HOMOMORPHISMS BETWEEN KÄHLER GROUPS

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This paper is an expanded version of my talk at the Jaca conference; as such it is somewhere in between a survey and a research article. Algebraic geometry and topology have, of course, been connected from almost the beginning of both subjects. The definition of the fundamental group is one of the first things one learns in a topology class, but ironically it appears to be one of the most subtle and mysterious invariants of an algebraic variety. My talk was about the groups that arise as fundamental groups of projective manifolds or more generally compact Kähler manifolds – the so called Kähler groups. The study of these groups came of age in the 1980’s and 90s. During this period, tools ranging from mixed Hodge theory, harmonic maps, rational homotopy theory and so on, were applied to obtain strong restrictions on the groups in this class. In spite of all of the progress, some difficult open questions remain. I will touch on a few of these later on.

In this paper, I want to propose one possible way forward by considering geometrically meaningful homomorphisms between Kähler groups. I define a homomorphism between such groups to be Kähler (or Kähler-surjective) if it comes from a holomorphic map (or a surjective holomorphic map with connected fibres) between Kähler manifolds. Note that Kähler-surjective maps are surjective, but not all surjective Kähler homomorphisms are Kähler-surjective. Some of the standard obstructions for a group to be Kähler extend to Kähler homomorphisms in a fairly straightforward fashion. A few examples are discussed below. However, I want to concentrate on a different sort of obstruction. I will say that a surjective homomorphism of groups \( h : H \to G \) splits if it has right inverse. There is a natural obstruction to splitting, namely the class \( e(h) \in H^2(G,K/[K,K]) \) associated to the extension

\[
0 \to K/[K,K] \to H/[K,K] \to G \to 1
\]

where \( K = \ker(h) \). The main result here is that \( e(h) \) vanishes in rationalized cohomology, when \( h \) is Kähler-surjective. The vanishing result leads to a new restriction for Kähler groups: If a Kähler group admits a map \( f \) onto a genus \( g \) surface group \( \Gamma_g \), with \( g \) maximal, then \( e(f) = 0 \) (rationally). Thus for example, the group

\[
\langle a_1, \ldots, a_{2g}, c \mid [a_1, a_{g+1}] \ldots [a_g, a_{2g}] = c, [a_i, c] = 1 \rangle
\]

is not Kähler.

When the Kähler-surjective homomorphism comes from a surjective map \( f : X \to Y \) of projective varieties, there is a fairly simple proof of the main theorem. This hinges on the fact that \( f \) has a multisection because the generic fibre has a rational point over a finite extension of \( \mathbb{C}(Y) \). Of course this argument is no longer valid in the general Kähler setting, so I use a different strategy. The first step is to

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show that \( f : X \to Y \) can be replaced by a map where \( \ker \pi_1(f) \) has a reasonable structure. There is a small price to be paid, in that \( Y \) is now an orbifold. Then the main step is to identify \( e(\pi_1(f)) \) as lying in the image of the differential of the Leray spectral sequence, and then deduce the result from the decomposition theorem [BBD, S3] suitably extended to Kähler orbifolds.

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1. Structure of Kähler groups

I want to say a few words about what is and is not known about the structure of Kähler groups, while ignoring many other interesting things pertaining to fundamental groups of varieties (see [H], [L], [K], [Scn], [Si2] and references therein). My summary will be fairly sketchy. The five author book [ABC] gives a much more complete account of the structure theory.

1.1. What we know.

(1) (Hodge) The abelianization \( \Gamma / D\Gamma \), \( (D\Gamma = [\Gamma, \Gamma]) \) of a Kähler group has even rank by the Hodge decomposition. While elementary, this remark eliminates 50% of all groups from the outset! However, it doesn’t say anything about a group such as the Heisenberg group, which is a nontrivial central extension of \( \mathbb{Z}^2 \) by \( \mathbb{Z} \).

(2) (Deligne-Griffiths-Morgan-Sullivan [DGMS]) The rational homotopy type of a compact Kähler manifold is as simple as possible. More precisely, it is \((1-)\)formal. This implies that secondary operations such as Massey products vanish. Furthermore the Malcev Lie algebra of a Kähler group

\[
L(\Gamma) = \frac{\Gamma}{[\Gamma, \Gamma]} \otimes \mathbb{Q} \times \frac{[\Gamma, \Gamma]}{[[\Gamma, \Gamma], \Gamma]} \otimes \mathbb{Q} \times \ldots,
\]

which is a kind of linearization, has a presentation with quadratic relations. This is quite a strong restriction. One can see, for example, that the Malcev algebra of the Heisenberg group is not quadratic, so it is not Kähler.

(3) (Goldman-Millson [GM]) The singularities of the representation variety \( \text{Hom}(\Gamma, GL_n(\mathbb{C})) \) of a Kähler group at points corresponding to semisimple representations are quadratic. This is closely related to the previous result, and one can redo the Heisenberg group example from this point of view.

(4) (A.-Nori [AN], Delzant [Dz]) Most solvable groups are not Kähler. The precise result of Delzant is that a solvable group is not Kähler unless it contains a nilpotent subgroup of finite index. The special case for polycyclic groups, proved in [AN], was based on a study of what has been variously called the first cohomology support locus or jumping locus, or characteristic variety. I ought to mention that this object has been studied quite extensively by Libgober and others.

(5) (Carlson-Toledo [CT1], Simpson [SI]) Recall that a discrete subgroup \( \Gamma \subset G \) of a Lie group is a lattice if the Haar measure \( \text{vol}(G/\Gamma) < \infty \). While some lattices are Kähler, many are not. These authors give specific obstructions for a lattice in a semisimple Lie group to be Kähler. For example, lattices in \( SO(1,n) \) or \( SL(n,\mathbb{R}) \), with \( n \geq 3 \), are not Kähler.

(6) (Gromov-Schoen [GS], A.-Bressler-Ramachandran [ABR]) “Big” groups, such as free products, are not Kähler.
1.2. **What we don’t know.** Here are a few open problems in the area. Obviously, this list reflects my own taste and biases.

1. **(Johnson-Rees [JR]) Is the class of Kähler groups the same as the class of fundamental groups of smooth projective varieties?** Inspired by work of Voisin [V], who constructed compact Kähler manifolds which are not homotopic to projective manifolds, I will conjecture the answer is no. Unfortunately, her examples don’t shed any light here, since they do not have interesting fundamental groups. In fact, I do not have any potential counterexamples; a possible candidate proposed in the introduction to [AN] has been taken out of the running by subsequent work of Campana [C2] and Delzant [Dz]. I do want to mention as evidence that Botong Wang has recently found a counterexample which settles the analogous problem for Kähler homomorphisms. I will say more about this later.

2. It seems unreasonable to try to characterize all Kähler groups; there is just no way to get a handle on all of them. A more reasonable goal would be to try to characterize the Kähler groups within a well understood subclass of all groups such as lattices in connected Lie groups. This can be specialized further to two subclasses with very distinct behaviours: Lattices in solvable groups and lattices in semisimple groups.
   
   (a) By [AN] and some structure theory [R], a lattice \( \Gamma \) in a solvable group is not Kähler unless the ambient Lie group is nilpotent, i.e., unless \( \Gamma \) is a torsion free finitely generated nilpotent group. The restrictions on the Kähler groups in this class coming from [DGMS] and other sources are so severe that only a few nonabelian examples are known [CI, CT2]. These are all (larger rank) Heisenberg groups. It is not clear whether these examples are in any sense typical. So the problem is to either construct more examples of nilpotent Kähler groups or find more constraints on them.

   (b) Carlson and Toledo [CT1] have conjectured that a lattice in a semisimple Lie group \( G \) is not in Kähler unless the associated symmetric space \( G/K \) (\( K \) = a maximal compact) is Hermitian symmetric. In more explicit terms, for the space to be Hermitian the simple factors of \( G \) must be one of
   
   \[ SU(p, q), SO^*(2n), SO(p, 2), Sp(n, \mathbb{R}) \]
   
   or among a finite list of exceptional cases (see [He]). Note that the converse is often true. That is if \( G/K \) is a Hermitian symmetric space, then a lattice \( \Gamma \subset G \) is Kähler if it is either cocompact, or arithmetic and the Baily-Borel boundary of \( \Gamma \backslash G/K \) has codimension at least 3. The results of [CT1, Si1] mentioned above go part of the way toward this conjecture, but it appears that new ideas are needed.

3. **Is the genus \( g \) mapping class group Kähler?** I want to emphasize that this is open in spite of some misleading statements in the literature. The mapping class group can be viewed as the orbifold fundamental group of the moduli space of curves \( \mathcal{M}_g \). An incorrect proof that this group is Kähler is to take the Satake compactification of \( \mathcal{M}_g \), and cut by hyperplanes. But for this to work, the codimension of the complement would have to be greater than 2 (otherwise the fundamental group changes when one cuts). However, if
one analyzes things carefully, one sees that the Satake boundary does in fact have a codimension 2 stratum. When \( g = 2 \), Veliche [Ve1, Ve2] showed that the mapping class group is not Kähler by reducing to the case of braid groups which were checked to be non-Kähler in [A1]. Veliche’s argument doesn’t generalize, so it is unclear what to expect for other genera. Farb [F] suggests that these are Kähler, but I see the glass as half empty.

(4) Is the image of \( H^*(\pi_1(X), \mathbb{Z}) \to H^*(X, \mathbb{Z}) \) a sub Hodge structure? When \( X \) is a variety, one can ask whether this is a submotive, and in particular whether the image \( H^*(\pi_1(X), \mathbb{Z}_p) \to H^*(X, \mathbb{Z}_p) \cong H^*(X_{et}, \mathbb{Z}_p) \) is invariant under the action of \( Aut(\mathbb{C}) \) on étale cohomology. The paper [A2] was motivated by the Hodge structure question, but it doesn’t directly address it. To attempt to answer it one ask: Does there exist a good analytic/algebraic model for the classifying map \( X \to B\pi_1(X) = K(\pi_1(X), 1) \) when \( X \) is a Kähler manifold/algebraic variety? I am purposely being vague about what a “good model” actually means; part of the problem is to make this precise. In fact, a certain analytic model is used in this paper, but it is not “good” as far as the above question is concerned. One thing is clear that the \( K(\pi_1(X), 1) \) cannot simply be a manifold or variety in general, because \( \pi_1(X) \) might have infinite cohomological dimension. As recent work of Dimca, Papadima and Suciu [DPS] shows, one cannot expect such a simple model even if one allows \( \pi_1(X) \) to be replaced by a commensurable group.

2. Elementary properties of Kähler homomorphisms

To reiterate a homomorphism of Kähler groups is Kähler (or Kähler-surjective) if it comes from a holomorphic map (or a surjective holomorphic map with connected fibres) between Kähler manifolds. It seems an interesting question of whether there are reasonable analogues of the statements in §1.1. For now, note that any constraint for Kähler groups which comes from the existence of some functorial structure automatically generalizes to Kähler homomorphisms. For example:

**Lemma 2.1.** If \( h : \Gamma_1 \to \Gamma_2 \) is Kähler then the image, kernel and cokernel of the induced map \( \Gamma_1/D\Gamma_1 \to \Gamma_2/D\Gamma_2 \) has even rank.

**Proof.** If \( h \) is realized by a holomorphic map \( f : X_1 \to X_2 \) of compact Kähler manifolds, the groups \( \Gamma_i/D\Gamma_i = H_1(X_i) \) carry Hodge structures of weight \(-1\) which are preserved by \( f_* \).

The Malcev Lie algebra \( L(\Gamma) \) has a filtration by the lower central series

\[
C^0L(\Gamma) = L(\Gamma), \quad C^{n+1}(\Gamma) = [L(\Gamma), C^nL(\Gamma)]
\]

The quotients of \( L(\Gamma)/C^n \) gives an inverse system of finite dimensional nilpotent Lie algebras, and \( L(\Gamma) \) can be identified with the limit.

**Proposition 2.2.** If \( h : \Gamma_1 \to \Gamma_2 \) is Kähler, then the induced map of Malcev Lie algebras \( L(\Gamma_1) \to L(\Gamma_2) \) strictly preserves the lower central series. That is \( h(C^nL(\Gamma_1)) = h(L(\Gamma_1)) \cap C^nL(\Gamma_2) \).

\(^1\)I’m not sure who to attribute this to. It’s come up at various times in conversations with Hain, Nori, and Srinivas.
Proof. By a theorem of Morgan [Mo], $L(\Gamma_i)/CN$ carries a functorial system of mixed Hodge structures, such that the weight filtration is exactly the lower central series. The proposition follows from the corresponding strictness statement in mixed Hodge theory. □

A more explicit obstruction is given by:

**Corollary 2.3.** Suppose $h : \Gamma_1 \to \Gamma_2$ is a homomorphism of Kähler groups such that the induced map of Malcev Lie algebras is nonzero. Then $h$ cannot be Kähler.

**Proof.** This is a consequence of the fact that a map strictly preserving a filtration is zero if the induced map on the associated graded is zero. □

I want to point out a serious limitation of the notion of Kähler morphism, which is that the class is not closed under composition. Thus the collection of Kähler groups and morphisms do not form a subcategory of the category of groups.

**Example 2.4.** The maps

$$\mathbb{Z}^2 \overset{p}{\longrightarrow} \mathbb{Z}^4 \overset{q}{\longrightarrow} \mathbb{Z}^2$$

given by the matrices

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

are both Kähler, but $q \circ p$ is not since it has a rank one image.

I want to end this section by explaining the analogue of problem 1 of §1.2. Call a group (respectively homomorphism) projective if can be realized as the fundamental group (respectively homomorphism between fundamental groups) of a smooth complex projective variety (respectively induced by a morphism of projective varieties).

**Theorem 2.5 (B. Wang).** There exists a Kähler morphism between projective groups which is not Kähler.

The proof, which is based on Voisin’s method [V], will be written up by him separately.

3. **Reduction to Orbifolds**

The rest of this paper will devoted to the proof of the theorem stated in the introduction about the vanishing of the splitting obstruction associated to a Kähler-surjective map $\pi_1(f)$. One of the difficulties that needs to be dealt with is that the kernel of $\pi_1(f)$ can be quite wild. In this section, I want to show how to reduce it to something more manageable.

Given a surjective homomorphism of groups $h : H \to G$, let $H^{(1)}(h) = H/D\ker(h)$ with canonical surjection $h^{(1)} : H^{(1)}(h) \to G$. For each $n > 1$, define $H^{(n)}(h)$ by
the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{K}/\text{DK} & \rightarrow & H^{(1)}(h) & \rightarrow & \text{G} & \rightarrow & 1 \\
\downarrow n. & & \downarrow & & \downarrow & & \downarrow & & = \\
0 & \rightarrow & \text{K}/\text{DK} & \rightarrow & H^{(n)}(h) & \rightarrow & \text{G} & \rightarrow & 1 \\
\end{array}
\]

where \(K = \ker(h)\), \(n\cdot\) is multiplication by \(n\), and the left hand square is a pushout. We will write \(H^{(n)}\) if \(h\) is understood. Let \(e(h)\) denote the class \(H^{1}(G, \text{K}/\text{DK})\) defined earlier. Observe that \(n \cdot e(h) = 0\) if and only if \(H^{(n)} \rightarrow \text{G}\) splits.

Call a continuous map connected if all of its fibres are connected. Fix a proper connected holomorphic map \(f : X \rightarrow Y\) of connected complex manifolds. Then we have a surjection \(\pi_{1}(f) : \pi_{1}(X) \rightarrow \pi_{1}(Y)\).

**Lemma 3.1.** Given a commutative diagram of topological spaces

\[
\begin{array}{ccccccccc}
X' & \rightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \rightarrow & Y \\
\end{array}
\]

where \(f'\) is proper and connected. If \(g\) induces an isomorphism of fundamental groups, then \(e(\pi_{1}(f')) = 0\) (respectively, is torsion) implies \(e(\pi_{1}(f)) = 0\) (respectively, is torsion). In particular, this holds if \(g\) is a bimeromorphic map of complex manifolds, or if \(g\) is the inclusion of the complement of an analytic subset of codim \(\geq 2\) in a complex manifold.

**Proof.** If \(ne(\pi_{1}(f')) = 0\) then \(\pi_{1}(X')(n) \rightarrow \pi_{1}(Y') = \pi_{1}(Y)\) has a splitting. By composition, we get a splitting of \(\pi_{1}(X)(n) \rightarrow \pi_{1}(Y)\). \(\square\)

A variant of the previous lemma will be needed later.

**Lemma 3.2.** Given a commutative diagram of topological spaces

\[
\begin{array}{ccccccccc}
X' & \rightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \rightarrow & Y \\
\end{array}
\]

Such that \(f,f'\) induce surjections on fundamental groups, and \(g : Y' \rightarrow Y\) is a generically finite proper map between oriented manifolds. If \(e(\pi_{1}(f'))\) is torsion then \(e(\pi_{1}(f))\) is torsion.

**Proof.** Suppose that \(\pi_{1}(X')(n) \rightarrow \pi_{1}(Y')\) has a splitting. Then \(\pi_{1}(X \times_{Y} Y')(n) \rightarrow \pi_{1}(Y')\) has a splitting. This implies that

\[
(\pi_{1}(X) \times_{\pi_{1}(Y)} \pi_{1}(Y'))(n) \rightarrow \pi_{1}(Y)
\]

splits. Therefore the pullback \(g^{*}e(\pi_{1}(f))\) to \(H^{2}(\pi_{1}(Y'), K/[K,K] \otimes \mathbb{Q})\) is zero. Now for any rational local system \(V\) on \(Y\), the maps labeled \(\alpha\) and \(\beta\) in the commutative diagram

\[
\begin{array}{cccccc}
H^{2}(\pi_{1}(Y), V) & \rightarrow & H^{2}(\pi_{1}(Y'), g^{*}V) \\
\downarrow \alpha & & \downarrow \\
H^{2}(Y, V) & \rightarrow & H^{2}(Y', g^{*}V) \\
\end{array}
\]

are easily seen to be injective. For α this is a standard result. It follows from
the Leray spectral sequence for the classifying map \( k : Y \rightarrow B\pi_1(Y) \) and the fact
that the fibres of \( k \) are simply connected. For \( \beta \), a left inverse is given by \( \frac{1}{\deg g} g \).
Therefore \( e(\pi_1(f)) \otimes \mathbb{Q} \) vanishes.

\( \square \)

**Lemma 3.3.** Let \( h : H \rightarrow F \) be a surjective group homomorphism, and let \( f : F \rightarrow G \) be another surjective homomorphism such that \( \ker(f)/D\ker(f) \) is a possibly infinite torsion group. If \( e(h) \) is torsion, then \( e(f \circ h) \) is torsion.

**Proof.** Let \( K = \ker(f \circ h) \), \( L = \ker(f) \) and \( P = F \rtimes_G H \) with its canonical projections \( p : P \rightarrow F \) and \( q : P \rightarrow H \). Observe that there is a commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{id} & H \\
\downarrow h & & \downarrow f \\
P & \xrightarrow{p} & F \quad \xrightarrow{f} \quad G & \xrightarrow{f \circ h} & H \\
\end{array}
\]

Suppose that \( ne(h) = 0 \) then \( h^{(n)} : H^{(n)}(h) \rightarrow F \) splits. Therefore \( P^{(n)} \rightarrow F \) also splits since \( h^{(n)} \) factors through it. Thus \( ne(p) = 0 \). Note that \( e(p) \) is the restriction of \( e(f \circ h) \) to \( H^2(F, K/\mathcal{D}K) \). Therefore this implies the image of \( e(f \circ h) \) in \( H^2(F, K/\mathcal{D}K \otimes \mathbb{Q}) \) vanishes.

The Hochschild-Serre spectral sequence yields an exact sequence

\[
H^0(G, H^1(L, K/\mathcal{D}K \otimes \mathbb{Q})) \rightarrow H^2(G, K/\mathcal{D}K \otimes \mathbb{Q}) \rightarrow H^2(F, K/\mathcal{D}K \otimes \mathbb{Q})
\]

Since \( L \) acts trivially on \( K/\mathcal{D}K \) and since \( L/\mathcal{D}L \) is torsion,
\( H^1(L, K/\mathcal{D}K \otimes \mathbb{Q}) = \text{Hom}(L/\mathcal{D}L, K/\mathcal{D}K \otimes \mathbb{Q})) = \text{Hom}(L/\mathcal{D}L, K/\mathcal{D}K \otimes \mathbb{Q}) = 0 \)
Therefore \( r \) is injective. Thus \( e(f \circ h) \otimes \mathbb{Q} = 0 \).

\( \square \)

Fix a reduced effective divisor \( D = \cup D_i \subset Y \) with simple normal crossings and positive integers \( m_i \) along each component. In \([\text{MO}], \text{thm 4.1}]\), it is shown how to construct a smooth Deligne-Mumford stack with a morphism \( p : Y^{\text{orb}} \rightarrow Y \) which is the minimal object for which \( p^*D_i = m_i D_i \). We will need to understand the structure to some extent, and in particular to see why this extends to the analytic category. So as to avoid too many abstractions, we describe this using the older orbifold language (c.f. \([\text{CR}], [\text{M}]\)). Recall that an analytic orbifold is given by a locally finite atlas \( \{ (U_i, G_i, \phi_{ij}) \} \) consisting of open sets \( U_i \subset \mathbb{C}^n \) stable under finite groups \( G_i \), and analytic gluing functions

\[
U_i \supset U_{ij} \xrightarrow{\phi_{ij}} U_{ij} \subseteq U_j
\]

respecting the actions of \( G_{ij} = \{ g \in G_i \mid g(U_{ij}) \subseteq U_{ij} \} \). This can be conveniently packaged by the groupoid

\[
\mathcal{G}_1 = \coprod_{ij} G_{ij} \times U_{ij} \xrightarrow{\mu} \coprod_{i} U_i \xrightarrow{s} \coprod_{i} U_i = \mathcal{G}_0
\]

\[
\mathcal{G}_1 \times s, \mathcal{G}_0, t \mathcal{G}_1 \xrightarrow{\mu} \mathcal{G}_1 \rightarrow \mathcal{G}_1
\]
with source \( s(g, y)_{ij} = (y)_i \), target \( t(g, y)_{ij} = (g \phi_{ij}(y))_j \), unit \( u(y)_1 = (1, y)_u \), and multiplication and inversion induced by the corresponding operations on the groups. The groupoid defines the corresponding analytic Deligne-Mumford stack.

To construct \( Y_{\text{orb}} \), choose an atlas \( \{V_i, \phi_{ij}\} \) for the manifold \( Y \), consisting of polydisks \( |y_j| < \epsilon \), such that for each \( i \) either \( V_i \cap D = \emptyset \) or \( D_j \) is defined by \( y_j = 0 \).

In the first case, let \( U_i = V_i \) with the old coordinates \( z_j = y_j \), and \( G_i = \{1\} \).

In the second case, using new coordinates \( z_j = y_j^{1/m_j} \), set \( U_i = \{|z_j| < \epsilon^{1/m_j}\} \) with \( G_i = \prod \mathbb{Z}/m_j \mathbb{Z} \) acting by \( z_j \mapsto \exp(2\pi \sqrt{-1}/m_j)z_j \). For \( \phi_{ij} \) we use the old transition functions in the \( y \)'s rewritten in terms of the \( z \)'s. This defines \( Y_{\text{orb}} \) as an orbifold. For our purposes a holomorphic map of orbifolds is a map which can be described locally by maps \( U_i \to U_i' \) and homomorphisms \( G_i \to G_i' \) together with the obvious compatibilities, or equivalently by a morphism of analytic groupoids (such maps are called good or strong in the above references). In particular, we have a map \( Y_{\text{orb}} \to Y \) defined by the map of atlases \( U_i \to V_i, G_i \to 1 \) given the change of variables.

Note that contrary to first appearances, the topology of \( Y \) and \( Y_{\text{orb}} \) should be regarded as different unless \( m_i = 1 \). This begs the question of what we even mean by the topology of an orbifold. The quickest answer is that we can build a topological space \( B\mathbb{G} \) by taking the geometric realization of the nerve of the above topological groupoid, which is the simplicial space

\[
B\mathbb{G} = \mathcal{G}_1 \times_{s, \mathbb{G}_0, t} \mathcal{G}_1 \times_{s, \mathbb{G}_0, t} \ldots \mathcal{G}_1
\]

with face maps described in [Se]. This involves choices, but the weak homotopy type of \( B\mathbb{G} \) depends only on \( Y_{\text{orb}} \), and so we will denote this by \([Y_{\text{orb}}]\). When this construction is applied to a manifold, we recover its homotopy type. The fundamental group \( \pi_1(Y_{\text{orb}}) := \pi_1([Y_{\text{orb}}]) \) can be understood in a more explicit fashion. It is defined so that its (set, abelian group,...)-valued representations correspond to locally constant sheaves (of sets, abelian groups...) on \( Y_{\text{orb}} \), which are given by collections of \( G_i \)-equivariant locally constant sheaves \( L_i \) on \( U_i \) with glueing isomorphisms \( \phi_{ij}^* L_i|_{U_i \cap U_j} \cong L_j|_{U_i \cap U_j} \) subject to the cocycle condition. So for instance a disk \( \Delta \) “modulo” \( \mathbb{Z}/m \mathbb{Z} \) is not contractible; it has fundamental group \( \mathbb{Z}/m \mathbb{Z} \).

To work out \( \pi_1(Y_{\text{orb}}) \) explicitly, choose simple loops \( \gamma_i \) around the components \( D_i \) of \( D \). Then \( \pi_1(Y) \) can be identified with \( \pi_1(Y - D) \) modulo the group generated by the \( \gamma_i \) and its conjugates. While \( \pi_1(Y_{\text{orb}}) \) is the quotient of \( \pi_1(Y - D) \) by the subgroup generated by the conjugates of \( \gamma_i^{m_i} \).

**Lemma 3.4.** The map \( Y_{\text{orb}} \to Y \) induces a surjection from \( \pi_1(Y_{\text{orb}}) \) to \( \pi_1(Y) \). The abelianization of the kernel of this map is a (possibly infinite) torsion group.

**Proof.** The first statement is clear. The kernel is generated by conjugates of the \( \gamma_i \) which are torsion elements. Therefore the abelianized kernel is torsion since it is generated by torsion elements.

The following seems well known when the base is a curve, e.g. [CKO] lemma 3].

**Lemma 3.5.** Let \( f : X \to Y \) be a surjective holomorphic map of complex manifolds. Suppose that the discriminant of \( f \) is a smooth divisor \( D \), and that \( f^{-1}D \) is a divisor with normal crossings such that the restriction of \( f \) to the intersections of components are submersions over \( D \). Let \( m_i \) denote the greatest common divisor of the multiplicities of the components of \( f^{-1}D_i \), and construct \( Y_{\text{orb}} \) as above. Let
y_0 \in Y - D. Then \( \pi_1(f) \) factors through a surjection \( \phi: \pi_1(X) \to \pi_1(Y_{\text{orb}}) \), and 
\( \pi_1(f^{-1}(y_0)) \) surjects onto \( \ker \phi \).

**Proof.** The map \( f \) is given locally by

\[
y_i = x_1^{k_{i1}} \cdots x_n^{k_{in}}
\]

The map \( f \) factors through a map \( X \to Y_{\text{orb}} \) given in the above coordinates by

\[
y_i^{1/m_i} = x_1^{k_{i1}/m_i} \cdots x_n^{k_{in}/m_i}
\]

where \( m_i = \gcd(k_{i1}, \ldots, k_{in}) \). This induces a factorization \( \pi_1(X) \to \pi_1(Y_{\text{orb}}) \to \pi_1(Y) \). Now consider the commutative diagram

\[
\begin{array}{ccc}
\ker(r_2) & \to & \ker(r_3) \\
\downarrow & & \downarrow \\
\pi_1(f^{-1}(y_0)) & \to & \pi_1(X - f^{-1}D) \\
\downarrow r_1 & & \downarrow r_2 \\
\ker(\phi) & \to & \pi_1(X) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \phi \\
\downarrow & & \downarrow \\
& & \pi_1(Y_{\text{orb}}) \\
\end{array}
\]

The middle row is exact since \( f \) is a fibration over \( Y - D \). Furthermore the arrows \( r_2, r_3 \) are surjective. Therefore \( \phi \) is also surjective. This also implies that any element of \( \ker(\phi) \) can be lifted to an element of \( \psi^{-1}\ker(r_3) \subset \pi_1(X - f^{-1}D) \).

To prove surjectivity of \( r_1 \), it suffices to prove that \( \psi|_{\ker(r_2)} \) surjects onto \( \ker(r_3) \). By definition \( \ker(r_3) \) is generated by conjugates of \( \gamma_i^{m_i} \), where \( \gamma_i \) is a loop around \( y_i = 0 \), using the coordinates of (1). A simple loop \( \delta_j \) around the divisor \( x_j = 0 \) maps to \( \gamma_i^{\pm k_{ij}} \) in \( Y - D \). Since \( \gcd(k_{ij}) = m_i \), we can lift \( \gamma_i^{m_i} \) to a word in the \( \delta_j \)'s. Therefore any conjugate of \( \gamma_i^{m_i} \) can be lifted to an element of \( \ker(r_2) \).

\( \square \)

### 4. Decomposition theorem for orbifolds

In [BBD], Beilinson, Bernstein, Deligne and Gabber developed the theory of perverse sheaves on algebraic varieties. These form an abelian subcategory of the derived category, and basic examples include the intersection cohomology complexes \( IC(L)[\dim Z] \) associated to locally constant sheaves \( L \) defined on locally closed sets \( Z \subseteq X \). (We find it convenient to index \( IC(L) \) so that \( IC(L) = L \) generically on \( Z \).) They proved a basic result called the decomposition theorem: If \( L \) is a semisimple perverse sheaf of geometric origin then for any proper map

\[
\mathbb{R}f_*L = \bigoplus_j IC(M_j)[m_j]
\]

for some \( m_j \in \mathbb{Z} \) and \( M_j \). Saito [S3] has shown that this holds when \( f \) is a proper holomorphic map of Kähler manifolds. This will play a crucial role in the proof of our theorem, however we will need a slight extension to orbifolds.

Suppose that \( G \) is a finite group acting on a complex manifold \( U \). A \( G \)-equivariant sheaf is a sheaf \( F \) equipped with a a collection of isomorphisms \( \phi_g : \):
$g^*\mathcal{F} \cong \mathcal{F}$, for each $g \in G$, such that $\phi_1 = id$ and

\[
(*) \quad h^*g^*\mathcal{F} \xrightarrow{h^*\phi_g} h^*\mathcal{F} \xrightarrow{\phi_h} \mathcal{F}
\]

commutes. For example, the pullback of a sheaf from $U/G$ is naturally $G$-equivariant. But, unless the action of $G$ is free, not every equivariant arises this way. Let $\text{Sh}_G(U)$ denote the category of $G$-equivariant sheaves of $\mathbb{Q}$-vector spaces. Suppose that $K$ is the stabilizer of the $G$-action on $U$. This will act on a $G$-equivariant $\mathcal{F}$ by sheaf automorphisms, and the invariants $\mathcal{F}^K$ is a naturally a $G/K$-equivariant sheaf. This gives an exact functor $\Gamma_K : \text{Sh}_G(U) \to \text{Sh}_{G/K}(U)$. If $H \subset G$ is subgroup, then we have an induction functor $\text{ind}_{H}^{G} : \text{Sh}_{H}(U) \to \text{Sh}_{G}(U)$ right adjoint to the obvious restriction. Given a homomorphism $f : H \to G$, we let $\text{ind}_{H}^{G}$ denote the composition $\text{ind}_{H}^{G} = \pi_{H}^{G} \circ \Gamma_{\ker(f)}$. Suppose that $X$ is an orbifold given by an atlas $\{(U_i, G_i), \phi_{ij}\}$. By a sheaf on $X$, we mean a collection of $G_i$-equivariant sheaves $\mathcal{F}_i$ together with isomorphisms $\phi_{ij}^* \mathcal{F}_i|_{U_i \cap U_j} \cong \mathcal{F}_j|_{U_i \cap U_j}$ subject to the cocycle condition. Let $D^+(X)$ denote the derived category of sheaves on $X$. Given a map $f : X \to Y$ of orbifolds the direct image $\mathcal{R}f_* : D^+(X) \to D^+(Y)$ can be constructed in couple of equivalent ways. Sheaves on $X$ correspond to simplicial sheaves on the nerve $BG\ast$, with respect to a given atlas, such that the structure maps are all isomorphisms. The direct image can then realized as a direct image of simplicial sheaves $[D2]$. Alternatively, here is an explicit recipe: If $f$ is given by $U_i \to V_i, G_i \to H_i$, and sheaf $\mathcal{F} = \{\mathcal{F}_i\}$ is a sheaf on $X$. Replace $\mathcal{F}_i$ by its Godement flasque resolution $\mathcal{G}(\mathcal{F}_i)$ which is equivariant, then form $\text{ind}_{H_i}^{G_i} f_* \mathcal{G}(\mathcal{F}_i)$. An object in the derived category of sheaves on $X$ is a perverse sheaf if it restricts to a perverse sheaf in the usual sense on each $U_i$. $IC$ also extends to this setting.

By a Kähler orbifold, we simply mean that an atlas $\{(U_i, G_i), \phi_{ij}\}$ can be chosen so that each $U_i$ possess a Kähler form preserved by the $\phi_{ij}$. We can assume, by averaging over the groups, that these forms are invariant. We therefore have a Kähler class in $H^2(X, \mathbb{R})$. So now we can state:

**Theorem 4.1** (Saito+). Suppose that $f : X \to Y$ is a proper holomorphic map of orbifolds with $X$ Kähler. Let $L$ be a geometric perverse sheaf on $X$ which means that it is a direct summand of $IC(R^p\pi_*\mathbb{Q})[\text{dim} X]$ for some proper surjective holomorphic map $\pi : X' \to X$ of Kähler orbifolds which is proper over an open set. Then

$$\mathcal{R}f_* L = \bigoplus_j IC(M_j)[m_j]$$

Restricting $f$ to the complement of the discriminant $D \subset Y$ allows us to identify some of these $M_j$’s and deduce that

**Corollary 4.2.** The perverse Leray spectral sequence decomposes as

$$E_2^{ij} = H^i(Y, IC(R^j f_* L|_{Y-D})) \oplus \text{(irrelevant stuff)}$$

and furthermore it degenerates at $E_2$.

We recall the basic properties of Saito’s theory of mixed Hodge modules $[S1]$ $[S2]$ $[S3]$. 
(1) To any complex manifold $X$, there is an abelian category of mixed Hodge modules $MHM(X)$ with a full semisimple abelian subcategory $MH(X)$ of pure Hodge modules. (We are including polarizability in the definition of $MH$.)

(2) There is a faithful exact forgetful map from $MHM(X)$ to the category of rational perverse sheaves.

(3) Given a closed irreducible analytic subvariety $Z \subseteq X$, a pure Hodge module is said to have strict support $Z$ if the support of the perverse sheaves corresponding to all of its simple factors are all exactly $Z$. Any Hodge module decomposes uniquely into a sum of submodules with strict support along various subvarieties.

(4) If $Z$ is as above, any polarizable variation of Hodge structure $L$ on a Zariski open subset of $Z$ extends to an object of $MH(X)$ such that the underlying perverse sheaf is the intersection cohomology complex $IC(L)[\dim Z]$. Conversely, any Hodge module with strict support $Z$ is of this form.

(5) All standard sheaf theoretic operations extend to $MHM$ and are compatible with the corresponding operations on perverse sheaves.

(6) If $f : X \to Y$ is proper holomorphic map of manifolds with $X$ Kähler, then the perverse sheaves $pH^iRf_*\mathbb{Q}$ lift to objects in $MH(Y)$. Moreover the hard Lefschetz holds, i.e. cupping with the Kähler class induces

$$pH^{-i}Rf_*\mathbb{R} \cong pH^iRf_*\mathbb{R}$$

in $MH(Y) \otimes \mathbb{R}$; the twist $(i)$ can be ignored for our purposes.

Given a manifold with a finite group action, an equivariant mixed Hodge module can be defined to be a mixed Hodge module $F$ with isomorphisms $\phi_g : g^*F \cong F$ satisfying the above compatibilities \([\dag]\). A mixed Hodge module on an orbifold $X$ with atlas $\{(U_i, G_i), \phi_{ij}\}$, is given by a collection of $G_i$-equivariant modules $F_i$ together with isomorphisms $\phi_{ij}^*: F_i|_{U_i \cap U_j} \cong F_j|_{U_i \cap U_j}$. It is fairly easy to see that $MHM(X)$ still forms an abelian category with a forgetful functor to the category perverse sheaves, which is defined in a similar fashion. A subvariety $Z \subset X$ corresponds to compatible family of $G_i$-invariant subvarieties $Z_i \subseteq U_i$. A pure Hodge module has strict support along $Z$ if each component has strict support along $Z_i$. We can see that this notion is well defined and that strict support decompositions hold. Moreover the perverse sheaves corresponding to Hodge modules with strict support along $Z$ are of the form $IC(L)[\dim Z]$ for a generic local system $L$ on $Z$.

The remaining statements are also true but require a bit more explanation.

**Proposition 4.3.** Let $G$ be a finite group. If $f : X \to Y$ is a $G$-equivariant proper holomorphic map of manifolds with $X$ Kähler. Then $pH^iRf_*\mathbb{Q}$ lifts to an equivariant pure Hodge module. This Hodge module is compatible with respect to restriction along open immersions.

**Proof.** All the essential ideas are due to Bernstein and Lunts \([BL]\). We can find a connected algebraic variety $V$ with free $G$-action, such that

$$H^i(V, \mathbb{Q}) = 0 \text{ for } 0 < i \leq N \gg 0$$

(for example, by fixing an embedding $G \subset GL_{n+N}(\mathbb{C})$ and taking the Stieffel variety $V$ of $n$-frames in $\mathbb{C}^{n+N}$). There are two key points. First, since the action of $G$ on $Y \times V$ is free, the category of equivariant Hodge modules on $Y \times V$ can be identified with $MH((Y \times V)/G)$. Secondly, under the projection $p : Y \times V \to Y$, we have an
embedding of the category of equivariant (mixed) Hodge modules on $Y$ to that on $Y \times V$ given by $p^!$. To see this, first note that $p_* p^! F = F$ and $R^i p_* p^! F = 0$ for and any sheaf $F$ and $0 < i \leq N$ by (2). Therefore, given a pair of perverse sheaves $L, M \in Perv(Y) \subset D\left[-\dim Y, 0\right](Y)$,

$$\text{Hom}(L, M) \to \text{Hom}(p^! L, p^! M)$$

is injective because it has a left inverse given by $\tau \leq 0 R^p$. Since the Hom's for the category of equivariant Hodge modules are contained in the Hom's of the underlying perverse sheaves, the earlier claim about embeddings follows.

Now consider the diagram

$$
\begin{array}{ccc}
X & \xleftarrow{p'} & X \times V \\
| & & | \\
Y & \xrightarrow{p} & Y \times V \\
| & & | \\
f & & f_V \\
(X \times V)/G & \xrightarrow{q'} & (X \times V)/G \\
F
\end{array}
$$

Then by [S3] $pH^i RF_* Q$ lifts naturally to $MH((X \times V)/G)$. By base change, we can identify

$$q^*(pH^i RF_* Q) = pH^i RF_V_* Q = p^*(pH^i RF_* Q)$$

So the module on the right inherits the structure of an equivariant Hodge module.

Given a commutative diagram

$$
\begin{array}{ccc}
X' & \xleftarrow{H} & X \\
| & & | \\
Y' & \xrightarrow{h} & Y \\
| & & | \\
f' & & f \\
(X \times V)/G & \xrightarrow{q} & (X \times V)/G \\
F
\end{array}
$$

of $G$-equivariant maps with $h, H$ open immersions, the map $h^* pH^i RF_* Q \to pH^i RF'_* Q$ is clearly compatible with Hodge module structures.

**Corollary 4.4.** If $f : X \to Y$ is a proper holomorphic map of orbifolds with $X$ Kähler, $pH^i RF_* Q$ lifts to a Hodge module.

Combining this with the above remarks.

**Corollary 4.5.** If $f : X \to Y$ is a proper holomorphic map of orbifolds with $X$ Kähler, $pH^i RF_* Q$ can be expressed as a sum $\oplus IC(M_j)[\dim Z_j]$.

**Lemma 4.6.** $f : X \to Y$ is a proper holomorphic map of orbifolds with $X$ Kähler, the hard Lefschetz theorem holds for $pH^i RF_* Q$.

**Proof.** This follows immediately from Saito’s result [S3].

**Proof of theorem [4.4]** By extending scalars to $\mathbb{R}$ and applying the previous lemma together with [D], we get

$$RF_* Q \cong \bigoplus pH^i RF_* Q[-i]$$

Thus by corollary 4.5, we can rewrite $RF_* Q$ as a sum of intersection cohomology complexes up to shift. This proves the result for $L = Q$. The general case can be deduced by applying the theorem to $f \circ \pi$ as in [S3].
5. Splitting theorem

We are ready to state the main theorem.

**Theorem 5.1.** Suppose that \( f : X \to Y \) is a proper connected holomorphic map of complex manifolds, with \( X \) Kähler. Then the splitting obstruction \( e(\pi_1(f)) \in H^2(\pi_1(Y), K/\pi DK) \) is torsion, where \( K = \ker(\pi_1(f)) \).

**Corollary 5.2.** If \( h \) is a Kähler-surjective homomorphism then \( e(h) \) is torsion.

We note that Kähler-surjective homomorphisms are indeed both Kähler and surjective, but not conversely. Furthermore, the hypothesis of this corollary cannot be replaced by the weaker condition.

**Example 5.3.** Campana [Cl] [CT2] has shown that certain Heisenberg groups \( \Gamma \), which are nontrivial extensions of \( \mathbb{Z}^2 \) by \( \mathbb{Z} \), are Kähler. The natural projection \( \alpha : \Gamma \to \Gamma/\Gamma = \mathbb{Z}^2 \) is Kähler but \( e(\alpha) \) is not torsion.

Before starting the proof, we need to recall some standard facts about classifying spaces. Given a discrete group \( G \), we can identify \( BG = K(G, 1) \). If \( X \) is a good topological space (e.g. a CW complex), then there is a canonical map \( k : X \to B\pi_1(X) \), unique up to homotopy, classifying the universal cover viewed as a principle bundle over \( X \). Given an orbifold \( Y \), we thus get a classifying map \( k : [X] \to B\pi_1([X]) = B\pi_1(Y) \), where \([X]\) is the associated homotopy type. We can realize this as in a more explicit fashion by a simple modification of the procedure given in [Sg] §4. Choose an atlas \( \{ U_i \} \) so that the intersections \( U_I = \cap_i \in I U_i \) are all simply connected. Consider the groupoid

\[
\mathcal{H} = \coprod_{I \subseteq J} G_J \times U_J \rightrightarrows \coprod_I G_I \times U_I
\]

where \( I, J \) run over finite subsets of the index set. The structure maps are similar to those of the groupoid \( G \) constructed in section 3. The groupoids \( \mathcal{H} \) and \( G \) are easily seen to be equivalent in the sense of [M] §2.4. Thus the weak homotopy type of \( B\mathcal{H} \) is also \([X]\). Let \( L \) be the locally constant sheaf of sets on \( X \) corresponding to \( \pi = \pi_1(X) \) with its left \( \pi \)-action. We get a morphism of groupoids \( \lambda : \mathcal{H} \to \pi \) with the elements \( \lambda_{I,J} \in \pi \) corresponding to the transition functions of \( L \) viewed as a flat bundle. The map \( \lambda \) induces a a map of simplicial spaces \( k_* : B\mathcal{H}_* \to B\pi_* \), whose geometric realization is precisely \( k \).

As noted in the introduction, theorem 5.1 is fairly elementary when \( X \) and \( Y \) are smooth algebraic varieties.

**Proof for algebraic varieties.** Since the generic fibre \( X_\eta \) has a rational point over some finite extension of \( \mathbb{C}(Y) \). There exists (by resolution of singularities) a smooth \( Y' \) and a proper generically finite map \( p : Y' \to Y \) such that \( X \times_Y Y' \to Y' \) has a section. Therefore \( \pi_1(X \times_Y Y') \to \pi_1(Y') \) splits, and so the theorem follows from lemma 3.2.

We employ a different strategy for the general case.

**Proof of theorem 5.1 in general.** By lemmas 3.1, 3.3, 3.4 and 3.5, we can reduce to the case where \( f : X \to Y \) is a connected holomorphic Kähler map of orbifolds satisfying the assumptions of lemma 5.3. Set \( G = \pi_1(Y) \), \( H = \pi_1(X) \) and \( K = \ker(\pi_1(f)) \). Let \( V = H_1(K, \mathbb{Q}) = K/\pi DK \otimes \mathbb{Q} \) with its natural \( G \)-action. We also
view this as a locally constant sheaf on $Y$. Then by [HS, thm 4], $e \otimes \mathbb{Q}$ is $\pm d_2(id)$, where

$$d_2 : \text{Hom}_G(V, V) \cong H^0(G, H^1(K, V)) \to H^2(G, H^0(K, V)) \cong H^2(G, V)$$

is the differential of the Hochschild-Serre spectral sequence.

After choosing compatible atlases for $X$ and $Y$, we can, for the purposes of sheaf theoretic calculations, replace $f$ by a map of simplicial spaces $f_* : X_* \to Y_*$. More precisely, we can identify sheaves on $X$ and $Y$ with certain simplicial sheaves on $X_*$ and $Y_*$. In addition, the Leray spectral sequences for $f_*$ and $f$ can be identified. Consider the diagram

$$
\begin{array}{c}
\text{BH} \quad k_* \quad X_* \\
\downarrow \phi_* \quad \quad \quad \quad \quad \quad \downarrow f_* \\
BG_* \quad k_* \quad Y_*
\end{array}
$$

where the maps labeled by $k_*$ are the canonical maps realized simplicially as above. The left hand square is Cartesian. The geometric realization $\phi$ of $\phi_*$ can be assumed to be a fibration. Consequently, the realization $F$ of $F_*$ is also a fibration.

The Hochschild-Serre spectral sequence can be identified with the Leray spectral for $\phi$ with coefficients in the local system $\phi^*V$. This in turn can be identified with the spectral sequence for $\phi_*$. Since the Leray spectral sequences for $\phi_*, F_*$ and $f_*$, and hence $f$ are compatible, we have a commutative diagram

$$
\begin{array}{cccc}
H^0(BG_*, R^1\phi_*\phi^*V) & \quad & H^0(Y_*, R^1F_* F^*V) & \quad & H^0(Y, R^1f_* f^*V) \\
\downarrow d_2 & & \downarrow & & \downarrow d_2 \\
H^2(BG_*, \phi_*\phi^*V) & \quad & H^2(Y_*, F_* F^*V) & \quad & \ell \to H^2(Y, f_* f^*V)
\end{array}
$$

We can identify $k_*^*$ with the map on geometric realizations $H^2(BG, \phi_* \phi^*V) \to H^2([Y], F_* F^*V)$. This is injective, since the (homotopy) fibre of $k$ is simply connected. Also since the fibres of the geometric realizations of $f_*$ and $F_*$ are connected, we have $F_* F^* V = f_* f^* V = V$ by the projection formula. Thus $\ell$ is an isomorphism. So it suffices to prove that $d_2^2$ is zero.

By our assumptions, we have an exact sequence

$$1 \to L \to \pi_1(U) \to G \to 1$$

where the group $L$ is normal subgroup generated by powers of loops $\gamma_i$ around components of the discriminant divisor $D$, and $U = Y - D$. Since the first homology of the general fibre of $f$ surjects onto $V$ by lemma 8.5, the restriction $V|_U$ can be identified with a locally constant quotient of $W = (R^1f_* Q)^\vee|_U$ on which $L$ acts trivially. $W$ carries a pure variation of Hodge structure, therefore the action of $\pi_1(U)$ on it is semisimple by Deligne’s theorem [Sc, 7.2.5]. Consequently $V|_U \subset W$ is a direct summand. Therefore $V = IC(V)$ is a direct summand of $IC(W) = \ldots$
IC(f, f*W). Consequently, we have a diagram

\[
\begin{array}{ccc}
H^0(Y, R^1 f_* f^* V) & \xrightarrow{d'_2} & H^2(Y, f_* f^* V) \\
\downarrow & & \downarrow \\
H^0(Y, IC(R^1 f_* f^* W)) & \xrightarrow{d''_2} & H^2(Y, IC(f_* f^* W))
\end{array}
\]

where the map labeled \(i\) is injective. The map \(d''_2\), which is a summand of the differential of the perverse Leray spectral sequence, vanishes by corollary 4.5. Therefore \(d'_2 = 0\) and the theorem is proven.

**Corollary 5.4.** The theorem holds when \(X\) is Kähler and \(f\) is assumed to be proper surjective such that the map \(\pi_1(X) \to \pi_1(Y)\) is surjective. (Connectedness is not assumed.)

**Proof.** By Stein factorization and resolution of singularities, we can find a commutative diagram of complex manifolds

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

where \(f'\) is as in the theorem and \(g\) is generically finite. The result now follows lemma 3.2 and the theorem. \(\square\)

Let \(\Gamma_g = \langle a_1, \ldots, a_{2g} \mid [a_1, a_{g+1}] \cdots [a_g, a_{2g}] = 1 \rangle\) be the surface group of genus \(g\). Call a homomorphism \(h : \pi \to \Gamma_g\) maximal, if it does not factor through any \(\Gamma_{g'}\) with \(g' > g\).

**Corollary 5.5.** If a Kähler group admits a maximal surjective homomorphism \(h : \pi \to \Gamma_g\) with \(g \geq 2\), then \(e(h)\) must be torsion.

**Proof.** By a theorem of Beauville-Siu [ABC, thm 2.11], \(h\) is can be realized as \(\pi_1(f)\) where \(f : X \to C\) is a surjection onto a Riemann surface. This uses maximality. We can now finish the proof by appealing to the previous corollary, although in fact it is not necessary. If \(f\) were not connected, we could Stein factor the map, and conclude that \(h\) is not maximal. \(\square\)

This implies that a nontrivial central extension of \(\Gamma_g\) by a torsion free abelian group is not Kähler. In particular, this rules out the group given in the introduction.

J. Amorós has pointed that to me that one can also see that this example has nontrivial Massey products, thus contradicting the formality theorem of [DGMS].

**Corollary 5.6.** Suppose that \(\pi\) is a Kähler group, and either \(b_1(\pi) = \dim H^1(\pi, \mathbb{Q}) = 2\), or \(b_1(\pi) = 4\) and cup product \(\wedge^2 H^1(\pi, \mathbb{Q}) \to H^2(\pi, \mathbb{Q})\) is injective. Then \(e(\alpha)\) is torsion, where \(\alpha : \pi \to \pi/D\pi\) is the Abelianization.

**Proof.** Let \(X\) be a compact Kähler manifold with fundamental group \(\pi\). The map \(\pi \to \pi/D\pi/(\text{torsion})\) can be realized as the map as \(\pi_1(alb)\), where \(alb : X \to Alb(X)\) is the Albanese map. We claim that \(alb\) is surjective. If \(b_1(\pi) = 2\), then \(Alb(X)\) is an elliptic curve, and so surjectivity is clear. When \(b_1(\pi) = 4\), \(Alb(X)\) is a two dimensional torus. So the image of \(alb(X)\) is either a curve or all of \(Alb(X)\).
The first case is ruled out by the injectivity assumption for the cup product map. Therefore we are done by corollary 5.4. □

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