Revisiting the Relation between Nonthermal Line Widths and Transverse MHD Wave Amplitudes

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Abstract

Observations and 3D MHD simulations of the transverse MHD waves in the solar corona have established that true wave energies hide in the nonthermal line widths of the optically thin emission lines. This displays the need for a relation between the nonthermal line widths and transverse wave amplitudes for estimating the true wave energies. In the past decade, several studies have assumed that the root mean square (rms) wave amplitudes are larger than the nonthermal line widths by a factor of \( \sqrt{2} \). However, a few studies have ignored this factor while estimating rms wave amplitudes. Thus, there appears to exist a discrepancy in this relation. In this study, we investigate the dependence of nonthermal line widths on wave amplitudes by constructing a simple mathematical model followed by 3D MHD simulations. We derive this relation for the linearly and circularly polarized oscillations, as well as oscillations excited by multiple velocity drivers. We note a fairly good match between mathematical models and numerical simulations. We conclude that the rms wave amplitudes are never greater than the nonthermal line widths, which raises questions about earlier studies claiming transverse waves carry enough energy to heat the solar corona.

Unified Astronomy Thesaurus concepts: Magnetohydrodynamics (1964); Solar corona (1483); Alfvén waves (23)

1. Introduction

The mechanism of heating of the solar corona can be broadly divided into two categories: heating due to the dissipation of waves, and heating by current dissipation due to the magnetic reconnection (Walsh & Ireland 2003; Parnell & De Moortel 2012). The propagation of MHD waves and their contribution to the coronal heating have been investigated for many decades (Klimchuk 2006; Banerjee et al. 2007; Hahn & Savin 2014; Arregui 2015). Perhaps the earliest signature of Alfvén(ic) waves in the solar atmosphere is the nonthermal broadening of optically thin emission lines due to unresolved wave amplitudes (Hollweg 1973; Doschek et al. 1976a, 1976b). After the launch of Solar and Heliospheric Observatory and Hinode, nonthermal broadening of the spectral lines has been unambiguously observed and reported in the solar atmosphere (Doschek et al. 1976a, 1976b; Feldman et al. 1976; Hassler et al. 1990; Banerjee et al. 1998, 2009; Doyle et al. 1998; O’Shea et al. 2005; Hahn et al. 2012). This nonthermal broadening is observed to vary with heights above the solar atmosphere (Doschek et al. 1976a; Doyle et al. 1998; Hahn et al. 2012). However, the nature of the variation is different in different regions of the Sun (Del Zanna et al. 2019; Gupta et al. 2019). These studies assume that the nonthermal broadening is produced by unresolved Doppler velocity amplitudes in time that are due to the presence of MHD waves in the solar corona. To complicate matters, several physical processes other than MHD waves affect nonthermal line widths in the solar atmosphere. Plasma upflows along the coronal loops and plumes can cause nonthermal broadenings at the footpoints of these structures (De Pontieu & McIntosh 2010; Tian et al. 2011a, 2012). Large-scale upflows (possibly responsible for the solar wind) in open magnetic field regions such as coronal holes can also contribute toward broadening of a spectral line (McIntosh et al. 2011; Tian et al. 2011b).

Both numerical simulations and observations have suggested the presence of counterpropagating waves in the solar atmosphere (Tomczyk & McIntosh 2009; Morton et al. 2015; van Ballegooijen et al. 2017). Such counterpropagating waves may lead to turbulence (also termed as Alfvén wave turbulence; AWT) and nonlinear cascade of energy to small spatial scales. This causes nonthermal broadening of emission lines (see Figure 8 in van Ballegooijen et al. (2017)). Apart from AWT, a new mechanism for generating turbulence by Alfvénic waves propagating in transversely inhomogeneous plasma has been reported by Magyar et al. (2017). This type of turbulence is termed as uniturbulence because it relaxes the need of counterpropagating waves to generate turbulence. A unidirectionally propagating wave in the presence of transverse inhomogeneities cause self-deformation and nonlinear cascade of energy (Magyar et al. 2019). Since the solar corona is highly structured (transversely inhomogeneous), uniturbulence is inevitable and could play an important role in the nonthermal broadening, especially in the open magnetic field regions where waves are predominantly unidirectional.

Finally, the superposition of structures swaying in different directions along the line of sight (LOS) also broadens an optically thin emission line (McIntosh & De Pontieu 2012; Pant et al. 2019). Recently, Pant et al. (2019) studied the role of LOS superposition and uniturbulence in explaining the observed spectral properties of transverse MHD waves propagating in the coronal holes. These authors have performed ideal 3D MHD simulations and reproduced the observed large nonthermal line widths, small resolved Doppler shifts, and a wedge-shaped correlation between them. These authors also reported that true wave amplitudes (and thus energy) are hidden in the nonthermal line widths.

In the present study, we ignore the effects of flows and (uni) turbulence, and we assume that the nonthermal broadening is caused by unresolved Doppler shifts generated due to the transverse MHD waves in the solar atmosphere. Often, the
The energy content of a transverse wave is computed by estimating the root mean square (rms) velocity of the wave amplitude (Hollweg 1981). Further, rms wave amplitudes can be estimated by measuring nonthermal broadening of optically thin emission lines. Thus, the observed magnitude of the nonthermal line widths can provide an estimate of the energy carried by transverse waves. Therefore, it is imperative to understand the relation between nonthermal line widths and rms wave amplitudes. In fact, such a relation was used to compute the Alfvénic wave energy flux (Hassler et al. 1990; Banerjee et al. 1998; Doyle et al. 1998). These studies assumed that the rms wave amplitude, \( v_{\text{rms}} \), is related to the nonthermal line width, \( \sigma_w \), as \( v_{\text{rms}} = \alpha \sigma_w \), considering different polarizations and direction of propagation of transverse waves relative to an LOS (Hassler et al. 1990). Here, \( \alpha \) was assumed to be approximately \( \sqrt{2} \). A similar relation was later reported by Doyle et al. (1998), accounting for two degrees of freedom for an Alfvén wave. Since then, this relation has been used extensively in many studies for estimating the energy carried by the Alfvénic wave using observed values of the nonthermal line widths in the solar corona (O’Shea et al. 2005; Banerjee et al. 2009; Hahn et al. 2012).

Surprisingly, a few studies have ignored \( \alpha \) while estimating the wave energies (Chae et al. 1998; Tu et al. 1998; Bemporad & Abbo 2012; Tian et al. 2012). Chae et al. (1998) argued that \( v_{\text{rms}} = \sigma_{\text{nt}} \) in a pure Alfvén wave because only the directions perpendicular to the wave motion contribute to the energy transport. On the other hand, Tu et al. (1998) argued that there are two degrees of freedom in the Alfvénic waves or turbulence, therefore \( v_{\text{rms}} = \sigma_{\text{nt}} \). Until today, there has been no convincing explanation of which one of these relations should be used to estimate the wave amplitude (and hence energy). This serves as a motivation for the study described in this paper. We present the relations between \( v_{\text{rms}} \) and \( \sigma_w \) for different velocity drivers by constructing a simple mathematical model. We then confirm the results of the mathematical model with MHD simulations. Finally, we discuss the implications of our study for studying the role of waves on coronal heating.

### 2. Mathematical Model

We assume a uniform plasma at temperature \( T \) oscillating along the LOS with a period \( P \) and velocity \( v \), as shown schematically in Figure 1(a). Further assuming that the emitting plasma is in thermal equilibrium, the shape of a spectral line \( (G(\lambda)) \) at an instant \( t \) is given by the following relation (Van Doorselaere et al. 2016):

\[
G(\lambda) = \frac{1}{\sigma_w \sqrt{2\pi}} \exp\left( -\frac{1}{2\sigma_w^2} \left( \lambda - \lambda_0 \left( 1 \pm \frac{v}{c} \right) \right)^2 \right). \tag{1}
\]

Here, \( v \) is the velocity of the emitting plasma along LOS, \( \lambda_0 \) is the central wavelength of the emission, and \( \lambda_0(1 \pm \frac{v}{c}) \) is the wavelength shift due to the velocity of emitting plasma along the LOS. We assume the Doppler broadening (due to the temperature of the plasma) to be the dominant broadening mechanism when plasma is at rest. The width of the Gaussian-shaped spectral line, \( \sigma_w \), is defined as \( \sqrt{2k_BT/M \lambda_0/c^2} \). Note that \( k_b \) is the Boltzmann constant, \( M \) is the mass of the emitting ion in the plasma, and \( c \) is the speed of light. In some studies, \( \sqrt{2k_BT/M} \) is termed as thermal line width or exponential line width (\( \sigma_{\text{th}} \)). For simplicity, we choose \( \sigma_v = \sqrt{2k_BT/M} \).\( \sqrt{2} \) such that \( \sigma_{\text{th}} = \sigma_{\text{nt}} / \sqrt{2} \).

Inserting the expression for \( \sigma_w \) in Equation (1), we get

\[
G(\lambda) = \frac{c}{\sigma_\lambda_0 \sqrt{2\pi}} \exp\left( -\frac{1}{2\sigma_\lambda_0^2} \left( \lambda - \lambda_0 \left( 1 \pm \frac{v}{\lambda_0} \right) \right)^2 \right). \tag{2}
\]

Often in literature, for simplicity, we assume the dependence of \( G(\lambda) \) on velocity instead of wavelength. To convert wavelength to velocity, we define a new variable \( v' \) such that when \( \lambda \) is replaced by \( \lambda_0(1 + \frac{v}{\lambda_0}) = \lambda' \), Equation (2) reduces to

\[
G(\lambda') = \frac{c}{\sigma_\lambda_0 \sqrt{2\pi}} \exp\left( -\frac{1}{2\sigma_\lambda_0^2} \left( \lambda' - \lambda_0 \left( 1 \pm \frac{v}{\lambda_0} \right) \right)^2 \right). \tag{3}
\]

It should be noted that \( \frac{c}{\sigma_\lambda_0 \sqrt{2\pi}} \) is a normalization constant such that \( \int G(\lambda')d\lambda' = 1 \). This constant will have no effect on the analysis presented in this paper. Equation (3) can be reformulated in terms of variable \( v \) as follows:

\[
G(v) = \frac{1}{\sigma_v \sqrt{2\pi}} \exp\left( -\frac{1}{2\sigma_v^2} \left( v - \lambda_0 \left( 1 \pm \frac{v}{\lambda_0} \right) \right)^2 \right). \tag{4}
\]

such that \( \int G(v)d\nu = 1 \). Equation (4) gives the shape of a spectrum, emitted by the plasma moving with velocity \( v \) along the LOS, in the Doppler-shifted velocity coordinate (\( v \)), such that if the emitting ion is at rest, the Gaussian spectrum will be centered around zero. Henceforth, we will employ Equation (4) instead of Equation (1) for further analysis.
2.1. Oscillations along the LOS and Effect of Finite Exposure Time

We assume a harmonic plane wave propagating along the background magnetic field. Following the analysis presented in Walker (2005), the velocity of a transverse wave can be assumed to be \( v = \frac{\sigma}{2}[v_0 \cos \omega t + v_0^* \exp(-\omega t)] e_z \). Taking only real amplitudes of the wave velocity, we get \( v = v_0 \cos \omega t e_z \). Here, \( e_z \) is the unit vector along the \( x \)-axis.

Now let us assume that the angle between LOS and \( e_z \) (as shown in Figure 1) vanishes. Thus, the LOS is in the \( e_z \) direction. Further, we assume that wave amplitudes are unresolved in time either due to the high-frequency nature of the oscillations or due to a large exposure time, \( t \), of an instrument such as SUMER (~300 s). In such a scenario, the spectrum recorded by the spectrograph will be approximately given by the following relations:

\[
\langle G(v, t') \rangle_t = \int_0^t \frac{1}{t \sigma} \exp\left(-\frac{(v - v_0 \cos(\omega t'))^2}{2 \sigma^2}\right) dt',
\]

\[
= \left( \int_0^P \frac{1}{t \sigma} \exp\left(-\frac{(v - v_0 \cos(\omega t'))^2}{2 \sigma^2}\right) dt' \right) + \int_0^P \frac{1}{t \sigma} \exp\left(-\frac{(v - v_0 \cos(\omega t'))^2}{2 \sigma^2}\right) dt',
\]

\[
(5)
\]

Here, \( P \) is the period of the oscillations and \( f \) is a positive integer such that \( fP \leq t < (f + 1)P \). If \( f \gg 1 \), we get \( t \sim fP \). Thus, the contribution of the second term in Equation (5) toward the estimation of \( \langle G(v, t') \rangle_t \) will be much less, compared to the first term. Therefore, we can neglect the second term in Equation (5). This means when the exposure time of an instrument is much larger than the period of oscillations, averaging the spectrum over one period is a good assumption. Note that we normalize Equation (5) by the exposure time, \( t \). The normalization will not affect the estimation of the nonthermal line widths. We solve Equation (5) analytically, assuming \( v_0 \ll \sigma \) (limiting case).

We find that the nonthermal line width, \( \sigma_{nt} \), of the period-averaged spectrum \( \langle G(v, t') \rangle_t \) is approximately equal to the wave amplitude \( v_0 \) (see Appendix A for the derivation). Further, we compare \( \sigma_{nt} \) with the rms wave velocity computed using the following relation:

\[
\sigma_{nt} = \sqrt{\frac{1}{P} \int_0^P v \cdot v^* dt}.
\]

(6)

For linearly polarized oscillations, \( v \cdot v^* = \frac{1}{2} \sigma^2 \cos^2(\omega t) \). Thus, we find that, \( \sigma_{nt} \sim \sqrt{2} \sigma_{rms} \). Note that, if \( v_0 \) is comparable to \( \sigma \), we can no longer ignore higher-order terms of \( v_0 \) (see Appendix A). Thus, we need to solve Equation (5) using numerical integration.

Next, we solve Equation (5) using realistic wave amplitudes observed in the solar corona. Since the mean value of the transverse wave amplitude in the coronal holes is \( 11 \sqrt{2} \text{ km s}^{-1} \) (Morton et al. 2015), we numerically integrate Equation (5) assuming \( v_0 = 11 \sqrt{2} \text{ km s}^{-1} \) and \( \sigma = 19 \sqrt{2} \text{ km s}^{-1} \) such that the thermal width \( (\sigma_{1/e}) \) is 19 km s\(^{-1}\) (for 1.2 MK plasma).

Throughout the manuscript, we use these values to perform the numerical integration. It is worth mentioning here that \( v_0 \) is comparable to the thermal line width, \( \sigma_{1/e} \), but larger than \( \sigma \). The period-averaged spectrum, \( \langle G(v, t') \rangle_t \), obtained by numerical integration is shown in Figure 2(a) in blue. Next, we fit a Gaussian curve to \( \langle G(v, t') \rangle_t \) and estimated the nonthermal line width, \( \sigma_{nt} = \sqrt{\sigma_{1/e}^2 - \sigma_{1/e}^2} \). Here, \( \sigma_{1/e} \) is the exponential line width of the best-fit Gaussian curve shown in red.

In this case, we find that \( \sigma_{nt}/v_0 \sim 1.1 \) or \( \sigma_{nt}/v_{rms} \sim 1.1 \times 1.1 \sqrt{2} \), as shown in Figure 2(a). This is different from the case when \( v_0 \ll \sigma \), because of the contribution of the higher-order terms in \( v_0 \). Therefore, we get \( \sigma_{nt}/v_{rms} \sim 1.1 \) instead of \( 1 \) (when \( v_0 \ll \sigma \)). Intuitively, one can say that the period-averaged spectrum is computed by averaging the spectra that are equally shifted in blueward and redward directions by \( v_0 \).

Thus, the nonthermal width of the period-averaged spectrum is equal to \( v_0 \). These relations differ from those used in the earlier studies where either \( \sigma_{nt}/v_{rms} = 1 \) or \( \sigma_{nt}/v_{rms} = 1/\sqrt{2} \).

The scenario we discuss in this section is too ideal to be realistic. The exposure time of the spectrographs such as the Extreme Ultraviolet Imaging Spectrometer on board Hinode and the Coronal Multichannel Polarimeter have cadences of less than 60 s, which is less than the characteristic transverse wave period (~5 minutes). In later sections, we discuss more general scenarios, where we include the inclination of LOS with oscillations, superposition of different polarization of oscillations, and superposition of many wave modes, and relax the requirement of a large exposure time.

2.2. LOS Inclined to the Oscillations’ Direction

If the directions of the oscillations are inclined to the LOS at an angle \( \theta \) (Figure 1(a)), the time-integrated spectrum is given by the following equation:

\[
\langle G(v, t') \rangle_t = \int_0^P \frac{1}{P \sigma} \exp\left(-\frac{(v - v_0 \cos(\theta) \cos(\omega t'))^2}{2 \sigma^2}\right) dt',
\]

Note that Equation (7) is similar to Equation (5), except that the wave amplitude is modulated along the LOS. Borrowing the relation from the previous scenario, we get \( \sigma_{nt} \sim \sqrt{2} v_{rms} \cos \theta \sim v_0 \cos \theta \) when \( v_0 \ll \sigma \). Taking \( v_0 = 11 \sqrt{2} \text{ km s}^{-1} \) and \( \sigma = 19 \sqrt{2} \text{ km s}^{-1} \), we get \( \sigma_{nt} \sim 1.1 \sqrt{2} v_{rms} \cos \theta \sim 1.1 v_0 \cos \theta \).

2.3. Effect of LOS Superposition of Oscillating Structures on the Nonthermal Line Widths

Now we assume that different structures are positioned along the same LOS but oscillating in different directions. At a given time \( t_0 \), the emitted spectra from the oscillating structures along the LOS will superimpose to generate a broader spectrum. Thus, the nonthermal line widths will increase due to unresolved wave amplitudes along the LOS. The shape of the normalized spectrum
at an instant \((t_0)\) is given by the following relation:

\[
\langle G(v, \theta') \rangle_\theta = \int_0^{2\pi} \frac{1}{2\pi \sigma \sqrt{2\pi}} \times \exp\left(-\frac{(v - v_0 \cos(\theta') \cos(\omega t_0))^2}{2\sigma^2}\right) d\theta'.
\]  

(8)

Here, wave amplitude \((v_0)\) is modulated depending on the angle between the direction of oscillations and LOS. Note that Equation (8) is similar to Equation (5) because \(t_0\) is a fixed constant. Therefore, in this case too, \(\sigma_{nt} \sim \sqrt{2} v_{rms} = v_0 \cos(\omega t_0)\) for the limiting case \((v_0 \ll \sigma)\). Otherwise, taking \(v_0 = 11 \sqrt{2} \text{ km s}^{-1}\) and \(\sigma = 19/2 \text{ km s}^{-1}\), we find \(\sigma_{nt} \sim 1.1 \sqrt{2} v_{rms} \sim 1.1 v_0 \cos(\omega t_0)\). Depending on the phase of oscillation \((t_0)\), the nonthermal line width will change. Here, we make a rigid assumption that all structures along the LOS are oscillating in (or out of) phase because \(t_0\) is assumed constant. No other value of phase difference is possible. This may not be a realistic scenario, but for the sake of completeness, we discuss it here.

2.4. Effect of Different Polarizations and Phase of Oscillations on Nonthermal Line Widths

Perhaps the most probable scenario in the optically thin corona is the occurrence of structures oscillating in different directions and different phases superposed along the LOS. Assuming a uniform probability of choosing oscillating structures in a given state of polarization and phase of oscillation, the time-integrated spectrum over one period can be computed by the following relation:

\[
\langle G(v) \rangle_\theta = \frac{1}{P \sigma (2\pi)^{3/2}} \times \int_0^{2\pi} \int_0^P \exp\left(-\frac{(v - v_0 \cos \theta' \cos(\omega t'))^2}{2\sigma^2}\right) dt' d\theta'.
\]

(9)

For simplicity, we are integrating up to one period of oscillation only. We also do not assume any constraint on the exposure time of the measuring instrument. It should be noted that, as long as different phases and polarizations are equally likely and the total number of structures oscillating along the

Figure 2. Best-fit Gaussian curves in red over the integrated spectra shown in blue for the (a) linear, (b) circular, and (c) multiple velocity drivers. Note that \(\frac{\sigma_{nt}}{v_{rms}} \sim 1.5, 1, 1\) respectively for these three scenarios. (d) Averaged spectrum when the strength of the velocity driver is greater than the thermal line width.
Therefore, we averaging of spectra of different nonthermal widths. Thus, the nonthermal line widths in this scenario is further reduced by $\sqrt{2}$, compared to the scenario described in Section 2.1. Therefore, we find that $\sigma_{nt}/v_{rms} \sim 1$. Next, we relax the assumption $v_0 \ll \sigma$ and choose $v_0 = 11\sqrt{2}$ km s$^{-1}$ and $\sigma = 19/\sqrt{2}$ km s$^{-1}$. We numerically integrate Equation (9) using the quadrature method and compute the normalized spectrum, which is shown in Figure 2(b). We compute the nonthermal line widths and estimate that $\sigma_{nt}/v_{rms} \sim 1.1$. Note that it is less than those described in above sections where $\sigma_{nt}/v_{rms} \sim 1.1\sqrt{2}$.

Until now, we have assumed only linearly polarized transverse oscillations. Now, we assume circularly polarized oscillations (Ferraro 1955). A circularly polarized oscillations can be expressed mathematically as $v = \frac{1}{\sqrt{2}}[(v_0 e^{-i\omega t} e_x + v_0^* e^{i\omega t} e_y)]$, where $e_\pm = \frac{1}{\sqrt{2}}(e_x \pm i e_y)$ (Goldstein 1978). Taking only real velocity amplitudes, we get $v = \frac{v_0}{\sqrt{2}}[\cos(\omega t) e_x + \sin(\omega t) e_y]$. Thus, for a circularly polarized oscillation, polarization direction along two perpendicular axes has a phase difference of $\pi/2$ shown schematically in Figure 1. The time-integrated spectrum due to such oscillations can be computed by the following expression:

$$\langle G(v) \rangle_{r, \theta} = \frac{1}{P \sigma (2\pi)^{3/2}} \times \int_0^{2\pi} \int_0^p \exp \left( -\frac{(v - v_0 \cos(\theta' - \omega t'))^2}{2\sigma^2} \right) dt' d\theta'. \quad (10)$$

Here, $v_0 = \frac{v_0}{\sqrt{2}}$. For consistency, we use $v_0$ instead of $v_0$, such that the energy in the linearly and circularly polarized driver remains equal. First, note that the integration along $\theta$ is redundant because a circularly polarized oscillation will cycle through all angles relative to the LOS. Therefore, Equation (10) is equivalent to Equation (5). Thus, for a circularly polarized velocity driver as described above, $\sigma_{nt} \sim v_0$ for the limiting case and $\sigma_{nt} \sim 1.1v_0$ when $v_0 = 11$ km s$^{-1}$ and $\sigma = 19/\sqrt{2}$ km s$^{-1}$ as shown in Figure 2(b). It is worth combining equations for different velocity drivers described above into a single equation. Equations (9) and (10) describe the shape of the period-averaged spectrum for a linearly and circularly polarized driver. To combine these two equations, we define a new variable $\phi$, which may or may not be a function of time depending on the nature of the velocity driver. Mathematically, $\phi(t)$ is defined as follows:

$$\phi(t) = \tan^{-1} \left( \frac{v_x \sin(\omega_t t + \delta)}{v_y \sin(\omega_t t)} \right). \quad (11)$$

where $\delta$ is the phase difference between the $x$- and $y$-components of the velocity driver. Here, $\omega_x$ and $\omega_y$ are the frequencies of oscillations of the velocity driver in the $x$- and $y$-directions. From Equation (11), we note that, for a linearly polarized driver with equal amplitudes and frequency in the $x$- and $y$-axes, $\delta$ vanishes. Thus, $\phi$ is equal to $\pi/4$ and is constant in time. However, for a circularly polarized $(v_x = v_y$ and $\omega_x = \omega_y$) polarized driver, $\delta = \pi/2$. Thus, $\phi = \omega t$. Here, $\phi$ can be considered as the angle that the velocity driver forms with $e_x$. The value of $\theta$ used in the equations above is the angle that the LOS forms with the $e_x$. Thus, the effective angle between LOS and direction of oscillation at an instant can be taken as $\theta - \phi(t)$.

We define total velocity of the driver as follows:

$$v_T(t) = \sqrt{(v_x \sin(\omega_t t))^2 + (v_y \sin(\omega_t t + \delta))^2}. \quad (12)$$

Expressions of total velocity $v_T(t)$ and $\phi(t)$ that can be used to obtain period-averaged spectrum can be computed by the following expression:

$$\langle G(v) \rangle_{r, \theta} = \frac{1}{P \sigma (2\pi)^{3/2}} \times \int_0^{2\pi} \int_0^p \exp \left( -\frac{(v - v_T(t) \cos(\theta' - \phi(t'))^2}{2\sigma^2} \right) dt' d\theta'. \quad (13)$$

Note that, under the assumption $v_x = v_y = \frac{v_0}{\sqrt{2}}$, $\omega_x = \omega_y = \omega$, for $\delta = 0$ and $\pi/2$, Equation (13) reduces to Equations (9) and (10), respectively.

Thus, we conclude that, for the limiting case when $v_0 \ll \sigma$, $\sigma_{nt}/v_{rms} \sim 1$. Taking wave amplitudes $(v_0)$ of $11\sqrt{2}$ km s$^{-1}$ and $\sigma = 19/\sqrt{2}$ km s$^{-1}$ in the solar corona, $\sigma_{nt}/v_{rms} \sim 1.1$ for a linearly and circularly polarized oscillation when averaged over period and direction of oscillations. Equation (13) is quite useful for deriving the relation between the nonthermal line widths and rms wave velocities if multiple wave drivers are assumed or if a random distribution is assumed instead of a uniform distribution. We again reiterate that the wave amplitude is assumed to be smaller than the thermal line width of the optically thin emission line. The derived relations may not be valid if wave amplitudes are larger than thermal line widths. Figure 2(d) shows the period-averaged spectrum (blue curve) for $v_{rms} = 22$ km s$^{-1}$ when line width, $\sigma$, is assumed to be $19/\sqrt{2}$ km s$^{-1}$. It can be seen that, for a large amplitude, the shape of the averaged spectrum is no longer a Gaussian.

2.5. Multiple Wave Drivers

Here, we investigate the effect of the multiple wave drivers on the rms wave velocities and nonthermal line widths. Such drivers were used in earlier studies (Magyar et al. 2017; Pant et al. 2019) to excite waves in 3D MHD simulations. We assume a superposition of ten different velocity drivers with different velocity amplitudes and periodicity given by the

\footnote{https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.dblquad.html#scipy.integrate.dblquad}
following relations:

\[ v_x(t) = \sum_{i=1}^{10} v_{xi} \sin(\omega_{xi} t), \]
\[ v_y(t) = \sum_{i=1}^{10} v_{yi} \sin(\omega_{yi} t). \]

Using Equations (11) and (12), we compute \( v_T = \sqrt{v_x^2 + v_y^2} \) and \( \phi(t) = \tan^{-1}(v_y/v_x) \). Note that the sum over different velocity drivers is computed at every instant of time. Also, the amplitude and frequency of the ten drivers do not change with time. We insert \( v_T \) and \( f \) in Equation (13) to compute the period-averaged spectrum shown in Figure 2 in blue. Fitting a Gaussian function over the numerically integrated period-averaged spectrum, we find that \( \sigma_{nt}/v_{rms} \approx 1.1 \).

We learn from these ideal cases that, when different structures with different polarizations are oscillating in different phases, the nonthermal line widths are at least equal to the rms wave velocities. Further, we note that when the wave amplitude is on the order of the thermal line widths, \( \sigma_{nt}/v_{rms} \approx 1.1 \). These results are in contrast with those used in the earlier studies by Hassler et al. (1990), Banerjee et al. (1998, 2009), O’Shea et al. (2005), and Hahn et al. (2012), where \( \sigma_{nt}/v_{rms} \approx 1/\sqrt{2} = 0.71 \) is used. Thus, we find that the Alfvénic wave energy flux was overestimated by at least a factor of two in these studies.

### 3. Numerical Simulations and Forward Modeling

We test the validity of the mathematical models described in Section 2 using physical models employing ideal 3D MHD simulations using MPI-AMRVAC that solves the following equations in the near-conservative form (Porth et al. 2014):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \mathbf{B} \mathbf{B}) + \nabla (p + \mathbf{B}^2/2) - \rho g = 0, \\
\frac{\partial E}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{E} + \mathbf{v} \cdot (p \mathbf{v} + \mathbf{B}^2/2)) = 0, \\
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \\
\nabla \cdot \mathbf{B} = 0.
\]

Here, \( g \) is the acceleration due to the gravity of the Sun pointing along the negative z-axis, and \( E \) is total energy density (defined as \( E = \rho + \frac{\mathbf{v}^2}{2} + \frac{\mathbf{B}^2}{2} \)). In these equations, \( \rho \) represents density, which is an exponentially decaying function of the height (z-axis) due to gravitational stratification. We choose a background magnetic field strength (\( B \)) of 5 G along the z-axis in all simulation runs. We perform numerical simulations for transversely homogeneous plasma for monoperiodic and multiperiodic velocity drivers. We choose a transversely homogeneous plasma because we want to study the effects of LOS superposition without generating the uniturbulence due to the transverse inhomogeneity in the density.

The set of equations described in Equation (15) are solved in the Cartesian geometry for a grid size of \( 64 \times 64 \times 128 \) that spans a physical dimension of \( 5 \text{ Mm} \times 5 \text{ Mm} \times 50 \text{ Mm} \) as shown in Figure 3. The geometry and size of the simulation setup are similar to those described in Pant et al. (2019). Since we assume transversely homogeneous plasma, we used a
coarse grid resolution along the x- and y-directions. The side boundaries of the simulation domain are periodic, while the top boundary is kept open so that waves can leave the simulation domain. Plasma beta ($\beta$) and temperature of the plasma are kept at 0.07 and 1.2 MK, respectively, at the start of the simulation. We let simulations reach a quasi-steady state before implementing the velocity drivers at the bottom boundary. In the quasi-steady state, the density varies exponentially (Figure 3(b)) and the scale height was estimated to be $\sim 51 \text{ Mm}$. Figure 3(a) shows the density distribution in the initial configuration of the simulation cube. Figure 3(b) shows the variation of density with height ($z$-axis).

First, we excite the bottom boundary of the simulation with a linearly polarized transverse velocity driver given by the following relations:

$$v_x(z = 0, t) = U_0 \sin(\omega_0 t),$$
$$v_y(z = 0, t) = V_0 \sin(\omega_0 t). \tag{16}$$

We choose $U_0 = V_0 = 11 \text{ km s}^{-1}$ and the period of oscillations, $P = 2\pi/\omega_0 = 400 \text{ s}$.

To study the relationship between the nonthermal line widths and r.m.s wave velocities, we perform forward modeling of the simulations for 12 different LOS using the Fe XIII emission line (10749 A) in FoMo-c$^2$ (Van Doorsselaere et al. 2016). We adopted the same methodology as described in Pant et al. (2019) for estimating the nonthermal line widths at all lines of sight. Using simulated data, we compute the variation of the r.m.s wave velocity with height (shown in black in Figure 4).

Further, we compute the nonthermal line widths for an LOS inclined at an angle $\theta$ to the direction of oscillations (shown in green in Figure 4). We also estimate the nonthermal line widths after averaging over the period and direction of the oscillations (curves in blue in Figure 4). Finally, we choose 100 random segments with random directions of oscillations and phases, and we then compute the nonthermal line widths using the spectrum averaged over these segments (overplotted in red in Figure 4). The right-hand panels of Figure 4 show the ratio of the nonthermal line widths and r.m.s wave amplitude with height. In Figure 4(a) and (b), we note that $\sigma_{\text{nt}}/v_{\text{rms}} \sim 1.2\sqrt{2}$ when the LOS is aligned in the direction of the oscillations (which is $135^\circ$) and the spectra are integrated over time. This relation matches fairly well with those predicted using the numerical integration in Section 2.1. Similarly, $\sigma_{\text{nt}}/v_{\text{rms}} \sim 1.2$ when spectra are integrated spatially and temporally (as shown in blue). It should be noted that the results for the random segments (overplotted in red) are similar to those obtained assuming a uniform probability of occurrence of different polarization and phases of the oscillation along the LOS. The small deviation from the blue could be due to randomness in choosing segments with different phase and polarization of oscillations. Again, these results match with those discussed in Section 2.

Next, we implement the velocity drivers with a phase difference of $\pi/2$ in the y- and z-directions leading to circularly polarized oscillation

$$v_x(z = 0, t) = U_0 \sin(\omega_0 t),$$
$$v_y(z = 0, t) = V_0 \cos(\omega_0 t). \tag{17}$$

The period and velocity amplitude of these oscillations are similar to those described above. Panels (c) and (d) of Figure 4 show the results of this simulation run. We find that, irrespective of the LOS chosen, $\sigma_{\text{nt}}/v_{\text{rms}} \sim 1.2$, which is in good agreement with those derived in Section 2 for the circularly polarized oscillations.

Finally, we implement the multiple (ten) velocity drivers described by the Equation (18) at the bottom boundary (see Magyar et al. 2017; Pant et al. 2019) and repeat the analysis described above:

$$v_i(x = 0, t) = \sum_{i=1}^{10} U_i \sin(\omega_i t),$$
$$v_j(x = 0, t) = \sum_{i=1}^{10} V_j \sin(\omega_j t). \tag{18}$$

We choose $\omega_i$ from the observed distribution of transverse oscillation period in the coronal holes (Morton et al. 2015). Here, $U_i$ and $V_j$ are randomly chosen such that $v_{\text{rms}}$ at the bottom boundary $\sim 16 \text{ km s}^{-1}$; see Pant et al. (2019) for details. The results for this velocity driver are presented in Figure 4 (e) and (f). We notice that $\sigma_{\text{nt}}/v_{\text{rms}} \sim 1.2$ for the scenarios when averaging over 100 random segments and over period and LOSs are performed. These results match with those presented in Section 2.5. It is worth noting that, for $\theta = 75^\circ$, $\sigma_{\text{nt}}/v_{\text{rms}}$ is between 1.2 and $1.2\sqrt{2}$. In fact, we note that the period-averaged spectrum for any LOS is $1.2 \leq \sigma_{\text{nt}}/v_{\text{rms}} \leq 1.2\sqrt{2}$. This happens because such multiple velocity drivers result in a velocity field that is neither circular nor linear but forms a Lissajous patterns (see online animation in Pant et al. (2019)). It is worth noting that, in all the above described scenarios, $v_{\text{rms}}$ is less than the thermal width of the emission line, which is 19 km s$^{-1}$.

A key result obtained from this study is that, for a transversely homogeneous and gravitationally stratified plasma, $\sigma_{\text{nt}}/v_{\text{rms}} \sim 1.2$ for a scenario where different polarizations and phases of oscillations occur along the LOS of an observer. We use both mathematical modeling and 3D MHD simulations together with forward modeling to verify these relations. These relations are valid for both single- and multifrequency velocity drivers.

4. Wave Amplitudes versus Height

We note that, regardless of the nature (linear or circular) of the polarization of the transverse MHD waves, the wave amplitudes (and hence line widths) increase and level off with height in our simulations. This behavior is noted for both transversely homogeneous (this study) and inhomogeneous simulations (see Pant et al. 2019). In this section, we try to understand the nature of this variation.

We eliminate the possibility of damping due to resonant absorption or numerical dissipation by performing the 3D MHD simulations of a homogeneous plasma without gravity. We did not find any significant damping of the wave amplitudes. Wave amplitudes are reduced by $\sim 2\%$–$3\%$. This means wave energy is reduced by $4\%$. In observations, it is often assumed that the r.m.s wave amplitude of Alfvénic waves increases with height, assuming a Wentzel–Kramers–Brillouin (WKB) approximation. Under this approximation, $v_{\text{rms}}$ varies

---

2 https://wiki.esat.kuleuven.be/FoMo/FoMo-C
Figure 4. Variation of $v_{\text{rms}}$ and $\sigma_{nt}$ with height for different scenarios. Black lines display $v_{\text{rms}}$ computed from simulation. Green lines indicate $\sigma_{nt}$ computed from a period-averaged spectrum for a given LOS. Overplotted in blue and red are $\sigma_{nt}$ for a period (and LOS) averaged spectrum and spectrum averaged over 100 random segments. Panels (a), (c), and (e) represent the results for linear, circular, and multifrequency velocity drivers. Panels (b), (d), and (f) present the variation of the ratio of $\sigma_{nt}$ and $v_{\text{rms}}$ for linear, circular, and multifrequency velocity drivers.
as $\rho^{-1/4}$ (Hollweg 1972). Using this expression, $v_{\text{rms}}$ at the base of the corona is estimated by comparing it with the observed values of the nonthermal line widths (Hassler et al. 1990; Banerjee et al. 1998). However, the line widths appear to increase more slowly than expected from WKB propagation (level-off) at higher heights (Hassler et al. 1990; Banerjee et al. 1998; Hahn et al. 2012). The deviation from WKB propagation is considered to be the signature of the damping of waves (Hahn et al. 2012). In our simulations, the damping of wave energy is insignificant; this means the wave amplitude should increase according to the WKB approximation and should be around $\sim 13.7$ km s$^{-1}$ at 50 Mm (dashed curve in red in Figure 5). However, we note from Figure 4(a) that the wave amplitudes are around 11.9 km s$^{-1}$. This corresponds to a 15% difference in the wave amplitudes between simulations and those expected from the WKB theory.

It should also be noted that the wavelength of the transverse oscillation is $\sim 300$ Mm in our simulations. Since the wavelength is much larger than the scale height of the simulations (50 Mm), the WKB approximation may not be valid in this scenario (Hollweg 1972, 1978; Heinemann & Obert 1980).

To understand the propagation of transverse MHD waves in gravitationally stratified medium, we use the expressions derived by Hollweg (1978) for propagating transverse waves without assuming WKB (or eikonal) approximation. We borrow the following relation from Hollweg (1978):

$$v = aH_0^{(1)}(\alpha) + bH_0^{(2)}(\alpha).$$  

(19)

Here, $v$ is the velocity amplitude, $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kinds, respectively, and

$$\alpha = \frac{2\omega}{\omega(z)}.\,$$

The scale height is represented by $H$, $\omega$ is the frequency, and $v_\alpha(z)$ is the Alfvén velocity, which is a function of height ($x$-direction). In this equation, $a$ and $b$ are unknowns to be derived from the boundary conditions.

The reflection coefficient, $r$, is defined as $r = \frac{|b|}{|a|}$. A value of $r = 0$ means no reflection (only outward propagating waves). Using Equation (19), we can show that (see Hollweg 1978)

$$\text{Re}(v) = R_2 \cos(\omega t) - I_2 \sin(\omega t).$$

(20)

This equation leads to the estimation of the rms velocity, which is given by:

$$v_{\text{rms}} = \sqrt{\left(R_2^2 + I_2^2\right)/2}.$$  

(21)

Here,

$$R_2 = \text{Re}(a)J_0 - \text{Im}(a)Y_0 + \text{Re}(b)J_0 + \text{Im}(b)Y_0,$$

$$I_2 = \text{Re}(a)Y_0 + \text{Im}(a)J_0 - \text{Re}(b)Y_0 + \text{Im}(b)J_0,$$

(22)

where $J$ and $Y$ are Bessel functions.

Using Equation (21), we computed the variation of $v_{\text{rms}}$ with height by iterating over several values of complex numbers $a$ and $b$ and performing chi-square minimization over rms wave amplitudes computed from simulations. We choose those values for which the analytical expression given by Equation (21) matches fairly well with those obtained from simulations. The best-fit curve in blue obtained using Equation (21) is overplotted on the rms wave amplitude for the linearly polarized velocity driver overplotted with green in Figure 5. We find a fairly good match between the two curves. Finally, we estimate the reflection coefficient, $r$, which was found to be 2.5% in the case of linearly polarized oscillations. For the circularly polarized and multi-frequency drivers, the reflection coefficient was found to be 1.5%–3%. It is worth noting here that a reflection coefficient of 2.5% leads to a difference of 15% in the wave amplitudes. This matches with the difference in the wave amplitudes between simulations and those expected from WKB.

We also note that, when wavelength of the wave is much less than the scale height, the rms wave velocity will match fairly with those computed assuming a WKB approximation. We notice that the deviation from the expected WKB propagation in our simulations could be due to the presence of incoming waves ($b \neq 0$) due to gravitational stratification leading to non-WKB effects. We also note that the wave amplitudes are smaller than those expected from the WKB approximation. This is consistent with the study of Hollweg (1972), where authors reported a similar nature of variation; see Figures 1 and 7 of Hollweg (1972, 1981), respectively. Thus, both large wavelength and reflection affect the observed wave amplitudes (and hence line widths) in the solar atmosphere.

This leads us to believe that the reflection coefficient as low as 2% can significantly change the nature of the variation of wave amplitudes with height. Thus, a slower increase of the wave amplitudes (and hence the nonthermal line widths) than expected from the WKB approximation is not only a signature of the damping but could also be due to the non-WKB nature of transverse wave propagation in a gravitationally stratified plasma. In the polar coronal holes, the departure from the WKB theory appears to happen at heights of 0.2 $R_\odot$ (from the photosphere) or 140 Mm (Banerjee et al. 1998; Hahn et al. 2012). In this study, the effect is seen after 10 Mm. There might be several reasons for this. First, our simulations assume an isothermal atmosphere with a scale height of $\sim 50$ Mm. The scale heights in the solar corona might be very large. For example, Banerjee et al. (1998), reported a scale height of 100 Mm (see Figure 4(a) in Banerjee et al. (1998)). Similarly, Dolla & Solomon (2008) reported a scale height of $\sim 70$ Mm. Recently, Pascoe et al. (2019) investigated the density and temperature variation along height in the coronal holes. They found that the density scale height changes with height. We have not considered these effects in our simulations, but they might affect the rms velocities and nonthermal line widths. Second, the nature of the variation of the nonthermal line...
widths is different in different emission lines (Dolla & Solomon 2008). Third, the observed nonthermal line widths, rather than wave amplitudes, are compared with the WKB calculations. Though the nonthermal line widths depend on the rms wave amplitudes, in a realistic atmosphere, there will be other effects (such as turbulence) that can change the nature of the variation of the nonthermal line widths with height. For example, Pant et al. (2019) reported that the nonthermal line widths did not level off significantly with height when $v_{\text{rms}} = 26$ km s$^{-1}$, as compared to the scenarios when $v_{\text{rms}} = 11$ and 7 km s$^{-1}$ (see Figure 10 in Pant et al. (2019)). Fourth, the mismatch between $v_{\text{rms}}$ and those obtained using WKB approximation might also depend on the nature of the velocity driver. For example, the nature of the variation of the nonthermal line widths of random segments for a multi-frequency driver is slightly different from the nature of the variation of rms wave amplitude (see Figure 4(e)). This difference might be due to the choice of the random segments. Finally, it should be noted that the difference in the wave amplitudes computed using WKB theory and the real wave amplitudes in Figure 5 is on the order of 1–2 km s$^{-1}$. In real observations, this difference might appear within the error bars of the measurements. Recently, Weberg et al. (2020) computed the transverse wave amplitudes using the Atmospheric Imaging Assembly/Solar Dynamics Observatory and noted a similar variation of the wave amplitudes with height. In their study, the flattening of wave amplitudes happens at 15 Mm. Interestingly, these authors also suggest the possibility of a reflection in the low corona. However, they do not attribute the flattening of wave amplitudes to that reflection. Thus, we believe that wave amplitudes, rather than the nonthermal line widths, are more meaningful to compare.

5. Summary and Conclusions

McIntosh & De Pontieu (2012), De Moortel & Pascoe (2012), and Pant et al. (2019) have shown that LOS superpositioning of oscillating structures greatly reduces the rms Doppler velocities and increases nonthermal line widths. Thus, real wave amplitudes are hidden in the nonthermal line widths. This motivated us to estimate a relation between nonthermal line widths and rms wave velocities that can be used to estimate the true energy carried by the Alfvén(ic) wave.

Furthermore, estimating wave amplitudes using the nonthermal line widths is useful because, if high-resolution spectral and imaging data are available, the resolved wave amplitudes either in the plane of sky (POS) or along the LOS (exhibiting the periodic variation in the Doppler velocities) can be combined to compute true wave amplitudes and hence energies. We built a mathematical model and found the relation between $\sigma_{nt}$ and $v_{\text{rms}}$ for different scenarios. The results are summarized in Table 1. For the limiting case, when $v_0 \ll \sigma$ and if a single structure is assumed to oscillate along the LOS, $\sigma_{nt}/v_{\text{rms}} \sim \frac{1}{\sqrt{2}}$, provided the wave amplitudes are unresolved in time. In other words, this relation is only valid if the period (frequency) of the wave is much smaller (higher) than the exposure time of the spectrograph. This relation is also valid if an observer performs smoothing of spectra in time. Next, we studied a generalized scenario when structures oscillating in different directions and different phases happen to lie along the LOS of the observer. Due to the optically thin nature of the solar corona, the spectrum recorded by an observer will be the superposition of the spectra of all oscillating structures along the LOS. In such a scenario, $\sigma_{nt}/v_{\text{rms}} \sim 1$, irrespective of uniform or random distribution of oscillating structures along the LOS. These results are valid for circularly polarized oscillations and oscillations driven by the multifrequency driver. We also tested the scenario where wave amplitude is of the order of $\sigma$. Taking wave amplitudes of $11\sqrt{2}$ km s$^{-1}$ and $\sigma = 19\sqrt{2}$ km s$^{-1}$, we note about 10% change from the limiting case in the estimated magnitude of the nonthermal line widths (see Table 1). To substantiate the mathematical model, we performed 3D MHD simulations and forward modeling of gravitationally stratified plasma and launched transverse MHD waves by driving the bottom boundary transversely. We note from Table 1 that the results of the numerical simulations are in good agreement with those obtained analytically.

This leads us to believe that, depending on the scenario, either $\sigma_{nt} > \frac{1}{\sqrt{2}} v_{\text{rms}}$ or $\sigma_{nt} > v_{\text{rms}}$, but never $\sigma_{nt} = v_{\text{rms}}/\sqrt{2}$ as used in previous studies. This raises questions about the estimation of the energy flux carried by these waves and claims made in earlier studies that the transverse waves carry enough energy to heat the solar corona using the observed nonthermal line widths. Table 2 quotes the earlier and revised (expressions derived in this study) estimates of the energy flux carried by the

**Table 1**

| Nature of Velocity Driver | Theoretical $\sigma_{nt}/v_{\text{rms}}$ | Numerical $\sigma_{nt}/v_{\text{rms}}$ |
|---------------------------|----------------------------------------|--------------------------------------|
| Monoperiodic linearly polarized | Oscillations along LOS | 1.1$\sqrt{2}$ | 1.2$\sqrt{2}$ |
|                            | Oscillations superimposed along LOS   | 1.1                               | 1.2                               |
| Monoperiodic circularly polarized |                          | 1.1                               | 1.2                               |
| Multifrequency driver      |                          | 1.1                               | 1.2                               |

**Table 2**

| References          | Energy Flux (erg cm$^{-2}$ s$^{-1}$) | Revised Energy Flux (erg cm$^{-2}$ s$^{-1}$) |
|---------------------|--------------------------------------|-----------------------------------------------|
| Hassler et al. (1990)| 4.3 × 10$^5$                         | <2.15 × 10$^5$                                |
| Doyle et al. (1998) | 3.1 × 10$^5$                         | <1.52 × 10$^5$                                |
| Banerjee et al. (1998)| 4.9 × 10$^5$                      | <2.45 × 10$^5$                                |
| Banerjee et al. (2009)| 1.85 × 10$^6$                     | <9.25 × 10$^5$                                |
| Hahn et al. (2012)  | 5.4 × 10$^5$                         | <2.7 × 10$^5$                                 |
Alfvénic waves. As discussed above, the nonthermal line widths might be larger than wave amplitudes \((\sigma_{\text{nt}} > \nu_{\text{ms}})\) if the wave amplitude is on the order of the thermal line widths. Therefore, the wave energy flux, \(F \propto \rho v^2 \nu_{\text{ms}}\). Furthermore, this leads us to believe that the hotter emission lines (with large thermal line widths) are more suitable for studying large- and small-amplitude transverse waves.

We find that the non-WKB propagation of the transverse wave in a gravitationally stratified plasma can cause the rms wave amplitudes to increase more slowly with height than expected from the WKB approximation. We also report that, for a million degree hot corona where scale heights are on the order of 50 Mm, low-frequency transverse waves of period \(\sim 400\) s do not propagate like those predicted by the WKB theory. In such a scenario, a slow increase of wave amplitudes (and hence nonthermal line widths) might be due to a non-WKB nature of propagation. It should be noted that we are not ruling out wave damping in the solar atmosphere as a mechanism for the deviation of the nonthermal line width from the WKB theory. In this study, we show that at least a part of the leveling off of rms wave amplitudes (and hence line widths) is due to the non-WKB nature of the propagation of transverse waves in the solar atmosphere.

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### Appendix A

**Estimating Nonthermal Line Width of a Period-averaged Spectrum**

Since the second term in Equation (5) can be neglected, we use the first term for obtaining an analytical expression. In Section 2.1, we mention that integrating over one period is a good assumption. Thus, we integrate over one period only. Equation (5) can be reformulated as follows:

\[
\langle G(v, t') \rangle = \int_0^b \frac{1}{P \sigma \sqrt{2\pi}} \exp\left(-\frac{(v - \nu_0 \cos(\omega't))^2}{2\sigma^2}\right) dt.
\]  

(6)

Equation (A1) can be written as:

\[
\langle G(v, t') \rangle = \int_0^b \frac{1}{P \sigma \sqrt{2\pi}} Q dt.
\]  

(A2)

Where \(Q\) can be expanded as:

\[
Q = 1 - \frac{(v - \nu_0 \cos(\omega't))^2}{2\sigma^2} + \frac{1}{2} \frac{(v - \nu_0 \cos(\omega't))^4}{4\sigma^4} - \frac{1}{6} \frac{(v - \nu_0 \cos(\omega't))^6}{8\sigma^6} + \ldots
\]  

(A3)

For simplicity, we consider only the first four terms for further analysis. The second term (say \(A\)) in Equation (A3) can be expanded as follows:

\[
A = \frac{v^2}{2\sigma^2} - \frac{2v\nu_0 \cos(\omega't)}{2\sigma^2} + \frac{\nu_0^2 \cos^2(\omega't)}{2\sigma^2}.
\]  

(A4)

Because we integrate over one period, the integration over time of the odd powers of the cosine function vanishes. Henceforth, we will consider only even powers of the cosine function.

Similarly, the third (say \(B\) and fourth terms (say \(C\) in Equation (A3) can be expanded to the following equations (after ignoring odd powers of the cosine function):

\[
B = \frac{v^4}{8\sigma^4} + \frac{6v^2 \nu_0^2 \cos^2(\omega't)}{8\sigma^4} + \frac{v_0^4 \cos^4(\omega't)}{8\sigma^4}
\]  

(A5)

\[
C = \frac{v_0^6}{48\sigma^6} + \frac{15v^4 \nu_0^2 \cos^2(\omega't)}{48\sigma^6} + \frac{15v^2 \nu_0^4 \cos^4(\omega't)}{48\sigma^6} + \frac{v_0^6 \cos^6(\omega't)}{48\sigma^6}.
\]  

(A6)

The period averages of \(\cos^2(\omega't)\), \(\cos^4(\omega't)\), and \(\cos^6(\omega't)\) are \(\frac{1}{2}\), \(\frac{3}{4}\), and \(\frac{5}{8}\), respectively. Note that the period average of the odd powers of the cosine function vanishes. We replace cosine functions with these values and drop the integration. Equation (A2) can be rearranged to the following equation after collecting the terms of same order in \(v\):

\[
\langle G(v, t') \rangle = \frac{1}{\sigma \sqrt{2\pi}} \left[1 - \frac{v_0^2}{4\sigma^2} + \frac{3v_0^4}{64\sigma^4} - \frac{5v_0^6}{768\sigma^6} + \ldots\right]
\]  

\[
- \frac{v^2}{2\sigma^2} \exp\left(-\frac{v_0^2}{4\sigma^2}\right) + \frac{v^4}{8\sigma^4} \exp\left(-\frac{15v_0^2}{12\sigma^2}\right) + \ldots
\]  

(A7)

Assuming \(v_0 \ll \sigma^2\), Equation (A7) can be reformulated as follows:

\[
\langle G(v, t') \rangle \approx \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{v_0^2}{4\sigma^2}\right)
\]  

- \(\frac{v^2}{2\sigma^2}\)\(\exp\left(-\frac{3v_0^2}{4\sigma^2}\right) + \frac{v^4}{8\sigma^4} \exp\left(-\frac{15v_0^2}{12\sigma^2}\right) + \ldots \)  

(A8)

\[
\langle G(v, t') \rangle \approx \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{v_0^2}{4\sigma^2}\right) \left[1 - \frac{v^2}{2\sigma^2} \exp\left(-\frac{v_0^2}{2\sigma^2}\right) + \ldots\right]
\]  

(A9)

Assuming \(\sigma' = \sigma \exp\left(\frac{v_0^2}{4\sigma^2}\right)\). Equation (A9) can be written as follows:

\[
\langle G(v, t') \rangle \approx \frac{1}{\sigma' \sqrt{2\pi}} \left[1 - \frac{v^2}{2\sigma'^2} + \frac{v^4}{8\sigma'^4} + \ldots\right]
\]  

(A10)

\[
\langle G(v, t') \rangle \approx \frac{1}{\sigma' \sqrt{2\pi}} \exp\left(-\frac{v^2}{2\sigma'^2}\right)
\]  

(A11)

We note that the period-averaged spectrum under the assumption described above is a Gaussian described by Equation (A11). The nonthermal width, \(\sigma_{\text{nt}}\), of \(\langle G(v, t') \rangle\) can
be estimated using the following relation:

\[
\sigma_{nt}^2 = \sigma_{1/e}^2 - \frac{1}{2} \sigma_{1/e}^2 - 2 \sigma^2
\]

\[
= 2 \sigma^4 \left( \exp \left( \frac{\nu_0^2}{2 \sigma^2} \right) - 1 \right) \approx \nu_0^2 \approx 2 \nu_{\text{inv}}^2. \tag{A12}
\]

Thus, we find that the nonthermal line width is approximately equal to the wave amplitude when \(v_0 \ll \sigma\).

**Appendix B**

**Estimating Nonthermal Line Width of a Period- and LOS-averaged Spectrum**

In order to obtain an analytical relation between rms wave velocity and the nonthermal line widths for a period- and LOS-averaged spectrum assuming \(v_0 \ll \sigma\), we use Equation \((A11)\) that describes the shape of a period-averaged spectrum for oscillations aligned along the LOS (meaning \(\theta = 0\); see Figure 1(a)). For a given \(\theta\), the period-averaged spectrum is described by Equation \((A11)\) but replacing \(\sigma^t\) with \(\sigma \exp \left( \frac{\nu_0^2 \cos^2 \theta}{4 \sigma^4} \right)\). Next, we integrate the resulting equation in \(\theta\) as shown below:

\[
\langle G(v) \rangle_{1,\theta} \approx \int_0^{2\pi} \frac{1}{\sigma \exp \left( \frac{\nu_0^2 \cos^2 \theta}{4 \sigma^4} \right) \sqrt{2\pi}}
\]

\[
\times \exp \left( -\frac{\nu^2}{2 \sigma^2 \exp \left( \frac{\nu_0^2 \cos^2 \theta}{2 \sigma^2} \right)} \right) \right) d\theta \tag{B1}
\]

\[
\langle G(v) \rangle_{1,\theta} \approx \int_0^{2\pi} \frac{\exp \left( -\frac{\nu_0^2 \cos^2 \theta}{4 \sigma^4} \right)}{\sigma \sqrt{2\pi}}
\]

\[
\times \exp \left( -\frac{\nu^2 \exp \left( -\frac{\nu_0^2 \cos^2 \theta}{2 \sigma^2} \right)}{2 \sigma^2} \right) \right) d\theta \tag{B2}
\]

\[
\langle G(v) \rangle_{1,\theta} \approx \int_0^{2\pi} \frac{\exp \left( -\frac{\nu_0^2 \cos^2 \theta}{4 \sigma^4} \right)}{\sigma \sqrt{2\pi}}
\]

\[
\times \left[ 1 - \frac{\nu^2 \exp \left( -\frac{\nu_0^2 \cos^2 \theta}{2 \sigma^2} \right)}{2 \sigma^2} + \frac{\nu^4 \exp \left( -\frac{\nu_0^2 \cos^2 \theta}{\sigma^2} \right)}{8 \sigma^4} - ... \right] d\theta \tag{B3}
\]

Next, we expand the first term in Equation \((B4)\), say \(A\), and ignore higher-order terms to obtain the following equation:

\[
A = 1 - \frac{\nu_0^2 \cos^2 \theta}{\sigma^2} + ... \tag{B5}
\]

Similarly, second (say \(B\)) and third (say \(C\)) term in Equation \((B4)\) can be expanded as follows:

\[
B = \frac{\nu^2}{2 \sigma^2} - \frac{3\nu_0^2 \cos^2 \theta}{4 \sigma^4} + ... \tag{B6}
\]

\[
C = \frac{\nu^4}{8 \sigma^4} - \frac{5\nu_0^2 \cos^2 \theta}{8 \sigma^2} + ... \tag{B7}
\]

Integrating Equations \((B5)\), \((B6)\), and \((B7)\) for \(\theta = 0\) and inserting these expressions back into Equation \((B4)\), we get

\[
\langle G(v) \rangle_{1,\theta} = \frac{1}{\sigma \sqrt{2\pi}} \left[ 1 - \frac{\nu^2}{2 \sigma^2} \right]
\]

\[
\times \left( 1 - \frac{3\nu_0^2}{8 \sigma^2} + ... + \frac{\nu^4}{8 \sigma^4} \left( 1 - \frac{5\nu_0^2}{8 \sigma^2} + ... \right) \right) \tag{B8}
\]

If \(v_0 \ll \sigma\), we get

\[
\langle G(v) \rangle_{1,\theta} = \frac{\exp \left( -\frac{\nu_0^2}{4 \sigma^4} \right)}{\sigma \sqrt{2\pi}} \left[ 1 - \frac{\nu^2}{2 \sigma^2} \right]
\]

\[
\times \left( 1 - \frac{3\nu_0^2}{4 \sigma^2} + ... + \frac{\nu^4}{8 \sigma^4} \left( 1 - \frac{5\nu_0^2}{8 \sigma^2} \right) \right) \tag{B9}
\]

Equation \((B9)\) can be reformulated as follows:

\[
\langle G(v) \rangle_{1,\theta} \approx \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{\nu^2}{2 \sigma^2} \right) \tag{B10}
\]

Where, \(\sigma'' = \sigma \exp \left( \frac{\nu_0^2}{4 \sigma^4} \right)\). Thus,

\[
\sigma_{nt}^2 = 2\sigma''^2 - 2\sigma^2 = 2\sigma^4 \left( \exp \left( \frac{\nu_0^2}{4 \sigma^4} \right) - 1 \right) \approx \frac{\nu_0^2}{2} \approx \nu_{\text{inv}}^2. \tag{B11}
\]

The difference between \(\sigma_{nt}\) computed using Equation \((B11)\) and that estimated using Equation \((A12)\) is a factor of 1/2 that appears because of the integration of \(\cos^2 \theta\) inside the exponent in Equation \((B1)\). From this analysis, we conclude that whenever \(v_0 \ll \sigma\), we can expand the exponential functions by keeping terms up to second order in \(v_0\). Finally, the integration over all directions leads to a factor of 1/2 appearing in Equation \((B11)\). This is somewhat similar to the rms type of averaging of spectra of different nonthermal line widths that leads to overall reduction in the \(\sigma_{nt}\) by \(\sqrt{2}\) in this scenario compared to that discussed in Appendix A.

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