1. Introduction

Nash has related the space of arcs centered in the singular locus of a variety to its resolution of singularities in 1968 (see [14]). Since the late nineties till nowadays, these schemes and their finite dimensional approximations – jet schemes – have generated much interest because of their appearance in motivic integration [8, 2] and their use in birational geometry [4].

Despite the appearance of these jet schemes in numerous articles and in many interesting questions, few is known about their geometry for specific classes of singularities, except for the following three classes: monomial ideals [5], determinantal varieties [3], plane branches [11].

While arcs on toric varieties have been intensively studied [7, 10, 1, 6], jet schemes of such varieties are still unknown. The subject of this note is the study of the jet schemes of toric surfaces. Beside being the simplest toric singularities, this class of singularities is interesting from two points of view: on one hand, these surfaces are examples of varieties having rational singularities, but which are not necessary local complete intersection, therefore we cannot characterize their rationality by [13] via their jet schemes; on the other hand, despite that these singularities are not complete intersections and therefore we do not have a definition of non-degeneration with respect to their Newton polyhedra in the sense of Kouchnirenko [9], they heuristically are non-degenerate because they are desingularized with one toric morphism, so from a jet-scheme theoretical point of view, their jet schemes should not give rise vanishing components [11] (i.e. projective systems of irreducible components whose limit in the arc space are included in the arc space of the singular locus); this follows from Remark 2.3. For $m \in \mathbb{N}$, $m \geq 1$, we determine the irreducible components of the $m$-th jet scheme of a toric surface $S$. For $m$ big enough, we connect the number of a class of these irreducible components to the number of exceptional divisors on the minimal resolution of $S$.

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that appear on the minimal resolution of $S$. This is to compare with a result that we have obtained in [12] for rational double point singularities.

2. Jet schemes of toric surfaces

Let $K$ be an algebraically closed field. Let $X$ be a $K$-scheme of finite type over $K$ and let $m \in \mathbb{N}$. The functor $F_m : K$-Schemes $\rightarrow$ Sets which to an affine scheme given by a $K$-algebra $A$ associates $F_m(\text{Spec } A) = \text{Hom}_K(\text{Spec } A[t]/(t^{m+1}), X)$ is representable by a $K$-scheme, $X_m$ called the $m$-jet scheme of $X$.

For $m, p \in \mathbb{N}$, $m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \rightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \rightarrow X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{n,m} = \pi_{n,p}$ for $p < m < q$. This yields an inverse system whose limit $\lim_{\leftarrow} X_n$ is a scheme called the arc space of $X$. Note that $X_0 = X$. We denote the canonical projections $X_m \rightarrow X_0$ by $\pi_m$ and $X_\infty \rightarrow X_m$ by $\psi_m$. See [4] for more about jet schemes.

Let $S$ be a singular affine toric surface defined over $K$ by the cone $\sigma \subset N_\mathbb{R} = \mathbb{R}^2$ generated by $(1,0)$ and $(p,q)$, where $0 < p < q$ and $p,q$ are relatively prime. Let $(c_2, \ldots, c_{e-1})$ be the entries greater than or equal to two occurring in the Hirzebruch–Jung continued fraction associated to $q/p$. Then the embedding dimension of $S$ is $e$ [15, Section 1.6]. We suppose that $e > 3$, the case $e = 3$, i.e. the rational double point $S = A_{2,1}$ is studied in [12]. Analyzing the convex hull of $\sigma^\vee \cap M$, where $M$ is the dual lattice of $N$, Riemenschneider has exhibited the generators of the ideal defining $S$ in $A^e = \text{Spec } K[x_1, \ldots, x_e]$ in [16]; these are:

$$E_{ij} = x_i x_j - x_{i+1} x_{j+1}^{c_{i+2} \cdot 2} x_{i+2}^{c_{i+3} \cdot 2} \cdots x_{j-2}^{c_{j-2} \cdot 2} x_{j-1}^{c_{j-1} \cdot 2} x_{j-1}, \text{ where } 1 \leq i < j \leq 1 \leq e - 1.$$ 

Let $f \in \mathbb{K}[x_1, \ldots, x_e]$; for $m, p \in \mathbb{N}$ such that $p \leq m$, we set:

$$\text{Cont}^{m}(f)_m (\text{resp. } \text{Cont}^{-p}(f)_m) := \{ \gamma \in S_m \mid \text{ord}_\gamma (f) = p (\text{resp. } > p) \},$$

$$\text{Cont}^0(f) = \{ \gamma \in S_\infty \mid \text{ord}_\gamma (f) = p \},$$

where $\text{ord}_\gamma (f)$ is the $t$-order of $f \circ \gamma$.

For $a, b \in \mathbb{N}$, $b \neq 0$, we denote by $[\frac{a}{b}]$ the ceiling of $\frac{a}{b}$. For $i = 2, \ldots, e - 1$, $s \in \{1, \ldots, [\frac{e}{2}]\}$ (i.e. $m \geq 2s - 1 \geq 1$) and $l \in \{s, \ldots, m\}$, where $m : = \min(i - 1, (m + 1) - s)$, we set

$$D^i_{i,m} := \text{Cont}^l(x_i^s) \text{ and } C^i_{i,m} := \text{Cont}^l(x_{i+1}^s).$$

If $R$ is a ring, $I \subset R$ an ideal and $f \in R$, we denote by $V(I)$ the subvariety of $\text{Spec } R$ defined by $I$ and by $D(f)$ the open set $D(f) := \text{Spec } R[f]$. 

Lemma 2.1. For $i = 2, \ldots, e - 1$, $s \geq 1$, the ideal defining $C^i_{i,2s-1}$ in $A^e_{2s-1}$ is $D^i_{i,2s-1} = (x^s_j, 1 \leq j \leq e, 0 \leq b < s)$. Note that $C^i_{i,2s-1}$ does not depend on $i$. For $j = 1, e$, we set $C^i_{1,2s-1} = C^i_{e,2s-1}, i = 2, \ldots, e - 1$.

Proof. Let's prove that $D^i_{i,2s-1} = V(D^i_{i,2s-1}) \cap D(x^s_i x^s_{i+1})$. Let $\gamma \in A^e_{2s-1}$ such that $\text{ord}_\gamma x_i = \text{ord}_\gamma x_{i+1} = s$. So, we have $\text{ord}_\gamma x_i = c_i > 2s - 1$ because $c_i \geq 2$. If moreover $\gamma$ lies in $S_{2s-1}$, then it satisfies $E_{i,-1,1} \mod t^{2s}$, which is equivalent to $\text{ord}_\gamma x_{i-1} > s$, because $x^s_i \circ \gamma = 0 \mod t^{2s}$ and $\text{ord}_\gamma x_{i+1} = s$. The same argument, using $E_{i-2,1}, E_{i+2,1}$ and so on by induction, using the other $E_{j}$'s and $E_{ij}$'s, gives that $\text{ord}_\gamma x_j > s$. We deduce

$$D^i_{i,2s-1} \subset V(D^i_{i,2s-1}) \cap D(x^s_i x^s_{i+1}).$$

The opposite inclusion comes from the fact that a jet in $V(D^i_{i,2s-1}) \cap D(x^s_i x^s_{i+1})$ satifies all the equations of $S$ modulo $t^{2s}$. Since $V(D^i_{i,2s-1}) \subset A^e_{2s-1}$ is irreducible, the lemma follows. □

Proposition 2.2. For $i = 2, \ldots, e - 1$, $m \in \mathbb{N}$, $s \in \{1, \ldots, [\frac{m}{2}]\}$ and $l \in \{s, \ldots, m\}$, $C^i_{i,m}$ is irreducible, and its codimension in $A^e_m$ is equal to $se + (m - (2s - 1))(e - 2)$.

Proof. A similar argument to the one used in Lemma 2.1 shows that $\pi_{m,2s-1}(D^i_{i,m}) \subset C^i_{i,2s-1}$. Using syzygies among $E_{j,h}, 1 \leq j < h < e - 1 \leq e - 1$, we prove that

$$D^i_{i,m} = \{ \gamma \in A^e_m \mid \text{ord}_\gamma x_i = s, \text{ord}_\gamma x_{i+1} = l, \text{ord}_\gamma E_{j,h} \geq m + 1 \text{ for } (j,h) = (i - 1, i + 1), j = l, h = i \}.$$ 

This explicit description of $D^i_{i,m}$ shows that its coordinate ring is isomorphic to a polynomial ring over $\text{Spec } \mathbb{K}[x^s_i, x^s_{i+1}]$ therefore its closure $C^i_{i,m}$ is irreducible. It also allows to compute its codimension. □
Remark 2.3. For \( i = 2, \ldots, e - 1 \) and \( m, s \in \mathbb{N} \) such that \( m \geq 2s - 1 \) and \( l \in \{ s, \ldots, m^l \} \), we have \( \Psi_m^{-1}(D_{i,m}^{i,l}) \neq \emptyset \).

**Proof.** Actually we prove that if \( s \leq l \leq (c_i - 1)s \), then \( \text{Cont}^t(x_i) \cap \text{Cont}^t(x_{i+1}) \neq \emptyset \). Let \( u_i, i = 1, \ldots, e \), be the system of minimal generators of \( v^r \cap M \), so we have that \( x_i = x^h_i \). First note that since \( (u_i, u_{i+1}) \) is a \( \mathbb{Z} \)-basis of \( M \), there exists a unique \( v \in \mathbb{N} \) such that \( (u_i, v) = s \) and \( (u_{i+1}, v) = l \). It is enough to prove that \( v \in \sigma \). For \( e = 4 \), this is easy to check, and the lemma follows by induction on \( e \). □

**Proposition 2.4.** Let \( m, s \in \mathbb{N} \) such that \( m \geq 2s - 1 \).

1. For \( i = 1, e \), we have that \( \pi_m^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_i)) \) is irreducible.

2. For \( i = 2, \ldots, e - 1, m \geq 2s - 1 \), the irreducible components of \( \pi_m^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_i)) \) are the \( c^{i,s}_{i,m,l} \) for \( l \in \{ s, \ldots, m^l \} \).

**Proof.** We sketch the proof of (2), the proof of (1) is similar. We have already seen in the proof of Proposition 2.2 that \( D_{i,m}^{i,l} \subset \pi_m^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_i)) \) for \( l \in \{ s, \ldots, m^l \} \). Using syzygies among \( E_{jk} \), \( 1 \leq j < h - 1 \leq e - 1 \), we prove that

\[
\pi_m^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_i)) = \{ y \in A_m : \text{ord}_y x_j > s \text{ for } j = 1, \ldots, e, \text{ord}_y x_i = s, \text{ord}_y E_{j,h} \geq m + 1 \text{ for } (j, h) = (i - 1, i + 1), j = i, h = i \}.
\]

This implies that the coordinate ring of the above set is isomorphic to a polynomial ring over the coordinate ring of the locally closed subset of the \( m \)-jets of the \( A_{i-1} \) singularity defined by \( E_{i-1,i+1} \), consisting of those \( y \) such that \( \text{ord}_y x_i = s \), \( \text{ord}_y x_{i+1} \) and \( \text{ord}_y x_{i+1} > s \). The claim follows from the description of this latter. □

**Lemma 2.5.** For \( i = 2, \ldots, e - 2 \), \( c^{i,s}_{i,m} = c^{i,m^l}_{i+1,m} \).

**Proof.** This follows from the fact that an \( m \)-jet should verify \( (E_{i,i+2}) \) modulo \( m + 1 \), and from the explicit description in Proposition 2.4. □

Let \( S^0_m := \pi_m^{-1}(O) \), where \( O \) is the singular point of \( S \). Note that \( \pi_m^{-1}(S - \{ 0 \}) \) is an irreducible component of \( S \) of codimension \( (m + 1)(e - 2) \) in \( A_m \). We will see that the irreducible components of \( S^0_m \) have codimension less than or equal to \( (m + 1)(e - 2) \), therefore they are irreducible components of \( S_m \).

**Proposition 2.6.** \( S^0_m = \bigcup_{i \in \{2, \ldots, e - 1\}, \sigma \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor\}} c^{i,s}_{i,2s-1} \).

**Proof.** We first look at the case \( m = 2n + 1 \), \( n \geq 0 \). We claim that

\[
S^0_{2n+1} = \bigcup_{i \in \{1, \ldots, e\}, \sigma \in \{1, \ldots, n\}} \pi_{2n+1,2s-1}^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_i)) \cup c^{n+1,n+1}_{i,2n+1}.
\]

The proof of the claim is by induction on \( n \). By Lemma 2.1, we have that \( S^0_0 = c^{1,1}_{1,1} \) for any \( i = 1, \ldots, e \), hence the case \( n = 0 \). Using the inductive hypothesis for \( n - 1 \), and the fact that for \( s \in \{1, \ldots, n-1\} \) we have that \( \pi_{2n,2s-1}^{-1} \circ \pi_{2n+1,2s-1}^{-1} = \pi_{2n+1,2s-1}^{-1} \) then we obtain:

\[
S^0_{2n+1} = \pi_{2n+1,2s-1}^{-1}(S^0_{2n-1}) = \bigcup_{i \in \{1, \ldots, e\}, \sigma \in \{1, \ldots, n-1\}} \pi_{2n+1,2s-1}^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_i)) \cup \pi_{2n,2s-1}^{-1}(c^{n+1,n+1}_{i,2n+1}).
\]

The claim follows from the stratification \( c^{n+1,n+1}_{i,2n+1} = \bigcup_{j = 1, \ldots, e}(c^{n+1,n+1}_{i,2n-1}\cap D(x^{(n)}_j)) \cup (c^{n+1,n+1}_{i,2n-1}\cap V(x^{(n)}_1, \ldots, x^{(n)}_e)) \), and from the fact that by Lemma 2.1 \( \pi_{2n+1,2s-1}^{-1}(c^{n+1,n+1}_{i,2n+1}\cap V(x^{(n)}_1, \ldots, x^{(n)}_e)) = c^{n+1,n+1}_{i,2n+1} \).

We then conclude the proposition for \( m = 2n + 1 \) in two steps: First, by using Proposition 2.4(2), Second, by deducing from the fact that the vector \( (s, s) \in \sigma \), hence \( \text{Cont}^t(x_1) \cap \text{Cont}^t(x_2) \neq \emptyset \), that \( \pi_{2n+1,2s-1}^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_2)) \cap \pi_{2n,2s-1}^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_1)) \neq \emptyset \). Since by 2.4(1) this latter is irreducible, its generic point coincides with the generic point of one of the irreducible components of \( \pi_{2n+1,2s-1}^{-1}(c^{i,s}_{i,2s-1}\cap D(x^{(s)}_1)) \).

Case \( m = 2(n + 1) \), \( n \geq 0 \): By (2) we just need to prove that

\[
\pi_{2(n+1),2s-1}^{-1}(c^{n+1,n+1}_{i,2n+1}) = \bigcup_{i = 2, \ldots, e - 1, l = n + 1, \ldots, m^l} c^{n+1,l}_{i,2(n+1)}.
\]

The proof is by induction on the embedding dimension. We show below the case \( e = 4 \):
If \( c_2 = c_3 = 2 \), then \( m_n^{n+1} = n + 1 \) and by Lemma 2.1, \( \pi_{2(n+1),2(n+1)−1}(C_{2(n+1),2(n+1)−1}) \) is defined in \( A_0^{0,1} \) whose generators are coordinates and the ideal
\[
(x_1^{(n+1)} - x_1 x_2^{(n+1)}) (x_1 x_2 - x_1^{(n+1)} x_3),
\]
Therefore \( \pi_{2(n+1),2(n+1)−1}(C_{2(n+1),2(n+1)−1}) \) is irreducible (the above ideal is isomorphic to the ideal which defines the surface \( S \)) and is equal to \( D_{2(n+1),2(n+1)−1} \). Since \( D_{2(n+1),2(n+1)−1} \) is dense in both. The subcases \( (c_2 = 2 \) and \( c_3 \neq 2 \) and \( c_2 \neq 2 \) and \( c_3 \neq 2 \)) follow also easily.

**Theorem 2.7.** Let \( m \in \mathbb{N} \), \( m \geq 1 \). Modulo the identifications \( C_{i,m}^{s,l} = C_{i+1,m}^{s,1} \), the irreducible components of \( S_m^0 := \pi_{m}^{-1}(0) \) are the \( C_{i,m}^{s,l} \), \( i = 2, \ldots, e - 1 \), \( s \in \{1, \ldots, \lceil \frac{m}{2} \rceil \} \) and \( l \in \{s, \ldots, m_t \} \).

**Proof.** By Proposition 2.6, \( S_m^0 \) is covered by the \( C_{i,m}^{s,l} \). But apart from the identifications above, \( C_{i,m}^{s,l} \neq C_{i,m}^{s',l'} \), because by Proposition 2.4, there exist hyperplane coordinates that contain the one but not the other, and by Proposition 2.2 they have the same dimension. On the other hand, \( C_{i,m}^{s,l} \neq C_{i,m}^{s',l'} \), because by Proposition 2.4 the \( C_{i,m}^{s,l} \) has non-empty intersection with \( D(x^{(i)}) \), but \( C_{i,m}^{s',l'} \neq C_{i,m}^{s',l'} \). Finally, \( C_{i,m}^{s,l} \neq C_{i,m}^{s',l'} \), because by Proposition 2.2 the codimension of the first one is less than or equal to the codimension of the second one, and the theorem follows.

**Remark 2.8.** Given Theorem 2.7, Remark 2.3 means that there are no vanishing components.

**Definition 2.9.** Let \( m \in \mathbb{N} \), \( m \geq 1 \), and let \( C \) be an irreducible component of \( S_m^0 \). By Theorem 2.7, there exist \( s \in \{1, \ldots, \lceil \frac{m}{2} \rceil \} \), \( l \in \{s, \ldots, m_t \} \) and \( i \in \{2, \ldots, e - 1 \} \) such that \( C = C_{i,m}^{s,l} \). We say that \( C \) has index of speciality \( s \). Note that \( s = ord_{y}(M) := \min \{f \in \mathcal{M} | ord_{y}(f) = \gamma \} \), where \( M \) is the maximal ideal of the local ring \( O_{S,0} \) and \( y \) the generic point of \( C \).

For \( a, b \in \mathbb{N} \), \( b \neq 0 \), we denote by \( \lfloor \frac{a}{b} \rfloor \) the integral part of \( \frac{a}{b} \). For \( c, m \in \mathbb{N} \), let \( m = q c + r \) be the Euclidean division of \( m \) by \( c \). We set \( N_{c,i}^c(m) := sc - (2s - 1) \), \( s = 1, \ldots, q_c \); \( N_{c,i}^c(m) := m - (2s - 2) \), \( s = q_c + 1, \ldots, \lceil \frac{m}{2} \rceil \).

For \( m \in \mathbb{N} \), \( m \geq 1 \), we call \( N(m) \) the number of irreducible components of \( S_m^0 \). Then counting the irreducible components in Theorem 2.7 we find

**Corollary 2.10.** If all the \( c_i \) are equal to 2, then \( N(m) = \lceil \frac{m}{2} \rceil \). Otherwise let \( c_{i_1}, \ldots, c_{i_{h}} \) be the elements in \( \{c_2, \ldots, c_e\} \) different from 2, then we have \( N(m) = \sum_{i=1}^{\lceil \frac{m}{2} \rceil} (N_{i_1}^c(m) + (N_{i_2}^c(m) - 1) + \ldots + (N_{i_{h}}^c(m) - 1)).\)

**Corollary 2.11.** For \( m \geq max\{c_i, i = 2, \ldots, e - 1\} \), the number of irreducible components of \( S_m^0 \), with index of speciality \( s = 1 \), is equal to the number of exceptional divisors that appear on the minimal resolution of \( S \).

**Proof.** This comes from the comparison of Corollary 2.10 with Corollary 1.23 in [15, p. 29].

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