Scalar field in the anisotropic universe

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We discuss the primordial spectrum of a massless and minimally coupled scalar field, produced during the initial anisotropic epoch before the onset of inflation. We consider two models of the anisotropic cosmology, the (planar) Kasner de Sitter solution (Bianchi I) and the Taub-NUT de Sitter solution (Bianchi IX), where the 3-space geometry is initially anisotropic, followed by the de Sitter phase due to the presence of a positive cosmological constant. We discuss the behavior of a quantized, massless and minimally coupled scalar field in the anisotropic stage. This scalar field is not the inflaton and hence does not contribute to the background dynamics. We focus on the quantization procedure and evolution in the pre-inflationary anisotropic background. Also, in this paper for simplicity the metric perturbations are not taken into account. The initial condition is set by the requirement that the scalar field is initially in an adiabatic state. Usually, in a quantum harmonic oscillator system, an adiabatic process implies the one where the potential changes slowly enough compared to its size, and the time evolution can be obtained from the zero-th order WKB approximation. In our case, such a vacuum state exists only for limited solutions of the anisotropic Universe, whose spacetime structure is regular in the initial times. In this paper, we call our adiabatic vacuum state the anisotropic vacuum. In the Kasner de Sitter model, for one branch of planar solutions there is an anisotropic vacuum unless \( k_3 \neq 0 \), where \( k_3 \) is the comoving momentum along the third direction, while in the other branch there is no anisotropic vacuum state. In the first branch, for the moderate modes, \( k_3 \sim k \), where \( k \) is the total comoving momentum, the scalar power spectrum has an oscillatory behavior and its direction dependence is suppressed. For the planar modes, \( k_3 \ll k \), in contrast, the direction dependence becomes more important, because of the amplification of the scalar amplitude during this interval of the violation of WKB approximation in the initial anisotropic stage. The qualitative behaviors in the Taub-NUT de Sitter models are very similar to the case of the first branch of the planar Kasner de Sitter model.

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I. INTRODUCTION

It is well-known that our Universe is filled with the thermal microwave radiation field, so-called the cosmic microwave background (CMB). CMB follows the almost idealized Planck distribution of 2.73 K. The intensity of CMB is distributed almost isotropically, but it contains small temperature fluctuations of order \( 10^{-5} \) to the background temperature, which contain much information about the history of the Universe. Recent measurements by WMAP satellite [1, 2] have shown that the observed map of CMB anisotropy is almost consistent with the Gaussian and statistically isotropic primordial fluctuations, which is nothing but the most important predictions of the inflation.

However, after the release of WMAP data, several groups have reported that there seem to be a few anomalies in the CMB temperature map on large angular scales. The most well-known fact is that there seems to be suppression of the power of CMB fluctuations on angular scales more than sixty degrees [3]. There are other observational facts that imply the effect which induces the violation of the rotational invariance. More precisely, probably there are the planarity of lower multipole moments, the alignment between the quadrupole (\( \ell = 2 \)) and the octopole (\( \ell = 3 \)), and the alignment of them with the equinox and the ecliptic plane [4]. There are other observational facts implying the large-scale anisotropy, i.e., odd correlations of \( \ell = 4 \sim 8 \) multipoles with \( \ell = 2, 3 \) multipoles [5], a very large, possibly non-Gaussian cold spot in 10 degree scale [6], asymmetry of angular map measured in north and south hemispheres [7], even though some authors claim that there is no significant evidence for primordial isotropy breaking in five-year WMAP data [8].

Indeed, to explain the origin of the anomalies, various solutions have been suggested, introducing a non-trivial
In this paper, we will revisit the possibility that such large scale anomalies may be the relics of the physics of preinflationary anisotropic Universe. Wald’s no-hair theorem ensures that in the presence of a positive cosmological constant an initially anisotropic Universe exponentially approaches the de Sitter spacetime at the later time under the strong or dominant energy condition. It implies that it is plausible that the Universe is highly anisotropic at the beginning. The cosmological perturbation theory in the pre-inflationary Kasner phase was formulated in Ref. \[21, 22\]. They showed that in general in an expanding (planar) Kasner phase one of two polarizations of gravitational waves is coupled with the scalar mode, but the other gravitational mode is decoupled. Of course, the tensor-scalar coupling vanishes in the isotropic limit. The latter mode can be amplified significantly before the onset of inflation. The instability of the gravitational wave mode may be deeply connected to the unstable mode found by the Belinskii-Khalatnikov-Lifshitz (BKL) analysis \[23\] (see e.g., \[24\] for review). In a contracting Kasner universe such an instability can be identified with the large scale anisotropic mode.

The problem is how to set the initial conditions. In the standard inflation, the initial condition is set inside the Hubble horizon, where the effects of the cosmic expansion can be ignored and the adiabatic condition is satisfied. Therefore, to compare with the prediction from the standard inflation, it is natural to apply similar arguments to the case of the anisotropic Universe. Usually, in a quantum harmonic oscillator system, an adiabatic process implies the one where the potential changes slowly enough compared to its size, and the time evolution can be obtained from the zero-th order WKB approximation. In this paper, we follow this definition for the term adiabatic. In the standard inflationary models, an adiabatic vacuum is also defined in the same way. Note that an isentropic process implies (adiabatic) + (reversible) process and therefore is slightly different from our adiabatic one. We call our adiabatic vacuum state an anisotropic vacuum. In an expanding Kasner solution, there are two branches of solutions with the planar symmetry. Then, the answer to the question on the initial adiabaticity depends on the choice of the branch. In one branch, initially the expansion rate along the planar directions vanishes while that along the third axis is finite. In this branch, the initial spacetime structure can be seen as (a patch of) the Minkowski spacetime, and thus the corresponding anisotropic vacuum has the physical meaning. The tensor-scalar coupling mentioned above vanishes, and therefore the initial dynamics reduces to the three-independent harmonic oscillators. However, in the other branch the tensor-scalar coupling diverges in going back to the initial time. In this branch, there is no adiabatic state. Thus, we mainly focus on the first branch. For a given set of initial conditions, the power spectrum was investigated, resorting to the numerical ways in Ref. \[21, 22\]. One of the important motivations of this paper is to aim more analytic understanding of the primordial power spectrum and in particular its direction-dependence. As the first step, we will consider a free scalar theory, which is the counterpart of the inflaton fluctuation. The other important purpose is to investigate the case of the other Bianchi model and to see how generic the prediction of the Bianchi I model is. In particular, we will consider the analytic solution of the Bianchi IX Universe, i.e., the Taub-NUT de Sitter solution.

The paper is constructed as follows: In Sec. II, the background solutions in the Bianchi I and Bianchi IX models are introduced, say Kasner de Sitter solutions and Taub-NUT de Sitter solutions, respectively. In Sec. III, we investigate the behavior of a massless and minimally coupled scalar field in the background of Kasner de Sitter solution with the planar symmetries. We discuss the existence of a well-defined anisotropic vacuum for each mode. In terms of the validity of the WKB approximation in the very early Universe and determine the initial mode functions. For the modes for which the anisotropic vacuum can be well-defined, we derive the final power spectrum and investigate its direction dependence. In Sec. IV, we repeat the similar discussions in the case of the Taub-NUT de Sitter solutions. In Sec. V, we close the article after giving a brief summary and discussion.

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1 Here, the tensor or scalar modes are defined in the isotropic limit, where perturbations are decomposed into modes on the maximally symmetric 3-space.
II. SET-UP

We are interested in the cosmological solutions of the Einstein gravity with a positive cosmological constant in the absence of the matter field with the action

$$S = M_{Pl}^2 \int d^4x \sqrt{-g} \left( \frac{1}{2} R - \Lambda \right),$$

(1)

where $g_{\mu\nu}$ is the spacetime metric, $\Lambda$ is a (positive) cosmological constant and $M_{Pl}$ is the Planck mass.

A. Kasner-de Sitter spacetime

Assuming the spatial part to be flat and homogeneous, the solution to the Einstein equation is given by

$$ds^2 = -d\tau^2 + e^{2\alpha(\tau)} \left[ e^{2(\beta_+ + \sqrt{3}\beta_-)} (dx^1)^2 + e^{2(\beta_+ - \sqrt{3}\beta_-)} (dx^2)^2 + e^{-4\beta_+} (dx^3)^2 \right],$$

(2)

where

$$\alpha = \ln \left( a_0 \sinh^{1/3} \left( 3H\tau \right) \right),$$

$$\beta_+ = \frac{m}{3} \ln \left( \tanh \left( \frac{3H\tau}{2} \right) \right),$$

$$\beta_- = \frac{1}{3} \sqrt{1 - m^2} \ln \left( \tanh \left( \frac{3H\tau}{2} \right) \right),$$

(3)

and $|m| \leq 1$ and $H := \sqrt{\Lambda/3}$. The solution initially behaves as a Kasner spacetime, followed by a de Sitter phase, as discussed in [21, 22]. At the initial time $\tau = 0$, there is the Big Bang singularity. In the later discussion, we will set $a_0 = 1$, which can always be done by an appropriate rescaling of the spatial coordinates $x^i$ ($i = 1, 2, 3$).

In the later discussions, we will focus on the case with a planar symmetry $m = +1$ or $m = -1$. In the branch of $m = -1$, in the early time limit $\tau \to 0$ the scale factor in the planar direction $e^{\alpha + \beta_+}$ approaches a constant, while that in the $x^3$ direction $e^{\alpha - 2\beta_+}$ linearly depends on $\tau$. Thus, in this branch, there is one dynamical direction in the early time. In the branch of $m = +1$, in the early time limit, $e^{\alpha + \beta_+} \sim \tau^{2/3}$ and $e^{\alpha - 2\beta_+} \sim \tau^{-1/3}$, and thus there are initially expanding planar directions and contracting $x^3$ direction. Thus, both the directions are dynamical.

B. Taub-NUT de Sitter spacetime

A Bianchi IX universe has three Killing vectors $\xi_i$ satisfying $[\xi_i, \xi_j] = \epsilon_{ijk} \xi_k$. The corresponding one form $\chi^i = (- \sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2, \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2, \cos x^1 dx^2 + dx^3)$ constitutes the spatial part of the metric. Therefore, the general spacetime metric of a Bianchi IX universe is given by

$$ds^2 = -n(t)^2 dt^2 + a(t)^2 (\chi^1)^2 + b(t)^2 (\chi^2)^2 + c(t)^2 (\chi^3)^2.$$  

(4)

For the case of a planar symmetry in the sense of Bianchi I model, $a(t) = b(t)$, the metric Eq. (4) becomes

$$ds^2 = -n(t)^2 dt^2 + a(t)^2 \left( (dx^1)^2 + \sin^2 x^1 (dx^2)^2 \right) + c(t)^2 \left( dx^3 + \cos x^1 dx^2 \right)^2.$$  

(5)

As a background solution, the Taub-NUT de Sitter solution is considered

$$n(t)^2 = \frac{t^2 + \ell^2}{4\Delta(t)}, \quad a(t)^2 = \frac{t^2 + \ell^2}{4}, \quad c(t)^2 = \frac{\ell^2 \Delta(t)}{t^2 + \ell^2},$$  

(6)

where

$$\Delta(t) := -t^2 + 2\Lambda t + \ell^2 - \frac{\Lambda}{3} \left( t^4 - 2\ell^2 t^2 - \frac{\ell^4}{3} \right).$$  

(7)

In the case of $\Lambda = 0$, the solution reduces to the Taub-NUT solution. In the late time limit $t \to \infty$, the spacetime approaches de Sitter (dS) with the Hubble expansion rate $H = (\Lambda/3)^{1/2}$. As for the Taub-NUT space, in the region
\[ \Delta(t) = \frac{1}{4\ell^2} \left( t^4 + 2\ell^2 t^2 + 8M\ell^2 t + \ell^4 \right). \]  

(8)

The function \( \Delta(t) \) can be expressed as

\[ \Delta(t) = \frac{1}{4\ell^2} (t - t_0) (t^3 + t_0 t^2 + (2\ell^2 + t_0^2) t + (t_0^3 + 2\ell^2 t_0 + 8M\ell^2)), \]

where \( t_0 \) satisfies

\[ t_0^4 + 2\ell^2 t_0^2 + 8M\ell^2 t_0 + \ell^4 = 0. \]

(9)

(10)

For \(|M| < 2\ell/(3\sqrt{3})\), there is no real root for Eq. (10) and for \(|M| > 2\ell/(3\sqrt{3})\) there are two real roots (For \(M > 2\ell/(2\sqrt{3})\), \( t_0 \) is negative and for \(M < -2\ell/(3\sqrt{3})\), \( t_0 \) is positive). For \(|M| < 2\ell/(3\sqrt{3})\), \( \Delta \) is always positive, and the spacetime geometry approaches de Sitter in the past and future \( t \to \pm \infty \), while it is anisotropic during the intermediate time. In the later discussions, we will focus on the case of \( M \geq 2\ell/(3\sqrt{3}) \), where \( \Delta(T) \) vanishes at two points, represented by \( t = t_{0+}, t_{0-}, \) where \( t_{0+} > t_{0-} \) (\( t_{0+} = t_{0-} \) for \( M = \pm 2\ell/(3\sqrt{3}) \)). We set the null surface \( t = t_{0+} \) as the initial surface (In the later discussions, we omit the subscript “+”). Note that for \( t_0 \ll \ell \),

\[ t_0 \approx -\frac{\ell^2}{8M}. \]

(11)

By introducing the new coordinate \( T := t - t_0 \), we obtain

\[ \Delta(T) = \frac{T}{4\ell^2} (T^3 + 4t_0 T^2 + (2\ell^2 + 6t_0^2) T + 4(t_0^3 + \ell^2 t_0 + 2M\ell^2)) \]

\[ = \frac{T}{4\ell^2} (T^3 + 4t_0 T^2 + (2\ell^2 + 6t_0^2) T + 3t_0^3 + 2\ell^2 t_0 - \frac{\ell^4}{t_0^2}). \]

(12)

In the new coordinate system, the spacetime metric can be written as

\[ ds^2 = -n(T)^2 dT^2 + a(T)^2 \left( (dx^1)^2 + \sin^2 x^1 (dx^2)^2 \right) + c(T)^2 \left( dx^3 + \cos x^1 dx^2 \right)^2. \]

(13)

To compare the time evolutions of two independent scale factors, it is useful to rewrite the metric

\[ ds^2 = -n(T)^2 dT^2 + e^{2\alpha} \left( (dx^1)^2 + \sin^2 x^1 (dx^2)^2 \right) + e^{-4\beta(T)} \left( dx^3 + \cos x^1 dx^2 \right)^2, \]

(14)

where \( \alpha = (1/3) \ln(a^2 c) \) and \( \beta = (1/3) \ln (a/c) \). The dynamics of Bianchi IX Universe and in particular the Taub-NUT de Sitter spacetime can be understood in terms of an analogy with the classical mechanics, as is discussed in Appendix A. See [25] for the other Bianchi IX solutions which are coupled to a scalar field.

### III. SCALAR FIELD IN THE BIANCHI I MODEL

We consider a massless, minimally coupled scalar field propagating on the background spacetime discussed in the previous section with action

\[ S_\phi = \frac{1}{2} \int d^4x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \]

(15)

Note that this scalar field is not the inflaton and hence does not contribute to the background dynamics. We focus on the quantization procedure and the evolution in the anisotropic stage. The effects of a scalar field onto the geometry of the anisotropic Universe were discussed in [26].
A. Equation of motion

Explicitly, the scalar action is written as

\[ S_\phi = \frac{1}{2} \int d\tau d^3x e^{3\alpha} \left( \phi^2_\tau - e^{-2\alpha} h^{ij} \partial_i \phi \partial_j \phi \right). \tag{16} \]

Introducing the Fourier decomposition for the mode function

\[ \phi = \int \frac{d^3k}{(2\pi)^{3/2}} \phi_\gamma e^{ikx}, \tag{17} \]

where \( \gamma = (k_1, k_2, k_3) \) denotes the mutually orthogonal components of the comoving momentum, the action can be reduced to

\[ S_\phi = \frac{1}{2} \int d^3k \eta \left[ \left( \frac{d\chi_\gamma}{d\eta} \right)^2 - \omega^2_\gamma \chi^2_\gamma \right], \tag{18} \]

where \( \chi_\gamma = e^{\alpha} \phi_\gamma \) and \( d\eta = e^{-\alpha} d\tau \), which corresponds to the conformal time in the isotropic case. The frequency squared is given by

\[ \omega_\gamma(t)^2 = -2H^2 e^{2\alpha} + e^{-4\alpha} \left[ H^2 + \Omega^2(t) \right], \tag{19} \]

where \( H = \sqrt{\frac{\Lambda}{3}} \) is the Hubble parameter for a large \( \tau \) and the momentum dependent part is

\[ \Omega^2(t) = \left( 1 + \sqrt{1 + e^{6\alpha}} \right)^{\frac{1}{2}} \frac{\alpha}{e^{2(m-2+\sqrt{3(1-m^2)})\alpha}} k^2_1 + \left( 1 + \sqrt{1 + e^{6\alpha}} \right)^{\frac{1}{2}} \frac{\alpha}{e^{2(m-2-\sqrt{3(1-m^2)})\alpha}} k^2_2 + \frac{e^{3(m+1)}\alpha}{(1 + \sqrt{1 + e^{6\alpha}}) k^3_3}. \tag{20} \]

The equation of motion for each Fourier mode becomes

\[ \frac{d^2 \chi_\gamma}{d\eta^2} + \omega^2_\gamma \chi_\gamma = 0. \tag{21} \]

In the following discussions, we will focus on the background solutions which have the planar symmetry, \( m = -1 \) or \( m = +1 \). For the branch \( m = -1 \), in the limit \( a \to 0 \),

\[ \omega^2_\gamma = \frac{H^2 + 2^{4/3} k^2_3}{a^4} + \left( -2H^2 + \frac{3k^2_1 + k^2_3}{3 \times 2^{2/3}} \right) e^{2\alpha} + \cdots, \tag{22} \]

where \( k^2 := k^2_1 + k^2_2 + k^2_3 = k^2_1 + k^2_3 \). In the later time limit \( e^\alpha \to \infty \),

\[ \omega^2_\gamma = \frac{-2H^2 e^{2\alpha} + k^2}{e^{4\alpha}} - \frac{k^2 - 3k^2_3}{3 e^{3\alpha}} + \frac{H^2}{e^{4\alpha}} + O(e^{-5\alpha}), \tag{23} \]

which approaches the one in the ordinary de Sitter universe for large scale factor \( e^\alpha \). The case of the branch of \( m = +1 \) will be discussed in subsection II. G.

B. Quantization and power spectrum

The canonical quantization of the scalar field is done in the standard manner:

\[ \phi = \int d^3k \left( u_k a_k + u^*_k a_k^\dagger \right), \tag{24} \]

where \( [a_k, a^\dagger_k] = \delta(k_1 - k_2) \) (others are zero) and \( u_k = e^{ikx} \phi_k / (2\pi)^{3/2} \), and the mode function in the time-direction can be normalized as \( \phi_k \partial_t \phi_k^* - (\partial_t \phi_k) \phi_k^* = i/e^{3\alpha} \). The power spectrum is defined by

\[ \langle 0 | \phi^2 | 0 \rangle := \int d\ln k \int \frac{d^3k}{2} P_\phi, \quad P_\phi = \frac{k^3}{2\pi^2} \left| \phi_k \right|^2. \tag{25} \]
Note that in contrast to the case of the standard inflation the direction dependence would be included in the power spectrum. The vacuum is chosen at the initial anisotropic era: \( \tau \to 0^+ \). Usually, in a quantum harmonic oscillator system, an adiabatic process implies the one where the potential changes slowly enough compared to its size, and the time evolution can be obtained from the zero-th order WKB approximation. In this paper, we follow this definition for the term adiabatic. In the standard inflationary models, an adiabatic vacuum is also defined in the same way. Note that an isentropic process implies \((\text{adiabatic}) + (\text{reversible})\) process and therefore is slightly different from our adiabatic one. In the anisotropic spacetime an adiabatic vacuum state can be found only in the special solutions of the anisotropic Universe, which are regular in the initial times. We call our adiabatic vacuum an anisotropic vacuum.

C. WKB solution for \( m = -1 \)

From now on, we will focus on the branch of \( m = -1 \) except in the subsection II F. By changing the coordinate \( dt = d\tau/e^{3\alpha} \), which leads to

\[
e^\alpha = \sinh^{1/3}(3H\tau) = \frac{1}{\sinh^{1/3}(-3Ht)},
\]

one obtains the equation of motion in the form of a time-dependent oscillator

\[
\left( \frac{d^2}{dt^2} + \Omega(t)^2 \right) \phi = 0.
\]

When \( \tau \) changes from \( 0^+ \) to \( +\infty \), \( t \) changes from \( -\infty \) to \( 0^- \). The frequency squared becomes

\[
\Omega^2(t) := \frac{2^{4/3}(k^2 \epsilon^{6Ht} + k_3^2)}{(1 - \epsilon^{6Ht})^{4/3}} = \frac{2^{4/3}k^2}{x^{4/3}} (1 - r_+^2 x), \tag{27}
\]

where \( r_\perp := \frac{k_\perp}{k} \) and \( x(t) = 1 - \epsilon^{6Ht} = e^{-6\alpha(\sqrt{e^{6\alpha} + 1} - 1)} \) monotonically varies from one to zero as time \( t \) increases from negative infinity to zero.

The WKB solution is given by

\[
\phi_{\text{WKB}} = \frac{1}{\sqrt{2\Omega(t)}} \exp \left[ -i \int_{t_0}^{t} dt' \tilde{\Omega}(t') + i\psi \right], \tag{28}
\]

where \( \psi \) is a phase factor and \( H|t_0| \gg 1 \) and \( \tilde{\Omega} \) satisfies the nonlinear equation

\[
\tilde{\Omega}(t) = \Omega(t)^2 - \frac{1}{2} \left( \frac{\dot{\tilde{\Omega}}}{\Omega} \right) \frac{3\tilde{\Omega}^3}{2\Omega^2}. \tag{29}
\]

The WKB wavefunction is valid up to the order of the correction term if

\[
\epsilon(t) := \left| \frac{d\Omega^2(t)}{dt} \frac{1}{\Omega^3(t)} \right| = \frac{H}{k} \frac{1 - x(t)}{(x/2)^{1/3} (1 - r_\perp^2 x(t))^{1/2}} \left( \frac{3}{1 - r_\perp^2 x(t)} + 1 \right) \ll 1, \tag{30}
\]

where \( \epsilon \) play the role of the adiabatic parameter and is plotted in Fig. 1 for several different values of \( r_\perp \). At \( x = 1 \), all the curves converge to zero and therefore WKB approximation is always valid at the beginning. Thus, it is possible to define an anisotropic vacuum. In the case of \( m = -1 \), the initial spacetime metric can be approximated as

\[
ds^2 \approx -d\tau^2 + d(x^1)^2 + d(x^2)^2 + \tau^2 d(x^3)^2, \tag{31}
\]

where unimportant constants are absorbed by the rescaling of \( x^i \). It is straightforward to see that this metric represents (a patch of) the Minkowski spacetime:

\[
ds^2 = -dU^2 + d(x^1)^2 + d(x^2)^2 + dR^2, \tag{32}
\]

by the coordinate transformation \( R = \tau \sinh x^3 \) and \( U = \tau \cosh x^3 \). Thus, it is possible to find the positive frequency modes and the associated vacuum state, for an observer moving toward the increasing \( \tau \)-direction along a constant \( x^3 \) curve on the \((U, R)\)-plane. This corresponds to our choice of the vacuum at the beginning time.
Near $x \sim 1$, the condition is always valid for all momentum modes except for the mode with $r_\perp = 1$. The condition is also valid for high momentum modes if the scale factor $e^\alpha$ satisfies $e^\alpha \ll \frac{k}{H}$, i.e., $-Ht \gg \left(\frac{H}{k}\right)^3$. On the other hand, the large $e^\alpha$ approximation [23] is valid if $e^\alpha \gg 1$. For high frequency modes, if we choose an intermediate time $t_*$ satisfying

$$1 \ll e^{\alpha(t_*)} \ll \frac{k}{H},$$

and choose the WKB wavefunction for $t < t_*$ and the large $e^\alpha$ approximation, for $t > t_*$ the approximation will be good and in fact its accuracy is almost independent of the choice of the intermediate time $t_*$ as long as it satisfies Eq. (33). Therefore, we choose

$$e^{\alpha(t_*)} = \sqrt{\frac{k}{H}}$$

and the adiabatic parameter $\epsilon_*$ at $t = t_*$ is order of $O(H^{1/2}/k^{1/2})$. For modes with $k/H \sim 1$ or $k \ll H$, the approximation is not valid. The WKB wavefunction is expanded up to an enough adiabatic order, to validate our matching scheme with the solutions in the quasi-de Sitter region. The assumption of $k \gg H$ is justified if we assume a slow rolling inflation for our universe. Note that however, as we will discuss later, for a given class of inflation models, the modes of $k \gtrsim H$ are observable in some cases.

Near $x \sim 0$, the WKB solution at the zeroth adiabatic order takes the form:

$$\phi_{WKB} = \frac{1}{\sqrt{2\Omega(t)}} \exp\left[\frac{\sqrt{2/3}k}{6H} \left(\psi_k + 3x^{1/3} + \frac{3}{4} \left(1 - r_\perp^2/2\right)x^{4/3} + \cdots\right)\right].$$

(35)

Note that the next order correction at $e^{\alpha(t_*)} = \sqrt{k/H}$ is $O(H/k)$ since $x_* := x(t_*) = 2(H/k)^{3/2}(1 - (H/k)^{3/2} + \cdots)$. The value of the frequency $\Omega_* := \Omega_k(t_*)$ is

$$\Omega_* = \frac{k^2}{H} \left[1 + \left(\frac{2}{3} - r_\perp^2\right) \left(\frac{H}{k}\right)^{3/2} + \cdots\right].$$

(36)

The integral on the exponent is evaluated as

$$k \int_{t_0}^{t} dt \frac{2^{2/3}\sqrt{1 - r_\perp^2(1 - e^{6Ht})}}{(1 - e^{6Ht})^{2/3}} = -\frac{2^{2/3}k}{6H} \int_{1-\epsilon}^{x} \frac{dx}{1 - x} \frac{(1 - r_\perp^2x)^{1/2}}{x^{2/3}},$$

(37)

where $\epsilon := e^{6Ht_0}$ is some cut-off scale at an initial time. The integration can be executed explicitly to give the Appell hypergeometric function of two variables,

$$\int \frac{dx}{1 - x} \frac{(1 - r_\perp^2x)^{1/2}}{x^{2/3}} = \frac{2r_\perp^{4/3}(1 - r_\perp^2x)^{3/2}}{3(1 - r_\perp^2)} F_1(3, \frac{2}{3}, 1, \frac{5}{2}, 1 - r_\perp^2x, \frac{1 - r_\perp^2x}{1 - r_\perp^2}).$$

FIG. 1: The behavior of $\epsilon(t)$ with respect to $x$. Here, $r_\perp^2 = 0.999, 0.99, 0.9, 0.5$, and $0.1$ respectively from the top and we choose $H/k = 0.05$. The WKB approximation is valid if $\epsilon(t) < 1$. The matching time $t_*$ is almost independent of $k$. 
D. Matching

In the (quasi-) de Sitter region, \((-t) \ll H^{-1}\), the equation of motion \([26]\) becomes
\[
\frac{d^2}{dt^2} \phi + \frac{k^2}{(-3Ht)^{4/3}} \phi = 0,
\]
whose solution becomes
\[
\phi_{\text{fin}} = A_+ \phi_+ (t) + A_- \phi_- (t),
\]
where the positive and negative frequency modes are given by
\[
\phi_+ (t) := \left( -1 + \frac{ik}{H} (-3Ht)^{1/3} \right) e^{i \frac{k}{H} (-3Ht)^{1/3}}, \quad \phi_- (t) := \left( -1 - \frac{ik}{H} (-3Ht)^{1/3} \right) e^{-i \frac{k}{H} (-3Ht)^{1/3}}.
\]
This approximation is valid for \(e^\alpha \gg 1\). The WKB solution is now matched to the one in the quasi-de Sitter region, at \(t = t_*\). Now, we should determine the adiabatic order where the matching is done. As an example, we post the frequency \(\Omega\) at \(t_*\) by using \((-3Ht_*) = \sinh^{-1} (e^{-3\alpha(t_*)}) = \sinh^{-1} \left( \left( \frac{H}{k} \right)^{3/2} \right)\) and Eq. \([36]\):
\[
\Omega_* = k \left( \frac{k}{H} \right) \left[ 1 + 2 \left( \frac{2}{3} - r_\perp^2 \right) \left( \frac{H}{k} \right)^{3/2} + \cdots \right].
\]
As seen here, the first direction dependence appears at the third order in \(\epsilon^3\) and we need the WKB approximation up to the fourth adiabatic order. Then, the WKB frequency is given by
\[
\tilde{\Omega} (t_*)^2 = \left[ \Omega(t)^2 - \frac{1}{2} \frac{\Omega_{tt}}{\Omega} - \frac{3 \Omega_t^2}{2 \Omega^2} \right]_{t_*} = \Omega_0^2 \left( 1 - \frac{2H}{k} \right) + O(\epsilon_4^4).
\]

The matching condition is given by
\[
A_+ \phi_+ (t_*) + A_- \phi_- (t_*) = \left( 1 + \frac{H}{2k} \right) \frac{\Phi_*}{\sqrt{2 \Omega_*}},
\]
\[
A_+ \phi_+ ' (t_*) + A_- \phi_- ' (t_*) = -i \left[ (1 - \frac{H}{2k}) - i \sqrt{\frac{H}{k} \left( 1 - \frac{H}{2k} \right)} \right] \sqrt{\frac{\Omega_*}{2}} \Phi_*.
\]
Now the solution is exact up to the adiabatic order of \(\epsilon_4^4\). The phase factor is given by
\[
\Phi_* = \exp \left[ -i \int_{t_0}^{t_*} \tilde{\Omega} dt + i \psi \right].
\]
It is straightforward to confirm that the coefficients \(A_+\) and \(A_-\) satisfy the normalization condition
\[
|A_+|^2 - |A_-|^2 = \frac{H^2}{2k^3}.
\]

As the reference, in the standard isotropic inflation, the normalized mode function is given by
\[
\phi_{\text{isotropic}} = \frac{iH}{\sqrt{2k^3}} \left( -1 + \frac{k}{H} (-3Ht)^{1/3} \right) e^{i \frac{k}{H} (-3Ht)^{1/3}}.
\]

E. Power spectrum

In the limit of the later times \(t \to 0^-\), the wavefunction behaves as
\[
\phi_{\text{fin}} \bigg|_{t \to 0^-} = -(A_+ + A_-).
\]
Explicitly, the wavefunction in the later time to the order $c_s^2$ becomes

$$\phi_{\text{fin}} = -\frac{\Phi_s}{\sqrt{2k}} \left( \frac{H}{k} \right) \left( 1 - \frac{H}{2k} \right) \left( 1 - \frac{H}{k} \right)^{3/2} \cos \left( \frac{k}{H} \right) \sqrt{\frac{k}{H}} \sin \left( \frac{k}{H} \right) .$$

Then, the power spectrum Eq. (25) in terms of the adiabatic order $O((H/k)^{3/2})$ is given by

$$P_\phi = \frac{H^2}{4\pi^2} \left\{ 1 - \frac{H}{k} + \left( \frac{2}{3} - r^2_{\perp} \right) \left( \frac{H}{k} \right)^{3/2} \right\} \cos \left( \frac{k}{H} \right) \sqrt{\frac{k}{H}} + O \left( \frac{H}{k} \right)^2 .$$

As a reference, we present the power spectrum from the standard inflation obtained from Eq. (25) and (46):

$$P^{(0)}_\phi = \frac{H^2}{4\pi^2} .$$

There are two modifications: One is an oscillatory behavior of the power spectrum, which seems to be observed in the CMB angular power spectrum around the first acoustic peak $\ell = 10 \sim 100$, and the other is an overall change of the amplitude given by $(1 - H/k)$. In the above expression by taking $k \to H$, the power spectrum seems to vanish. This would be consistent with the suppression of the angular power spectrum of CMB fluctuations on large scales if the anisotropy of the present Hubble horizon scale corresponds to the wavenumber of the order $k \gtrsim H$. For larger $k$, it approaches the scale invariant form with a small oscillatory behavior. The direction dependence appears only on the oscillatory term.

As discussed above, our WKB approximation is valid only for modes with $k/H \gg 1$. Indeed, one may ignore the contributions of modes such as $k \sim H$ or $k < H$. One may not observe modes whose wavelengths are larger than the present Hubble scale. Thus, there is the critical wavenumber given by $k_{\text{crit}} = a_0 H_0$, where $H_0^{-1} \sim 10^{26}\text{cm}$ is present day’s Hubble scale: one observes the modes of $k > k_{\text{crit}}$. If it is possible to take $k_{\text{crit}} > H$, our WKB approximation is justified for all the observable modes. On the other hand, $a_0 = (T_R/T_0)a_1$, where $T_R, T_0 \approx 2.7K$ and $a_1$ are the reheating temperature, the CMB temperature at present and the value of scale factor at the end of inflation, respectively. One finds the relation

$$k_{\text{crit}} = a_0 H_0 = \frac{a_1 T_R}{T_0} H_0 = e^N \frac{T_R}{T_0} H_0 = A H, \quad A := e^{N} \frac{T_R V_{\text{DE}}^{1/2}}{T_0 V_{\text{inf}}^{1/2}} ,$$

where $N = \ln(a_1/a_0) = \ln a_1$ is the number of e-folding of inflation, $V_{\text{inf}}$ and $V_{\text{DE}} \sim (10^{-3}\text{eV})^4$ are the values of inflation and dark energy potentials. Note that $H \approx \sqrt{(8\pi/3M_p^2)V_{\text{inf}}}$ is the Hubble constant during inflation and $H_0 \approx \sqrt{(8\pi/3M_p^2)V_{\text{DE}}}$ is that of the present Universe. One can estimate the value of $A$ as

$$A \approx e^{N-64} \left( \frac{T_R}{10^{14}\text{GeV}} \right) \left( \frac{10^{16}\text{GeV}}{V_{\text{inf}}^{1/4}} \right)^2 .$$

Thus, for inflation with GUT or a lower energy scale and if e-folding is around $N \gtrsim 64$, it may be possible to realize $A \gtrsim 1$, leading to $k_{\text{crit}} > H$. On the other hand, for sufficiently long duration of inflation, say $N \gg 64$, the observable modes are only those with $k \gg H$ since $A \gtrsim 1$, and therefore it seems to be very hard to find any deviation from the scale-invariant spectrum. In other words, to find the information about the preinflationary anisotropy, the number of e-folding must be tuned.

**F. Planar modes, $r_{\perp} \lesssim 1$**

As seen in the case of $r_{\perp}^2 = 0.999$ in Fig. 1, namely for modes on a plane, there appears a region where the WKB approximation may not be valid near $x \sim 1$. We divide the time into three separate regions divided by the times $t_1$ and $t_*$. In the region $t_1 < t < t_*$ the WKB approximation is valid. For other two regions, we may find an appropriate
approximate solution. Note that in the case of $r_\perp = 1$ exactly, the adiabaticity parameter diverges in the limit of $t \to -\infty$ and there is no anisotropic vacuum state.

The frequency squared for $x \simeq 1$ becomes

$$\Omega^2 = 2^{4/3}(k_3^2 + k_\perp^2 e^{6Ht}) + O(k^2 e^{12Ht}), \quad \text{for } Ht \ll -1.$$  

The corresponding solution for Eq. (26) is

$$\phi_1 = \sqrt{\frac{\pi}{6H \sinh(\pi q_3)}} J_{-i\Omega}(q_\perp e^{3Ht}), \quad \text{for } Ht \ll -1,$$

where

$$q_3 = \frac{2^{2/3}|k_3|}{3H}, \quad q_\perp = \frac{2^{2/3}k_\perp}{3H}.$$

Here we choose the solution so that it becomes an incoming wave form:

$$\phi_1 = \frac{1}{\sqrt{2 \cdot 2^{2/3}|k_3|}} \exp \left\{ -2^{2/3}i|k_3|t + i\psi \right\}, \quad \text{for } t \to -\infty,$$

where the initial phase becomes $e^{i\psi} = [\Gamma(1 + iq_3)/\Gamma(1 - iq_3)]^{1/2} (2\pi)^{i\psi}$. We choose the momentum dependent cutoff time $t_1$ so that both of the solution (48) and the WKB solution are equally good. The WKB solution is valid if $\epsilon < 1$. For $x \approx 1$, we check the validity of the WKB approximation,

$$\epsilon \approx \frac{3 \cdot 2^{1/3}H}{k} \left( \frac{e^{6Ht}}{k^{1/3}} \right)^{1/2} \simeq \frac{3 \cdot 2^{1/3}H}{k} \frac{1}{e^{3Ht}},$$

where in the second equality we assume $e^{3Ht} \gg r_3$ since we want $t_*$ to be placed over the first peak in $t$ as seen in Fig. 1. For the WKB approximation to be valid, we should choose the time so that $e^{3Ht_1} > \frac{3 \cdot 2^{1/3}H}{k}$. For example, we may choose the time $t_1$ satisfying

$$e^{3Ht_1} = \frac{3}{2} \frac{\sqrt{H}}{k},$$

where both of the WKB approximation and the solution (48) are equally good. Note that the time $t_1$ is direction independent. At time $t_1$, the assumption $e^{3Ht_1} \gg r_3$ constrains the orthogonal momentum value to

$$q_3 \ll \frac{\sqrt{k}}{H}.$$

For planar high-momentum modes, the argument of the Bessel function, $q_\perp e^{3Ht_1} = \sqrt{\frac{k}{H}} \simeq \sqrt{\frac{k}{H}}$, is very large. Therefore at $t_1$, we may employ the asymptotic form for the Bessel function:

$$\lim_{x \to \infty} J_{\alpha}(x) = \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi}{4} - \frac{\alpha \pi}{2} \right).$$

During $t_1 < t < t_*$, we use the WKB approximation where $t_*$ is given in Eq. (51). The WKB solution is

$$\phi_2 = \frac{B_+}{\sqrt{2\Omega}} \exp \left\{ -i \int_{t_1}^t dt' \Omega(t') \right\} + \frac{B_-}{\sqrt{2\Omega}} \exp \left\{ i \int_{t_1}^t dt' \Omega(t') \right\}.$$

For $t \gg t_*$, the WKB approximation is not valid. However, for $t > t_*$ the asymptotic solution (48) is valid. At $t_1$, to the zeroth order in $H/k$, the matching condition becomes

$$B_+ = \sqrt{\frac{\Omega_1}{2}} \left[ \phi_1(t_1) + i \frac{\phi'(t_1)}{\Omega_1} \right] \simeq \frac{i}{\sqrt{2 \sinh(\pi q_3)}} \left( 1 + \frac{Hq_3^2}{k} \right)^{-1/4} \cos \psi - i \left( 1 + \frac{Hq_3^2}{k} \right)^{1/4} \sin \psi,$$

$$B_- = \sqrt{\frac{\Omega_1}{2}} \left[ \phi_1(t_1) - i \frac{\phi'(t_1)}{\Omega_1} \right] \simeq \frac{-i}{\sqrt{2 \sinh(\pi q_3)}} \left( 1 + \frac{Hq_3^2}{k} \right)^{-1/4} \cos \psi + i \left( 1 + \frac{Hq_3^2}{k} \right)^{1/4} \sin \psi,$$
where $\Omega_3^2 \approx 9 \left( \frac{H}{k^2} k_+^2 + \frac{2H}{3y} k_3^2 \right) = 9H^2 \frac{k^2}{k^2} \left( 1 + \frac{H}{k_+^2} \frac{Hq}{k} \right)$ and $\psi = \frac{2\pi}{3} + \frac{k}{\sqrt{kH}}$. Since $q_3^2 \ll k_3^2/(kH)$ as in Eq. (50), the coefficient $B_\pm$ becomes of the zeroth order,

$$B_+ \approx \frac{1}{\sqrt{1 - e^{-2\pi q_3}}} e^{i\pi \frac{k}{\sqrt{kH}}} \quad \text{and} \quad B_- \approx \frac{e^{-\pi q_3}}{\sqrt{1 - e^{-2\pi q_3}}} e^{-i\pi \frac{k}{\sqrt{kH}}} \quad \text{for } t_+ < t < 0.$$

For $t_+ < t < 0$, the isotropic solution is valid and

$$\phi_{\pm} = C_+ \phi_+(t) + C_- \phi_-(t).$$

The primordial power spectrum is proportional to the square of the size of $C_+ + C_-$. Matching this solution with the WKB solution at the boundary $t_*$, we have

$$C_+ + C_- = \frac{H^2}{2k^3} \left[ \phi_2(t_*)(\phi_+(t_*) - \phi_+(t_*)) - \phi_2(t_*)(\phi_-(t_*) - \phi_+(t_*)) \right],$$

Using Eqs. (53) and (39), we get, in the limit $q_3^2 \ll k_3^2/(kH)$,

$$C_+ + C_- \approx - \frac{H}{\sqrt{2(1 - e^{-2\pi q_3})k^3}} \left\{ (-1 - e^{-\pi q_3}) \sin \left( \frac{\pi}{4} - \Phi(k) \right) + i(1 - e^{-\pi q_3}) \cos \left( \frac{\pi}{4} - \Phi(k) \right) \right\},$$

where $\Phi(k) = \sqrt{k_3^2/(kH)} + \int_{t_1}^{t_*} dt' \Omega(t') + \sqrt{kH}$ and we ignore terms of $O(\sqrt{H/k})$.

The primordial power spectrum to the zeroth order is given by

$$P_\phi = \frac{k_3}{k^2} \int \left[ \left( \text{Re}(C_+ + C_-) \right)^2 + \left( \text{Im}(C_+ + C_-) \right)^2 \right]$$

$$= \frac{H^2(1 - e^{-\pi q_3})}{4\pi^2} \left[ 1 + \frac{1}{2} \coth^2 \frac{\pi q_3}{2} (1 - \sin(2\Phi(k))) \right].$$

If one reduces $q_3$ keeping $k$ constant, the direction dependence of the power spectrum becomes un-negligible at $q_3 \sim 2/\pi$ or $k_3 \sim \frac{3^{2/3}}{\pi} H k \ll k$. Note that this result is quite different from the general expectation that the direction dependent term would be (approximately) scale invariant and second order in $k_3$. The phase factor $\Phi(k)$ is dominated by $\int_{t_1}^{t_*} dt' \Omega(t')$ and is becomes

$$2\Phi(k) = 2\sqrt{\frac{k_3}{kH}} + 2 \int_{t_1}^{t_*} dt' \Omega(t') + 2 \sqrt{\frac{k}{H}} \approx \frac{2^{2/3}\sqrt{\pi}(1/3)k}{3\Gamma(5/6)} \frac{k}{H}.$$  

G. The case of $m = +1$

In this subsection, we briefly discuss the case of the other solution with a planar symmetry, $m = +1$. We will see that the WKB approximation can not be applied at the beginning and there is no anisotropic vacuum. In this case, it is appropriate to change the variables as $\phi = a^{-1/2} \Phi$ and $dt = a^{-1} du$, to rewrite the equation of motion into the form of a time-dependent harmonic oscillator:

$$\left[ \frac{d^2}{du^2} + \Xi(u)^2 \right] \Phi(u) = 0,$$

where

$$\Xi(u)^2 = \frac{a''^2}{4a^2} - \frac{a''}{2a} + a^{-2}\Omega^2 = \frac{a''^2}{4a^2} - \frac{a''}{2a} + k^2 \left( 1 + \sqrt{1 + a(u)^6} + a(u)^6 (1 - r^2) \right),$$

and the prime denotes the derivative with respect to $u$. The coordinate $u$ is defined in the range $0 \leq u < u_0 := \sqrt{\pi(1/3)/(3H\Gamma(5/6))} \approx 1.402 H^{-1}$. In order to check the validity of the WKB approximation, one needs another coordinate transformation such that $Hv = (u/u_0)$ and $\Phi = e^{vH/2} \Psi$, where $v$ runs from $-\infty$ to zero, and then the equation of motion becomes

$$\left[ \frac{d^2}{dv^2} + \xi(v)^2 \right] \Psi = 0,$$
where

\[ \xi(v)^2 := H^2 u_0^2 e^{2v} \Xi(u(v))^2 - \frac{H^2}{4}. \]  

(60)

The validity of WKB approximation is determined by the parameter \( \epsilon_v := \left| \frac{d\xi^2}{dv} / \xi^3 \right| \). For \( t \to -\infty, \ u \to 0 \) and \( v \to -\infty \), then \( \approx H u \), and one can evaluate \( \xi^2 \approx k^2 \beta \gamma(1) \) \( H^2 u_0^2 \) \( e^{2Hv} \) and \( \epsilon_v \approx 2e^{-Hv} / k_1 u_0 \). Therefore, the adiabaticity parameter exceeds unity for \( v \ll -(1/H) \ln(k_1 u_0) \). In such a case, one cannot find a well-defined adiabatic vacuum state.

IV. SCALAR FIELD IN THE BIANCHI IX UNIVERSE

In this section, we investigate the primordial spectrum of a massless and minimally coupled scalar field in the Bianchi IX Universe. Such an investigation will give a deep insight about how generic the results obtained in the case of the Bianchi I model are. One reason is that recent WMAP data seem to favor a spatially closed Universe \(^2\), and the Bianchi IX Universe exactly has that geometry. The other important reason is that, similarly to the case of the Bianchi I model, in the Bianchi IX model there is an exact analytic solution, the Taub-NUT de Sitter spacetime whose metric is given by Eq. (5). The spacetime geometry is a Taub-NUT spacetime in the initially anisotropic era, and approaches de Sitter solution in the later time. Our discussion will come along the same line in the case of the Bianchi I. Note that as discussed in Sec. II, we restrict the value of \( M \) to be \( M \geq 2\gamma/(3\sqrt{5}) \), for which \( \Delta(t) \) vanishes at \( t = t_0 < 0 \ (T = 0) \). For such a choice, one can naturally choose the initial time to be \( T = 0 \).

A. Scalar field on the Bianchi IX Universe

The equation of motion of a massless and minimally coupled scalar field, \( \Box \phi = 0, \) explicitly reduces to

\[ -\frac{1}{n^2} \left[ \frac{\partial^2}{\partial x^2} + \left( 3\tilde{\alpha} - \frac{n}{n} \right) \frac{\partial}{\partial x} \right] \phi + \frac{1}{e^{2\alpha + 2\beta}} \left[ -L^2 + \left( -1 + e^{6\beta} \right) \left( -L_3^2 \right) \right] \phi = 0, \]

(61)

where the angular momentum operators are defined by

\[-L^2 := \left( \frac{\partial}{\partial x^i} \right)^2 + \cot x^1 \frac{\partial}{\partial x^i} + \frac{1}{\sin^2 x^1} \left( \frac{\partial}{\partial x^2} \right)^2 + \frac{2}{\sin^2 x^1} \frac{\partial^2}{\partial x^2 \partial x^3}, \quad -L_3^2 := \left( \frac{\partial}{\partial x^3} \right)^2.\]

The general classical solution can be decomposed by the basis of harmonic functions \( \phi(T) = \sum_\gamma \varphi_\gamma(T) Y_\gamma(x^1, x^2, x^3) \), where \( \gamma \equiv (J, K, M) \) represents a short-hand notation of a set of quantum numbers associated with the spatial directions \([27]\). The harmonic function on the three-space can be constructed by \( Y_\gamma(x^1, x^2, x^3) := e^{iKx^1} e^{iMx^2} \Theta_\gamma(x^1) \), where \( \Theta_{JKM} \) satisfies the ordinary differential equation

\[
\frac{d^2}{d(x^1)^2} + \cot x^1 \left[ \frac{1}{\sin^2 x^1} \left( \frac{d}{dx^2} \right)^2 - 2 \cos x^1 \frac{d^2}{dx^2 dx^3} \right] \Theta_\gamma(x^1) = -J(J + 1) \Theta_\gamma(x^1).
\]

The harmonic function is normalized as

\[
\int dx^1 dx^2 dx^3 \sin x^1 Y_\gamma(x^1, x^2, x^3) Y_\gamma^*(x^1, x^2, x^3) = \delta_{\gamma, \gamma'}.
\]

The equation in the time-like direction is given by

\[
0 = \left[ \frac{d^2}{dT^2} + \left( 3\tilde{\alpha} - \frac{n}{n} \right) \frac{d}{dT} \right] \varphi_\gamma(T) + \frac{n^2}{e^{2(\alpha - 2\beta)}} \left[ \frac{J(J + 1) - K^2}{e^{6\beta}} + K^2 \right] \varphi_\gamma(T).
\]

(62)

For a given vacuum \( |0\rangle \), the quantization of the scalar field can be done in the standard way

\[
\phi(T, x^i) = \sum_\gamma \left( \varphi_\gamma(T) Y_\gamma(x^i) a_\gamma + \varphi_\gamma^*(T) Y_\gamma^*(x^i) a_\gamma^{\dagger} \right).
\]

(63)

\(^2\) Simultaneous constraints on the spatial curvature and the (constant) equation of state parameter of dark energy by the five-year WMAP data seem to favor a positive curvature, although the flat Universe is also consistent at 95 percent confidence level \([3]\).
where the creation and annihilation operators, $a_i^\dagger$ and $a_i$, have the properties $a_i|0\rangle = 0$ and $\langle 0|a_i^\dagger = 0$, and satisfy the commutation relations $[a_{\gamma_i}, a_{\gamma_j}^\dagger] = \delta_{\gamma_i, \gamma_j}$, and $[a_{\gamma_i}, a_{\gamma_j}] = [a_{\gamma_i}^\dagger, a_{\gamma_j}^\dagger] = 0$. In our later discussions, a vacuum $|0\rangle$ will be chosen in order for the initial modes to be adiabatic. The mode functions satisfy the Wronskian normalization condition

$$\varphi_\gamma \partial_T \varphi_\gamma^* - \partial_T \varphi_\gamma \varphi_\gamma^* = \frac{in}{\epsilon^3}.$$  

For the case of the discrete spectrum, the power spectrum can be defined

$$\langle 0|\phi^2|0\rangle = \sum_{\gamma} P_\gamma, \quad P_\gamma := |\varphi_\gamma|^2.$$  

It is convenient to define the new time coordinate as

$$dx = n e^{-\alpha + 2\beta} dT = \frac{T^2 + 2t_0 T + t_0^2 + \ell^2}{2\ell \Delta} dT = \frac{2(T^2 + 2t_0 T + t_0^2 + 1)}{\ell(3 T^3 + 4 T^2 + (6 t_0^2 + 2) T + 3 t_0^2 + 2t_0 - 1/t_0)} dT,$$

and variable as $\varphi = e^{-\alpha - \beta} \chi$, where $\mathcal{T} := T/\ell$ and $\tilde{t}_0 := t_0/\ell$ are dimensionless. Then, the equation of motion is rewritten as

$$\left(\frac{d^2}{dx^2} + \Omega(x)^2\right) \chi(x) = 0,$$

where the frequency squared is given by

$$\Omega^2(x) := \Omega_0^2(x) + \left(J(J+1) - K^2\right) e^{-6\beta} + K^2.$$  

Here the momentum independent part is

$$\Omega_0^2(T) := - \left(\frac{d\alpha}{dx} + \frac{d\beta}{dx}\right)^2 - \frac{e^{2\alpha-4\beta}}{n^2} \left[\left(\frac{d^2\alpha}{dT^2} + \frac{d^2\beta}{dT^2}\right)^2 + \left(\frac{2d\alpha}{dT} - \frac{d\beta}{dT} - \frac{1}{n} \frac{dn}{dT}\right)\left(\frac{d\alpha}{dT} + \frac{d\beta}{dT}\right)\right]$$

$$= - \frac{4}{T F(T)} \frac{T^3 + 4T^2 \tilde{t}_0 T + 6\tilde{t}_0^2 T + 2 T + 3\tilde{t}_0^3 + 2\tilde{t}_0 - 1/\tilde{t}_0}{(T^2 + 2t_0 T + t_0^2 + 1)^2} = \frac{F(T) \Delta(T)}{\ell^2(T^2 + 2t_0 T + t_0^2 + 1)}.$$

where $\Delta$ is defined in Eq. (10) and

$$F(T) := 2T^6 + 12 T^5 \tilde{t}_0 + 30 T^4 \tilde{t}_0^2 + 5 T^4 + 41 T^3 \tilde{t}_0 + 22 T^3 \tilde{t}_0^2 + \frac{T^3}{\tilde{t}_0} + 33 \tilde{t}_0^3 T^2 + 36 \tilde{t}_0^2 T^2 + 7 T^2$$

$$+ 15 \tilde{t}_0^4 T + 24 \tilde{t}_0^3 T + 7 \tilde{t}_0 T - 2 \frac{T}{\tilde{t}_0} + 3 \tilde{t}_0^5 + 5 \tilde{t}_0^4 + \tilde{t}_0^2 - 1.$$  

For a small $T$,

$$\Omega_0^2 \approx \frac{(1 - 3\tilde{t}_0^2)^2 T}{4t_0(1 + \tilde{t}_0^2)} + \frac{(-2 + 25 \tilde{t}_0^2 + 64 \tilde{t}_0^3 - 66 \tilde{t}_0^4 - 18 \tilde{t}_0^5 + 27 \tilde{t}_0^6)}{4t_0^2(1 + \tilde{t}_0^2)^2} + O(T^2),$$

and for a large $T$,

$$\Omega_0^2 = - \frac{T^2}{2} - \tilde{t}_0 T + O(T^0).$$  

For a large $T$ $x(T) = -1/T + O(T^2)$, and for a small $T$ the relation becomes

$$x = \frac{2\left(1 - \frac{\tilde{t}_0}{1 - 3\tilde{t}_0^2}\right) \ln \mathcal{T}}{1 - 3\tilde{t}_0^2} T_0,$$

where $T_0(\mathcal{M}/\ell)$ can be determined if we integrate out the equation (66) numerically. Note that

$$e^{-6\beta} = \frac{4\Delta/\ell^2}{(T^2 + 2T \tilde{t}_0 + \tilde{t}_0^2 + 1)^2} = \frac{T(3 T^3 + 4T \tilde{t}_0 T^2 + (6 \tilde{t}_0^2 + 2) T + 3 \tilde{t}_0^3 + 2 \tilde{t}_0 - 1/\tilde{t}_0)}{(T^2 + 2T \tilde{t}_0 + \tilde{t}_0^2 + 1)^2}. $$
B. WKB solutions, matching and scalar power spectrum

We discuss the primordial spectrum in the Bianchi IX Universe. The new coordinate \( x \) runs from \(-\infty\) to \( 0^- \), where \( T \to 0^+ \) and \( T \to \infty \) correspond to \( x = -\infty \) and \( x = 0^- \), respectively. The high angular-momentum limit \( (J \to \infty) \) corresponds to the high momentum limit \( k \gg \hbar \) in the Bianchi I model, since \( J \) is dimensionless.

a. Adiabatic parameter \( \epsilon \) and the WKB approximation: We define the adiabaticity parameter,

\[
\epsilon_{K \neq 0} := |\epsilon|; \quad \epsilon = \frac{1}{|\Omega|^3} \frac{d\Omega^2}{dx} = e^{\alpha-2\beta} \frac{1}{n} \frac{d\Omega^2}{dT} = \frac{2T\Delta}{T^2 + 2aT + a^2 + \ell^2} \frac{1}{|\Omega|^3} \frac{d\Omega^2}{dT}.
\]  

(73)

![FIG. 2: The behavior of \( \epsilon(T) \). The horizontal axis represents \( T \). Here we take \( K = 2 \) and \( J = 30 \). The parameter \( \tilde{t}_0 \) is chosen to be \( \tilde{t}_0 = -0.5, -0.3, -0.1, -0.05, -0.03, -0.01 \) respectively from the top. As seen in this figure, the large \( J \) approximation gives the best result at \( T \sim \sqrt{J} \) and gives a good approximation even for time \( T \sim J/2 \).](image)

We focus on the case of a small \( T \) \( (|T| \ll 1) \). The value of \( \epsilon \) vanishes at \( T = 0 \) and varies with time as

\[
\epsilon = \frac{[4(J(J+1) - K^2) + 1]}{8(-\tilde{t}_0)^2 K^3} \left(1 + O(\tilde{t}_0^2)\right) T - \frac{3[4(J(J+1) - K^2) + 1]^2 + 32K^2}{64(-\tilde{t}_0)^3 K^5} \left(1 + O(\tilde{t}_0^2)\right) T^2 + O(T^3).
\]  

(74)

As in the \( m = -1 \) branch of the planar Kasner-de Sitter model, the WKB approximation is always valid at the beginning and therefore one can find a well-defined anisotropic vacuum state. This can be understood as follows. From Eq. (13), with use of Eqs. (6) and (12), it turns out that in the \( T \to 0 \), the initial metric functions behaves as \( n \propto T^{-1/2}, a \propto \text{constant} \) and \( c \propto T^{1/2} \). After defining the proper time \( d\tau = n(T) dT \approx dT/T^{3/2} \), one finds \( a \propto \text{constant} \) and \( c \propto \tau \), which is a patch of a static spacetime as Eq. (31).

As \( T \) increases \( \epsilon(T) \) has a maximum value

\[
\epsilon_{\text{max}} \approx \frac{[4(J(J+1) - K^2) + 1]^2}{4(-\tilde{t}_0)K \left[3[4(J(J+1) - K^2) + 1]^2 + 32K^2\right]} \approx \frac{1}{12K(-\tilde{t}_0)},
\]

where the last step is for large \( J \) at

\[
T = T_{\text{max}} \approx \frac{4K^2(-\tilde{t}_0) [4(J(J+1) - K^2) + 1]}{3[4(J(J+1) - K^2) + 1]^2 + 32K^2}
\]

and then start to decrease toward negative values crossing zero. For modes satisfying \( \epsilon_{\text{max}} < 1 \), we may use the WKB approximation around \( T = T_{\text{max}} \). Therefore, assuming \( J \gg 1 \), the WKB approximation is valid at the initial period of time for modes satisfying,

\[
\frac{1}{12(-\tilde{t}_0)} < K < J.
\]  

(75)
The subsequent behavior of the adiabatic parameter can be described by the large $J$ approximation.

$$\epsilon \simeq \frac{(\tilde{t}_0^2 + 1)^2}{2t_0 \sqrt{J(J + 1) T (T^2 + t_0 T + \tilde{t}_0^2 + 1)}} \frac{3\tilde{t}_0^2 + 6t_0 - 1 + 3T^2}{\sqrt{T^3 + 4\tilde{t}_0^2 T^2 + 6\tilde{t}_0^2 T + 2T + 3\tilde{t}_0^2 + 2t_0 - 1/\tilde{t}_0}}.$$ 

Starting from a positive value around $T \sim \tilde{t}_0$, the explicit value of $\epsilon$ decreases and vanishes as $T = 1/\sqrt{3} - \tilde{t}_0$. In small $\tilde{t}_0$ limit, it continually decreases to a negative number of the order $-1/(J\sqrt{-\tilde{t}_0})$ and then bounces back to zero as $T$ goes to infinity. Therefore, the WKB approximation in this intermediate region is valid if

$$J > \sqrt{\frac{1}{(-\tilde{t}_0)}}.$$ 

(76)

This large $J$ approximation, however, is not satisfied both for small and large $T$ limits. As for a small $T$, Eq. (74) is a good approximation. For a large $T$, the highest power terms in $T$ will become comparable to the large $J$ contributions, and the adiabatic parameter can be written as follows:

$$\epsilon \simeq \frac{\sqrt{2}/[1 + 3(-\tilde{t}_0)^{-1}(1 + \tilde{t}_0)^2(J(J + 1) - K^2)/T^2]}{[2J(J + 1)/(T^2) - 1]^{3/2}}.$$ 

The numerator will be dominated by the $J$ dependent term for a time $T$ smaller than $T_m = (J(J + 1)/\tilde{t}_0)^{1/5}$. On the other hand, if $T > T_m$, we may neglect the $J$ dependent term compared to unity. The denominator vanishes at time $T_c = \sqrt{2J(J + 1)}$, leading to the divergence of the adiabaticity parameter. For $T < T_c \sim J$, the first term rules and for $T > T_c$, the constant $(-1)$ rules the denominator. Noting these behaviors, the order of $\epsilon$ will be minimized at time $T \sim T_m$, where the accuracy of the WKB approximation will be maximized. At a time $T$ satisfying $T_m < T \ll T_c$, the adiabaticity parameter can be approximated to be

$$\epsilon \simeq \frac{T^3}{2[J(J + 1)]^{3/2}},$$

(77)

which is much smaller than unity.

The adiabaticity parameter then goes to negative infinity at $T = T_c$ (for a large $J$), and then bounces back to a finite negative value $-\sqrt{2}$ as $T$ goes to $\infty$. As $T \to \infty$, $\epsilon$ approaches $-\sqrt{2}$ which is larger than one and the WKB approximation is broken there. In this region, we use the large $T$ approximation. It is good to solve the later time evolution equation (72) in $T$ coordinate rather than in $x$ coordinate since the order of the frequency part is transparent:

$$\hat{\Omega}^2 = \frac{n^2}{\epsilon}$$. \[e^{2\Omega} \left[ \frac{J(J + 1) - K^2}{e^{2\Omega} + K^2 e^{4\Omega}} \right] \approx \frac{4J(J + 1)}{T^4} \left( 1 - \frac{4\tilde{t}_0}{T} + \frac{10\tilde{t}_0^2 - 2}{T^2} + O(T^{-3}) \right).$$

(78)

The first $T^{-4}$ term gives the asymptotic solution $\chi_\pm$ in Eq. (81) below and the second $T^{-5}$ term gives corrections. Therefore, at a time $T \gg 1(\gg \tilde{t}_0)$, the error of the solution $\chi_\pm$ is of order

$$\frac{4\tilde{t}_0}{T}.$$ 

The matching time $T_\ast$ is determined by the condition that this accuracy is the same as Eq. (74):

$$\frac{4(-\tilde{t}_0)}{T} = \frac{T^3}{2J(J + 1)^{3/2}}.$$ 

We get the matching time

$$T_\ast = 2[J(J + 1)]^{3/8}(-\tilde{t}_0)^{1/4}.$$ 

(79)

In summary, the WKB approximation is valid for $T \ll T_c$ for modes satisfying Eqs. (75) and (76). However, it fails to be satisfied for $T \geq T_c$ and we use the large $T$ approximation. These two solutions are matched at time $T_\ast$ and the accuracy of the whole solution is of $O((-\tilde{t}_0)^{3/4}J^{-3/4})$. 

Therefore the power spectrum, including the first non-vanishing direction dependent correction, becomes standard spectrum is suppressed by the factor \( O \) becomes scale-invariant. Thus, in the resultant power spectrum for the moderate modes, the deviation from the

As the reference, it is useful to show the scalar power spectrum in the standard de Sitter invariant vacuum is given

and thus multiplied by the factor \( \sim K^4 \), which is proportional to the density of state for a given \( J \), the spectrum becomes scale-invariant. Thus, in the resultant power spectrum for the moderate modes, the deviation from the standard spectrum is suppressed by the factor \( O(J^{-3/4}) \).

c. Planar modes with \( K \leq \frac{1}{12(\tilde{t}_0)} \): In the case of planar modes, the WKB approximation fails to be satisfied at \( T \sim t_{\max} \sim \frac{K^2(\tilde{t}_0)}{4\tilde{t}_0^2} \). Around these initial times, we try to find an approximate solution for the times \( t \ll \tilde{t}_0 \). We need to expand the frequency squared up to first order in \( \tilde{t}_0 T \):

\[
\tilde{\Omega}^2 \simeq \frac{4\tilde{t}_0^2}{(1 - 3\tilde{t}_0^2)^2} \left( K^2 + \frac{J(J + 1)(1 - 3\tilde{t}_0^2)}{(1 + \tilde{t}_0^2)(-\tilde{t}_0)} \frac{T}{T} \right),
\]

\[
P_{JKM} = \frac{1}{2^T(J(J + 1))^3/2} \frac{1}{|A_+ - A_-|^2}
\]

\[
P_{JKM} \approx \frac{1}{2^T(J(J + 1))^3/2} \left( 1 + \frac{8(-\tilde{t}_0)^{3/4}}{[J(J + 1)]^{3/8}} \sin \left[ \left( J(J + 1) \right)^{1/8}(\tilde{t}_0)^{-1/4} \right] \right).
\]

\[
P_{JKM}^{(0)} = \frac{1}{2^T(J(J + 1))^3/2}
\]
where we ignore $1/t_0$ and $K$ relative to $J$. Noting $d(3\alpha - \log p) \approx \frac{1}{J}$, we get the solution to the differential equation \([62]\),

$$
\phi_1(T) = \sqrt{\frac{2(-t_0)}{(1-3t_0^2)} \sinh(\pi qK) J^{-i qK} (A_J T)^{1/2}},
$$

where

$$
q_K := \frac{4(-t_0)}{1-3t_0^2} K, \quad A_J = 4 \sqrt{\frac{J(J+1)(-t_0)}{(1-3t_0^2)(1+t_0^2)}},
$$

and we choose the normalization of the solution so that it becomes an incoming wave in $x$ coordinate at $x \to -\infty$, noting the relation \([72]\) between $T$ and $x$. If we use the solution \([84]\) up to a time $T_1$, the accuracy of the above solution will be of order $T_1(1/t_0)$.

On the other hand, the WKB solution is still valid at a time $T_1$ satisfying $\frac{2(-t_0)^2 K^2}{J^2} \ll T_1 \ll 1$. At this time we may also use the high $J$ limit and the adiabaticity parameter becomes $\epsilon \approx \frac{2(-t_0)^2J}{4T_1^{1/2}}$. The matching time $T_1$ between the initial exact solution and the WKB solution can be determined by setting the accuracies to be the same:

$$
T_1(1/t_0) = \frac{1}{2\sqrt{-t_0 J T_1^{1/2}}},
$$

Therefore, the matching time becomes $T_1 = \frac{1}{2^{2/3}(-t_0)J^{2/3}}$. The WKB solution becomes

$$
\phi_2(T) = e^{-\alpha - \beta} \chi_{WKB}(x) = \sqrt{\frac{2}{\Omega(T^2 + 2t_0T + t_0^2 + \epsilon^2)}} \left( B_+ e^{-i \int_{T_1}^T \Omega(T') \frac{dx}{x^2} dT'} + B_- e^{i \int_{T_1}^T \Omega(T') \frac{dx}{x^2} dT'} \right).
$$

We use this solution during the period of time $T_1 < T < T_*$. The coefficient $B_\pm$ is determined to be

$$
B_\pm = \frac{\sqrt{\Omega_1(1 + t_0^2)}(\phi_1(T_1) \pm i(1 - 3t_0^2)T_1 d\phi_1(t_1))}{2(-t_0) \Omega_1} = \frac{1}{2} \sqrt{\frac{(1+t_0^2)e^{-\pi qK(1+\frac{1}{2})}}{1 - e^{-2\pi qK}}, e^{-i(A_J T_1^{1/2} - \frac{\pi}{4})}},
$$

where we write down the zeroth order term only.

After $T < T_*$, we may once again can use the asymptotic solution $\chi_\pm$ to compare it with the WKB solution $\chi_{WKB}$. The late time solution is given by

$$
\chi(x) = c_+ \chi_+(x) + c_- \chi_-(x).
$$

At $T_*$, $\Omega_1 = J(J+1)$ and $(J(J+1))^{1/2} x_* \approx -J^{1/4}/(-t_0)^{1/4}$.

The primordial power spectrum is proportional to the square of the size of $c_+ - c_-$. Matching at the boundary $x_*$, we have

$$
c_+ = -i \left( \chi_{WKB} \frac{d\chi_-}{dx} - \frac{d\chi_{WKB}}{dx} \chi_- \right)_{x=x_*}, \quad c_- = i \left( \chi_{WKB} \frac{d\chi_+}{dx} - \frac{d\chi_{WKB}}{dx} \chi_+ \right)_{x=x_*}.
$$

If we keep only the zeroth order, the power spectrum becomes

$$
P_{J K M} = \frac{1}{2 \ell^2 (J(J+1))^{3/2}} |c_+ - c_-|^2 = \frac{1 + t_0^2}{2 \ell^2 (J(J+1))^{3/2}} \left( 1 + e^{-2\pi qK} \frac{1 + e^{-2\pi qK}}{1 - e^{-2\pi qK}} \cos 2\Phi \right),
$$

where $\Phi = (J(J+1))^{1/2} x_* + \int_{x_*}^{x_K} \Omega dx + A_J T_1^{1/2} - \frac{\pi}{4}$ and $q_K = 4(-t_0)K/(1 - 3t_0^2)$ contains the direction dependence. Thus, the explicit direction dependence is not suppressed. Note that $\Psi \sim J$ rules the phase factor $\Phi$ for larger $J$. The $K=0$ limit is not well defined, however, this mode is not in our concern since the wavelength of such a perturbation mode is beyond our Hubble horizon.
V. CONCLUSION AND DISCUSSIONS

In this article, we considered the quantization of a massless and minimally coupled scalar field in the Universe, which is initially anisotropic and approaches the de Sitter spacetime. The motivation to consider the initially anisotropic Universe is two-fold: The first motivation is that even if the current Universe is almost isotropic, it does not mean that the Universe is isotropic from the beginning. In fact, Wald’s no-hair theorem ensures that in the presence of a positive cosmological constant an initially anisotropic Universe exponentially approaches a de Sitter spacetime at the later time under the strong or dominant energy condition. Therefore, it would be more generic that the initial Universe is anisotropic. The second motivation comes from the recent observations by WMAP satellite. WMAP measured the temperature fluctuations of cosmic microwave background (CMB) and almost confirmed the predictions from the inflation, during which the Gaussian and statistically isotropic primordial fluctuations are produced. But after the WMAP data were released, several groups have reported the so-called low-$\ell$ anomalies in large angular power of CMB fluctuations, e.g., the suppression of the power of quadrupole, the planarity of quadrupole and octopole CMB maps, the alignment of the preferred directions of quadrupole and octopole moments, and so on. They may not be satisfactorily explained by the standard isotropic initial state, but may be done by the direction-dependent primordial fluctuations.

In this paper, we considered the gravitational theory composed of the Einstein-Hilbert term and a positive cosmological constant. For simplicity, we assume that the late-time inflationary stage is exactly described by the de Sitter solution and ignored the dynamics of an inflaton field for simplicity. We considered two kinds of initially anisotropic Universe, say, the Bianchi I and Bianchi IX models. In each model, there is an exact solution. In the Bianchi I model, we considered the Kasner-de Sitter solution, in which the spacetime geometry is initially a Kasner spacetime and approaches the de Sitter spacetime. In the Bianchi IX model, similarly we considered the Taub-NUT de Sitter solution, in which the spacetime geometry is initially a Taub-NUT spacetime. Note that in the case of Bianchi I, we focused on the case that the Universe has an exact planar symmetry, which is isotropic along two of three spatial axes. In the Bianchi IX model, the exact solution has only two independent scale factors.

After giving the background geometry, we investigated the spectrum of a scalar field, which is the counterpart of the inflaton fluctuation. We discussed how a massless scalar field is quantized in the initial anisotropic stage. The qualitative behaviors of the resultant spectrum in both the Bianchi I and IX models are very similar and therefore, here we summarize our result mainly focusing on the Bianchi I model. We found that in the case with a planar symmetry, there is a well-defined adiabatic vacuum unless $k_3 \neq 0$, where $k_3$ is the comoving momentum along the preferred direction. As we mentioned previously, in a quantum harmonic oscillator system, an adiabatic process usually implies the one where the potential changes slowly enough compared to its size, and the time evolution can be obtained from the zero-th order WKB approximation. We followed this definition for the term adiabatic. In the standard inflationary models, an adiabatic vacuum is also defined in the same way. In our case an adiabatic vacuum state, called an anisotropic vacuum in this paper, was found only in the special solutions of anisotropic Universe, which are regular in the initial times. It was shown that for the moderate modes, $k_3 \sim k$, where $k$ is the total comoving momentum, the scalar power spectrum has an oscillatory behavior in the smaller value of $k$ and a suppression of a large scale power. For the planar mode, $k_3 \ll k$, the adiabaticity parameter vanishes in the earlier times the scalar power spectrum is well defined, but during the intermediate times it becomes greater than unity. Then, the effect of primordial anisotropy is enhanced and in the resultant spectrum the scale-dependence is unsuppressed. For the modes of $k_3 = 0$, the adiabaticity parameter diverges and the WKB approximation is not well defined. But such a mode is not observable.

In the case of the Bianchi IX model, by definition, the modes are discrete, but each mode exhibits a similar behavior to the corresponding mode in the case of Bianchi I model. For the moderate modes that satisfy $K > \ell/(-12t_0)$, where the quantum number $K$ characterizes the angular momentum along the preferred direction ($\sim k_3/H$ in the Bianchi I model), in the resultant power spectrum the angular dependence is suppressed. Note that roughly speaking $t_0$ characterizes the degree of the initial anisotropy, and $\ell$ is related to the late time expansion rate of de Sitter ($\ell = 1/H$). In contrast, for the planar modes $K < \ell/(-12t_0)$, the effect of the primordial anisotropy is not suppressed.

In summary, the anisotropy is not always suppressed in the present models. The anisotropy of the planar modes may leave non-negligible effects on the plane orthogonal to the preferred direction. They would be generic predictions from an initially anisotropic Universe in vacuum. There are important issues left for future studies. To obtain more reliable predictions, our method should be applied to solve the metric perturbations and in particular to find out the role of coupling of one of the tensor polarizations to the scalar mode. The evaluation of the bispectrum or trispectrum would also be important.
Appendix A: Evolution of Bianchi IX Universe

Employing the Hamiltonian formalism, one can study the cosmology in the Bianchi universe in analogy with the classical mechanics. By parameterizing the scale factors of three independent spatial directions as $e^{\alpha+\beta+\sqrt{3}\beta_-}$, $e^{\alpha+\beta-\sqrt{3}\beta_-}$ and $e^{\alpha-2\beta_-}$, the comoving evolution can be specified by two variables $(\beta_+, \beta_-)$. In analogy with the classical mechanism, the evolution of the Bianchi IX Universe can be described by the motion of a particle on the $(\beta_+, \beta_-)$ plane with the potential

$$V_{IX}(\beta_+, \beta_-) = 2e^{4\beta_+} \cosh (4\sqrt{3}\beta_-) + e^{-8\beta_-} - 2e^{4\beta_+} - 4e^{-2\beta_+} \cosh (2\sqrt{3}\beta_-).$$  \hspace{1cm} (A1)

The potential is almost flat near the origin ($\beta_+ = 0$, $\beta_- = 0$) but there is a steep exponential wall which forms an equilateral. Note that the corresponding potential in the Bianchi I model is vanishing, say $V_I(\beta_+, \beta_-) = 0$. The cosmic motion is almost confined inside this wall. There are three valleys into the potential along the straight lines, $\beta_- = 0$ ($\beta_+ > 0$), $\beta_- = \sqrt{3}\beta_+$ and $\beta_- = -\sqrt{3}\beta_+$ (for $\beta_- < 0$). Only along these valleys, the Bianchi IX Universe can evolve to have highly anisotropic geometry with large $\beta_+$ or $\beta_-$. These three valleys are equivalent under the exchange of label of the axes.

The evolution of Taub-NUT de Sitter spacetime is initially along one of these valleys on the $(\beta_+, \beta_-)$ plane, starting from the infinity. In fact, one can read off the correspondence of the variables $\alpha + \beta + \sqrt{3}\beta_- = \ln a$, $\alpha + \beta - \sqrt{3}\beta_- = \ln a$, and thus $\alpha - 2\beta_+ = \ln c$, and one can easily find $\alpha = (1/3) \ln(a^2\beta_-)$, $\beta_+ = (1/3) \ln(a/c)$ and $\beta_- = 0$.

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