I. INTRODUCTION

Thermodynamic observables including, e.g., density correlations are among the most prominent observables that provide information about the phase structure of heavy ion collisions. Their computations rely on access to the bulk thermodynamic information of the system. This information is encoded in the free energy, whose determination at finite temperature and density is of prime interest. At large densities functional approaches such as the Functional Renormalization Group (FRG) and Dyson-Schwinger equations (DSEs) circumvent the eminent sign problem that at present prevents lattice simulations in this regime. However, while the part of correlations and thermodynamics that comes from the matter fluctuations does not pose problems in functional approaches, the access to the thermodynamics of gauge fluctuations poses a formidable challenge beyond perturbation theory. At its root it is related to the relevant momentum-scale running of the thermodynamic potentials such as the free energy. Even though by now functional methods have reached high quantitative precision, such a computation still remains a demanding calculation in terms of resources. This asks for computational approaches that reduce the computational effort.

Such an alternative may be provided by a computation in terms of the Tan contact \cite{tan,local}, for a discussion in Yang-Mills theory see \cite{tanY}. Essentially, it boils down to the idea, rehearsed in section II, that the thermodynamic part could be encoded in a simple way in the high-momentum behavior of the two-point correlation functions. These are much simpler to determine reliably. While the extracted part still needs some processing to obtain the free energy, drastic features, e.g. phase transitions, should manifest themselves already directly in the unprocessed data. The aim of the present work is to explore exactly this possibility.

To this end, we use the Landau-gauge gluon propagator of SU(2) Yang-Mills theory at finite temperature. This theory undergoes a second-order phase transition at a well established critical temperature. Furthermore, its free energy is known quite well. It is thus an ideal testbed for a new method. We use for this purpose the gluon propagator as obtained using lattice methods, the functional renormalization group, and Dyson-Schwinger equations. The results are shown in section III.

The Tan contact term is related in a straightforward way to the thermodynamic anomaly. This is exploited in section IV, where in a proof-of-principle style the anomaly is determined. There it will also be discussed what further steps are required to make this a quantitatively competitive approach.

In fact, the results show that interesting features of the thermodynamics are already manifest for the unprocessed data. These results are encouraging in that this is possible and may be a promising future avenue, as is concluded in section V.

II. SETUP

The basic idea of how to transfer the Tan contact term formalism \cite{tan,local} from solid state physics and ultracold atoms, e.g. \cite{karbach,deGennes,parisi,bork,hecht}, to particle physics has been outlined in \cite{tanY}. It essentially boils down to that the high-momentum behavior of a propagator $D(T, p_0, \vec{p})$, i.e. at momenta $p \gg \Lambda_{YM}$, depending on both the temperature $T$ and the momentum $p$, should behave essentially as

$$D(T, p_0^2, \vec{p}^2) = \frac{Z}{Z_0(p_0^2 + \vec{p}^2)} + C(T) D_0(p_0^2 + \vec{p}^2)$$

(1)

where $C(T)$ is the Tan contact term, which is the only source of temperature dependence. $D_0$ is the vacuum propagator and $Z$ is a total normalization of the propagator. Here it is chosen such that $Z D_0(\mu^2) = 1/\mu^2$, i.e. we choose a temperature-independent renormalization scheme. By construction, the Tan contact term satisfies $C(T = 0) = 0$ if the fit form (1) describes the propagator perfectly. Note also that the Tan contact in (1) is not RG invariant, it runs with twice the anomalous
The dimension of the propagator. An RG-invariant form is

easily achieved by multiplication with the wave function

renormalization squared. As we concentrate on the com-

parison between functional approaches and the lattice,

this is not important for us.

In the present work we consider the gluon propagator.

As we are interested in high energies, we set \( D_0 \) to be the

one-loop resummed propagator

\[
D_0(p^2, \mu^2) = \frac{1}{p^2 \left( 1 + \omega^2 \ln \frac{p^2}{\mu^2} \right)^2},
\]

which entails \( Z = 1 \). The quantity \( \omega \) also involves

coupling \( g \). To accommodate for different renor-

malization prescriptions, we fit \( \omega^2 \) to the zero-temperature

propagator for the different methods rather than to use

some prescribed value. This approach describes the gluon

propagator above 2 GeV at zero temperature for all

methods at the 1-2\% level. In this regime also the propa-

gator for the different methods rather than to use

\( D \) propagator above 2 GeV at zero temperature for all

\( Z \) magnonic Tan contact,

\( C_\perp(T) \) independently. This yields agreement of the spatial-

diagonal lattice data and the DSE and FRG results from

the zero-temperature propagator.\( \mu = 2 \) GeV.

For the DSE, we use the data from [10] with

some additional statistics and two additional lattice dis-

cretizations at \( T/T_c = 0.9 \) and \( T/T_c = 1.1 \) with an

\( 8 \times 40^3 \) lattice. For the zero-temperature form (2)
data from [11] are used. This entails statistical errors on

the fit parameters \( Z \) and \( \omega \), which were propagated to

the fit of \( C(T) \). While \( \omega = 0.82^{+0.04}_{-0.03} \) is essentially \( \beta \)-

independent, \( Z \) was interpolated for different \( \beta \) values

by \( Z_01.50^{+0.06}_{-0.03} \ln(\beta) - 11^{-0.04}_{-0.01} \), where \( Z_0 \) is the arbi-

trarily chosen renormalization prescription at zero tempera-

ture fixed \( \mu = 2 \) GeV. In all cases fits where done along spa-

tial diagonals, which are least affected by discretization
effects at high momenta [11].

For the FRG, we use the results from [12]. The vacuum

results yield \( Z = 2.69Z_0 \) and \( \omega = 0.795 \) at \( \mu = 2 \) GeV.
The value for \( \omega \) agrees well with the lattice result.

We also extract the Tan contact term from DSE re-

sults. Although they are obtained from a much simpler

truncation than the FRG results, the high momentum

behavior is determined sufficiently well to extract the re-
vant information as shown below. Details of the DSE

calculations can be found in Appendix A. Their fit para-

terms are \( Z = 1.78 \) and \( \omega = 0.752 \), in good

agreement to the other methods.

The temperatures are taken from the respective works

as well, i.e. we did not additionally try to fix any scales

independently. This yields agreement of the spatial-
diagonal lattice data and the DSE and FRG results from

2 GeV up to 12 GeV at the percent level and thus for the

whole range of relevant momenta in this work.

III. RESULTS

At finite temperature, we find that the fits work at the

few percent level well in all cases. However, results from

the lattice for the two temperatures \( T/T_c = 0.9 \) and

\( T/T_c = 1.1 \) with ten times more statistics reveal that (1)
is insufficient if at that level of statistics a sub-percent

fit is desired. Rather, \( C(T) \) needs then to be replaced by

some extended form, e.g. \( C(T) + p^2 D(T) \). A similar re-

sult is obtained in the FRG and DSE cases. However, for

the present purpose, and without a major effort for cre-

ating more statistics for the lattice, we contend ourselves

here with fits at the 2-3\% level, which at low statistics is

also the statistical accuracy of the lattice results, al-

lowing for agreement within errors. Note that for the

continuum results the fit stops to work above roughly

\( T/T_c \approx 5 \). This is expected, as when \( T \) becomes larger,

eventually screening effects will propagate to larger mo-

menta which are not included in the fit ansatz (1).

The results are shown in Fig. 1. First of all, it is visible

that the general agreement between lattice and functional

methods is satisfactory, except for the DSE in the trans-

verse case at high temperatures. Also, at this level of

statistics no statistically significant dependency on latti-

ce parameters is visible. Then, there are a number of

visible trends which are quite different for the transverse

and the longitudinal Tan contact term.

The probably most significant one is the difference be-

tween the transverse one and the longitudinal one at high

temperatures. The transverse one starts to rise from

essentially zero somewhere around \( t = T/T_c \approx 0.8 \) for the

lattice data, levels off shortly after \( t \gtrsim 1 \), and stays con-

stant up to \( t \approx 3 \). There is no significant change happen-

ing at the phase transition. The functional results switch

on smoothly, but follow the same trend. However, above

\( t \gtrsim 2.5 \), the functional methods yield again a slow rise of

the Tan contact term.

The longitudinal one is quite different. Up to \( t \approx 1 \), the

lattice results are compatible with zero. There is a slight

systematic, though not statistically satisfactory trend to

non-zero values above \( t = 1 \). However, at large tempera-

tures the Tan contact term rises quicker than quadrat-

ically with temperature. Except for the smoothing of

the transition, this behavior is also seen in the functional

results, this time with no particular impact at \( t \gtrsim 3 \).

In comparison to the low-momentum behavior [10, 12],

this provides a consistent picture. There, also the trans-

verse propagator shows no substantial impact of the

phase transition, while the longitudinal one seems to do

so. At the same time, the impact at high temperatures is

also stronger for the longitudinal one.

This leads us to the following picture: The transverse

sector carries non-trivial thermodynamic behavior, which

is sensitive to the interactions which create a strongly-
interacting phase above the phase transition for a range of a few $T_c$. The bulk thermodynamics is manifested in the longitudinal degrees of freedom, including both the phase transition and the Stefan-Boltzmann trend at high temperatures.

IV. THE ANOMALY FROM THE TAN CONTACT TERM

While the Tan contact term in solid state physics and ultracold atoms encodes the thermodynamics, it is in itself not yet equivalent to a thermodynamic potential. However, it is linked to the thermodynamic anomaly $A(T)$, see e.g. $[13]$, $A(T) = \beta(g(T))C(T)$ (3)

wherein $\beta(g)$ is the $\beta$-function and $g(T)$ the temperature-dependent running coupling evaluated at the temperature. An analogous derivation in Yang-Mills theory faces several intricacies. First of all this concerns the unphysical nature of gluon fields in comparison to that in solid state and ultracold atomic systems. This leads us to negative norm states in the Fock space as well as the occurrence of ghost fields. Accordingly, a Yang-Mills analogue of the relation (3) will involve $C_{\perp}, C_L$ and $C_{\text{ghost}}$ and the respective $\beta$-functions $\beta_{\perp}, \beta_L, \beta_{\text{ghost}}$ as well additional normalisation factors. The latter differ in the strongly-correlated low temperature regime with $T \lesssim T_c$. Being short of the full resolution of the different ingredients of the Yang-Mills relation we here discuss the chromomagnetic and chromoelectric parts of this relation. They are given by $A_{\perp/L}(T) \approx \beta(g(T))C_{\perp/L}(T)$, (4)

where we will use the same $\beta$-function for chromomagnetic and chromoelectric parts and take the normalisation factors to unity.

Note that the left-hand side of (4) are related to an observable, the thermodynamic or trace anomaly in Yang-Mills theory. Thus, scheme-dependencies on the right-hand side need to cancel, implying that the Tan contact term is scheme-dependent. In addition, the miniMOM or Taylor scheme $[14, 15]$ employed in the calculation of the gluon propagators yields a multi-valued $\beta$-function and its precise determination in lattice calculations requires high statistics. While the former can be remedied by using the temperature-dependent correct branch, the latter precludes us yet from a full determination within each method separately. Also, as will be seen, the Tan contact term needs to be determined at much higher precision in the low-temperature domain.

However, as a proof-of-principle, we will use here an analytic, temperature-independent coupling motivated by analytic perturbation theory, $\alpha(p) = \pi \frac{\ln \Lambda^2_{\text{YM}}}{\ln \Lambda^2_{\text{YM}} + p^2}$

taking $\Lambda^2_{\text{YM}} = 0.81$ GeV$^2$ for the scale. Note that this will necessarily upset the overall scale of the result, as we do not use matched schemes.

The results are shown in figure 2. The lattice results, albeit with large errors, are consistent with the temperature dependence of the anomaly, showing a peak around the phase transition, and a slow decrease towards large temperatures. At low temperatures, where already the Tan contact term is compatible with zero within the er-
FIG. 2: The anomaly obtained from the Tan contact term from lattice, FRG & DSE.

errors, so is necessarily the anomaly. Moreover, we deduce from figure 2 that the overall normalisation of (4) is non-trivial as the trace anomaly $A_{YM}$ in Yang-Mills obeys $A_{YM} \lesssim 3$, see e.g. [16], while $A_{LT} \lesssim 10^3$. Both functional results show a quite similar behavior at high temperatures, but tend to have the peak at far too low temperatures. This is likely partly because this temperature regime is in the deep infrared, where the $\beta$-function is not dominated by its perturbative behavior. Here, a determination of the $\beta$-function in a consistent scheme would likely cure these problems.

Nonetheless, the anomaly shows qualitatively the expected behavior, indicating that the Tan contact term may indeed be a suitable approach to obtain thermodynamic information from propagators.

V. CONCLUSIONS

We have extracted for the first time the Tan contact term for Yang-Mills theory from the gluon propagator. We see that known thermodynamic features, the phase transition, the asymptotic Stefan-Boltzmann behavior, and the strongly-interacting liquid behavior imprint themselves qualitatively in the Tan contact term. We also see that the various effects distribute themselves among the transverse and longitudinal degrees of freedom differently. While the strong-interaction regime above the phase transition seems to be encoded in the chromomagnetic sector, the critical and bulk behavior seems to be carried by the chromoelectric sector. This agrees with observations in the infrared [10]. It has also been shown that it is, in principle, possible to use the Tan contact term to determine the anomaly and thus thermodynamic bulk properties.

The obvious steps to be taken from here are to improve statistics and systematics on the lattice and to compare to further results from other sources, e.g. hard-thermal loop calculations or results from dimensionally-reduced calculations [17, 18]. Another issue are contributions from the ghost, which at first sight seems to be inert to temperature [10, 12, 19, 20]. For a reconstruction of the thermodynamic potential in full it is required to find the correct normalisation, a suitable scheme and sufficient precision to determine the anomaly. Finally, an extension to finite density is of high interest. Here, also QCD-like theories without sign problem, e.g. 2-color QCD or $G_2$-QCD, could be interesting testing grounds.

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Appendix A: Propagators from Dyson-Schwinger equations

The truncation used for the calculation of the propagators from their DSEs is described in the following. The equations that were solved are the ones for the ghost and gluon propagators truncated to one-loop without tadpoles. The only remaining higher n-point functions are the ghost-gluon and three-gluon vertices. The former is taken as bare, which is within the context of the present work sufficient since we are only interested in the high-momentum behavior. The deviation from a bare vertex is known to be a bump around 1 GeV which falls off quickly [12, 20, 21].

The three-gluon vertex plays a crucial role for the gluon propagator. It is not only quantitatively relevant, but also the existence of a solution for the gluon propagator depends strongly on its properties. Here, the following model adapted from Ref. [22] was used for dressing the tree-level tensor,

\[ C^{AAA}(p_0, q_0, \vec{p}, q) = \frac{G(p^2)}{Z_T(p^2)} \frac{\vec{p}^2}{p^2 + A_s^2} \left( -G(p^2) \right)^3 A_{3g}^2 \frac{A_{3g}^2}{A_{3g}^2 + p^2 A_{3g}^2 + q^2 A_{3g}^2 + (p + q)^2} + \frac{G(p^2)}{Z_T(p^2)} \frac{\vec{p}^2}{p^2 + A_s^2}. \]  

(A1)

The momentum \( p^2 \) is \( (p^2 + q^2 + (p + q)^2)/2 \) and \( p \) and \( q \) are four-momenta. \( G \) and \( Z_T \) are the ghost and the transverse gluon dressing functions, respectively. The first term in the parentheses determines the IR behavior of the vertex, the second the UV behavior. The term in front of the parentheses accounts for missing perturbative higher loop contributions relevant for the resummed one-loop behavior [23–25]. The model contains two scales which are fixed as \( A_s = A_{3g} = 0.741 \) GeV.

The integral kernels for the Dyson-Schwinger equations can be found, e.g., in Ref. [26]. Here they were derived with DoFun [27–29], and the equations were solved with CrasyDSE [30]. Quadratic divergences in the gluon propagator DSE were renormalized via second renormalization conditions chosen as the value of the propagators at zero momentum [25, 31, 32]. For the employed truncation this leaves some ambiguity how to select these conditions, but at the scales of relevance here it is expected that such effects are subleading.

The overall scale was set for the lowest calculated temperature by matching the UV tail to FRG results. The relative scales for the other temperatures were set by matching the perturbative couplings.

The resulting dressing functions for the gluon propagators are shown in Fig. 3 for selected temperatures. Clearly, the present truncation cannot capture the IR behavior but reproduces the momentum and temperature dependencies qualitatively. Both, in the FRG and the DSE computation of the propagators the computation of the non-trivial \( A_0 \)-background has not been taken into account. This background \( A_0 \neq 0 \) is the equation of motion and is directly linked to the vanishing of the Polyakov loop in the confining phase, [33–37], for perturbative computations within the background see [38].

In [12] it has been argued that this should lead to deviations of the chromo-electric propagators in functional approaches from the chromo-electric lattice propagators (as they are computed on different a background) for temperatures with

\[ 0.5 T_c \lesssim T \lesssim 1.3 T_c. \]  

(A2)

Indeed this expectation is confirmed by the data, see figure 3b.

[1] S. Tan, Annals of Physics 323, 2952 (2008), cond-mat/0505200.
[2] S. Tan, Annals of Physics 323, 2987 (2008).
[3] L. Fister and J. M. Pawlowski, (2011), 1112.5440.
[4] J. T. Stewart, T. E. Gaebler, and D. S. Jin, Physical Review Letters 104, 235301 (2010).
[5] B. Mukherjee et al., Physical Review Letters 122, 203402 (2019).
[6] T. Enss, R. Haussmann, and W. Zwerger, Annals of Physics 326, 770 (2011).
[7] I. Boettcher, S. Diehl, J. M. Pawlowski, and C. Wetterich, Phys. Rev. A87, 023606 (2013), 1209.5641.
[8] E. Braaten, The BCS-BEC Crossover and the Unitary Fermi Gas volume 836 of Lecture Notes in Physics (Springer, 2012), chap. Universal Relations for Fermions with Large Scattering Length, p. 193.
[9] C. S. Fischer, A. Maas, and J. A. Müller, Eur. Phys. J. C68, 165 (2010), 1003.1960.
[10] A. Maas, J. M. Pawlowski, L. von Smekal, and D. Spielmann, Phys.Rev.D85, 034037 (2012), 1110.6340.
[11] A. Maas, Phys. Rev. D91, 034502 (2015), 1402.5050.
[12] A. K. Cyrol, M. Mitter, J. M. Pawlowski, and A. Maas, Phys. Rev. D96, 074504 (2017), 1704.03542.
FIG. 3: Gluon propagator dressings for selected temperatures from the FRG (continuous), DSEs (dashed) and lattice simulations [9, 10]. For reference the vacuum result from the FRG is shown in black.

N. Strodthoff, Phys. Rev. D97, 054015 (2018), 1708.03482.
[13] T. Enss, (2019), 1904.12772.
[14] L. von Smekal, K. Maltman, and A. Sternbeck, Phys. Lett. B681, 336 (2009), 0903.1696.
[15] P. Boucaud et al., Phys. Rev. D79, 014508 (2009), 0811.2059.
[16] S. Borsanyi, G. Endrodi, Z. Fodor, S. D. Katz, and K. K. Szabo, JHEP 07, 056 (2012), 1204.6184.
[17] J.-P. Blaizot and E. Iancu, Phys. Rept. 359, 355 (2002), hep-ph/0101103.
[18] A. Hietanen, K. Kajantie, M. Laine, K. Rummukainen, and Y. Schröder, Phys. Rev. D79, 045018 (2009), 0811.4664.
[19] M. Q. Huber and L. von Smekal, PoS LATTICE2013, 364 (2014), 1311.0702.
[20] L. Fister and A. Maas, Phys.Rev. D90, 056008 (2014), 1406.0638.
[21] M. Q. Huber, EPJ Web Conf. 137, 07009 (2017), 1611.06136.
[22] M. Q. Huber, Eur. Phys. J. C77, 733 (2017), 1709.05848.
[23] L. von Smekal, A. Hauck, and R. Alkofer, Ann. Phys. 267, 1 (1998), hep-ph/9707327.
[24] M. Q. Huber and L. von Smekal, JHEP 1304, 149 (2013), 1211.6092.
[25] M. Q. Huber, (2018), 1808.05227.
[26] A. Maas, The high-temperature phase of Yang-Mills theory in Landau gauge, PhD thesis, Darmstadt University of Technology, 2004, hep-ph/0501150.
[27] R. Alkofer, M. Q. Huber, and K. Schwenzer, Comput. Phys. Commun. 180, 965 (2009), 0808.2939.
[28] M. Q. Huber and J. Braun, Comput.Phys.Commun. 183, 1290 (2012), 1102.5307.
[29] M. Q. Huber, A. K. Cyrol, and J. M. Pawlowski, (2019), 1908.02760.
[30] M. Q. Huber and M. Mitter, Comput.Phys.Commun. 183, 2441 (2012), 1112.5622.
[31] J. C. Collins, Renormalization: An introduction to renormalization, the renormalization group, and the operator product expansion (Cambrdige University Press, Cambridge, 1984).
[32] J. Meyers and E. S. Swanson, Phys. Rev. D90, 045037 (2014), 1403.4350.
[33] J. Braun, H. Gies, and J. M. Pawlowski, Phys. Lett. B684, 262 (2010), 0708.2413.
[34] F. Marhauser and J. M. Pawlowski, (2008), 0812.1144.
[35] J. Braun, A. Eichhorn, H. Gies, and J. M. Pawlowski, Eur.Phys.J. C70, 689 (2010), 1007.2619.
[36] L. Fister and J. M. Pawlowski, Phys. Rev. D88, 045010 (2013), 1301.4163.
[37] T. K. Herbst, J. Luecker, and J. M. Pawlowski, (2015), 1510.03830.
[38] U. Reinoso, J. Serreau, M. Tissier, and A. Tresmontant, Phys. Rev. D95, 045014 (2017), 1606.08012.