Quantum theory of the real and the complexified projective line

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Abstract: Quantum deformations of sets of points of the real and the complexified projective line are constructed. These deformations depend on the deformation parameter $q$ and certain further parameters $\lambda_{ij}$. The deformations for which the subspace of polynomials of degree three has a basis of ordered monomials are selected. We show that the corresponding algebras of three points have "polynomiality". Invariant elements which turn out to be cross ratios in the classical limit are defined. For the special case $|\lambda_{ij}| = 1$ a quantum cross ratio with properties similar to the classical case is presented. As an application a quantum version of the real Euclidean distance is given.
I. Introduction

Noncommutative geometry can be viewed as the quantum deformation of an increasing set of mathematical objects and theories. In this article we introduce a quantized version of the projective geometry in one dimension.

First we consider the algebra of coordinates of points $t_i$, $i \in I$ of the real line. All possible Poisson brackets between these variables (which are covariant with respect to the action of the standard Poisson structure of the Möbius group) are parametrized by a large set of parameters $\lambda_{ij}$, $i,j \in I$ which obey certain second order polynomial conditions. These quasiclassical structures give a good insight into the quantized situation.

In section 3 we replace the algebra of coordinates $t_i$ by a $U_q(sl(2,\mathbb{R})$-modul algebra of noncommutative coordinates $v_i$. We show that (up to reordering the points) third degree polynomials have a basis of ordered polynomials if a condition in analogy to the Jacobi identity is satisfied. There is one exceptional structure, which has no quasiclassical limit.

For algebras of three points we give a proof that the subspaces of homogeneous polynomials of higher degrees have also the classical dimension. The condition (which plays the role of the Jacobi identity) takes the form of the Einstein addition law for velocities, where the light velocity corresponds to the quantization parameter $q$. We consider also the complexified case.

In section 4 we consider invariant elements $C_{ijkl}$ of four points, which turn out to be cross ratios in the classical limit. Invariant elements occurred first in Ref. 1 in a similar situation for the quantum sphere. Our class of invariant elements contain the elements of the type of Ref. 1 as a subclass. However these elements do not satisfy a reality condition ($\ast$-invariance) and have no simple behaviour with respect to permutations of the four points. We show that for parameters $|\lambda_{ij}| = 1$ certain rational functions $f(C_{ijkl}, q)$ have the classical properties of the cross ratio.

As an application we define a quantization of the Euclidean distance on the real line.
II. The quasiclassical limit

Consider the one-dimensional real projective line \( \mathbb{P} \), the action of the Möbius group \( G \) on it and an arbitrary set of variable points \( P_i \in \mathbb{P}, i \in I \) on it. Let \( t \) be a coordinate on \( \mathbb{P} \) and \( t_i \) the coordinates of the points \( P_i \) and let the Möbius group \( G \cong SL(2, \mathbb{R})/\{\pm 1\} \) act on the points \( P_i \) according to

\[
t_i \rightarrow \frac{at_i + b}{ct_i + d}.
\]

Endow \( G \) with the structure of a Poisson group:

\[
\{a, b\} = ab, \quad \{a, c\} = ac, \quad \{a, d\} = 2bc,
\]

\[
\{b, c\} = 0, \quad \{b, d\} = bd, \quad \{c, d\} = cd.
\]

Let \( I \) be an arbitrary set. The general \( G \)-covariant (cf. Ref. 2) pre-Poisson bracket on the algebra of polynomials of the variables \( t_i, i \in I \) is given by

\[
\{t_j, t_i\} = t_j^2 - t_i^2 - \lambda_{ij}(t_i - t_j)^2
\]

with arbitrary coefficients \( \lambda_{ij} = -\lambda_{ji} \).

Proof. The \( G \)-covariant pre-Poisson brackets are given as sums of a special \( G \)-covariant bracket and an arbitrary \( G \)-invariant bracket (cf. Ref. 2). The general \( G \)-invariant bracket is given by

\[
\{t_j, t_i\} = \lambda_{ij}(t_i - t_j)^2.
\]

Therefore it is sufficient to check the \( G \)-covariance for one of the above structures by explicit calculation.

A calculation yields that this pre-Poisson bracket becomes a Poisson bracket (i.e. it satisfies the Jacobi identity) if and only if

\[
\lambda_{ij}\lambda_{jk} - \lambda_{ij}\lambda_{ik} - \lambda_{ik}\lambda_{jk} = -1
\]

for arbitrary three different \( i, j, k \in I \). The relation is invariant with respect to a permutation of the indices.

Instead to give the general solution of this equation we prefer to describe some basic examples.

Example 1. Let \( I \) be a linear ordered set. \( I \) may have arbitrary cardinality. Set \( \lambda_{ij} = 1 \) for \( i < j, i, j \in I \). We obtain

\[
\{t_j, t_i\} = 2t_i(t_j - t_i), \quad (i < j).
\]
Conversely, let $|\lambda_{ij}| = 1$ for arbitrary different $i, j \in I$. Then we can define a linear order relation on $I$. We set $i < j$, if $\lambda_{ij} = 1$ (the transitivity law of the order relation corresponds to the Jacobi identity).

**Example 2.** Let $I = \mathbb{Z}$ be the set of integers, $(\alpha_k)_{k \in \mathbb{Z}}$ be a sequence of positive integers and let

$$\lambda_{ij} = \coth(\sum_{k=i}^{j-1} \alpha_k), \quad \alpha_k > 0, \quad i < j.$$ 

The Poisson property follows from the addition formula

$$\coth(\alpha + \beta) = \frac{1 + \coth(\alpha) \coth(\beta)}{\coth(\alpha) + \coth(\beta)}.$$ 

We have

$$\{t_j, t_i\} = t_j^2 - t_i^2 - \coth(\sum_{k=i}^{j-1} \alpha_k)(t_i - t_j)^2, \quad i < j.$$ 

Note that $|\lambda_{ij}| > 1$ because of $|\coth(\alpha)| > 1$ for $\alpha \neq 0$. We obtain again Example 1 in the limit $\alpha_k \to \infty$.

**Example 3.** In order to get examples with $|\lambda_{ij}| < 1$ for certain $i, j \in I$, let $I = \mathbb{Z}$ and

$$\lambda_{ij} = \coth(\sum_{k=i}^{j-1} (\alpha_k + \frac{i\pi}{2})), \quad \alpha_k > 0.$$ 

Then we have $|\lambda_{ij}| < 1$ for $i - j$ odd and $|\lambda_{ij}| > 1$ for $i - j$ even.

**Example 4.** Now we give an example with $\lambda_{ij} = 0$ for certain $i, j \in I$. We remark that if $i$ is fixed, there is only one different index $j$ with $\lambda_{ij} = 0$ (because of equation (2)). Let $I = \mathbb{Z} \times \mathbb{Z}$ and $z_i := v_{i,0}$, $z_i^* := v_{i,1}$ be the corresponding variables. One can check equation (2) for $\lambda_{ij} = 0$ and $\lambda_{ij} = \lambda_{ij}^* = \lambda_{ij}^* = 1$ for $i < j$. The Poisson bracket is given by

$$\{z_i^*, z_i\} = (z_i^2 - z_i^2),$$

$$\{z_j, z_i\} = 2z_i(z_j - z_i), \quad i < j,$$

$$\{z_i^*, z_j^*\} = 2z_i^*(z_j^* - z_i^*), \quad i < j,$$

$$\{z_j, z_i^*\} = 2z_i^*(z_j - z_i^*), \quad i < j,$$

$$\{z_i^*, z_i\} = 2z_i(z_j^* - z_i), \quad i < j.$$ 

Because $z_i$ and $z_i^*$ transform according to

$$z_i \to \frac{az_i + b}{cz_i + d}, \quad z_i^* \to \frac{az_i^* + b}{cz_i^* + d}$$
we identify $z_i$ and $z_i^\ast$ with complex and conjugate complex coordinates of the complexified projective line.

**Example 5.** Now we explain, how to put structures together to a new one. Let $I'$ be an ordered set. Consider variables $t_i^{(j)}$, $i \in I^{(j)}$, $j \in I'$ (i.e. $I = \bigcup_{j \in I'} I^{(j)}$). Suppose that for fixed $j$ the variables $t_i^{(j)}$, $i \in I^{(j)}$ carry an arbitrary Poisson structure. We combine these structures according to

$$\{t_{i'}^{(j')}, t_i^{(j)}\} = 2t_i^{(j)}(t_{i'}^{(j')} - t_i^{(j)}), \quad j < j'.$$

The verification uses the fact that equation (2) is satisfied, if two of the numbers $\lambda_{ij}$, $\lambda_{ik}$ and $\lambda_{jk}$ are equal to 1.

**Example 6.** An example with complex $\lambda_{ij}$ is given by $I = \{1, 2, 3, 4\}$ and $\lambda_{12} = \lambda_{23} = \lambda_{34} = i$, $\lambda_{14} = -i$ and $\lambda_{13} = \lambda_{24} = 0$. 


III. Quantization of point sets of the real and complex projective line

A. A class of $U_q(sl_2, \mathbb{R})$-modul algebras

Let $U_q(sl_2, \mathbb{R})$, $|q| = 1$, $q \not= \pm 1, \pm i$ be the unital $*$-algebra, given by generators $E, F, K, K^{-1}$ and relations

$$KK^{-1} = K^{-1}K = I, \quad KE = qEK, \quadKF = q^{-1}FK$$

$$[E,F] = \frac{K^2 - K^{-2}}{q - q^{-1}},$$

and the involution

$$E^* = F, \quad F^* = E, \quad K^* = K^{-1}.$$

For an arbitrary set $I$ consider the unital complex algebra $A$ which is freely generated by the elements $x_i, \ i \in I$.

An action of $U_q(sl(2, \mathbb{R}))$ on $A$ is given, if we set

$$E(1) = F(1) = 0, \quad K(1) = 1, \quad E(x_i) = 1, \quad F(x_i) = -x_i^2, \quad K(x_i) = q^{-1}x_i$$

and require for $x, y \in A$

$$K(xy) = K(x)K(y)$$

$$E(xy) = E(x)K(y) + K^{-1}(x)E(y),$$

$$F(xy) = F(x)K(y) + K^{-1}(x)F(y).$$

A proof can be given along the lines of Ref. 3, where a similar modul of two variables occurs.

Further we consider ideals $I$ of $A$, which are generated by all (nonvanishing) elements of the form

$$X_{ij} := a_{ij}x_ix_j + b_{ij}x_i^2 + c_{ij}x_j^2 + d_{ij}x_jx_i,$$

where $a_{ij} = d_{ji}, \ b_{ij} = c_{ji}$ (i.e. the elements are invariant with respect to a change of $i$ and $j$). The action of $U_q(sl(2, \mathbb{R}))$ on $A$ induces an action on the factor algebra $A/I$ if and only if $I$ is $U_q(sl(2, \mathbb{R}))-invariant$. We denote the image of the quotient map of $x_i$ by $v_i$. 

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Proposition 1  The ideal $I$ is $U_q(sl(2, \mathbb{R}))$-invariant if and only if

$$a_{ij}q^2 + b_{ij}(1 + q^2) + d_{ij} = 0$$

and

$$a_{ij} + c_{ij}(1 + q^2) + d_{ij}q^2 = 0. \quad (8)$$

The ideal $I$ is generated by the elements

$$I^i_j = [x_j, x_i] - \frac{q^2 - 1}{q^2 + 1}(x_j^2 - x_i^2 - \lambda_{ij}(x_i - x_j)^2),$$

with $\lambda_{ij} = \frac{q^2 + 1}{q^2 - 1} a_{ij} + d_{ij}$, ($\lambda_{ij} = -\lambda_{ji}$, $i, j \in I$) for $a_{ij} \neq d_{ij}$ and

$$I^\infty_{ij} = (x_i - x_j)^2.$$

for $a_{ij} = d_{ij}$.

Proof: First we consider the case $a_{ij} \neq d_{ij}$. Using (3), (4), (5), (6), we have

$$E(X_{ij}) := E(a_{ij}x_i x_j + b_{ij}x_i^2 + c_{ij}x_j^2 + d_{ij}x_i x_j)$$

$$= (a_{ij}q + b_{ij}(q + \frac{1}{q}) + d_{ij}\frac{1}{q})x_i + (a_{ij}\frac{1}{q} + c_{ij}(q + \frac{1}{q}) + d_{ij}q)x_j.$$

Because there are no linear relations between the generators, the last expression is contained in $I$, if and only if it is zero. Therefore the equations (8) are necessary.

Now we show, that the equations (8) are also sufficient. We have to show, that $K(X_{ij}), F(X_{ij}) \in I$. We have

$$K(X_{ij}) := q^{-2}X_{ij} \in I.$$

Further

$$F(X_{ij}) := -\frac{a_{ij}}{q} x_i^2 x_j - a_{ij}qx_i x_j^2 - \frac{b_{ij}}{q} x_i^3 - b_{ij}qx_i^3 - \frac{c_{ij}}{q} x_j^3 - c_{ij}qx_j^3 - \frac{d_{ij}}{q} x_i^2 x_j - d_{ij}qx_i x_j^2.$$

Now let $\alpha := \frac{1 + q^2}{q^2 - 1} a_{ij} - d_{ij}$, $\beta := \frac{1 + q^2}{q^2 - 1} a_{ij} - d_{ij}$. Because of (8) we have

$$\alpha(a_{ij}+b_{ij}) = -\frac{a_{ij}}{q}, \quad \alpha(a_{ij}+c_{ij}) = -a_{ij}q, \quad \beta(c_{ij}+d_{ij}) = -\frac{d_{ij}}{q}, \quad \beta(b_{ij}+d_{ij}) = -d_{ij}q,$$
\[(\alpha + \beta) = -(q + \frac{1}{q}), \quad (d_{ij}\alpha + a_{ij}\beta) = 0.\]

Therefore

\[
F(X_{ij}) = \alpha(a_{ij} + b_{ij})x_i^2x_j + \alpha(a_{ij} + c_{ij})x_i^2x_j + \beta(c_{ij} + d_{ij})x_j^2x_i + \beta(b_{ij} + d_{ij})x_jx_i^2 + \\
+ (\alpha + \beta)b_{ij}x_i^3 + (\alpha + \beta)c_{ij}x_j^3 + (d_{ij}\alpha + a_{ij}\beta)x_ix_jx_i + (d_{ij}\alpha + a_{ij}\beta)x_jx_ix_j
\]

\[= \alpha(x_iX_{ij} + X_{ij}x_j) + \beta(x_jX_{ij} + X_{ij}x_i) \in I.\]

I.e. the equations (8) are sufficient.

Further we confirm the second statement of the proposition. We show that the generators \(X_{ij}\) and \(I_{ij}^\lambda\) differ only by a constant nonzero factor.

\[
I_{ij}^\lambda = (-1 - \lambda_{ij}\frac{q^2 - 1}{q^2 + 1})x_ix_j + (1 + \lambda_{ij})(\frac{q^2 - 1}{q^2 + 1}x_i^2 + (-1 + \lambda_{ij})(\frac{q^2 - 1}{q^2 + 1}x_j^2 + (1 - \lambda_{ij}\frac{q^2 - 1}{q^2 + 1})x_jx_i)
\]

\[= \frac{-2}{a_{ij} - d_{ij}}(a_{ij}x_ix_j + b_{ij}x_i^3 + c_{ij}x_j^3 + d_{ij}x_jx_i).\]

The second statement of the proposition follows.

Finally, let \(a_{ij} = d_{ij} \neq 0\). We obtain \(b_{ij} = c_{ij} = -a_{ij} = -d_{ij}\). The proposition follows. \(\bullet\)

We exclude the case \(I_{ij}^\infty\) from the following considerations. Proposition 1 admits the following Definition.

**Definition 1** Let \(I\) be an arbitrary set. \(A'_{\lambda}, \lambda = (\lambda_{ij}; i, j \in I)\) is the left \(U_q(sl_2, \mathbb{R})\)-modul algebra given by generators \(v_i, i \in I\) and relations

\[
[v_j, v_i] = \frac{q^2 - 1}{q^2 + 1}(v_j^2 - v_i^2 - \lambda_{ij}(v_i - v_j)^2)
\]

\[(\lambda_{ij} = -\lambda_{ji}, i, j \in I).\]

Let \(\lambda_{ij}\) be fixed. In the quasiclassical limit \(q \to 1\) the bracket (9) becomes the pre-Poisson bracket (1). By Proposition 1 the moduls \(A'_{\lambda}\) are the most general \(U_q(sl(2, \mathbb{R}))\)-moduls given by generators and quadratic relations.

In order to define crossratios (see below) we need also elements of the form \((\sum \alpha_i v_i)^{-1}\).
**Definition 2** We obtain the extension $A_\lambda$ of $A'_\lambda$ if we adjoin the inverses of the nonvanishing finite sums $(\sum \alpha_i v_i)^{-1}$ to $A_\lambda$ and require that

$$(\sum \alpha_i v_i)^{-1}(\sum \alpha_i v_i) = (\sum \alpha_i v_i)(\sum \alpha_i v_i)^{-1} = 1. \quad (10)$$

The $U_q(sl_2,\mathbb{IR})$-modul structure extends by the requirements (2), (3), (4) to all $x, y \in A_\lambda$.

**B. Polynomials of degree three**

There is a principle about a correspondence between the Jacobi identity for Poisson brackets and the Poincaré-Birkhoff-Witt-Theorem for the quantized algebras. We will prove the statement of the PBW-Theorem only for polynomials of degree three. We obtain that condition (2) for the Jacobi identity characterizes also the classical behaviour of third order polynomials (with the exception of two special values for $\lambda_{i,j}$ and one structure where (2) is not satisfied). The main part of the proof is Lemma 3.

**Remark:** Let $b_{ij}, c_{ij} \neq 0$ (i.e. $|\lambda_{ij}| \neq 1$). Then on the right side in relation (12) occurs a monomial which is lexicographically smaller and a monomial which is lexicographically greater than the monomial on the left side. Therefore we can not use the Bergman theory (cf. Ref. 4) for a proof of the PBW-Theorem.

Let $i < j$ and suppose $d_{ij} \neq 0$ (or the equivalent condition $\lambda_{i,j} \neq \frac{q^2+1}{q^2-1}$). It follows

$$v_jv_i := \alpha_{ij}v_i v_j + \beta_{ij}v_i^2 + \gamma_{ij}v_j^2 \quad (11)$$

with

$$\alpha_{ij} := \frac{-a_{ij}}{d_{ij}} = \frac{-(q^2 + 1) - \lambda_{ij}(q^2 - 1)}{\lambda_{ij}(q^2 - 1) - (q^2 + 1)},$$

$$\beta_{ij} := \frac{-b_{ij}}{d_{ij}} = \frac{1 - q^2 \alpha_{ij}}{1 + q^2} = \frac{(q^2 - 1)(1 + \lambda_{ij})}{\lambda_{ij}(q^2 - 1) - (q^2 + 1)},$$

$$\gamma_{ij} := \frac{-c_{ij}}{d_{ij}} = \frac{q^2 - \alpha_{ij}}{1 + q^2} = \frac{(q^2 - 1)(-1 + \lambda_{ij})}{\lambda_{ij}(q^2 - 1) - (q^2 + 1)},$$

(cf. formulas (7),(8),(9)).
Lemma 2  (i) Let $i < j$, $\lambda_{i,j} \neq \frac{q^2+1}{q^2 - 1}$ and $v_i, v_j$ two of the generators of $A'$. Then

$$(1-\beta_{ij}\gamma_{ij})v_i^2v_j = (\beta_{ij}^2(1+\alpha_{ij}))v_i^3 + (\alpha_{ij}\beta_{ij}(1+\alpha_{ij}))v_i^2v_j + (\alpha_{ij}^2 + \alpha_{ij}\beta_{ij}\gamma_{ij})v_iv_j^2 + (\gamma_{ij}(1+\alpha_{ij}))v_j^3.$$  

We have $1 - \beta_{ij}\gamma_{ij} \neq 0$ for the coefficient on the left side if and only if $\lambda_{i,j} \neq \frac{q^4+1}{q^4 - 1}$.

(ii) Let $1 - \beta_{ij}\gamma_{ij} \neq 0$. Then we can reduce every monomial of degree three into a sum of ordered monomials.

Proof. (i) We have

$$v_jv_jv_i = \alpha_{ij}v_jv_i + \beta_{ij}v_i^2 + \gamma_{ij}v_j^3$$

$$= \alpha_{ij}v_jv_i + (\beta_{ij}\alpha_{ij}v_iv_jv_i + \beta_{ij}\beta_{ij}v_i^2 + \gamma_{ij}\beta_{ij}v_i^2v_i) + \gamma_{ij}v_j^3$$

$$= (\alpha_{ij}\alpha_{ij}v_i^2 + \alpha_{ij}\beta_{ij}v_i^2v_i + \alpha_{ij}\gamma_{ij}v_j^3) + (\beta_{ij}\alpha_{ij}\alpha_{ij}v_i^2v_i + \beta_{ij}\alpha_{ij}\gamma_{ij}v_j^3 + \beta_{ij}\beta_{ij}v_i^2 + \gamma_{ij}v_j^3).$$

The identity follows. Further

$$(1-\beta_{ij}\gamma_{ij}) = 1 - \frac{(q^2 - 1)(1 + \lambda_{ij})}{(\lambda_{ij}(q^2 - 1) - (q^2 + 1))}$$

$$\frac{(q^2 - 1)(-1 + \lambda_{ij})}{(\lambda_{ij}(q^2 - 1) - (q^2 + 1))} = \frac{2(q^4 + 1) - 2\lambda_{ij}(q^4 - 1)}{(\lambda_{ij}(q^2 - 1) - (q^2 + 1))^2}.$$

Therefore $1 - \beta_{ij}\gamma_{ij} = 0$ if and only if $\lambda_{i,j} = \frac{q^4+1}{q^4 - 1}$.

(ii) Using (11) we can replace every nonordered monomial by a sum of lexicographically smaller monomials or monomials of the type $v_jv_iv_i$. Because of Lemma 2(i) we can reduce $v_jv_iv_i, i < j$ to ordered monomials. Therefore we can reduce every polynomial of degree three with finite steps to a sum of ordered polynomials. 

Lemma 3  (i) Let $i < j < k$ be three indices, $x_i, x_j, x_k$ three of the generators of $A$, $X_{ij} \in I$ (cf. (7)), $V \subset I$ the subspace of all elements $x_aX_{ab}, x_bX_{ab}, X_{ab}x_a, X_{ab}x_b$ and

$$X_{ijk} := -a_{jk}a_{ik}X_{ij}x_k - a_{jk}x_jX_{ik} - X_{jk}x_i + x_kX_{ij} + a_{ij}x_kx_j + a_{ik}a_{ij}x_iX_{jk} \in I.$$  

Then we have

$$X_{ijk} = \frac{(\lambda_{ij}\lambda_{jk} - \lambda_{ij}\lambda_{ik} - \lambda_{ik}\lambda_{jk} + 1)(q^2 - 1)^2(q^2 + 1)}{k_{ij}k_{ik}k_{jk}}.$$
\[ (p_1 x_i^3 + p_2 x_j^3 + p_3 x_k^3 + p_4 x_i^2 x_j + p_5 x_i x_j^2 + p_6 x_i^2 x_k + p_7 x_i x_k^2 + p_8 x_j^2 x_k + p_9 x_j x_k^2) \mod(V) \]

with

\[ p_1 := \frac{4(\lambda_{ij} - \lambda_{ik})q^6(q^2 - 1)}{l_{ij} l_{ik}}, \quad p_2 := \frac{-2(q^2 - 1)K}{l_{ij} l_{jk}} \]
\[ p_3 := \frac{4(\lambda_{ik} - \lambda_{jk})(q^2 - 1)}{l_{ik} l_{jk}} \]
\[ p_4 := \frac{-2m_{ij}q^2}{l_{ij}}, \quad p_5 := \frac{2m_{ij}}{l_{ij}}, \quad p_6 := \frac{2m_{ik}q^2}{l_{ik}} \]
\[ p_7 := \frac{-2m_{ik}}{l_{ik}}, \quad p_8 := \frac{-2m_{jk}q^2}{l_{jk}}, \quad p_9 := \frac{2m_{jk}}{l_{jk}} \]

and

\[ k_{ij} := 1 + q^2 + \lambda_{ij}(1 - q^2), \quad l_{ij} := 1 + q^4 + \lambda_{ij}(1 - q^4), \]
\[ m_{ij} := 1 + q^2 + \lambda_{ij}(-1 + q^2), \]
\[ K := -(1 + q^2 + q^4 + q^6) + \lambda_{12}(1 - q^2 + q^4 + q^6) + \lambda_{23}(-1 - q^2 + q^4 - q^6) + \]
\[ \lambda_{12}\lambda_{23}(1 - q^2 - q^4 - q^6). \]

The right side in (12) vanishes if and only if \( \lambda_{ij} \lambda_{jk} - \lambda_{ij} \lambda_{ik} - \lambda_{ik} \lambda_{jk} = -1 \) or \( \lambda_{ij} = \frac{1 + q^2}{1 - q^2}, \forall i < j \).

**Proof.** One can check that in \( X_{ijk} \) all terms \( x_ax_bx_c \) with three different \( a, b, c \) cancel. The remaining monomials have only two different indices. Because we used in the reduction process of \( v_av bv, v_avv, v_avbv, v_avv \) in the proof of Lemma 2 only reduction rules which correspond to \( x_aX_{ab}, x_bX_{ab}, X_{ab}x_a, X_{ab}x_b \), we can replace the remaining monomials of type \( x_ax_bx_c, x_axaxb, x_axbx_a \) by sums of ordered polynomials \( \mod(V) \). This was done, using a computer calculation.

The right side of (12) vanishes if and only if (2) is satisfied or \( p_1, \ldots, p_9 = 0 \). From \( p_1 = p_3 = p_4 = 0 \) follows \( \lambda_{ij} = \lambda_{ik} = \lambda_{jk} = \frac{1 + q^2}{1 - q^2} \). It follows \( p_5, \ldots, p_9 = 0 \) and \( p_2 = 0 \) because of

\[ (1 - q^2)^2 K = -(1 - q^2)^2(1 + q^2 + q^4 + q^6) + (1 - q^2)(1 + q^2)(-2q^2 + 2q^4) + (1 + q^2)^2(1 - q^2 - q^4 + q^6) = 0. \]

I.e. \( p_1, \ldots, p_9 = 0 \) if and only if \( \lambda_{ij} = \frac{1 + q^2}{1 - q^2} \). \( \bullet \)
Theorem 4  Let $A'_\lambda$ be the algebra of $n$ points. We can represent every polynomial of degree less than three as a unique sum of ordered polynomials $v_i, v_iv_j, v_iv_jv_k, \ i \leq j \leq k$ if and only if the parameters $\lambda_{ij}, \lambda_{ik}, \lambda_{jk}$ satisfy the conditions (2):

$$\lambda_{ij}\lambda_{jk} - \lambda_{ij}\lambda_{ik} - \lambda_{ik}\lambda_{jk} = -1, \quad i < j < k$$

and the additional conditions $\lambda_{ij}, \lambda_{ik}, \lambda_{jk} \neq \frac{q^2+1}{q^2-1}, \frac{q^4+1}{q^4-1}, \quad i < j < k$ or

$$\lambda_{ij} = \lambda_{ik} = \lambda_{jk} = \frac{1+q^2}{1-q^2}, \quad i < j < k.$$

Proof. (i) Let $\lambda_{ij} = \frac{q^2+1}{q^2-1}$ (i.e. $d_{ij} = 0$). Because of (7) we obtain the linear dependence of ordered monomials $a_{ij}v_iv_j + b_{ij}v_i^2 + c_{ij}v_j^2 = 0$.

Let $\lambda_{ij} = \frac{q^4+1}{q^4-1}$. Because of Lemma 2 we obtain a linear dependence of ordered monomials.

Finally let $i, j, k$ be three indices, such that (2) is not satisfied and let $\lambda_{ij} \neq \frac{1+q^2}{1-q^2}$ for $i < j$. If we apply the quotient map $A \rightarrow A/I$ to (12) we obtain a linear dependence between ordered monomials.

Therefore the conditions of Theorem 4 are necessary.

(ii) Now let the conditions of the Theorem be satisfied. Because the relations (9) are homogeneous, $A'_\lambda$ is the direct sum of the subspaces of homogeneous polynomials.

Because of $\lambda_{ij} \neq \frac{q^2+1}{q^2-1}$ (i.e. $d_{ij} \neq 0$) we can represent every polynomial of degree $\leq 2$ as a sum of ordered monomials.

It remains to consider monomials of degree three. Because of $\lambda_{ij} \neq \frac{q^4+1}{q^4-1}$ and Lemma 2(ii) we can reduce every monomial of degree three in finite steps to a sum of ordered monomials.

We show that the ordered monomials are linear independent. The space of third degree polynomials in the free algebra $A$ is $n^3$-dimensional. The Ideal $I$ is generated by the $2n \binom{n}{2}$ elements $x_iX_{jk}, X_{jk}x_i, j < k$. Because (2) is satisfied or $\lambda_{ij} = \frac{1+q^2}{1-q^2}$ for $i < j$, the left side of (12) in Lemma 3 vanishes mod($V$). Therefore we have $\binom{n}{3}$ relations between the generators $x_iX_{jk}, X_{jk}x_i$ of $I$. These relations are independent, because elements $x_iX_{jk}, X_{jk}x_i$ with three fixed different indices occur only in one of the relations. Therefore $\dim I \leq 2n \binom{n}{2} - \binom{n}{3}$. It follows, that the dimension of the space of
third degree polynomials in the factor algebra $A'_{\lambda} = A/I$ is greater than or equal $n^3 - 2n \left( \begin{array}{c} n \\ 2 \end{array} \right) + \left( \begin{array}{c} n \\ 3 \end{array} \right) = n^2 + \left( \begin{array}{c} n \\ 2 \end{array} \right) + \left( \begin{array}{c} n \\ 3 \end{array} \right)$. We have shown above that this space is spanned by ordered polynomials, i.e. the dimension of this space is less than or equal $n + 2 \left( \begin{array}{c} n \\ 2 \end{array} \right) + \left( \begin{array}{c} n \\ 3 \end{array} \right) = n^2 + \left( \begin{array}{c} n \\ 2 \end{array} \right) + \left( \begin{array}{c} n \\ 3 \end{array} \right)$. Therefore the ordered monomials are linear independent.  

Remark: If (2) is satisfied and $\lambda_{ij} = \frac{q^{2i+1}}{q^{2}-1}$ or $\frac{q^{2i+1}}{q^{2}-1}$ for certain $i < j$ one can change the order of the indices in order to satisfy the conditions of Theorem 4. 

Remark: We expect an analogue of Theorem 4 for polynomials of higher degrees. One difficulty is to classify the additional conditions. For example, for fourth order polynomials one can derive the additional condition $\beta_{ij}\gamma_{ij}(1+\alpha_{ij}) = 1$.

Remark: If we introduce the new parameters $c := i\frac{q^{2i+1}}{q^{2}-1}$ and $\phi_{ij} := \frac{c}{\lambda_{ij}}$, $(\lambda_{ij} \neq 0)$ we have

$$[v_j, v_i] = \frac{i}{c}(v_j^2 - v_i^2) - \frac{i}{\phi_{ij}}(v_i - v_j)^2.$$

For condition (2) we get the form

$$\phi_{ik} = \frac{\phi_{ij} + \phi_{jk}}{1 + \frac{\phi_{ij}\phi_{jk}}{c^2}},$$

of the Einstein addition theorem for velocities.

In subsections C and D we give some examples.

C. Points of the real projective line

Consider $A_{\lambda}$ with real $\lambda_{ij}$ and endow $A_{\lambda}$ with the trivial involution $v_i^* = v_i$. The relations (9) are obviously $*$-invariant. Therefore the involution on $A$ induces an involution on $A'_{\lambda}$ and $A_{\lambda}$.

In view of Theorem 4 and the Remark after Theorem 4 we give the following definition.

Definition 3 The algebra $A_{\lambda}^R$ of coordinates of noncommutative points $v_i$ of the real line is the $U_q(sl(2, IR))$-modul $A_{\lambda}$, if $A_{\lambda}$ carries the involution $*$ and $A'_{\lambda}$ has the property (2) or is the exceptional structure of Theorem 4.
Example 7. Let \( I \) be an arbitrary linear ordered set and let \( \lambda_{ij} = 1 \) for \( i < j \) (cf. Example 1). It follows
\[
[v_j, v_i] = (q^2 - 1)v_i(v_j - v_i), \quad i < j.
\]
We denote this structure by \( A^\mathbb{R}_I \).

Example 8. Let \( I \) be the set \( \mathbb{Z} \) of integers and \( \lambda_{ij} = \coth\left(\sum_{k=i}^{j-1} \alpha_k\right), \quad \alpha_k > 0, \quad i < j \) (cf. Example 2). It follows
\[
[v_j, v_i] = \frac{q^2 - 1}{q^2 + 1}(v_j^2 - v_i^2 - \coth\left(\sum_{k=i}^{j-1} \alpha_k\right)(v_i - v_j)^2), \quad i < j.
\]

Example 9. Let \( I \) be the set of integers and \( \lambda_{ij} = 1 + \frac{q^2}{1 + q^2}, \quad i < j \) (cf. Theorem 4). It follows
\[
v_jv_i = \frac{v_i^2 + q^2v_j^2}{1 + q^2}.
\]
We denote this structure by \( A^\mathbb{R}_E \). Because of \( \lambda_{ij} \to \infty \) for \( q \to 1 \) this structure has no quasiclassical limit.

D. Points of the complexified projective line
Let \( I' \) be an arbitrary set and \( I = I' \times \mathbb{Z}_2 \). We use the notations \( w_i := v_{(i,0)} \) and \( w_i^\gamma := v_{(i,1)}, \quad i \in I' \). Let \( w_i, \ w_i^\gamma, \ i \in I' \) be connected by the involution \(*\) (i.e. \( w_i^{\gamma^*} = w_i \)). Then the relations (9) get the form
\[
[w_j, w_i] = \frac{q^2 - 1}{q^2 + 1}(w_j^2 - w_i^2 - \lambda_{ij}(w_i - w_j)^2),
\]
\[
[w_j, w_i^\gamma] = \frac{q^2 - 1}{q^2 + 1}(w_j^2 - w_i^{\gamma^2} - \lambda_{ij}^{\gamma^*}(w_i^{\gamma^*} - w_j)^2),
\]
\[
[w_j^\gamma, w_i^\gamma] = \frac{q^2 - 1}{q^2 + 1}(w_j^{\gamma^2} - w_i^{\gamma^2^2} - \lambda_{ij}^{\gamma^*}(w_i^{\gamma^*} - w_j)^2).
\]

Lemma 5 The above relations are \(*\)-invariant (and therefore induce an involution on \( A^\gamma \)), if and only if \( \lambda_{ij} = \overline{\lambda_{ij}} \) and \( \lambda_{ij} = \lambda_{ij}^{\gamma^*} \).

We denote the algebra generated by elements \( w_i, w_i^\gamma \) and the above relations with \( \overline{\lambda_{ij}} = \lambda_{ij}^{\gamma^*} \) and \( \lambda_{ij} = \lambda_{ij}^{\gamma^*} \) by \( A^\gamma \).

In order to check (2) for \( A^\gamma \) it is sufficient to check \( \lambda_{\alpha\beta}\lambda_{\beta\gamma} - \lambda_{\alpha\beta}\lambda_{\alpha\gamma} - \lambda_{\alpha\gamma}\lambda_{\beta\gamma} = -1 \) for \( (\alpha, \beta, \gamma) = (i, j, k), (\overline{i}, j, k), (i, \overline{j}, j), \forall i < j < k \). (For real coefficients one has to check the case \( (\alpha, \beta, \gamma) = (i, j, k) \) and \( \lambda_{ij} = \lambda_{ij}, \lambda_{ij} = \lambda_{ij}^{\gamma^*} = \lambda_{ij}^{-1} \) (i.e. \( \lambda_{ij} = 0 \)).)
Definition 4 By the algebra $A^A_{C^\lambda}$ of quantized complex points $w_i$ of the complexified projective line we denote the $U_q(sl_2,\mathbb{R})$-modul $A^A_{C^\lambda}$, if $A^A_{C^\lambda}$ satisfies (2) and $A^A_{C^\lambda}$ carries the involution $\ast$.

We consider the elements $w_i$ and $w_i^\ast$, respectively, as quantized complex and quantized complex conjugated points, respectively.

Example 10. Let $I'$ be a linear ordered set, $I = I' \times \mathbb{Z}_2$ and let $\lambda_{ij}$ be defined as in Example 4. We obtain

\[
[w_i^\ast, w_i] = \frac{q^2 - 1}{q^2 + 1} (w_i^2 - w_i^2),
\]
\[
[w_j, w_i] = (q^2 - 1) w_i(w_j - w_i), \quad i < j,
\]
\[
[w_i^\ast, w_j^\ast] = (q^2 - 1) w_j^\ast(w_i^\ast - w_j^\ast), \quad i < j,
\]
\[
[w_j, w_j^\ast] = (q^2 - 1) w_j^\ast(w_j - w_j), \quad i < j,
\]
\[
[w_j^\ast, w_i] = (q^2 - 1) w_i(w_j - w_j^\ast), \quad i < j.
\]

We denote this structure by $A^A_I$.

Example 11. Let $I' = \mathbb{Z}$, $I = I' \times \mathbb{Z}_2$ and $\lambda_{ij} = \coth(\sum_{k=i}^{j-1} \alpha_k)$, $\alpha_k > 0$, $i < j$. It follows

\[
[w_i^\ast, w_i] = \frac{q^2 - 1}{q^2 + 1} (w_i^2^\ast - w_i^2),
\]
\[
[w_j, w_i] = \frac{q^2 - 1}{q^2 + 1} (w_j^2 - w_i^2 - \coth(\sum_{k=i}^{j-1} \alpha_k)(w_i - w_j)^2), \quad i < j,
\]
\[
[w_j, w_j^\ast] = \frac{q^2 - 1}{q^2 + 1} (w_j^2^\ast - w_j^2 - \tanh(\sum_{k=i}^{j-1} \alpha_k)(w_i^\ast - w_j)^2), \quad i < j,
\]
\[
[w_j^\ast, w_i] = \frac{q^2 - 1}{q^2 + 1} (w_i^2^\ast - w_i^2 - \coth(\sum_{k=i}^{j-1} \alpha_k)(w_i - w_i^\ast)^2), \quad i < j.
\]

Example 10 arises again in the limit $\alpha_k \to \infty$.

Example 12. An example with complex coefficients is given by $I = \{1, 2, \mathbb{T}, \mathbb{Z}\}$ and $\lambda_{12} = \lambda_{2\mathbb{T}} = \lambda_{\mathbb{T}\mathbb{Z}} = \lambda_{\mathbb{Z}\mathbb{I}} = i$ and $\lambda_{1\mathbb{T}} = \lambda_{\mathbb{T}2} = 0$ (cf. Example 6).
E. Algebras of three points

We will say that $A_{\lambda}'$ is polynomial, if the subspaces of homogeneous polynomials have the classical dimensions.

From Theorem 4 and the Remark after Theorem 4 follows that the subspaces of homogeneous polynomials of degree $\leq 3$ of the algebras $A_{\lambda}'$ which satisfy (2) and of the exceptional algebra $A_{IR}^E$ have the classical dimensions.

We expect that these algebras are polynomial. We give a proof for algebras of three points.

Lemma 6 Let $A_{\lambda}'$ be the algebra with three generators $v_1, v_2, v_3$ and let (2) be satisfied. $A_{\lambda}'$ is equivalent to the algebra with generators $u_1, u_2, u_3$ and relations

\[
\begin{align*}
    u_2u_1 &= \frac{1}{q^2}u_1u_2, \\
    u_3u_1 &= \frac{1}{q^2}u_1u_3 + (\frac{1}{q^2} - 1)(\lambda_{1,2} - 1)(\lambda_{2,3} + 1)u_2^2, \\
    u_3u_2 &= \frac{1}{q^2}u_2u_3
\end{align*}
\]

The equivalence is given by a linear transformation between the generators $u_i$ and $v_i$.

Proof. Consider the transformation

\[
\begin{pmatrix}
    u_1 \\
    u_2 \\
    u_3
\end{pmatrix} =
\begin{pmatrix}
    \frac{1}{\lambda_{1,2} - 1} & \frac{-\lambda_{1,2} + \lambda_{2,3}}{(\lambda_{1,2} + 1)^2(\lambda_{2,3} + 1)} & \frac{\lambda_{2,3} + 1}{(\lambda_{2,3} - 1)^2(\lambda_{1,2} - 1)} \\
    \frac{1}{-(\lambda_{1,2} + 1)} & \frac{-\lambda_{1,2} + \lambda_{2,3}}{(\lambda_{1,2} + 1)^2(\lambda_{2,3} + 1)} & \frac{\lambda_{2,3} + 1}{(\lambda_{2,3} - 1)^2(\lambda_{1,2} - 1)} \\
    \frac{1}{(\lambda_{1,2} + 1)^2(\lambda_{2,3} + 1)} & \frac{-\lambda_{1,2} + \lambda_{2,3}}{(\lambda_{1,2} + 1)^2(\lambda_{2,3} + 1)} & \frac{\lambda_{2,3} + 1}{(\lambda_{2,3} - 1)^2(\lambda_{1,2} - 1)}
\end{pmatrix}
\begin{pmatrix}
    v_1 \\
    v_2 \\
    v_3
\end{pmatrix}.
\]

We denote the matrix by $(k_{ij})$. For the determinant one obtains

\[
\det((k_{ij})) = -8(\lambda_{1,2} + \lambda_{2,3})^2.
\]

(If $\det((k_{ij})) = 0$ there exists a permutation of the three indices such that $\det((k_{ij})) \neq 0$.)

Let

\[
\begin{align*}
    A_{12} &= 2u_1u_2 - 2q^2u_2u_1, \\
    A_{13} &= 2u_1u_3 - 2q^2u_3u_1 + (1 - q^2)(\lambda_{1,2} - 1)(\lambda_{2,3} + 1)u_2^2, \\
    A_{23} &= 2u_2u_3 - 2q^2u_3u_2.
\end{align*}
\]
Because of (13) we have $A_{12} = A_{13} = A_{23} = 0$. We form the vanishing expressions

\[ B_{12} := (k_{33}A_{12} - k_{23}A_{13} + k_{13}A_{23})/\det(K), \]
\[ B_{13} := (-k_{32}A_{12} + k_{22}A_{13} - k_{12}A_{23})/\det(K), \]
\[ B_{23} := (k_{31}A_{12} - k_{21}A_{13} + k_{11}A_{23})/\det(K). \]  

(16)

We insert (14). We obtain with a computer calculation

\[ B_{12} = -(1 + q^2)(v_2v_1 - v_1v_2) + (1 - q^2)(v_1^2 - v_2^2 + \lambda_{12}(v_1 - v_2)^2) = 0 \]
\[ B_{13} = -(1 + q^2)(v_3v_1 - v_1v_3) + (1 - q^2)(v_1^2 - v_3^2 + \lambda_{12}\lambda_{23}(v_1 - v_3)^2) = 0 \]
\[ B_{23} = -(1 + q^2)(v_3v_2 - v_2v_3) + (1 - q^2)(v_2^2 - v_3^2 + \lambda_{13}(v_2 - v_3)^2) = 0 \]

Using (2) the formulas (9) follow. •

**Proposition 7** Let (2) be satisfied. Then $A_{\lambda'}$ is polynomial.

**Proof.** The transformation between the $v_i$ and $u_i$ does not change the dimensions of the subspaces of homogeneous polynomials. We have to show the polynomiality of the algebra (13).

By the Diamond lemma (cf. Ref. 4) this algebra has the PBW-property (and is therefore ”polynomial”) if and only if the ”overlap”

\[ (u_3u_2)u_1 - u_3(u_2u_1) \]

gives zero when it is reduced by means of (13) to a linear combination of ordered monomials (cf. Ref. 4 for details). Let $a := \frac{1}{q^2}$ and $\epsilon := (\frac{1}{q^2} - 1)(\lambda_{1,2} - 1)(\lambda_{2,3} + 1)$. We obtain

\[ (u_3u_2)u_1 - u_3(u_2u_1) = au_2(u_3u_1) - a(u_3u_1)u_2 = a^2(u_2u_1)u_3 + a\epsilon u_2^3 - a^2u_1(u_3u_2) - a\epsilon u_2^3 = a^3u_1u_2u_3 - a^3u_1u_2u_3 = 0. \]

Finally we consider the exceptional structure $A_{\overline{E}}$. It is given by

\[ v_jv_i = \frac{v_i^2 + q^2v_j^2}{1 + q^2}, \quad i < j, \]

(17)

(cf. Example 9).
Proposition 8  The Algebra $A_{E}^{R}$ is polynomial.

Proof. Consider the transformation
\[
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix} =
\begin{pmatrix}
q^2 & q^2 & q^2 \\
1 & q^2 & q^2 \\
1 & 1 & q^2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}.
\]

(18)

We denote the matrix by $(k_{ij})$. We obtain for the determinant
\[
\det((k_{ij})) = q^2(q^2 - 1)^2 \neq 0.
\]

Let
\[
A_{12} := -(1 + q^2)v_2v_1 + v_1^2 + q^2v_2^2,
\]
\[
A_{13} := -(1 + q^2)v_3v_1 + v_1^2 + q^2v_3^2,
\]
\[
A_{23} := -(1 + q^2)v_3v_2 + v_2^2 + q^2v_3^2.
\]

(19)

Because of (17) we have $A_{12} = A_{13} = A_{23} = 0$. We form the vanishing expressions $B_{12}, B_{13}, B_{23}$ (cf. (16)) and insert (19). A calculation yields
\[
B_{12} = -(1 + q^2)u_2u_1 = 0,
\]
\[
B_{13} = u_1u_2 + u_2u_1 + u_1u_3 - q^2u_3u_1 = 0,
\]
\[
B_{23} = -u_1u_2 - u_2u_1 + u_2u_3 - q^2u_3u_2 = 0.
\]

Therefore the algebra has generators $u_1, u_2, u_3$ and relations
\[
u_2u_1 := 0,
\]
\[
u_3u_1 := \frac{1}{q^2}(u_1u_3 + u_1u_2),
\]
\[
u_3u_2 := \frac{1}{q^2}(u_2u_3 - u_1u_2).
\]

(20)

The monomials on the left side are lexicographically greater than the monomials on the right side. Therefore it is sufficient to show, that the “overlap”
\[
(u_3u_2)u_1 - u_3(u_2u_1)
\]
gives zero when it is reduced by means of (20) to a linear combination of ordered monomials (cf. Proof of Proposition 7). We have
\[(u_3 u_2) u_1 - u_3 (u_2 u_1)\]
\[= \frac{1}{q^2} (u_2 (u_3 u_1) - u_1 (u_2 u_1))\]
\[= \frac{1}{q^4} ((u_2 u_1) u_3 + (u_2 u_1) u_2) = 0.\]

F. Projective Coordinates
As the algebra of noncommutative projective coordinates $B_\mu$ for $A_\lambda$ we denote the $U_q(sl(2,\mathbb{R}))$-module algebra generated by elements $x_i, y_i, y_i^{-1}, (\sum \alpha_i x_i y_i^{-1})^{-1}$, $i, j \in I$ and relations
\[y_i y_i^{-1} = y_i^{-1} y_i = 1,\]
\[(\sum \alpha_i x_i y_i^{-1})^{-1} (\sum \alpha_i x_i y_i^{-1}) = (\sum \alpha_i x_i y_i^{-1}) (\sum \alpha_i x_i y_i^{-1})^{-1} = 1\]
and
\[x_i y_i = q y_i x_i, \quad \text{(21)}\]
\[x_j x_i = \mu_{ij}^{(1)} x_i x_j, \quad \text{(22)}\]
\[y_j y_i = \mu_{ij}^{(1)} y_i y_j, \quad \text{(23)}\]
\[x_j y_i = \mu_{ij}^{(2)} y_i x_j + (\mu_{ij}^{(1)} - \frac{1}{q} \mu_{ij}^{(2)}) x_i y_j, \quad \text{(24)}\]
\[y_j x_i = \mu_{ij}^{(2)} x_i y_j + (\mu_{ij}^{(1)} - q \mu_{ij}^{(2)}) y_i x_j \quad \text{(25)}\]
with
\[\mu_{ji}^{(1)} = \frac{1}{\mu_{ij}^{(1)}} \quad \text{and} \quad \mu_{ji}^{(2)} = \frac{1}{\mu_{ij}^{(2)} \left( \frac{\mu_{ij}^{(1)} - \mu_{ij}^{(2)}}{\mu_{ij}^{(1)} (q + \frac{1}{q} - \mu_{ij}^{(1)})} \right)}.\]
The $U_q(sl(2,\mathbb{R}))$-module structure on $B_\mu$ is given by
\[E(x_i) = q^{\frac{1}{2}} y_i, \quad E(y_i) = 0,\]
\[F(x_i) = 0, \quad F(y_i) = q^{-\frac{1}{2}} x_i,\]
\[K(x_i) = q^{-\frac{1}{2}} x_i, \quad K(y_i) = q^{\frac{1}{2}} y_i,\]
and equations (4), (5), (6).
Lemma 9  The elements $v_i := q^\frac{1}{2}x_i y_i^{-1}$, $i \in I$ satisfy (9) with

$$\lambda_{ij} = \frac{q^2 - 2q^{(1)}_{ij} + 1}{1 - q^2} \quad \text{or} \quad \mu_{ij} := \frac{\mu_{ij}^{(1)}}{\mu_{ij}^{(2)}} = \frac{1 + q^2 + \lambda_{ij}(q^2 - 1)}{2q}.$$ 

and $U_q(sl(2, \mathbb{R}))$ acts on $v_i$ according to (3).

This can be checked by an explicit calculation. Therefore we can map $A_\lambda$ onto a submodul of $B_\mu$. We call $x_i, y_i$ noncommutative coordinates of $v_i$.

Remark: One can show that the algebra of projective coordinates is polynomial if and only if $\mu_{ij}^{(1)} = q^{\pm 1}\mu_{ij}^{(2)}$, i.e. $\lambda_{ij} = \pm 1$. Therefore for $\lambda_{ij} \neq \pm 1$ the polynomiality of the algebra $A'_\lambda$ does not yield the polynomiality of the algebra of projective coordinates.
IV. Quantum cross ratios

A. Invariant elements

In this section we investigate a class of invariant elements in $A_\lambda$.

**Definition 5** We say that $x \in A_\lambda$ (or $B_\mu$) is invariant, if

$$E(x) = 0, \quad F(x) = 0, \quad K(x) = x.$$  

Following Ref. 1 we define

$$(ij) := q^{-\frac{1}{2}}x_iy_j - q^{\frac{1}{2}}y_ix_j, \quad \text{and} \quad [ij] := v_i - v_j.$$  

One can check that $(ij)$ is invariant and we have

$$(ij) = y_i[ij]y_j.$$  \hspace{1cm} (26)$$

Now we use the invariants $(ij) \in B_\mu$ to combine invariants of $A_\lambda$. We define

$$C_{ijkl} := (il)(kl)^{-1}(kj)(ij)^{-1}.$$  

**Proposition 10** (i) $C_{ijkl} \in B_\mu$, $(i, j, k, l \in I)$ is invariant.

(ii) We can represent $C_{ijkl}$ by elements $v_i$ (i.e. $C_{ijkl} \in A_\lambda$) and we have the formula

$$C_{ijkl} := (q + q^{-1} - \mu_{il})(v_i - v_l)((q + q^{-1} - \mu_{ik})v_k - (q + q^{-1} - \mu_{il})v_l + (\mu_{ik} - \mu_{il})v_i)^{-1} \times$$

$$(q + q^{-1} - \mu_{ik})v_k - (q + q^{-1} - \mu_{ij})v_j + (\mu_{ik} - \mu_{ij})v_i)(q + q^{-1} - \mu_{ij})^{-1}(v_i - v_j)^{-1}. \hspace{1cm} (27)$$

**Proof.** The element $(ij)^{-1}$ and products of invariant elements are invariant because of (4), (5), (6). The assertion follows.

(ii) From (21)-(25) follow the commutation rules

$$y_i v_i = q^{-1} v_i y_i, \quad y_i v_j = ((q + q^{-1} - \mu_{ij})v_j + (\mu_{ij} - q)v_i)y_i.$$  \hspace{1cm} (28)$$

Further, because of (26) we have

$$C_{ijkl} = y_i[il][kl]^{-1}[kj][ij]^{-1}y_i^{-1}. \hspace{1cm} (29)$$

The assertion follows, if we move $y_i$ to the right applying (28).
Proposition 11 In analogy to the classical case we have

\[
C_{iklj} = \frac{1}{C_{ijkl}}, \quad C_{ijkl} = 1 - C_{iklj}, \quad C_{ikjl} = \frac{C_{ijkl}}{C_{ijkl} - 1}, \\
C_{iklj} = \frac{1}{1 - C_{ijkl}}, \quad C_{ijlk} = 1 - \frac{1}{C_{ijkl}}.
\]

Proof. One checks the equations \(C_{iklj}C_{ijkl} = 1\) and \(C_{ijlk} + C_{ijkl} = 1\) using (29) and the rule \([ik] = [ij] + [jk]\). The five identities can combined from them. •

Remark: In the general case, the elements \(C_{ijkl}\) are not \(*\)-invariant and the identities \(C_{ijkl} = C_{lkji} = C_{klij} = C_{jilk}\) are in general not valid (see below).

B. The cross ratio

We restrict the consideration to the structure \(A_1^R\) of Example 7, i.e. \(\lambda_{ij} = 1\). (If \(|\lambda_{ij}| = 1\), we can reorder the indices such that \(\lambda_{ij} = 1\) (cf. Example 1).)

We can choose the algebra of projective coordinates with \(\mu^{(1)}_{ij} = q^2, \mu^{(2)}_{ij} = q\) (cf. Lemma 9), i.e.

\[
\begin{align*}
  x_iy_i &= qy_ix_i, \\
  x_jx_i &= q^2x_ix_j, \\
  y_jy_i &= q^2y_ix_i, \\
  y_jx_i &= qy_ix_j + (q^2 - 1)x_iy_j, \\
  y_jx_i &= qx_iy_j.
\end{align*}
\]

(30)

Lemma 12 Let \(i < j < k < l\) be four ordered indices. Then we have

\[
\begin{align*}
  (kl)(ij) &= q^6(ij)(kl), \\
  (jk)(il) &= (il)(jk), \\
  (jl)(ik) &= q^4(ik)(jl) + (q^6 - q^4)(ij)(kl), \\
  (jk)(ij) &= q^4(ij)(jk), \\
  (ik)(ij) &= q^2(ij)(ik), \\
  (jk)(ik) &= q^2(ik)(jk), \\
  (ij)^* &= (ij).
\end{align*}
\]
Proof. We prove the fourth rule. Let $i < j < k$. From (30) follows

\[
x_k (x_i y_j - q y_i x_j) = q^2 x_i x_k y_j - q^2 y_i x_k x_j - (q^3 - q) x_i y_k x_j
\]

\[
= q^3 x_i y_j x_k + (q^4 - q^2) x_i x_j y_k - q^4 y_i x_j x_k - (q^4 - q^2) x_i x_j y_k
\]

\[
= q^3 (x_i y_j - q y_i x_j) x_k,
\]

i.e. $x_k(ij) = q^3(ij)x_k$. Similar one derives $y_k(ij) = q^3(ij)y_k$, $x_j(ij) = q(ij)x_j$, $y_j(ij) = q(ij)y_j$. The identity $(jk)(ij) = q^4(ij)(jk)$ follows. The proof of the remaining rules is similar. 

In a similar situation some of the invariant elements $C_{ijkl}$ were first considered in Ref. 1. Let $i < j < k < l$ be four elements of $I$ in increasing order. With $C := C_{ijkl}$ we derive using Proposition 11 and Lemma 12.
\[ C_{ijkl} = C_{klij} = C, \quad C_{jilk} = C_{tkji} = C^{*} = q^{2}C. \]

\[ C_{ilkj} = C_{kjil} = \frac{1}{C}, \quad C_{jlki} = C_{lijk} = C_{ilkj}^{*} = \frac{1}{q^{2}C} = \frac{C_{ilkj}}{q^{2}}, \]

\[ C_{ijkl} = C_{klji} = 1 - C, \quad C_{jikl} = C_{tkij} = C_{ijkl}^{*} = 1 - q^{2}C = q^{2}C_{ijkl} + (1 - q^{2}), \]

\[ C_{ikjl} = C_{kitj} = \frac{C}{C_{ijkl} - 1}, \quad C_{jikl} = C_{ijkl} = C_{ikjl}^{*} = \frac{q^{2}C_{ijkl}}{q^{2}C_{ijkl} - 1} = \frac{q^{2}C_{ijkl}}{1+(q^{2}-1)C_{ijkl}}, \]

\[ C_{ijkl} = C_{klij} = \frac{1}{1-C}, \quad C_{ijkl} = C_{ijkl} = C_{ijkl}^{*} = \frac{1}{1-q^{2}C} = \frac{C_{ijkl}}{q^{2}+(1-q^{2})C_{ijkl}}, \]

\[ C_{ijkl} = C_{klij} = 1 - \frac{1}{C}, \quad C_{ijkl} = C_{likj} = C_{ijkl}^{*} = 1 - \frac{1}{q^{2}C} = \frac{1}{q}C_{ijkl} + (1 - \frac{1}{q^{2}}). \]

The Remark after Proposition 11 indicates that \( C_{ijkl} \) is not a good candidate of a Quantum cross ratio. Now we make the following observation:

**Lemma 13** Let \( i, j, k, l \) be four indices in arbitrary order and let \( C_{ijkl}^{*} = f(C_{ijkl}, q^{2}) \), where \( f \) is a certain rational function. Then \( f(C_{ijkl}, q) \) is a \( * \)-invariant element of \( A_{\lambda}^{R} \).

This can be checked with the above formulas. Now we give the following definition.

**Definition 6** We say that the \( * \)-invariant element \( f(C_{ijkl}, q) \) is the Quantum cross ratio of the Quadrupel \( (v_i, v_j, v_k, v_l) \). We use the notation \( C_{ijkl} \).

The elements \( C_{ijkl} \) have the desired properties. They tend to the classical cross ratio for \( q \to 1 \) and for \( i < j < k < l \) we obtain
\[ C_{ijkl} = qC = q[il][kl]^{-1}[kj][ij]^{-1}, \]
\[ C_{ikjl} = \frac{1}{C_{ijkl}} = \frac{1}{q} = \frac{1}{q} C_{iklj}, \]
\[ C_{ijlk} = 1 - C_{ijkl} = 1 - q = qC_{ijlk} + (1 - q), \]
\[ C_{ikjl} = \frac{C_{ijkl}}{C_{ijkl} - 1} = \frac{qC}{qC - 1} = \frac{qC_{ikjl}}{1 + (q - 1)C_{ikjl}}, \]
\[ C_{iklj} = \frac{1}{1 - C_{ijkl}} = \frac{1}{1 - qC} = \frac{C_{iklj}}{q + (1 - q)C_{iklj}}, \]
\[ C_{iljk} = 1 - \frac{1}{C_{ijkl}} = 1 - \frac{1}{qC} = \frac{1}{q} C_{iljk} + (1 - \frac{1}{q}), \]
and
\[ C_{ijkl} = C_{lkji} = C_{klji} = C_{jilk}. \]

C. The Euclidean distance
In the classical case an Euclidean distance on \( PIR \) can be defined, if we fix the three points \( t_0 \) (zero-point), \( t_1 \) (one-point) and \( t_\infty \) (the point at infinity). The distance \( d(t_j, t_i) \) of two points \( t_j, t_i \) is given by
\[ d(t_j, t_i) := C_{0,1,\infty,j} - C_{0,1,\infty,i}. \]
This observation leads us to a notion of a noncommutative Euclidean distance. Fix three different generators \( v_0, v_1 \) and \( v_\infty \) of \( A_1^R \). Then the noncommutative distance of the generators \( v_i \) and \( v_j \) is given by the \(*\)-invariant element
\[ D_{v_0, v_1, v_\infty}(v_j, v_i) := C_{0,1,\infty,j} - C_{0,1,\infty,i}. \]

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