THE DETERMINATION OF THE INTERNAL STRUCTURE OF THE SUN BY THE DENSITY DISTRIBUTION

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ABSTRACT

This paper examines a number of analytic models which can describe the gravitationally stabilized fusion reactor of the Sun. An analytical multiparameter model is shown to reproduce various structure parameters such as density, mass, pressure and temperature throughout the solar core, which have been obtained by solving numerically the system of solar structure differential equations. For simplicity, numerical and analytical results are discussed in detail for a two-parameter model.

1 INTRODUCTION

The numerical approach to the problem of studying solar structure is to go for numerical solutions of the underlying system of differential equations (A. Noels, R. Papy, and F. Remy 1993). Even for a simple main-sequence star in hydrostatic equilibrium, like the Sun, at least four nonlinear differential equations are to be dealt with to obtain a detailed picture of the run of physical variables throughout the star. Another approach is to search for analytical solutions of these differential equations under justified physical simplifications (Haubold and Mathai 1992).

In this paper we start with numerical data given in Sears (1964) for various analytical models which will fit the data on density in the gravitationally
stabilized solar core. Let $r$ be the distance from the center of the Sun to an arbitrary point in its interior, $R$ the solar radius, $\rho(r)$ the density at $r$, $\rho_c$ the density at the center and $y = r/R$. For the range $0 \leq y \leq 0.3$ consider the following models for the density $\rho(r)$.

\[
u = \frac{\rho(r)}{\rho_c} = 1 - 4.94y + 6.67y^2 - 2.73y^3 \quad (1.1)
\]

\[
u = 1 - 4y + 2y^2 + 2y^3 - y^4 \quad (1.2)
\]

\[
u = (1 - \sqrt{y})(1 - y^3)^{64} \quad (1.3)
\]

\[
u = (1 - y^{3/2})^{16} \quad (1.4)
\]

\[
u = (1 - \sqrt{y})(1 - y^3)^{64}(1 - y) \quad (1.5)
\]

\[
u = (1 - y^{1.48})^{14} \quad (1.6)
\]

\[
u = (1 - y^{1.28})^{10} \quad (1.7)
\]

\[
u = (1 - y^{1.28})^{10} \quad (1.8)
\]

The following table gives $y$, the numerically obtained data on $u = \rho(r)/\rho_c$ from Sear’s (1964) and the estimated values under models (1.1) to (1.8).

**Table 1 density distribution**

| $y = \frac{r}{R}$ | $\frac{\rho(r)}{\rho_c}$ (Sears, 1964) | Model (1.1) | Model (1.2) | Model (1.3) | Model (1.4) |
|-------------------|--------------------------------------|-------------|-------------|-------------|-------------|
| 0.0864            | 0.6519                               | 0.6213      | 0.6720      | 0.6795      | 0.6626      |
| 0.1153            | 0.5253                               | 0.5149      | 0.5690      | 0.5987      | 0.5283      |
| 0.1441            | 0.3856                               | 0.4185      | 0.4710      | 0.5722      | 0.4065      |
| 0.1873            | 0.2810                               | 0.2908      | 0.3340      | 0.3720      | 0.2588      |
| 0.2161            | 0.1994                               | 0.2164      | 0.2490      | 0.2796      | 0.1837      |
| 0.2450            | 0.1424                               | 0.1499      | 0.1700      | 0.1957      | 0.1264      |
| 0.2882            | 0.0962                               | 0.0649      | 0.0560      | 0.0985      | 0.0679      |
Table 1 continued

| Model | Model | Model | Model |
|-------|-------|-------|-------|
| (1.5) | (1.6) | (1.7) | (1.8) |
| 0.6208 | 0.6849 | 0.7037 | 0.6418 |
| 0.5297 | 0.5573 | 0.5811 | 0.5229 |
| 0.4435 | 0.4404 | 0.4669 | 0.4179 |
| 0.3023 | 0.2928 | 0.3196 | 0.2885 |
| 0.2192 | 0.2159 | 0.2409 | 0.2202 |
| 0.1478 | 0.1551 | 0.1772 | 0.1649 |
| 0.0701 | 0.0885 | 0.1053 | 0.1033 |

Model (1.1) results from a least square fit of a third degree polynomial to the data of Sears (1964). Model (1.2) comes from successive eliminations. Even though these models can adequately describe the data in the range \(0 \leq y \leq 0.3\) the equation \(u=0\) has real roots in \((0,1)\) besides the root at \(y = 1\). Hence these cannot be used as models for the density in \([0,1]\). From models (1.3) to (1.8) it is evident that a model of the type

\[
    u = (1 - y^{a_1})^{b_1}(1 - y^{a_2})^{b_2} \ldots (1 - y^{a_k})^{b_k},
\]

where \(a_i > 0, b_i = 0, 1, 2, \ldots, i = 1, \ldots, k\) can be an excellent fit to the data. For simplicity we will study a model of the type

\[
    u = (1 - y^{\delta})^{\gamma},
\]

where \(\delta > 0\) and \(\gamma\) a positive integer. Illustration will be given for the case \(\delta = 1.28\) and \(\gamma = 10\).

2 **A Solar Model**

We assume that rotation and magnetic fields have no impact on the internal structure of the Sun and the gravitational force directed inward and the gas pressure force directed outward keep the gas sphere in hydrostatic equilibrium. The ratio of gas pressure and radiation pressure increases towards the center of the Sun but never exceeds \(10^{-3}\). Thus radiation pressure can be
neglected for the purpose of considering hydrostatic equilibrium. Assume that the density \( \rho(r) \) varies from \( \rho_c \) at \( r = 0 \) to zero at \( r = R \) in the following fashion:

\[
\rho(r) = \rho_c [1 - \left( \frac{r}{R} \right)^\delta]^\gamma, \quad \delta > 0, \quad \gamma \text{ a positive integer.} \tag{2.1}
\]

Then the distribution of mass in terms of radius is given by

\[
\frac{dM(r)}{dr} = 4\pi r^2 \rho(r),
\]

so that

\[
M(r) = 4\pi \int_0^r dt t^2 \rho(t) = 4\pi \rho_c \int_0^r dt t^2 [1 - (\frac{t}{R})^\delta]^\gamma = 4\pi \rho_c \sum_{m=0}^\gamma (-1)^m \binom{\gamma}{m} R^3 \int_0^{r/R} dzz^{m\delta+2} = \frac{4\pi \rho_c R^3 (\frac{r}{R})^3}{3} \, _2F_1(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; (\frac{r}{R})^\delta), \tag{2.2}
\]

where \(_2F_1\) is a Gauss’ hypergeometric function, see for example, Mathai (1993). \( M(r) \) gives the total mass at \( r = R \), that is,

\[
M(R) = \frac{4\pi \rho_c R^3}{3} \, _2F_1(-\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; 1) = \frac{4\pi \rho_c R^3}{3} \, \frac{\gamma!}{(\frac{3}{\delta} + 1)(\frac{3}{\delta} + 2) \cdots (\frac{3}{\delta} + \gamma)}, \tag{2.3}
\]

which is evaluated by using the formula

\[
_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{2.4}
\]

From (2.2) and (2.3) we can get a formula for estimating \( \rho_c \) also. That is,

\[
\rho_c = \frac{3M(R) \left( \frac{3}{\delta} + 1 \right) \left( \frac{3}{\delta} + 2 \right) \cdots \left( \frac{3}{\delta} + \gamma \right)}{4\pi R^3 \gamma!}. \tag{2.5}
\]

The numerical values for the Sun from Sears (1964) are

\[
\begin{align*}
\rho_c &= 158 \text{ g cm}^{-3} \\
M(R) &= 1.991 \times 10^{33} \text{ g} \\
R &= 6.96 \times 10^{10} \text{ cm}.
\end{align*} \tag{2.6}
\]
Substituting these in (2.5) we have
\[
\frac{(3\delta + 1)(3\delta + 2) \cdots (3\delta + \gamma)}{\gamma!} = 112.08. \tag{2.7}
\]
Hence a pair of \((\delta, \gamma)\) which satisfy (2.7) is a good choice. We have many choices for \((\delta, \gamma)\) even if \(\gamma\) is kept as a positive integer. Keeping \(\gamma\) a positive integer, for simplicity, it is found that \((\delta = 1.28, \gamma = 10)\) is a convenient choice in the sense that these values give good estimates for \(M(R)\) and \(R\) also, close to the realistic values and the estimates for the density are not far off.

Note that (2.7) is obtained by using the readings in (2.6). From (2.7) one can get a set of positive values for \(\delta\) and \(\gamma\). These are given here for some integer values of \(\gamma\):

| \(\gamma\) | 2   | 3   | 4   | 5   | 6   | 7   |
|-------|-----|-----|-----|-----|-----|-----|
| \(\delta\) | 0.2225 | 0.4412 | 0.6265 | 0.7802 | 0.9098 | 1.0211 |
| 8     | 1.1182 | 1.2043 | 1.2814 | 1.3512 | 1.4149 | 1.4735 | 1.5276 |
| 9     | 1.5780 | 1.6251 | 1.6692 | 1.7108 | 1.75  | 1.7872 |

From (2.2) and (2.3) the relative mass is given by
\[
\frac{M(r)}{M(R)} = \frac{(3\delta + 1)(3\delta + 2) \cdots (3\delta + \gamma)}{\gamma!}\left(\frac{r}{R}\right)^{3\delta} \frac{\gamma!}{\gamma} \frac{\gamma}{R}^\delta \Gamma\left(-\gamma, \frac{3\delta}{\gamma}; \frac{3\delta}{\gamma} + 1; \left(\frac{r}{R}\right)^{\delta}\right). \tag{2.8}
\]
This is computed for \(\delta = 1.28\), \(\gamma = 10\) and compared with Sear's (1964) numerical results in the following table 3.
Table 3 mass distribution

\[ y = \frac{r}{R} \]

\[ \frac{M(r)}{M(R)} \] (analytic model) \quad \frac{M(r)}{M(R)} (Sears 1964)

| \( r \) | \( \frac{M(r)}{M(R)} \) | \( \frac{M(r)}{M(R)} \) |
|------|----------------|----------------|
| 0.0864 | 0.0533 | 0.05 |
| 0.1153 | 0.1106 | 0.1 |
| 0.1441 | 0.1866 | 0.2 |
| 0.1873 | 0.3249 | 0.3 |
| 0.2161 | 0.4241 | 0.4 |
| 0.2450 | 0.5225 | 0.5 |
| 0.2882 | 0.6576 | 0.6 |

Assuming that the pressure \( P(r) \) at the center of the Sun is \( P_c \) and that at the surface is zero, we have

\[ P(r) = P_c - G \int_0^r dt \frac{\rho(t)}{t^2} m(t) \]  

where \( G \) is the gravitational constant. That is,

\[ P(r) = P_c - \frac{4\pi G}{\delta} \rho_c^2 R \sum_{m=0}^{\gamma} \frac{(-\gamma)_m}{m!} \int_0^r dt \left( \frac{t}{R} \right)^{m\delta+1} \left[ 1 - \left( \frac{t}{R} \right)^\delta \right]^\gamma \]

\[ = P_c - \frac{4\pi G}{\delta^2} \rho_c^2 R^2 \sum_{m=0}^{\gamma} \frac{(-\gamma)_m}{m!} \left( \frac{3}{\delta} + m \right) \left( \frac{2}{\delta} + m \right) \times \]

\[ 2F_1(-\gamma, \frac{2}{\delta} + m; \frac{2}{\delta} + m + 1; \left( \frac{r}{R} \right)^\delta). \]  \hspace{1cm} (2.10)

Assuming that the pressure at the surface is zero, that is \( P(R) = 0 \), we obtain

\[ P_c = \frac{4\pi G}{\delta^2} \rho_c^2 R^2 \sum_{m=0}^{\gamma} \frac{(-\gamma)_m}{m!} \frac{1}{(\frac{3}{\delta} + m)(\frac{2}{\delta} + m)} \times \]

\[ \gamma! \left( \frac{2}{\delta} + m + 1 \right) \left( \frac{2}{\delta} + m + 2 \right) \ldots \left( \frac{2}{\delta} + m + \gamma \right). \]  \hspace{1cm} (2.11)

Then

\[ P(r) = \frac{4\pi G}{\delta^2} \rho_c^2 R^2 \sum_{m=0}^{\gamma} \frac{(-\gamma)_m}{m!} \frac{1}{(\frac{3}{\delta} + m)(\frac{2}{\delta} + m)} \times \]
\[
\begin{align*}
\left[ \frac{\gamma!}{(\frac{2}{\delta}+m+1)\ldots(\frac{2}{\delta}+m+\gamma)} - \frac{r}{R} \right]^{m\delta+2} \times \\
2F_1(-\gamma, \frac{2}{\delta}+m; \frac{2}{\delta}+m+1; \left(\frac{r}{R}\right)^{\delta})
\end{align*}
\] 

(2.12)

The above expansion can also be written as a double sum. In this case

\[
P(r) = P_c - \frac{4\pi G}{6} \rho_c^2 r^2 \sum_{m=0}^{\gamma} \sum_{n=0}^{(-\gamma)m} \frac{(-\gamma)_m (-\gamma)_n}{m! n!} \times \\
\frac{\left(\frac{2}{\delta}\right)_m \left(\frac{3}{\delta}\right)_m \left(\frac{2}{\delta}\right)_{m+n}}{\left(\frac{2}{\delta}+1\right)_m \left(\frac{3}{\delta}+1\right)_m \left(\frac{2}{\delta}+1\right)_{m+n}} \left[\left(\frac{r}{R}\right)^{\delta}\right]^m \left[\left(\frac{r}{R}\right)^{\delta}\right]^n.
\] 

(2.13)

This double sum can be written in terms of a Kampé de Fériet’s function. That is,

\[
P(r) = P_c - \frac{2}{3} \pi G \rho_c^2 R^2 F^{1:3:1}_{1:2:0} \left[\frac{\left(\frac{2}{\delta}\right)^{\delta}}{\left(\frac{3}{\delta}\right)^{\delta}}\right] \left[\frac{\left(\frac{2}{\delta}+\gamma\right)^{\delta}}{\left(\frac{3}{\delta}+\gamma\right)^{\delta}}\right] \left[\frac{\left(\frac{2}{\delta}+1\right)^{\delta}}{\left(\frac{3}{\delta}+1\right)^{\delta}}\right] \left[\frac{\left(\frac{2}{\delta}+\gamma\right)^{\delta}}{\left(\frac{3}{\delta}+1\right)^{\delta}}\right].
\] 

(2.14)

and then assuming that \(P(R)=0\),

\[
P_c = \frac{2}{3} \pi G \rho_c^2 R^2 F^{1:3:1}_{1:2:0} \left[\frac{1}{1}\right] \left[\frac{\left(\frac{2}{\delta}\right)^{\delta}}{\left(\frac{3}{\delta}+\gamma\right)^{\delta}}\right] \left[\frac{\left(\frac{2}{\delta}+1\right)^{\delta}}{\left(\frac{3}{\delta}+1\right)^{\delta}}\right] \left[\frac{\left(\frac{2}{\delta}+\gamma\right)^{\delta}}{\left(\frac{3}{\delta}+1\right)^{\delta}}\right].
\] 

(2.15)

where the Kampé de Fériet’s series is defined by the following:

\[
F^{p;q;k}_{r;m:n} \left[\frac{(a_p);(b_q);(c_k)}{(a_r);(b_m);(c_n)}\right] = \\
= \sum_{m'=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\Pi_{j=1}^{p} (a_j)_{m'+n'}}{\Pi_{j=1}^{m} (a_j)_{m'+n'}} \frac{\Pi_{j=1}^{q} (b_j)_{m'}}{\Pi_{j=1}^{m} (b_j)_{m'}} \frac{\Pi_{j=1}^{k} (c_j)_{n'}}{\Pi_{j=1}^{m} (c_j)_{n'}} 2^{m'n'} y^{n'}
\] 

(2.16)

For a discussion of Kampé de Fériet’s function see Srivastava and Karlsson (1985). Note that (2.14) is a polynomial since \(\gamma\) is a positive integer and hence convergence conditions do not arise in (2.14). The form in (2.10) is the most appropriate for computational purposes and the computations given later are done by using (2.10).

The temperature is given by the equation of state of the perfect gas

\[
T(r) = \frac{\mu}{kN_A} P(r) \rho(r),
\] 

(2.17)
where \( \mu \) is the mean molecular weight, \( k \) is Boltzmann’s constant and \( N_A \) is Avogadro’s number. Let
\[
g(r) = \frac{1}{\delta^2} \sum_{m=0}^{\gamma} \frac{(-\gamma)_m}{m!} \frac{1}{\left(\frac{2}{\delta} + m\right)\left(\frac{2}{\delta} + m\right)} \times \\
\left[ \frac{\gamma!}{\left(\frac{2}{\delta} + m + 1\right) \cdots \left(\frac{2}{\delta} + m + \gamma\right)} \right] \\
-\left(\frac{r}{R}\right)^{m\delta+2} {}_2F_1(-\gamma; \frac{2}{\delta} + m; \frac{2}{\delta} + m + 1; \left(\frac{r}{R}\right)^{\gamma})
\]
(2.18)

Then
\[
P(r) = 4\pi G \rho_c^2 R^2 g(r)
\]
(2.19)
and the temperature is then given by
\[
T(r) = \frac{\mu}{kN_A} \frac{4\pi G \rho_c^2 R^2}{v} \frac{g(r)}{1 - \left(\frac{r}{R}\right)^{\delta}}. 
\]
(2.20)

In the following table we tabulate \( g(r) \) for \( \delta = 1.28 \) and \( \gamma = 10 \) from which we can compute the pressure for the solar core of the sum ranging from \( 0 \leq \frac{r}{R} \leq 0.3 \).

Table 4 pressure = const. \( \times g(r) \), temperature = const. \( \times g(r)/u \),
\[
u = \left[1 - \left(\frac{r}{R}\right)^{\delta}\right]^{\gamma}, \text{ for } \delta = 1.28 \text{ and } \gamma = 10
\]

| \( y = \frac{r}{R} \) | \( g(r) \) | \( v \) | \( \frac{g(r)}{u} \) |
|---|---|---|---|
| 0.0864 | 1.7911 \times 10^{-3} | 0.6418 | 2.7907 \times 10^{-3} |
| 0.1153 | 1.4035 \times 10^{-3} | 0.5228 | 2.6842 \times 10^{-3} |
| 0.1441 | 1.0546 \times 10^{-3} | 0.4179 | 2.5234 \times 10^{-3} |
| 0.1873 | 6.4117 \times 10^{-4} | 0.2884 | 2.2227 \times 10^{-3} |
| 0.2161 | 4.4168 \times 10^{-4} | 0.2202 | 2.0055 \times 10^{-3} |
| 0.2450 | 2.9482 \times 10^{-4} | 0.1649 | 1.7876 \times 10^{-3} |
| 0.2882 | 1.5279 \times 10^{-4} | 0.1033 | 1.4788 \times 10^{-3} |
Hence the proportional decrease in temperature is:

Table 5 temperature distribution

| Sears (1964) | \( \frac{g(r)}{[\rho(r)/\rho_c]^{1/2}} \) | \( \frac{g(r)}{[\rho(r)/\rho_c]^{1/4}} \) |
|--------------|---------------------------------|---------------------------------|
| 0.8789       | 0.9428                          | 0.8987                          |
| 0.9275       | 0.9316                          | 0.8819                          |
| 0.8828       | 0.9219                          | 0.8385                          |
| 0.8938       | 0.9612                          | 0.8956                          |
| 0.8911       | 0.9709                          | 0.8992                          |
| 0.9000       | 0.7644                          | 0.8656                          |

Table 6 pressure = const. \( \times g(r) \), for \( \delta = 1.28 \) and \( \gamma = 10 \)

| \( y = \frac{r}{R} \) | \( g(r) \) | \( g(r) \) |
|---------------------|---------|---------|
| 0.0864              | 3.1681 \times 10^{-3} | |
| 0.1153              | 2.7142 \times 10^{-3} | |
| 0.1441              | 2.2664 \times 10^{-3} | |
| 0.1873              | 1.7286 \times 10^{-3} | |
| 0.2161              | 1.4425 \times 10^{-3} | |
| 0.2450              | 1.2012 \times 10^{-3} | |
| 0.2882              | 0.9129 \times 10^{-3} | |

3 ENERGY GENERATION

From Mathai and Haubold (1988) we have the net release of thermonuclear energy per gram per second given by

\[
\epsilon[\rho(r), T(r)] = \epsilon_0[\rho(r)]^n[T(r)]^m
\]

\[
= \epsilon_0[\rho(r)]^n[\frac{\mu}{kN_A \rho(r)}]^m
\]

\[
= \epsilon_0(\frac{\mu}{kN_A})^m[\rho(r)]^{n-m}[P(r)]^m
\] (3.1)

for the parameter \( n \) and \( m \) determined by the proton-proton chain and the CNO cycle. Since the \( g(r) \) of table 5 is proportional to \( P(r) \) we have

\[
\epsilon[\rho(r), T(r)] = k_1[\rho(r)]^{n-m}[g(r)]^m,
\] (3.2)
where \( k_1 \) is a constant. The equation of energy conservation is

\[
\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \epsilon[\rho(r), T(r)],
\]

where \( L(r) \) denotes the luminosity of the Sun. Hence the total luminosity is given by

\[
L = 4\pi \int_0^R drr^2 \rho(r) \epsilon[\rho(r), T(r)]
\]

for the model in (3.2). Substituting for \( P(r) \) from (2.10) one has

\[
L = (4\pi \epsilon_0)^{m+1} \left( \frac{GR^2}{\delta^2} \right)^m \rho_c^m \frac{\mu}{kN_A} \times
\int_0^R drr^2[1 - \left( \frac{r}{R} \right)^\delta]^{(1+n-m)[\Phi(\delta, \gamma) - \phi(\frac{r}{R})]^m},
\]

where

\[
\phi(\frac{r}{R}) = \sum_{m_1=0}^{\gamma} \frac{(-\gamma)_{m_1}}{m_1!} \frac{(\frac{r}{R})^{m_1,\delta+2}}{(\frac{\delta}{\delta} + m_1)(\frac{\delta}{\delta} + m_1)} \times
\]

\[
2F_1(-\gamma, \frac{2}{\delta} + m_1; \frac{2}{\delta} + 1 + m_1; (\frac{r}{R})^\delta)
\]

and

\[
\psi(\delta, \gamma) = \sum_{m_1=0}^{\gamma} \frac{(-\gamma)_{m_1}}{m_1!} \frac{1}{(\frac{\delta}{\delta} + m_1)(\frac{\delta}{\delta} + m_1)} \frac{\gamma!}{(\frac{\delta}{\delta} + m_1 + 1) \cdots (\frac{\delta}{\delta} + m_1 + \gamma)}.
\]

Note that \(|\frac{\phi(\frac{r}{R})}{\psi(\delta, \gamma)}| \leq 1\). If \( m \) and \( n \) are not assumed to be positive integers then the general binomial expansion can be used but one has to check the convergence conditions in the final sum. Assuming that \( m \) and \( n \) are positive integers we get polynomials from the two factors in the integral in (3.6). For a general positive integer \( m \) we get higher powers of \( \Phi(u) \) but \( \Phi(u) \) itself is a double sum. Hence the expression becomes complicated. For \( m=1 \) and \( n \) a positive integer one can get some simpler representations. Hence we consider this case. Then we have

\[
L = (4\pi \epsilon_0)^2 \left( \frac{GR^2}{\delta^2} \right)^n \rho_c^{n+2} \left( \frac{\mu}{kN_A} \right) R^3[\Phi(\delta, \gamma)I_0 - I_1],
\]
where

\[ I_0 = \int_0^1 duu^2[1 - u^\delta]^{n\gamma} = \frac{(n\gamma)!}{\delta} \frac{\Gamma\left(\frac{3}{\delta}\right)}{\Gamma\left(\frac{3}{\delta} + n\gamma + 1\right)} \]  

(3.10)

and

\[
I_1 = \int_0^1 duu^2[1 - u^\delta]^{n\gamma}\phi(u) \\
= \frac{\delta^2}{6} \sum_{m_1=0}^{\gamma} \sum_{m_2=0}^{\gamma} \frac{(-\gamma)^{m_1} (-\gamma)^{m_2}}{m_1! m_2!} \frac{(\frac{3}{\delta})^{m_1} (\frac{3}{\delta})^{m_1}}{\left(\frac{3}{\delta} + 1\right)m_1 (\frac{3}{\delta} + 1)m_1} \times \\
= \frac{(\frac{2}{\delta})^{m_1+m_2}}{(\frac{3}{\delta} + 1)^{m_1+m_2}} \int_0^1 duu^2[1 - u^\delta]^{n\gamma} u^{m_1+\delta+m_2+2}. 
\]  

(3.11)

But

\[
\int_0^1 duu^2[1 - u^\delta]^{n\gamma} u^{m_1+\delta+m_2+2} = \frac{(n\gamma)!}{\delta} \frac{\Gamma\left(\frac{3}{\delta} + m_1 + m_2\right)}{\Gamma\left(\frac{3}{\delta} + n\gamma + 1 + m_1 + m_2\right)} \\
= \frac{(n\gamma)!}{\delta} \frac{\Gamma\left(\frac{3}{\delta}\right)}{\Gamma\left(\frac{3}{\delta} + n\gamma + 1\right)} \times \\
= \frac{(\frac{2}{\delta})^{m_1+m_2}}{(\frac{3}{\delta} + n\gamma + 1)^{m_1+m_2}}. 
\]  

(3.12)

Substituting (3.12) in (3.11) and writing the resulting expansions as Kampé de Férier’s function we have

\[
I_1 = \frac{\delta}{6} \frac{(n\gamma)!}{\Gamma\left(\frac{3}{\delta} + n\gamma + 1\right)} \times \\
F_{2:3:1}^{2:2:0} \left[ \left(\frac{2}{\delta}, \frac{2}{\delta} - \gamma, \frac{3}{\delta} + \gamma; \frac{3}{\delta} + 1\right) \right]_{\frac{2}{\delta} + 1, \frac{3}{\delta} + n\gamma + 1, \frac{3}{\delta} + 1, \frac{3}{\delta} + 1}. 
\]  

(3.13)

The luminosity at any given point \( r \) is available from (3.3). That is,

\[
L(r) = (4\pi \epsilon_0)^{m+1} \left( \frac{GR^2}{\delta^2} \right)^m \rho_{\epsilon}^{m+n+1} \left( \frac{\mu}{kN_A} \right)^m \times \\
\int_0^r dt t^2 \left[ 1 - \left(\frac{t}{R}\right)^\delta \right]^{\gamma(1+n-m)} \left[ \Psi(\gamma, \delta) - \Phi\left(\frac{t}{R}\right) \right]^m. 
\]  

(3.14)

We will evaluate this for \( m=1 \). Then the two integrals to be evaluated are

\[
I_2 = \int_0^r dt t^2 \left[ 1 - \left(\frac{t}{R}\right)^\delta \right]^{n\gamma}
\]
and

\[ I_3 = \int_0^r dt t^2 \left[ 1 - \left( \frac{t}{R} \right)^\delta \right] \Phi \left( \frac{t}{R} \right). \]

That is,

\[ I_2 = R^3 \int_0^{r/R} du u^2 [1 - u^\delta]^{n\gamma} \]
\[ = R^3 \sum_{\alpha=0}^{n\gamma} \frac{(-n\gamma)_\alpha}{\alpha!} \int_0^{r/R} du u^{2+\alpha\delta} \]
\[ = R^3 \sum_{\alpha=0}^{n\gamma} \frac{(-n\gamma)_\alpha (r/R)^{3+\alpha\delta}}{\alpha! (3 + \alpha\delta)} \]
\[ = \frac{r^3}{3} \sum_{\alpha=0}^{n\gamma} \frac{(-n\gamma)_\alpha}{\alpha!} \frac{\left( \frac{2}{\delta} \right)_\alpha}{\left( \frac{2}{\delta} + 1 \right)_\alpha} \left[ \left( \frac{r}{R} \right)^\delta \right]^\alpha \]
\[ = \frac{r^3}{3} \, 2F_1 \left( -n\gamma, \frac{3}{\delta}; \frac{3}{\delta} + 1; \left( \frac{r}{R} \right)^\delta \right). \quad (3.15) \]

\[ I_3 = R^3 \sum_{m_1=0}^{\gamma} \sum_{m_2=0}^{\gamma} \sum_{m_3=0}^{n\gamma} \frac{(-\gamma)_{m_1} (-\gamma)_{m_2} (-n\gamma)_{m_3}}{m_1! \, m_2! \, m_3!} \times \]
\[ \left( \frac{2}{\delta} \right)_{m_1+m_2} \left( \frac{r}{R} \right)^{5+(m_1+m_2+m_3)\delta} \]
\[ \left( \frac{2}{\delta} + 1 \right)_{m_1+m_2} \frac{\delta \left( \frac{5}{\delta} \right)_{m_1+m_2}}{m_1 + m_2 + m_3} \]
\[ = \frac{r^5}{5R^2} \sum_{m_1=0}^{\gamma} \sum_{m_2=0}^{\gamma} \frac{(-\gamma)_{m_1} (-\gamma)_{m_2} \left( \frac{5}{\delta} \right)_{m_1+m_2}}{m_1! \, m_2! \, \left( \frac{2}{\delta} + 1 \right)_{m_1+m_2}} \times \]
\[ \left( \frac{2}{\delta} \right)_{m_1+m_2} \left[ \left( \frac{r}{R} \right)^\delta \right]^{m_1+m_2} \]
\[ \times \]
\[ 2F_1 \left( -n\gamma, \frac{5}{\delta} + m_1 + m_2; \frac{5}{\delta} + 1 + m_1 + m_2; \left( \frac{r}{R} \right)^\delta \right). \quad (3.16) \]

Note that (3.16) can also be written in terms of a Kampé de Fériet’s function of three variables. In that case the variables are 1, \((r/R)^\delta\), \((r/R)^\delta\), respectively. Note also that (3.15) and (3.16) are polynomials. Hence for \(m=1\), \(n\) a positive integer

\[ L(r) = (4\pi)^2 \left( \frac{GR^2}{\delta^2} \right) \rho_c^{n+2} \left( \frac{\mu}{kN_A} \right) [\Psi(\gamma, \delta) I_2 - I_3], \quad (3.17) \]
where $I_2$ and $I_3$ are given in (3.15) and (3.16) respectively. A particular case of (3.17) is also available from Haubold and Mathai (1992).

REFERENCES

H.J. Haubold and A.M. Mathai, Astrophys. Space Sci. 197, 153(1982).

A.M. Mathai and H.J. Haubold, Modern Problems in Nuclear and Neutrino Astrophysics. Akademie-Verlag, Berlin 1988.

A.M. Mathai, A Handbook of Generalized Special Functions for Statistical and Physical Sciences. Oxford University Press, Oxford 1993.

A. Noels, R. Papy, and F. Remy, Computers in Physics 7, 22(1993).

R.L. Sears, Astrophys. J. 140, 477 (1964).

H.M. Srivastava and P.W. Karlsson, Multiple Gaussian Hypergeometric Series. Ellis Horwood, Chichester 1985.