A new Capacity-Achieving Private Information Retrieval Scheme with (Almost) Optimal File Length for Coded Servers

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Abstract

In a distributed storage system, private information retrieval (PIR) guarantees that a user retrieves one file from the system without revealing any information about the identity of its interested file to any individual server. In this paper, we investigate \((N, K, M)\) coded server model of PIR, where each of \(M\) files is distributed to the \(N\) servers in the form of \((N, K)\) maximum distance separable (MDS) code for some \(N > K\) and \(M > 1\). As a result, we propose a new capacity-achieving \((N, K, M)\) coded linear PIR scheme such that it can be implemented with file length \(\frac{K(N-K)}{\gcd(N,K)}\), which is much smaller than the previous best result \(K\left(\frac{N}{\gcd(N,K)}\right)^{M-1}\). Notably, among all the capacity-achieving coded linear PIR schemes, we show that the file length is optimal if \(M > \left\lfloor\frac{K}{\gcd(N,K)} - \frac{K}{N-K}\right\rfloor + 1\), and within a multiplicative gap \(\frac{K}{\gcd(N,K)}\) of a lower bound on the minimum file length otherwise.

Index Terms

Private information retrieval, distributed storage system, file length, coded servers, capacity-achieving.

I. INTRODUCTION

With the rapid development of open source systems, protecting user’s privacy of retrieved information from public servers becomes a new challenge. This problem, referred to as private information retrieval (PIR) has direct applications in many scenarios such as cloud service, Internet of things, social networks, and so on. For examples, PIR can ensure the privacy of the identity of the stocks downloaded by an investor from the stock market, and the privacy of activists against oppressive regimes while browsing some information deeded to be anti-regime.

The PIR problem was first introduced by Chor et al. in [3] and has drawn much attention from computer science community subsequently [2, 6, 7, 16]. The classical formulation of PIR allows a user to query and download a file from \(N\) servers, each hosting the whole library of \(M\) files, without revealing any information about the identity of the desired file to any one of the servers. To retrieve a particular file, the user first sends a query string to each server. After receiving query, the server responds by sending an answer string to the user. Then the user decodes its requested file with these answers from the servers.

A native strategy is to download all the files of the library no matter which file the user needs. However, it is extremely inefficient in terms of the retrieval rate, which is defined as the number of bits that the user can privately retrieve per bit of downloaded data. In the practical systems, it is preferred to design PIR schemes achieving as large as possible retrieval rate, especially achieving its supremum, which is known as the capacity of the system. In the initial work [3], one bit length file was considered, while the cost was measured by the total lengths of all query and answer strings. In fact, the corresponding retrieval rate is \(\Theta(1/M^2)\). Nevertheless, the Shannon theoretic formulation allows the length of files to be arbitrarily large, and therefore the upload cost (total length of query strings) is insignificant compared to the download cost (total length of answer strings), which is the more practical scenario. Under this assumption, Shah et al. proposed a scheme achieving the retrieval rate \(1 - \frac{1}{N} \Theta\). In the later attractive work by Sun and Jafar [16], the PIR capacity was finally characterized as \(1 + \frac{1}{N} + \ldots + \frac{1}{M}\) for any \(N\) and \(M\).
The capacity-achieving PIR scheme in [10] requires the length of the files to be a multiple of $N^M$, which increases to infinity exponentially with number of files $M$. Although large file length can contribute to improve retrieval rate, it also arouses complexity increase in practical implementations. The problem of decreasing the file length, also known as sub-packetization in the literature, has been noted in many applications, for example coded caching [14, 15], while it becomes the bottleneck for practical application of the PIR scheme in [11]. The file length of capacity-achieving PIR schemes was reduced to $N^{M-1}$ in [11] subsequently. Most notably, the optimal length was proved to be $N - 1$ in [13] very recently, which turns out to be independent of $M$ and much smaller than previous schemes.

It should be noted that all the aforementioned results are obtained in the classical setup, which requires each sever to store all the files completely, i.e., a replication code is used to store the contents. Though repetition coding can offer the highest capability against sever failures and simplify the implementation of PIR schemes as well, it incurs pretty large storage cost. This impels the use of erasure coding techniques, especially the maximum distance separable (MDS) code that achieves the capability against sever failures and simplify the implementation of PIR schemes as well, it incurs pretty large storage cost.

For the coded servers, the exciting result [13] in the classical setup motivates us to explore the problem: Does there exist a capacity-achieving coded scheme under this more general setup such that, the length of files is independent of $M$? In this paper, we give an affirmative answer. The previous results in [1] and [17] indicates that, linear PIR scheme is sufficient to achieve the capacity even in the case of coded servers, i.e., the answers can be restricted to linear functions of the stored contents. So, we only focus on capacity-achieving coded linear PIR schemes due to their simplicity. Moreover, we discard the trivial cases of contents. So, we only focus on capacity-achieving coded linear PIR schemes due to their simplicity. Moreover, we discard the trivial cases of $N = K$ or $M = 1$, since the user can download all contents stored by the servers to guarantee privacy in the former case and essentially no file can be protected in the latter case. For that reason, the objective of this paper is to design capacity-achieving coded linear PIR scheme with practical file length for the coded server system in the cases of $N > K$ and $M > 1$. The contributions of this paper are two folds:

1) We propose a new capacity-achieving coded linear PIR scheme with file length $L = \frac{K(N-K)}{gcd(N,K)}$. The new scheme has obvious advantage with respect to file length over the state-of-the-art in [11] and [17]. For clarity, we compare them in Table I.

| Capacity-Achieving PIR Schemes | Banawan-Ulukus scheme | Xu-Zhang scheme | New scheme |
|-------------------------------|-----------------------|----------------|------------|
| Minimum File Length           | $KN^M$                | $K\left(\frac{N}{gcd(N,K)}\right)^{M-1}$ | $K\left(\frac{N-K}{gcd(N,K)}\right)$ |

2) We show that the file length $L$ is optimal among all the capacity-achieving $(N, K, M)$ coded linear PIR schemes when $M > \left\lfloor \frac{K}{gcd(N,K)} \right\rfloor - \frac{K}{N-K}$ + 1. In the other case, the file length $L$ is within a multiplicative gap $\frac{K}{gcd(N,K)}$ in contrast to its lower bound.

The rest of this paper is organized as follows. In Section III the MDS coded server model is formally described and linear PIR scheme is characterized. In Section IV a new capacity-achieving coded linear PIR scheme is proposed. In Section V the achievability of the new coded scheme is analyzed. In Section VI the optimality of the file length is established. Finally, the paper is concluded in Section VII.

Throughout this paper, the following notations are used.

- For two non-negative integers $n$ and $m$ with $n < m$, define $[n : m]$ as the set \{n, n + 1, \ldots, m - 1\};
- For a finite set $S$, $|S|$ denotes its cardinality;

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- Denote \( A_{0:m-1} \) an ordered set \( \{A_0, \ldots, A_{m-1}\} \), and define \( A_\Gamma \) as \( \{A_{\gamma_0}, \ldots, A_{\gamma_{k-1}}\} \) for any indices set \( \Gamma = \{\gamma_0, \ldots, \gamma_{k-1}\} \subseteq [0 : m] \) with \( \gamma_0 < \ldots < \gamma_{k-1} \);
- For a matrix \( Q \), define \( Q(i,:) \), \( Q(:,j) \) and \( Q(i,j) \) as its \( i \)-th row vector, its \( j \)-th column vector and the element in row \( i \) and column \( j \), respectively.

II. SYSTEM MODEL

Consider a distributed storage system that stores \( M \) encoded files \( W_0, W_1, \ldots, W_{M-1} \) across \( N \) servers by using a fixed \( (N, K) \) MDS code over \( \mathbb{F}_p \) for a prime power \( p \). The \( M \) files are independent of each other and each file \( W_i \) is of the form

\[
W_i = \begin{pmatrix}
W_i(0,0) & \cdots & W_i(0,N-1) \\
\vdots & \ddots & \vdots \\
W_i(\lambda-1,0) & \cdots & W_i(\lambda-1,N-1)
\end{pmatrix}, \quad i \in [0 : M),
\]

(1)

where the row vectors \( W_i(0,:) \), \( W_i(1,:) \), \ldots, \( W_i(\lambda-1,:) \) are \( \lambda \) independent codewords of the \( (N, K) \) MDS code. We refer to the quantity

\[
L \triangleq K \cdot \lambda
\]
as file length, since each file can be equivalently represented by a source file consisting of \( \lambda \) vectors of length \( K \) over \( \mathbb{F}_p \). As a consequence,

\[
H(W_0) = \ldots = H(W_{M-1}) = L,
\]

(2)

\[
H(W_0, \ldots, W_{M-1}) = \sum_{i=0}^{M-1} H(W_i),
\]

(3)

where the entropy function \( H(\cdot) \) is measured with logarithm \( p \).

At the \( t \)-th server, the stored contents \( y_t \in \mathbb{F}_p^{M \lambda} \) are the concatenation of the \( t \)-th column in all the encoded files \( W_{0:M-1} \) and are given by

\[
y_t = \begin{pmatrix}
W_0(:,t) \\
W_1(:,t) \\
\vdots \\
W_{M-1}(:,t)
\end{pmatrix}, \quad t \in [0 : N).
\]

Due to the property of the base \( (N, K) \) MDS code, the storage system can reconstruct all the files by connecting to any \( K \) servers to tolerate up to \( N - K \) server failures. Then, for any set \( \Gamma \subseteq [0 : N] \) with \( |\Gamma| \geq K \),

\[
H(y_{\overline{\Gamma}} | y_{\Gamma}) = 0,
\]

where \( \overline{\Gamma} \triangleq [0 : N]\backslash \Gamma \).

A user privately generates \( \theta \in [0 : M) \) and wishes to retrieve \( W_\theta \) from the storage system\(^1\) while ensuring that any one of the \( N \) servers will have no information about \( \theta \), by means of a private information retrieval (PIR) scheme consisting of the following phases:

1) **Query Phase**: The user randomly generates a set of queries \( Q_{0:N-1}^{[\theta]} \) according to some distribution over \( \mathcal{U}^N \), and sends \( Q_t^{[\theta]} \) to the \( t \)-th server, where the set \( \mathcal{U} \) will be referred to as query space. Indeed, the queries are generated independently of file realizations, i.e.,

\[
I(Q_t^{[\theta]} ; W_{0:M-1}) = 0.
\]

(4)

2) **Answer Phase**: Upon receiving the query \( Q_t^{[\theta]} \), the \( t \)-th server responds with an answer \( A_t^{[\theta]} \), which is a deterministic function of the received query and the stored contents at server \( t \). Thus, by the data-processing theorem,

\[
H(A_t^{[\theta]} | Q_t^{[\theta]} , y_t) = H(A_t^{[\theta]} | Q_t^{[\theta]} , W_{0:M-1}) = 0, \quad t \in [0 : N).
\]

(5)

3) **Decoding Phase**: The user must correctly decode the desired file \( W_\theta \) from answers \( A_{0:N-1}^{[\theta]} \).

\(^1\)Note that, the user can retrieve the encoded file \( W_\theta \) if and only if it can retrieve \( \theta \)-th source file.
As a PIR scheme, it has to satisfy

- **Correctness:** The file $W_0$ can be completely disclosed by the queries and answers, i.e.,

\[
H(W_0 \mid A_{0:N-1}^{[\theta]} , Q_{0:N-1}^{[\theta]}) = 0, \quad \theta \in [0 : M].
\] (6)

- **Privacy:** Each server $t \in [0 : N)$ should learn nothing about the index $\theta$, i.e.,

\[
I(Q_t^{[\theta]}, A_t^{[\theta]}, y_t; \theta) = 0, \quad (7)
\]

where $(Q_t^{[\theta]}, A_t^{[\theta]}, y_t)$ is all the information owned by the $t$-th server. Equivalently, its distribution is independent of the realization of $\theta$, i.e.,

\[
(Q_t^{[\theta]}, A_t^{[\theta]}, y_t) \sim (Q_t^{[\theta]}, A_t^{[\theta]}, y_t), \quad \forall \theta, \theta' \in [0 : M), \ t \in [0 : N),
\] (8)

where $X \sim Y$ indicates that $X$ and $Y$ are identically distributed.

We call the above PIR an $(N, K, M)$ coded PIR scheme.

The rate of a coded PIR scheme, denoted by $R$ is defined as

\[
R = \frac{L}{D},
\] (9)

where $D$ is download cost (the total length of the answers) from the $N$ servers averaged over random query realizations.

Notice that the $R$ and $D$ are independent of $\theta$ by the privacy constraint in (8).

**Definition 1** (The Capacity of $(N, K, M)$ Coded PIR). Given any $(N, K, M)$, a rate $R$ is said to be achievable if there exists a coded PIR scheme with rate greater than or equal to $R$. The capacity of the coded PIR, denoted by $C$, is defined as the supremum over all the achievable rates.

The capacity of the $(N, K, M)$ coded PIR scheme has been determined in [1] as

\[
C = \left( 1 + \frac{K}{N} + \left( \frac{K}{N} \right)^2 + \ldots + \left( \frac{K}{N} \right)^{M-1} \right)^{-1}.
\] (10)

The works in [1] and [17] indicate that coded linear PIR schemes suffice to achieve the capacity.

**Definition 2** (Coded Linear PIR Scheme). For an $(N, K, M)$ coded PIR scheme, let $\ell_t$ be the answer length of the received query $Q_t^{[\theta]}$ at server $t$. It is said to be an $(N, K, M)$ coded linear PIR scheme if the answers $A_t^{[\theta]}(t \in [0 : N))$ are formed by

\[
A_t^{[\theta]} = A_{t, [0:M]}^{[\theta]}y_t, \quad t \in [0 : N),
\] (11)

where $A_{t, [0:M]}^{[\theta]} = (A_{t,0}^{[\theta]}, \ldots, A_{t, M-1}^{[\theta]})$ is an answer matrix of order $\ell_t \times M \lambda$ over $\mathbb{F}_p$ only determined by the query $Q_t^{[\theta]}$, and $A_{t, i}^{[\theta]}(i \in [0 : M))$ is a sub-matrix of order $\ell_t \times \lambda$. In particular, we call

\[
A_{t, [0:M]}^{[\theta]} = \left( A_{t,0}^{[\theta]}, \ldots, A_{t, \theta-1}^{[\theta]}, A_{t, \theta+1}^{[\theta]}, \ldots, A_{t, M-1}^{[\theta]} \right)
\]

the answer-interference matrix.

Hence, the download cost $D$ of a linear PIR scheme can be calculated as

\[
D = \mathbb{E} \left[ \sum_{t=0}^{N-1} \ell_t \right].
\]

The objective of this paper is to design $(N, K, M)$ coded linear PIR schemes which simultaneously achieve the PIR capacity and possess the practical file length in the non-trivial cases of $N > K$ and $M > 1$. Obviously, for a linear PIR scheme, all the queries and answers can be viewed as matrices. Thus, from now on, for the sake of unification we use cursive capital letters to denote random matrices such as $Q, A, \tilde{Q}$ and $\tilde{A}$ as their realizations. Besides, we denote the column vectors by the bold letters, e.g. $\mathbf{A}, \mathbf{q}$.

### III. A New Capacity-Achieving Coded Linear PIR Scheme

In this section, we propose a new capacity-achieving coded linear PIR scheme with the file length $\frac{K(N-K)}{\gcd(N,K)}$ for $N > K$ and $M > 1$, and then give an illustrative example.
A. A New Coded Linear PIR Scheme

Define
\[ n \triangleq \frac{N}{\gcd(N, K)}, \]
\[ k \triangleq \frac{K}{\gcd(N, K)}. \]

Consider the case that each file has length \( L = K \cdot \lambda \), where
\[ \lambda = n - k. \tag{12} \]
That is to say, each column vector of all encoded files in (11) is of length \( \lambda = n - k \). For easy of exposition, we append \( k \) dummy zeros to the vectors of each file stored at each server, i.e., the dummy expanded vectors at server \( t \) are
\[ \mathcal{W}_i(:, t) \triangleq (\mathcal{W}_i(0, t), \ldots, \mathcal{W}_i(n - k - 1, t), 0, \ldots, 0)^\top, \quad i \in [0 : M), \]
where \( \top \) denotes the transpose operator.

Denote \( \Omega \) the set of vectors in \([0 : n) \times k\) with distinct entries in all coordinates, i.e.,
\[ \Omega = \{ q = (q(0), \ldots, q(k - 1))^\top \in [0 : n)^k : q(i) \neq q(j), \; \forall i, j \in [0 : k), \; i \neq j \}. \]
We are now ready to present the new coded linear PIR scheme.

1) **Query Phase:** Assume that file \( \mathcal{W}_0 \) is requested by the user. The user first generates a \( k \times M \) random matrix
\[ Q = (q_0, \ldots, q_{\theta - 1}, q_{\theta}, q_{\theta + 1}, \ldots, q_{M - 1}), \]
where each column vector \( q_i \) is drawn independently and uniformly from \( \Omega \) for all \( 0 \leq i < M \).

Next, the user constructs the queries based on the matrix \( Q \) as
\[ Q_i^{[\theta]}(t) = (q_0, \ldots, q_{\theta - 1}, (q_{\theta} + t)\bmod n, q_{\theta + 1}, \ldots, q_{M - 1}), \; t \in [0 : N), \]
where \( q_{\theta} + t \) denotes the vector that each element in the column vector \( q_{\theta} \) is added to, i.e., \( q_{\theta} + t = (q_{\theta}(0) + t, \ldots, q_{\theta}(k - 1) + t)^\top \) and \( (\cdot)\bmod n \) denotes the element-wise modulo \( n \) operation. Notice that \( k \times M \) matrix \( Q_i^{[\theta]}(0 \leq t < N) \) is almost the same as \( Q \) with the exception of the \( \theta \)-th column, and \( Q_i^{[\theta]}_{0:N-1} \) satisfies (4) since each \( Q_i^{[\theta]} \) only depends on \( Q, t, \theta \), which are all independent of \( \mathcal{W}_0 : M-1 \).

2) **Answer Phase:** The servers respond in \( k \) rounds, each round indexed by an integer \( s \in [0 : k) \). In particular, in the round \( s \), server \( t \) sends
\[ A_i^{[\theta]}(s) = \begin{cases} \text{NULL}, & \text{if } Q_i^{[\theta]}(s, :) \in [n - k : n)^M \\ \sum_{i \in [0 : M)} W_i(Q_i^{[\theta]}(s, i), t), & \text{else} \end{cases} \]
(16)
to the user, where the value NULL indicates that the server keeps silence, and the additions are operated on the finite field \( \mathbb{F}_p \).

3) **Decoding Phase:** We defer the reconstruction procedure in Section IV-A.

It is easy to see from (16) that there exists answer matrices \( A_i^{[\theta]}_{0:M} \) satisfying (11). That is, the proposed scheme is actually a linear PIR scheme.

For the above scheme, we have the following main results of this paper.

**Theorem 1.** Given any \((N, K, M)\) with \(N > K\) and \(M > 1\), there exists a capacity-achieving coded linear PIR scheme with file length \( L = \frac{K(N - K)}{\gcd(N, K)} \). The file length \( L \) is optimal among all the capacity-achieving \((N, K, M)\) coded linear PIR schemes if \( M > \left\lfloor \frac{K}{\gcd(N, K)} \right\rfloor + 1 \). Otherwise, the file length \( L \) is within a multiplicative gap \( \frac{K}{\gcd(N, K)} \) compared to a lower bound on the minimum file length of capacity-achieving \((N, K, M)\) coded linear PIR schemes.

**Proof:** The proofs of correctness, privacy, and performance of the new scheme are given in Section IV-A, B, and C respectively. The optimality and multiplicative gap of the file length are shown in Theorems 3 and 4. \( \blacksquare \)
B. An Illustrative Example

In this subsection, we illustrate an example of \((N, K, M) = (5, 3, 3)\) coded linear PIR scheme. According to [12], \(\lambda = 2\) and \(L = K \cdot \lambda = 6\). The three encoded files \(W_0, W_1, W_2\) are respectively denoted as

\[
W_i = \begin{pmatrix} W_i(0, 0) & W_i(0, 1) & W_i(0, 2) & W_i(0, 3) & W_i(0, 4) \\ W_i(1, 0) & W_i(1, 1) & W_i(1, 2) & W_i(1, 3) & W_i(1, 4) \end{pmatrix}, \quad i = 0, 1, 2,
\]

where each row vector forms a \((5, 3)\) MDS codeword. Then, they are stored across 5 servers. To better understand the corresponding relationship of the stored contents, the query matrices and the answers, the former at each sever are arranged as a matrix as shown in Fig. 1. Notably, the symbols 0 in the boxes with dotted lines are the appended dummy zeros and not stored at all.

Fig. 1. The stored contents at each server.

Let

\[
\Omega = \{q \in \{0 : 5\}^3 : q(0) \neq q(1), q(0) \neq q(2), q(1) \neq q(2)\}.
\]

In the query phase, the user generates a \(3 \times 3\) matrix \(Q\) with each column chosen independently and uniformly from \(\Omega\), for example,

\[
Q = (q_0, q_1, q_2) = \begin{pmatrix} 3 & 4 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}.
\]

Assume that \(W_0\) is the desired file. Then, the user sends the following query matrices to the servers:

\[
\begin{array}{cccc}
W_0(0, 0) & W_1(0, 0) & W_2(0, 0) & W_0(0, 1) \\
W_0(1, 0) & W_1(1, 0) & W_2(1, 0) & W_0(1, 1) \\
W_0(1, 1) & W_1(1, 1) & W_2(1, 1) & W_0(1, 2) \\
W_0(1, 2) & W_1(1, 2) & W_2(1, 2) & W_0(2, 0) \\
W_0(2, 0) & W_1(2, 0) & W_2(2, 0) & W_0(2, 1) \\
W_0(2, 1) & W_1(2, 1) & W_2(2, 1) & W_0(2, 2) \\
W_0(2, 2) & W_1(2, 2) & W_2(2, 2) & W_0(3, 0) \\
W_0(3, 0) & W_1(3, 0) & W_2(3, 0) & W_0(3, 1) \\
W_0(3, 1) & W_1(3, 1) & W_2(3, 1) & W_0(3, 2) \\
W_0(3, 2) & W_1(3, 2) & W_2(3, 2) & W_0(4, 0) \\
W_0(4, 0) & W_1(4, 0) & W_2(4, 0) & W_0(4, 1) \\
W_0(4, 1) & W_1(4, 1) & W_2(4, 1) & W_0(4, 2) \\
W_0(4, 2) & W_1(4, 2) & W_2(4, 2) \\
\end{array}
\]

Upon receiving the queries, the servers answer in \(k = 3\) rounds, which are referred to as round R0, R1 and R2 as shown in Fig. 2.

In decoding phase, the user decodes two symbols of \(W_0\) from the answers of each round:

1) In R0, get \(W_0(0, 2)\) and \(W_0(1, 3)\) directly;
2) In R1, decode the two interference symbols \(W_1(1, 0)+W_2(0, 0), W_1(1, 1)+W_2(0, 1)\) from \(W_1(1, 2)+W_2(0, 2), W_1(1, 3)+W_2(0, 3), W_1(1, 4)+W_2(0, 4)\) by MDS property and then get \(W_0(0, 0)\) and \(W_0(1, 1)\);
3) In R2, similarly to R1, get \(W_0(1, 0)\) and \(W_0(0, 4)\).

Then, the user is able to recover \(W_0(0,:)\) and \(W_0(1,:\) and thus the file \(W_0\) by means of MDS property again.

Privacy is guaranteed, since each column of the query matrix received by each server is uniformly and independently distributed on \(\Omega\), regardless of the file being requested.
In this realization, the download cost is 12. Among all the realizations, since the probability of each signal being NULL is \((\frac{3}{5})^3\), and there are \(3 \times 5\) signals sent in total, the average download cost is

\[
D = 3 \times 5 \times \left(1 - \left(\frac{3}{5}\right)^3\right) = \frac{294}{25}
\]

The file length is \(L = 3 \times 2 = 6\), thus the retrieve rate is

\[
R = \frac{L}{D} = \frac{25}{49}
\]

which achieves the capacity of \((5, 3, 3)\) coded PIR scheme in [10].

IV. THE ACHIEVABILITY OF NEW CODED SCHEME

In this section, we analyse the correctness, privacy, and performance of the scheme. Before that, we need three obvious facts from the matrix \(Q\) and \(Q_t^{[\theta]}\) in [14] and [15]:

F1: For each \(s \in [0 : k]\), \((Q(s, \theta) + t)_n\) takes each value in the set \([0 : n]\) exactly \(d = \gcd(N, K)\) times, with \(t\) ranging over the set \([0 : N]\);

F2: For \(s_1 \neq s_2 \in [0 : k]\), \(Q(s_1, \theta) \neq Q(s_2, \theta)\);

F3: For any given \(s \in [0 : k]\), \(\{Q(s, j) : j \in [0 : M]\}\) are independent and uniform in \([0 : n]\), and so does \(Q_t^{[\theta]}(s, j) : j \in [0 : M]\) for all \(\theta \in [0 : M]\).

A. Proof of Correctness

For any realizations of random variables \(\theta\) and \(Q\) in the new coded linear PIR scheme, the file \(\mathcal{W}_t\) can be reconstructed from \(A_t^{[\theta]}_{M, N-1}\) and \(Q_t^{[\theta]}_{M, N-1}\). We now describe the decoding processes in detail.

First of all, by (13) and (16), no matter the answer \(A_t^{[\theta]}(s)\) of the \(s\)-th round of the received query matrix \(Q_t^{[\theta]}(s, i)\) at server \(t\) is NULL or not, the user can rewrite it as

\[
A_t^{[\theta]}(s) = \mathcal{W}_t((Q(s, \theta) + t)_n, t) + n_t^{[\theta]}(s), \quad t \in [0 : N), s \in [0 : k),
\]

where

\[
n_t^{[\theta]}(s) = \sum_{i \in [0 : M] \setminus \{\theta\}} \mathcal{W}_i(Q(s, i), t).
\]

This is to say, if the answer \(A_t^{[\theta]}(s)\) is NULL, the user can interpret it as 0, since in this case, \(\mathcal{W}_t((Q(s, \theta) + t)_n, t) = 0\) and \(\mathcal{W}_i(Q(s, i), t) = 0\) for every \(i \in [0 : M] \setminus \{\theta\}\).

From (1) and (13), \((\mathcal{W}_t(Q(s, i), 0), \ldots, \mathcal{W}_t(Q(s, i), N - 1))\) constitutes a codeword of the linear \((N, K)\) MDS code, so does \((n_0^{[\theta]}(s), \ldots, n_{N-1}^{[\theta]}(s))\). Thus, the reconstruction phase can be depicted as follows.

Reconstruction Phase: Define

\[
\Delta_s := \{t \in [0 : N]: (Q(s, \theta) + t)_n \in [n - k : n]\}, \quad s \in [0 : k),
\]

\[
\Lambda_j := \{t \in [0 : N]: (Q(s, \theta) + t)_n = j, s \in [0 : k), j \in [0 : n - k]\}.
\]
Step 1. Given a fixed \( s \in [0 : k] \), the user recovers the residual \( N - K \) values \( n_t^{[\theta]}(s), t \in [0 : N) \setminus \Delta_s \), from the \( K \) answers \( n_t^{[\theta]}(s) = A_t^{[\theta]}(s), t \in \Delta_s \), by the MDS property of the codeword \( \{n_0^{[\theta]}(s), \ldots, n_{n-1}^{[\theta]}(s)\} \).

This is because (i) \( \mathcal{W}_0((Q(s, \theta) + t)_n, t) = 0 \) for \( t \in \Delta_s \); (ii) \( |\Delta_s| = d \cdot k = K \) by fact F1, and then the user has all the desired \( K \) values \( n_t^{[\theta]}(s) \) from the answers \( A_t^{[\theta]}(s) \), \( t \in \Delta_s \).

Step 2. The user cancels the term \( n_t^{[\theta]}(s) \) in the answers (17) from all severs \( t \in [0 : N) \setminus \Delta_s \) by those \( N - K \) values calculated in Step 1 to obtain
\[
\{\mathcal{W}_0((Q(s, \theta) + t)_n, t) : t \in [0 : N) \setminus \Delta_s \}, \; s \in [0 : k].
\] (18)

Step 3. Collecting all the values in (18) when \( s \) enumerates \([0 : k]\), the user respectively reconstructs all the vectors \( \mathcal{W}_0(j,:)(j \in [0 : n - k]) \) from \( \{\mathcal{W}_0(j, t) : t \in \Lambda_j\} \) by means of its MDS property.

This is because given \( j \in [0 : n - k] \), (i) \( \Lambda_{j,s} = \{t \in [0 : N) : (Q(s, \theta) + t)_n = j\} \subseteq [0 : N) \setminus \Delta_s \) is of cardinality \( d \) for any fixed \( s \in [0 : k] \), again by fact F1; and then (ii) \( |\Lambda_j| = d \cdot k = K \) by fact F2, i.e., there are \( K \) distinct coded symbols \( \{\mathcal{W}_0(j, t) : t \in \Lambda_j\} \) available from (18).

After Reconstruction Phase, the user is able to reconstruct the whole file \( \mathcal{W}_0 \).

B. Proof of Privacy

Denote the set of \( k \times M \) matrices with column vectors chosen from \( \Omega \) by \( \mathcal{U} \), i.e.,
\[ \mathcal{U} = \{ (q_0, q_1, \ldots, q_{M-1}) : q_i \in \Omega, \; \forall \; i \in [0 : M] \} . \]

Recall from (14) that \( q_i \) \((i \in [0 : M])\) are independent and uniform in \( \Omega \). So, according to fact F3, \((q_\theta + t)_n\) is uniform in \( \Omega \) for any \( \theta \in [0 : M) \) and \( t \in [0 : N] \). Consequently, \( q_0, q_0, \ldots, q_0, (q_\theta + t)_n, q_{\theta + 1}, \ldots, q_{M-1} \) are independent and uniform in \( \Omega \) for any \( t \in [0 : N] \). That is, the matrix \( Q_t^{[\theta]} \) in (15) has uniform distribution over \( \mathcal{U} \). Then, given \( \bar{Q} \in \mathcal{U} \),
\[ \Pr(\bar{Q}_t^{[\theta]} = \bar{Q}) = \frac{1}{|\mathcal{U}|}, \; \theta \in [0 : M), \; t \in [0 : N], \] (19)
which is independent of \( \theta \). Next, for any \( t \in [0 : N) \) and \( \theta \in [0 : M) \), we have
\[
0 \leq I(\bar{Q}_t^{[\theta]} , A_t^{[\theta]} , y_i ; \theta)
\leq I(\bar{Q}_t^{[\theta]} , A_t^{[\theta]} , \mathcal{W}_{0:M-1} ; \theta)
= I(\bar{Q}_t^{[\theta]} ; \theta) + I(\mathcal{W}_{0:M-1} ; \theta | \bar{Q}_t^{[\theta]} ) + I(A_t^{[\theta]} ; \theta | \bar{Q}_t^{[\theta]} , \mathcal{W}_{0:M-1} )
\overset{(a)}{=} H(\mathcal{W}_{0:M-1} | \bar{Q}_t^{[\theta]} ) - H(\mathcal{W}_{0:M-1} | \theta , \bar{Q}_t^{[\theta]} )
\overset{(b)}{=} H(\mathcal{W}_{0:M-1} ) - H(\mathcal{W}_{0:M-1} )
= 0,
\]
where (a) is because the query is independent of \( \theta \) by (19) and the answer is a determined function of the received query and the files by (16) such that \( I(\bar{Q}_t^{[\theta]} ; \theta) = 0 \) and \( I(A_t^{[\theta]} ; \theta | \bar{Q}_t^{[\theta]} , \mathcal{W}_{0:M-1} ) = 0 \); (b) is due to the fact that the files are independent of the desired file index and the query.

Thus, privacy of the new PIR scheme follows from (19).

C. Proof of Performance

Recall that \( \ell_t \) is the answer length of the query \( \bar{Q}_t^{[\theta]} \) at server \( t \), clearly which is
\[ \ell_t = \sum_{s=0}^{k-1} \ell(A_t^{[\theta]}(s)), \]
where \( \ell(A_t^{[\theta]}(s)) \) is the length of \( A_t^{[\theta]}(s) \), satisfying
\[ \ell(A_t^{[\theta]}(s)) = \begin{cases} 
0, & \text{if } Q_t^{[\theta]}(s,:) \in [n-k:n]^M, \; s \in [0 : k) \\
1, & \text{otherwise}
\end{cases} \] (20)
by (16).
It follows from fact F3 and (20) that
\[
E \left[ \ell(A_i^{[\theta]}(s)) \right] = 1 \cdot \Pr(\ell(A_i^{[\theta]}(s)) = 1) + 0 \cdot \Pr(\ell(A_i^{[\theta]}(s)) = 0) = 1 - \left( \frac{k}{n} \right)^M.
\]

Then, we have
\[
D = E \left[ \sum_{t=0}^{N-1} \ell_t \right] \\
= E \left[ \sum_{t=0}^{N-1} \sum_{s=0}^{k-1} \ell(A_t^{[\theta]}(s)) \right] \\
= \sum_{t=0}^{N-1} \sum_{s=0}^{k-1} E \left[ \ell(A_t^{[\theta]}(s)) \right] \\
= Nk \left( 1 - \left( \frac{k}{n} \right)^M \right).
\] (21)

Finally, substituting (12), (21) and the fact \( \frac{N}{n} = \frac{k}{n} \) into (2), we obtain
\[
R = \frac{L}{D} = \frac{K(n-k)}{Nk \left( 1 - \left( \frac{k}{n} \right)^M \right)} = \left( 1 + \frac{K}{N} + \left( \frac{K}{N} \right)^2 + \ldots + \left( \frac{K}{N} \right)^{M-1} \right)^{-1},
\]
which achieves the capacity of the coded PIR scheme in (10).

V. THE OPTIMALITY ON THE FILE LENGTH

A. Some Necessary Conditions for Capacity-achieving Coded linear PIR Schemes

The capacity of coded PIR has been determined in [1]. In this subsection, we provide a detailed analysis of the converse proof to emphasize on three necessary conditions P1-P3 for capacity-achieving coded linear PIR schemes. The similar approach was used in [13] for the replicated PIR scenario. The proofs of Lemma 1-3 are left in the Appendix.

**Lemma 1.** For any \( \Gamma \subseteq [0 : N], \| \Gamma \| = K, \Lambda \subseteq [0 : M], \theta \in [0 : M] \), the answers from servers in \( \Gamma \) are mutually and statistically independent conditioned on the files \( W_{\Lambda} \), i.e.,
\[
H(A_t^{[\theta]}|W_{\Lambda}, \bar{Q}_{t-1}^{[\theta]}) = \sum_{t \in \Gamma} H(A_t^{[\theta]}|W_{\Lambda}, \bar{Q}_t^{[\theta]}), \quad \theta \in [0 : M].
\] (22)

Equivalently, an \( (N, K, M) \) coded linear PIR scheme must have

**P1:** The answers from servers in \( \Gamma \) are statistically independent conditioned on the files \( W_{\Lambda} \) for any query realization \( \bar{Q}_{0:N-1}^{[\theta]} \) with positive probability, i.e.,
\[
H(A_t^{[\theta]}|W_{\Lambda}, \bar{Q}_t^{[\theta]} = \bar{Q}_t^{[\theta]} = \bar{Q}_t^{[\theta]}), \quad \theta \in [0 : M].
\] (23)

**Lemma 2.** For any \( \Lambda \subseteq [0 : M], \theta \in \Lambda, \theta' \in [0 : M]\setminus \Lambda, \Gamma \subseteq [0 : N], \| \Gamma \| = K, \)
\[
H(A_{0:N-1}^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]}) \geq \frac{KL}{N} + \frac{K}{N} H(A_{0:N-1}^{[\theta']}|W_{\Lambda}, W_{\theta'}, Q_{0:N-1}^{[\theta']}).
\] (24)

Moreover, to establish the equality in (24), the \( (N, K, M) \) coded linear PIR scheme must have

**P2:** The answers in any \( K \) servers determine all the answers conditioned on the files \( W_{\Lambda} \) for any query realization \( \bar{Q}_{0:N-1}^{[\theta]} \) with positive probability, i.e.,
\[
H(A_{0:N-1}^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \bar{Q}_{0:N-1}^{[\theta]} = \bar{Q}_{0:N-1}^{[\theta]}), \quad \theta \in \Lambda.
\] (25)
Lemma 3. For any \((N, K, M)\) coded linear PIR scheme,

\[
H(A_{0:N-1}^{[\theta]}|W_{\theta}, Q_{0:N-1}^{[\theta]}) \leq \frac{L}{R} - L. \tag{26}
\]

Moreover, to establish the equality in (26), the \((N, K, M)\) coded linear PIR scheme must have

P3: The \(N\) answers are mutually independent for any query realization \(\tilde{Q}_{0:N-1}^{[\theta]}\) with positive probability, i.e.,

\[
D_{rel} = H(A_{0:N-1}^{[\theta]}|Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}) = \sum_{t=0}^{N-1} H(A_t^{[\theta]}|Q_t^{[\theta]} = \tilde{Q}_t^{[\theta]}), \quad \theta \in [0 : M), \tag{27}
\]

where \(D_{rel}\) denotes the download cost for the query realization \(\tilde{Q}_{0:N-1}^{[\theta]}\).

Theorem 2. Any capacity-achieving coded linear PIR scheme must satisfy the three necessary conditions P1-P3.

Proof: First, P1 is satisfied for any coded linear PIR scheme by Lemma 1. Next, set \([0 : M)\setminus\theta = \{\theta_1, \ldots, \theta_{M-1}\}\). Then, by (26),

\[
\frac{L}{R} - L \geq H(A_{0:N-1}^{[\theta']}|W_{\theta'}, Q_{0:N-1}^{[\theta']}) \tag{28}
\]

where (a) follows from recursively applying Lemma 2 to \(\theta' = \theta_i, i = 1, \ldots, M-1\); (b) is because of (5), i.e., \(H(A_{0:N-1}^{[\theta]}|W_{0:M-1}, Q_{0:N-1}^{[\theta]} = 0)\).

According to the capacity \(C\) in (10), a capacity-achieving \((N, K, M)\) coded linear PIR scheme achieves the quality in (28). Consequently, any capacity-achieving \((N, K, M)\) coded linear PIR scheme must attain the equalities (24) and (26), i.e., P2 and P3 are necessary conditions for such PIR scheme.

B. Rank Properties of the Answer-Interference Matrix of Capacity-Achieving Coded Linear PIR Schemes

Lemma 4. ([18 Lemma 8]) In an \((N, K, M)\) coded linear PIR scheme, for any \(\Lambda \subseteq [0 : M), \theta \in [0 : M]\), and any realization \(\tilde{Q}_{0:N-1}^{[\theta]}\) of the random queries \(Q_{0:N-1}^{[\theta]}\), we have

\[
H(A_t^{[\theta]}|W_{\Lambda}, Q_t^{[\theta]} = \tilde{Q}_t^{[\theta]}) = \text{rank}(\tilde{A}_{t,[0:M]\setminus\theta}^{[\theta]}), \tag{29}
\]

where \(\tilde{A}_{t,[0:M]}^{[\theta]} = (\tilde{A}_{t[0:M]}^{[\theta]}, \tilde{A}_{t[0:M]}^{[\theta]}, \ldots, \tilde{A}_{t[M-1]}^{[\theta]})\) is the answer matrix of query realization \(Q_{0:N-1}^{[\theta]}\) at server \(t\).

In the following, Lemma 5 determines the relation between the specific download cost and the file length for any concrete realization \(\tilde{Q}_{0:N-1}^{[\theta]}\) of the random queries \(Q_{0:N-1}^{[\theta]}\). It is worth pointing out that the same relation between the average download cost and the file length for random queries \(Q_{0:N-1}^{[\theta]}\) has been established in [17]. Essentially, Lemma 5 implies the result in [17] by the fact that the relation holds for all the specific download cost and so does for average ones, but not vice versa.

Lemma 5. In a capacity-achieving \((N, K, M)\) coded linear PIR scheme, for any realization \(\tilde{Q}_{0:N-1}^{[\theta]}\) of the random queries \(Q_{0:N-1}^{[\theta]}\) with positive probability, \(\theta \in [0 : M]\), all the answer-interference matrices \(\tilde{A}_{t,[0:M]\setminus\theta}^{[\theta]}(t \in [0 : N])\) have the same rank \(r\), i.e.,

\[
r = \text{rank}(\tilde{A}_{0:[0:M]\setminus\theta}^{[\theta]}) = \ldots = \text{rank}(\tilde{A}_{N-1:[0:M]\setminus\theta}^{[\theta]}). \tag{29}
\]

and the download cost satisfies

\[
D_{rel} - L = K \cdot r. \tag{30}
\]

Proof: For any realization \(\tilde{Q}_{0:N-1}^{[\theta]}\) of the random queries \(Q_{0:N-1}^{[\theta]}\), we have

\[
L = H(W_{\theta}) \tag{31}
\]

\[
ev H(W_{\theta}|Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]} - H(W_{\theta}|A_{0:N-1}^{[\theta]}, Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}) = I(W_{\theta}; A_{0:N-1}^{[\theta]}|Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}). \tag{32}
\]
Further replacing $\Gamma$ in (32) by $\Gamma' = (\Gamma \setminus \{i\}) \cup \{j\}$ for any $j \in [0:N]\setminus \Gamma$, we have

$$H(A_{0:N-1}^t | W_\theta, Q_{0:N-1}^t = \overline{Q}_{0:N-1}^t) = \sum_{t \in \Gamma'} H(A_{0:N-1}^t | W_\theta, Q_{0:N-1}^t = \overline{Q}_{0:N-1}^t).$$

(33)

Then, combining (32) and (33), we obtain

$$H(A_{0:N-1}^t | W_\theta, Q_{0:N-1}^t = \overline{Q}_{0:N-1}^t) = \sum_{t \in \Gamma'} H(A_{0:N-1}^t | W_\theta, Q_{0:N-1}^t = \overline{Q}_{0:N-1}^t).$$

(34)

for any $i, j \in [0:N], i \neq j$.

Therefore, applying Lemma 4 to (34), we reach (29), which gives $H(A_{0:N-1}^t | W_\theta, Q_{0:N-1}^t = \overline{Q}_{0:N-1}^t) = K \cdot r$ by (32) and then (30) by (31).

C. A Lower Bound on the Minimum File Length of Capacity-Achieving Coded Linear PIR Schemes

In this subsection, we establish an information theoretic lower bound on the minimum file length among all capacity-achieving $(N, K, M)$ coded linear PIR schemes.

Lemma 6. For a capacity-achieving $(N, K, M)$ coded linear PIR scheme and an integer $\theta \in [0:M)$, let $\overline{Q}_{0:N-1}^t$ be a query realization with positive probability, and denote $r$ the rank of all its answer-interference matrices $\overline{A}_{t,[0:M]\setminus\{\theta\}}^t$, $t \in [0:N)$. Then, its file length satisfies

$$L \geq (N - K) \cdot r.$$

Proof: For the realization $\overline{Q}_{0:N-1}^t$, we have

$$K \cdot r + L \overset{(a)}{=} D_{rel} \overset{(b)}{=} \sum_{t=0}^{N-1} H(A_{0:N-1}^t | Q_{0:N-1}^t = \overline{Q}_{0:N-1}^t) \overset{(c)}{=} \sum_{t=0}^{N-1} \text{rank}(\overline{A}_{t,[0:M]}^t) \overset{(d)}{=} \sum_{t=0}^{N-1} \text{rank}(\overline{A}_{t,[0:M]\setminus\{\theta\}}^t) \overset{(e)}{=} N \cdot r,$$

(35)

where (a) and (b) respectively follow from (30) and (27); (c) is due to Lemma 4; (d) is because the rank of matrix is not less than the rank of its sub-matrix, i.e., $\text{rank}(\overline{A}_{t,[0:M]}^t) \geq \text{rank}(\overline{A}_{t,[0:M]\setminus\{\theta\}}^t) = r$. Therefore, we get $L \geq (N - K) \cdot r$. ■

Now, we are ready to derive our bound based on the following observation from the privacy constraint in (8) and the definition of coded linear PIR scheme in (11).

Observation: For any query realization $\overline{Q}_{0:N-1}^t$ with positive probability, the query $\overline{Q}_{0:N-1}^t$, which is sent by the user to the server $t$ for retrieving file $W_\theta$, can also be sent to the same server but for retrieving a distinct file $W_\theta$ in another query realization $\overline{Q}_{0:N-1}^t$ with positive probability, i.e., $\overline{Q}_{0:N-1}^t = \overline{Q}_{0:N-1}^t$. As a result, in the two query realizations $\overline{Q}_{0:N-1}^t$ and $\overline{Q}_{0:N-1}^t$, server $t$ will respond with the same answer matrix $\overline{A}_{t,[0:M]}^t = \overline{A}_{t,[0:M]}^t$.  

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Theorem 3. Given any $N, K$ and $M$ with $N > K$ and $M > 1$, the file length of any capacity-achieving $(N, K, M)$ coded linear PIR scheme satisfies

$$L \geq N - K.$$  \hfill (36)

Proof: Assume that (36) is not true for a capacity-achieving $(N, K, M)$ coded linear PIR scheme. Then, it follows from Lemmas 5 and 6 that all the answer-interference matrices will have the rank $r = 0$ for each of its query realization with positive probability.

Let $\mathcal{W}_\theta$ ($\theta \in [0 : M]$) be a desired file and $\mathcal{Q}_{0:N-1}^{[\theta]}$ be a query realization with positive probability. In this scenario, to ensure that the user can decode the desired file $\mathcal{W}_\theta$, the user must directly download the file from the servers, i.e., there exists at least one server $t \in [0 : N)$ such that its answer matrix

$$\tilde{A}_{t,0:M}^{[\theta]} = (\tilde{A}_{t,0}^{[\theta]}, \ldots, \tilde{A}_{t,M-1}^{[\theta]})$$

satisfies that $\tilde{A}_{t,0:M}^{[\theta]} \setminus \{\theta\} = 0$ and $\tilde{A}_{t,0:M}^{[\theta]} \neq 0$.

By the Observation, for any server $t$ and any $\theta' \neq \theta$, there exists a query realization $\mathcal{Q}_{0:M}^{[\theta]}$ with positive probability such that the server $t$ will respond with the same answer matrix, i.e., $\tilde{A}_{t,0:M}^{[\theta]} = \tilde{A}_{t,0:M}^{[\theta']}$; then, the answer-interference matrix

$$\tilde{A}_{t,0:M}^{[\theta]} \setminus \{\theta\} = (\tilde{A}_{t,0}^{[\theta]}, \ldots, \tilde{A}_{t,\theta}^{[\theta]}, \tilde{A}_{t,\theta}^{[\theta]}, \ldots, \tilde{A}_{t,M-1}^{[\theta]})$$

with

$$r = \text{rank}(\tilde{A}_{t,0:M}^{[\theta]} \setminus \{\theta\}) \geq 1,$$

which contradicts the assumption that the rank $r = 0$ for any query realization with positive probability. This completes the proof. \hfill \Box

Remark 1. When $K = 1$, the $(N, 1, M)$ coded PIR scheme is just a repetition scheme. In this case, our bound becomes $L \geq N - 1$, which is consistent with the bound in [13].

Specifically, when $M > \left\lfloor \frac{K}{\gcd(N,K)} \right\rfloor + 1$, we can further improve the lower bound to a tight one.

Theorem 4. Given any $N, K$ and $M$ with $N > K$ and $M > \left\lfloor \frac{K}{\gcd(N,K)} \right\rfloor + 1$, the file length of any capacity-achieving $(N, K, M)$ coded linear PIR scheme satisfies

$$L \geq \frac{K(N-K)}{\gcd(N,K)}.$$  \hfill (37)

Proof: Recall that $n = \frac{N}{\gcd(N,K)}$, $k = \frac{K}{\gcd(N,K)}$. Suppose that $L = K \cdot \lambda < \frac{K(N-K)}{\gcd(N,K)}$, i.e., $\lambda < n - k$. Let $\mathcal{Q}_{0:N-1}^{[\theta]}$ ($\theta \in [0 : M]$) be a realization of the random queries $\mathcal{Q}_{0:N-1}^{[\theta]}$ with positive probability. Then, all the answer-interference matrices have the same rank $r$ by Lemma 5. Using $L = K \cdot \lambda$ in (35), we have

$$r \leq \frac{K \cdot \lambda}{N - K} = \frac{k \cdot \lambda}{n - k},$$  \hfill (38)

which results in

$$r \leq \left\lfloor \frac{k \cdot (n - k - 1)}{n - k} \right\rfloor = \left\lfloor \frac{k}{n - k} \right\rfloor$$

by the fact that the rank $r$ must be an integer. In addition, note that, for any $\lambda < n - k$, $\frac{k \cdot \lambda}{n - k}$ cannot be an integer since $\gcd(k, n - k) = 1$. This is to say, the inequality in (e) of (37) always holds, and so does for the one in (d) of (35), i.e.,

$$\sum_{t=0}^{N-1} \text{rank}(\tilde{A}_{t,0:M}^{[\theta]}) > N \cdot r.$$  \hfill (39)
Again by \( \text{rank}(A[\theta]_{t,[0:M]}) \geq \text{rank}(A[\theta]_{t,[0:M]\setminus\{\theta}\}}) = r \), (39) indicates that there exists at least one server \( t' \in [0 : N] \) whose answer matrix satisfies
\[
\text{rank}(A[\theta]_{t',[0:M]}) \geq r + 1.
\]

Next, we write the matrix \( A[\theta]_{t,[0:M]}' \) as \( M - 1 \) sub-matrix, i.e.,
\[
A[\theta]_{t',[0:M]' \} = (A[\theta]_{t,'0}, A[\theta]_{t,'\theta-1}, A[\theta]_{t,'\theta+1}, \ldots, A[\theta]_{t,'M-1})
\]
which has the rank \( r < M - 1 \) by (38) if \( M > \left\lfloor k - \frac{k}{n-k} \right\rfloor + 1 \). Therefore, there must be a \( \theta' \in [0 : M]\{\theta}\) such that every column in \( A[\theta]_{t,'\theta} \) is a linear combination of the ones in \( A[\theta]_{t,[0:M]\{\theta',\theta}\}} \), which implies
\[
\text{rank}(A[\theta]_{t,'[0:M]\{\theta'})) = \text{rank}(A[\theta]_{t,[0:M]\{\theta'}}) \geq r + 1. \tag{40}
\]

By the Observation, there exists a new query realization \( Q[\theta']_{0:N-1} \) with positive probability such that the \( t' \)-th sever will respond with the same answer matrices, i.e., \( A[\theta']_{t',[0:M]} = A[\theta]_{t,[0:M]} \). Then, the answer-interference matrix \( A[\theta']_{t,[0:M]\{\theta'} \} \) satisfies
\[
\text{rank}(A[\theta']_{t',[0:M]\{\theta'}}) = \text{rank}(A[\theta]_{t,[0:M]\{\theta'}}) \geq r + 1
\]
by (40). Again by Lemma 5 all the answer-interference matrices \( A[\theta']_{t',[0:M]\{\theta'} (t' \in [0 : N]) \) have the same rank \( r + 1 \). That is, the query realization \( Q[\theta']_{0:N-1} \) still satisfies all the prerequisites. Hence, we can repeat the above procedures till
\[
\text{rank}(A[\theta']_{t'[0:M]\{\theta'}}) > \left\lfloor k - \frac{k}{n-k} \right\rfloor
\]
for some sever index \( t' \in [0 : N] \) and file index \( \theta' \in [0 : M] \), which contradicts (38). Therefore, we arrive at the conclusion.

Obviously, Theorem 4 degrades to the result in Theorem 3 in terms of the lower bound on the minimum file length if \( K|N \).

**Corollary 1.** Given any \( N, K \) and \( M \) with \( M > 1 \), the file length of the new coded linear PIR scheme achieves the lower bound \( N - K \) on the file length of capacity-achieving coded linear PIR schemes for \( K|N \).

**VI. CONCLUSION**

In this paper, for the setting with MDS coded servers, we considered the problem of minimizing file length among all capacity-achieving coded linear PIR schemes. Firstly, we proposed a new capacity-achieving \((N, K, M)\) coded scheme with the file length \( \frac{KN(N-K)}{\text{gcd}(N,K)} \), which has dramatically reduced the file length required for capacity-achieving PIR schemes in the literature. Secondly, we derived lower bound on the minimum file length for capacity-achieving coded linear PIR schemes. With respect to the bound, the file length of our scheme is shown to be optimal if \( M > \left\lfloor \frac{K}{\text{gcd}(N,K)} - \frac{K}{N-K} \right\rfloor + 1 \), and be within a multiplicative gap \( \frac{K}{\text{gcd}(N,K)} - \frac{K}{N-K} \) of the lower bound in the other case.

In this sense, the problem of minimizing file length remains open for \( M \leq \left\lfloor \frac{K}{\text{gcd}(N,K)} - \frac{K}{N-K} \right\rfloor + 1 \), which deserves further studies in the future.

**APPENDIX**

Before presenting the proofs of Lemma 13, we prove two lemmas. Though they can be proved similarly to [18, Lemma 2], we briefly summarize their proofs for the completeness.

**Lemma 7.** For any \( \Lambda \subseteq [0 : M] \),
\[
H(A[\theta]_{t,[0:M]} | \Lambda, Q[\theta]) = H(A[\theta']_{t,[0:M]} | \Lambda, Q[\theta']) \quad \forall \theta, \theta' \in [0 : M], t \in [0 : N]. \tag{41}
\]

*Proof:* From the privacy constraint in (7), for any \( \theta \in [0 : M] \), we have
\[
0 = I(Q[\theta], A[\theta]; Y_t; \theta) \geq I(Q[\theta]; \theta) \overset{(a)}{=} I(Q[\theta]; \theta) + I(W_{0:M-1}; \theta | Q[\theta]) + I(A[\theta]; \theta | Q[\theta], W_{0:M-1}) = I(Q[\theta], A[\theta]; W_{0:M-1}; \theta)
\]
where \((a)\) is because the files are independent of the desired file index and the query, i.e., \(I(W_{0:M-1}; θ|Q_t^{[θ]}) = 0\), and the answer is a determined function of the received query and the files by \((5)\), i.e., \(I(A_t^{[θ]}; θ|Q_t^{[θ]}, W_{0:M-1}) = 0\). Hence,

\[
0 = I(Q_t^{[θ]}, A_t^{[θ]}, W_{0:M-1}; θ) \\
\geq I(Q_t^{[θ]}, A_t^{[θ]}, W_A; θ) \\
\geq 0,
\]

which tell us \((Q_t^{[θ]}, A_t^{[θ]}, W_A)\) and \(θ\) are independent. Therefore, \((41)\) holds.

Lemma 8. In the \((N, K, M)\) coded linear PIR scheme, for any \(θ ∈ [0 : M), Γ ⊆ [0 : N), Λ ⊆ [0 : M)\) and given realization \(\bar{Q}_{0:N-1}^{[θ]}\) of random queries \(Q_{0:N-1}^{[θ]}\),

\[
H(A_{Γ}^{[θ]}|W_A, Q_{Γ}^{[θ]}) = H(A_{Γ}^{[θ]}|W_A, Q_{0:N-1}^{[θ]}), \tag{42}
\]

\[
H(A_{Γ}^{[θ]}|W_A, Q_{Γ}^{[θ]}) = H(A_{Γ}^{[θ]}|W_A, Q_{0:N-1}^{[θ]} = \bar{Q}_{0:N-1}^{[θ]}). \tag{43}
\]

Proof: As for \((42)\),

\[
\begin{align*}
H(A_{Γ}^{[θ]}|W_A, Q_{Γ}^{[θ]}) &- H(A_{Γ}^{[θ]}|W_A, Q_{0:N-1}^{[θ]}) \\
&= I(A_{Γ}^{[θ]}, Q_{0:N}^{[θ]}, Λ|W_A, Q_{Γ}^{[θ]}) \\
&\leq I(A_{Γ}^{[θ]}, W_{0:M}\setminus Λ; Q_{0:N}\setminus Λ|W_A, Q_{Γ}^{[θ]}) \\
&= I(W_{0:M}\setminus Λ; Q_{0:N}\setminus Λ|W_A, Q_{Γ}^{[θ]}) + I(A_{Γ}^{[θ]}, Q_{0:N}\setminus Λ|W_{0:M-1}, Q_{Γ}^{[θ]}) \\
&= I(W_{0:M}\setminus Λ; Q_{0:N}\setminus Λ|W_A, Q_{Γ}^{[θ]}) \\
&= 0,
\end{align*}
\]

where \((a)\) follows because \(A_{Γ}^{[θ]}\) is only determined by \(W_{0:M-1}\) and \(Q_{Γ}^{[θ]}\) in \((5)\), i.e., \(I(A_{Γ}^{[θ]}, Q_{0:N}\setminus Λ|W_{0:M-1}, Q_{Γ}^{[θ]}) = 0\); \((b)\) is due to the fact that the queries are independent of the files by \((4)\) such that

\[
0 = I(W_{0:M-1}; Q_{0:N-1}^{[θ]}) \\
= I(W_{0:M-1}; Q_{Γ}^{[θ]}) + I(W_{0:M-1}; Q_{0:N}\setminus Λ|Q_{Γ}^{[θ]}) \\
= I(W_{0:M-1}; Q_{Γ}^{[θ]}) + I(W_{A}; Q_{0:N}\setminus Λ|Q_{Γ}^{[θ]}) + I(W_{0:M}\setminus Λ; Q_{0:N}\setminus Λ|W_A; Q_{Γ}^{[θ]}) \\
\geq I(W_{0:M}\setminus Λ; Q_{0:N}\setminus Λ|W_A, Q_{Γ}^{[θ]}) \\
\geq 0.
\]

For \((43)\), we have

\[
H(A_{Γ}^{[θ]}|W_A, Q_{0:N-1}^{[θ]} = \bar{Q}_{0:N-1}^{[θ]}) = H(A_{Γ}^{[θ]}|W_A, Q_{0:N-1}^{[θ]} = \bar{Q}_{0:N-1}^{[θ]}) \\
= H(\bar{A}_{Γ}^{[θ]}|W_A, Q_{0:N-1}^{[θ]} = \bar{Q}_{0:N-1}^{[θ]}) \\
= H(\bar{A}_{Γ}^{[θ]}|W_A, Q_{Γ}^{[θ]} = \bar{Q}_{Γ}^{[θ]}),
\]

where we use \((11)\) and the fact that the answer matrices \(A_{Γ}^{[θ]}(t ∈ Γ)\) are completely determined to be \(\bar{A}_{Γ}^{[θ]}(t ∈ Γ)\) by the corresponding query realization \(\bar{Q}_{Γ}^{[θ]}\).

Proof of Lemma 7]

The proof of \((22)\) can be found in \((1)\) Lemma 1.

In fact, for any \(Γ ⊆ [0 : N), |Γ| = K, Λ ⊆ [0 : M)\), \((22)\) can be equivalently written as

\[
0 = \sum_{t ∈ Γ} H(A_{Γ}^{[θ]}|W_A, Q_{Γ}^{[θ]}) - H(A_{Γ}^{[θ]}|W_A, Q_{Γ}^{[θ]})
\]
\[ = \sum_{\widebar{Q}_{\Gamma}^{[\theta]}} \Pr(Q_{\Gamma}^{[\theta]} = \widebar{Q}_{\Gamma}^{[\theta]}) \left[ \sum_{t \in \Gamma} H(A_t^{[\theta]}|W_{\Lambda}, Q_t^{[\theta]} = \widebar{Q}_{t}^{[\theta]}) - H(A_t^{[\theta]}|W_{\Lambda}, Q_t^{[\theta]} = \widebar{Q}_{t}^{[\theta]}) \right]. \tag{44} \]

While for every query realization \( \widebar{Q}_{0:N-1}^{[\theta]} \), we always have

\[
\sum_{t \in \Gamma} H(A_t^{[\theta]}|W_{\Lambda}, Q_t^{[\theta]} = \widebar{Q}_{t}^{[\theta]}) \\
= (a) \sum_{t \in \Gamma} H(A_t^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \widebar{Q}_{0:N-1}^{[\theta]}) \\
\geq H(A_t^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \widebar{Q}_{0:N-1}^{[\theta]}) \\
= (b) H(A_t^{[\theta]}|W_{\Lambda}, Q_t^{[\theta]} = \widebar{Q}_{t}^{[\theta]}),
\]

where (a) and (b) follow from applying (43) to the set \( \{t\} \) and \( \Gamma \) respectively.

That is, the terms in square bracket of (44) are nonnegative. Therefore, they have to be zero for all the query realizations \( \widebar{Q}_{0:N-1}^{[\theta]} \) with positive probability, i.e., the necessary condition \( \textbf{P1} \) must be satisfied.

**Proof of Lemma 2**

In fact,

\[
H(A_{0:N-1}^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]}) \geq \frac{1}{K} \sum_{\Gamma \subseteq [0:N]-[\Gamma] = K} H(A_{\Gamma}^{[\theta]}|W_{\Lambda}, Q_{\Gamma}^{[\theta]}) \tag{45} \]

(b) and (f) are due to (42); (c) and (e) follow from (22); (d) is because of (41); (g) is due to the well-known Han’s inequality [4, Theorem 17.6.1]:

\[
\sum_{\Gamma \subseteq [0:N]-[\Gamma] = K} H(A_{\Gamma}^{[\theta]}|W_{\Lambda}, Q_{\Gamma}^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]}) \geq \frac{1}{K} H(A_{0:N-1}^{[\theta]}|W_{\Lambda}, Q_{0:N-1}^{[\theta]}),
\]

for any set \( \Lambda \subseteq [0:M] \); (h) follows from the correctness constraint in (6) such that \( A_{0:N-1}^{[\theta]} \) and \( Q_{0:N-1}^{[\theta]} \) can decode the requested file \( W_{\theta} \), i.e., \( H(W_{\theta}|W_{\Lambda}, A_{0:N-1}^{[\theta]}, Q_{0:N-1}^{[\theta]}) = 0 \); (i) is due to (4) where queries are independent of the files such that \( H(W_{\theta}|W_{\Lambda}, Q_{0:N-1}^{[\theta]}) = H(W_{\theta}) = L \) by (3).
If the equality in (a) holds, then the equality in (j) of (45) must hold, i.e., for every \( \Lambda \subseteq \{0 : M\} \), \( \Gamma \subseteq \{0 : N\} \), \( |\Gamma| = K \), and \( \theta \in \Lambda \),

\[
0 = H(A_{0:N-1}^{[\theta]} | W_{\Lambda}, Q_{0:N-1}^{[\theta]}) - H(A_{1}^{[\theta]} | W_{\Lambda}, Q_{0:N-1}^{[\theta]})
= \sum_{Q_{0:N-1}^{[\theta]}} \Pr(Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}) \left[ H(A_{0,N-1}^{[\theta]} | W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0,N-1}^{[\theta]}) - H(A_{1}^{[\theta]} | W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0,N-1}^{[\theta]} \right]. \tag{46}
\]

Whereas for every query realization \( \tilde{Q}_{0:N-1}^{[\theta]} \) with positive probability,

\[
H(A_{0:N-1}^{[\theta]} | W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}) \geq H(A_{1}^{[\theta]} | W_{\Lambda}, Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0,N-1}^{[\theta]} \]
\[
\geq H(A_{1}^{[\theta]} | W_{\Lambda}, Q_{1}^{[\theta]} = \tilde{Q}_{1}^{[\theta]}),
\]

where \( (k) \) is due to (43).

This is to say, the terms in square bracket of (46) are nonnegative. Consequently, to make the equality in (a) hold, they have to be zero for all the query realizations \( \tilde{Q}_{0:N-1}^{[\theta]} \) with positive probability, i.e., the necessary condition \( P2 \) must be satisfied.

\textit{Proof of Lemma 3}

Notice that,

\[
H(A_{0:N-1}^{[\theta]} | W_{\theta}, Q_{0:N-1}^{[\theta]}) = H(A_{0:N-1}^{[\theta]} | Q_{0:N-1}^{[\theta]}) - I(A_{0:N-1}^{[\theta]}; W_{\theta} | Q_{0:N-1}^{[\theta]})
= H(A_{0:N-1}^{[\theta]} | Q_{0:N-1}^{[\theta]}) - H(W_{\theta} | Q_{0:N-1}^{[\theta]}) + H(W_{\theta} | A_{0:N-1}^{[\theta]}, Q_{0:N-1}^{[\theta]})
\leq H(A_{0:N-1}^{[\theta]} | Q_{0:N-1}^{[\theta]}) - L
\leq \sum_{t=0}^{N-1} H(A_{t}^{[\theta]} | Q_{0:N-1}^{[\theta]}) - L
\leq \sum_{t=0}^{N-1} H(A_{t}^{[\theta]} | Q_{t}^{[\theta]}) - L
\leq \sum_{t=0}^{N-1} \sum_{Q_{t}^{[\theta]}} \Pr(Q_{t}^{[\theta]} = \tilde{Q}_{t}^{[\theta]}) H(A_{t}^{[\theta]} | Q_{t}^{[\theta]} = \tilde{Q}_{t}^{[\theta]}) - L
\leq \sum_{t=0}^{N-1} \sum_{Q_{t}^{[\theta]}} \Pr(Q_{t}^{[\theta]} = \tilde{Q}_{t}^{[\theta]}) (H(\tilde{A}_{t,0:M}^{[\theta]} y_{t}) - L)
\leq \sum_{t=0}^{N-1} \sum_{Q_{t}^{[\theta]}} \Pr(Q_{t}^{[\theta]} = \tilde{Q}_{t}^{[\theta]}) \ell_{t} - L
= \sum_{t=0}^{N-1} \mathbb{E} [\ell_{t}] - L
= \frac{L}{R} - L,
\]

where \( (a) \) follows from the correctness constraint in (6) and the fact that queries are independent from the files, i.e., \( H(W_{\theta} | Q_{0:N-1}^{[\theta]}) = H(W_{\theta}) = L \); \( (b) \) is due to the inequality of joint entropy; \( (c) \) follows from (42) by setting \( \Gamma = \{t\} \) and \( \Lambda = 0 \); \( (d) \) is by the definition of the linear coded PIR in (11) and then \( A_{t}^{[\theta]} = \tilde{A}_{t,0:M}^{[\theta]} y_{t} = \bar{A}_{t,0:M}^{[\theta]} y_{t} \) for the received query realization \( \tilde{Q}_{t}^{[\theta]} \); \( (e) \) is due to the principle of maximum entropy, i.e., \( H(\tilde{A}_{t,0:M}^{[\theta]} y_{t}) \leq \ell_{t} \), where \( \ell_{t} \) is the length of \( \tilde{A}_{t,0:M}^{[\theta]} y_{t} \).

In fact, the equality in (b) holds if and only if

\[
0 = \sum_{t=0}^{N-1} H(A_{t}^{[\theta]} | Q_{0:N-1}^{[\theta]}) - H(A_{0:N-1}^{[\theta]} | Q_{0:N-1}^{[\theta]})
= \sum_{Q_{0:N-1}^{[\theta]}} \Pr(Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}) \left[ \sum_{t=0}^{N-1} H(A_{t}^{[\theta]} | Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0:N-1}^{[\theta]}) - H(A_{0:N-1}^{[\theta]} Q_{0:N-1}^{[\theta]} = \tilde{Q}_{0,N-1}^{[\theta]} \right]. \tag{47}
\]
While

$$\sum_{t=0}^{N-1} H(A_t^{[\theta]} | Q_t^{[\theta]}) = H(A_t^{[\theta]} | Q_t^{[\theta]} = \tilde{Q}^{[\theta]}_{0:N-1}) = H(A_t^{[\theta]} | Q_t^{[\theta]} = \tilde{Q}^{[\theta]}_{0:N-1})$$

$$\geq H(A_t^{[\theta]} | Q_t^{[\theta]} = \tilde{Q}^{[\theta]}_{0:N-1}) = H(A_t^{[\theta]} | Q_t^{[\theta]} = \tilde{Q}^{[\theta]}_{0:N-1})$$

where (f) follows from (43) by setting $\Gamma = \{t\}$ and $\Lambda = \emptyset$.

This means that the terms in square bracket of (47) are nonnegative. Hence, to make the equality in (b) hold, they have to be zero for all the query realizations $\tilde{Q}^{[\theta]}_{0:N-1}$ with positive probability, i.e., the necessary condition $P3$ must be satisfied.

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