A degree bound for families of rational curves on surfaces

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February 12, 2014

Abstract

We give an upper bound for the degree of rational curves in a family that covers a given birational ruled surface in projective space. The upper bound is stated in terms of the degree, sectional genus and arithmetic genus of the surface. We introduce an algorithm for constructing examples where the upper bound is tight. As an application of our methods we improve an inequality on lattice polygons.

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1 Introduction

A parametrization of a rational surface $S \subset \mathbb{P}^n$ is a birational map

$$f : \mathbb{C}^2 \rightarrow Y \subset \mathbb{P}^n, \quad (s, t) \mapsto (f_0(s, t) : \ldots : f_n(s, t))$$

The parametric degree of $S$ is defined as the minimum of $\max\{\deg f_i | i \in [0, n]\}$ among all birational maps $f$.

A bound for the parametric degree over an algebraically closed field of characteristic 0 is given in Schicho [2000] in terms of the sectional genus and degree of $S$. In Schicho [2006] bounds for the parametric degree over perfect fields are expressed in terms of the level and keel. The upper bound in Schicho [2000] can be interpreted as an upper bound on the level. The analysis of Schicho [2006] applied to toric surfaces led to new inequalities for invariants of lattice polygons in Haase and Schicho [2009]. In subsection 2.7 of Castryck [2012] it is conjectured that the inequality can be improved using the number of vertices. In Haase and Schicho [2013] these inequalities for lattice polygons are translated to inequalities of rational surfaces. In the conclusion of Haase and Schicho [2013] the conjecture of Castryck [2012] is restated in the context of rational surfaces.

In this paper we pick up the torch and generalize the level and keel for rational surfaces in Schicho [2006] to birational ruled surfaces. This generalization is also posed as an open question in Haase and Schicho [2013]. Instead of the parametric degree we now consider the minimal family degree (see §4). We give an upper bound for the level of a birational ruled surface $S \subset \mathbb{P}^n$ in terms of the sectional genus, degree and arithmetic genus. As a corollary we obtain an upper bound for the minimal family degree. If $S$ is rational then our upper bound for the level coincides with the upper bound for the level in Schicho [2000]. However, in order to generalize this bound we give an alternative proof. This proof enables us to make a case distinction on the invariants of $S$, which improves the upper bound for the level. Moreover, these methods enables us to proof the correctness of an algorithm that outputs examples where our upper bound is attained. Thus we show that our upper bound for the level is tight. This algorithm is very simple but has a non-trivial correctness proof. We use the methods of our paper to generalize the inequality in Haase and Schicho [2013] to birational ruled surfaces. If we restrict our generalized inequality to toric surfaces, we obtain an improved inequality on lattice polygons as conjectured in Castryck [2012]. In line of the historical context, I would like to give the torch back, wondering whether this inequality can be improved in the language of lattice geometry.
I would like to end the introduction with some additional remarks on the degree of minimal parametrizations. Let $s(f) := \max\{\deg f_i | i \in [0, n]\}$, $t(f) := \max\{\deg t_i | i \in [0, n]\}$ and we assume without loss of generality that $t(f) \geq s(f)$. The parametric bi-degree of $S$ is defined as the minimum of $(s(f), t(f))$ among all birational maps $f$ with respect to the lexicographic order on ordered pairs of integers. If $S \subset \mathbb{P}^n$ attains at least 2 minimal families then from Theorem 17 in Lubbes [2013] it follows that the parametric bi-degree of $S$ equals $(v(S), v(S))$ where $v(S)$ is minimal family degree. Thus in this case our upper bound for the minimal family degree translates into an upper bound of the parametric bi-degree. If $S$ does only carry 1 minimal family then an upper bound on the parametric bi-degree is still open. In this case we also have to incorporate the keel aside the sectional genus, degree and arithmetic genus of $S$.

2 Intersection theory

We recall some intersection theory and this section can be omitted by the expert. We refer to chapter 2 and 5 in Hartshorne [1977] and chapter 1 in Matsuki [2002] for more details. See also Griffiths and Harris [1978]. The results in this section may be used implicitly in proofs of this paper.

The Neron-Severi group $N(X)$ of a non-singular projective surface $X$ can be defined as the group of divisors modulo numerical equivalence. This group admits a bilinear intersection product

$$N(X) \times N(X) \rightarrow \mathbb{Z}.$$ 

The Picard number of $X$ is defined as the rank of $N(X)$. The Neron-Severi theorem states that the Picard number is finite. For proofs in the next section we implicitly also consider $N(X) \otimes \mathbb{R}$. Moreover, we switch between the linear and numerical equivalence class of a divisor where needed.

The class of an exceptional curve $E$ in $N(X)$ is characterized by $E^2 = EK = -1$ where $K$ is the anticanonical divisor class of $X$. Castelnuovo’s contractibility criterion states that for all exceptional curves $E$ there exists a contraction map

$$X \xrightarrow{f} Y,$$

such that $f(E) = p$ is a smooth point and $(X \setminus E) \rightarrow (Y \setminus p)$ is an isomorphism via $f$. The assignment of Neron-Severi groups is functorial such that

$$N(Y) \xrightarrow{f^*} N(X).$$
The groups are related by $N(X) \cong N(Y) \oplus \mathbb{Z}(E)$. We have that $f^*(K) = K_Y - E$. Let $D \subset Y$ be a divisor and let $\tilde{D}$ be the strict transform of $D$ along $f$. We have that $f^*[D] = [\tilde{D}] + mE$ where $[D] \in N(Y)$, $[\tilde{D}] \in N(X)$ and $m$ is the order of $D$ at $p$. For the intersection product we have the projection formula

$$f^*(C)A = Cf_\ast(A),$$

and compatibility with the pullback

$$f^*(A)f^*(B) = AB,$$

for all $A, B \in N(X)$ and $C \in N(Y)$.

The Hodge index theorem states that if $A^2 > 0$ and $AB = 0$ then $B^2 < 0$ or $B = 0$ for all $A, B \in N(X)$. The adjunction formula implies that $A^2 + AK \geq -2$ for all $A \in N(X)$.

We denote by $p_a(X)$ the arithmetic genus of $X$ and it is a birational invariant. The Riemann-Roch theorem states that

$$h^0(D) - h^1(D) + h^2(D) = \frac{D(D-K)}{2} + p_a(X) + 1,$$

for a divisor class $D$ (up to linear equivalence) with associated sheaf $\mathcal{O}(D)$. Here $h^i(D)$ denotes the dimension of the $i$-th sheaf cohomology $\dim H^i(\mathcal{O}(D))$. Serre duality states that $h^2(D) = h^0(K - D)$.

## 3 Adjunction

For standard definition such as nef and big we refer to Matsuki [2002]. Adjunction works over any field.

We call a divisor class $D$ of a surface efficient if and only if $DE > 0$ for all exceptional curves $E$.

We define a ruled pair as a pair $(X, D)$ where $X$ is a non-singular birational ruled surface and $D$ is a nef and efficient divisor class of $X$.

If $D$ is effective then the polarized model of $(X, D)$ is defined as $\varphi_D(X) \subset \mathbb{P}^{h^0(D)-1}$ where $\varphi_D$ is the map associated to the global sections $H^0(\mathcal{O}(D))$.

If $(X, D)$ is a ruled pair then the canonical divisor class $K$ of $X$ is not nef. We recall that the nef threshold of $D$ is defined as

$$t(D) = \sup \{ q \in \mathbb{R} | D + qK \text{ is nef} \}.$$
We call a ruled pair \((X, D)\) non-minimal if \((D\text{ is big})\) and either \((t(D) = 1\) and \(D \sim -K\)) or \((t(D) > 1)\).

We call a ruled pair \((X, D)\) minimal if either \((t(D) = 1\) and \(D \sim -K\)) or \((t(D) < 1)\).

An adjoint relation is a relation

\[
(X, D) \xrightarrow{\mu} (X', D') := (\mu(X), \mu_*(D + K))
\]

where \((X, D)\) is a non-minimal ruled pair, and \(X \xrightarrow{\mu} X'\) is the birational morphism that contracts all exceptional curves \(E\) such that \((D + K)E = 0\).

**Proposition 1. (adjoint relation)**

If \((X, D) \xrightarrow{\mu} (X', D')\) is an adjoint relation then \((X', D')\) is either a non-minimal or a minimal ruled pair.

**Proof.** Let \((E_j)\) be the curves that are contracted by \(\mu\). From \((D + K)E_j = 0\) it follows that \(\mu^*D' = D + K\).

Suppose by contradiction that \(D'\) is not nef. It follows that there exists a curve \(C'\) such that \(D'C' = \mu^*D'\mu^*C' < 0\). From \(\mu^*D'\mu^*C' < 0\) it follows that \((D + K)C < 0\) where \(C\) is the strict transform of \(C'\). However, the nef threshold \(t(D) \geq 1\). Contradiction.

Suppose by contradiction that \(D'\) is not efficient. It follows that there exists exceptional curve \(E'\) such that \(D'E' = \mu^*D'\mu^*E' = (D + K)E = 0\) where \(E\) is the strict transform of \(E'\). From \(K'\mu_*E = \mu^*K'E = -1\) it follows that \(KE \leq -1\). From

\[
\mu^*K'\mu^*E' = \left( K - \sum_j E_j \right) \left( E + \sum_j m_j E_j \right) = KE - \sum_{a \neq b} m_a E_a E_b = -1,
\]

it follows that \(KE \geq -1\). From adjunction formula it follows that \(E^2 + EK = -2\). It follows that \(E^2 = EK = -1\) and thus \(E\) is an exceptional curve not contracted by \(\mu\). Contradiction.

We call a minimal ruled pair \((X, D)\) a weak Del Pezzo pair if and only if either \(D = -K\), \(D = -\frac{1}{2}K\), \(D = -\frac{1}{3}K\), or \(D = -\frac{2}{3}K\), with \(K\) the canonical divisor class of \(X\).

We call a minimal ruled pair \((X, D)\) a geometrically ruled pair if and only if \(X \xrightarrow{\phi_M} C\) is a geometrically ruled surface such that either \(M = aD\), or
\[ M = a(2D + K) \] for large enough \( a \in \mathbb{Z}_{>0} \). Here \( \varphi_M \) is the map associated to the global sections \( H^0(O(M)) \), \( C = \varphi_M(X) \) and \( K \) is the canonical divisor class of \( X \).

**Proposition 2. (Neron-Severi group of minimal ruled pair)**

Let \((X, D)\) be a minimal ruled pair, with \( K \) be the canonical divisor class of \( X \) and \( N(X) \) the Neron-Severi group.

**a)** If \((X, D)\) is a weak Del Pezzo pair with \( K^2 \neq 8 \) then \( N(X) \cong \mathbb{Z}\langle H, Q_1, \ldots, Q_r \rangle \) with \( 0 \leq r = 9 - K^2 \leq 8 \) and intersection product \( HQ_i = 0 \), \( Q_i^2 = 1 \) and \( Q_iQ_j = 0 \) for \( i \neq j \) in \([1, r]\). We have that \( -K = 3H - Q_1 - \ldots - Q_r \), and either \( D = -K \), \( D = -\frac{2}{3}K \), or \( D = -\frac{4}{3}K \).

**b)** If \((X, D)\) is a weak Del Pezzo pair with \( K^2 = 8 \) then \( N(X) \cong \mathbb{Z}\langle H, F \rangle \) with intersection product \( H^2 = r \), \( HF = 1 \) and \( F^2 = 0 \) for \( r \in \{0, 1, 2\} \). We have that \( K = -2H + (r - 2)F \) and either \( D = -K \) or \( D = -\frac{1}{2}K \).

**c)** If \((X, D)\) is a geometrically ruled pair then \( N(X) \cong \mathbb{Z}\langle H, F \rangle \) with intersection product \( H^2 = r \), \( HF = 1 \) and \( F^2 = 0 \) for \( r \in \mathbb{Z}_{\geq 0} \). Either \( D = kF \) or \( 2D + K = kF \) for \( k \in \mathbb{Z}_{\geq 0} \) and \( K = -2H + (r - 2p - 2)F \) where \( p \) is the arithmetic genus of \( X \).

**Proof.** For a) and b) see section 8.4.3 in Dolgachev [2012]. For c) see Beauville [1983], chapter 3, proposition 18, page 34. \( \Box \)

An adjoint chain of a ruled pair is defined as a chain of subsequent adjoint relations until a minimal ruled pair is obtained.

**Proposition 3. (adjoint chain)**

The adjoint chain is finite and a minimal ruled pair at the end is either a weak Del Pezzo pair or a geometrically ruled pair.

**Proof.** Let \((X, D)\) be a non-minimal ruled pair and let \( t := t(D) \) be the nef threshold. From Corollary 1-2-15 in Matsuki [2002] it follows that \( t \in \mathbb{Q}_{>0} \) with denominator bounded by 3. After a finite sequence of adjoint relations that do not contract curves we may assume that \( t \leq 1 \).

If \((D + tK)^2 > 0\) then there exists an irreducible curve \( C \) such that \((D + tK)C = 0\), \( DC > 0 \) and \( KC < 0 \). From Hodge index theorem and \((D + tK)^2 > 0\) it follows that \( C^2 < 0 \). From the adjunction formula it follows that
$C^2 + KC = -2$. From Lemma 1-1-4 in Matsuki [2002] it follows that $C$ is an exceptional curve. From §2 it follows that Picard number drops for each contracted exceptional curve and that this number is finite.

If $(D + tK)^2 = 0$ and $D \sim -tK$ then $(X, D)$ is a weak Del Pezzo pair.

We assume $(D + tK)^2 = 0$ and $D \sim -tK$. If $t = 1$ then we apply one extra adjoint relation so we may assume that $t < 1$. From Theorem 1-2-14 and Proposition 1-2-16 in Matsuki [2002] it follows that that the map associated to $l(D + tK)$ with large enough $l \in \mathbb{Z}_{<0}$ defines a Mori fibre space. From Theorem 1-4-4 it follows that a fibre $F$ of this morphism is isomorphic to $\mathbb{P}^1$ with $F^2 = 0$. From the adjunction formula it follows that $FK = -2$. From $(D + tK)F = 0$ it follows that $t = \frac{DF}{2} \in \frac{1}{2}\mathbb{Z}_{>0}$. Thus in this case $(X, D)$ is a geometrically ruled pair.

Let $(X_0, D_0) \xrightarrow{\mu_0} (X_1, D_1) \xrightarrow{\mu_1} \ldots \xrightarrow{\mu_{l-1}} (X_l, D_l)$ be an adjoint chain. The level of $(X_0, D_0)$ is defined by $l$.

The keel of $(X_0, D_0)$ is either:

- 0 if $(X_l, D_l)$ is a weak Del Pezzo pair, or
- $k$ as in Proposition 2.c) if $(X_l, D_l)$ is a geometrically ruled pair.

The level and keel were introduced in Schicho [2006] and are in accordance with our definitions.

**Proposition 4. (level and keel)**

The level and keel are well defined.

**Proof.** Let $(X, D) \xrightarrow{\mu} (X', D')$ be an adjoint relation.

From Hodge index theorem, $(D + K)^2 > 0$ and $(D + K)(E_1 + E_2) = 0$ it follows that $(E_1 + E_2)^2 < 0$ and thus $E_1E_2 = 0$. It follows that if $D'^2 > 0$ then then the contracted exceptional curves are disjoint. The blow down of an exceptional curve is an isomorphism outside this exceptional curve. Thus the order of contracting disjoint curves does not matter up to biregular isomorphism.

Since $D'^2 = 0$ can only occur at the last adjoint relation in an adjoint chain it follows that the level is well defined.

Suppose that $(X', D')$ is a geometrically ruled pair. From Proposition 2 it follows that if $D'^2 = (D+K)^2 = 0$ then $-2k = \mu^*D'\mu^*K' = (D+K)K$ defines
the keel $k$. Similarly, if $D^2 = (D + K)^2 > 0$ then $-2k = \mu^*(2D' + K')\mu^*K' = 2(D + K)K + K'^2$. From Proposition 2 it follows that $K'^2 = 8(p+1)$ where the arithmetic genus $p$ is birational invariant. It follows that the keel independent of the last adjoint relation.

\[ \square \]

4 Minimal families

A family of curves $F$ for ruled pair $(X, D)$ and indexed by a smooth curve $C$ is defined as a divisor $F \subset X \times C$ such that the 1st projection $F \rightarrow X$ is dominant. If the generic curve of $F$ is rational and if $DF$ is minimal with respect to all families of rational curves, then we call $F$ minimal. The minimal family degree $v(X, D)$ is defined as $DF$ for a minimal family $F$.

Note that since $(X, D)$ is a ruled pair, there always exists a minimal family.

We recall part of a theorem in Lubbes and Schicho [2010] concerning the degree of minimal families along an adjoint relation $(X, D) \xrightarrow{\mu} (X', D')$. If $X \cong X' \cong \mathbb{P}^2$ then $v(X, D) = v(X', D') + 3$, else $v(X, D) = v(X', D') + 2$.

If $(X, D)$ is a weak Del Pezzo pair and $X \cong \mathbb{P}^2$ then $v(X, D) \leq 3$. If $(X, D)$ is a weak Del Pezzo pair and $D^2 = 8$ then $v(X, D) \leq 2$. If $(X, D)$ is a weak Del Pezzo pair and $D^2 < 8$ then $v(X, D) = 2$. If $(X, D)$ is a geometrically ruled pair then $v(X, D) \leq 1$.

5 Upper bound for the level

Let $(X_0, D_0) \xrightarrow{\mu_0} (X_1, D_1) \xrightarrow{\mu_1} \ldots \xrightarrow{\mu_{l-1}} (X_l, D_l)$ be an adjoint chain. We introduce the following notation:

\[
\alpha(i) = D_i^2, \quad \beta(i) = D_iK_i, \quad \gamma(i) = K_i^2, \quad h(i) = D_i^2 - D_iK_i,
\]

and $n(i)$ denotes the number of curves contracted by $\mu_i$ for $i \in [0, l]$.

**Lemma 5. (adjoint intersection products)**

If $l > 0$ then

a) $\alpha(i + 1) = \alpha(i) + 2\beta(i) + \gamma(i)$,

b) $\beta(i + 1) = \beta(i) + \gamma(i)$,

c) $\gamma(i + 1) = \gamma(i) + n(i)$,
d) \( h(i + 1) = h(i) + 2\beta(i) \), for \( i \in [0, l - 1] \).

Proof. From \((D_i + K_i)E = 0\) for all exceptional curves \( E \) contracted by \( \mu_i \) it follows that \( \mu_i^*D_{i+1} = D_i + K_i \). Now a) and b) are straightforward (see §2). From §2 it also follows that \( K_i^2 = (\mu_i^*K_{i+1})^2 = K_{i+1}^2 - n(i) + \sum_{j \neq t} E_j E_t \). From

\[
K_{i+1} \mu_i^* E_t = \mu_i^* K_{i+1} E_t = \left( K_i - \sum_j E_j \right) E_t = -1 + 1 - \sum_j E_j E_t = 0
\]

it follows that c) holds. From \( h(i + 1) = \alpha(i + 1) - \beta(i + 1) = \alpha(i) + \beta(i) = h(i) + 2\beta(i) \) it follows that d) holds.

**Remark 6. (Castelnuovo)**

Lemma 5.d) is essentially Lemma 7 in Schicho [2000] and Josef Schicho in turn attributes this result to Castelnuovo. See remark 3 in Schicho [2000].

We distinguish between the following adjoint states:

| adjoint state | \( \gamma(i) \) | \( \beta(i) \) |
|---------------|-----------------|----------------|
| \( S_1(i) \)  | \( < 0 \)        | \( \geq 0 \)    |
| \( S_2(i) \)  | \( < 0 \)        | \( < 0 \)       |
| \( S_3(i) \)  | \( = 0 \)        | \( < 0 \)       |
| \( S_4(i) \)  | \( > 0 \)        | \( < 0 \)       |

for \( i \in [0, l] \).

**Lemma 7. (adjoint states)**

a) The adjoint states are all possible states.

b) If \( S_a(i) \) and \( S_b(i + 1) \) for \( i \in [0, l - 1] \) then \( a \leq b \).

Proof. a) We assume first that \( \gamma(i) = 0 \). Assume by contradiction that \( \beta(i) \geq 0 \). From Lemma 5 it follows that \( \alpha(j + 1) \geq \alpha(j) \) and \( \beta(j) = \beta(j + 1) \) for all \( j \geq i \). But then the adjoint chain is of infinite length. Contradiction.
Next we assume that $\gamma(i) > 0$. From Proposition 2 it follows that the arithmetic genus of $X_i$ is 0. From Riemann Roch theorem and Serre duality it follows that $h^0(-K_i) \geq \gamma(i) + 1 > 0$. From $D_i$ being nef it follows that $\beta(i) \leq 0$. From Hodge index theorem it follows that $\beta(i) < 0$.

b) From Lemma 5 it follows that $\gamma(i) < \gamma(i + 1)$ and if $\gamma(i) < 0$ then $\beta(i + 1) < \beta(i)$. \hfill $\Box$

**Lemma 8. (dimension)**

We have that $h(i) > 0$ for $1 \leq i \leq l$.

**Proof.** From Lemma 5 it follows that $h(0) + h(1) = 2\alpha(0) > 0$. It follows that the induction basis $h(0) > 0$ or $h(1) > 0$ holds. By induction hypothesis $h(i) > 0$. The induction step is to show that $h(i + 1) > 0$. If $\beta(i) \geq 0$ then from Lemma 5 it follows that $h(i + 1) = h(i) + 2\beta(i - 1) > 0$. If $\beta(i) < 0$ then from Lemma 7 it follows that $\beta(i + 1) < 0$ and thus $h(i + 1) > 0$. \hfill $\Box$

**Theorem 9. (upper bound level)**

Let $p$ be the arithmetic genus of $X_0$.

If $p = 0$ or $p = -1$ then

$$l \leq \frac{\beta(0)^2 + \alpha(0)}{2} + \beta(0),$$

and if $\beta(0) < 0$ then

$$l \leq \left[\frac{\alpha(0) - \beta(0)}{-2\beta(0)}\right] - 1.$$

If $p \leq -2$ then

$$l \leq \left[\frac{\beta(0)}{-8(p + 1)}\right] + \left[\frac{-8(p + 1) - \sqrt{\Delta}}{16(p + 1)}\right],$$

where

$$\Delta = (8(p + 1))^2 - 32(p + 1)\left(\frac{\beta(0)^2}{-8(p + 1)} + \alpha(0)\right),$$

and if $\beta(0) < 0$ then

$$l \leq \left[\frac{8(p + 1) - 2\beta(0) - \sqrt{\Delta}}{16(p + 1)}\right] - 1$$

where

$$\Delta = (2\beta(0) - 8(p + 1))^2 - 32(p + 1)(\alpha(0) - \beta(0)).$$
Proof. From b) and d) in Lemma 5 and from Lemma 8 it follows that
\[ h(i + 1) = h(i) + 2\beta(i), \quad \beta(i + 1) = \beta(i) + \gamma(i), \quad h(i) > 0, \]
for \( i \in [0, l] \). These formulas together with Lemma 7 for the adjoint states are the main ingredients of this proof.

We state upper bounds for each adjoint state.

1. If \( S_1(0) \) then \( S_{\geq 2}(j) \) and \( h(j) \leq \frac{1}{t}\beta(0)^2 + \alpha(0) \) for
\[ j \leq \left\lfloor \frac{\beta(0)}{t} \right\rfloor + 1. \]

where \( t = -\min(\gamma(l), -1) \).

Upper bound (1) is reached if \( \gamma(i) = -t \) for all \( i \leq j \).

Note that \( h(j) \leq h(0) + 2 \sum_{n=0}^{j-1} (\beta(0) + n\gamma(l)) \). See \( 0 \leq i \leq 2 \) in Table 1 of Example 12.

2a. If \( S_2(0) \) and \( p \geq -1 \) then \( S_3(j) \) and \( h(j) \leq h(0) + \beta(0)(\beta(0) + 1) - \beta(l)(\beta(l) + 1) \) for
\[ j \leq \beta(0) - \beta(l). \]

Upper bound (2a) is reached if \( \gamma(0) = -1 \) and \( \gamma(j) = 0 \) for minimal \( j \) such that \( \beta(j) = \beta(l) \). See \( 6 \leq i \leq 7 \) in Table 2 of Example 12.

2b. If \( S_2(0) \) and \( p \leq -2 \) then
\[ l \leq \left\lfloor \frac{-2\beta(0) - \gamma(l)) - \sqrt{\Delta}}{2\gamma(l)} \right\rfloor - 1 \]
where \( \Delta = (2\beta(0) - \gamma(l))^2 - 4\gamma(l)h(0) \)

Upper bound (2b) is reached if \( \gamma(i) = \gamma(l) \) for \( i \leq l \). It follows that
\[ h(x) \leq h(0) + 2 \sum_{n=0}^{x-1} (\beta(0) + n\gamma(l)) = \gamma(l)x^2 + (2\beta(0) - \gamma(l))x + h(0). \]

Now (2b) follows from the quadratic formula and \( h(x) > 0 \) for \( x \leq l \). See \( 5 \leq i \leq 9 \) in Table 4 of Example 12.
(3) If \( S_3(0) \) then
\[
l \leq \left\lceil \frac{h(0)}{-2\beta(l)} \right\rceil - 1.
\]
Upper bound (3) is reached if \( \gamma(i) = 0 \) for \( i < l \) and \( \beta(i) = \beta(l) \) for \( i \leq l \).
See \( 3 \leq i \leq 7 \) in Table 1 of Example 12.

(4) If \( S_4(0) \) then \( h(j) \leq 0 \) for
\[
j \leq \frac{h(0)}{-2\beta(0)}
\]
Upper bound (4) is reached if \( \gamma(i) = 0 \) for \( i < l \) (so it could be improved since \( \gamma(i) > 0 \)). See \( i = 8 \) in Table 1 of Example 12.

Suppose that \( p = 0 \) or \( p = -1 \). From Proposition 2 it follows that \( \beta(l) \leq -1 \) and \( \gamma(l) \geq 0 \). If we select \( \beta(l) = -1 \) then (2a) is minimized to zero, but (3) is maximized much more. It follows that the upper bound is maximized if we skip adjoint state \( S_2 \) as in Table 1 of Example 12 and choose \( \beta(l) = -1 \). For similar reasons we let \( S_4(i) \) only if \( p = 0 \) and \( i = l \). It follows that the upper bound is (1) plus (3) with the upper bound for \( h(j) \) from (1) substituted. If \( \beta(0) < 0 \) then (3) gives the required upper bound with \( \beta(0) \) substituted for \( \beta(l) \).

Suppose that \( p \leq -2 \). Recall that the arithmetic genus \( p \) is a birational invariant. From Proposition 2, c) it follows that \( \gamma(l) = 8(p + 1) \). Thus only adjoint states \( S_1 \) and \( S_2 \) are possible. If \( \beta(0) \geq 0 \) in (1) then assuming \( \beta(0) = -8(p+1) \) in (2b) gives the required upper bound. Note that although \( 0 > \beta(0) > -8(p + 1) \) is possible relative to (2b), we surely have \( \beta(1) = \beta(0) + \gamma(0) \leq -8(p + 1) \). In this case the estimate in (1) is assuming \( \beta(1) \) as the first step of adjoint state \( S_2 \), and the estimated \( h(0) \) relative to (2b) is an upper bound. The upper bound is (1) plus (2b) with the upper bound for \( h(j) \) from (1) substituted. If \( \beta(0) < 0 \) then (2b) gives the required upper bound.

Corollary 10. (upper bound minimal family degree)
Let \( v = v(X_0, D_0) \) be the minimal family degree. Let \( \tilde{l} \) be the upper bound for the level from Theorem 9.
If \( p = 0 \) then \( v \leq 2\tilde{l} + 2 \).
If \( p \leq -1 \) then \( v \leq 2\tilde{l} + 1 \).
Proof. From §4 it follows that if \( n(i+1) = 0 \) and \( X_i \cong \mathbb{P}^2 \) then \( v(X_{i+1}, D_{i+1}) = v(X_i, D_i) + 3 \). Otherwise \( v(X_{i+1}, D_{i+1}) = v(X_i, D_i) + 2 \).

We show that the construction of the upper bound of Theorem 9 for \( p = 0 \) enables us to assume that the level is increased by 2 and not 3.

Suppose that \( X_i \cong \mathbb{P}^2 \) and \( n(i+1) = 0 \). In this case \( -\beta(i) = \gamma(i) = 9 \) and \( S_4(j) \) for \( i \leq j \leq l \). Upper bound (4) in the proof of Theorem 9 increases the minimal family degree by at most \( 3\frac{h(0)}{-2}(-9) = \frac{h(0)}{6} \).

For the upper bound for the level when \( p = 0 \) we assumed that adjoint state \( S_4(i) \) if and only if \( i = l \). For upper bound (3) \( X_i \not\cong \mathbb{P}^2 \) and thus \( v(X_{i+1}, D_{i+1}) = v(X_i, D_i) + 2 \). It increases the minimal family degree with \( 2(\frac{h(0)}{2} - 1) = \frac{h(0)}{2} - 2 > \frac{h(0)}{6} \). It follows that we may assume that \( v(X_l, D_l) = 2 \).

From §4 it follows that if \( p \leq -1 \) then \( v(X_l, D_l) \leq 1 \).

Remark 11. (computing invariants)

Note that \( \alpha(0) \) is the degree of the (projection of the) polarized model of \( (X_0, D_0) \). From the adjunction formula it follows that the geometric genus of a generic hyperplane section of \( (X_0, D_0) \) is equal to

\[
p_a(D_0) = \frac{\alpha(0) + \beta(0)}{2} + 1.
\]

It follows that \( \alpha(0) \) and \( \beta(0) \) can computed from the degree and geometric genus of a generic hyperplane section.

If the ring of the initial surface represented by \( (X_0, D_0) \) is integrally closed then it is not a projection of the polarized model \( Y \subset \mathbb{P}^n \). From Proposition 3 it follows that

\[
n + 1 = h^0(D_0) = \frac{\alpha(0) - \beta(0)}{2} + p + 1
\]

and thus we can compute the arithmetic genus \( p \) of \( X_0 \).

Example 12. (adjoint chains)

In the following 4 tables we represent the invariants of an adjoint chain. We denote the upper bound of Theorem 9 by \( \tilde{l}(i) \). See the beginning of this section for the remaining notation. The heading of the table denotes the arithmetic genus \( p = p_a(X_0) \) and the number of different adjoint states that are reached. The transition between adjoint states in indicated by a vertical double line. These examples confirm that the upper bounds in Theorem 9 and Corollary 10 are tight. The tables were constructed using Algorithm 14 (see forward).
In Table 1 the minimal pair is a weak Del Pezzo pair of degree 1. The upper bound for the level is tight for this example and it follows the proof of Theorem 9. The polarized model of this surface is of degree 8. From §4 it follows that minimal family degree \( v(X_0) = 18 \) and thus the bound in Corollary 10 is tight.

In Table 2 the minimal pair is a weak Del Pezzo pair of degree 3. We see that the upper bound for the level is not tight in this example. All the adjoint states are reached in this example. If the arithmetic genus \( p = 0 \) then the upper bound is tight if adjoint state \( S_2 \) is not reached, as was the case in Table 1.

In Table 3 the minimal pair is a geometrically ruled surface such that \( p = -1 \) and \( 2D + K = kF \) as in Proposition 2. We find that the upper bound for the level in Theorem 9 is tight. The upper bound for the minimal family degree in Corollary 10 is also tight \( v(X_0) = 17 \).

In Table 4 the minimal pair is a geometrically ruled surface such that \( p = -2 \) and \( D = kF \) as in Proposition 2. We find that the upper bound for the level is tight. From Corollary 10 it follows that minimal family degree \( v(X_0) \leq 19 \). From §4 it follows that \( v(X_0) = 18 \).

Table 1 (arithmetic genus 0 and 3 adjoint states)

| \( i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|---|
| \( n(i) \) | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( \gamma(i) \) | -1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( \beta(i) \) | 2 | 1 | 0 | -1 | -1 | -1 | -1 | -1 | -1 |
| \( h(i) \) | 6 | 10 | 12 | 12 | 10 | 8 | 6 | 4 | 2 |
| \( \alpha(i) \) | 8 | 11 | 12 | 11 | 9 | 7 | 5 | 3 | 1 |
| \( \tilde{l}(i) \) | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
### Table 2 (arithmetic genus 0 and 4 adjoint states)

| $i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $n(i)$ | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 3  |
| $\gamma(i)$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | 0  | 0  | 0  | 0  | 0  | 3  |
| $\beta(i)$ | 5  | 4  | 3  | 2  | 1  | 0  | $-1$ | $-2$ | $-3$ | $-3$ | $-3$ | $-3$ | $-3$ |
| $h(i)$ | 6  | 16 | 24 | 30 | 34 | 36 | 36  | 34  | 30  | 24  | 18  | 12  | 6  |
| $\alpha(i)$ | 11 | 20 | 27 | 32 | 35 | 36 | 35  | 32  | 27  | 21  | 15  | 9   | 3  |
| $\tilde{l}(i)$ | 23 | 22 | 21 | 20 | 19 | 18 | 17  | 8   | 4   | 3   | 2   | 1   | 0  |

### Table 3 (arithmetic genus -1 and 2 adjoint states)

| $i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|----|
| $n(i)$ | 0  | 0  | 1  | 0  | 0  | 0  | 0  | 0  | 0  |
| $\gamma(i)$ | $-1$ | $-1$ | $-1$ | 0  | 0  | 0  | 0  | 0  | 0  |
| $\beta(i)$ | 2  | 1  | 0  | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $h(i)$ | 6  | 10 | 12 | 12 | 10 | 8  | 6   | 4   | 2  |
| $\alpha(i)$ | 8  | 11 | 12 | 11 | 9  | 7  | 5   | 3   | 1  |
| $\tilde{l}(i)$ | 8  | 7  | 6  | 5  | 4  | 3  | 2   | 1   | 0  |

### Table 4 (arithmetic genus -2 and 2 adjoint states)

| $i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|----|----|----|----|----|----|----|----|----|----|
| $n(i)$ | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| $\gamma(i)$ | $-8$ | $-8$ | $-8$ | $-8$ | $-8$ | $-8$ | $-8$ | $-8$ | $-8$ | $-8$ |
| $\beta(i)$ | 32 | 24 | 16 | 8  | 0  | $-8$ | $-16$ | $-24$ | $-32$ | $-40$ |
| $h(i)$ | 40 | 104 | 152 | 184 | 200 | 200 | 184 | 152 | 104 | 40 |
| $\alpha(i)$ | 72 | 128 | 168 | 192 | 200 | 192 | 168 | 128 | 72  | 0  |
| $\tilde{l}(i)$ | 9  | 8  | 7  | 6  | 5  | 4  | 3   | 2   | 1  | 0  |
6 Algorithm for constructing examples

We use the same notation as in the previous section.

Lemma 13. (formulas for intersection products)

a) $\alpha(0) = \gamma(l)l^2 - 2\beta(l)i + \alpha(l) - \sum_{i=0}^{l-1}(i + 1)^2n(i)$.

b) $\beta(0) = -\gamma(l)i + \beta(l) + \sum_{i=0}^{l-1}(i + 1)n(i)$.

c) $\gamma(0) = \gamma(l) - \sum_{i=0}^{l-1}n(i)$.

Proof. See §2. Let $R_i$ be the sum of exceptional curves that are contracted by $\mu_i$ for $0 \leq i < l$. We have that $K_{i-1} = \mu_{i-1}^*K_i + R_{i-1}$ and $D_{i-1} = \mu_{i-1}^*D_i - K_{i-1}$. By abuse of notation we will denote $\mu_{i-1}^*D_i$ as $D_i$ and $\mu_{i-1}^*K_i$ as $K_i$.

It follows that $D_{i-1} = D_i - K_i - R_{i-1}$. In the next iteration we obtain

$D_{i-2} = D_{i-1} - K_{i-1} - R_{i-2} = (D_i - K_i - R_{i-1}) - (K_i + R_{i-1}) - (R_{i-2}) = D_i - 2K_i - 2R_{i-1} - R_{i-1}$. Repeating this we obtain

$D_0 = D_l - lK_l - \sum_{i=0}^{l-1}(i + 1)R_i$.

Similarly we find

$K_0 = K_l + \sum_{i=0}^{l-1}R_i$.

From Lemma 5.c), it follows that $R_i^2 = -n(i)$. From the projection formula it follows that $R_iR_j = D_jR_i = K_iR_i = 0$ for $i, j \in [0, l]$. 

The following algorithm constructs an adjoint chain for a given level and minimal ruled pair. The level of the output adjoint chain approximates the upper bound of Theorem 9 as close as possible. The adjoint chains in Example 12 were constructed with this algorithm and proof that the upper bounds of Theorem 9 are tight.
Algorithm 14. *(construct adjoint chain)*

**input:** Level $l$, $\alpha(l)$, $\beta(l)$, $\gamma(l)$ and $c \in \mathbb{Z}_{\geq 1}$.

**output:** The number of contracted curves $n(i)$ for $i \in [0, l-1]$ such that the difference between $l$ and the upper bound in Theorem 9 is minimal, under the condition that $\alpha(0) \geq c$. If no such valid adjoint chain exists for given input then the output is $\emptyset$.

- We set tuple $\tilde{n}(i) := 0$ for $i \in [0, l-1]$ and we set $j := 0$.

- While $j \neq -1$ repeat the following:
  
  Set $n$ equal to $\tilde{n}$ except $n(j) := \tilde{n}(j) + 1$ where $j$ is the maximal index in $[0, l-1]$ such that $\alpha(0) \geq c$. Here $\alpha(0)$ is computed with the formula in Lemma 13.a).

  If no such $j$ exists then $j := -1$ else $\tilde{n} := n$.

- If $\alpha(0) \geq c$ return the tuple $n$ and $\emptyset$ otherwise.

**Remark 15. (geometric meaning of constant c)**

By increasing the input constant $c$ we also increase $h(0)$. If we choose the points that are blown up generically enough then we may assume that $D_0$ is ample. From Proposition 2 it follows that the arithmetic genus of $X_0$ equals $p = \min(0, \lceil \frac{1}{8} \gamma(l) - 1 \rceil)$. By Kodaira vanishing and Riemann-Roch it follows that

$$h^0(D_0) \geq \frac{h(0)}{2} + p + 1.$$  

If $D_0 - K_0$ is nef and big then alternatively we can use Kawamata-Viehweg vanishing theorem. See chapter 4 in Lazarsfeld [2004] for vanishing theorems.

Recall that by definition of nef and big only a high enough multiple of $D_0$ defines a birational morphism. Reiders theorem says that if $D_0^2 \geq 10$ and there exists no curve $C$ such that $(DC = 0$ and $C^2 = -1)$ or $(DC = 1$ and $C^2 = 0$) or $(DC = 2$ and $C^2 = 0$) then $D_0 + K_0$ defines a birational morphism. This can be used to ensure that the polarized model of $(X_1, D_1)$ is a surface.

For rational surfaces we can compute equations in the following way. From Proposition 2 and the proof of Lemma 13 we find that

$$D_0 = dH - \sum_i m_i E_i$$
where $H$ is the pullback of lines in the projective plane and the $E_i$ are the pullback of exceptional curves. We construct a linear series $|D_0|$ in the plane with polynomials of degree $d$ and generic base points with multiplicities $(m_i)_i$. We check whether the map associated to the linear series parametrizes a surface, otherwise we have to consider a multiple of $D_0$. After a generic projection we may assume that we have a parametrization a hypersurface in 3-space. We consider an implicit equation of degree $\alpha(0)$ with undetermined coefficients and substitute the parametrization. We obtain an implicit equation by solving the linear system of equations in the undetermined coefficients.

**Proposition 16. (algorithm)**

Algorithm 14 is correct.

**Proof.** We denote by $\tilde{l}$ the upper bound of Theorem 9. We proof that the algorithm minimizes $\tilde{l} - l$ under the constraint that $\alpha(0) \geq c$.

**Claim 1:** In order to minimize $\tilde{l} - l$ we need to minimize $\gamma(i) - \gamma(0)$ under the constraints that $\gamma(0) < 0$ and $\alpha(0) \geq c$.

From Lemma 7 it follows that we need $\gamma(0) < 0$. We refer to the upper bounds in the proof of Theorem 9 labeled (1), (2a), (2b), (3), (4) and we follow their analysis with corresponding notation. In order to approximate bound (1) we require that $\gamma(i)$ approximates $-\min(\gamma(l), -1)$ and this confirms this claim. If $p \geq -1$ then the adjoint state $S_2$ is skipped and we have $S_4(i)$ only if $i = l$; thus we don’t consider the bounds (2a) and (4). For bound (2b) we require that $\gamma(l) - \gamma(j)$ is minimal for $j \leq i < l$ where $j$ is the first index such that $S_2(j)$. If the arithmetic genus $p \geq -1$ then in order to approximate bound (3) as close as possible, we require that $|\gamma(i)|$ is as small as possible for $j \leq i < l$ where $j$ is the first index such that $S_3(j)$. Note that from Lemma 5 it follows that $\gamma(i+1) = \gamma(i) + n(i)$ for $i \in [0, l - 1]$. It follows that claim 1 holds.

From Lemma 13.a) it follows that the algorithm outputs $n(l - 1) \geq \gamma(l)$ if $\alpha(l) - 2\beta(l)l \geq c$. From Lemma 13.c) it follows that $\gamma(0) < 0$. The algorithm is minimizing

$$\Theta := \sum_{i=0}^{l-1} n(i).$$

From Lemma 13.c) it follows that $\gamma(l) - \gamma(0)$ is minimized as is required by claim 1.
From Lemma 13.a) it follows that the algorithm is maximizing
\[ \sum_{i=0}^{l-1} (i + 1)^2 n(i) \]
under the constraint that \( \alpha(0) \geq c \). Note that the algorithm is increasing
\( n(j) \) for maximal \( j \) such that \( \alpha(0) \geq c \). Since its coefficient in the above sum
is \((j + 1)^2\), it follows that \( \Theta \) is kept minimal.

It follows that the output of the algorithm is conform its specification. \( \square \)

7 Inequality for lattice polygons

Let \((X_0, D_0)\) be a toric surface with polarized model \( Y_0 \subset \mathbb{P}^n \). We define
the lattice polygon \( P_0 \) by taking the convex hull of the lattice points in the
lattice \( \mathbb{Z}^n \subset \mathbb{R}^n \) with coordinates defined by the exponents of a monomial
parametrization \((\mathbb{C}^*)^2 \longrightarrow Y_0\).

We denote \( \rho(0) \) for the Picard number of \( X_0 \). We define \( S(0) \) to be the number
of exceptional divisors in the minimal resolution of the isolated singularities
of \( Y_0 \). We introduce the following notation:
\[ v(0) := \rho(0) + 2 - S(0). \]

The adjoint of a lattice polygon is defined as the convex hull of its interior
lattice points. We call a lattice polygon minimal if its adjoint is either the
empty set, a point or a line segment. The level \( l(P_0) \) of a lattice polygon is
defined as the number of subsequent adjoint lattice polygons \( P_0 \longrightarrow \ldots \longrightarrow P_{l(P_0)} \) until a minimal lattice polygon \( P_{l(P_0)} \) is obtained.

We recall part of the dictionary in Haase and Schicho [2013] using the nota-
tion at the beginning of §5:

- \( \frac{\alpha(0)}{2} = a(P_0) \) (area),
- \( -\beta(0) = b(P_0) \) (number of boundary lattice points),
- \( v(0) = v(P_0) \) (number of vertices), and
- \( l = l(P_0) \) (level).
From Lemma 13 it follows that
\[ \alpha(0) + 2l\beta(0) = -\gamma(l)l^2 + \alpha(l) + \Phi \tag{1} \]
where
\[ \Phi := \sum_{i=0}^{l-1} (2l - i - 1)(i + 1)n(i). \]

As an immediate consequence we obtain the following inequality
\[ \alpha(0) + 2l\beta(0) + \gamma(l)l^2 \geq 0. \tag{2} \]

From Proposition 2 it follows that \( \gamma(l) \leq 9 \) and by substituting 9 for \( \gamma(l) \) in (2) we recover the inequality of Theorem 5 in Haase and Schicho [2013]. Moreover, we see that the inequality holds more generally for birationally ruled surfaces. Note that for irrational birationally ruled surfaces we have that \( \gamma(l) \leq 0 \).

We want to improve (2) by bounding \( \Phi \) in terms of \( v(0) \). From Proposition 2 it follows that \( \rho(l) \leq 9 \). From §2 it follows that the Picard number decreases by 1 for each contracted exceptional curve, and thus
\[ \sum_{i=0}^{l-1} n(i) \geq \rho(0) - 9 \geq v(0) - 11. \]

From \( (2l - i - 1)(i + 1) \geq 2l - 1 \) for all \( i \in [0, l-1] \) it follows that
\[ \Phi \geq (2l - 1)\sum_{i=0}^{l-1} n(i) \geq (2l - 1)(v(0) - 11) \tag{3} \]

Now from (1), (3) and \( \alpha(l) \geq 0 \) we obtain the following inequality on invariants of birationally ruled surfaces:

**Theorem 17.**
\[ \alpha(0) + 2l\beta(0) + 9l^2 \geq (2l - 1)(v(0) - 11). \]

Restricting to toric surfaces and applying the dictionary we obtain an improved inequality for lattice polygons, as was predicted in Castryck [2012]:

**Corollary 18.**
\[ 2a(P_0) - 2l(P_0)b(P_0) + 9l(P_0)^2 \geq (2l(P_0) - 1)(v(P_0) - 11). \]
8 Acknowledgements

I would like to thank Josef Schicho and RISC/RICAM in Linz, Austria, for their warm hospitality and interesting discussions.

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