Nonlocally-Correlated Disorder and Delocalization in One Dimension II: Localization Length

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Abstract

We study delocalization transition in a one-dimensional Dirac fermion system with random varying mass by using supersymmetric (SUSY) methods. In a previous paper, we calculated density of states and found that (quasi-)extended states near the band center are enhanced by nonlocal correlation of the random Dirac mass. Numerical studies support this conclusion. In this paper, we shall calculate localization length as a function of correlation length of the disorder. The result shows that the localization length is an increasing function of the correlation of the random mass.

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1 Introduction

Random disorderd system is one of the most interesting problem in condensed matter physics. Especially, localization phenomenon plays an important role in various problems, e.g., quantum Hall effect, transport properties of mesoscopic systems, quantum chaos, etc. In a previous paper (which will be refered to as paper I hereafter)[1], we studied random-mass Dirac fermion in one-spatial dimension, which is a low-energy effective field theory of random hopping tight-binding model, quantum spin chain, etc. This system has been studied extensively but in most of analytical studies only the δ-function type white noise limit is considered for technical reason[2, 3, 4]. In most of realistic systems however, disorders have nonlocal correlations, and moreover sometimes correlation length of disorders is larger than typical length scale of the system. An important example is the quantum Hall state where magnetic length is smaller than averaged length scale of random potential. Another example is quantum spin chain with the ground state of dimer structure whose low-energy excitations are described by a Dirac fermion with random varying mass, i.e., the system under investigation. There impurities generate low-energy excitations.

In paper I, we studied effects of nonlocal correlation of random mass, which is quite natural for application to realistic systems, and obtained density of states (DOS) as a function of correlation length of the disorder. The result indicates that (quasi-)extended states near the band center are enhanced by nonlocal correlation of the random-mass variables. Recently we have performed numerical studies of the model and verified that analytical results obtained by supersymmetric (SUSY) methods are in good agreement with numerical calculations[5].

In this paper, we shall study the same model and calculate localization length by using the SUSY methods. In Sect.2, we shall briefly review the model and the SUSY methods. In Sect.3, localization length is calculated by solving eigenvalue problem with respect to transfer ‘Hamiltonian’ for the spatial direction. Section 4 is devoted for discussion. Physical meanings of the result of the localization length
and the DOS obtained in paper I will be discussed. Numerical studies also give us useful informations on the system. As an application of the results, we shall discuss low-energy properties of quantum spin chains.

2 Model and SUSY

In this section, we shall review the model and SUSY methods mainly in order to fix our notations and make this paper self-contained. Reader who is familiar with the subjects can skip this section and go directly to Sect.3.

2.1 Model and Green’s functions

We shall study random system whose Hamiltonian is given by

\[ H_c = -\int dx \left[ \psi_R^\dagger i\partial_x \psi_R - \psi_L^\dagger i\partial_x \psi_L - im(x)(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R) \right] , \]

\[ = -\int dx \psi^\dagger h \psi , \]

\[ h = -i\sigma^x \partial_x + m(x)\sigma^y , \quad \psi = (\psi_R \psi_L)^t . \]

The random mass \( m(x) \) is decomposed into a uniform and random piece as

\[ m(x) = m_0 + \phi(x) , \]

where \( [\phi]_{\text{ens}} = 0 \) and

\[ [\phi(x)\phi(y)]_{\text{ens}} = g \frac{\exp(-|x-y|/\lambda)}{2\lambda} , \]

with positive parameters \( g \) and \( \lambda \). The symbol \( [A]_{\text{ens}} \) denotes the ensemble average of \( A \) over the randomness. It is easily verified

\[ \int dx \ [\phi(x)\phi(y)]_{\text{ens}} = g , \]

and

\[ [\phi(x)\phi(y)]_{\text{ens}} \to g\delta(x-y) \text{ as } \lambda \to 0. \]
It is obvious that the parameter $g$ controls magnitude of fluctuation and $\lambda$ is the correlation length of the disorder.

Single fermion Green’s function at energy $\omega$ is defined as

$$G_{\alpha\beta}(x, y; i\omega) = \langle x, \alpha | \frac{1}{\hbar - i\omega} | y, \beta \rangle, \quad \alpha, \beta = R, L,$$  \hspace{1cm} (2.4)

where $|x, \alpha\rangle$ is the normalized position eigenstate of fermion at $x$ and chirality $\alpha$. By functional integral,

$$G_{\alpha\beta}(x, y; i\omega) = \frac{1}{Z_\psi} \int \mathcal{D}\psi^\dagger \psi_{\alpha}(x) \psi_{\beta}^\dagger(y) e^{-S_\psi},$$

$$S_\psi = \int dx \psi^\dagger (ih + \omega) \psi,$$

$$Z_\psi = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger e^{-S_\psi}. \hspace{1cm} (2.5)$$

Ensemble averaged Green’s function is obtained from (2.4) as

$$\bar{G}_{\alpha\beta}(x - y; i\omega) = \left[ \langle x, \alpha | \frac{1}{\hbar - i\omega} | y, \beta \rangle \right]_{ens},$$  \hspace{1cm} (2.6)

where the ensemble average is taken with respect to $\phi(x)$ in (2.1) and (2.2) according to (2.3). Introducing the bosonic superpartner $\xi$,

$$\bar{G}_{\alpha\beta}(x - y; i\omega) = i \langle \psi_\alpha(x) \psi_{\beta}^\dagger(y) \rangle_S,$$  \hspace{1cm} (2.7)

where

$$\langle A \rangle_S = \left[ \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}\xi \mathcal{D}\xi^\dagger A e^{-S} \right]_{ens},$$  \hspace{1cm} (2.8)

with

$$S = \int dx \left[ \psi^\dagger (ih + \omega) \psi + \xi^\dagger (ih + \omega) \xi \right].$$  \hspace{1cm} (2.9)

The ensemble average can be converted into the functional integral form,

$$[\phi(x)\phi(y)...]_{ens} = \int \mathcal{D}\phi(\phi(x)\phi(y)... \exp(-S_\phi),$$

$$S_\phi = \int dx \frac{1}{4g} \phi(x)(-\lambda^2 \partial_x^2 + 1)\phi(x).$$  \hspace{1cm} (2.11)
From Eqs. (2.8), (2.9) and (2.10), the expectation value of operator $A$ is given by

$$\langle A \rangle_S = \int \mathcal{D}\psi \mathcal{D}\psi^\dagger \mathcal{D}\xi \mathcal{D}\xi^\dagger \mathcal{D}\phi \ A \exp(-(S + S_\phi)).$$  \hspace{1cm} (2.12)

The above total action is obviously invariant under SUSY transformation $\psi \leftrightarrow \xi$ and has such a form that the SUSY partners $\psi$ and $\xi$ couple to the ‘dynamical’ real scalar field $\phi$ which is ‘singlet’ under SUSY transformation.

Transfer Hamiltonian is obtained by regarding the spatial coordinate $x$ as time, and then the system reduces to a quantum mechanical system with the two bosonic and one fermionic variables. By solving the Schrödinger equations in the quantum system, the Green’s functions are calculated.

### 2.2 Transfer Hamiltonian

The transfer Hamiltonian is obtained by regarding the spatial coordinate $x$ as time in the functional integral representation (2.12). Then the system reduces to a quantum mechanical system. In Ref.[3], the following canonical creation and annihilation operators are introduced corresponding to the functional integral variables,

$$
\begin{align*}
\psi_R &\rightarrow F_\uparrow, \quad \psi_R^\dagger \rightarrow F_\uparrow^\dagger, \\
\psi_L &\rightarrow F_\downarrow, \quad \psi_L^\dagger \rightarrow F_\downarrow^\dagger, \\
\xi_R &\rightarrow B_\uparrow, \quad \xi_R^\dagger \rightarrow B_\uparrow^\dagger, \\
\xi_L &\rightarrow B_\downarrow, \quad \xi_L^\dagger \rightarrow B_\downarrow^\dagger.
\end{align*}
$$  \hspace{1cm} (2.13)

Fermionic and bosonic spin operators are defined by,

$$
\begin{align*}
\vec{S} &= \frac{1}{2} F^\dagger \vec{\sigma} F, \\
\vec{J} &= \frac{1}{2} \vec{\sigma} \vec{B},
\end{align*}
$$  \hspace{1cm} (2.14)

where

$$
\vec{B} = B^\dagger \sigma_\varepsilon.
$$  \hspace{1cm} (2.15)

The commutation relations followed by $F_\sigma$ and $B_\sigma$ is

$$
\begin{align*}
\{F_\alpha, F_\beta^\dagger\} &= \delta_{\alpha\beta}, \\
[B_\alpha, B_\beta]\ &= \delta_{\alpha\beta}.
\end{align*}
$$  \hspace{1cm} (2.16) (2.17)
It is proved that $\vec{S}$ and $\vec{J}$ satisfy $SU(2)$ and $SU(1,1)$ algebras, respectively, and they commute with SUSY charges $Q$ and $\bar{Q}$,

$$Q = BF, \quad \bar{Q} = F^\dagger B,$$  \hspace{1cm} (2.18)

which satisfy

$$Q^2 = \bar{Q}^2 = 0,$$  \hspace{1cm} (2.19)

$$\{Q, \bar{Q}\} = N = N_B + N_F,$$  \hspace{1cm} (2.20)

with $N_B = \bar{B}B$ and $N_F = F^\dagger F$. Note that SUSY charges $Q$ and $\bar{Q}$ are different from those in particle physics, where $\{Q, \bar{Q}\} = H$ with Hamiltonian $H$. $F_\sigma$ and $B_\sigma$ ($\sigma = \uparrow, \downarrow$) part of the transfer Hamiltonian is written in terms of the spin operators. Transfer Hamiltonian of $\phi$ is also obtained from Eq.(2.11). The system of $\phi$ is nothing but a simple harmonic oscillator linearly coupled with the SUSY spin $\mathcal{J} = S + J$. In terms of the spin operators and the canonical boson operators of the harmonic oscillator $a$, $a^\dagger$ which correspond to $\phi$, Hamiltonian of the system is given as,

$$H = 2\omega \mathcal{J}^x + 2m_0 \mathcal{J}^x + \sqrt{\frac{4g}{\lambda}} \mathcal{J}^x(a + a^\dagger) + \frac{1}{\lambda}(a^\dagger a + \frac{1}{2}).$$  \hspace{1cm} (2.21)

Fermionic states of $F_\sigma$ are specified by representations of $SU(2)$. Similarly bosonic states of $B_\sigma$ form multiplet of irreducible representations of $SU(1,1)$, which are specified by total spin

$$J^2 = (N_B^2 + 2N_B)/4, \quad N_B = \bar{B}B = B^\dagger_\uparrow B^\downarrow_\uparrow - B^\dagger_\downarrow B^\downarrow_\downarrow,$$  \hspace{1cm} (2.22)

and $z$-component of spin $J^z$, i.e.,

$$J^2|jn\rangle = j(j + 1)|jn\rangle,$$

$$J^z|jn\rangle = \left[\frac{1 + |2j + 1|}{2} + n\right]|jn\rangle,$$  \hspace{1cm} (2.23)

where $j = 0, \pm 1/2, \pm 1, \cdots$ and $n = 0, 1, \cdots$.

Quantum states of the system are specified by the quantum numbers $N_B$, $N_F$, $\Gamma = \bar{Q}Q$, $\bar{\Gamma} = Q\bar{Q}$, $\mathcal{J}^z$ and $N_\phi = a^\dagger a$, which commute with each other. The operators
Γ and \( \bar{\Gamma} \) satisfy identities, \( \Gamma^2 = \bar{\Gamma}^2 = 0 \), \( \Gamma + \bar{\Gamma} = N \) and \( \Gamma \bar{\Gamma} = \bar{\Gamma} \Gamma = 0 \). The state \( |N_B, N_F, \Gamma, \bar{\Gamma}, J^z, N_\phi\rangle \) is given by the direct product \( |N_B, N_F, \Gamma, \bar{\Gamma}, J^z\rangle \otimes |m\rangle_H \) where \( N_\phi |m\rangle_H = m |m\rangle_H \). In Ref.\[3\], the structure of \( |N_B, N_F, \Gamma, \bar{\Gamma}, J^z\rangle \) is studied in detail. The state with \( N_F = 0 \) is

\[
|2j, 0, 0, N, (|N + 1| + 1)/2 + n\rangle = |jn\rangle |\text{vac}\rangle,
\]

and the one with \( N_F = 2 \)

\[
|2j, 2, N, 0, (|N - 1| + 1)/2 + n\rangle = |jn\rangle |\uparrow\rangle,
\]

where the fermionic sector is given as

\[
|\uparrow\rangle = F_\uparrow |\text{vac}\rangle,
\]

\[
|\downarrow\rangle = F_\downarrow |\text{vac}\rangle,
\]

\[
|\downarrow\uparrow\rangle = F_\uparrow F_\downarrow |\text{vac}\rangle,
\]

with \( |\text{vac}\rangle \), the vacuum of the fermion \( F_\sigma \). The states with \( N_F = 1 \) can be constructed by acting \( Q \) and \( \bar{Q} \) on the above states. For \( N \neq 0 \), we have

\[
|2j, 1, (N - |N|)/2, (N + |N|)/2, |N|/2 + n\rangle = \sqrt{n + |N| / 2n + |N|} |jn\rangle |\downarrow\rangle + \sqrt{n / 2n + |N|} |jn\rangle |\uparrow\rangle,
\]

\[
|2j, 1, (N + |N|)/2, (N - |N|)/2, |N|/2 + n\rangle = \sqrt{n / 2n + |N|} |jn\rangle |\downarrow\rangle + \sqrt{n + |N| / 2n + |N|} |jn\rangle |\uparrow\rangle.
\]

For \( N = 0 \), however, these two sets of states coincide and are equal to

\[
| - 1, 1, 0, 0, n\rangle = \frac{1}{\sqrt{2}}(| - 1/2, n\rangle |\downarrow\rangle + | - 1/2, n - 1\rangle |\uparrow\rangle),
\]

where \( n \geq 0 \) and \(| - 1/2, -1\rangle = 0 \). Note that these are annihilated by both \( Q \) and \( \bar{Q} \). In order to complete the set of the quantum space, we introduce a set of states orthogonal to the states Eq.(2.31),

\[
| - 1, 1, *, *, n\rangle = \frac{1}{\sqrt{2}}(| - 1/2, n\rangle |\downarrow\rangle - | - 1/2, n - 1\rangle |\uparrow\rangle),
\]
which are not eigenstates of $\Gamma$ and $\bar{\Gamma}$.

In later discussions, we shall consider ‘right’ eigenstates of $H$,

$$H|N_B, N_F, \Gamma, \Gamma, E\rangle = E|N_B, N_F, \Gamma, \Gamma, E\rangle,$$  \hspace{1cm} (2.33)

which can be expanded in a basis of appropriate eigenstates of $J_z$ and $N_{\phi}$,

$$|N_B, N_F, \Gamma, \bar{\Gamma}, E\rangle = \sum_{J_z, N_{\phi}} \chi_{E}^{N_B, N_F, \Gamma}(J_z, N_{\phi})|N_B, N_F, \Gamma, \bar{\Gamma}, J_z, N_{\phi}\rangle. \hspace{1cm} (2.34)$$

Since $Q$ and $\bar{Q}$ do not commute with $N_B, N_F, \Gamma, \bar{\Gamma}$ but do with $H$, they relate degenerate eigenstates of $H$. For $N \neq 0$, there are two sets of SUSY doublet,

$$\bar{Q}|N, 0, 0, N, E\rangle = f_{0}|N - 1, 1, 0, N, E\rangle, \hspace{1cm} (2.35)$$

$$Q|N - 1, 1, 0, N, E\rangle = \frac{N}{f_{0}}|N, 0, 0, N, E\rangle, \hspace{1cm} (2.36)$$

$$Q|N, 0, 0, N, E\rangle = \bar{Q}|N - 1, 1, 0, N, E\rangle = 0, \hspace{1cm} (2.37)$$

and

$$Q|N - 2, 2, 0, N, E\rangle = f_{0'}|N - 1, 1, 0, N, E\rangle, \hspace{1cm} (2.38)$$

$$\bar{Q}|N - 1, 1, 0, N, E\rangle = \frac{N}{f_{0'}}|N - 2, 2, 0, N, E\rangle, \hspace{1cm} (2.39)$$

$$\bar{Q}|N - 2, 2, 0, N, E\rangle = Q|N - 1, 1, 0, N, E\rangle = 0, \hspace{1cm} (2.40)$$

where $f_{0}$ and $f_{0'}$ are constants to be determined by normalization. For $N = 0$, the space spanned by $| - 1, 1, *, *, n\rangle$ is not closed under the operation of $H$. The best one can do in this sector is to find an eigenstate of $H$ projected back onto the same sector, namely,

$$H| - 1, 1, *, *, E\rangle = E| - 1, 1, *, *, E\rangle + |\psi\rangle, \hspace{1cm} (2.41)$$

with $|\psi\rangle = \sum_{n} \psi_{n}| - 1, 1, 0, 0, n\rangle$. Including the above state, one finds a superquadruplet,

$$Q|- 1, 1, *, *, E\rangle = f_{1}|0, 0, 0, 0, E\rangle, \hspace{1cm} (2.42)$$

$$\bar{Q}|- 1, 1, *, *, E\rangle = f_{2}|- 2, 2, 0, 0, E\rangle, \hspace{1cm} (2.43)$$

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for $N \neq 0$ and

$$Q|0,0,0,0,E\rangle = f_3|1,1,0,0,E\rangle, \quad (2.44)$$

$$Q|2,2,0,0,E\rangle = f_4|1,1,0,0,E\rangle, \quad (2.45)$$

$$\bar{Q}|2,2,0,0,E\rangle = Q|0,0,0,0,E\rangle$$

$$= \bar{Q}|1,1,0,0,E\rangle = Q|1,1,0,0,E\rangle = 0, \quad (2.46)$$

for $N = 0$. Since the SUSY invariant ‘vacuum’ should be annihilated by $Q$ and $\bar{Q}$, the ‘vacuum’ has the eigenvalues $N_B = -N_F = 1$ and $\Gamma = \bar{\Gamma} = 0$, and therefore, it belongs to the subspace spanned by $|1,1,0,0,n\rangle \otimes |m\rangle_H$.

We dedicate the rest of this section to review the ‘left’ eigenstate of $H$, SUSY invariant identity operators and supertraces [3]. Please notice that the operator $H$ is not hermitian, since $J^x$ is anti-hermitian. ‘Left’ eigenstate of $H$, therefore, is not identical with ‘right’ eigenstate. Since the operator

$$U = e^{i\pi J^x + i\frac{\pi}{2} N_F} \quad (2.47)$$

transforms $J^x$ as

$$U U^\dagger J^x U = -J^x, \quad (2.48)$$

$U$ transforms

$$U U^\dagger H U = H^\dagger, \quad (2.49)$$

$$U U^\dagger Q U = \bar{Q}^\dagger, \quad (2.50)$$

$$U U^\dagger \bar{Q} U = Q^\dagger. \quad (2.51)$$

Given a right eigenstate of $H$, $|E\rangle_R$, the state,

$$|E\rangle_L = U^\dagger |E\rangle_R, \quad (2.53)$$

is left eigenstate of $H$, i.e.,

$$L \langle E|H = L \langle E|E. \quad (2.54)$$
We normalize the states as
\[ L(E|E)R = 1. \] (2.55)
This gives the constant \( f_0 = f'_0 = \sqrt{N} \) for \( N \neq 0 \) and \( f_1 = f_3, f_2 = f_4 \) for \( N = 0 \).
For \( N \neq 0 \), the SUSY invariant identity operators are
\[
1_{N;N_F=0,1} = \sum_E \left( |N, 0, 0, N, E\rangle_R L(N, 0, 0, N, E| \\
+ |N - 1, 1, N, 0, E\rangle_R L(N - 1, 1, N, 0, E| \right),
\] (2.56)
and
\[
1_{N;N_F=1,2} = \sum_E \left( |N - 1, 1, 0, N, E\rangle_R L(N - 1, 1, 0, N, E| \\
+ |N - 2, 2, N, 0, E\rangle_R L(N - 2, 2, N, 0, E| \right).\] (2.57)
Please notice \([Q, 1] = [\bar{Q}, 1] = 0\). For \( N = 0 \), the normalization condition is more complicated since \(| -1, 1, 0, 0, E\rangle_R\) and \(| -1, *, *, E\rangle_R\) are both orthogonal to their corresponding left eigenstates, i.e.,
\[
L(| -1, 1, 0, 0, E\rangle_R - 1, 1, 0, 0, E\rangle_R = 0, \] (2.58)
\[
L(| -1, 1, *, *, E\rangle_R - 1, 1, *, *, E\rangle_R = 0. \] (2.59)
We therefore require
\[
L(| -1, 1, 0, 0, E\rangle_R - 1, 1, *, *, E\rangle_R = 1, \] (2.60)
\[
L(| -1, 1, *, *, E\rangle_R - 1, 1, 0, 0, E\rangle_R = 1, \] (2.61)
with the usual normalization condition,
\[
L(| 0, 0, 0, 0, E\rangle_R | 0, 0, 0, 0, E\rangle_R = 1, \] (2.62)
\[
L(| -2, 2, 0, 0, E\rangle_R | -2, 2, 0, 0, E\rangle_R = 1. \] (2.63)
\[1\] In order to have this property, the definition of \( U = e^{i\pi J_z + i\pi N_F} \) is essential. This is the reason why we have chosen the different definition from that in Ref.\[3\]
The corresponding identity operator for $N = 0$ is therefore

\begin{align}
1_{N=0} &= \sum_E \left( | -2, 2, 0, 0, E \rangle_R \cdot L \langle -2, 2, 0, 0, E | + | 0, 0, 0, 0, E \rangle_R \cdot L \langle 0, 0, 0, 0, E | \\
&\quad + | -1, 1, *, *, E \rangle_R \cdot L \langle -1, 1, 0, 0, E | + | -1, 1, 0, 0, E \rangle_R \cdot L \langle -1, 1, *, *, E | \right).
\end{align}

(2.64)

In the following, we study the system in the long size limit, where the system size $L \rightarrow \infty$. In this case, the supertrace of an operator $O e^{-LH}$ is expected to reduce to an expectation value in the SUSY invariant vacuum,

\begin{equation}
\langle O \rangle = \text{Str} O e^{-LH} \rightarrow L \langle 0 | O | 0 \rangle_R e^{-LE_b},
\end{equation}

(2.65)

where $E_b$ is the lowest eigenvalue of $H$. As explained above, the vacuum $|0\rangle_R$ has eigenvalues $N_B = -N_F = 1, \Gamma = \bar{\Gamma} = 0$.

3 Localization length

3.1 Eigenvalue problem

The ensemble-averaged localization length is the parameter which measures how fast the averaged one-particle fermion Green’s function decays. The averaged fermion Green’s function is given as follows,

\begin{equation}
\bar{G}_{\alpha\alpha}(x; i\omega) = L \langle 0 | e^{xH} F_{\alpha} e^{-xH} F_{\alpha}^\dagger | 0 \rangle_R e^{-\frac{L}{x}}.
\end{equation}

(3.1)

Since the state $F_{\alpha}^\dagger | 0 \rangle_R$ has the quantum number $N_B = -1, N_F = 2, \Gamma = N = 1$ and $\bar{\Gamma} = 0$,

\begin{align}
\bar{G}_{\alpha\alpha}(x; i\omega) &= \sum_E L \langle 0 | F_{\alpha} | -1, 2, 1, 0, E \rangle_R \cdot L \langle -1, 2, 1, 0, E | F_{\alpha}^\dagger | 0 \rangle_R e^{-(E - \frac{1}{x})x} e^{-\frac{L}{x}} \\
&\simeq e^{-\frac{L}{x}},
\end{align}

(3.2)
where \( \xi_\omega \) is 'imaginary-time' localization length and the state \( |-1, 2, 1, 0, E\rangle_R \) satisfies the following equation,

\[
H|-1, 2, 1, 0, E\rangle_R = E|-1, 2, 1, 0, E\rangle_R.
\] (3.3)

It is obvious that the localization length is related with the eigenvalue \( E \). We shall obtain the genuine localization length \( \xi_\epsilon \) from \( \xi_\omega \) by the analytic continuation \( i\omega \rightarrow \epsilon \).

In the following, we rescale the parameters as \( \omega \rightarrow g\omega \), \( m_0 \rightarrow gm_0 \) and \( \lambda \rightarrow \frac{\lambda}{g} \).

\( |-1, 2, 1, 0, E\rangle_R \) is given by a linear combination of the states \( |n\rangle_1 \equiv |-1, 2, 1, 0, n\rangle_R \), and can be written as

\[
|-1, 2, 1, 0, E\rangle_R = \sum_{n=0}^{\infty} \chi_{n,m}|n\rangle_1|m\rangle_H,
\] (3.4)

with coefficients \( \chi_{n,m} \). Reminding that the wave function \( \{\phi_{n,m}\} \) for \( |0\rangle_R \),

\[
|0\rangle_R = \sum_{n,m} \phi_{n,m}|-1, 1, 0, 0, n\rangle \otimes |m\rangle_H,
\]

has the form[1],

\[
\phi_{n,m} = \lambda^{m/2}(\phi_{n,m}^{[1]} + \lambda \phi_{n,m}^{[2]}),
\]

where \( \phi_{n,m}^{[i]} (i = 1, 2) \) are independent of \( \lambda \), and the 'energy eigenvalue' of \( |0\rangle_R \) is \( \frac{1}{2\lambda} \), we assume that \( \chi_{n,m} \) and \( E \) are expanded as

\[
\chi_{n,m} = \lambda^{m/2}(\chi_{n,m}^{[0]} + \lambda \chi_{n,m}^{[1]} + \lambda^2 \chi_{n,m}^{[2]} \cdots),
\] (3.5)

\[
E = \frac{1}{2\lambda} + E^{[0]} + \lambda E^{[1]} + \lambda^2 E^{[2]} \cdots,
\] (3.6)

where \( \chi_{n,m}^{[i]} \) and \( E^{[i]} \) are independent of \( \lambda \). Equation (3.3) gives the Shrödinger equation of \( \chi_n \);

\[
\sum_{i,n} \left( H_0 + 4(J^x)^2 + \frac{m}{\lambda} \right) \lambda^{i+m/2} \chi_{n,m}^{[i]}|n\rangle_1
\]

\[
+ \sqrt{\frac{4}{\lambda}} J^x \sum_{i,n} \left( \sqrt{m} \lambda^{i+m/2} \chi_{n,m-1}^{[i]} + \sqrt{m + 1} \lambda^{i+m+1} \chi_{n,m+1}^{[i]} \right)|n\rangle_1
\]

\[
= \sum \lambda^{j} E^{[j]} \sum_{i,n} \lambda^{i+m/2} \chi_{n,m}^{[i]}|n\rangle_1,
\] (3.7)
where
\[ H_0 = 2\omega J_z + 2m_0 J^x - 2(J^x)^2. \]
Comparing terms of each order of \( \lambda \) leads us to the following series of equations, the \( O(\lambda^{-1}) \) terms;
\[ \sum_n \chi_{n,m}^{[0]} |n\rangle_1 + \sqrt{\frac{4}{m}} J^x \sum_n \chi_{n,m-1}^{[0]} |n\rangle_1 = 0, \quad (3.8) \]
the \( O(\lambda^0) \) terms;
\[ \sum_n H_0 \chi_{n,m}^{[0]} |n\rangle_1 \\
+4(J^x)^2 \sum_n \chi_{n,m}^{[0]} |n\rangle_1 \\
+2\sqrt{m + 1} J^x \sum_n \chi_{n,m+1}^{[0]} |n\rangle_1 \\
+m \sum_n \chi_{n,m}^{[1]} |n\rangle_1 \\
+2\sqrt{m} \sum_n \chi_{n,m}^{[1]} |n\rangle_1 \\
= E^{[0]} \sum_n \chi_{n,m}^{[0]} |n\rangle_1, \quad (3.9) \]
the \( O(\lambda^1) \) terms;
\[ \sum_n H_0 \chi_{n,m}^{[1]} |n\rangle_1 \\
+4(J^x)^2 \sum_n \chi_{n,m}^{[1]} |n\rangle_1 \\
+2\sqrt{m + 1} J^x \sum_n \chi_{n,m+1}^{[1]} |n\rangle_1 \\
+m \sum_n \chi_{n,m}^{[2]} |n\rangle_1 \\
+2\sqrt{m} \sum_n \chi_{n,m}^{[2]} |n\rangle_1 \\
= E^{[0]} \sum_n \chi_{n,m}^{[1]} |n\rangle_1 + E^{[1]} \sum_n \chi_{n,m}^{[0]} |n\rangle_1, \quad (3.10) \]
and so on. It is easily seen that Eq.(3.8) and Eq.(3.9) for \( m = 0 \) determine \( \chi_{n,m}^{[0]} \).
Equation(3.9) for \( m = 0 \) is identical with the equation of \( \chi_n \) which is considered by Balents and Fisher [3]. In a similar way, Eq.(3.9) and Eq.(3.10) for \( m = 0 \) determine \( \chi_{n,m}^{[1]} \) and \( E^{[1]} \).

3.2 Solution in \( O(\lambda^0) \)

The solution \( \chi_{n,m}^{[0]} \) and \( E^{[0]} \) are given by Balents and Fisher [3]. We review the derivation of them again for completeness. Equation to be solved is
\[ (2\omega J^z - 4M\omega J^x + 4(\omega J^x)^2) \sum_n \chi_{n,0}^{[0]} |n\rangle_1 = E^{[0]} \sum_n \chi_{n,0}^{[0]} |n\rangle_1, \quad (3.11) \]
with \( M = 2m_0 \). Using the representation of \( J^z \) and \( J^x \) for \( |n\rangle_1 \),

\[
J^z|n\rangle_1 = (n + \frac{1}{2})|n\rangle_1 \\
J^x|n\rangle_1 = \frac{n+1}{2}|n+1\rangle_1 - \frac{n}{2}|n-1\rangle_1,
\]

we have the discrete equation of \( \chi_{n,0}^{[0]} \),

\[
2\omega(n + \frac{1}{2})\chi_{n,0}^{[0]} + 2M(n\chi_{n-1,0}^{[0]} - (n + 1)\chi_{n+1,0}^{[0]}) - (n + 2)(n + 1)\chi_{n+2,0}^{[0]} \\
+ (2n^2 + 2n + 1)\chi_{n,0}^{[0]} - n(n - 1)\chi_{n-2,0}^{[0]} = E\chi_{n,0}^{[0]}.
\]

Since multiplying a solution for \( M \) by \((-1)^n\) yields a solution for \(-M\), we expect

\[
\chi_{n,0}^{[0]} = c_3\chi^{[0]}(n, M, E) + c_4(-1)^n\chi^{[0]}(n, -M, E),
\]

where \( \chi(n, M, E) \) is obtained by solving the continuum equation derived from Eq.(3.15),

\[
\left(2\omega n - 2M(2n\frac{\partial}{\partial n} + 1) - (2n\frac{\partial}{\partial n} + 1)^2\right)\chi^{[0]}(n, M, E) = E\chi^{[0]}(n, M, E).
\]

Changing variables \( n \) to \( z = \ln n \), Eq. (3.17) gives

\[
\left(-(2\frac{\partial}{\partial z} + 1)^2 - 2M(2\frac{\partial}{\partial z} + 1) + 2\omega e^z\right)\Phi_E(z, M) = E\Phi_E(z, M),
\]

where \( \Phi_E(z, M) = \chi^{[0]}(n, M, E) \).

Though it is possible to solve Eq.(3.18) exactly, we use the hard-wall approximation for simplicity, i.e., the ‘potential’ term \( \omega e^z \) is neglected and the boundary condition \( \Phi_E(|\ln \omega|, M) = 0 \) is imposed instead. Then the solution is obtained as

\[
\Phi_E(z, M) = e^{-\frac{1+M}{2}z}\sin \frac{\beta(z - z_\omega)}{2},
\]

with \( \beta = \sqrt{E - M^2} \) and \( z_\omega = |\ln \omega| \). In order to obtain a solution for \( \chi_{n,0}^{[0]} \), namely, to fix the constants \( c_3 \) and \( c_4 \) in Eq.(3.19), additional conditions have to be imposed. Since neither \( \chi^{[0]}(n, M, E) = 1 \) nor \((-1)^n\) are, unfortunately, solutions in the limit
\( \omega = M = 0 \), we are not able to use similar boundary conditions used in paper I in which we obtained the DOS.

Our strategy is to compare the solution (3.19) with the one obtained by solving the discrete equation (3.15) by neglecting the term \( \omega J^z \). Since the other terms in the Hamiltonian except \( \omega J^z \) depend on only \( J^x \), we consider eigenstate \( |\alpha\rangle \) of \( J^x \):

\[
J^x|\alpha\rangle = \alpha|\alpha\rangle. 
\] (3.20)

This state is the eigenstate of \( H \) with \( \omega = 0 \), because

\[
H|\alpha\rangle = (4M\alpha - 4\alpha^2)|\alpha\rangle = E|\alpha\rangle. 
\]

Expanding \( |\alpha\rangle \) in terms of \( |n\rangle_1 \),

\[
|\alpha\rangle = \sum_n \psi_n(\alpha)|n\rangle_1, 
\] (3.21)

we find \( \chi_{n,0}^{[0]} = c_+\psi_n(\alpha_+) + c_-\psi_n(\alpha_-) \) with \( \alpha_\pm = \frac{M \pm i\beta}{2} \). This approximation is valid for \( n \ll 1/\omega \). Eq.(3.21) leads us to the following equation of \( \psi_n(\alpha) \),

\[
(n + 1)\psi_{n+1}(\alpha) - n\psi_{n-1}(\alpha) = -2\alpha\psi_n(\alpha). 
\] (3.22)

Equation (3.22) can be solved easily by introducing the function \( \hat{\psi}(w, \alpha) \equiv \sum_n \psi_n(\alpha)w^n \).

By using the identity for \( \sigma(w) = \sum_n \sigma_n w^n \),

\[
\frac{1}{2} \sum_n ((n + 1)\sigma_{n+1} - n\sigma_{n-1})w^n = \frac{1}{2}((1 - w^2) \frac{\partial}{\partial w} - \omega) \sigma(w) 
\equiv D_w \sigma(w), 
\] (3.23)

the difference equation (3.22) is converted into the differential equation,

\[
\left( (1 - w^2) \frac{\partial}{\partial w} - w + 2\alpha \right) \hat{\psi}(w, \alpha) = 0, 
\]

and solution to this equation is obtained as

\[
\hat{\psi}(w, \alpha) = c(1 - w)^{-\frac{1}{2} + \alpha}(1 + w)^{-\frac{1}{2} - \alpha}, 
\]
with \( c \) being a constant. Thus we have
\[
\psi_n(\alpha) = \frac{c}{n!} \oint_C \frac{dw}{2\pi i} \frac{(1-w)^{\frac{1}{2}+\alpha}(1+w)^{-\frac{1}{2}-\alpha}}{w^{n+1}}
\]
\[
= \frac{c}{\Gamma(n+1)} \cos \pi \alpha \pi \sum_{l=0}^{n} (-1)^l \Gamma \left( \frac{1}{2} - l + n - \alpha \right) \Gamma \left( \frac{1}{2} + l + \alpha \right).
\]
(3.24)
The contour \( C \) is the circle whose center is located at the origin and radius is small enough. Dominant terms in Eq.(3.24) are \( l = 0 \) and/or \( l = n \) for large \( n \). Using the formula of the Gamma function,
\[
\lim_{n \to \infty} \frac{\Gamma(n+a)}{\Gamma(n)n^a} = 1,
\]
(3.25)
we find that, for large \( n \), Eq.(3.24) can be approximated as,
\[
\psi_n(\alpha) = n^{-\frac{1}{2} - \alpha} \frac{\Gamma(\frac{1}{2} + \alpha)}{\sqrt{\pi}} + (-1)^n n^{-\frac{1}{2} + \alpha} \frac{\Gamma(\frac{1}{2} - \alpha)}{\sqrt{\pi}},
\]
(3.26)
where we have fixed \( c = \frac{\sqrt{\pi}}{\cos \pi \alpha} \) without loss of generality. As we shall see, in the limit \( M \to 0 \) and \( \omega \to 0 \), \( \alpha \pm \) tend to vanish, and thus \( \Gamma(\frac{1}{2} \pm \alpha) \) tend to \( \sqrt{\pi} \). Therefore we have
\[
\chi_{n,0}^0 = e^{-\frac{1+M}{2}z}(c_+ e^{-i\frac{\theta}{2}z} + c_- e^{i\frac{\theta}{2}z}) + (-1)^n e^{-\frac{1-M}{2}z}(c_+ e^{i\frac{\theta}{2}z} + c_- e^{-i\frac{\theta}{2}z}).
\]
(3.27)
We shall compare Eq.(3.27) with Eq.(3.16). These two solutions should coincide with each other for \( 1/\omega \ll n \ll 1 \). This matching condition gives
\[
c_+ = c_- = c_3/2 = c_4/2, \quad \beta z_\omega/\pi = \text{an even integer},
\]
\[
c_+ = -c_- = c_3/2i = -c_4/2i, \quad \beta z_\omega/\pi = \text{an odd integer}
\]
(3.28)
From the above condition on \( \beta z_\omega/\pi \), \( E^0 \) is determined. So much is the review for the zeroth order solutions.

We summarize the solutions \( \chi^0 \) and the eigenvalues \( E^0 \) of Eq.(3.11) as follows;
\[
\chi_{n,0}^0 = c_3 \left( e^{-\frac{1+M}{2}z} \sin \frac{k\pi}{2z_\omega}(z - z_\omega) + (-1)^n e^{-\frac{1-M}{2}z} \sin \frac{k\pi}{2z_\omega}(z - z_\omega) \right),
\]
(3.29)
\[
E^0 = M^2 + \left( \frac{\pi k}{z_\omega} \right)^2,
\]
(3.30)
with an integer \( k > 1 \). The constant \( c_3 \) is determined to be \( z_\omega^{-1/2} \) by the normalization condition. By solving Eq.(3.28), \( \chi_{n,m}^0 (m > 0) \) is determined in terms of \( \chi_{n,0}^0 \).
3.3 Solution in O(λ)

Now let us turn to $E^{[1]}$, which gives the first-order correction of the localization length. From Eq.(3.9) and Eq.(3.10) for $m = 0$, we have,

$$
(H_0 - E^{[0]}) \sum_n \chi_{n,m}^{[1]} |n\rangle_1 = -\frac{4}{m+1} J^x (H_0 - E^{[0]}) J^z \sum_n \chi_n^{[0]} |n\rangle_1 + E^{[1]} \sum_n \chi_n^{[0]} |n\rangle_1.
$$

(3.31)

Since $E^{[1]}$ does not depend on $m$, it is sufficient to solve Eq.(3.31) for $m = 0$ in order to obtain $E^{[1]}$. We employ a similar strategy used above, namely, we shall obtain solutions by both the hard-wall approximation to the continuum equation and the approximation which neglects the term $\omega J^z$ in $H_0$ in the original discrete equation.

The solutions depend on the constant $E^{[1]}$. We obtain $E^{[1]}$ by matching the solutions obtained by these methods in the region where the above two approximations are both legitimate. Using the representation of $J^x$ and $J^z$, Eqs.(3.12) and (3.31) give

$$
\left(-(2\frac{\partial}{\partial z} + 1)^2 - 2M(2\frac{\partial}{\partial z} + 1) + 2\omega e^z - E^{[0]}\right) \Psi_E(z,M) = -4\omega e^z(2\frac{\partial}{\partial z} + 1) \Phi_E(z,M) + E^{[1]} \Phi_E(z,M),
$$

(3.32)

in the continuum approximation with $n = e^z$ and

$$
\Psi_E(z,M) = \chi^{[1]}(e^z, M) + 4(2\frac{\partial}{\partial z} + 1)^2 \Phi_E(z,M),
$$

where

$$
\chi_n^{[1]} = c_3 \chi^{[1]}(n, M) + c_4 (-1)^n \chi^{[1]}(n, -M).
$$

In the hard-wall approximation, we neglect the term $\omega e^z$ on the left hand side of Eq.(3.32), and we impose the boundary condition $\Psi_E(z, M) = 0$ instead. Making use of Eq.(3.19) and assuming $\Psi_E(z, M) = e^{-\frac{1+M}{2}z} \eta(z, M)$,

$$
(-4\frac{\partial^2}{\partial z^2} - \beta^2) \eta(z, M) = -8\omega e^z(\frac{\partial}{\partial z} - \frac{M}{2}) \sin \frac{\beta}{2}(z - z_\omega) + E^{[1]} \sin \frac{\beta}{2}(z - z_\omega).
$$

(3.33)

It is not so difficult to solve this equation,

$$
\eta(z, M) = -\frac{2}{\beta} \int_{z_\omega}^z dx \left(2\omega e^x \left(\frac{\beta}{2} \cos \frac{\beta}{2}(x - z_\omega) - \frac{M}{2} \sin \frac{\beta}{2}(x - z_\omega)\right) - \frac{E^{[1]}}{4} \sin \frac{\beta}{2}(x - z_\omega)\right)
$$
\[
\times \left( \sin \frac{\beta}{2} (x - z_\omega) \cos \frac{\beta}{2} (z - z_\omega) - \cos \frac{\beta}{2} (x - z_\omega) \sin \frac{\beta}{2} (z - z_\omega) \right)
= -\frac{1}{\beta} \left( \omega \beta (e^z (\frac{\beta}{1 + \beta^2} \cos \frac{\beta}{2} (z - z_\omega) + \frac{1}{1 + \beta^2} \sin \frac{\beta}{2} (z - z_\omega))) \right)
+ e^{z_\omega} \beta \left( \frac{\beta}{1 + \beta^2} \cos \frac{\beta}{2} (z - z_\omega) + \frac{1}{1 + \beta^2} \sin \frac{\beta}{2} (z - z_\omega) \right)
- \omega M (e^z (\frac{1}{1 + \beta^2} \cos \frac{\beta}{2} (z - z_\omega) + \frac{\beta}{1 + \beta^2} \sin \frac{\beta}{2} (z - z_\omega)) \right)
+ e^{z_\omega} (\frac{1}{1 + \beta^2} \cos \frac{\beta}{2} (z - z_\omega) - \frac{\beta}{1 + \beta^2} \sin \frac{\beta}{2} (z - z_\omega))
- \frac{E}{4} (z - z_\omega) \cos \frac{\beta}{2} (z - z_\omega) - \frac{1}{\beta} \sin \frac{\beta}{2} (z - z_\omega),
\tag{3.34}
\]

apart from the general solution, \( \sin \frac{\beta}{2} (z - z_\omega) \), of the homogeneous equation.

On the other hand, the discrete equation (3.31) with \( \omega J^z = 0 \) is
\[
(4M J^x - 4J^z) \sum_n \chi_n |n\rangle_1 = E^{[1]} \sum_n \chi_n |n\rangle_1.
\tag{3.35}
\]

In order to solve Eq. (3.35), we use Eqs. (3.12) and (3.23) as in Eq. (3.24), and we have the following equation,
\[
(4MD_w - 4D_w^2 - E^{[0]}) \chi(w) = E^{[1]} \chi(w),
\tag{3.36}
\]

with \( \chi(w) = \sum_n \chi_n w^n \) and \( \chi(w) = \sum_n \chi_n |n\rangle_1 \). Decomposing each power of \( w \) in \( \chi(w) \), solution of Eq. (3.36) gives us solution of Eq. (3.35), \( \chi_n \)'s. Equation (3.36) can be solved exactly. The solution is given by
\[
\chi(w) = E^{[1]} \int w \frac{\hat{\psi}(v, \alpha_+ \hat{\psi}(w, \alpha_-) - \hat{\psi}(w, \alpha_+ \hat{\psi}(v, \alpha_-)}{\psi(v, \alpha_+ \hat{\psi}(v, \alpha_-)} \left( - \frac{\chi(v)}{1 - v^2} \right).
\tag{3.37}
\]

This can be verified by directly substituting (3.37) into Eq. (3.36). Performing the integral in (3.37) and using the identity,
\[
\hat{\psi}(v, \alpha_+) \hat{\psi}'(v, \alpha_-) - \hat{\psi}'(v, \alpha_+) \hat{\psi}(v, \alpha_-) = -2i \beta \frac{1}{1 - v^2} \hat{\psi}(v, \alpha_+) \hat{\psi}(v, \alpha_-),
\tag{3.38}
\]
we have
\[
\chi(w) = \frac{i E^{[1]}}{2 \beta} \int w d\nu \left( \frac{\hat{\psi}(w, \alpha_+ \hat{\psi}(v, \alpha_-)}{\psi(v, \alpha_-)} \frac{\chi(v)}{1 - v^2} \right)
\tag{3.39}
\]
where some factor in Eq. (3.39) can be simplified by
\[
\hat{\psi}(v, \alpha_+) = \left(1 - v\right)^{i\beta}.
\] (3.40)

By using the identity,
\[
\int^w dv \left(1 - \frac{v}{1 + v}\right)^{i\beta} \frac{1}{1 - v^2} = \frac{1}{2i\beta} \frac{1}{\left(1 + w\right)^{i\beta}} + \text{const.}
\]
\[
\int^w dv \frac{1}{1 - v^2} = \frac{1}{2} \ln \left|\frac{w + 1}{w - 1}\right| + \text{const.},
\] (3.41)
we can calculate the above integral exactly and we have
\[
\bar{\chi}(w) = \frac{E^{[1]}}{4i\beta} \ln \left|\frac{w - 1}{w + 1}\right| \left(c_+ \hat{\psi}(w, \alpha_+) - c_- \hat{\psi}(w, \alpha_-)\right),
\] (3.42)
up to the general solution \(\chi^{[1]}(w)\). We hence obtain the exact form of \(\chi^{[1]}_{n,0}\) as
\[
\chi^{[1]}_{n,0} = \oint_C \frac{dw}{2\pi i} \frac{\bar{\chi}(w)}{w^{n+1}}
\]
\[
= \frac{E^{[1]}}{4i\beta} \sum_{l=0}^{n-1} \frac{1}{n - l} \left((-1)^{n-l} - 1\right) \left(c_+ \psi_l(\alpha_+) - c_- \psi_l(\alpha_-)\right).
\] (3.43)

In order to compare the above result with the solution obtained by the hard-wall approximation, we approximate \(\chi^{[1]}_{n,0}\) as follows,
\[
\chi^{[1]}_{n,0} \simeq -\frac{E^{[1]}}{4i\beta} \sum_{l=0}^{n-1} \frac{1}{n - l} \left((-1)^{n-l} - 1\right) \left(c_+ \psi_l(\alpha_+) - c_- \psi_l(\alpha_-)\right)
\]
\[
- (-1)^{n} \left(c_+ \frac{1}{lB(l, \frac{1}{2} - \alpha_+)} - c_- \frac{1}{lB(l, \frac{1}{2} + \alpha_-)}\right),
\] (3.44)
where we have picked out the leading terms in \(\psi_l(\alpha_\pm)\), i.e.,
\[
\psi_l(\alpha) \simeq \frac{1}{lB(l, \frac{1}{2} - \alpha)} + (-1)^l \frac{1}{lB(l, \frac{1}{2} + \alpha)}.
\] (3.45)

From the fact that \(\frac{1}{n - lB(l, \frac{1}{2} \pm \alpha)}\) has its maximum at \(l = n - 1\) and \(\frac{1}{lB(l, \frac{1}{2} \pm \alpha)}\) is a slowly varying function for large \(l\), we find
\[
\sum_{l=0}^{n-1} \frac{1}{n - l} \frac{1}{lB(l, \frac{1}{2} \pm \alpha)} \simeq \sum_{l=0}^{n-1} \frac{1}{n - l} \frac{1}{(n - 1)B(n - 1, \frac{1}{2} \pm \alpha)}
\]
\[
\simeq \frac{\ln n + \gamma}{\Gamma(n)} \frac{\Gamma(n - \frac{1}{2} \pm \alpha)}{\Gamma(n)}
\]
\[
\simeq (\ln n + \gamma) \frac{\Gamma(n - \frac{1}{2} \pm \alpha)}{\Gamma(\frac{1}{2} \pm \alpha)} n^{-\frac{1}{2} \pm \alpha}
\] (3.46)
for large $n$, where $\gamma$ is Euler’s gamma constant. We therefore have the first order correction of $\chi_{n,0}$ as

$$\chi_{n,0}^{[1]} \simeq -\frac{E^{[1]}}{4i\beta}(\ln n + \gamma)(n^{\frac{\beta}{2}} + (c_+ n^{-\frac{\beta}{2}} - c_- n^{\frac{\beta}{2}}) + (-1)^n n^{\frac{1+M}{2}}(c_- n^{-\frac{\beta}{2}} - c_+ n^{\frac{\beta}{2}}))$$

$$= -\frac{E^{[1]}}{4i\beta}(z + \gamma)(e^{-\frac{1+M}{2}z} \cos \frac{\beta}{2}(z - z_\omega) + (-1)^n(M \leftrightarrow -M), (3.47)$$

which should be compared with the solution obtained from the difference equation

$$e^{-\frac{1+M}{2}z} e^{-1+\frac{\beta}{2}(1+M)} z^{\frac{\beta}{2}} \cos \frac{\beta}{2}(z - z_\omega) + (-1)^n(M \leftrightarrow -M). (3.48)$$

We hence obtain the ‘energy eigenvalue’ in $O(\lambda)$,

$$E^{[1]} = -\frac{4\beta}{z_\omega + \gamma} \cdot \frac{\beta}{1+\beta^2}(1+M). (3.49)$$

In the limit $M \to 0$ and $\omega \to 0$, we can neglect the higher-order terms in $\beta$ and we have

$$E^{[1]} \simeq -\frac{4\beta^2}{z_\omega} = -\frac{4}{z_\omega^2} k^2. (3.50)$$

Thus the energy eigenvalue is

$$E_k = E_k^{[0]} + \lambda E_k^{[1]}$$

$$= \left(\frac{\pi^2}{z_\omega} - \frac{4\pi^2}{z_\omega^3}\right) k^2. (3.51)$$

Since the one-particle Green’s function has the form $\sum_k f(k) \exp(-E_k x)$ with certain calculable functions $f(k)$, the ‘imaginary-time’ localization length $\xi_\omega$ is the inverse of the coefficient of $k^2$ in $E_k$, namely,

$$\xi_\omega = \frac{z_\omega}{\pi^2} \frac{1}{1 - \frac{1}{z_\omega}} \simeq \frac{z_\omega}{\pi^2} (1 + \frac{4\lambda}{z_\omega}). (3.52)$$
The genuine localization length $\xi_\epsilon$ is the real part of $\xi_\omega$ after performing analytic continuation $i\omega \to \epsilon + i\omega$, and then Eq. (3.52) leads to

$$\xi_\epsilon \simeq \frac{1}{g} \left( \left| \frac{\ln \frac{\epsilon}{2g}}{\pi^2} \right|^2 + g\lambda \frac{4|\ln \frac{\epsilon}{2g}|}{\pi^2} \right), \quad (3.53)$$

where we have recovered the constant $g$ and use the fact that $\ln \frac{\epsilon}{g} \simeq \ln \frac{\epsilon}{2g}$ for small $\epsilon$. Equation (3.53) shows that the localization length $\xi_\epsilon$ is a increasing function of $(g\lambda)$. This means that the nonlocally-correlated disorder enhances the delocalization. Moreover the number of states with energy below $\epsilon$ is

$$N(\epsilon) = \int_0^\epsilon d\epsilon \rho(\epsilon) = \frac{g}{2} \left( \frac{1}{\left| \ln \frac{\epsilon}{2g} \right|^2} - \frac{4g\lambda}{\left| \ln \frac{\epsilon}{2g} \right|^3} \right) + \text{(higher-order terms)}, \quad (3.54)$$

and this shows that the product of the localization length and the number of states is constant up to the first order of $(g\lambda)$,

$$\xi_\epsilon N(\epsilon) = \frac{1}{2\pi^2} + O((g\lambda)^2). \quad (3.55)$$

Physical meaning of this result will be discussed in Sect.4 from microscopic point of view (see also Ref.[5]).

4 Discussion

In this paper, we have studied how the nonlocally correlated disorder affects the the mean localization length $\xi_\epsilon$ in the random-mass Dirac fermions by making use of SUSY methods. (In fact this is a low-energy effective model of the random hopping tight binding model[3].) We found that $\xi_\epsilon$ is an increasing function of the correlation length $\lambda$. This result is consistent with the behavior of the DOS $\rho(\epsilon)$ obtained in the previous paper[11] and the numerical studies in Ref.[5]. There exists a simple intuitive picture of the result: For the case of larger correlation length $\lambda$, average distance between kinks is longer and also average height of kinks is lower. Therefore,
finite correlation of disorders hinders the scattering of fermions by disorders and enhances the delocalization of fermions. Though this result is obtained in the specific one-dimensional model, we expect that similar behaviours hold in general random-disordered systems.

Let us consider relationship between the localization length \( \xi \) and the number of states \( N(\epsilon) \). Our SUSY calculation suggests that the following relation

\[
\xi = \text{const.} \frac{1}{N(\epsilon)}
\]  

holds (up to \( O(g\lambda) \)). This can be understood as below. As is shown by the numerical calculations of wave functions of random-mass Dirac fermions[3], fermions in the random potential localize within an interval between adjacent nodes. According to ‘one-dimensional node counting theorem’[4], the number of nodes \( N_d(\epsilon) \) is equal to \( LN(\epsilon) - 1 \). Let us suppose that the position of each node \( x_i (i = 1, 2, \ldots, N_d) \) is randomly distributed under the constraint \( 0 < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_{N_d} < L \). The expectation value \( \langle x_i \rangle \) is then given as

\[
\langle x_i \rangle = \frac{1}{N} \prod_{k=1}^{N_d} \int_0^L dx_k \frac{x_i}{L} \prod_{l=1}^{N_d-1} \theta(x_{l+1} - x_l) = \frac{L}{N_d + 1} = \frac{i}{N(\epsilon)},
\]  

with the normalization constant

\[
N = \prod_{k=1}^{N_d} \int_0^L dx_k \frac{dx_k}{L} \prod_{l=1}^{N_d-1} \theta(x_{l+1} - x_l).
\]  

Therefore it is found that the expectation value of the distance between adjacent nodes is \( \langle x_{i+1} \rangle - \langle x_i \rangle = \frac{1}{N} \). According to this estimation, we roughly understand that the localization length is to be inversely proportional to the number of states \( N(\epsilon) \).

In usual cases, it is not easy to obtain the localization length accurately by numerical calculations. However very recently, a very useful method for that was proposed, i.e., the non-Hermitian extension by introduction of an imaginary vector potential[6]. Then we hope that the localization length in the present model is also calculated
numerically by that method, and the analytical expression, which we obtained in this paper, is compared with numerical calculations. This is under study and results will be reported in a future publication.

While the results in this paper are obtained for the (effective) random hopping tight binding (RHTB) model, they give some important implications for closely related model — spin-Peierls model \cite{7}. Since the $z$-component of spin in the spin-Peierls model corresponds to the presence or absence of the tight-binding fermion, the Green’s function in the RHTB model can be considered as a correlation function of spins in the spin-Peierls model. In fact, the single-fermion Green’s function of the RHTB model,

$$\int_0^\infty dt \ e^{i\epsilon t} \langle v|c_n(t)c_{n'}^\dagger(0)|v\rangle,$$

coincides with the spin-spin correlation function

$$\int_0^\infty dt \ e^{i\epsilon t} \langle v|S^-(n,t)S^+(n',0)|v\rangle,$$

where $S^\pm(n,0)$ are the spin-up and spin-down operators at site $n$, and $S^-(n,t)$ is given by

$$S^-(n,t) = e^{iH_{SP}t}S^-(n,0)e^{-iH_{SP}t}$$

with the spin-Peierles Hamiltonian $H_{SP}$. The operators $S^+(n,0)$ and $c_n$ are related with each other by the Jordan-Wigner transformation. In the spin systems, the mean localization length $\xi_\epsilon$, which is calculated in this paper, can be considered as the mean correlation length of the spins. We therefore find that the spin-spin correlation is enhanced by the suppression of randomness of disorders, because the suppression enhances the (quasi-)extended states near the band center $\epsilon = 0$.

Let us compare the above result with the experimental observation in Ref.\cite{8}. In Ref.\cite{8}, a phase diagram of Cu$_{1-x}$Zn$_x$GeO$_3$ was obtained by the measurements of the magnetic susceptibility. The phase diagram shows that the antiferromagnetic (AF) order is enhanced by decrease of the impurity concentration at higher-doped region ($x > 0.03$). (Note that the spin-Peierls state collapses at around $x = 0.03$\cite{8}.) We
expect that the white-noise limit $\lambda = 0$ describes the system at around $x \sim 0.5$, i.e., extremely high-doped region. As increasing $\lambda$, we approach the lower-doped region. Then as the calculations in this paper show, the extended low-energy excitations are enhanced as $\lambda \to \text{large}$. This fact obviously means the enhancement of the AF order in the lower-doped region. This result is qualitatively in good agreement with the experiments. It is interesting to give more quantitative argument about this problem. This can be a useful check on our results.
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