Binary Relations in Mathematical Economics: On the Continuity, Additivity and Monotonicity Postulates in Eilenberg, Villegas and DeGroot

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Abstract: This chapter examines how positivity and order play out in two important questions in mathematical economics, and in so doing, subjects the postulates of continuity, additivity and monotonicity to closer scrutiny. Two sets of results are offered: the first departs from Eilenberg’s (1941) necessary and sufficient conditions on the topology under which an anti-symmetric, complete, transitive and continuous binary relation exists on a topologically connected space; and the second, from DeGroot’s (1970) result concerning an additivity postulate that ensures a complete binary relation on a σ-algebra to be transitive. These results are framed in the registers of order, topology, algebra and measure-theory; and also beyond mathematics in economics: the exploitation of Villegas’ notion of monotonic continuity by Arrow-Chichilnisky in the context of Savage’s theorem in decision theory, and the extension of Diamond’s impossibility result in social choice theory by Basu-Mitra. As such, this chapter has a synthetic and expository motivation, and can be read as a plea for inter-disciplinary conversations, connections and collaboration.

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It has often happened that a theory designed originally as a tool for the study of a physical problem came subsequently to have purely mathematical interest. When that happens the theory is generalized way beyond the point needed for applications, the generalizations make contact with other theories (frequently in completely unexpected directions), and the subject becomes established as a new part of pure mathematics. Physics is not the only external source of mathematical theories; other disciplines (such as economics and biology) can play a similar role.\(^1\)

Halmos (1956)

It is also possible that algebra, as a separate discipline within mathematics may not survive. The 20th century was a period of unification, with algebra invading other areas of math, and they counter-invading it. If I am engaged in studying a family of functions on multi-dimensional manifolds, those families having a group structure, am I working in analysis (the functions), topology (the manifolds) or algebra (the groups)?\(^2\)

Derbyshire (2006)

1 Introduction

In this chapter revolving around the ideas of positivity and order in mathematical economics, one can do worse than begin with Garett Birkhoff’s review of Eilenberg (1941): it is well-worth quoting in full.

An “ordered topological space” is, in effect, a simply ordered set whose topology is obtainable by a weakening of its intrinsic topology. The author proves that a topological connected space \(X\) can be ordered if and only if the subset of its square \(X^2\) obtained by deleting the diagonal of points \((x, x)\) is not connected; the same condition also characterizes those connected locally connected separable topological spaces which are homeomorphic with subsets of the linear continuum.

In this, his paper on “ordered topological spaces,” Eilenberg (1941) is justly celebrated for posing two questions of seminal importance for economic theory. First, can a continuous binary relation on a set be represented by a continuous function on the same set? Second, what are the conditions on the set under which a complete and continuous relation is necessarily transitive? Both questions, the second perhaps more than the other, investigate how technical topological conditions, assumed for tractability, necessarily translate into behavioral consequences. However, Eilenberg limited himself to the study of anti-symmetric relations, and thereby to studying agency in a context wherein distinct elements in the choice set are necessarily preferred one to another, a kind of extreme decisiveness. It remained for Debreu (1954, 1960) to place the first

\(^1\)(Halmos, 1956, p.419) The part of pure mathematics so created does not (and need not) pretend to solve the physical problem from which it arises; it must stand and fall on its own merits.

\(^2\)(Derbyshire, 2006, p.319)
question \(^3\) and for Sonnenschein (1965, 1967) the second, in a setting where the symmetric part of the given binary relation is not an equality, which is to say, the set of indifferent elements of the relation are not singletons. They and their followers have by now given rise to a rich and mature body of work.

Eilenberg also asked, and answered, two other questions that seem to have had less traction in economic theory, at least in the way that they were initially posed. He asked for conditions on the topology under which there exist “nice” relations (in the sense of being anti-symmetric, transitive, complete and continuous) on a given set, and furthermore, turning the matter on its head, how such relations disallow sets that are “rich” in the meaning endowed to the term through the topological and/or algebraic structures on the set over which they are defined. We shall think of these as Eilenberg’s third and fourth questions. Both questions are again natural ones. The third is in some sense analogous \(^4\) to the question concerning conditions on a topology under which non-constant continuous functions exist. If the topology is too ”sparse” then every continuous function is necessarily constant, and every reflexive, transitive and continuous relation is necessarily trivial in sense that no element is preferred to another. In the context of his fourth question, Eilenberg showed that the existence of a “nice” relation defined on a connected, locally connected and separable space necessarily renders the space to be a linear continuum. These results then are a testimony to the mutual imbrication of assumptions on a relation and the space on which the relation is defined, a two-way relationship that in recent work, Khan-Uyanık (2019) see and study as the Eilenberg-Sonnenschein (ES) research program.

In terms of the third and fourth questions concerning “nice” relations, to be sure, topologists have understood this mutual imbrication very well. Thus, for there to be a rich supply of continuous linear functions, the topology on the common domain of the functions must, of necessity, satisfy some properties, and cannot be too sparse. Alternatively, the only continuous functions on a set endowed with an indiscrete topology are the constant functions; and digging a little deeper, there is a plethora of (say) non-locally convex spaces with no continuous function at all other than the zero function. The question of the existence of a supporting hyperplane is explicitly studied by Klee (1963) in the context of an algebraic structure, and in the context of topological vector spaces, Kalton, Peck, and Roberts (1984, p. vii) write:

The role of the Hahn-Banach theorem may be said to be that of a universal simplifier whereby infinite-dimensional arguments can be reduced to the scalar case by the use of the

\(^3\)In his reproduction of Debreu’s theorem [Proposition 1] on the sufficiency of connectedness of a choice set in a finite-dimensional Euclidean space, Koopmans (1972) for example, observes that “Debreu credits a paper of Eilenberg (1941) as containing the mathematical essence of [his] Proposition 1.”

\(^4\)Eilenberg’s third question is entirely analogous to the existence of a one-to-one continuous function since he requires the anti-symmetry property. In Section 3 we introduce a result for binary relations that is exactly analogous to the existence of a non-constant continuous function.
ubiquitous linear functional. Thus the problem with non-locally convex spaces is that of “getting off the ground.”

The point is that there is some hiddenness in the mutual interaction of a function and set that needs to be flushed out. In terms of the origins, Urysohn (1925) studies the problem of determining the most general class of topological spaces in which non-constant real-valued continuous functions exist. Hewitt (1946) provides an example of a countable, connected Urysohn space in which every continuous function is constant. Following Hewitt’s work, there are results on the class of topological spaces on which every continuous function is constant; see Chittenden (1929) for the original paper, and the following, for example, for more modern work. Wallace (1962), Herrlich (1965), Lord (1995) and Iliadis and Tzannes (1986).

The question is of substantive consequence for functional analysis but also beyond it for economic theory and mathematical economics. In terms of this register, the problem gets translated into the question of the sustaining of technologically efficient program as in value maximization programs. Majumdar (1974) furnishes a complete characterization and refers to his result as follows:

One should recall that a major motivation behind research in this area comes from the need to determine whether efficient allocations can be attained by the use of a price mechanism in a decentralized system achieving economy of information and utilizing individual incentives. The implications of any result on complete characterization should be seriously considered in this context, and as far as [the result] goes, they seem to be somewhat negative in character. The equivalence established indicates that, in general, one would need a family of price systems to specify an efficient program. Indeed, the applicability of the criterion is rather restricted since one has to know too many prices.

It is then to this literature that we connect Eilenberg’s third and fourth questions. We see him asking this question: rather than the existence of a function from a “nice” class of functions, does there exist a binary relation from a “nice” class of binary relations? And the first contribution of this chapter is that it generalizes Eilenberg’s answer to this question by relaxing connectedness and anti-symmetry assumptions: in a nutshell, we do to Eilenberg in this context what Sonnenschein did in another and Debreu did in yet another. This is to say that we generalize Eilenberg’s result by dropping the anti-symmetry assumption, and then extend the

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5 A topological space in which any two distinct points can be separated by closed neighborhoods.
6 Majumdar continues, “But being a complete characterization, [the result] provides a new angle from which the difficulties faced by the earlier approaches can be viewed and tends to suggest that simpler criteria involving fewer price systems, in particular, the use of just one price system as is typically the case, may be incapable of isolating the set of efficient programs unless restrictive assumptions on technology are introduced.” For further work on the problem, see Stephan (1986), and the references to his chapters in Faber (1986).
generalization to $k$-connected spaces, and then to a setting that substitutes $k$-connectedness with local-connectedness. Finally, we note that our first two results can be analogously generalized to general preferences, and connect our results to the literature on the non-existence of non-constant continuous functions.

But we also make another connection that has been missed in the economic literature. This is the application of our results on the existence of a “nice” preference relation to Diamond’s (1965) impossibility theorem: what this economic literature sees as an impossibility result, we see simply as a question of the existence of a nice binary relation where the adjective nice has been given a meaning and an elaboration in terms of intergenerational equity. In introducing his own paper, Zame (2007, p. 188) documents the trajectory of this substantial economic literature.

Diamond (1965) shows that a complete transitive preference relation that displays intergenerational equity and respects the Pareto ordering cannot be continuous in the topology induced by the supremum norm. Basu and Mitra (2003) show that such a preference relation – whether continuous or not – cannot be represented by a (real-valued) utility function. On the other hand, Syrnsis (1980) proves that such preference relations do exist. Fleurbaey and Michel (2003), Hara, Suzumura, and Xu (2006), Basu and Mitra (2007), and Bossert, Sprumont, and Suzumura (2007) provide further results, both positive and negative.

Moreover, as already illustrated in Toranzo-Hervés-Beleso (1995), there are continuous, complete and transitive relations on non-separable spaces which are not representable. This connection that we make is important in that it sights Eilenberg (1941) as one of the originating papers of this substantial economic literature. This concludes our discussion of the first substantive section, Section 3, of the paper.

Section 4 of the paper returns to Eilenberg’s second question: to find a suitable topological condition which ensures the transitivity of a complete, reflexive and continuous binary relations. Khan and Uyanık (2019, 2020) frame this question in settings that remain squarely remain within the purely topological regime, but go considerably beyond Eilenberg. In his consideration of the relationship however, Sonnenschein (1965) move to a setting that also embrace linear structures. In a complementary result, Galabatatar-Khan-Uyanık (2018), henceforth GKU, show the existence of a mixture-continuous, anti-symmetric, transitive and complete relation defined on a mixture space renders the setting to be isomorphic to either a greater-than-or-equal-to relation, or its inverse, defined on the interval! These results are of substantive consequence for

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5It is worth noting that both Eilenberg (1941) and Sonnenschein (1965) limit their attention to one way of the two-way relationship, in that they examine the implication of assumptions on the choice set on the properties of the relation defined on that set; the backward direction exploring the implication of the properties of a class of preferences on the choice set over which they are defined is the signature of the Khan-Uyanık work.
social science since they pertain directly to the formalization of human agency. It mandates that in a sufficiently rich choice set, an agent in an economy, or a player in a game, cannot be simultaneously consistent (transitive) and extremely decisive (anti-symmetric and complete); or to put the matter in a contra-positive way, the choice-set of a sufficiently rational agent in the sense of satisfying the above two desiderata must of necessity be sparse and impoverished: a linear continuum in the case of Eilenberg and an interval in the case of GKU. Note that these results, while bearing obvious implication for results on the representation of binary relations, belong to an entirely different register. They concern the dove-tailing and mutual imbrication of a set of assumptions on one object for those on a different but not unrelated object.

The second contribution of this chapter is to make a further move from the register of mixture-spaces to a more abstract algebraic one. Our point of departure now is Villegas (1964, 1967): this work studied countably additive qualitative probability representations and showed that given a finite additive qualitative probability, monotone continuity is necessary and sufficient for a countably additive representation. It remained for DeGroot (1970) to flush out the abstract algebraic register grounding this result. The contribution of Section 4 below is (i) to introduce an sharper additivity postulate, one supplemented by monotone continuity postulates, on abstract algebraic structures that are analogous to Villegas’ additivity postulate, (ii) to obtain an equivalence result between additivity and transitivity without referring to completeness or continuity of the binary relation, (iii) to show that under additivity, different variants of the monotonicity concepts are equivalent, (iv) to relate our result to Villegas, DeGroot, de Finetti, Arrow and Chichinisky. In particular we highlight the hiddenness and redundancy of the transitivity assumption as these desiderata are emphasized in Khan and Uyanık (2019). As such, it contributes to the depth and maturity of the ES program. This concludes our discussion of Section 4 of the paper.

We began this introduction by reading Kakutani (1941) as a text revolving around four questions concerning binary relations: leaving Section for notational and conceptual preliminaries, we shall focus on the third and the fourth in Section 3, and on the second in Section 4. The reader may well wonder our silence about the (first) question that mathematical economists and economic theorists know him by. As mentioned, this pertains to the representation of a binary relation by a function, of a continuous relation by a continuous function, of a monotonic relation by a monotonic function, and of a concave relation by a quasi-concave function. We state the question in all these elaborated way simply to allude the river of work that has accumulated in mathematical economics and mathematical psychology on this question. But to keep to Elinberg’s parameters except that of his singleton indifference sets, we quote from Beardon (2020, p. 3).

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8See Krantz, Luce, Suppes, and Tversky (1971, Section 5.4.2) for further discussion.
In this expository essay we consider how much of the theory can be developed from a purely
topological perspective. We focus on those ideas which provide a link between utility theory
and topology, and we leave the economic interpretations to others. Briefly, we give priority
to results that seem to be topologically important, so we pay more attention to the quotient
space of indifference classes than is usual, and more attention to the order topology than
other topologies.

We refer the reader to the above article and to Bosi, Campión, Candeal, and Indurain (2020),
the book of which it is a chapter, and move on.

2 Mathematical and Conceptual Preliminaries

Let \( X \) be a set. A subset \( \succeq \) of \( X \times X \) denote a binary relation on \( X \). We denote an element
\((x, y) \in \succeq\) as \( x \succeq y \). The asymmetric part \( \succ \) of \( \succeq \) is defined as \( x \succ y \) if \( x \succeq y \) and \( y \not\succeq x \), and its
symmetric part \( \sim \) is defined as \( x \sim y \) if \( x \succeq y \) and \( y \succeq x \). The inverse of \( \succeq \) is defined as \( y \preceq x \). Its asymmetric part \( \prec \) is defined analogously and its symmetric part is \( \sim \). We provide
the descriptive adjectives pertaining to a relation in a tabular form for the reader’s convenience
in the table below.

| Property          | Description                                                                 |
|-------------------|-----------------------------------------------------------------------------|
| reflexive         | \( x \succeq x \ \forall x \in X \)                                       |
| complete          | \( x \succeq y \) or \( y \succeq x \ \forall x, y \in X \)                |
| non-trivial       | \( \exists x, y \in X \) such that \( x \succ y \)                          |
| transitive        | \( x \succ y \Rightarrow x \succ z \ \forall x, y, z \in X \)              |
| semi-transitive   | \( x \succ y \sim z \Rightarrow x \succ z \) and \( x \sim y \sim z \Rightarrow x \succ z \ \forall x, y, z \in X \) |
| anti-symmetric    | \( x \succ y \) and \( y \succeq x \Rightarrow x = y \ \forall x, y \in X \) |

Table 1: Properties of Binary Relations

Let \( \succeq \) be a binary relation on a set \( X \). For any \( x \in X \), let \( A_\succ(x) = \{ y \in X | y \succ x \} \) denote
the upper section of \( \succeq \) at \( x \) and \( A_\prec(x) = \{ y \in X | y \preceq x \} \) its lower section at \( x \). Now assume \( X \)
is endowed with a topology. We say \( \succeq \) is continuous if its upper and lower sections are closed
at all \( x \in X \) and the upper and lower sections of its asymmetric part \( \succ \) are open at all \( x \in X \).

A topological space \( X \) is said to be connected if it is not the union of two non-empty, disjoint
open sets. The space \( X \) is disconnected if it is not connected. A subset of \( X \) is connected if it is connected as a subspace. We say \( X \) is locally connected if for all \( x \in X \), every open neighborhood
of \( x \) contains a connected and open set containing \( x \). A component of a topological space is a
maximal connected set in the space; that is, a connected subset which is not properly contained

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9The reader interested in this (first) Eilenberg question can also see Wold (1943–44), Nachbin (1965),
Bridges and Mehta (1995), Herden (1989b); we shall return to Wold (1943–44) in Section 5.
in any connected subset. For any natural number $k$, a topological space is $k$-connected if it has at most $k$ components. The concept of $k$-connectedness provides a quantitative measure of the degree of disconnectedness of a topological space. It is easy to see that 1-connectedness is equivalent to connectedness and that any $k$-connected space is $l$-connected for all $l \geq k$.

3 On the Existence of a Continuous Binary Relation

Eilenberg (1941, Theorem I) provides a necessary and sufficient condition for the existence of an anti-symmetric, complete, transitive and continuous binary relation on a connected topological space $X$. In this section we start with introducing a generalization of Eilenberg’s result to $k$-connected spaces, and then show that when the space is locally connected, then cardinality of the components of the space does not matter. We continue by presenting a result which eliminates the anti-symmetry requirement in Eilenberg’s theorem. We end the section with a brief discussion of our results.

Before presenting our result, we need the following notation: for any set $X$, define

$$P(X) = \{(x, y) \in X \times X : x \neq y\}.$$ 

3.1 On Ordered Topological Spaces

Eilenberg (1941) calls a topological space ordered if there exists an anti-symmetric, complete, transitive and continuous binary relation on it. He then presents

**Theorem (Eilenberg).** A connected topological space $X$ which contains at least two elements can be ordered if and only if $P(X)$ is disconnected.

The following theorem generalizes Eilenberg’s theorem to $k$-connected spaces.

**Theorem 1.** For any natural number $k$, a $k$-connected topological space $X$ can be ordered if and only if $P(C)$ is disconnected for each non-singleton component $C$ of $X$.

**Proof of Theorem.** Let $\{C_i\}_{i=1}^{\ell}$ be the collection of the components of $X$ where $\ell \leq k$. First note that Theorem (Eilenberg) implies that for each component $C_i$ of $X$ which contains at least two elements, there exists an anti-symmetric, complete, transitive and continuous binary relation $\succ_i$ on $C_i$ if and only if $P(C_i)$ is disconnected.

In order to prove the forward direction, assume $\succ$ is an anti-symmetric, complete, transitive and continuous binary relation on $X$. Then, for each $C_i$, the restriction of $\succ$ on $C_i$, defined as

\[ \text{See Khan and Uyanık (2019) for a detailed discussion on $k$-connectedness.} \]
≽_i = ≽ ∩ (C_i × C_i), is an anti-symmetric, complete, transitive and continuous binary relation on C_i. Then, Theorem (Eilenberg) implies that P(C_i) is disconnected for all non-singleton C_i.

In order to prove the backward direction, assume P(C_i) is disconnected for each non-singleton C_i. It follows from Theorem (Eilenberg) that there exists an anti-symmetric, complete, transitive and continuous binary relation ≽_i on every non-singleton C_i. If C_i is a singleton, then define ≽_i = C_i × C_i. Then define a binary relation ≽ on X as follows: \( \bigcup_{i=1}^f ≽_i \subseteq ≽ \), and for all \( i > j \), \( C_i × C_j \subseteq ≽ \). Then, ≽ is anti-symmetric, complete, and transitive. Since each C_i is closed in X, therefore ≽_i has closed sections in both C_i and X, and hence ≽ has closed sections. Therefore, ≽ is continuous.

**Theorem 2.** A locally connected topological space X can be ordered if and only if P(C) is disconnected for each non-singleton component C of X.

**Proof of Theorem 2.** Let \( \{C_i\}_{i \in I} \) be the collection of the components of X. The proof of the forward direction is identical to the proof of the forward direction of Theorem 1. In order to prove the backward direction, assume P(C_i) is disconnected for each non-singleton C_i. It follows from Theorem (Eilenberg) that there exists an anti-symmetric, complete, transitive and continuous binary relation ≽_i on each non-singleton C_i. If C_i is a singleton, then define ≽_i = C_i × C_i. The well-ordering theorem (Munkres, 2000, Theorem, p.65) implies that there exists an anti-symmetric, complete and transitive binary relation ˆ≽ on I. Then define a binary relation ≽ on X as follows: \( \bigcup_{i \in I} ≽_i \subseteq ≽ \), and for all \( i \prec j \), \( C_i × C_j \subseteq ≽ \). Then, ≽ is anti-symmetric, complete, and transitive. Since X is locally connected, each C_i is both open and closed. Then, each ≽_i has closed sections in both C_i and X. Note that for all C_i and all \( x \in C_i \),

\[
A_{≽}(x) = A_{≽_i}(x) \cup \left( \bigcup_{j \prec i} C_j \right) = A_{≽_i}(x) \cup \left( \bigcap_{i \prec j} C_j^c \right).
\]

Then, it follows from C_i is open for all \( i \in I \) that ≽ has closed upper sections. An analogous argument implies that ≽ has closed lower sections. Therefore, ≽ is continuous.

### 3.2 On Weakly Ordered Topological Spaces

This subsection provides a necessary and sufficient condition for the existence of a non-trivial, complete, transitive and continuous binary relation on a connected topological space. This result is analogue to Theorem (Eilenberg), except that the binary relation is not necessarily anti-symmetric.

**Definition 1.** A topological space is weakly ordered if there exists a non-trivial, complete, transitive and continuous binary relation on it.
**Theorem 3.** A connected topological space $X$ which contains at least two elements can be weakly ordered if and only if $P(X | \sim)$ is disconnected for some equivalence relation $\sim$ on $X$.

**Proof of Theorem 3.** Let $X$ be a topological space with at least two elements. Assume there exists a non-trivial, complete, transitive and continuous binary relation $\succ$ on $X$. Let $\sim$ denote the symmetric part of $\succ$. Since $X$ is connected, the quotient space $X | \sim$ is connected. It is easy to show that the induced binary relation $\hat{\succ}$ on $X | \sim$, defined as $([x], [y]) \in \hat{\succ}$ if and only if $(x', y') \in \succ$ for all $x' \in [x]$ and all $y' \in [y]$, is non-trivial, anti-symmetric, complete, transitive and continuous. Then, it follows from Theorem (Eilenberg) that $P(X | \sim)$ is disconnected.

In order to prove the backward direction, assume there exists an equivalence relation $\tilde{\sim}$ on $X$ such that $P(X | \tilde{\sim})$ is disconnected. Then, $X | \tilde{\sim}$ contains at least two elements. Since $X | \tilde{\sim}$ is connected, it follows from Theorem (Eilenberg) that there exists an anti-symmetric, complete and continuous binary relation $\hat{\succ}$ on $X | \tilde{\sim}$. Define a binary relation $\succ$ on $X$ as $(x, y) \in \succ$ if and only if $([x], [y]) \in \hat{\succ}$. Then the symmetric part $\sim$ of $\succ$ is identical to $\tilde{\sim}$. It follows from $A_{\succ}(x)$ and $A_{\sim}(x)$ are closed in $X | \tilde{\sim}$ and the definition of the quotient topology that the sections

$$A_{\succ}(x) = \bigcup_{[y] \succ ([x])} [y] \quad \text{and} \quad A_{\sim}(x) = \bigcup_{[y] \sim ([x])} [y]$$

of $\succ$ are closed in $X$, hence $\succ$ is continuous. The non-triviality, completeness and transitivity of $\succ$ directly follow from its construction.

Note that in an ordered space, the indifference relation $\sim$ in Theorem 3 is assumed to be the equality relation. Hence, as expected, the requirement for the existence of an order is stronger than the requirement for that of a weak order. The following example illustrates a weakly ordered topological space which cannot be ordered.

**Example.** Let $X = [0, 2]$ and the following define a basis for the topology on $X$: $[0, x)$ for all $x \in (1, 2]$, $(x, 2]$ for all $x \in [1, 2]$, and $(x, y)$ for all $x, y \in [1, 2]$. Note that the smallest closed set containing any point in $[0, 1]$ is $[0, 1]$. It is clear that $X$ is connected. Since the topology is not Hausdorff, Eilenberg (1941. 1.4) implies that there does not exist an anti-symmetric, complete and continuous binary relation on $X$. However, the following is a non-trivial, complete, transitive and continuous binary relation on $X$: $(x, y) \in \preceq$ for all $x, y \in [0, 1]$, and $(x, y) \in \preceq$ for all $x, y \in X$ with $x < y$.

Finally, the methods of proofs presented in Theorems 1 and 2 can be used to provide generalizations of this result to disconnected spaces.
3.3 Discussion of the Results

We can apply our results to the literature on the non-existence of a non-constant function on topological spaces as follows. First, note that every non-constant continuous function induces a non-trivial, complete, transitive and continuous binary relation. Therefore, by Hewitt’s (1946) result we know that there does not exist a non-trivial, complete, transitive and continuous relation. Moreover, note that the space in Hewitt’s paper is countable, hence separable, and connected. Therefore, every non-trivial, complete, transitive and continuous relation has a non-constant, continuous real-valued representation. Therefore, Theorem 3 provides an equivalence condition for the existence of a non-constant function in Hewitt’s setting. Hence, Theorem 3 may provide a new perspective on Hewitt’s theorem and on the subsequent work in this line of work.

Moreover, Miller (1970), Golomb (1959), Kirch (1969) and Jameson (1974) provide countable spaces that are connected and satisfy the Hausdorff separation axiom. Since continuous functions take connected sets to connected sets, therefore there cannot exist a non-constant continuous function on these spaces. We next show that there does not exist a continuous, non-trivial, semi-transitive relation with a transitive symmetric part on these spaces. First, by appealing to the current authors’ earlier work\footnote{See Khan and Uyanık (2019, Theorem 2). We refer the reader to Khan and Uyanık (2020) and Uyanık and Khan (2019b) for generalizations to bi-preference structures and general parametrized topological spaces.} any such relation is complete and transitive. Since the space is countable, it is separable. Therefore, it follows from Debreu (1954, Theorem I) that there exists a continuous real-valued function representing the binary relation. Since the relation is non-trivial, therefore the function is non-constant. This furnishes us a contradiction.

The literature has focused on the existence, or non-existence, of a non-constant continuous function. For binary relations, different continuity postulates has been introduced and used in mathematical economics. The existence of a non-trivial binary relation satisfying different continuity assumptions may be of interest; see Uyanık and Khan (2019a) for an extended discussion on the continuity postulate.\footnote{There is a literature on different continuity postulates for functions; see Giesielski and Miller’s (2016) recent survey on this.}

4 On the Additivity Postulate

In this section we provide two results on the implications of the additivity postulate. We first show that a strong form of additivity postulate is equivalent to the transitivity postulate. Then we define three monotone continuity postulates on partially ordered sets, inspired by the pioneering work of Villegas on qualitative probability, and then show that under the additivity
postulate, the three continuity postulates are equivalent. We end this section by relating our results to the antecedent literature.

4.1 Additivity and Transitivity: A Two-Way Relationship

**Definition 2.** A binary relation $\succeq$ on an Abelian group $(X, +)$ is called additive if for all $x, y, z \in X$, $x \succeq y$ implies $x + z \succeq y + z$. Moreover, we say $\succeq$ is strongly additive if for all $x_1, x_2, y_1, y_2 \in X$, $x_i \succeq y_i$ for $i = 1, 2$ implies $x_1 + x_2 \succeq y_1 + y_2$.

We first present a result on the relationship between additivity and strong additivity.

**Proposition 1.** Every reflexive and strongly additive relation on an Abelian group is additive.

*Proof of Proposition 1.* Assume $\succeq$ is strongly additive relation on an Abelian group $(X, +)$. Pick $x, y, z \in X$ such that $x \succeq y$. Then $z \succeq z$, by reflexivity, and strong additivity of $\succeq$ imply $x + z \succeq y + z$. Hence $\succeq$ is additive.

Along with this observation, the next result shows that when a reflexive binary relation is transitive, the two additivity postulates are equivalent. Moreover, it shows that the transitivity of the relation is implied by strong additivity.

**Theorem 4.** An additive binary relation $\succeq$ on an Abelian group $(X, +)$ is transitive if and only if it is strongly additive.

*Proof of Theorem 4.* Let $\succeq$ be an additive binary relation on an Abelian group $(X, +)$. Assume $\succeq$ is transitive. Pick $x_1, x_2, y_1, y_2 \in X$ such that $x_i \succeq y_i$ for $i = 1, 2$. Then it follows from additivity that $x_1 + x_2 \succeq y_1 + x_2$ and $x_2 + y_1 \succeq y_2 + y_1$. Then commutativity of $+$ and transitivity of $\succeq$ implies that $x_1 + x_2 \succeq y_1 + y_2$.

Now assume $\succeq$ is strongly additive. Pick $x, y, z \in X$ such that $x \succeq y \succeq z$. Then strong additivity implies $x + y \succeq y + z$. Then additivity of $\succeq$ imply $x + y + (-y) \succeq y + z + (-y)$. Therefore $x \succeq z$.

The following is a direct corollary of Proposition 1 and Theorem 4.

**Corollary 1.** Every reflexive and strongly additive relation on an Abelian group is transitive.

4.2 Implications of Additivity for Monotone Continuity

Let $(X, \geq)$ be a partially ordered set. We say $X$ is order-complete if every non-empty subset of $X$ with an upper bound has a least upper bound. Note that a poset $X$ is order-complete if and only if every non-empty subset of $X$ with a lower bound has a greatest lower bound; see Fremlin (3.14B, vol3I).
Definition 3. Let \((X, \geq)\) be an order-complete poset and \(\succ\) a binary relation on \(X\). We define the following monotone continuity axioms for \(\succ\):

[C1'] For all \(y \in X\) and all bounded below sequence \(\{x_i\}_{i \in \mathbb{N}}\) in \(X\), \(x_i \geq x_{i+1}\) and \(x_i \succ y\) for all \(i\) imply \(\inf\{x_i\}_{i \in \mathbb{N}} \succ y\).

[C2'] For all \(y \in X\) and all bounded above sequence \(\{x_i\}_{i \in \mathbb{N}}\) in \(X\), \(x_{i+1} \geq x_i\) and \(y \succeq x_i\) for all \(i\) imply \(y \succeq \sup\{x_i\}_{i \in \mathbb{N}}\).

[C3'] For all \(y \in X\) and all bounded above sequence \(\{x_i\}_{i \in \mathbb{N}}, x_{i+1} \geq x_i\) and \(y < \sup\{x_i\}_{i \in \mathbb{N}}\) imply there exists an integer \(N > 0\) such that, for \(i \geq N\), we have \(y < x_i\).

Theorem 5. For any complete and strongly additive binary relation on an Abelian group which is also an order-complete poset, the continuity axioms C1', C2' and C3' are equivalent.

Proof of Theorem 5. Let \((X, +, \geq)\) be an order-complete poset on an Abelian group and \(\succ\) a complete and strongly additive binary relation on \(X\). It follows from Corollary 1 that \(\succ\) is transitive.

First, we show that C1' is equivalent to C2'. Note that additivity implies \(x \succeq y\) if and only if \(-y \succeq -x\). Assume C1'. Pick a bounded above sequence \(\{x_i\}_{i \in \mathbb{N}}\) and \(y \in X\) such that \(x_{i+1} \geq x_i\) and \(y \succeq x_i\) for all \(i\). Then, \(-x_i \geq -x_{i+1}\) and \(-x_i \succeq -y\) for all \(i\). It follows form C1' that \(\inf\{-x_i\}_{i \in \mathbb{N}} \succeq -y\). We now show that additivity implies \(\inf\{-x_i\}_{i \in \mathbb{N}} = -\sup\{x_i\}_{i \in \mathbb{N}}\). Define \(\underline{x} = \inf\{-x_i\}_{i \in \mathbb{N}}\) and \(\bar{x} = -\sup\{x_i\}_{i \in \mathbb{N}}\). Assume towards a contradiction that \(\underline{x} < \bar{x}\). By definition, \(\underline{x} \leq -x_i\) for all \(i\). Then additivity implies \(-\underline{x} \geq x_i\) for all \(i\). Then \(-\underline{x}\) is an upper bound of \(\{x_i\}_{i \in \mathbb{N}}\), hence \(-\underline{x} \geq \bar{x}\). This contradicts the assumption that \(\underline{x} < \bar{x}\). An analogous argument yields a contradiction for \(\underline{x} > \bar{x}\). Therefore, \(\underline{x} = \bar{x}\). Then \(-\sup\{x_i\}_{i \in \mathbb{N}} \succeq -y\), hence by additivity, \(y \succeq \sup\{x_i\}_{i \in \mathbb{N}}\). Therefore, C2' holds. The proof of the converse relationship is analogous.

We next show that C2' is equivalent to C3'. Assume C2'. Assume towards a contradiction that there exists a bounded above \(\{x_i\}_{i \in \mathbb{N}}\) and \(y \in X\) such that \(x_{i+1} \geq x_i\) and \(y < \sup\{x_i\}_{i \in \mathbb{N}}\), but for all \(N > 0\), there exists \(j \geq N\) such that \(y \succeq x_j\). Then there exists a subsequence \(\{x_{ik}\}_{k \in \mathbb{N}}\) such that for all \(k\), \(x_{ik+1} \geq x_{ik}\) and \(y \succeq x_{ik}\). Then C2' implies \(y \succeq \sup\{x_{ik}\}_{k \in \mathbb{N}}\). It is easy to see that \(\sup\{x_{ik}\}_{k \in \mathbb{N}} = \sup\{x_i\}_{i \in \mathbb{N}}\). This contradicts the assumption that \(y < \sup\{x_i\}_{i \in \mathbb{N}}\). Hence C3' holds. The converse relationship immediately follows from the definitions.

4.3 Discussion of the Results

Villegas [1964] introduced the following additivity concept for binary relations on a \(\sigma\)-algebra.

Definition 4. A preference relation \(\succ\) on a \(\sigma\)-algebra \(\mathcal{X}\) on a set \(X\) is Villegas-additive if for all \(A_1, A_2, B_1, B_2 \in \mathcal{X}\) with \(A_1 \cap A_2 = B_1 \cap B_2 = \emptyset\), \(A_i \succ B_i\) for \(i = 1, 2\) implies \(A_1 \cup A_2 \succeq B_1 \cup B_2\).
If, in addition, \( A_1 \succ B_1 \) or \( A_2 \succ B_2 \), then \( A_1 \cup A_2 \succ B_1 \cup B_2 \).

First note that the union operation is similar to the additivity operation\(^{13}\) but it does not satisfy all properties the addition in an Abelian group satisfies. Moreover, the usual additivity assumption is neither stronger nor weaker than Villegas-additivity: the latter imposes restriction on a smaller class of elements whereas additivity does not impose a restriction on the strict relation. DeGroot (1970, Theorem 1, p. 71) followed Villegas and proved a result analogous to Theorem 4 where the space is a \( \sigma \)-algebra with the usual inclusion relation.

**Theorem** (DeGroot). Every complete and Villegas-additive binary relation on a \( \sigma \)-algebra is transitive.

We next apply our results to de Finetti’s expected utility representation theorem. Let \( X = \mathbb{R}^n \) which is endowed with the usual topological, algebraic and order structures. A real valued function \( u \) is called monotone if for all \( x, y \in \mathbb{R}^n \) such that \( x > y \) (i.e. \( x_i \geq y_i \) for all \( i \) and \( x \neq y \)), \( u(x) > u(y) \). A preference relation \( \succ \) on \( \mathbb{R}^n \) is monotone if for all \( x, y \in \mathbb{R}^n \), if \( x > y \), then \( x \succ y \). The following theorem is due to de Finetti (1937, 1974)\(^{14}\).

**Theorem** (de Finetti). Let \( \succ \) be a binary relation on \( \mathbb{R}^n \). The following are equivalent.

(a) The binary relation \( \succ \) is complete, transitive, additive and continuous.

(b) There exist positive \((p_i)_{i=1}^n\), summing to one, such that \( u(x) = \sum p_i x_i \) represents \( \succ \).

The equivalence theorem of de Finetti can be restated as

**Corollary 2.** Let \( \succ \) be a binary relation on \( \mathbb{R}^n \). The following are equivalent.

(a) The binary relation \( \succ \) is complete, strongly additive and continuous.

(b) There exist positive \((p_i)_{i=1}^n\), summing to one, such that \( u(x) = \sum p_i x_i \) represents \( \succ \).

Therefore, we can drop the transitivity assumption in de Finetti’s theorem by replacing additivity with strong additivity, which are equivalent in the presence of the transitivity postulate. We can also drop the completeness assumption; see Uyanık and Khan (2019a) for a detailed exposition on the hiddenness and redundancy in mathematical economics\(^{15}\).

We next move to monotone continuity. The second subsection above is an attempt to understand this postulate introduced in Villegas, DeGroot, Arrow and Chichilnisky in order to study qualitative/subjective probability. As we illustrate above, monotone continuity neither

\(^{13}\)Note that Fishburn (1986, p. 336) calls Villegas-additivity the *additivity axiom*.

\(^{14}\)See Wakker (1989, Theorem A.2.1, p.161) for the statement and further details.

\(^{15}\)See also Krantz, Luce, Suppes, and Tversky (1971, Section 5.4.2) for an interesting discussion on hiddenness and redundancy. Moreover, it may be of interest to generalize this result to groupoids or semigroups; see Fishburn (1972, Chapter 11).
requires any topological property on the choice set, nor uses the structure of the unit interval, unlike the continuity assumption of Herstein and Milnor (1953). Hence, an investigation of the relationship between monotone continuity with the other continuity postulates, and its applications may be of interest, and our definitions and results can be considered as the first step for such investigation. Villegas (1964) and DeGroot (1970) provide the following monotone continuity postulates for binary relations defined on σ-algebras.

**Definition 5.** Let $X$ be a σ-algebra on a set and $\succsim$ a binary relation on $X$. We define the following monotone continuity axioms for $\succsim$.

[C1] For all sets $\{A_i\}_{i \in \mathbb{N}}, B$ in $X$, $A_1 \supseteq A_2 \supseteq \cdots$ and $A_i \succsim B$ for all $i$ imply $\bigcap_i A_i \succsim B$.

[C2] For all sets $\{A_i\}_{i \in \mathbb{N}}, B$ in $X$, $A_1 \subseteq A_2 \subseteq \cdots$ and $B \succsim A_i$ for all $i$ imply $B \succsim \bigcup_i A_i$.

[C3] For all sets $\{A_i\}_{i \in \mathbb{N}}, B$ in $X$, $A_1 \subseteq A_2 \subseteq \cdots$ and $B \prec \bigcup_i A_i$ imply there exists an integer $N > 0$ such that, for $i \geq N$, we have $B \prec A_i$.

The following is a result analogous to Theorem 5 above for the special case of σ-algebras.

**Theorem 6.** For any complete and Villegas-additive binary relation on a σ-algebra, the monotone continuity postulates C1, C2 and C3 are equivalent.

The equivalence between C2 and C3 is due to Villegas (1964, Theorem) and between C1 and C2 is due to DeGroot (1970, Theorem 5).

Villegas (1964, 1967) studied countably additive qualitative probability representation and showed that given a finitely additive qualitative probability, monotone continuity is necessary and sufficient for countably additive representation; see Krantz, Luce, Suppes, and Tversky (1971, Section 5.4.2) for further discussion. In particular, the following result is quoted.

**Theorem (Villegas).** A finitely additive probability representation of a structure of qualitative probability, on a σ-algebra, is countably additive if and only if the structure is monotonically continuous.

Finally, the following monotone continuity postulate is due to Arrow (1971).

**Definition 6.** Given $a$ and $b$, where $a \succ b$, a consequence $c$ and a vanishing sequence $\{E_i\}$, suppose sequence of actions satisfy the conditions that $(a^i, s)$ yield the same consequences as $(a, s)$ for all $s \in E_i^c$, and the consequence $c$ for all $s \in E_i$, while $(b^i, s)$ yield the same consequences.

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16See Khan and Uyanık (2019) and Ghosh, Khan, and Uyanık (2020) for a discussion on the relationship among different continuity postulates. Moreover, DeGroot’s assumption SP5 “There exists a random variable which has a uniform distribution on the interval [0, 1]” may have some relevance to the existence of a nice relation, or a continuous function. We leave this for future work.

17For definitions, we refer the reader to Krantz, Luce, Suppes, and Tversky (1971).
as \((b, s)\) for all \(s \in E^c_i\), and the consequence \(c\) for all \(s \in E_i\). Then, for all \(i\) sufficiently large, 
\(a^i \succ b\) and \(a \succ b^i\).

Chichilnisky (2010) interpreted Arrow’s definition as follows and showed that it is equivalent to the continuity postulate C1.

**Definition 7.** Let \(\mathcal{X}\) be a \(\sigma\)-algebra on a set and \(\succ\) a binary relation on \(\mathcal{X}\). We call \(\succ\) satisfies Monotone Continuity Axiom 4 (C4) if for all \(\{A_i\}_{i \in \mathbb{N}}, F, G \in \mathcal{X}\), \(A_1 \supseteq A_2 \supseteq \cdots, \bigcap_{i=1}^{\infty} A_i = \emptyset\) and \(F \succ G\) imply there exists \(N > 0\) such that altering arbitrarily the events \(F\) and \(G\) on the set \(A_i\), where \(i > N\), does not alter the ranking of the events, namely \(F' \succ G'\), where \(F'\) and \(G'\) are the altered events.

## 5 Order and Positivity in Mathematical Economics

We began this essay with Halmos’ take on how applied mathematics transits to pure mathematics; and Derbyshire’s take on how an important sub-field, with increasing importance, gets incorporated into the larger field of which it is a part, and thereby changes the identity of the larger field and loses its own. In this concluding section, we read, against the grain, these two texts and their claims on the incorporation of positivity and order-theoretic methods in Walrasian general equilibrium theory.

In classical Walrasian general equilibrium theory, as brought to fruition in Koopmans (1957), Debreu (1959), Nikaido (1968), Arrow and Hahn (1971) and McKenzie (2002), the agents in the economy are categorized as consumers and producers, with the former parametrized by preferences (a binary relation) defined on a (consumption) set and endowments being elements of such a set; and the latter drawing their signature simply by having an access to a production set. The vernacular of order and positivity is relevant in so far as it is relevant to its constituent conceptions of a consumer and a producer. The idea of monotonicity enters the theory of

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18 In his participation in the composition of this section, Khan should like to acknowledge his indebtedness to conversations with Malcolm King, and Niccolò Urbinati, and to JJ Grobler’s inspiring talk titled 101 years of vector lattice theory: A general form of integral: PJ Daniell (1918) at the Conference. He should also like to acknowledge the stimulus received from Schlesser’s readings of Foucault (2008 (French edition 2004).

19 Our choice of these four texts should perhaps be justified. For the texts of Debreu and McKenzie, we can appeal to Dippe and Weintraub (2014) who argue that the 1954 papers of Arrow-Debreu and McKenzie wrought fundamental changes in economic theory, a claim contested in Khan (2020) who urged the inclusion of Uzawa, Nikaido and Gale also as fellow-pioneers of what we are calling here “Walrasian general equilibrium theory.” The naming of the Arrow-Debreu model or the Arrow-Debreu-McKenzie model facilitated a homogeneous monolithic view and added to the confusion and to an unfortunate haste in canonization; see Footnote 22 below. A confounding factor in this is that many of the pioneers of Walrasian general equilibrium theory were also pioneers of linear and non-linear programming; Uzawa being one of the leaders. For this line of work, see Dorfman, Samuelson, and Solow (1958), and the recent application of Uzawa’s consequential extension of the Kuhn-Tucker-Karush theorem in Khan and Schled (2013).
production through the assumption of free disposal, an assumption delineated by Debreu (1954) in the context of a production set, say in a ordered normed space whose positive cone has a non-empty interior.

The assumption of free disposal for the technology means that if an input-output combination is possible, so is one where one where some outputs are smaller or some inputs larger; it is implied that a surplus can be freely disposed of. With this assumption, if the production set is non-empty, it has an interior point.

It is the existence of an interior that proves crucial for the sustainability of technologically efficient production plans and Pareto optimal allocations through individual value and profit maximization.

As far as the theory of the consumer is concerned, the ideas of order and positivity enter through the assumption of monotonicity of preferences which gets translated into “more is always preferred to less.” To be sure, it factors into the Eilenberg questions regarding binary relations with which we began the introduction. Thus (Arrow and Hahn, 1971, p. 106) write:

Wold seems to have been the first to see the need of specifying assumptions under which the representation of the continuous utility functions exists. Wold assumed that the [consumption set] is the entire non-negative orthant [of finite-dimensonal Euclidean space] and that preference is strictly monotone in each commodity. A very considerable generalization, based on a mathematical paper by Eilenberg (1941), was achieved with the deeper methods of Debreu (1954); he assumed only the continuity of preferences and the connectedness of the [consumption set] (a property weaker than convexity).

This is an important passage: its irony lies in the fact that it comes from two of the more distinguished and senior Walrasian theorists at the time who could not refrain from drawing arbitrary and needless distinctions between mathematicians and economists, and between mathematical and economic papers, and thereby in sighting Eilenberg (1941), and devaluing it at the same time. The point is that Eilenberg and Wold were independent pioneers of what later assumed the identity of an important subfield of “choice and decision theory.”

But returning to trajectories being implicitly charted by Halmos and to Derbyshire, the point is that the monotonicity assumption for consumers in the Walrasian conception comes rather late in its development: it is not there, for example, in Debreu (1959), or in McKenzie.

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20 We invite the reader to compare Debreu’s definition with corresponding definitions of the concept in the five texts to which Footnote 19 refers. The idea of “free disposal” is intimately tied to the non-negativity of prices; see Harr (2005), and compare Debreu (1959) and McKenzie (2002) on this issue.

21 We can recommend Fishburn (1972), Luce (2000), Gilboa (2004, 2009), Moscati (2016) and their references for this subject, which branches off also into mathematical psychology.
The more important question, however, is where the subject is in terms of these, their trajectories. This is a question that merits an investigation of its own, and is outside the scope of this technical essay: it suffices to make two observations. With respect to Halmos, classical Walrasian general equilibrium theory has neglected, by its very definitional conception, interdependencies between the parametrizations of what it sees as the relevant agents in the economy; and classical game theory, again by virtue of its definitional conception, has neglected the market in its formalizations. The applied problems of our time cry out for a formalization of these interdependencies in what perhaps ought to be a synthetic view of both subjects. Thus even after 70 years, mathematical economics (including game theory) has very much retained its dependence on both economics and mathematics. This is to say that it has remained pure and applied. As to Derbyshire on algebra, in terms of the algebraic approach to these subjects, it has yet to be incorporated into both Walrasian general equilibrium theory and in non-cooperative game theory. In the authors’ judgement, this cannot but be a fruitful task.

There is another, perhaps narrower, way to view the substance of these results. The question of the “right” commodity space for general Walrasian general equilibrium, or the “right setting” of the individual action sets in game theory, has not been explicitly posed. There has been little need to do so. Given the substantive questions at issue, the economic or game-theoretic formulations assume a strong-enough structure on the payoff functions and the choice sets by setting them either in a finite-dimensional Euclidean space, or in the context of game theory, a finite number of actions, to allow the question to be investigated and determinatively answered. When this rather arbitrary limitaion is removed, the question becomes of consequence, and notions of order and positivity began to take on colours that one may not have previously imagined.

This introduction has framed the results to follow as stemming from Eilenberg’s (1941)...

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22 This also suggests how much a reader of Walrasian general equilibrium theory loses by ascribing to it a monolithic conception. Each of these pioneers had their own ways of looking at their subject. In this connection one may also refer the interested reader to McKenzie’s conception of production in his (2002) Section 2.8, pages 77-82 on an “Economy of Activities.” It is also perhaps worth noting that Debreu’s resistance to the monotonicity assumption on consumers may be due to his having relaxed the monotonicity assumption in Wold (1943–44). To the authors knowledge, his first recourse to the assumption is in connection with the Debreu-Scarf theorem in 1963, and to be sure the assumption irrevocably enters into the field with Aumann and his Israeli school of Walrasian theory; see (1983) for the relevant papers and references. As emphasized in Khan (2020), the erasure of the production sector can also be ascribed to this school, and it becomes folded into the ideological divide between the “two Cambridges,” those of the UK and the US. Foucault’s emphasis on “governmentality” in the formulation of perfect competition and its normative properties is clearly relevant here.

23 This investigation remains an ongoing project of Khan and Urbinati, and in his talk in Pretoria, Khan made some room to expand at some length to report on Nikaido’s contributions to this question in keeping with this project.
seminal work. In this connection we observe that it is an interesting curiosum in the history of ideas that a piece of work entirely peripheral to an author’s oeuvre, written almost as a fragmentary passing thought, proves to be of such decisive and sustainable consequence in what may have been perceived at the time of its writing to be an unrelated discipline. Eilenberg’s paper, along with Kakutani’s (1941) fixed point theorem, coincidentally published in the same year, may well be two canonical examples. In any case, as far as mathematical economics is concerned, the belated recognition of this pioneering paper by Debreu (1954, 1964) and Sonnenschein (1965, 1967) has subsequently waned, and it is only recent work that has re-emphasized Eilenberg’s work and given it importance under the rubric of what it refers to as the Eilenberg-Sonnenschein program. It is a source of satisfaction to the authors that its importance can also be delineated in a chapter on a book on positivity and order.

6 Bibliography

Arrow, K. J. (1965): Aspects of the Theory of Risk-bearing. Helsinki: Yrjö Jahnsson Foundation.

——— (1966): “Exposition of the theory of choice under uncertainty,” Synthese, pp. 253–269.

——— (1971): Essays in the Theory of Risk-Bearing. Amsterdam: North-Holland.

——— (1972): “Exposition of the theory of choice under uncertainty,” in Decision and Organization: A Volume in Honor of Jacob Marschak, ed. by C. B. McGuire, and R. Radner. Amsterdam: North-Holland.

Arrow, K. J., and F. H. Hahn (1971): General Competitive Analysis. Amsterdam: North-Holland.

Asheim, G. B., T. Mitra, and B. Tungodden (2012): “Sustainable recursive social welfare functions,” Economic Theory, 49(2), 267–291.

Basu, K., and T. Mitra (2003): “Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian,” Econometrica, 71(5), 1557–1563.

——— (2007): “Possibility theorems for equitably aggregating infinite utility streams,” in Intergenerational Equity and Sustainability, pp. 69–84. Springer.

Beardon, A. F. (2020): “Topology and preference relations,” in Mathematical Topics on Representations of Ordered Structures and Utility Theory: Essays in Honor of Professor Ghan-

241941 was a particular productive years for Eilenberg, especially given the standards of the time: in addition to the paper being discussed here, he published at least six other papers; see (Derbyshire, 2006, p.302) for his 1940 meeting with Saunders MacLane, and his subsequent involvement with algebraic topology. His paper with Wilder on “uniform local connectedness and contractability” was to follow an year later, and the generalization of Kakutani’s fixed point theorem with Montgomery, five years later.
shyam B. Mehta, ed. by G. Bosi, M. J. Campión, J. C. Candeal, and E. Indurain, pp. 3–21. Springer: Switzerland.

Beardon, A. F., and G. B. Mehta (1994): “The utility theorems of Wold, Debreu, and Arrow-Hahn,” *Econometrica*, 62(1), 181–186.

Birkhoff, G. (1948): *Lattice Theory (Revised Edition)*, vol. XXV. New York: American Mathematical Society.

Bosi, G., M. J. Campión, J. C. Candeal, and E. Indurain (2020): *Mathematical Topics on Representations of Ordered Structures and Utility Theory*. Springer: Switzerland.

Bosi, G., and G. Herden (2019): “The structure of useful topologies,” *Journal of Mathematical Economics*, 82, 69–73.

Bosi, G., and M. Zuanon (2020): “Continuity and continuous multi-utility representations of nontotal preorders: some considerations concerning restrictiveness,” in *Mathematical Topics on Representations of Ordered Structures and Utility Theory: Essays in Honor of Professor Ghanshyam B. Mehta*, ed. by G. Bosi, M. J. Campión, J. C. Candeal, and E. Indurain, pp. 213–236. Springer: Switzerland.

Bossert, W., Y. Sprumont, and K. Suzumura (2007): “Ordering infinite utility streams,” *Journal of Economic Theory*, 135(1), 579–589.

Bridges, D. S., and G. B. Mehta (1995): *Representations of Preference Orderings*. Berlin: Springer-Verlag.

Chichilnisky, G. (2010): “The foundations of statistics with black swans,” *Mathematical Social Sciences*, 59(2), 184–192.

Chittenden, E. (1929): “On general topology and the relation of the properties of the class of all continuous functions to the properties of space,” *Transactions of the American Mathematical Society*, 31(2), 290–321.

Ciesielski, K. C., and D. Miller (2016): “A continuous tale on continuous and separately continuous functions,” *Real Analysis Exchange*, 41(1), 19–54.

de Finetti, B. (1937): “La prévision: ses lois logiques, ses sources subjectives,” in *Annales de l’institut Henri Poincaré*, vol. 7, pp. 1–68.

——— (1974): *Theory of Probability*, vol. 1. New York: Wiley.

Debreu, G. (1954): “Valuation equilibrium and Pareto optimum,” *Proceedings of the National Academy of Sciences*, 40(8), 588–592.

——— (1959): *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, no. 17. Connecticut: Yale University Press.

——— (1960): “Topological methods in cardinal utility theory,” in *Mathematical Methods in the Social Sciences*, ed. by K. Arrow, S. Karlin, and P. Suppes, pp. 16–26. California: Stanford University Press.
(1964): “Continuity properties of Paretian utility,” *International Economic Review*, 5(3), 285–293.

(1983): *Mathematical Economics: Twenty Papers of Gerard Debreu*, no. 4. Cambridge: Cambridge University Press, Econometric Society Monographs.

DeGROOT, M. H. (1970): *Optimal Statistical Decisions*. McGraw-Hill: New York.

DERBYSHIRE, J. (2006): *Unknown Quantity: A Real and Imaginary History of Algebra*. New York: Plume Books.

DIAMOND, P. A. (1965): “The evaluation of infinite utility streams,” *Econometrica*, 33(1), 170–177.

DORFMAN, R., P. A. SAMUELSON, AND R. M. SOLOW (1958): *Linear Programming and Economic Analysis*. New York: McGraw-Hill.

DÜPPE, T., AND E. R. WEINTRAUB (2014): *Finding Equilibrium: Arrow, Debreu, McKenzie and the Problem of Scientific Credit*. New Jersey: Princeton University Press.

DUSHNIK, B., AND E. W. MILLER (1941): “Partially ordered sets,” *American Journal of Mathematics*, 63(3), 600–610.

EILENBERG, S. (1941): “Ordered topological spaces,” *American Journal of Mathematics*, 63(1), 39–45.

FABER, M. (ed.) (1986): *Studies in Austrian Capital Theory, Investment and Time*. New York: Springer-Verlag.

FISHBURN, P. C. (1972): *Mathematics of Decision Theory*. The Hague: Mouton.

(1986): “The axioms of subjective probability,” *Statistical Science*, pp. 335–345.

FLEURBAEY, M., AND P. MICHEL (2003): “Intertemporal equity and the extension of the Ramsey criterion,” *Journal of Mathematical Economics*, 39(7), 777–802.

FOUCAULT, M. (2008 (French edition (2004))): *The Birth of Biopolitics: Lectures at the College de France*. New York: Palgrave MacMillan.

GALAABAATAR, T., M. A. KHAN, AND M. UYANIK (2018): “Completeness and transitivity of preferences on mixture sets,” *arXiv preprint arXiv:1810.02454*.

(2019): “Completeness and transitivity of preferences on mixture sets,” *Mathematical Social Sciences*, 99, 49–62.

GHOSH, A., M. A. KHAN, AND M. UYANIK (2020): “Solvability and continuity: on bridging two communities,” *Johns Hopkins University, mimeo*.

GILBOA, I. (ed.) (2004): *Uncertainty in Economic Theory*. Cambridge: Cambridge University Press.

(2009): *Theory of Decision Under Uncertainty*. Cambridge: Cambridge University Press.
Golomb, S. W. (1959): “A connected topology for the integers,” *The American Mathematical Monthly*, 66(8), 663–665.

Halmos, P. R. (1956): “The basic concepts of algebraic logic,” *American Mathematical Monthly*, 53(6), 217–249.

Hara, C. (2005): “Existence of equilibria in economies with bads,” *Econometrica*, 73(2), 647–658.

Hara, C., K. Suzumura, and Y. Xu (2006): “On the possibility of continuous, Paretian and egalitarian evaluation of infinite utility streams,” *Andrew Young School of Policy Studies Research Paper Series*, (07-12).

Herden, G. (1989a): “On the existence of utility functions,” *Mathematical Social Sciences*, 17(3), 297–313.

Herden, G. (1989b): “On the existence of utility functions II,” *Mathematical Social Sciences*, 18(2), 107–117.

Herden, G. (1991): “Topological spaces for which every continuous total preorder can be represented by a continuous utility function,” *Mathematical Social Sciences*, 22(2), 123–136.

Herrlich, H. (1965): “Wann sind alle stetigen Abbildungen in Y konstant?,” *Mathematische Zeitschrift*, 90(2), 152–154.

Herstein, I. N., and J. Milnor (1953): “An axiomatic approach to measurable utility,” *Econometrica*, 21(2), 291–297.

Hewitt, E. (1946): “On two problems of Urysohn,” *Annals of Mathematics*, 47, 503–509.

Iliadis, S., and V. Tzannes (1986): “Spaces on which every continuous map into a given space is constant,” *Canadian Journal of Mathematics*, 38, 1281–1298.

Jameson, G. J. (1974): *Topology and Normed Spaces*. London: Chapman and Hall.

Kakutani, S. (1941): “A generalization of Brouwer’s fixed point theorem,” *Duke Mathematical Journal*, 7(2), 457–459.

Kalton, N. J., N. T. Peck, and J. W. Roberts (1984): *An F-space Sampler*, vol. 89. New York: Cambridge University Press.

Khan, M. A. (2020): “On the Finding of an Equilibrium: Düppe-Weintraub and the Problem of Scientific Credit,” *Journal of Economic Literature*, online(1), 1–50.

Khan, M. A., and E. E. Schlee (2019): “Money-metrics in applied welfare analysis: A saddlepoint rehabilitation,” *ASU Working Paper*, Forthcoming, 169–171.

Khan, M. A., and M. Uyanık (2019): “Topological connectedness and behavioral assumptions on preferences: a two-way relationship,” *Economic Theory*, (published online, doi: 10.1007/s00199-019-01206-7).

Khan, M. A. (2020): “On an extension of a theorem of Eilenberg and a characterization of topological
connectedness,” *Topology and its Applications*, 273, 107–117.

Kirch, A. (1969): “A countable, connected, locally connected Hausdorff space,” *The American Mathematical Monthly*, 76(2), 169–171.

Klee, V. (1963): “On a question of Bishop and Phelps,” *American Journal of Mathematics*, 85(1), 95–98.

Koopmans, T. C. (1957): *Three Essays on the State of Economic Science*. New York: McGraw-Hill.

Koopmans, T. C. (1972): “Representation of preference orderings with independent components of consumption,” in *Decision and Organization*, ed. by C. B. McGuire, and R. Radner, pp. Chapter 4, 79–100. Minneapolis: University of Minnesota Press.

Krantz, D., D. Luce, P. Suppes, and A. Tversky (1971): *Foundations of Measurement, Volume I: Additive and Polynomial Representations*. New York: Academic Press.

Lauwers, L. (2010): “Ordering infinite utility streams comes at the cost of a non-Ramsey set,” *Journal of Mathematical Economics*, 46(1), 32–37.

Lord, H. (1995): “Connectednesses and disconnectednesses,” *Annals of the New York Academy of Sciences*, 767(1), 115–139.

Luce, D. (2000): *Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches*. New Jersey: Lawrence Erlbaum Associates.

Majumdar, M. (1974): “Efficient programs in infinite dimensional spaces: A complete characterization,” *Journal of Economic Theory*, 7(4), 355–369.

McKenzie, L. W. (2002): *Classical General Equilibrium Theory*, vol. 1. Cambridge: The MIT Press.

Miller, G. G. (1970): “Countable connected spaces,” *Proceedings of the American Mathematical Society*, 26(2), 355–360.

Moscati, I. (2016): *Measuring Utility*. Oxford: Oxford University Press.

Munkres, J. R. (2000): *Topology*. New Jersey: Prentice Hall.

Nachbin, L. (1965): *Topology and Order*, vol. 4. New Jersey: Krieger Pub Co.

Nikaido, H. (1968): *Convex Structures and Economic Theory*, Mathematics in Science and Engineering, Vol. 51. New York: Academic Press.

Sentilles, F. D. (1972): “Bounded continuous functions on a completely regular space,” *Transactions of the American Mathematical Society*, 168, 311–336.

Sonnenschein, H. (1965): “The relationship between transitive preference and the structure of the choice space,” *Econometrica*, 33(3), 624–634.

——— (1967): “Reply to “a note on orderings”,” *Econometrica*, 35(3, 4), 540.

Stephan, G. (1986): “Competitive finite value prices: a complete characterization,” in *Studies
in *Austrian Capital Theory, Investment and Time*, ed. by M. Faber, pp. 173–183. New York: Springer.

**Svensson, L.-G.** (1980): “Equity among generations,” *Econometrica*, 48(5), 1251–1256.

**Toranzo, M. E., and C. Hervés-Beloso** (1995): “On the existence of continuous preference orderings without utility representations,” *Journal of Mathematical Economics*, 24(4), 305–309.

**Urysohn, P.** (1925): “Über die Mächtigkeit der zusammenhängenden Mengen,” *Mathematische Annalen*, 94(1), 262–295.

**Uyanık, M., and M. A. Khan** (2019a): “The continuity postulate in economic theory: a deconstruction and an integration,” *Working Paper*.

——— (2019b): “On the consistency and the decisiveness of the double-minded decision-maker,” *Economics Letters*, 185, 108657.

**Villegas, C.** (1964): “On qualitative probability σ-Algebras,” *The Annals of Mathematical Statistics*, 35(4), 1787–1796.

——— (1967): “On qualitative probability,” *The American Mathematical Monthly*, 74(6), 661–669.

**Wakker, P. P.** (1989): *Additive Representations of Preferences: A New Foundation of Decision Analysis*. Boston: Kluwer Academic Publishers.

**Wallace, A. D.** (1962): “Relations on topological spaces,” in *General Topology and its Relations to Modern Analysis and Algebra*, Proceedings of the symposium held in Prague in September 1961, pp. 356–360. Prague: Academia Publishing House of the Czechoslovak Academy of Sciences.

**Ward, L.** (1954a): “Binary relations in topological spaces,” *An. Acad. Brasil. Ci*, 26, 357–373.

——— (1954b): “Partially ordered topological spaces,” *Proceedings of the American Mathematical Society*, 5(1), 144–161.

**Wilder, R. L.** (1949): *Topology of Manifolds*. Berlin: American Mathematical Society Colloquium Publications XXXII.

**Wold, H.** (1943–44): “A synthesis of pure demand analysis, I–III,” *Scandinavian Actuarial Journal*, 26, 85–118, 220–263; 27, 69–120.

**Zame, W. R.** (2007): “Can intergenerational equity be operationalized?,” *Theoretical Economics*, 2(2), 187–202.