Trace Dynamics as a model for emergent spacetime

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Abstract. Trace Dynamics is a classical theory of non-commuting matrices, which uses cyclic permutation inside a trace to define the derivative with respect to an operator. We have used the methods of Trace Dynamics to construct a non-commutative special relativity. We have defined a line-element using the trace over spacetime coordinates, which are assumed to be operators. The line-element is shown to be invariant under generalized Lorentz transformations, and is used to construct a non-commutative relativistic dynamics. We have been motivated for such an operator structure at a more fundamental level, and attempt to obtain an emergent picture of classical spacetime.

1. Introduction
Trace Dynamics [TD] is the classical dynamics of (generic $N \times N$) matrices (equivalently, operators) $\{q_r\}$, whose elements can be either even grade [Bosonic sector] or odd grade [Fermionic sector] elements of Grassmann algebra [1]. The Lagrangian $L[\{q_r\}, \{\dot{q}_r\}]$ in this programme has been defined as the trace of a polynomial function of the matrices $\{q_r\}$ and their time derivatives $\{\dot{q}_r\}$. The derivative with respect to an operator $O$, of the trace $P$ of a polynomial $P$ made out of non-commuting operators is defined as follows: Infinitesimal variation in $TrP$ is written and arranged as:

$$\delta TrP = \delta P = Tr \frac{\delta P}{\delta O} \delta O = Tr \frac{\delta P}{\delta O} \delta O,$$

with $\delta P/\delta O$ being designated as the trace derivative. By proceeding as in ordinary classical mechanics, one constructs an action; Lagrangian dynamics is derived from an action principle, and conjugate momenta $p_r$, a trace Hamiltonian $H$ and Hamilton’s equations of motion are constructed. Apart from the trace Hamiltonian, there are two other important conserved quantities: One is the ‘trace Fermionic number’ $N \equiv \sum_F q_r p_r$ obtained by summing over Fermionic variables. The other is the remarkable traceless and anti-self-adjoint Adler-Millard constant [2] given as:

$$\tilde{C} \equiv \sum_B [q_r, p_r] - \sum_F \{q_r, p_r\},$$

which is a result of the invariance of the Lagrangian under global unitary transformations of the $q_r$ and $p_r$. The subscripts $B/F$ denote sum over commutators/anti-commutators of Bosonic/Fermionic matrices. It is profound that such a conserved commutator should appear in a classical theory, in which the matrices and their commutators/anti-commutators take arbitrary values. The presence of this matrix-valued Noether charge, makes TD different from ordinary classical mechanics. However, at the effective level, when we do statistical mechanics with these
degrees of freedom, the statistical average of this charge is expected to throw some light on the effective commutation relations.

2. Non-commutative spacetime in Trace Dynamics approach

In TD, a background spacetime is given, and is made up of the usual spacetime coordinates \((x, t)\) of a Minkowski spacetime, and dynamics obeys Poincaré invariance. Here, we ask if the techniques of TD can be used to raise the coordinates \((x, t)\) to the level of non-commuting operators, and a non-commutative classical dynamics constructed, while still preserving the techniques of TD can be used to raise the coordinates \((x, t)\) to the level of non-commuting operators, and a non-commutative classical dynamics constructed, while still preserving the metric under certain spacetime diffeomorphisms (Poincaré like invariance). We have considered a set of four non-commuting finite dimensional operators \((\hat{x}, \hat{y}, \hat{z}, \hat{t})\) having arbitrary commutation relations (which are not fixed and are completely general as of now), and from which, a real line-element is defined as follows:

\[
ds^2 = T r d \hat{s}^2 \equiv T r [d\hat{t}^2 - d\hat{x}^2 - d\hat{y}^2 - d\hat{z}^2]. \tag{3}\]

We ask for the most general linear transformations as:

\[
\begin{align*}
\hat{t}' &= A_1 \hat{t} + B_1 \hat{x} + C_1 \hat{y} + D_1 \hat{z}, \\
\hat{x}' &= A_2 \hat{t} + B_2 \hat{x} + C_2 \hat{y} + D_2 \hat{z}, \\
\hat{y}' &= A_3 \hat{t} + B_3 \hat{x} + C_3 \hat{y} + D_3 \hat{z}, \text{ and} \\
\hat{z}' &= A_4 \hat{t} + B_4 \hat{x} + C_4 \hat{y} + D_4 \hat{z},
\end{align*}
\tag{4}\]

which keep the line element invariant. Here, \(A_i, B_i, C_i, \text{ and } D_i\) belong to a graded vector space given as:

\[
x = x_0 + \sum_i x_i \theta_i + \sum_{i<j} x_{ij} \theta_i \theta_j + ..., \tag{5}
\]

where \(\{x_0, x_i, x_{ij}, ...\} \in \mathcal{C}\). In the above equation, \(\theta_i\)s are anti-commuting Grassmann numbers.

Taking clue from the standard Lorentz transformation for a boost along the \(x\)-axis, and the symmetry between the \(y\)- and \(z\)-axes, we propose:

\[
\begin{align*}
\hat{t}' &= A \hat{t} + B \hat{x} + \alpha \hat{C} \hat{y} + \alpha \hat{C} \hat{z}, \\
\hat{x}' &= A \hat{x} + B \hat{t} + \alpha \hat{C} \hat{y} + \alpha \hat{C} \hat{z}, \\
\hat{y}' &= \hat{y} + \hat{B}_3 \hat{x} + \hat{C}_3 \hat{t} + \hat{D}_3 \hat{z}, \text{ and} \\
\hat{z}' &= \hat{z} + \hat{B}_3 \hat{x} + \hat{C}_3 \hat{t} + \hat{D}_3 \hat{y},
\end{align*}
\tag{6}\]

where \(\alpha\) is an ordinary real number. By substituting these transformations in Eq. (3), and by demanding the invariance of \(ds^2 = Tr d\hat{s}^2\), it can be shown that:

(i) All \(A, B, C, D\) must belong to Grassmann even sector.

(ii) Demanding that \(Tr d\hat{s}^2, Tr d\hat{t}^2, Tr d\hat{x}^2, Tr d\hat{y}^2, Tr d\hat{z}^2\) are real, and \(\hat{t}, \hat{x}, \hat{y}, \hat{z}\) are all forced to be either self-adjoint or anti-self-adjoint. Furthermore, for them to remain real in a generic transformation, and for invariance of adjointness type, it is required of all of them to be of same adjointness type and \(A, B, C, D\) to be Grassmann real.

(iii) \(A\) and \(B\) are Grassmann even elements, which satisfy \(A^2 - B^2 = 1\). This relation implies that generically \(A \text{ and } B\) are non-Grassmann real numbers (this assertion also requires that the transformations form a group).

(iv) \(\hat{C}_3 = -\hat{B}_3 \equiv E\) are those Grassmann even elements, which have their squares identically zero.
(v) $\tilde{D}_3$ is another Grassmann even element proportional to $E$, i.e., $\tilde{D}_3 = \kappa E$, where $\kappa$ is an ordinary real number.

(vi) $E = \alpha (B - A)\tilde{C}$, where $\tilde{C}$ is the Grassmann element, and $\alpha$ is a real number.

The requirement of existence of a symmetry group of these transformations (product of two transformations with the above properties must be a transformation with the same properties) further forces the choice $\alpha = 0$. Therefore, one is very rigidly constrained to restrict the transformation to the standard Lorentz transformation, with $\beta = -B/A$, and $A^2 - B^2 = 1$, as in ordinary special relativity. We have set $c = 1$; here, it is a universal constant without any further physical interpretation. With these choices of parameters, the transformation of symmetry is devoid of any Grassmann variable and with this structure, we clearly have obtained a symmetry group (the Lorentz group) over the field of real numbers as:

$$
\begin{pmatrix}
1 & \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\
\frac{-\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

We have shown next that it is possible to construct a Poincaré invariant dynamics. We have defined a four velocity $\hat{u}^\mu = d\hat{x}^\mu /ds$, and for a particle, a four-momentum $\hat{p}^\mu$ which satisfies $Tr[\hat{p}^\mu \hat{p}_\mu] = m^2$ (some invariant). An action operator $\hat{S}$ and the Trace Action are introduced as:

$$
S = Tr\hat{S} = \int ds \; Tr\hat{L}(\hat{x}, \dot{\hat{x}}),
$$

where dot represents derivative with respect to the parameter $s$. Extremization of the action leads to the Lagrange equations for the Trace Lagrangian $L = Tr\hat{L}$ as:

$$
\frac{\partial L}{\partial \hat{x}} - \frac{d}{ds} \frac{\partial L}{\partial \dot{\hat{x}}} = 0.
$$

Hence,

$$
\frac{\delta L}{\delta \hat{x}} = \int ds \frac{\delta L}{\delta \hat{x}} = \frac{\delta}{\delta \hat{x}} \int ds L.
$$

If we define $\delta L/\delta \dot{\hat{x}}$ as momentum conjugate to $\dot{\hat{x}}$, then we have:

$$
\hat{P}_x = \frac{\delta S}{\delta \hat{x}}.
$$

Derivatives with respect to operators are to be understood as trace derivatives. Momentum is the tangent to the trajectory in configuration space. Therefore, $\hat{P}_x^\mu$ is tangent to the curve drawn in the configuration space of $x^\mu$ co-ordinates. Hence,

$$
\hat{P}_x^\mu = \hat{p}_\mu, \quad and \quad \hat{P}_\mu = \frac{\delta S}{\delta \hat{x}^\mu}.
$$

Therefore, the Hamilton-Jacobi equation of motion is:

$$
Tr \left[ \left( \frac{\delta S}{\delta t} \right)^2 - \left( \frac{\delta S}{\delta \hat{x}} \right)^2 - \left( \frac{\delta S}{\delta \hat{y}} \right)^2 - \left( \frac{\delta S}{\delta \hat{z}} \right)^2 \right] = m^2.
$$
The Hamiltonian analogue of the dynamics is obtained by going to phase-space, and constructing the Hamiltonian as:

\[ \mathcal{H} = Tr \left( \sum_r \hat{p}_r \dot{x}_r - \hat{L} \right), \]

and the Hamilton’s equations of motion follow, by considering the variation as:

\[ \delta \mathcal{H} = Tr \sum_r (\epsilon_r \dot{x}_r \delta \hat{p}_r - \dot{\hat{p}}_r \delta \hat{x}_r), \]

where the Trace Hamiltonian is a trace functional of operators \( \hat{x}_r, \hat{p}_r \). Thus,

\[ \frac{\delta \mathcal{H}}{\delta \hat{x}_r} = -\dot{\hat{p}}_r, \quad \text{and} \quad \frac{\delta \mathcal{H}}{\delta \hat{p}_r} = \epsilon_r \dot{x}_r. \]

Now, the derivative can be performed element-wise, following Adler’s scheme and \( \epsilon_r = \pm 1 \), if the element belongs to Bosonic/Fermionic sector.

This phase-space can be equipped with a generalized Poisson bracket. For traced operators \( A(\hat{x}_r, \hat{p}_r, \tau), B(\hat{x}_r, \hat{p}_r, \tau) \), the generalized Poisson bracket is:

\[ \{ A, B \}_{GPB} = Tr \sum_r \epsilon_r \left( \frac{\delta A}{\delta \hat{x}_r} \frac{\delta B}{\delta \hat{p}_r} - \frac{\delta A}{\delta \hat{p}_r} \frac{\delta B}{\delta \hat{x}_r} \right). \]

With the help of Hamilton’s equations of motion given in Eq. (13), one can show that:

\[ \frac{dA}{d\tau} = \frac{\partial A}{\partial \tau} + \{ A, \mathcal{H} \}_{GPB}. \]

The generalized Poisson bracket satisfies the Jacobi-identity [1, 3], and the Lie algebras of symmetries of the theory can be represented as Lie algebras of trace functionals under generalized Poisson bracket operation.

Moreover, the line element defined above has its symmetry group as the Poincaré group, whose generators commute with the generators of global unitary transformations. Thus, Poincaré invariant theories will have, in complete analogy with the construction of Adler and Millard, a Noether charge corresponding to global unitary invariance of \( \mathcal{H} \), given by:

\[ Q = \sum_{r \in B} [\dot{x}_r, \hat{p}_r] - \sum_{r \in F} \{ \dot{x}_r, \hat{p}_r \}. \]

We have, thus, demonstrated that it is possible to construct a special relativity for non-commuting coordinate operators, by defining an infinitesimal distance using the Trace function.

As a more general extension of TD, we would like to treat space and time also as operators, and then derive the classical spacetime, and a quantum theory on the classical spacetime background, as thermodynamic approximation to this generalized Trace Dynamics. Further details and the motivation for such a construction can be found in [4].

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