Rigidly Rotating String Sticking in a Kerr Black Hole

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We analyze rigidly rotating Nambu–Goto strings in the Kerr spacetime, particularly focusing on the strings sticking in the horizon. From the regularity on the horizon, we find the condition for sticking in the horizon, which is consistent with the second law of the black hole thermodynamics. Energy extraction through the sticking string from a Kerr black hole occurs. We obtain the maximum value of the luminosity of the energy extraction.

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I. INTRODUCTION

Motion of classical objects is one of the most fundamental blocks to construct a theory of physics. Needless to say, the motion of a test particle, which is described by a geodesic motion, plays an important role in the theoretical framework of general relativity. The geodesic motion can be also used to probe the geometric properties of a background curved spacetime. In this paper, we focus on another fundamental object, the Nambu–Goto string. This is the simplest $1 + 1$ object, whose motion is governed by the action of the two-dimensional world sheet area. The classical motion of the string describes not only the motion of a macroscopic object like a cosmic string but also is used to analyze quantum aspects of a system through AdS/CFT correspondence (see, e.g., Refs. [1–3]).

The motion of the Nambu–Goto string is generally described by a set of non-linear wave equations on its world sheet. Therefore, general string motion can hardly be analyzed without a highly sophisticated numerical procedure. Nevertheless, if the background spacetime admits a Killing vector field, and the string world sheet is foliated by the integral curves of the Killing vector field, we can reduce the equations of the string motion to a set of ordinary differential equations that describes a particle motion in the orbit space of the Killing vector field [4, 5]. We call these strings co-homogeneity-one strings, c-1 strings for brevity. Classification and integrability of c-1 strings in maximally symmetric spacetimes were discussed in Refs. [6–9]. As a class of c-1 strings, rigidly rotating string motions were studied in the Minkowski background [10, 11] and stationary black hole backgrounds [10, 12, 13]. Gravitational perturbations sourced by a rigidly rotating string were analyzed in Ref. [14]. A generalization of c-1 strings to c-1 membranes was given in Ref. [15].

In particular, the system of a rigidly rotating string in a Kerr black hole is worthy of attention. As is reported in Ref. [13] (see also Ref. [16]), energy extraction from a Kerr black hole is possible through a stationary rotating string analogously to the widely known extraction processes: super-radiance [17–22], the Penrose process [23, 24], and the Blandford–Znajek process [25]. Furthermore, the system is interesting if the classical string motion can be related with some quantum aspects of field theories in the context of the Kerr/CFT correspondence [26].

In this paper, we clarify the natural question: whether a rigidly rotating string can stick into the black hole horizon or not. In the paper by Frolov et al. [10], a stationary string winds infinite times around a black hole just outside its horizon. We obtain the condition for a string to stick into the horizon by analyzing the equations of motion in the Kerr coordinates, which regularly cover the horizon.

This paper is organized as follows. In Sec. II, we derive the reduced equations of motion for a rigidly rotating Nambu–Goto string in the Kerr spacetime. We show that the allowed region for the motion of the string is bounded by the light surface and the centrifugal barrier surface. The configuration of the allowed region is classified in Sec. III for every parameter set, and we discuss the regularity on the light surface, where the norm of the associated Killing vector becomes null. In Sec. IV, we show that the string can regularly penetrate through the horizon, depending on the direction of its physical energy flux, and find it
consistent with the area law of the black hole thermodynamics. The maximum value of the luminosity of the energy extraction through a rigidly rotating string from a Kerr black hole is given in Sec. \textsection{V}. Section \textsection{VI} is devoted to a summary and discussion.

In this paper, we use geometrized units in which both the speed of light $c$ and Newton’s gravitational constant $G$ are one.

II. EQUATIONS OF MOTION

The line element of the Kerr spacetime in the Boyer–Lindquist coordinates is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar}{\Sigma}\sin^2\theta dtd\bar{\phi} + \frac{A}{\Sigma}\sin^2\theta d\bar{\phi}^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

with

$$\Delta(r) := r^2 - 2Mr + a^2,$$

$$\Sigma(r, \theta) := r^2 + a^2 \cos^2\theta,$$

$$A(r, \theta) := (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta,$$

where $M$ and $a$ are the mass of the black hole and the spin parameter satisfying $0 \leq |a| \leq M$. The radius $r_+$ of the outer horizon is given by

$$r_+ = M + \sqrt{M^2 - a^2}.$$ 

We consider rigidly rotating Nambu–Goto strings, that is, a class of c-1 strings associated with the Killing vector $\xi$ given by

$$\xi = \partial_t + \Omega \partial_{\bar{\phi}},$$

where the constant $\Omega$ is the angular velocity. Such a kind of Nambu–Goto string configuration is obtained by solving geodesic equations on the reduced space with the metric

$$\tilde{h}_{ij}dx^i dx^j := - (f g_{\mu\nu} - \xi_\mu \xi_\nu) dx^\mu dx^\nu = -f \left(\frac{\Sigma}{\Delta}d\theta^2 + \Sigma d\theta^2\right) + \Delta \sin^2\theta d\bar{\phi}^2,$$

where $\phi = \bar{\phi} - \Omega t$. The function $f(r, \theta)$ is the norm of $\xi$:

$$f(r, \theta) := g_{\mu\nu} \xi^\mu \xi^\nu = -1 + \frac{2Mr}{\Sigma} - \frac{4Mar \sin^2\theta}{\Sigma} \Omega + \frac{A \sin^2\theta}{\Sigma} \Omega^2.$$ 

The world sheet of the c-1 string is obtained by the foliation of integral curves of the Killing vector field $\xi$ along the geodesic curve. The spacetime outside the outer horizon is divided by the surface $f = 0$, we call it ‘light surface’. The rigidly rotating strings in the region where $\xi$ is timelike, $f < 0$, are stationary rotating strings, while the strings in the region where $\xi$ is spacelike, $f > 0$, are non-stationary rotating strings. Since the relative sign of $\Omega$ to $a$ is relevant in the following analyses, we assume $a \geq 0$ in this paper.
The action for a geodesic particle in the reduced space can be written in the form

$$ S = \frac{1}{2} \int \left( \frac{1}{N} \tilde{h}_{ij} \dot{x}^i \dot{x}^j + N \right) d\sigma, \quad (9) $$

where the dot ‘’ denotes the derivative with respect to a parameter $\sigma$ on the world line, and $N$ is the Lagrange multiplier. From the action (9), we obtain the Hamiltonian in the form

$$ H = \frac{N}{2} \left( \tilde{h}^{ij} p_i p_j - 1 \right) = \frac{N}{2} \left( \frac{\Delta}{f \Sigma} p_r^2 - \frac{1}{f \Sigma} p_\theta^2 + \frac{p_\phi^2}{\Delta \sin^2 \theta} - 1 \right), \quad (10) $$

where $p_i := N^{-1} \tilde{h}_{ij} \dot{x}^j$ is the conjugate momentum to $x^i$.

Setting the Lagrange multiplier $N$ as

$$ N = -f \Sigma, \quad (11) $$

we obtain the equations of motion as

$$ \dot{r} = \Delta p_r, \quad (12) $$

$$ \dot{p}_r = -\frac{1}{2} \left( \partial_r \Delta p_r^2 + \partial_r V \right), \quad (13) $$

$$ \dot{\theta} = p_\theta, \quad (14) $$

$$ \dot{p}_\theta = -\frac{1}{2} \partial_\theta V, \quad (15) $$

$$ \dot{\phi} = \frac{q f \Sigma}{\Delta \sin^2 \theta}. \quad (16) $$

Since $\phi$ is a cyclic coordinate, we can set

$$ p_\phi = -q = \text{const.}, \quad (17) $$

Further, the Hamiltonian constraint, obtained by the variation of $N$, is written as

$$ H = \Delta p_r^2 + p_\theta^2 + V \approx 0, \quad (18) $$

where

$$ V := f k \Sigma, \quad (19) $$

$$ k(r, \theta) := 1 - \frac{q^2}{\Delta \sin^2 \theta}. \quad (20) $$

We call the surface $k(r, \theta) = 0$ the centrifugal barrier because $k$ is characterized by $q = -p_\phi$. Since $\Delta$ is non-negative outside the horizon, then $V$ in Eq. (18) must be non-positive. Since $\Sigma$ is also positive, the allowed region of the string motion is given by

$$ f(r, \theta) k(r, \theta) \leq 0. \quad (21) $$
III. CLASSIFICATION OF THE ALLOWED REGION

We consider the spatial configuration of the light surface $f = 0$. It is obvious that the light surface has axial symmetry and reflection symmetry to the equatorial plane. Therefore, we pay attention to the curve $f(r, \theta) = 0$ on the $r$-$\theta$ plane. The topology of the light surface is classified by the number of intersection of $f = 0$ and $\theta = \pi/2$, the equatorial plane. Then, we consider the equation

$$f \left( r, \frac{\pi}{2} \right) = \frac{1}{r} \left[ \Omega^2 r^3 - (1 - a^2\Omega^2) r + 2M (1 - a\Omega)^2 \right] = 0. \quad (22)$$

As is shown in Appendix A, the number of roots of Eq. (22) in the region $r > r_+$ for $0 \leq a \leq M$ is summarized as follows:

- 2 roots for $\Omega^- < \Omega < \Omega^+$, \hspace{1cm} (23)
- 1 roots for $\Omega = \Omega^\pm$, \hspace{1cm} (24)
- 0 roots for $\Omega < \Omega^-_c$ or $\Omega^+_c < \Omega$, \hspace{1cm} (25)

where $\Omega^\pm_c$ are positive and negative roots of the equation

$$27(M\Omega_c)^2 (1 - a\Omega) = (1 + a\Omega)^3 \quad (26)$$

in the range of $|a\Omega_c^\pm| < 1/2$. The explicit forms of $\Omega^\pm_c$ are given in Appendix A.

The parameter $\Omega$ characterizes the shape of the light surface as shown in Fig. I. For a fixed $a$ in the range $|a| < M$, if $\Omega^-_c < \Omega < \Omega^+_c$, a cylindrical light surface and a spherical light surface appear, and two surfaces move closer to each other as $\Omega$ approaches to the critical values $\Omega^\pm_c$. If $\Omega = \Omega^\pm_c$, these surfaces touch each other at a point, say $X$, on the equatorial plane, and if $\Omega < \Omega^-_c$ or $\Omega^+_c < \Omega$, the two light surfaces merge and change to a bottle-like surface on the pole of the horizon, respectively (see Fig. I).

We evaluate $f(r, \theta)$ on the horizon as

$$f(r_+, \theta) = \frac{(r_+^2 + a^2)^2 \sin^2 \theta}{\Sigma(r_+, \theta)} (\Omega - \Omega_H)^2 \geq 0, \quad (27)$$

where

$$\Omega_H := \frac{a}{r_+^2 + a^2} \quad (28)$$

is the so-called horizon angular velocity. We see that the horizon is surrounded by the region $f > 0$ for $\Omega \neq \Omega_H$, and the light surface touches the horizon at the north and south poles. If $\Omega = \Omega_H$, the spherical light surface coincides with the horizon as the exceptional case.

A. Intersection between the light surface and the centrifugal barrier

The centrifugal barrier $k = 0$ in the Kerr geometry is a cylindrical surface. The configuration of the allowed region $f k \leq 0$ of the rigidly rotating strings is classified by inspecting the
FIG. 1. Parameter regions and the number of roots for Eq. (22) (left). Some examples of the light surfaces on a \( t = \text{const.} \phi = \text{const.} \) plane (right), where \( z := r \cos \theta \) and \( \rho := r \sin \theta \).

intersection of the light surface and the centrifugal barrier, or equivalently, the intersecting point \( S \) of curves \( f(r, \theta) = 0 \) and \( k(r, \theta) = 0 \) on the \( r-\theta \) plane. Because of the reflection symmetry with respect to the equatorial plane, we consider the position of the point \( S \), \( (r_S, \theta_S) \), in the range \( 0 \leq \theta_S \leq \pi/2 \).

We depict the parameter regions of \( q \) and \( \Omega \) which allow the intersection between the centrifugal barrier and the light surface in Fig. 2. A simple derivation of the boundary curves of the allowed region is given in Appendix B and detailed analyses on the shape of the parameter region are given in Appendix C. For some parameter sets of \( q \) and \( \Omega \) marked in the right panel in Fig. 2, the corresponding allowed regions are shown in Fig. 3.

For a fixed \( \Omega \) in the range \( \Omega^-_{\text{cr}} \leq \Omega \leq \Omega^+_{\text{cr}} \), where light surface consists of a spherical surface and a cylindrical surface, if \( |q| = 0 \), the curve \( k = 0 \) intersects with the spherical light surface at the north pole. As \( |q| \) increases, \( r_S \) and \( \theta_S \) increase along the spherical light surface, and the point \( S \) reaches the equatorial plane, \( \theta_S = \pi/2 \), and then the intersection disappears. Increasing \( |q| \) further, we see that \( k = 0 \) touches the cylindrical light surface on the equatorial plane, \( \theta_S = \pi/2 \), and \( r_S \) increases and \( \theta_S \) decreases along the cylindrical light surface. Finally, the point \( S \) goes as \( r_S \to \infty \) and \( \theta_S \to 0 \) at a critical value of \( |q| \).

For a fixed \( \Omega \) in the range \( \Omega < \Omega^-_{\text{cr}} \) or \( \Omega^+_{\text{cr}} < \Omega \), where the light surface has a bottle-like shape, as \( |q| \) increases from 0, the point \( S \) moves from the north pole to infinity, i.e., \( (r_S = r_+, \theta_S = 0) \to (r_S = \infty, \theta_S = 0) \). In the special case \( \Omega = \Omega^\pm_{\text{cr}} \), for special values of \( q_\pm \) in Fig. 2 the point \( S \) coincides with the point \( X \) on the equatorial plane. The intersecting point \( S \) appears for limited regions on the parameter space \( (\Omega, q) \). As is shown in Appendix B the regions are bounded by the curves:

\[ \Omega^2 q^2 = 1, \]  

(29)
and

\[ \Omega^4 q^6 - \Omega^2 \left( a^2 \omega^2 + 2 \right) q^4 \\
- \left( a^2 \left( 8M^2 \Omega^4 - 2\Omega^2 \right) - 24aM^2\Omega^3 + 16M^2\Omega^2 - 1 \right) q^2 \\
- \left( 4aM^2\Omega^2 + a - 4M^2\Omega \right)^2 = 0. \]

(30)

The region for the existence of S is divided by the point \((\Omega, q) = (\Omega_H, 0)\) in two parts, and the radius of S in the right region is given by

\[ r_S = \frac{M}{1 - |q|\Omega} + \sqrt{\left( \frac{M}{1 - |q|\Omega} \right)^2 - a^2 - |q|a}, \]

(31)

and the same in the left region is given by

\[ r_S = \frac{M}{1 + |q|\Omega} + \sqrt{\left( \frac{M}{1 + |q|\Omega} \right)^2 - a^2 + |q|a}. \]

(32)

The value \(\theta_S\) is obtained by \(k(r_S, \theta_S) = 0\) as

\[ \sin \theta_S = \frac{q}{\sqrt{\Delta(r_S)}} \]

(33)

Typical configurations of the centrifugal barrier and the light surfaces are shown in Fig. 3.

**FIG. 2.** The parameter regions which allow the intersection between the centrifugal barrier and the light surface are depicted for \(a = M/2\) (left panel), where “LS” is the abbreviation of light surface. The parameter region can be classified into three regions by the configuration of the light surface which intersects with the centrifugal barrier. The right panel is the scale-up figure of the upper right part of the left figure with the corresponding points to the configurations depicted in Fig. 3.

The intersecting point S is a saddle point of the potential \(V\) in the \(r-\theta\) plane. Stationary rotating strings exist in the region \(f < 0\) and \(k > 0\), and non-stationary rotating strings exist in the region \(f > 0\) and \(k < 0\). A rigidly rotating string passing through the point S from the outside to the inside is converted from a stationary rotating string to a non-stationary rotating string.
FIG. 3. The allowed region for the parameter values \( q \) and \( \Omega \) with \( a = M/2 \) marked in Fig. 2 are depicted. The length scale is normalized by \( r_+ \).

B. Regularity on the saddle point

Since \( f > 0 \) in the vicinity of the horizon, a string that is stationary rotating in the far region needs to pass through the point S and be converted to a non-stationary rotating string
for sticking into the horizon. We consider whether the string can pass through S regularly.

At the point S, we find that \( p_r \) and \( p_\theta \) must vanish because of the constraint (18). Then, Eqs. (12) and (14) reduce to

\[
\dot{r} = 0 \quad \text{and} \quad \dot{\theta} = 0
\]

(34)
at the point S. It means that the parameter \( \sigma \) is not appropriate to analyze the behavior of the strings passing through the point S.

Introducing the new parameter \( \chi \) defined by

\[
\frac{d\sigma}{d\chi} = \frac{1}{p_r},
\]

(35)
we obtain the equations with respect to \( \chi \) across the point S in the form

\[
r' = \Delta, \quad \theta' = \frac{p_\theta}{p_r}, \quad p_r' = -\frac{1}{2} \frac{p_r' \Delta \partial_r^2 V + p_\theta \partial_r \partial_\theta V}{p_r'^2}, \quad p_\theta' = -\frac{1}{2} \frac{p_\theta' \Delta \partial_\theta V + p_r \partial_r^2 V}{p_r'^2},
\]

(36)-(39)
where the prime ‘ ’ denotes the derivative with respect to \( \chi \). Since \( \Delta > 0 \) outside the horizon, \( r' > 0 \) means that the parameter \( \chi \) increases from the black hole side through S to the far side. From Eqs. (36) and (39), we obtain the following expressions for \( p_\theta' \) and \( p_r' \):

\[
p_\theta' = -\frac{p_r' \Delta \partial_r \partial_\theta V}{\partial_\theta^2 V + 2p_r'^2}, \quad p_r'^2 = \frac{1}{4} \left( -\partial_\theta^2 V - \Delta \partial_r^2 V + \sqrt{(\partial_\theta^2 V - \Delta \partial_r^2 V)^2 + 4\Delta (\partial_r \partial_\theta V)^2} \right). \quad \text{(41)}
\]

Evaluating Eqs. (40) and (41) at S, we find \( p_\theta' \) and \( p_r' \) are finite at the point S, and \( \theta' \) is also finite at the point S. We note that the sign of \( p_r \) changes across the point S, then the sign of \( f p_r \) does not change.

C. Induced Metric

The metric induced on the world sheet of the string is defined by

\[
\gamma_{AB} = g_{\mu\nu} \frac{\partial \zeta^A}{\partial \zeta^a} \frac{\partial \zeta^B}{\partial \zeta^b}, \quad \text{(42)}
\]

where \( \zeta^A = (\tau, \sigma) \) are parameters on the world sheet. If we take \( \sigma \) by Eq. (11) and \( \tau = t \) being a parameter along the integral curve of \( \xi \), using the equations of motion (12), (14), and (16), we have the induced metric in the form

\[
\gamma_{AB} d\zeta^A d\zeta^B = f d\tau^2 + \frac{2Mf}{\Delta} (\pm 2Mr + A\Omega) d\tau d\sigma + \left( -fk\Sigma^2 + \frac{q^2 f^2 \Sigma A}{\Delta^2 \sin^2 \theta} \right) d\sigma^2. \quad \text{(43)}
\]
We find that the determinant of the induced metric given by

$$\det \gamma = -f^2 \Sigma^2$$ (44)

is negative except at the point S.

We use $\chi$ introduced by Eq. (35) instead of $\sigma$ on the world sheet near the point S, then the induced metric is obtained by using Eqs. (36) and (38) with Eq. (40) in a regular form: $-f^2 \Sigma^2 / p^2$. The regularity on the light surface discussed in Sec. IIB guarantees that the world sheet is everywhere timelike.

**IV. REGULARITY ON THE HORIZON**

Let us show two cases of the string configuration on a constant $t$ slice in the Boyer–Lindquist coordinates. Solving Eqs. (12)–(16) with $a/M = 1$, $\Omega M = 1/4$, and $q/M = 0.2$, we obtain two cases: a string which connects a black hole neighborhood and infinity, and a string whose both ends exist in a horizon neighborhood (see Fig. 4). These strings wind infinitely on the horizon because $\dot{\phi}$ diverges in the limit to the horizon as is seen in Eq. (16). Note that, however, such infinite winding occurs at the bifurcation two-sphere, and it does not necessarily indicate the prohibition of horizon penetration in general.

![Diagram of string configurations](image)

**FIG. 4.** Two cases of the string configuration projected on the $\rho$-$z$ plane(left) and those three dimensional view(right), where $(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ and $\rho := r \sin \theta$(left). One is the string connecting a horizon neighbourhood and infinity, and the other is string with both ends that exist in a horizon neighbourhood.

In order to investigate possible configurations of the rigidly rotating string near the horizon and the condition for sticking into the horizon, let us rewrite the equations of
motion in the Kerr coordinate, which is regular on the horizon. The line element is given by

\[ ds^2 = - \left( 1 - \frac{2Mr}{\Sigma} \right) dv^2_{\pm} \pm 2dv_{\pm}dr + \Sigma d\theta^2 + \frac{A}{\Sigma} \sin^2 \theta d\varphi^2_{\pm} \pm 2a \sin^2 \theta dr d\varphi_{\pm} - \frac{4Mra}{\Sigma} \sin^2 \theta dv_{\pm}d\varphi_{\pm}, \tag{45} \]

where the subscripts + and - denote the advanced and retarded Kerr coordinates, respectively.

The Killing vector \( \xi \) is given by \( \partial v_{\pm} + \Omega \partial \bar{\varphi}_{\pm} \) in the present coordinate. By the same procedure as was used in Sec. II, we obtain the reduced space metric in the form

\[ \tilde{h}_{ij} dx^i dx^j = \left( 1 - a \Omega \sin^2 \theta \right)^2 dr^2 - f \Sigma d\theta^2 + \Delta \sin^2 \theta d\varphi^2_{\pm} + 2C_{\pm} \sin^2 \theta dr d\varphi_{\pm}, \tag{46} \]

where \( \varphi_{\pm} = \varphi_{\pm} - \Omega v_{\pm} \), and we have defined \( C_{\pm} \) as

\[ C_{\pm} := \pm \left[ \Omega \left( r^2 + a^2 \right) - a \right]. \tag{47} \]

Since the reduced metric (46) has the same form for the advanced and retarded Kerr coordinates, hereafter, we concentrate on the future event horizon and drop the subscript + for notational simplicity. Then, the Hamiltonian is given by

\[ \mathcal{H} = - \frac{N}{2f\Sigma} \left( \Delta p_r^2 + p_{\theta}^2 - 2C p_r p_{\varphi} + \frac{\left( 1 - a \Omega \sin^2 \theta \right)^2}{\sin^2 \theta} p_{\varphi}^2 \right) - \frac{N}{2}, \tag{48} \]

where, since \( \partial \phi = \partial \varphi \), we find \( p_{\phi} = p_{\varphi} = -q \). The constraint equation is given by

\[ \left( \Delta p_r + Cq \right)^2 + \Delta p_{\theta}^2 - f\Sigma \left( \frac{q^2}{\sin^2 \theta} - \Delta \right) = 0. \tag{49} \]

Setting \( N = -f\Sigma \), we obtain the following equations of motion

\[ \dot{r} = \Delta p_r + qC, \tag{50} \]

\[ \dot{p}_r = \left( M - r \right) p_r^2 - q p_r \partial_r C - \frac{1}{2} \partial_r \left( f\Sigma \right), \tag{51} \]

\[ \dot{\theta} = p_{\theta}, \tag{52} \]

\[ \dot{p}_{\theta} = -\frac{q^2}{2} \partial_{\theta} \left( \frac{\left( 1 - a \Omega \sin^2 \theta \right)^2}{\sin^2 \theta} \right) - \frac{1}{2} \partial_{\theta} \left( f\Sigma \right), \tag{53} \]

\[ \dot{\varphi} = -q \left( 1 - a \Omega \sin^2 \theta \right)^2 \frac{C p_r}{\sin^2 \theta} - C p_r, \tag{54} \]

where the dot ‘’ denotes the derivative with respect to the parameter along the geodesic on the metric (46).

It is different from the previous equations (12)–(16) with the Boyer–Lindquist coordinates that these equations can describe regular evolution of a string on the horizon. If we set \( \dot{r} < 0 \)
on the horizon, namely such a string is sticking into the horizon, where $\Delta = 0$ in Eq. (50), then $qC$ should be negative. It means that the following inequality holds:

$$q (\Omega_H - \Omega) \geq 0.$$  \hfill (55)

Since $\dot{r} < 0$ and $f > 0$ on the horizon, then the quantity $q$ is regarded as the outward angular momentum flux (see Appendix D). This inequality means the non-negativity of the energy flux across the horizon, which is measured by the stationary null observer on the horizon (i.e., the horizon generator $\partial_v + \Omega_H \partial_\varphi$).

Conversely, any rigidly rotating string that does not satisfy the condition (55) cannot penetrate the horizon. As is shown in Fig. 5, there are two possible behaviors near the horizon, i.e., the sticking into the horizon or infinitely winding around the horizon. It should be pointed out that both ends of a rigidly rotating string cannot stick into the horizon. Since $q$ is a constant that describes the constant flow of the angular momentum along the string, noting that the sign of $\dot{r}$ changes at the other end, we find that only one of the end can satisfy the condition (55). By using the retarded Kerr coordinates, we obtain the condition for sticking into the past event horizon as

$$q (\Omega_H - \Omega) \leq 0.$$  \hfill (56)

FIG. 5. Three dimensional view of the two cases of the string configuration, in the advanced Kerr coordinates(left) and retarded Kerr coordinates(right), where $(x, y, z) = (r \sin \theta \cos \varphi_\pm, r \sin \theta \sin \varphi_\pm, r \cos \theta)$. One is the string connecting a horizon neighbourhood and infinity, and the other is string with both ends that exist in a horizon neighbourhood.

V. ENERGY EXTRACTION FROM THE BLACK HOLE

We consider the energy extraction from the Kerr black hole by using a rigidly rotating string which sticks into the black hole [13]. We fix the direction of the parameter $\sigma$ as $\dot{r} < 0$ near the horizon.
Within the parameter range of a totally non-stationary rotating string, for example the case of A-2 in Fig. 3, the condition for the outward energy flux, \( q\Omega > 0 \), and the condition for sticking into the horizon, (55), are not compatible.

In the case of a stationary rotating string, on the other hand, as is noted above, a string needs to pass through the point S for sticking into the horizon. Since the sign of \( f \) flips at S, then the sign of \( \dot{r} \) does simultaneously. Therefore, \( q \) and \( q\Omega \) mean the outward flux of angular momentum and energy, respectively, throughout the string (see Appendix D). The energy extraction from the Kerr black hole by a stationary rotating string can be realized in the limited parameter region where \( q\Omega > 0 \) and the condition \((55)\) holds (see Fig. 2). The condition \((55)\) means the second law of the black hole thermodynamics:

\[
TdS = dM - \Omega_H dJ
= q(\Omega_H - \Omega) \geq 0, \tag{57}
\]

namely the area law \([13]\). Every rigidly rotating string passing through the horizon is compatible with the area law of the black hole.

The maximum value of the energy flux extracted from the black hole is realized for the case in which the curve \( \ell := q\Omega = \text{const.} \) is tangent to the curve \((30)\). Then we have

\[
\ell = \frac{q(a - q)}{2M^2 + 2M\sqrt{M^2 - a^2 + q^2} + q^2 - qa}. \tag{58}
\]

Since the right-hand side is an increasing function of \( a \) for \( q > 0 \) and \( \Omega > 0 \), we consider the extremal case, that is, \( a = M \).

Then Eq. \((58)\) can be rewritten as follows:

\[
\ell = \frac{q(M - q)}{2M^2 + Mq + q^2}. \tag{59}
\]

We can easily find that the maximum value is

\[
\ell_{\text{max}} = \frac{4\sqrt{2} - 5}{7}. \tag{60}
\]

Therefore, the maximum power from the Kerr black hole through a test Nambu–Goto string is given by \((4\sqrt{2} - 5)/7 \times G\mu/c^4 \simeq 9.38 \times 10^{-2} \times G\mu/c^4 \) times the Dyson luminosity\((c^5/G)\), for \( a = M \) and \( \theta_{P+} = \pi/2 \), where \( \mu \) is the tension of the string and we have restored the \( G \)'s and \( c \)'s in this equation.

VI. SUMMARY AND DISCUSSION

We have analyzed the rigidly rotating co-homogeneity-1 string which is foliated by a stationary Killing vector field in the Kerr spacetime. The problem to obtain the string configuration is reduced to solving geodesic equations on a reduced three-dimensional space. We
have shown that the world sheet is everywhere timelike. We have considered the regularity of the string configuration at two different positions, the light surface and the horizon. We have classified the configuration of the allowed region of the string motion by three parameters per unit black hole mass; the string angular velocity, angular momentum flux and black hole spin. From the regularity on the horizon, we have found that a rigidly rotating string can penetrate the black(white) hole horizon. All such configurations have non-negative physical energy flux through the horizon, and are consistent with the area law of the black hole thermodynamics. If the string with outward(inward) physical energy flux stretches to the black(white) hole horizon, it is infinitely wound around the horizon. On the bifurcation two sphere, the string is infinitely wound as was reported in Ref. [10]. It has also been shown that the maximum value of the luminosity of the energy extraction from a Kerr black hole through a Nambu–Goto string is given by \((4\sqrt{2} - 5)/7 \times \mu c)\.

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Appendix A: Number of roots of the equation \(f(r) = 0\)

On the restriction of the range \(r > r_+\), the roots of \(f = 0\) are the same as \(rf = 0\), which gives a cubic equation. The positive root \(r_*\) of the equation \((rf)' = 0\) is given by

\[
r_* = \sqrt{\frac{1 - a^2\Omega^2}{3\Omega^2}}.
\]  

(A1)

The condition that the equation \(rf = 0\) has two positive roots is

\[
r_* > r_+, \quad \text{and} \quad r_*f(r_*) < 0.
\]  

(A2)

For the existence of real \(r_*\), \(a^2\Omega^2 < 1\) must hold. These inequalities are equivalent to

\[
1 + 2a^2\Omega^2 - 6M^2\Omega^2 > 6M\Omega^2 \sqrt{M^2 - a^2},
\]  

(A3)

\[
(1 + a\Omega)^3 - 27M^2\Omega^2(1 - a\Omega) > 0.
\]  

(A4)

In Eq. [A3], since the right-hand side is non-negative, the left-hand side must be positive. Therefore, we obtain

\[
1 + 2a^2\Omega^2 > 6M^2\Omega^2.
\]  

(A5)

Taking the square of the inequality [A3], we obtain

\[
\frac{(1 + 2a^2\Omega^2)^2}{1 - a^2\Omega^2} > 12M^2\Omega^2.
\]  

(A6)
It is easy to find that, under the condition (A4), the inequality (A6) is automatically satisfied, and the condition (A5) can be replaced by
\[ |a\Omega| < \frac{1}{2}. \]  
(A7)

Then, the parameter region for the existence of two roots is given by Eq. (23). The parameters \( \Omega^\pm_{cr} \) are given by the positive and negative roots of Eq. (26) in the range (A7) as follows:
\[
\Omega^+_{cr} = 2p \cos \left( \frac{1}{3} \arccos \left( \frac{q}{2p} \right) + \frac{2\pi}{3} \right) - \frac{a^2 - 9M^2}{a(27M^2 + a^2)},
\]
(A8)
\[
\Omega^-_{cr} = 2p \cos \left( \frac{1}{3} \arccos \left( \frac{q}{2p} \right) + \frac{4\pi}{3} \right) - \frac{a^2 - 9M^2}{a(27M^2 + a^2)},
\]
(A9)

where
\[
p := \frac{3M\sqrt{9M^2 - 5a^2}}{a(27M^2 + a^2)}, \quad q := \frac{6(a^4 - 36M^2a^2 + 27M^4)}{a(9M^2 - 5a^2)(27M^2 + a^2)}.
\]
(A10)

Appendix B: Parameters for the intersection of the light surface and the centrifugal barrier: a simple derivation

The intersecting point S of the light surface \( f = 0 \) and the centrifugal barrier \( k = 0 \) exists for limited regions in the \( \Omega-q \) plane. The parameter regions for the existence of the intersecting point are bounded by two kinds of curves. The first kind curves specify that the intersecting point disappears at \( r_S \to \infty \) and \( \theta_S \to 0 \). The curves are given by
\[
r_f \to -1 + \Omega^2 r_S^2 \sin^2 \theta_S = 0, \tag{B1}
\]
\[
\Delta k \to 1 - \frac{q^2}{r_S^2 \sin^2 \theta_S} = 0. \tag{B2}
\]

Then, eliminating \( r_S^2 \sin^2 \theta_S \) from these equations, we have
\[
\Omega^2 q^2 = 1. \tag{B3}
\]
The second kind curves specify that the intersecting point S disappears on the equatorial plane. Then, curves are determined by
\[
r_f(r_S, \theta_S = \frac{\pi}{2}) = \Omega^2 r_S^3 - (1 - a^2\Omega^2)r_S + 2M(1 - a\Omega)^2 = 0, \tag{B4}
\]
\[
\Delta k(r_S, \theta_S = \frac{\pi}{2}) = r_S^2 - 2Mr_S + a^2 - q^2 = 0. \tag{B5}
\]
Eliminating \( r_S \) from these equations, we have
\[
\Omega^4 q^6 - \Omega^2 (a^2\Omega^2 + 2) q^4
- (a^2 (8M^2\Omega^4 - 2\Omega^2) - 24aM^2\Omega^3 + 16M^2\Omega^2 - 1) q^2
- (4aM^2\Omega^2 + a - 4M^2\Omega)^2 = 0. \tag{B6}
\]
This equation determines the curves implicitly in the \( \Omega-q \) plane once \( M \) and \( a \) are fixed.
Appendix C: Parameters for the intersection of the light surface and the centrifugal barrier: details

Eliminating $\theta$ by using equations $f = 0$ and $k = 0$, we obtain the following equation:

$$P(r; q)P(r; -q) = 0 \quad \text{(C1)}$$

with

$$P(r; q) := r^2 - \frac{2M}{1 + q\Omega}r + a^2 - aq, \quad \text{(C2)}$$

where we have assumed $|q\Omega| \neq 1$. The case of $|q\Omega| \neq 1$ will be clarified below. Since the equation is symmetric under flipping the sign of $q$, hereafter, we consider only the case $q \geq 0$ for a while.

a. Possible roots

From $k = 0$ we have the condition

$$0 \leq \sin^2 \theta_S = \frac{q^2}{\Delta(r_S)} \leq 1. \quad \text{(C3)}$$

The last inequality says

$$Q(r_S) := r_S^2 - 2Mr_S + a^2 - q^2 \geq 0. \quad \text{(C4)}$$

Let $r^{\text{ex}}_{P\pm}$, $r_{P\pm}$, $r^{\text{ex}}_Q$, and $r_Q$ denote roots for $P(r; \pm q) = 0$ and $Q(r) = 0$, respectively, as follows:

$$r^{\text{ex}}_{P\pm} = \frac{M}{1 \pm q\Omega} - \sqrt{\left(\frac{M}{1 \pm q\Omega}\right)^2 - a^2 \pm qa}, \quad \text{(C5)}$$  

$$r_{P\pm} = \frac{M}{1 \pm q\Omega} + \sqrt{\left(\frac{M}{1 \pm q\Omega}\right)^2 - a^2 \pm qa}, \quad \text{(C6)}$$  

$$r^{\text{ex}}_Q = M - \sqrt{M^2 - a^2 + q^2}, \quad \text{(C7)}$$  

$$r_Q = M + \sqrt{M^2 - a^2 + q^2}. \quad \text{(C8)}$$

Since $r^{\text{ex}}_Q < r_+$, the root $r^{\text{ex}}_Q$ is irrelevant to the analysis. The inequality (C4) and $r^{\text{ex}}_Q < r_+$ imply that $r_{P\pm}$ must satisfy $r_{P\pm} \geq r_Q$. As is shown in Ref. [13], $r^{\text{ex}}_{P\pm}$ is irrelevant in the sense that, if $r^{\text{ex}}_{P\pm} > 0$, then $r^{\text{ex}}_{P\pm} < r_Q$. Therefore, only $r_{P\pm}$ are candidates for $r_S$. In the case of $q\Omega = \pm 1$, we obtain the equation $P(r; \pm 1/\Omega) = 0$ instead of Eq. (C1). Therefore we may just ignore $r_{P\mp}$ for $q\Omega = \pm 1$. 

b. Region of $\Omega$ for the existence of the relevant root

Regarding $r_P$ as a function of $\Omega$ and differentiating the equation $P(r_P; \pm q) = 0$ by $\Omega$, we obtain

$$\pm \frac{\partial r_P}{\partial \Omega} = \frac{-qMr_P}{(1 \pm q\Omega)^2\sqrt{[M/(1 \pm q\Omega)]^2 - a^2 \pm qa}} \leq 0. \quad (C9)$$

Then, we can find the following monotonic dependence:

$$\pm \frac{\partial}{\partial \Omega} \left( q^2/\Delta(r_P) \right) = \pm \frac{\partial}{\partial \Omega} \sin^2 \theta_P \geq 0. \quad (C10)$$

Here we note that the minimum value of $r_P$ is given by

$$r_Q \geq r_+ \quad \text{for } \theta_P = \pi/2.$$  

Therefore, for the existence of the relevant roots, the region of $\Omega$ is restricted as

$$\Omega_{0+} < \Omega \leq \Omega_{\pi+} \quad \text{for } + \text{ branch,} \quad (C11)$$

$$\Omega_{\pi-} \leq \Omega < \Omega_{0-} \quad \text{for } - \text{ branch,} \quad (C12)$$

where $\Omega_{0\pm} = \mp 1/q$ and $\Omega_{\pi\pm}$ is given by solving $Q(r_P) = 0$ for $\Omega$ as

$$\Omega_{\pi\pm} = a \mp q \frac{2Mr_Q}{2M^2 + 2M\sqrt{M^2 - a^2 + q^2 + q^2 \mp qa}}. \quad (C13)$$

The value of $\Omega_{0\pm}$ is given by taking the limit $r_P \to \infty$ corresponding to $\sin \theta_P = 0$. Since we find

$$\Omega_{\pi-} - \Omega_{\pi+} = \frac{2qr_Q}{2M(r_Q^2 + a^2) + q^2r_Q} \geq 0, \quad (C14)$$

the two regions have intersection at $q = 0$ only if $\Omega = \Omega_H$. It is easy to show that the lines specified by $\Omega = \Omega_{\pi\mp}$ are equivalent to the lines given by Eq. (30).

c. Extremum of $\Omega_{\pi\pm}$ as a function of $q$

In order to clarify the shape of the region which allows the existence of the intersection in the parameter space of $q$ and $\Omega$, let us regard $\Omega_{\pi\pm}$ as a function of $q$. The equation for the extremum $d\Omega_{\pi\pm}/dq = 0$ can be written as

$$2M^2 - (a \mp q)^2 = -\frac{2M[M^2 - a(a \mp q)]}{\sqrt{M^2 - (a^2 - q^2)}}. \quad (C15)$$

For $+\text{branch}(upper \ sign)$, taking the square of this equation, we obtain the following equation:

$$q^3 - 3aq^2 - 3(M^2 - a^2)q + a(M^2 - a^2) = 0. \quad (C16)$$

This equation has three roots $-q_-, q_0$, and $q_+$ satisfying $-q_- \leq 0 \leq q_0 < q_+$ and $a < q_+$ for $a \leq M$, where the equality is given for $a = M$. It can be shown that each root of Eq. (C15)
is given by $q_{\pm}$ (note we have assumed $q \geq 0$). We can also show $\Omega_+ := \Omega_{\frac{\pi}{2}+}|_{q=q+} < 0$ from the following fact

$$\Omega_{\frac{\pi}{2}+}|_{q=a} = 0 \text{ and } \left. \frac{d\Omega_{\frac{\pi}{2}+}}{dq} \right|_{q=a} < 0. \quad (C17)$$

While, from $\Omega_- = \Omega_H$ for $a = M$ and

$$\Omega_{\frac{\pi}{2}-}|_{q=0} = \Omega_H \text{ and } \left. \frac{d\Omega_{\frac{\pi}{2}-}}{dq} \right|_{q=0} > 0 \text{ for } a < M, \quad (C18)$$

we find $\Omega_- := \Omega_{\frac{\pi}{2}-}|_{q=q_-} \geq \Omega_H$.

### Appendix D: Energy and angular momentum flux

Let us define the energy current $E^A$ on the string world sheet. We use the intrinsic coordinates $\zeta^A = (\tau, \sigma)$. The energy current is defined by

$$E^A := -T^{AB} \partial_{\zeta^B} (\partial_t)^\nu g_{\mu\nu}, \quad (D1)$$

where $T^{AB} := -\mu \gamma^{AB}$ is the string stress-energy tensor projected on the world sheet. The string tension $\mu$ is set to be unity hereafter for simplicity. Since $\partial_t$ is the Killing vector field, $E^A$ is conserved, that is,

$$D_A E^A = 1 \sqrt{-\gamma} \partial_\tau (\sqrt{-\gamma} E^\tau) + 1 \sqrt{-\gamma} \partial_\sigma (\sqrt{-\gamma} E^\sigma) = 0, \quad (D2)$$

where $D_A$ is the covariant derivative associated with the induced metric $\gamma_{AB}$. Then, the energy flux in the direction of increasing $\sigma$ is explicitly given by

$$\sqrt{-\gamma} E^\sigma = \sqrt{-\gamma} \left( \gamma^\sigma_\tau (g_{tt} + \Omega g_{\bar{t}\bar{t}}) + \gamma^\sigma_\sigma \dot{\phi} g_{\bar{t}\bar{t}} \right)$$

$$= -\text{sign}(f) q \Omega. \quad (D3)$$

In the same way, we can define the angular momentum current by

$$J^A := T^{AB} \partial_{\zeta^B} (\partial_\phi)^\nu g_{\mu\nu}. \quad (D4)$$

The angular momentum flux in the direction of increasing $\sigma$ is given by

$$\sqrt{-\gamma} J^\sigma = \sqrt{-\gamma} \left( \gamma^\sigma_\tau (g_{t\bar{t}} + \Omega g_{\bar{t}\bar{t}}) + \gamma^\sigma_\sigma \dot{\phi} g_{\bar{t}\bar{t}} \right)$$

$$= -\text{sign}(f) q. \quad (D5)$$

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