ON THE THETA OPERATOR FOR MODULAR FORMS MODULO PRIME POWERS

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Abstract. We consider the classical theta operator \( \theta \) on modular forms modulo \( p^m \) and level \( N \) prime to \( p \) where \( p \) is a prime greater than 3. Our main result is that \( \theta \mod p^m \) will map forms of weight \( k \) to forms of weight \( k + 2 + 2p^m(p - 1) \) and that this weight is optimal in certain cases when \( m \) is at least 2. Thus, the natural expectation that \( \theta \mod p^m \) should map to weight \( k + 2 + p^m-1(p - 1) \) is shown to be false.

The primary motivation for this study is that application of the \( \theta \) operator on eigenforms mod \( p^m \) corresponds to twisting the attached Galois representations with the cyclotomic character. Our construction of the \( \theta \)-operator mod \( p^m \) gives an explicit weight bound on the twist of a modular mod \( p^m \) Galois representation by the cyclotomic character.

1. Introduction

Let \( p \) be a prime number. We shall assume \( p \geq 5 \) throughout the paper in order to avoid certain technicalities when \( p \) is 2 or 3.

Further let \( m \in \mathbb{N} \) and denote by \( M_k(N, \mathbb{Z}_p) \) the \( \mathbb{Z}_p \)-module of modular forms of weight \( k \) for \( \Gamma_1(N) \) over \( \mathbb{Z}_p \) and let \( M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \) be the \( \mathbb{Z}/p^m\mathbb{Z} \)-module of modular forms for \( \Gamma_1(N) \) over \( \mathbb{Z}/p^m\mathbb{Z} \) as defined classically by \( M_k(N, \mathbb{R}) = M_k(N, \mathbb{Z}) \otimes \mathbb{R} \).

Note that this definition relies on the existence of an integral structure on \( M_k(N, \mathbb{C}) \) (see for instance [1]).

Let \( k_1, \ldots, k_t \) be a collection of weights and let \( f_i \in M_{k_i}(N, \mathbb{Z}_p) \). The \( q \)-expansion of an element in a direct sum of the \( M_k(N, \mathbb{Z}_p) \)'s or \( M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \)'s is defined by extending linearly on each component. When we write \( f_1 + \ldots + f_t \equiv 0 \pmod{p^m} \), we shall mean that the \( q \)-expansion \( f_1(q) + \ldots + f_t(q) \) lies in \( p^m\mathbb{Z}_p[[q]] \). Similarly for \( f_i \in M_{k_i}(N, \mathbb{Z}/p^m\mathbb{Z}) \), we write \( f_1 + \ldots + f_t \equiv 0 \pmod{p^m} \) if the \( q \)-expansion of \( f_1 + \ldots + f_t \) equals 0 in \( (\mathbb{Z}/p^m\mathbb{Z})[[q]] \). In such a case, we say that \( f_1 + \ldots + f_t \) is congruent to 0 modulo \( p^m \).

Let us recall the definition and basic properties of the standard Eisenstein series on \( \text{SL}_2(\mathbb{Z}) \), cf. §1 of [13], for instance: the series

\[
G_k := \frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n
\]

with \( B_k \) the \( k \)-th Bernoulli number and \( \sigma_t(n) := \sum_{d|n} d^t \) the usual divisor sum, is (with \( q := e^{2\pi i z} \)) for \( k \) an even integer \( \geq 4 \) a modular form on \( \text{SL}_2(\mathbb{Z}) \). Defining \( E_k \) as the normalization

\[
E_k := -\frac{2k}{B_k} \cdot G_k
\]

one has \( E_k \equiv 1 \pmod{p^t} \) when (and only when) \( k \equiv 0 \pmod{p^{t-1}(p - 1)} \).
There are natural inclusions (preserving $q$-expansions)
\[ M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \hookrightarrow M_{k+p^{-1}(p-1)}(N, \mathbb{Z}/p^m\mathbb{Z}), \]
induced by multiplication by $E_{p^{-1}}^{m-1}$, using the fact that $E_{p-1}^{m-1} \equiv 1 \pmod{p^m}$. Note that $E_{p^{-1}(p-1)}^{m-1} = E_{p-1}^{m-1}$ in $M_k(N, \mathbb{Z}/p^m\mathbb{Z})$, again by the $q$-expansion principle, so that the map can also be seen as induced by multiplication by $E_{p^{-1}(p-1)}^{m-1}$.

As is well-known, when we specialize the above series for $G_k$ to $k = 2$ and define
\[ G_2 := \frac{-B_2}{4} + \sum_{n=1}^{\infty} \sigma_1(n)q^n = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n \]
then $G_2$ does not represent a modular form in the usual sense, but does so in the $p$-adic sense, cf. [13], §2. One defines $E_2$ as the normalization of $G_2$, i.e., $E_2 := -24G_2$. Thus, $E_2$ is also a $p$-adic modular form.

Consider the classical theta operator $\theta f = \frac{1}{2}E_2 + \frac{1}{12}\partial f$ of Ramanujan. Its effect on $q$-expansions is $\sum_n a_n q^n \mapsto \sum_n n a_n q^n$. Since $E_2$ is a $p$-adic modular form it is for any $m \in \mathbb{N}$ congruent modulo $p^m$ to a classical modular form of some weight. Thus we have $E_2 \equiv E_{p+1} \pmod{p}$, for example, and since we classically know that $\partial$ maps modular forms of weight $k$ to modular forms of weight $k + 2$, one obtains the classical operator $\theta$ that maps $M_k(N, \mathbb{F}_p)$ to $M_{k+1}(N, \mathbb{F}_p)$. Studying this operator as well as its interaction with the ‘weight filtration’ (see below) is a key tool in the theory of modular forms modulo $p$; cf. for instance Jochnowitz’ proof of finiteness of systems of Hecke eigenvalues mod $p$ across all weights in [10], or Edixhoven’s results on the optimal weight in Serre’s conjectures [6].

As we have launched a framework for the study of modular forms mod $p^m$ in [2] and [1], it is natural to ask whether a $\theta$ operator with similar properties can be defined on such forms. We have discussed in [1] how to attach Galois representations to eigenforms mod $p^m$, and it is clear from the properties of those attached representations that applying the $\theta$ operator corresponds on the Galois side to twisting by the cyclotomic character mod $p^m$. Hence, the construction of the theta operator mod $p^m$ yields an immediate application to mod $p^m$ Galois representations (see Corollary 1).

Notice that Serre shows in [13, Théorème 5] that there exists a $\theta$ operator on $p$-adic modular forms (of level 1) whose effect on $q$-expansions is $\sum_n a_n q^n \mapsto \sum_n n a_n q^n$ and that sends a form of $(p$-adic) weight $k$ to a form of weight $k + 2$. One can view our results as finding a partially explicit expression with explicit weights for the mod $p^m$ reduction of this operator.

Hence, our results are then that on the one hand an extension of the $\theta$ operator from the mod $p$ to the mod $p^m$ situation is indeed possible, but that the interplay of the $\theta$ operator with the weights of the forms becomes much more complicated when $m > 1$ and that, in fact, there are certain genuine qualitative differences between the case $m = 1$ and the general cases $m > 1$. Let us explain in detail.

We show that a $\theta$ operator on modular forms mod $p^m$ can be defined such that $\theta$ maps $M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z})$ with $k(m) := 2 + 2p^{m-1}(p-1)$, such that the effect on $q$-expansions is $\sum_n a_n q^n \mapsto \sum_n n a_n q^n$, and such that $\theta$ satisfies simple commutation rules with Hecke operators $T_{\ell}$ for primes $\ell \neq Np$, cf. the first part of Theorem 1 below. The proofs use a number of results from [13] plus the observation that $f \mid V \equiv f^p \pmod{p}$ where $V$ is the classical $V$ operator.
Define the weight \( w_{p^m}(f) \) of a modular form \( f \mod p^m \) with \( f \neq 0 \mod p \) to be the smallest \( k \in \mathbb{Z} \) such that \( f \) is congruent modulo \( p^m \) to an element of \( M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \). A classical fact, crucial for instance in the work [10], is that when \( m = 1 \) we have \( w_p(\theta f) \leq w_p(f) + p + 1 \) with equality if (and only if) \( p \nmid w_p(f) \).

One might expect the generalization of this to be that \( w_{p^m}(\theta f) \leq w_{p^m}(f) + 2 + p^{m-1}(p-1) \) (perhaps with equality in some cases). However, as the second part of Theorem 1 shows, this is false:

**Theorem 1.** Let \( p \geq 5 \) be a prime. Put

\[
k(m) := 2 + 2p^{m-1}(p-1).
\]

(i) The classical theta operator \( \theta \) induces an operator

\[
\theta : M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z})
\]

whose effect on \( q \)-expansions is \( \sum a_n q^n \mapsto \sum n a_n q^n \).

(ii) If \( \ell \neq p \) is a prime and \( T_\ell \) denotes the \( \ell \)-th Hecke operator, then

\[
T_\ell \theta = \ell \cdot \theta T_\ell
\]
as linear maps \( M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z}) \).

(iii) Assume \( N \geq 5 \) is prime to \( p \). Let \( m \geq 2 \) and \( f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \) with \( f \neq 0 \mod p \). Suppose further that \( p \nmid k \) and \( w_p(f) = k \). Then

\[
w_{p^m}(\theta f) = k + 2 + 2p^{m-1}(p-1).
\]

The proof of the second part of the Theorem uses some results of [11], which are applicable to general level \( N \) prime to \( p \). In particular, a main point is that if we consider the Eisenstein series \( E_{p-1} \) and \( E_{p+1} \) as modular forms modulo \( p \) in the sense of Katz then \( E_{p-1} \) (Hasse invariant) has only simple zeros and no zero common with \( E_{p+1} \); cf. [11, Remark on p. 57]. This point allows one to compute the weight filtration of the last term of the expression below for the \( \theta \) operator mod \( p^m \), which is the controlling term. We would like to thank Nadim Rustom for a useful discussion about this point.

We have in fact conducted an in-depth study of the relation between and \( w_{p^m}(f) \) and \( w_{p^m}(\theta f) \) in the case of level \( N = 1 \) and for \( m = 2 \). However, the results are rather complicated and will not be stated in this article.

For simplicity, we have stated results in this paper for modular forms with coefficients in \( \mathbb{Z}_p \) and hence reductions with coefficients in \( \mathbb{Z}/p^n\mathbb{Z} \). The above theorem however is valid for coefficients in \( \mathbb{Z}/p^m\mathbb{Z} \) (see e.g. [1], section 2.4 for a definition of this ring) using the same proofs.

An immediate consequence of Theorem 1 to Galois representations is the following. We use the notation and terminology from [1].

**Corollary 1.** Let \( p \geq 5 \) be a prime, \( \rho : G_\mathbb{Q} \to \text{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) \) be a residually absolutely irreducible Galois representation, and \( \chi : G_\mathbb{Q} \to \mathbb{Z}/p^m\mathbb{Z}^\times \) be the reduction modulo \( p^m \) of the \( p \)-adic cyclotomic character. Suppose \( \rho \cong \rho_f \) for some weak eigenform \( f \in S_k(N)(\mathbb{Z}/p^m\mathbb{Z}) \). Then \( \rho \otimes \chi \cong \rho_g \) for some weak eigenform \( g \in S_{k+k(m)}(N)(\mathbb{Z}/p^m\mathbb{Z}) \).

**Proof.** Note first that \( \theta \) maps cusp forms to cusp forms. Suppose \( f \in S_k(\mathbb{Z}/p^m\mathbb{Z}) \) is a weak eigenform in the sense that \( T_\ell f \equiv f(T_\ell) f \mod p^m \) for all \( \ell \nmid D \) for some integer \( D \). Then \( g = \theta f \in S_{k+k(m)}(\mathbb{Z}/p^m\mathbb{Z}) \) is a weak eigenform in the sense
that $T_\ell g \equiv g(T_\ell)g \pmod{p^m}$ for all $\ell \nmid NDp$, and we have that $f(T_\ell)\chi(\ell) \equiv g(T_\ell) \pmod{p^m}$ for all $\ell \nmid Dp$. The hypothesis of residual absolute irreducibility then allows us to conclude that $\rho_f \otimes \chi \cong \rho_g$.

Regarding Corollary 1, the main result of the paper [3] implies an analogous statement about twisting with the mod $p^m$ reduction of the Teichmüller character, with some differences: In [3], they consider the twist of $f$ by the mod $p^m$ reduction of the Teichmüller character where $f$ is a strong eigenform (again using the terminology of [1]). One then finds a strong (and not merely weak) eigenform $g$ as in the Corollary, but apparently without control over the weight $k + k(m)$. The proof uses different methods (Coleman $p$-adic families of modular forms.)

2. The theta operator modulo prime powers

2.1. Eisenstein series. We shall now develop an explicit expression for the truncation modulo $p^m$ of the $p$-adic Eisenstein series $G_2$.

Proposition 1. Let $m \in \mathbb{N}$. Define the positive even integers $k_0, \ldots, k_{m-1}$ as follows: If $m \geq 2$, define:

$$k_j := 2 + p^{m-j-1}(p^{j+1}-1) \quad \text{for } j = 0, \ldots, m-2$$

and

$$k_{m-1} := p^{m-1}(p + 1)$$

and define just $k_0 := p + 1$ if $m = 1$.

Then $k_0 < \ldots < k_{m-1}$ and there are modular forms $f_0, \ldots, f_{m-1}$, depending only on $p$ and $m$, of level 1 and of weights $k_0, \ldots, k_{m-1}$, respectively, that have rational $q$-expansions, satisfy $v_p(f_j) = 0$ for all $j$, and are such that

$$G_2 \equiv \sum_{j=0}^{m-1} p^j f_j \pmod{p^m}$$

as a congruence between $q$-expansions.

The form $f_{m-1}$ can be chosen to be $f_{m-1} = G_{p+1}^{p-1}$.

When $m = 2$ we can, and will, be a bit more explicit:

Proposition 2. We have:

$$G_2 \equiv f_0 + p \cdot f_1 \pmod{p^2}$$

with modular forms $f_0$ and $f_1$ of weights $2 + p(p - 1)$ and $p(p + 1)$, respectively, explicitly:

$$G_2 \equiv G_{2+p(p-1)} + p \cdot G_{p+1}^p \pmod{p^2}.$$ 

It is amusing to note the following consequence of the Proposition: For $p \neq 2, 3$, we have the following congruence of Bernoulli numbers,

$$\frac{B_2}{2} \equiv \frac{B_{p(p-1)+2}}{p(p-1)+2} + p \frac{B_{p+1}}{p+1} \pmod{p^2}.$$

However, this can also be seen in terms of $p$-adic continuity of Bernoulli numbers (cf. [14], Cor. 5.14 on p. 61, for instance).

Before the proofs of these propositions we need a couple of preparations.

Let $k$ be an even integer $\geq 2$. Recall from [13] that if we choose a sequence of even integers $k_i$ such that $k_i \to \infty$ in the usual, real metric, but $k_i \to k$ in the...
Let \( G_k \) be a \( p \)-adic modular form of weight \( k \). It does not depend on the choice of the sequence \( k_i \). In particular, we can, and will, choose \( k_i := k + p^{i-1}(p-1) \), because if we chose another \( k'_i = k + \lambda p^{i-1}(p-1) \), where \( \lambda \geq 2 \), then \( G_{k'_i} \equiv G_{k_i} \pmod{p^i} \) so that \( G_{k'_i} = G_k p_p^{(\lambda-1)} \) in \( M_{k'_i}(\mathbb{Z}/p^i\mathbb{Z}) \) is not essentially different.

**Lemma 1.** Let \( k \) be an even integer \( \geq 2 \) and assume \( (p-1) \nmid k \). Let \( t \in \mathbb{N} \).

Then \( G_k^* \equiv G_{k+p^{t-1}(p-1)} \pmod{p^t} \).

**Proof.** Let \( u, v \geq t \). We claim that \( G_{k_v} \equiv G_{k_u} \pmod{p^t} \). Since the series \( G_{k_i} \) converges \( p \)-adically to \( G_k^* \), the claim clearly implies the Lemma.

If \( u = v \) the claim is trivial, so suppose not, say \( u < v \). Then \( k_v - k_u = p^{v-1}(p-1) - p^{u-1}(p-1) \) is a multiple of \( p^{t-1}(p-1) \), say \( k_v - k_u = s \cdot p^{t-1}(p-1) \). We also have \( k_v - k_u \geq 4 \). Hence, we find that \( G := G_{k_u} \cdot E^{p,v-1}_{p^{t-1}(p-1)} \) is a modular form of weight \( k_v \), and we have \( G_{k_u} \equiv G \pmod{p^t} \).

Now notice that, when \( i \geq t \), we have:

\[
\sigma_{k_i-1}(n) = \sum_{d|n} d^{k-1+p^{i-1}(p-1)} \equiv \sum_{d|n} d^{k-1} \pmod{p^t}
\]

as \( p^{t-1}(p-1) \equiv 1 \pmod{p^t} \) when \( p \nmid d \) and \( i \geq t \), and as \( p^{t-1}(p-1) \equiv 0 \pmod{p^t} \) when \( p \mid d \) and \( i \geq t \) (because \( p^{t-1}(p-1) \geq t \) as long as \( p \neq 2 \)).

We conclude that the nonconstant terms of the series \( G_{k_v} \) and \( G_{k_u} \) are termwise congruent modulo \( p^t \). The same is then true of the forms \( G \) and \( G_{k_v} \) that are both forms of weight \( k_v \). Hence, the nonconstant terms of the form \( (G - G_{k_v})/p^t \) are all \( p \)-integral. As \( k_v \equiv k \not\equiv 0 \pmod{(p-1)} \), it follows from Théorème 8 of [12] that the constant term of this form is in fact also \( p \)-integral. Hence,

\[
G_{k_u} \equiv G \equiv G_{k_v} \pmod{p^t}
\]
as desired. \( \square \)

Recall that the \( V \) operator is defined on formal \( q \)-expansions as

\[
(\sum a_n q^n) \mid V := \sum a_n q^{np}.
\]

**Corollary 2.** We have

\[
G_2 \equiv \sum_{j=0}^{m-1} p^j \cdot (G_{2+p^{m-j-1}(p-1)} \mid V^j) \pmod{p^m}
\]
as a congruence between formal \( q \)-expansions.

**Proof.** Recall from [13], §2, the identity, valid for any even integer \( k \geq 2 \), that

\[
G_k = G_k^* + p^{k-1} (G_k^* \mid V) + \ldots + p^{(k-1)} (G_k^* \mid V^i) + \ldots.
\]

The identity is first an identity of formal \( q \)-expansions, but then shows that \( G_k \) is a \( p \)-adic modular form as \( V \) acts on \( p \)-adic modular forms, cf. [13], §2.

If we specialize this identity to the case \( k = 2 \), reduce modulo \( p^m \), and note that the previous Lemma applies since \( (p-1) \nmid 2 \), the claim follows immediately. \( \square \)

In the next paragraph and lemma, we use the notation \( M_k(\Gamma, F) \) to mean the \( F \)-module of modular forms of weight \( k \) over \( F \), where \( \Gamma \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) and \( F \) is a subring of \( \mathbb{C} \) or \( \mathbb{C}_p \).
We can also see the $V$ operator as an operator on modular forms: Suppose that $f \in M_k(\text{SL}_2(\mathbb{Z}), \mathbb{C})$. Then $(f \mid V)(z) = f(pz)$, and as is well-known $f \mid V \in M_k(\Gamma_0(p); \mathbb{C})$. The proof of the next lemma is a simple application of section 3.2 of [13]. Recall that if $f \in M_k(\text{SL}_2(\mathbb{Z}), \mathbb{Q}_p)$ is nonzero with $q$-expansion $\sum a_n q^n$ we define $v_p(f) := \min\{v_p(a_n) \mid n \in \mathbb{N}\}$ where $v_p(a_n)$ is the usual (normalized) $p$-adic valuation of $a_n$.

**Lemma 2.** Let $f \in M_k(\text{SL}_2(\mathbb{Z}), \mathbb{Q})$ and suppose $v_p(f) = 0$. Let $t \in \mathbb{N}$ and suppose that $s \in \mathbb{Z}_{\geq 0}$ is such that

$$\inf(s + 1, p^s + 1 - k) \geq t.$$  

Then there is $h \in M_{k+p^s(p-1)}(\text{SL}_2(\mathbb{Z}), \mathbb{Q})$ with $v_p(h) = 0$, and such that

$$f \mid V \equiv h \pmod{p^t}.$$  

**Proof.** As we noted above, $f \mid V$ is a modular form of weight $k$ on $\Gamma_0(p)$. Since $f \mid V = \sum a_n q^n$ if $f = \sum a_n q^n$ we have $v_p(f \mid V) = 0$. Recall the Fricke involution for modular forms on $\Gamma_0(p)$ given by the action of the matrix

$$W = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.$$  

Since

$$f \mid V = p^{-k/2} f \mid k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

(recall that the weight $k$ action is normalized so that diagonal matrices act trivially), since

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$

and since $f$ is on $\text{SL}_2(\mathbb{Z})$ we see that $f \mid VW = p^{-k/2} f$ so that $v_p(f \mid VW) = -k/2$

Now let $E := E_{p-1}$ and put

$$g := E - p^{p-1}(E \mid V)$$

so that $g$ is a modular form of weight $p - 1$ on $\Gamma_0(p)$. Then, if we put

$$f_s := \text{tr}((f \mid V) \cdot g^p)$$

for $s \in \mathbb{Z}_{\geq 0}$ where $\text{tr}$ denotes the trace from $\Gamma_0(p)$ to $\text{SL}_2(\mathbb{Z})$, it follows from section 3.2 of [13] that $f_s$ is a modular form on $\text{SL}_2(\mathbb{Z})$ of weight $k + p^s(p - 1)$ and rational $q$-expansion. Furthermore, Lemma 9 of [13] implies that

$$v_p(f_s - (f \mid V)) \geq \inf(s + 1, p^s + 1 + v_p(f \mid VW) - k/2) = \inf(s + 1, p^s + 1 - k) \geq t.$$  

Hence, we can choose $h := f_s$. As $f \mid V \equiv h \pmod{p^t}$ and $v_p(f \mid V) = 0$, we must have $v_p(h) = 0$. \hfill \Box

**Proof of Proposition 1:** That the defined weights $k_0, \ldots, k_{m-1}$ satisfy $k_0 < \ldots < k_{m-1}$ is verified immediately.  

Thus, starting with Corollary 2 we see that it suffices to show that for each $j \in \{0, \ldots, m-1\}$ there is a modular form $f_j$ of weight $k_j$ with rational $q$-expansion and $v_p(f_j) = 0$, and such that

$$G_{2+p^{j-1}(p-1)} \mid V^j \equiv f_j \pmod{p^{m-j}}.$$  

If $m = 1$, $j = 0$ we just take $f_0 := G_{p+1}$, so assume $m \geq 2$. Then, if $j = m-1$, note that

$$G_{p+1} \mid V^{m-1} \equiv G_{p+1}^{p^{m-1}} \pmod{p}.$$
It is interesting to note that in the induction, the inequalities do not allow us to deal with the case $j = m - 1$. We claim that for $r = 0, \ldots, j$ there is a modular form $g_r$ of weight $2 + p^{m-j-1}(p^{r+1} - 1)$, rational $q$-expansion with $v_p(g_r) = 0$, and such that

$$G_{2+p^{m-j-1}(p-1)} \mid V \equiv g_r \pmod{p^{m-j}}$$

which is the desired when $r = j$.

We prove the last claim by induction on $r$ noting that the case $r = 0$ is trivial. So, suppose that $r < j$ and that we have already shown the existence of a modular form $g_r$ as above. Notice that

$$p^{m-j+r} + 1 - (2 + p^{m-j-1}(p^{r+1} - 1)) = p^{m-j-1} - 1 \geq m - j$$

holds because $m - j \geq 2$ (we used here that $p > 2$). Thus we see that Lemma 2 applies (taking $s = m - j + r$) and shows the existence of a modular form $g_{r+1}$ with rational $q$-expansion and $v_p(g_{r+1}) = 0$, such that

$$g_r \mid V \equiv g_{r+1} \pmod{p^{m-j}},$$

and such that $g_{r+1}$ has weight

$$2 + p^{m-j-1}(p^{r+1} - 1) + p^{m-j+r}(p - 1) = 2 + p^{m-j-1}(p^{r+2} - 1),$$

and we are done. \qed

**Remark 1.** It is interesting to note that in the induction, the inequalities do not allow us to deal with the case $j = m - 1$, but then we use the congruence $f \mid V \equiv f^p \pmod{p}$ to take care of the last term.

**Proof of Proposition 2:** Again by Corollary 2 we have:

$$G_2 \equiv G_{2+p^{m-1}(p-1)} + p \cdot (G_{p+1} \mid V) \pmod{p^2}.$$  

Noting again that $G_{p+1} \mid V \equiv G^p_{p+1} \pmod{p}$ so that

$$p \cdot (G_{p+1} \mid V) \equiv p \cdot G^p_{p+1} \pmod{p^2},$$

we are done. \qed

2.2. **Definition and properties of the $\theta$ operator.** Recall the classical $\theta$ operator acting on formal $q$-expansions as $q^{\frac{d}{24}}$, i.e.,

$$\theta \left( \sum a_n q^n \right) := \sum n a_n q^n,$$

and the operator $\partial$ defined by

$$\frac{1}{12} \partial f := \theta f - \frac{k}{12} E_2 \cdot f = \theta f + 2kG_2 \cdot f$$

when $f = \sum a_n q^n \in M_k(N, \mathbb{C})$ is a modular form of weight $k$ (we have $B_2 = \frac{1}{6}$ so that $E_2 = -24G_2$). Then $\partial f$ is in $M_{k+2}(N, \mathbb{C})$, and $\partial$ defines a derivation on $M(N, \mathbb{C}) := \oplus_k M_k(N, \mathbb{C})$ (as follows by writing $\theta = 2 \pi iz \frac{d}{dq}$ (as $q = e^{2\pi i z}$) and combining with the classical transformation properties of $E_2$ under the weight 2 action of $\text{SL}_2(\mathbb{Z})$ given in (4))

The definition of $\partial$ implies that $\partial$ defines a derivation on $\oplus_k M_k(N, \mathbb{Z})$ and hence also on $M(N, \mathbb{Z}_p) := \oplus_k M_k(N, \mathbb{Z}_p)$.
**Proof of Theorem 1.** (i) Retain the notation of Proposition 1 so that

\[ k_j := 2 + p^{m-j-1}(p^j+1) \quad \text{for } j = 0, \ldots, m-2, \]

and

\[ k_{m-1} := p^{m-1}(p+1). \]

Then \( k_0 < \ldots < k_{m-1} \) and by Proposition 1 we have modular forms \( f_0, \ldots, f_{m-1} \) (of level 1 and) of weights \( k_0, \ldots, k_{m-1}, \) respectively, that have rational \( q \)-expansions, satisfy \( v_p(f_j) = 0 \) for all \( j \), and are such that

\[ G_2 \equiv \sum_{j=0}^{m-1} p^j f_j \pmod{p^m}. \]

With \( k(m) := 2 + 2p^{m-1}(p-1) \) one checks that each number \( k(m) - k_j \) is a multiple of \( p^{m-j-1}(p-1) \), say \( k(m) = k_j + t_j \cdot p^{m-j-1}(p-1) \) for \( j = 0, \ldots, m-1 \).

Since \( E_{p-1}^{p^{m-j-1}} \equiv 1 \pmod{p^{m-j}} \) we find that \( p^j E_{p-1}^{p^{m-j-1}t_j} \equiv p^j \pmod{p^m} \), and so the above congruence for \( G_2 \) can also be written as

\[ G_2 \equiv \sum_{j=0}^{m-1} p^j E_{p-1}^{p^{m-j-1}t_j} f_j \pmod{p^m} \]

where now each summand is a form of weight \( k(m) \).

Hence, for any an element \( f \in M_k(N, \mathbb{Z}_p) \) we find that

\[(1) \quad \theta f = \frac{1}{12} \partial f - 2kG_2 \cdot f \]

\[(2) \quad \equiv \frac{1}{12} E_{p-1}^{2p^{m-1}} \partial f - 2k \sum_{j=0}^{m-1} p^j E_{p-1}^{p^{m-j-1}t_j} f_j \pmod{p^m} \]

\[(3) \quad =: \theta_{p^m} f \in M_{k+k(m)}(N, \mathbb{Z}_p) \]

where now each summand on the right hand side is an element of \( M_{k+k(m)}(N, \mathbb{Z}_p) \). Thus the classical theta operator induces a linear map

\[ \theta = \theta_{p^m} : M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z}) \]

the effect of which on \( q \)-expansions is \( \sum a_nq^n \mapsto \sum na_nq^n \). We still denote this operator (by abuse of notation) by \( \theta \), but later when we need to distinguish from \( \theta := \frac{1}{2\pi i} \cdot \frac{d}{dz} \), we will denote it by \( \theta_{p^m} \).

(ii) First assume for the prime \( \ell \) that we have \( \ell \nmid N \) (in addition to \( \ell \neq p \)). Recall that the diamond operator \( \langle \ell \rangle_k \) on a modular form \( f \) of weight \( k \) is defined by

\[ \langle \ell \rangle_k f = f \mid_k \gamma \quad \text{for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \text{ with } c \equiv 0 \pmod{N}, \quad d \equiv \ell \pmod{N}. \]

(Note that we write the action of \( \langle \ell \rangle \) from left, though it is given by the stroke operator which is from the right. This is fine because \( (\mathbb{Z}/N\mathbb{Z})^\times \) is abelian.) As \( \ell \neq p \) the operator \( \langle \ell \rangle_k \) induces a linear action on \( M_k(N, \mathbb{Z}_p) \) (as follows from the well-known formula \( \ell^{k-1} \langle \ell \rangle_k = T_{\ell^2}^2 - T_{\ell^2} \) and the fact that the Hecke operators \( T_n \) preserve \( M_k(N, \mathbb{Z}) \)), and hence also on \( M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \).

Using \( \frac{1}{12} \partial f = \theta f - \frac{1}{12} E_2 f \) as well as the transformation property of \( E_2 \) given by

\[(4) \quad (E_2 \mid_2 \gamma)(z) = E_2(z) + \frac{12}{2\pi i} cj(\gamma, z)^{-1} \]


have level 1 and thus are fixed under the action of the operators \( \langle E \rangle \) as well as the fact that the forms \( f \) (see for instance [5]), a computation shows that we have

\[
\langle \ell \rangle_k f = \langle \ell \rangle_{k+1} f + \theta_{p^m} \langle \ell \rangle_k f
\]

for all \( f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \).

Now recall the above definition of \( \theta_{p^m} : M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z}) \), as well as the fact that the forms \( E_{p^m-1} \) and the \( f_j \) occurring in the definition all have level 1 and thus are fixed under the action of the operators \( \langle \ell \rangle_{p-1} \) and \( \langle \ell \rangle_{k_j} \), respectively. We deduce that:

\[
\theta_{p^m} \langle \ell \rangle_k f = \langle \ell \rangle_{k+k(m)} \theta_{p^m} f
\]

for all \( f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \).

Let now \( f = \sum a_n q^n \in M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \). Then we have (with the usual convention that \( a_q := 0 \) if \( \ell \nmid n \)):

\[
\langle \ell \rangle_k f = \sum b_n q^n
\]

\[
T_t f = \sum \left( a_{\ell n} + \ell^{k-1} \sum b_q \right) q^n
\]

\[
\langle \ell \rangle_{k+k(m)} \theta_{p^m} f = \theta_{p^m} \langle \ell \rangle_k f = \sum n b_n q^n
\]

\[
T_t \theta_{p^m} f = \sum \left( \ell n a_{\ell n} + \ell^{k+k(m)-1} n b_q \right) q^n
\]

\[
\ell \theta_{p^m} T_t f = \ell n \sum \left( a_{\ell n} + \ell^{k-1} b_q \right) q^n.
\]

For \( \ell \mid N, \ell \neq p \), a similar calculation holds (the second term involving \( b_q \) is omitted throughout).

Thus, we have that

\[
T_t \theta_{p^m} f = \ell \theta_{p^m} T_t f
\]

for all \( f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \) using the fact that \( k(m) - 2 = 2 p^{m-1} (p-1) \) is divisible by \( p^{m-1} (p-1) \) so that

\[
e^{k(m) - 2} \equiv 1 \pmod{p^m}.
\]

(iii) Now suppose that \( m \geq 2 \), that \( f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \) with \( f \neq 0 \pmod{p} \), and suppose further that \( p \nmid k \) and \( w_p(f) = k \).

Assume that we had \( w_p(\theta f) < k + 2 + 2 p^{m-1} (p-1) = k + k(m) \), i.e., that there exist a form \( g \in M_{k'}(N, \mathbb{Z}/p^m\mathbb{Z}) \) with \( g = f \) as forms with coefficients in \( \mathbb{Z}/p^m\mathbb{Z} \), and where \( k' < k + k(m) \). We then know that

\[
k' \equiv k + k(m) \pmod{p^{m-1} (p-1)},
\]

say \( k + k(m) = k' + t \cdot p^{m-1} (p-1) \) with \( t \geq 1 \) (see [1, Corollary 22]; note we use the fact that modular forms in \( M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \) can be lifted to classical modular forms over \( \mathbb{Z}_p \) which is what is used in loc. cit. and that \( N \) is prime to \( p \)). Putting

\[
h := E_{p-1}^{p^{m-1}(t-1)} g
\]

we find that

\[
\theta f = E_{p-1}^{p^{m-1}} h
\]

as an equality of forms in \( M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z}) \). If we combine this with (1), we obtain:

\[
2kp^{m-1} E_{p-1}^{p^{m-1}} f_{m-1} f = -E_{p-1}^{p^{m-1}} h + \frac{1}{12} E_{p-1}^{2p^{m-1}} \partial f - 2k \sum_{j=0}^{m-2} p^j E_{p-1}^{p^{m-j-1} t_j} f_j.
\]
If we now use the fact that \( p \nmid k \), that \( p \) is odd, and that, as is easily checked, we have
\[
t_{m-1} = (k(m) - k_{m-1})/(p - 1) = (p^n - 3p^{m-1} + 2)/(p - 1) < p^{n-1},
\]
as well as \( t_{m-1} < p^{n-j-1}t_j \) for \( j = 0, \ldots, m - 2 \), we deduce that
\[
p^{n-1}M_{p-1}^{-1} f_{m-1} f = E_{p-1}^{t_{m-1}+1} h'
\]
for some \( h' \in M_{k+k(m)-(p-1)(t_{m-1}+1)}(N, \mathbb{Z}/p^n\mathbb{Z}) \). Hence we must have \( h' \in p^{n-1}M_{k+k(m)-(p-1)(t_{m-1}+1)}(N, \mathbb{Z}/p^n\mathbb{Z}) \), say \( h' = p^{m-1}h'' \), so that
\[
E_{p-1}^{t_{m-1}} f_{m-1} f \equiv E_{p-1}^{t_{m-1}+1} h'' \pmod{p},
\]
and hence
\[
f_{m-1} f \equiv E_{p-1} h'' \pmod{p}.
\]
It follows that
\[
w_p(f_{m-1} f) < k + k(m) - t_{m-1}(p - 1) = k + k_{m-1} = k + p^{m-1}(p + 1).
\]
Now recall (from Proposition 1) that \( f_{m-1} = G_{p+1}^{p^{m-1}} \). As \( G_{p+1} = -B_{p+1}/2(p+1)E_{p+1} \)
with \( B_{p+1} \) invertible modulo \( p \), we deduce
\[
w_p(E_{p+1}^{p^{m-1}} f) < k + p^{m-1}(p + 1).
\]
However, as \( w_p(f) = k \) by assumption, this conclusion contradicts the following general fact:
Suppose that \( N \geq 5 \) and let \( 0 \neq \phi \in M_\kappa(N, \mathbb{Z}/p\mathbb{Z}) \) where \( \kappa \geq 1 \), but \( \kappa \neq p \). Then, for \( a \in \mathbb{N} \) we have:
\[
w_p(E_{p+1}^a \phi) = w_p(\phi) + a(p + 1).
\]
To prove this general fact, by induction on \( a \) it is clearly enough to prove the case \( a = 1 \). Hence, let us assume \( a = 1 \).
Assume without loss of generality that \( w_p(\phi) = \kappa \). Assume for a contradiction that we had \( w_p(E_{p+1} \phi) < \kappa + p + 1 \). Then
\[
E_{p+1} \phi = E_{p-1} \psi
\]
for some \( \psi \in M_{1+p,\kappa}(N, \mathbb{Z}/p\mathbb{Z}) \).
Let \( \mathcal{M}_k(N, R) \) denote the space of modular forms of weight \( k \) on \( \Gamma_1(N) \) with coefficients in \( R \) as defined in [4] using Katz’s definition. One has an injection
\[
M_k(N, R) \to \mathcal{M}_k(N, R)
\]
sending classical modular forms over \( R \) to Katz modular forms over \( R \).
Under the hypothesis \( N \geq 5 \), we have from [4, Theorem 12.3.7]) that:

(K1) \( \mathcal{M}_k(N, \mathbb{Z}_p) \cong M_k(N, \mathbb{Z}_p) \) (as \( \mathbb{Z}_p \) is flat over \( \mathbb{Z} \))

(K2) \( \mathcal{M}_k(N, \mathbb{Z}/p^m\mathbb{Z}) \cong M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \) if \( k > 1 \) and \( N \) is prime to \( p \).

Regard the above identity as an identity of Katz modular forms on \( X_1(N) \) over \( \mathbb{Z}/p\mathbb{Z} \) and let us use some results from [11]: By the remark after Lemma 1 of [11], the forms \( E_{p-1} \) and \( E_{p+1} \) are without a common zero (the results in [11] are for modular forms on \( X(N) \) which implies the result for \( X_1(N) \)).

But, by a theorem of Igusa [9], \( E_{p-1} \) vanishes to order 1 at every supersingular point of \( X_1(N) \) (see [8, second paragraph following (4.6)]). Hence the equality \( E_{p+1} \phi = E_{p-1} \psi \) means that \( \phi \) vanishes at every zero of \( E_{p-1} \). Thus, we must have \((\kappa > p - 1) \) and \( \phi = E_{p-1} \eta \) for some \( \eta \in M_{\kappa-(p-1)}(N, \mathbb{Z}/p\mathbb{Z}) \).
The \( \eta \) cannot be classical in the sense that \( \eta \in M_{\kappa-(p-1)}(N,\mathbb{Z}/p\mathbb{Z}) \) or else \( w_p(\phi) < \kappa \), contrary to hypothesis. Hence, \( \eta \) is a non-classical modular form in the space \( M_{\kappa-(p-1)}(N,\mathbb{Z}/p\mathbb{Z}) \) of Katz modular forms. By (K2), this can only happen if \( \kappa - (p - 1) = 1 \), which means \( \kappa = p \), contrary to hypothesis. \( \square \)

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