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The stationary Dirac equation as a generalized Pauli equation for two quasiparticles

Abstract. By the example of the Dirac equation with static electric and magnetic fields it is shown that Dirac’s theory is nothing but a generalized one-particle quantum theory compatible with the special theory of relativity. This equation describes a quantum dynamics of a single relativistic fermion, and its solving is reduced to solving the generalized Pauli equation for two quasiparticles which move in the Euclidean space but their effective masses keep information about the Lorentzian symmetry of the four-dimensional space-time. We reveal the correspondence between the Dirac bispinor and Pauli spinor (two-component wave function), and show that all four components of the Dirac bispinor correspond to a fermion (or all of them correspond to its antiparticle). Mixing the particle and antiparticle states is prohibited. On this basis we discuss the paradoxical phenomena of Zitterbewegung and the Klein tunneling.

Keywords. Dirac equation · Klein tunneling · Dirac sea · potential step

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1 Introduction

As is known, because of paradoxical physical implications of the Dirac equation, its interpretation as a quantum-mechanical equation for a one-particle wave function faces the problems. As a consequence, now there is a widespread opinion that this equation cannot be considered as the generalization of the Schrödinger-Pauli equation onto single relativistic fermions; Dirac theory is treated, rather, as a field theory which is in need of quantization. As regards the Schrödinger formalism (non-relativistic quantum mechanics), it is believed that it is in principle incompatible with special relativity.

Perhaps, the most paradoxical implications of the Dirac equation are the Klein tunneling [1–8] and the so-called 'Zitterbewegung' phenomenon [11]. Both are often mentioned in the current literature on this equation and both give rise to controversy among researchers. Even the very nature of both these phenomena is differently understood in the current literature. For example, the Klein paradox for an electron scattering on a strong electric scalar step potential (when its energy lies, on the energy scale, below the step height in the so-called Klein zone) is understood by some authors as the appearance of a classically accessible region behind the step; while others say about the Klein paradox when, in this scattering problem, the probability flow associated with reflected particles exceeds the incident flow.

As regards Zitterbewegung, there are at least two main, principally different versions of this phenomenon, Schrödinger’s [10] and Hestenes’ [11] ones. The latter treats an electron as "a rapidly rotating electric dipole", i.e., as a particle having an internal substructure. This model of Zitterbewegung needs a reformulation of the standard Dirac equation, and what is more important is that it, in fact, lies...
2 Dirac dynamics as a generalized Schrödinger dynamics of two spin-1/2 quasiparticles with different effective masses

Let us consider the (3+1)-Dirac equation with the static electric scalar potential \( \phi(r) \) and vector potential \( \mathbf{A}(r) \):

\[
\left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} i\hbar \frac{\partial}{\partial t} - V(r) \end{pmatrix} + \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} (i\hbar \mathbf{\nabla} + e\mathbf{A}) - mc^2 \right] \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0
\]

where \( V = e\phi \); \( e \) is the electric charge of a particle; \( m \) is its (rest) mass; \( r \) is its radius-vector; \( \sigma^1, \sigma^2 \) and \( \sigma^3 \) are the Pauli matrices. The corresponding continuity equation is

\[
\frac{\partial W}{\partial t} + \nabla J = 0; \quad W = |\phi|^2 + |\chi|^2, \quad J = c(\phi^*\sigma\chi + \chi^*\sigma\phi).
\]

According to the current vision of Eq. (1), each component of the Dirac bispinor is associated with a given orientation of the spin along the axis \( OZ \) and with a given sign of the particle energy: if \( \Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix} \) and \( \chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \), then the pairs \( (\Phi_+,\chi_+) \) and \( (\Phi_-,\chi_-) \) describe a particle whose \( z \)-projection of spin is +1/2 and −1/2, respectively. As regards the spinors \( \Phi \) and \( \chi \) (referred to, in the non-relativistic limit, as ‘large’ and ‘small’ components, respectively), they are assumed to correspond to the positive and negative values of the particle energy [1].

However, there is every reason to believe that it is not. Indeed, since the scalar and vector potentials are static, one can search a particular (stationary) solution of Eqs. (1) in the form

\[
\begin{pmatrix} \phi(r,t) \\ \chi(r,t) \end{pmatrix} = \begin{pmatrix} \phi(r;E) \\ \chi(r;E) \end{pmatrix} e^{-iEt/\hbar}
\]

where \( E \) is the particle energy. For the stationary state we have

\[
(E - V - mc^2)\phi + (i\hbar \mathbf{\nabla} + e\mathbf{A})\sigma\chi = 0, \quad (E - V + mc^2)\chi + (i\hbar \mathbf{\nabla} + e\mathbf{A})\sigma\phi = 0;
\]

beyond the scope of quantum mechanics itself. It can be considered as a pre-quantum Zitterbewegung model. And, since our final aim is to study this phenomenon from the quantum-mechanical point of view, we shall consider only the Schrödinger version [10] of Zitterbewegung, where this phenomenon follows from the standard Dirac equation when one assumes that a particle might be in a quantum state representing the superposition of the particle and antiparticle states (the electron itself is considered as a point object).

The idea that such states may coexist with each other underlies also most approaches to the Klein tunneling, where this phenomenon is treated as a many-particle effect accompanied by creation of electron-positron pairs. This interpretation is evident to lie beyond the scope of Dirac theory, as quantum formalism that describes one particle. At the same time this equation remains valid at all energies of a relativistic particle (see, e.g., [12,13]) and hence there is no reason to discard the possibility to interpret the Klein tunneling as a single-particle phenomenon.

Instead of the pair-creation mechanism, some authors (see, e.g., [5,7]) attempt to resolve the Klein paradox with making use of different kinds of ‘ghost’ wave modes and virtual particles. In these approaches the probability flow associated with ‘ghost’ modes and virtual particles balance the electron flow at the step, and the Klein tunneling disappears. But this result, too, cannot be considered satisfactory, because it contradicts the studies of the Klein tunneling in graphene where this effect, predicted on the basis of the Dirac equation, really exists (see, e.g., [14]).

We consider that the source of all paradoxes that surround at present the Dirac equation is the existing practice to associate the ‘small’ component of the Dirac bispinor with an antiparticle (see, e.g., [1]). Our aim is to show that this practice, based on the assumption that the particle and antiparticle quantum states belong to the same Hilbert space, is unfounded. We present Dirac theory as a pre-quantum formalism that describes the dynamics of single relativistic fermions. On this basis we discuss the above paradoxical phenomena – the Klein tunneling and Zitterbewegung.
Let $\epsilon = E - mc^2$, $D = \sigma \nabla - \frac{ie}{mc} \sigma \mathcal{A}$. Then Eqs. (4) and Exp. (2) for $W$ and $J$ can be rewritten as

$$-\frac{\hbar^2}{2M} \nabla^2 \Phi + V \Phi = c\Phi; \quad \chi = -\frac{i\hbar}{2Mc} \nabla \Phi; \quad M(\epsilon, V) = m \left( 1 + \frac{\epsilon - V}{2mc^2} \right) \tag{5a}$$

$$W = |\Phi|^2 + \frac{\hbar^2}{4M^2 c^2} |\nabla \Phi|^2, \quad J = \frac{i\hbar}{2M} \left[ (\nabla \Phi)^* \sigma \Phi - \Phi^* \sigma \nabla \Phi \right]. \tag{5b}$$

Note that the set of Eqs. (5a) is exactly equivalent to Eqs. (4). Thus, solving the Dirac equation is reduced now to solving Eq. (8a) for the spin or reduced, in fact, to solving Eq. (5a) for the spinor $\chi$.

As is known, equations of such a kind (without the vector potential $\mathcal{A}$) play a large role in solid state physics (see, e.g., [15][16]), where they describe the quantum dynamics of a Bloch electron in superlattices. Namely, they arise within the effective-mass approximation as equations for the envelope of the wave function of a Bloch electron. In this approximation, the effective mass of this quasiparticle, in each layer of a superlattice, keeps information about the periodic potential in the layer. And, according to this approach, the envelop of the wave function must be everywhere continuous together with its first spatial derivative divided by the effective mass.

Essentially the same situation arises for a Dirac particle. Now, to ensure the continuity of the probability density $W$ and the probability current density $J$ at the points where the scalar and vector potentials are discontinuous, the spinor $\Phi$ must be everywhere continuous together with the spinor $\nabla \Phi$. It is evident that the last requirement provides also the continuity of the spinor $\chi$. Like the effective mass of a Bloch electron, the effective mass $M$ associated with the spinor $\Phi$ keeps information about the Lorentzian symmetry of the four dimensional space-time. However, unlike the equation for the envelop of the wave function of a Bloch electron, Eq. (5a) (or Eq. (7)) is exact.

As is known, equations of such a kind (without the vector potential $\mathcal{A}$) play a large role in solid state physics (see, e.g., p.934 in [1]) as a good cause in order to neglect the spinor $\chi$ in this limit. But this is mistaken in principle: the validity of the inequality $|\chi|^2 << |\Phi|^2$ does not at all mean that the 'small' component $\chi$ is inessential in this limit, in comparison with the 'large' component $\Phi$. Firstly, we have to recall that the second-order differential equation (6a) is equivalent to the system (4) of coupled first-order differential equations for $\Phi$ and $\chi$, where both these components are equally essential. Secondly, both are also equally essential in Exp. (2) for the probability current density.

To elucidate the role of the 'small' component $\chi$ it is useful to express Eqs. (5a) and (5b) in another equivalent form, where $\Phi$ and $\chi$ change roles:

$$-\frac{\hbar^2}{2\mu} \nabla^2 \chi + (V - 2mc^2) \chi = c\chi; \quad \Phi = -\frac{i\hbar}{2\mu \epsilon} \nabla \chi; \quad \mu(\epsilon, V) = \frac{\epsilon - V}{2mc^2} \tag{8a}$$

$$W = \frac{\hbar^2}{4\mu^2 c^2} |\nabla \chi|^2 + |\chi|^2, \quad J = \frac{i\hbar}{2\mu} \left[ (\nabla \chi)^* \sigma \chi - \chi^* \sigma \nabla \chi \right]. \tag{8b}$$

Thus, solving Eqs. (4) is reduced now to solving Eq. (8a) for the spinor $\chi$, and the analog of Eq. (7) is

$$-\frac{\hbar^2}{2\mu} \left[ \nabla^2 - \frac{ie}{\hbar c} \frac{\sigma \nabla \mu}{\mu} \cdot \nabla \right] \chi + \left( V - 2mc^2 - \frac{e\hbar}{2\mu \sigma H} \right) \chi = c\chi. \tag{9}$$
This equation represents the generalized Pauli equation for the quasiparticle that has the effective mass \( \mu \) and moves, like the heavy quasiparticle, in the same vector potential but the scalar potential is less now by \( 2mc^2 \). In this case the spinor \( \chi \) and the one \( \mu^{-1}\mathcal{D}\chi \) must be continuous. The last condition provides the continuity of \( \Phi \); hence \( W \) and \( J \) will be continuous too.

So, the quantum ensemble of a Dirac particle with the energy \( E \) consists of two subensembles: the subensemble of 'heavy' spin-1/2 quasiparticles with effective mass \( \mu \); and the subensemble of 'light' spin-1/2 quasiparticles with the effective mass \( \mu \); in this case \( M - \mu = m \) and \( M + \mu = (E - V)/c^2 \equiv M \). Such partitioning of the original ensemble of a Dirac particle is unique, because no effective mass can be assigned to any superposition \( c_1\Phi + c_2\chi \) with \( c_1 \neq 0 \) and \( c_2 \neq 0 \). Only the (stationary) spinors \( \Phi \) and \( \chi \) by themselves, arising within the standard representation of the Dirac equation, can be associated with quasiparticles possessing definite effective masses.

Thus, according to our approach, in the static external potentials \( V \) and \( \mathcal{A} \), a Dirac particle moves, with the probability \( |\Phi|^2 \), just as a Schrödinger spin-1/2 quasiparticle moving in these fields with the effective mass \( M \). And, with the probability \( |\chi|^2 \), it moves just as a Schrödinger spin-1/2 quasiparticle with the effective mass \( \mu \) moves in the same vector potential but in the reduced scalar potential \( V - 2mc^2 \). In fact, the effective masses \( M \) and \( \mu \) must be considered, together with the spin projections \( +h/2 \) and \( -h/2 \), as quantum numbers characterizing the components of the (stationary) Dirac bispinor.

Of importance is to stress once more that the generalized Pauli equations (7) and (9) are equivalent to the same set of coupled Eqs. (4). Thus, in fact they represent two different forms of the same second-order differential equation for the same energy \( E \). Or, more precisely, the components \( \Phi_+ \) and \( \chi_+ \) of the Dirac bispinor correspond to the component \( \psi_{+1/2} \) of the Pauli spinor; the components \( \Phi_- \) and \( \chi_- \) of the Dirac bispinor correspond to the component \( \psi_{-1/2} \) of the Pauli spinor. Thus, unlike the conventional approach (see, e.g., [1]) where the 'small' ('light', in our terms) component \( \chi \) is associated with negative energies, in our approach both the 'heavy' and 'small' quasiparticles have the same energy and move in the same vector potential. Hence both \( \Phi \) and \( \chi \) describe the particle states (or both describe the antiparticle states).

Such states are invariant with respect to the Lorentz transformations and represent a complete set of states of a relativistic particle. The fact that Eqs. (4) possess also solutions with the negative values of \( E \) means simply that these equations imply also, in addition to the particle states, the existence of the antiparticle states. Again, the set of the antiparticle states is invariant under the Lorentz transformations and, thus, it represents a complete set of states of a relativistic antiparticle.

Of course, the group of transformations of symmetry of the Dirac equation contains also the operation of charge conjugation that transforms the particle and antiparticle states into each other. But this transformation essentially changes the physical context that determines the properties of a quantum ensemble (it changes the signs of the external static fields \( V \) and \( \mathcal{A} \)) and hence it transforms one quantum ensemble into another. The particle and antiparticle states cannot be mixed with each other: a superposition of states with the positive and negative values of \( E \) is not a solution of the Dirac equation (and hence Schrödinger’s version of Zitterbewegung) is prohibited.

All this means that it is sufficient to solve Eqs. (4), with the potentials \( V \) and \( \mathcal{A} \), for a particle and then apply these solutions to the corresponding antiparticle moving under the potentials \( -V \) and \( -\mathcal{A} \). In doing so, we have to take into account that for the static electric field, for example, all particle states lie in the region \( \epsilon > V_{\text{min}} \), where \( V_{\text{min}} \) is the minimal value of the scalar potential \( V(r) \) for a given structure.

### 3 Scattering a Dirac particle on the potential step

Our next step is to study the Klein tunneling. Thus, it is sufficient to consider the scattering problem where the vector potential \( \mathcal{A} \) is zero and the scalar potential \( V \) depends only on \( z \), representing a piecewise constant function: \( V(z) = 0 \) for \( z < 0 \) and \( V(z) = V_0 \) for \( z > 0 \); \( V_0 \) is constant. We will also assume that a particle moves toward the potential step from the left, strictly in \( z \)-direction. Note that \( V_{\text{min}} = 0 \) in this problem: thus all states lie in the region \( \epsilon > 0 \).

Note that, in this scattering problem, equations for both spin components are separated from each other. Thus, it is sufficient to consider only the equations for the upper spin. Since Eqs. (7) and (9) are equivalent, the components \( \Phi_+ \) and \( \chi_+ \) are described by the same second-order differential equation.
which can be written for every layer as

\[-\frac{\hbar^2}{2M} \frac{d^2\psi}{dz^2} + V_0 \theta(z)\psi = e\psi,\]  \hspace{1cm} (10)

where \(\theta(z)\) is the Heaviside function. The wave function \(\psi(z; E)\) which represents the pair \((\Phi_+, \chi_+)\) (and the pair \((\Phi_-, \chi_-)\)) is continuous, at the points where the potential \(V(z)\) is discontinuous, together with the function \(\frac{d\Phi(z; E)}{dz}\). The corresponding probability density \(W\) and the probability current \(J_z\) are

\[W = |\psi|^2 + \frac{\hbar^2}{4M^2c^2} \left| \frac{d\psi}{dz} \right|^2, \hspace{1cm} J_z = \frac{i\hbar}{2M} \left( \psi \frac{d\psi^*}{dz} - \psi^* \frac{d\psi}{dz} \right).\]  \hspace{1cm} (11)

Note that the components \(\Phi_+\) and \(\Phi_-\) are described in general by different solutions of Eq. (10). As regards the components \(\chi_+\) and \(\chi_-\), they are determined by the equality \(\chi_{\pm} = \mp \frac{\hbar}{2Mc^2} \frac{d\Phi}{dz}\).

Since the effective masses of the heavy and light components are different and constant in the regions \(z < 0\) and \(z > 0\), let further

\[M_0 = M(\epsilon, 0) = m + \mu_0, \hspace{0.5cm} \mu_0 = \mu(\epsilon, 0) = \frac{\epsilon}{2c^2}, \hspace{0.5cm} M_V = M(\epsilon, V_0) = m + \mu_V, \hspace{0.5cm} \mu_V = \mu(\epsilon, V_0) = \frac{\epsilon - V_0}{2c^2}.\]

Then the general solution of Eq. (10) in the region \(z < 0\), where the particle is free and the effective masses \(M\) and \(\mu\) of both its components are positive, can be written as follows

\[\psi = A_1 e^{i\kappa z} + B_1 e^{-i\kappa z}, \hspace{1cm} \hbar \kappa = \sqrt{2M_0\epsilon} \equiv 2c\sqrt{\mu_0 M_0};\]  \hspace{1cm} (12)

\(A_1\) and \(B_1\) are constants to be determined.

For the region \(z > 0\) we have

\[\psi = A_2 e^{i\kappa z} + B_2 e^{-i\kappa z}, \hspace{1cm} \hbar \kappa = \beta \sqrt{2M_V(\epsilon - V_0)} \equiv 2\beta c\sqrt{\mu_V M_V} \hspace{1cm} (\mu_V M_V > 0);\]  \hspace{1cm} (13a)

\[\psi = A_2 e^{-i\kappa z} + B_2 e^{i\kappa z}, \hspace{1cm} \hbar \kappa = \sqrt{2M_V V_0 - \epsilon} \equiv 2\sqrt{-\mu_V M_V} \hspace{1cm} (\mu_V M_V < 0);\]  \hspace{1cm} (13b)

\(\beta = \text{sign}(M_V)\). \(A_2\) and \(B_2\) are arbitrary constants.

Now, when the effective masses of both quasiparticles can have opposite signs, when they move under the influence of different potentials, we are facing with a more complex situation which depends on the energy of the quasiparticles. Indeed, for \(V_0 > 2mc^2\) we have the following possibilities:

- When \(\epsilon > V_0\) (and hence \(\epsilon > V_0 - 2mc^2\)) both heavy and light quasiparticles have the positive effective masses and both move in the under-barrier regime.
- When \(V_0 - 2mc^2 < \epsilon < V_0\) the heavy quasiparticle has the positive effective mass \(M_V\) and moves, in the region \(z > 0\), in the under-barrier regime – this spatial region is classically forbidden for it. At the same time \(\mu_V < 0\) and, thus, the light quasiparticle behaves in the region \(z > 0\) like an anti-particle. As a consequence, though \(\epsilon > V_0 - 2mc^2\) as in the above case, the region \(z > 0\) is now classically forbidden for the light quasiparticle.
- When \(0 < \epsilon < V_0 - 2mc^2\) (the Klein zone) the effective masses of both quasiparticles are negative. As a consequence, they behave in the region \(z > 0\) like anti-particles; that is, this region is classically accessible for them.

As is seen from this analysis, despite the different effective masses, the heavy and light quasiparticles behave equally in all the energy intervals. Note that when \(\mu_V M_V < 0\) the region \(z > 0\) is classically forbidden for both quasiparticles; otherwise it is classically accessible for them.
3.1 Total reflection

Let us first consider the case when $\mu_V M_V < 0$. This takes place when $V_0 - 2mc^2 < \epsilon < V_0$ for $V_0 > 2mc^2$; otherwise, this condition can be written as $0 < \epsilon < V_0$. In both cases, $M_V > 0$ but $\mu_V < 0$.

Of course, since $\Psi$ should be everywhere bounded, $B_2 = 0$. Then, matching the solutions (12) and (13a) at the point $z = 0$, with making use of the continuity conditions

$$
\Psi|_{z=-0} = \Psi|_{z=+0}, \quad \left. \frac{1}{M_0} \frac{d\Psi}{dz} \right|_{z=-0} = \left. \frac{1}{M_V} \frac{d\Psi}{dz} \right|_{z=+0} \tag{14}
$$

we find the constants $B_1$ and $A_2$:

$$
B_1 = \frac{k - i\kappa}{k + i\kappa} A_1, \quad A_2 = \frac{2k}{k + i\kappa} A_1; \quad \tilde{\kappa} = \frac{k}{M_0} \equiv \frac{2c}{\hbar} \sqrt{\frac{\mu_0}{M_0}}, \quad \kappa = \frac{\kappa}{M_V} \equiv \frac{2c}{\hbar} \sqrt{\frac{\mu_V}{M_V}}. \tag{15}
$$

As it was expected, $|B_1| = |A_1|$. Exps. (12) and (13a), with the constants $B_1$ and $A_2$ (15), represent a standing wave. In this case the probability current density $J_z$ (see Exp. (11)) is zero; the incident and reflected flows coincide with each other – total reflection.

3.2 The Klein tunneling and passage of a particle above the potential step

Let now $\mu_V M_V > 0$. This takes place in the following two cases: when $\epsilon > V_0$ – the passage of a particle above the potential step; when $0 < \epsilon < V_0 - 2mc^2$ – the Klein tunneling (this implies that $V_0 > 2mc^2$).

Matching the solutions (12) and (13a) at the point $z = 0$, with making use of the continuity conditions (14), we obtain

$$
\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \alpha Y \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}; \quad Y = \begin{pmatrix} q & p \\ p & q \end{pmatrix}, \quad q = \frac{1}{\sqrt{T}} = \theta_+, \quad p = \sqrt{\frac{R}{T}} = \theta_-; \tag{16}
$$

$$
\theta_+ = \frac{1}{2} (\alpha^{-1} \pm \alpha), \quad \alpha = \sqrt{\frac{\kappa \tilde{\kappa}}{\kappa}}, \quad \tilde{\kappa} = \kappa \equiv \frac{2c}{\hbar} \sqrt{\frac{\mu_V}{M_V}}. \tag{17}
$$

Here $Y$ is the transfer matrix of the potential step; $T$ and $R$ are the transmission and reflection coefficients, respectively; $R = 1 - T$; note that $\alpha < 1$.

Since a particle source is located on the left of the step the wave $B_2 e^{-i\kappa z}$, associated with the negative current density, must be discarded: $B_2 = 0$. (Note that, in the Klein zone this wave has a positive phase velocity ($-\kappa > 0$) and sometimes namely this wave is erroneously considered as an essential.) As a consequence,

$$
\Psi = A_1 [\exp(i\kappa z) + \sqrt{R} \exp(-i\kappa z)], \quad (z < 0); \quad \Psi = A_1 \alpha^{-1} \sqrt{T} \exp(i\kappa z), \quad (z > 0).
$$

Note that the probability density $W_{tr}$ in the region $z > 0$ as well as the (total) probability current density $J_z$ (see Exps. (11)) are

$$
W_{tr} = \left(1 + \frac{\hbar^2 \kappa^2}{4c^2}ight) |A_2|^2 \equiv \left(1 + \frac{\mu V}{M_V} \right) |A_2|^2, \quad J_z = h\kappa |A_2|^2 \equiv 2c \frac{\mu V}{M_V} |A_2|^2.
$$

Thus, the 'flow' velocity $v_{flow}$ is $J_z/W_{tr}$ in the region $z > 0$ is

$$
v_{flow} = 2c \frac{\sqrt{\mu V}}{M_V + \mu V} \equiv \frac{h\kappa}{M}.
$$

As is seen, $0 < v_{flow} < c$ both for $\epsilon > V_0$ and in the Klein zone. The only peculiarity of the Klein zone is that now the flow and phase velocities of the wave $A_2 \exp(i\kappa z)$ have the opposite signs. This is so because the repulsive potential $V$ becomes attractive, in the Klein zone, both for the 'heavy' and 'light' components; their effective masses $M_V$ and $\mu_V$ are negative in this zone.

Here it is important also to note that both terms in Exp. (11) for $W$ – the first one that corresponds to the 'heavy' component of the Dirac bispinor, as well as the second one that corresponds to its 'light' component – are necessary in order to guarantee the fulfillment of the inequality $v_{flow} < c$. 
4 Discussion and conclusion

By the example of a Dirac particle with a given energy $E$, moving orthogonally to the layers of a spatial structure described by the static scalar potential $V$ and the vector potential $A$, it is shown that the set of two coupled first-order differential equations for the "large" ($\Phi$) and "small" ($\chi$) components of the Dirac bispinor can be presented in the following two equivalent forms: (i) in the form of the Pauli equation for the component $\Phi$ that describes the quantum dynamics, in these fields, of a "heavy" quasiparticle with the effective mass $M$; (ii) in the form of the Pauli equation for the component $\chi$ that describes the quantum dynamics, in the same vector potential but in the scalar potential $V - 2mc^2$, of a "light" quasiparticle with the effective mass $\mu$.

That is, by our approach the ensemble of Dirac particles with the energy $E$, moving in the four-dimensional space-time under the influence of the scalar potential $V$ and vector potential $A$, consists of two subensembles of 'heavy' and 'light' Pauli quasiparticles with the same energy $E$, moving in the Euclidian three-dimensional space under the same vector potential $A$. As regards the scalar potential $V$, the 'heavy' quasiparticle 'sees' it as it stands, while the 'light' quasiparticle 'sees' the reduced potential $V - 2mc^2$, rather than $V$. The effective mass of each Pauli quasiparticle keeps information about the Lorentzian symmetry of the four-dimensional space-time: $M$ and $\mu$ are dynamical rather than inertial or gravitational masses of the Dirac particle. Note that $\mu/M\rightarrow 1$, in the limit $E\rightarrow\infty$.

Contrary to the conventional approach, in ours the 'small' component $\chi$ remains essential even in the non-relativistic limit: firstly, both the 'large' ('heavy') and 'small' ('light') components are equally important in the expression (2) for the probability current density; secondly, both the components are equally important for transforming the system (1) of the first-order differential equations into the equivalent second-order differential (Pauli) equations (7) and (9).

We have to stress once more, since the Dirac bispinor like the Pauli spinor (two-component Schrödinger wave function) describes a quantum particle on the statistical level, there no reason to believe that the Dirac particle is a physical object consisting of these two quasiparticles. It is rather the ensemble of Dirac particles with the 'mass' $(E - V)/c^2$ that consists of two subensembles of quasiparticles with the effective masses $M$ and $\mu$, such that $M + \mu = (E - V)/c^2$. The Dirac particle can move either as a heavy quasiparticle or as a light quasiparticle, with the probabilities $|\Phi|^2$ and $|\chi|^2$, respectively. Any averaging is allowed only for both subensembles of quasiparticles.

As regards the Klein paradox, its old version where the incident flow of particles is less than the outgoing flows should be considered as a result of an incorrect statement of the scattering problem for the Klein zone (it is incorrect to set in (13) $A_2 = 0$ and $B_2 \neq 0$). In the correct statement of this problem the flow of incident particles is always equal to the sum of the absolute values of the outgoing flows. In this case the transmission coefficient for a Dirac particle scattering on the strong potential step, with the energy in the Klein zone, is not zero. By our approach this takes place because both quasiparticles have, in this zone, negative effective masses: a repulsive potential acts on them as an attractive potential.

Thus, according to our approach there is a close relationship between the Lorentzian Dirac’s dynamics and Euclidian Schrödinger’s dynamics. On the one hand, this means that (Schrödinger’s) quantum mechanics is compatible with special relativity. On the other hand, this means that Dirac theory is indeed a quantum theory of single fermions. Besides, this approach says once more that the four-dimensional space-time is not empty. The space is filled with a physical vacuum, and of importance is the nature on the statistical level. Rather it is the task of QED (quantum field theory) which should be treated as a sub-quantum theory.

This approach opens also the possibility to apply the mathematical methods of solving the stationary Schrödinger equation to the Dirac equation. In particular, this concerns the well-known transfer-matrix approach which is suitable for solving the Schrödinger equation with piecewise constant effective mass and potential function.

Among urgent tasks is the study of the temporal aspects of the Dirac quantum dynamics. In our opinion, this can help us to observe indirectly the individual quantum dynamics of the heavy and light components $\Phi$ and $\chi$ (and to indirectly measure their effective masses). Besides, an interesting task is to study the Dirac dynamics of wave packets consisting of the stationary solutions that correspond to energies from the Klein zone and the interval $E > V_0 + mc^2$; in this approach there is also enough room
for Zitterbewegung (but now this phenomenon does not imply a mixing of the particle and antiparticle states).

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