CP(N) model on regions with boundary

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Abstract

In this note we discuss the CP(N) model in large $N$ limit in saddle point approximation on
disc and annulus with various combinations of Dirichlet and Neumann boundary conditions. We
show that homogeneous condensate is not a saddle point in any of considered cases. Behavior of
inhomogeneous condensate near boundary is briefly discussed.

1 Introduction

Two-dimensional CP(N) model in the large $N$ limit was solved in [1, 2] (see [3] for a detailed review).
It was found that the model is asymptotically free and exhibits dynamical generation of mass via
dimensional transmutation, which makes it similar to QCD. The CP(N) model appears as an effective
low energy theory for worldsheet of non-Abelian string. It was found in supersymmetric case in [11, 12,
13, 14] and then in non-supersymmetric case in [15]. Both supersymmetric and non-supersymmetric
versions of the model on interval with periodical boundary conditions were studied in [17]. Recently
this theory on finite interval and on disc was discussed [4, 5, 7, 8, 10, 16]. It was shown that for a finite
interval with Dirichlet boundary conditions a homogeneous condensate is not a solution of saddle point
equation [7, 8]. In [5] a modification of boundary conditions consistent with homogeneous solution was
proposed. Namely, instead of CP(N) model CP(2N) model with $N$ components satisfying Dirichlet
boundary conditions and $N$ components satisfying Neumann boundary conditions was suggested.
This choice of boundary conditions leads to full cancellation of coordinate dependance in saddle
point equation and thus to possibility of homogeneous solution. Similar method was successfully
applied to Grassmannian model on the finite interval [6]. A short time ago an exact solution based
on correspondence with Gross-Neveu model was proposed [10]. Details of the correspondence are
presented in [9]. Behavior of this solution near boundary is different from one found in [7, 8]. The model
on the disc was discussed in [16] in connection with problem of non-Abelian string decay. However,
the case of inhomogeneous condensate was not analyzed. In this note we will focus on inhomogeneous
condensate on disc and annulus. Our main result is that there is no choice of boundary conditions
compatible with homogeneous condensate. We check this conclusion for some types of boundary
conditions and expect that similar analysis can be applied to more complicated choice of boundary
conditions. Nevertheless, we can find boundary conditions similar to one used in [5] for which the
leading terms in the divergences of condensate near boundary cancel.

This note is organized as follows. In the Section 2 we revise the one-loop effective action and the
gap equation for CP(N) model. In the Section 3 we discuss condensate on disc. We firstly consider
Dirichlet boundary conditions and then discuss combination of Dirichlet and Neumann boundary

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conditions similar to one used in [5]. In the Section 4 similar analysis is performed for the case of annulus.

2 Effective action and gap equation

The Euclidean action for $\mathbb{C}P(N)$ model is

$$S = \int d^2x \left( (D_\mu n_i)^* (D_\mu n_i) + \lambda \left( |n|^2 - 1 \right) \right), \quad i = 1, \ldots, N + 1$$

(1)

We presume that number of components $N$ is large. Here $\lambda$ is Lagrange multiplier that leads to constraint $n_i n^i = 1$. For the first component the boundary condition is chosen to be consistent with constraint $n_1 = \sqrt{1}$. On fields $n_i$ with $i = 2, \ldots, N + 1$ we impose Dirichlet or Neumann boundary conditions which will be specified later. $D_\mu = \partial_\mu - iA_\mu$ is covariant derivative. We do not consider dynamics of gauge field $A_\mu$ and suppose that its vacuum expectation value is zero in the leading order of $1/N$ expansion. At classical level it is just a dummy field that can be eliminated by its equation of motion. In the later analysis it is assumed that $A_\mu = 0$ and $n_1 = \sigma$ is real. We consider the theory on the disc $(\sqrt{x_0^2 + x_1^2} < R)$ and annulus $R_1 < \sqrt{x_0^2 + x_1^2} < R_2$ with different boundary conditions. As usual, we can integrate over components with trivial boundary conditions $n_i$, $i = 2, \ldots, N + 1$ and obtain effective action in terms of fields $\lambda$ and $\sigma$. Thus effective action

$$S_{eff} = N \text{Tr} \log (-\partial^2 + \lambda) + \int d^2x \left( (\partial \sigma)^2 + \lambda \left( \sigma^2 - 1 \right) \right)$$

(2)

Variation of the action with respect to $\lambda$ yields the gap equation

$$N \text{Tr} \frac{\delta \lambda}{-\partial^2 + \lambda} + \int d^2x \delta \lambda \left( \sigma^2 - 1 \right)$$

(3)

We can express this equation through the eigenfunction of operator $-\partial^2 + \lambda$:

$$(-\partial^2 + \lambda) f_\alpha(x) = \kappa_\alpha f_\alpha$$

(4)

$$N \sum_\alpha \frac{|f_\alpha(x)|^2}{\kappa_\alpha} + \sigma^2 - 1 = 0,$$

(5)

Here $f_\alpha$ must satisfy the same boundary conditions as corresponding field $n_i$ and normalization condition $\int d^2x |f_\alpha(x)|^2 = 1$, integration is over the considered region. Throughout this section $\alpha$ is index that enumerates all eigenvalues of particular differential operator so summation in (5) is over all eigenfunctions. Note that the sum is Green function of the operator $-\partial^2 + \lambda$ taken at coinciding points.

$$(-\partial^2 + \lambda) G(x, y) = \delta(x - y), \quad G(x, y) = \sum_\alpha \frac{f_\alpha(x) f_\alpha^*(y)}{\kappa_\alpha}$$

(6)

so

$$\sum_\alpha \frac{|f_\alpha(x)|^2}{\kappa_\alpha} = \lim_{y \to x} G(x, y)$$

(7)

Strictly speaking, this limit is infinite, so we need to regularize the sum by considering small but finite distance between points $x$ and $y$. The equation obtained by variation of action with respect to $\sigma$ is

$$(-\partial^2 + \lambda) \sigma = 0$$

(8)

The problem has rotational symmetry, so we assume that all fields depend only on the distance to the center:

$$\lambda = \lambda(\rho), \quad \sigma = \sigma(\rho), \quad f = \exp(i\ell \varphi) g(\rho),$$

$$\rho = \sqrt{x_0^2 + x_1^2}$$

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where \( x_1 = \rho \cos \varphi, \; x_2 = \rho \sin \varphi. \)

The sum over eigenfunctions in the equation (5) is divergent. However we can subtract from eigenfunctions their mean values and obtain conditionally convergent sum which contains all information about dependence on the coordinates and divergent sum which leads to renormalization of \( r. \) The transformed equation is

\[
N \sum \frac{1}{\kappa_\alpha} \left( |f_\alpha(x)|^2 - \frac{1}{A} \right) + \sigma^2 + \frac{N}{A} \sum \frac{1}{\kappa_\alpha} r = 0
\]

(9)

Here \( A \) is area of the region, \( A = \pi R^2 \) for a disk and \( A = \pi \left( R_2^2 - R_1^2 \right) \) for annulus. Now the first sum converges. It will be used in next sections to investigate the behavior of \( \sigma \) near boundary. We can not solve the system of equation (5) and (8) analytically. However it is possible to calculate eigenvalues and eigenfunctions for Dirichlet and Neumann boundary conditions for the case \( \lambda \approx \text{const} = m^2 \) in terms of zeros of Bessel functions. We assume that large eigenvalues for arbitrary \( \lambda \) are almost the same, so this allows to understand behavior of \( \sigma \) near boundaries using this eigenvalues and eigenfunctions.

So we are interested in the behavior of the sum

\[
\Sigma(x) = \sum \left( \frac{1}{A} - \frac{|f_\alpha(x)|^2}{\kappa_\alpha} \right)
\]

(10)

near the boundary, where \( f_\alpha \) are eigenfunctions of operator \(-\partial^2 + m^2\). Our hypothesis is that near the boundary \( \sigma^2 \sim N\Sigma \) at least in case when this sum tends to infinity as we approach to boundary so by considering (10) we can find out whether \( \sigma \) is finite near the boundary or not.

### 3 Model on disc

Firstly we consider model on disc with Dirichlet boundary conditions. The eigenfunctions and eigenvalues are

\[
f_{l,n} = A_{l,n} \exp (i l \varphi) J_l \left( \mu_{l,n} \frac{\rho}{R} \right), \quad \kappa_{l,n} = \mu_{l,n}^2 + m^2
\]

(11)

Here \( J_l(z) \) is Bessel function of the first kind, which is regular at \( z = 0 \) and \( J_l(\mu_{l,n}) = 0. \) Normalization constant can be found from condition

\[
A_{l,n}^2 \pi R^2 J_{l-1}^2 (\mu_{l,n}) = 1
\]

(12)

so the sum (10) becomes

\[
\Sigma_D(\rho) = \frac{1}{\pi} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\mu_{l,n}^2 + m^2 R^2} \left( 1 - \frac{J_l^2 (\mu_{l,n} \rho/R)}{J_{l-1}^2 (\mu_{l,n})} \right)
\]

(13)

In the same way for Neumann boundary conditions we obtain

\[
f_{l,n} = B_{l,n} \exp (i l \varphi) J_l \left( \tilde{\mu}_{l,n} \frac{\rho}{R} \right), \quad \kappa_{l,n} = \tilde{\mu}_{l,n}^2 + m^2
\]

(14)

where \( J_l'(\tilde{\mu}_{l,n}) = 0 \) and

\[
B_{l,n}^2 \pi R^2 \left( J_l^2 (\tilde{\mu}_{l,n}) - J_{l-1}^2 (\tilde{\mu}_{l,n}) \right) = 1
\]

(15)

so the sum is

\[
\Sigma_N(\rho) = \frac{1}{\pi} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\mu_{l,n}^2 + m^2 R^2} \left( 1 - \frac{J_l^2 (\tilde{\mu}_{l,n} \rho/R)}{J_{l-1}^2 (\tilde{\mu}_{l,n})} \right)
\]

(16)
Both sums have logarithmic divergence near the boundary. The easiest way to see this is consideration of Green function \( (10) \). Numerical computation of this sums confirms this conclusion. The leading terms obtained from Green functions are

\[
\Sigma_D (\rho) = -\Sigma_N (\rho) = \frac{1}{2\pi} \log \left( \frac{1}{R - \rho} \right)
\]

(17)

We can hope to cancel this divergence by considering \( CP (2N) \) model and imposing on \( N \) fields Dirichlet boundary conditions and on other \( N \) fields Neumann boundary conditions. Thus instead of sums (13) and (16) we will have

\[
\Sigma (x) = \Sigma_D (\rho) + \Sigma_N (\rho)
\]

(18)

This sum does not have logarithmic divergence. However, asymptotic of zeros of Bessel function and its derivative are (the first of this expressions is a case of general formula 8.547 from \[18\], the second can be derived similarly)

\[
\mu_{l, n} = \left( n + \frac{l}{2} - \frac{1}{4} \right) \pi - \frac{4l^2 - 1}{8\pi (n + l/2 - 1/4)}
\]

(19)

\[
\hat{\mu}_{l, n} = \left( n + \frac{l}{2} + \frac{1}{4} \right) \pi - \frac{4l^2 + 3}{8\pi (n + l/2 + 1/4)}
\]

(20)

so \( \mu_{l, n} - \hat{\mu}_{l, n} \to \pi/2 \) as \( n \to \infty \) and frequencies of oscillations of eigenfunctions with Dirichlet and Neumann boundary conditions are different, so we might expect slow oscillations of amplitude of the sum so \( \Sigma \neq \text{const} \). This conclusions are also confirmed by numerical summation.

4 Model on annulus

Now let us consider \( CP (2N) \) model on annulus. We want to impose different types of boundary conditions on \( n_i \) for \( i = 2, \ldots, N + 1 \) and for \( i = N + 2, \ldots, 2N + 1 \) in order to cancel divergence of \( \Sigma \) at the boundary. One of the possible choices is to use Dirichlet boundary condition for \( N \) fields and Neumann boundary conditions for other \( N \) fields. In both of these cases eigenfunctions are

\[
f_{l, n} = A_{l, n} \exp (il\varphi) \left( J_l \left( \frac{\mu_{l, n} \rho}{R_2} \right) \cos \alpha_{l, n} - Y_l \left( \frac{\mu_{l, n} \rho}{R_2} \right) \sin \alpha_{l, n} \right)
\]

(21)

where \( Y_l (z) \) is Bessel function of the second kind, \( \alpha_{l, n} \in [0, \pi] \) and \( \mu_{l, n} \) and the parameter \( \alpha_{l, n} \) are determined from boundary conditions. Corresponding eigenvalue is

\[
\kappa_{l, n} = \frac{\mu_{l, n}^2}{R_2^2} + m^2
\]

(22)

We will use following formulas for large zeros of function \( J_l (z) \) and \( Y_l (z) \) (this expression is formula 8.547 from \[18\] again)

\[
z_{l, n} = \left( n + \frac{l}{2} - \frac{1}{4} \right) \pi - \alpha_{l, n} - \frac{4l^2 - 1}{8 \{ \pi (n + l/2 - 1/4) - \alpha_{l, n} \}}
\]

(23)

and zeros of its derivative

\[
\tilde{z}_{l, n} = \left( n + \frac{l}{2} - \frac{1}{4} \right) \pi - \alpha_{l, n} - \frac{4l^2 + 3}{8 \{ \pi (n + l/2 + 1/4) - \alpha_{l, n} \}}
\]

(24)

We introduce dimensionless parameter \( \gamma = R_1 / R_2 \). Firstly consider Dirichlet boundary conditions

\[
n (R_1) = n (R_2) = 0
\]
We are going to find asymptotic of eigenvalues with large \( n \). From boundary conditions follow equation for \( \alpha \)

\[
\mu_{l, n} \gamma = \left( n_1 + \frac{l}{2} - \frac{1}{4} \right) \pi - \alpha_{l, n} - \frac{4l^2 - 1}{8 \{ \pi (n_1 + l/2 - 1/4) - \alpha_{l, n} \}}
\]

\[ (25) \]

\[
\mu_{l, n} = \left( n_2 + \frac{l}{2} - \frac{1}{4} \right) \pi - \alpha_{l, n} - \frac{4l^2 - 1}{8 \{ \pi (n_2 + l/2 - 1/4) - \alpha_{l, n} \}}
\]

\[ (26) \]

Here \( n_2 - n_1 = n \) because in one dimension the number of the root is equal to number of zeros of eigenfunction minus one due to oscillatory theorem. So for the large \( n \) we have approximately

\[
\mu_{l, n} = \frac{\pi n}{1 - \gamma}
\]

\[ (27) \]

\[
n_1 = \frac{\gamma n}{1 - \gamma} - \left( \frac{l}{2} - \frac{1}{4} - \frac{\alpha_{l, n}}{\pi} \right), \quad n_2 = \frac{\gamma n}{1 - \gamma} - \left( \frac{l}{2} - \frac{1}{4} - \frac{\alpha_{l, n}}{\pi} \right)
\]

\[ (28) \]

\( \alpha_{l, n} \) can be determined from the condition that \( n_1 \) is integer. Thus we can calculate correction for the \( \mu_{l, n} \)

\[
\mu_{l, n} = \frac{\pi n}{1 - \gamma} + \frac{4l^2 - 1 + \gamma}{8\pi n}
\]

\[ (29) \]

So approximate eigenvalues for large \( n \) are

\[
\kappa_{l, n} = \left( \frac{\pi n}{R_2 - R_1} + \frac{4l^2 - 1}{8\pi n} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right)^2 + m^2
\]

\[ (30) \]

Similarly for the Neumann boundary conditions

\[
\partial_{\rho} n (R_1) = \partial_{\rho} n (R_2) = 0
\]

eigenvalues are

\[
\tilde{\mu}_{l, n} = \frac{\pi n}{1 - \gamma} + \frac{4l^2 + 3 + \gamma}{8\pi n}
\]

\[ (31) \]

\[
\tilde{\kappa}_{l, n} = \left( \frac{\pi n}{R_2 - R_1} + \frac{4l^2 + 3}{8\pi n} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right)^2 + m^2
\]

\[ (32) \]

The first term in brackets in (30) and (32) corresponds to the case of an interval. For large \( l \) eigenvalues differ significantly from the case of interval. However, one can easily obtain that

\[
\tilde{\mu}_{l, n} - \mu_{l, n} = \frac{1 + \gamma}{4\pi n} \rightarrow 0, \text{ as } n \rightarrow \infty
\]

Note that the leading term of asymptotic does not contain the large number \( l \). It means that differences of spacial frequencies of eigenfunctions tend to zero uniformly so we might expect better cancellation of divergence than for the case of disc.

5 Discussion

In this note we have discussed \( CP(N) \) on disc and annulus in large \( N \) limit. It was shown that on disc with Dirichlet boundary conditions homogeneous solution is impossible. We also showed that mixing of Dirichlet and Neumann conditions does not yield constant in \( \sigma \) condensate neither in case of disc nor in case of annulus. However we claim that this combination of boundary condition makes \( \sigma \) finite near boundary. It might be useful for numerical analysis of the inhomogeneous condensates.
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