} = h(\sigma^p T_{F[-k,k]}^q)$ where $F^n$ denotes the n-th iteration of a function (or map) $F$ (cf. [7]). It is easy to show that $h_{p,0} < \infty$ for all $-\infty < p < \infty$.

2000 Mathematics Subject Classification: Primary 28D15; Secondary 37A15.
Key words and phrases: Cellular automata, measure directional entropy.
The question posed by Milnor [5] is: Does the limit
\[ \lim_{i \to \infty} \frac{1}{\sqrt{m_i^2 + n_i^2}} h_{m_i, n_i} \]
eexist for the sequence \( \{(m_i, n_i), i = 1, \infty\} \subset \mathbb{Z}^2 \), \( m_i \to \infty, n_i \to \infty, \frac{m_i}{n_i} \to \omega_0 \) as \( i \to \infty \)?

An affirmative answer to this question was given by Park [6] and Sinai [8] for an irrational number \( \omega_0 \). Sinai [8] and Park [6] also showed that the function \( h_{p, q} \) is a homogeneous function of the first degree, i.e., \( h_{u p, u q} = |u|h_{p, q} \).

In this paper under additional assumptions we show that \( h_\mu(T_{F[−k, k]}^u) = u h_\mu(T_{F[−k, k]}), h_\mu(\Phi^u) = u h_\mu(\Phi) \) and \( h_\nu(\Phi^u) = u h_\nu(\Phi) \).

2. Preliminaries

Let \( \mathbb{Z}_r = \{0, 1, ..., r-1\} \) be the set of integers modulo \( r \) and denotes a state set of each cell and \( \Omega = \prod_{i=-\infty}^{\infty} \mathbb{Z}_r = \mathbb{Z}_r^\mathbb{Z} \) be the space of all doubly infinite sequences \( x = \{x_i\}^\infty_{i=-\infty}, x_i \in \mathbb{Z}_r \). \( \Omega \) is compact in topology of direct product and a measurable space. We denote by \( \sigma \) the shift transformation on \( \Omega \), i.e., \( (\sigma x)_i = x_{i+1} \) for \( i \in \mathbb{Z} \).

It is obvious that \( \sigma \) is homeomorphism. Let \( \mathbf{M} \) be the product \( \sigma \)-algebra of \( \Omega \) and \( \mu \) be a probability invariant measure. The quadruplet \( (\Omega, \mathbf{M}, \mu, \sigma) \) is called symbolic dynamic system.

Let \( m \) be a fixed positive integer. We denote by \( \mathbb{Z}^m_r \) the \( m \)-fold direct product \( \mathbb{Z}_r \times \ldots \times \mathbb{Z}_r \).

An automaton rule \( F \) is said to be right permutative (cf. [7]) if for any \( (x_1, \ldots, x_{m-1}) \in \mathbb{Z}_r^{m-1} \) the mapping \( x_m \to F(x_1, \ldots, x_{m-1}, x_m) \) is a permutation of \( \mathbb{Z}_r \). Similarly we define a left permutative mapping.

We say that \( F \) is bipermutative if it is right and left permutative.

Any mapping \( F : \mathbb{Z}_r^{2k+1} \to \mathbb{Z}_r \) is called an automaton rule. Take any nonnegative integer \( k \) and consider a linear map \( F : \mathbb{Z}_r^{2k+1} \to \mathbb{Z}_r \) defined by formula

\[ F(z_{−k}, \ldots, z_k) = \sum_{i=−k}^{k} a_i z_i (mod r) \] 

(1)

where \( a_i \in \mathbb{Z}_r \), \( i = (−k, k) \). An automaton rule \( F \) in the form (1) is called an additive automaton rule.

The homeomorphism \( T_{F[−k, k]} : \Omega \to \Omega \) defined as

\[ (T_{F[−k, k]} x)_n = F(x_{n+k}, \ldots, x_{n−k}) = \sum_{i=−k}^{k} a_i x_{n+i} (mod r), n \in \mathbb{Z} \]
is said to be the additive one-dimensional cellular automata (CA) defined by \( F[−k, k] \).

It it clear that the additive CA-map \( T_{F[−k, k]} \) is surjective and non-invertible. Moreover, \( T_{F[−k, k]} \) preserves the uniform Bernoulli measure \( \mu \) [7].

In [7], Shereshevsyky has define inductively the u-th iteration \( F^u : \mathbb{Z}_r^{2ku+1} \to \mathbb{Z}_r \) of the rule \( F \) as follows:

\[ F^u(x_{−2ku}, \ldots, x_{−2ku+2k}, \ldots, x_{−2ku+4k}, \ldots, x_{2ku−2k}, \ldots, x_{2ku}) = F^{u−1}(F(x_{−2ku}, \ldots, x_{−2ku+2k}), F(x_{−2ku+1}, \ldots, x_{−2ku+2k+1}), \ldots, F(x_{2ku−2k}, \ldots, x_{2ku})). \]
Lemma 2.1. ([7], Lemma 1.6) The $u$-th iteration $T^u_F[-k,k]$ of CA-map $T_F[-k,k]$ generated by the rule $F$ coincides with the CA-map $T_{F^u[-k,k]}$.

It can be easily checked that the shift $\sigma$ and a cellular automaton map $T_F[-k,k]$ are commuted i.e. $\sigma \circ T_F[-k,k] = T_F[-k,k] \circ \sigma$. The $\mathbb{Z}^2$-action $\Phi$ generated by $\sigma$ and $T_F[-k,k]$, i.e. $\Phi^{(p,q)} = \sigma^n T^n_F[-k,k]$ is said to be a CA-action, if $T_F[-k,k]$ is invertible.

Let $(\Omega, \mathcal{M}, \mu, \sigma)$ be a symbolic dynamic system. Let $\prec$ denotes the lexicographical ordering of $\mathbb{Z}^2$. Denote by $O$ the zero of $\mathbb{Z}^2$. A sub $\sigma$-algebra $A$ is said to be invariant if $\Phi^{(p,q)}(A) \subset A$ for every $(p,q) \prec O$. It is clear that $A$ is invariant iff $\sigma^{-1}(A) \subset A$ and $T_F^{-1}[-k,k](A) \subset A$.

Let $\xi$ be a zero-time partition of $\Omega$:

$$\xi = \{ C_0(0), C_0(1), \ldots, C_0(r-1) \}$$

where $C_0(i) = \{ x \in \Omega ; x_0 = i \}$, $i \in \mathbb{Z}_r$, is a cylinder set.

We note that if cellular automata $T_F[-k,k]$ is permutative then the partition $\xi = \{ C_0(i), i \in \mathbb{Z}_r \}$ is a generating partition for CA-map $T_F[-k,k]$.

Now we introduce some necessary notations. Let $a \in \mathbb{R}^1$, $\omega \in \mathbb{R}^+$, and $I = \{a, \omega\}$ be a closed interval on the plane with endpoints $(a,0)$ and $(a+\omega^{-1},1)$, and $\Gamma(a,\omega)$ be a half-line $y = \omega(x-a)$, $y \leq 1$. Suppose that a probability measure $\mu$ on $\mathcal{M}$ is invariant with respect to the shift $\sigma$ and cellular automata $T_F[-k,k]$.

Define the following conditional properties:

$$H_r(I) = H( \Phi^{(p,1)}(\xi) \bigg| \bigcup_{q=0}^{\infty} \bigcup_{a+\omega^{-1} \leq p} \Phi^{(p,q)}(\xi))$$

$$H_l(I) = H( \Phi^{(p,1)}(\xi) \bigg| \bigcup_{q=0}^{\infty} \bigcup_{p \leq a + \omega^{-1}} \Phi^{(p,q)}(\xi))$$

where $H_r(I)$ and $H_l(I)$ are called the right and left entropies, respectively. In [8], it was shown that these entropies are finite.

Let $(p,q)$ be a point of $\mathbb{Z}^2$. Denote $h_{p,q} = h(\Phi^{(p,q)}) = h(\sigma^n T^n_F[-k,k])$. The value of $h_{p,q}$ is equal to the limit

$$h_{p,q} = \lim_{s \to \infty} H_p \left( \bigcup_{n=1}^{q} \bigcup_{|m-(a+\omega^{-1})n| \leq s} \Phi^{(m,n)}(\xi) \bigg| \bigcup_{n=0}^{\infty} \bigcup_{|m-(a+\omega^{-1})n| \leq s} \Phi^{(m,-n)}(\xi) \right)$$

with $\omega = \frac{q}{p}$.

Let $\omega_0$ be an irrational number, $\{(m_i, n_i), i = 0, \infty\}$ be a sequence of points of the lattice $\mathbb{Z} \times \mathbb{N}$ such that $m_i \to +\infty$ or $m_i \to -\infty$, $n_i \to +\infty$ and $\lim_{i \to \infty} \frac{m_i}{n_i} = \omega_0$.

Sinai has proved in [8] that there exists a finite limit

$$\lim_{i \to \infty} \frac{1}{m_i^2 + n_i^2} h_{m_i, n_i} = C$$

and it doesn’t depend on the choice of the sequence $\{(m_i, n_i)\}$.

Definition 2.2. The value $C$ of the limit (2) is called an entropy per unit of length in the direction $\omega_0$.

It is well known that the automaton map is not one-to-one, in general, so we should consider the natural extension of the automaton map (determined by an
automaton rule), we need to use the natural extension the semi-group action to a group action.

Let \((\Omega, M, \hat{\mu}, \hat{T})\) be a natural extension of the dynamical system \((\Omega, M, \mu, T)\) (cf. [2]).

Let us recall that \(\hat{T}\) is defined as follows:
\[
\hat{T}: (T^0, x(0), \ldots, x(\infty)) \rightarrow (T^0, T x(0), \ldots)
\]
where \(T x(i) = x(i-1), i \geq 1\). We put
\[
\hat{\tau} x = (\tau x(0), \tau x(1), \ldots).
\]
Obviously, \(\hat{T} \hat{\tau} x = \hat{\tau} \hat{T} x\). The \(\mathbb{Z}^2\) - action \(\Phi\) generated by \(\hat{\tau}\) and \(\hat{T}\):
\[
\Phi(p, q) = \hat{\tau}^p \hat{T}^q
\]
is said to be a CA-action. For a positive integer \(m\) and \(E \in M\) we put
\[
E(m) = \{\hat{x} \in \hat{\Omega}; x(m) \in E\}.
\]
It is clear that \(\hat{T}^{-1} E(m) E(m-1), m \geq 1\).

If \(\eta = \{E_1, \ldots, E_t\}\) is a measurable partition of \(\Omega\) then we denote by \(\eta^{(m)}\) the measurable partition of \(\hat{\Omega}\) defined by
\[
\eta^{(m)} = \{E_1^{(m)}, \ldots, E_t^{(m)}\}.
\]

Let \(\xi\) be the zero-time partition of \(\Omega\): \(\xi = \{C_0(i), \ldots, C_0(r-1)\}\) where \(C_0(i) = \{x \in \Omega; x_0 = i\}, i \in \mathbb{Z}\). For \(i, j \in \mathbb{Z}, i \leq j\) we put \(\xi(i, j) = \bigvee_{u=v}^{j} \sigma^{-u} \xi\).

Note that the corresponding entropies on \((\hat{\Omega}, M, \hat{\mu}, \hat{T})\) are coincide \((\Omega, M, \mu, T)\) (cf. [2], [8])

3. Main Results

Let \(\xi\) be a zero-time partition of \(\Omega\), i.e. \(\xi = \{C_0(i), i \in \mathbb{Z}\}\) and \(\{(m_i, n_i)\}\) be a sequence of the lattice \(\mathbb{Z} \times \mathbb{N}\). Define a sequence of partitions of space \(\Omega\) with respect to \(\mathbb{Z}^2\)-action \(\Phi\) by formula
\[
\xi_{(m, n)} = \Phi^{(m, n)} \xi, \quad i = 1, \infty.
\]

Lemma 3.1. Let \(\xi_{(m, n)} \nearrow \xi\) and \(\eta\) be an arbitrary measurable partition with \(H_\mu(\xi_{(m, n)} \mid \eta) < \infty\). Then
\[
H_\mu(\xi_{(m, n)} \mid \eta) \nearrow H_\mu(\xi \mid \eta).
\]

Proof. Put \(\alpha(n) = a + \omega^{-1}(n + 1) - [a + \omega^{-1}], \) where \([a]\) denotes the greatest integer \(\leq a\). Denote
\[
\eta = \bigvee_{\alpha(0) \leq m_i \leq \alpha(0) + r} \Phi_{(m_i, 1)} ^{(m, n)} \xi
\]
Let \(\xi_{(m, n)}\) and \(\xi\) be two partitions as
\[
\xi_{(m, n)} = \bigvee_{\alpha(0) \leq m_i \leq \alpha(0) + r + 2s} \Phi_{(m, 0)}^{(m, n)} \xi \vee \bigvee_{n_i < 0 \alpha(n_i) \leq m_i \leq \alpha(n_i) + 2s} \Phi_{(m_i, n_i)}^{(m, n)} \xi
\]
and
\[
\xi = \bigvee_{n_i < 0 \alpha(n_i) \leq m_i} \Phi_{(m, n_i)}^{(m, n)} \xi.
\]
Denote $C_\eta(x), C_\zeta(x)$ and $C_{\xi(m,n)}(x)$ elements of partitions $\eta$, $\zeta$ and $\xi(m,n)$ containing $x \in \Omega$, respectively. Using Doob’s theorem on convergence of conditional probabilities, we have if $\xi(m,n) \searrow \zeta$ then

$$\mu(C_{\xi(m,n)}(x)|C_\eta(x)) \to \mu(C_\zeta(x)|C_\eta(x)).$$

From this immediately follows that

$$\lim_{i \to \infty} \mu(\xi(m,n_i) | \eta) = \mu(\zeta | \eta).$$

From this using the properties of continuity of conditional entropy and logarithm we obtain that

$$\lim_{i \to \infty} H_\mu(\xi(m,n_i) | \eta) = H_\mu(\zeta | \eta).$$

Now define a transformation $Q$ in the space of segments $I(a, \omega)$ by

$$Q(I(a, \omega)) = I(a', \omega),$$

where $a' = a + \omega^{-1}$. Using properties of the measure-theoretical entropy of dynamical system we shall prove the following theorem.

**Theorem 3.2.** If $Q^i(I_1) \subset Q^i(I)$, then $H(Q^i(I_1)) \leq H(Q^i(I))$.

**Proof.** Let $(\Omega, M, \mu, \sigma)$ be symbolic dynamic system and $\Phi^{[p,q]} = \sigma^pT^q_{F[-k,k]}$ be a $\mathbb{Z}^2$-action on product space $\Omega$. Let $Q^i(I)$ and $Q^i(I_1)$, $i \geq 0$, be two transformations in the space of segments $I$ and $I_1$, respectively. We consider the case when $i = 0$. Other cases can be shown in the same way. We have $H(I_1) = H_r(I_1) + H_I(I_1)$.

Since $\xi$ is a partition of $\Omega$ we get

$$\eta = \bigvee_{a+\omega^{-1} \leq p} \sigma^pT^0_{F[-k,k]}\xi \leq \bigvee_{a+\omega^{-1} \leq p} \sigma^pT^1_{F[-k,k]}\xi = \bigvee_{a+\omega^{-1} \leq p} T^1_{F[-k+p,k+p]}\xi$$

From the continuity of conditional entropy and from Lemma 3.1 it follows

$$H(\bigvee_{a+\omega^{-1} \leq p} \sigma^pT^0_{F[-k,k]}\xi | \bigvee_{q=0}^{\infty} \sigma^pT^q_{F[-k,k]}\xi) \leq$$

$$\leq H(\bigvee_{a+\omega^{-1} \leq p} \sigma^pT^1_{F[-k,k]}\xi | \bigvee_{q=0}^{\infty} \sigma^pT^q_{F[-k,k]}\xi)$$

It means that $H_r(I_1) \leq H_r(I)$.

Similarly, it can be shown that $H_I(I_1) \leq H_I(I)$. From this and the fact that $Q^0(I_1) = I_1, Q^0(I) = I$ it is easily follows the assertion of theorem 3.2 for the case $i = 0$.

Here, we investigate the measure-theoretic entropy of $u$-th iteration of additive one-dimensional cellular automata. Recall that the CA-map $T_{F[-k,k]}$ preserves the Bernoulli measure and is non-invertible map of $\Omega$ generated by a block map. So we should consider the condition $u \geq 0$.

**Theorem 3.3.** Let $T_{F[-k,k]}$ be additive one-dimensional cellular automata. Then for every $u \geq 0$ we have

$$h_\mu T^u_{F[-k,k]} = uh_\mu(T_{F[-k,k]}).$$
Proof. Define the cylinder set \( x[i_1, \ldots, i_t] = \{ x \in \Omega : x_j = i_j, s \leq j \leq t, \ i_j \in \mathbb{Z}_r \} \). Using the definition of partition \( \xi(-k, k) \) it can be easily checked that \( \xi(-k, k) = \{ -k[i_{-k}, \ldots, i_k]k : i_j \in \mathbb{Z}_r \} \). Moreover, the partition \( \xi(-k, k) \) is a generator for \( T_{F[-k,k]} \), i.e.

\[
\bigvee_{i=0}^\infty T_{F[-k,k]}^{-i}\xi(-k,k) = \varepsilon
\]

Using the properties of the measure-theoretic entropy and Kolmogorov-Sinai theorem (cf. [9]) we get

\[
h_\mu(T_{F[-k,k]}^u) = h_\mu(T_{F[-k,k]}^u) \bigvee_{i=0}^{u-1} T_{F[-k,k]}^{-i}\xi(-k,k)
\]

\[
= \lim_{n \to \infty} \frac{1}{n} H_\mu \left( \bigvee_{j=0}^{n-1} T_{F[-k,k]}^{-uj} \left( \bigvee_{i=0}^{u-1} T_{F[-k,k]}^{-i}\xi(-k,k) \right) \right)
\]

\[
= \lim_{n \to \infty} \frac{u}{nu} H_\mu \left( \bigvee_{i=0}^{un-1} T_{F[-k,k]}^{-i}\xi(-k,k) \right)
\]

\[
= u.2k \log r = uh_\mu(T_{F[-k,k]}; \xi(-k,k)) = uh_\mu(T_{F[-k,k]}).
\]

\[\square\]

**Theorem 3.4.** Let \( \Phi = \sigma T_{F[-k,k]} \) be \( \mathbb{Z} \times \mathbb{N} \)-action. Then for all \( u \geq 0 \), \( h_\mu(\Phi^u) = uh_\mu(\Phi) \) and if the automaton rule \( F[-k,k] \) is bipermutative then \( h_{\bar{\nu}}(\Phi^u) = uh_{\bar{\nu}}(\Phi) \) for all \( \bar{\nu} \in \mathbb{Z}^2 \).

Proof. Again first it is easy to see that the partition \( \xi(-k,k) = \{ -k[i_{-k}, \ldots, i_k]k : i_j \in \mathbb{Z}_r \} \) is generator for \( \Phi = \sigma T_{F[-k,k]} \) that is \( \bigvee_{i=0}^\infty \sigma^{-i}T_{F[-k,k]}^{-i}\xi(-k,k) = \varepsilon \). So we have

\[
h_\mu(\Phi^u) = h_\mu(\Phi^u) \bigvee_{i=0}^{u-1} \Phi^{-i}\xi(-k,k)
\]

\[
= \lim_{s \to \infty} \frac{1}{s} H_\mu \left( \bigvee_{j=0}^{s-1} T_{F[-k,k]}^{-uj} \sigma^{-uj} \left( \bigvee_{i=0}^{u-1} T_{F[-k,k]}^{-i}\xi(-k,k) \right) \right)
\]

\[
= \lim_{s \to \infty} \frac{1}{s} H_\mu \left( \bigvee_{j=0}^{s-1} T_{F[-k,k]}^{-uj} \sigma^{-uj} \left( \bigvee_{i=0}^{u-1} T_{F[-k,k]}^{-i}\xi(-k - i, k - i) \right) \right)
\]

\[
= \lim_{s \to \infty} \frac{1}{s} H_\mu \left( \bigvee_{j=0}^{s-1} T_{F[-k,k]}^{-uj} \left( \bigvee_{i=0}^{u-1} T_{F[-k,k]}^{-i}\xi(-k - (i + ju), k - (i + ju)) \right) \right)
\]

\[
= u \lim_{s \to \infty} \frac{1}{us} H_\mu \left( \bigvee_{j=0}^{us-1} T_{F[-k,k]}^{-j}\xi(-k - (i + ju), k - (i + ju)) \right) = uh_\mu(\Phi)
\]
Now we consider the directional entropy of $\mathbb{Z}^2$-action. Here we only consider $\vec{v} \in \mathbb{Z}^2$. Using the definition of $h_{\vec{v}}(\Phi)$ (cf. [2]) we have

\[
h_{\vec{v}}(\Phi^u) = h_{\mu}(\tilde{\sigma}^{up}T_{-k,k}) = h_{\mu}(\sigma^{up}T_{-k,k}) = h_{\mu}(\sigma^{up}T_{-qk_u,qk_u}) = h_{\mu}(T_{F^{-eq}[qk_u+up,qk_u+up]}) = uh_{\vec{v}}(\Phi). \]

□

One can investigate for $\vec{v} \in \mathbb{R}^2$.

References

[1] H. Akın, On the measure entropy of additive cellular automata $f_\infty$, Entropy 2003, 5, 233-238.
[2] M. Courbage and B. Kaminski, On the directional entropy of $\mathbb{Z}^2$-actions generated by cellular automata, Studia Math. 153 (3) (2002).
[3] F. B. Coven and M. E. Paul, Endomorphisms of irreducible subshifts of finite type, Math. Systems Theory 3 (1974), 167-175.
[4] G. A. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, Math. Systems Theory 3 (1969), 320-375.
[5] J. Milnor, Directional entropies of Cellular autamation -maps, Nato ASI Series vol.F20, (1986), 133-115.
[6] K. K. Park, Continuity of directional entropy, Osaka J. Math. 31(1994), 613-628.
[7] M. A. Shereshevsky, Ergodic properties of certain surjective cellular automata, Monatsh. Math. 114 (1992), 305-316.
[8] Y. Sinai, An answer to a question by J. Milnor, Comment. Math. Helv. 60 (1985), 173-178.
[9] P. Walters, An Introduction to Ergodic Theory, New York, Springer, (1982).