A CLASSICAL APPROACH TO RELATIVE QUADRATIC EXTENSIONS

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Abstract. We show that we can develop from scratch and using only classical language a theory of relative quadratic extensions of a given number field $K$ which is as explicit and easy as for the well-known case that $K$ is the field of rational numbers. As an application we prove a reciprocity law which expresses the number of solutions of a given quadratic equation modulo an integral ideal $\mathfrak{a}$ of $K$ in terms of $\mathfrak{a}$ modulo the discriminant of the equation. We study various $L$-functions associated to relative quadratic extensions. In particular, we define, for totally negative algebraic integers $\Delta$ of a totally real number field $K$ which are squares modulo 4, numbers $H(\Delta, K)$, which share important properties of classical Hurwitz class numbers. In an appendix we give a quick elementary proof of certain deeper properties of the Hilbert symbol on higher unit groups of dyadic local number fields.

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1. Introduction

If $L$ is a quadratic extension of $\mathbb{Q}$ we can pick any integral quadratic irrationality $a$ in $L$, set $\Delta := \text{tr}(a) - 4n(a)$ and read off all important arithmetic properties of $L$ from the rational integer $\Delta$. We have $L = \mathbb{Q}(\sqrt{\Delta})$, the integer $\Delta$ is a square modulo 4, and if we divide out the largest perfect square $f^2$ such that $\Delta_0 := \Delta/f^2$ is still a square modulo 4, then $\Delta_0$ is the discriminant of $L$, which carries all information about the ramification of rational primes in $L$. The application which associates to a rational prime $p$ the Legendre symbol $(\frac{\Delta}{p})$ extends to a character modulo $\Delta$. It factors through the primitive Dirichlet character $(\frac{\Delta}{s})$. The Dedekind zeta functions $\zeta_L(s)$ of $L$ and $\zeta_Q(s) = \zeta(s)$ of $\mathbb{Q}$ are related by the identity $\zeta_L(s) = \zeta(s)L((\frac{\Delta}{s}), s)$, where $L((\frac{\Delta}{s}), s) = \sum_{n \geq 1} (\frac{\Delta}{n})n^{-s}$, and this identity encodes the information of the splitting of the rational primes in $L$.

These facts are well-known since the beginning of algebraic number theory in the 19th century, and the explicit character of this theory is helpful in many applications. In contrast to this one does not find such an easy and smooth description of the arithmetic theory of relative quadratic extensions. One finds some hints towards such a theory in Hecke’s “Vorlesungen über die Theorie der Algebraischen Zahlen” [Hec23, §39] and, in particular, in its last section [Hec23, §63]. However, Hecke’s treatment is not as completely developed as in the case of extensions of $\mathbb{Q}$. In modern algebraic number theory relative quadratic extensions are subsumed under the more abstract class field theory, which provides a conceptual background for the arithmetic of general abelian extensions, but lacks often explicitness even in simpler subclasses of abelian extensions as for instance quadratic extensions.

In this article we propose an explicit theory of relative quadratic extension which extends almost completely the classical and explicit theory of quadratic extensions of $\mathbb{Q}$. This depends, first of all, on a correct extension of the notion of discriminants and fundamental discriminants to arbitrary number fields $K$. By discriminant in $K$ we mean, similar to the case of rational numbers, any algebraic integer in $K$ which is a square modulo 4. However, if the class number of $K$ is larger than 1, there will be in general no algebraic integer $f$ in $K$ whose square divides $\Delta$ and such that $\Delta/f^2$ is still a square modulo 4. However, as it turns out, the maximal ideal $f_\Delta$ whose square divides $\Delta$ and such that

$$\Delta \equiv x^2 \mod 4f_\Delta^2$$

for some algebraic integer $x$ in $K$, serves very well as replacement for the missing $f$ for developing an explicit theory of relative quadratic extensions.

This theory will be developed in §2 and its subsections. We show in particular, that, for a given discriminant $\Delta$ in $K$, the relative discriminant $D_{L/K}$ of $L = K(\sqrt{\Delta})$ over $K$ is given by the formula (Theorem 2)

$$D_{L/K} = \Delta/f_\Delta^2.$$

We introduce a multiplicative function $(\frac{\Delta}{p})$ on ideals $a$ of $K$ which are relatively prime to $\Delta$ by setting, for any prime ideal $p$,

$$\left( \frac{\Delta}{p} \right) = \begin{cases} +1 & \text{if } \Delta \text{ is a square modulo } 4p \\ -1 & \text{otherwise.} \end{cases}$$

and show that it defines a Größencharakter modulo $\Delta$ whose conductor equals $\Delta/f_\Delta^2$ (Theorem 10). (For $p \nmid 2$ we can replace $4p$ by $p$ since $\Delta$ is a square modulo 4, and then the definition of the symbol $(\frac{\Delta}{p})$ is classical.) The associated primitive Größencharakter is of course nothing else than the Größencharakter commonly known as ‘the character of the quadratic extension $L/K$’; see §2.6)
For a discriminant $\Delta$ in the field of rational numbers, the Dirichlet $L$-function $L((\sqrt{\Delta}), s)$ and related Dirichlet series like $\sum_{a \geq 1} N_\Delta(a)a^{-s}$, where $N_\Delta(a)$ counts the number of solutions $x$ modulo $2a$ of $x^2 \equiv \Delta \mod 4a$, play an important role in the arithmetic in $\mathbb{Q}(\sqrt{\Delta})$. In §3 we define and study the analogues of these Dirichlet series for relative quadratic extensions. The main results are Theorem 14, 17, 18 and 19.

The first theorem states, for a given discriminant $\Delta$ and integral ideal $\mathfrak{a}$ in a number field $K$ with ring of integers $\mathfrak{o}$, a reciprocity law expressing the number

$$N_\Delta(\mathfrak{a}) = \text{card}\left(\{\mathfrak{o}/2\mathfrak{a} : x^2 \equiv \Delta \mod 4\mathfrak{a}\}\right)$$

in terms of a generalized quadratic residue symbol $\chi_\Delta$ modulo $\Delta$ (see (6) for its definition). Theorem 17 relates the Dirichlet series

$$\zeta(\Delta, s) := \sum_{\mathfrak{a}} \frac{N_\Delta(\mathfrak{a})}{n(\mathfrak{a})^s}$$

(the sum has to be taken over all integral ideals of $K$) to the Dirichlet series of the order $\Omega_{\mathfrak{a}} := \mathfrak{o} + \mathfrak{o}\sqrt{\Delta}$ of $L = K(\sqrt{\Delta})$, where $b$ is any solution in $\mathfrak{o}$ of $b^2 \equiv \Delta \mod 4\mathfrak{f}_\Delta^2$. Theorem 18 summarizes the important properties of the Dirichlet series $L(\chi_\Delta, s) = \sum_n \chi_\Delta(n)n^{-s}$. Finally, Theorem 19 proposes a generalization of Hurwitz class numbers to totally negative discriminants in totally real number fields $K$ as

$$H(\Delta, K) := L(\chi_\Delta, 0).$$

It gives an explicit finite formula for these numbers in terms of the class numbers of $K$ and $L = K(\sqrt{\Delta})$, which shows in particular, that these numbers are rational. These formulas generalize the classical formulas for the Hurwitz class numbers.

In Appendix 1 (§4) the reader finds tables of the numbers $H(\Delta, K)$ for various totally real number fields. We computed, for all fields in the range of the Bordeaux tables of number fields [Gro07] with not more than 8 genus classes, the discriminants $\Delta$ (modulo squares in $K$) such that $\Delta = \mathfrak{f}_\Delta^2$. These discriminants are exactly those for which the extension $K(\sqrt{\Delta})/K$ is unramified at the finite places, and their number modulo squares in $K$ equals the number of genus classes in $K$. The results of our computation is summarized in Table 4.

Finally we prove in Appendix 2 (Section 5) an orthogonality relation for the Hilbert symbol $(-, -)_K$ of a dyadic number field $K$ (cf. Theorem 24), which we needed in the study of the Größencharakters $\left(\frac{a}{\Delta}\right)$ in Theorem 10. The non-trivial part of its proof is to show, for the higher unit groups $U_i$ in $K$, that $(U_i, U_j)_K = 1$ whenever $i+j = 2e$, where $e$ is the ramification index of $K$. All proofs of this identity which we found in the literature make use of rather deep theorems of local class field theory. To comply with the explicit and self-contained nature of this article we sought for an elementary proof. Eventually we were able to to give a simple six-line-proof (Proof of Proposition 21), which the reader might find amazing.

2. Discriminants and relative quadratic extensions

2.1. Discriminants in a number field. For this section we fix a number field $K$ with ring of integers $\mathfrak{o}$. Let $L$ be a quadratic extension of $K$. We can obtain $L$ by adjoining the square root $\Delta = \text{tr}_{L/K}(a)^2 - 4n_{L/K}(a)$ to $K$, where $a$ is any integer in $L$ not in $K$. Note that $\Delta$ is a square modulo 4 (not necessarily relatively prime to 2). For obtaining the quadratic extensions of $K$ it suffices therefore to adjoin square roots of integers $\Delta$ in $K$ which are squares modulo 4.
We shall call those nonzero integers $\Delta$ of $K$ which are squares modulo 4 henceforth discriminants in $K$. For a discriminant $\Delta$ in $K$ we let $f_\Delta$ be the largest integral ideal in $K$ whose square divides $\Delta$ and such that

$$\Delta \equiv x^2 \mod 4f_\Delta$$

for some integer $x$ in $K$. We partition the discriminants of $K$ into classes, where $\Delta_1$ and $\Delta_2$ belong to the same class if and only if $K(\sqrt{\Delta_1}) = K(\sqrt{\Delta_2})$. It is also clear that a class of discriminants coincides with the set of discriminants in a class of $K^\times/K^\times$. Moreover, we have

**Proposition 1.** If $\Delta$ is not a square in $K$, then the class of $\Delta$ coincides with the set of all nonzero $\text{tr}_{L/K}(a)^2 - 4n_{L/K}(a)$, where $a$ runs through the integers of $L = K(\sqrt{\Delta})$.

**Proof.** If the discriminant $\Delta_1$ is in the class of $\Delta$, so that $K(\sqrt{\Delta_1}) = K(\sqrt{\Delta})$ and $\Delta_1 \equiv x^2 \mod 4$ for some integer $x$ of $K$, then $a := -\frac{x + \sqrt{\Delta_1}}{2}$ is an integer of $L$, and $\Delta_1 = \text{tr}_{L/K}(a)^2 - 4n_{L/K}(a)$. If vice versa $\Delta_1 := \text{tr}_{L/K}(a)^2 - 4n_{L/K}(a) \neq 0$, then $L = K(a)$ (since, $a$ in $K$ implies $\Delta_1 = 0$), and hence $a$ is a solution of $x^2 - \text{tr}_{L/K}(a)x + n_{L/K}(a) = 0$, which implies $K(a) = K(\sqrt{\Delta_1})$. \qed

### 2.2. The discriminant of a relative quadratic extension.

For an extension $L$ of $K$, we use $D_{L/K}$ for the relative discriminant of the extension $L/K$. For quadratic extensions, $D_{L/K}$ equals the gcd of all numbers $\text{tr}_{L/K}(a)^2 - 4n_{L/K}(a)$, where $a$ runs through the integers of $L$. In other words, if we write $L = K(\sqrt{\Delta})$ with a discriminant of $K$, it equals the ideal generated by all discriminants in the class of $\Delta$ (cf. Proposition 1). The relative discriminant of $L/K$ can be expressed in terms of $\Delta$ alone by a formula analogous to the case where $K = \mathbb{Q}$.

**Theorem 2.** For any given integer $\Delta$ in $K^\times$ (not necessarily a square mod 4), let $s$ be the largest integral ideal whose square divides $\Delta$, and let $t$ be the largest integral ideal dividing 2 such that $\Delta$ is a square mod $(st)^2$. Then

$$D_{K(\sqrt{\Delta})/K} = 4\Delta/(st)^2.$$  

In particular, if $\Delta$ is a discriminant in $K$, then one has

$$D_{K(\sqrt{\Delta})/K} = \Delta/f_\Delta^2.$$  

**Proof.** If $\Delta$ is a square in $K$ both sides of (1) are equal to the unit ideal. We can therefore assume that $\Delta$ is not a square, so that $L = K(\sqrt{\Delta})$ is a quadratic extension of $K$ with, say ring of integers $\mathcal{O}$. It suffices to prove $D_{L/K}\mathfrak{p} = (\Delta/(st)^2)\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of $K$, where $\mathfrak{p}$ denotes the localization of $\mathfrak{o}$ at $\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of $K$. Then

$$D_{L/K}\mathfrak{p} = \det \begin{pmatrix} 1 & \omega' \\ 1 & \omega \end{pmatrix}^2 \mathfrak{p} = (\omega - \omega')^2 \mathfrak{p},$$

where $1, \omega$ is an $\mathfrak{p}$-basis of $\mathcal{O}\mathfrak{p}$, and where $\omega'$ is the Galois conjugate of $\omega$ in the extension $L/K$ (see e.g. [Frö67, §3, Prop. 4]). For computing a basis we recall that the ring $\mathcal{O}\mathfrak{p}$ is the algebraic closure of $\mathfrak{p}$ in $L$ (see e.g. [Frö67, §4, Lemma 1]). In other words, we can write

$$\mathcal{O}\mathfrak{p} = \{a + b\sqrt{\Delta_1} : a, b \in K, \ 2a \in \mathfrak{p}, \ a^2 - b^2\Delta_1 \in \mathfrak{p}\},$$

The discriminants of $K$ which are relatively prime to 2 are occasionally called *Primärzahlen* in older literature; see [Hec23, §59].
where we set $\Delta_1 = \Delta/\pi^{21/2}$ with $p^i$ denoting the exact power dividing $\Delta$, and $\pi$ an element of $\mathfrak{o}_p$ such that $\mathfrak{o}_p \mathfrak{p} = \pi \mathfrak{o}_p$. It is quickly verified that we can take as second basis element $\omega$ of $D_p \mathfrak{o}_p$ over $\mathfrak{o}_p$ the element $\omega = \sqrt{\Delta_1}$ if $p$ is odd, and also if $p = 2$ and $\Delta_1$ is divisible by $\mathfrak{p}$. Otherwise one can take $\omega = (c + \sqrt{\Delta_1})/\pi^i$, where $t$ is maximal such that $\pi^t | 2$ and $\Delta_1$ is a square modulo $\pi^{2i} \mathfrak{o}_p$, and where $c$ is any solution in $\mathfrak{o}_p$ of $c^2 \equiv \Delta_1 \mod \mathfrak{p}^{2i}$. For this $\omega$ we have

$$(\omega - \omega')^2 = \begin{cases} 4\Delta_1 & \text{if } p \text{ is odd or } p | 2, \Delta_1, \\ 4\Delta_1/\pi^{2t} & \text{otherwise.} \end{cases}$$

The formula (1) becomes now obvious.

Assume that $\Delta$ is a square mod 4. We have to show that $\mathfrak{f} := \mathfrak{st}/2$ is integral and that $\mathfrak{f} = \mathfrak{f}_\Delta$. For proving the first statement, it suffices to note that $\Delta$ is a square mod 4 and mod $(\mathfrak{st})^2$, and that $\mathfrak{t}$ is the largest ideal (dividing 2) with this property. For the second statement we note, first of all, that $\mathfrak{f}^2$ divides $\Delta$ (since $(\mathfrak{st})^2$ divides $4\Delta$), and that $\Delta$ is a square mod $4\mathfrak{f}^2 = (\mathfrak{st})^2$. But $\mathfrak{f}$ is also the maximal ideal with these properties. Namely, if, for some prime ideal $\mathfrak{p}$, the square of $\mathfrak{f} := \mathfrak{fp}$ divides $\Delta$, then $v_p(\mathfrak{f}/2) \leq 0$ (by the maximality of $\mathfrak{s}$), i.e. $\mathfrak{tp} | 2$. But then $\Delta$ cannot be a square mod $4\mathfrak{f}^2 = (\mathfrak{st})^2$ by the maximality of $\mathfrak{t}$. This completes the proof of the theorem.

We note two corollaries.

**Corollary 3.** The discriminant of a relative quadratic extension represents a square in the class group.

Indeed, the square of the class of $D_{K(\sqrt{\Delta})/K}$ equals the square of the ideal class of $\mathfrak{f}_\Delta$. The corollary is due to Hecke [Hecke 23, §63, Satz 177], who proved it in fact for arbitrary (not necessarily quadratic) relative extensions.

For the $\mathfrak{f}_\Delta$ we find the following.

**Corollary 4.** The $\mathfrak{f}_\Delta$, for $\Delta$ in a given class of $K^\times/K^\times_2$ belong all to one and the same ideal class of $K$. More precisely, one has

$$\mathfrak{f}_{a^2\Delta} = a\mathfrak{f}_\Delta$$

for any $\Delta$ and $a$ in $K$ such that $\Delta$ and $a^2\Delta$ are discriminants.

**Proof.** If $\Delta$ and $\Delta'$ are in the same class of $K^\times/K^\times_2$, then, using (2), we find $\mathfrak{f}_{\Delta}^2 = a^2\mathfrak{f}_{\Delta'}$ for some $a$ in $K^\times$, which implies $\mathfrak{f}_{\Delta} = a\mathfrak{f}_{\Delta'}$. However, one can prove the corollary also more directly as follows. If $a$ is a nonzero integer in $K$ then $\mathfrak{f}_{a^2\Delta} = a\mathfrak{f}_\Delta$ (since $a\Delta$ is divisible by the square of $a\mathfrak{f}_\Delta$ a square modulo $4a^2\mathfrak{f}_\Delta^2$, and on the other hand $\mathfrak{f}_{a^2\Delta}$ can obviously not larger than $a\mathfrak{f}_\Delta$). Therefore, if $\Delta_i$ ($i = 1, 2$) are in the same class, i.e. if $a_i^2\Delta_1 = a_2^2\Delta_2$ for integers $a_i$, we conclude that $a_1\mathfrak{f}_{\Delta_1} = a_2\mathfrak{f}_{\Delta_2}$.

In fact, one can say more about the $\mathfrak{f}_{\Delta}$. The ring of integers $\mathfrak{O}$ of $K(\sqrt{\Delta})$, being in general not free as module over $\mathfrak{o}$, can be written as $\mathfrak{O} = \mathfrak{g} \mathfrak{a} \oplus \mathfrak{b}$ with a suitable fractional ideal $\mathfrak{g}$ and $\mathfrak{a}, \mathfrak{b}$ in $\mathfrak{O}$ if $\Delta$ is not a square (and otherwise as $\mathfrak{O} = \mathfrak{a} = \mathfrak{g} \mathfrak{a}$).

The ideal $\mathfrak{g}$ is not unique, but its ideal class is, which is called the *Steinitz invariant of the $\mathfrak{o}$-module $\mathfrak{O}$* [FT03, Thm. 13].

**Proposition 5.** The $\mathfrak{f}_\Delta^{-1}$, for $\Delta$ in a given class of $K^\times/K^\times_2$ belong to the Steinitz invariant of the ring of integers of $K(\sqrt{\Delta})$ as module over $\mathfrak{o}$.

**Proof.** The case that $\Delta$ is a square being trivial we assume that $L = K(\sqrt{\Delta})$ has degree two over $K$. For a prime ideal $\mathfrak{p}$ of $K$ we have $D_{\mathfrak{O}_p} = \mathfrak{g}\mathfrak{p}\mathfrak{a} + \mathfrak{a}\mathfrak{b}$, and as in the proof of Theorem 2 we find therefore $D_{L/K}\mathfrak{p} = \pi^{29}(ab^* - a'b)^2$, where
\[ \pi o_p = p o_p, \text{ and } g o_p = \pi^{2g} o_p. \] It follows \( D_{L/K} = g^2(ab' - a'b)^2 \). On the other hand \( D_{L/K} = \Delta/f_\Delta \), and hence \( (g/f_\Delta)^2 = \Delta/(ab' - a'b)^2 \). But \( \Delta/(ab' - a'b)^2 \) is a square in \( K^\times \) (since \( ab' - a'b \), being not invariant under the Galois group of \( L/K \), is in \( L \) but not in \( K \)), and hence \( g/f_\Delta \) is a principal ideal. This proves the proposition. \[ \square \]

2.3. Fundamental discriminants. There is still a dichotomy left since the notion “Fundamental discriminant” is missing for general relative quadratic extensions. More precisely, we would like to have, for a given class \( C \) in \( K^\times/K^\times^4 \), a discriminant \( \Delta_0 \) such that \( \Delta_0 f^2 \) runs through all discriminants in the given class when \( f \) runs through all nonzero integers of \( K \). Such a \( \Delta_0 \) would be uniquely determined by this property modulo \( \pi^2 \), and it would generate the discriminant ideal \( D_{L/K} \) of the quadratic extension of \( K \) determined by \( C \).

Such a \( \Delta_0 \) will not in general exist. In fact, it is not hard to see that it exists if and only if the ideal class of the \( f_\Delta \) (\( \Delta \) in \( C \)) is trivial (see Theorem 9). However, such a \( \Delta_0 \) exists for any class \( C \) as an idèle of \( K \), and it is useful to study its idèle construction since it will give us, amongst others, further criteria for the existence of fundamental discriminant for classes in \( K^\times/K^\times^2 \).

For a valuation \( v \) of \( K \) let \( K_v \) denote the completion of \( K \) at \( v \), and \( \sigma_v \) the ring of integers in \( K_v \) (with the convention \( \sigma_v = K_v \) if \( v \) is real or complex). We use \( I \) for the idèle group of \( K \), and \( U \) for the direct product over all \( \sigma_v^\times \). Finally we identify \( K \) with its image in \( I \) under the diagonal embedding.

Let \( C \) be a class in \( K^\times/K^\times^2 \). Let \( D_C U^2 \) be the element of \( H_K := K^\times I^2/U^2 \) which at a finite place \( v \) is defined as

\[
(D_C)_v = \Delta/\pi_v^{2v(f_\Delta)},
\]

at a real place as \((D_C)_v = \text{sign}(\sigma(\Delta))\) with the embedding \( \sigma \) corresponding to \( v \), and at a complex place as 1. Here \( \pi_v \) is a prime element of \( K_v \) (and therefore unique up to multiplication by a unit in \( \sigma_v^\times \)), and \( \Delta \) is an element of \( C \). Note that the coset \((D_C)_v\sigma_v^{-\varepsilon} \) does not depend on the choice of \( \pi \).

Lemma 6. The coset \( D_C U^2 \) does not depend on the choice of \( \Delta \) in \( C \).

Proof. We have \( D_C = \Delta/\varphi^2 \), where \( \varphi \) is the idèle associated to \( f_\Delta \), i.e. \( \varphi_v = \pi_v^{v(f_\Delta)} \). If \( \Delta' \) is another discriminant in \( C \), then \((2) \) implies \( \Delta/\varphi^2 = \varepsilon\Delta'/\varphi'^2 \), where \( \varphi' \) is the idèle associated to \( f_{\Delta'} \) and \( \varepsilon \) is in \( U \). Since \( \Delta \) and \( \Delta' \) differ by a square we conclude that \( \varepsilon \) is in fact in \( U^2 \), which proves the lemma. \[ \square \]

We call any representative \( D_C \) of \( D_C U^2 \) (by slight abuse of language) the fundamental discriminant of \( C \). This is justified by the following propositions.

First of all, as for the case that \( K \) is the field of rational numbers the fundamental discriminant \( D_C \) uniquely determines \( C \) (and hence the extension \( K(\sqrt{\Delta}) \)) since \( C = D_C I^2 \cap K^\times \), in other words:

Proposition 7. The map \( C \rightarrow D_C U^2 \) defines an injection \( K^\times/K^\times^2 \rightarrow H_K \).

Moreover, the usual properties of rational fundamental discriminants have also their equivalents as follows:

Proposition 8. Let \( C \) be a class in \( K^\times/K^\times^2 \) and \( D_C \) its fundamental discriminant.

1. Every discriminant \( \Delta \) in \( C \) can be written uniquely mod \( U^2 \) as \( \Delta = D_C \alpha^2 \) with an idèle \( \alpha \) whose valuation at a finite place is non-negative.
(2) $D_C$ is mapped onto $D_{K(\sqrt{\Delta})/K}$ under the natural map which takes $I/U^2$ onto the group of fractional ideals\(^2\) of $K$.

(3) $(D_C)\nu$ is a square mod $4\mathfrak{a}_v$ at every finite place of $K$.

(4) For every real place the sign of $(D_C)\nu$ equals the sign of $\sigma(\Delta)$ for every discriminant $\Delta$ in $C$, with the embedding $\sigma$ of $K$ corresponding to $v$.

Proof. These statements are immediate consequences of the definition (3) of $D_C$, where, for the second one, one needs also the formula (2).

It is not difficult to verify that our $D_C$ coincides with the enhanced notion of relative discriminant for arbitrary extensions as proposed in [Fro60] (whose definition is slightly different from the one given here).

We call $D_C$ principal if it is represented by a number in $K^\times$, i.e. if $D_CU^2 = \Delta_0U^2$ for some number $\Delta_0$. This number is then a discriminant in $C$ (by (3) of the preceding proposition). Moreover, if $D_C$ is principal, say represented by $\Delta_0$, then $\Delta_0a^2$ runs through all discriminants in $C$ when $a$ runs through the nonzero integers of $K$. (Namely, if $\Delta$ is a discriminant in $C$ then $\Delta/\Delta_0 = \alpha^2$ for some $\alpha$ in $I$ as in (1), and on the other hand $\Delta/\Delta_0$ is in $K^\times$, so that $\alpha$ is in fact in $\mathfrak{o}$.)

Theorem 9. Let $C$ be a class in $K^\times/K^\times^2$, let $D_C$ denote its fundamental discriminant, and let $\Delta$ in $C$. Then the following statements are equivalent:

(1) $D_C$ is principal.

(2) The ideal $\mathfrak{f}_\Delta$ is principal.

(3) The ring of integers of $K(\sqrt{\Delta})$ is a free module over $\mathfrak{o}$.

Proof. (1) implies $\Delta = D_Ca^2$ for some integer $a$ in $K$. Since $D_C\mathfrak{o} = D_{K(\sqrt{\Delta})/K}$ (by Prop. 8 (2)) and $\Delta = D_{K(\sqrt{\Delta})/K}\mathfrak{f}_\Delta$ we conclude $\mathfrak{f}_\Delta = a\mathfrak{o}$. Vice versa, if $\mathfrak{f}_\Delta = a\mathfrak{o}$ for some integer $a$, then $\Delta/a^2\mathfrak{o}_v^2 = D_CU^2$ for every $v$. (2) and (3) are equivalent since by Proposition 5 the ring of integers of $K(\sqrt{\Delta})$ is isomorphic as $\mathfrak{o}$-module to $\mathfrak{f}_\Delta^{-1} \oplus \mathfrak{o}$. □

For the reader who wishes to avoid the idèlic setting for the notion of fundamental discriminants the following remark may be useful. The natural map $H_K \rightarrow K^\times I^2/U^2 \cong K^\times/K^\times^2$ defines an exact sequence

$$1 \rightarrow I^2/U^2 \rightarrow H_K \rightarrow K^\times/K^\times^2 \rightarrow 1.$$  

This sequence splits, a section is given by $C \rightarrow D_CU^2$. We therefore obtain a isomorphism $H_K \cong I^2/U^2 \times K^\times/K^\times^2 \cong J_K^2 \times K^\times/K^\times^2 =: H'_K$, where $J_K$ is the group of nonzero fractional ideals of $K$. (For the latter isomorphism note that the natural map $I^2/U^2 \rightarrow (I/U)^2$ is an isomorphism since $I^2 \cap U = U^2$.) We leave it to the interested reader to go again through this section while replacing $H_K$ systematically by $H'_K$.

2.4. The Gröbchencharakter of a relative quadratic extension. We extend the Dirichlet characters $\left(\frac{\Delta}{\mathfrak{p}}\right)$ from the theory where $K$ is the field of rational numbers to arbitrary number fields $K$ as follows: Let $\Delta$ be a discriminant of $K$. For a prime ideal $\mathfrak{p} \mid \Delta$, we set

$$\left(\frac{\Delta}{\mathfrak{p}}\right) = \begin{cases} +1 \text{ if } \Delta \text{ is a square modulo } 4\mathfrak{p} \\ -1 \text{ otherwise.} \end{cases}$$

Of course, for $\mathfrak{p} \nmid 2$ the number $\Delta$ is a square modulo $4\mathfrak{p}$ if and only if it is square modulo $\mathfrak{p}$ as follows from the Chinese remainder theorem. We continue $\left(\frac{\Delta}{\mathfrak{p}}\right)$ to a

\(^2\)This is the map which takes an $\alpha U^2$ to the product of all $p^{e(\alpha_v)}$, where $p$ runs through the prime ideals of $K$ and $v$ is the place corresponding to $p$.  

homomorphism of the group of fractional ideals relatively prime\(^3\) to \(\Delta\) onto the group \(\{\pm 1\}\).

**Theorem 10.** The homomorphism \((\hat{\Delta})\) defines a Größencharakter modulo \(\Delta\) of infinity type \(\alpha \mapsto \prod_{\sigma \in M} \text{sign}(\sigma)\), where \(M\) is the set of real embeddings of \(K\) with \(\sigma(\Delta) < 0\). Its conductor equals \(\Delta/f_{\Delta}^2\).

We postpone the proof to the end of this section.

According to the theorem the character \((\hat{\Delta})\) is the restriction of a primitive Größencharakter modulo \(\Delta/f_{\Delta}^2\), which we denote in the sequel by \([\hat{\Delta}]\). Note that \([\hat{\Delta}]\) is uniquely determined by \((\hat{\Delta})\). Indeed, if the fractional ideal \(a\) is relatively prime to \(D := \Delta/f_{\Delta}^2\), then we can find an \(a\) in \(K^\times\) relatively prime to \(D\) and such that \(b := a/a\) is relatively prime to \(\Delta\). (One can take, for instance, for \(b\) any prime ideal in the same ideal class as \(a\) which does not divide \(\Delta\).)

If \(p \neq 0\) with suitable nonzero integers \(a\) and \(b\), and \([\hat{\Delta}]\prod_{\sigma \in M} \text{sign}(\sigma)\) depends only on \(a\) modulo \(D\sigma_D\) (where \(\sigma_D\) is the intersection of the localizations \(\sigma_p\) of \(\sigma\) at \(p\) \((p \mid D\))), and hence can be evaluated by replacing \(a\) by any \(b\) in \(a + D\sigma_D\) which is relatively prime to \(\Delta\).

**Proposition 11.** The primitive Größencharakter \([\hat{\Delta}]\) depends only on the class of \(\Delta\) in \(K^\times/K^\times\).

**Proof.** Let \(\Delta'\) be another discriminant in the same class as \(\Delta\). Then \(\Delta a^2 = \Delta'a'^2\) with suitable nonzero integers \(a\) and \(a'\). Let \(\Delta'' = \Delta a'^2 = \Delta'a^2\). It suffices to show that \([\hat{\Delta''}] = [\hat{\Delta}]\) and \([\hat{\Delta''}] = [\hat{\Delta}]\). But this is clear since \([\hat{\Delta''}]\) is the restriction of \((\hat{\Delta})\) to the group of fractional ideals relatively prime to \(\Delta''\). \(\Box\)

### 2.5. The decomposition of primes in relative quadratic extensions

Let

\[
L \left( \left[ \frac{\Delta}{\sigma} \right], s \right) = \sum_{\sigma} \left[ \frac{\Delta}{\sigma} \right] \frac{n(\sigma'^{-1})}{s^s},
\]

where the sum runs over all integral ideals of \(K\) with the convention that \([\hat{\Delta}]\) equals \(1\) if \(a\) is not relatively prime to conductor of \([\hat{\Delta}]\).

**Theorem 12.** If \(\Delta\) is not a square in \(K\), then

\[
\zeta_{K(\sqrt{\Delta})}(s) = \zeta_K(s)L \left( \left[ \frac{\Delta}{\sigma} \right], s \right)
\]

**Proof.** It is a basic fact \([\text{Hec23, Satz 117}]\) that every prime ideal \(p\) of \(K\) is inert (i.e. remains a prime in \(L\)), or splits (i.e. factors into a product of two different primes), or ramifies (i.e. is the square of a prime ideal in \(L\)). Moreover, which property holds true is given by the character criterion, namely, the first, second or third property holds true accordingly as \([\hat{\Delta}]\) equals \(-1\), \(+1\) or \(0\). For \(p \mid \Delta\) the character criterion is \([\text{Hec23, Satz 118, 119}]\) (for applying Satz 119 loc.cit. recall that \((\hat{\Delta})\) equals \(-1\) or \(+1\) according as \(\Delta\) is a square mod \(4p\), and that \(\Delta\) is a square mod \(4\)). But then the criterion is also true for any \(p \mid D_{L/K} = \Delta/f_{\Delta}^2\).

If \(p \mid \Delta\), but \(p \nmid \Delta/f_{\Delta}^2\), we can find a discriminant \(\Delta'\) in \(\Delta K^\times\) with \(p \mid \Delta'\) and apply Satz 118, 119 loc.cit. to \(\Delta'\). Indeed, choose an integral ideal \(b\) in the class of \(f_{\Delta}^2\) relatively prime to \(2\Delta\), let \(a = bf_{\Delta}\), and set \(\Delta' = \Delta a^2\).

If \(p \mid \Delta/f_{\Delta}^2\), i.e. if \([\hat{\Delta}] = 0\), then \(p\) is ramified according to Satz 118 loc.cit. if the exact \(p\)-power dividing \(\Delta/f_{\Delta}^2\) is \(p^f\) with odd \(f\) (which is always the case for \(p \mid 2\)). If \(f\) is even and \(p \mid 2\) the ideal \(p\) ramifies in \(L\) by Satz 119 loc.cit. (for applying Satz 119 we write \(L = K(\sqrt{d})\) with some integer \(d\) in \(\Delta K^\times\) which is not divisible

---

\(^3\)A fractional ideal is called relatively prime to \(\Delta\) is it is of the form \(a/b\) with integral ideals \(a\) and \(b\) both of which have no prime ideal common with \(\Delta\).
by \( p \) and observe that \( d \) cannot be a square modulo 4 since otherwise \( \Delta/f_\Delta^2 = d/f_\Delta^2 \), contradicting \( p \nmid d \); one can choose \( d = \Delta a^2 \) with \( a = b/f_\Delta p^{f/2} \) and \( b \) relatively prime to \( p \).

The claimed identity is now an easy consequence of the character criterion by comparing, for each prime ideal \( p \) of \( K \), the \( p \)th Euler factors on both sides of the claimed identity.

If the conductor of \( \frac{\Delta}{\Delta} \) is 1 no prime ideal ramifies and vice versa. In other words, we have

**Corollary 13.** The extension \( K(\sqrt{\Delta}) \) is unramified if and only if \( \Delta = f_\Delta^2 \).

### 2.6 Proof of Theorem 10

It remains to prove Theorem 10. The educated reader might have noticed that \( \frac{\Delta}{\Delta} \) is essentially nothing else than the Artin reciprocity map associated to \( K(\sqrt{\Delta}) \).

Indeed, let \( \Delta \) be a discriminant not a square, let \( L = K(\sqrt{\Delta}) \), let \( \mathcal{O} \) be the ring of integers of \( L \) and let \( \sigma \) be the nontrivial Galois substitution of \( L/K \), which maps \( \sqrt{\Delta} \) to \(-\sqrt{\Delta} \). As we saw in the proof of Theorem 12, any prime ideal \( p \) of \( K \) relatively prime to \( \Delta \) maps a prime ideal \( \mathfrak{p} \) of \( \mathcal{O} \) to \( \mathcal{O}_L \) in \( L \) in the form \( p\mathcal{O} = \mathfrak{p}\mathcal{O}_L \) with \( \mathfrak{p} \neq \sigma(\mathfrak{p}) \). The Frobenius \( F_p \) (i.e. the generator of the subgroup of \( \text{Gal}(L/K) \) mapping each prime ideal of \( L \) over \( p \) to itself) is therefore the identity. On the other hand, if \( \left( \frac{\Delta}{p} \right) = -1 \), then \( p\mathcal{O} \) is the only prime ideal in \( L \) over \( p \), and hence \( F_p = \sigma \). Therefore, if \( f \) denotes the isomorphism of \( \{ \pm 1 \} \) with \( \text{Gal}(L/K) \), then \( f \circ \left( \frac{\Delta}{\sigma} \right) \) equals the Artin reciprocity map \( \left( \frac{L/K}{\mathfrak{p}} \right) \) on the group of fractional ideals relatively prime to \( \Delta \), which maps a prime ideal \( \mathfrak{p} \) to \( F_p \). The fact that \( \left( \frac{\Delta}{\sigma} \right) \) is a Gr"{o}ssencharakter follows then from well-known facts of Global Classfield theory, which include also that the discriminant \( D_{L/K} \) is the conductor of \( \left( \frac{\Delta}{\sigma} \right) \) [CF67, Ch. VI §4.4].

However, we prefer to keep with the explicit character of this note and to give a more direct and elementary proof. We shall need the product formula for the quadratic Hilbert symbols though, which however can be proved without developing full Classfield Theory (see e.g. [O’M00, Ch. 7]).

**Proof of Theorem 10.** For a nonzero \( a \) in \( K^\times \) relatively prime to \( \Delta \), set

\[
\psi(a) := \left( \frac{\Delta}{a} \right) \prod_{\sigma \in M} \text{sign} \sigma(a).
\]

Note that \( \psi \) defines a linear character of the multiplicative group \( K^\times \) of elements in \( K^\times \) relatively prime to \( \Delta \). That \( \left( \frac{\Delta}{\sigma} \right) \) is a Grössencharakter mod \( \Delta \) means that \( \psi \) factors through a homomorphism of \( (\sigma/\Delta)\times \), i.e. that \( \psi \) is trivial on the kernel of the natural map \( K^\times \rightarrow (\sigma/\Delta)^\times \) (which sends \( a \) to \( a' + \Delta a \), where \( a' \) is any element in \( \sigma \) such that \( a' \equiv a \mod \Delta K_\Delta \), and where \( K_\Delta \) is the ring of elements in \( K \) relatively prime to \( \Delta \)). For this it suffices to show that, for integral \( a \), the value \( \psi(a) \) depends only on the residue class of \( a \) modulo \( \Delta \) (as one sees on writing any \( a \) in the kernel of the natural map in the form \( a = \gamma/\delta \) with integers \( \gamma \equiv \delta \mod \Delta \) and \( (\delta, \Delta) = 1 \)).

So, let \( a \) be integral and relatively prime to \( \Delta \). For the proof that \( \psi(a) \) depends only on \( a \) modulo \( \Delta \), we use the product formula (see e.g. [Neu99, Ch. VI, Thm. (8.1)] or [O’M00, 71:18 Thm.])

\[
(4) \quad \prod_{v} (\Delta, a)_v = 1,
\]

where \( v \) runs through all places of \( K \) (including the infinite ones) and \( (\omega, \cdot)_v \) denote the quadratic Hilbert symbol of the completion \( K_v \) of \( K \). Thus, for \( a, b \) in \( K_v \), we
have \((a, b)_v = +1\) if \(ax^2 + by^2 = 1\) has a solution \(x, y\) in \(K_v\) and 
\((a, b)_v = -1\), otherwise.

If \(v\) is infinite, corresponding to the embedding \(\sigma\) of \(K\) into \(\mathbb{C}\), then obviously 
\((\Delta, a)_v = -1\) if and only if \(\sigma\) is real and \(\sigma(\Delta)\) and \(\sigma(a)\) are both negative. In other words, the contribution to the left hand side of \((4)\) of the infinite places equals \(\prod_{v \in \mathcal{M}} \operatorname{sign}(\sigma(a))\). If \(v\) is a finite place corresponding to a prime ideal \(p\) not dividing \(2\), then \((\Delta, a)_v = 1\), or \((\Delta/\sigma(\Delta))^v(a)\), or \((\Delta/p^v(\Delta))^v\) according as \(p \not| \Delta\), or \(p \mid a\), or \(p \mid \Delta\) (as follows e.g. from [Neu99, Ch. V, Prop. (3.4)] or [O’M00, p. 165, 63:11a]). Therefore the product formula \((4)\) becomes

\[
\left(\frac{\Delta}{a}\right) \prod_{\sigma \in \mathcal{M}} \operatorname{sign}(\sigma(a)) \prod_{v \mid |2} (\Delta, a)_v = 1,
\]

where \(\mathcal{D}\) is the odd part of \(\Delta\) (i.e. the product of all prime ideal powers dividing \(\Delta\) and relatively prime to \(2\)), and \(a\) is the odd part of \(a\). In other words

\[
\psi(a) = \left(\frac{\Delta}{a/a}\right) \prod_{v \mid 2} (\Delta, a)_v = \left(\frac{a}{\mathcal{D}}\right) \prod_{v \mid |2} \left(\frac{\Delta}{p_v^v(a)}\right) (\Delta, a)_v,
\]

where \(p_v\) is the prime ideal corresponding to \(v\). If \(v(\Delta) = 0\) the \(vth\) factor on the right equals \(1\) (see [Ser79, Ch.XV, §3, Prop. 6]).

Finally, let \(v \mid 2\) and \(l := v(\Delta) \geq 1\) (and hence \(v(a) = 0\)). We need to prove that the quadratic character of \(U := \sigma^\lor\) defined by

\[
\kappa : a \mapsto (\Delta, a)_v
\]

factors through a Dirichlet character modulo \(p_v^l\). Since the natural map \(\sigma^\lor : (a/p_v^l) \rightarrow (a/\mathcal{D})\) is surjective and has kernel \(U_l := 1 + p_v^l \sigma_v\) the character \(\kappa\) factors through a Dirichlet character modulo \(p_v^l\) if and only if \(\kappa\) is trivial on \(U_l\).

Since \(U_{2e+1} \subseteq U^2\), where \(e = v(2)\), i.e. by the Local Square Theorem (see Section 5), \(\kappa\) is in any case a Dirichlet character modulo \(p_v^{2e+1}\). Hence we can assume that \(l \leq 2e\). But then \(l\) is even (since \(\Delta\) is a square mod \(4\)). Let \(\pi_v\) a uniformizer of \(K_v\) and write \(\Delta = \pi_v^2 b\). By assumption \(b\) is a square modulo \(p_v^{2e-1}\), i.e. \(b/c^2 \equiv 1\) mod \(p_v^{2e-1}\). In other words, \(\Delta = \pi_v^2 c^2 d\), where \(d\) is in \(U_{2e-1}\), with the convention \(U_2 = U\) for the case \(l = 2e\). It follows that \(\kappa(a) = (d,a)_v\). It remains therefore to show that, for any even integers \(i, j \geq 0\) with \(i + j = 2e\), one has \((U_i, U_j)_v = 1\).

Though this seems to be known (see the very last exercise [Ser79, Ch.XV, §3, Ex. 3], from which it follows) we did not find any reference for this providing a proof not depending on heavy machinery (see [Dai16], where a proof of a more general result is given based on Local Class Field Theory). For the convenience of the reader we give a self-contained and easy proof in the Appendix Section 5, Proposition 21.

The conductor of \((\Delta)\) equals the conductor of the Dirichlet character \(\psi\) (see e.g. [Neu99, Ch. 7, (6.2) Prop.]). But this is the product all local conductors \(p_v^{s_v}\) of the Dirichlet characters \(a + p_v \mapsto (\Delta, a)_v\), taken over all finite \(v\) with \(v(\Delta) \geq 1\). If \(p_v \not| 2\) then obviously \(s_v = 0\) or \(s_v = 1\) according as \(v(\Delta)\) is even or odd, and therefore \(s_v = v(\Delta/p_v^2)\). If on the contrary \(p_v \mid 2\) then \((\Delta, a)_v = (\Delta/p_v^2, a)_v\) for any integer \(k\), and as we saw \(\kappa(a) := (\Delta/p_v^{2k}, a)_v\) factors through a Dirichlet character modulo \(\Delta/p_v^{2k}\) if \(\Delta/p_v^{2k}\) is integral and a square modulo 4. Therefore, \(s_v\) is \(\leq\) the largest power \(p_v^{2k}\) dividing \(\Delta\) and such that \(\Delta' := \Delta/p_v^{2k}\) is a square modulo 4, i.e. \(s_v \leq v(\Delta')\). But \(s_v\) is even equal to \(v(\Delta')\). For this let \(l = v(\Delta')\). Clearly, \(l \leq 2e + 1\) (with \(e = v(2)\)). If \(l = 2e + 1\), Theorem 24 in Section 5 below implies \((\Delta, a)_v = (\pi, a)_v\) for all units \(a\), and another application of this theorem implies that there is a unit \(a\) in \(U_{2e}\) such that \((\pi, a)_v = -1\); we conclude \(s_v = 2e + 1\). If \(l \leq 2e\) (so that \(l\) is even), we proceed as in the last paragraph and write as before \(\Delta' = \pi_v^2 c^2 d\).
with units \( c, d \) and \( d \) in \( U_{2e-1} \). Again \( (\Delta', a) = (d, a)_v \). Therefore \( (d, U_{s_v})_v = 1 \), and then Theorem 24 implies that \( d \) is in \( U_{2e-s_v} \). But by the choice of \( k \) the unit \( d \) is not in any \( U_m \) for any \( m \geq 2e - l + 2 \), and therefore \( 2e - s_v \leq 2e - l + 1 \), i.e. \( s_v \geq l - 1 \). Since \( U_{l-1} U_{(l-1)/2} = U_{l-2} \) (see Section 5, Lemma 20 below) we conclude \( s_v \leq l \), which was to be shown. This proves the theorem. \( \square \)

3. Zeta functions associated to discriminants

3.1. A reciprocity law. As in the preceding section \( K \) denotes an arbitrary number field. For any discriminant \( \Delta \) of \( K \), we set\(^4\)

\[
\zeta(\Delta, s) := \sum_a \frac{N_\Delta(a)}{n(a)^s}
\]

where

\[
N_\Delta(a) = \text{card } \left\{ b \mod 2a : \Delta \equiv b^2 \mod 4a \right\},
\]

and where the sum is over all integral ideals \( a \) of \( K \).

The series \( \zeta(\Delta, s) \) converges for \( \Re(s) > 1 \). By the Chinese remainder theorem \( N_\Delta(a) \) is multiplicative in \( a \) and possesses hence an Euler product. For a prime ideal power \( p^k \) which is relatively prime to \( \Delta \) we have \( N_\Delta(p^k) = 1 + (\frac{\Delta}{p}) \). Accordingly, \( \zeta(\Delta, s) \) equals up to a finite Euler product the product of \( \zeta_K(s)/\zeta_K(2k) \) with \( L \left( \left( \frac{\Delta}{a}, s \right), s \right) = \sum_a (\frac{\Delta}{a}) n(a)^{-s} \), the sum being over all integral ideals of \( K \) relatively prime to \( \Delta \).

For explaining the precise connection between \( \Delta \) and \( \zeta_K(s)L \left( \left( \frac{\Delta}{a}, s \right)/\zeta_K(2k) \right) \), we define a function \( \chi_\Delta \) on the semigroup of all integral ideals \( a \) by setting

\[
\chi_\Delta(a) := \begin{cases} n(g)|\frac{\Delta}{a/g} & \text{if } (a, \Delta) = g^2 \text{ and } \Delta \text{ is a square mod } 4g^2, \\ 0 & \text{otherwise.} \end{cases}
\]

Note that, for an integral square \( g^2 \mid \Delta \), the condition that \( \Delta \) is a square mod \( 4g^2 \) is equivalent to \( g \mid f_\Delta \). Of course, \( \chi_\Delta \) is no longer a homomorphism, but it remains multiplicative in the sense that \( \chi_\Delta(ab) = \chi_\Delta(a)\chi_\Delta(b) \) whenever \( a \) and \( b \) are relatively prime. We use \( n(g) \) for the (absolute) norm of \( g \) (i.e. \( n(g) = \text{card } (a/g) \)).

If we associate to a function \( \psi \) defined on integral ideals of \( K \) the Dirichlet series \( L(\psi, s) := \sum_a \psi(a)n(a)^{-s} \) (the sum being over all integral ideals of \( K \)), then it is a bit tedious but straightforward to rewrite the definition of \( \chi_\Delta \) in the form

\[
L(\chi_\Delta, s) = L \left( \left[ \frac{\Delta}{\Delta}/s \right], s \right) \sum_{t \mid \Delta} \frac{\mu(t)|\frac{\Delta}{t}}{n(t)^s} \sigma_{1-2s}(f_\Delta/t),
\]

where we set \( |\frac{\Delta}{t}| = 0 \) if \( t \) is not relatively prime to \( \Delta/f_\Delta^2 \), where \( \mu(a) \) is the Möbius \( \mu \)-function of \( K \), and where \( \sigma_s(f) = \sum_d n(d)^s \). We leave this identity as an exercise; the interested reader may as well look up its simple proof in [Boy17, Lemma 6.1].

**Theorem 14.** For any discriminant \( \Delta \) in \( K \) and integral ideal \( a \) of \( K \), one has

\[
\text{card } \left\{ x \in o/2a : x^2 \equiv \Delta \mod 4a \right\} = \sum_{b \mid a \text{ squarefree}} \chi_\Delta(b).
\]

(The sum is over all integral ideals diving \( a \) and such that \( a/b \) is squarefree.) In other words,

\[
\zeta(\Delta, s) = \frac{\zeta_K(s)}{\zeta_K(2s)} L(\chi_\Delta, s),
\]

where \( \zeta(\Delta, s) \) is the zeta function (5).

\(^4\)Note that \( \zeta(\Delta, s) \) depends also on \( K \) since \( \Delta \) might be a discriminant in various fields. However, since we fixed \( K \) once and for all we suppress this dependence in the notation.
Remark. This formula (9) is a classical fact if $K = \mathbb{Q}$ and $\Delta$ is a negative fundamental discriminant of $\mathbb{Q}$ (so that $\chi_\Delta$ is simply the character of $\mathbb{Q}(\sqrt{\Delta})$). In [Zag77, Prop. 3] it is shown that the formula holds still true if $K = \mathbb{Q}$ and for all rational discriminants $\Delta$, where $L(\chi_\Delta, s)$ is defined by (7) (with $k = 0$). The theorem therefore shows that is holds true in any number field with the appropriate definition (6) of $\chi_\Delta$.

3.2. Proof of Theorem 14.

Proof. Denote the left hand side of the claimed identity by $S_\Delta(a)$. Note that $S_\Delta(a)$ is multiplicative in $a$. Indeed, the $\Delta \equiv s^2 \mod 4$ has a unique solution $s \mod 2$.

Hence $S_\Delta(a)$ equals the number of solutions mod $a2^a$ of $x^2 \equiv \Delta \mod a2^2$, where $2^a$ is the product of all prime powers $p^n$ exactly dividing 2 where $p \mid a$. Using this description of $S_\Delta(a)$ the claimed multiplicativity follows now from the Chinese remainder theorem.

It suffices therefore to prove (8) for prime ideal powers $p^k$ ($k \geq 1$), i.e. it suffices to prove, for $k \geq 1$,

$$S_\Delta(p^k) = \chi_\Delta(p^k) + \chi_\Delta(p^{k-1}).$$

Moreover, we can replace $a$ and $p$ in the definition of $S_\Delta(p^k)$ by the localization $\mathcal{O}_\wp$, and the principal ideal $\wp = \pi\mathcal{O}_\wp$, where $\pi$ is a uniformizing parameter for the maximal ideal of $\mathcal{O}_\wp$.

Assume first of all $p \nmid \Delta$. The right hand side of (10) equals then $1 + \chi_\Delta(p)$, which is 2 or 0 according as $\Delta$ is a square mod 4 or not. This proves (10) for $k = 1$.

But then (10) is also true for all $k \geq 1$ since $S_\Delta(p^k) = S_\Delta(p)$. For this it suffices to show that, for $k \geq 1$, the canonical reduction map $\rho : S_\Delta(p^{k+1}) \rightarrow S_\Delta(p^k)$ is a bijection. Indeed, let $x$ be in $S_\Delta(p^k)$, and let $y$ in $x + 2\wp^k$, say $y = x + 2\pi^kt$ for some $t$ in $\mathcal{O}_\wp$.

The congruence

$$(x + 2\pi^k t)^2 \equiv \Delta \mod 4\pi^{k+1}$$

is equivalent to $xt \equiv \frac{\Delta - x^2}{4\pi^2} \mod \pi$, which has exactly one solution $x \mod \pi$ (since $\pi \nmid x$).

Next, suppose that $p^l$ for some $l \geq 1$ is the exact power of $p$ dividing $\Delta$, and let $p^e$ be the exact $p$-power dividing 2.

For $2e + k \leq l$, the congruence $x^2 \equiv \Delta \mod 4p^k$ is equivalent to $x^{e + [k/2]} \equiv x \mod 2$ (where $x^2 \equiv \Delta \mod 4$), and hence has $n(p)^{[k/2]}$ solutions mod $2p^k$.

But $n(p)^{[k/2]}$ equals also the right hand side of (8) since one of the terms is zero and the other one equals $n(p)^{[k/2]}$. (For the verification note that $2e \leq l - k$, so that $\Delta/p^{2[k/2]}$ is divisible by $p^{2e}$, and hence still a square modulo 4).

For $2e + k > l$ and odd $l$, the congruence $x^2 \equiv \Delta \mod 4p^k$ has no solution (since a square $x^2$ cannot have the odd $p$-power $p^l$ as exact divisor). But the right hand side of (10) is also zero since, for any $k'$ with $2e + k' \geq l$, either the gcd of $\Delta$ and $p^{k'}$ equals $p^l$ (which is not a square), or else it equals $p^{k'}$, where $p^{k'}$ is not a square, or where $2e > l - k'$ and hence $\Delta/p^{k'}$ is not a square mod 4)

Finally, let $2e + k > l$ and $l$ be even. Then the congruence $x^2 \equiv \Delta \mod 4p^k$ is equivalent to $x \equiv \pi t/y \mod 2\pi^k$ and $y^2 \equiv \Delta' \mod 4\pi^{k-l}$. In other words,

$$S_\Delta(p^k) = \text{card} \left\{ y \in \mathcal{O}_\wp/2\pi^{k-l/2} : y^2 \equiv \Delta' \mod 4\pi^{k-l} \right\},$$

where $\Delta' = \Delta/\pi^l$.

Suppose $\Delta'$ is a square mod 4. Then, for $k > l$, we have

$$S_\Delta(p^k) = n(p)^{l/2}S_{\Delta'}(p^{k-l}) = n(p)^{l/2} \left( \chi_{\Delta'}(p^{k-l}) + \chi_{\Delta'}(p^{k-l-1}) \right),$$

where $\chi_{\Delta'}(p^{k-l})$ is defined by (7) (with $k = \Delta'$)
where the second identity follows from the already proven validity of (10) for squares \( \Delta \equiv 0 \pmod{4} \) not divisible by \( p \). But the right hand side of the last identity equals the right hand side of (10). If \( k \leq l \) then \( y^2 \equiv \Delta' \mod{4\pi^{k-l}} \) is equivalent to \( y \equiv s \mod{2\pi^{k-l}} \). Hence \( S_{\Delta}(\pi^k) = n(p)^{[k/2]} \), which again proves (10).

Suppose now that \( \Delta' \) is not a square mod 4. Then, for \( k \geq l \) both sides of (10) equal 0. If \( l > k \) and \( k > l - 2e \), so that in particular, \( e \geq 1 \) then \( y^2 \equiv \Delta' \mod{4\pi^{k-l}} \) might have a solution or not. Let \( \delta = \delta(\Delta') \) be the largest integer \( 1 \leq \delta \leq 2e - 1 \) such that \( \Delta' \) is a square mod \( \pi^\delta \). Note that \( \delta \) must be odd (see Lemma 20 below). Note also that \( \Delta' \) is a square mod \( p \) (since squaring is the Frobenius isomorphism in a field of characteristic 2). Note also that the congruence \( \Delta' \equiv x^2 \mod{\pi^t} \) with \( t < 2e \) determines \( x \) uniquely mod \( \pi^{(t/2)} \). We therefore find \( S_{\Delta}(\pi^k) = n(p)^{[k/2]} \) for \( 1 \leq 2e + k - l - \delta \), and \( S_{\Delta}(\pi^k) = 0 \) for \( \delta + l - 2e < k < l \).

Again this equals the right hand side of (10) since, for \( k' = k \) or \( k' = k - 1 \), we have \( (\Delta, \pi^k) = \pi^{k'} \) and exactly one of these \( k' \) is even, and for this \( k' \) we have \( \chi_{\Delta}(\pi^{k'}) = n(p)^{[k/2]} \) or \( \chi_{\Delta}(\pi^{k'}) = 0 \) according as \( \Delta/\pi^{k'} \) is a square mod 4, or not. But \( \Delta/\pi^{k'} \) being a square mod 4 is equivalent to \( \Delta' \) being a square mod \( 2e + k' - l \), i.e. \( 2e + k' - l \leq \delta \). Since \( \delta \) is odd, the latter is equivalent to \( 2e + k - l \leq \delta \). This completes the proof of Theorem 14.

3.3. The function \( \zeta(\Delta, s) \) in the non-relative case. If \( K \) equals the field of rational numbers it can be quickly verified that, for any pair \((m, n)\) of relatively prime integers, one has for the series (5) the identity

\[
\zeta(\Delta, s) = \sum_{\substack{f \in \mathcal{F}(\Delta) / \text{SL}(2, \mathbb{Z})_{(m, n)} \cap \text{SL}(2, \mathbb{Z})_{(m, n)} \geq 0}} \frac{1}{f(m, n)^s},
\]

where the sum is over a complete system of representatives for the set \( \mathcal{F}(\Delta) \) of integral binary quadratic forms \( f = [a, b, c] = ax^2 + bxy + cy^2 \) satisfying \( b^2 - 4ac = \Delta \) and \( f(m, n) > 0 \) modulo the stabilizer of \((m, n)\) in \( \text{SL}(2, \mathbb{Z}) \). The set of orbits is defined via the usual action \( (f, A) \mapsto f(A(x, y)^t) \) of \( \text{SL}(2, \mathbb{Z}) \) on binary forms. The stated identity is obvious for \((m, n) = (1, 0)\): Here we have \( \text{SL}(2, \mathbb{Z})_{(m, n)} = \langle \frac{1}{1} \rangle \), and hence the series on the right of the last identity becomes \( \sum_{\substack{[a, b, c] \in \mathbb{A} \cap \mathbb{A}}} a^{-s} \), the sum being over a set of representatives \([a, b, c]\) for \( \mathcal{F}^+(\Delta) / \langle \frac{1}{1} \rangle \), where the \( {}^+ \) indicates the subset of all \([a, b, c]\) in \( \mathcal{F}(\Delta) \) with \( a > 0 \). As representatives one can take the forms \([a, b, c]\) in \( \mathcal{F}^+(\Delta) \) with \( 0 \leq b < 2a \), and one recognizes the series \( \zeta(\Delta, s) \).

One can write the identity of the last paragraph also in another way. According to the well-known theory of binary quadratic forms (see e.g. [1, 1]) The application \([a, b, c] \mapsto \mathbb{A}_{[a, b, c]} := Z + Z \cdot \frac{1}{1} \Delta \) maps \( \mathcal{F}(\Delta) \) to the set of ideals of the order \( R_{\Delta} := Z + Z \cdot \frac{1}{1} \Delta \). In fact, it induces a bijection \( \mathcal{F}^+(\Delta) / \langle \frac{1}{1} \rangle \cong \mathcal{I}(\Delta) / \mathcal{Q}_0^* \), where \( \mathcal{I}(\Delta) \) is the set of all fractional ideal of \( R_{\Delta} \). The ideals of \( R_{\Delta} \) in \( \mathbb{A}_{[a, b, c]} \mathcal{Q}_0^* \) are \( \mathbb{A}_{[a, b, c]} \mathbb{Z}_{\geq 1} \), and hence the identity of the last paragraph can be stated as

\[
\zeta(2s)\zeta(\Delta, s) = \sum_{\mathbb{A}} \frac{1}{[R_{\Delta} : \mathbb{A}]^s},
\]

where the sum on the right is over all ideals of \( R_{\Delta} \). In terms of \( L(\chi_{\Delta}, s) \) this can also be written (using Theorem 14) as

\[
\zeta(s)L(\chi_{\Delta}, s) = \sum_{\mathbb{A}} \frac{1}{[R_{\Delta} : \mathbb{A}]^s}.
\]

As we shall show in the next section the last two identities generalize to relative quadratic extensions.
3.4. Orders and $\mathfrak{o}$-lattices of relative quadratic extensions. We fix again a number field $K$ with ring of integers $\mathfrak{o}$. Let $L$ be a quadratic extension of $K$ and $\mathfrak{D}$ the ring of integers of $L$. An $\mathfrak{o}$-lattice in $L$ is a finitely generated $\mathfrak{o}$-submodule of $L$ containing a $K$-basis of $L$. If $\mathfrak{L}$ is an $\mathfrak{o}$-lattice in $L$ we define the order of $\mathfrak{L}$ to be the set of elements $a$ in $L$ such $a\mathfrak{L} \subseteq \mathfrak{L}$. This order is then an order of $L$ in the usual sense, i.e. it is a subring of finite index in $\mathfrak{D}$. Note that $\mathfrak{D}$ contains $\mathfrak{o}$, and that every order of $L$ containing $\mathfrak{o}$ is the order of an $\mathfrak{o}$-module (namely the order of itself, viewed as $\mathfrak{o}$-module). If $\Delta$ is a discriminant in $K$ and $b$ in $\mathfrak{o}$ a solution of of $\Delta \equiv b^2 \bmod 4$ we set

$$\omega_{\Delta,b} := \frac{b + \sqrt{\Delta}}{2}.$$ 

If $L = K(\sqrt{\Delta})$ then $\omega_{\Delta,b}$ is an integral element of $L$.

**Lemma 15.** Every $\mathfrak{o}$-lattice in $L$ containing $\mathfrak{o}$ is of the form $\mathfrak{o} + a^{-1}\omega_{\Delta,b}$ with an integral ideal $\mathfrak{a}$ of $K$, a discriminant $\Delta$ of $K$, and an integer $b$ of $\mathfrak{o}$ such that $\Delta \equiv b^2 \bmod 4$.

**Proof.** Let $\mathfrak{L}$ denote a $\mathfrak{o}$-lattice in $L$ containing $\mathfrak{o}$. Since the $\mathfrak{o}$-module $\mathfrak{L}/\mathfrak{o}$ is of rank 1, it is isomorphic to a fractional ideal $b$. Multiplying $b$ by a suitable element of $K^\times$ we can assume that $b^{-1}$ is integral. We then have the exact sequence of $\mathfrak{o}$-modules

$$0 \to \mathfrak{o} \xrightarrow{\pi} \mathfrak{L} \xrightarrow{f} \mathfrak{L}/\mathfrak{o} \xrightarrow{g} b \to 0,$$

where $\pi$ is the canonical projection and $f$ and isomorphism. Since $b$ is projective we can find a section $s : b \to \mathfrak{L}$ (i.e. $f \circ \pi \circ s = 1$). It follows $\mathfrak{L} = \mathfrak{o} + s(b)$. Setting $\omega := s(1)$ we obtain $s(b) = b\omega$ (If $n$ is in $\mathfrak{o}$ such that $nb$ is integral, then $ns(b) = nbs(1)$ since $s$ is a $\mathfrak{o}$-homomorphism.) But $aw^2 - bw + c = 0$ for suitable $a, b, c$ in $\mathfrak{o}$, and hence $b\omega = (b/a)^{1+\sqrt{\Delta}}$ with $\Delta = b^2 - 4ac$. The lemma is now obvious. □

**Proposition 16.** Let $\Delta$ be a discriminant of $K$, $L = K(\sqrt{\Delta})$ and $b$ in $\mathfrak{o}$ a solution of $\Delta \equiv b^2 \bmod 4$.

(1) One has $\mathfrak{D} = \mathfrak{o} + f_{f_{\Delta}}^{-1}\omega_{\Delta,b}$.

(2) Every order of $L$ containing $\mathfrak{o}$ is of the form

$$\mathfrak{D}_{\epsilon} := \mathfrak{o} + c\mathfrak{D} = \mathfrak{o} + cf_{f_{\Delta}}^{-1}\omega_{\Delta,b}$$

for a unique integral ideal $\epsilon$ of $K$.

**Proof of Proposition 16.** For (1) we use the preceding lemma to write $\mathfrak{D} = \mathfrak{o} + a^{-1}\omega_{\Delta,b}$ with an integral ideal $\mathfrak{a}$, a discriminant $\Delta'$ of $K$ and integral $b'$ with $\Delta' \equiv b'^2 \bmod 4$. Since the elements of $\mathfrak{D}$ are integral we conclude $\mathfrak{o} | b'$, and $a^2 | b'^2$. It follows $\mathfrak{D} \subseteq \mathfrak{o} + f_{f_{\Delta}}^{-1}\omega_{\Delta,b}$, and then even equality since the elements of $f_{f_{\Delta}}^{-1}\omega_{\Delta,b}$ are integral. Finally one has $f_{f_{\Delta}}^{-1}\omega_{\Delta,b} = f_{f_{\Delta}}^{-1}\omega_{\Delta,b}$ as is quickly shown on writing $\Delta' = a'^2 \Delta$ with $a'$ in $K$ and using $f_{f_{\Delta}} = a'f_{f_{\Delta}}$ (see Corollary 4).

For (2) let $R$ be an order of $L$ containing $\mathfrak{o}$. Then $R$ is of finite index in $\mathfrak{D}$. Let $c$ be the annihilator of $\mathfrak{D}/R$. Note that by (1) $c$ equals the ideal of all $a$ in $\mathfrak{o}$ such that $af_{f_{\Delta}}^{-1}\omega_{\Delta,b} \subseteq R$ since $\mathfrak{D}$ is contained in $R$. We claim $R = \mathfrak{o} + c\mathfrak{D}$. The inclusion `'\subset`' is trivial. Let $a$ be in $R$. Then $a = x + y\omega_{\Delta,b}$ for suitable $x$ in $\mathfrak{o}$ and $y$ in $f_{f_{\Delta}}$. But then $\mathfrak{o}(a - x) = y\omega_{\Delta,b} \subseteq \mathfrak{D}$ which implies $yf_{f_{\Delta}} \subseteq c$. Finally, the $c$ in the representation $R = \mathfrak{o} + c\mathfrak{D}$ is unique since, for any integral ideal $\epsilon$ in $K$ the annihilator of $\mathfrak{D}/(\mathfrak{o} + c\mathfrak{D})$ equals $c$. □
For a discriminant $\Delta$ of $K$ let $\mathcal{Q}(\Delta)$ be the set of all pairs $(a, b)$, where $a$ is an integral ideal in $K$, where $b$ is in $\mathfrak{a}$ and where $\Delta \equiv b^2 \mod 4a$. For $(a, b)$ in $\mathcal{Q}(\Delta)$ set

$$J_{a,b,\Delta} := a + \mathfrak{a} \omega_{\Delta,b}.$$  

Suppose $L = K(\sqrt{\Delta})$. Then $J_{a, b, \Delta}$ is clearly an ideal of $\mathcal{O}_{\Delta}$ (as is immediately clear since by noting that the preceding proposition implies $\mathcal{O}_{\Delta} = \mathfrak{a} + \mathfrak{a} \omega_{\Delta,b}$ for every solution $b$ of $\Delta \equiv b^2 \mod 4$). We let $I_L(\Delta)$ and $I_K$ denote the monoids of fractional ideals of $\mathcal{O}_{\Delta}$ and $\mathfrak{a}$. (Recall that a fractional ideal of $\mathcal{O}_{\Delta}$ is a finitely generated $\mathcal{O}_{\Delta}$-submodule $\mathfrak{A}$ of $L$.)

**Theorem 17.** Let $\Delta$ be a discriminant in $K$ such that $L = K(\sqrt{\Delta})$. The application $(a, b) \mapsto J_{a,b,\Delta}$ induces a bijection

$$\mathcal{Q}(\Delta)/\sim \rightarrow I_L(\Delta)/I_K,$$

where $(a, b) \sim (a', b')$ if $a = a'$ and $b \equiv b' \mod 2a$. In particular, one has

$$\sum_{\mathfrak{A}} [\mathcal{O}_{\Delta}:\mathfrak{A}]^{-s} = \zeta_K(s)L(\chi_{\Delta}, s) = \zeta_K(2s)\zeta(\Delta, s),$$

with $\chi_{\Delta}$ as in (6), and where the sum on the left is over all ideals of $\mathcal{O}_{\Delta}$.

**Proof.** It is clear that $J_{a,b,\Delta}$ depends only on $b$ modulo $a$ so that the given application induces indeed a map $f : \mathcal{Q}(\Delta)/\sim \rightarrow I_L(\Delta)/I_K$.

The map $f$ is surjective: By Lemma 15 every element $I_L(\Delta)/I_K$ is represented by an $\mathfrak{a}$-lattice of the form $\mathfrak{A} := \mathfrak{a} + a^{-1} \omega_{\Delta', b'}$ with an integral ideal $a$, a discriminant $\Delta'$ and a solution $b'$ in $a$ of $\Delta' \equiv b'^2 \mod 4$. Since $\Delta'$ and $\Delta$ differ by a square in $K^×$ we can assume (after possibly multiplying $\mathfrak{A}$ and $\omega_{\Delta', \mathfrak{b}}$ by a suitable integer in $K$) that $\Delta' = a^2 \Delta$ for some integer $a$ in $\mathfrak{a}$. From $\mathcal{O}_{\Delta} = \mathfrak{A}$ we deduce (on writing, using Proposition 16, $\mathcal{O}_{\Delta} = \mathfrak{a} + a \omega_{\Delta, \mathfrak{b}}$ with any $\Delta \equiv b^2 \mod 4$) that $\omega_{\Delta, \mathfrak{b}} \in \mathfrak{A}$. But $\omega_{\Delta, \mathfrak{b}} = a^{-1} \omega_{\Delta', \mathfrak{b}} + b^{-1} b'/2a$, so that $\omega_{\Delta, \mathfrak{b}} \in \mathfrak{A}$ implies $a | a, b$ and $b \equiv b'/a \mod 2$.

Writing $a$ for $a/a$, replacing $b$ by $b'/a$, we find $\mathfrak{A} = \mathfrak{a} + a^{-1} \omega_{\Delta, \mathfrak{b}}$. Again from $\mathcal{O}_{\Delta} = \mathfrak{A}$ we deduce $\omega_{\Delta, \mathfrak{b}} \in \mathfrak{A}$, which finally implies $\frac{b}{a} \in \mathfrak{a}$. The map $f$ is injective: Every class in $I_L(\Delta)/I_K$ contains exactly one element of the form $\mathfrak{A} := J_{a', b', \Delta}$ for any $a'$ in $\mathfrak{A}$ and $b$ in $\mathfrak{a}$.

For proving (11) we note that $b \mapsto 5\mathcal{O}_{a,b,\Delta}$ defines a bijection between the set of integral ideals of $K$ and the subset of $\mathcal{O}_{\Delta}$-ideals in the class of $I_{a,b,\Delta}$ in $I_L(\Delta)/I_K$.

Moreover, $[\mathcal{O}_{\Delta} : bJ_{a,b,\Delta}] = n(b)^2 n(a)$. The left hand side of (11) equals therefore

$$\sum_{b} \sum_{(a, b) \in \mathcal{Q}(\Delta) / \equiv} n(b)^{-2s} n(a)^{-s},$$

and inserting (9) we recognize the first of the claimed identities. The second one is merely a restatement of (9) of Theorem 14. $\square$

### 3.5. Analytic properties of $L(\chi_{\Delta}, s)$.

From its definition it is clear that the $L$-series $L(\chi_{\Delta}, s)$ converges absolutely for $\mathfrak{R}(s) > 1$.

**Theorem 18.** Let $\Delta$ be a discriminant in $K$. The function $L(\chi_{\Delta}, s)$ in (7) has the following properties.

1. It can be analytically continued to $\mathbb{C} \setminus \{1\}$.

2. It is an entire function if $\Delta$ is not a square in $K$. 

(3) It has a simple pole at \( s = 1 \) with residue
\[
\kappa = \frac{2^{\nu_1(2\pi)^2}}{w_K|D_K|^{1/2}} h_K R_K
\]
if \( \Delta \) is a square in \( K \).

(4) It satisfies the functional equation
\[
L^*(\chi\Delta, s) := \gamma(s)L(\chi\Delta, s) = L^*(\chi\Delta, 1 - s),
\]
where
\[
\gamma(s) = \frac{n(\Delta^{\nu_1})^2 \Gamma(1 + s) \Gamma(\frac{2}{2} - s)}{\pi^{2s}} = \frac{n(\Delta^{\nu_1})^2 \Gamma(1 + s) \Gamma(\frac{2}{2} - s)}{\pi^{2s}}.
\]
Here \( a = \sum_{\sigma} \text{sign}(\sigma(\Delta)) \) with \( \sigma \) running over all real embeddings of \( K \), and
\( \Gamma_2(s) = \pi^{-s/2} \Gamma(s/2) \) and \( \Gamma_2(s) = 2(2\pi)^{-s} \Gamma(s) \).

Here \( n, D_K, \vartheta_K, h_K, R_K, w_K \) denote the degree, discriminant, different, class number, regulator and number of roots of unity of \( K \).

**Proof.** We have \( L(\Delta, s) = \zeta_L(s)/\zeta_K(s) \), where \( L = K(\sqrt{\Delta}) \) if \( \Delta \) is not a square in \( K \), and where \( \zeta_L(s) = \zeta_K(s)^2 \) otherwise. Using this we can write
\[
(12) \quad L(\chi\Delta, s) = \frac{\zeta_L(s)}{\zeta_K(s)} F(s),
\]
where \( F(s) \) is the finite Euler product on the right of (7).

It is a classical fact that the Dedekind zeta function of a number field can be analytically continued to \( \mathbb{C} \setminus \{1\} \), has a simple pole at \( s = 1 \) and satisfies a functional equation \( s \mapsto 1 - s \). For \( \Delta \) not a square, \( \zeta_L(s)/\zeta_K(s) \) is an entire function since \( L/K \) is Galois (cf. [1, 2]). From these facts and the preceding identity for \( L(\chi\Delta, s) \) the properties (1) to (4) become obvious. For the formula for the residue we note \( F(1) = 1 \), so that the and the residue of \( L(\chi\Delta, s) \) at \( s = 1 \) equals the residue \( \kappa \) of \( \zeta_K(s) \) at \( s = 1 \), which by a well-known formula (see e.g. [Neu99, Chap. VII, Cor. (5.11)]) is given by \( \rho = \frac{2^{\nu_1(2\pi)^2}}{w_K|D_K|^{1/2}} h_K R_K \).

For the formula for \( \gamma(s) \) we note that \( n(\Delta)^s F(s) \) is invariant under \( s \mapsto 1 - s \), that \( \zeta_K(s) = \frac{|D_K|^{s/2} \Gamma(1 + s) \Gamma(\frac{2}{2} - s)}{\pi^{2s}} \), where \( r_1 \) and \( r_2 \) are the real and complex places of \( K \), and that a similar formula holds for \( \zeta_L(s) \) if \( \Delta \) is not a square. Therefore (4) holds with
\[
\gamma(s) = \frac{n(\Delta)^s}{|D_K|^{s/2}} \frac{|D_K|^{s/2} \Gamma(1 + s) \Gamma(\frac{2}{2} - s)}{\pi^{2s}} R_1 - r_1 \Gamma(1 - s) R_2 - r_2
\]
\[
= n(\Delta^{\nu_1})^2 \Gamma(1 + s) \Gamma(\frac{2}{2} - s) R_1 - r_1 \Gamma(1 - s) R_2 - r_2,
\]
where \( R_1, r_1 \) are the real places and and \( R_2, r_2 \) are the complex places of \( L \) and \( K \). For the second identity we used \( D_L = D_K^2 \) (see e.g. [Neu99, Ch. III, Cor. (2.10)]), \( D_L/K = \Delta/p_3^2 \) (Theorem 1), and the duplication formula \( L(s) = L(s)L(s + 1) \). (These identities are literally true if \( \Delta \) is not a square in \( K \), but hold also true with the convention \( D_L = D_K^2 \) and \( R_j = 2r_j \) for squares \( \Delta \) in \( K \).) If \( \Delta \) is not a square then every complex place of \( K \) splits into two complex places of \( L \), and every real embedding \( \sigma \) can be continued to two real places of \( L \) if \( \sigma(\Delta) > 0 \), and can be continued to a pair of complex places \( \tau \) and \( \overline{\tau} \) in \( L \) if \( \sigma(\Delta) < 0 \). Accordingly, if we use \( r_1^+ \) and \( r_1^- \) for the real places \( \sigma \) of \( K \) with \( \sigma(\Delta) > 0 \) \( \sigma(\Delta) < 0 \), respectively, then \( R_1 = 2r_1^+ \) and \( R_2 = r_1^- + 2r_2 \). In other words, \( R_2 - r_2 = \frac{R_1}{2} \) and \( R_1 + R_2 - r_1 - r_2 = \frac{R_1}{2} \), and we recognize the claimed formula for \( \gamma(s) \).
3.6. **Special values.** There is another situation where relative quadratic extension continue smoothly properties of quadratic number fields. This is the case of a totally real field \( K \) and a totally imaginary extension.

**Theorem 19.** Let \( K \) be a totally real number field of degree \( n \) and \( \Delta \) a totally negative discriminant of \( K \). Then

\[
H(\Delta) = H(\Delta, K) := L(\chi_\Delta, 0)
\]

(with \( \chi_\Delta \) as defined in (6)) is a rational number. More precisely, one has

\[
H(\Delta) = \frac{2^{n-1} h_L/h_K}{w_L/2} \sum_{f \mid \Delta} n(f) \prod_{p \mid f} \left( 1 - \frac{\frac{\Delta}{f}}{n(p)} \right).
\]

Here \( h_L \) and \( h_K \) denote the class numbers of \( L = K(\sqrt{\Delta}) \) and \( K \), and we use \( \gamma = [\Omega_L^\times : W_L^\times] \) with \( \Omega_L^\times \) and \( W_L \) denoting the group of units and roots of unity in \( L \), and \( w_L = \text{card}(W_L) \).

**Remark.** The numbers \( H(\Delta, \mathbb{Q}) \) are the Hurwitz class numbers, i.e. the number of \( SL(2, \mathbb{Z}) \)-equivalence classes of binary positive definite integral quadratic forms of discriminant \( \Delta \), where forms equivalent to a multiple of \( x^2 + y^2 \) or \( x^2 + xy + y^2 \) are counted with weight \( \frac{1}{2} \) and \( \frac{1}{4} \), respectively. For \( K = \mathbb{Q} \) the identity (13) is classical.

The computation of \( H(\Delta) \) the following rearks are useful:

1. Since \( K \) is totally real and \( L := K(\sqrt{\Delta}) \) totally imaginary, the rank of the unit groups of \( K \) and \( L \) are equal and in fact equal to \( n - 1 \) by Dirichlet’s unit theorem. In particular, \( \gamma_L \) is well-defined.
2. As observed in [Rem54, §2] one has \( \gamma_L \leq 2 \). Indeed, \( W_L^\times \) equals the kernel of the homomorphism \( : \Omega_L^\times \to W_L/W_L^\times \cong \{\pm 1\}, u \mapsto (\overline{u}/u)W_L^\times \).
3. \( W_L \neq \{\pm 1\} \) can occur only for finitely many \( \Delta \) modulo \( K^{\times 2} \). Indeed, if \( W_L = \{z\} \) with a primitive \( k \)th root of unity \( z \), then \( \varphi(k) \mid 2n \) and, for \( k > 2 \), one has \( L = K(z) \). Note that \( k \) must then also be even (since \( W_L \) contains \( -1 \)), and \( K \) must contain the totally real subfield \( Q(z + \overline{z}) \) of the \( k \)th cyclotomic field.
4. One has \( \Omega_L^\times = \sigma^\times \) for all but finitely many \( \Delta \) modulo \( K^{\times 2} \). If \( \Omega_L^\times \neq \sigma^\times \) and \( W_L = \{\pm 1\} \) then \( \gamma_L = [\Omega_L^\times : \sigma^\times] = 2 \), and hence \( L = K(z) \) with a non-real \( z \) in \( \Omega_L^\times \). But then \( \overline{z} = -z \), i.e. \(-z^2 \) is a totally positive unit in \( K \). Thus \( L \) belongs to the (finite) image of the application which associates to each class \( u \sigma^\times \) in \( \sigma^\times \sigma^\times \) containing only totally positive units the field \( L = K(\sqrt{-u}) \).
5. Assume that every totally positive unit in \( K \) is in \( \sigma^\times \). Then one has \( \Omega_L^\times = \sigma^\times \) if and only if \( W_L \neq \{\pm 1\} \). Namely, if \( \sigma^\times \) is a proper subgroup of \( \Omega_L^\times \) then there exists a unit \( z \) in \( L \) and \( z \neq 1 \) in \( W_L \) such that \( \overline{z} = uz \). Then either \( u \neq -1 \) (and hence \( w_L > 2 \)) or \( z = yi \) for a unit \( y \) in \( \sigma^\times \) (and hence \( i \in W_L \)). In particular, \( \gamma_L = 2 \) at most if \( w_L > 2 \).
6. The quotient \( h_L/h_K (L = K(\sqrt{\Delta})) \) is an integer. This follows from the fact that the norm map from the class group of \( L \) to the class group of \( K \) is surjective. (This follows by translating this map via class field theory to a map \( \text{Gal}(\overline{L}/L) \to \text{Gal}(\overline{K}/K) \), where \( \overline{L} \supseteq \overline{K} \) are the Hilbert class fields of \( L \) and \( K \), and noticing that this map is essentially the restriction map; for the latter one uses that the infinite places of \( K \) ramify in the totally imaginary extension \( L \), so that \( \overline{K} \cap L = K \).) Therefore \( h_L/h_K \) equals the cardinality of the kernel of the norm map, i.e. the cardinality of the relative class group of \( L/K \).
Proof of Theorem 19. We note that the sum on the right hand side of the claimed formula for $H(\Delta)$ equals $F(0)$ with $F$ as in (12). Moreover, $\lim_{s \to \infty} s^{-r} \zeta_K(s) = h_K R_K/w_K$ and, setting $L = K(\sqrt{\Delta})$, $\lim_{s \to \infty} s^{-R} \zeta_L(s) = h_L R_L/w_L$, where $r = r_1 + r_2 - 1$, $R = R_1 + R_2 - 1$, and $R_L$, $R_K$ and $w_L$, $w_K$ denote the regulators and the number of roots of units in $K$ and $L$ (see [Neu99, ??]). Since assumption $K$ is totally real and $L$ is a totally complex extension of $K$, we have $r = R$ and $R_L w_K/R_K w_L = 2^{n-1}/[\Delta_L^2 : \sigma^\ast]$. But $[\Delta_L^2 : \sigma^\ast] = \gamma_L \cdot [W_L : W_K]$ and $W_K = \{\pm 1\}$. The formula for $H(\Delta)$ is now obvious. \hfill\Box

4. Appendix: Tables

In this appendix the reader finds tables for the numbers $H(\Delta, K)$ of Theorem 19 for the three fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{10})$ and $\mathbb{Q}[x]/(x^3 - x^2 - 9x + 10)$. The field $\mathbb{Q}(\sqrt{5})$ has class number 1, whereas the other two fields are the fields having smallest discriminant among all totally real fields of degree 2 and 3 whose class number is greater than 1; in fact the class number is 2 in both cases.

A fourth table lists for certain fields $K$ all discriminant classes $\Delta$ in $K$ such that $K(\sqrt{\Delta})$ is unramified at all finite places of $K$. By Theorem 2 the latter is equivalent to $\Delta/f_\Delta^2 = 1$. But if $\Delta/f_\Delta^2 = 1$, then $[\Delta]/f_\Delta$ has conductor 1 and is trivial on all principal ideals generated by a totally positive element (see Theorem 10), so induces a character of the narrow class group $\text{Cl}^+(K)$ of $K$ with $\psi^2 = 1$. Class field theory shows that the application $\Delta \mapsto [\Delta]/f_\Delta$ induces in fact a bijection of the (at finite primes) unramified quadratic extensions of $K$ and the group $\psi$ of characters of $\text{Cl}^+(K)$ with $\psi^2 = 1$ (for this recall from \S 2.6 that $[\Delta]/f_\Delta$ equals the Artin reciprocity map of $K(\sqrt{\Delta})/K$). In other words, the number of (at finite places) unramified extensions $K(\sqrt{\Delta})/K$ in is one to one correspondence with the group $\text{Cl}^+(K)/\text{Cl}^+(K)^2$ of genus classes of $K$, whence the number of such extensions equals the cardinality of the subgroup $\text{Cl}^+(K)/2[2]$ of elements $C$ in $\text{Cl}^+(K)$ such that $C^2 = 1$, or, equivalently, the number of even elementary divisors of the narrow class group. An unramified extension $K(\sqrt{\Delta})/K$ is unramified at the infinite places too (i.e. every real place of $K$ splits into two real places of the extension) if $\Delta$ is totally positive, which, according to Theorem 10, means that $[\Delta]/f_\Delta$ is trivial on the narrow ideal classes of principal ideals, i.e. factors through a character of the class group $\text{Cl}(K)$. The number of discriminant classes with totally positive $\Delta$ such that $\Delta = f_\Delta^2$ equals therefore $\text{card}(\text{Cl}(K)[2])$, i.e. the number of even elementary divisors of $\text{Cl}(K)$.

For the fields $K$ of Table 4 we chose all number fields of degree $\leq 5$ having smallest discriminant among all fields with card $\{\text{Cl}^+(K)[2]\} \in \{2, 4, 8\}$. We used the Bordeaux tables [Gro07] for finding these fields, and for the computations of this and the other tables we used [Dev15]. We did not find any such $K$ in the range of the Bordeaux tables of degree 5 and not totally real, or of degree 6 or 7. All computations were done using [Dev15].

For the computation of the numbers $H(\Delta, K)$ for $K = \mathbb{Q}(\sqrt{5})$ in Table 1 we note the following. Since the class number of $K$ is one every discriminant in $K$ decomposes as $\Delta_0 a^2$, where $\Delta_0$ is a fundamental discriminant and $a$ an integer in $K$. The field $K$ is the totally real subfield of the fifth cyclotomic field $K(\sqrt{-5})$ where $\Delta_5 = \frac{5}{2}$. Therefore, for a given totally negative fundamental discriminant $\Delta$, the field $L = K(\sqrt{\Delta})$ contains roots of unity different from $\pm 1$ exactly if $\Delta \in \{\Delta_5, -3, \Delta_4 = 2\sqrt{5} - 6\}$ up to multiplication by a square of a unit in $K$. Since the fundamental unit of $K$ has norm $-1$, every totally positive unit in $K$ is a square in $K$. Accordingly, the group of units of $L = K(\sqrt{-5})$ and of $K$ differ at most if $\Delta$ is one
of the described three kinds of discriminants. (see Remark (5) after Theorem 19). For these \( \Delta \) one has \( \gamma_L = 1 \).

Table 2 lists \( H(\Delta, K) \) for \( K = \mathbb{Q}(\sqrt{10}) \). This is the totally real quadratic field with smallest discriminant having class number greater 1, in fact, equal to 2. The field \( K \) is not the totally real subfield of any cyclotomic field. Therefore, for a given totally negative \( \Delta \), the field \( L = K(\sqrt{\Delta}) \) contains roots of unity different from \( \pm 1 \) exactly if \( L = \mathbb{Q}(\sqrt{10}, \sqrt{-3}) \) or \( L = \mathbb{Q}(\sqrt{10}, \sqrt{-4}) \), i.e. if and only if \( \Delta \in \{-3, -4\} \) (up to multiplication by a square in \( K \)). Since every totally positive unit in \( K \) is a square in \( K \), the group of units of \( L \) and of \( K \) differ at most if \( \Delta \in -3K^{\times 2} \) or \( \Delta \in -4K^{\times 2} \) (see Remark (5) after Theorem 19). For these \( \Delta \) one has \( \gamma_L = 1 \).

Finally, Table 3 lists \( H(\Delta, K) \) for \( K = \mathbb{Q}[x] / (x^3 - x^2 - 9x + 10) \). This is the totally real cubic field with smallest discriminant having class number greater than 1. The discriminant is \( D_K = 19 \cdot 103 \) and the class number is \( h_K = 2 \). For a given totally negative \( \Delta \), the field \( L = K(\sqrt{\Delta}) \) contains roots of unity different from \( \pm 1 \) exactly if and only if \( \Delta \in \{-3, -4\} \) (up to multiplication by a square in \( K \)) as follows from Remark (3) after Theorem 19 and the fact that \( K \) itself is not the subfield of a cyclotomic field (e.g. since \( K/\mathbb{Q} \) is not galois). We checked that in both cases \( \gamma_L = 1 \).

If \( L = K(\sqrt{\Delta}) \) has only \( \pm 1 \) as roots of unity, then \( \gamma_L \neq 1 \) (in fact, \( \gamma_L \neq 2 \)) if and only if \( L = K(\sqrt{u}) \) for a totally negative unit \( u \) in \( K \) (see Remark (4) after Theorem 19). But modulo \( \mathfrak{o}^{\times 2} \) the field \( K \) contains only one totally negative unit, which is not in \( -1 \cdot \mathfrak{o}^{\times} \), namely \( a - 3 \).
Table 1. The numbers $H(\Delta, \mathbb{Q}(\sqrt{5}))$ for all totally negative discriminants $\Delta$ in $\mathbb{Q}(\sqrt{5})$ modulo $\sigma^2$ with $n(\Delta) \leq 500$.

| $n(\Delta)$ | $\Delta$ | $f_\Delta$ | $H(\Delta)$ | $n(\Delta)$ | $\Delta$ | $f_\Delta$ | $H(\Delta)$ |
|-------------|----------|------------|-------------|-------------|----------|------------|-------------|
| 5           | $-1/2\sqrt{5} - 5/2$ | (1) | 2/5 | 261 | $-3/2\sqrt{5} - 33/2$ | (1) | 4 |
| 9           | $-3$ | (1) | 2/5 | 269 | $-2\sqrt{5} - 17$ | (1) | 2 |
| 16          | $-4$ | (1) | 1 | 269 | $-11/2\sqrt{5} - 41/2$ | (1) | 2 |
| 41          | $1/2\sqrt{5} - 13/2$ | (1) | 2 | 281 | $-7/2\sqrt{5} - 37/2$ | (1) | 6 |
| 41          | $-1/2\sqrt{5} - 13/2$ | (1) | 2 | 281 | $-4\sqrt{5} - 19$ | (1) | 6 |
| 49          | $-7$ | (1) | 2 | 304 | $-6\sqrt{5} - 22$ | (1) | 4 |
| 61          | $-2\sqrt{5} - 9$ | (1) | 2 | 304 | $-2\sqrt{5} - 18$ | (1) | 4 |
| 61          | $-3/2\sqrt{5} - 17/2$ | (1) | 2 | 320 | $-4\sqrt{5} - 20$ | (1) | 4 |
| 64          | $-8$ | (1) | 2 | 341 | $1/2\sqrt{5} - 37/2$ | (1) | 4 |
| 80          | $-2\sqrt{5} - 10$ | (2) | 12/5 | 341 | $-1/2\sqrt{5} - 37/2$ | (1) | 4 |
| 109         | $1/2\sqrt{5} - 21/2$ | (1) | 2 | 361 | $-19$ | (1) | 8 |
| 109         | $-1/2\sqrt{5} - 21/2$ | (1) | 2 | 389 | $-13/2\sqrt{5} - 49/2$ | (1) | 2 |
| 121         | $-11$ | (1) | 4 | 389 | $-7/2\sqrt{5} - 41/2$ | (1) | 2 |
| 125         | $-5/2\sqrt{5} - 25/2$ | $\left(-\sqrt{5}\right)$ | 12/5 | 400 | $-20$ | $\left(-\sqrt{5}\right)$ | 5 |
| 144         | $-12$ | (2) | 8/5 | 405 | $-9/2\sqrt{5} - 45/2$ | (3) | 22/5 |
| 145         | $-3/2\sqrt{5} - 25/2$ | (1) | 4 | 409 | $3/2\sqrt{5} - 41/2$ | (1) | 6 |
| 145         | $-4\sqrt{5} - 15$ | (1) | 4 | 409 | $-1/2\sqrt{5} - 41/2$ | (1) | 6 |
| 149         | $-2\sqrt{5} - 13$ | (1) | 2 | 421 | $-2\sqrt{5} - 21$ | (1) | 6 |
| 149         | $-7/2\sqrt{5} - 29/2$ | (1) | 2 | 421 | $2\sqrt{5} - 21$ | (1) | 6 |
| 176         | $-2\sqrt{5} - 14$ | (1) | 4 | 445 | $-7/2\sqrt{5} - 45/2$ | (1) | 4 |
| 176         | $-4\sqrt{5} - 16$ | (1) | 4 | 445 | $-6\sqrt{5} - 25$ | (1) | 4 |
| 209         | $-1/2\sqrt{5} - 29/2$ | (1) | 4 | 449 | $-11/2\sqrt{5} - 49/2$ | (1) | 6 |
| 209         | $1/2\sqrt{5} - 29/2$ | (1) | 4 | 449 | $-4\sqrt{5} - 23$ | (1) | 6 |
| 225         | $-15$ | $\left(-\sqrt{5}\right)$ | 14/5 | 464 | $2\sqrt{5} - 22$ | (1) | 4 |
| 241         | $-9/2\sqrt{5} - 37/2$ | (1) | 6 | 464 | $-2\sqrt{5} - 22$ | (1) | 4 |
| 241         | $-5/2\sqrt{5} - 33/2$ | (1) | 6 | 496 | $-6\sqrt{5} - 26$ | (1) | 8 |
| 256         | $-16$ | (2) | 5 | 496 | $-4\sqrt{5} - 24$ | (1) | 8 |
| 261         | $3/2\sqrt{5} - 33/2$ | (1) | 4 |          |            |          |            |
Table 2. The numbers $H(\Delta, \mathbb{Q}(\sqrt{10}))$ for all totally negative discriminants $\Delta$ in $\mathbb{Q}(\sqrt{10})$ modulo $\sigma^2$ with $n(\Delta) \leq 500$.

| $n(\Delta)$ | $\Delta$ | $f_\Delta$ | $H(\Delta)$ | $n(\Delta)$ | $\Delta$ | $f_\Delta$ | $H(\Delta)$ |
|------------|----------|-----------|-------------|------------|----------|-----------|-------------|
| 4          | $-2$     | (1)       | 2           | 265        | $-36\sqrt{10} - 115$ | (1)       | 4           |
| 9          | $-3$     | (1)       | 4/3         | 265        | $-6\sqrt{10} - 25$  | (1)       | 4           |
| 16         | $-4$     | (2, $\sqrt{10}$) | 3           | 321        | $-34\sqrt{10} - 109$ | (1)       | 44          |
| 36         | $-4\sqrt{10} - 14$ | (3, $\sqrt{10} + 2$) | 5           | 321        | $-8\sqrt{10} - 31$  | (1)       | 44          |
| 36         | $-6$     | (2, $\sqrt{10}$) | 8           | 324        | $4\sqrt{10} - 22$   | ($\sqrt{10} - 1$) | 18          |
| 36         | $-8\sqrt{10} - 26$ | (3, $\sqrt{10} + 1$) | 5           | 324        | $-12\sqrt{10} - 42$ | ($\sqrt{10} + 2$) | $64/3$       |
| 41         | $-2\sqrt{10} - 9$ | (1)       | 4           | 324        | $-18$           | (3)       | 18          |
| 41         | $2\sqrt{10} - 9$ | (1)       | 4           | 324        | $-24\sqrt{10} - 78$ | ($-\sqrt{10} + 2$) | $64/3$       |
| 49         | $-7$     | (1)       | 4           | 324        | $-4\sqrt{10} - 22$ | ($\sqrt{10} + 1$) | 18          |
| 64         | $-8$     | (2)       | 14          | 356        | $-16\sqrt{10} - 54$ | (2, $\sqrt{10}$) | 32          |
| 65         | $-4\sqrt{10} - 15$ | (1)       | 12          | 356        | $-20\sqrt{10} - 66$ | (2, $\sqrt{10}$) | 32          |
| 65         | $-14\sqrt{10} - 45$ | (1)       | 12          | 361        | $-19$           | (1)       | 4           |
| 81         | $-6\sqrt{10} - 21$ | (3, $\sqrt{10} + 2$) | 16          | 369        | $-32\sqrt{10} - 103$ | (3, $\sqrt{10} + 2$) | 40          |
| 81         | $-12\sqrt{10} - 39$ | (3, $\sqrt{10} + 1$) | 16          | 369        | $4\sqrt{10} - 23$   | (3, $\sqrt{10} + 1$) | 24          |
| 89         | $-8\sqrt{10} - 27$ | (1)       | 4           | 369        | $-4\sqrt{10} - 23$  | (3, $\sqrt{10} + 2$) | 24          |
| 89         | $-10\sqrt{10} - 33$ | (1)       | 4           | 369        | $-10\sqrt{10} - 37$ | (3, $\sqrt{10} + 1$) | 40          |
| 96         | $-20\sqrt{10} - 64$ | (1)       | 12          | 384        | $-40\sqrt{10} - 128$ | (2, $\sqrt{10}$) | 12          |
| 96         | $-4\sqrt{10} - 16$ | (1)       | 12          | 384        | $-8\sqrt{10} - 32$  | (2, $\sqrt{10}$) | 12          |
| 100        | $-10$    | (5, $\sqrt{10}$) | 5           | 400        | $-20$           | ($-\sqrt{10}$) | 42          |
| 129        | $-11$    | (1)       | 12          | 401        | $-2\sqrt{10} - 21$  | (1)       | 36          |
| 129        | $-2\sqrt{10} - 13$ | (1)       | 20          | 401        | $2\sqrt{10} - 21$   | (1)       | 36          |
| 129        | $2\sqrt{10} - 13$ | (1)       | 20          | 409        | $-30\sqrt{10} - 97$ | (1)       | 4           |
| 144        | $-8\sqrt{10} - 28$ | (\sqrt{10} + 2) | 18          | 409        | $-12\sqrt{10} - 43$ | (1)       | 4           |
| 144        | $-12$    | (2)       | 40/3        | 416        | $4\sqrt{10} - 24$   | (1)       | 20          |
| 144        | $-16\sqrt{10} - 52$ | (\sqrt{10} + 2) | 18          | 416        | $-4\sqrt{10} - 24$  | (1)       | 20          |
| 160        | $-12\sqrt{10} - 40$ | (1)       | 4           | 441        | $-14\sqrt{10} - 49$ | (3, $\sqrt{10} + 2$) | 24          |
| 164        | $-4\sqrt{10} - 18$ | (2, $\sqrt{10}$) | 16          | 441        | $-28\sqrt{10} - 91$ | (3, $\sqrt{10} + 1$) | 24          |
| 164        | $4\sqrt{10} - 18$ | (2, $\sqrt{10}$) | 16          | 445        | $-16\sqrt{10} - 55$ | (1)       | 40          |
| 196        | $-14$    | (2, $\sqrt{10}$) | 32          | 465        | $-26\sqrt{10} - 85$ | (1)       | 40          |
| 201        | $-4\sqrt{10} - 19$ | (1)       | 12          | 481        | $-18\sqrt{10} - 61$ | (1)       | 20          |
| 201        | $4\sqrt{10} - 19$ | (1)       | 12          | 481        | $6\sqrt{10} - 29$   | (1)       | 68          |
| 225        | $-15$    | (5, $\sqrt{10}$) | 20          | 481        | $-6\sqrt{10} - 29$  | (1)       | 68          |
| 240        | $-4\sqrt{10} - 20$ | (1)       | 8           | 481        | $-24\sqrt{10} - 79$ | (1)       | 20          |
| 240        | $4\sqrt{10} - 20$ | (1)       | 8           | 484        | $-22$           | (2, $\sqrt{10}$) | 16          |
| 249        | $-2\sqrt{10} - 17$ | (1)       | 4           | 489        | $-20\sqrt{10} - 67$ | (1)       | 12          |
| 249        | $2\sqrt{10} - 17$ | (1)       | 4           | 489        | $-22\sqrt{10} - 73$ | (1)       | 12          |
| 256        | $-16$    | (4, $2\sqrt{10}$) | 15          | 496        | $-36\sqrt{10} - 116$ | (1)       | 40          |
| 260        | $-8\sqrt{10} - 30$ | (2, $\sqrt{10}$) | 16          | 496        | $-12\sqrt{10} - 44$ | (1)       | 40          |
| 260        | $-28\sqrt{10} - 90$ | (2, $\sqrt{10}$) | 16          | 496        | $-12\sqrt{10} - 44$ | (1)       | 40          |
Table 3. The numbers $H(\Delta, K)$ for $K = \mathbb{Q}[x]/(x^3 - x^2 - 9x + 10)$ and for all totally negative discriminants $\Delta$ in $K$ modulo $\mathfrak{o}^{x^2}$ with $|n(\Delta)| \leq 500$.

| $n(\Delta)$ | $\Delta$ | $f_\Delta$ | $H(\Delta)$ | $n(\Delta)$ | $\Delta$ | $f_\Delta$ | $H(\Delta)$ |
|-------------|------------|-------------|--------------|-------------|------------|-------------|--------------|
| $-475$      | $-3a^2 - 2a + 5$ | $(5, a)$ | 40             | $-320$      | $-a^2 + 3a - 5$ | (1) | 40         |
| $-475$      | $-4a^2 + 16a - 15$ | $(5, a)$ | 56             | $-320$      | $-5a^2 + 23a - 25$ | (1) | 24         |
| $-432$      | $-3a^2 - 3a + 6$ | $(a - 2)$ | 64             | $-304$      | $-a^2 + 7a - 13$ | $(2, a^2 + a - 5)$ | 48 |
| $-432$      | $-3a^2 + 12a - 12$ | $(a - 2)$ | 16/3           | $-304$      | $-a^2 - a - 1$ | $(2, a^2 + a - 5)$ | 32 |
| $-412$      | $-3a^2 + a + 2$ | $(2, a)$ | 80             | $-236$      | $a^2 + a - 14$ | $(2, a)$ | 48 |
| $-412$      | $-7a^2 + 28a - 24$ | $(2, a)$ | 32             | $-236$      | $a^2 + 8a - 32$ | $(2, a)$ | 32 |
| $-404$      | $-2a^2 - 2a + 2$ | (1) | 32             | $-104$      | $a^2 - 12$ | (1) | 8         |
| $-404$      | $-3a^2 + 13a - 14$ | (1) | 32             | $-104$      | $2a^2 + 3a - 26$ | (1) | 24         |
| $-359$      | $-4a^2 + 15a - 13$ | (1) | 16             | $-95$       | $a^2 + 2a - 15$ | (1) | 40         |
| $-359$      | $-4a^2 - 4a + 9$ | (1) | 80             | $-95$       | $-a - 5$ | (1) | 8         |
| $-352$      | $-7a^2 + 29a - 26$ | $(2, a)$ | 72             | $-76$       | $-2a^2 + 7a - 6$ | $(2, a)$ | 32 |
| $-352$      | $-2a^2 + 3a - 2$ | $(2, a)$ | 24             | $-76$       | $-3a^2 - 4a + 8$ | $(2, a)$ | 16 |
| $-347$      | $-3a^2 + 2a + 1$ | (1) | 64             | $-64$       | $4a - 12$ | (1) | 8         |
| $-347$      | $-8a^2 + 32a - 27$ | (1) | 16             | $-64$       | $-4$ | (1) | 16         |
| $-343$      | $7a - 21$ | (1) | 16             | $-27$       | $3a - 9$ | (1) | 16         |
| $-343$      | $-7$ | (1) | 48             | $-27$       | $-3$ | (1) | $\frac{16}{3}$ |
Table 4. The first fields \( K = \mathbb{Q}[x]/(f) \) which contain “unit discriminants \( \Delta \neq 1 \), i.e. discriminants such that \( \Delta/\Delta_{K} \) is trivial. The columns list the signature, discriminant, defining polynomial \( f \), elementary divisors of the class group and the narrow class group, and the unit discriminants (modulo squares) of \( K \). Discriminants which are not totally positive are marked by a *.

| sign. | \( D_K \) | \( f \) | \( \text{Cl}(K) \) | \( \text{Cl}^{+}(K) \) | discriminants |
|-------|--------|--------|----------------|----------------|-----------|
| 0,1   | −15    | \( x^2 - x + 4 \) | [2] | [2] | 1, −a + 1 |
|       | −84    | \( x^2 + 21 \) | [2, 2] | [2, 2] | 1, −1, 3, −3 |
|       | −420   | \( x^2 + 105 \) | [2, 2, 2] | [2, 2, 2] | 1, −1, 3, −3, 5, −5, 7, −7 |
| 2,0   | 40     | \( x^2 - 10 \) | [2] | [2] | 1, 2 |
|       | 60     | \( x^2 - 15 \) | [2, 2] | [2, 2] | 1, −1*, 2a + 8, −2a − 8* |
|       | 780    | \( x^2 - 195 \) | [2, 2] | [2, 2] | 1, −1*, 2a + 28, −2a − 28*, 3, −3*, 5, −5* |
| 1,1   | −283   | \( x^3 + 4x - 1 \) | [2] | [2] | 1, a |
|       | −6571  | \( x^3 - x^2 - 9x - 16 \) | [2, 2] | [2, 2] | 1, −3a^2 − 8a + 84, a + 1, 4a^2 − 8a − 35 |
|       | −300551| \( x^3 + 99x - 169 \) | [2, 2, 2] | [2, 2, 2] | 1, a^2 − 4a − 36, a + 3, −a^2 + a + 61, a + 7, 5a^2 − 15a − 83, a + 11, 7a^2 − 31a − 227 |
| 3,0   | 1957   | \( x^3 - x^2 - 9x + 10 \) | [2] | [4] | 1, −a + 3 |
|       | 7537   | \( x^3 - x^2 - 24x + 35 \) | [2, 2] | [2, 2] | 1, −a − 3*, 2a^2 + 9a + 10, −17a^2 − 8a − 100 |
|       | 210649 | \( x^3 - x^2 - 140x - 587 \) | [2, 2, 2] | [2, 2, 2] | 1, −a − 7*, 2a^2 + 25a + 78, −41a^2 − 533a + 1720, a^2 − 6a − 71, 2a^2 − 9a + 10, −a + 7, 5a^2 + 66a + 217 |
| 0,2   | 1521   | \( x^4 - x^3 + 4x^2 + 3x + 9 \) | [2] | [2] | 1, −a + 1 |
|       | 18000  | \( x^4 + 15x^2 + 45 \) | [2, 2] | [2, 2] | 1, −1, −4a^2, 4^2 |
|       | 112806 | \( x^4 + 20x^2 + 121 \) | [2, 2, 2] | [2, 2, 2] | 1, −1, 2^2, 5^2, 9^2 |
| 2,1   | −6848  | \( x^4 - 2x^3 + 5x^2 - 2x - 1 \) | [2] | [2] | 1, −a + 1 |
|       | −12375 | \( x^4 - x^3 - 4x^2 - 11x - 29 \) | [2, 2] | [2, 2] | 1, −1^*a^3 + 2a^2 + a + 12, a^2 + 1^*a + a + 12 |
|       | −198000| \( x^4 - 15x^2 - 495 \) | [2, 2, 2] | [2, 2, 2] | 1, −1^*a^3 + 2a^2 + a + 12, a^2 + 1^*a + a + 12 |
| 4,0   | 21025  | \( x^4 - 17x^2 + 36 \) | [2] | [2] | 1, −1^*a^3 + 2a^2 + 5^2a + 1 |
|       | 32625  | \( x^4 - x^3 - 19x^2 + 4x + 76 \) | [2, 2] | [2, 2] | 1, −1^*a^3 + 2a^2 + 5^2a + 1^*a + 12, 2a^2 − 3a + 2^6* |
|       | 176400 | \( x^4 - 19x^2 + 64 \) | [2, 2, 4] | [2, 2, 4] | 1, −1^*a^3 + 2a^2 − 2a + 7, a^2 + 4^2a + 1^*a + 4, a^2 + a^2 − 4, a^2 + a^2 |
| 1,2   | 41381  | \( x^5 - x^4 - 2x^2 + 4x - 1 \) | [2] | [2] | 1, a |
| 3,1   | −243219| \( x^5 - 2x^4 + 2x^3 - 12x^2 + 21x - 9 \) | [2] | [2] | 1, a^4 + 2a^2 − 9a + 5 |
|       | −802663| \( x^5 - x^4 - 5x^3 - 6x^2 - 3x + 1 \) | [2, 2] | [2, 2] | 1, a^3 − 2a^2 − 4a, a + 2, a^4 − 8a^2 − 8a^* |
| 5,0   | 4010276| \( x^5 - 11x^4 - 9x^2 + 14x + 9 \) | [2] | [4] | 1, a^3 + 2a^2 − a − 1 |
|       | 5229109| \( x^5 - x^4 - 12x^3 + 26x^2 - 12x - 1 \) | [2, 2] | [2, 2] | 1, a^4 + a^3 − 3a^2 + 6a + 2, 2a^4 + 3a^3 − 20a^2 + 14a + 1 |
|       | 15216977| \( x^5 - 2x^4 - 9x^3 + 2x^2 + 8x - 1 \) | [2, 2, 2] | [2, 2, 2] | 1, a^2 + a, 3a^4 − 12a^3 − 3a^2 + 13a − 2^*, 6a^3 − 23a^2 + 7a^2 + 25a − 3^*, | 2a^4 + 16a^3 + 5a^2 − 16a + 1, −8a^4 + 32a^3 + 9a^2 − 35a + 4, a^3 − 2a^2 + a^2, a^3 − 3a^2 + 2a^* |
5. Appendix: Hilbert Symbol and Higher Unit Groups

We give a short self-contained proof of a property of the Hilbert symbol in dyadic number fields\(^5\) (Theorem 24 below) which we needed for the calculation of the conductors of the characters \((\pi)\) in Section 2.6.

In this section \(K\) denotes a finite extension of \(\mathbb{Q}_2\) with ring of integers \(\mathfrak{o}\) and prime element \(\pi\). We let \(e\) be the ramification index of \(K\), i.e. the largest integer such that \(\pi^e \mid 2\). We use \((\pi)\) for the quadratic Hilbert symbol of \(K\). Recall that this is the map \(K^\times \times K^\times \to \{\pm 1\}\) such that \((a, b)_K = 1\) if and only if \(ax^2 + by^2 = 1\) has solutions \(x, y \in K\). The Hilbert symbol is bilinear (see e.g. [O'M00, Prop. 57:10 and p. 166]), it obviously factors through a bilinear form on \(K^\times / K^\times 2\), and this form is non-degenerate, i.e. \((a, b)_K = 1\) for all \(b\) is only possible if \(a\) is a square in \(K\) (see e.g. [O'M00, 63:13] for a short proof).

We set \(U_0 := \mathfrak{o}^\times\), and, for \(n \geq 1\), let \(U_n = 1 + \pi^n\mathfrak{o}\) be the \(n\)th higher unit group of \(K\). Clearly \(U_n \supseteq U_{n+1}\) and \(U_k \subseteq U_2^k\). The Local Square Theorem states that \(U_{2^k+1} = U_{2^k+1}^2\). (Recall a simple proof: \(1 + 4\pi X = \sum_{n \geq 0} (1/4)(4\pi X)^n = 1 + 4\pi X + (4\pi X)^2\) in the ring of formal power series \(\mathfrak{o}[X]\), and the series converges for each \(x \in \mathfrak{o}\) with respect to the valuation \(v\) of \(K\) (and then towards an element of \(U_{e+1}\)) since \(v((1/4)(4\pi x)^n) \geq n(e + 1) - v(n)\) and, by Legendre’s formula, \(v(n) = (n - s_n)e\), where \(s_n\) is the sums of \(1\)'s is the binary expansion of \(n\), so that \(v(n) \leq (n-1)e\).

**Lemma 20.** For \(0 \leq k \leq e-1\), one has

\[
U_{2k} = U_{2k+1}U_k^2.
\]

**Proof.** Let \(a \in U_{2k}\). The congruence \(a \equiv (1 + \pi^{k}t)^2 \mod \pi^{2k+1}\) is equivalent to \(t^2 \equiv (a-1) / \pi^{2k}\) mod \(\pi^{2k+1}\) (since the assumption \(k < e\) implies \(\pi^{2k+1} \mid 2\pi^k\)), and this has a solution \(t\) (as the map \(t \mapsto t^2\) defines an automorphism of \(\mathfrak{o}/\pi\)). With such a solution \(t\), we have \(a / (1 + \pi^{k}t)^2 \equiv 1 \mod \pi^{2k+1}\), i.e. \(a \in U_{2k+1} (1 + \pi^{k}t)^2\). For proving the inverse inclusion it suffices to note that \(U_k^2 \subseteq U_{2k}\). Indeed, for any \((1 + \pi^{k}t) \in U_k\), we have \((1 + \pi^{k}t)^2 \equiv 1 \mod \pi^{2k}\), since \(\pi^k \mid 2\).

**Proposition 21.** For any pair of even integers \(i, j \geq 0\) with \(i + j = 2e\), one has \((U_i, U_j)_{K} = 1\).

**Proof.** We can assume that \(0 \leq i \leq e\). Let \(a \in U_i\) and \(b \in U_j\). We have to show that \(ax^2 + by^2 = 1\) has solutions \(x, y \in K\). We can assume by Lemma 20 that \(a \in U_{i+1}\). But then \(a + b - ab = 1 - (a-1)(b-1)\) is in \(U_{2e+1}\), hence, by the Local Square Theorem \(a + b - ab = w^2\) for a unit \(w\). But then again \(a(w + b)^2 + b(w - a)^2 = (a + b)^2\), which implies the claim.

**Lemma 22.** One has \(U_{2k}/U_k^2 \cong \mathbb{F}_2\).

**Proof.** The application \(1 + 4x \mapsto \text{tr}_{\mathfrak{o}/\pi}/\mathbb{F}_2 \overline{x}\) (where \(\overline{x}\) is the residue class of \(x\) mod \(\pi\)) defines an epimorphism of \(U_{2k}\) onto \(\mathbb{F}_2\). We claim that its kernel equals \(U_k^2\). It contains \(U_k^2\) since \((1 + cy)^2 = 1 + 4(y + y^2)\) and \(\text{tr}_{\mathfrak{o}/\pi}/\mathbb{F}_2(\overline{y} + \overline{y}^2) = 0\). Vice versa, if \(1 + 4x = 0\) then \(1 + 4x \equiv (1 + 2y)^2 \mod 4\pi\), i.e. \(x \equiv y + y^2 \mod \pi\), or equivalently \((1 + 4x)/(1 + 2y)^2 \in U_{2e+1} = U_{e+1}^2\), has a solution \(y\). Namely, \(\overline{y} \mapsto \overline{y} + \overline{y}^2\) defines an \(\mathbb{F}_2\)-linear endomorphism of \(\mathfrak{o}/\pi\) with kernel \(\mathbb{F}_2\) and image equal to the the kernel of \(\text{tr}_{\mathfrak{o}/\pi}/\mathbb{F}_2\).

We use \(\overline{U}_n\) for the image of \(U_n\) under the canonical map \(K^\times \to K^\times / K^\times 2\). Note that \(\overline{U}_n = U_n K^\times 2 / K^\times 2\). for \(0 \leq k \leq e-1\), Lemma 20 implies \(U_{2k} K^\times 2 = U_{2k+1} K^\times 2\), and we conclude

\[
\overline{U}_{2k} = \overline{U}_{2k+1}.
\]

\(^5\)We thank Chandan Singh Dalawat for a helpful discussion on this matter [hb].
We set
\[(14) \quad V_{-1} := K^\times / K^{x,2}, \quad V_k := U_{2k}K^{x,2}/K^{x,2} \quad (0 \leq k \leq e), \quad V_{e+1} := 1.\]

**Lemma 23.** One has
\[1 = \mathcal{V}_{e+1} \subseteq \mathcal{V}_e \subseteq \mathcal{V}_{e-1} \cdots \subseteq \mathcal{V}_2 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_0 = K^\times / K^{x,2},\]
where “\(\subseteq_n\)” means “is subgroup of index \(n\),” and where \(f\) is the degree of \(\pi\), i.e. \(2f = \text{card}(\mathfrak{a}/\mathfrak{p})\). One has, in particular, for \(0 \leq k \leq e,\)
\[(15) \quad \text{card} (V_k) = 2 \cdot 2^{f(e-k)}.\]

**Proof.** The first equality in the filtration chain, i.e. that the elements of \(U_{2e+1}\) are squares, is the Local Square Theorem.

For the first “\(\subseteq_2\)” we note that \(U_{2e} \cap K^{x,2} = U_{e}^2\), whence
\[U_{2e}/U_{e}^2 \cong U_{2e}K^{x,2}/K^{x,2},\]
and apply Lemma 22.

For the last “\(\subseteq_2\)” note that \(K^\times = (\pi) \times U_0\), from which we deduce \(K^\times / K^{x,2} = (\overline{\pi}) \times U_0\), where \((\overline{\cdot})\) is the canonical projection.

For the “\(\subseteq_2\)” we calculate
\[V_{k-1}/V_k = U_{2k-1}/U_{2k} \cong U_{2k-1}K^{x,2}/U_{2k}K^{x,2} \cong U_{2k-1}/U_{2k} \cong 0/\pi 0\]
The last two isomorphisms are from right to left: \(a + \pi 0 \mapsto (1 + a\pi 2^{k-1})U_{2k}\) and \(U_{2k} \mapsto aU_{2k}K^{x,2}\). Note that the second application defines indeed an isomorphism: It is obviously surjective. For proving that it is injective suppose, \(uU_{2k}\), for a given \(u\) in \(U_{2k-1}\), is mapped to \(U_{2k}K^{x,2}\), i.e. \(u = va^2\) for some \(v\) in \(U_{2k}\) and \(a\) in \(K^\times\). But then \(a\) must be unit, and since \(a^2 \equiv 1 \mod \pi 2^{k-1}\) and \(k \leq e\), we conclude that, in fact, \(a \equiv 1 \mod \pi V\), which in turn implies that \(a^2\) is in \(U_{2k}\) (again since \(k \leq e\); hence \(u = va^2\) is in \(U_{2k}\).

**Theorem 24.** For the subspaces of the filtration (14) of \(K^\times / K^{x,2}\), one has
\[V_k^# = V_{e-k},\]
for every \(-1 \leq k \leq e + 1\) (Here \(V_k^#\) denotes the subgroup of all \(aK^{x,2}\) in \(V_{-1} = K^\times / K^{x,2}\) such that \((a,b)_K = 1\) for all \(bK^{x,2}\) in \(V_k\).)

**Proof.** The Hilbert symbol defines a non-degenerate bilinear form on the \(\mathbb{F}_2\)-vector space \(V_{-1}\). From (15) we know
\[\dim_{\mathbb{F}_2} V_k + \dim_{\mathbb{F}_2} V_{e-k} = \dim_{\mathbb{F}_2} V_{-1},\]
and hence \(\dim_{\mathbb{F}_2} V_k^# = \dim_{\mathbb{F}_2} V_{e-k}\). It suffices therefore to prove \(V_k^# \supseteq V_{e-k}\). For \(k = -1\) or \(k = e + 1\) this is trivial. For \(0 \leq k \leq e\) the inclusion is equivalent to \((U_{2k}, U_{2e-2k})_{K} = 1\). But this is Proposition 21.

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