Research Article

The Asymptotic Solutions for a Class of Nonlinear Singular Perturbed Differential Systems with Time delays

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We study a kind of vector singular perturbed delay-differential equations. By using the methods of boundary function and fractional steps, we construct the formula of asymptotic expansion and confirm the interior layer at $t = \sigma$. Meanwhile, on the basis of functional analysis skill, the existence of the smooth solution and the uniform validity of the asymptotic expansion are proved.

1. Introduction

Singular perturbed differential equations are often used as mathematical models describing processes in biological sciences and physics, such as genetic engineering and the El Nino phenomenon of atmospheric physics [1]. In order to study natural and social processes more accurately, we often construct the models with small delay time and obtain much behavior of corresponding objects. The models are mostly expressed by singular perturbed delay-differential equations. So, singular perturbed delay-differential equations can express the processes more exactly. Studying the singular perturbed delay-differential problem is a very attractive object in the mathematical circle.

In addition, in the study of population models and propagation of epidemic virus, we sometimes require the construction of models. The models are often expressed by singular perturbed delay-differential equations. We can get the equilibrium points of singular perturbed delay-differential equations and confirm the laws of processes. Therefore, using the research methods and theoretical results of singular perturbed delay-differential problems to solve natural and social processes is essential.

In recent years, more and more attention was paid to the study of singular perturbed delay-differential problems [2], especially for scalar boundary value problems [3–5], but vector boundary value problems are rarely seen [6, 7]. Up to now, the vector theory of singular perturbed problems is still not mature. Wang and Ni study a class of semilinear singularly perturbed equations using the method of fractional steps [8]. By the method of boundary layer function [9], Wang considered a kind of nonlinear singularly perturbed boundary value problems [10].

In this paper, we will discuss the interior layer for a class of nonlinear singularly perturbed differential-difference equations and construct its asymptotic expansion formula. Then, the existence of the smooth interior layer solution and the uniform validity of the asymptotic expansion are proved. The results of this paper are new and complement the previous ones.

We consider the following nonlinear singularly perturbed differential-difference equations

\[
\mu^2 y'' = F\left(\mu y'(t), y(t), y(t-\sigma), t\right), \quad t \in [0, 2\sigma],
\]

\[
y(t, \mu) = \alpha(t), \quad -\sigma \leq t \leq 0, \quad y(2\sigma, \mu) = y^\star,
\]

where

\[
y(t) = \begin{pmatrix} y^1(t) \\ \vdots \\ y^n(t) \end{pmatrix} \in \mathbb{R}^n, \quad F = \begin{pmatrix} F^1 \\ \vdots \\ F^n \end{pmatrix} \in \mathbb{R}^n, \quad 0 < \mu \ll 1,
\]

and functions $F^i$ ($i = 1, \ldots, n$) are sufficiently smooth on the domain $D = \{(\mu y'(t), y(t), t) \mid \mu y'(t) \leq l_1, y(t) \leq l_2, 0 \leq t \leq 2\sigma\}$, and $l_1, l_2$ are given positive real numbers.
The restriction on $2\sigma$ will not influence the essence of the problems.

We will use the method of fractional steps to discuss the system (1). Let $\mu = 0$; then we can obtain the degenerate equations (3) and (4) of (1)

$$F(0, \tilde{y}^{(1)}(t), \alpha(t - \sigma), t) = 0, \quad t \in [0, \sigma], \quad (3)$$

$$F(0, \tilde{y}^{(2)}(t), \tilde{y}^{(1)}(t - \sigma), t) = 0, \quad t \in [0, 2\sigma]. \quad (4)$$

The degenerate problem (3) is solvable. We hypothesize that the solution of system (3) is $\tilde{y}^{(1)}(t) = \varphi(t)$; substituting $\tilde{y}^{(1)}(t) = \varphi(t)$ into (4) yields $\tilde{y}^{(2)}(t) = \psi(t)$. Thus, we have the degenerate solution $\tilde{y}(t)$ on the interval $[0, 2\sigma]$, namely, the following:

$$\tilde{y}(t) = \begin{cases} \varphi(t), & 0 \leq t \leq \sigma, \\ \psi(t), & \sigma \leq t \leq 2\sigma. \end{cases} \quad (5)$$

According to the truth of boundary layer functions and interior layer functions in [3], we can confirm that the interior layer may occur at $t = \sigma$ and boundary layers may occur at the two terminal points of interval $[0, 2\sigma]$.

2. The Construction of Asymptotic Expansion in $[0, \sigma]$

In $[0, \sigma]$, let $\mu y' = z$; the system (1) can be rewritten as

$$\mu y' = z,$$

$$\mu z' = F(\mu y'(t), y(t), \alpha(t - \sigma), t), \quad (6)$$

$$y(t, \mu) = \alpha(t), \quad -\sigma \leq t \leq 0, \quad y(2\sigma, \mu) = y^*.$$

Let $x^{(-)} = (y, z)^T$ and using the method of boundary function [9], we can construct a series satisfying (6) in $[0, \sigma]$:

$$x^{(-)}(t, \mu) = x_0(t) + \mu x_1(t) + \cdots + \mu^k x_k(t) + \cdots \quad (8)$$

is called regular series of (6), while

$$\Pi x(t_0, \mu) = \Pi_0 x(t_0) + \mu \Pi_1 x(t_0) + \cdots + \mu^k \Pi_k x(t_0) + \cdots \quad (9)$$

is called the boundary series for $t = 0$, and

$$Q^{(-)} x(t, \mu) = Q_0^{(-)} x(t) + \mu Q_1^{(-)} x(t) + \cdots + \mu^k Q_k^{(-)} x(t) + \cdots \quad (10)$$

is called the left boundary series for $t = \sigma$, and $\lim_{\tau_n \to +\infty} \Pi_1 x(\tau_n) = 0, \lim_{\tau_n \to -\infty} Q^{(-)} x(\tau) = 0$ hold. The system (6) has a continuous solution, so we assume that

$$y(\sigma, \mu) = p_0 + \mu p_1 + \cdots + \mu^k p_k + \cdots, \quad (11)$$

where $p_k (k = 0, 1, \ldots)$ are undetermined $n$-dimensional vector functions.

Put (7)--(10) into (6) and separate equations by measures $t, \tau_0, \tau$; then we can write the regular part

$$\tilde{y}_0(t) = 0, \quad \tilde{y}'_0(t) = \varphi_0(t), \quad (\tilde{y}'_0(t) = \psi_0(t), \quad (\tilde{y}'_0(t) = \alpha_0(t - \sigma), t) = 0, \quad (12)$$

where $\tilde{h}_k(t)$ is a known vector function about $\tilde{y}_0(t), \tilde{y}_1(t), \ldots, \tilde{y}_{k-1}(t), \tilde{y}_k(t), \tilde{y}_{k+1}(t), \ldots, \tilde{y}_2(0)$; elements of matrix $\tilde{F}_y, \tilde{F}_z$ take value at the point $(0, \psi(t), \alpha(t - \sigma), t)$.

The first equation of (12) coincides with the left degradation problem (3), so we have $\tilde{y}_0(t) = \varphi(t), \tilde{y}_0(t) = 0$. To determine vector functions $\tilde{y}_k(t), \tilde{z}_k(t)$, we need the following conditions.

(H1) Suppose that the determinant of $\tilde{F}_y = (\partial F_i/\partial y^j)|_{y=0}$ is not equal to zero at all times.

By (H1) and (12), $\tilde{y}_k(t), \tilde{z}_k(t)$ can be completely determined.

(H2) Suppose that the characteristic equation of the systems (6) given by $|\lambda^2 I - \tilde{F}_z \lambda - \tilde{F}_y|_{y=0,\psi=\varphi(t)} = 0$ has $2n$ real valued solutions $\lambda_i(t), for i = 1, \ldots, 2n$, for all $t \in [0, \sigma]$, where Re $\lambda_i(t) < 0, \lambda = 1, \ldots, n, \text{Re} \lambda_i(t) > 0, i = n+1, \ldots, 2n, and I$ is the $n \times n$ identity matrix.

For $\Pi_0 x(\tau_0)$, we have

$$\frac{d\Pi_0 y}{d\tau_0} = \Pi_0 z, \quad \frac{d\Pi_0 z}{d\tau_0} = F(\Pi_0 z, \varphi(0) + \Pi_0 y, \alpha(-\sigma), 0), \quad (13)$$

$$\Pi_0 y(0) = \alpha(0) - \varphi(0), \quad \Pi_0 y(+\infty) = 0, \quad (14)$$

$$\Pi_0 z(+\infty) = 0.$$

Let $\bar{y} = \varphi(0) + \Pi_0 y, \bar{z} = \Pi_0 z; then the problems (13) and (14) can be changed into the following equations:

$$\frac{d\bar{y}}{d\tau} = \bar{z}, \quad \frac{d\bar{z}}{d\tau} = F(\bar{z}, \bar{y}, \alpha(-\sigma), 0),$$

$$\bar{y}(0) = \alpha(0), \quad \bar{y}(+\infty) = \varphi(0).$$

By (H2), there exists an $n$-dimensional stable manifold near the point $(\varphi(0), 0)$ on the phase plane ($\bar{y}, \bar{z}$), which is in some region $G_1$ of vector function $\Pi_0 y$.

(H3) Suppose that the $n$-dimensional stable manifold is $\Pi_0 z = \Phi^F(\Pi_0 y)$, and $\alpha(0) - \varphi(0) \in G_1$.
By (H2) and (H3), systems (13) and (14) have a solution \( \Pi_0 y, \Pi_1 z \), which are both satisfied with exponential decay.

For \( \Pi_k x \), we have

\[
\frac{d\Pi_k y}{d\tau_0} = \Pi_k z, \quad \frac{d\Pi_k z}{d\tau_0} = \tilde{F}_z \Pi_k z + \tilde{F}_y \Pi_k y + G_k (\tau_0) ,
\]

\[
\Pi_k y (0) = -y_k (0), \quad \Pi_k y (+\infty) = 0, \quad \Pi_k z (+\infty) = 0,
\]

where \( \tilde{F}_z, \tilde{F}_y \) take value at \( (\Pi_0, \phi (0) + \Pi_0 y, \alpha (-\sigma), 0) \) and \( G_k (\tau_0) \) is a known vector function about \( \Pi_1 y (\tau_0), \ldots, \Pi_{k-1} y (\tau_0) \).

In fact, the homogeneous system of (16)

\[
\frac{d\Pi_k y}{d\tau_0} = \Pi_k z, \quad \frac{d\Pi_k z}{d\tau_0} = \tilde{F}_z \Pi_k z + \tilde{F}_y \Pi_k y
\]

is the variational equation of (13). Thus, it has a steady manifold

\[
\Pi_k z = \frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \Pi_k y.
\]

Substituting (18) into (16), we have

\[
\frac{d\Pi_k y}{d\tau_0} = \frac{\partial \Phi^L}{\partial y} \Pi_k y.
\]

Let \( \Pi_k y = \Phi_1 (\tau_0) \) be the general solution of (20), under the boundary conditions (17), we obtain the general solution of (18) as

\[
\left( \Pi_k y (\tau_0) \right)^G = -\Phi_1 (\tau_0) \Phi_1^{-1} (0) y_k (0),
\]

\[
\left( \Pi_k z (\tau_0) \right)^G = -\frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \Phi_1 (\tau_0) \Phi_1^{-1} (0) y_k (0).
\]

Next, set \( \Pi_k^* y, \Pi_k^* z \) be the particular solution of (16). Introducing a new transformation

\[
\Pi_k^* y = \delta_1, \quad \Pi_k^* z = \frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \delta_1 + \delta_2
\]

and substituting it into (16), we obtain the system

\[
\frac{d\delta_1}{d\tau_0} = \frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \delta_1 + \delta_2,
\]

\[
\frac{d\delta_2}{d\tau_0} = \left( \tilde{F}_z - \frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \right) \delta_2 + G_k (\tau_0).
\]

Let \( \delta_2 = \Psi_1 (\tau_0) \) be the general solution of \( \frac{d\delta_2}{d\tau_0} = \left( \tilde{F}_z - \frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \right) \delta_2 \); then we obtain a particular solution of (23) as

\[
\delta_2 (\tau_0) = \int_{\tau_0}^{0} \Psi_1 (\tau_0) \Psi_1^{-1} (s) G_k (s) \, ds,
\]

\[
\delta_1 (\tau_0) = \int_{0}^{\tau_0} \Phi_1 (\tau_0) \Phi_1^{-1} (s) \int_{\tau_0}^{s} \Psi_1 (t) \Psi_1^{-1} (t) G_k (t) \, dt \, ds.
\]

So a particular solution of (16) is given in the following form:

\[
\Pi_k^* y (\tau_0) = \int_{0}^{\tau_0} \Phi_1 (\tau_0) \Phi_1^{-1} (s) \int_{\tau_0}^{s} \Psi_1 (t) \Psi_1^{-1} (t) G_k (t) \, dt \, ds,
\]

\[
\Pi_k^* z (\tau_0) = \frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \Pi_k^* y (\tau_0) + \int_{\tau_0}^{0} \Psi_1 (\tau_0) \Psi_1^{-1} (s) G_k (s) \, ds.
\]

Hence, by virtue of (21) and (25), we obtain a solution of (16) and (17) as

\[
\Pi_k y (\tau_0) = -\Phi_1 (\tau_0) \Phi_1^{-1} (0) \tilde{y}_k (0) + \Pi_k^* y (\tau_0),
\]

\[
\Pi_k z (\tau_0) = -\frac{\partial \Phi^L (\Pi_0 y)}{\partial y} \Phi_1 (\tau_0) \Phi_1^{-1} (0) \tilde{y}_k (0) + \Pi_k^* z (\tau_0).
\]

Now, \( \Pi_k x (\tau_0) \) is completely determined. Obviously, \( \Pi_k x (\tau_0) \) decays exponentially as \( \tau \to \infty \).

**Lemma 1.** Under conditions (H1)–(H3), the boundary functions \( \Pi_k x (\tau_0) \) satisfy the following inequality:

\[
\| \Pi_k x (\tau_0) \| \leq C_1 e^{-k \cdot \tau}, \quad \tau_0 \geq 0, \quad l = 0, 1, 2, \ldots,
\]

where \( C_1, k_1 \) are all positive constants.

We first consider the system of \( Q_{0}^{(\cdot)} x \)

\[
\frac{dQ_{0}^{(\cdot)} y}{d\tau} = Q_{0}^{(\cdot)} z,
\]

\[
\frac{dQ_{0}^{(\cdot)} z}{d\tau} = F (Q_{0}^{\cdot} z, \varphi (\sigma), Q_{0}^{(\cdot)} y, \alpha (0), \sigma),
\]

\[
Q_{0}^{(\cdot)} y (0) = p_0 - \varphi (\sigma),
\]

\[
Q_{0}^{(\cdot)} y (-\infty) = Q_{0}^{(\cdot)} z (-\infty) = 0.
\]

Let \( \varphi (\sigma) + Q_{0}^{(\cdot)} y = \tilde{y}, Q_{0}^{(\cdot)} z = \tilde{z}, \) (28) can be written as

\[
\frac{d\tilde{y}}{d\tau} = \tilde{z}, \quad \frac{d\tilde{z}}{d\tau} = F (\tilde{z}, \tilde{y}, \alpha (0), \sigma),
\]

\[
\tilde{y} (0) = p (0), \quad \tilde{y} (-\infty) = \varphi (\sigma).
\]

By (H2), the equilibrium \( (\varphi (\sigma), 0) \) is a hyperbolical singular point on the plane \( (\tilde{y}, \tilde{z}) \). There exists an \( n \)-dimensional stable manifold passing through \( (\varphi (\sigma), 0) \).

(H4) Suppose that this \( n \)-dimensional stable manifold is \( \tilde{z} = \Phi^{(-)} (\tilde{y}) \), and \( \Phi^{(-)} (\tilde{y}) = p (0) \in G_2 \), where \( G_2 \) is a domain of \( \tilde{y} \).

By (H4) and (28), \( Q_{0}^{(\cdot)} x \) can be completely determined, but it contains the unknown vector \( p_0 \).
$Q^{(-)}_k x$ is determined by the following system:

$$\frac{dQ^{(-)}_k y}{d\tau} = Q^{(-)}_k z,$$

$$\frac{dQ^{(-)}_k z}{d\tau} = \bar{F}^{(-)}_x Q^{(-)}_k z + \bar{F}^{(-)}_y Q^{(-)}_k y + H^{(-)}_k (\tau),$$

$$Q^{(-)}_k y(0) = p_k - \bar{y}_k (\sigma), \quad Q^{(-)}_k y(-\infty) = Q^{(-)}_k z(-\infty) = 0. \quad (30)$$

Here $H^{(-)}_k (\tau)$ is a known vector; elements of matrix $\bar{F}^{(-)}_y, \bar{F}^{(-)}_z$ take value at the point $(Q^{(-)}_0 z, \varphi(\sigma) + Q^{(-)}_0 y, \alpha(0), \sigma)$. Because $\bar{z}(\tau) = Q^{(-)}_y y$ is the solution of homogeneous systems of (30). By virtue of Liouville formula, we can obtain

$$Q^{(-)}_k y = \frac{\bar{z}(\tau)}{\bar{z}(0)} [p_k - \bar{y}_k (\sigma)] + \bar{z}(\tau) \int_0^\tau \frac{d\eta}{\bar{z}^2(\eta)} h(\eta) \int_0^\eta \bar{z}(s) h(s) H^{(-)}_k (s) ds, \quad (31)$$

where $h(\tau) = \exp(- \int_0^\tau \bar{F}_y d\tau)$. Thus, $Q^{(-)}_k x(\tau)$ can be completely determined. We can easily obtain the exponential decay of $Q^{(-)}_k x(\tau)$.

**Lemma 2.** Under the condition (H4), the boundary functions $Q_0 x(\tau)$ satisfy the following inequality:

$$\|Q^{(-)}_0 x(\tau)\| \leq C_2 e^{k_2 \tau}, \quad -0, \quad l = 0, 1, 2, \ldots, \quad (32)$$

where $C_2, k_2$ are positive constants.

### 3. The Construction of Asymptotic Expansion in $[\sigma, 2\sigma]$

In $[\sigma, 2\sigma]$, let $\mu y' = z$; the system (1) can be rewritten as

$$\mu y' = z,$$

$$\mu z' = F\left(\mu y' (t), y(t), \varphi(t - \sigma), t\right), \quad (33)$$

$$y(t, \mu) = \alpha(t), \quad -\sigma \leq t \leq 0, \quad y(2\sigma, \mu) = y^*.$$

Let $x^{(+)} = (y, z)^T$ and using the method of boundary function [9], we can construct a series formally satisfying (33) in $[\sigma, 2\sigma]$:

$$x^{(+)} = \bar{x}(t, \mu) + Q^{(+)} x(t, \mu) + R x(\tau, \alpha),$$

$$\tau = \frac{t - \sigma}{\mu}, \quad \tau = \frac{t - 2\sigma}{\mu}, \quad (34)$$

where

$$\bar{x}(t, \mu) = \bar{x}_0 (t) + \mu \bar{x}_1 (t) + \cdots + \mu^k \bar{x}_k (t) + \cdots \quad (35)$$

is called regular series of (33), while

$$Q^{(+)} x(t, \mu) = Q^{(+)}_0 x(t) + \mu Q^{(+)}_1 x(t) + \cdots \quad (36)$$

is called the left boundary series for $t = \sigma$, and

$$R x(\tau, \mu) = R_0 x(\tau) + \mu R_1 x(\tau) + \cdots + \mu^k R_k x(\tau) + \cdots \quad (37)$$

is called the boundary series for $t = 2\sigma$, and $\lim_{t \to -\infty} Q^{(+)} x(\tau) = 0, \lim_{t \to -\infty} R_0 x(\tau) = 0$ hold.

Substituting (35)–(37) into (33), separating $t, \tau, \alpha$, and equating terms with same powers of $\mu$, we have $\bar{x}_0 (t) = 0$, $F(0, \bar{y}_0 (t), \varphi(t - \sigma), t) = 0$,

$$\bar{y}_0 (t) = \psi(t), \quad (38)$$

where $h_0 (t)$ is a known vector function about $\bar{x}_0 (t), \bar{y}_0 (t), \ldots, \bar{x}_{k-1} (t), \bar{y}_{k-1} (t)$; elements of matrix $\bar{F}_y, \bar{F}_z$ take value at the point $(0, \psi(t), \varphi(t - \sigma), t)$. To determine the vector functions $\bar{y}_i (t), \bar{z}_i (t)$, we need the following condition.

(H5) Suppose that the determinant of $\bar{F}_y = (\partial F_i / \partial y^j)_{nov}$ is not equal to zero at all times.

By (H5) and (38), $\bar{y}_k (t), \bar{z}_k (t)$ can be completely determined.

For the zeroth approximation of the left boundary layer $Q^{(+)}_0 x$, we have

$$\frac{dQ^{(+)}_0 y}{d\tau} = Q^{(+)}_0 z,$$

$$\frac{dQ^{(+)}_0 z}{d\tau} = F\left(Q^{(+)}_0 z, \psi(\sigma) + Q^{(+)}_0 y, \varphi(0) + \Pi_0 y(\tau), \sigma\right), \quad (39)$$

$$Q^{(+)}_0 y(0) = p_0 - \psi(\sigma), \quad Q^{(+)}_0 y(+\infty) = Q^{(+)}_0 z(+\infty) = 0, \quad (40)$$

where $Q^{(+)}_0 x$ is relevant to $\Pi_0 y$ and (39) is no longer an autonomous system. Combining with systems (13) and (14), we can discuss $Q^{(+)}_0 x(\tau)$. The combining equations and according conditions are

$$\frac{d\bar{x}_0}{d\tau} = F(\bar{x}, \bar{y}, \bar{y}, \sigma), \quad \frac{d\bar{y}_0}{d\tau} = \bar{z}, \quad (41)$$

$$\bar{x}_0 (0) = p_0, \quad \bar{y}_0 (+\infty) = \psi(\sigma), \quad (42)$$

where the phase space $(\bar{y}, \bar{z})$ is the direct sum of $(\bar{y}, \bar{z})$ and $(\bar{y}, \bar{z})$. 
(H6) Suppose that the characteristic equations of the systems (41) have $4n$ real valued solutions $\lambda_i(t)$, for $i = 1, \ldots, 4n$, for all $t \in [\sigma, 2\sigma]$, where $\text{Re} \lambda_i(t) < 0$, $i = 1, \ldots, 2n$, $\text{Re} \lambda_i(t) > 0, i = 2n + 1, \ldots, 4n$.

By condition (H6), there exists a $2n$-dimensional stable manifold passing the equilibrium point of $(\psi(\sigma), \varphi(0), 0)$.

(H7) Suppose that this $2n$-dimensional stable manifold is

$\mathbf{z} = \Phi^{(+)}(\mathbf{y}), \tag{42}$

where $\mathbf{z} = (\Pi_0 \Phi^0 \mathbf{x}, \Phi^1 \mathbf{y})^T$, $\mathbf{y} = (\Pi_0 \varphi, \Phi^1 \varphi)^T$, $\Phi^{(+)} = (\Phi^1_1, \Phi^1_2)^T$, and $(\alpha(0) - \varphi(0), \mathbf{p}_0 - \psi(\sigma)) \in G_2$.

By (H7) and (39), (40), $Q^+_0 x(\tau)$ can be determined.

$Q^{(+)}_k x(\tau) = \mathbf{p}_k - \overline{\Phi}^0_0 (\sigma), \quad Q^{(+)}_k (-\infty) = \Phi^{(+)}_k (\infty) = 0, \quad Q^{(+)}_k y(0) = \mathbf{p}_k - \overline{\Phi}^0_0 (\sigma), \quad Q^{(+)}_k y(\infty) = \Phi^{(+)}_k (\infty) = 0. \tag{43}

By analogy with the left boundary terms, we have

$\frac{dQ^{(+)}_k y}{d\tau} = Q^{(+)}_k \mathbf{z}, \tag{44}$

where

$Q^{(+)}_k y = (Q^{(+)}_k y)^G + Q^{(+)}_k z, \tag{45}$

$Q^{(+)}_k z = (Q^{(+)}_k z)^G + Q^{(+)}_k z,$

where

$(Q^{(+)}_k y)^G = (\mathbf{p}_k - \overline{\Phi}^0_0 (\sigma)) \Phi_2 (\tau) \Phi_2^{-1} (0), \tag{46}$

$Q^{(+)}_k z = \frac{\partial \Phi^{(+)}_k}{\partial y} (Q^{(+)}_k y, \Phi_2 (\tau) \Phi_2^{-1} (0), (\mathbf{p}_k - \overline{\Phi}^0_0 (0)), \tag{47}$

$Q^{(+)}_k y = \int_0^\tau \Phi_2 (\tau) \Phi_2^{-1} (s) \Psi_1 (s) \Psi_2^{-1} (t) H^{(+)}_k (t) \, dt \, ds, \tag{48}$

$Q^{(+)}_k z = \frac{\partial \Phi^{(+)}_k}{\partial y} (Q^{(+)}_k y, \Phi_2 (\tau) \Phi_2^{-1} (s) H^{(+)}_k (s) \, ds. \tag{49}$

The meaning of $\Phi_2 (\tau)$ and $\Psi_1 (\tau)$ is similar to that of $\Phi$ and $\Psi$, respectively, but $Q^{(+)}_k x(\tau)$ contains the unknown vector functions $\mathbf{p}_1, \ldots, \mathbf{p}_k$.

**Lemma 3.** Under conditions (H5)–(H7), the boundary functions $Q^{(+)}_k x(\tau)$ satisfy the following inequality:

$\|Q^{(+)}_k x(\tau)\| \leq C_3 e^{-k_3 \tau}, \quad \tau \geq 0, \tag{50}$

where $C_3, k_3$ are all positive constants.

For $R_0x(\tau_*)$, we have

$\frac{dR_0x}{d\tau_0} = R_0\mathbf{z}, \tag{51}$

$R_0z = F(R_0z, \varphi (\tau), \varphi (\sigma), 2\sigma), \tag{52}$

$R_0 \varphi (0) = \varphi (2\sigma), \quad R_0 \varphi (\infty) = 0, \quad R_0z (\infty) = 0. \tag{53}$

Consider the first approximate system of (47)

$\begin{pmatrix} \frac{dR_0y}{d\tau} \\ \frac{dR_0z}{d\tau} \end{pmatrix} = \begin{pmatrix} 0 \\ \begin{pmatrix} F_x (0, \varphi (2\sigma), \varphi (\sigma), 2\sigma) \\ F_z (0, \varphi (2\sigma), \varphi (\sigma), 2\sigma) \end{pmatrix} \end{pmatrix} \begin{pmatrix} R_0y \\ R_0z \end{pmatrix}. \tag{54}$

In the same way, we can confirm that there exists an $n$-dimensional stable manifold, which is in some region $G_4$ of vector function $R_0y$.

(H8) Suppose that the $n$-dimensional stable manifold is $R_0z = \Phi^\delta(R_0y)$, and $\varphi^\delta - \overline{\Phi}_0 (2\sigma) \in G_4$.

By (H8), the system (47) has a solution. $R_0y$ and $R_0z$ are both satisfied with exponential decay estimate.

For $R_0x$, we have

$\frac{dR_0x}{d\tau_0} = R_0x, \tag{55}$

$R_0x = F(R_0x, \varphi (\tau), \varphi (\sigma), 2\sigma), \tag{56}$

$R_0x (0) = x^\delta, \quad R_0x (\infty) = 0, \quad R_0x (\infty) = 0, \tag{57}$

where $F_x, F_z$ take value at $(R_0z, \overline{\Phi}_0 (2\sigma) + R_0x, \varphi (\sigma), 2\sigma)$ and $G_4 (\tau_0)$ is a known vector function.

The determination of $R_0x(\tau_*)$ is treated in the same way as $\Pi_1 \mathbf{x}(\tau_0)$ and is omitted here. Obviously, $R_0x(\tau_*)$ satisfy the following Lemma.

**Lemma 4.** Under the condition (H8), the boundary functions $R_0x(\tau_*)$ satisfy the following inequality:

$\|R_0x (\tau_*)\| \leq C_4 e^{k_3 \tau}, \quad \tau_* \leq 0, \tag{58}$

where $C_4, k_3$ are all positive constants.

Obviously, $Q^{(+)}_k x(\tau), Q^{(+)}_k x(\tau)$ contain the unknown vector functions $\mathbf{p}_0, \ldots, \mathbf{p}_k$. Next, we will use the continuous conditions to determine them.
As the solution of the original problem is continuous at \( t = \sigma \), the solution \( x(t, \mu) \) of equivalent system needs to satisfy
\[
z^{(-)}(\sigma, \mu) = z^{(+)}(\sigma, \mu),
\]
\[
Q_k^{(-)}z(0) = Q_k^{(+)}z(0),
\]
\[
\bar{y}_{k-1}^{(-)}(\sigma) + Q_k^{(-)}z(0) = \bar{y}_{k-1}^{(+)}(\sigma) + Q_k^{(+)}z(0).
\]
First, we will seek the value of \( p_0 \). (H4), (H7), and (51) show
\[
\Phi'(p_0 - \varphi(\sigma), 0) = \Phi_2^{(+)}(p_0 - \varphi(\sigma), 0).
\]
Let
\[
N_0(p_0) = \Phi'(p_0 - \varphi(\sigma), 0) - \Phi_2^{(+)}(p_0 - \varphi(\sigma), 0) = 0.
\]
(54)
(H9) Suppose that (54) has a solution \( p_0 = \bar{p}_0 \) and \((\partial N_0/\partial p_0)|_{y = \bar{p}_0} \neq 0.\)

Next, bringing (30) and (39) into (52), we have
\[
N_k(p_k) = \bar{y}_{k-1}^{(-)}(\sigma) + Q_k^{(-)}z(0) - \bar{y}_{k}^{(-)}(\sigma) - Q_k^{(+)}z(0) = 0,
\]
\[
Q_k^{(-)}z(0) = \frac{\bar{y}_{k}^{(-)}(0)}{\bar{y}_{0}^{(-)}}\left[ p_k - \bar{y}_{k}^{(-)}(\sigma) \right] + \frac{1}{2} \left[ \begin{array}{c}
\int_{-\infty}^{0} \bar{z}(s) H_k^{(-)(s)} ds,
\end{array} \right]^{-1}
\]
\[
Q_k^{(+)}z(0) = \frac{\partial \Phi_2^{(+)}(p_k, y(\sigma), 0)}{\partial y} \left[ p_k - \bar{y}_{k}^{(-)}(\sigma) \right] + \int_{-\infty}^{0} \Psi_2^{(r)}(r) \Psi_2^{(-1)}(s) H_k^{(+)(s)} ds.
\]
(55)
By (55), \( p_k \) can be completely determined.

4. The Main Result

Let
\[
X_n(t, \mu) = \left\{ \begin{array}{ll}
\sum_{i=0}^{n} f^i(t, \mu) + \Pi_1 x(\tau_i) + Q_k^{(-)}x(\tau), & 0 \leq t \leq \sigma,
\sum_{i=0}^{n} f^i(t, \mu) + Q_k^{(+)}x(\tau) + R_1 x(\tau_i), & \sigma \leq t \leq 2\sigma.
\end{array} \right.
\]
(56)

Theorem 5. Under conditions (H1)–(H9) and Lemma 1–Lemma 4, there exist positive constants \( \mu_0 > 0, c > 0 \) such that, for \( 0 < \mu \leq \mu_0 \), the solution \( x(t, \mu) \) of the systems (1) exists in the interval \([0, 2\sigma]\) and satisfies the inequality
\[
\|x(t, \mu) - X_n(t, \mu)\| \leq cm^{n+1}.
\]
(57)

5. Example

Let us consider the systems
\[
\mu^2 y'' = y(t) - y\left(t - \frac{1}{2}\right),
\]
\[
y(t) = t, \quad t \in \left[\frac{1}{2}, 0\right], \quad y(1) = \frac{1}{2},
\]
(58)
We can see that the zero order asymptotic solution is close to the reduced solution.

6. Conclusive Remarks

Using the boundary layer function method, we consider a class of \(n\)-dimensional singularly perturbed differential equations with time delay. Under some assumptions, we obtain the asymptotic solution of the system (1). In comparison with [7] and [9], the system we study is more general. We use functional method to solve the asymptotic solution of (1) in this paper. It is different from the numerical solution obtained by numerical method. The asymptotic solution of (1) can be used in analytic calculation and obtain the asymptotic behaviors for deeper physical quantities.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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