3-FOLDS CR-EMBEDDED IN 5-DIMENSIONAL REAL HYPERQUADRRICS

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Abstract. E. Cartan’s method of moving frames is applied to classify 3-dimensional manifolds $M$ which are CR-embedded in 5-dimensional real hyperquadrics $Q$, up to CR symmetries of $Q$ given by the action of one of the Lie groups $SU(3,1)$ or $SU(2,2)$. In the latter case, the CR structure of $M$ derives from a shear-free null geodesic congruence on Minkowski spacetime, and the relationship to relativity is discussed. In both cases, we compute which homogeneous CR 3-folds may be embedded in $Q$.

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1. Introduction

For several physically significant solutions to Einstein’s equations in general relativity, spacetime is a 4-dimensional Lorentzian manifold that is foliated by a family of curves called a shear-free null geodesic congruence (SFNGC), which induces a CR structure on the 3-dimensional leaf space of the foliation. Conversely, a 3-dimensional CR structure can be “lifted” to a spacetime admitting a SFNGC. The Robinson-Trautman metrics, for example, describe congruences which are hypersurface-orthogonal, and their corresponding CR structures are Levi-flat. These include models of electromagnetic and gravitational fields radiating along the foliating curves, generating wave-fronts orthogonal to their direction of...
equations, which are computed in §Case 1,2,3 of interest in physics. The final forms of the structure equations are labeled in §embeddable in $Q \subset \mathbb{C}^3$ transitive group of CR symmetries. $M$ those of the reduced bundle over $Q$ the (maximal) CR symmetry group of points. However, there is no danger of excluding any homogeneous CR structures; i.e., for curves away from points of vanishing curvature, or Darboux frames for surfaces away from umbilic this entails some loss of generality, analogous to the construction in Euclidean geometry of a Frenet frame we treat separately the cases that the function vanishes identically or is nowhere-vanishing. Of course, several times based on whether functions appearing in the structure equations vanish; in these instances, $SU(2,2)$ should be equivalent to applying Cartan’s method of moving frames (§2.1), which is the strategy of the present paper.

Once the frame bundle has been reduced as much as possible, an abstract CR 3-fold is (generically) embeddable in $Q$ if and only if it admits a coframing with structure equations of the same form as those of the reduced bundle over $M$. For homogeneous CR structures, invariance of the coframing under the (maximal) CR symmetry group of $Q$ implies that the structure equations have constant coefficients. Cartan classified homogeneous CR 3-folds in [Car32]. Their symmetry algebras were classified earlier by Bianchi (see, for example, [SKM+03] §8.2). In §2.2 we identify them by a pair of labels referring to their Bianchi type and corresponding Cartan model; e.g., (II, A).

The reduction procedure is carried out in §4. To work in full generality, we perform the calculation for hyperquadrics with either symmetry group $SU(3,1)$ or $SU(2,2)$, as indexed by the constant $\epsilon = \pm 1$. Geometrically, this makes for an interesting comparison, though it is the latter case $\epsilon = -1$ which is of interest in physics. The final forms of the structure equations are labeled in §3.3-3.5 as one of Flat Case 1,2,3 or Curved Cases 1,2,3 based on the homogeneous models that occur with those structure equations, which are computed in §3. Every flat CR 3-fold is locally equivalent to the 3-sphere, which appears in both hyperquadrics. Beyond that, they have no homogeneous models in common.
Bianchi types VIII and IX are the algebras $su(1, 1)$ and $su(2)$, respectively, and each is realized as the CR symmetry algebra for a parameter family of homogeneous 3-folds. For $\epsilon = 1$, there exist embeddings of the homogeneous models of type (IX, L) for every $t > 0$ and (VIII, K) for $t > 2$. For $\epsilon = -1$ we find (VI, E) for $t = \frac{1}{2}$ and $t = 9$, as well as (VIII, K) for $0 < t < 2$ ($t \neq 1$). It is, perhaps, of greater physical interest which models do \textit{not} appear when $\epsilon = -1$, as these correspond to SFNGCs in curved spacetimes.

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2. CR Structures

2.1. Basic Definitions, Adapted Coframings. For any fiber bundle $\pi : E \to M$, $E_x = \pi^{-1}(x)$ denotes the fiber of $E$ over $x \in M$ and $\Gamma(E)$ denotes the sheaf of smooth (local) sections of $E$. If $E$ is a vector bundle, $\mathbb{C}E$ is its complexification whose fibers are $\mathbb{C}E_x = E_x \otimes \mathbb{C}$. We use bold for the constants $i = \sqrt{-1}$ and $e$, the natural exponential.

Here, CR structure refers specifically to a hypersurface-type CR structure $(M, D, J)$, which is a $(2n+1)$-dimensional smooth manifold $M$ equipped with a corank-1 distribution $D \subset TM$ carrying an \textit{almost-complex structure}

$$J : D \to D,$$ 

where $J_x : D_x \to D_x$ is linear for every $x \in M$, and $1$ denotes the identity map on the fibers of $D$. The induced map on the complexified bundle splits

$$\mathbb{C}D = H \oplus \overline{H},$$

where the CR bundle $H$ is the $i$-eigenspace and the anti-CR bundle $\overline{H}$ the $-i$-eigenspace of $J$. The CR dimension of $M$ is $\text{dim}_\mathbb{C} H = n$. For two CR manifolds $(M_1, D_1, J_1), (M_2, D_2, J_2)$, a CR map between them is a smooth map $f : M_1 \to M_2$ whose pushforward $f_* : TM_1 \to TM_2$ satisfies $f_* D_1 \subset D_2$ and $f_* J_1 = J_2 \circ f_*$; in other words, $f_* H_1 \subset H_2$. $M_1, M_2$ are CR equivalent if there exists a CR map between them that is a diffeomorphism. Often we are merely concerned with local equivalence maps, defined on some neighborhood of any given point.

All CR structures in this paper are CR-integrable; i.e., sections of the (anti-)CR bundle are closed under the Lie bracket of vector fields,

$$[\Gamma(\overline{H}), \Gamma(\overline{H})] \subset \Gamma(\overline{H}).$$

The failure of integrability of the underlying real distribution $D$ is measured by the Levi form,

$$\ell : H_x \times H_x \to \mathbb{C}D_x \left( y_1, y_2 \right) \mapsto i[y_1, \nabla_2](x) \mod C D_x$$

for $Y_i \in \Gamma(H)$, $Y_i(x) = y_i$ ($i = 1, 2$).

$M$ is Levi-flat if $\ell$ vanishes identically. The Newlander-Nirenberg Theorem implies that Levi-flat CR manifolds are locally equivalent to $\mathbb{R} \times \mathbb{C}^n$.

CR structures can locally be encoded into an adapted coframing. Writing $D^\perp \subset T^* M, \overline{H}^\perp \subset \mathbb{C} T^* M$ for the annihilators of $D$ and $\overline{H}$, a $0$-adapted coframing is given by a collection of 1-forms

$$\varphi^j \in \Gamma(D^\perp), \quad 1 \leq j \leq n, \quad \text{such that} \quad \varphi^0 \wedge \left( \bigwedge_{j=1}^n \varphi^j \right) \wedge \left( \bigwedge_{j=1}^n \overline{\varphi}^j \right) \neq 0.$$ 

Equivalently, a 0-adapted coframing is a local section of the bundle $\pi : \mathcal{F}^0 \to M$ whose fiber over $x \in M$ consists of 0-adapted coframes, which are linear isomorphisms

$$\mathcal{F}^0_x = \{ \varphi_x : T_x M \xrightarrow{\cong} \mathbb{R} \oplus \mathbb{C}^n \mid \varphi_x(D_x) = \mathbb{C}^n, \varphi_x \circ J_x = i \varphi_x \}. $$
We call $\varphi^0$ a **characteristic form**, while $\varphi^j$ and $\overline{\varphi}^j$ are CR and anti-CR forms, respectively. The CR integrability condition (2.4) is expressed

$$d\varphi^i \equiv 0 \mod \{\varphi^0, \ldots, \varphi^n\}, \quad 0 \leq i \leq n.$$  

In particular, a characteristic form is real-valued, so using the summation convention we can write

$$d\varphi^0 \equiv \iota_{jk} \varphi^j \wedge \overline{\varphi}^k \mod \{\varphi^0\}, \quad \iota_{kj} = \overline{\iota}_{jk} \in C^\infty(M, \mathbb{C}); \quad 1 \leq j, k \leq n,$$

where $\iota_{jk}(x)$ is a local representation of the Levi form (2.2) as an $n \times n$ Hermitian matrix. The signature $(p, q)$ of this matrix is an invariant of $M$ under CR equivalence.

Of course, the 0-adapted coframing $\{\varphi^0, \varphi^j\}$ is not uniquely determined by (2.3), but only up to a transformation of the form

$$\begin{bmatrix} u & 0 \\ c & a \end{bmatrix} \begin{bmatrix} \varphi^0 \\ \varphi^j \end{bmatrix}, \quad 0 \neq u \in C^\infty(M), \quad a \in C^\infty(M, GL_n\mathbb{C}), \quad c \in C^\infty(M, \text{Mat}_{n \times n}\mathbb{C});$$

i.e., $c$ is an arbitrary $n \times n$ matrix of \mathbb{C}-valued functions while $a$ is invertible, and $u$ is a nowhere-vanishing \mathbb{R}-valued function. Equivalently, the bundle $\pi : F^0 \to M$ carries a natural $G_0$-principal action on its fibers (2.4),

$$G_0 = \left\{ \begin{bmatrix} u & 0 \\ c & a \end{bmatrix} \in GL(\mathbb{R} \oplus \mathbb{C}^n) \mid 0 \neq u \in \mathbb{R}, \quad a \in GL_n\mathbb{C}, \quad c \in \text{Mat}_{n \times n}\mathbb{C} \right\}.$$  

Thus we see that a CR structure is an example of a $G$-structure ([BGG03 Def.2.1]); there is a tautologically defined 1-form $\Phi \in \Omega^1(F^0, \mathbb{R} \oplus \mathbb{C}^n)$,

$$\Phi|_{\varphi_x} = \varphi_x \circ \pi_x,$$

and any local equivalence $f : M_1 \to M_2$ between CR manifolds lifts canonically to a diffeomorphism $f : F_1^0 \to F_2^0$ between their 0-adapted coframe bundles in a manner that identifies their tautological forms, $f^*\Phi_2 = \Phi_1$. To find all local invariants of a CR structure via Cartan’s method of equivalence ([Car89]), one attempts to complete the tautological form to a full coframing of a principal bundle over $M$ by choosing a complementary pseudoconection form ([BGG03 Def.2.2]). Such a choice depends on reducing the structure group of the coframe bundle as much as possible by successively adapting frames to higher order.

For example, if $M$ is not Levi-flat, we could define a 1-adapted coframing to be a 0-adapted coframing which has the additional property that the matrix entries (2.7) of the Levi form take constant, specified values (such as $\ell$ being diagonalized with $p$ positive ones and $q$ negative ones on the diagonal). This reduces $F^0$ to the subbundle of 1-adapted coframes whose structure group $G_1$ is matrices (2.7) with the additional constraint,

$$G_1 \subset G_0 : \quad \overline{\varphi}^j a = u \ell.$$

This reduction is not meaningful in the Levi-flat case; indeed, we have already noted that there are no local invariants for Levi-flat CR manifolds. In general, the degree of (non)degeneracy of the Levi form has substantial bearing on the application of the method of equivalence.

The opposite extreme of Levi-flatness is **Levi-nondegeneracy**, when $\ell$ has signature $(p, q)$, $p + q = n$. In CR dimension $n = 1$, Levi-nondegeneracy is the same as pseudo-convexity, and the corresponding equivalence problem was solved by Cartan ([Car32], [Jac90]). The general case was treated by Tanaka ([Tan62]) using his modified version of Cartan’s method that would later provide a valuable framework for understanding all parabolic geometries with canonical Cartan connections ([Tan79], [CS09]). Chern-Moser ([CM74]) offered an alternative solution using the standard method and emphasizing the link between the intrinsic geometry of CR manifolds and the extrinsic analysis of normal forms. The solution to the Levi-nondegenerate equivalence problem may be stated as follows.

**Theorem 2.1.** Let $M$ be a hypersurface-type CR manifold of dimension $2n + 1$ whose Levi form has signature $(p, q)$, $p + q = n$.  

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• There exists a canonically defined principal bundle $\mathcal{P} \rightarrow M$ whose structure group is isomorphic to the parabolic subgroup $P \subset SU(p + 1, q + 1)$ given by the stabilizer of a complex line in $\mathbb{C}^{n+2}$ which is null for a Hermitian form $h$ of signature $(p + 1, q + 1)$.

• There exists a canonical Cartan connection $\gamma \in \Omega^1(\mathcal{P}, su(p + 1, q + 1))$ whose curvature tensor $\Gamma = d\gamma + \frac{1}{2} [\gamma, \gamma] \in \Omega^2(\mathcal{P}, su(p + 1, q + 1))$ and its covariant derivatives determine a complete set of local invariants of $M$.

• The algebra of infinitesimal symmetries of $M$ has dimension $\leq n^2 + 4n + 3$, and the upper bound is only achieved where $\Gamma$ locally vanishes. In this case, $M$ is locally equivalent to the hyperquadric $Q \subset \mathbb{C}P^{n+1}$ given by the complex projectivization of the $h$-null cone in $\mathbb{C}^{n+2}$; i.e., $Q = SU(p + 1, q + 1)/P$.

The real hyperquadric $Q$ is the “flat model” of Levi-nondegenerate CR geometry in the sense that it is locally characterized by a vanishing curvature tensor. When $\dim M = 5$, a nondegenerate Levi form either has definite signature $(2, 0)$ or split signature $(1, 1)$, and the Cartan connection $\gamma$ takes values in $su(3, 1)$ or $su(2, 2)$, respectively. Thus, for the flat models $M = Q$, the principal bundle $\mathcal{P}$ is isomorphic to one of the Lie groups $SU(3, 1)$ or $SU(2, 2)$, and $\gamma$ is exactly the Maurer-Cartan form of $\mathcal{P}$.

In the next section, we will be more explicit about how $\mathcal{P}$ arises from bundles of adapted coframings on $M$ when $\dim M = 3$.

2.2. 3-dimensional CR Manifolds. This section closely follows Bryant’s [Bry04] with only minor changes to notation, and omitting several details. Fix $n = 1$ so that $\dim M = 3$ and a local 0-adapted coframing (2.3) consists of an $\mathbb{R}$-valued characteristic form $\varphi^0$ and a $\mathbb{C}$-valued CR form $\varphi^1$ such that $\varphi^0 \wedge \varphi^1 \neq 0$. This coframing is a local section of the bundle $\pi : \mathcal{F}^0 \rightarrow M$ of 0-adapted coframes, and as such it determines a local trivialization of $\mathcal{F}^0$ over which the tautological form (2.8) is

$$\Phi = \begin{bmatrix} \kappa \\ \eta \end{bmatrix} = \begin{bmatrix} u & 0 \\ c & a \end{bmatrix} \pi^* \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix}, \quad \pi^* \Theta^0 \neq 0 \neq a \in C^\infty(\mathcal{F}^0), \quad 0 \neq \Theta^0 \in C^\infty(\mathcal{F}^0, \mathbb{C}), \quad c \in C^\infty(\mathcal{F}^0, \mathbb{C}),$$

with the functions $u, a, c$ acting as $G_0$-valued fiber coordinates for $\mathcal{F}^0$.

CR integrability (2.5) is automatic in dimension three, and in particular (2.6) reads

$$d\varphi^0 \equiv i\ell\varphi^1 \wedge \overline{\varphi}^1 \mod \{\varphi^0\}, \quad \ell \in C^\infty(M).$$

Levi-nondegeneracy says $\ell$ is non-vanishing, so we can reduce to the bundle $\mathcal{F}^1 \subset \mathcal{F}^0$ of 1-adapted coframes with $\ell = 1$, which reduces the structure group $G_0$ to

$$G_1 = \left\{ \begin{bmatrix} \alpha^2 \\ c \\ a \end{bmatrix} \in GL(\mathbb{R} \oplus \mathbb{C}) \mid a \in \mathbb{C} \setminus \{0\}, \ c \in \mathbb{C} \right\}.$$

After pulling back the tautological form (2.9) along the inclusion $\mathcal{F}^1 \hookrightarrow \mathcal{F}^0$, its exterior derivative can be expressed in terms of a pseudoconnection form taking values in the Lie algebra $\mathfrak{g}_1$ of (2.10),

$$d \begin{bmatrix} \kappa \\ \eta \end{bmatrix} = -\begin{bmatrix} \alpha_0 + \alpha_0 \\ \beta_0 \\ \alpha_0 \end{bmatrix} \wedge \begin{bmatrix} \kappa \\ \eta \end{bmatrix} + \begin{bmatrix} 1 \eta \wedge \eta \\ \kappa \end{bmatrix}, \quad \alpha_0, \beta_0 \in \Omega^1(\mathcal{F}^1, \mathbb{C}).$$

However, the structure equations (2.11) do not uniquely determine $\alpha_0$ and $\beta_0$ as they remain the same after a replacement

$$\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} + \begin{bmatrix} s^1 \\ s^2 \end{bmatrix} \begin{bmatrix} s^1 \\ s^2 \end{bmatrix} \begin{bmatrix} \kappa \\ \eta \end{bmatrix}, \quad s^1, s^2 \in C^\infty(\mathcal{F}^1, \mathbb{C}).$$

For any $\alpha_0, \beta_0$ of the form (2.12), the 1-forms $\kappa, \eta, \alpha_0, \beta_0, \overline{\alpha_0}, \overline{\beta_0}$ are called a 1-adapted coframing of $\mathcal{F}^1$. The bundle $\hat{\pi} : \hat{\mathcal{F}}^1 \rightarrow \mathcal{F}^1$ of 1-adapted coframes of $\mathcal{F}^1$ features a tautological $\mathbb{R} \oplus \mathbb{C} \oplus \mathfrak{g}_1$-valued form whose $\mathbb{R} \oplus \mathbb{C}$-valued components are simply the $\hat{\pi}$ pullback of $\Phi$ (we recycle the names of the individual 1-forms),

$$\hat{\pi}^* \Phi = \begin{bmatrix} \kappa \\ \eta \end{bmatrix} \in \Omega^1(\hat{\mathcal{F}}^1, \mathbb{R} \oplus \mathbb{C}),$$
and whose \( g_1 \)-valued components \( \begin{bmatrix} \alpha + \bar{\pi} \alpha \\ \beta \end{bmatrix} \in \Omega^1(\hat{F}^1, g_1) \) satisfy the “lifted” structure equations,

\[
d \begin{bmatrix} \alpha \\ \eta \end{bmatrix} = -\begin{bmatrix} \alpha + \bar{\pi} \alpha \\ \beta \end{bmatrix} \wedge \begin{bmatrix} \kappa \\ \eta \end{bmatrix} + \begin{bmatrix} i \eta \wedge \bar{\eta} \\ 0 \end{bmatrix}; \quad \alpha, \beta \in \Omega^1(\hat{F}^1, \mathbb{C}).
\]

In particular, from (2.12) we see that

\[
\alpha = \pi^* \alpha_0 - s^1 \kappa, \quad \beta = \pi^* \beta_0 - s^2 \kappa - s^1 \eta, \quad s^1, s^2 \in C^\infty(\hat{F}^1, \mathbb{C}),
\]

where \( s^1, s^2 \) now serve as fiber coordinates for \( \pi : \hat{F}^1 \to F^1 \). Differentiating the structure equations (2.13) yields

\[
d \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\begin{bmatrix} \sigma_0^1 \\ \sigma_0^\beta \end{bmatrix} \wedge \begin{bmatrix} \kappa \\ \eta \end{bmatrix} + \begin{bmatrix} -i \beta \wedge \bar{\eta} - 2i \bar{\beta} \wedge \eta + R \eta \wedge \bar{\eta} \\ -\beta \wedge \bar{\pi} \end{bmatrix},
\]

for some \( R \in C^\infty(\hat{F}^1) \), where \( \sigma_0^1, \sigma_0^\beta, \kappa, \eta, \alpha, \beta \) and their conjugates furnish a coframing of \( \hat{F}^1 \). The identity \( d^2 \alpha = 0 \) then reveals that we can restrict to a subbundle \( F^2 \subset \hat{F}^1 \) whose sections are 2-adapted coframings defined by \( R = 0 \), which reduces the real dimension of the fibers over \( F^1 \) by one and forces \( s^1 \) and \( \sigma = \sigma_0^1 \) to be strictly \( \mathbb{R} \)-valued. Then the same identity shows that we can reduce further to 3-adapted coframings corresponding to a subbundle \( F^3 \subset F^2 \) where \( s^2 = 0 \), hence the real fiber dimension of \( F^3 \to F^1 \) is one.

The coframing of \( F^3 \) given by the complex forms \( \eta, \alpha, \beta \) and their conjugates, along with the real forms \( \kappa, \sigma \), is globally defined on \( F^3 \) and uniquely determined by the structure equations

\[
d \kappa = i \eta \wedge \bar{\eta} - (\alpha + \bar{\pi} \alpha) \wedge \kappa,
\]

\[
d \eta = -\beta \wedge \kappa - \kappa \wedge \eta,
\]

\[
d \alpha = -\sigma \wedge \kappa - i \beta \wedge \eta - 2i \bar{\beta} \wedge \eta,
\]

\[
d \beta = -\sigma \wedge \eta + \bar{\pi} \wedge \beta + b \kappa \wedge \bar{\eta},
\]

\[
d \sigma = (\alpha + \bar{\pi}) \wedge \sigma + i \beta \wedge \bar{\sigma} + \kappa \wedge (p \bar{\eta} + \bar{p} \eta),
\]

where \( b, p \in C^\infty(F^3, \mathbb{C}) \) have differential identities

\[
d b = (3 \pi + \alpha) b + u \kappa + p \eta + q \bar{\eta},
\]

\[
d p = (3 \pi + 2 \alpha) p - i b \bar{\pi} + u \kappa + r \eta + v \bar{\eta},
\]

for some additional \( u, q, v, w \in C^\infty(F^3, \mathbb{C}) \) and \( r \in C^\infty(F^3) \). \( F^3 \) realizes the bundle \( \mathcal{P} \) whose existence is guaranteed by Theorem 2.1, while the Cartan connection is given by

\[
\gamma = \begin{bmatrix} -\frac{1}{3}(2 \alpha + \bar{\pi}) & -i \beta & -i \sigma \\ \eta & \frac{1}{3}(\alpha - \bar{\pi}) & \bar{\eta} \\ -i \kappa & -\bar{\eta} & \frac{1}{3}(\alpha + 2 \bar{\pi}) \end{bmatrix},
\]

so that

\[
\eta^T h + h \gamma = 0,
\]

\[
h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},
\]

and \( \gamma \) is indeed \( su(2, 1) \)-valued. The equations (2.15) exhibit the Bianchi identities of the curvature tensor \( \Gamma = d \gamma + \gamma \wedge \gamma \). Furthermore, when \( b = 0 \Rightarrow p = 0 \) so that \( \Gamma \) locally vanishes, (2.14) are exactly the Maurer-Cartan equations of \( su(2, 1) \), as previously discussed.

**Remark 2.2.** Suppose \( M \) is a 3-dimensional CR manifold with a local characteristic form \( \kappa \in \Omega^1(M) \) and a CR form \( \eta \in \Omega^1(M, \mathbb{C}) \) such that \( \{ \kappa, \eta \} \) is a 1-adapted coframing. Along with \( \kappa, \eta \), a collection of local 1-forms \( \alpha, \beta \in \Omega^1(M, \mathbb{C}) \), \( \sigma \in \Omega^1(M) \) satisfying the equations (2.14) constitutes a local section \( s : M \to F^3 \), which we call a 3-adapted coframing of \( M \) in spite of the fact that the forms are not linearly independent. In general, the pullbacks \( s^* b, s^* p \in C^\infty(M, \mathbb{C}) \) depend on a choice of 3-adapted coframing. However, \( M \) is locally flat if and only if \( s^* b = 0 \) for a given 3-adapted coframing \( s \).
Furthermore, when \( \text{model.} \) (2.19) takes any 6-adapted coframing to one with structure equations \([\text{NT88}]\), we record the values \((\theta, \eta, \bar{\eta})\). Following \(\text{bras, which are the possible algebras of infinitesimal symmetries for a homogeneous CR 3-fold. Following } \) and make explicit the reliance of Cartan’s arguments on Bianchi’s classification of 3-dimensional Lie algebras, we have \( \text{structure, or a 6-adapted coframing for a non-flat homogeneous CR structure.} \) More generally, (2.19) and (2.20) either describe a 1-adapted coframing for a flat homogeneous CR structure, or a 6-adapted coframing for a non-flat homogeneous CR structure.

Homogeneous CR 3-folds are classified (locally and globally) in \([\text{Car32}]\), which exhibits local hypersurface realizations of each model. Nurowski and Tafel (\([\text{NT88}]\)) offer alternative coordinate realizations and make explicit the reliance of Cartan’s arguments on Bianchi’s classification of 3-dimensional Lie algebras, which are the possible algebras of infinitesimal symmetries for a homogeneous CR 3-fold. Following \([\text{NT88}]\), we record the values \((z_1, z_2, \delta)\) occurring in the homogeneous structure equations (2.19) for each model.
CR structures with symmetry algebras of Bianchi’s type I or V are Levi-flat. Type II is the Heisenberg Lie algebra, and the structure equations \( \Psi_{i j k} \) are those of Cartan’s (flat) A model, (II, A) \( (z_1, z_2, \delta) = (0, 0, 0) \).

Curvature also vanishes for (III, B) \( (z_1, z_2, \delta) = (0, 0, 1) \).

The first case with non-vanishing curvature is (IV, F) \( (z_1, z_2, \delta) = \left( \frac{1}{4}, \frac{1}{4}, 1 \right) \).

The next two are families of Lie algebras, each indexed by a choice of real parameter, (VI, E) \( (z_1, z_2, \delta) = \left( \frac{i(t - 1)}{4t}, \frac{i(t - 1)}{4t}, 1 \right) \) \( 0 < t \neq 1 \), (VII, H) \( (z_1, z_2, \delta) = \left( \frac{i(t + 1)}{4t}, \frac{i(t + 1)}{4t}, 1 \right) \) \( 0 < t \).

Note that III is VI for \( t = 1 \). One can also show (see [15.6]) that VI is flat for \( t = 9 \), but otherwise these are non-flat and they both approach (IV, F) in the limit \( t \to \infty \). For the final two Lie algebras, \( \delta = 0 \) so there is some freedom to scale \( \eta \) by \( \alpha \in \mathbb{C} \setminus \{0\} \). Though VIII and IX are individual algebras, each serves as the symmetry algebra of a parameter-family of inequivalent CR structures, (VIII, K, C (\( t = 0 \))) \( (z_1, z_2, \delta) = \left( \frac{i(t - 2)}{|\alpha|^2}, \frac{it}{|\alpha|^2}, 0 \right) \) \( 0 \leq t \neq 1 \), (IX, L, D (\( t = 0 \))) \( (z_1, z_2, \delta) = \left( \frac{i(t + 2)}{|\alpha|^2}, \frac{it}{|\alpha|^2}, 0 \right) \) \( 0 \leq t \).

Depending on the value of \( t \), one can choose \( \alpha \) so that \( z_2 = i \) or \( z_1 = \pm i \). However, the limiting case \( z_2 = i = \pm z_1 \) (as \( t \to \infty \) or \( t = 1 \) in VIII) does not occur among Cartan’s models.

3. Physical Motivation

3.1. Some History. The subject matter of this article is closely related to relativistic theories of radiation, both electromagnetic and gravitational. Spacetime is an oriented 4-dimensional manifold \( M \) with a Lorentzian metric \( g \), or more generally, a conformal class of such metrics. A Maxwell field is a 2-form \( F \in \Omega^2(M) \) which is closed – \( dF = 0 \) – and, away from electrical currents or sources of charge, co-closed – \( d * F = 0 \), where \( *F \in \Omega^2(M) \) is the Hodge dual of \( F \). \( F \) is null if it is \( g \)-orthogonal to itself and \( *F \). Null Maxwell fields are associated with electromagnetic radiation. Gravitational fields derive from metrics \( g \) which satisfy Einstein’s field equations. A gravitational field is null if the Weyl tensor of \( g \) has Petrov type N (see [18.95] Ch.5) for a pleasant introduction to the Petrov classification) – the most degenerate type among non-conformally-flat metrics. More generally, a metric is algebraically special if its Weyl tensor is at all degenerate; i.e., if it is of any Petrov type besides I. It is helpful to think of the motivation for the present work in the context of the search for a theoretical framework for gravitational radiation analogous to that of electromagnetic radiation.

The study of gravitational waves was initiated by Einstein in the beginning of the twentieth century; a brief history with references is given in [18.17]. For our purposes, it suffices to join the story in medias res, when Trautman showed in [18.58] that gravitational fields satisfying a Sommerfeld radiation condition are asymptotically null. The next year, Robinson reported to the Royaumont Conference that null Maxwell and gravitational fields determine a foliation of spacetime by a family of curves known as a shear-free null geodesic congruence, or SFNGC. These are discussed in detail in [18.83] Robinson also proved the converse for electromagnetic fields ([18.61]); i.e., a SFNGC gives rise to a null Maxwell field. Spacetimes admitting SFNGCs seemed to be natural candidates for a model of gravitational radiation, though the work [18.61] of Sachs established that these were not restricted to null gravitational fields. Indeed, joint work [18.62] with Goldberg would show that, away from sources of mass-energy – that is,
3-fold CR-embedded in 5-dimensional real hyperquadrics

in a vacuum spacetime with a Ricci-flat metric – every non-flat, algebraically special metric admits a SFNGC tangent to principal null directions of algebraic multiplicity > 1.

Robinson and Trautman ([R162]) produced a class of metrics corresponding to hypersurface-orthogonal SFNGC, including a model for radiation with spherical wavefronts. Then, Kerr sought metrics corresponding to SFNGC that were not hypersurface orthogonal ([Ker63]), and in the process generalized the Schwarzschild solution to incorporate angular momentum, generating a model for spinning black holes ([ON95]). The Kerr metrics have Petrov type D.

Kerr’s name is also attached to a theorem relating SFNGC of flat (Minkowski) spacetime to the objects of study of this article. In §3.2 we offer a geometric overview of the correspondence between subsets of Minkowski spacetime and those of a 5-dimensional real hyperquadric in the spirit of Penrose’s Twistor program ([Pen67 WW90]), emphasizing the various symmetry groups involved. Then §3.3 delves into SFNGC for general spacetimes and explains their connection to CR geometry, following [RT83] and [NT03]. Finally, a sketch of the proof of the Kerr Theorem appears in §3.4.

### 3.2. The Kerr Theorem.

\( \mathbb{R}^n \) equipped with a symmetric, bilinear form \( b \) of signature \( (p, q) \), \( p + q = n \), will be denoted \( \mathbb{R}^{p,q} \). The complex-linear extension of \( b \) to \( \mathbb{C}^n \) is also called \( b \), but in the complexification its signature is no longer significant. Hence, we reserve the notation \( \mathbb{C}^{p,q} \) for when \( \mathbb{C}^n \) carries a Hermitian form \( h \) of signature \( (p, q) \), \( p + q = n \), unrelated to any underlying real form.

The setting of special relativity is Minkowski spacetime \( \mathbb{M} \), an affine space with modeling vector space \( \mathbb{R}^{1,3} \). Therefore, the Lorentz group \( O(1, 3) \) – and its affine extension, the Poincaré group – plays a central role in relativistic theories. However, when a relativistic theory (such as the electrodynamics expressed in Maxwell’s equations) exhibits conformal invariance, the corresponding group of symmetries is larger.

Conformal compactification of \( \mathbb{R}^{p,q} \) is achieved by affixing a point “at infinity” for each one in the \( b \)-null cone in order that inversion may be globally defined. The resulting quadric is the real-projectivization \( \mathbb{R}^n_+ \) of \( \mathbb{R}^{p+1,q+1} \), whose bilinear form wears a hat to distinguish it from that of \( \mathbb{R}^{p,q} \). The group of (oriented) conformal symmetries of compactified Minkowski space \( \mathbb{M}^c \) is the symmetry group of the \( b \)-null cone in \( \mathbb{R}^{2,4} \); i.e., \( SO(2, 4) \).

The Plücker embedding sends the Grassmannian \( Gr(2, \mathbb{C}^4) \) of complex 2-planes in \( \mathbb{C}^4 \) into the complex-projective space \( \mathbb{P}(\wedge^2 \mathbb{C}^4) = \mathbb{C}^5 \), and its image is the quadric given by the projectivization of the \( b \)-null cone in \( \mathbb{C}^6 \). This may be considered a geometric analog of the Lie algebra isomorphism \( su(\mathbb{C}) \cong so_6(\mathbb{C}) \). Moreover, the Grassmannian \( Gr^0(2, \mathbb{C}^{2,2}) \subset Gr(2, \mathbb{C}^4) \) of totally \( h \)-null 2-planes embeds onto \( \mathbb{M}^c \) by analogy to the isomorphism \( su(2, 2) \cong so(2, 4) \).

Both \( \mathbb{C}P^3 \) and \( Gr(2, \mathbb{C}^4) \) are partial flag manifolds associated to \( \mathbb{C}^4 \); to these we add \( F_{1,2} \mathbb{C}^4 \) consisting of pairs \( (l, \Pi) \) of a complex line and plane (respectively) satisfying \( l \subset \Pi \subset \mathbb{C}^2 \). With the projection maps \( \lambda(l, \Pi) = l \) and \( \pi(l, \Pi) = \Pi \) we obtain the double fibration

\[
(3.1) \quad \begin{array}{ccc}
\mathbb{C}P^3 & \xleftarrow{\lambda} & F_{1,2} \mathbb{C}^4 \\
& \pi \swarrow & \downarrow \\
& Gr(2, \mathbb{C}^4) & 
\end{array}
\]

lying at the heart of Penrose’s Twistor theory, which concerns the correspondence between subsets of \( \mathbb{C}P^3 \) and \( Gr(2, \mathbb{C}^4) \) via the images of \( \lambda \circ \pi^{-1} \) and \( \pi \circ \lambda^{-1} \). To clarify some of the physical motivation for this framework, we restrict to \( h \)-isotropic flags \( F_{1,2}^0 \mathbb{C}^{2,2} \subset F_{1,2} \mathbb{C}^4 \), so that (3.1) becomes

\[
(3.2) \quad \begin{array}{ccc}
\mathbb{C}P^3 & \xleftarrow{\lambda} & F_{1,2}^0 \mathbb{C}^{2,2} \\
& \pi \swarrow & \downarrow \\
Q & \xrightarrow{\pi} & \mathbb{M}^c = Gr^0(2, \mathbb{C}^{2,2}), 
\end{array}
\]

where \( Q \subset \mathbb{C}P^3 \) is the 5-dimensional real hyperquadric given by the complex projectivization of the \( h \)-null cone \( N \subset \mathbb{C}^{2,2} \). The trajectory of a massless particle in \( \mathbb{M} \) is tangent to a \( b \)-null (affine) line, and
each such line corresponds to a point in $Q$. Physicists refer to a foliation of $\mathbb{M}$ by null lines as a null congruence, the relevance of which to the present work is stated in the

**Kerr Theorem.** A null congruence of $\mathbb{M}$ corresponds to a CR submanifold of $Q$ if and only if it is shear-free.

The Kerr Theorem first appeared in print in [Pen67, VIII], where it is stated that a shear-free null congruence is representable in $\mathbb{CP}^3$ as the intersection of $Q$ with a complex-analytic surface (or the limiting case of such an intersection); see also [PR88, Ch.6]. The proof sketch in [3.4] makes this construction explicit. The version we’ve stated is closer to [NT02, Thm.7].

3.3. **Shear-Free Null Geodesic Congruences.** In this section we follow [RTS3] and [NT02]. Let $M$ be a smooth, 4-dimensional manifold with a line bundle $K \subset TM$ whose fibers are spanned by a nowhere-vanishing vector field $k \in \Gamma(K)$, which determines a smooth flow

$$\phi: I \times M \to M,$$

where $I \subset \mathbb{R}$ is some open interval containing zero. For fixed $x \in M$ and variable $t \in I$, $\phi(t, x)$ is the integral curve of $k$ passing through $x$ when $t = 0$, and $M$ is foliated by these flow curves. For fixed $t \in I$,

$$\phi_t: M \to M
\quad x \mapsto \phi(t, x)$$

is a diffeomorphism whose pushforward $\phi_t*: TM \to TM$ satisfies

$$\phi_t*K_x = K_{\phi(t,x)},$$

and therefore descends to a well-defined map on the quotient bundle $TM/K \to M$. Thus, the family $\{\phi_t: t \in I\}$ of diffeomorphisms provides linear isomorphisms between the spaces $T_{\phi(t,x)}M/K_{\phi(t,x)}$ for any fixed $x \in M$, and we see that the quotient bundle $TM/K$ has the same fibers as the tangent bundle of the leaf space $\mathcal{M}$; i.e., the 3-dimensional quotient manifold of equivalence classes $[x]$ of points $x \in M$, where two points are equivalent if they lie in the same leaf of the foliation (the same flow curve),

$$T_{[x]}\mathcal{M} \cong T_{\phi(t,x)}M/K_{\phi(t,x)} \quad \forall t \in I.$$

**Remark 3.1.** In general, a 4-manifold need not admit a globally defined, non-vanishing tangent vector field, nor should the entire leaf space of a foliation necessarily inherit a global manifold structure. However, our considerations are local in nature and we will continue to implicitly assume that $M$ is such that our constructions are well-defined. In particular, we may also take $M$ to be orientable. If $\omega \in \Omega^4(M)$ is a volume form, then the contraction $k_\omega \in \Omega^3(M)$ vanishes on $K$ and descends to a 3-form on $TM/K$. Note that $k_\omega$ does not determine a well-defined volume form on $\mathcal{M}$ unless $\mathcal{L}_k \omega = 0$, where $\mathcal{L}_k$ denotes the Lie derivative along $k$. However, the sign of $k_\omega$ on any ordered basis of $\mathcal{M}$ is sufficient to determine whether a volume form on $M$ is positively or negatively oriented relative to $k_\omega$, and so determines a choice of orientation on $\mathcal{M}$.

Suppose that $M$ is equipped with a non-degenerate metric $g \in \bigwedge^2 T^*M$. For the moment, we make no assumptions about the signature of $g$. The one-form $\kappa \in \Omega^1(M)$ dual to $k$ has as its kernel a rank-3 distribution

$$\kappa = k_\omega g \quad \Rightarrow \quad \ker \kappa = K^\perp \subset TM.$$

**Definition 3.2.** The flow of $k$ is conformally geodesic if it preserves the distribution $K^\perp$,

$$\phi_t*K^\perp_x = K^\perp_{\phi(t,x)} \quad \forall t \in I, x \in M.$$

Equivalently, the flow of $k$ is conformally geodesic when

$$\kappa \wedge \phi_t*\kappa = 0 \quad \Rightarrow \quad \kappa \wedge \mathcal{L}_k\kappa = 0.$$
Hence, a conformally geodesic flow not only identifies the fibers of $K$ along a flow curve as in (3.4), but also the fibers of $K^\perp$. The implications of this for the leaf space $\mathcal{M}$ depend on the metric properties of $k$. If $g(k_x,k_x) \neq 0$ for every $x \in M$, then $TM = K \oplus K^\perp$ and $T_{[x]}M \cong K^\perp_{\phi(t,x)}$ for every $t \in I$. On the other hand, if $g$ has mixed signature and $g(k,k) = 0$, then $K \subset K^\perp$ and $\mathcal{M}$ inherits a well-defined, rank-2 distribution

$$D \subset TM \quad \text{with fibers} \quad D_{[x]} \cong K^\perp_{\phi(t,x)}/K_{\phi(t,x)} \quad \forall t \in I.$$  

We also have when $k$ is null that $\kappa$ descends to the quotient bundle $TM/K$, and the additional condition (3.6) that $k$ is conformally geodesic further implies that $\kappa$ determines a well-defined, non-vanishing one-form (of the same name) on $\mathcal{M}$, which annihilates (3.7).

**Remark 3.3.** Condition (3.6) is always invariant under conformal scaling of $\kappa$, and when $k$ is null it is even invariant under scaling of $k$ by a non-vanishing function, which effects a reparameterization of the flow curves of $k$.

Henceforth, we restrict to the case that $g$ has Lorentzian signature $(1,3)$ and $k$ is $g$-null with a conformally geodesic flow. The foliation of $M$ by flow curves is now called a null geodesic congruence, the fibers of the quotient bundle $K^\perp/K$ are called screen spaces, and the geometry of the null congruence may be understood intuitively in terms of the following illustration regarding optical scalars (\cite{ON95} §5.7). In relativity, light propagates in null directions; imagine a beam of light casting the shadow of an opaque disk onto a 2-dimensional screen placed orthogonal to its (null) direction. As the screen is moved along the flow curve, this circular image might be rotated, enlarged, or distorted into an ellipse of greater eccentricity. If the latter, non-conformal distortion does not occur, the null congruence is shear-free. The precise geometric definition applies to arbitrary conformally geodesic flows.

**Definition 3.4.** A conformally geodesic flow is shear-free if it preserves the conformal class of $g$ restricted to $K^\perp$; i.e., for any $t \in I$ and $x \in M$, there is some $s \in \mathbb{R}$, $s > 0$ such that

$$\phi_t^*(g|_{K^\perp_{\phi(t,x)}}) = sg|_K^\perp,$$

so that in particular,

$$\mathcal{L}_k g = ag + \kappa \circ \alpha$$

for some $a \in C^\infty(M)$ and $\alpha \in \Omega^1(M)$.

**Remark 3.5.** Using general properties of the Lie derivative, it is straightforward to confirm that (3.9) is maintained under rescaling of $k$ by a non-vanishing function, albeit for different $a, \alpha$. Along with Remark 3.3, this shows that a shear-free null geodesic congruence (SFNGC) is independent of the choice of $k$ spanning $K$. Note that if $k$ is $g$-null, it is also $\tilde{g}$-null, where

$$\tilde{g} = fg + \kappa \circ \xi,$$

and $\tilde{\kappa} = k_{,\tilde{g}}$ is a rescaling of $\kappa$. Here again, the properties of the Lie derivative show that $\tilde{g}$ satisfies (3.9) whenever $g$ does, so the class (3.10) of metrics associated to given SFNGC is manifestly larger than a conformal class of metrics.

For null $k$, $g|_K = 0$ and we see from (3.8) that a SFNGC determines a well-defined conformal structure on the subbundle (3.7) of the leaf space. As such, we can define an almost-complex structure on $\mathcal{M}$,

$$J : D \to D, \quad J^2 = -1,$$

by taking $J_{[x]}$ to be a rotation by $\frac{\pi}{2}$ in $D_{[x]}$. (There are two choices for the direction of the rotation – clockwise or counterclockwise – in each $D_{[x]}$. Take the one which is positively oriented for the orientation induced by the semi-Riemannian volume form on $M$; see Remark 3.1.)

Thus we see that a SFNGC induces a CR structure on the 3-dimensional leaf space $\mathcal{M}$, with $\kappa$ serving as a characteristic form. To this we may add a CR form $\eta \in \Omega^1(\mathcal{M}, \mathbb{C})$ so that $\kappa, \eta, \overline{\eta}$ is a 0-adapted CR coframing. CR integrability is automatic in dimension three, but pseudo-convexity is not. In the Levi-flat case, $\kappa \wedge d\kappa = 0$ and $\mathcal{M}$ is foliated by complex curves; the original curves of our SFNGC are hypersurface-orthogonal, as one would expect from a spacetime featuring radiating wave fronts. More interesting from
a CR perspective is the Levi-non-degenerate case corresponding to “twisting” congruences \( \kappa \wedge d\kappa \neq 0 \), including the Kerr spacetime which describes a rotating black hole.

Conversely, suppose that \( \mathcal{M} \) is a 3-dimensional CR manifold with a 0-adapted coframing \( \kappa, \eta, \overline{\eta} \), and set \( M = \mathbb{R} \times \mathcal{M} \). We use the same names \( \kappa, \eta, \overline{\eta} \) to denote their pullbacks along the projection \( M \rightarrow \mathcal{M} \). Take \( k \in \Gamma(TM) \) to be \( k = \frac{\partial}{\partial r} \) where \( r \) is the Cartesian coordinate of \( \mathbb{R} \), and choose any \( \rho \in \Omega^1(M) \) with \( \rho(k) = 1 \); i.e., \( \rho \equiv dr \mod \{\kappa, \eta, \overline{\eta}\} \). The metric

\[
g = \kappa \odot \rho \circ \eta \odot \overline{\eta}
\]

has signature (1, 3) and satisfies \( g(k, k) = 0 \) as well as \( \kappa = k \lrcorner g \). The flow curves of \( k \) are the \( r \)-coordinate curves of \( M \), and Lie derivatives along \( k \) can be computed via H. Cartan’s formula, yielding

\[
\mathcal{L}_k \kappa = \mathcal{L}_k \eta = \mathcal{L}_k \overline{\eta} = 0, \quad \mathcal{L}_k \rho \equiv 0 \mod \{\kappa, \eta, \overline{\eta}\},
\]

whence both conditions (3.6) and (3.9) are verified. This establishes a correspondence

\[
\{\text{SFNGC on 4-manifolds}\} \xrightarrow{\text{(local)}} \{\text{CR structure on 3-manifolds}\}
\]

Now suppose we submit our coframing on \( \mathcal{M} \) to a 0-adapted transformation,

\[
\begin{bmatrix} \kappa' \\ \eta' \end{bmatrix} = \begin{bmatrix} u & 0 \\ c & a \end{bmatrix} \begin{bmatrix} \kappa \\ \eta \end{bmatrix}; \quad u \in C^\infty(M), \ a, c \in C^\infty(M, \mathbb{C}), \ u, a \neq 0,
\]

and write the metric \( \tilde{g} \) as in (3.11). In terms of our original coframing, we obtain

\[
\tilde{g} = \kappa' \odot \rho \circ \eta' \odot \overline{\eta}'
\]

\[
= |c|^2 g + \kappa \circ ((u - |c|^2) \rho - |c|^2 \kappa - a \overline{\eta} - \overline{a c} \overline{\eta})
\]

\[
= f g + \kappa \circ \xi
\]

as in (3.10). Following our initial selection of \( \rho \), the ambiguity of the metric \( g \) due to our choice (3.14) of coframing on \( \mathcal{M} \) is measured by 5 real functions of 3 variables, rather than the 5 functions of 4 variables apparent in the full class of metrics discussed in Remark 3.5. However, if we allow the fiber coordinates \( u, a, c \) of our G-structure (3.14) to vary with \( r \), then the structure group of our bundle of 0-adapted CR frames exactly parameterizes the class of metrics associated to this SFNGC (note that the Lie derivatives along \( k \) of our CR coframing will no longer vanish identically as in (3.12) if \( u, a, c \) depend on \( r \); (3.6) and (3.9) will hold nonetheless).

The correspondence (3.13) raises several questions, the first of which is presented as Problem 1 in [NT02], and the second of which was communicated to the author by Paweł Nurowski:

- Which CR structures lift to a SFNGC whose class (3.10) of metrics contains a solution to Einstein’s field equations?
- The Goldberg-Sachs theorem says that there are two SFNGC associated to a metric of Petrov type D; are the two corresponding CR structures always equivalent?
- Which CR structures lift to a SFNGC whose class (3.10) of metrics contains one that is (conformally) flat?

The Kerr Theorem offers the answer to the final question: those that are embedded within the real hyperquadric \( Q \). The present article attempts to answer the inevitable follow-up question: which are those?

3.4. Kerr Theorem Proof Sketch. We argue as in [Taf85], §5. Remark 3.5 says that we can scale the vector field \( k \in \Gamma(K) \) tangent to our SFNGC at will, so we are less occupied with null vectors tangent to Minkowski spacetime \( \mathcal{M} \) than we are with null directions. The projectivized b-null cone in \( \mathbb{R}^{1,3} \) is the (Riemann) sphere \( S^2 = \mathbb{CP}^1 \), hence a single stereographic coordinate \( \zeta \in \mathbb{C} \) suffices to parameterize
all null tangent directions in each $T_x \mathcal{M}$, with the exception of one direction “at infinity.” In standard coordinates $(x_0, x_1, x_2, x_3) \in \mathcal{M}$, the metric is diagonal,

$$g = dx_0 \odot dx_0 - dx_1 \odot dx_1 - dx_2 \odot dx_2 - dx_3 \odot dx_3.$$  

Introducing null and complexified coordinates

$$u = x_0 - x_3, \quad v = x_0 + x_3, \quad w = x_1 + ix_2, \quad \overline{w} = x_1 - ix_2,$$

brings $g$ into the form

$$g = du \odot dv - dw \odot d\overline{w}.$$  

A general null vector field and its dual form are, up to real scale,

$$k = \frac{\partial}{\partial v} - \zeta \frac{\partial}{\partial w} - \overline{\zeta} \frac{\partial}{\partial \overline{w}} + |\zeta|^2 \frac{\partial}{\partial u}, \quad \kappa = du + \zeta dw + \overline{\zeta}(d\overline{w} + \zeta dv), \quad \zeta \in C^\infty(\mathcal{M}, \mathbb{C}),$$

while the SFNGC “at infinity” is given by the $u$-coordinate lines. In the latter case, the remaining coordinates $v, w, \overline{w}$ descend to the leaf-space of $u$-coordinate lines, which is the Levi-flat $\mathbb{R} \times \mathbb{C}$, and this corresponds to a CR structure in $Q$ that is tangent to a complex curve.

In the general case we can write

$$g = \kappa \odot dv - \eta \odot \overline{\eta}, \quad \eta = dw + \zeta dv,$$

and after computing Lie derivatives,

$$\mathcal{L}_k \kappa = d\zeta(k)\overline{w} + d\overline{\zeta}(k)dw + (d\zeta(k) + \zeta d\overline{\zeta}(k))dv, \quad \mathcal{L}_k \eta = d\zeta + d\zeta(k)dv,$$

we see that conditions (3.6) and (3.9) hold when

conformally geodesic:

$$0 = d\zeta(k) = d\overline{\zeta}(k),$$

shear-free:

$$0 = \kappa \wedge \eta \wedge \mathcal{L}_k \eta,$$

where the second is equivalent to

$$(3.15) \quad d(u + \zeta \overline{w}) \wedge d(w + \zeta v) \wedge d\zeta = 0.$$  

Name the three $\mathbb{C}$-valued functions

$$z_1 = u + \zeta \overline{w}, \quad z_2 = w + \zeta v, \quad z_3 = \zeta,$$

and observe that

$$(3.16) \quad i(z_1 - \overline{z}_1 + z_2 \overline{z}_3 - z_3 \overline{z}_2) = 0.$$  

If $Z_0, Z_1, Z_2, Z_3$ are coordinates for $C^{2,2}$ with the Hermitian form

$$h(Z, W) = i(Z_1 \overline{W}_0 - Z_0 \overline{W}_1 + Z_2 \overline{W}_3 - Z_3 \overline{W}_2),$$

then (3.16) describes the projectivization in $\mathbb{CP}^3$ of the $h$-null cone $h(Z, Z) = 0$ in the affine coordinate neighborhood $Z_0 \neq 0$, via projective coordinates $[Z_0 : Z_1 : Z_2 : Z_3] = [1 : z_1 : z_2 : z_3]$. Moreover, if $\zeta = z_3$ is implicitly defined by $H(z_1, z_2, z_3) = 0$, where $H$ is holomorphic (and not constant) in $z_1, z_2, z_3$, then the 3-form $dz_1 \wedge d\overline{z}_2 \wedge d\overline{z}_3$ vanishes on the subbundle $dH = 0$ of the complexified tangent bundle of $\mathbb{CP}^3$, and over the quadric $Q$ locally defined by (3.10), this is exactly the shear-free condition (3.15). The level set $H = 0$ is a complex-analytic surface in $\mathbb{CP}^3$ whose intersection with the real hyperquadric $Q$ defines a 3-dimensional CR submanifold of $Q$.

For the remaining details, consult [Pen67, §VIII], [Taf85, §5], or [PRS88] Ch.6].
4. Moving Frames Over Embedded 3-folds

4.1. Hermitian Frames of $\mathbb{C}^4$. Let $\mathfrak{g} = \{e_0, e_1, e_2, e_3\}$ denote the standard basis of column vectors for $\mathbb{C}^4$ and recall that $\mathfrak{e}$ is the natural exponential and $i = \sqrt{-1}$. Fix index ranges and constants

$$0 \leq i, j \leq 3, \quad \epsilon = \pm 1, \quad \delta_i = \begin{cases} 0, & \epsilon = 1 \\ 1, & \epsilon = -1 \end{cases} \Rightarrow \epsilon = (-1)^{\delta_i}.$$ 

The Hermitian form $h$ of signature $(3 - \delta_i, 1 + \delta_i)$ acts on vectors $z = z^i e_i$ and $w = w^i e_j$ via

$$h(z, w) = i(z^0\overline{w}^0 - z^3\overline{w}^3) + z^1\overline{w}^1 + \epsilon z^2\overline{w}^2.$$ 

A **Hermitian frame** is an ordered, complex basis $\mathbf{v} = (v_0, v_1, v_2, v_3)$ of $\mathbb{C}^4$ such that

$$h(v_i, v_j) = (\pm 1)^{\delta_i} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}.$$ 

Note that $(\pm 1)^{\delta_i}$ means the sign is allowed to change when $\epsilon = -1$ but not when $\epsilon = 1$. The vectors $v_1$ and $v_2$ are called “orthonormal” regardless of the value of $\epsilon$. Denote by $\mathcal{H}$ the collection of all Hermitian frames, and note that $\mathcal{H} \cong U(3 - \delta_i, 1 + \delta_i)$ by fixing $\mathfrak{g}$ as the identity and taking $\mathbf{v}$ to be the matrix whose column vectors are the basis vectors of $\mathfrak{g}$. In particular, if two Hermitian frames $\mathbf{u}$ and $\mathbf{v}$ share the same $v_0 = v_0$, then they differ by a transformation

$$g = \begin{bmatrix} 1 & i(\mp 1)^{\delta_i} (a\overline{e}_1 - e\overline{b}_2) & i(\mp 1)^{\delta_i} (e\overline{m}_1 + b\overline{e}_1) & c_0 \\ 0 & a & b & c_1 \\ 0 & -e\overline{b} & e\overline{m} & c_2 \\ 0 & 0 & 0 & (\pm 1)^{\delta_i} \end{bmatrix}, \quad a, b, c_1, c_2, \in \mathbb{C}, 
$$

$$a^2 + e^2 = (\pm 1)^{\delta_i}, \quad c_0 = (\pm 1)^{\delta_i} t + \frac{1}{2} (|c_1|^2 + \epsilon |c_2|^2), \quad r, t \in \mathbb{R}.$$ 

Name this subgroup $\tilde{G} \subset U(3 - \delta_i, 1 + \delta_i)$.

Using the same symbol for $v_i \in C^\infty(\mathcal{H}, \mathbb{C}^4)$ which maps $\mathbf{u} \mapsto v_i$, we differentiate these functions via the Maurer-Cartan (MC) forms of $U(3 - \delta_i, 1 + \delta_i)$,

$$d\psi = \omega^i v_i, \quad \omega \in \Omega^1(\mathcal{H}, U(3 - \delta_i, 1 + \delta_i)).$$

In our representation of this Lie algebra, we can write

$$\omega = \begin{bmatrix} \omega^0 & \phi_1 & \phi_2 & \psi \\ \omega^1 & \phi_1 & \phi_2 & 0 \\ \omega^2 & 0 & \phi_2 & 0 \\ \omega^3 & 0 & 0 & \phi_1 \end{bmatrix}, \quad \phi_i = -\overline{\phi}_j,$$

where $\omega^3, \psi \in \Omega^1(\tilde{G})$ and the rest are $\mathbb{C}$-valued. The MC equations are

$$\begin{align*}
d\omega^0 &= -\phi_1^3 \wedge \omega^1 - \phi_2^3 \wedge \omega^2 - \psi \wedge \omega^3, \\
d\omega^1 &= \omega^0 - \phi_1^3 \wedge \omega^1 - \phi_2^3 \wedge \omega^2 - \iota \phi_1^3 \wedge \omega^3, \\
d\omega^2 &= (\omega^0 - \phi_2^3) \wedge \omega^2 - \epsilon \phi_1^3 \wedge \omega^1 - \epsilon \iota \phi_2^3 \wedge \omega^3, \\
d\omega^3 &= \iota \omega^1 \wedge \overline{\omega}^1 + \iota \omega^2 \wedge \overline{\omega}^2 + (\omega^0 + \overline{\omega}^0) \wedge \omega^3, \\
d\phi_1^3 &= \phi_3^1 \wedge \omega^1 + \phi_1^3 \wedge \overline{\omega}^1 - \epsilon \phi_2^3 \wedge \phi_1^3, \\
d\phi_2^3 &= \phi_2^3 \wedge \omega^2 + \phi_3^2 \wedge \overline{\omega}^2 + \epsilon \phi_1^3 \wedge \phi_2^3, \\
d\phi_3^1 &= \phi_3^1 \wedge \omega^1 + \epsilon \phi_1^3 \wedge \omega^2 + \epsilon \phi_3^2 \wedge (\phi_1^3 - \phi_2^3), \\
d\phi_3^2 &= -i\psi \wedge \overline{\omega}^1 + \phi_3^2 \wedge (\omega^0 - \phi_1^3) + \epsilon \phi_2^3 \wedge \phi_3^2, \\
d\psi &= -\iota \psi \wedge \overline{\omega}^1 + \iota \phi_1^3 \wedge \phi_1^3 + \iota \phi_2^3 \wedge \phi_2^3.
\end{align*}$$

[14]
Define \( \det : \hat{\mathcal{H}} \rightarrow U(1) \subset \mathbb{C} \) as usual by
\[
v_0 \wedge v_1 \wedge v_2 \wedge v_3 = \det(v)e_0 \wedge e_1 \wedge e_2 \wedge e_3,
\]
and let \( \mathcal{H} \subset \hat{\mathcal{H}} \) denote the collection of oriented Hermitian frames satisfying \( \det(v) = 1 \). From any Hermitian frame \( \underline{v} \) one obtains an oriented Hermitian frame in a variety of ways; e.g.,
\[
\underline{v} \mapsto (v_0, v_1, \det(v)v_2, v_3),
\]
in this case preserving the vectors \( v_0, v_1, v_3 \). As we see from \( (4.12) \), the collection of oriented Hermitian frames with a fixed \( v_0 \) is parameterized by the subgroup \( G \subset \hat{G} \) with \( r = 0 \). Note that \( \mathcal{H} \cong SU(3 - \delta_e, 1 + \delta_e) \). Keeping the same names after pulling back the MC forms \( (4.1) \) along the inclusion \( \mathcal{H} \hookrightarrow \hat{\mathcal{H}} \) will now give
\[
\phi^1 + \phi^2 + \omega^0 - \overline{\omega}^0 = 0.
\]

4.2. First Adaptations. Let \( N \subset \mathbb{C}^4 \) be the null-cone of \( h \),
\[
N := \{ v \in \mathbb{C}^4 : h(v, v) = 0 \} \quad \implies \quad T_vN = \{ w \in \mathbb{C}^4 : \Re(h(v, w)) = 0 \}.
\]
We identify the following distinguished subbundles of \( TN \) by their fibers,
\[
L_v := \{ \lambda v : \lambda \in \mathbb{C} \} \quad \subset \quad L^\perp_v := \{ w \in \mathbb{C}^4 : h(v, w) = 0 \}.
\]
The fibration \( \pi : \hat{\mathcal{H}} \rightarrow N \) given by the projection \( \pi(\underline{v}) = v_0 \) identifies \( N \cong U(3 - \delta_e, 1 + \delta_e)/\hat{G} = SU(3 - \delta_e, 1 + \delta_e)/G \), and we have spanning sets
\[
\langle v_0 \rangle_C = L_{v_0}, \quad \langle v_0, v_1, v_2 \rangle_C = L^\perp_{v_0}, \quad \langle v_0, v_1, v_2, v_3, i v_0, i v_1, i v_2 \rangle_C = T_vN.
\]
Conversely, one can assign an adapted basis of \( T_vN \) to each \( v_0 \in N \) in order to define a section \( s : N \rightarrow \hat{\mathcal{H}} \) as follows. Take \( v_0 \) itself to span \( L_{v_0} \), choose two orthonormal vectors \( v_1, v_2 \in L^\perp_{v_0} \), and then \( v_3 \) is uniquely determined to complete the Hermitian frame.

With a section \( s \), we can pull back \( dv_0 \in \Omega^1(\hat{\mathcal{H}}, \mathbb{C}^4) \) from \( (4.8) \) to get
\[
s^*dv_0 = s^*\omega^0 v_0 + s^*\omega^1 v_1 + s^*\omega^2 v_2 + s^*\omega^3 v_3,
\]
but \( s^*v_0 \) is just the identity map on \( N \), so its differential is the identity on \( TN \). Thus, if we use superscripts to denote the dual coframe of \( \underline{v} \) (i.e., \( v^*(u_j) = \delta^*_j \)), then we see
\[
s^*\omega^i = v^i.
\]
The coframe dual to the frame \( \underline{v} \) in \( (4.2) \) differs by the transformation
\[
\begin{bmatrix}
v^0 \\
v^1 \\
v^2 \\
v^3
\end{bmatrix} = g^{-1} \begin{bmatrix}
v^0 \\
v^1 \\
v^2 \\
v^3
\end{bmatrix}.
\]
In this way we realize \( \hat{\mathcal{H}} \) as an adapted (co)frame bundle of \( N \) whose tautological forms are the real and imaginary parts of the \( \omega^j \) MC forms of \( U(3 - \delta_e, 1 + \delta_e) \) as in \( (4.11) \). From this perspective, the collection \( \mathcal{H} \) of oriented frames given by the image of the projection \( (4.6) \) is a subbundle with the reduced structure group \( G \subset \hat{G} \), over which we have \( (4.7) \). The following discussion is equally valid on this reduced bundle.

Rewriting (some of) the MC equations \( (4.3) \),
\[
\begin{bmatrix}
\omega^0 \\
\omega^1 \\
\omega^2 \\
\omega^3
\end{bmatrix} = - \begin{bmatrix}
0 & \phi^1 & \phi^2 & \psi \\
0 & \phi^1 & \phi^2 & \psi \\
0 & \phi^1 & \phi^2 & \psi \\
0 & 0 & 0 & 0
\end{bmatrix} \wedge \begin{bmatrix}
\omega^0 \\
\omega^1 \\
\omega^2 \\
\omega^3
\end{bmatrix} + \begin{bmatrix}
0 \\
\omega^0 \wedge \omega^1 \\
\omega^0 \wedge \omega^2 \\
\omega^0 \wedge \omega^3 + \psi \wedge \nu \wedge \omega^3 + \nu \wedge \nu \wedge \omega^3
\end{bmatrix},
\]
while committing the standard notational abuse of writing \( g \) for \( \hat{G} \)-valued fiber coordinates of \( \hat{\mathcal{H}} \) on a local trivialization determined by a section \( s \), \((4.9)\) and \((4.10)\) imply

\[
(4.12) \quad \begin{bmatrix} 0 & \phi^1_1 & \phi^2_1 & \psi \\ 0 & \phi^1_2 & \phi^2_2 & i\phi^3_1 \\ 0 & \epsilon \phi^1_3 & \phi^2_3 & i\phi^3_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \equiv g^{-1} dg \mod \{\omega^i, \bar{\omega}^j\}.
\]

i.e., the matrix \((4.12)\) of MC forms coincides with the pseudoconnection form \( g^{-1} dg \in \Omega^1(\hat{\mathcal{H}}, \hat{\mathcal{G}}) \) taking values in the Lie algebra of \( \hat{\mathcal{G}} \), at least up to \( \hat{\mathcal{G}} \)-valued combinations of the tautological forms \( \omega^i, \bar{\omega}^j \). Indeed, even the structure equations \((4.11)\) do not uniquely determine the pseudoconnection forms \((4.12)\), as \( \psi \) is apparently only determined up to \( \mathbb{R} \)-valued multiples of \( \omega^3 \). Of course, \( \psi \) is completely determined by the full gamut of MC equations \((4.5)\) on \( \hat{\mathcal{H}} \), but these correspond to a higher-order adaptation of (co)framings on \( N \) which controls how the vector field \( s^* v_3 \in \Gamma(TN) \) varies in the direction of \( L \subset TN \) for sections \( s : N \to \hat{\mathcal{H}} \). This is related to the fact that \( \hat{\mathcal{H}} \) also fibers over the real hyperquadric.

Let \( \lambda : \mathbb{C}^4 \to \mathbb{C}P^3 \) be the canonical projection and denote by \( Q \subset \mathbb{C}P^3 \) the real hyperquadric given by the image of \( \lambda \) when restricted to \( N \). Note that the fibers of \( TN \) vary along those of \( \lambda \), but it always holds that \( L_v = L_{\tilde{v}} \) and \( L_v^\perp = L_{\tilde{v}}^\perp \) when \( \tilde{v} = \lambda^{-1}(\lambda(v)) \). In particular, for any \( v \in N \), \( L_v = \ker \lambda_v \) and there is a well-defined subbundle

\[
D_{\lambda(v)} = \lambda_v L_v^\perp \subset T_{\lambda(v)}Q.
\]

Scalar multiplication by \( i \) in \( \mathbb{C}^4 \) defines an endomorphism \( J : L_v^\perp \to L_v^\perp \) satisfying \( J^2 = -1 \), of which \( L_v \) is an invariant subspace, hence \( J \) descends to a well-defined almost-complex structure \( J : D \to D \). If \( M \subset Q \) is a 3-dimensional CR submanifold of \( Q \), then there is a rank-2, \( J \)-invariant subbundle \( D_M \subset D \) tangent to \( M \). Let \( \hat{M} \subset N \) be the cone over \( M \) and \( L_M \subset \hat{T}M \) be \( \lambda^{-1}(D_M) \).

\( \hat{H} \) and \( \hat{\mathcal{H}} \) fiber over \( Q \) by composing the projection \( \pi : \hat{H} \to N \) with \( \lambda \). A section \( \varsigma : Q \to \hat{H} \) factors through a lift \( \sigma : Q \to N \) by a section \( s : N \to \hat{\mathcal{H}} \) such that \( s^* v_1 \) and \( s^* v_2 \) are constant along the fibers of \( \lambda \). By \((4.9)\), \( s^* \omega^3 \) annihilates \( L_v^\perp \), whence \( \sigma^* s^* \omega^3 \) defines a contact form which annihilates \( D \). As \( D \subset TQ \) is a rank-4 contact distribution, there cannot be a 3-dimensional manifold tangent to \( D \), which is to say that \( \varsigma^* \omega^3 \) does not vanish identically on \( TM \). Together, \((4.1), (4.2)\), and \((4.11)\) show that \(-i\varsigma^* \omega^3\) evaluates as \( h \) on the subbundle \( L^\perp \subset TN \) (up to a sign when \( \epsilon = -1 \)), and since the action of \( h|_{L^\perp} \) is the same along the fibers of \( \lambda \), it descends to \( D \). Therefore, up to a nonzero scalar determined by \( \sigma \), \( h|_D \) coincides with the Levi form \(-\varsigma^* \omega^3|_D \) of \( Q \). Our consideration of the Levi form of \( Q \) is strictly limited to its degeneracy properties, so we will refer to \( h|_D \) and \(-\varsigma^* \omega^3|_D \) collectively as \( \ell \) for the sake of adapting frames to \( M \).

Let \( \mathcal{H}^{11} \subset \hat{\mathcal{H}} \) denote the subbundle of Hermitian frames \( \mathcal{F} \) with \( v_0 \in \hat{M} \). The superscript indicates the real dimension of the fibers as a bundle over \( M \); in particular, as a bundle over \( \hat{M} \) the fibers possess the full \( \hat{G} \) structure \((4.2)\). When \( \epsilon = -1 \), it is possible that \( \ell|_{D_M} = 0 \). In this case, we could restrict to those Hermitian frames with \( L_M = \langle v_0, v_1 + v_2 \rangle_C \), which corresponds to a reduction of the structure group \((4.2)\) to matrices with \( a + e^i \vec{b} = b + e^i \vec{a} \). The fact that \( \varsigma^* \omega^3 \) does not vanish identically on \( TM \) would then (generically) imply \( T\hat{M} = \langle v_0, v_1 + v_2, iv_0, i(v_1 + v_2), v_3 \rangle_R \), which shows that \( D_M \) is necessarily integrable; i.e., \( \lambda \) is Levi-flat. It remains to consider \( M \) such that \( \ell|_{D_M} \neq 0 \), and in this case we can restrict to those frames \( \mathcal{H}^0 \subset \mathcal{H}^{11} \) with \( L_M = \langle v_0, v_1 \rangle_C \), an adaptation which reduces the structure group \((4.2)\) to matrices with \( b = 0 \). Once again, \( \varsigma^* \omega^3 \) is nonvanishing on \( TM \), so further reduction to frames \( \mathcal{H}^7 \subset \mathcal{H}^0 \) with

\[
T\hat{M} = \langle v_0, v_1, iv_0, iv_1, v_3 \rangle_R
\]

requires \( c_2 = 0 \) in \((4.2)\). Moreover, the identity map \((4.8)\) on \( T\hat{M} \) reveals that on \( \mathcal{H}^7 \) we have

\[
\omega^2 = 0,
\]

where we have suppressed the pullback along the inclusion \( \mathcal{H}^7 \hookrightarrow \hat{\mathcal{H}} \) and kept the same name for the MC form (an abuse of notation we will continue to commit).
Pulling back the MC forms (4.14) to $H^7$, we will continue to adapt frames to the geometry of $M$ to progressively higher order by examining the differential consequences of the condition (4.14) for the equations (4.15). As a final, obvious adaptation, however, we observe that the projection (4.10) does not alter (4.14), and we can reduce to oriented Hermitian frames $H^6 \subset H^7$, over which we have (4.17). As a result, the coframing we construct will encode invariants of $M$ under the CR symmetry group $SU(3 - \delta_e, 1 + \delta_e)$ of $Q$.

4.3. Higher-Order Adaptations. By (4.14), we also have $d\omega^2 = 0$ over $H^6$, so we apply Cartan’s lemma to the MC equation (4.15) for $d\omega^2$ to conclude

\[
\begin{bmatrix}
\phi_2^2 \\
\phi_3^2
\end{bmatrix} = \begin{bmatrix}
z_1 & z_0 \\
-1z_0 & Z
\end{bmatrix} \begin{bmatrix}
\omega^1 \\
\omega^3
\end{bmatrix} ;
\]
\[
z_0, z_1, Z \in C^\infty (H^6, \mathbb{C}).
\]

Use the MC equations (4.15) again to differentiate (4.15),

\[
dz_0 = 2z_0\phi_1^1 + iz_1\phi_1^3 - 2z_0\bar{\omega}^0 - Z\bar{\omega}^1 + b_0\omega^1 + p\omega^3,
\]
\[
dz_1 = 3z_1\phi_1^1 - z_1\bar{\omega}^0 + 2iz_0\bar{\omega}^1 + q\omega^1 + b_0\omega^3,
\]
\[
dZ = Z\phi_1^1 + 2z_0\phi_3^1 - 3Z\bar{\omega}^0 - ip\omega^1 + b_0\omega^3,
\]

for some $b_0, b, p, q \in C^\infty (H^6, \mathbb{C})$. We update the remaining MC equations,

\[
d\omega^0 = -\phi_3^1 \wedge \omega^1 - \psi \wedge \omega^3,
\]
\[
d\omega^1 = (\omega^0 - \phi_1^1) \wedge \omega^1 - i\phi_3^1 \wedge \omega^3,
\]
\[
d\omega^3 = i\omega^1 \wedge \bar{\omega}^1 + (\omega^0 + \bar{\omega}^0) \wedge \omega^3,
\]
\[
d\phi_1^1 = \phi_3^1 \wedge \omega^1 + \phi_1^3 \wedge \bar{\omega}^1 - c|z_1|^2 \omega^1 \wedge \bar{\omega}^1 + \epsilon\omega^3 \wedge (\bar{z}_0z_1\omega^1 - z_0\bar{z}_1\bar{\omega}^1),
\]
\[
d\phi_3^1 = -i\psi \wedge \bar{\omega}^1 + \phi_1^3 \wedge (\omega^0 - \phi_1^1) - i\bar{z}_0z_1\omega^1 \wedge \bar{\omega}^1 + \epsilon\omega^3 \wedge (\bar{z}_0\omega^1 - i|z_0|^2\bar{\omega}^1),
\]
\[
d\psi = \psi \wedge (\omega^0 + \bar{\omega}^0) + i\phi_3^1 \wedge \phi_1^3 - c|z_0|^2 \omega^1 \wedge \bar{\omega}^1 + \epsilon\omega^3 \wedge (\bar{z}_0\omega^1 + Z\bar{z}_0\bar{\omega}^1).
\]

The identities (4.16) show that if any of $z_0, z_1, Z$ vanishes identically on $H^6$, then they all vanish identically. We name this case and flag its structure equations, that they may be examined in (5.2).

\[
\begin{align*}
&d\omega^0 = -\phi_3^1 \wedge \omega^1 - \psi \wedge \omega^3, \\
&d\omega^1 = (\omega^0 - \phi_1^1) \wedge \omega^1 - i\phi_3^1 \wedge \omega^3, \\
&d\omega^3 = i\omega^1 \wedge \bar{\omega}^1 + (\omega^0 + \bar{\omega}^0) \wedge \omega^3, \\
&d\phi_1^1 = \phi_3^1 \wedge \omega^1 + \phi_1^3 \wedge \bar{\omega}^1, \\
&d\phi_3^1 = -i\psi \wedge \bar{\omega}^1 + \phi_1^3 \wedge (\omega^0 - \phi_1^1), \\
&d\psi = \psi \wedge (\omega^0 + \bar{\omega}^0) + i\phi_3^1 \wedge \phi_1^3.
\end{align*}
\]

(Flat Case 1)

To proceed, we implicitly work on the open subset of $H^6$ where $z_1 \neq 0$. The equation (4.16) for $dz_1$ implies that each of the fibers of $H^6 \to M$ (not $\tilde{M}$!) contains a frame $\nu$ such that $z_1(\nu) = 1$. We keep the same names of the functions and forms when we reduce to the subbundle $H^4 \subset H^6$ of such frames, over which

\[
\bar{\omega}^0 = 3\phi_1^1 + 2iz_0\bar{\omega}^1 + q\omega^1 + b_0\omega^3.
\]

After separating $b_0$ into its real and imaginary parts

\[
b_0 = r_0 + ir; \quad r_0, r \in C^\infty (H^4),
\]

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we differentiate \([4.18]\) and compare to the pullback of \(d\omega^3\) from \(\mathcal{H}^6\) to compute

\[
dq = 4q\phi_1^3 - 3\phi_1^3 - (3\xi + 3r + 2iqz_0 - 3ir_0 + |q|^2)\omega^1 + c\omega^1 + z\omega^3, \\
dr = -\psi + (\frac{1}{2}\tau + \tau_0)\phi_1^3 + \left(\frac{1}{2}\tau - z_0\right)\phi_1^3 - \left(\frac{1}{2}r_0 - \tau_0(3ir_0 + r) + i\tau + \frac{1}{2}(iqr - z)\right)\omega^1 \\
- \left(\frac{1}{2}r_0\tau + z_0(3ir_0 - r) - ip - \frac{1}{2}(i\tau + \tau)\omega^1 + s\omega^3, \\
dr = (\tau_0 + \frac{1}{2}q)\phi_1^3 + \left(\tau_0 - \frac{1}{2}\tau\right)\phi_1^3 - \left(\tau_0(6\xi - 6r - 2ir_0) + r_0q + 2i\tau - 3iqr + z\right)\omega^1 \\
+ \frac{1}{2}(z_0(6\xi - 6r + 2ir_0) + r_0\tau - 3iqr - 2i\tau + 3iqr + \tau)\omega^1 + tw^3,
\]

(4.19)

for some \(c, z \in C^\infty(\mathcal{H}^4, \mathbb{C})\) and \(s, t \in C^\infty(\mathcal{H}^4)\).

The formula \([4.16]\) for \(dz_0\) shows that the condition \(z_0 = 0\) defines a subbundle \(\mathcal{H}^2 \subset \mathcal{H}^4\), over which

\[
\phi_1^3 = i\left(-Z\omega^1 + (r_0 + ir)\omega^1 + p\omega^3\right).
\]

Differentiating \([4.20]\) and subtracting the pullback of the MC equation \(d\phi_1^3\) yields

\[
dp = -4p\phi_1^3 - ((\mathcal{Z})^2 - r_0^2 + 3qp - r^2 - s - it)\omega^1 + (3r_0 Z + i\mathcal{Z}(r - s) - \tau p - b)\omega^1 + w\omega^3,
\]

for some \(w \in C^\infty(\mathcal{H}^2, \mathbb{C})\). The identity \(0 \equiv d^2q \pmod{\omega^1}\) additionally provides

\[
dz = 4z\phi_1^3 + i(3r_0^2 - 2imz + iq\mathcal{Z} - i\mathcal{Z}(Z - 2|q|^2 r + 3it - 6\xi r - 2qp - 9r^2 + 3s)\omega^1 + mw^1 + nw^3,
\]

for some \(m, n \in C^\infty(\mathcal{H}^2, \mathbb{C})\).

As before, the equation \([4.19]\) for \(dr_0\) suggests that we reduce to the subbundle \(\mathcal{H}^1 \subset \mathcal{H}^2\) where \(r_0 = 0\) and we have

\[
\psi = \left(\frac{1}{2}(z + \mathcal{Z}) - i(qr + \mathcal{P})\right)\omega^1 + \left(\frac{1}{2}(\mathcal{Z} + \mathcal{Z}) + i(qr + p)\right)\omega^1 + (s - \frac{1}{2}(qpr + \mathcal{P}))\omega^3.
\]

Differentiating \([4.23]\) and comparing to the pullback of \(d\psi\) from \(\mathcal{H}^2\) gives

\[
ds = s_1\omega^1 + \tau_1\omega^1 + lw^3;
\]

(4.24) 
\(-s_1 = 7\tau r - Z\tau - 2i\tau - 3i\mathcal{Z}p - i\tau Z r + i\omega + \frac{1}{2}(3qs + |q|^2\tau + iq + 3i\tau - 2i\tau Z - pc + 3qr^2 - n),
\]

\(l \in C^\infty(\mathcal{H}^1)\).

Let us update the MC equations \([4.17]\) on \(\mathcal{H}^1\),

\[
d\omega^1 = -4\phi_1^1 \wedge \omega^1 - \tau \omega^1 \wedge \omega^1 + 2ir\omega^1 \wedge \omega^3 - Z\omega^1 \wedge \omega^3,
\]

\[
d\omega^3 = i\omega^1 \wedge \omega^1 + (q\omega^1 + \tau\omega^1) \wedge \omega^3,
\]

\[
d\phi_1^1 = -(2r + \xi)\omega^1 \wedge \omega^3 + \tau \omega^3 \wedge (\tau \omega^1 + p\omega^1).
\]

(4.25)

We also update and collect the differential identities \([4.10], [4.11], [4.12], [4.20]\), and \([4.24]\), leaving \([4.23]\) where it is,

\[
dZ = -8Z\phi_1^1 - (ip + 3Zq)\omega^1 + (b - 3iZr)\omega^3, \\
dq = 4q\phi_1^1 + (c + 3iZ)\omega^1 - (3\xi + 6r + |q|^2)\omega^1 + (z - 3i\tau)\omega^3, \\
dr = (\tau - 2q + \frac{1}{2}(\mathcal{Z} - z)\omega^1 + (p + 2\tau r - \frac{1}{2}(qZ - \tau))\omega^1 + (t + \frac{1}{2}(qpr - \tau))\omega^3, \\
dp = -4p\phi_1^1 + (r^2 + s + it - |Z|^2 - 3qp)\omega^1 - (b + i\tau (r - s) + \tau p)\omega^1 + w\omega^3, \\
dz = 4z\phi_1^1 + i(2i\tau z + iq\tau - i\mathcal{Z} - 2|q|^2 r + 3it - 6\xi r - 2qp - 9r^2 + 3s)\omega^1 + mw^1 + nw^3.
\]

(4.26)

At this point we must consider branching based on the possible values of \(Z\).
4.4. The Case $Z = 0$ on $H^1$. If $Z$ is identically zero on $H^1$ the equation (4.20) for $dZ$ shows $b = p = 0$, after which the identity $dp$ implies $u = t = 0$ (since $t$ is real), as well as $s = -r^2$. Differentiating this by way of the identities (4.21) and (4.26) subsequently yields $l = 0$ and $n = 6izr + 8qg^2$. In summary, we have structure equations

$$
d\omega^1 = -4\phi_1^1 \wedge \omega^1 - \overline{\gamma} \omega^1 \wedge \overline{\omega}^1 + 2i\nu \omega^1 \wedge \omega^3,\
d\omega^3 = i\omega^1 \wedge \overline{\nu}^1 + (q\omega^1 + \overline{\gamma} \omega^3) \wedge \omega^3,\
d\phi_1^1 = -(2r + c)\omega^1 \wedge \overline{\omega}^1,
$$

and identities

$$
dq = 4q\phi_1^1 + c\omega^1 - (3\epsilon + 6r + |q|^2)\overline{\nu}^1 + z\omega^3,\
dr = -(2qr + \frac{3}{2}z)\omega^1 - (2q\overline{r} - \frac{3}{2}x)\overline{\nu}^1,\
dz = 4z\phi_1^1 + i(2iz\overline{r} + iz\overline{c} - 2|q|^2r - 6cr - 12r^2)\overline{\nu}^1 + m\omega^1 + (6izr + 8qg^2)\omega^3.
$$

Evidently, if $q$ also vanishes identically on $H^1$, the rest of the functions vanish as well, with the exception of $r = -\frac{2}{z}$. In this case, which we study further in [5,3], the structure equations become

(Flat Case 2)

$$
d\omega^1 = -4\phi_1^1 \wedge \omega^1 - i\epsilon \omega^1 \wedge \omega^3,\
d\omega^3 = i\omega^1 \wedge \overline{\nu}^1,\
d\phi_1^1 = 0.
$$

It remains (in this subsection) to consider the case $Z = 0$, $q \neq 0$. We first invoke $d^2q = 0$ to calculate

$$
dc = 8c\phi_1^1 + a\omega^1 + (4iz - 2\overline{c} - 4q + 4qr)\overline{\nu}^1 + (qz + 2irc + m)\omega^3,\
$$

for some $a \in C^\infty(H^1, \mathbb{C})$. Next we split a few functions into their real and imaginary parts,

$$
q = u + iu_0, \\
z = x + iy, \\
u, u_0, x, y \in C^\infty(H^1),
$$

so that in particular,

$$
du_0 = -4iu\phi_1^1 - \frac{1}{2}(u^2 + u_0^2 + 3\epsilon + 6r + c)\omega^1 - \frac{1}{2}(u^2 + u_0^2 + 3\epsilon + 6r + c)\overline{\nu}^1 + y\omega^3,\
du = 4iu_0\phi_1^1 - \frac{1}{2}(u^2 + u_0^2 + 3\epsilon + 6r - c)\omega^1 - \frac{1}{2}(u^2 + u_0^2 + 3\epsilon + 6r - c)\overline{\nu}^1 + x\overline{\nu}^1,\
$$

which together imply that we can reduce to the subbundle $H^0 \subset H^1$ where $u_0 = 0$, $u \neq 0$, and

$$
\phi_1^1 = -\frac{1}{8u}(u^2 + 3\epsilon + 6r + c)\omega^1 + \frac{1}{8u}(u^2 + 3\epsilon + 6r + c)\overline{\nu}^1 - \frac{1}{4u}y\omega^3.
$$

With (4.31) in hand we can update the structure equations (4.27), and compile their final versions along with the identities (4.22), (4.27), (4.30), to be revisited in [5,4]

Curved Cases 1,2)

$$
d\omega^1 = \frac{1}{2\nu}(\overline{\nu} - u^2 + 6r + 3\epsilon)\omega^1 \wedge \omega^3 + \frac{1}{u}(2ur - y)\omega^1 \wedge \omega^3,\
d\omega^3 = i\omega^1 \wedge \overline{\nu}^1 + u(\omega^1 \wedge \overline{\nu}^1) \wedge \omega^3,\
d\omega^3 = \frac{1}{2}(u^2 + 3\epsilon - c + 6r)\omega^1 - \frac{1}{2}(u^2 + 3\epsilon - c + 6r)\overline{\nu}^1 + x\overline{\nu}^1,\
d\omega^3 = \frac{1}{2}(u^2 + 3\epsilon - c + 6r)\omega^1 - \frac{1}{2}(u^2 + 3\epsilon - c + 6r)\overline{\nu}^1 + x\overline{\nu}^1,\
dc = \frac{1}{2}(2i(cru - y) + u^2(x + iy) + mu)\omega^3 - \frac{1}{2}(c(u^2 + 3\epsilon + 6r + c) - au)\omega^1 + \frac{1}{4u}(4u(ix - y + ur - c) + |c|^2 + c(3\epsilon + 6r - u^2))\overline{\nu}^1,\
dx = \frac{1}{2}(8u^2 - 6ury + y^2)\omega^3 - \frac{1}{2}(3u(m - 3ux) - 2ru(3\epsilon + 6r + u^2) + y(3\epsilon + 6r + c))\omega^3 + \frac{1}{2}(3u(m - 3ux) - 2ru(3\epsilon + 6r + u^2) + y(3\epsilon + 6r + c))\omega^3 + \frac{1}{2}(2ru(3\epsilon + 6r + u^2) + u(u(2x + iy) + m)x(3\epsilon + 6r + c))\omega^1 + \frac{1}{2}(2ru(3\epsilon + 6r + u^2) + u(u(2x + iy) + m)x(3\epsilon + 6r + c))\omega^1.
4.5. The Case \( Z \neq 0 \) on \( H^1 \). Suppose \( Z \neq 0 \) and split up

\[
Z = u_0 + iu, \quad b = x + iy, \quad u_0, u, x, y \in C^\infty(H^1),
\]

which allows us to rewrite \( dZ \) from (4.26),

\[
du_0 = -8iue_1 - \frac{1}{2}(3q(u - iu_0) + p)e_1 + \frac{1}{2}(3\overline{q}(u + iu_0) + \overline{p})\overline{e}_1 + (3ru + x)e_3, \\
du = 8iue_0\phi_1^e - \frac{1}{2}(3q(u - iu_0) + p)e_1 - \frac{1}{2}(3\overline{q}(u + iu_0) + \overline{p})\overline{e}_1 + (y - 3ru_0)e_3,
\]

and these show that we can reduce to a subbundle \( H^0 \subset H^1 \) where \( u_0 = 0, u \neq 0 \) and

\[
\phi_1^e = -\frac{1}{16u}(3uq + p)e_1 + \frac{1}{16u}(3u\overline{q} + \overline{p})\overline{e}_1 - \frac{3}{8u}(3ru + x)e_3.
\]

Differentiating (4.33) and comparing to the pullback of \( dH^1 \) from \( H^1 \) will provide an expression for \( dx \) (up to an unknown \( \phi \in C^\infty(H^0) \)), which we include as we summarize the final form of the equations (4.25) and identities (4.24), (4.26), (4.32),

\[
d\omega^1 = \frac{1}{4u}(\overline{p} - u\overline{q})e_1 \wedge \overline{e}_1 + \frac{1}{2u}(ur - x)e_1 \wedge \omega^3 - iue_1 \wedge \omega^3, \\
d\omega^3 = i\omega^1 \wedge \overline{e}_1 + (q\omega^1 + \overline{q}\omega^1) \wedge \omega^3,
\]

\[
du = -\frac{1}{2}(3uq + p)e_1 - \frac{1}{2}(3u\overline{q} + \overline{p})\overline{e}_1 + y\omega^3, \\
dp = \frac{1}{4u}(3ipru + ipx + 2wu)\omega^3 + \frac{1}{4u}(4itu - 9qpu + 4r^2u - 4u^3 + p^2 + 4su)\omega^1 \\
- \frac{1}{16u}(4ipru + 7\overline{q}pu - 4e^2u + 4ru^2 + |p|^2 + 4xu)\overline{e}_1, \\
dq = \frac{1}{4u}(3q^2u + pq - 4cu - 12u^2)e_1 - \frac{1}{16u}(3qu^2 - pq + 12cu + 24ru)\overline{e}_1, \\
dz = -\frac{1}{2u}(3izru + izx - 2nu)e^3 - \frac{1}{4u}(3qzu + pz - 4mu)e^1, \\
dr = (\overline{p} - 2qr + \frac{1}{2}(u\overline{q} - u\overline{c})e_1 + (p - 2\overline{q}r + \frac{1}{2}(uq + u\overline{c}))\overline{e}_1 + (t + \frac{1}{2}(pq - \overline{p}q))\omega^3, \\
dx = -\frac{1}{2u}(iu(2pu + pr - 5qx) - py + wu)e_1 - \frac{1}{2u}(iu(2pu + pr - 5qx) + py - wu)\overline{e}_1 + \alpha\omega^3, \\
ds = \text{as in (4.24)} \text{ with } Z = \bar{i}u.
\]

It will be useful to subdivide case (4.34) into subcases where \( q = 0 \) (4.5.3) and \( q \neq 0 \) (4.5.4). When \( q \) vanishes identically the equation \( 0 = dq \) implies \( c = -3u, r = -\frac{r}{2} \) and \( z = 3\overline{u} \), after which \( dr = 0 \) shows \( p = t = 0 \). In particular, \( z = 0 \Rightarrow m = n = 0 \) as well as \( s = u^2 - \frac{1}{4} \). At this point, \( dp = 0 \) reveals \( w = y = 0 \) and \( x = \frac{4uq}{p} \). Differentiating \( s \) and \( x \) gives the final vanishing conditions \( l = o = 0 \), and the equations of (4.34) reduce in this subcase,

\[
d\omega^1 = -e\omega^1 \wedge \omega^3 - iu\omega^1 \wedge \omega^3, \\
d\omega^3 = i\omega^1 \wedge \overline{e}_1, \\
du = 0.
\]

We also recognize

(Flat Case 3) (4.5.4) with \( q \neq 0 \).

5. Homogeneous Embeddings

5.1. Moving Frames and CR Coframings. In section 4 we reduced the bundle \( H \) of Hermitian frames over the hyperquadric \( Q \) as much as possible over the generic points of any embedded, Levi-nondegnerate
CR 3-fold $M \subset Q$, arriving to $\mathcal{H}^m \rightarrow M$ for one of $m = 0, 1, 6$. Pulling back $\omega^1 \in \Omega^1(\mathcal{H}^m, \mathbb{C})$ and $\omega^3 \in \Omega^1(\mathcal{H}^m)$ along any section $\varsigma : M \rightarrow \mathcal{H}^m$ yields a 1-adapted CR coframing

$$\kappa = \varsigma^* \omega^3, \quad \eta = \varsigma^* \omega^1.$$ 

By construction, any ambient CR symmetry

$$A : Q \rightarrow Q, \quad A \in SU(3 - \delta_\epsilon, 1 + \delta_\epsilon)$$

leaves invariant the structure equations of the coframing on $\mathcal{H}^m$. In particular, if $M$ is homogeneous under the action of a subgroup of $SU(3 - \delta_\epsilon, 1 + \delta_\epsilon)$, these structure equations have constant coefficients. By imposing this condition on the equations, we calculate the possible values of these constants, thereby determining which homogeneous $M$ exist in $Q$.

### 5.2. Flat Case 1

The bundle $\mathcal{H}^6$ is 9-dimensional, and it has the structure equations of the Lie group $U(2, 1)$. In particular, using a section $\varsigma : M \rightarrow \mathcal{H}^6$ to pullback MC forms, we name

$$\alpha = \varsigma^*(-\omega^0 + \phi^1_1), \quad \beta = \varsigma^*(i\phi^1_1), \quad \sigma = \varsigma^*(-\psi),$$

which exactly satisfy the equations (2.14) of a 3-adapted coframing on $M$ (see Remark 2.2) for $b = p = 0$. Thus, $M \subset Q$ is a flat CR 3-sphere.

### 5.3. Flat Case 2

Name $\phi = \varsigma^* \phi^1_1$ for $\varsigma : M \rightarrow \mathcal{H}^4$ so that we have structure equations

$$d\kappa = i\eta \wedge \overline{\eta},$$

$$d\eta = -4\phi \wedge \eta - i\epsilon \eta \wedge \kappa,$$

$$d\phi = 0.$$

Setting

$$\alpha = -\epsilon \frac{3}{4} \kappa + 4\phi, \quad \beta = \epsilon \frac{1}{4} \eta, \quad \sigma = \epsilon \frac{1}{16} \kappa,$$

will recover equations (2.14) with $b = p = 0$ once again. Hence, $M \subset Q$ is a flat CR 3-sphere which is nonetheless inequivalent to that in Flat Case 1 under $SU(3 - \delta_\epsilon, 1 + \delta_\epsilon)$-symmetries of $Q$.

### 5.4. Curved Cases 1, 2

After pulling back forms from $\mathcal{H}^0$ we have structure equations

$$d\kappa = i\eta \wedge \overline{\eta} + u(\eta + \overline{\eta}) \wedge \kappa,$$

$$d\eta = \frac{1}{3u}(\overline{\eta} - u^2 + 6r + 3\epsilon)\eta \wedge \overline{\eta} + \frac{1}{2}(2ur - y)\eta \wedge \kappa,$$

with $c \in C^\infty(M, \mathbb{C})$ and $u, r, y \in C^\infty(M)$ where $u$ is nonvanishing, and identities

$$du = -\frac{1}{2}(u^2 + 3\epsilon - c + 6r)\eta - \frac{1}{2}(u^2 + 3\epsilon - \overline{\eta} + 6r)\overline{\eta} + x\kappa,$$

$$dr = -(2ur - \frac{1}{2}(y - ix))\eta - (2ur - \frac{1}{2}(y + ix))\overline{\eta},$$

$$dc = \frac{1}{u}(2ic(ru - y) + u^2(x + iy) + mu)\kappa - \frac{1}{u}(c(u^2 + 3\epsilon + 6r + c) + au)\eta$$

$$+ \frac{1}{u}(4u(ix - y + u(r - \epsilon)) + |c|^2 + c(3\epsilon + 6r - u^2))\overline{\eta},$$

$$dz = \frac{1}{u}((8r^2u^2 - 6ruy + y^2)\kappa - \frac{1}{2u}(1u(m - 3ux) - 2ru(3\epsilon + 6r + u^2) + y(3\epsilon + 6r + c))\eta$$

$$+ \frac{1}{u}(-iu(m - 3ux) - 2ru(3\epsilon + 6r + u^2) + y(3\epsilon + 6r + \overline{\eta}))\overline{\eta},$$

$$dy = \frac{1}{u}x(6ru - y)\kappa + \frac{1}{2u}(2iru(3\epsilon + 6r + u^2) + u(iy - 2x - m) + x(3\epsilon + 6r + c))\eta$$

$$+ \frac{1}{2u}(2iru(3\epsilon + 6r + u^2) + u(2x + iy) + m) - x(3\epsilon + 6r + \overline{\eta})\overline{\eta}.$$

Beginning with the equation $d\kappa$, we assume $u$ is constant so that $0 = du$ is satisfied if

$$x = 0, \quad c = u^2 + 3\epsilon + 6r.$$
Now we have
\[
d\eta = \frac{1}{3}(2r + \epsilon)\eta \wedge 7 + \frac{1}{u}(2ur - y)\eta \wedge \kappa,
\]
so for this to have constant coefficients requires \(dr = dy = 0\). In particular, from \(0 = dr\) we find
\[
y = 4ur.
\]
By (5.1) and the identities for \(dx\) and \(dc\) we see
\[
m = 2ir(u^2 + 9\epsilon + 18r), \quad a = \frac{2}{u}(u^4 + 6eu^2 + 12ru^2 + 9 + 36r + 36r^2).
\]
Also by \(dc\),
\[
eu^2 + 9 + 36\epsilon r + 36r^2 = 0,
\]
while \(dy\) and (5.2) provide
\[
(5.4) \quad r(-u^2 + 3\epsilon + 6r) = 0.
\]
Solutions of (5.4) are
\[
r = 0, \quad r = \frac{1}{6}u^2 - \frac{1}{3}\epsilon,
\]
and plugging these into (5.3) yields
\[
(5.5) \quad 9 + eu^2 = 0, \quad u^2(u^2 + \epsilon) = 0.
\]
Solutions satisfy \(0 \neq u \in \mathbb{R}\) only when \(\epsilon = -1\). In this case structure equations are
\[
(5.6) \quad dr = \mp \eta \wedge 7, \quad a = \mp \frac{3}{4}\kappa \wedge \eta.
\]
when \(r = 0\) and otherwise
\[
(5.7) \quad dr = \mp \eta \wedge 7 + \frac{3i}{4}\kappa \wedge \eta.
\]
Starting from (5.5), if we apply the 1-adapted transformation
\[
[ \begin{array}{cc} 4 & 0 \\ -6i & \pm 2 \end{array} ] \begin{bmatrix} \kappa \\ \eta \end{bmatrix}
\]
(where the sign of \(\pm 2\) is determined by that of (5.5)), our new 6-adapted coframing has structure equations of the form (2.19) with \((z_1, z_2, \delta) = (-\frac{3}{4}, -\frac{3}{4}, 1)\), coinciding with (VI, E) for \(t = \frac{1}{4}\). Transforming the coframing in (5.6) according to
\[
[ \begin{array}{cc} 1 & 0 \\ 1 & \pm i \end{array} ] \begin{bmatrix} \kappa \\ \eta \end{bmatrix},
\]
we arrive at a 6-adapted coframing whose structure equations are (2.19) for \((z_1, z_2, \delta) = (\frac{1}{4}, i, 0)\), which is (VIII, K) for \(a = \sqrt{t}\) and \(t = \frac{1}{4}\).

5.5. **Curved Cases 3** Here the structure equations are already fully reduced,
\[
(5.8) \quad dr = i\kappa \wedge (\epsilon\eta + u\eta), \quad u \in \mathbb{R} \setminus \{0\}.
\]
First suppose \(\epsilon = 1\). Every (IX, L) model appears twice for \(|u| < 1\); when \(0 < u < 1\), take \(a = \sqrt{1 + 2}\) and \(t = \frac{2u}{1-u}\), and when \(-1 < u < 0\) take \(a = i\sqrt{1 + 2}\) and \(t = \frac{2u}{1-u}\). On the other hand, \(|u| > 1\) produces some (VIII, K) models twice; when \(u > 1\), take \(a = \sqrt{1 - 2}\) and \(t = \frac{2u}{u-1}\), and when \(u < -1\) take \(a = i\sqrt{1 - 2}\) and \(t = \frac{2u}{u+1}\). Either way, \(t > 2\).

Now consider \(\epsilon = -1\), where we recover the remaining (VIII, K) models twice. For \(u > 0\), take \(a = \sqrt{2-t}\) and \(t = \frac{2u}{u+1}\) (omitting \(u = 1 = t\)), and for \(u < 0\), take \(a = i\sqrt{2-t}\) and \(t = \frac{2u}{u-1}\) (omitting \(u = -1 = -t\)). Either way, \(0 < t < 2\).
5.6. Flat Case 3 Structure equations are
\[
dx = i\eta \wedge \overline{\eta} + (q\eta + \overline{q}\eta) \wedge \kappa,
\]
\[
d\eta = \frac{1}{4u}(\overline{q} - u\overline{\eta})\eta \wedge \overline{\eta} + \frac{1}{iu}(ur - x)\eta \wedge \kappa - iu\overline{\eta} \wedge \kappa, \quad 0 \neq u \in \mathbb{R}.
\]
with differential identities
\[
du = -\frac{1}{4}(3uq + p)\eta - \frac{1}{4}(3u\overline{\eta} + \overline{\eta})\eta + y\kappa,
\]
\[
dp = \frac{1}{4u}(3ipru + ipx + 2wu)\kappa + \frac{1}{4u}(4itu - 9qpu + 4r^2u - 4w^2 + p^2 + q^2)\eta
\]
\[- \frac{1}{4u}((4iuy + 7\overline{py}) - 4eu^2 - 4ru^2 + \mid p \mid^2 + 4xu)\overline{\eta},
\]
\[
dq = \frac{1}{4u}(3q^2u + pq - 4cu - 12u^2)\eta - \frac{1}{iu}(u|q|^2 - \overline{pq} + 12eu + 24ru)\overline{\eta}
\]
\[- \frac{1}{2u}(3iqru + 6i\overline{ru} + iqx - 2zu)\kappa,
\]
\[
dz = -\frac{1}{2u}((3izru + izx - 2nu)\kappa - \frac{1}{4u}(3qzu + pz - 4mu)\eta
\]
\[- \frac{1}{4u}(4i(2)|q|^2 + 6er + 2qp - cu + 9r^2 - 3s) + 5\overline{q}zu + 4\overline{rz}u - \overline{pz} + 12tu)\overline{\eta},
\]
\[
dr = (\overline{p} - 2gr + \frac{1}{4}(u\overline{q} - iz))\eta + (p - 2\overline{p}r + \frac{1}{4}(uq + 16z))\overline{\eta} + (t + \frac{1}{4}(q - \overline{qr}))\kappa,
\]
\[
dz = -\frac{1}{2u}(iu(2pu + pr - 5qx) - py + wu)\eta - \frac{1}{2u}(1u(2pu + pr - 5qx) - py - wu)\overline{\eta} + \kappa,
\]
\[
ds = \text{as in (4.24)} \text{ with } Z = iu.
\]
Beginning with the equation for \(d\eta\), \(u\) constant means \(du = 0\), which implies
\[
y = 0, \quad p = -3uq.
\]
Differentiating the latter and comparing to \(dp\) yields
\[
c = -\frac{1}{36}(9uq^2 + it + r^2 + 8u^2 + s), \quad x = -8cu - 19ru, \quad w = 9iqru + 3iux - 27iu^2q - 3zu.
\]
We also require \(q\) to be constant, so we see
\[
(5.8) \quad r = -\frac{1}{4}|q|^2 - \frac{1}{4} \epsilon, \quad z = -9iuq - 4i\epsilon q - 8iqr,
\]
after which the identity for \(dz\) gives
\[
m = -\frac{24q^2}{36u}(1|q|^4 + 6i|q|^2 + 324q^2u + 9i - 36i(u^2 + s) - 36t),
\]
while the identity for \(dr\) shows
\[
t = \frac{3i}{2u}(q^2 - \overline{q}^2), \quad s = -\frac{1}{36}|q|^4 - \frac{3}{4}u(q^2 + \overline{q}^2) - \epsilon \frac{1}{4}|q|^2 + 12|q|^2u - \frac{1}{4} + 18cu - 125u^2,
\]
as well as
\[
(5.9) \quad (\overline{q} - q)(\frac{2}{3}|q|^2 + \epsilon) = 0.
\]
Returning to \(dq = 0\), we find
\[
(5.10) \quad 2|q|^2 + q^2 + 3\epsilon - 21u = 0.
\]
The identities \(dx\) and \(ds\) can be used to show \(o = l\), and the former also shows
\[
(5.11) \quad -\frac{1}{u}(38q^2 q + 2q^2 q + 5i\overline{q}e - 408q\overline{u}u - 9eq) = 0.
\]
In particular, we have reduced all of the invariants to \(u, q, \overline{q}\), subject to the equations (5.9), (5.10), and (5.11).

For either value of \(\epsilon\), \(\overline{q} = q\) is a solution to (5.9). When \(\epsilon = -1\) we should also consider \(|q|^2 = \frac{5}{4}\), but then (5.10) implies \(u = \frac{1}{4}q^2\) and (5.11) reduces to a multiple of \(q^3\), so the only viable solution to (5.9) is \(\overline{q} = q\). When \(q\) is real, (5.10) implies
\[
u = \frac{1}{4}(q^2 + \epsilon),
\]
and (5.11) becomes
\[
\frac{5}{12}q(q^2 + \epsilon)(16q^2 + 9\epsilon).
\]
This case is characterized by $u, q \neq 0$, so the only solutions are

$$q = \pm \frac{3}{4} \sqrt{-\epsilon}, \quad \Rightarrow u = \frac{1}{16} \epsilon.$$ 

Here again, homogeneous embeddings occur only when $\epsilon = -1$, and they have structure equations

$$(5.12) \quad d\kappa = i\eta \wedge \eta \pm \frac{4}{3} \kappa \wedge (\eta + \eta)$$

Setting

$$\alpha = -\frac{33}{32} \kappa \pm \frac{3}{4} \eta, \quad \beta = \pm \frac{1}{64} \kappa - \frac{5}{32} \eta - \frac{i}{16} \eta, \quad \sigma = -\frac{3}{1024} \kappa \pm \frac{1}{128} (\eta - \eta),$$

defines a 3-adapted coframing on $M$ satisfying (2.14) for $b = p = 0$, so we have another flat model.

Moreover, submitting our coframing $\kappa, \eta$ to the 1-adapted transformation

$$\frac{1}{8} \begin{pmatrix} 18 & 0 \\ -9i & \pm 12 \end{pmatrix} \begin{pmatrix} \kappa \\ \eta \end{pmatrix}$$

produces a 6-adapted coframing of the form (2.19) for $(z_1, z_2, \delta) = (i\frac{\eta}{\kappa}, i\frac{\eta}{\kappa}, 1)$, which coincides with that of $\text{VI}_1$ for $t = 9.$

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