SUMMARY OF SPECTRAL INVARIANCE RESULTS

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Abstract. The author’s recent results on spectral invariant dense subalgebras of $C^*$-algebras associated with dynamical systems are summarized. If $G$ is a compactly generated polynomial growth Type $R$ Lie group, and the action of $G$ on $S(M)$ (Schwartz functions on a locally compact $G$-space $M$) is tempered in a certain sense, then there is a natural smooth crossed product $S(G \times M)$ which is dense and spectral invariant in the $C^*$-crossed product $C^*(G \times M)$.

The theory of differential geometry on a $C^*$-algebra (or noncommutative space) Connes [3] requires the use of “differentiable structures” for these noncommutative spaces, or some sort of algebra of “differentiable functions” on the noncommutative space. Such algebras of functions have usually been provided by a dense subalgebra of smooth functions $A$ for which the $K$-theory $K_*(A)$ is the same as the $K$-theory of the $C^*$-algebra $K_*(B)$ (see for example Blackadar-Cuntz [1], and the recent works of J. Bost, G. Elliott, T. Natsume, R. Nest, R. Ji, P. Jolissaint, V. Nistor and many others).

One goal of both of the papers Schweitzer [10] [11] was to construct such dense subalgebras of smooth functions in the case that $B$ is a $C^*$-crossed product $C^*(G \times M)$ associated with a dynamical system, or more specifically with an action of a Lie group $G$ (not necessarily connected) on a locally compact space $M$. In these papers, we realize this goal by constructing smooth crossed products $S(G \times M)$ of Schwartz functions on $G \times M$, which are spectral invariant in the $C^*$-crossed product. Spectral invariant means that the spectrum of every element of $S(G \times M)$ is the same in $S(G \times M)$ and $C^*(G \times M)$. In the
language of Palmer [5], this is the same as saying that \( S(G \times M) \) is a spectral subalgebra of \( C^*(G \times M) \). By Schweitzer [8], Lemma 1.2, Corollary 2.3, and Connes [2], VI.3, spectral invariant subalgebras have the same \( K \)-theory as the \( C^* \)-algebra itself, so these smooth crossed products \( S(G \times M) \) provide us with the “noncommutative differentiable structures” we are looking for.

I will begin by describing the results obtained in [11]. The idea in that paper is to employ the following theorem, which is interesting in its own right. It gives a condition for a dense Fréchet subalgebra \( A \) to be a spectral invariant subalgebra of \( B \) when certain subrepresentations of topologically irreducible representations of \( A \) extend appropriately to \( B \). This is in contrast to the situation in [8], Theorem 1.4, Corollary 1.5, which says that algebraically irreducible representations extend iff the subalgebra is spectral invariant.

If \( E \) is an \( A \)-module, we say that \( E \) is algebraically cyclic iff there exists an \( e \in E \) such that the algebraic span \( Ae \) is equal to \( E \).

**Theorem 1** ([11], Theorem 1.4). Let \( A \) be a dense \( m \)-convex Fréchet subalgebra of a \( C^* \)-algebra \( B \) with continuous inclusion map \( A \hookrightarrow B \). Assume that every algebraically cyclic subrepresentation of every topologically irreducible representation of \( A \) on a Banach space is contained in a \( * \)-representation of \( B \) on a Hilbert space. Then \( A \) is spectral invariant in \( B \).

The smooth subalgebras \( S(G \times M) \) are shown to be \( m \)-convex Fréchet algebras in Schweitzer [9], §3, and their topologically irreducible representations are relatively accessible when the \( C^* \)-crossed product is CCR. Hence Theorem 1 gives many new interesting cases of spectral invariant smooth crossed products \( S(G \times M) \). For example, results are obtained when \( G \) is a closed subgroup of a connected, simply connected nilpotent Lie group with certain restrictions on the isotropy subgroups (that they be CCR for one), and when the action of \( G \) on \( M \) has closed orbits. (See the examples in [11], §18, §2, §16-17.)

A simple illustrative example is given by \( \mathbb{Z} \) acting by translation on \( \mathbb{R} \). The Schwartz
functions $S(\mathbb{Z} \times \mathbb{R})$ with convolution multiplication provide a dense subalgebra of smooth functions of the C*-crossed product $C^*(\mathbb{Z} \times \mathbb{R})$. Any topologically irreducible representation of $S(\mathbb{Z} \times \mathbb{R})$ must factor through an orbit to a representation of the convolution algebra $S(\mathbb{Z} \times \mathbb{Z})$ [11], Theorem 14.1. The latter algebra is a smooth version of the compact operators, whose representation theory is quite nice Du Cloux [4], Corollary 3.5 or [11], Theorem 15.1, Example 2.5. Theorem 1 may then be applied to obtain the spectral invariance of $S(\mathbb{Z} \times \mathbb{R})$ in $C^*(\mathbb{Z} \times \mathbb{R})$.

As one might speculate, when the C*-crossed product is not CCR (or at least when it is not GCR), the representation theory of the dense subalgebra becomes quite complicated as does the representation theory of the C*-algebra. For example, the dense subalgebra $A^\infty_\theta$, given by the canonical action of $T^2$ on the irrational rotation C*-algebra $A_\theta$, is spectral invariant but does not satisfy the hypothesis of Theorem 1. That is, there exists certain “bad” topologically irreducible representations of $A^\infty_\theta$, which have algebraically cyclic subrepresentations which do not extend to $A_\theta$ [11], Example 7.1.

In order to get results in the non-CCR case, a new method is needed to replace Theorem 1. Such a method, or methods, is introduced in [10], which I shall now describe.

We begin by trying to show that the smooth crossed product $S(G, A)$ is spectral invariant in $L^1(G, B)$ instead of in the C*-crossed product $C^*(G, B)$. Let $\| \|_0$ be the norm on $B$, and let $\{\| \|_n\}_{n=0}^\infty$ be a family of increasing submultiplicative norms giving the topology of $A$. In the paper Blackadar-Cuntz [1], the condition $\| ab \|_n \leq C \sum_{i+j=n} \| a \|_i \| b \|_j$ for all $a, b$ in $A$, is used to show that $A$ is a spectral invariant subalgebra of $B$. The commutative Fréchet algebra $S(M)$ of Schwartz functions on $M$ satisfies this condition in $C_0(M)$ [10], §2. Moreover, for some very nice actions of $G$ on $A$ (isometric on each norm), one can show that if the norms on $A$ satisfy the condition in $B$, then the norms on the smooth crossed product $S(G, A)$ satisfy the condition in $L^1(G, B)$. The following more general condition introduced in [10] does the same thing without requiring an isometric action.
We say that $A$ is **strongly spectral invariant** in $B$ if

\[(\exists C > 0)(\forall m)(\exists D_m > 0)(\exists p_m \geq m)(\forall a_1, \ldots, a_n \in A)\]

\[
\left\{\| a_1 \ldots a_n \|_m \leq D_m C^n \sum_{k_1+\ldots+k_n \leq p_m} \| a_1 \|_{k_1} \ldots \| a_n \|_{k_n}\right\}.
\]

Notice that in the summand of (*), at most $p_m$ of the natural numbers $k_j$ are nonzero, regardless of $n$. The idea behind showing that strong spectral invariance implies spectral invariance is given by setting $a_1 = \ldots = a_n = a$ in (*). We have

\[
\| a^n \|_m \leq DC^n \sum_{k_1+\ldots+k_n \leq p_m} \| a \|_{k_1} \ldots \| a \|_{k_n}
\]

\[
\leq K^n \| a \|_0^{-p} \| a \|_0^p
\]

where $p$ is fixed as $n$ runs. It follows that the series $(1 - a)^{-1} = 1 + a + a^2 + \ldots$ converges absolutely in the norm $\| \|_m$ when $\| a \|_0$ is sufficiently small. So $1 - a$ is invertible in the completion of $A$ in $\| \|_m$ when $a$ is sufficiently close to 0 in $B$. The rest of the argument is in Theorem 1.17 of [10].

There are also examples of spectral invariant dense subalgebras which are not strongly spectral invariant [10], Example 1.13. The following theorem and corollary illustrates the usefulness of the concept of strong spectral invariance.

We say that a Lie group $G$ (not necessarily connected) is **compactly generated** if $G$ has an open relatively compact neighborhood $U$ of the identity which satisfies $\bigcup_{n=0}^{\infty} U^n = G$ and $U^{-1} = U$. We call $\tau(g) = \min\{ n \mid g \in U^n \}$ the word gauge on $G$. (The smooth crossed product $S(G, A)$ is then defined to be the set of $G$-differentiable $\tau$-rapidly vanishing functions from $G$ to $A$.) We say that the action of $G$ on $A$ is $\tau$-tempered if for every $m$, $\| \alpha_g(a) \|_m$ is bounded by a polynomial in $\tau(g)$ times $\| a \|_n$ for some $n$. Finally, we say that $G$ is **Type R** if all the eigenvalues of $Ad_g$ lie on the unit circle for each $g \in G$.

**Theorem 2.** If $A$ is strongly spectral invariant in $B$ and $G$ is a compactly generated Type R Lie group, and the action of $G$ on $A$ is $\tau$-tempered, then the smooth crossed product $S(G, A)$ is strongly spectral invariant in $L^1(G, B)$.
Corollary 3. For compactly generated Type R Lie groups, for which the action of $G$ on $S(M)$ is $\tau$-tempered, the smooth crossed product $\mathcal{S}(G \times M)$ is strongly spectral invariant in $L^1(G, C_0(M))$.

(Note that $\mathcal{S}(G \times M)$ is shorthand for $\mathcal{S}(G, S(M))$.) It is the subadditivity of $\tau$ and the strong spectral invariance of $A$ in $B$ that play the essential role in the proof of Theorem 2 and Corollary 3. The hypotheses that $G$ be Type R and that the action is $\tau$-tempered are not used in the proof, but they are necessary to assure the existence of the smooth crossed product $\mathcal{S}(G \times M)$, and to assure that $\mathcal{S}(G \times M)$ is a Fréchet *-algebra. There are a wide variety of Type R Lie groups (see below or [9], §1.4, [10]), and also many examples of $\tau$-tempered actions of such groups $G$ on $S(M)$ [9], §5, [10], Examples 6.26-7, 7.20, [11].

Remark. We clarify what the $\tau$-tempered assumption can mean in practice. Let $G$ be the integers $\mathbb{Z}$, and let $G$ act on $\mathbb{R}$ via $\alpha_n(r) = e^{-nr}$. The word gauge $\tau$ is equivalent in an appropriate sense to the absolute value function $\tau(n) = |n|$. If we take $\mathcal{S}(M) = C_0(\mathbb{R})$, or $\mathcal{S}(M) = C_0^\infty(\mathbb{R})$, then $\alpha$ is an isometric action of $\mathbb{Z}$ on $\mathcal{S}(M)$, meaning that $\alpha$ leaves each seminorm invariant, and so $\alpha$ is $\tau$-tempered. However, if we take $\mathcal{S}(M)$ to be the standard Schwartz functions $\mathcal{S}(\mathbb{R})$, then for a fixed $\varphi \in \mathcal{S}(M)$, $\|\alpha_n(\varphi)\|_m$ will in general grow exponentially in $n$ as $n \to +\infty$, so $\alpha$ is no longer $\tau$-tempered. So the $\tau$-temperedness condition does place a restriction on what $\mathcal{S}(M)$ can be for a given action. If the action of $\mathbb{Z}$ on $\mathbb{R}$ were by translation $\alpha_n(r) = r + n$, then the action of $\mathbb{Z}$ on $\mathcal{S}(\mathbb{R})$ would be $\tau$-tempered. For general $M$ and $G$ as in Corollary 3, and regardless of the action, one can always get a $\tau$-tempered action by taking $\mathcal{S}(M) = C_0(M)$, or $\mathcal{S}(M) = C_0^\infty(M)$, where the superscript $\infty$ means “$G$-differentiable”.

Note that Corollary 3 makes no assumption about the crossed product being CCR. No restrictions on the action of $G$ or the isotropy subgroups are needed. However, we are left with the question of whether $\mathcal{S}(G \times M)$ is spectral invariant in the C*-crossed product $C^*(G \times M)$, and not just $L^1(G, C_0(M))$. To take care of this we generalize a result of Pytlik [7] which says that if $G$ has polynomial growth, then the rapidly vanishing $L^1$-functions on
$G$ form a symmetric Fréchet $*$-algebra, which consequently is spectral invariant in $C^*(G)$.

In particular, we show in §7 of [10] that the rapidly vanishing $L^1$-functions from $G$ to $B$ is spectral invariant in the C*-crossed product $C^*(G, B)$ when $G$ has polynomial growth. Since these rapidly vanishing $L^1$-functions are also spectral invariant in $L^1(G, B)$, and since they contain the smooth crossed product, we are able to conclude that the smooth crossed product is spectral invariant in the C*-crossed product when $G$ has polynomial growth. Our main result is then:

**Corollary 4.** For compactly generated polynomial growth Type R Lie groups $G$, and $\tau$-tempered actions of $G$ on $S(M)$, the smooth crossed product $S(G \times M)$ is spectral invariant in the C*-crossed product $C^*(G \times M)$.

Examples of such groups are given by finitely generated polynomial growth discrete groups, compact or connected nilpotent Lie groups, the group of Euclidean motions on the plane, any motion group, or any closed subgroup of one of these. Numerous examples of smooth crossed products which are spectral invariant because of Corollary 4 can be found in [10], Examples 2.6-7, 6.26-7, 7.20, [11], [9], §5.

We remark that in [6], methods are given to show that $S(G \times M) \hookrightarrow C^*(G \times M)$ is an isomorphism on $K$-theory without using spectral invariance, whenever $G$ is a closed subgroup of a connected, simply connected nilpotent Lie group, and the action of $G$ on $S(M)$ is $\tau$-tempered [6], Example 3.2.

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