The Ground Surface Displacement with a Circular Inclusion Buried in a Layered Half-space Impacted by SH Wave

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Abstract. To analysis the ground surface displacement with buried circular inclusion under the action of SH wave, the large-arc assumption method is used to approximate the straight boundary by a circle with large radius. A numerical example demonstrates that the ground surface displacement amplification coefficient increases with the down of incident wave number $k_1$ and the hardness of the surface covered layer.

1. Introduction
With the development of modern society, a large number of population settle in cities and the cities scale are bigger and bigger. The harm of earthquake also cause everyone's attention. Elastic wave theory is an important theoretical method in seismic analysis and research. When the seismic wave propagated in soil meet with discontinuous place, scattering will occurs in the discontinuous place and the surface ground motion will be affected. Studies of elastic wave scattering has been a lot. In 1973, MOW, C. C. and PAO, Y introduced the theory of elastic wave scattering and dynamic stress concentration problem in their monograph[1].In 1982, the complex function method used for the solution of static stress concentration was generalized to the case of dynamic loading [2]. Based on this method, a lot of researches was given about the scattering of circular defects in the infinite elastic plane and elastic half space, as well as the quarter space and double-phase medium. Scientist Trifunac studied the surface motion of semi-cylindrical and semi-elliptical alluvial valley impacted incident plane SH waves[3, 4].In 1993, Lee and Karl given the analytic solution of P Wave and SH wave scattering about the single circular cavity in half-space based on the large arc hypothesis method[5].On this basis, the problem of circular scattering in layer half space incidented by SH wave was studied by Qi Hui[6, 7].The ground surface motion of a semi-elliptical hill and an alluvial valley in layered half-space for incident plane SH waves was researched by Liang[8, 9].This paper presents the influence of shallow buried circular inclusion to the ground surface displacement impacted by SH-wave based on the complex method and the great circle hypothesis method. The numerical result is analyzed by example.

2. Theoretical analysis
2.1. Problem model
The simplified model of problem researched in this paper is shown in Figure 1. The problem could be described as a single circular inclusion located in the surface elastic covered layer and the SH wave incident from the elastic half-space. The circular inclusion is defined as the area I, and the overbarden is defined as the area II, the area III represents the half-space. Respectively, their physical parameters
are \( G_1, k_1, \rho_1, G_2, k_2, \rho_2, G_3, k_3, \rho_3 \), in which \( G \) means the shear modulus, \( k \) means the wave number of SH wave, \( \rho \) means the density of medium, and the subscript corresponds to the corresponding area. \( h \) represent thickness of the surface layer. \( r \) expresses the radius of the circular inclusion. \( TC \) stands for the edge of the circular inclusion, whose radius is \( r \). Respectively, the upper and lower boundary are defined as \( TU \) and \( TD \). \( h_1 \) and \( h_2 \) denote the distance from the central point of the circular inclusion to the upper and the lower boundary, respectively. To construct the scattering SH wave of the surface covered layer, the large arc assumption method is used to approximate \( TU \) and \( TD \), and the center of the great circle is defined as so the \( TU \) is approximately expressed as \( TTU_1 \), and the \( TD \) is described as \( TTD \). The rectangular coordinate system \( X_1O_1Y_1 \) is established for the circular inclusion centered on the point \( O_1 \), which is the center of the circular inclusion. Correspondingly, the point \( O_2 \) is defined as the center of the great arc, which is the original point of the rectangular coordinate system \( X_2O_2Y_2 \). Corresponding to the rectangular coordinate system, the complex function \( z_1=X_1+iY_1 \) and \( z_2=X_2+iY_2 \) are set up to describe the complex planes \( z_1 \) and \( z_2 \). The schematic diagram is shown as Figure 1 and the relationship between various parameters could be described as:

\[
h = h_1 + h_2, \quad R_U = h + R_D, \quad z_2 = z_1 + i(R_D + h_2)
\]

\[\text{Figure 1. Schematic diagram of the model.}\]

2.2. Control equations
This paper studies the out-of-plane steady shear SH-Wave scattering. In the conventional rectangular coordinate system, the SH-Wave displacement field could be expressed as \( W(X, Y, t) \), whose movement is perpendicular to the \( X-Y \) plane and has nothing to do with the \( Z \) axis. The fields of displacement \( W(X, Y, t) \) must be satisfied with the Helmholtz equation in the rectangular coordinate system:

\[
\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} + k^2 W = 0
\]

where \( k=\omega/c, \ c^2=G/\rho, \ \omega=\text{the circular frequency of displacement field} \ W(X, Y, t) \), The correlation relationship between the displacement field \( W(X, Y, t) \) and the time is \( \exp(-i\omega t) \), and the \( \exp(-i\omega t) \) will not be considered in the following study because of the problem studied is steady state. In the complex plane, the formula (2) could be represented as:

\[
4\frac{\partial^2 W}{\partial \zeta \partial \overline{\zeta}} + k^2 W = 0
\]

In the polar coordinates of complex plane, the stress-strain relationship could be described as:

\[
\tau_{r\theta} = G \left[ (\frac{\partial W}{\partial \zeta})e^{i\theta} + (\frac{\partial \overline{W}}{\partial \overline{\zeta}})e^{-i\theta} \right], \quad \tau_{\theta\theta} = iG \left[ (\frac{\partial W}{\partial \zeta})e^{i\theta} - (\frac{\partial \overline{W}}{\partial \overline{\zeta}})e^{-i\theta} \right]
\]

In which the \( \theta \) means the angle from any point on the circumference of a circle in the polar coordinates.

2.3. Incident wave field and stress
In the complex plane \( z_0 \), the incident wave displacement field and the stress could be expressed as:

\[
W^{(i)}(z_0, z_2) = W_0 \exp \left[ ik_z \text{Re} \left( z_2 e^{-i\omega t} \right) \right]
\]
\[
\begin{align*}
\tau_{iz_1(z_2,z_1)}^{(1)} &= ik_z G_z W_0 \exp \left[ ik_z \text{Re} \left( z_2 e^{-i\alpha_0} \right) \right] \text{Re} \left( z_2 e^{-i\alpha_0} / |z_1| \right) \\
\tau_{iz_1(z_2,z_1)}^{(2)} &= -ik_z G_z W_0 \exp \left[ ik_z \text{Re} \left( z_2 e^{-i\alpha_0} \right) \right] \text{Im} \left( z_2 e^{-i\alpha_0} / |z_1| \right)
\end{align*}
\]

Where the \( W_0 \) is the maximal displacement amplitude and the \( \alpha_0 \) is the incident angle.

### 2.4. Scattered wave field and stress
In the complex plane \( z_2 \), the displacement field and stress of scattered wave \( W^{(1)} \) caused by TT\( D \) in region III, the displacement field and stress of scattered wave \( W^{(2)} \) occurred in region II by TT\( D \), the displacement field and stress of scattered wave \( W^{(1)} \) occurred in region II could be shown as formula (8)–(16). In the complex plane \( z_1 \), the displacement field and stress of scattered wave \( W^{(2)} \) occurred in region II by TT\( C \), the displacement field and stress of scattered wave \( W^{(2)} \) occurred in region I by TT\( C \) could be shown as formula (17)–(22):

\[
W^{(1)}(z_1,z_2) = \sum_{n=-\infty}^{\infty} A_n H_n^{(2)}(k_z |z_1|)(z_2/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(1)} = 0.5k_z G_z \sum_{n=-\infty}^{\infty} A_n \left[ H_n^{(1)}(k_z |z_2|) - H_n^{(2)}(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(2)} = i0.5k_z G_z \sum_{n=-\infty}^{\infty} A_n \left[ H_n^{(2)}(k_z |z_2|) + H_n^{(2)}(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
W^{(2)}(z_1) = \sum_{n=-\infty}^{\infty} B_n H_n^{(1)}(k_z |z_1|)(z_2/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(2)} = 0.5k_z G_z \sum_{n=-\infty}^{\infty} B_n \left[ H_n^{(1)}(k_z |z_2|) - H_n^{(1)}(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(1)} = i0.5k_z G_z \sum_{n=-\infty}^{\infty} B_n \left[ H_n^{(1)}(k_z |z_2|) + H_n^{(1)}(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
W^{(3)}(z_1) = \sum_{n=-\infty}^{\infty} C_n H_n^{(1)}(k_z |z_1|)(z_1/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(3)} = 0.5k_z G_z \sum_{n=-\infty}^{\infty} C_n \left[ H_n^{(1)}(k_z |z_2|) - H_n^{(1)}(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(3)} = i0.5k_z G_z \sum_{n=-\infty}^{\infty} C_n \left[ H_n^{(1)}(k_z |z_2|) + H_n^{(1)}(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
W^{(4)}(z_1) = \sum_{n=-\infty}^{\infty} D_n J_n(k_z |z_1|)(z_1/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(4)} = 0.5k_z G_z \sum_{n=-\infty}^{\infty} D_n \left[ J_n(k_z |z_2|) - J_n(k_z |z_2|) \right](z_2/|z_1|)^n
\]

\[
\tau_{iz_1(z_2,z_1)}^{(4)} = i0.5k_z G_z \sum_{n=-\infty}^{\infty} D_n \left[ J_n(k_z |z_2|) + J_n(k_z |z_2|) \right](z_2/|z_1|)^n
\]

In the complex plane \( z_1 \), the formulas (11)–(16) could be shown as:

\[
W^{(2)}(z_1) = \sum_{n=-\infty}^{\infty} B_n H_n^{(1)}(k_z |z_1| + i(R_D + h_2) |z_1|)(z_1/|z_1|)^n
\]
\[ r_{S_{(2)}}^{(5)}(z, \varpi) = 0.5k_2G_2 \sum_{n=0}^{\infty} B_n \left[ H_0^{(1)} \left[ k_2 \right] \left\{ z_1 + i(R_D + h_2) \right\} \left\{ z_1 + i(R_D + h_2) \right\} + e^{i\theta_0} \right] \]  
(24)

\[ r_{D_{(2)}}^{(5)}(z, \varpi) = i0.5k_2G_2 \sum_{n=0}^{\infty} B_n \left[ H_0^{(1)} \left[ k_2 \right] \left\{ z_1 + i(R_D + h_2) \right\} \left\{ z_1 + i(R_D + h_2) \right\} + e^{i\theta_0} \right] \]  
(25)

\[ W_{(5)}(z, \varpi) = \sum_{n=0}^{\infty} E_n H_n^{(1)} \left[ k_2 \right] \left\{ z_1 + i(R_D + h_2) \right\} \left\{ z_1 + i(R_D + h_2) \right\} + e^{i\theta_0} \]  
(26)

In the complex plane \( z \), the formulas (17)–(19) could be shown as:

\[ W_{(5)}^{(5)}(z, \varpi) = \sum_{n=0}^{\infty} C_n H_n^{(1)} \left[ k_2 \right] \left\{ z_2 - i(R_D + h_2) \right\} \left\{ z_2 - i(R_D + h_2) \right\} + e^{i\theta_0} \]  
(29)

\[ r_{S_{(3)}}^{(3)}(z, \varpi) = 0.5k_2G_2 \sum_{n=0}^{\infty} C_n \left[ H_0^{(1)} \left[ k_2 \right] \left\{ z_2 - i(R_D + h_2) \right\} \left\{ z_2 - i(R_D + h_2) \right\} + e^{i\theta_0} \right] \]  
(30)

\[ r_{D_{(3)}}^{(3)}(z, \varpi) = i0.5k_2G_2 \sum_{n=0}^{\infty} C_n \left[ H_0^{(1)} \left[ k_2 \right] \left\{ z_2 - i(R_D + h_2) \right\} \left\{ z_2 - i(R_D + h_2) \right\} + e^{i\theta_0} \right] \]  
(31)

2.5. Equations set

Because the displacement fields and stress fields expressions of incident and scattering waves have been constructed, according to the boundary conditions that the radial stress of \( T_{T_{(1)}} \) and \( T_{C} \) is free and the continuous condition that the displacement and the radial stress of \( T_{T_{(2)}} \) are continuous, the set of equations could be expressed as equations set, multiply both sides by \( \exp(-\imath \theta_0) \), integrate them in \( (-\pi, \pi) \) respectively, So we can calculate the coefficient \( A_n \sim E_n \), take them to the formulas and intercept limited items, all the unknown quantities will be found out.

2.6. The ground surface displacement amplification coefficient (\( W^* \))

Define \( W^* \) as the displacement amplification coefficient:

\[ W^* = \left| \frac{W_{(5)}^{(5)}(z, \varpi) + W_{(5)}^{(3)}(z, \varpi)}{W_{(5)}(z, \varpi)} \right| \]  
(32)

3. Numerical Example

In this part, example of numerical computation was presented, the results are non-dimensionalized. It is supposed that \( r=1 \). Define the parameters combination \( G = G_2/G_3, G = G_2/G_3, k = k_2/k_1, k = k_2/k_3 \). If \( k \) is less than 1, it means Region III is harder than region II, as well as \( k' \). In this example, the incident angle is 90°, which means the incident wave vertical incidence. From figure 2 to figure 4 separately shows the ground surface displacement amplification coefficient \( W^* \) with \( x_3/r \) when the incident wave incoming from the hard area III, the circular inclusion is in the middle of the layer, whose thickness is 3 times of \( r \), \( \rho'=0.7 \), \( \rho'=0.8 \), the incident wave is in different frequency. Conversely, the figures 5, 6, 7 introduce the incident wave incoming from the soft area III.

It is observed that from figure 2 to figure 4, when the half space is harder than the layer area, the more soft of surface covered layer, the larger of the ground surface displacement amplification coefficient...
$W^*$. When the incident wave number is $0.1$, $W^*$ is largest and will reduce continuously with the increase of the incident wave number $k_3$. It can be perceived from figure 5 to figure 7, when the semi-space is softer than the surface elastic layer, the $W^*$ is less than the figure 2 to figure 4. The harder of the surface covered layer, the less of $W^*$. The $W^*$ will reduce with the increasement of incident wave number $k_3$.

![Figure 2](image2.png)  
**Figure 2.** $W^*$ with $x_2/r(k_3=0.1, \rho^*=0.8)$.  

![Figure 3](image3.png)  
**Figure 3.** $W^*$ with $x_2/r(k_3=1, \rho^*=0.8)$.  

![Figure 4](image4.png)  
**Figure 4.** $W^*$ with $x_2/r(k_3=2, \rho^*=0.8)$.  

![Figure 5](image5.png)  
**Figure 5.** $W^*$ with $x_2/r(k_3=0.1, \rho^*=1.2)$.  

![Figure 6](image6.png)  
**Figure 6.** $W^*$ with $x_2/r(k_3=1, \rho^*=1.2)$.  

![Figure 7](image7.png)  
**Figure 7.** $W^*$ with $x_2/r(k_3=2, \rho^*=1.2)$.
4. Conclusion
Based on the large-arc assumption method, complex function and wave function expansion method, the ground surface displacement of shallow buried circular inclusion in the surface covered layer of half-space impacted by SH wave is studied. The numerical example indicates that the ground surface displacement amplification coefficient will reduce with the increase of rigidity of the surface covered layer, and will be lower when the incident wave number increases from low frequency to high frequency.

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