Embedding of the Lie superalgebra $D(2, 1; \alpha)$ into the Lie superalgebra of pseudodifferential symbols on $S^{1|2}$

Elena Poletaeva

School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540 and Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78539 (permanent address)
Electronic mail: elena@math.ias.edu and elenap@utpa.edu

Abstract. We obtain an embedding of a one-parameter family of exceptional simple Lie superalgebras $D(2, 1; \alpha)$ into the Lie superalgebra of pseudodifferential symbols on the supercircle $S^{1|2}$. Correspondingly, there is an embedding of $D(2, 1; \alpha)$ into a nontrivial central extension of the derived contact superconformal algebra $K'(4)$ realized in terms of $4 \times 4$ matrices over a Weyl algebra.

I. Introduction

Recall that $D(2, 1; \alpha)$ with $\alpha \in \mathbb{C}\{0, -1\}$ is a one-parameter family of classical simple Lie superalgebras of dimension 17 (Ref. 1). The bosonic part of $D(2, 1; \alpha)$ is $sl(2) \oplus sl(2) \oplus sl(2)$, and the action of $D(2, 1; \alpha)_0$ on $D(2, 1; \alpha)_1$ is the product of 2-dimensional representations. These superalgebras are also denoted in the literature by $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i$ are nonzero complex numbers such that $\sigma_1 + \sigma_2 + \sigma_3 = 0$ (Refs. 2, 3). $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong D(2, 1; \alpha)$, where $\alpha = \sigma_1/\sigma_2$. In Ref. 4 M. Güngördin gave a differential operator realization of the action of $D(2, 1; \alpha)$ on a family of superspaces with 2 bosonic and 2 fermionic coordinates.

In this work we consider $\Gamma(2, -1 - \alpha, \alpha - 1)$ as a one-parameter family of deformations of the Lie superalgebra $spo(2|4) \cong osp(4|2)$ embedded into the Poisson superalgebra $P(4)$ of pseudodifferential symbols on the supercircle $S^{1|2}$ with even variable $t$ and odd variables $\xi_1$ and $\xi_2$. $P(4) = \Lambda(4)$, where $P$ is the Poisson algebra of functions on the cylinder $T^* S^{1|1}$ (which are formal Laurent series in $\tau = \frac{\partial}{\partial t}$ along the fibres, with coefficients periodic in $t$), and $\Lambda(4) = \Lambda(\xi_1, \xi_2, \eta_1, \eta_2)$ is the Grassmann algebra, see Ref. 5. If $\alpha = 0$, then $\Gamma(2, -1, -1) \cong spo(2|4)$, and it is naturally embedded into $P(4)$. Note that $spo(2|4)_1$ is spanned by the zero modes of the fermionic fields of two copies of a Lie superalgebra, which is isomorphic to the derived superalgebra $S'(2, 0)$ of divergence-free derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2)$. We obtain one copy, if we identify $\frac{\partial}{\partial \xi_i}$ with $\eta_i$ for $i = 1, 2$. To obtain the other copy we interchange $\xi_i$ with $\eta_i$ in all formulas. Then using the Schwimmer-Seiberg's
deformation $S'(2, \alpha)$ (see Refs. 6 and 7) of each copy of this superalgebra, we embed $\Gamma(2,-1-\alpha,\alpha-1)$ into $P(4)$ for each $\alpha \in \mathbb{C}$. There is also an embedding of $\Gamma(2,-1-\alpha,\alpha-1)$ into the family of Lie superalgebras of pseudodifferential symbols $P_h(4)$, where $h \in (0,1]$, which contracts to $P(4)$.

Note that $S'(2,\alpha)$ is spanned by 4 bosonic and 4 fermionic fields, and it is a subsuperalgebra of the derived contact superconformal algebra $K'(4)$, which is spanned by 8 bosonic and 8 fermionic fields (Refs. 7, 8, 9 and 10). $K'(4)$ is also known to physicists as the (centerless) “big $N=4$ superconformal algebra” (Refs. 11 and 12).

We have shown in Ref. 5 that there exists an embedding of one of three independent nontrivial central extensions $\hat{K}'(4)$ of $K'(4)$ into $P_h(4)$ for each $h \in (0,1]$. Note that this central extension is different from the one that corresponds to the Virasoro cocycle. Associated to these embeddings, there are spinor-like irreducible representations of $\hat{K}'(4)$ in the superspaces $V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2)$, where $(\frac{\partial}{\partial t})^{-1}$ acts as an antiderivative. This requires that $\mu \in \mathbb{C} \setminus \mathbb{Z}$. Nevertheless, a representation of $\hat{K}'(4)$ in $V^\mu$ is well-defined even if $\mu = 0$. In this case we obtain a realization of $\hat{K}'(4)$ in terms of $4 \times 4$ matrices over a Weyl algebra $W = \sum_{i \geq 0} A d^i$, where $A = \mathbb{C}[t, t^{-1}]$ and $d = t \frac{\partial}{\partial t}$. Then we describe $\Gamma(2,-1-\alpha,\alpha-1)$ as a subsuperalgebra of $\hat{K}'(4)$ for each $\alpha \in \mathbb{C}$.

In Ref. 14 (see also Ref. 13) we used the similar approach to realize the exceptional $N=6$ superconformal algebra, which is spanned by 32 fields (Refs. 9, 10 and 15-18), as a subsuperalgebra of $8 \times 8$ matrices over a Weyl algebra. This realization is analogous to the realization, given by C. Martinez and E. I. Zelmanov in Refs. 19 and 20, where they used a different method.

Note that the affine superalgebra $\hat{D}(2,1;\alpha)$ is closely related to the big $N=4$ superconformal algebra (see Ref. 21). It is an interesting problem to realize $\hat{D}(2,1;\alpha)$ in terms of pseudodifferential symbols and matrices over a Weyl algebra. We would also like to find such realizations for the exceptional Lie superalgebra $F(4)$ (Ref. 1).

II. Superconformal algebras

A superconformal algebra is a complex Lie superalgebra $\mathfrak{g}$ such that

1) $\mathfrak{g}$ is simple,
2) $\mathfrak{g}$ contains the Witt algebra $Witt = \text{der} \mathbb{C}[t, t^{-1}] = \oplus_{n \in \mathbb{Z}} \mathbb{C} L_n$ with the well-known commutation relations

$$[L_n, L_m] = (m - n)L_{n+m} \quad (2.1)$$

as a subalgebra,
3) $adL_0$ is diagonalizable with finite-dimensional eigenspaces:

$$\mathfrak{g} = \oplus_i \mathfrak{g}_i, \quad \mathfrak{g}_i = \{ x \in \mathfrak{g} \mid [L_0, x] = ix \}, \quad (2.2)$$

so that $\text{dim} \mathfrak{g}_i < C$, where $C$ is a constant independent of $i$, see Refs. 7, 8, 22 and 23.
Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$, and let $\Lambda(1,2N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)$ be an associative superalgebra with natural multiplication and with the following parity of generators: $p(t) = 0$, $p(\xi_i) = p(\eta_i) = 1$ for $i = 1, \ldots, N$. Let $W(2N)$ be the Lie superalgebra of all superderivations of $\Lambda(1,2N)$. Let $\partial_t, \partial_{\xi_i}$ and $\partial_{\eta_i}$ stand for $\frac{\partial}{\partial t}, \frac{\partial}{\partial \xi_i}$ and $\frac{\partial}{\partial \eta_i}$, respectively. Every $D \in W(2N)$ is represented by a differential operator,

$$D = f \partial_t + \sum_{i=1}^{N} (f_i \partial_{\xi_i} + g_i \partial_{\eta_i}),$$

(2.3)

where $f, f_i, g_i \in \Lambda(1,2N)$.

The Lie superalgebra $W(2N)$ contains a one-parameter family of Lie superalgebras $S(2N, \alpha)$. By definition

$$S(2N, \alpha) = \{ D \in W(2N) \mid \text{Div}(t^\alpha D) = 0 \} \text{ for } \alpha \in \mathbb{C},$$

(2.4)

see Refs. 6 and 7. Recall that

$$\text{Div}(D) = \partial_t(f) + \sum_{i=1}^{N} ((-1)^{p(f_i)} \partial_{\xi_i}(f_i) + (-1)^{p(g_i)} \partial_{\eta_i}(g_i)).$$

(2.5)

Let $S'(2N, \alpha) = [S(2N, \alpha), S(2N, \alpha)]$ be the derived superalgebra. Assume that $N \geq 1$. If $\alpha \notin \mathbb{Z}$, then $S(2N, \alpha)$ is simple, and if $\alpha \in \mathbb{Z}$, then $S'(2N, \alpha)$ is a simple ideal of $S(2N, \alpha)$ of codimension one defined from the exact sequence,

$$0 \rightarrow S'(2N, \alpha) \rightarrow S(2N, \alpha) \rightarrow \mathbb{C}t^{-\alpha} \xi_1 \cdots \eta_N \partial_t \rightarrow 0.$$  

(2.6)

Notice that

$$S(2N, \alpha) \cong S(2N, \alpha + n) \text{ for } n \in \mathbb{Z}. $$

(2.7)

There exists, up to equivalence, one nontrivial 2-cocycle on $S'(2N, \alpha)$ if and only if $N = 1$, see Ref. 7. The corresponding central extension $\hat{S}'(2,0)$ is also called the “$N = 4$ superconformal algebra” (Refs. 11, 12, 9 and 22). Let

$$\{L_n, E_n, H_n, F_n, h_n, p_n, x_n, y_n\}_{n \in \mathbb{Z}}$$

(2.8)

be the following basis of $S'(2,0)$:

$$L_n = -t^n (t \partial_t + \frac{1}{2} (n+1) (\xi_1 \partial_{\xi_1} + \eta_1 \partial_{\eta_1})), $$

$$E_n = t^n \eta_1 \partial_{\xi_1}, \quad H_n = t^n (\eta_1 \partial_{\eta_1} - \xi_1 \partial_{\xi_1}), \quad F_n = t^n \xi_1 \partial_{\eta_1}, $$

$$h_n = t^n \eta_1 \partial_t - nt^{n-1} \xi_1 \eta_1 \partial_{\xi_1}, \quad p_n = t^{n+1} \partial_{\eta_1}, $$

$$x_n = t^{n+1} \partial_{\xi_1}, \quad y_n = t^n \xi_1 \partial_t + nt^{n-1} \xi_1 \eta_1 \partial_{\eta_1}.$$  

(2.9)
The 2-cocycle in $\hat{S}'(2,0)$ is given as follows:

\[
c(L_n, L_k) = \frac{1}{12} (n^3 - n) \delta_{n+k,0},
\]
\[
c(E_n, F_k) = \frac{1}{6} n \delta_{n+k,0}, \quad c(H_n, H_k) = \frac{1}{3} n \delta_{n+k,0},
\]
\[
c(h_n, p_k) = -\frac{1}{6} (n^2 - n) \delta_{n+k,0}, \quad c(x_n, y_k) = -\frac{1}{6} (n^2 + n) \delta_{n+k,0}.
\]

(2.10)

III. Poisson superalgebra

The Poisson algebra $P$ of pseudodifferential symbols on the circle is formed by the formal series

\[
A(t, \tau) = \sum_{-\infty}^{n} a_i(t) \tau^i,
\]

(3.1)

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the even variable $\tau$ corresponds to $\partial_t$, Refs. 24-27. The Poisson bracket is defined as follows:

\[
\{A(t, \tau), B(t, \tau)\} = \partial_\tau A(t, \tau) \partial_t B(t, \tau) - \partial_t A(t, \tau) \partial_\tau B(t, \tau).
\]

(3.2)

An associative algebra $P_h$, where $h \in (0, 1]$, is a deformation of $P$. The multiplication in $P_h$ is given as follows:

\[
A(t, \tau) \circ_h B(t, \tau) = \sum_{n \geq 0} \frac{h^n}{n!} \partial^n_\tau A(t, \tau) \partial^n_t B(t, \tau).
\]

(3.3)

The Lie algebra structure on the vector space $P_h$ is given by

\[
[A, B]_h = A \circ_h B - B \circ_h A,
\]

(3.4)

so that

\[
\lim_{h \to 0} \frac{1}{h}[A, B]_h = \{A, B\}.
\]

(3.5)

The Poisson superalgebra of pseudodifferential symbols on $S^{1|N}$ is $P(2N) = P \otimes \Lambda(2N)$. The Poisson bracket is defined as follows:

\[
\{A, B\} = \partial_\tau A \partial_t B - \partial_t A \partial_\tau B + (-1)^{p(A)+1} \sum_{i=1}^{N} (\partial_{\xi_i} A \partial_{\eta_i} B + \partial_{\eta_i} A \partial_{\xi_i} B).
\]

(3.6)

Let $\Lambda_h(2N)$ be an associative superalgebra with generators $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$ and relations

\[
\xi_i \xi_j = -\xi_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \xi_j = h \delta_{i,j} - \xi_j \eta_i.
\]

(3.7)
Superalgebras $\Gamma(\sigma, \sigma)$

Let $P_h(2N) = P_h \otimes \Lambda_h(2N)$ be a superalgebra with the product given by

$$(A_1 \otimes X)(B_1 \otimes Y) = (A_1 \circ_h B_1) \otimes (XY),$$

where $A_1, B_1 \in P_h$ and $X, Y \in \Lambda_h(2N)$. The Lie bracket of $A = A_1 \otimes X$ and $B = B_1 \otimes Y$ is

$$[A, B]_h = AB - (-1)^{p(A)p(B)} BA,$$

and (3.5) holds. $P_h(2N)$ is called the Lie superalgebra of pseudodifferential symbols on $S^{1|N}$, see Refs. 5 and 13.

**IV. Superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$**

Recall the definition of $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, see Refs. 1, 2, 3. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, where $\mathfrak{g}_{\bar{0}} = sp(\psi_1) \oplus sp(\psi_2) \oplus sp(\psi_3)$ and $\mathfrak{g}_{\bar{1}} = V_1 \otimes V_2 \otimes V_3$, where $V_i$ are 2-dimensional vector spaces, and $\psi_i$ is a non-degenerate skew-symmetric form on $V_i, i = 1, 2, 3$. A representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is the tensor product of the standard representations of $sp(\psi_i)$ in $V_i$. Consider $sp(\psi_i)$ - invariant bilinear mapping

$$p_i : V_i \times V_i \rightarrow sp(\psi_i), \quad i = 1, 2, 3,$$

given by

$$p_i(x_i, y_i)z_i = \psi_i(y_i, z_i)x_i - \psi_i(z_i, x_i)y_i$$

for all $x_i, y_i, z_i \in V_i$. Let $\mathcal{P}$ be a mapping

$$\mathcal{P} : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \rightarrow \mathfrak{g}_{\bar{0}}$$

given by

$$\mathcal{P}(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3) =$$

$$\sigma_1 \psi_2(x_2, y_2)\psi_3(x_3, y_3)p_1(x_1, y_1) +$$

$$\sigma_2 \psi_1(x_1, y_1)\psi_3(x_3, y_3)p_2(x_2, y_2) +$$

$$\sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)p_3(x_3, y_3)$$

for all $x_i, y_i \in V_i, i = 1, 2, 3$, where $\sigma_1, \sigma_2, \sigma_3$ are some complex numbers. The Jacobi identity is satisfied if and only if $\sigma_1 + \sigma_2 + \sigma_3 = 0$. In this case $\mathfrak{g}$ is denoted by $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. Superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ and $\Gamma(\sigma'_1, \sigma'_2, \sigma'_3)$ are isomorphic if and only if there exists a nonzero element $k \in \mathbb{C}$ and a permutation $\pi$ of the set $\{1, 2, 3\}$ such that

$$\sigma'_i = k \cdot \sigma_{\pi_i}, \text{ for } i = 1, 2, 3.$$

Superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ are simple if and only if $\sigma_1, \sigma_2, \sigma_3$ are all different from zero (see Ref. 3). Note that $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong D(2, 1; \alpha)$, where $\alpha = \sigma_1/\sigma_2$. 

5
\textbf{Theorem 4.1:} Let $\Gamma_\alpha$, where $\alpha \in \mathbb{C}$, be the Lie superalgebra spanned by the following elements in $P(4)$:

\begin{align*}
E^1_\alpha &= t^2, & F^1_\alpha &= \tau^2 - 2\alpha t^{-2}\xi_1\xi_2\eta_1\eta_2, & H^1_\alpha &= t\tau, \\
E^2_\alpha &= \xi_1\xi_2, & F^2_\alpha &= \eta_1\eta_2, & H^2_\alpha &= \xi_1\eta_1 + \xi_2\eta_2, \\
E^3_\alpha &= \xi_1\eta_2, & F^3_\alpha &= \xi_2\eta_1, & H^3_\alpha &= \xi_1\eta_1 - \xi_2\eta_2, \\
T^1_\alpha &= t\eta_1, & T^2_\alpha &= t\eta_2, & T^3_\alpha &= t\xi_1, & T^4_\alpha &= t\xi_2, \\
D^1_\alpha &= \tau\xi_1 + \alpha^{-1}\xi_1\xi_2\eta_2, & D^2_\alpha &= \tau\xi_2 - \alpha^{-1}\xi_1\xi_2\eta_1, & D^3_\alpha &= \tau\eta_1 + \alpha^{-1}\xi_1\xi_2\eta_2, & D^4_\alpha &= \tau\eta_2 - \alpha^{-1}\xi_1\eta_1\eta_2. 
\end{align*}

(4.5)

Then $\Gamma_\alpha \cong \Gamma(2, -1 - \alpha, \alpha - 1)$ for each $\alpha \in \mathbb{C}$.

\textbf{Proof.} Consider the following fields:

\begin{align*}
H^{1}_{n,\alpha} &= t^{n+1}\tau, & H^{2}_{n,\alpha} &= t^{n}(\xi_1\eta_1 + \xi_2\eta_2), \\
E^{3}_{n,\alpha} &= t^{n}\xi_1\eta_2, & F^{3}_{n,\alpha} &= t^{n}\xi_2\eta_1, & H^{3}_{n,\alpha} &= t^{n}(\xi_1\eta_1 - \xi_2\eta_2), \\
T^{1}_{n,\alpha} &= t^{n+1}\eta_1, & T^{2}_{n,\alpha} &= t^{n+1}\eta_2, & T^{3}_{n,\alpha} &= t^{n+1}\xi_1, & T^{4}_{n,\alpha} &= t^{n+1}\xi_2, \\
D^{1}_{n,\alpha} &= t^{n}\tau\xi_1 + (\alpha + n)t^{n-1}\xi_1\xi_2\eta_2, & D^{2}_{n,\alpha} &= t^{n}\tau\xi_2 - (\alpha + n)t^{n-1}\xi_1\xi_2\eta_1, \\
D^{3}_{n,\alpha} &= t^{n}\tau\eta_1 + (\alpha + n)t^{n-1}\xi_1\xi_2\eta_2, & D^{4}_{n,\alpha} &= t^{n}\tau\eta_2 - (\alpha + n)t^{n-1}\xi_1\eta_1\eta_2. 
\end{align*}

(4.6)

Set

\begin{equation}
L^{1}_{n,\alpha} = H^{1}_{n,\alpha} + \frac{1}{2}(\alpha + n + 1)H^{2}_{n,\alpha}. 
\end{equation}

(4.7)

Let

\begin{equation}
S^{1}_{\alpha} \subset P(4) 
\end{equation}

(4.8)

be defined as follows:

\begin{equation}
S^{1}_{\alpha} = \text{Span}(L^{1}_{n,\alpha}, E^{3}_{n,\alpha}, F^{3}_{n,\alpha}, H^{3}_{n,\alpha}, T^{1}_{n,\alpha}, T^{2}_{n,\alpha}, D^{1}_{n,\alpha}, D^{2}_{n,\alpha}). 
\end{equation}

(4.9)

Then

\begin{equation}
S^{1}_{\alpha} \cong S'(2, \alpha) \text{ for all } \alpha \in \mathbb{C} 
\end{equation}

(4.10)

Note that (4.8) is the restriction of the obvious embedding of $W(2)$ into $P(4)$. We obtain an embedding of the second copy of $S'(2, \alpha)$:

\begin{equation}
S^{2}_{\alpha} \subset P(4) 
\end{equation}

(4.11)

by interchanging $\xi_i$ with $\eta_i$ in all formulas. Thus

\begin{equation}
S^{2}_{\alpha} = \text{Span}(L^{2}_{n,\alpha}, E^{3}_{n,\alpha}, F^{3}_{n,\alpha}, H^{3}_{n,\alpha}, T^{3}_{n,\alpha}, T^{4}_{n,\alpha}, D^{3}_{n,\alpha}, D^{4}_{n,\alpha}). 
\end{equation}

(4.12)
where

\[ L_{n,\alpha}^2 = H_{n,\alpha}^1 - \frac{1}{2}(\alpha + n + 1)H_{n,\alpha}^2. \]  

(4.13)

Let \( spo(2|4) \cong osp(4|2) \) be a Lie superalgebra which preserves an even nondegenerate superskew-symmetric form on the (2|4)-dimensional superspace. Note that if \( \alpha = 0 \), then the zero modes of the fermionic fields

\[ T^i_{n,\alpha} \text{ and } D^i_{n,\alpha} \text{ for } i = 1, 2, 3, 4 \]  

(4.14)

span \( spo(2|4)_0 \), hence these elements generate \( \Gamma_0 \cong spo(2|4) \cong \Gamma(2, -1, -1) \). Analogously, for each \( \alpha \in \mathbb{C} \), the zero modes of the fields (4.14) generate \( \Gamma_{\alpha} \) and it is isomorphic to \( \Gamma(2, -1 - \alpha, \alpha - 1) \).

Explicitly an isomorphism \( \varphi : \Gamma(2, -1 - \alpha, \alpha - 1) \to \Gamma_{\alpha} \) is given as follows. Let

\[ V_1 = \text{Span}(e_1, e_2), \quad V_2 = \text{Span}(f_1, f_2), \quad V_3 = \text{Span}(h_1, h_2), \]

and

\[ \psi_1(e_1, e_2) = -\psi_1(e_2, e_1) = 1, \]

\[ \psi_2(f_1, f_2) = -\psi_2(f_2, f_1) = 1, \]

\[ \psi_3(h_1, h_2) = -\psi_3(h_2, h_1) = 1. \]

(4.15)

Then

\[ \varphi(\mathcal{P}_1(e_1, e_1)) = -E^1_\alpha, \quad \varphi(\mathcal{P}_1(e_2, e_2)) = -F^1_\alpha, \quad \varphi(\mathcal{P}_1(e_1, e_2)) = -H^1_\alpha, \]

\[ \varphi(\mathcal{P}_2(f_1, f_1)) = -2F^2_\alpha, \quad \varphi(\mathcal{P}_2(f_2, f_2)) = -2E^2_\alpha, \quad \varphi(\mathcal{P}_2(f_1, f_2)) = H^2_\alpha, \]

\[ \varphi(\mathcal{P}_3(h_1, h_1)) = -2F^3_\alpha, \quad \varphi(\mathcal{P}_3(h_2, h_2)) = 2E^3_\alpha, \quad \varphi(\mathcal{P}_3(h_1, h_2)) = H^3_\alpha, \]

\[ \varphi(e_1 \otimes f_1 \otimes h_1) = -\sqrt{2}i T^1_\alpha, \quad \varphi(e_1 \otimes f_1 \otimes h_2) = \sqrt{2}i T^2_\alpha, \]

\[ \varphi(e_1 \otimes f_2 \otimes h_1) = -\sqrt{2}i T^1_\alpha, \quad \varphi(e_1 \otimes f_2 \otimes h_2) = \sqrt{2}i T^3_\alpha, \]

\[ \varphi(e_2 \otimes f_1 \otimes h_1) = \sqrt{2}i D^3_\alpha, \quad \varphi(e_2 \otimes f_1 \otimes h_2) = \sqrt{2}i D^1_\alpha, \]

\[ \varphi(e_2 \otimes f_2 \otimes h_1) = -\sqrt{2}i D^2_\alpha, \quad \varphi(e_2 \otimes f_2 \otimes h_2) = \sqrt{2}i D^1_\alpha. \]

(4.16)

Thus \( sp(\psi_i) \cong \text{Span}(E^i_\alpha, H^1_\alpha, F^i_\alpha) \) for \( i = 1, 2, 3 \).

\[ \square \]

Remark 4.2: We will use the following commutation relations:

\[ [T^1_\alpha, T^3_\alpha] = E^1_\alpha, \quad [T^1_\alpha, D^4_\alpha] = -(1 + \alpha)F^2_\alpha, \]

\[ [T^2_\alpha, T^4_\alpha] = E^1_\alpha, \quad [T^2_\alpha, D^3_\alpha] = (1 + \alpha)F^2_\alpha, \]

\[ [D^1_\alpha, T^4_\alpha] = (1 + \alpha)E^2_\alpha, \quad [D^1_\alpha, D^3_\alpha] = F^1_\alpha, \]

\[ [D^2_\alpha, T^3_\alpha] = -(1 + \alpha)E^2_\alpha, \quad [D^2_\alpha, D^1_\alpha] = F^1_\alpha. \]

(4.17)
V. Superalgebra $\hat{K}'(4)$

By definition,

$$K(2N) = \{ D \in W(2N) \mid D\Omega = f\Omega \text{ for some } f \in \Lambda(1,2N) \},$$  \hspace{1cm} (5.1)

where $\Omega = dt + \sum_{i=1}^{N} \xi_i d\eta_i + \eta_i d\xi_i$ is a differential 1-form, which is called a contact form, see Refs. 7-10, 16-18 and 28. The Euler operator is defined by $E = \sum_{i=1}^{N} \xi_i \partial_{\xi_i} + \eta_i \partial_{\eta_i}$. We also define operators $\Delta = 2 - E$ and $H_f = (-1)^{p(f)+1} \sum_{i=1}^{N} \partial_{\xi_i} f \partial_{\eta_i} + \partial_{\eta_i} f \partial_{\xi_i}$, where $f \in \Lambda(1,2N)$.

There is a one-to-one correspondence between the differential operators $D \in K(2N)$ and the functions $f \in \Lambda(1,2N)$. The correspondence $f \leftrightarrow D_f$ is given by

$$D_f = \Delta(f) \frac{\partial}{\partial t} + \frac{\partial f}{\partial t} E - H_f.$$  \hspace{1cm} (5.2)

The contact bracket on $\Lambda(1,2N)$ is

$$\{f, g\}_K = \Delta(f) \partial_t g - \partial_t f \Delta(g) - \{f, g\}_{P.b},$$  \hspace{1cm} (5.3)

where

$$\{f, g\}_{P.b} = (-1)^{p(f)+1} \sum_{i=1}^{N} (\partial_{\xi_i} f \partial_{\eta_i} g + \partial_{\eta_i} f \partial_{\xi_i} g)$$  \hspace{1cm} (5.4)

is the Poisson bracket. Thus $[D_f, D_g] = D_{\{f, g\}_K}$.

The superalgebras $K(2N)$ are simple, except when $N = 2$. If $N = 2$, then the derived superalgebra $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one defined from the exact sequence

$$0 \rightarrow K'(4) \rightarrow K(4) \rightarrow \mathbb{C}D_{t^{-1}\xi_1\xi_2\eta_1\eta_2} \rightarrow 0.$$  \hspace{1cm} (5.5)

The superalgebra $K'(4)$ has 3 independent central extensions, see Refs. 7, 10, 29 and 30. The following statement is proven in Refs. 5 and 13.

**Proposition 5.1:** There exists an embedding

$$i_0 : K'(4) \rightarrow P(4).$$  \hspace{1cm} (5.6)

The superalgebra $i_0(K'(4))$ is spanned by the 12 fields:

$$L_n = t^{n+1}\tau, \quad Q_n = t^{n+1}\tau\xi_1\xi_2,$$
$$X_n^i = t^{n+1}\tau\xi_i, \quad Y_n^i = t^n\eta_i,$$
$$R_n^{ij} = t^n\xi_j\eta_i, \quad Z_n^i = t^n\xi_1\xi_2\eta_i.$$  \hspace{1cm} (5.7)
where \( i, j = 1, 2 \), and 4 fields:

\[
\begin{align*}
G_n^0 &= t^{n-1} \tau^{-1} \eta_1 \eta_2, \\
G_n^i &= t^{n-1} \tau^{-1} \xi_i \eta_1 \eta_2, \quad i = 1, 2, \\
G_n^3 &= nt^{n-1} \tau^{-1} \xi_1 \xi_2 \eta_1 \eta_2, \quad n \neq 0.
\end{align*}
\]  

(5.8)

Note that \( L_n \) is a Virasoro field. Let \( \hat{K}'(4) = K'(4) \oplus \mathbb{C} \) be one of three independent central extensions of \( K'(4) \), such that the corresponding 2-cocycle is

\[
\begin{align*}
c(L_n, G_k^3) &= -n \delta_{n+k,0}, \\
c(X_n^i, G_k^j) &= (-1)^j \delta_{n+k,0}, \quad 1 \leq i \neq j \leq 2, \\
c(Q_n, G_k^0) &= \delta_{n+k,0}.
\end{align*}
\]  

(5.9)

For each \( h \in (0, 1] \), there exists an embedding

\[
i_h : \hat{K}'(4) \rightarrow P_h(4).
\]  

(5.10)

The superalgebra \( i_h(\hat{K}'(4)) \) is spanned by the 12 fields (5.7) and 4 fields:

\[
\begin{align*}
G_{n,h}^0 &= \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2, \\
G_{n,h}^i &= \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2 \xi_i, \quad i = 1, 2, \\
G_{n,h}^3 &= n \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2 \xi_1 \xi_2 + ht^n.
\end{align*}
\]  

(5.11)

Note that the central element in \( i_h(\hat{K}'(4)) \) is \( G_{0,h}^3 = h \), and

\[
\lim_{h \rightarrow 0} i_h(\hat{K}'(4)) = i_0(K'(4)) \subset P(4).
\]  

(5.12)

**Theorem 5.2:** Let \( \Gamma_{\alpha,h} \), where \( \alpha \in \mathbb{C} \) and \( h \in (0, 1] \), be spanned by the following elements in \( P_h(4) \):

\[
\begin{align*}
E^1_{\alpha,h} &= t^2, \quad H^1_{\alpha,h} = t \tau + \frac{\alpha + 1}{2} h, \\
F^1_{\alpha,h} &= \tau^2 - \alpha (2t^{-2} \xi_1 \xi_2 \eta_1 \eta_2 + t^{-2} (\xi_1 \eta_1 + \xi_2 \eta_2) h - t^{-1} \tau h), \\
E^2_{\alpha,h} &= \xi_1 \xi_2, \quad F^2_{\alpha,h} = \eta_1 \eta_2, \quad H^2_{\alpha,h} = \xi_1 \eta_1 + \xi_2 \eta_2 - h, \\
E^3_{\alpha,h} &= \xi_1 \eta_2, \quad F^3_{\alpha,h} = \xi_2 \eta_1, \quad H^3_{\alpha,h} = \xi_1 \eta_1 - \xi_2 \eta_2, \\
T^1_{\alpha,h} &= t \eta_1, \quad T^2_{\alpha,h} = t \eta_2, \quad T^3_{\alpha,h} = t \xi_1, \quad T^4_{\alpha,h} = t \xi_2, \\
D^1_{\alpha,h} &= \tau \xi_1 + \alpha t^{-1} \xi_1 \xi_2 \eta_2, \quad D^2_{\alpha,h} = \tau \xi_2 - \alpha t^{-1} \xi_1 \xi_2 \eta_1, \\
D^3_{\alpha,h} &= \tau \eta_1 + \alpha t^{-1} \eta_1 \eta_2 \xi_2, \quad D^4_{\alpha,h} = \tau \eta_2 - \alpha t^{-1} \eta_1 \eta_2 \xi_1.
\end{align*}
\]  

(5.13)

Then \( \Gamma_{\alpha,h} \cong \Gamma(2, -1 - \alpha, \alpha - 1) \), and \( \lim_{h \rightarrow 0} \Gamma_{\alpha,h} = \Gamma_\alpha \subset P(4) \).
Proof. We can obtain the second embedding
\[ j_h : \hat{K}'(4) \rightarrow P_h(4) \] (5.14)
for each \( h \in (0, 1] \), if we interchange \( \xi_i \) with \( \eta_i \) in all the formulas for the embedding (5.10). Then
\[ \lim_{h \to 0} j_h(\hat{K}'(4)) = j_0(K'(4)) \subset P(4). \] (5.15)
In (4.8) and (4.11) we obtained embeddings
\[ S^1_\alpha \subset i_0(K'(4)), \quad S^2_\alpha \subset j_0(K'(4)). \]
Naturally
\[ S^1_\alpha = S^1_{\alpha,h} \subset i_h(\hat{K}'(4)). \] (5.16)
To obtain embedding
\[ S^2_{\alpha,h} \subset j_h(\hat{K}'(4)) \] (5.17)
we interchange \( \xi_i \) with \( \eta_i \) in all formulas for the embedding (5.16). Thus
\[ S^2_{\alpha,h} = \text{Span}(L^2_{n,\alpha,h}, E^3_{n,\alpha,h}, F^3_{n,\alpha,h}, H^3_{n,\alpha,h}, T^3_{n,\alpha,h}, T^4_{n,\alpha,h}, D^3_{n,\alpha,h}, D^4_{n,\alpha,h}), \] (5.18)
where
\[ L^2_{n,\alpha,h} = t^{n+1} \tau + \frac{1}{2}(\alpha + n + 1)(\eta_1 \xi_1 + \eta_2 \xi_2), \]
\[ E^3_{n,\alpha,h} = E^3_{n,\alpha}, \quad F^3_{n,\alpha,h} = F^3_{n,\alpha}, \quad H^3_{n,\alpha,h} = H^3_{n,\alpha}, \]
\[ T^3_{n,\alpha,h} = T^3_{n,\alpha}, \quad T^4_{n,\alpha,h} = T^4_{n,\alpha}, \]
\[ D^3_{n,\alpha,h} = t^n \tau \eta_1 + (\alpha + n)t^{n-1} \eta_1 \eta_2 \xi_2, \]
\[ D^4_{n,\alpha,h} = t^n \tau \eta_2 - (\alpha + n)t^{n-1} \eta_1 \eta_2 \xi_1. \] (5.19)
For each \( \alpha \in \mathbb{C} \) and each \( h \in (0, 1] \), the zero modes of the fermionic fields
\[ T^i_{n,\alpha,h} \text{ and } D^i_{n,\alpha,h} \text{ for } i = 1, 2, 3, 4 \] (5.20)
generate \( \Gamma_{\alpha,h} \) and it is isomorphic to \( \Gamma(2, -1-\alpha, \alpha-1) \). One can use the commutation relations given in Remark 4.2 to find \( E^i_{\alpha,h}, F^i_{\alpha,h} \text{ and } H^i_{\alpha,h} \) for \( i = 1, 2 \).

\[ \square \]

VI. Realizations as matrices over a Weyl algebra

In this section we will describe \( \hat{K}'(4) \) in terms of matrices over a Weyl algebra.
By definition, a Weyl algebra is

\[ W = \sum_{i \geq 0} \mathcal{A} d^i, \] (6.1)

where \( \mathcal{A} \) is an associative commutative algebra and \( d : \mathcal{A} \to \mathcal{A} \) is a derivation of \( \mathcal{A} \), with the relations

\[ da = d(a) + ad, \quad a \in \mathcal{A}, \] (6.2)

see Refs. 19 and 20. Set

\[ \mathcal{A} = \mathbb{C}[t, t^{-1}], \quad d = L_0 = t \tau. \] (6.3)

Let \( M(2|2, W) \) be the Lie superalgebra of \( 4 \times 4 \) matrices over \( W \).

**Theorem 6.1:** There exists an embedding

\[ I : \hat{K}'(4) \longrightarrow M(2|2, W). \] (6.4)

The superalgebra \( I(\hat{K}'(4)) \) is spanned by the following elements:

\[
I(L_n) = \begin{pmatrix} dt^n & 0 & 0 & 0 \\ 0 & t^n d & 0 & 0 \\ 0 & 0 & t^n d & 0 \\ 0 & 0 & 0 & t^n d \end{pmatrix}, \quad I(G^3_n) = t^n 1_{2|2},
\]

\[
I(R^{11}_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t^n & 0 & 0 \\ 0 & 0 & t^n & 0 \\ 0 & 0 & 0 & t^n \end{pmatrix}, \quad I(R^{22}_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t^n & 0 & 0 \\ 0 & 0 & t^n & 0 \\ 0 & 0 & 0 & t^n \end{pmatrix},
\]

\[
I(R^{12}_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^n \\ 0 & 0 & t^n & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(R^{21}_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^n \\ 0 & 0 & t^n & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
I(Q_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(Q_n) = \begin{pmatrix} t^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\] (6.5)
Let \( V \) be an operator. Consider the following basis in \( \mathbb{C} \) with an antiderivative, and the central element \( \xi \) the operator of multiplication in \( \Lambda(\mathbb{C}) \).

Explicitly, the action of \( \hat{K}'(4) \) on \( V \) is given as follows:

\[
I(Y^1_n) = \begin{pmatrix} 0 & 0 & dt^n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & t^n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(Y^2_n) = \begin{pmatrix} 0 & 0 & 0 & dt^n \\ 0 & 0 & 0 & 0 \\ 0 & -t^n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
I(X^1_n) = \begin{pmatrix} 0 & 0 & 0 & t^n \\ 0 & 0 & 0 & 0 \\ t^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(X^2_n) = \begin{pmatrix} 0 & 0 & 0 & -t^n \\ 0 & 0 & 0 & 0 \\ t^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
I(G^1_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(G^2_n) = \begin{pmatrix} 0 & 0 & t^n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
I(Z^1_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & t^n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I(Z^2_n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Note that the central element is \( C = I(C^3_0) = 1_{2|2} \).

**Proof.** Consider the embedding

\[
i_h : \hat{K}'(4) \rightarrow P_h(4).
\]

Let \( V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2) \), where \( \mu \in \mathbb{C} \setminus \mathbb{Z} \). We fix \( h = 1 \), and define a representation of \( \hat{K}'(4) \) in \( V^\mu \) according to the formulas (5.7) and (5.11). Namely, \( \xi_i \) is the operator of multiplication in \( \Lambda(\xi_1, \xi_2) \), \( \eta_i \) is identified with \( \partial_{\xi_i} \), \( \tau^{-1} \) is identified with an antiderivative, and the central element \( C = 1 \in P_{h=1}(4) \) acts by the identity operator. Consider the following basis in \( V^\mu \):

\[
v^0_m(\mu) = t^{m+\mu}, \quad v^1_m(\mu) = t^{m+\mu} \epsilon_1,
\]

\[
v^2_m(\mu) = t^{m+\mu} \epsilon_2, \quad v^{12}_m(\mu) = t^{m+\mu} \epsilon_1 \epsilon_2 \quad \text{for all} \ m \in \mathbb{Z}.
\]

Explicitly, the action of \( \hat{K}'(4) \) on \( V^\mu \) is given as follows:

\[
L_n(v^0_m(\mu)) = (n + m + \mu)v^0_{m+n}(\mu),
\]

\[
L_n(v^i_m(\mu)) = (m + \mu)v^i_{m+n}(\mu), \quad i = 1, 2, 3,
\]

\[
X_n(v^0_m(\mu)) = v^i_{m+n}(\mu), \quad i = 1, 2,
\]

\[
X_n(v^2_m(\mu)) = (m + \mu)v^3_{m+n}(\mu),
\]

\[
X_n(v^1_m(\mu)) = -(m + \mu)v^3_{m+n}(\mu),
\]

\[
Q_n(v^0_m(\mu)) = v^3_{m+n}(\mu).
\]
These formulas remain valid for $\mu = 0$. Thus we obtain a representation of $\hat{K}'(4)$ in the superspace $V = \mathbb{C}[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2)$ with a basis
\[
\{ v^0_m, v^1_m, v^2_m, v^3_m \},
\]
where
\[
v^0_m = t^m, \quad v^3_m = t^m \xi_1 \xi_2, \quad v^i_m = t^m \xi_i, \quad i = 1, 2, \quad m \in \mathbb{Z}.
\]
We have
\[
\begin{align*}
L_n(v^0_m) &= dt^n v^0_m, & L_n(v^i_m) &= t^n dv^i_m, & i &= 1, 2, 3, \\
X^i_n(v^0_m) &= t^nu^i_m, & X^1_n(v^2_m) &= t^n dv^3_m, & X^2_n(v^1_m) &= -t^ndv^3_m, \\
Q_n(v^0_m) &= t^n v^3_m, & Y^i_n(v^0_m) &= dt^n v^0_m, & Y^i_n(v^1_m) &= t^n v^2_m, & i &= 1, 2, \\
Y^3_n(v^2_m) &= -t^n v^1_m, & R^i_n(v^0_m) &= t^n v^i_m, & R^i_n(v^2_m) &= t^n v^i_m, & i &= 1, 2, \\
R^{i j}_n(v^1_m) &= t^n v^3_m, & R^{i j}_n(v^3_m) &= t^n v^j_m, & i \neq j &= 1, 2, \\
Z^i_n(v^0_m) &= t^n v^i_m, & i &= 1, 2, \\
G^0_{n, 1}(v^0_m) &= -t^n v^0_m, & G^1_{n, 1}(v^2_m) &= -t^n v^0_m, & G^2_{n, 1}(v^1_m) &= t^n v^0_m, \\
G^3_{n, 1}(v^j_m) &= t^n v^i_m, & n \neq 0, & i &= 0, 1, 2, 3.
\end{align*}
\]

Thus we obtain the above-mentioned realization of $\hat{K}'(4)$ as a subsuperalgebra of matrices of size $4 \times 4$ over $\mathcal{W}$.

Remark 6.2. Naturally, $V = \oplus_m V_m$, where $V_m = t^m \otimes \Lambda(\xi_1, \xi_2)$. Recall that the element $L_0 = \tau$ of the Virasoro algebra defines a $\xi$-grading in $\hat{K}'(4)$: $\hat{K}'(4) = \oplus_i \mathfrak{g}_i$. It follows from (6.8) that
\[
\mathfrak{g}_i(V_m) \subset V_{m+i}.
\]
Note that \( g_0 \) is isomorphic to the universal central extension of \( sl(2|2) \), and it is realized as a superalgebra of \( 4 \times 4 \) matrices over \( W \) of type

\[
\left( \begin{array}{cc}
A & B + d\tilde{C} \\
C & D
\end{array} \right) \oplus \mathbb{C}d \cdot 1_{2|2},
\]

(6.10)

where \( A, B, C, D \in \mathfrak{gl}(2, \mathbb{C}) \) and \( trA = trD \). \( \tilde{C} \) is determined by the following conditions:

\[
\begin{align*}
\text{if } C &= E_{ii}, \text{ then } \tilde{C} = E_{jj}, \text{ where } i \neq j, \\
\text{if } C &= E_{ij}, \text{ } i \neq j, \text{ then } \tilde{C} = -E_{ij},
\end{align*}
\]

(6.11)

where \( E_{ij} \) is an elementary \( 2 \times 2 \)-matrix.

It was observed in Refs. 30 and 31 that the big \( N = 4 \) superconformal algebra contains \( D(2, 1; \alpha) \) as a subsuperalgebra. In the next theorem, we give a realization of \( D(2, 1; \alpha) \) inside \( I(\check{K}'(4)) \). Note that it is different from the realization of \( D(2, 1; \alpha) \) inside \( M(2|2, W) \), which one can directly obtain from Theorem 5.2.

**Theorem 6.3:** For each \( \alpha \in \mathbb{C} \) the superalgebra \( \Gamma(2, -1 - \alpha, \alpha - 1) \) is realized inside the superalgebra \( I(\check{K}'(4)) \) as follows:

\[
\begin{align*}
T_1^\alpha &= \left( \begin{array}{cccc}
0 & 0 & t(d + 1) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & t & 0 & 0
\end{array} \right), & T_2^\alpha &= \left( \begin{array}{cccc}
0 & 0 & t(d + 1) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -t & 0 & 0
\end{array} \right), \\
D_1^\alpha &= \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & t^{-1}(d + \alpha) \\
0 & 0 & 0 & 0 \\
t^{-1} & 0 & 0 & 0
\end{array} \right), & D_2^\alpha &= \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -t^{-1}(d + \alpha) \\
0 & 0 & 0 & 0 \\
t^{-1} & 0 & 0 & 0
\end{array} \right), \\
T_3^\alpha &= \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & t(d + 1) \\
0 & 0 & 0 & 0 \\
t & 0 & 0 & 0
\end{array} \right), & T_4^\alpha &= \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -t(d + 1) \\
0 & 0 & 0 & 0 \\
t & 0 & 0 & 0
\end{array} \right), \\
D_3^\alpha &= \left( \begin{array}{cccc}
0 & 0 & t^{-1}(d + \alpha) & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
t^{-1} & 0 & 0 & 0
\end{array} \right), & D_4^\alpha &= \left( \begin{array}{cccc}
0 & 0 & 0 & t^{-1}(d + \alpha) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -t^{-1} & 0 & 0
\end{array} \right),
\end{align*}
\]

(6.12)
Note that under both embeddings, \( \hat{T} \) describe the associated representations we choose the following basis in embedding

One can repeat the construction given in the proof of Theorem 6.1, and obtain an

Proof. Consider the embedding

\( j_h : \hat{K}'(4) \longrightarrow P_h(4). \) \hspace{1cm} (6.13)

To describe the associated representations we choose the following basis in \( V^\mu \):

\[
  v_m^0 (\mu) = t^{m+\mu}, \quad v_m^1 (\mu) = t^{m+\mu} \xi_1,
\]

\[
  v_m^2 (\mu) = t^{m+\mu} \xi_2, \quad v_m^{12} (\mu) = \frac{1}{m+\mu} t^{m+\mu} \xi_1 \xi_2, \quad m \in \mathbb{Z}.
\] \hspace{1cm} (6.14)

One can repeat the construction given in the proof of Theorem 6.1, and obtain an embedding

\( J : \hat{K}'(4) \longrightarrow M(2|2, W). \) \hspace{1cm} (6.15)

Note that under both embeddings, \( \hat{K}'(4) \) is realized as the same matrix superalgebra:

\( I(\hat{K}'(4)) = J(\hat{K}'(4)). \) \hspace{1cm} (6.16)

We have

\[
  J(L_n) = I(L_n) - nI(G_n^3) + nI(R_n^{11}) + nI(R_n^{22}), \quad J(Q_n) = I(G_n^0),
\]

\[
  J(R_n^{11}) = I(G_n^3) - I(R_n^{11}), \quad J(R_n^{22}) = I(G_n^3) - I(R_n^{22}),
\]

\[
  J(R_n^{12}) = -I(R_n^{21}) \quad J(R_n^{21}) = -I(R_n^{12}),
\]

\[
  J(G_n^0) = I(Q_n), \quad J(G_n^2) = I(G_n^2),
\]

\[
  J(Y_n^1) = I(X_n^1) + nI(Z_n^2), \quad J(Y_n^2) = I(X_n^2) - nI(Z_n^1),
\]

\[
  J(X_n^1) = I(Y_n^1) - nI(G_n^2), \quad J(X_n^2) = I(Y_n^2) + nI(G_n^1),
\]

\[
  J(G_n^4) = I(Z_n^1), \quad J(G_n^2) = I(Z_n^2), \quad J(Q_n^1) = I(G_n^1), \quad J(Q_n^2) = I(G_n^2).
\] \hspace{1cm} (6.17)
To find matrix realizations of $T^i_\alpha$ and $D^i_\alpha$, we use (6.4), if $i = 1, 2$, and we use (6.15), if $i = 3, 4$. Note that formulas (6.4) and (6.15) determine the same matrices for $E^3_\alpha, F^3_\alpha$ and $H^3_\alpha$. Finally, to find $E^i_\alpha, F^i_\alpha$ and $H^i_\alpha$ for $i = 1, 2$, we use the commutation relations given in Remark 4.2, and the relations in $\mathcal{W}$:

$$dt^n = t^n d + nt^n \text{ for all } n \in \mathbb{Z}. \quad (6.18)$$

Thus we have

$$T^1_\alpha = I(Y^1_1), \quad T^2_\alpha = I(Y^2_1),$$
$$T^3_\alpha = I(X^1_1) + I(Z^1_1), \quad T^4_\alpha = I(X^2_1) - I(Z^1_1),$$
$$D^1_\alpha = I(X^1_{-1}) + \alpha I(Z^2_{-1}), \quad D^2_\alpha = I(X^2_{-1}) - \alpha I(Z^1_{-1}),$$
$$D^3_\alpha = I(Y^1_{-1}) + (\alpha + 1)I(G^2_{-1}), \quad D^4_\alpha = I(Y^2_{-1}) - (\alpha + 1)I(G^1_{-1}),$$
$$E^1_\alpha = I(L_2) + I(R^{11}_2) + I(R^{22}_2),$$
$$E^2_\alpha = I(L_{-2}) + (\alpha + 1)I(G^3_{-1}) - I(R^{11}_{-2}) - I(R^{22}_{-2}),$$
$$H^1_\alpha = I(\mathbb{K}_0) + \frac{1}{2}(1 + \alpha)\mathbb{C},$$
$$E^2_\alpha = I(Q_0), \quad F^2_\alpha = I(G^0_0), \quad H^2_\alpha = I(R^{11}_0) + I(R^{22}_0) - \mathbb{C},$$
$$E^3_\alpha = I(R^{12}_0), \quad F^3_\alpha = I(R^{21}_0), \quad H^3_\alpha = I(R^{11}_0) - I(R^{22}_0). \quad (6.19)$$

Remark 6.4: In Theorem 5.2 we described an embedding of $\Gamma(2, -1 - \alpha, \alpha - 1)$ into $P_h(4)$. Note that it is actually an embedding of $\tilde{\Gamma}(2, -1 - \alpha, \alpha - 1)$ into the Lie superalgebra of differential operators on $S^{1/2}$. One can use the fields in (5.7) and (5.11) and formulas (6.19) to obtain a different embedding of this superalgebra into $P_h(4)$ such that

$$\Gamma_{\alpha,h} \subset i_h(\tilde{\Gamma}(4)) \quad (6.20)$$

for each $h \in (0, 1]$ and each $\alpha \in \mathbb{C}$. In this embedding the pseudodifferential symbols are essentially used. $\Gamma_{\alpha,h} \cong \Gamma(2, -1 - \alpha, \alpha - 1)$ is spanned by the following elements:

$$E^1_{\alpha,h} = t^3 \tau + t^2(\xi_1 \eta_1 + \xi_2 \eta_2),$$
$$F^1_{\alpha,h} = t^{-1} \tau + (\alpha + 1)(-2t^{-1} \circ_h t^{-3} \eta_1 \eta_2 \xi_1 \xi_2 + ht^{-2}) - t^{-2}(\xi_1 \eta_2 + \xi_2 \eta_2),$$
$$H^1_{\alpha,h} = t \tau + \frac{\alpha + 1}{2} h, \quad (6.21)$$
$$E^2_{\alpha,h} = t \tau \xi_1 \xi_2, \quad F^2_{\alpha,h} = \tau^{-1} \circ_h t^{-1} \eta_1 \eta_2, \quad H^2_{\alpha,h} = \xi_1 \eta_1 + \xi_2 \eta_2 - h,$$
$$E^3_{\alpha,h} = \xi_1 \eta_2, \quad F^3_{\alpha,h} = \xi_2 \eta_1, \quad H^3_{\alpha,h} = \xi_1 \eta_1 - \xi_2 \eta_2,$$
\[T^1_{\alpha,h} = t\eta_1, \quad T^2_{\alpha,h} = t\eta_2,\]
\[T^3_{\alpha,h} = t^2\tau_1 + t\xi_1\xi_2\eta_2, \quad T^4_{\alpha,h} = t^2\tau_2 - t\xi_1\xi_2\eta_1,\]
\[D^1_{\alpha,h} = \tau\xi_1 + \alpha t^{-1}\xi_1\xi_2\eta_2, \quad D^2_{\alpha,h} = \tau\xi_2 - \alpha t^{-1}\xi_1\xi_2\eta_1,\]
\[D^3_{\alpha,h} = t^{-1}\eta_1 + (\alpha + 1)\tau^{-1} \circ h t^{-2}\eta_1\eta_2\xi_2,\]
\[D^4_{\alpha,h} = t^{-1}\eta_2 - (\alpha + 1)\tau^{-1} \circ h t^{-2}\eta_1\eta_2\xi_1.\]

Then
\[\lim_{h \to 0} \Gamma_{\alpha,h} = \Gamma_{\alpha} \subset i_0(K'(4)) \subset P(4). \quad (6.22)\]
\[\Gamma_{\alpha} \cong \Gamma(2,-1-\alpha,\alpha-1)\] is spanned by the following elements:
\[E^i_{\alpha} = E^i_{\alpha,h}, \quad i = 1, 2, 3, \quad F^3_{\alpha} = F^3_{\alpha,h}, \quad H^3_{\alpha} = H^3_{\alpha,h},\]
\[T^i_{\alpha} = T^i_{\alpha,h}, \quad i = 1, 2, 3, 4, \quad D^i_{\alpha} = D^i_{\alpha,h}, \quad i = 1, 2,\]
\[F^1_{\alpha} = t^{-1}\tau - 2(\alpha + 1)t^{-3}\tau^{-1}\xi_1\xi_2\eta_1\eta_2 - t^{-2}(\xi_1\eta_1 + \xi_2\eta_2), \quad H^1_{\alpha} = t\tau, \quad (6.23)\]
\[F^2_{\alpha} = t^{-1}\tau^{-1}\eta_1\eta_2, \quad H^2_{\alpha} = \xi_1\eta_1 + \xi_2\eta_2,\]
\[D^3_{\alpha} = t^{-1}\eta_1 + (\alpha + 1)t^{-2}\tau^{-1}\eta_1\eta_2\xi_2, \quad D^4_{\alpha} = t^{-1}\eta_2 - (\alpha + 1)t^{-2}t^{-2}\tau^{-1}\eta_1\eta_2\xi_1.\]

Note that the matrix realization of \(\Gamma(2,-1-\alpha,\alpha-1)\) in Theorem 6.3 is associated to (6.20), where \(h = 1\): it is the restriction of the mapping \(I\) given in (6.4), to \(\Gamma(2,-1-\alpha,\alpha-1)\).

**Remark 6.5:** Recall that superalgebras \(\Gamma(2,-1-\alpha,\alpha-1)\) are not simple when \(\alpha = 1\) or \(-1\). Correspondingly, we have the following realizations of \(psl(2|2) = sl(2|2)/<1_{2|2}>\) as a subsuperalgebra of \(M(2|2,\mathcal{W})\). If \(\alpha = 1\), then
\[\text{Span}(E^i_{\alpha}, H^i_{\alpha}, F^i_{\alpha}, T^i_{\alpha}, D^i_{\alpha} \mid i = 1, 2 \text{ and } j = 1, \ldots, 4) \cong psl(2|2), \]
\[\Gamma(2,-2,0)/psl(2|2) \cong sl(2). \quad (6.24)\]
If \(\alpha = -1\), then
\[\text{Span}(E^i_{\alpha}, H^i_{\alpha}, F^i_{\alpha}, T^i_{\alpha}, D^i_{\alpha} \mid i = 1, 3 \text{ and } j = 1, \ldots, 4) \cong psl(2|2), \]
\[\Gamma(2,0,-2)/psl(2|2) \cong sl(2).\quad (6.25)\]

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