The Local Isometric Embedding of Two-Metrics of Low Differentiability in Euclidean Three Space.

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Abstract

We prove that the isometric embedding of any metric of differentiability class $C^1$ in $E^3$ exists. We use simplified notation for the given metric, namely geodesic parameters, and level parameters for the embedded surface in $E^3$. Central to our discussion will be solutions of initial value problems for two first order non-linear partial differential equations. We also make use of the classical theory of linear algebraic systems. We will prove local isometric embedding. An example is given for which the Gaussian curvature of the metric is equal to one but the embedded surface is non-analytic.

1 Introduction.

The Main Problem is to prove the existence of an isometric embedding in $E^3$ of the given metric

$$\omega = \bar{E}(\bar{u}, \bar{v})d\bar{u}^2 + \bar{F}d\bar{u}d\bar{v} + \bar{G}(\bar{u}, \bar{v})d\bar{v}^2$$

where $\bar{E} = 1$, $\bar{F} = 0$, $\bar{G} > 0$, $\bar{G} \in C^1$. We emphasize that this is a local problem. That is, the given metric is given locally and the existence is to be proved locally.

This is called the geodesic form of the metric. Any metric can be assumed to be in the geodesic form without loss of generality [?, [2].

The isometrically embedded surface will be presented in the form of a vector function

$$X(u, v) = x(u, v)i + y(u, v)j + vk$$

where $i, j, k$ is a right-handed orthonormal basis of $E^3$ and $X_u \cdot X_v = 0$.

We note that $i, j$ is a basis of $E^2$ as a subspace of $E^3$. Definitions. $u, v$ are level parameters of the surface.

$X_0=x(u, v)i + y(u, v)j$ is the principal part of the surface.
It follows that
\[ X(u, v) = X_0 + v k. \]

We need to define the function \( X(u, v) \) in order to find the function giving the embedding of the given metric \( \omega \). The embedding function will be
\[ \bar{X}(\bar{u}, \bar{v}) = X(u, v) \]
where the change of parameters from the barred parameters to the unbarred parameters will be given by solving the pair of initial value problems
\[ \sqrt{G(\bar{u}, \bar{v})} f_u + \sqrt{\frac{f_u^2}{1 - f_u^2}} f_\bar{v} = 0 \]
\[ \sqrt{\frac{f_u^2}{1 - f_u^2}} g_u \sqrt{G(\bar{u}, \bar{v})} - g_\bar{v} = 0 \]
with \( C^1 \) initial values \( f(\bar{u}, 0) = \bar{h}(\bar{u}) \) with \( \bar{h}'(\bar{u}) \neq 0 \);
\[ \sqrt{\frac{f_u^2}{1 - f_u^2}} g_u \sqrt{G(\bar{u}, \bar{v})} - g_\bar{v} = 0 \]
with \( C^1 \) initial values \( g(\bar{u}, 0) = \bar{k}(\bar{u}) \) with \( \bar{k}'(\bar{u}) \neq 0 \).

Remark. We may choose the initial values to be not of class \( C^2 \) so that the solutions of the PDEs will not have power series in a neighborhood of the initial point and hence will not be real analytic. The isometric embedding will then be not real analytic. This will be true even though the given metric is real analytic. An example is
\[ \omega = d\bar{u}^2 + (\cos^2 \bar{u}) d\bar{v}^2. \]

An easy calculation using the Theorema Egregium of Gauss shows that the curvature of the metric is equal to one in a neighborhood of \((0, 0)\).

Using the solutions of the initial value problems we define a change of parameters by
\[ u = f(\bar{u}, \bar{v}), \quad v = g(\bar{u}, \bar{v}). \]

Claim. The change of parameters (3) is invertible. For a proof see the Appendix of calculations.

We now define the composite vector function
\[ \bar{X}(\bar{u}, \bar{v}) = X(f(\bar{u}, \bar{v}), g((\bar{u}, \bar{v}))). \]

By the change of parameters \( \bar{X}(\bar{u}, \bar{v}) \) will be well-defined once \( X(u, v) \) is defined. We can then write
\[ \bar{X}(\bar{u}, \bar{v}) = X(u, v). \]

Suppose for now that the functions \( x(u, v), y(u, v) \) are known. This will enable us to compute the components of the metric of the immersion \( \bar{X}(\bar{u}, \bar{v}) \) induced by the ambient space \( E^3 \) and show that they are equal respectively to 1, 0, \( G \), i.e., to the components of the given metric \( \omega \).
2 Definition of System S.

The system $S$, to play an important role in our proof of embedding, is

$$1 = 1 f_u^2 + G g_v^2.$$  
$$0 = 1 f_u f_v + G g_u g_v.$$  
$$\bar{G} = 1 f_v^2 + G g_v^2.$$  

$\bar{G} = \bar{G}(\bar{u}, \bar{v})$ comes from the given metric to be embedded. $S$ is algebraically linear in unknowns $1, G$ with coefficients given by the partial derivatives which are known from the initial value problems solved above.

Theorem 1. $S$ has an algebraic solution for $1, G$ in terms of the partial derivatives and $\bar{G}$.

Proof. See the appendix of calculations subsection 5.1.

We can now solve $S$ for $1, G(u, v)$ by Cramer’s rule.

$G(u, v)$ is now determined in terms of $\bar{G}(\bar{u}, \bar{v})$. In fact, from the third equation,

$$G(u, v) = \frac{\bar{G} - f_v^2}{g_v^2}.$$  

The system $S$ says that $1, 0, \bar{G}$ and $1, 0, G$ are components of the same metric. Therefore we have

Theorem 2.

$$\omega = d\bar{u}^2 + \bar{G}(\bar{u}, \bar{v})d\bar{v}^2 = du^2 + G(u, v)dv^2.$$  

QED theorem 2.

3 The function $X(u, v)$ is well-defined.

Recall that

$$X(u, v) = x(u, v)i + y(u, v)j + vk.$$  

We first put the metric of the Euclidean plane in the geodesic form

$$\omega_0 = du^2 + G_0(u, v)dv^2.$$  

This can always be accomplished by choosing an arbitrary $C^1$ base curve as referenced above. If the base curve is a segment of the x axis of a standard coordinate system of $E^2$, then the metric is in the canonical form

$$\omega_0 = dx^2 + dy^2.$$  

The straight lines $x =$ constant are the geodesics. Let

$$X_0 = x(u, v)i + y(u, v)j$$
represent $E^2$ in terms of geodesic parameters $u, v$. Thus $x(u, v), y(u, v)$ are determined once the base curve is given. The metric induced by the ambient space $E^2$ on $X_0$ is given by

$$E_0 = X_{0,u} = (x_u \mathbf{i} + y_u \mathbf{j})^2 = x_u^2 + y_u^2.$$  

$$F_0 = X_{0,u} \cdot X_{0,v} = (x_u \mathbf{i} + y_u \mathbf{j}) \cdot (x_v \mathbf{i} + y_v \mathbf{j}) = x_u x_v + y_u y_v = 0.$$  

(since geodesic parameters are orthogonal)

$$G_0 = X_{0,v}^2 = (x_v \mathbf{i} + y_v \mathbf{j})^2 = x_v^2 + y_v^2,$$

where $E_0 = 1$, $F_0 = 0$, $G_0 > 0$.

Then the Euclidean metric $\omega_0$ of $E^2$ has components $E_0, F_0, G_0$ which satisfy the system $S_0$ defined as

$$1 = x_u^2 + y_u^2,$$

$$0 = x_u x_v + y_u y_v,$$

$$G_0 = x_v^2 + y_v^2.$$  

$G_0$ is known from the geodesic form $\omega_0$ above.

The system $S_0$ expresses the relation between the components 1, 1 of the Euclidean metric relative to the coordinates $x, y$ and its components relative to the parameters $u, v$. Therefore the Euclidean metric can be written

$$\omega_0 = dx^2 + dy^2 = du^2 + G_0(u, v)dv^2.$$  

Clearly the geodesics of $E^2$ are straight line segments given by $v = \text{constant}$. Since $x(u, v), y(u, v)$ are known by the definition of geodesic parameters, therefore

Theorem. $X(u, v)$ is well-defined as

$$X(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + v\mathbf{k}.$$  

This is an immersion of $u, v$ space in $E^3$. We now show that it is a surface:

Calculate the metric induced by the ambient space $E^3$.

$$E(u, v) = X_u^2 = x_u^2 + y_u^2,$$

$$F(u, v) = X_u \cdot X_v = x_u x_v + y_u y_v = 0,$$

$$G(u, v) = x_v^2 + y_v^2 + 1.$$  

Thus, $G(u, v) = G_0 + 1$, $E(u, v) = E_0(u, v) = 1$.

$$EG - F^2 \neq 0.$$  

Therefore we have the

Theorem 3. $X(u, v)$ is a surface.
4 Proof of Main Theorem.

Main theorem. The composite vector function defined as

\[ \bar{X}(\bar{u}, \bar{v}) = X(f(\bar{u}, \bar{v}), g(\bar{u}, \bar{v})) \]

is an isometric embedding of the given metric

\[ \omega = \bar{E}(\bar{u}, \bar{v})d\bar{u}^2 + \bar{F}d\bar{u}d\bar{v} + \bar{G}(\bar{u}, \bar{v})d\bar{v}^2 \]

in \( E^3 \).

Proof. We compute the components of the metric of the immersion \( \bar{X}(\bar{u}, \bar{v}) \) induced by the ambient space \( E^3 \) and show that they are equal respectively to 1, 0, \( G \), i.e., to the components of the given metric \( \omega \).

\[ \bar{X}_u^2 = (X_u u_u + X_v v_u)^2 = X_u^2 u_u^2 + 2X_u \cdot X_v u_u v_u + X_v^2 v_u^2 \]
\[ = X_u^2 u_u^2 + X_v^2 v_u^2 \]
\[ = E(u, v)u_u^2 + G(u, v)v_u^2 \]
\[ = (x_u^2 + y_u^2)u_u^2 + (x_v^2 + y_v^2 + 1)v_u^2. \]

But \( E = E_0 = 1 \), \( G = G_0 + 1 \) implies

\[ \bar{X}_u^2 = E_0(u, v)u_u^2 + (G_0(u, v) + 1)v_u^2 \]

by the first equation of system \( S \) and \( v = g(\bar{u}, \bar{v}) \). Therefore

\[ \bar{X}_u^2 = 1. \]

\[ \bar{X}_u \cdot \bar{X}_v = (X_u u_u + X_v v_u) \cdot (X_u u_v + X_v v_v) \]
\[ = X_u^2 u_u u_v + X_v^2 u_v v_v \]
\[ = (x_u^2 + y_u^2)u_u u_v + (x_v^2 + y_v^2 + 1)v_u v_v \]
\[ = E_0 u_u u_v + G u_v v_v \]
\[ = u_u u_v + G g_u g_v \]

using the change of parameters \( \bar{u}, \bar{v} \). The second equation of \( S \) may be written

\[ 0 = 1u_u u_v + G g_u g_v. \]

Thus,

\[ \bar{F}(\bar{u}, \bar{v}) = 0. \]

\[ \bar{X}_v^2 = (X_u u_v + X_v v_v)^2 \]
\[ = (x_u^2 + y_u^2)u_v^2 + (x_v^2 + y_v^2 + 1)v_v^2 \]
\[ = E u_v^2 + G(u, v)v_v^2 \]
\[ = E f_v^2 + G(u, v)g_v^2 \]
\[ = f_v^2 + G(u, v)g_v^2 = \bar{G}(\bar{u}, \bar{v}) \]

by system \( S \). QED main theorem.
5 Appendix of Calculations.

5.1 Proof of Theorem 1.

Theorem 1. S has an algebraic solution for $1, G$ in terms of the partial derivatives and $\bar{G}$.

Proof. In order to prove theorem 1 we must show that S is algebraically consistent. This will be true if and only if the rank of the augmented matrix of the system equals the rank of its coefficient matrix [3].

Calculation of the ranks of the augmented matrix and the coefficient matrix:

Lemma. The Jacobian $J$ of the parameter change (3) is not zero in some neighborhood of the initial point.

$$J = \begin{vmatrix} f_{\bar{u}} & f_{\bar{v}} \\ g_{\bar{u}} & g_{\bar{v}} \end{vmatrix}.$$

At the initial point $f_{\bar{u}}(\bar{u}, 0) = \bar{h}'(\bar{u}) \neq 0$. We may take $\bar{h}'(\bar{u}) > 0$.

Using initial value problem 1 [1] we calculate $f_{\bar{v}}(\bar{u}, 0)$:

$$\sqrt{G(\bar{u}, 0)} \bar{h}'(\bar{u}) + \sqrt{\frac{(\bar{h}'(\bar{u})^2}{1 - (\bar{h}'(\bar{u})^2}) f_{\bar{v}}(\bar{u}, 0) = 0.}$$

Cancelling $\bar{h}'(\bar{u})$ we obtain

$$\sqrt{G(\bar{u}, 0)} + \sqrt{\frac{1}{1 - (\bar{h}'(\bar{u})^2}) f_{\bar{v}}(\bar{u}, 0) = 0.}$$

Next we calculate $g_{\bar{v}}$ using the solution of the second initial value problem [2].

At the initial point this becomes

$$g_{\bar{v}} = \sqrt{\frac{(\bar{h}'(\bar{u})^2}{1 - (\bar{h}'(\bar{u})^2}) g_{\bar{u}} \sqrt{G(\bar{u}, 0)} = \sqrt{\frac{(\bar{h}'(\bar{u})^2}{1 - (\bar{h}'(\bar{u})^2}) \bar{k}'(\bar{u}) \sqrt{G(\bar{u}, 0)}.}$$

Therefore,

$$J(\bar{u}, 0) = \begin{vmatrix} f_{\bar{u}} - \sqrt{G(\bar{u}, 0) \sqrt{1 - (\bar{h}'(\bar{u})^2)} \\ g_{\bar{u}} \sqrt{\frac{(\bar{h}'(\bar{u})^2}{1 - (\bar{h}'(\bar{u})^2}) \bar{k}'(\bar{u}) \sqrt{G(\bar{u}, 0)} \end{vmatrix}$$

$$= \begin{vmatrix} \bar{h}'(\bar{u}) - \sqrt{G(\bar{u}, 0) \sqrt{1 - (\bar{h}'(\bar{u})^2)} \\ \bar{k}'(\bar{u}) \sqrt{\frac{(\bar{h}'(\bar{u})^2}{1 - (\bar{h}'(\bar{u})^2}) \bar{k}'(\bar{u}) \sqrt{G(\bar{u}, 0)} \end{vmatrix}$$

$$= \sqrt{1 - (\bar{h}'(\bar{u})^2) \begin{vmatrix} \bar{h}'(\bar{u}) - \sqrt{G(\bar{u}, 0) \sqrt{1 - (\bar{h}'(\bar{u})^2)} \\ \bar{k}'(\bar{u}) \sqrt{\frac{(\bar{h}'(\bar{u})^2}{1 - (\bar{h}'(\bar{u})^2}) \bar{k}'(\bar{u}) \sqrt{G(\bar{u}, 0)} \end{vmatrix}.$$

This can be made positive by taking $\bar{h}'(\bar{u})$ and $\bar{k}'(\bar{u})$ small positive. QED Lemma.
By the Lemma the parameter change is locally one-to-one, bicontinuous and continuously differentiable, i.e., is locally invertible.

We now compute the ranks of the coefficient and augmented matrices of system $S$.

Claim. The rank of the augmented matrix is two.

Proof of Claim. The coefficient matrix is

$$
\begin{vmatrix}
    f_{\bar{u}}^2 & g_{\bar{u}}^2 & 1 \\
    f_{\bar{u}} f_{\bar{v}} & g_{\bar{u}} g_{\bar{v}} & 0 \\
    f_{\bar{v}}^2 & g_{\bar{v}}^2 & \bar{G}(\bar{u}, \bar{v}) \\
\end{vmatrix}
$$

$$
= \begin{vmatrix}
    f_{\bar{u}} f_{\bar{v}} & g_{\bar{u}} g_{\bar{v}} & \bar{G}(\bar{u}, \bar{v}) \\
    f_{\bar{v}}^2 & g_{\bar{v}}^2 & 0 \\
\end{vmatrix}
+ \bar{G}(\bar{u}, \bar{v}) f_{\bar{u}} g_{\bar{v}}
$$

$$
= f_{\bar{v}} g_{\bar{v}} \begin{vmatrix}
    f_{\bar{u}} & g_{\bar{u}} \\
    f_{\bar{v}} & g_{\bar{v}} \\
\end{vmatrix} + \bar{G}(\bar{u}, \bar{v}) f_{\bar{u}} g_{\bar{v}}
$$

$$
= J(f_{\bar{v}} g_{\bar{v}} + \bar{G}(\bar{u}, \bar{v}) f_{\bar{u}} g_{\bar{v}}).
$$

Now substitute from the PDEs of the initial value problems to obtain zero. Therefore the rank of the augmented matrix is at most two. That the rank of the augmented matrix is at least two follows from the Lemma and the fact that the determinant has two by two minors that are not zero. Therefore the rank of the augmented matrix is two. QED Claim.

Claim'. The rank of the coefficient matrix is two.

Proof of Claim'. The coefficient matrix is

$$
\begin{vmatrix}
    f_{\bar{u}}^2 & g_{\bar{u}}^2 \\
    f_{\bar{u}} f_{\bar{v}} & g_{\bar{u}} g_{\bar{v}} \\
    f_{\bar{v}}^2 & g_{\bar{v}}^2 \\
\end{vmatrix}
$$

At the initial point, where $\bar{v} = 0$, this becomes

$$
J(\bar{u}, 0) h'(\bar{u}) k'(\bar{u}) \neq 0.
$$

Therefore in a neighborhood of the initial point the rank of the coefficient matrix is two. QED Claim'. QED Theorem 1.

5.2 Proof of the existence of the inverse of the parameter change.

Lemma. The Jacobian of the parameter change is not zero locally.

Proof. Define

$$
\lambda(\bar{u}, \bar{v}) = \sqrt{\frac{f_{\bar{u}}^2}{1 - f_{\bar{u}}^2}}.
$$
Then the PDEs can be written

\[ \sqrt{G(\bar{u}, \bar{v})} f_{\bar{u}} + \lambda(\bar{u}, \bar{v}) f_{\bar{v}} = 0. \]

\[ \lambda(\bar{u}, \bar{v}) g_{\bar{u}} \sqrt{G(\bar{u}, \bar{v})} - g_{\bar{v}} = 0. \]

\[
J = \begin{vmatrix}
    f_{\bar{u}} & f_{\bar{v}} \\
    g_{\bar{u}} & g_{\bar{v}}
\end{vmatrix}
= \begin{vmatrix}
    f_{\bar{u}} & \frac{\sqrt{G(\bar{u}, \bar{v})} f_{\bar{u}}}{\lambda} \\
    g_{\bar{u}} & \sqrt{G(\bar{u}, \bar{v})} g_{\bar{u}}
\end{vmatrix}
= f_{\bar{u}} g_{\bar{u}} \sqrt{G} \begin{vmatrix}
    1 & -\lambda^{-1} \\
    1 & \lambda
\end{vmatrix}.
\]

By a suitable choice of the initial conditions we can make \( J \neq 0 \) in a neighborhood of the initial point. Choose

\[ \bar{k}'(\bar{u}') > 0, \]
\[ \bar{h}'(\bar{u}) > 0. \]

Thus \( J > 0 \) taking care to choose \( \bar{h}'(\bar{u}') \) small enough that \( \lambda \) is real. By a textbook result the mapping is locally invertible. QED.

6 Topic for further investigation. Informal Remarks.

The existence of isometric embeddings of \( C^1 \) metrics which are \( C^2 \). Compatibility conditions are needed for a proof since the systems are overdetermined.

The principal part of the embedded surface, defined in the introduction, is two-dimensional and serves to determine a surface in a three-dimensional space. This essentially reduces a three-dimensional problem to a two-dimensional problem. It relates to an interesting discussion of Brian Greene in a chapter of his book (The Fabric of the Cosmos) entitled 'Is the Universe a Hologram?'

'Whereas Plato envisioned common perceptions as revealing a mere shadow of reality, the holographic principle concurs, but turns the metaphor on its head. The shadows—the things that are flattened out and hence live on a lower dimensional surface – are real, while what seems to be the more richly structured, higher-dimensional entities (us; the world around us) are evanescent projections of the shadows.

References

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