Abstract. In this study, we define the spaces $M_u(\Delta), C_p(\Delta), C_0p(\Delta), C_r(\Delta)$ and $L_q(\Delta)$ of double sequences whose difference transforms are bounded, convergent in the Pringsheim’s sense, null in the Pringsheim’s sense, both convergent in the Pringsheim’s sense and bounded, regularly convergent and absolutely $q$–summable, respectively, and also examine some inclusion relations related to those sequence spaces. Furthermore, we show that these sequence spaces are Banach spaces. We determine the alpha-dual of the space $M_u(\Delta)$ and the $\beta(v)$–dual of the space $C_q(\Delta)$ of double sequences, where $v, \eta \in \{p, bp, r\}$. Finally, we characterize the classes $(\mu : C_v(\Delta))$ for $v \in \{p, bp, r\}$ of four dimensional matrix transformations, where $\mu$ is any given space of double sequences.

1. Introduction and preliminaries

By $\omega$ and $\Omega$, we denote the sets of all real valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication. Any vector subspaces of $\omega$ and $\Omega$ are called as the single sequence space and double sequence space, respectively. By $M_u$, we denote the space of all bounded double sequences, that is

$$M_u := \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty \right\}$$

which is a Banach space with the norm $\|x\|_\infty$; where $\mathbb{N}$ denotes the set of all positive integers. Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$ then we call that the double sequence $x$ is convergent in the Pringsheim’s sense to the limit $l$ and write $p - \lim x_{mn} = l$; where $\mathbb{R}$ denotes the real field. By $C_p$, we denote the space of all convergent double sequences in the Pringsheim’s sense. It is well-known that there are such sequences in the space $C_p$ but not in the space $M_u$. Indeed following Boos [5], if we define the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} n & : m = 1, n \in \mathbb{N}, \\ 0 & : m \geq 2, n \in \mathbb{N}, \end{cases}$$

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then it is trivial that \( x \in C_p \setminus M_u \), since \( p - \lim x_{mn} = 0 \) but \( \|x\|_\infty = \infty \). So, we can consider the space \( C_{bp} \) of the double sequences which are both convergent in the Pringsheim’s sense and bounded, i.e., \( C_{bp} = C_p \cap M_u \). A sequence in the space \( C_p \) is said to be \emph{regularly convergent} if it is a single convergent sequence with respect to each index and denote the set of all such sequences by \( C_r \). Also by \( C_{bp} \) and \( C_r \), we denote the spaces of all double sequences converging to 0 contained in the sequence spaces \( C_{bp} \) and \( C_r \), respectively. Móricz [9] proved that \( C_{bp}, C_{bp0}, C_r \) and \( C_{r0} \) are Banach spaces with the norm \( \|\cdot\|_\infty \).

Let us consider the isomorphism \( T \) defined by Zeltser [14] as

\[
T : \Omega \to \omega
\]

\[
x \mapsto z = (z_i) := (x_{\varphi^{-1}(ij)}),
\]

where \( \varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is a bijection defined by

\[
\varphi[(1, 1)] = 1, \\
\varphi[(1, 2)] = 2, \varphi[(2, 2)] = 3, \varphi[(2, 1)] = 4, \\
\ldots
\]

\[
\varphi[(1, n)] = (n - 1)^2 + 1, \varphi[(2, n)] = (n - 1)^2 + 2, \\
\varphi[(n, n)] = (n - 1)^2 + n, \varphi[(n, n - 1)] = n^2 - n + 2, \\
\ldots
\]

Let us consider a double sequence \( x = (x_{mn}) \) and define the sequence \( s = (s_{mn}) \) which will be used throughout via \( x \) by

\[
s_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij}
\]

for all \( m, n \in \mathbb{N} \). For the sake of brevity, here and in what follows, we abbreviate the summation \( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \) by \( \sum_{i,j} \) and we use this abbreviation with other letters. Let \( \lambda \) be a space of a double sequences, converging with respect to some linear convergence rule \( v - \lim : \lambda \to \mathbb{R} \). The sum of a double series \( \sum_{i,j} x_{ij} \) with respect to this rule is defined by \( v - \lim x_{ij} = v - \lim_{m,n \to \infty} s_{mn} \). Let \( \lambda, \mu \) be two spaces of double sequences, converging with respect to the linear convergence rules \( v_1 - \lim \) and \( v_2 - \lim \), respectively, and \( A = (a_{mnl}) \) also be a four dimensional infinite matrix over the real or complex field.

The \( \alpha - \text{dual} \lambda^{\alpha}, \beta(v) - \text{dual} \lambda^{\beta(v)} \) with respect to the \( v \)-convergence for \( v \in \{p, bp, r\} \) and the \( \gamma - \text{dual} \lambda^{\gamma} \) of a double sequence space \( \lambda \) are respectively defined.
by

$$
\lambda^\alpha := \left\{ (a_{ij}) \in \Omega : \sum_{i,j} |a_{ij}x_{ij}| < \infty \text{ for all } (x_{ij}) \in \lambda \right\},
$$

$$
\lambda^{\beta(v)} := \left\{ (a_{ij}) \in \Omega : v - \sum_{i,j} a_{ij}x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\},
$$

$$
\lambda^\gamma := \left\{ (a_{ij}) \in \Omega : \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=1}^{k,l} a_{ij}x_{ij} \right| < \infty \text{ for all } (x_{ij}) \in \lambda \right\}.
$$

It is easy to see for any two spaces $\lambda, \mu$ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(v)}$ holds while the inclusion $\lambda^{\beta(v)} \subset \lambda^\gamma$ does not hold, since the $v-$convergence of the sequence of partial sums of a double series does not imply its boundedness.

The $v-$summability domain $\lambda_A^{(v)}$ of a four dimensional infinite matrix $A = (a_{mnkl})$ in a space $\lambda$ of a double sequences is defined by

$$
\lambda_A^{(v)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left( v - \sum_{k,l} a_{mnkl}x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and is in } \lambda \right\}.
$$

We say, with the notation (1.3), that $A$ maps the space $\lambda$ into the space $\mu$ if and only if $Ax$ exists and is in $\mu$ for all $x \in \lambda$ and denote the set of all four dimensional matrices, transforming the space $\lambda$ into the space $\mu$, by $(\lambda : \mu)$. It is trivial that for any matrix $A \in (\lambda : \mu)$, $(a_{mnkl})_{k,l \in \mathbb{N}}$ is in the $\beta(v)-$dual $\lambda^{\beta(v)}$ of the space $\lambda$ for all $m, n \in \mathbb{N}$. An infinite matrix $A$ is said to be $C_v-$conservative if $C_v \subset (C_v)_A$. Also by $(\lambda : \mu; p)$, we denote the class of all four dimensional matrices $A = (a_{mnkl})$ in the class $(\lambda : \mu)$ such that $v_2 - \lim Ax = v_1 - \lim x$ for all $x \in \lambda$.

Now, following Zeltser [15] we note the terminology for double sequence spaces. A locally convex double sequence space $\lambda$ is called a $DK-$space, if all of the seminorms $r_{kl} : \lambda \to \mathbb{R}$, $x = (x_{kl}) \mapsto |x_{kl}|$ for all $k,l \in \mathbb{N}$ are continuous. A $DK-$space with a Fréchet topology is called an $FDK-$space. A normed $FDK-$space is called a $BDK-$space. We record that $C_r$ endowed with the norm $\| \cdot \|_\infty : C_r \to \mathbb{R}$, $x = (x_{kl}) \mapsto \sup_{k,l \in \mathbb{N}} |x_{kl}|$ is a $BDK-$space.
Let us define the following sets of double sequences:

\[ M_u(t) := \left\{ (x_{mn}) \in \Omega : \sup_{m,n} |x_{mn}|^{t_{mn}} < \infty \right\}, \]

\[ C_p(t) := \left\{ (x_{mn}) \in \Omega : \exists l \in \mathbb{C} \ni p - \lim_{m,n \to \infty} |x_{mn} - l|^{t_{mn}} = 0 \right\}, \]

\[ C_{0p}(t) := \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 0 \right\}, \]

\[ L_u(t) := \left\{ (x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}|^{t_{mn}} < \infty \right\}, \]

\[ C_{bp}(t) := C_p(t) \cap M_u(t) \quad \text{and} \quad C_{0bp}(t) := C_{0p}(t) \cap M_u(t); \]

where \( t = (t_{mn}) \) is the sequence of strictly positive reals \( t_{mn} \) for all \( m, n \in \mathbb{N} \). In the case \( t_{mn} = 1 \) for all \( m, n \in \mathbb{N} \), \( M_u(t), C_p(t), C_{0p}(t), L_u(t), C_{bp}(t) \) and \( C_{0bp}(t) \) reduce to the sets \( M_u, C_p, C_{0p}, L_u, C_{bp} \) and \( C_{0bp} \), respectively. Now, we can summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [6, 7] have proved that \( M_u(t), C_p(t) \) and \( C_{bp}(t) \) are complete paranormed spaces of double sequences and gave the alpha-, beta-, gamma-duals of the spaces \( M_u(t) \) and \( C_{bp}(t) \). Quite recently, in her PhD thesis, Zeltser [14] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [10] have introduced the statistical convergence and statistical Cauchy for double sequences, and gave the relation between statistically convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [11] and Mursaleen and Edely [12] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the \( M \)-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence \( x = (x_{jk}) \) into one whose core is a subset of the \( M \)-core of \( x \). More recently, Altay and Başar [1] have defined the spaces \( BS, BS(t), CS_{bp}, CS_r \) and \( BV \) of double series whose sequence of partial sums are in the spaces \( M_u, M_u(t), C_p, C_{bp}, C_r \) and \( L_u \), respectively, and also examined some properties of those sequence spaces and determined the alpha-duals of the spaces \( BS, BV, CS_{bp} \) and the \( \beta(v) \)-duals of the spaces \( CS_{bp} \) and \( CS_r \) of double series. Quite recently, Başar and Sever [4] have introduced the Banach space \( L_q \) of double sequences corresponding to the well-known space \( \ell_q \) of absolutely \( q \)-summable single sequences and examined some properties of the space \( L_q \). Furthermore, they determine the \( \beta(v) \)-dual of the space and establish that the alpha- and gamma-duals of the space \( L_q \) coincide with the \( \beta(v) \)-dual; where

\[ L_q := \left\{ (x_{ij}) \in \Omega : \sum_{i,j} |x_{ij}|^q < \infty \right\}, \quad (1 \leq q < \infty), \]

\[ CS_v := \left\{ (x_{ij}) \in \Omega : (s_{mn}) \in C_v \right\}. \]

Here and after we assume that \( v \in \{ p, bp, r \} \).

The double difference matrix \( \Delta = (\delta_{mnkl}) \) of order one is defined by
\[
\delta_{mnkl} := \begin{cases} 
(-1)^{m+n-k-l} , & m - 1 \leq k \leq m, \ n - 1 \leq l \leq n, \\
0 , & \text{otherwise}
\end{cases}
\]

for all \( m, n, k, l \in \mathbb{N} \). Define the sequence \( y = (y_{mn}) \) as the \( \Delta \)-transform of a sequence \( x = (x_{mn}) \), i.e.,

\[
y_{mn} = (\Delta x)_{mn} = x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}
\]

for all \( m, n \in \mathbb{N} \). Additionally, a direct calculation gives the inverse \( \Delta^{-1} = S = (s_{mnkl}) \) of the matrix \( \Delta \) as follows:

\[
s_{mnkl} := \begin{cases} 
1 , & 0 \leq k \leq m, \ 0 \leq l \leq n, \\
0 , & \text{otherwise}
\end{cases}
\]

for all \( m, n, k, l \in \mathbb{N} \).

In the present paper, we introduce the new double difference sequence spaces \( \mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta) \) and \( \mathcal{L}_q(\Delta) \), that is,

\[
\mathcal{M}_u(\Delta) := \left\{(x_{mn}) \in \Omega : \sup_{m,n \in \mathbb{N}} |y_{mn}| < \infty \right\},
\]

\[
\mathcal{C}_p(\Delta) := \left\{(x_{mn}) \in \Omega : \exists \ell \in \mathbb{C} \ni p - \lim_{m,n \to \infty} |y_{mn} - \ell| = 0 \right\},
\]

\[
\mathcal{C}_{0p}(\Delta) := \left\{(x_{mn}) \in \Omega : p - \lim_{m,n \to \infty} |y_{mn}| = 0 \right\},
\]

\[
\mathcal{L}_q(\Delta) := \left\{(x_{ij}) \in \Omega : \sum_{m,n} |y_{mn}|^q < \infty \right\}, \quad (1 \leq q < \infty).
\]

By \( \mathcal{C}_{0p}(\Delta) \) and \( \mathcal{C}_r(\Delta) \), we denote the sets of all the \( \Delta \)-transforms convergent and bounded, and the \( \Delta \)-transforms regularly convergent double sequences. One can easily see that the spaces \( \mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_r(\Delta) \) and \( \mathcal{L}_q(\Delta) \) are the domain of the double difference matrix \( \Delta \) in the spaces \( \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{C}_{0p}, \mathcal{C}_r \) and \( \mathcal{L}_q \), respectively.

2. Some new double difference sequence spaces

In the present section, we deal with the sets \( \mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_r(\Delta) \) and \( \mathcal{L}_q(\Delta) \) consisting of the double sequences whose \( \Delta \)-transforms of order one are in the spaces \( \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{C}_{0p}, \mathcal{C}_r \) and \( \mathcal{L}_q \), respectively.

**Theorem 2.1.** The sets \( \mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_r(\Delta) \) and \( \mathcal{L}_q(\Delta) \) are the linear spaces with the coordinatewise addition and scalar multiplication, and \( \mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_r(\Delta) \) and \( \mathcal{L}_q(\Delta) \) are the Banach spaces with the norms

\[
\|x\|_{\mathcal{M}_u(\Delta)} = \sup_{m,n \in \mathbb{N}} |x_{mn} + x_{m-1,n-1} - x_{m,n-1} - x_{m-1,n}|,
\]

\[
\|x\|_{\mathcal{L}_q(\Delta)} = \left[ \sum_{m,n} |x_{mn} + x_{m-1,n-1} - x_{m,n-1} - x_{m-1,n}|^q \right]^{1/q}, \quad (1 \leq q < \infty).
\]
Theorem 2.2. The space $\lambda(\Delta)$ is linearly isomorphic to the space $\lambda$, where $\lambda$ denotes any of the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_p, \mathcal{C}_b, \mathcal{C}_q$ and $\mathcal{L}_q$.

Proof. We show here that $\mathcal{M}_u(\Delta)$ is linearly isomorphic to $\mathcal{M}_u$. Consider the transformation $T$ from $\mathcal{M}_u(\Delta)$ to $\mathcal{M}_u$ defined by $x = (x_{jk}) \mapsto y = (y_{mn})$. Then, clearly $T$ is linear and injective. Now, define the sequence $x = (x_{jk})$ by

$$x_{jk} = \sum_{m=0}^{j} \sum_{n=0}^{k} y_{mn}$$

(2.6)
for all \( j, k \in \mathbb{N} \). Suppose that \( y \in \mathcal{M}_u \). Then, since
\[
\|x\|_{\mathcal{M}_u(\Delta)} = \sup_{j, k \in \mathbb{N}} \left| \sum_{m=0}^{j} \sum_{n=0}^{k} y_{mn} \right|
= \sup_{j, k \in \mathbb{N}} |y_{jk}| = \|y\|_\infty < \infty,
\]
x = (x_{jk}) defined by (2.6) is in the space \( \mathcal{M}_u(\Delta) \). Hence, \( T \) is surjective and norm preserving. This completes the proof of the theorem. \( \square \)

Now, we give some inclusion relations between the double difference sequence spaces.

**Theorem 2.3.** \( \mathcal{M}_u \) is the subspace of the space \( \mathcal{M}_u(\Delta) \).

**Proof.** Let us take \( x = (x_{mn}) \in \mathcal{M}_u \). Then, there exists an \( K \) such that
\[
\sup_{m, n \in \mathbb{N}} |x_{mn}| \leq K
\]
for all \( m, n \in \mathbb{N} \), one can observe that
\[
|\Delta x_{mn}| = |x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}|
\leq |x_{mn}| + |x_{m,n-1}| + |x_{m-1,n}| + |x_{m-1,n-1}|. \tag{2.7}
\]
Then, we see by taking supremum over \( m, n \in \mathbb{N} \) in (2.7) that \( \|x\|_\infty \leq 4K \), i.e., \( x \in \mathcal{M}_u(\Delta) \).

Now, we see that the inclusion is strict. Let \( x = (x_{mn}) \) be defined by
\[
x_{mn} = mn
\]
for all \( m, n \in \mathbb{N} \). Then the sequence is in \( x \in \mathcal{M}_u(\Delta) \setminus \mathcal{M}_u \). This completes the proof of the theorem. \( \square \)

**Lemma 2.4.** [1, Theorem 1.2] \( \mathcal{L}_u \subset \mathcal{B} \mathcal{S} \subset \mathcal{M}_u \) strictly hold.

**Lemma 2.5.** [1, Theorem 2.9] Let \( v \in \{p, bp, r\} \). Then, the inclusion \( \mathcal{B} \mathcal{V} \subset \mathcal{C}_v \) and \( \mathcal{B} \mathcal{V} \subset \mathcal{M}_u \) strictly hold.

Combining Lemma 2.4, Lemma 2.5 and Theorem 2.3, we get the following corollaries.

**Corollary 2.6.** The inclusion \( \mathcal{L}_u \subset \mathcal{B} \mathcal{S} \subset \mathcal{M}_u(\Delta) \) strictly hold.

**Corollary 2.7.** The inclusion \( \mathcal{B} \mathcal{V} \subset \mathcal{M}_u(\Delta) \) strictly holds.

**Theorem 2.8.** The inclusion \( \mathcal{L}_q \subset \mathcal{L}_q(\Delta) \) strictly holds; where \( 1 \leq q < \infty \).

**Proof.** The prove the validity of the inclusion \( \mathcal{L}_q \subset \mathcal{L}_q(\Delta) \) for \( 1 \leq q < \infty \), it suffices to show the existence of a number \( K > 0 \) such that
\[
\|x\|_{\mathcal{L}_q(\Delta)} \leq K \|x\|_{\mathcal{L}_q}
\]
for every $x \in \mathcal{L}_q$. Let $x \in \mathcal{L}_q$ and $1 \leq q < \infty$. Then, we obtain

$$\|x\|_{\mathcal{L}_q(\Delta)} = \left\{ \sum_{m,n} |x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}|^q \right\}^{1/q} \leq 4\|x\|_{\mathcal{L}_q}.$$ 

This shows that the inclusion $\mathcal{L}_q \subset \mathcal{L}_q(\Delta)$ holds.

Additionally, since the sequence

$$x_{mn} := \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $m,n \in \mathbb{N}$ is in $\mathcal{L}_q(\Delta)$ but not in $\mathcal{L}_q$, as asserted. This completes the proof. \(\square\)

**Theorem 2.9.** The following statements hold:

(i) $\mathcal{C}_p$ is the subspace of the space $\mathcal{C}_p(\Delta)$.

(ii) $\mathcal{C}_{0p}$ is the subspace of the space $\mathcal{C}_{0p}(\Delta)$.

(iii) $\mathcal{C}_{bp}$ is the subspace of the space $\mathcal{C}_{bp}(\Delta)$.

(iv) $\mathcal{C}_r$ is the subspace of the space $\mathcal{C}_r(\Delta)$.

**Proof.** We only prove that the inclusion $\mathcal{C}_p \subset \mathcal{C}_p(\Delta)$ holds. Let us take $x \in \mathcal{C}_p$. Then, for a given $\varepsilon > 0$, there exists an $n_x(\varepsilon) \in \mathbb{N}$ such that

$$|x_{mn} - l| < \frac{\varepsilon}{4}$$

for all $m, n > n_x(\varepsilon)$. Then,

$$|x_{mn} - l| + |x_{m,n-1} - l| + |x_{m-1,n} - l| + |x_{m-1,n-1} - l| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

for sufficiently large $m, n$ which means that $p-\lim(\Delta x)_{mn} = 0$. Hence, $x \in \mathcal{C}_p(\Delta)$ that is to say that $\mathcal{C}_p \subset \mathcal{C}_p(\Delta)$ holds, as expected.

Now, we see that the inclusion is strict. Let $x = (x_{mn})$ be defined by

$$x_{mn} = (m+1)(n+1)$$

for all $m, n \in \mathbb{N}$. It is easy to see that

$$p-\lim(\Delta x)_{mn} = 1.$$ 

But

$$\lim_{m,n \to \infty} (m+1)(n+1)$$

which does not tend to a finite limit. Hence $x \notin \mathcal{C}_p$. This completes the proof. \(\square\)

**Lemma 2.10.** [1, Theorem 2.3] $\mathcal{CS}_p$ is the subspace of $\mathcal{C}_p$.

Combining Lemma 2.10 and Theorem 2.9, we get the following corollary.

**Corollary 2.11.** The inclusion $\mathcal{CS}_p \subset \mathcal{C}_p(\Delta)$ strictly holds.
3. The Alpha- and Beta-Duals of the New Spaces of Double Sequences

In this section, we determine the alpha-dual of the space $\mathcal{M}_u(\Delta)$ and the $\beta(r)$—dual of the space $\mathcal{C}_r(\Delta)$, and $\beta(\vartheta)$—dual of the space $\mathcal{C}_\vartheta(\Delta)$ of double sequences, $\vartheta, \eta \in \{p, bp, r\}$. Although the alpha-dual of a space of double sequences is unique, its beta-dual may be more than one with respect to $\vartheta$—convergence.

**Theorem 3.1.** $\{\mathcal{M}_u(\Delta)\}^\alpha = \mathcal{L}_u$.

**Proof.** Let $x = (x_{kl}) \in \mathcal{M}_u(\Delta)$ and $z = (z_{kl}) \in \mathcal{L}_u$. Hence, there is a sequence $y = (y_{ij}) \in \mathcal{M}_u$ related with $x = (x_{kl})$ from Theorem 2.2 and there is a positive real number $K$ such that $|y_{ij}| \leq \frac{K}{(k+1)(l+1)}$ for all $i, j \in \mathbb{N}$. So we use the relation (2.6) we have that,

$$
\sum_{k,l} |z_{kl}x_{kl}| = \sum_{k,l} \left| z_{kl} \sum_{i=0}^{k} \sum_{j=0}^{l} y_{ij} \right| \leq K \sum_{k,l} |z_{kl}| < \infty
$$

so $z \in \{\mathcal{M}_u(\Delta)\}^\alpha$, that is

$$
\mathcal{L}_u \subset \{\mathcal{M}_u(\Delta)\}^\alpha.
$$

Conversely, suppose that $z = (z_{kl}) \in \{\mathcal{M}_u(\Delta)\}^\alpha$, that is $\sum_{k,l} |z_{kl}x_{kl}| < \infty$ for all $x = (x_{kl}) \in \mathcal{M}_u(\Delta)$. If $z = (z_{kl}) \notin \mathcal{L}_u$, then $\sum_{k,l} |z_{kl}| = \infty$. Further, if we choose $y = (y_{kl})$ such that

$$
y_{kl} := \begin{cases} 
\frac{1}{(k+1)(l+1)}, & 0 \leq i \leq k, \quad 0 \leq j \leq l \\
0, & \text{otherwise}
\end{cases}
$$

for all $k, l \in \mathbb{N}$. Then, $y \in \mathcal{M}_u$ but

$$
\sum_{k,l} |z_{kl}x_{kl}| = \sum_{k,l} \left| z_{kl} \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{1}{(k+1)(l+1)} \right| = \sum_{k,l} |z_{kl}| = \infty.
$$

Hence, $z \notin \{\mathcal{M}_u(\Delta)\}^\alpha$, this is a contradiction. So, we have the following inclusion,

$$
\{\mathcal{M}_u(\Delta)\}^\alpha \subset \mathcal{L}_u.
$$

Hence, from the inclusions (3.1) and (3.2) we get

$$
\{\mathcal{M}_u(\Delta)\}^\alpha = \mathcal{L}_u.
$$

□

Now, we may give the $\beta$—duals of the spaces with respect to the $\vartheta$—convergence using the technique in [2] and [3] for the single sequences.

The conditions for a 4-dimensional matrix to transform the spaces $\mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{C}_p$ into the space $\mathcal{C}_{bp}$ are well known (see for example [8, 16]).
Lemma 3.2. The matrix $A = (a_{mnij})$ is in $(\mathcal{C}_r : \mathcal{C}_\vartheta)$ if and only if the following conditions hold:

$$\sup_{m,n \in \mathbb{N}} \sum_{i,j} |a_{mnij}| < \infty,$$  \hfill (3.3)

$$\exists v \in \mathbb{C} \ni \vartheta - \lim_{m,n \to \infty} \sum_{i,j} a_{mnij} = v,$$  \hfill (3.4)

$$\exists (a_{ij}) \in \Omega \ni \vartheta - \lim_{m,n \to \infty} a_{mnij} = a_{ij} \text{ for all } i,j \in \mathbb{N},$$  \hfill (3.5)

$$\exists u^{j_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n \to \infty} \sum_{i} a_{mnijo} = u^{j_0} \text{ for fixed } j_0 \in \mathbb{N},$$  \hfill (3.6)

$$\exists v_{i_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n \to \infty} \sum_{j} a_{mnioj} = v_{i_0} \text{ for fixed } i_0 \in \mathbb{N}. \hfill (3.7)$$

Lemma 3.3. The matrix $A = (a_{mnij})$ is in $(\mathcal{C}_{bp} : \mathcal{C}_\vartheta)$ if and only if the conditions (3.3)-(3.5) of Lemma 3.2 hold, and

$$\vartheta - \lim_{m,n \to \infty} \sum_{i} |a_{mnijo} - a_{ij}| = 0 \text{ for each fixed } j_0 \in \mathbb{N},$$  \hfill (3.8)

$$\vartheta - \lim_{m,n \to \infty} \sum_{j} |a_{mnioj} - a_{i0j}| = 0 \text{ for each fixed } i_0 \in \mathbb{N}. \hfill (3.9)$$

Lemma 3.4. The matrix $A = (a_{mnij})$ is in $(\mathcal{C}_p : \mathcal{C}_\vartheta)$ if and only if the conditions (3.3)-(3.5) of Lemma 3.2 hold, and

$$\forall i \in \mathbb{N} \exists J \in \mathbb{N} \ni a_{mnij} = 0 \text{ for } j > J \text{ for all } m,n \in \mathbb{N},$$  \hfill (3.10)

$$\forall j \in \mathbb{N} \exists I \in \mathbb{N} \ni a_{mnij} = 0 \text{ for } i > I \text{ for all } m,n \in \mathbb{N}. \hfill (3.11)$$

Theorem 3.5. Define the sets

$$F_1 = \left\{ a = (a_{ij}) \in \Omega : \sum_{i,j} (i+1)(j+1)|a_{ij}| < \infty \right\},$$

$$F_2 = \left\{ a = (a_{ij}) \in \Omega : r - \lim_{m,n \to \infty} \sum_{i} \sum_{p=i}^{m} \sum_{q=j_0}^{n} a_{pq} \text{ exists for each fixed } j_0 \right\},$$

$$F_3 = \left\{ a = (a_{ij}) \in \Omega : r - \lim_{m,n \to \infty} \sum_{j} \sum_{p=i_0}^{m} \sum_{q=j}^{n} a_{pq} \text{ exists for each fixed } i_0 \right\}.$$

Then, $\{\mathcal{C}_r(\Delta)\}_{\beta(r)} = F_1 \cap F_2 \cap F_3$. 
Proof. Let \( x = (x_{ij}) \in \mathcal{C}_r(\Delta) \). Then, there exists a sequence \( y = (y_{mn}) \in \mathcal{C}_r \). Consider the following equality

\[
 z_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x_{ij} = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \sum_{p=0}^{i} \sum_{q=0}^{j} y_{pq} \right) a_{ij}
 = \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \sum_{p=i}^{m} \sum_{q=j}^{n} a_{pq} \right) y_{ij}
 = \sum_{i=0}^{m} \sum_{j=0}^{n} b_{mnij} y_{ij} = (By)_{ij}
\]

for all \( m,n \in \mathbb{N} \). Hence we can define the four-dimensional matrix \( B = (b_{mnij}) \) as following

\[
b_{mnij} := \begin{cases} 
  \sum_{p=i}^{m} \sum_{q=j}^{n} a_{pq} , & 0 \leq i \leq m, \ 0 \leq j \leq n , \\
  0 , & \text{otherwise}.
\end{cases}
\] (3.12)

Thus we see that \( ax = (a_{mn}x_{mn}) \in \mathcal{CS}_r \) whenever \( x = (x_{mn}) \in \mathcal{C}_r(\Delta) \) if and only if \( z = (z_{mn}) \in \mathcal{C}_r \) whenever \( y = (y_{mn}) \in \mathcal{C}_r \). This means that \( a = (a_{mn}) \in \{ \mathcal{C}_r(\Delta) \}^{\beta(r)} \) if and only if \( B \in (\mathcal{C}_r : \mathcal{C}_r) \). Therefore, we consider the following equality and equation

\[
 r - \lim_{m,n \to \infty} \sum_{i,j} b_{mnij} = r - \lim_{m,n \to \infty} \sum_{i=0}^{m} \sum_{j=0}^{n} \left( \sum_{p=0}^{i} \sum_{q=0}^{j} a_{pq} \right)
 = \sum_{i,j} \left( \sum_{p=0}^{i} \sum_{q=0}^{j} a_{pq} \right).
\] (3.14)

Then, we derive from the condition (3.3)-(3.5) that

\[
 \sum_{i,j} (i + 1)(j + 1)|a_{ij}| < \infty.
\] (3.15)

Further, from Lemma 3.2 conditions (3.6) and (3.7),

\[
r - \lim_{m,n \to \infty} \sum_{i} b_{mnij_0} = r - \lim_{m,n \to \infty} \sum_{i} \sum_{p=i}^{m} \sum_{q=j_0}^{n} a_{pq}
\] (3.16)
exists for each fixed \( j_0 \in \mathbb{N} \) and
\[
\vartheta - \lim_{m,n \to \infty} \sum_{j} b_{mni_0} = r - \lim_{m,n \to \infty} \sum_{i} \sum_{p=i_0}^{m} \sum_{q=j_0}^{n} a_{pq} \tag{3.17}
\]
exists for each fixed \( i_0 \in \mathbb{N} \). This shows that \( \{C_r(\Delta)\}^{\beta(r)} = F_1 \cap F_2 \cap F_3 \) which completes the proof. \( \square \)

Now, we may give our theorem exhibiting the \( \beta(\vartheta) \)-dual of the series space \( C_\eta(\Delta) \) in the case \( \eta, \vartheta \in \{p, bp, r\} \), without proof.

**Theorem 3.6.** \( \{C_\eta(\Delta)\}^{\beta(\vartheta)} = \{a = (a_{mn}) \in \Omega : B = (b_{mnij}) \in (C_\eta : C_\vartheta)\} \) where \( B = (b_{mnij}) \) is defined by (3.12).

4. **Characterization of some classes of four dimensional matrices**

In the present section, we characterize the matrix transformations from the space \( C_r(\Delta) \) to the double sequence space \( C_\vartheta \). Although the theorem characterizing the class \( (\mu : C_\vartheta(\Delta)) \) are stated and proved, the necessary and sufficient conditions on a four dimensional matrix belonging to the classes \( (C_r : C_r(\Delta)) \) and \( (C_{bp} : C_{bp}(\Delta)) \) also given without proof.

**Theorem 4.1.** \( A = (a_{mnkl}) \in (C_r(\Delta) : C_\vartheta) \) if and only if the following conditions hold:
\[
\sup_{m,n} \sum_{k,l} \left| \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{mnpq} \right| < \infty, \tag{4.1}
\]
\[
\vartheta - \lim_{s,t \to \infty} \sum_{i=0}^{t} \sum_{j=0}^{s} a_{mnpq} \text{ exists for fixed } j_0, \tag{4.2}
\]
\[
\vartheta - \lim_{s,t \to \infty} \sum_{j=0}^{t} \sum_{i=0}^{s} a_{mnpq} \text{ exists for fixed } i_0, \tag{4.3}
\]
\[
\vartheta - \lim_{m,n} \sum_{p=i_0}^{\infty} \sum_{q=j_0}^{\infty} a_{mnpq} = a_{kl} \text{ for all } k, l \in \mathbb{N}, \tag{4.4}
\]
\[
\exists u_{l_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n} \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} a_{mnpq} = u_{l_0} \text{ for fixed } l_0 \in \mathbb{N}, \tag{4.5}
\]
\[
\exists v_{k_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n} \sum_{l} \sum_{p=k_0}^{\infty} \sum_{q=l_0}^{\infty} a_{mnpq} = v_{k_0} \text{ for fixed } k_0 \in \mathbb{N}, \tag{4.6}
\]
\[
\exists v \in \mathbb{C} \ni \vartheta - \lim_{m,n} \sum_{k,l} \sum_{p=k_0}^{\infty} \sum_{q=l_0}^{\infty} a_{mnpq} = v. \tag{4.7}
\]

**Proof.** Let us take any \( x = (x_{mn}) \in C_r(\Delta) \) and define the sequence \( y = (y_{kl}) \) by
\[
y_{kl} = x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}; \quad (k, l \in \mathbb{N}).
\]
Then, \( y = (y_{kl}) \in \mathcal{C}_r \) by Theorem 2.2. Now, for the \((s, t)\)th rectangular partial sum of the series \( \sum_{j,k} a_{mnjk} x_{jk} \), we derive that

\[
(Ax)_{mn}^{[s,t]} = \sum_{j=0}^{s} \sum_{k=0}^{t} a_{mnjk} x_{jk}
\]

\[
= \sum_{j=0}^{s} \sum_{k=0}^{t} \left( \sum_{p=0}^{j} \sum_{q=0}^{k} y_{pq} \right) a_{mnjk}
\]

\[
= \sum_{j=0}^{s} \sum_{k=0}^{t} \left( \sum_{p=j}^{s} \sum_{q=k}^{t} a_{mnpq} \right) y_{jk}
\]

(4.8)

for all \( m, n, s, t \in \mathbb{N} \). Define the matrix \( B_{mn} = (b_{mnjk}^{[s,t]}) \) by

\[
b_{mnjk}^{[s,t]} := \begin{cases} 
\sum_{p=j}^{s} \sum_{q=k}^{t} a_{mnpq}, & 0 \leq j \leq s, \quad 0 \leq k \leq t, \\
0, & \text{otherwise.}
\end{cases}
\]

(4.9)

Then, the equality (4.8) may be rewritten as

\[
(Ax)_{mn}^{[s,t]} = (B_{mn}y)_{s,t}.
\]

(4.10)

Then, the convergence of the rectangular partial sums \((Ax)_{mn}^{[s,t]}\) in the regular sense for all \( m, n \in \mathbb{N} \) and for all \( x \in \mathcal{C}_r(\Delta) \) is equivalent of saying that \( B_{mn} \in (\mathcal{C}_r : \mathcal{C}_\vartheta) \). Hence, the following conditions

\[
\sum_{k,l}(k+1)(l+1)|a_{mnl}| < \infty,
\]

(4.11)

\[
\vartheta - \lim_{s,t \to \infty} \sum_{i=0}^{s} \sum_{p=i}^{s} \sum_{p=j_0}^{t} a_{mnpq} - \text{exists for fixed } j_0,
\]

(4.12)

\[
\vartheta - \lim_{s,t \to \infty} \sum_{j=0}^{t} \sum_{p=j_0}^{t} \sum_{p=q}^{s} a_{mnpq} - \text{exists for fixed } i_0
\]

(4.13)

must be satisfied for every fixed \( m, n \in \mathbb{N} \). In this case,

\[
\vartheta - \lim_{s,t \to \infty} b_{mnjk}^{[s,t]} = \sum_{p=j}^{s} \sum_{q=k}^{t} a_{mnpq},
\]

\[
\vartheta - (Ax)_{mn}^{[s,t]} = \vartheta - (B_{mn}y)
\]

hold. Thus, we derive from the two-sided implication "\( Ax \) is in \( \mathcal{C}_r \) whenever \( x \in \mathcal{C}_r(\Delta) \) if and only if \( B = (\sum_{p=j}^{s} \sum_{q=k}^{t} a_{mnpq})_{mn} \in (\mathcal{C}_r : \mathcal{C}_\vartheta) \)”, we have Lemma
3.2 that

\[
\sup_{m,n} \sum_{k,l} \sum_{p=q}^{\infty} \sum_{q=l}^{\infty} a_{mnlpq} < \infty, \quad (4.14)
\]

\[
\vartheta - \lim_{m,n} \sum_{k,p}^{\infty} \sum_{q,l}^{\infty} a_{mnpq} = a_{kl} \text{ for all } k,l \in \mathbb{N}, \quad (4.15)
\]

\[
\exists u_{l_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n} \sum_{k=p}^{\infty} \sum_{q=l_0}^{\infty} a_{mnlpq} = u_{l_0} \text{ for fixed } l_0 \in \mathbb{N}, \quad (4.16)
\]

\[
\exists v_{k_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n} \sum_{l=q}^{\infty} \sum_{p=k_0}^{\infty} a_{mnlpq} = v_{k_0} \text{ for fixed } k_0 \in \mathbb{N}, \quad (4.17)
\]

\[
\exists v \in \mathbb{C} \ni \vartheta - \lim_{m,n} \sum_{k,l}^{\infty} \sum_{p=k}^{\infty} \sum_{q=l}^{\infty} a_{mnlpq} = v. \quad (4.18)
\]

Now, from the conditions (4.11)-(4.18), we have that \( A = (a_{mnkl}) \in (C_r(\Delta) : C_v) \) if and only if the conditions (4.1)-(4.7) hold. This completes the proof.

\[
\square
\]

**Theorem 4.2.** Suppose that the elements of the four dimensional infinite matrices \( E = (e_{mnkl}) \) and \( F = (f_{mnkl}) \) are connected with the relation

\[
f_{mnkl} = \sum_{i=m-1}^{m} \sum_{j=n-1}^{n} (-1)^{m+n-i-j} e_{ijkl} \quad (4.19)
\]

for all \( k,l,m,n \in \mathbb{N} \) and \( \mu \) be any given space of double sequences. Then, \( E \in (\mu : C_\vartheta(\Delta)) \) if and only if \( F \in (\mu : C_\vartheta) \).

\[\text{Proof.}\] Let \( x = (x_{kl}) \in \mu \) and consider the following equality with (4.19)

\[
\sum_{i=m-1}^{m} \sum_{j=n-1}^{n} \sum_{k=s-1}^{s} \sum_{l=t-1}^{t} (-1)^{m+n-i-j} e_{ijkl} x_{kl} = \sum_{k=s-1}^{s} \sum_{l=t-1}^{t} f_{mnkl} x_{kl} \quad (4.20)
\]

for all \( m,n,s,t \in \mathbb{N} \). By letting \( s,t \to \infty \) in (4.20) one can derive that

\[
\sum_{i=m-1}^{m} \sum_{j=n-1}^{n} (-1)^{m+n-i-j} (Ex)_{ij} = (Fx)_{mn} \text{ for all } m,n \in \mathbb{N}. \quad (4.21)
\]

Therefore, it is seen by (4.21) that \( Ex \in C_\vartheta(\Delta) \) if and only if \( Fx \in C_\vartheta \) whenever \( x \in \mu \). This step completes the proof.

\[
\square
\]

Of course, Theorem 4.2 has several consequences depending on the choice of the sequence space \( \mu \). Prior to giving some results as an application of this idea, we need the following lemmas:
Lemma 4.3. [8, 13, 15] A = \((a_{mnkl}) \in (C_r : C_r)\) if and only if
\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \quad (4.22)
\]
\[
\exists (a_{kl}) \in \Omega \ni r - \lim_{m,n \to \infty} a_{mnkl} = a_{kl} \quad \text{for each } k,l \in \mathbb{N}, \quad (4.23)
\]
\[
\exists v \in \mathbb{C} \ni r - \lim_{m,n \to \infty} \sum_{k,l} a_{mnkl} = v, \quad (4.24)
\]
\[
\exists u^{l_0}, v_{k_0} \in \mathbb{C} \ni r - \lim_{m,n \to \infty} \sum_{k} a_{mnkl_0} = u^{l_0} \quad \text{and} \quad (4.25)
\]
\[
r - \lim_{m,n \to \infty} \sum_{l} a_{mnk_0l} = v_{k_0} \quad \text{for any } k_0, l_0 \in \mathbb{N}.
\]

Lemma 4.4. [8, 13, 15] A = \((a_{mnkl}) \in (C_{bp} : C_{bp})\) if and only if
\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \quad (4.26)
\]
\[
\exists (a_{kl}) \in \Omega \ni bp - \lim_{m,n \to \infty} a_{mnkl} = a_{kl} \quad \text{for each } k,l \in \mathbb{N}, \quad (4.27)
\]
\[
\exists v \in \mathbb{C} \ni bp - \lim_{m,n \to \infty} \sum_{k,l} a_{mnkl} = v, \quad (4.28)
\]
\[
bp - \lim_{m,n \to \infty} \sum_{k} |a_{mnkl_0} - a_{kl_0}| = 0 \quad \text{and} \quad (4.29)
\]
\[
bp - \lim_{m,n \to \infty} \sum_{l} |a_{mnk_0l} - a_{k_0l}| = 0 \quad \text{for any } k_0, l_0 \in \mathbb{N}.
\]

Corollary 4.5. Suppose that the relation (4.19) holds between the elements of the four dimensional infinite matrices \(E = (e_{mnjk})\) and \(F = (f_{mnkt})\). Then, the following statements hold:

(i) \(E = (e_{mnjk}) \in (C_r : C_r(\Delta))\) if and only if the conditions (4.22)-(4.25) hold with \(f_{mnkt}\) instead of \(a_{mnkl}\).

(ii) \(E = (e_{mnjk}) \in (C_{bp} : C_{bp}(\Delta))\) if and only if the conditions (4.26)-(4.29) hold with \(f_{mnkt}\) instead of \(a_{mnkl}\).

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