CGC/saturation approach for high energy soft interactions: $v_2$ in proton-proton collisions

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In this paper we continue our program to construct a model for high energy soft interactions, based on the CGC/saturation approach. We demonstrate that in our model which describes diffractive physics as well as multi-particle production at high energy, the density variation mechanism leads to the value of $v_2$ which is about 60% ÷ 70% of the measured $v_2$. Bearing in mind that in CGC/saturation approach there are two other mechanisms present: Bose enhancement in the wave function and local anisotropy, we believe that the azimuthal long range rapidity correlations in proton-proton collisions stem from the CGC/saturation physics, and not from quark-gluon plasma production.

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INTRODUCTION

The large body of experimental data on soft interactions at high energy, presently, cannot be described in terms of theoretical high energy QCD (see [10] for a review).

In this paper we continue our effort [11–15] to comprehend such interactions, by constructing a model that incorporates the advantages of two theoretical approaches to high energy QCD: the CGC/saturation approach [16–22] and the BFKL Pomeron calculus [16, 23–32]. Both provide an effective theory of QCD at high energies. However, the interpretation of processes at high energy appear quite different in each, since they have different structural elements.

The CGC/saturation approach describes the high energy interactions in terms of colorless dipoles, their density, distribution over impact parameters, evolution in energy and so on. Such a description appears quite natural in perturbative QCD, and can be easily applied to multi-particle production at high energy. In this approach a new saturation scale $Q_s$ occurs which is much larger than the soft scale, and the description of the production of quarks and gluons can be attained theoretically, in an economical way. The transition from quarks and gluons to hadrons, has to be handled phenomenologically using data from hard processes. This approach leads to a good description of the experimental data on inclusive production, and the observation of some regularities in the data, such as geometric scaling behaviour [33–38].

BFKL Pomeron calculus which deals with BFKL Pomerons and their interactions, is similar to the old Reggeon theory [39], and is suitable for describing diffractive physics and correlations in multi-particle production, as we can use the Mueller diagram technique [40]. The relation between these two approaches has not yet been established, but they are equivalent [51] for rapidities ($\ln (s/s_0)$), such that

$$Y \leq \frac{2}{\Delta_{BFKL}} \ln \left( \frac{1}{\Delta_{BFKL}^2} \right)$$

where $\Delta_{BFKL}$ denotes the intercept of the BFKL Pomeron. As we have discussed [11], the parameters of our model are such that for $Y \leq 36$, we can trust our approach, based on the BFKL Pomeron calculus.

This paper is the next step in our program to construct a model for high energy soft scattering, based on an analytical calculation, without using Monte Carlo simulations.

One of the most intriguing experimental observations made at the LHC and RHIC, is the same pattern of azimuthal angle correlations in the three types of interactions: hadron-hadron, hadron-nucleus and nucleus-nucleus collisions. In all three reactions, correlations in the events with large density of produced particles, are observed between two charged hadrons, which are separated by the large values of rapidity. Further, we believe that the results of these experiments provide strong evidence, that the underlying physics is the same for all three reactions, and is due the high density partonic state that has been produced at high energies, in all three reactions. Due to causality arguments [47] two hadrons with large difference in rapidity between them, could only correlate at the early stage of the collision and, therefore, we expect that the correlations between two particles with
large rapidity difference (at least the correlations in rapidity) are due to the partonic state with large parton density. The parton (gluon) density is governed in QCD by the CGC/saturation non-linear equations and therefore, the CGC approach is the appropriate tool to study these correlations.

Unlike, the rapidity correlations at large values of the rapidity difference, which stem from the initial state interactions, the azimuthal angle correlations can originate from the collective flow in the final state [48]. Nevertheless, in this paper we would like to analyze the same mechanism for both correlations: i.e. the initial state interaction in the CGC phase of QCD. However, even in the framework of the saturation/CGC approach, we presently are not able to propose a unique mechanism for the azimuthal angle correlations. At the moment three sources for the azimuthal angle correlation have been suggested: [74] Bose enhancement in the wave function [50], local anisotropy [51, 52] and density variation [36]. We cite only a restricted number of papers for each approach. The reader can find more references, and more ideas on the origin of the correlations in the review papers Refs. [52–58].

The goal of this paper is to study in more detail the density variation mechanism proposed in Ref. [36]. In this approach, both rapidity and azimuthal angle correlations originate from two gluons production from two parton showers.

This process can be calculated using Mueller diagrams [40] (see Fig. 2.a). The difference between rapidity and azimuthal angle correlations is only in the form of the Mueller vertices in this figure. For rapidity correlations, such vertices can be considered as being independent of $Q_T$, while for the azimuthal angle correlation, this vertex is proportional to $(Q_T \cdot p_{1,\perp})^2$ or $(Q_T \cdot p_{2,\perp})^2$. The integration over the direction of $Q_T$ leads to the term $(p_{1,\perp} \cdot p_{2,\perp})^2$, which is proportional to $\cos 2\phi$, resulting in azimuthal angle correlations. The strength of the term $\cos 2\phi$ is proportional to $\langle Q_T^2 \rangle^2$, where averaging is over the wave function of the one parton shower, which is described by the BFKL Pomeron. In other words since $Q_T = iv\cdot b$, where $b$ is the impact factor for the scattering process, the magnitude of the azimuthal angle correlation depends on the gradient of the parton density.

**THE BRIEF REVIEW OF THE MODEL**

In this section we briefly review our model, which provides a successful description of the diffractive [11, 12], inclusive cross sections [13] and rapidity correlations [14]. For the description of the angular correlations, it is important to take into account the dependance of the scattering amplitude on the sizes of dipoles, hence in all formulae below, we include this dependance. In our description of the cross sections we previously took $r = 1/m$.

The main ingredient of our model is the BFKL Pomeron Green function, which we determined using the CGC/saturation approach [11, 32]. We calculated this function from the solution of the non-linear Balitsky-Kovchegov equation [19, 21], using the MPSI approximation [59] to sum enhanced diagrams shown in Fig. 1-a. It has the following form:

$$
G^{dressed}(T) = a^2(1 - \exp(-T)) + 2a(1-a)\frac{T}{1+T} + (1-a)^2G(T)
$$

with

$$
G(T) = 1 - \frac{1}{T} \exp \left( \frac{1}{T} \right) \Gamma_0 \left( \frac{1}{T} \right)
$$

$$
T(r_\perp, s, b) = \phi_0 \left( r_\perp^2 Q_s^2 (Y, b) \right) \tilde{\gamma}
$$

where the saturation momentum $Q_s$ is given by

$$
Q_s^2(b,Y) = Q_{0s}^2 (b, Y_0) e^{\lambda (Y - Y_0)}
$$

with

$$
Q_{0s}^2 (b, Y_0) = \left( m^2 \right)^{1-1/\tilde{\gamma}} \left( S(b, m) \right)^{1/\tilde{\gamma}}
$$

$$
S(b, m) = \frac{m^2}{2\pi} e^{-mb} \quad \text{and} \quad \tilde{\gamma} = 0.63
$$

$T(r_\perp, s, b)$ denotes the dipole - proton amplitude in the vicinity of the saturation scale [60].

In these formulae we take $a = 0.65$, this value was chosen, to reproduce the analytical form for the solution of the BK equation. Parameters $\lambda$ and $\phi_0$, can be estimated in the leading order of QCD, but due to large next-to-leading order corrections, we treat them as parameters of the fit. $m$ is a non-perturbative parameter, which characterizes the large impact parameter behavior of the saturation momentum, as well as the typical size of dipoles that take part in the interactions. The value of $m = 5.25 \text{GeV}$ in our model, justifies our main assumption, that BFKL Pomeron
calculus based on a perturbative QCD approach, is able to describe soft physics, since \( m \gg \mu_{\text{soft}} \), where \( \mu_{\text{soft}} \) denotes the natural scale for soft processes \( \mu_{\text{soft}} \sim \Lambda_{\text{QCD}} \) and/or pion mass.

Unfortunately, since the confinement problem is far from being solved, we assume a phenomenological approach for the structure of the colliding hadrons. We use a two channel model, which allows us to calculate the diffractive production in the region of small masses. In this model, we replace the rich structure of the diffractively produced states, by a single state with the wave function \( \psi_D \), a la Good-Walker[61]. The observed physical hadronic and diffractive states are written in the form

\[
\psi_h = \alpha \Psi_1 + \beta \Psi_2; \quad \psi_D = -\beta \Psi_1 + \alpha \Psi_2; \quad \text{where} \quad \alpha^2 + \beta^2 = 1; \tag{6}
\]

Functions \( \psi_1 \) and \( \psi_2 \) form a complete set of orthogonal functions \( \{\psi_i\} \) which diagonalize the interaction matrix \( T \)

\[
A_{i,k}^{b'} = \langle \psi_i | T | \psi_{b'} \rangle = A_{i,k} \delta_{i,b'} \delta_{k,b'}. \tag{7}
\]

The unitarity constraints take the form

\[
2 \text{Im} A_{i,k} (s,b) = |A_{i,k} (s,b)|^2 + G_{i,k}^{\text{in}} (s,b), \tag{8}
\]

where \( G_{i,k}^{\text{in}} \) denotes the contribution of all non diffractive inelastic processes, i.e. it is the summed probability for these final states to be produced in the scattering of a state \( i \) off a state \( k \). In Eq. (8) \( \sqrt{r} = W \) denotes the energy of the colliding hadrons, and \( b \) the impact parameter. A simple solution to Eq. (8) at high energies, has the eikonal form with an arbitrary opacity \( \Omega_{i,k} \), where the real part of the amplitude is much smaller than the imaginary part.

\[
A_{i,k} (s,b) = i\left(1 - \exp (-\Omega_{i,k} (s,b))\right), \tag{9}
\]

\[
G_{i,k}^{\text{in}} (s,b) = 1 - \exp (-2\Omega_{i,k} (s,b)). \tag{10}
\]

Eq. (10) implies that \( P_{i,k} = \exp (-2\Omega_{i,k} (s,b)) \), is the probability that the initial projectiles \( (i,k) \) reach the final state interaction unchanged, regardless of the initial state re-scatterings.

Note, that there is no factor 1/2, its absence stems from our definition of the dressed Pomeron.

| model | \( \lambda \) | \( \phi_0 \) (GeV\(^{-2}\)) | \( g_1 \) (GeV\(^{-1}\)) | \( g_2 \) (GeV\(^{-1}\)) | \( m_1 \) (GeV) | \( m_2 \) (GeV) | \( \beta \) | \( \alpha_{FP} \) |
|-------|------|-----------------|-----------------|-----------------|--------|--------|--------|--------|
| 2 channel | 0.38 | 0.0019 | 110.2 | 11.2 | 5.25 | 0.92 | 1.9 | 0.58 | 0.21 |

**TABLE I**: Fitted parameters of the model. The values are taken from Ref.[12].

In the eikonal approximation we replace \( \Omega_{i,k} (r_\perp,s,b) \) by

\[
\Omega_{i,k} (r_\perp,Y - Y_0,b) = \int d^2 b' d^2 b'' g_i (b') G^{\text{dressed}} (T (r_\perp,Y - Y_0,b'')) g_k (b - b' - b'') \tag{11}
\]

We propose a more general approach, which takes into account new small parameters, that come from the fit to the experimental data (see Table I and Fig. 1):

\[
G_{3FP}/g_i (b = 0) \ll 1; \quad m \gg m_1 \text{ and } m_2 \tag{12}
\]

The second equation in Eq. (12) leads to the fact that \( b'' \) in Eq. (11) is much smaller than \( b \) and \( b' \), therefore, Eq. (11) can be re-written in a simpler form

\[
\Omega_{i,k} (r_\perp,Y - Y_0,b) = \left( \int d^2 b'' G^{\text{dressed}} (T (r_\perp,Y - Y_0,b'')) \right) \int d^2 b' g_i (b') g_k (b - b') \tag{13}
\]

Selecting the diagrams using the first equation in Eq. (12), indicates that the main contribution stems from the net diagrams shown in Fig. 1b. The sum of these diagrams leads to the following expression for \( \Omega_{i,k} (s,b) \)

\[
\Omega (r_\perp,Y - Y_0;b) = \int d^2 b' \frac{g_i (b') g_k (b - b') \tilde{G}^{\text{dressed}} (r_\perp,Y - Y_0)}{1 + G_{3FP} \tilde{G}^{\text{dressed}} (r_\perp,Y - Y_0) [g_i (b') + g_k (b - b')]}, \tag{14}
\]

\[
g_i (b) = g_i S_p (b; m_i); \tag{15}
\]
FIG. 1: Fig. 1-a shows the set of the diagrams in the BFKL Pomeron calculus that produce the resulting (dressed) Green function of the Pomeron in the framework of high energy QCD. In Fig. 1-b the net diagrams which include the interaction of the BFKL Pomerons with colliding hadrons are shown. The sum of the diagrams reduces to Fig. 1-c after integration over positions of $G_{3P}$ in rapidity.

where

$$S_p(b,m_i) = \frac{1}{4\pi}m_i^3bK_1(m_ib)$$

$$\tilde{G}_\text{dressed}(r_\perp,Y-Y_0) = \int d^2bG_\text{dressed}(T(r_\perp,Y-Y_0,b))$$

where $T(r_\perp,Y-Y_0,b)$ is given by Eq. (3).

Note that $\tilde{G}_\text{dressed}(T)$ does not depend on $b$. In all previous formulae, the value of the triple BFKL Pomeron vertex is known: $G_{3P} = 1.29 \text{GeV}^{-1}$.

To simplify further discussion, we introduce the notation

$$N^{BK}(G_{3P}(r_\perp,Y,b)) = a(1 - \exp(-G_{3P}(r_\perp,Y,b))) + (1-a)\frac{G_{3P}^a(r_\perp,Y,b)}{1 + G_{3P}^a(r_\perp,Y,b)}$$

(17)

with $a = 0.65$. Eq. (17) is an analytical approximation to the numerical solution for the BK equation.

For the elastic amplitude we have

$$a_{el}(b) = (a^4A_{1,1} + 2a^2\beta^2A_{1,2} + \beta^4A_{2,2}).$$

(18)

CORRELATIONS BETWEEN TWO PARTON SHOWERS

In our previous paper, we discovered that in the framework of our model that has been described above, the main source of the long range rapidity correlation, is the correlation between two parton showers. The appropriate Mueller diagrams are shown in Fig. 2. Examining this diagram, we see that the contribution to the double inclusive cross section, differs from the product of two single inclusive cross sections. This difference generates the rapidity correlation function, which is defined as

$$R(y_1,y_2) = \frac{1}{\sigma_\text{in} \sigma_\text{in}} \frac{d^2\sigma}{dy_1 dy_2} - 1$$

(19)

There are two reasons for the difference between the double inclusive cross section due to production of two parton showers, and the products of inclusive cross sections: the first, is that in the expression for the double inclusive cross section, we integrate the product of the single inclusive inclusive cross sections, over $b$. The second, is that the summation over $i$ and $k$ for the product of single inclusive cross sections, is for fixed $i$ and $k$. 
Introducing the following new function, enables us to write the analytical expression for the double inclusive cross section:

\[ I^{i,k}(y, b) = a_{PP} \ln \left( \frac{W}{W_0} \right) \]

\[ \times \int d^2b' N^{BK} \left( g^{(i)} S_{m_i, b'} \tilde{G}^{dressed} \left( r_\perp = \frac{1}{m}, \frac{1}{2}Y + y \right) \right) \]

\[ N^{BK} \left( g^{(k)} S_{m_k, b - b'} \tilde{G}^{dressed} \left( r_\perp = \frac{1}{m}, \frac{1}{2}Y - y \right) \right) \]

Using Eq. (20) we can write the double inclusive cross section in the form

\[ \frac{d^2 \sigma^2 \text{ parton showers}}{dy_1 dy_2} = \int d^2p_{1T} d^2p_{2T} \frac{d^2 \sigma^2 \text{ parton showers}}{dy_1 dy_2 d^2p_{1T} d^2p_{2T}} = \int d^2b \left\{ \alpha^4 I^{(1,1)}(y_1, b) I^{(1,1)}(y_2, b) + \beta^4 I^{(2,1)}(y_1, b) I^{(2,1)}(y_2, b) + \beta^4 I^{(2,2)}(y_1, b) I^{(2,2)}(y_2, b) \right\} \]

Comparing Eq. (21) with the square of the single inclusive cross section (see below Eq. (23)), we note the different powers of \( \alpha \) and \( \beta \), which reflect the different summation over \( i \) and \( k \), as well as different integration over \( b \).
Other sources can contribute to the correlation function \( R(y_1, y_2) \) which is defined as

\[
R(y_1, y_2) = \sigma_{NSD} \left\{ \frac{d^2 \sigma^2 \text{ parton showers}}{dy_1 dy_2} + \frac{d^2 \sigma^1 \text{ parton shower}}{dy_1 dy_2} + \frac{d^2 \sigma^1 \text{ parton shower}}{dy_1 dy_2} \right\} \bigg/ \left( \frac{d\sigma}{dy_1 dy_2} \right) - 1
\]  

(22)

In Ref. [14] we showed that both semi-enhanced and enhanced diagrams which are related to the correlations in one patron shower, give negligible contributions, and can be neglected.

We have discussed in Ref. [14] the rapidity correlations that are generated by Eq. (21). In the present paper we wish to consider the correlations in the azimuthal angle between two momenta of produced gluons: \( p_{1,\bot} \) and \( p_{2,\bot} \).

The single inclusive cross section can be calculated using the following formula [13]:

\[
\frac{d\sigma}{dy} = \int d^2 p_T \frac{d\sigma}{dy d^2 p_T} = a_{F,F'} \ln (W/W_0) \left\{ \alpha^4 I_n^{(1)} \left( \frac{1}{2} Y + y \right) I_n^{(1)} \left( \frac{1}{2} Y - y \right) 
+ \alpha^2 \beta^2 \left( I_n^{(1)} \left( \frac{1}{2} Y + y \right) I_n^{(2)} \left( \frac{1}{2} Y - y \right) + I_n^{(2)} \left( \frac{1}{2} Y + y \right) I_n^{(1)} \left( \frac{1}{2} Y - y \right) \right) 
+ \beta^4 I_n^{(2)} \left( \frac{1}{2} Y + y \right) I_n^{(2)} \left( \frac{1}{2} Y - y \right) \right\}
\]

(23)

where \( Y \) denotes the total rapidity of the colliding particles, and \( y \) is the rapidity of produced hadron. \( I_n^{(i)}(y) \) is given by

\[
I_n^{(i)}(y) = \int d^2 b \, N^{BK} \left( g^{(i)} S (m_i, b) \tilde{G}_F (r_\perp = 1/m, Y - Y_0) \right)
\]

(24)

and \( a_{F,F'} \) is a fitted parameter, that was determined in Ref. [13] (see Table 1).

**CALCULATION OF THE FIRST DIAGRAM**

In this section we calculate the first Mueller diagram shown in Fig. 2-a. We reproduce in an alternative way, the main results of Ref. [39]. We start from the calculation of the inclusive production, from one BFKL Pomeron, which enters this diagram at fixed momentum transfer \( Q_T \). Note, that the inclusive cross section, shown in Fig. 3, is determined by the same BFKL Pomeron, but at \( Q_T = 0 \).

**Inclusive production from one BFKL Pomeron**

*The BFKL Pomeron: generalities*

The general solution to the BFKL equation for the scattering amplitude of two dipoles with the sizes \( r_1 \) and \( r_2 \), has been derived in Ref. [24], and has the form

\[
N_F \left( r_1, r_2; Y, b \right) = \sum_{n=0}^{\infty} \int \frac{d\gamma}{2\pi i} \phi_{in}^{(a)}(\gamma; r_2) \, d^2 R_1 \, d^2 R_2 \, \delta (R_1 - R_2 - b) \, e^{\omega(\gamma,n) Y} \, E^{\gamma,n} (r_1, R_1) \, E^{1-\gamma,n} (r_2, R_2)
\]

(25)

with

\[
\omega(\gamma,n) = \bar{\alpha}_S \chi(\gamma,n) = \bar{\alpha}_S \left( 2 \psi(1) - \psi(\gamma + |n|/2) - \psi(1 - \gamma + |n|/2) \right);
\]

(26)

where \( \psi(\gamma) = d \ln \Gamma(\gamma)/d\gamma \) and \( \Gamma(\gamma) \) is Euler gamma function. Functions \( E^{\gamma,n}(\rho_{1a}, \rho_{2a}) \) are given by the following equations.

\[
E^{\gamma,n}(\rho_{1a}, \rho_{2a}) = \left( \frac{\rho_{12}}{\rho_{1a} \rho_{2a}} \right)^{1-\gamma+n/2} \left( \frac{\rho_{12}}{\rho_{1a} \rho_{2a}} \right)^{1-\gamma-n/2},
\]

(27)
In Eq. (27) we use complex numbers to characterize the point on the plane

$$\rho_1 = x_{i,1} + i x_{i,2}; \quad \rho_1^* = x_{i,1} - i x_{i,2}$$

(28)

where the indices 1 and 2 denote two transverse axes. Notice that

$$\rho_{12} \rho_{12}^* = r_i^2; \quad \rho_{1a} \rho_{1a}^* = \left( R_i - \frac{1}{2} r_i \right)^2 \quad \rho_{2a} \rho_{2a}^* = \left( R_i + \frac{1}{2} r_i \right)^2$$

(29)

At large values of \( Y \), the main contribution stems from the first term with \( n = 0 \). For this term Eq. (27) can be re-written in the form

$$E^{\gamma,0}(r_1, R_i) = \left( \frac{r_i^2}{(R_i + \frac{1}{2} r_i)^2 (R_i - \frac{1}{2} r_i)^2} \right)^{1-\gamma}.$$  

(30)

The integrals over \( R_1 \) and \( R_2 \) were taken in Refs. [24, 63] and at \( n = 0 \) we have

$$H^{\gamma}(w, w^*) = \int d^2 R_1 E^{\gamma,0}(r_1, R_i) E^{1-\gamma,0}(r_2, R_1 - b) =$$

$$\frac{(\gamma - \frac{1}{2})^2}{(\gamma(1-\gamma))^2} \left\{ b_\gamma w^* w^{* \gamma} F(\gamma, \gamma, 2\gamma, w) F(\gamma, \gamma, 2\gamma, w^*) + b_{1-\gamma} w^{1-\gamma} w^{* 1-\gamma} F(1-\gamma, 1-\gamma, 2-2\gamma, w) F(1-\gamma, 1-\gamma, 2-2\gamma, w^*) \right\}$$

where \( F \) is hypergeometric function [64]. In Eq. (31) \( w w^* \) is equal to

$$w w^* = \frac{r_1^2 r_2^2}{(b - \frac{1}{2} (r_1 - r_2))^2 (b + \frac{1}{2} (r_1 - r_2))^2}$$

(32)

and \( b_\gamma \) is equal to

$$b_\gamma = \pi^3 2^{4(1/2-\gamma)} \left( \frac{\Gamma(\gamma)}{\Gamma(1/2-\gamma)} \right)^{1/2} \left( \frac{\Gamma(1-\gamma)}{\Gamma(1/2+\gamma)} \right).$$

(33)

Finally, the solution at large \( Y \) has the form

$$N^{\gamma}(r_1, r_2; Y, b) = \int \frac{d\gamma}{2 \pi i} e^{\omega(\gamma, 0) Y} H^{\gamma}(w, w^*)$$

(34)

In the vicinity of the saturation scale \( N^{\gamma} \) takes the form (see Refs. [60, 63])

$$N^{\gamma}(r_1, r_2; Y, b) = \frac{(\gamma_{cr} - \frac{1}{2})^2}{\gamma_{cr}(1 - \gamma_{cr})} b_{\gamma_{cr}} (ww^*)^{1-\gamma_{cr}}$$

$$= \frac{(\gamma_{cr} - \frac{1}{2})^2}{\gamma_{cr}(1 - \gamma_{cr})} b_{\gamma_{cr}} \left( \frac{r_1^2 r_2^2}{(b - \frac{1}{2} (r_1 - r_2))^2 (b + \frac{1}{2} (r_1 - r_2))^2} \right) e^{\frac{\lambda S}{\gamma_{cr}}} \chi_{\gamma_{cr}}(Y) (1-\gamma_{cr})$$

$$\overset{r_2 \gg r_1}{\approx} \phi_0 \left( \frac{r_1^2 Q_s^2(r_2, b; Y)}{b^2} \right)^{1-\gamma_{cr}}$$

(35)

with

$$Q_s^2(r_2, b; Y) = \frac{r_2^2 e^{\frac{\lambda S}{\gamma_{cr}}} \chi_{\gamma_{cr}}(Y)}{(b - \frac{1}{2} r_2)^2 (b + \frac{1}{2} r_2)^2}$$

(36)

where (see Refs. [60, 64, 62])

$$\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} = - \frac{d\chi(\gamma_{cr})}{d\gamma_{cr}} \quad \text{where} \quad \chi(\gamma) = 2 \psi(1) - \psi(\gamma) - \psi(1 - \gamma) \quad \text{\& \ kernel of the BFKL equation}$$

(37)

Below we denote by \( \bar{\gamma} = 1 - \gamma_{cr} \), and will use Eq. (33) and Eq. (36) in the momentum transfer representation, viz.

$$N^{\gamma}(r_1, r_2; Y, Q_T) = \int d^2 b e^{iQ_T \cdot b} N^{\gamma}(r_1, r_2; Y, b)$$

(38)
The integral of Eq. (38) with \( N_{\mathcal{F}}(r_1, r_2; Y, b) \) from Eq. (35) can be evaluated using the complex number description for the point on the plane, (see Eq. (23) and Eq. (29)). The integral has the form \( 24, 63 \)

\[
N_{\mathcal{F}}(r_1, r_2; Y, Q_T) = (r_1^2 r_2^2)^\gamma e^{\bar{\alpha}_s \chi(\gamma r)} Y \int d\rho_0 \ e^{i\bar{\alpha}_s \rho_0 \left( \frac{1}{\rho_0^2 - \rho_1^2} \right)^\gamma} \int d\rho_0^* \ e^{i\bar{\alpha}_s \rho_0^* \left( \frac{1}{\rho_0^* \rho_1^2 - \rho_1^2} \right)^\gamma} \tag{39}
\]

Using new variables \( t = \rho_0 / \rho_{12} \) and \( t^* = \rho_0^* / \rho_{12} \) and the integral representation of Hankel functions (see formulae 8.422(1,2) in Ref. [64])

\[
H^{(1,2)}_\nu (z) = \frac{\Gamma \left( \frac{3}{2} - \nu \right)}{\pi i \Gamma \left( \frac{1}{2} \right)} \left( \frac{1}{2} \right)^\nu \int_{C_{1,2}} dt e^{izt} \left( t^2 - 1 \right)^{\nu - \frac{1}{2}} \tag{40}
\]

where contours \( C_1 \) and \( C_2 \) are shown in Fig. 4 we obtain

\[
N_{\mathcal{F}}(r_1, r_2; Y, Q_T) = C^2(\gamma) r_2^2 e^{\bar{\alpha}_s \chi(\gamma Y)} \left( \frac{r_2 r_1}{r_{12}} \right)^\gamma \left( Q^2 r_{12}^2 \right)^{-\frac{1}{2} + \gamma} J_{\frac{1}{2} - \gamma} \left( \rho_0 \rho_{12} \right) J_{\frac{1}{2} - \gamma} \left( \rho_{0} \rho_{12} \right) \tag{41}
\]

where \( 2J_\nu(z) = H^{(1)}(z) + H^{(2)}(z) \); \( r_{12} = \frac{1}{2}(r_1 - r_2) \) and

\[
C(\gamma) = 2^{-\frac{1}{2} + \gamma} \pi \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma(\gamma)} \tag{42}
\]

Two limits will be useful for further presentation:

\[
N_{\mathcal{F}}(r_1, r_2; Y, Q_T) \xrightarrow{Q_T \to 0} C^2(\gamma) r_{12} e^{\bar{\alpha}_s \chi(\gamma Y)} \left( \frac{r_2 r_1}{r_{12}} \right)^\gamma \tag{43}
\]

\[
N_{\mathcal{F}}(r_1, r_2; Y, Q_T) \xrightarrow{Q_T \to 1, r_1 \ll r_2} \frac{2}{\pi} C^2(\gamma) r_{12} e^{\bar{\alpha}_s \chi(\gamma Y)} \left( \frac{r_2 r_1}{r_{12}} \right)^\gamma \left( Q^2 r_{12}^2 \right)^{-\frac{1}{2} + \gamma} \cos^2 \left( \frac{\pi \gamma}{2} \right) e^{iQ \cdot r_{12}} \tag{44}
\]

Eq. (43) can be re-written at \( r_1 \ll r_2 \) in the form:

\[
N_{\mathcal{F}}(r_1, r_2; Y, Q_T) \xrightarrow{Q_T \to 0, r_1 \ll r_2} C^2(\gamma) r_{12} (r_1^2 Q_s^2 (Y, r_2))^\gamma \text{ with } Q_s^2 = \frac{1}{r_2^2} e^{\bar{\alpha}_s \chi(\gamma Y)} \tag{45}
\]

**General formula**

In this subsection we calculate the cross section for the inclusive production of a gluon jet with transverse momentum \( p_\perp \) at rapidity \( Y_1 \), in the collision of two dipoles with sizes \( r_1 \) and \( r_2 \), at rapidity \( Y \), and at impact parameter \( b \). The general formula which shows \( k_T \)-factorization [66], has been derived in Ref. [67] and has the form

\[
\frac{d\sigma}{d^2 b dY d^2 p_\perp} = \frac{2C_F}{\alpha_s(2\pi)^2} \frac{1}{p_\perp^2} \int d^2 B \ d^2 r_\perp e^{i p_\perp \cdot r_\perp} \nabla_\perp^2 N_G \left( \frac{1}{2} Y - y_1; r_\perp, r_1; b \right) \nabla_\perp^2 N_G \left( \frac{1}{2} Y + y_1; r_\perp, r_2; |b - B| \right) \tag{46}
\]
where
\[ N^G(Y; r_\perp, r_i; b) = 2N(Y; r_\perp, r_i; b) - N^2(Y; r_\perp, r_i; b), \]

(47)

For one Pomeron exchange Eq. (47) reduces to the following equation
\[ N^G_P(Y; r_\perp, r_i; b) = 2N_P(Y; r_\perp, r_i; b) \]

(48)

Plugging Eq. (48) into Eq. (49) we have
\[
N^{\text{incl}}_P(Y, r_1, r_2, b, p_\perp; Y_1) = \frac{8CF}{\alpha_s(2\pi)^3} \frac{1}{p_1^2} \int d^2b \, d^2r_\perp \, e^{ip_\perp \cdot r_\perp} \nabla^2 N_P(Y; r_1, r_2; B) \nabla^2 N_P(Y - Y; r_\perp, r_2; |b - B|) \]

(49)

Note, that \( b \) is the difference of the impact parameters between scattering diopes, while \( B \) is the impact parameter of the produced gluon with respect to the dipole of size \( r_1 \).

It is more convenient to use \( \nabla^2 N_P(Y; r_\perp, r_1; b) \) in momentum representation, namely,
\[
\int d^2b \, e^{iQ \cdot b} \nabla^2 N_P(Y; r_\perp, r_1; b) = \nabla^2 N_P(Y; r_\perp, r_1; Q) \]

(50)

Using Eq. (51) for \( N_P(Y; r_\perp, r_1; Q) \) and \( \nabla^2 \gamma = 4 \partial_\gamma \partial_{\gamma'} \) we obtain (denoting \( r_1 \equiv r_0 \))
\[
\nabla^2 N_P(Y; r_\perp, r_1; Q) = \int_{e^{-\infty+i\epsilon}}^{e^{+\infty}} \frac{d\gamma}{2\pi i} \, 4 \, C^2(\gamma) r_0^2 e^{i\gamma b(x)} Y \left( \frac{r_0^2}{r_0^2} \right)^\gamma (Q^2 r_0^2)^{-\frac{1}{2}+\gamma} \]
\[
\times \left\{ \frac{\gamma - \frac{1}{2} + \gamma}{\rho_0} J_{\frac{1}{2}-\gamma} (\rho_0^2 \rho_0) + \frac{1}{2} \rho_0^2 \left( J_{\frac{1}{2}-\gamma} (\rho_0^2 \rho_0) - J_{\frac{1}{2}-\gamma} (\rho_0^2 \rho_0) \right) \right\} \]
\[
\times \left\{ \frac{\gamma - \frac{1}{2} + \gamma}{\rho_1} J_{\frac{1}{2}-\gamma} (\rho_0^2 \rho_0) + \frac{1}{2} \rho_0^2 \left( J_{\frac{1}{2}-\gamma} (\rho_0^2 \rho_0) - J_{\frac{1}{2}-\gamma} (\rho_0^2 \rho_0) \right) \right\} \]

(51)

We need to estimate
\[
N^{\text{incl}}_P(Y, r_1, r_2, Q_T, p_\perp, Y_1) = \int d^2b \, e^{iQ \cdot b} \nabla^2 N^{\text{incl}}_P(Y, r_1, r_2, b, p_\perp, Y_1) \]

(52)

for calculating the diagrams of Fig. 2a. From Eq. (41) and Eq. (51) we obtain
\[
N^{\text{incl}}_P(Y, r_1, r_2, Q_T, p_\perp, Y_1) = \frac{8CF}{\alpha_s(2\pi)^3} \frac{1}{p_1^2} \int d^2r_\perp \, e^{ip_\perp \cdot r_\perp} \nabla^2 N_P(y_1; r_\perp, r_1; Q) \nabla^2 N_P(Y - y_1; r_\perp, r_2; Q) \]

(53)

Note, the dependence on \( y_1 \) is very weak since \( N_P(y_1; r_\perp, r_1; Q) \propto \exp(\tilde{\alpha}_S \chi(\gamma) y_1) \).

Azimuthal angle dependence

As we have discussed in the introduction, the azimuthal angle correlation arises from the terms \((p_{\perp,1} \cdot Q_T)^2\) and \((p_{\perp,2} \cdot Q_T)^2\), after integration over \( Q_T \) in the Pomeron loop in the diagram of Fig. 2a, since
\[
\int d^2Q_T (p_{\perp,1} \cdot Q_T)^2 (p_{\perp,2} \cdot Q_T)^2 \rightarrow (p_{\perp,1} \cdot p_{\perp,2})^2 \]

Such terms in the coordinate representation that we are using here, stem from the terms \((r_{1z} \cdot Q_T)^2\) and \((r_{2z} \cdot Q_T)^2\) in \( N_P(r_1, r_2, Y, Q_T) \) and \( N_P(r'_1, r'_2, Y, Q_T) \) (see Eq. (41)).

These terms come from \( J_{\frac{1}{2}-\gamma} (\rho_0 \rho_1) \) \( J_{\frac{1}{2}-\gamma} (\rho_0 \rho_1) \). For small \( Q_T \) we can see how these terms appear by expanding \( J_{\frac{1}{2}-\gamma} \).

Indeed,
\[
(Q_{T_{12}}^2)^{-\frac{1}{2}+\gamma} J_{\frac{1}{2}-\gamma} (\rho_0 \rho_1) J_{\frac{1}{2}-\gamma} (\rho_0 \rho_1) = \]
\[
= \left( \frac{1}{2^{\frac{1}{2}-\gamma} \Gamma \left( \frac{3}{2} - \gamma \right)} \right)^2 \left\{ 1 + \frac{1}{2(-3+2\gamma)} Q^2 r_{12}^2 e^{2i(\phi-\psi)} \right\} \left\{ 1 + \frac{1}{2(-3+2\gamma)} Q^2 r_{12}^2 e^{-2i(\phi-\psi)} \right\} \]
\[
\rightarrow \left( \frac{1}{2^{\frac{1}{2}-\gamma} \Gamma \left( \frac{3}{2} - \gamma \right)} \right)^2 \left\{ 1 + \frac{1}{(-3+2\gamma)} Q^2 r_{12}^2 \cos(2(\phi-\psi)) + \left( \frac{1}{2(-3+2\gamma)} \right)^2 Q^4 r_{12}^4 \right\} \]

(54)
\[
\rightarrow \left( \frac{1}{2\pi - \gamma} \right)^2 \left\{ 1 + \frac{1}{(-3 + 2\gamma)} \left( 2(Q_T \cdot r_{12})^2 - Q_T^2 r_{12}^2 \right) + \left( \frac{1}{2(-3 + 2\gamma)} \right)^2 Q_T^2 r_{12}^2 \right\}
\]

In Eq. (54) we use the representation of complex numbers in the polar coordinates, for example, \(\rho_Q = Q e^{i\phi}\) and \(\bar{\rho}_Q = Q e^{-i\phi}\). The same type of contributions come from Eq. (51).

For \(Q_T r_{12} \gg 1\) we have the same features since

\[
N_{\Phi} (r_1, r_2; Y, Q_T) \rightarrow \frac{Q_T^2 r_{12}^2 \gg 1}{2(-3 + 2\gamma)} \left\{ (1/8)(\gamma (\gamma - 2)(1 - \gamma^2)) + Q_T^2 r_{12}^2 e^{2i(\phi - \psi)} \right\} \left\{ (1/8)(\bar{\gamma} (\bar{\gamma} - 2)(1 - \bar{\gamma}^2)) + Q_T^2 r_{12}^2 e^{-2i(\bar{\phi} - \bar{\psi})} \right\} / Q_T^2 r_{12}^2
\]

However, the largest contribution at \(r \ll r_1\) and \(r \ll r_2\) comes from Eq. (51), which can be re-written as

\[
\nabla_\perp^2 N_{\Phi} (Y_1; r_{1\perp}, r_1; Q_T) \rightarrow \int_{r_{1\perp} \ll r_1} C^2(\gamma) r_{01}^2 e^{\alpha_S} \chi(\gamma) Y \left( \frac{r_{01}^2}{\gamma} \right)^2 (Q_T r_{01})^{\gamma - \frac{3}{2}} \frac{\gamma^2}{r^2} J_{\frac{3}{2} - \gamma} (\rho_Q \rho_{01}) J_{\frac{3}{2} - \gamma} (\bar{\rho}_Q \bar{\rho}_{01})
\]

where \(r_{01} \equiv r_{1\perp}\).

Note, that at \(Q_T \rightarrow 0\) Eq. (56) reduces to

\[
\nabla_\perp^2 N_{\Phi} (Y_1; r_{1\perp}, r_1; Q_T) \rightarrow \int_{r_{1\perp} \ll r_1} C^2(\gamma) r_{01}^2 e^{\alpha_S} \chi(\gamma) Y \left( \frac{r_{01}^2}{\gamma} \right)^2 \frac{\gamma^2}{r^2} \left\{ 1 - \frac{1}{(3 - 2\gamma)} (\rho_Q^2 + \bar{\rho}_Q^2) \right\}
\]

To calculate the production of the gluon with the transverse momentum \(p_{1\perp}\), we need to plug Eq. (57) into Eq. (51) and Eq. (53).

**Angular dependence of the double inclusive cross section**

The contribution to the double inclusive production from the diagram of Fig. 2 takes the general form

\[
\frac{d^2 g^2}{dy_1 dy_2, d^2 p_{1\perp}, d^2 p_{2\perp}} = \frac{d^2 Q_T}{4\pi^2}
\]

where \(\alpha_1 = \alpha, \alpha_2 = \beta\) and \(g_i(Q_T) = g_i \int d^2 b e^{iQ_T \cdot b} S_{\Phi} (b, m_i) = g_i/(1 + Q_T^2/m_i^2)^2\), as it follows from Eq. (10).
Substituting Eq. (57) into Eq. (53) we obtain

$$
N_{\gamma}^{\text{incl}}(Y, y_1, r_1, r_2, Q_T, p_{1\perp}, Y_1) = \frac{8 C_F}{\alpha_s(2\pi)^4} \frac{1}{p^2_{1\perp}} \int d^2 r_0 e^{i p_{\perp} \cdot r_0} \left( \int_{-\infty}^{+i \phi_1} \frac{d \gamma}{2 \pi i} \right) \frac{4 C^2(\gamma_1)}{(2 \pi i)^2} \frac{d^2 r_0}{r^2_{01}} e^{i \gamma (\chi(1) \frac{1}{2} Y - y_1) + \chi(2) \frac{1}{2} Y + y_1)} \left( \frac{2 \gamma_1}{r^2_{01}} \right)^{\gamma_1} \left( \frac{2 - \gamma_1}{r^2_{01}} \right) \left( \frac{2 - \frac{1}{2} \gamma_2}{r^2_{02}} \right) \left( \frac{2 - \gamma_2}{r^2_{02}} \right) \left( \frac{1}{\Gamma(3/2 - \gamma_1)} \right) \left( \frac{1}{\Gamma(3/2 - \gamma_2)} \right) \left( 1 - \frac{2}{3 - 2 \gamma_1} \left( Q_T r_0 \right)^2 \right) \left( 1 - \frac{2}{3 - 2 \gamma_2} \left( Q_T r_0 \right)^2 \right)
$$

(59)

Integrating first over $d^2 r_0$ we obtain

$$
N_{\gamma}^{\text{incl}}(Y, y_1, r_1, r_2, Q_T, p_{1\perp}, Y_1) = \frac{128 C_F}{\alpha_s(2\pi)^4} \frac{d^2 r_0}{r^2_{1\perp}} \int_{-\infty}^{+i \phi_1} \frac{d \gamma}{2 \pi i} \int_{-\infty}^{+i \phi_2} \frac{d \gamma}{2 \pi i} C^2(\gamma_1) C^2(\gamma_2) \left( \frac{1}{\Gamma(3/2 - \gamma_1)^2} \right) \left( \frac{1}{\Gamma(3/2 - \gamma_2)^2} \right) \left( 1 - \frac{2}{3 - 2 \gamma_1} \right) \left( 1 - \frac{2}{3 - 2 \gamma_2} \right) \left( \frac{Q_T \cdot p_{\perp}}{p^2_{1\perp}} \right)^2
$$

(60)

In Eq. (60) we denote $\gamma_1 = \gamma_1 + \gamma_2$, consider $r_0 \ll r_1(r_2)$, and neglected the contributions of the order of $Q_T^4$.

For further estimates, we need to return to a general formula of Eq. (56). We know that as a result of the shadowing corrections $N_{\gamma}^{G}(Y; r_{1\perp}, r_1; b) \rightarrow 1$ for large values of $Y$. It means that $\nabla^2_{\perp} N_{\gamma}^{G}(Y; r_{1\perp}, r_1; b) \rightarrow 0$ in the saturation region, where $r^2_{1\perp} Q_s(Y) \gg 1$. Such behavior stems from the diagrams of Fig. 3b, but not from the first diagram that we are presently considering. Consequently, we have $\nabla^2_{\perp} N_{\gamma}^{G}(Y; r_{1\perp}, r_1; b) \rightarrow 0$ which vanishes both at $r^2_{1\perp} Q_s(Y) \gg 1$ and at $r^2_{1\perp} Q_s(Y) \ll 1$, and the main contribution originates for the value of $r$ in the vicinity of the saturation scale $r^2_{1\perp} Q_s(Y) \approx 1$. As we have discussed, in the vicinity of the saturation scale $\gamma_1 = \gamma_2 = \gamma = 1 - \gamma_{cr} = 0.63$ (see Eq. (57) and Eq. (60)). The second observation which simplifies the estimates, is that in our approach $r_1 = r_2 \sim 1/m$ and $m \gg m_1$ and $m_2$ (see Table 1). Since the typical $Q_T$ in the integration is approximately $m_1$ or $m_2$, we can neglect the $Q_T$ dependence of the BFKL Pomeron.

Therefore, we can write the double inclusive production cross section in the following form

$$
\frac{d^2 \sigma}{dy_1 \, dy_2 \, d^2 p_{1\perp} \, d^2 p_{2\perp}} / \int d \phi \, d^2 \sigma \, d^2 p_{1\perp} \, d^2 p_{2\perp} = 1 + \frac{\kappa}{p^2_{1\perp} p^2_{2\perp}} \cos(2\phi)
$$

(61)

where $\kappa$ is equal to

$$
\kappa = \frac{(8 \gamma (2 \gamma - 1))}{3 - 2 \gamma} \left( \frac{Q^4}{1 + Q^2 / m_Y^2} \right) \left( \frac{1 + Q^2 / m_Y^2}{1 + Q^2 / m_Y^2} \right)
$$

(62)

In our model $i = j = 1$ gives the largest contribution, due to large value of $g_1$ (see Table 1), and we obtain $\kappa = 0.04 GeV^4$. The contribution of the term proportional to $\cos(2\phi)$ depends on the value of $p_{1\perp}$. Actually, we can trust Eq. (61) for $p_{1\perp} \geq Q_s(Y)$. Integrating over $p_{1\perp}$ and $p_{2\perp}$ we expect that the contribution to the correlation function will be equal to

$$
R(y_1, y_2, \phi) = R(y_1, y_2) \left( \frac{\kappa}{Q^4_s(Y)} \right) \cos(2\phi) = 2 v^2 \cos(2\phi)
$$

(63)

leading to $R(\frac{1}{2} Y, \frac{1}{2} Y, \phi) = 2 \cos(2\phi)$ for $Q_s \approx 1 GeV$ or $v_2 = 0.23$. This value is in a good agreement with the estimates for this correlation from the elliptic flow and experimental data. Eq. (61) leads to $v_2 = R(y_1, y_2) / p^2_{1\perp} p^2_{2\perp}$. However, we can trust this $p_{1\perp}$ dependence only for $p^2_{1\perp} > Q_s$ and $p^2_{2\perp} > Q_s$. We should introduce the shadowing corrections to reproduce the behaviour of $v_2$ for $p^2_{1\perp} < Q_s$ and $p^2_{2\perp} < Q_s$. We will do this in the next section for our model, but here we estimate the influence of the shadowing correction by integrating Eq. (59) over $r_0$ in the limits $0 < r_0 < R = 1/Q_s$. Indeed, as has been mentioned, $N_{\gamma} \to 0$ for $r_0 \gg 1/Q_s$. In Fig. 4 we plot the $v_2$ dependence for $p_{1\perp} = p_{2\perp}$ for $Q_s = 1 GeV$ and choosing $R = 3/Q_s$. One can see that $v_2$ decreases at $p_{1\perp} < Q_s$. 
FORMULA FOR AZIMUTHAL CORRELATIONS OF TWO PARTICLES PRODUCED IN TWO PARTON SHOWERS

Based on our experience from calculating the first diagrams, we can estimate the two particle angular correlations that stem for the general diagram of Fig. 2-b. The general formulae have the same form as Eq. (46), which we used in estimating the contribution of the first diagram. However, we now need to calculate $N_G$ in Eq. (46) using our model described in section 2. As we have seen for angular correlations, it is essential to use the $r$ and $b$ dependence of the BFKL contribution. Bearing this in mind, we have to generalize Eq. (3) replacing it by the following formula

$$T(r, b, Y) \rightarrow T_W(r, b, Y) = \phi_0 \left(w w^* Q_s^2 (Y, b)\right)^\gamma$$

where $w w^*$ is given by Eq. (32) with the arguments $r = r_0$ and $r_2 = 1/m$, and $Q_s$ is given by Eq. (4). In Eq. (64) we replace $r_0^2$ in Eq. (46) by $w w^*$, since $T_W$ describes the behavior of the scattering amplitude in the vicinity of the saturation scale.

The expression of $N_G$ is the direct generalization of Eq. (24) and has the following form

$$N_G \left(r_1, r_2; Y, Q_T\right) = \int d^2 b e^{iQ_T \cdot b} N_BK \left(\int d^2 b' G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) g_i \left(b - b'\right)\right)$$

Calculating $\nabla^2 N_G \left(r_1, r_2; Y, Q_T\right)$ we see that we have two contributions

$$\nabla^2 N_G \left(r_1, r_2; Y, Q_T\right) = \int d^2 b d^2 b' e^{iQ_T \cdot b}$$

$$\left\{\nabla^2 G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) \frac{d}{dG^{dressed}} + \nabla_{r_1} G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) \cdot \nabla_{r_1} G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) \frac{d^2}{\left(dG^{dressed}\right)^2}\right\}$$

$$N_BK \left(\int d^2 b' G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) g_i \left(b - b'\right)\right)$$

The first term in Eq. (66) is proportional to $\left(r_1^2\right)^{\gamma - 2}$, while the second one is $\propto \left(r_1^2\right)^{2\gamma - 2}$. Therefore, for small $r_1$ we can neglect the second term.

We have that

$$\frac{d}{dG^{dressed}} N_BK \left(\int d^2 b' G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) g_i \left(b - b'\right)\right) = g_i \left(b - b'\right) \frac{dN_BK(Z)}{dZ} \left(Z = \int d^2 b' G^{dressed} \left(T_W \left(r_1, b', Y\right)\right) g_i \left(b - b'\right)\right)$$
with

\[ N^\text{NK}(Z) \equiv \frac{dN^\text{NK}(Z)}{dZ} = a e^{-z} + \frac{1 - a}{(1 + Z)^2} \]  

(68)

For small \( r_\perp \)

\[ \nabla^2_{r_\perp} G^{\text{dressed}} \left( T_W (r_\perp, b', Y) \right) = \nabla^2_{r_\perp} T_W \left( r_\perp, b', Y \right) \frac{d}{dT} G^{\text{dressed}} \left( T = T_W \left( r_\perp, b', Y \right) \right) \]

\[ \equiv \nabla^2_{r_\perp} N_g \left( r_\perp, b', Y \right) \frac{d}{dT} G^{\text{dressed}} \left( T = T_W \left( r_\perp, b', Y \right) \right) \]

(69)

with

\[ G' (T) \equiv \frac{d}{dT} G^{\text{dressed}} (T) = a^2 e^{-T} + \frac{(a - 1)^2 e^{1/T}(T + 1) \Gamma(0, \frac{1}{T})}{T^3} + (a - 1) \left( \frac{1}{T^2} - \frac{2a}{(T + 1)^2} \right) \]  

(70)

Plugging Eq. (67) - Eq. (70) into Eq. (66) and using that \( b' \approx 1/m \ll b \approx 1/m_i \) we obtain

\[ \nabla^2_{r_\perp} N^g_C \left( r_\perp, r_1; Y, Q_T \right) \equiv \nabla^2_{r_\perp} N^g_C \left( r_\perp, Y, Q_T \right) \left( \frac{G'}{r_\perp \rightarrow 1/Q_s} \right) \]  

\[ N^\text{NK} (r_\perp, Y, Q_T) = \int d^2 b \left[ e^{Q_T - b} g_i (b) \right] N^\text{NK} \left( \tilde{G} \left( r_\perp, Y \right) g_i (b) \right) \]  

(71)

where

\[ \tilde{G} (r_\perp, Y) = \int d^2 b' G \left( T \left( r_\perp, b', Y \right) \right) \]

\[ \tilde{G}' \left( r_\perp, Y \right) = \int d^2 b' S (b', m) G' \left( T \left( r_\perp, b', Y \right) \right) \]  

(72)

Using Eq. (57), in which we substitute \( \gamma = \tilde{\gamma} \) we obtain

\[ \nabla^2_{r_\perp} N^g_C \left( Y; r_\perp; Q_T \right) \rightarrow 4 C^2 (\gamma) r_{01}^2 e^{\tilde{\gamma} S \chi (\tilde{\gamma})} Y \]

\[ \times \left( \frac{r_{12}^2 r_{21}^2}{r_{01}^2} \right)^{\tilde{\gamma}} \left( \frac{2 - \frac{1}{2} \tilde{\gamma}}{\Gamma (3/2 - \tilde{\gamma})} \right)^{\frac{\tilde{\gamma}^2}{r^2}} \left\{ 1 - \frac{2}{3 - 2 \tilde{\gamma}} \left( Q_T r_\perp \right)^2 \right\} \tilde{G}' \left( r_\perp, Y \right) N^\text{NK} (r_\perp, Y, Q_T) \]

\[ \equiv C (\gamma) e^{\tilde{\gamma} S \chi (\tilde{\gamma})} Y r_{01}^2 \left( \frac{r_{12}^2 r_{21}^2}{r_{01}^2} \right)^{\tilde{\gamma}} \left\{ 1 - \frac{2}{3 - 2 \tilde{\gamma}} \left( Q_T r_\perp \right)^2 \right\} \tilde{G}' \left( r_\perp, Y \right) N^\text{NK} (r_\perp, Y, Q_T) \]  

(73)

Using Eq. (73) we can re-write the double inclusive cross section in the form (see Eq. (61))

\[ \frac{d^2 \sigma^2 \text{ parton showers}}{dy_1 dy_2 d^2 p_{1\perp} d^2 p_{2\perp}} \left/ \int \frac{d\phi}{2\pi} \frac{d^2 \sigma^2 \text{ parton showers}}{dy_1 dy_2 d^2 p_{1\perp} d^2 p_{2\perp}} \right. = 1 + 2 v_{22} (p_{1T}, p_{2T}) \cos (2\phi) \]  

(74)

where

\[ v_{22} (p_{1T}, p_{2T}) = N \left( p_{1\perp}, p_{2\perp}, Y \right) / D \left( p_{1\perp}, p_{2\perp}, Y \right) \]  

(75)

where

\[ N \left( p_{1\perp}, p_{2\perp}, Y, y_1, y_2 \right) = \]

\[ \frac{1}{4} \sum_{i,j=1}^{i,j} \alpha_i^2 \alpha_j^2 \left\{ \int \frac{dr_{1\perp}^2 J_2 \left( p_{1\perp} r_{1\perp} \right) \left( \frac{r_{1\perp}^2}{r_{1\perp}^2} \right)^{2\tilde{\gamma}} \left( \frac{4}{3 - 2 \tilde{\gamma}} \right) \tilde{G}' \left( r_{1\perp}, \frac{1}{2} Y - y_1 \right) \tilde{G}' \left( r_{1\perp}, \frac{1}{2} Y + y_1 \right) \right\} \]

\[ \times \left\{ \int \frac{dr_{2\perp}^2 J_2 \left( p_{2\perp} r_{2\perp} \right) \left( \frac{r_{2\perp}^2}{r_{1\perp}^2} \right)^{2\tilde{\gamma}} \left( \frac{4}{3 - 2 \tilde{\gamma}} \right) \tilde{G}' \left( r_{1\perp}, \frac{1}{2} Y - y_2 \right) \tilde{G}' \left( r_{1\perp}, \frac{1}{2} Y + y_2 \right) \right\} \int dQ_T^4 \]  

\[ N^\text{NK} \left( r_{1\perp}, \frac{1}{2} Y - y_1, Q_T \right) N_j^\text{NK} \left( r_{1\perp}, \frac{1}{2} Y + y_1, Q_T \right) N^\text{NK} \left( r_{1\perp}, \frac{1}{2} Y - y_2, Q_T \right) N_j^\text{NK} \left( r_{1\perp}, \frac{1}{2} Y + y_2, Q_T \right) \]  

(76)
and

\[
D(p_{1\perp}, p_{2\perp}, Y, y_1, y_2) =
\sum_{i=1,j=1}^{i,j} \alpha_s^2 \alpha_s^2 \left\{ \int \frac{dr_{1\perp}^2}{r_{1\perp}^2} J_0(p_{1\perp} r_{1\perp}) \left( \frac{r_{2\perp}^2}{r_{2\perp}^2} \right)^{2\gamma} \tilde{G}'(r_{1\perp}, \frac{1}{2} Y - y_1) \tilde{G}'(r_{2\perp}, \frac{1}{2} Y + y_1) \right\}
\times \left\{ \int \frac{dr_{1\perp}^2}{r_{1\perp}^2} J_0(p_{1\perp} r_{1\perp}) \left( \frac{r_{2\perp}^2}{r_{2\perp}^2} \right)^{2\gamma} \tilde{G}'(r_{1\perp}, \frac{1}{2} Y - y_2) \tilde{G}'(r_{2\perp}, \frac{1}{2} Y + y_2) \right\} \int dQ_T^2
\]

\[
N_i^{\text{MK}} \left( r_{\perp}, \frac{1}{2} Y - y_1, Q_T \right) N_j^{\text{MK}} \left( r_{\perp}, \frac{1}{2} Y + y_1, Q_T \right) N_i^{\text{MK}} \left( r_{\perp}, \frac{1}{2} Y - y_2, Q_T \right) N_j^{\text{MK}} \left( r_{\perp}, \frac{1}{2} Y + y_2, Q_T \right)
\]

DOUBLE INCLUSIVE CROSS SECTION IN THE EVENTS WITH FIXED MULTIPlicity

In this section we calculate the angular dependence of the double inclusive cross section, for the event with large multiplicity. In our approach, the large multiplicity event stems from the production of several parton showers, as it is shown in Fig. 6. Indeed, if \( N \) particles are produced in the collision, the \( n \) parton showers contribute, where \( n = N/\bar{n} \). \( \bar{n} \) is the multiplicity in the single parton shower, which can be estimated as being equal to the average multiplicity in the single inclusive production. Bearing this in mind, we see that

\[
\frac{d^2\sigma}{dy_1 dy_2 dp_{1T} dp_{2T}} = \frac{8 C_F}{\alpha_s (2\pi)^4} \frac{1}{p_{1T}^2} F_i^{\text{incl}} (p_T) ; \quad \frac{d^2\sigma_{\text{one parton shower}}}{dy_1 dy_2 dp_{1T} dp_{2T}} \big|_{y_1 = y_2} = \left( \frac{8 C_F}{\alpha_s (2\pi)^4} \right)^2 \frac{1}{p_{1T}^2 p_{2T}^2} F_i^{\text{incl}} (|p_{1T} + p_{2T}|)
\]

The calculation of the first term in Eq. (80) can be simplified for \( y_1 \approx y_2 \) by using the following relation (see Ref. [73])

\[
\frac{d\sigma}{dy dp_{T}} = \frac{8 C_F}{\alpha_s (2\pi)^4} \frac{1}{p_T^2} F_i^{\text{incl}} (p_T) ; \quad \frac{d^2\sigma_{\text{one parton shower}}}{dy_1 dy_2 dp_{1T} dp_{2T}} \big|_{y_1 = y_2} = \left( \frac{8 C_F}{\alpha_s (2\pi)^4} \right)^2 \frac{1}{p_{1T}^2 p_{2T}^2} F_i^{\text{incl}} (|p_{1T} + p_{2T}|)
\]

FIG. 6: The large multiplicity event in our approach: production from one parton shower (Fig. 6a) and from two parton showers (Fig. 6b).

The expression for the function \( F_i^{\text{incl}} \) we have found and it is equal to

\[
F_i^{\text{incl}} (p_T, Y, y_1) = \int \frac{d\gamma}{\gamma^2} J_0(p_{1\perp} r_{1\perp}) \left( \frac{r_{2\perp}^2}{r_{2\perp}^2} \right)^{2\gamma} \tilde{G}'(r_{\perp}, \frac{1}{2} Y - y_1) \tilde{G}'(r_{\perp}, \frac{1}{2} Y + y_1) N_i^{\text{MK}} (r_{\perp}, \frac{1}{2} Y - y_1, Q_T = 0) N_j^{\text{MK}} (r_{\perp}, \frac{1}{2} Y + y_1, Q_T = 0)
\]

Therefore, the first term in Eq. (80) reduces to the following expression

\[
n \frac{d^2\sigma_i^{\text{one parton shower}}}{dy_1 dy_2 dp_{1T} dp_{2T}} = n \left( \frac{8 C_F}{\alpha_s (2\pi)^4} \right)^2 \frac{1}{p_{1T}^2 p_{2T}^2} F_i^{\text{incl}} (|p_{1T} + p_{2T}|)
\]

where \( F_i^{\text{incl}} \) is given by Eq. (80).
The second term in Eq. (83) is almost equal to Eq. (77)

\[ n(n-1) \frac{d^2 \sigma_{\text{two parton showers}}}{dy_1 dy_2 d^2 p_{1T} d^2 p_{2T}} = n(n-1) \left( \frac{8 C_F}{\alpha_S (2 \pi)^2} \right)^2 \frac{1}{p_{1T}^2 p_{2T}^2} \]

\[ \left\{ \int \frac{d^2 r_{\perp}}{r_{\perp}^2} J_0 (p_{1T} r_{\perp}) \left( \frac{r_{\perp}^2}{r_{\perp}^2} \right)^{25} \mathcal{G}(r_{\perp} \cdot \frac{1}{2} Y - y_1) \mathcal{G}(r_{\perp} \cdot \frac{1}{2} Y + y_1) \right\} \times \left\{ \int \frac{d^2 r_{\perp}'}{r_{\perp}'^2} J_0 (p_{1T} r_{\perp}') \left( \frac{r_{\perp}'^2}{r_{\perp}'^2} \right)^{25} \mathcal{G}(r_{\perp}' \cdot \frac{1}{2} Y - y_2) \mathcal{G}(r_{\perp}' \cdot \frac{1}{2} Y + y_2) \right\} \int dQ_T^2 \]

\[ N_i^{\text{tik}} \left( r_{\perp}, \frac{1}{2} Y - y_1, Q_T \right) N_j^{\text{tik}} \left( r_{\perp}, \frac{1}{2} Y + y_1, Q_T \right) \]

Using our calculation for inclusive production, and for the rapidity correlation function we can re-write Eq. (82) in the form

\[ \frac{1}{\sigma_{\text{in}}} \frac{d^2 \sigma}{dy_1 dy_2 d^2 p_{1T} d^2 p_{2T}} = n \left( \frac{8 C_F}{\alpha_S (2 \pi)^2} \right) \frac{1}{\sigma_{\text{in}}} \frac{d \sigma}{p_{1T}^2 p_{2T}^2} \left\{ \frac{1}{\sigma_{\text{in}}} \frac{d \sigma}{dy_1 \sigma_{\text{in}} dy_2} \left( 1 + R(y_1, y_2) \right) \left( 1 + 2 \nu_{22} (p_{1T}, p_{2T}) \cos (2 \phi) \right) \right\} \]

\[ \sum_{i=1,j=1}^{i=2,j=2} \alpha_i^2 \alpha_j^2 \frac{d^2 \sigma_{\text{two parton showers}}}{dy_1 dy_2 d^2 p_{1T} d^2 p_{2T}} \]

\[ \sum_{i=1,j=1}^{i=2,j=2} \alpha_i^2 \alpha_j^2 \int d^2 p_{1T} d^2 p_{2T} \frac{d^2 \sigma_{\text{two parton showers}}}{dy_1 dy_2 d^2 p_{1T} d^2 p_{2T}} \]

\[ \frac{d^2 N}{dy_1 dy_2 d^2 p_{1T} d^2 p_{2T} d \phi} = Y_{\text{periph}} (\phi) + Y_{\text{ridge}} (\phi) \]

PREDICTIONS AND COMPARISON WITH THE EXPERIMENT

Eq. (83) has the form that has been used in the analysis of the experimental data [72]; viz.

\[ \int d y_1 d y_2 \int_{p_{1T}^\text{min}}^{p_{1T}^\text{max}} d p_{1T} \int_{p_{2T}^\text{min}}^{p_{2T}^\text{max}} d p_{2T} \frac{1}{\sigma_{\text{in}}} \frac{d \sigma}{dy_1 \sigma_{\text{in}} dy_2} d^2 \phi = \int d y_1 d y_2 \int_{p_{1T}^\text{min}}^{p_{1T}^\text{max}} d p_{1T} \int_{p_{2T}^\text{min}}^{p_{2T}^\text{max}} d p_{2T} \frac{d^2 N}{dy_1 dy_2 d^2 p_{1T} d^2 p_{2T} d \phi} = Y_{\text{periph}} (\phi) + Y_{\text{ridge}} (\phi) \]

where \( \phi \) denotes the difference between the azimuthal angles \( \phi = \phi_1 - \phi_2 \).

\( Y_{\text{periph}} (\phi) \) describes the production of two gluons from one parton shower (see Fig. 5a) while \( Y_{\text{ridge}} (\phi) \) stands for the second term, which is related to the emission of gluons from two different parton showers (see Fig. 5b). Eq. (83) shows several qualitative features which have been observed experimentally [71, 72]: (1) \( Y_{\text{periph}} (\phi) \) is smaller than \( Y_{\text{ridge}} (\phi) \) and (2) it decreases with increasing multiplicity of the event; (their ratio is proportional to \( 1/n \)); and (3) \( \nu_{22} \) does not depend on the multiplicity of the event and on the rapidity difference \( Y_{12} = Y_1 - Y_2 \).

From Eq. (76) and Eq. (77) we can see that

\[ N \propto \int d^2 Q_T Q_T^2 \left\{ \int \frac{d^2 r_{\perp}}{r_{\perp}^2} J_2 (p_{1T} r_{\perp}) N_{ij} \left( r_{\perp}, Q_T \right) \right\} \left\{ \int \frac{d^2 r_{\perp}'}{r_{\perp}'^2} J_2 (p_{2T} r_{\perp}') N_{ij} \left( r_{\perp}', Q_T \right) \right\}; \]

\[ D \propto \int d^2 Q_T \left\{ \int \frac{d^2 r_{\perp}}{r_{\perp}^2} J_0 (p_{1T} r_{\perp}) N_{ij} \left( r_{\perp}, Q_T \right) \right\} \left\{ \int \frac{d^2 r_{\perp}'}{r_{\perp}'^2} J_0 (p_{2T} r_{\perp}') N_{ij} \left( r_{\perp}', Q_T \right) \right\}; \]

As we have discussed, functions \( N_{ij} \) have maxima at \( r_{\perp} \approx 1/Q_s \). In Fig. 4 we plot function \( N_{11} \) at \( W = 13 \) TeV. One can see that \( N_{11} \) has a maximum at \( r_{\perp} \approx 3 \div 4 \) GeV\(^{-1}\). Therefore, our estimates in section 4.2 are justified.

The calculations of \( \nu_{22} \) using Eq. (75) with the parameters of Table 1, are shown in Fig. 8. In Eq. (84) we integrate over \( p_{2T} \) in the intervals shown in the figure: \( 0.5 \div 1 \) GeV, \( 2 \div 3 \) GeV and \( 0.5 \div 5 \) GeV.

First conclusion from these figures is that \( \nu_{22} \) does not depend on energy, which is in agreement with the experimental data of Ref. 72. The values of \( v_2(p_T) \) which is determined as

\[ v_2 (p_{1T}) = \frac{\nu_{22} (p_{1T}, p_{2T})}{\nu_{22} (p_{2T}, p_{2T})} \]
CONCLUSION

In this paper we estimate the contribution of the density variation mechanism to the value of $v_2$. This was done in a model based on CGC/saturation approach. It has been demonstrated in Refs. [11–14], that the model is able to describe both the diffraction-type reactions (the total and inelastic cross section, elastic cross section and its $t$-distribution, diffractive production) and the multi-particle production processes, such as inclusive cross sections
and rapidity correlations. We present here, the first attempt to describe the azimuthal angular correlation, in the framework of a unique approach based on an effective theory for high energy QCD.

Comparing with the experimental $v_2$ in proton-proton collision, we conclude that the density variation mechanism in the framework of CGC/saturation approach, provides a substantial contribution which cannot be neglected. Bearing in mind that in CGC/saturation approach there are other two mechanisms present: Bose enhancement in the wave function [50] and local anisotropy [51, 52], we believe that the azimuthal long range rapidity correlations in proton-proton collisions stem from the CGC/saturation physics, and not from quark-gluon plasma production. It should be noted that none of the models based on the quark-gluon plasma production are able to describe diffractive physics. Hence, at present, the CGC/saturation approach, appears to be the only effective theory that can provide such a description.

We plan to include other CGC sources in the description of $v_2$, as well as compare in more detail, the cross section for double inclusive production of two gluon jets in proton-proton collisions.

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FIG. 11: $v_2$ versus $p_T$ at $W = 13$ TeV with the choice $Q_0 = 0.2 m$ in Eq. [87].
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$v_2(p_T)$