CONJUGACY, ROOTS, AND CENTRALIZERS IN THOMPSON’S GROUP $F$

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ABSTRACT. We complete the program begun by Brin and Squier of characterising conjugacy in Thompson’s group $F$ using the standard action of $F$ as a group of piecewise linear homeomorphisms of the unit interval.

1. INTRODUCTION

The object of this paper is to extend the methods of Brin and Squier described in [3] to solve the conjugacy problem in Thompson’s group $F$, and to analyse roots and centralizers in $F$.

Let $\text{PL}^+(a,b)$ denote the group of piecewise linear order-preserving homeomorphisms of an open interval $(a,b)$. The points at which an element $f$ of $\text{PL}^+(a,b)$ is not locally affine are called the nodes of $f$. Thompson’s group $F$ is the subgroup of $\text{PL}^+(0,1)$ defined as follows: an element $f$ of $\text{PL}^+(0,1)$ lies in $F$ if and only if the nodes of $f$ lie in the ring of dyadic rational numbers, $\mathbb{Z}\left[\frac{1}{2}\right]$, and $f'(x)$ is a power of 2 whenever $x$ is not a node.

In [3] Brin and Squier analysed conjugacy in $\text{PL}^+(a,b)$ for $(a,b)$ any open interval. For $(a,b)$ equal to $(0,1)$ we can restate their primary result [3 Theorem 5.3] as follows: we have a simple quantity $\Sigma$ on $\text{PL}^+(0,1)$ such that two elements $f$ and $g$ of $\text{PL}^+(0,1)$ are conjugate if and only if $\Sigma_f = \Sigma_g$. If $f$ and $g$ are elements of $F$ then $\Sigma_f$ and $\Sigma_g$ can be computed and compared using a simple algorithm. Brin and Squier comment on their construction of $\Sigma$ that, “Our goal at the time was to analyze the conjugacy problem in Thompson’s group $F$.” In this paper we achieve Brin and Squier’s goal by defining a quantity $\Delta$ on $F$ such that the following theorem holds.

**Theorem 1.1.** Let $f, g \in F$. Then $f$ and $g$ are conjugate in $F$ if and only if

$$(\Sigma_f, \Delta_f) = (\Sigma_g, \Delta_g).$$

This is not the first solution of the conjugacy problem in $F$. In particular the conjugacy problem in $F$ was first solved by Guba and Sapir in [5] using diagram groups. More recently, Belk and Matucci [1, 2] have another solution using strand diagrams. Kassabov and Matucci [6] also solved the conjugacy problem, the simultaneous conjugacy problem, and analysed roots and centralizers in $F$. Our analysis is different to all of these as we build on the geometric invariants introduced by Brin and Squier.

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We will not prove Theorem 1.1 directly. Rather we prove the following proposition which, given \[3, \text{Theorem 5.3}\], implies Theorem 1.1:

**Proposition 1.2.** Two elements \(f\) and \(g\) of \(F\) that are conjugate in \(\text{PL}^+(0,1)\) are conjugate in \(F\) if and only if \(\Delta_f = \Delta_g\).

Our paper is structured as follows. In §2 we introduce some important background concepts, including the definition of Σ. In §3 we define ∆. In §4 we prove Proposition 1.2. In §5 and §6 we outline formulae which can be used to calculate ∆. Finally, in §7 we discuss roots and centralizers in \(F\).

It is likely that the results in this paper can be extended to hold in Thompson’s groups \(T\) and \(V\); we hope to describe such extensions in a forthcoming paper.

### 2. The definition of Σ

Let \(f\) be a member of Thompson’s group \(F\), embedded in \(\text{PL}^+(0,1)\). Following Brin and Squier \[3\] we define the invariant \(\Sigma_f\) to be a tuple of three quantities, \(\Sigma_1, \Sigma_2\) and \(\Sigma_3\), which depend on \(f\).

The first quantity, \(\Sigma_1\), is a list of integers relating to values of the *signature of \(f, \epsilon_f\).* We define \(\epsilon_f\) as follows:

\[
\epsilon_f : \mathbb{R} \to \{-1, 0, 1\}, \quad x \mapsto \begin{cases} 
1, & f(x) > x; \\
0, & f(x) = x; \\
-1, & f(x) < x.
\end{cases}
\]

If \(f\) is an element of \(\text{PL}^+(0,1)\) then there is a sequence of open intervals

\[
I_1, I_2, \ldots, I_m, \quad I_j = (p_{j-1}, p_j), \quad p_0 = 0, \quad p_m = 1,
\]

such that \(\epsilon_f\) is constant on each interval, and the values of \(\epsilon_f\) on two consecutive intervals differ. We define \(\Sigma_1 = (\epsilon_f(x_1), \ldots, \epsilon_f(x_m))\) where \(x_i \in I_i\) for \(i = 1, \ldots, m\).

Let \(\text{fix}(f)\) be the set of fixed points of \(f\) and observe that the points \(p_0, \ldots, p_m\) from (2.1) all lie in \(\text{fix}(f)\). We say that the interval \(I_j\) is a *bump domain of \(f\) if \(\epsilon_f\) is non-zero on that interval.* Our next two invariants consist of lists with entries for each bump domain of \(f\).

If \(k\) is a piecewise linear map from one interval \((a, b)\) to another, then the *initial slope* of \(k\) is the derivative of \(k\) at any point between \(a\) and the first node of \(k\), and the *final slope* of \(k\) is the derivative of \(k\) at any point between the final node of \(k\) and \(b\). The invariant \(\Sigma_2\) is a list of positive real numbers. The entry for a bump domain \(I_j = (p_{j-1}, p_j)\) is the value of the initial slope of \(f\) in \(I_j\).

Finally, the invariant \(\Sigma_3\) is a list of equivalence classes of *finite functions.* We calculate the entry for a bump domain \(I_j = (p_{j-1}, p_j)\) as follows. Suppose first of all that \(\Sigma_1 = 1\) in \(I_j\). Define, for \(x \in I_j\), the *slope ratio* \(f^*(x) = \frac{f'(x)}{f^0(x)}\). Thus \(f^*(x) = 1\) except when \(x\) is a node of \(f\). Now define

\[
\phi_{f,j} : I_j \to \mathbb{R}, \quad x \mapsto \prod_{n=-\infty}^{\infty} f^*(f^n(x)).
\]
Since \( f \) has only finitely many nodes, only finitely many terms of this infinite product are distinct from 1. Let \( p \) be the smallest node of \( f \) in \( I_j \) and let \( p_* \) be the smallest node of \( f \) in \( I_j \) such that \( \phi_{f,j}(p_*) \neq 1 \) (such a node must exist). Define, for \( s \in [0,1) \),

\[
\psi_{f,j}(s) = \phi_f(\lambda^s(r - p_{j-1}) + p_{j-1}).
\]

Here \( \lambda \) is the entry in \( \Sigma_2 \) corresponding to \( I_j \) and \( r \) is any point in the interval \((0,p)\) which satisfies the formula \( r = f^n(p_*) \) for \( n \) some negative integer.

The function \( \psi_{f,j} \) is a finite function, that is, a function \([0,1) \rightarrow \mathbb{R}^+\) which takes the value 1 at all but finitely many values. In our definition of \( \psi_{f,j} \), we have chosen a value for \( r \) which guarantees that \( \psi_{f,j}(0) \neq 1 \); we can do this by virtue of [3, Lemma 4.4].

The entry for \( \Sigma_3 \) corresponding to \( I_j \) is the equivalence class \([\psi_{f,j}]\), where two finite functions \( c_1 \) and \( c_2 \) are considered equivalent if \( c_1 = c_2 \circ \rho \) where \( \rho \) is a translation of \([0,1)\) modulo 1. If \( f(x) < x \) for each \( x \in I_j \) then the entry for \( \Sigma_3 \) corresponding to \( I_j \) is the equivalence class \([\psi_{f-1,j}]\).

3. The definition of \( \Delta \)

The quantity \( \Delta \) will also be a list, this time a list of equivalence classes of tuples of real numbers. To begin with we need the concept of a minimum cornered function.

3.1. The minimum cornered function. Take \( f \in \text{PL}^+(0,1) \) and focus on the restriction of \( f \) to one of its bump domains \( D = (a,b) \). We adjust one of the definitions of Brin and Squier [3]: for us, a cornered function in \( \text{PL}^+(0,1) \) is an element \( l \) which has a single bump domain \((a,b)\) and which satisfies the following property: \( \Sigma_1 \) takes value 1 (resp. \(-1\)) in relation to \((a,b)\) and there exists a point \( x \in (a,b) \) such that all nodes of \( l \) which lie in \((a,b)\) lie in \((x,l(x))\) (resp. \((l(x),x))\). We will sometimes abuse notation and consider such a cornered function as an element of \( \text{PL}^+(a,b) \).

We say that a cornered function \( l \) corresponds to a finite function \( c \) if \( \psi_l = c \) (we drop the subscript \( i \) here, since there is only one bump domain). Roughly speaking this means that the first node of \( l \) corresponds to \( c(0) \). For a given initial slope \( \lambda \) there is a unique cornered function in \( \text{PL}^+(a,b) \) such that \( \psi_l = c \) (this can be deduced from the proof of Lemma 5.2).

Now let \( c : [0,1) \rightarrow \mathbb{R} \) be a finite function such that \([c]\) is the entry in \( \Sigma_3 \) associated with \( D \). Within this equivalence class \([c]\) we can define a minimum finite function \( c_m \) as follows. First define \( C = \{c_1 \in [c]|c_1(0) \neq 1\} \) and define an ordering on \( C \) as follows. Let \( c_1, c_2 \in C \) and let \( x \) be the smallest value such that \( c_1(x) \neq c_2(x) \). Write \( c_1 < c_2 \) provided \( c_1(x) < c_2(x) \). We define \( c_m \) to be the minimum function in \( C \) under this ordering.

Suppose that \( \lambda \) is the entry in \( \Sigma_2 \) associated with \( D \). Suppose that \( l \) is the cornered function in \( \text{PL}^+(a,b) \), with initial slope \( \lambda \), which corresponds to \( c_m \). We say that \( l \) is the minimum cornered function associated with \( f \) over \( D \).

3.2. The quantity \( \Delta \). Suppose now that \( f \in F \). A bump chain is a subsequence \( I_1, I_{t+1}, \ldots, I_u \) of (2.1) such that each interval is a bump domain, and of the points \( p_{t-1}, p_t, \ldots, p_u \) only \( p_{t-1} \) and \( p_u \) are dyadic. Thus \( I_1, I_2, \ldots, I_m \), can be partitioned into
bump chains and open intervals of fixed points of \( f \) (which have dyadic numbers as end-points).

In [3], conjugating functions in \( \text{PL}^+(0, 1) \) are constructed by dealing with one bump domain at a time. We will construct conjugating functions in \( F \) by dealing with one bump chain at a time. Consider a particular bump chain \( D_1, \ldots, D_s \) and let \( f_j \) be the restriction of \( f \) to \( D_j = (a_j, b_j) \).

According to [3, Theorem 4.18], the centralizer of \( f_j \) within \( \text{PL}^+(a_j, b_j) \) is an infinite cyclic group generated by a root \( \hat{f}_j \) of \( f_j \). We define \( \lambda_j \) to be the initial slope of \( \hat{f}_j \) and \( \mu_j \) to be the final slope of \( \hat{f}_j \). (Let \( m_j \) be the integer such that \( \hat{f}_j^{m_j} = f_j \); then \( \lambda_j \) and \( \mu_j \) are the positive \( m_j \)th roots of the initial and final slopes of \( f_j \).)

Next, let \( k_j \) be a member of \( \text{PL}^+(a_j, b_j) \) that conjugates \( f_j \) to the associated minimum cornered function, \( l_j \), in \( \text{PL}^+(a_j, b_j) \). Thus \( k_j \) is some function satisfying \( k_j f_j k_j^{-1} = l_j \).

Consider the equivalence relation on \( \mathbb{R}^s \) such that \((x_1, \ldots, x_s)\) is equivalent to \((y_1, \ldots, y_s)\) if and only if there are integers \( m, n_1, \ldots, n_s \) such that
\[
2^m x_1 = \lambda_1^{n_1} y_1 \\
\mu_1^{n_1} x_2 = \lambda_2^{n_2} y_2 \\
\mu_2^{n_2} x_3 = \lambda_3^{n_3} y_3 \\
\vdots \\
\mu_{s-1}^{n_{s-1}} x_s = \lambda_s^{n_s} y_s.
\]
It is possible to check whether two \( s \)-tuples of real numbers are equivalent according to the above relation in a finite amount of time because the \( \lambda_i \) and \( \mu_j \) are rational powers of 2. We assign to the chain \( D_1, \ldots, D_s \) the equivalence class of the \( s \)-tuple
\[
\left( \frac{\alpha_1}{w_1}, \frac{\alpha_2 w_1}{w_2 \beta_1}, \ldots, \frac{\alpha_s w_{s-1}}{w_s \beta_{s-1}} \right)
\]
where \( w_j = b_j - a_j \). We define \( \Delta_f \) to consist of an ordered list of such equivalence classes; one per bump chain.

4. Proof of Proposition 1.2

We prove Proposition 1.2 after the following elementary lemma.

**Lemma 4.1.** Let \( f \) and \( g \) be maps in \( F \), and let \( h \) be an element of \( \text{PL}^+(0, 1) \) such that \( hf h^{-1} = g \). Let \( D = (a, b) \) be a bump domain of \( f \) and suppose that the initial slope of \( h \) in \( D \) is an integer power of 2. Then all slopes of \( h \) in \( D \) are integer powers of 2 and all nodes of \( h \) in \( D \) occur in \( \mathbb{Z}[\frac{1}{2}] \).

**Proof.** Let \( (a, a + \delta) \) be a small interval over which \( h \) has constant slope; suppose that this slope is greater than 1. We may assume that \( f \) has initial slope greater than 1 otherwise replace \( f \) with \( f^{-1} \) and \( g \) with \( g^{-1} \). Now observe that \( hf^n h^{-1} = g^n \) for all integers \( n \) and so \( h = g^n h f^{-n} \).
Now, for any \( x \in (a, b) \) there is an interval \((x, x + \epsilon)\) and an integer \( n \) so that \( f^{-n}(x, x + \epsilon) \subset (a, a + \delta) \). Then the equation \( h = g^nhf^{-n} \) implies that, where defined, the derivative of \( h \) over \((x, x + \epsilon)\) is an integral power of 2. Furthermore any node of \( h \) occurring in \((x, x + \epsilon)\) must lie in \( \mathbb{Z}^{[\frac{1}{2}]^2} \) as required.

If \( h \) does not have slope greater than 1 then apply the same argument to \( h^{-1} \) using the equation \( h^{-1}gfh = f \).

We have two elements \( f \) and \( g \) of \( F \) and a third element \( h \) of \( \text{PL}^+(0, 1) \) such that \( hfh^{-1} = g \). We use the notation for \( f \) described in the previous section, such as the quantities \( I_j, p_j, f_j, f_j', k_j, l_j, \alpha_j, \beta_j, w_j, \lambda_j, \) and \( \mu_j \). We need exactly the same quantities for \( g \), and we distinguish the quantities for \( g \) from those for \( f \) by adding a ' after each one. In particular, we choose a bump chain \( D_1, \ldots, D_s \) of \( f \) and define \( D_i' = h(D_i) \) for \( i = 1, \ldots, s \). Note that \( D_1', \ldots, D_s' \) are bump domains but need not form a bump chain for \( g \) according to our assumptions, because \( h \) is not necessarily a member of \( F \).

Let the function \( h_i = h|_{D_i} \) have initial slope \( \gamma_i \) and final slope \( \delta_i \). Let \( u \) be the member of \( \text{PL}^+(0, 1) \) which, for \( i = 1, \ldots, m \), is affine when restricted to \( I_i \), and maps this interval onto \( I_i' \). Notice that, restricted to \( D_i' \), \( ul_iu^{-1} \) is a cornered function which is conjugate to \( l_i' \) (by the map \( k_i'h_i^{-1}k_iu^{-1} \)), and which satisfies \( \psi_{l_i'} = \psi_{ul_iu^{-1}} \). Therefore \( ul_iu^{-1} = l_i' \). Combine this equation with the equations \( k_if_ik_i^{-1} = l_i \), \( k_igk_i^{-1} = l_i' \), and \( h_if_ih_i^{-1} = g_i \) to yield

\[
(k_i^{-1}u^{-1}k_i'h_i)f_i(k_i^{-1}u^{-1}k_i'h_i)^{-1} = f_i.
\]

Therefore \( k_i^{-1}u^{-1}k_i'h_i \) is in the centralizer of \( f \), so there is an integer \( N_i \) such that

\[
h_i = (k_i')^{-1}uk_ih_i^{N_i}
\]

for each \( i = 1, \ldots, s \). Then by comparing initial and final slopes in this equation we see that

\[
\gamma_i = \lambda_i^{N_i} \frac{\alpha_i}{w_i} \alpha_i', \quad \delta_i = \mu_i^{N_i} \frac{\beta_i}{w_i} \beta_i'.
\]

(4.1)

for \( i = 2, \ldots, s \). We are now in a position to prove Proposition 1.2.

**Proof of Proposition 1.2.** Suppose that \( h \in F \). Then there are integers \( M_1, \ldots, M_s \) such that \( \gamma_i = 2^{M_i} \) and \( \gamma_i = \delta_i = 2^{M_i} \) for \( i = 2, \ldots, s \). Substituting these values into (4.1) we see that

\[
2^{M_i} \frac{\alpha_i'}{w_i'} = \lambda_i^{N_i} \frac{\alpha_i}{w_i}, \quad \mu_i^{N_i} \frac{\alpha_i}{w_i} \beta_i^{-1} = \lambda_i^{N_i} \frac{\alpha_i}{w_i} \beta_i^{-1},
\]

for \( i = 2, \ldots, s \), as required.

Conversely, suppose that \( \Delta_f = \Delta_g \). We modify \( h \) so that it is a member of \( F \). If \( I_j \) is an interval of fixed points of \( f \) then modify \( h_j \) so that it is any piecewise linear map from \( I_j \) to \( I_j' \) whose slopes are integer powers of 2, and whose nodes occur in \( \mathbb{Z}^{[\frac{1}{2}]^2} \). (It is straightforward to construct such maps, see [1, Lemma 4.2].)

Now we modify \( h \) on a bump chain \( D_1, \ldots, D_s \). Since \( \Delta_f = \Delta_g \) we know that there are integers \( m \) and \( n_1, \ldots, n_s \) such that, for \( i = 2, \ldots, s \),

\[
2^m \frac{\alpha_i}{w_i} = \lambda_i^{n_1} \frac{\alpha_i'}{w_i'}, \quad \mu_i^{n_i} \frac{\alpha_i}{w_i} \beta_i^{-1} = \lambda_i^{n_1} \frac{\alpha_i}{w_i} \beta_i^{-1}.
\]

(4.2)
Consider the piecewise linear map $h'_i : D_i \to h_i(D_i)$ given by $h'_i = h_i f_i^{t_i - N_i}$. The initial slope $\gamma'_i$ of $h'_i$ is $\gamma_i \lambda_i^{t_i - N_i}$ and the final slope $\delta'_i = \delta_i \mu^{t_i - N_i}$. From (4.1) and (4.2) we see that

$$\gamma'_1 = 2^{-m}, \quad \gamma'_i = \delta'_{i-1}. $$

for $i = 2, \ldots, s$. We modify $h$ by replacing $h_i$ with $h'_i$ on $D_i$. Then $h$ does not have a node at any of the end-points of $D_1, \ldots, D_s$ other than the first and last end-point. By Lemma 4.1 the nodes of $h_1$ occur in $\mathbb{Z}\left[\frac{1}{d}\right]$ and the slopes of $h_1$ are all powers of 2. Since the initial slope of $h_1$ coincides with the final slope of $h_1$, the same can be said of $h_2$. Similarly, for $i = 2, \ldots, s$, the initial slope of $h_i$ coincides with the final slope of $h_{i-1}$. We repeat these modifications for each bump chain of $f$; the resulting conjugating map is a member of $F$. \qed

5. Calculating $\alpha_i$ and $\beta_i$

It may appear that, in order to calculate $\Delta$, it is necessary to construct various conjugating functions. In particular to calculate $\alpha_i$ one might have to construct the function in $PL^+(a_i, b_i)$ which conjugates $f_i$ to the conjugate minimum cornered function in $PL^+(a_i, b_i)$.

It turns out that this is not the case. The values for $\alpha_i$ and $\beta_i$ can be calculated simply by looking at the entries in $\Sigma_1, \Sigma_2$ and $\Sigma_3$ which correspond to $D_i$. In this section we give a formula for $\alpha_i$; we then observe how to use the formula for $\alpha_i$ to calculate $\beta_i$.

In what follows we take $f$ to be a function in $PL^+(a, b)$ such that $f(x) \neq x$ for $x \in (a, b)$. Let $l$ be the minimum cornered function which is conjugate to $f$ in $PL^+(a, b)$.

5.1. Calculating $\alpha_i$. Suppose first that $f(x) > x$ for $x \in (a, b)$. Let $y_j$, for $j = 0, \ldots, t$ be the points at which the finite function $\psi_f$ does not take value 1; let $\psi_f$ take the positive value $z_j$ at the point $y_j$ and assume that $0 = y_0 < y_1 < \cdots < y_t < 1$. We will denote $\psi_f$ by $c_t$ and define $c_j = c_t(x + y_{j+1})$. Then $c_j$ is a translation of $c_t$ under which $y_j$ is mapped to the last point of $c_j$ which does not take value 1.

Let $u_j$ be the cornered function corresponding to $c_j$ and let $x_j$ be the final node of $u_j$. Note that $u_j$ is conjugate to $f$ and, for $j$ equal to some integer $n$, $u_j$ equals $l$, the minimum cornered function. Define the elementary function $h_{x_j, r}$ to be the function which is affine on $(0, x_j)$ and $(x, 1)$ and which has slope ratio $r$ at $x$. We define $\zeta_j$ to be the initial slope of the elementary function $h_{x_j, r}$.

Let $p$ be the first node of $f$ and let $q$ be the first node of $u_t$.

Lemma 5.1. There exists $k$ in $PL^+(a, b)$ such that $kf k^{-1} = l$ and the initial slope of $k$ is

$$\left(\zeta_t \zeta_{t-1} \cdots \zeta_{n+1}\right) \left(\frac{q-a}{p-a}\right).$$

Note that, in the formula just given, $p$ and $q$ stand for the $x$-coordinates of the corresponding nodes. Before we prove Lemma 5.1 we observe that we can calculate values for the $\zeta_j$ and $q$ simply by looking at $\Sigma_2$ and $\Sigma_3$ and using the following lemma:
Lemma 5.2. Let $l$ be a cornered function in $\text{PL}^+(a,b)$ with initial slope $\lambda > 1$, and suppose that the corresponding finite function $c$ takes the value 1 at all points in $[0,1)$ except $0 = s_0 < s_1 < \cdots < s_k < 1$, at which $c(s_i) = z_i$. Then the first node $q_0$ of $l$ is given by the formula
\[
q_0 = a + \frac{(b-a)(1-[\lambda z_0])}{\lambda(1-z_0)+[\lambda s_1+1-1]+[\lambda s_2+1-2]+\cdots+[\lambda s_k+1-1]+[\lambda s_k+1-z_k]},
\]
and the initial slope $\zeta$ of the elementary function $h_{q_k,z_k}$, where $q_k$ is the final node of $l$, is given by
\[
\zeta = \frac{b-a}{\lambda^{s_k}(q_0-a)(1-z_k)+(b-a)z_k}.
\]

Proof. If $q_0, \ldots, q_k$ are the nodes of $l$ we have equations
\[
\lambda^{s_i}(q_0-a) + a = q_i, \quad i = 0, 1, 2, \ldots, k.
\]
Define $q_{k+1} = b$ and let $\lambda_i$ be the slope of $l$ between the nodes $q_{i-1}$ and $q_i$ for $i = 1, \ldots, k + 1$. Then $z_i = \lambda_i/\lambda_{i-1}$ for $i > 1$, and we obtain
\[
\lambda_i = \lambda z_0 \cdots z_{i-1}, \quad i = 1, \ldots, k + 1.
\]
If we substitute (5.3) and (5.4) into the equation
\[
b-a = \lambda(q_0-a) + \lambda_1(q_1-q_0) + \lambda_2(q_2-q_1) + \cdots + \lambda_{k+1}(b-q_k),
\]
then we obtain (5.1). To obtain (5.2), notice that $z_k\zeta$ is the final slope of $h_{q_k,z_k}$, therefore $b-a = \zeta(q_k-a) + z_k\zeta(b-q_k)$. Substitute the value of $q_k$ from (5.3) into this equation to obtain (5.2). \qed

Before we prove Lemma 5.1, we make the following observation. Let $g$ be a function such that $g(x) > x$ for all $x \in (a,b)$ and suppose that $g$ has nodes $p_1 < \cdots < p_s$. Now let $h = h_{p_i,g^*(p_s)}$. Then $hgh^{-1}$ has nodes $h(p_1), \ldots, h(p_{s-1}), h^{-1}(p_s)$ with $(hgh^{-1})^*$ taking on values $g^*(p_1), \ldots, g^*(p_s)$ at the respective nodes. If $h^{-1}(p_s) = h(p_i)$ for some $i$, then $(hgh^{-1})^*$ has value $g^*(p_i)g^*(p_s)$.

Proof of Lemma 5.1. The formula given in Lemma 5.1 arises as follows. We start by finding the conjugator from $f$ to the cornered function $u_t$; then we cycle through the cornered functions $u_j$ until we get to $u_n = l$. Thus the $\frac{q-a}{p-a}$ part of the formula arises from the initial conjugation to a cornered function, and the $\zeta_j$’s arise from the cycling.

Consider this cycling part first and use our observation above on the cornered functions, $u_j$: we have $h_{x_j,z_j}u_j(h_{x_j,z_j})^{-1} = u_{j-1}$. Thus in order to move from $u_t$ to $u_n$ we repeatedly conjugate by elementary functions with initial gradient $\zeta_t, \ldots, \zeta_{n+1}$.

We must now explain why we can use $\frac{q-a}{p-a}$ for the first conjugation which moves from $f$ to $u_t$. It is sufficient to find a function which conjugates $f$ to $u_t$ and which is linear on $[a,p]$.

Consider the effect of applying an elementary conjugation to a function $f$ that is not a cornered function. Suppose that $f$ has nodes $p_1 < \cdots < p_s$. So $p = p_1$. We consider the effect of conjugation by an elementary function $h = h_{p_i,f^*(p_s)}$ as above. To reiterate, we obtain a function with nodes
\[
h(p_1), \ldots, h(p_{s-1}), h^{-1}(p_s)
\]
Now observe that, since \( f^*(p_s) < 1 \), \( h(x) > x \) for all \( x \) and \( h \) is linear on \([a, p_a]\). So clearly \( h \) is linear on the required interval. There are three possibilities:

- If \( hf^{-1}(p_s) < h(p) \) then \( f \) was already a cornered function; in fact \( f = u_t \). We are done.
- If \( hf^{-1}(p_s) > h(p) \) then we simply iterate. We replace \( f \) with \( hfh^{-1}, p \) with \( h(p) \) etc. We conjugate by another elementary function exactly as before. It is clear that the next elementary conjugation will be linear on \([a, h(p)]\) which is sufficient to ensure that the composition is linear on \([a, p]\).
- If \( hf^{-1}(p_s) = h(p) \) then we need to check if \( hfh^{-1} \) is a cornered function. If so then \( hfh^{-1} = u_t \), the corner function we require. If \( hfh^{-1} \) is not a cornered function then we iterate as above, replacing \( f \) with \( hfh^{-1} \). It is possible that \( h(p) \) will no longer be the first node of \( hfh^{-1} \), but in this case we replace \( p \) by \( h(p_2) \). Since \([a, h(p_2)] \supset [a, h(p)]\) this is sufficient to ensure that the composition is linear on \([a, p]\).

We can proceed like this until the process terminates at a cornered function. Since conjugating a non-cornered function by \( h \) preserves \( \psi \) we can be sure that we will terminate at \( u_t \) as required. What is more the composition of these elementary functions is linear on \([a, p]\). \(\blacksquare\)

Suppose next that \( f(x) < x \) for all \( x \in (a, b) \) and \( kfk^{-1} = l \), a minimum cornered function. Observe that \( kf^{-1}k^{-1} = l^{-1} \) and \( f^{-1}(x) > x \) for all \( x \in (a, b) \). We can now apply the formula in Lemma 5.1, replacing \( f \) with \( f^{-1} \) and \( l \) with \( l^{-1} \), to get a value for the initial slope of \( k \).

5.2. **Calculating \( \beta_i \).** The method we have used to calculate \( \alpha_i \) can also be used to calculate \( \beta_i \). Define

\[
\tau : [a, b] \to [a, b], x \mapsto b + a - x.
\]

Now \( \tau \) is an automorphism of \( \text{PL}^+(a, b) \); the graph of a function, when conjugated by \( \tau \), is rotated \( 180^\circ \) about the point \((\frac{b+a}{2}, \frac{b+a}{2})\). Consider the function \( \tau f \tau \) and let \( k \) be the conjugating function from earlier, so that \( kfk^{-1} = l \). Then

\[
(\tau k \tau)(\tau f \tau)(\tau k \tau)^{-1} = (\tau l \tau).
\]

The initial slope of \( \tau k \tau \) equals the final slope of \( k \). Thus we can use the method outlined above – replacing \( f \) with \( \tau f \tau \) and \( l \) with \( \tau l \tau \) – to calculate the initial slope of \( \tau k \tau \). Note that, for this to yield \( \beta_i \), we must make an adjustment to the integer \( n \) in the formula in Lemma 5.1; the function \( \tau l \tau \) is not necessarily the *minimum* cornered function which is conjugate to \( \tau f \tau \). Thus we choose \( n \) to ensure that \( l \) is minimum rather than \( \tau l \tau \).

6. **Calculating \( \lambda_i \) and \( \mu_i \)**

Let \( f \) be a fixed-point free element of \( \text{PL}^+(a, b) \). Let \( \widehat{f} \) be a generator of the centralizer of \( f \) within \( \text{PL}^+(a, b) \). The formula for \( \Delta \) requires that we calculate the initial slope and the final slope of \( \widehat{f} \). It turns out that this is easy—thanks to the work of Brin and Squier [3].
Let $c, c' : [0, 1) \to \mathbb{R}$ be finite functions. We say that $c'$ is the $p$-th root of $c$ provided that, for all $x \in [0, 1)$, we have $c(x) = c'(px)$. The property of having a $p$-th root is preserved by the equivalence used to define $\Sigma_3$. Thus we may talk about the equivalence class $[c]$ having a $p$-th root, provided any representative of $[c]$ has a $p$-th root.

Now \cite{Brin-Squier-1998} Theorem 4.15] asserts that $f$ has a $p$-th root in $PL^+(a, b)$, for $p$ a positive integer, if and only if the single equivalence class in $\Sigma_3$ is a $p$-th power (following Brin and Squier we say that this class has $p$-fold symmetry). What is more \cite{Brin-Squier-1998} Theorem 4.18] asserts that $\hat{f}$ must be a root of $f$.

Thus if $p$ is the largest integer for which the single class in $\Sigma_3$ has $p$-fold symmetry then $\hat{f}$ is the $p$-th root of $f$. The initial slope of $\hat{f}$ is the positive $p$-th root of the initial slope of $f$, and the final slope of $\hat{f}$ is the positive $p$-th root of the final slope of $f$.

7. ROOTS AND CENTRALIZERS

We describe the roots and centralizers in $F$ by extending the work of Brin and Squier on $PL^+(0, 1)$. Let $f$ be a non-trivial element of $F$ with bump domains $E_1, \ldots, E_k$. Let $f_i$ denote the restriction of $f$ to $E_i = (a_i, b_i)$ and, for $i = 1, \ldots, k$, define $m_i$ to be the integer such that the initial slope of $f_i$ is $2^m_i$. Define $p_i$ to be the largest integer such that the entry in $\Sigma_3$ for $E_i$ has $p_i$-fold symmetry.

We are interested in giving conditions for an element to be a root, or a centralizer, of $f$. We also want to give the structure of the following groups:

$$ R(f) = \{ g \in F : g^p = f \text{ for some } a \in \mathbb{Z} \}, \quad C(f) = \{ g \in F : gf = fg \}. $$

**Theorem 7.1.** Let $f$ be a non-trivial element of $F$. Given an integer $p$, there is an element $g$ of $F$ such that $g^p = f$ if and only if $p$ divides each of $p_1, \ldots, p_k, m_1, \ldots, m_k$. Then $R(f) \cong \mathbb{Z}$.

**Proof.** Suppose $F$ contains an element $g$ such that $g^p = f$. Then $g$ shares the same bump domains as $f$ and \cite{Brin-Squier-1998} Theorem 4.15] implies that each entry of $\Sigma_3$ must have $p$-fold symmetry. Hence $p$ divides $p_i$ for $i = 1, \ldots, k$. Furthermore the initial slope $2^{m_i}$ of $g|_{E_i}$ satisfies $2^{m_i} = 2^m$. Therefore $p|m_i$.

Conversely suppose that $p$ is an integer dividing each of $p_1, \ldots, p_k, m_1, \ldots, m_k$. Then \cite{Brin-Squier-1998} Theorem 4.15] implies that there exists a map $g$, which is a $p$-th root of $f$ in $PL^+(0, 1)$. Again, $g$ shares the same bump domains as $f$, and on $E_i$, the initial slope $\gamma_i$ of $g|_{E_i}$ satisfies $\gamma_i^p = 2^{m_i}$. Therefore $\gamma_i = 2^{m_i/p}$. Lemma \ref{lem:2^n} applies to show that, within bump domains, all slopes of $g$ are powers of 2, with all nodes in $D$ occurring in $\mathbb{Z}[\frac{1}{2}]$. What is more, if $E_i$ and $E_{i+1}$ occur in the same bump chain in $f$ then the initial slope of $f$ in $E_{i+1}$ must equal the final slope of $f$ in $E_i$. Clearly this property will also transfer to $g$; hence $g$ is an element of $F$.

Now \cite{Brin-Squier-1998} Theorem 4.15] asserts that a $p$-th root of $f$ is unique in $PL^+(0, 1)$. Hence $R(f)$ is generated by $g$ where $g$ is the smallest root of $f$ in $F$; thus $R(f) \cong \mathbb{Z}$. \hfill \Box

Brin and Squier also proved in \cite{Brin-Squier-1998} Theorem 4.18] that the only maps in $PL^+(a, b)$ that commute with a fixed point free member $g$ of $PL^+(a, b)$ are roots of $g$. We can use this to describe the maps in $F$ which commute with $f$. 
For \( a, b \in \mathbb{Z}[\frac{1}{2}] \cap [0, 1] \), let \( C = (a, b) \) and define \( F_C \) to be the group consisting of those elements in \( F \) which fix \( x \) for \( x \not\in C \). Then [4] Lemma 4.4 implies that \( F_C \cong F \) provided \( b - a \) is a power of 2. In fact it is easy to extend the proof of [4] to prove that, even without this proviso, \( F_C \cong F \): break \( C \) into intervals \((p_{i-1}, p_i)\) for \( i = 1, \ldots, s \) such that \( p_0 = a, p_s = b \) and \( p_i - p_{i-1} \) is a power of 2. Similarly, break \((0, 1)\) into intervals \((q_{i-1}, q_i)\) for \( i = 1, \ldots, s \) such that \( q_0 = 0, q_s = 1 \) and \( q_i - q_{i-1} \) is a power of 2. Note that \( p_i \) and \( q_i \) are dyadic rational numbers for \( i = 0, \ldots, s \). Now define the map \( k : (0, 1) \to (a, b) \) such that \( k(q_i) = p_i \) for \( i = 1, \ldots, s \) and \( k \) is affine on the interval \((q_{i-1}, q_i)\). Then we define a map \( \phi : F \to F_C \) as follows: for \( f \in F \),

\[
(\phi(f))(x) = \begin{cases} 
(kf^{-1})(x), & x \in C; \\
 x, & x \not\in C.
\end{cases}
\]

It is easy to check that \( \phi \) is an isomorphism from \( F \) to \( F_C \).

**Theorem 7.2.** Suppose that \( g \in F \) commutes with \( f \in F \). Then

- the restriction of \( g \) to a bump chain of \( f \) is the root of a restriction of \( f^p \) for some integer \( p \);
- the restriction of \( g \) to a maximal connected open set, \( C \), of \( \text{fix}(f) \) is any element of \( F_C \).

Then \( C_F(f) \cong F^a \times \mathbb{Z}^b \) where \( a \) is the number of non-empty maximal connected open sets in \( \text{fix}(f) \) and \( b \) is the number of bump chains of \( f \).

**Proof.** Let \( D_1, \ldots, D_s \) be a bump chain of \( f \). Now, by [3] Theorem 4.18, we know that \( g|_{D_i} \) is a root of \( f|_{D_i} \) for \( i = 1, \ldots, s \). In order for \( g \) to lie in \( F \), the final slope of \( g \) in \( D_{i-1} \) must equal the initial slope in \( D_i \) for \( i = 1, \ldots, s - 1 \). Since the same is true of \( f \) we conclude that there exist integers \( p \) and \( q \) such that \( g|_{D_i}^q = f|_{D_i}^p \) for \( i = 1, \ldots, s \).

Clearly if \( C \) is a maximal connected open set in \( \text{fix}(f) \) then \( f|_C \) may coincide with any element of \( F_C \). The structure for \( C_F(f) \) follows easily. \( \square \)

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