Article

Cohen-Macaulay and (S₂) Properties of the Second Power of Squarefree Monomial Ideals

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Abstract: We show that Cohen-Macaulay and (S₂) properties are equivalent for the second power of an edge ideal. We give an example of a Gorenstein squarefree monomial ideal I such that S/I² satisfies the Serre condition (S₂), but is not Cohen-Macaulay.

Keywords: Stanley-Reisner ideal; edge ideal; Cohen-Macaulay; (S₂) condition

1. Introduction

Let K be a fixed field. Let S = K[x₁,...,xₙ] be a polynomial ring with deg xᵢ = 1 for all i ∈ [n] = {1,2,...,n}. Let I be a squarefree monomial ideal.

For a Stanley-Reisner ring S/I, the Cohen-Macaulay and (S₂) properties are different in general. For instance, consider the Stanley-Reisner ring of a non-Cohen-Macaulay manifold, e.g., a torus, which satisfies the (S₂) condition. However, for some special classes of such rings, they are known to be equivalent. The quotient ring of the edge ideal of a very well-covered graph (see [1]) and a Stanley-Reisner ring with “large” multiplicity (see [2] for the precise statement) are such examples. What about the powers of squarefree monomial ideals?

As for the third and larger powers, the following is proven in [3]:

Theorem 1. Let I be a squarefree monomial ideal. Then, the following conditions are equivalent for a fixed integer m ≥ 3:

1. S/I is a complete intersection.
2. S/I^m is Cohen-Macaulay.
3. S/I^m satisfies the Serre condition (S₂).

Then, what about the second power of a squarefree monomial ideal? This is the theme of this article. If the second power I^2 is Cohen-Macaulay, I is not necessarily a complete intersection. Gorenstein ideals with height three give such examples.

In Section 3, we prove that the Cohen-Macaulay and (S₂) properties are equivalent for the second power of a squarefree monomial ideal generated in degree two:

Theorem 2. Let I be a squarefree monomial ideal generated in degree two. Then, the following conditions are equivalent:

1. S/I^2 is Cohen-Macaulay.
2. $S/I^2$ satisfies the Serre condition ($S_2$).

In Section 4, we first give an upper bound of the number of variables in terms of the dimension of $S/I$ when $I$ is a squarefree monomial ideal generated in degree two and $S/I^2$ has the Cohen-Macaulay (equivalently $(S_2)$) property. Using a computer, we classify squarefree monomial ideals $I$ generated in degree two with $\dim S/I \leq 4$ such that $S/I^2$ have the Cohen-Macaulay (equivalently $(S_2)$) property. Since not many examples of squarefree monomial ideals $I$ generated in degree two such that $S/I^2$ are Cohen-Macaulay are known, new examples might be useful. See [4,5] for the two- and three-dimensional cases, respectively, and [6,7] for the higher dimensional case. See also [6,8] for the fact that for a very well-covered graph $G$, the second power $I(G)^2$ is not Cohen-Macaulay if the edge ideal $I(G)$ of $G$ is not a complete intersection.

In Section 5, we give an example of a Gorenstein squarefree monomial ideal $I$ such that $S/I^2$ satisfies the Serre condition ($S_2$), but is not Cohen-Macaulay. Hence, the Cohen-Macaulay and $(S_2)$ properties are different for the second power in general.

2. Preliminaries

2.1. Stanley-Reisner Ideals

We recall some notation on simplicial complexes and their Stanley-Reisner ideals. We refer the reader to [9–11] for the detailed information.

Set $V = [n] = \{1, 2, \ldots, n\}$. A nonempty subset $\Delta$ of the power set $2^V$ of $V$ is called a simplicial complex on $V$ if the following two conditions are satisfied: (i) $\{v\} \in \Delta$ for all $v \in V$, and (ii) $F \subseteq \Delta$ implies $H \subseteq F$ imply $H \in \Delta$. An element $F \in \Delta$ is called a face of $\Delta$. The dimension of $F$, denoted by $\dim F$, is defined by $\dim F = |F| - 1$. The dimension of $\Delta$ is defined by $\dim \Delta = \max\{\dim F : F \in \Delta\}$. We call a maximal face of $\Delta$ a facet of $\Delta$. Let $\mathcal{F}(\Delta)$ denote the set of all facets of $\Delta$. We call $\Delta$ pure if all its facets have the same dimension. We call $\Delta$ connected if for any pair $(p, q)$, $p \neq q$, of vertices of $\Delta$, there is a chain $p = p_0, p_1, p_2, \ldots, p_k = q$ of vertices of $\Delta$ such that $\{p_{i-1}, p_i\} \in \Delta$ for $i = 1, 2, \ldots, k$.

The Stanley-Reisner ideal $I_\Delta$ of $\Delta$ is defined by:

$$I_\Delta = \langle x_{i_1}x_{i_2}\cdots x_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n, \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta \rangle.$$ 

The quotient ring $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$ is called the Stanley-Reisner ring of $\Delta$.

We say that $\Delta$ is a Cohen-Macaulay (resp. Gorenstein) complex if $K[\Delta]$ is a Cohen-Macaulay (resp. Gorenstein) ring. A Gorenstein complex $\Delta$ is called Gorenstein* if $x_i$ divides some minimal monomial generator of $I_\Delta$ for each $i$.

For a face $F \in \Delta$, the link and star of $F$ are defined by:

$$\text{link}_\Delta F = \{H \in \Delta : H \cup F \in \Delta, H \cap F = \emptyset\},$$

$$\text{star}_\Delta F = \{H \in \Delta : H \cup F \in \Delta\}.$$ 

The Stanley-Reisner ideal $I_\Delta$ of $\Delta$ has the minimal prime decomposition:

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F,$$

where $P_F = (x \in [n] \setminus F)$ for each $F \in \mathcal{F}(\Delta)$. We call $I_\Delta$ unmixed if all $P_F$ have the same height for $F \in \mathcal{F}(\Delta)$. Note that $\Delta$ is pure if and only if $I_\Delta$ is unmixed. We define the $\ell$th symbolic power of $I_\Delta$ by:

$$I_\Delta^{(\ell)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^\ell.$$
For a Noetherian ring \( A \), the following condition \((S_i)\) for \( i = 1, 2, \ldots \) is called Serre’s condition:

\[
(S_i) \text{ depth } A_P \geq \min \{ \text{height } P, i \} \quad \text{for all } P \in \text{Spec}(A).
\]

See [12] for more information for Stanley-Reisner rings satisfying Serre’s condition \((S_i)\).

To introduce a characterization of the \((S_2)\) property for the second symbolic power of a Stanley-Reisner ideal, we first define the diameter of a simplicial complex. Let \( \Delta \) be a pure simplicial complex. Then, the following conditions are equivalent:

1. \( S / I^{(2)}_\Delta \) satisfies \((S_2)\).
2. \( \text{diam}(\text{link}_\Delta F) \leq 2 \) for any face \( F \in \Delta \) with \( \dim \text{link}_\Delta F \geq 1 \).

2.2. Edge Ideals

Let \( G \) be a graph, which means a finite simple graph, which has no loops and multiple edges. We denote by \( V(G) \) (resp. \( E(G) \)) the set of vertices (resp. edges) of \( G \). We call \( F \subseteq V(G) \) an independent set of \( G \) if any \( e \in E(G) \) is not contained in \( F \). The independence complex \( \Delta(G) \) of \( G \) is defined by:

\[
\Delta(G) = \{ F \subseteq V(G) : e \not\subseteq F \text{ for any } e \in E(G) \},
\]

which is a simplicial complex on the vertex set \( V(G) \). We define \( \alpha(G) \) by:

\[
\alpha(G) = \dim \Delta(G) + 1.
\]

We define the neighbor set \( N_G(a) \) of a vertex \( a \) of \( G \) by:

\[
N_G(a) = \{ b \in V : ab \in E(G) \}.
\]

Set \( N_G[a] := \{ a \} \cup N_G(a) \), which is called the closed neighbor set of a vertex \( a \) of \( G \). For \( S \subseteq V(G) \), we denote by \( G \setminus S \) the induced subgraph on the vertex set \( V(G) \setminus S \). Set \( G_S := G \setminus N_G[S] \), where \( N_G[S] := \cup_{x \in S} N_G[x] \). If \( S \in \Delta(G) \), then:

\[
\text{link}_{\Delta(G)}(S) = \Delta(G_S).
\]

See ([11], Lemma 7.4.3). For \( ab \in E(G) \), set \( G_{ab} := G \setminus (N_G(a) \cup N_G(b)) \).

Set \( V(G) = \{ 1, \ldots, n \} \). Then, the edge ideal of \( G \), denoted by \( I(G) \), is a squarefree monomial ideal of \( S = K[x_1, \ldots, x_n] \) defined by:

\[
I(G) = (x_{i}x_{j} : \{ x_{i}, x_{j} \} \subseteq E(G)).
\]

Note that \( I(G) = I^{(2)}_\Delta(G) \). We call \( G \) well-covered (or unmixed) if \( I(G) \) is unmixed.

Theorem 4 ([13,14]). Let \( G \) be a graph. Then, the following conditions are equivalent:

1. \( G \) is triangle-free.
2. \( I(G)^{(2)} = I(G)^{2} \).
Theorem 5 ([15]). Let $G$ be a graph. Then, the following conditions are equivalent:

1. $G$ is triangle-free, and $I(G)$ is Gorenstein.
2. $S/I(G)^2$ is Cohen-Macaulay.

3. The Second Power of Edge Ideals

In this section, we show that the Cohen-Macaulay and $(S_2)$ properties are equivalent for the second power of an edge ideal.

Lemma 1. Let $G$ be a graph with $a(G) \geq 2$. The following conditions are equivalent:

1. $S/I(G)^{(2)}$ satisfies the $(S_2)$ property.
2. $G$ is a well-covered graph and satisfies $\text{diam}(G_F) \leq 2$ for all the independent sets $F$ of $G$ such that $|F| \leq a(G) - 2$.
3. $G_{ab}$ is well-covered and satisfies $a(G_{ab}) = a(G) - 1$ for all $ab \in E(G)$.

Proof. (1) $\Rightarrow$ (2): By [12], Theorem 8.3, $I(G)$ satisfies the $(S_2)$ property if so does $S/I(G)^{(2)}$. Using [12], Corollary 5.4, we obtain that $\Delta(G)$ is pure. This means that $G$ is well-covered, and thus:

$$\text{dim link}_{\Delta(G)}(F) = \text{dim } \Delta(G) - |F|$$

and $\text{link}_{\Delta(G)}(F) = \Delta(G_F)$. The result is implied by Theorem 3.

(2) $\Rightarrow$ (3): For all $ab \in E(G)$, we have:

$$a(G_{ab}) \leq a(G) - 1.$$

Let $F$ be an independent set of $G_{ab}$. If $|F| < a(G) - 1$, then $|F| \leq a(G) - 2$. Recall that $G_{ab} = G' \setminus (N_G(a) \cup N_G(b))$ and $F \subseteq V(G_{ab})$. This implies that $a, b \notin N_G[F]$. Hence, we obtain that $\{a, b\}$ is an edge of $G_F$. In other words, $\{a, b\}$ is an independent set of $G_{ab}$. By the assumption, $\text{diam}(G_F) \leq 2$, there is a vertex $c \in V(G_F)$ such that $\{a, c\}, \{c, b\}$ are independent sets of $G_F$. Thus, $ac, bc \notin E(G_F)$. Hence, $c \in V(G_{ab})$. Therefore, $F \cup \{c\}$ is an independent set of $G_{ab}$. Then, $G_{ab}$ is well-covered, and moreover, $a(G_{ab}) = a(G) - 1$.

(3) $\Rightarrow$ (2): By [15], Lemma 4.1 (2), $G$ is a well-covered graph. We will prove that $\text{diam}(G_F) \leq 2$ for all independent set $F$ with $|F| \leq a(G) - 2$ by induction on $a(G)$.

If $a(G) = 2$, then we must prove $\text{diam}(G) \leq 2$. For all $a, b \in V(G)$, we assume $\{a, b\} \notin \Delta(G)$. Then, $ab \in E(G)$. By the assumption, $a(G_{ab}) = a(G) - 1 = 1 > 0$. Therefore, we can take a vertex $c$ in $G_{ab}$, and thus, $ac, bc \notin E(G)$. Hence, $\{a, c\}, \{b, c\} \in \Delta(G)$. Therefore, we conclude that $\text{diam}(G) \leq 2$.

Let $a(G) > 2$, and suppose that the assertion is true for all graphs $G'$ with the same structure as $G$ satisfying the condition “$G_{ab}$ is well-covered and satisfies $a(G_{ab}) = a(G) - 1$ for all $ab \in E(G')$” with $a(G') < a(G)$. For all independent set $F$ of $G$ such that $|F| \leq a(G) - 2$, we divide the proof into the following two cases:

Case 1: $F = \emptyset$. In this case, we need to prove that $\text{diam}(G) \leq 2$. In fact, using the same argument as above, we obtain $\text{diam}(G) \leq 2$.

Case 2: $F \neq \emptyset$. Let $x \in F$. Recall that $G$ is a well-covered graph, and thus, we have $a(G_x) = a(G) - 1$. Hence, $|F \setminus \{x\}| = |F| - 1 \leq a(G) - 3 = a(G_x) - 2$. Note that for all $ab \in E(G_x)$, we have that $G_{ab}$ and $G_{ab} \times x$ are two induced subgraphs of $G$ on vertex set $V(G) \setminus (N_G(x) \cup N_G(a) \cup N_G(b))$. Thus, $(G_x)_{ab} = (G_{ab})_x$. By the assumption and [15], Lemma 4.1 (1), $(G_{ab})_x$ is a well-covered graph with $a((G_{ab})_x) = a(G_{ab}) - 1$. Therefore, $(G_{ab})_x$ is also a well-covered graph. Moreover,

$$a((G_x)_{ab}) = a((G_{ab})_x) = a(G_{ab}) - 1 = a(G) - 2 = a(G_x) - 1.$$
Thus, \( G_s \) has the same structure as \( G \) satisfying the condition “\( G_{ab} \) is well-covered and satisfies \( \alpha(G_{ab}) = \alpha(G) - 1 \) for all \( ab \in E(G) \)” with \( \alpha(G_s) < \alpha(G) \). By the induction hypothesis, we obtain \( \text{diam} \Delta((G_s)_{F_x \setminus \{x\}}) \leq 2 \). Note that:

\[
(G_s)_{F_x \setminus \{x\}} = G_x \setminus N_G[F \setminus \{x\}] = G \setminus (N_G[x] \cup N_G[F \setminus \{x\}]).
\]

Therefore, \( \Delta(G_F) = \Delta((G_s)_{F_x \setminus \{x\}}). \) Therefore, we conclude that \( \text{diam} \Delta(G_F) \leq 2 \). \( \square \)

Then, we get the following theorem.

**Theorem 6.** Let \( G \) be a graph. The following conditions are equivalent:

1. \( S/I(G)^2 \) satisfies the \((S_2)\) property,
2. \( S/I(G)^2 \) is Cohen-Macaulay,
3. \( G \) is triangle-free, and \( G_{ab} \) is a well-covered graph with \( \alpha(G_{ab}) = \alpha(G) - 1 \) for all \( ab \in E(G) \).

**Proof.** By the statements of Conditions (1), (2) and (3), without loss of generality, we can assume that \( G \) contains no isolated vertices.

(2) \(\Leftrightarrow\) (3): By [15], Theorem 4.4, \( S/I(G)^2 \) is Cohen-Macaulay if and only if \( G \) is triangle-free and in \( W_2 \), which is a well-covered graph such that the removal of any vertex of \( G \) leaves a well-covered graph with the same independence number as \( G \). By [15], Lemma 4.2, this is equivalent to the condition that \( G \) is triangle-free and \( G_{ab} \) is a well-covered graph with \( \alpha(G_{ab}) = \alpha(G) - 1 \) for all \( ab \in E(G) \).

(2) \(\Rightarrow\) (1): It is obvious.

(1) \(\Rightarrow\) (3): If \( \alpha(G) = 1 \), then \( G \) is a complete graph. By the assumption, \( G \) is one edge. Therefore, the statement holds true. Now, we assume \( \alpha(G) \geq 2 \). We know that \( S/I(G)^2 \) satisfies that \((S_2)\) property if and only if \( S/I(G)^2 \) satisfies the \((S_2)\) property and \( I(G)^2 \) has no embedded associated prime, which means \( I(G)^2 = I(G)^{(2)} \). By Theorem 4 and Lemma 1, \( G \) is triangle-free, and \( G_{ab} \) is well-covered with \( \alpha(G_{ab}) = \alpha(G) - 1 \) for all \( ab \in E(G) \). \( \square \)

**Question.** If \( S/I(G)^{(2)} \) satisfies the \((S_2)\) property, then is it Cohen-Macaulay?

The question is affirmative if \( G \) is a triangle-free graph by Theorems 4 and 6.

### 4. Classification

The purpose of the section is to classify all graphs \( G \) such that \( S/I(G)^2 \) is Cohen-Macaulay with dimension less than five. First, we give an upper bound of the number of vertices of a graph \( G \) such that \( S/I(G)^2 \) is Cohen-Macaulay.

#### 4.1. Upper Bound of the Number of Vertices

**Theorem 7 (Upper bound).** Let \( G \) be a graph with the vertex set \([n]\). Suppose \( G \) has no isolate vertex. If \( S/I(G)^2 \) is \( d \)-dimensional Cohen-Macaulay, where \( d \geq 3 \), then we have \( n \leq \frac{d^2+3d-2}{2} \).

**Proof.** We prove this by induction on \( d \). For \( d = 3 \), we have \( n \leq 8 \) by [5] (see Proposition 3). Set \( N(d) = \frac{d^2+3d-2}{2} \). Let \( n \) be the number of vertices of \( G \) such that \( S/I(G)^2 \) is \( d \)-dimensional and Cohen-Macaulay. Let \( i \in [n] \). Then, we have\( n = |V(\text{star}_G\{i\})| + |[n] \setminus V(\text{star}_G\{i\})|\). Since \( G \) is triangle-free by Theorem 5, an edge among \( \{i, p\}, \{i, q\} \) and \( \{p, q\} \) belongs to \( \Delta(G) \) for any \( p, q \in ([n] \setminus V(\text{star}_G\{i\})) \), where \( p \neq q \). By the definition of \( \text{star}_G\{i\} \), we have \( \{i, p\}, \{i, q\} \notin \Delta(G) \). Then, we have \( \{p, q\} \in \Delta(G) \). By the fact that \( I(G) \) is generated in degree two, all minimal non-faces of \( \Delta(G) \) have cardinality two. Now, we know that \( \{p, q\} \in \Delta(G) \) for any \( p, q \in ([n] \setminus V(\text{star}_G\{i\})) \); hence, we have\( |n| \setminus V(\text{star}_G\{i\})| \leq d \). Since \( \Delta(G) \) is Gorenstein, so is \( \text{link}_{\Delta(G)}\{i\} \) by [10], Theorem
5.1. By Theorem 5, \( I_{\text{link}_{\Delta}(G)}^2 \) is Cohen-Macaulay. Hence, \( |V(\text{star}_{\Delta(G)} \{i\})| = |V(\text{link}_{\Delta(G)} \{i\})| + 1 \leq N(d - 1) + 1 \) by the induction hypothesis. Therefore, \( n \leq N(d - 1) + 1 = (d - 1)^2 + 3(d - 1) - 2 + 1 = \frac{d^2 + 3d - 2}{2} = N(d) \). □

4.2. Classification

In this subsection, we classify all graphs \( G \) such that \( S/I(G)^2 \) is Cohen-Macaulay with dimension less than five.

**Proposition 1.** (One-dimensional case) Let \( G \) be a graph with the vertex set \([n]\). Suppose \( G \) has no isolate vertex. Then, \( S/I(G)^2 \) is one-dimensional Cohen-Macaulay if and only if \( n = 2 \) and \( I(G) = (x_1x_2) \).

**Proposition 2** ([4]). (Two-dimensional case) Let \( G \) be a graph with the vertex set \([n]\). Suppose \( G \) has no isolate vertex. Then, \( S/I(G)^2 \) is two-dimensional Cohen-Macaulay if and only if \( I(G) \) is one of the following up to the permutation of variables:

1. If \( n = 4 \), then \((x_1x_3, x_2x_4)\).
2. If \( n = 5 \), then \((x_1x_3, x_1x_4, x_2x_3, x_2x_5, x_4x_5)\).

**Proposition 3** ([5]). (Three-dimensional case) Let \( G \) be a graph with the vertex set \([n]\). Suppose \( G \) has no isolate vertex. Then, \( S/I(G)^2 \) is three-dimensional Cohen-Macaulay if and only if \( I(G) \) is one of the following up to the permutation of variables:

1. If \( n = 6 \), then \((x_1x_4, x_2x_5, x_3x_6)\).
2. If \( n = 7 \), then \((x_1x_5, x_1x_6, x_2x_3, x_2x_7, x_3x_4, x_6x_7)\).
3. If \( n = 8 \), then \((x_1x_2, x_1x_5, x_1x_8, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8)\).

Using a computer with Nauty [16] and CoCoA [17], we classify four-dimensional case: By Theorem 7, it is enough to search for them up to \( n = 13 \).

**Theorem 8.** (Four-dimensional case) Let \( G \) be a graph with the vertex set \([n]\). Suppose \( G \) has no isolate vertex. Then, \( S/I(G)^2 \) is four-dimensional Cohen-Macaulay if and only if \( I(G) \) is one of the following up to the permutation of variables:

1. If \( n = 8 \), then \((x_1x_5, x_2x_6, x_3x_7, x_4x_8)\).
2. If \( n = 9 \), then \((x_1x_5, x_2x_6, x_3x_7, x_1x_8, x_4x_8, x_4x_9, x_5x_9)\).
3. If \( n = 10 \), then
   (a) \((x_1x_5, x_2x_6, x_3x_7, x_1x_8, x_4x_8, x_2x_9, x_4x_9, x_5x_9, x_4x_{10}, x_5x_{10}, x_6x_{10})\).
   (b) \((x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_3x_8, x_5x_8, x_2x_9, x_4x_9, x_4x_{10}, x_5x_{10})\).
4. If \( n = 11 \), then
   (a) \((x_1x_5, x_2x_6, x_3x_7, x_1x_8, x_4x_8, x_2x_9, x_4x_9, x_3x_{10}, x_4x_{10}, x_5x_{10}, x_6x_{10}, x_4x_{11}, x_5x_{11}, x_6x_{11}, x_7x_{11})\).
   (b) \((x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_3x_8, x_5x_8, x_2x_9, x_4x_9, x_1x_{10}, x_4x_{10}, x_6x_{10}, x_4x_{11}, x_5x_{11}, x_6x_{11}, x_7x_{11})\).
5. If \( n = 12 \), then
   \((x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_2x_8, x_4x_8, x_2x_9, x_3x_9, x_5x_9, x_1x_{10}, x_4x_{10}, x_6x_{10}, x_4x_{11}, x_5x_{11}, x_6x_{11}, x_7x_{11}, x_3x_{12}, x_5x_{12}, x_8x_{12})\).
6. If \( n = 13 \), then
   \((x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_2x_8, x_4x_8, x_2x_9, x_3x_9, x_3x_9, x_5x_9, x_1x_{10}, x_3x_{10}, x_4x_{10}, x_6x_{10}, x_3x_{11}, x_5x_{11}, x_6x_{11}, x_8x_{11}, x_2x_{12}, x_4x_{12}, x_5x_{12}, x_7x_{12}, x_4x_{13}, x_6x_{13}, x_7x_{13}, x_9x_{13})\).
See [18] for the concrete algorithm we used. By Theorem 6 in this case, the Cohen-Macaulay property is equivalent to the $(S_2)$ property, which is independent of the base field $K$.

5. Example

In this section, we give an example of a Gorenstein squarefree monomial ideal $I$ such that $S/I^2$ satisfies the Serre condition $(S_2)$, but it is not Cohen-Macaulay.

The Cohen-Macaulay property of $I^2$ implies the “Gorenstein” property of $I_\Delta$. More precisely:

**Theorem 9 ([7]).** Let $\Delta$ be a simplicial complex on $[n]$. Suppose that $S/I_\Delta^2$ is Cohen-Macaulay over any field $K$. Then, $\Delta$ is Gorenstein for any field $K$.

In [7], the authors asked the following question:

**Question.** Let $\Delta$ be a simplicial complex on $[n]$. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring for a fixed field $K$. Suppose $\Delta$ satisfies the following conditions:

1. $\Delta$ is Gorenstein.
2. $S/I_\Delta^2$ satisfies the Serre condition $(S_2)$.

Then, is it true that $S/I_\Delta^2$ is Cohen-Macaulay?

Using a list in [19] and CoCoA, we have the following counter-example:

**Example 1.** Let $K$ be a field of characteristic zero. Set:

$$I_\Delta = \langle x_1x_{10}, x_3x_9, x_2x_9, x_7x_8, x_2x_8, x_4x_7, x_5x_6, x_3x_6, x_4x_5, x_6x_8x_{10}, x_2x_5x_{10}, x_1x_4x_9, x_1x_3x_7 \rangle.$$

Then, the following conditions hold:

1. $\Delta$ is Gorenstein.
2. $S/I_\Delta^2$ satisfies the Serre condition $(S_2)$.
3. $S/I_\Delta^2$ is not Cohen-Macaulay.

We explain how to find the example. The manifold page of Lutz [19] gives a classification of all triangulations $\Delta$ of the three-sphere with 10 vertices, which shows that there are 247,882 types. Using Theorem 3, we checked the Serre condition $(S_2)$ for them, and there were only nine types such that $S/I_\Delta^2$ satisfies the Serre condition $(S_2)$. Among the nine types, there was only one simplicial complex $\Delta$ such that $S/I_\Delta^2$ is not Cohen-Macaulay, which is the above example. Note that a triangulation $\Delta$ of a sphere is always Gorenstein. See [18] for more information.

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