Sub-sampling and other considerations for efficient risk estimation in large portfolios

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April 6, 2022 ‡

Abstract
Computing risk measures of a financial portfolio comprising thousands of derivatives is a challenging problem because (a) it involves a nested expectation requiring multiple evaluations of the loss of the financial portfolio for different risk scenarios and (b) evaluating the loss of the portfolio is expensive and the cost increases with its size. In this work, we look at applying Multilevel Monte Carlo (MLMC) with adaptive inner sampling to this problem and discuss several practical considerations. In particular, we discuss a sub-sampling strategy whose computational complexity does not increase with the size of the portfolio. We also discuss several control variates that significantly improve the efficiency of MLMC in our setting.

Keywords: Risk estimation, Monte Carlo, Nested simulation, Multilevel Monte Carlo.

AMS Class: 65C05 (Monte Carlo methods), 65C30 (Stochastic differential and integral equations).

1 Introduction
Various risk measures are computed to assess the risk of a financial portfolio. These measures include the probability of a large loss, Value-At-Risk (VaR) and Conditional VaR (CVaR), also called expected shortfall. Computing these risk measures on a large portfolio usually involves two challenges: a nested expectation and a large sum. To be more precise, consider computing the probability that the expected loss exceeds some given $K_\eta \in \mathbb{R}$, that is, we want to compute

$$\eta \defeq \mathbb{P}[\mathbb{E}[\Lambda | R_\tau] > K_\eta] = \mathbb{E}[\mathbb{H}(\mathbb{E}[\Lambda | R_\tau] - K_\eta)],$$

where $\mathbb{E}[\Lambda | R_\tau]$ is the risk-neutral, expected loss given some risk scenario, $R_\tau$, at some short risk horizon, $\tau$, and $\mathbb{H}(\cdot)$ is the Heaviside function. For example, when considering market risk, the risk scenario is the values of the underlying assets at some risk horizon, $\tau$, which affect the loss incurred by the portfolio at maturity. The loss is usually an aggregate of many losses from different financial derivatives depending on a set of common underlying assets. That is

$$\Lambda \equiv \frac{1}{P} \sum_{i=1}^{P} \Lambda_i,$$

where $P$ is the total number of derivatives and $\Lambda_i$ is the loss incurred by the $i$’th derivative. The $1/P$ factor is a normalization factor that ensures boundedness as the number of derivatives in the
portfolio, $P$, increases. In realistic portfolios, the derivatives are heterogeneous in their evaluation. Some derivatives can be computed analytically, other derivatives have to be approximated by simulating the underlying assets, others still depend on assets which can only be sampled approximately. Moreover, the nominal values of these derivatives can vary greatly; a few derivatives might have large nominal values and thus contribute significantly to the total loss compared to the majority of derivatives.

A straightforward method to approximate the probability of a large expected loss is to simulate the nested expectation in (1) using Monte Carlo. That is, $M$ independent scenarios of the risk parameter, $R_x$, are sampled and, for each risk scenario, $N$ independent samples of the total loss $\Lambda$ are sampled by evaluating the sum in (2). This method was explored by Gordy & Juneja [12] who showed that the bias in the outer expectation is related to the variance of the estimator of the inner expectation. See also [10] for sharper and extended analysis of their results. Hence, using $N$ samples to estimate each inner expectation, $\mathbb{E}[\Lambda_i | R_x]$, the bias in the outer estimator is $O(N^{-1}P^{-1})$. Setting $N = O(\max(1, \varepsilon^{-1}P^{-1}))$ and $M = O(\varepsilon^{-2})$ to achieve a root mean-squared (RMS) error $\varepsilon$, and since evaluating $\Lambda$ is an $O(P)$ operation, the total computational complexity is $O(\max(P\varepsilon^{-2}, \varepsilon^{-3}))$. Additionally, Gordy & Juneja propose handling heterogeneous derivatives with different nominal values or different computational cost in the portfolio by proportionally dividing the $N$ samples amongst the different derivatives instead of evaluating the sum [12, Section 3.4], see also Subsection 2.1.

In a previous work [8], the authors showed how to combine Multilevel Monte Carlo (MLMC), as introduced by Giles [4], with adaptive sampling, as introduced by Broadie et. al. [1], to estimate quantities of the form $\mathbb{E}[H(\mathbb{E}[X | Y])]$ for two random variables $X$ and $Y$. Using this strategy, for $Y \equiv R_x$ and $X \equiv \Lambda - K$, the probability of a large expected loss can be estimated with a reduced computational complexity of $O(\max(P\varepsilon^{-2}, \varepsilon^{-2}\log\varepsilon^2))$. This computational complexity is an improvement compared to that of Monte Carlo but it still suffers from the dependence on the number of derivatives, $P$, which, as mentioned, can be significant for large portfolios.

The objective of this paper is two-fold: (i) to introduce random sub-sampling in the context of pricing derivatives or computing risk measures and (ii) to show how several computational strategies can be combined in a unified framework for efficient computation of risk measures in large financial portfolios. First, in Section 2 we discuss sub-sampling strategies to handle large sums of heterogeneous terms and present a method whose computational complexity does not depend on the number of terms in the sum. Then, in Section 3 we apply this method to our motivating problem involving a large portfolio, discuss several variance reduction techniques and show how to handle different computation models for $\mathbb{E}[\Lambda_i | R_x]$. In Section 4 we discuss how to apply Multilevel Monte Carlo and adaptive sampling to obtain a method whose computational complexity is $O(\varepsilon^{-2}\log\varepsilon^2)$ to achieve a RMS error $\varepsilon$, independently of the number of derivatives. Finally, in Section 5, we apply our results to fictitious portfolios with heterogeneous derivatives to illustrate the benefit of the methods that are presented in the current work.

## 2 Random Sub-sampling

In this section, we discuss unbiased methods to estimate an expectation involving a sum of terms $\{f_i\}_{i=1}^P$, for a large, fixed number of terms, $P$,

$$\mathbb{E}\left[\frac{1}{P} \sum_{i=1}^P f_i\right].$$

(3)

We focus on this generic problem in the current section and later apply the discussed strategies to approximate the inner conditional expectation in (1), for a given risk scenario $R_x$, and discuss how to relate the terms $\{f_i\}_{i=1}^P$ to the losses $\{\Lambda_i\}_{i=1}^P$, depending on the computational model of $\mathbb{E}[\Lambda_i]$.
We will initially assume that the terms \( \{ f_i \} \) are mutually independent (or, in the case of considering conditional expectation, conditionally independent) and discuss the general case later.

A naïve Monte Carlo estimator of (3) with \( N \geq 1 \) samples of the sum requires a minimum budget equal to the cost to compute the sum once. The minimum budget thus increases with the number of terms \( P \). Instead, we use a random sub-sampler based on the observation that

\[
\frac{1}{NP} \sum_{n=1}^{N} f_j^{(n)} p_{j^{(n)}},
\]

where \( j \) is a random integer with \( \mathbb{P}[ j = i ] = p_i \) for \( i \in \{1, \ldots, P\} \) and zero otherwise. Using \( N \) samples in a Monte Carlo estimator to estimate \( \mathbb{E}[ f_j / (P p_j) ] \), the resulting estimator is then

\[
\frac{1}{NP} \sum_{n=1}^{N} f_j^{(n)} p_{j^{(n)}}^{-1},
\]

where \( j^{(n)} \) is the \( n \)'th sample of the random integer \( j \) and \( f_j^{(n)} \) is the \( n \)'th sample of \( f_j \). The variance of this estimator, which is equal to the mean-square error (MSE) since the estimator is unbiased, is

\[
\text{Var} \left[ \frac{1}{NP} \sum_{n=1}^{N} f_j^{(n)} p_{j^{(n)}}^{-1} \right] = \frac{1}{NP^2} \text{Var}[ f_j / p_j ]
\]

\[
= \frac{1}{NP^2} \left( \frac{1}{P} \sum_{i=1}^{P} g_i^2 p_i^{-1} - \left( \frac{1}{P} \sum_{i=1}^{P} \mathbb{E}[ f_i ] \right)^2 \right),
\]

where \( g_i^2 \overset{df}{=} \mathbb{E}[ f_i^2 ] \). On the other hand, the expected total work is \( N \sum_{i=1}^{P} p_i W_i \) where \( W_i \) is the work required to sample the term \( f_i \). Minimizing the variance of the estimator subject to fixed expected total work leads to the choice \( p_i \propto g_i / W_i^{1/2} \). By using an estimate of \( g_i \), denoted by \( \tilde{g}_i \), and imposing the constraint of the probabilities summing up to 1, we set

\[
p_i = \frac{\tilde{g}_i / W_i^{1/2}}{\sum_{j=1}^{P} \tilde{g}_j / W_j^{1/2}}. \tag{4}
\]

The work of this random sub-sampler is

\[
N \frac{\sum_{i=1}^{P} \tilde{g}_i W_i^{1/2}}{\sum_{i=1}^{P} \tilde{g}_i / W_i^{1/2}}.
\]

Assuming we have a total budget \( B \) to approximate (3), we set

\[
N \equiv B \frac{\sum_{i=1}^{P} \tilde{g}_i W_i^{1/2}}{\sum_{i=1}^{P} \tilde{g}_i / W_i^{1/2}}.
\]

Here, we ignore the restriction of the number of samples, \( N \), to integers and treat it as a real number instead. Note that rounding the number of samples up increases the total computational cost by \( \max_i W_i \) at most. In any case, using the previous real value of \( N \), the optimal variance can then be bounded as

\[
\frac{1}{NP^2} \text{Var}[ f_j / p_j ] \leq \frac{1}{N} \left( \frac{1}{P} \sum_{i=1}^{P} \frac{g_i^2}{\tilde{g}_i} W_i^{1/2} \right) \left( \frac{1}{P} \sum_{i=1}^{P} \frac{\tilde{g}_i}{W_i^{1/2}} \right)
\]

\[
\leq \frac{1}{B} \left( \frac{1}{P} \sum_{i=1}^{P} \frac{g_i^2}{\tilde{g}_i} W_i^{1/2} \right) \left( \frac{1}{P} \sum_{i=1}^{P} \tilde{g}_i W_i^{1/2} \right). \tag{5}
\]
If we further assume that \( g_i \leq c \tilde{g}_i \) for some constant \( c > 0 \) and that \( P^{-1} \sum_{i=1}^{P} \tilde{g}_i W_i^{1/2} \leq C \), for some \( C > 0 \), then the variance of the estimator is \( O(B^{-1}) \), independently of \( P \), while the total cost of the estimator is \( B \), up to the rounding of \( N \). Under these same conditions, the previous discussion applies even in the limit as \( P \to \infty \). For finite \( P \), we note that in the typical case when, for every \( i \in \{1, \ldots, P\} \), we have that \( g_i \) and the work estimate \( W_i \) do not increase with \( P \) and \( \tilde{g}_i \) is bounded from below, we can simply use \( c = \max_i (g_i / \tilde{g}_i) \) and \( C = \max_i \tilde{g}_i W_i^{1/2} \).

### 2.1 Mixed sub-sampling

Another way to handle heterogeneous terms is to use deterministic, stratified sub-sampling. This was explored in the current context of computing probabilities of a large loss by Gordy & Juneja [12, Section 3.4]. Applied to our setting, we write

\[
\mathbb{E} \left[ \frac{1}{P} \sum_{i=1}^{P} f_i \right] \approx \frac{1}{P} \sum_{i=1}^{P} \frac{1}{PN_i} \sum_{n=1}^{N_i} f_i^{(n)}, \tag{6}
\]

where \( N_i \geq 1 \) is the number of samples of the \( i \)-th term. The variance of this unbiased estimator is 

\[
P^{-2} \sum_{i=1}^{P} \frac{\sigma_i^2}{\tilde{\sigma}_i W_i^{1/2}}, \tag{7}
\]

where \( \sigma_i^2 \equiv \text{Var}[f_i] \), while the work is \( \sum_{i=1}^{P} N_i W_i \). Similar to random sub-sampling, we minimize the variance subject to a budget constraint, \( B \), to find the optimal number of samples for the \( i \)-th term

\[
N_i = B \cdot \frac{\tilde{\sigma}_i W_i^{1/2}}{\sum_{j=1}^{P} \tilde{\sigma}_j W_j^{1/2}},
\]

assuming we have estimates of \( \sigma_i \) denoted by \( \tilde{\sigma}_i \). Note that we again ignore the integer constraints on \( N_i \) and treat it as a real number. The optimal variance is bounded by

\[
\frac{1}{B} \left( \frac{1}{P} \sum_{i=1}^{P} \frac{\sigma_i^2}{\tilde{\sigma}_i W_i^{1/2}} \right) \left( \frac{1}{P} \sum_{i=1}^{P} \tilde{\sigma}_i W_i^{1/2} \right), \tag{8}
\]

assuming \( N_i > 1 \) for all \( i \). If we further assume that \( \sigma_i \leq c \tilde{\sigma}_i \) for some constant \( c \) and that \( P^{-1} \sum_{i=1}^{P} \tilde{\sigma}_i W_i^{1/2} \leq C \) for some \( C > 0 \), then the variance is \( O(B^{-1}) \), independently of \( P \) and similar to random sampling. However, a crucial constraint is that the budget, \( B \), must be sufficiently large so that \( N_i \geq 1 \) in (7) for all \( i \); otherwise the estimator (6) is biased. In particular, the budget must be at least \( \sum_{i=1}^{P} W_i \) to have at least one sample per term. This leads to a computational complexity that depends on the number of terms in the sum, unlike random sub-sampling. On the other hand, the variance of the stratified sub-sampler in (8) is always smaller than the variance of the random sub-sampler in (5). The variance reduction roughly scales with \( P^{-1} \sum_{i=1}^{P} (\tilde{g}_i - \tilde{\sigma}_i) W_i^{1/2} \) which is bounded independently of \( P \). In other words, in our setting, using random sub-sampling rather than stratified sub-sampling increases the error by a constant independent of \( P \).

We can also combine random and stratified sub-sampling as follows

\[
\mathbb{E} \left[ \frac{1}{P} \sum_{i=1}^{P} f_i \right] = \mathbb{E} \left[ \frac{1}{P} \sum_{i=1}^{K} f_i \right] + \frac{1}{P} \mathbb{E}[f_j / p_j],
\]

where \( P[j = i] = p_i \) for \( j \in \{K + 1, \ldots, P\} \) and is zero otherwise. Then the sum of the first \( K \) terms is approximated using stratified sub-sampling while the sum of the remaining \((P - K)\) is approximated using random sub-sampling. Compared to random sub-sampling, this new sub-sampler evidently leads to smaller variance for a fixed budget when the \( K \) terms are themselves deterministic, i.e., \( \mathbb{E}[f_i] = f_i \) for \( i \leq K \). In this case, evaluating the sum of the \( K \) terms directly increases the work by \( \sum_{i=1}^{K} W_i \) but decreases the variance by \( \sum_{i=1}^{K} \tilde{g}_i W_i^{1/2} \), approximately. Assuming the
budget is larger than $\sum_{i=1}^{K} W_i$ and by picking those $K$ terms to have large $\tilde{g}_i/W_i^{1/2}$, i.e., large nominal value or small cost, we can ensure the increase in cost is small compared to the decrease in the error. To further illustrate this point, consider the case when $W_i = 1$ for all $i = 1, \ldots, P$ and $\{f_i\}_{i=1}^{P}$ are all deterministic, i.e., we are simply estimating the average $P^{-1}\sum_{i=1}^{P} f_i$ using a computational budget $B \leq P$; when $B \geq P$ we can compute the average directly. The mixed sub-sampler can then be written as

$$\frac{1}{P} \sum_{i=1}^{P} f_i \approx \frac{1}{P} \sum_{i=1}^{K} f_i + \frac{P - K}{P (B - K)} \sum_{n=1}^{B-K} f_{j(n)},$$

for $K \geq 0$ and where $j$ is a random integer over $\{K + 1, \ldots, P\}$. The variance is

$$\frac{(P - K)^2}{P^2 (B - K)} \cdot \text{Var}[f_j] \leq (\max_i f_i^2) \frac{(P - K)^2}{P^2 (B - K)}.$$  

The optimal value of $K$ which minimizes the variance is $\min(0, 2B - P)$ and the corresponding variance is bounded by

$$(\max_i f_i^2) \cdot \begin{cases} 4 (P - B)/P^2 & P/2 \leq B \leq P \\ 1/B & 0 < B \leq P/2. \end{cases}$$

This is consistent with intuition: when the computational budget passes a certain threshold, in this case $P/2$, sub-sampling some terms deterministically leads to smaller variance for the same computational budget.

More generally, determining if a particular term $f_i$ should be sub-sampled deterministically or randomly for a given budget $B$ requires good estimates of both $\tilde{g}_i \approx g_i$ and $\tilde{\sigma}_i \approx \sigma_i$ (compare (5) and (8)), and hence of $E[f_i]$, the quantity we are trying to estimate. If the optimal strategy is to sub-sample $f_i$, deterministically instead of randomly, the variance reduction roughly scales with the difference, $\frac{(\tilde{g}_i - \tilde{\sigma}_i)W_i^{1/2}}{W_i^{1/2}}$. Considering the need for additional estimates, the optimization of the sub-sampling strategy for a term, $f_i$, is worthwhile only when the budget is sufficiently large compared to the number of term $P$ and we know that $\frac{(\tilde{g}_i - \tilde{\sigma}_i)W_i^{1/2}}{W_i^{1/2}}$ is large, which is maximal when $f_i$ is deterministic. Hence, when considering a portfolio of terms, the variance reduction will be significant if the portfolio contains mostly deterministic terms or terms with small variability. Additionally, using mixed sub-sampling complicates analysis and precludes the application of other computational methods, such as using antithetic sub-sampling in MLMC, c.f. Section 4. Based on these observation, and several numerical experiments, we have found that mixed sub-sampling is not worthwhile in most practical cases, including the example that we consider in Section 5.

### 2.2 Dependent $f_i$

In the beginning of this section, we assumed that $\{f_i\}_{i=1}^{P}$ are mutually independent. In real applications, including the ones we consider in this work, some of these terms might depend on a set of common underlying random variables. Nevertheless, we can use independent samples of those underlying random variables when sampling $f_i$ to get independent samples of $f_i$ and the previous discussion applies. Clearly such re-sampling introduces additional overhead since we have to re-sample the common underlying random variables.

On the other hand, this re-sampling has several advantages. In addition to simplifying analysis and implementation and making the parallelization of the sampler easier, Gordy & Juneja [12, Section 3] argue that re-sampling the common random factors is advisable to ensure that the Monte Carlo errors cancel out at the portfolio level. Another advantage is that this re-sampling allows us to optimize the number of samples per term based on estimates of the second moments or variance of $\{f_i\}_{i=1}^{P}$. Because of these advantages, we argue that re-sampling is the prudent...
choice in most situations. It should be noted however that terms that are known to be negatively correlated should be sampled together to reduce the overall variance and hence the computational cost. In Section 3 we will see additional strategies to reduce the variability of the loss variables, $\Lambda_i$, in certain settings.

## 3 Probability of Loss as a Nested Expectation

In this section, we focus on our motivating problem of evaluating the probability of a large loss of a financial portfolio under market risk. We will focus on a model for the loss of a derivative that can be written as a difference between $V_i,\tau$, the discounted value of the derivative given the risk scenario, $R_{\tau}$, at the risk horizon, $\tau$, and $V_{i,0}$, the risk-neutral discounted value at initial time. That is

$$E_Q[\Lambda_i | R_{\tau}] = V_{i,0} - V_i,\tau$$

$$= E_Q[h_i(S)] - E_Q[h_i(S) | S(\tau) = R_{\tau}].$$

Here, $Q$ is the risk-neutral measure and $h_i$ is the discounted payoff functional which depends on the asset process, $S$. We will also assume that $S$ is a stochastic process satisfying an Itô stochastic differential equation (SDE)

$$dS(t) = a(t, S(t)) dt + b(t, S(t)) dB(t),$$

for some sufficiently smooth coefficients, $a$ and $b$, and a Brownian process, $\{B(t)\}_{t \geq 0}$. Recall that we are interested in computing

$$\eta = \mathbb{P}[E_Q[\Lambda_i | R_{\tau}] > \mathcal{K}_\eta]$$

$$= \mathbb{E}[H(E_Q[\Lambda_i] - \mathcal{K}_\eta | R_{\tau})]$$

$$= \mathbb{E}[H\left(E_Q\left[\frac{1}{P} \sum_{i=1}^{P} \Lambda_i - \mathcal{K}_\eta \bigg| R_{\tau}\right]\right)]$$,

for a given $\mathcal{K}_\eta \in \mathbb{R}$ and $Q$ and $\mathbb{P}$ being the risk-neutral and physical measures, respectively. Since we consider the market risk, the risk parameter, $R_{\tau}$, is the asset value, $S(\tau)$, in the physical measure, $\mathbb{P}$, at the risk horizon $\tau$.

We will consider three common categories of computation models for $E_Q[\Lambda_i | R_{\tau}]$ and, for each computation model, we will discuss different strategies to reduce the variability of $\Lambda_i$ which in turn reduces the bias of a Monte Carlo estimator of $\eta$, as discussed in the introduction. At the end of this section, we will construct a “portfolio of terms”, $\{f_i\}_{i=1}^{P}$, such that

$$E_Q\left[\frac{1}{P} \sum_{i=1}^{P} \Lambda_i \bigg| R_{\tau}\right] = E_Q\left[\frac{1}{P} \sum_{i=1}^{P} f_i \bigg| R_{\tau}\right].$$

Then we can apply the sub-sampling strategies that were discussed in the previous section when computing the inner expectation of the sum. Recall that when using a random sub-sampler to estimate the right hand side in the previous equation the optimal probabilities depend on estimates of the work required to sample $f_i$ and of $g_i^2 = E_Q[ f_i^2 ] | R_{\tau}$ for every $i \in \{1, 2, \ldots, P\}$, i.e., estimating $g_i$ ultimately depends on the risk scenario. For an estimator of $\eta$ which is based on sampling many risk scenarios this is clearly too costly, with a cost that grows with $P$ which is counter to our original objective of devising a method whose computational complexity does not depend on $P$. Instead, we propose to use estimates $\tilde{g}_i \approx g_i$ that do not depend on the risk scenario. For example, we may assign them to values that represent the relative importance of an derivative compared to the others, or we may assign $\tilde{g}_i = E_Q[ f_i^2 ]$ for all $i$ and all risk scenarios.
3.1 Exact, deterministic evaluation

For some derivatives, \(\Lambda_i\) might be deterministic when conditioned on the risk scenario \(R_\tau\), or we may be able to directly, with unit cost, compute \(\mathbb{E}_Q[\Lambda_i \mid R_\tau]\) exactly, or almost exactly, given the risk scenario \(R_\tau\). For example, when considering put or call options on assets that follow Geometric Brownian processes, we may be able to solve the Black-Scholes partial differential equation (PDE) analytically or numerically with sufficient accuracy. Note that, the Black-Scholes PDE needs to be solved only once to compute \(\mathbb{E}_Q[\Lambda_i \mid R_\tau]\) for all risk scenarios \(R_\tau\), hence we may consider approximating the solution to the PDE as offline work. In this case, we set \(f_i \equiv \mathbb{E}_Q[\Lambda_i \mid R_\tau]\) for a given \(R_\tau\). Note that for a given risk scenario \(R_\tau\), \(f_i\) is deterministic with zero variance and the cost to compute it is \(O(1)\).

Delta Control Variate Using the Delta Greek to construct a control variate for the probability of large loss is well-known, c.f. [11, 13], and we recall the basic idea here. Recall that the expected loss incurred by derivative \(i\) given a risk scenario, \(R_\tau\), is written as a difference, i.e., \(\mathbb{E}_Q[\Lambda_i \mid R_\tau] \equiv V_{i,0} - V_{i,\tau}\). Then, using an Itô expansion yields

\[
\mathbb{E}_Q[\Lambda_i^2 \mid R_\tau] = ((R_0 - R_\tau) \cdot \nabla_{R_0} V_{i,0})^2 + O(\tau^2),
\]

where \(R_0 \equiv S(0)\) and, for \(R_\tau\) being the price of the underlying asset, \(\nabla_{R_0} V_{i,0}\) is the Delta Greek. The first term dominates in the previous expression since the risk parameter is an Itô process, \(R_\tau \equiv S(\tau)\), yielding \(\mathbb{E}[|R_\tau - R_0|^2] = O(\tau)\). By subtracting this term, we can define a new loss variable, \(\hat{\Lambda}_i \equiv \Lambda_i - (R_0 - R_\tau) \cdot \nabla_{R_0} V_{i,0}\), for a given risk scenario, \(R_\tau\), and a new loss threshold, which depends on the risk scenario,

\[
\hat{\kappa}_\tau = \kappa_\tau - (R_0 - R_\tau) \cdot \nabla_{R_0} V_0 = \frac{1}{P} \sum_{i=1}^{P} \nabla_{R_0} V_{i,0},
\]

where \(\nabla_{R_0} V_0 = \frac{1}{P} \sum_{i=1}^{P} \nabla_{R_0} V_{i,0}\). So that

\[
\mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^{P} \hat{\Lambda}_i - \hat{\kappa}_\tau \mid R_\tau \right] = \mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^{P} \Lambda_i - \kappa_\tau \mid R_\tau \right],
\]

with \(\mathbb{E}_Q[\hat{\Lambda}_i^2 \mid R_\tau] = O(\tau^2)\). Hence, we have the deterministic term \(f_i \equiv \mathbb{E}_Q[\hat{\Lambda}_i \mid R_\tau]\) with a second moment \(O(\tau^2) \ll O(\tau)\) since \(\tau \ll 1\). Note that \(\nabla_{R_0} V_{i,0}\) is independent of the risk scenario, \(R_\tau\), for all \(i\) and can be computed once for all risk scenarios as offline work. If the portfolio is delta-hedged then \(\nabla_{R_0} V_0 = 0\).

3.2 Exact simulation

In some settings, we might be able to exactly sample \(\Lambda_i\) for a given risk scenario \(R_\tau\), but cannot compute \(\mathbb{E}_Q[\Lambda_i \mid R_\tau]\) exactly. This is the case for example for exotic options or underlying assets involving high dimensional Itô processes, but when we might still be able to solve the underlying SDEs analytically to exactly sample \(\Lambda_i\) for a given \(R_\tau\), e.g., when the SDE solution is a Geometric Brownian Motion. In this case, we simply set \(f_i \equiv \Lambda_i\). Note that, for a given risk scenario \(R_\tau\), the term \(f_i\) has non-zero variance and the cost to compute it is again \(O(1)\).

Reducing the variance of \(\Lambda_i\) Denote by \(S_{t,x}\) the solution of (9) given \(S(t) = x\), then we can write

\[
\Lambda_i = h_i(S_{0,R_0}) - h_i(S_{t,R_\tau}) = h_i(S_{t,S(\tau)}) - h_i(S_{t,R_\tau}).
\]
Hence, to sample \( \Lambda_i \) for a given risk scenario \( R_\tau \), we need to first sample \( S(\tau) \), which requires sampling a Brownian path \( \{ B(t) \}_{0 \leq t \leq \tau} \). Then, we sample \( \{ S_{\tau,R}(t) \}_{t \geq \tau} \) and \( \{ S_{\tau,R}(t) \}_{t \geq \tau} \) starting from \( S(\tau) \) and \( R_\tau \), respectively, which requires sampling one shared Brownian path \( \{ B(t) \}_{t \geq \tau} \). While we could use two independent Brownian paths to sample two independents paths \( S_{\tau,R}^{(1)} \) and \( S_{\tau,R}^{(2)} \), this would yield a larger second moment. For example when \( h_i(S) \equiv h_i(S(T)) \) for some maturity, \( T \gg \tau \), i.e., the payoff is a function of the asset value at maturity, and for a sufficiently smooth payoff functional, \( h_i \), we have

\[
\mathbb{E}_Q \left[ \Lambda_i^2 \mid R_\tau \right] = \mathbb{E}_Q \left[ |S(\tau) - R_\tau|^2 \mid R_\tau \right] + \mathcal{O} \left( \mathbb{E}_Q \left[ \left| S_{\tau,R}^{(1)}(T) - S_{\tau,R}^{(2)}(T) \right|^2 \mid R_\tau \right] \right).
\]

Here, the second term dominates since \( T \gg \tau \). Using a shared Brownian path to sample \( \{ S_{\tau,R}(t) \}_{t \geq \tau} \) and \( \{ S_{\tau,R}(t) \}_{t \geq \tau} \) and for a sufficiently smooth payoff functional, \( h_i \), we write

\[
\mathbb{E}[\Lambda_i^2 \mid R_\tau] \leq 2 \mathbb{E}[ (h_i(S_{0,R_0}) - h_i(S_{0,R_0}))^2 \mid R_\tau] + 2 \mathbb{E}[ (h_i(S_{0,R_0}) - h_i(S_{0,R_0}))^2 \mid R_\tau] 
= 2 \mathbb{E} \left[ ((R_0 - R_\tau) \nabla_{R_0} h_i(S_{0,R_0}))^2 \mid R_\tau \right] + \mathcal{O}(\mathbb{E}[|S_{0,R_0} - S_{0,R_0}|^2 \mid R_\tau]) + \mathcal{O}(\tau^2),
\]

where \( S_{0,R_0} \) is the solution of (9) given \( S(0) = R_\tau \). Here, both \( \mathbb{E}[|R_\tau - R_0|^2] \) and \( \mathbb{E}[|S_{0,R_0} - S_{0,R_0}|^2 \mid R_\tau] \) are \( \mathcal{O}(\tau) \). Hence, to reduce the variance of \( \Lambda_i \), we will use control variates to eliminate the terms involving these factors. Starting with the second term, where we use an antithetic variates approach. As a general methodology, this is a standard approach to variance reduction [11] which has been used previously for pricing American options [2] and also for nested simulation within Multilevel Monte Carlo [3, 6, 8]. However, the specific treatment used here for estimating portfolio losses does not appear to have been previously published. We denote by \( S^+(\tau) \) and \( S^-(\tau) \) the two antithetic Itô processes that both start from \( S^+(0) = S^-(0) = R_0 \) and depend on the Brownian paths \( \{ B(t) \}_{0 \leq t \leq \tau} \) and \( \{ -B(t) \}_{0 \leq t \leq \tau} \), respectively. Then we set

\[
\bar{A}_i \equiv \frac{1}{2} (h_i(S_{\tau,S^+(\tau)}) + h_i(S_{\tau,S^-(\tau)})) - h_i(S_{\tau,R_\tau}),
\]

where all three processes, \( S_{\tau,R_\tau} \), \( S_{\tau,S^+(\tau)} \) and \( S_{\tau,S^-(\tau)} \) use the same Brownian path \( \{ B(t) \}_{t \geq \tau} \). Then, we have that \( \mathbb{E}_Q \left[ \bar{A}_i \mid R_\tau \right] = \mathbb{E}_Q[\Lambda_i \mid R_\tau] \) and, defining \( S_{t,x}^+ \) to be the solution of (9) given \( S^+(t) = x \) and using the Brownian path \( \{ B(s) \}_{s \geq t} \), while \( S_{t,x}^- \) uses the Brownian path \( \{ -B(s) \}_{t \leq s \leq \tau} \) and \( \{ B(s) \}_{\max(\tau,t) \leq s} \), then for sufficiently smooth payoff, \( h_i \),

\[
\mathbb{E} \left[ \bar{A}_i^2 \mid R_\tau \right] = 2 \mathbb{E} \left[ \left( \frac{1}{2} (R_0 - R_\tau) \nabla_{R_0} h_i(S_{0,R_0}^+) + \nabla_{R_0} h_i(S_{0,R_0}^-) \right) \right] \mid R_\tau 
+ \mathcal{O} \left( \mathbb{E} \left[ \left( \frac{1}{2} (S_{0,R_\tau}^+ + S_{0,R_\tau}^-) - S_{\tau,R_\tau} \right)^2 \mid R_\tau \right] \right) + \mathcal{O}(\tau^2).
\]

Here, assuming the SDE coefficients are sufficiently smooth, the second term is now \( \mathcal{O}(\tau^2) \ll \mathcal{O}(\tau) \) since \( \tau \ll 1 \).

Finally, similar to Subsection 3.1, we can use the Delta control variate to eliminate the remaining \( \mathcal{O}(\tau) \) term by defining

\[
\hat{A}_i \equiv \bar{A}_i - \frac{1}{2} (R_0 - R_\tau) D_i
\]

\[
D_i \equiv \nabla_{R_0} h_i(S_{0,R_0}^+) + \nabla_{R_0} h_i(S_{0,R_0}^-),
\]

where we assume here that \( h_i(S) \) is differentiable with respect to the initial state, \( R_0 \). We also modify the loss threshold, \( \mathcal{K}_\eta \), as in (10) so that

\[
\mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^{P} \hat{A}_i - \hat{\mathcal{K}}_\eta \right] = \mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^{P} \Lambda_i - \mathcal{K}_\eta \right],
\]
since
\[ \nabla \theta \{ V \} = \mathbb{E}_\theta [ \nabla \theta \{ h_i (S_{0 \cdot R_T}) \} ] = \mathbb{E}_\theta [ \nabla \theta \{ h_i (S_{0 \cdot R_T}) \} ]. \]

Recall that \( \nabla \theta \{ V \} \) is independent of the risk scenario, \( R_T \), for all \( i \) and can be computed once for all risk scenarios as offline work. In summary, to sample \( \hat{\Lambda} \), we use all the variance reduction techniques that were discussed above: (a) the delta control variate (b) the antithetic pair \( S^+ (\tau) \), and \( S^- (\tau) \) and (c) the same Brownian path \( \{ B(t) \}_{t \geq \tau} \) when simulating \( S_{R_T}, S_{R_T} S^+ (\tau) \) and \( S_{R_T} S^- (\tau) \).

Indeed, all three variance reduction techniques ensure that \( \mathbb{E} \left[ (\hat{\Lambda}_i)^2 \mid R_T \right] = \mathcal{O}(\tau^2) \) compared to \( \mathbb{E} \left[ \Lambda_i^2 \mid R_T \right] = \mathcal{O}(\tau) \).

### 3.3 Approximate simulation

More generally, for some derivatives we might be only able to approximately sample \( \Lambda_i \) for a given risk scenario \( R_T \). This is the case for example if (9) cannot be solved analytically and we have to use a numerical scheme to approximate samples of the process, \( S \), and then compute the loss to obtain an approximate sample of \( \Lambda_i \). The cost per an approximate sample of \( \Lambda_i \) increases as the approximation error, and consequently the bias when estimating \( \mathbb{E}_\theta \{ \Lambda_i \mid R_T \} \), decreases.

Nevertheless, using Unbiased MLMC [15], we can, in certain cases, obtain an unbiased Monte Carlo estimator of \( \mathbb{E}_\theta \{ \Lambda_i \mid R_T \} \) using samples whose expected cost is \( \mathcal{O}(1) \). To briefly present Unbiased MLMC here, we denote by \( \Lambda_{i,l} \) the \( l \)th approximation-level of \( \Lambda_i \), for example using \( 4^l \) time steps\(^1\) in a Milstein scheme to approximate the samples of the solution of (9). Then define

\[
\Delta \Lambda_{i,l} \overset{\text{def}}{=} \Lambda_{i,l} - \Lambda_{i,l-1},
\]

with \( \Lambda_{i,l-1} = 0 \). As in standard Multilevel Monte Carlo [5], we assume that the cost of computing \( \Delta \Lambda_{i,l} \) grows like \( 4^l \) while its expectation and variance satisfy, \( \mathbb{E} \left[ \Delta \Lambda_{i,l} \mid R_T \right] = \mathcal{O}(4^{-\alpha l}) \) and \( \mathbb{E} \left[ (\Delta \Lambda_{i,l})^2 \mid R_T \right] = \mathcal{O}(4^{-\beta l}) \), respectively, for \( \alpha, \beta, \gamma > 0 \). Then, we write

\[
\mathbb{E}_\theta \{ \Lambda_i \mid R_T \} = \sum_{l=0}^{\infty} \mathbb{E}_\theta \{ \Delta \Lambda_{i,l} \mid R_T \} = \mathbb{E}_\theta \{ C_{\xi} 4^{\xi l} \Delta \Lambda_{i,l} \mid R_T \},
\]

where on the right hand side, with a slight abuse of notation, \( l \) is a random integer satisfying \( \mathbb{P} \{ l = j \} = 4^{-\xi j} / C_{\xi} \) where \( j \in \{ 0, 1, 2, \ldots \} \), \( \xi > 0 \) and \( C_{\xi} \overset{\text{def}}{=} 1/(1 - 4^{-\xi}) \) is a normalization constant. In other words, just like the random sub-sampling method introduced in Section 2, Unbiased MLMC is based on randomly sub-sampling the corrections \( \Delta \Lambda_{i,l} \) to compute the infinite sum in (14). The analysis of Unbiased MLMC is also similar to the one shown in Section 2. In this setting, the condition \( \gamma < \xi < \beta \leq 2\alpha \) is sufficient [15] to bound the expected cost and variance of \( C_{\xi} 4^{\xi l} \Delta \Lambda_{i,l} \), for random \( l \) as above, and hence we can estimate \( \mathbb{E}_\theta \{ \Lambda_i \mid R_T \} \) without bias by using standard Monte Carlo to estimate \( \mathbb{E}_\theta \{ C_{\xi} 4^{\xi l} \Delta \Lambda_{i,l} \mid R_T \} \). The optimal value for \( \xi \), obtained by minimizing the RMS error for a given computational budget, is \( (\beta + \gamma) / 2 \). As an example, if \( h_i (S) \equiv h_i (S (T)) \), for some maturity \( T > 0 \), i.e., the payoff is a function of the asset value at maturity, then if \( h_i \) is Lipschitz and a Milstein scheme is used to approximate samples of the solution of (9), then we have \( \beta = 2\alpha = 2\gamma \), [7]. On the other hand, if \( h_i \) is discontinuous then one can show that \( \beta = \gamma - \nu \) for any \( \nu > 0 \) using a similar analysis to [9, Section 3]. In this case, since \( \beta \leq \gamma \) we would need to truncate the sum of corrections in (14) at some maximum level \( L \) to ensure that Unbiased MLMC has finite work, introducing a bias of \( \mathcal{O}(4^{-\alpha L}) \). A modified Unbiased MLMC estimator [15, Section 4] can then be constructed with samples which have bounded variance but with expected cost that

\footnote{\text{The same discussion applies if } m^l \text{ time steps are used for the } l \text{'th approximation-level, for any } m \geq 1. \text{ The choice } m = 4 \text{ is motivated by the fact that when the variance of } \Delta \Lambda_{i,l} \text{ decreases like } m^{-2l} \text{ while is cost increases like } m^l, \text{ as we later assume, this choice minimizes the total cost of an MLMC estimator; see [14].}}
is $O(4^{(\gamma - \beta)L})$ for $\beta < \gamma$ or $O(L^2)$ for $\gamma = \beta$. In the current work, we will assume that we are always in the case $\beta > \gamma$. In the previous example with a discontinuous $h_i$, an estimator based on conditional expectation can be used to ensure faster variance convergence [7, Section 3.2.8].

In summary, in the case of approximate simulation we take $f_i \equiv C_\zeta 4^q \Delta \Lambda_{i,l}$ where $l$ is a random index. In this case, for a given risk scenario $R_\tau$, the term $f_i$ has non-zero variance and the expected cost to compute it is $O(1)$; since we assume $\beta > \gamma$.

**Remark 3.1 (Moments of unbiased estimator).** For the case $\beta > \gamma$, where we do not have to truncate the sum in (14) and we have an unbiased estimator of $E_Q[\Lambda_i | R_\tau]$, assume further that $E_Q[|\Delta \Lambda_{i,l}|^q] = O(4^{-q\beta l/2})$ for some $q > 2$. The $q$-moment of the unbiased estimator is then

$$E_Q\left[\|C_\zeta 4^q \Delta \Lambda_{i,l}\|^q\right] = C_q^q \sum_{l=0}^{\infty} 4^q (q-1)! E_Q[|\Delta \Lambda_{i,l}|^q]$$

$$= O\left(\sum_{l=0}^{\infty} 4^{-q\beta l/2 + (q-1)!}\right).$$

Hence, even if the $q$-moment of $\Delta \Lambda_{i,l}$ is finite for a given level $l$, the $q$-moment of $4^q \Delta \Lambda_{i,l}$, where $l$ is a random level, is finite only when $q < (1 - \beta/(2\zeta))^{-1}$. For example, when $\zeta = (\beta + \gamma)/2$, the $q$-moment of the unbiased estimator is finite for $q < 1 + \beta/\gamma$. In other words, if we require certain finite $q$-moments of the unbiased estimator, for example when using MLMC with adaptive sampling, c.f. Section 4, we might have to use a smaller, sub-optimal value of $\zeta$.

**Control variates** The discussion on control variates in Subsection 3.2 carries over to the case of approximate simulation. Seen another way, we assume we can approximately sample $\hat{\Lambda}_i$ in (12) along with the modified loss threshold, $\hat{K}_\eta$, in (10). Then, denoting the $l$th approximation-level by $\hat{\Lambda}_{i,l}$, and defining $\Delta \hat{\Lambda}_{i,l}$ as in (13), we set $f_i \equiv C_\zeta 4^q \Delta \hat{\Lambda}_{i,l}$.

One important observation to make here is that, depending on the payoff function, $h_i$, we might have the case where $\text{Var}[\Delta \hat{\Lambda}_{i,l} | R_\tau] > \text{Var}[\Delta \Lambda_{i,l} | R_\tau]$ for some $l$, where $\Delta \hat{\Lambda}_{i,l}$ and $\Delta \Lambda_{i,l}$ are defined as above for $\hat{\Lambda}$ in (11). In other words, using the Delta control variate leads to a larger variance for some approximation levels. As an example, consider $h_i(S) = h_i(S(T))$ and $h_i$ is Lipschitz but $\nabla h_i$ is discontinuous and assume that we use the Milstein scheme to approximate (9) with $4^l$ time steps. Then, denote by $D_{i,l}$ the $l$th approximation-level of $D_i$ in (12) and $\Delta D_{i,l}$ as in (13) and write

$$\Delta \hat{\Lambda}_{i,l} \equiv \Delta \Lambda_{i,l} - \frac{1}{2}(R_\tau - R_0)\Delta D_{i,l}.$$  

We see that while $\text{Var}[\Delta \hat{\Lambda}_{i,l} | R_\tau] = O(4^{-2l})$, we have $\text{Var}[\Delta D_{i,l} | R_\tau] = O(4^{-l(1+\nu)})$ for any $\nu > 0$; using again a similar analysis to [9, Section 3]. Hence, for sufficiently large $l$ we have that $\text{Var}[\Delta \hat{\Lambda}_{i,l} | R_\tau] < \text{Var}[\Delta \Lambda_{i,l} | R_\tau]$ in other words, applying the Delta control variate beyond a certain level $l$ might lead to an estimator with a larger variance, unless the payoff $h_i$ is sufficiently smooth; in this example requiring $\nabla h_i$ to be Lipschitz. An alternative is to use a modified Milstein scheme for the Delta control variate, [7, Section 3.2.8], so that the variance $\text{Var}[\Delta D_{i,l} | R_\tau]$ is sufficiently small compared to, or of the same order as, $\text{Var}[\Delta \hat{\Lambda}_{i,l} | R_\tau]$.

If $h_i$ is not sufficiently smooth, then we may apply the Delta control variate only up to some level, for example, at level $l = 0$ only. That is, we define

$$\hat{\Delta} \hat{\Lambda}_{i,l} \equiv \begin{cases} \hat{\Lambda}_{i,l} & l = 0 \\ \Delta \hat{\Lambda}_{i,l} & \text{otherwise} \end{cases}$$
and set \( f_i \equiv \hat{\Delta} \hat{\Lambda}_{i,t} \). In this case, the modification to the threshold value should also be approximated at level 0. That is, we define the new loss threshold

\[
\hat{\kappa}_\eta \overset{\text{def}}{=} \kappa_\eta + \frac{1}{2} (R_\tau - R_0) \mathbb{E}_Q[D_{i,0} | R_\tau],
\]

so that

\[
\mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^P \hat{\Delta} \hat{\Lambda}_{i,t} - \hat{\kappa}_\eta \bigg| R_\tau \right] = \mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^P \Lambda_i - \kappa_\eta \bigg| R_\tau \right].
\]

Finally, since the Delta control variate reduces the variance of the first level only, we should ensure that the variance at level \( l = 1 \), i.e., \( \text{Var}[ \hat{\Delta} \hat{\Lambda}_{i,1} | R_\tau \] is sufficiently smaller than the variance at level \( l = 0 \), i.e., \( \text{Var}[ \hat{X}_{i,0} | R_\tau \], otherwise refining the first level of approximation of (9) leads to overall smaller RMS; see the discussion in [8, Section 3] and the end of Section 4 for more details.

### 4 MLMC and Adaptive Sampling

The outcomes of the previous section are the terms \( \{ f_i \}_{i=1}^P \) and a new loss threshold, \( \hat{\kappa}_\eta \), depending on the risk scenario, \( R_\tau \), such that we can write

\[
\eta = \mathbb{P} \left[ \mathbb{E}_Q \left[ \frac{1}{P} \sum_{i=1}^P \Lambda_i \bigg| R_\tau \right] > \kappa_\eta \right] = \mathbb{E}_P \left[ H \left( \mathbb{E}_Q \left[ \frac{f_i}{P \theta_{p_j}} - \hat{\kappa}_\eta \bigg| R_\tau \right] \right) \right],
\]

where \( j \) is a random integer satisfying \( \mathbb{P}[j = i] = p_i \) for \( i \in \{1, 2, \ldots, P\} \). In this section, for notational convenience, we will drop the measures \( \mathbb{P} \) and \( Q \), and define the random variables \( Y \overset{\text{def}}{=} R_\tau \) and \( X \overset{\text{def}}{=} f_j/(P \theta_{p_j}) - \hat{\kappa}_\eta \) so that the objective is to simply compute \( \mathbb{E}[H(\mathbb{E}[X | Y])] \). Then, we will discuss using MLMC with adaptive inner sampling as we previously proposed in [8]. We start by defining

\[
\hat{\mathbb{E}}_\ell(y) = \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} X^{(n)}(y),
\]

which is a Monte Carlo estimator of \( \mathbb{E}[X | Y]\) using \( N_\ell \) samples. Here, \( X^{(n)}(y) \) denotes the \( n \)th sample of \( X \) conditioned on \( Y = y \) and the number of samples \( N_\ell \) may depend on \( y \). Then the MLMC estimator for \( \mathbb{E}[H(\mathbb{E}[X | Y])] \) is

\[
\sum_{\ell=0}^L \frac{1}{M_\ell} \sum_{m=1}^{M_\ell} \Delta H_\ell(Y^{(\ell,m)})
\]

where \( \Delta H_\ell(y) = H(\hat{\mathbb{E}}_\ell(y)) - H(\hat{\mathbb{E}}_{\ell-1}(y)) \) and \( \{Y^{(\ell,m)}\}_{\ell,m} \) are i.i.d. samples of \( Y \). Moreover, we set \( H(\hat{\mathbb{E}}_{-1}(\cdot)) = 0 \). We can choose \( N_\ell \) uniformly for all \( y \), for example \( N_\ell = N_0 2^\ell \) for some \( N_0 > 0 \). In this case, it can be shown, under certain moment and smoothness conditions [8, 10], that

\[
|\mathbb{E}[\Delta H_\ell(Y)]| = \mathcal{O}(2^{-\ell})
\]

and

\[
\text{Var}[\Delta H_\ell(Y)] = \mathcal{O}(2^{-\ell/2}).
\]

Assuming that the expected cost of evaluating \( X \) is \( \mathcal{O}(1) \) independently of \( \ell \), the optimal complexity of MLMC to achieve a RMS error, \( \epsilon \), can then be shown to be \( \mathcal{O}(\epsilon^{-5/2}) \), [5, Theorem 2.1].
To improve the computational complexity, we instead select \( N_\ell \) adaptively based on samples of \( Y \). Let

\[
\delta \overset{\text{def}}{=} \left| \mathbb{E}[X | Y] \right| \left( \text{Var}[X | Y] \right)^{1/2}
\]

and let \( \hat{\delta} \approx \delta \) be an estimate computed using Monte Carlo estimates of \( \mathbb{E}[X | Y] \) and \( \text{Var}[X | Y] \) for a given \( Y \). We then select \( N_\ell \) using Algorithm 1 which is an iterative algorithm that starts from a minimum number of samples \( N_\ell = N_0 2^\ell \) for a given \( Y = y \) and then, on every iteration, the number of samples is doubled until the inequality

\[
N_\ell \geq N_0 4^\ell \left( C^{-1} N_0^{1/2} 2^\ell \hat{\delta} \right)^{-r}, \tag{16}
\]

for given constants \( C > 0 \) and \( 1 < r < 2 \), is satisfied or the maximum number of samples \( N_0 4^\ell \) is reached. Algorithm 1, with (16), returns the minimum \( N_0 2^\ell \) when \( \hat{\delta} \) is sufficiently large and hence a Monte Carlo estimate of \( \mathbb{E}[X | Y] \) is likely to have the correct sign, leading to an exact evaluation of \( H(\cdot) \). When \( \hat{\delta} \) is small, estimating the sign of \( \mathbb{E}[X | Y] \) using a Monte Carlo estimator is more difficult and Algorithm 1 returns a larger number of samples, up to the maximum \( N_0 4^\ell \) to account for that; see [8] for a motivation of the exact form of (16). More concretely, assuming the following

**ALGORITHM 1:** Adaptive algorithm to determine \( N_\ell \).

**Input:** \( \ell, y, N_0 > 1, C > 0, 1 < r < 2 \)

**Output:** \( N_0 2^\ell \leq N_\ell \leq N_0 4^\ell \)

set \( N_\ell = N_0 2^\ell \)

set done := false

repeat

if \( 2N_\ell \geq N_0 4^\ell \) then

set \( N_\ell \equiv N_0 4^\ell \)

set done := true

else

Generate \( N_\ell \) new, and independent, inner samples of \( X \) given \( Y = y \)

Estimate \( \hat{\delta} \approx \delta \)

if (16) is satisfied then

set done := true

else

\( N_\ell \equiv 2N_\ell \)

end if

end if

until done

return \( N_\ell \)

mild conditions:

- \( \delta \) has a probability density function, \( \rho \), and there exists positive constants \( \rho_0 \) and \( \delta_0 \) such that \( \rho(\delta) \leq \rho_0 \) for all \( \delta \geq \delta_0 \).
- there exists \( q > 2 \) such that

\[
\sup_y \mathbb{E} \left[ \left( \frac{|X - \mathbb{E}[X | Y]|}{\left( \text{Var}[X | Y] \right)^{1/2}} \right)^q \right| Y = y < \infty
\]

- and \( r \) is chosen such that

\[
1 < r < 2 - \frac{(4q + 1)^{1/2} - 1}{q}, \tag{17}
\]
the analysis in [8, Theorem 2.7] proves the following two crucial properties

$$\mathbb{E}[N_\ell] = \mathcal{O}(2^\ell)$$

and

$$\text{Var}[\Delta H_\ell(Y)] = \mathcal{O}(2^{-\ell}).$$

(18)

Additionally assuming that the expected cost of evaluating $X$ is $\mathcal{O}(1)$ independently of $\ell$ guarantees that the optimal complexity of the MLMC method to achieve a RMS error, $\varepsilon$, is $\mathcal{O}(\varepsilon^{-2} \log \varepsilon^2)$, c.f. [5, 8].

**Antithetic sampling**  Recall that, given a risk scenario $Y$, we need to sample both $\hat{E}_\ell(Y)$ and $\hat{E}_{\ell-1}(Y)$. Sampling $\hat{E}_\ell$ requires sampling $N_\ell$ independent and identically distributed samples of $X$ given the risk scenario $Y$. Similarly, sampling $\hat{E}_{\ell-1}$ requires sampling $N_{\ell-1}$ samples of $X$ given the same risk scenario $Y$. Here, $\text{Var}[\Delta H_\ell(Y)]$ decreases with increasing $\ell$, i.e., with increasing number of internal samples, even if the internal samples used in $\hat{E}_\ell$ and $\hat{E}_{\ell-1}$ are mutually independent. This is because $\hat{E}_\ell(Y)$ converges almost surely to the expectation $\mathbb{E}[X \mid Y]$, due to the Strong Law of Large Numbers. However, by carefully using the same samples of $X$ in both $\hat{E}_\ell$ and $\hat{E}_{\ell-1}$, we can reduce the variance by a constant factor.

In particular, for a given risk scenario, $Y$, assume $N_\ell \geq N_{\ell-1}$ and let $N_\ell = sN_{\ell-1}$ for some integer $s > 0$. Such an integer exists since the adaptive algorithm always returns $N_0 2^\ell$ for some integer $\ell$. Then, let $\{X^{(n)}\}_{n=1}^{N_\ell}$ be $N_\ell$ samples of $X$ given $Y$ and define $\hat{E}_\ell(Y)$ as in (15). Additionally, define $s$ coarse approximations as

$$\hat{E}_{\ell-1}^{(i)}(Y) = \frac{1}{N_{\ell-1}} \sum_{n=1}^{N_{\ell-1}} X^{(n+(i-1)N_{\ell-1})}(Y),$$

for $i = \{1, 2, \ldots, s\}$. The MLMC estimator with antithetic sampling is

$$\sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{m=1}^{M_\ell} \tilde{\Delta} H_\ell(Y^{(\ell,m)})$$

where

$$\tilde{\Delta} H_\ell(y) = H(\hat{E}_\ell(y)) - \frac{1}{s} \sum_{i=1}^{s} H(\hat{E}_{\ell-1}^{(i)}(y)).$$

Note that since $E[\tilde{\Delta} H_\ell(Y)] = E[\Delta H_\ell(Y)]$, the MLMC estimator with antithetic sampling has the same expectation. Moreover, since $\Delta H_\ell = 0$ whenever $\hat{E}_\ell$ and all $\hat{E}_{\ell-1}^{(i)}$ for $i = \{1, 2, \ldots, s\}$ have the same sign, we have that $\text{Var}[\tilde{\Delta} H_\ell(Y)] \leq \text{Var}[\Delta H_\ell(Y)]$. When $N_\ell \leq N_{\ell-1}$, which may happen due to inaccurate estimates of $E[X \mid Y]$ and $\text{Var}[X \mid Y]$, the same discussion as above applies with the fine approximation having the antithetic estimators instead of the coarse one.

**Starting level of MLMC**  An important point to consider when using MLMC is the choice of the starting level. To explain this, let $V_\ell \equiv \text{Var}[\tilde{\Delta} H_\ell(Y)]$ and $V_\ell^f \equiv \text{Var}[H(\hat{E}_\ell(Y))]$ and let $W_\ell$ denote the expected work of sampling $\Delta H$, in the current setting we have $W_\ell \equiv E[N_\ell]$. Then, consider the MLMC estimator

$$\frac{1}{M_0} \sum_{m=1}^{M_0} H(\hat{E}_{\ell_0}(Y^{(\ell_0,m)})) + \sum_{\ell=\ell_0+1}^{L} \frac{1}{M_\ell} \sum_{m=1}^{M_\ell} \tilde{\Delta} H_\ell(Y^{(\ell,m)}).$$

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In other words, the previous MLMC estimator starts at some level \( \ell_0 \geq 0 \). It can be shown [5] that the expected work of MLMC is proportional to

\[
\left( (V_{\ell_0}^{f} W_{\ell_0})^{1/2} + \sum_{\ell=\ell_0+1}^{L} (V_{\ell} W_{\ell})^{1/2} \right)^2.
\]

Hence, given some level of approximation, \( L \), an optimal \( \ell_0 \) satisfies

\[
(V_{\ell_0}^{f} W_{\ell_0})^{1/2} + \sum_{\ell=\ell_0+1}^{\ell_0'} (V_{\ell} W_{\ell})^{1/2} < \left( V_{\ell_0'}^{f} W_{\ell_0'} \right)^{1/2},
\]

for all \( \ell_0 < \ell_0' \leq L \). Otherwise, starting at the level \( \ell_0' \) leads to overall less computational work. Since the quantities \( V_{\ell} \) and \( V_{\ell}^{f} \) for \( \ell = 0, 1, \ldots, L \) must be approximated using a sample variance estimator, we may relax the previous condition by multiplying the right hand side by some constant larger than one to increase the stability of the MLMC algorithm. We use the constant 1.5 in our numerical examples in Section 5.

Choosing an optimal starting level is especially relevant in nested simulation applications because the variance \( V_{\ell}^{f} \) may be large for small \( \ell \) but then decreases as more samples are used in the inner estimator, asymptotically converging to \( \text{Var}[H(E[X\mid Y])] \). See Section 5 and Fig. 3 for an illustration of this.

5 Numerical Experiments

In this section, using numerical experiments on fictitious portfolios of put and call options, we will illustrate the benefits of using random sub-sampling as discussed in Section 2, the control variates that were discussed in Section 3, and adaptive sampling as discussed in Section 4.

5.1 Test setup

Underlying assets We assume we have \( Q \) assets, \( S = \{S_k\}_{k=1}^{Q} \), modelled by Geometric Brownian Motions satisfying

\[
dS_k(t) = \mu_k S_k(t) \, dt + \sigma_k S_k(t) \left( \rho \, dB_0(t) + (1-\rho^2)^{1/2} \, dB_i(t) \right)
\]

in the physical measure. Here the Brownian process \( B_0 \) is the systematic noise, common to all assets, while \( \{B_i\}_{i=1}^{Q} \) are mutually independent Brownian processes and represent the idiosyncratic noise of each asset. We select the following parameters:

- Number of assets: \( Q \equiv 16 \),
- initial asset price: \( S_k(0) \in [90, 110] \),
- drift rate: \( \mu_k \in [0.05, 0.15] \),
- volatility: \( \sigma_k \in [0.01, 0.4] \),
- correlation coefficient: \( \rho \equiv 0.2 \).

Portfolio construction The loss of our example portfolio is an average of losses from \( P \) derivatives (2), i.e., \( \Lambda \equiv P^{-1} \sum_{i=1}^{P} \Lambda_i \), and we consider the market risk. For a short risk horizon, \( \tau = 0.02 \), we set the risk parameter to be the value of the underlying assets at \( \tau \), i.e, \( R_\tau \equiv S(\tau) \), and then set

\[
\Lambda_i \equiv w_i \left( h_i(S_{ki}(T_i)) - h_i(S_{ki,\tau,R_\tau}(T_i)) \right),
\]
for some weight $w_i$ and $h_i$ being the discounted payoff function for the $i$'th option. Here, $S_{k,\tau,R}$ is the $k$'th asset conditioned on $S(\tau) = R$. We assume that the risk-free interest rate is $r = 0.05$ and the discount factor at time $t$ is $\exp(-rt)$. Each option is characterized by its type, put or call, which determines the payoff function $h_i$, along with the following parameters:

| Parameter | Description |
|-----------|-------------|
| asset     | $k_i \in \{1, 2, \ldots, Q\}$ |
| maturity  | $T_i \in [0, 5]$ |
| strike    | $K_i \in [80, 120]$ |
| weight    | $w_i \equiv \begin{cases} \bar{w}_i & \text{put option} \\ \bar{w}_ib_{ki} & \text{call option} \end{cases}$ |

To get concrete values for the parameters above, we generate a random instance of the assets and the portfolio by taking the type to be put or call with equal probability (ensuring at least a single put and call options for each underlying asset), and $S_k(0), \mu_k, \sigma_k, k_i, T_i, K_i$ are sampled independently and uniformly in their respective ranges. On the other hand, the parameters $b_{ki}$ are balancing constants which are determined by the constraint that the portfolio should be delta-neutral with respect to the risk parameter at the initial time, $R_0 = \{R_{0,k}\}_{k=1}^Q = \{S_k(0)\}_{k=1}^Q$, i.e.,

$$\sum_{i=1}^P \frac{\partial V_{i,0}}{\partial R_{0,k}} = 0, \quad \forall k \in \{1, \ldots, Q\}.$$ 

More specifically, for $i = 1, \ldots, Q$, we set

$$b_k = \frac{\sum_{i=1}^P \frac{\partial}{\partial R_{0,k}} \mathbb{E}[\bar{w}_i V_{i,0}]}{\sum_{i=1}^P \frac{\partial}{\partial R_{0,k}} \mathbb{E}[\bar{w}_i V_{i,0}]}.$$

We will discuss the choice of $\{\bar{w}_i\}_{i=1}^P$ in our fictitious portfolios below. In any case, the last step is to normalize the weights, $\{w_i\}_{i=1}^P$, so that their average is 1.

**Computation Methods** We consider the three computational models for computing the value of the options: (a) exact, deterministic evaluation of the option value using the analytic solution of the Black-Scholes PDE, (b) exact simulation of the asset values by analytically solving the SDE, and (c) approximate simulation using the Milstein numerical scheme to estimate the asset values.

**5.2 Results**

All numerical experiments use MLMC with an initial number of samples of $M_0 = 1024$ to estimate the work and variance of the MLMC levels. Moreover, for the inner Monte Carlo estimator, we set $N_0 = 32$ and, when using the adaptive algorithm to select the number of inner samples, we set $r = 1.5$ and $C = 3$ in (16). The code was written in C++ and the experiments were carried out in single-precision on an NVIDIA Tesla K20m GPU with 2496 cores. Note that the embarrassingly parallel nature of Monte Carlo simulation makes it possible to fully exploit parallelization in addition to the computational savings provided by the sub-sampling approach.

To illustrate the benefit of uniform random sub-sampling we first consider large, delta-hedged portfolios comprising options with similar nominal values, i.e., $\bar{w}_i = 1$ for all $i$. The computation

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2The full code can be found on https://github.com/haji-ali/nested-risk-mlmc

3Provided by the Edinburgh Centre for Robotics’ Robotarium Cluster located at Heriot-Watt University, funded by Engineering and Physical Sciences Research Council (EPSRC) Centre for Doctoral Training in Robotics and Autonomous Systems through grant EP/L016834/1.
method to evaluate each option is chosen to be exact evaluation or exact simulation with probabilities 30% and 70%, respectively. We compare two methods: (a) in the first method we use random sub-sampling with uniform probabilities, i.e., setting $g_i = 1$ for all $i$, (b) and in the second method we do not use any sub-sampling and instead evaluate the full portfolio for every combination of risk scenarios and underlying asset values; making sure that options that can be exactly computed are evaluated only once for every risk scenario. Both methods use MLMC with adaptive sampling as discussed in Section 4, with appropriate redefinition of $X$ and $Y$, and use all the control variates that were discussed in Section 3. When estimating the work of these methods, we simply count the number of times the value of an option or a payoff function are evaluated; the work estimates are shown in Fig. 1a. For the considered tolerances, using random sub-sampling leads consistently to fewer evaluations and, for a fixed tolerance, the total number of payoff evaluations does not increase as the number of options increase. Fig. 1b shows the actual run-time for the numerical tests. Uniform, random sub-sampling has an overhead that make its advantage slightly less pronounced for small tolerances or small portfolios. To explain these results, recall that evaluating the full portfolio for every combination of risk scenarios and underlying asset values, i.e., not using sub-sampling, imposes a minimum budget which increases the computational complexity for large tolerances. Nevertheless, for sufficiently small tolerances or portfolios, and sufficiently large budgets, evaluating the full portfolio for every risk scenario does not add a significant computational overhead. On the other hand, random sub-sampling has an overhead not accounted for in the work estimate. Namely, the cost of sampling the random option index which entails sampling a uniform random variable and a table-lookup operation. While this additional cost is small in typical cases, especially since we use binary search to perform the table-lookup, it is not wholly insignificant compared to the cost of sampling the options in our simple numerical example.

Random sub-sampling is most useful when the financial derivatives in the portfolio are heterogeneous, even in moderate-sized portfolios. To illustrate this we consider a smaller portfolio of $10^3$ options with different nominal values. To model this, we sample the logarithm of the weight parameters, $\log(\tilde{w}_i)$, from a normal distribution with mean 0 and standard deviation 3. Moreover, when using random sub-sampling we use the estimates $\tilde{g}_i = \tilde{w}_i$. Like before, the computation method of each portfolio is chosen to be exact evaluation or exact simulation with probabilities 30% and 70%, respectively. We now test several methods and show their work estimates and runtimes in Fig. 2.

The first method, labelled “Full method”, uses MLMC with adaptive sampling as discussed in Section 4, all the control variates as discussed in Section 3 and random sub-sampling as discussed in Section 2. The second method, labelled “No sub-sampling” does not use random sub-sampling and instead evaluates the whole portfolio for every combination of risk scenarios and asset values; again making sure that options that can be exactly computed are evaluated once for every risk scenario. In this case, the work reduction measured by work estimates and total runtime is more than tenfold. The third method we consider, labelled “No CV”, is the same as “Full method” except that we do not use the Delta and antithetic control variates that were discussed in Section 3. In this example, by using these control variates, work estimate and runtime is again reduced by around 40-fold. Recall that this reduction is related to the risk horizon, $\tau = 0.02$, and we should expect that longer risk horizons, compared to the maturities of options, would reduce the savings of the antithetic and Delta control variates. The fourth method we consider, labelled “Non-adaptive”, is again the same as “Full method” except that it uses instead deterministic, non-adaptive number of inner samples, i.e. $N_\ell = N_0 4^\ell$ for all risk scenarios. Using adaptive sampling is two to seven times more efficient than non-adaptive sampling. Moreover, recall that to achieve RMS error $\varepsilon$, we expect MLMC with adaptive sampling to have a computational complexity of $O(\varepsilon^{-2}\log\varepsilon^{-1})^2$ while MLMC with non-adaptive sampling would have a complexity of $O(\varepsilon^{-5/2})$, approximately. The observed complexities in Fig. 2 are consistent with the expected complexities and with the variance and work estimates in Fig. 3.

To show that using the framework outlined above accommodates approximate simulation, we also include in these plots the runtime of the “Full method” when applied to a similar portfolio
with the same number of options and the same weights but with the computational method being exact evaluation, exact simulation or approximate simulation with probabilities 30%, 50% and 20%, respectively. Recalling the discussion in Theorem 3.1 and the notation used there, we note that setting $r = 1.5$ in the adaptive algorithm to select the number of inner samples would not work in this setting. This is because we use the Milstein scheme to approximate samples of the underlying assets for 20% of the options, which yields $\beta = 2\gamma$, and we use Unbiased MLMC with $\zeta = (\beta + \gamma)/2$ to approximate the expectation of the loss, as discussed in Subsection 3.3. Hence, the $q$-moments of the unbiased estimator are finite for $q < 3$ only while $r = 1.5$ requires finite $q$-moments for $q \geq 15$ to satisfy the condition (17). Instead, we set $r = 1.1$ in this case which requires finite $q$-moment for $q \approx 2.72$.

The starting levels, $\ell_0$, of MLMC for each of the methods in this section were selected based on the criteria (19). As discussed above, a correct choice of the starting level is crucial in nested simulation because the variance, $\mathbb{V}_\ell = \text{Var}[\mathbb{H}(E_\ell(X | Y))]$ may exhibit a pre-asymptotic behaviour with respect to $\ell$. This is illustrated in Fig. 3-(top).

6 Conclusions

This work has shown the application of MLMC with adaptive sampling to estimating the probability of a large loss of a large financial portfolio of heterogeneous derivatives. The key elements to reduced computational complexity are using MLMC with adaptive sampling, applying several control variates that exploit the short risk horizon and using sub-sampling strategies to obtain a computational complexity that does not depend on the number of derivatives in the portfolio. Using the methods above to efficiently compute probabilities of loss in a portfolio, other risk measures such as Value-at-Risk (VaR) or Conditional VaR (CVaR) can also be computed efficiently as discussed in detail in [8]. VaR can be computed by finding the root $K_\eta$ of the equation $\mathbb{P}[\mathbb{E}[\Lambda | R_\tau] > K_\eta] = \eta$ for a given risk level, $\eta$. Given an efficient method to solve the forward problem, i.e., computing $\eta$ given an estimate of $K_\eta$, the root can be approximated efficiently using a stochastic root finding algorithm, c.f. [8]. Since CVaR can be written as a minimization problem whose solution is VaR [8, 16], then we can write, denoting $X \equiv \mathbb{E}[\Lambda | R_\tau]$,

$$
\mathbb{E}[X | X > K_\eta] = K_\eta + \eta^{-1}\mathbb{E}[\max(0, X - K_\eta)] \\
= \min\{x + \eta^{-1}\mathbb{E}[\max(0, X - x)]\} \\
= \tilde{K}_\eta + \eta^{-1}\mathbb{E}[\max(0, X - \tilde{K}_\eta)] + O((\tilde{K}_\eta - K_\eta)^2),
$$

given an estimate of VaR, $\tilde{K}_\eta$. Hence, to approximate CVaR, we first approximate $\tilde{K}_\eta$ up to a RMS error $\varepsilon^{1/2}$ with work $o(\varepsilon^{-2})$. Then, $\mathbb{E}\left[\max(0, \mathbb{E}[\Lambda | R_\tau] - \tilde{K}_\eta)\right]$, involving a nested expectation, can be estimated with total work $O(\varepsilon^{-2})$ to achieve a RMS error $\varepsilon$ using MLMC with antithetic sampling for nested expectations [3, 6, 8] combined with random sub-sampling of the financial derivatives in the portfolio and the control variates that were discussed in Sections 2 and 3, respectively.

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Figure 1: The work estimate (left), measured in number of evaluations of option values and payoff functions, and runtime (right), measured in seconds, of MLMC with adaptive sampling when applied to large portfolios of options with similar nominal values, i.e., \( \tilde{\omega}_i = 1 \) for all \( i \). 30% of the options are computed using exact evaluation while 70% are computed using exact simulation. Here \( \varepsilon \) is the tolerance normalized by the exact value which was estimated using Monte Carlo to be 3-4% approximately for the considered portfolios. Note that the work estimates and running time are multiplied by \( \varepsilon^2 \) to normalize the work effort for different portfolios and to emphasize the difference of the computational effort when using random, uniform sub-sampling or not. In the (top) plots we fix the size of portfolio to \( P = 10^5 \) and vary \( \varepsilon \), while in the (bottom) plot we fix \( \varepsilon \approx 3 \times 10^{-3} \) and vary \( P \). We see that using random sub-sampling, even when applied to options with similar nominal value, reduces the computational complexity, particularly for large tolerances. Moreover the computational complexity is independent of the number of options in the portfolio.
Figure 2: The work estimate (left) and runtime (right) of MLMC with adaptive sampling when applied to a portfolio of $10^3$ heterogeneous options. Here $\varepsilon$ is the tolerance, normalized by the exact value which was estimated using Monte Carlo to be 1% approximately for our particular portfolio. Note that the work estimates and running time are multiplied by $\varepsilon^2$ to emphasize the differences between the methods, since $O(\varepsilon^{-2})$ is the computational complexity in the best-case when the inner expectation can be computed exactly at $O(1)$ cost. The full method, which uses MLMC with adaptive inner sampling, all control variates as discussed in Section 3 and random sub-sampling with non-uniform probabilities, clearly outperforms other the methods.
Figure 3: (top) The variance estimates of the MLMC levels where $V_{\ell} \equiv \text{Var}[\tilde{\Delta}H(Y)]$ and $V_{f,\ell} \equiv \text{Var}[H(\hat{E}_{\ell}(Y))]$. Note that $V_{f,\ell}$ has a pre-asymptotic behaviour where it asymptotically approaches $\text{Var}[H(\mathbb{E}[X|Y])]$ from above. Because of this, the starting level should be chosen carefully as discussed in Section 4. Note also that $V_{f,\ell}$ decreases like $\mathcal{O}(2^{-\ell})$ for all methods.

(bottom) Work estimate and runtime of the MLMC levels. Note that the work increases like $\mathcal{O}(2^\ell)$ for methods that use adaptive inner sampling for sufficiently large $\ell$, unlike the non-adaptive method where the work increases like $4^\ell$ for all $\ell$. Additionally, when not using the control variates and because of the increase of the variance per level, the region of pre-asymptotic behaviour where the work increases like $4^\ell$ is extended.