Multistage Utility Preference Robust Optimization

Jia Liu and Zhiping Chen
School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, Shaanxi, P. R. China,
Center for Optimization Technique and Quantitative Finance, Xi’an International Academy for Mathematics and
Mathematical Technology, Xi’an, P. R. China, jialiu@xjtu.edu.cn, zchen@mail.xjtu.edu.cn
Huifu Xu
Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Hong Kong,
hfxu@se.cuhk.edu.hk

In this paper, we consider a multistage expected utility maximization problem where the decision maker’s
utility function at each stage depends on historical data and the information on the true utility function is
incomplete. To mitigate adverse impact arising from ambiguity of the true utility, we propose a maximin
robust model where the optimal policy is based on the worst-case sequence of utility functions from an ambi-
guity set constructed with partially available information about the decision maker’s preferences. We then
show that the multistage maximin problem is time consistent when the utility functions are state-dependent
and demonstrate with a counter example that the time consistency may not be retained when the utility
functions are state-independent. With the time consistency, we show the maximin problem can be solved
by a recursive formula whereby a one-stage maximin problem is solved at each stage beginning from the
last stage. Moreover, we propose two approaches to construct the ambiguity set: a pairwise comparison
approach and a $\zeta$-ball approach where a ball of utility functions centered at a nominal utility function under
$\zeta$-metric is considered. To overcome the difficulty arising from solving the infinite dimensional optimization
problem in computation of the worst-case expected utility value, we propose piecewise linear approximation
of the utility functions and derive error bound for the approximation under moderate conditions. Finally,
we use the stochastic dual dynamic programming (SDDP) method and the nested Benders’ decomposition
method to solve the multistage state-dependent preference robust problem and the scenario tree method to
solve the state-independent problem, and carry out comparative analysis on the efficiency of the computa-
tional schemes as well as out-of-sample performances of the state-dependent and state-independent models.
The preliminary results show that the state-dependent preference robust model solved by SDDP algorithm
displays overall superiority.

Key words: Preference robust optimization, state-dependent utility, rectanglarity, time consistency,
Kantorovich ball, scenario tree method, SDDP, nested Benders’ decomposition method

1. Introduction

Decision making under uncertainty has two important elements: belief and taste. Belief is the
decision maker’s (DM for brevity) view on the state of nature of the underlying uncertainty whereas
taste is the DM’s preference. When there is an ambiguity about belief, one may base the optimal
decision on the worst-case scenario of the uncertainty or worst-case probability distribution

[22]
and this is the case of robust optimization or distributionally robust optimization. Over the past few decades, a lot of research have been conducted on robust optimization and distributionally robust optimization models, see monograph by Ben-Tal et al. [4] and a comprehensive overview by Rahimian and Mehrotra [42].

Ambiguity may also occur with regard to taste. An explicit utility function might not be available when information on the DM’s preference is incomplete. There is a multitude of ways about how to use preference information to construct a utility function. In the literature of decision analysis and behavioural economics, a popular method is to elicit the DM’s preferences with paired gambling approaches for preference comparisons [20], use the elicited information to identify the value of the utility function at a discrete set of points and construct an approximate utility function via some interpolation methods, see for instance [10].

Armbruster and Delage [1] argue that the interpolation approach has some drawbacks because not only it is often difficult to identify a non-parametric utility function purely based on the DM’s preferences over pairwise comparison lotteries but also it could be risky to use a single approximate utility function without considering other plausible ones nearby. Consequently, they propose an alternative approach, that is, instead of trying to find a single approximate von Neumann and Morgenstern’s utility function, they propose to use available information of the DM’s preferences such as preferring certain lotteries over other lotteries, being risk averse over gains and risk taking over losses to construct an ambiguity set of plausible utility functions and then base the optimal decision on the worst-case utility function from the ambiguity set. The approach is called preference robust optimization (PRO) as it follows the general philosophy of robust optimization. In the case that the ambiguity set is constructed through pairwise comparisons, the PRO model may be viewed as an extension of the well-known stochastic programs with stochastic dominance constraints (Dentcheva and Ruszczyński [15]). Hu and Mehrotra [27] also take a PRO approach to tackle the ambiguity of the true utility function but in a slightly different manner. They consider a probabilistic representation of the class of increasing concave utility functions by confining them to a compact interval and scaling them to being bounded by 1. In doing so, they propose a moment-type framework for constructing the ambiguity set of the DM’s utility functions which covers a number of important approaches such as the certainty equivalent and pairwise comparison.

Over the past few years, PRO has attracted increasing attentions. For instances, Haskell et al. [25] propose a robust model which handles the ambiguity of DM’s belief and taste. Hu and Stepanyan [28] propose a so-called reference-based almost stochastic dominance method for constructing a set of utility functions near a reference utility which satisfy certain stochastic dominance relationship and use the set to characterize the DM’s preference. Hu et al. [29] consider a PRO model with an ambiguity set of general utility functions and propose a Lagrangian function approach for solving
the resulting maximin problem. Guo and Xu [24] propose a piecewise linear approximation approach for solving a PRO model with the ambiguity set being specified by moment-type conditions and derive a bound for approximation error in terms of the ambiguity set, the optimal value and optimal solutions. The PRO approach has also been effectively applied to risk management problems where the DM’s risk preferences are ambiguous, see [14, 23, 35, 58, 62, 66].

In this paper, we extend this stream of research to multistage decision making process. There are several modelling approaches in multistage decision making such as multistage stochastic optimization, Markovian decision making, and approximate dynamic programming [41]. Among them, multistage stochastic optimization (MSO) has been widely studied and applied in long term financial planning, pension fund management, energy production and trading, supply chain management and inventory control [38], as it can flexibly characterize the dynamic dependent structure of random data process. A key component in the multistage decision making modelling is the dynamic decision criterion, i.e., the objective function for the multistage stochastic optimization model. One of the most widely adopted objective functions is the multistage expected utility models, which can be also understood as a kind of multistage risk function where the utility function at each stage characterizes the dynamic preference of the DM [16]. There are basically three types of multistage expected utility models: terminal utility model, additive utility model and recursive utility model [11]. Like terminal risk measures, the terminal utility model may lead to time inconsistent optimal policies as it only measures the utility of reward at the terminal stage [12].

The additive utility model, which is most extensively studied, considers the sum of utilities of rewards at different stages, thus the DM’s intertemporal preferences are risk neutral [16, 49]. The recursive utility model, also known as stochastic differentiable utility in continuous time setting, characterizes DM’s nonlinear intertemporal preferences. The model has a natural connection with time consistency of the optimal policy. Important contributions include the recursive expected utility [33] and the well-known Kreps-Porteus utility which is recursive, but not necessarily expected utility [34]. However, traditional expected utility theory has received many criticisms for its failure to explain some experimental observations and theoretical puzzles such as Allais paradox. Rank-dependent expected utility theory and cumulative prospect theory are subsequently proposed to address the drawbacks, see monograph by Puppe [39] for an overview of the development of the theories. In dynamic setting, Hu et al. [30] study a continuous-time portfolio selection model where a sequence of time-dependent probability weighting functions and rank-dependent utility functions are used to capture a DM’s overweighting and underweighting behaviours on tail losses/rewards at different stages, see also [51] for empirical studies.

In all these works, the DM’s utility functions are assumed to be known exactly and fixed in the decision making process. However, as we discussed earlier, the DM’s utility function may be
ambiguous and this motivates us to propose a PRO model for the multistage decision making process. Moreover, many studies argue that utility functions may be state-dependent. The most widely adopted approach is to consider a parametric form of utility functions where the parameters are state-dependent. For instance, Strub and Li [52] consider a sequence of S-shaped utility functions parameterized by a sequence of state-dependent reference points and show that failing to update the reference point as state changes may lead to time inconsistent investments. Likewise, He et al. [26] consider a series of state-dependent distortion functions when they apply the rank-dependent expected utility theory to continuous time investment problems. Björk et al. [8] adopt a state-dependent risk-aversion parameter in the multistage mean-risk model. There is also a specific stream of research on so-called habit formation utility where the DM’s consumption habit level and her/his utility at a particular stage and/or state is determined by the historical consumption process [16].

In this paper, we will also use the habit formation utility model and concentrate on a situation where the DM’s utility at each stage is ambiguous but it is possible to use partially available information to construct a set (called ambiguity set later on) of plausible utility functions which capture the DM’s preferences. Two ways are proposed to construct the ambiguity set. One is to use the pairwise comparison approach which are widely used in the literature of PRO models and behavioural economics. The other is to construct a ball of utility functions centered at a nominal utility function under some pseudo-metrics. The main challenge to be tackled is to develop efficient computational schemes for solving the resulting multistage PRO models.

As far as we are concerned, the main contributions of the paper can be summarized as follows.

First, we propose a multistage PRO model where the DM’s utility preferences at different stages depend on not only the current stage and state but also the history of the underlying random data process leading to the state. We introduce a definition of ambiguity set comprising certain sequences of state-wise utility functions and a maximin optimization model where the optimal policy is based on the worst-case summed expected utility values of the random reward functions at different stages computed with the ambiguity set.

Second, we introduce the concept of rectangularity of the ambiguity set of utility functions. Under some moderate conditions, we show the multistage maximin problem with state-dependent ambiguity set is time consistent and demonstrate through a simple example that the problem is time inconsistent when the utility functions are state-independent.

Third, by utilizing the time consistency, we derive a recursive formula for solving the multistage PRO problem when the utility functions are state-dependent. For the ambiguity of general utility functions, error bounds for both the ambiguity set and the optimal value are derived when the utility functions in the ambiguity set are approximated by piecewise linear utility functions at each stage.
To tackle time inconsistency and nonlinearity in solving maximin problem at each stage, we propose a scenario tree approach which reformulates the holistic maximin problem as a single mixed integer linear program, we propose to use the stochastic dual dynamic programming (SDDP) method and the nested Benders’ decomposition (NBD) method to solve the state-dependent multistage PRO models.

Fourth, we apply the proposed PRO model and the computational scheme to a multistage investment-consumption problem and carry out comparative analysis on the efficiency of the computational schemes as well as out-of-sample performances of the state-dependent and state-independent models. The preliminary results show that the state-dependent preference robust model solved by SDDP algorithm displays overall superiority.

The rest of the paper is organized as follows. Section 2 defines the multistage expected utility maximization models to be discussed in this paper. Section 3 introduces the robust counterparts and discusses rectangularity and time consistency of the models when the utility functions are state-dependent. Section 4 details construction of the ambiguity set with two approaches: pairwise comparisons and $\zeta$-ball. In the latter approach, a piecewise linear approximation approach is proposed to approximate the general utility functions and error bounds are derived. Section 5 discusses computational schemes for solving multistage PRO models by the scenario tree method and dynamic programming algorithms. Section 6 reports a number of numerical results and comparative analysis. Finally, Section 7 gives some concluding remarks. Due to the limitation of pages in the main body of the paper, all proofs of the technical results, some examples and the detailed algorithmic procedures of SDDP and NBD methods are moved to Electronic Companions.

2. Multistage expected utility models

We begin by introducing notions and notations that are commonly used in multistage stochastic optimization. Let $\xi = \{\xi_t\}_{t=1}^T$ be a stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$, where $\xi_t : \Omega \rightarrow \mathbb{R}^d_t$ is a random vector supported on $\Xi_t$ for $t = 1, \ldots, T$. For simplicity of notation, we write $\xi_{[t]}$ for historical information $(\xi_1, \ldots, \xi_t)$. Let $\mathcal{F}_t$ denote the sigma algebra in the sample space $\Omega$ generated (induced) by $\xi_{[t]}$, that is, $\mathcal{F}_t = \{(\xi_{[t]})^{-1}B : B \in \mathcal{B}(\Xi_{[t]})\}$, where $\mathcal{B}(\Xi_{[t]})$ denotes the Borel sigma algebra of set $\Xi_{[t]} := \Xi_1 \times \cdots \times \Xi_t$. By convention, we assume that there is an initial state $\xi_0$ which is deterministic and corresponds to the deterministic events $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Consequently, we have $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_t \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$. As $\mathcal{F}_t$ is generated by $\xi_{[t]}$, we denote $E_{\mathcal{F}_t} [\cdot] := E[\cdot | \xi_t]$ for simplicity and $E_{\mathcal{F}_0} [\cdot] := E[\cdot]$.

Let $L^p(\Omega, \mathcal{F}, P; \mathbb{R})$ denote the set of random variables $\psi : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ with finite $p$-th moments, i.e., $\int_\Omega |\psi(\omega)|^p dP(\omega) < \infty$, for $p \geq 1$. Let $L^p(\mathbb{R})$ denote the set of real functions $u : \mathbb{R} \rightarrow \mathbb{R}$ integrable to the $p$-th order and $L^p(\Omega, \mathcal{F}, P; L^p(\mathbb{R}))$ denote the set of random integrable functions $\hat{u}$:
We consider the following multistage expected utility maximization problem.

2.1. Models

We consider the following multistage expected utility maximization problem

$$
\max_{x_1 \in \mathcal{X}_1} \mathbb{E} \left[ u_1(h_1(x_1, \xi_1)) + \max_{x_2 \in \mathcal{X}_2(x_1, \xi_1)} \mathbb{E}_{|\mathcal{F}_1} \left[ u_2(h_2(x_2, \xi_2), \xi_1) + \cdots + \max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} \mathbb{E}_{|\mathcal{F}_{T-1}} \left[ u_T(h_T(x_T, \xi_T), \xi_{[T-1]}) \right] \right] \right],
$$

(1)

where $h_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \rightarrow \mathbb{R}$ is a continuous reward function at stage $t$, and $u_t : \mathbb{R} \times \mathbb{R}^{\Sigma_{i=1}^{T-1} d_t} \rightarrow \mathbb{R}$ is the utility function characterizing the DM's utility value of the reward at stage $t$, $x_t$ is the decision vector, $x_{[t]}$ is the historical decision process $(x_1, \ldots, x_t)$ till stage $t$, and $\mathcal{X}_t(x_{[t-1]}, \xi_{[t-1]})$ is the set of feasible decisions at stage $t$ for $t = 2, \ldots, T$, the expectation at stage 1 is taken with respect to the distribution of $\xi_1$, and the expectation at stage $t$ is taken w.r.t. the distribution of $\xi_t$ conditional on the filtration $\mathcal{F}_{t-1}$, i.e., the historical data $\xi_{[t-1]}$, for $t = 2, \ldots, T$. In this setup, the DM chooses an optimal decision $x_t$ from $\mathcal{X}_t(x_{[t-1]}, \xi_{[t-1]})$ so that

$$
\mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(h_t(x_t, \xi_t), \xi_{[t-1]}) + \cdots + \max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} \mathbb{E}_{|\mathcal{F}_{T-1}} \left[ u_T(h_T(x_T, \xi_T), \xi_{[T-1]}) \right] \right]
$$

is maximized. The utility function at stage $t$ depends not only on the current stage (indicated by the subscript) but also on the historical state (realization of) $\xi_{[t-1]}$. The choice of the optimal decision $x_t$ is independent of the realizations of $\xi_t$ which means the decision is made before the realization of uncertainty $\xi_t$, and it is not a recourse action. Of course, we can interpret $u_t(h_t(x_t, \xi_t), \xi_{[t-1]})$ as the optimal value arising from a recourse action. In particular, if the random reward function at stage $t$ depends on the current state $\xi_{t-1}$ rather than state $\xi_t$ at next stage (mathematically replacing $h_t(x_t, \xi_t)$ with $h_t(x_t, \xi_{t-1})$), then problem (1) can be written as

$$
\max_{x_1 \in \mathcal{X}_1} u_1(h_1(x_1, \xi_0)) + \mathbb{E} \left[ \max_{x_2 \in \mathcal{X}_2(x_1, \xi_1)} u_2(h_2(x_2, \xi_1), \xi_1) + \cdots + \max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} \mathbb{E}_{|\mathcal{F}_{T-1}} \left[ u_T(h_T(x_T, \xi_T), \xi_{[T-1]}) \right] \right].
$$
\[
\cdots + \mathbb{E}_{\mathcal{F}_{T-2}} \left[ \max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} u_T(h_T(x_T, \xi_T), \xi_{T-1}) \right].
\] (2)

In this formulation, a recourse action \( x_t \) is taken before realization of \( \xi_{t-1} \) is observed. We skip the details on recourse actions so that we may focus on the key issues in this paper. Note that if we interpret the utility function as the DM’s taste and the distribution of the future uncertainty and of the reward as belief in the literature of decision analytics, then we can see that the randomness in \( u_t \) (the taste) arises from historical data \( \xi_{t-1} \) whereas the belief is concerned with future uncertainty of rewards. Unless specified otherwise, we assume that \( \mathcal{X}_T(x_{T-1}, \xi_{T-1}) \) is a convex and compact subset of \( \mathbb{R}^n_t \) for \( t = 1, \ldots, T \).

A simplified version of (1) is that the utility functions at each stage are state-independent, that is,

\[
\max_{x_1 \in \mathcal{X}_1} \mathbb{E}[u_1(h_1(x_1, \xi_1))] + \max_{x_2 \in \mathcal{X}_2(x_1, \xi_1)} \mathbb{E}_{\mathcal{F}_1}[u_2(h_2(x_2, \xi_2))] + \\
\cdots + \max_{x_T \in \mathcal{X}_T(x_1, \xi_1)} \mathbb{E}_{\mathcal{F}_{T-1}}[u_T(h_T(x_T, \xi_T))].
\] (3)

In this model, the DM has the same utility preference in all states at stage \( t \) regardless of the overall wealth accumulated over the past \( t-1 \) stages. In the case when the DM takes an identical view on utilities over all stages, the model may be further simplified to

\[
\max_{x_1 \in \mathcal{X}_1} \mathbb{E}[u(h_1(x_1, \xi_1))] + \max_{x_2 \in \mathcal{X}_2(x_1, \xi_1)} \mathbb{E}_{\mathcal{F}_1}[u(h_2(x_2, \xi_2))] + \\
\cdots + \max_{x_T \in \mathcal{X}_T(x_1, \xi_1)} \mathbb{E}_{\mathcal{F}_{T-1}}[u(h_T(x_T, \xi_T))].
\] (4)

In the classical multistage stochastic programming (MSP) models, utility functions at different stages may be different but they are pre-determined at the beginning which means the DM cannot adjust her/his utility at later stages as dynamic stochastic environment changes and this is inconsistent with practical decision making process. Indeed, many theoretical and empirical studies show that the utility function should depend on the current and/or historical state [51, 30, 36]. For example, the DM’s utility of wearing a mask in the year of 2020 (at the peak of COVID-19 epidemic) must be totally different from the utility in normal circumstances. Even at different stages of the COVID-19 epidemic, the utility of wearing a mask varies. Here we assume that at stage \( t \), the DM can adjust her/his utility function according to the current and historical states. Of course, regardless of the stage that the DM is in, her/his utility function must be specified prior to the decision making at the stage, i.e., \( u_t \) does not depend on \( \xi_t \).

In this paper, we will focus on the case when the utility functions are ambiguous, and we will see that utility preference robust formulations of the above three models will have completely different properties.
2.2. Reformulations

In model (1), optimal decision at stage \( t \) is a vector in \( \mathbb{R}^{n_t} \). However, if we view the decision making from stage 1, then we may regard it as a random function of \( \xi_{[t-1]} \). Consequently, we may reformulate the multistage expected utility maximization problem (1) as

\[
\max_{x_{[T]}} E\left[u_1(h_1(x_1,\xi_1)) + u_2(h_2(x_1,\xi_2),\xi_1) + \cdots + u_T(h_T(x_T(\xi_{[T-1]}),\xi_T),\xi_{[T-1]})\right]
\]

s.t. \( x_1 \in \mathcal{X}_1, x_t(\xi_{[t-1]}) \in \mathcal{X}_t(x_{[t-1]}(\xi_{[t-2]}),\xi_{[t-1]}), \) for \( t = 2, \ldots, T \),

where the expectation is taken w.r.t. the distribution of \( \xi_{[T]} \) and we write \( x_{[1,T]} \) (or \( x_{[T]} \) when the decision process starts from the initial stage) for a sequence of decisions \((x_1, x_2(\cdot), \ldots, x_T(\cdot))\), which is also known as an implementable policy. We denote \( x_{[t-1]}(\xi_{[t-2]}):=(x_1, x_2(\xi_1), \ldots, x_{t-1}(\xi_{[t-2]})) \) the \( \xi_{[t-2]} \)-dependent historical decision process up to stage \( t-1 \). The reformulation is fundamentally related to Bellman’s principle in dynamic programming that an optimal policy at the initial planning stage is consistent with the optimal decisions at each of the remaining stages, we will come back to this in Section 3. The reformulation requires some moderate conditions, see Lemma 1 on Page 13 for the two stage case. Likewise, we can reformulate (3) as

\[
\max_{x_{[T]}} E\left[u_1(h_1(x_1,\xi_1)) + u_2(h_2(x_1,\xi_2),\xi_2) + \cdots + u_T(h_T(x_T(\xi_{[T-1]}),\xi_T))\right]
\]

s.t. \( x_1 \in \mathcal{X}_1, x_t(\xi_{[t-1]}) \in \mathcal{X}_t(x_{[t-1]}(\xi_{[t-2]}),\xi_{[t-1]}), \) for \( t = 2, \ldots, T \).

A practical application of the multistage utility maximization model is multistage portfolio selection problem, see an example in EC.1.

3. Robust models

In the multistage expected utility optimization models that we presented in the previous section, the true utility functions which capture the DM’s preferences at each stage are assumed to be known. This assumption may not be satisfied in practice as we discussed in the introduction section. It motivates us to consider a robust model where the optimal decision at each stage is based on the worst-case utility function from a set of plausible utility functions. Since the robust model is essentially built upon von Neumann-Morgenstern expected utility theory, we make a blanket assumption as follows.

**Assumption 1.** The DM’s preference can be represented by von Neumann-Morgenstern expected utility theory and is consistent at each state.

The assumption on the preference consistency means that at each state, there exists at least one VNM’s utility function which can be used to represent all of the elicited/observed preferences of the DM at the state. In practice, however, DM’s utility preferences may be inconsistent due to cognitive biases \[54\] and/or elicitation errors \[1\]. Bertsimas and O’Hair \[6\] and Armbruster and...
Delage [1] proposed some approaches to handle the issue. Here by introducing Assumption [1] we restrict our discussions to the consistent preference case so that we may focus on the key challenges arising from multistage maximin problems.

### 3.1. Multistage PRO models

In the expected utility maximization model (1) or its equivalent formulation (5), the sequence of dynamic decisions is made with respect to a sequence of utility functions \( \{u_t\} \). However, a DM may not have complete information to identify a sequence of true utility preferences but it is possible to use partial information to build an ambiguity set of plausible utility functions. We begin with a formal definition of the ambiguity set which captures DM’s utility preferences at each stage.

**Definition 1 (Ambiguity set of utility functions).** Let \( \mathcal{U} \) be the set of all continuous, bounded and monotonically increasing functions in \( L^p(\mathbb{R}) \) and \( \mathcal{U}_t(\xi_{[t-1]}) \) be a \( \mathcal{F}_{t-1} \)-measurable set-valued mapping. For any given \( \xi_{[t-1]} \), \( \mathcal{U}_t(\xi_{[t-1]}) \) is a subset of \( \mathcal{U}_t \), for \( t = 1, \ldots, T \). Define the ambiguity set

\[
\mathcal{U} := \{ \bar{u} \mid \bar{u} = [u_1, u_2, \ldots, u_T]^\top, \ u_t(\cdot, \xi_{[t-1]}) \in \mathcal{U}_t(\xi_{[t-1]}), \text{for any } \xi_{[t-1]}, \ t = 1, \ldots, T \},
\]

where \( u_1(\cdot, \xi_0) = u_1(\cdot) \) is a real-valued function in the deterministic ambiguity set \( U_1 \). We say that the sequence of utility functions \( \{u_t(\cdot, \xi_{[t]})\} \) is state-independent if \( \mathcal{U}_t \) is \( \mathcal{F}_0 \)-measurable, i.e., a deterministic set, for \( t = 1, \ldots, T \). In this case, we write \( u_t(\cdot) \) for \( u_t(\cdot, \xi_{[t-1]}) \) and \( \mathcal{U}_t \) for \( \mathcal{U}_t(\xi_{[t-1]}) \).

In this definition, each utility function in the set \( \mathcal{U}_t(\xi_{[t-1]}) \) depends on historical information \( \xi_{[t-1]} \), which means the DM’s utility preference at state \( \xi_{t-1} \) is affected not only by the current state \( \xi_{t-1} \) (at the point of the decision making) but also the earlier experiences. The \( \mathcal{F}_{t-1} \)-measurability of \( \mathcal{U}_t \) paves the way for the rectangularity of the ambiguity set to be stated in the forthcoming Proposition [1]. A classical example of such state-dependent utility function is the habit-formation utility model where a DM’s utility \( u_t(c_t, h_t) \) at stage \( t \) depends on both the current consumption \( c_t \) and the historical habit level of consumption \( h_t = \sum_{j=1}^t \alpha_j c_{t-j} \) where \( \alpha_j, j = 1, \ldots, t \) are positive numbers. The latter can be understood as historical path \( \xi_{[t-1]} \) dependent, see [16, 17]. Likewise, an investor who has experienced tough economic circumstances in the past may be more risk averse at the current stage. The structure of the ambiguity set depends on available information in concrete decision making problems, we will come back to details about this in Section [4].

To mitigate the model risk arising from ambiguity of the true utility functions in the decision-making process under model (5), we propose a robust counterpart where the optimal policy is based on the worst-case sequence of utility functions:

\[(\text{MS-PRO-SD})\]
Here the maximin robust formulation is based on a holistic view at the very beginning of the decision making process on both the optimal policy and the expected utility. Specifically, instead of considering the maximin robust formulation at each stage, we compute, for every sequence of feasible decisions $x_{[T]}$, the worst-case expected utility

$$\mathbb{E}\left[u_1(h_1(x_1, \xi_1)) + u_2(h_2(x_2(\xi_1, \xi_2, \xi_1)) + \cdots + u_T(h_T(x_T(\xi_{[T-1]}, \xi_T) ; \xi_{[T-1]}))\right]$$

with a sequence of utility functions $\vec{u}$ from the ambiguity set. The optimal policy is subsequently identified via the largest worst-case expected utility value. This kind of maximin robust approach is consistent with the philosophy of robust optimization, particularly the recent multistage distributionally robust optimization models \[48\]. We call it multistage utility preference robust optimization models.

In the case that the utility functions are independent of states, we may obtain a PRO counterpart for model (6):

$$(\text{MS-PRO-SID}) \max_{x_{[T]}} \inf_{\vec{u} \in U} \mathbb{E}\left[u_1(h_1(x_1, \xi_1)) + u_2(h_2(x_2(\xi_1, \xi_2, \xi_1)) + \cdots + u_T(h_T(x_T(\xi_{[T-1]}, \xi_T) ; \xi_{[T-1]}))\right]$$

where $\vec{u} = [u_1, u_2, \ldots, u_T]^T$ and $U \subset \mathcal{L}^p(\mathbb{R}) \times \cdots \times \mathcal{L}^p(\mathbb{R})$ is an ambiguity set of the vectors of utility functions in product form.

### 3.2. Time consistency

An important and widely accepted practice in multistage stochastic programming is that the optimal policy determined at stage 1 should be consistent with the optimal sub-policy to be set at stage $t$ for $t \geq 1$, which is known as time consistency or Bellman’s optimality principle \[2, 12, 57\]. The principle is not automatically fulfilled in the multistage PRO models unless the ambiguity set of utility functions is structured properly. This motivates us to introduce the next definition.

**Definition 2 (Time consistency of dynamic policy).** A multistage PRO model is said to be *time consistent* if any optimal policy for the multistage PRO model over the entire time horizon also satisfies the local optimality conditions of the sub-PRO model from period $t$ to period $T$, for any given historical $\xi_{[t-1]}$, for all $t = 2, \ldots, T$.

In multistage risk minimization problems, the time consistency of the optimal dynamic policy can be achieved if the corresponding multistage risk measure is time consistent. The concept of
time consistency on multistage risk measure characterizes an order keeping relationship among different stages: given two investment positions A and B, if A is at least as good as B under a specific risk measure at some future time $\tau$, and they are identical between now (time $t$) and the future time $\tau$, then A is at least as good as B under the same measure from today ($t$)’s perspective [10][41]. All time consistent risk measures can be written in a nested form [41].

In multistage distributionally robust optimization, time consistency of the optimal dynamic policy is related to the structure of the dynamic ambiguity set of probability distributions. If the distributionally robust counterpart can be written in a nested form of stage-wise conditional distributionally robust counterparts, known as the rectangular set or recursive multiple-priors set [18][48], then the optimal policy is time consistent and the dynamic programming equation may follow [48].

Likewise, the time consistency of the optimal dynamic policy of the multistage preference robust optimization problem relies on the structure of the preference ambiguity set. We shall define a property on the decomposability of the preference set. To this end, we introduce the concept of rectangularity of the ambiguity set of utility functions.

### 3.2.1. Rectangularity of the ambiguity set

To ease the exposition, we denote the reward function $h_t(x_t(\xi_{[t-1]}), \xi_t)$ by an $\mathcal{F}_t$-adaptable random variable $Z_t$.

**Definition 3 (Rectangularity of the ambiguity set).** Let $\mathcal{U}$ be a nonempty set of utility sequences $\vec{u}$, $\mathcal{U}$ is said to be rectangular if

$$
\inf_{\vec{u} \in \mathcal{U}} \mathbb{E} \left[ u_1(Z_1) + u_2(Z_2, \xi_1) + \cdots + u_T(Z_T, \xi_{[T-1]}) \right]
= \inf_{u_t \in \mathcal{U}_t} \mathbb{E} \left[ u_1(Z_1) + \inf_{u_2 \in \mathcal{U}_2(\xi_1)} \mathbb{E}_{\mathcal{F}_1} \left[ u_2(Z_2) + \cdots + \inf_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{\mathcal{F}_{T-1}} [u_T(Z_T)] \right] \right]
$$

holds for any $\{x_t\}$, $\{\xi_t\}$ and $\{Z_t := h_t(x_t(\xi_{[t-1]}), \xi_t)\}$, where

$$
\mathcal{U}_t(\xi_{[t-1]}) := \mathcal{U}_t \left( \vec{u}_{[1,t-1]}(\cdot, \xi_{[1-1]}), \xi_{[t-1]} \right) = \left\{ u_t \in \mathcal{L}^p(\mathbb{R}) \mid \exists \vec{u}_{[t+1,T]} \in \mathcal{L}^p(\mathbb{R}) \times \cdots \times \mathcal{L}^p(\mathbb{R}) \text{ such that } \begin{bmatrix} \vec{u}_{[1,t-1]}(\cdot, \xi_{[t-1]}) ; u_t ; \vec{u}_{[t+1,T]} \end{bmatrix}^\top \in \mathcal{U} \right\},
$$

The property has two important components: one is the interchangeability of infimum operation (with respect to the utility function) and the conditional expectation operation, which indicates the consistency between the global worst-case utility sequence $\vec{u}_{[1,T]}$ and the local worst-case utility functions $\vec{u}_{[t,T]}$; the other is the consistency that each of the current utility function $u_t \in \mathcal{U}_t(\xi_{[t-1]})$ can be paired up with the DM’s potential utility sequence at the remaining stages $[t+1,T]$ given the utility sequence over stage $[1,t-1]$, to form an element in the specified ambiguity set $\mathcal{U}$. This is similar to time consistency in [13][46] and local property in [44]. In the forthcoming discussions, we will show that the ambiguity set defined in Definition 3 satisfies the rectangularity.
Analogous to the rectangularity of distributionally ambiguity set for multistage DRO problems [48] and the conditional (state-dependent) decomposition of uncertainty set in multistage parametric robust optimization problem [13], the proposed rectangularity is built on a broad decomposable structure of the inner minimization problem without relying on a specific form of the ambiguity set. The concept differs from the rectangularity in some MSP literature [40] or MDP literature [31, 61], where a product form of sub-ambiguity sets in different stages or states are considered. As noted by Pichler and Shapiro [40], a product form of sub-ambiguity sets is not enough to guarantee the decomposability of a multistage DRO problem. In the multistage PRO problems, this means a product form with deterministic sub-ambiguity sets may lead to state-independent PRO problems which are not rectangular and time inconsistent (see Appendix EC.3). By adding state-dependent property to sub-ambiguity sets, we can show in Proposition 1 that the defined state-dependent preference ambiguity set in Definition 1 is rectangular.

To study the time consistency of the optimal policy of a PRO model, we shall investigate whether the global optimal solution is consistent with the local optimal solution of the sub-PRO problem over a sub-horizon. If we consider the sub-PRO model of (MS-PRO-SID) (9) from period $t$ to period $T$,

$$\max_{x_{[t,T]}} \inf_{\vec{u}_{[t,T]} \in U_{[t,T]}} \mathbb{E}_{|F_{t-1}} \left[ u_t(h_t(x_t(\xi_{[t-1]}),\xi_t)) + u_{t+1}(h_{t+1}(x_{t+1}(\xi_{[t]}),\xi_{t+1})) + \cdots + u_T(h_T(x_T(\xi_{[T-1]}),\xi_T)) \right]$$

s.t. $x_s(\xi_{[s-1]}) \in \mathcal{X}_s(x_s(\xi_{[s-2]}),\xi_{[s-1]})$, $s = t, \ldots, T$,

(12)

where $x_{[t,T]} := (x_t(\cdot), \ldots, x_T(\cdot))$, $U_{[t,T]} = \{ \vec{u}_{[t,T]} | \exists \vec{u}_{[1,t-1]} \text{ such that } [\vec{u}_{[1,t-1]}, \vec{u}_{(t,T)}]^T \in U \}$, we may find that the worst-case utility series $\vec{u}_{[t,T]}$ of the sub-PRO problem (12) depends on historical states $\xi_{[t-1]}$ and historical decisions $x_{[t-1]}$. However, the worst-case utility series of the global PRO problem (9) is a deterministic function series. Then, such an inconsistency of the worst-case utility series between the global PRO problem and the sub-PRO problem leads the inconsistency of their optimal solutions. An example which shows the point is given in Appendix EC.3.

In what follows, we will show that the ambiguity set $\mathcal{U}$ defined as in (7) is rectangular and the PRO model (MS-PRO-SD) is time consistent. To this end, we introduce an interchangeability principle for the preference robust counterpart. In the literature of stochastic programming and variational analysis, there have been several results on the principle of interchangeability, see for example [45, Proposition 5], [43, Theorem 14.60], [49, Proposition 6.37, Theorem 7.80] and [53, Theorem 2.1]. While these results are derived under some different conditions, they are all stated in the finite dimensional space. Here we need a principle of interchangeability which is in the infinite-dimensional space.

Let $\mathbb{Z}$ be a Polish space with Borel field $\mathcal{B}(\mathbb{Z})$ and $\Omega$ be a sample space associated with filtration $\mathcal{F}$ and measure $\mathbb{P}$. We say a random function $f : \mathbb{Z} \times \Omega \to \mathbb{R}$ is a Carathéodory function [49] if
\(\omega \rightarrow f(z,\omega)\) is \(\mathcal{F}\)-measurable for every fixed \(z \in \mathbb{Z}\) and the function \(z \rightarrow f(z,\omega)\) is continuous for almost every fixed \(\omega \in \Omega\).

**Lemma 1.** Consider a Polish space \(\mathbb{Z}\) and a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(Z : \Omega \rightarrow \mathbb{Z}\) be a \(\mathcal{F}\)-measurable set-valued mapping with closed values. Let \(\mathfrak{M}\) be a linear space of measurable functions \(\mathfrak{z} : \Omega \rightarrow \mathbb{Z}\) and \(\mathfrak{M}_{z} := \{\mathfrak{z} \in \mathfrak{M} : \mathfrak{z}(\omega) \in Z(\omega) \subset \mathbb{Z}, \text{ for a.e. } \omega \in \Omega\}\). Let \(f : \mathbb{Z} \times \Omega \rightarrow \mathbb{R}\) be a Carathéodory function. Suppose that either \(\mathbb{E}\left(\inf_{z \in Z(\omega)} f(z,\omega)\right) < \infty\) or \(\mathbb{E}\left(-\inf_{z \in Z(\omega)} f(z,\omega)\right) < \infty\), where \((a)_{+} = \max(0,a)\). Then

\[
\mathbb{E}\left[\inf_{z \in Z(\omega)} f(z,\omega)\right] = \inf_{\mathfrak{z} \in \mathfrak{M}_{z}} \mathbb{E}[F_{\mathfrak{z}}],
\]

where \(F_{\mathfrak{z}}(\omega) := f(\mathfrak{z}(\omega),\omega)\).

The main difference with existing results in the literature is that here the infinite dimensionality of variable \(z\) poses more rigorous requirements on the measurability. For this, we exploit some fundamental results about measurability of random functions in infinite-dimensional space from monograph [3]. Another main difference is that here we consider \(Z(\omega)\), which is a random set of functions in the space \(\mathbb{Z}\) rather than a deterministic set of functions as in [45, Proposition 5]), [43, Theorem 14.60], or [49, Proposition 6.37, Theorem 7.80]. Because of the differences, we include a proof in EC.2.1 for completeness.

With the new version of the principle of interchangeability, we are able to address the interchangeability in the expected utility case. The next lemma states this.

**Lemma 2.** Let \(\mathcal{U} := \{u \in \mathcal{L}^{p}(\mathbb{R} \rightarrow \mathbb{R}) \mid u\text{ is a bounded and continuous function}\}\) and \(\mathcal{U}(\tau)\) be a nonempty subset of \(\mathcal{U}\). Let \(\mathfrak{M}_{\mathcal{U}} := \{u \in \mathcal{L}^{p}(\mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}) \mid u(\cdot,\tau) \in \mathcal{U}(\tau), \text{ for any } \tau \in \mathbb{R}^{d}\}\), where \(\mathcal{L}^{p}(\mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R})\) denotes the set of all state-dependent Lebesgue integrable utility functions \(u(\cdot,\tau)\). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with sigma algebra \(\mathcal{F}\) and probability measure \(\mathbb{P}\). Let \(\eta : \Omega \rightarrow \mathbb{R}\) be a random variable representing reward and \(\xi : \Omega \rightarrow \mathbb{R}^{d}\) be a random vector representing state. Then

\[
\inf_{u \in \mathfrak{M}_{\mathcal{U}}} \mathbb{E}[u(\eta,\xi)] = \mathbb{E}\left[\inf_{u \in \mathcal{U}(\xi)} \mathbb{E}[u(\eta) | \mathcal{F}_{\xi}]\right],
\]

where \(\mathcal{F}_{\xi}\) is the minimal sub-sigma algebra of \(\mathcal{F}\) to which \(\xi\) is adapted.

We give an explanation about the relation (14). Observe first that \(\mathfrak{M}_{\mathcal{U}}\) is a set of deterministic utility functions in \(\mathcal{L}^{p}(\mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R})\) such that \(u(\cdot,\tau) \in \mathcal{U}(\tau)\) for any \(\tau \in \mathbb{R}^{d}\). The left-hand side of equation (14) denotes the worst-case expected utility value for a given pair of reward function \(\eta\) and state \(\xi\) when \(u\) is restricted to set \(\mathfrak{M}_{\mathcal{U}}\). The right-hand side of (14) is the expectation of the worst-case expected utility value of \(\eta\) conditional on \(\mathcal{F}_{\xi}\) when the utility function is taken from \(\mathcal{U}(\xi)\). Here the set \(\mathcal{U}(\xi)\) depends on the state \(\xi\). The difference between \(\mathfrak{M}_{\mathcal{U}}\) and \(\mathcal{U}(\xi)\) is that
the former stipulates a set-valued mapping from \( \mathbb{R}^d \) to a set of utility functions with the specific structure \((u(\cdot, \tau)) \in \mathcal{U}(\tau)\) for any state \(\tau\) whereas the latter is the image of the set-valued mapping when \(\tau = \xi\). We refer readers to [EC.2.2] for the details of the proof.

With Lemma 2, we are ready to deliver the rectangularity of the ambiguity set introduced in Definition 1 in the next proposition.

**Proposition 1.** Let \(\mathcal{U}\) be defined as in Definition 1. Then \(\mathcal{U}\) holds.

In problem (8), at each stage, the utility function is taken in the worst-case sense from a random set depending on historical information. By Proposition 1 problem (8) can be rewritten as

\[
\max_{x_{[T]}} \inf_{u_t \in U_t} \mathbb{E} \left[ u_1(h_1(x_1, \xi_1)) + \inf_{u_2 \in U_2(\xi_{[1]})} \mathbb{E}_{\mathcal{F}_1} \left[ u_2(h_2(x_2(\xi_1), \xi_2)) + \cdots \right. \right. \\
+ \left. \left. \inf_{u_T \in U_T(\xi_{[T-1]})} \mathbb{E}_{\mathcal{F}_{T-1}} \left[ u_T(h_T(x_T(\xi_{[T-1]}), \xi_T)) \right] \cdots \right]\right]
\]

(15)

s.t. \(x_1 \in \mathcal{X}_1, x_t(\xi_{[t-1]}) \in \mathcal{X}_t(x_{[t-1]}(\xi_{[t-2]}), \xi_{[t-1]}), t = 2, \ldots, T\).

Here, \(U_t = U_t(\xi_{[t]}\) relies only on \(\xi_{[t]}\) and thus is deterministic.

The reformulations from (8) to (15) rely on the interchangeability between operation \(\inf_{u_t \in U_t}\) and the expectation \(\mathbb{E}\), \(t = 2, \ldots, T\). However, the \(\inf_{u_t \in U_t(\xi_{[t-1]})}\) as the worst-case utility function \(u_t\) and the preference ambiguity set \(\mathcal{U}_t(\xi_{[t-1]}\) are determined by information \(\mathcal{F}_{t-1}\). At each stage, we would meet a single period preference robust optimization problem which can be viewed as the well-studied static PRO models.

**3.2.2. Time consistency of (MS-PRO-SD)** From (15), we can see that the multistage preference robust utility function can be described in a nested form. Analogous to the multistage risk aversion models [11, 44] and multistage distributionally robust optimization models [48], the nested form guarantees the time consistency of the optimal dynamic policy of problem (15), i.e., it can be solved in a recursive dynamic programming procedure.

**Theorem 1.** Let \(\mathcal{U}_t(\xi_{[t-1]}), t = 2, \ldots, T\), and \(\mathcal{U}\) be defined as those in Definition 1. Assume: (a) for \(t = 2, \ldots, T\), the utility functions in \(\mathcal{U}_t(\xi_{[t-1]}\) are Lipschitz continuous with modulus being bounded by \(\kappa(\xi_{[t-1]}\) and \(\mathcal{U}_t(\xi_{[t-1]}\) is a compact set for any \(\xi_{[t-1]}\); (b) the reward function \(h_t : \mathbb{R}^{n_t} \times \mathbb{R}^{d_t} \to \mathbb{R}\) is Lipschitz continuous in \(x_t\) with modulus \(\sigma_t\) where \(\mathbb{E}_{\mathcal{F}_{t-1}}[\sigma_t] < +\infty\), for \(t = 1, \ldots, T\); (c) for \(t = 2, \ldots, T\), the feasible set \(\mathcal{X}_t(x_{[t-1]}, \xi_{[t-1]}\) is compact for any fixed \(x_{[t-1]}\) and \(\xi_{[t-1]}\) and as set-value mapping of \(x_{[t-1]}\), \(\mathcal{X}_t(\cdot, \xi_{[t-1]}\) is Lipschitz continuous. Then the (MS-PRO-SD) problem has the following dynamic programming reformulation:

\[
V_t(x_{[t-1]}, \xi_{[t-1]}) = \max_{x_t \in \mathcal{X}_t(x_{[t-1]}, \xi_{[t-1]})} \inf_{u_t \in U_t(\xi_{[t-1]})} \mathbb{E}_{\mathcal{F}_{t-1}} \left[ u_t(h_t(x_t, \xi_t)) + V_{t+1}(x_{[t]}, \xi_{[t]}\right]
\]

(16)

for \(t = 1, \ldots, T\), where \(V_{T+1}(\cdot, \cdot) := 0\), and \(V_1\) coincides with the optimal value of problem (MS-PRO-SD). The optimal policy of (MS-PRO-SD) is time consistent.
The proof is given in EC.2.4. In some applications, construction of the scenario (historical path) dependent preference ambiguity set $U_t(\xi_{t-1})$ is a bit complicated for practical use. There are potentially two ways to simplify. One is to consider the Markovian preference ambiguity set $U_t(\xi_{t-1})$ which relies only on the randomness at current stage $\xi_{t-1}$. The other is to consider a discrete approximation of the random process $\xi_{[t-1]}$.

**Remark 1.** It is worth noting that, we only need to interchange the order of the conditional expectation with the infimum operator as the considered utility function is additive in both probability and temporal dimension. For some utility considering nonlinear elasticity of intertemporal substitution, such as recursive or temporal utility functions, Lemma 1 is not enough to guarantee time consistency as we need a stronger version which studies the interchangeability between the infimum operator and both conditional expectation operator and utility functions in previous stages.

### 4. Construction of the ambiguity set

The structure of the ambiguity set of utility functions is determined by available information on the DM's utility preferences at each stage. Here we follow two approaches which are widely used in the literature of PRO models: pairwise comparison [1, 24, 27] and nominal utility approach [28, 58, 59]. The former elicits DM’s preferences via pairwise comparison questionnaires such as a lottery vs a deterministic gain/loss and translates the preferences (answers) into a characterization/specification of the true utility function, whereas the latter constructs an ambiguity set of utility functions in a neighborhood of a plausible nominal utility function.

To simplify the discussion, here we restrict the domain of utility functions to $[a, b]$ which means the range of reward function $h_t(x_t, \xi_t)$ falls within the interval, and normalize the utility function with $u(a) = 0$ and $u(b) = 1$ for $t = 1, \cdots, T$. The normalization does not affect the utility preferences. Let $\mathcal{U}$ be the set of continuous and normalized non-decreasing utility functions in $L^p([a, b])$ with $u(a) = 0$, $u(b) = 1$, and $\mathcal{U}^c$ a subset where the utility functions are concave.

#### 4.1. Pairwise comparisons

We begin with the pairwise comparison approach which is based on Von Neumann-Morgenstern’s expected utility theory, that is, any preference between two random prospects by the DM can be represented by expected utility of the random prospects albeit such a utility is unknown. To narrow down the scope of the true utility function, one may design more pairwise comparison questionnaires and ask the DM to make a choice on each pair of them, see Armbruster and Delage [1] for details.

In a dynamic decision making process, the DM’s preference depends on not only the stage she/he is standing, but also the historical path. The latter is particularly important because the
DM’s preference may be affected by the current environment. For instance, an investor in a bull market may prefer high growth stocks with higher tolerance to volatility, while in a bear market, she/he may prefer less volatile stocks even with a lower return rate. This means her/his answer to the same questionnaires may be affected by her/his risk attitude under different macro-market conditions. This motivates us to introduce ambiguity set of state-dependent utility functions in Definition[1] by setting

$$U^c_t(\xi_{t-1}) := \left\{ u \in \mathbb{W}^c \mid \begin{array}{l}
z_k(\xi_{t-1}) E\left[ u(W_k) \mid \xi_{t-1} \right] \geq z_k(\xi_{t-1}) E\left[ u(Y_k) \mid \xi_{t-1} \right], \\
\text{Lin}(u) \leq L(\xi_{t-1})
\end{array} \right\}$$

(17)

where \{(W_k,Y_k), k = 1, \cdots, K\} is a set of prospects for pairwise comparison. Note that this set may be fixed or evolved over the process, which means the questionnaires used in stage \(t - 1\) will be used in stage \(t\), but the DM might have different answers due to the change of stage/state. Here \(z_k(\xi_{t-1}) \in \{+1, -1, 0\}\) is used to indicate the choice of the decision-maker at stage \(t\). If the DM prefers \(W_k\) to \(Y_k\), then \(z_k(\xi_{t-1}) = 1\), otherwise \(z_k(\xi_{t-1}) = -1\). In the case of no preference, \(z_k(\xi_{t-1}) = 0\). Consequently the ambiguity of the utility functions is time-dependent as opposed to static in one stage PRO models. Under Assumption[1] \(U^c_t(\xi_{t-1}) \neq \emptyset\). \(\text{Lin}(u) \leq L(\xi_{t-1})\) means that \(u\) is Lipschitz continuous with modulus bounded by \(L(\xi_{t-1})\), \(\mathbb{W}^c\) involves the concavity constraint of \(u\). It means that the DM is risk averse at all stages over the time horizon. Obviously \(U^c_t(\xi_{t-1})\) is a convex set.

Armbruster and Delage[1] show that a static PRO problem with pairwise comparison ambiguity set can be reformulated as an LP, when the supports of \(W_k\) and \(Y_k\) are finite. In Section[EC.4.1] we will derive a tractable LP reformulation of multistage PRO problem with the ambiguity set defined as \(U^c_t(\xi_{t-1})\). Note that the ambiguity set constructed as such in (17) has some limitations: the utility function is independent of past decisions or the current financial position (e.g. cumulative wealth up to date). The reformulation under the scenario tree will be much more complex if the ambiguity set is decision-dependent or wealth-dependent (when \(z_k(s)\) in [EC.4.1] is replaced by \(z_k,s(x(s^-))\), it contains bi-linear terms and 0-1 valued non-smooth functions).

Let

$$\mathcal{S} := \{a\} \cup \bigcup_{k=1}^{K} (\text{supp}(Y_k) \cup \text{supp}(W_k)) \cup \{b\}$$

and \(N := |\mathcal{S}|\) denotes the cardinality of set \(\mathcal{S}\), let \(\{y_j\}_{j=1, \cdots, N}\) be the ordered sequence of points in \(\mathcal{S}\) with fixed \(y_1 = a, y_N = b\). In the forthcoming discussions, we will use utility values at \(\mathcal{S}\) to characterize the property of the true unknown utility function. The details are given in EC.4.1 and EC.4.3.
4.2. ζ-ball approach

In some decision making problems, a DM may be able to “roughly” identify a nominal utility function which captures most of the DM’s preferences either elicited through empirical data, or based on subjective judgement or from partially elicited preference information, but there is incomplete information to tell whether the nominal utility is the true utility. Under such a circumstance, it might be sensible to consider a set of utility functions near the nominal utility and base the optimal decision on the worst-case utility function from the set. We call this a nominal approach.

We begin by defining a kind of semi-distance between any two utility functions. Let \( G \) be a set of measurable functions defined over \([a,b]\). For \( u, v \in U \), define the semi-distance between \( u \) and \( v \) by
\[
\|u - v\|_G := \sup_{g \in G} \left| \int_a^b g(z) du(z) - \int_a^b g(z) dv(z) \right|,
\]
where the integrals are in the sense of Lebesgue-Stieltjes integration, \( g \) might be viewed as a test function and \( \|u - v\|_G = 0 \) means that for all of the test functions in \( G \), there is no difference between \( u \) and \( v \) albeit that \( u \neq v \). In the case that the utility functions in \( U \) are normalized with \( u(a) = 0, u(b) = 1 \), \( \|u - v\|_G \) resembles the pseudo-metric of \( \zeta \)-structure in probability theory. In this paper, we are interested in two cases:

\[ G = G_L := \{ g : [a,b] \to \mathbb{R} | g \text{ is Lipschitz continuous with modulus bounded by 1} \} \quad (18) \]

and

\[ G = G_I := \{ g := \mathbb{1}_{(a,z]}(\cdot) | \text{where } \mathbb{1}_{(a,z]}(s) := 1 \text{ if } s \in (a,z] \text{ and 0 otherwise} \}. \quad (19) \]

The former corresponds to the Kantorovich metric, denoted by \( d_{KL}(u,v) \), and the latter corresponds to the uniform Kolmogorov metric. With the definition of the \( \zeta \)-metric, we are ready to introduce the definition of \( \zeta \)-ball in the space of the utility functions \( U \). We begin with the static case.

**Definition 4 (Static ζ-ball of utility functions).** Let \( U \) be the set of all continuous, non-decreasing utility functions defined over interval \([a,b]\), \( u(a) = 0, u(b) = 1 \) for all \( u \in U \). For a fixed \( \tilde{u} \in U \), the \( \zeta \)-ball of utility functions in \( U \) centered at \( \tilde{u} \) with radius \( r \) under metric \( d_G \) is defined as:

\[
B(\tilde{u}, r) := \{ u \in U | d_G(u, \tilde{u}) \leq r \}.
\]

(20)

In this paper, our focus is on the construction of an ambiguity set of a sequence of state-dependent utility functions specified in Definition 1.

**Definition 5 (Dynamic ζ-ball based ambiguity set of utility functions).** Consider the ambiguity set in (7). For given nominal state-dependent utility function \( \tilde{u}_t(\cdot; \xi_{t-1}) \in U \), define for all \( \xi_{t-1} \),

\[
U^0_t(\xi_{t-1}) := \left\{ u \in U | u \in B(\tilde{u}_t(\cdot; \xi_{t-1}), r_t(\xi_{t-1})), \text{Lip}(u) \leq L(\xi_{t-1}) \right\}.
\]

(21)
In this formulation, $\mathcal{U}_t^p(\xi_{[t-1]})$ is determined by the center $\bar{u}_t(\cdot, \xi_{[t-1]})$, the radius $r_t(\xi_{[t-1]})$ and the pseudo-metric $d_{\mathcal{G}}$. The choice of functions in set $\mathcal{G}$ may depend on historical data $\xi_{[t-1]}$. The nominal utility function $\bar{u}_t(\cdot, \xi_{[t-1]})$ may be identified from empirical data, that is, the utility function is inferred from the DM’s past utility preferences and the feedback (represented by historical path $\xi_{[t-1]}$). As the time goes on, we can collect more data/information about the DM’s preferences and subsequently a more accurate nominal utility as well as a smaller radius.

**Proposition 2.** Let $\mathcal{U}_t^p(\xi_{[t-1]})$ and $\mathcal{U}_t^p(\xi_{[t-1]})$ be defined as in (17) and (21). Then the following assertions hold.

(i) For each fixed $\omega$, $\mathcal{U}_t^p(\xi_{[t-1]}(\omega))$ and $\mathcal{U}_t^p(\xi_{[t-1]}(\omega))$ are compact sets.

(ii) If $\bar{u}_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})$ and $L(\xi_{[t-1]})$ are continuous in $\xi_{[t-1]}$, then $\mathcal{U}_t^p(\xi_{[t-1]}(\cdot))$ and $\mathcal{U}_t^p(\xi_{[t-1]}(\cdot))$ are $\mathcal{F}_{t-1}$-measurable.

(iii) The ambiguity $\mathcal{U}$ constructed from $\mathcal{U}_t^p(\xi_{[t-1]})\ (\mathcal{U}_t^p(\xi_{[t-1]}))$ in the form of (11) satisfies the rectangularity (the conditions in Definition 7).

The next proposition quantifies the difference between two $\zeta$-balls of utility functions with different nominals and radii under the Hausdorff distance. For any two sets $U, V \subset \mathcal{G}$, define $\mathbb{D}(U, V; d_{\mathcal{G}}) := \sup_{u \in U} \inf_{v \in V} d_{\mathcal{G}}(u, v)$, which quantifies the deviation of $U$ from $V$ and $\mathbb{H}(U, V; d_{\mathcal{G}}) := \max \{ \mathbb{D}(U, V; d_{\mathcal{G}}), \mathbb{D}(V, U; d_{\mathcal{G}}) \}$, the Hausdorff distance between the two sets under the pseudo-metric.

**Proposition 3.** Let $u, v \in \mathcal{G}$ and $r_1, r_2 \in \mathbb{R}_+$. Then

$$\mathbb{H}(\mathbb{B}(u, r_1), \mathbb{B}(v, r_2); d_{\mathcal{G}}) \leq d_{\mathcal{G}}(u, v) + |r_2 - r_1|.$$  \hspace{1cm} (22)

In particular, if $u^*$ is the true utility function and $u_{\text{ref}}$ is a nominal utility function, then

$$\mathbb{H}(u^*, \mathbb{B}(u_{\text{ref}}, r); d_{\mathcal{G}}) \leq d_{\mathcal{G}}(u^*, u_{\text{ref}}) + r.$$  

Inequality (22) means that the Hausdorff distance of two balls is bounded by the distance of their centers plus the difference of the radii. With the proposition, we are ready to present a multistage PRO model with the ambiguity set defined via (15) and (21) as follows:

$$\max \inf_{\mathbb{X}[T]} \mathbb{E} \left[ u_1(h_1(x_1, \xi_1)) + \inf_{u_2 \in \mathcal{U}_1(\xi_1)} \mathbb{E}_{x_1} \left[ u_2(h_2(x_2(\xi_1), \xi_2)) + \cdots \right] \right]$$

$$+ \inf_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{x_{T-1}} \left[ u_T(h_T(x_T(\xi_{[T-1]}), \xi_T)) \right] \right) \cdots \right] \right]$$

s.t. $x_1 \in \mathcal{X}_1, x_t(\xi_{[t-1]}) \in \mathcal{X}_t(\xi_{[t-1]}(\xi_{[t-2]}), \xi_{[t-1]}), t = 2, \ldots, T.$  \hspace{1cm} (23)
Here, $B(u_t, r_t)$ in $U_t^B$ relies only on deterministic nominal utility $u_t$ and radius $r_t$. By Theorem 4.2.1 can be computed by the following dynamic programming equation,

$$V_t(x_{[t-1]}, \xi_{[t-1]}) = \max_{x_t \in X_t(x_{[t-1]}, \xi_{[t-1]})} \inf_{y_t \in U_t^B(\xi_{[t-1]})} \mathbb{E}_{x_{t+1}} \left[ u_t(x_t, \xi_t) + V_{t+1}(x_{[t]}, \xi_{[t]}) \right] . \quad (24)$$

From computational point of view, problem (24) is still not easy to solve because the inner minimization problem is infinite dimensional. This motivates us to develop an approximation scheme where the ball of utility functions $B(u_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]}))$ is approximated by a ball of piecewise linear utility functions.

4.2.1. Piecewise-linear utility functions  Let $y_1 < \cdots < y_N$ be an ordered sequence of points in $[a, b]$ with $y_1 = a$ and $y_N = b$ and $Y := \{y_1, \cdots, y_N\}$. Let $\mathcal{U}_N$ be a class of continuous, non-decreasing, piecewise linear functions defined over the interval $[y_1, y_N]$ with breakpoints on $Y$. For a given $v \in \mathcal{U}_N$, let

$$B_N(v, r) := \{u \in \mathcal{U}_N \mid d_{\mathcal{U}}(u, v) \leq r\} \quad (25)$$

and

$$U_t^{B_N}(\xi_{[t-1]}) := \left\{ u \in \mathcal{U}_N \left| u \in B_N(u_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})) \right. \right\}$$

for a given nominal utility function $u_t(\cdot, \xi_{[t-1]}) \in \mathcal{U}_N$. We propose to solve (24) by solving

$$\tilde{V}_t(x_{[t-1]}, \xi_{[t-1]}) = \max_{x_t \in X_t(x_{[t-1]}, \xi_{[t-1]})} \inf_{u_t \in U_t^{B_N}(\xi_{[t-1]})} \mathbb{E}_{x_{t+1}} \left[ u_t(x_t, \xi_t) + \tilde{V}_{t+1}(x_{[t]}, \xi_{[t]}) \right] . \quad (26)$$

To justify this, we derive the error between $V_t(x_{[t-1]}, \xi_{[t-1]})$ and $\tilde{V}_t(x_{[t-1]}, \xi_{[t-1]})$.

**Remark 2.** By restricting the nominal utility function to be piecewise linear, it is easier to estimate the function from the customer/investor in practice. We preset two endpoints $a, b$ and some values in $[a, b]$, and then let the customer/investor score on these values under different scenarios. By collecting and normalizing the scores in different scenarios and linking the utility scores by a piecewise linear function, we obtain a normalized nominal utility function in each scenario. The radius describes the error in the scoring process which depends on the credibility of the scores.

Differing from $B(u, r)$ defined in (20), the $\zeta$-ball consists of piecewise linear utility functions only. In what follows, we quantify the difference between $B(u, r)$ and $B_N(v, r)$ under the $\zeta$-metric so that we will be able to assess the impact when we replace the former with the latter in the utility preference robust optimization model.
In what follows, we derive tractable formulation for computing 
where $\beta_{a,b}$ of all Lipschitz continuous functions defined on 

$$L_\beta = \mathcal{G}_L,$$ or $\mathcal{G}_I$. Then

$$\mathbb{H}(\mathbb{B}(u,r), \mathbb{B}(v,r); d_{\mathcal{G}}) \leq d_{\mathcal{G}}(u,v) + 4\max(2, L)\beta_N. \quad (27)$$

In the case when $u = v_N$ is a projection of $v$ on $\mathcal{U}_N$,

$$\mathbb{H}(\mathbb{B}(v_N,r), \mathbb{B}(v,r); d_{\mathcal{G}}) \leq 6\max(2, L)\beta_N, \quad (28)$$

where $L$ and $\beta_N$ are defined as in Proposition [EC.1].

We are now ready to present the main result of this section.

**Theorem 2 (Error bound).** Let $V_t(x_{[t-1]}, \xi_{[t-1]})$ and $\tilde{V}_t(x_{[t-1]}, \xi_{[t-1]})$ be defined as in (24) and (26), respectively. Let $\{\tilde{u}_i(\cdot, \xi_{[t-1]})\}$ be a sequence of nominal utility functions and $\{\tilde{u}_i^N(\cdot, \xi_{[t-1]})\}$ its piecewise linear approximations. Let

$$\beta_N(\xi_{[t-1]}) := \max_{i=2,\ldots,N} (y_i - y_{i-1}),$$

where the breakpoints are chosen according to historical data $\xi_{[t-1]}$. Assume that $\tilde{u}_i(\cdot, \xi_{[t-1]})$ is Lipschitz continuous with modulus $L(\xi_{[t-1]})$. Then

$$\left|V_t(x_{[t-1]}, \xi_{[t-1]}) - \tilde{V}_t(x_{[t-1]}, \xi_{[t-1]})\right| \leq \sum_{s=t}^{T} 6\mathbb{E} \left[\max(2, L(\xi_{[s-1]}))\beta_N(\xi_{[s-1]}) \mid \mathcal{F}_{t-1}\right] \quad (29)$$

for $t = 1, \ldots, T$. In the case when $\beta_N(\xi_{[s-1]})$ and $L(\xi_{[s-1]})$ are independent of states,

$$\left|V_t(x_{[t-1]}, \xi_{[t-1]}) - \tilde{V}_t(x_{[t-1]}, \xi_{[t-1]})\right| \leq 6(T-t+1)\max(2, L)\beta_N.$$

**4.2.2. Kantorovich ball** Let $\tilde{u} \in \mathcal{U}_N$. We consider a ball in the space of $\mathcal{U}_N$ with the Kantorovich metric

$$\mathbb{B}_K(\tilde{u}, r) = \{u \in \mathcal{U}_N \mid d_K(u, \tilde{u}) \leq r\}. \quad (30)$$

In what follows, we derive tractable formulation for computing $d_K(u, \tilde{u})$. Let $g \in \mathcal{G}$ where $\mathcal{G}$ consists of all Lipschitz continuous functions defined on $[a, b]$ with modulus bounded by 1. By definition

$$\int_a^b g(t)du(t) = \sum_{j=2}^{N} \beta_j \int_{y_{j-1}}^{y_j} g(t)dt,$$

where $\beta_j$ denotes the slope of $u$ at interval $[y_{j-1}, y_j]$. Since for each $g \in \mathcal{G}$, $-g \in \mathcal{G}$,

$$d_K(u, \tilde{u}) = \sup_{g \in \mathcal{G}} \sum_{j=2}^{N} (\beta_j - \tilde{\beta}_j) \int_{y_{j-1}}^{y_j} g(t)dt,$$
where \( \tilde{\beta}_j \) denotes the slope of \( \tilde{u} \) at interval \([y_{j-1},y_j]\). Note that in this formulation, \( d_{IK}(u,\tilde{u}) \) depends on the slopes of \( u,\tilde{u} \) rather than their function values, \( \sum_{j=2}^{N} \beta_j(y_j-y_{j-1}) = u(b)-u(a) = 1 \)
\( \sum_{j=2}^{N} \tilde{\beta}_j(y_j-y_{j-1}) = u(b)-u(a) = 1 \). Let \( w_j := \int_{y_{j-1}}^{y_j} g(t)dt \) and \( z_j = g(y_j), j = 2, \ldots, N \). Since \( |g(y) - g(y_{j-1})| \leq y - y_{j-1} \) for all \( y \in [y_{j-1},y_j] \), we have

\[
z_{j-1}(y_j - y_{j-1}) - \frac{1}{2}(y_j - y_{j-1})^2 \leq w_j \leq z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2
\]
for \( j = 2, \ldots, N \). Likewise, since \( |g(y_{j}) - g(y)| \leq y_j - y \) for all \( y \in [y_{j-1},y_j] \), we have

\[
z_j(y_j - y_{j-1}) - \frac{1}{2}(y_j - y_{j-1})^2 \leq w_j \leq z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2
\]
for \( j = 2, \ldots, N \). Consequently

\[
d_{IK}(u,\tilde{u}) = \max_{w_2, \ldots, w_N, z_1, \ldots, z_N} \sum_{j=2}^{N} (\beta_j - \tilde{\beta}_j)w_j
\]

\[
\begin{align*}
\text{s.t.} \quad & w_j \leq z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2, \quad j = 2, \ldots, N, \quad (31b) \\
& -w_j \leq -z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2, \quad j = 2, \ldots, N, \quad (31c) \\
& w_j \leq z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2, \quad j = 2, \ldots, N, \quad (31d) \\
& -w_j \leq -z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2, \quad j = 2, \ldots, N. \quad (31e)
\end{align*}
\]

Problem (31) is a linear program. Using Lagrange duality, we can reformulate it as

\[
\min_{\lambda, \mu, \rho, \phi} \frac{1}{2} \sum_{j=2}^{N} (\lambda_j + \mu_j + \rho_j + \phi_j)(y_j - y_{j-1})^2
\]

\[
\begin{align*}
\text{s.t.} \quad & \tilde{\beta}_j - \beta_j + \lambda_j - \mu_j + \rho_j - \phi_j = 0, \quad j = 2, \ldots, N, \quad (32b) \\
& (\mu_2 - \lambda_2)(y_2 - y_1) = 0, \quad (32c) \\
& (\mu_{j+1} - \lambda_{j+1})(y_{j+1} - y_j) + (\rho_j - \phi_j)(y_j - y_{j-1}) = 0, \quad j = 2, \ldots, N - 1, \quad (32d) \\
& (\rho_N - \phi_N)(y_N - y_{N-1}) = 0, \quad (32e) \\
& \mu_j, \lambda_j, \rho_j, \phi_j \geq 0, \quad j = 2, \ldots, N. \quad (32f)
\end{align*}
\]

The discussion above shows that we can obtain the Kantorovich distance \( d_{IK}(u,\tilde{u}) \) by solving a linear program. This will facilitate us to derive tractable formulations for solving problem (26) by imbedding (32) into the inner minimization problem.
4.2.3. Tractable formulation of dynamic program \[26\] We can easily incorporate the tractable formulations of the Kantorovich ball into the dynamic programming equation (26) and develop tractable formulations for the latter. To comply with the setting in Theorem 2, we need to impose Lipschitz continuity on the nominal utility function \(\hat{u}_t(\cdot, \xi_{t-1})\) and its derivative \(\hat{u}'_t(\cdot, \xi_{t-1})\) as well as the concavity of the utility function.

**Theorem 3.** Consider
\[
\mathcal{U}_t^K(\xi_{t-1}) := \left\{ u \in \mathcal{W}^c \middle| \text{Lip}(u) \leq L(\xi_{t-1}) \right\} \quad (33)
\]
for all \(\xi_{t-1}\). Suppose that the optimal value function at period \(t+1\) is \(\bar{V}_{t+1}(x_{t}, \xi_{t})\). Given historical data \(\xi_{t-1}\) and historical decision \(x_{t-1}\), \(\xi_t\) is discretely distributed with \(S\) scenarios \(\xi_{i}^{1}, \ldots, \xi_{i}^{S}\) and
appearing probability \(P(\xi_t = \xi_t^{i} | \xi_{t-1}^{i})\), \(i = 1, \ldots, S\), then the optimal decision \(x_t\) at stage \(t\) can be derived by solving the following programming problem,

\[
\begin{align*}
\max & \quad \theta_{N-1} + \sum_{i=1}^{S} \left( \mu_{t,N} + \mathbb{P}(\xi_t = \xi_t^{i} | \xi_{t-1}^{i}) \bar{V}_{t+1}(x_{t}, [\xi_{t-1}^{i}, \xi_{t}^{i}]) \right) - L(\xi_{t-1}) \sum_{j=1}^{N-1} \eta_j \\
\text{s.t.} \quad & \sum_{j=1}^{N} y_{j,\mu_{i,j}} \leq \mathbb{P}(\xi_t = \xi_t^{i} | \xi_{t-1}^{i}) h_t(x_{t}, \xi_t^{i}), \quad i = 1, \ldots, S, \\
& \mathbb{P}(\xi_t = \xi_t^{i} | \xi_{t-1}^{i}) - \sum_{j=1}^{N} \mu_{i,j} = 0, \quad i = 1, \ldots, S, \\
& \theta_{j-1} y_{j-1} - \theta_{j-1} y_{j} + v_{j-2} (y_{j-1} - y_{j-2}) + w_{j} + \eta_{j-1} + \tau_{j-1} - \tau_{j-2} + \sigma_{j-2} - \sigma_{j-1} \geq 0, \quad j = 3, \ldots, N-1, \\
& \theta_{1} y_{1} - \theta_{1} y_{2} + w_{2} + \eta_{1} + \tau_{1} - \sigma_{1} \geq 0, \\
& \theta_{N-1} y_{N-1} - \theta_{N-1} y_{N} + v_{N-2} (y_{N-1} - y_{N-2}) + w_{N} + \eta_{N-1} - \tau_{N-2} + \sigma_{N-2} \geq 0, \\
& \theta_{j-1} - \theta_{j} + \sum_{i=1}^{S} \mu_{i,j} - v_{j-1} + v_{j} = 0, \quad j = 2, \ldots, N-2 \\
& \theta_{N-2} - \theta_{N-1} + \sum_{i=1}^{S} \mu_{i,N-1} - v_{N-2} = 0, \\
& w_{j} \leq z_{j-1} (y_{j} - y_{j-1}) + \frac{1}{2} (y_{j} - y_{j-1})^2 \varsigma, \quad j = 2, \ldots, N, \\
& -w_{j} \leq -z_{j-1} (y_{j} - y_{j-1}) + \frac{1}{2} (y_{j} - y_{j-1})^2 \varsigma, \quad j = 2, \ldots, N, \\
& w_{j} \leq z_{j} (y_{j} - y_{j-1}) + \frac{1}{2} (y_{j} - y_{j-1})^2 \varsigma, \quad j = 2, \ldots, N, \\
& -w_{j} \leq -z_{j} (y_{j} - y_{j-1}) + \frac{1}{2} (y_{j} - y_{j-1})^2 \varsigma, \quad j = 2, \ldots, N, \\
\end{align*}
\]
\[ x_t \in \mathcal{X}_t \left( x_{t-1}, \xi_{t-1} \right), \quad \theta \in \mathbb{R}^{N-1}, v \in \mathbb{R}^{N-2}, \eta \in \mathbb{R}^{N-1}_+, \tau \in \mathbb{R}^{N-2}_+, \sigma \in \mathbb{R}^{N-2} \]  \tag{34o}

\[ \mu \in \mathbb{R}^{S \times N}_+, \varsigma \in \mathbb{R}_+, w \in \mathbb{R}^{N-1}_+, z \in \mathbb{R}^N \]  \tag{34p}

where the optimal value is \[ \tilde{V}_t \left( x_{t-1}, \xi_{t-1} \right) \].

Theorem 4 establishes a connection between the optimal value functions at the adjacent stages by solving an optimization problem. When the optimal value function at period \( t + 1 \) is concave, the reward function \( h_t(\cdot, \xi) \) is concave, the feasible set \( \mathcal{X}_t \left( x_{t-1}, \xi_{t-1} \right) \) is compact and convex, the optimization problem (34) becomes a convex programming problem which can be solved efficiently by the interior point method.

5. Computational schemes

In this section, we discuss computational schemes for solving the time consistent MS-PRO model \( \mathcal{S} \) and the time inconsistent MS-PRO model \( \mathcal{I} \). We proceed with two kinds of approaches: the scenario tree method and dynamic programming algorithms including SDDP/NBD methods. The scenario tree approach can be used to solve both \( \mathcal{S} \) and \( \mathcal{I} \) whereas dynamic programming type algorithms can only be applied to solve \( \mathcal{S} \) on the basis of \( \mathcal{I} \).

5.1. Scenario tree method

Let \( \Xi \) be a discrete support set and \( \{ \xi \}_{i=1}^T \) a scenario tree. Denote by \( B \) the set of all nodes in the scenario tree, \( S^- \) the set of all non-leaf nodes, and \( S(t) \) the set of nodes at stage \( t \). Denote by \( s^- \) the father node of \( s \), \( s^+ \) the set of son nodes of \( s \), \( \xi[s] \) the historical scenario from the root node to node \( s \). Denote by \( t(s) \) the stage of node \( s \), and by \( p_s \geq 0 \) the appearing probability of node \( s \). Denote the decision at node \( s \) by \( x(s) := x(\xi[s]) \) and the historical decision from the root node to node \( s \) by \( x[s] \). Notice that the decision at node \( s \) (at stage \( t(s) \)) is made according to the future realizations on the father node \( s^- \) at stage \( t(s) - 1 \). Thus, the realization of the reward function at node \( s \) is \( h_{t(s)}(x(s^-), \xi(s)) \). For the state-dependent problem, the ambiguity set of the utility functions upon historical samples at node \( s \) is denoted by \( U(s) := U_{t(s)}(\xi[s]) \). For the state-independent problem, the ambiguity set of the utility functions at stage \( t \) is \( U_t \), which is deterministic (independent of historical samples).

**Time consistent (MS-PRO-SD).** Problem \( \mathcal{S} \) can be reformulated as the following min-max problem:

\[
\max_{\{x(s), s \in S^-\}} \sum_{s \in S^-} p_s \inf_{u(s) \in U(s)} \left( \sum_{i \in \mathbb{I}^+} \mathbb{E}_p u_s \left( h_{t(i)}(x(s), \xi(i)) \right) \right) \tag{35}
\]

s.t. \( x(1) \in \mathcal{X}_1, x(s) \in \mathcal{X}_{t(s)}(x[s^-], \xi[s]), \forall s \in S^- \setminus \{1\} \),

where the inner minimization is to calculate the worst-case conditional expected utility value over the son nodes of \( s \) across all scenarios whereas the outer maximization is w.r.t. the optimal decision.
at node $s$. The objective function is an average of all worst-case utility values at all non-leaf nodes of the tree. We can reformulate it to indicate more clearly stages and nodes at each stage:

$$\sum_{s \in S^+} p_s \inf_{u_s \in U(s)} \left( \sum_{i \in S^+} \frac{p_i}{p_s} u_s(h_{t(i)}(x(s), \xi(i))) \right) = \sum_{i=1}^{T-1} \sum_{s \in S(t)} p_s \inf_{u_s \in U(s)} \left( \sum_{i \in S^+} \frac{p_i}{p_s} u_s(h_{t+1}(x(s), \xi(i))) \right).$$

**Time inconsistent (MS-PRO-SID).** Problem (9) can be reformulated as the following min-max problem:

$$\max_{\{x(s), s \in S^\ast\}} \sum_{t=1}^{T-1} \inf_{u_t \in U_t} \left[ \sum_{s \in S(t)} p_s \left( \sum_{i \in S^+} \frac{p_i}{p_s} u_t(h_{t+1}(x(s), \xi(i))) \right) \right]$$

s.t. $x(1) \in X_1, x(s) \in X(t(s))(x[s^{-}], \xi[s]), \forall s \in S(t), t = 2, \ldots, T - 1.$

In both (35) and (36), the decisions $x(s)$ are node-dependent. The only difference is that, in (35), the ambiguity sets are node-wise and we find the worst-case utility at each node; whereas in (36), the ambiguity sets are stage-wise and we find the worst-case utility for all nodes at each stage. Further detailed reformulations depend on the structure of the scenario tree and the stage-wise ambiguity set $U(s)/U_t$. We refer readers to Appendix EC.3 for details.

### 5.2. Dynamic programming methods

The reformulation of (8) as (16) paves the way for us to apply the NBD and SDDP methods for solving the problem. The basic idea of the DP-type algorithms is to develop an approximation of the cost-to-go function $V_t(x_{[t-1]}, \xi_{[t-1]})$, use the optimal solution based on the approximate problem as an approximate optimal solution of (16) (and ultimately (8)) and improve the approximations over an iterative forward and backward process. To this end, we need to make the following standard assumption.

**Assumption 2.** Denote $\xi_t = (c_t, W_t, b_t, D_t)$. (a) The decision $x_t$ is $\xi_{[t-1]}$-dependent, (b) the constraints at recourse stages in the MS-PRO problem have a linear block-diagonal structure, i.e., only consecutive stages can be linked by linear constraints, i.e., $X_t = \{x_t \mid W_{t-1}(\xi_{[t-1]})x_t = b_{t-1}(\xi_{[t-1]}) - D_{t-1}(\xi_{[t-1]})x_{t-1}\}$, $W_{t-1}$ is invertible or fixed, $t = 2, \ldots, T$, (c) the reward functions are linear, i.e., $h_t(\xi_t, x_t) = c_t(\xi_t) \top x_t$.

Assumption 2 ensures concavity of $V_t(x_{[t-1]}, \xi_{[t-1]})$ (see (16)) in $x_{[t-1]}$ and $\xi_{[t-1]}$ for $t = T, \ldots, 2$. This enables us to construct piecewise linear approximations of $V_t(x_{[t-1]}, \xi_{[t-1]})$, which underlies NBD algorithm and SDDP algorithm, and guarantees the strong duality of the inner minimization problem of (16) and thus the final convergence of the algorithms. Here we give a sketch of the algorithmic structure and refer readers to Appendix EC.5 for details.

**Algorithm 1** Outline of NBD/SDDP algorithms
Input: A finite set of scenarios $\mathcal{K}$

while $i < N_{\text{max}}$ do

- for $k \in \mathcal{K}$, $t = 1, \ldots, T$ do (forward pass)
  - solve (16) with current piecewise linear approximation of $V_t$, denoted by $V^i_t$, and trial decision $x_{t-1}^{k,i}$ at stage $t-1$ to obtain trial decision $x_{t}^{k,i}$ at stage $t$. Calculate a lower bound of the optimal value.
  
- for $k \in \mathcal{K}$, $t = T, \ldots, 1$ do (backward pass)
  - solve (16) with updated $V_{t+1}$ and trial decision $x_{t}^{k,i}$ to obtain the optimal value of dual variables.
  
- update $V_{t}^i$ by adding a cut constructed with the optimal values of the dual variables. Calculate an upper bound of the optimal value.

- terminate when the gap between the upper and lower bounds falls within the prescribed precision.

There are two ways to proceed. One is to use a large scenario tree of the multistage decision making process in the sample space and then find historical path-dependent optimal solutions by solving the optimization problem (26) in Theorem 3 embedded into each node on the large scenario tree. This is known as the NBD algorithm. The other is to take some i.i.d. samples from all scenarios in the finite-support case or the continuous distribution in the infinite-support case in solving (26), which is known as SDDP algorithm. Here we adopt both and compare them with the scenario tree algorithm. We will report comparative results in the next section.

Convergence of the two algorithms are guaranteed under some standard conditions. For instance, when the MS-PRO problem has relatively complete recourse and the distribution of the process $\{\xi_t\}$ is known, we can show that the NBD algorithm converges to an optimal solution of MS-PRO-SD in finitely many iterations following a similar analysis to that of [7, 21]. If, in addition, $\xi_t$ is independent of the history $\xi_{[t-1]}$ of the process, then we may follow [47, 21] to show that the SDDP algorithm converges with probability 1 to an optimal policy of MS-PRO-SD in a finite number of iterations. We skip the details as these are not the main focus of this paper.

6. Numerical tests

To examine the performance of the proposed multistage MS-PRO-SD model (8) and MS-PRO-SID model (6), as well as numerical schemes, we carry out a number of numerical tests on a multistage investment-consumption problem on the basis of [16, 19] with state-dependent utility functions.

6.1. An investment-consumption problem

Consider an investor who plans to use her/his wealth to purchase crude oil and make oil products over $T$ periods. At the beginning of each time period, the investor has two options: (a) consume
all of the wealth for the purchase, and (b) consume part of it and invest the remaining wealth in n risky assets of a security market. The objective of the investor is to maximize the overall expected utility of the oil products consumption.

Let \( w_0 = 1 \) denote the normalized initial wealth and \( q_t \) denote the quantity of crude oil that the investor plans to buy at beginning of time period \( t \) at price \( p_{t-1} \) which is the oil price at the end of time period \( t-1 \) (alternatively, at the beginning of period \( t \)). The total cost from the purchase is \( q_t \cdot p_{t-1} \) and the remaining wealth is \( w_{t-1} - q_t p_{t-1} \), where \( w_{t-1} \) is the wealth at the end of period \( t-1 \). The remaining wealth is invested in \( n \) risky assets with a portfolio \( x_t \), where \( x^i_t \) is the wealth invested in the \( i \)-th asset, \( i = 1, \ldots, n \), whose random return rate, denoted by \( r^i_t \), is calculated period-wise, i.e., a $1 investment at the beginning of period \( t \) will generate $\( 1 + r^i_t \) at the end of the period. Thus, the wealth of the investor at the end of period \( t-1 \) is \( w_{t-1} = (1 + r^i_t) x^i_t \). This wealth is divided into the consumption \( q_t p_{t-1} \) and the further investment \( e^\top x_t \), i.e., \( w_{t-1} = q_t p_{t-1} + e^\top x_t \). A combination of the two equations gives rise to the following wealth balance equation

\[
e^\top x_t = (e + r^i_t) x^i_{t-1} - q_t p_{t-1}, \quad t = 2, \ldots, T - 1.
\]

At the initial period \( t = 1 \), we have \( e^\top x_1 = w_0 - q_1 p_0 \) and at the final period \( T \), the investor must consume all of the wealth on purchase of oil, thus \( (e + r^i_{T-1}) x^i_{T-1} = q_T p_{T-1} \).

The utility of the oil products is calculated at the end of each period as follows. We assume that all of the \( q_t \) barrels of oil purchased at the beginning of period \( t \) is used to produce \( g_t(q_t) \) quantities of the oil products by the end of period \( t \) with unit value \( d_t \). Thus the total value from the production is \( g_t(q_t) d_t \) and the period-wise utility value is \( u_t(g_t(q_t) d_t, h_{t-1}^T) \). Here the investor’s utility function depends on all the historical information \( h_{t-1}^T = \{p_0, \ldots, p_{t-1}, d_1, \ldots, d_{t-1}, r_1, \ldots, r_{t-1}\} \). Based on the discussions above, we formulate the multistage investment-consumption problem as

\[
\max_{x_{[1,T-1]}; q_{[1,T]}} \mathbb{E} \left[ u_1(g_1(q_1) d_1, h_0) + u_2(g_2(q_2) d_2, h_{1|1}) + \cdots + u_T(g_T(q_T) d_T, h_{T|T-1}) \right] \tag{37a}
\]

\[
s.t. \quad e^\top x_1 = w_0 - q_1 p_0, \quad x_1 \in \mathbb{R}^n_+, \quad q_1 \in \mathbb{R}^n_+, \tag{37b}
\]

\[
e^\top x_t = (e + r^i_{t-1}) x^i_{t-1} - q_t p_{t-1}, \quad x^i(t) \in \mathbb{R}^n_+, q_t(\cdot) \in \mathbb{R}^n_+, t = 2, \ldots, T - 1, \tag{37c}
\]

\[
(e + r^i_{T-1}) x^i_{T-1} = q_T p_{T-1}, \quad q_T(\cdot) \in \mathbb{R}^n_+. \tag{37d}
\]

In the setup, we assume that short sales of the security assets and crude oil are forbidden, i.e., \( x_t \in \mathbb{R}^n_+ \) and \( q_t \in \mathbb{R}^n_+ \). Assume that the investor is ambiguous about the true utility function at each stage, we then propose a preference robust counterpart of the multistage investment-consumption problem to mitigate the risk arising from the ambiguity:

\[
\max_{x_{[1,T-1]}; q_{[1,T]}} \inf_{u \in U} \mathbb{E} \left[ u_1(g_1(q_1) d_1, h_0) + u_2(g_2(q_2) d_2, h_{1|1}) + \cdots + u_T(g_T(q_T) d_T, h_{T|T-1}) \right] \tag{38a}
\]
We carry out comparative numerical analysis on the model by considering the utility functions being state-dependent (with the ambiguity set being constructed via pairwise comparison and Kantorovich ball) and state-independent, respectively.

### 6.2. Setup of tests

To ease the exposition, we consider a simple case where \( g_t(x) = x \) and \( d_t = p_t \), for \( t = 1, \ldots, T \). This is based on the understanding that the productions of oil products are proportional to the purchased amount of the crude oil and the value of oil products is proportional to the crude oil price. We assume that the true utility of oil products depends on the crude oil price in two regimes. In the usual regime when the crude oil price is less than or equal to $60 per barrel, the investor has a linear utility function \( u_{\text{lin}}(x) = x \) defined over \([0, 1]\). In the other regime when the crude oil price is greater than $60 per barrel, the investor has a concave utility \( u_{\text{exp}}(x) = (1 - \exp(-3x))/(1 - \exp(-3)) \) defined over \([0, 1]\).

We set the risky assets pool with 9 exchange-traded-funds (ETF) in the US equity market corresponding to different industry sectors including Utilities (XLU), Energy (XLE), Finance (XLF), Technology (XLK), Health Care (XLV), Consumer Staples (XLP), Consumer Discretionary (XLY), Industry (XLI), and Materials (XLB) sectors. We collect weekly data of crude oil price (OK Crude Oil Future Contract) and the ETF prices over the period 2007/1/1 - 2021/3/29. ETF data are downloaded from Yahoo Finance\(^1\) and oil prices are downloaded from Energy Information Administration\(^2\). Before generating the scenario tree, the price data are transformed into log-return rate to pass the stationary test of the data series. We adopt an ARMA(0,1)-GARCH(1,1) model with Gaussian residuals to forecast the future return rate of oil and ETF prices and built a scenario tree with a symmetrical branching structure. The optimal orders for the ARMA and GARCH models were determined through maximum likelihood estimation. One can refer to \[64\] for detailed algorithms of the scenario tree generation. To reduce the computational complexity of DP-type algorithms, we consider the stage independent case.

The models to be tested in comparative analysis include: MSP-True: problem \([37]\) with the true utility functions, SP-PLN-SD: problem \([37]\) with piecewise linear nominal utility functions, MS-PRO-SD-Kan: problem \([38]\) with the state-dependent ambiguity set \( U^K_t(\xi_{t-1}) \) constructed via the Kantorovich ball centered at a piecewise linear nominal utility function at each node. MS-PRO-SD-PC: problem \([38]\) based on the state-dependent pairwise comparison ambiguity set \( U^P_t(\xi_{t-1}) \) with randomly generated questionnaires and answers at each node. MS-PRO-SID-Kan: problem \([38]\) with the state-independent ambiguity set \( U^K_t \) constructed via the Kantorovich ball.
centered at a piecewise linear nominal utility function at each stage. Details of preference elicitation and construction of the ambiguity sets are deferred to EC. All optimization problems in the deterministic reformulations are solved by Gurobi solver through CVX package in Matlab R2016a on a PC with 3.4GHz CPU and 16GB RAM.

6.3. Numerical results: validation of three solution approaches

In the first set of tests, we solve MS-PRO-SD-Kan with the scenario tree method, the NBD method, and the SDDP method for small instance problems with 2-6 stages. In order to compare the three solution methods in a same problem, we focus on a scenario tree with stagewise independence (corresponding to a recombining tree \([21, 49]\)) and state-dependent utilities. At each stage, we generate 5 samples of the oil price and return rates of the 9 ETF assets. For the scenario tree method and the NBD method, we generate a tree with 5\(^T\) scenarios with the stagewise independent samples, where \(T = 2, \cdots, 6\). Table 1 displays the optimal values and CPU times of the three approaches.

From the table, we can see that the scenario tree method and the NBD method generate the same optimal values when \(T = 2, \cdots, 5\). In the case that \(T = 6\), the lower and the upper bounds generated by the NBD method do not match in the last two digits within the specified algorithmic stopping criteria. The SDDP method generates slightly wider gaps between the lower bounds and upper bounds for \(T = 2, \cdots, 6\) where the lower bounds are heuristic. In terms of CPU time, the scenario tree method is very efficient when \(T \leq 4\) but its CPU time increases rapidly when \(T = 5, 6\) because the number of scenarios increases exponentially. In contrast, the SDDP method displays a kind of “linear” increase of CPU time w.r.t. \(T\). The NBD method displays the longest CPU time in all five cases \((T = 2, \cdots, 6)\).

| Table 1 | The Optimal values and CPU times of the scenario tree method, the NBD method and the SDDP method for MS-PRO-SD-Kan with 2-6 stages |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(T\) | Scenario tree method |  |  |  |  |  |
|  | Opt. Val. | CPU time (s) | Opt. Val. | CPU time (s) | Opt. Val. | CPU time (s) |
| 2 | 1.1977 | 2.6905 | 1.1977 | 46.128 | 1.1970 | 28.15 |
| 3 | 1.2281 | 13.2355 | 1.2281 | 224.34 | 1.2287 | 59.90 |
| 4 | 1.3607 | 716.96 | 1.3607 | 780.1 | 1.3620 | 99.21 |
| 5 | 1.6832 | 3738.8 | 1.6832 | 2411.4 | 1.6838 | 367.94 |
| 6 | 1.9859 |  | 1.9808, 1.9888 |  | 1.9858 |  |

In the second set of tests, we solve the same problem but for the case when \(T\) ranges from 10 to 50. Since the scenario tree method and the NBD method require unaffordable storage space and unacceptably long CPU time, we concentrate on SDDP only. Table 2 lists lower and upper bounds of the optimal values, the number of iterations of forward-backward processes, and CPU
time for $T = 10, 30, 50$ with different numbers of scenarios ($|\mathcal{X}| = 5, 10, 20$). We can see that the lower bounds are close to upper bounds in all of the cases which means the algorithm converges within the prescribed precision. The change of CPU time confirms our earlier observation that it increases at a “linear” rate w.r.t. the increase of $T$.

### Table 2
Upper bounds, lower bounds and CPU time of the SDDP method for MS-PRO-SD-Kan with 10-50 stages ($\text{tol}=z_{1-0.99/2}$)

| Stages | $|\mathcal{X}|$ | Iterations | Upper bound | Lower bound | CPU time (s) |
|--------|----------------|-------------|-------------|-------------|--------------|
| 5      | 13             | 2.4473      | 2.4513      | 333.44      |
| 10     | 10             | 2.5307      | 2.5297      | 551.16      |
| 20     | 13             | 2.3907      | 2.3921      | $1.44 \times 10^3$ |
| 30     | 10             | 2.9791      | 2.9718      | $4.31 \times 10^4$ |
| 20     | 13             | 2.9207      | 2.9186      | $6.57 \times 10^3$ |
| 5      | 44             | 3.4359      | 3.4462      | $9.95 \times 10^4$ |
| 30     | 10             | 3.4213      | 3.4194      | $1.61 \times 10^4$ |
| 20     | 23             | 3.3919      | 3.394       | $1.93 \times 10^4$ |

#### 6.4. Comparative analysis of the models

To examine the effects of different models, we have conducted comparative numerical analysis from the following four perspectives: (a) Compare the optimal values of MSP-True, MS-PRO-SD-Kan and MS-PRO-SID-Kan with respect to different numbers of time periods, $T = 2, 3, \ldots, 6$. We set the radius of the Kantorovich ball to $R = 0.001$ and the number of breakpoints to $N = 40$ under all scenarios, see Figure 1. (b) Compare the optimal values of MSP-PLN and MS-PRO-SD-Kan with different numbers of breakpoints $N$, where the optimal value of MSP-True is chosen as the benchmark. Here, we set $T = 4$, and $R = 0.001$ under all scenarios, see Figure 2. (c) Compare the optimal values of MS-PRO-SD-Kan and MS-PRO-SID-Kan with different radii of Kantorovich ball when $T = 4$, and $N = 40$, see Figure 3. (d) Compare the optimal values of MS-PRO-PC with different number of questionnaires with $T = 4$, see Figure 4.

From Figure 1, we can see that with more stages ($T$) to be included in the models, the investor has greater flexibility in setting future consumption and consequently obtaining higher optimal total expected utility values. This phenomenon is observed for both MSP and MS-PRO models. Moreover, the optimal value of the MS-PRO model is smaller than that of MSP, which can be interpreted as the price of robustness. The gap narrows down as $T$ increases. The optimal values of MS-PRO-SID-Kan with optimistic estimations are the highest while that with pessimistic estimations are the lowest and that with unbiased estimations are in the middle. This relationship reflects the nature of the estimations. Figure 2 depicts the variation trends of the optimal values.
Figure 1  Comparison of the optimal values of MSP-True, MS-PRO-SD-Kan and MS-PRO-SID-Kan models with increasing number of stages ($R = 0.001, N = 40$).

Figure 2  Comparison of the optimal values of MSP-True, MSP-PLN and MS-PRO-SD-Kan with increasing number of breakpoints ($T = 4, R = 0.001$).

Figure 3  Comparison of the optimal values of MS-PRO-SD-Kan and MS-PRO-SID-Kan with increasing radius of Kantorovich Ball ($T = 4, N = 40$).

Figure 4  Boxplot of the optimal values of MS-PRO-SD-PC with different numbers of questionaries (for each number, we randomly generate 100 groups of questionaries and plot the mean, maximum, minimum and 25−75% quantiles ($T = 4, N = 40$).

as the number of the breakpoints ($N$) of piecewise linear approximation increases. We can see that the optimal value of MSP-PLN approaches that of MSP-True when $N$ reaches 20; and the optimal value of MS-PRO-SD-Kan moves closer to that of MSP-True despite a gap exists due to $R > 0$. This is consistent with our theoretical results.

Figure 3 presents comparative analysis between state-dependent utility model and state-independent utility model under the framework of MS-PRO-Kan. We can observe that with the
decrease of the radius, the optimal value of MS-PRO-SD-Kan approaches that of MSP-PLN with the same piecewise linear nominal utility function. When the radius is greater than 0.25, MS-PRO-SD-Kan and MS-PRO-SID-Kan generate almost the same solution. This is because the constraint corresponding to the Kantorovich ball becomes inactive (the worst-case utility function becomes linear and the concavity constraint overrides the ball constraint, see Figure EC.14) and subsequently only the bounds on Lipschitz modulus and the convexity constraints are effective. Figure 4 shows that with the increase of the number of questionnaires, the optimal value of MS-PRO-SD-PC converges to that of MSP-True since the randomness of questionnaires recedes.

6.5. Out-of-sample performance of different models with randomly generated true utilities

We now turn to report our numerical test results on the out-of-sample performance of the proposed MS-PRO models. Specifically, we solve the MS-PRO models including MS-PRO-SD-Kan and MS-PRO-SID-Kan, obtain an optimal solution, and implement it in the out-of-sample tests with the true utility function. We begin by randomly generating a set of non-decreasing, piecewise linear and concave utility functions which are within an $\epsilon$-Kantorovich ball centered at a state-dependent reference utility function, see Figures EC.15-EC.17.

The first set of tests is carried out as follows. For the MS-PRO-SID-Kan model, we use each of the four estimation approaches outlined in Section 6.3 to figure out a state-independent nominal utility function, construct respective Kantorovich balls with three different radii ($R = 0.01, 0.1$ or $0.2$), and solve the resulting MS-PRO-SID-Kan models. For the MS-PRO-SD-Kan model, we use the unbiased estimation method to find a piecewise linear nominal utility function at each state and then construct a Kantorovich ball (with different radii $R = 0.01, 0.1, 0.2$). Here we assume that the number of states is known but the correspondence between the elicited scores and the states is unknown. For each of the optimal solutions, we calculate the returns in each scenario and then evaluate the out-of-sample expected utility value with one of the randomly generated utility functions (we call one simulation). We repeat the simulation 100 times and calculate the average of the expected utility values. The rationale behind the simulations is that the true utility function is unknown and we presume that each of the 100 utility functions could be the true. Table 3 displays the average of the mean values, the minimum value and the maximum value of the 100 out-of-sample tests.

From Table 3 we can see that MSP-True performs best in terms of the mean value and the maximum value. MS-PRO-SD-Kan ($R = 0.01$) gives the best of the worst-case expected utility value, which highlights the value of adopting the robust model. The MS-PRO-SID-Kan delivers the worst performance in all aspects. This is primarily because the true utility function is state-dependent. Figure 5 depicts the results in box-plots, we can see that when $\epsilon$ increases, the difference
Table 3  Comparisons of out-of-sample performances of MS-PRO-SD-Kan and MS-PRO-SID-Kan with $T = 4$, $N = 40$ and $\epsilon = 0.1$.

| Statistics | Model | MSP-True | MS-PRO-SD-Kan (Unbiased) | MS-PRO-SD-Kan ($R = 0.1$) | MS-PRO-SID-Kan (Pessimistic) | MS-PRO-SID-Kan (Optimistic) | MS-PRO-SID-Kan (Unbiased) | MS-PRO-SID-Kan (Best-fit) |
|------------|-------|----------|--------------------------|---------------------------|-------------------------------|-------------------------------|-----------------------------|--------------------------|
| Mean       |       | 1.3660   | 1.3654 1.3587 1.1542     | 1.0860 1.1467 1.1418 1.1480 |                               |                               |                             |                          |
| Min        |       | 1.1660   | 1.1663 1.1670 1.0962     | 1.0791 1.0899 1.0996 1.1003 |                               |                               |                             |                          |
| Max        |       | 1.5660   | 1.5645 1.5501 1.2113     | 1.0945 1.2028 1.1834 1.1946 |                               |                               |                             |                          |

of the performances in terms of minimum values becomes smaller. This is because the concavity constraint overrides the Kantorovich ball constraint (the worst-case utility function becomes linear). Moreover, when $R$ matches $\epsilon$ (0.1), the MS-PRO-SD-Kan performs best in terms of the minimum value. In this case, the ambiguity set in MS-PRO-SD-Kan covers the set of randomly generated utility functions (for out-of-sample tests).

Figure 5  Boxplots of out-of-sample utility values of MS-PRO-SD-Kan and MS-PRO-SID-Kan under three sets of randomly generated utility functions

7. Concluding remarks

In this paper, we present a full investigation of the PRO models for expected utility based multistage decision making. We begin with holistic maximin models (8) for state-dependent utility case and (9) for state-independent utility case, demonstrate time consistency and time inconsistency for them respectively, and derive the dynamic recursive formulation (16) for the former. We then use scenario-tree methods to solve both (8) and (9) with a given scenario tree structure of the underlying random process, and the SDDP and the NBD methods to solve (8) via (16). Finally, we carry out comparative numerical tests on state-dependent model (8) vs state-independent model (9), and scenario tree method vs dynamic programming method for solving (8). To derive dynamic reformulation of (8), we derive a new version of the principle of interchangeability in Banach space (Lemmas 11 and 2).

A clear benefit of beginning the robust model with (8) rather than (16) as some of the distributionally robust MSP models do in the literature (see e.g. [37, 65]) is that it allows us to apply both
the scenario tree algorithm and DP algorithms for solving the state-dependent MS-PRO model. Moreover, since (9) does not have a dynamic reformulation, the presence of (8) facilitates us to compare the performances of the two models by solving them with the same scenario tree methods.

Establishing a link between (8) and (16) is a key step given that our PRO model is non-parametric and establishing an equivalence relation requires a new interchangeability result in Banach space. While our focus in the paper is on the utility-based PRO models, our approach on both (8) and (16) may have some ramifications on other nonparametric multistage maximin (res. minimax) optimization problems ([63, 50]). To the best of our knowledge, the existing research only allows one to establish an equivalence relation (analogous to (16) and (8)) for the multistage parametric robust optimization problems where both the outer maximization (res. minimization) and the inner minimization (res. maximization) problems are essentially finite-dimensional or nonstochastic, see [5, 13] and the references therein. This is perhaps because the existing interchangeability results in the literature are established in the finite-dimensional spaces. We hope that our new interchangeability result (Lemmas 1 and 2) will help to make a breakthrough in these models.

Constructing a nominal utility function for the ambiguity set of Kantorovich ball is another important component of this work. The approaches outlined in EC.6 for state-dependent and state-independent utility cases provide a new avenue for estimating an approximate utility function based on incomplete information of scoring and may provide a new direction for general preference elicitation. It remains to be an open question how the model will perform if the number of states is incorrectly preset, how to design consumption trajectories for more effective preference elicitation (a main departure from one stage), and how to deal with errors occurring in scoring. The setting of the radius of the Kantorovich ball can also be improved by comprehensively considering the estimation error on the number of states, the estimation error on piecewise linear approximation and the errors in scoring, with appropriate statistical inference and guarantee.

Another aspect of our model which could be potentially strengthened is that instead of separating the preference elicitation/scoring and the optimization process, we may consider the dynamic interaction between the elicitation process and the optimization process on an online footing. Online optimization, reinforcement learning or meta-learning approaches may be further incorporated to improve the intelligence of our MS-PRO model. We leave all these issues for future research.

Acknowledgements

This work was funded by the National Key R&D Program of China (No. 2022YFA1004000, 2022YFA1004001), National Natural Science Foundation of China (No. 11991023 and 11901449), RGC grant (14204821) and CUHK startup grant.
References

[1] B. Armbruster and E. Delage, Decision making under uncertainty when preference information is incomplete, *Management Science*, 61: 111–128, 2015.

[2] P. Artzner, F. Delbaen, J. M. Eber, D. Heath and H. Ku, Coherent multi-period risk adjusted values and Bellman principle, *Annals of Operations Research*, 152: 5–22, 2007.

[3] J.P. Aubin, H Frankowska, *Set-valued Analysis*, Springer Science & Business Media, 2009.

[4] A. Ben-Tal, L El Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton University Press, NJ, 2009.

[5] D. P. Bertsekas. *Dynamic Programming and Optimal Control*, 4th Edition. Athena Scientific, Belmont, MA, 2017.

[6] D. Bertsimas and A. O’Hair, Learning preferences under noise and loss aversion: An optimization approach, *Operations Research*, 61: 1190–1199, 2013.

[7] J. R. Birge and F. Louveaux, *Introduction to Stochastic Programming*, Springer Series in Operations Research and Financial Engineering, Springer Science & Business Media, 2nd edition, 2011.

[8] T. Björk, A. Murgoci and X. Y. Zhou, Mean-variance portfolio optimization with state-dependent risk aversion, *Mathematical Finance*, 24(1):1-24, 2014.

[9] K. Boda and J. A. Filar, Time consistent dynamic risk measures, *Mathematical Methods of Operations Research*, 63(1): 169–186, 2006.

[10] R. T. Clemen and T. Reilly, *Making Hard Decisions with Decision Tools Suite*. Duxbury, Pacific Grove, CA, 2nd edition, 2001.

[11] Z. Chen, G. Consigli, J. Liu, G. Li, T. Fu and Q. Hu, Multi-period risk measures and optimal investment policies. In G. Consigli, D. Kuhn, P. Brandimarte (Eds.), *Optimal Financial Decision Making under Uncertainty*, Springer International Publishing, 2017, 1–34.

[12] X. Y. Cui, D. Li, S. Y. Wang and S. S. Zhu, Better than dynamic mean-variance: Time inconsistency and free cash flow stream, *Mathematical Finance*, 22(2), 346–378, 2012.

[13] E. Delage and D. A. Iancu, Robust multistage decision making, *INFORMS Tutorials in Operations Research*, The Operations Research Revolution: 20–46, 2015.

[14] E. Delage and J. Y. Li, Minimizing risk exposure when the choice of a risk measure is ambiguous, *Management Science*, 64: 327–344, 2018.

[15] D. Dentcheva and A. Ruszczyński, Optimization with stochastic dominance constraints, *SIAM Journal on Optimization* 14: 548-566, 2003.

[16] D. Duffie, *Dynamic Asset Pricing Theory*, Princeton University Press, 2010.

[17] K. Dunn and K. Singleton, Modelling the term structure of interest rates under nonseparable utility and durability of goods, *Journal of Financial Economics*, 17: 27–55, 1986.

[18] L. G. Epstein and M. Schneider, Recursive multiple-priors, *Journal of Economic Theory*, 113(1): 1-31, 2003.

[19] E. Fama, Multiperiod consumption-investment decisions, *The American Economic Review*, 60(1):163–174, 1970.

[20] P. H. Farquhar, Utility assessment methods, *Management Science*, 30: 1283–1300, 1984.

[21] C Füllner, S Rebennack, Stochastic dual dynamic programming and its variants, Preprint, 2021

[22] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18: 141–153, 1989.

[23] S. Guo and H. Xu, Robust spectral risk optimization when the subjective risk aversion is ambiguous: a moment-type approach, *Mathematical Programming*, 194: 305–340, 2022.

[24] S. Guo and H. Xu, Utility preference robust optimization with moment-type information structure, 2021.

[25] W. Haskell, L. Fu and M. Dessouk, Ambiguity in risk preferences in robust stochastic optimization, *European Journal of Operational Research*, 254: 214–225, 2016.

[26] X. D. He, M. S. Strub and T. Zariphopoulou, Forward rank-dependent performance criteria: Time-consistent investment under probability distortion, *Mathematical Finance*, 31(2): 683–721, 2021.

[27] J. Hu and S. Mehrrotra, Robust decision making over a set of random targets or risk-averse utilities with an application to portfolio optimization, *IEE Transaction*, 47: 358-372, 2015.

[28] J. Hu and G. Stepanyan, Optimization with reference-based robust preference constraints, *SIAM Journal on Optimization*, 27: 2230–2257, 2017.

[29] J. Hu, M. Bansal and S. Mehrrotra, Robust decision making using a general utility set. *European Journal of Operational Research*, 269(2): 699–714, 2018.

[30] Y. Hu, H. Jin and X.Y. Zhou, Consistent investment of sophisticated rank-dependent utility agents in continuous time, *arXiv:2006.01979*, 2020.

[31] G. N. Iyengar, Robust dynamic programming, *Mathematics of Operations Research*, 30(2): 257–280, 2005.

[32] U. S. Karmarkar, Subjectively weighted utility: A descriptive extension of the expected utility model, *Organizational Behavior and Human Performance*, 21: 61–72, 1978.

[33] T. C. Koopmans, Stationary ordinal utility and impatience, *Econometrica*, 28, 287–309, 1960.
[34] D. M. Kreps and E. L. Porteus, Temporal resolution of uncertainty and dynamic choice theory, *Econometrica*, 46: 185–200, 1978.
[35] J. Y. Li, Inverse optimization of convex risk functions, *Management Science*, 67(11): 7113–7141, 2021.
[36] L. Mononen, State-dependent utility and ambiguity. Working paper, Princeton University, 2020.
[37] J. Park and G. Bayraksan, A multistage distributionally robust optimization approach to water allocation under climate uncertainty, *European Journal of Operational Research*, 306(2): 849–871, 2023.
[38] G. Ch. Pflug and A. Pichler, *Multistage Stochastic Optimization*, Springer Series in Operations Research and Financial Engineering, Springer, 2014.
[39] C. Puppe, *Distorted Probabilities and Choice under Risk*, vol. 363. Springer Science & Business Media, 2012.
[40] A. Pichler and A. Shapiro, Mathematical foundations of distributionally robust multistage optimization, arXiv: 2101.02948, 2021.
[41] W. B. Powell, A unified framework for stochastic optimization. *European Journal of Operational Research*, 275(3): 795–821, 2019.
[42] H. Rahimian and S. Mehrrotra, Distributionally robust optimization: A review, arXiv:1908.05659, 2019.
[43] R.T. Rockafellar and R.J.B. Wets, *Variational Analysis*, Vol. 317. Springer Science & Business Media, 2009.
[44] A. Ruszczyński, Risk-averse dynamic programming for Markov decision processes, *Mathematical Programming*, 125(2): 235–261, 2010.
[45] A. Ruszczyński and A. Shapiro, Stochastic programming models, *Handbooks in Operations Research and Management Science*, 10: 1-64, 2003.
[46] A. Shapiro, On a time consistency concept in risk averse multistage stochastic programming, *Operations Research Letters*, 37(3): 143–147, 2009.
[47] A. Shapiro. Analysis of stochastic dual dynamic programming method, *European Journal of Operational Research*, 209: 63–72, 2011.
[48] A. Shapiro, Rectangular sets of probability measures. *Operations Research*, 64(2): 528–541, 2016.
[49] A. Shapiro, D. Dentcheva and A. Ruszczyński, *Lectures on Stochastic Programming: Modeling and Theory*, 1st Edition, SIAM, Philadelphia, 2009.
[50] A. Shapiro and A. Xu, Preference robust distortion risk measure and its application, 2021, Available at SSRN.
[51] Y. Shi, X. Y. Cui and X. Y. Zhou, Beta and coskewness pricing: Perspective from probability weighting, *Operations Research*, doi:10.1287/opre.2022.2421, 2023.
[52] Z. Yan, Z. Chen, G. Consigli, J. Liu and M. Jin, A copula-based scenario tree generation algorithm for multiperiod portfolio selection problems, *Annals of Operations Research*, 292: 849–881, 2020.
[53] X. Yu and S. Shen, Multistage distributionally robust mixed-integer programming with decision-dependent moment-based ambiguity sets, *Mathematical Programming*, 1196: 1025–1064, 2022.
[54] Y. Zhang, H. Xu and W. Wang, Preference robust models in multivariate utility-based shortfall risk minimization. *Optimization Methods and Software*, 37(2): 712-752, 2022.
Electronic Companion for “Multistage Utility Preference Robust Optimization”

**EC.1. An example of multistage portfolio selection problem with utility maximization**

**EXAMPLE EC.1.** Consider a financial market with $n$ risky assets. Suppose that an investor joins the market at time 0 with a positive initial wealth $w_0$ and plans to invest her/his wealth in the market for $T$ periods. At each period, the investor gains a reward which could be her/his end-of-period wealth or the increase of her/his wealth over this period, i.e., $h_t(r_t, x_t) = (e + r_t)\top x_t$ or $h_t(r_t, x_t) = r_t\top x_t$, where $x_t$ is the asset allocation vector, $r_t$ is the excess return rate vector of $n$ risky assets over period $t$, $e = [1, \cdots, 1]_T$ is the vector with all components being one. The investor presents a utility $u_t(\cdot)$ which measures her/his preferences on the reward over that period. Then the investor would like to maximize the overall expected utility over all of the $T$ periods by adjusting her/his portfolios at the beginning of each period. If the investor’s objective is to maximize the overall expected utility of the wealth, the decision making problem can be reformulated as a multistage expected utility maximization problem:

$$
\max_{\mathcal{Z}_T} \mathbb{E} \left[ u_1((e + r_1)\top x_1) + u_2((e + r_2)\top x_2) + \cdots + u_T((e + r_T)\top x_T) \right]
$$

s.t. $x_1 \in \{ x \in \mathbb{R}^n_+ | e\top x = w_0 \}$, $x_t(r_{t-1}) \in \{ x \in \mathbb{R}^n_+ | e\top x_t = (e + r_{t-1})\top x_{t-1} \}$, $t = 2, \ldots, T$,

where $x_t$ is the asset allocation vector of the current wealth invested in the $n$ risky assets at the beginning of period $t$, $r_t\top x_t$ is the wealth at the end of period $t$. $e\top x_t = (e + r_{t-1})\top x_{t-1}$ is the wealth balance equation, which together with the no-shorting constraint characterizes the feasible set $\mathcal{X}_t$ of portfolio $x_t$ at period $t$. If the investor’s utility is valued over the state-wise return rates, the objective could be set as

$$
\mathbb{E} \left[ u_1 \left( \frac{r_1\top x_1}{e\top x_1} \right) + u_2 \left( \frac{r_2\top x_2}{e\top x_2} \right) + \cdots + u_T \left( \frac{r_T\top x_T}{e\top x_T} \right) \right].
$$

In this case, by normalizing $\tilde{x}_t = x_t/(e\top x_t)$, we have an equivalent utility maximization problem:

$$
\max_{\tilde{x}_1(\tilde{x}_t(\cdot))} \mathbb{E} \left[ u_1(r_1\top \tilde{x}_1) + u_2(r_2\top \tilde{x}_2) + \cdots + u_T(r_T\top \tilde{x}_T) \right]
$$

s.t. $e\top \tilde{x}_1 = 1$, $\tilde{x}_1 \in [0, 1]^n$, $e\top \tilde{x}_t(r_{t-1}) = 1$, $\tilde{x}_t(\cdot) \in \mathcal{L}^0([0, 1]^n)$, $t = 2, \ldots, T$.

**EC.2. Proofs**

**EC.2.1. Proof of Lemma 1**

For any $\mathcal{Z} \in \mathcal{M}_Z$, we have that $\mathcal{Z}(\omega) \subseteq Z$ and hence $\inf_{z \in \mathcal{Z}(\omega)} f(z, \omega) \leq f(\mathcal{Z}(\omega), \omega)$ a.s.. Since $Z$ is measurable and $f$ is continuous in $z$, it follows by [2] Theorem 8.2.11] that $\inf_{z \in \mathcal{Z}(\omega)} f(z, \omega)$ is measurable. By taking mathematical expectation on both sides of the inequality, we obtain

$$
\mathbb{E} \left[ \inf_{z \in \mathcal{Z}(\omega)} f(z, \omega) \right] \leq \mathbb{E} \left[ f(\mathcal{Z}(\omega), \omega) \right] = \mathbb{E} [F_\mathcal{Z}] . \tag{EC.1}
$$
Moreover, by taking infimum w.r.t. \( z \) over \( \mathcal{M}_Z \) on both sides of the inequality, we have

\[
\mathbb{E} \left[ \inf_{z \in Z(\omega)} f(z, \omega) \right] \leq \inf_{z \in \mathcal{M}_Z} \mathbb{E} [F_z]. \tag{EC.2}
\]

Next, we show the inequality holds in the opposite direction. We first consider the case when \( \inf_{z \in Z(\omega)} f(z, \omega) \) is finite valued a.s.. For \( k = 1, 2, \ldots \), we consider the level-set mapping \( S_k : \Omega \to \mathbb{Z} \)

\[
S_k(\omega) = \left\{ z \in \mathbb{Z} : f(z, \omega) \leq \inf_{z \in Z(\omega)} f(z, \omega) + \frac{1}{k} \right\}.
\]

Since \( f \) is a Carathéodory function, we know from [2, Lemma 8.2.6] that \( f(z, \omega) \) is \( B(\mathbb{Z}) \otimes \mathcal{F} \)-measurable and for every \( \omega \in \Omega \), the function \( z \to f(z, \omega) \) is continuous. Thus \( S_k(\omega) \) is a closed set for every given \( \omega \). By [2, Theorem 8.1.4], the measurability of \( f \) ensures the measurability of \( S_k(\omega) \) w.r.t. \( \mathcal{F} \). Let

\[
Z_k(\omega) := S_k(\omega) \bigcap Z(\omega), \forall \omega \in \Omega.
\]

Since both \( S_k \) and \( Z \) are \( \mathcal{F} \)-measurable, by [2, Theorem 8.2.4], \( Z_k \) is also \( \mathcal{F} \)-measurable. Moreover, since \( \inf_{z \in Z(\omega)} f(z, \omega) \) is finite-valued a.s., then \( Z_k(\omega) \) is non-empty and

\[
f(z, \omega) \leq \inf_{z \in Z(\omega)} f(z, \omega) + \frac{1}{k}, \forall z \in Z_k(\omega), \text{ a.s.}
\]

Together with the closedness and \( \mathcal{F} \)-measurability of \( Z_k \), we know by virtue of Theorem 8.1.3 in [2] that there exists a \( \mathcal{F} \)-measurable selection \( \bar{z}_k \) of \( Z_k \) such that

\[
f(\bar{z}_k(\omega), \omega) \leq \inf_{z \in Z(\omega)} f(z, \omega) + \frac{1}{k}, \text{ a.e. } \omega \in \Omega. \tag{EC.3}
\]

By taking expectation on both sides of inequality (EC.3), we have

\[
\mathbb{E} [F_{\bar{z}_k}] = \mathbb{E}[f(\bar{z}_k(\omega), \omega)] \leq \mathbb{E} \left[ \inf_{z \in Z(\omega)} f(z, \omega) \right] + \frac{1}{k}.
\]

Since \( \bar{z}_k \) is a \( \mathcal{F} \)-measurable selection from \( Z_k = S_k \cap Z \), then \( \bar{z}_k \in \mathcal{M}_Z \). Letting \( k \to +\infty \) gives us that

\[
\inf_{z \in \mathcal{M}_Z} \mathbb{E} [F_z] \leq \mathbb{E} \left[ \inf_{z \in Z(\omega)} f(z, \omega) \right]. \tag{EC.4}
\]

Combining with inequality (EC.2), we arrive at (13) as desired.

Next, we move on to consider two extreme cases: (a) the event \( \{ \omega \mid \inf_{z \in Z(\omega)} f(z, \omega) = -\infty \} \) has a positive probability \( p_{-\infty} \) and (b) the event \( \{ \omega \mid \inf_{z \in Z(\omega)} f(z, \omega) = +\infty \} \) has a positive probability \( p_{+\infty} \). We first consider case (a). In this case, \( \mathbb{E} \left[ (-\inf_{z \in Z(\omega)} f(z, \omega))_+ \right] = +\infty \), which, by the assumption of the lemma, implies \( \mathbb{E} \left[ (\inf_{z \in Z(\omega)} f(z, \omega))_+ \right] < +\infty \). This gives rise to \( \mathbb{E} [\inf_{z \in Z(\omega)} f(z, \omega)] = -\infty \). We can use the same approach as that in the finite-valued case to show that the right hand
side of (EC.2) is also equal to \(-\infty\). Specifically, for any \(k \in \{1, 2, 3 \ldots\}\), we consider the level set mapping
\[
S_k(\omega) := \left\{ z \in \mathbb{Z} \mid \frac{f(z, \omega)}{f(\omega)} \leq \inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) + \frac{1}{k}, \quad \text{if} \quad \inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) > -\infty, \quad \frac{f(z, \omega)}{f(\omega)} \leq -k, \quad \text{if} \quad \inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) = -\infty \right\}.
\]
We can show that \(S_k(\omega) \cap Z(\omega)\) is \(\mathcal{F}\)-measurable and there exists a \(\mathcal{F}\)-measurable selection of \(z_k\) from \(Z_k := S_k \cap Z\) such that
\[
\inf_{z \in \mathbb{Z}} \mathbb{E}[F_3] \leq \mathbb{E}[F_3] \leq \int_{\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) = -\infty}(1 - p_{\infty}) \mathbb{E}(d\omega) + \int_{\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) > -\infty}(\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) + \frac{1}{k}) \mathbb{P}(d\omega)
\]
\[
= -p_{\infty} k + \int_{\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) > -\infty} \left[ \inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) \right] - \left[ - \inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) \right] \mathbb{P}(d\omega)
\]
\[
\leq -p_{\infty} k + \int_{\mathbb{Z}(\omega)} \left[ \inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) \right] \mathbb{P}(d\omega) + (1 - p_{\infty}) \frac{1}{k},
\]
where \(p_{\infty} = \mathbb{P}(\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega) = -\infty) > 0\). Since \(\int_{\mathbb{Z}(\omega)} \mathbb{E}(f(z, \omega)) < +\infty\), by letting \(k \to +\infty\), we arrive at \(\inf_{z \in \mathbb{Z}} \mathbb{E}[F_3] = -\infty\) as desired.

Consider now case (b). In this case, \(\mathbb{E}\left[\left(\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega)\right)_{+}\right] = +\infty\), which implies by the assumption of the lemma that \(\mathbb{E}\left[\left(\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega)\right)_{+}\right] < +\infty\). This gives rise to
\[
\mathbb{E}\left[\left(\inf_{z \in \mathbb{Z}(\omega)} f(z, \omega)\right)_{+}\right] = +\infty.
\]
By (EC.2), we know that both sides of (13) are equal to \(+\infty\). Note that the cases (a) and (b) cannot occur simultaneously due to the assumption on the positive/negative part of the expectation. \(\square\)

**EC.2.2. Proof of Lemma 2**

The thrust of the proof is to fit (14) in the framework of Lemma 1 so that the principle of the interchangeability established in Lemma 1 can be readily applied. To this effect, we introduce a new random function \(\hat{u} : \mathbb{R} \times \Omega \to \mathbb{R}\) such that \(\hat{u}(x, \omega) = u(x, \xi(\omega)), \forall x \in \mathbb{R}\) and a.e. \(\omega \in \Omega\), where \(u \in \mathcal{M}_{\mu}\). Define \(Z := \mathcal{L}^p(\mathbb{R} \to \mathbb{R})\) as a functional space. Let \(\Omega\) denote the space of measurable functions \(\hat{u} : \Omega \to \mathbb{Z}\) with finite \(p\)-th order moments and define
\[
\hat{\mathcal{M}}_{\mu} := \left\{ \hat{u} \in \Omega \mid \hat{u}(x, \omega) = u(x, \xi(\omega)), \forall x \in \mathbb{R}, \text{for a.e. } \omega, \quad u(\cdot, \xi) \in \mathcal{U}(\xi), \text{ for any } \xi \right\}.
\]
By letting \(\hat{\mathcal{U}} := \mathcal{U}(\xi)\), we have \(\hat{\mathcal{M}}_{\mu} = \left\{ \hat{u} \in \Omega \mid \hat{u}(x, \omega) \in \hat{\mathcal{U}}(\omega), \text{ for a.e. } \omega \in \Omega \right\}\). By changing the variable from \(u\) to \(\hat{u}\), we obtain
\[
\inf_{u \in \mathcal{M}_{\mu}} \mathbb{E}[u(\eta, \xi)] = \inf_{u \in \mathcal{M}_{\mu}} \int_{\Omega} u(\eta(\omega), \xi(\omega)) d\mathbb{P}(\omega) = \inf_{u \in \hat{\mathcal{M}}_{\mu}} \int_{\Omega} \hat{u}(\eta(\omega), \omega) d\mathbb{P}(\omega) = \inf_{u \in \hat{\mathcal{M}}_{\mu}} \mathbb{E}[\hat{u}(\eta(\omega), \omega)].
\]
Moreover, by the tower property of the expectation operator, we have

\[
\mathbb{E} [\hat{u}(\eta(\omega), \omega)] = \mathbb{E} \left[ \mathbb{E}_{F_\xi} [\hat{u}(\eta(\omega), \omega)] \right],
\]

where \(\mathbb{E}_{F_\xi}\) denotes the conditional expectation with respect to \(F_\xi\), and hence

\[
\inf_{u \in \mathcal{U}_{F_\xi}} \mathbb{E} [u(\eta, \xi)] = \inf_{u \in \mathcal{U}_{F_\xi}} \mathbb{E} \left[ \mathbb{E}_{F_\xi} [\hat{u}(\eta(\omega), \omega)] \right]. \tag{EC.5}
\]

To apply Lemma 1, we define

\[
f(\hat{u}(\cdot, \omega), \omega) := \mathbb{E}_{F_\xi} [\hat{u}(\eta(\omega), \omega)]. \tag{EC.6}
\]

Here, \(f : \mathbb{Z} \times \Omega \to L^p(\Omega, F_\xi, \mathbb{P}; \mathbb{R})\) is a functional with \(f(z, \omega) = \mathbb{E}_{F_\xi} [z(\eta(\omega))]\) for each \(z \in \mathbb{Z}\). By the definition, we can see that for each fixed \(\omega\), \(f(\cdot, \omega)\) is continuous in \(z\) since the conditional expectation is a linear operator. For each fixed \(z\), \(f(z, \cdot)\) is measurable. By the continuity of \(z(\cdot) \in \mathbb{Z}\), and the measurability of \(\mathbb{E}_{F_\xi}\) and \(\eta\), we know by virtue of \([2, Corollary\ 8.2.3]\) that \(f(z, \omega)\) is \(F\)-measurable. Thus, \(f(z, \omega)\) is a Carathéodory function. Thus,

\[
\mathbb{E} \left[ \mathbb{E}_{F_\xi} [\hat{u}(\eta(\omega), \omega)] \right] = \mathbb{E} \left[ f(\hat{u}(\cdot, \omega), \omega) \right].
\]

By the nonemptyness of \(\mathcal{M}_{F_\xi}\) and the boundedness of \(u \in \mathcal{M}_{F_\xi}\), \(\inf_{\hat{u} \in \mathcal{M}_{F_\xi}} \mathbb{E} [f(\hat{u}(\cdot, \omega), \omega)]\) is bounded. By Lemma 1 (here we require \(u\) to have finite \(p\)-th moment which is equivalent to \(\zeta\) having finite \(p\)-th moment. This additional condition does not affect the result in Lemma 1), we have that

\[
\inf_{\hat{u} \in \mathcal{M}_{F_\xi}} \mathbb{E} [f(\hat{u}(\cdot, \omega), \omega)] = \mathbb{E} \left[ \inf_{u \in U(\omega)} f(u, \omega) \right], \tag{EC.7}
\]

where \(U(\omega) = U(\zeta(\omega))\), for a.e. \(\omega \in \Omega\). Thus

\[
\mathbb{E} \left[ \inf_{u \in U(\omega)} f(u, \omega) \right] = \mathbb{E} \left[ \inf_{u \in U(\omega)} \mathbb{E} [u(\eta(\omega)) | F_\xi] \right] = \mathbb{E} \left[ \inf_{u \in U(\zeta(\omega))} \mathbb{E} [u(\eta(\omega)) | F_\xi] \right] = \mathbb{E} \left[ \inf_{u \in U(\zeta)} \mathbb{E} [u(\eta) | F_\xi] \right]. \tag{EC.8}
\]

Combining (EC.5)-(EC.8), we obtain (14) as desired.

\[\square\]

**EC.2.3. Proof of Proposition 1**

By Lemma 2

\[
\inf_{u \in U(\xi)} \mathbb{E} [u(Z_i(\xi), \xi_{t-1})] = \mathbb{E}_{F_0} \left[ \inf_{u \in U(\xi)} \mathbb{E}_{F_{\xi_{t-1}}} [u(Z_i(\xi))] \right], \tag{EC.9}
\]
where \( \mathcal{U}_t = \{ u_t \mid \exists \tilde{u}_{i,t-1} \text{ and } \tilde{u}_{t+1,T} \text{ such that } [\tilde{u}_{i,t-1}, u_t, \tilde{u}_{t+1,T}]^T \in \mathcal{U} \} \). From Definition 1, we can see that \( \mathcal{U}_t(\xi_{t-1}) \) is a decomposition of \( \mathcal{U} \). By the decomposability of the objective function and feasible set \( \mathcal{U} \), the tower property and the translation invariance property of the expectation operator, we have

\[
\inf_{\tilde{u} \in \mathcal{U}} \mathbb{E}_{|\mathcal{F}_0} \left[ u_1(1) + u_2(Z_2, \xi_1) + \cdots + u_T(Z_T, \xi_{T-1}) \right] \\
= \inf_{u_t \in \mathcal{U}_t} \sum_{t=1}^T \mathbb{E}_{|\mathcal{F}_0} \left[ u_t(Z_t, \xi_{t-1}) \right] \\
= \inf_{u_t \in \mathcal{U}_t} \sum_{t=1}^T \sum_{t=1}^T \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(Z_t, \xi_{t-1}) \right] \\
= \sum_{t=1}^T \inf_{u_t \in \mathcal{U}_t} \mathbb{E}_{|\mathcal{F}_0} \left[ u_t(Z_t, \xi_{t-1}) \right] \\
= \sum_{t=1}^T \inf_{u_t \in \mathcal{U}_t} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(Z_t, \xi_{t-1}) \right] \\
= \sum_{t=1}^T \mathbb{E}_{|\mathcal{F}_0} \left[ \inf_{u_t \in \mathcal{U}_t(\xi_{t-1})} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(Z_t) \right] \right] \tag{EC.10} \\
= \mathbb{E}_{|\mathcal{F}_0} \left[ \inf_{u_1 \in \mathcal{U}_1} \mathbb{E}_{|\mathcal{F}_0} \left[ u_1(Z_1) \right] + \mathbb{E}_{|\mathcal{F}_0} \left[ \inf_{u_2 \in \mathcal{U}_2(\xi_1)} \mathbb{E}_{|\mathcal{F}_1} \left[ u_2(Z_2) \right] \right] + \cdots + \mathbb{E}_{|\mathcal{F}_0} \left[ \inf_{u_T \in \mathcal{U}_T(\xi_{T-1})} \mathbb{E}_{|\mathcal{F}_{T-1}} \left[ u_T(Z_T) \right] \right] \right] \tag{EC.11} \\
= \inf_{u_1 \in \mathcal{U}_1} \mathbb{E}_{|\mathcal{F}_0} \left[ u_1(Z_1) \right] + \inf_{u_2 \in \mathcal{U}_2(\xi_1)} \mathbb{E}_{|\mathcal{F}_1} \left[ u_2(Z_2) \right] + \cdots + \inf_{u_T \in \mathcal{U}_T(\xi_{T-1})} \mathbb{E}_{|\mathcal{F}_{T-1}} \left[ u_T(Z_T) \right] \right] \tag{EC.12}
\]

which gives rise to (11). □

**EC.2.4. Proof of Theorem 1**

We divide the proof into three main steps.

**Step 1.** We begin by decomposing problem (MS-PRO-SD) into a stagewise maximin problem. By Proposition 1, for any fixed decision sequence \( x_1, \ldots, x_T \) and random process \( \xi_1, \ldots, \xi_T \), we have

\[
\inf_{u_t \in \mathcal{U}} \mathbb{E} \left[ u_1(h_1(x_1, \xi_1)) + u_2(h_2(x_2(\xi_1), \xi_2), \xi_1) + \cdots + u_T(h_T(x_T(\xi_{T-1}), \xi_T), \xi_{T-1}) \right] \\
= \sum_{t=1}^T \mathbb{E}_{|\mathcal{F}_0} \left[ \inf_{u_t \in \mathcal{U}_t(\xi_{t-1})} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(h_t(x_t(\xi_{t-1}), \xi_t)) \right] \right]. \tag{EC.13}
\]

Consequently

\[
V_1 := \max_{x_{[1,T]} \in \mathcal{X}_{[1,T]}^T} \inf_{u_t \in \mathcal{U}} \mathbb{E} \left[ u_1(h_1(x_1, \xi_1)) + u_2(h_2(x_2(\xi_1), \xi_2), \xi_1) + \cdots + u_T(h_T(x_T(\xi_{T-1}), \xi_T), \xi_{T-1}) \right] \\
= \max_{x_{[1,T]} \in \mathcal{X}_{[1,T]}^T} \mathbb{E}_{|\mathcal{F}_0} \left[ \sum_{t=1}^T \inf_{u_t \in \mathcal{U}_t(\xi_{t-1})} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(h_t(x_t(\xi_{t-1}), \xi_t)) \right] \right].
\]
Here, we denote \( \mathcal{X}_{[t,T]} := \{ x_{[t,T]} \mid x_s(\xi_{[s-1]}) \in \mathcal{X}_s(x_{[s-1],\xi_{[s-1]}}), s = t, \ldots, T \} \), \( t = 1, \ldots, T \), for short. At stage \( t = 1, \ldots, T \), for given \( x_{[t-1]} \) and \( \xi_{[t-1]} \), we denote the optimal value of the sub-optimization problem at remaining stages by

\[
V_t(x_{[t-1]}, \xi_{[t-1]}) := \max_{x_{[t,T]} \in \mathcal{X}_{[t,T]}} \mathbb{E}_{\mathcal{F}_{t-1}} \left[ \sum_{s=t}^{T} \inf_{u_s \in \mathcal{U}_s(\xi_{[s-1]})} \mathbb{E}_{\mathcal{F}_{s-1}} \left[ u_s \left( h_s(x_s(\xi_{[s-1]}), \xi_s) \right) \right] \right]. \tag{EC.14}
\]

Let \( V_{T+1}( \cdot, \cdot ) := 0 \). At the first stage, \( V_1 \) is the optimal value of problem (MS-PRO-SD). We then prove the dynamic equations \( \tag{16} \) between \( V_t \) and \( V_{t+1} \) by induction. At stage \( T \), we have \( \tag{16} \) directly by the definition above. We then prove \( \tag{16} \) at stage \( T - 1 \) in Step 2, and then prove that the equation at stage \( T - 1 \) implies the equation at stage \( T - 2 \) in Step 3. As the induction relationship between adjacent two stages holds, we can establish the results by induction.

Step 2. We consider the sub-optimization problem at the last two stages. On the basis of the right-hand side of \( \tag{EC.14} \), we prove the recursive formula for \( t = T - 1 \). Observe first that

\[
V_{T-1}(x_{[T-2]}, \xi_{[T-2]}) := \max_{x_{[T-1,T]} \in \mathcal{X}_{[T-1,T]}} \mathbb{E}_{\mathcal{F}_{T-2}} \left[ \sum_{t=T-1}^{T} \inf_{u_t \in \mathcal{U}_t(\xi_{[t-1]})} \mathbb{E}_{\mathcal{F}_{t-1}} \left[ u_t \left( h_t(x_t(\xi_{[t-1]}), \xi_t) \right) \right] \right].
\]

\[
= \max_{x_{T-1} \in \mathcal{X}_{T-1}(x_{[T-2],\xi_{[T-2]}})} \left[ \mathbb{E}_{\mathcal{F}_{T-2}} \left[ \inf_{u_{T-1} \in \mathcal{U}_{T-1}(\xi_{[T-2]})} \mathbb{E}_{\mathcal{F}_{T-2}} \left[ u_{T-1} \left( h_{T-1} \left( x_{T-1}, \xi_{T-1} \right) \right) \right] \right] \right]
\]

\[
+ \mathbb{E}_{\mathcal{F}_{T-2}} \left[ \inf_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{\mathcal{F}_{T-1}} \left[ u_T \left( h_T \left( x_T(\xi_{[T-1]}), \xi_T \right) \right) \right] \] \right]
\]

\[
= \max_{x_{T-1} \in \mathcal{X}_{T-1}(x_{[T-2],\xi_{[T-2]}})} \left[ \inf_{u_{T-1} \in \mathcal{U}_{T-1}(\xi_{[T-2]})} \mathbb{E}_{\mathcal{F}_{T-2}} \left[ u_{T-1} \left( h_{T-1} \left( x_{T-1}, \xi_{T-1} \right) \right) \right] \right]
\]

\[
+ \max_{x_T(\xi_{[T-1]}) \in \mathcal{X}_{T}(x_{[T-1],\xi_{[T-1]}})} \mathbb{E}_{\mathcal{F}_{T-2}} \left[ \inf_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{\mathcal{F}_{T-1}} \left[ u_T \left( h_T \left( x_T(\xi_{[T-1]}), \xi_T \right) \right) \right] \right]. \tag{EC.15}
\]

This is because the objective in the square brackets is separable and the first term is independent of \( x_T \). Let

\[
f_T(x_T, \xi_{[T-1]}) := \inf_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{\mathcal{F}_{T-1}} \left[ u_T \left( h_T \left( x_T, \xi_T \right) \right) \right]. \tag{EC.16}
\]

For fixed \( \xi_{[T-1]} \), since \( \mathcal{U}_T(\xi_{[T-1]}) \) is a compact set and \( \mathbb{E}_{\mathcal{F}_{T-1}} \left[ u_T \left( h_T \left( x_T, \xi_T \right) \right) \right] \) is continuous in \( x_T \) under conditions (a) and (b), then \( f_T(x_T, \xi_{[T-1]}) \) is finite-valued. Moreover, for any \( \bar{x}_T, \tilde{x}_T \in \mathcal{X}_{T}(x_{[T-1],\xi_{[T-1]}}) \),

\[
| f_T(\bar{x}_T, \xi_{[T-1]}) - f_T(\tilde{x}_T, \xi_{[T-1]}) | \leq \sup_{u_T \in \mathcal{U}_T(\xi_{[T-1]})} \mathbb{E}_{\mathcal{F}_{T-1}} \left[ | u_T \left( h_T \left( \bar{x}_T, \xi_T \right) \right) - u_T \left( h_T \left( \tilde{x}_T, \xi_T \right) \right) | \right]. \tag{EC.17}
\]

Since any \( u_T \in \mathcal{U}_T(\xi_{[T-1]}) \) is globally Lipschitz continuous under condition (a),

\[
| u_T \left( h_T \left( \bar{x}_T, \xi_T \right) \right) - u_T \left( h_T \left( \tilde{x}_T, \xi_T \right) \right) | \leq \kappa(\xi_{[T-1]}) | h_T \left( \bar{x}_T, \xi_T \right) - h_T \left( \tilde{x}_T, \xi_T \right) |,
\]
We show the recursive formula for $u_{x,T} \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})$.

\[
\forall \bar{x}_T, \tilde{x}_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1}), \quad (EC.18)
\]

Under condition (b),

\[
| h_T(\bar{x}_T, \xi_{T}) - h_T(\tilde{x}_T, \xi_{T}) | \leq \sigma(\xi_{T}) \| \bar{x}_T - \tilde{x}_T \|, \forall \bar{x}_T, \tilde{x}_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1}), \quad (EC.19)
\]

where $E_{\mathcal{X}_T(\xi_{T-1})}[\sigma(\xi_{T})] < +\infty$. Combining (EC.17)-(EC.19), we obtain

\[
| f_T(\bar{x}_T, \xi_{T-1}) - f_T(\tilde{x}_T, \xi_{T-1}) | \leq \sup_{u_T \in \mathcal{U}_T(\xi_{T-1})} \mathbb{E}_{\mathcal{X}_T(\xi_{T-1})} [\kappa(\xi_{T-1}) \sigma(\xi_{T}) \| \bar{x}_T - \tilde{x}_T \|] = \kappa(\xi_{T-1}) \mathbb{E}_{\mathcal{X}_T(\xi_{T-1})}[\sigma(\xi_{T})] \| \bar{x}_T - \tilde{x}_T \|. \quad (EC.20)
\]

Hence we obtain the continuity of $f_T$ in $x_T$ for fixed $\xi_{T-1}$ and $\xi_{T}$. Next, we can show that $f_T(x_T, \xi_{T-1})$ is a Carathéodory function, that is, for fixed $x_T$, $f_T(x_T, \xi_{T-1})$ is $\mathcal{F}_{\{T\}}$-measurable. To see this, we note that $\mathbb{E}_{\mathcal{X}_T(\xi_{T-1})} [u_T(h_T(x_T, \xi_{T}))]$ is continuous in $u_T$ and is $\mathcal{F}_{\{T\}}$-measurable for fixed $u_T$, and $\mathcal{U}_T(\xi_{T-1})$ is $\mathcal{F}_{\{T\}}$-measurable, by the marginal map theorem [2, Theorem 8.2.11], $f_T$ is $\mathcal{F}_{\{T\}}$-measurable for fixed $x_T$.

By Lemma [1] we have

\[
\max_{x_T(\xi_{T-1}) \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} \mathbb{E}_{\mathcal{X}_T(x_{T-1}, \xi_{T-1})} \left[ f_T(x_T(\xi_{T-1}), \xi_{T-1}) \right] = \mathbb{E}_{\mathcal{X}_T(x_{T-1}, \xi_{T-1})} \left[ \max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} f_T(x_T, \xi_{T-1}) \right] =: \mathbb{E}_{\mathcal{X}_T(x_{T-1}, \xi_{T-1})} \left[ V_T(x_{T-1}, \xi_{T-1}) \right]. \quad (EC.21)
\]

Note that $V_T(x_{T-1}, \xi_{T-1})$ is well-defined since $f_T(x_T, \xi_{T-1})$ is uniformly bounded under the uniform boundedness condition of $u_T$ and the fact that $\max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} f_T(x_T, \xi_{T-1})$ is $\mathcal{F}_{\{T\}}$-measurable. Combining (EC.13) and (EC.21) gives us that

\[
V_{T-1}(x_{T-2}, \xi_{T-2}) = \max_{x_{T-1} \in \mathcal{X}_{T-1}(x_{T-2}, \xi_{T-2})} \left[ \inf_{u_{T-1} \in \mathcal{U}_{T-1}(\xi_{T-2})} \mathbb{E}_{\mathcal{X}_{T-1}(\xi_{T-2})} \left[ u_{T-1}(h_{T-1}(x_{T-1}, \xi_{T-1})) \right] \right] + \mathbb{E}_{\mathcal{X}_{T-2}} \left[ \max_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_{T-1})} \left[ \inf_{u_T \in \mathcal{U}_T(\xi_{T-1})} \mathbb{E}_{\mathcal{X}_T(\xi_{T-1})} \left[ u_T(h_T(x_T, \xi_{T})) \right] \right] \right] = \max_{x_{T-1} \in \mathcal{X}_{T-1}(x_{T-2}, \xi_{T-2})} \left[ \inf_{u_{T-1} \in \mathcal{U}_{T-1}(\xi_{T-2})} \mathbb{E}_{\mathcal{X}_{T-1}(\xi_{T-2})} \left[ u_{T-1}(h_{T-1}(x_{T-1}, \xi_{T-1})) \right] \right] + V_T(x_{T-1}, \xi_{T-1}). \quad (EC.22)
\]

**Step 3.** We show the recursive formula for $t = T - 2$. Let

\[
f_{T-1}(x_{T-1}, \xi_{T-2}) := \inf_{u_{T-1} \in \mathcal{U}_{T-1}(\xi_{T-2})} \mathbb{E}_{\mathcal{X}_{T-2}} \left[ u_{T-1}(h_{T-1}(x_{T-1}, \xi_{T-1})) + V_T(x_{T-1}, \xi_{T-1}) \right]
\]
Observe first that \( V_T (x_{[T - 1]}, \xi_{[T - 1]}) := \max_{x_T \in \mathcal{X}_T (x_{[T - 1]}, \xi_{[T - 1]})} f_T (x_T, \xi_{[T - 1]}) \) is a Carathéodory function. To see this, we note that by assumption (c), the feasible set \( \mathcal{X}_T (x_{[T - 1]}, \xi_{[T - 1]}) \) is Lipschitz continuous w.r.t. \( x_{T - 1} \). Together with the continuity of \( f_T (x_T, \xi_{[T - 1]}) \) in \( x_T \), we obtain by virtue of [6] Theorem 1] that \( V_T (x_{[T - 1]}, \xi_{[T - 1]}) \) is continuous in \( x_{[T - 1]} \). The measurability follows from [2] Theorem 8.2.11] since \( \mathcal{X}_T (x_{[T - 1]}, \xi_{[T - 1]}) \) is \( \mathcal{F}_{[T - 1]} \)-measurable and \( f_T (x_T, \xi_{[T - 1]}) \) is a Carathéodory function. The conclusion follows since the conditional expectation preserves the above-mentioned continuity and measurability. We now show that \( \inf_{u_{T - 1} \in \mathcal{U}_{T - 1} (\xi_{[T - 2]})} \mathbb{E}_{\mathcal{F}_{T - 2}} \left[ u_{T - 1} (h_{T - 1} (x_{T - 1}, \xi_{T - 1})) \right] \) is also a Carathéodory function. This can be established following a proof analogous to that of \( f_T \).

Summarizing the discussions above, we conclude that \( f_{T - 1} (x_{T - 1}, \xi_{[T - 2]}) \) is a Carathéodory function. Thus the optimization problem at stage \( T - 1 \) can be written as

\[
V_{T - 1} (x_{[T - 2]}, \xi_{[T - 2]}) := \max_{x_{T - 1} \in \mathcal{X}_{T - 1} (x_{[T - 2]}, \xi_{[T - 2]})} \mathbb{E}_{\mathcal{F}_{T - 2}} \left[ f_{T - 1} (x_{T - 1}, \xi_{[T - 2]}) \right].
\]

Since the feasible set is assumed to be compact and the objective function is continuous in \( x_{T - 1} \), the optimal solution exists. Moreover, the Lipschitz continuity of \( \mathcal{X}_{T - 1} (x_{[T - 2]}, \xi_{[T - 2]}) \) in \( x_{[T - 2]} \) and the continuity of \( \mathbb{E}_{\mathcal{F}_{T - 2}} \left[ f_{T - 1} (x_{T - 1}, \xi_{[T - 2]}) \right] \) ensures that \( V_{T - 1} (x_{[T - 2]}, \xi_{[T - 2]}) \) is continuous in \( x_{[T - 2]} \) and for fixed \( x_{[T - 2]} \), we can show by [2] Theorem 8.2.11] that \( V_{T - 1} (x_{[T - 2]}, \xi_{[T - 2]}) \) is \( \mathcal{F}_{[T - 2]} \)-measurable.

Summarizing from the discussions above, the continuity and measurability can be established in the recursive manner. This shows that the recursive formula (16) holds.

Since the optimal solutions exist at individual stages and the recursive formula (16) holds, the global optimal solution is also optimal to the local problems, i.e., the time consistency of the policy holds.

**EC.2.5. Proof of Proposition 2**

Part (i). By definition, \( u \) is nondecreasing over \([a, b]\) with \( u(a) = 0, u(b) = 1 \), and both \( u \) is globally Lipschitz continuous with a uniformly bounded Lipschitz modulus. The monotonic increasing property and the normalization condition ensure the boundedness of the utility functions in the set, the globally Lipschitz continuity guarantees equicontinuity of the class of functions. By Arzelà-Ascoli Theorem (see e.g. [3] Theorem 2.3]), \( \mathcal{U}_T^\beta (\xi_{[t - 1]}) \) is a weakly compact set, that is, it is contained by a compact set in the space of continuous functions. To show the compactness of the set, it suffices to show that the set is closed. Let \( \{ u_k \} \subseteq \mathcal{U}_T^\beta (\xi_{[t - 1]}) \) be a sequence converging to \( u \) under some norm
topology in the $\mathcal{L}^p$ space. The uniform convergence ensures continuity of $u$. For any fixed points $x, y \in [a, b], \quad |u_k(x) - u_k(y)| \leq L(\xi_{t-1})|x - y|, \ \forall k.

By driving $k$ to infinity, we obtain

$$|u(x) - u(y)| \leq L(\xi_{t-1})|x - y|,$$

which means that $u$ is also Lipschitz continuous with modulus being bounded by $L(\xi_{t-1})$. Moreover, since $u_k$ is a concave function, its limit is also a concave function. This shows $u \in \mathcal{U}^L(\xi_{t-1})$.

Part (ii). Let us show first that $\tilde{g}$ is a Carathéodory function. By Lemma EC.1, $\tilde{g}(\cdot, \omega)$ is continuous jointly in $\xi_{t-1}$ and $u$. Let

$$\tilde{g}(u, \omega) := g(u, \xi_{t-1}(\omega)).$$

Then $\tilde{g} : \mathcal{Z} \times \Omega \rightarrow \mathbb{R}$ is $\mathcal{F}_{t-1}$ measurable for every $z \in \mathcal{Z}$, and $\tilde{g}(\cdot, \omega)$ is continuous for fixed $\omega$. This shows that $\tilde{g} : \mathcal{Z} \times \Omega \rightarrow \tilde{R}$ is a Carathéodory function. By Lemma EC.1, $\mathcal{U}^L(\xi_{t-1}(\cdot))$ is measurable in $\mathcal{F}_{t-1}$.

Next, let

$$f_k(u, \omega) := z_k(\xi_{t-1}(\omega))\mathbb{E}[u(Y_k)] - z_k(\xi_{t-1}(\omega))\mathbb{E}[u(W_k)], \ k = 1, \ldots, K.$$

Define level sets $\mathcal{L}_{f_k \leq 0}(\omega) := \{ u \in \mathcal{U}^c | f_k(u, \omega) \leq 0 \}$ and we can rewrite the pairwise comparison ambiguity set as

$$\mathcal{U}^P(\xi_{t-1})(\omega) = \bigcap_{k=1, \ldots, K} \mathcal{L}_{f_k \leq 0}(\omega) \cap \mathcal{U}^L(\xi_{t-1})(\omega).$$
Since $f_k$ is linear in $u$ and measurable w.r.t. $\omega$, then it is a Carathéodory function. By Lemma EC.1, we have that $L_{f_k \leq \theta}$ is closed-valued and measurable. By [2, Theorem 8.2.4], the intersection of those sets in $U^\theta_\omega(\xi_{[t-1]})$ is closed-valued and measurable.

The measurability of $\mathbb{B}(\tilde{u}_t(\cdot,\xi_{[t-1]}), r_t(\xi_{[t-1]}))$ can be observed by [2, Corollary 8.2.13] given that the center $\tilde{u}_t(\cdot,\xi_{[t-1]})$ and the radius $r_t(\xi_{[t-1]})$ are $\mathcal{F}_{t-1}$-measurable. Thus

$$U^\theta_\omega(\xi_{[t-1]}) = \mathbb{B}(\tilde{u}_t(\cdot,\xi_{[t-1]}), r_t(\xi_{[t-1]})) \bigcap \mathcal{U}^L(\xi_{[t-1]})$$

is measurable.

Part (iii). To show the rectangularity of $U$, we recall that in Proposition 1, we have demonstrated that the ambiguity set $U$ defined in Definition 1 satisfies the rectangularity. The key underlying reason is that $U$ is constructed by a series of conditional ambiguity sets $U_t(\xi_{[t-1]})$ which satisfies the following three properties:

- $U_t(\xi_{[t-1]})$ is $\mathcal{F}_{t-1}$-measurable;
- $U_t(\xi_{[t-1]})$ comprises continuous, bounded and monotonically increasing utility functions;
- for given $\xi_{[t-1]}$, $U_t(\xi_{[t-1]})$ is a compact set.

Thus, it suffices to show here that the ambiguity set $U$ constructed through conditional ambiguity sets $U^P_t(\xi_{[t-1]})$ and $U^B_t(\xi_{[t-1]})$ satisfies the three properties. The measurability is addressed in Part (ii) and the compactness is addressed in Part (i). Continuity, boundedness and monotonicity follow from the definition of $U^\omega$. Thus, the ambiguity $U$ constructed by $U^P_t(\xi_{[t-1]})$ or $U^B_t(\xi_{[t-1]})$ in the form of (7) satisfy the conditions in Definition 1 and is thus rectangular.

**Lemma EC.1 (Measurability of level set-mapping).** Let $Z$ be a Polish space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. For a random function $f : Z \times \Omega \rightarrow \mathbb{R}$, i.e., $f(\cdot, \omega)$ is lsc for any fixed $\omega \in \Omega$ and $f(z, \cdot)$ is measurable for any fixed $z \in Z$, the level set mapping $L_{f \leq \alpha} : \Omega \rightarrow Z$, defined by $L_{f \leq \alpha}(\omega) := \{z \in Z | f(z, \omega) \leq \alpha\}$, is closed-valued and measurable.

**Proof:** For fixed $\omega$, the closedness of $L_{f \leq \alpha}(\omega)$ follows directly from the lsc of $f$ in $z$. For any given closed set $Z \in Z$, we consider a closed-valued mapping $R : \omega \rightarrow Z \times [-\infty, \alpha]$. Thus a constant-valued mapping is naturally measurable. By the lsc of $f(\cdot, \omega)$ for any $\omega$, the epi-mapping $\text{epi } f(\omega) : \Omega \mapsto Z \times \mathbb{R}$ is closed-valued and measurable (see [49, Definition 7.35] when $z$ is finite dimensional).

Then we have that

$$L_{f \leq \alpha}^{-1}(Z) = \{\omega | \text{epi } f(\omega) \bigcap (Z \times [-\infty, \alpha]) \neq \emptyset\} = \text{dom}(\text{epi } f \bigcap (Z \times [-\infty, \alpha])).$$

By [2, Theorem 8.2.4], $\omega \rightarrow \text{epi } f(\omega) \bigcap (Z \times [-\infty, \alpha])$ is a closed-valued and measurable set-valued mapping. Thus, its domain $\text{dom}(\text{epi } f \bigcap (Z \times [-\infty, \alpha])) \in \mathcal{F}$. This shows that $L_{f \leq \alpha}^{-1}(Z) \in \mathcal{F}$. By [2, Theorem 8.1.4 (iii)], we know that the $L_{f \leq \alpha}$, as an inverse of $L_{f \leq \alpha}^{-1}(Z)$, is measurable. □
EC.2.6. Proof of Lemma 3

Before presenting a proof for the lemma, we need the following technical result which is drawn from [24 Proposition 4.1] and the proof of [24 Theorem 4.1].

**Proposition EC.1.** Let $u \in \mathcal{U}$. Assume that $u(\cdot)$ is Lipschitz continuous over an interval $[a, b]$ with modulus $L$ and $u_N$ is its piecewise linear approximation, that is,

$$ u_N(y) := u(y_{i-1}) + \frac{u(y_i) - u(y_{i-1})}{y_i - y_{i-1}}(y - y_{i-1}), \text{ for } y \in [y_{i-1}, y_i], \ i = 2, \ldots, N, \quad (\text{EC.25}) $$

where $y_1 = a, y_N = b$. Let $\beta_N := \max_{i=2, \ldots, N}(y_i - y_{i-1})$. Then the following assertions hold.

(i) $\|u_N - u\|_\infty := \sup_{y \in [a, b]} |u_N(y) - u(y)| \leq L\beta_N$.

(ii) For $\mathcal{G} = \mathcal{G}_L$, $d_{\mathcal{G}}(u, u_N) \leq 2\beta_N$.

(iii) For $\mathcal{G} = \mathcal{G}_1$, $d_{\mathcal{G}}(u, u_N) \leq L\beta_N$.

**Proof.** We call $u_N$ defined in (EC.25) a *projection* of $u$ on $\mathcal{U}_N$. Using the proposition, we are able to derive an upper bound for the Hausdorff distance between $\mathbb{B}(u, r)$ and $\mathbb{B}(v, r)$.

It suffices to show that

$$ \mathbb{D}(\mathbb{B}(u, r_1), \mathbb{B}(v, r_2); d_{\mathcal{G}}) \leq d_{\mathcal{G}}(u, v) + |r_2 - r_1| \quad (\text{EC.26}) $$

and

$$ \mathbb{D}(\mathbb{B}(v, r_2), \mathbb{B}(u, r_1); d_{\mathcal{G}}) \leq d_{\mathcal{G}}(u, v) + |r_2 - r_1|. \quad (\text{EC.27}) $$

We prove (EC.26). The conclusion is trivial if $\mathbb{B}(u, r_1) \subset \mathbb{B}(v, r_2)$, so we consider the case that $\mathbb{B}(u, r_1) \not\subset \mathbb{B}(v, r_2)$. Let $\tilde{v} \in \mathbb{B}(u, r_1) \setminus \mathbb{B}(v, r_2)$ and $\lambda = r_2/d_{\mathcal{G}}(\tilde{v}, v)$. Then $\lambda \in (0, 1)$. Let $v_\lambda = \lambda v + (1 - \lambda)\tilde{v}$. Then

$$ d_{\mathcal{G}}(v_\lambda, v) = d_{\mathcal{G}}((1 - \lambda)v + \lambda\tilde{v}, v) \leq \lambda d_{\mathcal{G}}(\tilde{v}, v) = r_2. $$

This shows $v_\lambda \in \mathbb{B}(v, r_2)$. Thus

$$ d_{\mathcal{G}}(\tilde{v}, \mathbb{B}(v, r_2)) \leq d_{\mathcal{G}}(\tilde{v}, v_\lambda) = (1 - \lambda)d_{\mathcal{G}}(\tilde{v}, v) = d_{\mathcal{G}}(\tilde{v}, v) - r_2 \leq d_{\mathcal{G}}(u, v) + r_1 - r_2. $$

Swapping the positions of the two balls in the above discussions, we obtain (EC.27). □

**Proof of Lemma 3.** Inequality (28) follows straightforwardly from (27) and (EC.29), so we only prove (27). By the triangle inequality,

$$ \mathbb{H}(\mathbb{B}(u, r), \mathbb{B}(v, r); d_{\mathcal{G}}) \leq \mathbb{H}(\mathbb{B}(u, r), \mathbb{B}(u, r); d_{\mathcal{G}}) + d_{\mathcal{G}}(u, v). $$

So it suffices to show that

$$ \mathbb{H}(\mathbb{B}(u, r), \mathbb{B}(u, r); d_{\mathcal{G}}) \leq 4 \max(2, L)\beta_N. $$


By definition, $B_N(u, r) \subset B(u, r)$, so it is enough to show that
\[
\mathbb{D}(B(u, r), B_N(u, r); d_{\mathcal{F}}) \leq 4 \max(2, L) \beta_N. \tag{EC.28}
\]

Let $\epsilon$ be a small positive number and $u^\epsilon \in B(u, r) \setminus B_N(u, r)$ be such that
\[
d_{\mathcal{F}}(u^\epsilon, B_N(u, r)) \geq \mathbb{D}(B(u, r), B_N(u, r); d_{\mathcal{F}}) - \epsilon.
\]

For the given $u^\epsilon$, we may find $u_N^\epsilon \in \mathcal{U}_N$ as that in Proposition EC.1 such that
\[
d_{\mathcal{F}}(u^\epsilon, u_N^\epsilon) \leq \max(2, L) \beta_N.
\]

If $u_N^\epsilon \in B_N(u, r)$, then
\[
\mathbb{D}(B(u, r), B_N(u, r); d_{\mathcal{F}}) \leq d_{\mathcal{F}}(u^\epsilon, u_N^\epsilon) + \epsilon \leq d_{\mathcal{F}}(u^\epsilon, u_N^\epsilon) + \epsilon \leq \max(2, L) \beta_N + \epsilon
\]
and hence EC.28 because $\epsilon$ can be driven to zero. So we are left with the case that $u_N^\epsilon \notin B_N(u, r)$. Let $\lambda = \frac{r_{\mathcal{F}}(x_N, u)}{d_{\mathcal{F}}(x_N, u)}$. Then $\lambda \in (0, 1)$. Let $u_\lambda = \lambda u_N^\epsilon + (1 - \lambda)u$. Then $u_\lambda \in \mathcal{U}_N$ and $d_{\mathcal{F}}(u_\lambda, u) = \lambda d_{\mathcal{F}}(u_N^\epsilon, u) = r$. This shows $u_\lambda \in B_N(u, r)$. Thus
\[
d_{\mathcal{F}}(u^\epsilon, B_N(u, r)) \leq d_{\mathcal{F}}(u^\epsilon, u_\lambda) \leq d_{\mathcal{F}}(u^\epsilon, u_N^\epsilon) + d_{\mathcal{F}}(u_N^\epsilon, u_\lambda)
\]
\[
\leq 2 \max(2, L) \beta_N + (1 - \lambda) d_{\mathcal{F}}(u_N^\epsilon, u) = 2 \max(2, L) \beta_N + d_{\mathcal{F}}(u_N^\epsilon, u) - r
\]
\[
\leq 2 \max(2, L) \beta_N + d_{\mathcal{F}}(u_N^\epsilon, u^\epsilon) + d_{\mathcal{F}}(u^\epsilon, u) - r \leq 4 \max(2, L) \beta_N + r - r
\]
\[
= 4 \max(2, L) \beta_N,
\]
and hence
\[
\mathbb{D}(B(u, r), B_N(u, r); d_{\mathcal{F}}) \leq d_{\mathcal{F}}(u^\epsilon, B_N(u, r)) + \epsilon
\]
which gives EC.28 by driving $\epsilon$ to zero.

**EC.2.7. Proof of Theorem 2**

We prove by induction. Observe that for any $x_{t-1}$ and $\xi_{[t-1]}$, $t = 2, \ldots, T$,
\[
|V_t(x_{[t-1]}, \xi_{[t-1]}) - \tilde{V}_t(x_{[t-1]}, \xi_{[t-1]})|
\]
\[
\leq \max_{x_t \in \mathcal{X}_T(x_{[t-1]}, \xi_{[t-1]})} \inf_{u_t \in \mathcal{U}_T(x_{[t-1]}, \xi_{[t-1]})} \mathbb{E}_{[T-1]} \left[ u_t \left( h_t(x_t, \xi_t) \right) + V_{t+1} \left( x_{[t]}, \xi_{[t]} \right) \right]
\]
\[
- \inf_{u_t \in \mathcal{U}_T(x_{[t-1]}, \xi_{[t-1]})} \mathbb{E}_{[T-1]} \left[ u_t \left( h_t(x_t, \xi_t) \right) + \tilde{V}_{t+1} \left( x_{[t]}, \xi_{[t]} \right) \right]
\]
\[
\leq \mathbb{H} \left( B(\tilde{u}_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})), B_N(\tilde{u}_t(\cdot, \xi_{[t-1]}), r_t(\xi_{[t-1]})); d_{\mathcal{F}} \right)
\]
The resulting robust dynamic programming equation can be written as
\[
+ \max_{x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t-1]})} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ \left| V_{t+1}(x_t,\xi_t) - \tilde{V}_{t+1}(x_t,\xi_t) \right| \right]
\]
and setting \( V_{T+1} = 0 \) and \( \tilde{V}_{T+1} = 0 \) and the last inequality follows from Lemma 3. Assume for stage \( t+1 \) that
\[
\left| V_{t+1}(x_t,\xi_t) - \tilde{V}_{t+1}(x_t,\xi_t) \right| \leq \sum_{s=t+1}^{T} 6 \mathbb{E} \left[ \max(2, L(\xi_{s-1}))) \beta_N(\xi_{s-1}) \mid \mathcal{F}_s \right],
\]
for any fixed \( x_t \) and \( \xi_t \). Then
\[
\left| V_{t}(x_{[t-1]},\xi_{[t-1]}) - \tilde{V}_{t}(x_{[t-1]},\xi_{[t-1]}) \right| \leq \sum_{s=t}^{T} 6 \mathbb{E} \left[ \max(2, L(\xi_{s-1}))) \beta_N(\xi_{s-1}) \mid \mathcal{F}_s \right]
\]
which gives rise to (29).

**EC.2.8. Proof of Theorem 3**

The resulting robust dynamic programming equation can be written as
\[
\tilde{V}_t(x_{[t-1]},\xi_{[t-1]}) = \max_{x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t-1]})} \inf_{u_t \in U_t(x_{[t-1]},\xi_{[t-1]})} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(h_t(x_t,\xi_t)) + \tilde{V}_{t+1}(x_t,\xi_t) \right]. \quad \text{(EC.29)}
\]
We can separate the maximin operations by writing \( \tilde{V}_t(x_{[t-1]},\xi_{[t-1]}) \) as
\[
\tilde{V}_t(x_{[t-1]},\xi_{[t-1]}) = \max_{x_t \in \mathcal{X}_t(x_{[t-1]},\xi_{[t-1]})} \hat{V}_t(x_t,\xi_t) + \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ \tilde{V}_{t+1}(x_t,\xi_t) \right],
\]
where
\[
\hat{V}_t(x_t,\xi_t) := \inf_{u_t} \mathbb{E}_{|\mathcal{F}_{t-1}} \left[ u_t(h_t(x_t,\xi_t)) \right], \quad \text{(EC.30a)}
\]
\[
\text{s.t. } d_K(u_t,\tilde{u}_t^N(\cdot,\xi_{[t-1]})) \leq r_t(\xi_{[t-1]}), \quad \text{(EC.30b)}
\]
\[
u_t \in \mathcal{U}_N, \quad \text{(EC.30c)}
\]
\[
\text{Lip}(u_t) \leq L, \quad \text{(EC.30d)}
\]
\[
u_t'' \leq 0. \quad \text{(EC.30e)}
\]
By utilizing the piecewise linear structure of \( u \) and setting \( \alpha_j = u_t(y_j) \) and \( \beta_j = u''_t(y_j) \) at the breakpoints \( y_j, \ j = 1, \ldots, N \), we can effectively write (EC.30) as
\[
\hat{V}_t(x_t,\xi_t) := \quad \text{(EC.31a)}
\]
\[
\inf_{\lambda, \mu, \rho, \phi, \alpha, \beta, \varepsilon, \varphi} \sum_{i=1}^{S} \mathbb{P}(\xi_t = \xi^i_t | \xi_{t-1}) \left( \varepsilon_i h_t(x_t, \xi^i_t) + \varphi_i \right) \quad \text{(EC.31b)}
\]
\[
\text{s.t.} \quad \frac{1}{2} \sum_{j=2}^{N} (\lambda_j + \mu_j + \rho_j + \phi_j) (y_j - y_{j-1})^2 \leq r_t(\xi_{t-1}) \quad \text{(EC.31c)}
\]
\[
\beta_j - \beta_j + \lambda_j - \mu_j + \rho_j - \phi_j = 0, \quad j = 2, \cdots, N, \quad \text{(EC.31d)}
\]
\[
(\mu_2 - \lambda_2) (y_2 - y_1) = 0, \quad \text{(EC.31e)}
\]
\[
(\mu_{j+1} - \lambda_{j+1}) (y_{j+1} - y_j) + (\rho_j - \phi_j) (y_j - y_{j-1}) = 0, \quad j = 2, \cdots, N-1, \quad \text{(EC.31f)}
\]
\[
(\rho_N - \phi_N) (y_N - y_{N-1}) = 0, \quad \text{(EC.31g)}
\]
\[
\mu_j, \lambda_j, \rho_j, \phi_j \geq 0, \quad j = 2, \cdots, N. \quad \text{(EC.31h)}
\]
\[
y_j \varepsilon_i + \varphi_i \geq \alpha_j, \quad i = 1, \ldots, S, \quad j = 1, \ldots, N, \quad \text{(EC.31i)}
\]
\[
\alpha_{j+1} - \alpha_j = \beta_{j+1} (y_{j+1} - y_j), \quad j = 1, \ldots, N-1, \quad \text{(EC.31j)}
\]
\[
\alpha_{j+1} - \alpha_j \geq \beta_{j+2} (y_{j+1} - y_j), \quad j = 1, \ldots, N-2, \quad \text{(EC.31k)}
\]
\[
0 \leq \beta_{j+1} \leq L(\xi_{t-1}), \quad j = 1, \ldots, N-1, \quad \text{(EC.31l)}
\]
\[
\alpha_1 = 0, \quad \alpha_N = 1, \quad \varepsilon_i \geq 0, \quad i = 1, \ldots, S, \quad \text{(EC.31m)}
\]

where constraints \((\text{EC.31c})-\text{(EC.31h)}\) characterize the Kantorovich ball \((\text{EC.30b})\) as we described in \((32)\). Constraint \((\text{EC.31i})\) characterizes the piecewise linear structure of \(u_t\) in \((\text{EC.30c})\) and constraints \((\text{EC.31j})-\text{(EC.31k)}\) imply that \(\beta_j \geq \beta_{j+1}\) and hence the concavity of the piecewise linear utility function \(u_t\). Constraint \((\text{EC.31l})\) is concerned with the non-decreasing property and Lipschitz continuity of \(u_t\) with modules bounded by \(L(\xi_{t-1})\) as in \((\text{EC.30d})\). As in the literature of PRO models in one-stage decision making, the evaluation of the utility function at point \(h_t(x_t, \xi^i_t)\) in the objective is carried out by a linear function passing through point \((h_t(x_t, \xi^i_t), u_t(h_t(x_t, \xi^i_t))\), with slope \(\varepsilon_i\) and intercept \(\varphi_i\). Constraint \((\text{EC.31l})\) requires that all those linear pieces upper bound \(u_t\) at those breakpoints. \(\ddot{\beta}_j = \frac{\ddot{u}(y_j \xi_{t-1}) - \ddot{u}(y_{j-1} \xi_{t-1})}{y_{j+1} - y_j}\) is the slope of nominal utility at those breakpoints.

By taking the duality of the linear program \((\text{EC.31l})\), we obtain
\[
\max \quad \theta_{N-1} + \sum_{i=1}^{S} \mu_{i,N} - L(\xi_{t-1}) \sum_{j=1}^{N-1} \eta_j - \sum_{j=2}^{N} \beta_j w_j - r_t(\xi_{t-1}) \zeta
\]
\[
\text{s.t.} \quad \sum_{j=1}^{N} y_j \mu_{i,j} \leq \mathbb{P}(\xi_t = \xi^i_t | \xi_{t-1}) h_t(x_t, \xi^i_t), \quad i = 1, \ldots, S,
\]
\[
\frac{p_1}{p_s} - \sum_{j=1}^{N} \mu_{i,j} = 0, \quad i = 1, \ldots, S,
\]
\[
\theta_{j-1} y_{j-1} - \theta_{j-1} y_j + v_{j-2} (y_{j-1} - y_{j-2}) + w_j + \eta_{j-1} \geq 0, \quad j = 3, \ldots, N-1,
\]
\[
\theta_{1} y_{1} - \theta_{1} y_{2} + w_{2} + \eta_{1} \geq 0,
\]
\[
\theta_{N-1} y_{N-1} - \theta_{N-1} y_{N} + v_{N-2} (y_{N-1} - y_{N-2}) + w_{N} + \eta_{N-1} \geq 0,
\]
\[\begin{align*}
\theta_{j-1} - \theta_j + \sum_{i=1}^S \mu_{i,j} - v_{j-1} + v_j &= 0, \quad j = 2, \cdots, N-2, \\
\theta_{N-2} - \theta_{N-1} + \sum_{i=1}^S \mu_{i,N-1} - v_{N-2} &= 0, \\
w_j &\leq z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \quad j = 2, \cdots, N, \\
-w_j &\leq -z_{j-1}(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \quad j = 2, \cdots, N, \\
w_j &\leq z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \quad j = 2, \cdots, N, \\
-w_j &\leq -z_j(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma, \quad j = 2, \cdots, N, \\
\theta &\in \mathbb{R}^{N-1}, v \in \mathbb{R}^{N-2}, \eta \in \mathbb{R}^{N-1}, \mu \in \mathbb{R}^{S \times N}, \varsigma \in \mathbb{R}_+, w \in \mathbb{R}^{N-1}, z \in \mathbb{R}^N,
\end{align*}\]

Taking this duality form back to (EC.29) gives the results. \hfill \Box

**EC.3. An example of time inconsistency**

In the Section 3, we have demonstrated the rectanglarity of the ambiguity set \(\mathcal{U}\) and subsequently time consistency of problem (MS-PRO-SD). It is natural to ask whether the same property is retained by the ambiguity set of state-independent utility functions and the robust model (MS-PRO-SID). The answer is no. In this section, we use a counter example to illustrate this fact.

Consider a preference robust counterpart of the stage-wise return rate utility maximization problem in Example [EC.1](Example) with three time points 0, 1, 2 and two investment stages 1, 2 between the time points. At each time point, there are two branches from the current state with probability 50% each. Thus, we have a two-stage scenario tree with an initial node at time point 0, two nodes at the end of the first stage and four leaf nodes at the end of the second stage. We assume that there are two risky assets with random excess return rates \(r_t = [r_{1,t}, r_{2,t}]\) in range \([0,1]\) at the two stages \(t = 1, 2\). We denote the realization of \(r_t\) on the \(k\)-th node at stage \(t\) by \(r_{t,k}\). We mark the return rates \(r_{t,k}\) around the nodes of each scenario on the scenario tree, see Figure [EC.1](Figure)

![Figure EC.1](Branching probability, realizations of two risky assets’ return rates and predictable portfolios on the two-stage scenario tree)
At the beginning of each stage, the investor may reallocate the wealth among the two risky assets. We assume that the portfolio at stage $t$ is $x_t = [x^1_t, x^2_t]$ with $x^i_t + x^2_t = 1$, where $x^i_t$, $t = 1, 2, i = 1, 2$ is the proportion of wealth invested in the $i$-th asset at stage $t$. The first stage portfolio is deterministic while the second stage portfolio is random and scenario dependent.

We assume that the DM is ambiguous about the true utility function which lies in the ambiguity set:

$$U = \{ u^1(y) := \min\{3y, 0.5y + 0.5\}, \quad u^2(y) = 2y - y^2 \},$$

where $u^i(0) = 0$, $u^i(1) = 1$ for $i = 1, 2$. It is easy to see that $u^i$ is strictly increasing and concave over $[0, 1]$ and $U$ is independent of state and stage.

![Figure EC.2](image)

**Figure EC.2** Plot of $u^1(y)$ and $u^2(y)$.

We consider a simple two-stage portfolio selection problem under the state-independent preference robust expected utility model (9):

$$\max_{x_1, x_2(\cdot)} \inf_{u_1 \in U, u_2 \in U} \mathbb{E}[u_1(x^T_1 r_1) + \mathbb{E}[u_2(x^T_2 r_2)|r_1]]$$

s.t. $e^T x_1 = 1$, $x_1 \in \mathbb{R}^2_+$, $e^T x_2(r_1) = 1$, $x_2(\cdot) \in L^0(\mathbb{R}^2_+)$, (EC.32)

where $L^0(\mathbb{R}^2_+)$ denotes the space of measurable functions taking finite values in $\mathbb{R}^2_+$, and discuss how the worst-case utility function is identified at each investment stage.

**EC.3.1. Non-rectangularity of the preference robust counterpart**

We begin by investigating rectangularity of the ambiguity set in problem (EC.32), which is essentially about the consistency between the global preference robust counterpart

$$f^*(x) := \inf_{u_1 \in U, u_2 \in U} \mathbb{E}[u_1(x^T_1 r_1) + \mathbb{E}[u_2(x^T_2 r_2)|r_1]]$$

with global worst-case utility functions $u^*_1, u^*_2$ and the nested local preference robust counterpart

$$\hat{f}^*(x) := \inf_{u_1 \in U} \mathbb{E} \left[ u_1(x^T_1 r_1) + \inf_{u_2 \in U} \mathbb{E}[u_2(x^T_2 r_2)|r_1] \right],$$

(EC.34)
with local worst-case utility functions $\hat{u}_1^*, \hat{u}_2^*(\cdot)$. Note that in both problems (EC.33) and (EC.34), the decision variables are fixed. Here we set $x_1 = [1, 0], x_{2,1} = [1, 0]$ and $x_{2,2} = [1, 0]$ and demonstrate that the worst-case utility functions of the two problems are different at some state in the second stage. Since both problems are decomposable, we may solve them by solving $f_1^* = \inf_{u_1 \in U} \mathbb{E}[u_1(x_1^T r_1)]$, $f_2^* = \inf_{u_2 \in U} \mathbb{E}[u_2(x_2^T r_2)|r_1]$, $\hat{f}_1^*(r_1) = \inf_{u_2 \in U} \mathbb{E}[u_2(x_2^T r_2)|r_1]$ and then setting $f^* = f_1^* + f_2^*$ and $\hat{f}^* = \hat{f}_1^* + \mathbb{E}[\hat{f}_2^*(r_1)]$.

For instance,

$$f_1^* = \inf_{u_1 \in U} \frac{1}{2} \left[ u_1([1, 0] \times [0, 0]^T) + u_1([1, 0] \times [0.8, 0.2]^T) \right]$$

$$= \inf_{u_1 \in (u^1, u^2)} \frac{1}{2} \left[ u_1(0) + u_1(0.8) \right]$$

$$= 0 + \min \left\{ \frac{1}{2}, \min \{3 \times 0.8, 0.5 \times 0.8 + 0.5\}, \frac{1}{2} (2 \times 0.8 - 0.8^2) \right\}$$

$$= \min \{0.45, 0.48\} = 0.45.$$

The worst-case utility value is attained by $u^1(\cdot)$. Likewise

$$f_2^* = \inf_{u_2 \in U} \frac{1}{2} \left[ u_2([1, 0] \times [0.6, 0.2]^T) + u_2([1, 0] \times [0.6, 0.8]^T) \right]$$

$$+ \frac{1}{2} \left[ u_2([1, 0] \times [0.4, 0.6]^T) + u_2([1, 0] \times [1, 0.6]^T) \right]$$

$$= \inf_{u_2 \in (u^1, u^2)} \frac{1}{2} \left[ u_2(0.4) + u_2(0.6) + \frac{1}{2} [u_2(0.4) + u_2(1)] \right]$$

$$= \min \left\{ \frac{1}{2} (0.8 + 0.85), \frac{1}{2} (0.84 + 0.82) \right\} = \min \{0.825, 0.83\} = 0.825.$$

The worst-case utility value is attained by $u^1(\cdot)$. Summing them up, we obtain $f^* = f_1^* + f_2^* = 1.275$.

The analysis is depicted at the left-hand side of Figure EC.3 where “PLU” denotes the piecewise linear utility function. We now move on to calculate $\hat{f}^*$.

$$\hat{f}_2^*(r_{1,1}) = \inf_{u_2 \in U} \frac{1}{2} \left[ u_2([1, 0] \times [0.6, 0.2]^T) + u_2([1, 0] \times [0.6, 0.8]^T) \right]$$

$$= \inf_{u_2 \in (u^1, u^2)} \frac{1}{2} [u_2(0.6) + u_2(0.6)]$$

$$= \min \{0.8, 0.84\} = 0.8.$$

The worst-case utility (locally) at the second-stage is attained by $u^1(\cdot)$ in the first node at stage 1.

$$\hat{f}_2^*(r_{1,2}) = \inf_{u_2 \in U} \frac{1}{2} \left[ u_2([1, 0] \times [0.4, 0.6]^T) + u_2([1, 0] \times [1, 0.6]^T) \right]$$

$$= \inf_{u_2 \in (u^1, u^2)} \frac{1}{2} [u_2(0.4) + u_2(1)]$$

$$= \min \{0.83, 0.82\} = 0.82.$$
The worst-case utility (locally) is attained by \( u^2(\cdot) \) in the second node at stage 1. Consequently
\[
\hat{f}_2^* = \frac{1}{2}(0.8 + 0.82) = 0.81 < 0.825 = f_2^*.
\]

From the analysis above, we can see that the DM would adopt a quadratic utility (QU) function \( u^2(\cdot) \) which is more risk-averse after she/he has earned some money (second node at stage 1) but would take a piecewise linear utility function \( u^1(\cdot) \) after she/he has failed to earn anything (first node at stage 1). The analysis is depicted at the right-hand side of Figure EC.3. The overall worst-case expected utility value in the two stages is \( \hat{f}^* = f_1^* + \hat{f}_2^* = f_1^* + \frac{1}{2}(\hat{f}_{2,1}^* + \hat{f}_{2,2}^*) = 1.26 \).

Summarizing the calculations of both problems (EC.33) and (EC.34), we conclude that \( \hat{f}^* = 1.26 < 1.275 = f^* \). This is because model (EC.33) chooses the worst-case utility function independent of scenarios in the second stage whereas model (EC.34) chooses the worst-case utility function after observing the scenarios and hence is more conservative. The underlying reason is that the utility functions in the ambiguity set are state-independent.

**EC.3.2. Time inconsistency of the preference robust optimization model**

We now turn to discuss time consistency of the preference robust optimization problem (EC.32). Let
\[
\{x_1^*, x_2^*(\cdot)\} = \arg\max_{x_1 \in X_1, x_2(\cdot) \in X_2} \inf_{u_1 \in U, u_2 \in \hat{U}} \mathbb{E}\left[u_1(x_1^T r_1) + \mathbb{E}\left[u_2(x_2^T r_2)|r_1\right]\right] \tag{EC.35}
\]
and
\[
\hat{x}_2^*(\cdot) = \arg\max_{x_2 \in X_2} \inf_{u_2 \in U} \mathbb{E}\left[u_2(x_2^T r_2)|r_1\right], \tag{EC.36}
\]
where \( X_1 = \{x_1 \in \mathbb{R}_+^2 | x_1^1 + x_1^2 = 1\}, \ X_2 = \{x_2(\cdot) \in \mathcal{L}^0(\mathbb{R}_+^2) | x_2^1(r_1) + x_2^2(r_1) = 1\} \). We want to show that \( x_2^*(\cdot) \neq \hat{x}_2^*(\cdot) \). Observe that due to the decomposable structure of problem (EC.35),
\[
\{x_1^*, x_2^*(\cdot)\} = \left\{ \arg\max_{x_1 \in X_1} \inf_{u_1 \in U} \mathbb{E}\left[u_1(x_1^T r_1)\right], \arg\max_{x_2(\cdot) \in X_2} \inf_{u_2 \in U} \mathbb{E}\left[u_2(x_2^T r_2)\right] \right\}. \tag{EC.37}
\]
For the first-stage optimization problem in (EC.37), the optimal portfolio is always \( x_1^* = [1, 0] \) as \( r^1 \geq r^2 \) in both scenarios and the utility function does not affect the optimal choice, given that the utility function is increasing. To identify the worst-case utility, let us compare the optimal values \( u_1(x_1^T r_1) \) under the two utility functions. It is easy to obtain that the optimal value is 0.48 under the quadratic utility function and 0.45 under the piecewise linear utility function. Thus the worst-case utility function \( u^*_1 \) at the first-stage is \( u^1(\cdot) \).

Let us now look at the second-stage optimization problem in (EC.37). By the finiteness of the preference robust set \( U \) and the scenario tree structure of the random return \( r \), the second-stage optimization problem in (EC.37) can be formulated as

\[
\begin{align*}
v_2^* &= \max_{z \in \mathbb{R}_+^2} z \\
&\text{s.t.} \quad z \leq \mathbb{E}[\mathbb{E}[\min\{3x_2^T r_2, 0.5x_2^T r_2 + 0.5\}|r_1]], \\
&\quad z \leq \mathbb{E}[\mathbb{E}[2x_2^T r_2 - (x_2^T r_2)^2]|r_1]] \\
&= \max_{x_2, y, z} z \\
&\text{s.t.} \quad z \leq \frac{1}{4}\sum_{i=1}^4 y_i, \\
y_1 \leq 3x_2^T r_{2,i}, \quad i = 1, 2, \\
y_2 \leq 0.5x_2^T r_{2,i} + 0.5, \quad i = 1, 2, \\
y_3 \leq 3x_2^T r_{2,i}, \quad i = 3, 4, \\
y_4 \leq 0.5x_2^T r_{2,i} + 0.5, \quad i = 3, 4, \\
z \leq \frac{1}{4}\sum_{i=1}^4 [2x_2^T r_{2,i} - (x_2^T r_{2,i})^2] + \frac{1}{4}\sum_{i=3}^4 [2x_2^T r_{2,i} - (x_2^T r_{2,i})^2], \\
z \in \mathbb{R}, y \in \mathbb{R}^4, x_2 \in \mathbb{R}_+^{4x2}, x_{2,i}^1 + x_{2,i}^2 = 1, \quad i = 1, 2.
\end{align*}
\]

by adding some auxiliary variables. Problem (EC.38) is a convex quadratic constrained quadratic programming problem which can be solved efficiently by CVX in Matlab.

Alternatively, we can solve (EC.38) in a closed-form. As the return rates in all scenarios are larger than 0.2, thus \( 3x_2^T r_2 \geq 0.5x_2^T r_2 + 0.5 \) in all scenarios. Then we can reformulate (EC.38) as

\[
\begin{align*}
v_2^* &= \max_{x_2, y, z} \min \left\{ \frac{1}{2} \left( 0.5x_2^T r_{2,i} \begin{bmatrix} 0.6 \\ 0.5 \end{bmatrix} + 0.5 + 0.5x_2^T r_{2,i} \begin{bmatrix} 0.7 \\ 0.6 \end{bmatrix} + 0.5 \right), \\
&\quad \frac{1}{2} \left( 0.6 - (x_2^T r_{2,i} \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix})^2 \right) + \frac{1}{2} \left( 2x_2^T r_{2,i} \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} - (x_2^T r_{2,i} \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix})^2 \right) \\
&\quad + \frac{1}{2} \left( 2x_2^T r_{2,i} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} - (x_2^T r_{2,i} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix})^2 \right) + \frac{1}{2} \left( 2x_2^T r_{2,i} \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} - (x_2^T r_{2,i} \begin{bmatrix} 1 \\ 0.6 \end{bmatrix})^2 \right) \right\} \\
&= \max_{x_2, y, z} \min \left\{ \frac{1}{2} \left( x_{2,i} \begin{bmatrix} 0.6 \\ 0.5 \\ 0.7 \\ 0.6 \end{bmatrix} \right) + 0.5, \\
x_{2,i} \begin{bmatrix} 0.6 \\ 0.5 \\ 0.18 \\ 0.15 \\ 0.15 \\ 0.17 \end{bmatrix} + x_{2,i} \begin{bmatrix} 0.7 \\ 0.6 \end{bmatrix} - x_{2,i} \begin{bmatrix} 0.29 \\ 0.21 \\ 0.21 \\ 0.18 \end{bmatrix} \right\} \\
&\quad \text{s.t.} \quad x_{2,i}^1 + x_{2,i}^2 = 1, \quad x_{2,i} \in [0, 1]^2, \quad i = 1, 2.
\end{align*}
\]

By eliminating variables \( x_{2,i}^2, i = 1, 2 \), we can obtain a reduced maximin problem

\[
\begin{align*}
v_2^* &= \max_{x_{2,i}^1, x_{2,i}^2} \min \left\{ 0.025(x_{2,i}^1 + x_{2,i}^2) + 0.775, -0.05(x_{2,i}^1)^2 + 0.14x_{2,i}^1 - 0.05(x_{2,i}^2)^2 + 0.04x_{2,i}^1 + 0.75 \right\} \\
&\quad \text{s.t.} \quad x_{2,i}^1 \in [0, 1], x_{2,i}^2 \in [0, 1],
\end{align*}
\]
where \( x_{2,i}^2 = 1 - x_{2,i}^1, \ i = 1,2. \) This is a maximization problem with a piecewise quadratic objective function. The optimum is attained potentially at two sets of points: the global maximizers of each piece, and the set of points where the two pieces intersect, that is,

\[
  v_2^* = \max \{ \min \{v_{\text{linear}}^*, v_{\text{quad}}^*\}, v_{\text{int}}^* \},
\]

where

\[
v_{\text{linear}}^* = \max_{x_{2,1}^1, x_{2,2}^1} 0.025(x_{2,1}^1 + x_{2,2}^1) + 0.775, \tag{EC.39}
\]

\[
v_{\text{quad}}^* = \max_{x_{2,1}^1, x_{2,2}^1} -0.05(x_{2,1}^1)^2 + 0.14x_{2,1}^1 - 0.05(x_{2,2}^1)^2 + 0.04x_{2,2}^1 + 0.75 \quad \text{s.t.} \quad x_{2,1}^1 \in [0,1], x_{2,2}^1 \in [0,1],
\]

\[
v_{\text{int}}^* = \max_{x_{2,1}^1, x_{2,2}^1} 0.025(x_{2,1}^1 + x_{2,2}^1) + 0.775 \quad \text{s.t.} \quad 0.025(x_{2,1}^1 + x_{2,2}^1) + 0.775 = -0.05(x_{2,1}^1)^2 + 0.14x_{2,1}^1 - 0.05(x_{2,2}^1)^2 + 0.04x_{2,2}^1 + 0.75, \tag{EC.40}
\]

and

\[
x_{2,1}^1 \in [0,1], x_{2,2}^1 \in [0,1].
\]

Problem \((\text{EC.39})\) achieves its maximum at the boundary \( x_{2,1}^1 = 1, x_{2,2}^1 = 1 \) with \( v_{\text{linear}}^* = 0.825. \) Problem \((\text{EC.40})\) achieves its maximum at the boundary of \( x_{2,1}^1 = 1 \) and stationary point of \( x_{2,2}^1 = 0.4 \) with \( v_{\text{quad}}^* = 0.848. \) Problem \((\text{EC.41})\) attains the maximum at the intersection point \( x_{2,1}^1 = 0.8, x_{2,2}^1 = 1 \) with \( v_{\text{int}}^* = 0.82. \) Thus \( v_2^* = \max \{v_{\text{linear}}^*, v_{\text{quad}}^*, v_{\text{int}}^* \} = 0.825 \) with the optimal solution \( x_{2,1}^* = [1,0], x_{2,2}^* = [1,0]. \) The PRO model has a piecewise linear worst-case utility function at its optimum \( u_2^* \). The analysis is depicted at the left-hand side of Figure \((\text{EC.4}).\)

We now turn to discuss solution of PRO problem \((\text{EC.36}).\) Suppose that at the beginning of the second-stage, the DM can predict different scenarios that would occur at the end of the second stage. Then the DM may consider the sub-PRO problem \((\text{EC.36}).\) at the second stage, which may have different optimal solutions and corresponding worst-case utility functions at the two different nodes.

As there are two nodes at the end of the first stage, we have to solve the two sub-optimization problems conditional on the historical information on the two nodes at stage 1, i.e.,

\[
\hat{u}_2^*(r_{1,1}) = \max_{x_2 \in X_2} \inf_{u_2 \in U} E[u_2(x_2 \ r_2) | r_{1,1}]
\]

\[
= \max_{x_{2,1}, x_{2,2} \in X_2} \min \left\{ 0.5x_{2,1}^1 \left[ 0.6 \atop 0.5 \right] + 0.5x_{2,1}^1 \left[ 0.6 \atop 0.5 \right] - \frac{1}{2} \left( x_{2,1}^1 \left[ 0.6 \atop 0.2 \right] \right)^2 - \frac{1}{2} \left( x_{2,1}^1 \left[ 0.6 \atop 0.8 \right] \right)^2 \right\} \tag{EC.42}
\]

\[
= \max_{x_{2,1} \in [0,1]} \min \left\{ 0.05x_{2,1}^1 + 0.75, -0.1(x_{2,1}^1)^2 + 0.28x_{2,1}^1 + 0.66 \right\}.
\]
where $x^2_{2,1} = 1 - x^1_{2,1}$ and
\[
\tilde{v}^*_2(r_{1,2}) = \max_{x_2 \in X_2} \inf_{u_2 \in U} \mathbb{E}[u_2(x_2^r, r_2) | r_{12}]
\]
\[
= \max_{x_{2,1}, x_{2,2} \in X_2} \min \left\{ 0.5 x_{2,2}^r \left[ \begin{array}{c} 0.7 \\ 0.6 \end{array} \right] + 0.5, 2 x_{2,2}^r \left[ \begin{array}{c} 0.7 \\ 0.6 \end{array} \right] - \frac{1}{2} \left( x_{2,2}^r \left[ \begin{array}{c} 0.4 \\ 0.6 \end{array} \right] \right)^2 - \frac{1}{2} \left( x_{2,2}^r \left[ \begin{array}{c} 1 \\ 0.6 \end{array} \right] \right)^2 \right\}
\]  
(EC.43)

Problem (EC.42) has an optimal solution $\hat{x}^*_{2,1} = [1, 0]$ with $\tilde{v}^*_2(r_{1,1}) = 0.8$. The corresponding worst-case utility $u^*_{2,1}$ is $u^1(\cdot)$. Problem (EC.43) has an optimal solution $\hat{x}^*_{2,2} = [0.8, 0.2]$ with $\tilde{v}^*_2(r_{1,2}) = 0.84$. At the optimum, the expected utility values of $u^1(\cdot)$ and $u^2(\cdot)$ are the same. The analysis is depicted at the right-hand side of Figure EC.4.

By comparing the solutions shown in Figure EC.4, we can see that the global optimal solution at the left-hand side and the local optimal solution at the right-hand side are not the same. Thus, the optimal solution of the state-independent PRO model (9) is not time consistent. This is because in model (EC.35) the worst-case utility $u^*_{2,1}$ must be equal to $u^*_{2,2}$ regardless of the reward at the end of stage one. In contrast, model (EC.36) allows one to choose worst-case utility $u^*_{2,1}$ or $u^*_{2,2}$ after viewing the outcome of reward at the end of stage one. The fundamental reason is that the worst-case utilities in sub-horizon model (12) are scenario ($\mathcal{F}_{t-1}$-adapted $\xi_{t-1}$) dependent whereas the worst-case utilities in (9) are all deterministic (independent of the stochastic process $\{\xi_t\}$).

**EC.4. Reformulations of the multistage PRO models under a scenario tree structure**

**EC.4.1. Time-consistent model with pairwise comparison-based ambiguity set**

If the state-dependent pairwise comparisons ambiguity set $\mathcal{U}^P_t(\xi_{t-1})$ defined in Section 4.1 is adopted, we can apply the tractable reformulation of the one-stage PRO model with pairwise com-
parison proposed in [1] to each non-leaf node of problem (35) and get the following reformulation of the time consistent model.

**Proposition EC.2.** Given the scenario tree structure of \(\{\xi_t\}\) and a series of pairwise comparisons ambiguity sets \(U^p(s) = U^p_t(\xi[s])\), problem (35) is equivalent to

\[
\max_{s \in S^-} \left( \theta_{N-1}(s) + \sum_{i \in s^+} \mu_{i,N} - L(s) \sum_{j=1}^{N-1} \eta_j(s) + \sum_{k=1}^{K} z_k(s) \left( \mathbb{P}[Y_k = y_N] - \mathbb{P}[W_k = y_N] \right) \lambda_k(s) \right)
\]

subject to

\[
\begin{align*}
\sum_{j=1}^{N} y_j \mu_{i,j} &\leq \frac{p_i}{p_i-} \sum_{j=1}^{N} \mu_{i,j} = 0, \; i \in S \setminus \{1\}, \\
\theta_{j-1}(s) y_{j-1} - \theta_{j-1}(s) y_j + v_{j-1}(s) (y_{j-1} - y_j) + \eta_{j-1}(s) &\geq 0, \; j = 3, \ldots, N-1, \; s \in S^-, \\
\theta_1(s) y_1 - \theta_1(s) y_2 + \eta_1(s) &\geq 0, \; s \in S^-, \\
\theta_{N-1}(s) y_{N-1} - \theta_{N-1}(s) y_N + v_{N-2}(s) (y_{N-1} - y_N) + \eta_{N-1}(s) &\geq 0, \; s \in S^-, \\
\theta_{j-1}(s) - \theta_j(s) + \sum_{i \in s^+} \mu_{i,j} - v_{j-1}(s) + v_j(s) \\
+ \sum_{k=1}^{K} z_k(s) \left( \mathbb{P}[Y_k = y_j] - \mathbb{P}[W_k = y_j] \right) \lambda_k(s) &\geq 0, \; j = 2, \ldots, N-2, \; s \in S^-, \\
\theta_{N-2}(s) - \theta_{N-1}(s) + \sum_{i \in s^+} \mu_{i,N-1} - v_{N-2}(s) \\
+ \sum_{k=1}^{K} z_k(s) \left( \mathbb{P}[Y_k = y_{N-1}] - \mathbb{P}[W_k = y_{N-1}] \right) \lambda_k(s) &\geq 0, \; s \in S^-, \\
x(1) &\in \mathcal{X}_1, x(s) \in \mathcal{X}_t(s) \left( x[s^+], \xi[s] \right), \; s \in S \setminus \{1\}, \\
\theta(s) &\in \mathbb{R}^{N-1}, v(s) \in \mathbb{R}^{N-2}, \eta(s) \in \mathbb{R}^{N-1}, \lambda(s) \in \mathbb{R}^K, \; s \in S^-, \mu(s) \in \mathbb{R}^N, s \in S \setminus \{1\}.
\end{align*}
\]

Given the concavity of \(h_t(\cdot, \cdot)\), \(t = 1, \ldots, T\), and the convexity of \(\mathcal{X}_t(\cdot)\), \(\mathcal{X}_t(\cdot, \cdot)\), \(t = 2, \ldots, T\), problem (EC.44) is a convex programming problem.

**Proof of Proposition EC.2.** Analogous to the proof of Theorem 1 in [1], we can show that the worst-case utility function is in a piecewise linear form with at most \(N\) breakpoints. Let \(S(s) = |s^+|\), here \(s^+\) stands for the set of all son nodes of \(s\). By taking the piecewise linear form in the functional infimum problem, we have

\[
\inf_{u_\alpha} \sum_{s \in S^+} \frac{p_s}{p_\alpha} u_s \left( h_{t(i)} \left( x(s), \xi(i) \right) \right) = \inf_{\alpha, \beta, \epsilon, \varphi} \sum_{i=1}^{T(s)} \mathbb{P}(\xi_t = \xi(i)|\xi_{t-1}) \left( \epsilon(h_{t(i)} \left( x(s), \xi(i) + \varphi_i \right) \right)
\]
The only difference between the studied model and the model in Theorem 1 of [1] is that, we replace the normalization constraint in [1] by bounded support constraints.

Taking the duality to the minimization LP problem gives an equivalent maximization LP reformulation. Applying the maximization LP reformulation to each inner infimum problem at node $s$ in (35), we obtain the deterministic reformulation of (35).

\[ \text{max} \quad \sum_{s \in S^-} p_s \left( \theta_{N-1}(s) + \sum_{i \in S^+} \mu_{i,N} - L(s) \sum_{j=1}^{N-1} \eta_j(s) - \sum_{j=2}^N \beta_j(s) w_j(s) - r(s) \varsigma(s) \right) \]

s.t. \[
\sum_{j=1}^N y_j \mu_{i,j} \leq \frac{p_i}{p_{j^-}} h_{i(j)}(x(i^-), \xi(i)), \quad i \in S \setminus \{1\}, \\
\frac{p_i}{p_{j^-}} - \sum_{j=1}^N \mu_{i,j} = 0, \quad i \in S \setminus \{1\}, \\
\theta_{j-1}(s) y_{j-1} - \theta_{j-1}(s) y_j + v_{j-2}(s)(y_{j-1} - y_{j-2}) + w_j(s) + \eta_{j-1}(s) \geq 0, \quad j = 3, \cdots, N-1, \quad s \in S^-, \\
\theta_1(s) y_1 - \theta_1(s) y_2 + w_2(s) + \eta_1(s) \geq 0, \quad s \in S^-, \\
\theta_{N-1}(s) y_{N-1} - \theta_{N-1}(s) y_N + v_{N-2}(s)(y_{N-1} - y_{N-2}) + w_N(s) + \eta_{N-1}(s) \geq 0, \quad s \in S^-, \\
\theta_{j-1}(s) - \theta_j(s) + \sum_{i \in S^+} \mu_{i,j} - v_{j-1}(s) + v_j(s) = 0, \quad j = 2, \cdots, N-2, \quad s \in S^-, \\
\theta_{N-2}(s) - \theta_{N-1}(s) + \sum_{i \in S^+} \mu_{i,N-1} - v_{N-2}(s) = 0, \quad s \in S^-, \\
w_j(s) \leq z_{j-1}(s)(y_j - y_{j-1}) + \frac{1}{2} (y_j - y_{j-1})^2 \varsigma(s), \quad j = 2, \cdots, N, \quad s \in S^-, \\
-w_j(s) \leq -z_{j-1}(s)(y_j - y_{j-1}) + \frac{1}{2} (y_j - y_{j-1})^2 \varsigma(s), \quad j = 2, \cdots, N, \quad s \in S^-, \tag{EC.45} \\
w_j(s) \leq z_j(s)(y_j - y_{j-1}) + \frac{1}{2} (y_j - y_{j-1})^2 \varsigma(s), \quad j = 2, \cdots, N, \quad s \in S^-, \\
-w_j(s) \leq -z_j(s)(y_j - y_{j-1}) + \frac{1}{2} (y_j - y_{j-1})^2 \varsigma(s), \quad j = 2, \cdots, N, \quad s \in S^-,
Given the scenario tree structure of \( \{\xi_t\} \) and \( T \) pairwise comparison based state-independent ambiguity sets \( U^P_t := U^P_t(\xi_0) \) at each stage \( t = 1, \ldots, T \), program (9) can be reformulated as

\[
\max_{i \in S(t)} \sum_{j=1}^{N} y_j \mu_{i,j} \leq p_i h_{i(t)}(x(i^-), \xi(i)), \quad i \in S \setminus \{1\},
\]

s.t.

\[
p_i - \sum_{j=1}^{N} \mu_{i,j} = 0, \quad i \in S \setminus \{1\},
\]

\[
\theta_{j-1}(t)y_{j-1} - \theta_{j-1}(t)y_j + v_{j-2}(t)(y_{j-1} - y_{j-2}) + \eta_{j-1}(t) \geq 0, \quad j = 3, \ldots, N - 1, \quad t = 1, \ldots, T,
\]

\[
\theta_{t}(t)y_{t} - \theta_{t}(t)y_2 + \eta_{t}(t) \geq 0, \quad t = 1, \ldots, T,
\]

\[
\theta_{t}(y_{N-1} - \theta_{N-1}(t)y_N + v_{N-2}(t)(y_{N-1} - y_{N-2}) + \eta_{N-1}(t) \geq 0, \quad t = 1, \ldots, T,
\]

\[
\theta_{t}(t) - \theta_{t-1}(t) + \sum_{i \in S(t)} \mu_{i,j} - v_{j-1}(t) + v_{j}(t)
\]

\[
+ \sum_{k=1}^{K} z_k(t) (\mathbb{P}[Y_k = y_j] - \mathbb{P}[W_k = y_j]) \lambda_k(t) = 0, \quad j = 2, \ldots, N - 2, \quad t = 1, \ldots, T,
\]

\[
\theta_{T}(t) - \theta_{T-1}(t) + \sum_{i \in S(t)} \mu_{i,N-1} - v_{N-2}(t)
\]
Finally, we apply the state-independent Kantorovich ball-based ambiguity set \( EC.4.4 \). Time-inconsistent model with Kantorovich ball-based ambiguity set each stage \( t \) state-independent utilities. In contrast, the slack variables \( \theta(t), v(t), \eta(t) \) and \( \lambda(t) \), \( t = 1, \ldots, T \), are node-dependent in the time consistent model as they are added at each non-leaf node to determine the worst-case state-dependent utilities. The convexity of the reformulation follows by the concavity of \( h_t(\cdot, \cdot) \), \( t = 1, \ldots, T \), and the convexity of \( \mathcal{X}_i(\cdot), \mathcal{X}_i(\cdot, \cdot), t = 2, \ldots, T \).

**EC.4.4. Time-inconsistent model with Kantorovich ball-based ambiguity set**

Finally, we apply the state-independent Kantorovich ball-based ambiguity set \( \mathcal{U}^K_t := \mathcal{U}^K_t(\xi_0) \) to each stage \( t = 1, \ldots, T \) of problem (9) and obtain the following reformulation.

**Proposition EC.5 (Kantorovich ball based ambiguity set).** Given the scenario tree structure of \( \{\xi_i\} \) and \( T \) Kantorovich ball based state-independent ambiguity sets \( \mathcal{U}^K_t := \mathcal{U}^K_t(\xi_0) \) at each stage \( t = 1, \ldots, T \), program (9) can be reformulated as

\[
\max_{\mathcal{T}} \sum_{t=1}^{T} \left( \theta_{N-1}(t) + \sum_{i \in S(t)} \mu_{i,N} - L(t) \sum_{j=1}^{N-1} \eta_j(t) - \sum_{j=2}^{N} \bar{\beta}_j(t) w_j(t) - r(t) \zeta(t) \right)
\]

s.t. \( \sum_{j=1}^{N} y_j \mu_{i,j} \leq p_i h_{t(i)}(x(i'), \xi(i)) \), \( i \in S \setminus \{1\} \),

\[
p_i - \sum_{j=1}^{N} \mu_{i,j} = 0, \quad i \in S \setminus \{1\},
\]

\[
\theta_{j-1}(t)y_{j-1} - \theta_{j-1}(t)y_j + v_{j-2}(t)(y_{j-1} - y_{j-2}) + w_j(t) + \eta_{j-1}(t) \geq 0, \quad j = 3, \ldots, N - 1, \quad t = 1, \ldots, T,
\]

\[
\theta_{t-1}(t)y_{t-1} - \theta_{t-1}(t)y_{t} + v_{t-2}(t)(y_{t-1} - y_{t-2}) + w_{t}(t) + \eta_{t-1}(t) \geq 0, \quad t = 1, \ldots, T,
\]

\[
\theta_{N-1}(t)y_{N-1} - \theta_{N-1}(t)y_{N} + v_{N-2}(t)(y_{N-1} - y_{N-2}) + w_{N}(t) + \eta_{N-1}(t) \geq 0, \quad t = 1, \ldots, T,
\]

\[
\theta_{j-1}(t) - \theta_{j}(t) + \sum_{i \in S^+} \mu_{i,j} - v_{j-1}(t) + v_{j}(t) = 0, \quad j = 2, \ldots, N - 2, \quad t = 1, \ldots, T,
\]

\[
\theta_{N-2}(t) - \theta_{N-1}(t) + \sum_{i \in S^+} \mu_{i,N-1} - v_{N-2}(t) = 0, \quad t = 1, \ldots, T,
\]

\[
w_{j}(t) \leq z_{j-1}(t)(y_{j} - y_{j-1}) + \frac{1}{2}(y_{j} - y_{j-1})^2 \zeta(t), \quad j = 2, \ldots, N, \quad t = 1, \ldots, T,
\]

\[
-w_{j}(t) \leq -z_{j-1}(t)(y_{j} - y_{j-1}) + \frac{1}{2}(y_{j} - y_{j-1})^2 \zeta(t), \quad j = 2, \ldots, N, \quad t = 1, \ldots, T,
\]

\[
w_{j}(t) \leq z_{j}(t)(y_{j} - y_{j-1}) + \frac{1}{2}(y_{j} - y_{j-1})^2 \zeta(t), \quad j = 2, \ldots, N, \quad t = 1, \ldots, T,
\]
\[-w_j(t) \leq -z_j(t)(y_j - y_{j-1}) + \frac{1}{2}(y_j - y_{j-1})^2 \varsigma(t), \; j = 2, \ldots, N, \; t = 1, \ldots, T,\]

\[x(1) \in \mathcal{X}_1, x(s) \in \mathcal{X}_{t(s)}(x[s^{-}], \xi[s]), \; s \in S^- \setminus \{1\},\]

\[\theta(t) \in \mathbb{R}^{N-1}, \; v(t) \in \mathbb{R}^{N-2}, \; \eta(t) \in \mathbb{R}^{N-1}, \; t = 1, \ldots, T,\]

\[\varsigma(t) \in \mathbb{R}^+, \; w(t) \in \mathbb{R}^{N-1}, \; z(t) \in \mathbb{R}^N, \; t = 1, \ldots, T, \; \mu(s) \in \mathbb{R}^+, \; s \in S \setminus \{1\}.\]

The convexity of the reformulation follows by the concavity of \(h_t(\cdot, \cdot), t = 1, \ldots, T\), and the convexity of \(\mathcal{X}_1(\cdot), \mathcal{X}_t(\cdot, \cdot), t = 2, \ldots, T\).

**EC.5. NBD algorithm and SDDP algorithm**

The scenario tree method is a generic solution approach which can handle nonlinear dependence structure between stages. However, it does not exploit the dynamic programming structure of the time consistent model established in Theorem 1. Here, we propose to use the efficient DP-type methods such as the NBD algorithm and SDDP algorithm to solve the state-dependent multistage PRO models based on the recursive equations in Theorem 1.

**EC.5.1. General principle of the DP-type algorithm**

For simple problems with finite states and finite actions, we can apply tabular solution methods by maintaining a state/policy-to-value mapping table and updating it with value/policy iteration schemes.

For problems with infinite actions (for instance, a polyhedral feasible set \(\mathcal{X}\)), a natural idea is to approximate the value function by a piecewise linear function, and then use the optimal values obtained from solving state-dependent problem in Theorem 3 to update the approximation function. This is known as the approximate dynamic programming approach.

For some particular problems, for instance in our MS-PRO problem, if the constraints at recourse stages have a linear block-diagonal structure, i.e., only consecutive stages can be linked by linear constraints, meanwhile, the reward functions are linear, and we have applied the piecewise linear approximation to the value function, then the optimization problem (26) in the dynamic programming equation is convex and thus strong duality holds. We can use the solution to its duality problem to generate some optimality cuts with tight approximation gap and good convergence property. This is known the Benders’ style algorithm.

**Approximate the expected value operator.** To solve problem (16), we first need to estimate or approximate the expected value operator. At stage \(t\) and scenario \(k\), we select \(S_{t,k}\) samples of \(\xi_t\), denoted by \(\xi_t(s)\) with appearing probability \(p_s\), \(s = 1, \ldots, S_{t,k}\). If we have a scenario tree representation of \(\xi_t\), we can use all the son nodes of \(\xi_t^s\) as the samples, which is exact reformulation
of (EC.47). This is known as the NBD algorithm. When \( \xi \) is continuously distributed or have a large number of realizations, we can draw finite i.i.d. samples instead to get a small approximation problem, which is known as the SDDP algorithm. With the finite samples of \( \xi \), we can reformulate/approximate the expected value operator, thus problem (16) as

\[
\max_{x_t} \inf_{u_t \in T_t(\xi_{[t-1]}^k)} \sum_{s=1}^{S_{t,k}} p_s \left[ u_t \left( h_t (x_t, \xi_t(s)) \right) + V_{t+1}^i \left( x_{t+1}^k; x_t, \xi_{[t-1]}^k, \xi_t \right) \right] \\
\text{s.t.} \quad W_{t-1} \left( \xi_{[t-1]}^k \right) x_t = b_{t-1} \left( \xi_{[t-1]}^k \right) - D_{t-1} \left( \xi_{[t-1]}^k \right) x_{t-1}^{k_{[t-1]}}.
\]

(EC.46)

**Piecewise linear approximations** The main challenge of a dynamic programming type algorithm is to find a good approximation to \( V_{t+1}^i \left( x_{[t]}, \xi_{[t]} \right) \). The Benders’ type algorithm uses a piecewise linear approximation to \( V_{t+1}^i \left( x_{[t]}, \xi_{[t]} \right) \), denoted by \( V_{t+1}^i \left( x_{[t]}, \xi_{[t]} \right) \), after \( (i-1) \)-th iteration/updating. Under Assumption 2 the dynamic equation has a block diagonal structure. Thus, the cost-to-go value function only depends on the current decision \( x_t \) rather than historical decisions \( x_{[t-1]} \). Moreover, we consider finite scenarios for \( \xi_{[t-1]} \) and finite samples for \( \xi \), which means that we can maintain a piecewise linear approximation function in \( x_t \) for each realization of \( \xi_{[t]} \). Denote the approximation function by

\[
V_{t+1}(x_t; \xi_{[t]}) := \min_{r \in R^i(\xi_{[t-1]})} \left( \beta_r^i(\xi_{[t]} \right) x_t + \alpha_r^{i+1}(\xi_{[t]}))
\]

where \( R^i(\xi_{[t]}) \) is the index set of linear pieces at the \( i \)-th iteration. \( \alpha_r^{i+1}(\xi_{[t]}) \) is the intercept and \( \beta_r^{i+1}(\xi_{[t]}) \) is the slope of the \( r \)-th piece.

**Forward pass** At the \( i \)-th iteration, with the approximation of the value function at the previous iteration, we solve approximately (EC.46) in sequence in some scenario to obtain some trial decision sequences. Specifically, for each scenario \( k \) in a selected scenario set \( \mathcal{K} \), we solve the following problem from stage 1 to stage \( T \),

\[
\max_{x_t, \delta_{t+1}} \left( \inf_{u_t \in T_t(\xi_{[t-1]}^k)} \sum_{s=1}^{S_{t,k}} p_s \left[ u_t \left( h_t (x_t, \xi_t(s)) \right) \right] + \sum_{s=1}^{S_{t,k}} p_s \left[ \delta_{t+1} \right] \right) \\
\text{s.t.} \quad W_{t-1} \left( \xi_{[t-1]}^k \right) x_t = b_{t-1} \left( \xi_{[t-1]}^k \right) - D_{t-1} \left( \xi_{[t-1]}^k \right) x_{t-1}^{k_{[t-1]}} : \pi_{t}^k \\
\delta_{t+1}^{k} \leq \beta_{t+1}^{i+1}(\xi_{[t-1]}^k, \xi_{[t]}^k(s)) x_t + \alpha_{t+1}^{i+1}(\xi_{[t-1]}^k, \xi_{[t]}^k(s)) : \rho_{t}^{k_{[t-1]}} s = 1, \ldots, S_{t,k}, r \in R(\xi_{[t-1]}^k, \xi_{[t]}),
\]

(EC.47)

for each realization of historical path \( \xi_{[t-1]}^k \), historical decisions \( x_{[t-1]}^k \) and \( R(\xi_{[t-1]}^k, \xi_{[t]}^k) = R^{-1}(\xi_{[t-1]}^k, \xi_{[t]}^k) \). We denote the optimal solution of (EC.47) by \( x_{t}^k \), which is the trial decision at stage \( t \). Here \( \pi_{t}^k \) is the optimal dual variable of the dynamic balance equation and \( \rho_{t} \) is the optimal dual variable of the linear cuts.
Kantorovich ball-based ambiguity set case. Under the Kantorovich ball ambiguity set, we can apply Theorem 3 to get a reformulation of (EC.47):

\[
\begin{align*}
\max & \quad \theta_{N-1} + \sum_{s=1}^{S_{t,k}} (\mu_{i,N} + p_s \delta_s) - L(\xi_{[t-1]}) \sum_{j=1}^{N-1} \eta_j \\
- \tilde{L}(\xi_{[t-1]}) & \sum_{j=1}^{N-2} (\tau_j + \sigma_j) (y_{j+2} - y_j) - \sum_{j=2}^{N} \tilde{\beta}_j w_j - r_t(\xi_{[t-1]}) s \\
\text{s.t.} & \quad W_{t-1} (\xi^k_{[t-1]}, \xi_{[t-1]}) x_t = b_{t-1} (\xi^k_{[t-1]}), D_{t-1} (\xi^k_{[t-1]}) x_{t-1}^k : \pi^i_{t,k} \\
& \quad \xi^k_{t+1} \leq \beta^r_{t+1} (\xi^k_{[t-1]}, \xi_t(s)) x_t + \alpha_{t+1}^r (\xi^k_{[t-1]}, \xi_t(s)) : \rho^i_{t,kr} s = 1, \ldots, S_{t,k}, \ r \in R(\xi^k_{[t-1]}, \xi_t), \\
& \quad \sum_{j=1}^{N} y_{j} \mu_{j,i,j} \leq \mathbb{P}(\xi_t = \xi_t(s)) | \xi_{[t-1]} \ h_t(x_t, \xi_t(s)), s = 1, \ldots, S, \\
& \quad \text{max}_{\pi_t} \quad \text{inf}_{u_t \in \mathcal{U}(\xi^k_{[t-1]})} \mathbb{E}_{\mathbb{P}_{t-1}^i} \left[ u_t (h_t(x_t, \xi_t)) + V_{t+1} (x_{t+1}, \xi^k_{[t-1]}, \xi^k_{t+1}) \right] \\
\text{s.t.} & \quad W_t (\xi^k_{[t-1]}, \xi_t(s)) x_t = b_t (\xi^k_{[t-1]}) - D_{t-1} (\xi^k_{[t-1]}) x_{t-1}^k
\end{align*}
\] (EC.48)

Similarly to the forward pass, we collect \(S_{t,k}\) samples as the set of all realizations of \(\xi_t\) conditional on the realization of \(\xi^k_{[t-1]}\). Thus, we can have a similar linear programming reformulation of (EC.49) similar to (EC.48). The only difference is that we set \(R(\xi^k_{[t-1]}, \xi_t) = R^i(\xi^k_{[t-1]}, \xi_t)\).

Generate new cuts. We first store the optimal dual variables \(\pi_t^i, \rho_t^i_{kr}, r \in R^i(\xi^k_{[t-1]}, \xi_t(s)), s = 1, \ldots, S_{t,k}\), in (EC.48) at stage \(t\). Then we create an optimality cut, indexed by \(\hat{r}\), for \(V_t(\cdot)\), with

\[
\beta_t^r(\xi^k_{[t-1]}) = - (\pi_t^k)^\top D_{t-1}(\xi^k_{[t-1]}),
\] (EC.50)

\[
\alpha_t^r(\xi^k_{[t-1]}) = (\pi_t^k)^\top b_t(\xi^k_{[t-1]}) + \sum_{s=1}^{S_{t,k}} \sum_{r \in R^i(\xi^k_{[t-1]}, \xi_t(s))} a^r_{t+1}(\xi^k_{[t-1]}, \xi_t(s)) \rho^i_{t,kr}
\] (EC.51)

by taking the optimal dual variables into the objective function of the dual problem of (EC.48) with \(R(\xi^k_{[t-1]}, \xi_t) = R^i(\xi^k_{[t-1]}, \xi_t)\). Then we obtain the piecewise linear approximation \(V_t(\cdot)\) at stage \(t - 1\) by adding the new cut into the index set \(R^i(\xi^k_{[t-1]}):= R^{i-1}(\xi^k_{[t-1]}) \cup \{ \hat{r} \}\).

Since the dual feasible set in (EC.48) is defined by finitely many linear constraints for each scenario, there exist only finitely many dual extreme points, which can be attained for the considered scenario. Hence, only finitely many different cut coefficients can be generated which guarantees the convergence of the Benders’ style algorithm [7, 4]. Using backward recursion, in a similar way, cuts can be derived for any stage \(t = T - 1, \ldots, 1\), using the already updated cut approximation \(R(\xi^k_{[t-1]}, \xi_t) = R^i(\xi^k_{[t-1]}, \xi_t)\).
EC.5.2. Algorithmic procedures of the NBD algorithm

We give the detailed algorithmic procedures when the classical NBD algorithm is adopted to solve problem (N).

Algorithm 2 Solving MS-PRO-SD by NBD

**Inputs:** Confidence level \( \alpha \in (0, 1) \), maximum number of iterations \( N_{\text{max}} \), tolerance \( tol \).

**Initialize:** (a) Initialize the lower bound \( \underline{v}^0 := -\infty \) and the upper bound \( \overline{v}^0 = \infty \), (b) set the counter of iterations to \( i \leftarrow 0 \), (c) set the scenario tree structure of \( \{\xi_i\} \) (including the set of scenarios \( K \), the appearing probability \( p_k \) of scenario \( k \)), (d) set \( V_{i,t}^0(\cdot, \xi_{[t]}):= 0 \), for \( t = 2, \ldots, T + 1 \). (e) set \( R_0^0(\xi_{[t]}):= \{r_0^0\} \) with \( \beta_{r_0^0}^t = 0 \) and a large enough \( \alpha_{r_0^0}^t \) for all non-leaf nodes.

**Output:** \( x_{ik}^t, k \in K, t = 1, \cdots, T \).

while \( i < N_{\text{max}} \) do
- Set \( i \leftarrow i + 1 \).
- **Forward Pass**
  - Solve the approximate first-stage problem
    \[
    \overline{v}^i := \max_{x_1 \in X_1} \inf_{u_1 \in U_1} \mathbb{E}_{|F_0}[u_1(h_1(x_1, \xi_1)) + V_{2}^{i-1}(x_{[1]}, \xi_{[1]})]
    \]
  to obtain trial point \( x_{ik}^1 = x_i^1 \) for all \( k \in K \) (We store the optimal value of the first part of the objective function \( \inf_{u_1 \in U_1} \mathbb{E}_{|F_0}[u_1(h_1(x_1, \xi_1))] \) as \( \underline{v}_i^1 \)).

  for stages \( t = 2, \ldots, T \) do
    for samples \( k \in K \) do
      - Solve the approximate stage-\( t \) subproblem (EC.47) (a conditional one-stage PRO problem) for \( x_{ik}^t \) to obtain trial point \( x_{ik}^t \). We store the optimal value of the first part of the objective function \( \inf_{u_t \in U_t} \mathbb{E}_{|F_{t-1}}[u_t(h_t(x_t, \xi_t))] \) as \( \underline{v}_i^k \).

    At the final stage \( T \), store the optimal dual values \( \pi_{i,T-1}^t, \rho_{i,T-1}^t \).
  end for
end for

- Substitute \( x_{ik}^t, k \in K \), into the objective function to derive a lower bound \( \overline{v}^i = v_i^1 + \sum_{t=2}^{T-1} \sum_{k \in K} p_k v_{ik}^t \), where \( p_k \) is the appearing probability of the \( k \)-th scenario.
- Check the stopping criterion
  if \( |\overline{v}^i - \overline{v}^i| \leq tol \)
    terminate loop and return output.
end if
- **Backward Pass**

Set \( V_{T+1}^i(x_{[T]}, \xi_{[T]}) := 0 \)
for stages $t = T - 1, \ldots, 2$ do

for samples $k \in \mathcal{K}$ do

- Load $\text{Son}(\xi^k_{[t-1]})$, the set of all realizations of $\xi_t$ conditional on the realization of $\xi^k_{[t-1]}$, i.e., the set of all son nodes of the $\xi_{[t-1]}$, set $S_{t,k}$ as the number of elements in $\text{Son}(\xi^k_{[t-1]})$.

- Solve the updated approximate stage-$t$ subproblem (EC.48) with $R(\xi^k_{[t-1]}, \xi_t) = R^i(\xi^k_{[t-1]}, \xi_t)$ for $x_{ik}^{t-1}$. Notice that for stage $T$, we do not solve the optimization problem. We directly use the dual variables recorded in solving (EC.47) in the forward pass

- Store the optimal dual values $\pi^i_{1,k}, \rho^i_{rk}, r \in R^i(\xi^k_{[t-1]}, \xi_t(s)), s = 1, \ldots, S_{t,k}$

- Create an optimality cut, indexed by $\hat{r}$, for $V^i(\cdot)$, with (EC.50) and (EC.51)

- Update the cut approximation $V^i(\cdot)$ at stage $t - 1$. $R^i(\xi^k_{[t-1]}) := R^{i-1}(\xi^k_{[t-1]}) \cup \{\hat{r}\}$

end for

end for

end while

EC.5.3. SDDP algorithm

The SDDP algorithm has the same procedures as the NBD algorithm (Algorithm 2), with three exceptions. First, the SDDP algorithm randomly selects a finite number of scenarios, denoted by $N$, to construct $\mathcal{K}$ at each loop rather than use all scenarios as in the NBD algorithm. Moreover, at each node in both the forward pass and the backward pass, the SDDP algorithm randomly generates $S$ samples to compute the conditional expected value approximately, rather than use the realizations of all son nodes in the NBD algorithm. Finally, since the lower bound provided by the SDDP algorithm is not exact but relies on random sampling, we usually set a stopping criterion based on a confidence interval, say, $\text{tol} = z_{1-0.99/2} \sigma_v/|K|$, where $\sigma_v$ is the sample standard deviation of $\sum_{t=2}^{T-1} v^k_t, k \in \mathcal{K}$ and $z_{1-0.99/2}$ is the $(1 - 0.99/2)$-quantile of the standard normal distribution $N(0,1)$, see [4]. To avoid repeat, we do not give the complete algorithmic procedures of the SDDP method.

EC.6. Details of preference elicitation and construction of the ambiguity sets

We use pairwise comparison approach and scoring approach to elicit the investor(DM)’s preference and use the elicited preference information to construct the ambiguity sets. The former is based on the random relative utility split scheme (RRUS) which is widely used in the literature of PRO models, see e.g. [1]. The key idea of RRUS is to generate a pair of lotteries and ask the investor to select, we refer readers to [1, 5] for the detailed procedures. The latter is to ask the investor to give scores at different levels of consumption and use them to construct an approximate utility function. Here we use RRUS to construct the pairwise comparison-based ambiguity sets and the scoring
Specifically, we use a scoring and fitting method to obtain a nominal utility function based on the historical data related to the investor. Suppose that we have observed \( N \) historical consumption trajectories (each of which consists of \( T \) stages) and the investor’s scores about the utility of the consumption at each stage for each trajectory. This means that we have collected \( N \times T \) pairs of consumption-score data.

**EC.6.1. Estimating nominal utility in state-independent case**

We consider \( N \) consumption levels, denoted by \( x_i \), for \( i = 1, \ldots, N \) at each stage of historical trajectory and let \( \hat{u}_i, i = 1, \ldots, N \) be the relevant utility scores by the investor. By assorting them if necessary, we assume that the consumption values are in an increasing order. A standard approach to construct an approximate utility function is to use a piecewise linear interpolation passing through the observed data points (pairs of \( (x_i, \hat{u}_i) \)). However, the approximated piecewise linear utility function constructed as such is not necessarily monotonically increasing since \( \hat{u}_i, i = 1, \ldots, N \) are not necessarily in an increasing order. This is primarily because these scores might be obtained at different states where the investor’s risk preferences are actually state-dependent. To tackle the
issue, we propose four optimization-based models to construct a piecewise linear nominal utility function. The procedures are illustrated in the flowchart in Figure EC.5.

**Best-fit estimation.** The first approach is to find a non-decreasing and concave function with least squares errors at the breakpoints from the scores by solving the following minimization problem

$$\min_u \sum_{i=2}^{N-1} (u_i - \hat{u}_i)^2$$

subject to:

$$\frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \leq \frac{(u_i - u_{i-1})}{(x_i - x_{i-1})}, \ i = 2, \ldots, N - 1,$$

$$0 \leq \frac{(u_N - u_{N-1})}{(x_N - x_{N-1})},$$

$$u_1 = 0, \ u_N = 1, \ 0 \leq u_i \leq 1, \ i = 2, \ldots, N - 1.$$  

This approach is analogous to the best-fit approach in [1], the main difference is that here $\hat{u}_i$, $i = 1, \ldots, N$ are obtained from scoring.

**Optimistic estimation.** The second approach is to find the upper non-decreasing and concave envelope of the graph of score points and construct a (piecewise linear) optimistic utility function by solving the following minimization problem:

$$\min_u \sum_{i=2}^{N-1} (u_i - \hat{u}_i)^2$$

subject to:

$$\frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \leq \frac{(u_i - u_{i-1})}{(x_i - x_{i-1})}, \ i = 2, \ldots, N - 1,$$

$$0 \leq \frac{(u_N - u_{N-1})}{(x_N - x_{N-1})},$$

$$u_1 \leq \hat{u}_i, \ i = 2, \ldots, N - 1.$$  

This estimation is optimistic because we consider the largest possible utility value of the utility function at each consumption level. We denote the optimal value by $u^U = [u^U_1, \ldots, u^U_N]$, with slope $\beta^U_i$ between $u^U_{i-1}$ and $u^U_i$.

**Pessimistic estimation.** The third approach is to use the lower non-decreasing and concave envelope of the graph of the score points by solving

$$\min_u \sum_{i=2}^{N-1} (u_i - \hat{u}_i)^2$$

subject to:

$$\frac{(u_{i+1} - u_i)}{(x_{i+1} - x_i)} \leq \frac{(u_i - u_{i-1})}{(x_i - x_{i-1})}, \ i = 2, \ldots, N - 1,$$

$$u_i \leq \hat{u}_i, \ i = 2, \ldots, N - 1.$$  

This estimation is pessimistic since we consider the smallest possible utility value of the utility function at each consumption level. We denote the optimal solution of $u$ as $u^L = [u^L_1, \ldots, u^L_N]$, with slope $\beta^L_i$ between $u^L_{i-1}$ and $u^L_i$.

To ensure the existence of pessimistic estimation and optimistic estimation, we assume that the scores are always larger than or equal to the consumption and smaller than or equal to 1.
Unbiased estimation. Instead of considering an optimistic or a pessimistic utility function based on the investor’s historical utility scores, we may consider a utility function which lies in the middle of the two under the Kantorovich distance. Specifically, we solve the following program:

\[
\min_{u, \beta, \zeta, \lambda} \max\{d_K(u, u^L), d_K(u, u^U)\} \\
\text{s.t. } \beta_i = (u_i - u_{i-1})/(x_i - x_{i-1}), \ i = 2, \ldots, N, \\
\beta_i \geq \beta_{i+1}, \ i = 1, \ldots, N = 1, \ \beta_N \geq 0, \\
u_1 = 0, \ u_N = 1, \ 0 \leq u_i \leq 1, \ i = 2, \ldots, N - 1.
\]

By utilizing the dual form of the Kantorovich distance in (32), we can reformulate problem (EC.55) as the following linear programming problem:

\[
\min_{u, \beta, \zeta, \lambda} \zeta \\
\text{s.t. } \zeta \geq \frac{1}{2} \sum_{i=2}^{N} (\lambda_i^k + \mu_i^k + \rho_i^k + \phi_i^k)(x_i - x_{i-1})^2, \ j = 1, 2, \\
\beta_i - \beta_i^L + \lambda_i^1 - \mu_i^1 + \rho_i^1 - \phi_i^1 = 0, \ i = 2, \ldots, N, \\
\beta_i - \beta_i^U + \lambda_i^2 - \mu_i^2 + \rho_i^2 - \phi_i^2 = 0, \ i = 2, \ldots, N, \\
(\mu_2^k - \lambda_2^k)(x_2 - x_1) = 0, \ k = 1, 2, \\
(\mu_{i+1}^k - \lambda_{i+1}^k)(x_{i+1} - x_i) + (\rho_i^k - \phi_i^k)(x_i - x_{i-1}) = 0, k = 1, 2, i = 2, \ldots, N - 1, \\
(\rho_N^k - \phi_N^k)(x_N - x_{N-1}) = 0, \\
\mu_i^k, \lambda_i^k, \rho_i^k, \phi_i^k \geq 0, \ k = 1, 2, \ i = 2, \ldots, N. \\
\beta_i = (u_i - u_{i-1})/(x_i - x_{i-1}), \ i = 2, \ldots, N, \\
\beta_i \geq \beta_{i+1}, \ i = 1, \ldots, N = 1, \ \beta_N \geq 0, \\
u_1 = 0, \ u_N = 1, \ 0 \leq u_i \leq 1, \ i = 2, \ldots, N - 1.
\]

The optimal value of the program corresponds to the radius of the Kantorovich ball. The components \((\beta_i, u_i), \ i = 1, \ldots, N,\) of the optimal solution of the above program are used to construct a piecewise linear nominal utility function.

Illustrating examples. We give two simple examples to show the piecewise linear nominal utility functions obtained through the four approaches outlined above. Figure EC.6 depicts these functions. 20 data points are used in Figure EC.6 and 40 are used in Figure EC.7. In both cases, consumption-utility score pairs are randomly generated with the true utility of the investor. We can see that the one based on unbiased estimation is much better than the one via the best-fit approach. As for the setting of the radius of the Kantorovich ball, we can base on subjective judgement or rely on (EC.55).
EC.6.2. Estimating state-dependent Kantorovich ball

We now turn to discuss construction of the nominal utility function of the Kantorovich ball when the investor’s utility is state-dependent. We consider two cases depending on whether the state information of the historical trajectories are observable or not by the modeller.

**States are known.** If the modeller has complete information about states, then we can separate the \( N \) data points (consumption levels and scores) at each stage in different groups, corresponding to different states (in our case study, the two states correspond to the high oil price state and the low oil price state). We apply one of the four approaches (best-fit for instance) in different states respectively, and derive the state-dependent nominal utility functions accordingly. The procedures are explained in the flowchart in Figure EC.8.

**States are not known.** If we do not have complete information about states, then we may divide the data points into several groups, each of which will be used to construct a piecewise linear utility function. Alternatively, we can sort out the state-dependent data points with all available data by solving a single optimization problem and construct the state-dependent utility functions accordingly with the optimal solutions. We can do so by the best-fit approach or unbiased estimation approach with some minor modifications. In the former case, we solve program:

\[
\begin{align*}
\min_u \sum_{i=1}^N \min_{k=1,\ldots,K} \{(u^k_i - \hat{u}_i)^2\} & \quad (EC.57a) \\
n & (u^k_{i+1} - u^k_i)/(x_{i+1} - x_i) \leq (u^k_i - u^k_{i-1})/(x_i - x_{i-1}), i = 2, \ldots, N - 1, \quad k = 1, \ldots, K, \quad (EC.57b)
\end{align*}
\]
Flowcharts of procedures for constructing a nominal utility function in MS-PRO-SD-Kan model.

For each state, collect consumption-score data in this state

For each stage, collect all consumption-score data, preset the number of unknown states K

Determine state-dependent nominal utility value $u^k_i$ by solving (EC.57) or (EC.60) for each state k

Form piecewise linear approximation and determine nominal slope $\beta_j(s)$ at node s for MS-PRO-SD-Kan

Determine state-dependent nominal utility value $u^k_i$ by solving (EC.58b) for each state k

Figure EC.8 Flowcharts of procedures for constructing a nominal utility function in MS-PRO-SD-Kan model.

$$0 \leq (1-u^k_N)/(x_1-x_N) \leq (u^k_N-u^k_{N-1})/(x_N-x_{N-1}), \quad k = 1, \ldots, K,$$

$$\text{(EC.57c)}$$

$$0 \leq u^k_i \leq 1, \quad i = 1, \ldots, N, \quad k = 1, \ldots, K.$$  

$$\text{(EC.57e)}$$

The key idea here is to use $u^k_i$ instead of $u_i$ where $k$ represents state $k$ for $k = 1, \ldots, K$, at each breakpoint (consumption level). Constraints (EC.58b)-(EC.58e) are imposed to ensure monotonicity and concavity of the utility function at state $k$, for $k = 1, \ldots, K$. The objective is to minimize the least squares errors/gaps between the state-dependent utility (to be decided) and the collected utility values. The modified unbiased estimation approach uses the same idea:

$$\min_{u^C, u^U, u^L, \beta^C, \beta^U, \beta^L, S_k} \sum_{k=1}^{K} \max \{ d_k(u^C_k, u^U_k), d_k(u^C_k, u^V_k) \}$$

$$\text{(EC.58a)}$$

s.t.  

$$\beta_{i,j}^k = (u_{i+1,j}^k - u_{i,j}^k)/(x_i-x_{i-1}), \quad i = 2, \ldots, N, \quad k = 1, \ldots, K, \quad j \in \{U, L, C\},$$

$$\text{(EC.58b)}$$

$$\beta_{i,j}^k \geq \beta_{i+1,j}^k, \quad i = 1, \ldots, N-1, \quad \beta_N \leq 0, \quad k = 1, \ldots, K, \quad j \in \{U, L, C\},$$

$$\text{(EC.58c)}$$

$$u_{i,j}^k = 0, \quad u_{i,j}^k = 1, \quad 0 \leq u_{i,j}^k \leq 1, \quad i = 2, \ldots, N-1, \quad k = 1, \ldots, K, \quad j \in \{U, L, C\},$$

$$\text{(EC.58d)}$$

$$u_{i,j}^k \leq u_i \leq u_{i,j}^k, \quad \text{if } i \in S_k, \quad i = 2, \ldots, N-1, \quad k = 1, \ldots, K.$$  

$$\text{(EC.58e)}$$

Here, we generate $K$ Kantorovich balls simultaneously with center $u^C_k$, upper boundary $u^U_k$ and lower boundary $u^L_k$ for the $k$-th ball at each breakpoint. The objective is to minimize the sum of the Kantorovich distance from the center of each ball to the upper or the lower boundary whichever is greater. Constraints (EC.58b)-(EC.58e) are imposed to ensure monotonicity and concavity of the utility function underlying the values $u^C_k$, $u^U_k$ and $u^L_k$ in each state $k$. (EC.58e) is to guarantee that $u^U_k$ and $u^L_k$ cover all the consumption-utility pairs under state $k$. Since in this case we do not know which the state each of the consumption-utility pairs belongs to, the index set $S_k$ is a decision
variable. To get rid of the index set, we may introduce variables \(z_{i,k}\) which takes a value of 0 or 1. Consequently we can reformulate (EC.58) as

\[
\min_{u^C,u^U,u^L,\beta^C,\beta^U,\beta^L} \sum_{k=1}^{K} \max \{ d_K(u^C_k, u^U_k), d_L(u^C_k, u^L_k) \}
\]

(\text{EC.59a})

s.t. \[ (\text{EC.58b}) - (\text{EC.58d}) \]

\[
u^{k,L}_i \leq u_i + (1 - z_{i,k})M, \quad i = 2, \ldots, N - 1, \quad k = 1, \ldots, K,
\]

(\text{EC.59c})

\[
u^{k,U}_i \leq u_i + (1 - z_{i,k})M, \quad i = 2, \ldots, N - 1, \quad k = 1, \ldots, K,
\]

(\text{EC.59d})

\[
\sum_{k=1}^{K} z_{i,k} = 1, \quad z_{i,k} \in \{0, 1\}^{(N-2) \times K}, \quad i = 2, \ldots, N - 1.
\]

(\text{EC.59e})

By the dual formulation of the Kantorovich distance in (32), we can recast (EC.59) as an MILP:

\[
\min_{u^C,u^U,u^L,\beta^C,\beta^U,\beta^L} \sum_{k=1}^{K} \zeta_k
\]

(\text{EC.60a})

s.t. \[ (\text{EC.58b}) - (\text{EC.58d}), \quad (\text{EC.59e}) - (\text{EC.59f}) \]

\[
\zeta_k \geq \frac{1}{2} \sum_{i=2}^{N} (\lambda_i^{k,j} + \mu_i^{k,j} + \phi_i^{k,j})(x_i - x_{i-1})^2, \quad k = 1, \ldots, K, \quad j \in \{U, L\},
\]

(\text{EC.60c})

\[
\beta_i^{k,C} - \beta_i^{k,U} + \lambda_i^{k,U} - \mu_i^{k,U} + \phi_i^{k,U} = 0, \quad i = 2, \ldots, N, \quad k = 1, \ldots, K,
\]

(\text{EC.60d})

\[
\beta_i^{k,C} - \beta_i^{k,L} + \lambda_i^{k,L} - \mu_i^{k,L} - \phi_i^{k,L} = 0, \quad i = 2, \ldots, N, \quad k = 1, \ldots, K,
\]

(\text{EC.60e})

\[
(\mu_{2,j} - \lambda_{2,j}^{2,j})(x_2 - x_1) = 0, \quad k = 1, \ldots, K, \quad j \in \{U, L\},
\]

(\text{EC.60f})

\[
(\mu_{i+1,j} - \lambda_{i,j}^{i+1,j})(x_{i+1} - x_i) + (\rho_{i,j}^{k,j} - \phi_i^{k,j})(x_i - x_{i-1}) = 0,
\]

(\text{EC.60g})

\[
i = 2, \ldots, N - 1, \quad k = 1, \ldots, K, \quad j \in \{U, L, C\},
\]

(\text{EC.60h})

\[
(\rho_N - \phi_N^{k,j})(x_N - x_{N-1}) = 0, \quad k = 1, \ldots, K, \quad j \in \{U, L, C\},
\]

(\text{EC.60i})

\[
\mu_i^{k,j}, \lambda_i^{k,j}, \rho_i^{k,j}, \phi_i^{k,j} \geq 0, \quad i = 2, \ldots, N, \quad k = 1, \ldots, K, \quad j \in \{U, L, C\}.
\]

(\text{EC.60j})

By solving (EC.60), we can figure out simultaneously the optimistic estimation \(u^U_k\), the pessimistic estimation \(u^L_k\) and the unbiased estimation \(u^C_k\) at each state \(k\). Subsequently, we can use the unbiased estimation as the center of the Kantorovich ball in each state and the optimal value of \(\zeta_k\) as the radius of the ball.

We use two examples to illustrate the estimations with different a-priori number of states \((K)\). In the first group of tests, we generate 40 score points with a true utility function which depends on two states, see the empty circles in Figures EC.9 EC.10. The true utility function is defined in Section 6.2. We work out the unbiased estimations with \(K = 1\) and \(K = 2\) by solving (EC.60). The resulting unbiased utility functions are displayed in Figure EC.9 and Figure EC.10.

In the second set of tests, we also generate 40 score points by a true utility which are dependent of three states, see the empty circles in Figures EC.11 EC.13. We figure out the unbiased estimations
for $K = 1$, $K = 2$ and $K = 3$ by solving (EC.60) respectively, the approximate piecewise linear utility functions are displayed in Figure EC.11, Figure EC.12 and Figure EC.13.

**Figure EC.9** Empty circles represent 40 sample scores generated by the true utility which are dependent of two states. The green dashed curve is the unbiased utility function obtained from solving (EC.60) with $K = 1$.

**Figure EC.10** Empty circles represent 40 sample scores generated by the true utility which are dependent of two states. The pink and yellow dashed curve are two unbiased utility functions obtained from solving (EC.60) with $K = 2$.

**EC.7. Figures from numerical tests**
Figure EC.11 Empty circles represent 40 sample scores generated by the true utility which are dependent of three states. The red dashed curve is an unbiased utility function obtained from solving (EC.60) with $K = 1$.

Figure EC.12 Empty circles represent 40 sample scores generated by the true utility which are dependent of three states. The yellow dashed curves and the red dashed line are two unbiased utility functions obtained from solving (EC.60) with $K = 2$.

Figure EC.13 Empty circles represent 40 sample scores generated by the true utility which are dependent of three states. The pink and red dashed curves and the yellow dashed line are three unbiased utility functions obtained from solving (EC.60) with $K = 3$. 
Figure EC.14  Worst-case utility functions in state 1 (high oil price) of MS-PRO-SD-Kan with different radii.

Figure EC.15  Utility functions randomly generated within 0.01 Kantorovich ball centered at the reference utility function in state 1 (high oil price) ($N = 40$).

Figure EC.16  Utility functions randomly generated within 0.1 Kantorovich ball centered at the reference utility function in state 1 (high oil price) ($N = 40$).

Figure EC.17  Utility functions randomly generated within 0.2 Kantorovich ball centered at the reference utility function in state 1 (high oil price) ($N = 40$).

References

[1] B. Armbruster and E. Delage, Decision making under uncertainty when preference information is incomplete, Management Science, 61: 111–128, 2015.
[2] J.P. Aubin, H Frankowska, Set-valued Analysis. Springer Science & Business Media, 2009.
[3] R. F. Brown, A Topological Introduction to Nonlinear Analysis, 3rd edition, Springer, New York, 2014.
[4] C. Füllner, S. Rebennack. Stochastic dual dynamic programming and its variants. Preprint, 2021,
[5] S. Guo and H. Xu, Utility preference robust optimization with moment-type information structure, 2021.
[6] D. Klatte, A note on quantitative stability results in nonlinear optimization. Seminarbericht, Sektion Mathematik, Humboldt-Universität zu Berlin, Berlin 90, 77–86, 1987.
[7] A. Shapiro. Analysis of stochastic dual dynamic programming method. *European Journal of Operational Research*, 209: 63–72, 2011.