Extremes of locally stationary Gaussian and chi fields on manifolds

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Abstract
Depending on a parameter $h \in (0, 1]$, let $\{X_h(t), t \in \mathcal{M}_h\}$ be a class of centered Gaussian fields indexed by compact manifolds $\mathcal{M}_h$. For locally stationary Gaussian fields $X_h$, we study the asymptotic excursion probabilities of $X_h$ on $\mathcal{M}_h$. Two cases are considered: (i) $h$ is fixed and (ii) $h \to 0$. These results are extended to obtain the limit behaviors of the extremes of locally stationary $\chi$-fields on manifolds.

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1 Introduction

We study the following two related problems in this manuscript.

(i) Let \( \{X(t), \ t \in \mathcal{M}\} \) be a centered Gaussian field indexed on a compact submanifold \( \mathcal{M} \) of \( \mathbb{R}^n \). We derive the asymptotic form of the excursion probability

\[
\mathbb{P}\left( \sup_{t \in \mathcal{M}} X(t) > u \right), \text{ as } u \to \infty. \tag{1.1}
\]

(ii) Let \( \{X_h(t), \ t \in \mathcal{M}_h\}_{h \in (0, 1]} \) be a class of centered Gaussian fields, where \( \mathcal{M}_h \) are compact submanifolds of \( \mathbb{R}^n \). Suppose that we have the structure \( \mathcal{M}_h = \mathcal{M}_{h,1} \times \mathcal{M}_{h,2} \) such that \( t = (t_1^T, t_2^T)^T \in \mathcal{M}_h \) means \( t_1 \in \mathcal{M}_{h,1} \) and \( t_2 \in \mathcal{M}_{h,2} \), where we allow \( \mathcal{M}_{h,2} \) to be a null set. The Gaussian fields \( X_h(t) \) we consider has a rescaled form \( X_h(t) = \underline{X}_h(t_1/h, t_2), t \in \mathcal{M}_h \) for some \( \underline{X}_h \) satisfying a local stationarity condition. We derive the following limit result

\[
\lim_{h \to 0} \mathbb{P}\left( a_h \left( \sup_{t \in \mathcal{M}_h} X_h(t) - b_h \right) \leq z \right) = e^{-e^{-z}}, \tag{1.2}
\]

for some \( a_h, b_h \in \mathbb{R}_+ \) and fixed \( z \in \mathbb{R} \).

While there is a large amount of literature on excursion probabilities of Gaussian processes or fields (see, e.g., Adler and Taylor [1], and Azaës and Wscherebor [3]), most of the existing work only considers index sets \( \mathcal{M} \) (or \( \mathcal{M}_h \)) of dimension \( n \) (the same as the ambient Euclidean space), while we focus on Gaussian fields indexed by manifolds that can be low-dimensional.

For problem (i), some relevant results can be found in Mikhaleva and Piterbarg [27], Piterbarg and Stamatovich [32], and Cheng [12]. Compared with these works, the framework of our result is more general in the following aspects: First of all, Cheng [12] studies the excursion probabilities of \textit{locally isotropic} Gaussian random fields on manifolds, where local isotropic means the variance between two local points only depends on their (geodesic) distance, while we consider \textit{locally stationary} Gaussian fields, for which not only the distance between the points but also their locations are involved in the variance. Furthermore, in Mikhaleva and Piterbarg [27] and Piterbarg and Stamatovich [32], the Gaussian fields are assumed to be indexed by \( \mathbb{R}^n \), while we only require the index sets to be the manifolds. As pointed out in Cheng [12], it is not clear whether one can always find a Gaussian field indexed by \( \mathbb{R}^n \) whose restriction on \( \mathcal{M} \) is \( X(t) \). Also see Cheng and Xiao [13] for some further arguments on this point. In addition, all the above works assume that the manifolds are smooth (\( C^\infty \)), while we consider a much larger class of manifolds (only satisfying a \textit{positive reach} condition). In fact, the properties of positive reach play a critical role in the geometric construction in our proofs.

For problem (ii), the study in Qiao and Polonik [34] corresponds to a special case of (1.2) when \( \mathcal{M}_h \equiv \mathcal{M} \) for some manifold \( \mathcal{M} \) independent of \( h \), and \( \mathcal{M}_{h,2} = \emptyset \). They use some ideas from Mikhaleva and Piterbarg [27] and also assume that \( X_h \) is indexed by a neighborhood of
higher dimensions around $\mathcal{M}$, while we only need $X_h$ to be indexed by the manifolds $\mathcal{M}_h$. This weaker requirement for the Gaussian fields finds broader applications when the Gaussian fields are observable or can be approximated only on low-dimensional manifolds. See (1.7) below for example. Also, by using the assumed structure of $\mathcal{M}_h$, only rescaling the parameters $t_1$ allows us to apply (1.2) to get asymptotic extreme value distributions of $\chi$-fields on manifolds, which in fact is one of the motivations of this work, as described below.

Let $\{X(s), s \in \mathcal{M}\}$ be a $p$-dimensional Gaussian vector field, where $X = (X_1, \cdots, X_p)^T$ has zero mean and identity variance-covariance matrix. Note that we have suppressed the possible dependence of $X$ and $\mathcal{M}$ on $h$. Define

$$\chi(s) = [X_1^2(s) + \cdots + X_p^2(s)]^{1/2}, \ s \in \mathcal{M},$$

which is called a $\chi$-field, where we allow the components $X_i(s_i)$ and $X_j(s_j)$ to be dependent, if $s_i \neq s_j$. Let $\mathbb{S}^{p-1} = \{x \in \mathbb{R}^p : \|x\| = 1\}$ be the unit $(p-1)$-sphere. Using the property of Euclidean norm, we have

$$\sup_{s \in \mathcal{M}} \chi(s) = \sup_{s \in \mathcal{M}, v \in \mathbb{S}^{p-1}} Y_h(s, v),$$

where $v = (v_1, \cdots, v_p) \in \mathbb{R}^p$ and

$$Y(s, v) = X_1(s)v_1 + \cdots + X_p(s)v_p, \ s \times v \in \mathcal{M} \times \mathbb{S}^{p-1}.$$  

Note that $Y(s, v)$ is a zero-mean and unit-variance Gaussian field on $\mathcal{M} \times \mathbb{S}^{p-1}$. Using the relation in (1.4) and by applying the results in (1.1) and (1.2), we can study the asymptotic excursion probabilities of $\sup_{s \in \mathcal{M}} \chi(s)$ as well as obtain a result in the form of

$$\lim_{h \to 0} \mathbb{P}\left( a_h \left( \sup_{s \in \mathcal{M}} \chi(s/h) - b_h \right) \leq z \right) = e^{-e^{-z}}.$$  

The result in (1.5) has the following two interesting applications. We consider a vector-valued signal plus noise model

$$\hat{f}_h(s) = f(s) + X(s/h), \ s \in \mathcal{M},$$

where $f(s)$ is a $p$-dimensional signal, $X(s)$ is the noise modeled by the Gaussian vector field considered above. We assume that only $\hat{f}_h(s)$ is directly observable. Given $\alpha \in (0, 1)$, let $z_\alpha$ be such that $\exp(-\exp(-z_\alpha)) = 1 - \alpha$.

(a) Suppose that $\mathcal{M}$ is known, and the inference for the signal $f(s)$ is of interest. We have the following asymptotic $(1 - \alpha)$ confidence tube for $f(s)$:

$$\mathcal{G}_h(s) := \left\{ g \in \mathbb{R}^p : a_h \left( \|\hat{f}_h(s) - g\| - b_h \right) \leq z_\alpha \right\}, \ s \in \mathcal{M}.$$  

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In other words, $\mathbb{P}(f(s) \in \mathcal{G}_h(s), \forall s \in \mathcal{M}) \to 1 - \alpha$, as $h \to 0$.

(b) Suppose that the manifold $\mathcal{M}$ is unknown but implicitly defined by $\mathcal{M} = \{s \in \mathcal{A} : f(s) = g_0\}$, where $\mathcal{A} \subset \mathbb{R}^n$ is a known neighborhood of $\mathcal{M}$ (say, a unit cube), and $g_0$ is a known $p$-dimensional vector so that $\mathcal{M}$ is the intersection of multiple level sets. Suppose that $\tilde{f}_h(s)$ is observable on $\mathcal{A}$, and the inference for the manifold $\mathcal{M}$ is of interest. We have the following asymptotic $(1 - \alpha)$ confidence region for $\mathcal{M}$:

$$\mathcal{F}_h := \{s \in \mathcal{A} : a_h \left(\|\tilde{f}_h(s) - g_0\| - b_h\right) \leq z_\alpha\}. \tag{1.8}$$

That is, $\mathbb{P}(\mathcal{M} \subset \mathcal{F}_h) \to 1 - \alpha$, as $h \to 0$.

In statistics the suprema of empirical processes can be approximated by the suprema of Gaussian processes or fields under regularity assumptions (see Chernozhukov et al. [14]). Applying results in (a) and (b) to the approximating Gaussian fields, one can study the statistical inference for a large class of objects including functions and geometric features (low-dimensional manifolds). In a form similar to (1.7), confidence bands for density functions are given in Bickel and Rosenblatt [7] and Rosenblatt [36]. Similar work for regression functions can be found in Konakov and Piterbarg [21]. We note that in these examples the study of the suprema of the approximating Gaussian processes or fields focuses on $\mathcal{M}$ being compact intervals or hypercubes. We expect that our result (1.7) is useful in studying functions supported on more general (low-dimensional) manifolds, especially in the context of manifold learning, which usually assumes that data lie on low-dimensional manifolds embedded in high-dimensional space. The result (1.8) is useful to infer the location of the manifolds. In fact, the results proved in this work provide the probabilistic foundation to our companion work Qiao [33], where the confidence regions for density ridges are obtained. Ridges are low-dimensional geometric features (manifolds) that generalize the concepts of local modes, and have been applied to model filamentary structures such as the Cosmic Web and road systems. See Qiao and Polonik [35] for a similar application for the construction of confidence regions for level sets.

The study of the asymptotic extreme value behaviors of $\chi$-processes and fields has drawn quite some interest recently. To our best knowledge, the study in the existing literature has only focused on $\chi$-processes and fields indexed by intervals or hypercubes, but not low-dimensional manifolds. See, for example, Albin et al. [2], Bai [4], Hashorva and Ji [19], Ji et al. [20], Konstantinides et al. [22], Lindgren [23], Ling and Tan [24], Liu and Ji [25, 26], Piterbarg [30, 31], Tan and Hashorva [38, 39], Tan and Wu [40]. Also it is worth mentioning that it is often assumed that $X_1, \ldots, X_r$ are independent copies of a Gaussian process or field $X$ in the literature, while the cross-dependence among $X_1, \ldots, X_r$ is allowed under certain constraints in this work. The cross-dependence structures of multivariate random fields have been important objects to study in multivariate geostatistics (see Genton and Kleiber [18]).

The manuscript is organized as follows. In Section 2 we introduce the concepts that we use in this work to characterize the manifolds (positive reach) and the Gaussian fields (local
stationarity). Then the result for (1.1) (called the unscaled case) is formulated in Theorem 2.1. As an application, a similar result for the \( \chi \)-fields in presented in Corollary 2.1. In Section 3 we give the result (1.2) (called the rescaled case) in Theorem 3.1 and its \( \chi \)-fields extension in Corollary 3.1. All the proofs are presented in Section 4 and Section 5 contains some miscellaneous results used in the manuscript.

2 Extremes of unscaled Gaussian and \( \chi \) fields on manifolds

We consider a centered Gaussian field \( X(t), t \in M \), where \( M \) is a \( r \)-dimensional submanifold of \( \mathbb{R}^n \) \((1 \leq r \leq n)\). Let \( r_X(t_1, t_2) = \text{Cov}(X(t_1), X(t_2)) \) for any \( t_1, t_2 \in M \). We first introduce some concepts we need to characterize the covariance \( r_X \) of the Gaussian field \( X \) and the manifold \( M \).

For a positive integer \( k \leq n \), let \( E = \{ e_1, \cdots, e_k \} \) be a collection of positive integers such that \( n = e_1 + \cdots + e_k \), and let \( \alpha = \{ \alpha_1, \cdots, \alpha_k \} \) be a collection of positive numbers. Then the pair \((E, \alpha)\) is called a structure. Let \( \| \cdot \| \) denote the Euclidean norm. Denote \( E(0) = 0 \) and \( E(i) = e_1 + \cdots + e_i, i = 1, \cdots, k \). For any \( t = (t_1, \cdots, t_n)^T \in \mathbb{R}^n \), its structure module is denoted by \( |t|_{E, \alpha} = \sum_{i=1}^k \| t(i) \|^{\alpha_i} \), where \( t(i) = (t_{E(i-1)+1}, \cdots, t_{E(i)})^T \).

Suppose that \( \alpha_i \leq 2, i = 1, \cdots, k, \) and consider a Gaussian field \( W(t), t \in \mathbb{R}^n \), with continuous trajectories such that \( \mathbb{E}W(t) = -|t|_{E, \alpha} \) and \( \text{Cov}(W(t), W(s)) = |t|_{E, \alpha} + |s|_{E, \alpha} - |t - s|_{E, \alpha} \).

It is known that such a field exists (see page 98, Piterbarg \[31\]). For any measurable subset \( T \subset \mathbb{R}^n \) define

\[
H_{E, \alpha}(T) = \mathbb{E}\exp\left( \sup_{t \in T} W(t) \right).
\]

For any \( T > 0 \), denote \([0, T]^n = \{ t \in \mathbb{R}^n : t_i \in [0, T] \} \). The generalized Pickands’ constant is defined as

\[
H_{E, \alpha} = \lim_{T \to \infty} \frac{H_{E, \alpha}([0, T]^n)}{T^n},
\]

which is a positive finite number. When \( k = 1, E = \{ 1 \} \) and \( \alpha = \alpha \in (0, 2] \), we denote \( H_{E, \alpha} = H_\alpha \).

**Definition 2.1** (local-\((E, \alpha, D_t)\)-stationarity). Let \( \{ Z(t), t \in M \} \) be a Gaussian random field with covariance function \( r_Z \), indexed on a submanifold \( M \) of \( \mathbb{R}^n \). \( Z \) is said to be locally-\((E, \alpha, D_t)\)-stationary on \( M \), if for all \( t \in M \) there exists a nonsingular matrix \( D_t \) such that

\[
r_Z(t_1, t_2) = 1 - |D_t(t_1 - t_2)|_{E, \alpha}(1 + o(1)),
\]

(2.1)
as \( \max\{\|t - t_1\|, \|t - t_2\|\} \to 0 \) for \( t_1, t_2 \in M \).
Let $\Phi$ denote the standard normal density and cumulative distribution function, respectively, and $\bar{\Phi}$\(\cdot\) = \Phi\(\cdot\) = 1 - \Phi\(\cdot\). Some notation: all minor determinants of order $m$ are denoted by $\Delta(\cdot, t) = \det(D_t \cdot)$, where $D_t \cdot$ is the matrix valued function $D_t \cdot$ is continuous in $t \in M$, for $i = 1, \ldots, k$. For $0 < \alpha_1, \ldots, \alpha_k \leq 2$, we assume that the Gaussian field $X(t)$ on $M$ has zero mean and is locally-$(E, \alpha, D_t)$-stationary.

Remark 2.1. Note that the local stationarity condition for the Gaussian field is given using the structure $(E, \alpha)$ for $\mathbb{R}^n$. The structural assumptions on $M$ and $D_t$ in (A1) and (A2) are used to guarantee that a similar structure $(R, \alpha)$ can be found when the local stationarity of the Gaussian field is expressed on a low-dimensional manifold, which locally resembles $\mathbb{R}^r$. Note that, however, in the special case of $k = 1$ we do not have these structural constraints for $M$ and $D_t$ any more.

Some notation: Let $1 \leq m \leq n$. For an $n \times m$ matrix $G$, let $\|G\|_m^2$ be the sum of squares of all minor determinants of order $m$. Let $\mathcal{H}_m$ denote the $m$-dimensional volume measure. For a $C^1$ manifold $M$, at each $u \in M$, let $T_u M$ denote the tangent space of $M$ at $u$. Let $\phi$ and $\Phi$ denote the standard normal density and cumulative distribution function, respectively, and let $\bar{\Phi}(u) = 1 - \Phi(u)$ and $\bar{\Psi}(u) = u^{-1} \bar{\phi}(u)$. Recall that $t = (t^T_{(1)}, \ldots, t^T_{(k)})^T$. The following is a result for the asymptotic behavior of the excursion probability of $X$ on the manifold $M$.

Theorem 2.1. For a Gaussian field $X(t)$, $t \in M$ satisfying assumptions (A1) and (A2), if $r_X(t, s) < 1$ for all $t, s$ from $M$, $t \neq s$, then

$$
\mathbb{P}\left(\sup_{t \in M} X(t) > u\right) = H_{R,\alpha} \int_M \prod_{j=1}^k \|D_j t P_j t(\cdot)\|_{r_j} d\mathcal{H}_r(t) \prod_{i=1}^k u^{2r_i/\alpha_i} \bar{\Psi}(u)(1 + o(1)),
$$

(2.2)
as \( u \to \infty \), where \( P_{j,t(i)} \) is an \( e_j \times r_j \) matrix whose columns are orthonormal and span the tangent space of \( T_{t(i)} M_j \).

**Remark 2.2.** The factorization lemma (Lemma 6.4, Piterbarg [31]) implies that \( H_{R,\alpha} = \prod_{i=1}^k H_{r_i,\alpha_i} \), where in the notation we do not distinguish between \( r_i \) (or \( \alpha_i \)) and \( \{ r_i \} \) (or \( \{ \alpha_i \} \)).

We will apply the above theorem to study the excursion probabilities of manifolds. Let \( \{ X(s), s \in \mathcal{L} \} \) be a centered \( p \)-dimensional \(( p \geq 2)\) Gaussian vector field, where \( X = (X_1, \cdots, X_p)^T \) with \( \text{Var}(X_i) = 1, i = 1, \cdots, p \), and \( \mathcal{L} \) is a \( m \)-dimensional submanifold of \( \mathbb{R}^n \) \(( 1 \leq m \leq n) \). We consider the asymptotics of

\[
\mathbb{P} \left( \sup_{s \in \mathcal{L}} \|X(s)\| > u \right), \text{ as } u \to \infty. \tag{2.3}
\]

Let \( v = (v_1, \cdots, v_p)^T \in \mathbb{R}^p \), \( t = (s^T, v^T)^T \in \mathbb{R}^{n+p} \), and

\[
Y(t) = Y(s,v) = X_1(s)v_1 + \cdots + X_p(s)v_p. \tag{2.4}
\]

Due to the relation in (1.4), it is clear that (2.3) is equivalent to

\[
\mathbb{P} \left( \sup_{t \in \mathcal{S} \times \mathbb{S}^{p-1}} Y(t) > u \right), \text{ as } u \to \infty. \tag{2.5}
\]

To study (2.3) through (2.5), we directly impose an assumption on the covariance function \( r_Y \) of \( Y \), which we find convenient because it allows us to encode the possible cross-dependence structure among \( X_1, \cdots, X_r \) into \( r_Y \). See example (ii) below. For \( i = 1, 2 \), denote \( t_i = (s_i^T, v_i^T)^T \), where \( v_i^T = (v_{i,1}, \cdots, v_{i,p}) \). Let \( r_Y(t_1, t_2) = \text{Cov}(Y(t_1), Y(t_2)) \). Then notice that

\[
r_Y(t_1, t_2) = \sum_{i=1}^p \sum_{j=1}^p \text{Cov}(X_i(s_1), X_j(s_2))v_{1,i}v_{2,j}
\]

\[
= v_1^T v_2 - \sum_{i=1}^p \sum_{j=1}^p [\delta_{ij} - \text{Cov}(X_i(s_1), X_j(s_2))]v_{1,i}v_{2,j}
\]

\[
= 1 - \frac{1}{2} \| v_1 - v_2 \|^2 - \sum_{i=1}^p \sum_{j=1}^p [\delta_{ij} - \text{Cov}(X_i(s_1), X_j(s_2))]v_{1,i}v_{2,j}, \tag{2.6}
\]

where \( \delta_{ij} = 1(i = j) \) is the Kronecker delta. The structure in (2.6) suggests the following assumption on \( r_Y(t_1, t_2) \).

(A3) We assume that \( Y(t) \) given in (2.4) is a local-\((E, \alpha, D_t)\)-stationary Gaussian field on \( \mathcal{S} \times \mathbb{S}^{p-1} \) with \( D_t = \text{diag}(B_t, \frac{1}{\sqrt{2}} I_p) \), where \( B_t \) is a nonsingular \( n \times n \) dimensional matrix for all \( t \in \mathcal{S} \times \mathbb{S}^{p-1}, E = \{ n, p \} \) and \( \alpha = \{ \alpha, 2 \} \), for some \( 0 < \alpha \leq 2 \). We assume that matrix-valued function \( B_t \) is continuous in \( t \in \mathcal{S} \times \mathbb{S}^{p-1} \).
Remark 2.3. Note that assumption (A3) implies that for $s \in \mathcal{L}$ and $1 \leq i, j \leq p$

$$\text{Cov}(X_i(s), X_j(s)) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}.$$ 

In other words, we are considering a Gaussian vector field $\mathbf{X}(s)$ whose variance-covariance matrix at any point $s \in \mathcal{L}$ has been standardized. However, cross-dependence between $X_i(s_i)$ and $X_j(s_j)$ is still possible under assumption (A3) for $s_i, s_j \in \mathcal{L}$, $s_i \neq s_j$ and $i \neq j$.

Corollary 2.1. Let $\{\mathbf{X}(s), s \in \mathcal{L}\}$ be a Gaussian $p$-dimensional ($p \geq 2$) vector field with zero mean on a compact $m$-dimensional submanifold $\mathcal{L} \subset \mathbb{R}^n$ of positive reach and positive $m$-dimensional Lebesgue measure, such that $\{Y(t), t \in \mathcal{L} \times S^{p-1}\}$ in (2.4) satisfies assumption (A3). If $r_Y(t_1, t_2) < 1$ for all $t_1, t_2$ from $\mathcal{L} \times S^{p-1}$, $t_1 \neq t_2$, then

$$\mathbb{P}\left(\sup_{s \in \mathcal{L}} \|\mathbf{X}(s)\| > u\right) = \frac{H_{m,\alpha}}{(2\pi)^{(p-1)/2}} \int_{\mathcal{L} \times S^{p-1}} \|B_tP_s\|_m dH_{m+p-1}(t)u^{2m/\alpha + p-1} \Psi(u)(1 + o(1)), \quad (2.7)$$

as $u \to \infty$, where $P_s$ is an $n \times m$ dimensional matrix whose columns are orthonormal and span the tangent space of $T_s \mathcal{L}$.

Remark 2.4.

a. This corollary is a direct consequence of Theorem 2.1 using $R = (m, p - 1)$. To see this, notice that $H_{R,\alpha} = H_{m,\alpha}H_{p-1,2} = H_{m,\alpha}(\sqrt{\pi})^{-(p-1)}$, because of the factorization lemma (see Remark 2.3) and the well known fact $H_2 = (\pi)^{-1/2}$ (see page 31, Piterbarg [31]). Also notice that $\|\frac{1}{\sqrt{2}}I_mP_s\|_{p-1} = 2^{-(p-1)/2}$, where $P_s$ is a $p \times (p - 1)$ dimensional matrix whose columns span the tangent space of $T_s S^{p-1}$.

b. Even though the result in this corollary is stated for $p \geq 2$, it can be easily extended to the case $p = 1$. When $p = 1$, we write $\mathbf{X}(s) = X(s) \in \mathbb{R}$ and $S^{p-1} = \{\pm 1\}$. Then using the same proof of this corollary, one can show that under the assumptions given in this corollary (in a broader sense such that $B_t = B_s$ only depends on $s \in \mathcal{L}$, because $S^{p-1}$ now is a discrete set), we have that as $u \to \infty$,

$$\mathbb{P}\left(\sup_{s \in \mathcal{L}} |X(s)| > u\right) = 2H_{m,\alpha} \int_{\mathcal{L}} \|B_sP_s\|_m dH_m(s)u^{2m/\alpha} \Psi(u)(1 + o(1)), \quad (2.8)$$

where the factor 2 on the right-hand side is the cardinality of the set $S^0$.

Examples. Below we give two examples of Gaussian vector fields $\mathbf{X}$ that satisfy assumption (A3).

(i) Let $X_1(s), \cdots, X_p(s)$ be i.i.d. copies of $\{X(s), s \in \mathcal{L}\}$, which is assumed to be locally-($n, \alpha, B_s$)-stationary, where $0 < \alpha \leq 2$, that is,

$$r_X(s_1, s_2) = 1 - \|B_s(s_1 - s_2)\|^\alpha(1 + o(1)),$$
as $\max\{\|s-s_1\|,\|s-s_2\|\} \to 0$. In this case, (A3) is satisfied because

$$r_Y(t_1, t_2) = r_X(s_1, s_2)v_1^Tv_2$$

$$= 1 - \left[ \|B_s(s_1 - s_2)\|_2^2 + \frac{1}{2}\|v_1 - v_2\|^2 \right] (1 + o(1)),$$

max{$\|t - t_1\|,\|t - t_2\|$} $\to 0$. In other words, $Y(t)$ is locally-$(E, \alpha, D_t)$-stationary, where $D_t = \text{diag}(B_s, \frac{1}{2}I_p)$, $E = \{n, p\}$ and $\alpha = \{\alpha, 2\}$.

(ii) Consider $X_i(s)$ as a locally-$(n, 2, (A_0^i)^{1/2})$ stationary field, where $A_0^i$ are positive definite $n \times n$ matrices, for $i = 1, \cdots, p$. Also for $1 \leq i \neq j \leq p$, suppose $\text{Cov}(X_i(s_1), X_j(s_2)) = (s_1 - s_2)^TA_s^{ij}(s_1 - s_2)(1 + o(1))$, as $\max\{\|s - s_1\|,\|s - s_2\|\} \to 0$, where $A_s^{ij}$ are $n \times n$ symmetric matrices. So overall we may write

$$\text{Cov}(X_i(s_1), X_j(s_2)) = \delta_{ij} - (s_1 - s_2)^TA_s^{ij}(s_1 - s_2)(1 + o(1)),$$

as $\max\{\|s - s_1\|,\|s - s_2\|\} \to 0$. Using (2.6), we have

$$r_Y(t_1, t_2) = 1 - \frac{1}{2}\|v_1 - v_2\|^2 - (s_1 - s_2)^T\left\{ \sum_{i=1}^p \sum_{j=1}^p [v_iv_jA_s^{ij}] \right\} (s_1 - s_2)(1 + o(1)).$$

Let $A_t = \sum_{i=1}^p \sum_{j=1}^p [v_iv_jA_s^{ij}]$. If $A_t$ is positive definite, then (A3) is satisfied with $B_t = \sqrt{A_t}$, $E = n + p$ and $\alpha = 2$. The matrix $A_t$ is positive definite under many possible conditions. For example, if for each $i$, $\lambda_{\min}(A_s^{ij}) > \sum_{j \neq i} |\lambda_{\min}(A_s^{ij})|$, where $\lambda_{\min}$ is the smallest eigenvalue of a matrix, then $A_t$ is positive definite because for any $u \in \mathbb{R}^n$ with $\|u\| > 0$ and any $v \in \mathbb{S}_{r-1}$,

$$u^TA_tu \geq \sum_{i=1}^p \sum_{j=1}^p \lambda_{\min}(A_t^{ij})v_iv_j\|u\|^2 = v^T\Lambda_{\min}v\|u\|^2 > 0,$$

where $\Lambda_{\min}$ is a matrix consisting of $\lambda_{\min}(A_t^{ij})$, which is positive definite.

### 3 Extremes of rescaled Gaussian and $\chi$ fields on manifolds

In this section, we consider a class of centered Gaussian fields $\{Z_h(t), t \in \mathcal{M}_h\}_{h \in (0, h_0]}$ for some $0 < h_0 < 1$, where $\mathcal{M}_h = \mathcal{M}_{h,1} \times \mathcal{M}_{h,2}$ are $r$-dimensional compact submanifolds of $\mathbb{R}^n$. The goal is to develop the result in (1.2), where the index $t$ is partially rescaled by multiplying $h^{-1}$. For simplicity of exposition, in the structure $(E, \alpha)$, we take $k = 2$ so that $\alpha = (\alpha_1, \alpha_2)$, $E = (n_1, n_2)$ and $R = (r_1, r_2)$, where $1 \leq r_1 \leq n_1$, $1 \leq r_2 \leq n_2$, $r = r_1 + r_2$, and $n = n_1 + n_2$. The results in this section can be generalized to use the same structure $(E, \alpha)$ as in Section 2.
We first give the following assumptions before formulating the main result. For \( t = (t_{(1)}^T, t_{(2)}^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n \), let \( \xi_h : \mathbb{R}^n \mapsto \mathbb{R}^n \) be a function such that \( \xi_h(t) = (ht_{(1)}^T, t_{(2)}^T)^T \) and \( \xi_h^{-1} \) be its inverse. Denote \( \mathcal{M}_h = \xi_h^{-1}(\mathcal{M}_h) = \{ t : \xi_h(t) \in \mathcal{M}_h \} \). Let \( \mathcal{Z}_h(t) = Z_h(\xi_h(t)), t \in \mathcal{M}_h \). Let \( \bar{r}_h(t_1, t_2) \) be the covariance between \( \mathcal{Z}_h(t_1) \) and \( \mathcal{Z}_h(t_2) \), for \( t_1, t_2 \in \mathcal{M}_h \).

\( \text{(B1)} \) Assume \( \mathcal{M}_h = \mathcal{M}_{h,1} \times \mathcal{M}_{h,2} \), where \( \mathcal{M}_{h,i} \) is a \( r_i \)-dimensional compact submanifold of \( \mathbb{R}^{n_i} \), with \( \inf_{0<h\leq \eta_0} \Delta(\mathcal{M}_{h,i}) > 0, i = 1, 2 \), and

\[
0 < \inf_{0<h\leq \eta_0} \mathcal{H}_{r_i}(\mathcal{M}_{h,i}) \leq \sup_{0<h\leq \eta_0} \mathcal{H}_{r_i}(\mathcal{M}_{h,i}) < \infty, i = 1, 2.
\]

\( \text{(B2)} \) \( \mathcal{Z}_h(t) \) is locally-(\( E, \alpha, D_{\xi_h(t), h} \))-stationary in the following uniform sense: for \( t, t_1, t_2 \in \mathcal{M}_h \), as \( \max\{\|t - t_1\|, \|t - t_2\|\} \to 0 \),

\[
\bar{r}_h(t_1, t_2) = 1 - |D_{\xi_h(t), h}(t_1 - t_2)|_{E, \alpha}(1 + o(1)),
\]

where the \( o(1) \)-term is uniform in \( t \in \mathcal{M}_h \) and \( 0 < h \leq \eta_0 \), and \( D_{s,h} = \text{diag}(D_{s,h}^{(1)}, D_{s,h}^{(2)}) \), \( s \in \mathcal{M}_h \) is a block diagonal matrix. Here for \( i = 1, 2 \), the dimension of \( D_{s,h}^{(i)} \) is \( e_i \times e_i \), and the matrix-valued function \( D_{s,h}^{(i)} \) of \( s \) has continuous components on \( \mathcal{M}_h \). Also

\[
0 < \inf_{0<h\leq \eta_0, s \in \mathcal{M}_h} \lambda_{\min}(D_{s,h}^{(i)})^T D_{s,h}^{(i)} \leq \sup_{0<h\leq \eta_0, s \in \mathcal{M}_h} \lambda_{\max}(D_{s,h}^{(i)})^T D_{s,h}^{(i)} < \infty, i = 1, 2.
\]

\( \text{(B3)} \) Suppose that, for any \( x > 0 \), there exists \( \eta > 0 \) such that \( Q(x) < \eta < 1 \), where

\[
Q(x) = \sup_{0<h\leq \eta_0} \left\{ \|\bar{r}_h(t, s)\| : t, s \in \mathcal{M}_h, \|t^{(1)} - s^{(1)}\| > x \right\}.
\]

\( \text{(B4)} \) There exist \( x_0 > 0 \) and a function \( v(\cdot) \) such that for \( x > x_0 \), we have

\[
Q(x) \left( \log x \right)^{2(r_1/2 + r_2/2)} \leq v(x),
\]

where \( v \) is monotonically decreasing, such that, for any \( q > 0 \), \( v(x^q) = O(v(x)) = o(1) \) and \( v(x)x^q \to \infty \) as \( x \to \infty \).

**Remark 3.1.** Assumptions (B1)-(B3) extends their counterparts used in Theorem 2.1 to some forms that are uniform for the classes of Gaussian fields and manifolds. Assumption (B4) is analogous to the classical Berman condition used for proving extreme value distributions. An example of \( v(x) \) in assumption (B4) is given by \( v(x) = (\log x)^{-\beta} \), for some \( \beta > 0 \).

**Theorem 3.1.** Suppose assumptions (B1)-(B4) hold. Let

\[
\beta_h = \left( 2r_1 \log \frac{1}{h} \right)^{-\frac{1}{2}} + \left( 2r_1 \log \frac{1}{h} \right)^{-\frac{1}{2}},
\]

then...
\[
\times \left[ \frac{r_1}{\alpha_1} + \frac{r_2}{\alpha_2} - \frac{1}{2} \right] \log \log \frac{1}{h} + \log \left\{ \frac{(2r_1)^{\frac{\alpha_1}{r_1}} + (2r_2)^{\frac{\alpha_2}{r_2}} - \frac{1}{2}}{\sqrt{2\pi}} \right\} H_{R,\alpha} I_h(M_h) \right].
\]

where \( I_h(M_h) = \int_{\mathcal{M}_h} \| D_{t,h}P_t \| \, dH_r(t) \) with \( P_t \) an \( n \times r \) matrix with orthonormal columns spanning \( T_t\mathcal{M}_h \). Then

\[
\lim_{h \to 0} \mathbb{P} \left\{ \sqrt{2r_1} \log \frac{1}{h} \left( \sup_{t \in \mathcal{M}_h} Z_h(t) - \beta_h \right) \leq z \right\} = e^{-e^{-z}}.
\]

Remark 3.2.

a. If there exists \( \gamma > 0 \) such that \( I_h(M_h) \to \gamma \) as \( h \to 0 \). Then obviously \( \gamma \) can replace \( I_h(M_h) \) in the theorem. Also if \( \mathcal{M}_h \equiv \mathcal{M} \) and \( D_{t,h} \equiv D_t \) (i.e. they are independent of \( h \)), then \( I_h(M_h) = \int_{\mathcal{M}} \| D_{t,M} \| \, dH_r(t) \).

b. In fact, it can be easily seen from the proof that the result in the theorem also holds for the case that \( \mathcal{M}_{h,1} = \emptyset \) (that is, \( r_2 = 0 \)) so that \( \mathcal{M}_h \equiv \mathcal{M}_{h,1} \).

Next we consider the asymptotic extreme value distribution of rescaled \( \chi \)-fields on manifolds. For some \( 0 < h_0 < 1 \), let \( \{ X_h(s), s \in \mathcal{L}_h \}_{h \in (0,h_0]} \) be a class of centered \( p \)-dimensional Gaussian random vector fields, where \( X_h = (X_{h,1}, \ldots, X_{h,p})^T \) and \( \mathcal{L}_h \) are \( m \)-dimensional compact submanifolds of \( \mathbb{R}^n \) \((1 \leq m \leq n)\). Let \( v = (v_1, \ldots, v_p)^T \in \mathbb{R}^p \) and \( t = (s^T, v^T)^T \in \mathbb{R}^{n+p} \).

\[
Z_h(t) = Z_h(s, v) = X_{h,1}(s)v_1 + \cdots + X_{h,p}(s)v_p, \quad t \in \mathcal{M}_h := \mathcal{L}_h \times \mathbb{S}^{p-1}
\]

Using the property of Euclidean norm, we have

\[
\sup_{s \in \mathcal{L}_h} \| X_h(s) \| = \sup_{t \in \mathcal{M}_h} Z_h(t).
\]

Corollary 3.1. Suppose \( p \geq 2 \) and \( \{ Z_h(t), t \in \mathcal{L}_h \times \mathbb{S}^{p-1} \}_{h \in (0,h_0]} \) in (3.7) satisfies assumptions (B1)-(B4) with \( E = \{ n, p \}, \quad R = \{ m, p - 1 \}, \quad \alpha = \{ \alpha, 2 \}, \) and \( D_{t,h} = \text{diag}(B_{t,h}, \frac{1}{2}I_p) \) where \( B_{t,h} \) is a nonsingular \( n \times n \) dimensional matrix. Let

\[
\beta_h = \left( 2m \log \frac{1}{h} \right)^{\frac{1}{2}} + \left( 2m \log \frac{1}{h} \right)^{-\frac{1}{2}} \left[ \left( \frac{m}{\alpha} + \frac{p - 2}{2} \right) \log \log \frac{1}{h} + \log \left\{ \frac{(2m)^{\frac{m}{\alpha}} + \frac{p - 2}{2}}{\sqrt{2\pi}} H_{m,\alpha} I_h(M_h) \right\} \right],
\]

where \( I_h(M_h) = \int_{\mathcal{L}_h \times \mathbb{S}^{p-1}} \| B_{t,h}P_s \| \, dH_{m+p-1}(t) \) with \( P_s \) an \( n \times m \) matrix with orthonormal columns spanning \( T_s\mathcal{L}_h \). Then

\[
\lim_{h \to 0} \mathbb{P} \left\{ \left( 2m \log \frac{1}{h} \right)^{\frac{1}{2}} \left( \sup_{s \in \mathcal{L}_h} \| X_h(s) \| - \beta_h \right) \leq z \right\} = e^{-e^{-z}}.
\]

Remark 3.3. The result in this corollary immediately follows from Theorem 3.1. See Remark 2.4 (a) for some relevant calculation. Also, similar to Remark 2.4 (b), the result in this corollary can be extended to the case \( p = 1 \), for which (3.10) holds with

\[
\beta_h = \left( 2m \log \frac{1}{h} \right)^{\frac{1}{2}} + \left( 2m \log \frac{1}{h} \right)^{-\frac{1}{2}} \left[ \left( \frac{m}{\alpha} - \frac{1}{2} \right) \log \log \frac{1}{h} + \log \left\{ \frac{(2m)^{\frac{m}{\alpha} - \frac{1}{2}}}{\sqrt{2\pi}} H_{m,\alpha} I_h(M_h) \right\} \right],
\]

where \( I_h(M_h) = 2 \int_{\mathcal{L}_h} \| B_{t,h}P_s \| \, dH_m(s) \).
4 Proofs

4.1 Geometric construction for the proof of Theorem 2.1

The proof of Theorem 2.1 relies on some geometric construction on manifolds with positive reach, which we present first. Let $M$ be a $r$-dimensional submanifold of $\mathbb{R}^n$. Suppose it has positive reach, i.e., $\Delta(M) > 0$. For $\varepsilon, \eta > 0$, a set of points $Q$ on $M$ is called a $(\varepsilon, \eta)$-sample, if

(i) $\varepsilon$-covering: for any $x \in M$, there exists $y \in Q$ such that $\|x - y\| \leq \varepsilon$;

(ii) $\eta$-packing: for any $x, y \in Q$, $\|x - y\| > \eta$.

For simplicity, we always use $\eta = \varepsilon$, and such an $(\varepsilon, \varepsilon)$-sample is called an $\varepsilon$-net. It is known that an $\varepsilon$-net always exists for any positive real $\varepsilon$ when $M$ is bounded (Lemma 5.2, Boissonnat, Chazal and Yvinec [9]). Let $N_\varepsilon$ be the cardinality of this $\varepsilon$-net. Let

$$P_\varepsilon = \max\{n : \text{there exists an } \varepsilon \text{-packing of } M \text{ of size } n\},$$

$$C_\varepsilon = \min\{n : \text{there exists an } \varepsilon \text{-covering over } M \text{ of size } n\},$$

which are called the $\varepsilon$-packing and $\varepsilon$-covering numbers, respectively. It is known that (see Lemma 5.2 in Niyogi et al. [28])

$$P_\varepsilon \leq C_\varepsilon \leq N_\varepsilon \leq P_\varepsilon.$$

Also it is given on page 431 of Niyogi et al. (2008) that when $\varepsilon < \Delta(M)/2$

$$P_\varepsilon \leq \frac{\mathcal{H}_r(M)}{[\cos^r(\theta)]^r B_r},$$

where $B_r$ is the volume of the unit $r$-ball, and $\theta = \arcsin(\varepsilon/2)$. This implies that $N_\varepsilon = O(\varepsilon^{-r})$, as $\varepsilon \to 0$, when $\mathcal{H}_r(M)$ is bounded.

Let $\{x_1, \ldots, x_{N_\varepsilon}\} \subset M$ be an $\varepsilon$-net. With this $\varepsilon$-net, we can construct a Voronoi diagram restricted on $M$ consisting of $N_\varepsilon$ Voronoi cells $V_1, \ldots, V_{N_\varepsilon}$, where $V_i = \{x \in M : \|x - x_i\| \leq \|x - x_j\|, \text{ for all } j \neq i\}$. The Voronoi diagram gives a partition of $M$, that is $M = \bigcup_{i=1}^{N_\varepsilon} V_i$.

Due to the definition of the $\varepsilon$-net, we have that

$$(B(x_i, \varepsilon/2) \cap M) \subset V_i \subset (B(x_i, \varepsilon) \cap M), \ i = 1, \ldots, N_\varepsilon.$$

In other words, the shape of all the Voronoi cells is always not very thin.

4.2 Proof of Theorem 2.1

We first give a lemma used in the proof of Theorem 2.1

**Lemma 4.1.** Suppose that the conditions in Theorem 2.1 hold. For any subset $U \subset M$, if there exists a diffeomorphism $\psi : U \mapsto \Omega \subset \mathbb{R}^r$, where $\Omega = \psi(U)$ is a closed Jordan set of positive $r$-dimensional Lebesgue measure, then as $u \to \infty$

\[
\mathbb{P}\left(\sup_{t \in U} X(t) > u\right) = H_{R, \alpha} \int_U \prod_{j=1}^k \|D_{j,t}P_{j,t}\| r_j d\mathcal{H}_r(t) \prod_{i=1}^k u^{2r_i/\alpha_i} \Psi(u)(1 + o(1)).
\]

(4.1)
Proof. Let $\tilde{X} = X \circ \psi^{-1}$, which is a Gaussian field indexed by $\Omega \subset \mathbb{R}^r$. Consider $\tilde{t}, \tilde{t}_1, \tilde{t}_2 \in \Omega$ such that $\max\{\|\tilde{t} - \tilde{t}_1\|, \|\tilde{t} - \tilde{t}_2\|\} \to 0$. Since $\psi$ is a differomorphism, we also have $\max\{\|\psi^{-1}(\tilde{t}) - \psi^{-1}(\tilde{t}_1)\|, \|\psi^{-1}(\tilde{t}) - \psi^{-1}(\tilde{t}_2)\|\} \to 0$. Let $J_{\psi^{-1}}$ be the Jacobian matrix of $\psi^{-1}$, whose dimension is $n \times r$. Using assumption (A1), we have

$$
\text{Cov}(\tilde{X}(\tilde{t}_1), \tilde{X}(\tilde{t}_2)) = \text{Cov}(X(\psi^{-1}(\tilde{t}_1)), X(\psi^{-1}(\tilde{t}_2)))
$$

$$
= 1 - |D_{\psi^{-1}}(\tilde{t})(\psi^{-1}(\tilde{t}_1) - \psi^{-1}(\tilde{t}_2))|_{E, \alpha}(1 + o(1))
$$

$$
= 1 - |D_{\psi^{-1}}(\tilde{t})J_{\psi^{-1}}(\tilde{t}_1 - \tilde{t}_2)|_{E, \alpha}(1 + o(1)),
$$

where in the last step we have used a Taylor expansion. Since the columns of the Jacobian matrix $J_{\psi^{-1}}$ span the tangent space $T_{\psi^{-1}(\tilde{t})}\mathcal{M}$, and the matrix $D_{\psi^{-1}}(\tilde{t})$ is assumed to be nonsingular, the matrix $D_{\psi^{-1}}(\tilde{t})J_{\psi^{-1}}(\tilde{t})$ is of full rank, and therefore

$$
A(\tilde{t}) := [J_{\psi^{-1}}(\tilde{t})]^T D_{\psi^{-1}}(\tilde{t})^T D_{\psi^{-1}}(\tilde{t}) J_{\psi^{-1}}(\tilde{t})
$$

is positive definite. Also note that $A(\tilde{t})$ is block diagonal matrix, where the diagonal blocks have dimension $r_i \times r_i$, $i = 1, \cdots, k$. Let $A_{\tilde{t}}^{1/2}$ be the principal square root matrix of $A(\tilde{t})$. We have that

$$
\text{Cov}(\tilde{X}(\tilde{t}_1), \tilde{X}(\tilde{t}_2)) = 1 - |A(\tilde{t})^{1/2}(\tilde{t}_1 - \tilde{t}_2)|_{R, \alpha}(1 + o(1)).
$$

Using Theorem 7.1 in Piterbarg [31], we obtain that $u \to \infty$

$$
\mathbb{P}\left( \sup_{\tilde{t} \in \Omega} \tilde{X}(\tilde{t}) > u \right) = H_{R, \alpha} \int_{\Omega} \det[A(\tilde{t})^{1/2}] dH_r(\tilde{t}) \prod_{i=1}^k u^{2r_i/\alpha_i} \Psi(u)(1 + o(1)).
$$

Using the area formula on manifolds (see page 117, Evans and Gariepy [16]) and noticing that $\sup_{\tilde{t} \in \Omega} \tilde{X}(\tilde{t}) = \sup_{t \in U} X(t)$, we have

$$
\mathbb{P}\left( \sup_{t \in U} X(t) > u \right) = H_{R, \alpha} \int_{U} \det[A(\psi(t))^{1/2}] dH_r(\tilde{t}) \prod_{i=1}^k u^{2r_i/\alpha_i} \Psi(u)(1 + o(1)),
$$

where $B(\psi(t)) = [J_{\psi^{-1}}(\psi(t))]^T J_{\psi^{-1}}(\psi(t))$. Let $\{p_1(t), \cdots, p_r(t)\}$ be an orthonormal basis of the tangent space $T_t\mathcal{M}$ and write $P_t = [p_1(t), \cdots, p_r(t)]$. There exists a $r \times r$ nonsingular matrix $Q_t$ such that $J_{\psi^{-1}}(\psi(t)) = P_t Q_t$. Hence

$$
\frac{\det[A(\psi(t))^{1/2}]}{\det[B(\psi(t))^{1/2}]} = \frac{\det[Q_t]}{\det[P_t]} \frac{\det[(P_t^T D_t^T D_t P_t)^{1/2}]}{\det[Q_t]} = \det[(P_t^T D_t^T D_t P_t)^{1/2}]
$$

For $j = 1, \cdots, k$, let $P_{j, t}$ be a $e_j \times r_j$ matrix whose columns span the tangent space of $M_j$. Then by the Cauchy-Binet formula (see Broida and Williamson [11], page 214), we have

$$
\det[P_t^T D_t^T D_t P_t^{1/2}] = \prod_{j=1}^k \det[(P_{j, t}^T D_t^T D_t P_{j, t})^{1/2}] = \prod_{j=1}^k \|D_{j, t} P_{j, t}\|_{r_j},
$$

Therefore we get (4.1). \qed
Proof of Theorem 2.1

Proof. For any \( t \in \mathcal{M} \), let \( \rho \equiv \rho_t : B(t, \epsilon) \cap \mathcal{M} \mapsto T_t \mathcal{M} \) be the projection map to the tangent space \( T_t \mathcal{M} \), that is, \( \rho \) is a restriction of the normal projection \( \pi_t \) to the set \( B(t, \epsilon) \cap \mathcal{M} \). When \( \epsilon < \Delta(\mathcal{M})/2 \), it is known that \( \rho \) is a diffeomorphism (see Lemma 5.4, Niyogi et al. [28]).

The Jacobian of \( \rho \), denoted by \( J_\rho \), is a differential map that projects the tangent space of \( B(t, \epsilon) \cap \mathcal{M} \) at any point in it onto \( T_t \mathcal{M} \). It is also known that the angles between two tangent spaces \( T_p \mathcal{M} \) and \( T_q \mathcal{M} \) is bounded by \( L \| p - q \| \) for \( p, q \in B(t, \epsilon) \cap \mathcal{M} \) when \( \epsilon < \Delta(\mathcal{M})/2 \) (see Propositions 6.2 and 6.3 of Niyogi et al. [28]), where \( L > 0 \) is a constant only depending on \( \Delta(\mathcal{M}) \). Hence \( J_\rho \) is Lipstchtz continuous on \( B(t, \epsilon) \cap \mathcal{M} \). Suppose that \( \{ e_1, \ldots, e_r \} \) is an orthonormal basis of \( T_t \mathcal{M} \). Let \( \iota : T_t \mathcal{M} \mapsto \mathbb{R}^r \) be a map such that \( \iota(y) = (y_1, \ldots, y_r) \in \mathbb{R}^r \) for \( y = e_1 + \cdots + e_r \in T_t \mathcal{M} \). Then \( \psi := \iota \circ \rho \) is the diffeomorphism we need to apply Lemma 4.1.

We choose \( \epsilon < \Delta(\mathcal{M})/10 \). Using the method in Section 4.1 we find an \( \epsilon \)-net \( \{ t_1, \ldots, t_{N_\epsilon} \} \) for \( \mathcal{M} \), and construct a partition of \( \mathcal{M} \) with Voronoi cells \( V_1, \ldots, V_{N_\epsilon} \), where \( N_\epsilon = O(\epsilon^{-r}) \). Since \( V_i \subset (B(t_i, \epsilon) \cap \mathcal{M}) \), \( \rho \equiv \rho_{t_i} \) is a diffeomorphism on \( V_i \), \( i = 1, \ldots, N_\epsilon \).

Using Lemma 4.1 we have that

\[
P \left( \sup_{t \in V_i} X(t) > u \right) = H_{R, \alpha} \int_{V_i} \prod_{j=1}^k \| D_j t P_j, t \| \| r_j \| d \mathcal{H}_r(t) \prod_{j=1}^k u^{2r_j/\alpha_j} \Psi(u)(1 + o(1)),
\]

as \( u \to \infty \), and hence

\[
\sum_{i=1}^{N_\epsilon} P \left( \sup_{t \in V_i} X(t) > u \right) = H_{R, \alpha} \int_\mathcal{M} \prod_{j=1}^k \| D_j t P_j, t \| \| r_j \| d \mathcal{H}_r(t) \prod_{j=1}^k u^{2r_j/\alpha_j} \Psi(u)(1 + o(1)). \tag{4.3}
\]

Using the Bonferroni inequality, we have

\[
\sum_{i=1}^{N_\epsilon} P \left( \sup_{t \in V_i} X(t) > u \right) - \sum_{i \neq j} \sum_{i=1}^{N_\epsilon} P \left( \sup_{t \in V_i} X(t) > u, \sup_{t \in V_j} X(t) > u \right) \leq P \left( \sup_{t \in V_i} X(t) > u \right) \leq \sum_{i=1}^{N_\epsilon} P \left( \sup_{t \in V_i} X(t) > u \right). \tag{4.4}
\]

For \( i \neq j \), define \( d_{\max}(V_i, V_j) = \sup\{\| x - y \| : x \in V_i, y \in V_j \} \) and \( d_{\min}(V_i, V_j) = \inf\{\| x - y \| : x \in P_i, y \in P_j \} \). We divide the set of indices \( S = \{(i, j) : 1 \leq i \neq j \leq N_\epsilon \} \) into \( S_1 \) and \( S_2 \), where \( S_1 = \{(i, j) \in S : d_{\max}(V_i, V_j) \leq 5\epsilon \} \) and \( S_2 = \{(i, j) \in S : d_{\max}(V_i, V_j) > 5\epsilon \} \). If \( (i, j) \in S_1 \), then there exists \( t \in \mathcal{M} \) such that \( (V_i \cup V_j) \subset (B(t, 5\epsilon) \cap \mathcal{M}) \subset (B(t, \Delta(\mathcal{M})/2) \cap \mathcal{M}) \), and therefore using Lemma 4.1 we have as \( u \to \infty \)

\[
P \left( \sup_{t \in V_i} X(t) > u, \sup_{t \in V_j} X(t) > u \right)
\]
The assumption in the theorem guarantees that 
\[ \rho(\cdot) \quad \text{and} \quad P \]
Now it remains to show that 
\[ \text{and hence} \]
\[ \text{In order to further bound the probability on the right-hand side, we will use the Borell inequality} \]
\[ \text{Therefore as} \ u \to \infty \]
\[ \sum_{(i,j) \in S_1} P \left( \sup_{t \in V_i} X(t) > u, \sup_{t \in V_j} X(t) > u \right) = o \left( \prod_{i=1}^{k} u^{2r_i/\alpha_i} \Psi(u) \right). \quad (4.5) \]
Next we proceed to consider \((i,j) \in S_2\). Let \(Y(t,s) = X(t) + X(s)\). Note that 
\[ P \left( \sup_{t \in V_i} X(t) > u, \sup_{t \in V_j} X(t) > u \right) \leq P \left( \sup_{t \in V_i, s \in V_j} Y(t,s) > 2u \right). \quad (4.6) \]
In order to further bound the probability on the right-hand side, we will use the Borell inequality \([10]\) (see Theorem D.1 in Piterbarg \([31] \)). Notice that \(d_{\text{min}}(V_i,V_j) \geq d_{\text{max}}(V_i,V_j) - 4\epsilon\), and hence 
\[ \min_{(i,j) \in S_2} d_{\text{min}}(V_i,V_j) \geq \epsilon. \]
The assumption in the theorem guarantees that \(\rho := \sup_{\|t-s\| \geq \epsilon} r_X(t,s) < 1\). This then yields that 
\[ \max_{(i,j) \in S_2} \sup_{(t,s) \in V_i \times V_j} \text{Var} \left( Y(t,s) \right) \leq 2 + 2\rho \]
and 
\[ \sup_{(i,j) \in S_2} \sup_{(t,s) \in V_i \times V_j} E \left( Y(t,s) \right) = 0. \]
Now it remains to show that 
\[ P \left( \sup_{t \in V_i, s \in V_j} Y(t,s) > b \right) \leq 1/2 \]
for some constant \(b\) for all \((i,j) \in S_2\) in order to apply the Borell inequality to \(Y(t,s)\). Such \(b\) exists because 
\[ P \left( \sup_{t \in V_i, s \in V_j} Y(t,s) > u \right) \leq P \left( \sup_{t \in M, s \in M} Y(t,s) > u \right) \leq P \left( \sup_{t \in M} X(t) > u/2 \right) \]
\[ \leq H_{R,a} \int_{M} \prod_{j=1}^{k} \|D_j,tP_{j,t}\|_{r_j} dH_r(t) \prod_{j=1}^{k} \left( \frac{u}{2} \right)^{2r_j/\alpha_j} \Psi \left( \frac{u}{2} \right) \left( 1 + o(1) \right), \]
which tends to zero as \(u \to \infty\). The application of the Borell inequality now gives that 
\[ P \left( \sup_{t \in V_i, s \in V_j} Y(t,s) > 2u \right) \leq 2\Phi \left( \frac{u - b/2}{\sqrt{(1 + \rho)/2}} \right). \quad (4.7) \]
Also note that the cardinality $|S_2| \leq N_\varepsilon^2 \leq C\varepsilon^{-2r}$, for some constant $C > 0$. Hence
\[
\sum_{(i,j) \in S_2} \mathbb{P} \left( \sup_{t \in V_i} X(t) > u, \sup_{t \in V_j} X(t) > u \right) \leq 2|S_2| \Phi \left( \frac{u - b/2}{\sqrt{(1 + \rho)}/2} \right) = o \left( \prod_{i=1}^k u^{2p_i/\alpha_i} \Phi(u) \right),
\]

as $u \to \infty$. Combining (4.3), (4.4), (4.5) and (4.8), we have the desired result.

\[\square\]

### 4.3 Geometric construction for the proof of Theorem 3.1

We first give some geometric construction used in the proof of Theorem 3.1.

(i) **Voronoi diagram on $\mathcal{M}_h$:** Let $\ell_1 = \inf_{h \in (0,h_0]} \Delta(\mathcal{M}_{h,1})/2$. It is known from Section 4.1 that there exists an $(h\ell_1)$-net $\{s_1, \cdots, s_{m_h}\}$ on $\mathcal{M}_h$, where $m_h = O((h\ell_1)^{-r_1})$ is the cardinality of the net. With this $(h\ell_1)$-net and using the technique described in Section 4.1, we construct a Voronoi diagram restricted on $\mathcal{M}_{h,1}$. The collections of the cells are denoted by $\{J_{k,h} : k = 1, \cdots, m_h\}$, which forms a partition of $\mathcal{M}_{h,1}$. Similarly for $\mathcal{M}_{h,2}$, with $\ell_2 = \inf_{h \in (0,h_0]} \Delta(\mathcal{M}_{h,2})/2$, there exists an $\ell_2$-net $\{u_1, \cdots, u_{n_h}\}$ on $\mathcal{M}_{h,2}$, where $n_h = O(\ell_2^{-r_2})$. The cells of the corresponding Voronoi diagram on $\mathcal{M}_{h,2}$ are denoted by $U_1, \cdots, U_{n_h}$.

(ii) **Separation of Voronoi cells:** The construction of the Voronoi diagram restricted on $\mathcal{M}_{h,1}$ guarantees that each cell $J_{k,h} \supset (\mathcal{M}_{h,1} \cap B(s_k, h\ell_1)/2)$. In other words, $J_{k,h}$ is not too thin. For $0 < \delta < \ell_1/2$, let $\partial J_k = \cup_{k=1}^{m_h}(\partial J_{k,h})$ be the union of all the boundaries of the cells. Let
\[
B^{h\delta} = \{x \in \mathcal{M}_h : d(x, \partial J_k) \leq h\delta\},
\]
which is the $(h\delta)$-enlarged neighborhood of $\partial J_k$. We obtain $J_k^{\delta} = J_{k,h} \setminus B^{h\delta}$ and $J_k^{-\delta} = J_{k,h} \setminus J_k^{\delta}$ for $1 \leq k \leq m_h$. The geometric construction ensures that if $k \neq k'$, $J_{k,h}^\delta$ and $J_{k',h}^{\delta}$ are separated by $B^{h\delta}$, which is partitioned as $\{J_k^{\delta}, k = 1, \cdots, m_h\}$.

(iii) **Discretization:** We construct a dense grid on $\mathcal{M}_h$ as follows. Let $\Pi_{k,j} = (\Pi_{s_k}, \Pi_{u_j})$ be the projection map from $J_{k,h} \times U_j$ to the tangent space $T_{s_k} \mathcal{M}_{h,1} \times T_{u_j} \mathcal{M}_{h,2}$. Let the image of $J_{k,h} \times U_j$ be $\tilde{J}_{k,h} \times \tilde{U}_j$. For a given $\gamma > 0$, consider the (discrete) set $\Xi_{h\gamma\theta^{-2/\alpha_1}}(\tilde{J}_{k,h}) = \{t \in \tilde{J}_{k,h} : t = s_k + (h\gamma\theta^{-2/\alpha_1}t_1)(e_iM_{s_k})_i, e_i \in \mathbb{Z}\}$ and let $\Xi_{h\gamma\theta^{-2/\alpha_1}}(J_{k,h}) = \Pi_{s_k}^{-1}(\Xi_{h\gamma\theta^{-2/\alpha_1}}(\tilde{J}_{k,h}))$, which is a subset of $J_{k,h}$. Similarly, let $\{M_{u_j}^i : i = 1, \cdots, r_2\}$ be orthonormal vectors spanning the tangent space $T_{u_j} \mathcal{M}_{h,2}$ and we discretize $\tilde{U}_j$ with $\Xi_{\gamma\theta^{-2/\alpha_2}}(\tilde{U}_j)$. We discretize $\tilde{U}_j$ with $\Xi_{\gamma\theta^{-2/\alpha_2}}(U_j) = \Pi_{u_j}^{-1}(\Xi_{\gamma\theta^{-2/\alpha_2}}(\tilde{U}_j))$.

We denote the union of all the grid points by
\[
\Gamma_{h,\gamma,\theta} = \bigcup_{k=1}^{m_h} \bigcup_{j=1}^{n_h} [\Xi_{h\gamma\theta^{-2/\alpha_1}}(J_{k,h}) \times \Xi_{\gamma\theta^{-2/\alpha_2}}(U_j)]
\]

(4.9)
For any \( \theta \), \( \beta \), for any \( \xi \) and \( J \), it is easy to see that (Lemma 4.2).

To prove Theorem 3.1, we need to establish a sequence of approximations using the above geometric construction, detailed in Lemmas 4.2-4.7 as follows.

Let \( N_h^{(1)} \) be the cardinality of the set \( \cup_{k=1}^{m_h} \Xi_{h \theta^{-2/\alpha_1}} (J_{k,h}) \). Then obviously,

\[
N_h^{(1)} = \left| \cup_{k=1}^{m_h} \Xi_{h \theta^{-2/\alpha_1}} (J_{k,h}) \right| = O \left( \frac{\sum_{k=1}^{m_h} \mathcal{H}_{\mathcal{R}_1} (\mathcal{J}_{k,h})}{(h \theta^{-2/\alpha_1})^{r_1}} \right) = O \left( \frac{\mathcal{H}_{\mathcal{R}_1} (\mathcal{M}_{h,1})}{(h \theta^{-2/\alpha_1})^{r_1}} \right) = O(\theta^{2r_1/\alpha_1} h^{-r_1} \gamma^{-r_1}).
\]

Similarly, the cardinality of \( \cup_{j=1}^{n_h} \Xi_{\gamma \theta^{-2/\alpha_2}} (U_j) \) is given by

\[
N_h^{(2)} := \left| \cup_{j=1}^{n_h} \Xi_{\gamma \theta^{-2/\alpha_2}} (U_j) \right| = O(\theta^{2r_2/\alpha_2} \gamma^{-r_2}).
\]

It is easy to see that \( (\mathcal{J}_h^{\delta} \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta} = [\cup_{k=1}^{m_h} \Xi_{h \theta^{-2/\alpha_1}} (J_{k,h})] \times [\cup_{j=1}^{n_h} \Xi_{\gamma \theta^{-2/\alpha_2}} (U_j)] \), and

\[
N_{h,\delta}^{(1)} := \left| \cup_{k=1}^{m_h} \Xi_{h \theta^{-2/\alpha_1}} (J_{k,h}) \right| = O(N_h^{(1)}) = O(\theta^{2r_1/\alpha_1} h^{-r_1} \gamma^{-r_1}).
\]

### 4.4 Proof of Theorem 3.1

For a random process or field \( X(t) \), \( t \in \mathcal{S} \subset \mathbb{R}^n \) and \( \theta \in \mathbb{R} \), we denote

\[
\mathbb{P}_X (\theta, \mathcal{S}) = \mathbb{P}(\sup_{t \in \mathcal{S}} X(t) \leq \theta), \quad \mathbb{Q}_X (\theta, \mathcal{S}) = 1 - \mathbb{P}_X (\theta, \mathcal{S}).
\]

With \( \beta_h \) in (3.9), let

\[
\theta_{h,z} = \beta_h + \frac{1}{\sqrt{2r_1 \log(1/h)}} z.
\]

With this notation, we can rewrite (3.10) as

\[
\lim_{h \to 0} \mathbb{P}_h (\theta_{h,z}, \mathcal{M}_h) = e^{-e^{-z}}.
\]

To prove Theorem 3.1 we need to establish a sequence of approximations using the above geometric construction, detailed in Lemmas 4.2-4.7 as follows.

Recall that \( I_h (\mathcal{A}) = \int_{\mathcal{A}} \| D_{t,h} P_t \|_{r_1} d\mathcal{H}_{r_1} (t) \) for any measurable set \( \mathcal{A} \subset \mathcal{M}_h \). In the following lemma we consider \( \theta \) as a large number with \( \theta = \theta_{h,z} \) as a special case in mind.

**Lemma 4.2.** For any \( \epsilon > 0 \), there exist \( \theta_0 > 0 \) such that for all \( \theta \geq \theta_0, 0 < h \leq h_0 \), and \( J_k \in \{ J_{k,h}, J^\delta_{k,h}, J^{-\delta}_{k,h} \} \) with \( 1 \leq k \leq m_h (J) \), we have for some \( \epsilon_{k,h} \) with \( |\epsilon_{k,h}| \leq \epsilon \),

\[
\frac{\mathbb{Q}_Z_h (\theta, J_k \times \mathcal{M}_{h,2})}{\theta^{2(r_1/\alpha_1 + r_2/\alpha_2)} \Psi (\theta)} = (1 + \epsilon_{k,h}) h^{-r_1} H_R \mathcal{A} I_h (J_k \times \mathcal{M}_{h,2}).
\]
Proof. For $J_k \in \{J_{k,h}^+, J_{k,h}^{-}, J_{k,h}^0\}$, denote $\mathcal{J}_k = \{t(1)/h : t(1) \in J_k\}$. Then notice that $\mathcal{J}_k$ has a positive diameter and volume. Recall that $\xi_h(t) = (h^T_{(1)}, t^T_{(2)})^T$ for $t = (t^T_{(1)}, t^T_{(2)})^T \in \mathcal{J}_k \times \mathcal{M}_{h,2}$ and the Gaussian field $\mathcal{G}_h(t) = Z_h(\xi_h(t))$ is locally-(E, $\alpha$, $\mathcal{D}_{\xi_h(t),h}$)-stationary on $\mathcal{J}_k \times \mathcal{M}_{h,2}$. Let $\mathcal{T}_h(\mathcal{A}) = \int_{\mathcal{A}} \|D\xi_h(t),h P_t\|_{r_1} d\mathcal{H}_r(t)$ for any measurable set $\mathcal{A} \subset \xi_h^{-1}(\mathcal{M}_h)$. Then using Theorem 2.1, we obtain that

$$Q_{\mathcal{G}_h}(\theta, \mathcal{J}_k \times \mathcal{M}_{h,2}) = H_{R,\alpha} \mathcal{T}_h(\mathcal{J}_k \times \mathcal{M}_{h,2})(1 + o(1)),$$

where the $(1)$-term is uniform in $1 \leq k \leq m_h$ and $0 < h \leq h_0$, because of assumption (B2). Noticing that $\mathcal{T}_h(\mathcal{J}_k \times \mathcal{M}_{h,2}) = h^{-r_1} I_h(J_k \times \mathcal{M}_{h,2})$, we get the desired result. \hfill \Box

**Lemma 4.3.** For any $\epsilon > 0$, there exist $\gamma_0 > 0$, $\theta_0 > 0$ such that for all $\gamma \leq \gamma_0$, $\theta \geq \theta_0$, $0 < h \leq h_0$, and $J_k \in \{J_{k,h}^+, J_{k,h}^{-}, J_{k,h}^0\}$ with $1 \leq k \leq m_h$, we have for some $\epsilon_{k,h}$ with $|\epsilon_{k,h}| \leq \epsilon$,

$$Q_{\mathcal{G}_h}(\theta, (J_k \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}) = (1 + \epsilon_{k,h}) h^{-r_1} \tilde{H}_{R,\alpha}(\gamma) I_h(J_k \times \mathcal{M}_{h,2}), \quad (4.15)$$

where $\tilde{H}_{R,\alpha}(\gamma)$ only depends on $\gamma$ such that $\tilde{H}_{R,\alpha}(\gamma) \to H_{R,\alpha}$ as $\gamma \to 0$.

**Proof.** The proof is similar to that of Lemma 4.2. The difference is that, instead of applying Theorem 2.1, we use Lemma 5.3 in the appendix. Note that in order to apply Lemma 5.3, we need to find a diffeomorphism $\psi_k$ from $J_k$ to $\mathbb{R}^r$, for each $k = 1, \ldots, m_h$. This diffeomorphism is constructed in the same way as shown at the beginning of the proof of Theorem 2.1. \hfill \Box

**Lemma 4.4.** For $\theta = \theta_{h,z}$ given in (4.13) with any fixed $z$, we have that as $h \to 0$,

$$h^{-r_1} \theta^{2(r_1/\alpha_1 + r_2/\alpha_2)} \Psi(\theta) = e^{-z} H_{R,\alpha} I_h(\mathcal{M}_h)(1 + o(1)) = O(1). \quad (4.16)$$

**Proof.** Observe that the first equality in (4.16) follows from a direct calculation using (4.13). Next we show (4.16) is bounded. Recall that $\|D_{t,h} P_t\|_{r_1} = \det([P_t^T D_{t,h}^T g D_{t,h} P_t]^{1/2})$ (see (4.2)), where the columns of $P_t$ are orthonormal and span the tangent space $T_{t,h} \mathcal{M}_h$. Since $D_{t,h}$ is non-singular, there exists an orthogonal matrix $E_{t,h}$ such that the columns of $P_t$ are the eigenvectors of $E_{t,h} D_{t,h}$, whose associated eigenvalues are denoted by $\lambda_{t,1}, \ldots, \lambda_{t,r_1}$. Let $\Lambda_t = diag(\lambda_{t,1}, \ldots, \lambda_{t,r_1})$. Then

$$\|D_{t,h} P_t\|_{r_1} = \det([P_t^T D_{t,h}^T g D_{t,h} P_t]^{1/2}) = \det([\Lambda_t P_t^T P_t \Lambda_t]^{1/2}) = \prod_{j=1}^{r_1} |\lambda_{t,j}|.$$
The above calculation also shows that \( \lambda^2_{t,1}, \cdots, \lambda^2_{t,r_1} \) are eigenvalues of \( D^T_{t,h} E^T_{t,h} E_{t,h} D_{t,h} = D^T_{t,h} D_{t,h} \). It then follows that

\[
[\lambda_{\min}(D^T_{t,h} D_{t,h})]^{r_1/2} \leq \|D_{t,h} P_t\|_{r_1} \leq [\lambda_{\max}(D^T_{t,h} D_{t,h})]^{r_1/2}.
\]

The left-hand side in (4.16) is bounded because with assumption (B2) we have

\[
0 < \inf_{0<h\leq h_0} [\lambda_{\min}(D^T_{t,h} D_{t,h})]^{r_1/2} \inf_{0<h\leq h_0} \mathcal{H}_{r_1}(M_h)
\]

\[
\leq \inf_{0<h\leq h_0} I_h(M_h) \leq \sup_{0<h\leq h_0} I_h(M_h)
\]

\[
\leq \sup_{0<h\leq h_0} [\lambda_{\max}(D^T_{t,h} D_{t,h})]^{r_1/2} \mathcal{H}_{r_1}(M_h) < \infty.
\]

Denote \( J^\delta_h = \bigcup_{k\leq m_h} J^\delta_{k,h} \). Recall that \( M_h = \mathcal{M}_{h,1} \times \mathcal{M}_{h,2} \). Approximating \( M_h \) by \( J^\delta_h \times M_{h,2} \) leads to the approximation of \( \mathbb{P}_{Z_h}(\theta, M_h) \) by \( \mathbb{P}_{Z_h}(\theta, J^\delta_h \times M_{h,2}) \). The volume of \( \bigcup_{k\leq m_h} J^\delta_{k,h} \), i.e., the difference between the volumes of \( M \) and \( J^\delta_h \), is of the order \( O(\delta) \) uniformly in \( h \). As the next lemma shows, the order of the difference \( \mathbb{P}_{Z_h}(\theta, M_h) - \mathbb{P}_{Z_h}(\theta, J^\delta_h \times M_{h,2}) \) turns out to be of the same order.

**Lemma 4.5.** With \( \theta = \theta_{h,z} \) given in (4.13), there exists \( 0 < C < \infty \) such that for \( \delta \) and \( h \) small enough,

\[
0 < \mathbb{P}_{Z_h}(\theta, J^\delta_h \times M_{h,2}) - \mathbb{P}_{Z_h}(\theta, M_h) \leq C\delta,
\]

and

\[
0 < \sum_{k=1}^{m_h} \mathbb{P}_{Z_h}(\theta, J^\delta_{k,h} \times M_{h,2}) - \sum_{k=1}^{m_h} \mathbb{P}_{Z_h}(\theta, J^\delta_{k,h} \times M_{h,2}) \leq C\delta.
\]

**Proof.** Using (3.2), we have that

\[
\sup_{0<h\leq h_0, t\in \mathcal{M}_h} \|D_{t,h} M_t\|_{r_1} \leq C_1 := \sup_{0<h\leq h_0, t\in \mathcal{M}_h} [\lambda_{\max}(D^T_{t,h} D_{t,h})]^{r_1/2} < \infty.
\]

Also note that for all \( h \in (0, h_0] \), there exists \( 0 < C_2 < \infty \) such that \( \max_{1\leq k\leq m_h} \mathcal{H}_r(J^\delta_{k,h} \times M_h) \leq C_2 \delta h^{-r_1} \). Our construction of the partition of the \( M_h \) guarantees that there exists \( 0 < C_3 < \infty \) such that \( m_h \leq C_3 h^{-r_1} \). Therefore

\[
\sum_{k=1}^{m_h} I_h(J^\delta_{k,h} \times M_{h,2}) \leq m_h \sup_{0<h\leq h_0, t\in \mathcal{M}_h} \|D_{t,h} M_t\|_{r_1} \max_{1\leq k\leq m_h} \mathcal{H}_r(J^\delta_{k,h} \times M_h) \leq C_1 C_2 C_3 \delta.
\]

Using Lemma 4.2 for any \( \epsilon > 0 \), we have for \( h \) small enough that
\[ 0 \leq Q_{Z_h}(\theta_{h,z}, M_h) - Q_{Z_h}(\theta_{h,z}, J_h^{\delta} \times M_{h,2}) \]
\[ \leq \sum_{k=1}^{m_h} Q_{Z_h}(\theta_{h,z}, J_{k,h}^{\delta} \times M_{h,2}) \]
\[ \leq (1 + \epsilon) h^{-r_1} H_{R,\alpha} \theta^{(r_1/\alpha_1 + r_2/\alpha_2)} \Psi(\theta) \sum_{k=1}^{m_h} I_h(J_{k,h}^{\delta} \times M_{h,2}). \]

Then (4.18) follows from Lemma 4.4 and (4.20). Also (4.17) holds because
\[ 0 < P_{Z_h}(\theta, J_h^{\delta} \times M_{h,2}) - P_{Z_h}(\theta, M_h) \leq \sum_{k=1}^{m_h} Q_{Z_h}(\theta_{h,z}, J_{k,h}^{\delta} \times M_{h,2}). \]

With \( \Gamma_{h,\gamma,\theta} \) given in (4.9), \( (J_h^{\delta} \times M_{h,2}) \cap \Gamma_{h,\gamma,\theta} \) is a grid over \( J_h^{\delta} \times M_{h,2} \). Next we show that excursion probabilities over these two sets are close, by choosing both \( h \) and the grid size to be sufficiently small.

**Lemma 4.6.** With \( \theta = \theta_{h,z} \) given in (4.13), we have that
\[ P_{Z_h}(\theta, J_h^{\delta} \times M_{h,2}) = P_{Z_h}(\theta, (J_h^{\delta} \times M_{h,2}) \cap \Gamma_{h,\gamma,\theta}) + o(1) \] (4.21)
and
\[ \sum_{k=1}^{m_h} Q_{Z_h}(\theta, J_{k,h}^{\delta} \times M_{h,2}) = \sum_{k=1}^{m_h} Q_{Z_h}(\theta, (J_{k,h}^{\delta} \times M_{h,2}) \cap \Gamma_{h,\gamma,\theta}) + o(1), \] (4.22)
as \( \gamma, h \to 0 \).

**Proof.** Lemmas 4.2 and 4.3 imply that for any \( \epsilon > 0 \), there exist \( \gamma_0 > 0 \) and \( \theta_0 > 0 \) such that for all \( \gamma \leq \gamma_0 \) and \( \theta \geq \theta_0 \),
\[ 0 \leq Q_{Z_h}(\theta, J_{k,h}^{\delta} \times M_{h,2}) - Q_{Z_h}(\theta, (J_{k,h}^{\delta} \times M_{h,2}) \cap \Gamma_{h,\gamma,\theta}) \]
\[ \leq \sum_{j=1}^{n_h} \sum_{i=1}^{N_h} \left[ Q_{Z_h}(\theta, S_i^h \times U_j) - Q_{Z_h}(\theta, (S_i^h \times U_j) \cap \Gamma_{h,\gamma,\theta}) \right] \]
\[ \leq \epsilon h^{-r_1} \theta^{2(r_1/\alpha_1 + r_2/\alpha_2)} \Psi(\theta) H_{R,\alpha} I_h(J_{k,h}^{\delta} \times M_{h,2}). \]
As a result,
\[ 0 \leq Q_{Z_h}(\theta, J_h^{\delta} \times M_{h,2}) - Q_{Z_h}(\theta, (J_h^{\delta} \times M_{h,2}) \cap \Gamma_{h,\gamma,\theta}) \]
By Lemma 4.1 of Berman [6] (also see Lemma A4 of Bickel and Rosenblatt [7]), we have

\[ \eta^2 h^{-1} \theta^{2(\alpha_1+\alpha_2/2)} \Psi(\theta) H_{R,\alpha} I_h(\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \]

\[ \leq e h^{-1} \theta^{2(\alpha_1+\alpha_2/2)} \Psi(\theta) H_{R,\alpha} I_h(\mathcal{M}_{h,i}). \]

Then (4.21) and (4.22) immediately follows from (4.16). \[ \square \]

Recall that \((\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}\) gives a set of dense grid points in \(\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}\). For any \(1 \leq k \leq m_h\), denote the set \(T_{k,h,\gamma,\theta} = (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}\). Define a probability measure \(\overline{P}\) such that under \(\overline{P}\) the vectors \((Z_h(t) : t \in T_{k,h,\gamma,\theta})\) and \((Z_h(t') : t' \in T_{k',h,\gamma,\theta})\) are independent for \(k \neq k'\). In other words, \(\overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}) = \prod_{k \leq m_h} \overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}).\)

As the next lemma shows, the probability \(\overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta})\) can be approximated by using the probability measure \(\overline{P}\), if \(\delta\) and \(\gamma\) are small.

**Lemma 4.7.** For \(\delta > 0\) fixed and small enough, there exists \(\gamma = \gamma(h) \rightarrow 0\) as \(h \rightarrow 0\), such that with \(\theta = \theta_{h,z}\) given in (4.13), we have

\[ \overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}) = \prod_{k \leq m_h} \overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}) + o(1). \] (4.23)

**Proof.** Denote \(t = (t_1^T, t_2^T)^T\) and \(t' = (t'_1^T, t'_2^T)^T\), where \(t_1, t'_1 \in \mathbb{R}^n_1\) and \(t_2, t'_2 \in \mathbb{R}^n_2\). For \(t \in T_{k,h,\gamma,\theta}\) and \(t' \in T_{k',h,\gamma,\theta}\) with \(k \neq k'\), we have \(t_1, t'_1 \in J_{k,h}\) and \(t'_2 \in J_{k',m_h}\), and hence for all \(0 < h \leq h_0\), we have

\[ \|\xi^{-1}_h(t) - \xi^{-1}_h(t')\| \geq \|(t_1(t) - t'_1(t))/h\| \geq (2h\delta)/h = 2\delta > 0. \]

Let \(r_h(t_1, t_2)\) be the covariance between \(Z_h(t_1)\) and \(Z_h(t_2)\), for \(t_1, t_2 \in \mathcal{M}_{h}\). Then assumption (B3) implies that there exists \(\eta = \eta(\delta) > 0\), such that

\[ \sup_{0 < h \leq h_0} \sup_{k \neq k'} \sup_{t \in T_{k,h,\gamma,\theta}} \sup_{t' \in T_{k',h,\gamma,\theta}} |r_h(t, t')| < \eta < 1. \] (4.24)

By Lemma 4.1 of Berman [6] (also see Lemma A4 of Bickel and Rosenblatt [7]), we have

\[ \left| \overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}) - \prod_{k \leq m_h} \overline{P}Z_h(\theta, (\mathcal{J}_h^\delta \times \mathcal{M}_{h,2}) \cap \Gamma_{h,\gamma,\theta}) \right| \]

\[ \leq 8 \sum_{1 \leq k \neq k' \leq m_h} \sum_{t \in T_{k,h,\gamma,\theta}} \sum_{t' \in T_{k',h,\gamma,\theta}} \int_{0}^{r_h(t, t')} \frac{1}{2\pi(1 - \lambda^2)^{1/2}} \exp \left( - \frac{\theta^2}{1 + \lambda} \right) d\lambda \]

\[ \leq \sum_{1 \leq k \neq k' \leq m_h} \sum_{t \in T_{k,h,\gamma,\theta}} \sum_{t' \in T_{k',h,\gamma,\theta}} \zeta_h(t, t'). \] (4.25)
where
\[
\zeta_h(t, t') = \frac{4|r_h(t, t')|}{\pi(1 - \eta^2)^{1/2}} \exp \left( - \frac{\theta^2}{1 + |r_h(t, t')|} \right).
\]

We take \( \gamma = [v(h^{-1})]^{1/(3r_1 + 3r_2)} \). Let \( \omega \) be such that \( 0 < \omega < \frac{2}{(1 + \eta)} - 1 \), and define
\[
\mathcal{G}_{h, \gamma, \theta}^{(1)} = \{(t, t') \in T_h \times T_h : \|t(1) - t'(1)\| < h(N_{h, \delta}^{(1)})^{\omega/r_1 \gamma \theta^{-2/\alpha_1}}, 1 \leq k \neq k' \leq m_h\},
\]
\[
\mathcal{G}_{h, \gamma, \theta}^{(2)} = \{(t, t') \in T_h \times T_h : \|t(1) - t'(1)\| \geq h(N_{h, \delta}^{(1)})^{\omega/r_1 \gamma \theta^{-2/\alpha_1}}, 1 \leq k \neq k' \leq m_h\},
\]
where \( N_{h, \delta}^{(1)} \) is given in (4.12). Then the triple sum on the right-hand side of (4.25) can be written as
\[
\sum_{(t, t') \in \mathcal{G}_{h, \gamma, \theta}^{(1)}} \zeta_h(t, t') + \sum_{(t, t') \in \mathcal{G}_{h, \gamma, \theta}^{(2)}} \zeta_h(t, t'). \tag{4.26}
\]
Note that the cardinality of \( \mathcal{G}_{h, \gamma, \theta}^{(1)} \) is of the order \( O((N_{h, \delta}^{(1)})^{\omega+1}(N_{h}^{(2)})^2) \), where \( N_{h}^{(2)} \) is given in (4.11). Hence for the first sum in (4.26) we have
\[
\sum_{(t, t') \in \mathcal{G}_{h, \gamma, \theta}^{(1)}} \zeta_h(t, t') = O\left((N_{h, \delta}^{(1)})^{\omega+1}(N_{h}^{(2)})^2 \exp \left\{ - \frac{\theta^2}{1 + \eta} \right\} \right)
\]
\[
= O\left(\left(\frac{\theta^{2r_1/\alpha_1}}{h^{r_1} \gamma^{r_1}}\right) \frac{\gamma^{2r_2}}{\gamma^{2r_2}} \exp \left\{ - \frac{\theta^2}{1 + \eta} \right\} \right)
\]
\[
= O\left(\left(\frac{\log \frac{1}{h}}{h^{r_1} \gamma^{r_1+2r_2/(1+\omega)}}\right)^{1+\omega} \exp \left\{ - \frac{2r_1 \log \frac{1}{h}}{1 + \eta} \right\} \right)
\]
\[
= O\left(h^{2r_1/r_1 \gamma r_1(1+\omega)} \left(\frac{\log \frac{1}{h}}{h^{r_1} \gamma^{r_1+2r_2/(1+\omega)}}\right)^{1+\omega} \exp \left\{ - \frac{2r_1 \log \frac{1}{h}}{1 + \eta} \right\} \right)
\]
\[
= o(1) \quad \text{as} \quad h \to 0. \tag{4.27}
\]
Now we consider the second sum in (4.26). Due to (4.24) and \((1 + |r_h(t, t')|)^{-1} \geq 1 - |r_h(t, t')|\), we have
\[
\zeta_h(t, t') \leq \frac{4|r_h(t, t')|}{\pi(1 - \eta^2)^{1/2}} \exp \left( - (1 - |r_h(t, t')|) \theta^2 \right).
\]
Since \( \theta^2 = O(\log \frac{1}{h}) \) and \( \exp(-\theta^2) = O(h^{-2r_1}) \), we have \( \exp(-\theta^2) = O(h^{-2r_1}) \) for \((t, t') \in \mathcal{G}_{h, \gamma, \theta}^{(2)} \) by using (3.4). Hence when \( h \) is sufficiently small, there exists a constant \( C > 0 \) such that
\[
sup_{(t, t') \in \mathcal{G}_{h, \gamma, \theta}^{(2)}} \zeta_h(t, t') \leq C h^{2r_1} \frac{v((N_{h, \delta}^{(1)})^{\omega/r_1 \gamma \theta^{-2/\alpha_1}})}{\log((N_{h, \delta}^{(1)})^{\omega/r_1 \gamma \theta^{-2/\alpha_1}})^{2r_1/r_1 \gamma r_1(1+\omega)}}. \tag{4.28}
\]
Therefore it follows from (3.4) that
\[ \sum_{(t, t') \in \mathcal{G}_{h, \gamma, \theta}^{(2)}} \zeta_h(t, t') = O \left( h^{2r_1/N_{h, \delta}^{(1)} N_{h, \delta}^{(2)} v \left( \frac{(N_{h, \delta}^{(1)} \omega/r_1 \gamma - 2/\alpha_1)}{\log ((N_{h, \delta}^{(1)} \omega/r_1 \gamma - 2/\alpha_1))^{2r_1/\alpha_1 + 2r_2/\alpha_2}} \right) \right) \]

Combining (4.25), (4.27) and (4.29), we obtain (4.23).

Proof of Theorem 3.1

Proof. We choose the same \( \gamma = \gamma(h) \) in Lemma 4.7, and use \( \theta = \theta_{h, \epsilon} \) given in (4.13). Fix a small \( \delta > 0 \). By using (4.17), (4.21), and (4.23), we have that as \( h \to 0 \),

\[ P_{Z_h}(\theta, \mathcal{M}_h) = \exp \left\{ - \left( 1 + o(1) \right) h^{2r_1/\alpha_1 + 2r_2/\alpha_2} \Psi(\theta) H_{R, \alpha_p} I_h(\mathcal{M}_h) \right\} + o(1) \]

Then by using (4.22), (4.18), and (4.14), we get

\[ P_{Z_h}(\theta, \mathcal{M}_h) = \exp \left\{ - (1 + o(1)) h^{2r_1/\alpha_1 + 2r_2/\alpha_2} \Psi(\theta) H_{R, \alpha_p} I_h(\mathcal{M}_h) \right\} + o(1) \]

The proof is completed by noticing (4.16).

5 Appendix

In this appendix, we collect some miscellaneous results that are straightforward extensions from some existing results in the literature, and have been used in our proofs.

For an integer \( \ell > 0 \) and \( \gamma > 0 \), let \( C(\ell, \gamma) = \{ t \gamma : t \in [0, \ell] \cap \mathbb{Z} \} \). Given a structure \((E, \alpha)\), let \( H_{E, \alpha}(\ell, \gamma) = H_{E, \alpha}(C(\ell, \gamma)) \) and

\[ H_{E, \alpha}(\gamma) = \lim_{\ell \to \infty} \frac{H_{E, \alpha}(\ell, \gamma)}{\ell^{n}}. \]

The existence of this limit follows from Pickands [29]. Using the factorization lemma (Lemma 6.4 of Piterbarg [31]) and Theorem B3 of Bickel and Rosenblatt [8], we have
Lemma 5.1. \( H_{E,\alpha} = \lim_{r \to 0} \frac{H_{E,\alpha}(r)}{r^\alpha} \).

Let \( \Gamma_{E,\alpha}(\gamma, u) = \{(x_1, \cdots, x_k) \in \mathbb{R}^n : x_i = \gamma u^{-2/\alpha_i} \ell_i, \ell_i \in \mathbb{Z}^{\ell_i}, i = 1, \cdots, k\} \). The following result extends Lemma 4.2 in Qiao and Polonik [33] from assuming a simple structure with \( E = \{n\} \) and a scalar \( 0 < \alpha \leq 2 \) to a more general structure. The proof uses similar ideas and therefore is omitted. Also see Lemma 3 of Bickel and Rosenblatt [8], and Lemma 7.1 of Piterbarg [31].

Lemma 5.2. Given a structure \((E, \alpha)\), let \( X(t), t \in \mathbb{R}^n \), be a centered homogeneous Gaussian field with covariance function \( r(t) = \mathbb{E}(X(t + s)X(s)) = 1 - |t|_{E,\alpha}(1 + (1)) \), as \( t \to 0 \). Then there exists \( \delta_0 > 0 \) such that for any closed Jordan measurable set \( A \) of positive \( n \)-dimensional Lebesgue measure with diameter not exceeding \( \delta_0 \), the following asymptotic behavior occurs:

\[
\mathbb{P}\left( \sup_{t \in A_{\gamma, u}} X(t) > u \right) = \frac{H_{E,\alpha}(\gamma)}{\gamma^n} \mathcal{H}_n(A) \prod_{i=1}^k u^{2\alpha_i/\alpha} \Psi(u)(1 + o(1)),
\]

as \( u \to \infty \), where \( A_{\gamma, u} = A \cap \Gamma_{E,\alpha}(\gamma, u) \).

The next theorem is similar to Theorem 7.1 of Piterbarg [31], except that the supremum is over a dense grid. The proof is similar, where one need to replace the role of Lemma 7.1 of Piterbarg [31] by our Lemma 5.2 above.

Theorem 5.1. Let \( X(t), t \in A \subset \mathbb{R}^n \) be a locally-(\(E, \alpha, D_t\))-stationary Gaussian field with zero mean, where \( A \) is a closed Jordan set of positive \( n \)-dimensional Lebesgue measure. Assume also that the matrix-valued function \( D_t \) is continuous in \( t \) and non-singular everywhere on \( A \). Then if \( r_X(t, s) < 1 \) for all \( t, s \) from \( A \), \( t \neq s \), the following asymptotic behavior occurs:

\[
\mathbb{P}\left( \sup_{t \in A_{\gamma, u}} X(t) > u \right) = \frac{H_{E,\alpha}(\gamma)}{\gamma^n} \int_A |\det D_t| dt \prod_{i=1}^k u^{2\alpha_i/\alpha} \Psi(u)(1 + o(1)),
\]

as \( u \to \infty \), where \( A_{\gamma, u} = A \cap \Gamma_{E,\alpha}(\gamma, u) \).

The following lemma is analogous to Lemma 4.1 with the index set being a grid. The proof is also similar to that of Lemma 4.1 except that in the proof we use Theorem 5.1 to replace the role of Theorem 7.1 of Piterbarg [31].

Lemma 5.3. Suppose that the conditions in Theorem 5.1 hold. For any subset \( U \subset \mathcal{M} \), if there exists a diffeomorphism \( \psi : U \to \Omega \subset \mathbb{R}^r \), where \( \Omega = \psi(U) \) is a closed Jordan set of positive \( r \)-dimensional Lebesgue measure, then we have that as \( u \to \infty \),

\[
\mathbb{P}\left( \sup_{t \in M_{\gamma, u}} X(t) > u \right) = \frac{H_{R,\alpha}(\gamma)}{\gamma^r} \int_M \int_U \prod_{j=1}^k \|D_j t P_j t\| d\mathcal{H}_r(t) \prod_{i=1}^k u^{2\alpha_i/\alpha} \Psi(u)(1 + o(1)),
\]

where \( M_{\gamma, u} = \psi^{-1}(\Omega \cap \Gamma_{R,\alpha}(\gamma, u)) \).

(5.1)
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