Analysis of the Continued Logarithm Algorithm

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Abstract. The Continued Logarithm Algorithm – CL for short– introduced by Gosper in 1978 computes the gcd of two integers; it seems very efficient, as it only performs shifts and subtractions. Shallit has studied its worst-case complexity in 2016 and showed it to be linear. We here perform the average-case analysis of the algorithm: we study its main parameters (number of iterations, total number of shifts) and obtain precise asymptotics for their mean values. Our “dynamical” analysis involves the dynamical system underlying the algorithm, that produces continued fraction expansions whose quotients are powers of 2. Even though this CL system has already been studied by Chan (around 2005), the presence of powers of 2 in the quotients ingrains into the central parameters a dyadic flavour that cannot be grasped solely by studying the CL system. We thus introduce a dyadic component and deal with a two-component system. With this new mixed system at hand, we then provide a complete average-case analysis of the CL algorithm, with explicit constants\textsuperscript{4}.

1 Introduction

In an unpublished manuscript \cite{Gosper1978}, Gosper introduced the continued logarithms, a mutation of the classical continued fractions. He writes “The primary advantage is the conveniently small information parcel. The restriction to integers of regular continued fractions makes them unsuitable for very large and very small numbers. The continued fraction for Avogadro’s number, for example, cannot even be determined to one term, since its integer part contains 23 digits, only 6 of which are known. (…) By contrast, the continued logarithm of Avogadro’s number begins with its binary order of magnitude, and only then begins the description equivalent to the leading digits – a sort of recursive version of scientific notation”.

The idea of Gosper gives rise to an algorithm for computing gcd’s, described by Shallit in \cite{Shallit2016}. This algorithm has two advantages: first, it can be calculated starting from the most representative bits, and uses very simple operations (subtractions and shifts); it does not employ divisions. Second, as the quotients which intervene in the associated continued fraction are powers of two $2^a$, we can store

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Pablo Rotondo, Brigitte Vallée, and Alfredo Viola

each of them with \(\log_2 a\) bits. Then, the algorithm seems to be of small complexity, both in terms of computation and storage.

Shallit \cite{Shallit} performs the worst-case analysis of the algorithm, and studies the number of steps \(K(p, q)\), and the total number of shifts \(S(p, q)\) that are performed on an integer input \((p, q)\) with \(p < q\): he proves the inequalities

\[
K(p, q) \leq 2 \log_2 q + 2, \quad S(p, q) \leq (2 \log_2 q + 2) \log_2 q,
\]

and exhibits instances, namely the family \((2^{n-1}, 1)\), which show that the previous bounds are nearly optimal,

\[
K(1, 2^{n-1}) = 2n - 2, \quad S(1, 2^{n-1}) = n(n - 1)/2 + 1.
\]

In a personal communication, Shallit proposed us to perform the average-case analysis of the algorithm. In this paper, we answer his question. We consider the set \(\Omega_N\) which gathers the integer pairs \((p, q)\) with \(0 \leq p \leq q \leq N\), endowed with the uniform probability, and we study the mean values \(E_N[K]\) and \(E_N[S]\) as \(N \to \infty\). We prove that these mean values are asymptotically linear in the size \(\log N\), and exhibit their precise asymptotic behaviour for \(N \to \infty\),

\[
E_N[K] \sim \frac{2}{H} \log N, \quad E_N[S] \sim \log 3 - \log 2 - \log \frac{\pi^2}{6} + 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2 2^k} = (\log 2)(3 \log 3 - 4 \log 2) \frac{\log N}{N}.
\]

The constant \(H\) is related to the entropy of an associated dynamical system and

\[
H = \frac{1}{2 \log 2 - \log 3} \left[ \frac{\pi^2}{6} + 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2 2^k} = (\log 2)(3 \log 3 - 4 \log 2) \right].
\]

This entails numerical estimates (validated by experiments) for the mean values

\[
E_N[K] \sim 1.49283 \log N, \quad E_N[S] \sim 1.40942 \log N.
\]

Then, from \cite{Shallit}, the mean number of divisions is about half the maximum.

Our initial idea was to perform a dynamical analysis along the lines described in \cite{Chan}. The \(CL\) algorithm is defined as a succession of steps, each consisting of a pseudo-division which transforms an integer pair into a new one. This transformation may be read first on the associated rationals and gives rise to a mapping \(T\) that is further extended to the real unit interval \(I\). This smoothly yields a dynamical system \((I, T)\), the \(CL\) system, already well studied, particularly by Chan \cite{Chan} and Borwein \cite{Borwein}. The system has an invariant density \(\psi\) (with an explicit expression described in \cite{Shallit}) and is ergodic. Thus we expected this dynamic analysis to follow general principles described in \cite{Chan}.

However, the analysis of the algorithm is not so straightforward. The binary shifts, which make the algorithm so efficient, cause many problems in the analysis. Even on a pair of coprime integers \((p, q)\), the algorithm creates intermediate pairs \((q_i+1, q_i)\) which are no longer generally coprime, as their gcd is a non-trivial power of 2. These extra gcd’s are central in the analysis of the algorithm, as they have an influence on the evolution of the sizes of the pairs \((q_i+1, q_i)\) which may grow due these extra factors. As these extra factors may only be powers of 2, they are easily expressed with the dyadic absolute value on \(\mathbb{Q}\), at least when the input is rational. However, when extending the algorithm into a dynamical system on the unit interval, we lose track of these factors.
The (natural) idea is thus to add to the usual CL dynamical system (on the unit interval $I$) a new component in the dyadic field $\mathbb{Q}_2$. The dyadic component is just added here to deal with the extra dyadic factors, as a sort of accumulator, but it is the former real component that dictates the evolution of the system. As the initial CL system has nice properties, the mixed system inherits this good behaviour. In particular, the transfer operator of the mixed system presents a dominant eigenvalue, and the dynamical analysis may be performed successfully. The constant $H$ in Eqn (2) is actually the entropy of this extended system.

After this extension, the analysis follows classical steps, with methodology mixing tools from analytic combinatorics (generating functions, here of Dirichlet type), Tauberian theorems (relating the singularities of these generating functions to the asymptotics of their coefficients), functional analysis (which transfers the geometry of the dynamical system into spectral properties of the transfer operator).

**Plan of the paper and notation.** The paper is structured into three sections. Section 2 introduces the algorithm and its associated dynamical system, as well as the probabilistic model, the costs of interest and their generating functions. Then, Section 3 defines the extended dynamical system, allowing us to work with dyadic costs; it explains how the corresponding transfer operator provides alternative expressions for the generating functions. Finally, Section 4 describes the properties of the transfer operator, namely its dominant spectral properties on a convenient functional space. With Tauberian theorem, it provides the final asymptotic estimates for the mean values of the main costs of interest.

For an integer $q$, $\delta(q)$ denotes the dyadic valuation, i.e., is the greatest integer $k$ for which $2^k$ divides $q$. The dyadic norm $|\cdot|_2$ is defined on $\mathbb{Q}$ with the equality $|a/b|_2 := 2^{\delta(b)−\delta(a)}$. The dyadic field $\mathbb{Q}_2$ is the completion of $\mathbb{Q}$ for this norm. See [7] for more details about the dyadic field $\mathbb{Q}_2$.

## 2 The CL algorithm and its dynamical system.

We kick off this section with a precise description of the CL algorithm, followed by its extension to the whole of the unit interval $I$, giving rise to a dynamical system, called the CL system, whose inverse branches capture all of our costs of interest. Then we recall the already known features of the CL system and we present the probabilistic model, with its generating functions.

**Description of the algorithm.** The algorithm, described by Shallit in [8], is a sequence of (pseudo)–divisions: each division associates to a pair $(p, q)$ with $p < q$ a new pair $(r, p')$ (where $r$ is the “remainder”) defined as follows

$$q = 2^a p + r, \quad p' = 2^a p,$$

with $a = a(p, q) := \max\{k \geq 0 \mid 2^k p \leq q\}$.

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5. Our notations are not the same as in the paper of Shallit as we reverse the roles of $p$ and $q$.

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The CL algorithm and its dynamical system.
This transformation rewrites the old pair \((p, q)\) in terms of the new one \((r, 2^ap)\) in matrix form,

\[
\begin{pmatrix} p \\ q \end{pmatrix} = N_a \begin{pmatrix} r \\ 2^a p \end{pmatrix}, \quad \text{with} \quad N_a = \begin{pmatrix} 0 & 2^{-a} \\ 1 & 1 \end{pmatrix} = 2^{-a} M_a, \quad M_a = \begin{pmatrix} 0 & 1 \\ 2^a & 2^a \end{pmatrix}. \tag{3}
\]

The CL algorithm begins with the input \((p, q)\) with \(p < q\). It lets \((q_1, q_0) := (p, q)\), then performs a sequence of divisions

\[
(q_i + 1, q_i) = N_{a_{i+1}} (q_{i+2}, 2^a q_{i+1})^T,
\]

and stops after \(k = K(p, q)\) steps on a pair of the form \((0, 2^a q_k)\). The complete execution of the algorithm uses the set of matrices \(N_a\) defined in (3), and writes the input as

\[
(p, q)^T = N_{a_1} \cdot N_{a_2} \cdots N_{a_k} \begin{pmatrix} 0 \\ 2^a q_k \end{pmatrix}^T.
\]

The rational input \(p/q\) is then written as a continued fraction according to the LFTs (linear fractional transformations) \(h_a\) associated with matrices \(N_a\) or \(M_a\),

\[
p/q = \frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{1 + \frac{2^{-a_3}}{1 + \cdots \frac{2^{-a_k}}{1}}}} = h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_k}(0), \quad \text{with} \quad h_a : x \mapsto \frac{2^{-a}}{1 + x}. \tag{4}
\]

Moreover, it is possible to choose the last exponent \(a_k\) to be 0. (and the last quotient to be 1). This is a gcd algorithm: as \(q_k\) is equal to \(\gcd(p, q)\) up to a power of 2, the CL algorithm determines the odd part of \(\gcd(p, q)\) whereas the even part is directly determined by the dyadic valuations of \(p\) and \(q\).

Shallit \[8]\ proves that this algorithm indeed terminates and characterizes the worst-case complexity of the algorithm. Figure 1 describes the execution of the algorithm on the pair \((31, 75)\).

| \(i\) | \(a_i\) | \(2^a q_i\) | \(q_{i+1}\) | \(2^a q_{i+1}\) | \(\delta(2^a q_i)\) | \(\delta(q_{i+1})\) | \(\delta(\hat{g}_i)\) |
|------|--------|-----------|-------|--------|-----------|--------|--------|
| 0    | 1      | 31        | 10010111 | 1111 | 0         | 0       | 0       |
| 1    | 1      | 62        | 01111110 | 1101 | 1         | 0       | 0       |
| 2    | 2      | 52        | 101100   | 1010 | 2         | 1       | 1       |
| 3    | 2      | 40        | 101000   | 1100 | 3         | 2       | 2       |
| 4    | 1      | 24        | 1100     | 1000 | 3         | 4       | 3       |
| 5    | 0      | 16        | 10000    | 1000 | 4         | 3       | 3       |
| 6    | 0      | 8         | 10000    | 1000 | 3         | 3       | 3       |
| 7    | 0      | 8         | 10000    | 1000 | 3         | 3       | 3       |

Fig. 1. Execution for the input \((31, 75)\). Here \(\hat{g}_i = \gcd(2^a q_i, q_{i+1})\). The dyadic valuation \(\delta(\hat{g}_i)\) seems to linearly increase with \(i\), with \(\delta(\hat{g}_i) \sim \delta(\hat{g}_{i+1}) \sim i/2 (i \to \infty)\).
**Dynamical system.** The relations

\[(r, 2^a p)^T = N_a^{-1}(p, q)^T, \quad (p, q)^T = N_a(r, 2^a p)^T,\]

are first transformed into relations on the associated rationals \(p/q, r/(2^a p)\) via the LFT’s \(T_a, h_a\) associated to matrices \(N_a^{-1}, N_a\),

\[
T_a(x) := \frac{1}{2^a x} - 1, \quad h_a(x) = \frac{1}{2^a(1 + x)}, \quad a \geq 0. \tag{5}
\]

They are then extended to the reals of the unit interval \(I := [0, 1]\). This gives rise to a dynamical system \((I, T)\), denoted \(CL\) in the sequel, defined on the unit interval \(I\), with fundamental intervals \(I_a := [2^{-a-1}, 2^{-a}]\), the surjective branches \(T_a : I_a \to I_a\), and their inverses \(h_a : I \to I_a\).

The \(CL\) system \((I, T)\) is displayed on the left of the figure below, along with the shift \(S : I \to I\) which gives rise to the \(CL\) system by induction on the first branch. The map \(S\) is a mix of the Binary and Farey maps, as its first branch comes from the Binary system, and the second one from the Farey system. On the right, the usual Euclid dynamical system (defined from the Gauss map) is derived from the Farey shift by induction on the first branch.

With each \(k\)-uple \(a := (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k\) we associate the matrix \(M_a := M_a_1 \cdots M_a_k\) and the LFT \(h_a := h_{a_1} \circ h_{a_2} \circ \cdots \circ h_{a_k}\). Then the set \(H\) of the inverse branches, and the set \(H^k\) of the inverse branches of \(T^k\) (of depth \(k\)) are

\[
H := \{h_a \mid a \geq 0\}, \quad H^k := \{h_a \mid a \in \mathbb{N}^k\}.
\]

As the branches \(T_a\) are surjective, the inverse branches are defined on \(I\) and the images \(h_a(I)\) for \(a \geq 0\) form a topological partition of \(I\). This will be true at any depth, and the intervals \(h_a(I)\), called fundamental intervals of depth \(k\), form a topological partition for \(a \in \mathbb{N}^k\).

**Properties of the CL system.** The Perron Frobenius operator

\[
H[f](x) := \sum_{h \in H} |h'(x)| f(h(x)) = \left(\frac{1}{1 + x}\right)^2 \sum_{a \geq 0} 2^{-a} f \left(\frac{2^{-a}}{1 + x}\right). \tag{6}
\]

describes the evolution of densities: If \(f\) is the initial density, \(H[f]\) is the density after one iteration of the system \((I, T)\). The invariant density \(\psi\) is a fixed point for \(H\) and satisfies the functional equation

\[
\psi(x) = \left(\frac{1}{1 + x}\right)^2 \sum_{a \geq 0} 2^{-a} \psi \left(\frac{2^{-a}}{1 + x}\right). \tag{7}
\]
Chan [3] obtains an explicit form for $\psi$

$$\psi(x) = \frac{1}{\log(4/3)} \frac{1}{(x + 1)(x + 2)}. \quad (8)$$

He also proves that the system is ergodic with respect to $\psi$, and entropic. However, he does not provide an explicit expression for the entropy. We obtain here such an expression, with a precise study of the transfer operator of the system. We introduce two (complex) parameters $t, v$ in $[6]$, and deal with a perturbation of the operator $H$, defined by

$$H_{t,v}[f](x) := \sum_{h \in H} |h'(x)|^t d(h)^v f(h(x)) = \left(\frac{1}{1 + x}\right)^{2t} \sum_{a \geq 0} 2^{a(v-t)} f\left(\frac{2^{-a}}{1 + x}\right). \quad (9)$$

Such an operator $H_{t,v}$ is called a transfer operator. When $(t, v)$ satisfies $\Re(t-v) > 0$, we prove the following: the operator $H_{t,v}$ acts nicely on the space $C^1(I)$ endowed with the norm $\|\cdot\|_1$, defined by $\|f\|_1 := \|f\|_0 + \|f'\|_0$, where $\|\cdot\|_0$ denotes the sup norm. In particular, it has a dominant eigenvalue $\lambda(t, v)$ separated from the remainder of the spectrum by a spectral gap, for $(t, v)$ close to $(1, 0)$. The Taylor expansion of $\lambda(t, v)$ near $(1, 0)$

$$\lambda(t, v) \approx 1 - A(t-1) + Dv$$

involves the two constants $A = -\partial \lambda / \partial t(1, 1, 0)$, $D = \partial \lambda / \partial v(1, 1, 0)$, that are expressed as mean values with respect to the invariant density $\psi$,

$$A = E - D, \quad E = \mathbb{E}_\psi[2 \log x], \quad D = (\log 2) \mathbb{E}_\psi[a], \quad (10)$$

(here, the function $a$ associates with $x$ the integer defined with the Iverson bracket $a(x) := a \cdot \lfloor x \in h_a(I) \rfloor$. The constants $A$ is the entropy of the system, and $E,D$ admit explicit expressions

$$E = \frac{1}{\log(4/3)} \left[ \frac{\pi^2}{6} + 2 \sum_{k \geq 1} (-1)^k \frac{k^2}{2^k} \right], \quad D = \frac{\log(3/2)}{\log(4/3)}. \quad (11)$$

Then, with $[10]$ and $[11]$, there is an explicit value for the entropy $A$, and

$$A \approx 1.62352 \ldots, \quad D \approx 0.97693 \ldots, \quad E \approx 2.60045 \ldots.$$  

**Main costs associated to a truncated expansion.** Each real number of the unit interval admits an infinite continued fraction expansion derived from the dynamical system, which we call its CLCF expansion. When truncated at a finite depth, its expansion becomes finite, as in $[4]$, and defines a LFT $h := h_a$. This expansion gives rise to a rational $p/q$, (assumed to be irreducible) which is thus written as $p/q = h_a(0)$.

On the other hand, the $k$-uple $a$ defines a matrix $M_a$ and an integer pair $(P, Q)$, called the continuant pair, defined by $(P, Q)^T := M_a(0, 1)^T$. The equality $P/Q = p/q$, holds, but, as the integers $P$ and $Q$ are not necessarily coprime, the pair $(P, Q)$ does not coincide with the pair $(p, q)$. The integer $R(Q) := Q/\gcd(P, Q)$,
called the reduced continuant, is an important parameter that actually dictates the quality of the rational approximation given by the truncation of the CLCF. We will see that it also plays a central role in the analysis of the CL algorithm. As \( \gcd(P, Q) \) divides \( |\det(M_A)| \) that is a power of two, it is itself a power of two. It then proves fundamental to deal with dyadic tools.

The main interesting costs associated with a finite expansion, as in (4), are expressed in terms of the quadruple \((C, Q)\) to the depth of the CLCF. We will see that it also plays a central role in the analysis of the algorithm. As \( \gcd(P, Q) \) divides \( |\det(M_A)| \) that is a power of two, it is itself a power of two. It then proves fundamental to deal with dyadic tools.

**Proposition 1.** Consider the function \( G_2 : \mathbb{Q}_2 \to \mathbb{R}^+ \) (called the \( \gcd \) map) equal to \( G_2(y) = \min(1, |y|_2^2) \), namely
\[
G_2(y) = 1 \quad \text{for } |y|_2 \leq 1, \quad G_2(y) = |y|_2^2 \quad \text{for } |y|_2 > 1.
\] (12)

The main costs associated with the CLCF expansion of a rational \( h(0) \)
\[
Q, \quad g(P, Q) := \gcd(P, Q), \quad R(P, Q) = Q/\gcd(P, Q), \quad |Q|_2.
\]
are all expressed in terms of the quadruple \((|h'(0)|, |h'(0)|_2, d(h), G_2[h(0)]\) as
\[
Q^{-2} = |h'(0)|/d(h), \quad |Q|_2^{-2} = d(h) |h'(0)|_2,
\]
\[
R^{-2}(Q) = |h'(0)| |h'(0)|_2 G_2[h(0)], \quad g^2(P, Q) = d(h) |h'(0)|_2 G_2[h(0)].
\]

**Proof.** One has (by definition)
\[
P/Q = h(0), \quad Q^{-2} = |h'(0)|/d(h), \quad r(Q)^{-2} = g^2(P, Q)/Q^2.
\]
As \( g(P, Q) \) is a power of 2, and using the function \( G_2 \) defined in (12), one has
\[
g^2(P, Q) = \min(|P|_2, |Q|_2)^{-2} = |Q|_2^{-2} \min(1, |P|/|Q|_2^2) = |Q|_2^{-2} G_2(P/Q).
\]
We conclude with the equalities:
\[
|Q|_2^{-2} = |h'(0)|/|d(h)|_2, \quad d(h) \cdot |d(h)|_2 = 1.
\]

Any cost \( C \) of Proposition 1 admits an expression of the form
\[
|h'(0)|^t \cdot |h'(0)|_2^u \cdot d(h)^v \cdot G_2[h(0)]^z.
\]

The quadruple \((t, u, v, z)\) associated with the cost \( C \) is denoted as \( \gamma_C \). Moreover, as these costs \( C \) are expected to be of exponential growth with respect to the depth of the \( CF \), we will work with their logarithms \( c = \log C \). Figure 2 summarizes the result.

**Generating functions.** We deal with sets of coprime integer pairs
\[
\Omega := \{(p, q) \mid 0 < p < q, \gcd(p, q) = 1\}, \quad \Omega_N := \Omega \cap \{(p, q) \mid q \leq N\}.
\]
The set \( \Omega_N \) is endowed with the uniform measure, and we wish to study the mean values \( \mathbb{E}_N[c] \) of parameters \( c \) on \( \Omega_N \). We focus on parameters which describe the execution of the algorithm and are “read” from the \( CF(p/q) \) built by the algorithm as in (4). They are defined in Proposition 1 and depend on the continuant pair \((P, Q)\); as already explained, the reduced continuant \( R(P, Q) \) plays a fundamental role here.

\[^6\] This restriction can be easily removed and our analysis extends to the set of all integer pairs.
Cost \( C \) \hspace{1cm} e = \log C \hspace{1cm} Quadruple \( \gamma_C \) \hspace{1cm} Constant \( M(c) \) \hspace{1cm} Numerical value of \( M(c) \)
\begin{array}{|c|c|c|c|c|}
\hline
\( d(h) \) \hspace{0.5cm} \sigma \hspace{0.5cm} (0, 0, 1, 0) \hspace{0.5cm} D \hspace{0.5cm} \pm 0.97693 \ldots \\
\hline
\( Q^2 \) \hspace{0.5cm} q \hspace{0.5cm} (1, 1, 0, 0) \hspace{0.5cm} A + D \hspace{0.5cm} \pm 2.60045 \ldots \\
\hline
\( g^2(P, Q) \) \hspace{0.5cm} q \hspace{0.5cm} (0, 1, 1, 1) \hspace{0.5cm} B + D \hspace{0.5cm} \pm 1.26071 \ldots \\
\hline
\( R^2(P, Q) \) \hspace{0.5cm} r \hspace{0.5cm} (1, 1, 0, -1) \hspace{0.5cm} A - B \hspace{0.5cm} \pm 1.33973 \ldots \\
\hline
\|Q\|_2^{-2} \hspace{0.5cm} q_2 \hspace{0.5cm} (0, 1, 1, 0) \hspace{0.5cm} B + D \hspace{0.5cm} \pm 1.26071 \ldots \\
\hline
\end{array}

Fig. 2. Main costs of interest, with their quadruple, and the constants which intervene in the analysis of their mean values. (see Thm 1).

We deal with analytic combinatorics methodology and work with (Dirichlet) generating functions (dgf in short). Here is the plain Dirichlet generating function

\[
S(s) := \sum_{(p,q) \in \Omega} \frac{1}{q^{2s}} = \frac{\zeta(2s - 1)}{\zeta(2s)}.
\]  

There are also two generating functions that are associated with a cost \( C : \Omega \rightarrow \mathbb{R}^+ \) (and its logarithm \( c \)), namely the bivariate dgf and the cumulative dgf,

\[
S_C(s,w) := \sum_{(p,q) \in \Omega} e^{wc(p,q)} q^{2s}, \quad \hat{S}_C(s) := \sum_{(p,q) \in \Omega} \frac{c(p,q)}{q^{2s}} = \frac{\partial}{\partial w} S_C(s,w) \bigg|_{w=0}.
\]

The expectation \( E_N[c] \) is now expressed as a ratio which involves the sums \( \Phi_N(S), \Phi_N(\hat{S}_C) \) of the first \( N \) coefficients of the Dirichlet series \( S(s) \) and \( \hat{S}_C(s) \),

\[
E_N[c] = \Phi_N[\hat{S}_C]/\Phi_N[S].
\]

From principles of Analytic Combinatorics, we know that the dominant singularity of a dgf (here its singularity of largest real part) provides precise information (via notably its position and its nature) about the asymptotics of its coefficients, here closely related to the mean value \( E_N[c] \) via Eqn (15). Here, in the Dirichlet framework, this transfer from the analytic behaviour of the dgf to the asymptotics of its coefficients is provided by Delange’s Tauberian Theorem [5].

We now describe an alternative expression of these series, from which it is possible to obtain information regarding the dominant singularity, which will be transferred to the asymptotics of coefficients.

**Proposition 2.** The Dirichlet generating \( S(s) \) and its bivariate version \( S_C(s, w) \) relative to a cost \( C : \Omega \rightarrow \mathbb{R} \), admit alternative expression:\footnote{We recall that the last exponent is 0 by convention, and the last LFT is thus \( J = h_0 \).}

\[
S(s) = S_C(s, 0), \quad S_C(s, w) = \sum_{h \in H \cdot \cdot \cdot J} e^{wc(h)} |h'(0)|^s |h'(0)|_2^t G_2^s \circ h(0).
\]
For any cost \( C \) described in Figure 2, the general term of \( S_C(s,w) \) is of the form
\[
|h'(0)|^t \quad |h'(0)|^u \quad d(h)^v \quad G_2 \circ h(0),
\]
and involves a quadruple of exponents \((t,u,v,z)\), denoted as \( \gamma_C(s,w) \), that is expressed with the quadruple \( \gamma_C \) defined in Figure 2 as
\[
\gamma_C(s,w) = s(1,1,0,1) + w \gamma_C.
\]

Proof. By definition, the denominator \( q \) equals \( R(P,Q) \). With Figure 2, the quadruple relative to \( q^{-2s} \) is then \( s(1,1,0,1) \), whereas the quadruple relative to \( e^{wc} = C^w \) is just \( w \gamma_C \).

We have thus described the general framework of our paper. We now look for an alternative form for the generating functions: in dynamical analysis, one expresses the dgf in terms of the transfer operator of the dynamical system which underlies the algorithm. Here, it is not possible to obtain such an alternative expressions if we stay in the real “world”. This is why we will add a component to the CL system which allows us to express parameters with a dyadic flavour. It will be possible to express the dgf’s in term of a (quasi-inverse) of an (extended) transfer operator, and relate their dominant singularity to the dominant eigenvalue of this extended transfer operator.

We then obtain our main result, precisely stated in Theorem 1, at the end of the paper: we will prove that the mean values \( E_N[\log C] \) associated with our costs of interest are all of order \( \Theta(\log N) \), and satisfy precise asymptotics that involve three constants \( A,B,D \): the constants \( D \) and \( A \) come from the real word, and have been previously defined in (11) and (10), but there arises a new constant \( B \) that comes from the dyadic word.

3 The extended dynamical system.

In this section, we extend the CL dynamical system, adding a new component to study the dyadic nature of our costs. We then introduce transfer operators, and express the generating functions in terms of the quasi-inverses of the transfer operators.

Extension of the dynamical system. We will work with a two-component dynamical system: its first component is the initial CL system, to which we add a second (new) component which is used to “follow” the evolution of dyadic phenomena during the execution of the first component.

We consider the set \( \mathcal{I} := \mathbb{I} \times \mathbb{Q}_2 \). We define a new shift \( T : \mathcal{I} \to \mathcal{I} \) from the characteristics of the old shift \( T \) defined in \( \mathcal{I} \). As each branch \( T_a \), or its inverse \( h_a \), is a LFT with rational coefficients, it is well-defined on \( \mathbb{Q}_2 \); it is moreover a bijection from \( \mathbb{Q}_2 \cup \{\infty\} \) to \( \mathbb{Q}_2 \cup \{\infty\} \). Then, each branch \( T_a \) of the new shift \( T \) is defined via the equality \( T_a(x,y) := (T_a(x),T_a(y)) \) on the fundamental domain \( \mathcal{I}_a := \mathcal{I}_a \times \mathbb{Q}_2 \), and the shift \( T_a \) is a bijection from \( \mathcal{I}_a \) to \( \mathcal{I} := \mathcal{I} \times \mathbb{Q}_2 \) whose inverse branch \( h_a : (x,y) \mapsto (h_a(x),h_a(y)) \) is a bijection from \( \mathcal{I} \) to \( \mathcal{I}_a \).
Measures. We consider the three domains
\[ B := \mathbb{Q}_2 \cap \{|y|_2 < 1\}, \quad U := \mathbb{Q}_2 \cap \{|y|_2 = 1\}, \quad C := \mathbb{Q}_2 \cap \{|y|_2 > 1\}. \]
There exists a Haar measure \( \nu_0 \) on \( \mathbb{Q}_2 \) which is finite on each compact of \( \mathbb{Q}_2 \), and can be normalized with \( \nu_0(B) = \nu_0(U) = 1/3 \) (see [7]). We will deal with the measure \( \nu \) with density \( G_2 \) wrt to \( \nu_0 \), for which \( \nu(C) = 1/3. \) The measure \( \nu[2^kU] \) equals \((1/3)2^{-|k|}\) for any \( k \in \mathbb{Z} \) and \( \nu \) is a probability measure on \( \mathbb{Q}_2 \).

On \( \mathcal{I} \), we deal with the probability measure \( \rho := \mu \times \nu \) where \( \mu \) is the Lebesgue measure on \( \mathcal{I} \) and \( \nu \) is defined on \( \mathbb{Q}_2 \) as previously.

For integrals which involve a Haar measure, there is a change of variables formula. As \( \nu_0 \) is a Haar measure, and \( d\nu = G_2 \, d\nu_0 \), this leads to the following change of variables formula, for any \( F \in L^1(\mathbb{Q}_2, \nu), \)
\[
\int_{\mathbb{Q}_2} |h'(y)|^2 \, F(h(y)) \left[ \frac{G_2(h(y))}{G_2(y)} \right] \, d\nu(y) = \int_{\mathbb{Q}_2} F(y) \, d\nu(y). \tag{17}
\]

Density transformer and transfer operator. We now consider the operator \( \mathcal{H} \) defined as a “density transformer” as follows: with a function \( F \in L^1(\mathcal{I}, \rho), \)
it associates a new function defined by
\[
\mathcal{H}[F](x, y) := \sum_{h \in \mathcal{H}} |h'(x)| \, |h'(y)|^2 \, F(h(x), h(y)) \left[ \frac{G_2(h(y))}{G_2(y)} \right].
\]
When \( F \) is a density in \( L^1(\mathcal{I}, \rho) \), then \( \mathcal{H}[F] \) is indeed the new density on \( \mathcal{I} \) after one iteration of the shift \( \mathcal{T} \). This just follows from the change of variables formula \([17]\) applied to each inverse branch \( h \in \mathcal{H} \).

Proposition 2 leads us to a new operator that depends on a quadruple \((t, u, v, z)\),
\[
\mathcal{H}_{t,u,v,z}[F](x, y) := \sum_{h \in \mathcal{H}} |h'(x)|^t \, |h'(y)|^z \, \nu(h(x), h(y)) \left[ \frac{G_2(h(y))}{G_2(y)} \right]^v. \tag{18}
\]
We will focus on costs described in Figure 2 we thus deal with operators associated with quadruples \( \gamma_C(s, w) \) defined in Proposition 2 and in particular with the quadruple \((s, s, 0, s)\), and its associated operator \( \mathcal{H}_s \), \( := \mathcal{H}_{s,s,0,s}. \)

Alternative expressions of the Dirichlet generating functions. We start with the expressions of Proposition 1 consider the three types of dfg defined in (13) and (14), use the equality \( G_2(0) = 1 \), and consider the operator \( \mathcal{J}_s \), relative to the branch \( J \) used in the last step. For the plain dfg in (13), we obtain
\[
S(s) = \sum_{h \in \mathcal{H} \cap J} |h'(0)|^s \, |h'(0)|^s \, G_2^s \circ h(0) = \mathcal{J}_s \circ (I - \mathcal{H}_s)^{-1}[1](0, 0), \tag{19}
\]
We now consider the bivariate dfg’s defined in (14). For the depth \( K \), one has
\[
S_K(s, w) = e^{sw} \mathcal{J}_s \circ (I - e^{sw} \mathcal{H}_s)^{-1}[1](0, 0); \tag{19}
\]
This general result can be found for instance in Bourbaki [2], chapitre 10, p.36.
For costs $C$ of Figure[2], the bivariate dgf involves the quasi-inverse of $H_{\gamma C}(s, w)$,

$$S_C(s, w) = J_{\gamma C}(s, w) \circ (I - H_{\gamma C}(s, w))^{-1}[1](0, 0),$$

except for $C = |Q|^2$, where the function $1$ is replaced by the function $G_w$.

The dgf $\tilde{S}_C(s)$ defined in (14) is obtained with taking the derivative of the bivariate dgf wrt $w$ (at $w = 0$); it is thus written with a double quasi inverse which involves the plain operator $H_s$, separated “in the middle” by the cumulative operator $H_s$, namely

$$\tilde{S}_C(s) \equiv J_s \circ (I - H_s)^{-1} \circ H_{s, C}(s, w) \circ (I - H_s)^{-1}[1](0, 0), \quad (20)$$

and the cumulative operator is itself defined by $H_{s, C}(s, w) := \frac{\partial}{\partial w} H_{\gamma C}(s, w) \bigg|_{w=0}$.

4 Functional Analysis

This section deals with with a delicate context, which mixes the specificities of each world –the real one, and the dyadic one–. It is devoted to the study of the quasi-inverses $(I - H_s)^{-1}$ intervening in the expressions of the generating functions of interest. We first define an appropriate functional space on which we prove the operators to act and admit dominant spectral properties. This entails that the quasi-inverse $(I - H_s)^{-1}$ admits a pole at $s = 1$, and we study its residue, which gives rise to the constants that appear in the expectations of our main costs.

**Functional space.** The delicate point of the dynamical analysis is the choice of a good functional space, that must be a subset of $L^1(\mathbb{I}, \rho)$. Here, we know that, in the initial CL system, the transfer operator $H_s$ acts in a good way on $C_1(\mathbb{I})$. Then, for a function $F$ defined on $\mathbb{I}$, the main role will be played by the family of “sections” $\tilde{F}_y : x \mapsto \max(1, \log |y|^2) F(x, y)$ which will be asked to belong to $C^1(\mathbb{I})$, under the norm $|\cdot|_{1, 1}$, defined as $|\tilde{F}_y|_{1, 1} := |\tilde{F}_y|_0 + |\tilde{F}_y|_1$ with

$$|\tilde{F}_y|_0 := \sup_{x \in \mathbb{I}} |\tilde{F}(x, y)|, \quad |\tilde{F}_y|_1 := \sup_{x \in \mathbb{I}} \left| \frac{\partial}{\partial x} \tilde{F}(x, y) \right| .$$

We work on the Banach space

$$\mathcal{F} := \left\{ F : \mathbb{I} \rightarrow \mathbb{C} \mid F_y \in C^1(\mathbb{I}), \quad y \mapsto \tilde{F}_y \text{ bounded} \right\} ,$$

endoowed with the norm $||F|| := ||F||_0 + ||F||_1$, with

$$||F||_0 := \int_{\mathbb{Q}_2} |\tilde{F}_y|_0 \, dv(y), \quad ||F||_1 := \int_{\mathbb{Q}_2} |\tilde{F}_y|_1 \, dv(y) . \quad (21)$$

The next Propositions[3][4] will describe the behaviour of the operator $H_{s,u,v,z}$ on the functional space $\mathcal{F}$. Their proofs are quite technical and are omitted here.

9 There is another term which involves only a quasi-inverse. It does not intervene in the analysis.
The first result exhibits a subset of quadruples \((t, u, v, z)\) which contains \((1, 1, 0, 1)\) for which the resulting operator \(H_{t,u,v,z}\) acts on \(F\).

**Proposition 3.** The following holds:
(a) When the complex triple \((t, u, v)\) satisfies the constraint \(\Re(t - v - |u - 1|) > 0\), the operator \(H_{t,u,v,u}\) acts on \(F\) and is analytic with respect to the triple \((t, u, v)\).
(b) The operator \(H_s := H_{s,s,0,s}\) acts on \(F\) for \(\Re s > 1/2\), and the norm \(|| \cdot ||_0\) of the operator \(H_s\) satisfies \(||H_s||_0 < 1\) for \(\Re s > 1\).

**Dominant spectral properties of the operator.** The next result describes some of the main spectral properties of the operator on the space \(F\). Assertion (a) entails that the \(k\)-th iterate of the operator behaves as a true \(k\)-th power of its dominant eigenvalue. Then, as stated in (c), its quasi-inverse behaves as a true quasi-inverse which involves its dominant eigenvalue.

**Proposition 4.** The following properties hold for the operator \(H_{t,u,v,u}\), when the triple \((t, u, v)\) belongs to a neighborhood \(\mathcal{V}\) of \((1, 1, 0)\).

(a) There is a unique dominant eigenvalue, separated from the remainder of the spectrum by a spectral gap, and denoted as \(\lambda(t, u, v)\), with a (normalized) dominant eigenfunction \(\Psi_{t,u,v}\) and a dominant eigenmeasure \(\mu_{t,u,v}\) for the dual operator.
(b) At \((t, u, v, u) = (1, 1, 0, 1)\), the operator \(H_{t,u,v,u}\) coincides with the density transformer \(H_1\). At \((1, 1, 0)\) the dominant eigenvalue \(\lambda(t, u, v)\) equals 1, the function \(\Psi_{t,u,v}\) is the invariant density \(\Psi\) and the measure \(\mu_{t,u,v}\) equals the measure \(\rho\).
(c) The estimate holds for any function \(F \in L^1(\mathbb{Z}, \rho)\) with \(\rho[F] \neq 0\),

\[
(I - H_{t,u,v,u})^{-1}[F](x, y) \sim \frac{\lambda(t, u, v)}{1 - \lambda(t, u, v)} \Psi_{t,u,v}(x, y) \mu_{t,u,v}[F].
\]
(d) For \(\Re s = 1, s \neq 1\), the spectral radius of \(H_{s,s,0,s}\) is strictly less than 1.

The third result describes the Taylor expansion of \(\lambda(t, u, v)\) at \((1, 1, 0)\), and makes precise the behaviour of the quasi-inverse described in (c).

**Proposition 5.** The Taylor expansion of the eigenvalue \(\lambda(t, u, v)\) at \((1, 1, 0)\), written as \(\lambda(t, u, v) \sim 1 - A(t - 1) + B(u - 1) + Dv\), involves the constants

\[
A = -\partial \lambda / \partial t(1, 1, 0), \quad B = \partial \lambda / \partial u(1, 1, 0), \quad D = \partial \lambda / \partial v(1, 1, 0)
\]

(a) The constants \(A\) and \(D\) already appear in the context of the plain dynamical system, and are precisely described in [11] and [10]. In particular \(A - D\) is equal to the integral \(E := \mathbb{E}_{\Psi}[2 \log |y|]\);
(b) The constant \(B\) is defined with the extension of the dynamical system and its invariant density \(\Psi = \Psi_{1,1,0}\). The constant \(B + D\) is equal to the dyadic analog \(E_2\) of the integral \(E\), namely, \(B + D = E_2 := \mathbb{E}_{\Psi}[2 \log |y|_2]\);
(c) The constant \(A - B\) is the entropy of the extended dynamical system.
Final result for the analysis of the CL algorithm. We then obtain our final result:

**Theorem 1.** The mean values $E_N[c]$ for $c \in \{K, \sigma, q, r, q_2\}$ on the set $\Omega_N$ are all of order $\Theta(\log N)$ and admit the precise following estimates,

$$E_N[K] \sim \frac{2}{H} \log N, \quad E_N[c] \sim M(c) \cdot E_N[K], \quad \text{for} \ c \in \{\sigma, q, r, q_2\}.$$  

The constant $H$ is the entropy of the extended system. The constants $H$ and $M(c)$ are expressed with a scalar product that involves the gradient $\nabla \lambda$ of the dominant eigenvalue at $(1, 1, 0)$ and the beginning $\tilde{\gamma}_{C}$ of the quadruple $\gamma_{C}$ associated with the cost $c$. More precisely

$$H = -\langle \nabla \lambda, (1, 1, 0) \rangle, \quad M(c) = \langle \nabla \lambda, \tilde{\gamma}_C \rangle.$$  

The constants $M(c)$ are exhibited in Figure 2.

**Proof.** Now, the Tauberian Theorem comes into play, relating the behaviour of a Dirichlet series $F(s)$ near its dominant singularity with asymptotics for the sum $\Phi_N(F)$ of its first $N$ coefficients. Delange’s Tauberian Theorem is stated as follows (see [3]):

Consider for $\sigma > 0$ a Dirichlet series $F(s) := \sum_{n \geq 1} a_n n^{-2s}$ with non negative coefficients which converges for $\Re s > \sigma$. Assume moreover:

(i) $F(s)$ is analytic on $\{\Re s, s \neq \sigma\}$,

(ii) near $\sigma$, $F(s)$ satisfies $F(s) \sim A(s)(s - \sigma)^{-(k+1)}$ for some integer $k \geq 0$.

Then, as $N \to \infty$, the sum $\Phi_N(F)$ of its first $N$ coefficients satisfies

$$\Phi_N(F) := \sum_{n \leq N} a_n \sim 2^k A(\sigma) [\sigma \Gamma(k+1)]^{-1} N^{2\sigma} \log^k N.$$  

We now show that the two dgf’s $S(s)$ and $\tilde{S}_C(s)$ satisfy the hypotheses of the Tauberian theorem. The two expressions obtained in (19) and (20) involve quasi-inverses $(I - \tilde{H}_s)^{-1}$, a simple one in (19), a double one in (20).

First, Propositions 3(b) and 4(d) prove that such quasi-inverses are analytic on $\Re s \geq 1, s \neq 1$. Then Proposition 4(c), together with Eqn (20), shows that $S(s)$ and $\tilde{S}_C(s)$ have a pole at $s = 1$, of order 1 for $S(s)$, of order 2 for $\tilde{S}_C(s)$.

We now evaluate the dominant constants: first, the estimate holds,

$$1 - \lambda(s, 0) \sim (A - B)(s - 1) = H(s - 1), \text{ with } H = -\langle \nabla \lambda, (1, 1, 0) \rangle.$$  

Second, with Proposition 4(c), the dgf’s $S(s)$ and $\tilde{S}_C(s)$ admit the following estimates which both involve the constant $a = \frac{1}{2}(|\Psi|(0, 0), namely,

$$S(s) \sim \frac{a}{H(s - 1)^{1}}, \quad \tilde{S}_C(s) \sim \frac{a}{H^2(s - 1)^{2}} \rho \left[ \tilde{H}_{1,0}(\Psi) \right].$$  

We now explain the occurrence of the constant $M(c)$: we use the definition of the triple $\gamma_{C}(s, w)$, the definition of the cumulative operator $\tilde{H}_{1,0}(\Psi)$ as the derivative of the bivariate operator $H_{1,0}(s, w)$ at $(s, w) = (1, 0)$, and the fact that $\tilde{H}_{1,0,1} = \tilde{H}_1$, is the density transformer. This entails the sequence of equalities,

$$\rho \left[ \tilde{H}_{1,0}(\Psi) \right] = \frac{\partial}{\partial w} \lambda(\gamma_{C}(1, w)) \bigg|_{w=0} = \langle \nabla \lambda, \tilde{\gamma}_C \rangle = M(c).$$
About the constant $B$. The invariant density $\Psi$—more precisely the function $\hat{\Psi} := \Psi \cdot G_2$—satisfies a functional equation of the same type as the invariant function $\psi$, (described in Eqn (7)), namely,

$$
\hat{\Psi}(x, y) = \left(\frac{1}{1+x}\right)^2 \left(\frac{1}{1+y}\right)^2 \sum_{a \geq 0} \hat{\Psi} \left(\frac{2^{-a}}{1+x}, \frac{2^{-a}}{1+y}\right).
$$

Comparing to Eqn (7), we “lose” the factor $2^{-a}$ in the sum, and so we have not succeed in finding an explicit formula for $\Psi$. We do not know how to evaluate the integral $E_2$ defined in Proposition 5.(b).

However, we conjecture the equality $D - B = \log 2$, from experiments of the same type as those described in Figure 1. This would entail an explicit value for the entropy of the extended system,

$$
\frac{1}{2\log 2 - \log 3} \left[ \frac{\pi^2}{6} + 2 \sum_{k \geq 1} \frac{(-1)^k}{k^2 2^k} - (\log 2)(3 \log 3 - 4 \log 2) \right] \equiv 1.33973\ldots
$$

Conclusions and Extensions. We have studied the Continued Logarithm Algorithm and analyzed in particular the number of pseudo divisions, and the total number of shifts. It would be nice to obtain an explicit expression of the invariant density, that should entail a proven expression of the entropy of the dynamical system. It is also surely possible to analyze the bit complexity of the algorithm, notably in the case when one eliminates the rightmost zeroes when are shared by the two $q_i$’s (as suggested by Shallit). Such a version of this algorithm may have a competitive bit complexity that merits a further study.

There exist two other gcd algorithm that are based on binary shifts, all involving a dyadic point of view: the Binary Algorithm, and “the Tortoise and the Hare” algorithm, already analyzed in [9] and [4]; however, the role of the binary shifts is different in each case. The strategy of the present algorithm is led by the most significant bits, whereas the strategy of the “Tortoise and the Hare” is led by the least significant bits. The Binary algorithm adopts a mixed strategy, as it performs both right-shifts and subtractions. We have the project to unify the analysis of these three algorithms, and better understand the role of the dyadic component in each case.

References

1. Jonathan M. Borwein, Kevin G. Hare and Jason G. Lynch. Generalized Continued Logarithms and Related Continued Fractions, Journal of Integer Sequences, vol. 20, 2017.
2. Nicolas Bourbaki, Variétés différentielles et analytiques. Springer 2007
3. Hei-Chi Chan. The Asymptotic Growth Rate of Random Fibonacci Type Sequences. Fibonacci Quarterly, vol. 43, no. 3, pp. 243–255, 2005.

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This will be explained in the long paper.
4. Benoît Daireaux, Véronique Maume-Deschamps, Brigitte Vallée. The Lyapounov Tortoise and the Dyadic hare. Proceedings of AofA’05, DMTCS, pp. 71–94, 2005.
5. Hubert Delange. Généralisation du Théorème d’Ikehara, Ann. Sc. ENS 71, pp. 213–242, 1954
6. Bill Gosper. Continued fraction arithmetic, Unpublished manuscript, ca. 1978.
7. Neal Koblitz. \( p \)-adic Numbers, \( p \)-adic analysis and Zeta functions. 2nd edition, Springer Verlag, 1984
8. Jeffrey Shallit. Length of the continued logarithm algorithm on rational inputs. https://arxiv.org/abs/1606.03881v2 [arXiv:1606.03881v2], 2016.
9. Brigitte Vallée. Dynamics of the Binary Euclidean Algorithm: Functional analysis and operators, Algorithmica, vol. 22, no. 4, pp. 660–685, 1998.
10. Brigitte Vallée. Euclidean Dynamics. Discrete and Continuous Dynamical Systems, vol. 15, no. 1, pp. 281–352, 2006.