S₄ Flavor Symmetry Embedded into SU(3) and Lepton Masses and Mixing

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Abstract

Based on the assumption that an S₄ flavor symmetry is embedded into SU(3), a lepton mass matrix model is investigated. A Froggatt-Nielsen type model is assumed, and the flavor structures of the masses and mixing are caused by VEVs of SU(2)ₗ-singlet scalars φᵤ and φ₅ which are nonets (8+1) of the SU(3) flavor symmetry, and which are broken into 2+3+3' and 1 of S₄. If we require the invariance under the transformation (φ⁸, φ¹) → (−φ⁸, +φ¹) for the superpotential of the nonet field φ⁸+1, the model leads to a beautiful relation for the charged lepton masses. The observed tribimaximal neutrino mixing is understood by assuming two SU(3) singlet right-handed neutrinos νᵣ(±) and an SU(3) triplet scalar χ.

1 Introduction

The observed mass spectra and mixings of the fundamental particles will provide promising clues to unified understanding of the quarks and leptons. Especially, in the lepton sector, the following characteristic features have been observed [1]:

(i) The observed charged lepton masses (mₑ, mₘ, mₜ) satisfy the relation [2, 3]

\[ mₑ + mₘ + mₜ = \frac{2}{3}(\sqrt{mₑ} + \sqrt{mₘ} + \sqrt{mₜ})^2, \]  

(1.1)

with remarkable precision;

(ii) The observed neutrino mixing Uₜ is approximately given by the so-called tribimaximal mixing [4]

\[ UₜB = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} 
\end{pmatrix}. \]  

(1.2)

Such characteristic features have not been seen in the quark sector. For example, the mixing form (1.2) suggests that the mixing can be described by Clebsh-Gordan-like coefficients, while, for the Cabibbo-Kobayashi-Maskawa mixing in the quark sector, such a characteristic feature has not been seen, although we have known some relations among the mixing angles and quark mass ratios. Therefore, for a start, in the present paper, we investigate the lepton masses and mixings.

In order to understand the relation (1.1), for example, we assume that there are three scalars φᵢ (i = 1, 2, 3), and the values of the charged lepton masses mₑᵢ are proportional to the square of the vacuum expectation values (VEVs) vᵢ = ⟨φᵢ⟩ of the scalars φᵢ, mₑᵢ = kvᵢ² (in the
Ref. [3, 5, 6], for instance, a seesaw type model \((M_e)_{ij} = \delta_{ij}v_1(M_E)^{-1}v_2\) has been assumed. We define singlet \(\phi_\sigma\) and doublet \((\phi_\pi, \phi_\eta)\) of a permutation symmetry \(S_3\) [7] by

\[
\begin{pmatrix}
\phi_\pi \\
\phi_\eta \\
\phi_\sigma
\end{pmatrix}
= \begin{pmatrix}
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix},
\]  

(1.3)

from the three objects \((\phi_1, \phi_2, \phi_3)\), and we consider the following \(S_3\) invariant scalar potential \(V(\phi)\) [3, 8, 9]:

\[
V(\phi) = m^2(\phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2) + \lambda_1(\phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2)^2 + \lambda_2\phi_\pi^2(\phi_\pi^2 + \phi_\eta^2).
\]  

(1.4)

The minimizing condition of the potential (1.4) leads to the relation

\[
v_\pi^2 + v_\eta^2 = v_\sigma^2.
\]  

(1.5)

The relation (1.5) means

\[
v_1^2 + v_2^2 + v_3^2 = \frac{2}{3}(v_1 + v_2 + v_3)^2,
\]  

(1.6)

because

\[
v_1^2 + v_2^2 + v_3^2 = v_\pi^2 + v_\eta^2 + v_\sigma^2 = 2v_\sigma^2 = 2\left(\frac{v_1 + v_2 + v_3}{\sqrt{3}}\right)^2.
\]  

(1.7)

Therefore, we can obtain the mass relation (1.1). Here, note that although the scalar potential (1.4) is invariant under the \(S_3\) symmetry, but it is not a general one of the \(S_3\) invariant form. As pointed out in Ref. [9], the scalar potential with a general form cannot lead to the relation (1.5). For the derivation of the VEV relation (1.5), it is essential to choose the specific form (1.4) of the \(S_3\) invariant terms. Similar formulation is also possible for other discrete symmetries \(A_4\) [10] and \(S_4\) (see below). However, in such a symmetry, we still need an additional specific selection rule. What is the meaning of such a specific selection? In the present paper, we investigate this problem by assuming that the \(S_4\) flavor symmetry is embedded into SU(3).

Recently, a superpotential which leads to the relation (1.5) has proposed by Ma [11] on the basis of a symmetry \(\Sigma(81)\). Stimulated by the Ma’s idea, the author [10] has also investigated a similar superpotential on the basis of a symmetry \(A_4\). Here, based on an \(S_4\) flavor symmetry instead of the \(A_4\) symmetry, let us review the superpotential \(W\) which gives the relation (1.5).

We denote singlet and doublet of \(S_4\) as \(\phi_\sigma\) and \(\phi_D = (\phi_\pi, \phi_\eta)^T\), respectively, as well as those in \(S_3\). In order to write the superpotential for the scalar fields \(\phi_\sigma\) and doublet \(\phi_D\) of \(S_4\), we put the following phenomenological rule [10]: the field \(\phi_a\) \((a = \sigma, D)\) to the power \(n\)th, \((\phi_a)^n\) \((n = 1, 2, 3)\), appears always accompanied with the factor \(1/n!\) in the superpotential \(W\). Under this phenomenological rule, we can uniquely write the superpotential of \(\phi_\sigma\) and \(\phi_D\) as

\[
W(\phi) = \frac{1}{2!}m(\phi_\sigma^2 + \phi_D^T\phi_D) + \lambda\left(\frac{1}{2!}\phi_\sigma\phi_D^T\phi_D + \frac{1}{3!}\phi_\sigma^3\right)
\]
\begin{equation}
\frac{1}{2} m \left( \phi_\sigma^2 + \phi_\pi^2 + \phi_\eta^2 \right) + \frac{1}{2} \lambda \left[ \left( \phi_\pi^2 + \phi_\eta^2 \right) \phi_\sigma + \frac{1}{3} \phi_\sigma^3 \right].
\end{equation}

The potential (1.8) can also lead the relation (1.5). What is the meaning of this phenomenological rule?

On the other hand, we have to consider a mechanism which yields the charged lepton masses $m_{ei} \propto v_i^2$, i.e. the effective Hamiltonian for the charged lepton sector

\begin{equation}
H_e^{eff} = \left[ \bar{e}_{L1} \phi_1^2 e_{R1} + \bar{e}_{L2} \phi_2^2 e_{R2} + \bar{e}_{L3} \phi_3^2 e_{R3} \right].
\end{equation}

We will propose a Froggatt-Nielsen type model [12], $H_e^{eff} = (\ell_L H_d^T \phi \ell_R)$ in Sec.4.

Now, let us return the topic of the tribimaximal mixing. From the definition (1.2), we can denote the fields $(\psi_1, \psi_2, \psi_3)$ as

\begin{equation}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix} = U_{TB} \begin{pmatrix}
\psi_\eta \\
\psi_\sigma \\
\psi_\pi
\end{pmatrix}.
\end{equation}

The observed neutrino mixing (1.2) means that when the mass eigenstates of the charged leptons are given by the $(\psi_1, \psi_2, \psi_3)$ basis, the mass eigenstates of the neutrinos are given by the $(\psi_\eta, \psi_\sigma, \psi_\pi)$ basis. Therefore, the problem is to find a model where the charged lepton mass eigenstates are $(e_1, e_2, e_3)$, while the neutrino mass eigenstates are given by $(\nu_\eta, \nu_\sigma, \nu_\pi)$ with the mass hierarchy $m_\eta^2 < m_\sigma^2 < m_\pi^2$ (or $m_\eta^2 < m_\eta^2 < m_\pi^2$). In the present paper, we will investigate such a model based on an $S_4$ model. Here, note that the fermions $(\psi_1, \psi_2, \psi_3)$ is a triplet of $S_4$, but the basis $(\psi_\eta, \psi_\sigma, \psi_\pi)$ is not in any irreducible representations of $S_4$, while the scalar $(\phi_1, \phi_2, \phi_3)$ is not irreducible representation of $S_4$, but $(\phi_\pi, \phi_\eta)$ and $\phi_\sigma$ are doublet and singlet of $S_4$.

Thus, the characteristic features (1.1) and (1.2) in the lepton sector may be understood from the language of $S_4$ (also $S_3$ or $A_4$). However, as seen from the above review, the characteristic features (1.1) and (1.2) cannot be understood from the $S_4$ symmetry only. We need some additional assumptions. In this paper, we will investigate these problems under an assumption that the present $S_4$ symmetry is embedded into an SU(3) symmetry [13]. In the next section, the singlet $\phi_\eta$ and doublet $(\phi_\pi, \phi_\eta)$ will be understood as members of a nonet scalar $\phi$ [SU(3)], and the VEV relation (1.5) will be derived by requiring that $W(\phi)$ is invariant under a $Z_2$ symmetry.

We know that the three masses in any sectors of quarks and leptons are completely different among them. Therefore, if we assume a flavor symmetry, the symmetry must finally be broken completely. Usually, a relation which we derive in the exact symmetry limit is only approximately satisfied under the symmetry breaking. Although we derive the VEV relation (1.5) under the $S_4$ symmetry, the problem is whether the VEV relation (1.5) which is obtained under the $S_4$ symmetry is spoiled or not when we introduce such a symmetry breaking. In Sec.3, we will demonstrate that such a symmetry breaking term without spoiling the relation (1.5) is indeed possible.
In Sec. 4, in order to give the charged lepton masses and tribimaximal neutrino mixing, we will discuss the effective Hamiltonian by assuming an Froggatt-Nelsen \cite{12} type model. Finally, Sec. 5 will be devoted to the summary and concluding remarks.

2 VEVs of SU(3) nonet scalars

The goal in the present section is to obtain the VEV relation (1.6) \[i.e. \ (1.5)\]. As seen in the previous section, in order to obtain the desirable results (1.5), we need assume an equal weight between the doublet and singlet terms of $S_4$. In the present paper, we assume that the $S_4$ symmetry is embedded into an SU(3) symmetry. The doublet $(\phi_\pi, \phi_\eta)$ and singlet $\phi_\sigma$ of $S_4$ are embedded in the 6 and $(8 + 1)$ of SU(3) \cite{13}. In the present paper, we assume that the doublet $(\phi_\pi, \phi_\eta)$ and singlet $\phi_\sigma$ originate in SU(3) octet and singlet, respectively. The essential assumption in the present paper is that the fields $\phi_u$ and $\phi_d$ always appear in the theory with the form of the nonet of U(3):

$$\phi = \begin{pmatrix} \phi_1^1 & \phi_1^2 & \phi_1^3 \\ \phi_2^1 & \phi_2^2 & \phi_2^3 \\ \phi_3^1 & \phi_3^2 & \phi_3^3 \end{pmatrix},$$

where

$$\begin{align*}
\phi_1^1 &= \frac{1}{\sqrt{3}} \phi_\sigma + \frac{1}{\sqrt{6}} \phi_\eta, \\
\phi_2^1 &= \frac{1}{\sqrt{3}} \phi_\sigma - \frac{1}{\sqrt{6}} \phi_\eta - \frac{1}{\sqrt{2}} \phi_\pi, \\
\phi_3^1 &= \frac{1}{\sqrt{3}} \phi_\sigma - \frac{1}{\sqrt{6}} \phi_\eta + \frac{1}{\sqrt{2}} \phi_\pi,
\end{align*}$$

and the index $f \ (f = u, d)$ has been dropped.

The outline to obtain the superpotential form (1.8) in the present scenario is as follows: The SU(3) invariant superpotential for the nonet fields $\phi_f \ (f = u, d)$ are given by

$$W(\phi_f) = \frac{1}{2} m_f \text{Tr}(\phi_f \phi_f) + \frac{1}{2\sqrt{3}} \lambda_f \text{Tr}(\phi_f \phi_f \phi_f).$$

Since, in the next section, we want to assign charges +1 and −1 of a $Z_3$ symmetry to the fields $\phi_u$ and $\phi_d$, respectively, we also assign the $Z_3$ charges +1 and −1 to the mass parameters $m_u$ and $m_d$ in Eq. (2.3), respectively. However, since we do not consider a mass term $\text{Tr}(\phi_u \phi_d)$, we do not consider a mass parameter with the $Z_3$ charge zero. Hereafter, in the present section, for convenience, we will drop the index $f$, since the cross terms between $\phi_u$ and $\phi_d$ do not appear. In the superpotential (2.3), although the term $\text{Tr}(\phi \phi)$ gives the desirable term $\phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2 + \cdots$ of $S_4$, the cubic term $\text{Tr}(\phi \phi \phi)$ gives

$$\text{Tr}(\phi \phi \phi) = \sqrt{3} \left[ \frac{1}{\sqrt{2}} \left( -\phi_\pi^2 + \frac{1}{3} \phi_\eta^2 \right) \phi_\eta + (\phi_\pi^2 + \phi_\eta^2) \phi_\sigma + \frac{1}{3} \phi_\sigma^3 \right] + \cdots,$$

where the terms “…” denote terms which include 3 and 3′ of the subgroup $S_4$. Therefore, the potential (2.3) cannot give the relation (1.5). We must drop the first term in the cubic terms...
For this purpose, we introduce a $Z_2$ symmetry, and we assign the $Z_2$ parities $-1$ and $+1$ (the $Z_2$ charge $+1$ and 0) for the octet part $\phi^{(8)}$ and singlet part $\phi^{(1)}$ of the nonet field $\phi$, respectively. The symmetry $Z_2$ breaks U(3) into SU(3). (In other words, in the present model, the flavor symmetry U(3) is explicitly broken from the beginning by the $Z_2$ symmetry.) Under the requirement of the $Z_2$ invariance, i.e. the invariance under the transformation

$$ (\phi^{(8)}, \phi^{(1)}) \rightarrow (-\phi^{(8)}, +\phi^{(1)}), \quad (2.5) $$

the terms $\text{Tr}(\phi^{(8)}\phi^{(8)}\phi^{(8)})$, i.e. $(-\phi^{2}_\pi + (1/3)\phi^{2}_\eta)\phi_\eta + \cdots$, are forbidden. Thus, the superpotential (2.3) with the $Z_2$ invariance leads to

$$ W(\phi) = \frac{1}{2} m \left[ \text{Tr}(\phi^{(8)}\phi^{(8)}) + \phi^2_\sigma \right] + \frac{1}{2} \lambda \phi_\sigma \left[ \text{Tr}(\phi^{(8)}\phi^{(8)}) + \frac{1}{3} \phi^3_\sigma \right] $$

$$ = \frac{1}{2} m \left( \phi^2_\sigma + \phi^2_\pi + \phi^2_\eta \right) + \frac{1}{2} \lambda \left[ (\phi^2_\pi + \phi^2_\eta)\phi_\sigma + \frac{1}{3} \phi^3_\sigma \right] + \cdots. \quad (2.6) $$

The form (2.6) is just identical with (1.8) except for the “···” terms. As we show below, the potential (2.6) can give the desirable VEV relation (1.5) together with $\langle \phi^2_i \rangle = 0$ ($i \neq j$).

From the superpotential (2.6) with the $Z_2$ invariance, we obtain the VEV relation (1.5) as follows: From the condition

$$ \frac{\partial W}{\partial (\phi^{(8)})^i_j} = m(\phi^{(8)})^i_j + \lambda \phi_\sigma (\phi^{(8)})^i_j = 0, \quad (2.7) $$

we obtain

$$ m + \lambda \phi_\sigma = 0, \quad (2.8) $$

for $(\phi^{(8)})^i_j \neq 0$. By eliminating $m$ from Eq.(2.8) and the condition

$$ \frac{\partial W}{\partial \phi_\sigma} = m \phi_\sigma + \frac{1}{2} \lambda \left[ \text{Tr}(\phi^{(8)}\phi^{(8)}) + \phi^2_\sigma \right] = 0, \quad (2.9) $$

we obtain the relation

$$ \phi^2_\sigma = \text{Tr}(\phi^{(8)}\phi^{(8)}) = \phi^2_\pi + \phi^2_\eta + \cdots, \quad (2.10) $$

where “···” denotes the contributions of 3 and 3′ of $S_4$.

The result (2.10) is still not our goal, because the relation contains the VEVs of the 3 and 3′ of $S_4$. So far, we have not discussed the splitting among the $S_4$ multiplets. Now, we bring a soft symmetry breaking of SU(3) into $S_4$ with an infinitesimal parameter $\varepsilon$ into the mass term of $W(\phi)$ as

$$ \text{Tr}(\phi^{(8)}\phi^{(8)}) \Rightarrow \phi_\pi \phi_\pi + \phi_\eta \phi_\eta + (1 + \varepsilon) \sum_{i \neq j} (\phi^{(8)})^i_j (\phi^{(8)})^i_j, \quad (2.11) $$

by hand. (At present, we do not refer the origin of the symmetry breaking. The SU(3) flavor symmetry is explicitly (not spontaneously) broken with the order of $\varepsilon$.) Recall that when we
obtain the relation (2.8), we have assumed $(\phi^{(8)})^i_j \neq 0$. Now, the conditions (2.7) are modified into the following conditions:

\begin{align}
(1 + \varepsilon)m + \lambda \phi_{\sigma} [\phi^{(8)}]_j^i = 0 & \quad (i \neq j), \\
(m + \lambda \phi_{\sigma}) \phi_a = 0 & \quad (a = \pi, \eta),
\end{align}

Therefore, we must take either $(\phi^{(8)})^i_j = 0 \quad (i \neq j)$ or $\phi_a = 0 \quad (a = \pi, \eta)$ for $\varepsilon \neq 0$. When we choose the solution

\begin{equation}
\langle (\phi^{(8)})^i_j \rangle = 0 \quad (i \neq j),
\end{equation}

we can obtain the desirable relation (1.5). (However, it is possible that we can also take another solution with $\phi_{\pi} = \phi_{\eta} = 0$ and $(\phi^{(8)})^i_j \neq 0$. The VEV solutions are not unique. The result (1.5) is merely one of the possible solutions.)

Thus, we have obtained not only the desirable VEV relation (1.5), but also the results (2.14). It should be worthwhile noticing that if we have assume the superpotential (2.3) without requiring the $Z_2$ invariance, we could obtain neither (1.5) nor (2.14).

### 3 Superpotential with symmetry breaking

Since we know that the three masses in any sectors of quarks and leptons are completely different among them, we must consider that any flavor symmetry which we introduced should finally be broken completely. Although the superpotential (1.8) can give the VEV relation (1.5), it cannot fix the ratio $v_{\pi}/v_{\eta}$. In order to fix the ratio $v_{\pi}/v_{\eta}$, we consider the existence of an $S_4$ symmetry breaking term $W_{SB}$. Then, the problem is whether the VEV relation (1.5) which has been obtained under the $S_4$ symmetry which is embedded into SU(3) is spoiled or not by introducing such a symmetry breaking, because, usually, a relation which we have derived under an exact symmetry is only approximately satisfied under the symmetry breaking. In the present section, we will demonstrate that such a symmetry breaking term without spoiling the relation (1.5) is indeed possible.

We consider that the $S_4$ invariant superpotential (1.8) is softly broken. Since we want $v_{\pi}/v_{\eta} \neq 1$, the breaking should appear in the doublet part of $S_4$. In order to express the $S_4$ symmetry breaking term explicitly, we define the following symmetry breaking parameters $B^{(8)}$ and $B^{(1)}$ with $3 \times 3$ matrix forms,

\begin{align}
B^{(8)} &= \text{diag} \left( \frac{2}{\sqrt{6}} b_{\eta}, -\frac{1}{\sqrt{6}} b_{\eta} - \frac{1}{\sqrt{2}} b_{\pi}, -\frac{1}{\sqrt{6}} b_{\eta} + \frac{1}{\sqrt{2}} b_{\pi} \right), \\
B^{(1)} &= \text{diag} \left( \frac{1}{\sqrt{3}} b_{\sigma}, \frac{1}{\sqrt{3}} b_{\sigma}, \frac{1}{\sqrt{3}} b_{\sigma} \right),
\end{align}

which behave as if those were octet and singlet of SU(3), respectively, where

\begin{equation}
b_{\eta} = \sqrt{2} \sin \beta, \quad b_{\pi} = \sqrt{2} \cos \beta, \quad b_{\sigma} = 1,
\end{equation}

and the factor $\sqrt{2}$ in Eq.(3.2) has been chosen as $b_{\pi}^2 + b_{\eta}^2 = 2$ compared with $b_{\sigma}^2 = 1$. Then, we can express the symmetry breaking term as the form

\[ W_{SB} = \frac{\sqrt{3}}{2} \varepsilon m \left[ \text{Tr}(B^{(8)} \phi^{(8)} \phi^{(8)}) + \text{Tr}(B^{(1)} \phi^{(1)} \phi^{(1)}) \right] \]
\[ W = \frac{1}{2} \sum m \left[ -2 \phi_\pi \phi_\eta \cos \beta - (\phi_\pi^2 - \phi_\eta^2) \sin \beta + \phi_\sigma^2 \right], \quad (3.3) \]

where the factor \( \sqrt{3}/2 \) has been chosen as the coefficients in the expression (3.3) correspond to those in the unbroken form (1.8). Although the term \( \text{Tr}(B(1)\phi(1)^2) = \phi_\sigma^2/\sqrt{3} \) in (3.3) does not break the \( S_4 \) symmetry, it has been added by hand in order that the term \( W_{SB} \) (in other words, the parameter \( \varepsilon \)) does not affect the VEV relation (1.5).

As the result, we can write the superpotential including the symmetry breaking term as follows:

\[ W = \frac{1}{2} \sum m \left\{ \phi_\pi^2 + \phi_\eta^2 + (1 + \varepsilon) \phi_\sigma^2 - \varepsilon [2 \phi_\phi_\eta \cos \beta + (\phi_\pi^2 - \phi_\eta^2) \sin \beta] \right\} + \frac{1}{2} \lambda \phi_\sigma \left( \phi_\eta^2 + \phi_\pi^2 + \frac{1}{3} \phi_\sigma^2 \right). \quad (3.4) \]

Since

\[ \frac{\partial W}{\partial \phi_\pi} = [m + \lambda \phi_\sigma - \varepsilon m \sin \beta] \phi_\pi - \varepsilon m \phi_\eta \cos \beta, \quad (3.5) \]

\[ \frac{\partial W}{\partial \phi_\eta} = [m + \lambda \phi_\sigma + \varepsilon m \sin \beta] \phi_\eta - \varepsilon m \phi_\pi \cos \beta, \quad (3.6) \]

\[ \frac{\partial W}{\partial \phi_\sigma} = m(1 + \varepsilon) \phi_\sigma + \frac{1}{2} \lambda (\phi_\pi^2 + \phi_\eta^2 + \phi_\sigma^2), \quad (3.7) \]

the minimizing conditions of the potential leads to the relations

\[ \tan \beta = \frac{v_\pi^2 - v_\eta^2}{2v_\pi v_\eta}, \quad (3.8) \]

\[ v_\pi^2 + v_\eta^2 = v_\sigma^2, \quad (3.9) \]

\[ m(1 + \varepsilon) + \lambda v_\sigma = 0. \quad (3.10) \]

Note that the derivation of the relation (3.8) is independent of the explicit values of \( m, \lambda \) and \( \varepsilon \), and the derivation of the relation (3.9) is independent of the explicit values of \( m, \lambda, \varepsilon \) and \( \beta \). Thus, we can fix the value of \( v_\pi/v_\eta \) by the parameter \( \beta \) in \( W_{SB} \) without spoiling the VEV relation (1.5) [(3.9)]. Also note that the limit \( m_\varepsilon \to 0 \) corresponds to the limit \( v_\eta \to -v_\sigma/\sqrt{2} \) (i.e. \( v_\eta^2 = v_\pi^2 \)), so that the limit \( m_e \to 0 \) corresponds to \( \beta \to 0 \).

When we define the parameters \( z_i = \sqrt{m_{ei}}/\sqrt{m_\varepsilon + m_\mu + m_\tau} \), from the observed values \( \mathbf{M} \) of the charged lepton masses, we obtain the numerical values \( z_1 = 0.016473, z_2 = 0.236869 \) and \( z_3 = 0.971402 \), so that, for the VEVs of \( \phi_\alpha \) defined by Eq.(1.3) [(2.2)], we obtain \( z_\pi = 0.519393, z_\eta = -0.479824 \) and \( z_\sigma = 1/\sqrt{2} = 0.707106 \). Therefore, we can estimate the value of \( \beta \) as follows:

\[ \sin \beta = \frac{z_\eta^2 - z_\pi^2}{z_\eta^2 + z_\pi^2} = 4z_\eta^2 - 1 = -0.079078, \quad \beta = -4.5355^\circ, \quad (3.11) \]

where we have chosen the phase convention of \( \beta \) as \( \cos \beta = -2z_\pi z_\eta/(z_\eta^2 + z_\pi^2) > 0 \).
From the point of view of the parameter physics, the introducing the symmetry breaking term (3.3) is merely replacing the parameter $v_\pi/v_\eta$ by another parameter $\beta$. What is important is that we can indeed introduce a symmetry breaking term without spoiling the relation (1.5).

### 4 Effective Hamiltonian

If we regard the scalars $\phi_u$ and $\phi_d$ as $\text{SU}(2)_L$ doublets, such a model with multi-Higgs doublets causes a flavor changing neutral current (FCNC) problem. Therefore, we must consider that the fields $\phi_u$ and $\phi_d$ are $\text{SU}(2)_L$ singlets. In the present paper, we assume a Froggatt-Nielsen type model

$$H^{eff} = y_e \ell_L H_d^d \phi_d \xi + y_\nu \ell_L H_u^d \phi_u \chi + y_R \nu_R \Phi_R \nu_R^*,$$

where $\ell_{iL}$ are $\text{SU}(2)_L$ doublet leptons $\ell_{iL} = (\nu_{iL}, e_{iL})$, $H_d^d$ and $H_u^d$ are conventional $\text{SU}(2)_L$ doublet Higgs scalars, $\phi_f$ ($f = u, d$), $\xi$ and $\chi$ are $\text{SU}(2)_L$ singlet scalars, and $\Lambda$ is a scale of the effective theory. We consider that $\langle \phi_f \rangle / \Lambda$, $\langle \xi \rangle / \Lambda$ and $\langle \chi \rangle / \Lambda$ are of the order of 1. The scalar $\Phi_R$ has been introduced in order to generate the Majorana mass $M_R$ of the right-handed neutrino $\nu_R$. As we note later, in the present model, the right-handed neutrinos $\nu_R = (\nu_R^+, \nu_R^-) / \sqrt{2}$ are singlets of the $\text{SU}(3)$ flavor. The role of $\xi = (\xi^+ + \xi^-) / \sqrt{2}$ and $\chi$ will be explained later. In order to understand the appearance of the combinations $H_d^d \phi_d \phi_d \xi$ and $H_u^d \phi_u \chi$, we assume two $Z_3$ symmetries ($Z_3$ and $Z_3'$ in Table 1). Those quantum number assignments are given in Table 1. However, even with those quantum numbers, we cannot distinguish the state $\phi_f^1$ from $\phi_f \phi_f$. For example, the interaction $\bar{\ell}_L H_d \phi_d \xi e_R$ is possible in addition to the interaction $\bar{\ell}_L H_d \phi_d \phi_d \xi e_R$. Although we have started from an SUSY scenario in the previous section, now, we have adopted an effective Hamiltonian which is not renormalizable. Therefore, in principle, the interaction $\bar{\ell}_L H_d \phi_d \phi_d \xi e_R$ cannot be ruled out. For the moment, in order to forbid such an undesirable term, we assume that the fields which can appear in the effective Hamiltonian are confined to holomorphic ones.

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**Table 1** SU(3) and $S_4$ assignments of the fields

| Fields | SU(2)$_L$ | SU(3) | $S_4$ | $Z_3$ | $Z_3'$ | $Z_2$ |
|--------|-----------|--------|-------|-------|--------|-------|
| $\ell_L$ | 2 | 3 | 3' | 0 | 0 | 0 |
| $e_R$ | 1 | 3 | 3' | 0 | 0 | 0 |
| $\nu_R^{(\pm)}$ | 1 | 1 | 1 | 0 | 0 | 0/+1 |
| $\phi_u$ | 1 | 1+8 | 1 + (2 + 3 + 3') | +1 | +1 | 0/+1 |
| $\phi_d$ | 1 | 1+8 | 1 + (2 + 3 + 3') | -1 | -1 | 0/+1 |
| $\xi^{(\pm)}$ | 1 | 1 | 1 | 0 | -1 | 0/+1 |
| $\chi$ | 1 | 3 | 3' | +1 | -1 | 0 |
| $H_u^d$ | 2 | 1 | 1 | +1 | 0 | 0 |
| $H_d^d$ | 2 | 1 | 1 | -1 | 0 | 0 |
| $\Phi_R$ | 1 | 1 | 1 | 0 | 0 | 0 |
(a) Charged lepton sector

Recall that we have already assumed the invariance of the superpotential under the $Z_2$ transformation (2.5) in order to drop the cubic part of the octet $\phi^{(8)}$. Therefore, the term $\phi \phi$ means $\phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(1)}$ under the $Z_2$ invariance. However, in order to give $m_{ei} \propto \langle \phi^i \rangle^2$, what we want is not $\phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(1)}$, but $\phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(1)} + \phi^{(8)} \phi^{(8)} + \phi^{(1)} \phi^{(8)}$. In order to evade this problem, we introduce additional fields $\xi^{(+)}$ and $\xi^{(-)}$ whose $Z_2$ parity are +1 and −1, respectively. The effective interactions in the charged lepton sector are given by

$$H_{\text{eff}}^e = \frac{y_e}{\sqrt{2}} \bar{e}_L (\phi_d)^i_j (\phi_d)^{k} (\xi^{(+)}) + (\phi^{(+)}) e_R,$$

where we have dropped the Higgs scalar $H^d_L$ since we discuss flavor structure only. The expression (4.2) becomes

$$H_{\text{eff}}^e = \frac{y_e}{\sqrt{2}} \bar{e}_L [(\phi_d^{(8)} \phi_d^{(8)} + \phi_d^{(1)} \phi_d^{(1)}) \xi^{(+)}} + (\phi_d^{(8)} \phi_d^{(8)} + \phi_d^{(1)} \phi_d^{(8)}) \xi^{(-)}] e_R.$$

Since we have assumed that $\xi^{(+)}$ and $\xi^{(-)}$ appear symmetrically in the theory, we also assume

$$\langle \xi^{(+)} \rangle = \langle \xi^{(-)} \rangle \equiv v_\xi.$$

Then, we obtain the effective Hamiltonian for the charged leptons

$$H_{\text{eff}}^e = \frac{y_e v_d v_\xi}{\sqrt{2} \Lambda^d} \sum_i \bar{e}_L (\phi_d^{(8+1)})^i j e_R,$$

where $v_d = \langle H^d_L \rangle$. Since the fields $(\phi_d)^i_j$ are defined by Eq.(2.2), we can obtain the charged lepton mass relation (1.1) from the VEV relation (1.6).

However, the present mechanism to obtain $m_{ei} \propto \langle \phi^i \rangle^2$ is somewhat artificial. The present mechanism will be improved in the future model. [Of course, there is a possibility that the superpotential (2.3) must exactly be invariance under the $Z_2$ symmetry, but the effective Hamiltonian (4.1) does not need to be invariance under the $Z_2$ symmetry. Then, we can consider a model without $\xi^{(\pm)}$.]

(b) Neutrino sector

In the present model, the right-handed neutrinos $\nu^{(\pm)}$ are singlets of SU(3). Therefore, in the neutrino seesaw mass matrix $M_\nu = m^\nu_L M_R^{-1}(m^\nu_R)^T$, $M_R$ is a $1 \times 1$ matrix and $m^\nu_R$ is a $3 \times 1$ matrix. In order to compensate for the absence of the conventional triplet neutrinos $\nu_R$, a new scalar $\chi$ which is a triplet of SU(3) has been introduced. The neutrino Dirac mass terms are given by the following effective Hamiltonian

$$H_{\text{eff}}^{\nu R} = \frac{y_\nu v_u}{\Lambda^u} \bar{\nu}_L (\phi_u)^i_j \langle \chi_j \rangle (\nu^{(+)}_R + \nu^{(-)}_R),$$

where $v_u = \langle H^u_L \rangle$. It is likely that the scalar potential $V(\chi)$ for the SU(3) triplet $\chi$ has a specific VEV solution

$$\langle \chi_1 \rangle = \langle \chi_2 \rangle = \langle \chi_3 \rangle \equiv v_\chi.$$
We can choose a solution by a soft $S$ when we assume the VEVs (4.7), we obtain

$$
H_{Dirac}^{eff} = y_{\nu} \frac{v_{\nu} v_{\chi}}{\sqrt{2} \Lambda^2} (\bar{\nu}_{\eta} v_{\sigma} \bar{v}_{\pi}) L \left[ \begin{pmatrix} v_{\eta} \\ 0 \\ v_{\pi} \end{pmatrix} \nu_{R}^{(-)} + \begin{pmatrix} 0 \\ v_{\sigma} \\ 0 \end{pmatrix} \nu_{R}^{(+)} \right],
$$

where $v_{a} = \langle \phi_{a} \rangle$ ($a = \pi, \eta, \sigma$) (for convenience, we have dropped the index $u$). Therefore, we obtain the effective neutrino mass matrix on the $(\eta, \sigma, \pi)$ basis,

$$
U_{TB}^{T} M_{\nu} U_{TB} \equiv M_{\nu}^{(\eta \sigma \pi)} = \frac{1}{M_{R}^{(-)}} \begin{pmatrix} v_{\eta}^2 & 0 & v_{\pi} v_{\eta} \\ 0 & 0 & 0 \\ v_{\pi} v_{\eta} & 0 & v_{\pi}^2 \end{pmatrix} + \frac{1}{M_{R}^{(+)}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_{\sigma}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

where $M_{R}^{(\pm)} = y_{\nu}^{(\pm)} \langle \Phi_{R} \rangle$, and we have dropped the common factors $(y_{\nu} v_{\nu} v_{\chi} / \sqrt{2} \Lambda^2)^2$. By the way, the ratio $v_{\pi}/v_{\eta}$ cannot be determined from the potential (2.6), and the ratio is determined by a soft $S_4$ symmetry breaking term $W_{SB}$ which has been discussed in the previous section. We can choose a solution $v_{\pi} = 0$ in the superpotential $W(\phi_{a})$ by adjusting the parameter $\beta$ in $W_{SB}$, differently from the case of $W(\phi_{d})$. Then, the neutrino mass matrix (4.9) becomes a diagonal form $D_{\nu} = (1/M_{R}^{(-)}) \text{diag}(v_{\eta}^2, 0, 0) + (1/M_{R}^{(+)}) \text{diag}(0, v_{\sigma}^2, 0)$. Since the mass matrix $M_{\nu}$ on the $(\nu_{1}, \nu_{2}, \nu_{3}) = (\nu_{e}, \nu_{\mu}, \nu_{\tau})$ basis is given by

$$
M_{\nu} = U_{TB}^{T} M_{\nu}^{(\eta \sigma \pi)} U_{TB}^{T} = U_{TB}^{T} D_{\nu} U_{TB},
$$

we can obtain the tribimaximal mixing

$$
U_{\nu} = U_{TB},
$$

and the neutrino masses

$$
m_{\nu1} = k v_{\eta}^2, \quad m_{\nu2} = k v_{\sigma}^2, \quad m_{\nu3} = 0,
$$

for the case of $M_{R}^{(+)} = M_{R}^{(-)} \equiv M_{R}$, where $k = (y_{\nu} v_{\nu} v_{\chi})^2 / 2 M_{R} \Lambda^4$ and $(\nu_{\eta}, \nu_{\sigma}, \nu_{\pi})$ has been renamed $(\nu_{1}, \nu_{2}, \nu_{3})$ according to the conventional naming.

However, since we have taken $v_{\pi} = 0$, the value of $v_{\eta}$ satisfies $v_{\eta}^2 = v_{\sigma}^2$ from the relation (1.5), so that the result (4.12) gives $m_{\nu1} = m_{\nu2}$. The observed value $\Delta m_{2\text{sol}}^2$ is small, but it is not zero. Therefore, we must consider a small deviation between the first and second terms in (4.9) (i.e. $M_{R}^{(+)} \neq M_{R}^{(-)}$). Since the value $M_{R}^{(-)}/M_{R}^{(+)}$ is free in the present model, we cannot predict an explicit value of the ratio $\Delta m_{2\text{sol}}^2 / \Delta m_{atm}^2$.

Since the present model gives an inverse hierarchy of the neutrino masses, the predicted effective electron neutrino mass

$$
\langle m_{\nu e} \rangle = \left| \sum_{i} U_{ei} m_{\nu i} \right| \simeq |m_{\nu1}| \simeq |m_{\nu2}| \simeq \sqrt{\Delta m_{atm}^2} = 5.23^{+0.25}_{-0.40} \times 10^{-2} \text{eV},
$$

(4.13)
where we have used the value $\Delta m_{\text{atm}}^2 = 2.74^{+0.44}_{-0.26} \times 10^{-3} \text{eV}^2$. This value (4.13) is sufficiently sensitive to the next generation experiments of the neutrinoless double beta decay.

5 Summary

In conclusion, on the basis of the $S_4$ symmetry which is embedded into SU(3), we have investigated a lepton mass model with the effective Hamiltonian of the Froggatt-Nielsen type (4.1). We have assumed that the singlet and doublet of $S_4$ originate in the singlet and octet of SU(3), and we have obtained the VEV relation (1.5). In the derivation of the VEV relation (1.5), the essential assumptions for the superpotential $W(\phi_f)$ are the following two: (i) the scalar fields $\phi_f$ always appear in terms of the nonet form (2.1) of U(3); (ii) the superpotential $W(\phi_f)$ is invariant under the $Z_2$ transformation (2.5). Then, we have obtained not only the VEV relation (1.5), but also $\langle (\phi^{(8)})^i_j \rangle = 0$ ($i \neq j$) for the other components of $\phi^{(8)}$ (i.e. $\langle 3 \rangle = \langle 3' \rangle = 0$).

In the charged lepton sector, in order to give $m_{ei} \propto \langle (\phi_d)^i \rangle^2$, we have assumed new scalars $\xi^{(\pm)}$. Although it has been required to compensate for the $Z_2$ invariance, the model seems to leave the door open to further improvement.

For the neutrino sector, we have obtained the tribimaximal mixing (1.2) by introducing an SU(3) triplet scalar $\chi$ and the two SU(3) singlet right-handed neutrinos $\nu_R^{(\pm)}$ in addition to the nonet scalar $\phi_u$. In the present model, the right-handed neutrinos $\nu_R^{(\pm)}$ are singlets of SU(3), the Majorana neutrino mass matrices $M_R^{(\pm)}$ have no flavor structure. For the neutrino mass spectrum, since the model gives $m_{\nu 1} = m_{\nu 2}$ in the limit of $M_R^{(+)} = M_R^{(-)}$, we must consider a small deviation $M_R^{(+)} \neq M_R^{(-)}$. Since the value of $M_R^{(-)} / M_R^{(+)}$ is a free parameter in the present model, we cannot predict the value $\Delta m_{\text{solar}}^2 / \Delta m_{\text{atm}}^2$ at present, although the smallness of the ratio $\Delta m_{\text{solar}}^2 / \Delta m_{\text{atm}}^2$ can be understood. Since the present model gives an inverse hierarchy of the neutrino masses, we can predict the effective electron neutrino mass $\langle m_{\nu_e} \rangle \simeq 0.05 \text{eV}$, which is sufficiently sensitive to the next generation experiments of the neutrinoless double beta decay.

The present model seems to provide suggestive hints on seeking for a model which leads to the tribimaximal mixing (1.2) and the charged lepton mass relation (1.1), although the model has still many points which should be improved. At the same time, the model will provide a clue to the quark mass matrix model from a point of unified view of the quarks and leptons. The extension of the present model to the quark mass matrix model will be given elsewhere.

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