THE FAN THEOREM, ITS STRONG NEGATION, AND THE DETERMINACY OF GAMES

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Abstract. In the context of a weak formal theory called Basic Intuitionistic Mathematics BIM, we study Brouwer’s Fan Theorem and a strong negation of the Fan Theorem, Kleene’s Alternative (to the Fan Theorem). We prove that the Fan Theorem is equivalent to contrapositions of a number of intuitionistically accepted axioms of countable choice and that Kleene’s Alternative is equivalent to strong negations of these statements. We discuss finite and infinite games and introduce a constructively useful notion of determinacy. We prove that the Fan Theorem is equivalent to the Intuitionistic Determinacy Theorem. This theorem says that every subset of Cantor space \(2^\omega\) is, in our constructively meaningful sense, determinate. Kleene’s Alternative is equivalent to a strong negation of a special case of this theorem. We also consider a uniform intermediate value theorem and a compactness theorem for classical propositional logic. The Fan Theorem is equivalent to each of these theorems and Kleene’s Alternative is equivalent to strong negations of them. We end with a note on ‘stronger’ Fan Theorems. The paper is a sequel to [44].

1. Introduction

1.1. Intuitionistic reverse mathematics. L.E.J. Brouwer did not present his intuitionistic mathematics as a formal axiomatic theory. He did not like formalism and formalization and anxiously maintained the distinction between a mathematical proof and the linguistic expression that should help us to recover the proof but may fail to do so. The challenge to develop formal theories coming close to Brouwer’s intentions was taken up by A. Heyting, G. Gentzen, S.C. Kleene, G. Kreisel, J. Myhill, A.S. Troelstra, and others.

Given a preferably weak formal basic theory \(\Gamma\) and a formal proof in \(\Gamma\) of a statement \(T\) from some extra assumption \(A\), one may ask: is there also a formal proof in \(\Gamma\) of this extra assumption \(A\) from the statement \(T^*\)? The study of such questions, as far as they belong to the field of classical analysis or second-order arithmetic, is called reverse mathematics, see [27]. The weak basic theory there is RCA_0.

1.2. The basic theory BIM. Our subject is intuitionistic reverse mathematics.

The weak basic theory we use is the two-sorted first-order intuitionistic theory BIM (Basic Intuitionistic Mathematics), introduced in [44]. The domain of discourse of BIM consists of two kinds of objects: natural numbers and infinite sequences of natural numbers. The axioms express some basic assumptions like the (full) principle of induction on the set \(\omega\) of the natural numbers, and the fact that the set \(\omega^\omega\) of the infinite sequences of natural numbers is closed under the recursion-theoretic operations.

The reason that we use a basic theory different in spirit from the basic theory used in classical reverse mathematics is that, in intuitionistic analysis, one prefers the notion of an infinite sequence of natural numbers as a primitive notion above the notion of a subset of the set of the natural numbers, see [44] Section 5.
For the intuitionistic mathematician, the set $\omega^\omega$ of all infinite sequences of natural numbers is not, as one sometimes says when explaining the notion of ‘set’ that lies at the basis of classical set theory, the result of taking together the earlier constructed and completed items that are to be the ‘elements’ of the set. The set $\omega^\omega$ is a realm of possibilities: it is a framework for constructing, in the future, in all kinds of possibly as yet unforeseen ways, the objects that will be called the elements of the set. There are several kinds of infinite sequences $\alpha = (\alpha(0), \alpha(1), \ldots)$ of natural numbers. Sometimes, $\alpha$ is the result of executing a program, a finitely formulated algorithm. It is also possible that $\alpha$ is the result of a more or less free step-by-step construction that is not governed by a rule formulated at the start.

1.3. Two interpretations. The axioms of BIM hold for their intended interpretation, the interpretation given to them by the intuitionistic mathematician. The axioms of BIM also become true for her if she assumes that the elements of $\omega^\omega$ are just the Turing-computable functions from $\omega$ to $\omega$. Turing-computable functions may be represented by the natural number coding their program, and BIM may be seen to be a conservative extension of first-order intuitionistic arithmetic HA, Heyting arithmetic.

The model given by the computable functions thus is the second interpretation of BIM. Our study of this model is a contribution to intuitionistic recursive analysis.

In the weak context given by BIM one may study the further assumptions of the intuitionistic mathematician. They fall into three groups:

(1) Axioms of Countable Choice,
(2) Brouwer’s Continuity Principle and the Axioms of Continuous Choice, and
(3) Brouwer’s Thesis on bars in Baire space $\omega^\omega$ and the Fan Theorem.

The intuitionistic mathematician is prepared to argue the plausibility of these assumptions for her intended interpretation.

She defends the Axioms of Countable Choice, for instance, by explaining that the functions promised by the axioms may be constructed step by step.

She has no argument for the truth of the further assumptions under the second interpretation, where every function is assumed to be given by an algorithm. It is not clear to her if the Axioms of Countable Choice then are true.

Brouwer’s Continuity Principle and its extensions, the Axioms of Continuous Choice, certainly fail in the second interpretation.

The Thesis on bars in $\omega^\omega$ was introduced by Brouwer for the sake of the Fan Theorem. The Fan Theorem itself, dating from 1924, see [5], might be called the Thesis on Bars in Cantor space $2^\omega$, see [40], [41], and [44, Subsection 2.3].

In 1950, see [12], Kleene saw that, in our second interpretation, also the Fan Theorem, and, a fortiori, the Thesis on bars in $\omega^\omega$, do not hold. Actually, a positively formulated strong negation of the Fan Theorem becomes true. In [14], we called this statement Kleene’s Alternative (to the Fan Theorem).

1.4. Strong negations. The (strict) Fan Theorem, FT, see Subsubsection 2.2.4, is the statement

$$\forall \alpha [Bar_{2^\omega} (D_\alpha) \rightarrow \exists n [Bar_{2^\omega} (D_{\alpha n})]],$$

and Kleene’s Alternative (to the Fan Theorem), KA, see Subsubsection 2.2.8, is the statement

$$\exists \alpha [Bar_{2^\omega} (D_\alpha) \land \forall n [\neg Bar_{2^\omega} (D_{\alpha n})]].$$

We want to call KA the strong negation $\neg \neg FT$ of FT. In general, if we decide to call a statement $B$ the strong negation of a statement $A$, $B$ will be a statement more positive than the negation of $A$ that constructively implies the negation of $A$. The reader may consult Section 13 for the notations used.
A. We do not require that the statement $B$ is completely positive in the sense that
the corresponding formula does not contain $\neg$ and $\rightarrow$. We do not introduce strong
negation as a syntactical operation on formulas. It is important to realize that,
onece we have understood that statement $A$ is equivalent to statement $B$, it may be
the case that statements $C, D$, which have been chosen to be called the strong
negations of $A, B$, respectively, fail to be equivalent.

If we have decided to call the formula (denoted by) $B$ the strong negation of
the formula (denoted by) $A$, we will write $B = \neg!A$, but note that this notation
belongs to the meta-language of $\text{BIM}$. $\neg!$ is neither a connective belonging to
the language of $\text{BIM}$ nor a syntactical operation on formulas.

We shall prove a number of results of the form:

In $\text{BIM}$, $A$ is equivalent to $B$ and $\neg!B$ is equivalent to $\neg!A$.

When we do so, we try to explain that the conclusions $A \rightarrow B$ and
$\neg!B \rightarrow \neg!A$ have a common ground and that also the conclusions $B \rightarrow A$ and
$\neg!A \rightarrow \neg!B$ have a common ground.

1.5. Contrapositions or reversals. We may compare Weak K"onig’s Lemma,
$\text{WKL}$, see 2.2.12
\[
\forall \alpha \exists n[\neg\text{Bar}_2\omega(D_{\alpha n})] \rightarrow \exists \gamma \in 2^\omega \forall n[\alpha(\gamma n) = 0].
\]
with the (strict) \textit{Fan Theorem}, $\text{FT}$:
\[
\exists n[\text{Bar}_2\omega(D_{\alpha n})]\rightarrow \forall \alpha \exists n[\text{Bar}_2\omega(D_{\alpha n})].
\]
We like to say that $\text{WKL}$ is a reversal or contraposition of $\text{FT}$ and also that $\text{FT}$
is a reversal or contraposition of $\text{WKL}$.

We like to write: $\text{WKL} = \leftarrow\text{FT}$ and $\text{FT} = \leftarrow\text{WKL}$.

In general, if we call a statement $B$ the contraposition or reversal $\leftarrow A$ of a state-
ment $A$, both $A$ and $B$ will be largely positively formulated statements and the
classical mathematician would think $A$ and $B$ are equivalent, but, constructively,
$A$ and $B$ will have quite different meanings.

This is clear from the above example as $\text{FT}$ is intuitionistically true (under our
first interpretation) and $\text{WKL}$ is false (in both interpretations).

It may happen also that both $A$ and $B$ are intuitionistically true (at least under
the first interpretation), although they make different sense. An important example
of this phenomenon is given by $\Pi^0_1\text{-AC}_{\omega,2}$, see Subsection 4.7 and $\Sigma^0_1\text{-AC}_{\omega,2}$, see
Subsection 5.4 and Lemma 5.3.

Note that Kleene’s Alternative (to the Fan Theorem), $\text{KA}$, might be called the
strong negation $\neg!\text{WKL}$ of $\text{WKL}$ as well as the strong negation $\neg!\text{FT}$ of $\text{FT}$.

We do not claim that, given a statement $A$, there always is a unique candidate
for being called the contraposition of $A$. We do not introduce contraposition as a
syntactical operation on formulas and use the term only in certain specific cases. It
is important to realize that, once we have understood that statement $A$ is equivalent
to statement $B$, it may be the case that statements $C, D$, which one would like to
call contrapositions of $A, B$, respectively, fail to be equivalent.

1.6. Non-intuitionistic assumptions. The reader may wonder why we pay at-
tention to statements that fail in both our models, like Weak K"onig’s Lemma,
$\text{WKL}$, and Bishop’s Omniscience Principles, $\text{LPO}$, see 2.2.15 and $\text{LLPO}$, see
2.2.14. Doing so, however, we come to understand that certain other statements,
being equivalent, in $\text{BIM}$, to one of them, also do not make sense in either one of
our two interpretations.

\footnote{See, for instance, the sentences 7.4 and 7.6 and Subsection 9.8}
1.7. **Our aim.** As in [43] and [44], it is our aim, in this paper, to find statements that are, in BIM, equivalent to either \( \text{FT} \) or \( \neg \text{FT} = \text{KA} \).

1.8. **The contents of the paper.** Apart from this introduction, the paper contains Sections 2-13.

Section 2 is divided into two Subsections. In Subsection 2.1 we introduce the formal system BIM. Subsection 2.2 lists a number of assumptions one might study in the context of BIM.

In Section 3 we prove that, in BIM, the \( \Sigma^0_1 \)-Separation Principle \( \Sigma^0_1 \)-Sep is equivalent to WKL.

In Section 4 we formulate some special cases of the First Axiom of Choice \( \text{AC}_{\omega,\omega} \), among them \( \Pi^0_1 \)-AC\(_{\omega,\omega} \), the \( \Pi^0_1 \)-Axiom of Countable Binary Choice.

In Section 5 we introduce \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega} \), a contraposition of \( \Pi^0_1 \)-AC\(_{\omega,\omega} \), and we prove that, in BIM, \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega} \) is equivalent to \( \text{FT} \), and a strong negation of \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega} \) is equivalent to \( \neg \text{FT} = \text{KA} \).

Section 5 thus shows that a contraposition of \( \Pi^0_1 \)-AC\(_{\omega,\omega} \) fails in our second interpretation. This gives us no conclusion about the validity of \( \Pi^0_1 \)-AC\(_{\omega,\omega} \) itself in our second interpretation.

In Section 6 we formulate some special cases of the Second Axiom Scheme of Countable Choice \( \text{AC}_{\omega,\omega,\omega} \), among them \( \Pi^0_1 \)-AC\(_{\omega,\omega,\omega} \), the \( \Pi^1_1 \)-Axiom of Countable Compact Choice.

In Section 7 we introduce \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega,\omega} \), a contraposition of \( \Pi^0_1 \)-AC\(_{\omega,\omega,\omega} \), and we prove that, in BIM, \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega,\omega} \) is equivalent to \( \text{FT} \), and a strong negation of \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega,\omega} \) is equivalent to \( \neg \text{FT} \).

In Section 8 we introduce \( \Sigma^0_1 \)-\( \text{AC} \)\(_{2,\omega} \), a contraposition of a statement provable in BIM, to wit, the \( \Pi^1_1 \)-“axiom” of Twofold Compact Choice.

We prove that, in BIM + \( \Pi^0_1 \)-AC\(_{\omega,\omega} \), \( \Sigma^0_1 \)-\( \text{AC} \)\(_{2,\omega} \) is equivalent to \( \text{FT} \). There is no companion result for \( \neg \text{FT} \).

In Section 9 we consider finite and infinite games. We explain in what sense we want to call such games \( I \)-determinate or \( II \)-determinate. We see that \( \Sigma^0_1 \)-\( \text{AC} \)\(_{\omega,\omega} \) can be read as the statement that certain 2-move games are \( I \)-determinate. We prove: in BIM, \( \text{FT} \) is equivalent to the statement that every subset of Cantor space \( 2^\omega \) is (weakly) \( I \)-determinate, and \( \neg \text{FT} \) is equivalent to the statement that there exists an open subset of \( 2^\omega \) that positively fails to be \( I \)-determinate.

In Section 10 we consider a Uniform Contrapositive Intermediate Value Theorem \( \text{UIVT} \) and we prove: in BIM, \( \text{FT} \) is equivalent to \( \text{UIVT} \) and \( \neg \text{FT} \) is equivalent to a strong negation of \( \text{UIVT} \).

In Section 11 we see that, if one formulates the compactness theorem for classical propositional logic carefully and contrapositively, one obtains a statement that, in BIM, is equivalent to \( \text{FT} \). \( \neg \text{FT} \) is equivalent to a strong negation of this statement.

In Section 12 we ask the reader’s attention for the Approximate-Fan Theorem \( \text{AppFT} \), a statement stronger than \( \text{FT} \). We did so already in [44 Subsection 10.2], see also [45]. \( \text{AppFT} \) is studied further in [45].

Section 13 contains a list of defined notions. This Section may be used as a reference by the reader of the preceding Sections.

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2. The formal system BIM

2.1. The basic axioms. BIM, introduced in [44] Section 6], has two kinds of variables, numerical variables \(m, n, p, \ldots\), whose intended range is the set \(\omega\) of the natural numbers, and function variables \(\alpha, \beta, \gamma, \ldots\), whose intended range is the set \(\omega^\omega\) of all infinite sequences of natural numbers. There is a numerical constant 0. There are five unary function constants: \(\text{Id}\), a name for the identity function, \(\mathbb{0}\), a name for the zero function, \(S\), a name for the successor function, and \(K, L\), names for the projection functions. There is one binary function symbol \(J\), a name for the pairing function on \(\omega\). From these symbols numerical terms are formed in the usual way. The basic terms are the numerical variables and the numerical constant and other terms are obtained from earlier constructed terms by the use of a function symbol with parentheses indicating function application. The function constants \(\text{Id}, \mathbb{0}, S, K\) and \(L\) and the function variables are the only function terms.

BIM has two equality symbols, \(=\) and \(\neq\). The first symbol may be placed between numerical terms and the second one between function terms. When confusion seems improbable we simply write \(=\) and not \(=\) or \(\neq\). The usual axioms for equality are part of BIM. A basic formula is an equality between numerical terms or an equality between function terms. A basic formula in the strict sense is an equality between numerical terms. We obtain the formulas of the theory from the basic formulas by using the connectives, the numerical quantifiers and the function quantifiers.

The logic of the theory is intuitionistic logic.

Our first axiom is

**Axiom 1** (Axiom of Extensionality).

\[\forall \alpha \forall \beta (\alpha =_1 \beta \iff \forall n[\alpha(n) =_0 \beta(n)])\]

The Axiom of Extensionality guarantees that every formula will be provably equivalent to a formula built up by means of connectives and quantifiers from basic formulas in the strict sense.

The second axiom is the axiom on the unary function constants \(\text{Id}, \mathbb{0}, S, K, L\), and the binary function constant \(J\).

**Axiom 2.**

\[\forall n[\text{Id}(n) = n] \land \forall n[\neg (S(n) = 0)] \land \forall m \forall n[S(m) = S(n) \rightarrow m = n] \land \forall n[\mathbb{0}(n) = 0] \land \forall m \forall n[K(J(m, n)) = m \land L(J(m, n)) = n \land J(K(m), L(n)) = n]\]

Thanks to the presence of the pairing function we may treat binary, ternary and other non-unary operations on \(\omega\) as unary functions. “\(\alpha(m, n, p)\)”, for instance, will be an abbreviation of \(\alpha(J(J(m, n), p))\).

We also introduce the following notation: for each \(n, n' \equiv K(n)\) and \(n'' \equiv L(n)\), and, for all \(m, n, (m, n) \equiv J(m, n)\). The last part of Axiom 2 now reads as follows:

\[\forall m \forall n[(m, n)' = m \land (m, n)'' = n \land (n', n'') = n]\]

The next axiom asks for the closure of the set \(\omega^\omega\) under composition, pairing, primitive recursion and unbounded search.

**Axiom 3.**

\[\forall \alpha \forall \beta \exists \gamma \forall n[\gamma(n) = \alpha(\beta(n))] \land \forall \alpha \forall \beta \exists \gamma \forall n[\gamma(n) = (\alpha(n), \beta(n))] \land \]

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\(^3\)A referee made us see that this Axiom 3, as formulated in [44], is a little bit too weak.
\[\forall \alpha \forall \beta \exists \gamma \forall m \forall n [\gamma (m, 0) = \alpha (m) \land \gamma (m, S(n)) = \beta (m, n, \gamma (m, n))] \land
\forall \alpha [\forall n \exists m [\alpha (n, m) = 0] \to \exists \gamma \forall n [\alpha (n, \gamma (n)) = 0]]\]

We also need the Axiom Scheme of Induction:

Axiom 4.

\[(P(0) \land \forall n [P(n) \to P(S(n))]) \to \forall n [P(n)]\]

Instances of this axiom scheme are obtained by substituting a formula
\[\phi = \phi (m_0, m_1, \ldots, m_k, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}, n)\] for \(P\) and taking the universal closure of the resulting formula. The reader should understand the further axiom schemes we are to mention in this paper in the same way.

The axioms 1-4, together with the usual axioms for equality, define the system BIM.

Note BIM has the full induction scheme whereas RCA_0 has only \(\Sigma^0_1\)-induction, see [27, Definition II.1.5], a fact that is indicated by the suffix \(0\). We did not study the possibility of restricting induction likewise and we do not answer the question if our results might have been obtained in a system that probably should be called \(\text{BIM}_0\).

We form a conservative extension of BIM by adding constants for all primitive recursive functions and relations and making their defining equations into axioms. Primitive recursive relations are present via their characteristic functions. ‘\(x < y\)’, for instance, will be short for: ‘\(\chi_<(x, y) \neq 0\)’. Somewhat loosely, we also denote this conservative extension of BIM by the acronym BIM although one might decide to use the acronym BIM\(^+\), see [31].

BIM may be compared to the system \(H\) introduced in [9] and to the system \(\text{EL}\) occurring in [30] and to the system \(\text{IRA}\), proposed by J.R. Moschovakis and G. Vafeiadou, see [23]. A precise proof of the fact that BIM and these systems are essentially equivalent may be found in [31].

2.2. Possible further assumptions.

2.2.1. First Axiom Scheme of Countable Choice, \(\text{AC}_{\omega, \omega} (= \text{AC}_{0,0})\):

\[\forall n \exists m [R(n, m)] \to \exists \gamma \forall n [R(n, \gamma (n))].\]

The intuitionistic mathematician accepts \(\text{AC}_{\omega, \omega}\). If \(\forall n \exists m [R(n, m)]\), she builds the promised \(\gamma\) step by step, first choosing \(\gamma (0)\) such that \(R(0, \gamma (0))\), then choosing \(\gamma (1)\) such that \(R(1, \gamma (1))\), and so on. In her view, there is no need to give a finite description or algorithm that determines the infinitely many values of \(\gamma\) at one stroke.

2.2.2. Second Axiom Scheme of Countable Choice, \(\text{AC}_{\omega, \omega} = \text{AC}_{0,1}\):

\[\forall n \exists \gamma [R(n, \gamma)] \to \exists \gamma \forall n [R(n, \gamma^{(0)})].\]

The intuitionistic mathematician accepts \(\text{AC}_{\omega, \omega}\). If \(\forall n \exists \gamma [R(n, \gamma)]\), she first starts a project for building \(\gamma^{(0)}\) with the property \(R(0, \gamma^{(0)})\) and determines \(\gamma^{(0)}(0)\), she then starts a project for building \(\gamma^{(1)}\) with the property \(R(1, \gamma^{(1)})\) and determines \(\gamma^{(1)}(0)\), and also, continuing the project started earlier, \(\gamma^{(0)}(1)\), she then starts a project for building \(\gamma^{(2)}\) with the property \(R(2, \gamma^{(2)})\) and determines \(\gamma^{(2)}(0)\) and also, continuing the projects started earlier, \(\gamma^{(1)}(1)\) and \(\gamma^{(0)}(2), \ldots\).

\[^{4}\text{As for the notations used, the reader is advised to consult Section 13 in particular Subsections 13.1, 13.2 and 13.3. For the notation } \gamma^n \text{ see Subsection 13.4.}\]
2.2.3. The Fan Theorem as an Axiom Scheme, **FAN:**
\[\forall \beta ([F\alpha(\beta) \land Bar_{\mathcal{F}}(B) \rightarrow \exists \alpha [D_{\alpha} \subseteq B \land Bar_{\mathcal{F}}(D_{\alpha})]]).
\]
\(\beta\) is an explicit fan-law if and only if \(Fan(\beta)\) and, in addition, \(\exists \gamma \forall s \forall m \beta(s \cdot m) = 0 \rightarrow m \leq \gamma(s)\). We then write \(Fan^+(\beta)\). If \(Fan^+(\beta)\), then \(\mathcal{F}_\beta\) is called an explicit fan.

**Lemma 2.1.** **BIM** \(\vdash \forall \beta [Fan^+(\beta) \leftrightarrow (Spr(\beta) \land \exists \gamma \forall n \forall s \in \omega^n [\beta(s) = 0 \rightarrow s \leq \gamma(n)]]).
\]

**Proof.** Let \(\beta\) be given such that \(Fan^+(\beta)\). Find \(\gamma\) such that \(\forall s \forall m [\beta(s \cdot m) = 0 \rightarrow m \leq \gamma(s)]\). Define \(\delta\) such that \(\delta(0) = 0 = 0\), and, for each \(n\), \(\delta(n + 1) = \max\{s \cdot m \mid \beta(s \cdot m) = 0 \land s \leq \delta(n) \land m \leq \gamma(s)\}\). One proves by induction that \(\forall n \forall s \in \omega^n [\beta(s) = 0 \rightarrow s \leq \delta(n)]\).

Conversely, let \(\beta, \gamma\) be given such that \(Spr(\beta) \land \forall n \forall s \in \omega^n [\beta(s) = 0 \rightarrow s \leq \gamma(n)]\). Define \(\delta\) such that, for each \(n\), for each \(s\) in \(\omega^n\) such that \(\beta(s) = 0\), \(\delta(s) = \max\{m \mid \beta(s \cdot m) = 0 \land s \cdot m \leq \gamma(n + 1)\}\). Note that \(\forall s \forall m [\beta(s \cdot m) = 0 \rightarrow m \leq \delta(s)]\) and conclude that \(Fan^+(\beta)\). \(\square\)

2.2.4. The (strict) Fan Theorem, **FT:**
\[\forall \alpha [Bar_{\omega^2}(D_{\alpha}) \rightarrow \exists \alpha [Bar_{\omega^2}(D_{\alpha})]]\text{, or, equivalently,}\]
\[\forall \beta [Fan^+(\beta) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha}) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha})]]\text{, or, equivalently,}\]
\[\forall \beta [Fan^+(\beta) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha}) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha})]].\]

**Theorem 2.2.** **BIM** \(\vdash\) **FT** \(\leftrightarrow\)
\[\forall \alpha [(\text{Thinbar}_{\omega^2}(D_{\alpha}) \land D_{\alpha} \subseteq 2^{<\omega}) \rightarrow \exists \alpha [\exists m > n[m \notin D_{\alpha}]]].\]

**Proof.** The proof is left to the reader. \(\square\)

2.2.5. The strengthened (strict) Fan Theorem, **FT\(^+\):**
\[\forall \beta [Fan(\beta) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha}) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha})]]\text{, or, equivalently,}\]
\[\forall \beta [Fan^+(\beta) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha}) \rightarrow \exists \alpha [Bar_{\mathcal{F}}(D_{\alpha})]].\]

**Theorem 2.3.** **BIM** \(\vdash\) **FT\(^+\)\(\leftrightarrow\)
\[\forall \beta [Fan(\beta) \rightarrow \exists \alpha [(\text{Thinbar}_{\mathcal{F}}(D_{\alpha}) \land \forall s \in D_{\alpha} [\beta(s) = 0]) \rightarrow \exists \alpha [\exists m > n[m \notin D_{\alpha}]]].\]

**Proof.** The proof is left to the reader. \(\square\)

Note that **BIM + FAN** \(\vdash\) **FT\(^+\).**

2.2.6. Brouwer’s Thesis: Bar Induction as an Axiom Scheme, **BARIND:**
\[(\text{Bar}_{\omega^2}(B) \land \forall \alpha [\exists \beta [\forall s \cdot s \in B \rightarrow s \in E \land \forall \alpha [\forall s \cdot \forall \alpha [\forall s \cdot n \in E \leftrightarrow s \in E] \rightarrow \forall s \cdot n \in E]) \rightarrow (\forall s \cdot n \in E).\]

\(E \subseteq \omega\) is called an initial if and only if \(\forall s \forall \alpha [\forall s \cdot n \in E \rightarrow s \in E]\) and monotone if and only if \(\forall s \forall \alpha [s \in E \rightarrow s \cdot n \in E]\).

Brouwer derived **FAN** from **BARIND**, see [43] Subsections 2.2 and 2.3. We repeat the proof, in order to prepare the reader for Theorem [12].

**Theorem 2.4.** **BIM + BARIND** \(\vdash\) **FAN.**

**Proof.** Let \(\beta\) be given such that \(Fan(\beta)\) and \(\beta((\cdot)) = 0\).\footnote{If \(\beta((\cdot)) \neq 0\), then \(\mathcal{F}_\beta = 0\) and there is nothing to prove.} Assume \(Bar_{\mathcal{F}}(B)\). Define \(B' := B \cup \{s \mid \beta(s) \neq 0\}\). We will prove that \(Bar_{\omega^2}(B')\). Let \(\gamma\) be given. Define \(\gamma^+\) such that, for each \(n\), if \(\beta(\gamma(n + 1)) = 0\), then \(\gamma^+(n) = \gamma(n)\), and, if not, then \(\gamma^+(n) = \mu \gamma^+(\gamma(n) * p)\). Note that \(\gamma^+ \in \mathcal{F}_\beta\) and find \(n\) such that \(\gamma^+n \in B\). Either \(\gamma^+n = \gamma^+n \land \gamma^+n \in B\) or \(\gamma^+n \notin \gamma^+n\) and \(\beta(\gamma^+n) \neq 0\). In both cases, \(\gamma^+n \in B'\). We thus see that \(\forall \gamma \exists n [\gamma^+n \in B']\), i.e. \(Bar_{\omega^2}(B')\).
Let $E$ be the set of all $s$ such that either $\beta(s) \neq 0$ or $\beta(s) = 0$ and $\exists a[D_a \subseteq B \land Bar_{\mathcal{F}_\beta}(D_a)]$.

For every $s$, if $\beta(s) = 0$ and $s \in B$, define $a := \overline{s} * (1)$ and note that $\{s\} = D_a \subseteq B$ and $Bar_{\mathcal{F}_\beta}(D_a)$. Conclude that $B \subseteq E$.

Now let $s$ be given such that $\forall m[s * \langle m \rangle \in E]$. Find $n$ such that $\forall m \geq n[\beta(s * \langle m \rangle) \neq 0]$. Find $b$ in $\omega^\omega$ such that $\forall m < n[\beta(s * \langle m \rangle) = 0 \rightarrow \{D_b(m) \subseteq B \land Bar_{\mathcal{F}_\beta}(D_b(m))\}]$ and $\forall m < n[\beta(s * \langle m \rangle) \neq 0 \rightarrow b(m) = \langle \rangle]$. Find $a$ such that $p := length(a) = \max_{m < n} length(b(m))$ and $\forall t < p[a(t) \neq 0 \rightarrow \exists m < n[b(m)](t) \neq 0]$. Note that $D_a \subseteq B$ and $Bar_{\mathcal{F}_\beta}(D_a)$, and conclude that $s \in E$.

We thus see that $\forall s[\forall m[s * \langle m \rangle \in E] \rightarrow s \in E]$, i.e. $E$ is inductive.

Note that $\forall s[\forall m[s * \langle m \rangle \in E] \rightarrow s \in E]$, i.e. $E$ is monotone.

Using $\text{BARIND}$, conclude that $\langle \rangle \in E$, i.e. $\exists a[D_a \subseteq B \land Bar_{\mathcal{F}_\beta}(D_a)]$.  

2.2.7. Church’s Thesis, CT:

\[ \exists \tau \exists \psi \exists \exists \forall n \exists z[z = \mu][\tau(e, n, i) \neq 0] \land \psi(z) = (a(n)). \]

Kleene has shown that CT is true in the model of BIM given by the computable functions. He provided Kálmar-elementary functions $\tau, \psi$ satisfying the above conditions. Note that our formulation of CT is cautious and somewhat weaker than the usual one. We do not require that the set $\{(e, n, z) \mid \tau(e, n, z) \neq 0\}$ coincides with Kleene’s set $T$, but only ask that the set behaves appropriately. A similar ‘abstract’ approach to Church’s Thesis has been advocated by F. Richman, see [25 and 3 Chapter 3, Section 1].

2.2.8. Kleene’s Alternative (to the Fan Theorem), $\neg \text{FT}$:

\[ \exists a[Bar_{\mathcal{F}_\beta}(D_a) \land \forall m[\neg Bar_{\mathcal{F}_\beta}(D_{\beta(m)})]]. \]

Theorem 2.5. BIM + CT $\vdash \neg \text{FT}$.

Proof. Let $\tau, \psi$ be as in CT. Define $\alpha$ such that, for all $m$, for all $c$ in $2^{<\omega}$ such that $length(c) = m$,

\[ \alpha(c) = 0 \leftrightarrow \forall e < m \forall z < m[z = \mu][\tau(e, n, i) \neq 0] \rightarrow c(e) = 1 - \psi(z). \]

Let $\gamma$ in $2^{\omega}$ be given. Find $e$ such that, for all $n$, $\gamma(n) = \psi(\mu)[\tau(e, n, i) \neq 0])$. Define $z = \mu[\tau(e, n, i) \neq 0]$. Note that $\gamma(e) = \psi(z)$. Define $m := \max(e, z) + 1$ and note that $\alpha(\gamma(m)) \neq 0$. Conclude that $\forall \gamma \in 2^{\omega} \exists m[\alpha(\gamma(m)) \neq 0]$, i.e. $Bar_{\mathcal{F}_\beta}(D_a)$.

Let $m$ be given. Find $c$ in $2^{\omega}$ such that $length(c) = m$ and $\forall e < m \forall z < m[z = \mu][\tau(e, n, i) \neq 0] \rightarrow c(e) = 1 - \psi(z)]$. Note that $\forall n \leq m[\alpha(\gamma(n)) = 0]$. Define $\gamma := c \ast \emptyset$ and note that $\forall n[e \in \overline{m} > m]$ and conclude that $\neg \exists k[\gamma(k) < m \land \alpha(\gamma(k)) \neq 0]$ and $\neg Bar_{\mathcal{F}_\beta}(D_{\beta(m)})$. Conclude that $\forall n[\neg Bar_{\mathcal{F}_\beta}(D_{\beta(m)})]$.  

Theorem 2.5 is due to Kleene, see [12 §3] and [13 Lemma 9.8]. There is a proof in [20] vol. 1, Chapter 4, Subsection 7.6. In [14 Section 3] one finds two more proofs.

We do not know the answer to the question if BIM + $\neg \text{FT} \vdash \text{CT}$.

2.2.9. Brouwer’s (unrestricted) Continuity Principle as an Axiom Scheme, BCP:

\[ \forall \alpha \exists n[\alpha Rn] \rightarrow \forall \alpha \exists m \exists \beta[\mathcal{F}_\beta \subseteq \beta \rightarrow \beta Rm]. \]

Brouwer used this principle for the first time in 1918, see [1 Section 1, page 13]. The intuitionistic mathematician believes the axiom to be plausible. She argues as follows. If $\forall \alpha \exists n[\alpha Rn]$, I must be able, given any $\alpha$, to find effectively an $n$ as promised, also if the values of $\alpha$ are disclosed one by one and nothing is told about a law governing the development of $\alpha$ as a whole. I will decide upon $n$ at some point of time and, at that point of time, only finitely many values of $\alpha$, say
\(\alpha(0), \alpha(1), \ldots, \alpha(m-1)\), will be known to me. The number \(n\) will satisfy any infinite sequence that is a continuation of \(\alpha(0), \alpha(1), \ldots, \alpha(m-1)\).

The Continuity Principle is revolutionary and changes one’s mathematical perspective, see [35].

The classical mathematician may ask for a consistency proof for BCP. Kleene proved, using realizability methods, that his formal system FIM for intuitionistic analysis, actually an extension of BIM + FT + BCP, is (simply) consistent, see (13] Chapter II, Subsection 9.2. Kleene’s consistency proof should convince both the classical mathematician and the constructive mathematician, should the latter accept BARIND but be plagued by doubts concerning BCP. It is not difficult to see that BIM + CT + BCP is inconsistent, as it implies \(\forall m \exists \beta \mid \beta m = \overline{\alpha m} \rightarrow \beta = \alpha\), see [30, vol. I, Chapter 4, Theorem 6.7].

The next two axioms strengthen BCP. Kleene’s consistency proof extends to these stronger forms of the Continuity Principle.

2.2.10. The First Axiom Scheme of Continuous Choice, \(\mathbf{AC}_{\omega, \omega}(= \mathbf{AC}_{1,0})\):
\[
\forall \alpha \exists \bar{n}[\alpha R \bar{n}] \rightarrow \exists \varphi : \omega^\omega \rightarrow \omega \forall \alpha[\alpha R \varphi(\alpha)].
\]

2.2.11. The Second Axiom Scheme of Continuous Choice, \(\mathbf{AC}_{\omega^\omega, \omega^\omega}(= \mathbf{AC}_{1,1})\):
\[
\forall \alpha \exists \bar{\beta}[\bar{\alpha} R \bar{\beta}] \rightarrow \exists \varphi : \omega^\omega \rightarrow \omega^\omega \forall \alpha[\alpha R \varphi(\alpha)].
\]

2.2.12. Weak König’s Lemma, WKL:
\[
\forall \alpha[\forall n \bar{a}_n \in 2^{\omega}[\text{length}(a) = n \land \forall m \leq n[a(\overline{\gamma m}) = 0] \rightarrow \exists n \in 2^\omega \forall n[a(\overline{\gamma n}) = 0]]],
\]

or, equivalently, \(\forall \alpha[\forall n[\neg \text{Bar}_{2\omega}(D_{\alpha n})] \rightarrow \exists n \in 2^\omega \forall n[\overline{\gamma n} \notin D_{\alpha n}]]\).

WKL, as a classical theorem, dates from 1927, see [13]. It is a contraposition of FT and, from a classical point of view, the two are equivalent. The following result may be found in [10], and also in [15] and [11].

Theorem 2.6. \(\mathbf{BIM} \vdash \mathbf{WKL} \rightarrow \mathbf{FT}\).

Proof. Assume WKL.

Let \(\alpha\) be given such that \(\text{Bar}_{2\omega}(D_{\alpha})\). We intend to prove that \(\exists m[\text{Bar}_{2\omega}(D_{\overline{\alpha m}})]\).

Define \(\alpha^*\) such that \(\forall s \in 2^{\omega}[\alpha^*(s) = 0 \leftrightarrow \forall t \subseteq s[\alpha(t) = 0]]\). Define \(\alpha^**\) such that, for all \(s\) in \(2^{\omega}\), \(\alpha^**(s) = 0\) if and only if either \(\alpha^*(s) = 0\) or \(\neg \exists t \in 2^{\omega}[\text{length}(t) = \text{length}(s) \land \alpha^*(t) = 0]\). Note that, for each \(n\), there exists \(s\) in \(2^{\omega}\) such that \(\text{length}(s) = n\) and \(\alpha^**(s) = 0\). Applying WKL, find \(\gamma\) in \(2^{\omega}\) such that \(\forall n[\alpha^**(\overline{\gamma n}) = 0]\). Find \(n\) such that \(\alpha(\overline{\gamma n}) \neq 0\). Conclude that \(\neg \exists t \in 2^{\omega}[\text{length}(t) = n \land \alpha^*(t) = 0]\) and \(\forall \delta \in 2^\omega \exists j \leq n[\alpha(\overline{\delta j}) \neq 0]\) and \(\exists m[\text{Bar}_{2\omega}(D_{\overline{\alpha m}})]\).

We thus see that \(\forall \alpha[\text{Bar}_{2\omega}(D_{\alpha}) \rightarrow \exists m[\text{Bar}_{2\omega}(D_{\overline{\alpha m}})]\), i.e. FT. \(\square\)

2.2.13. Weak König’s Lemma with a uniqueness condition, WKL!:
\[
\forall \alpha[[(\forall m[\neg \text{Bar}_{2\omega}^2(D_{\overline{\alpha m}})] \land
\forall \gamma \in 2^\omega[\forall \delta \in 2^\omega[\gamma \perp \delta \rightarrow \exists m[\alpha(\overline{\gamma m}) \neq 0 \lor \alpha(\overline{\delta m}) \neq 0]]] \rightarrow \exists \gamma \forall n[\alpha(\overline{\gamma n}) = 0]]\).
\]

The next two theorems, apart from being of interest in themselves, are useful for the discussion in Subsubsection 8.1.1.

Theorem 2.7. \(\mathbf{BIM} \vdash \mathbf{WKL!} \rightarrow \mathbf{FT}\).

Proof. Assume WKL!.

\(^6The equivalence of FT and WKL! is a result due to J. Berger and H. Ishihara, see [2]. Another proof has been given by H. Schwichtenberg, see [26].\)
Let \( \alpha \) be given such that \( \text{Bar}_{2^\omega}(D_\alpha) \). We will prove that \( \exists m[\text{Bar}_{2^\omega}(D_{\bar{\sigma}_m})] \). If \( \alpha(0) = \alpha(\langle \rangle) \neq 0 \), then \( D_{\bar{\sigma}_1} = \{ \langle \rangle \} \) and \( \text{Bar}_{2^\omega}(D_{\bar{\sigma}_1}) \), and we are done. Now assume that \( \alpha(\langle \rangle) = 0 \), i.e. \( \langle \rangle \notin D_\alpha \). Define \( \alpha^* \) such that \( \forall s \in 2^{<\omega}[\alpha^*(s) = 0 \leftrightarrow \forall t \subseteq s[\alpha(t) = 0]] \). Define \( \alpha^{**} \) such that \( \alpha^{**}(\langle \rangle) = 0 \) and, for all \( s \in 2^{<\omega}\setminus\{\langle \rangle\} \), \( \alpha^{**}(s) = 0 \) if and only if either \( \alpha^*(s) = 0 \) and \( \forall t \in 2^{<\omega}[[\text{length}(t) = \text{length}(s) \land \alpha^*(t) = 0] \rightarrow s \leq t \searrow t] \) or \( \neg \exists t \in 2^{<\omega}[[\text{length}(t) = \text{length}(s) \land \alpha^*(t) = 0] \land \exists \delta[\alpha^{**}(t) = 0 \land s = t \searrow 0]] \). Using induction, one proves that, for each \( n \), there is exactly one \( s \in 2^{<\omega} \) such that \( \text{length}(s) = n \) and \( \alpha^{**}(s) = 0 \). Let \( \gamma, \delta \in 2^{<\omega} \) be given such that \( \gamma \perp \delta \). Find \( n \) such that \( \gamma \leq n \) and \( \delta \leq n \). Define \( \alpha^*(\langle \rangle) = \) 0 \( \land \alpha^{**}(\langle \rangle) = 0 \). We thus see that \( \exists m[\text{Bar}_{2^\omega}(D_{\bar{\sigma}_m})] \).

\[ \begin{align*}
\text{Theorem 2.8.} & \quad \text{BIM} \vdash \text{FT} \rightarrow \text{WKL}!.
\end{align*} \]

\[ \begin{align*}
\text{Proof.} & \quad \text{Assume FT.}
\end{align*} \]

Let \( \alpha \) be given such that \( \forall n[\neg \text{Bar}_{2^\omega}(D_{\bar{\sigma}_n})] \) and \( \gamma \in 2^{<\omega}[\gamma \perp \delta \rightarrow \exists n[\alpha(\gamma_n) \neq 0 \lor \alpha(\delta_n) \neq 0]] \). We will prove that \( \exists \gamma \in 2^{<\omega}[\forall n[\alpha(\gamma_n) = 0]] \). Define \( \alpha^* \) such that \( \forall s \in 2^{<\omega}[\alpha^*(s) = 0 \leftrightarrow \forall t \subseteq s[\alpha(t) = 0]] \). Let \( s \in 2^{<\omega} \) be given. Note that \( \forall \gamma \in 2^{<\omega}[\alpha^*(s) = 0 \land \forall t \subseteq s[\alpha(t) = 0]] \). Conclude that \( \forall \gamma \in 2^{<\omega}[\alpha^*(s) = 0 \land \forall t \subseteq s[\alpha(t) = 0]] \). Define \( \text{FT} \), find \( m \) such that \( \forall \gamma \in 2^{<\omega}[\alpha^*(s) = 0] \land \forall t \subseteq s[\alpha(t) = 0]] \). Conclude that \( \forall \gamma \in 2^{<\omega}[\alpha^*(s) = 0] \land \forall t \subseteq s[\alpha(t) = 0]] \). Using induction, one proves the following schema:\n
\[ \forall n[k < n \land k \leq n[A(k) \lor B(k)] \rightarrow (\forall k < n[A(k) \lor B(k)])] \]

Now define \( \gamma \) such that, for every \( s \in 2^{<\omega}, \gamma(s) = \mu k[\forall \gamma \in 2^{<\omega}[\text{length}(c) = k \rightarrow \alpha^*(s * (0) * c) \neq 0] \land \forall d \leq 2^{<\omega}[\text{length}(d) = k \rightarrow \alpha^*(s * (1) * d) \neq 0)] \). Then define \( \delta \) in \( 2^{<\omega} \) such that, for every \( s \in 2^{<\omega}, \delta(s) = 1 \) if \( \forall \gamma \in 2^{<\omega}[\text{length}(c) = \gamma(s) \rightarrow \alpha^*(s * (0) * c) \neq 0] \). Finally, define \( \gamma \) such that \( \forall n[\gamma(n) = 1 - \delta(\gamma_n)] \). Using this fact, \( \forall n[\neg \text{Bar}_{2^\omega}(D_{\bar{\sigma}_n})] \), one may prove, by induction, that, for each \( n \), \( \forall \gamma \in 2^{<\omega}[\gamma(n) = 0] \).\n
\[ \text{2.2.14. The Lesser Limited Principle of Omniscience, LLPO:} \]

\[ \forall n[k < 2^{<\omega}[\forall p[p^2 + i \neq \mu n[\alpha(n) \neq 0]]] \]

\[ \begin{align*}
\text{Theorem 2.9.} & \quad \text{BIM} \vdash \text{BCP} \rightarrow \neg \text{LLPO}.
\end{align*} \]

\[ \begin{align*}
\text{Proof.} & \quad \text{Assume LLPO. Using BCP, find m,i such that i < 2 and } \forall m[\forall n[\alpha(n) \neq 0]] \Rightarrow \forall p[p^2 + i \neq \mu n[\alpha(n) \neq 0]]]. \text{Define } \alpha := \neg(2m + i) \land (1) \land i \text{ and note that } \forall n[\forall n[\alpha(n) \neq 0] \land 2m + i \neq \mu n[\alpha(n) \neq 0]] \Rightarrow \neg \text{BCP}. \text{Contradiction.}
\end{align*} \]

\[ \begin{align*}
\text{Theorem 2.10.} & \quad \text{BIM} \vdash \text{WKL} \rightarrow \text{LLPO}.
\end{align*} \]
Proof. Assume WKL. Let α be given. Define β such that, for every s, β(s) = 0 if and only if ∃q∃i < 2|s = 2i ∧ ∀p[2p + i < q → 2p + i ̸= μn[α(n) ̸= 0]]. Note that ∀m∃i < 2q ≤ m[β(2i) = 0]. Using WKL, find γ such that ∀n[β(γn) = 0]. Define i := γ(0) and conclude: γ = 1 and ∀p[2p + i ̸= μn[α(n) ̸= 0]]. We thus see that ∀α∀n 2p + i < 2νp[α(n) ̸= 0], i.e. LLPO.

Using a weak axiom of countable choice, one may also prove LLPO → WKL, see Theorem 4.3.

2.2.15. The Limited Principle of Omniscience, LPO:

∀α[∃n[α(n) ̸= 0] ∨ ∀n[α(n) = 0]].

LPO and LLPO were introduced by E. Bishop, see [3] Chapter 1, Section 1]. The following result is not difficult and well-known, see [21, Section 2.6] and [1, Theorem 3.1].

Theorem 2.11. BIM ⊢ LPO → LLPO.

Proof. Let α be given. Apply LPO and distinguish two cases.

Case (1). ∃n[α(n) ̸= 0]. Find q, k such that k < 2 and 2q + k = μn[α(n) ̸= 0].

Conclude: ∀p[2p + 1 − k ̸= μn[α(n) ̸= 0]].

Case (2). ∀n[α(n) = 0]. Then ∀k < 2∀p[2p + k ̸= μn[α(n) ̸= 0]]. □

The following Lemma shows that, in BIM + BCP, not every closed subset of ωω is a spread, see also [17 Theorem 2.10 (vi)].

Lemma 2.12. BIM ⊢ ∀β∃γ[Spr(γ) ∧ Fβ = Fγ] → LPO.

Proof. Assume ∀β∃γ[Spr(γ) ∧ Fβ = Fγ]. We will prove LPO. Let α be given. Define β such that β(0) = 0 and ∀n∀s[β(γn) = 0 ↔ α(n) ̸= 0]. Assume we find γ such that Spr(γ) and Fβ = Fγ. If γ(0) = 0, then ∃n[α(n) ̸= 0] and, if γ(0) ̸= 0, then ∀n[α(n) = 0]. We thus see ∀α[∃n[α(n) ̸= 0] ∨ ∀n[α(n) = 0]], i.e. LPO. □

2.2.16. Markov’s principle, MP:

∀α[¬¬∃n[α(n) ̸= 0] → ∃n[α(n) ̸= 0]].

For some discussion of this principle, see [30, Volume I, Chapter 4, Section 5]. In this paper, the principle figures only in Subsection 2.1.

We would like to make a philosophical observation. For a constructive mathematician, the assumptions LPO, WKL, LLPO and MP make no sense, as she does not know a situation in which these assumptions are true. Theorem 2.10 WKL → LLPO concludes something which is never true from something which is never true. Nevertheless, the proof of Theorem 2.10 makes sense. It shows us how to find, given any α, a suitable β such that if β has the WKL-property, then α has the LLPO-property. This part of the argument does not use the assumption that every β has the WKL-property. Theorems 2.6 and 2.11 deserve a similar comment.

The reader may find more information on the axioms of intuitionistic analysis in [5], [6], [8], [13], [9], [30] and [40].

3. The Σ0-separation principle

3.1. In classical reverse mathematics, weak König’s Lemma is equivalent to a principle called Σ0-separation, see [27, Lemma IV.4.4]. We call X ⊆ ω enumerable if and only if ∃α[X = Eα]. The following statement, formulated in the intuitionistic language of BIM, comes close to the just-mentioned classical principle.

Σ0-separation principle, Σ0-Sep:

∀α[¬¬∃n∀i < 2|(n, i) ∈ Eα] → ∃γ∀n[(n, γ(n)) /∈ Eα]].
The next Theorem seems to confirm the just-mentioned classical result.

**Theorem 3.1.** BIM $\leftrightarrow$ WKL $\leftrightarrow$ $\Sigma_1^0$-Sep.

**Proof.** (i) Assume WKL. We will prove $\Sigma_1^0$-Sep.

Let $\alpha$ be given such that $\neg\exists n \forall i < 2 \{(n, i) \in E_\alpha\}$. Define $\beta$ such that $\forall a \in \Sigma^\omega_2(\beta(a) = 0 \leftrightarrow \forall m < \text{length}(a)(\{m, a(m)\} \notin E_{\beta|m|})].$ Let $n$ be given. Note that $\forall m < n \exists i < 2 \{(m, i) \notin E_{\beta|m|}\}$ and find $a$ in $\Sigma^\omega_2$ such that $\text{length}(a) = n$ and $\forall m < n(\{m, a(m)\} \notin E_{\beta|m|})].$ Conclude that $\forall j \leq m(\beta|j|)] = 0$. We thus see that $\forall n \exists a \in \Sigma^\omega_2(\beta(a) = n \land \forall m \leq n[\beta(\bar{\alpha}n)] = 0]$. Using WKL, find $\gamma$ in $\Sigma^\omega_2$ such that $\forall n[\beta(\bar{\alpha}n)] = 0]$. Conclude that $\forall n \forall m < n[\{m, \gamma(m)\} \notin E_{\gamma|m|}]$ and $\forall n \{(n, \gamma(n)) \notin E_\gamma\}$. We thus see that $\forall a[\neg\exists n \forall i < 2 \{(n, i) \in E_\gamma\}] \rightarrow \exists \gamma \forall n \{(n, \gamma(n)) \notin E_\gamma\}$, i.e., $\Sigma_1^0$-Sep.

(ii) Assume $\Sigma_1^0$-Sep. We will prove WKL.

Let $\alpha$ be given such that $\forall m[\neg\text{Bar}_{\omega_2}(D_{\alpha|m|})].$ Define $\beta$ such that, for both $i < 2$, for all $n$, if $s \in \Sigma^\omega_2$ and $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$ and $\neg\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$, then $\beta(n, s) = (s, i) + 1$, and, if not, then $\beta(n, s) = 0$. Note that $E_\gamma$ is the set of all $(s, i)$ such that $s \in \Sigma^\omega_2$ and $i < 2$ and $\forall n[\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})] \land \neg\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})].$ Note that, for all $s$ in $\Sigma^\omega_2$ for all $i < 2$, if $(s, i) \notin E_\beta$, then $\forall n[\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|}) \rightarrow \neg\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})].$ Note that $\neg\exists \gamma \forall i < 2 \{(s, i) \in E_\beta\}$. Using $\Sigma_1^0$-Sep, find $\gamma$ in $\Sigma^\omega_2$ such that $\forall n[\{(s, \gamma(s)) \notin E_\beta\}$. Define $\delta$ in $\Sigma^\omega_2$ such that $\forall n[\alpha(\delta n) = \gamma(\delta n)]$. Suppose we find $n$ such that $\alpha(\delta n) \neq 0$. Define $q := \delta n + 1$ and note that $\delta n \in D_{\alpha|m|}$, i.e., $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$. We prove, using backwards induction, that $\forall j \leq q(\delta n) \notin n[\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})].$ Observe that $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$. Now suppose $j + 1 \leq n$ and $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$. Note that $\delta(j + 1) = \delta j \ast \langle \gamma \delta j \rangle$. Also $\langle \delta j, \gamma(\delta j) \rangle \notin E_\beta$. Conclude that $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$ and $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$. This completes the proof of the induction step. After $n$ steps we find $\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})$. This contradicts the assumption $\forall n[\neg\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})].$ Conclude that $\forall n[\alpha(\delta n) = 0].$

We thus see that $\forall n[\neg\text{Bar}_{\omega_2 \rightarrow \gamma + 1}(D_{\alpha|m|})] \rightarrow \exists \delta \in \Sigma^\omega_2 \forall n[\alpha(\delta n) = 0]$, i.e., WKL. □

Theorem 3.1 shows that $\Sigma_1^0$-Sep, like WKL, is not constructive, see Theorem 2.10. It is not true in intuitionistic analysis and it also fails in the model of BIM given by the recursive functions.

4. AC$\omega_\omega$, some special cases

The following restricted version of AC$\omega_\omega$ is provable in BIM as it is a consequence of Axiom BIM see Section 2.

4.1. Minimal Axiom of Countable Choice, $\Delta_1^0$-AC$\omega_\omega$:

$$\forall \alpha[\forall n(\exists m[\alpha(n, m) = 0] \rightarrow \exists \gamma \forall n[\alpha(n, \gamma(n)) = 0]].$$

4.2. Axiom Scheme of Countable Unique Choice, AC$\omega_\omega$! = AC$0_0$!:

$$\forall \alpha \exists m[R(n, m)] \rightarrow \exists \gamma \forall n[R(n, \gamma(n))].$$

where $\forall n \exists m[R(n, m)]$’ abbreviates $\forall n \exists m[R(n, m) \land \forall p[R(n, p) \rightarrow m = p]]$. AC$\omega_\omega$! is not a theorem of BIM, see Subsection 4.3 and 31.

4.3. $\Sigma_1^0$-First Axiom of Countable Choice, $\Sigma_1^0$-AC$\omega_\omega$:

$$\forall \alpha[\forall n \exists m[n, m) \in E_\alpha] \rightarrow \exists \gamma \forall n[(n, \gamma(n)) \in E_\alpha]].$$

**Theorem 4.1.** BIM $\vdash$ $\Sigma_1^0$-AC$\omega_\omega$.

**Proof.** Assume $\forall n \exists m[(n, m) \in E_\alpha]$. Then $\forall n \exists p[\alpha(p) = (n, m) + 1]$. Find $\delta$ such that $\forall n[\delta(n) = \mu[q(\alpha(q')) = (n, q'') + 1]]$. Define $\gamma$ such that $\forall n[\gamma(n) = \delta''(n)]$ and note that $\forall n[(n, \gamma(n)) \in E_\alpha]$. □
Proof. (i) Assume, in
\[ \forall i < 2 \left[ \forall n \exists m \left[ (n, m) \notin E_\alpha \right] \rightarrow \exists \gamma \forall n \left[ (n, \gamma(n)) \notin E_\alpha \right] \right]. \]

\( \Pi^0_1 \)-AC\(_{\omega, \omega} \) is unprovable in BIM, see Subsection 4.8. In \([44, \text{Section 6}]\), we introduced the following special case of this axiom.

4.5. Weak \( \Pi^0_1 \)-First Axiom of Countable Choice, Weak-\( \Pi^0_1 \)-AC\(_{\omega, \omega} \):
\[ \forall \alpha \forall \exists m \left[ (n, m) \notin E_\alpha \right] \rightarrow \exists \gamma \forall n \left[ (n, \gamma(n)) \notin E_\alpha \right]. \]

Weak-\( \Pi^0_1 \)-AC\(_{\omega, \omega} \) follows from AC\(_{\omega, \omega} \). Weaker-\( \Pi^0_1 \)-AC\(_{\omega, \omega} \) is a special case of the axiom scheme AC\(_{0,0}^0 \) introduced in \([23, \text{Subsection 3.1}]\). We suspect that, in BIM, Weak-\( \Pi^0_1 \)-AC\(_{\omega, \omega} \) does not imply \( \Pi^0_1 \)-AC\(_{\omega, \omega} \), but we have no proof.

**Theorem 4.2.**
(i) BIM + Weak-\( \Pi^0_1 \)-AC\(_{\omega, \omega} \) \( \vdash \forall \beta [\text{Fan}(\beta) \rightarrow \text{Fan}^+(\beta)] \).
(ii) BIM + Weak-\( \Pi^0_1 \)-AC\(_{\omega, \omega} \) \( \vdash \text{FT} \rightarrow \text{FT}^+ \).

Proof. The proof is left to the reader. □

One may also study statements one obtains from AC\(_{\omega, \omega} \) by limiting the number of alternatives one has at each choice.

4.6. Axiom Scheme of Countable Binary Choice, AC\(_{\omega, \omega}^2 \):
\[ \forall n \exists m < 2 \left[ R(n, m) \right] \rightarrow \exists \gamma \in 2^{\exists \forall n \left[ R(n, \gamma(n)) \right]}. \]

Here is a restricted version of AC\(_{\omega, \omega}^2 \):

4.7. \( \Pi^0_1 \)-Axiom of Countable Binary Choice, \( \Pi^0_1 \)-AC\(_{\omega, \omega}^2 \):
\[ \forall n \exists m < 2 \left[ (n, m) \notin E_\alpha \right] \rightarrow \exists \gamma \in 2^{\exists \forall n \left[ (n, \gamma(n)) \notin E_\alpha \right]}. \]

A result related to the following Theorem has been proven by U. Kohlenbach, see \([16, \text{Theorem 3}]\). A similar result is mentioned in \([1, \text{Subsection 2.2}]\).

**Theorem 4.3.** BIM \( \vdash \left( \Pi^0_1 \right)-\text{AC}_{\omega, \omega} \land \text{LLPO} \leftrightarrow \text{WKL} \).
**Proof.** (i) Assume, in BIM, \( \Pi^0_1 \)-AC\(_{\omega, \omega} \) and LLPO. It suffices to prove \( \Sigma^0_1 \)-Sep, as, according to Theorem 5.1, BIM \( \vdash \Sigma^0_1 \)-Sep \( \leftrightarrow \text{WKL} \).

Let \( \alpha \) be given such that \( \forall n \forall \forall m < 2 \left[ (n, m) \in E_\alpha \right]. \) Let \( n \) be given. Define \( \beta \) such that \( \forall q \forall i < 2 \left[ \beta(2q + i) \neq 0 \leftrightarrow q = \mu(p) = (n, i + 1) \right]. \) Apply LLPO and find \( i < 2 \) such that \( \forall q \forall i < 2 \left[ \beta(2q + i) \neq 0 \right]. \) Assume \( (n, i) \in E_\alpha \). Then \( (n, 1 - i) \notin E_\alpha \) and \( \exists q(\mu(p) = (n, 1 - i + 1) \land \forall q(\beta(2q + 1 - i) = 0) \]. Find \( q := \mu(p) = (n, i + 1) \). Note \( \beta(2q + i) \neq 0 \) and \( \forall m < 2q + 1 \beta(m) = 0 \), so \( 2q + i = \mu(m)[\beta(m) \neq 0] \). Contradiction. Conclude that \( (n, i) \notin E_\alpha \). Conclude that \( \forall n \exists i < 2 \left[ (n, i) \notin E_\alpha \right]. \) Apply \( \Pi^0_1 \)-AC\(_{\omega, \omega} \) and find \( \gamma \) in \( 2^\omega \) such that \( \forall n \left[ (n, \gamma(n)) \notin E_\alpha \right]. \) Conclude that \( \forall n \left[ \forall \exists m < 2 \left[ (n, m) \in E_\alpha \right] \rightarrow \exists \gamma \in 2^{\exists \forall n \left[ (n, \gamma(n)) \notin E_\alpha \right]}. \) i.e. \( \Sigma^0_1 \)-Sep.

(ii) Note that \( \neg(P \land Q) \leftrightarrow \neg \neg(\neg P \lor \neg Q) \) is a valid scheme of intuitionistic logic. Conclude that BIM \( \vdash \Sigma^0_1 \)-Sep \( \leftrightarrow \forall \alpha \forall \exists m \left[ (n, m) \notin E_\alpha \right] \rightarrow \exists \gamma \in 2^{\exists \forall n \left[ (n, \gamma(n)) \notin E_\alpha \right]}. \) Conclude that BIM \( \vdash \Sigma^0_1 \)-Sep \( \rightarrow \Pi^0_1 \)-AC\(_{\omega, \omega} \), and, using Theorem 3.1, that BIM \( \vdash \text{WKL} \rightarrow \Pi^0_1 \)-AC\(_{\omega, \omega} \). The conclusion BIM \( \vdash \text{WKL} \rightarrow \text{LLPO} \) has been drawn in Theorem 2.10.

From a constructive point of view, \( \Sigma^0_1 \)-Sep, or equivalently, WKL, is an axiom of countable choice that is formulated too strongly.

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\footnote{J. R. Moschovakis made me see that S. Weinstein proved (arguing classically) a result from which one may conclude that, indeed, this implication does not hold, see \([19, \text{Theorem 3.10}]\).}
4.8. $\text{AC}_{\omega,2}$ and $\Pi^0_1-\text{AC}_{\omega,2}$ are unprovable in $\text{BIM}$.

Note that the theory $\text{BIM} + \text{CT}$ may be translated into intuitionistic arithmetic $\text{HA}$, by interpreting functions from $\omega$ to $\omega$ as indices of total computable functions. The negative translation due to Gödel and Gentzen, see [30], vol. I, Ch. 3, Subsection 3.4], shows that first order classical (Peano) arithmetic $\text{PA}$, the theory that results from $\text{HA}$ by adding the axiom scheme $X \lor \neg X$, is consistent. It follows that also the theory $\text{BIM} + \text{CT}$ remains consistent upon adding the axiom scheme $X \lor \neg X$.

4.8.1. Note that the theory $\text{BIM} + \text{CT} + X \lor \neg X + \text{AC}_{\omega,2}$! is inconsistent. The argument is as follows.

Let $\tau, \psi$ be as in $\text{CT}$. Define $H := \{ n \mid \exists z[\tau(n, n, z) \neq 0]\}$. Using classical logic, conclude: $\forall n \exists i < 2[i = 0 \leftrightarrow n \in H]$. Using $\text{AC}_{\omega,2}$, find $\alpha$ such that $\forall n[\alpha(n) \neq 0 \leftrightarrow n \in H]$. Define $\beta$ such that, for each $n$, if $\alpha(n) \neq 0$, then $\beta(n) = \psi(\mu z[\tau(n, n, z) \neq 0]) + 1$, and, if $\alpha(n) = 0$, then $\beta(n) = 0$. Using $\text{CT}$, find $n_0$ such that $\forall n[\beta(n) = \psi(\mu z[\tau(n_0, n, z) \neq 0])]$. Note that $\alpha(n_0) \neq 0$ and: $\beta(n_0) = \beta(n_0) + 1$. Contradiction.

Conclude that, if $\text{HA}$ is consistent, then $\text{AC}_{\omega,2}$! is not derivable in $\text{BIM}$.

4.8.2. Theorem 4.3 implies that $\text{BIM} + \neg \Pi^0_1-\text{LLPO} + \Pi^0_1-\text{AC}_{\omega,2}$ is not consistent, as $\neg \Pi^0_1-\text{FT}$ contradicts $\text{WKL}$. Conclude that $\text{BIM} + \neg \Pi^0_1-\text{FT} + \Pi^0_1-\text{LLPO} + \neg \Pi^0_1-\text{AC}_{\omega,2}$.

On the other hand, $\Pi^0_1-\text{AC}_{\omega,2}$ is not derivable in $\text{BIM} + \neg \Pi^0_1-\text{FT} + \Pi^0_1-\text{LLPO}$ and, a fortiori, not derivable in $\text{BIM}$. The stronger axiom $\Pi^0_1-\text{AC}_{\omega,2}$ and the even stronger axiom $\Pi^0_1-\text{AC}_{\omega,\omega}$, to be introduced in Section 6, also are not derivable in $\text{BIM}$.

From Theorem 4.3 we also conclude that $\text{BIM} + \neg \Pi^0_1-\text{FT} + \Pi^0_1-\text{AC}_{\omega,2} \vdash \neg \text{LLPO}$.

In the context of $\text{HA}$, Church’s Thesis is sometimes introduced an axiom scheme, $\text{CT}_0$, see [30], vol. I, Ch. 4, Sect. 3):

$$\forall n \exists m[A(n, m)] \rightarrow \exists \forall n \exists z[T(\langle e, n, z \rangle)] \land \forall i < z[\neg T(\langle e, n, i \rangle)] \land A(n, (U(z)))$$

It is not difficult to see that $\text{BIM} + \text{CT} + \text{AC}_{\omega,\omega}$ and also $\text{BIM} + \text{CT} + \text{AC}_{\omega,\omega}$ translate into $\text{HA} + \text{CT}_0$. There is no straightforward model for $\text{HA} + \text{CT}_0$ but, using realizability, one may show that, if $\text{HA}$ is consistent, then so is $\text{HA} + \text{CT}_0$, see [30], vol. I, Ch. 4, Sect. 4. It follows that, if $\text{HA}$ is consistent, then $\text{BIM} + \neg \Pi^0_1-\text{FT} + \Pi^0_1-\text{AC}_{\omega,2}$ is consistent.

The following axiom scheme may be compared to the axiom $\text{BC}_{0,0}$ of bounded countable choice occurring in [23, Section 3.2].

4.9. Axiom Scheme of Countable Finite Choice, $\text{AC}_{\omega,\omega}$:

$$\forall \beta \forall n \exists m \leq \beta(n)[R(n, m)] \rightarrow \exists \gamma \forall n[\gamma(n) \leq \beta(n) \land R(n, \gamma(n))].$$

4.10. $\Pi^0_1$-Axiom of Countable Finite Choice, $\Pi^0_1-\text{AC}_{\omega,\omega}$:

$$\forall n \exists m \leq \alpha^{10}(n)[(n, m) \notin E_{\alpha^{10}}] \rightarrow \exists \gamma \forall n[\gamma(n) \leq \alpha^{10}(n) \land (n, \gamma(n)) \notin E_{\alpha^{10}}].$$

We conjecture that $\Pi^0_1-\text{AC}_{\omega,\omega}$ is not provable in $\text{BIM} + \Pi^0_1-\text{AC}_{\omega,2}$, but we have no proof of this conjecture. There might be many statements intermediate in strength like $\Pi^0_1-\text{AC}_{\omega,3}$, the $\Pi^0_1$-Axiom of Countable Ternary Choice.

5. Contrapositions of some special cases of $\text{AC}_{\omega,\omega}$

The following statement is a contraposition of $\text{AC}_{\omega,\omega}$:

\[11\] $T$ is Kleene’s $T$-predicate and $U$ his result-extracting function.
5.1. **First Axiom Scheme of Reverse Countable Choice, \( \overline{\text{AC}}_{\omega,\omega} \):**

\[
\forall \gamma \exists n[R(n, \gamma(n))] \rightarrow \exists n \forall m[R(n, m)].
\]

A special case is:

5.2. **Minimal Axiom of Reverse Countable Choice, \( \Delta^0_1 \overline{\text{AC}}_{\omega,\omega} \):**

\[
\forall \alpha[\forall \gamma \exists n[(n, \gamma(n)) \in D_\alpha] \rightarrow \exists n \forall m[(n, m) \in D_\alpha]].
\]

The following result may be found in [33 Section 2].

**Theorem 5.1.** \( \text{BIM} + \Delta^0_1 \overline{\text{AC}}_{\omega,\omega} \vdash \text{LPO} \).

**Proof.** Assume \( \Delta^0_1 \overline{\text{AC}}_{\omega,\omega} \). We will prove LPO.

Let \( \beta \) be given. Define \( \alpha \) such that \( \forall n \forall m[\alpha(n, m) = 0 \leftrightarrow (\overline{\beta} n = \overline{\beta} m) \wedge \overline{\beta} m \neq \overline{\beta} n] \). Note that, for every \( \gamma \), either \( \overline{\beta}(\gamma(0)) = \overline{\beta}(\gamma(0)) \) and \( \alpha(0, \gamma(0)) \neq 0 \), i.e. \( (0, \gamma(0)) \in D_\alpha \), or \( \overline{\beta}(\gamma(0)) \neq \overline{\beta}(\gamma(0)) \), and \( \alpha(\gamma(0), \gamma(\gamma(0))) \neq 0 \), i.e. \( (\gamma(0), \gamma(\gamma(0))) \in D_\alpha \). Conclude that \( \forall \gamma \exists n[(n, \gamma(n)) \in D_\alpha] \). Applying \( \Delta^0_1 \overline{\text{AC}}_{\omega,\omega} \), find \( n \) such that \( \forall m[(n, m) \in D_\alpha] \). Either \( \overline{\beta} n \neq \overline{\beta} m \) and \( \exists j[\beta(j) \neq 0] \), or \( \overline{\beta} n = \overline{\beta} m \). In the latter case, for each \( m \), \( \overline{\beta} m = \overline{\beta} m \) and \( \exists j[\beta(j) = 0] \). We thus see \( \forall \gamma \exists n[\beta(n) \neq 0] \) or \( \forall n[\beta(n) = 0] \), i.e. LPO.

5.3. **Axiom Scheme of Reverse Countable Binary Choice, \( \overline{\text{AC}}_{\omega,2} \):**

\[
\forall \gamma \in 2^n \exists \forall i[R(n, \gamma(i))] \rightarrow \exists n \forall i < 2[R(n, i)].
\]

In [33 Section 4], \( \overline{\text{AC}}_{\omega,2} \) has been shown to be a consequence of \( \text{FT} + \overline{\text{AC}}_{\omega,\omega} \). We introduce a restricted version:

**\( \Delta^0_1 \)-Axiom of Reverse Countable Binary Choice, \( \Delta^0_1 \overline{\text{AC}}_{\omega,2} \):**

\[
\forall \alpha[\forall \gamma \in 2^n \exists \forall i[(n, \gamma(i)) \in D_\alpha] \rightarrow \exists n \forall i < 2[(n, i) \in D_\alpha]].
\]

**Theorem 5.2.** \( \text{BIM} \vdash \Delta^0_1 \overline{\text{AC}}_{\omega,2} \).

**Proof.** Let \( \alpha \) be given such that \( \forall \gamma \in 2^n \exists \forall i[(n, \gamma(i)) \in D_\alpha] \). Define \( \gamma \) in \( 2^\omega \) such that \( \forall n[0, n) \in D_\alpha \leftrightarrow \gamma(n) = 1 \). Find \( n \) such that \( (n, \gamma(n)) \in D_\alpha \). Note that \( \gamma(n) = 1 \) and \( \forall i < 2[(n, i) \in D_\alpha] \).

We now introduce a less restricted version:

5.4. **\( \Sigma^0_1 \)-Axiom of Reverse Countable Binary Choice, \( \Sigma^0_1 \overline{\text{AC}}_{\omega,2} \):**

\[
\forall \alpha[\forall \gamma \in 2^n \exists \forall i[(n, \gamma(i)) \in E_\alpha] \rightarrow \exists n \forall i < 2[(n, i) \in E_\alpha]].
\]

We define a formula that we want to call its strong negation:

5.5. **\( \neg \Sigma^0_1 \overline{\text{AC}}_{\omega,2} \):**

\[
\exists \alpha[\forall \gamma \in 2^n \exists \forall i[(n, \gamma(i)) \in E_\alpha] \wedge \neg \exists n \forall i < 2[(n, i) \in E_\alpha]].
\]

Note that BIM proves\( \forall \alpha[\neg \exists n \forall i < 2[(n, i) \in E_\alpha] \leftrightarrow \forall n \forall \nu \nu[p(n) = (n, 0) + 1 \rightarrow \alpha[q(n) = (n, 1) + 1]] \].

**Lemma 5.3.** BIM proves the following:

(i) \( \text{FT} \rightarrow \Sigma^0_1 \overline{\text{AC}}_{\omega,2} \) and \( \neg \Sigma^0_1 \overline{\text{AC}}_{\omega,2} \rightarrow \neg \text{FT} \).

(ii) \( \Sigma^0_1 \overline{\text{AC}}_{\omega,2} \rightarrow \text{FT} \) and \( \neg \text{FT} ightarrow \neg \Sigma^0_1 \overline{\text{AC}}_{\omega,2} \).
Proof. (i) We prove, in BIM, that, for each $\alpha$, there exists $\beta$ such that

1. $\forall \gamma \in 2^\prec \exists n[(n, \gamma(n)) \in E_\alpha] \to \text{Bar}_{2^\prec}(D_\alpha)$ and
2. $\exists m[\text{Bar}_{2^\prec}(D_{\alpha m})] \rightarrow \exists \nu i < 2[(i, n) \in E_\alpha]$.

The two promised conclusions then follow easily.

Let $\alpha$ be given. Define $\beta$ such that

$$\forall \alpha \in 2^{\prec \omega} \beta(\alpha) \neq 0 \iff \exists n < \text{length}(\alpha)[(n, a(n)) \in E_{\text{length}(\alpha)}].$$

1. Assume $\forall \gamma \in 2^\prec \exists n[(n, \gamma(n)) \in E_\alpha]$. Let $\gamma$ in $2^{\prec \omega}$ be given. Find $n, p$ such that $(n, \gamma(n)) \in E_p$ and $n < p$. Note that $\beta(\gamma p) \neq 0$. We thus see that $\forall \gamma \in 2^{\prec \omega} \beta(\gamma p) \neq 0$, i.e. $\text{Bar}_{2^\prec}(D_\alpha)$.

2. Let $m$ be given such that $\text{Bar}_{2^\prec}(D_{\alpha m})$. Note that $\forall a \in 2^{\prec \omega}[\text{length}(a) < a]$ and $\forall \gamma \in 2^\prec \exists n < m[\beta(\gamma n) \neq 0]$. It follows that $\forall a \in 2^{\prec \omega}[\text{length}(a) = m \rightarrow \exists n < m[\beta(\gamma n) \neq 0]]$. Assume $\forall n < m \exists i < 2[(i, n) \notin E_{\alpha m}]$. Define $a$ in $2^{\prec \omega}$ such that $\text{length}(a) = m$ and $\forall n < m[\forall (n, 0) \notin E_{\alpha m} \leftrightarrow a(n) = 0]$. Conclude that $\forall n < m[\exists i < 2[(i, n) \in E_{\alpha m} \subseteq E_{\alpha}]$.

(ii) We prove, in BIM: for each $\alpha$, there exists $\beta$ such that

1. $\text{Bar}_{2^\prec}(D_{\alpha}) \rightarrow \forall \gamma \in 2^\prec \exists n[(n, \gamma(n)) \in E_\beta]$ and
2. $\exists \nu i < 2[(i, n) \in E_\beta] \to \exists m[\text{Bar}_{2^\prec}(D_{\alpha m})]$.

The two promised conclusions then follow easily.

Let $\alpha$ be given. Define $\beta$ such that, for all $n$, for every $s$ in $2^{\prec \omega}$, for all $i < 2$, if either (a) $\text{Bar}_{2^\prec}(D_{\alpha m})$, or (b) $\text{Bar}_{2^\prec} r(i, s)[D_{\alpha m}]$ and not $\text{Bar}_{2^\prec} r(i, s)[1 - i, D_{\alpha m}]$, then $\beta(n, s * (i)) = (s, i) + 1$, and, (c) if both (a) and (b) fail, then $\beta(n, s * (i)) = 0$.

Furthermore, for all $n$, all $s$, if $s \notin 2^{\prec \omega}$, then $\beta(n, s) = 0$. Note that $E_\beta$ is the set of all pairs $(s, i)$ such that $s \in 2^{\prec \omega}$ and $i < 2$ and either $\exists n[\text{Bar}_{2^\prec}(D_{\alpha n})]$ or $\exists n[\text{Bar}_{2^\prec} r(i, s)[D_{\alpha m}] \land \neg \text{Bar}_{2^\prec} r(i, s)[1 - i, D_{\alpha m}]]$. Note that, for all $s$ in $Bim$, if $\nu i < 2[(i, s) \in E_{\beta}]$, then $\exists n[\text{Bar}_{2^\prec}(D_{\alpha n})]$.

1. Assume $\text{Bar}_{2^\prec}(D_{\alpha})$. We will prove that $\forall \gamma \in 2^\prec \exists n[(n, \gamma(n)) \in E_{\beta}]$. Let $\gamma$ in $2^{\prec \omega}$ be given. Define $\delta$ in $2^{\prec \omega}$ such that $\forall n[\delta(n) = \gamma(\delta n)]$. Find $n$ such that $\alpha(\delta n) \neq 0$. Define $q := \delta n + 1$ and note: $\delta n \in D_{\alpha n}$. We claim that for all $j \leq n$, either $\exists i \leq n[\delta j, \gamma(\delta j)] \in E_{\beta}$, or $\text{Bar}_{2^\prec} r(i, j)[D_{\alpha m}]$. We prove this claim by backwards induction, starting from $j = n$. Note that $\text{Bar}_{2^\prec} r(i, j)[D_{\alpha m}]$. Now assume $j < n$ and $\text{Bar}_{2^\prec} r(i, j + 1)[D_{\alpha m}]$, i.e. $\text{Bar}_{2^\prec} r(i, j + 1)[D_{\alpha m}]$. Find out if also $\text{Bar}_{2^\prec} r(i, j + 1)[D_{\alpha m}]$. If so, then $\text{Bar}_{2^\prec} r(i, j)[D_{\alpha m}]$ and, if not, then $\exists j \neq \delta j \in \mathbb{R}$. We may conclude that either $\exists j \neq \delta j \in \mathbb{R}$ or $\text{Bar}_{2^\prec}(D_{\alpha n})$. Note that, if $\text{Bar}_{2^\prec}(D_{\alpha n})$, then $\forall s \in 2^{\prec \omega} \nu i < 2[\beta(q, s + (i)) = (s, i) + 1$ and $\forall s \in 2^{\prec \omega}[(s, \gamma(s)) \in E_{\beta}]$. We thus see that $\forall \gamma \in 2^\prec \exists n[(n, \gamma(n)) \in E_{\beta}]$.

2. Assume that $\exists \nu i < 2[(i, n) \in E_{\beta}]$. Conclude, using the observation we made just after the definition of $\beta$, that $\exists n[\text{Bar}_{2^\prec}(D_{\alpha n})]$.

\[\square\]

**Theorem 5.4.** BIM proves: $\text{FT} \leftrightarrow \Sigma_1^\mathcal{A}\mathcal{C}_\omega$ and: $\neg \text{FT} \leftrightarrow \neg \Sigma_1^\mathcal{A}\mathcal{C}_\omega$.

**Proof.** Use Lemma 5.3. \[\square\]

\[\text{12The reader should compare this proof to the proof of Theorem 5.1 (ii).}\]
5.6. Axiom Scheme of Reverse Countable Finite Choice, $\text{AC}_{ω,ω}$:

$$\forall \beta \forall \gamma \exists n[\gamma(n) ≤ β(n) \rightarrow R(n, \gamma(n))] \rightarrow \exists n \forall m ≤ β(n)[R(n, m)].$$

$\text{AC}_{ω,ω}$ may be concluded from $\text{FT} + \text{AC}_{ω,ω}$, by a slight extension of the argument given in Section 4.

The following is a restricted version:

$$\Sigma_1^0 \text{AC}_{ω,ω}:$$

$$\forall a[\forall \gamma \exists n[\gamma(n) ≤ \alpha^{10}(n) \rightarrow (n, \gamma(n)) \in E_{ω^{11}}] \rightarrow \exists n \forall i ≤ \alpha^{10}(n)[(n, i) \in E_{ω^{11}}]].$$

We introduce a strong negation of this restricted version:

$$\neg[(\Sigma_1^0 \neg\text{AC}_{ω,ω})]:$$

$$\exists a[\forall \gamma \exists n[\gamma(n) ≤ \alpha^{10}(n) \rightarrow (n, \gamma(n)) \in E_{ω^{11}}] \land \neg \exists n \forall i \leq \alpha^{10}(n)[(n, i) \in E_{ω^{11}}]].$$

Note that BIM proves that $\forall a[\neg \exists n \forall i \leq \alpha^{10}(n)[(n, i) \in E_{ω}] \leftrightarrow \forall \forall \forall \in ω^{α^{10}(n)+1} \exists i \leq \alpha^{10}(n)[α(t(i)) \neq (n, i) + 1]].$

**Lemma 5.5. BIM proves:**

(i) $\Sigma_1^0 \text{AC}_{ω,ω} → \Sigma_1^0 \text{AC}_{ω,ω}$ and $\neg[(\Sigma_1^0 \neg\text{AC}_{ω,ω})] → \neg[(\Sigma_1^0 \neg\text{AC}_{ω,ω})]$

(ii) $\text{FT} → \Sigma_1^0 \text{AC}_{ω,ω}$ and $\neg[(\Sigma_1^0 \neg\text{AC}_{ω,ω})] → \neg \text{FT}.$

**Proof.** (i) We prove, in BIM: for each $α$, there exists $β$ such that

(1) $\forall γ \in 2^{ω} \exists n[(n, γ(n)) \in E_{α}] \rightarrow \forall γ \exists n[γ(n) ≤ β^{10}(n) \rightarrow (n, γ(n)) \in E_{β^{11}}]$ and

(2) $\exists n \forall i ≤ β^{10}(n)[(n, i) \in E_{β^{11}}] → \exists n \forall i < 2^{[n], i} \in E_{α}].$

The two promised conclusions then follow easily.

Let $α$ be given. Define $β$ such that $β^{10} = 1$ and $β^{11} = α$.

(1) Assume $∃ γ \in 2^{ω} \exists n[(n, γ(n)) \in E_{α}].$ Let $γ$ be given. Define $γ^*$ such that $γ^*(n) = \min(1, γ(n))].$ Note: $γ^* ∈ 2^{ω}$ and find $n$ such that $(n, γ^*(n)) \in E_{α}.$ If $γ^*(n) \neq γ(n),$ then $γ(n) > 1 = β^{10}(n),$ and if $γ^*(n) = γ(n),$ then $(n, γ(n)) \in E_{α}.$

Conclude that $\forall γ \exists n[γ(n) ≤ β^{10}(n) → (n, γ(n)) \in E_{β^{11}}].$

(2) Let $n$ be given such that $∀ i ≤ β^{10}(n)[(n, i) \in E_{β^{11}}].$ Conclude that $∀ i < 2^{[n], i} \in E_{α}].$

(ii) We prove, in BIM: for each $α$, there exists $β$ such that

(1) $\forall γ \exists n[γ(n) ≤ α^{10}(n) → (n, γ(n)) \in E_{α^{11}}] → \text{Bar}_{2ω}-(D_β)$ and

(2) $\exists m[\text{Bar}_{2ω}-(D_α)] → \exists n \forall m ≤ α^{10}(n)[(n, m) \in E_{α^{11}}].$

The two promised conclusions then follow easily.

Define $\text{Cod}_{2ω} : ω → 2^{ω+ω}$ such that $\text{Cod}_{2ω}(β) = γ$ and $∀ γ \exists n[\text{Cod}_{2ω}(s * γ(n)) = \text{Cod}_{2ω}(s * γ(n)) = \text{Cod}_{2ω}(s * \bar{γ}(n))].$ Let $α$ be given. Define $β$ such that, for all $s, i$, $β(\text{Cod}_{2ω}(s * \bar{γ}(n)) \neq 0$ if and only if $∃ n < \text{length}(s)[s(n) > α^{10}(n) \lor (n, s(n)) \in E_{α^{11}}].$

(1) Assume that $\forall γ \exists n[γ(n) ≤ α^{10}(n) → (n, γ(n)) \in E_{α^{11}}].$ Assume that $δ ∈ 2^{ω}.$ Define $γ$, by induction, such that, for each $n$, if $∃ i ≤ α^{10}(n)[\text{Cod}_{2ω}(γ(n) * i) \subseteq δ]$, then $γ(n) = μ[\text{Cod}_{2ω}(γ(n) * i) \subseteq δ]$, and, if not, then $γ(n) = 0.$ Note that $∀ n[γ(n) ≤ α^{10}(n)].$ Find $n, p$ such that $(n, γ(n)) \in E_{α^{11}}.$ Define $γ := μ(n).$ Note that $β(\text{Cod}_{2ω}(γ(n + 1))) \neq 0$ and distinguish two cases. Case (a). $γ(n) ≤ q(n) ≤ α^{10}(m)[\text{Cod}_{2ω}(γ(n) * i) \subseteq δ]$ and $β(i) \neq 0.$ Case (b). $γ(n) ≤ q(n) ≤ α^{10}(m)[\text{Cod}_{2ω}(γ(n) * i) \subseteq δ]$ and $β(i) = 0.$ Find $m_0 := μm ≤ γ(n) ≤ α^{10}(m)[\text{Cod}_{2ω}(γ(n) * i) \subseteq δ]$ and note that $d := \text{Cod}_{2ω}(m_0) * \bar{o}(α^{10}(m_0) + 1) \subseteq
\[ \forall \delta(d) \neq 0. \text{ In both cases } \exists p[\beta(\delta p) \neq 0]. \text{ We thus see that } \forall \delta \in 2^{\omega} \exists p[\beta(\delta p) \neq 0], \text{ i.e. } Bar_2(D_3). \]

(2) Let \( m \) be given such that \( Bar_2(D_{\Sigma m}) \). Suppose that

\[ \forall \alpha < m \forall j < \alpha^{\omega} \exists \beta \psi(j, \beta) \notin E_{\Sigma m}^{\omega}. \]

Find \( s \) such that \( \text{length}(s) = m \) and \( \forall \alpha < m \exists i \in \alpha \forall j < \alpha^{\omega} \exists \beta \psi(i, \beta) \notin E_{\Sigma m}^{\omega}. \) Note that \( \text{Cod}_2(s) > m \) and \( \forall \alpha < \text{Cod}_2(s)[\beta(t) = 0], \text{ so } \neg Bar_2(D_{\Sigma m}). \) Contradiction. Conclude that

\[ \exists i < m \forall j < \alpha^{\omega} \exists \beta \psi(i, \beta) \notin E_{\Sigma m}^{\omega}. \]

\[ \square \]

**Theorem 5.6.** BIM proves: \( FT \leftrightarrow \Sigma_1^0 \neg \text{AC}_{\omega, \omega} \) and \( \neg FT \leftrightarrow \neg ! \Sigma_1^0 \neg \text{AC}_{\omega, \omega} \).

**Proof.** These statements follow from Lemma [5.5] and Theorem [5.5]. \( \square \)

5.7. No Double Negation Shift.

Assume \( \neg ! FT \). Using \( ! \Sigma_1^0 \neg \text{AC}_{\omega, \omega} \), find \( \alpha \) such that \( \forall \gamma \in 2^{\omega} \exists \gamma [(n, \gamma(n)) \in E_{\alpha}] \) and \( \exists \gamma \forall m < 2^{\omega} \exists (n, m) \in E_{\alpha} \). Then, for each \( n \), \( \neg \forall m < 2^{\omega} \exists (n, m) \in E_{\alpha} \) and \( \forall \gamma \exists m < 2^{\omega} \exists (n, m) \in E_{\alpha} \). Conclude that \( \forall \gamma \exists \gamma \exists m < 2^{\omega} \exists (n, m) \notin E_{\alpha} \). Note that \( \neg \exists \gamma \in 2^{\omega} \forall \gamma [(n, \gamma(n)) \notin E_{\alpha}] \). Using \( \Pi_1^0 \neg \text{AC}_{\omega, \omega} \), conclude that \( \neg \forall \gamma \exists m < 2^{\omega} \exists (n, m) \notin E_{\alpha} \). We thus see that if we assume both \( \neg ! FT \) and \( \Pi_1^0 \neg \text{AC}_{\omega, \omega} \) we can find \( \Pi_1^0 \)-subsets \( P = \{ n \mid (n, 0) \notin E_{\alpha} \} \) and \( Q = \{ n \mid (n, 1) \notin E_{\alpha} \} \) of \( \omega \) such that \( \forall \gamma [\neg \exists \gamma ((P(n) \cup Q(n))] \) and \( \forall \gamma [\neg \exists \gamma (P(n) \cup Q(n))] \).

S. Kuroda’s scheme of Double Negation Shift \( \forall \gamma [\neg \exists \gamma ] \rightarrow \neg \exists \gamma [\neg \exists \gamma ] \) (see [19 page 45] and [3 page 105]) thus is refuted.

In [30 vol. 1, Chapter 4, Proposition 3.4.1], the same conclusion is obtained in HA from CT_0.

6. \( AC_{\omega, \omega} \), some special cases

6.1. \( \Sigma_1^0 \)-Second Axiom of Countable Choice, \( \Sigma_1^0 \neg \text{AC}_{\omega, \omega} \):

\[ \forall \alpha [\forall \gamma \exists \gamma [(\gamma \in G_{\alpha^{\omega}}) \rightarrow \exists \gamma \forall \gamma [(\gamma)^{\alpha^{\omega}} \in G_{\alpha^{\omega}}]]. \]

**Theorem 6.1.** BIM \( \vdash \Sigma_1^0 \neg \text{AC}_{\omega, \omega} \).

**Proof.** Assume \( \forall \gamma \exists \gamma [(\gamma \in G_{\alpha^{\omega}}) \rightarrow \exists \gamma \forall \gamma [(\gamma)^{\alpha^{\omega}} \in G_{\alpha^{\omega}}]]. \) Then \( \exists \gamma \exists \gamma [\alpha^{\omega}(\gamma) \neq 0]. \) Find \( \delta \) such that \( \forall \gamma [\delta(\gamma) = 0] \). \( \square \)

6.2. \( \Pi_1^0 \)-Second Axiom of Countable Choice, \( \Pi_1^0 \neg \text{AC}_{\omega, \omega} \):

\[ \forall \alpha [\forall \gamma \exists \gamma [(\gamma \notin G_{\alpha^{\omega}}) \rightarrow \exists \gamma \forall \gamma [(\gamma)^{\alpha^{\omega}} \notin G_{\alpha^{\omega}}]]. \]

**Theorem 6.2.** BIM \( \vdash \Pi_1^0 \neg \text{AC}_{\omega, \omega} \rightarrow \Pi_1^0 \neg \text{AC}_{\omega, \omega} \).

**Proof.** Let \( \alpha \) be given such that \( \forall \gamma \exists m[\forall \gamma \exists m(n, m) \notin E_{\alpha}]. \) Define \( \beta \) such that

\[ \forall \gamma [\beta^{\omega} \neq 0 \leftrightarrow \exists \gamma \exists \gamma [\exists \gamma \exists m(n, m) = \alpha^{\omega}(m, 1 + a = (m) * b)]. \]

Note that \( \forall \gamma [\exists \gamma \exists m(n, m) \in E_{\alpha} \rightarrow \forall \gamma [(\gamma)^{\omega} \notin G_{\alpha^{\omega}}]]. \) Conclude that \( \forall \gamma \exists \gamma [(\gamma \notin G_{\alpha^{\omega}})]. \) Using \( \Pi_1^0 \neg \text{AC}_{\omega, \omega} \), find \( \gamma \) such that \( \forall \gamma [(\gamma)^{\omega} \notin G_{\alpha^{\omega}}]]. \) Define \( \delta \) such that \( \forall \gamma [\delta(\gamma) = 0] \) and note: \( \forall \gamma [\delta(n) \in G_{\alpha^{\omega}}]. \) \( \square \)

One may conclude that \( \Pi_1^0 \neg \text{AC}_{\omega, \omega} \) is unprovable in BIM, see Subsection [4.8].

Not every \( \Pi_1^0 \)-subset of \( \omega^{\omega} \) is a spread, see Lemma [2.12]. For spreads, which are a special kind of \( \Pi_1^0 \)-sets, countable choice is easier:

**Theorem 6.3.** BIM \( \vdash \forall \alpha [\forall \gamma [\exists \gamma [(\gamma \notin G_{\alpha^{\omega}}) \rightarrow \exists \gamma \forall \gamma [(\gamma)^{\alpha^{\omega}} \notin G_{\alpha^{\omega}}]]. \]

**Proof.** Let \( \alpha \) be given such that \( \forall \gamma [\exists \gamma [(\gamma)^{\alpha^{\omega}}] \) and \( \forall \gamma \exists \gamma [(\gamma \notin G_{\alpha^{\omega}})]. \) Define \( \gamma \) such that, for each \( n \), \( \forall \gamma \exists \gamma [\gamma(m) = \mu k[\alpha^{\omega}(\gamma^{\alpha^{\omega}}(m) * k)] = 0]. \) \( \square \)
6.3. **Axiom Scheme of Countable Compact Choice,** \( \mathbf{AC}_{\omega,2^\omega} \):

\[
\forall n \exists \gamma \in 2^\omega[R(n, \gamma)] \rightarrow \exists \gamma \in 2^\omega \forall n[R(n, \gamma)]
\]

Here is a restricted version of \( \mathbf{AC}_{\omega,2^\omega} \):

6.4. **\( \Pi_1^0 \)-Axiom of Countable Compact Choice,** \( \Pi_1^0 \mathbf{AC}_{\omega,2^\omega} \):

\[
\forall \alpha[\forall n \exists \gamma \in 2^\omega[\gamma \notin G_{\alpha}] \rightarrow \exists \gamma \in 2^\omega \forall n[\gamma \notin G_{\alpha}]]
\]

**Theorem 6.4.** \( \text{BIM} \vdash \Pi_1^0 \mathbf{AC}_{\omega,2^\omega} \rightarrow \Pi_1^0 \mathbf{AC}_{\omega,2^\omega} \).

**Proof.** The proof is almost the same as the proof of Theorem 6.2 and is left to the reader. \( \square \)

We may conclude: \( \Pi_1^0 \mathbf{AC}_{\omega,2^\omega} \) is unprovable in \( \text{BIM} \), see Subsection 13.8.

The treatment of real numbers in \( \text{BIM} \) is sketched in Subsection 13.7.

6.5. **\( \Pi_1^0 \)-\( \mathbf{AC}_{\omega,\{0,1\}} \):**

\[
\forall \alpha[\forall n \exists \delta \in [0,1][\delta \notin H_{\alpha}] \rightarrow \exists \delta \in [0,1] \forall n[\delta \notin H_{\alpha}]]
\]

**Theorem 6.5.** \( \text{BIM} \vdash \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \leftrightarrow \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \).

**Proof.** First assume \( \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \).

Using Lemma 13.2, find \( \sigma : 2^\omega \rightarrow [0,1] \) and \( \psi : \omega^\omega \rightarrow \omega^\omega \) such that

1. \( \forall \delta \in [0,1] \exists \gamma \in 2^\omega[\delta = R \sigma \gamma] \) and
2. \( \forall n \forall \gamma \in 2^\omega[\gamma \notin G_{\psi}] \rightarrow \sigma \gamma \in H_{\alpha}] \).

Let \( \alpha \) be given such that \( \forall n \exists \delta \in [0,1][\delta \notin H_{\alpha}] \). Then \( \forall n \exists \gamma \in 2^\omega[\sigma \gamma \notin H_{\alpha}] \) and \( \forall n \exists \gamma \in 2^\omega[\gamma \notin G_{\psi}] \). Find \( \gamma \) in \( 2^\omega \) such that \( \forall n[\gamma \notin G_{\psi}] \). Conclude that \( \forall n[\gamma \notin H_{\alpha}] \) and \( \exists \delta \forall n[\delta \notin H_{\alpha}] \). We thus see that, for all \( \alpha \), if \( \exists \delta \in [0,1][\delta \notin H_{\alpha}] \), then \( \exists \delta \forall n[\delta \notin H_{\alpha}] \), i.e. \( \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \).

Now assume \( \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \).

Using Lemma 13.3, find \( \tau : 2^\omega \rightarrow [0,1] \) and \( \chi : \omega^\omega \rightarrow \omega^\omega \) such that

1. \( \forall \gamma \in 2^\omega \forall \delta \in 2^\omega[\gamma \neq \delta \neq \tau \gamma \neq R \tau \delta] \) and
2. \( \forall n \forall \gamma \in 2^\omega[\gamma \notin G_{\chi}] \rightarrow \chi \gamma \in H_{\alpha}] \).

Let \( \alpha \) be given such that \( \forall n \exists \gamma \in 2^\omega[\gamma \notin G_{\alpha}] \). Conclude that \( \forall n \exists \gamma \in 2^\omega[\tau \gamma \notin H_{\chi}] \) and \( \forall n \exists \delta \in [0,1][\delta \notin H_{\chi}] \). Using \( \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \), find \( \delta \in [0,1] \) such that \( \forall n[\delta \notin H_{\chi}] \). Using (3), find \( \gamma \) in \( 2^\omega \) such that \( \forall n[\delta \neq \tau \gamma \neq H_{\chi}] \). Conclude that \( \exists \delta \forall n[\delta \neq \tau \gamma \neq H_{\chi}] \), i.e. \( \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \).

Clearly, \( \exists \gamma \in 2^\omega \forall n[\gamma \notin G_{\alpha}] \). We thus see that, for all \( \alpha \), if \( \exists \gamma \in 2^\omega[\gamma \notin G_{\alpha}] \), then \( \exists \gamma \in 2^\omega \forall n[\gamma \notin G_{\alpha}] \), i.e. \( \Pi_1^0 \mathbf{AC}_{\omega,\{0,1\}} \).

\( \square \)

7. **Contrapositions of Some Special Cases of** \( \mathbf{AC}_{\omega,\omega^\omega} \)

Let us consider the following axiom scheme, the **Second Axiom Scheme of Reverse Countable Choice.**

7.1. **\( \mathbf{AC}_{\omega,\omega^\omega} \):**

\[
\forall \gamma \in \omega^\omega \exists n[R(n, \gamma)] \rightarrow \exists n \forall \gamma \in \omega^\omega[R(n, \gamma)]
\]

The axiom scheme \( \mathbf{AC}_{\omega,\omega^\omega} \) implies \( \Delta^0_1 \mathbf{AC}_{\omega,\omega^\omega} \) and, therefore, LPO, see Theorem 5.1. Let us consider a restricted version, the **Axiom Scheme of Reverse Countable Choice:**

\[
\forall n \exists \gamma \in \omega^\omega[R(n, \gamma)] 
\]
7.2. $\text{AC}_{\omega^2}$: $\forall \gamma \in 2^\omega \exists n[R(n, \gamma^n)] \rightarrow \exists n\forall \gamma \in 2^\omega[R(n, \gamma)]$.

In [33] it is shown that $\text{AC}_{\omega^2}$ is a consequence of the First Axiom of Continuous Choice $\text{AC}_{\omega \cdot \omega}$ and FAN.

We now require the relation $R$ to be $\Sigma^0_1$ and obtain the $\Sigma^0_1$-Axiom of Reverse Countable Compactness:

7.3. $\Sigma^0_1\text{AC}_{\omega^2}$: $\forall \alpha \forall \gamma \in 2^\omega \exists n[\gamma^n \in G_{\alpha^{1n}}] \rightarrow \exists n[2^\omega \subseteq G_{\alpha^{1n}}]$.

We also introduce a strong negation:

7.4. $\neg(\Sigma^0_1\text{AC}_{\omega^2})$: $\forall \alpha \forall \gamma \in 2^\omega \exists n[\gamma^n \in G_{\alpha^{1n}}] \land \neg\exists n[2^\omega \subseteq G_{\alpha^{1n}}]$.

We also introduce a ‘real’ version:

7.5. $\Sigma^0_1\text{AC}_{\omega(0,1)}$: $\forall \alpha \forall \delta \in [0,1]^{\omega} \exists n[\delta^n \in H_{\alpha^{1n}}] \rightarrow \exists n[[0,1] \subseteq H_{\alpha^{1n}}]$.

and a strong negation:

7.6. $\neg(\Sigma^0_1\text{AC}_{\omega(0,1)})$: $\forall \alpha \forall \delta \in [0,1]^{\omega} \exists n[\delta^n \in H_{\alpha^{1n}}] \land \neg\exists n[[0,1] \subseteq H_{\alpha^{1n}}]$.

The treatment of real numbers in BIM is sketched in Subsection [33.7]

Lemma 7.1. BIM proves:

(i) $\text{FT} \rightarrow \Sigma^0_1\text{AC}_{\omega^2}$ and $\neg(\Sigma^0_1\text{AC}_{\omega^2}) \rightarrow \neg\text{FT}$.

(ii) $\Sigma^0_1\text{AC}_{\omega^2 \to \omega}$ and $\neg(\Sigma^0_1\text{AC}_{\omega^2 \to \omega}) \rightarrow \neg\text{FT}$.

(iii) $\Sigma^0_1\text{AC}_{\omega(0,1)} \rightarrow \Sigma^0_1\text{AC}_{\omega^2 \to \omega}$ and $\neg(\Sigma^0_1\text{AC}_{\omega(0,1)}) \rightarrow \neg(\Sigma^0_1\text{AC}_{\omega^2 \to \omega})$.

Proof. (i) We prove, in BIM: for each $\alpha$, there exists $\beta$ such that

$\forall \gamma \in 2^\omega \exists n[\gamma^n \in G_{\alpha^{1n}}] \rightarrow Bar_{\omega}(D_{\beta})$ and $\exists m[Bar_{\omega}(D_{\beta_m})] \rightarrow \exists n[2^\omega \subseteq G_{\alpha^{1n}}]$.

The two promised statements then follow easily.

Let $\alpha$ be given. Define $\beta$ such that, for every $s$, $\beta(s) \neq 0 \leftrightarrow (s \in 2^{<\omega} \land \exists n < length(s) \exists p \leq length(s^n)[\alpha^{1n}(s^n[p]) \neq 0])$.

Assume that $\forall \gamma \in 2^\omega \exists n[\beta(\gamma) \neq 0]$. Conclude that $\forall \gamma \in 2^\omega \exists n[\beta(\gamma) \neq 0]$, i.e. $Bar_{\omega}(D_{\beta})$.

Now let $m$ be such that $Bar_{\omega}(D_{\beta_m})$. Conclude that $\forall s \in 2^{<\omega}[s > m \rightarrow \exists \gamma[t \leq s \land t \in D_{\beta_m}]]$. We have to prove that, for some $n$, $2^\omega \subseteq G_{\alpha^{1n}}$, i.e. $\forall \gamma \in 2^\omega \exists n[\alpha^{1n}(s^n[p]) \neq 0]$. We will prove the stronger statement that, for some $n < m$, $\forall n < m [\beta(n) = 0]$. We argue by contradiction. Assume that, for each $n < m$, there exists $u$ in $2^{<\omega}$ such that $\beta(u) = m$ and $\neg\exists p < m[\alpha^{1n}(u^n[p]) \neq 0]$. Let $s$ be an element of Bin such that, for each $n < m$, $s^n \in 2^{<\omega}$ and $length(s^n) \geq m$ and $\neg\exists p < m[\alpha^{1n}(u^n[p]) \neq 0]$. Note that $s > m$ and $\neg\exists \gamma \in 2^\omega[\beta(\gamma) \neq 0]$. Contradiction. Thus we see there must exist $n < m$ such that $\forall n < m[\beta(n) = 0]$.

(ii) We prove, in BIM: for each $\alpha$, there exists $\beta$ such that

$\forall \delta \in [0,1]^{\omega} \exists n[\delta^n \in H_{\alpha^{1n}}] \rightarrow \forall \gamma \in 2^\omega \exists n[\gamma^n \in G_{\beta^{1n}}]$ and $\exists n[2^\omega \subseteq G_{\beta^{1n}}] \rightarrow \exists n[[0,1] \subseteq H_{\alpha^{1n}}]$.

Using Lemma [33.2] find $\sigma : 2^\omega \rightarrow [0,1]$ and $\psi : \omega^\omega \rightarrow \omega^\omega$ such that $\forall \delta \in [0,1] \exists \gamma \in 2^\omega[\sigma(\gamma) \equiv \delta]$ and $\forall \gamma \in 2^\omega[\gamma \in G_{\psi(\alpha)} \leftrightarrow \sigma(\gamma) \in H_{\alpha}]$.

Let $\alpha$ be given. Define $\beta$ such that, for every $n$, $\beta(n) = \psi(\alpha^n)$. Assume that $\forall \delta \in [0,1] \exists \gamma \in 2^\omega[\gamma^n \in H_{\alpha^{1n}}]$. Then $\forall \gamma \in 2^\omega \exists n[\gamma^n \in H_{\alpha^{1n}}]$ and $\forall \gamma \in 2^\omega \exists n[\gamma^n \in G_{\beta^{1n}}]$.

15The argument may be compared to the arguments for Lemma [33.1] and for Lemma [33.4].
16The argument may be compared to the argument for the first half of Theorem [33.3].
Let \( n \) be given such that such that \( 2^\omega \subseteq \mathcal{G}_\beta^\omega = \mathcal{G}_{\sigma(\alpha^n)} \). Note that \( \forall \gamma \in 2^{\omega \setminus \sigma} \cap \mathcal{H}_{\alpha^\omega} \). Conclude that \([0,1] \subseteq \mathcal{H}_{\alpha^\omega} \).

(iii) We prove in BIM that, for each \( \alpha \), there exists \( \beta \) such that
\[
\forall \gamma \in 2^\omega \exists n ((n, \gamma(n)) \in E_\alpha) \rightarrow \forall \delta \in [0,1] \exists n [\delta^\omega \in \mathcal{H}_\beta^\omega] \text{ and }
\exists n ((0,1) \subseteq \mathcal{H}_\beta^\omega) \rightarrow \exists n \forall i < 2((n, i) \in E_\alpha).
\]
Let \( \alpha \) be given. Define \( \beta \) such that \( \forall \forall \forall s \in \mathcal{B}^\omega (s) \neq 0 \leftrightarrow \exists i < \mathcal{S} \{(\alpha(i) = (n, 0) + 1 \land s^{(i \land < 1)} < 1^i) \vee (\alpha(i) = (n, 1) + 1 \land 0_q < s^{(i \land < 1)})\} \).

Note that \( \forall \forall \forall \{(n, 0) \in E_\alpha \leftrightarrow [0, 1) \subseteq \mathcal{H}_\beta^\omega \} \land (\{ (n, 1) \in E_\alpha \leftrightarrow (0, 1) \subseteq \mathcal{H}_\beta^\omega \}) \).
Assume that \( \forall \forall \forall \gamma \in 2^{\omega \setminus \delta} \{(n, \gamma(n)) \in E_\alpha \} \), and \( \delta \in [0,1]^\omega \). Define \( \varepsilon \) such that \( \forall \forall \forall \varepsilon (n) = \mu \mu [0_q < (\delta^\omega (m))] \leftrightarrow (\delta^\omega (m))^{(i \land < 1)} < 1^i \). Define \( \gamma \) in \( 2^\omega \) such that \( \forall \forall \forall \gamma(n) = 0 \leftrightarrow (\delta^\omega (\varepsilon(n)))^{(i \land < 1)} < 1^i \). Note that \( \forall \forall \forall \gamma(n) = 1 \rightarrow 0 \subseteq \delta^\omega \). Find \( n \) such that \( (n, \gamma(n)) \in E_\alpha \) and conclude that either \( \gamma(n) = 0 \) and \( \delta^\omega \subseteq \delta^\omega \) and \( (n, 1) \subseteq \mathcal{H}_\beta^\omega \), or \( \gamma(n) = 1 \) and \( 0 \subseteq \delta^\omega \). Conclude that \( \forall \forall \forall \exists n [\delta^\omega \in \mathcal{H}_\beta^\omega] \).

Let \( \alpha \) be given such that \([0,1] \subseteq \mathcal{H}_\beta^\omega \). Conclude that \( \forall i < 2((n, i) \in E_\alpha) \). \( \square \)

**Theorem 7.2.** (i) \( \text{BIM} \vdash \text{FT} \leftrightarrow \Sigma^0_1 \text{AC}_{\omega,2^\omega} \leftrightarrow \Sigma^0_1 \text{AC}_{\omega, [0,1]} \).
(ii) \( \text{BIM} \vdash \neg \text{FT} \leftrightarrow \neg (\Sigma^0_1 \text{AC}_{\omega,2^\omega}) \leftrightarrow \neg (\Sigma^0_1 \text{AC}_{\omega, [0,1]}) \).

**Proof.** These statements follow from Lemmas 7.1 and 5.3 \( \square \)

## 8. ON THE CONTRAPOSITION OF TWOFOLD COMPACT CHOICE

We introduce a limited version of \( \Sigma^0_1 \text{AC}_{\omega,2^\omega} \):

8.1. \( \Sigma^0_1 \text{AC}_{2^\omega}: \forall \alpha [\forall \forall \forall \gamma \in 2^\omega \exists i < 2 | \gamma | \subseteq G_{\alpha} ] \rightarrow \exists i < 2 [ 2^\omega \subseteq G_{\gamma} ] \).

This statement should be called the \( \Sigma^0_1 \text{AC}_{\omega} \) of \textit{Reverse Twofold Compact Choice}. It is a contraposition of a special case of the following scheme:

\[ \forall i < 2 \exists \gamma \in 2^\omega [ R(i, \gamma) ] \rightarrow \exists \gamma \in 2^\omega \forall i < 2 [ R(i, \gamma) ] \]

and the latter scheme is provable in BIM.

For each \( \alpha \), we define the following statement, called \textit{LLPO}^{\alpha}:

\[ \forall \forall \forall [ \forall \forall [ 2p = \mu \mu [ \varepsilon (m) \neq 0 ] \rightarrow \text{Bar}_{2^\omega} (D_{\mathcal{P}}) ] \rightarrow \text{Bar}_{2^\omega} (D_{\mathcal{P}}) ] \leftrightarrow \text{LLPO}^{\alpha} \]

**Lemma 8.1.** (i) \( \text{BIM} \vdash \text{LLPO} \leftrightarrow \forall \alpha [ \text{LLPO}^{\alpha} ] \).
(ii) \( \text{BIM} + \Sigma^0_1 \text{AC}_{2^\omega} \vdash \forall \alpha [ \text{Bar}_{2^\omega} (D_{\alpha}) \rightarrow \text{LLPO}^{\alpha} ] \).
(iii) \( \text{BIM} + \Pi^0_1 \text{AC}_{\omega,2^\omega} \vdash \forall \alpha [ \text{Bar}_{2^\omega} (D_{\alpha}) \rightarrow \text{LLPO}^{\alpha} ] \rightarrow \text{FT} \).
(iv) \( \text{BIM} \vdash \text{FT} \rightarrow \Sigma^0_1 \text{AC}_{2^\omega} \).

**Proof.** (i) Assume LLPO and let \( \alpha, \varepsilon \) be given. \textit{Either} \( \forall \forall [ 2p = \mu \mu [ \varepsilon (m) \neq 0 ] ] \) and, therefore, \( \forall \forall [ 2p = \mu \mu [ \varepsilon (m) \neq 0 ] \rightarrow \text{Bar}_{2^\omega} (D_{\mathcal{P}}) ] \), \textit{or} \( \forall \forall [ 2p + 1 \neq \mu \mu [ \varepsilon (m) \neq 0 ] ] \) and \( \forall \forall [ 2p + 1 = \mu \mu [ \varepsilon (m) \neq 0 ] \rightarrow \text{Bar}_{2^\omega} (D_{\mathcal{P}}) ] \). We thus see: LLPO^{\alpha}.

For the converse, note that LLPO \( \leftrightarrow \text{LLPO}_{\omega}^{\mathcal{P}} \).

(ii) Let \( \alpha \) be such that \( \text{Bar}_{2^\omega} (D_{\alpha}) \). Using \( \Sigma^0_1 \text{AC}_{2^\omega} \), we will prove LLPO^{\alpha}. Let \( \varepsilon \) be given. Define \( \eta \) such that, for each \( p \),

\begin{enumerate}
  \item if \( \bar{p}(2p + 2) \subset \varepsilon \), then \( \eta^{\alpha}(p) = \eta^{\alpha}(p) = \alpha(p) \), and,
  \item if \( 2p = \mu \mu [ \varepsilon (m) \neq 0 ] \), then \( \forall \forall [ \eta^{\alpha}(m) = 0 \land \eta^{\alpha}(m) = \alpha(m) ] \), and
  \item if \( 2p + 1 = \mu \mu [ \varepsilon (m) \neq 0 ] \), then \( \forall \forall [ \eta^{\alpha}(m) = 0 \land \eta^{\alpha}(m) = \alpha(m) ] \).
\end{enumerate}
Note that, if \( \eta^0 \neq \alpha \), then \( \eta^1 = \alpha \).

Let \( \gamma \) in \( 2^\omega \) be given. Find \( n \) such that \( \alpha (\gamma(n)) \neq 0 \). Either \( \eta^0 (\gamma(n)) = \alpha (\gamma(n)) \neq 0 \), or \( \eta^0 \neq \alpha \) and \( \eta^1 = \alpha \) and \( \exists m (\eta^1 (\gamma(m)) = \alpha (\gamma(m)) \neq 0) \). We thus see that \( \forall \gamma \in 2^\omega [\eta^0 (\gamma(n)) = \alpha (\gamma(n)) \neq 0] \lor \forall \gamma \in 2^\omega [\eta^1 (\gamma(m)) = \alpha (\gamma(m)) \neq 0] \). Use \( S_1^0 \vdash \text{AC}_{2,2^\omega} \) and find \( i < 2 \) such that \( 2^\omega \subseteq \mathcal{G}_{\eta^i} \), i.e. \( Bar_2 (D_{\eta^i}) \). Let \( p \) be given such that \( 2p + i = \mu m (\varepsilon (m) \neq 0) \). Note that \( \forall m \geq p (\eta^1 (m)) \neq 0 \). Conclude that \( Bar_2 (D_{\eta^p}) \) and that \( Bar_2 (D_{\mathcal{G}_{\eta^i}}) \).

We thus see that \( \forall \gamma (2p + i = \mu m (\varepsilon (m) \neq 0) \rightarrow Bar_2 (D_{\mathcal{G}_{\eta^p}})) \). Conclude that \( \exists i < 2p (2p + i = \mu m (\varepsilon (m) \neq 0) \rightarrow Bar_2 (D_{\mathcal{G}_{\eta^p}})) \), i.e. \( LLPO^\alpha \).

(iii) Assume \( \forall \alpha [Bar_2 (D_{\alpha}) \rightarrow LLPO^\alpha] \). Using \( \Pi^0_1 \text{-AC}_{\omega,2} \), we will prove \( \text{FT} \).

Let \( \alpha \) be given such that \( Bar_2 (D_{\alpha}) \) and, therefore, \( LLPO^\alpha \). Let \( s \in \delta^\omega \) be given. Define \( \varepsilon \) such that, \( \forall \varepsilon (2n + i) \neq 0 \leftrightarrow Bar_2 (\gamma^{(i)} (D_m)) \). Using \( LLPO^\alpha \), find \( i < 2 \) such that \( \forall \gamma (2n + i) \neq 0 \rightarrow Bar_2 (D_{\gamma^{(i)}}) \). Assume we find \( n \) such that \( Bar_2 (\gamma^{(i)} (D_m)) \). Then \( \varepsilon (2n + i) \neq 0 \). Find \( p = \mu j (\varepsilon (j) \neq 0) \).

Find \( q \leq n \) such that \( p = 2q \) or \( p = 2q + 1 \). Either \( p = 2q + i \) and \( Bar_2 (D_{\gamma^{(i)}}) \), or \( p = 2q + 1 - i \) and \( Bar_2 (\gamma^{(i)} (D_m)) \). In both cases, \( Bar_2 (\gamma^{(i)} (D_m)) \). We thus see that \( \forall n [Bar_2 (\gamma^{(i)} (D_m)) \rightarrow Bar_2 (\gamma^{(i)} (D_m)) \). Conclude that \( \forall \gamma (2n + i) \neq 0 \rightarrow Bar_2 (D_{\gamma^{(i)}}) \). Now use \( \Pi^0_1 \text{-AC}_{\omega,2} \) and find \( \gamma \) in \( 2^\omega \) such that \( \forall \gamma (2n + i) \neq 0 \rightarrow Bar_2 (\gamma^{(i)} (D_m)) \). Observe that, for each \( s \) in \( \delta^\omega \), for all \( n \), if \( Bar_2 (\gamma^{(i)} (D_m)) \), then also \( Bar_2 (\gamma^{(i)} (D_m)) \). And, therefore, \( Bar_2 (\gamma^{(i)} (D_m)) \). Define \( \delta \) in \( 2^\omega \) such that, for each \( n \), \( \delta(n) = \gamma (\delta(n)) \). Find \( p \) such that \( \alpha (\delta p) \neq 0 \) and define \( n = \delta p + 1 \). Note that \( Bar_2 (\gamma^{(i)} (D_m)) \). One now proves, by backwards induction, that, for each \( j < p \), \( Bar_2 (\gamma^{(i)} (D_m)) \). The induction step goes as follows. Assume \( j + 1 \leq n \) and \( Bar_2 (\gamma^{(i)} (D_m)) \). As \( \delta (j + 1) = \delta j * (\gamma (\delta j)) \), one conclude that \( Bar_2 (\gamma^{(i)} (D_m)) \). After \( n \) steps one sees \( Bar_2 (\gamma^{(i)} (D_m)) \). Conclude that \( \forall \alpha [Bar_2 (D_{\alpha}) \rightarrow \exists \eta [Bar_2 (D_{\eta^p})]] \), i.e. \( \text{FT} \).

(iv) Assume \( \text{FT} \). Use Theorem 7.2 and conclude \( \Sigma^0_1 \text{-AC}_{\omega,2^\omega} \) and its corollary \( \Sigma^0_1 \text{-AC}_{2,2^\omega} \).

**Theorem 8.2.** \( \text{BIM} + \Pi^0_1 \text{-AC}_{\omega,2} \vdash \Sigma^0_1 \text{-AC}_{2,2^\omega} \leftrightarrow \text{FT} \).

*Proof.* Use Lemma 8.1. □

8.1. Mark Bickford called my attention to the fact that \( \Sigma^0_1 \text{-AC}_{2,2^\omega} \) occurs in [22] §2 and is called there the *separation principle* \( \text{SP} \).

After having proved \( \text{FT} \rightarrow \Sigma^0_1 \text{-AC}_{2,2^\omega} \), see our Lemma 8.1 (iv), the author of [22] gives a proof of \( \text{FT} \rightarrow \text{WKL} \) using \( \Pi^0_1 \text{-AC}_{\omega,\omega} \) and quotes the result \( \text{WKL} \rightarrow \text{FT} \), see our Theorem 2.7. Our proof of Theorem 2.8 shows that no choice is needed for a proof of \( \text{FT} \rightarrow \text{WKL} \).

We introduce a 'real' version of \( \Sigma^0_1 \text{-AC}_{2,2^\omega} \):

\[ \forall \alpha [\forall \delta (\alpha (\delta^\omega) < 2^\delta \exists i \in \mathcal{H}_{\alpha^{(i)}}] \rightarrow \exists i < 2^\alpha \in \mathcal{H}_{\alpha^{(i)}}] \]

**Theorem 8.3.** \( \text{BIM} \vdash \Sigma^0_1 \text{-AC}_{2,2^\omega} \leftrightarrow \Sigma^0_1 \text{-AC}_{2,2^\omega} \).

*Proof.* Using Lemma 13.2, find \( \sigma : 2^\omega \rightarrow [0,1] \) and \( \psi : \omega^\omega \rightarrow \omega^\omega \) such that \( \forall \delta \in [0,1] [\exists \varepsilon (\delta^\omega) \in \mathcal{H}_{\alpha^{(i)}}] \rightarrow \exists i < 2^\alpha \in \mathcal{H}_{\alpha^{(i)}}] \).

First assume \( \Sigma^0_1 \text{-AC}_{2,2^\omega} \). Let \( \alpha \) be given such that \( \forall \delta \in [0,1] [\exists i < 2^\delta \in \mathcal{H}_{\alpha^{(i)}}] \). Define \( \beta \) such that, for both \( i < 2, \beta^{(i)} = \psi (\alpha^{(i)}) \). Then \( \forall \gamma (2^\delta \exists i < 2^\gamma \in \mathcal{H}_{\alpha^{(i)}}] \). Conclude that \( \gamma (2^\omega \exists i < 2^\gamma \in \mathcal{H}_{\alpha^{(i)}}] \).

Confirm that \( \exists i < 2^\omega (2^\delta \exists i < 2^\gamma \in \mathcal{G}_{\beta^{(i)}}] \). Find \( i < 2 \) such that \( 2^\omega \subseteq \mathcal{G}_{\alpha^{(i)}} \). Conclude that \( \exists i < 2^\omega \in \mathcal{H}_{\alpha^{(i)}} \). We thus see \( \Sigma^0_1 \text{-AC}_{2,2^\omega} \).
Now assume \( \Sigma^0_1\text{AC}_{2,[0,1]} \). Using Lemma 13.3, find \( \tau : 2^\omega \rightarrow [0,1] \) and \( \chi : \omega^\omega \rightarrow \omega^\omega \) such that

1. \( \forall \gamma \in 2^\omega \forall \delta \in 2^{[\gamma \# \delta \rightarrow \tau(\gamma) \# \tau(\delta)]}, \) and
2. \( \forall \alpha \forall \gamma \forall \beta \in 2^\gamma \forall \gamma \in \mathcal{G}_\alpha \leftrightarrow \tau(\gamma) \in \mathcal{H}_{\chi(\alpha)} \) and
3. \( \forall \delta \in [0,1] \exists \gamma \in 2^\delta \forall \tau(\gamma) \rightarrow \forall \alpha [\delta \in \mathcal{H}_{\chi(\alpha)}] \).

Let \( \alpha \) be given such that \( \forall \gamma \in 2^{3 \exists i < 2[\gamma \in H_{\mathcal{G}_\alpha}]} \). Let \( \delta \) in \([0,1]^2\) be given. Find \( \gamma \) in \(2^\omega\) such that \( \forall i < 2[\delta(\gamma) \# \gamma \rightarrow \tau(\gamma) \in H_{\mathcal{H}_{\chi(\alpha)}}] \). Find \( i < 2 \) such that \( \gamma \in H_{\mathcal{G}_\alpha} \). Let \( s, n \) such that \( (\chi(\gamma))s \neq 0 \)
and \( (\tau(\gamma))s \subseteq s \). Using Lemma 13.3, find \( p \) such that either \( \delta(\gamma)[p] \subseteq s \), and
\( \delta(\gamma)[p] \# \gamma \rightarrow \tau(\gamma))s \), and again, \( \delta(\gamma) \in H_{\mathcal{H}_{\chi(\alpha)}} \).
We see that \( \forall \delta \in [0,1]^2 \exists i < 2[\delta(\gamma) \in H_{\mathcal{H}_{\chi(\alpha)}}] \).
Applying \( \Sigma^0_1\text{AC}_{2,[0,1]} \), find \( i < 2 \) such that \( [0,1] \subseteq H_{\mathcal{H}_{\chi(\alpha)}} \). Conclude that \( \forall \gamma \in 2^\omega \tau(\gamma) \in H_{\mathcal{H}_{\chi(\alpha)}} \) and \( 2^\omega \subseteq \mathcal{G}_\alpha \). We thus see \( \Sigma^0_1\text{AC}_{2,[0,1]} \).

**Corollary 8.4.** \( \text{BIM} + \Pi^0_1\text{AC}_{\omega,2} \vdash \text{FT} \iff \Sigma^0_1\text{AC}_{2,[0,1]} \).

**Proof.** Use Theorems 8.2 and 8.3. \( \square \)

### 8.3. Some logical consequences.

In this Subsection, we want to formulate the result of Theorem 8.2 in model-theoretic terms and draw an even sharper conclusion.

Tarski’s truth definition makes sense intuitionistically as well as classically. For every structure \( \mathbb{A} = (A,...) \), for every formula \( \varphi = \varphi(x_0,x_1,...,x_{n-1}) \) in the elementary language of the structure \( \mathbb{A} \), for all \( a_0,a_1,...,a_{n-1} \) in \( A \),

\[ \mathbb{A} \models \varphi[a_0,a_1,...,a_{n-1}] \]

if and only if

the formula \( \varphi \) is true in the structure \( \mathbb{A} \), provided we interpret the individual variables \( x_0,x_1,...,x_{n-1} \) by \( a_0,a_1,...,a_{n-1} \), respectively.

For every structure \( \mathbb{A} = (A,...) \), for every sentence \( \varphi \) in the elementary language of the structure \( \mathbb{A} \),

\[ \mathbb{A} \models \varphi \]

if the sentence \( \varphi \) is true in the structure \( \mathbb{A} \).

For every \( \delta \), we define a proposition \( Pr_\delta \), as follows. \( Pr_\delta := \exists n[\delta(n) \neq 0] \).

**Theorem 8.5.** The following statements are equivalent in \( \text{BIM} \):

1. \( \forall \alpha \forall i < 2[\mathcal{G}_{\alpha(i)} \models \forall x[P(x) \lor A] \rightarrow (\forall x[P(x)] \lor A)] \).
2. \( \forall \alpha \forall i < 2[\mathcal{G}_{\alpha(i)} \models \forall x[P(x) \lor Q(x)] \rightarrow \forall x[P(x)] \lor \exists y Q(x)] \).

**Proof.** (i) \( \Rightarrow \) (ii).

Let \( \alpha \) be given such that \( 2^\omega \models \forall \alpha \forall i < 2[\mathcal{G}_{\alpha(i)} \models \forall x[P(x) \lor A]] \).

Note that \( (2^\omega, \mathcal{G}_{\alpha(i)}, \mathcal{G}_{\alpha(i)}) \models \forall x[P(x) \lor Q(x)] \).
Define \( \beta \) such that \( \beta(\gamma) = \alpha(\gamma) \)
and \( \forall n[\beta(n) \neq 0 \leftrightarrow (\alpha(n) \neq 0 \land n \in 2^{<\omega})] \).
Note that \( 2^\omega, \mathcal{G}_{\bar{\beta}(i)}, \mathcal{G}_{\bar{\beta}(i)} \models \forall x[P(x) \lor A] \).
Using (i), conclude that \( 2^\omega, \mathcal{G}_{\bar{\beta}(i)}, \mathcal{G}_{\bar{\beta}(i)} \models \forall x[P(x)] \lor A \) and
that \( (2^\omega, \mathcal{G}_{\alpha(i)}, \mathcal{G}_{\alpha(i)}) \models \forall x[P(x)] \lor \exists y Q(x)] \).

(ii) \( \Rightarrow \) (i).

Let \( \alpha \) be given such that \( 2^\omega, \mathcal{G}_{\alpha(i)}, \mathcal{G}_{\alpha(i)} \models \forall x[P(x) \lor A] \).
Define \( \beta \) such that \( \beta(\gamma) = \alpha(\gamma) \) and \( \forall n[\beta(n) \neq 0 \leftrightarrow (\exists i < n[\alpha(i) \neq 0] \land n \in 2^{<\omega})] \).
Note that, for each \( \gamma \) in \( 2^\omega \), \( \beta \) in \( \mathcal{G}_{\alpha(i)} \) if and only if \( Pr_{\alpha(i)} \). Conclude that \( 2^\omega, \mathcal{G}_{\alpha(i)}, \mathcal{G}_{\alpha(i)} \models \forall x[P(x) \lor Q(x)] \) and, using (ii), that \( 2^\omega, \mathcal{G}_{\alpha(i)}, \mathcal{G}_{\alpha(i)} \models \forall x[P(x)] \lor \exists y Q(x)] \) and that \( (2^\omega, \mathcal{G}_{\alpha(i)}, \mathcal{G}_{\alpha(i)} \models \forall x[P(x)] \lor A \).

For each \( \alpha \), we define the following statement:

\( \text{LPO}^\alpha : \forall \varepsilon \exists y \exists p(2p+2) \varepsilon \rightarrow \text{Bar}_2(\text{D}_{2p}) \lor \exists n[\varepsilon(n) \neq 0] \).

**Lemma 8.6.** (i) \( \text{BIM} \vdash \text{LPO} \rightarrow \forall \alpha \text{LPO}^\alpha \).
(ii) $\text{BIM} \vdash \forall \alpha [\text{LPO}^\alpha \rightarrow \text{LLPO}^\alpha]$.

Proof. The proof is left to the reader. $\square$

**Theorem 8.7.** The following statements are equivalent in $\text{BIM} + \Pi^0_1^-\text{AC}_{\omega,2}$:

(i) FT.

(ii) $\forall \alpha (\exists \gamma (\forall \beta [\text{LPO}^\beta \rightarrow \text{LLPO}^\beta]), \forall \gamma (\forall x (P(x) \lor Q(y)) \rightarrow (\forall x (P(x) \lor \forall y (Q(y))))].$

(iii) $\forall \alpha (\exists \gamma (\forall \beta [\text{LPO}^\beta \rightarrow \text{LLPO}^\beta]), \forall \gamma (\forall x (P(x) \lor A) \rightarrow (\forall x (P(x) \lor A))).$

(iv) $\forall \alpha [\text{Bar}_2 (D_\alpha) \rightarrow \text{LPO}^\alpha]$.

Proof. (i) $\Rightarrow$ (ii). Note that, in $\text{BIM} + \Pi^0_1^-\text{AC}_{\omega,2}$, FT implies $\Sigma^1_0\Delta^1_2$:

$\forall \alpha (\exists \gamma (\forall \beta [\text{LPO}^\beta \rightarrow \text{LLPO}^\beta]), \forall \gamma (\forall x (P(x) \lor A) \rightarrow (\forall x (P(x) \lor A))].$

(ii) $\Rightarrow$ (iii). Let $\alpha$ be given such that $(\forall \beta [\text{LPO}^\beta \rightarrow \text{LLPO}^\beta])$, i.e. $\forall \gamma (\forall x (P(x) \lor A) \rightarrow (\forall x (P(x) \lor A)).$ Define $\beta$ such that $\beta_0 = \alpha_0$ and $\exists \gamma (\forall x (P(x) \lor Q(y)) \land (\forall \beta [\text{LPO}^\beta \rightarrow \text{LLPO}^\beta]).$

(iii) $\Rightarrow$ (iv). Assume (iii).

Let $\alpha$ be given such that $\text{Bar}_2 (D_\alpha)$. We have to prove $\text{LPO}^\alpha$. Let $\varepsilon \in \alpha$ be given. Define $\eta$ in $2^\alpha$ such that $\eta_\varepsilon = \varepsilon$, and, for each $p$, if $\exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A)).$ Then, $\exists \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A).$

(iv) $\Rightarrow$ (i). Use Lemma 8.6(iii) and Theorem 8.2(iii).

The sentences mentioned in Theorems 8.3 and 8.7 are true in every structure $\{(0, 1, \ldots, n), P, Q, A\}$ where $n$ is a natural number, $P, Q$ are arbitrary subsets of $\{0, 1, \ldots, n\}$ and $A$ is an arbitrary proposition, that is, these sentences hold in every finite structure. They sometimes fail to be true in countable structures, as appears from the next two theorems.

**Theorem 8.8.** The following statements are equivalent in $\text{BIM}$.

(i) LLPO.

(ii) $\forall \alpha (\exists \gamma (\forall x (P(x) \lor A) \rightarrow (\forall x (P(x) \lor A)).$

Proof. (i) $\Rightarrow$ (ii). Let $\alpha$ be given. Define $\beta$ such that $\forall \gamma (\forall x (P(x) \lor A) \rightarrow (\forall x (P(x) \lor A))].$

(ii) $\Rightarrow$ (i). Assume (ii). Let $\alpha$ be given. Define $\beta$ such that, for each $p$, if $\exists \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A).$

In both cases $(\forall \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A).$ Conclude that $(\forall \gamma (\forall x (P(x) \lor A) \lor \exists x_i < 2^{\forall \gamma (\forall x (P(x) \lor A).$
Theorem 8.9. The following statements are equivalent in BIM.

(i) LPO.
(ii) \( \forall n[\omega, D_{\alpha^{[0]}}, D_{\alpha^{[1]}}] \models \forall x[P(x) \lor Q(x)] \rightarrow (\forall x[P(x)] \lor \exists x[Q(x)]) \).
(iii) \( \forall n[\omega, D_{\alpha^{[0]}}, Pr_{\alpha^{[1]}}] \models \forall x[P(x) \lor A] \rightarrow (\forall x[P(x)] \lor A) \).

Proof. (i) \( \Rightarrow \) (ii). Let \( \alpha \) be given such that \( \omega, D_{\alpha^{[0]}}, D_{\alpha^{[1]}} \models \forall x[P(x) \lor Q(x)] \), i.e. \( \forall n[\alpha^{[0]}(n) \neq 0 \lor \alpha^{[1]}(n) \neq 0] \). Using LPO, distinguish two cases. Either \( \forall n[\alpha^{[0]}(n) \neq 0] \) and \( \omega, D_{\alpha^{[0]}}, D_{\alpha^{[1]}} \models \forall x[P(x)] \), or \( \exists n[\alpha^{[0]}(n) = 0] \) and \( \exists n[\alpha^{[1]}(n) \neq 0] \) and \( \omega, D_{\alpha^{[0]}}, D_{\alpha^{[1]}} \models \exists x[Q(x)] \).

(ii) \( \Rightarrow \) (i). Let \( \alpha \) be given. Define \( \beta \) such that \( \forall n[\beta^{[0]}(n) = 0 \leftrightarrow \alpha(n) \neq 0] \) and \( \beta^{[1]} = \alpha \). Note that \( \forall n[\beta^{[0]}(n) \neq 0 \land \beta^{[1]}(n) \neq 0] \), i.e. \( \omega, D_{\beta^{[0]}}, D_{\beta^{[1]}} \models \forall x[P(x) \lor Q(x)] \). Use (ii) and distinguish two cases. Either \( \omega, D_{\beta^{[0]}}, D_{\beta^{[1]}} \models \forall x[P(x)] \) and \( \forall n[\alpha(n) = 0] \), or \( \omega, D_{\beta^{[0]}}, D_{\beta^{[1]}} \models \exists x[Q(x)] \) and \( \exists n[\alpha(n) \neq 0] \). Conclude \( \forall n[\forall \alpha(n) = 0 \lor \exists n[\alpha(n) \neq 0]] \), i.e. LPO.

(i) \( \Rightarrow \) (iii). Let \( \alpha \) be given such that \( \omega, D_{\alpha^{[0]}}, Pr_{\alpha^{[1]}} \models \forall x[P(x) \lor A] \), i.e. \( \forall n[\alpha^{[0]}(n) \neq 0 \lor \exists m[\alpha^{[1]}(m) \neq 0] \). Using LPO, distinguish two cases. Either \( \forall n[\alpha^{[0]}(n) \neq 0] \) and \( \omega, D_{\alpha^{[0]}}, Pr_{\alpha^{[1]}} \models \forall x[P(x)] \), or \( \exists n[\alpha^{[0]}(n) = 0] \) and \( \exists n[\alpha^{[0]}(n) \neq 0] \) and \( \exists n[\alpha^{[1]}(m) \neq 0] \) and \( \omega, D_{\alpha^{[0]}}, Pr_{\alpha^{[1]}} \models A \).

(iii) \( \Rightarrow \) (i). Let \( \alpha \) be given. Define \( \beta \) such that \( \forall n[\beta^{[0]}(n) = 0 \leftrightarrow \alpha(n) \neq 0] \). Note that \( \forall n[\beta^{[0]}(n) \neq 0 \land \alpha(n) \neq 0] \), i.e. \( \omega, D_{\beta^{[0]}}, Pr_{\alpha^{[1]}} \models \forall x[P(x)] \lor A \). Use (iii) and distinguish two cases. Either \( \omega, D_{\beta^{[0]}}, Pr_{\alpha^{[1]}} \models \forall x[P(x)] \) and \( \forall n[\beta(n) \neq 0] \) and \( \forall n[\alpha(n) = 0] \), or \( \omega, D_{\beta^{[0]}}, Pr_{\alpha^{[1]}} \models A \) and \( \exists n[\alpha(n) \neq 0] \). Conclude \( \forall n[\forall \alpha(n) = 0 \lor \exists n[\alpha(n) \neq 0]] \), i.e. LPO. \( \square \)

8.4. A note. According to Theorem 8.7(iii), BIM + \( \Pi_1^0 \)-AC\( \omega \cdot 2 \) proves that FT is equivalent to:

For all \( \alpha \), if \( \forall \gamma \in 2^\omega \exists n[\alpha^{[0]}(\gamma n) \neq 0 \lor \alpha^{[1]}(n) \neq 0] \), then either \( \forall \gamma \in 2^\omega \exists n[\alpha^{[0]}(\gamma n) \neq 0] \lor \exists n[\alpha^{[1]}(n) \neq 0] \).

One might ask if BIM + \( \Pi_1^0 \)-AC\( \omega \cdot 2 \) proves that \( \neg \forall \text{FT} \) is equivalent to the following statement:

There exists \( \alpha \) such that \( \forall \gamma \in 2^\omega \exists n[\alpha^{[0]}(\gamma n) \neq 0 \lor \alpha^{[1]}(n) \neq 0] \) and
\( \neg \forall \gamma \in 2^\omega \exists n[\alpha^{[0]}(\gamma n) \neq 0] \) and \( \alpha^{[1]}(n) \neq 0 \).

i.e.

\((*)\): There exists \( \alpha \) such that
\( \forall \gamma \in 2^\omega \exists n[\alpha(\gamma n) \neq 0] \) and \( \neg \forall \gamma \in 2^\omega \exists n[\alpha(\gamma n) \neq 0] \).

The formula \( (*) \) is an outright contradiction!

9. The determinacy of finite and infinite games

We consider games of perfect information for players I, II. First finite games, then games with finitely many moves where the players may choose out of infinitely many alternatives, and then games of infinite length. In the last Subsection we prove that, in BIM, FT is an equivalent of the Intuitionistic Determinacy Theorem. This theorem says that every subset of \( \omega \times 2^\omega \) is weakly determinate.

9.1. Finite games.

9.1.1. Finite Choice and a contraposition of Finite Choice.

Lemma 9.1. BIM proves the following scheme:

\[ \forall m[\forall n < m[A(n) \lor B)] \rightarrow (\forall n < m[A(n)] \lor B) \]

Proof. The proof is straightforward, by induction. \( \square \)

Lemma 9.2. BIM proves the following schemes:
(i) $\forall k[\forall n < k \exists m[R(n, m)] \rightarrow \exists s \forall n < k[R(n, s(n))]]$.
(ii) $\forall k[\forall i : k \rightarrow \exists n < k[R(n, s(n))]] \rightarrow \exists n < k \forall m < l[R(n, m)]$.

Proof. (i) The proof is by induction on $k$ and left to the reader.

(ii) The proof is by induction on $k$ and uses Lemma 9.1. Note that there is nothing to prove if $k = 0$. Now assume the statement holds for a certain $k$.

Assume $\forall s : (k + 1) \rightarrow \exists n < k + 1[R(n, s(n))]$. Note that

$\forall j < l \forall s : k \rightarrow l[\exists n < k[R(n, s(n))] \lor R(k, j)]$,

and, therefore, by Lemma 9.1

$\forall j < l[R(k, j) \lor \forall s : k \rightarrow l[\exists n < k[R(n, s(n))]])$.

Using the induction hypothesis, we conclude that

$\forall j < l[R(k, j) \lor \exists n < k \forall m < l[R(n, m)],$

i.e. $\exists n < k + 1 \forall m < l[R(n, m)]$. $\square$

9.1.2. Finite games. We want to study finite and infinite games for players $I$ and $II$ of perfect information. We first consider finite games. In such games, there are finitely many moves, and for each move there are only finitely many alternatives.

Assume $X \subseteq \omega$. Let $n, l$ be given such that $l > 0$.

Players $I$ and $II$ play the $I$-game for $X$ in $\text{Seq}(n, l)$ and the $II$-game for $X$ in $\text{Seq}(n, l)$ in the same way, as follows. First, player $I$ chooses $i_0 < l$, then player $II$ chooses $i_1 < l$, and they continue until a finite sequence $\langle i_0, i_1, \ldots, i_{n-1} \rangle$ of length $n$ has been formed, a so-called final position. Player $I$ wins the play $\langle i_0, i_1, \ldots, i_{n-1} \rangle$ in the $I$-game for $X$ if and only if $\langle i_0, i_1, \ldots, i_{n-1} \rangle \in X$. Player $II$ wins the play $\langle i_0, i_1, \ldots, i_{n-1} \rangle$ in the $II$-game for $X$ if and only if $\langle i_0, i_1, \ldots, i_{n-1} \rangle \in X$.

We define $\varphi$ such that, for all $n$, for all $l > 0$,

$\varphi(n, l) := \mu a \forall i < n \forall c \in \text{Seq}(i, l)[c < a]$.

Every number coding a position in $\text{Seq}(n, l)$ that is not a final position is smaller than $\varphi(n, l)$.

We define $\psi$ such that, for all $n$, for all $l > 0$,

$\psi(n, l) := \mu a \forall s \in \text{Seq}(\varphi(n, l), l)[s < a]$.

When studying games in $\text{Seq}(n, l)$ it suffices to consider strategies $s, t$ for player $I, II$, respectively, such that $s, t < \psi(n, l)$.

The reader should consult Subsection 13.10 for some of the notations we are going to use, like ‘$\in_I$’ and ‘$\in_{II}$’.

We define: $X \subseteq \omega$ is $I$-determinate in $\text{Seq}(n, l)$, $\text{Det}_I^{\text{Seq}(n, l)}(X)$, if and only if $\forall t < \psi(n, l) \exists c \in_I t \land c \in X \rightarrow \exists s \forall c \in \text{Seq}(n, l)[c \in_I s \rightarrow c \in X]$,

and: $X \subseteq \omega$ is $II$-determinate in $\text{Seq}(n, l)$, $\text{Det}_{II}^{\text{Seq}(n, l)}(X)$, if and only if $\forall s < \psi(n, l) \exists c \in_{II} s \land c \in X \rightarrow \exists t \forall c \in \text{Seq}(n, l)[c \in_{II} t \rightarrow c \in X]$.

Theorem 9.3 (Determinacy of finite games). Let $X \subseteq \omega$ be given.

For every $l > 0$, for every $n$, $\text{Det}_I^{\text{Seq}(n, l)}(X)$ and $\text{Det}_{II}^{\text{Seq}(n, l)}(X)$.

Proof. Let $X \subseteq \omega$ and $l > 0$ be given.

For every $u$, we define: $X_u := \{c \in \omega \mid u * c \in X\}$.

We intend to prove a statement seemingly stronger than the statement of the theorem:

$(\ast)$: for every $n$, for every $u$, $\text{Det}_I^{\text{Seq}(n, l)}(X_u)$ and $\text{Det}_{II}^{\text{Seq}(n, l)}(X_u)$.

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The proof is by induction on \( n \). If \( n = 0 \), then, for every \( t \), the statements: ‘\( X_u \) is \( I \)-determinate in Seq(0,1)’ and ‘\( X_u \) is \( II \)-determinate in Seq(0,1)’ both assert: ‘\( \{ \} \in X_u \), then \( \{ \} \in X_u \)’, and thus are obviously true.

Now assume the statement (*) has been established for a certain \( n \). We prove that (*) is also true for \( n + 1 \).

Let \( u \) be given.

First, assume that \( \forall t < \psi(n + 1,1) \exists c \in \{ \} \land c \in X_u \), or, equivalently \( \forall t < \psi(n + 1,1) \exists k \in \{ \} \land \langle k \rangle \ast c \in X_u \). Note that \( \forall k < \psi(n,1) \exists k \in \{ \} \land \langle k \rangle \ast c \in X_u \). Use Lemma 1.2.11) and find \( k < l \) such that \( \forall s < \psi(n,1) \exists c \in \{ \} \land c \in X_u \). Use the induction hypothesis and find \( t < \psi(n,1) \) such that \( \forall c \in \{ \} \land \langle k \rangle \ast c \in X_u \). Define \( s < \psi(n + 1,1) \) such that \( s(\{ \}) = k \) and \( s^{[k]} = t \). Note that \( \forall c \in \{ \} \land \langle k \rangle \ast c \in X_u \). We thus see that \( X_u \) is \( I \)-determinate.

Next, assume that \( \forall s < \psi(n + 1,1) \exists c \in \{ \} \land c \in X_u \), or, equivalently, \( \forall s < \psi(n + 1,1) \exists c \in \{ \} \land (s(\{ \}) = k \land s^{[k]} = t) \). Conclude that \( \forall k < \psi(n,1) \exists c \in \{ \} \land \langle k \rangle \ast c \in X_u \). Note that \( \forall k < \psi(n,1) \exists c \in \{ \} \land \langle k \rangle \ast c \in X_u \). Use Lemma 1.2.1) and find \( t < \psi(n + 1,1) \) such that \( \forall c \in \{ \} \land \langle k \rangle \ast c \in X_u \). Note that \( \forall c \in \{ \} \land \langle k \rangle \ast c \in X_u \). We thus see that \( X_u \) is \( II \)-determinate. \( \square \)

9.1.3. Comparison with the classical theorem. Note that, in classical mathematics, the \( I \)-determinacy of finite games is stated as follows:

For every \( X \subseteq \omega \), for every \( t > 0 \), for every \( n \),

either \( \exists t < \psi(n,1) \forall c \in \{ \} \land c \in X \),

or \( \exists s < \psi(n,1) \forall c \in \{ \} \land c \in X \),

i.e. either player II has a strategy ensuring that the result of the game will not be in \( X \), or player I has a strategy ensuring that it does.

Taken constructively, this statement fails to be true already in the case \( n = 0 \), because it then implies: for every subset \( X \) of \( \{ \{ \} \} \), either \( \{ \} \notin X \) or \( \{ \} \in X \), and therefore, for any proposition \( P \), \( \neg P \lor P \), the principle of the excluded third.

9.2. Infinitely many alternatives.

9.2.1. Infinitely many alternatives for player II.

Let \( X \subseteq 2 \times \omega \) be given. Player I and II play the I-game for \( X \) in \( 2 \times \omega \) in the following way. First, player I chooses \( i < 2 \), then player II chooses \( n \) and the play is finished. Player I wins the play \( \langle i, n \rangle \) if and only if \( \langle i, n \rangle \in X \).

We define: \( X \) is \( I \)-determinate in \( 2 \times \omega \), \( Det^{I}_{2 \times \omega}(X) \), if and only if

\[
\forall t \exists c \in 2 \times \omega \land c \in X \rightarrow \exists s < 2 \forall n[\{s, n\} \in X].
\]

**Theorem 9.4.** \( \text{BIM} \vdash \forall \alpha[Det^{I}_{2 \times \omega}(D_{\alpha})] \leftrightarrow \text{LLPO} \).

**Proof.** This Theorem is a reformulation of Theorem 8.8. In order to see this, make two observations.

(i) For each \( \alpha \), there exists \( \beta \) such that \( Det^{I}_{2 \times \omega}(D_{\alpha}) \leftrightarrow (\omega, D_{\beta}^{\geq 0}, D_{\beta}^{\vee}) \vdash \forall \alpha \forall y[P(x) \lor Q(y)] \rightarrow (\forall x[P(x)] \lor \forall y[Q(y)]) \). Given any \( \alpha \), define \( \beta \) such that, for each \( n \), \( \beta^{10}(n) = \alpha(0, n) \) and \( \beta^{11}(n) = \alpha(1, n) \).

\(^{17}\text{For the notation } t_{[k]} \text{, see Subsection 13.3.} \)
9.2.2. Infinitely many alternatives for player 1.

Let $X ⊆ ω × 2$ be given. Players I and II play the I-game for $X$ in $ω × 2$ in the following way. First, player I chooses a natural number $n$, then player II chooses a number $i$ from $\{0, 1\}$. Player I wins the play $⟨n, i⟩$ if and only if $⟨n, i⟩ ∈ X$.

Note that a strategy for player I in such a two-move-game coincides with his first move and thus is a natural number. A strategy for player II, on the other hand, is an infinite sequence $τ$ in $2^ω$ that expresses player II’s intention to play $τ(⟨n⟩)$ once player I has brought them to the position $⟨n⟩$.

We define: $X ⊆ ω × 2$ is $I$-determinate in $ω × 2$, $Det_{I,ω×2}^I(X)$, if and only if:

$$∀τ ∈ 2^ω∃c ∈ ω × 2[c ∈ IIτ ∧ c ∈ X] → ∃s∀i < 2[(s, i) ∈ X].$$

Theorem 9.5. BIM ⊢ $\forallα[Det_{I,ω×2}^I(D_α)]$.

Proof. Let $α$ be given. Assume that $∀τ ∈ 2^ω∃n[⟨n, τ(⟨n⟩)⟩ ∈ D_α]$. Find $τ$ in $2^ω$ such that $∀n[τ(⟨n⟩)] = 1 ↔ ⟨n, 0⟩ ∈ D_α$. Find $n$ such that $⟨n, τ(⟨n⟩)⟩ ∈ D_α$. Note that $τ(⟨n⟩) = 1$ and $∀i < 2[⟨n, i⟩ ∈ D_α]$. □

We define: $X ⊆ ω × ω$ is $II$-determinate in $ω × ω$, $Det_{II,ω×ω}^{II}(X)$, if and only if:

$$∀m∃n[⟨m, n⟩ ∈ X] → ∃τ∀m[⟨m, τ(⟨m⟩)⟩ ∈ X].$$

Theorem 9.6. BIM ⊢ $\forallα[Det_{II,ω×ω}^{II}(E_α)]$.

Proof. Let $α$ be given such that $∀m∃n[⟨m, n⟩ ∈ E_α]$, that is $∀m∃n[α(p) = ⟨m, n⟩ + 1]$. Find $γ$ such that $∀m[α(γ(⟨m⟩))] = ⟨m, γ''(⟨m⟩)⟩ + 1]$. Define $τ$ such that $∀m[τ(⟨m⟩)] = γ''(⟨m⟩)].$ □

We define: $X ⊆ ω × 2$ is $II$-determinate in $ω × 2$, $Det_{II,ω×2}^{II}(X)$, if and only if:

$$∀m < 2∃n[⟨m, n⟩ ∈ X] → ∃s∀m < 2[⟨m, t(⟨m⟩)⟩ ∈ X].$$

Note that BIM proves the scheme $Det_{II,ω×2}^{II}(X)$.

9.2.3. Longer games. We also consider games in which players I, II make more than one move. Which of those games are determinate from the viewpoint of Player I? Because of Theorem 23, we restrict ourselves to games in which player I has, for each one of his moves, countably many alternatives, whereas player II always has to choose one of two possibilities.

For every $n$, for every $X ⊆ (ω × 2)^n$, we define:

$X$ is $I$-determinate in $(ω × 2)^n$, $Det_{(ω × 2)^n}^I(X)$, if and only if

$$∀τ ∈ 2^n∃c[τ ∈ IIτ ∧ c ∈ X] → ∃s∀c ∈ (ω × 2)^n[c ∈ I s → c ∈ X].$$

This definition extends in the obvious way to subsets $X$ of $(ω × 2)^n × ω$.

9.3. Infinitely many moves. We also want to consider games of infinite length. We imagine players I, II to build together an infinite sequence $γ$ in $ω^ω$, as follows. First, player I chooses $γ(0)$, then player II chooses $γ(1)$, then player I chooses $γ(2)$, and so on.

We define a number of notions of determinacy.

$X ⊆ (ω × 2)^ω$ is $I$-determinate in $(ω × 2)^ω$, $Det_{(ω × 2)^ω}^I(X)$, if and only if

$$∀τ ∈ 2^ω∃γ[τ ∈ IIτ ∧ γ ∈ X] → ∃s∀γ ∈ (ω × 2)^ω[γ ∈ I s → γ ∈ X].$$

$X ⊆ (ω × 2)^ω$ is finitely $I$-determinate in $(ω × 2)^ω$, $Det_{(ω × 2)^ω}^{I,f}(X)$ if and only if

$$∀τ ∈ 2^ω∃γ[τ ∈ IIτ ∧ γ ∈ X] → ∃s∀γ ∈ (ω × 2)^ω[γ ∈ I s → γ ∈ X].$$
\[ X \subseteq 2^\omega \text{ is } I\text{-determinate in } 2^\omega, \ Det_1^I(X), \text{ if and only if} \]
\[ \forall \tau \in 2^\omega \exists \gamma \in \chi \land \gamma \in X \to \exists \sigma \in 2^\omega \exists \gamma \in 2^\omega [\gamma \in I, \sigma \to \gamma \in X]. \]
\[ X \subseteq 2^\omega \text{ is } \text{finitely } I\text{-determinate in } 2^\omega, \ Det^I_2(X), \text{ if and only if} \]
\[ \forall \tau \in 2^\omega \exists \gamma \in \chi \land \gamma \in X \to \exists \sigma \in 2^\omega \exists \gamma \in 2^\omega [\gamma \in I, \sigma \to \gamma \in X]. \]
\[ X \subseteq 2^\omega \text{ is } \text{II-determinate in } 2^\omega, \ Det^I_1(X), \text{ if and only if} \]
\[ \forall \tau \in 2^\omega \exists \gamma \in \chi \land \gamma \in X \to \exists \sigma \in 2^\omega \exists \gamma \in 2^\omega [\gamma \in II, \tau \to \gamma \in X]. \]
\[ X \subseteq 2^\omega \text{ is } \text{finitely } \text{II-determinate in } 2^\omega, \ Det^I_2(X), \text{ if and only if} \]
\[ \forall \sigma \in 2^\omega \exists \gamma \in \chi \land \gamma \in X \to \exists \sigma \in 2^\omega \exists \gamma \in 2^\omega [\gamma \in II, \tau \to \gamma \in X]. \]

We are going to study the following statements:

\[
\Sigma^0_1 Det^I_{\omega \times 2^2} : \forall \alpha [Det^I(E_{\alpha})].
\]
\[
\Delta^0_1 Det^I_{\omega \times 2 \times \omega} : \forall \alpha [Det^I_{\omega \times 2 \times \omega}(D_{\alpha})].
\]
\[
\Delta^0_1 Det^I_{(\omega \times 2^2) / \omega} : \forall \alpha [Det^I_{(\omega \times 2^2) / \omega}(D_{\alpha})].
\]
\[
\Sigma^0_1 Det^I_{\omega / \omega} : \forall \alpha [Det^I(\omega / \omega)(G_{\alpha})].
\]
\[
\Sigma^0_1 - \ast Det^I_{\omega / \omega} : \forall \alpha [\ast Det^I_{\omega / \omega}(G_{\alpha})].
\]
\[
\Sigma^0_1 Det^I_{\omega / \omega^2} : \forall \alpha [Det^I_{\omega / \omega^2}(G_{\alpha})].
\]
\[
\Sigma^0_1 - \ast Det^I_{\omega / \omega^2} : \forall \alpha [\ast Det^I_{\omega / \omega^2}(G_{\alpha})].
\]
\[
\Sigma^0_1 Det^I_{\omega / \omega^2} : \forall \alpha [Det^I_{\omega / \omega^2}(G_{\alpha})].
\]
\[
\Sigma^0_1 - \ast Det^I_{\omega / \omega^2} : \forall \alpha [\ast Det^I_{\omega / \omega^2}(G_{\alpha})].
\]

Each of the above formulas \( X \) has the form: \( \forall \alpha [P(\alpha) \to Q(\alpha)] \). For each of these nine formulas \( X \), we define the statement \( \neg X \), the strong negation of \( X \), as follows:

\[
\neg X := \neg(\forall \alpha [P(\alpha) \to Q(\alpha)]) := \exists \alpha [P(\alpha) \land \neg Q(\alpha)].
\]

Note that these strong negations contain the negation symbol \( \neg \), a possibility we mentioned in Subsection 1.4.

9.4. Simulating a game in \((\omega \times 2)^\omega\) by a game in \(2^\omega\). From the point of view of player \( I \), a game in \((\omega \times 2)^\omega\) may be simulated by a game in Cantor space \(2^\omega\). Where player \( I \) would play \( n \) in \((\omega \times 2)^\omega\), he will play \( n \) times 0 and one time 1 in \(2^\omega\). So he plays the finite sequence \( \mathbb{N} \ast \langle 1 \rangle \). Every time he plays 0, he makes what we call a postponing move. Player \( II \) has to react, in \(2^\omega\), to these postponing moves of player \( I \), but these reactions do not matter. As soon as player \( I \) plays 1 and completes \( \mathbb{N} \ast \langle 1 \rangle \), player \( II \) gives, in the play in \(2^\omega\), the answer he would give to player \( I \)'s move \( n \) in \((\omega \times 2)^\omega\). The reader should keep this in mind when reading the following definitions.

Define \( Bin := 2^\omega = \bigcup \mathbb{N} \{\mathbb{N} \mid \mathbb{N} \in 2^\omega\} \), the set of (code numbers) of finite binary sequences.

Define \( Halfbin := (\omega \times 2)^{\omega^\omega} \cup ((\omega \times 2)^{\omega^\omega} \times \omega) = \bigcup \mathbb{N} \{\mathbb{N} \mid \mathbb{N} \in (\omega \times 2)^\omega\} \).

Define \( \pi_{bin} \) in \( Bin^{\omega^\omega} \) such that

1. \( \pi_{bin}(\langle \rangle) = (\langle \rangle) \), and,
2. for each \( c \), if \( length(c) \) is even, then, for each \( n \), \( \pi_{bin}(c \ast \langle n \rangle) = \pi_{bin}(c) \ast \mathbb{N} \ast \langle 1 \rangle \), and, for both \( i < 2 \), \( \pi_{bin}(c \ast \langle n, i \rangle) = \pi_{bin}(c \ast \langle n \rangle) \ast \langle \rangle \).

The function \( \pi_{bin} \) associates to every position in \( Halfbin \) a position in \( Bin \).

Note that, for each \( c \), \( length(\pi_{bin}(c)) \geq length(c) \).

Define \( \rho_{bin} \) in \( \omega^{\omega^\omega} \) such that

1. \( \rho_{bin}(\langle \rangle) = (\langle \rangle) \), and,
2. for each $d$ in $\text{Bin}$, if $\text{length}(d)$ is even, then $\rho_{\text{bin}}(d \ast (0)) = \rho_{\text{bin}}(d \ast (0, 0)) = \rho_{\text{bin}}(d \ast (1)) = \rho_{\text{bin}}(d \ast (0, 1))$, and $\rho_{\text{bin}}(d \ast (1)) = \rho_{\text{bin}}(d \ast (0))$, where $n$ satisfies:

- for $2n = \text{length}(d)$ and $\forall i \in n[d(2i) = 0]$, or
- for some $k > 0$, $\text{length}(d) = 2k + 2n$ and $d(2k - 2) = 1$ and $\forall i < n[d(2k + 2i) = 0]$. 

The function $\rho_{\text{bin}}$ associates to every position in $\text{Halfbin}$. Note that, for every $c$ in $\text{Halfbin}$, $\rho_{\text{bin}} \circ \pi_{\text{bin}}(c) = c$.

Note that, for each $c$ in $\text{Halfbin}$, $\text{length}(c)$ is even if and only if $\text{length}(\pi_{\text{bin}}(c))$ is even.

**Lemma 9.7.** The following is provable in BIM.

For each $\alpha$, there exists $\beta$ such that

$$\forall \tau \in 2^\omega \exists \eta \in (\omega \times 2)^\omega | \gamma \in II \tau \land \gamma \in G_\alpha \rightarrow \forall \tau \in 2^\omega \exists \delta \in 2^\omega | \delta \in II \tau \land \delta \in G_\beta$$

$$\exists \delta \in 2^\omega | \delta \in II \sigma \rightarrow \delta \in G_\alpha$$

$$\exists \forall \gamma \in (\omega \times 2)^\omega | \gamma \in II \tau \land \gamma \in G_\alpha$$

$$\exists \forall \gamma \in (\omega \times 2)^\omega | \gamma \in II \tau \land \delta \in G_\alpha$$

**Proof.** Let $\alpha$ be given. Define $\beta := \alpha \circ \rho_{\text{bin}}$.

Assume $\forall \tau \in 2^\omega | \gamma \in II \tau \land \gamma \in G_\alpha$. Let $\tau$ be given as a strategy for player $II$ in $2^\omega$. We have to prove that $\exists \delta \in 2^\omega | \delta \in II \tau \land \delta \in G_\beta$. We first define $\tau^1$ as a strategy for player $II$ in $(\omega \times 2)^\omega$. We define $\tau^1$ on all positions in $\text{Halfbin}$ of odd length, by induction on the length of the position. It suffices to define $\tau^1$ on positions $c$ such that $c \in II \tau^1$. We take care that, for each $c$ in $\text{Halfbin}$, if $c \in II \tau^1$, then there exists $d$ in $\text{Bin}$ such that $\rho_{\text{bin}}(d) = c$ and $d \in II \tau^1$.

We first define $\tau^1$ on positions of length $1$. Let $n$ be given. We have to define $\tau^1(\langle n \rangle)$. Find $d$ in $\text{Bin}$ such that $\text{length}(d) = 2n + 1$ and $d \in II \tau^1$ and $d(2n) = 1$ and $\forall i < n[d(2i) = 0]$. Define $\tau^1(\langle n \rangle) := \tau(d)$. Note that $\rho_{\text{bin}}(d) = \langle n \rangle$ and $\rho_{\text{bin}}(d \ast \langle \tau(d) \rangle) = \langle n, \tau^1(\langle n \rangle) \rangle$.

Now suppose $k > 0$. Let $c$ in $(\omega \times 2)^k$ be given such that $c \in II \tau^1$. Let $n$ be given. We have to define $\tau^1(\langle k \rangle \langle n \rangle)$. First find $d$ in $\text{Bin}$ such that $d \in II \tau$ and $\rho_{\text{bin}}(d) = c$. Find $l := \text{length}(d)$. Find $e$ in $\text{Bin}$ such that $d \subseteq e$ and $\text{length}(e) = l + 2n + 1$ and $e \in II \tau$ and $e(l + 2n) = 1$ and $\forall i < n[e(l + 2i) = 0]$. Note that $\rho_{\text{bin}}(e) = c \ast \langle n \rangle$. Define $\tau^1(\langle k \rangle \langle n \rangle) := \tau(e)$. Note that $\rho_{\text{bin}}(e) = c \ast \langle n \rangle$ and $\rho_{\text{bin}}(e \ast \langle \tau(e) \rangle) = c \ast \langle \tau^1(e) \rangle$.

This completes the definition of $\tau^1$.

Using our assumption, find $\gamma$ in $(\omega \times 2)^\omega$ such that $\gamma \in II \tau^1 \land \gamma \in G_\alpha$. Note that $\forall n \exists d \in \text{Bin}| d \in II \tau \land \rho_{\text{bin}}(d) = \gamma$. Note that $\forall d \in \text{Bin} \forall c \in (\omega \times 2)^\omega | (d \in II \tau \land \rho_{\text{bin}}(d) = \rho_{\text{bin}}(\langle n \rangle)]$. Using this fact, construct $\delta$ in $2^\omega$ such that $\delta \in II \tau$ and $\forall n \exists m[\gamma = \rho_{\text{bin}}(\langle m \rangle)]$. Find $n$ such that $\gamma = \rho_{\text{bin}}(\langle m \rangle)$ in $D_\alpha$ and $\langle m \rangle = \delta$. We thus see that $\forall \tau \in 2^\omega | \exists \delta \in 2^\omega | \delta \in II \tau \land \delta \in G_\beta$.

Now assume $\exists \forall \gamma \in (\omega \times 2)^\omega | \gamma \in II \tau \land \gamma \in G_\alpha$. We have to prove that $\exists \forall \gamma \in (\omega \times 2)^\omega | \gamma \in II \tau \land \gamma \in G_\beta$. We will define $\sigma^*$ as a strategy for player $I$ in $(\omega \times 2)^\omega$ such that, for each $c$ in $\text{Halfbin}$, if $c \in II \sigma^*$, then either $\pi_{\text{bin}}(c) \in II \sigma$ or $\exists e \subseteq c[e \in D_\alpha]$. We first define $\sigma^*(\langle \rangle)$. Define $\delta$ in $2^\omega$ such that $\delta \in II \sigma$ and $\forall i \delta(2i + 1) = 0$. Find $m$ such that $\beta\langle \delta m \rangle \neq 0$ and distinguish two cases.

**Case (a).** $\exists n[2n < m \land \delta(2n) = 1]$. Define $\alpha : = \mu[n \delta(2n) = 1]$ and $\sigma^*(\langle \rangle) := n_0$. Note that $\langle n_0 \rangle \in I \sigma^*$ and $\pi_{\text{bin}}(\langle n_0 \rangle) = \langle 0(2n_0) \rangle \ast \langle 1 \rangle = \langle 0(2n_0) + 1 \rangle \in I \sigma$.

**Case (b).** $\forall n[2n < m \rightarrow \delta(2n) = 0]$. Conclude that $\rho_{\text{bin}}(\langle m \rangle) = \langle \rangle$ and $\beta(\langle \rangle) \neq 0$ and also $\alpha(\langle \rangle) \neq 0$. Define $\sigma^*(\langle \rangle) := 0$. Note that $\langle 0 \rangle \in I \sigma^*$ and $\exists e \subseteq \langle 0 \rangle[e \in D_\alpha]$. 30
Now suppose $k > 0$. Let $c$ in $(\omega \times 2)^k$ be given such that $c \in \sigma^*$. We have to define $\sigma^*(c)$ and distinguish two cases.

Case 1. $\exists c \in [c \in D_\alpha]$. We then define $\sigma^*(c) := 0$. Note that $\exists c \in c \ast (0)[c \in D_\alpha]$. 

Case 2. $\forall \exists c \in [c \in D_\alpha]$. Then $\pi_{bin}(c) \in I \sigma$. Note that $\text{length}(\pi_{bin}(c))$ is even and find $l$ such that $2l := \text{length}(\pi_{bin}(c))$. Define $\delta$ in $2^\omega$ such that $\delta \in \exists \sigma$ and $\pi_{bin}(c) \subseteq \delta$ and $\forall \exists |2l + 1 > 2l \rightarrow (2l + 1) = 0$. Find $m$ such that $\beta(\delta m) \neq 0$ and distinguish two cases.

Case (2a). $\exists n [2l < 2n < m \land \delta(2n + 1) = 0]$. Define $n_0 := \mu n[2l < 2n < m \land \delta(2n + 1) = 0]$ and $\sigma^*(c) := n_0 - l$. Note that $c \ast (n_0 - l) \in I \sigma^*$ and $\pi_{bin}(c \ast (n_0 - l)) = \pi_{bin}(c) \ast \tilde{\delta}(2n_0 - 2l) \ast (1) = \tilde{\delta}(2n_0 + 1) \in I \sigma$. 

Case (2b). $\forall n [2n < m \land \delta(2n) = 0]$. Conclude: $\rho_{bin}(\tilde{\delta} m) = c$ and $\beta(\delta m) \neq 0$ and $\alpha(c) = \beta(\delta m) \neq 0$. Define $\sigma^*((c)) := 0$. Note that $c \ast (0) \in I \sigma^*$ and $\exists c \in c \ast (0)[c \in D_\alpha]$. 

This completes the definition of $\sigma^*$.

Now assume $\gamma \in (\omega \times 2)^\omega$ and $\gamma \in I \sigma^*$. Find $\delta \in 2^\omega$ such that $\forall n[\pi_{bin}(\tilde{\gamma} n) \subseteq \delta]$. Find $\varepsilon$ in $2^\omega$ such that such that $\varepsilon \in \exists \sigma$ and, for each $n$, if $\delta n \in I \sigma$, then $\pi n = \delta n$. Find $m$ such that $\pi m \in D_\beta$ and distinguish two cases.

Case (\ast). $\delta m = \pi m$. Conclude that $\delta m \in D_\beta$ and $\rho_{bin}(\pi m) \in D_\alpha$ and $\gamma \in G_\alpha$. 

Case (\ast\ast). $\delta m \neq \pi m$. Then $\delta m \notin I \sigma$ and $\pi_{bin}(\pi m) \notin I \sigma$ and $\exists c \subseteq \pi m[c \in D_\alpha]$. 

Conclude that $\gamma \in G_\alpha$.

We thus see that $\forall \gamma \in (\omega \times 2)^\omega[\gamma \in I \sigma^* \rightarrow \gamma \in G_\alpha]$. 

Assume $\exists \forall \delta \in 2^\omega[\delta \in \exists \sigma \rightarrow \delta \in G_\beta]$. We have to prove that $\exists \forall \gamma \in (\omega \times 2)^\omega[\gamma \in s^* \rightarrow \gamma \in G_\alpha]$. First find $s$ such that $\forall \delta \in 2^\omega[\delta \in \exists \sigma \rightarrow \delta \in G_\beta]$. Consider $p := \text{length}(s)$. Find $q$ such that, for all $c$ in $\text{Halfbin}$, if $c \geq q$, then $\pi_{bin}(c) \geq p$. Define $s^*$ such that $\text{length}(s^*) = q$, inductively. For each $c < q$ in $\bigcup_k (\omega \times 2)^k$ such that $c \in s^*$, $s^*(c)$ is defined just as, in the previous paragraph, where we were given $s \in 2^\omega$, $\alpha(c)$ was defined for each $c \in s \in 2^\omega$, such that $c \in I \sigma^*$. One then may prove that $\forall \gamma \in (\omega \times 2)^\omega[\gamma \in s^* \rightarrow \gamma \in G_\alpha]$. 

Lemma 9.8. One may prove the following statements in BI\m.

(i) $\Sigma_1^0 \text{-Det}_{\omega \times 2} \Sigma_1^0 \text{-Det}_{\omega \times 2}$ and $\neg(\Sigma_1^0 \text{-Det}_{\omega \times 2}) \rightarrow \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2})$.

(ii) $\Delta_1^0 \text{-Det}_{\omega \times 2 \omega} \rightarrow \Sigma_1^0 \text{-Det}_{\omega \times 2} \land \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2}) \rightarrow \neg(\Delta_1^0 \text{-Det}_{\omega \times 2 \omega})$.

(iii) $\forall n[\Delta_1^0 \text{-Det}_{\omega \times 2 \omega}] \rightarrow \Delta_1^0 \text{-Det}_{\omega \times 2 \omega}$ and $\neg(\Delta_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \exists m[\neg(\Delta_1^0 \text{-Det}_{\omega \times 2 \omega})]$. 

(iv) $\forall n[\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}] \rightarrow \Delta_1^0 \text{-Det}_{\omega \times 2 \omega}$ and $\neg(\Delta_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega})$. 

(v) $\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \rightarrow \Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \land \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega})$, and $\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \rightarrow \Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}$ and $\neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega})$. 

(vi) $\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \rightarrow \Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \land \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega})$, and $\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \rightarrow \Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}$ and $\neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega})$. 

(vii) $\text{FT} \rightarrow \Sigma_1^0 \text{-Det}_{\omega \times 2 \omega} \land \neg(\Sigma_1^0 \text{-Det}_{\omega \times 2 \omega}) \rightarrow \neg(\text{FT})$. 

Proof. (i) We prove: given any $\alpha$, one may construct $\beta$ such that 

$\forall \gamma \in 2^\omega[\exists \gamma \in (n, \gamma(n)) \in E_\alpha] \rightarrow \forall \tau \in 2^\omega[\exists \gamma \in (n, \tau(n)) \in E_\beta]$ and 

$\exists \gamma \in (n, 0) \in E_\beta \land (n, 1) \in E_\beta \rightarrow \exists \gamma \in (n, 0) \in E_\alpha \land (n, 1) \in E_\alpha$. 

The two promised conclusions then follow easily. 

Given $\alpha$, define $\beta$ such that $\forall n \forall i < 2 \alpha(p) = (n, i) + 1 \iff \beta(p) = (n, i) + 1$ and $\forall p \forall n \forall i < 2 \alpha(p) = (n, i) + 1 \implies \beta(p) = 0$. 

Clearly, $\beta$ satisfies the requirements. 

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The two promised conclusions then follow easily.

Given \( \alpha \), define \( \beta \) such that \( \forall n \forall i < 2 \beta \left( \langle n, i \rangle, p \right) \neq 0 \iff \alpha(p) = \langle n, i \rangle + 1 \).

Note that \( \forall n \forall i < 2 \left[ \langle n, i \rangle \in E_{\alpha} \iff 3\beta \left( \langle n, i \rangle, p \right) \in D_{\beta} \right] \).

Clearly, \( \beta \) satisfies the requirements.

(iii) Note that BIM proves \( \Delta^0_1 - \text{Det}^t_{\langle \omega \times 2 \rangle} \rightarrow \Delta^0_1 - \text{Det}^t_{\omega \times 2 \times \omega} \) and

\[ \neg \Delta^0_1 - \text{Det}^t_{\langle \omega \times 2 \rangle} \rightarrow \neg \Delta^0_1 - \text{Det}^t_{\omega \times 2 \times \omega} \].

(iv) Let \( m \) be given. We prove: given any \( \alpha \) one may construct \( \beta \) such that

\[ \forall \tau \in 2^\omega \exists \varsigma \in (\omega \times 2)^m [c \in I_\tau \land \varsigma \in D_{\alpha}] \rightarrow \forall \tau \in 2^\omega \exists \gamma \in (\omega \times 2)^m [\gamma \in I_\tau \land \gamma \in G_{\beta}] \]

and \( \exists \nu \gamma \in (\omega \times 2)^m [\gamma \in I_\sigma \rightarrow \gamma \in G_{\beta}] \rightarrow \exists \nu c \in (\omega \times 2)^m [c \in I_\tau \land c \in D_{\alpha}] \).

The two promised conclusions then follow easily.

Given \( \alpha \), define \( \beta \) such that \( \forall \nu \nu \beta(s) \neq 0 \iff \nu \in (\omega \times 2)^m \land \alpha(s) \neq 0 \).

Observe that, if \( \forall \tau \in 2^\omega \exists \varsigma \in (\omega \times 2)^m [c \in I_\tau \land \varsigma \in D_{\alpha}] \), then

\[ \forall \tau \in 2^\omega \exists \gamma \in (\omega \times 2)^m [\gamma \in I_\tau \land \beta(\gamma) \neq 0] \], i.e. \( \forall \tau \in 2^\omega \exists \gamma \in (\omega \times 2)^m [\gamma \in I_\tau \land \gamma \in G_{\beta}] \).

Let \( \sigma \in 2^\omega \) be given such that \( \nu \gamma \in (\omega \times 2)^m [\gamma \in I_\sigma \rightarrow \nu \in (\omega \times 2)^m \beta(\gamma) \neq 0] \). Conclude that \( \forall \nu \gamma \in (\omega \times 2)^m [\gamma \in I_\sigma \rightarrow \nu \in (\omega \times 2)^m \beta(\nu \gamma) \neq 0] \). Find \( N \) such that \( \forall \nu c \in \bigcup_{n \leq 2m} \omega^m [c \in I_\gamma \rightarrow c < N] \) and define \( s := \nu \gamma N \).

We thus see that \( \beta \) satisfies the requirements.

(v) For each \( \alpha \), one may construct \( \beta \) such that

\[ \forall \sigma \in 2^\omega \exists \varsigma \in (\omega \times 2)^m [c \in I_\sigma \land \alpha(c) \neq 0] \rightarrow \forall \tau \in 2^\omega \exists \delta \in 2^\omega [\delta \in I_\tau \land \delta \in G_{\beta}] \]

and \( \exists \nu \gamma \in (\omega \times 2)^m [\nu \gamma \in I_\sigma \rightarrow \nu \gamma \in G_{\beta}] \). The proof has been given in Lemma 0.7. The promised conclusions follow easily.

(vi) We prove that, given any \( \alpha \), one may construct \( \beta \) such that

\[ \forall \sigma \in 2^\omega \exists \varsigma \in (\omega \times 2)^m [c \in I_\sigma \land \alpha(c) \neq 0] \rightarrow \forall \tau \in 2^\omega \exists \delta \in 2^\omega [\delta \in I_\tau \land \delta \in G_{\beta}] \]

and \( \exists \nu \gamma \in (\omega \times 2)^m [\nu \gamma \in I_\sigma \rightarrow \nu \gamma \in G_{\beta}] \). The promised conclusions then follow easily.

Given \( \alpha \), define \( \beta \) such that \( \beta(0) = 0 \) and \( \forall c \in (\omega \times 2)^m [\beta(c) = \beta((0) \ast c)] \). Assume that \( \forall \sigma \in 2^\omega \exists \varsigma \in (\omega \times 2)^m [\varsigma \in I_\sigma \land \alpha(\varsigma) \neq 0] \). Let \( \tau \) in \( 2^\omega \) be given. Define \( \sigma \) such that \( \forall \nu \nu \sigma(c) = \tau((0) \ast c) \). Find \( c \in (\omega \times 2)^m \) such that \( c \in I_\sigma \land \alpha(c) \neq 0 \). Define \( d : = (0) \ast c \) and note \( d \in I_\tau \land \beta(d) \neq 0 \). Conclude that \( \forall \tau \in 2^\omega \exists \delta \in 2^\omega [\delta \in I_\tau \land \delta \neq 0] \).

Let \( \tau \) in \( 2^\omega \) be given such that \( \forall \nu \delta \in 2^\omega [\nu \delta \in I_\tau \rightarrow \nu \in (\omega \times 2)^m \beta(\nu \delta) \neq 0] \). Define \( \sigma \) such that \( \forall \nu \nu \sigma(c) = \tau((0) \ast c) \). Note that \( \forall \delta \in 2^\omega [\delta \in I_\tau \land \delta(0) \neq 0] \) and \( \forall \delta \in 2^\omega [\delta \in I_\tau \land \delta(0) \neq 0] \).

Let \( t \) be given such that \( \forall \nu \nu \delta \in 2^\omega [\nu \delta \in I_\tau \rightarrow \nu \in (\omega \times 2)^m \beta(\nu \delta) \neq 0] \). Define \( s \) such that \( \forall \nu \nu s(c) = t((0) \ast c) \). Conclude, as above, that \( \forall \nu \nu \delta \in 2^\omega [\nu \delta \in I_\tau \rightarrow \nu \in (\omega \times 2)^m \beta(\nu \delta) \neq 0] \).

We thus see that \( \beta \) satisfies the requirements.
Conclude that $\forall c \in 2^{<\omega} [\beta(c) \neq 0 \iff \exists s < c [s \in I \land \alpha(s) \neq 0]]$.

Given $\alpha$, define $\beta$ such that $\forall c \in 2^{<\omega} [\beta(c) \neq 0 \iff \exists s < c [s \in I \land \alpha(s) \neq 0]]$. Clearly, $\forall \gamma \in 2^{<\omega} \exists n [\beta(\gamma n) \neq 0]$, i.e.

$\text{Bar}_2(\Delta_3)\,$.

Let $m$ be given such that $\text{Bar}_2(\Delta_{m+1})$. Define $X := \{ s \in 2^{<\omega} \mid \text{length}(s) = m \land \exists n \leq m [\alpha(\gamma n) \neq 0] \}$. Note that $\forall b \exists s [s \in I \land s \in X]$. According to Theorem 9.3, $\text{Det}^{\text{II},\text{Det}}$ is equivalent in $\text{BIM}$.

Theorem 9.9. (i) A (continuous) function $f : (\omega \times 2)^{<\omega} \Rightarrow (\omega \times 2)^{<\omega}$ is weakly $I$-determinate in $(\omega \times 2)^{<\omega}$ if and only if

$$\forall \tau \in 2^{<\omega} [\forall s \in I \tau \land \gamma \in \mathcal{X}] \Rightarrow \forall \forall \gamma \in 2^{<\omega} [\exists \gamma \in I \sigma \rightarrow \gamma \in \mathcal{X}].$$

We now define: $\mathcal{X} \subseteq (\omega \times 2)^{<\omega}$ is weakly $I$-determinate in $(\omega \times 2)^{<\omega}$ if and only if

$$\forall \varphi : 2^{<\omega} \rightarrow (\omega \times 2)^{<\omega} [\forall \tau \in 2^{<\omega} [\varphi \tau \in I \tau \land \varphi \tau \in \mathcal{X}] \rightarrow$$

$$\exists \forall \gamma \in (\omega \times 2)^{<\omega} [\exists \gamma \in I \sigma \rightarrow \gamma \in \mathcal{X}].$$

A (continuous) function $\varphi : 2^{<\omega} \rightarrow (\omega \times 2)^{<\omega}$ such that $\forall \tau \in 2^{<\omega} [\varphi \tau \in I \tau]$ will be called an anti-strategy for player I in $(\omega \times 2)^{<\omega}$. Note that the Second Axiom of Continuous Choice $\text{AC}_{\omega,\omega} = \text{AC}_{I,1}$, implies, for every subset $\mathcal{X} \subseteq (\omega \times 2)^{<\omega}$: if $\forall \tau \in 2^{<\omega} \exists \gamma [\forall \tau \in 2^{<\omega} [\gamma \in I \tau \land \gamma \in \mathcal{X}]]$, then $\exists \varphi : 2^{<\omega} \rightarrow (\omega \times 2)^{<\omega} [\forall \tau \in 2^{<\omega} [\varphi \tau \in I \tau \land \varphi \tau \in \mathcal{X}].$ $\text{AC}_{\omega,\omega}$ thus implies that, if $\mathcal{X} \subseteq (\omega \times 2)^{<\omega}$ is weakly $I$-determinate in $(\omega \times 2)^{<\omega}$, then $\mathcal{X}$ is $I$-determinate in $(\omega \times 2)^{<\omega}$.

Earlier versions of the next Theorem may be found in [42] Chapter 16] and [42].

Theorem 9.10 (Intuitionistic Determinacy Theorem). The following statements are equivalent in BIM:

(i) $\text{FT}$. (ii) For every anti-strategy $\varphi$ for player I in $(\omega \times 2)^{<\omega}$ there exists a strategy $\sigma$ for player I in $(\omega \times 2)^{<\omega}$ such that $\forall \gamma \in (\omega \times 2)^{<\omega} [\exists \gamma \in I \sigma \rightarrow \exists \tau \in 2^{<\omega} [\varphi(\tau) = \varphi(\gamma)]].$

Proof. (i) $\Rightarrow$ (ii). Let $\varphi : 2^{<\omega} \rightarrow (\omega \times 2)^{<\omega}$ be given such that $\forall \tau \in 2^{<\omega} [\varphi \tau \in I \tau]$. We intend to develop a strategy $\sigma$ for player I in $(\omega \times 2)^{<\omega}$ such that Player I, whenever he obeys $\sigma$ and develops, together with player II, $\gamma$ in $(\omega \times 2)^{<\omega}$, will be able to construct, simultaneously, a strategy $\tau$ for player II in $(\omega \times 2)^{<\omega}$ such that $\gamma = \varphi(\tau)$. The infinite sequence $\gamma$ must be the answer given by the anti-strategy $\varphi$ to player II's strategy $\tau$. So, while playing $\gamma$, player I conjectures a strategy $\sigma$ that player II may be assumed to follow during this very play.

Let $c, t$ be given such that $c \in \bigcup_k (\omega \times 2)^{k}$ and $t \in 2^{<\omega}$. We define: with respect to the given anti-strategy $\varphi$, $t$ is, at the position $c$, a safe (partial) conjecture by player I about the strategy followed by player II; notation: $\text{Safe}(c, t)$, if and only

\footnote{See Subsubsection 2.7.11}
We want to define a function taking these decisions, i.e., we want to define $\nu$ such that $\forall c \in \bigcup_k (\omega \times 2)^k$ and in $2^{<\omega}$, $\nu(c, t) \neq 0 \iff Safe(c, t)$. We first define $\psi$ such that, for all $c$ in $\bigcup_k (\omega \times 2)^k$, for all $t$ in $2^{<\omega}$, $\psi(c, t) = \mu n [n > length(t) \land \nu_u \in 2^{<\omega}[\nu_u] \to \nu_u \in 2^{<\omega}[\nu_u]]$, and then define $\nu$ such that, for all $c$ in $\bigcup_k (\omega \times 2)^k$, for all in $2^{<\omega}$, $\nu(c, t) \neq 0 \iff \forall r \in 2^{<\omega}[\nu_u] \to \nu_u \in 2^{<\omega}[\nu_u]$. We also define $\chi$ such that, for all $c$ in $\bigcup_k (\omega \times 2)^k$, for all $t$ in $2^{<\omega}$, if $\nu(c, t) = 0$, then $\chi(c, t) = \mu |r| \in 2^{<\omega} \land length(r) = \psi(c, t) \land \nu_u < 2^{<\omega}[\nu_u] \to \nu_u \in 2^{<\omega}[\nu_u]$. Note that, if $\nu(c, t) = 0$, then $\nu_u \neq 2^{<\omega}[\chi(c, t) \in \rho \land c \subseteq \nu_u]$. Let $n, t, c$ be given such that $c \in (\omega \times 2)^n$ and in $2^{<\omega}$ and $Safe(c, t)$. We shall prove that $\forall \rho \in 2^{\omega}$, $\exists \delta \in \bigcup_k (\omega \times 2)^k \exists [d \in \bigcup_k (\omega \times 2)^k \times \omega] (d \neq 0 \iff Safe(c, d, t * (i)) \in \nu_u)$. Let $\rho \in 2^{\omega}$ be given. Find $u$ in $2^{<\omega}$ such that $\forall t \in u \land \forall c \subseteq \nu_u \land \forall \rho \neq 0 \iff Safe(c, d, t * (i)) \in \nu_u$. Find $t$ such that $\forall t \in u \land \forall c \subseteq \nu_u \land \forall \rho \neq 0 \iff Safe(c, d, t * (i)) \in \nu_u$. Find $t$ such that, for all $d$ in $\bigcup_k (\omega \times 2)^k \times \omega$, if $d < length(c)$, then length($d) < 2p$. Define $D := \{d \in (\omega \times 2)^k \times \omega \mid d \neq 0 \iff Safe(c, d, t * (i)) \in \nu_u\}$.

It follows from FT that $E := \{\nu_u \in 2^{<\omega} | t \in 2^{<\omega}\}$ is a finite subset of $\omega$. Note that, for all $d$ in $(\omega \times 2)^k \times \omega$ and any $\rho \neq 0$, there exists a finite subset $\omega$ such that $\exists \nu_u \in 2^{<\omega} | t \in 2^{<\omega}$, $\forall d \in D\{d \neq 0 \iff Safe(c, d, t * (i)) \in \nu_u\}$. As $D$ is finite, conclude that $\exists d \in D\{d \neq 0 \iff Safe(c, d, t * (i)) \in \nu_u\}$. We thus see that, for all $c$ in $\bigcup_k (\omega \times 2)^k$, for all $t$ in $2^{<\omega}$, $\forall \rho \in 2^{\omega}$, $\exists \delta \in \bigcup_k (\omega \times 2)^k | d \in \bigcup_k (\omega \times 2)^k \times \omega (\nu_u \in \nu_u \land Safe(c, d, t * (i)) \in \nu_u)$.

This key fact enables us to develop the promised strategy for player $I$. Let $c, t$ be given such that $c \in \bigcup_k (\omega \times 2)^k$ and in $2^{<\omega}$ and $Safe(c, t)$, then $\forall \rho \in 2^{\omega}$, $\exists \delta \in \bigcup_k (\omega \times 2)^k | d \in \bigcup_k (\omega \times 2)^k \times \omega (\nu_u \in \nu_u \land Safe(c, d, t * (i)) \in \nu_u)$.

\footnote{For the notation $\nu_u$, see Subsection 13.3.}

\footnote{For the notation $\nu_u$, see Subsection 13.4.}

\footnote{For the notation $\nu_u$, see Subsection 13.2.}
Using We thus see that

\[ n \in \mathbb{N} \quad \text{and} \quad \lambda \text{ is weakly } \text{Safe}\]

Note that

\[ \forall \alpha \in [2k+2] \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \lambda \text{ is weakly } \text{Safe}\]

Find \( 
\[ \frac{3}{2} \] \text{ such that } \lambda(c, \tau(i)) = \lambda(c, \tau(j)) \text{, where } j \text{ is the least such } j, \text{ and, if not, then } \lambda(c, \tau(i)) = \lambda(c) \text{. Note that } \forall \in \lambda(2k+2) \text{ is an explicit fan. Using } \lambda \text{ as } \sigma \text{, we see that } \forall \in 2^\omega\text{[Safe}(c, \tau(i))]. \]

Now define a strategy \( \sigma \) for player I in \( (\omega \times 2)^\omega \) as follows. Let \( c \in (\omega \times 2)^\omega \) be given. Find \( n_0 := \mu[k, 2k+2, n(\cdot, \tau(n))] \), Define \( \sigma(\cdot, \cdot) := \mu[k, 2k+2, n(\cdot, \tau(n))] \), and \( \forall \in n(\cdot, i) \in \mathbb{N}_0 \text{ if } \lambda(c, \tau(i)) = \lambda(c, \tau(j)) \text{, where } j \text{ is the least such } j, \text{ and, if not, then } \lambda(c, \tau(i)) = \lambda(c) \text{. Note that } \forall \in (\omega \times 2)^\omega\text{[Safe}(c, \tau(i))]. \]

Now define a strategy \( \sigma \) for player I in \( (\omega \times 2)^\omega \) as follows. Let \( c \in (\omega \times 2)^\omega \) be given. Find \( n_0 := \mu[k, 2k+2, n(\cdot, \tau(n))] \), Define \( \sigma(\cdot, \cdot) := \mu[k, 2k+2, n(\cdot, \tau(n))] \), and \( \forall \in n(\cdot, i) \in \mathbb{N}_0 \text{ if } \lambda(c, \tau(i)) = \lambda(c, \tau(j)) \text{, where } j \text{ is the least such } j, \text{ and, if not, then } \lambda(c, \tau(i)) = \lambda(c) \text{. Note that } \forall \in (\omega \times 2)^\omega\text{[Safe}(c, \tau(i))]. \]

Now define a strategy \( \sigma \) for player I in \( (\omega \times 2)^\omega \) as follows. Let \( c \in (\omega \times 2)^\omega \) be given. Find \( n_0 := \mu[k, 2k+2, n(\cdot, \tau(n))] \), Define \( \sigma(\cdot, \cdot) := \mu[k, 2k+2, n(\cdot, \tau(n))] \), and \( \forall \in n(\cdot, i) \in \mathbb{N}_0 \text{ if } \lambda(c, \tau(i)) = \lambda(c, \tau(j)) \text{, where } j \text{ is the least such } j, \text{ and, if not, then } \lambda(c, \tau(i)) = \lambda(c) \text{. Note that } \forall \in (\omega \times 2)^\omega\text{[Safe}(c, \tau(i))]. \]
As in the proof of Theorem 10.10 (ii) ⇒ (i), find an anti-strategy \( \varphi \) for player \( I \) in \( (\omega \times 2)^\omega \) such that \( \forall \tau \in 2^n [\varphi(\tau)2 \in E_\varphi] \). Assume \( \sigma \) is a strategy for player \( I \) in \( (\omega \times 2)^\omega \) such that \( \forall \gamma \in (\omega \times 2)^\omega \cdot [\gamma \in \sigma \Rightarrow \exists \tau \in 2^n [\gamma = \varphi(\tau)] \]. Consider \( n := \sigma(\gamma) \) and conclude that \( \forall i < 2^n, (n, i) \in E_\varphi \). Contradiction.

We did not find an argument proving \( \neg \text{FT} \) from the assumption of the existence of an anti-strategy for player \( I \) in \( (\omega \times 2)^\omega \) that fails to translate into a strategy for player \( I \) in \( (\omega \times 2)^\omega \).

10. THE (UNIFORM) INTERMEDIATE VALUE THEOREM

10.1. The Intermediate Value Theorem, IVT

For all \( \varphi \in R^{[0,1]} \),

\[ \exists \gamma \in [0,1]^2 [ \varphi R (\gamma^{(0)}) \leq R \varphi R (\gamma^{(1)}) \land \exists \gamma \in [0,1] [\varphi R (\gamma) = R 0] \].

IVT fails constructively. The next two theorems are similar to [3, Chapter 3, Theorem 2.4] and [25, Theorem 6(iv) and (iii)].

Theorem 10.1. BIM + IVT \( \vdash \) LLPO.

Proof. Assume \( \varphi \) be given. Define \( \delta \) in \( R \) such that, for each \( n \), if \( \bar{\delta} \subseteq \beta \), then \( \delta(n) = \left(-\frac{1}{\beta_0 \cdot \gamma} \right) \) and, if \( \bar{\delta} \subseteq \beta \) and \( \beta_0 := \mu(p) [\beta(p) \neq 0] \), then \( \delta(n) = \left(\frac{1}{\beta_0 \cdot \gamma} \right)^{\neg \forall} + \frac{1}{\beta_0 \cdot \gamma} \). Note that \( \delta R 0 \land \exists \delta [\beta(p) = 0] \) and \( \delta R 0 \Leftrightarrow \exists \delta [\beta(p) + 1 = \mu(n)] \). Find \( \varphi \) in \( R^{[0,1]} \) such that \( \varphi R (0) = R (1 - \gamma) \), and \( \varphi R (\delta) = R \delta \) and \( \varphi R (1 - \gamma) = R 1 \) and \( \varphi \) is linear on \( \left[\frac{1}{3}, \frac{2}{3}\right] \) and on \( \left[\frac{2}{3}, 1\right] \). Note that \( \varphi R (0) \leq R 0 \leq R \varphi R (1-R) \).

Using IVT, find \( \gamma \) in \( [0,1] \) such that \( \varphi R (\gamma) = R 0 \). Either \( \gamma > R \frac{1}{3} \) or \( \gamma < R \frac{2}{3} \). If \( \gamma > R \frac{1}{3} \), then \( \neg (\delta > R 0) \) and \( \forall \varphi R (2p \neq \mu(n)] \). And, if \( \gamma < R \frac{2}{3} \), then \( \neg (\delta < R 0) \) and \( \forall \varphi R (2p + 1 \neq \mu(n)] \). We thus see that \( \forall \varphi R (2p \neq \mu(n)] \).\( \square \)

Theorem 10.2. BIM + \( \Pi_1^0 \)-AC\( \omega \), \( \omega \) \( \vdash \) LLPO \( \vdash \) IVT.

Proof. Let \( \varphi \in R^{[0,1]} \) and \( \gamma \in [0,1]^2 \) be given such that \( \varphi R (\gamma^{(0)}) \leq R 0 \leq R \varphi R (\gamma^{(1)}) \). Define \( \beta \) such that, for all \( n \), \( \beta(2n) \neq 0 \) and \( \beta(2n) \neq 0 \) and \( \beta(2n + 1) \neq 0 \) and \( \beta(2n + 1) \neq 0 \) and \( \forall \varphi R (2p + 1 \neq \mu(n)] \).\( \square \)

Assume \( \gamma_0 \leq R \gamma_1 \). Using LLPO like we used it just now, conclude that, for all \( n \), for all \( m \leq 2^n \), either \( \varphi R (\frac{2m - n}{2^n} \cdot R \gamma_0 + R \frac{n}{2^n} \cdot R \gamma_1 \) \( \leq R \varphi R (\gamma_1) \). Using \( \Pi_1^0 \)-AC\( \omega \), find \( \beta \) such that \( \beta(0) = 0 \) and \( \beta(1) = 0 \) and \( \beta(0) = 0 \) and \( \beta(1) = 0 \) and \( \beta(2n) \neq 0 \) and \( \beta(2n) = 0 \) and \( \beta(2n + 1) \neq 0 \) and \( \beta(2n + 1) \neq 0 \) such that, for all \( n \), for all \( m \leq 2^n \), either \( \varphi R (\frac{2m - n}{2^n} \cdot R \gamma_0 + R \frac{n}{2^n} \cdot R \gamma_1 \) \( \leq R \varphi R (\gamma_1) \).\( \square \)

Define \( \varepsilon \) such that, for each \( n, \varepsilon(n) = \left(\frac{2^n - \delta(n)}{2^n} \right) \cdot R \gamma_0 + R \left(\frac{2^n - \delta(n)}{2^n}\right) \cdot R \gamma_1 \) \( \leq R \gamma_1 \). Note \( \varepsilon \in [0,1] \). Assume \( \varphi R (\varepsilon) \) \( \vdash \) LLPO. Find \( n, p \) such that \( p \in E_\varphi \) and \( \varepsilon(p) \leq R \gamma_0 \) and \( \gamma_0 R \gamma_1 \) \( \leq R \gamma_1 \). Note that \( \varepsilon(p) \leq \varepsilon(n) \) and \( \varepsilon(n) \leq R \gamma_1 \). Contradiction. Therefore, \( \varphi R (\rho) \leq R 0 \).\( \square \)
The case $\gamma^1 \geq_R \gamma^{10}$ is treated similarly. \qed

**Corollary 10.3.** BIM + $I^0_2 \leftrightarrow$ IVT $\leftrightarrow$ LLPO $\leftrightarrow$ WKL.

**Proof.** Use Theorems 4.3, 10.1 and 10.2. \qed

10.2. A contraposition of the Intermediate Value Theorem, $\neg$ IVT:

For each $\varphi$ in $R^{[0,1]}$, if $\forall \gamma \in [0,1][\varphi^R(\gamma) \#_R 0_R]$, then either $\forall \gamma \in [0,1][\varphi^R(\gamma) <_R 0_R]$, or $\forall \gamma \in [0,1][\varphi^R(\gamma) <_R 0_R]$.

**Theorem 10.4.** BIM + $\neg$ IVT.

**Proof.** Assume $\varphi \in R^{[0,1]}$ and $\forall \gamma \in [0,1][\varphi^R(\gamma) \#_R 0_R]$. Assume that $\varphi^R(0_R) <_R 0_R$. Suppose we find $\gamma$ in $[0,1]$ such that $0_R <_R \varphi^R(\gamma)$. We will obtain a contradiction by the method of successive bisection. Define $q$ in $Q$ such that $0_R <_R \varphi^R((q)\bar{R})$. Define $\delta$ such that, for each $n$, $\delta(n) \in \mathcal{S}$ and $\delta(n) \subseteq (0,1)$, as follows, by induction. Define $\delta(0) := (0,q)$. Let $n,s$ be given such that $\delta(n) = s$. Note that $\varphi^R((\frac{s + s'}{2})\bar{R}) \neq 0_R$. Find $(r,t) \in \mathcal{F}_\varphi$ such that $r' <_Q \frac{1}{2}(s' + s'') <_Q r''$ and either $0_Q <_Q t'$ or $t'' <_Q 0_Q$. If $0_Q <_Q t'$, define $\delta(n + 1) = (s', s' + s'')$, and, if $t'' <_Q 0_Q$, define $\delta(n + 1) = (s', s)$. Note that, for each $n$, $\delta(n + 1) \subseteq \mathcal{S}$, and $\varphi^R((\delta(n))\bar{R}) <_R 0_R <_R \varphi^R((\delta''(n))\bar{R})$. Note that $\delta \in [0,1]$ and $\varphi^R(\delta) \#_R 0_R$. Determine $(r,s)$ in $E_\varphi$ and $n$ in $\omega$ such that $\delta(n) \subseteq r$ and either $s'' <_Q 0_Q$ or $0_Q <_Q s'$, that is, either $\varphi^R((\delta''(n))\bar{R}) <_R 0_R$ or $0_R <_R \varphi^R((\delta''(n))\bar{R})$. Contradiction. Conclude that $\neg \varphi^R(\gamma) <_R 0_R$. As $\varphi^R(\gamma) \#_R 0_R$, conclude that $\varphi^R(\gamma) <_R 0_R$. We thus see that, if $\varphi^R(0_R) <_R 0_R$, then $\forall \gamma \in [0,1][\varphi^R(\gamma) < R 0_R]$. One may prove also that, if $\varphi^R(0_R) <_R 0_R$, then $\forall \gamma \in [0,1][\varphi^R(\gamma) >_R 0_R]$. Conclude that either $\forall \gamma \in [0,1][\varphi^R(\gamma) <_R 0_R]$ or $\forall \gamma \in [0,1][\varphi^R(\gamma) <_R 0_R]$. \qed

10.3. FT is unprovable in BIM + IVT. As we observed in Subsection 1.8, BIM + CT + $X \vee \neg X$ is consistent. According to Theorem 10.3, BIM + $\neg$ IVT. Conclude that BIM + $X \vee \neg X \vdash$ IVT. Assume that BIM + IVT $\vdash$ FT. Then BIM + $X \vee \neg X \vdash$ FT. As we know from Theorem 2.3, BIM + CT $\vdash$ $\neg$ FT, and, therefore, BIM + CT $\vdash$ $\neg$ FT. Conclude that BIM + IVT $\not\vdash$ FT, and also, in view of Theorem 2.6, BIM + IVT $\not\vdash$ WKL. Note that, in view of Corollary 10.3, this gives another proof of BIM $\not\vdash I^0_2 \leftrightarrow$ AC$\omega$, a fact established in Subsection 1.3. One may even conclude that BIM + IVT $\not\vdash I^0_2 \leftrightarrow$ AC$\omega$.

One may ask if BIM + LLPO $\vdash$ IVT, i.e. if the proof of Theorem 10.2 can be given without recourse to $I^0_2 \leftrightarrow$ AC$\omega$, but we do not know the answer to this question.

10.4. The Uniform Intermediate Value Theorem, UIVT:

For each $\varphi$ such that $\forall n [\varphi^n \in R^{[0,1]}]$, if $\forall n \exists \gamma \in [0,1][\varphi^R(\gamma) \leq R 0_R \leq R (\varphi^n)^R(\gamma)]$, then $\exists \gamma \in [0,1]\exists n [\varphi^R(\gamma) \leq R 0_R]$.

In 27 Exercise IV.2.12, page 137], the reader is asked to prove that, in the classical system RCA$_0$, UIVT is an equivalent of WKL. As RCA$_0 \vdash$ IVT, RCA$_0$ proves the equivalence of UIVT and the next principle.

10.5. Uzero:

For all $\varphi$ such that $\forall n [\varphi^n \in R^{[0,1]}]$, if $\forall n \exists \gamma \in [0,1][\varphi^R(\gamma) = R 0_R]$, then $\exists \gamma \in [0,1]\forall n [\varphi^R(\gamma) = R 0_R]$.

We want to study Uzero in BIM. We need the following Lemma.

**Lemma 10.5.** BIM proves:

(i) $\exists \psi : \omega_\omega \rightarrow \omega_\omega \forall \varphi \in R^{[0,1]}[\mathcal{H}_{\psi \varphi} = \{ \gamma \in [0,1] | \varphi^R(\gamma) \#_R 0_R \}]$, and
(ii) \( \exists \tau : \omega^\omega \rightarrow \omega^\omega \forall \alpha \in R^{[0,1]} \land H_\alpha = \{ \gamma \in [0,1] \mid (\tau(\alpha))^R(\gamma) \not\equiv 0_R \} \).

**Proof.** (i) Define \( \psi : \omega^\omega \rightarrow \omega^\omega \) such that, for each \( \varphi \), each \( s \),

\[
(\psi(\varphi))(s) \neq 0 \Leftrightarrow \exists \nu \in E_{\varphi}R[s \subseteq \nu' \land (0_Q \leq (p''(\nu'))') \lor (p''(\nu'')) < 0_Q)]
\]

Note that \( \forall \varphi \in R^{[0,1]} \forall \gamma \in [0,1][\varphi(\gamma))^R(\gamma) \not\equiv 0_R \Leftrightarrow \gamma \in H_{\psi(\varphi)} \).

(ii) Define \( \rho \) such that, for each \( s \in S \), \( \rho^s \in R^{[0,1]} \) and, if \( \not\equiv -1_Q \leq s' \leq s'' \leq 0_Q \), then, for all \( \gamma \) in \( [0,1] \), \( \rho^s(\gamma) = 0_R \), and, if \( -1_Q \leq s' \leq s'' \leq 0_Q \), then, for all \( \gamma \) in \( [0,1] \),

1. if \( \gamma \equiv_R (s')_R \) or \( (s'')_R \equiv_R \gamma \), then \( \rho^s(\gamma) = 0_R \), and,
2. if \( (s')_R \leq \gamma \leq (s'')_R \), then \( \rho^s(\gamma) = \text{inf}(\gamma - \bigwedge (s')_R, (s'')_R - \gamma) \).

(Here \( \rho^s \) codes the restriction to \( [0,1] \) of the \( '\text{tent}' \) function from \( R \) to \( R \) that is zero outside of \([s', s'']\) and linear on both \([s', s'']\) and \([s'' + s' - s''] \) and \( s' - s'' \) and that takes the value 0 at \( s' \) and \( s'' \) and the value \( s'' - s' \) at \( s' + s'' - s' \) and the value \( 0_R \) at \( s'' \). Note that, for all \( s \in S \), for all \( \gamma \) in \( [0,1] \), \( \rho^s(\gamma) \equiv_R (\rho^s(\gamma)) \equiv_R 0_R \).

Define \( \tau : \omega^\omega \rightarrow \omega^\omega \) such that, for all \( \alpha, \tau(\alpha) \in R^{[0,1]} \), and, for each \( \gamma \) in \( [0,1] \),

\( (\tau(\alpha))^R(\gamma) = R \sum_{s \in S, \alpha(\gamma) \neq 0} (s')_R \cdot \tau(\gamma) \cdot (\rho^s(\gamma))^R(\gamma) \).

Note that \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma) = 0_R] \). □

**Theorem 10.6. BIM + \( \Pi_n^I - AC_{\omega, [0,1]} \) \Leftrightarrow \text{Uzero}.**

**Proof.** First, assume \( \Pi_n^I - AC_{\omega, [0,1]} \). Let \( \varphi \) be given such that \( \forall \gamma \in [0,1][\varphi|^n(\gamma) = 0_R] \). Using Lemma 10.5(ii), find \( \beta \) such that, \( \forall \gamma \in [0,1][\varphi|^n(\gamma) = 0_R] \).

Conclusion. \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma) = 0_R] \). Using \( \Pi_n^I - AC_{\omega, [0,1]} \) conclude that \( \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma) = 0_R] \).

We thus see \text{Uzero}. □

Secondly, assume \text{Uzero}. Let \( \alpha \) be given such that \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \).

Using Lemma 10.5(ii), \( \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \).

Conclusion. \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \). Using \text{Uzero} conclude that \( \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \), and \( \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \).

We thus see \( \Pi_n^I - AC_{\omega, [0,1]} \). □

10.6. A uniform contrapositive Intermediate Value Theorem \( \text{UIVT} \):

For each \( \varphi \) such that \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \), if \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \), then \( \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \).

As \( \text{BIM} \vdash \text{UIVT} \), \( \text{BIM} \) proves the equivalence of \( \text{UIVT} \) and the next statement.

10.7. \( \lnot \text{Uzero} \):

For all \( \varphi \) such that \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \),

if \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \), then \( \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \).

We define a strong negation of this statement. This strong negation itself contains a negation sign, a possibility mentioned in Subsection 1.3.

10.8. \( \lnot \lnot \text{Uzero} \):

There exists \( \varphi \) such that \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \) and \( \forall \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \) and \( \lnot \exists \gamma \in [0,1][\gamma \not\equiv_R (\tau(\alpha))^R(\gamma)] \).

**Lemma 10.7.** One may prove in \( \text{BIM} \):

(i) \( \Pi_n^I - AC_{\omega, [0,1]} \rightarrow Uzero \) and \( \lnot Uzero \rightarrow \lnot \Pi_n^I - AC_{\omega, [0,1]} \).

(ii) \( Uzero \rightarrow \Pi_n^I - AC_{\omega, [0,1]} \) and \( \lnot \Pi_n^I - AC_{\omega, [0,1]} \rightarrow \lnot Uzero \).
Note that $\forall = \forall^\gamma$. First, define $\gamma$ positively unrealizable $s$ such that

$$\exists n \in [0, 1] \subseteq H_{\beta|n} \rightarrow \exists n \forall \gamma \in [0, 1] \exists n [(\varphi|n)^R (\gamma) \#_R 0_R].$$

Using Lemma [10.5], one finds $\beta$ such that

$$\forall a \in H_{\beta|n} = \{ \gamma \in [0, 1] | (\varphi|n)^R (\gamma) \#_R 0_R \}. \quad \text{The two statements now are obvious.}$$

(ii) The two promised conclusions follow if one may prove in BIM that, for every $\alpha$, there exists $\varphi$ such that

$$\forall \gamma \in [0, 1] \exists n [\gamma | n \in H_{\alpha|n}] \rightarrow \forall \gamma \in [0, 1] \exists n [(\varphi|n)^R (\gamma) \#_R 0_R]$$

Let $\alpha$ be given. Using Lemma [10.5 ii], find $\varphi$ such that $\forall n \in \mathbb{R} [0, 1] \wedge H_{\alpha|n} = \{ \gamma \in [0, 1] | (\varphi|n)^R (\gamma) \#_R 0_R \}$. The two statements now are obvious. \hfill \Box

**Theorem 10.8.** BIM proves $\text{Uzero} \leftrightarrow \text{UIVT} \leftrightarrow \text{FT}$ and $\neg \text{Uzero} \leftrightarrow \neg \text{UIVT} \leftrightarrow \neg \text{FT}$. \hfill \Box

11. **The compactness of classical propositional logic**

In this Section, we prove that FT is equivalent to a contraposition of a restricted version of the compactness theorem for classical propositional logic. We also prove the corresponding result for $\neg \text{FT}$.

We introduce the symbols $\neg, \wedge$ and $\vee$ as natural numbers: $\neg := 1, \wedge := 2$ and $\vee := 3$. We define (the characteristic function of) Form $\subseteq \omega$, as follows, by recursion. For each $n, n \in \text{Form}$ if and only if

$$n' = 0 \lor (n' = \neg \wedge n'' \in \text{Form}) \lor$$

$$(n' = \wedge \lor n' = \vee) \land \forall i < \text{length}(n')|n''(i) \in \text{Form}|.$$

We define $\mathcal{T} := (\wedge, \{ \})$ and $\perp := (\vee, \{ \})$.

Assume $\gamma \in 2^\omega$. We define $\tilde{\gamma}$ in $2^\omega$ such that, for every $n$,

(i) if $n \notin \text{Form}$, then $\tilde{\gamma}(n) = 0$, and,

(ii) if $n \in \text{Form}$ and $n' = 0$, then $\tilde{\gamma}(n) = \gamma(n''),$ and,

(iii) if $n \in \text{Form}$ and $n' = \neg$, then $\tilde{\gamma}(n) = 1 - \tilde{\gamma}(n''),$ and,

(iv) if $n \in \text{Form}$ and $n' = \wedge$, then $\tilde{\gamma}(n) = \min\{\tilde{\gamma}(n''(i))|i < \text{length}(n'')\},$ and

(v) if $n \in \text{Form}$ and $n' = \vee$, then $\tilde{\gamma}(n) = \max\{\tilde{\gamma}(n''(i))|i < \text{length}(n'')\}.$

Note that $0 = \{ \}$. We define $\min(\emptyset) = 1$ and $\max(\emptyset) = 0$. Note that $\tilde{\gamma}(\perp) = \tilde{\gamma}(\wedge, 0) = 1$ and $\tilde{\gamma}(\perp) = \tilde{\gamma}(\vee, 0) = 0$. For all $m, n$ in Form, we define: $m \equiv n$ if and only if $\forall \gamma \in 2^\omega[\tilde{\gamma}(m) = \tilde{\gamma}(n)].$

Assume $c \in 2^{<\omega}$. We define $\tilde{c}$ in $2^{<\omega}$ such that $\text{length}(c) = \text{length}(\tilde{c}),$ as follows.

First, define $\gamma = c \ast \emptyset$. Then define, for all $m < \text{length}(c)$, $\gamma(m) := \tilde{\gamma}(m).

X \subseteq \omega$ is realizable, Real$(X)$, if and only if $\exists \gamma \in 2^{<\omega} \forall n \in X[\gamma(n) = 1],$ and positively unrealizable, Unreal$(X)$, if and only if $\forall \gamma \in 2^{<\omega} \exists n \in X[\gamma(n) = 0].$

We define a mapping $\text{Fm}$ from $2^{<\omega}$ to Form, as follows. Assume $a \in 2^{<\omega}$. Find $s$ such that $\text{length}(s) = \text{length}(a)$, and, for all $i < \text{length}(a)$, if $a(i) = 0$, then $s(i) = (\neg, (0, i))$, and, if $a(i) = 1$, then $s(i) = (0, i)$. Define $\text{Fm}(a) = (\wedge, s)$. \hfill \Box

**Lemma 11.1.**

(i) $\forall a \in 2^{<\omega} \forall \gamma \in 2^\omega[\tilde{\gamma}(\text{Fm}(a)) = 1 \leftrightarrow a \subseteq \gamma].$
Lemma 11.2. The following statements are provable in $\mathsf{BIM}$ and $\mathsf{WKL}$.

(i) For all $m, n$, for all in $2^{<\omega}$ such that
\[
\forall n \in \text{Form} \land \forall \gamma \in 2^{<\omega}[\gamma(m) = 1 \leftrightarrow \forall \eta \leq m[\alpha(\eta) = 0]].
\]

Proof. (i) We prove, by induction, that, for each $n, \forall a \in 2^{<\omega}[\text{length}(a) = n \rightarrow \forall \gamma \in 2^{<\omega}[\gamma(Fm(a)) = 1 \leftrightarrow a \in \gamma])$. Note that $Fm(\gamma) = \top$ and $\forall \gamma \in 2^{<\omega}[\gamma(\top) = 1$ and $\forall \gamma \in 2^{<\omega}[\gamma(Fm(a)) = 1 \leftrightarrow a \in \gamma]$. Note that, for each $\gamma$ in $2^{<\omega}$, for each $i < 2$, $\gamma(Fm(a \ast (i))) = 1 \leftrightarrow \hat{\gamma}(Fm(a)) = 1 \land \gamma(n) = i \leftrightarrow a \ast (i) \in \gamma$.

(ii) The proof is an exercise in calculating codes of formulas.

(iii) Let $\alpha$ be given. We define the promised $\delta$ as follows, by induction. If $\alpha() = 0$, define $\delta(0) := \top$, and, if $\alpha() \neq 0$, define $\delta(0) := \bot$. Note that $\delta(0)$ satisfies the requirements. Let $m$ be given such that $\delta(m)$ has been defined and $\delta(m + 1) \in [\alpha]^{m+1}$. Find $t$ such that \{i | i < \text{length}(t)\} = \{a \in 2^{<\omega} \mid \text{length}(a) = m + 1 \land \forall n \leq m + 1[\alpha(\eta) = 0]\}. Then find $s$ such that $\text{length}(s) = \text{length}(t)$ and $\forall i < \text{length}(s)[s(i) = Fm(t(i))].$ Note that, for each $\gamma$ in $2^{<\omega}$, $Fm(\gamma, s) = 1 \leftrightarrow \exists n \in 2^{<\omega}[\text{length}(a) = m + 1 \land \forall n \leq m[\alpha(\eta) = 0] \land \gamma(Fm(a)) = 1 \leftrightarrow \exists n \in 2^{<\omega}[\text{length}(a) = m + 1 \land \forall n \leq m[\alpha(\eta) = 0] \land a \in \gamma] \leftrightarrow \forall n \leq m[\alpha(\eta) = 0]$.

Define $\delta(m + 1) = \beta((\gamma, s), \delta(m))$, where $\beta$ is the function we found in (ii). Note that $\delta(m + 1)$ satisfies the requirements.

\[\square\]

Lemma 11.2. The following statements are provable in $\mathsf{BIM}$.

(i) $\mathsf{FT} \rightarrow \exists\alpha[\text{Unreal}(E_\alpha) \rightarrow \exists n[\text{Unreal}(E_{\alpha n})]].$

(ii) $\exists\alpha[\text{Unreal}(E_\alpha) \land \forall n[\text{Real}(E_{\alpha n})] \rightarrow \neg\mathsf{FT}]$.

(iii) $\mathsf{WKL} \rightarrow \forall\alpha[\forall n[\text{Real}(E_{\alpha n})] \rightarrow \text{Real}(E_\alpha)]$.

(iv) $\mathsf{WKL} \rightarrow \exists\alpha[\forall n[\neg\text{Real}(E_{\alpha n})] \rightarrow \text{Real}(E_\alpha)]$.

(v) $\neg\mathsf{FT} \rightarrow \forall\alpha[\forall n[\neg\text{Real}(E_{\alpha n})] \land \forall n[\neg\text{Real}(E_{\alpha n})] \rightarrow \mathsf{WKL}]$.

(vi) $\forall\alpha[\forall n[\neg\text{Real}(E_{\alpha n})] \rightarrow \mathsf{WKL}]$.

Proof. (i), (ii) and (iii). We argue in $\mathsf{BIM}$.

Let $\alpha$ be given. Define $\beta$ such that, for all $m$, for all $c$ in $2^{<\omega}$ such that $\text{length}(c) = m$, $\beta(c) \neq 0 \leftrightarrow \exists n < m[n \in E_{\alpha m} \land c(n) = 0]$. We shall prove that
\[\text{Unreal}(E_\alpha) \rightarrow \text{Bar}_{2^{<\omega}}(D_\beta)\] and
\[\exists m[\text{Bar}_{2^{<\omega}}(D_{\alpha m}) \rightarrow \exists n[\text{Unreal}(E_{\alpha m})]].\]

Assume that $\text{Unreal}(E_\alpha)$. Let $\gamma$ in $2^{<\omega}$ be given. Find $n, p$ such that $n \in E_{\alpha p}$ and $\gamma(n) = 0$. Define $m := \max\{n, p\} + 1$ and note that $\beta(\eta) \neq 0$. Conclude that $\forall \gamma \in 2^{<\omega}[\exists m[\beta(\eta) = 0]$ and $\text{Bar}_{2^{<\omega}}(D_\beta)$. Let $m$ be given such that $\text{Bar}_{2^{<\omega}}(D_{\alpha m})$. For all $c$ in $2^{<\omega}$ such that $\text{length}(c) = m$, $\exists n < m[\beta(\eta) \neq 0]$ and $\forall n < m[n \in E_{\alpha m} \land c(n) = 0]$. Conclude that $\text{Unreal}(E_{\alpha m})$ and $\exists n[\text{Unreal}(E_{\alpha m})]$. Note that, if $\text{Unreal}(E_\alpha)$, then $\text{Bar}_{2^{<\omega}}(D_\beta)$, and, by $\mathsf{FT}$, there exist $m$ such that $\text{Bar}_{2^{<\omega}}(D_{\alpha m})$ and $n$ such that $\text{Unreal}(E_{\alpha m})$. This establishes (i).

Note that, if $\text{Unreal}(E_\alpha) \land \forall n[\text{Real}(E_{\alpha n})]$, then $\text{Bar}_{2^{<\omega}}(D_\beta)$ and $\forall n[\neg\text{Bar}_{2^{<\omega}}(D_{\alpha n})]$, i.e. $\neg\mathsf{FT}$. This establishes (ii).

Note that, if $\forall n[\text{Real}(E_{\alpha n})]$, then $\forall n[\neg\text{Bar}(D_{\alpha n})]$, and, by $\mathsf{WKL}$, there exists $\gamma$ such that $\forall n[\beta(\eta) \neq 0]$. Conclude that $\forall n[\beta(\eta) \neq 0]$. Conclude that $\forall n[\beta(\eta) \neq 0]$ and $\text{Real}(E_\alpha)$. This establishes (iii).

\[\square\]
Use Lemma 11.2 and the fact that
\( \forall \) 
\( \exists \)(i). We argue in BIM.

Let \( \alpha \) be given. Using Lemma 11.2(iii), find \( \delta \) in \([\omega]^{\omega}\) such that
\( \forall n[\delta(m) = 1] \iff \forall n \leq m[\alpha(m) = 0] \). Note that \( \delta \) is strictly increasing and \( \forall n[\exists m[\delta(m) = \delta(n)]] \iff \exists n \leq m[\exists m = \delta(n)] \). Define \( \beta \) such that \( \forall n[\beta(m) \neq 0] \iff \exists n[m = \delta(n)] \). We shall prove that
\[
\text{Bar}_2(\alpha) \rightarrow \text{Unreal}(\beta) \text{ and }
\exists n[\text{Unreal}(\text{Bar}_2(\alpha))] \rightarrow \exists n[\text{Bar}_2(\text{Bar}_2(\alpha))].
\]

Assume that \( \text{Bar}_2(\alpha) \). Given any \( \gamma \in 2^{\omega} \), find \( m \) such that \( \alpha(\gamma) = \alpha(\gamma) \neq 0 \) and
therefore, \( \exists n \in D_\beta[\gamma(\gamma) 
eq 1] \). Conclude that \( \text{Unreal}(\beta) \).

Let \( n \) be given such that \( \text{Unreal}(\text{Bar}_2(\alpha)) \). Let \( m_0 \) be the largest \( m \) such that \( \delta(m) < n \). Note: \( \exists \gamma \in 2^{\omega}[\gamma(\delta(m_0)) = 1] \), and, therefore, \( \forall a \in 2^{\omega}[\text{length}(a) = m_0 + 1 \rightarrow \exists n \leq m_0[\alpha(\gamma) \neq 0] \). Find \( k \) such that \( \forall a \in 2^{\omega}[\text{length}(a) = m_0 + 1 \rightarrow a \leq k] \) and conclude that \( \text{Bar}_2(\text{Bar}_2(\alpha)) \) and \( \exists n[\text{Bar}_2(\text{Bar}_2(\alpha))] \).

Note that, if \( \text{Bar}_2(\alpha) \) and \( \text{Unreal}(\beta) \rightarrow \exists n[\text{Unreal}(\text{Bar}_2(\alpha))] \),
then \( \exists n[\text{Bar}_2(\text{Bar}_2(\alpha))] \). This establishes (iv).

Note that, if \( \text{Bar}_2(\alpha) \) and \( \exists n[\text{Bar}_2(\text{Bar}_2(\alpha))] \), then \( \text{Unreal}(\beta) \)
and \( \forall n[\text{Real}(\text{Bar}_2(\alpha))] \). This establishes (v).

Note that, if \( \forall n[\exists \gamma[\delta(\gamma) = 1] \rightarrow \exists n[\exists n \leq m[\alpha(\gamma) = 0]] \),
then there exists \( \gamma \in 2^{\omega} \) realizing \( \beta \), so \( \forall n[\exists n \leq m[\alpha(\gamma) = 0]] \).
\( \forall n[\exists n \leq m[\alpha(\gamma) = 0]] \) establishes (vi).

\[\square\]

**Theorem 11.3.**
(i) \( \text{BIM} \vdash \text{FT} \iff \forall \alpha[\text{Unreal}(E_\alpha) \rightarrow \exists n[\text{Unreal}(E_{\alpha n})]] \iff \forall \alpha[\text{Unreal}(E_\alpha) \rightarrow \exists n[\text{Unreal}(E_{\alpha n})]].
\]
(ii) \( \text{BIM} \vdash \neg \text{FT} \iff \exists \alpha[\text{Unreal}(E_\alpha) \wedge \forall n[\text{Real}(E_{\alpha n})]] \iff \exists \alpha[\text{Unreal}(E_\alpha) \wedge \forall n[\text{Real}(E_{\alpha n})]].
\]
(iii) \( \text{BIM} \vdash \text{WKL} \iff \forall \alpha[\forall n[\text{Real}(E_{\alpha n})] \rightarrow \text{Real}(E_\alpha)] \iff \forall \alpha[\forall n[\text{Real}(E_{\alpha n})] \rightarrow \text{Real}(E_\alpha)].
\]

**Proof.** Use Lemma 11.2 and the fact that \( \forall \alpha \exists \beta[\alpha = \beta] \). \( \square \)

**Theorem 11.3(i)** is also a consequence of \( \mathbf{20} \) Theorem 6.5.

V.N. Krivtsov has shown, among other things, that \( \text{FT} \) is an equivalent of an intuitionistic (generalized) completeness theorem for intuitionistic first-order predicate logic, see [18].

12. Other ‘Fan Theorems’?

In this Section, we indicate what, on our opinion, should be the subject of the next chapter in intuitionistic reverse mathematics. The Fan Theorem may be seen as a replacement, for the intuitionistic mathematician, of that enviable tool of the classical mathematician: (Weak) König’s Lemma. We hope to make clear that the greater subtlety of the language of the intuitionistic mathematician allows for many other possible replacements.

12.1. **Notions of finiteness.** Let \( \alpha \) be given. We consider the set \( \text{D}_\alpha := \{n \mid \alpha(n) \neq 0\} \), the subset of \( \omega \) decided by \( \alpha \). We define the following.

\( \text{D}_\alpha \) is **finite** if and only if \( \exists n \forall m \geq n[\alpha(m) = 0] \).\[24\]

\( \text{D}_\alpha \) is **bounded-in-number** if and only if \( \exists n \forall t \in [\omega]^{n+1}[\exists i < n + 1[\alpha \circ t(i) = 0]] \).

\( \text{D}_\alpha \) is **almost-finite** if and only if \( \forall \zeta \in [\omega]^{\omega}[\exists n[\alpha \circ \zeta(n) = 0]] \).

\( \text{D}_\alpha \) is **not-finite** if and only if \( \exists n \forall m \geq n[\alpha(m) = 0] \).

\( \text{D}_\alpha \) is **infinite** if and only if \( \exists n \forall m \geq n[\alpha(m) \neq 0] \).

\( \text{D}_\alpha \) is **not-infinite** if and only if \( \forall n \exists m \geq n[\alpha(m) \neq 0] \).

\[24\] A referee suggested to consider this notion too.
Note that $D_\alpha$ is infinite if and only if $\exists \xi \in [\omega]^\omega \forall n [\alpha \circ \xi(n) \neq 0]$, and that $D_\alpha$ is not-infinite if and only if $\exists \xi \in [\omega]^\omega \forall n [\alpha \circ \xi(n) \neq 0]$. Decidable subsets of $\omega$ that are bounded-in-number are introduced and discussed in [34]. Almost-finite decidable subsets of $\omega$ were introduced in [31] and [33], and are also studied in [39].

**Lemma 12.1.**
(i) $\Sigma_0 \vdash \forall \alpha \{D_\alpha \text{ is finite} \rightarrow D_\alpha \text{ is bounded-in-number}\}$. 
(ii) $\Sigma_0 \vdash \forall \alpha \{D_\alpha \text{ is bounded-in-number} \rightarrow D_\alpha \text{ is finite}\} \rightarrow \text{LPO}$. 
(iii) $\Sigma_0 \vdash \forall \alpha \{D_\alpha \text{ is bounded-in-number} \rightarrow D_\alpha \text{ is almost-finite}\}$. 
(iv) $\Sigma_0 \vdash \forall \alpha \{D_\alpha \text{ is almost-finite} \rightarrow D_\alpha \text{ is bounded-in-number}\} \rightarrow \text{LPO}$. 
(v) $\Sigma_0 \vdash \forall \alpha \{D_\alpha \text{ is almost-finite} \rightarrow D_\alpha \text{ is not-infinite}\}$. 
(vi) $\text{BIM + BARIN} \vdash \forall \alpha \{D_\alpha \text{ is almost-finite} \rightarrow D_\alpha \text{ is not-finite}\}$.

**Proof.** (i) Let $\alpha, n$ be given such that $\forall m \geq n [\alpha(m) = 0]$. Note that $\forall t \in [\omega]^{\omega+1} t(n) \geq n$ and conclude that $\forall t \in [\omega]^{\omega+1} [\alpha \circ t(n) = 0]$. 

(ii) Assume $\forall \alpha \{D_\alpha \text{ is bounded-in-number} \rightarrow D_\alpha \text{ is finite}\}$. Let $\alpha$ be given. Define $\alpha^*$ such that $\forall n [\alpha^*(n) \neq 0 \leftrightarrow n = \mu \alpha([\alpha(m) = 0])$. Note that, for all $k$, if $k = \mu \alpha([\alpha(m) = 0])$, then $\forall n [k \leq n < 2 \cdot k \leftrightarrow \alpha^*(n) \neq 0]$ and $\exists \eta \in [\omega]^\omega \forall i < k [\alpha^* \circ t(i) \neq 0]$ and $\forall t \in [\omega]^{\omega+1} \exists \eta < k + 1 [\alpha^* \circ t(i) = 0]$. Let $\zeta$ in $[\omega]^\omega$ be given. We want to prove that $\exists \eta [\alpha^* \circ \zeta(n) = 0]$ and distinguish two cases. Case (a). $\alpha^* \circ \zeta(0) = 0$. Then we are done. Case (b). $\alpha^* \circ \zeta(0) \neq 0$. Then $\exists m [\alpha(m) = 0]$. Define $k := \mu \alpha([\alpha(m) = 0])$ and note: $\forall m \geq 2 \cdot k [\alpha(m) = 0]$, and, in particular, $\alpha^* \circ \zeta(2 \cdot k) = 0$. Conclude that $\forall \zeta \in [\omega]^{\omega+1} \exists \eta [\alpha^* \circ \zeta(n) = 0]$, i.e. $D_\alpha$ is almost-finite. Using the assumption, conclude that $D_\alpha$ is bounded-in-number. Find $n$ such that $\forall t \in [\omega]^{\omega+1} \exists \eta < n + 1 [\alpha^* \circ t(i) = 0]$. Conclude that, for all $k$, if $k = \mu \alpha([\alpha(m) = 0])$, then $k \leq n < 1 [\alpha \circ k = 0]$, and, therefore, either $\exists k [\alpha(k) = 0]$ or $\forall k (\alpha(k) = 0)$. We thus see that $\forall \alpha \{\exists k [\alpha(k) = 0] \vee \forall k (\alpha(k) = 0)\}$, i.e. LPO.

(v) The proof is left to the reader.

(vi) Let $\alpha$ be given such that $D_\alpha$ is almost-finite, i.e. $\forall \zeta \in [\omega]^{\omega+1} \exists \eta [\alpha \circ \zeta(n) = 0]$. Define $B := \bigcup_n \{s \in [\omega]^{\omega+1} \mid s \notin [\omega]^\omega \vee \exists i < n [\alpha \circ s(i) = 0]\}$. We now prove that $B$ is a bar in $[\omega]^{\omega+1}$. Let $\gamma$ be given. Define $\gamma^*$ such that $\gamma^*(0) = \gamma(0)$, and, for each $n$, if $\gamma(n+2) \in [\omega]^{\omega+1}$, then $\gamma^*(n+1) = \gamma(n+1)$, and, if not, then $\gamma^*(n+1) = \gamma^*(n) + 1$. Note: $\gamma^* \in [\omega]^{\omega+1}$ and find $n$ such that $\forall \gamma^* \in B$. Either $\gamma^* \models \text{Bar} \text{$_\omega$}_B$ or $\gamma^* \notin [\omega]^\omega$, and, in both cases; $\gamma^* \models B$. We thus see that $\forall \gamma \exists n [\gamma^* \models B]$ i.e. $\text{Bar} \text{$_\omega$}_B$. Define $E := \bigcup_n \{s \in [\omega]^{\omega+1} \mid s \notin [\omega]^\omega \vee \exists i < n [\alpha \circ s(i) = 0] \vee D_\alpha \text{ is not-finite}\}$. 

Note that $B \subseteq E$.

We now prove that $E$ is inductive. Let $s, n$ be given such that $s \in [\omega]^\omega$ and $\forall m \exists n [s \circ m = E]$. We have to prove that $s \in E$. We may assume that $s \in [\omega]^\omega$ and $\exists \eta [\alpha \circ s(i) = 0]$, and first consider two special cases. Case (a). $\exists m [s \circ m \in [\omega]^{\omega+1} \wedge \alpha(m) = 0]$. Finding such $m$, we consider $s \circ \langle m \rangle$ and conclude that $s \circ \langle m \rangle \in E$ and $D_\alpha$ is not-finite. Case (b). $\nexists m [s \circ m \in [\omega]^{\omega+1} \wedge \alpha(m) = 0]$. Conclude that $\forall m [s \circ m \in [\omega]^{\omega+1} \wedge \alpha(m) = 0]$ and that $D_\alpha$ is finite. Defining $P := \exists m [s \circ m \in [\omega]^{\omega+1} \wedge \alpha(m) = 0]$, we may conclude.

\[\text{See 2.2.0.}\] We do not know if the use of this principle here is unavoidable.
that \((P \lor \neg P) \rightarrow D_\alpha\) is not-not-finite. By intuitionistic logic\(^{24}\), we conclude that \(D_\alpha\) is not-not-finite and \(s \in E\). We thus see that \(\forall s[\forall m[s \ast (m) \in E] \rightarrow s \in E]\), i.e. \(E\) is inductive.

Obviously, \(E\) is monotone, i.e. \(\forall s \forall m[s \in E \rightarrow s \ast (m) \in E]\).

Using \(\text{BARIND}\), we conclude that \(\langle \rangle \in E\) and \(D_\alpha\) is not-not-finite.

We thus see that \(\text{BIM} + \text{BARIND} \vdash \forall \alpha[D_\alpha\ is\ almost-finite \rightarrow D_\alpha\ is\ not-not-finite]\).

\Lemma{12.2} \(\text{BIM} + \text{MP}^{25}\) proves \(\forall \alpha[D_\alpha\ is\ not-infinite \leftrightarrow D_\alpha\ is\ not-not-finite] \leftrightarrow D_\alpha\ is\ almost-finite\).

\Proof{} The proof is left to the reader. \qed

We now extend the notion ‘almost-finite’ from decidable subsets of \(\omega\) to enumerable subsets of \(\omega\). For every \(\alpha\), \(E_\alpha := \{n \mid \exists m[\alpha(m) = n + 1]\}\), is the subset of \(\omega\) enumerated by \(\alpha\). We define: \(E_\alpha\) is almost-finite if and only if \(\forall \zeta \in [\omega]^\omega\exists m \exists n[m < n \land \alpha \circ \zeta(m) = \alpha \circ \zeta(n)]\).

The first item of the next Lemma shows that the definition is a good definition indeed as it does not depend on the enumeration \(\alpha\) of \(E_\alpha\). The second item shows that this definition is consistent with the definition given earlier for decidable subsets of \(\omega\). The fifth item shows that an almost-finite union of almost-finite enumerable subsets of \(\omega\) is enumerable and almost-finite.

\Lemma{12.3} \(\text{BIM}\) proves the following.

(i) \(\forall \alpha \forall \beta(E_\beta \subseteq E_\alpha \land \forall \zeta \in [\omega]^\omega\exists m \exists n[m < n \land \alpha \circ \zeta(m) = \alpha \circ \zeta(n)\}) \rightarrow \forall \zeta \in [\omega]^\omega\exists m \exists n[m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)].\)

(ii) \(\forall \alpha \forall \beta[D_\alpha = E_\beta \rightarrow (\forall \zeta \in [\omega]^\omega\exists m[\alpha \circ \zeta(n) = 0] \leftrightarrow \forall \zeta \in [\omega]^\omega\exists m[m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)].\)

(iii) \(\forall \alpha \forall \eta[\forall i < 2[E_{\alpha \!|\! i}\ is\ almost-finite] \rightarrow \bigcup_{i < 2} E_{\alpha \!|\! i\ is\ almost-finite].\)

(iv) \(\forall \alpha \forall \eta[\forall i < n[E_{\alpha \!|\! i}\ is\ almost-finite] \rightarrow \bigcup_{i < n} E_{\alpha \!|\! i\ is\ almost-finite].\)

(v) \(\forall \alpha \forall \eta[\forall n[E_{\alpha \!|\! n}\ is\ almost-finite] \land \forall \zeta \in [\omega]^\omega\exists m[\alpha(\zeta(n)) = \emptyset] \rightarrow \bigcup_{n} E_{\alpha \!|\! n\ is\ almost-finite].\)

\Proof{} (i) Let \(\alpha, \beta\) be given such that \(E_\beta \subseteq E_\alpha\) and \(\forall \zeta \in [\omega]^\omega\exists m[m < n \land \alpha \circ \zeta(m) = \alpha \circ \zeta(n)].\) Let \(\zeta\) in \([\omega]^\omega\) be given. We will prove that \(\exists m \exists n[m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)].\)

Define \(\zeta^*\) such that \(\zeta^*(0) = \zeta(0)\) and, for all \(n\), if \(\forall i < n + 1[\beta \circ \zeta(i) \neq 0]\) and \(\forall i < n + 1[\beta \circ \zeta(i) = 0] \rightarrow \beta(i) \neq \beta(j)\), then \(\zeta^*(n + 1) = \mu k[\alpha(k) = \beta \circ \zeta(n + 1)]\) and, if not then \(\zeta^*(n + 1) = \max_{i \leq n} \zeta^*(i) + 1\). Note that \(\forall n \forall m[m < n \rightarrow \zeta^*(m + 1) \neq \zeta^*(n + 1)]\). Find \(\eta\) in \([\omega]^\omega\) such that \(\forall n[\zeta^* \circ \eta(n + 1) > \zeta^* \circ \eta(n)]\). Find \(m, n\) such that \(m < n\) and \(\alpha \circ \zeta \circ \eta(m) = \alpha \circ \zeta \circ \eta(n)\).

Either \(\exists \beta \leq n[\beta \circ \zeta(i) = 0]\) or \(\beta \circ \zeta \circ \eta(m) = \alpha \circ \zeta \circ \eta(n)\). We thus see that \(\forall \zeta \in [\omega]^\omega\exists m[m < n \land (\beta \circ \zeta(m) = 0 \lor \beta \circ \zeta(m) = \beta \circ \zeta(n))].\)

One easily concludes that \(\forall \zeta \in [\omega]^\omega\exists m[m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)]\).

(ii) Let \(\alpha, \beta\) be given such that \(D_\alpha = E_\beta\). Define \(\gamma\) as follows. For each \(n\), if \(\alpha(n) = 0\), then \(\gamma(n) = 0\) and, if \(\alpha(n) \neq 0\), then \(\gamma(n) = \alpha(n) + 1\). Note that \(D_\alpha = E_\gamma\). \(\gamma\) might be called the canonical enumeration of \(D_\alpha\).

Assume \(\forall \zeta \in [\omega]^\omega\exists m[\alpha \circ \zeta(n) = 0]\). Let \(\zeta\) in \([\omega]^\omega\) be given. Find \(m, n\) such that \(m < n\) and \(\alpha \circ \zeta(m) = \alpha \circ \zeta(n) = 0\). Conclude that \(\gamma \circ \zeta(m) = \gamma \circ \zeta(n) = 0\). We thus see that \(\forall \zeta \in [\omega]^\omega\exists m[m < n \land \gamma \circ \zeta(m) = \gamma \circ \zeta(n)]\). Use (i) and conclude that \(\forall \zeta \in [\omega]^\omega\exists m[m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)].\)

\footnote{\(\neg\neg\neg(P \lor \neg P)\) is provable, and from \(A \rightarrow B\) one may conclude \(\neg B \rightarrow \neg A\) and also \(\neg\neg A \rightarrow \neg\neg B\). Furthermore, \(\neg\neg\neg C\) is equivalent to \(\neg C\) and \(\neg\neg\neg\neg C\) is equivalent to \(\neg C\).}

\footnote{For Markov’s Principle \(\text{MP}\), see Subsubsection 22.16.}
Now assume $\forall \zeta \in [\omega]^2 \exists m \exists n [m < n \land \beta \circ \zeta(m) = \beta \circ \zeta(n)]$. Use (i) and conclude that $\forall \zeta \in [\omega]^2 \exists m \exists n [m < n \land \gamma \circ \zeta(m) = \gamma \circ \zeta(n)]$. We will prove that $\forall \zeta \in [\omega]^2 \exists \alpha \circ \zeta(n) = 0$. Let $\zeta$ in $[\omega]^\omega$ be given. If $\alpha \circ \zeta(0) = 0$ we are done. Now assume $\alpha \circ \zeta(0) \neq 0$. Define $\zeta^*$ such that $\zeta^*(0) = 0$ and, for each $n$, if $\forall i \leq n + 1 [\alpha \circ \zeta(i) \neq 0]$, then $\zeta^*(n + 1) = \zeta(n) + 1$, and, if not, then $\zeta^*(n + 1) = \zeta^*(n) + 1$. Note that $\zeta^* \in [\omega]^\omega$ and find $m, n$ such that $m < n$ and $\gamma \circ \zeta^*(m) = \gamma \circ \zeta^*(n)$. Conclude that $\exists i \leq n [\alpha \circ \zeta(n) = 0]$. We thus see that $\forall \zeta \in [\omega]^2 \exists \alpha \circ \zeta(n) = 0$.

(iii) Let $\alpha$ be given such that $E_{\alpha^{10}}, E_{\alpha^{11}}$ are almost-finite. Define $\alpha^*$ such that, for each $n$, $\alpha^*(2n) = \alpha^{10}(n)$ and $\alpha^*(2n + 1) = \alpha^{11}(n)$ and note that $E_{\alpha^*} = E_{\alpha^{10}} \cup E_{\alpha^{11}}$. Let $\zeta$ in $[\omega]^\omega$ be given. We will prove $QED := \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)]$. We first prove that $\forall k \exists l > k [\exists \zeta(l) = 2p + 1] \lor QED$. Let $k$ be given. Define $\zeta^*$ such that, for each $i, j$, if $\forall i \leq i\exists \zeta(k + 1 + j) = 2p$, then $\zeta^*(i) = \zeta(k + 1 + i)$, and, if not, then $\zeta^*(i) = 2 \cdot \zeta(k + 1 + i)$. Note that $\forall \zeta \exists \zeta^*(i) = 2p$. Define $\zeta^**$ such that $\forall i [\zeta^*(i) = 2 \cdot \zeta^*(i)]$ and note that $\forall \zeta [\alpha^* \circ \zeta(i) = \alpha^0 \circ \zeta^*(i)]$. Find $m, n$ such that $m < n$ and $\alpha^0 \circ \zeta^*(m) = \alpha^0 \circ \zeta^*(n)$ and note that either $\zeta^*(m) = \zeta(k + 1 + m)$ and $\zeta^*(n) = \zeta(k + 1 + n) \land QED$, or $\exists j \leq n \exists \zeta(k + 1 + j) = 2p + 1$. Using $\Sigma^0_1$-$\mathbf{AC}_{\omega\omega}$, see Theorem 4.1, find $\delta$ such that $\forall k [\delta(k) + k \land [\exists \zeta \circ \delta(k) = 2p + 1] \lor QED]$. Define $\zeta^i$ such that $\zeta^i(0) = \delta(0)$ and, for each $k, \zeta^i(k + 1) = \delta(k)$). Note that, for each $k, \delta(2p^i(k) + 2).\land QED$. Define $\zeta^i$ such that, for each $i, j$, if $\forall j \leq i \exists \zeta^i(j) = 2p + 1$, then $\zeta^i(i) = 2 \cdot \zeta^i(i) + 1$. Note that $\forall \zeta \exists \zeta^i(i) = 2p + 1$. Define $\zeta^**$ such that $\forall i [\zeta^i(i) = 2 \cdot \zeta^*(i) + 1]$ and note that $\forall \zeta [\alpha^0 \circ \zeta^*(i) = \alpha^0 \circ \zeta^*(i)]$. Find $m, n$ such that $m < n$ and $\alpha^0 \circ \zeta^*(m) = \alpha^0 \circ \zeta^*(n)$ and note that either $\zeta^*(m) = \zeta^*(n)$ and $\zeta^*(n) = \zeta^*(n)$ or, $\exists j \leq n \exists \zeta^i(j) = 2p + 1$ and $QED$, so in any case $QED$. We thus see that $\forall \zeta \in [\omega]^2 \exists \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)]$, i.e. $E_{\alpha^*} = E_{\alpha^{10}} \cup E_{\alpha^{11}}$ is almost-finite.

(iv) Use (iii) and induction.

(v) Let $\alpha$ be given such that, for all $n, E_{\alpha^{1+n}}$ is almost-finite, and $\forall \zeta \in [\omega]^2 \exists n [\alpha \circ \zeta(n) = 0]$. We will prove that $\bigcup_n E_{\alpha^{1+n}}$ is almost-finite. Define $\alpha^*$ such that, for all $p, \alpha^0(p) = \alpha^{10}(p)$ and note that $E_{\alpha^*} = \bigcup_n E_{\alpha^{1+n}}$. Let $\zeta$ in $[\omega]^\omega$ be given. We will prove $QED := \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)]$. We first prove that $\forall k \exists l > k [\exists \zeta(l) = 2p + 1] \lor QED$. Let $k, n$ be given. If $\zeta(k + 1) > n$, there is nothing to prove. Assume $\zeta(k + 1) \leq n$. Define $\zeta^*$ such that $\zeta^*(0) = \zeta(k + 1)$ and, for all $i, j$, if $\forall i \leq i \exists \zeta(k + 1 + j) \leq n$, then $\zeta^*(i + 1) = \zeta(k + 2 + i)$, and, if not, then $\zeta^*(i + 1) = \mu \exists p > \zeta^*(i) \land \exists \kappa [\zeta^*(i) + 1)$ and $QED$, or $\exists j \leq n \exists \zeta(k + 1 + j) = 2p + 1$. Using (iii), find $p, q$ such that $p < q$ and $\alpha^0 \circ \zeta^*(p) = \alpha^0 \circ \zeta^*(q)$ and note that either $\zeta^*(p) = \zeta^*(k + 1 + p)$ and $\zeta^*(q) = \zeta(k + 1 + q)$ or $\zeta(k + 1 + q)] \lor QED$. We thus see that $\forall \zeta \in [\omega]^2 \exists \exists m \exists n [m < n \land \alpha^* \circ \zeta(m) = \alpha^* \circ \zeta(n)]$, i.e. $E_{\alpha^*} = \bigcup_n E_{\alpha^{1+n}}$ is almost-finite.

\[ \square \]

12.2. Almost-fans and approximate fans. Recall that, for each $\beta, F_\beta := \{ \alpha \mid \forall \eta [\beta(\eta) = 0] \}$. Let $\beta$ be given. We define the following.

$\beta$ is an almost-fan-law, $\text{Almfan}(\beta)$, if and only if $\text{Spr}(\beta)$ and
\[ \forall s \forall \zeta \in [\omega]^s \exists m[\beta(s \ast (\zeta(m))) \neq 0]. \] If \( \beta \) is an almost-fan-law, then \( \mathcal{F}_\beta \) is an almost-fan.

\( \beta \) is an approximate-fan-law, \( \text{Appfan}(\beta) \), if and only if \( \text{Spr}(\beta) \) and \( \forall n \exists k t \in [\omega]^{k+1} \exists i \leq k t(i) \notin \omega^n \lor \beta(t(i)) \neq 0 \). If \( \beta \) is an approximate-fan-law, then \( \mathcal{F}_\beta \) is an approximate fan.

\( \beta \) is an explicit approximate-fan-law, \( \text{Appfan}^+(\beta) \), if and only if \( \text{Spr}(\beta) \) and \( \exists \gamma \forall n t \in [\omega]^{\gamma(n)+1} i \leq \gamma(n) t(i) \notin \omega^n \lor \beta(t(i)) \neq 0 \). If \( \beta \) is an explicit approximate-fan-law, then \( \mathcal{F}_\beta \) is an explicit approximate fan.

Note that \( \text{BIM} + \text{Weak-\Pi^1_1} - \text{AC}_\omega, \omega \vdash \forall \beta[\text{Appfan}(\beta) \to \text{Appfan}^+(\beta)] \).

12.3. The Almost-fan Theorem as a Scheme, \( \text{ALMFAN} \),

\[ \forall \beta[(\text{Almfan}(\beta) \land \text{Bar}_{\mathcal{F}_\alpha}(B)) \to \exists \alpha[E_\alpha \subseteq B \land E_\alpha \text{ is almost-finite} \land \text{Bar}_{\mathcal{F}_\alpha}(E_\alpha)]] \]

The following theorem may be compared to Theorem 2.4.

**Theorem 12.4.** \( \text{BIM + BARIND} + \text{AC}_\omega, \omega \vdash \text{ALMFAN} \).

**Proof.** Let \( \beta \) be given such that \( \text{Almfan}(\beta) \) and \( \beta(\langle \rangle) = 0 \). Assume \( \text{Bar}_{\mathcal{F}_\alpha}(B) \). Define \( B' := B \cup \{s \mid \beta(s) = 0 \} \). In the proof of Theorem 2.4, we have seen how to prove that \( \text{Bar}_{\omega}(B') \).

Let \( E \) be the set of all \( s \) such that either \( \beta(s) \neq 0 \) or \( \beta(s) = 0 \) and \( \exists \alpha[E_\alpha \subseteq B \land E_\alpha \text{ is almost-finite} \land \text{Bar}_{\mathcal{F}_\alpha}(E_\alpha)] \).

We prove that \( B \subseteq E \). For every \( s \), if \( \beta(s) = 0 \) and \( s \in B \), define \( \alpha \) such that \( \forall n \exists (n) = s + 1 \) and note that \( \{s\} = E_\alpha \subseteq B \) and \( E_\alpha \) is finite and \( \text{Bar}_{\mathcal{F}_\alpha}(E_\alpha) \).

We prove that \( E \) is inductive. Let \( s \) be given such that \( \forall m[s \ast r \in m] \in E \). Using \( \text{AC}_\omega, \omega \), find \( \alpha \) such that, for all \( m \), if \( \beta(s \ast \langle m \rangle) = 0 \), then \( E_\alpha \subseteq B \) and \( E_\alpha \) is almost-finite and \( \text{Bar}_{\mathcal{F}_\alpha}(E_\alpha) \), and, if \( \beta(s \ast \langle m \rangle) \neq 0 \), then \( \alpha^m = 0 \) and \( E_\alpha \neq 0 \). Note that \( \text{Almfan}(\beta) \) and \( \forall \zeta \in [\omega]^s \exists m[\alpha^m(\zeta) = 0] \). Use Lemma 12.3(v) and conclude that \( \bigcup_m E_\alpha \) is almost-finite. Note that \( \text{Bar}_{\mathcal{F}_\alpha}(\bigcup_m E_\alpha) \) and conclude that \( s \in E \). We thus see that \( \forall s \exists m[s \ast r \in m] \in E \rightarrow s \in E \), i.e. \( E \) is inductive.

Note that \( E \) is also monotone, i.e. \( \forall s \forall m[s \in E \rightarrow s \ast \langle m \rangle \in E] \).

Using \( \text{BARIND} \), conclude: \( \langle \rangle \in E \), i.e. \( \exists \alpha[E_\alpha \subseteq B \land E_\alpha \text{ is almost-finite} \land \text{Bar}_{\mathcal{F}_\alpha}(E_\alpha)] \). \( \square \)

12.4. The Almost-fan Theorem, \( \text{AlmfT} \):

\[ \forall \beta[\text{Almfan}(\beta) \to \forall \alpha[\langle \text{Thinbar}_{\mathcal{F}_\alpha}(D_\alpha) \land \forall s \in D_\alpha[\beta(s) = 0] \rangle \to \forall \zeta \in [\omega]^s \exists m[\zeta(m) \notin D_\alpha]]], \]

**Theorem 12.5.** \( \text{BIM + ALMFAN} \vdash \text{AlmfT} \).

**Proof.** Let \( \beta, \alpha \) be given such that \( \text{Appfan}(\beta) \) and \( \text{Thinbar}_{\mathcal{F}_\alpha}(D_\alpha) \) and \( \forall s \in D_\alpha[\beta(s) = 0] \). Applying Theorem 12.4 find \( \gamma \) such that \( E_\gamma \subseteq D_\alpha \) and \( \text{Bar}_{\mathcal{F}_\alpha}(E_\gamma) \) and \( E_\gamma \) is almost-finite. As \( \forall s \in D_\alpha \forall t \in D_\alpha[s \subseteq t \rightarrow s = t] \), conclude that \( E_\gamma = D_\alpha \) and, by Lemma 12.3(ii), \( \forall \zeta \in [\omega]^s \exists m[\zeta(m) \notin D_\alpha] \). \( \square \)

The Almost-fan Theorem occurs in 27 and 38.

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28It is not true that the union of a collection of sets that is bounded-in-number and whose members are bounded-in-number is itself bounded-in-number, consider \( \{\{k_0, k_0 + 1, \ldots, 2 \cdot k_0\}\} = \bigcup_{n=0}^{k_0} C_n \) where, for each \( n, C_n = \{n, n+1, \ldots, 2n\} \). This is why the definition of an approximate fan is not completely parallel to the definition of a fan, see also Lemma 2.4.

29We do not know if the use of the Axiom \( \text{AC}_\omega, \omega \) is avoidable.

30If \( \beta(\langle \rangle) \neq 0 \), then \( \mathcal{F}_\beta = \emptyset \) and there is nothing to prove.
12.5. The Approximate-fan Theorem, AppFT:
\[ \forall \beta \forall \alpha [(Appfan^+(\beta) \land Thinbar_(D_o) \land \forall s \in D_o[\beta(s) = 0]) \rightarrow \forall \zeta \in \omega_1 \exists n(\zeta(n) \notin D_o)]. \]

Theorem 12.6. BIM + AlmFT ⊢ AppFT.

Proof. Obvious, as every approximate fan is an almost-fan. □

Theorem 12.7. BIM + AppFT ⊢ FT.

Proof. Assume AppFT. Using Theorem 2.2 we will prove FT. Let α be given such that Thinbar_(2c)(D_o) and D_o ⊆ 2^<ω. Using AppFT, conclude that D_o is almost-finite. Define ζ in [ω]ω such that, for each n, if \( \neg \forall \gamma \in 2^<i < n[i(i) \cap \gamma] \), then ζ(n) = \( \mu_{p[p \in D_o \land \forall i < n[p \notin \zeta(i)]]} \). Find p such that ζ(p) ∈ D_o. Conclude that \( \forall n > ζ(p)[n \notin D_o] \). We thus see that \( \forall \alpha[Thinbar_(2c)(D_o) \rightarrow \exists m \forall n > m[n \notin D_o]] \). Using Theorem 2.2, we conclude FT.

BIM does not prove FT → AppFT, see [15] Corollary 10.6.

In BIM, AppFT has a number of important equivalents. Two of them are the intuitionistic Infinite Ramsey theorem and a contrapositive form of the Bolzano-Weierstrass Theorem, see [15] Theorem 11.2 and Corollary 9.8 and [18] Sections 7.5 and 7.6.

The relation between FT and AppFT in BIM may be compared to the relation between WKL and KL in RCA_0. In the classical context of RCA_0, one studies two extensions of WKL. The first one is Bounded König’s Lemma BKL, that (for a classical reader) would coincide with (a contraposition of) the second formulation of FT [2.2.4]. The second one is König’s Lemma KL, that, similarly, would coincide with FT [2.2.5]. BKL is, in RCA_0, equivalent to WKL, see [27] Lemma IV.1.4, just as, in BIM, the first and second formulation of FT are equivalent. KL, on the other hand, is definitely stronger than WKL, as RCA_0 + KL is equivalent to ACA_0.

As we observed before, see Theorem 1.12 BIM + Weak-Π^0_1-AC_ωω ⊢ FT ↔ FT^+. □

From a constructive point of view, the axiom Weak-Π^0_1-AC_ωω is weak indeed, as the antecedent is read constructively. Note that BIM + AC_ωω! ⊢ Weak-Π^0_1-AC_ωω. In a classical context, the rôle of a countable axiom of choice is very different from the rôle of such an axiom in a constructive context, see 3 Appendix 1. It seems that FT^+ is too close to FT for being a good candidate to play, in the intuitionistic context, the rôle played by KL in the classical context. Note that our ‘axiom’ AppFT is a possibly better candidate, as, for a classical reader, AppFT is also indistinguishable from KL.

Some authors have called our FT the Weak Fan Theorem WF, see for instance [20], but we decided not to follow them. There is a constructive version of Weak Weak König’s Lemma WWKL, see [27] Definition X.1.7, that is called WWFT, see [24].

13. Notation and conventions

In this Section we explain how, in BIM, some useful notation is introduced and some elementary results are proven.

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31I do not know if BIM proves AppFT → AlmFT.
32One may learn from [17] Theorem 9.20 that, using countable choice, in fact only Weak-Π^0_1-AC_ωω, one may constructively derive KL from WKL. On the problem of treating non-constructive assumptions in a constructive context, see the end of Section 2.
13.1. Finite and infinite sequences of natural numbers.

BIM contains a constant $p$ denoting the function enumerating the prime numbers:

\[ p(0) = 2, p(1) = 3, \ldots \]

We code finite sequences of natural numbers by natural numbers:

\[
\langle \rangle := 0 \text{ and, for each } k > 0, \text{ for all } m_0, m_1, \ldots m_{k-1},
\]

\[
\langle m_0, \ldots, m_{k-1} \rangle = p(k - 1) \cdot \prod_{i<k} p(i)^{m_i} - 1
\]

\[ \text{length}(0) := 0 \text{ and,}
\]

for each $s > 0$, \[ \text{length}(s) := 1 + \text{the largest } k \text{ such that } p(k) \text{ divides } s + 1.\]

If $i < \text{length}(s) - 1$, then $s(i) := \text{the largest } m \text{ such that } p(i)^m \text{ divides } s + 1$, and,

if $i = \text{length}(s) - 1$, then $s(i) := \text{the largest } m \text{ such that } p(i)^{m+1} \text{ divides } s + 1$, and,

if $i \geq \text{length}(s)$ then $s(i) := 0$.

Note: if $\text{length}(s) = k$, then $s = (s(0), s(1), \ldots, s(k - 1))$. Also note: $\forall s[s \geq \text{length}(s)]$.

$a * b$ is the number $s$ satisfying: \[ \text{length}(s) = \text{length}(a) + \text{length}(b) \text{ and, for each } n, \text{ if } n < \text{length}(a), \text{ then } s(n) = a(n) \text{ and,}
\]

if $\text{length}(a) \leq n < \text{length}(s)$, then $s(n) = b(n - \text{length}(a))$.

$a * a$ is the element $\beta$ of $\omega^\omega$ satisfying: for each $n$, \[ \text{if } n < \text{length}(a), \text{ then } \beta(n) = a(n), \text{ and, if } \text{length}(a) \leq n, \text{ then } \beta(n) = a(n - \text{length}(a)) .
\]

For $n \leq \text{length}(a)$, $\alpha(n) := (a(0), \ldots, a(n - 1))$. If confusion seems unlikely, we sometimes write: \[ \alpha^n \text{ and not: } (\alpha(n))^n .
\]

We extend the language of BIM by introducing $\in$ and terms denoting subsets of $\omega$. This is not a real extension of the language of BIM. Formulas in which the new symbols occur are abbreviations of formulas in which they do not occur.

Given a formula $\varphi = \varphi(n)$ we may introduce a ‘set term’ $T_\varphi$. The basic formula $t \in X_\varphi$ is an abbreviation of $\varphi(a(t))$. Here is a first example.

\[ 2^{< \omega} := \text{Bin} := \{ a \mid \forall n < \text{length}(a) a(n) = 0 \lor a(n) = 1 \} . \]

\[ \alpha \in 2^{< \omega} \text{ is an abbreviation of } \forall n < \text{length}(a)[a(n) = 0 \lor a(n) = 1] . \]

\[ \omega^m := \{ s \mid \text{length}(s) = m \} . \]

\[ [\omega]^m := \{ s \in \omega^m \mid \forall[i + 1 < m \rightarrow s(i) < s(i + 1)] \} . \]

Given terms $A, B$ denoting subsets of $\omega$, \[ ‘A \subseteq B’ \text{ is an abbreviation of } \forall n[n \in A \rightarrow n \in B] \).

We also introduce terms denoting subsets of $\omega^\omega$, for instance:

\[ 2^{< \omega} := \{ \gamma \mid [\forall n[\gamma(n) = 0 \lor \gamma(n) = 1] \} . \]

\[ \alpha \in 2^{< \omega} \text{ is an abbreviation of } \forall n[\alpha(n) = 0 \lor \alpha(n) = 1] . \]

\[ [\omega]^m := \{ \zeta \mid \forall n[\zeta(s) < \zeta(s + 1)] \} . \]

Given terms $X, Y$ denoting subsets of $\omega^\omega$, \[ ‘X \subseteq Y’ \text{ abbreviates } \forall \alpha[\alpha \in X \rightarrow \alpha \in Y] \)."
Given a term $X$ denoting a subset of $\omega^\omega$, we introduce, for all $s$, $X \cap s := \{ \alpha \in X \mid s \subseteq \alpha \}$.

Given a term $X$ denoting a subset of $\omega^\omega$, and a term $B$ denoting a subset of $\omega$, we let $\text{Bar}_X(B)$ be an abbreviation of $\forall \alpha \in X \exists n[\exists m \in B]$.

Note that $\text{Bar}_X(B)$ is a formula scheme, that becomes a formula if one substitutes formulas defining $X$, $B$, respectively.

From now on, we will express ourselves more informally, as in the following example:

For each $X \subseteq \omega^\omega$, for each $B \subseteq \omega$,

\[ \text{Thin} \text{Bar}_X(B) \leftrightarrow (\text{Bar}_X(B) \land \forall s \in B \forall t \in B[s \subseteq t \rightarrow s = t]). \]

Note that we are not extending the language of BIM by second-order variables.

13.2. Decidable and enumerable subsets of $\omega$.

$D_\alpha := \{ i \mid \alpha(i) \neq 0 \}$. $D_\alpha$ is the subset of $\omega$ decided by $\alpha$.

The expression $i \in D_\alpha$ is an abbreviation of $\alpha(i) \neq 0$.

$X \subseteq \omega$ is decidable or $\Delta_0^0$ if and only if $\exists \alpha[X = D_\alpha]$.

$D_\alpha$ is the subset of $\omega$ decided by $\alpha$.

Given any $\alpha$, define $\beta$ such that

$\forall n[\alpha(n) = \beta(n) = 0 \lor (\alpha(n) \neq 0 \land \beta(n) = n + 1)]$, and note: $D_\alpha = E_\beta$.

We thus see: $\text{BIM} \vdash \forall \alpha \exists \beta[D_\alpha = E_\beta]$.

13.3. Open and closed subsets of $\omega^\omega$, spreads and fans.

$G_\beta := \{ \gamma \mid \exists n[\beta(\gamma n) \neq 0] \}$.

$G \subseteq \omega^\omega$ is open or $\Sigma_1^0$ if and only if $\exists \beta[G = G_\beta]$.

$F_\beta := \omega^\omega \setminus G_\beta = \{ \gamma \mid \forall n[\beta(\gamma n) = 0] \}$.

$F \subseteq \omega^\omega$ is closed or $\Pi_1^0$ if and only if $\exists \beta[F = F_\beta]$.

$\text{Spr}(\beta) \leftrightarrow \forall s[\beta(s) = 0 \lor \exists n[\beta(s \cup \{ n \}) = 0]]$.

$X \subseteq \omega^\omega$ is a spread if and only if $\exists \beta[\text{Spr}(\beta) \land X = F_\beta]$.

In intuitionistic mathematics, not every closed subset of $\omega^\omega$ is a spread, see Lemma 2.12.

\[ \text{Fan}(\beta) \leftrightarrow (\text{Spr}(\beta) \land \forall s \exists m[\beta(s \cup \{ m \}) = 0 \rightarrow m \leq n]) \]
and

\[ \text{Fan}^+(\beta) \leftrightarrow (\text{Spr}(\beta) \land \exists \gamma \forall s \exists m[\beta(s \cup \{ m \}) = 0 \rightarrow m \leq \gamma(s)]). \]

$X \subseteq \omega^\omega$ is a fan if and only if $\exists \beta[\text{Fan}(\beta) \land X = F_\beta]$.

$X \subseteq \omega^\omega$ is an explicit fan if and only if $\exists \beta[\text{Fan}^+(\beta) \land X = F_\beta]$.

13.4. Subsequences.

$\forall n \forall m[\alpha^{\langle n \rangle}(m) := \alpha((n) \ast m)]$.

$\alpha^n$ is called the $n$-th subsequence of the infinite sequence $\alpha$.

---

33 A referee called my attention to the fact that, in general, $\neg \exists \zeta \in [\omega]^{\omega} \forall m[\alpha^{\langle n \rangle}(m) = \alpha \circ \zeta(m)]$. The use of the term ‘subsequence’ therefore is slightly misleading. Also, in general, $\neg \exists n \exists m[\alpha^{\langle n \rangle}(m) = \alpha(0)]$. One might feel these are disadvantages of the definition we use, but they do not harm our arguments.
Partial continuous functions from $\omega^\omega$ to $\omega$ and from $\omega^\omega$ to $\omega^\omega$. 

\(Fun_0(\varphi) \leftrightarrow \forall a \in E_\varphi \forall b \in E_\varphi[a' \sqsubseteq b' \Rightarrow a'' = b'']\).

\(Dom_0(\varphi) := \{ \alpha \mid \exists a \in E_\varphi[a' \sqsubseteq a]\}\).

Assume: \(Fun_0(\varphi)\) and \(\alpha \in Dom_0(\varphi)\).

Then \(\varphi(\alpha) := \text{the element } \gamma \text{ of } \omega\) such that \(\exists n[[\overline{\alpha} n, \gamma] \in E_\varphi]\).

For every \(X \subseteq \omega^\omega\), for every \(\varphi, \varphi : X \rightarrow \omega \leftrightarrow (Fun_0(\varphi) \land X \subseteq Dom_0(\varphi))\).

\(Fun_1(\varphi) \leftrightarrow \forall a \in E_\varphi \forall b \in E_\varphi[a' \sqsubseteq b' \Rightarrow a'' \sqsubseteq b'']\).

\(Dom_1(\varphi) := \{ \alpha \mid \forall n \exists a \in E_\varphi[a' \sqsubseteq \alpha \land \text{length}(a'') \geq n]\}\).

\(\varphi[\alpha] := \text{max}(\{ t \mid \exists b \in E_\varphi[b' \sqsubseteq \alpha \land b'' = t] \})\).

Assume: \(Fun_1(\varphi)\) and \(\alpha \in Dom_1(\varphi)\).

Then \(\varphi[\alpha] := \text{the element } \gamma \text{ of } \omega^\omega\) such that \(\varphi : \alpha \mapsto \gamma\).

For every \(X \subseteq \omega^\omega\), for every \(\varphi, \varphi : X \rightarrow \omega^\omega \leftrightarrow (Fun_1(\varphi) \land X \subseteq Dom_1(\varphi))\).

Integers and rationals.

\(m +_\mathbb{Z} n \leftrightarrow m' + n'' = m'' + n'\).

\(m <_\mathbb{Z} n \leftrightarrow m'' < m'' + n'\).

\(0_\mathbb{Z} := (0, 0)\).

\(m +_\mathbb{Z} n = (m' + n', m'' + n'').\)

\(m -_\mathbb{Z} n = (m' + n'', m'' + n').\)

\(m \cdot_\mathbb{Z} n := (m' \cdot n' + m'' \cdot n'', m'' \cdot n' + m'' \cdot n').\)

\(Q := \{ n \mid n'' >_\mathbb{Z} 0_\mathbb{Z}\}\).

\(m =_\mathbb{Q} n \leftrightarrow m' = n'' =_\mathbb{Q} m'' \cdot n'\).

\(m <_\mathbb{Q} n \leftrightarrow m'' < m'' \cdot n'\).

\(m \leq_\mathbb{Q} n \leftrightarrow m' \cdot n'' \leq_\mathbb{Q} m'' \cdot n'\).

\(m \leq_\mathbb{Q} n \leftrightarrow \text{max}_\mathbb{Q}(m, n) =_\mathbb{Q} n \leftrightarrow \text{min}_\mathbb{Q}(m, n) =_\mathbb{Q} m\).

\(m +_\mathbb{Q} n = (m' \cdot n' + m'' \cdot n', m'' \cdot n'').\)

\(m -_\mathbb{Q} n = (m' \cdot n'' - n_\mathbb{Z} n', m'' \cdot n'').\)

\(m \cdot_\mathbb{Q} n = (m' \cdot n', m'' \cdot n'').\)

\(S := \{ s \mid s' \in \mathbb{Q} \land s'' <_\mathbb{Q} s''\}\).

\(s \sqsubseteq t \leftrightarrow (t' <_\mathbb{Q} s' \land s'' <_\mathbb{Q} s'')\).

\(s \sqsubseteq_\mathbb{Q} t \leftrightarrow (t' \leq_\mathbb{Q} s' \land s'' \leq_\mathbb{Q} s'')\).

\(s \sqsubseteq t \leftrightarrow s'' <_\mathbb{Q} t''\).

\(s \leq_\mathbb{Q} t \leftrightarrow s'' \leq_\mathbb{Q} t''\).

\(s \neq_\mathbb{Q} t \leftrightarrow (s <_\mathbb{Q} t \lor t <_\mathbb{Q} s)\).

For each \(n\), we define \(n_\mathbb{Q}\) in \(\mathbb{Q}\) by: \(n_\mathbb{Q} = ((n, 0), (1, 0))\).

For all \(s\) in \(S\), \(\overline{\mathbb{Q}}(s)\) is the element \(u\) of \(S\) satisfying:

\(u' +_\mathbb{Q} u'' =_\mathbb{Q} s' +_\mathbb{Q} s''\) and \(u'' -_\mathbb{Q} u' =_\mathbb{Q} 2_\mathbb{Q} \cdot Q(s'' -_\mathbb{Q} s').\)

Note that \(\forall s \in S \exists t \in S[s \sqsubseteq t \rightarrow \overline{\mathbb{Q}}(s) \sqsubseteq \overline{\mathbb{Q}}(t)]\).

\(s +_\mathbb{Q} t := (s' +_\mathbb{Q} t', s'' +_\mathbb{Q} t'')\).

\(^{34}\text{See [14] Subsections 7.2 and 7.5}\)
that

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right-two-third part of

One may prove in Lemma 13.1.

Proof. H

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One may prove in

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1

].

This is not the same definition as in [44, Subsubsection 8.1.4]. We replaced ‘∈ S’ by ‘∈ S’.

Lemma 13.1. One may prove in BIM:

∀s ∈ S[t ∈ S|s ⊆ t → ∀α ∈ R∃n[s #∈ S α(n) ∨ α(n) ⊆ t]]

Proof. The proof is left to the reader.

13.8. [0, 1] and 2ω.

[0, 1] := {α ∈ R | 0R ≤ R α ≤ R 1R}.

[0, 1] := {α ∈ R | 0R < R α ≤ R 1R}, and [0, 1] := {α ∈ R | 0R ≤ R α < R 1R}.

For all α, β in R, [α, β) := {γ ∈ R | α ≤ R γ < R β}

[0, 1]^2 := {γ | ∀i < 2|γ|^i ∈ [0, 1]} and [0, 1]^ω := {γ | ∀n[γ]^n ∈ [0, 1]}.

H_α := {γ ∈ [0, 1] | ∃n ∈ S[α(n) = 0 ∧ γ(n) ⊆ s]}

H ⊆ R is open if and only if ∃α[∈ H = H_α].

Lemma 13.2. One may prove in BIM:

There exist σ,ψ such that

(i) σ : 2ω → [0, 1] and ∀δ ∈ [0, 1]∃γ ∈ 2ω[σ]γ = R δ].

(ii) ψ : ω→ ω and ∀αψγ ∈ 2ω[γ ∈ G(σ|α) σ ∈ H_α].

Proof. (i) Define λ and ρ such that, for each s in S, λ(s) = (′s, ′′s + Q ′′s′′) and ρ(s) = (′s + Q ′′s′′, ′′s′′). For each s in S, λ(s) is the left-two-third part of s and ρ(s) is the right-two-third part of s. Define ν such that ν( ) = (0Q, 1Q) and, for all s in Bin, ν(s + (0)) = λ(ν(s)) and ν(s + (1)) = ρ(ν(s)). Define σ : 2ω → [0, 1] such that ∀γ ∈ 2ω[(σγ)]n = ν(γn)]. One may prove that ∀δ ∈ [0, 1]∃γ ∈ 2ω[σ]γ = R δ].

(ii) Define ψ : ω→ ω such that ∀αψs[(ψ|α)(s) = 0 ↔ (s ∈ 2ω ∧ ∃t < s[α(s) ⊆ t ∧ α(t) = 0])]. One may prove that ∀αψγ ∈ 2ω[γ ∈ G(σ|α) σ ∈ H_α].

Lemma 13.3. One may prove in BIM:

There exist τ,χ such that

(i) τ : 2ω → [0, 1] and ∀γ ∈ 2ω[τδ = τγ]δ ∈ τ[γ # R τ|δ].

(ii) χ : ω→ ω and ∀αψγ ∈ 2ω[γ ∈ G(σ|α) σ ∈ H_α].

(iii) ∀δ ∈ [0, 1]∀γ ∈ 2ω[γ # R δ → ∀α[δ ∈ H_α]].

(iv) ∀δ ∈ [0, 1]∀γ ∈ 2ω[γ # R δ → ∀α[δω ∈ H_α]].

This is not the same definition as in [35] Subsection 8.1.4]. We replaced ‘∈ S’ by ‘∈ S’.
Proof. (i) Define \( \pi_0, \pi_1, \pi_2, \pi_3 \) such that, for each \( s \) in \( \mathcal{S} \), for each \( i \leq 5 \), \( \pi_i(s) := (\frac{3}{5}s' + \frac{3}{5}s'' \frac{1}{\gamma} + \frac{3}{5}s + \frac{3}{5}s''' + s''') \). For each \( s \) in \( \mathcal{S} \), for each \( i \leq 5 \), \( \pi_i(s) \) is the \( i \)-th fifth part of \( s \). Define \( \varepsilon \) such that \( \varepsilon(\varepsilon(s)) = (0, Q, 0) \) and, for all \( a \) in \( 2^{<\omega} \), \( \varepsilon(a+0) = \varepsilon(\varepsilon(a)) \) and \( \varepsilon(s+1) = \pi_3(\varepsilon(a)) \). Define \( \tau : 2^\omega \rightarrow [0, 1] \) such that \( \forall \gamma \in 2^\omega \langle \gamma \rangle(n) = \varepsilon(\gamma(n)). \) One may prove that \( \forall \gamma \in 2^\omega \forall \delta \in 2^\omega \langle \gamma \rangle \neq \delta \rightarrow \tau(\gamma) \neq \tau(\delta). \)

(ii) Define \( \chi : \omega^* \rightarrow \omega^* \) such that, for all \( \alpha \), for all \( s \), \( \langle \chi(\alpha) \rangle(s) \neq 0 \) if and only if \( \exists \tau < s[t \in 2^{<\omega} \land s \in \varepsilon(t) \land \alpha(t) \neq 0] \lor \exists n < s[t \in 2^{<\omega}[\text{length}(t) = n \rightarrow s \notin \varepsilon(t)] \). We now prove that \( \forall \alpha \forall \gamma \in 2^\omega \langle \gamma \rangle \neq 0 \rightarrow \tau(\gamma) \in H_\alpha[n] \). Let \( \alpha \) be given such that \( \gamma \in 2^\omega \). Assume that \( \gamma \in G_\alpha \). Find \( n \) such that \( \alpha(n) \neq 0 \). Find \( k > n \) such that \( \varepsilon(\gamma(k)) / \gamma(n) \). Note that \( \varepsilon(\gamma(k)) \in \gamma(n) \) and conclude \( (\chi(\alpha)) \neq 0 \). As \( \langle \tau(\gamma) \rangle(k + 1) \in \langle \tau(\gamma) \rangle(k) = \varepsilon(\gamma(k)), \) conclude that \( \tau(\gamma) \in H_\alpha \). Conversely, assume that \( \tau(\gamma) \in H_\alpha \). Note that \( \forall n \langle \gamma(n) \rangle \in \tau(\gamma) \). Conclude that \( \forall \alpha \in \tau(\gamma) \land \forall n \in 2^{<\omega}[\text{length}(t) = n \rightarrow \alpha(n) \neq 0] \). Find \( n \) such that \( \tau(\gamma(n)) \neq 0 \). Find \( t \in 2^{<\omega} \) such that \( s \in \varepsilon(t) \) and \( \alpha(t) \neq 0 \). Note that \( \forall \alpha \neq 0 \in \tau(\gamma) \) and conclude that \( \gamma \in G_\alpha \). We thus see that \( \forall \gamma \in 2^\omega \langle \gamma \rangle \neq 0 \rightarrow \tau(\gamma) \in H_\alpha \).

(iii) Assume that \( \delta \in [0, 1] \). Note that \( \forall a \in 2^{<\omega}[\varepsilon(a+0) \neq Q \varepsilon(a+1)] \). Define \( \gamma \) in \( 2^\omega \) such that, for all \( m, p \), if \( p = \mu p \varepsilon(\gamma(m) + 0) \neq Q \varepsilon(\gamma(m) + 1) \), then \( \gamma(m) \neq 0 \rightarrow \delta(m) \neq Q \varepsilon(\gamma(m) + 1) \). One may prove, by induction on \( n \), that, for each \( n \), there exists \( p \) such that \( \forall t \in 2^{<\omega}[\langle \text{length}(t) = n \land t \in \gamma(n) \rightarrow \delta(p) \neq \varepsilon(t)] \). Assume that \( \forall \delta \neq \tau(\gamma) \). Find \( s \) such that \( \forall t \in 2^{<\omega}[\langle \text{length}(t) = n \land t \in \gamma(n) \rightarrow \delta(p) \neq \varepsilon(t)] \). Find \( n \) such that \( \forall a \neq 0 \in \tau(\gamma) \land \forall n \in 2^{<\omega}[\text{length}(t) = n \rightarrow \delta(p) \neq \varepsilon(t)] \). Conclude that \( \forall \delta \neq \tau(\gamma) \) and, for each \( \alpha \), \( \delta \neq \tau(\gamma) \). We thus see that, if \( \delta \neq \tau(\gamma) \), then \( \forall n \delta \neq \tau(\gamma) \).

(iv) Assume: \( \delta \in [0, 1]^\omega \). Define \( \gamma \) in \( 2^\omega \) such that, for all \( n, m, p \), if \( p = \mu p \varepsilon(\gamma(m) + 0) \neq Q \varepsilon(\gamma(m) + 1) \), then \( \gamma(m) = 0 \rightarrow \delta(m) \neq Q \varepsilon(\gamma(m) + 1) \). Conclude, following the argument for (iii), that \( \forall n \delta \neq \tau(\gamma) \rightarrow \delta(n) \neq \tau(\gamma) \in H_\alpha \).

\[ 13.9. \text{Real functions from } [0, 1] \rightarrow \mathcal{R}. \]

\( \varphi : [0, 1] \rightarrow \mathcal{R} \) if and only if

(i) \( \forall a \in E_\varphi \exists n \in \mathcal{S} \langle \alpha(n), s \rangle \in E_\varphi \land s'' - Q s' = Q \frac{3}{5} \).

\( \mathcal{R}[0, 1] := \{ \varphi \mid \varphi : [0, 1] \rightarrow \mathcal{R} \}. \)

Assume: \( \varphi : [0, 1] \rightarrow \mathcal{R} \).

We define, for each \( \alpha \in [0, 1] \), for each \( \beta \in \mathcal{R} \), \( \varphi : \alpha \rightarrow \beta \) if and only if \( \forall n \exists m \in E_\varphi \langle \alpha(n), m \rangle \in \mathcal{S} \land p'' \in \mathcal{B}(n) \).

For each \( \alpha \in [0, 1] \), we let \( \varphi^R(\alpha) \) be the element \( \beta \in \mathcal{R} \) such that, for each \( n \in \mathcal{B}(s) \), where \( s \) is the least \( t \) such that \( t \in \mathcal{S} \) and \( t^\omega = t^\omega \).

\( \text{Note: } \varphi : \alpha \rightarrow \varphi^R(\alpha). \)

\[ 13.10. \text{Game-theoretic terminology.} \]

\( s : n \rightarrow k \leftrightarrow \langle \text{length}(s) = n \land \forall j < n[s(j) < k] \rangle. \)

\( \text{Seq}(n, l) := \{ s \mid s : n \rightarrow l \}. \)

Note that \( \text{Seq}(n, 2) = \{ s \in 2^{<\omega} \mid \text{length}(t) = n \rangle = 2^{<\omega} \cap \omega^n \). \)

\( c \in E_\varphi \leftrightarrow \forall i [2i < \text{length}(c) \rightarrow c(2i) = s(\overline{\tau}(2i))] \).

\( c \in E_{\overline{t}} \leftrightarrow \forall i [2i + 1 < \text{length}(c) \rightarrow c(2i + 1) = t(\overline{\tau}(2i + 1))] \).

\[ \text{The definition slightly deviates from the one used in Subsection 8.4.} \]

\[ \text{Note that, if } \overline{\tau}(2i) = 0. \]

\[ \text{Note that, if } \overline{\tau}(2i + 1) \geq \text{length}(t), \text{ then } t(\overline{\tau}(2i)) = 0. \]
\[
\begin{align*}
    \textbf{(The numbers } s, t \text{ should be thought of as strategies for player } I, II, \text{ respectively.)}
    \\
    c \in \sigma \iff \forall i [2i < \text{length}(c) \rightarrow \text{c}(2i) = \sigma(\text{c}(2i))].
    \\
    c \in \tau \iff \forall i [2i + 1 < \text{length}(c) \rightarrow \text{c}(2i + 1) = \tau(\text{c}(2i + 1))].
    \\
    \gamma \in I \sigma \iff \forall i [\gamma(2i) = \sigma(\gamma(2i))].
    \\
    \gamma \in I \tau \iff \forall i [\gamma(2i + 1) = \tau(\gamma(2i + 1))].
    \\
    \gamma \in s \iff \forall i [\gamma(2i) < \text{length}(s) \rightarrow \gamma(2i) = s(\gamma(2i))].
    \\
    \gamma \in t \iff \forall i [\gamma(2i + 1) < \text{length}(t) \rightarrow \gamma(2i + 1) = t(\gamma(2i + 1))].
    \\
    \omega \times \omega := \{ s \mid \text{length}(s) = 2 \land s(0) < 2 \},
    \\
    \omega \times 2 := \{ s \mid \text{length}(s) = 2 \land s(1) < 2 \}.
\end{align*}
\]

For each } n \text{ > 0, } (\omega \times 2)^n := \{ s \mid \text{length}(s) = 2n \land \forall i < n[s(2i + 1) < 2] \} \text{ and }\]

\[
(\omega \times 2)^n \times \omega := \{ s \mid \text{length}(s) = 2n + 1 \land \forall i < n[s(2i + 1) < 2] \}.
\]

\[
(\omega \times 2)^n := \bigcup_n (\omega \times 2)^n.
\]

\[
H alfbin := (\omega \times 2)^{<\omega} \cup ((\omega \times 2)^{<\omega} \times \omega) = \bigcup_n (c_n \mid \gamma \in (\omega \times 2)^n).
\]

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