Calculation of three-layer composite beams using difference equations of successive approximation method

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Abstract. Beams are considered the most popular bending elements used in building structures. With the load-bearing capacity and production characteristics, it is possible to use many different materials to combine them into beams to receive high economic-technical efficiency. Besides, because of the development of structural material industry, the manufacturing and construction technology, the form of beams is as diverse as composite wood beams, foundation beams, force-sensitive composite beams prior to installation grafting, steel-concrete combination beams. So there are many researches about solving the problems of multi-layer structures in general and multi-layer beams in particular by different calculation methods. In this paper, based on the theory of multi-layer composite rods and plates of A.R. Rzhanitsyn the solutions of calculation analysis of three-layer beams that is subjected to discontinuous loads using difference equations of successive approximation method (MSA) are presented. The obtained results with good convergence show high accuracy of the numerical method with the use of difference equations of successive approximation method.

1. Introduction

From the beginning of the 20th century to till now the widespread use of multi-layer beams in many industries, such as aerospace and marine industries, transport and building construction, etc.... requires continuous enhancement and development of calculation methods with taking into account all the real factors in practical structures. Many studies about multilayer structures and beams were carried out on the basis of various calculation models. In the studies it can be seen that the works of Lei, Galuppi [1-3], which indicated the behavior of multilayer beams with a viscoelastic interlayer based on the Euler-Bernoulli theory, and the works of Lenci, Tessler, Arzikogl [4-6], analyzed the dynamic characteristics of the sandwich beam using the first-order shear deformation of S.I. Timoshenko. Besides, according to the theory of composite rods and plates A.R. Rzhanitsyn [7] many papers were published about reinforced concrete beams [8-12], steel perforated beams [13-15] and wooden beams [16], [17]. However, because of the difficulty in obtaining the general root of differential equations to determine internal forces and displacements of the problems of the multi-layer beam [7] subjected to any load, the above problems only solved in the case of simple loading and in the case of complex loading it is not considered particularly.

In this article, the authors propose an approach to solve this problem using difference equations of successive approximation method [18-20] to develop differential equations for three-layer beams subjected to any load. An established algorithm and software for modeling on a computer are used to...
solve these problems. To demonstrate the effectiveness of the proposed method, the calculation results of three-layer beams using proposed approach show good convergence of the solution on several proposed grids.

2. Theory background
The analysis of three-layer composite beam subjected to transverse load is simplified into solving four ordinary differential equations of second order, formulated with respect to shear forces $T$, deflections $y$ and bending moments $M$.

Based on the equations (5.18), (5.8), (5.6) of Rzhanitsyn [7], the differential equations are written in following form:

\[
\begin{align*}
\frac{T_1}{\xi_1} - \Delta_{11} T_1 - \Delta_{12} T_2 &= \Delta_{10}; \\
\frac{T_2}{\xi_2} - \Delta_{21} T_1 - \Delta_{22} T_2 &= \Delta_{20}; \\
y' &= -\frac{M}{\Sigma EI}; \\
d^2M_0 \frac{d^2y}{dx^2} &= -q,
\end{align*}
\]

in which:

\[
M = M_0 - T_1 \xi_1 - T_2 \xi_2.
\]

the coefficients $\Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22}$ and free members $\Delta_{10}, \Delta_{20}$ in (5.14) [7] have following values:

\[
\begin{align*}
\Delta_{11} &= \frac{1}{E_2 F_2} + \frac{1}{E_1 F_1} + \frac{c_1^2}{E_1 I_1 + E_2 I_2 + E_3 I_3}; \\
\Delta_{12} &= -\frac{1}{E_2 F_2} + \frac{c_1 c_2}{E_1 I_1 + E_2 I_2 + E_3 I_3}; \\
\Delta_{21} &= -\frac{1}{E_2 F_2} + \frac{c_2 c_1}{E_1 I_1 + E_2 I_2 + E_3 I_3}; \\
\Delta_{22} &= \frac{1}{E_3 F_3} + \frac{1}{E_2 F_2} + \frac{c_2^2}{E_1 I_1 + E_2 I_2 + E_3 I_3}; \\
\Delta_{10} &= -\frac{M_0 c_1}{E_1 I_1 + E_2 I_2 + E_3 I_3}; \\
\Delta_{10} &= -\frac{M_0 c_2}{E_1 I_1 + E_2 I_2 + E_3 I_3},
\end{align*}
\]

$\xi_1, \xi_2$ - stiffness coefficients of seam;
$q$ - distributed load;
$M_0$ - total bending moment in the cross section of the composite rod subjected to external load without consideration of the forces transmitted from transverse and shear joints;
$M$ - total bending moment in the system.
$c_1, c_2$ - distances between the center of the cross section of the two rods of 1 and 2, of 2 and 3 respectively (figure 1);
$T_1, T_2$ - total shear forces in the seams of 1 and 2, accumulated along the length of the rod from its beginning to the considered section;
$E_1 I_1, E_2 I_2, E_3 I_3$ - bending section stiffness of rods of 1, 2 and 3 (figure 1);
$E_1 F_1, E_2 F_2, E_3 F_3$ - section stiffness of rods of 1, 2 and 3 for the vertical deformation (figure 1);
$\Sigma EI$ - sum of the section stiffness of the three rods.
Applying the dimensionless parameters:
$$\Psi = \frac{x}{l}; \omega = \frac{y}{l}; m_0 = M_0 \frac{1}{EI}; t = T \frac{l^2}{EI},$$
(11)
In which $l$ - length of beam.
Using (11), (10), (9), (8), (7), (6), (5) it will be obtained system of equations (1), equations (2) and (3) in the dimensionless form:
In which $l$ - length of beam.
$$\left\{ \begin{array}{l}
\frac{d^2t_1}{d\Psi^2} = \xi_1 \cdot \alpha t_1 + \beta t_2 - \gamma m_0 ; \\
\frac{d^2t_2}{d\Psi^2} = \xi_2 \cdot \beta t_1 + \theta t_2 - \mu m_0 ; \\
\frac{d^2\omega}{d\Psi^2} = -\varepsilon \left( m_0 - t_1 \frac{c_1}{l} - t_2 \frac{c_2}{l} \right); \\
\frac{d^2m_0}{d\Psi^2} = -p,
\end{array} \right.$$  
(12)
(13)
(14)
in which:
$$\xi_1 = \xi_1 \cdot \frac{l^4}{EI} \cdot \alpha = \left( \frac{1}{E_2 F_2} + \frac{1}{E_1 F_1} + \frac{c_1^2}{E_1 I_1 + E_2 I_2 + E_3 I_3} \right) \cdot \frac{EI}{l^2};$$
$$\beta = \left( \frac{1}{E_2 F_2} + \frac{c_1 c_2}{E_1 I_1 + E_2 I_2 + E_3 I_3} \right) \cdot \frac{EI}{l^2};$$
$$\gamma = \frac{c_1}{E_1 I_1 + E_2 I_2 + E_3 I_3} \cdot \frac{EI}{l};$$
$$\xi_2 = \xi_2 \cdot \frac{l^4}{EI} \cdot \theta = \left( \frac{1}{E_3 F_3} + \frac{1}{E_2 F_2} + \frac{c_2^2}{E_1 I_1 + E_2 I_2 + E_3 I_3} \right) \cdot \frac{EI}{l^2};$$
$$\mu = \frac{c_2}{E_1 I_1 + E_2 I_2 + E_3 I_3} \cdot \frac{EI}{l};$$
$$\varepsilon = \frac{EI}{E_1 I_1 + E_2 I_2 + E_3 I_3};$$
$$p = \frac{l^2}{EI} \cdot q.$$  
(15)
(16)
(17)
(18)
The above-obtained system of equations (12) and equations (13), (14) can be presented with using differential formulation of successive approximations [18] in the following form:
\[
\left\{\begin{array}{l}
\left(1 - \frac{h^2}{12} \xi_1 \alpha \right) t_{i-1}^l - 2 \left(1 + \frac{5h^2}{12} \xi_1 \alpha \right) t_i^l + \left(1 - \frac{h^2}{12} \xi_1 \alpha \right) t_{i+1}^l - \frac{h^2}{12} \xi_{1, \beta}. t_{i-1}^l + 10t_i^l + t_{i+1}^l = \\
- \frac{h^2}{12} \xi_{1, \gamma} \left[m_{i-1}^R + 10m_i^L + m_{i+1}^L - 5. \Delta m_i - h. \Delta m_i^l\right]
\end{array}\right.
\]
\begin{equation}
(19)
\end{equation}

\[
\left\{\begin{array}{l}
\left(1 - \frac{h^2}{12} \xi_2 \beta \right) t_i^l - \frac{h^2}{12} \xi_{2, \beta} . t_i^l + 10t_i^l + t_{i+1}^l + \left(1 - \frac{h^2}{12} \xi_2 \beta \right) t_{i-1}^l - 2 \left(1 + \frac{5h^2}{12} \xi_2 \beta \right) t_2^l + \\
+ \left(1 - \frac{h^2}{12} \xi_{2, \gamma} \right) t_2^l = \frac{-h^2}{12} \xi_{2, \gamma} \left[m_i^R + 10m_i^L + m_{i+1}^L - 5. \Delta m_i - h. \Delta m_i^l\right];
\end{array}\right.
\]
\begin{equation}
(20)
\end{equation}

\[
m_{i-1}^l - 2m_i^L + m_{i+1}^L + \Delta m_i + h. \Delta m_i^l = \frac{-h^2}{12} . p_i^R + 10p_i^L + p_{i+1}^L + \frac{5}{12} h^2 \Delta p_i + \frac{h^3}{12} \Delta p_i^l.
\]
\begin{equation}
(21)
\end{equation}

Since \( m, \omega, t_1, t_2 \) are main unknowns, to calculate the simply supported beams, it can be sufficiently written equations (19)-(21) for all regular points. According to [7] with hinged support at the boundary points \( m = \omega = t_i = 0 \). With other boundary conditions it is necessary to consider the approximation of these conditions in following form.

For left boundary points of beam:
\[
\left\{\begin{array}{l}
\left(1 + \frac{5h^2}{12} \xi_1 \alpha \right) t_i^R - \left(1 - \frac{h^2}{12} \xi_1 \alpha \right) t_i^{R+1} - \frac{h^2}{12} . \Delta m_i^R + \frac{5h^2}{12} . \Delta m_i^L = \\frac{5h^2}{12} \xi_{1, \gamma} \left[m_i^R + \frac{m_{i+1}^L}{2} + \frac{h^2}{12} . 5p_i^R + p_{i+1}^L + \frac{h^3}{48} . p_i^R\right];
\end{array}\right.
\]
\begin{equation}
(22)
\end{equation}

\[
\left\{\begin{array}{l}
\left[1 + \frac{5h^2}{12} \xi_2 \beta \right] t_i^L + \frac{h^2}{12} \xi_{2, \beta} \left[1 + \frac{5h^2}{12} \xi_2 \beta \right] t_i^{L+1} + \frac{h^2}{12} \xi_{2, \gamma} \left[1 + \frac{5h^2}{12} \xi_2 \beta \right] t_i^{L+1} = \\frac{5h^2}{12} \xi_{2, \gamma} \left[m_i^L + \frac{m_{i+1}^R}{2} + \frac{h^2}{12} . \left(p_i^R + p_{i+1}^L\right) + \frac{h^3}{48} . \left(p_i^R\right)\right];
\end{array}\right.
\]
\begin{equation}
(23)
\end{equation}

\[
\left\{\begin{array}{l}
- h. \omega_i^R - \omega_i^L + \omega_i^{L+1} = \frac{-h^2}{12} . h. \left[m_i^R - \frac{c_1}{l} . t_i^{R+1} - \frac{c_2}{l} . t_i^{R+1}\right] + \\
+ 5 \left[m_i^R - \frac{c_1}{l} . t_i^{R+1} - \frac{c_2}{l} . t_i^{R+1}\right] = \frac{-h^2}{12} . 5 \left[p_i^R + 10p_{i+1}^L - \frac{h^3}{12} p_i^R\right].
\end{array}\right.
\]
\begin{equation}
(24)
\end{equation}

Similarly, for right boundary points of beam, equations (22)-(24) are written in a “mirror image”, in which the first order derivative of the functions in these equations at a point (i) change their sign.
In which \( h \) - step of grid; \( R \) - right; \( L \) - left.

3. Numerical example

To illustrate the above-developed algorithm, the calculation analysis of three-layer beams is considered in the following example.

Three-layer beam subjected to load is analyzed as calculation example, \( q = 0.02kN/cm \), \( P = 10kN \) (figure 1). Elastic modules of layers \( E_1 = E_2 = E_3 = 21.10^3kN/cm^2 \); inertial moments of the cross-section \( I_1 = 144cm^4, I_2 = 288,054cm^4, I_3 = 431,476cm^4 \); distance between two layers \( c_1 = 13,56cm, c_2 = 16,21cm \); height of layers \( h_1 = 12cm, h_2 = 15,12cm, h_3 = 17,3cm \); length of beam \( l = 600cm \); stiffness coefficient of seams \( \xi_1 = \xi_2 = 3kN/cm^2 \).

![Figure 1](image)

**Figure 1.** Simple supported three-layer beam subjected to distributed load (a) and combined load (b); fixed three-layer beam subjected to distributed load (c); three-layer beam with one fixed end and roller support subjected to combined load (d).

It will obtain the convergence of the example results of maximum shear force \( T_{(\text{max})} \), bending moment \( M_{(\text{max})} \) and displacement \( y_{(\text{max})} \) from four grids changed from one into another in the following tables 1, 2, 3, 4.

**Table 1.** Calculation results of example in the case a.

| h     | MSA   |
|-------|-------|
|       | 1/4   | 1/8   | 1/16  | 1/32  |
| \( T_{(\text{max})} \) (kN) | 18.45 | 18.46 | 18.47 | 18.47 |
| \( T_{(\text{max})} \) (kN) | 22.16 | 22.18 | 22.18 | 22.18 |
| \( M_{1(\text{max})} \) (kN.cm) | 131.61 | 131.6 | 131.59 | 131.59 |
| \( M_{2(\text{max})} \) (kN.cm) | 197.84 | 197.77 | 197.76 | 197.76 |
| \( M_{3(\text{max})} \) (kN.cm) | 353.89 | 353.82 | 353.81 | 353.81 |
| \( y_{(\text{max})} \) (cm) | 0.6211 | 0.6214 | 0.6214 | 0.6214 |

**Table 2.** Calculation results of example in the case b.

| h     | MSA   |
|-------|-------|
|       | 1/4   | 1/8   | 1/16  | 1/32  |
| \( T_{(\text{max})} \) (kN) | 34.67 | 34.81 | 34.82 | 34.82 |
| \( T_{(\text{max})} \) (kN) | 41.63 | 41.8 | 41.81 | 41.81 |
| \( M_{(\text{max})} \) (kN.cm) | 290.45 | 290.32 | 290.31 | 290.31 |
Table 3. Calculation results of example in the case c.

|        | MSA     |
|--------|---------|
|        | 1/4     | 1/8     | 1/16    | 1/32    |
| $T_1^{(\text{max})}$ (kN) | -4.59   | -4.74   | -4.75   | -4.75   |
| $T_2^{(\text{max})}$ (kN) | -5.5    | -5.67   | -5.69   | -5.69   |
| $M_1^{(\text{max})}$ (kN.cm) | -95.45  | -95.31  | -95.3   | -95.3   |
| $M_2^{(\text{max})}$ (kN.cm) | -174.73 | -173.91 | -173.85 | -173.85 |
| $M_3^{(\text{max})}$ (kN.cm) | -276    | -275.24 | -275.2  | -275.2  |
| $y^{(\text{max})}$ (cm)    | 0.228   | 0.231   | 0.231   | 0.231   |

Table 4. Calculation results of example in the case d.

|        | MSA     |
|--------|---------|
|        | 1/4     | 1/8     | 1/16    | 1/32    |
| $T_1$ (kN) | -9.62   | -9.82   | -9.83   | -9.83   |
| $T_2$ (kN) | -11.49  | -11.72  | -11.73  | -11.73  |
| $M_1^{(\text{max})}$ (kN.cm) | -217.94 | -216.87 | -216.78 | -216.78 |
| $M_2^{(\text{max})}$ (kN.cm) | -402.12 | -399.29 | -399.1  | -399.1  |
| $M_3^{(\text{max})}$ (kN.cm) | -632.23 | -628.59 | -628.31 | -628.3  |
| $y^{(\text{max})}$ (cm)    | 0.702   | 0.709   | 0.71    | 0.712   |

4. Summary

It is shown that the proposed methodology is not difficult to generalize in calculation analysis of the three-layer beams. Beams with an asymmetrical cross-section are not considered in [19] and the analysis results of three-layer beams according to the proposed methodology are illustrated in first publication. To ensure the quality of the results, the convergence of the analysis result on several proposed grids was checked.

The simplicity of the methodology, the high accuracy of the analysis results with small nodes number of beams allow to develop the proposed methodology for further research in universities of construction industry and application for design calculation process of construction structures.

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